Spotlighting phase separation in Rashba spin-orbit coupled Bose–Einstein condensates in two dimensions

S Bhuvaneswari, K Nithyanandan1,2, and P Muruganandam1,3

1 Department of Physics, Bharathidasan University, Palkalaiperur, Tiruchirappalli 620024, India
2 Laboratoire Interdisciplinaire Carnot de Bourgogne, Université de Bourgogne Franche-Comté, Dijon, France
3 CNRS/Université Joseph Fourier, Laboratoire Interdisciplinaire de Physique (LIPHY), Grenoble, France

E-mail: anand@cnld.bdu.ac.in

Abstract

We study the system of spin-orbit (SO) coupled Bose–Einstein condensates (BEC) with Rabi coupling in quasi-two dimensions characterized by unequal Rashba and Dresselhaus couplings. The ground state properties and the phase diagram of the system are studied within the mean-field approximation at $T = 0$. The energy-momentum dispersion relation corresponding to the single-particle ground state exhibits an infinitely degenerate Rashba ring, whose degeneracy is destroyed by the Rabi coupling, leading to a single-point lowest energy. The effect of Rabi coupling and interaction in the system show up three different phases, namely stripe, plane wave and zero momentum phase. At a particular parametric condition, the system admits a critical point separating all three different phases, known as the tri-critical point. For further insight, the momentum distribution, the energy and the longitudinal/transverse spin polarization corresponding to the quantum phases are comprehensively discussed.

1. Introduction

A large class of interesting phenomena in condensed matter physics (CMP) can be possible when electrons interact with the external gauge field or in the presence of strong spin–orbit (SO) coupling [1]. The spin–orbit coupling describes the interaction between particle’s spin and its orbital momentum remains an important topic of current research, especially in the context of macroscopic quantum phenomena [2–5]. This interesting physical effect brought many exotic features like topological insulators [6, 7], spin–Hall effect [8], Majorana fermions [9], spintronic devices [10], quantum computing applications [11] etc. Particularly, the experimental realization of the Bose–Einstein condensation (BEC) has opened new research avenue in the ultracold atoms to investigate these interesting quantum phenomena. An extended overview on the recent developments in the synthetic SO coupling in the ultracold atomic gases can be inferred from the review by Zhai [12]. The upfront difficulties in possibly employing the ultracold atoms as a test-bed to simulate the above CMP phenomena lies in the fact that the atoms are neutral and thus ruled-out any possibilities of coupling with the gauge field. To overcome this difficulty of neutrality, several interesting proposals were suggested to realize the synthetic gauge fields for quantum gases [13–16].

One of the successful means of realizing synthetic gauge field is by exploiting the space-dependent coupling of the tailor-made laser configurations with the atoms. By this way, one can engineer an artificial external abelian or non-abelian gauge potential coupled to neutral atoms based on the so-called Berry phase effect [17–19]. In the recent past, a variety of synthetic gauge fields can be engineered using laser beams. This field of research is identified to be very promising, owing to its striking features of realizing exotic configurations of nontrivial topology, and also the possibilities to simulate many important electronic phenomena in CMP. A seminal contribution to this exciting field was given by Spielman group at NIST, where they engineer the SO coupling in neutral BEC by dressing two atomic spin states using a pair of lasers [20]. The momentum-sensitive coupling was realized in $^{87}$Rb (equal contribution of Rashba and Dresselhaus) by a pair of Raman lasers using two of its
We consider a SO coupled BECs in which the atomic cloud along one or two directions is coupled to two directions in the momentum space and can have relatively higher symmetry than the standard Raman-induced SO coupling.

