HEAT FLOW ON THE MODULI SPACE OF FLAT CONNECTIONS AND YANG-MILLS THEORY

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Abstract. It is known that there is a bijection between the perturbed closed geodesics, below a given energy level, on the moduli space of flat connections $\mathcal{M}$ and families of perturbed Yang-Mills connections depending on a parameter $\varepsilon$. In this paper we study the heat flow on the loop space on $\mathcal{M}$ and the Yang-Mills $L^2$-flows for a 3-manifold $N$ with partial rescaled metrics. Our main result is that the bounded Morse homology of the loop space on $\mathcal{M}$ is isomorphic to the bounded Morse homologies of the connections space of $N$.

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1. Introduction

In 1983 Atiyah and Bott (cf. [4]) introduced the moduli space of flat connections for a principal bundle $P \to \Sigma$ over a surface $(\Sigma, g)$. This moduli space can be seen as an infinite dimensional symplectic reduction; in fact, the conformal structure on the surface defines an almost complex structure and with it and with the scalar product on the 1-forms one can also obtain a symplectic form. If we pick a principal non trivial SO(3)-bundle, then the moduli space $\mathcal{M}^\theta(P)$, defined as the quotient between the space of the flat connections $\mathcal{A}_0(P) \subset \mathcal{A}(P)$ and the identity component of the gauge group $G_0(P)$, is a smooth compact symplectic manifold of dimension $6g - 6$ (cf. [4]) where $g$ denotes the genus of the surface and $\mathcal{A}(P)$ the set of the connections of the bundle.

**Critical connections.** On the one side, we consider the loop space $\mathcal{L}(\mathcal{M}^\theta(P))$ of the manifold $\mathcal{M}^\theta(P)$ and if we want to compute the perturbed energy functional of a loop $\gamma$, where the perturbation comes from an equivariant Hamiltonian map, we can pick a lift $A(t) \in \mathcal{A}_0(P)$ of $\gamma$ and the unique loop $\Psi(t) \in \Omega^0(\Sigma, g_P)$ of 0-forms satisfying the condition

\[
d_A^\ast (\partial_t A - d_A \Psi) = 0;
\]

$\Omega^k(\Sigma, g_P)$ denotes the space of $k$-forms on $\Sigma$ with values in the adjoint bundle $g_P := P \times_M g$ defined by the equivalence classes $[pg, \xi] \equiv [p, \text{Ad}_g \xi] \equiv [p, g \xi g^{-1}]$ for $p \in P$, $g \in G$ and $\xi \in g$. In this case, we can see $A(t) + \Psi(t)dt$ as a connection on $\Sigma \times S^1$ and the perturbed energy functional can be written as

\[
E^H(A) = \frac{1}{2} \int_0^1 \left( \|\partial_t A - d_A \Psi\|_{L^2(\Sigma)}^2 - H_t(A) \right) dt
\]

where $H_t : \mathcal{A}(P) \to \mathbb{R}$ is a generic equivariant Hamiltonian map. The critical loops of (1.2) have to satisfy the equation

\[
\pi_A (-\nabla_t (\partial_t A - d_A \Psi) - *X_t(A)) = 0,
\]

where $\nabla_t := \partial_t + [\Psi_t, \cdot]$, where $\pi_A$ denotes the projection of the 1-forms in to the linear space of the harmonic 1-forms which corresponds to the tangent space of the manifold $\mathcal{M}^\theta(P)$ at the point $[A]$ and where the time-dependent Hamiltonian vector field $X_t$ is defined such that, for any connection $A \in \mathcal{A}(P)$ and any 1-form $\alpha$, $dH_t(A)\alpha = \int_{\Sigma} (X_t(A) \wedge \alpha)$. The equation (1.3) can be therefore also written as

\[
-\nabla_t (\partial_t A - d_A \Psi) - *X_t(A) - d_A^\ast \omega = 0;
\]

the 2-form $\omega(t) \in \Omega^2(\Sigma, g_P)$ is defined uniquely by the identity

\[
d_A d_A^\ast \omega = [(\partial_t A - d_A \Psi) \wedge (\partial_t A - d_A \Psi)] - d_A *X_t(A)
\]

and $d_A^\ast \omega$ corresponds to the non-harmonic part of $-\nabla_t (\partial_t A - d_A \Psi) - *X_t(A)$.

On the other side, if we pick the 3-manifold $\Sigma \times S^1$ with the metric $\varepsilon^2 g_{\Sigma} \oplus g_{S^1}$ for a positive parameter $\varepsilon$ and if we consider the principal SO(3)-bundle $P \times S^1 \to \Sigma \times S^1$ such that the restriction $P \times \{s\} \to \Sigma \times \{s\}$ is non trivial, then the perturbed Yang-Mills functional is

\[
\mathcal{YM}^\varepsilon,H(\Xi) = \frac{1}{2} \int_0^1 \left( \frac{1}{\varepsilon^2} \|F_A\|_{L^2(\Sigma)}^2 + \|\partial_t A - d_A \Psi\|_{L^2(\Sigma)}^2 - H_t(A) \right) dt;
\]

where the connection $\Xi \in \mathcal{A}(P \times S^1)$ can be written as $A + \Psi dt$ with $A(t) \in \mathcal{A}(P)$, $\Psi(t) \in \Omega^0(\Sigma, g_P)$; in fact, the curvature of $\Xi$ is $F_\Xi = F_A - (\partial_t A - d_A \Psi) \wedge dt$. A
p perturbed Yang-Mills connection \( \Xi^\varepsilon \in \mathcal{A}(P \times S^1) \) has therefore to satisfy the two conditions

\[
\frac{1}{\varepsilon^2} d_{A^\varepsilon}^* F_{A^\varepsilon} - \nabla_i (\partial_t A^\varepsilon - d_A \Psi^\varepsilon) - \ast X_i(A^\varepsilon) = 0,
\]

\[
d_{A^\varepsilon}^* (\partial_t A^\varepsilon - d_A \Psi^\varepsilon) = 0.
\]

Next, we consider the sets of critical connections below an energy level \( b \), i.e.

\[
\text{Crit}^b_{E,H} := \{ \Xi + \Psi dt \in \mathcal{L}(\mathcal{A}_0(P) \otimes \Omega^0(\Sigma, g_P) \wedge dt) \mid E^H(\Xi) \leq b, (1.1), (1.3) \},
\]

\[
\text{Crit}^b_{YM^\varepsilon,H} := \{ \Xi^\varepsilon \in \mathcal{A}(P \times S^1) \mid YM^\varepsilon,H(\Xi^\varepsilon) \leq b, (1.6), (1.7) \}.
\]

We can now define a map between the perturbed geodesics, \( \text{Crit}^b_{E,H} \), and the set of the perturbed Yang-Mills connections, \( \text{Crit}^b_{YM^\varepsilon,H} \), with energy less than \( b \) provided that the parameter \( \varepsilon \) is small enough (cf. [8], [9]); this map can also be defined uniquely, it is bijective and maps perturbed geodesics to perturbed Yang-Mills connections with the same Morse index (cf. [10]):

**Theorem 1.1** (cf. [10], theorem 1.1). We assume that the Jacobi operators of all the perturbed geodesics are invertible and we choose a regular value of the energy \( E^H \) and \( p \geq 2 \). Then there are two positive constants \( \varepsilon_0 \) and \( c \) such that the following holds. For every \( \varepsilon \in (0, \varepsilon_0) \) there is a unique gauge equivariant map

\[
\mathcal{T}^\varepsilon,b : \text{Crit}^b_{E,H} \to \text{Crit}^b_{YM^\varepsilon,H}
\]

satisfying, for \( \Xi^0 \in \text{Crit}^b_{E,H} \),

\[
d^\ast_{\Xi^0}(\mathcal{T}^\varepsilon,b(\Xi^0) - \Xi^0) = 0, \quad \| \mathcal{T}^\varepsilon,b(\Xi^0) - \Xi^0 \|_{\mathbb{E}^0,2,p,\varepsilon,S \times S^1} \leq \varepsilon_c^2.
\]

Furthermore, this map is bijective and

\[
\text{index}_{E,H}(\Xi^0) = \text{index}_{YM^\varepsilon,H}(\mathcal{T}^\varepsilon,b(\Xi^0)).
\]

The norm \( \| \cdot \|_{\mathbb{E}^0,2,p,\varepsilon,S \times S^1} \) is introduced in the appendix [10].

**Remark 1.2.** With the same assumptions of the last theorem we can also conclude the following estimates (cf. [10], theorem 9.1 and lemma 9.6). We consider the unique solution

\[
\alpha_0^\varepsilon(t) \in \text{im} \left( d^\ast_{A^0(t)} : \Omega^2(\Sigma, g_P) \to \Omega^1(\Sigma, g_P) \right)
\]

of the equation

\[
d^\ast_{A^0} d_{A^0} \alpha_0^\varepsilon = \varepsilon^2 \nabla_i (\partial_t A^0 - d_{A^0} \Psi^0) + \varepsilon^2 \ast X_i(A^0),
\]

then, for \( \alpha + \psi dt := \mathcal{T}^\varepsilon,b(\Xi^0) - \Xi^0 \),

\[
\| (1 - \pi_{A^0})(\mu - \alpha_0^\varepsilon) \|_{\mathbb{E}^0,2,p,\varepsilon,S \times S^1} + \varepsilon \| \psi dt \|_{\mathbb{E}^0,2,p,\varepsilon,S \times S^1} \leq c \varepsilon^4,
\]

\[
\| \pi_{A^0}(\mu) \|_{\mathbb{E}^0,2,p,1,S \times S^1} + \| \alpha_0^\varepsilon \|_{\mathbb{E}^0,2,p,1,S \times S^1} \leq c \varepsilon^2.
\]

**Bijection between the flows.** The theorem [11] allows us to identify the critical connections and thus the next step is to prove a bijection between the flow lines.

On the one side, every map \( [\Xi] : S^1 \times \mathbb{R} \to \mathcal{M}^q(P) \) can be seen as a connection \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{A}(P \times S^1 \times \mathbb{R}) \) which satisfies

\[
F_A = 0, \quad \partial_t A - d_A \Psi \in H^1_A, \quad \partial_s A - d_A \Psi \in H^1_A
\]
and if we have a map $A : S^1 \times \mathbb{R} \to \mathcal{A}_0(P)$, the second and the third condition of (1.12) yield to unique 0-forms $\Phi, \Psi \in \Omega^0(\Sigma \times S^1 \times \mathbb{R}, g_P)$. In order to achieve the transversality condition for the heat flow we need to choose a generic abstract perturbation on the loop space instead of the Hamiltonian one. Furthermore, $[\Xi]$ is a heat flow between the perturbed geodesics $\Xi_{\pm} \in \text{Crit}_b^{\mathcal{H}_r}$, $b \in \mathbb{R}$, if it satisfies the flow equation for the functional $\mathcal{H}_r$, i.e.

$$\partial_s A - d_A \Phi - \pi_A (\nabla_t (\partial_t A - d_A \Psi)) = 0, \quad (1.13)$$

$$\lim_{s \to \pm \infty} \Xi(s) = \Xi_{\pm}.$$  

On the other side, a perturbed, $\varepsilon$-dependent, Yang-Mills flow between two perturbed Yang-Mills connections $\Xi_{\pm} \in \text{Crit}_b^{\mathcal{M}^{\varepsilon}}$ can be considered as a connection $\Xi := A + \Psi dt + \Phi ds$ on the 4-manifold $\Sigma \times S^1 \times \mathbb{R}$, where $\Phi \in \Omega^0(\Sigma \times S^1 \times \mathbb{R}, g_P)$ makes the equations gauge invariant, and it satisfies the equations

$$\partial_s A - d_A \Phi + \frac{1}{\varepsilon^2} \partial_t^* d_A F_A - \nabla_t (\partial_t A - d_A \Psi) - \ast X_t(A) = 0,$$

$$\partial_s \Psi - \nabla_t \Phi - \frac{1}{\varepsilon^2} \partial_t^* (\partial_t A - d_A \Psi) = 0, \quad \lim_{s \to \pm \infty} \Xi = \Xi_{\pm}. \quad (1.14)$$

In the following we denote by $\mathcal{M}^{\varepsilon}(\Xi_{\pm}, \Xi_{\pm})$ (respectively by $\mathcal{M}^{\varepsilon}(\Xi_{\pm}, \Xi_{\pm})$) the moduli space of the solutions of (1.13) (respectively of (1.14)). We can therefore expect a bijective relation also between the flows of the two functionals for $\varepsilon$ small enough.

**Theorem 1.3.** We assume that the energy functional $\mathcal{H}_r$ is Morse-Smale and we choose $p > 2$ and a regular value $b > 0$ of $\mathcal{H}_r$. There are constants $\varepsilon_0, c > 0$ such that the following holds. For every $\varepsilon \in (0, \varepsilon_0)$, every pair $\Xi_{\pm} := A_{\pm}^0 + \Psi_{\pm}^0 dt \in \text{Crit}_b^{\mathcal{H}_r}$ with index difference 1, there exists a unique map

$$\mathcal{R}^{\varepsilon, b} : \mathcal{M}^0 (\Xi_{\pm}^0, \Xi_{\pm}^0) \to \mathcal{M}^{\varepsilon} (\mathcal{T}^{\varepsilon, b}(\Xi_{\pm}^0), \mathcal{T}^{\varepsilon, b}(\Xi_{\pm}^0))$$

satisfying for each $\Xi^0 \in \mathcal{M}^0 (\Xi_{\pm}^0, \Xi_{\pm}^0)$

$$d_{\Xi^0} (\mathcal{R}^{\varepsilon, b}(\Xi^0) - \mathcal{K}^\varepsilon(\Xi^0)) = 0, \quad \mathcal{R}^{\varepsilon, b}(\Xi^0) - \mathcal{K}^\varepsilon(\Xi^0) \in \text{im} \left( \mathcal{D}^* (\mathcal{K}^\varepsilon(\Xi^0)) \right)^*, \quad (1.15)$$

$$\left\| \mathcal{R}^{\varepsilon, b}(\Xi^0) - \mathcal{K}^\varepsilon(\Xi^0) \right\|_{1,2;p,1} \leq c\varepsilon^2. \quad (1.16)$$

**Furthermore,** $\mathcal{R}^{\varepsilon, b}$ is bijective.

In the last theorem the connection $\mathcal{K}^\varepsilon(\Xi^0)$ should be seen as a first approximation of the Yang-Mills flow and the norm $\| \cdot \|_{1,2;p,1}$ is defined in the section 5.

**Isomorphism between the homologies.** The theorem 1.3 assures a bijection between the critical connections with the same index and the theorem 1.3 between the flows and thus we can compare the Morse homologies defined using the $L^2$-flow of the two functionals below a energy level $b$. In the loop space case the homology is well defined by the work of Weber (cf. [17]) and in the Yang-Mills case we know that the flow exists in the case when the base manifold is two or three dimensional (cf. [11]) or when we have a symmetry of codimension 3 on the base manifold of higher dimension (cf. [12]), but no results about the Morse-Smale transversality or the orientation of the unstable manifolds are known and therefore a priori $\mathcal{H}_r (\mathcal{A}^{\varepsilon, b}(P \times S^1) / \mathcal{G}_0 (P \times S^1))$ might even not be defined; in our case, it makes sense because the unstable manifolds of $\mathcal{A}^{\varepsilon, b}(P \times S^1) / \mathcal{G}_0 (P \times S^1)$ inherit these properties from the unstable manifolds of $L^b \mathcal{M}^0 (P)$. Here $\mathcal{G}_0 (P \times S^1)$
denotes the loop group on the gauge group $G_0(P)$; $L_0^b M^g(P) \subset L M^g(P)$ and $A^{c,b} (P \times S^1) \subset A (P \times S^1)$ are respectively the subsets where $E^H \leq b$ and $\gamma M^{c,H} \leq b$.

**Theorem 1.4.** We assume that the energy functional $E^H$ is Morse-Smale. For every regular value $b > 0$ of $E^H$ there is a positive constant $\varepsilon_0$ such that, for $0 < \varepsilon < \varepsilon_0$, the inclusion $L_0^b M^g(P) \to A^{c,b} (P \times S^1)/G_0 (P \times S^1)$ induces an isomorphism

$$HM_* (L_0^b M^g(P), \mathbb{Z}_2) \cong HM_* (A^{c,b} (P \times S^1)/G_0 (P \times S^1), \mathbb{Z}_2).$$

Another way to approach this problem could have been to consider the $W^{1,2}$-flows; in this case both homologies are well defined since the Palais-Smale condition is satisfied in both cases (cf. [19]) and thus by the general Morse theory (cf. [1]) the transversality may be achieved. It is also interesting to remark that the Morse homology of $L_0^b M^g(P)$ correspond to the Floer homology of the cotangent bundle $T^* M^g(P)$ using the Hamiltonian $H_V$ given by the kinetic plus the potential energy and considering only orbits with the action bounded by $b$. This result was proved, for a more general case, by Viterbo (cf. [16]), by Salamon and Weber (cf. [15]) and by Abbondandolo and Schwarz (cf. [3], [2]) in three different ways. Furthermore, Weber (cf. [17]) announced that the Morse homology of the loop space defined by the heat flow is isomorphic to its singular homology. We have therefore the following identities

$$HM_* (A^{c,b} (P \times S^1)/G_0 (P \times S^1)) \cong H_* (A^{c,b} (P \times S^1)/G_0 (P \times S^1))$$

$$HM_* (L_0^b M^g(P)) \cong H_* (L_0^b M^g(P))$$

$$HF_* (T^* M^g(P), H_V)$$

We would like to conclude this discussion remarking that it is an open question whether the Morse homology of $A^{c,b} (P \times S^1)/G_0 (P \times S^1)$ defined using the $L^2$-flow is isomorphic to its singular homology.

**Outline.** In the sections [24] we will discuss the heat flow and the Yang-Mills flow equations; after that we will define the $\varepsilon$-dependent norms (section [13]) and introduce the Morse homologies (section [6]). In the sections [7] and [8] we will show respectively some linear and quadratic estimates that we will need in the section [9] to construct an approximation of a perturbed Yang-Mills flow starting from a perturbed geodesic flow and in the section [10] in order to define uniquely the map $R^{c,b}$ using a contraction argument. The next four sections are of preparatory nature for the proof of the surjectivity of $R^{c,b}$ (section [13]); in fact in the section [11] we will show some a priori estimates for the perturbed Yang-Mills flow and then we will prove an estimate for the $L^\infty$-norm of their curvature terms (section [12]), the uniformly exponential convergence of the flows (section [13]) and two theorems that allow to choose the right relative Coulomb gauge (section [14]). The surjectivity (section [15]) will be showed using an indirect argument; in fact we will prove that any sequence of perturbed Yang-Mills flows $\Xi^{c,\varepsilon}$, $\varepsilon_\nu \to 0$, which is not in the image of the map $R^{c,b}$, has a subsequence which converges (modulo gauge) by the implicit function theorem to a perturbed geodesic flow and thus, by the uniqueness property of $R^{c,b}$, it is in the image of this map which is a contradiction. In the last section
we will prove first the theorem \[1.3\] which follows easily from the definition \[10.3\] of the map \(R\) and its surjectivity (theorem \([15.1]\)), and then the theorem \([1.4]\).

2. Geodesic flow

Every continuously differentiable map \([\Xi]: S^1 \times \mathbb{R} \to \mathcal{M}^\partial(P)\) can be seen as a connection \(\Xi = A + \Psi dt + \Phi ds \in \mathcal{A}(P \times S^1 \times \mathbb{R})\) which satisfies the following conditions

\[
(2.1) \quad F_A = 0, \quad \partial_t A - d_A \Psi \in H^1_A, \quad \partial_s A - d_A \Phi \in H^1_A.
\]

In fact, for any \([\Xi]\) we can choose a lift \(A: S^1 \times \mathbb{R} \to \mathcal{A}_0(P)\); the second and the third condition of \((2.1)\) yield to unique 0-forms \(\Psi(t,s), \Phi(t,s) \in \Omega^0(P, g_P)\). One can also consider \(\Phi\) to have an exponential convergence as \(|s| \to \infty\) (cf. \([5]\)). The connection \(\Xi\) is clearly not uniquely defined, but for every two connections \(\Xi_1, \Xi_2\) with the above properties there is a map \(u \in \mathcal{G}_0(P \times S^1 \times \mathbb{R})\) such that \(u^* \Xi_1 = \Xi_2\), the existence and the uniqueness of \(u\) follow from the definition of \(\mathcal{M}^\partial(P)\) and from the equivariance of the conditions \((2.1)\). The gauge group \(\mathcal{G}_0(P \times S^1 \times \mathbb{R})\) is defined as the set of smooth maps \(g: S^1 \times \mathbb{R} \to \mathcal{G}_0(P)\).

Furthermore, \([\Xi]\) is a heat flow between the perturbed geodesics \(\Xi_{\pm} \in \text{Crit}_{E^H}, b \in \mathbb{R}\), if it satisfies the flow equation for the functional \(E^H\), i.e.

\[
(2.2) \quad \partial_s A - d_A \Phi - \pi_A(\nabla_t (\partial_t A - d_A \Psi) + \star X_t(A)) = 0,
\]

where \(\nabla_t := \partial_t + [\Psi, \cdot]\) and the perturbation term \(X_t\) will be discussed in the next section. Since

\[
d^*_A (\nabla_t (\partial_t A - d_A \Psi) + \star X_t(A)) = \nabla_t d^*_A (\partial_t A - d_A \Psi) + \star d_A X_t(A)
\]

\[
= \star [\partial_t A - d_A \Psi) \wedge \star (\partial_t A - d_A \Psi)] = 0,
\]

we can write the first line of \((2.2)\) as the pair of equations

\[
(2.3) \quad \partial_s A - d_A \Phi - \nabla_t (\partial_t A - d_A \Psi) - \star X_t(A) + d_A^* \omega = 0,
\]

\[
d_A \omega = - [\partial_t A - d_A \Psi) \wedge \partial_t A - d_A \Psi] + d_A \star X_t(A)
\]

where \(\omega(t,s) \in \Omega^2(\Sigma, g_P)\) is uniquely defined by the second condition which is obtained deriving the first one by \(d_A\) using the commutation formula \((2.4)\) and \((2.1)\).

**Lemma 2.1.** We have the following two commutation formulas:

\[
(2.4) \quad [d_A, \nabla_t] = -[(\partial_t A - d_A \Psi) \wedge \cdot];
\]

\[
(2.5) \quad [d_A^*, \nabla_t] = \star((\partial_t A - d_A \Psi) \wedge \star \cdot).
\]

**Proof.** The lemma follows from the definitions of the operators using the Jacobi identity for the super Lie bracket operator. \(\square\)

The linearised operator for a heat flow \(\Xi\) is

\[
D^\partial(\Xi)(\pi_A(\alpha)) := \pi_A(\nabla_s \pi_A(\alpha) - 2[\psi_0, (\partial_t A - d_A \Psi)] - \nabla_t \pi_A(\alpha) - d \star X_t(A) \pi_A(\alpha) + \star [\alpha \wedge \omega])
\]
for any $\alpha : S^1 \times \mathbb{R} \to \Omega^1(\Sigma, g_P)$ and where $\nabla_a := \partial_a + [\Phi, \cdot]$ and $\psi_0$ is defined uniquely by $d_A^* d_A \psi_0 = -2 \ast [\varpi_A(\alpha) \wedge (\partial_t A - d_A \Psi)]$. This last formula can be obtained in the same way as the Jacobi operator for perturbed geodesics (cf. [8] or [9]).

3. Perturbation

In order to achieve the transversality we have to perturb the equations and for this purpose we choose an abstract perturbation on $\mathcal{L}(\mathcal{M}^g(P))$.

First, we choose a perturbation $\bar{\mathcal{V}} : \mathcal{L}(\mathcal{M}^g(P)) \to \mathbb{R}$ on the loop space of $\mathcal{M}^g(P)$. We assume that $\bar{\mathcal{V}}$ satisfies the following condition (see condition (V4), section 2 of [13] or condition (V3), section 1 of [17]). For any two integers $k > 0$ and $l \geq 0$ there is a constant $c = c(k, l)$ such that

$$\left| \nabla_{t\bar{}}_{j\bar{}} \bar{\mathcal{V}}(A) \right| \leq c \sum_{k_j, l_j > 0} \left( \Pi_{j_k, l_j} \left| \nabla_{t\bar{}}_{j\bar{}} \bar{\mathcal{V}}(A) \right| \Pi_{j_k, l_k} \left( |\nabla_{t\bar{}}_{j\bar{}} A| + |\nabla_{t\bar{}}_{j\bar{}} A|_{L^p(\bar{\mathcal{V}})} \right) \right)$$

for every smooth map $A : \mathbb{R} \to \mathcal{L}(\mathcal{A}_0(P)) : s \mapsto A(s, \cdot)$ and every $(s, t) \in \mathbb{R} \times S^1$; here $p_j \geq 1$ and $\sum_{l_j = 0} = 1$; the sum runs over all partitions $k_1 + \cdots + k_m = k$ and $l_1 + \cdots + l_m \leq l$ such that $k_j + l_j \geq 1$ for all $j$. For $k = 0$ the same inequality holds with an additional summand $c$ on the right. In addition, $\nabla_{t\bar{}}_{j\bar{}} A$ and $\nabla_{t\bar{}}_{j\bar{}} A$ should be interpreted as $\nabla_{t\bar{}}_{j\bar{}} - 1(\partial_t A - d_A \Phi)$ and $\nabla_{t\bar{}}_{j\bar{}} - 1(\partial_t A - d_A \Phi)$.

Using the results of Weber (cf. [17], theorem 1.13) we can consider the following. For any regular value $b$ of the energy functional $E^H$ there is a Banach manifold $\mathcal{O}^b_{reg}$ of perturbations $\mathcal{V}$ that satisfy the above condition (3.1) and such that $E^H + \mathcal{V}$ have the same critical loops as $E^H$. Moreover there is a residual subset $\mathcal{O}^b_{reg} \subset \mathcal{O}^b$ such that the perturbed functional $E^H + \mathcal{V}$ is Morse-Smale below the energy level $b$ if $\mathcal{V} \in \mathcal{O}^b_{reg}$. From now on all the computations are done using a generic perturbation of this kind. Next, we define $\ast X_t(A), \partial_s A_0(t_0, 0)) dt := \left. \frac{d}{ds} \right|_{s=0} \left( \mathcal{V}(A(s)) + \int_0^1 H_t(A(s)) dt \right)$ for every smooth variation $A(t, s)$ of $A(t)$.

Furthermore, using the results of Weber (cf. [17], theorems 1.7, 1.8), for any constant $b$ there are positive constants $c, \rho, c_0, c_1, c_2, \ldots$ such that the following holds. If the connection $\Xi = A + \Psi dt + \Phi ds$ satisfies (2.2) and $E^H(A(\cdot, s)) \leq b$, then for every $s$

$$\left| \partial_t A - d_A \Phi \right|_{L^\infty} + \left| \nabla_t (\partial_t A - d_A \Phi) \right|_{L^\infty} \leq c,$n

$$\left| \partial_s A - d_A \Phi \right|_{L^\infty} + \left| \nabla_t (\partial_s A - d_A \Phi) \right|_{L^\infty} \leq c,$n

$$\left| \partial_s A - d_A \Phi \right|_{C^k(S^1 \times [T, \infty))} \leq c_k e^{-\rho T},$$n

$$\left| \partial_s A - d_A \Phi \right|_{C^k(S^1 \times (-\infty, -T])} \leq c_k e^{-\rho T},$$n

$^1\Psi(t, s), \Phi(t, s) \in \Omega^0(P, g_P)$ are uniquely defined by $d_A^*(\partial_t A - d_A \Psi) = 0$ and $d_A^*(\partial_t A - d_A \Phi) = 0$. 

for every \( T \geq 1 \). Moreover \([A]\) converges to a perturbed geodesic in \( C^2(S^1)\) as \( s \to \pm \infty \).

Next, we need to choose an extension \( V : \mathcal{L}(A(P)/G_0(P)) \to \mathbb{R}, V(A) = \tilde{V}(A)\) for \( A \in \mathcal{L}(A_0(P))\), such that \( V \) satisfies (3.1) for any smooth map \( A : \mathbb{R} \to \mathcal{L}(A(P))\) with \( \|F_A(s)\|_{L^2(\Sigma)} \leq \delta_0\) for every \( s \in \mathbb{R}\), where \( \delta_0\) is chosen such that the lemmas \( B.1\) and \( B.2\) hold for \( p = 2\) and \( q = 4\).

Another possibility is to extend \( \bar{B}\) and \( B.2\) hold for \( p = 2\) and \( q = 4\).

Another viewpoint to see the last equation is to consider (4.1) for the connection \( \bar{A}(t,s) + \bar{\Phi}(t,s)dA - \nabla_t(\partial_t A - dA \Psi) - \bar{X}_t(A) = 0\),

where \( \bar{\Phi}(t,s) \in \Omega^0(\Sigma, g_P)\) in order to make the equations gauge invariant. We can consider the \( s\)-dependent connection \( A(s) + \bar{\Psi}(s)dt\) together with the 0-form \( \bar{\Phi}\) as a connection \( \Xi := A + \psi dt + \Phi ds\) on the 4-manifold \( \Sigma \times S^1 \times \mathbb{R}\).

In our case we shrink the metric on \( \Sigma\) by \( \varepsilon^2\) and therefore the adjoint of the exterior derivative \( d_A\) contribute with a factor \( \frac{1}{\varepsilon^2}\) to the flow equation and if we consider the flow lines between two perturbed Yang-Mills connections \( \Xi_{\pm} \in \text{Crit}_{E_H}^0\) we have the equation

\[
\begin{align*}
\partial_s A - d_A \Phi &+ \frac{1}{\varepsilon^2}d_A^* F_A - \nabla_t(\partial_t A - d_A \Psi) - \bar{X}_t(A) = 0, \\
\partial_s \Psi - \nabla_t \Phi &+ \frac{1}{\varepsilon^2}d_A^* (\partial_t A - d_A \Psi) = 0, \\
\lim_{s \to \pm \infty} \Xi = \Xi_{\pm}.
\end{align*}
\]

Another viewpoint to see the last equation is to consider (4.1) for the connection

\[
\tilde{A}(t,s) + \tilde{\Phi}(t,s)dt + \tilde{\Psi}(t,s)ds = A(\xi t, \varepsilon^2 s) + \varepsilon \Psi(\xi t, \varepsilon^2 s)dt + \varepsilon^2 \Phi(\xi t, \varepsilon^2 s)ds,
\]

which is equivalent to (4.2) for \((t,s) \in [0, \frac{1}{\varepsilon}] \times \mathbb{R}\).

5. \textsc{Norms}  

We choose a reference connection \( \Xi := A + \psi dt + \Phi ds \in \mathcal{A}(\Sigma \times S^1 \times \mathbb{R})\); let \( \xi := \alpha + \psi dt + \phi ds \in \Omega^1(\Sigma \times S^1 \times \mathbb{R}, g_P), \alpha(t,s), \psi(t,s), \phi(t,s) \in \Omega^j(\Sigma, g_P), j = 0, 1\), then

\[
\|
\begin{align*}
\|\alpha + \psi dt + \phi ds\|_{\infty,\varepsilon} := &\|\alpha\|_{L^\infty} + \varepsilon\|\psi\|_{L^\infty} + \varepsilon^2\|\phi\|_{L^\infty}, \\
\|\alpha + \psi dt + \phi ds\|_{0,p,\varepsilon} := &\int_{S^1 \times \mathbb{R}} \left(\|\alpha\|_{L^p(\Sigma)} + \varepsilon^p\|\psi\|_{L^p(\Sigma)} + \varepsilon^{2p}\|\phi\|_{L^p(\Sigma)}\right) dt ds,
\end{align*}
\]

\[
\|
\]
\[
\begin{align*}
&\|\alpha + \psi dt + \phi ds\|_{1,p},e := \|\alpha + \psi dt + \phi ds\|_{0,p},e \\
&+ \int_{S^1 \times \mathbb{R}} \left( \|d_A'\alpha\|_{L^p}(\Sigma) + \|d_A\phi\|_{L^p}(\Sigma) + \varepsilon^p\|\nabla_t\alpha\|_{L^p}(\Sigma) + \varepsilon^{2p}\|\nabla_s\alpha\|_{L^p}(\Sigma) \right) \, dt \, ds \\
&+ \int_{S^1 \times \mathbb{R}} \left( \varepsilon^p\|d_A\psi\|_{L^p}(\Sigma) + \varepsilon^{2p}\|\nabla_t\psi\|_{L^p}(\Sigma) + \varepsilon^{3p}\|\nabla_s\psi\|_{L^p}(\Sigma) \right) \, dt \, ds \\
&+ \int_{S^1 \times \mathbb{R}} \left( \varepsilon^{2p}\|d_A\phi\|_{L^p}(\Sigma) + \varepsilon^{3p}\|\nabla_t\phi\|_{L^p}(\Sigma) + \varepsilon^{4p}\|\nabla_s\phi\|_{L^p}(\Sigma) \right) \, dt \, ds,
\end{align*}
\]
where \(\nabla_t := \partial_t + [\Psi, \cdot]\) and \(\nabla_s := \partial_s + [\Phi, \cdot]\). The \(\varepsilon\)-dependent norms are created using the following simple rule that is given from the linearisation \(D^e\) of the Yang-Mills flow equations. For every \(\nabla_t\) and every 0-form \(\psi\), which descends from a 1-form in the \(t\)-direction, we put an \(\varepsilon\) in front of the norm; for every \(\nabla_s\) and every 0-form \(\phi\), coming from a 1-form in the \(s\)-direction, we multiply by \(\varepsilon^2\). The definition [6.1] contains, in the first line, all the 0-order \(L^p\)-norms and the \(L^p\)-norms of all the first derivatives; in the last two lines we can find the \(L^p\)-norms of some second derivatives. These can be interpreted in the following way. We split \(\alpha + \psi dt\) in two orthogonal components \(\alpha_k + \psi_k dt \in \text{im } d_\Sigma\) and \(\alpha_\psi + \psi_k dt \in \ker d_\Sigma^*\); on the one side, if \(\alpha + \psi dt \in \ker d_A^* + \Psi dt\), then

\[
\varepsilon\|d_A^*d_A\psi - \varepsilon^2\nabla_t\nabla_s\psi\|_{L^p} = \varepsilon\|d_A^*d_A\psi - \nabla_t\alpha\|_{L^p} \\
\leq \varepsilon\|d_A^*(d_A\psi - \nabla_t\alpha)\|_{L^p} + \varepsilon\|[(\partial_t - d_A) - d_A\Psi] \wedge \ast \alpha\|_{L^p},
\]

and thus \(\varepsilon\|d_A^*d_A\psi\|_{L^p}, \varepsilon^p\|\nabla_t\nabla_s\psi\|_{L^p}, \|d_A^*d_A\alpha\|_{L^p}\) and \(\varepsilon^2\|\nabla_t\nabla_s\alpha\|_{L^p}\) can be estimates by \(\|\alpha + \psi dt + \phi ds\|_{1,2,p,e}\) as we will discuss in the section [7]. On the other side, if \(\alpha + \psi dt = d_A\gamma + \nabla_t\gamma dt\), \(\gamma \in \Omega^0(\Sigma \times S^1 \times \mathbb{R}, g_\Sigma)\), then

\[
d_A^*d_A\alpha = d_A^*[F_A, \gamma], \quad d_A^*(d_A\psi - \nabla_t\alpha) = -d_A^*[\partial_t A - d_A\Psi, \gamma],
\]

\[
\nabla_t d_A\alpha = \nabla_t[F_A, \gamma], \quad \nabla_t(d_A\psi - \nabla_t\alpha) = -\nabla_t[(\partial_t A - d_A\Psi)\gamma],
\]

therefore, under some extra conditions on the curvature \(F_A - (\partial_t A - d_A\Psi)dt\), for example that \(F_A = 0\) and \(\partial_t A - d_A\Psi\) is smooth, the last two lines of (6.1) can be estimate with the first two if \(\alpha + \psi dt \in \text{im } d_\Sigma\). Thus, \(\|\cdot\|_{1,2,p,e}\) considers the \(L^p\)-norm of \(\xi\) and of its derivatives, but the \(L^p\)-norm of the second derivatives in the \(\Sigma \times S^1\)-directions only for the \(\ker d_A^* + \Psi dt\)-part of \(\xi\). This orthogonal splitting plays a fundamental role in the proof of the linear estimates of the section [7]. Next, we can define the Sobolev spaces

\[
\begin{align*}
W^{1,2,p} := & W^{1,2,p}(\Sigma \times S^1 \times \mathbb{R}, T^*(\Sigma \times S^1 \times \mathbb{R}) \otimes g_{P \times S^1 \times \mathbb{R}}), \\
W^{1,p} := & W^{1,p}(\Sigma \times S^1 \times \mathbb{R}, T^*(\Sigma \times S^1 \times \mathbb{R}) \otimes g_{P \times S^1 \times \mathbb{R}})
\end{align*}
\]

as the completion, respect to the norm \(\|\cdot\|_{1,2,p,1}\) and \(\|\cdot\|_{1,p,1}\), of the 1-forms

\[
\Omega^1(\Sigma \times S^1 \times \mathbb{R}, g_{P \times S^1 \times \mathbb{R}})
\]
with compact support; we denote by \( W^{1,2;p}(\Xi,\Xi) \) (respectively \( W^{1,2} \)) the space of all connections \( \Xi \) that satisfy \( \Xi - \Xi_0 \in W^{1,2;p} \) (respectively \( \Xi - \Xi_0 \in W^{1,2} \)) for a smooth connection \( \Xi_0 \in \mathcal{A}(P \times S^1 \times \mathbb{R}) \) and the limit conditions \( \lim_{r \to \pm \infty} \Xi = \Xi_\pm \) (respectively without limit condition). Furthermore, we denote by \( G_0^{2;p}(P \times S^1 \times \mathbb{R}) \) the completion of \( G_0(P \times S^1 \times \mathbb{R}) \) with respect to the Sobolev \( W^{1,p} \)-norm on \( 1 \)-forms, i.e. \( g \in G_0^{2;p}(P \times S^1 \times \mathbb{R}) \) if \( g^{-1}d_{\Xi \times S^1 \times \mathbb{R}}g \in W^{1,p} \) and by \( G_0^{1,2;p}(P \times S^1 \times \mathbb{R}) \) the completion respect to the norm \( \| \cdot \|_{1,2;p,\varepsilon} \) on the \( 1 \)-forms. In addition, we denote by \( \tilde{G}_0^{1,2,2}(P \times S^1 \times \mathbb{R}) \) the gauge group such that an element \( g \) is locally in \( G_0^{1,2,2}(P \times S^1 \times \mathbb{R}) \), i.e. we allow also elements that do not vanish at \( \pm \infty \). We conclude this section proving the following Sobolev estimates.

**Theorem 5.1 (Sobolev estimate).** We choose \( 1 \leq p, q < \infty \). Then there is a constant \( c_S \) such that for any \( \xi \in W^{1,p}, 0 < \varepsilon \leq 1 \):

1. If \( -\frac{4}{q} \leq 1 - \frac{4}{p} \), then

\[
\|\xi\|_{0,q,\varepsilon} \leq c_S \varepsilon^{\frac{4}{p} - \frac{4}{q}}\|\xi\|_{1,p,\varepsilon}.
\]

2. If \( 0 < 1 - \frac{4}{p} \), then

\[
\|\xi\|_{\infty,\varepsilon} \leq c_S \varepsilon^{-\frac{4}{p}}\|\xi\|_{1,p,\varepsilon}.
\]

**Proof.** Analogously as for the lemma 4.1 in [7], we can define \( \tilde{\xi} = \tilde{\alpha} + \tilde{\psi}dt + \tilde{\phi}ds \) by \( \tilde{\alpha}(t,s) = \alpha(\varepsilon t, \varepsilon^2 s), \tilde{\psi}(t,s) = \varepsilon \psi(\varepsilon t, \varepsilon^2 s) \) and \( \tilde{\phi}(t,s) = \varepsilon^2 \phi(\varepsilon t, \varepsilon^2 s) \). Thus, \( \|\tilde{\xi}\|_{1,p,\varepsilon} = \varepsilon^2 \|\xi\|_{W^{1,p},\varepsilon} \). Therefore the theorem follows from the standard Sobolev’s inequality.

\[ \square \]

### 6. Morse homologies

In this section we want to define the Morse homologies defined using the heat flow and the Yang-Mills \( L^2 \)-flow. We start with the Morse homology of the loop space on \( \mathcal{M}^b(P) \). First, we introduce the moduli spaces

\[
\mathcal{M}^0(\Xi,\Xi_\pm) := \{ \Xi \in W^{1,2;2}(\Xi,\Xi_\pm); \Xi \text{ satisfies } (2.1), (2.2) \},
\]

\[
\tilde{\mathcal{M}}^0(\Xi,\Xi_\pm) := \mathcal{M}^0(\Xi,\Xi_\pm)/PG_\infty
\]

where

\[
PG_\infty := \{ g \in G_0^{1,2;2}(P \times S^1 \times \mathbb{R}); \exists S > 0 \text{ such that for } |s| \geq S, g(s) = 1 \}.
\]

Now, we choose a regular value \( b \) of the energy functional \( E^H \). In order to define the Morse homology of the loop space \( \mathcal{L}^b\mathcal{M}^b(P) \) in our Morse-Bott setting, where a critical loop is an equivalence class \( [A + \Psi dt] \) of perturbed geodesics with \( A + \Psi dt \in \text{Crit}_H^b \), we need to count the flow lines between the critical loops with Morse index difference 1 in the following way. We define \( \text{Crit}_H^b := \text{Crit}_H^b/G_0(P \times S^1) \) and we consider the space of flow lines between two loops \( \gamma_\pm \in \text{Crit}_H^b \):

\[
\mathcal{F}L^0(\gamma_-,\gamma_+) = \{ \Xi \in W^{1,2;2}(\Xi,\Xi_\pm); \Xi \text{ satisfies } (2.1), (2.2), [A_\pm] = \gamma_\pm \}
\]

and thus the moduli space

\[
\mathcal{M}^0(\gamma_-,\gamma_+) := \mathcal{F}L^0(\gamma_-,\gamma_+)/\tilde{G}_0^{1,2,2}(P \times S^1 \times \mathbb{R});
\]
Then we organise the critical loops of $[\text{Crit}^b E^H]$ in a chain complex where

$$C^E_k := \oplus_{\gamma \in [\text{Crit}^b E^H], \text{index}_{E^H}(\gamma) = k} \mathbb{Z}_2 \gamma$$

and the boundary operator $\partial^E_k : C^E_k \to C^E_{k-1}$ by

$$\partial^E_k \gamma := \sum_{\gamma_+ \in C^E_{k-1}} (\sharp_{\mathbb{Z}_2} (\mathcal{M}^0(\gamma_, \gamma_+) / \mathbb{R})) \gamma_+.$$ 

If the functional $E^H$ satisfies the transversality condition, then $\partial^E_k \partial^E_{k+1} = 0$ and in this case we can define the Morse homology

$$(6.1) \quad HM_* (\mathcal{L}^b \mathcal{M}^0(P), \mathbb{Z}_2) := \ker \partial^E_k / \text{im} \partial^E_{k+1}.$$ 

As we have already mentioned, by the work of Weber (cf. [17]), for a generic perturbation, the transversality condition is satisfied and thus the Morse homology of the loop space $HM_* (\mathcal{L}^0 \mathcal{M}(P), \mathbb{Z}_2)$ is well defined.

**Remark 6.1.** For any two perturbed geodesics $\gamma_\pm \in [\text{Crit}^b E^H]$ and any two representatives $\Xi_\pm$ we can identify the moduli spaces $\mathcal{M}^0(\gamma_-, \gamma_+)$ and $\mathcal{M}^0(\Xi_-, \Xi_+)$, in particular we have

$$\sharp_{\mathbb{Z}_2} (\mathcal{M}^0(\gamma_-, \gamma_+) / \mathbb{R}) = \sharp_{\mathbb{Z}_2} (\mathcal{M}^0(\Xi_-, \Xi_+) / \mathbb{R}).$$

Next, we define the Morse homology for the Yang-Mills case. First, we denote by $\mathcal{M}^c(\Xi_-, \Xi_+)$ and the moduli spaces

$$\mathcal{M}^c(\Xi_-, \Xi_+) := \{ \Xi \in W^{1,2,2} (P \times S^1 \times \mathbb{R}) ; \Xi \text{ satisfies } (12) \},$$

$$\bar{\mathcal{M}}^c(\Xi_-, \Xi_+) := \mathcal{M}^c(\Xi_-, \Xi_+) / PG_\infty.$$ 

Also in this case we can define a Morse homology for

$$A^{\gamma,b}_* (P \times S^1) / G_0 (P \times S^1);$$

in order to do that we consider the chain complex $C^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k := \oplus_{\gamma \in [\text{Crit}^b \mathcal{Y} \mathcal{M}^{\gamma,b}, H]} \mathbb{Z}_2 \gamma,$

where $[\text{Crit}^b \mathcal{Y} \mathcal{M}^{\gamma,b}, H] := \text{Crit}^b \mathcal{Y} \mathcal{M}^{\gamma,b}, H / G_0 (\Sigma \times S^1),$ with the boundary operator $\partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k : C^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k \to C^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_{k-1}$ defined by

$$\partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k \gamma := \sum_{\gamma_+ \in C^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_{k-1}} (\sharp_{\mathbb{Z}_2} (\mathcal{M}^c(\gamma_, \gamma_+) / \mathbb{R})) \gamma_+,$$

where $\mathcal{M}^c(\theta_-, \theta_+)$ is the moduli space

$$\mathcal{M}^c(\theta_-, \theta_+) := \mathcal{F} \mathcal{L}^c(\theta_-, \theta_+) / G_0^{1,2,2} (P \times S^1 \times \mathbb{R}),$$

$$\mathcal{F} \mathcal{L}^c(\theta_-, \theta_+) := \{ \Xi \in W^{1,2,2} (\Xi_-, \Xi_+); \Xi \text{ satisfies } (12), |\Xi_k| = \theta \pm \},$$

The functional $\mathcal{Y} \mathcal{M}^{\gamma,b}, H$ will inherits the transversality property of $E^H$ provided that $\varepsilon$ is small enough, in this case $\partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k \partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k = 0$ and thus we can define the Morse homology

$$(6.2) \quad HM_* (A^{\gamma,b}_* (P \times S^1) / G_0 (P \times S^1), \mathbb{Z}_2) := \ker \partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_k / \text{im} \partial^{\mathcal{Y} \mathcal{M}^{\gamma,b}, H}_{k+1}.$$
Remark 6.2. Also in this case, for any two orbits of perturbed Yang-Mills connections \( \theta \in \text{Crit}^b_{\mathcal{M}^s} \) and any two representatives \( \Xi \in \text{Crit}^b_{\mathcal{M}^s} \) we can identify the moduli spaces \( \mathcal{M}^s(\theta, \Xi) \) and \( \mathcal{M}^s(\Xi, \Xi) \); in particular we have
\[
\sharp_{\mathbb{Z}}(\mathcal{M}^s(\theta, \Xi)/\mathbb{R}) = \sharp_{\mathbb{Z}}(\mathcal{M}^s(\Xi, \Xi)/\mathbb{R}).
\]

The aim of this paper is to show that the two Morse homologies \( (6.1) \) and \( (6.2) \) are isomorph and we give the proof in the section 10. In order to do this we need to show that there is a bijective map
\[
\mathcal{R}^{b, \epsilon} : \mathcal{M}^0(\Xi, \Xi) \to \mathcal{M}^s \left( T^{b, \epsilon}(\Xi), T^{b, \epsilon}(\Xi) \right)
\]
for each regular value \( b \) of \( E^H \), every pair \( \Xi, \Xi \in \text{Crit}^b_{\mathcal{M}^s} \) with index difference 1 and for \( \epsilon \) sufficiently small; for this purpose, we will proceed in the following way. In section 7 we will prove some linear estimates using the linear operator, for a 1-form \( \xi = \alpha + \psi dt + \phi ds \in W^{1,2;p} \)
\[
D^s(\Xi)(\xi) := D^s_1(\Xi)(\xi) + D^s_2(\Xi)(\xi) + D^s_3(\Xi)(\xi) ds
\]
where the first two terms are the linearization of \( (6.2) \), i.e.
\[
D^s_1(\Xi)(\xi) := \nabla_s \alpha - d_A \phi + \frac{1}{\epsilon^2} d_A^s d_A \alpha + \frac{1}{\epsilon^2} \ast [\alpha \wedge * F_A] - \partial_t \nabla_t \alpha + d_A \nabla_t \psi - 2[\psi, (\partial_t A - d_A \Psi)] - d * X_t(A) \alpha,
\]
\[
D^s_2(\Xi)(\xi) := \nabla_s \psi - \nabla_t \phi + \frac{2}{\epsilon^2} [\alpha \wedge (\partial_t A - d_A \Psi)] - \frac{1}{\epsilon^2} \nabla_t d_A^s \alpha + \frac{1}{\epsilon^2} d_A^s d_A \psi.
\]
and the third one is, for a fixed reference connection \( \Xi^0 = A^0 + \Psi^0 dt + \Phi^0 ds \),
\[
D^s_3(\Xi)(\xi) := \nabla_s^0 \phi - \frac{1}{\epsilon^2} d_A^s \alpha + \frac{1}{\epsilon^2} \nabla_t^0 \psi.
\]
The linear operator \( D^s(\Xi) \) can also be seen as the linearisation of the map
\[
\mathcal{F}^s(\Xi) := \mathcal{F}^s_1(\Xi) + \mathcal{F}^s_2(\Xi) dt + \mathcal{F}^s_3(\Xi) ds
\]
where
\[
\mathcal{F}^s_1(\Xi) := \partial_s A - d_A \Phi + \frac{1}{\epsilon^2} d_A^s F_A - \nabla_t (\partial_t A - d_A \Psi),
\]
\[
\mathcal{F}^s_2(\Xi) := \partial_s \psi - \nabla_t \phi - \frac{1}{\epsilon^2} d_A^s (\partial_t A - d_A \Psi),
\]
\[
\mathcal{F}^s_3(\Xi) := \nabla_s^0 (\Phi - \Phi_2) - \frac{1}{\epsilon^2} d_A^s (A - A_2) + \frac{1}{\epsilon^2} \nabla_t^0 (\Psi - \Psi_2)
\]
and \( A_2 + \Psi_2 dt + \Phi_2 ds := \mathcal{K}_2(A^0 + \Psi^0 dt + \Phi^0 ds) \); we will discuss the map \( \mathcal{K}_2 \) in the section 9.

