Search versus Search for Collapsing Electoral Control Types*

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July 7, 2022; revised February 24, 2024

Abstract

Electoral control types are ways of trying to change the outcome of elections by altering aspects of their composition and structure [BTT92]. We say two compatible (i.e., having the same input types) control types that are about the same election system $E$ form a collapsing pair if for every possible input (which typically consists of a candidate set, a vote set, a focus candidate, and sometimes other parameters related to the nature of the attempted alteration), either both or neither of the attempted attacks can be successfully carried out (see the Preliminaries for a more formal definition) [HHM20]. For each of the seven general (i.e., holding for all election systems) electoral control type collapsing pairs found by Hemaspaandra, Hemaspaandra, and Menton [HHM20] and for each of the additional electoral control type collapsing pairs of Carleton et al. [CCH†22] for veto and approval (and many other election systems in light of that paper’s Theorems 3.6 and 3.9), both members of the collapsing pair have the same complexity since as sets they are the same set. However, having the same complexity (as sets) is not enough to guarantee that

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*Work supported in part by NSF grant CCF-2006496 and CIFellows grant CIF2020-UR-36.
†Work done in part while at the University of Rochester’s Department of Computer Science.
as search problems they have the same complexity. In this paper, we explore the relationships between the search versions of collapsing pairs. For each of the collapsing pairs of Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carleton et al. [CCH+22], we prove that the pair’s members’ search-version complexities are polynomially related (given access, for cases when the winner problem itself is not in polynomial time, to an oracle for the winner problem). Beyond that, we give efficient reductions that from a solution to one compute a solution to the other. For the concrete systems plurality, veto, and approval, we completely determine which of their (due to our results) polynomially-related collapsing search-problem pairs are polynomial-time computable and which are NP-hard.

1 Introduction

Algorithms for problems on elections are extremely important combinatorial algorithms, where attackers generally have available an exponential number of potential actions. Despite the combinatorially explosive number of potential actions available, in many cases polynomial-time algorithms can be developed. Thus the study of manipulative actions against elections is a beautiful showcase of an area with real-world importance, where the outcome of the clash between finding efficient algorithms for combinatorially explosive problems, and proving the impossibility (if $\text{P} \neq \text{NP}$) of finding such algorithms, is of vivid interest.

“Control” attacks on elections try to make a focus candidate win/lose/uniquely-win/not-uniquely-win through such actions as adding, deleting, or partitioning candidates or voters. Control was first studied in the seminal work of Bartholdi, Tovey, and Trick [BTT92], and has been further explored in many papers (see the survey chapter by Faliszewski and Rothe [FR16]).

Hemaspaandra, Hemaspaandra, and Menton [HHM20] noted that, surprisingly, seven pairs among the 44 (relatively) “standard” control types collapse: For each election system (i.e., each mapping from candidates and votes to a winner set among the candidates) $E$ whose vote type is linear orders, for each of those seven pairs of types $T_1$ and $T_2$ it holds that for each input there is a successful action on that input under the $T_1$ control type if and only if there is a successful action on that input under the $T_2$ control type. Viewed as sets, $T_1$ and $T_2$ are identical! Carleton et al. [CCH+22] noted that Hemaspaandra, Hemaspaandra, and Menton’s proof tacitly in fact established that those seven collapses hold for all election systems regardless of their vote type.

Carleton et al. [CCH+22] showed that those seven pairs are the only collapsing pairs among the 44 standard control types, at least if one wants the given collapse to hold for every election system. However, Carleton et al. [CCH+22] discovered some additional collapses that hold specifically for veto or specifically for approval voting, and also found some additional collapses that hold for all election systems that satisfy certain axiomatic properties.

For each election system to which a collapse applies, the complexity of the two collapsing control types is the same (since due to collapsing, the two control types yield the exact same set). However, Carleton et al. [CCH+22] raised (even in its initial technical report version, which was the main motivation for the present paper) as a challenge the issue of whether the
search complexity of the collapsing pairs is also the same. That is the issue that is the focus of the present paper.

Why is it even plausible that sets with the same decision complexity might have different search complexities (relative to some certificate/solution schemes)? Well, it indeed can and does happen if \( P \neq NP \cap coNP \) (and so certainly happens if integer factorization is not in polynomial time, since the natural decision version of that is in \( NP \cap coNP \)). In particular, if \( P \neq NP \cap coNP \), let \( L \) be any fixed set in \( (NP \cap coNP) - P \). Suppose \( L \)'s finite alphabet is \( \Sigma \). Consider a nondeterministic polynomial-time Turing machine (NPTM) \( N \) over input alphabet \( \Sigma \) that immediately halts and accepts. So \( L(N) = \Sigma^* \) and the certificate of acceptance (basically, the accepting computation path) of \( N \) on any input \( x \) is the empty string. Thus the function \( s(x) = \epsilon \) solves this search problem. However, consider an NPTM \( N' \) over the alphabet \( \Sigma \) that on each input \( x \) nondeterministically simulates both the NPTM \( N_L \) for \( L \) and the NPTM \( N_{\overline{L}} \) for \( \overline{L} \). Clearly, \( L(N') = \Sigma^* \), since each \( x \in \Sigma^* \) is either in \( L \) or in \( \overline{L} \). However, the search complexity of \( \Sigma^* \) relative to the certificate scheme of \( N' \) is not polynomial, since on input \( x \) any accepting path of \( N' \) will (after removing its initial guess bit regarding simulating \( N_L \) or \( N_{\overline{L}} \)) yield an accepting path for exactly one of \( N_L \) or \( N_{\overline{L}} \) on input \( x \), and so determines whether \( x \in L \) or \( x \in \overline{L} \). Thus, since \( L \notin P \), no search function for \( \Sigma^* \) relative to \( N' \) can be polynomial-time computable. This argument is basically due to a construction of Borodin and Demers [BD76] (see also [Val76], [HH88], and the proof in [Hem22, p. 39] that (in an earlier appearance) is pointed to by the discussion in Footnote 9 of [HHM20]).

This paper explores the issue of whether the collapsing (as decision problems) pairs of Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carlet et al. [CCH+22] have the same search complexity. We prove that in every case the answer is Yes (if given access, for the case where the election’s winner problem itself is not even in \( P \), to an oracle for the election’s winner problem), and indeed we show that in each case a solution for one can (again, given access to the winner problem) be polynomial-time transformed into a solution for the other. Thus the complexities are polynomially related (given access to the election system’s winner problem). Additionally, for the concrete cases of plurality’s, veto’s, and approval’s collapsing pairs, we explore whether those polynomially equivalent search complexities are clearly “polynomial time,” or are NP-hard, and we resolve every such case.

Why is this important? In reality, one typically—e.g., if one is a campaign manager—wants not merely to efficiently compute whether there exists some action that will make one’s candidate win, but rather one wants to get one’s hands, efficiently, on an actual such successful action. Unfortunately, the literature’s existing collapsing control type pairs are each about and proved for the “exists” case. In contrast, this paper’s results are establishing that the literature’s existing collapsing control type pairs even have the property that—given access to the winner problem for the election system in question\(^1\)—for both members of the pair the “getting one’s hands on a successful action when one exists” issue is of the same complexity for both. We also show that one usually can efficiently use access to solutions for one to get solutions for the other.

\(^1\)Although for the three concrete election systems we cover as examples that is not even needed as their winner problems are each in polynomial time.
2 Preliminaries

This section covers the preliminaries on election systems, control types/collapses, and search problems.

2.1 Elections and Election Systems

An election consists of a finite candidate set $C$ and a finite collection of votes $V$ over the candidates in $C$. The “type” of the votes depends on the election system one is considering. Most typically, each vote is a linear ordering—a complete, transitive, asymmetric binary relation—over the candidates (e.g., $3 > 1 > 2$). Another common vote type (called an approval vector) is that a vote is bit-vector of length $|C|$, with each bit typically indicating approval (1) or disapproval (0) of a candidate. No two candidates can have the same name.\(^2\)

An election system maps from an election, $(C, V)$, to a (possibly nonstrict) subset of $C$ (the set of winners). In pure social choice theory, empty winner sets are usually excluded. However, in computational social choice theory empty winner sets are often allowed, and in this paper we do allow empty winner sets. In fact, Bartholdi, Tovey, and Trick’s [BTT92] model of run-off partition of candidates, which is one focus of this paper, would not even be well-defined on one-candidate elections if one viewed zero-winner elections as illegal. That is, allowing empty winner sets is compelled unless one wants to change—in ways that in fact would open other difficulties—definitions that have been broadly accepted and used for three decades.

Three particular election systems that we will discuss are plurality, veto, and approval elections. In plurality elections, each vote is a linear ordering over $C$, and all candidates for whom the number of votes in which they were ranked first is the highest (possibly tied) among the candidates are winners. In veto elections, each vote is a linear ordering over $C$, and all candidates for whom the number of votes in which they were ranked last is the lowest (possibly tied) among the candidates are winners. In approval elections, each vote is an approval vector, and all candidates who garner the most (possibly tied) approvals are the election’s winners.

2.2 Control Types and Collapses

Definition 2.1 and the text following it define the 24 partition-based control types (for each election system $\mathcal{E}$). For uniformity, we take the definition essentially verbatim from the papers we are most closely related to [HHM20, CCH+22], which themselves were drawing on the line

\(^2\)By allowing candidates to have names, we are following the model of the papers that this paper is most closely related to [HHM20, CCH+22]. It is possible that election systems in this model may exploit candidate names in complicated ways. However, adopting this model in fact makes our results stronger than they would be if our model for example required election systems to be (candidate-)neutral and/or to have the candidate identities in $(C, V)$ always be 1, 2, \ldots, $|C|$. Also, studies on control are incompatible with assuming that candidate names are always 1, 2, \ldots, $|C|$, since many control types, including all partition-based ones, change the candidate set, thus resulting in elections that do not satisfy that condition; that issue has been implicit ever since the seminal work of Bartholdi, Tovey, and Trick [BTT92] that initiated the study of control attacks on elections.
of earlier papers—starting with Bartholdi, Tovey, and Trick [BTT92]—that developed the current set of control notions (for more history and citations, see [FR16,HHM20,CCH+22]). One must be clear as to the handling of candidates who are tied winners in first-round elections. The two tie-handling rules typically studied as to partition-based control types are ties-eliminate (TE)—in which a candidate must uniquely win its subelection to proceed to the next round—and ties-promote (TP)—in which all winners of a subelection proceed to the next round. In Definition 2.1 and later in the paper, we will at times have elections whose candidate set is $C'$ but whose votes, due to candidate partitioning and/or first-round candidate eliminations, are over a set $C \supseteq C'$. As is standard in the literature, in such cases we always take this to mean that the votes are each masked down to just the candidates in $C'$.

