Poletsky–Stessin Hardy Spaces on Complex Ellipsoids in $\mathbb{C}^n$

Sibel Şahin

Abstract We study Poletsky–Stessin Hardy spaces on complex ellipsoids in $\mathbb{C}^n$. Different from one variable case, classical Hardy spaces are strictly contained in Poletsky–Stessin Hardy spaces on complex ellipsoids so boundary values are not automatically obtained in this case. We have showed that functions belonging to Poletsky–Stessin Hardy spaces have boundary values and they can be approached through admissible approach regions in the complex ellipsoid case. Moreover, we have obtained that polynomials are dense in these spaces. We also considered the composition operators acting on Poletsky–Stessin Hardy spaces on complex ellipsoids and gave conditions for their boundedness and compactness.

Keywords Hardy space · Complex ellipsoid · Approach region · Composition operator

Mathematics Subject Classification Primary 32C15; Secondary 47B33

1 Introduction

The aim of this paper is to study the behavior of Hardy spaces introduced by Poletsky–Stessin in [7] in the case of complex ellipsoids, $\mathbb{B}^n$. Unlike the one variable case, for $n > 1$ Poletsky–Stessin Hardy spaces on complex ellipsoids strictly contain the clas-
sical Hardy spaces $H^p(\mathbb{B}^p)$. Hence, in this case we do not inherit the existence of boundary values from the classical theory. In this paper, we show the existence of boundary values through admissible approach regions. Moreover, we obtain a polynomial approximation as in the classical Hardy spaces and we also consider the boundedness and compactness properties of composition operators acting on Poletsky–Stessin Hardy spaces of complex ellipsoid.

The organization of this paper is as follows: in Sect. 1, we recall the classical Hardy spaces given in [10] and review the construction of the Poletsky–Stessin Hardy spaces $H^p_\phi(\Omega)$, for a hyperconvex domain $\Omega$ and a continuous, negative, plurisubharmonic exhaustion function $\phi$. The main results of this study are given in the following sections, in Sect. 2 we first examine the existence of radial limit values for Poletsky–Stessin Hardy spaces $H^p_\mu(\mathbb{B}^p)$. In addition, we will give a discussion of comparison between classical Hardy spaces and Poletsky–Stessin Hardy classes. Then, we consider a generalization of a method given by Stein and using this rather general method in the case of complex ellipsoid with Cauchy–Fantappie kernel, we show the existence of boundary values through admissible approach regions. Moreover, we will give a brief discussion about the relation between the admissible approach regions and Kobayashi approach regions given by the invariant Kobayashi–Royden metric. In this section we also show that polynomials are dense in the Poletsky–Stessin Hardy spaces $H^p_\mu(\mathbb{B}^p)$. Finally, in Sect. 3 we consider the composition operators, with holomorphic symbols, acting on $H^p_\mu(\mathbb{B}^p)$ and give the necessary and sufficient conditions for the boundedness and compactness of these operators.

2 Poletsky–Stessin Hardy Spaces on Complex Ellipsoids

In this section we will give the preliminary definitions and some important results that we will use throughout this paper. Before proceeding with Poletsky–Stessin Hardy spaces let us first recall the classical Hardy spaces given by [10]. Let $\Omega$ be a smoothly bounded, hyperconvex domain in $\mathbb{C}^n$ and $\lambda$ be a characterizing function for $\Omega$ which is defined in a neighborhood of $\overline{\Omega}$ i.e. $\lambda$ is smooth, $\lambda(x) < 0$ if and only if $x \in \Omega$, $\partial\Omega = \{\lambda(x) = 0\}$ and $|\nabla\lambda(x)| > 0$ if $x \in \partial\Omega$. (The last condition is equivalent to $\frac{\partial\lambda}{\partial\nu_x} > 0$ where $\nu_x$ is the outward normal at $x$.) Let $\Omega_r = \{z : \lambda(z) < r : r < 0\}$ and $\partial\Omega_r = \{z : \lambda(z) = r\}$.

In [10], E. M. Stein defines the class $H^p$ as:

$$H^p = \left\{ f \mid f \text{ holomorphic in } \Omega, \sup_{r < 0} \int_{\partial\Omega_r} |f|^p d\sigma_r < \infty \right\}$$

where $d\sigma_r$ is the induced surface area measure on $\partial\Omega_r$. This space is equipped with the norm

$$\|f\|_p^p = \sup_{r < 0} \int_{\partial\Omega_r} |f|^p d\sigma_r.$$ 

The space $H^p(\Omega)$ does not depend on the characterizing function $\lambda$ used to define $\Omega$ and one gets equivalent norms for different characterizing functions. In [7], Poletsky and Stessin introduced new Hardy type classes of holomorphic functions on hyper-
convex domains in $\mathbb{C}^n$. Before defining these new classes let us first give some preliminary definitions. Let $\varphi : \Omega \to [-\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion function for $\Omega$. Following [2] we define the pseudoball:

$$B(r) = \{ z \in \Omega : \varphi(z) < r \}, \quad r \in [-\infty, 0),$$

and pseudosphere:

$$S(r) = \{ z \in \Omega : \varphi(z) = r \}, \quad r \in [-\infty, 0),$$

and set

$$\varphi_r(z) = \max\{\varphi(z), r\}, \quad r \in (-\infty, 0).$$

In [2], Demailly introduced the Monge–Ampère measures in the sense of currents as:

$$\mu_{\varphi, r} = (dd^c \varphi_r)^n - \chi_{\Omega \setminus B(r)}(dd^c \varphi)^n \quad r \in (-\infty, 0).$$

It is clear from the definition that these measures are supported on $S(r)$. Demailly in [3], proved the so-called Lelong–Jensen formula which we use throughout the sequel. Lelong–Jensen formula is stated as follows:

**Theorem 2.1** Let $r < 0$ and $\varphi$ be a plurisubharmonic function on $\Omega$ then for any negative, continuous, plurisubharmonic exhaustion function $u$

$$\int_{S_u(r)} \varphi d\mu_{u,r} - \int_{B_u(r)} \varphi (dd^c u)^n = \int_{B_u(r)} (r - u)dd^c \varphi (dd^c u)^{n-1}. \quad (1)$$

