Hawking radiation from covariant anomalies in (2+1)-dimensional black holes

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Abstract

In an insightful approach, Robinson and Wilczek proposed that Hawking radiation can be obtained as the compensation of a breakdown of general covariance and gauge invariance and the radiation is a black body radiation at Hawking temperature. We apply this method to two types of black holes in three-dimensional spacetime, both of which have the form of a metric such that the $tt$ component of the metric is not inverse of the $rr$ component of the metric. The first one is the warped AdS$_3$ black hole in three-dimensional topologically massive gravity with the negative cosmological constant, and the second one is the charged rotating black hole in three dimensions.

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1. Introduction

Hawking radiation is one of the most important quantum effects in the study of the quantum gravity in a black hole background spacetime. Classically, a black hole does not radiate, but when we consider quantum effects around a generic black hole geometry we have radiation. Hawking’s original work [1] has shown that the calculation of the Bogoliubov coefficients gives a Planck distribution of radiation as one of the thermodynamical properties. There are other derivations of Hawking radiation taking account of the quantum effect in the black hole background as a tunneling process based on particles in a dynamical geometry [2, 3]. It was found that Hawking radiation is determined by the horizon properties. Another approach to understand Hawking radiation is the calculation of the energy–momentum tensor in black hole backgrounds. The energy–momentum tensor should be conserved in a curved background in classical field theory with general covariance. However, quantum mechanically, there is breaking of symmetry which gives rise to the anomaly term.

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Recently, Robinson and Wilczek [4], and also Iso, Umetsu and Wilczek [5], proposed another derivation of the energy–momentum tensor which describes the Hawking radiation as the compensation of the anomaly to break the classical symmetry. There are two types of anomalous currents, i.e. the consistent and the covariant. The anomaly equations with the consistent currents satisfy the Wess–Zumino consistency condition but do not transform covariantly under a gauge transformation. On the other hand, Banerjee and Kulkarni [6, 7] made use of another method involving covariant currents which transforms under gauge transformation, but which does not satisfy the Wess–Zumino consistency condition. This method was extended to the case of charged black holes, rotating black holes and other types of black holes [5, 8–14].

In this paper, we calculate the energy–momentum flux of the warped AdS3 black hole in three-dimensional topologically massive gravity with the negative cosmological constant. In three dimensions, ordinary Einstein gravity is trivial, but with the gravitational Chern–Simons term there is a propagating massive graviton mode [15]. In light of this, recently there have been some developments in understanding topologically massive gravity with the negative cosmological constant $-1/\ell^2$ [16–30]. We also consider another type of black hole in three dimensions: the charged rotating black hole, which has some interesting properties of its own such as unstable inner horizon.

In three-dimensional topologically massive gravity, there is an AdS3 vacuum solution for any coefficient $\mu$ of the Chern–Simons action term. Recently, it was shown that these solutions are unstable except at the chiral point $\mu \ell = 1$ [20]. More recently, for $\mu \ell \neq 3$, two other types of black hole solutions which have $SL(2, \mathbb{R}) \times U(1)$-invariant warped AdS3 geometries were considered with a timelike or spacelike $U(1)$ isometry, and at $\mu \ell = 3$ there are also a pair of solutions with a null $U(1)$ isometry [21].

It is known that for $\mu \ell > 3$ warped black hole solutions are asymptotic to warped AdS3. It is also shown that these black hole solutions can be regarded as discrete quotients of warped AdS3, just as Bañados–Teitelboim–Zanelli (BTZ) black holes are discrete quotients of AdS3. Guica et al [22] have argued that in the very near-horizon region of an extreme Kerr black hole, this geometry becomes warped AdS3 with an $SL(2, \mathbb{R}) \times U(1)$ isometry [23] and the geometry of a three-dimensional slice of fixed polar angle $\theta$ gives a quotient of warped AdS3 with the identification of azimuthal angle $\varphi$ [24, 25]. Such quotients are warped AdS black holes. With consistent boundary conditions, the asymptotic symmetry generators give one copy of Virasoro algebra with a chiral half of central charge. So they have conjectured that extreme Kerr black holes are dual to a chiral conformal field theory. At $\mu \ell = 3$ null warped AdS3 is given as a solution and it can be related to the dual theory between gravity and conformal quantum mechanical theory [26].

Here, we will concentrate on a type of spacelike stretched warped black hole solution [21]. We will apply the Robinson and Wilczek method to the warped stretched AdS3 black hole which has different metric forms for the ‘tt’ component and the inverse of the ‘rr’ component.

There is another type of three-dimensional black hole which shares a similar form of the metric. It is the charged rotating BTZ black hole [31]. The BTZ black hole solution can be extended to the Einstein–Maxwell theory with the negative cosmological constant [32]. The only modification of the metric is the sum of an extra term which gives the logarithmic function. This solution can be applied when the angular momentum vanishes [33].

There have been some other works related to the charged BTZ black hole. It has some unexpected properties, quite different from higher dimensional ones.

The first is the problem of stability of the inner horizon. Martinez et al [31] have shown that the inner horizon of the rotating uncharged black hole becomes unstable with the addition of a small electric charge. When $Q = 0$ the rotating black hole is obtained from the solution
with $J \neq 0$ by a Lorentz transformation in the $\varphi$–$t$ plane. Due to the change in the physical parameters of the solution, this is an 'illegitimate coordinate transformation'.

