Convergence rates for the
three state contact process

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Abstract

The basic contact process with infection parameter $\mu$ altered so that infections of never infected sites occur at rate proportional to $\lambda$ instead is considered. It is known that in dimension one the epidemic started from one infected cannot survive when $\mu$ is less than the contact process' critical value, while survival is possible when $\mu$ is greater than that value. In the former case the span of the epidemic is shown to decay exponentially in space and in time. In the latter case and for $\lambda$ less than $\mu$, the ratio of the endmost infected site’s velocity to that of the contact process is shown to be no greater than $\lambda/\mu$.

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1 Introduction and main results

The three state contact process is a continuous-time Markov process $\zeta_t$ on the space of configurations $\{-1, 0, 1\}^\mathbb{Z}$ with transition rates corresponding to the following local prescription. Flips of $\zeta_t(x)$, the state of site $x \in \mathbb{Z}$ at time $t$, occur according to the rules: $1 \to 0$ at rate 1, $-1 \to 1$ at rate $\lambda n(x)$, and $0 \to 1$ at rate $\mu n(x)$, where $n(x)$ takes values 0, 1, or 2, with regard to the number of $y = x-1, x+1$ such

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that $ζ_t(y) = 1$, and the parameters $λ, μ$ are positive and finite. The epidemiological interpretation of the process derives from considering sites in state 1 as infected, sites in state 0 as susceptible and previously infected, and sites in state $-1$ as susceptible and never infected. The process falls into the subject of interacting particles, general information about which may be found in [3, 7].

Let $ζ_t^{η_0}$ denote the process with initial configuration $η_0$ such that the origin is infected and all other sites are susceptible and never infected. The process is said to survive if $P(ζ_t^{η_0}$ survives $) > 0$, where $\{ζ_t^{η_0}$ survives $\}$ is a shorthand for $\{∀ t ≥ 0, ζ_t^{η_0}(x) = 1$ for some $x\}$, while otherwise it is said to die out. Supposing the process survives and letting $r_t$ denote the rightmost infected site in $ζ_t^{η_0}$, the limit of $t^{-1}r_t$ as $t → ∞$ on $\{ζ_t^{η_0}$ survives $\}$ is referred to as the asymptotic velocity of the rightmost infected.

The basic one-dimensional contact process, which is the case $λ = μ$ here, and the forest fire model, which is the case $μ = 0$ here, are extensively studied in the literature, see [2, 7] for a review. A fundamental fact about the former needed to state our results is the existence of a positive and finite value $μ_c$ at which the following dichotomy occurs. The contact process dies out for values of the parameter less than the so-called critical value $μ_c$, and survives for values greater that $μ_c$.

For any $μ$ such that $μ < μ_c$, as shown in [4], the process dies out. Taking a different approach here permits showing that the following stronger statement holds.

**Theorem 1.1.** For all $λ$ and $μ$ such that $μ < μ_c$ there exists $δ < 1$ such that $P(∃ t$ s.t., $ζ_t^{η_0}(n) = 1$ or $ζ_t^{η_0}(-n) = 1) ≤ δ^n$, for all $n ≥ 1$; further, there exist $C$ and $γ > 0$ such that $P(∃ x$ s.t., $ζ_t^{η_0}(x) = 1) ≤ Ce^{-γt}$, for all $t ≥ 0$.

Some remarks about the proof of Theorem 1.1 are as follows. That of the first part involves showing that the probability that the span of infected sites of a subcritical contact process never expands is bounded away from zero uniformly over finite initial configurations. Ad-hoc arguments based on the extension of this for the three state contact process are used in the proof of the second part for circumventing difficulties stemming from the lack of monotonicity of the process.

Turning to the case that $μ > μ_c$ and $μ ≥ λ$, it is shown in [4] that the process with parameters as such survives. (In fact, the analogous result for the $d$-dimensional lattice process for a notion stronger than that of survival is established in Theorem 3 there). Further in this case, as shown in [10], the asymptotic velocity of the
rightmost infected exists and is almost surely a positive constant. We prove the following comparison result.

**Theorem 1.2.** Let $\lambda$ and $\mu$ be such that $\mu > \mu_c$ and $\mu > \lambda$. Let $\alpha$ be the asymptotic velocity of the rightmost infected of the process with parameters $(\lambda, \mu)$, and $\beta$ be that of the contact process with parameter $\mu$. Then, for any such $\lambda$ and $\mu$, $\alpha \leq (\lambda/\mu)\beta$.

Under the assumptions of the last theorem above, known upper bounds of $\beta$ in terms of $\mu$ (see p.289 in [7]) yield corresponding upper bounds of $\alpha$ in terms of $\lambda$ and $\mu$ as corollaries of this result. The technique of proof of this theorem relies on comparison of the growth of the rightmost infected sites of a sequence of contact processes defined iteratively on the trajectory of the rightmost infected site of the process.

In the following preparatory section the graphical representation is explained and some known results we use are stated. The remainder of the paper is then devoted to proofs; that of Theorem 1.1 is in Section 3, while that of Theorem 1.2 is given in Section 4.

