ROAST: Rapid Orthogonal Approximate Slepian Transform

Zhihui Zhu, Santhosh Karnik, Michael B. Wakin, Mark A. Davenport, Justin Romberg

September 12, 2018

Abstract

In this paper, we provide a Rapid Orthogonal Approximate Slepian Transform (ROAST) for the discrete vector that one obtains when collecting a finite set of uniform samples from a baseband analog signal. The ROAST offers an orthogonal projection which is an approximation to the orthogonal projection onto the leading discrete prolate spheroidal sequence (DPSS) vectors (also known as Slepian basis vectors). As such, the ROAST is guaranteed to accurately and compactly represent not only oversampled bandlimited signals but also the leading DPSS vectors themselves. Moreover, the subspace angle between the ROAST subspace and the corresponding DPSS subspace can be made arbitrarily small. The complexity of computing the representation of a signal using the ROAST is comparable to the FFT, which is much less than the complexity of using the DPSS basis vectors. We also give non-asymptotic results to guarantee that the proposed basis not only provides a very high degree of approximation accuracy in a mean squared error sense for bandlimited sample vectors, but also that it can provide high-quality approximations of all sampled sinusoids within the band of interest.

1 Introduction

The Nyquist-Shannon sampling theorem guarantees that real world signals that are bandlimited (or can be made bandlimited by filtering) can be replaced by a discrete sequence of their samples without the loss of any information. These samples can then be processed digitally. In particular, the discrete Fourier transform (DFT) for digital signals has been widely used for many applications in engineering, mathematics, and science thanks to the fast Fourier transform (FFT), an efficient algorithm for computing the DFT.

Due to the fact that finite windowing in the time domain will spread out a signal’s spectrum in the frequency domain, however, the DFT suffers from frequency leakage when used to represent a finite-length vector arising from a bandlimited signal with a narrowband spectrum, or even a pure sinusoid. This problem can be mitigated to some degree by applying a smooth windowing function in the sampling system. Alternatively, one can compactly represent the signals using a basis of timelimited discrete prolate spheroidal sequences (DPSS’s). DPSS’s, first introduced by Slepian in 1978 [2], are a collection of orthogonal bandlimited sequences that are most concentrated in time to a given index range. When limited in the time domain, they provide a compact (and again orthogonal) representation for sampled bandlimited signals.

Owing to their concentration in the time and frequency domains, the DPSS’s have been successfully used in numerous signal processing applications. For instance, DPSS’s can be applied to find the minimum energy, infinite-length bandlimited sequence that extrapolates a given finite timelimited vector of samples [2]; bandlimited extrapolation is a classical signal processing problem and appears in applications such as spectral estimation and image processing [3,4]. Another problem involves estimating time-varying
channels in wireless communication systems. In [5,6], Zemen et al. showed that expressing the time-varying subcarrier coefficients with a DPSS basis yields better estimates than those obtained with a DFT basis, which suffers from frequency leakage. In through-the-wall radar imaging using stepped-frequency synthetic aperture radar (SAR) [7], the DPSS basis can be utilized for efficiently mitigating wall clutter and for detecting targets behind the wall [8–10]. In addition, DPSS’s are useful for multiband signal identification [11] and narrowband and multiband signal recovery from compressive measurements [12, 13]. Building on this, DPSS’s have been used to enable compressive sensing of physiological signals [14]. More broadly, the ability to recover multiband signals is beneficial for developing high-bandwidth radio receivers for cognitive radio and communications intelligence [15].

Unfortunately, unlike the DFT which can be computed efficiently with the FFT algorithm, there exists no algorithm that can efficiently compute the DPSS representation for a very large signal. Recently, we proposed [16] a fast Slepian transform (FST), a fast method for computing approximate projections onto the leading DPSS vectors and compressing a signal to the corresponding low dimension. Despite its favorable properties, the fast algorithm presented in [16] did not correspond to an orthogonal projection. In this paper, we illustrate an alternative orthonormal basis that provides an approximate but sufficiently accurate representation of the subspace spanned by the leading DPSS vectors and compactly captures most of the energy in oversampled bandlimited signals. The representation of an arbitrary vector in this basis can again be computed efficiently (with complexity comparable to that of the FFT), and we refer to this procedure as the Rapid Orthogonal Approximate Slepian Transform (ROAST).

One of the main contributions of this paper is to confirm that such an orthonormal basis not only provides a very high degree of approximation accuracy in a mean squared error (MSE) sense for baseband sample vectors, but also that it can provide high-quality approximations for all sample vectors of sinusoids with frequencies in the band of interest. After Section 1.1 provides background on DPSS’s, Section 1.2 provides details on the ROAST construction, fast computations, and theoretical approximation guarantees. The orthogonality of this transform also extends its relevance to new applications, as we describe in Section 1.3. Section 2 contains proofs of the main results. Experiments in Section 3 confirm that ROAST offers signal approximation quality that is comparable to the DPSS, but with a much lower computational burden.

1.1 DPSS bases

To begin, we briefly review some important definitions and properties of DPSS’s.

1.1.1 Definitions

For any \( W \in (0, \frac{1}{2}) \), let \( B_W : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) \) denote a bandlimiting operator that bandlimits the discrete-time Fourier transform (DTFT) of a discrete-time signal to the frequency range \([-W, W]\) (and returns the corresponding signal in the time domain). In addition, for any \( N \in \mathbb{N} \), let \( T_N : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) \) denote the timelimiting operator that zeros out all entries outside the index range \( \{0, 1, \ldots, N-1\} \).

**Definition 1.** (DPSS’s [2]) Given \( W \in (0, \frac{1}{2}) \) and \( N \in \mathbb{N} \), the Discrete Prolate Spheroidal Sequences (DPSS’s) \( \{s_{N,W}^{(0)}, s_{N,W}^{(1)}, \ldots, s_{N,W}^{(N-1)}\} \) are real-valued discrete-time sequences that satisfy \( B_W (T_N(s_{N,W}^{(l)})) = \lambda_{N,W}^{(l)} s_{N,W}^{(l)} \) for all \( l \in \{0, \ldots, N-1\} \). Here \( \lambda_{N,W}^{(0)}, \ldots, \lambda_{N,W}^{(N-1)} \) are the eigenvalues of the operator \( B_{[-W,W]} T_N \) with order \( 1 > \lambda_{N,W}^{(0)} > \lambda_{N,W}^{(1)} > \cdots > \lambda_{N,W}^{(N-1)} > 0 \).

**Definition 2.** (DPSS vectors [2]) Given \( W \in (0, \frac{1}{2}) \) and \( N \in \mathbb{N} \), the DPSS vectors\(^1\) \( s_{N,W}^{(0)}, s_{N,W}^{(1)}, \ldots, s_{N,W}^{(N-1)} \in \mathbb{R}^N \) are defined by limiting the DPSS’s to the index range \( \{0, 1, \ldots, N-1\} \) and satisfy

\[
B_{N,W} s_{N,W}^{(l)} = \lambda_{N,W}^{(l)} s_{N,W}^{(l)},
\]

\(^1\)Throughout the paper, finite-dimensional vectors and matrices are indicated by bold characters, while the other variables such as infinite-length sequences are not in bold typeface.
Theorem 1. (Concentration of the spectrum \([2, 13, 16, 17]\).) For any \(n\)als.
\[\lambda \begin{bmatrix} \ldots & 1 \end{bmatrix}, \ldots, N - 1 \end{bmatrix}\) and \(\Lambda_{N,W}\) be an \(N \times N\) diagonal matrix with diagonal entries being the DPSS eigenvalues \(\lambda_{N,W}^{(0)}, \ldots, \lambda_{N,W}^{(N-1)}\). The prolate matrix \(B_{N,W}\) can be factorized as
\[B_{N,W} = S_{N,W} \Lambda_{N,W} S_{N,W}^{*},\]
which is an eigendecomposition of \(B_{N,W}\). Here \(A^{*}\) represents the adjoint of \(A\). The DPSS’s are orthogonal on \(Z\) and on \(\{0, \ldots, N - 1\}\), and they are normalized so that
\[\langle T_{N} (s_{N,W}^{(k)}), T_{N} (s_{N,W}^{(l)}) \rangle = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}\]

Consequently, it can be shown \([2]\) that \(\|s_{N,W}^{(l)}\|_{2}^{2} = \frac{1}{\lambda_{N,W}^{(l)}}\). Thus, when \(\lambda_{N,W}^{(l)}\) is close to 1, the corresponding DPSS vector \(s_{N,W}^{(l)}\) has energy mostly concentrated in the frequency range \([-W, W]\). On the other hand when \(\lambda_{N,W}^{(l)}\) is close to 0, the corresponding DPSS vector \(s_{N,W}^{(l)}\) has most of its energy outside the frequency range \([-W, W]\). These properties, along with the following result on the distribution of the eigenvalues \(\lambda_{N,W}^{(l)}\), make the DPSS’s a suitable basis to provide a compact representation for sampled bandlimited signals.

**Theorem 1.** (Concentration of the spectrum \([2, 13, 16, 17]\).) For any \(W \in (0, \frac{1}{2})\), \(N \in \mathbb{N}\), and \(\epsilon \in (0, \frac{1}{2})\), we have
\[\lambda_{N,W}^{([2NW] - 1)} \geq \frac{1}{2} \geq \lambda_{N,W}^{([2NW])}\]

and
\[\#\{\epsilon \leq \lambda_{N,W}^{(l)} \leq 1 - \epsilon\} \leq 2C_{N} \log \left(\frac{15}{\epsilon}\right),\]
where \(C_{N} = \frac{1}{\pi} \log(8N) + 6\).

Here \([a]\) denotes the largest integer that is not greater than \(a\) and \([a]\) denotes the smallest integer that is not smaller than \(a\). Theorem 1 implies that the first \(\approx 2NW\) eigenvalues tend to cluster very close to 1, while the remaining eigenvalues tend to cluster very close to 0, after a narrow transition of width \(O(\log(N) \log(\frac{1}{\epsilon}))\).