The second is the problem of defining the mass of the black hole. The same transformation could be used when $Q \neq 0$ to make the solution with the angular momentum from the one with $J = 0$. By the nature of the logarithmic function, the total energy with the electromagnetic field diverges. This prevents us from defining the total energy of a self-gravitating system. The idea of quasilocal energy has been introduced [34]. The quasilocal energy is defined as the difference between the value of this Hamiltonian and that of the background with the same boundary. There are many investigations for the conserved quantities with the consideration of various choices of boundary conditions and backgrounds [35].

The third has to do with the mass bound in charged black holes. Unlike charged black holes in dimensions four or higher whose mass has a lower bound determined by the electric charge, there is no such bound. As mentioned before, the mass of the charged BTZ black hole is a poorly defined concept, due to the logarithmic divergent boundary term [31]. Extremal charged black holes in four or higher dimensions have 'supersymmetry' and can sometimes be studied in the context of string theory [36]. However, the charged BTZ black hole cannot be studied as such.

In spite of these problems, it was considered as a fruitful testing ground of various formalisms of black hole physics. For example, thermodynamic properties of the charged BTZ black hole were considered in [37], where its non-thermal nature such as frequency-dependent Hawking temperature was discussed. It was also obtained as a result of the collapse of a charged spinning ring [38].

The charged BTZ black hole falls into a class of black holes whose asymptotic behavior does not fall off sufficiently. This is due to the logarithmic behavior of the electric field in a two-dimensional space. The reason why such a case is of interest is that asymptotic symmetries which govern conserved charges can then be modified. Another type of black hole which has similar asymptotic behavior was considered in [39].

More recently, it was shown that the three-dimensional charged BTZ black hole solution interpolates between an asymptotic AdS$_3$ and a near-horizon AdS$_2 \times S^1$ geometry [40, 41]. The dimensional reduction with circular symmetry permits us to describe AdS$_3$ as AdS$_2$ with a linear dilaton. A solution describing AdS$_2$ with a constant dilaton and an electric field is the dimensional reduction of the near-horizon limit of the charged BTZ black hole. When this interpolation is applied to the calculation of the microscopic black hole entropy [41, 42], different results are obtained depending on how you calculate it [42]. Therefore, the charged BTZ black hole has very interesting thermodynamic properties and can be considered as an interesting example to apply using the Robinson and Wilczek method which gives us some insight into the Hawking radiation, a thermodynamic property.

On the more practical side, applying the Robinson and Wilczek method to the charged BTZ black hole is interesting because the metric function for the 'tt' component differs from the 'rr' component. In fact, Gangopadhyay and Kulkarni [27] have calculated the energy–momentum flux for a black hole with $g_{rr} g_{tt} \neq 1$ in a different context. Their method can be applied to the black holes studied in this paper.

2. Hawking radiation and covariant anomalies

In this section, we will review how the Hawking radiation can be obtained from the covariant anomaly. Consider the Reissner–Nordström black hole metric of the following form:

$$\text{d}s^2 = -f(r) \text{d}t^2 + \frac{\text{d}r^2}{f(r)} + r^2 \Omega_{d-2}^2.$$

(1)
Robinson and Wilczek [4] have considered that the energy flux of the Hawking radiation can be given by the effective two-dimensional theory having gravitational anomalies at the black hole horizon. This was further generalized for the case of charged black holes where gauge anomaly was also considered for the flux of charge [5]. We can also consider anomaly analysis of Hawking radiation for more general cases [27] of the following type,

\[ ds^2 = -h(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\phi + N^\phi(r) \, dt \right)^2, \]

where \( \phi \) is an angle, \( \phi \in [0, 2\pi] \), and the horizon is located at \( r = r_H \) satisfying \( f(r_H) = 0 \) and \( h(r_H) = 0 \).

To consider the Hawking radiation via anomaly, let us first consider the scalar field on this metric background. The action of this scalar field is

\[
S[\psi] = -\frac{1}{2} \int d^3x \sqrt{-g} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi
\]

Performing the partial wave decomposition, \( \psi(t, r, \phi) = \sum_m \varphi_m(t, r) \, e^{im\phi} \), this action is reduced to an infinite set of the scalar fields on a two-dimensional space. If we go near the horizon, \( r \to r_H \), the above action near the horizon can be effectively described in the following form:

\[
S[\psi] = -\frac{1}{2} \int dr \, \left[ \frac{h}{f} \psi H^2 \psi - \frac{1}{h} \psi \partial_r \left( \sqrt{fh} \partial_r \psi \right) \right]
\]

From this action, \( \varphi_m(t, r) \) is a two-dimensional complex scalar field in the background \( U(1) \) static gauge field \( A_t(r) = N^\phi(r) \) with charge \( m \). When we consider the charged rotating black hole in three dimensions, this two-dimensional effective theory has two types of \( U(1) \) charges. One is from the gauge field and the other from rotation [8, 9]. These two types of currents are identical in nature, so we consider one type of current and then we can use the same result.

The idea of Robinson and Wilczek is to divide the region outside the horizon of the black hole into two. One is the near horizon and the other sufficiently far away. The near-horizon region \( (H) \) is described by two-dimensional effective field theory. The radial direction \( r \) and the time \( t \) are the two-dimensional coordinates. We can put the outgoing modes near the horizon as right-moving modes and ingoing modes as left-moving modes. With the horizon being a null hypersurface, the ingoing modes at the horizon do not affect physics outside the horizon. It is a chiral theory and, therefore, there is chiral anomaly. The region away from the horizon \( (O) \) is four dimensional and is free of anomaly.