## 2 Preliminaries

Graphical representations, introduced by Harris [6], are an important tool in the study of particle systems which aids visualizing their imbeding in space-time. Here it is intended for constructing three state contact processes with parameters $(\lambda, \mu)$ and contact processes with parameter $\mu$, started from different configurations at different times, on the same probability space. Abundant use of this representation will be made in proofs below.

Suppose that $\mu > \lambda$ and, for all integer $x$ and $y = x - 1, x + 1$, let $\{T_n^{(x,y)}, n \geq 1\}$ and $\{U_n^{(x,y)}, n \geq 1\}$ be the event times of Poisson processes at rates $\lambda$ and $\mu - \lambda$ respectively. (The case that $\lambda > \mu$ is similar by considering Poisson processes at rates $\mu$ and $\lambda - \mu$ instead). Let also $\{S_n^z, n \geq 1\}$ be the event times of a Poisson process at rate 1. All Poisson processes introduced are independent.

Consider $\mathbb{Z} \times [0, \infty)$, where $\times$ denotes Cartesian product, thought of as giving a time line to each site in $\mathbb{Z}$. The graphical representation for parameters $(\lambda, \mu)$ and $\xi_{t}^{[n,s]}$, $t \geq s$, the three state contact process started from a configuration $\eta$ at time
\( s \geq 0 \), for a given realization of the before-mentioned ensemble of Poisson processes are defined as follows. At all times \( T_n^{(x,y)} = t \) place a directed \( \lambda \)-arrow from \( x \times t \) to \( y \times t \), and, for \( t \geq s \), if \( \zeta_t^{[n,s]}(x) = 1 \) and \( \zeta_t^{[n,s]}(y) = 0 \) or \( \zeta_t^{[n,s]}(y) = -1 \) then set \( \zeta_t^{[n,s]}(y) = 1 \). (Here \( \zeta_{t-\epsilon}(x) \) denotes the limit of \( \zeta_{t-\epsilon}(x) \) as \( \epsilon \to 0 \). At all times \( U_n^{(x,y)} = t \) place a directed \( (\mu - \lambda) \)-arrow from \( x \times t \) to \( y \times t \), and, for \( t \geq s \), if \( \zeta_t^{[n,s]}(x) = 1 \) and \( \zeta_t^{[n,s]}(y) = 0 \) then set \( \zeta_t^{[n,s]}(y) = 1 \). Finally, at all times \( S_n^x = t \) place a recovery mark at \( x \times t \), and, for \( t \geq s \), if \( \zeta_t^{[n,s]}(x) = 1 \) then set \( \zeta_t^{[n,s]}(x) = 0 \). Further, \( \xi_t^{A \times s}, t \geq s \), the (set valued) contact process parameter \( \mu \) started from \( A \) at time \( s \geq 0 \), is defined via paths of the graphical representation, which are defined as follows. The existence of a connected oriented path from \( x \times s \) to \( y \times t \), \( t \geq s \), that moves along arrows (of either type) in the direction of the arrow and along vertical segments of time-axes without passing through a recovery mark is denoted as \( x \times s \rightarrow y \times t \), while, that \( x \times s \rightarrow y \times t \) for some \( x \in A \) and \( y \in B \), is denoted as \( A \times s \rightarrow B \times t \). It is immediate that letting \( \xi_t^{A \times s} = \{ x : A \times s \rightarrow x \times t \} \) gives the correct transition rates (where the equivalence with the configuration valued contact process can easily be seen by noting that the two types of susceptibility merge when \( \lambda = \mu \) and regarding sites of \( \xi_t^{A \times s} \) as infected and others as susceptible). To simplify notation we will write \( \zeta_t^n \) for \( \zeta_t^{[n,0]} \) and \( \xi_t^A \) for \( \xi_t^{A \times 0} \) and, further, for an integer \( I \), we will write \( I \times s \) instead of \( \{I\} \times s \), and \( \xi_t^A \cap I \neq \emptyset \) instead of \( \xi_t^A \cap \{I\} \neq \emptyset \).

A property known as monotonicity in the starting set of the contact process is a well known immediate consequence of the definition by the graphical representation. Here, monotonicity will simply refer to the following form of this property that facilitates its applications below. If a path constrained on \( D \) from \( A \times s \) to \( B \times t \) exists, then a path constrained on \( D' \) from \( A' \times s \) to \( B \times t \) exists for any \( D' \supseteq D \) and \( A' \supseteq A \), where a path is said to be constrained on a subset of the integers if it includes vertical segments of time axes of sites of that subset only. The other basic property of the contact process we use is known as self duality. To state it, let \( (\xi_t^A) \) and \( (\xi_t^B) \) be two contact processes with the same parameter started from \( A \) and \( B \) respectively, then,

\[
\mathbf{P}(\xi_t^A \cap B \neq \emptyset) = \mathbf{P}(\xi_t^B \cap A \neq \emptyset),
\]

for all \( t \geq 0 \). This equality can be seen by considering paths of the graphical representation that move along time axes in decreasing time direction and along
arrows in direction opposite to that of the arrow, and noting that the law of these paths is the same as that of the paths going forward in time defined above. See [3] and [7, 8] for more information on duality.

