Index of Bipolar Surfaces to Otsuki Tori

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Abstract

For each rational number $p/q \in (1/2, \sqrt{2}/2)$ one can construct an $S^1$-equivariant minimal torus in $S^3$ called Otsuki torus and denoted by $O_{p/q}$. The Lawson’s bipolar surface construction applied to $O_{p/q}$ gives a minimal torus $\tilde{O}_{p/q}$ in $S^3$. In this paper we give upper and lower bounds on the Morse index and the nullity of these tori for $p/q$ close to $\sqrt{2}/2$. We also state a numerically assisted conjecture concerning the general case.

1 Introduction

In the present paper we study the index and the nullity of $S^1$-equivariant minimal tori in $S^3$ called bipolar surfaces to Otsuki tori. These surfaces are obtained as a result of a two-step construction. The first step is the construction of Otsuki tori, which are $S^1$-equivariant minimal tori in $S^3$. These tori were introduced by Otsuki in [Ots70] but later the original definition was significantly simplified by Penskoi [Pen13b] with the help of Hsiang-Lawson construction of equivariant minimal surfaces [HLJ71]. In particular, it turns out that there is a natural bijection between Otsuki tori and rational numbers from the interval $(1/2, \sqrt{2}/2)$. Following [Pen13b], we denote by $O_{p/q}$ the Otsuki tori corresponding to the number $p/q \in (1/2, \sqrt{2}/2)$.

The second step is the application of Lawson’s bipolar surface construction [LJ70, sec. 11] to $O_{p/q}$. Generally, starting from a minimal immersion $u: \Sigma \rightarrow S^3$, this construction gives a minimal immersion $\tilde{u}: \Sigma \rightarrow S^5$. However, for some immersions $u$ the image of $\tilde{u}$ is contained in an equatorial subsphere $S^1 \subset S^5$ and this is the case for Otsuki tori. Thus, we end up with a family of minimal surfaces in $S^4$, which are naturally called bipolar surfaces to Otsuki tori. These surfaces are extensively studied in [Kar14]. Following this paper, we denote by $\tilde{O}_{p/q}$ the bipolar surface to $O_{p/q}$, that is, the image of the corresponding bipolar immersion $\tilde{u}$.

All the surfaces $\tilde{O}_{p/q}$ are tori as well. However, if $q$ is even, then the corresponding bipolar immersion $\tilde{u}$ covers its own image twice, and this turns out to be very important in the sequel. In particular, it is convenient to introduce the surface $\tilde{O}_{p/q}$, which is the two-sheeted cover of $\tilde{O}_{p/q}$ if $q$ is even and coincides with $\tilde{O}_{p/q}$ otherwise. In practice one makes all the computations for $\tilde{O}_{p/q}$ at first and then descends on $\tilde{O}_{p/q}$ if $q$ is even.

Initially the study of the surfaces $O_{p/q}$ and $\tilde{O}_{p/q}$ was motivated by spectral geometry. Recall that by a celebrated result of Nadirashvili, El Soufi, and Ilias [Nad96, EI08], any metric $g$ induced by a minimal immersion of $\Sigma$ in a unit sphere is critical for the functional $\lambda_{\Sigma}(g)$, where $\lambda_k(g)$ is the $k$-th normalized eigenvalue of the Laplace-Beltrami operator on $\Sigma$ and $N_{\Sigma}(2)$ is the number of Laplace-Beltrami eigenvalues of $\Sigma$ less than 2. However, it takes some work to compute $N_{\tilde{O}_{p/q}(2)}$ and $N_{O_{p/q}(2)}$. These computations are actually the main purpose of the works [Pen13b] and [Kar14]. The main result of [Kar14] states that

$$N_{\tilde{O}_{p/q}(2)} = \begin{cases} 2q + 4p - 2, & q \text{ is odd;} \\ q + 2p - 2, & q \text{ is even.} \end{cases}$$

In fact, it is shown that $N_{\tilde{O}_{p/q}(2)} = 2q + 4p - 2$ and then analyzed which eigenfunctions descend on $\tilde{O}_{p/q}$ if $q$ is even. Critical metrics (also called extremal metrics) are studied in [Kar15, Pen15], see also reviews [Pen13a, Pen19].

Other important geometric quantities associated with a minimal surface are the (Morse) index and the nullity. The index of $O_{p/q}$ is computed in [MP23]. A general fact proved there implies that

$$\text{Ind}(O_{p/q}) = N_{\tilde{O}_{p/q}(2)} = 2q + 4p - 2.$$
The aim of the current paper is to obtain estimates on the index and the nullity of the surfaces $\tilde{O}_{p/q}$. Although this turns out to be a much more complicated problem, a rough upper bound can be obtained as follows. It follows from [EM08, Theorem 1.1] and [Kar21, Proposition 1.6] that

$$\text{Ind} \tilde{O}_{p/q} \leq 5N\tilde{O}_{p/q}(2) + 2. \quad (2)$$

Combining this with (1), we obtain

$$\text{Ind} \tilde{O}_{p/q} \leq \begin{cases} 10q + 20p - 8, & q \text{ is odd;} \\ 5q + 10p - 8, & q \text{ is even.} \end{cases}$$

Our main result is that if $p/q$ is sufficiently close to $\sqrt{2}/2$ (speaking informally, this means that $\tilde{O}_{p/q}$ is “close to the Clifford torus”), then one can improve the bound (2) and also obtain a lower bound.

**Theorem 1.1.** There exists $\varepsilon > 0$ such that if $\frac{\sqrt{2}}{2} - \frac{p}{q} < \varepsilon$, then the following inequalities hold

$$\begin{align*}
6q + 8p - 3 &\leq \text{Ind} \tilde{O}_{p/q} \leq 10q + 4p - 5, & q \text{ is odd;} \\
3q + 4p - 3 &\leq \text{Ind} \tilde{O}_{p/q} \leq 5q + 2p - 5, & q \text{ is even,}
\end{align*}$$

$$9 \leq \text{Nul} \tilde{O}_{p/q} \leq 13.$$ 

We expect that the computation of the exact value of $\text{Ind}(\tilde{O}_{p/q})$ (at least, for all $p/q \in (1/2, \sqrt{2}/2)$) is a very complicated problem. See Remark 3.22 for a detailed discussion of the difficulties encountered and a conjecture regarding the general case.

The plan of the proof of Theorem 1.1 is the following. Recall that the index can be defined as the number of negative eigenvalues of the Jacobi stability operator $L$. First we compute the operator $L$ on $\tilde{O}_{p/q}$ in appropriate local coordinates. Then, using separation of variables, we reduce the counting of negative eigenvalues of $L$ to the counting of negative eigenvalues of a family of matrix Sturm-Liouville operators $S_l$, depending on a non-negative integer parameter $l$. For $l = 0$ the matrix Sturm-Liouville problem reduces to a pair of scalar Sturm-Liouville problems. We then apply the technique based on the Sturm Oscillation Theorem to determine the number of negative eigenvalues. This is exactly the method used in [Pen13b, Kar14] for computing $N\tilde{O}_{p/q}(2)$ and $N\tilde{O}_{p/q}(2)$ and previously applied for Lawson tau-surfaces [Pen12].

Unfortunately, for $l > 0$ the matrix Sturm-Liouville problem does not split into scalar problems and we cannot apply Sturm Oscillation Theorem in this case. Instead we look at the limit case $p/q = \sqrt{2}/2$, which is a Clifford torus in equatorial $S^3 \subset S^5$. In this case all eigenvalues can be computed explicitly and we use this to get the desired bounds. A price to pay is that the estimates hold only for $p/q$ sufficiently close to $\sqrt{2}/2$. In particular, we are not able even to point out some specific $p/q$ for which our estimates hold. However we hope that the proposed method provides some intuition and can be easily adopted for numerical computations. For example, we are able to compute numerically the index of $\tilde{O}_{2/3}$ (see Conjecture 1).

