Trapped submanifolds in Lorentzian geometry

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Abstract

In Lorentzian geometry, the concept of trapped submanifold is introduced by means of the mean curvature vector properties. Trapped submanifolds are generalizations of the standard maximal hypersurfaces and minimal surfaces, of geodesics, and also of the trapped surfaces introduced by Penrose. Selected applications to gravitational theories are mentioned.

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1 Introduction

The concept of closed trapped surface, first introduced by Penrose [1], is extremely useful in many physical problems and mathematical developments, with truly versatile applications. It was a cornerstone for the achievement of the singularity theorems, the analysis of gravitational collapse, the study of the cosmic censorship hypothesis, or the numerical evolution of initial data, just to mention a few, see e.g. [1, 2, 3] (a more complete list of references can be found in [4].) Trapped surfaces are usually introduced as co-dimension 2 imbedded spatial surfaces such that all its local portions have, at least initially, a decreasing (increasing) area along any future evolution direction. However, it has been seldom recognized that the concept of trapped surface is genuinely and purely geometric, closely related to the traditional concepts of geodesics, minimal surfaces and variations of submanifolds. The purpose of this short note is to present this novel view, which may be clarifying for, and perhaps arouse interest of, the mathematical community.
2 Basics on semi-Riemannian submanifolds.

Let \((V, g)\) be any \(D\)-dimensional semi-Riemannian manifold with metric tensor \(g\) of any signature. An imbedded submanifold is a pair \((S, \Phi)\) where \(S\) is a \(d\)-dimensional manifold on its own and \(\Phi : S \to V\) is an imbedding [5]. As is customary in mathematical physics, for the sake of brevity \(S\) will be identified with its image \(\Phi(S)\) in \(V\). \(D - d\) is called the co-dimension of \(S\) in \(V\).

At any \(p \in \Phi(S)\) one has the decomposition of the tangent space
\[
T_p V = T_p S \oplus T_p S^\perp
\]
if and only if the inherited metric (or first fundamental form) \(\Phi^* g \equiv \gamma\) is non-degenerate at \(p\). Henceforth, I shall assume that \(\gamma\) is non-degenerate everywhere. Let us note in passing that \(\Phi(S)\) is called spacelike if \(\gamma\) is also positive definite. Thus, \(\forall p \in S, \forall \vec{v} \in T_p V\) we have \(\vec{v} = \vec{v}^T + \vec{v}^\perp\) which are called the tangent and normal parts of \(\vec{v}\) relative to \(S\).

Obviously, \((S, \gamma)\) is a semi-Riemannian manifold on its own, and its intrinsic structure as such is inherited from \((V, g)\). However, \((S, \gamma)\) inherits also extrinsic properties. Important inherited intrinsic objects are (i) the canonical volume element \(d\)-form \(\eta_S\) associated to \(\gamma\); (ii) a Levi-Civita connection \(\nabla\) such that \(\nabla \gamma = 0\). An equivalent interesting characterization is
\[
\forall \vec{x}, \vec{y} \in T_S, \quad \nabla_{\vec{x}} \vec{y} = (\nabla_{\vec{x}} \vec{y})^T
\]
(1)

(where \(\nabla\) is the connection on \((V, g)\)); and (iii) of course, the curvature of \(\nabla\) and all derived objects thereof.

Concerning the extrinsic structure, the basic object is the shape tensor \(K : TS \times TS \to TS^\perp\), also called extrinsic curvature of \(S\) in \(V\), defined by
\[
\forall \vec{x}, \vec{y} \in T_S, \quad K(\vec{x}, \vec{y}) = - (\nabla_{\vec{x}} \vec{y})^\perp.
\]
(2)
The combination of (1) and (2) provides
\[
\forall \vec{x}, \vec{y} \in TS, \quad \nabla_{\vec{x}} \vec{y} = \nabla_{\vec{x}} \vec{y} - K(\vec{x}, \vec{y}).
\]

An equivalent way of expressing the same is
\[
\forall \omega \in T^* S, \quad \Phi^* (\nabla \omega) = \nabla (\Phi^* \omega) + \omega(K)
\]
where by definition \(\omega(K)(\vec{x}, \vec{y}) = \omega(\vec{y})\) for all \(\vec{x}, \vec{y} \in TS\).

The shape tensor contains the information concerning the “shape” of \(\Phi(S)\) within \(V\) along all directions normal to \(\Phi(S)\). Observe that \(K(\vec{x}, \vec{y}) \in \)
If one chooses a particular normal direction \( \vec{n} \in TS^\perp \), then one defines a 2-covariant symmetric tensor field \( K_{\vec{n}} \in T_{(0,2)}S \) by means of

\[
K_{\vec{n}}(\vec{x}, \vec{y}) = n(K)(\vec{x}, \vec{y}) = g(\vec{n}, K(\vec{x}, \vec{y})), \quad \forall \vec{x}, \vec{y} \in TS
\]

which is called the second fundamental form of \( S \) in \((V, g)\) relative to \( \vec{n} \).