After computing some quadratic estimates in section 8 we will prove the existence and the local uniqueness of the map \( \mathcal{R}^{b, \epsilon} \) (section 10). In the following sections 11, 12 and 13 we will prove some a priori estimates that we will use in the section 15 in order to prove the surjectivity of \( \mathcal{R}^{b, \epsilon} \). The section 14 is devoted to prove the Coulomb gauge condition theorem.
7. Linear estimates for the Yang-Mills flow operator

As we already mentioned, by the Weber’s regularity theorem (cf. [17], theorem 1.13), we can assume that the energy functional $E^H$ is Morse-Smale. In this section we will prove a linear estimate, theorem [7.8] for the operator $D^s(\Xi)$ for a perturbed geodesic flow $\Xi = A + \Phi dt + \Psi ds$. The main idea is to divide the linear operator respect to the orthogonal splitting ker $d_A^* + \Phi dt \oplus \im d_A + \Psi dt$ and to use different linear estimates on the two parts. In order to investigate this we need to decompose, \[ \xi \]

*\[ \xi \]

in the other hand, using (7.4), the second component becomes

\[ \text{geodesic flow } \Xi = A \]

which has a unique solution $\gamma$ whose existence and uniqueness can be proved as in the lemma 6.4 of [7]; then we define

\[ \alpha_i + \psi_i dt := d_A \gamma + \nabla_t \gamma dt, \quad \alpha_k + \psi_k dt := (\alpha + \psi dt) - (\alpha_i + \psi_i dt) \]

and since the splitting is orthogonal,

\[ \|d_A \gamma + \nabla_t \gamma dt\|_{0,p,\varepsilon} \leq \|\alpha + \psi dt\|_{0,p,\varepsilon}, \quad \|\alpha_k + \psi_k dt\|_{0,p,\varepsilon} \leq \|\alpha + \psi dt\|_{0,p,\varepsilon}. \]

By definition and using the commutation formulas (2.4), (2.5) we have also that

\[ d_A \nabla_t \psi_k = \frac{1}{\varepsilon^2} d_A^* d_A \alpha_k, \]

\[ d_A \nabla_t \psi_i = \nabla_t \nabla_t \alpha_i - 2[\{\partial_t A - d_A \Psi\}, \psi_i] - \nabla_t [\partial_t A - d_A \Psi, \gamma], \]

\[ \nabla_t d_A^* \alpha_i = \{d_A^* \nabla_t A + \star[\alpha_i \wedge \star(\partial_t A - d_A \Psi)]\} + d_A^* [\{\partial_t A - d_A \Psi\}, \gamma] \]

\[ = d_A^* d_A \psi_i + \star[\alpha_i \wedge \star(\partial_t A - d_A \Psi)] + d_A^* [\{\partial_t A - d_A \Psi\}, \gamma] \]

\[ = d_A^* d_A \psi_i + 2 \{\alpha_i \wedge \star(\partial_t A - d_A \Psi)\} - \star[d_A \star(\partial_t A - d_A \Psi), \gamma]. \]

Now, we can write the components of the linear operator using this splitting. On the one hand, the first component of $D^s(\Xi)(\xi + \phi ds)$, defined by (6.4), is, using the identities (7.4),

\[ D^s_1(\Xi)(\xi + \phi ds) = \nabla s \alpha_k + \frac{1}{\varepsilon^2} d_A^* d_A \alpha_k + \frac{1}{\varepsilon^2} d_A^* d_A \alpha_k - \nabla_t \nabla_t \alpha_k \]

\[ + \nabla_t \alpha_i - d_A \phi - \nabla_t (\partial_t A - d_A \Psi), \gamma \]

\[ - 2[\psi_k, (\partial_t A - d_A \Psi)] - d \star X_t(A) \alpha; \]

in the other hand, using (7.4), the second component becomes

\[ D^s_2(\Xi)(\xi + \phi ds) = \nabla s \psi_k - \nabla_t \nabla_t \psi_k + \frac{1}{\varepsilon^2} d_A^* d_A \psi_k + \frac{2}{\varepsilon^2} \star[\alpha_k \wedge \star(\partial_t A - d_A \Psi)] \]

\[ + \nabla_s \psi_i - \nabla_t \phi. \]

The third component is the easiest to investigate, because it depends only on $\alpha_i + \psi_i dt$ and on $\phi$:

\[ D^s_3(\Xi)(\xi + \phi ds) = \nabla s \phi - \frac{1}{\varepsilon^4} d_A^* \alpha + \frac{1}{\varepsilon^2} \nabla_t \psi = \nabla s \phi - \frac{1}{\varepsilon^4} d_A^* \alpha_i + \frac{1}{\varepsilon^2} \nabla_t \psi. \]
Next, the idea is to consider \( \mathcal{D}^\xi(\Xi)(\xi + \phi ds) \) as the sum of the following three operators

\[
\mathcal{D}^{\xi_1}(\Xi)(\xi + \phi ds) := \nabla_s \alpha_k + \frac{1}{\varepsilon^2} d^*_A d_A \alpha_k + \frac{1}{\varepsilon^2} d_A d^*_A \alpha_k - [\psi_k, (\partial_t A - d_A \Psi)]
- \nabla_t \nabla_t \alpha_k + \left( \nabla_s \psi_k - \nabla_t \nabla_t \psi_k + \frac{1}{\varepsilon^2} d^*_A d_A \psi_k \right) dt
+ \frac{1}{\varepsilon^2} \ast \left( \alpha_k \Lambda \ast (\partial_t A - d_A \Psi) \right) dt,
\]

\[
\mathcal{D}^{\xi_2}(\Xi)(\xi + \phi ds) := \nabla_s \alpha_k - d_A \phi + \left( \nabla_s \psi_k - \nabla_t \phi \right) dt
+ \left( \nabla_s \phi - \frac{1}{\varepsilon^2} d^*_A \alpha + \frac{1}{\varepsilon^2} \nabla_t \psi \right) ds,
\]

\[
\text{Rest}^\xi(\Xi)(\xi + \phi ds) := - \left[ \nabla_t (\partial_t A - d_A \Psi), \gamma \right] - [\psi_k, (\partial_t A - d_A \Psi)]
- d \ast X_1(A) \alpha + \frac{1}{\varepsilon^2} \ast \left( \alpha_k \Lambda \ast (\partial_t A - d_A \Psi) \right) dt
\]

and to project them in to the two parts of the orthogonal splitting im \( d_{A^+} \Psi dt \oplus \ker d_{A^+}^* \Psi dt \). The result is that the important part of \( \mathcal{D}^{\xi_1}(\Xi) \) lies in \( \ker d_{A^+}^* \Psi dt \) and that of \( \mathcal{D}^{\xi_2}(\Xi) \) in \( \text{im} d_{A^+} \Psi dt \) as is showed in the next lemma; in other words, we interchange the operator \( \mathcal{D}^\xi(\Xi) \) with the projection in the two parts of the splitting.

We recall that, by (3.3) and (3.4), we can assume that
\[
\text{(7.5)} \quad \| \partial_t A - d_A \Psi \|_{L^\infty} + \| \nabla_t (\partial_t A - d_A \Psi) \|_{L^\infty} + \| \partial_s A - d_A \Psi \|_{L^\infty} \leq c_0
\]

for a positive constant \( c_0 \).

**Lemma 7.1.** We choose \( b, p > 0 \). For any geodesic flow \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+), \Xi_\pm \in \text{Crit}_{E^H}^b, \) there exists a positive constant \( c \) such that
\[
\| \Pi \_\text{im} d_{A^+} \Psi dt \mathcal{D}^{\xi_1}(\Xi)(\xi + \phi ds) \|_{0, p, \varepsilon} \leq c \left( \| \alpha_k \|_{L^p} + \| \psi_k \|_{L^p} + \| \nabla_t \alpha_k \|_{L^p} \right),
\]

\[
\left\| \left(1 - \Pi \_\text{im} d_{A^+} \Psi dt \right) \left( \nabla_s \alpha_k - d_A \phi + \left( \nabla_s \psi_k - \nabla_t \phi \right) dt \right) \|_{0, p, \varepsilon} \leq c \| \alpha_k \|_{L^p},
\]

\[
\varepsilon^2 \| \text{Rest}^\xi(\Xi)(\xi + \phi ds) \|_{0, p, \varepsilon} \leq \varepsilon \| \xi \|_{0, p, \varepsilon}
\]

for all \( \xi + \phi ds \in W^{1,2;p} \) and using the splitting \( \xi =: (\alpha_k + \psi_k dt) + (\alpha_k + \psi_k dt) \) defined by (7.7) and (7.3). We denote by \( \Pi \_\text{im} d_{A^+} \Psi dt \) the projection in to the linear subspace im \( d_{A^+} \Psi dt \).

**Proof.** First, we remark that
\[
\langle \nabla_s \alpha_k + \frac{1}{\varepsilon^2} d^*_A d_A \alpha_k, d_A \omega \rangle + \varepsilon^2 \langle \nabla_s \psi_k, \nabla_t \omega \rangle
= \langle \nabla_s (d^*_A \alpha_k - \varepsilon^2 \nabla_t \psi_k), \omega \rangle + \frac{1}{\varepsilon^2} \ast \left[ F_A \Lambda \ast d_A \alpha_k \right]
+ \langle \ast [\partial_s A - d_A \phi] \Lambda \ast \alpha_k \rangle + \varepsilon^2 [(\partial_s A \Psi - \partial_t \Phi + [\Psi, \Phi]), \psi_k, \omega]
= \langle \ast [\partial_s A - d_A \phi] \Lambda \ast \alpha_k \rangle + \varepsilon^2 [(\partial_s A \Psi - \partial_t \Phi + [\Psi, \Phi]), \psi_k, \omega],
\]

where we used the commutation formulas (2.5) and
\[
(7.6) \quad \langle \nabla_s, \nabla_t \rangle \omega = [(\partial_s \Psi - \partial_t \Phi + [\Psi, \Phi]), \omega]
\]
for any 0-form \( \omega \in \Omega^0(\Sigma \times S^1 \times \mathbb{R}, g_P) \).

\[
\frac{1}{\varepsilon^2} d_A^* d_A \alpha_k - d_A \nabla_i \psi_k, d_A \omega = 0,
\]

\[
\frac{1}{\varepsilon^2} \nabla_i d_A^* \alpha_k - \nabla_i \nabla_i \psi_k, \nabla \omega = 0,
\]

\[
(\nabla_i \nabla_i \alpha_k, d_A \omega) + (-d_A^2 \nabla_i \alpha_k, \nabla \omega) = - \ast \left( \partial_i A - d_A \Psi \right) \wedge \ast \nabla_i \alpha_k, \omega),
\]

\[
(\nabla_i d_A \psi_k, d_A \omega) + (d_A^2 d_A \psi_k, \nabla \omega) = \ast \left( (\partial_i A - d_A \Psi) \wedge \ast d_A \psi_k \right), \omega) = \langle \ast \left( (\partial_i A - d_A \Psi) \wedge \ast d_A \psi_k \right), \nabla \omega \rangle
\]

where for the last step we used that \( d_A^2 (\partial_i A - d_A \Psi) = 0 \). Next, we choose \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
\left\| \Pi_{im} d_A + \psi_i d d \mathcal{E}^{-1}(\Xi)(\xi + \phi ds) \right\|_{0, p, q}
\]

\[
\leq \sup_{\omega \in \Omega^0(\Sigma \times S^1 \times \mathbb{R})} \left( \mathcal{D}^{\mathcal{E}^{-1}}(\Xi)(\xi + \phi ds), d_A \omega + \nabla_i \omega dt \right)
\]

\[
\leq c \left( \|\alpha_k\|_{L^p} + \varepsilon^2 \|\psi_k\|_{L^p} + \|\nabla_i \alpha_k\|_{L^p} \right).
\]

The last estimate follows directly from the next identities, from the Hölder’s inequality, from \( \|\omega\|_{L^p} \leq \|\omega\|_{L^q} \) and from lemma 7.2. The second estimate of the lemma follows from the identity

\[
\nabla_i \alpha_k - d_A \phi + (\nabla_i \phi - \nabla_i \phi) dt = \nabla_i \alpha_k - d_A \phi + (\nabla_i \nabla_i \gamma - \nabla_i \phi) dt
\]

\[
= d_A + \psi_i dt (\nabla_i \gamma - \phi) + [(\partial_i A - d_A \Psi), \gamma] - ([\partial_i \Phi - \partial_i \Psi - [\Phi, \Psi]), \gamma] dt,
\]

from the a priori estimate (7.3) and from \( \|\gamma\|_{L^p} \leq c \|d_A \gamma\|_{L^p} = c \|\alpha_k\|_{L^p} \). The third estimate follows directly from the definition of \( \text{Rest}^e \) and the \( L^\infty \)-bound (7.6) for the curvature terms \( \partial_i A - d_A \Psi, \nabla_i (\partial_i A - d_A \Psi) \).

Lemma 7.2. We choose a regular value \( b \) of \( E^H \). There is a positive constant \( c \) such that for any perturbed geodesic flow \( A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+), \Xi_\pm \in \text{Crit}_{\pm}^{\psi_H}, \)

\begin{equation}
(7.7)
\end{equation}

\[
d_A^* d_A (\partial_i \Psi - \nabla_i \Phi) = 2 \ast [B_s \wedge \ast B_t],
\]

\begin{equation}
(7.8)
\end{equation}

\[
\|\partial_i \Psi - \nabla_i \Phi\|_{L^\infty} \leq c
\]

hold.

Proof. We define \( B_t = \partial_t A - d_A \Psi \) und \( B_s = \partial_s A - d_A \Phi \) then \( d_A^* B_t = d_A^* B_s = 0 \) and therefore

\[
\nabla_i d_A^* B_t = \ast [B_s \wedge \ast B_t] + d_A^2 \nabla_i B_t = 0
\]

\[
\nabla_i d_A^* B_s = \ast [B_t \wedge \ast B_s] + d_A^2 \nabla_i B_s
\]

\[
= - \ast [B_s \wedge \ast B_t] + d_A^2 \nabla_i B_s = 0
\]

yields to \( d_A^2 \nabla_i B_s - d_A^2 \nabla_i B_t = 2 \ast [B_s \wedge \ast B_t] \) where

\[
d_A^2 \nabla_i B_s - d_A^2 \nabla_i B_t = d_A^2 (\nabla_i \Psi - \nabla_i \Phi - [\Phi, \Psi]).
\]

Finally, we can finish the proof of the lemma, i.e.

\[
d_A^* d_A (\partial_i \Psi - \nabla_i \Phi) = d_A^* d_A (\nabla_i \Psi - \nabla_i \Phi - [\Phi, \Psi]) = 2 \ast [B_s \wedge \ast B_t].
\]

Furthermore, for a positive constant \( c \)

\[
\|\partial_i \Psi - \nabla_i \Phi\|_{L^\infty} \leq 8 \|B_s\|_{L^\infty} \|B_t\|_{L^\infty} \leq c
\]
Theorem 7.3. We choose a regular value $b$ of $E^H$, then there are two positive constants $c$ and $\varepsilon_0$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+)$, $\Xi \in \text{Crit}_{E}^b$, any 1-form $\xi = \alpha + \psi dt + \phi ds \in W^{1,2,p}$ and for $0 < \varepsilon < \varepsilon_0$

$\left(7.9\right)$

$$\|\xi\|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \|D^\phi(\Xi)(\xi)\|_{0,\varepsilon} + c\|\pi_A(\alpha)\|_{L^p}.$$  

$\left(7.10\right)$

$$\|(1 - \pi_A)(\xi)\|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \|D^\phi(\Xi)(\xi)\|_{0,\varepsilon} + c\varepsilon\|\pi_A(\alpha)\|_{L^p} + c\varepsilon^2 \|\nabla \pi_A(\alpha)\|_{L^p} + c\varepsilon^2 \|\nabla_t \pi_A(\alpha)\|_{L^p},$$

$\left(7.11\right)$

$$\|(1 - \pi_A)\alpha\|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \left(\|D^\phi(\Xi)(\xi)\|_{0,\varepsilon} + \|\nabla \pi_A(\alpha)\|_{L^p} + \|\pi_A(\alpha)\|_{L^p}\right)$$

$$+ c\varepsilon^2 \left(\|\nabla_t \pi_A(\alpha)\|_{L^p} + \|\nabla_t \pi_A(\alpha)\|_{L^p}\right).$$

In order to prove the last statement we need the next two theorems which will be proven in the subsections $\S 7.1$ $\S 7.2$

Theorem 7.4. We choose a regular value $b$ of $E^H$, then there are two positive constants $c$ and $\varepsilon_0$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+)$, $\Xi \in \text{Crit}_{E}^b$, any 1-form $\alpha \in W^{2,1,p}$, any 0-form $\psi \in W^{2,1,p}$ and for $0 < \varepsilon < \varepsilon_0$

$\left(7.12\right)$

$$\left(\begin{array}{l}
\|\alpha\|_{L^p} + \|d_A \alpha\|_{L^p} + \|d_A^\phi \alpha\|_{L^p} + \|d_A d_A \alpha\|_{L^p} + \|d_A d_A^\phi \alpha\|_{L^p} \\
+ \varepsilon \|\nabla \alpha\|_{L^p} + \varepsilon^2 \|\nabla \nabla t \alpha\|_{L^p} + \varepsilon \|d_A \nabla_t \alpha\|_{L^p} + \varepsilon \|\nabla_t d_A \alpha\|_{L^p} + \varepsilon \|\nabla \nabla_t \pi_A(\alpha)\|_{L^p}
\end{array}\right)$$

$$\leq c\left(\varepsilon^2 \|\nabla \nabla s - \varepsilon^2 \nabla t + \Delta_A\right) \alpha\|_{L^p} + c\|\pi_A(\alpha)\|_{L^p},$$

$\left(7.13\right)$

$$\left(\begin{array}{l}
\|\psi\|_{L^p} + \|d_A \psi\|_{L^p} + \|d_A^\phi \psi\|_{L^p} + \varepsilon \|\nabla \nabla \psi\|_{L^p} + \varepsilon \|\nabla_t \nabla \psi\|_{L^p} \\
+ \varepsilon \|d_A \nabla_t \psi\|_{L^p} + \varepsilon \|\nabla_t d_A \psi\|_{L^p} + \varepsilon \|\nabla_s \alpha\|_{L^p}
\end{array}\right)$$

$$\leq c\left(\varepsilon^2 \|\nabla \nabla s - \varepsilon^2 \nabla t + \Delta_A\right) \psi\|_{L^p}. $$

Theorem 7.5. We choose a regular value $b$ of $E^H$, then there are two positive constants $c$ and $\varepsilon_0$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+)$, $\Xi \in \text{Crit}_{E}^b$, any 1-form $\alpha + \psi dt \in W^{1,1,p} \cap \text{im} d_A + \Psi dt$, any 0-form $\phi \in W^{1,1,p}$ and for any $0 < \varepsilon < \varepsilon_0$

$\left(7.14\right)$

$$\left(\begin{array}{l}
\|\alpha\|_{L^p} + \|d_A \alpha\|_{L^p} + \varepsilon \|\nabla \alpha\|_{L^p} + \varepsilon \|\nabla_t \alpha\|_{L^p} + \varepsilon \|\nabla_t \psi\|_{L^p} \\
+ \varepsilon^3 \|\nabla_s \psi\|_{L^p} + \varepsilon^2 \|\phi\|_{L^p} + \varepsilon^2 \|d_A \phi\|_{L^p} + \varepsilon^3 \|\nabla_t \phi\|_{L^p} + \varepsilon^4 \|\nabla_s \phi\|_{L^p}
\end{array}\right)$$

$$\leq c\varepsilon^2 \|\nabla \nabla s - d_A \phi\|_{L^p} + c\varepsilon^3 \|\nabla_s \psi - \nabla_t \phi\|_{L^p}$$

$$+ c\varepsilon^4 \left\|\nabla_t \phi - \frac{1}{\varepsilon^2} d_A^\phi + \frac{1}{\varepsilon^2} \nabla_s \psi\right\|_{L^p}. $$

$^2$A 0-form $\gamma$, $i = 0, 1$, is an element of $W^{1,1,k,p}$, if $j$ derivatives of $\gamma$ in the $\Sigma$-direction, $l$ derivatives in the $S^1$-direction and $k$ derivatives in the $\mathbb{R}$ direction are in $L^p$. 

Proof of theorem 7.3 By theorem 7.4 and the lemma 7.1
\[ \|\alpha_k + \psi dt\|_{1,2, p,\epsilon} \leq c\|\varepsilon^2 \nabla_s \psi_k - \varepsilon^2 \nabla_s (1 - \pi_A)\alpha_k + \Delta_A \alpha_k\|_{L^p} + c\|\pi_A(\alpha)\|_{L^p} + c\varepsilon \|\nabla \psi_k\|_{L^p} + c\varepsilon^2 \|\nabla \alpha_k\|_{L^p} \]
and by theorem 7.5 and the lemma 7.1
\[ \|\alpha_i + \psi dt\|_{1,2, p,\epsilon} \leq c\|\nabla \psi_k\|_{L^p} + c\|\nabla \alpha_k\|_{L^p} + c\varepsilon^2 \|\nabla \psi_k\|_{L^p} + c\varepsilon^2 \|\nabla \alpha_k\|_{L^p} ; \]
for \( \varepsilon \) small enough we can conclude therefore
\[ \|\xi\|_{1,2, p,\epsilon} \leq c\|\nabla \psi_k\|_{L^p} + c\|\pi_A(\alpha)\|_{L^p} . \]
The second estimate of the theorem follows from (7.13) and
\[ \|\nabla \psi_k\|_{L^p} + c\|\nabla \alpha_k\|_{L^p} + c\varepsilon^2 \|\nabla \psi_k\|_{L^p} + c\varepsilon^2 \|\nabla \alpha_k\|_{L^p} + c\pi_A(\alpha)\|_{L^p} . \]
In order to prove the third estimate we need the following one. We choose \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[ \|\Pi_{\text{im} d_A + d_A} d^* A \nabla s \pi_A(\alpha) dt\|_{0, p,\epsilon} \leq \sup_{\gamma \in \Omega^p} \frac{\varepsilon^2 (d^* A \nabla s \pi_A(\alpha), \nabla_t \gamma)}{\|d^* A t + \nabla_t dt\|_{L^q}} \]
\[ = \sup_{\gamma \in \Omega^p} \frac{\varepsilon^2 (\nabla_t \pi_A(\alpha), d^* A \nabla_t \gamma)}{\|d^* A t + \nabla_t dt\|_{L^q}} \]
\[ = \sup_{\gamma \in \Omega^p} \frac{\varepsilon^2 (\nabla_t \pi_A(\alpha), \nabla_t d^* A \gamma)}{\|d^* A \gamma + \nabla_t \gamma dt\|_{L^q}} \]
\[ \leq \sup_{\gamma \in \Omega^p} \frac{\varepsilon^2 \|\nabla_t \nabla s \pi_A(\alpha)\|_{L^p} \|d^* A \gamma + \nabla_t \gamma dt\|_{L^q} + c\|\nabla_t \pi_A(\alpha)\|_{L^p} \|\gamma\|_{L^q}}{\|d^* A \gamma + \nabla_t \gamma dt\|_{L^q}} \]
where the last inequality follows from the estimate \( \|\gamma\|_{L^q} \leq c\|d^* A \gamma\|_{L^q} \). Thus, since by (7.3)
\[ \varepsilon^2 \|\Pi_{\text{im} d_A + d_A} \text{Rest} (\nabla s \psi)\|_{0, p,\epsilon} \leq c\varepsilon^2 \|\nabla \psi\|_{L^p} + c\varepsilon^2 \|\psi\|_{L^p} + c\varepsilon^2 \|\nabla s (1 - \pi_A)\alpha\|_{L^p} + c\|\Pi_{\text{im} d_A + d_A} d^* A \nabla s \pi_A(\alpha) dt\|_{0, p,\epsilon} , \]
\[ \|\alpha_i + \psi dt\|_{1,2, p,\epsilon} \leq c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} \text{Rest} (\nabla s \psi)\|_{0,2,\epsilon} + c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} \text{Rest} (\nabla s \psi)\|_{0,2,\epsilon} + c\varepsilon^2 \|\nabla \psi\|_{L^p} + c\varepsilon^2 \|\psi\|_{L^p} + c\varepsilon^2 \|\nabla s (1 - \pi_A)\alpha\|_{L^p} + c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} d^* A \nabla s \pi_A(\alpha) dt\|_{0, p,\epsilon} , \]
and
\[ \|\alpha_i + \psi dt\|_{1,2, p,\epsilon} \leq c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} \text{Rest} (\nabla s \psi)\|_{0, p,\epsilon} + c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} \text{Rest} (\nabla s \psi)\|_{0, p,\epsilon} + c\varepsilon^2 \|\nabla \psi\|_{L^p} + c\varepsilon^2 \|\psi\|_{L^p} + c\varepsilon^2 \|\nabla s (1 - \pi_A)\alpha\|_{L^p} + c\varepsilon^2 \|\Pi_{\text{im} d_A + d_A} d^* A \nabla s \pi_A(\alpha) dt\|_{0, p,\epsilon} , \]
\[ \|\alpha_k - \pi_A(\alpha)\|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \left\| (1 - \Pi_{\text{im} d_A + \phi dt} D^*_1 \left( (1 - \pi_A) \xi + \phi ds \right) \right\|_{L^p} \]
\[ \leq c\varepsilon^2 \left( \|D^*_1 (\xi + \phi ds)\|_{L^p} + \|\psi\|_{L^p} + \|\alpha\|_{L^p} \right) \]
\[ + c\varepsilon^2 \left( \|\nabla_s \pi_A(\alpha)\|_{L^p} + \|\nabla_t \pi_A(\alpha)\|_{L^p} + \|\nabla_t \nabla_s \pi_A(\alpha)\|_{L^p} \right) \]
and finally we can conclude that
\[ \| (1 - \pi_A) \alpha \|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \left( \|D^*(\xi)\|_{0,p,\varepsilon} + \|\nabla_s \pi_A(\alpha)\|_{L^p} + \|\pi_A(\alpha)\|_{L^p} \right) \]
\[ + c\varepsilon^2 \left( \|\nabla_t \pi_A(\alpha)\|_{L^p} + \|\nabla_t \nabla_s \pi_A(\alpha)\|_{L^p} \right). \]

\[ \Box \]

The next goal is to improve the theorem 7.3 in the sense that we want to estimate the norms using only the operator \( D^*(\xi) \) (theorem 7.8) and in order to do this we need to use the properties of the geodesic flow (lemma 7.6). We define
\[ \omega(A) := d_A (d^*_A d_A)^{-1} (\nabla_t (\partial_t A - d_A \Psi) + * X_t(A)). \]

**Lemma 7.6.** We choose a regular value \( b \) of \( E^H \), then there are two positive constants \( c \) and \( \varepsilon_0 \) such that the following holds. For any \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+) \), \( \Xi_+ \in \text{Crit}^b \mathcal{H}^H \), any 1-form \( \xi = \alpha + \psi dz + \phi ds \in W^{1,2,p} \) and for \( 0 < \varepsilon < \varepsilon_0 \)
\[ \| \pi_A (D^*(\xi))(\alpha) + *[\alpha, *\omega(A)] - D^0(\Xi) \pi_A(\alpha) \|_{L^p} \]
\[ \leq c\varepsilon \||1 - \pi_A(\alpha) + \psi dt\|_{L^p} + c\varepsilon \|\nabla_t (1 - \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\psi\|_{L^p} \]
\[ + c\varepsilon \|\nabla_t \nabla_s \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\nabla_t \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\pi_A(\alpha)\|_{L^p} \]
\[ + c\varepsilon \|\pi_A(\nabla_s \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\pi_A(\alpha)\|_{L^p} \]
\[ \leq c\varepsilon \||1 - \pi_A(\alpha) + \psi dt\|_{L^p} + c\varepsilon \|\nabla_t (1 - \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\psi\|_{L^p} \]
\[ + c\varepsilon \|\nabla_t \nabla_s \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\nabla_t \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\pi_A(\alpha)\|_{L^p} \]
\[ + c\varepsilon \|\pi_A(\nabla_s \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\pi_A(\alpha)\|_{L^p} \].

**Proof.** By definition we have that
\[ \pi_A D^*(\xi)(\alpha) := \pi_A (\nabla_s \alpha - \nabla_t \nabla_s \alpha - 2[\psi, (\partial_t A - d_A \Psi)] - d * X_t(A) \alpha), \]
\[ D^0(\Xi)(\pi_A(\alpha)) := \pi_A (\nabla_t \pi_A(\alpha) - 2[\psi_0, (\partial_t A - d_A \Psi)]) \]
\[ - \nabla_t \nabla_s \pi_A(\alpha) - d * X_t(A) \pi_A(\alpha) + * [\pi_A(\alpha) \& * \omega(A)] \]
where \( d^*_A d_A \psi_0 = -2 * [\pi_A(\alpha) \& * (\partial_t A - d_A \Psi)] \); therefore
\[ \| \pi_A (D^*(\xi))(\alpha) + *[\alpha, *\omega] - D^0(\Xi) \pi_A(\alpha) \|_{L^p} \]
\[ \leq \|\pi_A (\nabla_s (1 - \pi_A(\alpha) \alpha - \nabla_t \nabla_s (1 - \pi_A)\alpha) ||_{L^p} + c\varepsilon \|\alpha\|_{L^p} \]
\[ + c\varepsilon \|\pi_A (2[\psi - \psi_0, (\partial_t A - d_A \Psi)])\|_{L^p} \]
\[ \leq c\varepsilon \|\pi_A(\alpha)\|_{L^p} + c\varepsilon \|\nabla_t (1 - \pi_A(\alpha)\|_{L^p} + c\varepsilon \|\psi - \psi_0\|_{L^p} \]
where we used the commutation formula and the uniform \( L^\infty \) bound of the curvatures in order to drop the derivative \( \nabla_s \) and a derivative \( \nabla_t \). Next, we split the 1-form \( \alpha + \psi dt = (\alpha_k + \psi_k dt) + (\alpha_i + \psi_i dt) \) in the same way as (7.2). We can easily remark that
\[ \|\alpha_i + \psi_i dt\|_{L^p} + \|\alpha_k + \psi_k dt\|_{L^p} \leq 2\|\alpha + \psi dt\|_{L^p}, \]
\[ \|\alpha_i + \psi_i dt\|_{L^p} \leq \|(1 - \pi_A)\alpha + \psi dt\|_{L^p}. \]
Furthermore, since \( \|\psi_0\|_{L^p} \leq c|d_A\psi_0|_{L^p} \), by the commutation formula
\[
\|\psi_0\|_{L^p} \leq (1 - \pi_A)\|\alpha\|_{L^p} + \|\nabla_t \alpha\|_{L^p}
\leq (1 - \pi_A)\|\alpha\|_{L^p} + \|\nabla_t (1 - \pi_A)\|_{L^p} + \|d^*_A\nabla_t (1 - \pi_A)\|_{L^p}
\]
\( \alpha_k + \psi_k dt \) lies in the kernel of \( d_A + \psi ds \), thus
\[
\|\psi_0\|_{L^p} \leq (1 - \pi_A)\|\alpha\|_{L^p} + \|\nabla_t (1 - \pi_A)\|_{L^p} + \varepsilon^2 \|\nabla_t \nabla_t (\psi_k - \psi_0)\|_{L^p}
\]
by the definition of \( \psi_0 \)
\[
\|\psi_k - \psi_0\|_{L^p} + \varepsilon^2 \|\nabla_t \nabla_t (\psi_k - \psi_0)\|_{L^p}
\]
by the definition of \( \psi_0 \) and by lemma 7.1
\[
\leq c\|\varepsilon^2 \nabla_s - \varepsilon^2 \nabla_t + d^*_A d_A (\psi_k - \psi_0)\|_{L^p}
\]
Moreover, the theorem [13] yields to
\[
\|\psi_k - \psi_0\|_{L^p} + \varepsilon^2 \|\nabla_t \nabla_t (\psi_k - \psi_0)\|_{L^p}
\]
by the definition of \( D^\tau_\varepsilon \) and by lemma 7.1
\[
\leq c\|\varepsilon^2 \nabla s - \varepsilon^2 \nabla t + \nabla^*_A d_A (\psi_k - \psi_0)\|_{L^p}
\]
**Theorem 7.8.** We choose a regular value $b$ of $E^H$, then there are two positive constants $c$ and $\varepsilon_0$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^b(\Xi_-, \Xi_+)$, $\Xi_\pm \in \text{Crit}_E^b$, and any $0 < \varepsilon < \varepsilon_0$ the following estimates hold.

$$
\|\pi_A(\alpha)\|_{1,2;p,1} \leq c\varepsilon\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_A(D^\varepsilon(\Xi)(\xi) + *[\alpha, *\omega(A)])\|_{L^p}
$$

(7.20) $\pi_A(\xi - (D^\varepsilon(\Xi))^*(\pi_A(\eta)))\|_{L^p},$

(7.21) $\|(1 - \pi_A)\xi\|_{1,2;p,\varepsilon} \leq c\varepsilon^2\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_A(D^\varepsilon(\Xi)(\xi) + *[\alpha, *\omega(A)])\|_{L^p}
$+
$$+ c\varepsilon\|\pi_A(\xi - (D^\varepsilon(\Xi))^*(\pi_A(\eta)))\|_{L^p},
$$

(7.22) $\|(1 - \pi_A)\alpha\|_{1,2;p,\varepsilon} \leq c\varepsilon^2\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon^2\|\pi_A(\xi - (D^\varepsilon(\Xi))^*(\pi_A(\eta)))\|_{L^p}$

for all compactly supported 1-forms $\xi, \eta \in W^{1,2;p}$ and $0 < \varepsilon < \varepsilon_0$.

**Proof.** By theorem [7.3] and by lemma [7.7] we have that

$$
\|\xi\|_{1,2;p,\varepsilon} + \|\nabla_x \pi_A(\xi)\|_{L^p} + \|\nabla_t \pi_A(\alpha)\|_{L^p} \leq c\varepsilon\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon}
$$+

$$+ c\varepsilon\|\pi_A(\alpha - D^\varepsilon(\Xi)^*(\pi_A(\alpha)))\|_{L^p},
$$

and thus always by lemma [7.7] and by the last estimate

$$
\|(1 - \pi_A)\xi\|_{1,2;p,\varepsilon} \leq c\varepsilon^2\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_A(\alpha - D^\varepsilon(\Xi)^*(\pi_A(\alpha)))\|_{L^p}
$$

(7.23) $\leq c\varepsilon^2\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon^2\|\pi_A(\alpha - D^\varepsilon(\Xi)^*(\pi_A(\alpha)))\|_{L^p}$

and thus the theorem follows combining these last three estimates.

\[\square\]

In the same way as theorem [7.8] one can prove the following theorem.

**Theorem 7.9.** We choose a regular value $b$ of $E^H$, then there are two positive constants $c$ and $\varepsilon_0$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^b(\Xi_-, \Xi_+)$, $\Xi_\pm \in \text{Crit}_E^b$, and any $0 < \varepsilon < \varepsilon_0$ the following estimates hold.

$$
\|(1 - \pi_A)\xi\|_{1,2;p,\varepsilon} \leq c\varepsilon^2\|D^\varepsilon(\Xi)(\xi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_A(\alpha - D^\varepsilon(\Xi)^*(\pi_A(\alpha)))\|_{L^p},
$$

(7.23) $\leq c\varepsilon^2\|D^\varepsilon(\Xi)^*(\xi)\|_{0,p,\varepsilon} + c\varepsilon\|\pi_A((D^\varepsilon(\Xi))^*(\xi) + *[\alpha, *\omega(A)])\|_{L^p},$
The multipliers $m$ given by
\[ m(x) = \frac{\partial^s m}{\partial x_1 \partial x_2 \ldots \partial x_n} \leq c_0 \]
for all integers $0 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n$. We define $T_m : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by
\[ T_m f := F^{-1}(mF(f)) \]
where $f \in L^2(\mathbb{R}^2)$ and $F : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ is the Fourier transformation given by
\[ (Ff)(y_1, \ldots, y_n) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\left(-i(y_1 x_1 + \ldots + y_n x_n)\right) f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \]
for $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$. Then $m$ is an $L^p$-multiplier for all $1 < p < \infty$, i.e. there exists a constant $c$ such that whenever $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^2)$ then $T_m f \in L^p(\mathbb{R}^n)$ and
\[ \|T_m f\|_{L^p} \leq c \|f\|_{L^p}. \]

**Theorem 7.10** (Marcinkiewicz, Mihlin). Let $m : \mathbb{R}^n \to \mathbb{C}$ be a measurable function that for some constant $c_0$ satisfies
\[ \| (1 - \pi_A)\alpha \|_{1,2,p,\varepsilon} \leq c\varepsilon^2 \| (D^p(\Xi))^* (\xi) \|_{0,p,\varepsilon} \]
for all compactly supported $1$-forms $\xi \in W^{1,2,p}$ and $0 < \varepsilon \leq \varepsilon_0$.

**7.1. Proof of the theorem 7.4.** We will use the following criterion to prove our estimates (cf. [18], theorem C.2).

**Theorem 7.11.** For every $p > 1$ there is a positive constant $c$ such that
\[ \| \partial_s u \|_{L^p} + \sum_{i,j=0}^{n-1} \| \partial_x_i \partial_x_j u \|_{L^p} \leq c \left\| \left( \partial_s - \sum_{i=0}^{n-1} \partial_x_i \partial_x_i \right) u \right\|_{L^p} \]
for every $u \in W_0^{1,2,p}(\mathbb{R} \times \mathbb{R}^n) \cap W_0^{1,2,2}(\mathbb{R} \times \mathbb{R}^n)$.

**Proof.** We define $f \in L^p(\mathbb{R} \times \mathbb{R}^n) \cap L^2(\mathbb{R} \times \mathbb{R}^n)$ by $f = \left( \partial_s - \sum_{i=0}^{n-1} \partial_x_i \partial_x_i \right) u$ and thus
\[ F(f) = \left( i\sigma + \sum_{i=0}^{n-1} y_i^2 \right) F(u) \]
and therefore
\[ F(\partial_s u) = \frac{i\sigma}{i\sigma + \sum_{i=0}^{n-1} y_i^2} F(f) =: m_s(\sigma, y_0, \ldots, y_{n-1}) F(f), \]
\[ F(\partial_x_i \partial_x_j u) = \frac{y_i y_j}{i\sigma + \sum_{i=0}^{n-1} y_i^2} F(f) =: m_{ij}(\sigma, y_0, \ldots, y_{n-1}) F(f). \]
The multipliers $m_s(\sigma, y_0, \ldots, y_{n-1})$ and $m_{ij}(\sigma, y_0, \ldots, y_{n-1})$ satisfy the condition (7.25) and therefore we can apply the theorem 7.10 and conclude the proof. \hfill \Box

We denote $d_A d_A^* + d_A^* d_A$ by $\Delta_A$. 
Lemma 7.12. We choose a connection $A_0 \in \mathcal{A}_0(P)$, then there is a positive constant $c$ such that for any 0- or 1-form $\alpha$ with compact support:

$$
\begin{align*}
\|\partial_s \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} &+ \|\partial_t^2 \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \|d_{A_0} d^*_{A_0} \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} \\
&+ \|d_{A_0} \partial_t \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \|\partial_t d_{A_0} \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)}
\end{align*}
$$

(7.27)

Proof. The previous corollary continues to holds if we consider a metric closed to a constant metric. Therefore we can pick a finite atlas $\{\mathbb{R}^2 \times V_i, \varphi_i : \mathbb{R}^2 \times V_i \to \mathbb{R}^2 \times \Sigma\}_{i \in I}$ and a partition of the unity $\{\rho_i\}_{i \in I} \subset C^\infty(\mathbb{R}^2 \times \Sigma, [0,1])$, $\sum_{i \in I} \rho_i(x) = 1$ for every $x \in \mathbb{R}^2 \times \Sigma$ and supp($\rho_i$) $\subset \varphi_i^{-1}(\mathbb{R}^2 \times V_i)$ for any $i \in I$, and apply the corollary (7.11) for $(\rho_i \circ \varphi_i)\alpha_i$ where $\alpha_i$ is the local representations of $\alpha$ on $\mathbb{R}^2 \times V_i$. Summing all the estimates and considering the smooth constant connection $A_0(t, s)$ we obtain (7.24).

Lemma 7.13. We choose a flat connection $A_0 \in \mathcal{A}_0(P)$, then there is a positive constant $c$ such that for any 0- or 1-form $\alpha$ with compact support:

$$
\begin{align*}
\varepsilon^2\|\partial_s \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon^2\|\partial_t^2 \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \|d_{A_0} d^*_{A_0} \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} \\
&+ \|d_{A_0} \partial_t \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon\|\partial_t d_{A_0} \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon\|\partial_t d_{A_0} \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)}
\end{align*}
$$

(7.28)

Proof. The lemma follows from the previous lemma (7.12) using the rescaling $\tilde{\alpha}(t, s) := \alpha(\varepsilon t, \varepsilon^2 s)$.

Lemma 7.14. We choose a flat connection $A_0 \in \mathcal{A}_0(P)$ and a constant $c_0$, then there is a positive constant $c$ such that following holds. For any connection $A \in \mathcal{A}_0^2(P \times \mathbb{R}^2)$ which satisfies

$$
\sup_{(s, t) \in \mathbb{R}^2} (\|A(s, t) - A_0\|_{C^1} + \varepsilon\|\partial_t A\|_{L^\infty}) \leq c_0
$$

and for any 0- or 1-form $\alpha$ with compact support:

$$
\begin{align*}
\varepsilon^2\|\partial_s \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon^2\|\partial_t^2 \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \|d_A d^*_A \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} \\
&+ \|d_A \partial_t \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon\|\partial_t d_A \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)} + \varepsilon\|\partial_t d_A \alpha\|_{L^p(\Sigma \times \mathbb{R}^2)}
\end{align*}
$$

(7.30)

Proof. This lemma follows directly from the lemma (7.12) using the assumption (7.25) and the lemma (3.3).

Lemma 7.15. We choose a regular value $b$ of $E^H$, then there is a positive constant $c$ such that the following holds. For any $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}_0(\Xi_-, \Xi_+)$, $\Xi_-, \Xi_+ \in \text{Crit}^b_{E_n}$, and any 0- or 1-form $\alpha \in W^{2,1,p}$

$$
\begin{align*}
\|\alpha\|_{L^p} + \|d_A \alpha\|_{L^p} + \|d_A^* \alpha\|_{L^p} + \|d_A^* d_A \alpha\|_{L^p} + \|d_A d_A^* \alpha\|_{L^p} + \varepsilon\|\nabla_t \alpha\|_{L^p} + \varepsilon\|\nabla_t d_A \alpha\|_{L^p}
\end{align*}
$$

(7.31)

$$
\begin{align*}
\leq c \left( \epsilon^2 \|\nabla_s \alpha - \epsilon^2 \nabla_s^2 \Delta \alpha\|_{L^p} + c\|\alpha\|_{L^p} \right).
\end{align*}
$$
Proof. We choose a finite atlas \( \{ B_i, \varphi_i : B_i \to S^1 \times \mathbb{R} \}_{i \in I} \) of \( S^1 \times \mathbb{R} \) such that the condition (7.20) is satisfied for every chart; we can cover the two ends of the cylinder with two chart each because \( A(t, s) \) converges exponentially to \( A_\pm \) as \( s \to \pm \infty \) and thus for \( s_0 \) big enough

\[
\sup_{(s, t) \in S^1 \times (s_0, \infty)} (\| A(s, t) - A_+ \|_{C^1} + \varepsilon \| \partial_t A \|_{L^\infty}) \leq c_0,
\]

\[
\sup_{(s, t) \in S^1 \times (-\infty, s_0)} (\| A(s, t) - A_- \|_{C^1} + \varepsilon \| \partial_t A \|_{L^\infty}) \leq c_0.
\]

Furthermore, we take a partition of the unity \( \sum_{i \in \mathbb{N}} \rho_i(t, s) = 1, \rho_i(t, s) \in [0, 1] \) and \( \text{supp}(\rho_i) \subset \varphi(B_i) \); next, collecting the estimate given by the lemma (7.14) on every chart \( B_i \times \Sigma \) for \( (\rho_i \circ \varphi_i) \alpha_i \), where \( \alpha_i \) is the representation of \( \alpha \) on \( B_i \times \Sigma \), we obtain

\[
\varepsilon^2 \| \partial_s \alpha \|_{L^p} + \varepsilon^2 \| \partial_t \partial_s \alpha \|_{L^p} + \| d^*_A d_A \alpha \|_{L^p} + \| d_A d^*_A \alpha \|_{L^p}
\]

\[
+ \varepsilon \| \partial_t d^*_A \alpha \|_{L^p} + \partial_t \| d_A d^*_A \alpha \|_{L^p}
\]

\[
\leq c \left( (\varepsilon^2 \| \partial_s - \varepsilon^2 \partial_t^2 + \Delta_A \) \alpha \|_{L^p} + c_\| \alpha \|_{L^p} + c \varepsilon^2 \| \partial_t \alpha \|_{L^p}.
\]

Since \( \| \Psi \|_{L^\infty} + \| \partial_t \Psi \|_{L^\infty} + \| \Phi \|_{L^\infty} \leq c_1 \), we have

\[
\varepsilon^2 \| \nabla_s \alpha \|_{L^p} + \varepsilon^2 \| \nabla_t \nabla_s \alpha \|_{L^p} + \| d^*_A d_A \alpha \|_{L^p} + \| d_A d^*_A \alpha \|_{L^p}
\]

\[
+ \varepsilon \| \nabla_t d^*_A \alpha \|_{L^p} + \varepsilon \| d_A d^*_A \alpha \|_{L^p}
\]

\[
\leq c \left( (\varepsilon^2 \| \nabla_s - \varepsilon^2 \nabla_t^2 + \Delta_A \) \alpha \|_{L^p} + c_\| \alpha \|_{L^p}
\]

\[
+ c \varepsilon^2 \| \nabla_t \alpha \|_{L^p} + c \| \partial_t d^*_A \alpha \|_{L^p} + c \| d_A d^*_A \alpha \|_{L^p}.
\]

The estimate (7.31) follows then from the lemmas (7.4) and (7.5) \( \square \)

Lemma 7.16. We choose a regular value \( b \) of \( E^H \), then there are two positive constants \( c \) and \( \varepsilon_0 \) such that the following holds. For any \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi_-, \Xi_+) \), \( \Xi \in \text{Crit}_{E^H} \), any \( i \)-form \( \xi \in W^{2,2,1,p}, i = 0, 1 \) and \( 0 < \varepsilon < \varepsilon_0 \),

\[
\int_{S^1 \times \mathbb{R}} \| \xi \|^2_{L^2(\Sigma)} dt ds \leq c \int_{S^1 \times \mathbb{R}} \varepsilon^2 \| \partial_s \xi - \varepsilon^2 \partial_t^2 \xi + \Delta_A \xi \|_{L^2(\Sigma)}^p dt ds
\]

\[
+ c \int_{S^1 \times \mathbb{R}} \| \pi_A (\xi) \|_{L^2(\Sigma)}^p dt ds.
\]

Proof. In this proof we denote the norm \( \| \cdot \|_{L^2(\Sigma)} \) by \( \| \cdot \| \). If we consider only the Laplace part of the operator, we obtain that

\[
\int_{S^1 \times \mathbb{R}} \| \xi \|^p (\varepsilon^2 \| \partial_t \xi \|^2 + \| d_A \xi \|^2 + \| d^*_A \xi \|^2) dt ds
\]

\[
= \int_{S^1 \times \mathbb{R}} \| \xi \|^p (\varepsilon^2 \| \partial_t \xi \|^2 + \| d_A \xi \|^2 + \| d^*_A \xi \|^2) dt ds
\]

\[
+ \int_{S^1 \times \mathbb{R}} \| \pi_A (\xi) \|^p dt ds.
\]
and thus

\[
\int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \left( \varepsilon^2 \|\partial_s \xi\|^2 + \|dA\xi\|^2 + \|d^*A\xi\|^2 \right) \, ds \, dt \\
\leq \int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \langle \xi, \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \rangle \, ds \, dt \\
= \int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \langle \xi, \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \rangle \, ds \, dt \\
\leq \int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \, ds \, dt \\
\leq \left( \int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \right)^{\frac{p-1}{p}} \left( \int_{S^1 \times \mathbb{R}} \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \|^p \, ds \, dt \right)^{\frac{1}{p}} 
\]

(7.35)

where the second step follows from

\[
\int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \langle \xi, \partial_s \xi \rangle \, ds \, dt = \frac{1}{p} \int_{S^1 \times \mathbb{R}} \partial_s \|\xi\|^p \, ds \, dt = 0,
\]

the third from the Cauchy-Schwarz inequality and the fourth from the Hölder’s inequality. Therefore, by lemma B.3

\[
\int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \leq \int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \left( \|dA\xi\|^2 + \|d^*A\xi\|^2 + \|\pi_A(\xi)\|^2 \right) \, ds \, dt 
\]

by (7.35) we have that

\[
\leq \left( \int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \right)^{\frac{p-1}{p}} \left( \int_{S^1 \times \mathbb{R}} \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \|^p \, ds \, dt \right)^{\frac{1}{p}} \\
+ \int_{S^1 \times \mathbb{R}} \|\xi\|^{p-2} \|\pi_A(\xi)\| \, ds \, dt 
\]

and by the Hölder’s inequality

\[
\leq \left( \int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \right)^{\frac{p-1}{p}} \left( \int_{S^1 \times \mathbb{R}} \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \|^p \, ds \, dt \right)^{\frac{1}{p}} \\
+ \left( \int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \right)^{\frac{p-1}{p}} \left( \int_{S^1 \times \mathbb{R}} \|\pi_A(\xi)\|^p \, ds \, dt \right)^{\frac{1}{p}} ;
\]

thus, we can conclude that

\[
\int_{S^1 \times \mathbb{R}} \|\xi\|^p \, ds \, dt \leq c \int_{S^1 \times \mathbb{R}} \left( \varepsilon^2 \partial_s \xi - \varepsilon^2 \partial_s^2 \xi + \Delta_A \xi \right|^p + \|\pi_A(\xi)\|^p \right) \, ds \, dt.
\]

and hence we finished the proof of the lemma. \( \square \)
**Proof of theorem (7.4)** By lemma [B.3] for any $\delta > 0$ there is a $c_0$ such that

$$
\|\alpha\|^p_{L^p} \leq \delta \left( \|d_A^p\|^p_{L^p} + \|d_A^*\|^p_{L^p} + \int_{S^1 \times \mathbb{R}} \|\alpha\|^p_{L^2} dt ds \right)
$$

$$
\leq \delta \left( \|d_A^p\|^p_{L^p} + \|d_A^*\|^p_{L^p} + c_0 \int_{S^1 \times \mathbb{R}} \|\pi_\alpha^p\|^p_{L^2} dt ds \right)
$$

$$
+ c_0 c_1 \int_{S^1 \times \mathbb{R}} \|\epsilon^2 \partial_\alpha \psi - \epsilon^2 \partial_\alpha^2 \psi + \Delta \alpha\|^p_{L^2} dt ds
$$

where the second step follows from the lemma [7.14] and the third by the Hölder’s inequality with $c_2 := (\int_{S^1} \text{dvol}_{S^1})^{\frac{2}{p}}$. Therefore if we choose $\delta$ and $\epsilon$ small enough we can improve the estimate (7.31) of the corollary [7.15] using the last estimate, i.e.

$$
\|\alpha\|^p_{L^p} + \|d_A\|_{L^p} + \|d_A^*\|_{L^p} + \|d_A^p\|_{L^p} + \|d_A^*\|_{L^p} + \epsilon \|\partial_\alpha\|_{L^p} + \epsilon \|\partial_\alpha^2\|_{L^p} + \epsilon \|\partial_\alpha\|_{L^p} + \epsilon \|\partial_\alpha^2\|_{L^p} + \epsilon \|\partial_\alpha^3\|_{L^p}
$$

$$
\leq \left( \|\epsilon^2 \partial_\alpha \psi - \epsilon^2 \partial_\alpha^2 \psi + \Delta \alpha\|_{L^p} + \epsilon \|\partial_\alpha\|_{L^p} \right) + \epsilon \|\partial_\alpha\|_{L^p},
$$

because $\|\partial_\alpha\|_{L^p} + \|\partial_\alpha^2\|_{L^p} + \|\psi\|_{L^\infty}$ is bounded by a constant. Furthermore, the terms $\epsilon \|d_A\|_{L^p}$ and $\epsilon \|d_A^*\|_{L^p}$ can be estimated by

$$
d_A \partial_\alpha \partial_\alpha^p + \epsilon \|d_A^p \partial_\alpha \|_{L^p} \leq \epsilon \|\partial_\alpha \partial_\alpha^p \|_{L^p} + \epsilon \|\partial_\alpha \partial_\alpha^p \|_{L^p} + \epsilon \|\partial_\alpha \partial_\alpha^p \|_{L^p}
$$

using the commutation formulas and because the curvature $\partial_\alpha A - d_A \Psi$ is bounded; we proved therefore (7.12) and the second inequality of the theorem can be proved in the same way. \[\square\]

### 7.2. Proof of the theorem (7.5)

Before proving the theorem we will show some preliminary results, in fact the theorem (7.5) will then follow from the corollary (7.18) and the lemmas [B.3] and (7.19).

