NAVIER-STOKES EQUATIONS INTERACTING WITH A NONLINEAR ELASTIC FLUID SHELL

C.H. ARTHUR CHENG, DANIEL COUTAND, AND STEVE SHKOLLER

ABSTRACT. We study a moving boundary value problem consisting of a viscous incompressible fluid moving and interacting with a nonlinear elastic fluid shell. The fluid motion is governed by the Navier-Stokes equations, while the fluid shell is modeled by a bending energy which extremizes the Willmore functional and a membrane energy that extremizes the surface area of the shell. The fluid flow and shell deformation are coupled together by continuity of displacements and tractions (stresses) along the moving material interface. We prove existence and uniqueness of solutions in Sobolev spaces.

1. Introduction

1.1. The problem statement and background. We are concerned with establishing the existence and uniqueness of solutions to the time-dependent incompressible Navier-Stokes equations interacting with a nonlinear elastic fluid shell (bio-membrane). Recently, there have been many experimental and analytic studies on the configurations and deformations of elastic bio-membranes (see, for example, [5], [15], [18], [23], [24], [26], and [27]), but the basic analysis of the coupled fluid-structure interaction remains open. The fundamental difficulties arise from the degenerate elliptic operators that arise as the shell tractions. As we detail below, the bending energy of the shell is the well-known Willmore function, the integral over the moving surface of the square of the mean curvature. The degenerate elliptic operator arising from the first variation of this functional is a fourth order nonlinear operator that smoothes only in the direction which is normal to the moving domain. Our analysis will provide a well-posedness theorem and explain the interesting interaction between the shape of the shell and the flow of the fluid.

Fluid-structure interaction problems involving moving material interfaces have been the focus of active research since the nineties. The first problem solved in this area was for the case of a rigid body moving in a viscous fluid (see [12], [20] and also the early works of [25] and [26] for a rigid body moving in a Stokes flow in the full space). The case of an elastic body moving in a viscous fluid was considerably more challenging because of some apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic solid phase. The first existence results in this area were for regularized elasticity laws, such as in [13] for a finite number of elastic modes, or in [2], [4], and [3] for hyperviscous elasticity laws, or in [22] in which a phase-field regularization “fattens” the sharp interface via a diffuse-interface model.

The treatment of classical elasticity laws for the solid phase, without any regularizing term, was only considered recently in [10] for the three-dimensional linear St. Venant-Kirchhoff constitutive law and in [11] for quasilinear elastodynamics coupled to the Navier-Stokes equations. Some of the basic new ideas introduced in those works concerned a functional framework that scales in a hyperbolic fashion...
(and is therefore driven by the solid phase), the introduction of approximate problems either penalized with respect to the divergence-free constraint in the moving fluid domain, or smoothed by an appropriate parabolic artificial viscosity in the solid phase (chosen in an asymptotically convergent and consistent fashion), and the tracking of the motion of the interface by difference quotients techniques.

In our companion paper [6], we study the interaction of the Navier-Stokes equations with an elastic solid shell. Herein, we treat the case of a fluid shell or biomembrane. This is a moving boundary problem that models the motion of a viscous incompressible Newtonian fluid inside of a deformable elastic fluid structure. Let \( \Omega \subset \mathbb{R}^3 \) denote an open bounded domain with boundary \( \Gamma := \partial \Omega \). For each \( t \in (0,T] \), we wish to find the domain \( \Omega(t) \), a divergence-free velocity field \( u(t,\cdot) \), a pressure function \( p(t,\cdot) \) on \( \Omega(t) \), and a volume-preserving transformation \( \eta(t,\cdot) : \Omega \rightarrow \mathbb{R}^3 \) such that

\[
\begin{align*}
\Omega(t) &= \eta(t,\Omega), \\
\eta_t(t,x) &= u(t,\eta(t,x)), \\
u_t + \nabla u - \nu \Delta u &= -\nabla p + f \quad \text{in} \; \Omega(t), \\
\text{div} u &= 0 \quad \text{in} \; \Omega(t), \\
(\nu \text{Def } u - p \text{Id})n &= t_{\text{shell}} \quad \text{on} \; \Gamma(t), \\
u(0,x) &= u_0(x) \quad \forall x \in \Omega, \\
\eta(0,x) &= x \quad \forall x \in \Omega,
\end{align*}
\]

where \( \nu \) is the kinematic viscosity, \( n(t,\cdot) \) is the outward pointing unit normal to \( \Gamma(t) \), \( \Gamma(t) := \partial \Omega(t) \) denotes the boundary of \( \Omega(t) \), \( \text{Def } u \) is twice the rate of deformation tensor of \( u \), given in coordinates by \( u_{i,j} + u_{j,i} \), and \( t_{\text{shell}} \) is the traction imparted onto the fluid by the elastic shell, which we describe next.

We shall consider a thin elastic shell modeled by the nonlinear Saint Venant-Kirchhoff constitutive law. With \( \epsilon \) denoting the thickness of the shell, the hyperelastic stored energy function has the asymptotic expansion

\[
E_{\text{shell}} = \epsilon E_{\text{mem}} + \epsilon^3 E_{\text{ben}} + O(\epsilon^4).
\]

The membrane energy satisfies

\[
E_{\text{mem}} = \gamma \int_{\Gamma(t)} dS = \gamma \text{ times the surface area of } \Gamma(t)
\]

where \( \gamma > 0 \) is the surface tension, while the bending energy \( E_{\text{ben}} \) is given by

\[
E_{\text{ben}} = \int_{\Gamma(t)} \left[ (4\mu + 2\lambda)H^2 - 2\mu K \right] dS,
\]

where \( H, K \) denote the mean and Gauss curvatures on \( \Gamma(t) \), respectively, and \( \lambda/2 \) and \( \mu/2 \) are the Lamé constants (see, for example, [17]).

The traction vector

\[
t_{\text{shell}} = \epsilon t_{\text{mem}} + \epsilon^3 t_{\text{ben}} + O(\epsilon^4)
\]

is computed from the first variation of the energy function \( E_{\text{shell}} \); the traction vector associated to the membrane energy is well-known to be

\[
t_{\text{mem}} = \gamma H n,
\]
while the traction associated to the bending energy has a simple intrinsic characterization given by
\[ t_{\text{ben}} = \sigma(\Delta_g H - 2HK + 2H^3)n, \]
where \( \sigma \) is a function of the Lamé constants and \( \Delta_g \) denotes the Laplacian with respect to the induced metric \( g \) on \( \Gamma(t) \):
\[ \Delta_g f = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{\det(g)} g^{\alpha \beta} \frac{\partial f}{\partial x^\beta} \right). \]

1.2. Outline of the paper. In Section 2, in addition to the use of Lagrangian variables, we introduce a new coordinate system near the boundary (shell) and three new maps, \( \eta^\nu \), \( \eta^\tau \), and \( h \), which facilitate the computation of the membrane and bending tractions \( t_{\text{mem}} \) and \( t_{\text{ben}} \). A key observation is the symmetry relation (2.7) which reduces the derivative count on the tangential reparameterization map \( \eta^\tau \).

The space of solutions is introduced in Section 3 and the main theorem is stated in Section 4. Section 5 defines our notation, and Section 6 provides some useful lemmas.

In Section 7, we introduce the linearized and regularized problems. The regularization requires smoothing certain variables as well as the introduction of a certain artificial viscosity term on the boundary of the fluid domain. Weak solutions of this linear problem are established via a penalization scheme which approximates the incompressibility of the fluid.

In Section 8, we establish a regularity theory for our weak solution using energy estimates for the mollified problem with constants that depend on the mollification parameters. In Section 9, we improve these estimates so that the constants are independent of the artificial viscosity as well as other regularization parameters. This requires an elliptic estimate, arising from the boundary condition (1.1e), which provides additional regularity for the shape of the boundary.

In Section 10, the Tychonoff fixed-point theorem is used to prove the existence of solutions to the original nonlinear problem (1.1). Uniqueness, following required compatibility conditions, is established in Sections 4 and Section 10.

The inclusion of the inertial term \( \epsilon \eta_{tt} \) into the membrane traction \( t_{\text{mem}} \) will be studied in a future publication.

2. Lagrangian formulation

2.1. A new coordinate system near the shell. Consider the isometric immersion \( \eta_0 : (\Gamma, g_0) \to (\mathbb{R}^3, \text{Id}) \). Let \( B = \Gamma \times (-\epsilon, \epsilon) \) where \( \epsilon \) is chosen sufficiently small so that the map
\[ B : B \to \mathbb{R}^3 : (y, z) \mapsto y + zN(y) \]
is itself an immersion, defining a tubular neighborhood of \( \Gamma \) in \( \mathbb{R}^3 \). We can choose a coordinate system \( \frac{\partial}{\partial y^\nu} \), \( \alpha = 1, 2 \) and \( \frac{\partial}{\partial z} \) on \( B \) where \( \frac{\partial}{\partial y^\nu} \) denotes the tangential derivative and \( \frac{\partial}{\partial z} \) denotes the normal derivative.

Let \( G = B^*(\text{Id}) \) denote the induced metric on \( B \) from \( \mathbb{R}^3 \) so that
\[ G(y, z) = G_z(y) + dz \otimes dz, \]
where \( G_z \) is the metric on the surface \( \Gamma \times \{z\} \); note that \( G_0 = g_0 \).
Remark 1. By assumption, \( g_{\alpha \beta} = \frac{\partial}{\partial y^\alpha} \cdot \frac{\partial}{\partial y^\beta} \), where \( \cdot \) denotes the usual Cartesian inner-product on \( \mathbb{R}^n \). Let \( C_{\alpha \beta} \) denote the covariant components of the second fundamental form of the base manifold \( \Gamma \), so that \( C_{\alpha \beta} = -N_{,\alpha} \frac{\partial}{\partial y^\beta} \). Then, \( G_z \) is given by

\[
(G_z)_{\alpha \beta} = (g_0)_{\alpha \beta} - 2z C_{\alpha \beta} + z^2 g_0^{\gamma \delta} C_{\alpha \gamma} C_{\beta \delta}.
\]

Let \( h : \Gamma \to (-\epsilon, \epsilon) \) be a smooth height function and consider the graph of \( h \) in \( B \), parameterized by \( \phi : \Gamma \to B \) : \( y \mapsto (y, h(y)) \). The tangent space to graph\((h)\), considered as a submanifold of \( B \), is spanned at a point \( \phi(x) \) by the vectors

\[
\phi_* \left( \frac{\partial}{\partial y^\alpha} \right) = \frac{\partial \phi}{\partial y^\alpha} = \frac{\partial h}{\partial y^\alpha} \frac{\partial}{\partial y^\alpha} + \frac{\partial}{\partial z} \frac{\partial}{\partial \gamma^\alpha},
\]

and the normal to graph\((h)\) is given by

\[
n(y) = J_h^{-1}(y) \left( -G_{\beta \gamma} \frac{\partial h}{\partial y^\beta} \frac{\partial}{\partial y^\gamma} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) \quad (2.1)
\]

where \( J_h = (1 + h_{,\alpha} C_{\alpha \beta} h_{,\beta})^{1/2} \). The mean curvature \( H \) of graph\((h)\) is defined to be the trace of \( \nabla n \) where

\[
(\nabla n)_{ij} = G(\nabla_{y^i} n, \frac{\partial}{\partial \omega^j}) \quad \text{for } i, j = 1, 2, 3
\]

where \( \frac{\partial}{\partial \omega^\alpha} = \frac{\partial}{\partial y^\alpha} \) for \( \alpha = 1, 2 \) and \( \frac{\partial}{\partial \omega^3} = \frac{\partial}{\partial z} \), and \( \nabla B \) denotes the covariant derivative. Using (2.1),

\[
(\nabla n)_{\alpha \beta} = G \left( \nabla_{y^\alpha} \left[ -J_h^{-1} G^\gamma_h h_{,\gamma} \frac{\partial}{\partial y^\gamma} + J_h^{-1} \frac{\partial}{\partial z} \right], \frac{\partial}{\partial y^\beta} \right)
\]

or (in the divergence form)

\[
H = -(J_h^{-1} G^\gamma_h h_{,\gamma})_{,\delta} + J_h^{-1} (\gamma^\delta_h h_{,\gamma} \Gamma^j_{\gamma j} + \Gamma^j_{\beta j}),
\]

or (in the quasilinear form)

\[
H = -J_h^{-1} G_h^{\alpha \beta} \left[ \delta_{\beta \gamma} - J_h^{-2} G_h^{\gamma \delta} h_{,\gamma} h_{,\delta} \right] h_{,\alpha \gamma} + G_h^{\alpha \beta} F_{\alpha \beta}(y, h, \nabla h),
\]

where \( F_{\alpha \beta} \) denotes a smooth generic function of \( y, h \) and \( \nabla h \).

Remark 2. Note that \( G_h \) denotes the metric \( G_{z=h(y)} \), and not the metric on the submanifold graph\((h)\).

Remark 3. If the initial height function is zero, i.e., \( h(0) = 0 \), then \( H(0) = \Gamma^j_{\gamma j}(0) \), which is the mean curvature of the base manifold \( \Gamma \) as required.
2.2. Tangential reparameterization symmetry. Let $\mathcal{N}$ denote the normal bundle to $\Gamma$, so that for each $y \in \Gamma$, we have the Whitney sum $\mathbb{R}^3 = T_y \Gamma \oplus \mathcal{N}_y$.

Given a signed height function $h : \Gamma \times [0, T) \to \mathbb{R}$, for each $t \in [0, T)$, define the normal map

$$\eta^\nu : \Gamma \times [0, T) \to \Gamma, \quad (y, t) \mapsto y + h(y, t)N(y), \quad N(y) \in \mathcal{N}_y.$$ 

Then, there exists a unique tangential map $\eta^\tau : \Gamma \times [0, T) \to \Gamma$ (a diffeomorphism as long as $h$ remains a graph) such that the diffeomorphism $\eta(t)$ has the decomposition

$$\eta(t, \cdot) = \eta^\nu(\cdot, t) \circ \eta^\tau(\cdot, t), \quad \eta(y, t) = \eta^\tau(y, t) + h(\eta^\tau(y, t), t)N(\eta^\tau(y, t)).$$

The tangent vector $\eta,^\alpha$ to $\Gamma(t)$ can be decomposed with respect to the Whitney sum as $\eta,^\alpha = \eta^\nu,^\alpha \partial / \partial y^\alpha + h,^\alpha \eta^\nu,^\alpha \partial / \partial z^\alpha$ and hence the induced metric $g_{\alpha \beta} = \eta,^\alpha \cdot \eta,^\beta$ may be expressed as

$$g_{\alpha \beta} = \left[ (G_h,_{\kappa \sigma} + h,_{\kappa \sigma}) \circ \eta^\nu \right] \eta^\nu,^\alpha \eta^\nu,^\beta := \left[ \mathcal{G}_{\kappa \sigma} \circ \eta^\nu \right] \eta^\nu,^\alpha \eta^\nu,^\beta. \quad (2.4)$$

Note that $\mathcal{G}_{\kappa \sigma}$ is the induced metric with respect to the normal map $\eta^\nu$. Furthermore, we have the following useful relationship between the determinant of the two induced metrics:

$$\det(g) = \det(\nabla_0 \eta^\nu)^2 \left[ \det(G_h) \eta^\nu \right] \circ \eta^\nu = \det(\nabla_0 \eta^\nu)^2 \left[ \det(\mathcal{G}) \right] \circ \eta^\nu \quad (2.5)$$

where $\nabla_0$ denotes the surface gradient.

**Remark 4.** The identity (2.4) can also be read as $(\eta^\nu)^* g = \mathcal{G}$.

Let $y$ and $\tilde{y} = \varphi(y)$ denote two different coordinate systems on $\Gamma$ with associated metrics

$$g_{\alpha \beta} = \partial \eta^\alpha / \partial y^\alpha \cdot \partial \eta^\beta / \partial y^\beta, \quad \tilde{g}_{\alpha \beta} = \partial \tilde{\eta}^\alpha / \partial \tilde{y}^\alpha \cdot \partial \tilde{\eta}^\beta / \partial \tilde{y}^\beta.$$ 

It follows that $\varphi^* \tilde{g} = g$. Let $H$, $\tilde{H}$, $K$, $\tilde{K}$, $n$ and $\tilde{n}$ denote the mean curvature, Gauss curvature, and the unit normal vector computed with respect to $y$ and $\tilde{y}$, respectively. Since $H$, $K$, and $n$ depend only on the shape of $\Gamma(t)$, these geometric quantities are invariant to tangential reparameterization; thus, the identity

$$\tilde{H} = H \circ \varphi, \quad \tilde{K} = K \circ \varphi, \quad \tilde{n} = n \circ \varphi. \quad (2.6)$$
Similarly, computing the first variation of \( \int_{\Gamma(t)} H^2 dS \) in our two coordinate systems yields
\[
\left[ (\Delta g H + H(H^2 - K)) n \right](y) = \left[ \frac{\partial}{\partial y} \left( \Delta_g \tilde{H} + \tilde{H}(H^2 - \tilde{K}) \right) n \right](\tilde{y}) \quad \forall \tilde{y} = \varphi(y).
\]
By (2.6), we have the following important identity
\[
\left[ \Delta_{\varphi^*g} H \right](y) = \left[ \Delta_{\varphi}(H \circ \varphi) \right](\tilde{y}) \quad \forall \tilde{y} = \varphi(y)
\] (2.7)
and hence
\[
\left[ \Delta_{\varphi}(H \circ \eta^{-\gamma}) \right] \circ \eta^\gamma = \Delta_\gamma H
\] (2.8)
where by (2.3),
\[
H \circ \eta^{-\gamma} = -J_h^{-1} G_0^{\alpha\beta} \left[ \delta_{\beta \gamma} - J_h^{-2} G_0^{\gamma\delta} h_{\beta \delta} \right] h_{\alpha \gamma} + G_0^{\alpha\beta} F_{\alpha\beta}(y, h, \nabla h).
\] (2.9)

2.3. Bounds on \( \eta^\gamma \). Let \( \eta^\gamma \) denote the tangential velocity defined by \( \eta^\gamma = u^\gamma \circ \eta^{-\gamma} \). Time-differentiating the relation \( \eta = \eta^\gamma \circ \eta^{-\gamma} \) and using the definition of \( \eta^\gamma \), we find that
\[
u \cdot \eta^\gamma = (\nabla \varphi \eta^\gamma)^{-1} \left[ (\nabla \eta) - h \frac{\partial}{\partial z} \right].
\] (2.10)
\[\] From the trace theorem, it follows that
\[
\| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} \leq \mathcal{C} \mathcal{P} \left[ \| \eta \|_{L^2(\Omega)} \right]
\] (2.11)
for some polynomial \( \mathcal{P} \). Since, \( \eta^\gamma(y, t) = \eta + \int_0^t \nabla \varphi (y, s) ds \), it follows that
\[
\| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} \leq C \left[ 1 + \int_0^t \| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} ds \right]^4
\] (2.12)
and hence by Gronwall’s inequality,
\[
\| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} \leq C \left[ 1 + \int_0^t \| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} ds \right]^4
\] (2.13)
for \( t \in [0, T] \) sufficiently small. Furthermore, we also have
\[
\| \eta^\gamma \|_{H^{1.5}(\Gamma)} \leq C \left[ 1 + \| \nabla \varphi \eta^\gamma \|_{H^{1.5}(\Gamma)} \right]^4.
\] (2.13)

2.4. An expression for \( t_{\text{ben}} \) in terms of \( h \) and \( \eta^\gamma \). Now we can compute \( t_{\text{ben}} \) in terms of \( h \) and \( \eta^\gamma \); the highest order term of \( \Delta_\gamma H \) is
\[
\left\{ \frac{1}{\sqrt{\text{det}(g)}} \frac{\partial}{\partial y^\alpha} \left[ \sqrt{\text{det}(g)} G^{\gamma \delta} \frac{\partial}{\partial y^\gamma} \left( J_h^{-1} (G_h^{\alpha \beta} - J_h^{-2} G_0^{\alpha \gamma} G_h^{\delta \gamma} h_{\alpha \beta} h_{\gamma \delta} ) h_{\alpha \beta} \right) \right] \right\} \circ \eta^\gamma.
\]
Since \( G_{\alpha \beta} = (G_h)_{\alpha \beta} + h_{\alpha \beta} h_{\delta \gamma} \), the inverse of \( G^{\gamma \delta} \) is
\[
\frac{1}{\text{det}(g)} \left[ \begin{array} {ccc} (G_h)_{22} + h_{22}^2 & -(G_h)_{12} - h_{12} \frac{h_{22}}{2} & (G_h)_{11} - h_{11} \frac{h_{22}}{2} \\ -(G_h)_{12} - h_{12} \frac{h_{22}}{2} & (G_h)_{11} - h_{11} \frac{h_{22}}{2} & -(G_h)_{22} + h_{22}^2 \end{array} \right]
\]
which can also be written as
\[
G^{\alpha \beta} = J_h^{-2} \left[ G_h^{\alpha \beta} - (1 - \delta_{\alpha \gamma} (1 - \delta_{\beta \delta})) h_{\alpha \beta} h_{\gamma \delta} \right].
\]
Therefore, the highest order term of \( \Delta_\gamma H \) can be written as
\[
\frac{1}{\sqrt{\text{det}(g)}} \left[ \sqrt{\text{det}(g)} A^{\alpha \beta \gamma \delta} h_{\alpha \beta} \right] \circ \eta^\gamma
\]
where
\[ A^{\alpha\beta\gamma\delta} = J_h^{-3} \left[ G_h^{\alpha\gamma} - (-1)^{\kappa+\sigma} \text{det}(G_h)^{-1} (1 - \delta_{\alpha\kappa})(1 - \delta_{\gamma\sigma})h,\alpha h,\sigma \right] \] (2.14)
\[ \times (G_h^{\beta\delta} - J_h^{-2} G_h^{\beta\kappa} G_h^{\kappa\sigma} h,\alpha h,\sigma) \]
is a fourth-rank tensor.

2.5. **Lagrangian formulation of the problem.** Let \( \eta(t, x) = x + \int_0^t u(s, x)ds \) denote the Lagrangian particle placement field, a volume-preserving embedding of \( \Omega \) onto \( \Omega(t) \subset \mathbb{R}^3 \), and denote the cofactor matrix of \( \nabla \eta(x, t) \) by
\[ a(x, t) = [\nabla \eta(x, t)]^{-1}. \] (2.15)

Let \( v = u \circ \eta \) denote the Lagrangian or material velocity field, \( q = p \circ \eta \) the Lagrangian pressure function, and \( F = f \circ \eta \) the forcing function in the material frame. In the following discussion, we also set \( \epsilon = 1 \). Then the system (1.1) can be reformulated as
\[ \eta_t = v \quad \text{in} \ (0, T) \times \Omega, \] (2.16a)
\[ v_i^t - \nu(a_i^k D_\eta(v_i^t))_j = -(a_i^k q)_k + F^i \quad \text{in} \ (0, T) \times \Omega, \] (2.16b)
\[ a_i^k v_i^t = 0 \quad \text{in} \ (0, T) \times \Omega, \] (2.16c)
\[ (\nu D_\eta(v_i^t) - q_\eta N_j = \sigma \Theta \left[ L(h) B_*(\text{I} - G_h^{\alpha\beta} h,\alpha,1) \right] \circ \eta^\tau \quad \text{on} \ (0, T) \times \Gamma, \] (2.16d)
\[ h_t = B_*(\text{I} - G_h^{\alpha\beta} h,\alpha,1) \cdot (v \circ \eta^{-\tau}) \quad \text{on} \ (0, T) \times \Gamma, \] (2.16e)
\[ v = u_0 \quad \text{on} \ \{ t = 0 \} \times \Omega, \] (2.16f)
\[ h = 0 \quad \text{on} \ \{ t = 0 \} \times \Gamma, \] (2.16g)
\[ \eta = \text{Id} \quad \text{on} \ \{ t = 0 \} \times \Omega, \] (2.16h)

where \( D_\eta(v_i^t) := (a_i^k v_i^t + a_i^k v_i^t) \), \( N \) denotes the outward-pointing unit normal to \( \Gamma \), \( \Theta \) is defined in Remark 5 and \( B_* \) is the push-forward of \( B \) defined as
\[ B_*(\gamma'(0)) = (B \circ \gamma)'(0) \quad \forall \gamma(t) \subset \Gamma. \]

\( L(h) \) is the representation of \( t_{\text{shell}} \cdot n \) using the height function \( h \). It is defined as follows
\[ L(h) = \frac{1}{\sqrt{\text{det}(g_0)}} \left[ \sqrt{\text{det}(g_0)} A^{\alpha\beta\gamma\delta} h,\alpha h,\beta h,\gamma h,\delta \right] + L_1^{\alpha\beta\gamma}(y, h, Dh, D^2h) h,\alpha h,\beta h,\gamma \]
\[ + L_2(y, h, Dh, D^2h) \]
where \( L_1 \) and \( L_2 \) are polynomials of their variables with \( L_1(y, 0) = 0, g_0 \) is the metric tensor on \( \Gamma \). Note that \( t_{\text{mem}} \) is included in \( L_2 \) since it is a second order operator of \( h \).

**Remark 5.** For a point \( \eta(y, t) \in \Gamma(t) \), there are two ways of defining the unit normal \( n \) to \( \Gamma(t) \):
1. Let \( n = \sqrt{J_h^{-1}} a^T N \) where \( N \) is the unit normal to \( \Gamma \).
2. Let \( n = \left[ J_h^{-1} \left( - G_h^{\alpha\beta} h,\alpha \frac{\partial}{\partial y^\rho} + \frac{\partial}{\partial z^\sigma} \right) \right] \circ \eta^\tau \) (denoted by \( J_h^{-1}(-\nabla_\eta h, 1) \circ \eta^\tau \)).

The function \( \Theta \) is defined by
\[ \Theta(-\nabla_\eta h \circ \eta^\tau, 1) = a^T N. \]
Equating the modulus of both sides, by (2.46) we must have
\[ \Theta = \sqrt{\det(g)(J_h^{-1}) \circ \eta^*} = \det(\nabla \eta^*) \sqrt{\det(G_h) \circ \eta^*}. \]

**Remark 6.** An equivalent form of (2.16d) is given by
\[ h_t = - h_{\alpha}(v \circ \eta^*)_\alpha + (v \circ \eta^*)_z. \]
This equation states that the shape of the boundary moves with the normal velocity of the fluid.

**Remark 7.** For many of the nonlinear estimates that appear later, it is important that \( L(h) \) is linear in the third derivative \( h_{\alpha\beta\gamma} \).

**Remark 8.** Without using the symmetry (2.8), we can still compute \( \Delta_g \mathcal{H} \) in terms of \( h \) and \( \eta^* \) by using (2.4) and (2.5); however, \( L_1 \) would then depend on \( \nabla^2 \eta^* \) and thus lose one derivative of regularity, preventing the closure of our energy estimate.

### 3. Notation and conventions

For \( T > 0 \), we set
\[
\mathcal{V}^1(T) = \left\{ v \in L^2(0, T; H^1(\Omega)) \mid v_t \in L^2(0, T; H^1(\Omega')) \right\};
\]
\[
\mathcal{V}^2(T) = \left\{ v \in L^2(0, T; H^2(\Omega)) \mid v_t \in L^2(0, T; L^2(\Omega)) \right\};
\]
\[
\mathcal{V}^k(T) = \left\{ v \in L^2(0, T; H^k(\Omega)) \mid v_t \in L^2(0, T; H^{k-2}(\Omega)) \right\} \quad \text{for } k \geq 3;
\]
\[
\mathcal{H}(T) = \left\{ h \in L^2(0, T; H^{5.5}(\Gamma)) \mid h_t \in L^2(0, T; H^{2.5}(\Gamma)); h_{tt} \in L^2(0, T; H^{0.5}(\Gamma)) \right\}
\]

with norms
\[
\|v\|_{\mathcal{V}^1(T)}^2 = \|v\|_{L^2(0,T;H^1(\Omega))}^2 + \|v_t\|_{L^2(0,T;H^1(\Omega'))}^2;
\]
\[
\|v\|_{\mathcal{V}^2(T)}^2 = \|v\|_{L^2(0,T;H^2(\Omega))}^2 + \|v_t\|_{L^2(0,T;L^2(\Omega))}^2;
\]
\[
\|v\|_{\mathcal{V}^k(T)}^2 = \|v\|_{L^2(0,T;H^k(\Omega))}^2 + \|v_t\|_{L^2(0,T;H^{k-2}(\Omega))}^2 \quad \text{for } k \geq 3;
\]
\[
\|h\|_{\mathcal{H}(T)}^2 = \|h\|_{L^2(0,T;H^{5.5}(\Gamma))}^2 + \|h_t\|_{L^2(0,T;H^{2.5}(\Gamma))}^2 + \|h_{tt}\|_{L^2(0,T;H^{0.5}(\Gamma))}^2.
\]

We then introduce the space (of “divergence free” vector fields)
\[
\mathcal{V}_v = \left\{ w \in H^1(\Omega) \mid a_i^j(t)w^j = 0 \quad \forall \ t \in [0,T] \right\}
\]
and
\[
\mathcal{V}_v(T) = \left\{ w \in L^2(0,T;H^1(\Omega)) \mid a_i^j(t)w^j = 0 \quad \forall \ t \in [0,T] \right\},
\]
where the cofactor matrix \( a \) is defined by (2.15). We use \( X_T \) to denote the space \( \mathcal{V}^3(T) \times \mathcal{H}(T) \) with norm
\[
\|(v,h)\|_{X_T}^2 = \|(v,h)\|_{\mathcal{V}^3(T)}^2 + \|h\|_{\mathcal{H}(T)}^2
\]
and use \( Y_T \), a subspace of \( X_T \), to denote the space
\[
Y_T = \left\{ (v,h) \in \mathcal{V}^3(T) \times \mathcal{H}(T) \mid h_t \in L^\infty(0,T;H^2(\Gamma)) \right\}
\]
with norm
\[
\|(v,h)\|_{Y_T}^2 = \|(v,h)\|_{X_T}^2 + \|v\|_{L^\infty(0,T;H^2(\Omega))}^2 + \|h\|_{L^\infty(0,T;H^4(\Gamma))}^2 + \|h_t\|_{L^\infty(0,T;H^2(\Gamma))}^2.
\]

We will solve (2.16) by a fixed-point method in an appropriate subset of \( Y_T \).
4. THE MAIN THEOREM

Before stating the main theorem, we define the following quantities. Let \( q_0 \) be defined by

\[
\Delta q_0 = -\nabla u_0 : (\nabla u_0)^T + \nu [a_k^i \delta_{ij} a_j^k u_0 \cdot u_0]_{ij} (0) + \text{div} F(0) \quad \text{in} \quad \Omega, \quad (4.1a)
\]

\[
q_0 = \nu (\text{Def} u_0 \cdot N) \cdot N - \sigma L(0) \quad \text{on} \quad \Gamma \quad (4.1b)
\]

and

\[
u \Delta u_0 - \nabla q_0 + F(0).
\]

We also define the projection operator \( P_{ij}(x) : \mathbb{R}^3 \to T_{\eta(t)} \Gamma(t) \) by

\[
P_{ij}(x) = [(a_i^k \delta_{ij} a_j^k u_0 \cdot u_0 + \text{div} F(0))]_{ij} (0) = \begin{bmatrix}
\delta_{ij} - a_i^k N_k(x) \frac{a_j^k N_l(x)}{|a_i^k N_k(x)| |a_j^k N_l(x)|}
\end{bmatrix}.
\]

THEOREM 4.1. Let \( \nu > 0, \sigma > 0 \) be given, and

\[ F \in L^2(0, T; H^2(\Omega)), \quad F_t \in L^2(0, T; L^2(\Omega)), \quad F(0) \in H^1(\Omega). \]

Suppose that the shell traction satisfies the compatibility condition

\[
(\text{Def} u_0 \cdot N)_{tan} = 0. \quad (4.3)
\]

There exists \( T > 0 \) depending on \( u_0 \) and \( F \) such that there exists a solution \((v, h) \in Y_T \) of problem \((2.10)\). Moreover, if \( u_0 \in H^{5.5}(\Omega) \cap H^{7.5}(\Gamma) \) and the associated \( u_1 \), \( q_0 \) also satisfy the compatibility condition

\[
CP := \left[ g_0^k u_0^i N_t j + g_0^k u_0^i N_t j \right] \nu (\text{Def} u_0)_{i}^j - q_0 \delta_{ij} \right] N_j
+ \nu (\delta_{ij} - N_i N_t) \left[ (\text{Def} u_1)_{i}^j - \left( (\nabla u_0 \nabla u_0) + (\nabla u_0 \nabla u_0)^T \right)_{i}^j \right] N_j
- (\delta_{ij} - N_i N_t) \left[ (\nu (\text{Def} u_0)_{i}^j + q_0 \delta_{ij}) \right] u_0^j N_k = 0
\]

then the solution \((v, h) \in Y_T \) is unique.

