New phenomena with the $f(R)$-theory of gravitation in a central gravitational field

Pham Van Ky$^1$, Nguyen Thi Hong Van$^{2,3}$ and Nguyen Anh Ky$^2$

$^1$Graduate university of science and technology, Vietnam academy of science and technology, Hanoi, Viet Nam.
$^2$Institute of physics, Vietnam academy of science and technology, Hanoi, Viet Nam.
$^3$Institute for interdisciplinary research in science and education, ICISE, Quy Nhon, Viet Nam.

E-mail: phamkyvatly@gmail.com, nhvan@iop.vast.ac.vn, anhky@iop.vast.ac.vn

Abstract. The $f(R)$-theory (of gravitation) is an extension of Einstein’s general theory of relativity (GR) but if a spherically symmetric vacuum solution of the Einstein equation in the GR is always stationary, a spherically symmetric vacuum solution of an $f(R)$-theory is not necessary stationary. This may have interesting consequences. In comparison with the GR, a process such as a planet’s motion (its orbital precession and parameters) and a gravitational deflection of light now get a correction which is a constant for a static central field and varies with time for a non-static central field even from a source of a constant mass, unlike the corresponding GR value not changing in the same situation. In particular, a spherically symmetric source may radiate gravitational waves. This phenomenon cannot happen in the GR. The present work is an extended version based on a presentation in the 44th Vietnam conference on theoretical physics (Dong Hoi, 29 July - 01 August 2019).

1. Introduction

The General theory of Relativity (GR) of A. Einstein is a very successful theory of gravitation [1, 2]. This theory has been verified very precisely, in particular, it was once again confirmed triumphantly by recent detections of gravitational waves [3, 4].

The heart of the GR is Einstein’s equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k T_{\mu\nu},$$  \hspace{1cm} (1)

$k = \frac{8\pi G}{c^4}$, derived from the Lagrangian $L_G = R$. However, some cosmological problems require the GR to be extended or modified. One of the modified theories of the GR is called $f(R)$-theory of gravitation.

The $f(R)$-theory of gravitation is based on the Lagrangian $L_G = f(R)$ leading to the equation

$$f'(R)R_{\mu\nu} - g_{\mu\nu} f'(R) + \nabla_\mu \nabla_\nu f'(R) - \frac{1}{2} f(R) g_{\mu\nu} = -k T_{\mu\nu},$$  \hspace{1cm} (2)

where $f(R)$ is a scalar function of the scalar curvature $R$, while $\nabla_\mu$ is a covariant derivative and $\Box = \nabla_\mu \nabla^\mu$. The function $f(R)$ could be, for example, $f(R) = R + \lambda R^2$ or $f(R) = R - \frac{\lambda}{R}$, etc.
2. Perturbative spherically symmetric solutions of an $f(R)$-theory

Starting with

$$f(R) = R + \lambda h(R), \quad \lambda h(R) \ll R,$$

where $h(R)$ is a scalar function of $R$ and $\lambda$ is a parameter (with an appropriate dimension which will be tacit here), we obtain from Eq. (2)

$$R_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} R + \lambda h'(R) R_{\nu}^{\mu} = -\frac{\lambda}{2} \delta_{\nu}^{\mu} h(R) - \lambda \delta_{\nu}^{\mu} \Delta h'(R) + \lambda \nabla^{\mu} \nabla_{\nu} h'(R) = -kT_{\nu}^{\mu}.$$  (3)

Using Einstein’s equation in the form $R = kT$ and $R_{\nu}^{\mu} = -k(T_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} T)$ and solving (3) perturbatively, we find the following perturbative solution

$$g_{00}(r, t) = 1 - \frac{kc^2(M - \lambda M_1(r, t) - \lambda M_2(r, t))}{4\pi r},$$

$$g_{11}(r, t) = \frac{-1}{g_{00}(r, t)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta,$$  (4)

where

$$M_1(r, t) = -\frac{2\pi}{kc^2} \int_0^r \left[ h(kT_0^0) + kT_0^0 h'(kT_0^0) \right] r^2 dr', \quad M_2(t) = -\frac{4\pi h''(kT_0^0)}{kc^2} \left[ \frac{\partial}{\partial t} \left[ \frac{M}{R(t)} \right] \right] \Gamma \alpha(t).$$  (5)

with

$$\alpha(t) = \frac{3\arcsin[\xi(t)R_0(t)] - \xi(t)R_0(t) \left[ 3 + 2[\xi(t)R_0(t)]^2 \right]}{256\pi^2[\xi(t)]^5 (3k^2c^2)^{-1} (1 - [\xi(t)R_0(t)]^2)^{3/2}}, \quad \xi^2(t) = \frac{kMc^2}{4\pi[R_0(t)]^3}.$$  (6)

Here $R_0(t)$ is the radius of the gravitational source at time $t$. Let us apply this result to some special cases.