One of the main concerns of this study is to understand the boundary behavior of Poletsky–Stessin Hardy spaces. For this we also need boundary measures which were introduced by Demailly in [3]. Now let $\varphi : \Omega \to [-\infty, 0)$ be a continuous, plurisubharmonic exhaustion for $\Omega$ and suppose that the total Monge–Ampère mass is finite that is, we assume that

$$MA(\varphi) = \int_{\Omega} (dd^c \varphi)^n < \infty. \quad (2)$$

Then as $r$ approaches to 0, $\mu_{\varphi, r}$ converges to a positive measure $\mu_\varphi$ weak*-ly on $\Omega$ with total mass $\int_{\Omega} (dd^c \varphi)^n$ and supported on $\partial \Omega$. This measure $\mu_\varphi$ is called the Monge–Ampère measure on the boundary associated with the exhaustion $\varphi$.

Now we can introduce the Poletsky–Stessin Hardy classes, which will be our main focus throughout this study. In [7], Poletsky and Stessin gave the definition of new Hardy spaces using Monge–Ampère measures as:

**Definition 1** $H^p_\varphi(\Omega)$ for $p > 0$, is the space of functions $f \in \mathcal{O}(\Omega)$ such that

$$\limsup_{r \to 0^-} \int_{S_{\varphi,r}} |f|^p d\mu_{\varphi,r} < \infty.$$
The norm on these spaces is given by:

\[ \| f \|_{H^p_\phi} = \left( \lim_{r \to 0^+} \int_{S_\phi(r)} |f|^p \, d\mu_{\phi,r} \right)^{\frac{1}{p}} \]

and with respect to these norm the spaces \( H^p_\phi(\Omega) \) are Banach spaces \([7]\).

The next theorem gives us the comparison between Poletsky–Stessin Hardy spaces and the classical Hardy spaces:

**Theorem 2.2** Suppose that \( \Omega \) is a smoothly bounded, hyperconvex domain with a plurisubharmonic characterizing function \( \rho \). Then \( H^p(\Omega) \subseteq H^p_\rho(\Omega) \), \( 1 \leq p < \infty \).

**Proof** Since \( \rho \) is a smooth function we have the Monge–Ampère measure \( d\mu_\rho = dc^\rho \wedge (dd^c |\,z\,|^2)^{n-1} |S(r)\) \([2, \text{Proposition 3.3}]\) and the surface area measure induced by \( \rho \) is \( d\sigma = dc^\rho \wedge (dd^c |\,z\,|^2)^{n-1} |S(r)\) \([8, \text{Corollary 3.5}]\). These are both \( (2n-1)\)-dim differential forms on the \( (2n-1)\)-dim manifold so we have \( d\mu_\rho = c(z) d\sigma(z) \). In a neighborhood of \( \overline{\Omega} \), \( \rho \) is smooth and \( \Omega \subset \subset \mathbb{C}^n \) so \( c(z) \) is a bounded function. Hence,

\[ \int_{S(r)} \phi d\mu_{\rho,r} = \int_{S(r)} \phi(z) c(z) d\sigma(z) \leq K \int_{S(r)} \phi(z) d\sigma(z) \]

Thus, we have \( H^p(\Omega) \subseteq H^p_\rho(\Omega) \). \( \Box \)

From now on we will focus on Poletsky–Stessin Hardy spaces on the complex ellipsoids in \( \mathbb{C}^n \) which are considered as model cases for domains of finite type. It should be noted that although complex ellipsoids are convex domains they are not strictly pseudoconvex since they have Levi flat points at the boundary. The complex ellipsoid \( \mathbb{B}^p \in \mathbb{C}^n \) is given as

\[ \mathbb{B}^p = \left\{ z \in \mathbb{C}^n, \rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1 < 0 \right\} \]

where \( p = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \). One can easily see that \( u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \cdots + |z_n|^{2p_n}) \) is a continuous, plurisubharmonic exhaustion function for \( \mathbb{B}^p \) so we can consider the Poletsky–Stessin Hardy spaces \( H^p_u(\mathbb{B}^p) \) associated with this exhaustion function.

### 3 Boundary Behavior of Poletsky–Stessin Hardy Spaces on Complex Ellipsoids

In this section we will show that unlike the one variable case, for \( n > 1 \) Poletsky–Stessin Hardy spaces \( H^p_u(\mathbb{B}^p) \) are not included in the classical Hardy spaces \( H^p(\mathbb{B}^p) \) on complex ellipsoids. Hence in this case we do not automatically inherit the existence of boundary values from the theory of classical Hardy spaces. Now we start with exhibiting the existence of the radial limits for holomorphic functions in \( H^p_u(\mathbb{B}^p) \), \( p \geq 1 \).
Theorem 3.1 Let \( f \in H^p_u(\mathbb{B}^P) \) be a holomorphic function. Then the radial limit function \( f^*(\xi) = \lim_{r \to 1} f(\tilde{r}\xi) \), \( \xi \in \partial \mathbb{B}^P \) exists \( \mu_u \)-almost everywhere and \( f^* \in L^p_{\mu_u}(\partial \mathbb{B}^P) \), \( p \geq 1 \).

Proof Let \( \mathbb{B}^P \) be the complex ellipsoid determined by the exhaustion function \( u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \cdots + |z_n|^{2p_n}) \) and let \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \partial \mathbb{B}^P, t \in \mathbb{D} \).