### 3 The mean curvature vector.

The main object to be used in this contribution is the mean curvature vector \( \vec{H} \) of \( S \) in \((V, g)\). This is an averaged version of the shape tensor defined by

\[
\vec{H} = \text{tr} \, K, \quad \vec{H} \in TS^\perp
\]

where the trace \( \text{tr} \) is taken with respect to \( \gamma \), of course. Each component of \( \vec{H} \) along a particular normal direction, that is to say, \( g(\vec{H}, \vec{n}) (= \text{tr} \, K_{\vec{n}}) \) is termed “expansion along \( \vec{n} \)” in some physical applications.

The classical interpretation of \( \vec{H} \) can be understood as follows. Let us start with the simplest case \( d = 1 \), so that \( S \) is a curve in \( V \). Then there is only one independent tangent vector, say \( \vec{x} \), and \( (\nabla_{\vec{x}} \vec{x})^\perp = -K = -\vec{H} \) is simply (minus) the proper acceleration vector of the curve. In other words, \( S \) is a geodesic if and only if \( K = 0 \) (equivalently in this case, \( \vec{H} = \vec{0} \)). Hence, an immediate and standard generalization of a geodesic to arbitrary codimension \( d \) is: “\( S \) is totally geodesic if and only if \( K = 0 \)”. Totally geodesic submanifolds are those such that all geodesics within \((S, \gamma)\) are geodesics on \((V, g)\).

Nevertheless, one can also generalize the concept of geodesic to arbitrary \( d \) by assuming just that \( \vec{H} = \vec{0} \). To grasp the meaning of this condition, let us first consider the opposite extreme case: \( d = D - 1 \) or codimension 1. Then, \( S \) is a hypersurface and there exists only one independent normal direction, say \( \vec{n} \), so that necessarily \( \vec{H} = \theta \vec{n} \) where \( \theta \) is the (only) expansion, or divergence. Classical results imply that the vanishing of \( \vec{H} \) (ergo \( \theta = 0 \)) defines the situation where there is no local variation of volume along the normal direction. Actually, this interpretation remains valid for arbitrary \( d \). Indeed, let \( \xi \) be an arbitrary \( C^1 \) vector field on \( V \) defined on a neighbourhood of \( S \), and let \( \{\varphi_\tau\}_{\tau \in I} \) be its flow, that is its local one-parameter group of local transformations, where \( \tau \) is the canonical parameter and \( I \ni 0 \) is a real interval. This defines a one-parameter family of surfaces \( S_\tau \equiv \varphi_\tau(S) \) in \( V \), with corresponding imbeddings \( \Phi_\tau : S \to V \) given by \( \Phi_\tau = \varphi_\tau \circ \Phi \). Observe that \( S_0 = S \). Denoting by \( \eta_{S_\tau} \) their associated canonical volume element
$d$-forms, it is a matter of simple calculation to get
\[
\frac{d\eta_{S_\tau}}{d\tau} \bigg|_{\tau=0} = \frac{1}{2} \text{tr} \left[ \Phi^* (\mathcal{L}_\xi g) \right] \eta_S
\]
where $\mathcal{L}_\xi$ is the Lie derivative with respect to $\xi$. Another straightforward computation using the standard formulae relating the connections on $\nabla$ and $\nabla$ leads to
\[
\frac{1}{2} \text{tr} \left[ \Phi^* (\mathcal{L}_\xi g) \right] = \text{div}(\varphi^* \xi) + g(\xi, \bar{H})
\]
where $\text{div}$ is the divergence operator on $S$. Combining the two previous formulas one readily gets the expression for the variation of $d$-volume:
\[
\frac{dV_{S_\tau}}{d\tau} \bigg|_{\tau=0} = \int_S \left( \text{div} \xi + g(\xi, \bar{H}) \right) \eta_S
\]
where $V_{S_\tau} = \int_{S_\tau} \eta_{S_\tau}$ is the volume of $S_\tau$. In summary:

Among the set of all submanifolds without boundary (or with a fixed boundary under appropriate restrictions) those of extremal volume must have $\bar{H} = 0$.

4 Lorentzian case. Future-trapped submanifolds.

If $(V, g)$ is a proper Riemannian manifold, then $g(\bar{H}, \bar{H}) \geq 0$ and the only distinguished case is $g(\bar{H}, \bar{H}) = 0$ which is equivalent to $\bar{H} = \bar{0}$: a extremal submanifold. However, in general semi-Riemannian manifolds $g(\bar{H}, \bar{H})$ can be also negative, as well as zero with non-vanishing $\bar{H}$. Thus, new possibilities and distinguished cases arise.

