Triply-graded link homology and Hochschild homology of Soergel bimodules

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October 14, 2005

Abstract

We trade matrix factorizations and Koszul complexes for Hochschild homology of Soergel bimodules to modify the construction of triply-graded link homology and relate it to Kazhdan-Lusztig theory.

Hochschild homology.

Let $R$ be a $k$-algebra, where $k$ is a field, $R^e = R \otimes_k R^{op}$ be the enveloping algebra of $R$, and $M$ be an $R$-bimodule (equivalently, a left $R^e$-module). The functor of $R$-coinvariants associates to $M$ the factorspace $M_R = M/[R, M]$, where $[R, M]$ is the subspace of $M$ spanned by vectors of the form $rm - mr$. We have $M_R = R \otimes_{R^e} M$. The $R$-coinvariants functor is right exact and its $i$-th derived functor takes $M$ to $\text{Tor}^R_i(R, M)$. The latter space is also denoted $\text{HH}_i(R, M)$ and called the $i$-th Hochschild homology of $M$. The Hochschild homology of $M$ is the direct sum

$$\text{HH}(R, M) \overset{\text{def}}{=} \bigoplus_{i \geq 0} \text{HH}_i(R, M).$$

To compute Hochschild homology, we choose a resolution of the $R$-bimodule $R$ by projective $R$-bimodules

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow R \longrightarrow 0,$$

and tensor the resolution with $M$:

$$\ldots \longrightarrow P_2 \otimes_{R^e} M \longrightarrow P_1 \otimes_{R^e} M \longrightarrow P_0 \otimes_{R^e} M \longrightarrow 0.$$

Homology of this complex is isomorphic to the Hochschild homology of $M$. 

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Hochschild homology is a covariant functor from the category of $R$-bimodules to the category of $\mathbb{Z}_{\geq 0}$-graded $k$-vector spaces. In particular, a homomorphism of $R$-bimodules induces a map on their Hochschild homology. If $R$ has a grading, the Hochschild homology $\text{HH}(M, R)$ of a graded $R$-bimodule $M$ is bigraded.

Any $k$-algebra $R$ has the standard “bar” resolution by free $R$-bimodules. The polynomial algebra $R = k[y_1, \ldots, y_m]$ admits a much smaller “Koszul” resolution by free $R^e = R \otimes R$-modules (since $R$ is commutative, $R = R^{op}$), given by the tensor product (over $R^e$) of complexes

$$0 \longrightarrow R^e \xrightarrow{y_i \otimes 1 - 1 \otimes y_i} R^e \longrightarrow 0$$

for $i = 1, \ldots, m$. The resolution has length $m$ (spanning homological degrees between 0 and $m$) and its total space is naturally the tensor product of $R^e$ and the exterior algebra on $m$ generators.

Thus, the Hochschild homology of a bimodule $M$ over the polynomial algebra $R$ is the homology of the complex built out of $2^m$ copies of $M$ (the $i$-th term of the complex consists of $m$ choose $i$ copies of $M$), with the differential built out of multiplications by $y_i \otimes 1 - 1 \otimes y_i$. More precisely, denote by $C(M)$ the chain complex

$$0 \longrightarrow C_m(M) \longrightarrow \ldots \longrightarrow C_1(M) \longrightarrow C_0(M) \longrightarrow 0$$

where

$$C_j(M) = \bigoplus_{I \subset \{1, \ldots, m\}, |I| = j} M \otimes_{\mathbb{Z}} \mathbb{Z}[I],$$

the sum over all subsets $I$ of cardinality $j$ and $\mathbb{Z}[I]$ being rank 1 free abelian group generated by the symbol $[I]$. The differential has the form

$$d(m \otimes [I]) = \sum_{i \in I} \pm (y_i m - m y_i) \otimes [I \setminus \{i\}],$$

where we choose the minus sign if $I$ contains an odd number of elements less than $i$. Then $\text{HH}(R, M) \cong H(C(M))$. Clearly,

$$\text{HH}_m(R, M) = \{ m \in M | rm = mr \ \forall r \in R \} = M^R,$$

the $R$-invariants subspace of $M$.

The Hochschild cohomology groups of $M$ are the derived functors of the left exact functor of $R$-invariants, evaluated on $M$. For the polynomial algebra $R$, Hochschild homology and cohomology are isomorphic,

$$\text{HH}_i(R, M) \cong \text{HH}^{m-i}(R, M)$$
for any bimodule $M$. This property, which can be explained by the self-duality of the Koszul resolution of $R$, does not extend to arbitrary algebras. It implies that in all constructions described below we can substitute Hochschild cohomology for Hochschild homology without any gain or loss.

Hochschild cohomology of $R$-bimodules $M$, for any $R$, are covariant (rather than contravariant) in $M$, just like Hochschild homology. For a thorough treatment of Hochschild (co)homology we refer the reader to Loday’s book [L], and to Kassel [Ka] for a very brief introduction.

Figure 1: From left to right: $R$-bimodules $R, M, N \otimes_R M$; Hochschild homology of $M$.

For our purposes the figure\[\text{\scalebox{0.5}{\includegraphics{example.pdf}}}\]diagrammatical calculus will come in handy. Depict the $R$-bimodule $R$ by an oriented line, an $R$-bimodule $M$ by a box extended by two lines symbolizing the left and right actions of $R$. Concatenation of boxes is interpreted as the tensor product $N \otimes_R M$, and the closure of two ends of boxed $M$ as taking the Hochschild homology of $M$.

**Soergel bimodules.**

Let $R' = \mathbb{Q}[x_1, \ldots, x_m]$ and $R'_i = \mathbb{Q}[x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots, x_m]$. The ring $R'_i$ is a subring of $R'$ of polynomials which are symmetric in $x_i$ and $x_{i+1}$. Equivalently, $R'_i$ consists of polynomials invariant under the action of the symmetric group $S_2$ which exchanges $x_i$ and $x_{i+1}$. The ring $R'$ is free $R'_i$-module of rank 2.

