Ward–Schwinger–Dyson equations in $\phi^3_6$ Quantum Field Theory

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Abstract

We develop a system of equations for the propagators and three point functions of the $\phi^3$ quantum field theory in six dimensions. Inspired from a refinement by Ward on the Schwinger–Dyson equations, the main characteristics of this system are to be formulated purely in terms of renormalized quantities and to give solutions satisfying renormalisation group equations. These properties were difficult to get together, due to the overlapping divergences in the propagator. The renormalisation group equations are an integral part of any efficient resolution scheme of this system and will be instrumental in the study of the resurgent properties of the solutions.

It is our belief that this method can be generalized to the case of gauge fields, shedding some light on their quantum properties.

Keywords: Renormalisation, Schwinger-Dyson equation

Introduction

In this work we introduce a system of generalized Schwinger–Dyson equations for a massless $\phi^3$ model in 6 dimensions.

Usual works on Schwinger–Dyson equations miss one of their fundamental interest for us, the possibility to solve them purely in terms of renormalized Green functions. This possibility was first encountered in the case of a linear Schwinger–Dyson equation in [1], developed in [2] and found a powerful illustration in [3], from which a whole series of refinements have emerged, see e.g., [4, 5, 6].

The Schwinger–Dyson equations start from the implication of field equations on the field correlators and involve at first a bare vertex, but in many cases can be converted to ones depending only on dressed vertices. However, this is not possible for the propagator corrections, which are written asymmetrically, with a bare vertex on one side and a dressed vertex on the other side. In the language of renormalisation Hopf algebras, the resulting combinatorial Schwinger–Dyson equation is not based on a Hopf algebra cocycle. It has been proposed to put suitable combinatorial factors to ensure this cocycle property [7], but this does not solve all problems. First of all, as has been remarked by K. Yeats in her thesis [8], this cannot solve the problem in theories where different particles can run in the one loop corrections for a single propagator, as in QCD. Furthermore, even in cases where such a formulation is valid at the combinatorial level, the dependence of the combinatorial factors on the internal structure of the graphs precludes its transformation into an analytic Schwinger–Dyson equation, where one simply use the values up to some order of the vertex functions in the right hand side of the equation. The problem is directly linked to the presence of overlapping divergences. The Schwinger–Dyson equations build complex diagrams by assembling simpler parts, but in the presence of overlapping divergences, the obtained diagram have alternative ways of being dismantled. Since each of the ways to disassemble a diagram imply a possible chain of counterterms, the counterterms associated to the constituents are not sufficient for a full preparation of the diagram: subdivergences remain.

The solution we adopt has quite a long history, since we found its first expression in a paper by J.C. Ward [9], completed in [10]. The most complete exposition in these early times appears in a conference report of Symanzik [11] and some further application of these ideas appear for example in [12] and especially [13], where the case of QCD is worked out. However the complexity of the proposed solution did not really allow for applications. An important further advantage of
these methods is that they are naturally compatible with the Ward or Slavnov–Taylor identities expressing gauge invariance, when no consistent truncation of the usual Schwinger–Dyson equations can be found with this property. This will not appear in this paper, where we limit ourselves to scalar interactions, but we hope to go back to this question in future works.

We have two additional ingredients in our proposal. First, since the general analytic dependence of three point functions on the kinematic invariants rapidly becomes cumbersome, we will reduce to the use of a single scale version, which has the same type of functional dependence as the propagator. We will sketch a systematic computation of the corrections to this first approximation. The second one is that renormalization group equations are a direct consequence of our Ward–Schwinger–Dyson equations. The proof we give depends only on the fact that we deal with a massless theory and apply also to the Wess–Zumino model considered in previous works. It has the double advantage to simplify the proof with respect to the one used in [3] and to make it independent on previous works on renormalisability. This avoids to make a detour by the corresponding combinatorial Schwinger–Dyson equations and their renormalisation. Since the renormalization group equations were an essential part in our analysis of the asymptotic properties of the perturbative solutions in the case of the Wess–Zumino model, we surmise that the same kind of analysis can be done for the singularities of the Borel transform as in [14] and that, as in [5], higher order terms in the Ward–Schwinger–Dyson equations only bring higher order corrections to the properties of these singularities.

The rest of the paper is structured as follows: in section §1 we will present the Schwinger–Dyson equations we will be studying and introduce the problem of overlapping divergences that until now has precluded the use of Schwinger–Dyson equations in the presence of vertex corrections; in section §1.3 following [15, 16] we will present a deformation of Feynman rules that resembles the IR rearrangement used in QCD to have single-scale vertices, paving the way to treat them as solutions of renormalisation equation; section §2 is a curious intermezzo that tangles Schwinger–Dyson equations and renormalization group equations providing us with a recipe for the $\beta$-function; and in section §4.3 and §5 we will provide results for the various anomalous dimensions and thus reconstructing up to order $a^2$ the 2-point and 3-point functions; then we conclude.

1 The Ward–Schwinger–Dyson equations

After the introduction of Hopf algebra methods to deal with the combinatorics of renormalisation, it has been recognized that Schwinger–Dyson equations can be formulated in terms of 1-cocycles $B_+$ in Hochschild cohomology [17, 18]. In the case of the Hopf algebra of (decorated) trees, $B_+(F)$ is the tree made by putting the roots of the trees in the forest $F$ as direct descendants of a new root (which in the decorated case, will have the decoration associated to $B_+$). One obvious question is whether the solution of the said Schwinger–Dyson equations satisfies renormalization group equations. An important stepping stone is to know whether the series coefficients of the solution would generate a sub Hopf algebra of the Hopf algebra of graphs. The possible forms of such Schwinger–Dyson equations were investigated in [19] and one of the possible class of Schwinger–Dyson equations seems to correspond to the situation in quantum field theory. They indeed have variables which can be interpreted as propagators or vertex functions and the different $B_+$ act on products of powers of the variables, with the exponents corresponding to the number of vertices and propagators in a diagram.

However, Feynman rules for the evaluation do not apply to rooted trees but to Feynman diagrams which do not always have a nice translation in a tree structure, due to the overlapping divergences. To make things more concrete, we start from the Schwinger–Dyson equations for our model $\phi^3_6$:

\begin{align}
\begin{array}{c}
\text{black} \quad \times \quad \text{gray} \quad \times
\end{array}
& =
\begin{array}{c}
\text{black} \quad \times \quad \text{gray} \quad \times
\end{array}
\quad \frac{1}{2}
\end{align}

(1)

\begin{align}
\begin{array}{c}
\text{black} \quad \times \quad \text{gray} \quad \times
\end{array}
& =
\begin{array}{c}
\text{black} \quad \times \quad \text{gray} \quad \times
\end{array}
+ 1
\end{align}

(2)

where the black and gray elements denote respectively dressed propagators and vertices. The big oval denotes a four particle kernel, the Bethe–Salpeter kernel, which can be expanded as a sum of two-particle irreducible graphs. In the following equation, the internal lines denote the full
propagator, without an additional dot for the sake of readability:

\[
\begin{align*}
\mathcal{G} & = \mathcal{G}_0 + \frac{1}{2} \mathcal{G}_1 + \mathcal{G}_2 + \cdots \\
\end{align*}
\]

(3)

The factor before the second diagram is a symmetry factor linked to the invariance of this diagram through the exchange of its two outputs. Since we will only make explicit computations at rather low order, the first term will be generally sufficient. Combining the three previous equations, one may obtain solutions as series of one-particle irreducible graphs, but it is not the path we will follow.

The asymmetry in the first equation might puzzle at first glance. Indeed, it is not possible to write an equation with a single diagram and a constant combinatorial factor. To restore its symmetry we use the expression in (2) at the cost of introducing a second object:

\[
\begin{align*}
\mathcal{G} & = \mathcal{G}_0 - \frac{1}{2} \mathcal{G}_1 + \frac{1}{2} \mathcal{G}_2 \\
\end{align*}
\]

(4)

Each term in this equation produces multiple counting as can be seen already from the first few terms of their expansion:

\[
\begin{align*}
\mathcal{G}_0 & = 1 + 2 + 3 + 2 + \cdots \\
\mathcal{G}_1 & = 2 + 1 + \cdots \\
\mathcal{G}_2 & = 3 + 2 + \cdots , \\
\end{align*}
\]

but the combination of the two restore the proper count.