The particular feature of the Rashba type SO coupling is the existence of infinite degeneracy in the minima of single particle energy spectrum in the momentum space [24, 31, 32]. The system also admits two kinds of phases namely, the plane wave (PW) phase and the stripe wave phase (SW) as a consequence of the competition between the inter- and intra-component interaction. Moreover, a fully polarized phase known as the zero momentum (ZM) phase has also been identified at some parametric conditions. In reference [33], Li et al theoretically demonstrated the phase diagram encompassing different phase and reported a tri-criticality with an equal contribution of Rashba and Dresselhaus SO coupling in one-dimensional BEC [33]. Following this striking observation, there were few interesting results in the directions of exploring the different phase information associated with the SO coupled BEC [2, 34–37]. However, most of the studies on SO coupled BECs were primarily centered on the one-dimensional systems and only a few studies were devoted on two-dimensional SO coupled BECs. For instance, the dynamics of vortices, the existence of vortex-antivortex pair, modulation instability in quasi-two-dimensional BEC [38–43] are few notable works concerning higher dimensional SO coupled-BEC. One way of controlling the dimensionality of the system is by using a tight optical confinement of the atomic cloud along one or two directions [14–16].

Motivated by the recent advances in the higher dimensional SO coupled BEC, in what follows, we consider a two dimensional SO coupled BEC characterized by the different contribution of Rashba and Dresselhaus SO coupling [14–16, 58, 39, 41–43]. In this work, we have investigated the ground state properties of a SO coupled BEC and the quantum phases exhibited by the system with a particular emphasis on the influence of Rabi coupling. Firstly, we discussed the situation at the single-particle level, showing the peculiar features of the single-particle energy spectrum. Further, the properties are computed within the mean-field Gross-Pitaevskii (GP) theory through a variational approach. We point out certain interesting configurations, including stripe phase, spin-polarized phase and zero momentum phase. Then, we extend the analysis on the possibilities of phase transition between various phases and reported the existence of tri-criticality.

The paper is organized as follows. Section 2 introduces the theoretical model of the SO-coupled BECs. Section 3 features the results of our systematic analysis calculating three different quantum phases of the SO coupled BEC. Section 4 is devoted to the discussion of results and a summary is included in section 5.

2. The Model equation

We consider a SO coupled BECs confined in a quasi-2D harmonic trap, with respective longitudinal and transverse frequencies \( \omega_z \) and \( \omega_{\perp} \), such that \( \omega_z \gg \omega_{\perp} \). By assuming different contributions of Rashba and Dresselhaus coupling, within the framework of mean-field theory, the system can be described by the energy functional,

\[
E = \int \mathcal{E} d^3r
\]

where,

\[
\mathcal{E} = \Psi^\dagger H_0 \Psi + \frac{g_{11}}{2}|\psi_1|^4 + \frac{g_{22}}{2}|\psi_2|^4 + g_{12}(|\psi_1|^2|\psi_2|^2),
\]

\[\Psi = (\psi_1, \psi_2)^T, \text{ with } \psi_1 \text{ and } \psi_2 \text{ are the pseudo-spin components of the condensate wavefunction, and } H_0 \text{ is the single particle Hamiltonian which assumes the following form,} \]

\[
H_0 = \frac{1}{2m}\left\{ (p_x - k_x \sigma_x)^2 + (p_y - k_y \sigma_y)^2 + p_z^2 \right\} + \Omega \sigma_x + \delta \sigma_z + V(\vec{r}),
\]

where \((p_x, p_y, p_z) = -i\hbar(\partial_x, \partial_y, \partial_z)\) are the momentum operators, \(k_i\) is the wavenumber of the Raman laser which couples the two hyperfine states, \(\sigma_j (j = x, y, z)\) are the Pauli spin matrices, \(\Omega\) characterize the strength of the Rabi coupling, \(\delta\) is the Raman detuning parameter, and \(V(\vec{r}) = m[\omega_z^2(x^2 + y^2) + \omega_{\perp}^2z^2]/2\) is a quasi-2D harmonic trapping potential. The effective two dimensional coupling constants \(g_{ij} = 4\pi\hbar^2a_{ij}/m, (i,j = 1,2)\).
represent, the intra-($g_{11}, g_{22}$) and inter-component ($g_{12}$) interaction strengths, which are defined by means of $s$-wave scattering lengths $a_j$ and atomic mass $m$. Energy is measured in units of the radial trap frequency ($\omega_\parallel$), i.e., $\hbar\omega_\parallel$, length in units of harmonic oscillator length, $a_i = \sqrt{\hbar/(m\omega)}$, and time in units of $\omega_\parallel^{-1}$. The SO nature captured by the Hamiltonian is a consequence of the non-commutation of the kinetic energy and the position-rotation, while the re-normalization of the effective field $\delta = 0$ results from the additional time dependence exhibited by the wave function in the rotating frame [35, 36]. One of the principal aspect of the SO coupled Bose gases is the ability to create synthetic spin-half system by means of laser-dressed bosonic states. In the conventional BEC, as the temperature is lowered, the atoms condense at the ground state, often recognized as non-degenerate zero-momentum state. This would not be the case of SO-coupled BEC, where unlike the conventional counterpart, the spin-orbit bosons tend to condense with multiple lowest degenerate energy states, leading to energy-momentum dispersion [2, 34–36].