The paper is organised as follows. In section 2.1 we fix the notation and recall the basic definitions. In section 2.2 we describe a parametrization of $\tilde{O}_{p/q}$ used throughout the paper. In section 3.1 we compute the Jacobi stability operator on $\tilde{O}_{p/q}$ together with some of its eigensections. In section 3.2 we separate variables and introduce a family of matrix Sturm-Liouville operators $S_l$. In section 3.3 we analyse the “easy” case $l = 0$ and in section 3.4 we analyse the “difficult” cases $l = 1, 2$ and prove Theorem 1.1. The paper is concluded with Remark 3.22 containing a short discussion of the result.

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2 Preliminaries

2.1 Notation and definitions

In this section we fix the notation and recall the definition of the Morse index and the nullity of a minimal surface.

Let $\Sigma$ be an oriented minimal surface in the unit 4-dimensional sphere $S^4$ with the standard round metric $\langle \cdot, \cdot \rangle$. Throughout this paper $T\Sigma$ and $N\Sigma$ denote the tangent bundle of $\Sigma$ and the normal bundle to $\Sigma$ in $\mathbb{R}^5$ respectively. For any vector $v \in T\Sigma$ let $v^\top$ and $v^\perp$ denote the orthogonal projections of $v$ on $T\Sigma$ and $N\Sigma$ respectively. The Levi-Civita connection in $S^4$ is denoted by $\nabla$ and the induced connections in $T\Sigma$ and $N\Sigma$ are denoted by $\nabla^\top$ and $\nabla^\perp$ respectively. The space of all smooth sections of a vector bundle $E$ is denoted by $\Gamma(E)$.

Let $e_1, e_2$ be a local orthonormal basis in $T\Sigma$. Then the Laplace-Beltrami operator on $\Sigma$ is given by

$$\Delta f = \sum_{i=1}^2 (\nabla_{e_i} e_i f - e_i(e_i f)), \quad f \in C^\infty(\Sigma),$$

the Laplace-Beltrami operator in the normal bundle $\Delta^\perp: \Gamma(N\Sigma) \rightarrow \Gamma(N\Sigma)$ is given by

$$\Delta^\perp X = \sum_{i=1}^2 (\nabla^\perp_{e_i} e_i X - \nabla^\perp_{e_i} \nabla^\perp_{e_i} X), \quad X \in \Gamma(N\Sigma),$$

and the Simons operator $B: \Gamma(N\Sigma) \rightarrow \Gamma(N\Sigma)$ is given by

$$B(X) = \sum_{i=1}^2 (b_{ij} X) b_{ij}, \quad X \in \Gamma(N\Sigma),$$

where $b_{ij} := B(e_i, e_j)$ and $B(X, Y) = (\nabla X Y)^\perp$, $X, Y \in \Gamma(T\Sigma)$ is the second fundamental form of $\Sigma$ in $S^4$. Finally, the Jacobi stability operator $L: \Gamma(N\Sigma) \rightarrow \Gamma(N\Sigma)$ is given by

$$L = \Delta^\perp - 2 - B.$$

It is well-known that the operator $L$ is elliptic. In particular, the spectrum of $L$ is discrete and has the form

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty,$$

where each eigenvalue is listed as many times as its multiplicity is. Then the quantities

$$\text{Ind} \Sigma = \#\{\lambda_k < 0\} \quad \text{and} \quad \text{Nul} \Sigma = \#\{\lambda_k = 0\}$$

are called respectively the (Morse) index and the nullity of the minimal surface $\Sigma$. Hereafter we adhere to the following conventions concerning the collections of eigenvalues: 1) all collections of eigenvalues are considered as multisets, i.e., each eigenvalue is counted with multiplicity; 2) the range of $k$ is maximal possible (here, for instance, $k \geq 1$).

2.2 Bipolar surfaces to Otsuki tori

In this section we shortly describe a convenient parametrization of $\tilde{O}_{p/q}$ from [Kar14]. Here we give only the most necessary definitions. In particular, we do not describe neither Hsiang-Lawson construction nor Lawson’s bipolar surface construction. We refer to [Kar14, sec. 2.4] for the details.

Fix $b \in (-\frac{\pi}{2}, 0)$ and define the function $t(\varphi)$ on $[b, -b]$ by

$$t(\varphi) := \int_b^\varphi \frac{2\pi \cos^3 \psi d\psi}{\sqrt{\cos^4 \psi - \cos^4 b}}$$
Note that the singularities at $\psi = \pm b$ are integrable so that $t(\varphi)$ is well-defined and increasing. Let $T = t(-b)$ and define the function $\varphi(t)$ on $[0, T]$ as the inverse of $t(\varphi)$. Then the function $\varphi(t)$ satisfies

$$\varphi(0) = b, \quad \dot{\varphi}(0) = 0, \quad \varphi(T - t) = -\varphi(t).$$

In particular, $\varphi(t)$ extends to a smooth $T$-antiperiodic function on $\mathbb{R}$ satisfying

$$\dot{\varphi}(t)^2 = \frac{\cos^4 \varphi(t) - \cos^4 b}{4\pi^2 \cos^6 \varphi(t)}.$$  \hspace{1cm} (4)

Define the function $\theta(t)$ by

$$\dot{\theta}(t) = \frac{\cos^2 b}{2\pi \cos^4 \varphi(t)}, \quad \theta(0) = 0.$$ \hspace{1cm} (5)

**Proposition 2.1.** 1) For each $b \in (-\frac{\pi}{2}, 0)$ the image of the immersion $u_b: (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \rightarrow \mathbb{R}^5$ given by

$$u_b(\alpha, t) = \begin{pmatrix}
\cos \alpha \cos \varphi(t) \sin \theta(t) \\
\sin \alpha \cos \varphi(t) \sin \theta(t) \\
\cos \alpha \cos \varphi(t) \cos \theta(t) \\
\sin \alpha \cos \varphi(t) \cos \theta(t) \\
\sin \varphi(t)
\end{pmatrix}$$ \hspace{1cm} (6)

is a minimal surface in $S^4$.

2) The map $u_b$ is periodic in $t$ if and only if the number

$$\Xi(b) := \int_{-b}^{b} \frac{\cos^2 b \, d\varphi}{\cos \varphi \sqrt{\cos^4 \varphi - \cos^4 b}}$$

is a rational multiple of $\pi$. More precisely, if $\Xi(b) = (p/q)\pi$, where $\gcd(p, q) = 1$, then the map $u_b$ is $t_0$-periodic with $t_0 = 2qT$. The number $p/q$ can be any rational number from the interval $(1/2, \sqrt{2}/2)$.

3) In the conditions of 2), if $q$ is even, then the immersion $u_b$ is invariant under the map $(\alpha, t) \mapsto (\alpha + \pi, t + \frac{t_0}{2})$.

One can easily obtain the proof of this proposition from [Kar14, sec. 2.4 and 2.5]. In this paper we define $\hat{O}_{p/q}$ to be the image of $u_b$, where $\Xi(b) = (p/q)\pi$. Let $\hat{O}_{p/q}$ be the orientable two-sheeted cover of $\tilde{O}_{p/q}$ if $q$ is even and coincide with $\tilde{O}_{p/q}$ if $q$ is odd.

**3 Proof of Theorem 1.1**

**3.1 Jacobi operator on $\hat{O}_{p/q}$**

In this section we compute the Jacobi stability operator $L$ on $\hat{O}_{p/q}$ in local coordinates $\alpha, t$. We put $u = u_b$ and suppress the argument $t$ in $\varphi(t)$ and $\theta(t)$ for simplicity.
Proposition 3.1. For any point \( x \in \tilde{\Omega}_{p/q} \) the following vectors form an orthonormal basis of \( T_x\mathbb{R}^5 \),

\[
N = u = \begin{pmatrix}
\cos \alpha \cos \varphi \sin \theta \\
\sin \alpha \cos \varphi \sin \theta \\
\cos \alpha \cos \varphi \cos \theta \\
\sin \alpha \cos \varphi \cos \theta \\
\sin \varphi
\end{pmatrix},
\]

\[
e_1 = \frac{\partial_x u}{\cos \varphi} = \begin{pmatrix}
-\sin \alpha \sin \theta \\
\cos \alpha \sin \theta \\
-\sin \alpha \cos \theta \\
\cos \alpha \cos \theta \\
0
\end{pmatrix},
\]

\[
e_2 = 2\pi \cos \varphi \partial_x u = 2\pi \cos \varphi \begin{pmatrix}
\cos \alpha (-\sin \varphi \sin \theta \hat{\varphi} + \cos \varphi \cos \theta \hat{\vartheta}) \\
\sin \alpha (-\sin \varphi \sin \theta \hat{\varphi} + \cos \varphi \cos \theta \hat{\vartheta}) \\
\cos \alpha (-\sin \varphi \cos \theta \hat{\varphi} - \cos \varphi \sin \theta \hat{\vartheta}) \\
\sin \alpha (-\sin \varphi \cos \theta \hat{\varphi} - \cos \varphi \sin \theta \hat{\vartheta}) \\
\cos \varphi \hat{\varphi}
\end{pmatrix},
\]

(7)

Moreover, \( e_1, e_2 \) is a basis of \( T_x\tilde{\Omega}_{p/q} \) and \( n_1, n_2 \) is a basis of \( N_x\tilde{\Omega}_{p/q} \). If \( q \) is even, then both \( n_1 \) and \( n_2 \) descend on \( N\tilde{\Omega}_{p/q} \).