**Corollary 7.17.** For every $p > 1$ there is a positive constant $c$ such that the following holds. For every two maps $\gamma \in W_0^{2,p}(\mathbb{R}^4, \mathbb{R})$, $\phi \in W_0^{1,p}(\mathbb{R}^4, \mathbb{R})$ we have that

$$
\|\partial_\alpha \alpha_1\|_{L^p} + \lambda \|\partial_\alpha_1 \alpha_2\|_{L^p} + \|\partial_\alpha_2 \alpha_2\|_{L^p} + \lambda \|\partial_\alpha_2 \alpha_1\|_{L^p}
$$

$$
+ \|\partial_\alpha_2 \psi\|_{L^p} + \lambda \|\partial_\alpha_2 \psi\|_{L^p} + \|\partial_\alpha \phi\|_{L^p}
$$

$$
\leq c \|\partial_\alpha_1 \alpha_1 - \partial_\alpha_2 \alpha_1 + \partial_\alpha_2 \alpha_2 - \partial_\alpha_2 \phi\|_{L^p} + c \|\partial_\alpha_2 \psi - \partial_\alpha \phi\|_{L^p}
$$

(7.36)

where $\alpha_1 = \partial_\alpha_1 \gamma$, $\alpha_2 = \partial_\alpha_2 \gamma$, $\psi = \partial_\alpha \gamma$ and $\lambda \in [0, 1]$.

**Proof.** In order to prove this corollary we need to apply the theorem (7.10) of Marcinkiewicz and Mihlin stated in the previous subsection and in order to do...
this we have to define the multipliers and prove the assumption (7.25); therefore, we look at the following system of equations

\[ f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} := \begin{pmatrix} \partial_x & 0 & 0 & -\partial x_1 \\ 0 & \partial_x & 0 & -\partial x_2 \\ 0 & 0 & \partial_x & -\partial \xi \\ \partial x_1 & \partial x_2 & \partial \xi & \partial \xi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \psi \\ \phi \end{pmatrix}. \]

One can remark that the four lines of (7.37) correspond to the first four terms in the \(L^p\)-norm in the right side of the estimate (7.36). Applying the Fourier transformation to (7.36) we obtain

\[
\mathcal{F}(f) = \begin{pmatrix} \sigma i & 0 & 0 & -y_1 i \\ 0 & \sigma i & 0 & -y_2 i \\ 0 & 0 & \sigma i & -\tau i \\ y_1 i & y_2 i & \tau i & \sigma i \end{pmatrix} \begin{pmatrix} \mathcal{F}(a_1) \\ \mathcal{F}(a_2) \\ \mathcal{F}(\psi) \\ \mathcal{F}(\phi) \end{pmatrix},
\]

and thus computing its inverses:

\[
\mathcal{F}(a_1) = -i \frac{(\sigma^2 + y_2^2 + \tau^2)}{(\sigma^2 + y_1^2 + y_2^2 + \tau^2) \sigma} \mathcal{F}(f_1) + \frac{iy_2 y_1}{(\sigma^2 + y_1^2 + y_2^2 + \tau^2) \sigma} \mathcal{F}(f_2)
\]

\[
+ \frac{i\tau y_1}{(\sigma^2 + y_1^2 + y_2^2 + \tau^2) \sigma} \mathcal{F}(f_3) + \frac{-iy_1}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_4),
\]

\[
\mathcal{F}(\phi) = \frac{iy_1}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_1) + \frac{iy_2}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_2)
\]

\[
+ \frac{i\tau}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_3) + \frac{-i\sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_4),
\]

and then

\[
\mathcal{F}(\partial_x a_1) = \sigma^2 + y_1^2 + y_2^2 + \tau^2 \mathcal{F}(f_1) + \frac{-y_1 \sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_2)
\]

\[
+ \frac{-\tau y_1}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_3) + \frac{y_1 \sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_4),
\]

\[
\mathcal{F}(\partial_x \phi) = \frac{-y_1 \sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_1) + \frac{-y_2 \sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_2)
\]

\[
+ \frac{-\tau \sigma}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_3) + \frac{\sigma^2}{\sigma^2 + y_1^2 + y_2^2 + \tau^2} \mathcal{F}(f_4).
\]

The formulas for \(\mathcal{F}(a_2)\) and for \(\mathcal{F}(\phi)\), respectively for \(\mathcal{F}(\partial_x a_2)\) and for \(\partial_x \mathcal{F}(\phi)\), are similar to that of \(\mathcal{F}(a_1)\), respectively to that of \(\mathcal{F}(\partial_x a_1)\). Since the multipliers for \(\mathcal{F}(\partial_x a_1)\), \(\mathcal{F}(\partial_x a_2)\), \(\mathcal{F}(\partial_x \psi)\) and \(\mathcal{F}(\partial_x \phi)\) satisfy the assumption (7.25) of the theorem 1.1, we can conclude that

\[
\|\partial_x a_1\|_{L^p} + \|\partial_x a_2\|_{L^p} + \|\partial_x \psi\|_{L^p} + \|\partial_x \phi\|_{L^p} + \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p}
\]

\[
+ \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p} + \|\partial_x \xi\|_{L^p}
\]

\[
\leq c \|\partial_x a_1 - \partial_x a_2\|_{L^p} + c \|\partial_x a_2 - \partial_x a_2\|_{L^p} + c \|\partial_x \psi - \partial_x \phi\|_{L^p} + c \|\partial_x \phi + \partial_x a_2 + \partial_x a_2 + \partial_x \psi\|_{L^p}.
\]
Next, we use that \( \alpha_1 = \partial_x \gamma, \alpha_2 = \partial_x \gamma, \psi = \partial_t \gamma \) and thus
\[
\| \partial_x \gamma - (\partial_x^2 + \partial_t^2 + \partial_t^2) \gamma \|_{L_p} \leq \||\partial_x^2 + \partial_t^2 + \partial_t^2\| \gamma \|_{L_p} + \| \partial_x \gamma \|_{L_p}
\]
\[
\leq \| \partial_x \alpha_1 + \partial_x \alpha_2 + \partial_t \psi \|_{L_p} + \| \partial_t \gamma \|_{L_p}.
\]
Therefore by corollary 7.11 it follow that
\[
\lambda \| \partial_x \alpha_1 \|_{L_p} + \lambda \| \partial_x \alpha_2 \|_{L_p} + \lambda \| \partial_t \alpha_1 \|_{L_p} + \lambda \| \partial_t \alpha_2 \|_{L_p} + \lambda \| \partial_t \psi \|_{L_p}
\]
\[
\leq c \lambda \| \partial_x \alpha_1 - \partial_x \phi \|_{L_p} + c \lambda \| \partial_x \alpha_2 - \partial_x \phi \|_{L_p} + c \lambda \| \partial_t \psi - \partial_t \phi \|_{L_p}
\]
\[
+ c \lambda \| \partial_t \psi + \partial_t \alpha_1 + \partial_t \alpha_2 + \partial_t \phi \|_{L_p} + c \lambda \| \partial_t \gamma \|_{L_p}.
\]
(7.39)

Therefore the theorem follows combining (7.38) and (7.38).

\[\square\]

**Lemma 7.18.** We choose a regular value \( b \) of \( E^H \), then there is a positive constant \( c \) such that the following holds. For any \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi, \Xi_+), \Xi_+ \in \text{Crit}^1_{E^H} \), and any 1-form \( \alpha + \psi dt = d_A + \Psi dt \gamma \in W^{1,2;p} \cap \text{im} d_A + \Psi dt \)
\[
\| \alpha \|_{L_p} + \| d^* \alpha \|_{L_p} + \epsilon^2 \| \nabla \alpha \|_{L_p} + \epsilon \| \nabla \psi \|_{L_p} + \epsilon^2 \| \nabla_t \psi \|_{L_p}
\]
\[
+ \epsilon^2 \| \nabla_s \psi \|_{L_p} + \epsilon^2 \| \phi \|_{L_p} + \epsilon^2 \| d_A \phi \|_{L_p} + \epsilon^2 \| \nabla_t \phi \|_{L_p}
\]
\[
\leq c \epsilon \| \nabla_s \gamma - d_A \phi \|_{L_p} + c \epsilon \| \nabla_t \phi \|_{L_p} + c \| \alpha \|_{L_p} + c \| \phi \|_{L_p}. \tag{7.40}
\]

**Proof.** In the same way that the lemma 7.15 follows from the corollary 7.11 the corollary 7.17 implies
\[
\| \alpha \|_{L_p} + \| d^* \alpha \|_{L_p} + \lambda \| \nabla \phi \|_{L_p} + \lambda \| \nabla \psi \|_{L_p} + \lambda \| \nabla_t \phi \|_{L_p}
\]
\[
+ \| \phi \|_{L_p} + \| \phi_t \|_{L_p} + \| d_A \phi \|_{L_p} + \| \nabla_t \phi \|_{L_p}
\]
\[
\leq c \epsilon \| \nabla_s \gamma - d_A \phi \|_{L_p} + c \epsilon \| \nabla_t \phi \|_{L_p} + c \| \alpha \|_{L_p} + c \| \phi \|_{L_p}.
\]
The term \( \epsilon^2 \| \nabla_s \gamma \|_{L_p} \) can be estimate by \( c \epsilon \| \nabla_s \alpha \|_{L_p} + c \epsilon^2 \| \alpha \|_{L_p} \) by the lemma B.1 and the commutation formula and thus using the last estimate for \( \lambda = 0 \)
\[
\epsilon^2 \| \nabla_s \gamma \|_{L_p} \leq c \epsilon \| \nabla_s \alpha - d_A \phi \|_{L_p} + c \epsilon \| \nabla_t \phi \|_{L_p}
\]
\[
+ c \epsilon^4 \| \nabla_s \phi + \frac{1}{\epsilon^4} d_A \alpha_1 + \frac{1}{\epsilon^2} \nabla_t \psi \|_{L_p} + c \| \alpha \|_{L_p} + c \| \phi \|_{L_p}.
\]
Furthermore, by the lemma B.1 and the commutation formula
\[
\epsilon \| \psi \|_{L_p} \leq c \epsilon \| d_A \phi \|_{L_p} \leq c \| \alpha \|_{L_p} + c \| \nabla_t \alpha \|_{L_p}
\]
and finally collecting the last thee estimates we obtain (7.38).

\[\square\]

**Lemma 7.19.** We choose a regular value \( b \) of \( E^H \) and \( a, \delta > 0 \), then there are two positive constants \( c \) and \( \epsilon_0 \) such that the following holds. For any \( \Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^0(\Xi, \Xi_+), \Xi_+ \in \text{Crit}^1_{E^H} \), any 1-form \( \xi := \alpha + \psi dt + \phi ds \in W^{1,1;p} \), where
\( \alpha + \psi dt \in \text{im } d_{A+\Psi dt}, \) and any \( 0 < \varepsilon < \varepsilon_0 \)

\[
\int_{S^1 \times \mathbb{R}} \left\| \xi \right\|_{L^2(S)}^{-2} \left( \left\| \alpha \right\|_{L^2(S)}^2 + \varepsilon^4 \left\| \phi \right\|_{L^2(S)}^2 \right) dt \, ds
\]

(7.41)

\[
\leq \int_{S^1 \times \mathbb{R}} \left( c \varepsilon^2 \partial_s \alpha - \varepsilon^2 d_A \phi \right)^p_{L^2(S)} + c \varepsilon^p \left\| \varepsilon^2 \partial_s \psi - \varepsilon^2 \partial_t \phi \right\|^p_{L^2(S)} dt \, ds
\]

\[
+ \int_{S^1 \times \mathbb{R}} \left( c \varepsilon^p \varepsilon^2 \partial_s \phi - \frac{1}{\varepsilon^2} d_A^* \alpha + \partial_t \psi \right)_{L^2(S)}^p + \delta \left\| \xi \right\|_{L^2(S)}^p dt \, ds.
\]

**Proof.** In this proof we denote the norm \( \left\| \cdot \right\|_{L^2(S)} \) by \( \left\| \cdot \right\| \). We consider \( \xi = \alpha + \psi dt + \phi ds \) where \( \alpha + \psi dt = d_{A+\Psi dt} \gamma \) and

\[
\tilde{\xi} = \tilde{\alpha} + \tilde{\psi} dt + \tilde{\phi} ds = D\xi = \begin{pmatrix} \varepsilon^2 \partial_s & 0 & -\varepsilon^2 d_A \\ 0 & \varepsilon^2 \partial_t & -\varepsilon^2 d_A \\ -\frac{1}{\varepsilon^2} d_A^* & 0 & \varepsilon^2 \partial_t \end{pmatrix} \begin{pmatrix} \alpha \\ \psi \\ \phi \end{pmatrix}.
\]

thus \( D^* D\xi \) can be written in the following way

\[
D^* \xi = \begin{pmatrix} -\varepsilon^2 \partial_s & 0 & -\varepsilon^2 d_A \\ 0 & -\varepsilon^2 \partial_t & -\varepsilon^2 d_A \\ -\frac{1}{\varepsilon^2} d_A^* & 0 & \varepsilon^2 \partial_t \end{pmatrix} \begin{pmatrix} \varepsilon^2 \partial_s & 0 & -\varepsilon^2 d_A \\ 0 & \varepsilon^2 \partial_t & -\varepsilon^2 d_A \\ -\frac{1}{\varepsilon^2} d_A^* & 0 & \varepsilon^2 \partial_t \end{pmatrix} \begin{pmatrix} \alpha \\ \psi \\ \phi \end{pmatrix}
\]

\[
= \begin{pmatrix} -\varepsilon^2 \partial_s & 0 & -\varepsilon^2 d_A \\ 0 & -\varepsilon^2 \partial_t & -\varepsilon^2 d_A \\ -\frac{1}{\varepsilon^2} d_A^* & 0 & \varepsilon^2 \partial_t \end{pmatrix} \begin{pmatrix} \varepsilon^4 (\partial_s d_A - d_A \partial_s) \\ \varepsilon^4 (\partial_t d_A - d_A \partial_t) \\ 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \psi \\ \phi \end{pmatrix}.
\]

We define

\[
B := \frac{1}{2} \left\| d_A^* \alpha - \varepsilon^2 \partial_t \psi \right\|^2 + \varepsilon^4 \left\| \partial_s \alpha \right\|^2 + \varepsilon^6 \left\| \partial_t \psi \right\|^2 + 4 \left\| \partial_t \phi \right\|^2
\]

\[
+ \frac{1}{2} \left( \left\| \partial_s \alpha \right\|^2 + \left\| \partial_t \psi \right\|^2 \right) + \varepsilon^6 \left\| \partial_t \psi \right\|^2 + \varepsilon^8 \left\| \partial_t \phi \right\|^2
\]

\[
G^p := \left\| \varepsilon^2 \partial_s \alpha - \varepsilon^2 d_A \phi \right\|^p + \varepsilon^p \left\| \varepsilon^2 \partial_s \psi - \varepsilon^2 \partial_t \phi \right\|^p + \varepsilon^{2p} \left\| \varepsilon^2 \partial_s \phi - \frac{1}{\varepsilon^2} d_A^* \alpha + \partial_t \psi \right\| ^p
\]

Using the partial integration we obtain

(7.42)

\[
\int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} \left( \alpha, \varepsilon^4 \partial_s^2 + d_A d_A^* \right) \alpha - \varepsilon^2 d_A \partial_t \psi \right) dt \, ds
\]

\[
+ \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} \varepsilon^2 \left( \varepsilon^2 \partial_s^2 - \varepsilon^2 \partial_t^2 \right) \psi + \partial_t d_A^* \alpha \right) dt \, ds
\]

\[
+ \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} \varepsilon^4 \left( \phi, \partial_s d_A - \varepsilon^2 \partial_t^2 \right) \phi \right) dt \, ds
\]

\[
= \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} B \right) dt \, ds
\]

\[
+ \left( p - 2 \right) \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-4} \left( \alpha, \varepsilon^2 \partial_s \alpha \right) + \varepsilon^2 \left( \varepsilon^2 \partial_s \psi \right) + \varepsilon^4 \left( \phi, \varepsilon^2 \partial_t \phi \right) \right) \right) \right) \right) dt \, ds
\]

\[
+ \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} \varepsilon^2 \partial_t \psi \right) dt \, ds + \int_{S^1 \times \mathbb{R}} \left\| \xi \right\|^{p-2} \varepsilon^2 \left( d_A \psi, \partial_t \alpha \right) dt \, ds
\]
whose last term, using that $\alpha + \psi dt = d_A \gamma + \partial_t \gamma dt$, can be estimate as follows

\begin{equation}
\int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \varepsilon^2 \langle d_A \psi, \partial_t \alpha \rangle dt \, ds = \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \varepsilon^2 \langle d_A \nabla_1 \gamma, \partial_t d_A \gamma \rangle dt \, ds \\
\geq \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \varepsilon^2 \| d_A \psi \|^2 - c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-1} \| \alpha \| + \| d_A^* \alpha \| \rangle dt \, ds.
\end{equation}

Since the penultimate line of (7.42) is positive, (7.42) and (7.43) yield

\[
\int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} B dt \, ds \\
\leq \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \left( \langle \alpha, (D^* \tilde{\xi})^1 \rangle + \varepsilon^2 \langle \psi, (D^* \tilde{\xi})^2 \rangle + \varepsilon^4 \langle \phi, (D^* \tilde{\xi})^3 \rangle \right) dt \, ds \\
- \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \varepsilon^4 \left( \langle \alpha, [\partial_x A, \phi] \rangle - \langle \phi, *[\partial_x A \wedge *\alpha] \rangle \right) dt \, ds \\
+ c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-1} \| \alpha \| + \| d_A^* \alpha \| dt \, ds
\]

Integrating by parts the first line after the inequality and using the Cauchy-Schwarz inequality, we obtain

\[
\leq c \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} B^{\frac{1}{2}} \left( \| \varepsilon^2 \tilde{\alpha} \| + \varepsilon \| \varepsilon^2 \tilde{\psi} \| + \varepsilon^2 \| \varepsilon^2 \tilde{\phi} \| \right) dt \, ds \\
+ c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} dt \, ds + c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-1} \| d_A^* \alpha \| \, dt \, ds
\]

Since $2ab \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, choosing $\varepsilon$ small enough

\[
\leq c \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \left( \| \varepsilon^2 \tilde{\alpha} \| + \varepsilon \| \varepsilon^2 \tilde{\psi} \| + \varepsilon^2 \| \varepsilon^2 \tilde{\phi} \| \right)^2 dt \, ds \\
+ \frac{1}{2} \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} B dt \, ds + c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} dt \, ds.
\]

The last estimate implies that

\[
\frac{1}{2} \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} B \, dt \, ds \leq c \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} G^2 \, ds + \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} \, ds \\
\leq c \left( \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} \, ds \right)^{1 - \frac{1}{r}} \left( \int_{S^1 \times \mathbb{R}} G^p \, ds \right)^{\frac{1}{r}} + c \varepsilon^2 \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} \, ds \\
\leq c \int_{S^1 \times \mathbb{R}} G^p \, ds + \delta \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} \, ds,
\]

where in the second step we use the Hölder’s estimate and in the third the estimate $ab \leq \frac{a^2}{r} + \frac{b^2}{q} \cdot \frac{1}{r} + \frac{1}{q} = 1$, with $r = \frac{p}{2}$. Finally,

\[
\int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \left( \| \alpha \|^2 + \varepsilon^4 \| \phi \|^2 \right) dt \, ds \leq \int_{S^1 \times \mathbb{R}} \| \xi \|^{p-2} \left( \| d_A^* \alpha \|^2 + \varepsilon^4 \| d_A \phi \|^2 \right) \, ds \\
\leq c \int_{S^1 \times \mathbb{R}} G^p \, ds + \delta \int_{S^1 \times \mathbb{R}} \| \xi \|^{p} \, ds
\]

and thus the lemma is proved. \(\square\)
**Proof of Theorem 7.4** By Lemma 3.3, for any \( \delta > 0 \) there is a positive constant \( c_0 \) such that

\[
\|\alpha\|_{L^p}^{p} + \varepsilon^{2p}\|\phi\|_{L^p}^{p} \leq \delta \left( \|d_{A\alpha}\|_{L^p}^{p} + \|d_{A\alpha}\|_{L^p}^{p} + \varepsilon^{2p}\|\phi\|_{L^p}^{p} \right)
\]

\[
+ c_0 \int_{S^1 \times \mathbb{R}} \left( \|\alpha\|_{L^2}^{p} + \varepsilon^{2p}\|\phi\|_{L^2}^{p} \right) dt \, ds
\]

since \( \alpha = d_{A\gamma} \) and the connection is flat on \( \Sigma \), \( \|d_{A\alpha}\|_{L^p} \) vanishes. By the lemma 3.19, we have then

\[
\leq \delta \left( \|d_{A\alpha}\|_{L^p}^{p} + \varepsilon^{2p}\|\phi\|_{L^p}^{p} \right)
\]

\[
+ c_0 \int_{S^1 \times \mathbb{R}} \left( \|\phi\|_{L^2}^{p} + \varepsilon^{2p}\|\phi\|_{L^2}^{p} \right) dt \, ds
\]

and by the H"{o}lder’s inequality with \( c_2 = \left( \int_{\Sigma} d \text{vol}_\Sigma \right)^{\frac{p-2}{2}} \)

\[
\leq \delta \left( \|d_{A\alpha}\|_{L^p}^{p} + \varepsilon^{2p}\|\phi\|_{L^p}^{p} \right)
\]

\[
+ c_0 c_1 \varepsilon^{2p}\|\partial_t \phi\|_{L^p}^{p} + c_0 c_1 \varepsilon^{2p}\|\partial_t \phi\|_{L^p}^{p}
\]

since \( \|\Psi\|_{L^\infty} + \|\Phi\|_{L^\infty} \leq c_3 \), for \( c_0 c_1 c_3 \varepsilon^p \leq \delta \)

\[
\leq \delta \left( \|d_{A\alpha}\|_{L^p}^{p} + \varepsilon^{2p}\|\phi\|_{L^p}^{p} + 2c_2 \|\phi\|_{L^p}^{p} \right)
\]

\[
+ c_0 c_1 c_2 \varepsilon^{2p}\|\partial_t \phi\|_{L^p}^{p} + c_0 c_1 c_2 \varepsilon^{2p}\|\partial_t \phi\|_{L^p}^{p}
\]

Therefore, the theorem follows from the lemma 3.18, and the last estimate choosing \( \delta \) small enough.

\[\blacksquare\]

### 8. Quadratic Estimates

In this section we prove the following quadratic estimates.

**Lemma 8.1.** For any \( c_0 > 0 \) there are two positive constants \( c \) and \( \varepsilon_0 \) such that, for any \( 0 < \varepsilon < \varepsilon_0 \), the following holds. If two connections \( \Xi = A + \Psi dt + \Phi ds \), \( \bar{\Xi} = \bar{A} + \bar{\Psi} dt + \bar{\Phi} ds \in W^{1,p} \), with \( \alpha + \bar{\alpha} dt + \bar{\phi} ds := \Xi - \Xi \), satisfies \( \|\alpha + \bar{\phi} dt\|_{\infty,\varepsilon} \leq c_0 \), then

\[
\varepsilon^2 \left\| (\mathcal{D}^\varepsilon(\Xi) - \mathcal{D}^\varepsilon(\bar{\Xi})) (\alpha + \psi dt + \phi ds) \right\|_{0,p,\varepsilon}
\]

\[
\leq c \|\alpha + \bar{\psi} dt + \bar{\phi} ds\|_{\infty,\varepsilon} \|\alpha + \psi dt + \phi ds\|_{1,p,\varepsilon}
\]

\[
+ c \|\alpha + \psi dt + \phi ds\|_{\infty,\varepsilon} \|\bar{\alpha} + \bar{\psi} dt\|_{1,p,\varepsilon},
\]

\[
\varepsilon^2 \left\| (\mathcal{D}^\varepsilon(\Xi) - \mathcal{D}^\varepsilon(\bar{\Xi})) (\alpha + \psi dt + \phi ds) \right\|_{0,p,\varepsilon}
\]

\[
\leq c \|\alpha + \psi dt + \phi ds\|_{\infty,\varepsilon} \|\alpha + \psi dt + \phi ds\|_{1,p,\varepsilon}
\]

\[
+ c \|\alpha + \psi dt + \phi ds\|_{\infty,\varepsilon} \|\bar{\alpha} + \bar{\psi} dt + \bar{\phi} ds\|_{0,p,\varepsilon}
\]

\[
+ c \|\alpha + \psi dt + \phi ds\|_{0,p,\varepsilon} \|\bar{\alpha} + \bar{\psi} dt + \bar{\phi} ds\|_{L^\infty} + \|\bar{\alpha} + \bar{\psi} dt + \bar{\phi} ds\|_{L^\infty}
\]

\[
+ c \|\alpha + \psi dt + \phi ds\|_{0,p,\varepsilon} \left( \varepsilon \|d_A \bar{\psi}\|_{L^\infty} + \varepsilon^2 \|\nabla_t \bar{\psi}\|_{L^\infty} \right),
\]
Lemma 8.2. In this case we have the following estimates.

\[ \epsilon^2 \left\| \pi_A(D'(\Xi) - \lambda * [\alpha \wedge \omega(A)] - D'(\Xi)) (\alpha + \psi dt + \phi ds) \right\|_{L^p} \]

\[ \leq c \| \tilde{\alpha} \| + \tilde{\psi} \| + \| \phi \|_{L^p} \| \| (1 - \pi_A) \alpha + \psi dt + \phi ds \|_{1,p,\epsilon} + \epsilon^2 \| \| \alpha \|_{L^p} \| \| \psi \|_{L^p} \]

\[ + \epsilon^2 \| \| \phi \|_{L^p} \| \| \tilde{\psi} \|_{L^p} \| \| \tilde{\phi} \|_{L^p} \| \| \alpha \|_{L^p} \| \| \psi \|_{L^p} \leq \| \alpha \|_{L^p} \| \| \psi \|_{L^p} \| \| \phi \|_{L^p} \]

\[ + \epsilon^2 \| \| (\| \tilde{\phi} \|_{L^p} + \| \tilde{\alpha} \|_{L^p}) \| \pi_A(\alpha) \|_{L^p} + \| \| \alpha \|_{L^p} \| \| (\| \tilde{\alpha} - \lambda \epsilon^2 \omega(A) \|_{L^p} \]

For any \( \alpha + \psi dt + \phi ds \in W^{1,2p} \) and where \( \lambda \in \{0, 1\} \).

Proof. The lemma can be proved directly estimating term by term the following identities.

\[ \left( D'_1(\Xi) - D'_2(\Xi) \right) (\alpha + \psi dt + \phi ds) = \pi_A \left( [\tilde{\phi}, \alpha] - [\tilde{\alpha}, \phi] \right) \]

\[ + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] \]

\[ + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] \]

\[ - \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] \]

\[ - \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] + \epsilon^2 \left[ [\phi, \psi] + \left( d_A \tilde{\alpha} + \frac{1}{2} [\tilde{\alpha} \wedge \tilde{\alpha}] \right) \right] \]

We choose a connection \( \Xi = A + \Psi dt + \Phi ds \in W^{1,p} \) and a 1-form \( \xi = \alpha + \psi dt + \phi ds \in W^{1,2p} \), then \( F^c(\Xi + \xi) = F^c(\Xi) + D^c(\Xi) + C^c(\Xi)(\xi) \) and we denote by \( C^c_1(\Xi), C^c_2(\Xi) \) and \( C^c_3(\Xi) \) the three components of \( C^c(\Xi) = C^c_1(\Xi) + C^c_2(\Xi) dt + C^c_3(\Xi) ds \); in this case we have the following estimates.

Lemma 8.2. For any \( c_0 > 0 \) there are constants \( c > 0 \) and \( \epsilon_0 > 0 \) such that, for any \( 0 < \epsilon < \epsilon_0 \),

\[ \epsilon^2 \left\| C^c(\Xi)(\xi) \right\|_{0,p,\epsilon} \leq c \left\| \xi \right\|_{\infty,\epsilon} \left\| \xi \right\|_{1,p,\epsilon} \]

\[ \epsilon^2 \left\| \pi_A \left( C^c_1(\Xi)(\xi) \right) \right\|_{0,p,\epsilon} \leq c \left\| (1 - \pi_A) \xi \right\|_{\infty,\epsilon} \left\| (1 - \pi_A) \alpha + \psi dt \right\|_{1,p,\epsilon} + \epsilon \left\| \nabla_4 \alpha \right\|_{L^p} \]

\[ + c \left\| \pi_A \alpha \right\|_{L^\infty} \left\| (1 - \pi_A) \alpha + \psi dt \right\|_{1,p,\epsilon} \]

\[ + c \left\| \pi_A \alpha \right\|_{L^\infty} \left\| (\pi_A \alpha \right\|_{L^\infty} \left\| \epsilon^2 \right\|_{L^p} \]

for any \( \xi := \alpha + \psi dt + \phi ds \in W^{1,2p} \) and where we assume that \( \left\| \alpha + \psi dt \right\|_{\infty,\epsilon} \leq c_0 \).
Proof. Also this lemma can be showed estimating term by term the identities:

\[ C_1^s(\Xi)(\xi) = [\phi, \alpha] - [\alpha, \phi] + \frac{1}{\varepsilon^2} d^*_\alpha [\alpha \wedge \alpha] \]

\[ - \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + [\alpha \wedge \alpha])] + \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + \frac{1}{2} [\alpha \wedge \alpha])] \]

\[- \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + [\alpha \wedge \alpha])] + \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + \frac{1}{2} [\alpha \wedge \alpha])] \]

\[- 2[\psi, (\nabla_t \alpha + [\alpha, \psi])] - \nabla_t[\psi, \alpha] + [\alpha, \nabla_t \psi] - \nabla_t[\psi, \alpha] + [\alpha, \nabla_t \psi] \]

\[- \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + [\alpha, \psi])] + \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + \frac{1}{2} [\alpha \wedge \alpha])] \]

\[ \pi_A(C_1^s(\Xi)(\xi)) = \pi_A(\xi) \]

\[ - \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + [\alpha, \psi])] + \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + \frac{1}{2} [\alpha \wedge \alpha])] \]

\[ C_2^s(\Xi)(\xi) = [\phi, \psi] - [\phi, \psi] + \frac{2}{\varepsilon^2} [\alpha \wedge * (\nabla_t \alpha - d_A \psi - [\alpha, \psi])] \]

\[ - \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + [\alpha, \psi])] + \frac{1}{\varepsilon^2} [\alpha \wedge * (d_A \alpha + \frac{1}{2} [\alpha \wedge \alpha])] \]

\[ C_2^s(\Xi)(\alpha + \psi dt + \phi ds) = 0. \]

\[ \Box \]

9. The map \( \mathcal{K}^e_2 \): A first approximation

In the next section, we will construct, using a Newton’s iteration, a perturbed Yang-Mills flow \( \Xi^e \in \mathcal{M}^e \left( T^{e,b}(\Xi^-), T^{e,b}(\Xi^+) \right) \) for any perturbed geodesic flow \( \Xi^0 \in \mathcal{M}^0(\Xi^-,\Xi^+) \) and every pair of connections \( \Xi^\pm \in \mathcal{B}^{e,b} \) with index difference 1, where \( b \) is a regular value of \( \mathcal{E}^e \). For this purpose, we need to define a connection \( \mathcal{K}^e_2(\Xi^0) \) which is an approximate solution of the perturbed Yang-Mills flow equation. \( \mathcal{K}^e_2(\Xi^0) \) is constructed in two steps: First, we add to \( \Xi^0 \) a 1-form \( \alpha^e_0(s) + \psi^e_0(s)dt \) which satisfies the limit conditions \( \lim_{s \to \pm \infty} \Xi^0 + \alpha^e_0 + \psi^e_0dt = T^{e,b}(\Xi^\pm) \) and then we add a second 1-form in order to have an approximated solution of the Yang-Mills flow equations.

First, we recall that a connection \( \Xi^0 := A^0 + \Psi^0 dt + \Phi^0 ds \) descends to a flow line between two perturbed geodesics \( \Xi^\pm = A^\pm + \Psi^\pm dt \) when it satisfies the equations (2.2) and (2.1), i.e.

\[ \partial_s A^0 - d_A^0 \Phi^0 = \pi_A \left( \nabla_t (\partial_s A^0 - d_A^0 \Psi^0) + * X_s(A^0) \right) = 0, \]

\[ d_A^0 (\partial_s A^0 - d_A^0 \Psi^0) = d_A^0 (\partial_s A^0 - d_A^0 \Phi^0) = 0. \]

Next, we choose a smooth positive function \( \theta \) such that \( \theta(s) = 0 \) for \( s \leq 1 \), \( \theta(s) = 1 \) when \( s \geq 2 \), \( 0 \leq \theta \leq 1 \) and \( 0 \leq \partial_s \theta \leq c_0 \) with \( c_0 > 0 \). Thus, we define \( \alpha^e_0 + \psi^e_0 dt \) as

\[ \alpha^e_0(s) + \psi^e_0(s)dt := \theta(-s) g(s)^{-1} (T^{e,b}(A^- + \Psi^- dt) - (A^- + \Psi^- dt)) g(s) \]

\[ + \theta(s) g(s)^{-1} (T^{e,b}(A^+ + \Psi^+ dt) - (A^+ + \Psi^+ dt)) g(s), \]

where \( g \in \mathcal{C}^0(P \times S^1 \times \mathbb{R}) \) satisfies

\[ g^{-1} \partial_s g = \Phi^0, \quad \lim_{s \to -\infty} g = 1; \]
we introduce $g$ in order to make the definition of $\Xi^0 + \alpha^0 + \psi^0 dt$ gauge-invariant.

**Lemma 9.1.** We choose two constants $b > 0$, $p > 2$. There are positive constants $\varepsilon_0, \varepsilon$ such that the following holds. For every $\varepsilon \in (0, \varepsilon_0)$, every pair $\Xi_\pm := A_\pm + \Psi_\pm dt \in \text{Crit}^1_{\varepsilon, \varepsilon}$ that are perturbed closed geodesics of index difference one, there exists a unique equivariant map

$$K_2^\varepsilon : \mathcal{M}^0(\Xi_-, \Xi_+) \to W^{1,2;p}(\mathcal{T}^{\varepsilon,b}(\Xi_-), \mathcal{T}^{\varepsilon,b}(\Xi_+))$$

such that for any $\Xi := A^0 + \Psi^0 dt + \Phi^0 ds \in \mathcal{M}^0(\Xi_-, \Xi_+)$, with $\alpha^0 + \psi^0 dt$ and $g$ defined as in (9.1) and in (9.2),

$$K_2^\varepsilon(\Xi^0) - (\Xi^0 + \alpha^0 + \psi^0 dt) \in \text{im } d^*_{A^0}$$

and

$$\frac{1}{\varepsilon^2} d^*_{A^0} d_{A^0} \left( K_2^\varepsilon(\Xi^0) - (\Xi^0 + \alpha^0 + \psi^0 dt) \right)$$

In addition, it satisfies

$$\|K_2^\varepsilon(\Xi^0) - (\Xi^0 + \alpha^0 + \psi^0 dt)\|_{1,2;p,1} \leq \varepsilon^2,$$

$$\|\mathcal{F}_1(K_2^\varepsilon(\Xi^0))\|_{L^p} \leq \varepsilon^2, \quad \|\mathcal{F}_2(K_2^\varepsilon(\Xi^0))\|_{L^p} \leq \varepsilon.$$
Proof of lemma [9.4] $K_p^2(\Xi^0)$ is uniquely defined by (9.3) and (9.4) because $F_{A^0} = 0$ and
\[ d_{A^0}^*d_{A^0} : \text{im} \ d_{A^0}^*\Omega^2(\Sigma, g_p) \to \text{im} \ d_{A^0}^*\Omega^2(\Sigma, g_p) \]
is bijective. Furthermore, the lemma [3.1] the commutation formulas (2.24), (2.5) and the estimates of the geodesic flow (3.3)-(3.6) yield to (9.5). Therefore we need only to prove (9.6).

We define $\Xi_1 := A^*_1 + \Psi^*_1 dt + \Phi^*_1 ds := \Xi^0 + \alpha^*_0 + \psi^*_0 dt$ and we consider
\[ A_{\pm}(s) + \Psi_{\pm}(s)dt = g(s)^*(A_{\pm} + \Psi_{\pm} dt), \]
\[ (9.7) \quad \alpha(s) + \psi(s)dt = \theta(-s)((A^0(s) + \Psi^0(s)dt) - (A_-(s) + \Psi_-(s)dt)) \]
\[ + \theta(s)((A^0(s) + \Psi^0(s)dt) - (A_+(s) + \Psi_+(s)dt)) \]
and
\[ (9.8) \quad \bar{\alpha}_1 := \theta(-s) \left( d_{A_-}^*d_{A_-}^* \right)^{-1} \left( \nabla^\Psi \left( \partial_t A_- - d_{A_-}^* \Psi_- \right) + *X_t(A_-) \right) \]
\[ \theta(s) \left( d_{A_+}^*d_{A_+}^* \right)^{-1} \left( \nabla^\Psi \left( \partial_t A_+ - d_{A_+}^* \Psi_+ \right) + *X_t(A_+) \right). \]

Furthermore, we consider $A_2 - \Psi_2 dt + \Phi_2 ds := \Xi_2 := K_p^2(\Xi^0)$. If we look at the expansion
\[ F_1^* \left( K_p^2(\Xi^0) \right) = \partial_s A^*_2 - d_{A^*_2}^*\Psi^*_2 + \frac{1}{\varepsilon^2} d_{A^*_2}^*F_{A^*_2}^* + \frac{1}{\varepsilon^2} \nabla^\Psi \left( \partial_t A_2 - d_{A_2}^* \Psi_2 \right) - *X_t(A_2^*) \]
\[ = \partial_s A^0 - d_{A^0}^*\Psi^0 - \pi_{A^0} \left( \nabla_t (\partial_t A^0 - d_{A^0}^*\Psi^0) + *X_t(A^0) \right) \]
\[ + \left( \nabla_s - \nabla_t^* \right) \bar{\alpha}_1 + *X_t(A^2) + *X_t(A^0 + \alpha^*_0) \]
\[ - \frac{1}{\varepsilon^2} \left[ \alpha^*_0 \wedge * \left( d_{A^0}^*\alpha^*_1 + \frac{1}{2} [\alpha^*_1 \wedge \alpha^*_1] \right) \right] + \frac{1}{\varepsilon^2} d_{A^0}^*d_{A^0}^* \alpha^*_1 - (1 - \pi_{A^0}) \left( \nabla_t (\partial_t A^0 - d_{A^0}^*\Psi^0) + *X_t(A^0) \right) \]
\[ + F_2^* + F_3^* \]

we remark that the second line vanishes because $\Xi^0$ is a geodesic flow, the third and the fourth can be estimates by $c\varepsilon^2$ because the norm $\| \cdot \|_{1,2,p,1}$ of $\bar{\alpha}_1$ can be estimated by the same factor by (9.4). The fifth line of (9.9) can be written by the definitions as
\[ - \frac{1}{\varepsilon^2} d_{A^0}^*d_{A^0}^* \alpha^*_1 - \theta(-s)d_{A^0}^*d_{A^0}^*\pi_{A_-}(\alpha^*_0) - \theta(s)d_{A^0}^*d_{A^0}^*\pi_{A_+}(\alpha^*_0). \]

Therefore we have that
\[ \left\| F_1^* \left( K_p^2(\Xi^0) \right) \right\|_{L^p} \leq c\varepsilon^2 + \left\| \frac{1}{\varepsilon^2} d_{A^0}^*d_{A^0}^* \alpha^*_1 + \pi_{A_-}(\alpha^*_0) + \pi_{A_+}(\alpha^*_0) \right\|_{L^p} - F_1^* \]
\[ + \left\| F_2^* \right\|_{L^p} + \left\| F_3^* \right\|_{L^p}. \]

For this purpose, we need to investigate $F_1^*$, $F_2^*$ and $F_3^*$. In order to simplify the exposition we evaluate $F_1^*(s)$ for $s \leq 0$; for $s > 0$ the computation is the same, we
only need to substitute \( A_- + \Psi_- dt \) with \( A_+ + \Psi_+ dt \) and \( \theta(-s) \) with \( \theta(s) \). If we denote \( \partial_t A_- - A_- \Psi_- \) by \( B_t^- \), we have

\[
F_1^\varepsilon := \frac{1}{\varepsilon^2} d_{A_-}^* \left( \left[ \frac{1}{2} \varepsilon_0 - \alpha_0 \right] + \frac{1}{2} \left[ \alpha_0 \wedge \alpha_0^\varepsilon \right] \right) - [\Psi_0, B_t^-]
\]

and since \( T_{\varepsilon, b}(\Xi_-) \) is a perturbed Yang-Mills connection and by the definition of \( \alpha_0^\varepsilon + \Psi_0 dt \) we get that

\[
0 = \theta(-s) \left( -\nabla_t^\varepsilon B_t^- - *X_t(\alpha_-^\varepsilon) \right) + \frac{1}{\varepsilon^2} d_{A_-}^* \left( \left[ \frac{1}{2} \varepsilon_0 - \alpha_0 \right] + \frac{1}{2} \left[ \alpha_0 \wedge \alpha_0^\varepsilon \right] \right) - [\Psi_0, B_t^-]
\]

where \( F_{[-2,0]}^\varepsilon \) contain only quadratic terms with support in \([-2,0]\). Therefore we can write \( F_1^\varepsilon \) in the following way by the definition of \( \alpha_0^\varepsilon \)

\[
F_1^\varepsilon = \theta(-s) \left( \nabla_t^\varepsilon B_t^- + *X_t(\alpha_-^\varepsilon) \right) + F_{[-2,0]}^\varepsilon
\]

The two terms are

\[
F_2^\varepsilon := -\frac{1}{\varepsilon^2} \left[ \alpha \wedge \left[ \frac{1}{2} \varepsilon_0 - \alpha_0 \right] \right]
\]
\[ F^r_3 := - \nabla_t [\psi_0^r, \alpha^r_1] - [\psi_0^r, \nabla_t \alpha^r_1] \]
\[ - \frac{1}{\varepsilon^2} * \left[ \alpha^r_0 \wedge * \left( d_A \alpha^r_1 + \frac{1}{2} [\alpha^r_1 \wedge \alpha^r_1] \right) \right] + \frac{1}{\varepsilon^2} d^r_A [\alpha^r_0 \wedge \alpha^r_1] \]
\[ - \frac{1}{\varepsilon^2} * [\alpha^r_0 \wedge * (d_A \alpha^r_0 + [\alpha^r_0 \wedge \alpha^r_1])]. \]

This allows us to conclude that, by the a priori estimates (3.3)-(3.9),
\[
\| F^r_1 (K^r_2(\Xi^0)) \|_{L^p} \leq c \varepsilon^2 + \left\| \frac{1}{\varepsilon^2} d^r_A d_A \alpha^r_1 - F^r_1 \right\|_{L^p} + \| F^r_2 \|_{L^p} + \| F^r_3 \|_{L^p} \leq c \varepsilon^2
\]
in order to see this we need that
\[
\| d_{A_-}(\alpha^r_0 - \tilde{\alpha}^r_1) \|_{L^p(\Sigma \times \Sigma^1)} + \| (1 - \pi_{A_-}) \alpha^r_0 - \tilde{\alpha}^r_1 \|_{L^p(\Sigma \times \Sigma^1)} \leq c \varepsilon^4
\]
which holds by the remark (1.2). The second Yang-Mills flow equation can be written as
\[
F^r_2 (K^r_2(\Xi^0)) = \partial_t \Phi^2 - \nabla_t \Phi^2 - \frac{1}{\varepsilon^2} d^r_A (\partial_t A^2 - d_A \Phi^2) \]
\[ = \partial_t \Phi^0 - \nabla_t \Phi^0 - \frac{1}{\varepsilon^2} d^r_A (\partial_t A^0 - d_A \Phi^0) \]
\[ + \frac{1}{\varepsilon^2} * \left[ (\alpha^r_0 + \alpha_1) \wedge * (\partial_t A^0 - d_A \Phi^0) \right] \]
\[ + \frac{1}{\varepsilon^2} * \left[ (\alpha^r_0 + \alpha_1) \wedge \left( \nabla^\psi \alpha^r_0 - d_A \psi_0^r - [\alpha^r_0, \psi_0^r] \right) \right] \]
\[ + \frac{1}{\varepsilon^2} * \left[ (\alpha^r_0 + \alpha_1) \wedge \left( \nabla^\psi \alpha^r_0 \wedge (\partial_t + \partial_0) \right) \right] \]
\[ + d_A (\partial_t A^0 - d_A \Phi^2) - (\partial_t A^0 - d_A \Phi^0) \]

and therefore by the lemma (7.2) and the identity \( d^r_A (\partial_t A^0 - d_A \Phi^0) = 0 \)
\[
\| F^r_2 (K^r_2(\Xi^0)) \|_{L^p} \leq \| 2 \left[ (\partial_t A^0 - d_A \Phi^0) \wedge * (\partial_t A^0 - d_A \Phi^0) \right] \|_{L^p} \]
\[ + \frac{c}{\varepsilon^2} \| \alpha^r_0 + \alpha_1 \|_{L^p} + \| d^r_A d_A \psi_0^r \|_{L^p} \leq c. \]

\[
\Box
\]

**Theorem 9.2.** We choose a regular value \( b \) of \( E^H \), \( p > 2 \), then there are two positive constants \( c \) and \( \varepsilon_0 \) such that the following holds. For any \( \Xi^0 \in \mathcal{M}^0(\Xi_-, \Xi_+) \) with \( \Xi_{\pm} \in \text{Crit}_{E^H} \) the estimates
\[
\| \pi_A(\xi) \|_{L^2, p, 1} \leq c \varepsilon \| D^r(\Pi^r_2(\Xi^0)) \|_{L^p} + c \varepsilon \| \pi_A D^r(\Pi^r_2(\Xi^0)) \|_{L^p},
\]
(9.10) \[ \| (1 - \pi_A) \xi \|_{L^2, p, \varepsilon} \leq c \varepsilon^2 \| D^r(\Pi^r_2(\Xi^0)) \|_{L^p} + c \varepsilon \| \pi_A D^r \xi \|_{L^p}, \]
\[ \| (1 - \pi_A) \alpha \|_{L^2, p, \varepsilon} \leq c \varepsilon^2 \| D^r(\Pi^r_2(\Xi^0)) \|_{L^p} + c \varepsilon \| \pi_A D^r \xi \|_{L^p}, \]
hold for all compactly supported 1-form \( \xi = \alpha + \psi dt + \phi ds \in W^{1,2,p}, \eta \in W^{1,2,p}, \xi \in im (D^r(\Pi^r_2(\Xi^0)) \|^* \) and \( 0 < \varepsilon \leq \varepsilon_0. \)

**Proof.** By the theorem (1.1) by the remark (1.2) and by the Sobolev theorem (8.1) we have that
\[
\varepsilon \| (1 - \pi_A) \alpha_0^r + \psi_0^r dt \|_{L^\infty, \varepsilon} + \| d_A \alpha_0^r \|_{L^\infty} + \| d^r_A \alpha_0^r \|_{L^\infty} \leq c \varepsilon^4 + \frac{\varepsilon}{2},
\]
\[
\varepsilon \| \psi_0^r \|_{L^\infty} + \varepsilon^2 \| \nabla_t \psi_0^r \|_{L^\infty} \leq c \varepsilon^{3 - \frac{1}{2}}, \| \nabla_t \alpha_0^r \|_{L^\infty} + \| \pi_A(\alpha_0^r) \|_{L^\infty} \leq c \varepsilon^2;\]
in addition, by the previous lemma \([9.1]\) we know that
\[
\|K_\varepsilon(\Xi^0) - (\Xi^0 + \alpha_0^\varepsilon + \psi_0^\varepsilon dt)\|_{1,2,p,1} \leq c\varepsilon^2.
\]
Thus by the quadratic estimates of the lemma \([8.1]\) we obtain
\[
\varepsilon^2 \| (D^\varepsilon (K_\varepsilon(\Xi^0)) - D^\varepsilon (\Xi^0)) \|_{0,p,e} \\
\leq \varepsilon^2 \| (D^\varepsilon (K_\varepsilon(\Xi^0)) - D^\varepsilon (\Xi^0 + \alpha_0^\varepsilon + \psi_0^\varepsilon dt)) \|_{0,p,e} \\
+ \varepsilon^2 \| (D^\varepsilon (\Xi^0 + \alpha_0^\varepsilon + \psi_0^\varepsilon dt) - D^\varepsilon (\Xi^0)) \|_{0,p,e} \\
\leq c\varepsilon^2 \frac{1}{\varepsilon} \| \xi \|_{1,2,p,\varepsilon},
\]
(9.11) \[
\|\pi_{A^0}(D^\varepsilon (K_\varepsilon(\Xi^0)) \xi - D^\varepsilon (\Xi^0) \xi - \ast [\alpha, \ast \omega(A^0)])\|_{0,p,e} \leq c\varepsilon^{1 - \frac{1}{p}} \| \xi \|_{1,2,p,\varepsilon},
\]
where we used that
\[
\omega(A^0) = d_{A^0} (d_{A^0}^* d_{A^0})^{-1} (\nabla_t (\partial_t A^0 - d_{A^0} \Psi^0) + \ast X_t(A^0))
\]
and, with the notation of the proof of theorem \([9.1]\)
\[
\left| \frac{1}{\varepsilon^2} d_{A^0} (K_\varepsilon(\Xi^0) - \Xi^0) - \omega(A^0) \right|_{L^\infty} \\
\leq \theta(-s) \| d_{A^0} (T^{\varepsilon,b}(\Xi_+) - \Xi_+ - \alpha_1) \|_{L^\infty} + c\theta(s) \| \pi_{A^0} (T^{\varepsilon,b}(\Xi_+) - \Xi_+) \|_{L^\infty} \\
+ \theta(s) \| d_{A^0} (T^{\varepsilon,b}(\Xi_-) - \Xi_- - \alpha_1) \|_{L^\infty} + c\theta(s) \| \pi_{A^0} (T^{\varepsilon,b}(\Xi_-) - \Xi_-) \|_{L^\infty}
\]
which is smaller than \(c\varepsilon^2\). Furthermore, for \(\xi = (D^\varepsilon (K_\varepsilon))^* \eta, \eta = \eta_1 + \eta_2 dt + \eta_3 ds\), we have by the lemmas \([8.1]\) (for the adjoint operators) and \([7.4]\) as well as by the theorem \([7.8]\) that
(9.12) \[
\|\pi_A(\xi - (D^0(\Xi^0))^* (\pi_A(\eta)))\|_{L^p} \\
\leq \|\pi_A((D^\varepsilon(\Xi^0))^* (\eta) - \ast [\eta_1 \wedge \ast \omega(A^0)] - (D^0(\Xi^0))^* (\pi_A(\eta)))\|_{L^p} + c\varepsilon^{1 - \frac{1}{p}} \| \xi \|_{0,p,e} \\
\leq c\varepsilon^{1 - \frac{1}{p}} \| \xi \|_{0,p,e}.
\]
The theorem follows then from the theorem \([7.8]\) and the last computations.  \(\square\)

10. The map \(R^{\varepsilon,b}\) between flows

In this section we will show that, for any pair \(\Xi^0_{\pm} \in \text{Crit}_{E^H}\), any geodesic flow \(\Xi^0 \in \mathcal{M} (\Xi^0_{\pm}, \Xi^0_{\pm})\) can be approximated by a Yang-Mills flow
\[
\Xi^\varepsilon \in \mathcal{M} (T^{\varepsilon,b}(\Xi^0), T^{\varepsilon,b}(\Xi^0))
\]
provided that \(\varepsilon\) is small enough. In addition, \(\Xi^\varepsilon\) will turn out to be the unique Yang-Mills flow in a ball around \(\Xi^0\) of radius \(\delta\varepsilon\). Therefore we can define an injective map \(R^{\varepsilon,b}\) between the flows of the two functionals provided that we choose an energy bound \(b\) for the critical connections and \(\varepsilon\) small enough.