A miscellany of known results that are used in the proofs is collected together in the remainder of this section. First, an observation regarding monotonicity of the three state contact process which can be found within the last section in [9] is presented. Additional information regarding this property along with a different proof of this one can be found in the last chapter of [11].

Proposition 2.1. Endow the space of configurations with the natural partial ordering, ζ ≤ ζ' if and only if ζ(x) ≤ ζ'(x) for all x. Consider the graphical representation for (λ, μ) such that μ ≥ λ. If η and η' are such that η ≤ η', then ζ^n ≤ ζ'^n, for all t.

Two well known results for the contact process are presented next; for proofs see [2, 7]. To state them, let ξ^A_t denote the contact process parameter μ started from A.

Lemma 2.2. Let R^A_t = sup ξ^A_t. For any infinite B such that B ⊆ (−∞, 0], E(R^A_t ∪ {1} − R^B_t) ≥ 1, for all t ≥ 0.

For the final statement, recall that μ_c denotes the critical value of the contact process and that |B| denotes the cardinality of a set B.

Theorem 2.3. For all μ < μ_c there exists ψ > 0 independent of A such that P(ξ^A_t ≠ ∅) ≤ |A|e^−ψt, for all t ≥ 0.

3 Proof of Theorem 1.1

The theorem is obtained as a compound of the two separate propositions in this section.

Lemma 3.1. Let ξ^A_t be the contact process parameter μ on {min A, ..., max A} started from A, |A| < ∞. For all μ < μ_c there exist C, γ > 0 independent of A such that

\[ P \left( \exists s ≥ t \text{ s.t.}, \xi^A_s \cap \min A ≠ ∅ \text{ or } \xi^A_s \cap \max A ≠ ∅ \right) ≤ Ce^{−γt}, \text{ for all } t ≥ 0. \]
Proof. By monotonicity and translation invariance it is sufficient to prove that there exist \( C, \gamma > 0 \) independent of \( N \geq 0 \) such that
\[
P \left( \exists s \geq t \text{ s.t., } \xi_s^{[0,N]} \cap 0 \neq \emptyset \text{ or } \xi_s^{[0,N]} \cap N \neq \emptyset \right) \leq Ce^{-\gamma t}, \quad \text{for all } t \geq 0. \tag{3.1}
\]
Define \( E_{N,t} = \{ \xi_t^{[0,N]} \cap N \neq \emptyset \text{ or } \xi_t^{[0,N]} \cap 0 \neq \emptyset \}, \ t \geq 0. \) We first show that there exists \( \psi > 0 \) such that, for any \( N, \)
\[
P(E_{N,t}) \leq 2e^{-\psi t}, \quad \text{for all } t \geq 0. \tag{3.2}
\]
To this end, we have that there exists a \( \psi > 0 \) such that, for any \( N, \)
\[
P(\xi_t^{[0,N]} \cap N \neq \emptyset) = P(\xi_t^0 \cap [-N,0] \neq \emptyset) \leq P(\xi_t^0 \cap \mathbb{Z} \neq \emptyset) \leq e^{-\psi t} \tag{3.3}
\]
t \geq 0, where the equality comes from duality, equation (2.1), and translation invariance, while the two inequalities come from monotonicity and Theorem 2.3 respectively. Thus, since \( \xi_t^{[0,N]} \) stochastically dominates \( \xi_t^{[0,N]} \) by monotonicity, (3.2) follows from (3.3) and translation invariance.

For every integer \( k \geq 1 \) define the event \( D_{N,k} \) to be such that \( \omega \in D_{N,k} \) if and only if \( \omega \in E_{N,s} \) for some \( s \in (k-1,k] \). Because the probability of no recovery mark on the time axis of \( N \) or 0 after the first time \( s \in (k-1,k] \) such that \( \omega \in E_{N,s} \) and before time \( k \) is at least \( e^{-1} \), the Markov property for \( \xi_t^{[0,N]} \) gives that, for all \( k \geq 1, \)
\[
e^{-1}P(D_{N,k}) \leq P(E_{N,k}). \tag{3.4}
\]
Considering the event \( \bigcup_{t \geq 0} D_{N,t+[t]} \), where \( [ \cdot ] \) denotes the floor function, Boole’s inequality gives that
\[
P \left( \exists s \geq t \text{ s.t., } \xi_s^{[0,N]} \cap 0 \neq \emptyset \text{ or } \xi_s^{[0,N]} \cap N \neq \emptyset \right) \leq \sum_{t \geq 0} P(D_{N,t+[t]})
\]
t \geq 0. The proof is thus completed since by (3.4) and then (3.2) the last display implies (3.1).

The preceding lemma is used in the proof of the next one as well as in that of Lemma 3.4 below.