Introduce a grading on $R'$ and $R'_i$ by placing each $x_i$ in degree two. Then $R' \cong R'_i \{2\} \oplus R'_i$ as a graded $R'_i$-module, where $\{2\}$ is the grading shift up by two.

Let $B'_i = R' \otimes_{R'_i} R'$. It’s a graded $R'$-bimodule, free of rank two as a left $R'$-module and as a right $R'$-module.

We will call $B'_i$, their tensor products, and other related bimodules *Soergel*
bimodules, after Wolfgang Soergel, who introduced them and explained their importance for the infinite-dimensional representation theory of simple Lie algebras and (closely related) Kazhdan-Lusztig theory [S1], [S2].

Consider degree-preserving homomorphisms of graded $R'$-bimodules:

$$br'_i : B'_i \rightarrow R', \quad rb'_i : R'\{2\} \rightarrow B'_i,$$

where $br'_i(1 \otimes 1) = 1$ and

$$rb'_i(1) = (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}).$$

We would like to consider arbitrary tensor products (over $R'$)

$$B'_{i_1} \otimes B'_{i_2} \otimes \cdots \otimes B'_{i_n}$$

Bimodules $B'_i$ and all their tensor products have a trivial direction, in the following sense. Let

$$R = \mathbb{Q}[x_1 - x_2, \ldots, x_{m-1} - x_m]$$

be the polynomial ring generated by consecutive differences of variables $x_1, \ldots, x_n$ (note that in [KR2] $R$ denotes another ring). This is a subring of $R'$ and we can write $R' = R \otimes_{\mathbb{Q}} \mathbb{Q}[x_j]$, for any $j$ (we choose $j = 1$ from now on). The permutation action $x_i \leftrightarrow x_{i+1}$ on $R'$ restricts to that on $R$ and we define $R_i \subset R$ to be the ring of $S_2$-invariants. Further, set

$$B_i = R \otimes_{R_i} R$$

We have

$$R' \cong R \otimes \mathbb{Q}[x_1], \quad R'_i \cong R_i \otimes \mathbb{Q}[x_1], \quad B'_i \cong B_i \otimes \mathbb{Q}[x_1],$$

and bimodule homomorphisms $br'_i, rb'_i$ restrict to bimodule homomorphisms, denoted $br_i$ and $rb_i$, between $R$-bimodules $B_i$ and $R$.

**Example:** $m = 2$. Let $y = x_1 - x_2$. Then $R = \mathbb{Q}[y]$, $R_1 = \mathbb{Q}[y^2]$ and $B_1 = \mathbb{Q}[y] \otimes_{\mathbb{Q}[y^2]} \mathbb{Q}[y]$. The bimodule homomorphism $rb_1$ is defined by $rb_1(1) = y \otimes 1 + 1 \otimes y$.

We have an $R'$-bimodule isomorphism, for an arbitrary tensor product:

$$B'_{i_1} \otimes_{R'} B'_{i_2} \otimes \cdots \otimes_{R'} B'_{i_n} \cong (B_{i_1} \otimes R B_{i_2} \otimes R \cdots \otimes R B_{i_n}) \otimes_{\mathbb{Q}} \mathbb{Q}[x_1].$$

Thus, the left hand side can be recovered from the tensor product $B_{i_1} \otimes R B_{i_2} \otimes R \cdots \otimes R B_{i_n}$ and vice versa.
Soergel bimodules and a braid group action (after Raphaël Rouquier).

Raphaël Rouquier [R] pointed out and explored an explicit relation between Soergel bimodules $B_i$ and the braid group. We recall his results, taking the liberty to use our conventions. Assign to the braid generator $\sigma_i$ the complex $F(\sigma_i)$ of graded $R$-bimodules

$$F(\sigma_i) : \quad 0 \rightarrow R\{2\} \xrightarrow{\text{rb}_i} B_i \rightarrow 0,$$

with $B_i$ placed in cohomological degree 0. To the braid generator $\sigma_i^{-1}$ assign the complex $F(\sigma_i^{-1})$ of graded $R$-bimodules

$$F(\sigma_i^{-1}) : \quad 0 \rightarrow B_i\{-2\} \xrightarrow{\text{br}_i} R\{-2\} \rightarrow 0,$$

with $B_i\{-2\}$ placed in cohomological degree 0 (we suggest that the reader compares these complexes with figure 6 in [KR2] which assigns certain complexes of matrix factorizations to braid generators.)

To a braid word

$$\sigma = \sigma_{j_1}^{\epsilon_1} \sigma_{j_2}^{\epsilon_2} \cdots \sigma_{j_n}^{\epsilon_n}, \quad \epsilon_i \in \{1, -1\},$$

assign the tensor product (over $R$) of the above complexes and denote it by $F(\sigma)$. For instance, to $\sigma_2 \sigma_3^{-1} \sigma_1$ we assign the complex of bimodules

$$F(\sigma_2) \otimes_R F(\sigma_3^{-1}) \otimes_R F(\sigma_1).$$

We consider the category $\mathcal{B}(R)$ of complexes of graded $R$-bimodules up to chain homotopies and view $F(\sigma)$ as an object of $\mathcal{B}(R)$.

**Proposition 1** If braid words $\sigma$ and $\tilde{\sigma}$ represent the same element of the braid group then complexes $F(\sigma)$ and $F(\tilde{\sigma})$ are isomorphic in $\mathcal{B}(R)$.

See Rouquier [R, Section 3] for a proof of this proposition and of more general results. In particular, the tensor product $F(\sigma_i) \otimes_R F(\sigma_i^{-1})$ is chain homotopy equivalent to the complex $0 \rightarrow R \rightarrow 0$ of $R$-bimodules.

The tensor product over $R$ is a bifunctor

$$\mathcal{B}(R) \times \mathcal{B}(R) \rightarrow \mathcal{B}(R),$$

and each object $N$ of $\mathcal{B}(R)$ gives rise to an endofunctor of the category $\mathcal{B}(R)$ which takes a complex $M$ to the tensor product $M \otimes_R N$.