Thus, it is interesting to understand the energy-momentum dispersion relation corresponding to the single-particle ground state. Let us assume a homogeneous non-interacting SO coupled Bose gas by turning down the trapping potential and interatomic interactions. The derived eigenspectrum accounting for the energy-momentum dispersion can be written as

$$
\omega_\pm(k) = k_x^2 + \frac{k_y^2}{2} \pm \sqrt{\Omega^2 - 2\Omega k_x k_y + k_y^2},
$$

where $k_x^2 = k_{li}^2 + k_y^2$ with $k_x$ and $k_y$ are components of the wavevectors along the respective directions. The energy spectrum corresponding to the single-particle Hamiltonian consists of two branches. The $\pm$ sign denotes the different helicity basis corresponding to either parallel or anti-parallel spin-index with reference to the wave vector. Here, we study three different cases of energy spectrum based on the combination of Rashba SO and Rabi coupling. In figure 1(a) we show the typical energy momentum dispersion spectrum in the absence of SO and Rabi coupling. It is straightforward to note that the energy spectrum is a simple free-particle parabolic dispersion, indicating the existence of non-degenerate single-point lowest energy ground state. Figure 1(b) is the case with only Rashba SO coupling and the single-particle energy spectrum consists of ring of infinitely degenerate lowest energy states in momentum space, which is typically identified as Rashba ring [24, 31, 32].

While considering Rabi coupling into the picture, the ground state eigenspectrum of the spinor Bose gases shows some remarkable features as depicted in figure 1(c). For any finite value of $\Omega$, the rotating symmetry of the Hamiltonian is destroyed and the Rashba ring vanishes, which results in a single global minimum at $(-k_{li}, 0)$ as shown in figure 1(c). Thus it is evident that the Rabi coupling modifies the energy spectrum corresponding to a single-particle ground state from an infinitely degenerate Rashba ring to a non-degenerate state of single-point lowest energy at $(k_L, 0)$. For further insight into the effect of Rabi coupling, we plot the eigenspectrum with reference to a range of Rabi frequencies in figure 2. In our present case, the lowest energy (global minimum) ground state is given by $\omega_\pm(k)$ branch, however, for completeness, we also consider the other branch $\omega_\pm(k)$. In figure 2(a) we show the single-particle ground state for different values of $\Omega$, and figure 2(b) depicts the variation of lowest energy as a function Rabi frequency. It is very obvious that any increase in the strength of the Rabi coupling inherently decreases the minimum energy corresponding to the single particle ground state. Also, any change of sign of $\Omega$, meaning that $\Omega \rightarrow -\Omega$ is equivalent to postulating $\psi \rightarrow -\psi$, $x \rightarrow -x$ and $y \rightarrow -y$, therefore one can expect corresponding single particle energy minimum at $(-k_{li}, 0)$.
3. Phase diagram

We carry out a variational analysis by assuming the following ansatz for the spin components

\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = \sqrt{n} \begin{bmatrix}
\cos \theta \\
-\sin \theta e^{i\phi}
\end{bmatrix} e^{i k \cdot r} + C_2 \begin{bmatrix}
\sin \theta \\
-\cos \theta e^{i\phi}
\end{bmatrix} e^{-i k \cdot r},
\]

