PT Symmetry and Hermitian Hamiltonian in the Local Supercritical Pomeron Model

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Abstract

The local reggeon field theory is studied perturbatively taking advantage of the PT symmetry in the Hamiltonian formulation. In the lowest non trivial order we show that the pomeron interactions renormalize the slope. In the same order we find a non local pair potential acting between pomerons, which has a singular structure. However the analysis of the scattering operator shows that at small coupling constant bound states do not appear so that the two-particle spectrum is not changed.

1 Introduction

In recent years the study of strong interactions in the so called Regge kinematics has experienced a rebirth of interest due to advances in the Quantum Chromodynamics studies, originated from the seminal work about the BFKL \cite{1} pomeron. Subsequent investigations have unveiled the basic effective interaction \cite{2,3,4} of propagating BFKL pomerons. Equations to resum tree diagrams at large $N_c$ were derived for onium-nucleus scattering \cite{5,6} (BK equation) and nucleus-nucleus scattering \cite{7}. An effective quantum field theory describing the pomeron interaction at large $N_c$ was constructed in \cite{4,8}, which in principle allows to find quantum corrections related to pomeron loop diagrams. Similar studies were conducted in the alternative dipole and JIMWLK techniques, where also attempts to sum pomeron loops were made under certain drastic approximations \cite{9}.

Pomeron loops play a secondary role in the scattering with nuclei at not too high energies. But for the proton scattering and at asymptotic energies their contribution cannot be neglected. Summing pomeron loops is a formidable task. So, as a first step, it is worthwhile to study some features of a much simpler quantum field theory of the local supercritical pomerons, introduced by V.N.Gribov many years ago to sum reggeon diagrams which describe interacting pomerons in the phenomenological approach. This local reggeon field theory (LRFT) possesses many features similar to the QCD pomeron theory. In particular, for the supercritical pomeron with the intercept $\alpha(0) > 1$, it is also non-perturbative at high energies, so that loop contributions have to be summed by some technique, which may prove to be useful also for the QCD. Some very beautiful results in this direction were obtained for the even simpler LRFT living in zero transverse dimension (“a toy model”), which in fact reduces to the quantum mechanics. In particular it was shown that loops indeed play a decisive role at high energies and essentially transform the initially supercritical pomeron into a weakly subcritical. These results were in fact obtained long ago \cite{10,11} and were recently re-analyzed in \cite{12,13,14}. Unfortunately
it is not straightforward to extend these findings to the realistic LRFT living in two transverse dimensions. Different approximations have lead different authors to either predict a phase transition or claim complete inconsistency of the model. More studies are needed to clarify the situation.

In this paper we draw attention to the fact that in the Hamiltonian approach LRFT belongs to the class of models with a non-Hermitian Hamiltonian, which possesses a certain symmetry equivalent to the \( PT \) symmetry in the ordinary quantum mechanics. This was noted in [14] for the toy model in zero transverse dimension but remains true also for the realistic case. The \( PT \)-symmetric non-Hermitian Hamiltonians have been extensively studied for some time [15, 16] and we are going to apply some of the techniques developed in this study for the LRFT with a supercritical pomeron. In particular using the \( PT \) invariance we demonstrate that the non-Hermitian Hamiltonian of LRFT has only real eigenvalues and transform it to an Hermitian one by a suitable similarity transformation. This does not allow us to find the true asymptotic of the amplitudes in LRFT at high energies with all loops taken into account but gives results valid up to rapidities \( y \sim 1/\lambda^{2N} \) when terms up to order \( \lambda^{3N} \) are included. Thus, at small \( \lambda \), higher orders in the perturbation expansion of the Hamiltonian allow to study higher and higher rapidities.

In the course of our study we have to perform renormalization of the loop correction to the pomeron intercept, which is divergent in the lowest order, as is well known. As a result we find that the slope of the pomeron trajectory is decreased by interaction. We find that this is the only change in the supercritical pomeron spectrum at order \( \lambda^2 \).

The plan of the paper is as follows. In the next section we briefly review some features of \( PT \) symmetric quantum systems relevant to our study. Then we introduce the LRFT model and its Hamiltonian formulation and discuss The \( PT \) and \( CPT \) symmetries of the Hamiltonian. Next two chapters are devoted to the calculation of the metric operator and the Hermitian Hamiltonian in the lowest non-trivial order. In the fifth section we study and renormalize the single particle term of the Hamiltonian and derive the effective two particle non-local potential. In the sixth section we study the scattering states associated with this potential and show that no bound states are present at perturbative level. We then draw our conclusions. Some more technical details have been added in a few appendices.

\section{PT symmetric Quantum Mechanics and QFT}

Since it was noted [15] that there exist non hermitian Hamiltonians \( H \) having a real spectrum bounded from below, provided boundary conditions for the wave functions of the associated Sturm-Liouville differential problem are properly defined, a lot of investigations have been done.

An important observation has been that a class of such Hamiltonians possesses an unbroken \( PT \) symmetry, so that eigenstates of \( H \) can be chosen to be eigenstates of \( PT \). After investigating the scalar product for the state space it has been found that there is a natural choice \( (f,g) = \int dx[PTf]^t(x)g(x) \), \( t \) denoting the transpose operation, which gives an associated norm conserved in time. The problem of this choice is that it is not leading to an Hilbert space with a positive norm. The states are splitted in two classes with positive and negative \( PT \)-norms. The good news has been that it is generally possible to find a new symmetry operator, denoted by \( C \), whose eigenvalues are precisely the sign of the \( PT \)-norm.

Having \( [C,PT] = 0 \) and \( [C,H] = 0 \), one can define a new scalar product by \( \langle f|g \rangle = \int dx[CPTf]^t(x)g(x) \), which leads to a positive norm conserved in time and therefore to a physically acceptable probabilistic interpretation. In such a case one can also define observables to be operators such that \( O^t = CPTOCPT \), relation which coincides with the usual hermiticity condition in the conventional quantum mechanics where \( C = P \). We stress that the main conceptual
point here has been that if one insists to define a quantum theory with conserved probability using a $PT$-symmetric non-Hermithean Hamiltonian one has to use the new mathematically well defined $CPT$-scalar product, which depends on the Hamiltonian itself.

On the other hand in many physical situations the scalar product of the wave functions is well defined from the start and cannot be changed. With this scalar product the probabilities given by the norm of the wave functions will not be conserved in time. However this does not present any difficulty, since either they are not conserved physically, as for decaying channels, or evolution in fact goes not in time but in rapidity, as in LRFT, when requirement that the norm be conserved is absent. Thus the new norm introduced in earlier studies is not relevant for LRFT. However other points have a direct application.

In the next section we shall find that the Hamiltonian of the LRFT indeed possesses the $PT$ symmetry and has real eigenvalues. Construction of the new symmetry operator $C$ depends on the Hamiltonian. If one knows eigenstates of $H$ then one can use them to explicitly find a representation of $C$. A more convenient way, which admits a perturbative approach [17], is the following. Consider a system whose Hamiltonian has the form

$$H = H_0 + \lambda H_I,$$  

(1)

where the free part is given by a Hermithean $H_0$ and the interaction part by an anti-Hermithean $H_I$. Define the parity operator $P$ with $P^2 = 1$ to transform $H$ into $H^\dagger$, which implies

$$[H_0, P] = 0, \quad \{H_I, P\} = 0, \quad P^2 = 1.$$  

(2)

Now one looks for the symmetry operator $C$ satisfying

$$[C, H] = [C, PT] = 0$$  

(3)

in the form

$$C = e^Q P$$  

(4)

where $Q$ is an Hermithean operator. From [3] we find a relation [17]

$$2\lambda e^Q H_I = [e^Q, H],$$  

(5)

which can be solved perturbatively, using the expansion $Q = \lambda Q_1 + \lambda^3 Q_3 + ...$, by requiring

$$[H_0, Q_1] = -2H_I, \quad [H_0, Q_3] = -\frac{1}{6}[[H_I, Q_1]Q_1]$$  

(6)

and so on.