5. A BOUNDED CONVEX CLOSED SET OF \( Y_T \)

DEFINITION 5.1. Given \( M > 0 \). Let \( \mathcal{C}_T(M) \) denote the subset of \( Y_T \) consisting of elements of \((v, h) \in Y_T\) such that

\[
\| (v, h) \|_{Y_T} \leq M \quad (5.1)
\]

and such that \( v(0) = u_0, \ h(0) = 0 \) and \( h(t) = (B_0)_{\tau}((0, 1)) \cdot u_0. \)

REMARK 9. For \((v, h) \in \mathcal{C}_T(M)\), define \( u^\tau \) by \((2.10)\) and let \( \eta^\tau \) be the associated flow map. Also define \( v^\tau \) as \( u^\tau \circ \eta^\tau \). By \((2.12)\) and \((2.13)\), we have

\[
\sup_{t \in [0, T]} \| \nabla_0 \eta^\tau(t) \|_{H^{5.5}(\Gamma)} + \| v^\tau \|_{L^2(0, T; H^{2.5}(\Gamma))} \leq C(M) \quad (5.2)
\]

for some constant \( C(M) \).

We will make use of the following lemmas (proved in \( [10] \)):

LEMMA 5.1. There exists \( T_0 \in (0, T) \) such that for all \( T \in (0, T_0) \) and for all \( v \in \mathcal{C}_T(M) \), the matrix \( a \) is well-defined (by \((2.13)\)) with the estimate (independent of \( v \in \mathcal{C}_T(M) \))

\[
\| a \|_{L^\infty(0, T; H^2(\Omega))} + \| a_{ii} \|_{L^\infty(0, T; H^1(\Omega))} + \| a_{ii} \|_{L^2(0, T; H^2(\Omega))}
+ \| a_{ii} \|_{L^\infty(0, T; L^2(\Omega))} + \| a_{ii} \|_{L^2(0, T; H^1(\Omega))} \leq C(M). \quad (5.3)
\]
Lemma 5.2. There exists $T_1 \in (0, T)$ and a constant $C$ (independent of $M$) such that for all $T \in (0, T_1)$ and $v \in C_T(M)$, for all $\phi \in H^1(\Omega)$ and $t \in [0, T]$

$$C\|\phi\|_{H^1(\Omega)}^2 \leq \int_\Omega \left[|v|^2 + |D_\eta(v)|^2\right] dx$$  \hspace{1cm} (5.4)

where

$$|D_\eta(v)|^2 := D_\eta(v)^T D_\eta(v)^i = (a_j^k v_j^i + a_j^k v_j^i)(a_j^k v_j^i + a_j^k v_j^i).$$

In the remainder of the paper, we will assume that

$$0 < T < \min\{T_0, T_1, \bar{T}\}$$

for some fixed $\bar{T}$ where the forcing $F$ is defined on the time interval $[0, \bar{T}]$.

6. Preliminary results

6.1. Pressure as a Lagrange multiplier. In the following discussion, we use $H^{1,2}(\Omega; \Gamma)$ to denote the space $H^1(\Omega) \cap H^2(\Gamma)$ with norm

$$\|u\|_{H^{1,2}(\Omega; \Gamma)} = \|u\|_{H^1(\Omega)} + \|u\|_{H^2(\Gamma)}$$

and $\tilde{V}_0 (\tilde{V}_0(T))$ to denote the space

$$\left\{v \in V_0 \mid v \in H^2(\Gamma)\right\} \cap \left\{v \in V_0(T) \mid v \in L^2(0, T; H^2(\Gamma))\right\}.$$

Lemma 6.1. For all $p \in L^2(\Omega)$, $t \in [0, T]$, there exists a constant $C > 0$ and $\phi \in H^{1,2}(\Omega; \Gamma)$ such that $a_j^i(t) \phi_j^i = p$ and

$$\|\phi\|_{H^{1,2}(\Omega; \Gamma)} \leq C \|p\|_{L^2(\Omega)}.$$  \hspace{1cm} (6.1)

Proof. We solve the following problem on the time-dependent domain $\Omega(t)$:

$$\text{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} \quad \text{in} \quad \eta(t, \Omega) := \Omega(t).$$

The solution to this problem can be written as the sum of the solutions to the following two problems

$$\text{div}(\phi \circ \eta(t)^{-1}) = p \circ \eta(t)^{-1} - \bar{p}(t) \quad \text{in} \quad \eta(t, \Omega),$$  \hspace{1cm} (6.2)

$$\text{div}(\phi \circ \eta(t)^{-1}) = \bar{p}(t) \quad \text{in} \quad \eta(t, \Omega),$$  \hspace{1cm} (6.3)

where $\bar{p}(t) = \frac{1}{|\Omega|} \int_\Omega p(t, x) dx$. The existence of the solution to problem (6.2) with zero boundary condition is standard (see, for example, [10] Chapter 3), and the solution to problem (6.3) can be chosen as a linear function (linear in $x$) for example, $\bar{p}(t)x_1$. The estimate (6.1) follows from the estimates of the solutions to (6.2). \hfill \Box

Define the linear functional on $H^{1,2}(\Omega; \Gamma)$ by $(p, a_j^i(t) \varphi_j^i)_{L^2(\Omega)}$ where $\varphi \in H^{1,2}(\Omega; \Gamma)$. By the Riesz representation theorem, there is a bounded linear operator $Q(t) : L^2(\Omega) \to H^{1,2}(\Omega; \Gamma)$ such that for all $\varphi \in H^{1,2}(\Omega; \Gamma)$,

$$(p, a_j^i(t) \varphi_j^i)_{L^2(\Omega)} = (Q(t)p, \varphi)_{H^{1,2}(\Omega; \Gamma)} := (Q(t)p, \varphi)_{H^1(\Omega)} + (Q(t)p, \varphi)_{H^2(\Gamma)}.$$  \hspace{1cm} (6.4)

Letting $\varphi = Q(t)p$ shows that

$$\|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)} \leq C \|p\|_{L^2(\Omega)}$$

for some constant $C > 0$. By Lemma 6.1

$$\|p\|_{L^2(\Omega)} \leq \|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)} \|\varphi\|_{H^{1,2}(\Omega; \Gamma)} \leq C \|Q(t)p\|_{H^{1,2}(\Omega; \Gamma)} \|p\|_{L^2(\Omega)}$$
which shows that \( R(Q(t)) \) is closed in \( H^{1/2}(\Omega; \Gamma) \). Since \( \mathcal{V}_v(t) \subset R(Q(t)) \) and 
\( R(Q(t)) \subset \overline{\mathcal{V}_v(t)} \), it follows that 
\[
H^{1/2}(\Omega; \Gamma)(t) = R(Q(t)) \oplus H^{1/2}(\Omega; \Gamma) \mathcal{V}_v(t).
\] (6.4)

We can now introduce our Lagrange multiplier

**Lemma 6.2.** Let \( \mathcal{L}(t) \in H^{1/2}(\Omega; \Gamma)' \) be such that \( \mathcal{L}(t)\varphi = 0 \) for any \( \varphi \in \mathcal{V}_v(t) \). Then there exist a unique \( q(t) \in L^2(\Omega) \), which is termed the pressure function, satisfying 
\[
\forall \varphi \in H^{1/2}(\Omega; \Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), a^i_j(t)\varphi^j_j)_{L^2(\Omega)}.
\]

Moreover, there is a \( C > 0 \) (which does not depend on \( t \in [0,T] \) and \( \epsilon \) and on the choice of \( v \in C_T(M) \)) such that 
\[
\|q(t)\|_{L^2(\Omega)} \leq C\|\mathcal{L}(t)\|_{H^{1/2}(\Omega; \Gamma)'}.
\]

**Proof.** By the decomposition (6.4), for given \( \tilde{\psi} \), let \( \varphi = v_1 + v_2 \), where \( v_1 \in \mathcal{V}_v(t) \) and \( v_2 \in R(Q(t)) \). It follows that 
\[
\mathcal{L}(t)(\varphi) = \mathcal{L}(t)(v_2) = (\psi(t), v_2)_{H^{1/2}(\Omega; \Gamma)} = (\psi(t), \varphi)_{H^{1/2}(\Omega; \Gamma)}
\]
for a unique \( \psi(t) \in R(Q(t)) \).

From the definition of \( Q(t) \) we then get the existence of a unique \( q(t) \in L^2(\Omega) \) such that 
\[
\forall \varphi \in H^{1/2}(\Omega; \Gamma), \quad \mathcal{L}(t)(\varphi) = (q(t), a^i_j(t)\varphi^j_j)_{L^2(\Omega)}.
\]
The estimate stated in the lemma is then a simple consequence of (6.1). \( \square \)

### 6.2. Estimates for \( a \) and \( h \)

We make use of near-identity transformations. The following lemmas can be found in [9] and [10].

**Lemma 6.3.** There exists \( K > 0, T_0 > 0 \) such that if \( 0 < t \leq T_0 \), then, for any \( (\tilde{v}, \tilde{h}) \in C_{T_0}(M) \), 
\[
\|\tilde{a}^T - Id\|_{L^\infty(0,T_0; C^0(\Gamma))} \leq K\sqrt{t}; 
\]
(6.5a)
\[
\|\tilde{a}^T - Id\|_{L^\infty(0,T_0; C^1(\Gamma))} \leq K\sqrt{t}; 
\]
(6.5b)
\[
\|\tilde{a}_t - \tilde{a}_t(0)\|_{L^\infty(0,T_0; C^1(\Omega))} \leq C(M) t; 
\]
(6.5c)
\[
\|\tilde{a}_t\|_{L^\infty(0,T_0; C^1(\Omega))} \leq K. 
\]
(6.5d)

We also need the following

**Lemma 6.4.** For any \( (\tilde{v}, \tilde{h}) \in C_{T_0}(M) \), 
\[
\|\tilde{h}\|_{H^{3/5}(\Gamma)} \leq CMT^{1/4}
\]
(6.6)
for all \( 0 < t \leq T_0 \).

**Proof.** For \( (\tilde{v}, \tilde{h}) \in C_T(M) \), \( \|\tilde{h}\|_{H^{3/5}(\Gamma)}^2 + \|\tilde{h}_t\|_{H^{2}(\Gamma)}^2 \leq M \). By \( \tilde{h}(0) = 0 \), 
\[
\|\tilde{h}(t)\|_{H^2(\Gamma)} \leq \int_0^t \|\tilde{h}_t\|_{H^2(\Gamma)} ds \leq \sqrt{Mt}.
\]
Finally, the interpolation inequality 
\[
\|\nabla^2_0 f\|_{H^{1/5}(\Gamma)} \leq C\big\| \nabla^2_0 f\|_{L^2(\Gamma)}^{3/4}\big\| \nabla^2_0 f\|_{L^2(\Gamma)}^{1/4},
\]
implies 
\[
\|\tilde{h}\|_{H^{3/5}(\Gamma)} \leq C\big\| \tilde{h}\|_{H^1(\Gamma)}^{3/4}\big\| \tilde{h}\|_{H^2(\Gamma)}^{1/4} \leq CMT^{1/4}.
\]
\( \square \)
Corollary 6.1. \( \|L_1(t)\|_{H^{1/3}(\Gamma)} \) and \( \|L_2(t)\|_{H^{1/3}(\Gamma)} \) converge to zero as \( t \to 0 \), uniformly in \( (v, h) \in C_T(\mathcal{M}) \). Furthermore, for \( t \leq 1 \),
\[
\|L_1(t)\|_{H^{1/3}(\Gamma)} + \|L_2(t)\|_{H^{1/3}(\Gamma)} \leq C(M)t^{1/4}.
\]

By the fact that \( \|\hat{h}_t\|_{H^2(\Gamma)}^2 \leq M \) and \( \|\hat{h}_u\|_{L^2(0,T;H^{1/3}(\Gamma))} \leq M \) if \( (\hat{v}, \hat{h}) \in C_T(\mathcal{M}) \), similar computations lead to the following lemma.

Lemma 6.5. For all \((\hat{v}, \hat{h}) \in C_T(\mathcal{M})\),
\[
\|\hat{h}_t(t)\|_{H^{1/3}(\Gamma)} \leq C M t^{1/8}
\]
for all \( 0 < t \leq T \).

7. The linearized problem

Suppose that \((\hat{v}, \hat{h}) \in C_T(\mathcal{M})\) is given. Let \( \hat{\eta}(t) = \text{Id} + \int_0^t \hat{v}(s)ds \) and \( \hat{a} = (\nabla \hat{\eta})^{-1} \).

We are concerned with the following time-dependent linear problem, whose fixed-point \( v = \hat{v} \) provides a solution to \([2.16]\):
\[
\begin{align*}
\nu i^i &- \nu [\partial_\ell^i D_\eta(v)]_k, j = - (\partial_\ell^i q), k + F^i \quad \text{in} \quad (0, T) \times \Omega, \quad (7.1a) \\
\partial_\ell^i v^j, j & = 0 \quad \text{in} \quad (0, T) \times \Omega, \quad (7.1b) \\
[\nu D_\eta(v)]_i^j - \gamma_\ell^i \gamma_\ell^j N_i & = \sigma \Theta \left[ \mathcal{L}_h(h)(-\nabla_0 \hat{h}, 1) \right] \circ \hat{\eta}^r \quad \text{on} \quad (0, T) \times \Gamma, \quad (7.1c) \\
& \quad + \sigma \hat{\Theta} \left[ \left[ \mathcal{M}(h)(-\nabla_0 \hat{h}, 1) \right] \circ \hat{\eta}^r \right] \\
h_t \circ \hat{\eta}^r & = \left[ \hat{h}_\alpha \circ \hat{\eta}^r \right] v_\alpha - v_z \quad \text{on} \quad (0, T) \times \Gamma, \quad (7.1d) \\
v & = u_0 \quad \text{on} \quad \{ t = 0 \} \times \Omega, \quad (7.1e) \\
h & = 0 \quad \text{on} \quad \{ t = 0 \} \times \Gamma. \quad (7.1f)
\end{align*}
\]

where \( D_\eta(v)_i^j = \partial_\ell^k v^j, k + \partial_\ell^j v^i, k \), \( \hat{\Theta} = \text{det}(\nabla_0 \hat{\eta}) \), and
\[
\mathcal{L}_h(h) = \frac{1}{\sqrt{\text{det}(g_0)}} \sqrt{\text{det}(g_0)} \hat{A}_{\alpha\beta\gamma\delta} h_{,\alpha\beta} \eta_{,\gamma\delta}
\]
with
\[
\hat{A}_{\alpha\beta\gamma\delta} = J_h^{-2} \sqrt{\text{det}(G_h)} \left[ G^\alpha_\gamma - (-1)^{\kappa + \sigma} \text{det}(G_h)^{-1} (1 - \delta_{\alpha\alpha})(1 - \delta_{\gamma\sigma}) \hat{h}_k \hat{h}_\sigma \right] \\
\times (G_{\alpha\gamma}^\delta - J_h^{-1} G_{\gamma\ell}^\delta \hat{h}_\ell \hat{h}_\delta)
\]
and
\[
\mathcal{M}(h) = \sqrt{\text{det}(G_h)} \circ \hat{\eta}^r \left[ L_1^{1/2}(y, \hat{h}, \hat{D} \hat{h}, D^2 \hat{h}) h_{,\alpha\beta} + L_2(y, \hat{h}, \hat{D} \hat{h}, D^2 \hat{h}) \right].
\]

Here the thickness \( \epsilon \) is assumed to be 1.

We will also use \( L_\ell(h) \) to denote \( L_\ell(h) + M(h) \).

Remark 10. \( L_\ell(h) \) is a coercive fourth order operator for small \( \hat{h} \leq \epsilon \). Actually, it is easy to see that \( L_\ell(h) \) is coercive at time \( t = 0 \), and the coercivity of \( L_\ell(h) \) for \( t > 0 \) (but sufficiently small) follows from the continuity of \( \hat{h} \) in time into the space \( H^2(\Gamma) \).

Moreover, by Lemma 6.7, we have the following corollary.

Corollary 7.1. There exists a \( \nu_1 > 0 \) and \( 0 < T \leq T_0 \) such that for all \( 0 < t \leq T \),
\[
\nu_1 \| \nabla_0^2 f(t) \|_{L^2(\Gamma)}^2 \leq \int_\Gamma \hat{A}_{\alpha\beta\gamma\delta} f_{,\alpha\beta}(t) f_{,\gamma\delta}(t) dS.
\]
Definition 7.1. (Mollifiers on $\Gamma$) For $\epsilon > 0$, let

$$K^\epsilon_{\sigma} := (1 - \epsilon \Delta_\sigma)^{-\frac{1}{2}} : H^s(\Gamma) \to H^{s+\epsilon}(\Gamma)$$

denote the usual self-adjoint Frederich mollifier on the compact manifold $\Gamma$, where $\Delta_\sigma$ is the surface Laplacian defined on $\Gamma$.

By the Sobolev extension theorem, there exist bounded extension operators

$$E_s : H^s(\Omega) \to H^s(\mathbb{R}^n), \quad s \geq 1.$$ 

For fixed (but small) $\epsilon$ and $\epsilon_1 > 0$, let $\rho_\sigma$ be a (positive) smooth mollifier on $\mathbb{R}^n$. Set $\tilde{v} = \rho_\sigma \ast (v_1, \tilde{F}_1, \tilde{u}_0), \tilde{F} = \rho_\sigma \ast (F_1, \tilde{F}_2), \tilde{u}_0 = \rho_\sigma \ast (u_0)$, where $\ast$ denotes the convolution in space, and $\tilde{h} = K^\epsilon_{\sigma} \tilde{h}$ for large enough $m$. Define $\tilde{\eta}$ and $\tilde{a}$ in the same fashion as $\eta$ and $a$. Note that $\tilde{v} \to v \in V(T), \tilde{F} \to F$ in $\mathcal{V}^2(T), \tilde{u}_0 \to u_0$ in $H^{2,5}(\Omega)$ and $\tilde{h} \to h$ in $\mathcal{H}(T)$ as $\epsilon \to 0$.

The regularized problem takes the form

$$v^i_t - \nu [\tilde{a} \cdot D\eta(v^i_1), k] = -(\tilde{a}_\eta q)_k + \tilde{F}^i \quad \text{in } (0, T) \times \Omega, \quad (7.2a)$$

$$\tilde{a} \cdot v^i_j = 0 \quad \text{in } (0, T) \times \Omega, \quad (7.2b)$$

$$[\nu D\eta(v^i_1) - q\delta^i]_j = \sigma \tilde{M}^c_{\tilde{h}} (h^c) (-\nabla_0 \tilde{h} \circ \tilde{\eta}^c, 1)$$

$$+ \sigma \tilde{M}^c_{\tilde{h}} (\nabla_0 \tilde{h} \circ \tilde{\eta}^c, 1) + \kappa \Delta_\sigma v \quad \text{on } (0, T) \times \Gamma, \quad (7.2c)$$

$$h \circ \tilde{\eta}^c = [(\tilde{h}_\alpha) \circ \tilde{\eta}^c] v_\alpha = v_2$$

$$v = \tilde{u}_0 \quad \text{on } \{t = 0\} \times \Omega, \quad (7.2e)$$

$$h = 0 \quad \text{on } \{t = 0\} \times \Gamma, \quad (7.2f)$$

where

$$\tilde{L}^c_{\tilde{h}}(f) = \frac{\tilde{\Theta}}{\sqrt{\text{det}(g_0)}} \left[ \left( \sqrt{\text{det}(g_0)} \tilde{A}^{\alpha\beta\gamma} f_{,\alpha\beta} \right) ,_{\gamma\delta} \right]^{\epsilon_1} \circ \tilde{\eta}^c,$$

$$\tilde{M}^c_{\tilde{h}} = \tilde{\Theta} \left[ \left( L_{1}^{\alpha\beta\gamma}(\tilde{h}, \tilde{D}\tilde{h}) h_{,\alpha\beta\gamma} + L_{2}(\tilde{h}, \tilde{D}\tilde{h}) \right)^{\epsilon_1} \circ \tilde{\eta}^c \right].$$

Note that $\tilde{L}^c_{\tilde{h}}(f) + \tilde{M}^c_{\tilde{h}} = \tilde{\Theta} \left[ L_{\tilde{h}}(f) \right]^{\epsilon_1} \circ \tilde{\eta}^c$.

7.1. Weak solutions.

Definition 7.2. A vector $v \in \tilde{V}_c(T)$ with $v_t \in \tilde{V}_c(T)'$ for almost all $t \in (0, T)$ is a weak solution of (7.2a) provided that

$$\langle (v_t, \varphi) + \nu \int_\Omega D\eta v : D\eta \varphi dx + \sigma \int_\Gamma \tilde{A}^{\alpha\beta\gamma} h_{,\alpha\beta\gamma}^{\epsilon_1} [-\tilde{h}, \sigma] (\varphi^c \circ \tilde{\eta}^{-c}) + (\varphi^c \circ \tilde{\eta}^{-c})^{\epsilon_1} dS + \kappa \int_\Gamma \Delta_0 v \cdot \Delta_0 \varphi dS = (\tilde{F}_t, \varphi) - \sigma (\tilde{M}^c_{\tilde{h}}, \varphi) \Gamma$$

$$v(0, \cdot) = \tilde{u}_0$$

(7.3b)
for almost all \( t \in [0, T] \), where \( \langle ., \cdot \rangle \) denotes the duality product between \( V \) and its dual \( V' \), and \( h \) is given by the evolution equation (7.2d) and the initial condition (7.2f):

\[
h(y, t) = \int_0^t \left[ -\tilde{h}_{\cdot \alpha}(y, s)v^\alpha(\tilde{\eta}^{-\tau}(y, s), 0, s) + v^\tau(\tilde{\eta}^{-\tau}(y, s), 0, s) \right] ds \quad (7.4)
\]

### 7.2. Penalized problems.

Letting \( \theta > 0 \) denote the penalized parameter, we define \( \tilde{w}_\theta \) (with also \( \epsilon \) and \( \epsilon_1 \) dependence in mind) to be the “unique” solution of the problem (whose existence can be obtained via a modified Galerkin method which will be presented in the following sections):

\[
\begin{align*}
(\text{i}) \quad & \langle \tilde{w}_\theta, \varphi \rangle + \dfrac{\nu}{2} \int_0^T \int_\Omega D_{\tilde{w}_\theta} D_{\tilde{w}} \varphi dx \, dt + \sigma \int_{\Gamma} A_{\gamma \delta} \gamma_{\sigma} \tilde{w}_{\cdot \sigma} \varphi ds \\
& + \langle \varphi, \tilde{\eta}^{-\tau} \rangle \int_{\Gamma} \tilde{\eta}^{-\tau} ds + \kappa \int_{\Gamma} \Delta \varphi \cdot \Delta \tilde{w}_\theta ds + \left( \frac{1}{\theta} \tilde{a}^i_j v^i, \tilde{a}^k_j \varphi^j \right)_{L^2(\Omega)} \\
& = \left( \tilde{F}, \varphi \right) - \sigma(\mathcal{M}_h^{\epsilon_1}(-\nabla_0 \tilde{h} \circ \tilde{\eta}^{-\tau}, 1), \varphi)\Gamma \\
(\text{ii}) \quad & \varphi(0, \cdot) = \tilde{u}_0 \quad (7.5a)
\end{align*}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( H^1(\Omega) \) and its dual, and \( h \) in (7.5b) satisfies (7.4) with \( v \) replaced by \( w_\theta \).

### 7.3. Weak solutions for the penalized problem.

The goal of this section is to establish the existence of \( \tilde{v} \) to the problem (7.2) (or the weak formulation (7.3)), as well as the energy inequality satisfied by \( \tilde{v} \) and \( \tilde{v}_t \). Before proceeding, we introduce variable \( \tilde{q}_0 \) and \( \tilde{w}_1 \) as follows: let \( \tilde{q}_0 \) be the solution of the following Laplace equation

\[
\Delta \tilde{q}_0 = \nabla \tilde{u}_0 : (\nabla \tilde{u}_0)^t - \text{div} \, \tilde{F}(0) \quad \text{in} \, \Omega, \quad (7.6a)
\]

\[
\tilde{q}_0 = \nu(\text{Def} \, \tilde{u}_0)_j^i N_i N_j - \sigma \mathcal{M}_h^{\epsilon_1}(0) + \kappa \Delta \tilde{u}_0 \cdot N \quad \text{on} \, \Gamma, \quad (7.6b)
\]

and \( \tilde{w}_1 \) be defined by

\[
\tilde{w}_1 = \nu \Delta \tilde{u}_0 - \nabla \tilde{q}_0 + \tilde{F}(0). \quad (7.7)
\]

By elliptic regularity,

\[
\|\tilde{q}_0\|_{H^1(\Omega)} \leq C \left[ \|\tilde{u}_0\|_{H^2(\Omega)} + \|\tilde{F}(0)\|_{L^2(\Omega)} + \|\mathcal{M}_h^{\epsilon_1}(0)\|_{L^2(\Omega)} + \|\Delta \tilde{u}_0\|_{L^2(\Omega)} + 1 \right],
\]

and hence

\[
\|\tilde{w}_1\|_{L^2(\Omega)} \leq C(M) \left[ \|\tilde{u}_0\|_{H^2(\Omega)} + \|\tilde{u}_0\|_{H^4(\Gamma)} + \|\tilde{F}(0)\|_{L^2(\Omega)} + 1 \right].
\]

**Remark 12.** By (6.6), the constant \( C(M) \) in the estimates above can also be refined as a constant independent of \( M \) if \( T \) is chosen small enough.

By introducing a (smooth) basis \( (e_\ell)_{\ell=1}^\infty \) of \( H^{1;2}(\Omega; \Gamma) \), and taking the approximation at rank \( m \geq 2 \) under the form \( w_\ell(t, x) = \sum_{k=1}^\ell d_k(t)e_k(x) \) with

\[
h_\ell(y, t) = \int_0^t \left[ -\tilde{h}_{\cdot \alpha}(y, s)w^\alpha_t(\tilde{\eta}^{-\tau}(y, s), 0, s) + w^\tau_t(\tilde{\eta}^{-\tau}(y, s), 0, s) \right] ds, \quad (7.8)
\]
and satisfying on \([0, T]\),

(i) \((w_{t\ell}, \varphi)_{L^2(\Omega)} + \nu (\bar{a}_{ij}^k w_{t,ij}, \bar{a}_{ij}^k \varphi, k)_{L^2(\Omega)} + \nu ((\bar{a}_{ij}^k \bar{a}_{ij}^k), w_{t\ell}, \varphi, k)_{L^2(\Omega)} + \nu \int_\Omega \left[ \bar{a}_{ij}^k w_{t,ij} w_{t\ell,ij} + (\bar{a}_{ij}^k \bar{a}_{ij}^k) w_{t\ell,ij} \right] \varphi, k dx + \kappa \int_\Gamma \Delta_0 w_{t\ell} \cdot \Delta_0 \varphi dS - ((\bar{a}_{ij}^k \varphi, k)_{L^2(\Omega)} + \nu \int_\Omega \left[ \bar{a}_{ij}^k \varphi, k \right] dx + \kappa \int_\Gamma \Delta_0 \varphi dS - ((\bar{a}_{ij}^k \varphi, k)_{L^2(\Omega)} \right)

(ii) \(w_{t\ell}(0) = (w_{1\ell}, w_{2\ell}, w_{3\ell}) \in \Omega\), \((\bar{u}_0)_{t\ell}\) denote the respective \(H^{1/2}(\Omega; \Gamma)\) projections of \(u_0\) on \(\text{span}(e_1, e_2, \ldots, e_\ell)\).