The covariant gauge current is anomalous because modes interior to the horizon cannot affect physics outside the horizon. The covariant gauge current conservation is given by

\[
\nabla\mu \tilde{J}^\mu = -\frac{e^2}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} = -\frac{e^2}{2\pi \sqrt{-g}} F_{\mu\nu} \eta^{\nu},
\]

where we take \( \varepsilon^{\mu\nu} = \varepsilon^{\mu\nu}/\sqrt{-g} \), \( \varepsilon_{\mu\nu} = \sqrt{-g} \varepsilon^{\mu\nu} \) and \( \epsilon^{tr} = 1 = -\epsilon_{tr} \). The current outside the horizon \( (r \in [r_H + \epsilon, \infty]) \) is anomaly free and satisfies the conservation law

\[
\nabla\mu \tilde{J}^\mu_{(O)} = 0.
\]
Near the horizon \((r \in [r_H, r_H + \epsilon])\), there are only outgoing fields and the current satisfies the anomalous equation
\[
\nabla_{\mu} \tilde{J}_{\mu}(H) = \frac{e^2}{2\pi \sqrt{-g}} \partial_r A_t.
\]
(9)

It appears that we have introduced a very sharp boundary between the two regions \((H)\) and \((O)\). Such a sharp boundary brings in delta functions in the analysis. There have been some works [43] which did away with such delta functions, but in this paper we follow the original approach.

Under gauge transformation, we obtain the following equation from the current conservation equation:
\[
J^r_{(O)} = J^r_{(H)} - \frac{e^2}{2\pi \sqrt{-g}} A_t.
\]
(10)

Solving two equations (8) and (9), the above equation gives the following condition,
\[
c_O = c_H - \frac{e^2}{2\pi} A_t(r_H),
\]
where \(c_O\) and \(c_H\) are the integration constants describing the charge flux. The vanishing of the covariant current \(J^r_{(H)}\) at the horizon results in the constant \(c_H\) being zero. Therefore, the value of the charge flux is given by
\[
c_O = \frac{e^2}{2\pi} A_t(r_H).
\]
(12)

If we consider another \(U(1)\) charge \(m\), then we have two \(U(1)\) gauge symmetries and currents. The gauge potential \(B_t\) should be the sum of two gauge fields \(A_t\) and \(N^\phi\),
\[
B_t = eA_t + mN^\phi.
\]
(13)

With these \(U(1)\) gauge currents, the anomalous equation can be described as [8]
\[
\nabla_{\mu} J^\mu = \frac{1}{2\pi \sqrt{-g}} \partial_r \mathcal{B}_t,
\]
(14)

where \(J^\mu\) is the sum of \(U(1)\) currents \(J^\mu_i/e_i\) \((i = 1, 2)\) associated with the \(U(1)\) charges \(e_1 = e, e_2 = m\), respectively.

In the presence of gauge fields, the energy–momentum tensor does not preserve the current conservation law but appears as the Lorentz force law. So the corresponding anomalous Ward identity is given by
\[
\nabla_{\mu} \tilde{T}^\mu = \mathcal{F}_{\mu\nu} J^\nu + \tilde{A}_\nu = \mathcal{F}_{\mu\nu} J^\mu - \frac{1}{96\pi} \epsilon_{\mu\nu\rho} \partial^\rho R,
\]
(15)

where \(\tilde{A}_\nu\) is the covariant gravitational anomaly. For the metric (2), the covariant anomaly is purely timelike
\[
\tilde{A}_t = 0,
\]
(16)
\[
\tilde{A}_r = \frac{1}{\sqrt{-g}} \partial_r \tilde{N}^\tau_{(t)},
\]
(17)

where
\[
\tilde{N}^\tau_{(t)} = \frac{1}{96\pi} \left( \frac{1}{2} f'h' + f''h + \frac{fh'^2}{h} \right).
\]
(18)
Outside the region of the horizon, the energy–momentum tensor satisfies the conservation law
\[ \nabla_\mu \tilde{T}^\mu_{(O)\nu} = 0. \] (19)

Near the horizon, the covariant energy–momentum tensor has the gravitational anomaly [6] and this anomaly equation is derived from equation (15)
\[ \nabla_\mu \tilde{T}^\mu_{(H)t} = \frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} \tilde{T}^r_{(H)t}) + \frac{1}{\sqrt{-g}} \partial_r \tilde{N}^r_{t}. \] (20)

Under the diffeomorphism invariance at the horizon, the conservation of the energy–momentum tensor gives the following equation:
\[ \tilde{T}^r_{(O)t} = \tilde{T}^r_{(H)t} - \frac{1}{4\pi} \sqrt{-g} \tilde{B}^2_{rt} + \frac{1}{\sqrt{-g}} \tilde{N}^r_{t}. \] (21)