The operator $e^Q$ can be used to define a similarity transformation which maps the $PT$-symmetric Hamiltonian $H$ onto an Hermithean Hamiltonian $h$ with the same set of eigenvalues. It is easy to show that also

$$e^{-Q} H e^Q = H^\dagger.$$  

(7)

Indeed we have $e^Q P H - H e^Q P = 0$ Multiplying by $P$ and using $PHP = H^\dagger$ we find [7]. As a result, we find

$$h = e^{-Q/2} H e^{Q/2} = e^{Q/2} H^\dagger e^{-Q/2} = h^\dagger,$$  

(8)

so that $h$ is the equivalent Hermithean Hamiltonian. Once $Q$ is known as a power series in $\lambda$ the Hamiltonian $h$ can also be found in the same form: $h = h^{(0)} + \lambda^2 h^{(2)} + \lambda^4 h^{(4)} + ...$ where

$$h^{(0)} = H_0,$$  

$$h^{(2)} = \frac{1}{4}[[H_I, Q_1]],$$  

$$h^{(4)} = \frac{1}{24}[[H_I, Q_3]],$$  

$$...$$
\[ h^{(4)} = \frac{1}{4} [H_I, Q_3] + \frac{1}{32} [H_0, Q_3 Q_1] \] (9)

and so on.

In the next sections we show that it is possible to apply these general results to the LRFT and find the symmetry operator \( Q \) and the Hermitian Hamiltonian \( h \) in the first non-trivial order of the perturbation theory.

### 3 PT symmetry and operator \( Q \) in the LRFT

The LRFT can be defined as a theory of two fields \( \phi(y, x) \) and \( \phi^\dagger(y, x) \) depending on rapidity \( y \) and transverse coordinates \( x \) with a Lagrangian density

\[ \mathcal{L} = \phi^\dagger (\partial_y - \alpha' \nabla_x^2) \phi + \lambda \phi^\dagger(x) \left[ \phi^\dagger(x) + \phi(x) \right] \phi(x), \] (10)

where \( \mu > 0 \) is the intercept minus unity and \( \alpha' \) is the slope of the pomeron trajectory. With \( \mu > 0 \) the corresponding functional integral is divergent and the only way to define the theory beyond the set of perturbative Feynman diagrams is the analytic continuation from \( \mu < 0 \) when the theory is well defined. A constructive way to do this continuation is the Hamiltonian approach. One sets up a quasi-Schrödinger equation for the wave function \( \Psi \):

\[
\frac{d\Psi(y)}{dy} = -H \Psi(y),
\] (11)

where the Hamiltonian has the form

\[ H = H_0 + \lambda H_I \] (12)

with the free part given by

\[ H_0 = \int d^2x (-\mu \phi^\dagger(x) \phi(x) + \alpha' \nabla \phi^\dagger(x) \nabla \phi(x)), \] (13)

the interaction part

\[ H_I = i \int d^2x \phi^\dagger(x) \left[ \phi^\dagger(x) + \phi(x) \right] \phi(x), \] (14)

and the standard commutation relations between \( \phi \) and \( \phi^\dagger \):

\[ [\phi(x), \phi^\dagger(x')] = \delta^2(x - x'). \] (15)

The scattering amplitude with the target (‘initial’) state \( \Psi_i(y_1) \) at rapidity \( y_1 \) and the projectile (‘final’) state \( \Psi_f(y_2) \) at rapidity \( y_2 > y_1 \) is defined as

\[ iA_{fi}(y_2 - y_1) = \langle \Psi_f(y_2) | e^{-H(y_2 - y_1)} | \Psi_i(y_1) \rangle. \] (16)

One can demonstrate that the perturbation expansion in powers of \( \lambda \) of this expression reproduces the standard Reggeon diagrams of the LRFT and also that (16) satisfies the requirement of symmetry between the target and projectile (see [4]).

The Hamiltonian is not Hermitian and our first task is to demonstrate that its energy levels are all real. This is of course trivially seen in the perturbation theory. Since \( H_0 \) has its eigenstates with a fixed number \( n \) of pomerons, the energy change can only be accomplished by action of an even number of interactions \( H_I \). However we can also prove it on more general grounds. To this end we consider symmetry operations applied to \( H \). In correspondence with the definitions in the previous section we introduce parity \( P \) as a transformation of the fields

\[ \phi(y, x) \rightarrow -\phi(-x), \quad \phi^\dagger(y, x) \rightarrow -\phi^\dagger(-x). \] (17)
It follows that indeed \( PHP = H^\dagger \) and of course \( P^2 = 1 \). Next we introduce the ‘time reflection’ \( T \) as taking the complex conjugate of all coefficient functions without changing the fields.

It is evident that \( P \) and \( T \) commute and that their product \( PT \) will leave both parts of the Hamiltonian intact. So we indeed find a \( PT \) symmetry of the LRFT Hamiltonian

\[
[PT,H] = 0. \tag{18}
\]

As a result, if an eigenfunction of \( H \) is presented as the action of some operator depending on \( \phi^\dagger \) on the vacuum \( \Psi_0 \), then it has to be of the form

\[
F(i\phi^\dagger)\Psi_0, \quad \text{with} \quad F(z^*) = F^*(z), \tag{19}
\]

i.e. \( F \) has to be a real function of its complex argument. The eigenvalue is then

\[
\langle \Psi | H | \Psi \rangle = \langle \Psi_0 | F(-i\phi)HF(i\phi^\dagger) | \psi_0 \rangle
\]

and is obviously real, since the Hamiltonian can also be written as a real function of \( i\phi^\dagger \) and \(-i\phi \) and the total number of operators \( \phi \) and \( \phi^\dagger \) has obviously to be the same.

Following the technique discussed in the previous section our aim is to find an appropriate \( Q \)-operator, which will allow us to pass to a Hermithean Hamiltonian \( h \) by transformation \([\,]\). Once we find the latter, the amplitude will be given by

\[
i A_{f_1}(y_2 - y_1) = \langle e^{Q/2}\Psi_f(y_2)|e^{-h(y_2-y_1)}|e^{-Q/2}\Psi_i(y_1)\rangle. \tag{20}
\]

Thus evolution will be accomplished by the Hermithean \( h \) and operators \( e^{±Q/2} \) will transform (differently) the initial and final states. In particular the pomeron Green function at rapidity \( y \) and momentum \( k \) will be given as as

\[
\delta^2(k - k')G(y,k) = \langle 0|\phi(k)e^{Q/2}e^{-yh}e^{-Q/2}\phi^\dagger(k')|0 \rangle. \tag{21}
\]

To construct \( Q \) we shall use perturbative equations \([\,]\) presented in the previous section.

In this paper we shall restrict ourselves to the first non-trivial order in the triple pomeron coupling constant \( \lambda \), that is constructing \( Q_1 \) and \( h^{(2)} \). To accommodate to our notations in \([\,]\) we seek \( Q_1 \) in the form

\[
Q_1 = -2i\frac{\sqrt{3}}{\mu} \int d^2x_1d^2x_2d^2x_3 \left(f_1(x_1,x_2,x_3)\phi^\dagger_1\phi_2\phi_3 + f_2(x_1,x_2,x_3)\phi^\dagger_1\phi_2\phi_3\right), \tag{22}
\]

where we denote \( \phi_1 ≡ \phi(x_1) \) etc. To calculate the part of the commutator \([Q_1,H_0] \) which contains function \( f_1 \) we need to know

\[
[\phi^\dagger_1\phi_2\phi_3,\phi^\dagger(x)\phi(x)] = \delta^2(x_2 - x)\phi^\dagger_1\phi_3\phi(x) + \delta^2(x_3 - x)\phi^\dagger_1\phi_2\phi(x) - \delta^2(x_1 - x)\phi^\dagger(x)\phi_2\phi_3
\]

and

\[
[\phi^\dagger_1\phi_2\phi_3, \nabla\phi^\dagger(x)\nabla\phi(x)]
\]

\[
= \nabla\delta^2(x_2 - x)\phi^\dagger_1\phi_3\nabla\phi(x) + \nabla\delta^2(x_3 - x)\phi^\dagger_1\phi_2\nabla\phi(x) - \nabla\delta^2(x_1 - x)\nabla\phi^\dagger(x)\phi_2\phi_3,
\]

where \( \nabla \) refers to differentiation in \( x \). So this part of the commutator \([Q_1,H_0] \) takes the form

\[
[Q_1^{(1)},H_0] = -2i\frac{\sqrt{3}}{\mu} \int \prod_{i=1}^3 d^2x_i\phi^\dagger_1\phi_2\phi_3 \left(-\mu + \alpha'(-2\nabla_3^2 + \nabla_1^2)\right)f_1(x_1,x_2,x_3). \tag{23}
\]

