Fermionic impurities in the unquenched ABJM

Georgios Itsios$^{1,3}$, Konstadinos Sfetsos$^{2,1}$, Dimitrios Zoakos$^3$

1 Department of Engineering Sciences, University of Patras, 26110 Patras, Greece
2 Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK
3 Centro de Física do Porto & Departamento de Física e Astronomia, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail: gitsios@upatras.gr, k.sfetsos@surrey.ac.uk, dimitrios.zoakos@fc.up.pt

ABSTRACT: We study, in a holographic setup, the effect of adding localized fermionic impurities to the three-dimensional Chern–Simons-matter theories with unquenched fields in the fundamental representation of the gauge group. The impurities are introduced as probe D6-branes extending along the radial direction and wrapping a five-dimensional submanifold inside a squashed $\mathbb{CP}^3$. We analyze the straight flux tube embeddings and study the corresponding fluctuation modes of the D6-branes. The conformal dimensions of the operators dual to such fluctuations depend non-trivially on the ratio of the flavor number to the Chern–Simon level of the unquenched ABJM.

KEYWORDS: Gauge-gravity correspondence
1 Introduction

One of the developments emerging from string theory explorations is the idea of a
gauge/gravity correspondence [1]. The remarkable feature of the correspondence is
the relation of the strongly coupled regime of the gauge theory to the weakly coupled
regime of the string theory and vice-versa. Consequently, it has become a powerful
tool in studying strongly interacting systems, allowing for novel computations that go
beyond the standard perturbative techniques of quantum field theories.
In this theme, the recent developments on the $AdS_4/CFT_3$ constitute a rich framework in which fundamental questions about the correspondence can be posted. In particular, the ABJM theory [2] is a $U(N) \times U(N)$ Chern–Simons gauge theory with levels $(k, -k)$ and bifundamental matter fields. In the large $N$ limit the theory acquires a supergravity description in terms of the $AdS_4 \times S^7/Z_k$ geometry. When the Chern–Simons level is large the size of the fiber is small and the system acquires a ten-dimensional description in terms of $AdS_4 \times \mathbb{CP}^3$ with fluxes, that preserves 24 supersymmetries. The addition of flavor to the ABJM theory (fields transforming in the fundamental representations $(N, 1)$ and $(1, N)$ of the $U(N) \times U(N)$ gauge group) is realized through D6-branes filling the $AdS_4$ space and wrapping a submanifold inside the $\mathbb{CP}^3$, while preserving a fraction of the initial supersymmetry [3]. The addition of a large number of flavor branes, continuously smeared in the transverse space, produces a backreaction of the original geometry and induces a deformation. Utilizing techniques developed in [4, 5] and reviewed in [6], this unquenched solution was computed in [7] and depends non-trivially on the number of flavors.

The addition of an extra set of branes interacting with the colored ones creates a defect in the gauge theory. The characteristic example of this class is $\mathcal{N} = 4$ Super-Yang-Mills (SYM) with fermionic impurities. The brane implementation of such a construction is through the addition of D5-branes into the $AdS_5 \times S^5$ background. The D5-branes extend along the radial direction and wrap a four-dimensional submanifold inside the five-dimensional sphere, with the corresponding worldvolume being $AdS_2 \times S^4$ [8] (for other embeddings with lower dimensional spheres see [9]). These configurations have been used recently in [10, 11] to holographically construct dimer models, through D-branes that connect impurities on the boundary of $AdS$. The holographic setup of D3- and D5-branes also realizes the maximally supersymmetric Kondo model [12–14] (see also [15, 16]).

In this paper we consider D6-branes extending along an $AdS_2 \subset AdS_4$ and a wrapping of a five-dimensional submanifold inside the squashed $\mathbb{CP}^3$, in order to construct the holographic dual of a Chern–Simons-matter theory with flavor and fermionic impurities. We will build our solution on the unquenched background solution [7] (for other type of related impurities using D8-branes see also [17]).
An overview of the paper is as follows: In section 2 we give a short, self contained, review of the gravity dual of a three-dimensional Chern–Simons-matter theory with unquenched fields in the fundamental representation of the gauge group. In section 3 we introduce impurities as probe D6-branes wrapping a five-dimensional submanifold inside the squashed $\mathbb{CP}^3$ and analyze straight flux tube embeddings. In section 4 we analyze in detail the fluctuation modes of the D6-branes around the straight flux tube configurations and compute the conformal dimensions of the operators dual to such fluctuations. Due to the presence of the unquenched ABJM there is an explicit dependence on the number of flavors. In section 5 we conclude and discuss lines of possible future related research. In the appendix A we analytically derive the Lagrangian for the fluctuations of the probe brane and in appendix B the explicit solution of the radial type differential eq. arising in these fluctuations. In appendix C we compute the spectrum of the Laplacian corresponding to the angular part of the operator entering into the fluctuation analysis.

2 Review of the ABJM with unquenched massless flavor

In this section, following [7], we will provide a self contained review of type-IIA supergravity solutions, dual to three-dimensional Chern–Simons-matter theories, after the addition of unquenched flavor. This is implemented through D6-branes that extend along the Minkowski directions and smear in the internal space, in a way that preserves $\mathcal{N} = 1$ supersymmetry. The geometry is $AdS_4 \times \mathcal{M}_6$, where $\mathcal{M}_6$ is the squashed Fubini–Study metric of $\mathbb{CP}^3$ [18] and the squashing factors depend on the number of flavors. The metric of the flavored ABJM background (in the string frame) is given by

\begin{equation}
\text{ds}^2 = L^2 \text{ds}_{AdS_4}^2 + \text{ds}_6^2,
\end{equation}

with the standard parametrization for the $AdS_4$ metric\textsuperscript{1}

\begin{equation}
\text{ds}_{AdS_4}^2 = r^2 \left( -dt^2 + dx^2 + dy^2 \right) + \frac{dr^2}{r^2},
\end{equation}

while the six-dimensional metric is written in terms of the $SU(2)$ instanton of $S^4$

\begin{equation}
\text{ds}_6^2 = \frac{L^2}{b^2} \left[ q \text{ds}_{S^4}^2 + \left( dx^i + \epsilon^{ijk} A_j x^k \right)^2 \right],
\end{equation}

\textsuperscript{1}We have rescaled the Minkowski coordinates as $x^\mu \rightarrow L^2 x^\mu$, while working in units $\alpha' = 1$. 

- 3 -
where $b$ and $q$ are constant squashing factors. The metric for the unit $S^4$ is denoted by $ds^2_{S^4}$ and $x^i$ ($i = 1, 2, 3$) are Cartesian coordinates parametrizing the unit $S^2$ whereas $A^i$, $i = 1, 2, 3$ are the components of the non-Abelian one-form connection corresponding to the $SU(2)$ instanton. The solution depends on two integers $N$ and $k$ which, on the gauge theory side, represent the rank of the gauge group and the Chern–Simons level, respectively. In string units, the $AdS_4$ radius $L$ can be written as

$$L^4 = 2\pi^2 \frac{N}{k} \frac{(2 - q) b^4}{q(q + \eta q - \eta)}.$$  

(2.4)

Introducing a set of $SU(2)$ left-invariant one-forms, which satisfy the usual relation $d\omega_i = \frac{1}{2} \epsilon_{ijk} \omega_j \wedge \omega_k$, together with the coordinate $\alpha$ ($0 \leq \alpha \leq \pi$) one parametrizes the unit $S^4$ as

$$ds^2_{S^4} = d\alpha^2 + \frac{\sin^2 \alpha}{4} \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right).$$  

(2.5)

The parameterization for the $\omega$’s is

$$\begin{align*}
\omega_1 &= \cos \psi \, d\theta_2 + \sin \psi \, \sin \theta_2 \, d\phi_2 \\
\omega_2 &= \sin \psi \, d\theta_2 - \cos \psi \, \sin \theta_2 \, d\phi_2 \\
\omega_3 &= d\psi + \cos \theta_2 \, d\phi_2,
\end{align*}$$  