**Theorem 10.1 (Existence).** We assume that the energy functional \(E^H\) is Morse-Smale and we choose two constants \(b > 0, p > 2\). There are constants \(\varepsilon_0, c > 0\) such that the following holds. For every \(\varepsilon \in (0, \varepsilon_0),\) every pair \(\Xi^0_{\pm} := A^0_{\pm} + \Psi^0_{\pm} dt \in \text{Crit}^b_{E^H}\) that are perturbed closed geodesics of index difference one and
Figure 2. Existence and uniqueness.

every \( \Xi^0 \) := \( A^0 + \Psi^0 dt + \Phi^0 ds \in M^0(\Xi^0_-, \Xi^0_+) \), there exists a connection \( \Xi^\varepsilon \in M^\varepsilon(\mathcal{T}^{\varepsilon,b}(\Xi^0_0), \mathcal{T}^{\varepsilon,b}(\Xi^0_0)) \) which satisfies

\[
(10.1) \quad d_{\Xi^0}^* (\Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0)) = 0, \quad \Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \in \text{im} \left( \mathcal{D}^\varepsilon (\mathcal{K}_2 (\Xi^0)) \right)^*,
\]

\[
(10.2) \quad \| (1 - \pi_{A^0}) (\Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0)) \|_{1,2,p,\varepsilon} + \varepsilon \| \pi_{A^0} (\Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0)) \|_{1,2,p,1} \leq c \varepsilon^3.
\]

**Theorem 10.2** (Local uniqueness). We choose \( p > 3 \). For every pair \( \Xi^0_- := A^0_+ + \Psi^0_+ dt \in \text{Crit}_{E^b} \) that are perturbed closed geodesics of index difference one, every \( \Xi^0 := A^0 + \Psi^0 dt + \Phi^0 ds \in M^0(\Xi^0_-, \Xi^0_+) \) and any \( c > 0 \) there are an \( \varepsilon_0 > 0 \) and a \( \delta > 0 \) such that the following holds. If \( \Xi^\varepsilon, \tilde{\Xi}^\varepsilon \in M^\varepsilon(\mathcal{T}^{\varepsilon,b}(\Xi^0_0), \mathcal{T}^{\varepsilon,b}(\Xi^0_0)) \) satisfy

\[
(10.3) \quad \Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0), \tilde{\Xi}^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \in \text{im} \left( \mathcal{D}^\varepsilon (\mathcal{K}_2^\varepsilon (\Xi^0)) \right)^*.
\]

and the estimates

\[
(10.4) \quad \| \Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \|_{1,2,p,\varepsilon} + \| \tilde{\Xi}^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \|_{L^\infty} \leq \delta \varepsilon,
\]

then \( \Xi^\varepsilon = \tilde{\Xi}^\varepsilon \).

**Definition 10.3.** We choose \( p > 3 \). For every regular value \( b > 0 \) of the energy \( E^H \) there are positive constants \( \varepsilon_0, \delta \) and \( c \) such that the assertion of the theorems \([10.1]\) and \([10.2]\) hold with these constants. Shrink \( \varepsilon_0 \) such that \( c_0 \varepsilon_0 + c_0 \varepsilon_0^{1-\frac{1}{p}} < \delta \), where \( c_0 \) is the constant of the Sobolev’s theorem \([6.1]\). Theorems \([10.1]\) and \([10.2]\) assert that, for every pair \( \Xi^0_+ := A^0_+ + \Psi^0_+ dt \in \text{Crit}_{E^b} \) that are perturbed closed geodesics of index difference one, every \( \Xi^0 \in M^0(\Xi^0_-, \Xi^0_+) \) and every \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \), there is a unique \( \Xi^\varepsilon \in M^\varepsilon(\mathcal{T}^{\varepsilon,b}(\Xi^0_0), \mathcal{T}^{\varepsilon,b}(\Xi^0_0)) \) satisfying

\[
(10.5) \quad d_{\Xi^0}^* (\Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0)) = 0, \quad \Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \in \text{im} \left( \mathcal{D}^\varepsilon (\mathcal{K}_2 (\Xi^0)) \right)^*.
\]

and

\[
(10.6) \quad \| \Xi^\varepsilon - \mathcal{K}_2^\varepsilon (\Xi^0) \|_{1,2,p,\varepsilon} \leq c \varepsilon^2.
\]

We define the map

\[
\mathcal{R}^{\varepsilon,b} : M^0(\Xi^0_-, \Xi^0_+) \to M^\varepsilon(\mathcal{T}^{\varepsilon,b}(\Xi^0_0), \mathcal{T}^{\varepsilon,b}(\Xi^0_0))
\]
by $R^{\varepsilon,b}(\Xi^0) := \Xi^\varepsilon$ where $\Xi^\varepsilon \in \mathcal{M}^{\varepsilon} \left( T^{\varepsilon,b}(\Xi^0), T^{\varepsilon,b}(\Xi^0) \right)$ is the unique Yang-Mills flow satisfying \( (10.5) \) and \( (10.5) \).

**Proof of theorem \( 10.4 \).** We choose $\Xi^\varepsilon_k := K^\varepsilon_2(\Xi^0)$. By induction we define, for $k \geq 3$, $\Xi^\varepsilon_k := \Xi^\varepsilon_{k-1} + \eta^\varepsilon_{k-1}, \eta^\varepsilon_k = \alpha^\varepsilon_k + \psi^\varepsilon_k dt + \phi^\varepsilon_k ds$, where $\eta^\varepsilon_k$ is defined by

\[
D^\varepsilon(K^\varepsilon_2(\Xi^0))(\eta^\varepsilon_{k-1}) = -F^\varepsilon(\Xi^\varepsilon_{k-1}), \quad \eta^\varepsilon_{k-1} \in \text{im} \left( D^\varepsilon(K^\varepsilon_2(\Xi^0)) \right)^*.
\]

In addition, one can remark that

\[
F^\varepsilon(\Xi^\varepsilon_{k-1}) = F^\varepsilon(\Xi^\varepsilon_{k-2}) + D^\varepsilon(\Xi^\varepsilon_{k-2})(\eta^\varepsilon_{k-2}) + C^\varepsilon(\Xi^\varepsilon_{k-2})(\eta^\varepsilon_{k-2}).
\]

By theorem \( 9.2 \) we have the estimate

\[
\|(1 - \pi_{A^0})\eta^\varepsilon_{k-1} \|_{1,2,p,\varepsilon} + \varepsilon \|\pi_A(\alpha^\varepsilon_{k-1})\|_{1,2,p,1} \leq c \varepsilon^2 \|D^\varepsilon(K^\varepsilon_2(\Xi^0))(\eta^\varepsilon_{k-1})\|_{0,p,\varepsilon} + c \varepsilon \|\pi_{A^0}(D^\varepsilon(K^\varepsilon_2(\Xi^0))(\eta^\varepsilon_{k-1}))\|_{L^p}
\]

where the last step follows from \( (10.7) \) and by \( (10.8) \) we obtain

\[
\varepsilon^2 \|F^\varepsilon(\Xi^\varepsilon_{k-1})\|_{0,p,\varepsilon} + c \varepsilon \|\pi_{A^0}(F^\varepsilon(\Xi^\varepsilon_{k-1}))\|_{L^p} \leq c \varepsilon^2 \|C^\varepsilon(\Xi^\varepsilon_{k-2})(\eta^\varepsilon_{k-2})\|_{0,p,\varepsilon} + c \varepsilon \|\pi_{A^0}(C^\varepsilon(\Xi^\varepsilon_{k-2})(\eta^\varepsilon_{k-2}))\|_{L^p}
\]

and finally using the lemmas \( 8.1 \) and \( 8.2 \) we can conclude

\[
\leq c \varepsilon \|\eta^\varepsilon_{k-2}\|_{\infty,\varepsilon} \|\alpha^\varepsilon_{k-2} + \psi^\varepsilon_{k-2}\|_{1,1,p,\varepsilon}
\]

Next, by the estimates of lemma \( 9.1 \)

\[
\|F^\varepsilon_2(K^\varepsilon_2(\Xi^0))\|_{L^p} \leq c \varepsilon^2, \quad \|F^\varepsilon_2(K^\varepsilon_2(\Xi^0))\|_{L^p} \leq c,
\]

there is a positive constant $c_0$ such that

\[
\|(1 - \pi_{A^0})\eta^\varepsilon_2\|_{1,2,p,\varepsilon} + \varepsilon \|\pi_A(\alpha^\varepsilon_2)\|_{1,2,p,1} \leq c_0 \varepsilon^3.
\]

Using the Sobolev’s theorem \( 5.1 \) one can easily see that, for $\varepsilon$ small enough and $k > 3$, by induction there are two positive constants $c_1, c$ such that

\[
\|(1 - \pi_{A^0})\eta^\varepsilon_k\|_{1,2,p,\varepsilon} + \varepsilon \|\pi_A(\alpha^\varepsilon_k)\|_{1,2,p,1} \leq c \varepsilon^{4-k},
\]

\[
\|F^\varepsilon_2(\Xi^\varepsilon_k)\|_{0,p,\varepsilon} + \varepsilon \|\pi_{A^0}(F^\varepsilon(\Xi^\varepsilon_k))\|_{L^p} \leq 2^{-k} c_1 \varepsilon^3
\]

Therefore $F^\varepsilon(\Xi^\varepsilon_k)$ converges to 0 and we can choose $\Xi^\varepsilon := K^\varepsilon_2(\Xi^0) + \sum_{k=2}^{\infty} \eta^\varepsilon_k$. Then

$\Xi^\varepsilon - K^\varepsilon_2(\Xi^0) \in \text{im} \left( D^\varepsilon(K^\varepsilon_2(\Xi^0)) \right)^*$, \( F^\varepsilon(\Xi^\varepsilon) = 0 \)

and

\[
\|(1 - \pi_{A^0})(\Xi^\varepsilon - K^\varepsilon_2(\Xi^0))\|_{1,2,p,\varepsilon} + \varepsilon \|\pi_{A^0}(\Xi^\varepsilon - K^\varepsilon_2(\Xi^0))\|_{1,2,p,1} \leq c \varepsilon^3.
\]
Since $\mathcal{F}^\varepsilon(\Xi^\varepsilon) = 0$, by the definition of $\mathcal{F}^\varepsilon_{\Xi_3}$, $d^\varepsilon_{\Xi_3}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)) = 0$ holds and thus we concluded the proof of the theorem \[10.1\].

**Proof of theorem \[10.2\]** First, we improve the estimate \[10.3\] and we show that

\[
(10.9) \quad \|\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)\|_{1,2,p,\varepsilon} + \varepsilon\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,2,p,1} \leq c\varepsilon^3.
\]

In order to fulfill this task, we consider the identity

\[
(10.10) \quad D^\varepsilon_i(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)) = -C^\varepsilon_0(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)) - \mathcal{F}^\varepsilon_i(\mathcal{K}^\varepsilon_2(\Xi^0))
\]

then, by theorem \[9.2\]

\[
\|\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)\|_{1,2,p,\varepsilon} + \varepsilon\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,2,p,1} 
\leq c\varepsilon^2\|D^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{0,p,\varepsilon} 
+ c\varepsilon\|\pi_{\mathcal{A}_0}D^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p}
\]

next, we can apply \[10.10\], i.e.

\[
\leq c\varepsilon^2\|C^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{0,p,\varepsilon} 
+ c\varepsilon\|\pi_{\mathcal{A}_0}C^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p} 
+ c\varepsilon^2\|\mathcal{F}^\varepsilon_i(\mathcal{K}^\varepsilon_2(\Xi^0))\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{\mathcal{A}_0}\mathcal{F}^\varepsilon_i(\mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p}
\]

and the quadratic estimates \[8.1\] and \[8.2\]:

\[
\leq c\varepsilon^3 + c\|\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)\|_{\infty,\varepsilon}\|\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0)\|_{1,1,p,\varepsilon} 
+ \frac{c}{\varepsilon}\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{\infty,\varepsilon}\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,1,p,\varepsilon} 
+ c\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{\infty,\varepsilon}\|
\mathring{\nabla}_\varepsilon\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p} 
+ \frac{c}{\varepsilon}\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^\infty}\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,1,p,\varepsilon} 
+ \frac{c}{\varepsilon}\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^\infty}\|
\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p} 
+ c\varepsilon\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^\infty}\|
\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{L^p} 
\leq c\varepsilon^3 + c\delta\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,1,p,\varepsilon} 
+ c\delta\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \mathcal{K}^\varepsilon_2(\Xi^0))\|_{1,1,p,1}
\]

where the last inequality follows from the assumptions. Therefore for $\delta$ small enough \[10.1\] holds. Furthermore, by \[10.3\] and \[10.4\], $\|\Xi^\varepsilon - \Xi^{\varepsilon}\|_{1,2,p,\varepsilon} \leq c\varepsilon^2$. Thus, always by theorem \[9.2\]

\[
\|\Xi^\varepsilon - \Xi^{\varepsilon}\|_{1,2,p,\varepsilon} + \varepsilon\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{1,2,p,1} 
\leq c\varepsilon^2\|D^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{0,p,\varepsilon} + c\varepsilon\|\pi_{\mathcal{A}_0}D^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0))(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{L^p}
\]

and since $\mathcal{F}^\varepsilon_i(\Xi^{\varepsilon})\mathcal{F}^\varepsilon_i(\Xi^\varepsilon) = 0$, $D^\varepsilon_i(\Xi^{\varepsilon} - \Xi^{\varepsilon}) = -C^\varepsilon_0(\Xi^{\varepsilon} - \Xi^{\varepsilon})$ and thus we obtain

\[
\leq c\left(\varepsilon^2\|C^\varepsilon_0(\Xi^{\varepsilon})(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{0,p,\varepsilon} + \varepsilon\|\pi_{\mathcal{A}_0}C^\varepsilon_0(\Xi^{\varepsilon})(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{L^p}\right) 
+ c\varepsilon^2\|((\mathcal{D}^\varepsilon(\Xi^{\varepsilon}) - \mathcal{D}^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0)))(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{0,p,\varepsilon} 
+ c\varepsilon\|\pi_{\mathcal{A}_0}(\mathcal{D}^\varepsilon(\Xi^{\varepsilon}) - \mathcal{D}^\varepsilon(\mathcal{K}^\varepsilon_2(\Xi^0)))(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{L^p}
\leq c\varepsilon^3 + c\delta\left(\|(1 - \pi_{\mathcal{A}_0})(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{1,1,p,\varepsilon} + \varepsilon\|\pi_{\mathcal{A}_0}(\Xi^\varepsilon - \Xi^{\varepsilon})\|_{1,1,p,1}\right)
\]
where the last inequality follows from the quadratic estimates of the lemmas 8.1 and 8.2. Hence for \( p > 3 \) and \( \varepsilon \) small enough \( \bar{\Xi} = \Xi \).

\[ \square \]

11. A priori estimates for the Yang-Mills flow

In this section we will prove some a priori estimates, that will be stated in the theorem 11.1, on the curvature for a perturbed Yang-Mills flow. These will then be used to prove the surjectivity of the map \( R_{\varepsilon, b} \) in the section 15.

In order to simplify the exposition we denote by \( \| \cdot \| \) the \( L^2 \)-norm over \( \Sigma \) and we introduce the following notation. We choose two perturbed Yang-Mills connections \( \Xi_{+}^\varepsilon \in \text{Crit}^b_{YM, H} \) where \( b > 0 \). For any Yang-Mills flow \( \Xi_{\varepsilon} := A + \Psi dt + \Phi ds \in \mathcal{M}^\varepsilon(\Xi_{-}, \Xi_{+}) \) we define

\[
(11.1) \quad B_t := \partial_t A - dA\Psi, \quad B_s := \partial_s A - dA\Phi, \quad C := \partial_t \Psi - \partial_s \Phi - [\Psi, \Phi];
\]

thus, the Yang-Mills flow equations (4.1) can be written as

\[
(11.2) \quad B_s + \frac{1}{\varepsilon^2} d_A F_A - \nabla_t B_t - *X_t(A) = 0, \quad C - \frac{1}{\varepsilon^2} d_A B_t = 0;
\]

with this notation, the Bianchi identities are

\[
(11.3) \quad \nabla_t F_A = d_A B_t, \quad \nabla_s F_A = d_A B_s, \quad \nabla_t B_s - \nabla_s B_t = d_A C
\]

and the commutation formulas

\[
(11.4) \quad [\nabla_t, d_A] = B_t, \quad [\nabla_s, d_A] = B_s, \quad [\nabla_s, \nabla_t] = C.
\]

Furthermore, we have the identity

\[
(11.5) \quad \| B_s + C dt \|^2_{0, 2, \varepsilon} = \mathcal{Y}M^{\varepsilon, H}(\Xi_{-}) - \mathcal{Y}M^{\varepsilon, H}(\Xi_{+})
\]

which can be showed by the direct computation:

\[
\| B_s + C dt \|^2_{0, 2, \varepsilon} = \int_\mathbb{R} \| B_s + C dt \|^2_{0, 2, \varepsilon} ds
\]

and by the Yang-Mills flow equations (4.1)

\[
= \int_\mathbb{R} \int_{\Sigma \times S^1} \langle B_s, -\frac{1}{\varepsilon^2} d_A F_A + \nabla_t B_t + *X_t(A) \rangle \, dvol_{\Sigma} \, dt \, ds
\]

\[
+ \int_\mathbb{R} \int_{\Sigma \times S^1} \langle C, d_A B_t \rangle \, dvol_{\Sigma} \, dt \, ds
\]

\[
= \int_\mathbb{R} \int_{\Sigma \times S^1} \left( -\frac{1}{\varepsilon^2} \langle d_A B_s, F_A \rangle - \langle \nabla_t B_s, B_t \rangle \right) \, dvol_{\Sigma} \, dt \, ds
\]

\[
- \int_\mathbb{R} \partial_s H(A) \, ds + \int_\mathbb{R} \int_{\Sigma \times S^1} \langle d_A C, B_t \rangle \, dvol_{\Sigma} \, dt \, ds
\]

and by the Bianchi identity (11.3)

\[
= -\int_\mathbb{R} \int_{\Sigma \times S^1} \frac{1}{\varepsilon^2} \langle \nabla_s F_A + F_A \rangle \, dvol_{\Sigma} \, dt \, ds
\]

\[
- \int_\mathbb{R} \int_{\Sigma \times S^1} \langle \nabla_s B_s, B_t \rangle \, dvol_{\Sigma} \, dt \, ds
\]

\[
= \mathcal{Y}M^{\varepsilon, H}(A_- + \Psi_- dt) - \mathcal{Y}M^{\varepsilon, H}(A_+ + \Psi_+ dt).
\]
Furthermore, we denote by \(a^1_i, i = 1, 2, 3\), the three operators \(d_A, d_A^*\) and \(\varepsilon\nabla\), by \(a^2_i, i = 1, \ldots, 9\), the nine operators defined combining two operators between \(d_A, d_A^*\) and \(\varepsilon\nabla\), by \(a^3_i, i = 1, \ldots, 27\), the 27 operators defined combining three operators between \(d_A, d_A^*, \varepsilon\nabla\) and finally we denote by \(a^4_i, i = 1, \ldots, 81\), the 81 operators defined combining four operators between \(d_A, d_A^*, \varepsilon\nabla\). In addition we denote by \(\mathcal{N}_j(t, s)\) the norms

\[
\mathcal{N}_j(t, s) := \sum_{i=1,...,3^j} \left( ||a^j_i B_s||^2 + \varepsilon^4 ||a^j_i C||^2 \right).
\]

**Theorem 11.1.** We choose an open interval \(\Omega \subset \mathbb{R}\), a compact set \(Q \subset \Omega\), \(p > 4\) and two constants \(b, c_0 > 0\). There are two positive constants \(\varepsilon_0, c\) such that the following holds. If a perturbed Yang-Mills flow \(\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^p(\Xi_-, \Xi_+)\), with \(\Xi_-, \Xi_+ \in \text{Crit}_{\mathcal{YM}_{b, s}}\), and \(0 < \varepsilon < \varepsilon_0\), satisfies

\[
(11.6) \quad \sup_{(t, s) \in S^1 \times Q} \left( ||\partial_t A - d_A \Psi||_{L^4(\Sigma)} + ||\partial_s A - d_A \Phi||_{L^\infty(\Sigma)} \right) \leq c_0,
\]

then

\[
(11.7) \quad \int_{S^1 \times Q} \left( ||F_A||^2 + \varepsilon^2 ||\nabla F_A||^2 + ||d_A^* F_A||^2 \right) dt \, ds \leq c \varepsilon^4,
\]

\[
(11.8) \quad \sup_{S^1 \times Q} \left( \varepsilon^2 ||B_t||^2 + ||d_A^* d_A B_t||^2 + \varepsilon^4 ||d_A^* F_A||^2 \right)
\]

\[
(11.9) \quad \sup_{S^1 \times Q} \left( \varepsilon^2 ||B_s||^2 + ||d_A^* d_A B_s||^2 + \varepsilon^4 ||d_A^* F_A||^2 \right)
\]

\[
(11.10) \quad \sup_{S^1 \times Q} \left( ||F_A|| + \varepsilon ||\nabla F_A|| + \varepsilon^2 ||\nabla \nabla F_A|| \right) \leq c \varepsilon^2,
\]

\[
(11.11) \quad \sup_{(t, s) \in S^1 \times Q} ||F_A||_{L^\infty(\Sigma)} \leq c \varepsilon^2
\]

where \(c_{\mathcal{X}_i(A)}\) is a constant which estimates the norm \(||\mathcal{X}_i(A)||_{L^\infty(\Sigma)}^2\). The constant \(c\) depends on \(\Omega\) and on \(Q\), but only on their length and on the distance between their boundaries. Furthermore,

\[
(11.12) \quad \sup_{(t, s) \in S^1 \times Q} \left( \varepsilon^2 ||B_s||^2 + \varepsilon^4 ||C||^2 + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4 \right)
\]

\[
(11.12) \quad \leq \varepsilon^2 c \int_{S^1 \times \Omega} \left( ||B_s||^2 + \varepsilon^2 ||C||^2 \right) dt \, ds.
\]

**Remark 11.2.** In the theorem we assume that the \(L^4(\Sigma)\)-norm of the curvature term \(\partial_t A - d_A \Psi\) and the \(L^\infty(\Sigma)\)-norm of \(\partial_s A - d_A \Phi\) are uniformly bounded; this condition is, for a Yang-Mills flow, always satisfied if we choose \(\varepsilon\) small enough as we will see in section 12.

Before starting to prove the last theorem we consider the following three lemmas. The first one show a regularity result for the curvature terms \(B_s, C\). The last two
are the lemmas B.1. and B.4. that Salamon and Weber proved in [13]. For an interval $Q \subset \mathbb{R}$ and a 0- or 1-form $\alpha$ we define the norm $\| \cdot \|_{1,2,2,\varepsilon,Q}$ by

$$
\| \alpha \|_{1,2,2,\varepsilon,Q}^2 := \int_{S^1 \times Q} (\| \alpha \|^2 + \varepsilon^4 \| \nabla_s \alpha \|^2 + \| d_A \alpha \|^2 + \| d_A^* \alpha \|^2 ) \, dt \, ds
+ \int_{S^1 \times Q} (\varepsilon^2 \| \nabla_t \alpha \|^2 + \| d_A^* d_A \alpha \|^2 + \| d_A d_A^* \alpha \|^2 + \varepsilon^4 \| \nabla_t \nabla_t \alpha \|^2 ) \, dt \, ds.
$$

**Lemma 11.3.** We choose a positive constant $b$, then there are two positive constants $\varepsilon_0$, $c$ such that the following holds. For any perturbed Yang-Mills flow $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^c(\Xi_-, \Xi_+)$, with $\Xi_-, \Xi_+ \in \text{Crit}_b(\mathcal{M}, \mu)$ and $0 < \varepsilon < \varepsilon_0$, satisfies

$$
\| B_s \|_{1,2,2,\varepsilon,r}^2 + \varepsilon^2 \| C \|_{1,2,2,\varepsilon,r}^2 \leq c.
$$

**Proof.** By the Yang-Mills flow equation (11.2), the Bianchi identity (11.3) and the commutation formula (11.2) we have

$$
0 = \nabla_s B_s + \frac{1}{\varepsilon^2} \nabla_s d_A^* F_A - \nabla_s \nabla_t B_t - \nabla s \ast X_t(A)
= \nabla_s B_s + \frac{1}{\varepsilon^2} d_A^* d_A B_s - \nabla_t \nabla_s B_s - \nabla_t d_A C
- d \ast X_t(A) B_s - \frac{1}{\varepsilon^2} \ast [B_s, * F_A] - [C, B_t]
= \nabla_s B_s + \frac{1}{\varepsilon^2} d_A^* d_A B_s - \nabla_t \nabla_s B_s - d_A d_A^* B_s
- d \ast X_t(A) B_s - \frac{1}{\varepsilon^2} \ast [B_s, * F_A] - 2[C, B_t],
$$

$$
0 = \nabla_s C - \frac{1}{\varepsilon^2} \nabla_s d_A^* B_t
= \nabla_s C + \frac{1}{\varepsilon^2} d_A^* d_A C + \frac{1}{\varepsilon^2} d_A^* \nabla_t B_s + \frac{1}{\varepsilon^2} \ast [B_s, \ast B_t]
= \nabla_s C + \frac{1}{\varepsilon^2} d_A^* d_A C + \nabla_t \nabla_t C + \frac{2}{\varepsilon^2} \ast [B_s, \ast B_t].
$$

Furthermore, choosing $s_0 \in \mathbb{R}$ and a smooth cut-off function with support in $[s_0 - 1, s_0 + 2]$ and with value 1 on $[s_0, s_0 + 1]$, one can prove that, for $\Omega_t(s_0) := \Sigma \times S^1 \times [s_0 - 1, s_0 + 2]$,

$$
\| B_s \|_{1,2,2,\varepsilon,[s_0,s_0+1]}^2 + \varepsilon^2 \| C \|_{1,2,2,\varepsilon,[s_0,s_0+1]}^2
\leq \varepsilon^4 \left\| \nabla_s B_s + \frac{1}{\varepsilon^2} d_A^* d_A B_s - \nabla_t \nabla_s B_s + \frac{1}{\varepsilon^2} d_A d_A^* B_s \right\|^2_{L^2(\Omega_t(s_0))}
+ \varepsilon^6 \left\| \nabla_s C + \frac{1}{\varepsilon^2} d_A^* d_A C + \nabla_t \nabla_t C \right\|^2_{L^2(\Omega_t(s_0))}
+ \| B_s \|^2_{L^2(\Omega_t(s_0))} + \varepsilon^2 \| C \|^2_{L^2(\Omega_t(s_0))}
\leq -\varepsilon^2 d \ast X_t(A) B_s - [B_s, \ast F_A] - 2\varepsilon^2 [C, B_t] \|_{L^2(\Omega_t(s_0))}^2
+ \varepsilon^2 \| [B_s, \ast B_t] \|^2_{L^2(\Omega_t(s_0))} + \| B_s \|^2_{L^2(\Omega_t(s_0))} + \varepsilon^2 \| C \|^2_{L^2(\Omega_t(s_0))}
\leq \| B_s \|^2_{L^2(\Omega_t(s_0))} + \varepsilon^2 \| C \|^2_{L^2(\Omega_t(s_0))}
+ c\varepsilon \| B_s \|^2_{1,2,2,\varepsilon,[s_0-1,s_0+2]} + \varepsilon^3 \| C \|^2_{1,2,2,\varepsilon,[s_0-1,s_0+2]}
$$

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In order to prove the last estimate we use the energy bound
\[ \int_{\Sigma \times S^1} \left( \frac{1}{\varepsilon^2} \| F_A \|^2 + \| B_i \|^2 \right) dt \leq b \]
combined with the Sobolev inequality for 1-forms on \( \Sigma \times S^1 \): For example for the term \( \| \ast [B_s, * B_t] \|_{L^2(\Omega_1, (s_0))} \) we proceed in the following way.

\[ \varepsilon^2 \| [B_s, * B_t] \|^2_{L^2(\Omega_1)} \leq c \varepsilon^2 \int_{[s_0 - 1, s_0 + 2]} \| B_t \|^2_{L^2(\Sigma \times S^1)} \| B_s \|^2_{L^\infty(\Sigma \times S^1)} ds \]
\[ \leq c \varepsilon^2 \int_{[s_0 - 1, s_0 + 2]} \| B_s \|^2_{L^\infty(\Sigma \times S^1)} ds \]
\[ \leq c \varepsilon^2 \int_{[s_0 - 1, s_0 + 2]} \frac{1}{\varepsilon} \left( \| B_s \|^2_{L^2(\Sigma \times S^1)} + \| d_J A B_s \|^2_{L^2(\Sigma \times S^1)} \right) ds \]
\[ + \varepsilon^2 \int_{[s_0 - 1, s_0 + 2]} \frac{1}{\varepsilon} \left( \| d_J d_J A B_s \|^2_{L^2(\Sigma \times S^1)} + \varepsilon^4 \| \nabla_t \nabla_i B_s \|^2_{L^2(\Sigma \times S^1)} \right) ds \]
\[ \leq c \| B_s \|_{1, 2; 2; 2, \varepsilon, [s_0 - 1, s_0 + 2]} \]

Finally since

\[ \sum_{i=0}^{\infty} \varepsilon \frac{\| [B_s, * B_t] \|^2_{L^2(\Omega_1)}}{s_0 + i, s_0 + i + 1} + \varepsilon^2 \| C \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i, s_0 + i + 1]} \]
\[ \leq \sum_{i=0}^{\infty} \varepsilon \frac{\| B_s \|^2_{L^2(\Sigma \times S^1 \times [s_0 + i - 1, s_0 + i + 2])}}{s_0 + i, s_0 + i + 1} + \varepsilon^2 \| C \|^2_{L^2(\Sigma \times S^1 \times [s_0 + i - 1, s_0 + i + 2])} \]
\[ + c \varepsilon \sum_{i=0}^{\infty} \varepsilon \frac{\| [B_s, * B_t] \|^2_{L^2(\Sigma \times S^1 \times [s_0 + i - 1, s_0 + i + 2])}}{s_0 + i, s_0 + i + 1} + \varepsilon^2 \| C \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i - 1, s_0 + i + 2]} \]
\[ \leq 2 \| B_s \|^2_{L^2(\Omega_1)} + 2c^{2} \| C \|^2_{L^2(\Omega_1)} \]
\[ + c \varepsilon \sum_{i=0}^{\infty} \varepsilon \frac{\| B_s \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i, s_0 + i + 1]}}{s_0 + i, s_0 + i + 1} + \varepsilon^2 \| C \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i, s_0 + i + 1]} \]

we can conclude that

\[ \| B_s \|^2_{1, 2, 2; 2, \varepsilon, \mathbb{R}} + \varepsilon^2 \| C \|^2_{1, 2, 2; 2, \varepsilon, \mathbb{R}} \]
\[ \leq \sum_{s_0 \in \mathbb{Z}} \sum_{i=0}^{\infty} \varepsilon \frac{\| B_s \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i, s_0 + i + 1]}}{s_0 + i, s_0 + i + 1} + \varepsilon^2 \| C \|^2_{1, 2, 2; 2, \varepsilon, [s_0 + i, s_0 + i + 1]} \]
\[ \leq 5 \| B_s \|^2_{L^2} + 5 \varepsilon^2 \| C \|^2_{L^2} \leq 5b \]

for \( \varepsilon \) small enough.

\[ \square \]

**Remark 11.4.** We can prove that the curvature of a connection, which represents a Yang-Mills flow, is smooth with an analogous argument as for the previous lemma. Our connection satisfies the perturbed Yang-Mills equation and thus, by the Bianchi
derivatives (step 4). In the last two steps we will prove the estimates (11.11) and 

\[ \sum \] for some curvature terms (step 2), their derivatives (step 3) and their second 

\[ \text{(11.17)} \]

Thus in the same way as for the last lemma we can estimate all the first derivatives of \( F_A \) and \( B_t \) in a set \( \Omega_{s_0} := \Sigma \times S^1 \times [s_0, s_0 + 1], s_0 \in \mathbb{R} \). Then, by a bootstrapping argument one can prove that the curvature terms are in \( W^{k,2} \) for any \( k \).

We denote by \( \Delta := \partial_1^2 + \partial_2^2 + \cdots + \partial_n^2 \) the standard Laplacian on \( \mathbb{R}^n, n > 0 \), and we define \( P_r := (-r^2, 0) \times B_r(0) \).

**Lemma 11.5.** For every \( n \in \mathbb{N} \) there is a constant \( c_n > 0 \) such that the following holds for every \( r \in (0, 1] \). If \( a \geq 0 \) and \( w : \mathbb{R} \times \mathbb{R}^n \supset P_r \to \mathbb{R} \) is \( C^1 \) in the \( s \)-variable and \( C^2 \) in the \( x \)-variable such that

\[ (\Delta - \partial_s)w \geq -aw, \quad w \geq 0, \]

then

\[ w(0) \leq \frac{c_n e^{ar^2}}{r^{n+2}} \int_{P_r} w. \]

**Lemma 11.6.** Let \( R, r > 0 \) and \( u : \mathbb{R} \times \mathbb{R}^n \supset P_{R+r} \to \mathbb{R} \) be \( C^1 \) in the \( s \)-variable and \( C^2 \) in the \( x \)-variable and \( f, g : P_{R+r} \to \mathbb{R} \) be continuous functions such that

\[ (\Delta - \partial_s)u \geq g - f, \quad u \geq 0, \quad f \geq 0, \quad g \geq 0. \]

Then

\[ \int_{P_R} g \leq \int_{P_{R+r}} f + \left( \frac{4}{r^2} + \frac{1}{Rr} \right) \int_{P_{R+r} \setminus P_R} u. \]

**Proof of the theorem**. The proof will be divided in seven steps. In the first one we will prove that the \( L^2 \)-norm over \( \Sigma \) of \( F_A \) can be bounded by any positive constant provided we choose \( \varepsilon \) small enough. This allows us to apply the lemmas \( \text{B.1} \) and \( \text{B.2} \) for \( p = 2 \). The next three steps provide bounds of the \( L^2 \)-norm over \( \Sigma \) for some curvature terms (step 2), their derivatives (step 3) and their second derivatives (step 4). In the last two steps we will prove the estimates \( (11.11) \) and \( (11.10) \).
Step 1. We choose a positive constant \(\delta_0 < 1\). There is a constant \(\varepsilon_0 > 0\) such that the following holds. If \(0 < \varepsilon < \varepsilon_0\), then \(\sup_{(t,s) \in S^1 \times \mathbb{R}} \|F_A\|_{L^2(\Sigma)} \leq \delta_0\).

Proof of step 1. The idea is to use lemma 11.3 and therefore we need an estimate from below of \((\partial_t^2 - \partial_s) \|F_A\|^2\). First, using the Bianchi identity (11.3) in the second and in the fourth equality, the commutation formula (11.4) in the third and the Yang-Mills flow equation (11.2) in the fourth, we obtain that

\[
\frac{1}{2}(\partial_t^2 - \partial_s) \|F_A\|^2 = \|\nabla_t F_A\|^2 + \langle F_A, \nabla_t \nabla_s F_A \rangle - \langle F_A, \nabla_s F_A \rangle
\]

\[
= \|\nabla_t F_A\|^2 + \langle F_A, \nabla_t d_A B_t \rangle - \langle F_A, \nabla_s F_A \rangle
\]

\[
= \|\nabla_t F_A\|^2 + \langle F_A, d_A \nabla_t B_t \rangle + \langle F_A, [B_t \wedge B_t] \rangle - \langle F_A, \nabla_s F_A \rangle
\]

\[
= \|\nabla_t F_A\|^2 + \frac{1}{\varepsilon^2} \langle F_A, d_A d_A^* F_A \rangle + \langle F_A, d_A B_s \rangle
\]

\[
- \langle F_A, d_A X_t(A) \rangle + \langle F_A, [B_t \wedge B_t] \rangle - \langle d_A^* F_A, X_t(A) \rangle;
\]

therefore, applying the Cauchy-Schwarz inequality

\[
|\langle d_A^* F_A, X_t(A) \rangle| \leq c \|d_A^* F_A\| \leq \frac{1}{2\varepsilon^2} \|d_A^* F_A\|^2 + 2c^2 \varepsilon^2,
\]

for any positive \(\gamma_1\) we have

\[
\frac{1}{2}(\partial_t^2 - \partial_s) \|F_A\|^2 \geq - \|B_t\|^2_{L^4(\Sigma)} \|F_A\| - \varepsilon \|F_A\|^2 - \varepsilon^2 \geq - \frac{c}{\gamma_1} \|F_A\|^2 - \gamma_1 - \varepsilon^2
\]

\[
\geq - \frac{c}{\gamma_1} \left( \|F_A\|^2 + \frac{\gamma_1^2}{c} - \varepsilon^2 \right)
\]

where \(c \geq 1\) depends on \(\|B_t\|_{L^4(\Sigma)}\) and on \(\|X_t\|_{L^2(\Sigma)}\). Thus, by lemma 11.5 for every \(r \in (0,1]\), \(P_r := (-r^2,0) \times B_r(0)\),

\[
\|F_A\|^2 \leq \frac{c_1 e^{\frac{\gamma_1 r^2}{r^3}}}{r^3} \int_{P_r} \left( \|F_A\|^2 + \frac{\gamma_1^2}{c} + \varepsilon^2 \right) dt ds
\]

\[
\leq \frac{4c_1 e^{\frac{\gamma_1 r^2}{r^3}}}{r^3} + 2c_1 e^{\frac{r^2}{r^3}} \left( \frac{\gamma_1^2}{c} + \varepsilon^2 \right)
\]

and where we use that \(\int_0^1 \|F_A\|^2 dt \leq 2b \varepsilon^2\); next, we choose \(\gamma_1 := \frac{\delta_0}{2(c_1 e)^2}\) and \(r = (\frac{\gamma_1}{2})^{\frac{1}{2}}\); then \(\|F_A\|^2 \leq 4 \sqrt{2c_1} \frac{e^{\gamma_1 r^2}}{r^3} b \varepsilon^2 + \frac{1}{2} \delta_0^2 + (c_1 e)^2 \delta_0 \varepsilon^2\) and finally with \(\varepsilon^2 < \frac{1}{4} \frac{\delta_0^2}{4 \sqrt{2c_1} e^{\gamma_1 r^2}} + (c_1 e)^2 \delta_0^2\), it follows that \(\|F_A\|_{L^2(\Sigma)} < \delta_0\) and we end the proof of the first step.

For the rest of the proof we choose \(\delta_0\) satisfying the condition of the theorems B.1 and B.2 for \(p = 2\).
Thus, since by the Cauchy-Schwarz inequality and the Sobolev estimate
Yang-Mills flow equation (11.2) and the Bianchi identity (11.3) we finally obtain
the third the commutation formula (11.4) and the Bianchi identity (11.3). By the
where for the second equality we use the Yang-Mills flow equation (11.2) and for

**Step 2.** There are two constants $\varepsilon_0, c > 0$ such that the following holds. If
$0 < \varepsilon < \varepsilon_0$, then

$$
\sup_{(s,t) \in S^1 \times Q} (\|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|B_s\|^2) 
\leq c \int_{S^1 \times \Omega} (\varepsilon^2 \|B_t\|^2 + \varepsilon^2 c \chi_s + \|F_A\|^2 + \varepsilon^2 \|B_s\|^2 + \varepsilon^2 \|C\|^2) \, dt \, ds,
$$

$$
\int_{S^1 \times Q} (\|F_A\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 + \|d_A^* F_A\|^2) \leq c \varepsilon \int_{S^1 \times \Omega} (\|F_A\|^2 + \varepsilon^2 \|B_t\|^2) \, dt,
$$

$$
\int_{S^1 \times Q} (\varepsilon^2 \|\nabla_t B_t\|^2 + \|d_A B_t\|^2 + \|d_A^* B_t\|^2) \leq c \int_{S^1 \times \Omega} (\|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|C\|^2) \, dt \, ds,
$$

**Proof of step 2.** Analogously to the first step we need to compute $\frac{1}{2} (\partial_t^2 - \partial_s) \|B_t\|^2$, $\frac{1}{2} (\partial_t^2 - \partial_s) \|B_s\|^2$ and $\frac{1}{2} (\partial_s^2 - \partial_t) \|C\|^2$ in order to apply the lemmas [11.5] and [11.6].