Here it has been taken into account that \( f_1(x_1,x_2,x_3) \) is symmetric in \( x_2 \) and \( x_3 \).
In full analogy we calculate the commutators related to the part with $f_2$.

\[
[\phi_1^\dagger \phi_2 \phi_3, \phi^\dagger (x) \phi(x)] = \delta^2(x_3 - x)\phi_1^\dagger \phi_2 \phi(x) - \delta^2(x_2 - x)\phi_1^\dagger \phi(x) \phi_2^\dagger \phi(x) - \delta^2(x_1 - x)\phi_1^\dagger \phi_2^\dagger \phi_3
\]

and

\[
[\phi_1^\dagger \phi_2 \phi_3, \nabla \phi^\dagger (x) \nabla \phi(x)]
\]

\[
= \nabla \delta^2(x_3 - x)\phi_1^\dagger \phi_2 \nabla \phi(x) - \nabla \delta^2(x_2 - x)\nabla \phi^\dagger (x) \phi_1^\dagger \phi(x) - \nabla \delta^2(x_1 - x)\nabla \phi^\dagger (x) \phi_2^\dagger \phi_3,
\]

which gives the second part of $[Q_1, H_0]$:

\[
[Q_1^{(2)}, H_0] = 2i \int \frac{1}{\mu} \prod_{i=1}^{3} d^2 x_i \phi_1^\dagger \phi_2^\dagger \phi_3 \left( -\mu + \alpha \left(-2 \nabla_1^2 + \nabla_3^2\right) \right) f_2(x_1, x_2, x_3).
\]

It has been taken into account that $f_2(x_1, x_2, x_3)$ is symmetric in $x_1$ and $x_2$.

To satisfy the first of the conditions (25) which determines the form of $Q_1$ we have to require

\[
\left( -\mu + \alpha \left(-2 \nabla_3^2 + \nabla_1^2\right) \right) f_1(x_1, x_2, x_3) = -\mu \delta^2(x_1 - x_2) \delta^2(x_3 - x_1)
\]

and

\[
\left( -\mu + \alpha \left(-2 \nabla_1^2 + \nabla_3^2\right) \right) f_2(x_1, x_2, x_3) = \mu \delta^2(x_3 - x_1) \delta_2(x_3 - x_2).
\]

These equations are trivially solved in the momentum space. We define the Fourier transforms by

\[
f_i(k_1, k_2, k_3) = \int \prod_{i=1}^{3} \left( d^2 x_i e^{-i k_i x_i} \right) f_i(x_1, x_2, x_3), \quad i = 1, 2.
\]

Then we find

\[
\left( -\mu + \alpha \left(2 k_3^2 - k_1^2\right) \right) f_1(k_1, k_2, k_3) = -\mu (2\pi)^2 \delta^2(k_1 + k_2 + k_3)
\]

and

\[
\left( -\mu + \alpha \left(2 k_1^2 - k_3^2\right) \right) f_2(k_1, k_2, k_3) = \mu (2\pi)^2 \delta^2(k_1 + k_2 + k_3).
\]

So taking into account the symmetry properties of the functions $f_1$ and $f_2$ one has

\[
f_1(k_1, k_2, k_3) = \frac{(2\pi)^2 \delta(k_1 + k_2 + k_3)}{\mu - \alpha \left(k_1^2 + k_2^2 - k_3^2\right)}
\]

and

\[
f_2(k_1, k_2, k_3) = -\frac{(2\pi)^2 \delta(k_1 + k_2 + k_3)}{\mu - \alpha \left(k_1^2 + k_2^2 - k_3^2\right)} = -f_1(k_3, k_2, k_1).
\]

Note that the denominators in (30) and (31) may vanish. The requirement that $Q_1$ be a Hermitian operator implies that the singularities in (30) and (31) be circumvented from opposite sides or taken both in the principle value prescription. The study of the transformed Hamiltonian in the next two sections reveals that at order $\lambda^2$ there appears a pairwise interaction of pomeronas similar to the pairwise interaction of normal non-relativistic particles. In the latter case the interaction potential is standardly real, which leads to invariance under time reflection. The pomeronas are not propagating in real time, but rather in the imaginary one corresponding to the rapidity. Still, as mentioned above there exists a similar invariance consisting in changing signs of $\phi$ and $\phi^\dagger$ and taking complex conjugate of the coefficient functions. This requires the pair potential to be real as in the normal theory. As we shall see in the next section this requirement requires that the functions $f_1$ and $f_2$ be real, so that their singularities be taken in the principal value sense. This circumstance will be implicitly understood in the following.
4 The transformed Hermitian Hamiltonian $h^{(2)}$

In this section we shall find the second order Hamiltonian $h^{(2)}$ determined by second of Eqs. (9). We present the interaction term $H_I$ as an integral over three momenta $q_i$, $i = 1, 2, 3$ using

$$
\phi(x) = \int \frac{d^2k}{2\pi} e^{ikx} \phi(k)
$$

and similarly for $\phi^\dagger(x)$. With this normalization the fields will obey the standard commutation relations in the momentum space

$$
[\phi(k), \phi^\dagger(k')] = \delta(k - k') .
$$

The interaction Hamiltonian acquires the form

$$
H_I = i \int \prod_{i=1}^{3} \frac{d^2q_i}{2\pi} \left[ (2\pi)^2 \delta^2(q_1 - q_2 - q_3) \phi^\dagger(q_1) \phi(q_2) \phi(q_3) \\
+ (2\pi)^2 \delta^2(q_1 + q_2 - q_3) \phi^\dagger(q_1) \phi^\dagger(q_2) \phi(q_3) \right].
$$

In the same manner we get operator $Q_1$ as

$$
Q_1 = -2i \mu \int \prod_{i=1}^{3} \frac{d^2k_i}{2\pi} \left( f_1(-k_1, k_2, k_3) \phi^\dagger(k_1) \phi(k_2) \phi(k_3) + f_2(-k_1, -k_2, k_3) \phi^\dagger(k_1) \phi^\dagger(k_2) \phi(k_3) \right).
$$

To find the second order Hamiltonian $h^{(2)}$ we have to calculate the following 4 commutators:

$$
C_1 = [\phi^\dagger(k_1) \phi(k_2) \phi(k_3), \phi^\dagger(q_1) \phi(q_2) \phi(q_3)], \\
C_2 = [\phi^\dagger(k_1) \phi(k_2) \phi(k_3), \phi^\dagger(q_1) \phi^\dagger(q_2) \phi(q_3)], \\
C_3 = [\phi^\dagger(k_1) \phi^\dagger(k_2) \phi(k_3), \phi^\dagger(q_1) \phi(q_2) \phi(q_3)], \\
C_4 = [\phi^\dagger(k_1) \phi^\dagger(k_2) \phi(k_3), \phi^\dagger(q_1) \phi^\dagger(q_2) \phi(q_3)].
$$

They are all trivially found:

$$
C_1 = -\delta^2(q_2 - k_1) \phi^\dagger(q_1) \phi(q_3) \phi(k_2) \phi(k_3) - \delta^2(q_3 - k_1) \phi^\dagger(q_1) \phi(q_2) \phi(k_2) \phi(k_3) \\
+ \delta^2(q_1 - k_2) \phi^\dagger(k_1) \phi(k_3) \phi(q_2) \phi(q_3) + \delta^2(q_1 - k_3) \phi^\dagger(k_1) \phi(k_2) \phi(q_2) \phi(q_3) ,
$$

$$
C_2 = -\delta^2(k_1 - q_3) \phi^\dagger(q_1) \phi^\dagger(q_2) \phi(k_2) \phi(k_3) \\
+ \delta(k_2 - q_2) \phi^\dagger(q_1) \phi(k_3) \phi(q_3) + \delta(k_3 - q_2) \phi^\dagger(q_1) \phi^\dagger(k_1) \phi(k_2) \phi(q_3) \\
+ \delta(k_2 - q_1) \phi^\dagger(k_1) \phi(k_3) \phi^\dagger(q_2) \phi(q_3) + \delta(k_3 - q_1) \phi^\dagger(k_1) \phi^\dagger(k_2) \phi(q_2) \phi(q_3) ,
$$