**Definition 2.1 (see [HHM20,CCH+22] and the references/history therein)** Let $E$ be an election system.

1. In the constructive control by partition of voters problem for $E$, in the TP or TE tie-handling rule model (denoted by $E$-CC-PV-TP-NUW or $E$-CC-PV-TE-NUW, respectively), we are given an election $(C, V)$, and a candidate $p \in C$. We ask if there is a partition\(^3\) of $V$ into $V_1$ and $V_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C, V_1)$ that survive the tie-handling rule compete (with respect to vote collection $V$) against the winners of subelection $(C, V_2)$ that survive the tie-handling rule. Each election (in both stages) is conducted using election system $E$.

2. In the constructive control by run-off partition of candidates problem for $E$, in the TP or TE tie-handling rule model (denoted by $E$-CC-RPC-TP-NUW or $E$-CC-RPC-TE-NUW, respectively), we are given an election $(C, V)$, and a candidate $p \in C$. We ask if there is a partition of $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete (with respect to vote collection $V$) against the winners of subelection $(C_2, V)$ that survive the tie-handling rule. Each election (in both stages) is conducted using election system $E$.

3. In the constructive control by partition of candidates problem for $E$, in the TP or TE tie-handling rule model (denoted by $E$-CC-PC-TP-NUW or $E$-CC-PC-TE-NUW, respectively), we are given an election $(C, V)$, and a candidate $p \in C$. We ask if there is a partition of $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete (with respect to vote collection $V$) against all candidates in $C_2$. Each election (in both stages) is conducted using election system $E$.

In each of the six control types defined above we can replace “is a winner” with “is a unique winner” to denote constructive control in the unique-winner (UW) model. The

\(^3\)A partition of a multiset $V$ is a pair of multisets $V_1$ and $V_2$ such that $V_1 \cup V_2 = V$, where $\cup$ denotes multiset union. A partition of a set $C$ is a pair of sets $C_1$ and $C_2$ such that $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = \emptyset$, where $\cup$ and $\cap$ are standard set union and intersection.
resulting control types are appended with “-UW” rather than “-NUW.” NUW denotes the so-called nonunique-winner model, aka, the cowinner model, in which merely being an overall winner, whether tied or not, is the goal. This completes the definition of the 12 standard constructive partition-control types (for each election system; in text discussions, we will often omit the “for each election system,” and so may speak of control of a given type generically).

The standard 12 additional destructive partition-control types differ only in that instead of seeking to make a focus candidate a winner or unique winner, the goal is to prevent the focus candidate from being a winner or unique winner. To denote those, we replace the “CC” with a “DC,” e.g., the destructive control by partition of voters problem for E using the TE handling rule and the UW model is denoted E-DC-PV-TE-UW. Thus the 24 standard types of partition-based control are defined. As is standard, each control type is quietly defining—in some sense is—a set, namely, the set of all inputs on which the given control attack can be successfully carried out. Thus for example Veto-DC-PV-TP-NUW is a set. We will immediately use this view in the next sentence.

For any two control types T1 and T2 that are compatible (i.e., have the same collection of input fields), Carleton et al. [CCH+22] say that T1 and T2 collapse if T1 = T2. If T1 and T2 are both about the same election system E (i.e., each has as its prefix “E-”) and T1 and T2 are compatible, we will say that control types T1 and T2 are E-matched. If control types T1 and T2 are both about the same election system E and collapse (and so also are compatible), we say that they are a collapsing pair. Since all 24 control types just defined are mutually compatible, and none of the other 44 − 24 = 20 standard control types are known to be involved in any of the collapses found in Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carleton et al. [CCH+22], we will not further mention compatibility in this paper. (See [CCH+22] for the standard definitions of the 20 control types not defined here.)

Here are the collapses found in Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carleton et al. [CCH+22]. For every election system E, E-DC-RPC-TP-NUW = E-DC-PC-TP-NUW and E-DC-RPC-TE-NUW = E-DC-PC-TE-NUW = E-DC-RPC-TE-UW = E-DC-PC-TE-UW ([HHM20] supplemented by [CCH+22]’s observation that the [HHM20] collapse proofs work not just over linear orders but in fact work over any election system regardless of its vote type). Property Unique-α [HHR07], the unique version of Chernoff’s Property α ([Che54], see [Sen71]), is said to hold for election system E if in every election (C, V), a candidate p being a unique winner implies that p remains a unique winner whenever the election is restricted to a subset C′ ⊆ C of candidates such that p ∈ C′. For every election system E satisfying Property Unique-α, E-DC-PC-TE-UW = E-DC-RPC-TP-UW = E-DC-PC-TP-UW and E-CC-PC-TP-UW = E-CC-RPC-TP-UW [CCH+22].4 Since approval

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4What we call Property Unique-α (and for brevity we often drop the “Property”) has been previously referred to as the “Unique-WARP” property. We discuss below our reasons for moving away from that name.

Bartholdi et al. [BTT92] first used the term WARP (Weak Axiom of Revealed Preferences) in the domain of computational social choice by stating that it required that p being a winner of election (C, V) implies that p remains a winner of every election (C′, V), where p ∈ C′ ⊆ C, and they stated that “this” was also known as Property α. Unfortunately, Hemaspaandra et al. [HHR07] apparently read that as saying that a definition of WARP was being given. And so they used the term “Unique-WARP” when they defined the variant of that in which p is required to be a unique winner. In fact, WARP and Property α are not equivalent. (Sen [Sen71],
voting satisfies Unique-WARP, these four additional pair-collapses hold for approval voting [CCH+22]. Note that Approval-DC-PC-TE-UW participates in both the four-type and the three-type above collapses with $\mathcal{E} =$ Approval, so by transitivity we have the six-type collapse Approval-DC-RPC-TE-NUW = Approval-DC-PC-TE-NUW = Approval-DC-RPC-TE-UW = Approval-DC-PC-TE-UW = Approval-DC-RPC-TP-UW = Approval-DC-PC-TP-UW [CCH+22].

The remaining election-specific collapses from Carleton et al. [CCH+22] are Veto-DC-PV-TE-NUW = Veto-DC-PV-TE-UW, Approval-DC-PV-TE-NUW = Approval-DC-PV-TE-UW, Approval-CC-PC-TE-NUW = Approval-CC-RPC-TE-NUW, Approval-CC-PC-TE-UW = Approval-CC-RPC-TE-UW, and Approval-CC-PC-TP-NUW = Approval-CC-RPC-TP-NUW.

### 2.3 Search Problems and Their Interreductions and Complexity

For each election system $\mathcal{E}$, define the winner problem, $W_\mathcal{E}$, by $W_\mathcal{E} = \{(C', V', p') \mid p' \in C'$ and $p'$ is a winner of the $\mathcal{E}$ election $(C', V') \}$. Our control types are each defined as a language problem—a set. However, each has a clear “NP search problem” associated with it (so an “NP search problem” if $\mathcal{E}$’s winner problem is in P). Namely, the problem of, on input $(C, V, p)$, outputting a partition $(C_1, C_2)$ of the candidate set (or if the type is a voter partition type, a partition $(V_1, V_2)$ of the vote collection) such that under that partition, the control action regarding $(C, V, p)$ is successful under election system $\mathcal{E}$.

To be able to speak clearly of search problems and their interrelationships and complexity it is important to be specific as to what we mean, both in terms of relating the solutions of two search problems and as to classifying the complexity of a search problem. These two different tasks are related, yet differ in how they are formalized.

Megiddo and Papadimitriou [MP91] give a formalization of search problems along with a notion of reductions between them. We will use that for connecting solutions of search problems in this paper, except we will see that our problems will be connected in a way substantially tighter than their framework anticipated. On the other hand, as we want to interconnect even electoral-control search problems whose winner problems $W_\mathcal{E}$ are potentially not in P, we will not require our relations to be polynomial-time decidable. (Rather, we will in effect require them to be polynomial-time decidable given a $W_\mathcal{E}$ oracle. When $W_\mathcal{E} \in P$, however, proved that WARP—in its long-settled and standard sense, see [HG22, Section 5.1]—is, with respect to the standard notion of choice functions (election systems) used in pure social choice theory, equivalent to the combination of Property $\alpha$ and a so-called Property $\beta$ (from [Sen69]) that we will not define here. The reason we mention that the equivalence holds for the pure social choice notion of election systems is that in our model, where empty winner sets are allowed, the equivalence in fact fails. Consider the following election system $\mathcal{E}$. If three or more candidates run, then the two candidates with the lexicographically smallest names win. Otherwise, no one wins. Clearly, this election system does not satisfy Property $\alpha$ (and certainly not Properties $\alpha$ and $\beta$ combined), but it does satisfy WARP.)

To summarize briefly, a misreading by [HHR07] of an ambiguous sentence in [BTT92] led to an infelicitous naming in [HHR07] that we feel should be abandoned. We thus use “Unique-$\alpha$” to denote what to date has been called “Unique-WARP” (e.g., in [HHR07,Erd09,ENR09,EFRS15,FHH20,CCH+22]). We do that so the terminology is analogous to what has long been used in social choice for the notion that inspired the Unique-$\alpha$ notion.
that requirement of course is the same as requiring them to be polynomial-time decidable.)

Following Megiddo and Papadimitriou [MP91], let $\Sigma$ be a finite alphabet with at least two symbols and suppose $R \subseteq \Sigma^* \times \Sigma^*$ is a relation, decidable in polynomial time relative to oracle $A$, that is polynomially-balanced. (Note: A relation is polynomially-balanced if there is a polynomial $p$ such that $(x, y) \in R$ implies $|y| \leq p(|x|)$.) That relation $R$ (which is decidable in $P^A$) defines what Megiddo and Papadimitriou call the computational problem $\Pi_R$: Given an $x \in \Sigma^*$, find some $y \in \Sigma^*$ such that $(x, y) \in R$ if such a $y$ exists, and reply “no” otherwise. We will call the class of all such problems $\text{FNP}^A$ ([MP91] in fact covered only the case $A = \emptyset$, so their class was called $\text{FNP}$).