Lemma 3.2. Let \( \hat{\xi}_t^A \) be the contact process parameter \( \mu \) on \( \{ \min A - 1, \ldots, \max A + 1 \} \) started from \( A, |A| < \infty \). For all \( \mu < \mu_c \) there exists \( \epsilon > 0 \) independent of \( A \) such that \( P(\forall t \geq 0, \hat{\xi}_t^A \subseteq [\min A, \max A]) \geq \epsilon. \)
Proof. By monotonicity and translation invariance it is sufficient to show that there exists $\epsilon > 0$ independent of $N$ such that

$$\mathbb{P}(\forall t \geq 0, \xi_t^{[0,N]} \subseteq [0,N]) \geq \epsilon. \tag{3.5}$$

Define $\tilde{E}_{N,t} = \{\tilde{\xi}_t^{[0,N]} \cap N + 1 \neq \emptyset \text{ or } \tilde{\xi}_t^{[0,N]} \cap -1 \neq \emptyset\}$. We have that there exists $\psi > 0$ such that, for any $N \geq 0$,

$$\mathbb{P}(\tilde{E}_{N,t}) \leq 2e^{-\psi t}, \quad \text{for all } t \geq 0, \tag{3.6}$$

where (3.6) follows from (3.2) by noting that $\tilde{\xi}_t^{[0,N]}$ is stochastically smaller than $\hat{\xi}_t^{[-1,N+1]}$ by monotonicity, and thus $\tilde{E}_{N,t}$ is bounded above in distribution by $E_{N+2,t}$ from translation invariance. (Alternatively, (3.6) can be proved by arguments akin to those used for showing (3.2) in the proof of the previous statement).

Define $\tilde{D}_{N,k} = \{\omega : \omega \in \tilde{E}_{N,s} \text{ for some } s \in (k-1,k]\}$, for integer $k \geq 1$. Clearly, $\bigcap_{k \geq 1} \tilde{D}_{N,k}$ is equal to $\{\forall t \geq 0, \tilde{\xi}_t^{[0,N]} \subseteq [0,N]\}$ and $\mathbb{P}(\bigcap_{k \geq 1} \tilde{D}_{N,k}) = \lim_{K \to \infty} \mathbb{P}(\bigcap_{k \geq 1} \tilde{D}_{K,k})$. Further, $\{\tilde{D}_{N,k}\}_{k \geq 1}$ are monotone decreasing and hence positively correlated. Thus, the Harris-FKG inequality (see [2, 5]) gives that, for any $N \geq 0$,

$$\mathbb{P}(\forall t \geq 0, \xi_t^{[0,N]} \subseteq [0,N]) \geq \prod_{k \geq 1} \mathbb{P}(\tilde{D}_{N,k}).$$

However, from (3.6) and elementary properties of infinite products we have that there exists $\epsilon > 0$ independent of $N$ such that $\prod_{k \geq 1} (1 - e\mathbb{P}(\tilde{E}_{N,k})) > \epsilon$. Since also we have that $\mathbb{P}(\tilde{D}_{N,k}) \leq e\mathbb{P}(\tilde{E}_{N,k})$, shown similarly to (3.4), the proof is complete from (3.5) which thus follows from the last display. \hfill \Box

We return to consideration of the three state contact process.

**Definition 1.** Let $\mathcal{I}(\zeta)$ denote the set of infected sites in a configuration $\zeta$, that is, $\mathcal{I}(\zeta) = \{y \in \mathbb{Z} : \zeta(y) = 1\}$.

Let $\eta_N$ be such that $\mathcal{I}(\eta_N) = \{-N, \ldots, N\}$ and $\eta_N(x) = -1$ for all $x \notin \mathcal{I}(\eta_N)$, $N \geq 0$. For $N = 0$ the next result reduces to the first part of Theorem 1.1.

**Proposition 3.3.** For all $\lambda$ and $\mu$ such that $\mu < \mu_c$ there exists $\epsilon > 0$ independent of $N$ such that

$$\mathbb{P}(\exists t \text{ s.t., } \xi_t^{\eta_N}(N+n) = 1 \text{ or } \xi_t^{\eta_N}(-N-n) = 1) \leq (1 - \epsilon)^n, \quad \text{for all } n \geq 1.$$
Proof. Let \( I_t^{N} := \mathcal{I}(\zeta_t^{\eta,N}) \). We first show that there exists \( \epsilon > 0 \) such that, for any \( N \geq 0 \),
\[
P(\forall t \geq 0, \ I_t^{N} \subseteq [-N, N]) \geq \epsilon. \tag{3.7}
\]
Define the events \( B_N = \{ \forall s \in (0,1), F_s^{N} \subseteq [-N, N] \} \cap \{ I_1^{N} \subseteq [-N + 1, N - 1] \} \) and \( F_N = \{ \forall t \geq 1, I_t^{N} \subseteq [-N + 1, N - 1] \} \). Since \( \{ \forall t \geq 0, I_t^{N} \subseteq [-N, N] \} \supseteq F_N \cap B_N \) and, by Lemma 3.2 and the Markov property at time 1, there exists \( \epsilon > 0 \) independent of \( N \) such that \( P(F_N|B_N) \geq \epsilon \), it is sufficient to show that \( P(B_N) \) is bounded away from zero uniformly in \( N \). For this consider the event \( B_N' \) that:
- a) no arrow exists from \( N \times s \) to \( N + 1 \times s \) and from \( -N \times s \) to \( -N - 1 \times s \) for all times \( s \in (0,1) \),
- b) a recovery mark exists within \( N \times (0,1) \) and \( -N \times (0,1) \),
- and, c) no arrow exists from \( -N - 1 \times s \) to \( N \times s \) and over \( -N + 1 \times s \) to \( -N \times s \), for all times \( s \in (0,1) \).
Note that b) implies that there is a \( t \in (0,1) \) such that \( I_t^{N} \subseteq [-N + 1, N - 1] \) and c) assures that this holds for \( t = 1 \), and hence by a) we have that \( B_N \supseteq B_N' \). This proves (3.7) because \( B_N' \) has strictly positive probability which by translation invariance is independent of \( N \).