1.1 The problem of overlapping divergences

A Feynman diagram \( \Gamma \) could have divergent evaluation despite a negative superficial degree of divergence \( \omega(\Gamma) \). This happens whenever a sub-graph \( \gamma \subset \Gamma \) is divergent, thus with \( \omega(\gamma) \geq 0 \). It was the main contribution of Bogolubov [20] to show that a recursive subtraction of counterterms could suppress such divergences. Alternatively, in the context of Schwinger–Dyson equations, we make use of renormalised vertices.

But how do we generalize this approach to the case of overlapping divergences? Take as an example

\[
\begin{align*}
\mathcal{G} & \quad \text{or} \quad \\
\end{align*}
\]

with its two possible subdivergence structures, which overlap. We would prefer a simple tree structure, clearly separating all primitive contributions in an expression while avoiding double counting. If we look at equation (4), the first correction is fine, with a simple loop integration if we replace all vertices with their renormalized value, but the second one has visibly overlapping subdivergences, and the compensations between the two show that each one should be more difficult to evaluate than it seems.

The rather old solution, first proposed by Ward [9] is to derive with respect to the external momentum. The remark that the derivative of a propagator has better ultraviolet behavior is quite old and is at the base of the Bogolubov method [20]. Once the subdivergences are properly dealt with, a sufficient number of derivations makes the diagram convergent and its renormalized evaluation is obtained through integration of this finite result. The integration constants correspond exactly to the parameters of the Lagrangian and their arbitrariness corresponds to the choice of renormalisation scheme.

For us, it will be sufficient that the inclusion of the derivative of a propagator in a three point function makes it superficially convergent. If we mark the propagator with a derivation by a heavy line, one see that now the following diagram has only one possible subdivergence:

\[
\begin{align*}
\end{align*}
\]

(6)
The highlighted line splits the subdivergences, in the sense that there are those coming before it, and those coming after it in the flow of momentum through the diagram. For the derivative of the two-point function, there are many possible primitive diagrams beyond the simple one loop one. Here are some examples

\[ (7) \]

It remains however to know how these diagrams contribute to the derivative of the two-point function.

The first step is to derive the equation (8) for the two-point function:

\[ \mu := \partial \mu - \frac{1}{2} - \frac{1}{2} \] (8)

This equation involves the derivative of the vertex, for which one obtain an equation by deriving the equation (2):

\[ (9) \]

where the squares represent the derived objects and the round ones the original ones. We can now use equation (2) to re-express the right bare vertices in the derived propagator (8) and obtain

\[ (10) \]

which ultimately, by use of equation (9) in the first term of the parenthesis, simplifies to:

\[ (11) \]

If we expand the last term by deriving the expression for the four-point kernel (3) we have

\[ (12) \]

Whenever we have a derivative of a vertex, we should make use of equation (9) to obtain a more explicit value, with the remaining parts involving derivatives of the vertex or the 4-point kernel having to be recursively expanded. The final objects we want to evaluate should have the derivative only on one propagator. We could in this way obtain certain of the diagrams in equation (7) with determined weights. However, since the first contributions beyond the one-loop one appear at three loop order, we will not need them in this work, but should keep in mind that the full solution will require an infinite set of primitives.

This concludes the determination of the equations that we will use. As in many cases in quantum field theory, it is not clear whether this derivation can be made rigorous, since we should at least put some regularization. In a sense, this is not really important if we can show that the solutions of these equations have the properties of the Green functions of a quantum field theory.

### 1.2 Renormalised propagators

As renormalised quantities, all our \( n \)-point functions depend on the coupling constant \( g \) and a renormalisation momentum scale \( \mu \). The two point function \( P \) has more natural variables: \( a := g^2 / (2\sqrt{\pi})^D \) since all the graphs involved have an even number of vertices, and \( L := \log(p^2 / \mu^2) \) since the general solution to a Callan-Symanzik equation is a series in \( g \) with coefficients that are polynomials in \( L \) (see for instance [21], chap 5). We can represent \( P(a, L) \) as

\[ (13) \]
From now on, we will not indicate the evaluation at $x = 0$ but it remains implied. The derivation $D$ with respect to $L$ will be important for us:

$$D G(a, L) = \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(a)}{n!} L^n. \quad (15)$$

In the context of the corresponding pseudodifferential operators, the iterations of $D$ have a simple expression

$$G(a, \partial_x) (x^m \cdot f) = (D^n G)(a, \partial_x) f$$

as we can easily show by direct inspection

$$G(a, \partial_x) (x^m \cdot f) = \sum_{n=0}^{\infty} \frac{\gamma_n(a)}{n!} \partial_x^n (x^m \cdot f) = \sum_{n=0}^{\infty} \frac{\gamma_n(a)}{(n-m)!} \partial_x^{n-m} f$$

using that $\partial_x^m x^m = m! \delta_{km}$ when evaluated at $0$ and the binomial expansion of the higher derivatives of a product.

The Ward-Schwinger-Dyson equations depend on other quantities for which we will also need convenient expression. First we have the relation between the 2-point function (the propagator) and the corresponding vertex function (the 1PI two-point function)

$$P(a, L) \cdot \Gamma_2(a, L) = 1,$$

where the 1 comes from the fact that we are working in Euclidean space. The first few terms of $\Gamma_2(a, L)$ are given by

$$\Gamma_2(a, L) = p^2 \left( 1 - \gamma_1 L + (2 \gamma_1^2 - \gamma_2) L^2/2 + O(L^3) \right). \quad (19)$$

We will need the derivatives with respect to $p_\nu$ of both $P$ and $\Gamma_2$. This derivation will be abbreviated to $\partial^\nu$ in many cases. We have

$$K^\nu(a, L) := \frac{\partial}{\partial p_\nu} P(a, L) = \frac{2 p_\nu}{(p^2)^2} (D G(a, L) - G(a, L)) \quad (20)$$

since $\partial^\nu L = 2 p_\nu / p^2$. This can be expressed in terms of the $\gamma_n$:

$$K^\nu(a, L) = \frac{2 p_\nu}{(p^2)^2} \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(a) - \gamma_n(a)}{n!} L^n. \quad (21)$$

Likewise we define the derivative of the $\Gamma_2$ function,

$$C^\nu(a, L) := \partial^\nu \Gamma_2(a, L) = 2 p_\nu \left( D G^{-1}(a, L) + G^{-1}(a, L) \right) \quad (22)$$

with its coefficients obtained by adding two subsequent coefficients in the development of $G^{-1}$. The derivation of equation (23) gives a relation between $K^\nu$ and $C^\nu$

$$K^\nu(a, L) = -P(a, L) \cdot C^\nu(a, L) \cdot P(a, L) \quad (23)$$

which graphically becomes

$$\nu \quad = \quad - \quad \nu \quad \bullet \quad \nu \quad \bullet \quad \nu \quad (24)$$
1.3 Infrared rearrangement

We still need a way to treat the renormalised vertex function $\Gamma_3$. The full three point function depends on all the possible scalar products of the external momenta minus the delta function condition, so in our case 3 variables. Their functional dependence on these kinematic invariants are unknown at higher loop order, but even the known cases do not easily allow for the evaluation of diagrams including vertices with the full dependence on all invariants.

A guide for a solution is that the renormalisation functions only depend on the three point function evaluated at a multiple of the renormalisation condition. We will therefore try to only use this special case of the three-point function, which will have the same kind of functional dependence as the two-point one. This will use the trick called infrared rearrangement, where one transform a diagram in one with the same subdivergences, but with simpler evaluation. A nice characteristics of this six-dimensional theory is that we do not introduce infrared divergences in the process.

In the definition of the vertex in equation (2) we consider the case where one incoming momentum is set to zero. In Euclidean space, the null momentum condition is equivalent to the one of a zero norm for the momentum. A dotted line signals the zero momentum input.