In the case of our 24 electoral-control problems, the “$x$” here is the problem input, $(C, V, p)$. And if $(C, V, p)$ is a yes instance of the (viewed as a set) control type, an acceptable output $y$ in the sense of the above definition is (the encoding of) any partition $(C_1, C_2)$ of $C$ (or if the type is a voter partition, then any partition $(V_1, V_2)$ of $V$) that leads to success under the control type, the input $(C, V, p)$, and the election system. (We in this paper will not worry about encoding details, since such details are not a key issue here. We as is typical merely assume reasonable, natural encodings.)

For each election system $E$ and each $T$ that is one of our 24 partition control types for $E$, let $R_T$ be the natural $P^{W_E}$-decidable, polynomially-balanced relation for its search problem (i.e., $R_T$ is the set of pairs $((C, V, p), (P_1, P_2))$ where $(P_1, P_2)$ is a partition of the problem’s sort that in the setting $(C, V, p)$, under election system $E$, succeeds for the given type of control action). Then we will sometimes write $\Pi_T$ as a shorthand for $\Pi_{R_T}$. Note that for each of our 24 partition control types involving $E$, we clearly have $\Pi_T \in \text{FNP}^{W_E}$, e.g., $\Pi_{\text{Approval-DC-PC-TP-NUW}} \in \text{FNP}^{W_E}$ holds (and, for example, from that it follows immediately that, since $W_{\text{Approval}} \in \text{P}$, $\Pi_{\text{Approval-DC-PC-TP-NUW}} \in \text{FNP}$).

We wish to show that the known collapsing electoral control types also have polynomially related (given access to an oracle for the winner problem for $E$) search complexity, and indeed we wish to show even that a solution for one can efficiently be used to generate a solution for the other. To do this, we will need the notion of reductions between search problems. Fortunately, Megiddo and Papadimitriou defined a reduction notion between search problems that is close to what we need. Megiddo and Papadimitriou [MP91] say that a reduction from problem $\Pi_S$ to problem $\Pi_R$ is a pair of polynomial-time computable functions $f$ and $g$ such that, for any $x \in \Sigma^*$, $(x, g(y)) \in R \iff (f(x), y) \in S$. This in spirit is trying to say that we can map via $f$ to an instance (such that given a solution relative to $S$ of $f(x)$ we can via $g$ map to a solution relative to $R$ of $x$). Unfortunately, as written, it does not seem to do that, regardless of whether one takes the omitted quantification over $y$ to be existential or to be universal. Either way the definition leaves open a loophole in which on some $x$ for which there does exist a solution relative to $R$, the value of $f(x)$ will be some string that has no solution relative to $S$, and all $g(y)$’s will be strings that are not solutions to $x$ relative to $R$. So the “$\iff$” will be satisfied since both sides evaluate to False, but no solution transfer will have occurred. In the nightmare case, a given “reduction” could exploit this loophole on every $x$ that has a solution relative to $R$.

In what follows, we will reformulate the definition more carefully, for our case, to close that loophole (which is easily closed). However, more interestingly, we will in effect alter
the definition in two additional ways. First, notice that in Megiddo and Papadimitriou’s definition (assuming one fixes the above loophole first) the function \( f(x) \) allows the solution to \( x \) on the \( R \) side to be obtained via demanding a solution to a different (than \( x \)) instance \( f(x) \), on the \( S \) side. But in our setting, collapsing types are the same set, just with differing “witnessing” relations. And our goal is to make connections via those witnesses. So in the definitions we are about to give of our \( \leq_{\text{search}} \) reduction family, we in effect require their \( f(x) \) to be the identity function! Second, since we wish to connect the solutions of collapsing pairs even when the winner problem of \( \mathcal{E} \) is not in \( \mathbf{P} \), our reductions will have the winner problem as an oracle. (The reason for this is that even given a \((C, V, p)\) and a partition \((C_1, C_2)\) of \( C \) (or for voter partition types, a partition \((V_1, V_2)\) of \( V \)), to evaluate whether the partition is a solution generally needs calls to \( W_\mathcal{E} \).)

We will not in these definitions explicitly write “\( \Pi \)” expressions, but the solutions we speak of are with respect to the witnesses/actions that the given control type is about (they are the search component of \( \Pi \), i.e., in our case they are those second components that appear in pairs, belonging to the underlying relation \( R_T \), having “\( x \)” or “\( I \)” (i.e., the given \((C, V, p)\)) as their first component), and so we view these definitions as a reformulated version (with some changes for our particular situation) of the Megiddo and Papadimitriou “\( \Pi \)” approach.

Before we introduce our notation, let us first give some intuition behind it. \( T_1 \leq_{\text{search}}^\mathcal{E} T_2 \) means \( T_2 \)’s solutions are so powerful that for problem instance \( I \), given any solution for \( T_2 \) with respect to \( I \) one can quickly, given oracle \( W_\mathcal{E} \), build a solution to \( T_1 \) with respect to \( I \). That is why the notation has \( T_1 \) on the left.

**Definition 2.2**

1. For an election system \( \mathcal{E} \) and \( \mathcal{E} \)-matched, collapsing control types \( T_1 \) and \( T_2 \), we say that “\( T_1 \) is polynomially search-reducible to \( T_2 \) with respect to \( \mathcal{E} \)” (denoted by \( T_1 \leq_{\text{search}}^\mathcal{E} T_2 \)) if there is a reduction that, given an oracle for the winner problem for \( \mathcal{E} \), runs in polynomial time and on each input \((I, S)\), where \( I \) is an input to \( T_1 \) and \( S \) is a solution for \( I \) with respect to \( T_2 \), outputs a solution \( S’ \) for \( I \) with respect to \( T_1 \).

2. For an election system \( \mathcal{E} \) and \( \mathcal{E} \)-matched, collapsing control types \( T_1 \) and \( T_2 \), we say that “\( T_1 \) is polynomially search-equivalent to \( T_2 \) with respect to \( \mathcal{E} \)” if \( T_1 \) is polynomially search-reducible to \( T_2 \) with respect to \( \mathcal{E} \) and \( T_2 \) is polynomially search-reducible to \( T_1 \) with respect to \( \mathcal{E} \).

**Definition 2.3**

1. For an election system \( \mathcal{E} \) and \( \mathcal{E} \)-matched, collapsing control types \( T_1 \) and \( T_2 \), we say that “\( T_1 \) is polynomially search-reducible to \( T_2 \)” (denoted by \( T_1 \leq_{\text{search}}^\mathcal{E} T_2 \)) if there is a reduction that runs in polynomial time and on each input \((I, S)\), where \( I \) is an input to \( T_1 \) and \( S \) is a solution for \( I \) with respect to \( T_2 \), outputs a solution \( S’ \) for \( I \) with respect to \( T_1 \).

2. For an election system \( \mathcal{E} \) and \( \mathcal{E} \)-matched, collapsing control types \( T_1 \) and \( T_2 \), we say that “\( T_1 \) is polynomially search-equivalent to \( T_2 \)” if \( T_1 \) is polynomially search-reducible to \( T_2 \) and \( T_2 \) is polynomially search-reducible to \( T_1 \).

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*The reason we include “collapsing” in the definition is that if it is omitted, then one would trivially satisfy the notion on a given \( I \) whenever \( I \) was not in \( T_2 \). That is, our notion is focused on pairs of types that collapse—where each input is either in both or not in both.*
The closest notion in the literature to our Definitions 2.2 and 2.3 is the notion known as Levin reductions. Its definition can be found in such sources as Arora and Barak’s 2009 textbook [AB09], Piterman and Fisman’s notes [PF98] on a lecture from a 1998 course of Goldreich, and PlanetMath [Hen13]. That notion much differs from ours since it requires not just backward, but also forward transference of solutions. However, Goldreich’s 2008 textbook [Gol08] has a conflicting notion/definition of Levin reductions, and that notion is close to our notion. They are the same except we are focusing on collapsing electoral control types and so sometimes make the winner problem available as an oracle, and our problem-to-problem reduction is simply the identity function.

When a polynomial-time reduction has a polynomial-time computable oracle, one can alter the reduction machine $M$’s action to have $M$ itself do without the oracle by itself simulating the oracle’s work. We thus have the following observation.

**Proposition 2.4** Let $E$ be an election system that has a polynomial-time winner problem, and let $T_1$ and $T_2$ be $E$-matched control types.

1. If $T_1 \preceq_{pE}^\text{search} T_2$, then $T_1 \preceq_{\text{search}}^p T_2$.
2. If $T_1$ and $T_2$ are polynomially search-equivalent with respect to $E$, then $T_1$ and $T_2$ are polynomially search-equivalent.

The definitions we just gave provide the tools we will use to show that the complexities of the two members of each known pair of collapsing standard types are polynomially related to each other (given oracle access to $W_E$), and indeed that the members of the pair are closely related in terms of us being able to very efficiently build a solution to one from a solution to the other.\(^6\)

Note that even if $E$-matched collapsing types $T_1$ and $T_2$ have polynomially related search complexities (in the above sense), that does not tell us whether they both are easy, or they both are hard. Indeed, since for example our “close complexity relationship between the two members of the collapsing pair” results regarding the seven Hemaspaandra, Hemaspaandra, and Menton [HHM20] collapses for all election systems, it is completely possible that though (as we will show) the search complexity of such $\Pi_{T_1}$ and $\Pi_{T_2}$ are polynomially related, for some election system $E'$ both could be easy and for some election system $E''$ both could be hard (indeed, even undecidable).