Suppose that \( E \) is the ellipse which is the intersection of the complex line joining 0 to \( \xi \) and the ellipsoid \( \mathbb{B}^P \). An exhaustion function for \( E \) is \( g_E(t) = \log(A_1|t|^{2p_1} + A_2|t|^{2p_2} + \cdots + A_n|t|^{2p_n}) \) where \( A_i = |\xi_i|^{2p_i}, 1 \leq i \leq n \). The Monge–Ampère measure associated with the exhaustion function \( u \) is \( d\mu_{u,r} = d^c u \wedge d^c u|_{S_g(r)} \) and let \( A_0 \) be the \( n - 1 \)-dimensional manifold of complex lines passing through the point \( 0 \in \mathbb{B}^P \) [11]. Now take \( f \in H^p_u(\mathbb{B}^P) \) then

\[
\int_{S_u(r)} |f|^p d\mu_{u,r} = \int_{S_u(r)} |f|^p (d^c u \wedge d^c u) = \int_{A_0} \left( \int_{I_\omega \cap S_u(r)} |f|^p d^c u \right) \omega
\]

where we have the pull-back measure \( \pi^* \omega = dd^c u \) and \( \pi : \tilde{\mathbb{B}^P} \to A_0 \) is the function given by \( \pi(z) = [0, z] = l_z \) with \( l_z \) being the line joining 0 and \( z \).

We can use the above generalization of Fubini theorem since \( \pi \) is a submersion and \( \pi|_{\supp f} \) is proper [4, pg: 17].

The measure \( d^c u \) on \( l_z \cap S_u(r) \) is equal to \( d^c g_E(t) \) on \( S_g(r) \) and since it is a smoothly bounded domain \( d^c g_E(t) \) on \( S_g(r) = d\mu_{g,r} \) so

\[
\int_{S_u(r)} |f|^p d\mu_{u,r} = \int_{A_0} \left( \int_{S_g(r)} |f|^p d\mu_{g,r} \right) \omega
\]

and by Fatou’s lemma \( \int_{A_0} \left( \liminf_{r \to 0} \int_{S_g(r)} |f|^p d\mu_{g,r} \right) \omega < \infty \) for \( f \in H^p_u(\mathbb{B}^P) \).

This implies that for \( \omega \)-a.e. \( \liminf_{r \to 0} \int_{S_g(r)} |f|^p d\mu_{g,r} < \infty \) so \( f \in H^p_g(E) \) and it has radial boundary values \( d\sigma (\sim d\mu_{g}) \) almost everywhere [10]. Since \( f^* \) is the pointwise limit of measurable functions it is measurable and consider the set \( A = \{ \xi \in \partial \mathbb{B}^P, f^*(\xi) \text{ does not exist} \} \), then

\[
\int_{\partial \mathbb{B}^P} \chi_A d\mu_u = \int_{A_0} \left( \int_{\partial E} \chi_A(\eta) d\mu_{g}(\eta) \right) \omega.
\]

Since \( f \in H^1_g(E) \), it has radial limit values \( d\mu_{g} \)-a.e. so the integral inside is 0 and we have \( \int_{\partial \mathbb{B}^P} \chi_A d\mu_u = 0 \). Therefore \( f^*(\xi) \) exists \( \mu_u \)-a.e. Moreover for an analytic function \( f \in H^1_g(E) \) we know that the boundary function \( f^* \in L^p(\partial E) \) so we have

\[
\int_{\partial \mathbb{B}^P} |f^*|^p d\mu_u = \int_{A_0} \left( \int_{\partial E} |f^*|^p d\mu_g \right) \omega < \infty
\]

hence \( f^* \in L^p_{\mu_u}(\partial \mathbb{B}^P) \). \( \square \)
Now we have two Hardy type spaces on \( \mathbb{B}^p \), the first one is the Poletsky–Stessin Hardy space \( H^1_b(\mathbb{B}^p) \) and the other one is \( H^1(\mathbb{B}^p) \) which is defined with respect to surface area measure in accordance with Stein’s definition. We will now show that these spaces are not equal. In fact in contrast to the one variable case Poletsky–Stessin Hardy class strictly contains the classical Hardy space.

**Proposition 3.1** Let \( \mathbb{B}^p \) be the complex ellipsoid. Then there exists an exhaustion function \( u \) such that \( H^1(\mathbb{B}^p) \not\subset H^1_b(\mathbb{B}^p) \).

**Proof** We will explicitly construct the exhaustion function \( u \) by taking \( n = 2 \) and \( p = (1, 2) \). First of all the relation between \( d\sigma \) and \( d\mu_u \) on \( \partial\mathbb{B}^2 \) is given by \( K_1 |\xi_2|^2 d\sigma \leq d\mu_u \leq K_2 |\xi_2|^2 d\sigma \) for some \( K_1, K_2 > 0 \) [depending only on dimension and \( p = (1, 2) \)], now consider the analytic function \( f(z_1, z_2) = \frac{1}{(1 - z_1^{2\alpha})^2} \) where \( \frac{4}{16} < \alpha < \frac{4}{16} \).

We have

\[
\int_{\partial\mathbb{B}^2} |f^*| |\xi_2|^2 d\sigma = \int_{|\xi_2|^4 < 1} \left( \int_{|\xi_1| = \sqrt{1 - |\xi_2|^4}} |f^*| d\xi_1 \right) |\xi_2|^2 d\xi_2
\]

\[
= \int_{|\xi_2|^4 < 1} \left( \int_0^{2\pi} \frac{1}{|1 - (\sqrt{1 - |\xi_2|^4})^2|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2
\]

\[
= \int_{|\xi_2|^4 < 1} \left( \int_0^{2\pi} \frac{1}{|1 - e^{2i\theta} + |\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2
\]

Now we will consider the behavior of the inside integral near the point \( \{1\} \) i.e. as \( \theta \to 0 \) (this is the only problematic point as \( |\xi_2| \to 0 \)).