From (3.7) and monotonicity (of the contact process) we have that indeed for any \( \eta \) such that \( \eta(x) \neq -1 \), \( \forall x \in [\min \mathcal{I}(\eta), \max \mathcal{I}(\eta)] \), \( P(\forall t \geq 0, \ \mathcal{I}(\zeta_t^{\eta}) \subseteq \mathcal{I}(\eta)) \geq \epsilon \), and the proof is completed by repeated applications of the Strong Markov Property.

In the proof of Proposition 3.5 below we need to use the preceding proposition as well as the next corollary. To state the latter, let \( H \) be the collection of configurations \( \eta \) such that \( |\mathcal{I}(\eta)| < \infty \) and \( \eta(x) \neq -1 \), \( \forall x \in [\min \mathcal{I}(\eta), \max \mathcal{I}(\eta)] \), and further define the stopping time \( T^\eta := \inf\{ t \geq 0 : \mathcal{I}(\zeta_t^{\eta}) \subseteq [\min \mathcal{I}(\eta), \max \mathcal{I}(\eta)] \} \), \( \eta \in H \).

**Lemma 3.4.** For all \( \lambda \) and \( \mu \) such that \( \mu < \mu_c \) there exist \( C \) and \( \theta > 0 \) independent of \( \eta \in H \) such that \( E(e^{\theta T^\eta 1_{\{T^\eta < \infty\}}}) \leq C \).

**Proof.** This follows from Lemma 3.1 by the integral representation of expectation since, for any \( \eta \in H \), \( \{ t \leq T^\eta < \infty \} \) is bounded above in distribution by \( \{ \exists s \geq t \ s.t., \ \xi_s^{\mathcal{I}(\eta)} \cap \min \mathcal{I}(\eta) \neq \emptyset \text{ or } \xi_s^{\mathcal{I}(\eta)} \cap \max \mathcal{I}(\eta) \neq \emptyset \} \).

Consider \( \zeta_t^{\eta_0} \) with parameters \( (\lambda, \mu) \) and let \( I_t = \mathcal{I}(\zeta_t^{\eta_0}) \). The final statement of this section is the second part of Theorem 1.1.
Proposition 3.5. For all \( \lambda \) and \( \mu \) such that \( \mu < \mu_c \) there exist \( C, \gamma > 0 \) such that 
\[ P(I_t \neq \emptyset) \leq Ce^{-\gamma t}, \text{ for all } t \geq 0. \]

Proof. Define the stopping times \( \tau_k = \inf\{t : \max I_t - \min I_t = k\} \), define also \( K = \inf\{k : \tau_k = \infty\} \) and \( \sigma_K = \inf\{s \geq 0 : I_{s+\tau_{K-1}} = \emptyset\} \). Clearly \( \{I_t \neq \emptyset\} \) equals \( \{\tau_{K-1} + \sigma_K \geq t\} \), thus, showing that \( \tau_{K-1} \) and \( \sigma_K \) are exponentially bounded implies the statement since the sum of two exponentially bounded random variables is itself exponentially bounded (a simple proof of this fact can be seen by using the integral representation of expectation, the Chernoff bound and then the Cauchy-Schwartz inequality). Towards this, because \( K \) is exponentially bounded by Proposition 3.3 and by set theory we have that, for all \( a > 0 \),
\[ P(\tau_{K-1} > t) \leq P(K > \lceil at \rceil) + P(\tau_{K-1} > t, K \leq \lceil at \rceil), \]
\( t \geq 0 \), it suffices to show that (i) there is \( a > 0 \) such that \( \tau_{K-1} \) is exponentially bounded on \( \{K \leq \lceil at \rceil\} \) and, by repeating the argument in the last display, that (ii) \( \sigma_K \) is exponentially bounded on \( \{K \leq \lceil t \rceil\} \).

Towards (i), let \( H \) and \( C, \theta > 0 \) be as in Lemma 3.4. By the Strong Markov Property and because \( \zeta_{\tau_{k-1}}^{\eta_0} \in H \), we have that
\[ E(e^{\theta \tau_{k-1} 1_{(\tau_{k-1} < \infty)}}) \leq E(e^{\theta \tau_{k-1} 1_{(\tau_{k-1} < \infty)}} e^{\theta (\tau_k - \tau_{k-1}) 1_{(\tau_k - \tau_{k-1}) < \infty}}) \]
\[ \leq CE(e^{\theta \tau_{k-1} 1_{(\tau_{k-1} < \infty)}}) \]

\( k \geq 1 \), which by iteration gives that \( E(e^{\theta \tau_{k-1} 1_{(\tau_{k-1} < \infty)}}) \leq C^k \). Using this and set theory gives that, for all \( a > 0 \),
\[ P(\tau_{K-1} > t, K \leq \lceil at \rceil) \leq \sum_{k=1}^{\lceil at \rceil} e^{-\theta t} E(e^{\theta \tau_{k-1} 1_{(\tau_{k-1} < \infty)}}) \]
\[ \leq \lceil at \rceil e^{-\theta t} C^{\lceil at \rceil}, \]
\( t \geq 0 \), and the claim follows from the last display by choosing \( a > 0 \) such that \( e^{-\theta C^{\lceil a \rceil}} < 1 \).