2.1. Model $f(R) = R - 2\lambda$

In this case $h(R) = -2$ and, therefore,

$$g_{00}(r, t) = 1 - \frac{kc^2M}{4\pi r} - \frac{\lambda^2}{3}, \quad g_{11}(r, t) = \frac{-1}{1 - \frac{k^2c^2M}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta.$$  (7)

This solution coincides with the exact solution with $\lambda$ being the cosmological constant.

2.2. Model $f(R) = R + \lambda R^b$, ($b > 0$)

Here $h(R) = R^b$, we have

$$g_{00}(r, t) = 1 - \frac{kc^2M_f(t)}{4\pi r}, \quad g_{11}(r, t) = \frac{-1}{1 - \frac{k^2c^2M_f(t)}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta.$$  (8)

with

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t),$$

$$M_1(t) = \frac{(b + 1)(Mkc^2)^b}{2[\frac{4}{3}\pi R_0^3(t)]^{b-1}kc^2}, \quad M_2(t) = \frac{4\pi b(b - 1)(Mkc^2)^{b-2}2^{b-2}}{[\frac{4}{3}\pi R_0^3(t)]^{b-2}kc^2}\left[ \frac{\partial}{\partial t} \left[ \frac{M}{R_0(t)} \right] \right] \Gamma \alpha(t).$$  (9)
2.3. Model  $f(R) = R^{1+\varepsilon}$ with $\varepsilon$ very small

In this case $\lambda h(R) = R^{1+\varepsilon} - R$, we obtain

$$
g_{00}(r,t) = 1 - \frac{kc^2 M_f(t)}{4\pi r}, \quad g_{11}(r,t) = \frac{-1}{1 - \frac{kc^2 M_f(t)}{4\pi r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta$$

where

$$M_f(t) = M - \lambda M_1(t) - \lambda M_2(t),$$

$$\lambda M_1(t) = -M + \frac{(\varepsilon+2)(Mkc^2)^{\varepsilon+1}}{2[\frac{3}{2} \pi R_0^2(t)]^{\varepsilon+1}kc^2}, \quad \lambda M_2(t) = \frac{4\pi\varepsilon(\varepsilon+1)(Mkc^2)^{\varepsilon-1}}{[\frac{3}{2} \pi R_0^2(t)]^{\varepsilon-1}kc^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t).$$

3. Motion in a central field of the $f(R)$-theory

Applying metrics $g_{\mu\nu}$ obtained to the Hamilton-Jacobi equation

$$g_{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2,$$

we get a general equation of motion in a central field for an $f(R)$-theory, and, then a planet’s orbital precession [6]

$$\Delta \varphi(n) = \frac{6\pi m^2 G^2 [M - \lambda M_1(t_n) - \lambda M_2(t_n)]^2}{c^2 L^2},$$

where $L$ is the planet’s angular momentum. Using [2]

$$\frac{L^2}{m^2} = a(1 - e^2)GM_f(t)$$

we get

$$\Delta \varphi_{f(R)} = \frac{6\pi GM_f(t)}{c^2 a(1 - e^2)}.$$  \hspace{0.5cm} (13)

(for the deflection angle of light, see [5]).

4. Examination of the $f(R)$-theory

We will examine the $f(R)$-theory for $f(R) = R + \lambda R^2$ and $f(R) = R + \frac{\lambda}{R}$ in two cases: in a static central (gravitational) field and in a non-static central field. For this goal, let us apply the theory to some real gravitational systems, for example, Sun-Mercury (small system) and Sgr A*-S2 (big system).
4.1. Static central field