The above proposition says that bimodule complexes $F(\sigma_i)$ give rise to a (weak) braid group action on $\mathcal{F}$. Rouquier shows that the action is ”genuine”, i.e. comes with a transitive system of isomorphisms [D].
Figure 2: To left and right pictures we assign $R$-bimodules $R$ and $B_i$, respectively.

Figure 3: To the left picture we assign $R_3$-bimodule $B_2 \otimes B_1 \otimes B_2$; to the right picture—the Hochschild homology of this bimodule.

**Graphical presentation.**

We now refine the figure 1 diagrammatics to fit our situation. To $m$ parallel oriented vertical lines we assign the $R'$-bimodule $R'$. The lines symbolize generators $x_1, \ldots, x_m$ of $R'$. The same notation will be used to depict $R$. Bimodules $B'_i$ and $B_i$ will be assigned to a diagram with a wide edge as in [KR2] bounded by four oriented lines with endpoints in the $i$-th and $(i+1)$-st positions, the rest of the diagram consisting of oriented lines, see figure 2 right.

To a composition of diagrams with wide edges we assign the tensor product of corresponding bimodules, and to the closure of a composition—the Hochschild homology of the tensor product, see figure 3.

**Link homology.**

Let $\sigma$ be a braid word (see formula (1)). The Rouquier complex $F(\sigma)$:

$$\cdots \xrightarrow{\partial} F^j(\sigma) \xrightarrow{\partial} F^{j+1}(\sigma) \xrightarrow{\partial} \cdots$$

has $n + 1$ nontrivial terms, where $n$ is the length of $\sigma$. Each term $F^j(\sigma)$ is a
direct sum of graded bimodules which are tensor products of $B_i$'s (tensored with $R$ doesn’t do anything to a bimodule). One of the summands, for a suitable $j$, is $R$, which we view as the tensor product of zero number of $B_i$'s.

The Hochschild homology $\text{HH}(R, F^j(\sigma))$ of the bimodule $F^j(\sigma)$ is a bigraded $\mathbb{Q}$-vector space. Taking the Hochschild homology of each term, we obtain a complex of bigraded vector spaces

$$
\cdots \xrightarrow{\text{HH}^{(\partial)}} \text{HH}(R, F^j(\sigma)) \xrightarrow{\text{HH}^{(\partial)}} \text{HH}(R, F^{j+1}(\sigma)) \xrightarrow{\text{HH}^{(\partial)}} \cdots
$$

Its cohomology, which we denote $\text{HHH}(\sigma)$, is a triply-graded $\mathbb{Q}$-vector space.

**Theorem 1** Up to an overall shift in the grading, $\text{HHH}(\sigma)$ is an invariant of oriented links and, up to isomorphism, depends only on the closure of $\sigma$. This homology theory is isomorphic to the reduced homology $\overline{H}(\sigma)$ as defined in [KR2, end of Section 1].

The theorem implies that the Euler characteristic of $\text{HHH}(\sigma)$ is the HOMFLYPT link polynomial [HOMFLY], [PT]. By introducing a fractional $\frac{1}{2}\mathbb{Z}$-trigrading and a suitable shift, as in Wu [W], the grading indeterminacy can be renormalized away.

**Sketch of proof.** We assume familiarity with [KR2]. Homology groups $\overline{H}(\sigma)$ and $\text{HHH}(\sigma)$ have similar definitions. In both cases we resolve each crossing of the braid in two ways and obtain $2^n$ resolutions of $\sigma$. Each resolution $D$ is a braid diagram of a planar graph which is the closure of a concatenation of wide edges, see [KR2] and figures 2, 3 above.

In this paper we assign to $D$ the Hochschild homology $\text{HH}(R, B(D))$, where $B(D)$ denotes the $R$-bimodule which is the tensor product of $B_i$'s over all wide edges of $D$. In [KR2] to $D$ we assigned $CH(D)$, the cohomology of the tensor product of matrix factorizations over all wide edges and arcs of $D$. In both cases we finish by arranging these groups, over all resolutions $D$, into a complex and taking its cohomology. Cohomology groups

$$
H(\sigma) = H(\bigoplus_D CH(D), \partial)
$$

have a "trivial" variable and can be written as

$$
H(\sigma) = \overline{H}(\sigma) \otimes \mathbb{Q}[x],
$$

with all the complexity carried by $\overline{H}(\sigma)$.

Also, the variable $a$ of [KR2] is nearly superfluous and was used, for the most part, to keep track of the grading. Setting $a = 0$ and then following
the construction of [KR2] results in link homology which is the direct sum of two copies of \( H(\sigma) \), with a relative shift in the trigrading.

When \( a = 0 \), all matrix factorizations in [KR2] turn into Koszul complexes. The factorization associated with a wide edge [KR2, figure 2] becomes the Koszul complex of the sequence \( (x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4) \) in the polynomial ring \( \mathbb{Q}[x_1, x_2, x_3, x_4] \). The Koszul complex of this regular sequence has cohomology only in the rightmost degree. The cohomology is the quotient

\[
\mathbb{Q}[x_1, \ldots, x_4]/(x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4),
\]

naturally isomorphic to the bimodule \( B'_1 \) over the polynomial algebra \( R' = \mathbb{Q}[x_1, x_2] \) (see the section “Soergel bimodules” above). Multiplication by \( x_1 \) and \( x_2 \) corresponds to the right action of \( R' \) on \( B'_1 \), multiplication by \( x_3 \) and \( x_4 \) to the left action. Equalities \( x_1 + x_2 = x_3 + x_4 \) and \( x_1x_2 = x_3x_4 \) match the definition of \( B'_1 \) as the tensor product \( R' \otimes_{R'_1} R' \) over the subalgebra \( R'_1 \) of symmetric polynomials in \( x_1, x_2 \).