Towards (ii), let \( \xi_{l}^{[1,k]} \) denote the contact process parameter \( \mu \) on \( \{1, \ldots, k\} \) started from all sites infected, it then follows from Theorem 2.3 that the \( \sum_{k=1}^{\lceil t \rceil} P(\sigma_k > t, K = k) \) is exponentially bounded in \( t \), since \( \{\sigma_k 1_{(K=k)} \geq t\} \) is stochastically bounded above by \( \{\xi_{l}^{[1,k]} \neq \emptyset\} \).

\[ \square \]
Remark 2. It follows from Proposition 3.3 and bounded dominated convergence that $E|\zeta_{\eta}^n| \to 0$, as $t \to \infty$. Neither the technique of the proof of Theorem 6.1 in [5] nor that of Proposition 1.1 in [1] adapt to extend this conclusion to Proposition 3.5 due to lack of properties of $\zeta$, analogous to monotonicity and (sub)additivity of the contact process respectively.

4 Proof of Theorem 1.2

Let $\zeta^\eta_{\bar{\eta}}$ be the three state contact process with parameters $(\lambda, \mu)$ and initial configuration $\bar{\eta}$ such that $\bar{\eta}(x) = 1$ for all $x \leq 0$ and $\bar{\eta}(x) = -1$ for all $x \geq 1$. Let also $\bar{r}_t = \sup I(\zeta^\eta_{\bar{\eta}})$ and $\bar{x}_t = \sup_{s \leq t} \bar{r}_s$. In this section we concentrate on the study of $\bar{r}_t$, where the necessary connection between $\zeta^\eta_{\bar{\eta}}$ and $\zeta^\eta_0$ for establishing Theorem 1.2 is given by Corollary 4.2 below. The following lemma is required in the latter’s proof.

Lemma 4.1. If $\mu \geq \lambda$, then $\bar{x}_n/n \to a$ almost surely, where $a = \inf_{n \geq 0} \frac{E(\bar{x}_n)}{n}$ and $a \in [-\infty, \infty)$. If additionally $a > -\infty$, then $\bar{x}_n/n \to a$ in $L^1$.

Proof. Let $\eta_y$ denote the configuration such that $\eta_y(x) = 1$ for all $x \leq y$, and $\eta_y(x) = -1$ for all $x \geq y + 1$. For any times $s$ and $u$ such that $s \leq u$, define

$$\bar{x}_{s,u} = \max\{y : \zeta^{[\eta_y,s]}(y) = 1, \text{ for some } t \in [s, u]\} - \bar{x}_s,$$

where, note that, $\bar{x}_{0,u} = \bar{x}_u$. We aim to show that $\{\bar{x}_{m,n}, m \leq n\}$ satisfies the conditions of Theorem 2.6, Chapter VI in [7], known as the subadditive ergodic theorem. We have that

a) $\bar{x}_{0,s} + \bar{x}_{s,u} \geq \bar{x}_{0,u}$

since, by monotonicity in the initial configuration, Proposition 2.1 $\zeta^{[\eta_y,s]}(y) \geq \zeta^\eta_t$, for all $t \geq s$. We further have that $\bar{x}_{s,u}$ is equal in distribution to $\bar{x}_{0,u-s}$ and is independent of $\bar{x}_{0,s}$ by translation invariance and independence of Poisson processes at disjoint parts of the graphical representation respectively. Thus,

b) $\{\bar{x}_{(n-1)k,nk}, n \geq 1\}$ are i.i.d. for each $k \geq 1$,

and, furthermore,

c) $\{\bar{x}_{m,m+k}, k \geq 0\} = \{\bar{x}_{m+1,m+k+1}, k \geq 0\}$ in distribution, for each $m \geq 1$.  

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By ignoring recovery marks in the representation, \( \bar{x}_t \) is bounded above in distribution by the number of arrivals of a Poisson process at rate \( \lambda \) in \( (0, t] \), and thus, from standard properties of Poisson processes we also have that

\[
d) \quad \mathbb{E}(\max\{\bar{x}_{0,1}, 0\}) < \infty.
\]

The result’s statement follows from the conclusion of the before-mentioned theorem since the conditions under which it holds correspond to a)–d) above.

Let \( \lambda \) and \( \mu \) be such that \( \mu > \mu_c \) and \( \mu > \lambda \) and, further, let \( \alpha > 0 \) be the corresponding value of the asymptotic velocity of the rightmost infected. The next statement is obtained based on results in [10].