$$
C_3 = -\delta^2(k_1, k_2, k_3 \leftrightarrow q_1, q_2, q_3) \\
= \delta^2(q_1 - k_3) \phi^\dagger(k_1) \phi^\dagger(k_2) \phi(q_2) \phi(q_3) - \delta(q_2 - k_3) \phi^\dagger(q_1) \phi^\dagger(k_1) \phi(q_3) \phi(k_3) \\
- \delta(q_3 - k_2) \phi^\dagger(k_1) \phi^\dagger(q_1) \phi(q_2) \phi(k_3) - \delta(q_2 - k_1) \phi^\dagger(q_1) \phi(q_3) \phi^\dagger(k_2) \phi(k_3) \\
- \delta(q_3 - k_1) \phi^\dagger(q_1) \phi(q_2) \phi^\dagger(k_2) \phi(k_3) ,
$$

$$
C_4 = C_1^\dagger(k_1, k_2, k_3 \leftrightarrow q_3, q_2, q_1) \\
= -\delta^2(q_2 - q_3) \phi^\dagger(q_1) \phi^\dagger(q_2) \phi^\dagger(k_1) \phi(k_3) - \delta(q_1 - q_3) \phi^\dagger(q_1) \phi^\dagger(q_2) \phi^\dagger(k_2) \phi(k_3) \\
+ \delta^2(k_3 - q_2) \phi^\dagger(k_1) \phi^\dagger(k_2) \phi^\dagger(q_1) \phi(q_3) + \delta^2(k_3 - q_1) \phi^\dagger(k_1) \phi^\dagger(k_2) \phi^\dagger(q_2) \phi(q_3) .
$$
Passing to $h^{(2)}$ we get the following 4 terms

\[ h_1^{(2)} = -\frac{1}{2} \int \prod_{i=1}^{3} \frac{d^2 k_i d^2 q_i (2\pi)^2 \delta^2(q_1 - q_2 - q_3)(2\pi)^2 \delta(k_1 - k_2 - k_3)}{(2\pi)^2 \mu - \alpha'(k_2^2 + k_3^2 - k_1^2)} C_1(k_1, k_2, k_3|q_1, q_2, q_3) \]

\[ = -\frac{1}{(2\pi)^2} \int \frac{d^2 k_2 d^2 k_3 d^2 q_2}{\mu - \alpha'(k_2^2 + k_3^2 - (k_2 + k_3)^2)} \left( \phi^\dagger(k_2 + k_3) \phi(k_2) \phi(k_3) \phi(k_2 - q_2) - \phi^\dagger(q_2 + k_2 + k_3) \phi(q_2) \phi(k_2) \phi(k_3) \right), \quad (40) \]

\[ h_2^{(2)} = -\frac{1}{2} \int \prod_{i=1}^{3} \frac{d^2 k_i d^2 q_i (2\pi)^2 \delta^2(q_1 + q_2 - q_3)(2\pi)^2 \delta(k_1 - k_2 - k_3)}{(2\pi)^2 \mu - \alpha'(k_2^2 + k_3^2 - k_1^2)} C_2(k_1, k_2, k_3|q_1, q_2, q_3) \]

\[ = \frac{1}{(2\pi)^2} \int \frac{d^2 k_2 d^2 k_3 d^2 q_1}{\mu - \alpha'(k_2^2 + k_3^2 - (k_2 + k_3)^2)} \left( 2 \phi^\dagger(q_1) \phi^\dagger(k_2 + k_3) \phi(k_3) \phi(q_1 + k_2) \right. \]

\[ + 2 \phi^\dagger(k_2 + k_3) \phi(k_3) \phi^\dagger(q_1) \phi(q_1 + k_2) - \phi^\dagger(q_1) \phi^\dagger(k_2 + k_3 - q_1) \phi(k_2) \phi(k_3) \), \quad (41) \]

\[ h_3^{(2)} = +\frac{1}{2} \int \prod_{i=1}^{3} \frac{d^2 k_i d^2 q_i (2\pi)^2 \delta^2(q_1 - q_2 - q_3)(2\pi)^2 \delta(k_1 + k_2 - k_3)}{(2\pi)^2 \mu - \alpha'(k_2^2 + k_3^2 - k_1^2)} C_3(k_1, k_2, k_3|q_1, q_2, q_3) \]

\[ = -\frac{1}{(2\pi)^2} \int \frac{d^2 k_1 d^2 k_2 d^2 q_3}{\mu - \alpha'(k_1^2 + k_2^2 - (k_1 + k_2)^2)} \left( 2 \phi^\dagger(k_1) \phi^\dagger(k_2 + q_3) \phi(q_3) \phi(k_1 + k_2) \right. \]

\[ + 2 \phi^\dagger(k_2 + q_3) \phi(q_3) \phi^\dagger(k_1) \phi(k_1 + k_2) - \phi^\dagger(k_1) \phi^\dagger(k_2) \phi(k_1 + k_2 - q_3) \phi(q_3) \), \quad (42) \]

\[ h_4^{(2)} = +\frac{1}{2} \int \prod_{i=1}^{3} \frac{d^2 k_i d^2 q_i (2\pi)^2 \delta^2(q_1 + q_2 - q_3)(2\pi)^2 \delta(k_1 + k_2 - k_3)}{(2\pi)^2 \mu - \alpha'(k_1^2 + k_2^2 - k_3^2)} C_4(k_1, k_2, k_3|q_1, q_2, q_3) \]

\[ = -\frac{1}{(2\pi)^2} \int \frac{d^2 k_2 d^2 k_3 d^2 q_2}{\mu - \alpha'(k_2^2 + k_3^2 - (k_2 + k_3)^2)} \left( \phi^\dagger(k_2 - q_2) \phi^\dagger(q_2) \phi^\dagger(k_3) \phi(k_2 + k_3) - \phi^\dagger(k_3) \phi^\dagger(k_2) \phi^\dagger(q_2) \phi(k_2 + k_3 + q_2) \right), \quad (43) \]

We have obviously

\[ h_4^{(2)} = (h_1^{(2)})^\dagger, \quad h_3^{(2)} = (h_2^{(2)})^\dagger, \]

so that the total second order Hamiltonian $h$ is Hermitian.

## 5 Single- and two-particles terms in $h$ and renormalization

All terms in $h^{(2)}$ split into the pomeron number conserving, $h_2^{(2)} + h_3^{(2)}$, and pomeron number changing: $h_1^{(2)}$, with $\Delta N = -2$ and $h_4^{(2)}$ with $\Delta N = +2$. The two latter terms will contribute to energy levels only in the order $\lambda^4$ and can be neglected in the order $\lambda^2$ with we restrict ourselves here.

The pomeron number conserving terms may be rewritten in the normal form. As a result we get two contributions with two or four field operators, describing single- and double-particle parts. The single particle part is found to be

\[ h_{single}^{(2)} = -\frac{2}{(2\pi)^2} Re \int \frac{d^2 k_2 d^2 k_3}{\mu - \alpha'(k_2^2 + k_3^2 - (k_2 + k_3)^2)} \phi^\dagger(k_2 + k_3) \phi(k_2 + k_3) \]
\[
\Delta^{(2)}\epsilon(k) = -2 \frac{(2\pi)^2 \text{Re}}{\mu - \alpha'(k^2 + k_3^2 - k^2)} \int d^2k d^2k_3 \delta^2(k_2 + k_3 - k)
\]

is the shift in the pomeron energy in order \(\lambda^2\), so that the total pomeron energy is

\[
\epsilon(k) = -\mu + \alpha'k^2 + \lambda^2\Delta^{(2)}\epsilon(k).
\]

One easily finds that \(\Delta^{(2)}\epsilon(k)\) can be presented (see Appendix A) as

\[
\Delta^{(2)}\epsilon(k) = +\frac{1}{4\pi\alpha'} \int_0^\infty \frac{dx}{x} e^{(\mu + \alpha'k^2)/2}.
\]

The integral obviously exists only for \(\mu < -\alpha'k^2/2\) and diverges at \(x = 0\). To regularize it we require that at \(k = 0\) the change of pomeron energy vanishes, that is \(\epsilon(0) = -\mu\). In a way this is a definition of the renormalized intercept. One can obtain the same result choosing the standard dimensional regularization approach. With this condition the regularized \(\Delta\epsilon(k)\) is

\[
\Delta^{(2)}\epsilon_{\text{reg}}(k) = +\frac{1}{4\pi\alpha'} \int_0^\infty \frac{dx}{x} \left( e^{(\mu + \alpha'k^2)/2} - e^{\mu}\right)
\]

\[
= -\frac{1}{4\pi\alpha'} \ln \left(1 + \frac{\alpha'k^2}{2\mu}\right).
\]