(5)

where \( \mathbf{r} \equiv (x, y) \), and \( \mathbf{k} \equiv (k_x, k_y) \) is the canonical momentum, \( \bar{n} = N/V \) is the average density, and \( N \) be the total number of atoms in volume \( V \). \( C_1, C_2, \theta \) and \( \phi \) are the variational parameters whose values can be fixed by the procedure of energy minimization of equation (3) with the help of normalization constraint

\[
\sum_{i=1,2} \int d^3r |\psi_i|^2 = N,
\]

which implies \( |C_1|^2 + |C_2|^2 = 1 \). The minimization with respect to \( \theta \) and \( \phi \) yields

\[
\theta = \frac{1}{2} \arccos(k_x/k_z), \quad 0 \leq \theta \leq \pi/4,
\]

(6a)

\[
\phi = \arctan(k_y/k_z), \quad 0 \leq \phi \leq \pi,
\]

(6b)

and can be fixed by the single-particle Hamiltonian (3). The density \( n(\mathbf{r}) \), can be obtained from the variational parameters in terms of the key physical quantities of the system such as the momentum \( \mathbf{k} \) and the total density \( \bar{n} \) as follows

\[
n(\mathbf{r}) = \bar{n}[1 + 2|C_1C_2|(1 - \Delta^2)^{1/2} \cos(2\mathbf{k} \cdot \mathbf{r} + \varphi)],
\]

(7)

with \( \Delta = k_x/k_z \) and \( \varphi \) is the relative phase between \( C_1 \) and \( C_2 \). The spin densities are given by

\[
s_x(\mathbf{r}) = - \bar{n} \left[ (1 - \Delta^2)^{1/2} + 2|C_1C_2|(\cos(2\mathbf{k} \cdot \mathbf{r} + \varphi) + \Delta \sin(2\mathbf{k} \cdot \mathbf{r} + \varphi) \tan \phi) \right] \cos \phi,
\]

(8a)

\[
s_y(\mathbf{r}) = - \bar{n} \left[ (1 - \Delta^2)^{1/2} + 2|C_1C_2|(\cos(2\mathbf{k} \cdot \mathbf{r} + \varphi) - \Delta \sin(2\mathbf{k} \cdot \mathbf{r} + \varphi) \cot \phi) \right] \sin \phi,
\]

(8b)

\[
s_z(\mathbf{r}) = \bar{n} \Delta(|C_1|^2 - |C_2|^2).
\]

(8c)

Using equations (8) one can determine the corresponding spin polarizations given by the expression

\[
\sigma_j = (1/\bar{n}) \int s_j(\mathbf{r}) \, d\mathbf{r}
\]

with \( j = x, y, z \). Before going on, we notice that equations (8a) and (8b) hold in the spin-rotated frame. Since the operators \( \sigma_x \) and \( \sigma_y \) do not commute with \( \sigma_z \), the transverse spin densities are computed in the original laboratory frame and exhibits an additional oscillatory behaviour of \( \sigma_x \) and \( \sigma_y \) given by equations (8a) and (8b).

By substituting equation (5) in equation (1), we find that the energy per particle \( \varepsilon = E/N \) takes the form

\[
\varepsilon = k_x^2 + G_1 + \frac{k_y^2}{2} + 2G_1|C_1|^2|C_2|^2 \sin^2 2\theta + G_2(|C_1|^2 - |C_2|^2)^2 \cos^2 2\theta
\]

\[+ (|C_1|^2 - |C_2|^2)k_z \sin 2\theta (k_x \cos \phi + k_y \sin \phi) - \Omega \sin 2\theta \cos \phi,
\]

(9)

where \( G_1 = \bar{n}(g + g_{22})/4 \) and \( G_2 = \bar{n}(g - g_{11})/4 \) with \( g = g_{11} = g_{22} \). By minimizing equation (9) with respect to \( k \), we can deduce \( k_x = -k_z \sin 2\theta \cos \phi (|C_2|^2 - |C_1|^2) \) and \( k_y = -k_z \sin 2\theta \sin \phi (|C_2|^2 - |C_1|^2) \). The mean-field ground state of the system can then be obtained by fixing \( |C_1|^2 = (1 - \tau)/2 \) and \( |C_2|^2 = (1 + \tau)/2 \), where \( \tau \in [-1, 1] \).