### 1.1.2 Representations of sampled sinusoids and oversampled bandlimited signals

Define
\[e_{f} := \begin{bmatrix} e^{2\pi f_{0}} \\ e^{2\pi f_{1}} \\ \vdots \\ e^{2\pi f_{N-1}} \end{bmatrix} \in \mathbb{C}^{N}\]
for all \(f \in [-\frac{1}{2}, \frac{1}{2}]\) as the sampled complex exponentials. For any integer \(K \in \{1, 2, \ldots, N\}\), let \(S_{K} := (S_{N,W})_{K}\) denote the \(N \times K\) matrix formed by taking the first \(K\) DPSS vectors (where \(N\) and \(W\) are clear from the context and typically \(K \approx 2NW\)). Note that for any orthonormal matrix \(Q \in \mathbb{C}^{N \times K}\),
\[
\int_{-W}^{W} \|e_{f} - QQ^{*}e_{f}\|_{2}^{2} df = \int_{-W}^{W} \text{trace} \left( (e_{f}e_{f}^{*} - QQ^{*}e_{f}e_{f}^{*}) df \right) = \text{trace} \left( B_{N,W} - QQ^{*}B_{N,W} \right). \tag{1}
\]
For any value of $K$, the quantity in (1) is minimized by the choice of $Q = S_K$. This implies that $S_K$ is the best basis of $K$ columns to represent (in an MSE sense) the collection of sampled sinusoids $\{e_f\}_{f \in [-W, W]}$. Formally,

$$\int_{-W}^{W} \|e_f - S_K S_K^* e_f\|_2^2 df = \sum_{i=K}^{N-1} \lambda_{N,W}^{(i)},$$

where for each $f \in [-W, W]$, $\|e_f\|_2^2 = N$. It follows from Theorem 1 that $S_K$ provides very accurate approximations (in an MSE sense) for all sampled sinusoids $\{e_f\}_{f \in [-W, W]}$ if one chooses $K$ slightly larger than $2NW$. We note that this efficiency is in contrast to the DFT, where certain “on-grid” sinusoids (those whose frequencies are harmonic multiples of $1/N$) can be represented using just one DFT basis vector, but all other “off-grid” sinusoids require $O(N)$ DFT basis vectors due to frequency leakage.

We note that any representation guarantee for sampled sinusoids $\{e_f\}_{f \in [-W, W]}$ can also be used for finite-length sample vectors arising from sampling random bandlimited baseband signals. Suppose $x$ is a continuous-time, zero-mean, wide sense stationary random process with power spectrum

$$P_x(F) = \begin{cases} \frac{1}{B_{\text{band}}}, & F \in \left[-\frac{B_{\text{band}}}{2}, \frac{B_{\text{band}}}{2}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Let $x = [x(0) \ x(T_s) \ \cdots \ x((N-1)T_s)]^T \in \mathbb{C}^N$ denote a finite vector of samples acquired from $x(t)$ with a sampling interval of $T_s \leq 1/B_{\text{band}}$. Let $f_c = F_c T_s$ and $W = \frac{B_{\text{band}}T_s}{2}$. We have [13]

$$\mathbb{E} \left[ \|x - QQ^* x\|_2^2 \right] = \frac{1}{2W} \int_{-W}^{W} \|e_f - QQ^* e_f\|_2^2 df.$$  

Finally, let $F_{N,W}$ denote the partial normalized DFT matrix with the lowest $2\lfloor NW \rfloor + 1$ frequency DFT vectors of length $N$, i.e.,

$$F_{N,W} = \frac{1}{\sqrt{N}} \begin{bmatrix} e_{-\lfloor NW \rfloor /N} & \cdots & e_{\lfloor NW \rfloor /N} \end{bmatrix}.$$  

It follows that $F_{N,W} F_{N,W}^*$ is an orthogonal projector onto the column space of $F_{N,W}$. The following result states that the difference between the prolate matrix $B_{N,W}$ and $F_{N,W} F_{N,W}^*$ is effectively low rank.

**Theorem 2.** [16] Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Then for any $\epsilon \in (0, \frac{1}{2})$, there exist $N \times N$ matrices $L$ and $E$ such that

$$B_{N,W} = F_{N,W} F_{N,W}^* + L + E,$$

where

$$\text{rank}(L) \leq C_N \log \left( \frac{15}{\epsilon} \right), \quad \|E\| \leq \epsilon.$$  

Here $C_N$ is the constant specified in Theorem 1.

This result is a key factor in fast computing an approximate Slepian transform in [16] and will play an important role in the construction of the ROAST, which can be used for computing fast orthogonal approximations of sampled sinusoids and bandlimited signals.

### 1.2 ROAST: Rapid Orthogonal Approximate Slepian Transform

#### 1.2.1 Construction and relation to the DPSS subspace

In [16], we demonstrated a fast method to approximately project an arbitrary vector onto the subspace spanned by the first slightly more than $2NW$ eigenvectors of $B_{N,W}$ (i.e., the DPSS vectors) by utilizing the fact that the difference between $B_{N,W}$ and $F_{N,W} F_{N,W}^*$ approximately has a rank of $O(\log N)$ (see Theorem 2). Note that, in [16], the approximate projection is not a true orthogonal projection onto any
subspace. Here, we exhibit a subspace that captures most of the energy in the first $2NW$ DPSS vectors (and also the energy in sampled sinusoids within the band of interest), and this subspace has an orthogonal projector that can be applied efficiently to an arbitrary vector. By utilizing the result that $B_{N,W} - F_{N,W} F_{N,W}^*$ is approximately low rank and also that $F_{N,W}$ can be applied to a vector efficiently with the FFT, we build an orthonormal basis for our subspace by concatenating $F_{N,W}$ with a certain matrix $Q'$ as follows:

$$Q = [F_{N,W} \quad Q'],$$

where $Q'$ is an $N \times R$ (for some $R$ that we can choose as desired) orthonormal matrix that is also orthogonal to $F_{N,W}$. Let $\mathbf{F}_{N,W}$ denote the $N \times (N - 2\lfloor NW \rfloor - 1)$ matrix with the highest frequency $N - 2\lfloor NW \rfloor - 1$ DFT vectors of length $N$. Thus $F_N := [F_{N,W} \quad \mathbf{F}_{N,W}]$ is the normalized $N \times N$ DFT matrix. Since $Q'$ must be orthogonal to $F_{N,W}$ and the columns of $Q'$ must be orthonormal, we can write $Q'$ as $Q' = \overline{F}_{N,W} V$, for some $V \in \mathbb{C}^{(N - 2\lfloor NW \rfloor - 1) \times R}$ that is orthonormal (one can verify that $F_{N,W}^* Q' = 0$ and $(Q')^* Q' = I$). Thus, the desired orthogonal approximate Slepian basis is given as

$$Q = [F_{N,W} \quad \overline{F}_{N,W} V], \quad V^T V = I. \quad (4)$$

The optimal $V$ is chosen such that the subspace spanned by $Q$ captures the important DPSS vectors. (Since all the DPSS vectors $s_{N,W}^{(0)}, \ldots, s_{N,W}^{(N-1)}$ form an orthonormal basis for $\mathbb{C}^N$, no subspace of $\mathbb{C}^N$ can capture all of them except $\mathbb{C}^N$ itself.) To illustrate how we obtain $V$, consider the following weighted least squares problem

$$\text{minimize}_{Q} \varrho(Q) := \sum_{\ell=0}^{N-1} \lambda_{N,W}^{(\ell)} \left\| s_{N,W}^{(\ell)} - Q Q^* s_{N,W}^{(\ell)} \right\|_2^2. \quad (5)$$

Here we use the DPSS eigenvalue $\lambda_{N,W}^{(\ell)}$ to weight the energy in the DPSS vector $s_{N,W}^{(\ell)}$ that is not captured by $Q$. The reason is that the larger the DPSS eigenvalue, the more concentration the corresponding DPSS vector has in the frequency domain, implying that the DPSS vector is more important in practical applications such as representing sampled bandlimited signals (see (2)). To solve (5), we rewrite $\varrho(Q)$ as

$$\varrho(Q) = \text{trace} \left( \sum_{\ell=0}^{N-1} \lambda_{N,W}^{(\ell)} s_{N,W}^{(\ell)} (s_{N,W}^{(\ell)})^T - QQ^* \lambda_{N,W}^{(\ell)} s_{N,W}^{(\ell)} (s_{N,W}^{(\ell)})^T \right)$$

$$= \text{trace} (B_{N,W} - QQ^* B_{N,W})$$

$$= \int_{-W}^{W} \| e_f - QQ^* e_f \|_2^2 df,$$

where the last line follows from (1). In other words, an orthonormal basis $Q$ obtained by minimizing $\varrho(Q)$ is also an optimal basis to represent sampled sinusoids (and thus also certain bandlimited signals) in the MSE sense.

Plugging $Q = [F_{N,W} \quad \mathbf{F}_{N,W} V]$ into the above equation yields

$$\varrho(Q) = \text{trace}(\mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W} - V V^* \mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W}),$$

which suggests that setting $V$ equal to the $R$ dominant left singular vectors of $\mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W}$ results in a relatively small representation residual $\varrho(Q)$ as long as $\mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W}$ has an effective rank of $R$. In fact, we find that certain numerical issues can be avoided by adopting the $R$ dominant left singular vectors of $\mathbf{F}_{N,W}^* B_{N,W}$ (rather than $\mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W}$), and that the same strong theoretical guarantees can be established for this construction. The following result provides such a guarantee for the standard ROAST construction involving the singular vectors of $\mathbf{F}_{N,W}^* B_{N,W}$; we briefly revisit the idea of a constructing involving the singular vectors of $\mathbf{F}_{N,W}^* B_{N,W} \overline{F}_{N,W}$ in Section 2.3.

5
Theorem 3. (Representation guarantee for DPSS vectors) Fix $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$. For any $\epsilon \in (0, \frac{1}{2})$, fix $K$ to be such that $\lambda^{(K-1)}_{N,W} \geq \epsilon$ and set $R = \lfloor C_N \log (15/\epsilon) \rfloor$, where $C_N$ is the constant specified in Theorem 1. Then the orthonormal basis $Q = [F_{N,W} \ F_{N,W} V]$ with $V \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1) \times R}$ containing the $R$ dominant left singular vectors of $F_{N,W} B_{N,W}$ satisfies

$$\|S_K S_K^* - QQ^* S_K S_K^*\|_2^2 \leq \epsilon,$$

$$\|s^{(f)}_{N,W} - QQ^* s^{(f)}_{N,W}\|_2^2 \leq \epsilon,$$

for all $f = 0, 1, \ldots, K - 1$. By slightly increasing $R$ to $R = \lfloor C_N \log (15N/\epsilon) \rfloor$, the subspace angle $\Theta_{S_K, Q}$ between the columns spaces of $S_K$ and $Q$ satisfies

$$\cos (\Theta_{S_K, Q}) \geq \sqrt{1 - \epsilon}.$$

The formal definition of (the largest principal) angle between two subspaces is given in Definition 3. Informally, if the subspace angle $\Theta$ is small, the two subspaces are nearly linearly dependent and one subspace is almost “contained” in the other subspace. Here, to guarantee that the column space of $S_K$ is almost “contained” in the column space of $Q$, one can take $\Theta_{S_K, Q}$ arbitrary small by increasing $R$. However, we note that we are not guaranteed that $\|QQ^* - S_K S_K^*\|$ is small since in general $\|QQ^* - S_K S_K^*\|_2 = 1$ if $Q$ and $S_K$ have a different number of columns. Instead, we are guaranteed that the subspace spanned by the columns of $S_K$ is approximately within the column space of $Q$ and the angle between the two subspaces is small by Theorem 3. We also note that the bound on $\|S_K S_K^* - QQ^* S_K S_K^*\|$ is useful since for any vector $a \in \mathbb{C}^N$

$$\|a - QQ^* a\|_2 \leq \|a - S_K S_K^* a\|_2 + \|S_K S_K^* - QQ^* S_K S_K^*\|_2 \|a\|_2 \leq \|a - S_K S_K^* a\|_2 + \sqrt{\epsilon} \|a\|_2,$$

which implies any representation guarantee for $S_K$ can be utilized for $Q$.