It exists for any values of \(\mu\) and is real for \(\mu > 0\). Taking into account that the initial pomeron trajectory is determined only up to terms linear in \(k^2\), we approximate the energy shift as

\[
\Delta^{(2)}\epsilon_{\text{reg}}(k) = -\frac{1}{8\pi\mu} k^2,
\]

so that the net effect of the single-particle term in \(h^{(2)}\) is to renormalize the slope:

\[
\alpha' \rightarrow \alpha'_{\text{ren}} = \alpha' - \lambda^2 \frac{1}{8\pi\mu}.
\]

The two-pomeron interaction term \(h^{(2)}_{\text{pair}}\) can be presented in the form corresponding to transition \(k_1, k_2 \rightarrow q_1, q_2\)

\[
h^{(2)}_{\text{pair}} = \int d^2k_1 d^2k_2 d^2q_1 d^2q_2 \delta^2(q_1 + q_2 - k_1 - k_2) V^{(2)}(q_1, q_2|k_1, k_2) \phi^\dagger(q_1) \phi^\dagger(q_2) \phi(k_1) \phi(k_2),
\]

where

\[
V^{(2)}(q_1, q_2|k_1, k_2) = -\frac{1}{2\pi^2} \frac{1}{\mu - \alpha'(k_1^2 + (k_2 - q_1)^2 - q_2^2)} - \frac{1}{2\pi^2} \frac{1}{\mu - \alpha'(q_1^2 + (q_2 - k_1)^2 - k_2^2)}
\]

\[
+ \frac{1}{8\pi^2} \frac{1}{\mu - \alpha'(k_1^2 + k_2^2 - (k_1 + k_2)^2)} + \frac{1}{8\pi^2} \frac{1}{\mu - \alpha'(q_1^2 + q_2^2 - (q_1 + q_2)^2)}
\]

(symmetrization in \(q_1, q_2\) and \(k_1, k_2\) is implied). As mentioned, poles in the two terms have to be understood in the principal value sense for the potential to be not only Hermitian but also real.

This potential is non-local and degenerate. To more clearly see its properties consider a case when \(q_1 + q_2 = k_1 + k_2 = 0\) corresponding to the two-pomerom exchange for the forward scattering amplitude. Then denoting \(q_1 = -q_2 = q\) and \(k_1 = -k_2 = k\) we have a potential

\[
V^{(2)}(q|k) = -\frac{1}{2\pi^2} \frac{1}{\mu - \alpha'(k^2 + (k + q)^2 - q^2)} - \frac{1}{2\pi^2} \frac{1}{\mu - \alpha'(q^2 + (k + q)^2 - k^2)}
\]
\[ \frac{1}{8\pi^2} \frac{1}{\mu - 2\alpha'k^2} + \frac{1}{8\pi^2} \frac{1}{\mu - 2\alpha'q^2}. \] (53)

The last two terms depend only on the initial or only on the final momenta. Since the potential falls rather slowly at high momenta, its integration over \( q \) or \( k \) meets with a logarithmic divergence. (See the form of the kernel in the coordinate space in Appendix 2.)

The effect of this degenerate pair potential is not quite clear in the general case. Considerations in the next section, restricted to the case \( q_1 + q_2 = k_1 + k_2 = 0 \), tell that the spectrum of the pomeron states will not be changed by this interaction. So in its presence the two-pomeron states will continue to have their total energy

\[ E_2(k_1, k_2) = \epsilon(k_1) + \epsilon(k_2), \] (54)

with \( \epsilon(k) \) given by (56) but the wave functions will become changed by the standard scattering operator. To the second order in \( \lambda \)

\[ \Psi_{k_1,k_2}(q_1,q_2) = \text{Sym} \left\{ \delta^2(k_1 - q_1)\delta^2(k_2 - q_2) + \lambda^2 \frac{V(q_1,q_2|k_1,k_2)}{\epsilon(q_1) + \epsilon(q_2) - \epsilon(k_1) - \epsilon(k_2) \pm i0} \right\}, \] (55)

where symbol Sym means symmetrization in \( q_1 \) and \( q_2 \) and signs of \( i0 \) correspond to in- or -outgoing waves.

For the asymptotic of the Green function we shall find from the two-pomeron states

\[ \delta^2(k - k') \int d^2k_1d^2k_2 \langle 0|\phi(k')e^{Q/2}\Psi_{k_1,k_2}\rangle e^{-yE_2(k_1,k_2)}\langle \Psi_{k_1,k_2}|e^{-Q/2}\phi^0(k)|0 \rangle, \] (56)

where the matrix elements can be easily computed by perturbations in \( \lambda \). This asymptotics will be true at large \( y \) until \( y \sim 1/\lambda^4 \). Of course one also will have similar contributions from states with the number of pomerons greater than two, but the corresponding matrix elements will be of the higher order in \( \lambda \).

6 The Schroedinger equation with a degenerate potential

6.1 Problem

Consider the Schroedinger equation in the 2-dimensional momentum space

\[ (\epsilon(q) - E)\psi(q) = -\int d^2kV(q|k)\psi(k). \] (57)

To simplify notation we rescale \( E \) to exclude all terms independent of \( q \) in \( \epsilon(q) \) and have in our case \( \epsilon(q) = 2\alpha'q^2 \). Our potential has a structure (53):

\[ V(q,k) = v(q) + v(k) + V_1(q,k), \] (58)

where \( V_1(q,k) \) has the normal properties and vanishes as any of the arguments go to infinity. Our aim is to study the spectrum \( E \) of the solutions to Eq. (57).

Obviously Eq. (57) can have both solutions corresponding to the scattering states and to bound states. In the former case we standardly convert this equation into the Lippman-Schwinger equation presenting

\[ \psi_1(q) = \delta^2(q - l) + \frac{T(q|l)}{\epsilon(l) - \epsilon(q) \pm i0}, \] (59)

where \( T(q|l) \) (\( T \) from now is no more the “time reflection” operator) satisfies

\[ T(q|l) = V(q|l) + \int d^2k \frac{V(q,k)T(k|l)}{\epsilon(l) - \epsilon(k) \pm i0}. \] (60)
If this equation can be solved it corresponds to the scattering state with the incident momentum \( l \) and energy \( E = \epsilon(l) \) which belongs to the continuous positive spectrum.

For the bound state \( \psi_E(q) \) with energy \( E < 0 \) we analogously present

\[
\psi_E(q) = \frac{t_E(q)}{E - \epsilon(q)}, \quad \text{(61)}
\]

with an equation for \( t_E \)

\[
t_E(q) = \int d^2k \frac{V(q|k)t_E(k)}{E - \epsilon(k)}. \quad \text{(62)}
\]

Our aim is to study possible solutions of Eqs. (60) and (62) with a degenerate potential (58).

6.2 Continuous spectrum

Putting (58) into (60) and suppressing the fixed argument \( l \) in \( T \) we have

\[
T(q) = v(q) + c + V_1(q, l) + v(q) \int d^2k \frac{T(k)}{\epsilon(l) - \epsilon(k)}
\]

\[
+ \int d^2k \frac{v(k)T(k)}{\epsilon(l) - \epsilon(k)} + \int d^2k \frac{V_1(q, k)T(k)}{\epsilon(l) - \epsilon(k)}.
\quad \text{(63)}
\]

where we have denoted the part of \( V \) independent of \( q \)

\[
v(l) = c. \quad \text{(64)}
\]

We also denote

\[
d = \int d^2k \frac{T(k)}{\epsilon(l) - \epsilon(k)}, \quad e = \int d^2k \frac{v(k)T(k)}{\epsilon(l) - \epsilon(k)}.
\quad \text{(65)}
\]

We find an equation

\[
T(q) = c + e + (1 + d)v(q) + V_1(q, l) + \int d^2k \frac{V_1(q, k)T(k)}{\epsilon(l) - \epsilon(k)}. \quad \text{(66)}
\]

Correspondingly we present

\[
T(q) = c + e + (1 + d)v(q) + T_1(q). \quad \text{(67)}
\]

The equation for \( T_1(q) \) is

\[
T_1(q) = (c + e)\chi_1(q) + (1 + d)\chi_2(q) + V_1(q, l) + \int d^2k \frac{V_1(q, k)T_1(k)}{\epsilon(l) - \epsilon(k)}, \quad \text{(68)}
\]

with

\[
\chi_1(q) = \int d^2k \frac{V_1(q, k)}{\epsilon(l) - \epsilon(k)} \quad \text{(69)}
\]

and

\[
\chi_2(q) = \int d^2k \frac{V_1(q, k)v(k)}{\epsilon(l) - \epsilon(k)}. \quad \text{(70)}
\]