On the one hand, we consider the following computation

$$
\frac{1}{2} (\partial_t^2 - \partial_s) \|B_t\|^2 = \|\nabla_t B_t\|^2 - \langle \nabla_s B_t, B_t \rangle + \langle \nabla_t \nabla_t B_t, B_t \rangle
$$

$$
= \|\nabla_t B_t\|^2 - \langle \nabla_s B_t, B_t \rangle + \frac{1}{\varepsilon^2} \langle \nabla_t d_A^* F_A, B_t \rangle
$$

$$
+ \langle \nabla_t B_s, B_t \rangle - \langle \nabla_t \chi_t(A), B_t \rangle
$$

$$
= \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2} \langle d_A^* \nabla_t F_A, B_t \rangle + \frac{1}{\varepsilon^2} \langle [B_t, \ast F_A], B_t \rangle
$$

$$
+ \langle d_A C, B_t \rangle - \langle \ast X_t(A) B_t + \hat{X}_t(A), B_t \rangle
$$

where for the second equality we use the Yang-Mills flow equation (11.2) and for
the third the commutation formula (11.4) and the Bianchi identity (11.3). By the
Yang-Mills flow equation (11.2) and the Bianchi identity (11.3) we finally obtain that

$$
\|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2} \|d_A B_t\|^2 + \frac{1}{\varepsilon^2} \|d_A^* B_t\|^2 + \frac{1}{\varepsilon^2} \langle [B_t, \ast F_A], B_t \rangle
$$

$$
- \langle d \ast X_t(A) B_t + \hat{X}_t(A), B_t \rangle.
$$

Thus, since by the Cauchy-Schwarz inequality and the Sobolev estimate

$$
|\langle [B_t, \ast F_A], B_t \rangle| \leq c (\|B_t\| + \|d_A B_t\| + \|d_A^* B_t\|) \|F_A\|_{L^2(S)} \|F_A\|
$$

$$
\leq c \varepsilon^2 \|B_t\|^2 + \frac{1}{\varepsilon^2} \|F_A\|^2 + \frac{1}{2} \|d_A B_t\|^2 + \frac{1}{2} \|d_A^* B_t\|^2
$$

$$
|\langle d \ast X_t(A) B_t + \hat{X}_t(A), B_t \rangle| \leq c \|B_t\|^2 + \|X_t(A)\|^2,
$$
we have
\begin{equation}
(\partial_t^2 - \partial_s) \|B_t\|^2 \geq \|\nabla_t B_t\|^2 + \frac{1}{\varepsilon^2} \|d_A B_t\|^2 + \frac{1}{\varepsilon^2} \|d_A^* B_t\|^2 - \frac{c}{\varepsilon^4} \|F_A\|^2 - c\|B_t\|^2 - c\|\dot{X}_t(A)\|^2.
\end{equation}
(11.21)

On the other hand, using the Bianchi identity (11.3) in the second equality and the commutation formula (11.4) in the third, it follows that
\begin{align*}
\frac{1}{2}(\partial_t^2 - \partial_s) \|B_t\|^2 &= \|\nabla_t B_t\|^2 + \langle B_s, \nabla_t \nabla_t B_t \rangle - \langle B_s, \nabla_s B_t \rangle \\
&= \|\nabla_t B_s\|^2 + \langle B_s, \nabla_t \nabla_s B_t \rangle + \langle B_s, \nabla_t d_A C \rangle - \langle B_s, \nabla_s B_t \rangle \\
&= \|\nabla_t B_s\|^2 + \langle B_s, \nabla_s \nabla_t B_t \rangle - \langle B_s, [C, B_t] \rangle \\
&\quad + \langle d_A^* B_s, \nabla_t C \rangle + \langle B_s, [B_t, C]\rangle - \langle B_s, \nabla_s B_t \rangle
\end{align*}
the next equality follows from the Yang-Mills flow equation (11.2) and the one from the commutation formula (11.4)
\begin{align*}
&= \|\nabla_t B_s\|^2 + \frac{1}{\varepsilon^2} \langle B_s, \nabla_s d_A^* A \rangle - \langle B_s, \nabla_s * X_t(A) \rangle + \langle B_s, \nabla_s B_s \rangle \\
&\quad - 2\langle B_s, [C, B_t] \rangle + \frac{1}{\varepsilon^2} \langle d_A^* B_s, \nabla_t d_A^* B_t \rangle - \langle B_s, \nabla_s B_s \rangle \\
&= \|\nabla_t B_s\|^2 + \frac{1}{\varepsilon^2} \langle d_A B_s, \nabla_s F_A \rangle - \frac{1}{\varepsilon^2} \langle B_s, * [B_s, * F_A] \rangle \\
&\quad - \langle B_s, \nabla_s * X_t(A) \rangle - 2\langle B_s, [C, B_t] \rangle + \frac{1}{\varepsilon^2} \langle d_A^* B_s, d_A^* \nabla_t B_t \rangle
\end{align*}
finally, applying the Bianchi identity (11.3) and the Yang-Mills flow equation (11.2) one more time, we can conclude that
\begin{align*}
&= \|\nabla_t B_s\|^2 + \frac{1}{\varepsilon^2} \|\nabla_s F_A\| - \frac{1}{\varepsilon^2} \langle B_s, * [B_s, * F_A] \rangle - \langle B_s, \nabla_s * X_t(A) \rangle \\
&\quad - 2\langle B_s, [C, B_t] \rangle + \frac{1}{\varepsilon^2} \langle d_A^* B_s, d_A^* B_s \rangle \\
&\quad + \frac{1}{\varepsilon^2} \langle d_A^* B_s, d_A^* d_A^* A \rangle + \frac{1}{\varepsilon^2} \langle d_A^* B_s, d_A X_t(A) \rangle \\
&= \|\nabla_t B_s\|^2 + \frac{1}{\varepsilon^2} \|\nabla_s F_A\| + \frac{1}{\varepsilon^2} \|d_A^* B_s\|^2 - 2\langle B_s, [C, B_t] \rangle \\
&\quad - \frac{1}{\varepsilon^2} \langle B_s, * [B_s, * F_A] \rangle - \langle B_s, d * X_t(A) B_s \rangle
\end{align*}
where we use that \([F_A, * F_A] = 0\) and \(d_A X_t(A) = 0\). Thus, since \(\|B_s\|_{L^\infty(\Sigma)}\) is bounded by a constant by the assumptions,
\begin{equation}
(\partial_t^2 - \partial_s) \|B_s\|^2 \geq \|\nabla_t B_s\|^2 + \frac{1}{\varepsilon^2} \|d_A B_s\|^2 + \frac{1}{\varepsilon^2} \|d_A^* B_s\|^2 - \frac{c}{\varepsilon^4} \|F_A\|^2 - c\|B_s\|^2 - c\|B_t\|^2 - \|C\|^2.
\end{equation}
(11.22)
If we consider
\begin{align*}
\frac{1}{2}(\partial_t^2 - \partial_s) \varepsilon^4 \|C\|^2 &= \frac{1}{2}(\partial_t^2 - \partial_s) \|d_A^* B_t\|^2 \\
&= \|\nabla_t d_A^* B_t\|^2 - \langle \nabla_s d_A^* B_t, d_A^* B_t \rangle + \langle \nabla_t \nabla_t d_A^* B_t, d_A^* B_t \rangle
\end{align*}
by the Yang-Mills flow equation (11.2) and the commutation formula (11.4) we have
\[ = \| \nabla_t d_A^* B_t \|^2 - \langle \nabla_s d_A^* B_t, d_A^* B_t \rangle + \frac{1}{\varepsilon^2} \langle \nabla_t d_A^* d_A^* F_A, d_A^* B_t \rangle + \langle \nabla_t d_A^* B_s, d_A^* B_t \rangle - \langle \nabla_t d_A^* \ast X_t(A), d_A^* B_t \rangle - \langle \ast \nabla_t [B_t \wedge \ast B_t], d_A^* B_t \rangle \]
and using the commutation formula (11.2) one more time
\[ = \| \nabla_t d_A^* B_t \|^2 - \langle \nabla_s d_A^* B_t, d_A^* B_t \rangle + \langle d_A^* \nabla_t B_s, d_A^* B_t \rangle - \langle \ast [B_t \wedge \ast B_s], d_A^* B_t \rangle - \langle \ast \nabla_t d_A X_t(A), d_A^* B_t \rangle \]
where the last step follows from the Bianchi identity (11.3) and the commutation formula (11.4). Using the Cauchy-Schwarz inequality and that \( \| B_s \|_{L^\infty(\Sigma)} \) is uniformly bounded, one can easily see that
\[ (\partial_t^2 - \partial_s) \varepsilon^4 \| C \|^2 \geq \varepsilon^2 \| d_A C \|^2 + c_0 \varepsilon^2 \| B_t \|^2 + c_0 \varepsilon^2 \| \dot{X}_t(A) \|^2 \]
and therefore, by (11.21), (11.22), (11.23) and \( \| C \| \leq c \| d_A C \| \) by the lemma B.1 we have for two positive constants \( c, c_0 \)
\[ (\partial_t^2 - \partial_s) \left( \| B_t \|^2 + \| B_s \|^2 + c_0 \varepsilon^2 \| C \|^2 \right) \geq \| \nabla_t B_t \|^2 + \frac{1}{\varepsilon^2} \| d_A B_t \|^2 + \frac{1}{\varepsilon^2} \| d_A B_t \|^2 + \| \nabla_t B_s \|^2 + \frac{1}{\varepsilon^2} \| d_A B_s \|^2 \]
\[ + \frac{1}{\varepsilon^2} \| d_A B_s \|^2 - \frac{c}{\varepsilon} \| d_A F_A \|^2 - c \| B_t \|^2 - c \| B_s \|^2 - c \| \dot{X}_t(A) \|^2. \]
Since \( \| F_A \| \leq \delta, \| F_A \|_{L^\infty(\Sigma)} \leq c \| d_A F_A \| \) for a constant \( c \) by the theorems B.1 and B.2 there is an \( A_1 \in \mathcal{A}^0(P) \) such that \( \| A - A_1 \| \leq \| F_A \| \) and thus we can write
\[ d_A \ast X_t(A) = d_A (\ast X_t(A) - \ast X_t(A_1)) \]
\[ + d_A_1 \ast X_t(A_1) + [(A - A_1) \wedge \ast X_t(A_1)] \]
where \( d_A_1 \ast X_t(A_1) = 0 \). Therefore, by (11.18)
\[ \frac{1}{2} (\partial_t^2 - \partial_s) \| F_A \|^2 \geq \frac{1}{4 \varepsilon^2} \| d_A^* F_A \|^2 + \frac{1}{4} \| \nabla_t F_A \|^2 - c \varepsilon^2 \| B_t \|^2 \]
and with (11.24) it follows that for a constant \( c_0 \) big enough
\[ \frac{1}{2} (\partial_t^2 - \partial_s) \left( c_0 \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| B_s \|^2 + \varepsilon^2 \| \dot{X}_t(A) \|^2 + c_0 \varepsilon^4 \| C \|^2 \right) \geq - c \left( c_0 \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| B_s \|^2 + \varepsilon^2 \| \dot{X}_t(A) \|^2 + c_0 \varepsilon^4 \| C \|^2 \right). \]
Finally by lemma (11.5) for a fix \( r \) we can conclude that
\[ \sup_{(t,s) \in S^1 \times Q} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| B_s \|^2 + c_0 \varepsilon^4 \| C \|^2 \right) \]
\[ \leq c \int_{S^1 \times \Omega} \left( \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| \dot{X}_t \|^2 + \| F_A \|^2 + \varepsilon^2 \| B_s \|^2 + c_0 \varepsilon^4 \| C \|^2 \right) \, dt \, ds. \]
Next, we can apply lemma [11.6] to the inequality [11.26] and we obtain
\[
\int_{S^1 \times Q} \left( \| F_A \|^2 + \varepsilon^2 \| \nabla_t F_A \|^2 + \| d_A^* F_A \|^2 \right) \, dt \, ds
\]
(11.27)
\[
\leq \varepsilon \varepsilon^2 \int_{S^1 \times \Omega} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 \right) \, dt \, ds \leq \varepsilon \varepsilon^4,
\]
where the estimate of $\| F_A \|_{L^2(S^1 \times Q)}^2$ follows from $\| F_A \| \leq c \| d_A^* F_A \|$. Using the lemma [11.6] for the inequalities (11.21) and (11.24) we can conclude the second last two estimates of the third step:
\[
\int_{S^1 \times Q} \left( \varepsilon^2 \| \nabla_t B_t \|^2 + \| d_A B_t \|^2 + \| d_A^* B_t \|^2 \right) \, dt \, ds
\]
(11.28)
\[
\leq \varepsilon \varepsilon \int_{S^1 \times \Omega} \left( \frac{1}{\varepsilon^2} \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 c_{\hat{X}_t(A)} \right) \, dt \, ds,
\]
\[
\int_{S^1 \times Q} \left( \varepsilon^2 \| \nabla_t B_t \|^2 + \| d_A B_t \|^2 + \| d_A^* B_t \|^2 \right) \, dt \, ds
\]
(11.29)
\[
\leq \varepsilon \varepsilon \int_{S^1 \times \Omega} \left( \frac{1}{\varepsilon^2} \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| B_t \|^2 + c_0 \varepsilon^4 \| C \|^2 \right) \, dt \, ds
\]
and the second step follows combining the last two estimates with (11.27). \qed

**Step 3.** There are two constants $\varepsilon_0, c > 0$ such that the following holds. If $0 < \varepsilon < \varepsilon_0$, then
\[
\int_{S^1 \times Q} \left( \varepsilon^2 \| \nabla_t d_A B_t \|^2 + \| d_A^* d_A B_t \|^2 + \varepsilon^2 \| \nabla_t d_A^* B_t \|^2 + \| d_A d_A^* B_t \|^2 \right)
\]
(11.30)
\[
\leq \varepsilon \varepsilon^2 \int_{S^1 \times \Omega} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 c_{\hat{X}_t(A)} \right) \, dt \, ds,
\]
\[
\int_{S^1 \times Q} \left( \varepsilon^2 \| \nabla_t d_A B_t \|^2 + \| d_A^* d_A B_t \|^2 + \varepsilon^2 \| \nabla_t d_A^* B_t \|^2 + \| d_A d_A^* B_t \|^2 \right)
\]
(11.31)
\[
\leq \varepsilon \varepsilon \int_{S^1 \times \Omega} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^4 c_{\hat{X}_t(A)} + \varepsilon^6 \| C \|^2 \right) \, dt \, ds.
\]

**Proof of step 3.** Like in the previous steps we will prove this one using the lemmas [11.5] and [11.6] and therefore we need to compute $\frac{1}{2} \left( \partial^2_t - \partial_s \right) \| d_A B_t \|^2$, $\frac{1}{2} \left( \partial^2_t - \partial_s \right) \| d_A^* B_t \|^2$, $\frac{1}{2} \left( \partial^2_t - \partial_s \right) \| d_A^* B_t \|^2$ and $\frac{1}{2} \left( \partial^2_t - \partial_s \right) \| d_A F_A \|^2$. First, analogously to the previous steps, we obtain that
\[
\frac{1}{2} \left( \partial^2_t - \partial_s \right) \left( \| d_A B_t \|^2 + \| d_A^* B_t \|^2 \right) \geq \frac{1}{2} \| \nabla_t d_A B_t \|^2 + \frac{1}{2 \varepsilon^2} \| d_A d_A^* B_t \|^2
\]
(11.32)
\[
+ \frac{1}{2 \varepsilon^2} \| d_A d_A^* B_t \|^2 - \frac{1}{\varepsilon^2} \| d_A^* F_A \|^2
\]
\[
- \varepsilon \varepsilon^2 \| B_t \|^2 - \varepsilon \varepsilon^2 \| \hat{X}_t(A) \|^2 - \varepsilon \varepsilon^2 \| \nabla_t B_t \|^2.
\]
and combined with (11.21) yield to
\[
\left( \partial^2_t - \partial_s \right) \left( \| d_A B_t \|^2 + \| d_A^* B_t \|^2 + c_1 \| F_A \|^2 + c_0 \varepsilon^2 \| B_t \|^2 \right)
\]
(11.33)
\[
\geq \| \nabla_t d_A B_t \|^2 + \frac{1}{2 \varepsilon^2} \| d_A d_A^* B_t \|^2 + \| \nabla_t d_A^* B_t \|^2 + \frac{1}{2 \varepsilon^2} \| d_A d_A^* B_t \|^2
\]
\[
- \varepsilon \varepsilon^2 \| B_t \|^2 - \varepsilon \varepsilon^2 \| \hat{X}_t(A) \|^2_{L^\infty(\Sigma)}
\]
for two positive constants \( c_0 \) and \( c_1 \). Therefore by lemma 11.6 we have, for an open set \( \Omega_1 \) with \( Q \subset \Omega_1 \subset \subset \Omega \)

\[
(11.34) \quad \int_{S^1 \times Q} \left( \varepsilon^2 \| \nabla^t d_A B_t \|^2 + \| d_A^* d_A B_t \|^2 + \varepsilon^2 \| \nabla d_A B_t \|^2 + \| d_A d_A^* B_t \|^2 \right)
\]

\[
\leq c \varepsilon^2 \int_{S^1 \times \Omega_1} \left( \| d_A B_t \|^2 + \| d_A^* B_t \|^2 + \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 c_{\mathcal{X}_t(A)} \right) \, dt \, ds
\]

\[
\leq c \varepsilon^2 \int_{S^1 \times \Omega} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 c_{\mathcal{X}_t(A)} \right) \, dt \, ds
\]

where the second estimate follows from step 2. Moreover, by the Bianchi identity \( d_A^* B_s = \nabla_s F_A \) and by the identity \( \nabla_t C = \frac{1}{\varepsilon} d_A^* B_s \) which follows from the Yang-Mills flow equation, for a positive \( c_1 \)

\[
\frac{1}{2} (\partial_t^2 - \partial_s) (\| d_A B_s \|^2 + c_1 \| d_A^* B_s \|^2) \geq \| \nabla_t d_A B_s \|^2 + \frac{1}{\varepsilon^2} \| d_A^* d_A B_s \|^2
\]

\[
+ \| \nabla_t d_A^* B_s \|^2 + \frac{1}{\varepsilon^2} \| d_A d_A^* B_s \|^2
\]

\[
- \frac{c}{\varepsilon^2} \| d_A^* B_s \|^2 - c \varepsilon^2 \| \nabla_t B_s \|^2 - c \varepsilon^2 \| \nabla_t B_t \|^2 - \frac{1}{\varepsilon^2} \| d_A d_A^* B_t \|^2
\]

\[
- c^2 \| B_t \|^2 - c \varepsilon^2 \| B_s \|^2 - \frac{c}{\varepsilon^2} \| d_A^* F_A \|^2 - c \varepsilon^2 \| d_A d_A^* B_s \|^2.
\]

Collecting all the estimates, for a constant \( c_0 > 0 \), we obtain

\[
\frac{1}{2} (\partial_t^2 - \partial_s) \left( c_0 \| F_A \|^2 + c_0 \varepsilon^2 \| B_t \|^2 + c_0 \varepsilon^2 \| B_s \|^2 \right)
\]

\[
+ \frac{1}{2} (\partial_t^2 - \partial_s) \left( c_1 \| d_A^* B_s \|^2 + \| d_A B_s \|^2 + \| d_A^* B_t \|^2 \right)
\]

\[
\geq \frac{1}{\varepsilon^2} \| d_A d_A^* B_s \|^2 + \| d_A^* d_A B_s \|^2 - c \varepsilon^2 \| B_t \|^2
\]

\[
- c \varepsilon^2 \| \mathcal{X}_t(A) \|_{L^\infty} - c \varepsilon^2 \| B_s \|^2.
\]

By the lemma 11.6 we have then,

\[
\int_{S^1 \times Q} \left( \| d_A d_A^* B_s \|^2 + \| d_A^* d_A B_s \|^2 \right)
\]

\[
\leq c \varepsilon^2 \int_{S^1 \times \Omega_1} \left( \varepsilon^2 \| B_t \|^2 + \varepsilon^2 \| \mathcal{X}_t \| + \varepsilon^2 \| B_s \|^2 \right)
\]

\[
+ c \varepsilon^2 \int_{S^1 \times \Omega_1} \left( \| F_A \|^2 + \| d_A^* B_s \|^2 + \| d_A B_s \|^2 + \| d_A^* B_t \|^2 \right)
\]

\[
\leq c \int_{S^1 \times \Omega} \left( \varepsilon^2 \| B_t \|^2 + \varepsilon^4 \| \mathcal{X}_s \| + \varepsilon^2 \| B_s \|^2 + \varepsilon^2 \| F_A \|^2 + \varepsilon^6 \| C \|^2 \right).
\]

\[
\square
\]

**Step 4.** There are two constants \( \varepsilon_0, c > 0 \) such that the following holds. If \( 0 < \varepsilon < \varepsilon_0 \), then

\[
\sup_{(t,s) \in S^1 \times Q} \left( \| d_A d_A^* B_t \|^2 + \| d_A^* d_A B_t \|^2 \right)
\]

\[
\leq c \int_{S^1 \times \Omega} \left( \| F_A \|^2 + \varepsilon^2 \| B_t \|^2 + \varepsilon^2 c_{\mathcal{X}_t(A)} \right) \, dt \, ds,
\]
Thus, by the lemma 11.5 and the previous steps we can conclude that
\[
\sup_{(t,s) \in S^1 \times Q} \left( \|d_A^* d_A B_t\|^2 + \|d_A d_A^* B_s\|^2 \right)
\]
\[
\leq \int_{S^1 \times \Omega} \left( c \|F_A\|^2 + c\varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|B_s\|^2 + c\varepsilon \|\dot{X}_{s}(A)\| + \varepsilon^4 \|C\|^2 \right) dt \, ds,
\]
\[
\sup_{(t,s) \in S^1 \times Q} \|d_A^* d_A d_A^* B_t\|^2
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \varepsilon^2 \|B_s\|^2 + \varepsilon^2 \|B_t\|^2 + \|F_A\|^2 + \varepsilon^4 \|\dot{X}_{s}\| + \varepsilon^4 \|C\|^2 \right) dt \, ds.
\]

**Proof of step 4.** Analogously to the first two steps we can estimate
\[
(\partial_t^2 - \partial_s) \left( \|d_A d_A^* B_t\|^2 + \|d_A^* d_A B_t\|^2 \right)
\]
and we obtain that there are two positive constants \(c\) and \(c_0\) such that
\[
(\partial_t^2 - \partial_s) \left( \|d_A d_A^* B_t\|^2 + \|d_A^* d_A B_t\|^2 + c_0 \|F_A\|^2 + c_0 \varepsilon^2 \|B_t\|^2 + c_0 \varepsilon^2 \|B_s\|^2 \right)
\]
\[
\geq - c \varepsilon^2 \|B_t\|^2 - c \varepsilon^2 \|B_s\|^2 + c \varepsilon \|\dot{X}_{s}\|.
\]
We can therefore conclude by the lemma [11.5] and the previous step that
\[
\sup_{(t,s) \in S^1 \times Q} \left( \|d_A d_A^* B_t\|^2 + \|d_A^* d_A B_t\|^2 \right)
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|\dot{X}_{s}(A)\| \right) dt \, ds.
\]

Furthermore, for two positive constants \(c\) and \(c_0\)
\[
\frac{1}{2} \left( \partial_t^2 - \partial_s \right) \left( \|d_A^* d_A B_s\|^2 + \|d_A d_A^* B_s\|^2 + c_0 \|F_A\|^2 + c_0 \varepsilon^2 \|B_t\|^2 + c_0 \varepsilon^2 \|B_s\|^2 + c_0 \varepsilon^2 \|B_t\|^2 \right)
\]
\[
\geq c \left( \|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|B_s\|^2 + c \varepsilon \|\dot{X}_{s}(A)\| \right)
\]
Thus, by the lemma [11.5] and the previous steps we can conclude that
\[
\sup_{(t,s) \in S^1 \times Q} \left( \|d_A^* d_A B_s\|^2 + \|d_A d_A^* B_s\|^2 \right)
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \|d_A^* d_A B_s\|^2 + \|d_A d_A^* B_s\|^2 + \|F_A\|^2 + \|d_A B_t\|^2 \right) dt \, ds
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \|d_A^* d_A B_s\|^2 + \|d_A^* B_t\|^2 \right) dt \, ds
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|B_s\|^2 + \varepsilon^2 \|\dot{X}_{s}(A)\| \right) dt \, ds
\]
\[
\leq c \int_{S^1 \times \Omega} \left( \|F_A\|^2 + \varepsilon^2 \|B_t\|^2 + \varepsilon^2 \|B_s\|^2 + \varepsilon^2 \|\dot{X}_{s}(A)\| \right) dt \, ds.
\]

Moreover,
\[
\frac{1}{2} \left( \partial_t^2 - \partial_s \right) \|d_A^* d_A d_A^* B_t\|^2 \geq \|\nabla_t d_A^* d_A d_A^* B_t\|^2 + \frac{1}{\varepsilon^2} \|d_A^* d_A d_A^* B_t\|^2
\]
\[
- c \varepsilon^2 \left( \|B_s\|^2 + \|d_A B_t\|^2 + \|d_A B_t\|^2 \right) - c \varepsilon^2 \|d_A d_A^* B_s\| - \frac{c}{\varepsilon^2} \|d_A F_A\|^2
\]
and hence by the lemma [11.5] we can conclude the proof of the fourth step:

\[
\sup_{(t,s) \in S^1 \times Q} \| d_A^* d_A^* B_t \|^2 \\
\leq c \int_{S^1 \times \Omega_1} (\varepsilon^2 \| B_s \|^2 + \varepsilon^2 \| d_A B_s \|^2 + \varepsilon^2 \| d_A B_t \|^2) \, dt \, ds \\
+ c \int_{S^1 \times \Omega_1} (\varepsilon^2 \| B_t \|^2 + \| F_A \|^2 + \| d_A^* d_A^* B_t \|^2) \, dt \, ds \\
\leq c \int_{S^1 \times \Omega} (\varepsilon^2 \| B_s \|^2 + \varepsilon^2 \| B_t \|^2 + \| F_A \|^2 + \varepsilon^4 c_X + \varepsilon^4 \| C \|^2) \, dt \, ds.
\]

\[ \square \]

**Step 5.** There are two constants \( \varepsilon_0, c > 0 \) such that the following holds. If \( 0 < \varepsilon < \varepsilon_0 \), then

\[ (11.37) \quad \sup_{S^1 \times Q} (\| F_A \| + \varepsilon \| \nabla_t F_A \| + \| d_A d_A^* F_A \|) \leq c \varepsilon^{-\frac{1}{4}}. \]

Proof of step 5. In order to prove the fifth step we need the following observation. For

\[ f(t,s) := \| F_A \|^2 + \varepsilon^2 \| \nabla_t F_A \|^2 + \varepsilon^4 \| d_A d_A^* F_A \|^2 + \varepsilon^2 \| d_A^* B_t \|^2 \]

we have that

\[ \varepsilon^2 (\partial_t^2 - \partial_s) f \geq c_0 f - \varepsilon^4 \]

and since for \( p \in \mathbb{N}, p \leq 2 \)

\[ \partial_t^2 f^p = p f^{p-1} \partial_t^2 f + p(p-1) (\partial_t f)^2 \geq p f^{p-1} \partial_t^2 f, \]

\[ \varepsilon^2 (\partial_t^2 - \partial_s) f^p \geq p f^{p-1} (\partial_t^2 - \partial_s) f \geq p c_0 f^p - p c_0 f^{p-1} \geq c_0 f^p - \varepsilon^4 f^p \]

where we used that \( ab \leq \frac{(p-1)a^p}{p} + \frac{b^p}{p} \) for any positive numbers \( a \) and \( b \). Therefore by lemma [11.6] for a sequence of open sets \( \Omega_i \subset \mathbb{R}, i = 1, \ldots, 2p, Q \subset \Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \cdots \subset \subset \Omega_{2p} \subset \subset \Omega \)

\[ \int_{S^1 \times \Omega_0} f^p \, dt \, ds \leq \varepsilon^{4p} + \varepsilon^2 \int_{S^1 \times \Omega_1} f^p \, dt \, ds \leq \varepsilon^{4p} + \varepsilon^2 \int_{S^1 \times \Omega_{2p}} f^p \, dt \, ds \leq \varepsilon^{4p} \]

where the last step follows by iterating the first one and from \( \sup_{\Omega_{2p}} f \leq \varepsilon \). By lemma [11.5]

\[ \sup_{S^1 \times Q} \varepsilon^2 f^p \leq \varepsilon^{4p} + \varepsilon^2 \int_{S^1 \times \Omega_0} f^p \, dt \, ds \leq \varepsilon^{4p} \]

and therefore

\[ \sup_{S^1 \times Q} (\| F_A \| + \varepsilon \| \nabla_t F_A \| + \| d_A d_A^* F_A \|) \leq c \varepsilon^{-\frac{1}{4}}. \]

\[ \square \]

**Step 6.** There are two constants \( \varepsilon_0, c > 0 \) such that the following holds. If \( 0 < \varepsilon < \varepsilon_0 \), then

\[ (11.38) \quad \sup_{S^1 \times Q} (\| F_A \|_{L^\infty(\Sigma)} + \| d_A d_A^* F_A \| + \varepsilon^2 \| \nabla_t \nabla_F A \|) \leq c \varepsilon^2. \]
Proof. One can show that, by the previous steps, there are three constants $c, c_0$ and $c_1$ such that

$$\frac{1}{2} \left( \partial_t^2 - \partial_s \right) \left( \| \nabla_t \nabla_s F_A \|_2^2 + \frac{c_0}{\varepsilon^2} \| F_A \|_2^2 + c_1 \| B_s \|_2^2 \right)$$

$$\geq -c \left( \| \nabla_t \nabla_s F_A \|_2^2 + \frac{c_0}{\varepsilon^2} \| F_A \|_2^2 + c_1 \| B_s \|_2^2 \right)$$

and thus by the lemma \[11.5\]

$$\sup_{s \in Q} \| \nabla_s \nabla_t F_A \|^2 \leq c \int_{S^1 \times \Omega} \left( \| \nabla_t \nabla_s F_A \|^2 + \frac{c_0}{\varepsilon^2} \| F_A \|^2 + c_1 \| B_s \|^2 \right) dt \, ds \leq c.$$  

Furthermore, by the Bianchi identity \[11.3\] and the Yang-Mills flow equation \[11.2\]

$$\sup_{s \in Q} \| d_A d_A^* F_A \|^2 \leq \sup_{s \in Q} \varepsilon^4 \| d_A \nabla_t B_t \|^2 + c \varepsilon^4$$

$$\leq \sup_{s \in Q} \varepsilon^4 \| \nabla_t d_A B_t \|^2 + \varepsilon^4 \| [B_t, d_A B_t] \|^2 + c \varepsilon^4$$

$$\leq \sup_{s \in Q} \varepsilon^4 \| \nabla_t \nabla_s F_A \|^2 + c \varepsilon^4 \leq c \varepsilon^4$$

and by the lemma \[3.1\]

$$\| F_A \|_{L^\infty(\Sigma)} \leq c \| d_A d_A^* F_A \|_{L^\infty(\Sigma)} \leq c \varepsilon^2.$$  

\qed

Step 7. There are two constants $\varepsilon_0, c > 0$ such that the following holds. If $0 < \varepsilon < \varepsilon_0$, then

$$\sup_{(t, s) \in S^1 \times Q} \left( \varepsilon^2 \| B_s \|^2 + \varepsilon^4 \| C \|^2 + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4 \right)$$

$$\leq \varepsilon^2 c \int_{S^1 \times \Omega} \left( \| B_s \|^2 + \varepsilon^2 \| C \|^2 \right) dt \, ds.$$  

Proof. Since we know now that, for a positive constant $c_0$,  

$$\varepsilon^2 \| B_t \|_{L^\infty(\Sigma)} + \| F_A \|_{L^\infty(\Sigma)} \leq c_0 \varepsilon^2,$$

using the computations of the second step we can obtain the following estimate a positive constant $c$

$$\frac{1}{2} \left( \partial_t^2 - \partial_s \right) \left( \| B_s \|^2 + \varepsilon^2 \| C \|^2 \right)$$

$$\geq \| \nabla_t B_s \|^2 + \frac{1}{\varepsilon^2} \| d_A B_s \|^2 + \frac{1}{\varepsilon^2} \| d_A^* B_s \|^2 + \varepsilon^2 \| \nabla_t C \|^2 + \| d_A C \|^2$$

$$- c \left( \| B_s \|^2 + \varepsilon^2 \| C \|^2 \right).$$

Analogously, for seven positive constants $c_i, i = 1, \ldots, 7$, we have

$$\left( \partial_t^2 - \partial_s \right) \left( \mathcal{N}_1 + c_1 \varepsilon^2 \| B_s \|^2 + c_1 \varepsilon^4 \| C \|^2 \right)$$

$$\geq \frac{c_5}{\varepsilon^2} \mathcal{N}_2 - c \left( \mathcal{N}_1 + c_1 \varepsilon^2 \| B_s \|^2 + c_1 \varepsilon^4 \| C \|^2 \right),$$

$$\left( \partial_t^2 - \partial_s \right) \left( \mathcal{N}_2 + c_2 \mathcal{N}_1 + c_2 c_1 \varepsilon^2 \| B_s \|^2 + c_2 c_1 \varepsilon^4 \| C \|^2 \right)$$

$$\geq \frac{c_6}{\varepsilon^2} \mathcal{N}_3 - c \left( \mathcal{N}_2 + c_2 \mathcal{N}_1 + c_2 c_1 \varepsilon^2 \| B_s \|^2 + c_2 c_1 \varepsilon^4 \| C \|^2 \right),$$
(\partial_t^2 - \partial_s) (N_3 + c_3N_2 + c_3c_2N_1 + c_3c_2c_1\varepsilon^2\|B_s\|^2 + c_3c_2c_1\varepsilon^4\|C\|^2) \\
\geq \frac{c_7}{\varepsilon^2} N_4 - c (N_4 + c_4N_3 + c_4c_3N_2 + c_4c_3c_2N_1) \\
- c (c_4c_3c_2c_1\varepsilon^2\|B_s\|^2 + c_4c_3c_2c_1\varepsilon^4\|C\|^2)

(\partial_t^2 - \partial_s) (N_4 + c_4N_3 + c_4c_3N_2 + c_4c_3c_2N_1) \\
+ (\partial_t^2 - \partial_s) (c_4c_3c_2c_1\varepsilon^4\|B_s\|^2 + c_4c_3c_2c_1\varepsilon^4\|C\|^2) \\
\geq - c (N_4 + c_4N_3 + c_4c_3N_2 + c_4c_3c_2N_1) \\
- c (c_4c_3c_2c_1\varepsilon^2\|B_s\|^2 + c_4c_3c_2c_1\varepsilon^4\|C\|^2).

The seventh step follows then from the lemmas 11.5 and 11.6 and the last estimates.

With the seventh step we concluded the proof of the theorem 11.1.

12. \(L^\infty\)-bound for a Yang-Mills flow

In this section we want to show that for any value \(b > 0\), any Yang-Mills flow \(\Xi^\varepsilon \in M^\varepsilon(\Xi^\varepsilon_-, \Xi^\varepsilon_+)\), \(\Xi^\varepsilon_+ \in \text{Crit}^b_{YM^\varepsilon_+, \varepsilon_+}\) satisfies the \(L^\infty\)-estimate (12.1). This result allows us to apply the theorem 11.1 needed in the proof of the surjectivity in the section 15.

**Theorem 12.1.** We choose a regular value \(b > 0\). Then there are two positive constants \(c\) and \(\varepsilon_0\) such that the following holds. For any \(\varepsilon, 0 < \varepsilon < \varepsilon_0\), any \(\Xi^\varepsilon_-, \Xi^\varepsilon_+ \in \text{Crit}^b_{YM^\varepsilon_-, \varepsilon_+}\) and any \(\Xi^\varepsilon = A^\varepsilon + \Psi^\varepsilon dt + \Phi^\varepsilon ds \in M^\varepsilon(\Xi^\varepsilon_-, \Xi^\varepsilon_+), (12.1)\)

\[
\|\partial_s A^\varepsilon - d_A^* \Phi^\varepsilon\|_{L^\infty(\Sigma)} + \|\partial_t A^\varepsilon - d_A^* \Psi^\varepsilon\|_{L^\infty(\Sigma)} \leq c.
\]

**Proof.** Since the two estimates (11.8) and (11.9) in the theorem 11.1 hold also is we assume

\[
(12.2) \quad \sup_{(t,s) \in S^1 \times \mathbb{R}} \left( \|\partial_s A^\varepsilon - d_A^* \Phi^\varepsilon\|_{L^\infty(\Sigma)} + \|\partial_t A^\varepsilon - d_A^* \Psi^\varepsilon\|_{L^1(\Sigma)} \right) \leq c
\]

then by the Sobolev estimate for 1-forms on \(\Sigma\), (12.2) implies (12.3). In order to prove the theorem we assume therefore that there are sequences \(\varepsilon_\nu \to 0\) and \(\Xi^\varepsilon_\nu := A^\varepsilon + \Psi^\varepsilon dt + \Phi^\varepsilon ds \in M^\varepsilon(\Xi^\varepsilon^-_\nu, \Xi^\varepsilon^+_\nu), \Xi^\varepsilon^+_\nu \in \text{Crit}^b_{YM^\varepsilon^-_\nu, \varepsilon^+_\nu}\) such that

\[
(12.3) \quad m_\nu := \sup_{(t,s) \in S^1 \times \mathbb{R}} \left( \|\partial_s A^\varepsilon - d_A^* \Phi^\varepsilon\|_{L^\infty(\Sigma)} + \|\partial_t A^\varepsilon - d_A^* \Psi^\varepsilon\|_{L^1(\Sigma)} \right) \to \infty,
\]

furthermore we assume that there is a sequence \((t_\nu, s_\nu)\) such that

\[
(12.4) \quad \|\partial_s A^\varepsilon(t_\nu, s_\nu) - d_A^*(t_\nu, s_\nu) \Phi^\varepsilon(t_\nu, s_\nu)\|_{L^\infty(\Sigma)} \\
+ \|\partial_t A^\varepsilon(t_\nu, s_\nu) - d_A^*(t_\nu, s_\nu) \Psi^\varepsilon(t_\nu, s_\nu)\|_{L^1(\Sigma)} = m_\nu.
\]

We will consider the following three cases. We denote by \(\|\cdot\|\) the \(L^2\)-norm on \(\Sigma\).
**Case 1:** $\lim_{\nu \to \infty} \varepsilon_{\nu} m_{\nu} = 0$. We take the sequence of connections $\Xi^\nu = A^\nu + \Psi^\nu dt + \Phi^\nu ds$ defined by

$$
A^\nu(t, s) := A^\nu \left( \frac{1}{m_{\nu}} t + t_{\nu}, \frac{1}{m_{\nu}} s + s_{\nu} \right),
$$

$$
(12.5)
$$

$$
(12.6)
$$

$$
\Psi^\nu(t, s) := \frac{1}{m_{\nu}} \Psi^\nu \left( \frac{1}{m_{\nu}} t + t_{\nu}, \frac{1}{m_{\nu}} s + s_{\nu} \right),
$$

$$
\Phi^\nu(t, s) := \frac{1}{m_{\nu}} \Phi^\nu \left( \frac{1}{m_{\nu}} t + t_{\nu}, \frac{1}{m_{\nu}} s + s_{\nu} \right);
$$

then $\Xi^\nu$ satisfies the Yang-Mills flow equations

$$
(12.7)
$$

If we define $B^\nu_s := \partial_s A^\nu - d_{A^\nu} \Phi^\nu$, $\tilde{B}^\nu_s := \partial_s \tilde{A}^\nu - d_{\tilde{A}^\nu} \tilde{\Phi}^\nu$ and $\tilde{C}^\nu := \partial_s \tilde{\Psi}^\nu - \partial_t \tilde{\Phi}^\nu - [\tilde{\Psi}^\nu, \tilde{\Phi}^\nu]$, then we have the following estimates for the norms:

$$
(12.8)
$$

$$
(12.9)
$$
The theorem [11] tell us that for every open interval \( \Omega \subset \mathbb{R}, 0 \in \Omega \), and every compact set \( Q \in \Omega \) there is a positive constant \( c \) such that

\[
\sup_{(t,s) \in S^1 \times Q} \left( m^2 \left( B^\nu_s \right)^2 + \left\| \sigma A \cdot B^\nu_s \right\| + \left\| d_A \cdot B^\nu_s \right\| \right) \\
+ \sup_{(t,s) \in S^1 \times Q} \left( m^2 \left( B^\nu_s \right)^2 + \left\| \sigma A \cdot d_A \cdot B^\nu_s \right\| + \left\| d_A \cdot d_A \cdot B^\nu_s \right\| \right)
\leq c \int_{S^1 \times \Omega} \left( \left\| \sigma \cdot F^\nu \right\|^2 + \epsilon \sigma \left\| \sigma \cdot B^\nu_s \right\|^2 + \epsilon \sigma \left\| \dot{\sigma} \cdot \Psi^\nu \right\|^2 \right) dt \\
+ c \int_{S^1 \times \Omega} \left( \epsilon \sigma \left\| \dot{\sigma} \cdot \Psi^\nu \right\|^2 + \epsilon \sigma \left\| \dot{\sigma} \cdot \Psi^\nu \right\|^2 \right) dt
\leq c \epsilon \sigma m^2 \left( \frac{c}{m^2} \frac{1}{\epsilon} \int_{S^1 \times \Omega} \left( \sigma \cdot \Psi^\nu \right) dt + \frac{1}{m^2} \right)
\]

where for the last inequality we used that

\[
\int_{S^1 \times \Omega} \left\| \sigma \cdot F^\nu \right\|^2 dt \leq m \epsilon \sigma \sup_{s} s \in \Omega \int_{S^1} \left\| \sigma \cdot F^\nu \right\|^2 dt \leq c \epsilon \sigma m^2.
\]

Since (12.10)

\[
\left\| \frac{1}{m^2} \bar{X}^\nu_{\epsilon \sigma \cdot t + \epsilon \sigma \cdot \nu} (\bar{A}^\nu) \right\|_{L^\infty} = \frac{1}{m^2} \left\| \bar{X}^\nu (\bar{A}^\nu) \right\|_{L^\infty} \leq \frac{c}{m^2} \left( \frac{c}{m^2} \frac{1}{\epsilon} \int_{S^1 \times \Omega} \left( \sigma \cdot \Psi^\nu \right) dt + \frac{1}{m^2} \right),
\]

it follows that

\[
\sup_{(t,s) \in S^1 \times \Omega} \left\| B^\nu_t \right\|_{L^2 (\Sigma)} \leq c \sup_{(t,s) \in S^1 \times \Omega} \left( \left\| B^\nu_t \right\| + \left\| \sigma A \cdot B^\nu_s \right\| + \left\| d_A \cdot B^\nu_s \right\| \right)
\leq c \sqrt{m^2} \to 0 \quad (\nu \to \infty),
\]

\[
\sup_{(t,s) \in S^1 \times \Omega} \left\| \bar{B}^\nu_t \right\|_{L^\infty (\Sigma)} \leq c \sup_{(t,s) \in S^1 \times \Omega} \left( \left\| \bar{B}^\nu_t \right\| + \left\| \sigma A \cdot \bar{B}^\nu_s \right\| + \left\| d_A \cdot \bar{B}^\nu_s \right\| \\
+ \left\| \sigma d_A \cdot \bar{B}^\nu_s \right\| + \left\| d_A \cdot \sigma d_A \cdot \bar{B}^\nu_s \right\| \right)
\leq c \sqrt{m^2} \to 0 \quad (\nu \to \infty),
\]

and this is a contradiction.

**Case 2:** \( \lim_{\nu \to \infty} \epsilon \nu m_{\nu} = c_1 > 0 \). We consider the sequence of connections \( \Xi^\nu = \bar{A}^\nu + \bar{\Psi}^\nu dt + \bar{\Phi}^\nu ds \) defined by

\[
\bar{A}^\nu (t,s) := A^\nu (\epsilon \nu t + t, \epsilon \nu t + s), \quad \bar{\Psi}^\nu (t,s) := \epsilon \nu \psi^\nu (\epsilon \nu t + t, \epsilon \nu t + s),
\]

\[
\dot{\bar{\Phi}}^\nu (t,s) := \epsilon \nu \dot{\phi}^\nu (\epsilon \nu t + t, \epsilon \nu t + s);
\]

then \( \Xi^\nu \) satisfies the Yang-Mills flow equations

\[
\partial_\nu \dot{\bar{A}}^\nu - \partial_\nu \bar{A}^\nu \bar{\Psi}^\nu + d_\nu A \cdot \bar{\Psi}^\nu = -\dot{\bar{\Phi}}^\nu (\partial_\nu \dot{\bar{A}}^\nu - \partial_\nu \bar{A} \cdot \bar{\Psi}^\nu) - \epsilon \nu \bar{X}_{\epsilon \nu t + \epsilon \nu s} (\bar{A}^\nu) = 0,
\]

\[
\partial_\nu \bar{\Psi}^\nu - \bar{\Psi}^\nu \bar{\Phi}^\nu - d_\nu \bar{A} \cdot (\partial_\nu \bar{A} - \partial_\nu \bar{\Psi}^\nu) = 0.
\]

In addition, if define \( \dot{\bar{B}}^\nu_t := \partial_\nu \dot{\bar{A}}^\nu - \partial_\nu \bar{A} \cdot \bar{\Psi}^\nu, \bar{B}^\nu_s := \partial_\nu \bar{A}^\nu - \partial_\nu \bar{\Psi}^\nu \) and \( \bar{C}^\nu := \partial_\nu \bar{\Psi}^\nu - \partial_\nu \dot{\bar{\Psi}}^\nu - \partial_\nu \bar{\Psi}^\nu \), for \( \nu \to \infty \), we have that

\[
\sup_{(t,s) \in S^1 \times \Omega} \left( \left\| \dot{\bar{B}}^\nu_t \right\|_{L^2 (\Sigma)} + \left\| \bar{B}^\nu_s \right\|_{L^2 (\Sigma)} \right) = c_1 \to c_1,
\]
estimates (12.7)-(12.7). In addition, we denote $B_1$ and hence the new connection satisfies the Yang-Mills equations (12.6) and the Yang-Mills equations (12.5).

Case 3: $\lim_{\nu \to \infty} \varepsilon_\nu m_\nu = \infty$. We consider the substitution \((12.5)\) as in the case 1 and hence the new connection satisfies the Yang-Mills equations \((12.6)\) and the estimates \((12.7)\)-\((12.7)\). In addition, we denote $B^\nu := \partial_\nu A^\nu - d_{A^\nu} \Phi^\nu$, $B^\nu := \partial_\nu A^\nu - d_{A^\nu} \Phi^\nu$. 

We can compute the estimates \((11.8)\) and \((11.9)\) of the theorem \((11.1)\) also for $\varepsilon = 1$ exactly in the same way as we did in the proof of that theorem. Thus, for every open interval $\Omega \subset \mathbb{R}$, $0 \in \Omega$, and every compact set $Q \in \Omega$ there is a positive constant $c$ such that

$$\sup_{(t,s) \in S^1 \times Q} \left( \| \tilde{B}_t^\nu \|^2 + \| d_{\tilde{A}_t^\nu} \tilde{B}_t^\nu \|^2 + \| d_{\tilde{A}_t^\nu} \tilde{B}_s^\nu \|^2 \right)$$

$$\sup_{(t,s) \in S^1 \times Q} \left( \| \tilde{B}_s^\nu \|^2 + \| d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \|^2 + \| d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \|^2 \right)$$

\[ \leq c \int_{S^1 \times \Omega} \left( \| F_{A^\nu} \|^2 + \| \tilde{B}_t^\nu \|^2 + c_{x_\nu} \mathcal{Q}_{x_\nu + t_\nu} (A^\nu) + \| \tilde{B}_s^\nu \|^2 + \| \tilde{C}_s^\nu \|^2 \right) dt \]

\[ \leq c\varepsilon_\nu + \frac{1}{\varepsilon_\nu} c_{x_\nu} \mathcal{Q}_{x_\nu + t_\nu} (A^\nu) \]

where for the last estimate we used that

$$\int_{S^1 \times \Omega} \| F_{A^\nu} \|^2 dt \leq \frac{c}{\varepsilon_\nu} \sup_{s \in \Omega} \int_{S^1} \| F_{A^\nu} \|^2 dt \leq c\varepsilon_\nu.$$

Next, we consider

$$\left\| \varepsilon_\nu^2 \tilde{X}_{e_{x_\nu + t_\nu}} \right\| \leq \varepsilon_\nu^2 \| \tilde{X}_{\nu} (A) \| \leq c\varepsilon_\nu^2 = \left( c_{x_\nu} \mathcal{Q}_{x_\nu + t_\nu} (A^\nu) \right)^\frac{1}{2},$$

then

$$\sup_{(t,s) \in S^1 \times R} \| \tilde{B}_t^\nu \|_{L^1 (\Sigma)} \leq c \sup_{(t,s) \in S^1 \times R} \left( \| \tilde{B}_t^\nu \| + \| d_{\tilde{A}_t^\nu} \tilde{B}_t^\nu \| + \| d_{\tilde{A}_t^\nu} \tilde{B}_s^\nu \| \right)$$

\[ \leq c \varepsilon_\nu \rightarrow 0 \quad (\nu \to \infty), \]

$$\sup_{(t,s) \in S^1 \times R} \| \tilde{B}_s^\nu \|_{L^\infty (\Sigma)} \leq c \sup_{(t,s) \in S^1 \times R} \left( \| \tilde{B}_s^\nu \| + \| d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \| + \| d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \| \right.$$ \[ + \| d_{\tilde{A}_s^\nu} d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \| + \| d_{\tilde{A}_s^\nu} d_{\tilde{A}_s^\nu} \tilde{B}_s^\nu \| \right) \]

\[ \leq \frac{1}{\varepsilon_\nu} \rightarrow 0 \quad (\nu \to \infty), \]

and this is a contradiction.
$d_{\tilde{A}^\nu} \tilde{\Psi}_\nu, \tilde{B}_s^\nu := \partial_t \tilde{A}^\nu - d_{\tilde{A}^\nu} \tilde{\Phi}_\nu$. We recall that by the computations in the first and in the second step of the proof of theorem 11.1 we have

$$
\frac{1}{2} (\partial_t^2 - \partial_s) \| \mathcal{B}_t^\nu \|^2 = \| \nabla_t^\nu \tilde{B}_t^\nu \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| d_{\tilde{A}^\nu} \tilde{B}_t^\nu \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2
$$

$$
+ \frac{1}{\varepsilon_t^2 m_\nu^2} (\langle [\tilde{B}_s^\nu, *F_{\tilde{A}^\nu}], \tilde{B}_t^\nu \rangle) - \langle d_{\tilde{A}^\nu} \tilde{B}_t^\nu, * \frac{1}{m_\nu} X_{\frac{m_\nu}{m_\nu t + t_s}} (\tilde{A}^\nu) \rangle, \quad
eq \langle d_{\tilde{A}^\nu} \tilde{B}_t^\nu, * \frac{1}{m_\nu} X_{\frac{m_\nu}{m_\nu t + t_s}} (\tilde{A}^\nu) \rangle,
$$

$$
\frac{1}{2} (\partial_t^2 - \partial_s) \| F_{\tilde{A}^\nu} \|^2 = \| \nabla_t^\nu F_{\tilde{A}^\nu} \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| d_{\tilde{A}^\nu} F_{\tilde{A}^\nu} \|^2
$$

$$
+ \langle F_{\tilde{A}^\nu}, [\tilde{B}_s^\nu \wedge \tilde{B}_t^\nu] \rangle - \langle d_{\tilde{A}^\nu} F_{\tilde{A}^\nu}, * \frac{1}{m_\nu} X_{\frac{m_\nu}{m_\nu t + t_s}} (\tilde{A}^\nu) \rangle, \quad
eq \langle F_{\tilde{A}^\nu}, [\tilde{B}_s^\nu \wedge \tilde{B}_t^\nu] \rangle - \langle d_{\tilde{A}^\nu} F_{\tilde{A}^\nu}, * \frac{1}{m_\nu} X_{\frac{m_\nu}{m_\nu t + t_s}} (\tilde{A}^\nu) \rangle,
$$

$$
\frac{1}{2} (\partial_t^2 - \partial_s) \| B_s^\nu \|^2 = \| \nabla_t^\nu B_s^\nu \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| d_{\tilde{A}^\nu} B_s^\nu \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| d_{\tilde{A}^\nu} B_s^\nu \|^2
$$

$$
+ \frac{1}{\varepsilon_t^2 m_\nu^2} (\tilde{B}_s^\nu, [d_{\tilde{A}^\nu} \tilde{B}_t, \tilde{B}_t^\nu]) - \frac{2}{\varepsilon_t^2 m_\nu^2} \langle \tilde{B}_s^\nu, * [\tilde{B}_s^\nu, * F_{\tilde{A}^\nu}] \rangle
$$

$$
- 2 \langle \tilde{B}_s^\nu, \nabla_t^\nu * \frac{1}{m_\nu} X_{\frac{m_\nu}{m_\nu t + t_s}} (\tilde{A}^\nu) \rangle,
$$

thus for a constant $c_0 > 0$

$$
(\partial_t^2 - \partial_s) \left( \frac{c_0}{\varepsilon_t^2 m_\nu^2} \| F_{\tilde{A}^\nu} \|^2 + \| \tilde{B}_t^\nu \|^2 + \| \tilde{B}_s^\nu \|^2 \right)
$$

$$
\geq - \frac{1}{\varepsilon_t^2 m_\nu^2} \| F_{\tilde{A}^\nu} \|^2 - c \| \tilde{B}_t^\nu \|^2 - c \| \tilde{B}_s^\nu \|^2 - \frac{c}{m_\nu}
$$

and hence by the lemma 11.5 there is, for any open set $\Omega \subset \mathbb{R}$, $0 \in \Omega$, and any compact interval $Q \subset \Omega$, a positive constant $c$ such that

$$
sup_{(t,s) \in S^1 \times Q} (\| \tilde{B}_t^\nu \|^2 + \| \tilde{B}_t^\nu \|^2)
$$

$$
\leq c \int_{S^1 \times \Omega} (\| \tilde{B}_s^\nu \|^2 + \| \tilde{B}_s^\nu \|^2 + \frac{1}{\varepsilon_t^2 m_\nu^2} \| F_{\tilde{A}^\nu} \|^2 + \frac{1}{m_\nu}) \, dt \, ds
$$

$$
\leq \frac{c}{m_\nu} + c \sup_{s \in \Omega} \int_{S^1} \| \tilde{B}_s^\nu \|^2 \, dt \leq \frac{c}{m_\nu}.
$$