**Remark 13.** The existence of \(w_k\) follows from the solution of

\[
d_{t\ell}^k(t) + d_{t\ell}^k(t)A_{t\ell}(t) + d_{t\ell}(t)B_{t\ell}(t) + \int_0^t d_{t\ell}(s)C_{t\ell}(s, t) ds = F(t)
\]

for functions \(A, B, C\) and \(F;\) however, the existence of the solution \(d_k\) does not immediately follow from the fundamental theorem of ODE due to the presence of the time-integral. A straightforward fix-point argument can be implemented, whose details we leave to interested reader.

The use of the test function \(\varphi = w_{t\ell}\) in this system of ODE gives us in turn the energy law

\[
\frac{1}{2} \frac{d}{dt} \|w_{t\ell}\|^2_{L^2(\Omega)} + \frac{\nu}{2} \|D_\eta(w_{t\ell})\|^2_{L^2(\Omega)} + \frac{\sigma}{2} \frac{d}{dt} E_h(h_{t\ell,\alpha\beta}) + \theta \|q_{t\ell}\|^2_{L^2(\Omega)} + \nu \int_\Omega \left[ \bar{a}_{ij}^k w_{t,ij} w_{t\ell,ij} \right] \varphi, k dx + \kappa \int_\Gamma \Delta_0 w_{t\ell} \cdot \Delta_0 \varphi dS - ((\bar{a}_{ij}^k \varphi, k)_{L^2(\Omega)} + \nu \int_\Omega \left[ \bar{a}_{ij}^k \varphi, k \right] dx + \kappa \int_\Gamma \Delta_0 \varphi dS - ((\bar{a}_{ij}^k \varphi, k)_{L^2(\Omega)} \right)
\]

for \(q_{t\ell}, \bar{a}_{ij}^k w_{t,ij}\) on \(\Omega\), \((\bar{u}_0)_{t\ell}\) denote the respective \(H^{1/2}(\Omega; \Gamma)\) projections of \(u_0\) on \(\text{span}(e_1, e_2, \ldots, e_\ell)\).
For the tenth term (the integral with \( \sigma \) as its coefficient), we have

\[
\left| \int (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS \right| \leq C(M) \| \bar{h}_t \|_{H^2(\Gamma)} \| \nabla^2_0 h_t \|_{L^2(\Gamma)}^2.
\]

By \( \epsilon_1 \)-regularization and the identity

\[
\int (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS = \int \frac{1}{\sqrt{\text{det}(g_0)}} \sqrt{\text{det}(g_0)} (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS + \int \frac{2}{\sqrt{\text{det}(g_0)}} \sqrt{\text{det}(g_0)} (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS + \int (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS,
\]

we find that

\[
\left| \int (\tilde{A}^{\alpha^\beta\delta})_t h_{\ell,\alpha,\beta}^\gamma h_{\ell,\gamma,\delta}^i dS \right| \leq C(\epsilon_1) \left[ 1 + \| \bar{h}_t \|_{H^2(\Gamma)} \right] \| \nabla^2_0 h_t \|_{L^2(\Gamma)} \| w_t \|_{H^1(\Omega)} + \| w_t \|_{H^1(\Omega)}.
\]

Similarly, the second part of the eleventh term and the last term of the left-hand side can be bounded by

\[
C(\epsilon_1) \| \bar{h}_t \|_{H^2(\Gamma)} \| \nabla^2_0 h_t \|_{L^2(\Gamma)} \| w_t \|_{H^1(\Omega)}
\]

where we also use the \( \epsilon_1 \)-regularization to control \( \nabla^2_0 w_t \). It also follows that the last two terms on the right-hand side can be bounded by

\[
C(M) \left[ 1 + \| \bar{h}_t \|_{H^2(\Gamma)} \right] \| w_t \|_{H^1(\Omega)}.
\]

With positive \( \theta \), the fourth term of the left-hand side involving the square of \( q_{\ell t} \) acts as a viscous energy term. Integrating (7.10) in time from 0 to \( t \), we then get

\[
\| w_{\ell t} \|_{L^2(\Omega)}^2 + \| \nabla_0^2 h_{\ell t} \|_{L^2(\Gamma)}^2 \geq t \int_0^t \left[ \| \nabla_0 w_{\ell t} \|_{L^2(\Omega)}^2 + \kappa \| w_{\ell t} \|_{H^2(\Gamma)}^2 + \theta \| q_{\ell t} \|_{H^2(\Gamma)}^2 \right] dt + C(\epsilon_1) \int_0^t \left[ 1 + \| \bar{h}_t \|_{H^2(\Gamma)} \right] \| \nabla_0^2 h_{\ell t}(s) \|_{L^2(\Gamma)}^2 ds
\]

\[
+ C(\theta) \int_0^t \| \bar{v}(t') \|_{H^2(\Gamma)}^2 \int_0^{t'} \left[ \| \nabla_0 w_{\ell t}(s) \|_{L^2(\Omega)}^2 + \| q_{\ell t}(s) \|_{L^2(\Omega)}^2 \right] ds dt',
\]

where \( C(\epsilon_1), C(\theta) \to \infty \) as \( \epsilon_1, \theta \to 0 \), and we use

\[
\| f(t) \|_{X} \leq \| f(0) \|_{X} + \int_0^t \| f_t(s) \|_{X} ds \leq \| f(0) \|_{X} + \sqrt{t} \int_0^t \| f_t(s) \|_{X}^2 ds
\]

for \( f = w_{\ell t}, f = h_{\ell t} \) and \( f = g_{\ell t} \) to obtain (7.11).

**Remark 14.** The \( \theta \)-dependence follows from estimating the terms \( (q_{\ell t}, \bar{v}_{\ell, i}^j w_{\ell, i}^j)_{L^2(\Omega)} \):
By the Gronwall inequality, for $0 \leq t \leq T$,
\[
\|w_{\ell}(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 h_{\ell}(t)\|_{L^2(\Gamma)}^2 + \int_0^t \left[ \|\nabla w_{\ell}\|_{L^2(\Omega)}^2 + \kappa \|w_{\ell}\|_{H^2(\Gamma)}^2 \right] ds \leq C(\epsilon_1, \theta) N_0(u_0, F)
\]
(7.12)
where
\[
N_0(u_0, F) := \|u_0\|_{H^{2.5}(\Omega)}^2 + \|u_0\|_{H^{1.5}(\Gamma)}^2 + \|F_1\|_{L^2(0,T;H^1(\Omega))}^2 + \|F(0)\|_{H^{0.5}(\Omega)}^2 + 1.
\]
We can then infer that $w_{\ell}$ defined on $[0, T]$, and that there is a subsequence, still denoted with the subscript $\ell$, satisfying
\[
w_{\ell} \to w_0 \quad \text{in } L^2(0, T; H^{1.2}(\Omega; \Gamma))
\]
(7.13a)
\[
w_{\ell t} \to w_{\ell t} \quad \text{in } L^2(0, T; H^{1.2}(\Omega; \Gamma))
\]
(7.13b)
\[
\nabla_0^2 h_{\ell t} \to \nabla_0^2 h_{\ell t} \quad \text{in } L^2(0, T; L^2(\Gamma))
\]
(7.13c)
\[
\nabla_0^2 h_{\ell t} \to \nabla_0^2 h_{\ell t} \quad \text{in } L^2(0, T; L^2(\Gamma))
\]
(7.13d)
\[
q_{\ell t} \to q_{\ell t} \quad \text{in } L^2(0, T; L^2(\Omega))
\]
(7.13e)
where
\[
q_0 = \tilde{q}_0 - \frac{1}{\theta} \hat{a}_l^j w_{\ell j}.
\]
From the standard procedure for weak solutions, we can now infer from these weak convergences and the definition of $w_{\ell}$ that $w_{\ell t} \in L^2(0, T; H^1(\Omega)')$. In turn, $w_{\ell t} \in C^0([0, T]; H^1(\Omega))$, $w_\ell \in C^0([0, T]; L^2(\Omega))$ with $w_\ell(0) = u_0$, $w_{\ell t}(0) = w_1$.

Moreover, (7.13) implies that $w_\ell$ satisfies
\[
\int_0^T \left[ (w_{\ell t}, \varphi)_{L^2(\Omega)} + \nu(\hat{a}_l^j w_{\ell j}, \hat{a}_l^k \varphi_{,k})_{L^2(\Omega)} + \nu((\hat{a}_l^j \hat{a}_l^k)_{,t} w_{\ell j}, \varphi_{,k})_{L^2(\Omega)} \right] dt
\]

\[+ \nu \int_0^T \left[ \int_{\Gamma} \hat{a}_l^j \hat{a}_l^k w_{\ell j} \varphi_{,k} dx + \int_{\Omega} (\hat{a}_l^j \hat{a}_l^k)_{,t} w_{\ell j} \varphi_{,k} dx \right] dt + \sigma \int_0^T \int_{\Gamma} A^{\alpha_\beta\gamma\delta} \times \left[ -\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + w_\ell^\sigma \circ \bar{\eta}^{-\tau} \right]_{\alpha_\beta} \left[ -\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\sigma \circ \bar{\eta}^{-\tau} \right]_{\gamma\delta} dS dt
\]

\[+ \sigma \int_0^T \int_{\Gamma} (A^{\alpha_\beta\gamma\delta})_{,t} \left[ -\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\sigma \circ \bar{\eta}^{-\tau} \right]_{\gamma\delta} dS dt
\]

\[+ \sigma \int_0^T \int_{\Gamma} \left[ -\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \bar{\nu}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) \right]_{\gamma\delta} dS dt
\]

\[+ \kappa \int_0^T \int_{\Gamma} \Delta_0 w_{\ell t} \cdot \Delta_0 \varphi dS dt - \int_0^T \left[ (\hat{a}_l^j q_{\ell 0})_{,t} \varphi_{,j} \right]_{L^2(\Omega)} dt
\]

\[= \int_0^T \left\{ (F_t, \varphi) - \sigma \int_{\Gamma} \left[ L_1^{\alpha_\beta\gamma} \bar{h}_{,\alpha\beta\gamma} + L_2 \right] \left[ \bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\sigma \circ \bar{\eta}^{-\tau} \right]_{\gamma\delta} dS
\]

\[\quad - \sigma \int_{\Gamma} \left[ L_1^{\alpha_\beta\gamma} \bar{h}_{,\alpha\beta\gamma} + L_2 \right] \left[ \bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) \right]_{\gamma\delta} dS \right\} dt
\]

(7.14b)

(i) $w_{\ell t}(0) = \bar{w}_1$, $w_\ell(0) = \tilde{u}_0$ in $\Omega$.
for all $\varphi \in L^2(0,T; H^{1/2}(\Omega; \Gamma))$. Choosing $\varphi$ to be independent of time, we find that for all $t \in [0,T]$,

$$
(w_{0t}, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_{\Omega} D_\theta(w_\theta) : D_\theta(\varphi) dx + \kappa \int_{\Gamma} \Delta_0 w_\theta \cdot \Delta_0 \varphi d\Gamma \\
+ \sigma \int_{\Gamma} \bar{A}^{\alpha \beta \gamma \delta} h^{\epsilon_1}_{\alpha \beta \gamma \delta} [-\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\tau \circ \bar{\eta}^{-\tau}] d\Gamma - (\bar{u}_t^i q_\theta, \varphi)_{L^2(\Omega)}
$$

$$
= (\bar{F}, \varphi) + \sigma \int_{\Gamma} \left[ L^1 \alpha^{\beta \gamma \delta} h^{\epsilon_1}_{\alpha \beta \gamma \delta} + L_2 \right]^{\epsilon_1} [-\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\tau \circ \bar{\eta}^{-\tau}] d\Gamma + c(\varphi)
$$

for all $\varphi \in H^{1/2}(\Omega; \Gamma)$, where $c(\varphi) \in \mathbb{R}$ is given by

$$
c(\varphi) = (\bar{w}_1, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_{\Omega} \text{Def}(\bar{u}_0) : \text{Def} \varphi dx - (\tilde{q}_0 - \frac{1}{\theta} \text{div} \bar{u}_0, \text{div} \varphi)_{L^2(\Omega)} - (\bar{F}(0), \varphi)_{L^2(\Omega)} - \sigma (\bar{M}_0^i(0,0,1), \varphi)_{L^2(\Gamma)} + \kappa (\Delta_0 \bar{u}_0, \Delta_0 \varphi)_{L^2(\Gamma)}.
$$

By compatibility conditions (7.6) and (7.7), $c(\varphi) = 0$. Therefore, the weak limit $(w_\theta, h_\theta)$ satisfies, for all $t \in [0,T]$,

$$
(w_{0t}, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_{\Omega} D_\theta(w_\theta) : D_\theta(\varphi) dx + \kappa \int_{\Gamma} \Delta_0 w_\theta \cdot \Delta_0 \varphi d\Gamma \\
- (\bar{u}_t^i q_\theta, \varphi)_{L^2(\Omega)} + \sigma \int_{\Gamma} \bar{A}^{\alpha \beta \gamma \delta} h^{\epsilon_1}_{\alpha \beta \gamma \delta} [-\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\tau \circ \bar{\eta}^{-\tau}] d\Gamma
$$

$$
= (\bar{F}, \varphi) - \sigma \int_{\Gamma} \left[ L^1 \alpha^{\beta \gamma \delta} h^{\epsilon_1}_{\alpha \beta \gamma \delta} + L_2 \right]^{\epsilon_1} [-\bar{h}_{,\sigma}(\varphi^\sigma \circ \bar{\eta}^{-\tau}) + \varphi^\tau \circ \bar{\eta}^{-\tau}] d\Gamma,
$$

for all $\varphi \in H^{1/2}(\Omega; \Gamma)$.

Since $w_\theta \in L^2(0,T; H^{1/2}(\Omega; \Gamma))$, we can use it as a test function in (7.13) and obtain (after time integration)

$$
\frac{1}{2} \|w_\theta\|^2_{L^2(\Omega)} + \frac{\nu}{2} E_\theta(h^*_\theta) + \int_0^t \frac{\nu}{2} \|D_\theta w_\theta\|^2_{L^2(\Omega)} + \kappa \|\Delta_0 w_\theta\|^2_{L^2(\Gamma)} \\
+ \theta \|q_\theta\|^2_{L^2(\Omega)} ds - \theta \int_0^t (q_\theta, \bar{q}_0) dt - \frac{\sigma}{2} \int_0^t \int_{\Gamma} (\bar{A}^{\alpha \beta \gamma \delta})_{\alpha \beta \gamma \delta} h^{\epsilon_1}_{\alpha \beta \gamma \delta} \bar{h}^{\epsilon_1}_{\alpha \beta \gamma \delta} d\Sigma ds
$$

$$
= \frac{1}{2} \|\bar{u}_0\|^2_{L^2(\Omega)} + \int_0^t \langle \bar{F}, \varphi \rangle + \sigma \langle \bar{M}^i_{\bar{h}}(-\nabla\bar{h} \circ \bar{\eta}^\tau, 1), \varphi \rangle_{L^2(\Gamma)} dt.
$$

Consequently,

$$
\left[ \|w_\theta(t)\|^2_{L^2(\Omega)} + \|\nabla h^*_\theta(t)\|^2_{L^2(\Gamma)} \right] + \int_0^t \|\nabla w_\theta\|^2_{L^2(\Omega)} ds + \kappa \int_0^t \|w_\theta\|^2_{H^2(\Gamma)} ds \\
+ \theta \int_0^t \|q_\theta\|^2_{L^2(\Omega)} ds
\leq C(M) \left[ \|\bar{u}_0\|^2_{L^2(\Omega)} + \theta \|\bar{q}_0\|^2_{L^2(\Omega)} + \|\bar{F}\|^2_{H^1(\Omega)} + \|\bar{M}^i_{\bar{h}}(-\nabla\bar{h} \circ \bar{\eta}^\tau, 1)\|^2_{L^2(\Gamma)} \right] \\
+ C(M) \int_0^t \|\bar{h}_t\|_{H^2(\Gamma)} \|\nabla^2 h^*_\theta\|^2_{L^2(\Gamma)} ds
\leq C(M) \left[ N_1(u_0, F) + \int_0^t \|\bar{h}_t\|_{H^2(\Gamma)} \|\nabla^2 h^*_\theta\|^2_{L^2(\Gamma)} ds \right]
where 
\[ N_1(u_0, F) = \|u_0\|_{L^2(\Omega)}^2 + \|u_0\|_{H^{1,2}(\Gamma)}^2 + \|F\|_{L^2(0,T;H^1(\Omega))}^2 + \|F_t\|_{L^2(0,T;H^1(\Omega))}^2 + \|F(0)\|_{H^1(\Omega)}^2 + 1. \]

By the Gronwall inequality,
\[ \sup_{0 \leq t \leq T} \left[ \|w_0(t)\|_{L^2(\Omega)}^2 + \|\nabla^2 h_\theta^\dagger(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^T \left[ \|\nabla w_\theta\|_{L^2(\Omega)}^2 + \theta \|q_\theta\|_{L^2(\Omega)}^2 \right] \, ds \leq C(M)N_1(u_0, F). \]  
(7.17)

### 7.4. Improved pressure estimates

By \(c_1\)-regularization, we can rewrite (7.15) as, for a.a. \(t \in [0,T]\),
\[ (w_\theta, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_{\Omega} D_\theta(w_\theta) : D_\theta(\varphi) \, dx + \kappa \langle \Delta_0 w_\theta, \Delta_0 \varphi \rangle_{L^2(\Gamma)} - (\bar{a}_i^i \varphi_\theta, \varphi_\theta)_{L^2(\Omega)} + \sigma \int \mathcal{L}^e_h \left( h_\theta^\dagger \right) \left[ -\bar{h}_\theta \circ \bar{\eta} \varphi^\sigma + \varphi^\sigma \right] \, dS = \langle \bar{F}, \varphi \rangle + \sigma \langle \mathcal{M}^e_h(-\nabla_0 \bar{h} \circ \bar{\eta}, 1), \varphi \rangle_{\Gamma}. \]

Therefore, by the Lagrange multiplier lemma, we conclude that
\[ \|q_\theta\|_{L^2(\Omega)} \leq C(M) \left[ \|w_\theta\|_{H^{1,2}(\Omega)}^2 + \|\nabla w_\theta\|_{L^2(\Omega)}^2 + \|\bar{F}\|_{H^{1,2}(\Omega)}^2 + \kappa \|\Delta_0^2 w_\theta\|_{H^{-2}(\Gamma)}^2 + \|\mathcal{L}^e_h(h_\theta^\dagger) + \mathcal{M}^e_h(-\nabla_0 \bar{h} \circ \bar{\eta}, 1)\|_{H^{-2}(\Gamma)} \right] \]
and hence
\[ \|q_\theta\|_{L^2(\Omega)}^2 \leq C(M) \left[ \|w_\theta\|_{L^2(\Omega)}^2 + \|\nabla w_\theta\|_{L^2(\Omega)}^2 + \kappa \|w_\theta\|_{H^{1,2}(\Omega)}^2 + \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \|F\|_{H^{1,2}(\Omega)}^2 + 1 \right]. \]  
(7.18)

### 7.5. Weak limits as \(\theta \to 0\)

Since \(w_\theta \in L^2(0,T;H^{1,2}(\Omega;\Gamma))\), we can use it as a test function in (7.14). Similar to the way we obtain (7.11), we find that
\[ \frac{1}{2} \|w_\theta\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \int_0^t \|D_\theta w_\theta\|_{L^2(\Omega)}^2 \, ds + \frac{\sigma}{2} E_h(h_\theta^\dagger) + \kappa \int_0^t \|\Delta_0^2 w_\theta\|_{L^2(\Gamma)}^2 \, ds + \theta \int_0^t \|q_\theta\|_{L^2(\Omega)}^2 \, ds + \int_0^t (q_\theta, a_i^i w_\theta^i)_{L^2(\Omega)} \, ds - \int_0^t (q_\theta, a_i^i w_\theta^i)_{L^2(\Omega)} \, ds \]
\[ \leq C(M)N_0(u_0, F) + C(M) \int_0^t \|\bar{v}(\theta')\|_{H^{1,2}(\Omega)}^2 \int_0^t \|\nabla_{\theta'}(s)\|_{L^2(\Omega)}^2 \, ds \, dt' + C(\epsilon) \int_0^t \left[ 1 + \|\bar{h}_\theta\|_{H^{1,2}(\Gamma)} \right] \|\nabla^2 h_{\theta^\dagger}\|_{L^2(\Gamma)}^2 \, ds. \]

By (7.18),
\[ \left| \int_0^t (q_\theta, a_i^i w_\theta^i) \, ds \right| \leq C(M, \delta) \int_0^t \|q_\theta\|_{L^2(\Omega)}^2 \, ds + \delta \int_0^t \|\nabla w_\theta\|_{L^2(\Omega)}^2 \, ds \]
\[ \leq C(M) \left[ N_1(u_0, F) + \int_0^t \left( \|w_\theta\|_{L^2(\Omega)}^2 + \kappa \|w_\theta\|_{H^1(\Gamma)}^2 + \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 \right) \right] \]
\[ + \delta \int_0^t \|\nabla w_\theta\|_{L^2(\Omega)}^2 \, ds \]  
(7.19)
Integrating by parts,
\[
\int_0^t (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)} ds = (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)}(t) + (\bar{q}_\theta, \bar{u}^i_{\partial, j})_{L^2(\Omega)}
\]
\[
- \int_0^t (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)} ds - \int_0^t (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)} ds.
\]
By \(\epsilon\)-regularization, the last two term can be bounded by
\[
C(M) \int_0^t \|q_\theta\|_{L^2(\Omega)} \left[ C(\epsilon) \|\nabla w_\theta\|_{L^2(\Omega)} + \|\nabla w_\theta\|_{L^2(\Omega)} \right] ds
\]
and hence
\[
\left| \int_0^t (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)} ds \right| + \int_0^t (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)} ds
\]
\[
\leq C(M, \delta) \int_0^t \|q_\theta\|_{L^2(\Omega)}^2 ds + C(\epsilon) \int_0^t \|\nabla w_\theta\|_{L^2(\Omega)}^2 ds + \delta \int_0^t \|\nabla w_\theta\|_{L^2(\Omega)}^2 ds
\]
\[
\leq C(\epsilon, \delta) N_1(u_0, F) + C(M, \delta) \int_0^t \|w_{\partial t}\|_{L^2(\Omega)}^2 ds + C(\epsilon_1) \int_0^t \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 ds
\]
\[
+ \delta \int_0^t \|\nabla w_{\partial t}\|_{L^2(\Omega)}^2 ds.
\]
(7.20)
For \((q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)}(t)\), it is easy to see that
\[
\left| (q_\theta, \bar{u}^i_{\theta t} w_{\partial, j}^i)_{L^2(\Omega)}(t) \right| \leq \delta_1 \|w_{\partial t}\|_{L^2(\Omega)}^2 + C(\epsilon, \delta_1) \|\nabla w_\theta\|_{L^2(\Omega)}^2
\]
\[
\leq C(\epsilon, \delta_1) \|\nabla w_\theta\|_{L^2(\Omega)}^2 + \delta_1 C(\epsilon_1) \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \delta_1 \|w_{\partial t}\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)} + 1
\]
while for \((q_\theta, \bar{u}^i_{\partial, j})_{L^2(\Omega)}\), it is bounded by \(C(M) N_1(u_0, F)\). Combining (7.19), (7.20) and the estimates above, by choosing \(\delta > 0\) and \(\delta_1 > 0\) small enough,
\[
\|w_{\partial t}\|_{L^2(\Omega)}^2 + \|\nabla h_\theta\|_{L^2(\Gamma)}^2 + \int_0^t \left[ \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + (1 + \|h_\theta\|_{H^{2.5}(\Gamma)}) \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \|\bar{v}\|_{H^1(\Omega)}^2 \int_0^s \|\nabla w_{\partial t}\|_{L^2(\Omega)}^2 ds \right] ds
\]
\[
\leq C(\epsilon_1) \left[ N_2(u_0, F) + \int_0^t \left( \|w_{\partial t}\|_{L^2(\Omega)}^2 + (1 + \|h_\theta\|_{H^{2.5}(\Gamma)}) \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 \right) + C(\epsilon_1, \epsilon) \|\nabla w_\theta\|_{L^2(\Omega)}^2 \right]
\]
where \(N_2(u_0, F) = N_1(u_0, F) + \|F\|_{L^\infty(0,T;L^2(\Omega))}\). By the Gronwall inequality,
\[
\|w_{\partial t}\|_{L^2(\Omega)}^2 + \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \int_0^t \left[ \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \|\nabla w_{\partial t}\|_{L^2(\Omega)}^2 \right] ds
\]
\[
\leq C(\epsilon_1, \epsilon) N_2(u_0, F) + C(\epsilon_1, \epsilon) \|\nabla w_\theta\|_{L^2(\Omega)}^2,
\]
(7.21)
By using \(w_\theta(t) = \bar{u}_0 + \int_0^t w_\theta ds\), we see that
\[
\|w_{\partial t}\|_{L^2(\Omega)}^2 + \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \int_0^t \left[ \|\nabla^2 h_\theta\|_{L^2(\Gamma)}^2 + \|\nabla w_{\partial t}\|_{L^2(\Omega)}^2 \right] ds
\]
\[
\leq C(\epsilon_1, \epsilon) N_2(u_0, F) + C(\epsilon_1, \epsilon) t \int_0^t \|\nabla w_{\partial t}\|_{L^2(\Omega)}^2 ds.
\]
Therefore, for any $0 \leq t \leq t_1 = \min \left\{ T, \frac{1}{2C_1} \right\}$, we have
\[
\|w_{\theta t}\|_{L^2(\Omega)}^2 + \|
abla^2_{\partial\Gamma} h_{\theta t}\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_0^t \left[ \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 + \kappa \|w_{\theta t}\|_{H^2(\Gamma)}^2 \right] ds 
\leq C(\epsilon, \epsilon) N_2(u_0, F).
\]
By $w_{\theta}(t_1) = \tilde{u}_0 + \int_0^{t_1} w_{\theta t} ds$, we also have
\[
\|\nabla w_{\theta}(t_1)\|_{L^2(\Omega)}^2 \leq C(\epsilon, \epsilon) N_2(u_0, F).
\tag{7.22}
\]
For $t \geq t_1$, since $w_{\theta}(t) = w_{\theta}(t_1) + \int_{t_1}^t w_{\theta t} ds$, we have from (7.21) and (7.22) that
\[
\|w_{\theta t}\|_{L^2(\Omega)}^2 + \|
abla^2_{\partial\Gamma} h_{\theta t}\|_{L^2(\Gamma)}^2 + \int_0^t \left[ \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 + \kappa \|w_{\theta t}\|_{H^2(\Gamma)}^2 \right] ds 
\leq C(\epsilon, \epsilon) N_2(u_0, F) + C(\epsilon, \epsilon) \left[ \|w_{\theta}(t_1)\|_{L^2(\Omega)}^2 + (t - t_1) \int_{t_1}^t \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 ds \right] 
\leq C(\epsilon, \epsilon) N_2(u_0, F) + C(\epsilon, \epsilon)(t - t_1) \int_{t_1}^t \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 ds .
\]
Therefore, for any $t_1 \leq t \leq 2t_1$, we also have
\[
\|w_{\theta t}\|_{L^2(\Omega)}^2 + \|
abla^2_{\partial\Gamma} h_{\theta t}\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_0^t \left[ \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 + \kappa \|w_{\theta t}\|_{H^2(\Gamma)}^2 \right] ds 
\leq C(\epsilon, \epsilon) N_2(u_0, F)
\]
which with $w_{\theta}(2t_1) = \tilde{u}_0 + \int_0^{2t_1} w_{\theta t} ds$ gives
\[
\|\nabla w_{\theta}(2t_1)\|_{L^2(\Omega)}^2 \leq C(\epsilon, \epsilon) N_2(u_0, F).
\]
By induction, for any $t \in [0, T]$,
\[
\|w_{\theta t}\|_{L^2(\Omega)}^2 + \|
abla^2_{\partial\Gamma} h_{\theta t}\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_0^t \left[ \|\nabla w_{\theta t}\|_{L^2(\Omega)}^2 + \kappa \|w_{\theta t}\|_{H^2(\Gamma)}^2 \right] ds 
\leq C(\epsilon, \epsilon) N_2(u_0, F).
\tag{7.23}
\]
We also get a $\theta$-independent bound for $\|q_{\theta}\|_{L^2(0,T;L^2(\Omega))}$ by (7.15):
\[
\|q_{\theta}\|_{L^2(0,T;L^2(\Omega))} \leq C(\epsilon, \epsilon) N_2(u_0, F).
\tag{7.24}
\]
Let $\theta = \frac{1}{m}$. Energy inequalities (7.17), (7.23) and (7.24) show that there exists a subsequence $\frac{w}{\frac{1}{m}}$ such that
\[
\begin{align*}
\frac{w}{\frac{1}{m}} & \to \nu \quad \text{in} \quad L^2(0,T;H^{1,2}(\Omega;\Gamma)) \tag{7.25a} \\
\frac{w}{\frac{1}{m}} & \to \nu_t \quad \text{in} \quad L^2(0,T;H^{1,2}(\Omega)) \tag{7.25b} \\
\nabla^2_{\partial\Gamma} h_{\frac{1}{m}} & \to \nabla^2_{\partial\Gamma} h \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{7.25c} \\
\nabla^2_{\partial\Gamma} h_{\frac{1}{m}} & \to \nabla^2_{\partial\Gamma} h_t \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{7.25d} \\
q_{\frac{1}{m}} & \to q \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{7.25e}
\end{align*}
\]
Moreover, \((7.17)\) also shows that \(\|a_i^j w^1\|_{L^2(0,T;L^2(\Omega))} \to 0\) as \(m \to \infty\). Therefore the weak limit \(v\) satisfies the “divergence-free” condition \((7.2)\), i.e.,

\[
v \in \mathcal{V}_b(T).
\]

Since \((7.17)\) is independent of \(\theta\) and \(\epsilon_1\), by the property of lower-semicontinuity of norms,

\[
sup_{0 \leq t \leq T} \left[ \|v(t)\|_{L^2(\Omega)}^2 + \|\nabla_v^2 \theta(t)\|_{L^2(\Gamma)}^2 \right] + \|\nabla v\|_{L^2(0,T;L^2(\Omega))}^2 + \kappa\|v\|_{H^2(\Gamma)}^2 \leq C(M) N_1(u_0,F). \tag{7.27}
\]

By \((7.23)\) and \(\epsilon_1\)-regularization, the weak limit \((v, h, q)\) satisfies, for all \(\varphi \in L^2(0,T;H^{1,2}(\Omega;\Gamma))\),

\[
\int_0^T (v_t, \varphi)_{L^2(\Omega)} dt + \frac{\nu}{2} \int_0^T \int_\Omega D_\theta(v) : D_\theta(\varphi) dx dt + \kappa \int_0^T \int_\Gamma \Delta_0 v \cdot \Delta_0 \varphi dS dt
\]

\[
- \int_0^T (\vec{a}_i^j q, \varphi_{ij})_{L^2(\Omega)} dt + \sigma \int_0^T \int_\Gamma \vec{A}_{\alpha\beta\gamma\delta} \eta_{\alpha\beta \gamma\delta} [-\vec{h}, \sigma(\varphi \circ \vec{h}^{-1}) + \varphi^\gamma \circ \vec{h}^{-1}] \varphi_{ij} dS dt
\]

\[
= \int_0^T \left\{ (\bar{F}, \varphi) - \sigma \int_\Gamma \left[ L_{1\alpha\beta\gamma} \eta_{\alpha\beta \gamma} + L_{2\alpha\beta\gamma} \eta_{\alpha\beta \gamma} \right] \varphi_{ij} \right\} dt.
\]

By the density argument, we find that for a.a. \(t \in [0,T]\), \(\varphi \in H^{1,2}(\Omega;\Gamma)\),

\[
(v_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_\Omega D_\theta(v) : D_\theta(\varphi) dx + \kappa \int_\Gamma \Delta_0 v \cdot \Delta_0 \varphi dS
\]

\[
+ \sigma \int_\Gamma \vec{A}_{\alpha\beta\gamma\delta} \eta_{\alpha\beta \gamma\delta} [-\vec{h}, \sigma(\varphi \circ \vec{h}^{-1}) + \varphi^\gamma \circ \vec{h}^{-1}] \varphi_{ij} dS \tag{7.28}
\]

\[
= (\bar{F}, \varphi) - \sigma \int_\Gamma \left[ L_{1\alpha\beta\gamma} \eta_{\alpha\beta \gamma} + L_{2\alpha\beta\gamma} \eta_{\alpha\beta \gamma} \right] \varphi_{ij} dS,
\]

or after a change of variable \(y = \vec{h}(y, t)\),

\[
(v_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_\Omega D_\theta(v) : D_\theta(\varphi) dx + \kappa \int_\Gamma \Delta_0 v \cdot \Delta_0 \varphi dS - (\vec{a}_i^j q, \varphi_{ij})_{L^2(\Omega)} \tag{7.29}
\]

\[
+ \sigma \int_\Gamma L_{1\alpha\beta\gamma} (h)(-\nabla_0 \vec{h} \circ \vec{h}^{-1}, 1) \cdot \varphi_{ij} dS = (\bar{F}, \varphi) - \sigma \int_\Gamma \vec{M}_h^{(1)} (-\nabla_0 \vec{h} \circ \vec{h}^{-1}, 1) \cdot \varphi_{ij} dS.
\]

Furthermore, if \(\varphi \in \mathcal{V}_b\), then

\[
(v_t, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_\Omega D_\theta(v) : D_\theta(\varphi) dx + \kappa \int_\Gamma \Delta_0 v \cdot \Delta_0 \varphi dS
\]

\[
+ \sigma \int_\Gamma L_{1\alpha\beta\gamma} (h)(-\nabla_0 \vec{h} \circ \vec{h}^{-1}, 1) \cdot \varphi_{ij} dS = (\bar{F}, \varphi) - \sigma \int_\Gamma \vec{M}_h^{(1)} (-\nabla_0 \vec{h} \circ \vec{h}^{-1}, 1) \cdot \varphi_{ij} dS
\]

for a.a. \(t \in [0,T]\). In other words, \((v, h, q)\) is a weak solution of \((7.2)\).