**Corollary 4.2.** \( \frac{\mathbb{E} \bar{r}_t}{t} \to \alpha. \)

**Proof.** It is deduced by use of the restart argument in Lemma 4.4 in [10] that

\[
\frac{\bar{r}_t}{t} \to \alpha, \text{ almost surely, } \tag{4.1}
\]

since both \( Y_N \) and \( T_{Y_N} \) in its statement are almost surely finite from Proposition 4.2 of the same paper. From Lemma 4.1 and because \( \bar{x}_n \geq \bar{r}_n \), the last display gives that \( \bar{x}_n / n \to a \) in \( L^1 \) for \( a > 0 \), so that by the direct part of the theorem in section 13.7 in [12] it follows that \( \bar{x}_n / n \) are uniformly integrable and thus, using that \( \bar{r}_n \leq \bar{x}_n \) again, \( \bar{r}_n / n \) also are. The latter along with (4.1) imply from the reverse part of the before-mentioned theorem in [12] that \( \frac{\bar{r}_n}{n} \to \alpha \) in \( L^1 \). The extension along real times then comes elementarily by using that \( \max_{t \in [n,n+1]} (\bar{r}_t - \bar{r}_n) \) and \( \max_{t \in [n,n+1]} (\bar{r}_{n+1} - \bar{r}_t) \) are bounded above in distribution by the number of arrivals of a Poisson process at rate \( \mu \) in \( (0, 1] \).

**proof of Theorem [L2]** Let \( \xi^0, t \geq 0 \), be such that \( \xi^0 = \{\ldots, -1, 0\} \) and let also \( R^0_t = \sup \xi^0_t \). We prove the following stronger statement

\[
\mathbb{E} \bar{r}_t \leq \frac{\lambda}{\mu} \mathbb{E} R^0_t, \tag{4.2}
\]

for all \( t \geq 0 \), which implies the result from Corollary 4.2.

The first step of the iterative definitions following is outlined with remarks for purposes of illustration. By coupling, \( \bar{r}_t = R^0_t \) for all \( t \) up until the first time \( s \) such that...
\( \bar{r}_s = \bar{x}_s \) and a \((\mu - \lambda)\)-arrow exists from \( \bar{r}_s \) to \( \bar{r}_s + 1 \). Observe that the rightmost infected of the contact process started at time \( s \) from \( I(\zeta_0^s) \) coincides with \( \bar{r}_t \) up until the first time \( u, u > s \), at which \( \bar{r}_u = \bar{x}_u \) and a \((\mu - \lambda)\)-arrow from \( \bar{r}_u \) to \( \bar{r}_u + 1 \) is present, and further observe that \( I(\zeta_0^s) \), the starting set of this contact process, equals \( \xi_0^s \cup \sup \xi_s^0 \).

Define iteratively the stopping times

\[
v_n = \inf \{ t \geq v_{n-1} : R_{t}^{n-1} = \bar{r}_t + 1 \}, \quad (4.3)
\]

where \( v_0 = 0 \) and \( n \geq 1 \); define further \( \xi_n^t := \xi_t^{(\zeta_0^s)^{x_{\nu_n}}} \), \( t \geq v_n \), and \( R_{t}^{n} = \sup \xi_n^t \).

Then,

\[
\bar{r}_t = R_{t}^{n-1}, \text{ for all } t \in [v_{n-1}, v_n), \quad (4.4)
\]

\[
\xi_{v_n}^{n-1} = \xi_{v_n}^{n} \cup \{ \bar{r}_{v_n} + 1 \}, \text{ for all } n \geq 1, \quad (4.5)
\]

which can be seen to hold from the first and second observation respectively in the outline above. Define also \( F_t = \sup \{ n : v_n \leq t \} \). We will show that

\[
\mathbb{E}(F_t) = \frac{\mu - \lambda}{\lambda} \mathbb{E}(\bar{x}_t) \quad (4.6)
\]

and, further, that

\[
\mathbb{E}(R^0_t - \bar{r}_t) \geq \mathbb{E}F_t \quad (4.7)
\]

\( t \geq 0 \). Note that, since \( \bar{x}_t \geq \bar{r}_t \), (4.6) gives that \( \mathbb{E}(F_t) \geq \frac{\mu - \lambda}{\lambda} \mathbb{E}(\bar{r}_t) \), which, combined with (4.7), implies (4.2). Thus, showing the two last displays above gives (4.2) from which the proof is complete.

Let \( \mathcal{F}_t \) denote the sigma algebra associated to the Poisson processes in the graphical representation up to time \( t \) and recall that \( 1_E \) denotes the indicator of event \( E \).

We first prove (4.7). From (4.3) we have that

\[
R^{0}_t - \bar{r}_t = \sum_{n=1}^{\infty} (R^{n-1}_t - R^{n}_t) 1_{\{F_t \geq n\}}.
\]

This and the monotone convergence theorem, which applies because \( R^{n-1}_t \geq R^{n}_t \) by monotonicity of the contact process, give that

\[
\mathbb{E}(R^{0}_t - \bar{r}_t) = \sum_{n=1}^{\infty} \mathbb{E} \left( (R^{n-1}_t - R^{n}_t) 1_{\{F_t \geq n\}} \right), \quad (4.8)
\]

\( t \geq 0 \). Further, Lemma 2.2 and (4.5) by use of the Strong Markov Property give that

\[
\mathbb{E} \left( (R^{n-1}_t - R^{n}_t) 1_{\{F_t \geq n\}} \right) \geq \mathbb{P}(F_t \geq n), \quad (4.9)
\]

and, further, that

\[
\mathbb{E}(R^{0}_t - \bar{r}_t) \geq \mathbb{E}F_t \quad (4.7)
\]
Let \( \tau \) be the stopping times \( A \) from \( \bar{E} \).