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram1.png}
\end{array}
\]

Since the bare vertex term is the same in the two equations (2) and (25), we have

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram2.png}
\end{array}
\]

We would like to express the vertices appearing in, e.g., equation (11) using this equality, but this seems to entail strong constraints on the rest of the diagram. Inspired by the work of [15], we consider a deformation of the Feynman diagram where the line which one would like to put to zero is instead routed to the side of the three-point subdiagram:

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram3.png}
\end{array}
\]

We will adopt the following convention for the sake of readability

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram4.png}
\end{array}
\]

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram5.png}
\end{array}
\]

to denote the one scale object. It can be characterised by a formal series, again in $\log(p^2/\mu^2)$:

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram6.png}
\end{array}
\]

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram7.png}
\end{array}
\]

with again the convention that the first term of the series $\nu_0$ is set to 1, ensuring the renormalisation condition when $p^2 = \mu^2$. If we look at the difference

\[
\begin{array}{c}
\includegraphics[width=1.0\textwidth]{diagram8.png}
\end{array}
\]

we see that the same subdivergences appear in both terms and thus cancel. Both terms also produce the same leading log term. A particular care should be paid to the third term in equation (27), since it seems to involve a elementary four-particle vertex. Care must be taken on the way the expansion from equation (3) is inserted in it. Since our aim was to deform the vertex to have a single scale object, this should be seen as a 3-point function in which one of the leg has a momentum given by the sum of two different legs in the 4-point function. This means that we will have diagrams which are no longer proper diagrams of the $\phi^4$ theory.
We can now go back to equation (4) and insert equation (27) in it to obtain

\[
\begin{align*}
\left(\begin{array}{c}
\text{\includegraphics{equation4}}
\end{array}\right) &= \left(\begin{array}{c}
\text{\includegraphics{equation5}}
\end{array}\right) + 2 \left(\begin{array}{c}
\text{\includegraphics{equation6}}
\end{array}\right) + \left(\begin{array}{c}
\text{\includegraphics{equation7}}
\end{array}\right)
\end{align*}
\]

(30)

where all propagators are supposed to be full propagators, even if we do not mark them by black circles. We see that the first term will be the dominant one in the expansion in \( a \). The next parenthesis (with a factor of two indicating that you have the drawn diagrams plus their mirror images) will give a correction starting at order \( a^2 \) that will be addressed in section 5. The last group start contributing at order \( a^3 \) and is neglected in this work. We remark that in all these groupings, subdivergences are compensated between the different members, so that they behave as primitive divergences. We will not try to give an all order ansatz for the generation of all the primitive terms which will be generated by the recursive application of equation (27) in the diagrams. We suppose that they could be generated by a combinatorial Schwinger–Dyson equations for the full three-point functions followed by what should look akin to a renormalization where the counterterms are obtained by moving one of the exterior link of the subdiagram to make it single scale.

At each stage of our approximation scheme, we only deal with single scale versions of the vertices, with a logarithmic dependence on one the momentum entering it. This means that in our diagrams, the vertices do not introduce any new analytic difficulty, since they just add logarithmic factors similar to the ones coming from one of the neighbouring lines. New kind of dressed propagators will appear, like

\[
\begin{align*}
\text{\includegraphics{dressed_propagator}}
\end{align*}
\]

Their description in terms of series or pseudo-differential operators will be given just by the multiplication of the associated series \( G \) and \( Y \). Even though we distinguish, at first, the action of these operators by applying them on different dummy variables, in the end a single variable can be associated to the internal line, since the product of exponents of the same \( p^2/\mu^2 \) can be combined in the exponentiation by the sum of the variables. Take as an example the case of the operator \( YGY \), associated to the third example in equation (31): by taking the Cauchy product of the series and by using the sum

\[
\begin{align*}
Y(a, \partial x_1)G(a, \partial x_2)Y(a, \partial x_3)f(x_{123}) &= \sum_{n\geq 0} (v\gamma v) \frac{\partial^n}{\partial x_{123}} f(x_{123})
\end{align*}
\]

Furthermore, we will see in the next section that, since each of the factor satisfies the same kind of renormalisation group equation, the product satisfies a renormalisation group equation which allows for the easy evaluation of its higher orders in \( L \), so that, in fine, solving our equations written at a given order is no more complex than for preceding studies of Schwinger–Dyson equations.

## 2 WSD equations and the renormalisation group

Our strategy to solve the model is to use the Ward–Schwinger–Dyson equations to extract the anomalous dimensions of the 2 and 3 point functions and then generate all their higher orders in \( L \) by means of the renormalisation group equations. This is specially important for the propagator, since our equation (11) does not directly give the anomalous dimension \( \gamma \) as its coefficient of \( L \), but a sum of the two first coefficients of the propagator. Higher order terms in \( L \) will likewise give sums of two terms. It is only because, at a given order in the coupling, the right hand side of the equation is polynomial in \( L \) that we can have definite values for all these coefficients. A non zero value for the higher orders in \( L \) of the inverse propagator \( \Gamma_2 \) would give an \( \exp(-L) \) contribution, which would produce a constant term in \( \Gamma_2 \): such a constant is in contradiction with the renormalisation condition of our massless theory. The fact that a constant term is allowed by our equation can be easily understood, since it is an equation for the derivative of \( \Gamma_2 \).

Deducing the renormalisation group equations from the usual renormalisation procedure is however no longer the option it was in preceding works like [2, 3], since the propagators are deduced
from their derivatives and do no longer correspond to the evaluation of a set of Feynman diagrams. However we will see that the renormalization group equations can be seen as consequences of the Schwinger–Dyson equations, in a rather simple way, without any need for special properties of the combinatorial solution of the Schwinger–Dyson equations. In particular, the $\beta$-function appears naturally as a combination of different anomalous dimensions and an “effective coupling” can be obtained as a combination of two- and three-point functions.

We will start from a rather naive observation: we consider a formal series $A(a, L)$ over the ring of functions in the variable $a$, which is supposed to satisfy a renormalisation group like equation with the anomalous dimension $\gamma_A$:

$$DA(a, L) = (\gamma_A(a) + \beta a \partial_a) A(a, L).$$

We have the following lemma: let $P(D)$ be a polynomial with constant coefficients in the variable $D$, then if $A(a, L)$ satisfies the preceding equation

$$D \circ P(D) A^m(a, L) = (m \gamma_A(a) + \beta a \partial_a) P(D) A^m(a, L).$$

We see that $P(D) A^m(a, L)$ satisfies a renormalization group equations with $m \gamma_A$ as anomalous dimension. This happens simply because the coefficients of $P(D)$ do not depend on $a$ and the $D$ commutes with $(\gamma_A(a) + \beta a \partial_a)$. The lemma could be even generalised if we consider another formal series $B(a, L)$ that satisfies with an anomalous dimension $\gamma_B(a)$ but with the same $\beta$-function. We have

$$D \circ P(D) A^m(a, L) B^n(a, L) = (m \gamma_A(a) + n \gamma_B(a) + \beta a \partial_a) P(D) A^m(a, L) B^n(a, L)$$

and now the anomalous dimension is $m \gamma_A(a) + n \gamma_B(a)$.

We will now apply that lemma to the two series we have used to represent the 2 and the 3 point functions, respectively $G(a, L)$ and $Y(a, L)$. Supposing that they satisfy renormalisation group equations with anomalous dimensions $\gamma$ and $\nu$ up to some order in $a$ and $L$, we will show that the Ward–Schwinger–Dyson equations allow to extend them to a higher order.