1.2.2 Representations of sampled sinusoids and oversampled bandlimited signals

As illustrated in (6), the orthonormal matrix obtained by minimizing $\varrho(Q)$ is also expected to accurately represent sampled sinusoids within the band of interest in the MSE sense. This is formally established in the following results.

Theorem 4. (Average representation error) Fix $W \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$. For any $\epsilon \in (0, \frac{1}{2})$, set

$$R = \max \left\{ \left\lfloor C_N \log \left( \frac{15C_N}{N\epsilon} \right) \right\rfloor + 1, 0 \right\},$$

where $C_N$ is the constant specified in Theorem 1. Then the orthonormal basis $Q = [F_{N,W} \ F_{N,W} V]$ with $V \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1) \times R}$ containing the $R$ dominant left singular vectors of $F_{N,W} B_{N,W}$ satisfies

$$\int_{-W}^{W} \|e_f - QQ^* e_f\|^2 \|e_f\|^2 \mathbb{d}f \leq \epsilon$$

A similar approximation guarantee can be established for vectors arising from sampling random bandlimited signals by using (3).

---

2Here the first inequality holds because $QQ^* a$ is the orthogonal projection of $a$ onto $\text{Span}(Q)$ (the column space of $Q$) and thus is closest to $a$ among all points in $\text{Span}(Q)$, in which $QQ^* S_K S_K^* a$ also lies.
In [17], we rigorously show that every discrete-time sinusoid with a frequency \( f \in [-W, W] \) is well-approximated by the DPSS basis \( S_K \) when \( K \) is slightly larger than \( 2NW \). The proof is based on an asymptotic result on the DTFT of the DPSS basis functions (which are known as discrete prolate spheroidal wave functions (DPSWF’s)) and the result is thus asymptotic. Here we use a different approach to obtain a non-asymptotic guarantee for approximating every discrete-time sinusoid with a frequency \( f \in [-W, W] \). Noting that \( \|e_f - QQ^* e_f\|_2^2 \) is differentiable everywhere, we first show that its derivative is bounded above by \( 2\pi N^2 \). Then by utilizing the previous result on \( \int_{-W}^W \|e_f - QQ^* e_f\|_2^2 df \), one obtains a similar bound on \( \|e_f - QQ^* e_f\|_2^2 \).

**Theorem 5.** (Representation guarantee for pure sinusoids) Let \( N \in \mathbb{N} \) and \( W \in (0, \frac{1}{2}) \) be given such that \( W \geq \frac{1}{4\pi N} \). For any \( \epsilon \in (0, \frac{1}{2}) \), set

\[
R = \max \left( \left\lfloor C_N \log \left( \frac{60\pi C_N}{\epsilon^2} \right) \right\rfloor + 1, \left\lfloor C_N \log \left( \frac{15C_N}{NW\epsilon} \right) \right\rfloor + 1 \right),
\]

where \( C_N \) is the constant specified in Theorem 1. Then the orthonormal basis \( Q = [F_{N,W} \quad F_{N,W}V] \) with \( V \in \mathbb{C}^{(N-2)[NW]-1} \times R \) containing the R dominant left singular vectors of \( F_{N,W}B_{N,W} \) satisfies

\[
\frac{\|e_f - QQ^* e_f\|_2^2}{\|e_f\|_2^2} \leq \epsilon
\]

for all \( f \in [-W, W] \).

**Remark 1.** In [17], we show a similar but asymptotic result for the Slepian basis as follows. Fix \( W \in (0, \frac{1}{2}) \) and \( \delta \in (0, \frac{1}{2W} - 1) \). Let \( K = 2NW(1 + \delta) \). Then there exist constants \( \overline{C}_1, \overline{C}_2 \) and \( N_0 \in \mathbb{N} \) (which may depend on \( W \) and \( \delta \)) such that

\[
\frac{\|e_f - SS_K^* e_f\|_2^2}{\|e_f\|_2^2} \leq \overline{C}_1 N^{3/2}e^{-\overline{C}_2 N}
\]

for all \( N \geq N_0 \) and \( f \in [-W, W] \). Compared with this result, Theorem 5 is non-asymptotic and provides detail on the constants involved.

Finally, we remark that for \( Q = [F_{N,W} \quad F_{N,W}V] \) with \( V \in \mathbb{C}^{(N-2)[NW]-1} \times R \), both \( Q \) and \( Q^* \) can be applied to a vector with computational complexity \( O(N \log N + NR) \). As an example, for any \( a \in \mathbb{C}^N \), \( \tilde{a} = [F_{N,W} \quad F_{N,W}V]^H a \) can be efficiently computed using the FFT with complexity \( O(N \log N) \). Then \( V^* \tilde{a}_2 \) can be computed via conventional matrix-vector multiplication with complexity \( O(NR) \), where \( \tilde{a}_2 \) is the sub-vector obtained by taking the last \( N - 2\lfloor NW \rfloor - 1 \) entries of \( \tilde{a}_2 \). Thus the total computational complexity for computing \( Q^* a \) is \( O(N \log N + NR) \).

### 1.2.3 ROAST construction with a randomized algorithm

We note that the DPSS vectors are not involved in constructing \( V \) and \( Q \). Directly computing \( V \) with the Businger-Golub algorithm [18] has complexity \( O(N(N - 2\lfloor NW \rfloor - 1)R) \). Noting that \( F_{N,W}^* B_{N,W} \) is effectively low rank, however, we can apply a fast randomized algorithm [19] to compute an approximate basis for the range of \( F_{N,W}^* B_{N,W} \). Let \( \Omega \) be an \( N \times P \) standard Gaussian matrix. We construct a matrix \( \tilde{V} \) whose columns form an orthonormal basis for the range of \( F_{N,W}^* B_{N,W} \). Then the complexity of computing \( F_{N,W}^* B_{N,W} \) is \( O(NP^2) \) flops. The following results establish the dimensionality of \( \tilde{V} \) needed and the representation guarantee with the corresponding basis.

**Theorem 6.** (Guarantee for randomized algorithm) Fix \( N \in \mathbb{N} \) and \( W \in (0, \frac{1}{2}) \). For any \( \epsilon \in (0, \frac{1}{2}) \), fix \( K \) to be such that \( \lambda_{(K-1)}^{(N,W)} \geq \epsilon \). Let \( \Omega \) be an \( N \times P \) standard Gaussian matrix, with \( P \) specified as below. Also let
\( \mathbf{V} \) be an orthonormal basis for the column space of the sample matrix \( \mathbf{F}_{N,W}^* \mathbf{B}_{N,W} \mathbf{\Omega} \). Then the orthonormal basis \( \mathbf{Q} = [\mathbf{F}_{N,W} \mathbf{V}] \) has the following expression ability in expectation.

(i) Setting

\[
P = \left\lceil 2C_N \log \left( \frac{30 + 15e}{\epsilon} \right) \right\rceil + 3,
\]

we are guaranteed that

\[
\mathbb{E} \left[ \| \mathbf{S}_K \mathbf{S}_K^* - \mathbf{Q} \mathbf{Q}^* \mathbf{S}_K \mathbf{S}_K^* \| \right] \leq \epsilon,
\]

\[
\mathbb{E} \left[ \| \mathbf{s}_{N,W}^{(l)} - \mathbf{Q} \mathbf{Q}^* \mathbf{s}_{N,W}^{(l)} \| \right] \leq \epsilon
\]

for all \( l = 0, 1, \ldots, K - 1 \). By slightly increasing \( P \) to

\[
P = \left\lceil 2C_N \log \left( \frac{30 + 15e}{\epsilon} \right) \right\rceil + 3,
\]

we have

\[
\mathbb{E} \left[ \cos (\Theta \mathbf{S}_K, \mathbf{Q}) \right] \geq \sqrt{1 - N\epsilon}.
\]

(ii) Sampled sinusoids within the band of interest are well-approximated by \( \mathbf{Q} \) in expectation:

\[
\mathbb{E} \left[ \int_{-W}^{W} \| \mathbf{e}_f - \mathbf{Q} \mathbf{Q}^* \mathbf{e}_f \|_2^2 \right] \leq \epsilon
\]

with

\[
P = \left\lceil \frac{4}{3} C_N \log \left( \frac{15\sqrt{2C_N}}{\epsilon} \right) + \frac{7}{3} \right\rceil.
\]

(iii) The orthonormal basis \( \mathbf{Q} \) can also capture most of the energy in each pure sinusoid:

\[
\mathbb{E} \left[ \frac{\| \mathbf{e}_f - \mathbf{Q} \mathbf{Q}^* \mathbf{e}_f \|_2^2}{\| \mathbf{e}_f \|_2^2} \right] \leq \epsilon
\]

for all \( f \in [-W, W] \) with

\[
P = \max \left( \left\lceil \frac{4}{3} C_N \log \left( \frac{60\pi N\sqrt{2C_N}}{\epsilon^2} \right) + \frac{7}{3} \right\rceil, \left\lceil \frac{4}{3} C_N \log \left( \frac{15\pi \sqrt{2C_N}}{W \epsilon} \right) + \frac{7}{3} \right\rceil \right).
\]

Here \( \mathbb{E} \) denotes expectation with respect to the random matrix \( \mathbf{\Omega} \).

Remark 2. Using concentration of measure [19], we can argue that the results above hold for a particular sampling matrix \( \mathbf{\Omega} \) with high probability.

In summary, the ROAST offers a computationally efficient alternative to the DPSS with virtually the same approximation performance. The ROAST could therefore be considered for use in many of the applications involving DPSS’s that were described earlier in this introduction. For example, in through-the-wall radar imaging using stepped-frequency SAR [8,9], the wall return is modeled as a sampled bandpass signal and thus the ROAST can be used to efficiently mitigate the wall return at each antenna.
1.3 Benefits of an orthonormal basis

For any \( \delta \in (0, \frac{1}{3}) \), fix \( K \) to be such that \( \lambda_{N,W}^{(K-1)} \geq \delta \). In [16], we demonstrated a fast factorization of \( S_K S_K^* \) by constructing \( T_1 = [F_{N,W} \ D_1] \) and \( T_2 = [F_{N,W} \ D_2] \) (with \( D_1, D_2 \in \mathbb{R}^{N \times r} \)) such that

\[
\|S_K S_K^* - T_1 T_2^*\| \leq 2\delta, \quad \text{with} \quad r \leq 3C_N \log \left( \frac{15}{\delta} \right).
\]

We utilize FST to denote the approximate projection \( T_1 T_2^* \).