For the two constants \( d \) and \( e \) we obtain equations following from their definition

\[
d = (1 + d)I_1 + (c + e)I_0 + \int d^2k \frac{T_1(k)}{\epsilon(l) - \epsilon(k)} \quad \text{(71)}
\]
and
\[ e = (c + e)I_1 + (1 + d)I_2 + \int d^2k \frac{v(k)T_1(k)}{\epsilon(l) - \epsilon(k)}, \]  
where
\[ I_n = \int d^2k \frac{v^n(k)}{\epsilon(l) - \epsilon(k)}. \]  

To solve Eq. (68) with additional conditions (71) and (72) we first solve this equation for three different inhomogeneous terms
\[ T_1^{(i)}(q) = T_0^{(i)}(q) + \int d^2k \frac{V(q,k)T_1^{(i)}(k)}{\epsilon(l) - \epsilon(k)}, \]  
where \( i = 1, 2, 3 \) and
\[ T_0^{(1,2)}(q) = \chi_{1,2}(q), \quad T_0^{(3)}(q) = V_1(q,l). \]  
From these three solutions we obtain the solution to Eq. (68) as
\[ T_1(q) = (c + e)T_1^{(1)}(q) + (1 + d)T_1^{(2)}(q) + T_1^{(3)}(q). \]  

Now we put this solution into the equations (71), (72) to obtain a system of two linear equations for \( d \) and \( e \):
\[ d(I_1 + J_2 - 1) + e(I_0 + J_1) + c(I_0 + J_1) + I_1 + J_2 + J_3 = 0, \]
\[ d(I_2 + K_2) + e(I_1 + K_1 - 1) + c(I_1 + K_1) + I_2 + K_2 + K_3 = 0, \]  
where \( I_n \) are defined by (73) and
\[ J_n = \int d^2k \frac{T_1^{(n)}(k)}{\epsilon(l) - \epsilon(k)}, \quad K_n = \int d^2k \frac{v(k)T_1^{(n)}(k)}{\epsilon(l) - \epsilon(k)}. \]  

All quantities in fact depend on the fixed momentum \( l \). Solution of the linear system (77) is of course trivial. The determinant is
\[ D = (I_1 + J_2 - 1)(I_1 + K_1 - 1) - (I_0 + J_1)(I_2 + K_2) \]  
and so
\[ d = \frac{1}{D} \left[ \left( c(I_1 + K_1) + I_2 + K_2 + K_3 \right)(I_0 + I_1) - \left( c(I_0 + J_1) + I_1 + J_2 + J_3 \right)(I_1 + K_1 - 1) \right], \]
\[ e = \frac{1}{D} \left[ \left( c(I_0 + J_1) + I_1 + J_2 + J_3 \right)(I_2 + K_2) - \left( c(I_1 + K_1) + I_2 + K_2 + K_3 \right)(I_1 + J_2 - 1) \right]. \]  

Of course the resulting \( d \) and \( e \) are functions of the fixed momentum \( l \). With thus determined \( d(l) \) and \( e(l) \) Eq. (76) gives the final solution to the Lippmann-Schwinger problem. The only difficulty may generally arise in case \( D(l) = 0 \) which may only happen at some specific values of \( l \) and leads to certain singularities of the scattering matrix at these values of momentum, which we consider improbable.

For future reference, here we present orders in \( \lambda \) for different quantities defined in the previous derivation. Obviously
\[ D = 1 + \mathcal{O}(\lambda^2); \quad I_n \sim \lambda^{2n}; \quad \chi_{1,2} \sim \lambda^{2(4)} \]
\[ T_1^{(1)}, T_1^{(3)}, J_1, J_3 \sim \lambda^2; \quad T_1^{(2)}, J_2 \sim \lambda^4; \quad K_1, K_3 \sim \lambda^4; \quad K_2 \sim \lambda^6. \]  

(82)
Then it follows from (80) and (81) that $d \sim \lambda^2$ and $e \sim \lambda^4$ and in the lowest approximation (order $\lambda^2$) the scattering matrix $T$ is given just by the total potential $V$, as expected.

However in our case there is a new problem. The constant $I_0$ is in fact divergent. In the limit $I_0 \to \infty$ we find that the determinant grows linearly with $I_0$:

$$D = -I_0(I_2 + K_2).$$

(83)

The denominators of (80) and (81) also grow linearly with $I_0$. So in the limit $I_0 \to \infty$ we find finite values for both $d$ and $e$:

$$1 + d = -\frac{c + K_3}{I_2 + K_2}, \quad e + c = 0.$$

(84)

With these values we find in this limiting case

$$T_1(q) = T_1^{(3)}(q) - \frac{c + K_3}{I_2 + K_2}T_1^{(2)}(q).$$

(85)

So again the solution in all probability exists but the constant $e$ is automatically adjusted to exclude the constant term $c = v(l)$ from the original Lippmann-Schwinger equation (60).

As a result, we find that even for divergent $I_0$ the Lippmann-Schwinger equation has a solution for any positive energy, so that the spectrum is continuous and covers all positive values of energy.

Note that in the limit $I_0 \to \infty$ orders of $d$ and $e$ in powers of $\lambda$ are radically changed. Now the determinant $D \sim \lambda^4$ and as a result $1 + d \sim 1/\lambda^2$ and $e \sim \lambda^2$. As a result already in the lowest order $\lambda^2$ the scattering matrix $T_1$ acquires additional terms from $T_1^{(2)}$:

$$T_1(q) = V_1(q|l) - \frac{v(l)}{I_2}\chi_2(q).$$

(86)

Turning to the full scattering matrix (57) we find that it acquires a term of the order unity

$$T(q|l) = -\frac{v(q)v(l)}{I_2} + T_1(q).$$

(87)

It may be considered as a renormalization term for the scattering matrix in the limit $I_0 \to \infty$. Of course appearance of this term is due to the implicitly made assumption that $\lambda^4I_0 >> 1$ as $I_0 \to \infty$. Different relations between the small $\lambda$ and large $I_0$ will lead to different results.

To illustrate the described procedure for the solution of Lippmann-Schwinger equation in Appendix C we calculate the scattering matrices $T$ and $T_1$ up to order $\lambda^2$ for the pair pomeron potential $V^{(2)}$, Eq. (53), for the forward case $q_1 + q_2 = k_1 + k_2 = 0$. The found expressions are long and not very interesting but they show that the procedure is quite feasible and does not encounter any new complications.

### 6.3 Discrete spectrum

Putting (58) into (62) we now obtain

$$t_E(q) = v(q) \int d^2k \frac{t_E(k)}{E - \epsilon(k)} + \int d^2k \frac{v(k)t_E(k)}{E - \epsilon(k)} + \int d^2k \frac{V_1(q,k)t_E(k)}{E - \epsilon(k)}.$$

(88)

As before we denote

$$d = \int d^2k \frac{t_E(k)}{E - \epsilon(k)}, \quad e = \int d^2k \frac{v(k)t_E(k)}{E - \epsilon(k)}.$$

(89)
We find an equation

\[ t_E(q) = e + dv(q) + \int d^2k \frac{V_1(q,k) t_E(k)}{E - \epsilon(k)}. \]  

We present

\[ t_E(q) = e + dv(q) + t_{1E}(q) \]  

to find an equation for \( t_{1E} \)

\[ t_{1E}(q) = e\chi_1(q) + d\chi_2(q) + \int d^2k \frac{V_1(q,k) t_{1E}(k)}{E - \epsilon(k)}, \]

where similarly to the continuous spectrum case

\[ \chi_1(q) = \int d^2k \frac{V_1(q,k)}{E - \epsilon(k)} \]  

and

\[ \chi_2(q) = \int d^2k \frac{V_1(q,k)v(k)}{E - \epsilon(k)}, \]

with the two conditions to determine \( d \) and \( e \)

\[ d = dI_1 + eI_0 + \int d^2k \frac{t_{1E}(k)}{E - \epsilon(k)} \]

and

\[ e = eI_1 + dI_2 + \int d^2k \frac{v(k)t_{1E}(k)}{E - \epsilon(k)}. \]

The integrals \( I_n \) are the same as in (73) with \( \epsilon(l) \to E \).