By substituting the expression for \( k_x \) and \( k_y \) in equation (9), we observed that the momentum distribution has been modified explicitly by the interactions. Hence, the resulting ground state energy per particle is given as

![Figure 2. (a) Single particle spectrum as a function of \( k_x \) with fixed SO coupling (\( k_z = 2 \)) for various Rabi coupling strengths (\( \Omega \)) and (b) Minimum energy as a function of \( \Omega \).](image-url)
\[ \varepsilon = G_2 \tau^2 (1 - \sin^2 2\theta) + \frac{1}{2} G_1 (1 - \tau^2) \sin^2 2\theta - \Omega \sin 2\theta \cos \phi - \frac{1}{2} k_f^2 \tau^2 \sin^2 2\theta + k_f^2 + G_3. \]  

(10)

It is quite obvious from the above equation (10) that the ground state of the system is a function of SO and Rashba coupling. In order to find the ground state, we minimize equation (10) with respect to \( \sin 2\theta \) for \( \tau > 0 \) (note: both \( \tau > 0 \) and \( \tau < 0 \) cases correspond to the same minimum). The ground state is then compatible with three distinct quantum phases, namely, stripe, plane-wave and single-minimum phase, which will be discussed in the following sections.

### 3.1. Stripe wave phase

One of the important manifestations of the SO coupled BECs is the formation of stripe phase also known popularly as the spin-mixed phase (Phase I) due to the formation of standing wave pattern (SW phase). This phase is particularly attractive as it bears an analogy with the spatial pattern typical to smectic liquid crystals. In this case, the ground state is characterized by a linear combination of the two plane wave states, and the atom tends to condense by means of superposition of the respective two wave vectors \( \pm k_L \). The wave function can be written as

\[ \Psi_{SW} = \sqrt{\frac{n}{2G_2}} \left( \sqrt{G_2 - \Omega} \left( \begin{array}{c} \cos \theta \\ -\sin \theta e^{i\omega} \end{array} \right) e^{ik r} + \sqrt{G_2 + \Omega} \left( \begin{array}{c} \sin \theta \\ -\cos \theta e^{i\omega} \end{array} \right) e^{-ik r} \right), \]

(11)

and the associated energy reads

\[ \varepsilon_{SW} = k_f^2 - \Omega - \frac{G_1 (\Omega^2 - 3G_2^2)}{2G_2^2}k_f^2. \]

(12)

The system is endowed with SW phase at lower values of Rabi coupling \( \Omega \) with phase separation from PW phase defined by the SW-PW phase transition. One of the signatures of the SW phase is the existence of the density modulation as given by the expression

\[ n(r) = \hat{n} \left( 1 + \left( 1 - \frac{\Omega^2}{G_2 (2G_2 + k_f^2)} \right) \left( 1 - \frac{\Omega^2}{G_2^2} \right) \cos (2k \cdot r + \varphi) \right), \]

(13)

with the position of the fringes is given by the relative phase \( \phi \). The periodicity of the interfering fringe pattern is estimated to be \( \pi / k \), which depends on the wave vector as

\[ k_x = -\frac{k_f \Omega^2}{G_2 (2G_2 + k_f^2)}. \]

(14)

The calculated transverse and longitudinal spin polarizations corresponding to SW phase is given by

\[ \langle \sigma_x \rangle = -\sqrt{1 - \Delta^2}, \quad \text{and} \quad \langle \sigma_z \rangle = -\frac{\Omega \Delta}{G_2}, \]

(15)

respectively. The appearance of fringes as a result of density modulation is an interesting subject of recent investigation. As it is known, the phases are distinguished by the symmetries they possess and therefore, the phase transition between the phases are typically associated with some changes in the symmetries. It is evident from our study the stripes are a result of the breaking of two continuous symmetries, namely, gauge invariance and translational invariance, the former account for the superfluidity, while the latter yields the crystalline structure. As a consequence of this concurrent existence of the two broken symmetries, the SW phase is said to exhibit a close analogy with the supersolids, and therefore, naturally earn the wide interest among the researchers [36, 44, 45].