8. Estimates independent of \(\epsilon_1\)

8.1. Partition of unity. Since \(\Omega\) is compact, by partition of unity, we can choose two non-negative smooth functions \(\zeta_0\) and \(\zeta_1\) so that

\[
\zeta_0 + \zeta_1 = 1 \quad \text{in} \quad \Omega;
\]

\[
\text{supp}(\zeta_0) \subset \subset \Omega;
\]

\[
\text{supp}(\zeta_1) \subset \subset \Gamma \times (-\epsilon, \epsilon) := \Omega_1.
\]
We will assume that $\zeta_1 = 1$ inside the region $\Omega'_1 \subset \Omega_1$ and $\zeta_0 = 1$ inside the region $\Omega' \subset \Omega$. Note that then $\zeta_1 = 1$ while $\zeta_0 = 0$ on $\Gamma$.

### 8.2. Higher regularity.

#### 8.2.1. $\epsilon_1$-independent bounds for $q$. Similar to (7.18), we have
\[
\|q\|_{L^2(\Omega)}^2 \leq C(M) \left[ \|v_t\|_{L^2(\Omega)}^2 + \|\nabla q\|_{L^2(\Omega)}^2 + \|\nu a_{\ell,j} \circ \eta^{-1} u_{,\ell} - p a_{\ell,j} \circ \eta^{-1}\|_{L^2(\Omega)}^2 \right] + \|F\|_{L^2(\Omega)}^2 + 1, \tag{8.1}
\]

#### 8.2.2. Interior regularity. Converting the fluid equation (7.2) into Eulerian variables by composing with $\eta^{-1}$, we obtain a Stokes problem in the domain $\tilde{\eta}(\Omega)$:
\[
\begin{aligned}
-\nu \Delta u + \nabla p &= \tilde{F} \circ \eta^{-1} - v_t \circ \eta^{-1} + \nu a_{\ell,j} \circ \eta^{-1} u_{,\ell} - p a_{\ell,j} \circ \eta^{-1}, \\
\text{div} u &= 0,
\end{aligned} \tag{8.2a}
\]
where $u = v \circ \eta^{-1}$ and $p = q \circ \eta^{-1}$. By the regularity results for the Stokes problem,
\[
\|u\|_{H^2(\tilde{\eta}(\Omega))}^2 + \|p\|_{H^1(\tilde{\eta}(\Omega))}^2 \leq C \left[ \|\tilde{F} \circ \eta^{-1}\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|v_t \circ \eta^{-1}\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|\nabla u\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|p\|_{L^2(\tilde{\eta}(\Omega))}^2 \right] + \|u\|_{H^1(\Gamma)}^2
\]

or
\[
\|u\|_{H^2(\tilde{\eta}(\Omega))}^2 + \|q\|_{H^1(\Gamma)}^2 \leq C \left[ \|F\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|v_t\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|v\|_{H^1(\tilde{\eta}(\Omega))}^2 \right] + C(M) \left[ \|\nabla q\|_{L^2(\tilde{\eta}(\Omega))}^2 + \|q\|_{L^2(\tilde{\eta}(\Omega))}^2 \right]
\]

for some constant $C$ independent of $M, \epsilon$. By (8.1),
\[
\|v_t\|_{H^2(\Omega)}^2 + \|q\|_{H^1(\Omega)}^2 \leq C(M) \left[ \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right] + \|\nabla a_{\ell,j}\|_{L^2(\Gamma)}^2 + F\|_{L^2(\Omega)}^2 + 1 \tag{8.3}
\]

Similarly,
\[
\|v\|_{H^3(\Omega)}^2 + \|q\|_{H^2(\Omega)}^2 \leq C \left[ \|F\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right] + C(M) \left[ \|\nabla v\|_{H^1(\Omega)}^2 + \|q\|_{H^1(\Omega)}^2 \right]
\]

and therefore by (8.1) and (8.3),
\[
\|v\|_{H^3(\Omega)}^2 + \|q\|_{H^2(\Omega)}^2 \leq C(M) \left[ \|v\|_{H^1(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla a_{\ell,j}\|_{H^2(\Gamma)}^2 \right] + \|\nabla a_{\ell,j}\|_{L^2(\Gamma)}^2 + \|F\|_{H^1(\Omega)}^2 + 1. \tag{8.4}
\]

For the regularized problem, because the $\epsilon$-regularization ensures that the forcing and the initial data are smooth, while the $\epsilon_1$-regularization ensures that the right-hand side of (7.2) is smooth, by standard difference quotient technique, it is also easy to see that
\[
\nabla a_{\ell,j} v \in L^2(0, T; H^1(\Omega) \cap H^2(\Gamma)) \quad \text{for } k = 1, 2, 3, 4 \tag{8.5}
\]
Since (7.25b) implies that $v_i \in L^2(0, T; H^1(\Omega))$, by $\epsilon_1$-regularization and (8.4) we conclude that

$$v \in L^2(0, T; H^3(\Omega)), \quad q \in L^2(0, T; H^2(\Omega)).$$

(8.6)

8.3. Estimates for $v_i(0)$ and $q(0)$. By (5.6) and $\epsilon_1$-regularization, $(v, h, q)$ satisfies the strong from (7.2). Taking the “divergence” of (7.2a) and then making use of condition (7.2b), we find that

$$-\bar{a}_i^k a_i^j v_i - \nu \bar{a}_i^k [a_i^j D_q(v)]_{jk} = -\bar{a}_i^k (a_i^j q)_{jk} + \bar{a}_i^k \bar{F}_i.$$  

(8.7)

Let $t = 0$, by the identity $\bar{a}_i^k = -\bar{a}_k^j \bar{a}_i^j,$

$$\Delta q(0) = (\nabla \bar{u}_0) : (\nabla \bar{u}_0)^T - \text{div}(\bar{F}(0)) \quad \text{in} \quad \Omega$$

with

$$q(0) = \nu(\text{Def} \bar{u}_0)_{ij} N_i N_j - \sigma M_0 v_i^i(0) + \kappa \Delta^2 \bar{u}_0$$

on $\Gamma$.

while (7.2a) gives us

$$v_i(0) = \nu \Delta \bar{u}_0 - \nabla q(0) + \bar{F}(0) \quad \text{in} \quad \Omega.$$

By standard elliptic regularity result,

$$\|v_i(0)\|_{L^2(\Omega)}^2 + \|q(0)\|_{H^1(\Omega)}^2 \leq CN_0(u_0, F)$$

(8.8)

for some constant independent of $M, \epsilon$ and $\epsilon_1$.

8.4. $L^2 - L^2$-estimates for $v_i$. Since $v_i \in L^2(0, T; H^1(\Omega))$, we can use it as a test function in (7.2a). By (7.2a), we find that

$$\|v_i\|_{L^2(\Omega)}^2 + \nu \frac{d}{dt} \int_\Omega |D_q v|^2 dx - \frac{\nu}{2} \int_\Omega (D_q v)^2_{ij} a_i^j v_i^j dx + \kappa \int_\Gamma \Delta a \cdot \Delta q \cdot \Phi dS$$

$$+ \int_\Omega q \bar{a}_i^k a_i^j v_i^j dx + \sigma \int_\Gamma \nu \nu(\text{Def} \bar{u}_0)_{ij} N_i N_j (1,1) \cdot v_i dS$$

$$= \langle \bar{F}, v_i \rangle - \sigma \int_\Gamma (M_0^e)_{ij} (1,1) \cdot v_i dS.$$

By (5.3),

$$\int_\Omega (D_q v)^2_{ij} a_i^k a_i^j v_i^j dx \leq C(M)C(\delta) \|\nabla v\|_{L^2(\Omega)}^2 + \delta \|v\|_{H^2(\Omega)}^2$$

and by (8.1) and the interpolation inequality,

$$\left| \int_\Omega q \bar{a}_i^k a_i^j v_i^j dx \right| \leq C(M)C(\delta) \left[ \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla^2 \Phi_{ij}^i\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 + 1 \right]$$

$$+ \delta \|v\|_{H^2(\Omega)}^2 + \frac{1}{2} \|v_i\|_{L^2(\Omega)}^2$$

for some $C(\delta)$. Also, the last term on the left hand side is bounded by

$$C(M) \left[ \|\nabla^2 \Phi_{ij}^i\|_{L^2(\Omega)}^2 + 1 \right] \|v_i\|_{H^1(\Omega)}$$

$$\leq C(M)C(\delta_1) \left[ \|\nabla^2 \Phi_{ij}^i\|_{L^2(\Omega)}^2 + 1 \right] + \delta_1 \|v_i\|_{H^1(\Omega)}.$$

Combining all the estimates above,

$$\frac{1}{2} \|v_i\|_{L^2(\Omega)}^2 + \nu \frac{d}{dt} \int_\Omega |D_q v|^2 dx + \frac{\kappa}{2} \frac{d}{dt} \int_\Gamma |\Delta a|^2 dS$$

$$\leq C \left[ \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla^2 \Phi_{ij}^i\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 + 1 \right] + \delta \|v\|_{H^2(\Omega)}^2 + \delta_1 \|v_i\|_{H^1(\Omega)}.$$
for some constant $C$ depending on $M$, $\delta$ and $\delta_1$. Therefore by (8.24),
\[
\int_0^t \|\nu\|_{H^2(\Omega)}^2 ds + \|\nabla \nu(t)\|_{H^2(\Omega)}^2 + \kappa \|\nu\|_{H^2(\Gamma)}^2
\]
\[
\leq C \left[ N_2(u_0, F) + \int_0^t \|\nabla h_f\|_{L^2(\Gamma)}^2 ds \right] + \delta \int_0^t \|\nu\|_{H^2(\Gamma)}^2 ds + \delta_1 \int_0^t \|\nu\|_{H^1(\Gamma)}^2 ds.
\]  
(8.9)

8.5. Energy estimates for $\nabla_0^2 \nu$ near the boundary. Because of (8.5), $\nabla_0^2 (\zeta^2 \nabla_0^2 \nu)$ in (7.28) can be used as a test function in (7.29). It follows that
\[
\left| \int_{\Gamma} \left[ \hat{\mathcal{L}}^2_h (h^{*1}) + \hat{\mathcal{M}}^2_h \right] (\nabla \nu \circ \gamma, 1) \cdot \nabla_0 \nu dS \right| \leq C(M) \left[ \|\nabla_0^2 h^{*1}\|_{H^2(\Gamma)} + 1 \right] \|\nu\|_{H^4(\Gamma)}
\]
\[
\leq C(M, \delta_3) \left[ 1 + \|\nu\|_{H^4(\Gamma)}^2 \right] + \delta_3 \|\nu\|_{H^4(\Gamma)}^2.
\]
By (7.4), we find that
\[
\|\nu\|_{H^4(\Gamma)}^2 \leq C(\epsilon) \left[ \int_0^t \|\nabla \nu \|_{H^2(\Gamma)} \|\nu\|_{H^4(\Gamma)} ds \right] \leq C(\epsilon) \int_0^t \|\nu\|_{H^2(\Gamma)}^2 ds
\]
and hence
\[
\left| \int_{\Gamma} \left[ \hat{\mathcal{L}}^2_h (h^{*1}) + \hat{\mathcal{M}}^2_h \right] (\nabla \nu \circ \gamma, 1) \cdot \nabla_0 \nu dS \right| \leq \tilde{C} \left[ 1 + \int_0^t \|\nu\|_{H^4(\Gamma)}^2 \right] + \delta_3 \|\nu\|_{H^4(\Gamma)}^2.
\]
for some constant $\tilde{C}$ depending on $M$, $\epsilon$ and $\delta_3$. Since
\[
\Delta_0 f = \frac{1}{\sqrt{\text{det}(g_0)}} \frac{\partial}{\partial y^\alpha} \left[ \sqrt{\text{det}(g_0)} g_0^{\alpha \beta} \frac{\partial}{\partial y^\beta} f \right].
\]
by the regularity on $\Gamma$ (and hence on $g_0$),
\[
\left| \int_{\Gamma} \Delta_0 \nabla_0^2 \nu \right|^2 dS \leq \int_{\Gamma} \Delta_0^2 \nu \cdot (\nabla_0^2 \nu) dS + C \|\nu\|_{H^4(\Gamma)} \|\nu\|_{H^4(\Gamma)}
\]
\[
\leq \int_{\Gamma} \Delta_0^2 \nu \cdot (\nabla_0^2 \nu) dS + C(\delta) \|\nu\|_{H^4(\Gamma)}^2 + \delta \|\nu\|_{H^4(\Gamma)}^2
\]
which implies, by choosing $\delta > 0$ small enough, that
\[
\nu_2 \|\nu\|_{H^4(\Gamma)}^2 \leq \int_{\Gamma} \Delta_0^2 \nu \cdot (\nabla_0^2 \nu) dS + C \|\nu\|_{H^4(\Gamma)}^2.
\]
By the identity
\[
(q, \delta_k^{\ell} (\nabla_0^2 (\zeta^2 \nabla_0^2 \nu), \ell) = (q, \nabla_0^2 (\zeta^2 \nabla_0^2 \nu), \ell) + 4(\zeta_1 \nabla_0 q, \nabla_0 \delta_k^{\ell} \zeta_1 \ell \nabla_0^2 \nu) + 2(\nabla_0 q, \zeta_1^2 \nabla_0 \delta_k^{\ell} \nabla_0^2 \nu)
\]
\[
- 2(\zeta_1 \nabla_0 q, \nabla_0 (\delta_k^{\ell} \zeta_1 \ell \nabla_0 \nu) + 2(q, \nabla_0 (\delta_k^{\ell} \zeta_1 \ell \nabla_0 \nu) + \nabla_0 q, \nabla_0 (\zeta_1^2 \nabla_0 \delta_k^{\ell} \nabla_0 \nu))
\]
\] (8.10)
and \( \|q\|_{H^1(\Omega)} \|v\|_{H^3(\Omega)} \) imply that

\[
(q, \hat{a}^k (\nabla^2_{\xi} (\hat{a}^k \nabla^2_\xi v))) \leq C(M) \|q\|_{H^1(\Omega)} \|v\|_{H^3(\Omega)}
\]

\[
\leq C(M)C(\delta) \left[ \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla \nabla_0 v\|_{L^2(\Omega)}^2 + \kappa \|v\|_{H^2(\Gamma)}^2 + \|\nabla^2_{\nu^*} v\|_{L^2(\Gamma)}^2 + F_{L^2(\Gamma)}^2 + 1 \right] + \delta \|v\|_{H^3(\Omega)}^2.
\]

For the viscosity term,

\[
\int_{\Omega} D_\theta v : D_\theta (\nabla_0^2 (\zeta^1 \nabla^2_\xi v)) dx
\]

\[
= \| \zeta_1 D_\theta \nabla_0^2 v \|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} \left[ \nabla_0^2 (\hat{a}^k \hat{a}^j) v_{x}^j + \nabla_0^2 (\hat{a}^k \hat{a}^j) v_{x}^j \right] \zeta_1^2 \nabla^2_\xi v dx
\]

\[
+ \int_{\Omega} \left[ \nabla_0 (\hat{a}^k \hat{a}^j) \nabla_0 v_{x}^j + \nabla_0 (\hat{a}^k \hat{a}^j) \nabla_0 v_{x}^j \right] \zeta_1^2 \nabla^2_\xi v dx
\]

\[
+ \int_{\Omega} D_\theta (\nabla_0^2 v) \zeta_1 \zeta_1 \zeta_1 \zeta_1 \nabla^2_\xi v dx
\]

and hence by interpolation

\[
\frac{1}{2} \| \zeta_1 D_\theta \nabla_0^2 v \|_{L^2(\Omega)}^2 \leq \int_{\Omega} D_\theta v : D_\theta (\nabla_0^2 (\zeta_1 \nabla^2_\xi v)) dx
\]

\[
+ C(M)C(\delta) \left[ \|v\|_{L^2(\Omega)}^2 + \|\nabla \nabla_0 v\|_{L^2(\Omega)}^2 \right] + \delta \|v\|_{H^3(\Omega)}^2.
\]

Summing all the estimates, by letting \( \delta_3 = \frac{\nu_2}{\kappa} \), we conclude that

\[
\frac{1}{2} \frac{d}{dt} \| \zeta_1 \nabla_0^2 v \|_{L^2(\Omega)}^2 + \frac{\nu}{4} \| \zeta_1 D_\theta \nabla_0^2 v \|_{L^2(\Omega)}^2 + \frac{\nu \kappa}{2} \|v\|_{H^4(\Gamma)}^2
\]

\[
\leq \bar{C} \left[ \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{H^1(\Omega)}^2 + \|\nabla \nabla_0 v\|_{L^2(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 + \|\nabla^2_{\nu^*} v\|_{L^2(\Gamma)}^2 + F_{H^3(\Omega)}^2 + 1 \right] + \bar{C} \int_0^t \|v\|_{H^4(\Gamma)}^2 ds + \delta \|v\|_{H^3(\Omega)}^2
\]

for some constant \( \bar{C} \) depending on \( M, \kappa, \epsilon \) and \( \delta \). Integrating the inequality above in time from 0 to \( t \), by (7.24) we find that

\[
\| \nabla_0^2 v(t) \|_{L^2(\Omega)}^2 + \int_0^t \left[ \|\nabla \nabla_0^2 v\|_{L^2(\Omega)}^2 + \kappa \|v\|_{H^4(\Gamma)}^2 \right] ds
\]

\[
\leq \bar{C} N_2(u_0, F) + \bar{C} \int_0^t \left[ \|v\|_{L^2(\Omega)}^2 + \|\nabla \nabla_0 v\|_{L^2(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right] ds
\]

\[
(8.11)
\]

\[
+ \bar{C} \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr + \delta \int_0^t \|v\|_{H^3(\Omega)}^2 ds.
\]

By using \( \nabla_0 (\zeta_1^2 \nabla_0 v) \) as a testing function in (7.24), similar computations leads to

\[
\| \nabla_0 v(t) \|_{L^2(\Omega)}^2 + \int_0^t \left[ \|\nabla \nabla_0 v\|_{L^2(\Omega)}^2 + \kappa \|v\|_{H^4(\Gamma)}^2 \right] ds
\]

\[
\leq C(M) N_2(u_0, F) + C(M, \delta) \int_0^t \left[ \|v\|_{L^2(\Omega)}^2 + \kappa \|v\|_{H^2(\Gamma)}^2 \right] ds
\]

\[
+ C(M) \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr ds + \delta \int_0^t \|v\|_{H^3(\Omega)}^2 ds.
\]

\[
(8.12)
\]
8.6. **Energy estimates for \( v_t - L^2 H^1 \)-estimates.** In this section, we time-differentiate \((7.20)\) and then use \( v_t \) as a test function to obtain

\[
\langle \nu, v_t \rangle + \nu \int_{\Omega} \left[ \hat{a}_k (D_n \nu_v)_{t,k} \right] v_t^j dx + \sigma \int_{\Gamma} \left[ \bar{\mathcal{C}}_t^j (\theta^j) (-\nabla_0 \bar{h} \circ \eta^t, 1) \right] \cdot v_t \, dS
\]

\[+ \kappa \int_{\Omega} |\Delta_0 v_t|^2 dS - \int_{\Omega} (\bar{a}^k_q)v_t^k \, dx = \langle F_t, v_t \rangle - \sigma \int_{\Gamma} \left[ \bar{\mathcal{M}}_t^j (-\nabla_0 \bar{h} \circ \eta^t, 1) \right] \cdot v_t \, dS.
\]

By the chain rule,

\[
\int_{\Gamma} \left[ \left( \bar{\mathcal{C}}_t^j(\theta^j) + \bar{\mathcal{M}}_t^j(-\nabla_0 \bar{h} \circ \eta^t, 1) \right) \cdot v_t \right] \, dS
\]

\[= \int_{\Gamma} \Theta_t \left[ L_{\bar{h}}(\theta^j) \right]^t \cdot \eta^t \left( -\nabla_0 \bar{h} \circ \eta^t, 1 \right) \cdot v_t \, dS
\]

\[+ \int_{\Gamma} \Theta \eta^t \cdot \left[ \nabla_0 [L_{\bar{h}}(\theta^j)]^t (-\nabla_0 \bar{h}, 1) \right] \cdot \eta^t \cdot v_t \, dS
\]

\[+ \int_{\Gamma} \bar{\Theta} \left[ [L_{\bar{h}}(\theta^j)]^t (-\nabla_0 \bar{h}, -1) \right] \cdot \eta^t \cdot v_t \, dS.
\]

By using \( H^2(\Gamma)-H^{-2}(\Gamma) \) duality pairing with \( \epsilon \)-regularization on \( \bar{\Theta} \) and \( \bar{v} \), it follows that

\[
\left| \int_{\Gamma} \left[ \left( \bar{\mathcal{C}}_t^j(\theta^j) + \bar{\mathcal{M}}_t^j(-\nabla_0 \bar{h} \circ \eta^t, 1) \right) \cdot v_t \right] \, dS \right|
\]

\[\leq C(\epsilon) \left[ \| \nabla_0^3 \bar{h} \|_{L^2(\Gamma)} + \| \nabla_0^2 \bar{h} \|_{L^2(\Gamma)} + 1 \right] \| \nu_t \|_{H^2(\Gamma)},
\]

\[\leq C(\epsilon, \delta_3) \left[ \int_0^t \| \nu \|_{H^4(\Gamma)}^2 \, ds + \| \nu \|_{H^2(\Gamma)}^2 + 1 \right] + \delta_3 \| \nu_t \|_{H^2(\Gamma)}^2
\]

\[\leq C \left[ \int_0^t \| \nu \|_{H^4(\Gamma)}^2 \, ds + \| \nu \|_{H^2(\Gamma)}^2 + 1 \right] + \delta \| \nu \|_{H^2(\Gamma)}^2 + \delta_3 \| \nu_t \|_{H^2(\Gamma)}^2
\]

for some constant \( C \) depending on \( M, \epsilon, \delta \) and \( \delta_3 \), where we estimate \( \| \nu \|_{H^2(\Gamma)}^2 \) by interpolation.

Also by interpolation

\[
\int_{\Omega} |D_n \nu_t|^2 \, dx = 2 \int_{\Omega} \left[ \hat{a}^k \partial_{ik} \nu_v^j\right] v_t^j \, dx - 2 \int_{\Omega} \left[ (\hat{a}^k \partial_{ik}) v_t^j + (\hat{a}^k \partial_{ik}) v_t^j \right] v_t^j \, dx
\]

\[\leq 2 \int_{\Omega} \left[ \hat{a}^k \partial_{ik} \nu_v^j\right] v_t^j \, dx + C(M) C(\delta, \delta_1) \| \nabla_0 \nu \|_{L^2(\Omega)}^2
\]

\[- \int_{\Omega} (\hat{a}^k_q) v_t^k \, dx + \delta \| \nu \|_{H^2(\Omega)}^2 + \delta_1 \| \nu_t \|_{H^2(\Omega)}^2.
\]

Note that

\[
\langle F_t, v_t \rangle \leq C(\| F_t \|_{H^1(\Omega)}^2) \| v_t \|_{H^1(\Omega)} \leq C(\delta_1) \| F_t \|_{H^2(\Omega)}^2 + \delta_1 \| \nu_t \|_{H^2(\Omega)}^2.
\]

Summing all the estimates above,

\[
\frac{1}{2} \frac{d}{dt} \| v_t \|_{L^2(\Omega)}^2 + \nu \| \nabla_0 \nu_t \|_{L^2(\Gamma)}^2 + \kappa \| \Delta_0 \nu_t \|_{L^2(\Gamma)}^2
\]

\[\leq C \left[ \int_0^t \| \nu \|_{H^4(\Gamma)}^2 \, ds + \| \nu \|_{H^2(\Gamma)}^2 + 1 \right] + C(\delta_1) \| F_t \|_{H^2(\Gamma)}^2
\]

\[+ \delta \| \nu \|_{H^2(\Gamma)}^2 + \delta_1 \| \nu_t \|_{H^2(\Gamma)}^2 + \delta_3 \| \nu_t \|_{H^2(\Gamma)}^2 + \int_{\Omega} (\hat{a}^k_q) v_t^k \, dx.
\]
for some constant $\bar{C}$ depending on $M$, $\kappa$, $\delta$ and $\delta_1$. As in (10) and (11), the integral involving the pressure $q$ has the following estimate:

$$
\int_0^t \int_\Omega (\partial_t^2 q) L_t dx ds \leq C(M)C(\delta, \delta_1)N_3(u_0, F) + \delta \int_0^t \|v\|_{H^3(\Omega)}^2 ds + \delta_1 \int_0^t \|v_t\|_{H^1(\Omega)}^2 ds
$$

where

$$
N_3(u_0, F) := \|u_0\|_{H^2(\Omega)}^2 + \|u_0\|_{H^4(\Gamma)}^2 + \|F\|_{L^2(0, T; H^4(\Omega))}^2 + \|F_t\|_{L^2(0, T; H^4(\Gamma))}^2 + \|F(0)\|_{H^4(\Omega)}^2 + 1.
$$

Integrating (8.13) in time from 0 to $t$ and choosing $\delta_1, \delta_3 > 0$ small enough, (7.27) and (8.9) imply that, for all $t \in [0, T]$,

$$
\|v_t(t)\|_{L^2(\Omega)}^2 + \int_0^t \left[ \|\nabla v_t\|_{L^2(\Omega)}^2 + \kappa \|v_t\|_{H^2(\Gamma)}^2 \right] ds \leq \bar{C}N_3(u_0, F) + \bar{C} \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr ds + \delta \int_0^t \|v\|_{H^3(\Omega)}^2 ds
$$

for some constant $\bar{C}$ depending on $M$, $\kappa$, $\delta$ and $\delta_2$. In (8.14), (8.3) is used to bound $\|v_t(0)\|_{L^2(\Omega)}$.