Nonetheless, at least for specific, concrete systems that collapse results exist for, it would be nice to be able to prove results about whether their related search complexities are both easy or are both hard. The $\Pi_T$ formalism, even in the reformulated $\Pi$-free version in our

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\(^6\) Definitions 2.2 part 1 and 2.3 part 1 do not require that if $S$ is not a solution to $T_2$ then the reduction declares that fact. Rather, the definitions are simply about efficiently obtaining a solution to $T_1$ given a solution to $T_2$. However, we mention that if one changed Definition 2.2 part 1 to require detection of nonsolution-hood, the set of pairs $(T_1, T_2)$ for which the reduction held would not change at all, since with the $W_E$ oracle one can check whether $S$ is a solution to $T_2$. Although Definition 2.3 part 1 is not in general guaranteed to be unchanged if it is altered to require detection of the case where $S$ is not a solution to $T_2$, it clearly does remain unchanged by that alteration whenever $W_E \in \text{P}$. Most of the cases to which we apply Definition 2.3 indeed satisfy $W_E \in \text{P}$; in particular, plurality, veto, and approval voting each satisfy $W_E \in \text{P}$. 
definitions, is not ideal for doing this (since the Megiddo and Papadimitriou definition speaks of outputting “no,” which is not a typical NP-machine action, and that definition also is speaking about various y outputs without giving a framework for making clear how to speak of the instantiated (as to y) versions). One could approach the issue in various ways, but we will do so by drawing on the flavor of some relations (sometimes called multivalued “functions”) work that preceded that of Megiddo and Papadimitriou [MP91]. In particular, Book, Long, and Selman [BLS84] built a broad theory of multivalued NP functions. Rather than presenting it here—it is a bit more general than Megiddo and Papadimitriou [MP91] in that it maps to path outputs rather than to certificates (certificates are in flavor closer to capturing a path’s nondeterministic choices), and we do not need that generality—we draw just on one key notion. The Megiddo and Papadimitriou ΠR model says that the task is to output “any” appropriate y. So in some sense, ΠR is speaking of an entire family of maps. Viewed in the Book, Long, and Selman [BLS84] lens, ΠR is related to a multivalued function, call it ˆΠR, that on a given input x is undefined (i.e., maps to some special output ⊥) if x has no solution y (no appropriate-length y with (x, y) ∈ R) and otherwise maps to the set of all y with (x, y) ∈ R. And, crucially, a “single-valued refinement” of that multivalued function is any function that on each input x:

(a) is undefined on x if x has no solution relative to R, and

(b) maps to exactly one solution of x relative to R if x has at least one solution relative to R.

For example, if Σ = {0, 1} and R = {(0x, 1) | x ∈ Σ∗} ∪ {(0x, 0) | x ∈ Σ∗}, then the refinements of ˆΠR are each of the uncountable number of functions that on inputs in {ε}∪{1x | x ∈ Σ∗} map to ⊥ and on each input of the form {0x | x ∈ Σ∗} map to exactly one of 0 or 1.

We can now define what we mean by the search problem for T to be easy or hard. We say that the search problem for T is polynomial-time computable if there exists at least one refinement of ˆΠT (i.e., of ˆΠR,T) that is a polynomial-time computable function. We say that the search problem for T is NP-hard if for every refinement f of ˆΠT, it holds that NP ⊆ P^f. We say that the search problem for T is NP-easy if there exists at least one refinement f of ˆΠT that can be computed by a polynomial-time function given SAT as its oracle, i.e., is in the function class sometimes called “FPNP” or “PFNP.” If the search problem for T is both NP-hard and NP-easy, we say that the search problem is SAT-equivalent. Given two SAT-equivalent search problems, this definition does not promise (as, in contrast, the second parts of Definition 2.3 and, in some sense, Definition 2.2 do) that one can produce a solution for one of the problems from any solution, for the same instance, to the other, and vice versa. However, the definitions just given do give us a clean way to make clear that one of our search problems is hard, or is easy.

3 Related Work

The papers most related to this one are those discussed already: Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carleton et al. [CCH+22] collapsed existing control types as
decision problems. And Carleton et al. [CCH+22] posed as an open issue whether such collapses also collapse the associated search problems as to their complexity. Addressing that issue is the focus of the present paper.

Megiddo and Papadimitriou [MP91] and Book, Long, and Selman [BLS84] set frameworks that we use and adapt, as to the study of search problems, and refining multivalued functions.

Control was created by Bartholdi, Tovey, and Trick [BTT92], and the 24 control types we study were developed in that paper and (for the introduction of destructive control types and stating the TP/TE rules) in the work of Hemaspaandra, Hemaspaandra, and Rothe [HHR07]. The shift over time from the earliest papers’ focus on the UW model to a focus on either both UW and NUW, or sometimes even just NUW, is discussed, with citations, by Carleton et al. [CCH+22, Related Work].

Many papers have studied control. Faliszewski and Rothe’s survey [FR16] is an excellent resource, with a rich range of citations to papers investigating the control complexities of specific election systems. Among the many systems that have been studied and seem to do somewhat well in resisting control attacks are Schulze and ranked pair elections [PX12, HLM16], Llull and Copeland elections [FHHR09, FHH15], normalized range voting [Men13], Bucklin and fallback elections [ER10, EPR11] (see also [EFRS15]), and SP-AV elections [ENR09].

Among the most popular election systems whose control complexity has been studied are approval [HHR07, BEH+10], plurality [BTT92, HHR07], and veto elections [Lin12, MR18, ERY19]. We will draw on some decision-problem algorithms from these papers to establish some of our search results. For example, Theorem 4.8 part 1 follows directly from the properties of an algorithm of Maushagen and Rothe [MR18].

4 Results

This section will show that all known collapsing control-type pairs even have the same search complexity, given access to the election system’s winner problem. We will for the concrete cases determine when that shared complexity level is “polynomial-time computability,” and when it is “SAT-equivalence.” And we also explore, via our search-equivalence notions, how solutions from one can be used to obtain solutions to the other. We will even obtain new decision-case results (Proposition 4.10) that will help us in our quest to discover search complexities.

In the theorems and proofs that follow, \( \text{Winners}_{\mathcal{E}}(C, V) \) will denote the set of winners of the election \((C, V)\) under election system \( \mathcal{E} \). \( \text{UniqueWinnerIfAny}_{\mathcal{E}}(C, V) \) will be \( \text{Winners}_{\mathcal{E}}(C, V) \) if the latter has cardinality one, and will be the empty set otherwise. To compute these sets, we will typically leverage access to (in settings where one has such access) the oracle, \( W_{\mathcal{E}} = \{(C, V, p) \mid p \in C \text{ and } p \text{ is a winner of the } \mathcal{E} \text{ election } (C, V)\} \), since clearly each of the new functions can be computed in polynomial time given access to \( W_{\mathcal{E}} \).

**Theorem 4.1** For every election system \( \mathcal{E} \), \( \mathcal{E}-\text{DC-RPC-TP-NUW} \) and \( \mathcal{E}-\text{DC-PC-TP-NUW} \) are polynomially search-equivalent with respect to \( \mathcal{E} \).

**Proof.** Let \( \mathcal{T}_1 = \mathcal{E}-\text{DC-PC-TP-NUW} \) and \( \mathcal{T}_2 = \mathcal{E}-\text{DC-RPC-TP-NUW} \).
\( T_1 \preceq_{\text{p, search}} T_2 \): We give a polynomial-time algorithm performing the reduction, given access to oracle \( W_\mathcal{E} \). On input \((I, S)\), where \( I = (C, V, p) \) and \( p \in C \), do the following. If the syntax is bad, halt. Otherwise, check that \( S \) is a solution to \( I \) under \( T_2 \). This may involve up to three calls to the oracle. If \( S \) is not a good solution, then simply halt as the definition does not require anything in this case (although note also the comments in Footnote 6). Since \( S \) is a solution and our model is NUW, \( p \) must have participated and lost either in one of the two first-round elections or (in fact, exclusive-or) in the final-round election. In that election, let \( C' \) be the candidate set (note that \( p \in C' \) and \( p \) is not a winner of \((C', V)\)). Then output as the successful \( T_1 \) solution \( C_1 = C' \) and \( C_2 = C - C' \). In \( T_1 \) on input \( I \), \( p \) will be eliminated in the \((C', V)\) first-round election.\(^7\)

\( T_2 \preceq_{\text{p, search}} T_1 \): We give the algorithm. On input \((I, S)\), where \( I = (C, V, p) \) and \( S = (C_1, C_2) \), check for bad syntax and by using the \( W_\mathcal{E} \) oracle up to two times as per the nature of \( T_1 \), check that \( S \) is a solution to \( I \) under \( T_1 \). If the syntax is bad or \( S \) is not a solution under \( T_1 \), then simply halt. Otherwise, since \( S \) is a solution under \( T_1 \), \( p \) was a participant in and eliminated either in the \((C_1, V)\) contest or the \((C_2 \cup \text{Winners}_\mathcal{E}(C_1, V), V)\) contest. In the former case, output \((C_1, C_2)\), and in the latter case let \( D = C_2 \cup \text{Winners}_\mathcal{E}(C_1, V) \) and output \((D, C - D)\). This is a solution for \( T_2 \).

Proposition 4.2 If \( \mathcal{E} \) is an election system, \( T_1 \), \( T_2 \), and \( T_3 \) are pairwise \( \mathcal{E} \)-matched control types, \( T_1 \preceq_{\text{p, search}} T_2 \), and \( T_2 \preceq_{\text{p, search}} T_3 \), then \( T_1 \preceq_{\text{p, search}} T_3 \). That is, for \( \mathcal{E} \)-matched types \( T_1 \), \( T_2 \), and \( T_3 \), \( \preceq_{\text{p, search}} \) is transitive.

Proof. Let \( \mathcal{E} \) be an election system and let \( T_1 \), \( T_2 \), and \( T_3 \) be pairwise \( \mathcal{E} \)-matched control types such that \( T_1 \preceq_{\text{p, search}} T_2 \) via \( f \) and \( T_2 \preceq_{\text{p, search}} T_3 \) via \( g \) (with \( f \) and \( g \) both running in polynomial time given oracle \( W_\mathcal{E} \)). We will show that \( T_1 \preceq_{\text{p, search}} T_3 \). On input \((I, S)\), where \( I = (C, V, p) \), if \( S \) is a solution for \( T_3 \) on input \( I \), then on input \((I, S, (C, V, p), S) \) \( g \) outputs a solution \( S' \) for \( T_2 \) on input \( I \). Additionally, if \( S' \) is a solution for \( T_2 \) on input \( I \), then on input \((I, S', (C, V, p)) \) \( f \) outputs a solution \( S'' \) for \( T_1 \) on input \( I \). Thus if \( S \) is a solution for \( T_3 \) on input \( I \), then by applying \( g \) and then \( f \) in the manner just described we obtain, running in polynomial time with oracle \( W_\mathcal{E} \), a solution \( S'' \) for \( T_1 \) on input \( I \). Thus \( T_1 \preceq_{\text{p, search}} T_3 \).

Theorem 4.3 For every election system \( \mathcal{E} \), and each pair \((T_1, T_2)\) among the four control types \( \mathcal{E} \)-DC-RPC-TE-NUW, \( \mathcal{E} \)-DC-PC-TE-NUW, \( \mathcal{E} \)-DC-RPC-TE-UW, and \( \mathcal{E} \)-DC-PC-TE-UW (these decision-problem pairs are by [HHM20] known to be collapsing), we have that \( T_1 \) and \( T_2 \) are polynomially search-equivalent with respect to \( \mathcal{E} \).