The proof of this proposition is a direct verification. In the sequel, all computations are made w.r.t. the basis (7). The (local) section \( f_1 n_1 + f_2 n_2 \) of \( N\tilde{\Omega}_{p/q} \) is denoted by \( \left[ \begin{array}{c} f_1 \\ f_2 \end{array} \right] \).

Proposition 3.2. The matrix of the Simons operator \( B: \Gamma(\tilde{\Omega}_{p/q}) \to \Gamma(\tilde{\Omega}_{p/q}) \) in the basis \( n_1, n_2 \) is given by

\[
B = \begin{bmatrix}
8\pi^2 \cos^2 \varphi \hat{\theta}^2 \\
0 \\
8\pi^2 \sin^2 \varphi \cos^2 \varphi \hat{\theta}^2
\end{bmatrix}.
\]

Proof. The entry \( B_{\mu\nu} \) of the matrix \( B \) equals \( \sum_{i,j=1}^{2} (b_{ij}, n_\mu) (b_{ij}, n_\nu) \). We have

\[
b_{11} = (\nabla e_1 e_1) = \frac{1}{\cos^2 \varphi} (\partial_n n_1) \big|_{n_1} = 2\pi (\partial_n n_1),
\]

and \( b_{22} = -b_{11} \) by the minimality of \( u \). Hence,

\[
(b_{11}, n_\mu) = -(b_{22}, n_\mu) = \frac{1}{\cos^2 \varphi} (\partial_n n_1) = 2\pi (\partial_n n_1),
\]

\[
(b_{12}, n_\mu) = (b_{21}, n_\mu) = 2\pi (\partial_n n_1), \quad \mu = 1, 2,
\]

and the rest is a direct computation using (7). \( \square \)

Proposition 3.3. The operator \( \Delta^\perp: \Gamma(\tilde{\Omega}_{p/q}) \to \Gamma(\tilde{\Omega}_{p/q}) \) in the basis \( n_1, n_2 \) has the form

\[
\Delta^\perp \left[ \begin{array}{c} f_1 \\ f_2 \end{array} \right] = \begin{bmatrix}
\Delta f_1 + 4\pi^2 \varphi^2 f_1 - \frac{4\pi^2 \varphi^2}{\cos^2 \varphi} \partial_n f_2 \\
\Delta f_2 + 4\pi^2 \varphi^2 f_2 + \frac{4\pi^2 \varphi^2}{\cos^2 \varphi} \partial_n f_1
\end{bmatrix}.
\]

Proof. It easily follows from (3) that for any \( n \in \Gamma(\tilde{\Omega}_{p/q}) \) and \( f \in C^\infty(\tilde{\Omega}_{p/q}) \) we have

\[
\Delta^\perp f = f \Delta^\perp n + (\Delta f) n - 2 \sum_{i=1}^{2} (e_i f) \nabla e_i n.
\]

(8)

Therefore it suffice to calculate \( \Delta^\perp n_\mu \) and \( \nabla e_i n_\mu \) for \( i, \mu = 1, 2 \). We have

\[
\partial_x n_1 = \frac{1}{\cos \varphi} \partial_n n_1 = \frac{1}{\cos \varphi} \begin{pmatrix}
\cos \alpha \cos \theta \\
\sin \alpha \cos \theta \\
-\cos \alpha \sin \theta \\
-\sin \alpha \sin \theta \\
0
\end{pmatrix}, \quad \partial_x n_2 = 2\pi \cos \varphi \partial_n n_1 = 2\pi \cos \varphi \begin{pmatrix}
\cos \alpha (-\sin \varphi \sin \theta \hat{\varphi} + \cos \varphi \cos \theta \hat{\vartheta}) \\
\sin \alpha (-\sin \varphi \sin \theta \hat{\varphi} + \cos \varphi \cos \theta \hat{\vartheta}) \\
\cos \alpha (-\sin \varphi \cos \theta \hat{\varphi} - \cos \varphi \sin \theta \hat{\vartheta}) \\
\sin \alpha (-\sin \varphi \cos \theta \hat{\varphi} - \cos \varphi \sin \theta \hat{\vartheta}) \\
\cos \varphi \hat{\varphi}
\end{pmatrix},
\]

(9)
From this it is easy to see that
\[ \langle \partial_{e_1}, n_2, n_1 \rangle = -\langle \partial_{e_1}, n_1, n_2 \rangle = 2\pi \varphi \quad \text{and} \quad \langle \partial_{e_1}, n_\mu, n_\nu \rangle = 0 \text{ for all other } i, \mu, \nu = 1, 2. \]
Hence,
\[ \nabla^\perp_{e_1} n_1 = \langle \partial_{e_1}, n_1, n_2 \rangle n_2 = -2\pi \varphi n_2, \quad \nabla^\perp_{e_1} n_2 = \langle \partial_{e_1}, n_2, n_1 \rangle n_1 = 2\pi \varphi n_1, \]
\[ \nabla^\perp_{e_i} n_1 = \nabla^\perp_{e_i} n_2 = 0. \quad (10) \]
Further, using (7), we get
\[ \nabla^\top_{e_1} e_1 = \langle \partial_{e_1}, e_1, e_2 \rangle e_2 = 2\pi \sin \varphi \varphi e_2, \quad \nabla^\top_{e_2} e_2 = -\langle \partial_{e_2}, e_1, e_2 \rangle e_1 = 0, \quad (11) \]
and using (3), we find
\[ \Delta^\perp n_1 = -\nabla^\perp_{e_1} \nabla^\perp_{e_1} n_1 = 4\pi^2 \varphi^2 n_1, \quad \Delta^\perp n_2 = -\nabla^\perp_{e_1} \nabla^\perp_{e_1} n_2 = 4\pi^2 \varphi^2 n_2, \quad (12) \]
where all other terms in (3) vanish because of (10) and (11). The proposition now follows from (8), (10), (12).

**Proposition 3.4 ([Kar14, Proposition 6]).** The Laplace-Beltrami operator on $\hat{O}_{p/q}$ is given by the formula
\[ \Delta f = -\frac{1}{\cos^2 \varphi} \partial^2 f_\alpha - \partial_\alpha (4\pi^2 \cos^2 \varphi \partial_\alpha f). \]

**Proposition 3.5.** The Jacobi stability operator $L: \Gamma(N\hat{O}_{p/q}) \to \Gamma(N\hat{O}_{p/q})$ in the basis $n_1, n_2$ has the form
\[ L \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \Delta f_1 - 4\pi^2 \varphi^2 f_1 - \frac{4\pi^2 \varphi}{\cos \varphi} \partial_\alpha f_2 - 2f_1 - 8\pi^2 \cos^2 \varphi \partial^2 f_1 \\ \Delta f_2 - 4\pi^2 \varphi^2 f_2 + \frac{4\pi^2 \varphi}{\cos \varphi} \partial_\alpha f_1 - 2f_2 - 8\pi^2 \cos^2 \varphi \sin^2 \varphi \partial^2 f_2 \end{bmatrix}. \]

**Proof.** This follows from Propositions 3.2–3.4.

Some eigensections of $L$ can be found from geometrical considerations. In the sequel, the following proposition turns out to be very useful.