However, neither \( T_1 \) nor \( T_2 \) is orthonormal and in general \( \|T_1^* x\| \neq \|T_1^* T_2^* x\| \) and \( \|T_2^* x\| \neq \|T_2^* T_2^* x\| \). Moreover, neither \( T_1 \) nor \( T_2 \) is well conditioned (i.e., both have a large condition number). In some applications like orthogonal precoding for wireless communication [20], an orthonormal transform \( Q \) is required or preferred, in order to ensure that \( \|P_Q x\| = \|Q^* x\| \) or that \( Q \) is well conditioned. We list two more stylized applications in signal processing below.

1.3.1 Signal recovery

Suppose \( x \in \mathbb{C}^N \) is a sampled bandlimited signal with digital frequencies within the band \([-W, W]\) and we observe it through

\[
y = \Phi x,
\]

where \( \Phi \in \mathbb{C}^{M \times N} \) (\( 2NW \leq M \leq N \)) is the sensing matrix. Knowing that \( x \) approximately lives in the subspace spanned by \( S_K \), we recover \( x \) by solving

\[
\min_{\alpha} \|y - \Phi S_K \alpha\|^2_2,
\]

which is also a key part in a compressive sensing recovering algorithm for multiband analog signals [13] (see also [15]). The above least-squares problem is equivalent to the following system of linear equations

\[
S_K^* \Phi^* \Phi S_K \alpha = S_K^* \Phi^* y,
\]

which can be solved by numerical algorithms such as conjugate gradient descent (CGD) [21]. The computational complexity of the CGD method depends on two factors: the convergence speed, which depends on the condition number of the system \( A := S_K^* \Phi^* \Phi S_K \) and determines the number of iterations required, and the computational burden in each iteration, mainly involving the application of \( A \) to a length-\( M \) vector. Utilizing a structured sensing matrix \( \Phi \) that has a fast implementation (such as the fast Johnson-Lindenstrauss transform [22]), we can efficiently implement \( A \) if we replace \( S_K \) by the fast transform \( T_1 \) or \( T_2 \) [16] or the ROAST \( Q \) of the form (4). Unfortunately, both \( T_1 \) and \( T_2 \) have large condition number, resulting in slow convergence of the CGD method since the corresponding system \( A \) in general also has large condition number. Thus, in this case, the orthonormal basis \( Q \) is preferable.

1.3.2 Line spectral estimation

Consider a measurement vector \( y \) consisting of a superposition of \( r \) sampled exponentials:

\[
y = \sum_{i=1}^{r} a_i^* e_{f_i^*},
\]

where \( \{f_i^*\} \) are the frequencies and \( \{a_i^*\} \) are the corresponding coefficients. We may attempt to recover the frequencies \( \{f_1, \ldots, f_r\} \) by solving the following nonlinear least squares problem

\[
\{\tilde{f}_i, \tilde{a}_i\} := \arg \min_{f_i, a_i} \left\| y - \sum_{i=1}^{r} a_i e_{f_i} \right\|^2_2.
\]

(9)
Suppose we are given a priori knowledge that the frequencies $f^*_i \in [-W, W]$ for all $i \in \{1, \ldots, r\}$. Then we can reduce the computational cost of solving by (9) by projecting the measurements $y$ onto the range space of $Q$ [23]:

$$
\{f_i, \alpha_i\} := \arg \min_{f_i, \alpha_i} \|P_Q \left( y - \sum_{i=1}^{r} \alpha_i e_{f_i} \right) \|_2^2 = \arg \min_{f_i, \alpha_i} \|Q^* \left( y - \sum_{i=1}^{r} \alpha_i e_{f_i} \right) \|_2^2.
$$

It is shown in [23] that the projected problem (10) has the same stationary points as the full problem (9) under certain conditions on the range space of $Q$. When applying an optimization method like Gauss-Newton, the advantage of the projected problem (10) over the full problem (9) is that each optimization step is much cheaper since the projected Jacobian has much smaller size.

Based on this observation, for the general case where the frequencies lie in multiple bands, [23] provides an iterative algorithm that in each iteration finds one underlying band and projects the signal onto this band, then applies Gauss-Newton to solve the projected problem. We also note that our $Q$ can be further reduce the computational cost in [23] since $Q$ can be efficiently applied to a vector, while the orthonormal basis utilized in [23] is a numerical approximation (obtained by performing PCA on a set of sinusoids) to the Slepian basis $S_K$.

### 1.4 Comparison of ROAST and FST [16]

There are some similarities and differences between ROAST and FST (i.e., $T_1T_2^*$ in (7)). With respect to the similarities, both ROAST and FST consist of two parts: the partial DFT matrix (which can be applied to a vector efficiently via the FFT) and a skinny matrix (which can also be efficiently applied to any vector with standard matrix-vector multiplication since the number of columns is $O(\log N)$). Aside from the fact that ROAST corresponds to an orthonormal basis while FST is not an exact orthogonal projection, ROAST and FST also differ in the following respects.

(i) FST explicitly attempts to approximate the operator $S_K S^*_K$, while ROAST is motivated by the goal of approximating the subspace spanned by the DPSS basis vectors. To better reveal the subtle difference between these two goals, let us take a closer look at the objective function (5) corresponding to ROAST:

$$
\text{minimize}_{Q} \sum_{\ell=0}^{N-1} \lambda^{(\ell)}_{N,W} \|s^{(\ell)}_{N,W} - QQ^* s^{(\ell)}_{N,W}\|_2^2 = \text{trace} \left( SS^* - QQ^* SS^* \right).
$$

In this expression, note that we use the DPSS eigenvalue $\lambda^{(\ell)}_{N,W}$ to weight the energy in the DPSS vector $s^{(\ell)}_{N,W}$ that is not captured by $Q$. Such an eigenvalue-based weighting (most of the weights are either very close to 1 or 0) is not present in the FST objective $\|S_K S^*_K - T_1 T_2^*\|$. To see why it may not be appropriate to approximate $S_K S^*_K$ with $QQ^*$, we first note that for two orthogonal projectors $P_A$ and $P_B$, $\|P_A - P_B\| = 1$ if the dimension of subspace $A$ does not equal the dimension of subspace $B$. Therefore, $\|S_K S^*_K - QQ^*\|$ will always equal 1 unless the number of columns in $Q$ is set exactly equal to $K$. Even if we set the number of columns we have for $Q$ equal to $K$, let us take a closer look at what form $\|S_K S^*_K - QQ^*\|$ would take if $Q$ has the form $[F_{N,W} F_{N,W}^*] V$:

$$
\|S_K S^*_K - QQ^*\| = \left\| \begin{bmatrix} F_{N,W}^* S_K S^*_K F_{N,W} - I & F_{N,W}^* S_K S^*_K F_{N,W} \\ F_{N,W}^* S_K S^*_K F_{N,W} & F_{N,W}^* S_K S^*_K F_{N,W} - V V^* \end{bmatrix} \right\|.
$$

In (11), we see that regardless of the choice of $V$ (even if we could make the bottom right block equal to zero), the overall quantity $\|S_K S^*_K - QQ^*\|$ could be still large since the other three blocks in the right hand side of (11) are not negligible (in terms of the spectral norm), though probably all of them are low-rank.

(ii) Although [16] focuses on approximating $S_K S^*_K$, it is also possible to derive signal approximation guarantees for the FST. However, these will be slightly weaker than those for ROAST in that we require a
slightly larger size (number of columns in $T_1$ and $T_2$) for FST to have a similar approximation guarantee. In particular, using a similar approach to that used for establishing Theorem 4, we have

$$\int_{-W}^{W} \|e_f - T_1T_2^*e_f\|_2^2 df \leq 2\epsilon, \text{ with } r = 3 \max \left\{ C_N \log \left( \frac{15C_N}{N\epsilon} \right) + 1, 0 \right\}. \quad (12)$$

Comparing (12) and Theorem 4, we see that FST requires a slightly larger size for a comparable approximation guarantee. In practice, we observe that ROAST requires much smaller size than FST to achieve a similar approximation quality (see Section 3), since ROAST is constructed by explicitly minimizing the subspace gap.

(iii) We note that although the approximation guarantee (12) for FST and the one in Theorem 4 for ROAST provide similar upper bounds on the number of columns in the skinny matrices $D_1$, $D_2$ and $V$, the construction of these matrices is different. For FST, given $K$ and $\delta$, we provided an explicit construction for the skinny matrices $D_1, D_2 \in \mathbb{R}^{N \times r}$ in [16] with an upper bound on $r$ given in (7). For ROAST, we construct $V$ by computing the singular vectors of $F_{N,W}^* B_{N,W}$ and thus there is freedom to choose the number of singular vectors to be utilized. \footnote{Though the number of columns for $V$ in Theorems 4-6 matches the information-theoretical bound, it is still quite conservative compared to experimental results. The simulation results in Section 3 indicate that ROAST with $R = 4 \log N$ gives very accurate representations for most sampled sinusoids and bandlimited signals.}

We compare the speed and approximation performance of ROAST and FST using numerical experiments in Section 3.

2 Proof of main results

2.1 Supporting results

We first establish the following definition of angle between subspaces to compare subspaces of possibly different dimensions.

**Definition 3.** Let $S_A$ and $S_B$ be the subspaces formed by the columns of the matrices $A$ and $B$ respectively. The subspace angle $\Theta_{A,B}$ between $S_A$ and $S_B$ is given by

$$\cos(\Theta_{A,B}) := \inf_{a \in S_A, \|a\|_2 = 1} \|P_B a\|_2$$

if $\dim(S_B) \geq \dim(S_A)$, or

$$\cos(\Theta_{A,B}) := \inf_{b \in S_B, \|b\|_2 = 1} \|P_A b\|_2$$

if $\dim(S_B) < \dim(S_A)$. Here $P_B$ (or $P_A$) denotes the orthogonal projection onto the column space of $B$ (or $A$).

We remark that when the subspaces $S_A$ and $S_B$ have the same dimension, our definition of subspace angle coincides with the subspace gap [24], defined as $\sin(\Theta_{A,B})$. Smaller $\Theta_{A,B}$ indicates a smaller gap between $S_A$ and $S_B$. We also connect our definition of subspace angle to principal angles between two subspaces defined as follows.