\(^7\)This case shows why it is important that our definition is making an oracle to the winner problem of \( \mathcal{E} \) available. Suppose we tried to claim that this direction held without any oracle use by, if \( S \) is \((C_1, C_2)\), just outputting \( S' = (C_1, C_2) \). But then if \( p \) participated in and lost in the \((C_2, V)\) first-round \( T_2 \) case, \( S' \) might not be a solution with respect to \( T_1 \). Can we fix that by, if \( p \in C_2 \), just outputting \((C_2, C_1)\)? No. Maybe in that case under \( T_2 \) candidate \( p \) lost in \((C_2, V)\), or maybe it lost in \((\text{Winners}_\mathcal{E}(C_2, V) \cup \text{Winners}_\mathcal{E}(C_1, V), V)\). But if it was the latter, in \( T_1 \) our second-round election is \((\text{Winners}_\mathcal{E}(C_2, V) \cup C_1, V)\) and it is possible that \( p \) wins in that.
Proof. We will make a closed cycle of $\leq^{p.E}$ reductions involving these four types. In light of Proposition 4.2, this suffices to establish the theorem. In each part, as per the reduction definition, we will assume our input is $(I, S)$, with $I = (C, V, p)$.

$\mathcal{E}$-DC-RPC-TE-UW $\leq^{p.E}_{\text{search}}$ $\mathcal{E}$-DC-RPC-TE-NUW: A solution $(C_1, C_2)$ for $I$ under $\mathcal{E}$-DC-RPC-TE-NUW is always a solution for $I$ under $\mathcal{E}$-DC-RPC-TE-UW, since UW is stricter in the first round. If $p$ was eliminated under $\mathcal{E}$-DC-RPC-TE-NUW, then $p$ is also eliminated under $\mathcal{E}$-DC-RPC-TE-UW. So our reduction here can simply output the purported solution it is given. (Even if that is not a correct solution to the right-hand side, the reduction’s action is legal, since if given a nonsolution as input all the reduction has to do, under the definition of this reduction type, is not run for too long.)

$\mathcal{E}$-DC-RPC-TE-NUW $\leq^{p.E}_{\text{search}}$ $\mathcal{E}$-DC-PC-TE-NUW: Say we are for $I = (C, V, p)$ given a purported solution $S = (C_1, C_2)$ to $\mathcal{E}$-DC-PC-TE-NUW. If $S$ is not a successful solution, immediately reject. Otherwise, either (a) $p \in C_1$ but $p$ is not a unique winner of $(C_1, V)$, or (b) $p \in \text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2$ yet $p$ is not a winner of $(\text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2, V)$. Using our oracle, determine which of (a) or (b) holds (exactly one must hold if $S$ was a solution). If (a) holds, output $(C_1, C_2)$. This is then a successful solution on $I$ to $\mathcal{E}$-DC-RPC-TE-NUW. If (b) holds, then set $D = \text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2$ and output $(D, C - D)$ and this is a successful solution of $I$ to $\mathcal{E}$-DC-RPC-TE-UW, since we know that $p$ is not a winner of $(D, V)$, so it certainly is not a unique winner of $(D, V)$, and so in our $\mathcal{E}$-DC-RPC-TE-NUW first round $p$ participates and is eliminated.

$\mathcal{E}$-DC-PC-TE-NUW $\leq^{p.E}_{\text{search}}$ $\mathcal{E}$-DC-RPC-TE-UW: Say we are given for $I = (C, V, p)$ a purported solution $S = (C_1, C_2)$ for $I$ to $\mathcal{E}$-DC-PC-TE-UW. If $S$ is not a solution, immediately reject. Otherwise, we know that (a) $p \in C_1$ and $p$ does not uniquely win in $(C_1, V)$, exclusive-or (b) $p \in \text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2$ yet $p$ is not a unique winner of $(\text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2, V)$. If (a) holds, output $(C_1, C_2)$. This is a successful solution of $I$ for $\mathcal{E}$-DC-PC-TE-NUW as $p$ is eliminated in the first round. If (b) holds, set $D = \text{UniqueWinnerIfAny}_E(C_1, V) \cup C_2$ and output $(D, C - D)$. This is a successful solution of $I$ for $\mathcal{E}$-DC-PC-TE-UW as $p$ will be eliminated in the first round.

$\mathcal{E}$-DC-PC-TE-UW $\leq^{p.E}_{\text{search}}$ $\mathcal{E}$-DC-RPC-TE-UW: Say we are given $I = (C, V, p)$ and a purported solution $S = (C_1, C_2)$ for $I$ to $\mathcal{E}$-DC-RPC-TE-UW for $I$. If $S$ is not a solution, immediately reject. Otherwise, since $C_1$ and $C_2$ are symmetric in RPC, w.l.o.g. assume $p \in C_1$ (otherwise, if $p \in C_2$, rename $C_1$ and $C_2$ so that $p \in C_1$). So exactly one of (a) and (b) holds, where (a) and (b) are: (a) $p \in C_1$ and $p$ is not a unique winner of $(C_1, V)$, and (b) $p \in \text{UniqueWinnerIfAny}_E(C_1, V) \cup \text{UniqueWinnerIfAny}_E(C_2, V)$ and $p$ is not a unique winner of $(\text{UniqueWinnerIfAny}_E(C_1, V) \cup \text{UniqueWinnerIfAny}_E(C_2, V), V)$. If (a) holds, output $(C_1, C_2)$ and that is a successful solution for $I$ of $\mathcal{E}$-DC-PC-TE-UW as $p$ is eliminated in the first round. If (b) holds, set $D = \text{UniqueWinnerIfAny}_E(C_1, V) \cup \text{UniqueWinnerIfAny}_E(C_2, V)$, and output $(D, C - D)$ and that is a successful solution for $I$ of $\mathcal{E}$-DC-PC-TE-UW as $p$ is eliminated in the first round.

In light of Proposition 2.4, from Theorems 4.1 and 4.3, we have, respectively, the two parts of the following corollary.

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Corollary 4.4 For each $\mathcal{E} \in \{\text{Plurality, Veto, Approval}\}$ the following hold.

1. $\mathcal{E}$-DC-RPC-TP-NUW and $\mathcal{E}$-DC-PC-TP-NUW are polynomially search-equivalent.

2. For each pair $(\mathcal{T}_1, \mathcal{T}_2)$ among the four types $\mathcal{E}$-DC-RPC-TE-NUW, $\mathcal{E}$-DC-PC-TE-NUW, $\mathcal{E}$-DC-RPC-TE-UW, and $\mathcal{E}$-DC-PC-TE-UW, it holds that $\mathcal{T}_1$ and $\mathcal{T}_2$ are polynomially search-equivalent.

We now explore the system-specific collapses from Carleton et al. [CCH+22] in the same order that they appear in that paper. We observe that the way they establish their new decision-problem collapses for veto, approval, and election systems satisfying Unique-WARP is so constructive that the proofs implicitly show, for each new collapsing decision-problem pair $(\mathcal{T}_1, \mathcal{T}_2)$ that they find, that $\mathcal{T}_1$ and $\mathcal{T}_2$ are polynomially search-equivalent, thus giving us the following corollary.

Corollary (to the proofs of Thms./Cor. 3.2/3.6/3.9/3.12/3.14/3.16–3.18 of [CCH+22]) 4.5

1. Veto-DC-PV-TE-UW and Veto-DC-PV-TE-NUW are polynomially search-equivalent.

2. For each election system $\mathcal{E}$ that satisfies Unique-WARP, $\mathcal{E}$-DC-PC-TP-UW and $\mathcal{E}$-DC-PC-TE-UW are polynomially search-equivalent.$^8$

3. For each election system $\mathcal{E}$ that satisfies Unique-WARP, $\mathcal{E}$-CC-PC-TP-UW and $\mathcal{E}$-CC-RPC-TP-UW are polynomially search-equivalent.

4. Approval-DC-RPC-TP-UW and Approval-DC-PC-TP-UW are polynomially search-equivalent.

5. Approval-CC-PC-TP-NUW and Approval-CC-RPC-TP-NUW are polynomially search-equivalent.

6. Approval-DC-PV-TE-UW and Approval-DC-PV-TE-NUW are polynomially search-equivalent.

7. Approval-CC-PC-TE-NUW and Approval-CC-RPC-TE-NUW are polynomially search-equivalent.

8. Approval-CC-PC-TE-UW and Approval-CC-RPC-TE-UW are polynomially search-equivalent.

Proof. For all parts except 2 and 3 we have $W_\mathcal{E} \in \mathbb{P}$ so we can check if the $S$ of the input is a valid solution, and so in those parts we below assume that input always is a solution for

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$^8$One might expect here and in part 3 of this corollary the weaker conclusion “polynomially search-equivalent with respect to $\mathcal{E}$.” But in both these parts we mean and prove “polynomially search-equivalent.”
\( I = (C, V, p) \) of the problem on the right-hand side of the reduction. For parts 2 and 3 we cannot and do not make that assumption.

1. Veto-DC-PV-TE-UW \( \leq_{\text{search}}^{p} \) Veto-DC-PV-TE-NUW: A solution \((C_1, C_2)\) for \( I \) under Veto-DC-PV-TE-NUW is always a solution for \( I \) under Veto-DC-PV-TE-UW, so we just output \((C_1, C_2)\).

Veto-DC-PV-TE-NUW \( \leq_{\text{search}}^{p} \) Veto-DC-PV-TE-UW: The proof of Theorem 3.2 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and a solution \( S = (C_1, C_2) \) for Veto-DC-PV-TE-UW, a solution for \( I \) to Veto-DC-PV-TE-NUW, and we note that this construction can easily be done in polynomial time.

2. Let \( E \) be an election system that satisfies Unique-WARP. The proof of Theorem 3.6 of Carleton et al. [CCH+22] shows that \( E-DC-PC-TP-UW = E-DC-PC-TE-UW = \{ (C, V, p) \mid p \in C \text{ and } p \text{ is not a unique winner of the } E \text{ election } (C, V) \} \). They also show that for each \( I \in \mathcal{B}_E \), \((\emptyset, C)\) is a solution to \( I \) for both \( E-DC-PC-TP-UW \) and \( E-DC-PC-TE-UW \).