\[
\lim_{\theta \to 0} \frac{(1 - 2(1 - |\xi_2|^4) \cos 2\theta + (1 - |\xi_2|^4)^2)^\alpha}{|\xi_2|^{8\alpha}} = 1
\]

so our integral becomes for \( t > 0, \delta > 0 \)

\[
= \int_{|\xi_2|^4 < 1} \left( 2 \int_t^{\pi - t} \frac{1}{|1 - e^{2i\theta} + |\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2 + 2 \int_{|\xi_2|^2 < 1 \setminus B_t(0)} \frac{2t}{|\xi_2|^{8\alpha}} |\xi_2|^2 d\xi_2
\]

since we are away from the singularity first and third integrals are finite and if we take \( \frac{4}{16} < \alpha < \frac{4}{16} \) then second integral is also finite and we have \( f \in H^1_b(\mathbb{B}^2) \) but \( f \not\in H^1(\mathbb{B}^2) \) since for this choice of \( \alpha \)

\[
\int_{|\xi_2|^4 < 1} \left( \int_0^{2\pi} \frac{1}{|1 - e^{2i\theta} + |\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) d\xi_2
\]

diverges. \( \square \)
In the previous results we have shown that for the functions in the Poletsky–Stessin Hardy class $H^p_u(\mathbb{B}^p)$ we have the radial limit values and throughout the following arguments we will study the behavior of these boundary values in detail. In the classical Hardy space theory on strictly pseudoconvex domains, Stein showed the existence of boundary values along admissible approach regions that are more general than the radial approach. Throughout the rest of the section we will show that for the functions in the Poletsky–Stessin Hardy class $H^p_u(\mathbb{B}^p)$ boundary values along admissible approach regions exist. Although we use the general idea in Stein’s classical method, our approach differs in two aspects, respectively the use of Cauchy–Fantappie kernel instead of Poisson kernel and the use of radial limits. In the study of the boundary behavior of holomorphic functions, having the boundary of the domain as a space of homogenous type seems to be a leitmotif because one of the most commonly used methods in order to understand boundary behavior is to use maximal functions \cite[Theorem 3]{10} and the natural setting for this type of analysis is homogenous spaces. Therefore we will start with recalling the properties of homogenous spaces and then as an application of this classical method we will show that polynomials are dense in the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^p)$ on complex ellipsoids. Before proceeding our arguments in $\mathbb{C}^n$ with maximal functions, let us first mention the spaces of homogenous type in $\mathbb{C}^n$:

**Definition 2** Suppose that we are given a space $X$ which is equipped with a quasi-metric $\rho$ \cite[pg:145]{6} and a regular Borel measure $\mu$ on $X$. Denote the balls in this quasi-metric by $B(x, r) = \{y \in X : \rho(x, y) < r\}$. We say that $(X, \rho, \mu)$ is a space of homogenous type if the following conditions are satisfied:

- For each $x \in X$ and $r > 0$, $0 < \mu(B(x, r)) < \infty$
- (Doubling Condition) There is a constant $C_2 > 0$ such that for any $x \in X$ and $r > 0$ we have $\mu(B(x, 2r)) \leq C_2 \mu(B(x, r))$.

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain such that we have a quasi-metric $\rho$ on $\overline{\Omega}$ and a regular Borel measure $\mu$ on $\partial \Omega$. Let $K(z, \xi) : \Omega \times \partial \Omega \to \mathbb{C}$ be a kernel such that $K(z, \xi) \in L^1(d\mu)$ for $z \in \Omega$, $\xi \in \partial \Omega$. Let us consider the integral operator determined by $K(z, \xi)$ for an $L^p(d\mu)$ function $f^*$, 

$$Kf^*(z) = \int_{\partial \Omega} f^*(\xi) K(z, \xi) d\mu(\xi)$$

and define the associated maximal function as 

$$Mf^*(\xi) = \sup_{\varepsilon > 0} \frac{1}{\mu(B(\xi, \varepsilon))} \int_{B(\xi, \varepsilon)} |f^*| d\mu.$$ 

From the corresponding results in literature (see e.g. \cite[Theorem 2]{10}; \cite[chapter 14]{13}) the fundamental theorem of the theory of singular operators which is adopted to our setting can be stated as:

**Theorem 3.2** Suppose $f^* \in L^p(d\mu_u)$ and $1 \leq p \leq \infty$  

(a) $\|Mf^*\|_p \leq A_p \|f^*\|_p$ for $1 < p \leq \infty$
The mapping \( f^* \to Mf^* \) is of weak type (1-1) i.e. \( \mu_u \{ \xi : Mf^*(\xi) > \alpha \} \leq \frac{K}{\alpha} \| f^* \|_1 \) if \( f^* \in L^1(d\mu_u) \).

Now we further suppose that the following conditions are satisfied:

- \( \rho \) is a quasi-metric on \( \overline{\Omega} \)
- \((\partial\Omega, \rho, \mu)\) is a space of homogenous type
- For all \( z \in \Omega, \xi \in \partial\Omega \) with \( \eta = \rho(z, \xi) > 0 \) we have
  \[ |K(z, \xi)| \leq C \frac{1}{\mu(B(\xi, \eta))} \]
  for some \( C \) independent of \( \xi \) and \( \eta \) and dependence of \( C \) to the point \( z \) is given as in [5, (3.3)]. Such a kernel is called a standard kernel.

Following the method given in [10, Theorem 3], which was applied for the Poisson integrals of \( L^p \) functions, we can now estimate the integral operator given above in this general setting:

**Theorem 3.3** Suppose \( Kf^*(z) \) is the \( K(z, \xi) \)-integral of an \( L^p(d\mu) \) function \( f^* \) where \( K(z, \xi) \) satisfies the conditions given above. Let \( Q_\alpha(y) = \{ z \in \overline{\Omega}, \rho(y, z) < \alpha \delta(y) \} \) for \( y \in \partial\Omega, z \in \Omega \) with \( \delta(y) = \min \{ \rho(z, \partial\Omega), \rho(z, T_y) \} \) (\( T_y \) is the tangent plane at \( y \)), \( \alpha > 0 \), be the admissible approach region. Then

- When \( \rho(y, z) = \epsilon \) and \( z \in Q_\alpha(y) \) the following inequality holds
  \[ |Kf^*(z)| \leq \tilde{A} \sum_{k=1}^{\infty} (\mu(B(y, 2^k \epsilon)))^{-1} \int_{B(y, 2^k \epsilon)} |f^*|d\mu \]

- \( \sup_{z \in Q_\alpha(y)} |Kf^*(z)| \leq \tilde{A} Mf^*(y) \).