### 8.7. $\epsilon_1$-independent estimates.

Integrating (8.3) in time from 0 to $t$, (7.27), (8.9) and (8.12) imply that

$$
\int_0^t \left[ \|v\|_{H^2(\Omega)}^2 + \|q\|_{H^1(\Omega)}^2 \right] ds \leq C(M)N_1(u_0, F) + \int_0^t \left[ \|v_t\|_{L^2(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right] ds \leq \bar{C}N_3(u_0, F) + \bar{C} \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr ds + \bar{C} \int_0^t \int_0^s \|v\|_{H^3(\Omega)}^2 ds
$$

for some constant $\bar{C}$ depending on $M$, $\kappa$ and $\delta$. Integrating (8.4) in time from 0 to $t$, making use of (8.11), (8.12), (8.14), (8.15), and then choosing $\delta > 0$ small enough and $T$ even smaller, we find that

$$
\int_0^t \left[ \|v\|_{H^2(\Omega)}^2 + \|q\|_{H^2(\Omega)}^2 \right] ds \leq \bar{C}N_3(u_0, F) + \bar{C} \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr ds
$$

for some constant $\bar{C}$ depending on $M$, $\kappa$ and $\epsilon$.

Having (8.10), by choosing $\delta_2 > 0$ small enough, the estimates (8.11) can be rewritten as

$$
\|\nabla^2 u(t)\|_{L^2(\Omega)}^2 + \int_0^t \left[ \|\nabla^2 u\|_{L^2(\Omega)}^2 + \kappa \|v\|_{H^4(\Gamma)}^2 \right] ds \leq \bar{C}N_3(u_0, F) + \bar{C} \int_0^t \int_0^s \|v(r)\|_{H^4(\Gamma)}^2 dr ds
$$

for some constant $\bar{C}$ depending on $M$, $\kappa$ and $\epsilon$. Therefore,

$$
X(t) \leq \bar{C} \left[ \int_0^t X(s) ds + N_3(u_0, F) \right]
$$
where

\[ X(t) = \int_0^t \|v\|_{H^4(\Gamma)}^2 ds. \]

By the Gronwall inequality,

\[ \int_0^T \int_0^t \|v(r)\|_{H^4(\Gamma)}^2 dr ds \leq \tilde{C} N_3(u_0, F) \]  \hfill (8.18)

for all \( t \in [0, T] \) for some constant \( \tilde{C} \) depending on \( M, \kappa, \) and \( \epsilon. \) Having (8.18), estimates (8.1), (8.14), (8.16) and (8.17) along with the standard embedding theorem lead to

\[ \sup_{0 \leq t \leq T} \left[ \|v(t)\|_{H^2(\Omega)}^2 + \|v(t)\|_{L^2(\Gamma)}^2 \right] + \|v\|_{L^2(0,T;H^2(\Omega))}^2 + \kappa \|v\|_{L^2(0,T;H^4(\Omega))}^2 \leq \tilde{C} N_3(u_0, F) \]  \hfill (8.19)

for some constant \( \tilde{C} \) depending on \( M, \kappa, \) and \( \epsilon. \)

8.8. Weak limits as \( \epsilon_1 \to 0. \) Since the estimate (8.19) is independent of \( \epsilon_1, \) the weak limit as \( \epsilon_1 \to 0 \) of the sequence \((v, h, q)\) exists. We will denote the weak limit of \((v, h, q)\) by \((v_\kappa, h_\kappa, q_\kappa)\). By lower semi-continuity, (8.8) and thus (8.19) hold for the weak limit \((v_\kappa, h_\kappa, q_\kappa)\). Furthermore,

\[ \langle v_{\kappa t}, \varphi \rangle + \frac{\nu}{2} \int_\Omega D\eta v_{\kappa} : D\eta \varphi dx + \sigma \int_\Gamma \Theta \left[ \mathcal{L}_h(h_\kappa)(-\nabla_0 h, 1) \circ \bar{\eta}^7 \right] \cdot \varphi dS \]

\[ + \kappa \int_\Gamma \Delta_0 v_{\kappa} \cdot \Delta_0 \varphi dS - (q_\kappa, a_\kappa^i \varphi_{\kappa}^i)_{L^2(\Omega)} \]  \hfill (8.20)

for all \( \varphi \in H^{1,2}(\Omega; \Gamma) \) and a.a. \( t \in [0, T]. \)

9. Estimates independent of \( \kappa \) and \( \epsilon. \)

9.1. Energy estimates which are independent of \( \kappa. \) Although (8.19) doesn’t imply that \( h_\kappa \in H^4(\Gamma), \) \( h_\kappa \) is indeed in \( H^4(\Gamma) \) by (7.4). Therefore, we have that \((v_\kappa, h_\kappa, q_\kappa)\) satisfies

\[ v_{\kappa t} - \nu [a_\kappa^i D_h(v_{\kappa})_{,i}]_{,k} = -(a_\kappa^k q_\kappa)_{,k} + \bar{F}^i \]  \hfill (9.1a)

\[ a_\kappa^i v_{\kappa, i} = 0 \]  \hfill (9.1b)

\[ [\nu D_h(v_{\kappa})_{,i} - q_\kappa \delta^i_k]a_\kappa^j N_\kappa = \sigma \Theta [\mathcal{L}_h(h_\kappa)(-\nabla_0 h, 1) \circ \bar{\eta}^7 \rangle - \sigma \Theta [\mathcal{M}_h(-\nabla_0 h, 1) \circ \bar{\eta}^7 + \kappa \Delta_0^2 v_\kappa \]  \hfill (9.1c)

\[ h_\kappa \circ \bar{\eta}^7 = [(h_\kappa) \circ \bar{\eta}^7]v_\alpha - v_z \]  \hfill (9.1d)

\[ v = \bar{u}_0 \]  \hfill (9.1e)

\[ h = 0 \]  \hfill (9.1f)

Having (9.1c), (A.1) in Appendix A implies that \( h_\kappa \) is in \( H^3(\Gamma) \) for a.a. \( t \in [0, T] \) with estimate

\[ \int_0^t \|\nabla_0^2 h_\kappa\|_{H^3(\Gamma)}^2 ds \leq C(\epsilon) \int_0^t \left[ \|\nabla_0^4 h_\kappa\|_{L^2(\Gamma)}^2 + \|v_{\kappa t}\|_{H^3(\Omega)}^2 + \|q_{\kappa t}\|_{H^2(\Omega)}^2 + 1 \right] ds, \]
where the forcing $f$ in (A.7) is given by
\[ [\nu D\eta(v) - \eta_0] \delta_1 N_t - \sigma \Theta[M_h(-\nabla_0 h, 1)] \circ \tilde{\eta}^\tau. \]

By the same argument, (I.15) holds with all $\theta$ replaced by $\kappa$. Therefore, by (8.4) (which follows from (7.18)),
\[
\int_0^t \| \nabla_0^2 h_\kappa \|_{L^2(\Gamma)}^2 \, ds \leq C(\epsilon) \int_0^t \left[ \| v_{\kappa t} \|_{H^1(\Omega)}^2 + \| \nabla_0^2 h_\kappa \|_{L^2(\Gamma)}^2 + \| \nabla_0^2 v_\kappa \|_{H^1(\Omega_1)}^2 \right] \, ds + C(\epsilon) N_2(u_0, F).
\]

(9.2)

With this extra regularity of $h_\kappa$, the energy estimate (8.10) can be made independent of $\kappa$. In Appendix B2, we prove that
\[
\frac{\nu_1}{2} \| \nabla_0^4 h_\kappa(t) \|_{L^2(\Gamma)}^2 \leq \int_0^t \int_\Gamma \Theta \left[ L \tilde{h}(h_\kappa)(-\nabla_0 h, 1) \circ \tilde{\eta}^\tau \right] \cdot \nabla_0^2 (\zeta^2 \nabla_0^2 v_\kappa) \, dS \, ds + C' \int_0^t \left[ \| \tilde{v} \|_{H^2(\Gamma)}^2 + \| \tilde{h} \|_{H^2(\Gamma)}^2 + \| \tilde{h} \|_{H^2(\Gamma)}^2 \right] \| \nabla_0^4 h_\kappa \|_{L^2(\Gamma)}^2 \, ds + C' N_2(u_0, F) + C' \int_0^t \left[ \| \nabla_0^2 v_\kappa \|_{H^1(\Omega_1)}^2 + K(s) \| \nabla_0^4 h_\kappa \|_{L^2(\Gamma)}^2 \right] \, ds,
\]

for some constant $C'$ depending on $M$, $\epsilon$, $\delta$ and $\delta_1$. By (9.2),
\[
\frac{\nu_1}{2} \| \nabla_0^4 h_\kappa(t) \|_{L^2(\Gamma)}^2 \leq \int_0^t \int_\Gamma \Theta \left[ [L \tilde{h}(h_\kappa)(-\nabla_0 h, 1) \circ \tilde{\eta}^\tau] \cdot \nabla_0^2 (\zeta^2 \nabla_0^2 v_\kappa) \right] \, dS \, ds + C' N_2(u_0, F) + C' \int_0^t \left[ \| \nabla_0^2 v_\kappa \|_{H^1(\Omega_1)}^2 + K(s) \| \nabla_0^4 h_\kappa \|_{L^2(\Gamma)}^2 \right] \, ds + \delta \int_0^t \| v_{\kappa t} \|_{H^1(\Omega)}^2 \, ds + \delta_1 \int_0^t \| v_{\kappa t} \|_{H^2(\Omega)}^2 \, ds.
\]

(9.3)

where
\[ K(s) := 1 + \| \tilde{v} \|_{H^2(\Gamma)}^2 + \| \tilde{h} \|_{H^2(\Gamma)}^2 + \| \tilde{h} \|_{H^2(\Gamma)}^2. \]

With (9.3), (8.11) now is replaced by
\[
\left[ \| \nabla_0^2 v_\kappa(t) \|_{L^2(\Gamma)}^2 \right] + \left[ \| \nabla_0^3 h_\kappa(t) \|_{L^2(\Gamma)}^2 \right] + \left[ \| \nabla_0^2 \nabla_0^2 v_\kappa \|_{L^2(\Gamma)}^2 \right] \, ds \leq C' N_2(u_0, F) + C' \int_0^t \left[ \| v_{\kappa t} \|_{H^1(\Omega_1)}^2 + \| \nabla_0^3 v_\kappa \|_{L^2(\Gamma)}^2 \right] \, ds + \delta \int_0^t \| v_{\kappa t} \|_{H^1(\Omega)}^2 \, ds + \delta_1 \int_0^t \| v_{\kappa t} \|_{H^2(\Omega)}^2 \, ds.
\]

(9.4)

for some $C'$ depending on $M$, $\epsilon$, $\delta$ and $\delta_1$, where (A.5) is applied to bound $\kappa \| v_\kappa \|_{H^2(\Gamma)}^2$ (this is where $\| v_{\kappa t} \|_{L^2(\Gamma)}^2$ comes from). Similar computations leads to
\[
\left[ \| \nabla_0^2 v_\kappa(t) \|_{L^2(\Gamma)}^2 \right] + \left[ \| \nabla_0^3 h_\kappa(t) \|_{L^2(\Gamma)}^2 \right] + \left[ \| \nabla_0^2 \nabla_0^2 v_\kappa \|_{L^2(\Gamma)}^2 \right] \, ds \leq C N_2(u_0, F) + C \int_0^t \| \nabla_0^3 h_\kappa \|_{L^2(\Gamma)}^2 \, ds + \delta \int_0^t \| v_{\kappa t} \|_{H^1(\Omega)}^2 \, ds.
\]

(9.5)

for some constant $C$ depending on $M$ and $\delta$. 
In Appendix C we establish the following $\kappa$- and $\epsilon$-independent inequality for the time-differentiated problem:

$$
\int_0^t \left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|q_\kappa\|_{H^2(\Omega)}^2 \right] ds \leq \int_0^t \int_{\Gamma} \left[ [L_h(\kappa)(\nabla \varphi - 1) \circ \eta^\gamma]_t \cdot v_\kappa \right] dS \\
+ C N_3(u_0, F) + C \int_0^t K(s) \left[ \|\nabla_0^2 v_\kappa\|_{L^2(\Gamma)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right] ds \\
+ (\delta + C T^{1/2}) \int_0^t \|v_\kappa\|_{H^3(\Omega)}^2 ds + (\delta_1 + C T^{1/2}) \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds + \delta_2 \|\nabla_0^4 v_\kappa\|_{L^2(\Gamma)}^2
$$

for some constant $C$ depending on $M$, $\delta$, $\delta_1$ and $\delta_2$. Therefore, (8.14) can be replaced by the following estimate:

$$
\left[ \|v_\kappa\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right] + \int_0^t \left[ \|\nabla v_\kappa\|_{L^2(\Omega)}^2 + \kappa^2 \Delta_0 v_\kappa\|_{L^2(\Gamma)}^2 \right] ds \\
\leq C N_3(u_0, F) + C \int_0^t K(s) \left[ \|\nabla_0^2 v_\kappa\|_{L^2(\Gamma)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right] ds \\
+ (\delta + C T^{1/2}) \int_0^t \|v_\kappa\|_{H^3(\Omega)}^2 ds + (\delta_1 + C T^{1/2}) \int_0^t \|v_{\kappa t}\|_{H^1(\Omega)}^2 ds + \delta_2 \|\nabla_0^4 v_\kappa\|_{L^2(\Gamma)}^2
$$

9.2. $\kappa$-independent estimates. Just as in Section 8.7 we find that

$$
\int_0^t \left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|q_\kappa\|_{H^2(\Omega)}^2 \right] ds \\
\leq C(M) N_2(u_0, F) + C(M) \int_0^t \left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{H^1(\Omega)}^2 \right] ds.
$$

By choosing $\delta = \delta_1 = \delta_2 = 1/8$ and $T > 0$ so that $CT^{1/2} < 1/8$ in (9.6), we find that

$$
\int_0^t \left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|q_\kappa\|_{H^2(\Omega)}^2 \right] ds \leq C N_3(u_0, F) + \frac{1}{8} \|\nabla_0^4 v_\kappa\|_{L^2(\Gamma)}^2 \\
+ C(M) \int_0^t \left[ \|\nabla_0^2 v_\kappa\|_{H^1(\Omega)}^2 + K(s) \left( \|\nabla_0^4 v_{\kappa t}\|_{L^2(\Gamma)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right) \right] ds.
$$

Combining the estimates (7.27), (8.9), (9.4) and (9.5) with (9.6),

$$
\left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 + \|v_{\kappa t}\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right] (t) \\
+ \int_0^t \left[ \|\nabla v_{\kappa t}\|_{L^2(\Omega)}^2 + \|\nabla_0 v_{\kappa t}\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Omega)}^2 + \|v_{\kappa t}\|_{H^1(\Omega)}^2 \right] ds \\
\leq C' N_3(u_0, F) + C' \int_0^t \left[ \|v_{\kappa t}\|_{L^2(\Omega)}^2 + K(s) \left( \|\nabla_0^4 v_{\kappa t}\|_{L^2(\Gamma)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 \right) \right] ds
$$

for some constant $C'$ depending on $M$ and $\epsilon$. By the Gronwall inequality and (8.4),

$$
\sup_{0 \leq t \leq T} \left[ \|v_\kappa\|_{H^1(\Omega)}^2 + \|v_{\kappa t}\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v_{\kappa t}\|_{L^2(\Gamma)}^2 + \|\nabla_0^4 v_\kappa\|_{L^2(\Gamma)}^2 \right] (t) + \|v_{\kappa t}\|_{H^3(\Omega)}^2 + \|q_{\kappa t}\|_{L^2(\Omega;H^2(\Omega))}^2 \leq C(\epsilon) N_3(u_0, F).
$$
9.3. Weak limits as $\kappa \to 0$. Just as in Section 8.8, the weak limit $(v_\epsilon, h_\epsilon, q_\epsilon)$ of $(v_\epsilon, h_\epsilon, q_\epsilon)$ as $\kappa \to 0$ exists in $V(T) \times L^2(0,T;H^2(\Gamma)) \times L^2(0,T;H^2(\Omega))$ with estimate
\begin{align}
\sup_{0 \leq t \leq T} \left[ ||v_\epsilon(t)||^2_{H^2(\Omega)} + ||v_\epsilon(t)||^2_{L^2(\Omega)} + ||\nabla^2 h_\epsilon(t)||^2_{L^2(\Gamma)} + ||\nabla^2 h_\epsilon(t)||^2_{L^2(\Gamma)} 
\right. \\
+ ||q_\epsilon(t)||^2_{H^1(\Gamma)} + ||q_\epsilon||^2_{L^2(0,T;H^2(\Omega))} \leq C(\epsilon)N_3(u_0, F). \quad (9.9)
\end{align}

(9.9) implies that for a.a. $t \in [0,T]$, 
\[ ||v_\epsilon(t)||_{H^2,\gamma(\Gamma)} \leq \tilde{C}(t) \]
for some $\tilde{C}(t)$ independent of $\kappa$, and therefore for a.a. $t \in [0,T]$, 
\[ \kappa \int_{\Gamma} \Delta_0 v_\epsilon \cdot \Delta_0 \varphi dS \to 0 \]
as $\kappa \to 0$. This observation with (8.20) shows that $(v_\epsilon, h_\epsilon, q_\epsilon)$ satisfies, for a.a. $t \in [0,T]$, 
\begin{align}
(v_\epsilon, \varphi)_{L^2(\Omega)} + \frac{\nu}{2} \int_\Omega D\eta v_\epsilon : D\eta(\varphi) dx + \sigma \int_{\Gamma} \Theta \mathcal{L}_h(h_\epsilon)[-\bar{h}_\sigma \circ \bar{\eta} \varphi^\sigma + \varphi^\sigma] dS \\
- (\bar{a}_i^j q_\epsilon, \varphi^j)_{L^2(\Omega)} = (\bar{F}, \varphi) + \sigma (\bar{\Theta} M_\bar{h}(-\nabla_0 \bar{h} \circ \bar{\eta}^\tau, -1), \varphi)_\Gamma. \quad (9.10)
\end{align}

for all $\varphi \in H^{1,2}(\Omega; \Gamma)$. Since (9.10) also defines a linear functional on $H^1(\Omega)$, by the density argument, we have that (9.10) holds for all $\varphi \in H^1(\Omega)$. As $(v_\epsilon, h_\epsilon, q_\epsilon)$ are smooth enough, we can integrate by parts and find that $(v_\epsilon, h_\epsilon, q_\epsilon)$ satisfies (7.2) with (7.2) replaced by 
\[ [\nu D\bar{h}(v_\epsilon)_i^j - q_\epsilon \delta_i^j] \bar{a}_j^j N_t = \sigma [\bar{\Theta}[(\mathcal{L}_h(h_\epsilon) + \mathcal{M}(\bar{h}))(\nabla_0 \bar{h}, -1)] \circ \bar{\eta}^\tau] \]
on $(0,T) \times \Gamma$. \quad (9.11)

9.4. $H^{5.5}$-regularity of $h_\epsilon$. By (9.11), we have

**Lemma 9.1.** For a.a. $t \in [0,T]$, $h_\epsilon(t) \in H^{5.5}(\Gamma)$ with 
\[ ||h_\epsilon||_{H^{5.5}(\Gamma)}^2 \leq C(M) \left[ ||v_\epsilon||^2_{H^1(\Omega)} + ||\nabla v_\epsilon||^2_{L^2(\Omega)} + ||\nabla^2 v_\epsilon||^2_{L^2(\Omega)} + ||\nabla^2 h_\epsilon||^2_{L^2(\Gamma)} + ||F||^2_{L^2(\Omega)} + 1 \right], \quad (9.12) \]
and hence 
\[ ||h_\epsilon||^2_{H^{5.5}(\Gamma)} \leq C(M) e^{C(M)+T} N_3(u_0, F). \quad (9.13) \]

**Proof.** We write the boundary condition (9.11) as 
\[ \mathcal{L}_h(h_\epsilon) = \frac{1}{\sigma} J_h^{-2}(-\nabla_0 \bar{h}, 1) \cdot \left( \Theta^{-1} \left[ [\nu D\bar{h}(v_\epsilon)_i^j - q_\epsilon \delta_i^j] \bar{a}_j^j N_t \right] \right) \circ \bar{\eta}^\tau - \mathcal{M}(\bar{h}). \quad (9.14) \]

By Corollary (4.2), $\mathcal{L}_h$ is uniformly elliptic with the elliptic constant $\nu_1$ which is independent of $M$ which defines our convex subset $C_T(M)$. Since $h \in H(\Gamma)$, $\mathcal{M}(\bar{h}) \in L^2(0,T;H^{2.5}(\Gamma)) \cap L^\infty(0,T;H^1(\Gamma))$, and hence by (8.19), the right-hand side of (9.14) is bounded in $H^{1.5}(\Gamma)$. The important point is that these bounds are independent of $\epsilon$. Thus, elliptic regularity of $\mathcal{L}_h$ proves the estimate 
\[ ||h_\epsilon||_{H^{5.5}(\Gamma)} \leq C(M) \left[ ||\nabla h_\epsilon||^2_{H^{1.5}(\Gamma)} + ||q_\epsilon||^2_{H^{1.5}(\Gamma)} + 1 \right], \]
so that with (8.4), (9.12) is proved. \qed
9.5. **Energy estimates which are independent of \( \epsilon \).** Having estimate (6.12), one can follow exactly the same procedure as in Section 9.2 to show that the constant \( C' \) appearing in (9.9) is independent of \( \epsilon \), provided that we have an \( \epsilon \)-independent version of (9.4). By Appendix B.2, we indeed have such an estimate:

\[
\frac{\nu}{2} \left\| \nabla \partial_t \right\|_{L^2(\Gamma)}^2 \leq \int_0^t \int_{\Gamma} \bar{\Theta} \left[ L_5(h_c)(-\nabla \partial_t, 1) \circ \eta^\tau \right] \cdot \nabla \delta (\zeta^2 \nabla^2 \psi_c) dSds
\]

\[
+ C N_2(u_0, F) + C \int_0^t K(s) \left\| \nabla \partial_t h_c \right\|_{L^2(\Gamma)}^2 ds + (\delta + Ct^{1/2}) \int_0^t \left\| \psi_c \right\|_{H^3(\Omega)}^2 ds
\]

\[
+ (\delta_1 + Ct^{1/2}) \int_0^t \left\| v_c \right\|_{H^1(\Omega)}^2 ds
\]

for some constant \( C \) depending on \( M, \delta \) and \( \delta_1 \). Therefore, we can conclude that

\[
\sup_{0 \leq t \leq T} \left[ \left\| v_c \right\|_{H^2(\Omega)}^2 + \left\| \psi_c \right\|_{L^2(\Omega)}^2 + \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 + \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 \right]
\]

\[
+ \left\| \phi_c \right\|_{H^1(\Omega)}^2 (t) + \left\| v_c \right\|_{L^2(\Omega)}^2 + \left\| \phi_c \right\|_{L^2(0, T; H^2(\Omega))} \leq C(M)e^{C(M) + T} N_3(u_0, F).
\]

**Remark 15.** Literally speaking, we cannot use \( \nabla \delta (\zeta^2 \nabla^2 \psi_c) \) as a test function in (9.10) since it is not a function in \( H^1(\Omega) \). However, since \( h_c \in H^{3, 5}(\Gamma) \) for a.a. \( t \in [0, T] \), (9.10) also holds for all \( \varphi \in H^1(\Omega)^{\prime} \cap H^{-1, 3}(\Gamma) \) and \( \nabla \delta (\zeta^2 \nabla^2 \psi_c) \) is a function of this kind.

9.6. **Weak limits as \( \epsilon \to 0 \).** The same argument leads to that weak limits of \( (v_c, h_c, q_c) \) (denoted by \( (v, h, q) \)) as \( \epsilon \to 0 \) exists and \( (v, h, q) \) satisfies (7.1).

9.7. **Uniqueness.** In this section, we show that for a given \((\tilde{v}, \tilde{h}) \in Y_T\), the solution to (7.1) is unique in \( Y_T \). Suppose \((v_1, h_1)\) and \((v_2, h_2)\) are two solutions in \( Y_T \) to (7.3). Let \( w = v_1 - v_2 \) and \( g = h_1 - h_2 \), then \( w \) and \( g \) satisfy

\[
\langle w, \varphi \rangle + \frac{\nu}{2} \int_{\Omega} D_\eta w : D_\eta \varphi dx + \sigma \int_{\Gamma} \bar{\Theta} \left[ \tilde{L}_5 \left( \int_0^t \tilde{h}_c \circ w_{\alpha} - w_{\tau} ds \right) \right] \circ \eta^\tau \times
\]

\[
\times (-\tilde{h}_c \circ \tilde{\tau} \varphi^\alpha + \varphi^\tau) dS = 0
\]

for all \( \varphi \in \mathcal{V}_c(T) \) with \( w(0) = 0 \), where \( \tilde{L} \) equals \( L \) except \( L_1 = L_2 = 0 \). Since \( w \) is in \( \mathcal{V}_c(T) \), letting \( w = \varphi \) in (9.10) leads to

\[
\left[ \left\| v_c \right\|_{H^2(\Omega)}^2 + \left\| \nabla \delta \left( \zeta^2 \nabla^2 \psi \right) \right\|_{L^2(\Omega)}^2 + \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 \right](t)
\]

\[
+ \int_0^t \left[ \left\| \nabla v_c \right\|_{L^2(\Omega)}^2 + \left\| \nabla \partial_t v_c \right\|_{L^2(\Omega)}^2 + \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 + \left\| v_c \right\|_{H^3(\Omega)}^2 \right] ds
\]

\[
\leq C(M) \int_0^t K(s) \left[ \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 + \left\| \nabla^2 h_c \right\|_{L^2(\Gamma)}^2 \right] ds.
\]

Therefore, by the Gronwall inequality and the zero initial condition \((w(0) = 0)\), we have that \( w \) (and hence \( g \)) is identical to zero.

10. **Fixed-Point argument**

From previous sections, we establish a map \( \Theta_T \) from \( Y_T \) into \( Y_T \), i.e., given \((\tilde{v}, \tilde{h}) \in C_T(M)\), there exists a unique \( \Theta_T(\tilde{v}, \tilde{h}) = (v, h) \) satisfying (7.1). Theorem (11.1) is then proved if this mapping \( \Theta_T \) has a fixed point. We shall make use of the Tychonoff Fixed-Point Theorem which states as follows:
Theorem 10.1. For a reflexive Banach space \( X \), and \( C \subset X \) a closed, convex, bounded subset, if \( F : C \to C \) is weakly sequentially continuous into \( X \), then \( F \) has at least one fixed-point.

In order to apply the Tychonoff Fixed-Point Theorem, we need to show that \( \Theta(\tilde{v}, \tilde{h}) \in C_T(M) \) and this is the case if \( T \) is small enough. In the following discussion, we will always assume \( T \) is smaller than a fixed constant (for example, say \( T \leq 1 \)) so that the right-hand side of (\ref{eq:0.13}) can be written as \( C(M)N_3(u_0, F) \).

Remark 16. The space \( Y_T \) is not reflexive. We will treat \( C_T(M) \) as a convex subset of \( X_T \) and applied the Tychonoff Fixed-Point Theorem on the space \( X_T \).

Before proceeding the fixed-point proof, we note that lemma \ref{lem:5.3} implies that for a short time, the constant \( C(M) \) in (\ref{eq:8.1}) and (\ref{eq:8.4}) can be chosen to be independent of \( M \). To be more precise, for almost all \( 0 < t \leq T_1 \),

\[
\| q \|^2_{L^2(\Omega)} \leq C \left[ \| v \|^2_{L^2(\Omega)} + \| \nabla v \|^2_{L^2(\Omega)} + \| \nabla^2 h \|^2_{L^2(\Gamma)} + \| F \|^2_{L^2(\Omega)} + 1 \right], \tag{10.1}
\]

and

\[
\| v \|^2_{H^1(\Omega)} + \| q \|^2_{H^2(\Omega)} \leq C \left[ \| v \|^2_{H^1(\Omega)} + \| \nabla v \|^2_{H^1(\Omega)} + \| \nabla^2 v \|^2_{H^1(\Omega_1)} + \| \nabla v \|^2_{H^1(\Omega_1)} + \| \nabla^2 v \|^2_{H^1(\Omega_1)} + \| \nabla^2 h \|^2_{L^2(\Gamma)} + \| F \|^2_{H^1(\Omega)} + 1 \right], \tag{10.2}
\]

and

\[
\| h \|^2_{H^{2.5}(\Gamma)} \leq C \left[ \| v \|^2_{H^1(\Omega)} + \| \nabla v \|^2_{L^2(\Omega)} + \| \nabla^2 v \|^2_{H^1(\Omega_1)} + \| \nabla^2 h \|^2_{L^2(\Gamma)} + \| F \|^2_{H^1(\Omega)} + 1 \right] \tag{10.3}
\]

for some constant \( C \) independent of \( M \).

10.1. Continuity in time of \( h \). By the evolution equation (\ref{eq:7.1d}) and the fact that \( v \in V^3(T_1) \), \( h_t \in L^2(0, T_1; H^{2.5}(\Gamma)) \). Since \( h \in L^2(0, T_1; H^{2.5}(\Gamma)) \), we have that \( h \in C^0([0, T_1]; H^4(\Gamma)) \) by standard interpolation theorem. Although there is no uniform rate that \( h \) converges to zero in \( H^4(\Gamma) \), we have the following.

Lemma 10.1. Let \((v, h) = \Theta_T(\tilde{v}, \tilde{h})\). Then \( \| h(t) \|_{H^{2.5}(\Gamma)} \) converges to zero as \( t \to 0 \), uniformly for all \((\tilde{v}, \tilde{h}) \in C_T(M)\).

Proof. By the evolution equation (\ref{eq:7.1d}),

\[
\| h(t) \|^2_{H^{2.5}(\Gamma)} \leq \int_0^t \| h_{\alpha\alpha} v_\alpha - v_z \|^2_{H^{2.5}(\Gamma)} dS \leq C(M)N_3(u_0, F)^{1/2} t^{1/2}.
\]

The lemma follows directly from the inequality.