Solving equations (19) and (20) and substituting into the above equation, we obtain
\[ a_O = a_H + \frac{1}{4\pi} \tilde{B}^2_{rt}(r_H) - \tilde{N}^r_{t}(r_H), \] (22)
where \( a_O \) and \( a_H \) are the integration constants representing the energy–momentum flux. The integration constant \( a_H \) can be fixed by requiring the covariant energy–momentum tensor to vanish at the horizon and this gives \( a_H = 0 \). Therefore, the final result is given by
\[ a_O = \frac{1}{4\pi} \tilde{B}^2_{rt}(r_H) - \tilde{N}^r_{t}(r_H) \] (23)
\[ = \frac{1}{4\pi} \tilde{B}^2_{rt}(r_H) - \frac{1}{96\pi} \left( \frac{1}{2} f'h' + f h'' - \frac{f h'^2}{h} \right)_{r=r_H}, \] (24)

where we have used equation (18). This value can be interpreted as the energy–momentum flux which coincides with the flux from black body radiation with a chemical potential at temperature \( T_H \) [4, 5]. This form of the energy–momentum flux has already been mentioned in [27]. The second term of (24) can be represented by the Hawking temperature \( T_H \) which is related to the surface gravity. The surface gravity can be calculated as follows:
\[ \kappa^2 = -\frac{1}{2} (\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu) \bigg|_{N'} = \frac{f}{h} \left( \frac{h'}{2} \right)^2 \bigg|_{r=r_H}, \] (25)
\[ \therefore \kappa = \sqrt{\frac{f}{h}} \frac{h'}{2} \bigg|_{r=r_H}. \] (26)

The relationship between the Hawking temperature and surface gravity is given by \( T_H = 1/\beta = \frac{\kappa}{\sqrt{\pi}} \). Coming back to equation (24), we focus on the second term. For simplicity, we consider the case where \( h(r) = f(r)g(r) \) and \( g(r_H) \neq 0 \). In this case, the flux becomes
\[ a_O = \frac{1}{4\pi} \tilde{B}^2_{rt}(r_H) + \frac{1}{192\pi} g f'^2 \] (27)
\[ = \frac{1}{4\pi} \tilde{B}^2_{rt}(r_H) + \frac{\pi}{12\beta^2}. \] (28)

This form is the key formula in discussions of Hawking radiation.
3. Hawking radiation of a warped AdS_3 black hole

The action of the three-dimensional gravity with the negative cosmological constant $\Lambda = -1/l^2$ and the gravitational Chern–Simons term is [44]

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) - \frac{1}{32\pi G\mu} \int \Gamma \wedge \left( d\Gamma + \frac{2}{3} R \right).$$ (29)

The equations of motion are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{l^2} g_{\mu\nu} = -\frac{1}{\mu} C_{\mu\nu},$$ (30)

where $C_{\mu\nu}$ is the Cotton tensor defined by

$$C_{\mu\nu} \equiv \epsilon_{\mu\rho\sigma} \nabla_{\rho} \left( R_{\sigma\nu} - \frac{1}{4} g_{\sigma\nu} R \right).$$ (31)

Clearly, all the vacuum solutions of Einstein’s equations automatically satisfy the equation of motion of topologically massive gravity. Recently, Anninos et al [21] have found that there exist two other vacuum solutions which are given by $SL(2, \mathbb{R}) \times U(1)$—invariant warped AdS_3 geometries with a timelike or spacelike $U(1)$ isometry for $\mu \ell \neq 3$. The warping transition from a stretching solution to a squashing one occurs at critical point $\mu \ell = 3$. Two null warped AdS_3 solutions with a null $U(1)$ isometry also exist. There are known warped black hole solutions which are asymptotic to AdS_3 for $\mu \ell > 3$.

We will only consider the solution for the asymptotically spacelike stretched case ($\mu \ell > 3$) which is free of naked closed timelike curves [45]. We can put the coefficient of the Chern–Simons action as $-\frac{l}{(96\pi G\nu)}$ in terms of the dimensionless coupling $\nu = \mu \ell / 3$. Then the metric describing the spacelike stretched black holes for $\nu^2 > 1$ is given in the Schwarzschild coordinates by

$$\frac{ds^2}{\ell^2} = dr^2 + \frac{dr^2}{(\nu^2 + 3)(r - r_+)(r - r_-)} + \left( 2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)} \right) dt d\theta$$

$$+ \frac{r}{4} \left( 3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\sqrt{r_+ r_- (\nu^2 + 3)} \right) d\theta^2,$$ (32)

where $r \in [0, \infty]$, $t \in [-\infty, \infty]$ and $\theta \sim \theta + 2\pi$. In the ADM formalism, this metric can be written as

$$ds^2 = -N(r)^2 dt^2 + \ell^2 R(r)^2 (d\theta + N^\theta(r) dt)^2 + \frac{\ell^4 dr^2}{4R(r)^2 N(r)^2},$$ (33)

where

$$R(r)^2 = \frac{r}{4} \left( 3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\sqrt{r_+ r_- (\nu^2 + 3)} \right),$$ (34)

$$N(r)^2 = \frac{\ell^2(\nu^2 + 3)(r - r_+)(r - r_-)}{4R(r)^2},$$ (35)

$$N^\theta(r) = \frac{2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}}{2R(r)^2}.$$ (36)