Thus, the relation between [KR2] and Soergel bimodules becomes clear, at least locally, for a single wide edge. Globally, a resolution \( D \) consists of several wide edges and a number of arcs, as shown in figure 4. Put marks \((i, j)\) on arcs of \( D \), for \( 0 \leq i \leq r \) and \( 1 \leq j \leq m \), where \( r \) is the number of wide edges.

Consider the polynomial ring \( \tilde{R} = \mathbb{Q}[x_{i,j}] \) in \((r + 1)m\) variables. The Koszul complex of \( D \) (in the \( a = 0 \) case) corresponds to the following sequence of \( (r + 1)m \) elements of \( \tilde{R} \). For each \( 1 \leq i \leq r \) we have \( m \) elements, two of which are \( x_{i,s} + x_{i,s+1} - x_{i-1,s} - x_{i-1,s+1} \), \( x_{i,s}x_{i,s+1} - x_{i-1,s}x_{i-1,s+1} \) and the rest are \( x_{i,j} - x_{i-1,j} \) for \( 1 \leq j \leq m, j \neq s, s + 1 \). Here \( s \) is the position (counting from the left) of the \( i \)-th wide edge of \( D \); \( s \) is a function of \( i \). The remaining \( m \) elements of the sequence are \( x_{0,j} - x_{r,j} \) for \( 1 \leq j \leq m \).

**Lemma 1** The first \( rm \) elements on the above list constitute a regular sequence in \( \tilde{R} \).

To prove the lemma, start with the bottom \( m \) variables \( x_{0,1}, \ldots, x_{0,m} \) and work your way up. Each time we add a new layer of \( m \) variables, we encounter \( m \) new elements \( x_{i,s} + x_{i,s+1} - x_{i-1,s} - x_{i-1,s+1}, x_{i,s}x_{i,s+1} - x_{i-1,s}x_{i-1,s+1} \) and \( x_{i,j} - x_{i-1,j} \) for \( 1 \leq j \leq m, j \neq s, s + 1 \). These constitute a regular sequence, for they can be matched with the new variables (for any \( \mathbb{Q} \)-algebra \( S \) and any \( f \in S \), the element \( x^k - f \) is not a zero divisor in \( S[x] \)). □

By the time we reach the top layer of variables, we take the quotient of \( \tilde{R} \) by the above \( rm \) variables. The quotient ring is naturally isomorphic to the
Soergel bimodule $B'(D)$ assigned to $D$, the tensor product of $r$ bimodules $B'_s$ ($s$ is as before, and depends on the layer of the diagram). The Koszul complex of $D$ is quasiisomorphic to the Koszul complex of the quotient ring $B'(D)$ assigned to the sequence of the remaining $m$ elements $x_{0,j} - x_{r,j}$, for $1 \leq j \leq m$. The latter complex computes the Hochschild homology of $B'(D)$. Thus, the Hochschild homology of $B'(D)$ is naturally isomorphic to the homology of the Koszul complex of $D$, the latter isomorphic to the direct sum of two copies of $CH(D)$.

Downsizing from $m$ variables $x_{i,1}, \ldots, x_{i,m}$ to their differences gives us an isomorphism between the reduced homology $\overline{CH}(D)$ and the Hochchild homology of $B(D)$. These isomorphisms, over all $D$, respect the differentials in the complexes computing $\overline{H}(\sigma)$ and $\overline{HHH}(\sigma)$, leading to the desired isomorphism

$$\overline{HHH}(\sigma) \cong \overline{H}(\sigma).$$

It was shown in [KR2] that $\overline{H}(\sigma)$ is a link invariant, up to a grading shift. Therefore, the same is true of $\overline{HHH}(\sigma)$.

The isomorphism between $\overline{H}$ and $\overline{HHH}$ is nontrivial on their trigradings. First of all, a not so natural shift by $(-1, 1, 0)$ was built into the definition of $\overline{H}$ due to the presence of $a$. After shifting $\overline{H}$ back by $(1, -1, 0)$, both

Figure 4: A resolution $D$ of a braid’s closure
homology groups $\overline{H}(\sigma_*)$ and $HHH(\sigma_*)$ of the trivial 1-strand braid $\sigma_*$ become one-dimensional vector spaces sitting in tridegree $(0,0,0)$. After this shift is accounted for, the trigradings of the two theories relate as follows.

The third gradings of $\overline{H}$ and $HHH$ perfectly match. The third grading is the “cohomological” grading of both theories which comes last into the definition and not visible on the homology groups $CH(D)$ and $H(R,B(D))$ assigned to resolutions.

The Hochschild grading on $HHH$ matches the Koszul grading on $\overline{H}$ taken with the minus sign, the sign due to the difference in conventions.

Finally, the second grading on $\overline{H}$ equals the grading on $HHH$ by $\deg(x_i)$ minus the Hochschild grading, due to the normalization in [KR2, page 2] giving $d$ bidegree $(1,1)$.

**Hecke algebras, Soergel bimodules and Kazhdan-Lusztig theory.**

The Iwahori-Hecke algebra $H_m$ of the symmetric group is a $\mathbb{Z}[q,q^{-1}]$-algebra with generators $T_1,\ldots, T_{m-1}$ and relations

$$
(T_i - q^2)(T_i + 1) = 0,
$$

$$
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},
$$

$$
T_iT_j = T_jT_i \quad \text{for } |i-j| > 1.
$$

(our $q$ is usually denoted $q^{1/2}$ in the literature). $H_m$ is a free $\mathbb{Z}[q,q^{-1}]$-module of rank $n!$ and has a natural basis $\{T_w\}$, over all permutations $w \in S_m$, where $T_w = T_{i_1}\ldots T_{i_r}$ if $s_{i_1}\ldots s_{i_r}$ is a reduced presentation of $w$ as the product of transpositions $s_i = (i, i+1)$. We denote $r$ by $l(w)$ and call it the length of $w$. In our notations, $T_{s_i} = T_i$. Left multiplication by $T_i$ in this basis has the form

$$
T_iT_w = \begin{cases}
T_{s_{i}w} & \text{if } l(s_{i}w) > l(w) \\
q^2T_{s_{i}w} + (q^2 - 1)T_w & \text{if } l(s_{i}w) < l(w)
\end{cases}
$$

The involution $\iota$ of $H_m$ takes $q$ to $q^{-1}$, $T_w$ to $(T_{w^{-1}})^{-1}$ and acts on generators by

$$
\iota(T_i) = q^{-2}T_i + (q^{-2} - 1).
$$

The element $C'_i = q^{-1}(1 + T_i)$ is fixed by $\iota$.