(2.6)

while the $SU(2)$ instanton one-forms $A_i$ are given by

$$A_i = -\sin^2 \frac{\alpha}{2} \omega_i.$$  

(2.7)

Parametrizing the $x^i$ coordinates of the $S^2$ by means of the angles $\theta_1$ and $\phi_1$, the following relation holds$^2$

$$\left( dx^i + \epsilon^{ijk} A_j A_k \right)^2 = E_1^2 + E_2^2,$$  

(2.8)

where $E_1$ and $E_2$ are the following one-forms

$$\begin{align*}
E_1 &= d\theta_1 + \sin^2 \frac{\alpha}{2} \left( \omega_1 \sin \phi_1 - \omega_2 \cos \phi_1 \right) \\
E_2 &= \sin \theta_1 \left( d\phi_1 - \omega_3 \sin^2 \frac{\alpha}{2} \right) + \sin^2 \frac{\alpha}{2} \cos \theta_1 \left( \omega_1 \cos \phi_1 + \omega_2 \sin \phi_1 \right).
\end{align*}$$  

(2.9)$^2$

$^2$Explicitly we have $x^1 = \sin \theta_1 \cos \phi_1$, $x^2 = \sin \theta_1 \sin \phi_1$, $x^3 = \cos \theta_1$ with $0 \leq \theta_1 < \pi$ and $0 \leq \phi_1 < 2\pi$. For completeness we also note that $0 \leq \theta_2 < \pi$, $0 \leq \phi_2 < 2\pi$ and $0 \leq \psi < 4\pi$. 

- 4 -
Putting all these ingredients together we rewrite the six-dimensional metric (2.3) as
\[
ds_6^2 = \frac{L^2}{b^2} \left[ q ds_{S^4}^2 + E_1^2 + E_2^2 \right].
\]
(2.10)

In order to write the expression for the $F_2$ we will introduce a new set of one-forms
\[
S_1 = \sin \phi_1 \omega_1 - \cos \phi_1 \omega_2,
\]
\[
S_2 = \sin \theta_1 \omega_3 - \cos \theta_1 (\cos \phi_1 \omega_1 + \sin \phi_1 \omega_2),
\]
\[
S_3 = - \cos \theta_1 \omega_3 - \sin \theta_1 (\cos \phi_1 \omega_1 + \sin \phi_1 \omega_2).
\]
(2.11)

Then, the ansatz for the $F_2$ is the following
\[
F_2 = \frac{k}{2} \left[ E_1 \wedge E_2 - \eta (S_\alpha \wedge S_3 + S_1 \wedge S_2) \right],
\]
(2.12)

where the one-forms $S_\alpha$ and $S_i$ are
\[
S_\alpha = d\alpha, \quad S_i = \frac{\sin \alpha}{2} S_i, \quad i = 1, 2, 3
\]
(2.13)

and $\eta$ is a squashing parameter, directly related to the number $N_f$ of flavors as
\[
\eta = 1 + \frac{3N_f}{4k}, \quad 1 \leq \eta < \infty.
\]
(2.14)

The internal squashing $q$ is related to $\eta$ through the relation
\[
q = \frac{3(1+\eta) - \sqrt{9\eta^2 - 2\eta + 9}}{2},
\]
(2.15)

while $b$ can be written in terms of $q$ and $\eta$ as
\[
b = \frac{q(\eta + q)}{2(\eta q - \eta)}.
\]
(2.16)

The dilaton and the $F_4$ have the following expressions
\[
e^{-\Phi} = \frac{b \eta + q}{4} \frac{k}{2 - q \frac{L}{b}}, \quad F_4 = \frac{3k}{4} \frac{(\eta + q)b}{2 - q} L^2 \Omega_{AdS_4}.
\]
(2.17)

The value of $q$ ranges from 1 to $5/3$, while the case $q = 1$ (no flavors) corresponds to the $\mathcal{N} = 6$ ABJM background. We also note that had we taken the positive square root in (2.15) it would have corresponded to a different branch for which $q \geq 5$. For the value $q = 5$ the background has reduced supersymmetry, whose metric is the sum of the $AdS_4$ space with the squashed $\mathbb{CP}^3$. This metric is not the one corresponding to an Einstein space [19, 20]. This would have corresponded to the value $q = 2$, which nevertheless is not allowed in either branch.
3 The Hamiltonian density of D6-brane probes

In this section we will consider a D6-brane probe extending along the radial direction $r$ and wrapping a five-dimensional submanifold inside the squashed $\mathbb{CP}^3$ at constant values of the spatial Minkowski directions $x$ and $y$. The background coordinates $X^M$ and the worldvolume coordinates $\zeta^\mu$ are

\begin{align*}
\text{Background: } X^M &= (t, x, y, r, \alpha, \phi_2, \psi, \theta_1, \phi_1), \\
\text{Brane: } \zeta^\mu &= (t, r, \gamma^i) = (t, r, \theta_2, \phi_2, \psi, \theta_1, \phi_1).
\end{align*}

We consider embeddings in which the angle $\alpha$ depends only on the radial direction, i.e.

$$\alpha = \alpha(r).$$

These embeddings correspond to configurations in which the flux tube starts from the boundary of the $AdS_4$ and reaches the origin of the holographic coordinate.\footnote{That excludes hanging flux tubes, namely configurations that reach a minimal value of $r$ and return to the boundary. In such cases a non-constant Cartesian embedding coordinate is needed.} We will also turn on an electric worldvolume gauge field component $F_{0r}$, whose source is the RR potential $C_5$ through the Wess–Zumino (WZ) term of the D6-brane action.

The Dirac-Born-Infeld (DBI) part of the D6-brane action is given by

$$S_{DBI} = -T_6 \int d^7 \zeta e^{-\Phi} \sqrt{-\det(g + F)},$$

where $g$ is the induced metric on the worldvolume of the D6-brane and $T_6$ is the brane tension. After integrating over all the angles of the internal space we arrive at the following expression for the DBI contribution to the action

$$S_{DBI} = \int dt \, dr \, \mathcal{L}_{DBI},$$

with

$$\mathcal{L}_{DBI} = -\frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \sin^3 \alpha \sqrt{1 + \frac{q}{b^2} r^2 \alpha' - L^{-4} F_{0r}^2},$$

where $\alpha'$ denotes $d\alpha/dr$. The WZ part of the action is given by

$$\mathcal{L}_{WZ} = T_6 \int C_5 \wedge F \equiv \int dt \, dr \, \mathcal{L}_{WZ}.$$
The definition for the RR six-form is through the Hodge dual of the RR four-form, namely $\star F_4 = - F_6$, and in this way it is possible to obtain an expression for the five-form potential $C_5$

$$C_5 = -\frac{\pi^2}{8} NC(\alpha) \sin \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\psi , \quad (3.7)$$

where

$$C(\alpha) = \cos \alpha (\sin^2 \alpha + 2) - 2 . \quad (3.8)$$

After integrating over the above five angles $\gamma^i$ we obtain

$$L_{WZ} = -\frac{N}{8\pi} C(\alpha) F_{0r} . \quad (3.9)$$

The Lagrangian is the sum of the (3.5) and (3.9)

$$\mathcal{L} = -\frac{N}{8\pi} \frac{b}{\sqrt{q}} \left[ L^2 \sin^3 \alpha \sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2} - L^{-4} F_{0r}^2 + \frac{\sqrt{q}}{b} C(\alpha) F_{0r} \right] . \quad (3.10)$$

The equation of motion for the gauge field implies the following

$$\frac{\partial \mathcal{L}}{\partial F_{0r}} = \text{constant} . \quad (3.11)$$