Here $r = r_+$ and $r = r_-$ are the locations of outer and inner horizons, respectively. In order to obtain the energy–momentum flux, we change the above metric to the form (2). If we put $\ell R(r) = \rho$, then the metric (33) becomes

$$ds^2 = -N(\rho)^2 d\rho^2 + \rho^2 (d\theta + N^\theta(\rho) d\tau)^2 + \frac{\ell^2 d\rho^2}{N(\rho)^2 (\rho^2)^2},$$ (37)
where \((R^2)'\) is a function of \(\rho\) which is described by a differentiation of \(R^2(r)\) with respect to \(r\). Comparing this metric with
\[
ds^2 = -h(\rho) \, dt^2 + d\rho^2 + \frac{d^2}{f(\rho)} + \rho^2 (d\theta + N(\rho)) \, dt^2,
\]
we can define
\[
h(\rho) = N(\rho)^2, \quad f(\rho) = \frac{(R^2)^2 N(\rho)^2}{\ell^2}.
\]
With these functions, we can find the energy–momentum flux which is given by (24). The anomaly term at the horizon \(\rho_H = \rho_\pm = \ell R(r_\pm) = \ell R_\pm\) gives
\[
\tilde{N}_\rho (\rho_\pm) = \frac{\ell^3 (v^2 + 3)}{4v} \rho_\pm - \rho_-, \quad (N(\rho_\pm'))|_{\rho_\pm} = \frac{(v^2 + 3) (\rho_\pm - \rho_-)^2}{4v} \rho_\pm - \rho_+.
\]
Using the above two formulae (43) and (44) we have
\[
\kappa = \frac{1}{2} \sqrt{\frac{1}{h}} \bigg|_{\rho_\pm} \bigg|_{\rho_\pm} = \frac{1}{\ell^3} \rho (N^2)' \bigg|_{\rho_\pm} = \left(\frac{v^2 + 3}{4v} \rho_\pm - \rho_- \right) \frac{2\pi}{\beta}.
\]
We can now read off the Hawking temperature from the surface gravity equation (44)
\[
\kappa = 2\pi/\beta = 2\pi T_H \quad \text{in the Schwarzschild coordinates} \quad T_H = \frac{(v^2 + 3)(r_+ - r_-)}{4\pi (2v_\pm - \sqrt{(v^2 + 3)r_+r_-})}.
\]
The gauge potential $B_t$ from (13) at the horizon $r_+ = r(\rho_+)$ is given by

$$B_t(\rho_+) = \frac{n}{\Omega_1}\left(\frac{\rho_+}{2\nu r_+ - \sqrt{(\nu^2 + 3)r_+}r_-}\right).$$

(49)

Using (36) and (46) we can read off the angular velocity in the Schwarzschild coordinate system:

$$N_\theta(\rho_+) = \frac{2}{\nu r_+ - \sqrt{(\nu^2 + 3)r_+}r_-}.$$

(50)

In the $(t, \rho, \theta)$ coordinate system, this angular velocity reduces to

$$N_\theta(\rho_+) = \frac{\ell}{\rho} + \frac{1}{R^2}.$$

So, the total flux of the energy–momentum tensor becomes

$$a_O = \frac{n}{4\pi} \frac{\ell^2}{\rho^2} + \frac{\pi}{12\beta^2} = \frac{n}{4\pi} \left(\frac{2}{2\nu r_+ - \sqrt{(\nu^2 + 3)r_+}r_-}\right)^2 + \frac{\pi}{12\beta^2}.$$

(51)

At $\nu^2 = 1$, the metric (33) reduces to the BTZ black hole in a rotating frame. If we set

$$d\phi = d\theta + d\tau/\ell \quad \text{and} \quad d\tau/\ell = dt/(\sqrt{r_+} - \sqrt{r_-})^2,$$

then we can find the angular velocity at $r_+$ to be $\Omega_\tau = \ell/\rho_+$. However, at $\nu^2 > 1$ if we take a similar rotating frame which has

$$d\phi = d\theta + \nu r dt/R(r)^2$$

then the angular velocity becomes $\Omega_\tau = \sqrt{\rho_+^2 - \rho_-^2}/(\ell\rho_+)$. As $\nu^2$ goes to 1, the warped AdS$_3$ black hole does not go to the BTZ black hole.

4. Hawking radiation of charged rotating black hole in (2+1)D

In this section, we consider yet another (2+1)-dimensional black hole. Let us now consider the case of a charged rotating black hole in three-dimensional spacetime. Its metric is given in the following form [31],

$$ds^2 = -N^2 F^2 dt^2 + F^{-2} dr^2 + R^2 (d\phi + N^\phi dt)^2,$$

(52)

where

$$R^2 = \frac{r^2 - \omega^2 f^2}{1 - \omega^2},$$

(53)

$$F^2 = \left(\frac{dR}{dr}\right)^2 f^2,$$

(54)

$$N = \frac{r}{R} \left(\frac{dr}{dR}\right) = \left(\frac{dr^2}{dR^2}\right),$$

(55)

$$N^\phi = \frac{\omega f^2}{(1 - \omega^2)R^2}.$$

(56)

In order to obtain this rather complicated looking metric, one can ‘rotation boost’ a charged non-rotating black hole solution. Consider the following boost along the $\phi$ direction, making the black hole to rotate,

$$\tilde{t} = \frac{t - \omega \phi}{\sqrt{1 - \omega^2}}, \quad \tilde{\phi} = \frac{\phi - \omega t}{\sqrt{1 - \omega^2}},$$