**Proposition 2** The Hecke algebra has a basis $\{C'_w\}$, over all $w \in S_m$, with the elements uniquely determined by the conditions $\iota(C'_w) = C'_w$ and

$$
C'_w = q^{-l(w)} \sum_{y \leq w} P_{y,w}(q)T_y,
$$

where $P_{y,w}(q) \in \mathbb{Z}[q^2]$, $P_{w,w} = 1$ and $\deg P_{y,w} < l(w) - l(y)$ if $y \neq w$. The inequality sign under the sum sign refers to the Bruhat partial order on $S_m$. 
This proposition is due to Kazhdan and Lusztig [KL1] and admits a direct combinatorial proof. The basis \( \{ C'_w \} \) is called the Kazhdan-Lusztig basis (they also introduced a related basis \( \{ C_w \} \) which won’t appear in this exposition). The incredibly hard result, though, is the following.

**Proposition 3** Coefficients of polynomials \( P_{y,w} \) are nonnegative integers. The multiplication in the basis \( \{ C'_w \} \) has coefficients in \( \mathbb{N}[q, q^{-1}] \).

Kazhdan-Lusztig’s proof of this result [KL2] is essentially a categorification: coefficients of \( P_{y,w} \) are realized as dimensions of cohomology groups associated to simple perverse sheaves on the flag variety \( G/B \), where \( G = SL(m, \mathbb{C}) \) and \( B \) a Borel subgroup of \( G \). Multiplication in the basis \( \{ C'_w \} \) corresponds to the convolution of correspondences on the product variety \( G/B \times G/B \). Both positivity results rely on Beilinson, Bernstein and Deligne’s theory of perverse sheaves [BBD] (see also [GM]), including the decomposition theorem, which, in turn, requires Deligne’s theory of weights and mixed \( l \)-adic sheaves (an outgrowth of Deligne’s proof of the Weil conjectures). The latter is based on Grothendieck’s étale cohomology theory of varieties in finite characteristic. A characteristic zero alternative approach, via mixed Hodge modules, was developed by M. Saito (see [Sa], [T] and references therein).

Moreover, the work of Beilinson-Bernstein [BB] and Brylinski-Kashiwara [BK] on localization relates the whole story to infinite-dimensional representations of \( \mathfrak{sl}_m \) and implies that \( P_{y,w}(1) \) describe multiplicities of simple modules in the Verma modules for \( \mathfrak{sl}_m \).

Here are some additional references on these topics. See [H] for Hecke algebras and Kazhdan-Lusztig polynomials, [Sp], [Mi] for localization, [M], [Ki] for intersection homology and perverse sheaves, [KW] for \( l \)-adic and perverse sheaves and relation to KL polynomials.

To explain how Soergel bimodules fit into the picture, we rewrite the defining relations for the Hecke algebra via the generators \( C'_i = C'_{s_i} \). The relations become

\[
\begin{align*}
C'^2_i &= (q + q^{-1})C'_i, \\
C'_iC'_{i+1}C'_i + C'_{i+1} &= C'_{i+1}C'_iC'_{i+1} + C'_i, \\
C'_iC'_j &= C'_jC'_i \quad |i - j| > 1.
\end{align*}
\]

To match the bimodule \( B_i \) with the generator \( C'_i \), shift its grading down by 1 and denote by

\[
B_i = B_i\{-1\}.
\]

Now recall one of Soergel’s results [S1].
Proposition 4 There are isomorphisms of graded $R$-bimodules

$$B_i \otimes_R B_i \cong B_i \{1\} \oplus B_i \{-1\},$$

$$(B_i \otimes_R B_{i+1} \otimes_R B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes_R B_i \otimes_R B_{i+1}) \oplus B_i,$$

$$B_i \otimes_R B_j \cong B_j \otimes_R B_i \mid i - j > 1.$$  

Only the middle isomorphism is non-trivial. Let

$$B_{i,i+1} = R \otimes_{R_{i,i+1}} R\{-3\},$$

where $R_{i,i+1}$ is the ring of invariants under the action of $S_3$ on $R$ permuting $x_i, x_{i+1}, x_{i+2}$. Soergel shows

$$B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i,i+1} \oplus B_i,$$

$$B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus B_{i+1},$$

which implies the middle isomorphism.

The above isomorphisms between tensor products of Soergel bimodules lift defining relations in the Hecke algebra, when we associate bimodule $B_i$ to the element $C'_i$. Multiplication by $q$ corresponds to the grading shift up by 1.

Furthermore, the Kazhdan-Lusztig basis in $H_3$ is given by

$$C'_e = 1, \quad C'_{s_1} = C'_1, \quad C'_{s_2} = C'_2,$$

$$C'_{s_1s_2} = C'_1C'_2, \quad C'_{s_2s_1} = C'_2C'_1,$$

$$C'_{w} = C'_1C'_2C'_1 - C'_1 = C'_2C'_1C'_2 - C'_2,$$

where $w = s_1s_2s_1 = s_2s_1s_2$ is the longest element in $S_3$ and $e$ denotes the unit element of $S_3$.

Arbitrary tensor products of bimodules $B_1, B_2$ can have only 6 different indecomposable summands, up to isomorphism and grading shifts:

$$R, \quad B_1, \quad B_2, \quad B_1 \otimes_R B_2, \quad B_2 \otimes_R B_1, \quad B_{1,2}.$$  

Tensor products of these bimodules match the multiplication in the Kazhdan-Lusztig basis of $H_3$.