This constant is related to the number $n$ of strings (quarks) of the flux tube, through the quantization condition of [8]

$$\frac{\partial \mathcal{L}}{\partial F_{0r}} = n T_f , \quad (3.12)$$

where $T_f$ is the tension and $n \in \mathbb{Z}$ is the charge of the fundamental string attached to the D6-brane. Exploiting the above quantization condition, we obtain

$$\frac{\sin^3 \alpha}{\sqrt{1 - L^{-4} F_{0r}^2 + \frac{4}{b^2} r^2 \alpha'^2}} = \frac{\sqrt{\sin^6 \alpha + C_n(\alpha)^2}}{\sqrt{1 + \frac{4}{b^2} r^2 \alpha'^2}} , \quad (3.13)$$

where we have defined

$$C_n(\alpha) \equiv \frac{\sqrt{q}}{b} \left( C(\alpha) + \frac{4n}{N} \right) . \quad (3.14)$$

Then from (3.13) we obtain for the field strength

$$F_{0r} = \frac{L^2 \sqrt{1 + \frac{4}{b^2} r^2 \alpha'^2}}{\sin^6 \alpha + C_n(\alpha)^2} C_n(\alpha) . \quad (3.15)$$
In order to eliminate the electric field from the equations of motion we compute the Hamiltonian of the system by performing a Legendre transformation in (3.10)

\[ \mathcal{H} = F_{0r} \frac{\partial \mathcal{L}}{\partial F_{0r}} - \mathcal{L}. \]  

Using the above results together with (3.13) we end up with the following formula for the Hamiltonian density

\[ \mathcal{H} = \frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \sqrt{1 + \frac{q}{b^2} \alpha'^2 \sqrt{\sin^6 \alpha + C_n(\alpha)^2}}. \]  

It remains to determine \( \alpha(r) \) by integrating the corresponding Euler-Lagrange equations. In the next subsection we will constrain our analysis to configurations with constant \( \alpha \). Embeddings with \( \alpha \) depending on the holographic coordinate are related to the baryon vertex of the ABJM theory and will be analyzed in future work (for the similar analysis in the \( \text{AdS}_5 \times S^5 \) case see [21, 22]).

3.1 Flux tube configurations

In this subsection we will calculate the energy density of the configurations with constant \( \alpha \). Such configurations must satisfy the condition

\[ \frac{\partial \mathcal{H}}{\partial \alpha} \bigg|_{\alpha'=0} = 0. \]  

Since

\[ \frac{\partial \mathcal{H}}{\partial \alpha} \bigg|_{\alpha'=0} = 3 \frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \frac{\sin^3 \alpha \Lambda_n(\alpha)}{\sqrt{\sin^6 \alpha + C_n(\alpha)^2}}, \]  

with

\[ \Lambda_n(\alpha) \equiv \sin^2 \alpha \cos \alpha - \frac{\sqrt{q}}{b} C_n(\alpha), \]  

the non-trivial configurations with constant \( \alpha \) are the solutions of the following algebraic equation

\[ \Lambda_n(\alpha) = 0, \]  

which can be written as a cubic equation in \( \cos \alpha \) as

\[ \left( 1 - \frac{q}{b^2} \right) \cos^3 \alpha - \left( 1 - \frac{3q}{b^2} \right) \cos \alpha - \frac{2q}{b^2} \left( 1 - \frac{2n}{N} \right) = 0. \]
Due to Bolzano’s theorem\textsuperscript{4} the function $\Lambda_n(\alpha)$ has at least one root in the interval $\alpha \in [0, \pi]$ for every $n$ in the range $0 \leq n \leq N$, while the monotonicity of the function in this interval tells us that the root is unique. After using (3.20) to express $C_n(\alpha_n)$ in terms of $\alpha_n$

$$C_n(\alpha_n) = \frac{b}{\sqrt{q}} \sin^2 \alpha_n \cos \alpha_n, \quad (3.24)$$

as well as (3.17), we obtain the energy density of the configurations with constant $\alpha$

$$E_n = \frac{NL^2}{8\pi} \frac{b^2}{q} \sin^2 \alpha_n \sqrt{\cos^2 \alpha_n + \frac{q}{b^2} \sin^2 \alpha_n}, \quad (3.25)$$

where $\alpha_n$ is a solution of (3.22).\textsuperscript{5} The constant electric field $F_{0r}$ corresponding to such configurations is computed from (3.15). One finds that

$$\bar{f}_{0r} = \frac{L^2 \cos \alpha_n}{\sqrt{\cos^2 \alpha_n + \frac{q}{b^2} \sin^2 \alpha_n}}. \quad (3.26)$$

Notice that from (3.22) we have

$$\Lambda_n(\alpha_n) = - \Lambda_{N-n}(\pi - \alpha_n) \quad \Rightarrow \quad \alpha_{N-n} = \pi - \alpha_n, \quad (3.27)$$

which combined with (3.25), is telling us that $E_n$ is invariant under the change $n \rightarrow N-n$, as it should if an object is to transform in the fully anti-symmetric representation of the gauge group, with $n$ being the number of boxes in the corresponding Young tableaux. The induced metric on the D6-brane worldvolume is

$$ds^2 = L^2 \left[ - r^2 dt^2 + \frac{dr^2}{r^2} + ds^2_{M_5} \right], \quad (3.28)$$

which is of the form $AdS_2 \times M_5$, with the line element of $M_5$ having the following expression

$$ds^2_{M_5} = g_{ij} d\gamma^i d\gamma^j = \frac{q}{4b^2} \sin^2 \alpha_n \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right) + \frac{1}{b^2} \left( E_1^2 + E_2^2 \right). \quad (3.29)$$

Under the change of $\alpha_n$ described in (3.27) the line element of (3.29) remains invariant as well.

\textsuperscript{4}Note that $q/b^2 \in \left[1, \frac{16}{75}\right]$, $\Lambda_n(\alpha)$ runs monotonically as $\alpha \in [0, \pi]$ and

$$\Lambda_n(0) = -4 \frac{q}{b^2} \frac{n}{N} \leq 0, \quad \Lambda_n(\pi) = 4 \frac{q}{b^2} \left(1 - \frac{n}{N}\right) \geq 0. \quad (3.23)$$

\textsuperscript{5}For $n \ll N$ the energy turns out to be simply the sum of the energies of the individual fundamental strings, i.e. $E_n \simeq \frac{n}{N} L^2$, where the extra factor $L^2$ is related to the overall appearance of the same factor in (2.1) due to a rescaling of the world-volume coordinates. It is worth noting that in this dilute type approximation there is no dependence of the energy on the flavor number.
4 Fluctuations of the impurities

Moving one step forward, we will study in this section fluctuations around the static configurations we have computed. For this reason we consider the following

\[\alpha = \alpha_n + \xi(\zeta), \quad F = \bar{f} + f(\zeta), \quad x = \bar{x} + \chi(\zeta),\]

(4.1)

where \(\alpha_n\) is a solution of the condition \(\Lambda_n(\alpha) = 0\), \(\bar{x}\) is the constant Cartesian coordinate of the unperturbed D6-brane and \(\bar{f}\) is the background gauge field strength with non-vanishing component given by (3.26). The fluctuations around these constant values, namely \(\xi, \chi\) and \(f\), depend, as indicated, on the D-brane coordinates \(\zeta^\mu\) in (3.1).

The total perturbed Lagrangian density is the sum of the DBI and the WZ parts, and a detailed derivation is presented in the appendix A. Indeed, if in (A.23) we neglect constant and total derivative terms we find the following Lagrangian density for the quadratic fluctuations

\[L = -T_6 \frac{\pi^2 N b^6 L^2}{q^2} P^{1/2} \sqrt{g} \left\{ \frac{1}{2} L^2 r^2 G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{1}{2} \frac{q}{b^2} L^2 G^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + \frac{1}{4} G^{\mu\rho} G^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} + V \xi^2 - W \xi f_{0r} \right\},\]

(4.2)

where we have for notational convenience set

\[P = \frac{\sin^6 \alpha_n}{\sin^6 \alpha_n + C_n(\alpha_n)^2}, \quad V = -\frac{3}{2 \sin^2 \alpha_n}, \quad W = \frac{3}{L^2 P^{1/2} \sin^3 \alpha_n} \left( C_n(\alpha_n) \cot \alpha_n + \sqrt{q} \frac{\sin^3 \alpha_n}{b} \right)\]

(4.3)

and the seven-dimensional metric \(G\) is defined in (A.7).