To fix ideas, let us concentrate in the physically relevant case of a Lorentzian manifold $(V, g)$ with signature $(-, +, \ldots, +)$. Let $(S, \gamma)$ be spacelike. Then, $\bar{H}$ can be classified according to its causal character:

\[
g(\bar{H}, \bar{H}) = \begin{cases} 
> 0 & \bar{H} \text{ is spacelike} \\
= 0 & \bar{H} \text{ is null (or zero)} \\
< 0 & \bar{H} \text{ is timelike}
\end{cases}
\]

Of course, this sign can change from point to point of $S$. Recall that non-spacelike vectors can be subdivided into future- and past-pointing. Hence, $S$ can be classified as (omitting past duals) $4$ $6$:

1. future trapped if $\bar{H}$ is timelike and future-pointing all over $S$.  

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2. **nearly future trapped** if \( \vec{H} \) is non-spacelike and future-pointing all over \( S \), and timelike at least at a point of \( S \).

3. **marginally future trapped** if \( \vec{H} \) is null and future-pointing all over \( S \), and non-zero at least at a point of \( S \).

4. **extremal or symmetric** if \( \vec{H} = \vec{0} \) all over \( S \).

5. **absolutely non-trapped** if \( \vec{H} \) is spacelike all over \( S \).

The original definition of “closed trapped surface”, which is of paramount importance in General Relativity \((D = 4)\), is due to Penrose [1, 2, 3] and was for codimension two, in which case points 1, 4 and 5 coincide with the standard nomenclature; point 2 was coined in [4], while 3 is more general than the standard concept in GR (e.g. [2, 3]) — still, all standard marginally trapped \((D - 2)\)-surfaces are included in 3 —. On the other hand, the above terminology is unusual for the cases \( d = D - 1 \) or \( d = 1 \), see [4, 6] for explanations.

5 Applications

One of the advantages of having defined trapped submanifolds via \( \vec{H} \) is — apart from being generalizable to arbitrary codimension and thereby comparable with well-known cases such as maximal hypersurfaces and geodesics — that many simple results and applications can be derived. As an example, let us consider the case in which \( \vec{\xi} \) is a conformal Killing vector \( \mathcal{L}_{\vec{\xi}}g = 2\Psi g \) (including the particular cases of homotheties \((\Psi = \text{const.})\) and proper Killing vectors \((\Psi = 0)\)). Then formula (3) specializes to \( \Psi d = \text{div}(\varphi^*\vec{\xi}) + g(\vec{\xi}, \vec{H}) \) so that, integrating over any closed \( S \) (i.e. compact without boundary) we get

\[
\int_S \Psi \eta_S = \frac{1}{d} \int_S g(\vec{\xi}, \vec{H}) \eta_S.
\]

Therefore, if \( \Psi|_S \) has a sign, then \( g(\vec{\xi}, \vec{H}) \) must have the same sign, clearly restricting the possibility of \( \vec{H} \) being non-spacelike. For instance, if \( \vec{\xi} \) is timelike, then \( \vec{H} \) (if non-spacelike) must be oppositely directed to \( \text{sign}(\Psi|_S)\vec{\xi} \); in particular, if \( \Psi = 0 \), then there cannot be closed (nearly, marginally) trapped submanifolds at all [4, 6]. Analogously, if \( \vec{\xi} \) is null on \( S \) and \( \Psi|_S = 0 \), then the only possibility for a non-spacelike \( \vec{H} \) is that the mean curvature vector be null and proportional to \( \vec{\xi} \).

Specific consequences of the above are, for example, [4, 6, 7].
• that in Robertson-Walker spacetimes (where there is a conformal Killing vector), closed spacelike geodesics are forbidden (!), and closed submanifolds can only be past-trapped if the model is expanding [4, 6]; furthermore, there cannot be maximal closed hypersurfaces, nor minimal surfaces [4, 6].

• in stationary regions of \((V, g)\), any marginally trapped, nearly trapped, or trapped submanifold is necessarily non-closed and non-orthogonal to the timelike Killing vector [4, 6].

• in regions with a null Killing vector \(\vec{\xi}\), all trapped or nearly trapped submanifolds must be non-closed and non-orthogonal to \(\vec{\xi}\), and any marginally trapped submanifold must have a mean curvature vector parallel (and orthogonal!) to the null Killing vector.

• the impossibility of existence of closed trapped surfaces (co-dimension 2) in spacetimes (arbitrary dimension) with vanishing curvature invariants [7]. This includes, in particular, the case of pp-waves [4, 6, 7]. This has applications to modern string theories, implying that the spacetimes with vanishing curvature invariants, which are in particular exact solutions of the full non-linear theory, do not possess any horizons.

More details and applications can be found in [4, 6, 7, 8].

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