Towards (1.10) some additional definitions are necessary. Recall the setting of the graphical representation from Section 2. Let \( \tilde{T}_1 := T_{1}^{0,1}, \tilde{S}_1 := S_{1}^{0}, \tilde{U}_1 := U_{1}^{0,1} \) and also define the events \( A_1 = \{\min\{\tilde{T}_1, \tilde{S}_1, \tilde{U}_1\} = \tilde{U}_1\} \) and \( B_1 = \{\min\{\tilde{T}_1, \tilde{S}_1, \tilde{U}_1\} = \tilde{T}_1\} \). At time \( \tau_0 := 0 \) the first competition takes place in the sense that on \( A_1 \), \( \tilde{r}_\tilde{U}_1 = 0 \) and \( R_{\tilde{U}_1}^0 = 1 \) (and hence \( v_1 = \tilde{U}_1 \)); while on \( B_1 \), \( \tilde{r}_\tilde{T}_1 = \bar{x}_\tilde{T}_1 = 1 \). We repeat these inductively as follows. For all \( n \geq 1 \) consider

\[
\tau_n = \inf\{t \geq \min\{\tilde{T}_n, \tilde{S}_n, \tilde{U}_n\} : \tilde{r}_t = \bar{x}_t\},
\]

and let \( \tilde{T}_{n+1} = \inf\{T_k^{(\bar{r}_n, \bar{r}_n+1)} : T_k^{(\bar{r}_n, \bar{r}_n+1)} > \tau_n\} \), i.e. the first time a \( \lambda \)-arrow exists from \( \bar{r}_n \) to \( \bar{r}_n + 1 \) after \( \tau_n \), and \( \tilde{U}_{n+1} = \inf\{U_k^{(\bar{r}_n, \bar{r}_n+1)} : U_k^{(\bar{r}_n, \bar{r}_n+1)} > \tau_n\} \), i.e. the first such time a \( (\mu - \lambda) \)-arrow exists, and further \( \tilde{S}_{n+1} = \inf\{S_k^{\bar{r}_n} : S_k^{\bar{r}_n} > \tau_n\} \), i.e. the first time that a recovery mark exists on \( \tilde{r}_{\tau_n} \) after \( \tau_n \). Define also the events \( A_{n+1} := \{\tilde{U}_{n+1} < \min\{\tilde{T}_{n+1}, \tilde{S}_{n+1}\}\} \) and \( B_{n+1} := \{\tilde{T}_{n+1} < \min\{\tilde{U}_{n+1}, \tilde{S}_{n+1}\}\} \). The stopping times \( \tau_n \) can be thought of as the time that the \( n \) competition, in the sense explained above, takes place.

Letting \( N_t = \sup\{n : \tau_n < t\} \), we have that \( \bar{x}_t = \sum_{n=1}^{N_t} 1_{B_n} \) and also that \( F_t = \sum_{n=1}^{N_t} 1_{A_n} \), where the latter can be seen by noting that \( v_n \) can also be expressed as the first \( \tilde{U}_k \) after \( v_{n-1} \) such that \( \tilde{U}_k < \min\{\tilde{T}_k, \tilde{S}_k\} \). The two last equalities and assuming that \( \mathbb{E}(N_t) < \infty \) imply (1.6) as follows. Since conditional on \( \zeta_{\tau_n} \) the events \( A_{n+1} \) and \( B_{n+1} \) are independent of \( \{N_t \geq n+1\} = \{N_t \leq n\}^c \in \mathcal{F}_{\tau_n} \) from the Strong Markov Property, emulating the proof of Wald’s lemma and then using a basic result about competing Poisson processes gives that \( \mathbb{E}(\bar{x}_t) = \mathbb{E}(N_t) \frac{\lambda}{\mu + 1} \), and also that \( \mathbb{E}(F_t) = \mathbb{E}(N_t) \frac{\mu - \lambda}{\mu + 1} \), hence, (1.6) follows by combining these two last equalities.

It remains to show that \( \mathbb{E}(N_t) < \infty \). Ignoring recovery marks gives that \( R_t^0 \) is bounded above (in distribution) by \( \Lambda_{\mu}[0, t) \), the number of arrivals of a Poisson process at rate \( \mu \) in \( [0, t) \), and further that \( \bar{x}_t \) is bounded above by \( \Lambda_{\lambda}[0, t) \), while also \( D_t \), the total number of recovery marks on the trajectory of the rightmost infected site by time \( t \), equals \( \Lambda_{\lambda}[0, t) \). From these and noting that \( N_t \leq R_t^0 + \bar{x}_t + D_t \), the proof is complete by elementary Poisson processes results.

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