Obviously the solution to Eq. (92) can be presented as a sum

\[ t_{1E}(q) = e^{(1)}_{1E}(q) + d^{(2)}_{1E}(q), \]

where the two functions \( t^{(1,2)}_{1E} \) satisfy

\[ t^{(i)}_{1E}(q) = \chi_1(q) + \int d^2k \frac{V_1(q,k)t^{(i)}_{1E}(k)}{E - \epsilon(k)}, \quad i = 1, 2 \]

After functions \( t^{(1,2)}_{1E} \) are known, one finds a homogeneous system of linear equations to determine \( d \) and \( e \):

\[ d = eI_0 + dI_1 + eJ_1 + dJ_2, \quad e = eI_1 + dI_2 + eK_1 + dK_2, \]

where now, similarly to (78),

\[ J_n = \int d^2k \frac{t^{(n)}_{1E}(k)}{E - \epsilon(k)}, \quad K_n = \int d^2k \frac{v(k)t^{(n)}_{1E}(k)}{E - \epsilon(k)}. \]

All the coefficients in the system (99) depend on \( E \). The value of the bound energy \( E \) is found from the condition of existence of solutions to the system (99):

\[ (I_1 + J_2 - 1)(I_1 + K_1 - 1) - (I_0 + J_1)(I_2 + K_2) = 0. \]

Now consider the case \( I_0 \to \infty \). Then Eq. (101) reduces to

\[ I_2(E) + K_2(E) = 0, \]

which determines a possible bound state in this limiting case.

As we have seen, at small values of \( \lambda I_2 \) is of order \( \lambda^4 \) and \( K_2 \) is of order \( \Lambda^6 \). Since \( I_2(E) \) cannot vanish (it is strictly negative for \( E < 0 \)) Eq. (102) cannot be satisfied. So for small values of \( \lambda \) there are no bound states when \( I_0 \to \infty \).
7 Conclusions

We have generalized the technique of constructing an Hermithean Hamiltonian for the PT symmetric LRFT model, developed in [4] for the toy model in zero transverse dimensions, to the realistic case of two transverse dimensions. The complexity of the latter model makes both the derivation and analysis of the found Hamiltonian not straightforward already in the lowest non-trivial order in the coupling constant $\lambda$ of the triple pomeron interaction. In particular the divergence of the pomeron intercept has to be eliminated by renormalization. Also the found pair interaction between pomerons is both singular and degenerate. It has required a separate study of the Schrödinger equation with degenerate potentials, which may have a wider scope of applicability.

As a result we have found that at small $\lambda$ the total impact of the pomeron interaction at order $\lambda^2$ is reduced to the change of slope. These results allow to study the asymptotic of any scattering amplitude at large rapidities $y$ in the region

$$1 << y << (\alpha'/\lambda^2)^2$$

However in the course of our study it became clear that at finite $\lambda$ the pomerons may form bound states, whose presence will drastically change this asymptotics.

Note that the experimental values for $\alpha'$ and $\lambda$ are roughly $\alpha' \sim 0.25$ (GeV/c)$^{-2}$ and $\lambda \sim 0.33$ (GeV/c)$^{-1}$, so that the actual parameter of the perturbative expansion is $\lambda^2/\alpha' \sim 0.4$, which is not so small. So to estimate the possibility to apply our results to the realistic processes one has to study the NNLO (terms of order $\lambda^4$). If they happen to be relatively small then one can study possible pomeron bound states using our potential and, say, the variational methods.

We stress that the perturbative study of our Hermithean Hamiltonian cannot give the true asymptotic of the theory at $y \to \infty$, which remains unperturbative. To find it one has to search for non-perturbative techniques to construct this Hamiltonian.

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A Single pomeron energy shift

We have to calculate the integral

$$I = \int \frac{d^2k_2d^2k_3\delta^2(k_2 + k_3 - k)}{\mu - \alpha'(k_2^2 + k_3^2 - k^2)}. \quad (103)$$

We present the $\delta$-function as an integral over $r$ and the denominator at $\mu < -\alpha' k^2/2$ as

$$\frac{1}{\mu - \alpha'(k_2^2 + k_3^2 - k)^2} = -\int_0^\infty dx e^{\mu - \alpha'(k_2^2 + k_3^2 - k^2))} \quad (104)$$

to get

$$I = -\frac{1}{(2\pi)^2} \int_0^\infty dx \int d^2k_2d^2k_3d^2re^{ir(k_2 + k_3 - k)}e^{x(\mu - \alpha'(k_2^2 + k_3^2 - k^2))}. \quad (105)$$

Integrals over $k_2$ and $k_3$ give the same result:

$$\int d^2k_2e^{irk_2 - x\alpha'k_2^2} = \frac{\pi}{x\alpha'} e^{-\frac{r^2}{4x\alpha'}}. \quad (106)$$
So we find
\[ I = -\frac{1}{4} \int_0^\infty dx \int d^2r \frac{1}{(x\alpha')^2} e^{-irk} e^{x(\mu+\alpha'k^2-\frac{r^2}{2x\alpha'})}. \]  
(107)

Next we integrate over \( r \)
\[ \int d^2r e^{ikr-\frac{r^2}{2x\alpha'}} = 2\pi x\alpha' e^{-x\alpha'k^2/2} \]
(108)

to finally find an integral over \( x \)
\[ I = -\frac{\pi}{2\alpha'} \int_0^\infty \frac{dx}{x} e^{x(\mu+\alpha'k^2/2)}. \]
(109)

B The coordinate space potential between pomeron

The two-pomeron potential is non-local in the coordinate space. We shall limit ourselves with the case \( q_1 + q_2 = k_1 + k_2 \) when the in the momentum space the potential is given by \( \text{[54]} \). Then in the coordinate space its kernel is defined as
\[ V^{(2)}(y|x) = \int \frac{d^2qd^2k}{(2\pi)^4} e^{-iqy+ikx} V^{(2)}(q|k). \]
(110)

The non-degenerate part of the kernel comes from the first two terms in \( \text{[53]} \) and is
\[ V_1^{(2)}(y|x) = -\frac{1}{2\pi^2} \int \frac{d^2qd^2k}{(2\pi)^4} \frac{e^{-iqy+ikx}}{\mu - \alpha'(k^2+(k+q)^2-q^2)} + (x \leftrightarrow -y). \]
(111)

We rewrite the first term in this expression as
\[ V_{11}^{(2)}(y|x) = -\frac{1}{2\pi^2} \int \frac{d^2qd^2kd\kappa}{(2\pi)^4} \delta^2(k+q-\kappa) \frac{1}{\mu - \alpha'(k^2+\kappa^2-q^2)} \]
\[ = \frac{2}{(2\pi)^8\alpha'} \int d^2qd^2kd\kappa re^{-iqy+ikx+ir(q+\kappa)} \frac{1}{m+ak^2+bq^2+c\kappa^2}. \]
(112)

where \( a = c = -1, b = 1 \) and \( m = \mu/\alpha' \) We further present
\[ \frac{1}{m+ak^2+bq^2+c\kappa^2} = \int_0^\infty d\xi e^{-\xi(m+ak^2+bq^2+c\kappa^2)} \]
(113)

and do the Gaussian integrals in \( k, q \) and \( \kappa \) assuming that they exist, that is for positive \( a, b \) and \( c \). Transition to the desired values of \( a, b \) and \( c \) will be achieved by analytic continuation. We get
\[ V_{11}^{(2)}(y|x) = -\frac{2}{(2\pi)^8\alpha'} \frac{\pi^3}{abc} \int_0^\infty \frac{d\xi}{\xi^3} e^{-m\xi} \int d^2re^{-(x+r)^2/4\alpha\xi-(y-r)^2/4\beta\xi-y^2/4c\xi}. \]
(114)

The exponent in the integrand in \( r \) has the form
\[-(\beta r^2-2sr+x^2/a+y^2/b)/4\xi, \]

where
\[ \beta = 1/a + 1/b + 1/c, \quad s = (-x/a+y/b), \]

so that after the integration over \( r \) we find
\[ V_{11}^{(2)}(y|x) = \frac{1}{32\pi^4\alpha'abc\beta} \int_0^\infty \frac{d\xi}{\xi^2} e^{-m\xi-t/\xi}, \]  
(115)
where
\[ t = \frac{1}{4} \left( -\frac{s^2}{\beta} + \frac{x^2}{a} + \frac{y^2}{b} \right). \] (116)