**Proposition 3.6.** (i) The following sections of $N\hat{O}_{p/q}$ are in the kernel of $L$:
\[
\begin{bmatrix}
\cos \varphi \sin 2\theta \\
0
\end{bmatrix},
\begin{bmatrix}
\cos \varphi \cos 2\theta \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
2\pi \cos^2 \varphi \varphi
\end{bmatrix},
\begin{bmatrix}
-\sin \alpha \sin \varphi \cos \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
2\pi \cos \alpha \sin \varphi \sin \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
0 \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
-\sin \alpha \sin \varphi \cos \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
2\pi \cos \alpha \sin \varphi \sin \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
-\sin \alpha \sin \varphi \cos \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
2\pi \cos \alpha \sin \varphi \sin \theta \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix},
\begin{bmatrix}
0 \\
2\pi \cos \alpha \cos \varphi \sin \theta \cos \varphi \theta
\end{bmatrix}.
\]

(ii) The following sections of $N\hat{O}_{p/q}$ are eigensections of $L$ with eigenvalue $-2$:
\[
\begin{bmatrix}
-\sin \alpha \cos \theta \\
\cos \alpha (\cos \theta \varphi + \sin \theta \sin \varphi \varphi \theta)
\end{bmatrix},
\begin{bmatrix}
\cos \alpha \cos \theta \\
\sin \alpha (\cos \theta \varphi + \sin \theta \sin \varphi \varphi \theta)
\end{bmatrix},
\begin{bmatrix}
-\sin \alpha \sin \theta \\
\cos \alpha \sin \theta
\end{bmatrix},
\begin{bmatrix}
\cos \alpha \cos \theta \\
\sin \alpha (\cos \theta \varphi - \sin \theta \sin \varphi \varphi \theta)
\end{bmatrix},
\begin{bmatrix}
-\sin \alpha \sin \theta \\
\cos \alpha \sin \theta
\end{bmatrix},
\begin{bmatrix}
\cos \alpha \cos \theta \\
\sin \alpha (\cos \theta \varphi - \sin \theta \sin \varphi \varphi \theta)
\end{bmatrix},
\begin{bmatrix}
0 \\
\cos^3 \varphi \theta
\end{bmatrix}.
\]

**Proof.**
1) It follows from [Sim68, Lemma 5.1.7] that the image of any Killing vector field on $\mathbb{S}^4$ under the orthogonal projection on $N\hat{O}_{p/q}$ belongs to the kernel of $L$. The space of Killing vector fields on $\mathbb{S}^4$ is spanned by the fields $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$, where $i, j = 1, \ldots, 5$. The rest of the proof is a direct computation using (7).

2) It follows from [Sim68, Lemma 5.1.4] that the image of any constant vector field on $\mathbb{R}^5$ under the orthogonal projection on $N\hat{O}_{p/q}$ is an eigensection of $L$ with eigenvalue $-2$. The rest of the proof is again a direct computation using (7).
3.2 Separation of variables

In this section we reduce the spectral problem for $L$ to a family of matrix Sturm-Liouville eigenvalue problems.

Proposition 3.7. For each $l = 0, 1, 2, \ldots$, consider the following matrix Sturm-Liouville eigenvalue problem on $[0, t_0]$ with periodic boundary conditions

\[
\begin{align*}
S_l h &= \lambda h, \\
h(0) &= h(t_0), \quad h'(0) = h'(t_0),
\end{align*}
\]

(13)

where $h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$ is a vector function and $S_l = -\partial_t (p(t) \partial_t) + A(l, t)$ is a matrix Sturm-Liouville operator with

\[
p(t) = 4\pi^2 \cos^2 \varphi,
\]

\[
A(l, t) = \begin{pmatrix}
\frac{t^2}{\cos^2 \varphi} + 4\pi^2 \varphi^2 - 2 - 8\pi^2 \cos^2 \varphi \hat{\theta}^2 & -\frac{4\pi l \hat{\theta}}{\cos \varphi} \\
-\frac{4\pi l \hat{\theta}}{\cos \varphi} & \frac{t^2}{\cos^2 \varphi} + 4\pi^2 \varphi^2 - 2 - 8\pi^2 \sin^2 \varphi \cos^2 \varphi \hat{\theta}^2
\end{pmatrix}.
\]

Then the $\lambda$-eigenspace of the operator $L$ has a basis consisting of the eigensections of the form

\[
\begin{pmatrix}
h_1(t) \cos \lambda t \\
h_2(t) \sin \lambda t
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
-h_1(t) \sin \lambda t \\
h_2(t) \cos \lambda t
\end{pmatrix},
\]

(14)

where $h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$ solves the problem (13) and $l \geq 0$ is integer. Moreover, if $q$ is even, then one can choose this basis of eigensections in such a way that for each eigensection the corresponding vector eigenfunction $h(t)$ of (13) is either $\frac{l}{q}$-periodic or $\frac{l}{q}$-antiperiodic. In this case, the eigensections (14) descend on $\hat{O}_{p/q}$ exactly when either $h(t)$ is $\frac{l}{q}$-periodic and $l$ is even or $h(t)$ is $\frac{l}{q}$-antiperiodic and $l$ is odd.

Proof. Since the operator $L$ commutes with $\partial_x$, we see that the $\lambda$-eigenspace of the operator $L$ has a basis such that each basis eigensection has the form (14), where $h(t)$ is a vector function and $l \geq 0$ is integer. Then a direct computation involving Proposition 3.5 shows that $h(t)$ solves (13).

If $q$ is even, then the coefficients of the operator $S_l$ are $\frac{l}{q}$-periodic. In other words, the operator $S_l$ commutes with the map $\iota: h(t) \mapsto h(t + \frac{l}{q})$, where $h(t)$ is considered as a vector function on the circle $\mathbb{R}/t_0\mathbb{Z}$. Hence there exists a joint basis of eigenfunctions for $S_l$ and $\iota$. Since $\iota^2 = \text{id}$, we obtain that each $\iota$-eigenvalue is $\pm 1$, which means that the corresponding vector eigenfunction $h(t)$ is either $\frac{l}{q}$-periodic or $\frac{l}{q}$-antiperiodic. For the last claim it suffices to remark that a section $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \Gamma(N\hat{O}_{p/q})$ descends on $\hat{O}_{p/q}$ exactly when both $f_1$ and $f_2$ are invariant under the map $(\alpha, t) \mapsto (\alpha + \pi, t + \frac{l}{q})$ and apply the last claim of Proposition 3.1. \hfill $\square$

Let

\[
\lambda_1(l) \leq \lambda_2(l) \leq \ldots \leq \lambda_k(l) \leq \ldots \to +\infty
\]

be the eigenvalues of the problem (13). If $q$ is even, then each eigenvalue is one of two types, depending on whether the corresponding vector eigenfunction is $\frac{l}{q}$-periodic or $\frac{l}{q}$-antiperiodic. We call them periodic and antiperiodic eigenvalues respectively. Let

\[
\lambda_{i_1}(l) \leq \lambda_{i_2}(l) \leq \ldots \leq \lambda_{i_k}(l) \leq \ldots \to +\infty \quad \text{and} \quad \lambda_{j_1}(l) \leq \lambda_{j_2}(l) \leq \ldots \leq \lambda_{j_k}(l) \leq \ldots \to +\infty
\]

be all the periodic and antiperiodic eigenvalues of (13) respectively. In particular,

\[
\{i_1, i_2, \ldots\} \cup \{j_1, j_2, \ldots\} = \mathbb{N}.
\]

Put $\lambda^+_k(l) := \lambda_{i_k}(l)$ and $\lambda^-_k(l) := \lambda_{j_k}(l)$ for each $k \geq 1$.