Soergel extends this patterns to all $m$. For technical reasons he uses $\mathbb{C}$ as the ground field instead of $\mathbb{Q}$. He shows the existence of graded indecomposable $R$-bimodules $B_w$, for $w \in S_m$, with the following properties:

- $B_w$ is a finitely-generated projective left $R$-module and a finitely-generated projective right $R$-module.
• $B_{si} = B_i$, and $B_e = R$.

• This collection of bimodules is closed under tensor product:

$$B_w \otimes_R B_y \cong \bigoplus_{z \in S_m} B^z_{xy},$$

where $n^z_{wy} \in \mathbb{N}[q, q^{-1}]$ are structure coefficients of the multiplication in the Kazhdan-Lusztig basis $\{C'_w\}$ of $H_m$,

$$C'_w C'_y = \sum_z n^z_{wy} C'_z.$$

This construction, together with the original work of Kazhdan and Lusztig, can be viewed as a categorification of the Hecke algebra. The Kazhdan-Lusztig basis lifts to a collection of bimodules, multiplication in the Hecke algebra lifts to the tensor products of bimodules, etc.

More precisely, Soergel bimodules can be used to produce a categorification of the Hecke algebra action on its regular representation. The regular representation becomes the Grothendieck group of the graded version of the category of Harish-Chandra bimodules for $\mathfrak{sl}_m$ (with generalized trivial central character on both sides). To each $w \in S_m$ there is assigned an exact endofunctor in this category, which acts on the Grothendieck group in the same way as $C'_w$ acts by left multiplication on $H_m$. The endofunctors can be reconstructed from bimodules $B_w$, composition of endofunctors matching the tensor product of these bimodules. Some details can be found in [S1], others follow from Soergel’s results.

The group $G = SL(m, \mathbb{C})$ acts transitively on the flag variety $G/B$. The diagonal action of $G$ on $G/B \times G/B$ has finitely many orbits, which are in a natural bijection with elements of the symmetric group. The diagonal orbit corresponds to the trivial element, and the open orbit—to the maximal length permutation. Let $O_w$ be the orbit associated with $w \in S_m$. There exists a complex of sheaves $IC(\overline{O}_w)$ (the intersection cohomology sheaf), supported on the closure of $O_w$. This complex of sheaves is $G$-equivariant and its stalk cohomology groups are constant along each orbit. According to Soergel [S2], bimodule $B_w$ is isomorphic to the $G$-equivariant cohomology of this complex,

$$B_w \cong H_G(IC(\overline{O}_w)).$$

The bimodule structure comes from identifying $R$ with the $G$-equivariant cohomology of $G/B$. 

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Example: when \( m = 2 \), the flag variety is \( \mathbb{P}^1 \) and \( G \) acts on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with two orbits: the diagonal and its complement. The closure of each orbit is smooth, and the IC sheaf is the constant sheaf on the orbit’s closure, shifted by the dimension of the orbit. Consequently, the equivariant cohomology groups of these IC sheaves are the equivariant cohomology groups of the diagonal and of \( \mathbb{P}^1 \times \mathbb{P}^1 \). They are

\[
H_G(\mathbb{P}^1) \cong H_B(\cdot) \cong H_{SO(2)}(\cdot) \cong \mathbb{C}[y] \cong R,
\]

\[
H_G(\mathbb{P}^1 \times \mathbb{P}^1) \cong H_B(\mathbb{P}^1) \cong H_{SO(2)}(\mathbb{P}^1) \cong B_1,
\]

where dot denotes a point, and \( y \) a generator of \( H^2(\mathbb{C}\mathbb{P}^\infty, \mathbb{Q}) \). The group \( SO(2) \) acts on \( \mathbb{P}^1 \cong S^2 \) by rotations about the north pole-south pole axis. The action is free except at the poles, and the quotient by the action is naturally the interval \([−1, 1]\). The cohomology can be rewritten as \( H(S^2 \times_{SO(2)} ESO(2)) \). The space \( S^2 \times_{SO(2)} ESO(2) \) maps onto the orbit space \([−1, 1]\) of \( S^2 \) under the \( SO(2) \) action. The fiber over each point other than \(-1, 1\) is contractible, while over \(-1\) and \(1\) the fiber is isomorphic to \( \mathbb{C}\mathbb{P}^\infty \). Therefore, the equivariant cohomology is naturally isomorphic to the cohomology of the 1-point union of two copies of \( \mathbb{C}\mathbb{P}^\infty \). This cohomology can be identified with \( R \otimes_{R_1} R \) as a graded \( R \)-bimodule, and even as a ring.

For an arbitrary \( m \), the bimodule \( B_i \) is isomorphic to the \( G \)-equivariant cohomology groups of the “thickened” flag variety

\[
Y = \{(L_1, \ldots, L_i, \ldots, L_{m-1}, L'_i) \mid \dim(L_j) = j, \dim(L'_i) = i, 0 \subset L_1 \subset L_2 \subset \cdots \subset L_{m-1} \subset \mathbb{C}^m, L_{i-1} \subset L'_i \subset L_{i+1}\}.
\]

Bimodules \( B_w \) and link homology.
Our definition of link homology \( \text{HHH}(\sigma) \) used only tensor products of \( B_i \) rather than all \( B_w \), which are the indecomposable summands of the tensor products. However, the Koszul resolution of indecomposable bimodule \( B_{i_1 \cdots i_{l+1}} \) appear in the proof of the invariance of \( \overline{\text{HH}}(\sigma) \) under the third Reimeister move [KR2, Section 6].