### 3.2. Plane-wave phase

Another common form of phase exhibited by the quantum gas is the spin-polarized or demixed phase, often known as the plane-wave (PW) phase (Phase II). At relatively higher values of Rabi coupling \( \Omega \), the system transits to PW phase from SW-PW transition boundary. In the chosen parametric region, the atoms condense in a single plane-wave state with \( k_x \) is given by

\[ k_x = -\frac{k_f \Omega}{2G_2 + k_f^2}, \]

(16)

and \( k_y = 0 \). Choosing \( |C_0| = 1 \) and \( |C_3| = 0 \), the PW wave function can be written as

\[ \Psi_{PW} = \sqrt{\hat{n}} \left( \begin{array}{c} \cos \theta \\ -\sin \theta e^{i\omega} \end{array} \right) e^{ik r}, \]

(17)
and the corresponding energy is given by

\[ \varepsilon_{PW} = -\Omega + G_1 + \frac{k_t^2}{2}. \] (18)

The system tends to confine in PW phase for a range of Rabi frequency defined by SW-PW phase transition, before reaching to ZM phase for any further increase in the strength of Rabi coupling. This is indeed, an excellent exhibition of the role of Rabi frequency, where the phase transition is critically governed by the Raman coupling. Further, it is interesting to note that based on the choice of the spontaneous symmetry breaking mechanism, there exists another energetically equivalent configuration with \(-k (C_1 = 0)\). Unlike the stripe phase, the density is uniform (\(\partial n / \partial k_i = 0, i = x, y\)) and the transverse and longitudinal spin polarization, respectively, read

\[ \langle \sigma_x \rangle = -\sqrt{1 - \Delta^2}, \quad \text{and} \quad \langle \sigma_z \rangle = \Delta. \] (19)

3.3. Single-minimum phase

The last particular case is the so-called single minimum or zero-momentum phase (Phase III), where the condensate has zero momentum \((k_x = k_y = 0)\). In the parametric region of choice, we identify ZM as the dominant phase, and it is observable only at a higher value of Rabi frequency or in other words at the cost of strong Raman coupling. The zero-momentum wave function \((k_x = k_y = 0)\) is given by

\[ \Psi_{ZM} = \sqrt{\hbar} \begin{pmatrix} \cos \theta \\ -\sin \theta e^{i\phi} \end{pmatrix}. \] (20)

The corresponding energy can be written as

\[ \varepsilon_{ZM} = G_1 + G_2 + k_t^2 + \frac{\Omega^2}{4G_2 + 2k_t^2}. \] (21)

The longitudinal spin polarization vanishes in this case, while the transverse spin polarization can take \(\langle \sigma_y \rangle = -1\). As in the case of PW phase there is no appreciable density modulation as observed in ZM phase.

4. Results and discussion

In order to provide a quantitative picture on the different phases exhibited by the system, the effect of system parameters as a function of Rabi coupling for the three different phases as in figure 3. It is quite evident that the variation of the physical quantities such as momentum, energy, and spin polarizations are markedly different for different phases. For instance, in figure 3(a) we show the variation of momentum with Rabi coupling whose behavior is significantly different from one phase to the other. The Phase I and Phase II shows, respectively, both decreasing and increasing trends of momentum, while Phase III is confined to zero momentum. Further, the energy decreases with the increase in the coupling strength, regardless of the nature of phase as evident from figure 3(b). Figures 3(c) and (d) demonstrate the transverse \(\langle \sigma_y \rangle\) and longitudinal spin polarizations \(\langle \sigma_z \rangle\), respectively. In the case of \(\langle \sigma_y \rangle\), Phase I and II shows an upward trend with \(\Omega\), while Phase III remains saturated at \(\langle \sigma_y \rangle = -1\). The behavior of the longitudinal spin polarization is shown in figure 3(d), where all the three phases vary quite differently with \(\Omega\).