**Proof** Let \( Kf^*(z) \) be the \( K(z, \xi) \)-integral of the \( L^p(d\mu) \) function \( f^* \),

\[
|Kf^*(z)| \leq \int_{\partial\Omega} |f^*||K(z, \xi)|d\mu(\xi)
= \int_{\rho(\xi, y) < 2\epsilon} |f^*||K(z, \xi)|d\mu(\xi)
+ \sum_{k=2}^{\infty} \int_{2^{k-1} \epsilon \leq \rho(\xi, y) < 2^k \epsilon} |f^*||K(z, \xi)|d\mu(\xi)
\]

first,

\[
\int_{\rho(\xi, y) < 2\epsilon} |f^*||K(z, \xi)|d\mu(\xi) \leq \frac{C}{\mu(B(y, 2\epsilon))} \int_{B(y, 2\epsilon)} |f^*(\xi)|d\mu(\xi)
\]

by the condition on the kernel and the construction of approach region. Similarly since \( \rho \) is a pseudometric we have \( \rho(z, \xi) \geq \tilde{C}(\rho(\xi, y) - \rho(y, z)) \geq \tilde{C}2^{k-1} \epsilon -
\[ \hat{C} \varepsilon \geq \hat{C} 2^{k-2} \varepsilon \text{ if } k \geq 2 \text{ whenever } 2^{k-1} \varepsilon \leq \rho(\xi, y) < 2^k \varepsilon \text{ and } \rho(z, y) = \varepsilon, \text{ so } |K(z, \xi)| \leq \frac{2^{2k} \hat{C}}{\mu(B(y, 2^k \varepsilon))}. \text{ Hence for all } k, \]

\[
\int_{2^{k-1} \varepsilon < \rho(\xi, y) < 2^k \varepsilon} |f^*||K(z, \xi)|d\mu(\xi) \leq \frac{\hat{A}_{\alpha, n}}{2^k \mu(B(y, 2^k \varepsilon))} \int_{B(y, 2^k \varepsilon)} |f^*(\xi)|d\mu(\xi).
\]

Upon summing in \( k \) we get the first assertion and the second inequality is an immediate consequence of the first.

In [5], Hansson considered the boundedness of Cauchy–Fantappie integral operator, \( H \), from \( L^2(\partial\mathbb{B}^n) \) into \( H^2_u(\mathbb{B}^P) \). In his work he applied an operator theory result known as T1-Theorem and in order to use that result he showed the homogeneity of the boundary of the complex ellipsoid with respect to the quasi-metric \( d \) and the boundary measure \( \partial \rho \wedge (\partial \rho)^{n-1} \) where the function \( \partial \rho \) is defined as \( \rho(z) = \sum_{j=1}^n |z_j|^{2p_j} - 1 \).

In fact an easy calculation shows that this measure is the boundary Monge–Ampère measure associated with the exhaustion function \( u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \cdots + |z_n|^{2p_n}) \), \( \mathbb{P} = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \) of the complex ellipsoid \( \mathbb{B}^P \). Now let \( d(\xi, z) \equiv |v(\xi, z)| + |v(\xi, z)| \) be the quasi-metric defined on \( \mathbb{B}^P \) where \( v(\xi, z) = \langle \partial \rho(\xi), \xi - z \rangle \).

Then explicitly \( v(\xi, z) = \sum_{j=1}^n p_j |\xi_j|^{2(p_j - 1)} \bar{\xi}_j (\xi_j - z_j) \) and define the boundary balls as \( B(z, \varepsilon) = \{ \xi \in \partial\mathbb{B}^P, d(\xi, z) < \varepsilon \} \). It is shown that \((\partial\mathbb{B}^P, d, d\mu_u)\) is a space of homogenous type [5, pg:1483] and \( \frac{1}{(v(\xi, z))^{n}} \) is a standard kernel. In the following argument we will use his homogeneity result to apply the previous rather general procedure on the complex ellipsoid case with the so called Cauchy–Fantappie kernel: The Cauchy–Fantappie integral (from now on we will refer as CF-integral) of an \( L^p(d\mu_u) \) function \( f^* \) is defined as:

\[
Hf(z) = \left( \frac{1}{2\pi i} \right)^n \int_{\partial\mathbb{B}^P} f^*(\xi)d\mu_u(\xi) \left( \frac{v(\xi, z)}{(v(\xi, z))^{n}} \right)^n.
\]

Before proceeding to further results let us briefly discuss the Cauchy–Fantappie kernel. In the theory of holomorphic functions in one variable a fundamental tool is Cauchy integral formula and in the case of several variables one wants a suitable generalization to Cauchy integral. One of the possible choices for the generalization is the so called Szegö kernel however except for a few domains Szegö kernel has no explicit formulation. One other choice is the well known Bochner–Martinelli kernel but the major shortcoming of this kernel is that it is not holomorphic in \( z \) variable (For details see [8]). Contrary to Bochner–Martinelli kernel, Cauchy–Fantappie kernel is holomorphic in \( z \) hence it is a natural generalization of Cauchy kernel to multivariable case and it has reproducing property for the functions in the algebra \( A(\mathbb{B}^P) \) [8, Theorem 3.4]. Hardy spaces which are examined in [5] are exactly the Poletsky–Stessin Hardy spaces \( H^2_u(\mathbb{B}^P) \) that are generated by the exhaustion function \( u \). At the beginning of this section it is shown that for the functions in \( H^2_u(\mathbb{B}^P) \) the boundary value function \( f^* \in L^p(d\mu_u) \) exists so the CF-integral of \( f^* \) is well-defined. Now we will show that CF-integral has reproducing property for the functions in \( H^2_u(\mathbb{B}^P) \):
Proposition 3.2 Let \( f \in H^p_u(B^p) \) be a holomorphic function then

\[
f(z) = Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial B^p} f^*(\xi)d\mu_u(\xi).
\]

Proof By the Fubini type integral formula that we used in Theorem 3.1 we get that

\[
Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{A_0} \left(\int_{\partial E} f^*(\eta)\omega(\eta, v(\eta, z))\right)d\mu_g(\eta)
\]

and on every ellipse \( E \) by [10, 9.7] we have reproducing property as a consequence of one variable Cauchy integral formula. Hence the result follows.