Proposition 3.8. If $q$ is odd, then

\[
\text{Ind}(\hat{O}_{p/q}) = \#\{\lambda_k(0) < 0\} + 2 \sum_{l=1}^{\infty} \#\{\lambda_k(l) < 0\}, \quad \text{Nul}(\hat{O}_{p/q}) = \#\{\lambda_k(0) = 0\} + 2 \sum_{l=1}^{\infty} \#\{\lambda_k(l) = 0\},
\]

(15)
and if \( q \) is even, then

\[
\text{Ind}(\tilde{O}_{p/q}) = \# \{ \lambda_k^- (0) < 0 \} + 2 \sum_{l=1}^{\infty} \# \{ \lambda_k^{(-1)} (l) < 0 \}, \quad \text{Nul}(\tilde{O}_{p/q}) = \# \{ \lambda_k^- (0) = 0 \} + 2 \sum_{l=1}^{\infty} \# \{ \lambda_k^{(-1)} (l) = 0 \}. \tag{16}
\]

**Proof.** Let \( q \) be odd. If \( l > 0 \), then for each eigenfunction of (13) the corresponding eigensections (14) are linearly independent. Hence, in this case each negative (respectively, zero) eigenvalue \( \lambda_k (l) \) contributes 2 to \( \text{Ind}(\tilde{O}_{p/q}) \) (respectively, \( \text{Nul}(\tilde{O}_{p/q}) \)). If \( l = 0 \), then the matrix equation (13) separates into two scalar Sturm-Liouville equations and each negative (respectively, zero) eigenvalue of each of these equations contributes 1 to \( \text{Ind}(\tilde{O}_{p/q}) \) (respectively, \( \text{Nul}(\tilde{O}_{p/q}) \)). This proves (15).

Let \( q \) be even. Then by the last point of Proposition 3.7, if \( l \) is even, then the eigensections (14) both descend on \( \tilde{O}_{p/q} \) exactly when the corresponding eigenvalue \( \lambda_k (l) \) is periodic, and if \( l \) is odd, then the same happens exactly when the corresponding eigenvalue is antiperiodic. This proves (16).

**Proposition 3.9.** For \( l \geq 3 \) the inequality \( \lambda_1 (l) > 0 \) holds.

**Proof.** By the variational characterization of eigenvalues for the problem (13) we have

\[
\lambda_1 (l) = \inf_{h \in C^1 (\mathbb{R}/t_0 \mathbb{Z}, \mathbb{R}^2)} \frac{\int_0^{t_0} (p(t) |h'(t)|^2 + (h(t), A(l, t) h(t)) dt)}{\int_0^{t_0} |h(t)|^2 dt} \geq \inf_{h \in C^1 (\mathbb{R}/t_0 \mathbb{Z}, \mathbb{R}^2)} \frac{\int_0^{t_0} (h(t), A(l, t) h(t)) dt}{\int_0^{t_0} |h(t)|^2 dt},
\]

Hence, it suffices to show that if \( l \geq 3 \), then the matrix \( A(l, t) \) is positive definite for each \( t \). This holds as soon as the inequality

\[
\frac{l^2}{\cos^2 \varphi} + 4 \pi^2 \varphi^2 - 2 - 8 \pi^2 \cos^2 \varphi \vartheta^2 > \frac{4 \pi l \varphi}{\cos \varphi}
\]

holds for each \( t \). Using (4) and (5), one can rewrite this inequality in the form

\[
\left( l - \sqrt{1 - \frac{\cos^4 b}{\cos^4 \varphi}} \right)^2 > 2 \left( \cos^2 \varphi + \frac{\cos^4 b}{\cos^4 \varphi} \right),
\]

which holds for \( l \geq 3 \) because in this case we have l.h.s. \( \geq 4 > r.h.s. \)

**3.3 Case \( l = 0 \)**

This case is easy because the matrix problem (13) splits into two scalar Sturm-Liouville problems

\[
\begin{cases}
- (4 \pi^2 \cos^2 \varphi h_1')' + (4 \pi^2 \varphi^2 - 2 - 8 \pi^2 \cos^2 \varphi \vartheta^2) h_1 = \lambda_1 h_1, \\
h_1 (t_0) = h_1 (0), \quad h_1' (t_0) = h_1' (0).
\end{cases} \tag{17}
\]

and

\[
\begin{cases}
- (4 \pi^2 \cos^2 \varphi h_2')' + (4 \pi^2 \varphi^2 - 2 - 8 \pi^2 \sin^2 \varphi \cos^2 \varphi \vartheta^2) h_2 = \lambda_2 h_2, \\
h_2 (t_0) = h_2 (0), \quad h_2' (t_0) = h_2' (0).
\end{cases} \tag{18}
\]

Our aim is the following

**Proposition 3.10.** We have

\[
\# \{ \lambda_k (0) < 0 \} = 2q + 4p - 1, \quad \# \{ \lambda_k (0) = 0 \} = 3.
\]

If \( q \) is even, then, in addition, we have

\[
\# \{ \lambda_k^+ (0) < 0 \} = q + 2p - 1, \quad \# \{ \lambda_k^+ (0) = 0 \} = 3.
\]
Let
\[ \lambda_0^{(1)} \leq \lambda_1^{(1)} \leq \ldots \leq \lambda_k^{(1)} \leq \ldots \nearrow + \infty \quad \text{and} \quad \lambda_0^{(2)} \leq \lambda_1^{(2)} \leq \ldots \leq \lambda_k^{(2)} \leq \ldots \nearrow + \infty \]
be the eigenvalues of the problems (17) and (18) respectively. As in the paragraph after Proposition 3.7, for even \( q \) denote by \( \lambda_k^{(1)+} \) and \( \lambda_k^{(2)+} \) the \( k \)-th, \( k \geq 0 \), periodic eigenvalues of the problems (17) and (18) respectively (that is, those eigenvalues for which the corresponding eigenfunction is \( \frac{2\pi}{T} \)-periodic).

**Proposition 3.11.** We have
\[
\#\{\lambda_k^{(1)} < 0\} = 4p - 1, \quad \#\{\lambda_k^{(1)} = 0\} = 2.
\]
If \( q \) is even, then, in addition, we have
\[
\#\{\lambda_k^{(1)+} < 0\} = 2p - 1, \quad \#\{\lambda_k^{(1)+} = 0\} = 2.
\]

*Proof.* It follows from Proposition 3.6 that the functions \( \cos \varphi \cos 2\theta \) and \( \cos \varphi \sin 2\theta \) are the eigenfunctions of the problem (17) with eigenvalue 0. These functions are linearly independent and by [Kar14, Proposition 4] both have exactly \( 4p \) zeros on \([0, t_0]\). The Periodic Sturm Oscillation Theorem (see, for example, [CL95, Theorem 3.1 in Chapter 8]) implies that the only zero eigenvalues of the problem (17) are \( \lambda_k^{(1)_{2p-1}} \) and \( \lambda_k^{(1)p} \). This proves the first part of the proposition. The second part follows from the fact that for even \( q \) both mentioned 0-eigenfunctions are \( \frac{2\pi}{T} \)-periodic. \( \square \)

Unfortunately, for the problem (18) Proposition 3.6 gives only one 0-eigenfunction. This means that we cannot determine the number of this eigenvalue using only the Periodic Sturm Oscillation Theorem. To override this difficulty, consider the following problem on a short interval
\[
\begin{cases}
-(4\pi^2 \cos^2 \varphi \hat{h}_2)^\prime + (4\pi^2 \hat{\varphi}^2 - 2 - 8\pi^2 \sin^2 \varphi \cos^2 \varphi \hat{\theta}^2) \hat{h}_2 = \lambda \hat{h}_1, \\
\hat{h}_2(T) = -\hat{h}_2(0), \quad \hat{h}_2'(T) = -\hat{h}_2'(0).
\end{cases}
\]
(19)

Let
\[ \lambda_1^{(2')} \leq \lambda_2^{(2')} \leq \ldots \leq \lambda_k^{(2')} \leq \ldots \nearrow + \infty \]
be the eigenvalues of this problem.