Analogously, using the computations of the proof of theorem 11.1 we can obtain that there is a constant $c_0$ such that

$$
(\partial_t^2 - \partial_s) \left( \| d_{\tilde{A}^\nu} \tilde{B}_t^\nu \|^2 + \| d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 + \| d_{\tilde{A}^\nu} \tilde{B}_t^\nu \|^2 + \| d_{\tilde{A}^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 + \| d_{\tilde{A}^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 \right)
$$

$$
+ (\partial_t^2 - \partial_s) \left( \frac{c_0}{\varepsilon_t^2 m_\nu^2} \| F_{\tilde{A}^\nu} \|^2 + \| \tilde{B}_t^\nu \|^2 + \| \tilde{B}_s^\nu \|^2 \right)
$$

$$
\geq - \frac{c}{\varepsilon_t^2 m_\nu^2} \| F_{\tilde{A}^\nu} \|^2 - c \| d_{\tilde{A}^\nu} \tilde{B}_t^\nu \|^2 - c \| d_{\tilde{A}^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 - c \| \tilde{B}_t^\nu \|^2
$$

$$
- c \| d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 - c \| d_{\tilde{A}^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|^2 - c \| \tilde{B}_s^\nu \|^2
$$

(12.11)
thus, by the Sobolev estimates for 1-forms on $\Sigma$ and the lemma 11.13 we get, for $\Omega_1 = \Sigma \times S^1 \times \Omega$,

$$\sup_{(t,s) \in S^1 \times Q} \left( \| \tilde{B}_t^\nu \|_{L^2(\Sigma)}^2 + \| \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 \right)$$

\[ \leq \sup_{(t,s) \in S^1 \times Q} \left( \| \tilde{B}_t^\nu \|_{L^2(\Sigma)}^2 + \| d_{A^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 + \| d_{A^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 \right) \]

\[ + \sup_{(t,s) \in S^1 \times Q} \left( \| \tilde{B}_t^\nu \|_{L^2(\Sigma)}^2 + \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 + \| d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 \right) \]

\[ \leq c \int_{S^1 \times \Omega} \left( \| d_{A^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 + \| d_{A^\nu} d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 \right) dt \, ds \]

\[ + c \int_{S^1 \times \Omega} \left( \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 + \| d_{\tilde{A}^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma)}^2 \right) dt \, ds + \frac{c}{m_\nu} \]

\[ \leq \frac{c}{m_\nu} \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Omega_1)}^2 \]

< \frac{c}{m_\nu} + c \varepsilon_\nu \to 0 \]

where the last estimate follows from the lemma 11.13 and the third by

\[
\| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Omega_1)}^2 + \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Omega_1)}^2 \\
\leq \frac{c}{m_\nu} + c \varepsilon_\nu + c \varepsilon_\nu \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Omega_1)}^2 + \frac{c \varepsilon_\nu}{m_\nu} \| \nabla \tilde{B}_s^\nu \|_{L^2(\Omega_1)} \]

and therefore we have a contradiction. In order to finish the proof of the theorem it remains only to prove (12.12). By the identities

\[ \varepsilon^2 \| \tilde{B}_s^\nu + \frac{1}{\varepsilon^2} d_{A^\nu} F_{A^\nu} - \nabla \tilde{B}_s^\nu - \frac{1}{m_\nu} \frac{1}{t + \varepsilon^2} \left( \tilde{A}^\nu \right) \|_{L^2(\Sigma \times S^1)}^2 \]

\[ \leq \| \tilde{B}_s^\nu \|_{L^2(\Sigma \times S^1)}^2 + \| d_{A^\nu} \tilde{B}_s^\nu \|_{L^2(\Sigma \times S^1)}^2 + \| \nabla \tilde{B}_s^\nu \|_{L^2(\Sigma \times S^1)}^2 \]

\[ - 2 \varepsilon^2 \left( \frac{1}{\varepsilon^2} d_{A^\nu} F_{A^\nu} - \nabla \tilde{B}_s^\nu \right) \]

\[ + 2 \left( \tilde{B}_s^\nu, d_{A^\nu} F_{A^\nu} - \varepsilon^2 \nabla \tilde{B}_s^\nu - \frac{1}{m_\nu} \frac{1}{t + \varepsilon^2} \left( \tilde{A}^\nu \right) \right) \]

\[ - 2 \left( d_{A^\nu} F_{A^\nu}, \varepsilon^2 \nabla \tilde{B}_s^\nu \right) - \left( \nabla F_{A^\nu}, d_{A^\nu} \tilde{B}_s^\nu \right) \]

\[ - 2 \left( d_{A^\nu} F_{A^\nu}, \tilde{B}_s^\nu \right) - 2 \left( \nabla F_{A^\nu}, d_{A^\nu} \tilde{B}_s^\nu \right) = 2 \left( F_{A^\nu}, [\tilde{B}_s^\nu, \tilde{B}_s^\nu] \right), \]

\[ 2 \left( \tilde{B}_s^\nu, d_{A^\nu} F_{A^\nu} - \varepsilon^2 \nabla \tilde{B}_s^\nu \right) = 2 \left( d_{A^\nu} \tilde{B}_s^\nu, F_{A^\nu} \right) - \varepsilon^2 \left( \nabla \tilde{B}_s^\nu, \tilde{B}_s^\nu \right) \]
by the Bianchi identity $\nabla_t F_{A^t} - d_A \bar{B}_t = 0$ and by the perturbed Yang-Mills equation \((1.0)\) we have:

$$
\left\| d_{A^t} \bar{B}_t^\nu \right\|_{L^2(\Omega_t)}^2 + \left\| d_{A^t}^* \bar{B}_t^\nu \right\|_{L^2(\Omega_t)}^2 + \varepsilon \frac{\mathrm{tr}^2}{m^2} \left\| \nabla_t \bar{B}_t^\nu \right\|_{L^2(\Omega_t)}^2 \\
\leq \frac{c}{m^2} \varepsilon \int_{\Omega} \left( \mu^2_m \| d_{A^t} \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)}^2 + \frac{1}{\varepsilon \mu_m} \| F_{A^t} \|_{L^2(\Sigma \times S^1)} \right) \text{d}s \\
+ c \int_{\Omega} \left( \varepsilon \mu_m \| d_{A^t} \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)}^2 + \| \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)} \right) \text{d}s \\
+ c \int_{\Omega} \left( \frac{\varepsilon^2}{m^2} \| d_{A^t} \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)}^2 + \sup_{s \in \Omega} \| \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)} \right) \text{d}s \\
+ c \int_{\Omega} \left( \frac{\varepsilon^2}{m^2} \| \nabla \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)}^2 + \varepsilon \| \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)} \right) \text{d}s
$$

where we use the Hölder inequality and the Sobolev estimate in the first estimate. Thus choosing $\varepsilon \mu$ small enough

$$
\left\| d_{A^t} \bar{B}_t^\nu \right\|_{L^2(\Omega_t)}^2 + \left\| d_{A^t}^* \bar{B}_t^\nu \right\|_{L^2(\Omega_t)}^2 \\
\leq \frac{c}{m^2} \varepsilon \int_{\Omega} \left( \mu^2_m \| d_{A^t} \bar{B}_t^\nu \|_{L^2(\Sigma \times S^1)}^2 + \frac{1}{\varepsilon \mu_m} \| F_{A^t} \|_{L^2(\Sigma \times S^1)} \right) \text{d}s
$$

With this last estimate we conclude the discussion of the third case and thus, also the proof of the theorem \([12.1]\). \hfill \square

13. Exponential convergence

In this section we will prove the exponential convergence of the curvature terms $B_s$ and $C$ stated in the next theorem. In order to simplify the exposition, for this section, we denote by $\| \cdot \|_L$ the norm $\| \cdot \|_{L^2(\Sigma \times S^1)}$.

**Theorem 13.1.** We assume that every perturbed closed geodesic on $M^q(P)$ is non-degenerate. Then for every $c_0, b > 0$ there are four positive constants $\delta, \varepsilon_0, \varepsilon, \rho$ such that the following holds. If $\Xi \in \mathcal{M}^q(\Xi^\nu_0, \Xi^\nu_0)$, $\Xi^\nu_0, \Xi^\nu_0 \in \text{Crit}_\nu \mathcal{M}^q$, with $0 < \varepsilon \leq \varepsilon_0$, satisfies, for $B_t := \partial_t A - d_A \Psi$, $B_s := \partial_s A - d_A \Phi$, $C := \partial_t \Psi - \partial_s \Phi - [\Psi, \Phi]$,

\begin{align}
\| B_s \|_{L^\infty(\Sigma \times S^1 \times \mathbb{R})} + \| B_t \|_{L^\infty(\Sigma \times S^1 \times \mathbb{R})} + \varepsilon \| C \|_{L^\infty(\Sigma \times S^1 \times \mathbb{R})} & \leq c_0 \\
\| B_s \|_{L^2(\Sigma \times S^1 \times [0, \infty))} + \varepsilon^2 \| C \|_{L^2(\Sigma \times S^1 \times [0, \infty))} & \leq \delta,
\end{align}

and

"
then, for $S \geq 1$,
\begin{equation}
\|B_s\|_{L^2(\Sigma \times S^1 \times [S,\infty))} + \varepsilon^2 \|C\|_{L^2(\Sigma \times S^1 \times [S,\infty))} \leq ce^{-\rho S}.
\end{equation}

**Lemma 13.2.** We choose a positive constant $c_0$ and we assume that every perturbed closed geodesic on $M^g(P)$ is non-degenerate. Then there are positive constants $\delta_0, \varepsilon_0$ and $c$ such that the following holds. If $A + \Psi dt$ is a connection on $P \times S^1$ which satisfies, for $0 < \varepsilon \leq \varepsilon_0$, $B_t := \partial_t A - d_A \Psi$,
\begin{equation}
\left\| \frac{1}{\varepsilon^2} d^*_A F_A - \nabla_t B_t - \ast X_t(A) \right\|_{L^\infty} + \frac{1}{\varepsilon} \left\| d^*_A B_t \right\|_{L^\infty} + \sup_{t \in S^1} \| F_A \|_{L^2(\Sigma)} \leq \delta_0,
\end{equation}
with $B_t := \partial_t A - d_A \Psi$, then
\begin{equation}
\|A + \Psi dt\|_{2,2,\varepsilon} \leq c \left( \varepsilon \|D^\epsilon (A, \Psi)(\alpha, \psi)\|_{0,2,\varepsilon} + \|\pi_\alpha D^\epsilon (A, \Psi)(\alpha, \psi)\|_{L^2(\Sigma)} \right),
\end{equation}
\begin{equation}
\|(1 - \pi_\alpha) \alpha + \psi dt\|_{2,2,\varepsilon} \leq c\varepsilon^2 \|\pi_\alpha D^\epsilon (A, \Psi)(\alpha, \psi)\|_{0,2,\varepsilon} + c \|\pi_\alpha D^\epsilon (A, \Psi)(\alpha, \psi)\|_{L^2(\Sigma)},
\end{equation}
for every 1-form $\alpha + \psi dt$ on $P \times S^1$.

**Proof of Lemma 13.2** We suppose that the lemma does not hold. Then there are sequences $A^\nu + \Psi^\nu dt$, $\varepsilon_\nu \to 0$, such that, for $B^\nu_t := \partial_t A^\nu - d_A \Psi^\nu$, and
\begin{equation}
\delta_\nu := \left\| \frac{1}{\varepsilon^2} d^*_A F_A - \nabla_t B_t - \ast X_t(A^\nu) \right\|_{L^\infty} + \frac{1}{\varepsilon} \left\| d^*_A B_t \right\|_{L^\infty} + \sup_{t \in S^1} \| F_A \|_{L^2(\Sigma)},
\end{equation}
with $\delta_\nu \to 0$ and (13.10) or (13.17) are not satisfied for $A + \Psi dt = A^\nu + \Psi^\nu dt$ and $\varepsilon = \varepsilon_\nu$.

**Claim:** The following two estimates hold:
\begin{align*}
\|F_A\|_{1,2,\varepsilon} &\leq \|d^*_A F_A - \varepsilon^2 \nabla_t B_t - \varepsilon^2 \ast X_t(A)\| + c\varepsilon^2, \\
\|B_t\|_{1,2,1} &\leq \|d^*_A B_t\| + \sup_{t \in S^1} \| F_A \|_{L^2(\Sigma)} + \| B_t \|. 
\end{align*}

**Proof.** By the identity
\begin{align*}
\|d^*_A F_A - \varepsilon^2 \nabla_t B_t - \varepsilon^2 \ast X_t(A)\|^2 + \varepsilon^2 \|\nabla_t F_A - d_A B_t\|^2 &\leq \|d^*_A F_A\|^2 + \varepsilon^4 \|\nabla_t B_t\|^2 + \varepsilon^4 \|X_t(A)\|^2 + \varepsilon^2 \|\nabla_t F_A\|^2 \\
&\quad + \varepsilon^2 \|d_A B_t\|^2 - 2 \left( \varepsilon^2 \ast X_t(A), d^*_A F_A - \varepsilon^2 \nabla_t B_t \right) - 2\varepsilon^2 \langle d^*_A F_A, \nabla_t B_t \rangle - 2\varepsilon^2 \langle \nabla_t F_A, d_A B_t \rangle, \\
\|d^*_A B_t\|^2 &\leq \|d^*_A F_A - \nabla_t B_t - \ast X_t(A)\|^2 + \frac{1}{\varepsilon^2} \|\nabla_t F_A - d_A B_t\|^2 \\
&\quad - \frac{1}{\varepsilon^2} \left( d^*_A F_A, \nabla_t B_t \right) - 2 \frac{1}{\varepsilon^2} \left( \nabla_t F_A, d_A B_t \right), \\
-2 \langle d^*_A F_A, \nabla_t B_t \rangle - 2 \langle \nabla_t F_A, d_A B_t \rangle &\leq 2 \langle F_A, [B_t \wedge B_t] \rangle.
\end{align*}
and since \( \nabla F_A - d_A B_t = 0 \) by the Bianchi identity, the lemma holds.

Thus, the \( L^p \)-norm of the curvatures \( F_{A'} + B_t^\nu dt \) is uniformly bounded for any \( p \) which satisfies the Sobolev’s condition \(-\frac{2}{p} < 1 - \frac{2}{3} \) and hence for \( p < 6 \). Therefore, by the weak Uhlenbeck compactness theorem (see [15] or theorem 7.1. in [18]), we can assume that \( A' + \Psi' dt \) converges weakly to a \( A^0 + \Psi^0 dt \) in \( W^{1,p} \) and hence also strongly in \( L^\infty \). In addition we have that, for \( B_t^0 := \partial_t A^0 - d_A^0 \Psi^0 \),

\[
F_{A^0} = 0, \quad d_A^0 B_t^0 = d_A^0 = 0, \quad \nabla F_t^0 B_t^0 \neq 0 \in \text{im} d_A^0;
\]

thus, \( A^0 + \Psi^0 dt \) satisfies the equations of a perturbed closed geodesic and therefore for any 1-form \( \alpha \) on \( P \times S^1 \) with \( \alpha(t) \in \Omega^1(\Sigma, \mathfrak{g}_P) \)

\[
\| \pi_{A^0}(\alpha) \|_{1,2,1} \leq c \| D^0(A^0)(\pi_{A^0}(\alpha), \psi_0) \|_{L^2}
\]

where \( \psi \) satisfies \(-2 * [\pi_{A^0}(\alpha) \wedge *B_t^0] - d_A^0 d_A^0 \psi_0 = 0 \). Then

\[
\| \pi_{A^0}(\alpha) \|_{1,2,1} \leq c \left[ \| \pi_{A^0}(2[\psi_0, B_t^0]) + d * X_t(A') \pi_{A^0}(\alpha) \right.
\]

\[
+ \nabla_t \psi' \nabla_t \pi_{A^0}(\alpha) + \frac{1}{\varepsilon_\nu^2} * [\pi_{A^0}(\alpha) \wedge *F_{A'}] \right]\|_{L^2}
\]

where we used

\[
\left\| \left[ \pi_{A^0}(\alpha) \wedge * \left( d_{\partial A^0} (d_{A^0} \circ d_{A^0})^{-1} (\nabla F_t^0 B_t^0 - *X_t(A^0)) - \frac{1}{\varepsilon_\nu^2} F_{A'} \right) \right] \right\|
\]

\[
\leq \left\| \pi_{A^0}(\alpha) \right\|_{L^\infty} \left\| \frac{1}{\varepsilon_\nu^2} d_{\partial A^0} F_{A'} - \nabla F_t^0 B_t^0 - *X_t(A^0) \right\|
\]

\[
+ \left\| \pi_{A^0}(\alpha) \right\|_{L^\infty} \left\| \nabla F_t^0 B_t^0 + *X_t(A^0) \right\|_{L^\infty} \| A' - A^0 \|
\]

\[
\leq \left\| \pi_{A^0}(\alpha) \right\|_{L^\infty} \left\| \frac{1}{\varepsilon_\nu^2} d_{\partial A^0} F_{A'} - \nabla F_t^0 B_t^0 - *X_t(A^0) \right\|
\]

\[
+ \left\| \pi_{A^0}(\alpha) \right\|_{L^\infty} \left\| \nabla F_t^0 B_t^0 + *X_t(A^0) - \nabla F_t^0 B_t^0 - *X_t(A^0) \right\|
\]

\[
\leq \frac{1}{2} \left\| \pi_{A^0}(\alpha) \right\|_{1,2,1}
\]

for \( \nu \) big enough. Therefore, analogously to the theorems 7.1 and 7.2 of [10], one can prove that

\[
\| \alpha \|_{2,2,\varepsilon} \leq c \left( \varepsilon_\nu \| D^\nu(A^0, \Psi')(\alpha, \psi) \|_{0,2,\varepsilon} + \| \pi_{A^0} D^\nu_1(A^0, \Psi')(\alpha, \psi) \|_{L^2} \right),
\]

\[
\| \alpha + \psi ds - \pi_{A^0}(\alpha) \|_{2,2,\varepsilon} \leq c \left( \varepsilon_\nu^2 \| D^\nu(A^0, \Psi')(\alpha, \psi) \|_{0,2,\varepsilon}
\]

\[
+ \varepsilon_\nu \| \pi_{A^0} D^\nu_1(A^0, \Psi')(\alpha, \psi) \|_{L^2} \right).
\]

Thus, we have a contradiction and hence we finished the proof of the lemma.

\[ \square \]

**Proof of theorem 13.7.** To prove this theorem we proceed as Dostoglou and Salamon did for the theorem 7.4 in [17]. The idea is to find a positive bounded function \( f(s) \) such that it satisfies

\[
f''(s) \geq \rho^2 f(s)
\]

(13.8)
for \( s \geq 1 \). Then, this implies that \( f \) has an exponential decay, because, since
\[
\partial_s \left( e^{-\rho s} (f'(s) + \rho f(s)) \right) = e^{-\rho s} \left( -\rho^2 f(s) + f''(s) \right) \geq 0,
\]
\( f'(s) + \rho f(s) < 0 \) (otherwise \( e^{-\rho s} (f'(s) + \rho f(s)) \) would be positive and increase; thus, since \( f(s) \) is bounded, \( e^{-\rho s} f(s) \) would decrease and hence \( f'(s) \) would increase. Therefore \( f(s) \) would be unbounded which is a contradiction.) and hence \( e^{\rho s} f(s) \) is decreasing. Therefore, if a function \( f \) satisfies (13.8), then
(13.9)
\[
f(s) \leq e^{-\rho(s-1)}c_1.
\]
with \( c_1 = f(1) \). By the a priori estimate (11.12) and the lemma B.1 for any \( \delta > 0 \)
(13.10)
\[
\|B_s\|_{L^\infty(\Sigma \times S^1)} + \varepsilon \|C\|_{L^\infty(\Sigma \times S^1)} \leq \delta
\]
holds for \( s \) sufficiently big. We define
(13.11)
\[
f(s) := \frac{1}{2} \int_0^1 \left( \|B_s(t, s)\|_{L^2(\Sigma)}^2 + \varepsilon^2 \|C(t, s)\|_{L^2(\Sigma)}^2 \right) dt;
\]
then, as we will show later,
\[
f''(s) = \frac{1}{2} \partial_s^2 \left( \|B_s\|_{L^2(\Sigma \times S^1)}^2 + \varepsilon^2 \|C\|_{L^2(\Sigma \times S^1)}^2 \right)
\]
\[
\geq \frac{1}{\varepsilon^2} \left( d_A^* d_A B_s + d_A d_A^* B_s * * [B_s, *F_A] \right)
\]
(13.12)
\[
- \nabla t \nabla s B_s - d * X_t(A) B_s - 2[C, B_t] \right)^2 + \frac{1}{\varepsilon^2} \left( d_A^* d_A C - \varepsilon^2 \nabla t \nabla s C + * [B_s \wedge *B_t] \right)^2.
\]
Next, for \( s \geq 1 \) and \( \delta \) sufficiently small we apply the lemma [13.1] for \( \alpha + \psi dt := B_s + C dt \) and thus,
\[
f(s) = \frac{1}{2} \left( \|B_s\|_{L^2(\Sigma \times S^1)}^2 + \varepsilon^2 \|C\|_{L^2(\Sigma \times S^1)}^2 \right)
\]
\[
\leq \varepsilon^2 \left( \frac{1}{\varepsilon^2} \left( d_A^* d_A B_s + d_A d_A^* B_s * * [B_s, *F_A] \right)
\]
\[
- \nabla t \nabla s B_s - d * X_t(A) B_s - 2[C, B_t] \right)^2 + \frac{2}{\varepsilon^2} \left( * [B_s \wedge *B_t] \right)^2
\]
\[
\leq ce^6 \left( \frac{1}{\varepsilon^2} d_A^* d_A C - \nabla t \nabla s C \right)^2.
\]
Therefore, \( \rho^2 f(s) \leq f''(s) \) for \( \rho > 0 \) small enough. Thus, by (13.8),
\[
\int_{S_1}^\infty \left( \|B_s\|_{L^2(\Sigma \times S^1)}^2 + \varepsilon^2 \|C\|_{L^2(\Sigma \times S^1)}^2 \right) ds \leq ce^{-\rho s}
\]
for \( S \geq 1 \).

**Proof of (13.12).** First, we consider the following two short computations that are consequence of the commutation formula (11.4) and of the Yang-Mills flow equation (11.2):
\[
\nabla t d_A C = d_A \nabla t C + [B_t, C] = \frac{1}{\varepsilon^2} d_A \nabla t d_A B_t + [B_t, C]
\]
(13.13)
\[
= \frac{1}{\varepsilon^2} d_A d_A^* \nabla t B_t + [B_t, C] = \frac{1}{\varepsilon^2} d_A d_A^* B_s + [B_t, C].
\]
\begin{align}
\tag{13.14}
d_A^* \nabla_t B_s &= \nabla_t d_A^* B_s + * [B_t \wedge * B_s] = \nabla_t d_A^* \nabla_t B_t + * [B_t \wedge * B_s] \\
&= \nabla_t \nabla_t d_A^* B_t + * [B_t \wedge * B_s] = \varepsilon^2 \nabla_t \nabla_t C + * [B_t \wedge * B_s],
\end{align}
for in both cases we use $[B_t \wedge * B_t] = 0$ and $[F_A, * F_A] = 0$. In the following we use the notation
\begin{align}
D_1 &:= \frac{1}{\varepsilon^2} d_A^* d_A B_s - \nabla_t \nabla_t B_s + \nabla_t d_A C - d * X_t(A) B_s - \frac{1}{\varepsilon^2} * [B_s, * F_A] + [B_t, * C], \\
D_2 &:= \varepsilon^2 \nabla_t \nabla_t C - d_A^* d_A C - * [B_s \wedge * B_t].
\end{align}

Next, we can compute the second derivative of $\|B_s\|^2 + ||C||^2$, i.e.
\begin{align}
\frac{1}{2} \partial_s^2 (\|B_s\|^2 + \varepsilon^2 ||C||^2) &= \|\nabla_s B_s\|^2 + \varepsilon^2 \|\nabla_s C\|^2 + \langle \nabla_s \nabla_s B_s, B_s \rangle + \frac{1}{\varepsilon^2} \langle \nabla_s \nabla_s B_t, B_t \rangle \\
&= \left\| \frac{1}{\varepsilon^2} \nabla_s d_A^* F_A - \nabla_t \nabla_t B_t - * \nabla_s X_t(A) \right\|^2 + \frac{1}{\varepsilon^2} \left\| \nabla_s d_A^* B_t \right\|^2 \\
&\quad - \left\langle \nabla_s \nabla_s \left( \frac{1}{\varepsilon^2} d_A^* F_A - \nabla_t B_t - * X_t(A) \right), B_s \right\rangle + \langle \nabla_s \nabla_s d_A^* B_t, C \rangle
\end{align}

where in the second step we use the Yang-Mills flow equation (11.2). Then by the commutation formula (11.4)
\begin{align}
&= \left\| \frac{1}{\varepsilon^2} d_A^* \nabla_s F_A - \nabla_t \nabla_s B_t - d * X_t(A) B_s - \frac{1}{\varepsilon^2} * [B_s, * F_A] - [C, B_t] \right\|^2 \\
&\quad + \frac{1}{\varepsilon^2} \left\| d_A^* \nabla_s B_t - * [B_s \wedge * B_t] \right\|^2 \\
&\quad - \left\langle \nabla_s \left( \frac{1}{\varepsilon^2} d_A^* \nabla_s F_A - \nabla_t \nabla_s B_t - d * X_t(A) B_s \right), B_s \right\rangle \\
&\quad - \left\langle \nabla_s \left( - \frac{1}{\varepsilon^2} * [B_s, * F_A] - [C, B_t] \right), B_s \right\rangle \\
&\quad + \langle \nabla_s (d_A^* \nabla_s B_t - * [B_s \wedge B_t]), C \rangle
\end{align}

and by the Bianchi identity
\begin{align}
&= \|D_1\|^2 + \frac{1}{\varepsilon^2} \|d_A^* \nabla_t B_s - d_A^* d_A C - * [B_s \wedge * B_t]\|^2 \\
&\quad - \left\langle \nabla_s \left( \frac{1}{\varepsilon^2} d_A^* d_A B_s - \nabla_t \nabla_t B_s + \nabla_t d_A C - d * X_t(A) B_s \right), B_s \right\rangle \\
&\quad - \left\langle \nabla_s \left( - \frac{1}{\varepsilon^2} * [B_s, * F_A] - [C, B_t] \right), B_s \right\rangle \\
&\quad + \langle \nabla_s (d_A^* \nabla_t B_s - d_A^* d_A C - * [B_s \wedge B_t]), C \rangle
\end{align}
in addition, if we apply (13.3) and (13.14), then
\begin{align}
&= \|D_1\|^2 + \frac{1}{\varepsilon^2} \|D_2\|^2 - \langle \nabla_s D_1, B_s \rangle + \langle \nabla_s D_2, C \rangle
\end{align}
if we permute the derivatives in $D_1$ and $D_2$ with $\nabla_s$ and we apply the partial integration, then

$$
= \|D_1\|^2 + \frac{1}{\varepsilon^2} \|D_2\|^2 - \langle \nabla_s B_s, D_1 \rangle + \langle \nabla_s C, D_2 \rangle
$$

$$
- \left\langle \left[ \nabla_s, \left( \frac{1}{\varepsilon^2} d_A d_A - \nabla_t \nabla_t + \frac{1}{\varepsilon^2} d_A d_A^* \right) \right] B_s, B_s \right\rangle
$$

$$
+ \left\langle \frac{1}{\varepsilon^2} * [B_s, * \nabla_s F_A] + 2 [C, \nabla_s B_t] + d^2 * X_t(A)[B_s, B_s], B_s \right\rangle
$$

$$
+ \left\langle \nabla_s, (\varepsilon^2 \nabla_t - d^*_A d_A) \right\rangle C, C \right\rangle - 2 (\ast [B_s \wedge * \nabla_s B_s], C)
$$

The last three lines can be estimates by

$$
\left\langle \left[ \nabla_s, \frac{1}{\varepsilon^2} d_A d_A^* \right] B_s, B_s \right\rangle = \frac{1}{\varepsilon^2} \|B_s \| \cdot \|d_A B_s\| \leq \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2 + \delta \|B_s\|^2,
$$

$$
|\langle \nabla_s, (\frac{1}{\varepsilon^2} d_A d_A) B_s, B_s \rangle| = \|\langle C, \nabla_t B_s \rangle, B_s \rangle + \langle \nabla_t (C, B_s), B_s \rangle
$$

$$
= 2 \|\langle C, \nabla_t B_s \rangle, B_s \rangle| \leq c_\delta \|C\| \cdot \|B_s\| \leq c_\delta \|C\|^2 + \delta \|B_s\|^2,
$$

$$
\left\langle \left[ \nabla_s, \frac{1}{\varepsilon^2} d_A d_A^* \right] B_s, B_s \right\rangle = 0,
$$

$$
\left\langle \left[ \nabla_s, \frac{1}{\varepsilon^2} d_A d_A^* \right] B_s, B_s \right\rangle \leq \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2 \cdot \|B_s\| \leq \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2 + \delta \|B_s\|^2,
$$

$$
|\langle 2 \langle C, \nabla_t B_s \rangle, B_s \rangle| \leq c_\delta \|d_A B_s\|^2 \cdot \|C\| \leq \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2 + \varepsilon^2 \delta \|C\|^2,
$$

$$
|\langle [\nabla_s, \frac{1}{\varepsilon^2} d_A d_A^* \rangle C, C \rangle| = 2 \|\langle C, d_A C \rangle, B_s \rangle| \leq c_\delta \|d_A C\|^2,
$$

$$
|\langle 2 * [B_s \wedge * \nabla_s B_s], C \rangle| \leq c_\delta \|C\| \cdot \|d_A B_s\| \leq c_\delta \|C\|^2 + \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2,
$$

$$
|\langle d^2 * X_t(A)[B_s, B_s], B_s \rangle| \leq c_\delta \|B_s\|^2;
$$

we can therefore conclude that

$$
\frac{1}{2} \partial^2_t (\|B_s\|^2 + \varepsilon^2 \|C\|^2) \geq 2 \|D_1\|^2 \geq \frac{2}{\varepsilon^2} \|D_2\|^2 - \frac{c_\delta}{\varepsilon^2} \|d_A B_s\|^2
$$

$$
- \delta \|B_s\|^2 + \varepsilon^2 \|C\|^2 + c_\delta \|d_A C\|
$$

$$
\geq \|D_1\|^2 + \frac{1}{\varepsilon^2} \|D_2\|^2
$$

where the last step follows from the lemma[13.2] and choosing $\delta$ and $\varepsilon_0$ small enough and thus, we concluded the proof of the identity[13.12]. □

We concluded therefore the proof of the theorem[13.1] □

Next, we use the notation of the section [11]

**Theorem 13.3.** We choose four constants $b, c_0 > 0$, $p, s_1 \geq 2$. There are three positive constants $c_0$, $c$ and $\rho$ such that the following holds. If a perturbed Yang-Mills flow $\Xi = A + \Psi dt + \Phi ds \in \mathcal{M}^e(\Xi_-, \Xi_+)$, with $\Xi_\pm = A_\pm + \Psi_\pm dt \in \text{Crit}_Y \mathcal{M}^e$
and $0 < \varepsilon < \varepsilon_0$, satisfies

$$\|\partial_t A - d_A \Phi\|_{L^4(\Sigma)} + \|\partial_s A - d_A \Phi\|_{L^\infty(\Sigma)} \leq c_0,$$

then

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} \left( \|B_s\|_{L^\infty(\Sigma)} + \varepsilon \|C\|_{L^\infty(\Sigma)} \right) \leq ce^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (N_1 + N_2 + N_3 + N_4) \leq c\varepsilon^2 e^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (\|\alpha\|_{L^\infty(\Sigma)} + \varepsilon \|\nabla \alpha\|_{L^\infty(\Sigma)}) \leq ce^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (\|d^*_A \alpha\|_{L^\infty(\Sigma)} + \|d_A \alpha\|_{L^\infty(\Sigma)}) \leq ce^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (\varepsilon \|\nabla \partial_t A\|_{L^p(\Sigma)} + \varepsilon^2 \|\nabla \partial_s A\|_{L^p(\Sigma)}) \leq ce^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (\varepsilon \|\nabla \nabla \partial_t A\|_{L^p(\Sigma)} + \varepsilon^2 \|\nabla \nabla \partial_s A\|_{L^p(\Sigma)}) \leq ce^{-\rho s_0},$$

$$\sup_{S^1 \times [s_0, \infty)} (\|F_A - F_{A_+}(s)\|_{L^\infty(\Sigma)} + \varepsilon \|\nabla \partial_t (F_A - F_{A_+}(s))\|_{L^p(\Sigma)}) \leq ce^{-\frac{1}{\varepsilon} e^{-\rho s_0}},$$

$$\sup_{S^1 \times [s_0, \infty)} \left( \varepsilon^2 \|\nabla \nabla \partial_t F_A\|_{L^p(\Sigma)} + \varepsilon^2 \|\nabla \nabla \partial_s (F_A - F_{A_+}(s))\|_{L^p(\Sigma)} \right) \leq ce^{-\frac{1}{\varepsilon} e^{-\rho s_0}},$$

$$\sup_{S^1 \times [s_0, \infty)} \left( \|B_t - B_t^+\|_{L^\infty} + \varepsilon \|\nabla \partial_t (B_t - B_t^+)\|_{L^p} \right) \leq ce^{-\rho s_0},$$

$$\sup_{S^1 \times [s_0, \infty)} \left( \varepsilon^2 \|\nabla \delta (B_t - B_t^+)\|_{L^p} \right) \leq ce^{-\rho s_0},$$

where $s_0 > s_1$ and, for $g \in G^2_0(\Sigma \times S^1 \times \mathbb{R})$ defined by $g^{-1} \partial_s g = \Phi$,

$$A_+(s) + \Psi_+(s) \, dt := g(s)^* (A_+ + \Psi_+ \, dt),$$

$$\alpha(s) + \phi(s) \, dt := (A(s) + \Psi(s)) \, dt - (A_+(s) + \Psi_+(s)) \, dt,$$

$$B^+_t(s) := \partial_t A_+(s) + d_A(s) \Psi_+(s).$$

**Proof.** By the estimate (11.12), the theorem (13.1) and the lemma (13.1) we know that there are two constants $\rho$ and $c$ such that

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} \left( \|B_s\|_{L^\infty(\Sigma)} + \varepsilon \|C\|_{L^\infty(\Sigma)} \right) \leq ce^{-\rho s_0},$$

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (N_1 + N_2 + N_3 + N_4) \leq c\varepsilon^2 e^{-\rho s_0}.$$

Thus, if we integrate the first estimate of (13.27) we have

$$\sup_{(t,s) \in S^1 \times [s_0, \infty)} (\|\alpha\|_{L^\infty(\Sigma)} + \varepsilon \|\psi\|_{L^\infty(\Sigma)}) \leq ce^{-\rho s_0}. $$
and if we pick $s_2 \in [s_0, \infty)$, then the third estimate of the theorem follows from the computation

$$
\varepsilon \| \nabla_t \alpha(s_2) \|_{L^\infty(\Sigma)} \leq \int_{s_0}^{s_2} \varepsilon \| \nabla_s \nabla_t (A(s) - A_+(s)) \|_{L^\infty(\Sigma)} ds
$$

$$
\leq \int_{s_0}^{s_2} \varepsilon \| C \|_{L^\infty} \| A(s) - A_+(s) \|_{L^\infty(\Sigma)} ds
$$

$$
+ \int_{s_0}^{s_2} \varepsilon \| \nabla_t \nabla_s (A(s) - g(s)^+ A_+) \|_{L^\infty(\Sigma)} ds
$$

$$
\leq ce^{-p \varepsilon^2} + \int_{s_0}^{s_2} \varepsilon \| \nabla_t \partial_s A(s) + [\Psi(s), A(s)]
$$

$$
- \Phi(s), g(s)^+ A_+ - d_{g(s)^+ A_+} \Phi(s) \|_{L^\infty(\Sigma)} ds
$$

$$
\leq ce^{-p \varepsilon^2} + \int_{s_0}^{s_2} \varepsilon \| \nabla_t B_s \|_{L^\infty(\Sigma)} ds
$$

$$
\leq ce^{-p \varepsilon^2} + c \int_{s_0}^{s_2} (\varepsilon \| \nabla_t B_s \|_{L^2(\Sigma)} + \varepsilon \| d_A^* d_A \nabla_t B_s \|_{L^2(\Sigma)}) ds
$$

$$
+ \int_{s_0}^{s_2} \varepsilon \| d_A d_A^* \nabla_t B_s \|_{L^2(\Sigma)} ds
$$

$$
\leq ce^{-p \varepsilon^2}
$$

where the constant $c$ does not depend on $(t, s_2)$ for $(t, s_2) \in S^1 \times [s_0, \infty)$. The second step of the computation follows from the commutation formula (11.4), the third by the definition of $g(s)$ and the previous estimates, the fifth by the lemma B.1 and the last one by (13.27). The estimates (13.19)-(13.22) follows in the same way. Next we prove the first part of (13.23). By (11.11)

$$
\sup_{(t,s) \in S^1 \times [s_0, \infty)} \| F_A \|_{L^\infty(\Sigma)} \leq c \varepsilon^2
$$

and by the Bianchi identity $d_A B_s = \nabla_s A$ and by the lemma B.1

$$
\sup_{(t,s) \in S^1 \times [s_0, \infty)} \| \nabla_s F_A \|_{L^\infty(\Sigma)} = c \sup_{(t,s) \in S^1 \times [s_0, \infty)} \| d_A d_A^* d_A B_s \|_{L^2(\Sigma)} \leq c \varepsilon e^{-p \varepsilon^2}.
$$

Thus, integrating the last estimate, $\sup_{(t,s) \in S^1 \times [s_0, \infty)} \| F_A - F_{A_+} \|_{L^\infty(\Sigma)} \leq c \varepsilon e^{-p \varepsilon^2}$ and hence

$$
\sup_{(t,s) \in S^1 \times [s_0, \infty)} \| F_A - F_{A_+} \|_{L^\infty(\Sigma)}^p
$$

$$
\leq c \varepsilon e^{-p \varepsilon^2} \sup_{(t,s) \in S^1 \times [s_0, \infty)} \left( \| F_A \|_{L^\infty(\Sigma)} + \| F_{A_+} \|_{L^\infty(\Sigma)} \right)^{p-1}
$$

$$
\leq c \varepsilon^{2p-1} e^{-p \varepsilon^2}
$$

and finally we obtain

$$
\sup_{(t,s) \in S^1 \times [s_0, \infty)} \| F_A - F_{A_+} \|_{L^\infty(\Sigma)} \leq c \varepsilon^{2p} e^{-p \varepsilon^2}.
$$

The other estimates of (13.23)-(13.26) follow in the same way using the Bianchi identity, the Yang-Mills equation (11.2) in order to commute the operators and the estimates (13.27).
14. Relative Coulomb gauge

Theorem 14.1. Assume $q \geq p > 2$, $q > 4$ and $qp/(q-p) > 4$. We choose $\Xi_0 = A_0 + \Psi_0 dt + \Phi_0 ds \in A^{1,p}(\mathbb{R},\Xi_+)\times \mathbb{R}$ such that $F_{A_0} = 0$. Then for every constant $c_0 > 0$ there exist constants $\delta > 0$ and $c > 0$ such that the following holds for $0 < \varepsilon \leq 1$. If $\Xi \in A^{1,p}(\mathbb{R},\Xi_+)$ satisfies

\begin{equation}
\varepsilon^2 \left\| d^{\ast}_{\varepsilon_0} (\Xi - K_{2}^{\ast}(\Xi_0)) \right\|_{L^p} \leq c_0 \varepsilon^\frac{3}{p}, \quad \left\| \Xi - K_{2}^{\ast}(\Xi_0) \right\|_{1,q,c} \leq \delta \varepsilon^\frac{3}{p},
\end{equation}

then there exists a gauge transformation $g \in \mathcal{C}^{2,p}_0(\mathbb{R}\times S^1 \times \mathbb{R})$ such that

\begin{equation}
d^{\ast}_{\varepsilon_0} (g^\ast \Xi - K_{2}^{\ast}(\Xi_0)) = 0
\end{equation}

and

\begin{equation}
\left\| g^\ast \Xi - \Xi \right\|_{1,p,c} \leq c_0 \varepsilon^2 \left( 1 + \varepsilon^{-\frac{2}{p}} \left\| \Xi - K_{2}^{\ast}(\Xi_0) \right\|_{1,p,c} \right) \left\| d^{\ast}_{\varepsilon_0} (\Xi - K_{2}^{\ast}(\Xi_0)) \right\|_{L^p}.
\end{equation}

This last theorem is analogous to the proposition 6.2 in [7], but if we compare them, we will remark some differences. The first one is given by different rescaling in the $s$ direction induced by the equations (we have an $\varepsilon^2$ factor instead of $\varepsilon$) and this also induces a difference in the Sobolev’s estimates and it causes the change in some exponents: $\frac{3}{2}$, $\frac{3}{p}$ and $-\frac{2}{p}$ become $\frac{1}{2}$, $\frac{3}{p}$, and $-\frac{1}{2}$. The second difference is an extra $\varepsilon^2$ factor in the first estimate of (14.1) and in (14.2); this is given by the difference in the definitions of $d^{\ast}_{\varepsilon_0}$. In [7] it is defined by, with $\Xi_0 =: A + \Psi dt + \Phi ds$,

\begin{equation}
d^{\ast}_{\varepsilon_0}(\alpha + \Psi dt + \Phi ds) = d^{\ast}_A \alpha - \varepsilon^2 \nabla \psi \nabla \phi,
\end{equation}

our definition is instead

\begin{equation}
\varepsilon^2 d^{\ast}_{\varepsilon_0}(\alpha + \Psi dt + \Phi ds) = d^{\ast}_A \alpha - \varepsilon^2 \nabla \psi \nabla \phi.
\end{equation}

The third difference is that we use the difference $\Xi - K_{2}^{\ast}(\Xi_0)$ instead of $\Xi - \Xi_0$ and this is needed in order to have finite norms.

For any $\sigma \in \mathbb{R}$ we define $\rho_\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ by $\rho_\sigma(t,s) = (t, s + \sigma)$.

Theorem 14.2. We choose $p > 10$ and $b > 0$. Let $\Xi_0 \in \mathcal{M}^b(\mathbb{R},\Xi_+)\times \mathbb{R} \in \text{Crit}^{b}_{\theta}(\mathbb{R})$ with index difference 1. Then there exist three positive constants $\varepsilon_0, \delta$ and $c$ such that the following holds. If $0 < \varepsilon < \varepsilon_0$ and $\Xi \in \mathcal{M}^b(T^{\varepsilon,b}(\mathbb{R}^{-1},\mathbb{R}^{b+1}(\mathbb{R}^+))$ such that

\begin{equation}
\| \Xi - K_{2}^{\ast}(\Xi_0) \|_{1,p,c} \leq \delta \varepsilon^2, \quad \varepsilon^2 \| \nabla \Xi \|_{0,p,c} \leq c \varepsilon^2
\end{equation}

then there exist $\sigma \in \mathbb{R}$ and $g \in \mathcal{C}^{2,p}_0(\mathbb{R}\times S^1 \times \mathbb{R})$ such that $\Xi^\varepsilon = g^\ast (\Xi \circ \rho_\sigma)$ satisfies

\begin{equation}
d^{\ast}_{\varepsilon_0}(\Xi^\varepsilon - K_{2}^{\ast}(\Xi_0)) = 0, \quad \Xi^\varepsilon - K_{2}^{\ast}(\Xi_0) \in \text{im} \left( D^\varepsilon(K_{2}^{\ast}(\mathbb{R})) \right)^\ast
\end{equation}

and

\begin{equation}
\| \Xi^\varepsilon - K_{2}^{\ast}(\Xi_0) \|_{1,p,c} \leq c \| \Xi - K_{2}^{\ast}(\Xi_0) \|_{1,p,c}.
\end{equation}

Furthermore, for $\Xi^\varepsilon - K_{2}^{\ast}(\Xi_0) := \alpha^\varepsilon + \psi^\varepsilon dt + \phi^\varepsilon ds$ and $\Xi^\varepsilon - K_{2}^{\ast}(\Xi_0) := \alpha + \psi dt + \phi ds$, then

\begin{equation}
\| \nabla \alpha^\varepsilon \|_{L^p} \leq \| \nabla \alpha \|_{L^p} + c \| \Xi - K_{2}^{\ast}(\Xi_0) \|_{1,p,c},
\end{equation}

\begin{equation}
\| \nabla \phi^\varepsilon \|_{L^p} \leq \| \nabla \phi \|_{L^p} + c \| \Xi - K_{2}^{\ast}(\Xi_0) \|_{1,p,c},
\end{equation}

\begin{equation}
\| \nabla \psi^\varepsilon \|_{L^p} \leq \| \nabla \psi \|_{L^p} + c \| \Xi - K_{2}^{\ast}(\Xi_0) \|_{1,p,c}.
\end{equation}
Proof. The lemma follows exactly as the lemma 6.6 in [7] using the estimate (9.5).

(14.11) \[ \|\nabla s\phi\|_{L^p} \leq \|\nabla s\phi\|_{L^p} + c \|\nabla - K_2^\varepsilon(\Xi_0)\|_{1,p,\varepsilon}, \]

(14.12) \[ \|\nabla_t\phi\|_{L^p} \leq \|\nabla_t\phi\|_{L^p} + c \|\nabla - K_2^\varepsilon(\Xi_0)\|_{1,p,\varepsilon}, \]

(14.13) \[ \|\nabla s\phi\|_{L^p} \leq \|\nabla s\phi\|_{L^p} + c \|\nabla - K_2^\varepsilon(\Xi_0)\|_{1,p,\varepsilon}. \]

In order to prove the theorem [14.4] we need the following lemma

Lemma 14.3. Assume \( q \geq p > 2, \ q > 4, \) and \( pq/(q-p) > 4. \) Given \( c_0 > 0 \) there exists a constant \( c > 0 \) such that, if \( \|\eta\|_{L^\infty} \leq c_0 \) and \( g = \exp(\eta), \) then

(14.14) \[ \varepsilon^2 \left\| d^\varepsilon_{\Xi_0}(g^\varepsilon(\Xi - d\Xi_\eta)) \right\|_{L^p} \leq c_{\varepsilon^{-\frac{q}{2}}} \left( \|\eta\|_{1,q,\varepsilon} + \|\Xi - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon} + \varepsilon^2 \right) \|\eta\|_{2,p,\varepsilon} + c_{\varepsilon^{-\frac{q}{2}}} \left( \varepsilon^2 \left\| d^\varepsilon_{\Xi_0}(\Xi - K_2^\varepsilon(\Xi_0)) \right\|_{L^p} + \varepsilon^2 \right) \|\eta\|_{1,q,\varepsilon}, \]

and if \( \|\eta\|_{1,q,\varepsilon} + \|\Xi - \Xi_0\|_{0,q,\varepsilon} \leq c_0 \varepsilon^{\frac{q}{2}}, \) then

(14.15) \[ \|g^\varepsilon(\Xi - \Xi_0)\|_{0,q,\varepsilon} \leq c \left( \|g\|_{2,p,\varepsilon} + \varepsilon^{-\frac{q}{2}} \|\Xi - \Xi_0\|_{1,p,\varepsilon} \right). \]

Proof. The lemma follows exactly as the lemma 6.6 in [7] using the estimate (9.5).

Proof of theorem [14.4] We choose \( \Xi_1 = \Xi \) and we define the sequence \( \Xi_{\nu}, \) for \( \nu \geq 2, \)

\[ \Xi_{\nu+1} = g^\varepsilon_{\nu} \Xi_{\nu}, \quad g^\varepsilon_{\nu} = \exp(\eta_{\nu}), \quad d^\varepsilon_{\Xi_0}(d^\varepsilon_{\Xi_0}\eta_{\nu} + \Xi_{\nu} - K_2^\varepsilon(\Xi_0)) = 0, \]

by the definition of \( \eta_{\nu} \) and the lemma 6.4 in [7] and the Sobolev theorem 5.1 we have that

(14.17) \[ \|\eta_{\nu}\|_{2,p,\varepsilon} + \varepsilon^{\frac{q}{2}-\frac{q}{4}} \|\eta_{\nu}\|_{1,q,\varepsilon} \leq c_1 \varepsilon^2 \left\| d^\varepsilon_{\Xi_0}(\Xi - K_2^\varepsilon(\Xi_0)) \right\|_{L^p}, \]

(14.18) \[ \|\eta_{\nu}\|_{1,q,\varepsilon} \leq c_1 \|\Xi_{\nu} - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon}. \]

In order to conclude the proof of the theorem we need first to show by induction that there are three positive constants \( c_2, \) \( c_3 \) and \( c_4 \) such that the following estimates hold.