\( E-DC-PC-TP-UW \leq_{\text{search}}^{p} E-DC-PC-TE-UW \): Let our input be \((I, S)\). If \( S \) is a solution to \( I \) for \( E-DC-PC-TE-UW \), then \( I \in \mathcal{B}_E \). So output \((\emptyset, C)\). (Note that there is no guarantee on the output if \( S \) is not a solution of \( I \) for \( E-DC-PC-TE-UW \) and that is fine since our reduction type does not require us to make any such guarantee. This fact implicitly holds throughout the rest of this proof and so we do not mention it again.)

\( E-DC-PC-TE-UW \leq_{\text{search}}^{p} E-DC-PC-TP-UW \): Let our input be \((I, S)\). If \( S \) is a solution to \( I \) for \( E-DC-PC-TP-UW \), then \( I \in \mathcal{B}_E \). So output \((\emptyset, C)\).

3. Let \( E \) be an election system that satisfies Unique-WARP. The proof of Theorem 3.9 of Carleton et al. [CCH+22] shows that \( E-CC-PC-TP-UW = E-CC-RPC-TP-UW = \{ (C, V, p) \mid p \in C \text{ and } p \text{ is not a unique winner of the } E \text{ election } (C, V) \} \). They also show that for each \( I \in \mathcal{B}_E \), \((\emptyset, C)\) is a solution to \( I \) for both \( E-CC-PC-TP-UW \) and \( E-CC-RPC-TP-UW \).

\( E-CC-PC-TP-UW \leq_{\text{search}}^{p} E-CC-RPC-TP-UW \): Let our input be \((I, S)\). If \( S \) is a solution to \( I \) for \( E-CC-RPC-TP-UW \), then \( I \in \mathcal{A}_E \). So output \((\emptyset, C)\).

\( E-CC-RPC-TP-UW \leq_{\text{search}}^{p} E-CC-PC-TP-UW \): Let our input be \((I, S)\). If \( S \) is a solution to \( I \) for \( E-CC-PC-TP-UW \), then \( I \in \mathcal{A}_E \). So output \((\emptyset, C)\).

4. The proof of Theorem 3.12 of Carleton et al. [CCH+22] shows that Approval-DC-PC-TP-UW = Approval-DC-RPC-TP-UW = \( \{ (C, V, p) \mid p \in C \text{ and } p \text{ is not a unique winner of the approval election } (C, V) \} \). They also show that for each \( I \in \mathcal{B} \), \((\emptyset, C)\) is a solution to \( I \) for both Approval-DC-PC-TP-UW and Approval-DC-RPC-TP-UW.

Approval-DC-PC-TP-UW \( \leq_{\text{search}}^{p} \) Approval-DC-RPC-TP-UW: Let our input be \((I, S)\). Since \( S \) is a solution to \( I \) for Approval-DC-RPC-TP-UW, \( I \in \mathcal{B} \). So output \((\emptyset, C)\).

Approval-DC-RPC-TP-UW \( \leq_{\text{search}}^{p} \) Approval-DC-PC-TP-UW: Let our input be \((I, S)\). Since \( S \) is a solution to \( I \) for Approval-DC-PC-TP-UW, \( I \in \mathcal{B} \). So output \((\emptyset, C)\).
5. The proof of Theorem 3.14 of Carleton et al. [CCH+22] shows that Approval-CC-PC-TPNUW = Approval-CC-RPC-TPNUW = \( A = \{(C, V, p) \mid p \in C \text{ and } p \text{ is a winner of the approval election } (C, V)\} \). They also show that for each \( I \in A \), \((\emptyset, C)\) is a solution to \( I \) for both Approval-CC-PC-TPNUW and Approval-CC-RPC-TPNUW.

Approval-CC-PC-TPNUW \( \leq_{\text{search}} \) Approval-CC-RPC-TPNUW: Let our input be \((I, S)\). Since \( S \) is a solution to \( I \) for Approval-CC-RPC-TPNUW, \( I \in A \). So output \((\emptyset, C)\).

Approval-CC-RPC-TPNUW \( \leq_{\text{search}} \) Approval-CC-PC-TPNUW: Let our input be \((I, S)\). Since \( S \) is a solution to \( I \) for Approval-CC-PC-TPNUW, \( I \in S \). So output \((\emptyset, C)\).

6. Approval-DC-PV-TEUW \( \leq_{\text{search}} \) Approval-DC-PV-TENUW: On input \( I = (C, V, p) \) and \( S \), a solution for \( I \) to Approval-DC-PV-TENUW is a solution for \( I \) to Approval-DC-PV-TEUW, output \( S \).

Approval-DC-PV-TENUW \( \leq_{\text{search}} \) Approval-DC-PV-TEUW: The proof of Theorem 3.16 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and a solution \( S \) for Approval-DC-PV-TEUW, a solution to \( I \) for Approval-DC-PV-TENUW, and we note that the construction can be done in polynomial time.

7. Approval-CC-PC-TENUW \( \leq_{\text{search}} \) Approval-CC-RPC-TENUW: The proof of Theorem 3.17 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and a solution \( S \) for Approval-CC-RPC-TENUW, a solution to \( I \) for Approval-CC-PC-TENUW, and we note that the construction can be done in polynomial time.

Approval-CC-RPC-TENUW \( \leq_{\text{search}} \) Approval-CC-PC-TENUW: The proof of Theorem 3.17 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and a solution \( S \) for Approval-CC-PC-TENUW, a solution to \( I \) for Approval-CC-RPC-TENUW, and we note that the construction can be done in polynomial time.

8. Approval-CC-PC-TEUW \( \leq_{\text{search}} \) Approval-CC-RPC-TEUW: The proof of Corollary 3.18 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and a solution \( S \) for Approval-CC-RPC-TEUW, to construct a solution to \( I \) for Approval-CC-PC-TEUW, and we note that the construction can be done in polynomial time.

Approval-CC-RPC-TEUW \( \leq_{\text{search}} \) Approval-CC-PC-TEUW: The proof of Corollary 3.18 of Carleton et al. [CCH+22] shows how to construct, given \( I = (C, V, p) \) and solution \( S \) for Approval-CC-PC-TEUW, to construct a solution to \( I \) for Approval-CC-RPC-TEUW, and we note that the construction can be done in polynomial time.

\[ \square \]

**Corollary 4.6**

1. For each pair \((\mathcal{T}_1, \mathcal{T}_2)\) among the six types Approval-DC-RPC-TE-NUW, Approval-DC-RPC-TE-UW, Approval-DC-PC-TE-NUW, Approval-DC-PC-TE-UW, Approval-DC-RPC-TP-UW, and Approval-DC-PC-TP-UW, it holds that \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are polynomially search-equivalent.
2. Approval-CC-PC-TP-UW and Approval-CC-RPC-TP-UW are polynomially search-equivalent.

Proof. 1. This follows directly from Corollaries 4.4 and 4.5 (Parts 2 and 4 of the latter) since approval satisfies Unique-WARP.
2. This follows directly from Corollary 4.5 since approval satisfies Unique-WARP. ☐

We now, for each concrete search-equivalence above about veto and approval, pinpoint the exact search complexity of the search problems.

In the proof of Theorem 4.7 we will draw on the notion of, and known results about, immunity. The notion dates back to the seminal work of Bartholdi, Tovey, and Trick [BTT92] (see also [HHR07], which introduced the destructive cases and fixed a minor flaw in the original formulation for the unique-winner case). We take the definitions’ statements essentially verbatim from Carleton et al. [CCH+22]. In the unique-winner model, we say an election system is immune to a particular type of control if the given type of control can never change a candidate from not uniquely winning to uniquely winning (if the control type is constructive) or change a candidate from uniquely winning to not uniquely winning (if the control type is destructive). In the nonunique-winner model, we say an election system is immune to a particular type of control if the given type of control can never change a nonwinner to a winner (if the control type is constructive) or change a winner to a nonwinner (if the control type is destructive).

Theorem 4.7 For each \( T \in \{\text{Approval-DC-PC-TP-NUW}, \text{Approval-DC-PC-TE-UW}, \text{Approval-CC-PC-TP-UW}, \text{Approval-CC-PC-TP-NUW}\} \), it holds that the search problem for \( T \) is polynomial-time computable.

Proof. Consider the case of Approval-DC-PC-TE-UW. Approval is known to be immune to DC-PC-TE-UW [HHR07]. So on each input \((C, V, p)\), we have: If \( p \) is a unique winner of election \((C, V)\) under approval voting, then \( p \not\in \text{Approval-DC-PC-TE-UW} \) (i.e., there exists no candidate partition under which \( p \) is not a final-round unique winner in the PC-TE two-stage election process under approval voting). Our polynomial-time search-problem algorithm thus is the following: On input \((C, V, p)\), in polynomial time determine whether \( p \) is a unique winner under approval of election \((C, V)\). If it is, output \( \bot \) (indicating there is no partition that will prevent \( p \) from being the unique winner in the PC-TE two-stage election process under approval, regarding \((C, V)\)). Otherwise output as our solution the partition \((\emptyset, C)\), since this will make the final-round election be \((C, V)\), and we know in this “otherwise” case that \( p \) does not uniquely win there.

It is not hard to see that approval is also immune to DC-PC-TP-NUW: Let \( p \) be a winner of an election \((C, V)\). Let \( k \) denote the number of ballots (votes) that approve of \( p \) in \( V \). It follows that no candidate is approved by more than \( k \) votes (else \( p \) would not be a winner). Since the TP handling rule is used, \( p \) can never be eliminated from a subelection as no candidate is approved by more votes than \( p \). Thus \( p \) always proceeds to the final round, and is a winner. So the Approval-DC-PC-TP-NUW case follows by the above proof with
each instance of the word “unique” removed and each TE and UW respectively changed to TP and NUW.

Since approval is immune to both CC-PC-TP-UW \[HHR07\] and CC-PC-TP-NUW \[CCH+22\], the two constructive cases are analogous, of course by asking respectively as to the Approval-CC-PC-TP-UW and Approval-CC-PC-TP-NUW cases whether \(p\) is not a unique winner of \((C, V)\) or not a winner of \((C, V)\), and proceeding in the obvious way, again using \((\emptyset, C)\) as our output partition in those cases that do not output \(\bot\).

\[
\text{Theorem 4.8} \quad \text{The search problem for each of the following control problems is polynomial-time computable:}
\]

1. Veto-DC-PV-TE-UW.
2. Approval-DC-PV-TE-UW.
3. Approval-CC-RPC-TE-UW.
4. Approval-CC-RPC-TE-NUW.

\textbf{Proof.}

1. Maushagen and Rothe \[MR18\] show that Veto-DC-PV-TE-UW \(\in P\) and their algorithm detects whether a solution exists and explicitly constructs a solution in polynomial time when a solution exists.