**Proposition 3.12.** We have \( \lambda_1^{(2')} < 0 \) and \( \lambda_2^{(2')} = 0 \).

*Proof.* Consider the substitution
\[ \hat{h}_2(t) = \sqrt{p(t)}h_2(t) = 2\pi \cos \varphi(t)h_2(t). \]
It is well-known that this substitution transforms the first equation of (19) to an equation of the form
\[ -\hat{h}_2''(t) + q(t)\hat{h}_2(t) = \lambda w(t)\hat{h}_2(t), \]
where \( w(t) = \frac{1}{p(t)} \) (this is so-called Hill differential equation). Let us figure out what is \( q(t) \). From Proposition 3.6(ii) we know that the function \( 2\pi \cos^4 \varphi \hat{\theta} \) solves this equation with \( \lambda = -2 \). But this function is a nonzero constant (see (5)), that is, we actually have \( q(t) = -2w(t) \). Thus the problem (19) becomes
\[
\begin{cases}
-\hat{h}_2''(t) + 2w(t)\hat{h}_2(t) = \lambda w(t)\hat{h}_2(t), \\
\hat{h}_2(T) = -\hat{h}_2(0), \quad \hat{h}_2'(T) = -\hat{h}_2'(0).
\end{cases}
\]
(20)

We now need the following easy claim, which is a continuous version of the 'permutation inequality'.

**Claim 3.13.** Let \( f(t) \) and \( g(t) \) be two strictly decreasing functions on \([0, \tau]\). Then
\[ \int_0^\tau f(t)g(t)dt > \int_0^\tau f(t)g(\tau - t)dt. \]
Proof. Put \( f^*(t) = f(t) - f(\tau - t) \) and define \( g^* \) similarly. Then \( f^* \) and \( g^* \) are both positive on \([0, \tau]\) and

\[
\int_0^\tau f(t)g^*(t)dt = \int_0^{\tau/2} f^*(t)g^*(t) > 0,
\]

which proves the claim. \( \square \)

Now from Proposition 3.6(i) we know that the function \( \eta(t) := 2\pi \cos^3 \varphi \dot{\varphi} \) solves the problem (20) with \( \lambda = 0 \). By the variational characterization of the eigenvalues of the problem (20) with the function \( \eta(t) \) as a test function, we find

\[
\lambda_k^{(2)} + 2 \leq \frac{\int_0^T \eta'(\frac{T}{2} - t)^2 dt}{\int_0^T \eta(t)\eta(\frac{T}{2} - t)^2 dt} = \frac{\int_0^{\tau/2} \eta'(t)^2 dt}{\int_0^{\tau/2} \eta(t)\eta(t)^2 dt} \leq \frac{\int_0^T \eta'(t)^2 dt}{\int_0^T \eta(t)\eta(t)^2 dt} = 2,
\]

where Claim 3.13 is applied with \( \tau = T/2, f(t) = \eta(t), \) and \( g(t) = \eta(\frac{T}{2} - t)^2 \). \( \square \)

Remark 3.14. The trick with Claim 3.13, used in the proof of Proposition 3.12, can be also used to simplify the proof in [Kar14, sec. 3.4].

**Proposition 3.15.** We have

\[
\#\{\lambda_k^{(2)} < 0\} = 2q, \quad \#\{\lambda_k^{(2)} = 0\} = 1.
\]

If \( q \) is even, then, in addition, we have

\[
\#\{\lambda_k^{(2)+} < 0\} = q, \quad \#\{\lambda_k^{(2)+} = 0\} = 1.
\]

**Proof.** Actually, this follows from [Kar14, Proposition 11]), but let us give a proof here for the sake of completeness. Consider the \( \lambda_1^{(2)}, \lambda_2^{(2)} \)-eigenfunctions of the problem (19). By Periodic Sturm Oscillation Theorem, both these eigenfunctions have one zero on \([0, T]\). Moreover, these eigenfunctions extend by antiperiodicity to the eigenfunctions of the problem (18) with \( 2q \) zeroes. Again by Periodic Sturm Oscillation Theorem and Proposition 3.12 we obtain \( \lambda_{2q-1}^{(2)} = \lambda_k^{(2)} < 0 \) and \( \lambda_{2q}^{(2)} = \lambda_k^{(2)} = 0 \), which proves the first claim. The second claim follows from the fact that for even \( q \) both eigenfunctions extend to \( \frac{T}{2} \)-periodic eigenfunctions of (18). \( \square \)

**Proof of Proposition 3.10** follows from Propositions 3.11 and 3.15. \( \square \)

### 3.4 Cases \( l = 1 \) and \( l = 2 \)

This case is much more complicated because now the matrix problem 13 does not split into scalar problems. Our aim is the following.

**Proposition 3.16.** For each \( b \) sufficiently close to zero, we have

\[
2q + 2p - 1 \leq \#\{\lambda_k^-(1) < 0\} \leq 4q - 2, \quad 2 \leq \#\{\lambda_k(1) = 0\} \leq 4, \quad 2 \leq \#\{\lambda_k(2) = 0\} = 1.
\]

If, in addition, \( q \) is even, then

\[
q + p - 1 \leq \#\{\lambda_k^-(1) < 0\} \leq 2q - 2, \quad 2 \leq \#\{\lambda_k^-(1) = 0\} \leq 4, \quad \#\{\lambda_k^-(2) = 0\} = 1.
\]

All the coefficients of the problem (13) are either \( T \)-periodic or \( T \)-antiperiodic (recall that \( T = t_0/2q \)). Thus it makes sense to consider the following matrix Sturm-Liouville problem with quasiperiodic boundary conditions:

\[
\begin{cases}
S'h = \lambda h, \\
h(T) = \omega Jh(0), \quad h'(T) = \omega Jh'(0),
\end{cases}
\]

(21)
where $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\omega$ is some $2q$-th root of unity, and $h(t)$ is now a complex-valued vector function. Let

$$\lambda_k^{[\omega]}(l) \leq \lambda_k^{[\omega]}(l) \leq \ldots \leq \lambda_k^{[\omega]}(l) \leq \ldots \to +\infty$$

be the eigenvalues of the problem (21).

**Proposition 3.17.** Denote $\omega_0 = e^{\pi i / q}$. Then

$$\{\lambda_k(l) \mid k \geq 1\} = \bigcup_{r=0}^{2q-1} \{\lambda_k^{[\omega_0]}(l) \mid k \geq 1\},$$

and if $q$ is even, then

$$\{\lambda_k^+(l) \mid k \geq 1\} = \bigcup_{r=0}^{q-1} \{\lambda_k^{[\omega_0]}(l) \mid k \geq 1\}, \quad \{\lambda_k^-(l) \mid k \geq 1\} = \bigcup_{r=0}^{q-1} \{\lambda_k^{[\omega_0^{2r+1}]}(l) \mid k \geq 1\},$$

where all equalities are equalities of multisets, i.e., each eigenvalue is counted with its multiplicity.

**Proof.** The argument is almost the same as in the proof of the second part of Proposition 3.7. In this case the operator $S_l$ commutes with the map $\sigma: h(t) \mapsto Jh(t + T)$. Hence there exists a joint basis of eigenfunctions for $S_l$ and $\sigma$. Since $\sigma^{2q} = \text{id}$, we obtain that each $\sigma$-eigenvalue $\omega$ satisfies $\omega^{2q} = 1$ and hence equals $\omega_0$ for some $r = 0, \ldots, 2q - 1$. This proves the first claim. For the second claim note that if $q$ is even, then an eigenfunction of (21) is $\frac{T}{q}$-periodic if $\omega = 1$ and $\frac{T}{q}$-antiperiodic if $\omega = -1$. \qed

Consider the quadratic form, associated with the Sturm-Liouville operator $S_l$ on the segment $[0, T]$,

$$Q_l[h] = \int_0^T (p(t)|h'(t)|^2 + q(t), A(l, t)h(t))dt = (h(t), p(t)h'(t))|_0^T + \int_0^T (h(t), S_l h(t))dt,$$

where $(\cdot, \cdot)$ is the standard Hermitian product in $C^2$ (linear in first argument). Let $\Sigma_l$ be the space of solutions of the equation $S_l h = 0$ on the segment $[0, T]$. Consider the map

$$\Sigma_l \to C^2 \oplus C^2, \quad h(t) \mapsto (h(0), h(T)).$$

Since $\dim \Sigma_l = 4$, this map is an isomorphism provided that it is injective or, equivalently, that zero is not an eigenvalue of the Dirichlet problem

$$\begin{cases}
S_l h = \mu h, \\
h(0) = 0, \quad h(T) = 0.
\end{cases} \quad (22)$$

In this case one can define the following quadratic form $\alpha_l$ on $C^2 \oplus C^2$

$$\alpha_l[(v_0, v_T)] = Q_l[g] = (g(t), p(t)g'(t))|_0^T,$$

where $v_0, v_T \in C^2$ and $g$ is a unique element of $\Sigma_l$ such that $g(0) = v_0, g(T) = v_T$. Let $e_1, e_2$ be the standard basis of $C^2$.