Indecomposable bimodules \( B_w \) might prove useful in computations of link homology. Each term \( F^j(\sigma) \) of the complex \( F(\sigma) \) decomposes as a direct sum of \( B_w \), with various shifts and multiplicities,

\[
F^j(\sigma) \cong \bigoplus_w B_w^{n_w(\sigma, j)}, \quad n_w(\sigma, j) \in \mathbb{N}[q, q^{-1}].
\]

Suppose we know these multiplicities \( n_w(\sigma, j) \) and have the formula for the differential with respect to these direct sum decompositions of \( F^3(\sigma) \). For
each $j$ the differential is described by an $n! \times n!$ matrix, with rows and columns enumerated by $w$. The $(y, w)$-entry is itself a matrix with $n_y(\sigma, j+1)|_{q=1}$ rows and $n_w(\sigma, j)|_{q=1}$ columns describing the direct summand

$$B^w_{nw}(\sigma, j) \longrightarrow B^w_y(\sigma, j+1)$$

of the differential

$$\partial : F^j(\sigma) \longrightarrow F^{j+1}(\sigma).$$

Each entry of the latter matrix is a homomorphism of bimodule $B_w \longrightarrow B_y$ of a particular degree.

This horrendous complex can be simplified, by stripping off contractible summands of the form

$$0 \longrightarrow B_w\{i\} \xrightarrow{1} B_w\{i\} \longrightarrow 0.$$

Such a summand exists whenever there is an entry in the $j$-th matrix of matrices which is a nonzero complex multiple (recall we are working over $\mathbb{C}$) of the identity map of $B_w\{i\}$.

Throwing out all contractible summands from $F(\sigma)$ results in a much smaller complex which we denote $F_{min}(\sigma)$. Up to isomorphism, $F_{min}(\sigma)$ does not depend on the order and choices of removed contractible summands. The reduction to $F_{min}(\sigma)$ is best done inductively on the length of $\sigma = \sigma_1^{i_1} \ldots \sigma_r^{i_r}$

Once we found $F_{min}(\sigma_1^{i_1} \ldots \sigma_r^{i_r})$, for some $r < n$, tensor it with $F(\sigma_{r+1}^{i_{r+1}})$ and reduce to minimal size. We start with $r = 1$ and proceed until $r = n$. The resulting minimal complex is isomorphic to $F_{min}(\sigma)$.

To determine $\text{HHH}(\sigma)$, we take the Hochschild homology of each term $F_{min}(\sigma)$, arrange them into a complex and take its cohomology.

**Remark:** A similar algorithm to compute the $\text{sl}(2)$ link homology was found by Dror Bar-Natan and implemented by him and Jeremy Green [BN1,2,3]. They represent a link as a composition of elementary tangles $L = t_1 \ldots t_n$. In the language of [K], the invariant of a tangle $t_1 \ldots t_r$ is a complex of graded projective $H^s$-modules where $2s$ is the number of endpoints of $t_1 \ldots t_r$. After splitting off all contractible summands from the complex (thus reducing it to minimal size), tensor it with the complex assigned to $t_{r+1}$, reduce the product to minimal size, and so on. We note that Bar-Natan and Green use a more refined and, at the same time, more geometric framework than that of rings $H^s$.

**Example:** When $m = 2$ and $\sigma = \sigma_1^n$, the reduction leads to a simple computation of homology groups $\text{HHH}(\sigma)$. The complex $F(\sigma)$ consists of $2^n$...
terms which are tensor powers of $B_1$. The minimal complex has only $n+1$ indecomposable bimodules:

$$0 \longrightarrow R\{2n\} \overset{d^0}{\longrightarrow} B_1\{2n-2\} \overset{d^1}{\longrightarrow} B_1\{2n-4\} \overset{d^2}{\longrightarrow} \cdots \overset{d^n}{\longrightarrow} B_1 \longrightarrow 0$$

The differential is

$$d^0(1) = 1 \otimes y + y \otimes 1, \quad d^i(1 \otimes 1) = 1 \otimes y - y \otimes 1 \quad \text{odd } i > 0, \quad d^i(1 \otimes 1) = 1 \otimes y + y \otimes 1 \quad \text{even } i > 0,$$

where $R = \mathbb{Q}[y], y = x_1 - x_2, R_1 = \mathbb{Q}[y^2]$ and $B_1 = \mathbb{Q}[y] \otimes_{\mathbb{Q}[y^2]} \mathbb{Q}[y]$. We take the Hochschild homology of each term, and the resulting complex splits into the direct sum of two complexes of graded $R$-modules (for $\text{HH}_0$ and $\text{HH}_1$). The complex of Hochschild homologies of zero degree is

$$0 \longrightarrow R\{2n\} \overset{2y}{\longrightarrow} R\{2n-2\} \overset{0}{\longrightarrow} R\{2n-4\} \overset{2y}{\longrightarrow} R\{2n-6\} \overset{0}{\longrightarrow} \cdots.$$ 

It has cohomology $\mathbb{Q}$ in bidegrees $(2n-2,1), (2n-6,3), \ldots, (0,n)$, where the first grading corresponds to the degrees of variables $x_i$ and the second is cohomological.

The Hochschild homology complex of degree one is

$$0 \longrightarrow R\{2n+2\} \overset{1}{\longrightarrow} R\{2n+2\} \overset{0}{\longrightarrow} R\{2n\} \overset{2y}{\longrightarrow} R\{2n-2\} \overset{0}{\longrightarrow} R\{2n-4\} \overset{2y}{\longrightarrow} \cdots.$$ 

It has cohomology $\mathbb{Q}$ in bidegrees $(2n-2,3), (2n-6,5), \ldots, (4,n)$.

Hence, for the $(2,n)$-torus knot $L$, homology $\text{HHH}(L)$ has rank $n$, as previously computed by Rasmussen (private communication). It was predicted in [GSV], [DGR] that the suitable HOMFLYPT homology groups of the $(2,n)$ torus knot have rank $n$. If $\overline{\text{HH}} \cong \text{HHH}$ is isomorphic to the theory conjectured to exist in [GSV] and [DGR], one would have an interesting link between perverse sheaves on flag varieties and the Gromov-Witten theory on the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$.