By lemma \ref{lem:10.1} and the interpolation inequality, we also have

Lemma 10.2. \( \| \nabla^2 h(t) \|_{H^{1.5}(\Gamma)} \) converges to zero as \( t \to 0 \), uniformly for all \( \tilde{h} \in C_T(M) \) with estimate

\[
\| \nabla^2 h(t) \|^2_{H^{1.5}(\Gamma)} \leq C(M)N_3(u_0, F)t^{1/4} \tag{10.4}
\]

for all \( 0 < t \leq T_1 \).
10.2. Improved energy estimates. In order to apply the fixed-point theorem, we have to use the fact that the forcing \( F \) is in \( \mathcal{V}^2(T) \). We also define a new constant

\[
N(u_0, F) := \|u_0\|^2_{H^2(\Omega)} + \|F\|^2_{\mathcal{V}^2(T)} + \|F\|^2_{L^\infty(0,T;L^2(\Omega))} + \|F(0)\|^2_{H^1(\Omega)} + 1.
\]

Note that \( N_3(u_0, F) \leq N(u_0, F) \).

Remark 17. For the linearized problem \((T, T)\), we only need \( F \in \mathcal{V}^1(T) \) to obtain a unique solution \((v, h) \in Y_T\).

10.2.1. Estimates for \( \nabla_h^2 v \) near the boundary. Note that

\[
\frac{1}{2} \frac{d}{dt} \left\| \zeta_1 \nabla_0^2 v \right\|^2_{L^2(\Omega)} + \sigma \int_{\Gamma} \Theta B \tilde{A}^{a\beta\gamma\delta} \nabla_0^2 h_{\alpha\delta} \nabla_0^2 h_{\gamma\delta} dS + \frac{\nu}{2} \left\| \zeta_1 \mathcal{D}_\theta(\nabla_0^2 v) \right\|^2_{L^2(\Omega)}
\]

\[
= \langle F, \nabla_0^2 (\zeta_1^2 \nabla_0^2 v) \rangle - \frac{\nu}{4} \int_{\Omega} \left[ \nabla_0^2 (\tilde{a}_i^k \tilde{a}_j^l) v_{i,j}^l + \nabla_0^2 (\tilde{a}_i^k \tilde{a}_j^l) v_{i,j}^l \right] (\zeta_1^2 \nabla_0^2 v)^j , k dx
\]

\[
- \frac{\nu}{2} \int_{\Omega} \left[ \nabla_0 (\tilde{a}_i^k \tilde{a}_j^l) \nabla_0 v_{i,j}^l + \nabla_0 (\tilde{a}_i^k \tilde{a}_j^l) \nabla_0 v_{i,j}^l \right] (\zeta_1^2 \nabla_0^2 v)^j , k dx
\]

\[
- \frac{\nu}{2} \int_{\Omega} D\mathcal{D}_\theta(\nabla_0^2 v)^j \tilde{a}_i^k \zeta_1^j \nabla_0^2 v^i dx + \int_{\Omega} q a_k (\nabla_0^2 v)^j (\zeta_1^2 \nabla_0^2 v)^j dx - \sigma \left( \sum_{k=1}^3 I_k + \sum_{k=1}^8 J_k \right)
\]

where \( I_k \)’s and \( J_k \)’s are defined in Appendix 31 (with \( - \) replaced by \( \tilde{\ } \), and no \( \epsilon \) and \( \epsilon_1 \).

As in [10] and [11], we study the time integral of the right-hand side of the identity above in order to prove the validity of the requirement of applying Tychonoff Fixed-Point Theorem. By interpolation and (9.9),

\[
\int_0^t \int_{\Omega} \left[ \nabla_0^2 (\tilde{a}_i^k \tilde{a}_j^l) v_{i,j}^l + \nabla_0^2 (\tilde{a}_i^k \tilde{a}_j^l) v_{i,j}^l \right] (\zeta_1^2 \nabla_0^2 v)^j , k dx ds
\]

\[
\leq C \int_0^t \| \tilde{a} \|_{H^2(\Omega)} \| \nabla v \|_{L^\infty(\Omega)} \| v \|_{H^3(\Omega)} ds
\]

\[
\leq C(M) C(\delta) \int_0^t \| v \|^2_{H^1/2(\Omega)} \| v \|^2_{H^3(\Omega)} ds + \delta \| v \|^2_{L^2(0,T;H^3(\Omega))}
\]

\[
\leq C(M) C(\delta) N(u_0, F)^{1/2} \int_0^t \| v \|^2_{H^1/2(\Omega)} ds + \delta C(M) N(u_0, F)
\]

\[
\leq C(M) N(u_0, F) \left[ C(\delta) t^{3/4} + \delta \right].
\]

Similarly,

\[
\int_0^t \int_{\Omega} \left[ \nabla_0 (\tilde{a}_i^k \tilde{a}_j^l) \nabla_0 v_{i,j}^l + \nabla_0 (\tilde{a}_i^k \tilde{a}_j^l) \nabla_0 v_{i,j}^l \right] (\zeta_1^2 \nabla_0^2 v)^j , k dx ds
\]

\[
+ \int_0^t \int_{\Omega} D\mathcal{D}_\theta(\nabla_0^2 v)^j \tilde{a}_i^k \zeta_1^j \nabla_0^2 v^i dx ds \leq C(M) N(u_0, F) \left[ t^{1/2} + C(\delta) t + \delta \right].
\]
For the pressure term, by interpolation and (8.10),
\[
\int_0^t \int_\Omega q \Delta_k [\nabla_0^2 (\zeta^2 \nabla_0^2 v^k)]_e dx ds \\
\leq C(M) \int_0^t \|q\|_{L^\infty(\Omega)} + \|q\|_{W^{1,4}(\Omega)} + \|q\|_{H^1(\Omega)} \|v\|_{H^8(\Omega)} ds \\
\leq C(M) C(\delta) \int_0^t \|q\|_{H^2(\Omega)}^2 ds + \delta \left[ \|v\|_{L^2(0,T;H^3(\Omega))}^2 + \|q\|_{L^2(0,T;H^2(\Omega))}^2 \right] \\
\leq C(M) N(u_0,F) \left[ C(\delta) t^{1/2} + \delta \right].
\]
By the estimates already established in Appendix B with the help of (6.6), it is also easy to see that
\[
\int_0^t \left( \sum_{k=1}^3 I_k + \sum_{k=1}^8 J_k \right) ds \leq C(M) N(u_0,F) \left[ t^{1/4} + t^{1/2} + C(\delta) t^{2/3} + \delta \right].
\]
Finally, for the forcing term, by the extra regularity we assume on $F$,
\[
\int_0^t \langle F, \nabla_0^2 (\zeta^2 \nabla_0^2 v) \rangle ds \leq \int_0^t \|F\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)} ds \leq N(u_0,F) + \int_0^t \|v\|_{H^2(\Omega)}^2 ds \\
\leq N(u_0,F) + C(M) N(u_0,F) t.
\]
Therefore,
\[
\left[ \|\nabla_0^2 v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 h(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^t \|\nabla_0^2 v\|_{H^1(\Omega)}^2 ds \\
\leq \|u_0\|_{H^2(\Omega)}^2 + CN(u_0,F) + C(M) N(u_0,F) \left[ C(\delta) (t^{3/4} + t^{2/3} + t^{1/2} + t) + \delta \right].
\]
By Corollary 7.1
\[
\left[ \|\nabla_0^2 v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 h(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^t \|\nabla_0^2 v\|_{H^1(\Omega)}^2 ds \\
\leq CN(u_0,F) + C(M) N(u_0,F) \left[ C(\delta) \mathcal{O}(t) + \delta \right] \quad \text{as} \quad t \to 0 \quad (10.5)
\]
where $C$ depends on $\nu$, $\sigma$, $\nu_1$ and the geometry of $\Gamma$.

By similar computations, we can also conclude (the (7.27), (8.9) and (9.5) variants) that
\[
\left[ \|v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 h(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^t \|v\|_{H^1(\Omega)}^2 ds \\
\leq CN(u_0,F) + C(M) N(u_0,F) \mathcal{O}(t) \quad \text{as} \quad t \to 0 \quad ; \quad (10.6)
\]
\[
\left[ \|\nabla_0 v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0 ^3 h(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^t \|\nabla_0 v\|_{H^1(\Omega)}^2 ds \\
\leq CN(u_0,F) + C(M) N(u_0,F) \mathcal{O}(t) \quad \text{as} \quad t \to 0 \quad ; \quad (10.7)
\]
\[
\|\nabla_0 v(t)\|_{L^2(\Omega)}^2 + \int_0^t \|v_1\|_{L^2(\Omega)}^2 ds \\
\leq CN(u_0,F) + C(M) N(u_0,F) \mathcal{O}(t) \quad \text{as} \quad t \to 0 \quad (10.8)
\]
where $C$ depends on $\nu$, $\sigma$, $\nu_1$ and the geometry of $\Gamma$. 
10.2.2. $L^2_t H^1_x$-estimate for $v_t$. For the time-differentiated problem, we are not able to use estimates such as those in sections 8.6 and 10.2.1 since no $\epsilon$-regularization is present; nevertheless, we can obtain estimates at the $\epsilon$-regularization level and then pass $\epsilon$ to the limit once the estimate is found to be $\epsilon$-independent. We have that

$$
\frac{1}{2} \frac{d}{dt} \|v_t\|_{L^2_x(\Omega)}^2 + \frac{\nu}{2} \|D_\eta v_t\|_{L^2_x(\Omega)}^2 + \frac{\sigma^2}{2} \frac{d}{dt} \int_\Gamma \bar{\Theta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{t,\gamma\delta} dS
$$

$$
= (F_1, v_t) - \nu \int_\Omega \left[ \left( \bar{a}_k^\ell a_j^\ell \right) v_i^j + \left( \bar{a}_k^\ell a_j^\ell \right) v_i^j \right] v_i^j dx + \int_\Omega q_k^\ell v_i^j dx
$$

$$
+ \frac{1}{2} \int_\Gamma \left( \bar{\Theta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{t,\gamma\delta} dS - \int_\Gamma \frac{\Theta}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} (A^{\alpha\beta\gamma\delta})_{h_{t,\alpha\beta}} \right]_{\gamma\delta} h_{tt} dS
$$

$$
- 2 \int_\Gamma \Theta_{\gamma\delta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt} dS - \int_\Gamma \left( L^2_{\alpha\beta\gamma\delta} \bar{h}_{\alpha\beta} \right)_{tt} dS - \int_\Gamma (L_2)_{tt} h_{tt} dS + K_1 + K_5 + K_4 + K_5 + K_6
$$

where $K_i$'s are defined in Appendix C (without $\epsilon_1$).

As in the previous section, the time integral of the right-hand side of the identity above is studied. It is easy to see that

$$
\int_0^t \left[ (F_1, v_t) - \nu \left( \bar{a}_k^\ell a_j^\ell \right) v_i^j + \left( \bar{a}_k^\ell a_j^\ell \right) v_i^j \right] v_i^j dx + \int_\Omega q_k^\ell v_i^j dx ds
$$

$$
\leq C(M) N(u_0, F) \left[ t^{1/4} + t^{1/2} + C(\delta) (t^{1/2} + t) + \delta \right]
$$

and by Appendix C particularly Remark 21

$$
\int_0^t \int_\Omega \left[ \frac{1}{2} \left( \bar{\Theta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{t,\gamma\delta} - \frac{\Theta}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} (A^{\alpha\beta\gamma\delta})_{h_{t,\alpha\beta}} \right]_{\gamma\delta} h_{tt} dS dS
$$

$$
- 2 \Theta_{\gamma\delta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt} dS - \Theta_{\gamma\delta} A^{\alpha\beta\gamma\delta} h_{t,\alpha\beta} h_{tt} dS + K_1 + K_5 + K_4 + K_5 + K_6
$$

$$
\leq C(M) N(u_0, F) t^{1/2}.
$$

Special treatment needed to be done for the rest terms, and we break this procedure into several steps.

**Step 1.** Let $B_1 = \int_0^t \int_\Omega (q a_k^\ell) v_i^j dx ds$. By the “divergence free” condition 7.2b,

$$
B_1 = \int_0^t \int_\Omega a_k^\ell q v_i^j dx ds - \int_0^t \int_\Omega a_k^\ell q v_i^j dx ds.
$$

By interpolation and \[8.1\],

$$
\left| \int_0^t \int_\Omega a_k^\ell q v_i^j dx ds \right|
$$

$$
\leq C(M) C(\delta) \int_0^t \|q\|_{L^2_x(\Omega)} dx ds + \delta \left[ \|q\|_{L^2_x(0,T;H^1_x(\Omega))} + \|v_i\|_{L^2_x(0,T;H^1_x(\Omega))} \right]
$$

$$
\leq C(M) N(u_0, F) C(\delta) t + \delta.
$$
For the second integral, we have the following identity:
\[
\int_0^t \int_\Omega \bar{a}^t_{kt} q v^k_t \, dx \, ds = \int_0^t (\bar{a}^t_{kt} q v^k_t)(t) \, dx - \int_0^t \bar{a}^t_{kt}(0) q(0) u^{0, t}_k \, dx \\
- \int_0^t \int_\Omega (\bar{a}^t_{kt} v^k_t) q \, dx \, ds.
\]
By the identity \( \bar{a}^t_{kt} = -\bar{a}^{t^2} j \),
\[
\left| \int_0^t \int_\Omega (\bar{a}^t_{kt} v^k_t) q \, dx \, ds \right| \leq \int_0^t \int_\Omega \left[ \bar{a}^{t^2} j + \bar{a}^t_{kt} v^k_t \right] q \, dx \, ds \\
\leq C(M) \int_0^t (1 + \| \bar{v}_t \|_{H^1(\Omega)}) \| \nabla v \|_{L^2(\Omega)} \| q \|_{L^2(\Omega)} \, ds.
\]
Therefore,
\[
\left| \int_0^t \int_\Omega (\bar{a}^t_{kt} v^k_t) q \, dx \, ds \right| \\
\leq C(M) C(\delta) N(u_0, F) \int_0^t \| q \|_{L^2(\Omega)}^{2\alpha} \| q \|_{L^2(\Omega)}^{2(1-\alpha)} \, ds \\
+ \delta \int_0^t (1 + \| \bar{v}_t \|_{H^1(\Omega)}) \| q \|_{L^2(\Omega)} \, ds.
\]
where \( \alpha = \frac{3}{4} \) if \( n = 3 \) and \( \alpha = \frac{1}{2} \) if \( n = 2 \).

The second integral equals \( \int_\Omega \nabla u_0 : (\nabla u_0)^T q(0) dx \), which is bounded by \( CN(u_0, F) \).

It remains to estimate the first integral. By adding and subtracting \( \int_\Omega \bar{a}^t_{kt}(0) q v^k_t \, dx \), we find, by \( \bar{a}_t(0) \in H^2(\Omega) \), that
\[
\left| \int_\Omega (\bar{a}^t_{kt} q v^k_t)(t) \, dx \right| \\
\leq C\| \bar{a}_t(t) - \bar{a}_t(0) \|_{L^2(\Omega)} \| q \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} \\
+ C(\delta) \| \nabla v \|_{L^2(\Omega)}^2 + \delta_1 \| q \|_{L^2(\Omega)}^2.
\]
Noting that
\[
\| \nabla v \|_{L^2(\Omega)}^2 = \| \nabla u_0 \|_{L^2(\Omega)} + \int_0^t \| \nabla v_t \|_{L^2(\Omega)} \, ds \leq \left[ \| \nabla u_0 \|_{L^2(\Omega)} + \int_0^t \| \nabla v_t \|_{L^2(\Omega)} \, ds \right]^2 \\
\leq 2 \left[ \| u_0 \|_{H^1(\Omega)}^2 + C(M) N(u_0, F) t \right],
\]
(10.9), (10.10) and (10.11) imply
\[
\left| \int_\Omega (\bar{a}^t_{kt} q v^k_t)(t) \, dx \right| \leq C(M) N(u_0, F) t^{1/2} + C(\delta) N(u_0, F) \\
+ \delta_1 \left[ \| v_t \|_{L^2(\Omega)}^2 + \| \nabla h \|_{L^2(\Gamma)}^2 \right].
\]
Summing all the estimates above, we find that
\[
|B_1| \leq C(\delta) N(u_0, F) + C(M) N(u_0, F)^2 \left[ C(\delta)(t + t^{1/2}) + \delta \right] \\
+ \delta_1 \left[ \| v_t \|_{L^2(\Omega)}^2 + \| \nabla h \|_{L^2(\Gamma)}^2 \right].
\]
Remark 18. It may be tempting to use an interpolation inequality to show that 
\( q \in C([0,T];X) \) for some Banach space \( X \) by analyzing \( q_t \) via Laplace’s equation. 
The problem, however, is that the boundary condition for \( L \) (by the fact that \( h_t \in L^2(0,T;H^{2.5}(\Gamma)) \)), and thus standard elliptic estimates do not provide the desired conclusion that \( q_t \in L^2(0,T;H^1(\Omega')) \) (and hence by interpolation, \( q \in C([0,T];H^{0.5}(\Omega)) \)). However, suppose that \( q_t \in L^2(0,T;H^1(\Omega')) \); then we can estimate \( \int_0^t \int_0^t a_{kt}q_t v_t dx ds \) by the following method:

\[
\left| \int_0^t \int_0^t a_{kt}q_t v_t dx ds \right| \leq \int_0^t \| a_k v_t \|_{H^1(\Omega)} \| q_t \|_{H^1(\Omega')} ds 
\leq C(M)N(u_0, F) \left[ t + t^{5/8} \right].
\]

Step 2. Let \( B_2 = \int_0^t \int_\Gamma \tilde{\Theta} \left[ L_1^{\alpha\beta\gamma} \tilde{h}_{\alpha\beta\gamma} \right]_t h_{tt} + (L_2)_t h_{tt} \] ds. It is easy to see that

\[
\left| \int_0^t \int_\Gamma \tilde{\Theta} (L_2)_t h_{tt} ds \right| \leq C(M) \int_0^t \left[ \| v \|_{L^\infty(\Gamma)} + \| v_t \|_{L^2(\Gamma)} \right] ds 
\leq C(M)N(u_0, F)^{1/2}(t + t^{3/4}).
\]

For parts involving \( L_1 \), we have

\[
\int_0^t \int_\Gamma \tilde{\Theta} \left[ L_1^{\alpha\beta\gamma} \tilde{h}_{\alpha\beta\gamma} \right]_t h_{tt} ds = \int_0^t \int_\Gamma \tilde{\Theta} \left[ L_1^{\alpha\beta\gamma} \right]_{tt} h_{tt} ds \quad (\equiv B_2^1)
\]

\[
+ \int_0^t \int_\Gamma \tilde{\Theta} L_1^{\alpha\beta\gamma} \tilde{h}_{tt\alpha\beta\gamma} h_{tt} ds. \quad (\equiv B_2^2)
\]

By interpolation,

\[
|B_2^1| \leq C(M) \int_0^t \| \tilde{\Theta} \|_{L^\infty(\Gamma)} \| \tilde{h} \|_{W^{1.5}(\Gamma)} \| h_{tt} \|_{L^4(\Gamma)} ds 
\leq C(M) \int_0^t \left[ \| v \|_{H^2(\Omega)} + \| v_t \|_{H^1(\Omega)} \right] ds 
\leq C(M)N(u_0, F)^{1/2} \sqrt{t}
\]

while by (6.6) and Corollary 6.1

\[
|B_2^2| \leq \int_0^t \| \tilde{\Theta} \|_{H^1(\Gamma)} \| \tilde{h} \|_{H^2.5(\Gamma)} \| L_1^{\alpha\beta\gamma} \|_{H^{1.5}(\Gamma)} \| h_{tt} \|_{H^{0.5}(\Gamma)} ds 
\leq C(M) \| L_1^{\alpha\beta\gamma} \|_{H^{1.5}(\Gamma)} \int_0^t \| \tilde{h} \|_{H^2.5(\Gamma)} \left[ \| v \|_{H^2(\Omega)} + \| v_t \|_{H^1(\Omega)} \right] ds 
\leq C(M)N(u_0, F)t^{1/4}.
\]

Therefore,

\[
|B_2| \leq C(M)N(u_0, F)(t + t^{3/4} + t^{1/4}).
\]

Step 3. Let \( B_3 = \int_0^t K_3 ds = \int_0^t \int_\Gamma \tilde{\Theta} [L_1(\tilde{h})]_{tt} [(\tilde{v} \circ \tilde{\eta}^{-1}) \cdot (\nabla_0 h_t)] ds ds \). The \( L_1 \) and \( L_2 \) part of \( B_3 \) is bounded by

\[
C(M) \int_0^t \| \tilde{\Theta} \|_{H^1(\Gamma)} \| \tilde{v} \|_{H^1(\Gamma)} \| \tilde{h} \|_{H^2(\Gamma)} \| h_t \|_{H^2(\Omega)} ds
\]
and hence
\[
\left| \int_0^t \Theta \left[ \int \lambda_1 \partial_t \lambda_2 + L_2 \right] \left( \nabla_0 h_t \right) \right| dSds \leq C(M)N(u_0, F)t^{1/4}.
\]

By the integration by parts formula, the highest order part of \( B_3 \) can be expressed as
\[
\int_0^t \int_{\Gamma} \frac{\partial \Theta}{\partial n} \left( \bar{v} \circ \bar{n}^{-\tau} \right) \left( \nabla_0 h_t \right) \cdot \left( \nabla_0 h_t \right) dSds (\equiv B_3^1)
\]
\[
+ \int_0^t \int_{\Gamma} \frac{\partial \Theta}{\partial n} \left( \bar{v} \circ \bar{n}^{-\tau} \right) A_{\alpha \beta \gamma \delta} h_t,_{\alpha \beta} \left( \nabla_0 h_t \right),_{\gamma \delta} dSds (\equiv B_3^2)
\]
\[
+ 2 \int_0^t \int_{\Gamma} \frac{\partial \Theta}{\partial n} \left( \bar{v} \circ \bar{n}^{-\tau} \right) A_{\alpha \beta \gamma \delta} h_t,_{\alpha \beta} \left( \nabla_0 h_t \right),_{\gamma \delta} dSds (\equiv B_3^3)
\]
\[
+ \int_0^t \int_{\Gamma} \left[ \Theta \left( \bar{v} \circ \bar{n}^{-\tau} \right) \right] A_{\alpha \beta \gamma \delta} h_t,_{\alpha \beta} \left( \nabla_0 h_t \right),_{\gamma \delta} dSds. (\equiv B_3^4)
\]

It is easy to see that
\[
|B_3^1| \leq C(M) \int_0^t \left\| \frac{\partial \Theta}{\partial n} \left( \bar{v} \circ \bar{n}^{-\tau} \right) \right\|_{H^{1.5}(\Gamma)} \left\| \nabla_0 h_t \right\|_{H^2(\Gamma)} \left\| h_t \right\|_{H^4(\Gamma)} dS
\leq C(M)N(u_0, F)t
\]
and
\[
|B_3^2| \leq C(M) \int_0^t \left\| \frac{\partial \Theta}{\partial n} \left( \bar{v} \circ \bar{n}^{-\tau} \right) \right\|_{W^{1.4}(\Gamma)} \left\| \nabla_0 h_t \right\|_{H^2(\Gamma)} \left\| h_t \right\|_{W^{2.4}(\Gamma)} dS
\leq C(M)N(u_0, F)t^{1/2}.
\]

For \( B_3^3 \), by the integration by parts formula,
\[
B_3^3 = \int_0^t \int_{\Gamma} \frac{1}{\sqrt{\det(g_0)}} \left( \nabla_0 \Theta \right) \left( \bar{v} \circ \bar{n}^{-\tau} \right) A_{\alpha \beta \gamma \delta} h_t,_{\alpha \beta} h_t,_{\gamma \delta} dSds
\]
\[
= -\int_0^t \int_{\Gamma} \frac{1}{\sqrt{\det(g_0)}} \left( \nabla_0 \Theta \right) \left( \bar{v} \circ \bar{n}^{-\tau} \right) A_{\alpha \beta \gamma \delta} h_t,_{\alpha \beta} h_t,_{\gamma \delta} dSds
\]
and hence
\[
|B_3^3| \leq \int_0^t \left[ \left\| \nabla_0 \Theta \right\|_{L^4(\Gamma)} \left\| \bar{v} \nabla_0 h_t \right\|_{L^\infty(\Gamma)} + \left\| \Theta \right\|_{L^\infty(\Gamma)} \left\| \bar{v} \nabla_0 h_t \right\|_{W^{1.4}(\Gamma)} \right] \times \left\| h_t \right\|_{W^{2.4}(\Gamma)} dS
\leq C(M)N(u_0, F)^{1/2} \int_0^t \left\| v \right\|_{H^3(\Omega)} ds
\leq C(M)N(u_0, F)t^{1/2}.
\]

For \( B_3^4 \), noting that
\[
\Theta_{\gamma \delta} = \det(\nabla_0 \bar{n}^{-\tau})_{\gamma \delta} \sqrt{\det(G_h) \circ \bar{n}^{-\tau}} + \det(\nabla_0 \bar{n}^{-\tau})_{\gamma} \sqrt{\det(G_h) \circ \bar{n}^{-\tau}}_{\gamma \delta}
\]
\[
+ \det(\nabla_0 \bar{n}^{-\tau})_{\delta} \sqrt{\det(G_h) \circ \bar{n}^{-\tau}}_{\gamma \delta} + \det(\nabla_0 \bar{n}^{-\tau}) \sqrt{\det(G_h) \circ \bar{n}^{-\tau}}_{\gamma \delta}
\]

and $\|\nabla_0 \det(\nabla_0 \tilde{\eta})\|_{H^0.5(\Gamma)} \leq C(M) t^{1/2}$, we find that

$$|B_3| \leq C(M) \int_0^t \| \nabla_0 \det(\nabla_0 \tilde{\eta}) \|_{H^0.5(\Gamma)} \| \nabla_0 h_t \|_{H^0.5(\Gamma)} \| \nabla_0 h_t \|_{H^1.5(\Gamma)} ds$$

$$+ C(M) \int_0^t \| \det(\nabla_0 \tilde{\eta}) \|_{L^\infty(\Gamma)} \| \nabla_0 \tilde{\eta} \|_{L^2(\Gamma)}^2 \| \nabla_0^2 h_t \|_{L^2(\Gamma)} \| \nabla_0 h_t \|_{L^2(\Gamma)} ds$$

$$\leq C(M) N(u_0, F) t^{1/2} + C(M) N(u_0, F)^{3/4} \int_0^t \| v \|_{H^{3/2}(\Omega)}^{1/2} ds$$

$$\leq C(M) N(u_0, F) (t^{1/2} + t^{3/4}).$$

Combining all the estimates, we find that

$$|B_3| \leq C(M) N(u_0, F)(t + t^{1/2} + t^{3/4}).$$

**Step 4.** Let $B_4 = \int_0^t K_4 ds = \int_0^t \int_{\Gamma} \tilde{\Theta} \left[ L_h(h) \right] t [\nabla_0 \tilde{\eta}, -1]_t \cdot (v \circ \tilde{\eta}^-) dS ds$. Integrating by parts,

$$B_4 = -\int_0^t \int_{\Gamma} L_h(h) \left[ \tilde{\Theta}(\nabla_0 \tilde{\eta}, -1)_t \cdot (v \circ \tilde{\eta}^-) + \tilde{\Theta}(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \tilde{\eta}^-)_t \right.$$

$$+ \tilde{\Theta}(\nabla_0 \tilde{h}, -1)_{tt} \cdot (v \circ \tilde{\eta}^-) \big] dS ds + \int_{\Gamma} \tilde{\Theta} L_h(h) [\nabla_0 \tilde{h}, -1]_t \cdot (v \circ \tilde{\eta}^-)] dS.$$

For the first integral, $\| \tilde{\Theta} \|_{L^\infty(\Gamma)} \| L_h(h) \|_{L^2(\Gamma)} \| \nabla_0 \tilde{h} \|_{L^2(\Gamma)} \| v \circ \tilde{\eta}^- \|_{L^4(\Gamma)}$

$\leq C(M) N(u_0, F) \| \tilde{h} \|_{H^{1.5}(\Gamma)}$

$\leq C(M) N(u_0, F) t^{1/2}.$

It is also easy to see that

$$\left| \int_0^t \int_{\Gamma} L_h(h) \left[ \tilde{\Theta}(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \tilde{\eta}^-) + \tilde{\Theta}(\nabla_0 \tilde{h}, -1)_t \cdot (v \circ \tilde{\eta}^-)_t \right] dS ds \right|$$

$\leq C(M) \int_0^t \left[ \| v \|_{L^\infty(\Gamma)} + \| v_t \|_{L^4(\Gamma)} \right] \| L_h(h) \|_{L^2(\Gamma)} \| \nabla_0 \tilde{h} \|_{L^2(\Gamma)} ds$

$\leq C(M) N(u_0, F)^{1/2} \int_0^t \left[ \| v \|_{H^3(\Omega)} + \| v_t \|_{H^1(\Omega)} \right] ds$

$\leq C(M) N(u_0, F)^{1/2}.$

For the remaining terms, $H^{0.5}(\Gamma)-H^{-0.5}(\Gamma)$ duality pairing leads to

$$\left| \int_0^t \int_{\Gamma} \tilde{\Theta} L_h(h)(\nabla_0 \tilde{h}, -1)_{tt} \cdot v dS ds \right|$$

$\leq \int_0^t \| \tilde{\Theta} \|_{H^{1.5}(\Gamma)} \| L_h(h) \|_{H^{0.5}(\Gamma)} \| v \|_{H^{1.5}(\Gamma)} \| \tilde{h} \|_{H^{0.5}(\Gamma)} ds.$

By interpolation,

$$\| L_h(h) \|_{H^{0.5}(\Gamma)} \leq C(M) \left[ \| h \|_{H^{1/2}(\Omega)}^{1/2} \| h \|_{H^{1/2}(\Gamma)}^{1/2} + 1 \right]$$
and hence
\[
\left| \int_0^t \int_\Gamma \tilde{\Theta}L(h)(\nabla_\alpha \tilde{h}, -1)_{tt} \cdot (v \circ \tilde{\eta}^{-1})dSds \right| \\
\leq C(M)N(u_0, F) \int_0^t \|\tilde{h}_{tt}\|_{H^{5/2}\Gamma} \left[ \|\nabla_0^3 h\|_{L^2(\Gamma)}^{1/2} + 1 \right] ds \\
\leq C(M)C(\delta)N(u_0, F) \int_0^t \left[ \|\nabla_0^3 h\|_{L^2(\Gamma)} + 1 \right] ds + \delta C(M)N(u_0, F) \\
\leq C(M)N(u_0, F) \left[ C(\delta)(t^{1/2} + t) + \delta \right].
\]

All the inequalities above give us
\[
|B_4| \leq C(M)N(u_0, F) \left[ C(\delta)(t^{1/2} + t) + t^{1/8} + \delta \right].
\]

Summing all the estimates above, we find that
\[
\left[ \|v_t\|_{L^2(\Omega)}^2 + \sigma \int_\Gamma \tilde{\Theta}A^{\alpha \beta }h_{tt,\alpha \beta}^{\gamma \delta}ds \right](t) + \nu \int_0^t \|D_\delta v_t\|_{L^2(\Omega)}^2 ds \\
\leq \|v_t(0)\|_{L^2(\Omega)}^2 + \sigma \int_\Gamma |G_0^{\alpha \beta}h_{tt,\alpha \beta}(0)|^2 ds + (C + C(\delta_1))N(u_0, F) \\
+ C(M)N(u_0, F) \left[ C(\delta)(t + t^{3/4} + t^{1/2} + t^{1/4} + t^{1/8} + t^{1/8} + t^{1/8}) + \delta \right] \\
+ \delta_1 \left[ \|v_t\|_{L^2(\Omega)}^2 + \|\nabla_0^4 h\|_{L^2(\Gamma)}^2 \right]
\]

and by Corollary 7.1
\[
\left[ \|v_t(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 h_t(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^t \|v_t\|_{H^1(\Omega)}^2 ds \\
\leq \left( C + C(\delta_1) \right)N(u_0, F) + C(M)N(u_0, F) \left[ C(\delta)\mathcal{O}(t) + \delta \right] + \delta_1 \left[ \|v_t\|_{L^2(\Omega)}^2 + \|\nabla_0^4 h\|_{L^2(\Gamma)}^2 \right] \quad (10.9)
\]

where \( C \) depends on \( \nu, \sigma, \nu_1 \) and the geometry of \( \Gamma \). Since this estimate is independent of \( \epsilon \), we pass \( \epsilon \) to zero and conclude that the solution \((v, h)\) to (7.1) also satisfies (10.9).