In what follows, we briefly discuss the phase transition driven by the Rabi frequency. As illustrated before, the phase transition at a critical point where the system transit from one phase to another is typically associated with some symmetry breaking or invariance. For instance, the continuous translational symmetry of the PW phase spontaneously breaks at the phase transition point resulting in a stripe phase with density modulation resembling like standing wave pattern. The critical value of Rabi coupling corresponding to the phase transition between the two phases can be estimated by imposing a condition that the energy per particle \(\varepsilon\) be equal. Accordingly, the phase transition between the different phases can be defined as follows. The phase boundary from Phase-I to II (i.e., SW to PW phase) can be obtained by equating equations (12) and (18) as

\[ \Omega^{I-\,II} = -G_2, \] (22a)

and for the transition from Phase-II to III (PW to ZM phase), equating equations (18) and (21).

\[ \Omega^{II-\,III} = -2G_2 - k_t^2. \] (22b)

Similarly the phase boundary between Phase-III to I (ZM to SW phase) can be obtained by equating equations (21) and (12) as

\[ \Omega^{III-\,I} = -\frac{G_2^2(2G_2 + k_t^2) + \sqrt{G_2^2(G_1 + k_t^2)(2G_2 + k_t^2)(G_1 - 2G_2)k_t^2 + G_2(2G_1 - 3G_2))}}{(G_2 + k_t^2)^2 + G_1(2G_2 + k_t^2)}. \] (22c)
It is noteworthy that the individual phases are characterized by different values of the densities and also, the transition frequency is found to be independent of the density. With the help of these phase transitions, one can construct the phase diagram for three different phases. Figure 4 depicts the phase diagram at $\Omega-G_2$ plane. It is quite evident that the existence of the different phase crucially depends on the combination of the Rabi coupling and interaction strengths, and at the particular phase transition, the system leap from one phase to the other. As discussed in the preceding section, the strength of the Rabi coupling determines the phase transition and thereby defining the phases of the system. The most striking feature in the phase diagram is the existence of a critical point connecting all the three phases, known popularly as the tri-critical point, characterized by a well-defined value of the Rabi coupling $\Omega$. It is this existence of tri-criticality in the phase diagram makes the system unique, and rich in terms of fundamental studies on the quantum gases. Thus, the system admits tri-critical point as reported in the one-dimensional system by Li et al [33], and also the proportion of Rashba and Dresselhaus coupling is not necessarily required to be equal to realize the tri-criticality. We have demonstrated that by properly controlling the strength of Rabi coupling one can realize the coexistence of three different phases, as illustrated in figure 4. Further, it may be noted that in [33] the mapping of the tri-critical point has been shown in the $\Omega-G_2$ plane. The phase transition and the tri-critical point are all attributed to the control parameters namely, the strength of the Rabi coupling and the density. This has been illustrated in figure 2 of [33], and also reiterated in the review by Zhai (see figure 4 in [12]). Whereas, in

![Figure 3. Momentum $k_x$, energy per particle $E$, transverse and longitudinal spin polarizations $\langle \sigma_z \rangle$ and $\langle \sigma_y \rangle$ as a function of $\Omega$. Stripe phase $k_x = 0$, $|C| = (1 + |\Omega/G_2|)/2$ and $|C| = (1 - |\Omega/G_2|)/2$; Plane wave phase $k_x = 0$, $|C| = 1$ and $|C_2| = 0$; Zero momentum phase $k_x = k_y = 0$; The parameters: $G_1/k_L^2 = 0.2$ and $G_2/k_L^2 = 0.05$ (a)-(d).]

![Figure 4. Phase diagram in $G_2-\Omega$ illustrating phase boundaries and tricriticality in SO coupled BECs.]
the above choice parametric space, the two-dimensional system did not naturally admit tri-criticality in the $\Omega - \nu$ plane. However, we still demonstrate that it is possible to realize tri-criticality, provided the interaction parameters are taken into account. Thus by proper choice of $\Omega$ and $G_z$, it is possible for one to precisely control the phase transition and also the tri-critical point corresponding to the coexistence of three different phases. Indeed, it is worth to discuss the possible experimental realization of the proposed theoretical results. Recently, Huang et al experimentally realized the 2D synthetic SO coupling in ultracold Fermi gases using a configuration of three lasers, in a way that each of the lasers dresses one atomic hyperfine spin state [46]. This benchmark experimental work could lead to new interesting experiments in the direction of exploring many exotic quantum phenomena, including the possible experimental realization of phase transition and different phases associated with the quantum system as discussed in the present work. We believe that the results of this work could add new insights in this field, particularly in exploring two- and three-dimensional topological phases, which are in the developing stages in both theories and experiments.