The approach to computing $\text{HHH}(\sigma)$ via bimodules becomes significantly more challenging already for $m = 3$. With six indecomposable bimodules $B_w$ the endomorphism algebra $\text{End}_{R^e}( \oplus_{w \in S_3} B_w)$ looks quite complicated. If we don’t exclude trivial cases and use various symmetries, describing the multiplication

$$\text{Hom}(B_w, B_y) \otimes \text{Hom}(B_y, B_z) \longrightarrow \text{Hom}(B_w, B_z)$$

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directly requires dealing with $6^3 = 216$ possibilities for $w, y, z \in S_3$. Also, knowing the endomorphism algebra is only the first (and not the most difficult) step in the algorithm. It’s almost certain that describing $\overline{H}(\sigma)$ via the limit $N \to \infty$ of $SL(N)$ link homologies, as suggested by Gornik, Rasmussen and [DGR], together with Rasmussen’s methods [Ra] for computing the latter, will get the job done much faster.

**Acknowledgements:** This paper is a write-up of the talk the author gave at the knot homology conference at UQAM in Montreal in September, 2005. I am grateful to Olivier Collin for organizing the conference and giving me an opportunity to present the above observations. Several ideas implicit or explicit in this paper (setting $a$ of [KR2] to 0, passing from the Koszul complex of a wide edge to its zeroth cohomology) had been independently suggested by Jacob Rasmussen. Theorem II clarifies Maxim Vybornov’s conjecture [V] that the link homology of [KR2] should be related to the category $\mathcal{O}$ and perverse sheaves on the flag variety. We were also influenced by Jozef Przytycki’s comparison of the Hochschild homology of truncated polynomial algebras and the link homology of $(2, n)$-torus knots [P]. The author would like to thank Charles Frohman, Sergei Gukov, Lev Rozansky and Ilya Shapiro for useful discussions. Partial support came from the NSF grant DMS-0407784.

**References**

[BN1] D. Bar-Natan, Khovanov’s homology for tangles and cobordisms, *Geom. Topol.* 9 (2005) 1443-1499, arxiv math.GT/0410495.

[BN2] D. Bar-Natan, I’ve computed Kh(T(9,5)) and I’m happy, talk at George Washington University, February 2005, available online.

[BN3] D. Bar-Natan, Knot Atlas, [http://katlas.math.toronto.edu/wiki](http://katlas.math.toronto.edu/wiki)

[BB] A. Beilinson and J. Bernstein, Localization de $g$-modules, *Ç. R. Acad. Sci. Paris* (1) 292 (1981), 15–18.

[BBD] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, *Astérisque* 100 (1982).

[BK] J. L. Brylinski, M. Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems, *Invent. Math.* 64 (1981), 387–410.

[D] P. Deligne, Action du groupe des tresses sur une catégorie, *Inv. Math.* 128 (1997), 159–175.

[DGR] N. Dunfield, S. Gukov and J. Rasmussen, The superpolynomial for knot homologies, arxiv math.GT/0505662.

[GM] M. Goresky and R. MacPherson, Intersection homology II, *Invent. Math.* 71, 77–129 (1983).

[GSV] S. Gukov, A. Schwarz and C. Vafa, Khovanov-Rozansky homology
and topological strings, arXiv:hep-th/0412243.

[H] J. Humphreys, Reflection groups and Coxeter groups, *Cambridge studies in adv. math.* **29** 1990.

[HOMFLY] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, A new polynomial invariant of knots and links, *Bull. AMS. (N.S.)* **12** 2, 239–246, 1985.

[KL1] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1980), 191–213.

[KL2] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, *Proc. Symp. Pure Math.* **36**, 1980.

[Ka] C. Kassel, Homology and cohomology of associative algebras: a concise introduction to cyclic homology. Online notes.

[K] M. Khovanov, A functor-valued invariant of tangles, *Algebr. Geom. Topol.* **2** (2002) 665-741. [math.QA/0103190]

[KR1] M. Khovanov and L. Rozansky, Matrix factorizations and link homology, arxiv:math.QA/0401268

[KR2] M. Khovanov and L. Rozansky, Matrix factorizations and link homology II, arxiv:math.QA/0505056

[KW] R. Kiehl and R. Weissauer, Weil conjectures, perverse sheaves and l’adic Fourier transform, Springer-Verlag, Berlin, 2001.

[Ki] F. Kirwan, An introduction to intersection homology theory, *Pitman Research Notes in Mathematics Series, vol. 187*, Longman Scientific and Technical, Harlow, 1988.

[L] J.-L. Loday, Cyclic homology, 2nd edition. Springer-Verlag, Berlin, 1998.

[M] R. MacPherson, Global questions in the topology of singular spaces, *Proc. Int. Cong. Math. Warszawa*, 1983.

[Mi] D. Miščić, Localization and representation theory of reductive Lie groups, http://www.math.utah.edu/ftp/u/ma/milicic/math/book.dvi

[P] J. Przytycki, When the theories meet: Khovanov homology as Hochschild homology of links, arxiv:math.GT/0509334

[PT] J. Przytycki and P. Traczyk, Conway Algebras and Skein Equivalence of Links, *Proc. AMS* **100** 744-748, 1987.

[Ra] J. Rasmussen, Khovanov-Rozansky homology of two-bridge knots and links, arxiv:math.GT/0508510

[R] R. Rouquier, Categorification of the braid group, arxiv:math.RT/0409593.

[Sa] M. Saito, Mixed Hodge modules and applications, *Proceedings of the International Congress of Mathematicians, vol. I, II (Kyoto, 1990)*, 725–734, Math. Soc. Japan, Tokyo, 1991.

[S1] W. Soergel, The combinatorics of Harish-Chandra bimodules, *J. reine angew. Math.* **429** (1992), 49–74.
[S2] W. Soergel, Grading on representation categories, in: Proceedings of the ICM94 in Zrich, Birkhuser (1995), 800–806.
[Sp] T.A. Springer, Quelques applications de la cohomologie d’intersection, Séminaire Bourbaki 1981/82, 249–273.
[T] T. Tanisaki, Hodge modules, equivariant K-theory and Hecke algebras, Publ. RIMS, Kyoto Univ. 23 (1987), 841–879.
[V] M. Vybornov, Private communication, June 2005.
[W] H. Wu, Braids and the Khovanov-Rozansky cohomology, arxiv [math.GT 0508064]

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