**Definition 4.** [25] Suppose $A \in \mathbb{R}^{N \times p}$ and $B \in \mathbb{R}^{N \times q}$ are orthonormal bases for the subspaces $S_A \subset \mathbb{R}^{N \times N}$ and $S_B$, respectively. Suppose $p \geq q$. Then the principal angles between $S_A$ and $S_B$, $\theta_1(A,B) \leq \theta_2(A,B) \leq \cdots \leq \theta_q(A,B)$, are defined as

$$\cos(\theta_i(A,B)) = \sigma_i(A^*B)$$

for all $i \in \{1, 2, \ldots, q\}$, where $\sigma_i(\cdot)$ denotes the $i$-th largest singular value.
We note that the subspace angle $\Theta_{A,B}$ is equivalent to the largest principal angle $\theta_q(A,B)$. To see this, we rewrite the smallest singular value:

$$\cos(\theta_q(A,B)) = \sigma_q(A^*B) = \inf_{\|\alpha\|_2=1} \|A^*B\alpha\|_2 = \inf_{b \in S_B \|b\|_2=1} \|A^*b\|_2 = \inf_{b \in S_B \|b\|_2=1} \|P_Ab\|_2,$$

where the last inequality follows because by assumption $A$ is an orthonormal basis for $S_A$. Thus, our definition of subspace angle captures the largest possible principal angle between two subspaces.

Before moving on to prove the main result, we present several results which will also be useful in the remaining proofs. We start with the following result, a variant of Von Neumann’s trace inequality [26].

**Lemma 1.** [26] For any $M \times N$ (suppose $M \leq N$) matrices $A$ and $B$ with singular values $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{M-1} \geq 0$ and $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_{M-1} \geq 0$, we have

$$|\text{trace}(AB^*)| \leq \sum_{m=0}^{M-1} \alpha_m \beta_m.$$

*Proof of Lemma 1.* We enlarge $A$ and $B$ into $N \times N$ matrices $A'$ and $B'$ with zero rows, i.e.,

$$A' = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B' = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$ Let $\alpha_0' \geq \alpha_1' \geq \cdots \alpha_{N-1}'$ and $\beta_0' \geq \beta_1' \geq \cdots \geq \beta_{N-1}'$ be the singular values of $A'$ and $B'$, respectively. Note that $\alpha_n = \alpha_n'$, $\beta_n = \beta_n'$ for all $n \leq M-1$ and $\alpha_n' = 0$, $\beta_n' = 0$ for all $n > M$. It follows from Von Neumann’s trace inequality [26] that

$$|\text{trace}(AB^*)| = |\text{trace}(A'(B')^H)| \leq \sum_{n=0}^{N-1} \alpha_n' \beta_n' = \sum_{m=0}^{M-1} \alpha_m \beta_m.$$

\[\square\]

The following result establishes an upper bound on $q(Q)$ in terms of the singular values of $F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W}$.

**Lemma 2.** Let $V \in \mathbb{C}^{(N-2|NW|)-1 \times R}$ be an orthonormal basis with $R \leq (N-2|NW|-1)$. Let $\pi_0 \geq \pi_1 \geq \cdots \geq \pi_{N-2|NW|-2}$ denote the singular values of $F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W}$. Then

$$q(Q) = \int_{-W}^{W} \|e_f - QQ^* e_f\|_2^2 df \leq \sum_{l=0}^{N-2|NW|-2} \pi_l,$$

where $Q = \begin{bmatrix} F_{N,W} & F_{N,W}^* \end{bmatrix}$.

*Proof of Lemma 2.* Recall (6) that

$$q(Q) = \int_{-W}^{W} \|e_f - QQ^* e_f\|_2^2 df = \text{trace}((I - QQ^*) B_{N,W}).$$

Plugging in $Q = \begin{bmatrix} F_{N,W} & F_{N,W}^* \end{bmatrix}$, we have

$$q(Q) = \text{trace}\left(\left(I - \begin{bmatrix} F_{N,W} & F_{N,W}^* \end{bmatrix}\begin{bmatrix} F_{N,W} & F_{N,W}^* \end{bmatrix}^*\right)B_{N,W}\right)$$

$$= \text{trace}\left(F_{N,W}^* B_{N,W} F_{N,W} - F_{N,W}^* F_{N,W}^* B_{N,W} F_{N,W}\right)$$

$$= \text{trace}\left(F_{N,W}^* B_{N,W} F_{N,W} - V V^* F_{N,W}^* B_{N,W} F_{N,W}\right)$$

$$\leq \sum_{l=0}^{N-2|NW|-2} \pi_l \|F_{N,W}\| \leq \sum_{l=0}^{N-2|NW|-2} \pi_l,$$

\[13\]
where the first inequality follows from Lemma 1 by setting \( A = F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W} \) and \( B = F_{N,W}^* \).

In order to utilize Lemma 2, we need the distribution of the singular values of \( F_{N,W}^* B_{N,W} \). This is established by the following result, whose proof is given in Appendix A.

**Lemma 3.** (singular value decay) Let \( \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-2[NW]-2} \) denote the singular values of \( F_{N,W}^* B_{N,W} \). Then

\[
\sigma_\ell \leq \epsilon \quad \text{when} \quad \ell = C_N \log \left( \frac{15}{\epsilon^2} \right) \quad \text{for any} \quad \epsilon \in (0, 1).
\]

Also

\[
\sigma_\ell \leq 15 e^{-\frac{\ell}{CN}}.
\]

Now we are well equipped to prove the main results.

### 2.2 Proof of Theorem 3

**Proof of Theorem 3.** We first provide the following results on the representation guarantee for the leading DPSS vectors and the subspace angle between the column spaces of \( S_K \) and \( Q \). The proof of Lemma 4 is given in Appendix B.

**Lemma 4.** Let \( V \in \mathbb{C}^{(N-2[NW]-1) \times R} \) be an orthonormal basis with \( R \leq (N-2[NW]-1) \). For any \( \epsilon \in (0, \frac{1}{2}) \), fix \( K \) to be such that \( \lambda_{(K-1)}^{(N,W)} \geq \epsilon \). Let

\[
\eta := \frac{\| F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W} \|}{\epsilon}
\]

Then the orthonormal basis \( Q = \left[ F_{N,W} \quad F_{N,W}^* V \right] \) satisfies

\[
\| S_K S_K^* - Q Q^* S_K S_K^* \| \leq \eta,
\]

\[
\cos (\Theta_{S_K, Q}) \geq \sqrt{1 - N \eta},
\]

\[
\| s^{(l)}_{N,W} - Q Q^* s^{(l)}_{N,W} \| \leq \eta
\]

for all \( l = 0, 1, \ldots, K-1 \).

Since \( V \) contains the first \( R \) principal eigenvectors of \( F_{N,W}^* B_{N,W} \), using Lemma 3, we obtain

\[
\| F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W} \| \leq 15 e^{-\frac{\eta}{CN}}.
\]

If we set \( R = C_N \log \left( \frac{15}{\epsilon^2} \right) \), we have

\[
\| F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W} \| \leq \epsilon^2.
\]

Alternatively, if one set \( R = C_N \log \left( \frac{15N}{\epsilon^2} \right) \):

\[
\| F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W} \| \leq \frac{\epsilon^2}{N}.
\]

The proof of Theorem 3 completed by utilizing Lemma 4. \( \square \)
2.3 Proof of Theorem 4

Proof of Theorem 4. Let \( \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-2|NW|-2} \) denote the singular values of \( \mathbf{F}_{N,W}^* B_{N,W} \). Since \( V \) consists of the \( R \) dominant left singular vectors of \( \mathbf{F}_{N,W}^* B_{N,W} \), the singular values of \( \mathbf{F}_{N,W}^* B_{N,W} - VV^* \mathbf{F}_{N,W}^* \) are \( \sigma_R, \sigma_{R+1}, \ldots, \sigma_{N-2|NW|-2} \) and \( R \) zeros. It follows from Lemma 3 that

\[
\sum_{\ell=R}^{N-2|NW|-2} \sigma_{\ell} \leq 15e^{-\frac{R}{CN}} (1 - e^{-\frac{N-2|NW|-R-1}{CN}}).
\] (14)

where the last line holds because \( e^{a-1} \geq a \) for all \( a \geq 0 \).

If \( CN \log \left( \frac{15e^{CN}}{N\epsilon} \right) + 1 \leq 0 \), which implies that

\[
\sum_{\ell=0}^{N-2|NW|-2} \sigma_{\ell} \leq N\epsilon,
\]

then by setting \( R = 0 \) and \( Q = \mathbf{F}_{N,W} \) we are guaranteed that

\[
\int_{-W}^{W} \frac{\| e_f - QQ^* e_f \|^2}{\| e_f \|^2} df \leq \frac{N-2|NW|-2}{N} \sum_{\ell=0}^{N-2|NW|-2} \sigma_{\ell} \leq \frac{1}{N} N\epsilon = \epsilon.
\]

Otherwise, choosing \( R = CN \log \left( \frac{15e^{CN}}{N\epsilon} \right) + 1 \), we have

\[
\sum_{\ell=R}^{N-2|NW|-2} \sigma_{\ell} \leq N\epsilon.
\]

Now applying Lemma 2, we have

\[
\int_{-W}^{W} \frac{\| e_f - QQ^* e_f \|^2}{\| e_f \|^2} df \leq \frac{N-2|NW|-2}{N} \sum_{\ell=R}^{N-2|NW|-2} \sigma_{\ell} \leq \frac{1}{N} N\epsilon = \epsilon,
\]

where we utilize the fact that each sinusoid has energy \( \| e_f \|^2 = N \). This completes the proof of Theorem 4.

Remark 3. By (13), we have

\[
\varrho(Q) = \text{trace} \left( \mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W} - VV^* \mathbf{F}_{N,W}^* \right).
\]

Directly solving

\[
\min_{V \in \mathbb{C}^{(N-2|NW|-1) \times R}} \text{trace} \left( \mathbf{F}_{N,W}^* B_{N,W} \mathbf{F}_{N,W} - VV^* \mathbf{F}_{N,W}^* \right),
\]
we obtain an alternative optimal solution $\tilde{V}$ consisting of the first $R$ principal eigenvectors of $F^*_{N,W} B_{N,W} F_{N,W}$. The orthonormal basis $Q' = \left[ F_{N,W} \quad \tilde{V} \right]$ is optimal in terms of minimizing $\varrho(Q)$ and also for representing all discrete-time sinusoids with a frequency $f \in [-W, W]$ in the least square sense. Similar to Theorem 4, we can also establish an approximation guarantee for $\tilde{V}$. Note that

$$F^*_{N,W} (B_{N,W} - F_{N,W} F^*_{N,W}) F_{N,W} = \left[ F^*_{N,W} B_{N,W} F_{N,W} - I \quad F^*_{N,W} B_{N,W} F_{N,W} \right].$$

By utilizing the result that $B_{N,W} = F_{N,W} F^*_{N,W} + L + E$, where

$$\text{rank}(L) \leq C_N \log \left( \frac{15}{\epsilon} \right)$$

and $\|E\| \leq \epsilon$, we can rewrite $F^*_{N,W} B_{N,W} F_{N,W} = L_2 + E_2$, where

$L_2 := F^*_{N,W} L F_{N,W}$ and $E_2 := F^*_{N,W} E F_{N,W}$.