Now define the maximal function for the functions in \( L^p(d\mu_u) \) as follows:

\[
Mf^*(\xi) = \sup_{\varepsilon > 0} \frac{1}{\mu_u(B(\xi, \varepsilon))} \int_{B(\xi, \varepsilon)} |f^*|d\mu_u.
\]

The next result is a consequence of the general method given in Theorem 3.3 for complex ellipsoid case and it gives the relation between the CF-integral and the maximal function of an \( L^p(d\mu_u) \) function \( f^* \):

Corollary 3.1 Suppose \( Hf(z) \) is the CF-integral of an \( L^p(d\mu_u) \) function \( f^* \). Let \( Q_\alpha(y) = \{z \in B^p, |v(y, z)| < \alpha \delta_y(z)\} \) for \( y \in \partial B^p, z \in B^p \) with \( \delta_y(z) = \min\{d(z, \partial X), d(z, T_y)\} \) (\( T_y \) is the tangent plane at \( y \), \( \alpha > 0 \), be the admissible approach region. Then

- When \( d(y, z) = \varepsilon \) and \( z \in Q_\alpha(y) \) the following inequality holds

\[
|Hf(z)| \leq \hat{A} \sum_{k=1}^{\infty} (\mu_u(B(y, 2^k \varepsilon)))^{-1} \int_{B(y, 2^k \varepsilon)} |f^*|d\mu_u.
\]

- \( \sup_{z \in Q_\alpha(y)} |Hf(z)| \leq \hat{A} Mf^*(y) \).

Next using this maximal function tools we will see the existence of boundary values on the admissible approach regions \( Q_\alpha(y), y \in \partial B^p \):

Theorem 3.4 Let \( f \in H^p_u(B^p) \) be a holomorphic function and \( 1 \leq p < \infty \). Suppose that \( f^* \) is the radial limit function then

\[
\lim_{Q_\alpha(\xi) \ni z \to \xi} f(z) = f^*(\xi)
\]

exists for almost every \( \xi \in \partial B^p \).
Preliminaries

Values of Poletsky–Stessin Hardy spaces on complex ellipsoids. Let us first give the
Hence polynomials are dense in
∀∗ fr. Therefore
Hence the result follows.
Proof If ε > 0 then choose g ∈ C(∂Bp) so that ∥f* − g∥Lp(∂Bp) < ε2. Then we
know that lim Qa(ξ)∋z→ξ Hg(z) = g(ξ) for all ξ ∈ ∂Bp. Therefore

\[
\mu_u[\xi, \limsup_{Qa(ξ)∋z→ξ} |f(z) − f^*(ξ)| > \varepsilon] \leq \mu_u[\xi, \limsup_{Qa(ξ)∋z→ξ} |f(z) − Hg(z)| > \varepsilon/3]
+ \mu_u[\xi, \limsup_{Qa(ξ)∋z→ξ} |Hg(z) − g(ξ)| > \varepsilon/3]
+ \mu_u[\xi, \limsup_{Qa(ξ)∋z→ξ} |g(ξ) − f^*(ξ)| > \varepsilon/3]
\leq \mu_u[\xi, C_α M(f^* − g) > \varepsilon/3] + (\|f^* − g\|L^p(∂Bp)/\varepsilon/3)^p \leq C_α \varepsilon^p
\]

Hence the result follows.

Next, we will give an invariant form of the Fatou type theorem for the boundary
values of Poletsky–Stessin Hardy spaces on complex ellipsoids. Let us first give the
preliminaries:

Let kBp be the Kobayashi–Royden metric on Bp. Let U be a tubular neighborhood
of ∂Bp and take ε0 to be the one fourth of the distance of Uc to ∂Bp. Let νp be the unit
outward normal vector to ∂Bp at a boundary point P ∈ ∂Bp. Take a positive constant
β > 0. If P ∈ ∂Bp, then we let np = {P − tνp : 0 < t < ε0}. We set

\[K_β(P) = \{z ∈ Bp : k_{Bp}(z, n_p) < β\} .\]

We know that ∂Bp is strongly pseudoconvex at all points z ∈ (∂Bp) ∩ (C*)n so
μu almost all points on ∂Bp are strongly pseudoconvex points (*). Now combining
Theorem 3.4 with [1, Theorem 1] and using (*), we obtain the invariant form of the
Fatou type result that we proved in the previous theorem:

**Theorem 3.5** Let f ∈ H^p_u(Bp) be a holomorphic function and 1 ≤ p < ∞. Suppose
that f* is the radial limit function then for β > 0,

\[\lim_{K_β(P)∋z→P} f(z) = f^*(P)\]

exists μu-almost every point P ∈ ∂Bp.

As another application of the result given in Corollary 3.1, we will show an approx-
imation result on the Poletsky–Stessin Hardy spaces:

**Theorem 3.6** Polynomials are dense in H^p_u(Bp).

Proof Let f ∈ H^p_u(Bp) be a holomorphic function, 1 ≤ p < ∞ and let fr(ξ) =
f(rξ) for ξ ∈ ∂Bp. Then we have f(rξ) → f*(ξ) μu almost everywhere. By
the previous proposition we know that Hf(z) = f(z) when f ∈ H^p_u(Bp). Using
this and the previous results on maximal function we have |f(rξ)| ≤ Mf*, where
Mf* ∈ L^p_u(∂Bp) then by the Lebesgue Dominated Convergence Theorem we have
that fr → f* in L^p_u(∂Bp). Furthermore the complex ellipsoid is a complete Reinhardt
domain so as a consequence of series expansion we deduce that polynomials are dense
in A(Bp) in the topology of uniform convergence on compact subsets [12, Lemma 2].
Hence polynomials are dense in H^p_u(Bp).
4 Composition Operators: Boundedness and Compactness

Let $\phi : \mathbb{B}^p \to \mathbb{B}^p$ be a holomorphic self map of $\mathbb{B}^p$. The linear composition operator induced by the symbol $\phi$ is defined by $C_\phi(f) = f \circ \phi$, $f \in \mathcal{O}(\mathbb{B}^p)$. In [7], Poletsky and Stessin gave necessary and sufficient conditions for the boundedness and compactness of a composition operator acting on Poletsky–Stessin Hardy spaces in terms of generalized Nevanlinna counting functions. In this section we will characterize the boundedness and compactness of composition operators acting on Poletsky–Stessin Hardy spaces on complex ellipsoids in terms of Carleson conditions.