Let us take Starobinsky’s model \[ f(R) = R + \lambda R^2 \] as an example (for other models, see [5, 6]). Using the data [7]

\[
\begin{align*}
c &= 299792458 \, m/s; \\
g &= 6.67259 \times 10^{-11} \, kg^{-1} m^2 s^{-2}; \\
k &= \frac{8\pi G}{c^4} = 2.0761154 \times 10^{-43} \, kg^{-1} m^{-1} s^2; \\
M &\equiv M_\odot = 1.988919 \times 10^{30} kg; \\
2GM &\equiv c^2 = 2.95325008 \times 10^3 m; \\
a &= 5.7909175 \times 10^{10} m; \\
e &= 0.20563069; \\
\Delta\varphi_{\text{obs}} &= 2\pi(7.98734 \pm 0.00037) \times 10^{-8} \, \text{radian/revolution}, \\
\frac{6\pi G}{c^2a(1 - e^2)} &= 2.523307 \times 10^{-37}. 
\end{align*}
\]

we obtain the orbital precession of Mercury orbiting around the Sun (the Sun-Mercury system) as follows

- Einstein’s value:
  \[ \Delta\varphi_E = 1.59748694\pi \times 10^{-7} \, \text{radian/rev}. \]
- Observed value:
  \[ \Delta\varphi_{\text{obs}} = 1.597468\pi \times 10^{-7} \, \text{radian/rev}. \]
Deviation:

\[ \delta \varphi^{\text{Mer}} = \Delta \varphi_{\text{obs}} - \Delta \varphi_E = -0.1906\pi \times 10^{-11} \text{ radian/rev}. \]

Roughly, if the measurement’s error is smaller than the latter, \( \lambda \) should take the value

\[ \lambda = 0.296631 \times 10^{18} \] (16)

in order to explain the deviation

\[ \Delta \varphi_f(R) = \Delta \varphi_{\text{obs}} = 1.597468\pi \times 10^{-7} \text{ radian/rev}. \] (17)

and

\[ \delta \varphi^{\text{Mer}} = \Delta \varphi_f(R) - \Delta \varphi_E = \Delta \varphi_{\text{obs}} - \Delta \varphi_E = -0.1906\pi \times 10^{-11} \text{ radian/rev}. \] (18)

It is worth noting that the value of \( \lambda \) in (16) satisfies the perturbation condition \( \lambda h(R) \ll R \) (see [6]), that is

\[ \lambda h(kT_0^0) \ll \frac{6GM}{c^2[R_0]^3}, \] (19)

or, equivalently,

\[ \lambda \ll \frac{c^2[R_0]^3}{6GM} = 0.380053 \times 10^{23}. \] (20)

Applying \( \lambda \) given above to a stronger gravitational system, e.g., the system Sgr A*-S2 (with mass \( M = 4.31 \times 10^6 M_\odot \)) we obtain

- \( \Delta \varphi^{S2}_f(R) = 1.149305\pi \times 10^{-3} \text{ radian/rev}. \)
- \( \Delta \varphi^{S2}_E = 1.15114\pi \times 10^{-3} \text{ radian/rev}. \)
- \( \delta \varphi^{S2} = \Delta \varphi^{S2}_f(R) - \Delta \varphi^{S2}_E = -1.835\pi \times 10^{-6} \text{ radian/rev}. \)

We do not consider \( f(R) = R + \lambda'/R \) for this case as it, compared with the GR, does not give a new correction (to the orbital precession). To check the theory it is necessary to work in a non-static field.

4.2. Non-static central field

The theory \( f(R) = R + \lambda R^2 \) is examined above in a static central field and its examination can be repeated straightforwardly for a non-static case. In this case we will examine one more theory, namely, the theory \( f(R) = R + \lambda'/R. \) As the value of \( \lambda \), assumed to be universal and, therefore, applicable to the present case, is already estimated in (16), now we estimate the value of \( \lambda' \). The perturbation condition \( \lambda h(R) \ll R \) applied to the theory \( f(R) = R + \lambda'/R \) gives

\[ \lambda' \ll \frac{9}{[R_0]^6} \left( \frac{2GM}{c^2} \right)^2 = 6.923265 \times 10^{-46}. \] (21)

Next, we estimate the correction to the orbital parameters (eccentricity and axes), compared with their classical values, of a planet moving around a collapsing star with the beginning radius \( R_0 \) (before the collapse) and the final radius \( R_0(t) \) in a moment \( t \) (during the collapsing process or at the end of the collapse). Here, we consider a collapsing star as example but it is possible to consider an expanding (exploding) star. In order to have data for reference we will take a
Sun-similar star and its Mercury. Below, we will see how an orbit of a planet (Mercury) would change under the star (Sun) contraction keeping its spherical form. In this situation the GR gives no effect unlike the $f(R)$ theory predicting new interesting phenomena (such as corrections to a planet’s orbital eccentricity and axes, gravitational waves, etc.) which can be used for testing an $f(R)$ theory.