2. Hemaspaandra, Hemaspaandra, and Rothe \[HHR07\] show that Approval-DC-PV-TE-UW \(\in P\) and their construction detects whether a solution exists and explicitly constructs a solution in polynomial time when a solution exists.

3. Hemaspaandra, Hemaspaandra, and Rothe \[HHR07\] show that Approval-CC-RPC-TE-UW \(\in P\) and their construction detects whether a solution exists and explicitly constructs a solution in polynomial time when a solution exists.

4. Our algorithm, which is a modified version of one of Hemaspaandra, Hemaspaandra, and Rothe \[HHR07\], proceeds as follows: On input \((C, V, p)\), for each \(a \in C\), let \(y_a\) be the number of votes in \(V\) that approve \(a\) and let \(Y = \max\{y_a \mid a \in C\}\). If \(y_p \neq Y\) and \(\|\{a \in C \mid y_a = Y\}\| = 1\), then output \(\bot\). Otherwise, output \((\{p\}, C - \{p\})\). Why does \((\{p\}, C - \{p\})\) work? Note that in this “otherwise” case, we have that either (a) \(y_p = Y\) or (b) \(\|\{a \in C \mid y_a = Y\}\| \geq 2\). If (a) holds, then the partition \((\{p\}, C - \{p\})\) makes \(p\) a winner in the final round, since \(p\) will uniquely win and move forward from its subelection, and no candidate has more approvals than \(p\) so even if a candidate moves forward from the other subelection it cannot prevent \(p\) from being a final-round winner. If (a) fails and (b) holds, then the partition \((\{p\}, C - \{p\})\) will make \(p\) a winner (indeed, a unique winner) in the final round, since in the other subelection there will be at least two candidates that are approved by \(Y\) votes, so they tie as winners of that subelection and are eliminated; thus no candidates will move forward from that first-round election.
The algorithm just given clearly runs in polynomial time.

The following theorem will help us determine the search complexity of the remaining control problems.

**Theorem 4.9** Given an election system $\mathcal{E}$ satisfying $W_{\mathcal{E}} \in \mathsf{P}$, if $\mathcal{T}$ is one of our partition-control types involving $\mathcal{E}$ and (the decision problem) $\mathcal{T}$ is NP-complete, then the search problem for $\mathcal{T}$ is SAT-equivalent.

**Proof.** Assume $\mathcal{T}$ is one of our partition control types, that $\mathcal{E}$ is the election system of $\mathcal{T}$, that $W_{\mathcal{E}} \in \mathsf{P}$, and that (the decision problem) $\mathcal{T}$ is NP-complete.

NP-hard: Let $f$ be any refinement of $\hat{\Pi}_{\mathcal{T}}$. Since $\mathcal{T}$ is NP-complete, it suffices to show $\mathcal{T} \in \mathsf{P}^f$ to show that $\mathsf{NP} \subseteq \mathsf{P}^f$. Our polynomial-time algorithm to decide $\mathcal{T}$ with function oracle $f$ proceeds as follows: If $x$ is not a triple of the form $(C, V, p)$ where $C$ is a set of candidates, $V$ is a vote collection over $C$ of the vote-type of $\mathcal{E}$, and $p \in C$, then reject. Otherwise, query the function oracle with $x$ and verify, in polynomial time (using the fact that $W_{\mathcal{E}} \in \mathsf{P}$), whether the oracle response is a solution to the control problem. If it is, accept, and otherwise reject.

NP-easy: It is easy to see that there is a refinement of $\hat{\Pi}_{\mathcal{T}}$, $h$, such that $h \in \mathsf{FP}^{\mathsf{NP}}$. Let us focus on $\mathcal{T}$ being a voter partition type. The candidate cases are exactly analogous except we build the candidate rather than the voter partition. On input $(C, V, p)$, $h$ makes sure that the input is of the form $(C, V, p)$, that $p \in C$, and that the votes in $V$ are of the type appropriate for $\mathcal{E}$; if not output $\bot$. Otherwise, use a single call to SAT to determine if $(C, V, p) \in \mathcal{T}$. If not, output $\bot$. Otherwise, by an easy binary search, with an NP oracle, we will construct a set $V_1$ such that $(V_1, V - V_1)$ is a solution to $\mathcal{T}$ for $(C, V, p)$. In fact, if we naturally encode partitions into binary strings, we can binary search, with any NP-complete set such as SAT as our oracle, to find the lexicographically least encoding of a $V_1 \subseteq V$ such that $(V_1, V - V_1)$ is a solution to $(C, V, p)$ with respect to $\mathcal{T}$, and then we will output $(V_1, V - V_1)$. The “helper” NP set for the binary search is simply $\{(C, V, p, C') \mid (\exists C')|C' \geq_{\text{lex}} C\text{ and }C'\text{ encodes a }V_1\text{ such that } (V_1, V - V_1)\text{ is a solution to } (C, V, p)\text{ with respect to }\mathcal{T}\}$. Since this is an NP set, questions to it can be polynomial-time transformed into questions to SAT.)

Theorem 4.9 is a powerful tool that will help us determine the search complexity of many of our problems of interest as many of those decision problems have been shown to be NP-complete in the literature. Unfortunately, much research on electoral control types has been in the unique-winner model, and so we need to establish the following new decision-complexity result before proceeding.

**Proposition 4.10** As a decision problem, Plurality-DC-PC-TP-Nuw (and equivalently, Plurality-DC-RPC-TP-Nuw) is NP-complete.

**Proof.** The analogous result in the unique-winner model was established by Hemaspaandra, Hemaspaandra, and Rothe [HHR07]. Our proof, which we include for completeness,
closely follows their proof. We use the same construction (i.e., reduction), but our correctness argument involves small yet important modifications.

Membership in NP is immediately clear. We now prove NP-hardness. We in particular provide a reduction from the Hitting Set problem, a known NP-complete problem [GJ79].

The Hitting Set problem is defined as follows. Given a set \( B = \{b_1, b_2, \ldots, b_m\} \), a family \( S = \{S_1, S_2, \ldots, S_n\} \) of subsets of \( B \), and a positive integer \( k \), does \( S \) have a hitting set of size at most \( k \) (That is, is there a set \( B' \subseteq B \) with \( \|B'\| \leq k \) such that, for each \( i \), \( S_i \cap B' \neq \emptyset \)?)

We now state that construction that [HHR07] used for their NP-hardness reduction for Plurality-DC-PC-TP-UW, since we will use the same construction as our P-hardness reduction for Plurality-DC-PC-TP-NUW.

**Construction 4.11 ([HHR07])**

Given a triple \((B, S, k)\), where \( B = \{b_1, b_2, \ldots, b_m\} \), \( S = \{S_1, S_2, \ldots, S_n\} \) is a family of subsets of \( B \), and \( k \leq m \) is a positive integer, construct the following election:

1. The candidate set is \( C = B \cup \{c, w\} \).
2. The vote set \( V \) is defined as:
   
   a) There are \( 2(m-k) + 2n(k+1) + 4 \) votes of the form \( c > w > \cdots \), where the “…” means that the remaining candidates are in some arbitrary order.
   
   b) There are \( 2n(k+1) + 5 \) votes of the form \( w > c > \cdots \).
   
   c) For each \( i \in \{1, \ldots, n\} \), there are \( 2(k+1) \) votes of the form \( S_i > c > \cdots \), where “\( S_i \)” denotes the elements of \( S_i \) in some arbitrary order.
   
   d) For each \( j \in \{1, \ldots, m\} \), there are two votes of the form \( b_j > w > \cdots \).
3. The distinguished candidate is \( c \).

We now state two claims that will be used to prove that the reduction works in our case.

**Claim 4.12 ([HHR07])** If \( B' \) is a hitting set of \( S \) of size \( k \), then \( w \) is the unique winner of the plurality election \((B' \cup \{c, w\}, V)\).

**Claim 4.13** Let \( D \subseteq B \cup \{w\} \). If \( c \) is not a winner of plurality election \((D \cup \{c\}, V)\), then there exists a set \( B' \subseteq B \) such that

1. \( D = B' \cup \{w\} \),
2. \( w \) is the unique winner of plurality election \((B' \cup \{c, w\}, V)\), and
3. \( B' \) is a hitting set of \( S \) of size less than or equal to \( k \).

**Proof of Claim 4.13.** Fix \( D \subseteq B \cup \{w\} \) such that \( c \) is not a winner of plurality election \((D \cup \{c\}, V)\). We will show that the above three properties hold, by using a modified version of the argument used in [HHR07].

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For a given election, we will let \( \text{score}(d) \) denote the number of votes that rank candidate \( d \) first in that election.

First, notice that for each \( b \in D \cap B \), \( \text{score}(b) < \text{score}(c) \) in \( (D \cup \{c\}, V) \). Since \( c \) is not a winner of that election, it must hold that \( w \) is the unique winner of that election, and thus \( \text{score}(w) > \text{score}(c) \).

Let \( B' \subseteq B \) be such that \( D = B' \cup \{w\} \). Then \( D \cup \{c\} = B' \cup \{c, w\} \). Thus it follows that \( w \) is the unique winner of the election \( (B' \cup \{c, w\}) \), proving the first two properties.

Finally, observe that in \( (B' \cup \{c, w\}, V) \), it holds that

1. \( \text{score}(w) = 2n(k + 1) + 5 + 2(m - \|B'\|) \), and
2. \( \text{score}(c) = 2(m - k) + 2n(k + 1) + 4 + 2(k + 1)\ell \),

where \( \ell \) is the number of sets in \( S \) that have an empty intersection with \( B' \), i.e., are not “hit by \( B' \).” Since \( w \) is the unique winner of the election, it follows that

\[
\text{score}(c) < \text{score}(w),
\]

\[
2(m - k) + 2(k + 1)\ell < 1 + 2(m - \|B'\|),
\]

\[
(k + 1)\ell + \|B'\| - k < 1/2.
\]

Since \( \ell \) is a nonnegative integer, the only value that it can have here is 0, and so it follows that \( B' \) is a hitting set of \( S \) of size at most \( k \).