(14.19) \[ \|\Xi_{\nu} - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon} \leq c_2 \|\Xi - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon} + c_2 \|\Pi_{\text{im}} d_{\Xi_0}(\Xi - K_2^\varepsilon(\Xi_0))\|_{0,p,\varepsilon}, \]

(14.20) \[ \varepsilon^2 \left\| d^\varepsilon_{\Xi_0}(\Xi_{\nu} - K_2^\varepsilon(\Xi_0)) \right\|_{L^p} \leq c_3 \varepsilon^{2-\frac{q}{2}} \|\Xi_{\nu-1} - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon} \left\| d^\varepsilon_{\Xi_0}(\Xi_{\nu-1} - K_2^\varepsilon(\Xi_0)) \right\|_{L^p} + c_3 \varepsilon^2 \|\eta_{\nu-1}\|_{2,p,\varepsilon}, \]

(14.21) \[ \left\| d^\varepsilon_{\Xi_0}(\Xi_{\nu} - K_2^\varepsilon(\Xi_0)) \right\|_{L^p} \leq 2^{1-\nu} \left\| d^\varepsilon_{\Xi_0}(\Xi - K_2^\varepsilon(\Xi_0)) \right\|_{L^p}, \]

(14.22) \[ \|\eta_{\nu}\|_{1,q,\varepsilon} \leq c_4 \varepsilon^{-\nu} \left( \|\Xi - K_2^\varepsilon(\Xi_0)\|_{0,q,\varepsilon} + \|\Pi_{\text{im}} d_{\Xi_0}(\Xi - K_2^\varepsilon(\Xi_0))\|_{0,p,\varepsilon} \right). \]
For \( \nu = 1 \) (14.19) and (14.21) are satisfied by definition and with \( c_2 \geq 1 \). (14.18) implies (14.22) for \( c_4 \geq 2c_1 \) and (14.20) is empty. Next, we consider \( \nu \geq 2 \). By the assumptions of the theorem and by (14.18), for \( \delta \) small enough, we have that
\[
\| \eta_j \|_{1,q,\nu} + \| \Xi - K_j^\nu(\Xi_0) \|_{0,q,\nu} \leq \varepsilon^{\frac{1}{2}}, \quad j = 1, \ldots, \nu - 1.
\]
By lemma (14.13) and (14.22),
\[
\| \Xi_{j+1} - \Xi_j \|_{0,q,\nu} \leq c_5 \| \eta_j \|_{1,q,\nu}
\]
and thus we have (14.19). Next,
\[
d^\nu_{\Xi_0}(\Xi_{\nu+1} - K_j^\nu(\Xi_0)) = d^\nu_{\Xi_0}(g_{\nu}^\ast \Xi_{\nu} - \Xi_{\nu} - d_{\Xi_{\nu}} \eta_{\nu})
\]
\[
= d^\nu_{\Xi_0}(g_{\nu}^\ast \Xi_{\nu} - \Xi_{\nu} - d_{\Xi_{\nu}} \eta_{\nu}) + [d^\nu_{\Xi_0}(\Xi_{\nu} - K_j^\nu(\Xi_0)) \land \eta_{\nu}]
\]
\[
+ [d^\nu_{\Xi_0}(K_j^\nu(\Xi_0) - \Xi_0) \land \eta_{\nu}] - *_{\epsilon} [d^\nu_{\Xi_0}(\Xi_{\nu} - K_j^\nu(\Xi_0)) \land d_{\Xi_{\nu}} \eta_{\nu}]
\]
and hence by the lemma (14.17) and by (14.18), (14.20), we can conclude (14.20).

By (14.17) and (14.20) we get
\[
\| \eta_{\nu} \|_{2,p,\epsilon} + \varepsilon^{\frac{1}{2}} \| \eta_{\nu} \|_{1,q,\nu}
\]
and hence
\[
\| \eta_{\nu} \|_{2,p,\epsilon} + \varepsilon^{\frac{1}{2}} \| \eta_{\nu} \|_{1,q,\nu}
\]
\[
\leq c_1 c_3 \varepsilon^{2 - \frac{1}{2}} \| \Xi_{\nu-1} - K_j^\nu(\Xi_0) \|_{0,q,\nu} \| d^\nu_{\Xi_0}(\Xi_{\nu-1} - K_j^\nu(\Xi_0)) \|_{L^p}
\]
\[
+ c_2 c_3 \varepsilon^{2 - \frac{1}{2}} \| d_{\Xi_{\nu}} \eta_{\nu} \|_{0,p,\nu}
\]
and hence
\[
\| \eta_{\nu} \|_{2,p,\epsilon} + \varepsilon^{\frac{1}{2}} \| \eta_{\nu} \|_{1,q,\nu}
\]
\[
\leq c_1 c_3 \varepsilon^{2 - \frac{1}{2}} \| \Xi_{\nu-1} - K_j^\nu(\Xi_0) \|_{0,q,\nu} \| d^\nu_{\Xi_0}(\Xi_{\nu-1} - K_j^\nu(\Xi_0)) \|_{L^p}
\]
\[
+ c_2 c_3 \varepsilon^{2 - \frac{1}{2}} \| d_{\Xi_{\nu}} \eta_{\nu} \|_{0,p,\nu}
\]
and
\[
\varepsilon^2 \| d^\nu_{\Xi_0}(\Xi_{\nu} - K_j^\nu(\Xi_0)) \|_{L^p}
\]
\[
\leq 2 c_3 \varepsilon^{2 - \frac{1}{2}} \| \Xi_{\nu-1} - K_j^\nu(\Xi_0) \|_{0,q,\nu} \| d^\nu_{\Xi_0}(\Xi_{\nu-1} - K_j^\nu(\Xi_0)) \|_{L^p}
\]
\[
+ c_2 c_3 \varepsilon^{2 - \frac{1}{2}} \| d_{\Xi_{\nu}} \eta_{\nu} \|_{0,p,\nu}.
\]

By (14.18) and (14.19)
\[
\| \eta_2 \|_{1,q,\nu} \leq c_0 c_3 \| \Xi - K_j^\nu(\Xi_0) \|_{0,q,\nu}
\]
using (14.23)
\[
\| \eta_{\nu} \|_{1,q,\nu} \leq 16 c_1 c_2 c_3 \delta^{2 - \nu} \| \Xi_{\nu-1} - K_j^\nu(\Xi_0) \|_{0,q,\nu}
\]
\[
+ c_2 c_3 \varepsilon^{2 - \frac{1}{2}} \| d_{\Xi_{\nu}}(\Xi_{\nu} - \Xi_0) \|_{L^p}.
\]
(14.22) therefore holds for \( c_4 = 1 \) whenever \( \delta \) and \( \varepsilon \) are small enough. The lemma (14.13) with (14.17) and (14.18) implies
\[
\| \Xi_{\nu+1} - \Xi_0 \|_{1,p,\nu} \leq c_0 \varepsilon^2 \left( 1 + \varepsilon^{-\frac{1}{2}} \| \Xi_{\nu} - K_j^\nu(\Xi_0) \|_{1,p,\nu} \right) \| d^\nu_{\Xi_0}(\Xi_{\nu} - \Xi_0) \|_{L^p}
\]
and thus, for \( \delta \) sufficiently small,
\[
\| \Xi_{\nu} - \Xi_0 \|_{1,p,\nu} \leq \varepsilon^{\frac{3}{2}} + 2 \| \Xi - K_j^\nu(\Xi_0) \|_{1,p,\nu}.
\]
The sequence converges therefore in $W^{1,p}$ to a connection $\Xi_\varepsilon$ which satisfies the condition $d_{\varepsilon_0}^\ast (\Xi_\varepsilon - \Xi_0) = 0$ and the estimate (14.2). In addition, the sequence $h_\nu := g_1 g_2 \cdots g_\nu$ satisfies $h_\nu \Xi = \Xi_\varepsilon$ and converges in $G^2_r (P \times S^1 \times \mathbb{R})$ to a gauge transformation $g$ which satisfies $g^* \Xi = \Xi_\varepsilon$. □

**Proof of theorem 14.2.** We follow the proof of the proposition 6.3 in [7] adapting it first, we consider the estimates

\begin{equation}
(14.24) \quad \|\Xi \circ \tau_\sigma - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} \leq \|\Xi - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} + \|K^2_\sigma (\Xi_0) \circ \tau_\sigma - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon}
\end{equation}

\begin{equation}
(14.25) \quad \|\nabla_s (\Xi \circ \tau_\sigma - K^2_\sigma (\Xi_0))\|_{0,p,\varepsilon} \leq \|\nabla_s (\Xi - K^2_\sigma (\Xi_0))\|_{0,p,\varepsilon} + c|\sigma| \cdot \|\Xi - K^2_\sigma (\Xi_0)\|_{L^p}
\end{equation}

where, by the definitions of the section 9

\begin{equation}
(14.26) \quad \|\partial_s K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} \leq \|\partial_s \Xi_0\|_{1,p,\varepsilon} + \|\nabla_s (K^2_\sigma (\Xi_0) - \Xi_0)\|_{1,p,\varepsilon}
\end{equation}

\begin{equation}
(14.27) \quad \|\partial_s \nabla_s K^2_\sigma (\Xi_0)\|_{0,p,\varepsilon} \leq \|\partial_s \nabla_s \Xi_0\|_{0,p,\varepsilon} + \|\nabla_s \nabla_s (K^2_\sigma (\Xi_0) - \Xi_0)\|_{L^p}
\end{equation}

Therefore for $|\sigma| \leq \delta_0^{\frac{3}{2}}$, $\|\Xi \circ \tau_\sigma - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} \leq \delta_0^{\frac{3}{2}}$ for $\varepsilon$ small enough, and thus by theorem 14.1 there is a gauge transformation $g_\sigma \in G^2_r (P \times S^1 \times \mathbb{R})$ such that for $\Xi_\sigma = g_\sigma (\Xi \circ \tau_\sigma)$, $d_{\varepsilon_0}^\ast (\Xi_\sigma - K^2_\sigma (\Xi_0)) = 0$ and

\begin{equation}
\|\Xi_\sigma - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} \leq c_1 \left( |\sigma| + \|\Xi - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon} \right).
\end{equation}

We assume that $g_0 = 1$. We need to show that there is a $\sigma$ such that $\Xi_\sigma - K^2_\sigma (\Xi_0) \in \text{im} \ D^r (K^2_\sigma (\Xi_0))^\ast$ and $|\sigma| \leq c_2 \|\Xi - K^2_\sigma (\Xi_0)\|_{1,p,\varepsilon}$. The estimates (14.24, 14.25) follows as (14.24) and (14.25) computing separately every component.

$D^0 (\Xi_0)$ is onto and has index 1; therefore its kernel is spanned by $\xi_0 = \partial_3 \Xi_0 \in W^{1,p}$. Then $D^r (K^2_\sigma (\Xi_0))$ has index 1 with the kernel spanned by

$\xi_\varepsilon = \xi_0 - D^r (K^2_\sigma (\Xi_0))^\ast (D^r (K^2_\sigma (\Xi_0)) D^r (K^2_\sigma (\Xi_0))^\ast)^{-1} D^r (K^2_\sigma (\Xi_0)) \xi_0$.

Consider now the function $\theta (\sigma) = \langle \xi_\varepsilon, \Xi_\sigma - K^2_\sigma (\Xi_0) \rangle_\varepsilon$; thus, if $\theta (\sigma) = 0$, then $\Xi_\sigma - K^2_\sigma (\Xi_0) \in \text{im} \ D^r (K^2_\sigma (\Xi_0))^\ast$. We assume that there are positive constants $\delta_0$,
\( \varepsilon_0 \) and \( \rho_0 \) such that, for \( 0 < \varepsilon < \varepsilon_0 \),

\[
(14.26) \quad |\sigma| + \|\Xi - \mathcal{K}_2^*(\Xi_0)\|_{1,p,\varepsilon} \leq \delta_0 \varepsilon^{1-\frac{4}{p}} \quad \Rightarrow \quad \theta'(\sigma) \geq \rho_0.
\]

Then the existence of a zero for \( \theta(\sigma) \) follows from

\[
|\theta(0)| = |\langle \xi, \Xi - \mathcal{K}_2^*(\Xi_0) \rangle| \leq \|\xi\|_{0,q,\varepsilon} \|\Xi - \mathcal{K}_2^*(\Xi_0)\|_{0,p,\varepsilon} \leq c_3 \delta \varepsilon^{1-\frac{4}{p}}
\]

where \( q = \frac{p}{p-1} \). In fact, if \( c_3 \delta < \frac{1}{2} \delta_0 \rho_0 \), \( \delta \leq \frac{1}{2} \delta_0 \), then \( \|\Xi - \mathcal{K}_2^*(\Xi_0)\|_{1,p,\varepsilon} \leq \frac{1}{2} \delta_0 \varepsilon^{1-\frac{4}{p}} \).

Therefore, by (14.26), there is a \( \sigma \in \mathbb{R} \) with \( |\sigma| \leq \frac{\theta(0)}{\rho_0} \leq \frac{1}{2} \delta_0 \varepsilon^{1-\frac{4}{p}} \) such that \( \theta(\sigma) = 0 \). For this \( \sigma \), we have

\[
|\sigma| \leq c \|\Xi - \mathcal{K}_2^*(\Xi_0)\|_{1,p,\varepsilon}, \quad \Xi_0 - \mathcal{K}_2^*(\Xi_0) \in \text{im} \mathcal{D}'(\mathcal{K}_2^*(\Xi_0))^*.
\]

Thus, in order to finish the proof of the theorem we need only to show (14.26).

**Proof of (14.26).** We define \( \eta_\sigma = g_\sigma^{-1}(\partial_\sigma g_\sigma - \partial_\delta g_\sigma) \), then

\[
(14.27) \quad \theta'(\sigma) = \langle \xi, \partial_\sigma \Xi_\sigma + d_{\Xi_\sigma} \eta_\sigma \rangle \varepsilon, \quad \partial_\sigma d_{\Xi_\sigma}^* (\Xi_\sigma - \mathcal{K}_2^*(\Xi_0)) = 0.
\]

Thus,

\[
(14.28) \quad d_{\Xi_\sigma}^* \partial_\delta \Xi_\sigma + d_{\Xi_\sigma}^* d_{\Xi_\sigma} \eta_\sigma + d_{\Xi_\sigma}^* [\langle \Xi_\sigma - \Xi_0 \rangle \wedge \eta_\sigma] = 0.
\]

If \( \varepsilon^{-\frac{4}{p}} \|\Xi_\sigma - \mathcal{K}_2^*(\Xi_0)\|_{1,p,\varepsilon} \) and \( \|\mathcal{K}_2^*(\Xi_0) - \Xi_0\|_{1,\infty,\varepsilon} \) are sufficiently small, then there is a unique \( \eta_\sigma \) which satisfies (14.28), furthermore

\[
\|\eta_\sigma\|_{1,p,\varepsilon} \leq c \|\partial_\delta \Xi_\sigma\|_{0,p,\varepsilon} \leq c \left( 1 + \|\partial_\varepsilon (\Xi_\sigma - \Xi_0)\|_{0,p,\varepsilon} \right)
\]

\[
(14.29) \quad \leq c \left( 1 + \|\nabla_\varepsilon (\Xi_\sigma - \mathcal{K}_2^*(\Xi_0))\|_{0,p,\varepsilon} + \|\Xi_\sigma - \mathcal{K}_2^*(\Xi_0)\|_{0,p,\varepsilon} \right).
\]

where the last step follows from the definition of \( \mathcal{K}_2^* \) and the estimate (9.5). Since \( d_{\Xi_\sigma}^* \xi_\varepsilon = 0 \), we have

\[
|\langle \xi_\varepsilon, d_\Xi_\sigma \eta_\sigma \rangle| = |\langle \xi_\varepsilon, [(\Xi_\sigma - \Xi_0) \wedge \eta_\sigma] \rangle| \leq c \|\Xi_\sigma - \Xi_0\|_{\infty,\varepsilon} \|\eta_\sigma\|_{0,p,\varepsilon}
\]

\[
\leq c \varepsilon^{-\frac{4}{p}} \|\Xi_\sigma - \Xi_0\|_{1,p,\varepsilon} \left( 1 + \|\nabla_\varepsilon (\Xi_\sigma - \mathcal{K}_2^*(\Xi_0))\|_{0,p,\varepsilon} \right)
\]

\[
+ c \varepsilon^{-\frac{4}{p}} \|\Xi_\sigma - \Xi_0\|_{1,p,\varepsilon} \|\Xi_\sigma - \mathcal{K}_2^*(\Xi_0)\|_{0,p,\varepsilon}
\]

\[
\leq c_\delta + c_\delta \varepsilon^{-\frac{4}{p}} \varepsilon^{1-\frac{4}{p}} \|\nabla_\varepsilon (\Xi_\sigma - \mathcal{K}_2^*(\Xi_0))\|_{0,p,\varepsilon}.
\]

where the second inequality follows from the Sobolev theorem 5.1 and from (14.29) and the last one is a consequence of the assumptions. Next we consider

\[
\langle \xi_\varepsilon, \partial_\varepsilon \Xi_\sigma \rangle = \langle \partial_\varepsilon \xi_\varepsilon, \mathcal{K}_2^*(\Xi_0) - \Xi_\sigma \rangle \varepsilon + \langle \xi_\varepsilon, \partial_\varepsilon \mathcal{K}_2^*(\Xi_0) \rangle \varepsilon,
\]

Since \( \|\partial_\varepsilon \Xi_0\|_{0,2,\varepsilon} \geq 3 \rho_0 > 0 \) for some \( \rho_0 \), \( \langle \xi_\varepsilon, \partial_\varepsilon \mathcal{K}_2^*(\Xi_0) \rangle \varepsilon \geq 2 \rho_0 \) by the definition of \( \xi_\varepsilon \) and thus \( \langle \xi_\varepsilon, \partial_\varepsilon \Xi_\sigma + d_\Xi_\sigma \eta_\sigma \rangle \varepsilon > \rho_0 \) for \( \delta_0 \) small enough. Hence, by (14.27) \( \theta'(\sigma) > \rho_0 \).
15. Surjectivity of $R^{\varepsilon,b}$

In this section we will show that the map $R^{\varepsilon,b}$ defined in the section 10 is also surjective.

**Theorem 15.1.** We assume that the energy functional $E^H$ is Morse-Smale and we choose $b > 0$ to be a regular value of $E^H$. Then there is a constant $\varepsilon_0 > 0$ such that the following holds. For every $\varepsilon \in (0, \varepsilon_0)$, every pair $\Xi_{\pm} := A_{\pm} + \Psi_{\pm} dt \in \text{Crit}^b_{E^H}$, the map

$$R^{\varepsilon,b} : M^0(\Xi_-, \Xi_+) \to M^\varepsilon(T^{\varepsilon,b}(\Xi_-), T^{\varepsilon,b}(\Xi_+))$$

is surjective.

**Proof.** We prove the theorem indirectly. We assume that there is a sequence $\Xi_{\nu} \in M^\varepsilon(T^{\varepsilon,b}(\Xi_-), T^{\varepsilon,b}(\Xi_+))$, $\varepsilon_{\nu} \to 0$, that is not in the image of $R^{\varepsilon_{\nu},b}$. Hence by the theorems 11.1 and 12.1, for a positive constant $c_0$,

$$
\|\partial_s A^\nu - d_{A^\nu} \Phi^\nu\|_{L^\infty(\Sigma)} + \varepsilon_{\nu} \| \partial_s \Psi - \partial_t \Phi - [\Psi, \Phi]\|_{L^\infty(\Sigma)} \leq c_0,
$$

and all the estimates of the theorem 13.3. The proof is structured in the following way. In the first step, see figure 15.1, we will define a sequence $\Xi_{\nu}^\prime$ which converges to $\Xi_{\pm}$ for $s \to \pm \infty$ and in the following step we will project it on the space $A_0(P)$ defining a new sequence $\Xi_{\nu}^\prime$; then by the implicit function theorem 15.3 there is a subsequence of $\Xi_{\nu}^\prime$ which converges to a geodesic flow (step 3). Finally, after choosing appropriate gauge transformations and time shifts (steps 4-6) we can show that the sequence satisfies the assumptions of the local uniqueness theorem 10.2 (step 7) and therefore a subsequence turns out to lie in the image of $R^{\varepsilon,b}$. This yields to a contradiction.

Furthermore, we assume that there is a positive $S_0$ such that $\Phi^\nu = 0$ for $|s| \geq S_0$. In the general case, the $\Phi^\nu$ converge to 0 for $|s| \to \infty$ in an exponential way and hence we can find a sequence $\{g_{\nu}\}_{\nu \in \mathbb{N}}$ of gauge transformations such that $g^*_{\nu} \Xi^\nu$ has the above property. First, we pick the sequence $\{g_{\nu}\}_{\nu \in \mathbb{N}}$ of gauge transformations defined by $g_{\nu}^{-1} \partial_s g_{\nu} := \Phi_{\nu}$ and we define

$$A^\nu_{\pm}(s) + \Psi^\nu_{\pm}(s) dt := g(s)^* (A^\nu_{\pm} + \Psi^\nu_{\pm} dt),$$

**Figure 3.** Idea of the proof of theorem 15.1.
We define $\Xi_0 := A_0^\nu + \Psi_0^\nu dt := \lim_{s \to \pm \infty} \Xi^\nu$. Second, like in the section 10 we choose a smooth positive function $\theta(s) = 0$ for $s \leq 1$ and $\theta(s) = 1$ for $s \geq 2$, such that $0 \leq \theta \leq 1$ and $0 \leq \partial_s \theta \leq c_0$ with $c_0 > 0$ and we define a family of 1-forms $\alpha^\nu_0 + \psi^\nu_0 dt$ as
\[
\Xi^\nu_0 = \alpha^\nu_0 + \psi^\nu_0 dt := \theta(-s) \left( (A^\nu_0 + \Psi^\nu_0 dt) - (T^0, b)^{-1} (A^\nu - \Psi^\nu dt) \right) + \theta(s) \left( (A^\nu_0 + \Psi^\nu_0 dt) - (T^0, b)^{-1} (A^\nu_0 + \Psi^\nu_0 dt) \right);
\]
(15.3)
in addition we denote
\[
\alpha^\nu + \psi^\nu dt + \phi^\nu ds := \theta(-s) \left( \Xi^\nu - \Xi^\nu_0 \right) + \theta(s) \left( \Xi^\nu - \Xi^\nu_0 \right)
\]
which satisfies the uniformly exponential convergence estimates of the theorem 13.3

**Step 1.** We define $\Xi^\nu_1 = A^\nu_1 + \Psi^\nu_1 dt + \Phi^\nu_1 ds := \Xi^\nu - \Xi^\nu_0$ then there is a constant $c > 0$ such that
\[
\| \partial_t A^\nu_1 - d_A \Psi^\nu_1 \|_{L^\infty(\Sigma)} + \| \partial_s A^\nu_1 - d_A \Psi^\nu_1 \|_{L^\infty(\Sigma)} \leq c
\]
(15.4)
and for $s \geq 0$
\[
\partial_s A^\nu_1 - d_A \Psi^\nu_1 - \nabla_t^\nu \left( \partial_t A^\nu - d_A \Psi^\nu \right) - \ast X_t(A^\nu)
\]
\[
= -\frac{1}{\varepsilon^2} d^\ast_{A^\nu} d_{A^\nu} - \partial_0^\ast \left( \begin{array}{c}
\psi^\nu_0, \\
\left( \partial_t A^\nu - d_A \Psi^\nu \right) - \left( \partial_t A^\nu_0 - d_A \Psi^\nu_0 \right) \end{array} \right)
\]
\[
- \frac{1}{\varepsilon^2} d^\ast_{A^\nu} \left( F^\nu - F^\nu_0 \right) + \frac{1}{\varepsilon^2} \ast \left[ \begin{array}{c}
\alpha^\nu, \\
\left( d_{A^\nu} - d_{A^\nu_0} \right) + \frac{1}{2} [\alpha^\nu_0 \wedge \alpha^\nu_0] \end{array} \right] + \ast \left[ \begin{array}{c}
\left( \partial_t A^\nu - d_A \Psi^\nu \right) + \nabla_t^\nu \left( \begin{array}{c}
\left( \partial_t A^\nu - d_A \Psi^\nu \right) - \left( \partial_t A^\nu_0 - d_A \Psi^\nu_0 \right) \end{array} \right) + \nabla_t^\nu \left( \begin{array}{c}
\left( \partial_t A^\nu - d_A \Psi^\nu \right) - \left( \partial_t A^\nu_0 - d_A \Psi^\nu_0 \right) \end{array} \right) \end{array} \right]
\]
where $\alpha^\nu_0(s) \in im d_{A^\nu_0(s)}$ are defined uniquely by
\[
d^\ast_{A^\nu} d_{A^\nu} \partial_0^\ast = \varepsilon^2 \nabla_t^\nu \left( \partial_t A^\nu - d_{A^\nu} \Psi^\nu \right).
\]

**Proof of step 1.** The estimates (15.3) and (15.4) follow from (15.2), from the inequalities of the theorem 1.4 and the remark 1.12 and the Sobolev theorem A.1
\[
\| \partial_s A^\nu_1 - d_A \Psi^\nu_1 \|_{L^\infty(\Sigma)} = \| \partial_s A^\nu - d_A \Psi^\nu \|_{L^\infty(\Sigma)} \leq c
\]

\[
\| \partial_t A^\nu_1 - d_A \Psi^\nu_1 \|_{L^\infty(\Sigma)} \leq \| \partial_t A^\nu - d_A \Psi^\nu \|_{L^\infty(\Sigma)}
\]
\[
+ \| \nabla_t^\nu \alpha^\nu_0 \|_{L^\infty(\Sigma)} + \| d_{A^\nu_0} \psi^\nu_0 \|_{L^\infty(\Sigma)} \leq c
\]

\[
\| F^\nu_1 \|_{L^\infty(\Sigma)} \leq \| F^\nu_1 \|_{L^\infty(\Sigma)} + \| d_{A^\nu_0} \alpha^\nu_0 \|_{L^\infty(\Sigma)} + c \| \alpha^\nu_0 \|_{L^\infty(\Sigma)} \leq c
\]

\[
\| \nabla_t^\nu \left( \partial_t A^\nu_1 - d_A \Psi^\nu_1 \right) \|_{L^p(\Sigma)} \leq \| \nabla_t^\nu \left( \partial_t A^\nu - d_A \Psi^\nu \right) \|_{L^p(\Sigma)} + c
\]
\[
\leq \frac{1}{\varepsilon^2} \| d^\ast_{A^\nu} F^\nu_1 \|_{L^p(\Sigma)} + \| \partial_s A^\nu - d_A \Psi^\nu \|_{L^p(\Sigma)} + c \leq c
\]
where for the last estimate we use also the Yang-Mills flow equation \([1, 2]\). In order to prove the identity, we first remark that

\[
\partial_s A_1^\nu - d_{A_1^\nu} \Phi_1^\nu - \nabla_1^\nu t (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) - \ast X_1(A_1^\nu) = \partial_s A_1^\nu - d_{A_1^\nu} \Phi_1^\nu - \nabla_1^\nu t (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) - \ast X_1(A_1^\nu) + X_1(A_1^\nu) + \frac{1}{2} \langle \psi_0^\nu, (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) \rangle + \nabla_1^\nu t ((\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) - (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu)) ;
\]

(15.6)

next, in order to simplify the exposition, we consider \(s \geq 0\), for a negative \(s\) the proof is analogous. Since \(\Sigma^\nu\) is a Yang-Mills flow, we have

\[
\partial_s A_1^\nu - d_{A_1^\nu} \Phi_1^\nu - \nabla_1^\nu t (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) = \ast X_1(A_1^\nu)
\]

Further, since

\[
\left[ \psi_0^\nu, \left( \partial A_1^\nu - d_{A_1^\nu} \Psi_1^\nu \right) \right] + \nabla_1^\nu t \left( \left( \partial A_1^\nu - d_{A_1^\nu} \Psi_1^\nu \right) - \left( \partial A_1^\nu - d_{A_1^\nu} \Psi_1^\nu \right) \right) = \partial A_1^\nu - d_{A_1^\nu} \Psi_1^\nu \eta_1 + \frac{1}{2} \left[ \psi_0^\nu, (\partial A_1^\eta - d_{A_1^\nu} \Psi_1^\nu) \right]
\]

\[
= \frac{1}{\varepsilon_0} d_{A_1^\nu} F_0^\nu - \ast X_1(A_1^\nu) + \frac{1}{\varepsilon_0} d_{A_1^\nu} \tilde{\alpha}_0^\nu
\]

where the last identity follows from the equations for the perturbed geodesics \([1, 3]\) and for the perturbed Yang-Mills connections \([1, 0]\). Thus,

\[
\partial_s A_1^\nu - d_{A_1^\nu} \Phi_1^\nu - \nabla_1^\nu t (\partial_t A_1^\nu - d_{A_1^\nu} \Psi_1^\nu) = \ast X_1(A_1^\nu)
\]

Next, the first step then follows directly using the identities

\[
d_{A_1^\nu} F_1^\nu = d_{A_1^\nu} (F_0^\nu - F_{A_1^\nu}) = - \frac{1}{\varepsilon_0} d_{A_1^\nu} \tilde{\alpha}_0^\nu + \frac{1}{2} \left[ \alpha_0^\nu, \ast \left( d_{A_1^\nu} \tilde{\alpha}_0^\nu + \frac{1}{2} \left[ \alpha_0^\nu, \alpha_0^\nu \right] \right) \right] + d_{A_1^\nu} F_0^\nu,
\]

\[
- \frac{1}{\varepsilon_0} d_{A_1^\nu} d_{A_1^\nu} \tilde{\alpha}_0^\nu = - \frac{1}{\varepsilon_0} d_{A_1^\nu} d_{A_1^\nu} \tilde{\alpha}_0^\nu - \frac{1}{\varepsilon_0} \ast \left[ \alpha_0^\nu, \ast d_{A_1^\nu} \tilde{\alpha}_0^\nu \right].
\]
Since $F_{A_1^s} + d_{A_1^s} \alpha_0^s + \frac{1}{2}[\alpha_0^s \wedge \alpha_0^s] = F_{A^s}$, $d_{A_+} \alpha_0^s + \frac{1}{2}[\alpha_0^s \wedge \alpha_0^s] = F_{A_+^s}$ for $s \geq 2$, we have
\[(15.7) \quad F_{A_1^s} = F_{A^s} - F_{A_+^s} - [\alpha^s \wedge \alpha_0^s], \quad \text{for } s \geq 2,\]
and thus we can estimate the norm of $F_{A_1^s}$, for any $q \geq 2$, by
\[(15.8) \quad \|F_{A_1^s}\|_{L^q(\Sigma)} \leq \|F_{A^s} - F_{A_+^s}\|_{L^q(\Sigma)} + \|\alpha^s \wedge \alpha_0^s\|_{L^q(\Sigma)}\]
for $s \geq 2$.

**Step 2.** There are two positive constants $c$ and $\delta$ such that the following holds. For every $\Xi^\nu$ there is a connection $\Xi_2^\nu := A_2^\nu + \Psi_2^\nu dt + \Phi_2^\nu ds = \Xi_1^\nu + \alpha_1^\nu + \psi_1^\nu dt + \phi_1^\nu ds$, $\alpha_1^\nu \in \im d_{A_1^s}$, which satisfies
\[i) \quad F_{A_2^s} = 0, \quad ii) \quad d_{A_2^s} (\partial_s A_2^\nu - d_{A_2^s} \Phi_2^\nu) = 0,\]
\[iii) \quad d_{A_2^s} (\partial_s A_2^\nu - d_{A_2^s} \Phi_2^\nu) = 0, \quad iv) \quad \|\alpha_1^s (s)\|_{L^\infty(\Sigma)} \leq c e^{(\theta(s) - \theta(s))\delta s \frac{2-\beta}{\nu}}\]
\[v) \quad \|\pi_{A_2^s} (F_0 (\Xi_2^\nu))\|_{L^p(\Sigma)} \leq e^{1-\frac{1}{\nu}}, \quad vi) \quad \|\pi_{A_2^s} (F_0 (\Xi_2^\nu))\|_{L^p(\Sigma)} \leq e^{1-\frac{1}{\nu}}\]
\[vii) \quad \lim_{s \to \pm \infty} \Xi_2^\nu = \Xi_1^\nu.\]

**Proof of step 2.** By the the first step we know that $\|F_{A_1^s}\|_{L^\infty} \leq c \epsilon^2$ and thus, for $\epsilon$ small enough, the lemma [B.2] allows us to find a positive constant $c$ such that for any $A_2^\nu$ there is a unique $0$-form $\gamma^\nu$ such that
\[(15.9) \quad F_{A_1^s} + d_{A_1^s} \gamma^\nu = 0, \quad \|d_{A_1^s} \gamma^\nu\|_{L^\infty(\Sigma)} \leq c \|F_{A_1^s}\|_{L^\infty(\Sigma)} \leq c \epsilon^2.\]
and $\Psi_2^\nu, \Phi_2^\nu$ are then uniquely given by
\[(15.10) \quad d_{A_2^s} (\partial_s A_2^\nu - d_{A_2^s} \Phi_2^\nu) = 0, \quad d_{A_2^s} (\partial_s A_2^\nu - d_{A_2^s} \Phi_2^\nu) = 0.\]
Therefore $i)$, $ii)$ and $iii)$ are satisfied. By (15.5), (15.9) one can remark that
\[\|\alpha_1^s (s)\|_{L^\infty(\Sigma)} \leq c e^{(\theta(s) - \theta(s))\delta s \frac{2-\beta}{\nu}}\]
using the a priori estimate of the theorem [13.3]. In order to show the inequalities $v)$ and $vi)$ of the second step, we consider
\[
\partial_s A_2^s - d_{A_2^s} \Phi_2^\nu - \nabla_t \Phi_2^\nu (\partial_s A_1^s - d_{A_2^s} \Psi_2^s) - \ast X_t (A_2^s)
\]
\[
= \partial_s A_1^s - d_{A_1^s} \Phi_1^s - \nabla_t \Phi_1^s (\partial_s A_1^s - d_{A_1^s} \Psi_1^s) - \ast X_t (A_1^s)
\]
\[
+ \nabla_t \Phi_2^s (\partial_s A_1^s - d_{A_2^s} \Phi_2^s) - \nabla_t \Phi_1^s (\partial_s A_1^s - d_{A_1^s} \Psi_1^s)
\]
\[
- \nabla_t \Phi_2^s (\partial_s A_1^s - d_{A_1^s} \Psi_1^s - [\alpha_1^s, \psi_1^s]) - (\ast X_t (A_2^s) - \ast X_t (A_1^s))\]
and we remark that
\[
\nabla_t \Phi_1^s (\partial_s A_1^s - d_{A_1^s} \Psi_1^s)
\]
\[
= d_{A_1^s} \frac{1}{2} \sigma_1^s \nabla_t \Phi_1^s + \ast \nabla_t \Phi_1^s (\partial_s A_1^s - d_{A_1^s} \Psi_1^s, \gamma^\nu)
\]
\[
+ \ast \left(\partial_s A_1^s, \nabla_t \Phi_1^s, \gamma^\nu\right) - \ast \left[\alpha_1^s, \nabla_t \Phi_1^s, \gamma^\nu\right],
\]
\[
\nabla_t A_1^s \psi_1^s = d_{A_1^s} \nabla_t \Psi_1^s - \left[\alpha_1^s, \nabla_t \psi_1^s\right] + \left(\partial_s A_1^s, \Phi_1^s, \psi_1^s\right),
\]
\[ \nabla^\Psi_t \alpha_t^\nu = *d_{A_1^\nu} \nabla^\Psi_t \alpha_1^\nu + * \left[ (\partial_t A_1^\nu - d_{A_1^\nu} \Phi_1^\nu), \gamma^\nu \right] \]

Using the first step, the next lemma and the the uniformly exponential convergence estimates of the theorem 13.3 we can conclude that

\[
\left\| \pi_{A_2^\nu} \left( \partial_t A_1^\nu - d_{A_2^\nu} \Psi_2^\nu - \nabla^\Psi_t \alpha_1^\nu \left( \partial_t A_1^\nu - d_{A_2^\nu} \Psi_2^\nu \right) - *X_t(A_2^\nu) \right) \right\|_{L^p} \leq c \varepsilon_{1/2} \nu.
\]

\[
\left\| \pi_{A_2^\nu} \left( \partial_t A_1^\nu - d_{A_2^\nu} \Psi_2^\nu - \nabla^\Psi_t \alpha_1^\nu \left( \partial_t A_1^\nu - d_{A_2^\nu} \Psi_2^\nu \right) - *X_t(A_2^\nu) \right) \right\|_{L^p(\Sigma)} \leq c \varepsilon_{1/2} \nu.
\]

\[ \Box \]

**Lemma 15.2.** There are two positive constants \( c \) and \( \varepsilon_0 \) such that

\[ \varepsilon_\nu \left\| \nabla^\Psi_t \alpha_1^\nu \right\|_{L^p} + \varepsilon_\nu \left\| \nabla^\Psi_t \gamma^\nu \right\|_{L^p} \leq c \varepsilon_{2/2} \nu, \]

\[ \varepsilon_\nu \left\| \nabla^\Psi_t \gamma^\nu_1^\nu \right\|_{L^p(\Sigma)} + \varepsilon_\nu \left\| \nabla^\Psi_t \gamma^\nu \right\|_{L^p(\Sigma)} + \varepsilon_\nu \left\| \nabla^\Psi_t \nabla^\Psi_t \gamma^\nu \right\|_{L^p(\Sigma)} \leq c \varepsilon_{2/2} \nu, \]

\[ \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} + \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} \leq c \varepsilon_\nu, \]

\[ \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} + \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} \leq c \varepsilon_\nu, \]

\[ \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} + \|d_{A_2^\nu} \phi_2^\nu\|_{L^p(\Sigma)} \leq c \varepsilon_\nu, \]

\[ \|\partial_t A_2^\nu - d_{A_2^\nu} \Phi_2^\nu\|_{L^p(\Sigma)} \leq c \varepsilon_{\delta(\theta(-s)-\theta(s))}, \]

for any \( 0 < \varepsilon_\nu < \varepsilon_0 \).

**Proof of lemma 15.2** The first estimate of the lemma can be obtained deriving by \( \nabla^\Psi_t \) the identity

\[ d_{A_1^s} \partial_t A_1^s \gamma^s = F_{A_1^s} - \frac{1}{2} \left[ *d_{A_1^s} \gamma^s \wedge \ast d_{A_1^s} \gamma^s \right]; \]

Then, by the commutation formula 2.4 and the theorem 11.1

\[ \left\| d_{A_1^s} \partial_t A_1^s \nabla^\Psi_t A_1^s \right\|_{L^2(\Sigma)} \leq c \left( \|\nabla^\Psi_t A_1^s\|_{L^2(\Sigma)} + \|\alpha_1^s\|_{L^\infty(\Sigma)} \right) \leq c \varepsilon_{1/2} \nu \]

or by the theorem 13.3 for \( |s| \geq 2 \)

\[ \left\| d_{A_1^s} \partial_t A_1^s \nabla^\Psi_t A_1^s \right\|_{L^2(\Sigma)} \leq c \left( \|\nabla^\Psi_t A_1^s\|_{L^2(\Sigma)} + \|\alpha_1^s\|_{L^\infty(\Sigma)} \right) \leq c \varepsilon_{1/2} \nu, \]

\[ \left( \|\nabla^\Psi_t A_1^s\|_{L^2(\Sigma)} + \|\alpha_1^s\|_{L^\infty(\Sigma)} \right) \]

\[ \leq c \varepsilon_{1/2} \nu, \]

where for the second estimate we used 16.7 and for the last one from the theorem 13.3. Analogously, if we derive \( 15.11 \) twice by \( \nabla^\Psi_t \), then we obtain

\[ \left\| d_{A_1^s} \partial_t A_1^s \nabla^\Psi_t A_1^s \right\|_{L^2(\Sigma)} \leq c \varepsilon_{1/2} \nu, \]

\[ \left\| d_{A_1^s} \partial_t A_1^s \nabla^\Psi_t A_1^s \right\|_{L^2(\Sigma)} \leq c \varepsilon_{1/2} \nu, \]

\[ \left\| d_{A_1^s} \partial_t A_1^s \nabla^\Psi_t A_1^s \right\|_{L^2(\Sigma)} \leq c \varepsilon_{1/2} \nu. \]
In fact since

\[ \left\| \nabla_t^{s_t} \nabla_t^{s_t} F_{A_t} \right\|_{L^2(\Sigma)} = \left\| \nabla_t^{s_t} \nabla_t^{s_t} \left( F_{A_t} - F_{A_t} - [\alpha^s, \alpha^s] \right) \right\|_{L^2(\Sigma)}, \]

by theorem \[ \text{[13.3]} \] and the estimates on the 1-forms \( \alpha_0 + \psi_0 \) and \( \alpha_1 \) and if we consider the estimates of the theorem \[ \text{[13.3]} \] we get

\[ \left\| \nabla_t^{s_t} \nabla_t^{s_t} F_{A_t} \right\|_{L^2(\Sigma)} \leq c \]

Thus, we can also conclude that by \[ \text{[15.12]} \] and \[ \text{[15.14]} \] and the commutation formula \[ \text{[2.4]} \]

\[ (15.16) \quad \varepsilon_\nu \left\| \nabla_t^{s_t} \alpha_t^{s_t} \right\|_{L^p(\Sigma)} + \varepsilon_\nu^2 \left\| \nabla_t^{s_t} \nabla_t^{s_t} \alpha_t^{s_t} \right\|_{L^p(\Sigma)} \leq c \varepsilon_\nu^{2 - \frac{1}{p}} \]

and by \[ \text{[15.13]} \] and \[ \text{[15.15]} \] and the commutation formula \[ \text{[2.4]} \]

\[ \varepsilon_\nu \left\| \nabla_t^{s_t} \alpha_t^{s_t} \right\|_{L^p(\Sigma)} + \varepsilon_\nu^2 \left\| \nabla_t^{s_t} \nabla_t^{s_t} \alpha_t^{s_t} \right\|_{L^p(\Sigma)} \leq c \varepsilon_\nu^{2 - \frac{1}{p}}. \]

Next, we can derive

\[ F_{A_t} + d_{A_t} \left( *d_{A_t} \gamma^s - \alpha^s_0 \right) + \frac{1}{2} \left( *d_{A_t} \gamma^s - \alpha^s_0 \right) \cap \left( *d_{A_t} \gamma^s - \alpha^s_0 \right) = 0 \]

by \( \nabla_s \) and we obtain

\[
\begin{align*}
    d_{A_t} * d_{A_t} \nabla_s^{\gamma^s} & = - \nabla_s^{\gamma^s} F_{A_t} - d_{A_t} \nabla_s^{\gamma^s} \left[ \alpha^s, \gamma^s \right] \\
    & + \left[ \left( \partial_s \left( \nabla_t^{s_t} \Phi_t^s \right) \right), \alpha^s_0 \right] \\
    & - \left[ \left( \partial_s A_t^{s_t} - d_{A_t} \left( \Phi_t^s \right) \right) \wedge *d_{A_t} \gamma^s \right] - d_{A_t} * \left[ \left( \partial_s A_t^{s_t} - d_{A_t} \Phi_t^s \right), \gamma^s \right] \\
    & - \left( \nabla_s^{\gamma^s} d_{A_t} \gamma^s \right) \cap \left( *d_{A_t} \gamma^s - \alpha^s_0 \right) \\
\end{align*}
\]

and thus we can conclude that

\[ \left\| d_{A_t} * d_{A_t} \nabla_s^{\gamma^s} \right\|_{L^2(\Sigma)} \leq c \left\| \nabla_s F_{A_t} \right\|_{L^2(\Sigma)} + c \varepsilon_\nu e^{\theta(-s) \delta_s} e^{-\theta(s) \delta_s} \leq c \varepsilon_\nu e^{\theta(-s) \delta_s} e^{-\theta(s) \delta_s} \]

by theorem \[ \text{[13.3]} \] and hence \[ \left\| \nabla_s \alpha_t^{s_t} \right\|_{L^p} \leq c \varepsilon_\nu. \] In order to prove the other estimates we need to use the definitions of \( \Psi_2 \) and \( \Phi_2 \) and to expand the identity. On the one hand,

\[
\begin{align*}
0 & = d_{A_t} \left( \partial_t A_t^{s_t} - d_{A_t} \Psi_t^s \right) \\
& = - d_{A_t} * d_{A_t} \Psi_t^s - \left[ \alpha_t^{s_t}, \left( \partial_t A_t^{s_t} - d_{A_t} \Psi_t^s \right) \right] \\
& + d_{A_t} \nabla_t^{s_t} \alpha_t^{s_t} + d_{A_t} \left( \partial_t A_t^{s_t} - d_{A_t} \Psi_t^s \right). \\
\end{align*}
\]
where the last term can be written in the following way:
\[
d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) \\
= d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) + d_{A_v}^* \left( \nabla^{\nu}_{A_v^\nu} \alpha_0^\nu - d_{A_v}^* \Psi_v^\nu - [\alpha_v^\nu, \Psi_v^\nu] \right) \\
- * [\alpha_v^\nu \wedge * (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu)] \\
= d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) - * \left[ \alpha_v^\nu \wedge * \left( \nabla^{\nu}_{A_v^\nu} \alpha_0^\nu - d_{A_v}^* \Psi_v^\nu - [\alpha_v^\nu, \Psi_v^\nu] \right) \right] \\
+ d_{A_v}^* \left( - \nabla^{\nu}_{A_v^\nu} \alpha_0^\nu + d_{A_v}^* \Psi_v^\nu \right) - * \left[ \alpha_v^\nu \wedge * \left( \partial_t A_v - d_{A_v}^* \Psi_v \right) \right] \\
+ d_{A_v}^* \left( - [\Psi_v^\nu, \alpha_0^\nu] + [\alpha_v^\nu, \Psi_v^\nu] \right) \\
+ * \left[ \alpha_v^\nu \wedge * \left( (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) - (\partial_t A_v - d_{A_v}^* \Psi_v) \right) \right]
\]

where the second and the third line of the last expression vanish because they can be written as
\[
d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) - d_{A_v}^* (\partial_t A_v - d_{A_v}^* \Psi_v) = 0
\]
by the condition for the perturbed geodesics \ref{eq:1.1} and the equation for the perturbed Yang-Mills connections \ref{eq:1.17}. Thus
\[
d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) \\
= d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) - * \left[ \alpha_v^\nu \wedge * \left( \nabla^{\nu}_{A_v^\nu} \alpha_0^\nu - d_{A_v}^* \Psi_v^\nu - [\alpha_v^\nu, \Psi_v^\nu] \right) \right] \\
+ d_{A_v}^* \left( - [\Psi_v^\nu, \alpha_0^\nu] + [\alpha_v^\nu, \Psi_v^\nu] \right) \\
+ * \left[ \alpha_v^\nu \wedge * \left( (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) - (\partial_t A_v - d_{A_v}^* \Psi_v) \right) \right] \\
=: d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) + D^\nu
\]
and
\[
(15.18) \quad \left\| d_{A_v}^* (\partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu) \right\|_{L^p} \leq c \varepsilon_v^{1 - \frac{1}{p}}.
\]

Furthermore for the term \( d_{A_v}^* \nabla^{\nu}_{A_v^\nu} \alpha_v^\nu \) we have
\[
d_{A_v}^* \nabla^{\nu}_{A_v^\nu} \alpha_v^\nu = \nabla^{\nu}_{A_v^\nu} d_{A_v}^* \psi_v^\nu - * \left[ \alpha_v^\nu \wedge * \nabla^{\nu}_{A_v^\nu} \alpha_v^\nu \right] - * \left[ \left( \partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu \right) \wedge \alpha_v^\nu \right] \\
= * \left[ \left[ F_{A_v^\nu}, \nabla^{\nu}_{A_v^\nu} \gamma_v^\nu \right] + * \left[ \nabla^{\nu}_{A_v^\nu} F_{A_v^\nu}, \gamma_v^\nu \right] - * \left[ \alpha_v^\nu \wedge * \nabla^{\nu}_{A_v^\nu} \alpha_v^\nu \right] \\
- * \left[ \left( \partial_t A_v^\nu - d_{A_v}^* \Psi_v^\nu \right) \wedge \alpha_v^\nu \right].
\]

Therefore, estimating term by term \ref{eq:15.17} we obtain
\[
(15.19) \quad \left\| d_{A_v}^* d_{A_v}^* \psi_v^\nu \right\|_{L^p(\Sigma)} \leq c \varepsilon_v^{1 - \frac{1}{p}} \quad \text{or} \quad \left\| d_{A_v}^* d_{A_v}^* \psi_v^\nu \right\|_{L^p(\Sigma)} \leq c \varepsilon_v^{(\theta(-s) - \theta(s)) \delta s} \varepsilon_v^{1 - \frac{1}{p}}
\]
and hence by \ref{eq:15.14}, \ref{eq:15.19}, \ref{eq:15.10} and \ref{eq:15.14}
\[
\left\| d_{A_v}^* \psi_v^\nu - d_{A_v^2}^* \psi_v^\nu \right\|_{L^p(\Sigma)} \leq c.
\]
Analogously, deriving $d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Psi_2 \right)$ by $\nabla_t^\alpha$ we can obtain $\varepsilon_2^2 \left| \nabla_t^\alpha \Psi_2 \right|_{L^p} \leq \varepsilon_2^{2-\frac{1}{p}}$ using

$$\left\| \nabla_t^\alpha \partial_t A_2 - d_{A_2} \Phi_2 \right\|_{L^p} \leq \left\| d^\alpha_{A_2} \nabla_t^\alpha \left( \partial_t A_2 - d_{A_2} \Psi_2 \right) \right\|_{L^p}$$

by the commutation formula (2.5), by the identities

$$[\partial_t A_2 - d_{A_2} \Phi_2] = 0, \quad [F_{A_2}, + F_{A_2}] = 0,$$

by the Yang-Mills flow equation (4.2) and by the theorem 13.3. On the other hand, since $\Xi$ is a Yang-Mills flow and $d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Phi_2 \right) = 0$, we can estimate

$$\left\| d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Phi_2 \right) \right\|_{L^p} \leq c\varepsilon^{(\theta(s) - \theta(s))\varepsilon_x\varepsilon_\nu}$$

by the theorem 13.3 and the estimates computed so far; thus

$$d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Phi_2 \right) = -d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Phi_2 \right) + d^\alpha_{A_2} \left( \partial_t A_2 - d_{A_2} \Phi_2 \right)$$

and this implies that $\left\| d^\alpha_{A_2} \phi_2 \right\|_{L^p} \leq c\varepsilon^{1-\frac{1}{p}}$ for $|\alpha| \geq 2$ and $\left\| \partial_t \Phi_2 \right\|_{L^p} \leq c\varepsilon^{(\theta(s) - \theta(s))\delta \varepsilon_x\varepsilon_\nu}$.