4.1 Fluctuation of the Cartesian coordinate

Here we study the fluctuations of the Cartesian coordinate which do not couple to those of the gauge field and of the embedding angular coordinate. The equation of motion for \(\chi\) can be easily derived from (4.2) and it is

\[\partial_r (r^4 \partial_r \chi) - \partial_0^2 \chi + r^2 P \nabla_{M_5}^2 \chi = 0.\]

(4.4)

To obtain the above equation of motion we explicitly used the components of \(G^{\mu\nu}\), while \(\nabla_{M_5}^2\) is the Laplacian operator of the five-dimensional manifold in (3.29) (see appendix
The actual expression for this operator is quite complicated. Remarkably, we were able to find explicit solutions of the form either $\chi = \chi(t, r, \theta_2, \phi_2, \psi)$ or $\chi = \chi(t, r, \theta_1, \phi_1)$ (see appendices C.1 and C.2, respectively). Without loss of generality in the following analysis of the conformal dimensions we will focus on the first class of solutions. Using the separation of variables

$$\chi = e^{\text{i}E t} R(r) \Omega(\theta_2, \phi_2, \psi),$$

(4.5)

and equation (C.5) from appendix C.1

$$\nabla^2_{M_5} \Omega = \frac{b^2}{q \sin^2 \alpha_n} \nabla_{S^3} \Omega = -\frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} \Omega, \quad l = 0, 1, 2, \ldots \quad (4.6)$$

the equation of motion for the radial function $R(r)$ becomes

$$\partial_r (r^4 \partial_r R) + \left( E^2 - \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} P r^2 \right) R = 0. \quad (4.7)$$

This equation can be solved exactly, but since we are interested in the asymptotic behavior of the solution we put all the details on the analytic derivation in appendix B. Assuming that $R(r) \sim r^\lambda$ at large $r$, we arrive to the following quadratic equation

$$\lambda(\lambda + 3) = \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} P, \quad (4.8)$$

with solutions that we will denote as $\lambda_1$ and $\lambda_2$. We would like to associate them with the dimensions $\Delta$ of the operators of the defect theory. The fluctuations are not canonically normalized since there is a factor of $r^2$ in front of the kinetic term for the field $\chi$ in (4.2). Hence, we cannot simply use the usual relation [24]

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}, \quad (4.9)$$

with $d = 1$. Instead one may employ an approach that gives the result immediately [25]. According to this, if a scalar field in $AdS_2$ at large $r$ behaves as

$$\chi \sim d_1 r^{-2 \lambda_1} + d_2 r^{-2 \lambda_2}, \quad \lambda_2 > \lambda_1, \quad (4.10)$$

then the dimension of the operator dual to the normalizable mode is

$$\Delta = \frac{1}{2} + \lambda_2 - \lambda_1. \quad (4.11)$$
In our case the conformal dimension becomes
\[ \Delta = \frac{1}{2} + \sqrt{\frac{9}{4} + \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n}} P = \frac{1}{2} + \sqrt{\frac{9}{4} + \frac{l(l+2)}{q/b^2 \sin^2 \alpha_n + \cos^2 \alpha_n}}. \] (4.12)

In general, the conformal dimension is not a rational number and \( \Delta = 2 \) for \( l = 0 \). Moreover the dimension depends on the filling fraction \( \nu \), through the dependence of the angle \( \alpha_n \), and is invariant under the change of \( (3.27) \). Due to the range of the parameter \( q/b^2 \), defined in footnote 2, the prefactor multiplying \( l(l+2) \) is of order one.

The dependence of the conformal dimension on the number of flavors becomes in general complicated and it admits a power series expansion around the unquenched result. In particular, in the half filling fraction case, \( \nu = \frac{1}{2} \) the cubic equation \( (3.22) \) has the following solution in the interval \( \alpha_n \in [0, \pi] \)
\[ \cos \alpha_n = 0 \Rightarrow \alpha_n = \frac{\pi}{2}. \] (4.13)

Expanding (4.12) around the unquenched result we have
\[ \Delta = \frac{1}{2} + \sqrt{\frac{9}{4} + l(l+2)} - \frac{9}{512} \frac{l(l+2)}{\sqrt{\frac{9}{4} + l(l+2)}} \left( \frac{N_f}{k} \right)^2 + O \left( \frac{N_f}{k} \right)^3. \] (4.14)

### 4.2 Coupled modes

In this subsection we will focus our attention on the fluctuations of the gauge and scalar fields, which through their equations of motion appear to be coupled. The equation of motion for the gauge field is given by
\[ \frac{1}{\sqrt{\tilde{g}}} \partial_\mu \left( \sqrt{\tilde{g}} G^{\mu\nu} f_{\nu\sigma} \right) + W ( \partial_\mu \delta_0^\sigma - \partial_0 \delta_\mu^\sigma ) = 0, \] (4.15)
while that for the scalar is
\[ \frac{1}{\sqrt{\tilde{g}}} \partial_\mu \left( \sqrt{\tilde{g}} G^{\mu\nu} \partial_\nu \xi \right) - \frac{2}{L^2} \frac{b^2}{q} \frac{1}{L^2} W f_{0r} = 0. \] (4.16)

We consider the following ansatz for the fluctuations of the gauge fields and the scalar (the only non-vanishing components of the gauge field are \( \hat{A}_r \) and \( \hat{A}_i \))
\[ \hat{A}_r = e^{iEt} \Omega(\theta_2, \phi_2, \psi) \phi(r), \quad \hat{A}_i = e^{iEt} \partial_i \Omega(\theta_2, \phi_2, \psi) \phi(r) \]
\[ \xi_r = e^{iEt} \Omega(\theta_2, \phi_2, \psi) z(r). \] (4.17)

---

6Another way to arrive at the same result is to redefine the fluctuations by absorbing the factor \( r \) in the kinetic for \( \chi \) in (4.2) into a new scalar field \( \phi = r \chi \). Then \( \phi \) becomes canonically normalized but after some algebraic manipulations one sees that \( m^2 \) is shifted by a factor of 2. Then after using (4.9), with \( d = 1 \), one derives (4.12).
Note that the fact that the vector index of $\hat{A}_i$ is due to the derivative on $\Omega$, turns out after substitution into the equations of motion. Then, from (4.15) for $\sigma = 0$ we obtain
\[
\frac{b^2 l(l + 2)}{q \sin^2 \alpha_n} \tilde{\phi} = \frac{r^2}{P} \phi' - \frac{i}{E} L^4 P W r^2 z',
\]
while if we set $\sigma = r$ in the same equation we have that
\[
\frac{E^2}{L^4 P^2} \tilde{\phi} + \frac{b^2 l(l + 2)}{q \sin^2 \alpha_n} \frac{r^2}{L^4 P} \left( \phi' - \phi \right) - i E W z = 0.
\]
Also from (4.15) for $\sigma = i$ we have
\[
E^2 \tilde{\phi} = -r^2 \frac{d}{dr} \left[ r^2 \left( \phi' - \phi \right) \right],
\]
an equation that can be easily derived from (4.18) and (4.19). Using (4.18) in order to eliminate $\tilde{\phi}$ from (4.19), we have
\[
\frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) + \frac{E^2}{r^2} \phi - \frac{b^2 l(l + 2)}{q \sin^2 \alpha_n} P \phi - \frac{i}{E} L^4 P^2 W \left\{ \frac{d}{dr} \left( r^2 \frac{dz}{dr} \right) + \frac{E^2}{r^2} z \right\} = 0,
\]
while the equation of motion for the scalar, using (4.17), becomes
\[
\frac{d}{dr} \left( r^2 \frac{dz}{dr} \right) + \frac{E^2}{r^2} z - \frac{b^2}{q} \left( 2V + \frac{l(l + 2)}{\sin^2 \alpha_n} \right) P z + \frac{i b^2}{q} E P W \phi = 0.
\]
At this point, we define the differential operator $\hat{O}$, which acts on functions as follows
\[
\hat{O}f = \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) + \frac{E^2}{r^2} f
\]
and make the following field redefinitions
\[
\tilde{z} = \frac{L^4}{E} P^2 W z, \quad \eta = \phi - \frac{i}{E} \frac{L^4}{P^2} W z = \phi - \tilde{z}.
\]
Then (4.21) and (4.22) can be written in a more compact form as
\[
\left( \hat{O} \mathbb{I}_{2 \times 2} - \mathcal{M} \right) \begin{pmatrix} \tilde{z} \\ \eta \end{pmatrix} = 0,
\]
where the entries of the matrix $\mathcal{M}$ are
\[
\mathcal{M}_{11} = \frac{b^2}{q} \left( 2V + L^4 W^2 P^2 + \frac{l(l + 2)}{\sin^2 \alpha_n} \right) P,
\]
\[
\mathcal{M}_{12} = \frac{b^2}{q} L^4 W^2 P^3, \quad \mathcal{M}_{21} = \mathcal{M}_{22} = \frac{b^2 l(l + 2)}{q \sin^2 \alpha_n} P,
\]
while its eigenvalues are