In order to make the formulas less cumbersome, we will write $G$ instead of $G(a, \partial_x)$, with the dependency on $a$ understood. We can write the general Schwinger-Dyson equation as a series of primitive graph contributions:

$$C^{\nu}(a, L) = 2p_j \sum_n a^n \sum_{p \in \mathcal{P}_n} \left( \prod_{i=1}^{I-1} G_{x_i} \right) \left( DG_{x_I} - G_{x_I} \right) \left( \prod_{i=I+1}^{I+V} Y_{x_j} \right) e^{-\omega_p L} F_p \{ \{ x_i \}, \mu^2 \}$$

with in the $I$-th position the special link with a derivative of the propagator (marked by a square in the graph). The terms on the right hand side are monomials of pseudo-differential operators applied to some characteristic integrals depending on the chosen model. These integrals are similar to those usually found in the literature, but they are evaluated in fixed dimension and appear with arbitrary decorations of the propagators and not the usual ones, integers eventually shifted by multiples of the dimensional parameter. They are generically divergent for $\omega_p = \sum x_I$ equal to 0, where the evaluation finally takes place, but this divergence is a simple pole which is compensated by taking at least one derivative with respect to $L$. This is coherent with the renormalization approach, since the value of the left-hand part for $L$ null can be fixed at will.

Our aim is to express the derivative with respect to $L$ of equation (35). First of all, we can get rid of the special rôle of $x_I$ by using equation (16) to rewrite $(DG_{x_I} - G_{x_I}) = G_{x_I} (x_I - 1)$ and include the term $(x_I - 1)$, as well as a factor $\omega_p$, to suppress the divergence, intro $F_p$ to define $\tilde{F}_p$. Furthermore let us denote

$$O_x := \prod_{i=1}^{I} G_{x_i} \prod_{j=I+1}^{I+V} Y_{x_j}$$

and rewrite (35) as

$$DC^{\nu}(a, L) = 2p_j \sum_{n \geq 1} a^n \sum_{p \in \mathcal{P}_n} O_x e^{\sum x_I L} \tilde{F}_p \left( x, \mu^2 \right).$$

or equivalently

$$D^2 G^{-1} + DG^{-1} = \sum_{n \geq 1} a^n \sum_{p \in \mathcal{P}_n} O_x e^{\sum x_I L} \tilde{F}_p \left( x, \mu^2 \right).$$
If we now take $\mathcal{D}$ on both sides, we have

$$\mathcal{D}^3 G^{-1} + \mathcal{D}^2 G^{-1} = \sum_n a^n \sum_{p \in \mathcal{P}_n} \mathcal{O}_x \left( \sum x_i \right) e^{\Sigma x_i L \tilde{F}_p} (x, \mu^2)$$

(39)

$$= \sum_n a^n \sum_{p \in \mathcal{P}_n} \sum_k \mathcal{O}_{x_k} \mathcal{D} \mathcal{O}_{x_k} e^{\Sigma x_i L \tilde{F}_p} (x, \mu^2).$$

(40)

Here $\mathcal{O}_{x_k}$ denotes either $G$ or $Y$ according to $k$ and since both of them satisfy a renormalisation group equation with the same $\beta$ according to our recurrence hypothesis, then

$$\sum_k \mathcal{O}_{x_k} \mathcal{D} \mathcal{O}_k = (-\gamma + (3\gamma + 2\nu)n + \beta a \partial_a) \mathcal{O}_x$$

(41)

since at a given order $n$ we have $I = 3n - 1$ and $V = 2n$. Defining $\beta = 3\gamma + 2\nu$ and using that $a \partial_a a^n \mathcal{O}_x = a^n (n + a \partial_a) \mathcal{O}_x$, we have

$$\mathcal{D}^2 C^\nu (a, L) = (-\gamma + \beta a \partial_a) \mathcal{D} C^\nu (a, L).$$

(42)

We need to now recognise that $C$ is as an inverse propagator, so that the minus sign in front of $\gamma$ in the preceding equation is just the case $m = -1$ of equation (43). A similar computation for the three point function would give similarly, using that in this case, at order $n$, we have that $I = 3n$ and $V = 2n + 1$,

$$\mathcal{D}^2 Y (a, L) = (\nu + \beta a \partial_a) \mathcal{D} Y (a, L).$$

(43)

It seems that we are missing the case where the Green functions are taken at $L = 0$, but since the renormalisation conditions imply that the function at $L = 0$ are simply constants independent on $a$, this part of the renormalisation group equations are used to define $\gamma$ and $\nu$. It is the very fact that the Schwinger–Dyson equations can only define the derivative with respect to $L$ of the Green functions, due to the divergence in the constant part, which introduces the possibility of a breaking of the scale invariance of the theory through the introduction of the anomalous dimensions $\gamma$ and $\nu$, which will produce the $\beta$-function. Higher point functions, which cannot be primitively divergent, do not entail the introduction of new renormalisation group functions.

We therefore have accomplished our aim of proving that knowing the renormalization group equation up to a given order in the coupling $a$ allows to get them on the following order and therefore to arbitrary order through the recurrence principle.

The derivation in this section should allow to consider the Schwinger–Dyson equations as an alternative approach to the renormalisation of a theory. The only regularisation we use is a simple consequence of our need to invoke propagators with arbitrary exponents in order to be able to consider the logarithmic corrections to the propagators and vertices, in the spirit of analytic regularisation [22, 23, 24], with the simplification that we only consider it for primitively divergent diagrams with a single pole in the neighborhood of 0. The consideration of a scale invariant, massless theory simplified the argument, but we are confident that the same approach could be used in more general cases.

3 Consistency check: no vertex correction

Before delving in the computations for our model, let us indulge first in a case already studied by one of the author in [4, 25], where no vertex correction is needed. It will be useful to set up notations and to verify that we can recover usual results.

We consider a complex field with an interaction Lagrangian density proportional to $\phi^3 + \phi^4$. In this case we do not have vertex corrections at the one loop approximation and it is consistent to only consider the Schwinger–Dyson equation for the propagator. We start from the equation

$$\Gamma_2 (a, L) = p^2 - \frac{q^2}{2} \int_{k^2} d^4 u \frac{P(a, \log u^2/\mu^2)}{(2\pi)^4} P(a, \log (p + u)^2/\mu^2),$$

(45)
This enforces an equation for the $G^{-1}(a, L)$ series:

$$G^{-1}(a, L) = 1 - \frac{a}{2} G_x G_y e^{(x+y)l} \frac{\Gamma(-1-x-y) \Gamma(2+x) \Gamma(2+y)}{\Gamma(1-x) \Gamma(1-y) \Gamma(4+x+y)}.$$  \hspace{1cm} (46)

This expression is manifestly divergent in the neighbourhood of the origin due to the pole of $\Gamma(-1-x-y)$ for $x+y = 0$. This divergence could be compensated by a mere differentiation with respect to $L$, but other terms can be added. In particular, we can follow \[4\] and add a second derivative with respect to $L$ while in \[25\], a third derivative was used. To simplify notations, $\partial_L$, the partial derivative with respect to $L$, will imply an evaluation at $L = 0$. Applying $\partial_L + \partial_L^2$ to equation (46) result in the multiplication of the second hand by $(x+y) + (x+y)^2$ and we get

$$(\partial_L + \partial_L^2)G^{-1}(a, L) = -\frac{a}{2} G_x G_y H(x, y)$$  \hspace{1cm} (47)

with $H(x, y)$ regular in the neighbourhood of the origin given by

$$H(x, y) = \frac{\Gamma(1-x-y) \Gamma(2+x) \Gamma(2+y)}{\Gamma(1-x) \Gamma(1-y) \Gamma(4+x+y)}.$$  \hspace{1cm} (48)

In our new scheme, we consider an equation for the derivative $C^\nu$, with a divergent second member, so that we still have to do a further differentiation with respect to $L$ to get a finite equation. The left hand side, using the property (22) of $C^\nu$, will also be given as $(\partial_L + \partial_L^2)G^{-1}$. We therefore start from

$$\nu = \nu - \frac{1}{2}$$  \hspace{1cm} (49)

or in integral form

$$C^\nu(a, L) = 2p^\nu - \frac{g^2}{2} \int du (2\pi)^{1/2} P(a, \log u^2/\mu^2) K^\nu(a, \log (p + u)^2/\mu^2).$$  \hspace{1cm} (50)

In this particular one loop case, the result can be obtained simply by taking the derivative $\partial^\nu$ of the equation (46) to obtain

$$C^\nu(a, L) = 2p^\nu \left[ 1 - a G_x G_y e^{(x+y)l} \frac{\Gamma(-1-x-y) \Gamma(2-x) \Gamma(2-y)}{\Gamma(1-x) \Gamma(1-y) \Gamma(4+x+y)} \right] L.$$  \hspace{1cm} (51)

We still have a divergence coming from the pole in $\Gamma(-x-y)$, which can be cancelled by a differentiation with respect to $L$. Using equation (22) to express $C^\nu$, we have

$$\partial_L^2 G^{-1} + \partial_L G^{-1} = -\frac{a}{2} G_x G_y H(x, y)$$  \hspace{1cm} (52)

with the same $H(x, y)$ defined in equation (48), so that we obtain the same equations for the propagator as in the previous case. In the case of all other graphs, simple primitive ones described in equation (7) or combinations coming from the corrections to the infrared rearrangement introduced in equation (30), such a simple derivation is not possible and therefore we present a direct computation in appendix \[A\] which can be generalised to these other cases. The Mellin transform of the graph with a propagator $p^\nu(p^2)^{y-2}$ is the same function appearing in equation (51), but for a $\Gamma(2-y)$ in the denominator. However the expression of $K^\nu$ in terms of $G$ in equation (20) introduces a factor $y-1$ which gives back the preceding result.