Thus,

$$\text{rank}(L_2) \leq C_N \log \left( \frac{15}{\epsilon} \right) \quad \text{and} \quad \|E_2\| \leq \epsilon.$$

It follows from the Eckart-Young-Mirsky theorem [27] that

$$\|F^*_{N,W} B_{N,W} F_{N,W} - \tilde{V} \tilde{V}^* F^*_{N,W} B_{N,W} F_{N,W}\| \leq \|E_2\| \leq \epsilon.$$

Therefore, choosing $R = C_N \log \left( \frac{15C_N}{\epsilon'} \right) + 1$, with a similar argument we also have

$$\int_{-W}^{W} \frac{\|e_f - UU^* e_f\|^2}{\|e_f\|^2} df \leq \frac{1}{N} \text{trace} \left( F^*_{N,W} B_{N,W} F_{N,W} - \tilde{V} \tilde{V}^* F^*_{N,W} B_{N,W} F_{N,W} \right) \leq \epsilon.$$

We note that all other results in this paper involving $V$ can also be applied to $\tilde{V}$ with similar or slightly different guarantees.

### 2.4 Proof of Theorem 5

By Theorem 4, we are guaranteed that the pure sinusoids have, on average, a small representation residual in the basis $Q$. Intuitively, the representation error for each pure sinusoid is also guaranteed to be small. The following result provides an upper bound on the representation error for each pure sinusoid in terms of the average representation error. Its proof is given in Appendix C.

**Lemma 5.** For any $q \in \{1, 2, \ldots, N\}$, suppose $U \in \mathbb{C}^{N \times q}$ is an orthonormal basis such that $U^* U = I$. Also suppose $W \geq \frac{1}{4\pi N}$. Then

$$\frac{\|e_f - UU^* e_f\|^2}{\|e_f\|^2} \leq \max \left( 2\sqrt{\pi} \sqrt{\int_{-W}^{W} \|e_f - UU^* e_f\|^2 df}, \frac{1}{NW} \int_{-W}^{W} \|e_f - UU^* e_f\|^2 df \right).$$

**Proof of Theorem 5.** It follows from (14) that by choosing $R = C_N \log \left( \frac{15C_N}{\epsilon'} \right) + 1$, we have

$$\sum_{l=R}^{N-2|NW|-1} \sigma_l \leq \epsilon'.$$

15
Utilizing Lemma 2 gives
\[ \int_{-W}^{W} \| e_f - QQ^* e_f \|^2 \, df \leq \sum_{l=R}^{N-2|NW|-1} \sigma_l \leq \epsilon'. \]

The proof of Theorem 5 is completed by setting
\[ \epsilon' = \frac{e^2}{4\pi}, \quad R = C_N \log \left( \frac{60\pi C_N}{e^2} \right) + 1, \]
or
\[ \epsilon' = NW\epsilon, \quad R = C_N \log \left( \frac{15C_N}{NW\epsilon} \right) + 1. \]

2.5 Proof of Theorem 6

We first present the following guarantees on randomized algorithms for computing orthonormal bases from [19].

Theorem 7. [19, Theorem 10.5] (Average Frobenius norm) Let \( A \) be an \( M \times N \) (suppose \( M \leq N \)) matrix with singular values \( \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{M-1} \). Choose a target rank \( R \geq 2 \) and an oversampling parameter \( p \geq 2 \), where \( P = R + p \leq M \). Let \( \Omega \) be an \( N \times P \) standard Gaussian matrix. Let \( P_Y \) be an orthogonal projector onto the column space of the sample matrix \( Y = A\Omega \). Then the expected approximation error
\[ \mathbb{E} \left[ \| A - P_Y A \|_F \right] \leq \left( 1 + \frac{R}{p-1} \right)^{1/2} \left( \frac{M-1}{\sum_{m=R}^{M-1} \alpha_m^2} \right)^{1/2}, \]

where \( \mathbb{E} \) denotes expectation with respect to the random matrix \( \Omega \).

Theorem 8. [19, Theorem 10.6] (Average spectral error) Under the setup of Theorem 7,
\[ \mathbb{E} \left[ \| A - P_Y A \| \right] \leq \left( 1 + \sqrt{\frac{R}{p-1}} \right) \alpha_R + \frac{e\sqrt{P}}{p} \left( \sum_{m=R}^{M-1} \alpha_m^2 \right)^{1/2}. \]

Proof of Theorem 6. Let \( \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-2|NW|-2} \) denote the singular values of \( F^*_{N,W}B_{N,W} \). Utilizing Lemma 3, we have
\[ \sum_{l=R}^{N-2|NW|-2} \sigma_l^2 \leq \sum_{l=R}^{N-2|NW|-2} \left( 15e^{-\frac{2}{CN}} \right)^2 \]
\[ = 225 \frac{e^{-2\frac{R}{CN}}(1 - e^{-2\frac{N-2|NW|-2-1}{CN}})}{1 - e^{-\frac{2}{CN}}} \]
\[ \leq 225 \frac{e^{-2\frac{R}{CN}}}{1 - e^{-\frac{2}{CN}}} = 225 \frac{e^{-2\frac{R-1}{CN}}}{e^{\frac{2}{CN}} - 1} \]
\[ \leq 225e^{-2\frac{R-1}{CN}} \frac{C_N}{2}. \]

Note that here \( V \) is an orthonormal basis for the column space of the sample matrix \( F^*_{N,W}B_{N,W} \). Let \( \pi_0 \geq \pi_1 \geq \cdots \geq \pi_{N-2|NW|-2} \) denote the singular values of \( F^*_{N,W}B_{N,W} - \overline{V}V^*F^*_{N,W}B_{N,W} \).
Show (i): Utilizing Theorem 8, we have

\[
\mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] \\
\leq \left( 1 + \sqrt{\frac{R}{P - R - 1}} \right) \sigma_R + \frac{e \sqrt{P}}{P - R} \left( \sum_{i=R}^{M-1} \sigma_i^2 \right)^{1/2} \\
\leq \left( 1 + \sqrt{\frac{R}{P - R - 1}} \right) 15 e^{-\frac{R}{P - R} N} + \frac{e \sqrt{P}}{P - R} \left( 225 e^{-2 \frac{R}{P - R} C_N} \right)^{1/2} \\
= \left( 1 + \sqrt{\frac{R}{P - R - 1}} \right) 15 e^{-\frac{R}{P - R} N} + 15 e \sqrt{P} e^{-\frac{R}{P - R} N} \sqrt{C_N}.
\]

Setting \( R = C_N \log \left( \frac{30+15e}{e^2} \right) + 1 \) and \( P = 2R + 1 \), we have

\[
\mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] \leq 30 e^2 / 30 + 15 e \sqrt{C_N / R + 1} 30 + 15 e \leq e^2
\]

since \( C_N \leq R \) for any \( e^2 \in (0,1) \). It follows from Lemma 4 that

\[
\mathbb{E} \left[ \| S_K S_K^* - QQ^* S_K S_K^* \| \right] \leq \mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] \leq \epsilon, \quad \mathbb{E} \left[ \| s_{N,W}^{(l)} - QQ^* s_{N,W}^{(l)} \| \right] \leq \epsilon
\]

for all \( l = 0, 1, \ldots, K - 1 \). Alternatively, setting \( R = C_N \log \left( \frac{30+15e}{e^2} \right) + 1 \) and \( P = 2R + 1 \), we have

\[
\mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] \leq \frac{e^2}{N}.
\]

Thus applying Lemma 4 gives

\[
\mathbb{E} \left[ \cos (\Theta_{S_K} Q) \right] \geq \sqrt{1 - N \mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] / \epsilon} \geq \sqrt{1 - \epsilon}.
\]

Show (ii): Set \( p = \frac{R}{4} + 1 \), i.e., \( P = \frac{3}{4} R + 1 \). It follows from Theorem 7 that

\[
\mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] _F \leq \left( 1 + \frac{R}{P - 1} \right)^{1/2} \left( \sum_{i=R}^{N-2 |NW| - 2} \sigma_i^2 \right)^{1/2} \\
\leq 2 \sqrt{225 e^{-2 \frac{R}{P - R} C_N} \frac{C_N}{2}} = 15 e^{-\frac{R}{P - R} C_N} \sqrt{2 C_N}.
\]

By applying Lemma 2 and utilizing the inequality between the Frobenius norm and the nuclear norm, we have

\[
\mathbb{E} \int_{-W}^{W} \frac{\| f_J - QQ^* f_J \|_2^2}{\| f_J \|_2^2} df = \frac{1}{N} \mathbb{E} \sum_{m=0}^{N-2 |NW| - 2} \pi_m \\
\leq \frac{1}{N} N \mathbb{E} \left[ \| F_{N,W}^* B_{N,W} - \mathbf{V} \mathbf{V}' F_{N,W}^* B_{N,W} \| \right] _F \\
\leq 15 e^{-\frac{R}{P - R} C_N} \sqrt{2 C_N}.
\]
Setting $R = C_N \log \left( \frac{15 \sqrt{2} C_N}{\epsilon^2} \right) + 1$, we obtain

$$E \int_{-W}^{W} \frac{\|e_f - QQ^* e_f\|^2}{\|e_f\|^2} df \leq \epsilon.$$ 

Show (iii): Set $p = R + 1$, i.e., $P = \frac{R}{4} + 1$. From (15), it follows that

$$E \left[ \int_{-W}^{W} \frac{\|e_f - QQ^* e_f\|^2}{\|e_f\|^2} df \right] \leq 15 Ne^{-\frac{R-1}{6N}} \sqrt{2} C_N.$$ 

Utilizing Lemma 5, we have

$$E \left[ \frac{\|e_f - QQ^* e_f\|^2}{\|e_f\|^2} \right] \leq \max \left( E \left[ 2 \sqrt{\frac{\pi}{6}} \int_{-W}^{W} \|e_f - QQ^* e_f\|^2 df \right], E \left[ NW \int_{-W}^{W} \|e_f - QQ^* e_f\|^2 df \right] \right)$$

$$\leq \max \left( 2 \sqrt{\frac{15}{6} \pi N} e^{-\frac{R-1}{6N}}, 15 Ne^{-\frac{R-1}{6N}} \sqrt{2} C_N \right).$$

Setting

$$R = \max \left( C_N \log \left( \frac{60 \pi \sqrt{2} C_N}{\epsilon^2} \right) + 1, C_N \log \left( \frac{15 \pi \sqrt{2} C_N}{W \epsilon} \right) + 1 \right)$$

yields

$$E \left[ \frac{\|e_f - QQ^* e_f\|^2}{\|e_f\|^2} \right] \leq \epsilon.$$ 

\[\square\]

### 3 Simulations

In this section, we present some experiments to illustrate the effectiveness of our proposed ROAST and ROAST-R (which is short for ROAST with a Randomized algorithm for computing $V$—see Section 1.2.3). Throughout this section, we use $R$ (which is typically $O(\log(N))$) to denote the the dimensionality of $V$ for ROAST. For ROAST-R, we set $P$, the dimensionality of $V$, as $P = R$ here.