$$\lambda_{\pm} = \frac{P b^2}{q} \left[ 2V + L^4 P^2 W^2 + 2 \frac{l(l+2)}{\sin^2 \alpha_n} \pm \sqrt{4 \frac{l(l+2)}{\sin^2 \alpha_n} L^4 P^2 W^2 + (2V + L^4 P^2 W^2)^2} \right].$$

(4.27)

Now, if $\psi_{\pm}$ are the eigenvectors of $M$ with eigenvalues $\lambda_{\pm}$, then (4.25) takes the form

$$\frac{d}{dr} \left( r^2 \frac{d\psi_{\pm}}{dr} \right) + \left( \frac{E^2}{r^2} - \lambda_{\pm} \right) \psi_{\pm} = 0,$$

(4.28)

and in order to study the behavior of $\psi_{\pm}$ at large $r$ we assume that $\psi_{\pm} \sim r^s$. Then from the above differential equation we find that $s$ should obey the following quadratic equation

$$s(s+1) - \lambda_{\pm} = 0,$$

(4.29)

with solutions

$$s_{\pm} = \frac{-1 \pm \sqrt{1 + 4 \lambda_{\pm}}}{2}, \quad s_- < s_+.$$

(4.30)

Noting that in the kinetic term in (4.2) for these type of fluctuations there is no extra overall factor of $r$, we may safely use for the conformal dimension the expression [24] with $d = 1$ and $m^2 = \lambda_+$, i.e.

$$\Delta = \frac{1}{2} \left( 1 + \sqrt{1 + 4 \lambda_+} \right).$$

(4.31)

In general, it is not a rational number and depends on the filling fraction $\nu$. Notice, that unlike (4.12), for $l = 0$ the conformal dimension $\Delta_+$ does depend on the flavor number.

As also argued in the appendix C, for angular dependence of the form $\Omega(\theta_1, \phi_1)$ we may use the previous results by simply performing the replacement (C.8). In the half filling fraction case $\nu = \frac{1}{2}$ and expanding (4.31) around the unquenched result we obtain that

$$\Delta = 3 + l - \frac{9}{512} \frac{(l+3)(l+2)(2l-1)}{(l+1)(5+2l)} \left( \frac{N_f}{k} \right)^2 + \mathcal{O} \left( \frac{N_f}{k} \right)^3$$

(4.32)

5 Conclusions and future directions

In this paper we studied localized fermionic impurities to the unquenched ABJM Chern–Simons-matter theory, which are realized through the addition of fields transforming in the fundamental representations $(N, 1)$ and $(1, N)$ of the $U(N) \times U(N)$ gauge
group. In the holographic approach the impurities are added by introducing probe D6-branes, extending along the holographic coordinate and wrapping a five-dimensional submanifold inside a squashed $\mathbb{CP}^3$ at constant values of the Minkowski directions. The background RR field induces an electric gauge-field on the world-volume of the probe branes, giving rise to a bundle of strings that form a flux tube which prevents the collapse of the wrapped brane.

We concentrated on the simplest of the configurations in which the flux tube starts from the boundary of the $AdS_4$ and reach the origin of the holographic coordinate. More general solutions including the baryon vertex of the unquenched ABJM theory and hanging flux tubes, namely configurations that reach a minimal value in the holographic coordinate and return to the boundary, are left as open problems for future work.

The natural step forward was the investigation of the stability for the probe D6-branes, that introduce the holographic impurities. We presented an analytic study for the fluctuations of those probes in the unquenched ABJM. The fluctuations are separated in two categories. The first contains just the decoupled fluctuations of the Cartesian coordinates while the second the coupled fluctuations of the angular embedding function and the world-volume gauge field. The coupled modes were shown to decouple by appropriate field redefinitions. In this way we were able to determine the spectrum of conformal dimensions of the dual operators in the defect theory. The novel feature of our analysis is that using the unquenched ABJM background we obtained expressions of the conformal dimension that explicitly depend on the number of fundamental flavors, thus generalizing the previously obtained results for the ABJM background [11].

There are many interesting questions that follow from the analysis we presented in the recent paper, and we would like to pursue some of them in the near future. In the quenched ABJM background there is the robust proposition of [23] that D6-branes on $AdS_2 \times \mathcal{M}_5$, where $\mathcal{M}_5$ is a five-dimensional submanifold of $\mathbb{CP}^3$, holographically parametrize the Wilson lines in the antisymmetric representation of the gauge group. In turn these are natural candidates for the construction of a gravity dual for an ABJM theory with fermionic impurities. Contrary to the quenched case, there is no such proof for the unquenched ABJM, though we note the invariance under $n \to N - n$ in (3.27). It would be very interesting to pursue this issue further.
The addition of fermionic impurities in the unquenched ABJM background at finite temperature [26] will create a much richer structure. The analysis of the thermodynamic properties, of both straight and hanging flux tubes, is expected to unveil a competition between the two configurations. This in turn will lead to the existence of a dimerization transition similar to the one presented in [10, 11], but now in a background that will include the non-trivial presence of fundamental flavors.

Acknowledgments

We are grateful to A. Ramallo for sharing with us his unpublished notes and for valuable comments. The research of G. Itsios has been co-financed by the ESF and Greek national funds through the Operational Program “Education and Lifelong Learning” of the NSRF - Research Funding Program: “Heracleitus II. Investing in knowledge in society through the European Social Fund”. He would also like to thank the CFP of University of Porto for hospitality within the framework of the LLP/Erasmus Placements 2011-2012. D. Z. is funded by the FCT fellowship SFRH/BPD/62888/2009. Centro de Física do Porto is partially funded by FCT through the projects PTDC/FIS/099293/2008 and CERN/FP/116358/2010. This research is implemented under the “ARISTEIA” action of the “operational programme education and lifelong learning” and is co-funded by the European Social Fund (ESF) and National Resources.

A Fluctuation analysis

In this appendix we will analyze the small perturbations around the flux tube configurations, derived in section 3.1. We will analytically obtain the second order lagrangian for those fluctuations, which is the starting point of section 4.