(57)

where $\omega^2 \leq 1$. The charged non-rotating black hole solution we start with is of the form

$$ds^2 = -f^2(r) dt^2 + f^{-2}(r) dr^2 + r^2 d\tilde{\phi}^2,$$

(58)

where

$$f^2(r) = r^2 - \tilde{M} - \frac{1}{2} \tilde{Q}^2 \ln r^2.$$

(59)
Here $\tilde{M}$ and $\tilde{Q}$ are the ADM mass and the total charge of the black hole. After the rotation boost, one obtains the following forms for the functions in the metric:

$$R^2 = r^2 + \frac{\omega^2}{1 - \omega^2} \left(\tilde{M} + \frac{\tilde{Q}^2}{4} \ln r^2\right),$$

(60)

$$F^2 = \frac{\left(r^2 + \frac{\omega^2 \tilde{Q}^2}{4(1 - \omega^2)}\right)^2}{R^2r^2} \left(r^2 - \tilde{M} - \frac{1}{4} \tilde{Q}^2 \ln r^2\right),$$

(61)

$$N = \frac{r^2}{r^2 + \frac{\omega^2 \tilde{Q}^2}{4(1 - \omega^2)}},$$

(62)

$$N^\phi = -\omega \frac{\tilde{M} + \frac{1}{4} \tilde{Q}^2 \ln r^2}{(1 - \omega^2)R^2}. $$

(63)

The electric potential $A = -\tilde{Q} \ln r \, dt$ transforms into a mixture of electric and magnetic potentials,

$$A = -\frac{\tilde{Q}}{\sqrt{1 - \omega^2}} \ln r \, (dt - \omega \, d\phi).$$

(64)

Considering the metric (52), we can replace functions $h(r)$ and $f(r)$ by $N^2(R)F^2(R)$ and $F^2(R)$, respectively. Then we can obtain

$$\kappa = \frac{1}{2} \left. N(F^2)^{\prime}\right|_{R=R_H} = \frac{2\pi}{\beta} = 2\pi T_H,$$

(66)

where $\beta$ is the inverse of the Hawking temperature $T_H$. The total flux of the energy–momentum tensor is given by [8]

$$a_O = \frac{1}{4\pi} \left(\mathcal{E}(R_H) + \mathcal{W}(R_H)\right)^2 + \frac{\pi}{12\beta^2},$$

(67)

where $\mathcal{W}(R_H) = -N^\phi(R_H)$. From equations (60) and (61), we can find the event horizons at

$$R_H^2 = -\frac{\tilde{Q}^2}{4(1 - \omega^2)} \mathcal{W}_k \left(-\frac{4}{\tilde{Q}^2} e^{-\frac{1}{\tilde{Q}^2} \tilde{M}}\right),$$

(68)

where function $\mathcal{W}_k(x)$ has $k = 0$ and $-1$. There has been some work calculating the Hawking radiation of (2+1)-dimensional black holes [46]. In that paper, non-rotating but charged and rotating but chargeless black hole cases were considered separately. The formulation in this paper, since we consider a rotating and charged black hole, can be regarded as a unified formulation.

In the above formula, $\mathcal{W}_k(x)$ is the Lambert function which is defined as the inverse of the function $f(y) = ye^y$. So we have

$$x = ye^y = f(y), \quad y = f^{-1}(x) = \mathcal{W}(x),$$

(69)

(70)

where integer $k$ denotes the branch of $\mathcal{W}_k(z)$ when we consider the extension to complex plane $x \rightarrow z$. $\mathcal{W}_0(x)$ and $\mathcal{W}_{-1}(x)$ are the only branches of Lambert function $\mathcal{W}$ which has real values.
From (68), we find that $W_k(x)$ should have negative values. $W_0(x)$ is an increasing function from $W_0(-1/e) = -1$ to infinity, passing through the point $W_0(0) = 0$. On the other hand, $W_{-1}(x)$ decreases from $W_{-1}(-1/e) = -1$ to $W_{-1}(0^-) = -\infty$. If $\tilde{M} \to \infty$, then the outer horizon from equation (68) goes to infinity and the inner horizon becomes 0. So, in general, for a given value of $x$, $W_{-1}(x)$ and $W_0(x)$ correspond to outer and inner horizons, respectively. When the argument of equation (70) is equal to $x = -1/e$, $W_0(-1/e)$ and $W_{-1}(-1/e)$ have the same value of $-1$ and it corresponds to the extremal case where $\tilde{M} = \tilde{Q}^2(1 - \ln(\tilde{Q}^2/4))/4$. Since the outer horizon has to stay outside of the inner horizon, the range of the argument $x$ is $x \in [-1/e, 0]$.

The Lambert function occurs in the solution of time-delayed differential equation $y'(t) = ay(t-1)$. This solution appears in the form $y(t) = \exp(V_k(a)t)$. For more details, one can see [47]. In figures 1 and 2 we show two Lambert functions $-W_0(x)$ and $-W_{-1}(x)$.