To solve equation (51), we must express the derivatives of the inverse propagator in terms of $\gamma$ with the help of equation (19) and use the renormalisation group equation for $G(a, L)$, which can be spelted out as the following recursion for the coefficients

$$\gamma_{n+1} = \gamma(1 + 3a \partial_a) \gamma_n$$  \hspace{1cm} (53)

since $\beta = 3\gamma$ when there is no correction to the vertices. This ultimately brings us to

$$\gamma = \gamma(1 - 3a \partial_a) \gamma + \frac{a}{2} G_x G_y H(x, y).$$  \hspace{1cm} (54)

Using the first derivative of $H(x, y)$, we have

$$G_x G_y H(x, y) = \left( \frac{1}{6} - \frac{5}{18} \gamma + \ldots \right)$$  \hspace{1cm} (55)

and the solution starts as

$$\gamma = \frac{1}{12} a - \frac{11}{432} a^2 + \ldots$$  \hspace{1cm} (56)
4 Computations

It is now time to apply all the machinery to the full $\phi^3$ theory and check that it can reproduce known results. This will involve calculating corrections to the propagator and to the vertex.

4.1 Propagator: The Cat

Let’s consider the first non-trivial term in the full Schwinger-Dyson equation (11):

\[
\int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} \Gamma_3(-u)P(u)\Gamma_3(u)K^{\nu}(u+p)
\]

The integral is identical to the one encountered in the previous section and gives rise to the same Mellin transform $H(x,y)$, given in equation (48). We therefore obtain a quite similar equation for $\gamma$ apart for two differences. The second derivative of the propagator now involves a $\beta$ function which depends on the vertex corrections and the vertex corrections introduce further terms proportional to $\log(u^2/\mu^2)$. We get a contribution for the anomalous dimension

\[
-\gamma_2 + 2\gamma^2 - \gamma = -\frac{a}{2} (GY)_x G'_y H(x,y).
\]

where we see now the product of series along the line with variable $x$. It is important to notice that we could not have moved the vertex corrections to the line with a derivative of the propagator, since the trick of obtaining the part with $DG$ in $K^{\nu}$ simply by a multiplication by $y$ would no longer work. It is interesting to notice that the natural separation of variables allows us to think of the diagram as composed of the two following propagator like objects:

4.2 Vertex: The rising sun

We must also look at the vertex contribution. In equation (25), we look for the lowest order correction and use infrared rearrangement to obtain a one loop diagram quite similar to the one studied in the previous section, only simpler since it does not involve $K^{\nu}$. We look therefore at

\[
\int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} P(u)\Gamma_3(u)P(u)\Gamma_3(u+p)P(u+p)\Gamma_3(u+p)
\]

where the dashed line describes a 0-momentum coming in. We will see that the position of the vertex corrections does not modify the value of this diagram at the approximation level we compute in this work, but we argue that it is nevertheless the good one. A first, quite mundane argument is that it allows to reuse a propagator like object already appearing in the previous section. The more serious one is that in the asymptotic analysis we plan to do in the near future [26], this is the form which allows for the clearer derivation. This diagram is divergent by power counting, so we expect a contribution to the anomalous dimension $\nu$. Looking at the first coefficient in $L$ of equation (25), we have

\[
u = -a (GY)_x (GY)_y H_2(x,y).
\]

with the new Mellin transform $H_2(x,y)$ given by

\[
H_2(x,y) = \frac{\Gamma(1-x-y)\Gamma(1+x)\Gamma(2+y)}{\Gamma(2-x)\Gamma(1-y)\Gamma(3+x+y)}
\]

Again in this case we can imagine the diagram made of two composite objects:

where we have omitted the dashed lines.
4.3 First primitive diagram approximation

We are now ready to set up a system of equations that will allow us to solve the theory in the approximation of one primitive diagram for either of the propagator and vertex corrections. We have

\[
\begin{align*}
\text{SD} : \quad & \mu = \mu - \frac{1}{2} \\
\text{SD} : \quad & \quad \quad \quad \\
\text{RG} : \quad & \gamma_{n+1} = (\gamma_1 + \beta a \partial_a) \gamma_n \\
\text{SD}/\text{RG} : \quad & \beta = 3\gamma + 2v
\end{align*}
\]

Using the results of the preceding subsections, this can be converted in the following system of equations:

\[
\begin{align*}
2\gamma^2 - \gamma_2 - \gamma &= -\frac{a}{2} (GYG)_x G_y H(x, y) \\
v &= -a (GYG)_x (GYG)_y H_2(x, y) \\
\beta &= 3\gamma + 2v.
\end{align*}
\]

We do not write explicitly the equations needed to express the higher orders of \(G\) or \(Y\), since they would only appear in computations at higher order, which could only be made exact through the introduction of many more primitives than the one we will consider here. At this stage, we simply indicate that it would be much more efficient to directly compute the products \(GYG\) and \(YGY\) from appropriate version of the renormalisation group equations than to compute explicitly the products. The first step is to expand the right hand sides in the system (63). We limit ourselves to the linear terms in the functions \(H\) and \(H_2\):

\[
\begin{align*}
(YGY)_x (G)_y H(x, y) &= (1 + (2v + \gamma) \partial_x + \ldots) (1 + \gamma \partial_y + \ldots) H(x, y) \\
&= \frac{1}{6} \frac{5}{18} (v + \gamma) + \ldots; \\
(GYG)_x (GYG)_y H_2(x, y) &= (1 + (v + 2\gamma) \partial_x + \ldots) (1 + (\gamma + 2v) \partial_y + \ldots) H_2(x, y) \\
&= \frac{1}{2} - \frac{3}{4} (v + \gamma) + \ldots.
\end{align*}
\]

We can now write the equations in \(63\) as

\[
\begin{align*}
\gamma &= (\gamma - \beta a \partial_a) \gamma + \frac{a}{12} \left(1 - \frac{5}{3} (v + \gamma) + \ldots\right) \\
v &= -\frac{a}{2} \left(1 - \frac{3}{2} (v + \gamma) + \ldots\right) \\
\beta &= 3\gamma + 2v.
\end{align*}
\]

The first two steps of the solution of this system are

\[
\begin{align*}
1st : \quad & \gamma = \frac{a}{12}; \quad v = -\frac{a}{2}; \quad \beta = -\frac{3}{4} a \\
2nd : \quad & \gamma = \frac{a}{12} + \frac{55}{432} a^2; \quad v = -\frac{a}{2} - \frac{5}{16} a^2; \quad \beta = -\frac{3}{4} a - \frac{35}{144} a^2
\end{align*}
\]

5 Corrections to the order \(a^2\)

The results from the previous paragraph are not complete at order \(a^2\). We have two corrections to consider: one to the anomalous dimension \(\gamma\) and one to the anomalous dimension \(v\).
5.1 From the propagator

We first have the second term in (30), which is of minimum order 2:

\[ -a^2 \sim (69) \]

To estimate this contribution let us close the diagrams by connecting the external lines with such a propagator that they become conformal:

\[ (70) \]

The propagator with the square is of the form \( p^\mu / (p^2)^2 \) so the line we added will carry a similar factor \( q^\mu \) to make the diagram a scalar. To fix notations, the lines 1, 2 and 3 form the triangular subdivergence, 5 is the line carrying the box decoration and 6 is the line added to close the graph.