Now to conclude the proof of Proposition 4.10, we will leverage the two claims above to show that the following two items are equivalent:

1. There is a set \( B' \subseteq B \) of size at most \( k \) that is a hitting set of \( S \).

2. There is a partition of \( C \) such that \( c \) can be prevented from being a winner of the two-stage plurality election conducted under the PC-TP-NUW model.

Let \( B' \) be a hitting set of \( S \) of size \( k \). Let \( C_1 = B' \cup \{c, w\} \) and \( C_2 = C - C_1 \). By Claim 4.12 it holds that \( w \) is the unique winner of the subelection \( (C_1, V) \), and thus \( c \) does not proceed to the final round and is not a winner.

Suppose there is a partition of \( C \) such that \( c \) is not a winner of the corresponding two-stage plurality election. It must hold that \( c \) is eliminated either in the first-round election or in the final-round election. Then it holds that there is a set \( D \subseteq B \cup \{w\} \) such that \( c \) is not a winner of plurality election \( (D \cup \{c\}, V) \). It directly follows from Claim 4.13 that \( S \) has a hitting set of size at most \( k \). This concludes the proof that Plurality-DC-PC-TP-NUW is NP-complete.

The “equivalently” follows from the fact that for every election system \( \mathcal{E} \), \( \mathcal{E}-\text{DC-RPC-TP-NUW} = \mathcal{E}-\text{DC-PC-TP-NUW} \) [HHM20].

**Corollary 4.14** The search-problem versions of the following are SAT-equivalent.

1. Plurality-DC-RPC-TP-NUW
2. Plurality-DC-RPC-TE-UW
3. Veto-DC-RPC-TP-NUW
4. Veto-DC-RPC-TE-NUW

**Proof.**
Each of these four is NP-complete, the first by Proposition 4.10, the second is from [HHR07], and the remaining two are from Maushagen and Rothe [MR18]. The result follows from those four NP-completenesses, by Theorem 4.9.

Table 1 provides, for the concrete election systems discussed in this paper, a summary of our results on the search complexity of each equivalence class of decision-collapsing control problems. This paper’s results establish that within each such equivalence class the search complexities must be polynomially related; the table is summarizing our results on whether those linked complexities are both “polynomial(-time computability)” or are both “SAT-equivalence.” (The leftmost column’s classification of which control types, for the three studied election systems, collapse as decision problems is due to Carleton et al. [CCH+22], building on Hemaspaandra, Hemaspaandra, and Menton [HHM20].)

Note that Table 1 gives various results that are not explicitly stated in any of the theorems of this paper, but that in fact follow from combining results in the paper. To help explain why those results are indeed validly included in the table, we first need to state two results, and then will come back to how they help populate the table.

While the following two results may seem intuitive, for the sake of completeness we provide their proofs, since we are quietly using them in populating the table.

**Proposition 4.15** Let search problems $A$ and $B$ be polynomially search-equivalent. If $A$ is SAT-equivalent, then $B$ is SAT-equivalent.

**Proof.** Fix two polynomially search-equivalent search problems $A$ and $B$, with $A$ being SAT-equivalent. We need to show that, per our definitions, $B$ is both NP-hard and NP-easy. NP-hard: Let $g$ be an arbitrary refinement of $\hat{\Pi}_B$. We need to show that $\text{NP} \subseteq P^g$. It suffices to show that $\text{SAT} \in P^g$.

Let $r$ be a function that witnesses $A \leq_{\text{search}}^p B$. It follows that the function $h$ that on arbitrary input $x$ maps to $\bot$ if $g(x)$ maps to $\bot$, and that otherwise maps to $r(x; g(x))$, is a refinement of $\hat{\Pi}_A$.

Since $A$ is NP-hard, we have that $\text{SAT} \in P^h$. Let $T$ denote the polynomial-time algorithm (with oracle $h$) that decides SAT. Here is the $P^g$ algorithm $T'$ for SAT. Our polynomial-time algorithm $T'$ (with oracle $g$) simulates $T$, and each time $T$ asks a question, $y$, to its oracle, $T'$ asks the same question to its oracle $g$, and if the answer is $\bot$ then $T'$ acts as if the answer $T$ got is $\bot$, and otherwise $T'$ acts as if the answer $T$ got is $h(y)$. So the outcome of $T'$ with oracle $g$ is precisely the same as that of $T$ with oracle $h$, and so we have proven that $\text{SAT} \in P^g$.

NP-easy: It suffices to show that $\hat{\Pi}_B$ has a refinement in $\text{FP}^{\text{SAT}}$. Let $s$ denote a function witnessing $B \leq_{\text{search}}^p A$. Since $A$ is SAT-equivalent, there is a refinement $t$ of $\hat{\Pi}_A$ such that
| Collapsing Decision Control Types | Complexity of Their Search Versions | References          |
|----------------------------------|-------------------------------------|---------------------|
| Plurality-DC-RPC-TP-NUW,        | SAT-equivalent                      | Corollaries 4.4 and 4.14 |
| Plurality-DC-PC-TP-NUW          |                                     |                     |
| Plurality-DC-RPC-TE-NUW,        | SAT-equivalent                      | Corollaries 4.4 and 4.14 |
| Plurality-DC-PC-TE-NUW,         |                                     |                     |
| Veto-DC-PV-TE-NUW,              | polynomial                          | Corollary 4.5 and Theorem 4.8 |
| Veto-DC-PV-TE-UW                |                                     |                     |
| Veto-DC-RPC-TP-NUW,             | SAT-equivalent                      | Corollaries 4.4 and 4.14 |
| Veto-DC-RPC-TE-NUW,             |                                     |                     |
| Approval-DC-RPC-TP-NUW,         | polynomial                          | Corollary 4.4 and Theorem 4.7 |
| Approval-DC-PC-TP-NUW           |                                     |                     |
| Approval-DC-RPC-TP-UW,          | polynomial                          | Corollary 4.6 and Theorem 4.7 |
| Approval-DC-PC-TP-UW,           |                                     |                     |
| Approval-DC-RPC-TE-NUW,         |                                     |                     |
| Approval-DC-RPC-TE-UW,          |                                     |                     |
| Approval-DC-PV-TE-UW,           | polynomial                          | Corollary 4.5 and Theorem 4.8 |
| Approval-CC-RPC-TP-UW,          | polynomial                          | Corollary 4.6 and Theorem 4.7 |
| Approval-CC-RPC-TP-UW           |                                     |                     |
| Approval-CC-RPC-TE-NUW,         | polynomial                          | Corollary 4.5 and Theorem 4.7 |
| Approval-CC-RPC-TE-UW           |                                     |                     |
| Approval-CC-PV-TE-UW,           | polynomial                          | Corollary 4.5 and Theorem 4.8 |
| Approval-CC-PC-TE-UW,           |                                     |                     |

Table 1: For each collection of (decision-problem) collapsing partition control types for plurality, veto, and approval elections we have shown that their search problems are also of the same complexity. This table, for each, states whether the search problems are polynomial-time solvable (denoted “polynomial”) in the table or are SAT-equivalent. Proposition 4.15 and Proposition 4.16 are not listed in the References column, since it is clear when they are being drawn on (see the discussion of this near the end of Section 4).
\( t \in \text{FP}^{\text{SAT}} \). But it also holds, for each \( x \), that \( t(x) \) is a solution for \( x \) under \( A \) if and only if (\( t \) is not \( \bot \) and) \( s(x, t(x)) \) is a solution for \( x \) with respect to \( B \). Thus the function, \( h \), that on input \( x \) is \( \bot \) if \( t(x) \) is \( \bot \) and otherwise is \( s(x, t(x)) \) is a refinement of \( \hat{\Pi}_B \). But since \( t \in \text{FP}^{\text{SAT}} \), clearly so also is \( h \).

Proposition 4.16 Let search problems \( A \) and \( B \) be polynomially search-equivalent. If \( A \) is polynomial-time computable, then \( B \) is polynomial-time computable.

Proof. Fix two polynomially search-equivalent search problems \( A \) and \( B \), with \( A \) being polynomial-time computable. We need to show that \( B \) is polynomial-time computable, by providing a refinement of \( \hat{\Pi}_B \) in FP.

Let \( f \) be a polynomial-time computable refinement of \( \hat{\Pi}_A \) and let \( g \) be a function witnessing \( B \leq^p \text{search} A \). The function \( h \) that on input \( x \) is \( \bot \) if \( f(x) \) is \( \bot \) and otherwise is \( g(x, f(x)) \) is clearly a polynomial-time computable refinement of \( \hat{\Pi}_B \).

As mentioned earlier, Table 1 gives various results that are not explicitly stated in any of the theorems of this paper, but that in fact follow from combining results in the paper. For example, consider Plurality-DC-RPC-TE-NUW. The table states that it is SAT-equivalent, and that is true by the reasoning implicit in the table’s right-hand column. In particular, by Corollary 4.14 Plurality-DC-RPC-TE-UW is SAT-equivalent, and so by Corollary 4.4 and Proposition 4.15 it follows that Plurality-DC-RPC-TE-NUW is SAT-equivalent. In fact, Proposition 4.15 is broadly used in the table as to problems stated to be SAT-equivalent, namely to “inherit” the SAT-equivalence from one problem to other problems that we have shown to be search-equivalent to it. Likewise, Proposition 4.16 is analogously used in that table, regarding many cases of polynomial-time computable search problems.

In this section, we have shown that for all known collapsing control-type pairs, even their search complexities are polynomially equivalent, given oracle access to the election system’s winner problem.

5 Conclusion

In this paper, for all collapsing electoral control types found in Hemaspaandra, Hemaspaandra, and Menton [HHM20] and Carleton et al. [CCH+22]—those are all such collapses for the domains and groupings studied there (see [HHM20] and especially [CCH+22])—we proved that even the search-problem complexities of the collapsing types are polynomially equivalent (given oracle access to the election system’s winner problem).

In doing this—and building on earlier notions of relating search problems—we defined reductions that for the case of collapsing electoral control problems express how solutions to one can be efficiently converted to solutions to the other.

Also, for the key concrete systems plurality, veto, and approval, we establish—as polynomial-time computable or as SAT-equivalent—the complexity of their (now collapsed) search problems. An interesting open direction would be, for that part of our work, to seek
more general results, such as dichotomy theorems covering broad collections of election systems. However, that may be difficult since not much is known as to dichotomy theorems even for the decision cases of (unweighted) control problems (however, see [HHS14, HS16, FHH15]), though the few known such cases would be natural starting points to look at in this regard.

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