Weber proved the following theorem (cf. [17], theorem 1.12)

**Theorem 15.3** (Implicit function theorem). **Fix a perturbation** $H : \mathcal{L} \rightarrow \mathcal{R} \quad$ that satisfies (3.1). **Assume** $L^H$ is Morse and that $D^0_u$ is onto for every $u \in \mathcal{M}^0(x_-, x_+; H)$ and every pair $x_{\pm} \in \text{Crit}^b_{E^H}$. **Fix two critical points** $x_{\pm} \in \text{Crit}^b_{E^H}$ **with Morse index difference one.** **Then, for all** $c_0 > 0$ **and** $p > 2$, **there exist positive constants** $\delta_0$ **and** $c$ **such that the following holds. If** $u : \mathbb{R} \times S^1 \rightarrow M$ **is a smooth map that** $\lim_{s \rightarrow \pm \infty} u(s, \cdot) = x_{\pm}(\cdot)$ **exists, uniformly in** $t$, **and such that**

$$\left| \partial_s u(s, t) \right| \leq \frac{c_0}{1 + s^2}, \quad \left| \partial_t u(s, t) \right| \leq c_0, \quad \left| \nabla_t \partial_t u(s, t) \right| \leq c_0$$

**for all** $(s, t) \in \mathbb{R} \times S^1$ **and**

$$\left\| \partial_s u - \nabla_t \partial_t u - \text{grad} H(u) \right\|_{L^p} \leq \delta_0.$$

**Then there exist elements** $u_0 \in \mathcal{M}^0(x_-, x_+; H)$ **and** $\xi \in \text{im} (D^0_u)^* \cap \mathcal{W}_{u_0}$ **satisfying**

$$u = \exp_{u_0}(\xi), \quad \left\| \xi \right\|_{\mathcal{W}_{u_0}} \leq c\left\| \partial_s u - \nabla_t \partial_t u - \text{grad} H(u) \right\|_{L^p}.$$
Remark 15.4. The third condition of (15.20) follows from the first one and, for a positive constant \( c_1 \),
\[
|\partial_s u - \nabla c\partial_t u - \text{grad} H(u)| \leq c_1;
\]
therefore, in our case all the assumptions are satisfied by the second step and by the lemma [15.2]

**Step 3.** We choose \( p > 4 \). There are \( \varepsilon_0, c > 0 \) such that the following holds. If \( \varepsilon_\nu < \varepsilon_0 \), then there is a smooth map \( A_4' : \mathbb{R}^2 \to A_0(P) \) such that \( [A_3'] \in \mathcal{M}^0(\Xi_-^-, \Xi_+^+) \),

\[
d'_{A_3'} (A_3' - A_2') = 0,
\]

(15.30)
\[
\|d'_{A_3'} (A_3' - d_{A_3} \Phi_3') - (\partial_s A_3' - d_{A_3} \Phi_3')\|_{L^p} \leq c\varepsilon_\nu^{-\frac{1}{p}}
\]

where \( \Psi_4' \) and \( \Phi_3' \) are defined uniquely by

\[
d'_{A_3'} (\partial_s A_3' - d_{A_3} \Phi_3') = 0 \quad \text{and} \quad d'_{A_3'} (\partial_s A_3' - d_{A_3} \Phi_3') = 0.
\]

**Proof of step 3.** The third step follows directly from the theorem [15.3]. The condition (15.23) can be reached using the local slice theorem (theorem 8.1 in [13]). □

**Step 4.** We choose \( p > 4 \). There are \( \varepsilon_0, c > 0 \) such that the following holds. If \( \varepsilon_\nu < \varepsilon_0 \), then there is a smooth map \( A_4' : \mathbb{R}^2 \to A_0(P) \) such that \( [A_3'] \in \mathcal{M}^0(\Xi_-^-, \Xi_+^+) \),

\[
d'_{A_3'} (A_3' - A_1') = 0,
\]

(15.74)
\[
\|d'_{A_3'} (A_3' - A_1')\|_{L^p} + \|A_3' - A_1'\|_{L^\infty} \leq c\varepsilon_\nu^{-\frac{1}{p}}
\]

(15.75)
\[
\|d_3' \phi_3' (A_3' - A_1')\|_{L^p} \leq c\varepsilon_\nu^{-\frac{1}{p}}
\]

where \( \Psi_4' \) and \( \Phi_3' \) are defined uniquely by

\[
d'_{A_3'} (\partial_s A_3' - d_{A_3} \Phi_3') = 0 \quad \text{and} \quad d'_{A_3'} (\partial_s A_3' - d_{A_3} \Phi_3') = 0.
\]

**Proof of step 4.** By the previous two steps and the lemma [15.2] we can conclude that

\[
\|A_3' - A_1'\|_{L^p} + \|A_3' - A_1'\|_{L^\infty} \leq c\varepsilon_\nu^{-\frac{1}{p}},
\]

\[
\|d_3' \phi_3' (A_3' - A_1')\|_{L^p} \leq c\varepsilon_\nu^{-\frac{1}{p}}.
\]

Since
\[
d'_{A_3'} (A_3' - A_1') = d'_{A_3'} (A_3' - A_2') + d'_{A_3'} \alpha_1',
\]

\[
= * d_{A'} d_{A'} \gamma' - *[(A_3' - A_2') \wedge \alpha_1'],
\]

\[
d_{A'} (A_1' - A_2') = F_{A_1'} - \frac{1}{2} [(A_1' - A_2') \wedge (A_1' - A_2')]
\]

hold, we obtain
\[
\|d'_{A_3'} (A_3' - A_1')\|_{L^p(\Sigma)} + \varepsilon_\nu \|d_{A_3'} (A_3' - A_1')\|_{L^p(\Sigma)} \leq c\varepsilon_\nu^{-\frac{3}{p}}
\]
Thus, by the local gauge theorem there are maps \(g_\nu : \mathbb{R}^2 \to G_0^{2,\nu}(\mathcal{P})\) such that
\[
d^*_\nu(g_\nu^* A_3^\nu - A_1^\nu) = 0, \quad \|g_\nu^* A_3^\nu - A_1^\nu\|_{W^{1,\infty}(\Sigma)} \leq c \|A_3^\nu - A_1^\nu\|_{W^{1,\infty}(\Sigma)};
\]
then we conclude the proof of the fourth step defining \(A_4^\nu := g_\nu^* A_2^\nu\).

**Step 5.** For two positive constants \(c, \varepsilon_0, 0 < \varepsilon < \varepsilon_0, \Xi_4^\gamma := A_4^\nu + \Psi_4^\nu dt + \Phi_4^\nu ds \in \mathcal{M}^0(\Xi_-, \Xi_+^\gamma)\) satisfies

\[
(15.31) \quad \| (1 - \pi_{A_2^\nu})(\Xi_4^\nu - \Xi_3^\nu) \|_{\Xi_4^\gamma, 1, p, \varepsilon_\nu} + \varepsilon_\nu \| d_{A_4^\nu} \nabla_\tau^\nu (\Psi_4^\nu - \Psi_3^\nu) \| \leq c \varepsilon_\nu^{2-\frac{2}{p}};
\]

\[
(15.32) \quad \| \pi_{A_4^\nu}(A_4^\nu - A_3^\nu) \|_{\Xi_4^\gamma, 1, p, 1} \leq c \varepsilon_\nu^{1-\frac{2}{p}}.
\]

**Proof of step 5.** Since \(d^*_{A_4^\nu}(A_1^\nu - A_4^\nu) = 0\) and
\[
d_{A_4^\nu}(A_1^\nu - A_4^\nu) = F_{A_1^\nu} - \frac{1}{2} \left[ (A_1^\nu - A_4^\nu) \wedge (A_1^\nu - A_4^\nu) \right],
\]
by lemma [3.1]
\[
\left\| (1 - \pi_{A_2^\nu})(A_1^\nu - A_4^\nu) \right\|_{L^p} + \left\| d_{A_4^\nu}(A_1^\nu - A_4^\nu) \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{2}{p}}.
\]

By [15.27] we have
\[
d^*_{A_4^\nu} \nabla_\tau^\nu (A_4^\nu - A_4^\nu) = * \left[ (\partial_\nu A_4^\nu - d_{A_4^\nu} \Psi_4^\nu) \wedge (A_1^\nu - A_4^\nu) \right]
\]
\[
d^*_{A_4^\nu} \nabla_{A_4^\nu} (A_4^\nu - A_4^\nu) = \left[ (\partial_\nu A_4^\nu - d_{A_4^\nu} \Psi_4^\nu) \wedge (A_1^\nu - A_4^\nu) \right],
\]
and by the properties and definitions of the connections
\[
d^*_{A_4^\nu} (\partial_\nu A_4^\nu - d_{A_4^\nu} \Psi_4^\nu) = 0, \quad d^*_{A_4^\nu} (\partial_\nu A_4^\nu - d_{A_4^\nu} \Phi_4^\nu) = 0,
\]
\[
d^*_{A_4^\nu} (\partial_\nu A_4^\nu - d_{A_4^\nu} \Psi_4^\nu) = d^*_{A_4^\nu} (\partial_\nu A_4^\nu + d_{A_4^\nu} \Psi_4^\nu) + D^\nu,
\]
\[
d^*_{A_4^\nu} (\partial_\nu A_4^\nu - d_{A_4^\nu} \Phi_4^\nu) = d^*_{A_4^\nu} (\partial_\nu A_4^\nu + d_{A_4^\nu} \Phi_4^\nu) - [\alpha_0^\nu, (\partial_\nu A_4^\nu - d_{A_4^\nu} \Phi_4^\nu)],
\]
where \(D^\nu\) is defined in the proof of the lemma [15.2]. Hence we have
\[
d^*_{A_4^\nu} d_{A_4^\nu} (\Psi_4^\nu - \Psi_1^\nu) = d^*_{A_4^\nu} (\partial_\nu A_1^\nu - d_{A_4^\nu} \Psi_1^\nu)
\]
\[
+ * \left[ (A_4^\nu - A_1^\nu) \wedge (\partial_\nu A_1^\nu - d_{A_4^\nu} \Psi_1^\nu) \right]
\]
\[
- d^*_{A_4^\nu} \left( \nabla_\tau^\nu (A_1^\nu - A_4^\nu) - [(A_4^\nu - A_1^\nu), (\Psi_1^\nu - \Psi_4^\nu)] \right)
\]
\[
(15.33)
\]
\[
d^*_{A_4^\nu} d_{A_4^\nu} (\Phi_4^\nu - \Phi_1^\nu) = d^*_{A_4^\nu} (\partial_\nu A_1^\nu - d_{A_4^\nu} \Phi_1^\nu)
\]
\[
+ * \left[ (A_4^\nu - A_1^\nu) \wedge (\partial_\nu A_1^\nu - d_{A_4^\nu} \Phi_1^\nu) \right]
\]
\[
- * \left[ (\partial_\nu A_1^\nu - d_{A_4^\nu} \Phi_1^\nu) \wedge (A_1^\nu - A_4^\nu) \right]
\]
\[
+ * \left[ (A_1^\nu - A_4^\nu) \wedge d_{A_4^\nu} (\Phi_1^\nu - \Phi_4^\nu) \right],
\]
\[
(15.34)
\]
and thus by the first step, [15.18], the a priori estimates for a geodesics flow [15.3]-[3.3] and the a priori estimates of the theorem [13.3] we obtain
\[
\left\| d^*_{A_4^\nu} d_{A_4^\nu} (\Psi_4^\nu - \Psi_1^\nu) \right\|_{L^p} + \left\| d^*_{A_4^\nu} d_{A_4^\nu} (\Phi_4^\nu - \Phi_1^\nu) \right\|_{L^p} \leq c \varepsilon_\nu^{1-\frac{2}{p}}
\]
By the lemma [15.1] the estimates (15.29), (15.30) and the triangular inequality, we have also that
\[ \varepsilon_\nu \left\| \Psi_{t_1}^\nu - \Psi_{t_1}^\nu \right\|_{L^p} + \varepsilon_\nu \left\| \Phi_{t_1}^\nu - \Phi_{t_1}^\nu \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{1}{p}}, \]
\[ \varepsilon_\nu \left\| d_{A^t_1} (\Psi_{t_1}^\nu - \Psi_{t_1}^\nu) \right\|_{L^p} + \varepsilon_\nu \left\| d_{A^t_1} (\Phi_{t_1}^\nu - \Phi_{t_1}^\nu) \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{1}{p}}, \]
\[ \varepsilon_\nu \left\| \nabla_{\nu}^\psi (A^t_1 - A^t_1) \right\|_{L^p} + \varepsilon_\nu \left\| \nabla_{\nu}^\psi (A^t_1 - A^t_1) \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{1}{p}}. \]
Furthermore, deriving by \( \nabla_{\nu}^\psi \) and by \( \nabla_{\nu}^\psi \) the identities \( (15.33) \) and \( (15.32) \), we can obtain the other estimates needed for \( (15.31) \).

**Step 6.** We choose \( p > 10 \). Then there are \( \varepsilon_0, c > 0 \) such that the following holds. There are two sequences \( g_\nu \in \mathcal{G}_0^{2,p}(P \times S^1 \times \mathbb{R}) \) and \( s_\nu \in \mathbb{R} \) such that
\[
\Xi_\nu^\nu := g_\nu \Xi_\nu^\nu (t, s + s_\nu) = A^t_\nu + \Psi_{t_1}^\nu dt + \Phi_{t_1}^\nu ds \text{ satisfy}
\]
\[
d_{\Xi_\nu^\nu} (\Xi^\nu - \mathcal{K}_2(\Xi^\nu)) = 0, \quad \Xi^\nu - \mathcal{K}_2(\Xi^\nu) \in \text{im } \mathcal{D}^\nu (\mathcal{K}_2(\Xi^\nu))^* \]
\[
\left\| (1 - \pi A^t_\nu) (\Xi^\nu - \Xi^\nu) \right\|_{\Xi^\nu_{1,p,\varepsilon\nu}} \leq c \varepsilon_\nu^{2-\frac{1}{p}},
\]
\[
\left\| \pi A^t_\nu (A^t_\nu - A^t_\nu) \right\|_{\Xi^\nu_{1,p,1}} \leq c \varepsilon_\nu^{1-\frac{1}{p}}.
\]

**Remark 15.5.** In the sixth step we use the connection \( \mathcal{K}_2(\Xi^\nu) \) introduced in the section [9] the definition of the 1-form \( \alpha_0^\nu + \psi_0^\nu dt \) in that section is not the same as \( (15.3) \) even if we consider that this holds. In fact, one can replace the definition \( [9.1] \) by
\[
\alpha_0^\nu(s) + \psi_0^\nu(dt) := \theta(-s) (h(s) g(s))^{-1} (T^{c,h}(A_\nu + \Psi_{t_1}^\nu) - (A_\nu + \Psi_{t_1}^\nu)) h(s) g(s)
\]
\[
+ \theta(s) (h(s) g(s))^{-1} (T^{c,h}(A_\nu + \Psi_{t_1}^\nu) - (A_\nu + \Psi_{t_1}^\nu)) h(s) g(s),
\]
where \( g(s) \) is defined as in \( [9.2] \) and \( h \) by \( h^{-1} \partial_s h = g(\Phi^\nu - \Phi_5^\nu) g^{-1} \). In this case, \( (hg)^{-1} \partial_s h = g^{-1} (h^{-1} \partial_s h) g + g^{-1} \partial_s g = \Psi^\nu \). With this change all the theorems proved for \( \mathcal{K}_2 \) continues to hold.

**Proof of step 6.** Step 5 and the theorem [14.1] tells us that
\[
\varepsilon_\nu \left\| \nabla_{\nu}^\psi (\Xi^\nu - \mathcal{K}_2(\Xi^\nu)) \right\|_{0,p,\varepsilon\nu} \leq c \varepsilon_\nu^{2-\frac{1}{p}} \leq \delta \varepsilon_\nu^{1-\frac{1}{p}},
\]
\[
\varepsilon_\nu \left\| \nabla_{\nu}^\psi (\Xi^\nu - \mathcal{K}_2(\Xi^\nu)) \right\|_{0,p,\varepsilon\nu} \leq c \varepsilon_\nu^{2-\frac{1}{p}} \leq c \varepsilon_\nu^{1+\frac{1}{p}}
\]
for \( c \delta^{\frac{1}{p}} \leq \delta \) where \( \delta \) is given by the theorem [14.2]. Then by theorem [14.2] there is a sequence \( g_\nu \in \mathcal{G}_0^{2,p}(P \times S^1 \times \mathbb{R}), \pi_\nu \in \mathbb{R} \) such that
\[
d_{\Xi_\nu^\nu} (g_\nu (\Xi^\nu \circ \pi_\nu) - \mathcal{K}_2(\Xi^\nu)) = 0,
\]
\[
g_\nu (\Xi^\nu \circ \pi_\nu) - \mathcal{K}_2(\Xi^\nu) \in \text{im } (\mathcal{D}^\nu (\mathcal{K}_2(\Xi^\nu))^*).
\]
We define \( \Xi_\nu^\nu \) by \( (g_\nu)^{-1} \Xi_\nu^\nu \circ \pi_\nu \). By the step 5, the theorem [14.2] and the triangular inequality we have
\[
\varepsilon_\nu \left\| \nabla_{\nu}^\psi (A^t_\nu - A^t_\nu) \right\|_{L^p} + \varepsilon_\nu \left\| \nabla_{\nu}^\psi (\Psi_{t_1}^\nu - \Psi_{t_1}^\nu) \right\|_{L^p} + \varepsilon_\nu \left\| \nabla_{\nu}^\psi (\Phi_{t_1}^\nu - \Phi_5^\nu) \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{1}{p}},
\]
\[
\varepsilon_\nu \left\| \nabla_{\nu}^\psi (A^t_\nu - A^t_\nu) \right\|_{L^p} + \varepsilon_\nu \left\| \nabla_{\nu}^\psi (\Phi_{t_1}^\nu - \Phi_5^\nu) \right\|_{L^p} \leq c \varepsilon_\nu^{2-\frac{1}{p}},
\]

\[\square\]
i.e. if we define Ξν by the definition of F and since (15.38)

\[ \| \pi_{A^\nu} (\Xi_1 - \Xi_2) \|_{\Xi_k^{-1},p,1} \leq c\varepsilon^{1 - \frac{1}{p}}; \]

in order to improve the estimates for the non-harmonic part, we use the identity

\[ d_{\Xi_k}^{\nu} (\Xi_1 - K_2^{\nu} (\Xi_2)) = 0, \]

i.e. if we define Ξν = αν + ψν dt + ψμ ds and Ξν - K_2^{\nu} (Ξ_0) =: αν + ψν dt + ψμ ds, by the definition of K_2^{\nu}

\[ \| d^{\nu}_A \tilde{\alpha}^{\nu} \|_{L^p} = \| d^{\nu}_A \alpha^{\nu} \|_{L^p} \leq \varepsilon^2 \| \nabla \psi^{\nu} \|_{L^p} + \varepsilon^4 \| \nabla_s \phi^{\nu} \|_{L^p} \leq c\varepsilon^{2 - \frac{1}{p}}; \]

and since \( F^\nu_A = d^{\nu}_A \alpha^{\nu} + \frac{1}{2} [\alpha^{\nu} \land \tilde{\alpha}^{\nu}] \) and

\[ \| d^{\nu}_A \alpha^{\nu} \|_{L^p} \leq c\varepsilon^{2 - \frac{1}{p}} + \| \alpha^{\nu} \|_{L^2}^2 \leq c\varepsilon^{2 - \frac{1}{p}}. \]

**Step 7.** We choose \( p > 13 \). There are three positive constants \( \delta_1, \varepsilon_0, c \) such that for any \( \varepsilon_0 < \varepsilon_0 \)

(15.38) \[ \| \pi_{A^\nu} (A^\nu - A^\nu_0) \|_{L^p} + \| \pi_{A^\nu} (A^\nu_0 - A^\nu_0) \|_{L^\infty} \leq c\varepsilon^{1 + \delta_1}. \]

**End of the proof:** Finally we can apply the theorem [10.2] choosing \( \varepsilon_0 \) such that \( c\varepsilon^{\delta_1} < \delta \) for the \( \delta \) needed in the theorem [10.2] thus, we can conclude that for \( \nu \) big enough \( \Xi^{\nu} = R^{\nu,b}(\Xi^{\nu}_k) \) which is a contradiction to the fact that the \( \Xi^{\nu} \) are not in the image of \( R^{\nu,b} \). Therefore the proof of theorem [15.1] is concluded.

**Proof of step 7.** The idea is to consider the situation in the figure [4] in order to improve the norm of \( \pi_{A^\nu} (A^\nu - K_2^{\nu} (\Xi_k)) \). In particular, we use that \( \alpha^\nu_1 \in \text{im} d^{\nu}_A \) and the fact that the norm of \( \Pi_{\text{im} d^{\nu}_A} (\tilde{\alpha}^{\nu}) \) can be estimate using the identity \( d^{\nu}_A \tilde{\alpha}^{\nu} = -\frac{1}{2} [\alpha^{\nu} \land \tilde{\alpha}^{\nu}] \) deduced from \( F^\nu_A = F^\nu_A + d^{\nu}_A \alpha^{\nu} + \frac{1}{2} [\alpha^{\nu} \land \tilde{\alpha}^{\nu}] \). We denote by \( A^\nu_k + \Psi^k d^t + \Phi^k ds \) the connection \( K_2^{\nu} (\Xi_k) \).

**Figure 4.** The splitting of the seventh step.
Let $\nabla_t := \nabla^{\Phi^\nu}_{t}$ and $\nabla_s := \nabla^{\Phi^\nu}_{s}$. By the lemmas [10.11] and the estimates [9.12] we have

\begin{align}
\left\| \pi_{A^\nu} (A^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} &+ \left\| \nabla_t \pi_{A^\nu} (A^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
&+ \left\| \nabla_t \pi_{A^\nu} (A^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} + \left\| \nabla_t \pi_{A^\nu} (A^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
&\leq c \left\| \pi_{A^\nu} D_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
&+ c \left\| (1 - \pi_{A^\nu}) (A^\nu - A_1^\nu) \right\|_{1,p,\nu} + c \varepsilon^{3/2} \\
&+ c \left\| \nabla_t (1 - \pi_{A^\nu}) (A^\nu - A_1^\nu) \right\|_{L^p} \\
&+ c \varepsilon^2 \left\| D_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p}.
\end{align}

(15.39)

and thus our task is to estimate all the norms on the right hand side of the inequality: the second one can be estimate by $c \varepsilon^{2-\frac{3}{2}}$ by the previous step and the lemma [9.1]. The last term of (15.39) can be estimate by

\begin{align}
\varepsilon^2 \left\| D_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
\leq \varepsilon^2 \left\| F_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
&+ \varepsilon^2 \left\| C_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \leq c \varepsilon^{2-\frac{3}{2}}
\end{align}

(15.40)

where the first inequality follows from

\begin{align}
D_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) = - F_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) - C_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s))
\end{align}

because of $F_2^{\nu} (\Xi^\nu_s) = 0$ and the second estimating $C_2^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s))$ term by term using the formula [8.3]. Next, we define $\tilde{\alpha}^\nu + \tilde{\psi}^\nu dt + \tilde{\phi}^\nu ds := \Xi^\nu - K_2^{\nu} (\Xi^\nu_s)$ and

\begin{align}
A^\nu - A_1^\nu = (A^\nu - A_2^\nu) + (A_2^\nu - A_0^\nu) + (A_0^\nu - A_1^\nu) = \alpha^\nu + \tilde{\alpha}^\nu - \tilde{\alpha}_0^\nu
\end{align}

where $\tilde{\alpha}_0^\nu := K_2^{\nu} (\Xi^\nu_s) - \Xi^\nu_s$. We remark that $0 = F_{A^\nu + \tilde{\alpha}^\nu} = d_{A^\nu + \tilde{\alpha}^\nu} (\alpha^\nu + \frac{1}{2} [\tilde{\alpha}^\nu, \tilde{\alpha}^\nu])$. Furthermore, since by $F_{\tilde{\alpha}^\nu} (\Xi^\nu_s) = 0$

\begin{align}
D_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) = - F_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) - C_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s))
\end{align}

and by [9.6]

\begin{align}
\left\| \pi_{A^\nu} F_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \leq c \varepsilon^2,
\end{align}

\begin{align}
\left\| \pi_{A^\nu} D_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\tilde{\alpha}^\nu + \tilde{\psi}^\nu dt + \tilde{\phi}^\nu ds) \right\|_{L^p} \\
\leq \left\| \pi_{A^\nu} F_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) \right\|_{L^p} \\
&+ \left\| \pi_{A^\nu} C_1^{\nu} (K_2^{\nu} (\Xi^\nu_s)) (\tilde{\alpha}^\nu + \tilde{\psi}^\nu dt + \tilde{\phi}^\nu ds) \right\|_{L^p} \\
\leq c \varepsilon^{2-\frac{3}{2}} + \frac{1}{\varepsilon^2} \left\| \pi_{A^\nu} \left( \left[ \tilde{\alpha}^\nu \wedge \ast (d_{A^\nu} \tilde{\alpha}^\nu + \frac{1}{2} [\tilde{\alpha}^\nu, \tilde{\alpha}^\nu]) \right] \right) \right\|_{L^p} \\
\leq c \varepsilon^{2-\frac{3}{2}} + \frac{1}{\varepsilon^2} \left\| \pi_{A^\nu} \left( \left[ \tilde{\alpha}^\nu \wedge \ast (d_{A^\nu} (\tilde{\alpha}^\nu - \tilde{\alpha}_0^\nu)) \right] \right) \right\|_{L^p} + c \varepsilon^{1-\frac{3}{2}} \left\| \pi_{A^\nu} (\tilde{\alpha}^\nu) \right\|_{L^p}
\end{align}

(15.42)

where the second step follows estimating [8.3] term by term. Next, we consider the following operator

\begin{align}
Q^{\nu} (\Xi^\nu_s) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) := & D^{\nu} (\Xi^\nu_s) (\Xi^\nu - K_2^{\nu} (\Xi^\nu_s)) + \frac{1}{2 \varepsilon^2} d_{A^\nu}^{\nu} [\tilde{\alpha}^\nu \wedge \tilde{\alpha}^\nu]
\end{align}

(15.43)
where the last step follows estimating (8.3) term by term. Hence, by the last component can be written as

\[
Q_1^\nu (\Xi_5^\nu) \left( \tilde{\alpha}^\nu + \tilde{\psi}^\nu dt + \tilde{\phi}^\nu ds \right) = \nabla_s \left( \tilde{\alpha}^\nu - \Pi_{im} d_5^\nu (\tilde{\alpha}^\nu) \right) - d A_5 \tilde{\phi}^\nu
\]

\[
+ \frac{1}{\varepsilon_\nu^2} d_A^* d_A^\nu (\tilde{\alpha}^\nu - \tilde{\alpha}^\nu) - d * X_1 (A_5^\nu) \left( \tilde{\alpha}^\nu - \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right)
\]

\[
- \nabla_t \nabla_t \left( \tilde{\alpha}^\nu - \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) - 2 \left[ \tilde{\psi}^\nu, (\partial_t A_5^\nu - d_A \tilde{\psi}^\nu) \right]
\]

\[
+ \nabla_s \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) - d * X_t (A_5^\nu) \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) - \nabla_t \nabla_t \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right).
\]

By theorem [12.3] we obtain that

\[
\left\| d_A^* d_A^\nu (\tilde{\alpha}^\nu - \tilde{\alpha}^\nu) \right\|_{L_\nu} + \varepsilon_\nu^2 \left\| \nabla_t \nabla_t (1 - \Pi_A^\nu) \left( \tilde{\alpha}^\nu - \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu}
\]

\[
+ \varepsilon_\nu \left\| \nabla_t (1 - \Pi_A^\nu) \left( \tilde{\alpha}^\nu - \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu}
\]

\[
\leq \varepsilon_\nu^2 \left\| D_1^\nu (\Xi_5^\nu) \left( \tilde{\alpha}^\nu - \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu), \tilde{\psi}^\nu, \tilde{\phi}^\nu \right) \right\|_{L_\nu} + \varepsilon_\nu^2 \left\| \nabla_s \Pi_A^\nu (\tilde{\alpha}^\nu - \tilde{\alpha}^\nu) \right\|_{L_\nu}
\]

\[
+ \varepsilon_\nu^2 \left\| \nabla_t \nabla_t \Pi_A^\nu (\tilde{\alpha}^\nu - \tilde{\alpha}^\nu) \right\|_{L_\nu} + \varepsilon_\nu^2 \tilde{\beta}^\nu
\]

and by [15.44], step 2, the lemma [15.2] and the theorem [10.1]

\[
\leq \varepsilon_\nu^2 \left\| F_1^\nu (K_2^\nu (\Xi_5^\nu)) + Q_1^\nu (K_2^\nu (\Xi_5^\nu)) (\tilde{\alpha}^\nu, \tilde{\psi}^\nu, \tilde{\phi}^\nu) + C_1^\nu (K_2^\nu (\Xi_5^\nu)) (\tilde{\alpha}^\nu, \tilde{\psi}^\nu, \tilde{\phi}^\nu) - \frac{1}{2\varepsilon_\nu} d_A^\nu \left[ \tilde{\alpha}^\nu \wedge \tilde{\alpha}^\nu \right] = 0,
\]

\[
\leq \varepsilon_\nu^2 \left\| F_1^\nu (K_2^\nu (\Xi_5^\nu)) \right\|_{L_\nu} + \varepsilon_\nu^2 \tilde{\beta}^\nu
\]

\[
+ \varepsilon_\nu^2 \left\| C_1^\nu (K_2^\nu (\Xi_5^\nu)) \left( \tilde{\alpha}^\nu, \tilde{\psi}^\nu, \tilde{\phi}^\nu \right) \right\|_{L_\nu}
\]

\[
+ \varepsilon_\nu^2 \left\| \nabla_t \nabla_t \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu} + \varepsilon_\nu^2 \left\| \nabla_s \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu}
\]

\[
\leq \varepsilon_\nu^2 \tilde{\beta}^\nu + \varepsilon_\nu^2 \left\| \nabla_t \nabla_t \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu} + \varepsilon_\nu^2 \left\| \nabla_s \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu}
\]

where the last step follows estimating (8.3) term by term. Hence, by the last estimate, (15.39), (15.40), and the next claim:

\[
\left\| \Pi_A^\nu (A^\nu - K_2^\nu (\Xi_5^\nu)) \right\|_{L_\nu} + \left\| \nabla_t \Pi_A^\nu (A^\nu - K_2^\nu (\Xi_5^\nu)) \right\|_{L_\nu}
\]

\[
+ \left\| \nabla_t \nabla_t \Pi_A^\nu (A^\nu - K_2^\nu (\Xi_5^\nu)) \right\|_{L_\nu}
\]

\[
+ \left\| \nabla_s \Pi_A^\nu (A^\nu - K_2^\nu (\Xi_5^\nu)) \right\|_{L_\nu}
\]

\[
\leq \varepsilon_\nu^2 \tilde{\beta}^\nu + \varepsilon_\nu^2 \left\| \nabla_t \nabla_t \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu}
\]

\[
+ \varepsilon_\nu^2 \left\| \nabla_s \left( \Pi_{im} d_A^\nu (\tilde{\alpha}^\nu) \right) \right\|_{L_\nu} \leq \varepsilon_\nu^2 \tilde{\beta}^\nu.
\]
Therefore, for $p > 10$ and by the Sobolev’s theorem 5.1 for $\varepsilon = 1$, there is a $\delta_1 > 0$, such that
\[
\left\| \pi_{A_5}^r (A_1^r - A_2^r) \right\|_{L^p} + \left\| \pi_{A_5}^r (A_1^r - A_2^r) \right\|_{L^\infty} \leq c\varepsilon^{1+\delta_1}
\]
holds for $\varepsilon$, small enough. Thus we concluded the proof of the seventh step.

\section*{Claim.}
There are two positive constants $c$ and $\varepsilon_0$ such that, for $0 < \varepsilon < \varepsilon_0$,
\[
\left\| \nabla_t \nabla_t \left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p} + \left\| \nabla_s \left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p} 
\leq c\varepsilon^{1+\frac{2}{p}} \left( 1 + \left\| \nabla_t \nabla_t \pi_{A_5} (\tilde{\alpha}^r) \right\|_{L^p} + \left\| \nabla_s \pi_{A_5} (\tilde{\alpha}^r) \right\|_{L^p} \right)
\]

\section*{Proof of the claim.}
We write $\left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) = d^*_{A_5} \omega^r$ for 2-form $\omega^r$ and hence
\[
\left\| \nabla_t \nabla_t d^*_{A_5} \omega^r \right\|_{L^p} \leq \left\| d_{A_5}^r \nabla_t \nabla_t d_{A_5}^r \omega^r \right\|_{L^p} + \left( \left\| 1 - \Pi_{im} d^*_{A_5} \right\| \nabla_t \nabla_t d^*_{A_5} \omega^r \right\|_{L^p}
\]
using the commutation formulas, the $L^\infty$-bound on the curvature terms and the lemma 1.3, we obtain
\[
\left\| \nabla_t \nabla_t d_{A_5}^r \tilde{\alpha}^r \right\|_{L^p} + c \left\| d^*_{A_5} \omega^r \right\|_{L^p} + c \left\| \nabla_t d^*_{A_5} \omega^r \right\|_{L^p}
\]
and by the identity $d_{A_5} \tilde{\alpha}^r + \frac{1}{2} \left[ \tilde{\alpha}^r \wedge \tilde{\alpha}^r \right]$
\[
\leq \frac{1}{2} \left\| \nabla_t \nabla_t \left[ \tilde{\alpha}^r \wedge \tilde{\alpha}^r \right] \right\|_{L^p} + c \left\| \tilde{\alpha}^r \right\|_{L^p} + c \left\| \nabla_t \tilde{\alpha}^r \right\|_{L^p}
\leq c \left\| \tilde{\alpha}^r \right\|_{L^p} \left\| \nabla_t \nabla_t \tilde{\alpha}^r \right\|_{L^p} + c \left\| \nabla_t \tilde{\alpha}^r \right\|_{L^2^p} \left\| \nabla_t \tilde{\alpha}^r \right\|_{L^2^p}
\leq c\varepsilon^{1+\frac{2}{p}} \left( 1 + \left\| \nabla_t \nabla_t \left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p} \right)
\leq c\varepsilon^{1+\frac{2}{p}} \left\| \nabla_t \nabla_t \left( \pi_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p}
\]
In the same way, we can show that
\[
\left\| \nabla_s \left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p} \leq c\varepsilon^{1+\frac{2}{p}} \left( 1 + \left\| \nabla_s \left( \Pi_{im} d^*_{A_5} (\tilde{\alpha}^r) \right) \right\|_{L^p} \right)
\]
and thus the claim holds for $\varepsilon$ sufficiently small.

With the last claim we concluded also the proof of the theorem 1.3.

\section*{16. Proofs of the main theorems}

The definition 10.3 of the map $R^{e;b}$ and the theorem 1.5.1 which assures its surjectivity, allow us to conclude that the theorem 1.3 holds. Thus, we need only to explain the proof of theorem 1.4.

\section*{Proof of theorem 1.4}
If we fix a regular value $b$ of $E^H$, then by the theorem 1.3 there is a positive constant $\varepsilon_0$ such that the map $T^{e;b}$ is a bijection for $0 < \varepsilon < \varepsilon_0$. In addition, since $g_0(P)$ acts freely, $T^{e;b}$ descends to the map
\[
T^{e;b} : \text{Crit}_{E^H}^b / g_0(P \times S^1) \to \text{Crit}_{\mathcal{M}^{e; b}}^b / g_0(P \times S^1)
\]
which maps perturbed closed geodesics to orbits of perturbed Yang-Mills connections with the same Morse index and therefore we can see it as chain complex homeomorphism
\[ \mathcal{T}^{\varepsilon, b} : C^*_s \to C^*_s. \]

For any two perturbed geodesics \( \gamma_\pm \in \text{Crit}^b_{\mathcal{E}^H} \) with index difference 1, the map \( \mathcal{R}^{\varepsilon, b} \) is, for two lifts \( \Xi_\pm \in \text{Crit}^b_{\mathcal{E}^H} \) with \( [\Xi_\pm] = \gamma_\pm \), by theorem [13] bijective and thus
\[ \bar{\mathcal{E}}^0_{\mathcal{E}^H} (\Xi_-, \Xi_+) / \mathbb{R} = \bar{\mathcal{E}}^0_{\mathcal{E}^H} (\mathcal{T}^{\varepsilon, b}(\Xi_-), \mathcal{T}^{\varepsilon, b}(\Xi_+)) / \mathbb{R} \]
which yields that the following diagram commutes
\[
\begin{array}{cccc}
\ldots & \longrightarrow & C^{\mathcal{E}^H}_k & \longrightarrow & C^{\mathcal{E}^H}_{k-1} & \longrightarrow & \ldots \\
\downarrow \mathcal{T}^{\varepsilon, b} & & \downarrow \mathcal{T}^{\varepsilon, b} & & \downarrow \mathcal{T}^{\varepsilon, b} & & \\
\ldots & \longrightarrow & C^{\mathcal{Y}_{1, 2} \mathcal{M}^\varepsilon, H}_k & \longrightarrow & C^{\mathcal{Y}_{1, 2} \mathcal{M}^\varepsilon, H}_{k-1} & \longrightarrow & \ldots \\
\downarrow (\mathcal{T}^{\varepsilon, b})^{-1} & & \downarrow (\mathcal{T}^{\varepsilon, b})^{-1} & & \downarrow (\mathcal{T}^{\varepsilon, b})^{-1} & & \\
\ldots & \longrightarrow & C^{\mathcal{E}^H}_k & \longrightarrow & C^{\mathcal{E}^H}_{k-1} & \longrightarrow & \ldots \\
& \end{array}
\]
and hence
\[ (\mathcal{T}^{\varepsilon, b})_* : H\text{Mas}_{*}(\mathcal{L}^\varepsilon \mathcal{E}^0(P), \mathbb{Z}_2) \to H\text{Mas}_{*}(\mathcal{A}^{\varepsilon, b}(P \times S^1) / \mathcal{G}_0 (P \times S^1), \mathbb{Z}_2) \]
is an isomorphism.

\[ \square \]

**APPENDIX A. NORMS FOR 1-FORMS ON \( \Sigma \times S^1 \)**

We fix a connection \( \Xi_0 = A_0 + \Psi_0 dt \in \mathcal{A}(\Sigma \times S^1) \) and we define the following norms on \( \Omega^1(\Sigma \times S^1, \mathfrak{g}_P) \), \( i = 1, 2 \). Let \( \xi(t) = \alpha(t) + \psi(t) \wedge dt \) such that \( \alpha(t) \in \Omega^1(\Sigma, \mathfrak{g}_P) \) and \( \psi(t) \in \Omega^0(\Sigma, \mathfrak{g}_P) \) or \( \alpha(t) \in \Omega^2(\Sigma, \mathfrak{g}_P) \) and \( \psi(t) \in \Omega^1(\Sigma, \mathfrak{g}_P) \), then
\[
\begin{align*}
\| \xi \|_{0, p, \varepsilon, \Sigma \times S^1}^p & := \int_0^1 \left( \| \alpha \|^p_{L^p(\Sigma)} + \varepsilon^p \| \psi \|^p_{L^p(\Sigma)} \right) dt, \\
\| \xi \|_{\infty, \varepsilon, \Sigma \times S^1} & := \| \alpha \|_{L^\infty(\Sigma \times S^1)} + \varepsilon \| \psi \|_{L^\infty(\Sigma \times S^1)}
\end{align*}
\]
and
\[
\begin{align*}
\| \xi \|_{0, 1, p, \varepsilon, \Sigma \times S^1}^p & := \int_0^1 \left( \| \alpha \|^p_{L^p(\Sigma)} + \| dA_0 \alpha \|^p_{L^p(\Sigma)} + \| d^\ast A_0 \alpha \|^p_{L^p(\Sigma)} \right) dt \\
& + \int_0^1 \varepsilon^p \left( \| \nabla_t \alpha \|^p_{L^p(\Sigma)} + \| \psi \|^p_{L^p(\Sigma)} + \| dA_0 \psi \|^p_{L^p(\Sigma)} + \varepsilon^p \| \nabla_t \psi \|^p_{L^p(\Sigma)} \right) dt.
\end{align*}
\]
Inductively,
\[
\begin{align*}
\| \xi \|_{0, k+1, p, \varepsilon, \Sigma \times S^1}^p & := \| \alpha \| + \varepsilon \| dt \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1} + \| dA_0 \alpha \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1} \\
& + \| d^\ast A_0 \alpha \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1} + \varepsilon^p \| \nabla_t \alpha \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1} \\
& + \| dA_0 \psi \| dt \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1} + \varepsilon^p \| \nabla_t \psi \|^p dt \|^p_{0, k+1, p, \varepsilon, \Sigma \times S^1}.
\end{align*}
\]

For \( i = 1, 2 \), we can define by \( W^{k, p}(\Sigma \times S^1, \Lambda^i \mathcal{T}^*(\Sigma \times S^1) \otimes \mathfrak{g}_{P \times S^1}) \) the Sobolev space of the \( i \)-forms respect to the norm \( \| \|_{0, k+1, 1, \Sigma \times S^1} \). We now choose a reference connection \( \Xi_0 \) and all the Sobolev inequalities hold as follows by the Sobolev embedding theorem (cf. [10]).
Theorem A.1 (Sobolev estimates). We choose $1 \leq p, q < \infty$ and $l \leq k$. Then there is a constant $c_\delta$ such that for every $\xi \in W^{k,p}(\Sigma \times S^1, \Lambda^j T^* (\Sigma \times S^1) \otimes g_{\Sigma 	imes S^1})$, $i = 1, 2$, and any reference connection $\Xi_0$:

1. If $l - \frac{3}{q} \leq k - \frac{3}{p}$, then
   \[
   \|\xi\|_{\Xi_0 \cdot \Omega, l, q, \Sigma \times S^1} \leq c_\delta \varepsilon^{1/q - 1/p} \|\xi\|_{\Xi_0 \cdot k, p, \Sigma \times S^1}.
   \]

2. If $0 < k - \frac{3}{p}$, then
   \[
   \|\xi\|_{\Xi_0 \cdot \Omega, \infty, q, \Sigma \times S^1} \leq c_\delta \varepsilon^{1/p} \|\xi\|_{\Xi_0 \cdot k, p, \Sigma \times S^1}.
   \]

Appendix B. Estimates on the Surface

The first two lemmas were proved by Dostoglou and Salamon (cf. [7], lemma 7.6 and lemma 8.2) for $p > 2$ and $q = \infty$; the proofs in the case $p = 2$ and $2 \leq q < \infty$ are similar.

Lemma B.1. We choose $p > 2$ and $q = \infty$ or $p = 2$ and $2 \leq q < \infty$. Then there exist two positive constants $\delta$ and $c$ such that for every connection $A \in \mathcal{A}(P)$ with

\[
\|F_A\|_{L^p(\Sigma)} \leq \delta
\]

there are estimates

\[
\|\psi\|_{L^p(\Sigma)} \leq c\|d_A \psi\|_{L^p(\Sigma)}, \quad \|d_A \psi\|_{L^q(\Sigma)} \leq c\|d_A * d_A \psi\|_{L^p(\Sigma)},
\]

for $\psi \in \Omega^0(\Sigma, g_P)$.

Lemma B.2. We choose $p > 2$ and $q = \infty$ or $p = 2$ and $2 \leq q < \infty$. Then there exist two positive constants $\delta$ and $c$ such that the following holds. For every connection $A \in \mathcal{A}(P)$ with

\[
\|F_A\|_{L^p(\Sigma)} \leq \delta
\]

there exists a unique section $\eta \in \Omega^0(\Sigma, g_P)$ such that

\[
F_{A + d_A \eta} = 0, \quad \|d_A \eta\|_{L^q(\Sigma)} \leq c\|F_A\|_{L^p(\Sigma)}.
\]

The following lemma is a simplified version of the lemma B.2. in [14] where Salamon allows also to modify the complex structure on $\Sigma$ if it is $C^1$-closed to a fixed one.

Lemma B.3. Fix a connection $A^0 \in \mathcal{A}_0(P)$. Then, for every $\delta > 0$, $C > 0$, and $p \geq 2$, there exists a constant $c = c(\delta, C, A^0) \geq 1$ such that, if $A \in \mathcal{A}(P)$ satisfy $\|A - A^0\|_{L^\infty(\Sigma)} \leq C$ then, for every $\psi \in \Omega^0(\Sigma, g_P)$ and every $\alpha \in \Omega^1(\Sigma, g_P)$,

\[
\|\psi\|_{L^p(\Sigma)} \leq \delta \|d_A \psi\|_{L^p(\Sigma)} + c\|\psi\|_{L^2(\Sigma)},
\]

\[
\|\alpha\|_{L^p(\Sigma)} \leq \delta \left(\|d_A \alpha\|_{L^p(\Sigma)} + \|d_A * \alpha\|_{L^p(\Sigma)}\right) + c\|\alpha\|_{L^2(\Sigma)}.
\]

Lemma B.4. We choose $p \geq 2$. There is a positive constant $c$ such that the following holds. For any connection $A \in \mathcal{A}_0(P)$ and any $\alpha \in \Omega^0(\Sigma, g_P)$

\[
\|\alpha\|_{L^p(\Sigma)} \leq \|d_A \alpha\|_{L^p(\Sigma)} + \|d_A * \alpha\|_{L^p(\Sigma)} + \|d_A^* d_A \alpha\|_{L^p(\Sigma)} + \|d_A^* d_A \alpha\|_{L^p(\Sigma)}
\]

\[
\leq c\|d_A d_A + d_A d_A \alpha\|_{L^p(\Sigma)} + \|\pi_A(\alpha)\|_{L^p(\Sigma)}.
\]
Proof. For any flat connection $A$, the orthogonal splitting of $\Omega^1(\Sigma) = \text{im} \, d_A \oplus \text{im} \, d_A^* \oplus H^1_A(\Sigma, \mathfrak{g}_P)$ implies that there is a positive constant $c_0$ such that
\[
\|d_A^*d_A\alpha\|_{L^p(\Sigma)} + \|d_A^*d_A\alpha\|_{L^p(\Sigma)} \leq c_0\|\delta A d_A^* + d_A^*d_A\alpha\|_{L^p(\Sigma)};
\]
thus, we can conclude the proof applying the lemma [B.1].

\[\text{Lemma B.5. We choose } p \geq 2. \text{ There is a positive constant } c \text{ such that the following holds. For any } \delta > 0, \text{ any connection } A \in \mathcal{A}_0(P), \alpha \in \Omega^1(\Sigma, \mathfrak{g}_P) \text{ and } \psi \in \Omega^1(\Sigma, \mathfrak{g}_P), \]
\begin{align*}
\|d_A\alpha\|_{L^p(\Sigma)} &\leq c \left( \delta^{-1} \|\alpha\|_{L^p(\Sigma)} + \delta \|d_A^*d_A\alpha\|_{L^p(\Sigma)} \right), \\
\|d_A\alpha\|_{L^p(\Sigma)} &\leq c \left( \delta^{-1} \|\alpha\|_{L^p(\Sigma)} + \delta \|d_A^*d_A\alpha\|_{L^p(\Sigma)} \right), \\
\|d_A\psi\|_{L^p(\Sigma)} &\leq c \left( \delta^{-1} \|\psi\|_{L^p(\Sigma)} + \delta \|d_A^*d_A\psi\|_{L^p(\Sigma)} \right).
\end{align*}

Furthermore, for any $\delta > 0$, any connection $A + \Psi dt \in \mathcal{A}(P \times S^1)$, $\alpha + \psi dt \in \Omega^1(\Sigma \times S^1, \mathfrak{g}_P)$
\begin{align*}
\varepsilon \|\nabla \alpha\|_{L^p(\Sigma \times S^1)} &\leq c \left( \delta^{-1} \|\alpha\|_{L^p(\Sigma \times S^1)} + \delta \varepsilon^2 \|\nabla \alpha\|_{L^p(\Sigma \times S^1)} \right), \\
\varepsilon^2 \|\nabla \psi\|_{L^p(\Sigma \times S^1)} &\leq c \left( \delta^{-1} \varepsilon \|\psi\|_{L^p(\Sigma \times S^1)} + \delta \varepsilon^3 \|\nabla \psi\|_{L^p(\Sigma \times S^1)} \right).
\end{align*}

Proof. The last two estimates follow analogously to the lemma D.4. in [13]. The first can be proved as follows. We choose $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then
\[
\|d_A\alpha\|_{L^p(\Sigma)} = \sup_{\bar{\alpha}} \langle d_A\alpha, \delta^{-1} \bar{\alpha} + \delta d_A^*d_A\bar{\alpha} \rangle \\
\leq \sup_{\bar{\alpha}} c \langle \delta^{-1} \alpha + \delta d_A^*d_A\alpha, \delta d_A^*d_A\bar{\alpha} \rangle \\
\leq \delta^{-1} \|\alpha\|_{L^q(\Sigma)} + \delta \|d_A^*d_A\alpha\|_{L^q(\Sigma)} + \|d_A^*d_A\bar{\alpha}\|_{L^q(\Sigma)} \\
\leq \delta^{-1} \|\alpha\|_{L^p(\Sigma)} + \delta \|d_A^*d_A\alpha\|_{L^p(\Sigma)} \sup_{\bar{\alpha}} \frac{c \|d_A^*d_A\bar{\alpha}\|_{L^q(\Sigma)}}{\|d_A^*d_A\alpha\|_{L^q(\Sigma)}}.
\]
where the supremum is taken over all non-vanishing 1-forms $\bar{\alpha} \in L^q$ with $d_A^*d_A\bar{\alpha} \in L^q$. The norm $\|\delta^{-1} \alpha + \delta d_A^*d_A\alpha\|_{L^q(\Sigma)}$ is never 0 because
\[
\|\delta^{-1} \bar{\alpha} + \delta d_A^*d_A\bar{\alpha}\|_{L^2(\Sigma)}^2 = \delta^{-2} \|\bar{\alpha}\|_{L^2(\Sigma)}^2 + \delta^2 \|d_A^*d_A\bar{\alpha}\|_{L^2(\Sigma)}^2 + 2 \|d_A^*\bar{\alpha}\|_{L^2(\Sigma)}^2 \neq 0,
\]
otherwise we would have a contradiction by the Hölder inequality and the operator $\delta^{-1} + \delta d_A^*d_A$ is surjective. The second and the third estimate of the lemma can be shown exactly in the same way. \[\square\]

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