From (60), function $R^2$ as a function of $r^2$ has two branches separated by a singularity which makes an infinite throat at $r^2 = 0$. The region of the black hole corresponds to $R^2 \geq 0$. In the limit $\tilde{Q} \to 0$, $R^2$ becomes a linear function of $r^2$ and $R^2 = \tilde{R}^2 = \omega^2 \tilde{M}/(1 - \omega^2)$, which represents the inner horizon of the uncharged rotating black hole at $r^2 = 0$. 
When the electric charge is turned on, this throat disconnects inner horizon $R_-$ from the black hole space. Furthermore, a new inner horizon appears in the black hole space. This describes the perturbative instability of the inner horizon. In (68), outer and inner horizons are represented by $W_{-1}$ and $W_0$ respectively. In order to have two horizons, the argument of the Lambert function from (68) has to satisfy the condition $-1/e < x \leq 0$. This gives the condition between mass $\tilde{M}$ and charge $\tilde{Q}$,

$$\tilde{M} > \frac{\tilde{Q}^2}{4} \left( 1 - \ln \frac{\tilde{Q}^2}{4} \right),$$

(71)

and we can see the same condition in [31]. In figures 1 and 2, there are two points describing values of the Lambert functions $W_{-1}$ and $W_0$ as inner and outer horizons, respectively. So we can see that the outer horizon becomes

$$R_{H_{\text{out}}}^2 = -\frac{\tilde{Q}^2}{4(1 - \omega^2)} W_{-1} \left( -\frac{4}{\tilde{Q}^2} e^{-\frac{4\tilde{M}}{\tilde{Q}^2}} \right).$$

(72)

With the above outer horizon, we can obtain electric potential and angular momentum as follows:

$$\Lambda_t (R_{H_{\text{out}}}) = -\frac{\tilde{Q}}{\sqrt{1 - \omega^2}} \ln (R_{H_{\text{out}}}) \sqrt{1 - \omega^2},$$

(73)

$$\Omega (R_{H_{\text{out}}}) = -N^\phi (R_{H_{\text{out}}}) = \omega.$$  

(74)

Therefore, the final result for the flux of Hawking radiation becomes

$$a_O = \frac{1}{4\pi} \left( \frac{e\tilde{Q}}{\sqrt{1 - \omega^2}} \ln (R_{H_{\text{out}}}) \sqrt{1 - \omega^2} + m\omega \right)^2 + \frac{\pi}{12\beta^2},$$

(75)

which is the form of equation (24) with $\mathcal{B}_t$ in the form of (13). We have to consider the distribution of fermions in order to avoid superradiance in the case of rotating black holes [48]. The Hawking radiation is given by the Planck distribution with chemical potentials for electric charge $e$ of the field and an azimuthal angular momentum $m$ from the black hole. The fermion distribution for this black hole is given by

$$J_{e,\pm m}(\epsilon) = \frac{1}{e^\beta (e\epsilon + \Phi) + 1},$$

(76)

where $\Phi$ and $\Omega$ are defined by equations (73) and (74). From these distributions, we can also recover the same result (75)

$$a_O = \int_0^\infty \frac{d\epsilon}{2\pi} (J_{e,m}(\epsilon) + J_{-e,-m}(\epsilon)) = \frac{1}{4\pi} (e\Phi + m\Omega)^2 + \frac{\pi}{12\beta^2}.$$  

(77)

The first term of this value of the flux represents the chemical potential and the last term means the Hawking flux of this black hole.

5. Summary and outlook

In this paper, we studied the Hawking radiation from black holes in three-dimensional spacetime using methods of covariant anomaly cancelation [6] based on the works of Robinson and Wilczek [4]. In particular, we have calculated the total flux of the energy–momentum tensor from the warped spacelike stretched AdS$_3$ black hole in topologically massive gravity with negative cosmological constant, and also for the case of a charged rotating black hole in three-dimensional spacetime. The metric form of these two kinds of black holes is not
the same as that of the Reisner–Nordström black hole. The warped spacelike-stretched \( AdS_3 \) vacuum solution can be written as a spacelike Hopf fibration over Lorentzian \( AdS_2 \) with the fiber as a real line. In the case of the warped spacelike-stretched \( AdS_3 \) black hole, the metric form (33) is the same as that of the warped spacelike stretched \( AdS_3 \) vacuum solution, but the warped factor as a function of \( r \) is not constant. The metric form (2) can be obtained by the ‘rotation boost’ [31]. The radius of the event horizon can be determined by the solution of equation (61) which can be described using the Lambert function. The outer horizon is described by the Lambert function with its branch \( k = -1, W_{-1}(x) \), and the inner horizon by the Lambert function with \( k = 0, W_0(x) \). The flux of Hawking radiation (75), which is dependent on the outer horizon, is obtained in terms of \( W_{-1} \). There are many other higher dimensional black holes [10] and black rings [13] which were studied by this method. However, there are even more black objects which can, in principle, be studied by this method. It would be interesting to further our analysis to these objects.