We perturb a D6-brane probe as in (4.1) and expand the DBI+WZ action to second order in the perturbations $\xi$, $f$ and $\chi$. Starting with the components of the perturbed induced metric we write

$$g = \bar{g} + \hat{g},$$

where $\bar{g}$ is the zeroth order induced metric and the perturbation $\hat{g}$ has the form

$$\hat{g}_{\mu\nu} = L^2 \left[ r^2 \partial_\mu \chi \partial_\nu \chi + \frac{q}{b^2} \partial_\mu \xi \partial_\nu \xi + \xi \hat{g}^{(1)}_{\mu\nu} + \xi^2 \hat{g}^{(2)}_{\mu\nu} \right] + \ldots,$$
where the \( \tilde{g}_{ij}^{(1)} \) and \( \tilde{g}_{ij}^{(2)} \) are given by (their indices take values only in the angular part)

\[
\tilde{g}_{ij}^{(1)} d\gamma^i d\gamma^j = \frac{q}{4b^2} \sin 2\alpha_n \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right) + \frac{2}{b^2} \left[ E_1 \frac{\partial E_1}{\partial \alpha} + E_2 \frac{\partial E_2}{\partial \alpha} \right] \bigg|_{\alpha=\alpha_n},
\]

\[
\tilde{g}_{ij}^{(2)} d\gamma^i d\gamma^j = \frac{g}{4b^2} \cos 2\alpha_n \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right) + \frac{1}{b^2} \left[ \left( \frac{\partial E_1}{\partial \alpha} \right)^2 + \left( \frac{\partial E_2}{\partial \alpha} \right)^2 + E_1 \frac{\partial^2 E_1}{\partial \alpha^2} + E_2 \frac{\partial^2 E_2}{\partial \alpha^2} \right] \bigg|_{\alpha=\alpha_n}.
\]

The determinant of the DBI part can be written as

\[
\det(g + F) = \det(\tilde{g} + \tilde{f}) \det(\mathbb{I} + X), \quad X = (\tilde{g} + \tilde{f})^{-1} (\tilde{g} + f).
\]

Hence the important step is to compute the components of matrix \( X \) in the expansion

\[
\sqrt{\det(\mathbb{I} + X)} = 1 + \frac{1}{2} \text{Tr}X - \frac{1}{4} \text{Tr}X^2 + \frac{1}{8} (\text{Tr}X)^2 + \mathcal{O}(X^3).
\]

The matrix \( (\tilde{g} + \tilde{f})^{-1} \) can be written in a block diagonal form

\[
(\tilde{g} + \tilde{f})^{-1} = \begin{pmatrix} G^{-1} + \mathcal{J}_{0r} \bigg|_{0} & 0 \\ 0 & G^{ij} \end{pmatrix},
\]

where \( G^{-1} \) and \( \mathcal{J} \) are its symmetric and antisymmetric parts, respectively. The non-zero elements of those matrices are

\[
G^{00} = -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n}, \quad G^{rr} = \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{L^2 \sin^6 \alpha_n}, \quad G^{ij} = L^{-2} \tilde{g}^{ij}
\]

and

\[
\mathcal{J}^{0r} = -\mathcal{J}^{r0} = \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{L^2 \sin^6 \alpha_n}.
\]

The matrix elements that contribute to the \( \text{Tr}X \) are

\[
X_0^0 = -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 \sin^6 \alpha_n} \left\{ r^2 (\partial_0 \chi)^2 + \frac{q}{b^2} (\partial_0 \xi)^2 \right\},
\]

\[
+ \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \left\{ r^2 \partial_0 \chi \partial_\tau \chi + \frac{q}{b^2} \partial_0 \xi \partial_\tau \xi - L^{-2} f_{0r} \right\},
\]

\[
X_r^r = \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} r^2 \left\{ r^2 (\partial_\tau \chi)^2 + \frac{q}{b^2} (\partial_\tau \xi)^2 \right\},
\]

\[
- \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \left\{ r^2 \partial_0 \chi \partial_\tau \chi + \frac{q}{b^2} \partial_0 \xi \partial_\tau \xi + L^{-2} f_{0r} \right\},
\]

\[
X_j^i = r^2 \tilde{g}^{ik} \partial_k \chi \partial_j \chi + \frac{q}{b^2} \tilde{g}^{ik} \partial_k \xi \partial_j \xi + \xi (M^{(1)})^i_j + \xi^2 (M^{(2)})^i_j + L^{-2} \tilde{g}^{ik} f_{kj}.
\]
where we have defined the matrices

\[
(M^{(i)})^i_j = \tilde{g}^{ik} \hat{g}^{(i)}_{kj}, \quad i = 1, 2.
\]  

(A.10)

The metric \(\tilde{g}_{ij}\) corresponds to the five-dimensional space whose line element is given by (3.29). In order to calculate the trace of \(X^2\), we need to compute the non-diagonal elements of \(X\) up to first order in the fluctuations. The matrix elements that contribute to \(\text{Tr} X^2\) are

\[
X^0_r = -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n} f_{0r},
\]

\[
X^i_0 = -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n} f_{0i} + \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{L^2 \sin^6 \alpha_n} f_{ri},
\]

(A.11)

\[
X^i_r = L^{-2} \tilde{g}^{ij} f_{jr},
\]

\[
X^r_i = L^{-2} \tilde{g}^{ij} f_{j0}.
\]

Putting everything together we find that the \(\text{Tr} X\) is given by

\[
\text{Tr} X = -2 \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r} + 6 \cot \alpha_n \xi + \frac{1}{q \sin^2 \alpha_n} \xi^2
\]

\[
+ L^2 r L^2 G^{\mu \nu} \partial_{\mu} \chi \partial_{\nu} \chi + \frac{q}{b^2} L^2 G^{\mu \nu} \partial_{\mu} \xi \partial_{\nu} \xi,
\]

(A.12)

where we have used the following equations

\[
\text{Tr} M^{(1)} = 6 \cot \alpha_n, \quad \text{Tr} M^{(2)} = \frac{1 + (3q - 1) \cos 2 \alpha_n}{q \sin^2 \alpha_n}.
\]

(A.13)

The \(\text{Tr}(X^2)\) is given by

\[
\text{Tr}(X^2) = 2L^{-4} \frac{\sin^6 \alpha_n + 2C_n(\alpha_n)^2}{\sin^6 \alpha_n} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} f_{0r}^2 + 2\xi^2 \frac{1 + 3q + (3q - 1) \cos 2 \alpha_n}{q \sin^2 \alpha_n}
\]

\[
+ L^{-4} \left\{ \frac{2}{r^2} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} f_{0r}^2 - \frac{2}{r^2} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} f_{ri}^2 - f_{kj}^2 \right\},
\]

(A.14)
while for the \((\text{Tr} X)^2\) we have

\[
(\text{Tr} X)^2 = 4L^{-4} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^2 \alpha_n} C_n(\alpha_n)^2 f_{0r}^2 + 36 \cot^2 \alpha_n \xi^2
- 24L^{-2} \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \cot \alpha_n \xi f_{0r}.
\]

Putting everything together we calculate the expression of \(\sqrt{\det(1 + X)}\)

\[
\sqrt{\det(1 + X)} = 1 - \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r}^2 + 3 \cot \alpha_n \xi + \frac{1}{2} L^2 r^2 G^{\mu \nu} \partial_\mu \chi \partial_\nu \chi
+ \frac{1}{2} \frac{q}{b^2} L^2 G^{\mu \nu} \partial_\mu \xi \partial_\nu \xi + \frac{3}{2} (2 \cot^2 \alpha_n - 1) \xi^2 + \frac{1}{4} G^{\mu \nu} G^{\nu \sigma} f_{\mu \nu} f_{\rho \sigma}
- 3L^{-2} \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \cot \alpha_n \xi f_{0r} + \ldots.
\]

Since

\[
\det(\bar{g} + \vec{f}) = - \frac{L^{14} \sin^6 \alpha_n}{\sin^6 \alpha_n + C_n(\alpha_n)^2} \cdot \det \bar{g},
\]

the DBI part of the Lagrangian density is

\[
\mathcal{L}_{DBI} = - T_6 \frac{\pi^2 N b^6 L^2}{q^2} \frac{\sin^3 \alpha_n}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \sqrt{\bar{g}} \sqrt{\det(1 + X)},
\]

where \(\bar{g}\) is the determinant of \((3.29)\)

\[
\sqrt{\bar{g}} = \frac{g^{3/2}}{8 b^2} \sin^3 \alpha_n \sin \theta_1 \sin \theta_2.
\]