10.3. Mapping from \( C_T(M) \) into \( C_T(M) \). In this section, we are going to choose \( M \) so that \( \tilde{\Theta}(\tilde{v}, \tilde{h}) \in C_T(M) \) if \((\tilde{v}, \tilde{h}) \in C_T(M)\).

Summing (10.5), (10.6), (10.7), (10.8) and (10.9), by (6.3) we find that
\[
\left[ \|v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0 v(t)\|_{L^2(\Omega)}^2 + \|\nabla_0^2 v(t)\|_{L^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 \right] \\
+ \left[ \|\nabla_0^3 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla_0^3 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla_0^4 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla_0^3 h(t)\|_{L^2(\Gamma)}^2 \right] \\
+ \int_0^t \left[ \|v\|_{H^1(\Omega)}^2 + \|\nabla_0 v\|_{H^1(\Omega)}^2 + \|\nabla_0^2 v\|_{H^1(\Omega)}^2 + \|v_t\|_{H^1(\Omega)}^2 \right] ds \\
\leq \left( C + C(\delta_1) \right)N(u_0, F) + C(M)N(u_0, F) \left[ C(\delta)\mathcal{O}(t) + \delta \right] \\
+ \delta_1 \left[ \|v_t\|_{L^2(\Omega)}^2 + \|\nabla_0^4 h\|_{L^2(\Gamma)}^2 \right]
where $C$ depends on $\nu$, $\sigma$, $\nu_1$ and the geometry of $\Gamma$. Choose $\delta_1 = \frac{1}{2}$,

\[
\begin{align*}
&\left[\|v(t)\|_{L^2(\Omega)}^2 + \|\nabla v(t)\|_{L^2(\Omega)}^2 + \|\nabla^2 v(t)\|_{L^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 \right. \\
&\quad + \|\nabla^2 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla^3 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla^4 h(t)\|_{L^2(\Gamma)}^2 + \|\nabla^5 h(t)\|_{L^2(\Gamma)}^2 \\
&\quad \left. + \int_0^t \left[\|v\|_{H^1(\Omega)}^2 + \|\nabla v\|_{H^1(\Omega)}^2 + \|\nabla^2 v\|_{H^1(\Omega)}^2 + \|v_t\|_{H^1(\Omega)}^2\right] \, ds \right]
\leq C_1 N(u_0, F) + C(M) N(u_0, F)^2 \left[ C(\delta) \mathcal{O}(t) + \delta \right]
\end{align*}
\]

where $C_1$ depends on $\nu$, $\sigma$, $\mu$ and the geometry of $\Gamma$. Similar to Section 8.7 for almost all $0 < t \leq T$,

\[
\begin{align*}
&\left[\|v(t)\|_{H^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 + \|\nabla^2 h(t)\|_{H^2(\Gamma)}^2 + \|\nabla^3 h(t)\|_{L^2(\Gamma)}^2 \\
&\quad + \int_0^t \left[\|v\|_{H^3(\Omega)}^2 + \|\nabla v\|_{H^3(\Omega)}^2 + \|\nabla^2 v\|_{H^3(\Omega)}^2 + \|v_t\|_{H^3(\Omega)}^2\right] \, ds \right] \\
&\leq C_2 N(u_0, F) + C(M) N(u_0, F)^2 \left[ C(\delta) \mathcal{O}(t) + \delta \right]
\end{align*}
\]

for some constant $C_1$ depending on $C_1$.

By (10.7), (10.8) and (10.11),

\[
\int_0^t \|h_t\|_{H^1(\Gamma)}^2 \, ds \leq \int_0^t \left[ 1 + \|\tilde{h}\|_{H^1(\Gamma)}^2 \right] \|v\|_{H^2(\Gamma)}^2 \, ds
\leq C(M) N(u_0, F) t^{1/4}
\]

and

\[
\int_0^t \|h_t\|_{H^2(\Gamma)}^2 \, ds \leq C(M) \int_0^t \left[ \|\tilde{h}\|_{H^3(\Gamma)}^2 \|v\|_{H^2(\Gamma)}^2 + \|\tilde{h}\|_{H^2(\Gamma)} \|v_t\|_{H^1(\Omega)}^2 \right] \, ds
\leq C(M) N(u_0, F) \left[ t^{1/4} + t^{1/2} \right].
\]

Also, by (10.3) and (10.10),

\[
\int_0^t \|h\|_{H^3(\Gamma)}^2 \, ds \leq C \int_0^t \left[ \|v_t\|_{H^3(\Gamma)}^2 + \|\nabla v\|_{H^2(\Gamma)}^2 + \|\nabla^2 v\|_{H^1(\Omega)}^2 + \|\nabla^3 v\|_{H^1(\Omega_1)}^2 + \|\nabla^4 h\|_{L^2(\Gamma)}^2 \\
+ \|\tilde{h}\|_{H^2(\Gamma)} + 1 \right] \, ds
\leq C_3 N(u_0, F) + C(M) N(u_0, F)^2 \left[ C(\delta) \mathcal{O}(t) + \delta \right]
\]

for some constant $C_3$ depending on $C_2$.

Combining (10.10), (10.11), (10.12) and (10.13), we have the following inequality:

\[
\begin{align*}
&\left[\|v(t)\|_{H^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 + \|h(t)\|_{H^4(\Gamma)}^2 + \|h_t(t)\|_{H^2(\Gamma)}^2 \right. \\
&\quad + \int_0^t \left[\|v\|_{H^3(\Omega)}^2 + \|v_t\|_{H^3(\Omega)}^2 + \|h\|_{H^2(\Gamma)}^2 + \|h_t\|_{H^2(\Gamma)}^2 \right] \, ds \\
&\leq (C_2 + C_3) N(u_0, F) + C(M) N(u_0, F)^2 \left[ C(\delta) \mathcal{O}(t) + \delta \right].
\end{align*}
\]

Let $M = 2(C_2 + C_3) N(u_0, F) + 1$ (and hence corresponding $T_0$ and $T$ in Lemma 6.3 and Corollary 7.1 are fixed). Choose $\delta > 0$ small enough (but fixed one such $\delta$) so that

\[
C(M) N(u_0, F)^2 \delta \leq \frac{1}{4}
\]
and then choose $T > 0$ small enough so that

$$C(M)N(u_0, F)^2 C(\delta)T \leq \frac{1}{4}. $$

Then for almost all $0 < t \leq T$,

$$\left[\|v(t)\|_{L^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 + \|h(t)\|_{H^1(\Gamma)}^2 + \|h_t(t)\|_{H^2(\Gamma)}^2\right]$$

$$+ \int_0^t \left[\|v\|_{H^2(\Omega)}^2 + \|v_t\|_{H^1(\Omega)}^2 + \|h_t\|_{H^2(\Gamma)}^2 + \|h_{tt}\|_{H^0(\Gamma)}^2\right]ds$$

$$\leq C_2 N(u_0, F) + \frac{1}{2}$$

and therefore

$$\sup_{0 \leq t \leq T} \left[\|v(t)\|_{H^2(\Omega)}^2 + \|v_t(t)\|_{L^2(\Omega)}^2 + \|h(t)\|_{H^1(\Gamma)}^2 + \|h_t(t)\|_{H^2(\Gamma)}^2\right]$$

$$+ \|v\|_{L^2(\Omega)}^2 + \|h\|_{H^1(\Omega)}^2 \leq 2C_2 N(u_0, F) + 1, \quad (10.14)$$

or in other words,

$$\|(v, h)\|_{Y(\Omega)}^2 \leq 2C_2 N(u_0, F) + 1. $$

Remark 19. (10.14) implies that for $(\tilde{v}, \tilde{h}) \in C_T(M)$ (with $M$ and $T$ chosen as above), the corresponding solution to the linear problem (10.1) $(v, h) = \Theta_T(\tilde{v}, \tilde{h})$ is also in $C_T(M)$.

10.4. Weak continuity of the mapping $\Theta_T$.

Lemma 10.3. The mapping $\Theta_T$ is weakly sequentially continuous from $C_T(M)$ into $C_T(M)$ (endowed with the norm of $X_T$).

Proof. Let $(v_p, h_p)_{p \in \mathbb{N}}$ be a given sequence of elements of $C_T(M)$ weakly convergent (in $Y_T$) toward a given element $(v, h) \in C_T(M)$ ($C_T(M)$ is sequentially weakly closed as a closed convex set) and let $(v_{\sigma(p)}, h_{\sigma(p)})_{p \in \mathbb{N}}$ be any subsequence of this sequence.

Since $V^2(T)$ is compactly embedded into $L^2(0, T; H^2(\Omega))$, we deduce the following strong convergence results in $L^2(0, T; L^2(\Omega))$ as $p \to \infty$:

$$\left(\alpha_i^2\right)_p \to \alpha_i^2 \quad \text{and} \quad \left(\alpha_i^2\right)_p \to \alpha_i^2 \alpha_i^2, \quad (10.15a)$$

$$\left[[\alpha_i^2(\partial^2_w p)\right] \to \left(\alpha_i^2\right)_p \alpha_i^2 \quad \text{and} \quad \left[[\alpha_i^2(\partial^2_w p)\right] \to \left(\alpha_i^2\right)_p \alpha_i^2, \quad (10.15b)$$

$$\left(\alpha_i^2\right)_p \to \alpha_i^2. \quad (10.15c)$$

Now, let $(w_p, g_p) = \Theta_T(v_p, h_p)$ and let $q_p$ be the associated pressure, so that $(q_p)_{p \in \mathbb{N}}$ is in a bounded set of $V^2(T)$. Since $X_T$ is a reflexive Hilbert space, let $(w_{\sigma(p)}, g_{\sigma(p)}, q_{\sigma(p)})_{p \in \mathbb{N}}$ be a subsequence weakly converging in $X_T \times V^2(T)$ toward an element $(w, g, q) \in X_T \times V^2(T)$. Since $C_T(M)$ is weakly closed in $X_T$, we also have $(w, g) \in C_T(M)$.

For each $\phi \in L^2(0, T; H^1(\Omega))$, we deduce from (2.3) (and Remark 6) that

$$\int_0^T \left[\|w_t\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \int_\Gamma D_\eta w : D_\eta \phi dx + \sigma \int_\Gamma L_h(g) (g, \phi, \phi - \phi_0) dS + \int_\Omega q a_i^2 \phi_i^2 dx \right] dt = \int_0^T (F, \phi) dt$$

which with the fact that, from (10.15), for all $t \in [0, T]$, $w \in V_e$, provides that $(w, g)$ is a solution of (2.16) in $C_T(M)$, i.e., $(w, g) = \Theta_T(v, h)$. 

Therefore, we deduce that the whole sequence $((\Theta_T(\nu_n, h_n)))_{n \in \mathbb{N}}$ weakly converges in $C_T(M)$ toward $\Theta_T(v, h)$, which concludes the lemma.

10.5. Uniqueness. For the uniqueness result, we assume that $u_0$, $F$ and $\Gamma$ are smooth enough (e.g. $u_0 \in H^{5,5}(\Omega)$, $F \in \mathcal{V}^4(T)$, $\Gamma$ is a $H^{8,5}$ surface) so that $u_0$ and the associated $u_1$, $q_0$ satisfy compatibility condition (14.4). Therefore, the solution $(v, h, q)$ are such that $v \in \mathcal{V}^6(T)$, $q \in L^2(0, T; H^5(\Omega))$ and $h \in L^\infty(0, T; H^7(\Gamma)) \cap L^2(0, T; H^{8,5}(\Gamma))$, $h_t \in L^\infty(0, T; H^5(\Gamma)) \cap L^2(0, T; H^{5,5}(\Gamma))$. This implies $a \in L^\infty(0, T; H^5(\Omega))$ and hence by studying the elliptic equation

\[
(a'_t a^k_t q_{t,k})_t = \left[\nu a'_t (a^k_t a^l_t)_{,kl} + a'_t v^i_{,t} + a'_t F_d\right]_t - \left[(a'_t a^k_t),q_{,t}\right]_t \quad \text{in } \Omega,
\]

\[
q_t = \frac{J^{-1}}{h^2} \left[\sigma L_h(h)N_t - h D_t(v)_t a^k_t N_j\right] - \left[(a^k_t N_j) q\right] a^l_t N_t \quad \text{on } \Gamma,
\]

we find that $q_t \in L^2(0, T; H^2(\Omega))$ and this implies $v_t \in L^2(0, T; H^3(\Omega))$. By the interpolation theorem, we also conclude that $v_t \in C^0([0, T]; H^{2,5}(\Omega))$.

Suppose $(v, h, q)$ and $(\tilde{v}, \tilde{h}, \tilde{q})$ are two set of solutions of (11.1). Then

\[
(v - \tilde{v})_t - \nu \left[a^k_t D_t(v - \tilde{v})_t\right]_t - a^k_t (q - \tilde{q})_t + \delta F = 0 \quad (10.16a)
\]

\[
ad^k_t (v - \tilde{v})_t = \delta a \quad (10.16b)
\]

\[
\left[\nu D_t(v - \tilde{v})_t\right]_t - \left[(q - \tilde{q}) T\right] a^j_t N_j = \sigma \Theta \left[L_h(h - \tilde{h})(-\nabla v h, 1)\right] \circ \eta^r + \delta L_1 + \delta L_2 + \delta L_3 \quad (10.16c)
\]

\[
(h - \tilde{h})_t \circ \eta^r = [h_{\alpha} \circ \eta^r](v_{\alpha} - \tilde{v}_{\alpha}) - (v_z - \tilde{v}_z) + \delta h_1 + \delta h_2 + \delta h_3 \quad (10.16d)
\]

\[
(v - \tilde{v})(0) = 0 \quad (10.16e)
\]

\[
(h - \tilde{h})(0) = 0 \quad (10.16f)
\]

where

\[
\delta F = f - f \circ \eta - f \circ \tilde{\eta} + \nu \left[(a^k_t a^l_t - \tilde{a}^k_t \tilde{a}^l_t) v^i_{,i}\right]_t + \nu \left[(a^k_t a^l_t - \tilde{a}^k_t \tilde{a}^l_t) \tilde{v}^i_{,i}\right]_t \quad (10.17a)
\]

\[
d\alpha = (a^k_t - \tilde{a}^k_t) \tilde{v}^i_{,i} \quad (10.17b)
\]

\[
\delta L_1 = \sigma \Theta \left[L_h(h)(\nabla v h - \nabla \tilde{v} h, 0)\right] \circ \eta^r - \nu \left[(a^k_t a^l_t - \tilde{a}^k_t \tilde{a}^l_t) \tilde{v}^i_{,i}\right]_t \quad (10.17c)
\]

\[
\delta L_2 = \tilde{\Theta} \left[L_h(h)(\nabla \tilde{v} h - \nabla \tilde{v} h, 0)\right] \circ \eta^r - \nu \left[(a^k_t a^l_t - \tilde{a}^k_t \tilde{a}^l_t) \tilde{v}^i_{,i}\right]_t \quad (10.17d)
\]

\[
\delta L_3 = \left[L_h(h) - L_h(h)\right] (\nabla \tilde{v} h, -1) \circ \tilde{\eta}^r \quad (10.17e)
\]

\[
\delta h_1 = (h_{\alpha} \circ \eta - h_{\alpha} \circ \tilde{\eta}) \tilde{v}_{\alpha} \quad (10.17f)
\]

\[
\delta h_2 = \left[(h_{\alpha} - \tilde{h}_{\alpha}) \circ \eta^r\right] \tilde{v}_{\alpha} \quad (10.17g)
\]

\[
\delta h_3 = - (\tilde{h}_{\alpha} \circ \eta^r - \tilde{h}_{\alpha} \circ \tilde{\eta}^r) \quad (10.17h)
\]

We will also use $\delta L$ and $\delta h$ to denote $\sum_{k=1}^{3} L_k$ and $\sum_{k=1}^{3} \delta h_k$ respectively.
Similar to (11.3) in [11], we also have the following estimates.

**Lemma 10.4.** For \( f \in H^2(\Omega) \) and \( g \in H^{1.5}(\Gamma) \),

\[
\|f \circ \eta - f \circ \tilde{\eta}\|_{L^2(\Omega)} \leq C \sqrt{\|f\|_{H^2(\Omega)}} \left[ \int_0^t \|v - \tilde{v}\|_{H^1(\Omega)}^2 \, ds \right]^{1/2},
\]

\[
\|g \circ \eta^\gamma - g \circ \tilde{\eta}^\gamma\|_{L^2(\Gamma)} \leq C \sqrt{\|g\|_{H^{1.5}(\Gamma)}} \left[ \int_0^t \|v - \tilde{v}\|_{H^1(\Omega)}^2 \, ds \right]^{1/2}.
\]

for some constant \( C \).

**Remark 20.** Assuming the regularity of \( h, h_t \) and \( h_{tt} \) given in the beginning of this section, we have

\[
\|\delta L_2\|_{H^2(\Gamma)} + \|\delta h_1 + \delta h_3\|_{H^2(\Gamma)} \leq C \sqrt{t} \left[ \int_0^t \|v - \tilde{v}\|_{H^2(\Omega)}^2 \, ds \right]^{1/2}
\]

and

\[
\|((\delta L_2)_t)\|_{L^2(\Gamma)} + \|((\delta h_1 + \delta h_3)_t)\|_{H^1(\Gamma)} \leq C \left[ \|v - \tilde{v}\|_{H^2(\Omega)} + \sqrt{t} \left( \int_0^t \|v - \tilde{v}\|_{H^2(\Omega)}^2 \, ds \right)^{1/2} \right]
\]

and

\[
\|\nabla^2(\delta h_3)_t\|_{L^2(\Gamma)} \leq C \left[ \|v - \tilde{v}\|_{H^2(\Omega)} + \|v - \tilde{v}\|_{H^3(\Omega)} + \sqrt{t} \|h_{tt}\|_{H^3(\Gamma)} \left( \int_0^t \|v - \tilde{v}\|_{H^3(\Omega)}^2 \, ds \right)^{1/2} \right].
\]

By using (10.18) to estimate \( \|\delta F\|_{L^2(\Omega)} \), we find that

\[
\|\nabla (v - \tilde{v})(t)\|_{L^2(\Omega)}^2 + \int_0^t \|v - \tilde{v}\|_{L^2(\Omega)}^2 \, ds
\]

\[
\leq C(\delta) \int_0^t \left[ \|v - \tilde{v}\|_{H^1(\Omega)}^2 + \|h - \tilde{h}\|_{H^2(\Gamma)}^2 \right] \, ds + (C(\delta)t^2 + \delta) \int_0^t \|v - \tilde{v}\|_{H^2(\Omega)}^2 \, ds
\]

\[+ \delta \int_0^t \left[ \|\nabla (v - \tilde{v})_t\|_{H^1(\Omega)}^2 + \|q - \tilde{q}\|_{H^1(\Omega)}^2 \right] \, ds.
\]

(10.23)

For the \( L^2(\Omega) \) estimate for \( v - \tilde{v} \) and the \( L^2(\Omega) \) estimate for \( (v - \tilde{v})_t \), we have

\[
\frac{1}{2} \frac{d}{dt} \left[ \|\nabla^2(v - \tilde{v})\|_{L^2(\Omega)}^2 + 2\sigma E_h(\nabla^2(h - \tilde{h})) \right] + \frac{\nu}{4} \|\nabla^2(h - \tilde{h})\|_{L^2(\Omega)}^2
\]

\[\leq C \left[ \|\nabla^2 F\|_{H^1(\Omega)}^2 + \|(v - \tilde{v})_t\|_{L^2(\Omega)}^2 + \|\nabla (v - \tilde{v})\|_{L^2(\Omega)}^2 + \|\nabla (v - \tilde{v})\|_{L^2(\Omega)}^2 + \|\nabla^2 F\|_{H^1(\Omega)}^2 \right] + D_1 + D_2 + D_3
\]

and

\[
\frac{1}{2} \frac{d}{dt} \left[ \|(v - \tilde{v})_t\|_{L^2(\Omega)}^2 + 2\sigma E_h((h - \tilde{h})_t) \right] + \frac{\nu}{4} \|\nabla (v - \tilde{v})\|_{L^2(\Omega)}^2
\]

\[\leq C \left[ \|\nabla^2(h - \tilde{h})\|_{L^2(\Gamma)}^2 + \|\nabla^2(h - \tilde{h})_t\|_{L^2(\Gamma)}^2 + \|\delta F\|_{H^1(\Omega)}^2 \right] + \delta \|v - \tilde{v}\|_{H^2(\Omega)}^2
\]

\[+ E_1 + E_2 + E_3.
\]
By letting
\[ D_1 := \int_\Omega \xi_1^2 \nabla_0^2 (q - \tilde{q}) \nabla_0^2 \delta dx, \quad D_2 := \int_\Gamma \Theta \left[ L_h (h - \tilde{h}) \circ \eta^T \right] (\nabla_0^4 \delta h) dS, \]
\[ D_3 := \int_\Gamma \delta L \cdot \nabla_0^2 (v - \tilde{v}) dS, \]
and
\[ E_1 := \int_\Omega (q - \tilde{q})_t (\delta a)_t dx, \quad E_2 := \int_\Gamma \left[ \Theta [L_h (h - \tilde{h}) \circ \eta^T]_t \right] (\delta h)_t dS, \]
\[ E_3 := \int_\Gamma (\delta L)_t \cdot (v - \tilde{v})_t dS. \]

By using (10.20) to estimate \( D_1 \) and (10.21), (10.22) to estimate \( E_1 \), we obtain
\[
\left[ \| \nabla_0^2 (v - \tilde{v}) (t) \|^2_{L^2 (\Omega)} \right] + \int_0^t \| \nabla_0^2 (v - \tilde{v}) (t) \|^2_{L^2 (\Gamma)} dS \leq C(\delta) \int_0^t \left[ \| v - \tilde{v} \|^2_{H^1 (\Omega)} + \| \nabla_0^2 (h - \tilde{h}) \|^2_{L^2 (\Gamma)} \right] + \int_0^t \| v - \tilde{v} \|^2_{L^2 (\Gamma)} dS + C(\delta) (t^2 + \delta) \int_0^t \| v - \tilde{v} \|^2_{H^2 (\Omega)} dS + \delta \int_0^t \| q - \tilde{q} \|^2_{H^2 (\Omega)} dS \tag{10.24}
\]
and
\[
\left[ \| (v - \tilde{v})_t \|^2_{L^2 (\Omega)} + \| \nabla_0^2 (h - \tilde{h})_t \|^2_{L^2 (\Gamma)} \right] + \int_0^t \| (v - \tilde{v})_t \|^2_{L^2 (\Gamma)} dS \leq C(\delta) \int_0^t \left[ \| v - \tilde{v} \|^2_{H^1 (\Omega)} + \| \nabla_0^2 (h - \tilde{h}) \|^2_{L^2 (\Gamma)} + (1 + \| \tilde{h} \|^2_{H^{1,5} (\Gamma)}) \times \| \nabla_0^2 (h - \tilde{h})_t \|^2_{L^2 (\Gamma)} \right] dS \tag{10.25}
\]
\[ + (C(\delta) (t^2 + \delta) + \delta) \int_0^t \| v - \tilde{v} \|^2_{L^2 (\Omega)} dS + \delta \int_0^t \| q - \tilde{q} \|^2_{L^2 (\Omega)} dS. \]

Summing (10.23), (10.24) and (10.25), we find that
\[ Y(t) + \int_0^t Z(s) ds \leq C(\delta) \int_0^t k(s) Y(s) ds + (C(\delta) (t^2 + \delta) + \delta) \int_0^t Z(s) ds \tag{10.26} \]
where
\[ k(t) = 1 + \| \tilde{h} \|^2_{H^{1,5} (\Gamma)} \]
\[ Y(t) = \left[ \| v - \tilde{v} \|^2_{H^1 (\Omega)} + \| \nabla_0^2 (v - \tilde{v}) (t) \|^2_{L^2 (\Omega)} + \| (v - \tilde{v})_t \|^2_{L^2 (\Omega)} \right] + \| h - \tilde{h} \|^2_{H^2 (\Gamma)} + \| (h - \tilde{h})_t \|^2_{H^2 (\Gamma)}, \]
\[ Z(t) = \| (v - \tilde{v})_t \|^2_{H^1 (\Omega)} + \| \nabla_0^2 (v - \tilde{v}) (t) \|^2_{L^2 (\Omega)}. \]

By letting \( \delta = 1/4 \) and choosing \( T_u \leq T \) so that
\[ C(\delta) (T_u^2 + T_u) \leq 1/4, \]
\[ Y(t) + \int_0^t Z(s) ds \leq C \int_0^t k(s) Y(s) ds \tag{10.27} \]
for all \( 0 < t \leq T_u \). Since \( Y(0) = 0 \), the uniqueness of the solution follows from that \( Y(t) = 0 \) for all \( 0 < t \leq T_u \).
Appendix A. Elliptic regularity

We establish a $\kappa$-independent elliptic estimate for solutions of

$$
\frac{\Theta}{\sqrt{\det(g_0)}} \left( \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \right)_{\gamma\delta} (\nabla_0 \bar{h}, 1) \circ \bar{\eta}^T + \kappa \Delta_0^2 v_\kappa = f
$$

(A.1)

where $h_\kappa$ and $v_\kappa$ satisfy (A.3) with $h_\kappa \in H^4(\Gamma)$, $v_\kappa \in H^4(\Gamma)$, and $f \in H^{1,5}(\Gamma)$. Letting $w = v_\kappa \circ \bar{\eta}^T$, (A.1) is equivalent to

$$
\frac{\Theta}{\sqrt{\det(g_0)}} \left( \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \right)_{\gamma\delta} (\nabla_0 \bar{h}, 1) + \kappa \Delta_0^2 w = f \circ \bar{\eta}^T
$$

(A.2)

which implies

$$
\frac{\Theta}{\sqrt{\det(g_0)}} \left( \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta} h_{\kappa,\alpha\beta} \right)_{\gamma\delta} w + \kappa J_h^{-2} \Delta_0^2 w \cdot (\nabla_0 \bar{h}, 1)
$$

(A.3)

Recall that $w \cdot (\nabla_0 \bar{h}, 1) = h_{\kappa t}$.