Integration over \( \xi \) gives
\[ V_{11}^{(2)}(y|x) = -\frac{1}{8\pi^4\alpha'} \frac{1}{\alpha' z_1} \frac{m}{c} K_1(z_1) \] (117)
where \( z_1 = 2\sqrt{mt} \) Inserting the desired values \( a = -1, b = 1 \) and \( c = -1 \) we find \( abc\beta = -1 \) so that
\[ z_1 = \sqrt{\frac{2\mu y(y+x)}{\alpha'}} \] (118)
and thus
\[ V_{11}^{(2)}(y|x) = \frac{1}{16\pi^4 \alpha'^2} \mu \frac{1}{\alpha'} K_1(z_1) \] (119)

If \( z_1 \) is real then the potential is falls exponentially. If \( z_1 \) is pure imaginary, then putting \( z = -i|z| \) and taking the real part according to the principal value prescription the potential is expressed via the oscillating Neumann function:
\[ V_{11}^{(2)}(y|x) = \frac{1}{16\pi^3 \alpha'^2 \alpha' |z_1|} N_1(|z_1|) \] (120)

The second part of the potential \( V_1 \) is obtained as
\[ V_{12}^{(2)}(y|x) = V_{11}^{(2)}(-x|y) = \frac{1}{8\pi^4 \alpha'^2} \mu \frac{1}{z_2} K_1(z_2), \] (121)
where
\[ z_2 = \sqrt{\frac{2\mu x(x+y)}{\alpha'}}. \] (122)

The degenerate part of the potential is simpler. We have
\[ V_{2}^{(2)}(y|x) = \frac{1}{8\pi^2} \int \frac{d^2 q d^2 k}{(2\pi)^4} e^{-iqy + ikx} \frac{1}{\mu - 2\alpha'k^2 + i\nu} + (x \leftrightarrow -y). \] (123)

We calculate the first part by taking \( \mu = -2\alpha'\nu \) and assuming \( \nu > 0 \) to subsequently continue to negative \( \nu \). So
\[ V_{21}^{(2)}(y|x) = -\frac{1}{4\alpha'(2\pi)^2} \delta^2(y) \int d^2 k e^{ikx} \frac{1}{k^2 + \nu}. \] (124)

The integral in \( k \) is trivially done to give
\[ V_{21}^{(2)}(y|x) = -\frac{1}{4\alpha'(2\pi)^2} \delta^2(y) K_0(\sqrt{\nu x^2}). \] (125)

Continuing to negative \( \nu \) and putting \( \sqrt{\nu x^2} = -i\sqrt{|\nu|x^2} \) we finally find
\[ V_{21}^{(2)}(y|x) = \frac{1}{16\alpha'(2\pi)^2} \delta^2(y) N_0 \left( \sqrt{\frac{\mu y^2}{2\alpha'}} \right). \] (126)

Taking into account the requirement of hermiticity, the second part of the degenerate potential will be given by
\[ V_{22}^{(2)}(y|x) = \frac{1}{16\alpha'(2\pi)^2} \delta^2(x) N_0 \left( \sqrt{\frac{\mu y^2}{2\alpha'}} \right). \] (127)
C  Lowest order scattering matrix for the pomeron interaction

As in Section 4 we restrict ourself to the forward case \(q_1 + q_2 = k_1 + k_2\) where the full potential \(V(q, k)\) is given by (53), so that \(V_1(q|k)\) is given by the first two terms in (53) and

\[
v(k) = \frac{1}{8\pi^2} \frac{1}{\mu - 2\alpha' k^2}.
\]

(128)

Obviously in our case \(I_0\) is divergent. We assume that it is regularized in some manner (say restricting integration by \(k < \Lambda\)). As \(\Lambda \to \infty\) \(I_0 \to \infty\) logarithmically. This leads us to the expressions (86) and (87) for \(T_1\) and \(T\) respectively valid up to order \(\lambda^2\). To calculate these scattering matrices all we need is to find \(\chi_2(q)\) and \(I_2\), keeping in mind the reality of the potential and the prescription given in Eq. (60).

Calculation of \(I_2\) is of course trivial. We find

\[
I_2(l) = -\frac{\lambda^4}{512\pi^3 \alpha'^3 l^2 - m^2} \left[ \frac{1}{l^2 - m^2} \left( \ln \frac{m^2}{l^2} - i\pi \right) + \frac{1}{m^2} \right],
\]

(129)

where we use a convenient notation

\[
m^2 = \frac{\mu}{2\alpha'}.
\]

(130)

Calculation of \(\chi_2(q|l)\) is a bit more complicated. It consists of two terms coming from the two terms in the potential \(V_1\). The first term can be written as

\[
\chi_2^{(1)}(q|l) = \frac{\lambda^4}{128\pi^4 \alpha'^3} \int \frac{d^2k}{[(k + q/2)^2 - p^2](k^2 - m^2)(k^2 - l^2)}
\]

where

\[
p^2 = m^2 + \frac{1}{4}q^2
\]

(131)

and

\[
A(l^2) = \int \frac{d^2k}{[(k + q/2)^2 - p^2](k^2 - l^2)}.
\]

(132)

This latter integral is conveniently calculated using the Feynman parametrization to finally give

\[
\text{Re} A(l^2) = \frac{\pi}{\sqrt{\Delta}} \ln \frac{(l^2 - m^2 - q^2/2 - \sqrt{\Delta})(l^2 - m^2 + \sqrt{\Delta})}{(l^2 - m^2 - q^2/2 + \sqrt{\Delta})(l^2 - m^2 - \sqrt{\Delta})},
\]

(133)

where

\[
\Delta = (l^2 - m^2)^2 - q^2 l^2
\]

(134)

and in (133) it is assumed that \(\Delta > 0\). For \(\Delta < 0\) we find

\[
\text{Re} A(l^2) = \frac{2\pi}{\sqrt{-\Delta}} \left[ \arctg \frac{l^2 - m^2 - q^2/2}{\sqrt{-\Delta}} - \arctg \frac{l^2 - m^2}{\sqrt{-\Delta}} \right].
\]

(135)

The imaginary part is

\[
\text{Im} A(l^2) = -\frac{\pi^2}{\sqrt{\Delta}} \theta(\Delta).
\]

(136)

The expression for \(A(m^2)\) is simpler due to cancellations between \(p^2\) and \(m^2\). It is real:

\[
A(m^2) = -\frac{2\pi}{qm} \arctg \frac{q}{2m}.
\]

(137)
The second term in $\chi$ can be written in a manner similar to (131):

$$\chi_2^{(2)}(q|l|) = \frac{\lambda^4}{128\pi^4\alpha'^3} \int \frac{d^2k}{(qk + q^2 - m^2)(k^2 - l^2)^2} = \frac{\lambda^4}{128\pi^4\alpha'^3} \frac{1}{l^2 - m^2} \left( B(l^2) - B(m^2) \right),$$

(139)

where now

$$B(l^2) = \int \frac{d^2k}{(qk + q^2 - m^2)(k^2 - l^2)}. $$

(140)

Integration over the azimuthal angle gives

$$B(l^2) = 2\pi \int_0^{k_m} \frac{kdk}{(k^2 - l^2)\sqrt{\Delta_1}},$$

(141)

where

$$\Delta_1 = (q^2 - m^2)^2 - q^2k^2$$

and $k_m$ is defined by the condition that $\Delta_1 > 0$. Here we have used the principal value prescription which tells that at $\Delta < 0$ the real part of the azimuthal integral is zero. Subsequent integration over $k$ gives

$$B(l^2) = \text{Re} \frac{\pi}{\sqrt{\Delta_2}} \ln \frac{\sqrt{\Delta_2} + |q^2 - m^2|}{\sqrt{\Delta_2} - |q^2 - m^2|} - \frac{\pi^2}{\sqrt{\Delta_2}} \theta(\Delta_2),$$

(143)

where

$$\Delta_2 = (q^2 - m^2)^2 - q^2l^2$$

and $\Delta_2 > 0$. For $\Delta_2 < 0$ $B(l^2)$ is real and given by

$$B(l^2) = -\frac{2\pi}{\sqrt{-\Delta_2}} \text{arctg} \frac{|q^2 - m^2|}{\sqrt{-\Delta_2}}.$$

(145)

Calculation of $B(m^2)$ obviously gives the same expressions (143) and (145) in which $l^2$ is substituted by $m^2$. This finishes calculation of $\chi_2(q|l|)$.

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