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References

[1] Hawking S W 1975 Commun. Math. Phys. 43 199
Hawking S W 1976 Commun. Math. Phys. 46 206 (erratum)
[2] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2738
[3] Parikh M K and Wilczek F 2000 Phys. Rev. Lett. 85 5042 (arXiv:hep-th/9907001)
[4] Robinson S P and Wilczek F 2005 Phys. Rev. Lett. 95 011303 (arXiv:gr-qc/0502074)
[5] Iso S, Umetsu H and Wilczek F 2006 Phys. Rev. Lett. 96 151302 (arXiv:hep-th/0602146)
[6] Banerjee R and Kulkarni S 2008 Phys. Rev. D 77 024018 (arXiv:0707.2449)
[7] Banerjee R and Kulkarni S 2008 Phys. Lett. B 659 827 (arXiv:0709.3916)
[8] Iso S, Umetsu H and Wilczek F 2006 Phys. Rev. D 74 044017 (arXiv:hep-th/0606018)
[9] Setare M R 2007 Eur. Phys. J. C 49 865 (arXiv:hep-th/0608080)
[10] Iso S, Morita T and Umetsu H 2007 J. High Energy Phys. JHEP04(2007)068 (arXiv:hep-th/0612286)
[11] Vagenas E C and Das S 2006 J. High Energy Phys. JHEP10(2006)025 (arXiv:hep-th/0606077)
Xu Z and Chen B 2007 Phys. Rev. D 75 024041 (arXiv:hep-th/0612261)
Jiang Q-Q and Wu S-Q 2007 Phys. Lett. B 647 200 (arXiv:hep-th/0701002)
[12] Iso S, Morita T and Umetsu H 2007 Phys. Rev. D 75 124004 (arXiv:hep-th/0701272)
Iso S, Morita T and Umetsu H 2007 Nucl. Phys. B 799 60 (arXiv:0710.0453)
Iso S, Morita T and Umetsu H 2008 Phys. Rev. D 77 045007 (arXiv:0710.0456)
Iso S 2008 Int. J. Mod. Phys. A 23 2082 (arXiv:0804.0652)
[13] Chen B and He W 2008 Class. Quantum Grav. 25 135011 (arXiv:0705.2984)
[14] Jiang Q-Q, Wu S-Q and Cai X 2007 Phys. Rev. D 75 064029 (arXiv:hep-th/0701235)
Jiang Q-Q, Wu S-Q and Cai X 2007 Phys. Lett. B 651 65 (arXiv:0705.3871)
Kui X, Liu W and H-b Zhang 2007 Phys. Lett. B 647 482 (arXiv:hep-th/070102199)
Shin H and Kim W 2007 J. High Energy Phys. JHEP06(2007)012 (arXiv:0705.0265)
Peng J-J and Wu S-Q 2008 Chin. Phys. B 17 825 (arXiv:0705.1225)
Peng J-J and Wu S-Q 2007 Class. Quantum Grav. 24 5123 (arXiv:0706.0983)
Peng J-J and Wu S-Q 2008 Phys. Lett. B 661 300 (arXiv:0801.0185)
Jiang Q-Q 2007 Class. Quantum Grav. 24 4391 (arXiv:0705.2068)
Yang S-Z and Jiang Q-Q 2007 Int. J. Theor. Phys. 46 2138
Jiang Q-Q, Li H-L, Yang S-Z and Chen D-Y 2007 Mod. Phys. Lett. A 22 891
Kim W and Shin H 2007 J. High Energy Phys. JHEP07(2007)070 (arXiv:0706.3563)
Murata K and Miyamoto U 2007 Phys. Rev. D 76 084038 (arXiv:0707.0168)
Huang C-G, Sun J-R, X-n Wu and Zhang H-Q 2008 Mod. Phys. Lett. A 23 2957 (arXiv:0710.4766)
García A 1999 arXiv:hep-th/9909111

[34] Regge T and Teitelboim C 1974 Ann. Phys.,Lpz. 88 286
Sudarsky D and Wald R 1992 Phys. Rev. D 46 1453
Brown J and York J 1993 Phys. Rev. D 47 1407
Hayward S 1994 Phys. Rev. D 49 831
Hawking S and Horowitz G 1996 Class. Quantum Grav. 13 1487
Katz J, Bičák J and Lynden-Bell D 1997 Phys. Rev. D 55 5957
Booth I 2000 PhD Thesis University of Waterloo (arXiv:gr-qc/0008030)
Brown J, Lau S and York J 2000 arXiv:gr-qc/0010024

[35] Abbott L and Deser S 1982 Nucl. Phys. B 195 76
Deser S and Tekin B 2002 Phys. Rev. Lett. 89 101101
Clement G 2003 Phys. Rev. D 68 024032 (arXiv:gr-qc/0301129)

[36] See for example, Peet A W 2000 TASI lectures on black holes in string theory arXiv:hep-th/0008241

[37] Medved A J M 2002 Class. Quantum Grav. 19 589 (arXiv:hep-th/0110289)

[38] Olea R 2005 Mod. Phys. Lett. A 20 2649 (arXiv:hep-th/0401109)

[39] Henneaux M, Martinez C, Troncoso R and Zanelli J 2004 Phys. Rev. D 70 044034 (arXiv:hep-th/0404236)

[40] Cadoni M and Setare M 2008 arXiv:0806.2754

[41] Cadoni M, Melis M and Pani P 2008 arXiv:0812.3362

[42] Myung Y, Kim Y and Park Y 2009 arXiv:0903.2109

[43] Umetsu K 2008 Prog. Theor. Phys. 119 849 (arXiv:0804.0963)

[44] Deser S and Tekin B 2002 Class. Quantum Grav. 19 197 (arXiv:hep-th/0203273)

[45] Bouchareb A and Clement G 2007 Class. Quantum Grav. 24 5581 (arXiv:0706.0263)

[46] Jiang Q-Q, Wu S-Q and Cai X 2007 Phys. Lett. B 651 58 (arXiv:hep-th/0701048)

[47] Corless R M, Gonnet G H, Hare D E G, Jeffrey D J and Knuth D E 1996 Adv. Comput. Math. 5 329

[48] G W Gibbons G W 1975 Commun. Math. Phys. 44 245