For comparison, we also compute the projection onto the column space of $F_{N,W+\frac{R}{NW}}$ which is the $N \times (2 \lfloor NW \rfloor + 1 + R)$ DFT matrix with frequencies in $[-W - \frac{R}{NW}, W + \frac{R}{NW}]$. Such a projection is simply denoted by Sub-DFT. Note that the dimension of the column space of $F_{N,W+\frac{R}{NW}}$ is $2 \lfloor NW \rfloor + 1 + R$ and is equal to the dimension of the column space of $Q$ in ROAST and ROAST-R. We also compare with DPSS since it provides the gold standard in approximation performance. Specifically, the projection onto the column space of the leading DPSS vectors $S_K$ is computed and denoted simply by DPSS in the legends of the figures. We also choose $K = 2 \lfloor NW \rfloor + 1 + R$ so that all these subspaces have the same dimensionality.

We quantify the ability of the different projections to capture a given signal $x \in \mathbb{C}^N$ in terms of

$$\text{SNR} = 20 \log_{10} \left( \frac{\|\hat{x}\|^2}{\|x - \hat{x}\|^2} \right) \text{ dB},$$

where $\hat{x}$ is the resulting projection of $x$ by the above mentioned methods.

Figure 1(a) shows the SNR captured by different projections for various pure sinusoids $e_f$. We observe that the DPSS basis, ROAST, ROAST-R and provide almost equal approximation performance for the pure
sinusoids with frequencies in the band of interest. Also as guaranteed by Theorems 5, 6 and [17, Theorem 3.9], any sinusoid in the band of interest can be well represented by the DPSS basis, ROAST, and ROAST-R.

We also generate a sampled bandlimited signal \( x \) by adding \( 10^5 \) complex exponentials with frequencies selected uniformly at random within the frequency band \([-W, W]\). Figure 1(b) shows the ability of the different projections to capture this vector in terms of SNR. Again, it can be observed that the DPSS basis, ROAST, and ROAST-R provide almost equal approximation performance for sampled bandlimited signals.

We now compare ROAST with FST (see (7)) which involves two skinny matrices \( D_1, D_2 \in \mathbb{R}^{N \times r} \) with

\[
  r \leq 3C_N \log \left( \frac{15}{\delta} \right),
\]

where \( \delta \) is the approximation accuracy and is chosen as \( \delta = 10^{-5} \) unless stated otherwise. As we explained in Section 1.4, in some applications \( r \) is prescribed instead of the approximation accuracy \( \delta \). For these cases, we modify the FST such that \( D_1 \) and \( D_2 \) have the same number of columns as \( V \) (i.e., \( r = R \)). The corresponding transform is denoted by FST-FR (shorted for FST with Fixed Rank)\(^4\).

We compare the size, speed, and approximation performance the six projection methods. In these experiments, we fix \( R = \lfloor 3\log(N) \rfloor \) and \( \delta = 10^{-5} \). Figures 2(a) and (b) respectively plot SNR as a function of dimension \( N \) and the relationship between the run time and \( N \) for the six projection methods. As observed, the DPSS has the best approximation performance as guaranteed by (1) and (2), but the running time of DPSS has a quadratic increase. FST, FST-FR, ROAST and ROAST-R\(^5\) are nearly as fast as the DFT, but with much better approximation performance (except FST-FR which only has slightly better approximation quality than the DFT). Figure 2(c) shows the precomputation time needed for the five projection methods. For the DPSS basis, the first \( K \) DPSS vectors are precomputed with the Matlab command \texttt{dpss} (which actually computes the eigenvectors of a tridiagonal matrix with computational complexity of \( O(N^2) \)). As can be seen in Figure 2(c), the precomputation time required by the DPSS grows roughly quadratically with \( N \), while the precomputation time required by other fast transforms grows just faster than linearly in \( N \). Figure 2(d) compares the value of \( r \) (the number of columns of \( D_1 \) and \( D_2 \) for FST) and \( R \) (the number of columns of the skinny matrices in ROAST, ROAST-R, and FST-FR). In a nutshell, we see that FST has a similar approximation quality, but at the expense of a larger and slower transform. On the other hand, when we fix the size of FST the same as ROAST and ROAST-R, as depicted in Figure 2(a), FST-FR has inferior approximation quality to ROAST and ROAST-R.

\[\text{A Proof of Lemma 3}\]

\[\text{Proof of Lemma 3.}\] Note that

\[
  F_{N,W}^* (B_{N,W} - F_{N,W} F_{N,W}^*) = \begin{bmatrix} F_{N,W}^* B_{N,W} - F_{N,W}^* F_{N,W}^* \\ F_{N,W}^* B_{N,W} \end{bmatrix}.
\]

By utilizing the result that

\[
  B_{N,W} = F_{N,W} F_{N,W}^* + \mathbf{L} + \mathbf{E},
\]

where

\[
  \text{rank}(\mathbf{L}) \leq C_N \log \left( \frac{15}{\epsilon} \right) \quad \text{and} \quad \|\mathbf{E}\| \leq \epsilon,
\]

we can rewrite \( F_{N,W}^* B_{N,W} = \mathbf{L}_1 + \mathbf{E}_1 \), where

\[
  \mathbf{L}_1 := F_{N,W}^* \mathbf{L} \quad \text{and} \quad \mathbf{E}_1 := F_{N,W}^* \mathbf{E}.
\]

\(^4\)We note that the code for this transform is not optimized. For FST-FR, we set \( K = 2\lfloor NW \rfloor + 1 + \lfloor 1/4 R \rfloor \) as we require that \( D_1 \) and \( D_2 \) have the same number of columns as \( V \).

\(^5\)FST-FR, ROAST, and ROAST-R are expected to have the same running time since these three transforms have the same dimensionality and form.
Thus,
\[
\text{rank}(L_1) \leq C_N \log \left( \frac{15}{\epsilon} \right) \quad \text{and} \quad \|E_1\| \leq \epsilon.
\]
It follows from the Eckart-Young-Mirsky theorem [27] that
\[
\sigma_{\text{rank}(L_1)} \leq \|E_1\| \leq \epsilon
\]
for any \( \epsilon \in (0, 1) \). Noting that \( \|F_{N,W}^* B_{N,W}\| \leq \|F_{N,W}^*\| \|B_{N,W}\| < 1 \), we have
\[
\sigma_\ell \leq 15e^{-\frac{\ell}{CN}}.
\]
for all \( \ell = 0, 1, \ldots, N - 2[NW] - 2 \). Otherwise, suppose \( \sigma_\ell > 15e^{-\frac{\ell}{CN}} \). If \( 15e^{-\frac{\ell}{CN}} \geq 1 \), then this is in contradiction to the fact that \( \sigma_\ell < 1 \). If \( 15e^{-\frac{\ell}{CN}} < 1 \), let \( \epsilon = 15e^{-\frac{\ell}{CN}} \). Then we have a contradiction to the fact that \( \sigma_{\text{rank}(L_1)} \leq \epsilon \) and \( \text{rank}(L_1) \leq C_N \log \left( \frac{15}{\epsilon} \right) = \ell \).

\section{B Proof of Lemma 4}

\textbf{Proof of Lemma 4.} Fix \( K \) to be such that \( \lambda_{N,W}^{(K-1)} > \epsilon \). Utilizing \( B_{N,W} = S_{N,W} \Lambda_{N,W} S_{N,W}^* \), we have
\[
\|B_{N,W} - QQ^* B_{N,W}\|
\]
\[
= \|S_{N,W} \Lambda_{N,W} S_{N,W}^* - QQ^* S_{N,W} \Lambda_{N,W} S_{N,W}^*\|
\]
\[
= \|\Lambda_{N,W} - S_{N,W}^* QQ^* S_{N,W} \Lambda_{N,W}\|
\]
\[
\geq \|\Lambda_K - S_{K}^* QQ^* K \Lambda_K\| = \|(I - S_{K}^* QQ^* S_{K}) \Lambda_K\|
\]
\[
\geq \|I - S_{K}^* QQ^* S_{K}\| \epsilon.
\]
On the other hand,
\[
\|B_{N,W} - QQ^* B_{N,W}\|
\]
\[
= \|B_{N,W} - [F_{N,W} F_{N,W} V] [F_{N,W} F_{N,W} V]^H B_{N,W}\|
\]
\[
= \|F_{N,W}^* B_{N,W} - V V^* F_{N,W}^* B_{N,W}\|.
\]
Combining the above two set of equations yields

$$\| I - S_K^* Q Q^* S_K \| \leq \eta = \frac{\| (I - V V^*) P_{N,W} B_{N,W} \|}{\epsilon}.$$  

Now exploit the relationship between $S_K S_K^* - Q Q^* S_K S_K^*$ and $I - S_K^* Q Q^* S_K$ as follows

$$\| S_K S_K^* - Q Q^* S_K S_K^* \|^2 = \| (S_K S_K^* - Q Q^* S_K S_K^*)^T (S_K S_K^* - Q Q^* S_K S_K^*) \|$$

$$= \| S_K (I - S_K^* Q Q^* S_K) S_K^* \|$$

$$\leq \| (I - S_K^* Q Q^* S_K) \| \leq \eta.$$
Then, utilizing the inequality $\|I - S_K^* QQ^* S_K\|_{\max} \leq \|I - S_K^* QQ^* S_K\|$, where $\|S_K^* QQ^* S_K\|_{\max}$ is the maximum absolute entry of $I - S_K^* QQ^* S_K$, we have

$$\left| \left( s_{N,W}^{(l)} \right)^{\text{H}} QQ^* s_{N,W}^{(l)} \right| \leq \|I - S_K^* QQ^* S_K\| \leq \eta$$

for all $l \neq l', l, l' = 0, 1, \ldots, K - 1$, and

$$\left\| s_{N,W}^{(l)} - QQ^* s_{N,W}^{(l)} \right\|_2^2 = 1 - \left\| QQ^* s_{N,W}^{(l)} \right\|_2^2 \leq \|I - S_K^* QQ^* S_K\| \leq \eta$$

for all $l = 0, 1, \ldots, K - 1$.