Let $d$ be the quasi-metric given in Sect. 2, then given a boundary point $\xi$ and a positive constant $\varepsilon > 0$ we set the balls as follows:

$$Q(\xi, \varepsilon) = \{z \in \mathbb{B}^p : d(z, \xi) < \varepsilon\},$$

$$B(\xi, \varepsilon) = Q(\xi, \varepsilon) \cap \partial \mathbb{B}^p.$$

As in Sect. 2, for a function $f^* \in L^p_u(\mathbb{B}^p)$, $Hf$ denotes the Cauchy–Fantappie integral of $f^*$. Now with this notation we have the following result:

**Theorem 4.1** Let $\mu$ be a positive, finite measure on $\mathbb{B}^p$. Then $\mu$ is bounded in $L^p_u(\mathbb{B}^p)$, $1 < p < \infty$ i.e. for some positive constant $C > 0$,

$$\int_{\mathbb{B}^p} |Hf|^p \, d\mu \leq C \int_{\partial \mathbb{B}^p} |f^*|^p \, d\mu_u \quad (*)$$

if and only if

$$\mu(Q(\xi, \varepsilon)) \leq C \mu_u(B(\xi, \varepsilon)) \quad (**).$$

for all $\xi$ and $\varepsilon$.

**Proof** Taking $f^* = \chi_{Q(\xi, \varepsilon)}$ in $(*)$ we immediately obtain $(**).$ For the converse direction, let $f^* \in L^p_u(\partial \mathbb{B}^p)$ be given. For $\lambda > 0$, note that by lower semicontinuity the set $\{\xi \in \partial \mathbb{B}^p : Mf^*(\xi) > \lambda\}$ is open, so it consists of countably many open balls $A_j$ on $\partial \mathbb{B}^p$. For $\lambda$ large enough that this set is not the whole boundary, let $Q_j$ be the ball $Q(\xi, \varepsilon)$ for which $\xi$ is the center of $A_j$ and the radius $\varepsilon$ is such that $d(z, \xi) = \varepsilon$ contains the boundary of $A_j$. If $z = r\xi_0$, $\xi_0 \in \partial \mathbb{B}^p$, and $|Hf(z)| > \lambda$, let $J_z$ be the set $\{\eta \in \partial \mathbb{B}^p : z \in Q_3(\eta)\}$ where $Q_3(\xi)$ is the admissible approach region defined in Sect. 2. The point $\xi_0$ is the center of $J_z$ and if $\gamma$ is a boundary point of $J_z$, from the definition of $Q_3(\eta)$ we have,

$$d(z, \gamma) = 3\delta_{\gamma}(z) = 3d(z, \xi_0).$$

Thus,

$$d(\xi_0, \gamma) \geq C(d(z, \gamma) - d(z, \xi_0)) = 2Cd(z, \xi_0)$$

which means that $z \in Q(\xi_0, d(\xi_0, \gamma))$. Now $z \in Q_3(\eta)$ implies $Mf^*(\eta) > \lambda$ for $\eta \in J_z$. This means $J_z$ is contained in $A_j$ for some $j$ so $Q(\xi_0, d(\xi_0, \gamma))$ is a subset of $Q_j$ and $z$ is in $Q_j$. Now by $(**)$ for some constant $K > 0$, ...
\[ \mu(\{ z : |Hf(z)| > \lambda \}) \leq \sum \mu(Q_j) \leq K \sum \mu_u(A_j) = K \mu_u(\{ \eta : Mf^{*}(\eta) > \lambda \}) \]

Now we have that,

\[
\int_{\mathbb{B}^p} |Hf(z)|^p d\mu = \int_0^\infty p\lambda^{p-1} \mu(\{ z : |Hf(z)| > \lambda \}) d\lambda \\
\leq K \int_0^\infty p\lambda^{p-1} \mu_u(\{ \eta : Mf^{*}(\eta) > \lambda \}) d\mu_u(\eta). 
\]

Then by the maximal function result [9, Corollary 3.3.1] we obtain that,

\[
\int_{\mathbb{B}^p} |Hf(z)|^p d\mu \leq \tilde{C} \| f^* \|_{L_p^u(\partial\mathbb{B}^p)}. 
\]

As a consequence of this result, one can deduce the following characterization for the boundedness of the composition operators:

**Theorem 4.2** Let \( \phi : \mathbb{B}^p \to \mathbb{B}^p \) be a holomorphic self map of \( \mathbb{B}^p \). For \( 1 \leq p < \infty \), the composition operator \( C_\phi(f) = f \circ \phi \) is bounded on \( H^p_u(\mathbb{B}^p) \) if and only if \( \mu(Q(\xi, \epsilon)) \leq C \mu_u(B(\xi, \epsilon)) \) for all \( \xi \in \partial\mathbb{B}^p \) and \( \epsilon > 0 \) where \( \mu(E) = \mu_u((\phi)^{-1}(E)) \) for all measurable \( E \subset \mathbb{B}^p \).

Lastly, we will give necessary and sufficient conditions for the compactness of composition operators acting on \( H^p_u(\mathbb{B}^p) \):

**Theorem 4.3** The composition operator \( C_\phi(f) = f \circ \phi \) is compact on \( H^p_u(\mathbb{B}^p) \) if and only if \( \mu(Q(\xi, \epsilon)) = o(\mu_u(B(\xi, \epsilon))) \) as \( \epsilon \to 0 \) uniformly on \( \xi \in \partial\mathbb{B}^p \).