We call \( \Gamma_\tilde{i} \) the completed graphs and \( \Gamma_i \) the propagator-like ones. Massless vacuum diagrams have a vanishing second Symanzik polynomial and the first one is related to the Symanzik polynomials of the uncompleted graph. Calling the Schwinger parameters \( t_i \), we have

\[ \psi_\tilde{1} = t_6 \psi_1 + \phi_1. \]  
(71)

and

\[ \psi_\tilde{2} = t_6 \psi_2 + \phi_2. \]  
(72)

We point out that \( \psi_1 = \psi_2 \), thus

\[ \psi_\tilde{1} - \psi_\tilde{2} = \phi_1 - \phi_2. \]  
(73)

so in particular it does not depend on \( t_6 \). In the absence of any decoration, for a zero momentum insertion on the line 5, we would calculate

\[ \int dt_1 \ldots dt_5 t_6 t_6^2 \left( \frac{1}{\psi_1^4} - \frac{1}{\psi_2^4} \right) \delta H \]  
(74)

while with the numerators we have

\[ \int dt_1 \ldots dt_5 t_5 t_6^3 \left( \frac{C_1}{\psi_1^4} - \frac{C_2}{\psi_2^4} \right) \delta H. \]  
(75)

The \( \delta H \) factors in these integrals represent the restriction of these integrations to a hyperplane necessary to make these scale invariant integrals finite. We will not further precise these factors since these homogeneous integrals are independent on their precise choice. The exact formulas involve \( \Gamma \) factors which become important when considering modified propagators, but at this stage we can ignore them, because they are either taken for an index 1 or 2 and give 1, or there is a compensation between a \( \Gamma(D/2) \) factor in the numerator and the same factor in the denominator coming from the exponent of \( t_6 \). The computation of the numerators was completely described in a book by Nakanishi [28]: they could be calculated as a minor of a matrix associated to the graph, or by finding subgraphs with exactly one cycle including the lines 5 and 6. We have

\[ C_1 = t_4(t_1 + t_2 + t_3) + t_3 t_2 \]  
(76)

\[ C_2 = t_4(t_1 + t_2 + t_3) + (t_1 + t_2)t_3 \]  
(77)

\[ ^1 \text{In a forthcoming paper there will be a more complete characterisation of the method.} ^{27} \]
and therefore
\[ C_1 - C_2 = -t_1 t_3. \]  

We can therefore write the parenthesis in equation (75) as:
\[
\frac{C_1}{\psi_1^2} - \frac{C_2}{\psi_2^2} = C_1 \left( \frac{1}{\psi_1^2} - \frac{1}{\psi_2^2} \right) - \frac{t_1 t_3}{\psi_1^2} \psi_2^2 = C_1 \left( \psi_2 - \psi_1 \right) \left( \frac{1}{\psi_1^2 \psi_2^2} + \frac{1}{\psi_1^2 \psi_2^2} + \frac{1}{\psi_1^2 \psi_2^2} + \frac{1}{\psi_1^2 \psi_2^2} \right) - \frac{t_1 t_3}{\psi_1^2} \psi_2^2. \]  

From equation (73) we know that \( (\psi_2 - \psi_1) \) does not depend on \( t_6 \) so that the big first term in the preceding equation behaves as \( 1/t_6 \) for large \( t_6 \), making therefore a convergent contribution in equation (75) which will be absorbed in the renormalisation condition. At order \( a^2 \), we are left with the contribution coming from \( -t_1 t_3 t_5 t_6/\psi_2^2 \). This integral is the one we would obtain from the graph \( \Gamma_2 \) in \( D = 8 \) since the exponent of the denominator is 4. The factors in the numerator imply however that it must be taken with propagators with different indices.

\[
\begin{align*}
&\approx \frac{\Gamma(0)}{\Gamma(4)} \approx \frac{\Gamma(0)}{24} \\
&\approx \Gamma(0) / 24
\end{align*}
\]

We consider the last graph as the natural by-product of closing the diagram and therefore evaluate it to 1. The factor \( \Gamma(0) \) represents the divergence which will be compensated by taking the derivative with respect to \( L \), so that we remain with a residue \( 1/24 \) which is the period of the graph. Finally, the \( 1/2 \) symmetry factor and the doubling associated to the presence of the two symmetric possibilities for the corrected vertex cancel, so that we end up with a contribution \( -1/24a^2 \) for \( \gamma \).

5.2 From the vertex

In the non trivial term appearing in equation (25), the 4-point function can be expanded as in equation (3) to give
\[
\ldots = \ldots + \frac{1}{2} \ldots \ldots
\]

A correction comes from the second term, in which, according to our approximation scheme, we can shift the decorations of the vertices to obtain
\[
\begin{align*}
&\approx \frac{\Gamma(0)}{\Gamma(4)} \approx \frac{\Gamma(0)}{24} \\
&\approx \Gamma(0) / 24
\end{align*}
\]

\footnote{Since the difference we look at makes a primitive divergence, this global divergence is the only one which has to be checked, but one can also explicitly see that there are no subdivergences associated to the \((1,2,3)\) subdiagram.}
In fact, at the order $a^2$ we can simply ignore the decorations. We now close the diagram to obtain a vacuum one without any divergent subgraph

\[ \frac{1}{2} \int_{\mathbb{R}_+} dt_1 \ldots dt_6 \frac{t_5 t_6}{\tilde{\psi}^3} \delta_{H} \]  

where again

\[ \tilde{\psi} = t_6 \psi + \phi \]

The divergence takes the form of a pole with a residue linked to the factorisation of the diagram in the additional line 6 and the original diagram without its exterior lines. In the context of fixed dimension analytic regularisation, such poles have been studied in the general case in [5] and the residue is obtained as $1/\Gamma(D/2)$ times the product of the evaluation of the parts. The single loop as well as the remaining two loop graphs with all propagators with index 2 evaluate to 1, so that this residue is just $1/\Gamma(D/2) = 1/2$. With the symmetry factor $1/2$ of this graph, we have therefore a contribution of $a^2/4$ for the anomalous dimension $v$.

At the same order, we seem to have an analog of the contribution studied in the preceding subsection, coming from a correction to the infrared rearrangement. However this contribution turns out to be finite, like the term proportional to $C_1$ in equation (79), so that it does not affect the anomalous dimension $v$.

5.3 Comparison with known results

Our results are easily generalisable to include multicomponent fields. The structure of the diagrams remains unchanged and likewise the structure of the Schwinger–Dyson equations. The dependence on a single coupling can be obtained if we have a symmetry group. We would need to include Casimir factors that in the literature are usually denoted by $T_i$. Starting from an interaction term where $\phi^3$ is replaced by $d_{ijk} \phi i \phi j \phi k$, the $T_i$ relate the trace of the $i$-fold product of tensors $d_{ijk}$ and $\delta_{ij}$ (in the case of $T_2$) or $d_{ijk}$ (in the case of all others $T_j$, where $j$ is always odd). The first two Casimir $T_2$ and $T_3$ are defined by

\[ d_{ijk} d_{ijk} = T_2 \delta_{il}, \quad d_{ilm} d_{ijn} d_{kmn} = T_3 d_{ijk}. \]  

In fact, the order is not sufficient to characterize these Casimir factors, starting at order 7, but since they would only appear in vertex corrections with at least three loops or propagator corrections with at least one additional loop, this is of no concern to us in this work. These group factors change our equations in the following way

\[ \gamma = (\gamma - \beta a \partial_a) \gamma + \frac{T_2 a}{2} (GYG)_x G_y H(x, y) \]  
\[ v = -T_3 a (GYG)_x (GYG)_z H_2(x, z) \]  
\[ \beta = 3\gamma + 2v \]