Let $s$ be an arbitrary unit vector in the subspace spanned by $S_K$, i.e., $s = \sum_{\ell=0}^{K-1} \alpha_\ell s_{N,W}^{(\ell)}$ with $\|s\|_2 = \sum_{\ell=0}^{K-1} \alpha_\ell^2 = 1$. We have

$$\|s - QQ^* s\|_2 = \left\| \sum_{\ell=0}^{K-1} \alpha_\ell \left( s_{N,W}^{(\ell)} - QQ^* s_{N,W}^{(\ell)} \right) \right\|_2 \leq \sum_{\ell=0}^{K-1} |\alpha_\ell| \left\| s_{N,W}^{(\ell)} - QQ^* s_{N,W}^{(\ell)} \right\|_2 \leq \sqrt{\eta} \sum_{\ell=0}^{K-1} |\alpha_\ell| \leq \sqrt{K\eta} \leq \sqrt{N\eta}$$

where the last line follows from the inequality between the $\ell_1$-norm and the $\ell_2$-norm: $\|a\|_1 \leq \sqrt{K} \|a\|_2$ for any $a \in \mathbb{R}^K$. Thus, we obtain

$$\|QQ^* s\|_2^2 = 1 - \|s - QQ^* s\|_2^2 \geq 1 - N\eta.$$ 

Since this result holds for an arbitrary unit vector $s$ in the subspace spanned by $S_K$, we finally have

$$\cos(\Theta_{S_K, q}) \geq \sqrt{1 - N\eta}.$$ 

\[\square\]

### C Proof of Lemma 5

**Proof of Lemma 5.** Let $\Pi$ be an $N \times N$ diagonal matrix with diagonal entries $j2\pi 0, j2\pi, \ldots, j2\pi (N - 1)$. The derivative of $\|e_f - UU^* e_f\|_2^2$ in terms of $f$ can be computed as

$$\frac{d}{df} \|e_f - UU^* e_f\|_2^2 = 2 \Re \left( e_f^* (I - UU^*) \Pi e_f \right).$$

We first obtain an upper bound for its derivative

$$\left| \frac{d}{df} \|e_f - UU^* e_f\|_2^2 \right| \leq 2 |e_f^* (I - UU^*) \Pi e_f| \leq 2 |e_f^* \Pi e_f| \|I - UU^*\| \leq 2\pi N (N - 1) \leq 2\pi N^2$$

for all $f \in [-\frac{1}{2}, \frac{1}{2}]$. Since $\|e_f - UU^* e_f\|_2^2$ is nonnegative and its derivative is bounded above, $\|e_f - UU^* e_f\|_2^2$ cannot be too large if $\int_{-W}^{W} \|e_f - UU^* e_f\|_2^2 df$ is very small.
Figure 3: Illustration of (16). The area below the black curve is always larger than or equal to the area of each red triangle.

Suppose \( \| e_f - UU^* e_f \|^2 \leq 2W \). As illustrated in Figure 3, for any \( f \in [-W, W] \), we can always find a triangle with area either

\[
\frac{\| e_f - UU^* e_f \|^4}{2 \sup_{f \in [-W, W]} \left| \frac{d}{df} \| e_f - UU^* e_f \|_2 \right|^2} (\text{the area of the left and right red triangles}) \text{ or } \frac{\| e_f - UU^* e_f \|^4}{\sup_{f \in [-W, W]} \left| \frac{d}{df} \| e_f - UU^* e_f \|_2 \right|^2} (\text{the area of the middle red triangle})
\]

that is smaller than \( \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df \) (the area under the black curve). This is made more precise as

\[
\frac{\| e_f - UU^* e_f \|^4}{4\pi N^2} \leq \frac{\| e_f - UU^* e_f \|^4}{2 \sup_{f \in [-W, W]} \left| \frac{d}{df} \| e_f - UU^* e_f \|_2 \right|^2} \leq \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df
\]

for all \( f \in [-W, W] \). Thus, we have

\[
\frac{\| e_f - UU^* e_f \|^2}{\| e_f \|_2^2} = \frac{\| e_f - UU^* e_f \|^2}{N} \leq 2\sqrt{\pi} \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df
\]

for all \( f \in [-W, W] \).

On the other hand, suppose \( \| e_f - UU^* e_f \|^2 \geq 2W \). With a similar argument, as illustrated in Figure 4, for any \( f \in [-W, W] \), we can always find a region of area at least \( W \| e_f - UU^* e_f \|_2^2 \) (the area indicated by red dashed lines) that is smaller than \( \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df \) (the area under the black curve). This is made more precise as

\[
W \| e_f - UU^* e_f \|_2^2 \leq \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df
\]

for all \( f \in [-W, W] \). Thus, we have

\[
\frac{\| e_f - UU^* e_f \|^2}{\| e_f \|_2^2} \leq \frac{1}{NW} \int_{-W}^W \| e_f - UU^* e_f \|_2^2 df
\]

for all \( f \in [-W, W] \).
Figure 4: Illustration of (17). The area below the black curve is always larger than or equal to the area shaded by the red dashed lines.

References

[1] Z. Zhu, S. Karnik, M. B. Wakin, M. A. Davenport, and J. K. Romberg, “Fast orthogonal approximations of sampled sinusoids and bandlimited signals,” in IEEE Conf. Acous., Speech, Signal Process. (ICASSP), pp. 4511–4515, 2017.

[2] D. Slepian, “Prolate Spheroidal Wave Functions, Fourier analysis, and uncertainty. V- The discrete case,” Bell Syst. Tech. J., vol. 57, no. 5, pp. 1371–1430, 1978.

[3] A. Papoulis, “A new algorithm in spectral analysis and band-limited extrapolation,” IEEE Trans. Circuits, Systems, vol. 22, no. 9, pp. 735–742, 1975.

[4] M. Hayes and R. Schafer, “On the bandlimited extrapolation of discrete signals,” in Proc. IEEE Int. Conf. Acoust., Speech, and Signal Processing (ICASSP), vol. 8, pp. 1450–1453, IEEE, 1983.

[5] T. Zemen and C. F. Mecklenbräuker, “Time-variant channel estimation using Discrete Prolate Spheroidal Sequences,” IEEE Trans. Signal Process., vol. 53, no. 9, pp. 3597–3607, 2005.

[6] T. Zemen, C. F. Mecklenbräuker, F. Kaltenberger, and B. H. Fleury, “Minimum-energy band-limited predictor with dynamic subspace selection for time-variant flat-fading channels,” IEEE Trans. Signal Process., vol. 55, no. 9, pp. 4534–4548, 2007.

[7] M. G. Amin and F. Ahmad, “Wideband synthetic aperture beamforming for through-the-wall imaging [lecture notes],” IEEE Signal Process. Magazine, vol. 25, no. 4, 2008.

[8] F. Ahmad, Q. Jiang, and M. G. Amin, “Wall clutter mitigation using Discrete Prolate Spheroidal Sequences for sparse reconstruction of indoor stationary scenes,” IEEE Trans. Geosci. Remote Sens., vol. 53, no. 3, pp. 1549–1557, 2015.

[9] Z. Zhu and M. B. Wakin, “Wall clutter mitigation and target detection using Discrete Prolate Spheroidal Sequences,” in 3rd Int. Workshop on Compressed Sensing Theory and its Applications to Radar, Sonar and Remote Sensing (CoSeRa), June 2015.

[10] Z. Zhu and M. B. Wakin, “On the dimensionality of wall and target return subspaces in through-the-wall radar imaging,” in 4th Int. Workshop on Compressed Sensing Theory and its Applications to Radar, Sonar and Remote Sensing (CoSeRa), September 2016.

[11] Z. Zhu, D. Yang, M. B. Wakin, and G. Tang, “A super-resolution algorithm for multiband signal identification,” in 51st Asilomar Conference on Signals, Systems and Computers, (Pacific Grove, California), Oct. 2017.

[12] M. Davenport, S. Schnelle, J. P. Slavinsky, R. Baraniuk, M. Wakin, and P. Boufounos, “A wideband compressive radio receiver,” in Proc. Military Comm. Conf. (MILCOM), (San Jose, California), Oct. 2010.
[13] M. A. Davenport and M. B. Wakin, “Compressive sensing of analog signals using discrete prolate spheroidal sequences,” *Appl. Comput. Harmon. Anal.*, vol. 33, no. 3, pp. 438–472, 2012.

[14] E. Sejdić, A. Can, L. F. Chaparro, C. M. Steele, and T. Chau, “Compressive sampling of swallowing accelerometry signals using time-frequency dictionaries based on modulated Discrete Prolate Spheroidal Sequences,” *EURASIP J. Adv. Signal Process.*, vol. 2012, no. 1, pp. 1–14, 2012.

[15] M. Wakin, S. Becker, E. Nakamura, M. Grant, E. Sovero, D. Ching, J. Yoo, J. Romberg, A. Emami-Neyestanak, and E. Candes, “A nonuniform sampler for wideband spectrally-sparse environments,” *IEEE J. Emerg. Sel. Topic Circuits Syst.*, vol. 2, no. 3, pp. 516–529, 2012.

[16] S. Karnik, Z. Zhu, M. B. Wakin, J. K. Romberg, and M. A. Davenport, “The fast Slepian transform,” to appear in *Appl. Comp. Harm. Anal.*, arXiv preprint arXiv:1611.04950.

[17] Z. Zhu and M. B. Wakin, “Approximating sampled sinusoids and multiband signals using multiband modulated DPSS dictionaries,” *J. Fourier Anal. Appl.*, vol. 23, pp. 1263–1310, Dec 2017.

[18] P. A. Businger and G. H. Golub, “Algorithm 358: Singular value decomposition of a complex matrix [f1, 4, 5],” *Comm. ACM*, vol. 12, no. 10, pp. 564–565, 1969.

[19] N. Halko, P. Martinsson, and J. A. Tropp, “Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions,” *SIAM Rev.*, vol. 53, no. 2, pp. 217–288, 2011.

[20] T. Zemen, M. Hofer, D. Loeschenbrand, and C. Pacher, “Orthogonal precoding for ultra reliable wireless communication links,” arXiv preprint arXiv:1710.09912, 2017.

[21] Y. Saad, *Iterative methods for sparse linear systems*. SIAM, 2003.

[22] N. Ailon and B. Chazelle, “The fast Johnson–Lindenstrauss transform and approximate nearest neighbors,” *SIAM J. Comput.*, vol. 39, no. 1, pp. 302–322, 2009.

[23] J. M. Hokanson, “Projected nonlinear least squares for exponential fitting,” arXiv preprint arXiv:1508.05890.

[24] B. P. Duggal, “Subspace gaps and range-kernel orthogonality of an elementary operator,” *Linear Algebra Appl.*, vol. 383, pp. 93–106, 2004.

[25] A. Björck and G. H. Golub, “Numerical methods for computing angles between linear subspaces,” *Math. Comput.*, vol. 27, no. 123, pp. 579–594, 1973.

[26] L. Mirsky, “A trace inequality of John von Neumann,” *Monatshefte für Mathematik*, vol. 79, no. 4, pp. 303–306, 1975.

[27] C. Eckart and G. Young, “The approximation of one matrix by another of lower rank,” *Psychometrika*, vol. 1, no. 3, pp. 211–218, 1936.