5. Conclusions

In summary, we theoretically studied the ground state properties of spin-orbit-coupled Bose–Einstein condensates in quasi-two dimensions with unequal Rashba and Dresselhaus couplings at zero temperature. Firstly, we briefly studied the ground state properties of the system, particularly the energy–momentum dispersion for different combinations of Rashba SO and Rabi coupling. For zero Rabi coupling, the single-particle energy spectrum consists of a ring of infinitely degenerate lowest energy states in momentum space, known as Rashba ring. For any non-zero value of the Rabi coupling, the degeneracy is destroyed leading to a single-point lowest energy.

A comprehensive theoretical description of the phase diagram and the characteristic phase separation has been studied within the mean-field picture. The effect of Rabi coupling and interaction in the system shows up three different phases, namely, stripe, plane wave and zero momentum phase. The characteristic features of each phase are explained in detail, with a particular emphasis on the stripe phase. We found that at a particular combination of Rabi coupling and interaction parameter, the system admits a critical point separating all three different phases, known as the tri-critical point. We further studied in detail, the momentum distribution, energy, the longitudinal and transverse spin polarizations corresponding to the quantum phases. Thus we believe that our present theoretical results on quantum phase formation in the two-dimensional SO-coupled BEC can simulate experiments in the direction of realizing new and original exotic quantum phenomena on higher dimensional systems.

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ORCID iDs

K Nithyanandan  https://orcid.org/0000-0002-5531-7668
P Muruganandam  https://orcid.org/0000-0002-3246-3566

References

[1] Juzeliūnas G and Öhberg P 2004 Phys. Rev. Lett. 93 033602
[2] Juzeliūnas G, Ruseckas J, Öhberg P and Fleischhauer M 2006 Phys. Rev. A 73 022506
[3] Lin Y-J, Compton R L, Perry A R, Phillips W D, Porto J V and Spielman I B 2009 Phys. Rev. Lett. 102 130401
[4] Dalibard J, Gerbier F, Juzeliūnas G and Öhberg P 2011 Rev. Mod. Phys. 83 1523
[5] Galitski V and Spielman I B 2013 Nature (London) 494 49
[6] Dall’O blo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Rev. Mod. Phys. 71 463
[7] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[8] Qi X-L and Zhang S-C 2011 Rev. Mod. Phys. 83 1057
[9] Von Klitzing K 1986 Rev. Mod. Phys. 58 519
[10] Wilczek F 2009 Nat. Phys. 5 614
[11] Koralesk J D, Weber C P, Orenstein J, Bernevig B A, Zhang S-C, Mack S and Awschalom D D 2009 Nature 458 610
[12] Alicea J, Oreg Y, Refael G, Oppen F V and Fisher M P A 2011 Nat. Phys. 7 412
[13] Zhai H 2015 Rep. Prog. Phys. 78 026001
[13] Lin Y-J, Compton R L, Jiménez-García K, Porto J V and Spielman I B 2009 *Nature (London)* 462 628
[14] Paredes B, Widera A, Murg V, Mandel O, Fölling S, Cirac I, Shlyapnikov G V, Hänsch T W and Bloch I 2004 *Nature (London)* 429 277
[15] Kinoshita T, Wenger T and Weiss D S 2004 *Science* 305 1125
[16] Hadzibabic Z, Krüger P, Cheneau M, Battelier B and Dalibard J 2006 *Nature (London)* 441 1118
[17] Berry M V 1984 *Proc. R. Soc. Lond. A* 392 45
[18] Lin Y-J, Compton R L, Jiménez-García K, Phillips W D, Porto J V and Spielman I B 2011 *Nat. Phys.* 7 531
[19] Hadzibabic Z, Krüger P, Cheneau M, Battelier B and Dalibard J 2006 *Nature (London)* 441 1118
[20] Gao C, Jian C-M and Zhai H 2010 *Phys. Rev. Lett.* 105 160403
[21] Achilleos V, Stockhofe J, Kevrekidis P G, Frantzeskakis D J and Schmelcher P 2013 *EPL* 103 20002