Since the operators contain themselves products of $\gamma$ and $v$ we expect products of $T_i$ appearing all over. Our first correction from (69) will intervene with a factor $T_2 T_3$ while the second one from (83) will come with a factor $T_3$. With these corrections we have

\[ \gamma = \frac{T_2}{12} a + (-11T_2 + 48T_3) \frac{T_2}{432} a^2 \]  
\[ v = -\frac{T_3}{2} a + \left( -\frac{T_3}{4} + \frac{T_3}{16} (T_2 - 6T_3) \right) a^2 \]  
\[ \beta = (T_2 - 4T_3) \frac{a}{4} + \left( -11T_2^2 + 66T_2 T_3 - 108T_3^2 - 72T_5 \right) \frac{a^2}{144} \]
We can compare it with previous results for the \( \beta \)-function \([29, 30]\). This comparison is not immediate, since our definition of the \( \beta \)-function is unusual, as can be seen from the way we have written the renormalisation group equations: \( \beta(a) := \mu \frac{d}{d \mu} \log a = \frac{\epsilon}{\nu} \mu \frac{d}{d \mu} g \). The coefficient of \( a \) is twice the coefficient of \( g^3 \) in other works and the one of \( a^2 \) is twice the coefficient of \( g^5 \).

Furthermore, in the work \([29]\), the conventions are such that the terms with odd powers of \( a \) have an additional minus sign. With these provisions, we recover the usual \( \beta \)-function, but the \( T_2 T_3 \) term in \( \gamma \) is off by a factor of 2: this is not completely unexpected, since our renormalisation conditions are different and only the \( \beta \)-function is scheme independent at this order.

Conclusions

In this paper we have established and solved the Ward–Schwinger–Dyson equations for a massless \( \phi^3 \) in 6 dimensions. Our interest in this model is motivated by the presence of vertex corrections, in the simple context of a scalar theory, allowing to go beyond the Wess–Zumino model studied in our previous works. The presence of overlapping divergences was a major block for the expression of Schwinger–Dyson equations solvable in terms of renormalised Green functions. We have deployed a method inspired by Ward \([9]\) that consisted in studying a version of the Schwinger–Dyson equations for the derivative of the 2-point functions with respect to the external momentum. The reduction to analytically simple versions of these equations was achieved by a deformation of the diagrams akin to the infrared rearrangements used in other contexts (section \( \S \)1.3).

An interesting part of this work is that in section 2 we have shown that the Schwinger–Dyson equations imply the renormalization group equations, by passing the need to deduce them from the usual renormalisation procedure. We could not develop it in this letter, but this could be the base of a proof of the renormalized perturbative series independent on the methods of BPHZ \([20, 31, 32]\) and their more recent avatars \([33, 34]\). As in the works of Epstein and Glaser \([35]\), fully renormalized lower order results are used to produce the next step in the computation.

Finally, in sections \( \S \)4.3 and \( \S \)5 we have given the solution up to the order \( a^2 \), fully compatible with known results. It would be possible to proceed to higher orders, but we think that the present computations are sufficient to illustrate the general procedure. We suppose that computations at higher orders would be competitive in complexity with usual methods based on dimensional regularisation, especially for the contributions coming from deeply nested divergences. We believe these methods to be applicable to many other theories and we look forward to apply them to the case of theories with a gauge symmetry.

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A Parametric representations

It is well known that it is possible to rewrite a Feynman integral as a projective integral. The first step is to use the so called “Schwinger trick”:

\[
x^{-\alpha} \Gamma(\alpha) = \int_{\mathbb{R}_+} dt \ t^{\alpha-1} e^{-tx}
\]

on each propagator.

Then we need some definitions. A graph is a collection of vertices and edges, a tree (\( T \)) is a graph with no loops and a forest is a disjoint union of trees. A spanning tree (forest) is a tree (forest) which contains all the vertices of the graph. Let \( \mathcal{T} \) be the set of spanning trees and \( \mathcal{F}_k \) the set of spanning forests with exactly \( k \) components. Let \( I \) be the set of internal edges \( e_i \), \( V \) the set of vertices. We associate to each edge a mass parameter \( m_i \) and a power of the propagator \( \alpha_i \). Finally let \( p_T \) be the sum of the external momenta entering the subgraph \( T \). In the case of a graph with only one connected component, \( h_I(\Gamma) = |I| - |V| + 1 \) is the number of loops.

It is then possible to associate to a given graph \( \Gamma \) the two Symanzik polynomials \( \psi_T \) and \( \phi_T \) with the following definitions:

\[
\psi_T(\{t_i\}) := \sum_{T \in \mathcal{T}} \prod_{e_i \notin T} t_i
\]

\[
\phi_T(\{t_i\}) := \sum_{T \in \mathcal{T}} \prod_{e_i \notin T} t_i
\]
The Feynman integral for the graph $\Gamma$ can then be written as

$$\mathcal{F}(\{p_i, p_j\}, \{\alpha_i\}, \{m_i\}, D) = \int_{\mathbb{R}_+^{|I|}} \frac{dt_1 \ldots t_{|I|}}{\Gamma(\alpha_i)} \frac{1}{\psi^{D/2}} \frac{1}{t_1 \ldots t_{|I|}} e^{\phi/\psi}.$$

where we have omitted the variables on which $\psi$ and $\phi$ depend for the sake of readability. From their definition, we can see that the Symanzik polynomials $\psi$ and $\phi$ are homogeneous in the $t_i$ variables with respective degrees $h_1(\Gamma)$ and $h_1(\Gamma) + 1$. If we insert the equality

$$1 = \int_0^{+\infty} d\lambda \delta(\lambda - H(t))$$

for any non-zero hyperplane equation $H(t) = H^j t_j$, we obtain:

$$\mathcal{F}(\{p_i, p_j\}, \{\alpha_i\}, \{m_i\}, D) = \prod_i \int_{\mathbb{R}_+^{|I|}} \frac{dt_1 \ldots t_{|I|}}{\Gamma(\alpha_i)} \frac{1}{\psi^{D/2}} \frac{1}{t_1 \ldots t_{|I|}} e^{\phi/\psi} \int_0^{+\infty} d\lambda \lambda^{\sum \alpha_i - D/2 + h_1(\Gamma) - 1} e^{-\lambda \phi/\psi}$$

with a small abuse of notation due to the scaling of $t_i$. In terms of the superficial degree of divergence $\omega$,

$$\omega := h_1 \frac{D}{2} - \sum_i \alpha_i,$$

the integral on $\lambda$ gives $\Gamma(-\omega)(\phi/\psi)^\omega$. The presence of the hyperplane in the delta function induces a split on the domain and on the volume form which brings us to

$$\frac{1}{\prod_i \Gamma(\alpha_i)} \int_{\mathbb{R}^{|I| - 1}_+} \Omega_H \frac{1}{\psi^{D/2}} \left(\frac{\phi}{\psi}\right)^\omega \prod_i \frac{D}{2} - \alpha_i - 1$$

written as an integral on the projective space $\mathbb{RP}^{|I| - 1}_+$ with the volume form $\Omega_H$ given by

$$\Omega_H := \sum_i (-1)^i \left(1 - \frac{\alpha_i}{2}\right) \frac{t_i}{H} d\left(\frac{t_1}{H}\right) \wedge \ldots \wedge d\left(\frac{t_i}{H}\right) \wedge \ldots \wedge d\left(\frac{t_{|I|}}{H}\right).$$

This integral is independent on the particular choice of the hyperplane thanks to its homogeneity, since for a different choice $H'$, $\Omega_H = (H/H')^{|I|} \Omega_H$.