The following proposition is our main tool in the study of cases $l = 1$ and $l = 2$. It is a direct consequence of [Edw64, Proposition 2.6]. Let

$$\mu_1(l) \leq \mu_2(l) \leq \ldots \leq \mu_k(l) \leq \ldots \to +\infty$$

be the eigenvalues of the problem (22).

**Proposition 3.18.** Suppose that zero is not an eigenvalue of the problem (22). Then

$$\#\{\lambda_k^{[\omega]}(l) < 0\} = \#\{\mu_k(l) < 0\} + i_l(\omega),$$

$$\#\{\lambda_k^{[\omega]}(l) = 0\} = n_l(\omega),$$

where $i_l(\omega)$ and $n_l(\omega)$ denote, respectively, the index and the nullity of the quadratic form $\alpha_l$ restricted to a subspace of $C^2 \oplus C^2$ spanned by $(e_1, -\omega e_1)$ and $(e_2, \omega e_2)$.
Let $Gr_l(\omega)$ be the Gram matrix of the restriction of $\alpha_l$ on span\{$(e_1, -\omega e_1), (e_2, \omega e_2)$\}.

**Proposition 3.19.** For each $l$ the determinant of $Gr_l(\omega)$ is a polynomial in $\text{Re}\omega$ with real coefficients of degree at most 2.

**Proof.** Consider the following basis in $\mathbb{C}^2 \oplus \mathbb{C}^2 \simeq \mathbb{C}^4$:

$$e_1 = (e_1, 0), \quad e_2 = (e_2, 0), \quad e_3 = (0, e_2), \quad e_4 = (0, e_1).$$

Then $(e_1, -\omega e_1) = e_1 - \omega e_4$ and $(e_2, \omega e_2) = e_2 + \omega e_3$. We have

$$Gr_l(\omega) = \begin{pmatrix} a_{11} - \omega a_{41} - \bar{\omega} a_{14} + a_{44} & a_{12} - \omega a_{42} + \bar{\omega} a_{13} - a_{43} \\ a_{21} + \omega a_{31} - \bar{\omega} a_{24} - a_{34} & a_{22} + \omega a_{32} + \bar{\omega} a_{23} + a_{33} \end{pmatrix},$$

where $a_{ij} = \alpha_l[\varepsilon_i, \varepsilon_j]$ $(i, j = 1, 2, 3, 4)$. Note that $a_{ij} \in \mathbb{R}$ and $a_{ij} = \overline{a_{ji}} = a_{ji}$ since $S_l$ is a real operator and the form $\alpha_l$ is Hermitian. Finally, $\alpha_l = a_{5-1, 5-j}$ since the operator $S_l$ is invariant under the map $t \mapsto T - t$.

With all these relations we get

$$Gr_l(\omega) = \begin{pmatrix} 2a_{11} - 2a_{14} \text{Re}\omega & -2i a_{13} \text{Im}\omega \\ 2i a_{13} \text{Im}\omega & 2a_{22} + 2a_{23} \text{Re}\omega \end{pmatrix}.$$ 

Since $|\omega| = 1$, the proposition follows. \qed

Our next step is to consider the limit case $b \to 0$. In this case we have

$$\varphi(t) \equiv 0, \quad \theta(t) = \frac{t}{2\pi}, \quad T = \lim_{b \to 0} T(b) = 2\pi^2;$$

and

$$p(t) \equiv 4\pi^2, \quad A(l, t) = \begin{pmatrix} t^2 - 4 & 0 \\ 0 & t^2 - 2 \end{pmatrix}.$$ 

Thus the operator $S_l$ splits into a pair of scalar second-order differential operators with constant coefficients. In particular, $\mu_k(l)$ and $Gr_l(\omega)$ can be calculated explicitly.

**Proposition 3.20.** Let $b = 0$. Then

$$\mu_1(1) = -1, \quad \mu_2(1) = 1 > 0, \quad \mu_1(2) = 2 > 0,$$

and

$$Gr_1(\omega) = \begin{pmatrix} \frac{4\sqrt{\pi}}{\sin \frac{\sqrt{\pi}}{2}} (\cos \frac{\sqrt{\pi}}{2} + \text{Re}\omega) & 0 \\ 0 & \frac{4\pi}{\sin \frac{\sqrt{\pi}}{2}} (\cos \frac{\sqrt{\pi}}{2} - \text{Re}\omega) \end{pmatrix},$$

$$Gr_2(\omega) = \begin{pmatrix} 4\sqrt{2}(1 + \text{Re}\omega) & 0 \\ 0 & \frac{4\sqrt{2}\pi}{\sinh \pi} (\cosh \pi - \text{Re}\omega) \end{pmatrix}.$$ 

**Proof.** This is a tedious but very straightforward computation. Let us compute, for example, the first entry of the matrix $Gr_1(\omega)$. This entry equals $\alpha_l[(e_1, -\omega e_1)] = Q_1(g)$, where

$$g(t) = \frac{1}{\sin \frac{\sqrt{\pi}}{2}} \begin{pmatrix} \sin \frac{\sqrt{\pi}}{2} t - t \omega \sin \frac{\sqrt{\pi}}{2} \\ 0 \end{pmatrix}$$

is a unique solution of the equation $S_l h = 0$ such that $g(0) = e_1, g(T) = -\omega e_1$. We have

$$Q_1(g) = \langle g(t), p(t) g'(t) \rangle_{t=0} = \langle g(t), 4\pi^2 g'(t) \rangle_{t=0} = \frac{4\sqrt{3}\pi}{\sin \frac{6\pi}{2}} (\cos \frac{\sqrt{6\pi}}{2} + \text{Re}\omega).$$

All other entries of $Gr_1(\omega)$ and $Gr_2(\omega)$ are computed similarly. \qed
In order to pass to the limit $b \to 0$, we need the following technical proposition.

**Proposition 3.21.** Fix $l, \omega$ and consider the values $\mu_k(l)$ and (the entries of) the matrix $\text{Gr}_l(\omega)$ as functions in $b$. Then these functions are continuous at $b = 0$.

**Proof.** To show continuity of the matrix $\text{Gr}_l(\omega)$ it suffices to establish the continuity of $a_{ij}$ from the proof of Proposition 3.19. This is essentially the continuity of the solution of a system of ODEs on the parameter. Indeed, by (23) it suffices to check the continuous dependence on $b$ of $g'(0)$ and $g'(T)$, where $g$ is a unique solution of the equation $S_\omega g = 0$ such that $g(0) = \varepsilon_i$, $g(T) = \varepsilon_j$. For each $b$ consider a fundamental system of solutions $h^i(t)$ ($i = 1, 2, 3, 4$) of the equation $S_{\omega} h = 0$ with fixed (i.e., independent of $b$) initial conditions at $t = 0$. Then $g(t)$ is a linear combination of these fundamental solutions. Moreover, the coefficients of this linear combinations are rational functions in $h^i(T)$. Thus, the continuity of $g'(0)$ and $g'(T)$ follows from the continuity of $h^i(T)$, $\{h^i\}'(T)$. This last one is exactly continuous dependence of the solution on the parameter; see, for example, [CL95, Theorem 7.4 in Chapter 1]. The continuity of $\mu_k(l)$ follows from [HLWZ19, Theorem 6.1]. □

**Proof of Proposition 3.16.** Case $l = 1$. Let $b$ be sufficiently close to zero. Then by (24) and continuity of $\mu_k(1)$ at $b = 0$ we have $\mu_1(1) < 0$ and $\mu_2(1) > 0$ so that

$$\#\{\mu_k(1) < 0\} = 1.$$  \hspace{1cm} (27)