**Proof** Assume for a contradiction that \( \mu(Q(\xi, \epsilon)) \neq o(\mu_u(B(\xi, \epsilon))) \) so that we can find \( \xi_n \in \partial\mathbb{B}^p \), positive numbers \( h_n \) decreasing to 0 and \( \beta > 0 \) with \( \mu(Q(\xi_n, h_n)) \geq \beta \mu_u(B(\xi_n, h_n)) \). Set \( a_n = (1 - h_n)\xi_n \) and define \( f_n \) so that \( f_n(z) = (1 - a_nz)^{-\frac{1}{p}} \) then by the Fubini type formula that we obtained in the proof of Theorem 3.1 we see that,

\[
\| f_n \|_p^p = \int_{\partial\mathbb{B}^p} |f_n|^p d\mu_u = \int_{A_0} \left( \int_E |f_n|^p d\mu_g \right) \omega. 
\]

Now for all \( \xi \in \partial\mathbb{B}^p \), in the inner integral we make the change of variables \( \gamma(\xi_n^j) = (\xi_n^j)^{p_j} \) then if we define \( \tilde{f}(\xi^p) = f(\xi) \) we obtain that

\[
\int_{\partial\mathbb{B}^p} |f_n|^p d\mu_g \sim \int_0^{2\pi} |\tilde{f} |_{T} |^p d\theta \sim (1 - a_n^2)^{-3n} \sim h_n^{-3n} 
\]
so we have \( \| f_n \|_p \sim h_n^{-3n} \). Thus if \( g_n = \frac{f_n}{\| f_n \|_p} \in H^p_u(\mathbb{B}^p) \), we have \( g_n \) converges to 0 weakly since \( h_n \to 0 \) as \( n \to \infty \). However,

\[
\| g_n \circ \phi \|_p^p = \int_{\partial \mathbb{B}^p} |g_n \circ \phi|^p d\mu_u = \int_{\mathbb{B}^p} |g_n|^p d\mu \\
\geq \| f_n \|_p^p \int_{Q(\xi_n, h_n)} |f_n|^p d\mu.
\]

If \( z \in Q(\xi_n, h_n) \), then since \( d(z, \xi_n) \leq h_n \) implies \( |\xi_n - z| \leq h_n \) and

\[
|1 - \overline{a_n}z| = |1 - (1 - h_n)\overline{\xi_n}z| \leq |\overline{\xi_n}(\xi_n - z)| + |h_n\overline{\xi_n}z| \leq 2h_n
\]

we have \( |f_n|^p \geq (2h_n)^{-4} \) on \( Q(\xi_n, h_n) \). Thus \( \| g_n \circ \phi \|_p^p \) is bounded away from 0 and \( C_\phi \) cannot be compact.

For the converse direction assume \( \mu(Q(\xi, \epsilon)) = o(\mu_u(B(\xi, \epsilon))) \) uniformly in \( \xi \). Then given \( \epsilon > 0 \), there exists a \( \delta_0 > 0 \) so that for all \( \delta < \delta_0 \) we have that

\[
\mu(Q(\xi, \delta)) \leq 2\epsilon \mu_u(B(\xi, \delta)) \tag{***}
\]

Now suppose \( \{ f_n \} \) is a bounded sequence in \( H^p_u(\mathbb{B}^p) \) and \( f_n \to f \) uniformly on compact subsets of \( \mathbb{B}^p \). Then,

\[
\int_{\partial \mathbb{B}^p} |(f_n - f) \circ \phi|^p d\mu_u = \int_{\mathbb{B}^p} |f_n - f|^p d\mu.
\]

Now decompose \( \mu \) so that \( \mu = \mu_1 + \mu_2 \) where \( \mu_1 \) is the restriction of \( \mu \) to \( (1 - \delta_0)\overline{\mathbb{B}^p} \) and \( \mu_2 = \mu - \mu_1 \). Then,

\[
\int_{\mathbb{B}^p} |f_n - f|^p d\mu = \int_{\mathbb{B}^p} |f_n - f|^p d\mu_1 + \int_{\mathbb{B}^p} |f_n - f|^p d\mu_2. \tag{3}
\]

Since \( \mu_2 < \mu, \mu_2 \) satisfies (***) whenever \( \mu \) does. We claim that \( \mu_2 \) satisfies the condition (***) in the previous theorem. To see this claim, note that if \( Q(\xi, \eta) \subset N = \overline{\mathbb{B}^p \setminus (1 - \delta_0)\overline{\mathbb{B}^p}} \), the claim is immediate from (***) For an arbitrary \( B(\xi, \eta) \) decompose \( B(\xi, \eta) \) into a union of open balls \( B(\xi_j, \delta_j) \) so that \( \mu_u(B(\xi_j, \delta_j)) < \delta_0 \) and \( \sum_j \mu_u(B(\xi_j, \delta_j)) \leq 2\mu_uB(\xi, \delta) \). Then \( Q(\xi_j, \delta_j) \subset N \) and \( Q(\xi, \delta) \cap N = \bigcup_j Q(\xi_j, \delta_j) \). Hence,

\[
\mu_2(Q(\xi, \delta)) = \mu(Q(\xi, \delta) \cap N) \leq \sum_j \mu_u(Q(\xi_j, \delta_j)) \\
\leq 2\epsilon \mu_uB(\xi_j, \delta_j) \leq 4\epsilon \mu_uB(\xi, \delta).
\]

Therefore, \( \mu_2(Q(\xi, \delta)) \leq 4\epsilon \mu_uB(\xi, \delta) \) for all \( B(\xi, \delta) \) and thus in the Eq. (3) the first integral can be made arbitrarily small by choosing \( n \) sufficiently large and for the second integral we have that
\[ \int_{B^p} |f_n - f|^p \, d\mu_2 \leq C \varepsilon \| f_n - f \|_p \]

for some constant \( C \) by the previous theorem so \( \varepsilon \) can be chosen arbitrarily small and the result follows. \( \Box \)

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