In the case of the one-loop massless diagram ($m_i = 0 \; \forall i \in I$)

```
  \[ \Phi \]
```

the polynomials are very simple:

$$\psi = t_1 + t_2$$

$$\phi = t_1 t_2 p^2.$$

By the choice of the hyperplane $H(t) = t_2$ we get

$$\mathcal{F}(p^2, \alpha_1, \alpha_2, D) := \left(p^2\right)^\omega \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int dt_1 dt_2 \delta(1 - t_2) \frac{1}{(t_1 + t_2)^{D/2 - 2 \alpha_1 - 1}} =$$

$$= \left(p^2\right)^\omega \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} B(D/2 - 2 \alpha_1, D/2 - 2 \alpha_2) =$$

$$= \left(p^2\right)^{D/2 - 2 \alpha_1 - 2} \frac{\Gamma(D/2 - 2 \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \Gamma(D/2 - \alpha_1) \Gamma(D/2 + \omega)$$

Since we are studying a model in 6 dimensions, in the main text we will use this value.
Let us also report here the explicit calculation mentioned in section §B. We can complete the square in the exponent

\[ \int_{\mathbb{R}^6} dq \left( \frac{1}{(p + q)^2} \right)^{1-x} 2q^\nu \left( \frac{1}{q^2} \right)^{2-y}. \]  

(96)

We can perform the Schwinger trick [21] and have

\[ \frac{2}{\Gamma(1-x)\Gamma(2-y)} \int_{\mathbb{R}^+} dt_1 \int_{\mathbb{R}^+} dt_2 t_1 \Gamma(1-t_1) \int_{\mathbb{R}^6} dq \ q^\nu e^{-t_1(p+q)^2-2t_2q^2}. \]  

(97)

We can complete the square in the exponent

\[ t_1(p + q)^2 + t_2q^2 = t_{12} \left( q + \frac{t_1}{t_{12}} p \right)^2 + \frac{t_1 t_2}{t_{12}} p^2 \]  

(98)

and then perform a change of variable \[ u = q + \frac{t_1}{t_{12}} p \] to get

\[ \int_{\mathbb{R}^6} dq \ q^\nu e^{-t_1(p+q)^2-2t_2q^2} = \int_{\mathbb{R}^6} du \left( u^\nu - \frac{t_1}{t_{12}} p^\nu \right) e^{-t_{12}u^2} e^{-\frac{t_1 t_2}{t_{12}} p^2}. \]  

(99)

In all these formula, we use the abbreviation \[ t_{12} := t_1 + t_2. \] The term proportional to \[ u^\nu \] vanishes since it is the integral of an odd function and the second term is a gaussian integral which gives

\[ \int_{\mathbb{R}^6} dq \ q^\nu e^{-t_1(p+q)^2-2t_2q^2} = -\pi^2 \pi \frac{t_1}{t_{12}} e^{-\frac{t_1 t_2}{t_{12}} p^2}. \]  

(100)

Going back to equation (96)

\[ \int_{\mathbb{R}^6} dq \left( \frac{1}{(p + q)^2} \right)^{1-x} 2q^\nu \left( \frac{1}{q^2} \right)^{2-y} = -\pi^3 \pi \frac{t_1}{t_{12}} e^{-\frac{t_1 t_2}{t_{12}} p^2}. \]  

(101)

As before, we can insert the equality (95) and choose the hyperplane \[ H(t) = \lambda t_{12} \] to get

\[ \int_{\mathbb{R}^6} dt_1 \int_{\mathbb{R}^+} dt_2 t_{12}^{-x} t_{12}^{-y} e^{-\frac{t_1 t_2}{t_{12}} p^2} = (p^2)^{x+y} \Gamma(-x-y) \int_0^1 dt_1 t_1^{y}(1-t_1)^{1+x} \]  

(102)

which finally brings us to

\[ \int_{\mathbb{R}^6} dq \left( \frac{1}{(p + q)^2} \right)^{1-x} 2q^\nu \left( \frac{1}{q^2} \right)^{2-y} = -2\pi^3 \pi \frac{t_1}{t_{12}} (p^2)^{x+y} \Gamma(-x-y) \Gamma(2+x) \Gamma(2+y). \]  

(103)

### B Massless loop trick

In this article we are studying a massless model, so the propagators that appear in the whole text are powers of \( p^2 \). With the Feynman rules for momentum space a simple loop is the convolution of two propagators. In the massless case though, the Fourier transform back in position space will also be a power of \( x^2 \) by homogeneity reasons and the convolution can be evaluated as a multiplication in position space of the Fourier transforms. Apart from some \( \pi \) factors independent on \( a \), we have that

\[ \mathcal{F} \left( \frac{1}{p^2} \right) (x) \propto \frac{\Gamma(D/2-a)}{\Gamma(a)} \left( \frac{1}{x^2} \right)^{D/2-a} \]  

(104)

so that in the case of a loop

\[ a \quad \begin{array}{c} \infty \end{array} \quad b \quad \frac{1}{p^2} \]  

\[ \propto \left( \frac{1}{p^2} \right)^{a} \ast \left( \frac{1}{p^2} \right)^{b} = \mathcal{F}^{-1} \left( \mathcal{F} \left( \frac{1}{p^2} \right)^{a} \cdot \mathcal{F} \left( \frac{1}{p^2} \right)^{b} \right) \]  

(105)

\[ \propto \frac{\Gamma(D/2-a)\Gamma(D/2-b)}{\Gamma(a)\Gamma(b)} \mathcal{F}^{-1} \left( \frac{1}{p^2} \right)^{D-a-b} \]  

(106)

\[ \propto \frac{\Gamma(D/2-a)\Gamma(D/2-b)\Gamma(a+b-D/2)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)} \left( \frac{1}{p^2} \right)^{a+b-D/2} \]  

(107)
where we have used $F(\alpha \ast \beta) = F(\alpha) \cdot F(\beta)$. Using repetitively these transformations allow to evaluate the diagram appearing in section §5.1.

In this section, we also have used the following construction: starting from the diagram

```
  a
 /|
/ |
/  |
b  c
```

we can add another line of index $c$

```
  a
 /|
/ |
/  |
   b
  c
```

Now, if we want to consider this graph as a vacuum one, the dependence on the exterior momentum $p$ should disappear. We set therefore $c = D - a - b$, which is also the condition that $\omega = 0$ for the completed graph, making it logarithmically divergent. The preceding formula becomes

$$\frac{\Gamma(D/2 - a)\Gamma(D/2 - b)\Gamma(D/2 - c)\Gamma(a + b + c - D)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(3D/2 - a - b - c)} \left( \frac{1}{p^2} \right)^{a+b+c-D}$$

The first factor is the same we would have found if we had transformed the $a, b$ loop. There is however a $\Gamma(0)/\Gamma(D/2)$ additional factor, which is infinite, reflecting the divergence of the whole diagram. This factor can however be interpreted as the integral

$$\int_{\mathbb{R}^D} \frac{d^D u}{(u^2)^{D/2}}$$

the simplest scale invariant integral in $D$ dimensions. This same integral can appear either in $x$-space, since the product of the three propagators gives the power $D/2$ of $x^2$ or in $p$-space, where the combination of any two of the propagators and the last one combine to give $(p^2)^{D/2}$. The scale invariance can be broken by fixing the momentum in any of the propagators while giving the same number, giving another approach to the completion invariance of the residues of propagator graphs.

In this work, we have been interested in the pole structure when one of the propagator of a completed graph becomes scale invariant. Since the whole structure is scale invariant, the complementary of this line becomes also scale invariant, giving two infinite factors. We therefore understand that there should be a pole in the evaluation of the diagram, but it is not so clear how to evaluate the residue of this pole. In a previous work [5], a procedure was devised from the parametric representation, which has the advantage of generalising to poles associated to a propagator with a power larger than $D/2$ by any positive integer, but another approach is possible in the simple case. We consider the diagram in $x$-space: the scale invariance is broken by fixing the distance between the two vertices while the almost scale invariant link contributes only by its normalisation, $\Gamma(\varepsilon)/\Gamma(D/2 - \varepsilon)$. In the limit of vanishing $\varepsilon$, $\Gamma(\varepsilon)$ gives a pole of residue 1 while all other terms have a smooth limit. Differentiating with respect to $L$ compensate the pole, so that we end up with $1/\Gamma(D/2)$ times the residue of the remaining scale invariant diagram. From its scale invariance, it comes that the choice of any two fixed vertices will give the same value for the residue, allowing for a simpler evaluation through suitable choices of these vertices.

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