What remains is the computation of the WZ part. Using the conventions of section 3 we have

\[
\mathcal{L}_{WZ} = - T_6 \frac{\pi^2 N}{8} \sin \theta_1 \sin \theta_2 C(\alpha) F_{0r},
\]

where \(F_{0r} = \bar{f}_{0r} + f_{0r}\) and the function \(C(\alpha)\) has to be expanded around \(\alpha_n\)

\[
C(\alpha) = C(\alpha_n) - 3 \sin^3 \alpha_n \xi - \frac{9}{2} \sin^2 \alpha_n \cos \alpha_n \xi^2 + \ldots.
\]

Putting everything together, the WZ part becomes

\[
\mathcal{L}_{WZ} = - T_6 \frac{\pi^2 N}{8} \sin \theta_1 \sin \theta_2 \left\{ \frac{L^2 C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} C_n(\alpha_n) + C_n(\alpha_n) f_{0r}ight.
- \frac{3L^2 \sin^3 \alpha_n C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \xi - 3 \sin^3 \alpha_n f_{0r} \xi - \frac{9}{2} \frac{L^2 \sin^2 \alpha_n \cos \alpha_n C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \xi^2 \left\} + \ldots.
\]
Finally, summing the DBI and WZ parts we obtain the result
\[
\mathcal{L}_{DBI} + \mathcal{L}_{WZ} = -T_0 \frac{\pi^2 N b^4 L^2}{q^2} \frac{\sin^3 \alpha_n}{\sin^6 \alpha_n + C_n(\alpha_n)^2} \sqrt{g} \left\{ 1 + \frac{\sqrt{q} C_n(\alpha_n)}{b \sin^6 \alpha_n} C(\alpha_n) \right\}
\]
\[+ \left( \frac{\sqrt{q}}{b} C(\alpha_n) - C_n(\alpha_n) \right) \frac{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r} \quad (A.23)
\]
\[+ 3 \left( \cot \alpha_n - \frac{\sqrt{q} C_n(\alpha_n)}{b \sin^3 \alpha_n} \right) \xi + \frac{1}{2} L^2 r^2 G^\mu\nu \partial_\mu \chi \partial_\nu \chi + \frac{1}{2 b^2} L^2 G^{\mu\nu} \partial_\mu \xi \partial_\nu \xi
\]
\[+ \left( \frac{3}{2} (2 \cot^2 \alpha_n - 1) - \frac{9}{2} \frac{\sqrt{q} \cos \alpha_n}{b \sin^3 \alpha_n} C_n(\alpha_n) \right) \xi^2 + \frac{1}{4} G^\mu\rho G^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma}
\]
\[- 3 L^{-2} \frac{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)}}{\sin^6 \alpha_n} \left( \frac{\sqrt{q}}{b} \sin^3 \alpha_n + C_n(\alpha_n) \cot \alpha_n \right) \xi f_{0r} \right\}.
\]

In the above action the first two lines can be dropped containing either constant or linear or total derivative terms. The term linear in \(\xi\) also vanishes upon using the condition (3.21). The remaining terms constitute the action (4.2) used in the main text.

**B Solution of the differential equation for \(R(r)\)**

The differential equation (4.7) is of the form
\[
\frac{d}{dr} \left( r^4 \frac{dR}{dr} \right) + (E^2 - Ar^2) R = 0,
\]
where \(A\) is a constant. To solve this differential equation we first study the asymptotic behavior of \(R(r)\) at large \(r\). Setting \(R(r) \sim r^\lambda\) we end up with the condition
\[
\lambda(\lambda + 3) = A \quad \implies \quad \lambda_\pm = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + A}.
\]

Now that we know the asymptotic behavior of \(R(r)\), we make the ansatz \(R(r) = r^\lambda f(\frac{1}{r})\) and obtain the following differential equation for the function \(f = f(u) = f(1/r)\)
\[
u^2 \frac{d^2 f}{du^2} - 2(\lambda + 1) u \frac{df}{du} + E^2 u^2 f = 0.
\]
(B.3)

Now we observe that if we substitute \(f(u) = u^{\lambda+\frac{2}{2}} g(u)\) the above equation becomes
\[
u^2 \frac{d^2 g}{du^2} + u \frac{dg}{du} + (E^2 u^2 - t^2) g = 0,
\]
(B.4)
where \( t^2 = (\lambda + \frac{3}{2})^2 = \frac{9}{4} + A \), which is the Bessel differential equation with solution

\[
g(u) = C_1 J_{\frac{3}{2} + A}(Eu) + C_2 Y_{\frac{3}{2} + A}(Eu) .
\] (B.5)

Finally, the solution for \( R(r) \) is

\[
R(r) = r^{-3/2} \left( C_1 J_{\frac{3}{2} + A} \left( \frac{E}{r} \right) + C_2 Y_{\frac{3}{2} + A} \left( \frac{E}{r} \right) \right) ,
\] (B.6)

which has the correct asymptotic behavior at large \( r \)

\[
R(r) = C_1 r^{-\frac{3}{2} + \frac{2}{4} + A} + C_2 r^{-\frac{3}{2} - \frac{2}{4} + A} .
\] (B.7)

In our case the constant \( A \) appearing in (B.1) is

\[
A = \frac{b^2}{q} \left( l(l + 2) \right) P .
\] (B.8)

C The Laplacian on the five dimensional manifold \( M_5 \)

The action of the Laplacian of the five-dimensional manifold \( M_5 \), with metric \( \tilde{g}_{ij} \) (3.29) on a scalar depending on all the coordinates of \( M_5 \) is

\[
\nabla^2_{M_5} f = \frac{c_1}{4 \sin^2 \alpha_n} \nabla^2_{S^3} f + \frac{c_1 c_2}{2} \sin \theta_1 \left( \frac{1}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \theta_1} f \right) + \frac{1}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1^2} f \right) \\
- \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \theta_2} f \right) - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_1} f \right) \\
- \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \phi_2} f \right) - \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_2} \frac{\partial}{\partial \phi_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_1} f \right) \\
+ \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_2}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \psi} f \right) + c_1 \cot \theta_2 \frac{\partial}{\partial \phi_2} \left( \cos(\phi_1 - \psi) \frac{\partial}{\partial \theta_1} f \right) \\
- c_1 \cot \theta_1 \frac{\partial}{\partial \phi_1} \left( \cos(\phi_1 - \psi) \frac{\partial}{\partial \theta_2} f \right) - \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \phi_1} f \right) \\
+ c_1 \frac{\cot \theta_1}{\sin \theta_2} \frac{\partial}{\partial \theta_1} \left( \sin(\phi_1 - \psi) \frac{\partial}{\partial \phi_2} f \right) + c_1 \frac{\cot \theta_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \frac{\partial}{\partial \phi_2} \frac{\partial}{\partial \theta_1} f \\
+ \frac{c_1}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \phi_1} \left( (\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \frac{\partial}{\partial \psi} f \right) \\
+ \frac{c_1}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \psi} \left( (\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \frac{\partial}{\partial \phi_1} f \right)
\] (C.1)
where the last term is the Laplacian of the 3-sphere
\[ \nabla_{S^3}^2 f = 4 \left\{ \frac{1}{\sin \theta_2} \partial_{\theta_2} \left( \sin \theta_2 \partial_{\theta_2} f \right) + \left( \frac{1}{\sin \theta_2} \partial_{\phi_2} - \cot \theta_2 \partial_{\psi} \right)^2 f + \partial_{\psi}^2 f \right\} , \tag{C.2} \]
and the constants \( c_1, c_2 \) are given by
\[ c_1 = b^2 q \frac{1}{\cos^2 \frac{\alpha_n}{2}} , \quad c_2 = 1 + q + (q - 1) \cos \alpha_n . \tag{C.3} \]

Solving the corresponding eigenvalue problem in full generality seems very difficult. Instead, we look for configurations of the function \( f \), in which the Laplacian acts on, that do not depend on all variables. We found two consistent truncations that simplify our eigenvalue problem and we present them in the following subsections.