A star of the Sun’s size having a radius of the order

$$R_0 \approx 6.957 \times 10^8 \text{ m},$$

would collapse to a white dwarf \(^1\) of the Earth’s size (or smaller) with a radius of the order

$$R_0(t) \approx 6.371 \times 10^6 \text{ m}.$$  

(23)

The radius change

$$\Delta R_0(t) = R_0(t) - R_0 = -689329000 \text{ m}.$$  

(24)

would happen for the free falling (assumed) time interval \(^9\)

$$\Delta t = -\left(\frac{8\pi G\rho_0}{3}\right)^{-1/2} \int_1^{R_0(t)} \left(\frac{\zeta}{1 - \zeta}\right)^{1/2} d\zeta$$  

$$= \left(\frac{3\pi}{32G\rho_0}\right)^{1/2} \left(1 + 3.45 \times 10^{-4}\right)$$  

$$= \left(\frac{4\pi^2(R_0)^3}{32GM}\right)^{1/2} \left(1 + 3.45 \times 10^{-4}\right)$$  

$$= 1769.83 \text{ s},$$  

(25)

with \(\frac{R_0(t)}{R_0} = 9.158 \times 10^{-3}\). From here we can estimate the average free falling speed as

$$\frac{|\Delta R_0(t)|}{\Delta t} = 389488.82 \text{ m/s}.$$  

(26)

Now, let us calculate some parameters of a planet’s (Mercury’s) orbit in an $f(R)$ theory, where, the star’s (Sun’s) mass $M$ is replaced by an effective mass $M_f$ \(^5\). This mass would change with a value $\Delta M_f$ during the star contraction until its total collapse. The semi-major axis $a$ and the eccentricity $e$ of the planet’s elliptical orbit in an $f(R)$ theory are calculated by the formulas

$$a = \frac{GmM_f(t)}{2|E|},$$  

(27)

$$e \approx \sqrt{1 - \frac{2|E|L^2}{G^2m^3M_f^2}},$$  

(28)

where $L$ is the angular momentum which is a conserved quantity but the energy $E$ is not conserved for a non-static field. To calculate the energy change during the star contraction (until the total collapse) we use the approximation $E \approx -\frac{m(GmM_f)^2}{2L^2}$ (for a circular orbit), thus,

$$\Delta|E| = \frac{2|E|}{M_f} \Delta M_f.$$  

(29)

\(^1\) More precisely, according to the standard theory of the star evolution, a Sun-type star would first become a red giant before collapsing to a white dwarf.
Therefore, the planet’s orbital semi-major axis and eccentricity would change by the quantities

\[ \Delta a = \frac{Gm}{2|E|} \left( \Delta M_f - \frac{M_f \Delta |E|}{|E|} \right) = -\frac{Gm}{2|E|} \Delta M_f = -\frac{L^2 \Delta M_f}{Gm^2 M_f^2}, \quad (30) \]

\[ \Delta e = -\frac{L^2}{G^2 m^3 M_f^2} \left( \Delta |E| - \frac{2|E| \Delta M_f}{M_f} \right) = 0, \quad (31) \]

or with (12) taken into account these changes become

\[ \Delta a(t) = a(t) - a_0 = -a_0 (1 - e^2) \frac{\Delta M_f(t)}{M_f}, \quad (32) \]

\[ \Delta e = 0, \quad (33) \]

where \( a_0 \) and \( a(t) \) are the semi-major axis before the contraction and after the collapse of the star, respectively.

### 4.2.1. Model \( f(R) = R + \lambda R^2 \):

Assuming that the star’s mass \( M \) remains unchanged during the collapse process (in reality, some gravitational radiation and other matter loss may be possible), we calculate the change of the components \( M_1 \) and \( M_2 \) of \( M_f \) during the collapsing (in this \( f(R) \)-theory \( M_1 \) and \( M_2 \) have dimension of [mass/length^2] which will be tacit below).