Recall from Proposition 3.19 that $\det \text{Gr}_1(\omega)$ is a quadratic polynomial in $\Re \omega$ with real coefficients. For $b = 0$, the roots of this polynomial are $\cos \frac{\pi}{2} \omega$ and $-\cos \frac{\pi}{2} \omega$ by (25). By continuity of $\text{Gr}_1(\omega)$ at $b = 0$ (Proposition 3.21), the polynomial $\det \text{Gr}_1$ has two real roots $s_1$ and $s_2$, which are close to $\cos \frac{\pi}{2} \omega$ and $-\cos \frac{\pi}{2} \omega$ respectively. Further, it follows from Proposition 3.6(i) that the complex-valued vector function

$$e^{i\theta} \left( \begin{array}{c} \sin \varphi \\ 2\pi \cos \varphi (\sin \varphi \phi - i \cos \varphi \dot{\theta}) \end{array} \right)$$

is a vector eigenfunction of the problem (21) with $l = 1, \lambda = 0, \omega = \omega_0^p$, where $\omega_0 = e^{i \frac{\pi}{4}}$. This means that in fact

$$s_1 = \Re \omega_0^p = \Re \omega_0^{q-p}.$$  \hspace{1cm} (28)

Also note that $|\cos \frac{\pi}{2} \omega| < -\cos \frac{\pi}{2} \omega$ and hence

$$s_2 > |s_1| = \Re \omega_0^{q-p} \quad \text{for } b \text{ close to zero}.$$ \hspace{1cm} (29)

In the notation of Propositions 3.17 and 3.18 we have

$$i_1(\omega) = \begin{cases} 0, & \Re \omega \in [-1, s_1], \\
1, & \Re \omega \in (s_1, s_2], \\
2, & \Re \omega \in (s_2, 1], \end{cases} \quad n_1(\omega) = \begin{cases} 1, & \Re \omega = s_1, s_2, \\
0, & \text{otherwise}, \end{cases}$$ \hspace{1cm} (30)

and

$$\#\{\lambda_k(1) < 0\} = \sum_{r=0}^{2q-1} \#\{\lambda_k^{[r]}(1) < 0\}$$  \hspace{1cm} (by Proposition 3.17)

$$= \sum_{r=0}^{2q-1} \#\{\mu_k(1) < 0\} + i_1(\omega_0^p)$$ \hspace{1cm} (by (27) and (30))

$$= 2q + \#\{r \mid \Re \omega_0^p \in (s_1, 1]\} + \#\{r \mid \Re \omega_0^p \in (s_2, 1]\}$$ \hspace{1cm} (by (28))

$$\in [2q + 2p - 1, 4q - 1],$$ \hspace{1cm} (by (29)),

13
and similarly

\[
\#\{\lambda_k(1) = 0\} = \sum_{r=0}^{2q-1} \#\{\lambda_k^{[r]}(1) = 0\} = \sum_{r=0}^{2q-1} n_1(\omega_0^r) = \#\{r \mid \text{Re } \omega_0^r = s_1 \text{ or } s_2 \} \in [2, 4].
\]

For even \(q\) the computations are similar and therefore are omitted.

**Case** \(l = 2\) is similar but easier. By (24) and continuity of \(\mu_k(2)\) at \(b = 0\) we have \(\mu_1(2) > 0\) so that

\[
\#\{\mu_k(2) < 0\} = 0.
\]

For \(b = 0\), the roots of this polynomial are \(-1\) and \(\cosh \pi\) by (26). By continuity of \(\text{det } \text{Gr}_2(\omega)\) at \(b = 0\) (Proposition 3.21), the polynomial \(\text{det } \text{Gr}_2\) has two real roots \(s_1\) and \(s_2\), which are close to \(-1\) and \(\cosh \pi\) respectively. It follows from Proposition 3.6(i) that the vector function

\[
\left(\frac{\cos \varphi}{2\pi \cos^2 \varphi}, \frac{\sin \varphi}{2\pi \cos \varphi}\right)
\]

is a vector eigenfunction of the problem (21) with \(l = 2, \lambda = 0, \omega = -1\). This means that in fact \(s_1 = -1\). Since the second root \(s_2\) is close to \(\cosh \pi > 1\), we obtain

\[
i_2(\omega) = 0, \quad n_2(\omega) = \begin{cases} 1, & \omega = -1, \\ 0, & \text{otherwise}, \end{cases}
\]

and finally

\[
\#\{\lambda_k(2) < 0\} = \sum_{r=0}^{2q-1} \#\{\lambda_k^{[r]}(2) < 0\} = \sum_{r=0}^{2q-1} \#\{\mu_k(2) < 0\} + i_2(\omega_0^r) = 0,
\]

\[
\#\{\lambda_k(2) = 0\} = \sum_{r=0}^{2q-1} \#\{\lambda_k^{[r]}(2) = 0\} = \sum_{r=0}^{2q-1} n_2(\omega_0^r) = 1.
\]

Again, we omit the case of even \(q\) since it is similar. \(\square\)

**Proof of Theorem 1.1.** Let \(b\) be small enough as required in Proposition 3.16. Let \(q\) be odd. By Propositions 3.8, 3.9, 3.10, and 3.16 we have

\[
\text{Ind}(\widetilde{O}_{p,q}) = \#\{\lambda_k(0) < 0\} + 2\#\{\lambda_k(1) < 0\} + 2\#\{\lambda_k(2) < 0\} \in [6q + 8p - 3, 10q + 4p - 5],
\]

\[
\text{Nul}(\widetilde{O}_{p,q}) = \#\{\lambda_k(0) = 0\} + 2\#\{\lambda_k(1) = 0\} + 2\#\{\lambda_k(2) = 0\} \in [9, 13].
\]

If \(q\) is even, then, similarly,

\[
\text{Ind}(\widetilde{O}_{p,q}) = \#\{\lambda_k^+(0) < 0\} + 2\#\{\lambda_k^-(1) < 0\} + 2\#\{\lambda_k^+(2) < 0\} \in [3q + 4p - 3, 5q + 2p - 5],
\]

\[
\text{Nul}(\widetilde{O}_{p,q}) = \#\{\lambda_k^+(0) = 0\} + 2\#\{\lambda_k^-(1) = 0\} + 2\#\{\lambda_k^+(2) = 0\} \in [9, 13].
\]

\(\square\)

**Remark** 3.22 (Discussion). It seems that any exact computation of \(\text{Ind}(\widetilde{O}_{p,q})\) for \(b\) close to zero should contain a very accurate control over the root \(s_2\) from the proof of Proposition 3.16. This root (in contrast to \(s_1\), which comes from a Jacobi field) has not any clear geometric meaning and hardly can be calculated exactly. However, numerical experiments in Wolfram Mathematica suggest that when \(b\) varies from 0 to \(-\pi/2\), the root \(s_2\) increases and passes through 1 at some moment. More precisely, we have the following conjecture.
Conjecture 1. There exists \( r_0 \in (2/3, \sqrt{2}/2) \) such that for each \( p/q \in (1/2, r_0) \) one has

\[
\text{Ind}(\tilde{O}_{p/q}) = \begin{cases} 
6q + 8p - 3, & \text{if } q \text{ is odd;} \\
3q + 4p - 3, & \text{if } q \text{ is even.}
\end{cases}
\]

\( \text{Nul}(\tilde{O}_{p/q}) = 9. \)

In particular,

\[
\text{Ind}(\tilde{O}_{2/3}) = 31, \quad \text{Nul}(\tilde{O}_{2/3}) = 9.
\]

One can try to verify this conjecture at least for \( b \) close to \(-\pi/2\) using the same method as in the proof of Theorem 1.1. The problem here is that the limit matrix Sturm-Liouville problem in this case is somewhat very singular: the function \( p(t) \) vanishes at the ends of the interval and the matrix potential \( A(l, t) \) contains a Dirac delta function. It is not clear how one should treat the eigenvalues of this problem.

Finally, let us note that similar difficulties appear when one tries to compute the indices of some similar codimension 2 minimal surfaces. For example, the indices of bipolar Lawson \( \tau \)-surfaces \( \tilde{\tau}_{m,k} \) in \( S^4 \) and Fraser-Sargent free boundary minimal surfaces \( IFS_{k,l} \) in \( B^4 \) are known only for \( m = 3, k = 1 \) (see [KNPS21, proof of Corollary 6.4]) and \( k = 2, l = 1 \) (see [Med21, MM22]) respectively.

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