Let $D_h$ denote the difference quotients (w.r.t. the surface coordinate system). Taking the inner-product of (A.3) with $D_{-h} D_h \nabla_0^4 h_\kappa$, by Corollary 7.1 we find that

$$
\nu_1 \int_0^t \| D_h \nabla_0^4 h_\kappa \|^2_{L^2(\Gamma)} ds \leq C(\epsilon) \int_0^t \left( \| h_\kappa \|^2_{H^2(\Gamma)} + \| f \|^2_{H^1(\Gamma)} + \kappa \| w \|^2_{H^4(\Gamma)} \right) ds.
$$

Since the right-hand side is independent of difference parameter $h$, it follows that $h_\kappa \in H^2(\Gamma)$ (as it is already a $H^4$-function) with the estimate

$$
\int_0^t \| \nabla_0^2 h_\kappa \|^2_{L^2(\Gamma)} ds \leq C(\epsilon) \int_0^t \left( \| h_\kappa \|^2_{H^2(\Gamma)} + \| f \|^2_{H^1(\Gamma)} + \kappa \| w \|^2_{H^4(\Gamma)} \right) ds.
$$

(A.4)

Next, we obtain a $\kappa$-independent estimate of $\kappa \| w \|^2_{H^4(\Gamma)}$. By taking the inner-product of (A.2) with $\nabla_0^2 w$ and $\nabla_0^4 w$, we find that

$$
\| \nabla_0^3 h_\kappa(t) \|^2_{L^2(\Gamma)} + \kappa \int_0^t \| w \|^2_{H^4(\Gamma)} ds
$$

\( \leq C(\epsilon) \int_0^t \left( \| \nabla_0^3 h_\kappa \|^2_{L^2(\Gamma)} + \| f \|^2_{L^2(\Gamma)} + \| w \|^2_{H^2(\Gamma)} \right) ds. \quad \text{(A.5)}

and

$$
\| \nabla_0^2 h_\kappa(t) \|^2_{L^2(\Gamma)} + \kappa \int_0^t \| w \|^2_{H^4(\Gamma)} ds
$$

\( \leq C(\epsilon, \delta_1) \int_0^t \left( \| \nabla_0^2 h_\kappa \|^2_{L^2(\Gamma)} + \| f \|^2_{H^1(\Gamma)} + \| w \|^2_{H^3(\Gamma)} \right) ds + \delta_1 \int_0^t \| \nabla_0^2 h_\kappa \|^2_{L^2(\Gamma)} ds. \quad \text{(A.6)}

where we use (A.5) to estimate $\kappa \int_0^t \| w \|^2_{H^4(\Gamma)} ds$. (A.6) provides a $\kappa$-independent estimate for $\kappa \| w \|^2_{H^4(\Gamma)}$; hence by choosing $\delta_1 > 0$ small enough, (A.4) implies that for all $t \in [0, T],$

$$
\int_0^t \| \nabla_0^2 h_\kappa \|^2_{H^4(\Gamma)} ds \leq C' \int_0^t \left( \| \nabla_0^2 h_\kappa \|^2_{L^2(\Gamma)} + \| f \|^2_{H^1(\Gamma)} + \| w \|^2_{H^3(\Gamma)} \right) ds
$$

(A.7)

for some constant $C'$ depending on $\epsilon$. 
**APPENDIX B. INEQUALITIES IN THE ESTIMATES FOR $\nabla_0^2 \nu$ NEAR THE BOUNDARY**

**B.1. $\kappa$-independent estimates.** Since $\zeta_1 \equiv 1$ on $\Gamma$ and

$$(-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^4 \nu = \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nu) - \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nu) - 4 \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0 \nu \cdot 6 \nabla_0^2 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^2 \nu),$$

we find that

$$\int_{\Gamma} \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^2 (\zeta_1^2 \nabla_0^2 \nu)) dS$$

$$= -\int_{\Gamma} \nabla_0^2 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nu) + 4 \nabla_0^2 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0 \nu$$

$$+ 6 \nabla_0^2 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^2 \nu) dS \quad (\equiv I_1)$$

$$- 4 \int_{\Gamma} \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^3 \nu) dS \quad (\equiv I_2)$$

$$+ \int_{\Gamma} \nabla_0^4 \nabla_0^2 \left[ \sqrt{\text{det}(g_0)} (L_1^\alpha \beta \gamma h,_{\alpha \beta \gamma} + L_2) \circ \bar{\eta} \right] \nabla_0^2 (h_{\kappa \ell} \circ \bar{\eta}) dS \quad (\equiv I_3)$$

$$+ \int_{\Gamma} \frac{2 \nabla_0^4 \Theta}{\sqrt{\text{det}(g_0)}} \nabla_0 \left[ \sqrt{\text{det}(g_0)} (L_1^\alpha \beta \gamma h,_{\alpha \beta \gamma} + L_2) \circ \bar{\eta} \right] \nabla_0^2 (h_{\kappa \ell} \circ \bar{\eta}) dS \quad (\equiv I_4)$$

$$+ \int_{\Gamma} \nabla_0^4 \left[ \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^4 \nu) \int_{\Gamma} \frac{\Theta}{\sqrt{\text{det}(g_0)}} \left[ \sqrt{\text{det}(g_0)} (L_1^\alpha \beta \gamma h,_{\alpha \beta \gamma} + L_2) \circ \bar{\eta} \right] \nabla_0^4 (h_{\kappa \ell} \circ \bar{\eta}) dS \quad (\equiv I_5)$$

$$+ \int_{\Gamma} \frac{\Theta}{\sqrt{\text{det}(g_0)}} \left[ \nabla_0^4 ((-\nabla_0 \bar{h} \circ \bar{\eta}, 1) \cdot \nabla_0^4 \nu) \int_{\Gamma} \frac{B}{\sqrt{\text{det}(g_0)}} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^4 (h_{\kappa \ell} \circ \bar{\eta}) dS \right].$$

The last term of the identity above, by a change of coordinates, can be written as

$$\int_{\Gamma} \frac{\Theta}{\sqrt{\text{det}(g_0)}} \nabla_0^4 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^4 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$= \int_{\Gamma} \frac{B}{\sqrt{\text{det}(g_0)}} \nabla_0^4 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^4 (h_{\kappa \ell} \circ \bar{\eta}) dS \right] + R_1$$

$$+ 2 \int_{\Gamma} \frac{\nabla_0 \Theta}{\sqrt{\text{det}(g_0)}} \nabla_0 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^2 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} \frac{\nabla_0^2 \Theta}{\sqrt{\text{det}(g_0)}} \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^2 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Gamma} B_{\alpha \beta \gamma \delta} \nabla_0^2 (h_{\kappa \alpha \beta}) \nabla_0^2 (h_{\kappa \gamma \delta}) dS + R_1$$

where $B = b^t \otimes b^t \otimes b^t \otimes b^t$ with $b = \nabla_0 \bar{\eta}$, and

$$R_1(t) = \int_{\Gamma} b^t \otimes b^t \otimes (\nabla_0 b^t) \otimes (\nabla_0 b^t) \nabla_0 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} b^t \otimes b^t \otimes b^t \otimes (\nabla_0 b^t) \nabla_0 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0^2 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} b^t \otimes b^t \otimes b^t \otimes (\nabla_0 b^t) \nabla_0^2 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} b^t \otimes b^t \otimes b^t \otimes (\nabla_0 b^t) \nabla_0 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} b^t \otimes b^t \otimes b^t \otimes (\nabla_0 b^t) \nabla_0^2 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$

$$+ \int_{\Gamma} b^t \otimes b^t \otimes b^t \otimes (\nabla_0 b^t) \nabla_0^2 \left[ \sqrt{\text{det}(g_0)} (A_{\alpha \beta \gamma \delta} h,_{\kappa \alpha \beta}) \cdot \nabla_0 (h_{\kappa \ell} \circ \bar{\eta}) dS \right]$$
and

\[ R'_1(t) = R_1(t) + J_1(t) + J_2(t) - \frac{1}{2} \int_{\Gamma} (B A^{\alpha\beta\gamma\delta})_t \nabla_0^2 h_{\kappa,\alpha\beta\gamma} \nabla_0^2 h_{\kappa,\gamma\delta} dS \quad (\equiv J_6) \]

\[ + 2 \int_{\Gamma} \frac{B}{\sqrt{\det(g_0)}} \nabla_0 (\sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta} \nabla_0^2 h_{\kappa,\gamma\delta} dS \quad (\equiv J_7) \]

\[ + \int_{\Gamma} \frac{B}{\sqrt{\det(g_0)}} \nabla_0^2 (\sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta} \nabla_0^2 h_{\kappa,\gamma\delta} dS \quad (\equiv J_8) \]

\[ + 2 \int_{\Gamma} \frac{B}{\sqrt{\det(g_0)}} \nabla_0^2 (\sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta} \nabla_0 h_{\kappa,\gamma\delta} dS \quad (\equiv J_9) \]

\[ + \int_{\Gamma} \frac{B}{\sqrt{\det(g_0)}} \nabla_0^2 (\sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta})_t h_{\kappa,\alpha\beta} \nabla_0 h_{\kappa,\gamma\delta} dS. \quad (\equiv J_{10}) \]

It follows that

\[ |I_1| \leq C(\epsilon)(1 + \|\nabla_0^2 h_{\kappa}\|_{L^2(\Gamma)}) \|\nabla_0^2 v_{\kappa}\|_{H^1(\Omega)}, \]

\[ |I_3| + |I_4| + |I_5| \leq C(M) |(1 + \|\hat h\|_{H^2(\Gamma)}) \|\nabla_0^2 v_{\kappa}\|_{H^1(\Omega)}, \]

and hence that

\[ |I_1| + |I_3| + |I_4| + |I_5| \leq C(\epsilon) \left[ \|\nabla_0^4 h_{\kappa}\|_{L^2(\Gamma)}^2 + \|\hat h\|_{H^2(\Gamma)}^2 + 1 \right] + \delta \|v_{\kappa}\|_{H^3(\Omega)}^2. \]

It follows that

\[ |J_2| + |J_3| + |J_5| + |J_{10}| \leq C(\epsilon) \|\nabla_0^2 h_{\kappa}\|_{L^2(\Gamma)} \|\nabla_0^2 h_{\kappa}\|_{L^2(\Gamma)} \]

\[ |J_6| \leq C(M)(\|\hat h\|_{H^3(\Omega)} + \|\hat h\|_{H^2(\Gamma)}^2) \|\nabla_0^4 h_{\kappa}\|_{L^2(\Gamma)}^2. \]

We need only obtain \(\kappa\)-independent estimates for the terms \(I_2, J_1, J_4, J_7, J_8\) and \(J_9\). By the \(H^{-0.5}(\Gamma)-H^{0.5}(\Gamma)\) duality pairing,

\[ |I_2| \leq C(M) \left[ \|\nabla_0^2 h_{\kappa}\|_{H^{2.5}(\Gamma)} + 1 \right] \|v_{\kappa}\|_{H^{2.5}(\Gamma)}. \]

Therefore, by interpolation and Young’s inequality,

\[ |I_2| \leq C \left[ \|h_{\kappa}\|_{H^4(\Gamma)}^2 + 1 \right] + \delta_1 \|\nabla_0^2 h_{\kappa}\|_{H^3(\Gamma)}^2 + \delta \|v_{\kappa}\|_{H^3(\Omega)}^2 \quad \text{(B.1)} \]

for some \(C\) depending on \(M, \delta\) and \(\delta_1\).

For \(J_1, J_4\) and \(J_9\), we find that

\[ |J_1| + |J_4| + |J_9| \leq C(\epsilon) \|h_{\kappa}\|_{H^{4.5}(\Gamma)} \|v_{\kappa}\|_{H^{2.5}(\Gamma)} \]

\[ \leq C' \left[ \|\nabla_0^2 h_{\kappa}\|_{H^2(\Gamma)}^2 + 1 \right] + \delta_1 \|\nabla_0^2 h_{\kappa}\|_{H^3(\Gamma)}^2 + \delta \|v_{\kappa}\|_{H^3(\Omega)}^2 \]

for some constant \(C'\) depending on \(M, \epsilon, \delta\) and \(\delta_1\).

For \(J_7\) and \(J_8\), by the \(H^{-1.5}(\Gamma)-H^{1.5}(\Gamma)\) duality pairing,

\[ |J_7| + |J_8| \leq C(M) \|h_{\kappa}\|_{H^{1.5}(\Gamma)} \|\hat h\|_{H^{3.5}(\Gamma)} \|h_{\kappa}\|_{H^{4.5}(\Gamma)} \|v_{\kappa}\|_{H^{2.5}(\Gamma)}. \]

Similarly to the estimate in \(\text{B.1}\), we find that

\[ |J_7| + |J_8| \leq C(M) \left[ \|h_{\kappa}\|_{H^4(\Gamma)} + 1 \right] + \delta_1 \|\nabla_0^2 h_{\kappa}\|_{H^3(\Gamma)}^2 + \delta \|v_{\kappa}\|_{H^3(\Omega)}^2. \]
Summing all the estimates and then integrating in time from 0 to $t$, by Corollary 7.1 and the fact that $B$ is close to 1 in the uniform norm for $T$ small,
\[
\frac{\nu_1}{2} \|\nabla_0^4 h_\epsilon(t)\|^2_{L^2(\Gamma)} \leq \int_0^t \int_\Gamma \Theta \left[ [L_{\tilde{h}}(h_\epsilon)(-\nabla_0 \tilde{h}, 1)] \circ \tilde{\eta}^r \right] \cdot \nabla_0^2(\zeta^2 \nabla_0^2 v_\epsilon) dS ds \\
+ C' \int_0^t K(s) \|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma)} ds + C' \int_0^t \left[ \|\tilde{h}\|^2_{H^5(\Gamma)} + 1 \right] ds \\
+ \delta \int_0^t \|v_\epsilon\|^2_{H^3(\Omega)} ds + \delta_1 \int_0^t \nabla_0^2 h_\epsilon\|^2_{H^3(\Gamma)} ds
\]
for some constant $C'$ depending on $M$, $\epsilon$, $\delta$ and $\delta_1$.

$$K(s) := 1 + \|\tilde{v}\|^2_{H^5(\Omega)} + \|\tilde{h}\|^2_{H^5(\Gamma)} + \|\hat{h}_t\|^2_{H^{2,5}(\Gamma)}.$$  

**B.2. $\epsilon$-independent estimates.** We next obtain $\epsilon$-independent estimates for the first two terms of $J_1$, as well as those for $J_2$, $J_4$, $J_5$, $J_9$, and $J_{10}$ with $h_\epsilon$ replaced by $h_\epsilon'$. Let

$$I_1^1 = - \int_\Gamma \Theta \left[ L_{\tilde{h}}(h_\epsilon) \circ \tilde{\eta}^r \right] \left[ \nabla_0^4 \left( -\nabla_0 \tilde{h} \circ \tilde{\eta}^r, 1 \right) \right] \cdot v_\epsilon dS,$$

$$I_1^2 = - 4 \int_\Gamma \Theta \left[ L_{\tilde{h}}(h_\epsilon) \circ \tilde{\eta}^r \right] \left[ \nabla_0^2 \left( -\nabla_0 \tilde{h} \circ \tilde{\eta}^r, 1 \right) \right] \cdot \nabla_0 v_\epsilon dS$$

By the $H^{-1,5}(\Gamma)$-$H^{1,5}(\Gamma)$ duality pairing,

$$|I_1^1| + |I_1^2| \leq C(M) \|L_{\tilde{h}}(h_\epsilon)\|_{H^{1,5}(\Gamma)} \|v_\epsilon\|_{H^{2,5}(\Gamma)} \|(\nabla_0 \tilde{h}) \circ \tilde{\eta}^r\|_{H^{2,5}(\Gamma)}.$$  

Therefore, by (6.6) and (9.12),

$$|I_1^1| + |I_1^2| \leq C(M) t^{1/4} \left[ \|h_\epsilon\|^2_{H^{2,5}(\Gamma)} + 1 \right] \|v_\epsilon\|_{H^3(\Omega)}$$

(B.2)

$$\leq Ct^{1/2} \left[ \|v_\epsilon\|^2_{H^1(\Omega)} + \|\nabla_0^2 h_\epsilon\|^2_{L^2(\Gamma)} + \|F\|^2_{H^1(\Omega)} + 1 \right] + (\delta + Ct^{1/2}) \|v_\epsilon\|^2_{H^3(\Omega)}$$

for some constant $C$ depending on $M$ and $\delta$.

For $J_1$, we use an $L^4$-$L^4$-$L^2$ type of Hölder’s inequality and conclude that

$$|J_1| \leq C(M) t^{1/2} \|h_\epsilon\|_{H^{2,5}(\Gamma)} \|v_\epsilon\|_{H^{2,5}(\Gamma)}$$

while for the other $J$ terms, we use the $H^{0,5}(\Gamma)$-$H^{-0,5}(\Gamma)$ duality pairing to obtain

$$|J_2| + |J_3| + |J_4| + |J_5| + |J_9| + |J_{10}| \leq C(M) t^{1/2} \|h_\epsilon\|_{H^{2,5}(\Gamma)} \|v_\epsilon\|_{H^{2,5}(\Gamma)},$$

and hence all the $J$ terms are bounded by the same right-hand side of the inequality in (B.2).

Therefore,

$$\frac{\nu_1}{2} \|\nabla_0^4 h_\epsilon(t)\|^2_{L^2(\Gamma)} \leq \int_0^t \int_\Gamma \Theta \left[ [L_{\tilde{h}}(h_\epsilon)(-\nabla_0 \tilde{h}, 1)] \circ \tilde{\eta}^r \right] \cdot \nabla_0^2(\zeta^2 \nabla_0^2 v_\epsilon) dS ds \\
+ CN_2(u_0, F) + C \int_0^t K(s) \|\nabla_0^4 h_\epsilon\|^2_{L^2(\Gamma)} ds + (\delta + Ct^{1/2}) \int_0^t \|v_\epsilon\|^2_{H^3(\Omega)} ds \\
+ (\delta_1 + Ct^{1/2}) \int_0^t \|v_\epsilon\|^2_{H^3(\Omega)} ds$$

for some constant $C$ depending on $M$, $\delta$ and $\delta_1$.  

APPENDIX C. $L^2_t H^1_{x}$ estimates for $v_t$

By the chain rule and integrating by parts,

\[
\int_{\Gamma} \left[ \Theta [L_h(h_{\kappa})(-\nabla_0 \tilde{h}, 1) \circ \tilde{\eta}^r] \right]_t \cdot v_{\kappa t} dS = \int_{\Gamma} \left[ \Theta [L_h(h_{\kappa})] \circ \tilde{\eta}^r (-\nabla_0 \tilde{h} \circ \tilde{\eta}^r, 1) \cdot v_{\kappa t} dS + \int_{\Gamma} \Theta [\eta^r] \cdot \left( \nabla_0 [L_h(h_{\kappa})] (-\nabla_0 \tilde{h}, 1) \right) \circ \eta^r \cdot v_{\kappa t} dS \quad (\equiv K_1) \\
+ \int_{\Gamma} \Theta [L_h(h_{\kappa})] (\nabla_0 \tilde{h}, -1) \right]_t \circ \eta^r \cdot v_{\kappa t} dS. \quad (\equiv K_2)
\]

The first term is bounded by

\[
C(M) \| \tilde{v} \|_{H^2(\Omega)} \| \nabla_0^4 h_{\kappa} \|_{L^2(\Gamma)} + 1 \| v_{\kappa t} \|_{L^2(\Gamma)}.
\]

After integrating by parts, the most difficult term to estimate in $K_1$ consists of the integral

\[
\int_{\Gamma} \frac{\tilde{v}}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta}_{h_{\kappa,\alpha\beta}}(\nabla_0 \tilde{h}, -1) \right] \circ \tilde{\eta}^r \nabla_0 v_{\kappa t} dS.
\]

Integrating from 0 to $t$ and integrating by parts in time, we find that

\[
\int_{0}^{t} \int_{\Gamma} \frac{\tilde{v}}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta}_{h_{\kappa,\alpha\beta}}(\nabla_0 \tilde{h}, -1) \right] \circ \tilde{\eta}^r \nabla_0 v_{\kappa t} dSds
\]

= \(-\int_{0}^{t} \int_{\Gamma} \frac{\tilde{v}}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} A^{\alpha\beta\gamma\delta}_{h_{\kappa,\alpha\beta}}(\nabla_0 \tilde{h}, -1) \right] \circ \tilde{\eta}^r \nabla_0 v_{\kappa} dSds + R_3\)

where $R_3$ is bounded by

\[
C \int_{0}^{t} \left[ 1 + \| \tilde{v} \|_{H^1(\Omega)}^2 \right] \| \nabla_0^4 h_{\kappa} \|_{L^2(\Gamma)}^2 + \delta_2 \| \nabla_0^4 h_{\kappa} \|_{L^2(\Gamma)}^2 + \delta \int_{0}^{t} \| v_{\kappa} \|_{H^2(\Omega)}^2 ds + (\delta + C\delta^{1/2}) \int_{0}^{t} \| v_{\kappa t} \|_{H^1(\Omega)}^2 ds
\]

for some constant $C$ depending on $M$, $\delta$ and $\delta_2$. Next, using that

\[
[-\nabla_0 \tilde{h}, 1) \circ \tilde{\eta}^r] \cdot \nabla_0 v_{\kappa} = b'(\nabla_0 h_{\kappa t}) \circ \tilde{\eta}^r + b'(\nabla_0^2 \tilde{h} \circ \tilde{\eta}^r, 0) \cdot v_{\kappa},
\]

and integrating by parts, we find that the integral on the right-hand side is identical to

\[
\frac{1}{2} \int_{0}^{t} \int_{\Gamma} \frac{1}{\sqrt{\det(g_0)}} \nabla_0 \left[ \sqrt{\det(g_0)} \Theta b' A^{\alpha\beta\gamma\delta}_{h_{\kappa,\alpha\beta}} \right] h_{\kappa t,\alpha\beta} h_{\kappa t,\gamma\delta} dSds + R_4
\]

where

\[
|R_4| \leq C(M) C(\delta) \int_{0}^{t} \| \nabla_0^4 h_{\kappa} \|_{L^2(\Gamma)}^2 ds + \delta \int_{0}^{t} \| v_{\kappa} \|_{H^2(\Omega)}^2 ds
\]

By interpolation, the integral part is bounded by

\[
C \left[ N(u_0, F) + \int_{0}^{t} \| \nabla_0^4 h_{\kappa} \|_{L^2(\Gamma)}^2 ds \right] + \delta \int_{0}^{t} \| v_{\kappa} \|_{H^2(\Omega)}^2 ds + C t \int_{0}^{t} \| v_{\kappa t} \|_{H^1(\Omega)}^2 ds
\]
for some constant $C$ depending on $M$ and $\delta$. Therefore, $K_1$ satisfies
\[
\left| \int_0^t K_1 \, ds \right| \leq C \int_0^t \left[ K(s) \left( \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 \right) + 1 \right] \, ds + \delta_2 \| \nabla_0^4 h_{\kappa t} \|_{L^2(\Gamma)}^2 \\
+ (\delta + C t^{1/2}) \int_0^t \| v_{\kappa t} \|_{H^3(\Omega)}^2 \, ds + (\delta + C t^{1/2}) \int_0^t \| v_{\kappa t} \|_{H^3(\Omega)}^2 \, ds
\]  
(C.1)
for some constant $C$ depending on $M$, $\delta$ and $\delta_2$.

For $K_2$, by time differentiating the evolution equation, we find that
\[
(-\nabla_0 \tilde{h} \circ \tilde{\eta}^T, 1) v_{\kappa t} = h_{\kappa tt} \circ \tilde{\eta}^T + \tilde{v}^T \cdot (\nabla_0 h_{\kappa t}) \circ \tilde{\eta}^T - \tilde{v}^T \cdot (\nabla_0^2 \tilde{h} \circ \tilde{\eta}^T, 0) \cdot v_{\kappa}
\]
and hence (after a change of coordinates)
\[
K_2 = \int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \, h_{\kappa tt} \, dS + \int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \cdot (\nabla_0 h_{\kappa t}) \right] \, dS \right] (\equiv K_3)
\]
\[
- \int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \cdot (v_{\kappa} \circ \tilde{\eta}^T) \right] \, dS \quad (\equiv K_4)
\]
\[
- \int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \cdot (\nabla_0^2 \tilde{h}) \right] \cdot (v_{\kappa} \circ \tilde{\eta}^T) \, dS. \quad (\equiv K_5)
\]
\[
+ \int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \cdot (v_{\kappa} \circ \tilde{\eta}^T) \right] \, dS \quad (\equiv K_6).
\]

For the first term, we have
\[
\int_{\Gamma} \left[ (L_{\tilde{h}}(h_{\kappa t}))_t \, h_{\kappa tt} \right] \, dS = \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \tilde{A}^{\alpha \beta \gamma \delta} h_{\kappa t, \alpha \beta} h_{\kappa t, \gamma \delta} \, dS
\]
\[
+ \int_{\Gamma} \frac{1}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} (\tilde{A}^{\alpha \beta \gamma \delta})_t \right]_{\gamma \delta} h_{\kappa, \alpha \beta} h_{\kappa t, \gamma \delta} \, dS \quad (\equiv K_7) + R_5
\]
where $R_5$ is bounded by
\[
C \left[ 1 + \| \tilde{h}_t \|_{H^2(\Gamma)}^2 \right] \left[ 1 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 \right] + \delta \left[ \| v_{\kappa} \|_{H^2(\Omega)}^2 + \| \nabla_0^2 v_{\kappa} \|_{H^1(\Omega)}^2 \right]
\]
\[
+ \delta_1 \| v_{\kappa} \|_{H^2(\Omega)}^2
\]
for some constant $C$ depending on $M$, $\delta$ and $\delta_1$. Also, by the inequality $\| h_{\kappa tt} \|_{L^4(\Gamma)} \leq C(M) \left( \| v_{\kappa} \|_{H^2(\Omega)} + \| v_{\kappa} \|_{H^2(\Omega)} \right)$,
\[
| K_7 | \leq C \left[ \sqrt{\det(g_0)} (\tilde{A}^{\alpha \beta \gamma \delta})_t, \gamma \delta \right]_{H^0(\Gamma)} \left[ \frac{1}{\sqrt{\det(g_0)}} h_{\kappa, \alpha \beta} h_{\kappa t} \right]_{H^0(\Gamma)}
\]
\[
\leq C(M) C(\delta, \delta_1) \| \tilde{h}_t \|_{H^2(\Gamma)}^2 \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 + \delta \| v_{\kappa} \|_{H^2(\Omega)}^2 + \delta_1 \| v_{\kappa} \|_{H^2(\Omega)}^2
\]
REMARK 21. The bound for $K_7$ can be refined even further as
\[
| K_7 | \leq C(M) C(\delta) \| \tilde{h}_t \|_{H^2(\Gamma)}^2 \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 + \delta \| v_{\kappa} \|_{H^2(\Omega)}^2 + \delta_1 \| v_{\kappa} \|_{H^2(\Omega)}^2
\]

It remains to estimate $K_3$ to $K_6$. By proper use of Hölder’s inequality,
\[
| K_3 | + | K_5 | + | K_6 | \leq C \left[ 1 + \| \tilde{h}_t \|_{H^2(\Gamma)}^2 \right] \left[ 1 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 \right]
\]
\[
+ (\delta + C t^{1/2}) \| v_{\kappa} \|_{H^2(\Omega)}^2 + \delta \| v_{\kappa} \|_{H^2(\Omega)}^2
\]
for some constant $C$ depending on $M$ and $\delta$. For $K_4$, most of the terms can be estimated in the same fashion except the term

$$\int_{\Gamma} \frac{1}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} A^\alpha\beta\delta h_{\kappa t, \alpha\beta} \right] |(\nabla_0 \bar{h}_{t, \gamma\delta}, 0) \cdot (v_\kappa \circ \bar{\eta}^-)| dS$$

which is identical to

$$\int_{\Gamma} \left\{ \frac{1}{\sqrt{\det(g_0)}} \left[ \sqrt{\det(g_0)} A^\alpha\beta\delta h_{\kappa t, \alpha\beta} \right] |(\nabla_0 \bar{h}_{t, \gamma\delta}, 0) \cdot (v_\kappa \circ \bar{\eta}^-)| \right\} t dS \quad (\equiv K_8) + R_6$$

where

$$|R_6| \leq C \| \bar{h} \|_{H^2(\Gamma)}^2 \left[ \| v_\kappa \|_{L^2(\Omega)}^2 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Omega)}^2 \right] + \delta \| v_\kappa \|_{H^1(\Omega)}^2 + \delta_1 \| v_\kappa \|_{H^1(\Omega)}^2$$

for some constant $C$ depending on $M$, $\delta$ and $\delta_1$. Time integrating $K_8$ and use the interpolation inequality together with Young’s inequality, we find that

$$\left| \int_0^t K_8(s) ds \right| \leq C(M) \left[ \| u_0 \|_{H^2(\Omega)}^2 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Omega)} \| v_\kappa \|_{L^2(\Omega)} \right]$$

$$\leq C(M) C(\delta_1, \delta_2) N_3(u_0, F) + \delta_2 \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 + \delta_1 \int_0^t \| v_\kappa \|_{H^1(\Omega)}^2 ds \quad (C.3)$$

where

$$N_3(u_0, F) := \| u_0 \|_{H^2(\Omega)}^2 + \| u_0 \|_{H^1(\Omega)}^2 + \| F \|_{L^2(\Omega)}^2$$

and we use $\| v_\kappa \|_{H^1(\Omega)} \leq C \left[ \int_0^t \| v_\kappa \|_{H^1(\Omega)}^2 ds + \| u_0 \|_{H^1(\Omega)}^2 \right]$ to obtain (C.3) and hence

$$\sum_{i=3}^{6} \left| K_i \right| \leq C \left[ 1 + \| \bar{h} \|_{H^2(\Gamma)}^2 + \| \bar{h}_t \|_{H^2(\Gamma)}^2 \right] \left[ 1 + \| v_\kappa \|_{L^2(\Gamma)}^2 + \| \nabla_0^4 h_{\kappa t} \|_{L^2(\Gamma)}^2 \right]$$

$$+ (\delta + C t^{1/2}) \| v_\kappa \|_{H^1(\Omega)}^2 + \delta_1 \| v_\kappa \|_{H^1(\Omega)}^2 \quad (C.4)$$

with $K_8$ satisfying inequality (C.3). Finally, combining all the estimates,

$$\int_0^t \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 ds \leq \int_0^t \left[ \int_{\Gamma} \left[ \left( \frac{1}{\sqrt{\det(g_0)}} \sqrt{\det(g_0)} A^\alpha\beta\delta h_{\kappa t, \alpha\beta} \right) |(\nabla_0 \bar{h}_{t, \gamma\delta}, 0) \cdot (v_\kappa \circ \bar{\eta}^-)| \right] t dS + C N_3(u_0, F)$$

$$+ C \int_0^t K(s) \left[ \| v_\kappa \|_{L^2(\Omega)}^2 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 + \| \nabla_0^2 h_{\kappa t} \|_{L^2(\Gamma)}^2 \right] ds \quad (C.5)$$

$$+ (\delta + C t^{1/2}) \int_0^t \| v_\kappa \|_{H^1(\Omega)}^2 ds + (\delta_1 + C t^{1/2}) \int_0^t \| v_\kappa \|_{H^1(\Omega)}^2 ds + \delta_2 \| \nabla_0^4 h_{\kappa t} \|_{L^2(\Gamma)}^2$$

for some constant $C$ depending on $M$, $\delta$, $\delta_1$ and $\delta_2$.

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E-mail address: cchsiao@math.ucdavis.edu
E-mail address: coutand@math.ucdavis.edu
E-mail address: shkoller@math.ucdavis.edu

Department of Mathematics, University of California, Davis, CA 95616