Let us start with calculating the change \( \Delta M_1 \) of \( M_1 \). It is easy to see that

\[ \Delta M_1(t) = M_1(t) - M_1(0) = 1.02213108 \times 10^{14} \quad (34) \]

where

\[ M_1(0) = \frac{9M_f^2 k c^2}{8\pi [R_0(0)]^3} = 78498929.12, \quad (35) \]

is the value of \( M_1 \) before the collapse starting \( (t = 0) \), and

\[ M_1(t) = \frac{9M_f^2 k c^2}{8\pi [R_0(t)]^3} = 1.02213186 \times 10^{14}, \quad (36) \]

is the value of \( M_1 \) immediately before the end of the collapse. In order to calculate the change \( \Delta M_2 \) of \( M_2 \) we must first calculate \( \xi(t) \) and \( \alpha(t) \). Using (6), (14) and (23) we get

\[ \xi(t) = \sqrt{\frac{kM c^2}{4\pi [R_0(t)]^3}} = 3.37939 \times 10^{-9} \quad (37) \]

and

\[ \alpha(t) = 4.5029183 \times 10^{-36}. \quad (38) \]

With the approximation \( \frac{\partial}{\partial t} R_0(t) \approx \frac{\Delta R_0(t)}{\Delta t} \),

\[ \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t) = \left( \frac{-3M}{[R_0(t)]^4} \frac{\Delta R_0(t)}{\Delta t} \right)^2 \alpha(t) = 8.959833 \times 10^{-18}, \quad (39) \]

\( M_2 \) takes the value

\[ M_2 = \frac{8\pi}{k c^2} \left[ \frac{\partial}{\partial t} \frac{M}{[R_0(t)]^3} \right]^2 \alpha(t) = 1.206832 \times 10^{10}. \quad (40) \]
Combining (36) and (40) we have

\[ M_f(t) = M(t) - \lambda M_1(t) - \lambda M_2(t) = -2.833426 \times 10^{31} \text{ kg}, \tag{41} \]
\[ \Delta M_f = M_f(t) - M_f = -3.03231558 \times 10^{31} \text{ kg}. \tag{42} \]

Inserting this result in (30) we get

\[ \Delta a(t) = a(t) - a = 0.4166264 \times 10^{10} \text{ m}. \tag{43} \]

In comparison with the GR, i.e., comparing (43) with (14), we see that the semi-major axis \( a(t) \) of the Mercury-like planet in the \( f(R) \)-theory would increase with 7.19\% after the collapse of the Sun-like star. The sign minus in (41) and (42) shows something like an “anti-gravitational” effect which could be a strong argument for verifying the present model.

4.2.2. Model \( f(R) = R + \frac{\lambda}{M} \) :

In this model, following similar calculations as above it is not difficult to get for a Sun-Mercury-like system

\[ \Delta M_f(t) \simeq 0, \tag{44} \]

that is, there is no sensitive correction to the GR.

5. Conclusion

The \( f(R) \)-theory is a modified theory of gravitation which makes correction to the general theory of relativity and may replace the latter in explaining new cosmological observations.

We have shown that the \( f(R) \)-theory allows a non-static spherically symmetric solution and predicts (non-static in general) corrections to cosmological observations (such as orbital precessions and deformations, deflections of light, etc.). Another prediction which is about gravitational radiations of a (non-static) spherically symmetric source, a phenomenon not possible in the general theory of relativity, can be also considered, but it is a subject of a later work being in progress. A testing measurement (observation) may not be easy at the present technical level but we hope it can be done in a not very far future.

Acknowledgments

This work is funded by the Vietnam academy of science and technology (VAST) under the grant NVCC05.09/20-20.

References

[1] Weinberg S 1972 Gravitation and cosmology: Principles and applications of the general theory of relativity (John Wiley & Son, New York)
[2] Landau L D and Lifshitz E M 1994 The classical theory of fields vol. 2 (Elsevier Oxford)
[3] Abbott B P et al [LIGO scientific and Virgo collaborations] 2016 Phys. Rev. Lett. 116 061102 (Preprint arXiv:1602.03837 [gr-qc])
[4] Abbott B P et al [LIGO scientific and Virgo collaborations] 2017 Phys. Rev. Lett. 119, 161101 (Preprint arXiv:1710.05832 [gr-qc])
[5] Nguyen Anh Ky, Pham Van Ky and Nguyen Thi Hong Van 2018 Eur. Phys. J. C 78 539; Erratum: 2018 Eur. Phys. J. C 78 664 (Preprint arXiv:1807.04628 [gr-qc])
[6] Nguyen Anh Ky, Pham Van Ky and Nguyen Thi Hong Van 2019 Commun. Phys. 29 35 (Preprint arXiv:1904.04013 [physics.gen-ph])
[7] Majumder B *Preprint* arXiv: 1105.2428
[8] Starobinsky A A 1980 *Phys. Lett.* B 91 99 (1987 *Adv. Ser. Astrophys. Cosmol.* 3 130)
[9] Kippenhahn R and Weigert A and Weiss A 2012 *Stellar structure and evolution* (Springer-Verlag)