**C.1 Solutions of the form \( f = \Omega(\theta_2, \phi_2, \psi) \)**

In the analysis that we presented in the previous sections of this paper we focused on configurations of the form \( f = \Omega(\theta_2, \phi_2, \psi) \). For such configurations the above expression for the Laplacian becomes
\[ \nabla_{M_5}^2 \Omega = \frac{c_1}{4 \sin^2 \frac{\alpha_n}{2}} \nabla_{S^3}^2 \Omega - \frac{c_1 \sin(\phi_1 - \psi) }{\sin \theta_1} \partial_{\theta_1} \left( \sin \theta_1 \partial_{\theta_2} \Omega \right) \]
\[ - \frac{c_1 \cos(\phi_1 - \psi) }{\sin \theta_1 \sin \theta_2} \partial_{\theta_1} \left( \sin \theta_1 \partial_{\phi_2} \Omega \right) + \frac{c_1 \cos(\phi_1 - \psi) }{\sin \theta_1} \cot \theta_2 \partial_{\theta_1} \left( \sin \theta_1 \partial_{\psi} \Omega \right) \]
\[ - c_1 \cot \theta_1 \partial_{\phi_1} \left( \cos(\phi_1 - \psi) \partial_{\phi_2} \Omega \right) + c_1 \cot \theta_1 \sin \theta_2 \partial_{\phi_1} \left( \sin(\phi_1 - \psi) \partial_{\phi_2} \Omega \right) \]
\[ + \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\phi_1} \left( \left( \sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi) \right) \partial_{\psi} \Omega \right) . \tag{C.4} \]

Notice that there are still derivatives to be taken with respect to the angles \( \theta_1 \) and \( \phi_1 \) as well as explicit dependence on these angles. Consistency of our truncation requires that all such dependencies drop out completely which indeed turns out to be the case as we are left with
\[ \nabla_{M_5}^2 \Omega = \frac{c_1}{4 \sin^2 \frac{\alpha_n}{2}} \nabla_{S^3}^2 \Omega . \tag{C.5} \]

In this way our eigenvalue problem is reduced to that of \( \nabla_{S^3}^2 \), a well known operator, which has eigenvalues \( \lambda = -l(l + 2) \) where \( l = 0, 1, 2, \ldots \)
C.2 Solutions of the form \( f = \Omega(\theta_1, \phi_1) \)

Another truncation that simplifies the eigenvalue problem is an ansatz of the form \( f = \Omega(\theta_1, \phi_1) \). Then (C.1) simplifies as

\[
\nabla^2_{M_5} \Omega = \frac{c_1c_2}{2} \frac{1}{\sin \theta_1} \left( \sin \theta_1 \frac{1}{\sin \theta_1} \partial^2_{\theta_1} \Omega \right) + \frac{c_1c_2}{2} \frac{1}{\sin^2 \theta_1} \partial^2_{\phi_1} \Omega - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \frac{1}{\sin \theta_2} \partial_{\theta_2} \left( \sin \theta_2 \partial_{\theta_2} \Omega \right)
- \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_2} \cot \theta_1 \partial_{\theta_2} \left( \sin \theta_2 \partial_{\phi_2} \Omega \right) + c_1 \cot \theta_2 \partial_{\psi} \left( \cos(\phi_1 - \psi) \partial_{\theta_1} \Omega \right)
+ \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\psi} \left( (\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \partial_{\phi_1} \Omega \right). \quad (C.6)
\]

Again all dependence on \( \theta_1, \phi_1 \) and \( \psi \), finally obtaining that

\[
\nabla^2_{M_5} \Omega = \frac{c_1c_2}{2} \frac{1}{\sin \theta_1} \left( \sin \theta_1 \frac{1}{\sin \theta_1} \partial^2_{\theta_1} \Omega \right) + \frac{c_1c_2}{2} \frac{1}{\sin^2 \theta_1} \partial^2_{\phi_1} \Omega = \frac{c_1c_2}{2} \nabla^2_{S^2} \Omega. \quad (C.7)
\]

We see that configurations of the form \( f = f(\theta_1, \phi_1) \) lead to the well known eigenvalue problem of the Laplace operator on the unit \( S^2 \), with eigenvalues \( \lambda = -l(l + 1) \), where \( l = 0, 1, 2, \ldots \)

The results we have obtained for the fluctuations using the ansatz \( f = \Omega(\theta_2, \phi_2, \psi) \) can be trivially extended to the case when \( f = \Omega(\theta_1, \phi_1) \). We simply have to compare (C.5) and (C.7) and make the replacement

\[
l(l + 2) \rightarrow 4 \sin^2 \frac{\alpha_n}{2} \left( \sin^2 \frac{\alpha_n}{2} + q \cos^2 \frac{\alpha_n}{2} \right) l(l + 1). \quad (C.8)
\]

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/971120.

[2] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern–Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810 (2008) 091, arXiv:0806.1218 [hep-th].

[3] S. Hohenegger and I. Kirsch, “A Note on the holography of Chern–Simons matter theories with flavour,” JHEP 0904 (2009) 129, arXiv:0903.1730 [hep-th].

[4] F. Bigazzi, R. Casero, A.L. Cotrone, E. Kiritsis and A. Paredes, “Non-critical holography and four-dimensional CFT’s with fundamentals,” JHEP 0510 (2005) 012, hep-th/0505140.
[5] R. Casero, C. Nunez and A. Paredes, “Towards the string dual of N=1 SQCD-like
theories,” Phys. Rev. D73 (2006) 086005, hep-th/0602027.

[6] C. Nunez, A. Paredes and A.V. Ramallo, “Unquenched Flavor in the Gauge/Gravity
Correspondence,” Adv. High Energy Phys. 2010 (2010) 196714,
arXiv:1002.1088 [hep-th].

[7] E. Conde and A. V. Ramallo, “On the gravity dual of Chern–Simons-matter theories
with unquenched flavor,” JHEP 1107 (2011) 099, arXiv:1105.6045 [hep-th].

[8] J. M. Camino, A. Paredes and A. V. Ramallo, “Stable wrapped branes,” JHEP 0105
(2001) 011, hep-th/0104082.

[9] N. Karaiskos, K. Sfetsos and E. Tsatis, “Brane embeddings in sphere submanifolds,”
Class. Quant. Grav. 29 (2012) 025011, arXiv:1106.1200 [hep-th].

[10] S. Kachru, A. Karch and S. Yaida, “Holographic Lattices, Dimers, and Glasses,” Phys.
Rev. D81 (2010) 026007, arXiv:0909.2639 [hep-th].

[11] P. Benincasa and A.V. Ramallo, “Fermionic impurities in Chern–Simons-matter
theories,” JHEP 1202 (2012) 076, arXiv:1112.4669 [hep-th].

[12] W. Mueck, “The Polyakov Loop of Anti-symmetric Representations as a Quantum
Impurity Model,” Phys. Rev. D83 (2011) 066006, [Erratum-ibid. D84 (2011) 129903
(2011)], arXiv:1012.1973 [hep-th].

[13] S. Harrison, S. Kachru and G. Torroba, “A maximally supersymmetric Kondo model,”
arXiv:1110.5325 [hep-th].

[14] P. Benincasa and A. V. Ramallo, “Holographic Kondo Model in Various Dimensions,”
JHEP 1206 (2012) 133, arXiv:1204.6290 [hep-th].

[15] A. Faraggi and L.A. Pando Zayas, “The Spectrum of Excitations of Holographic
Wilson Loops,” JHEP 1105 (2011) 018, arXiv:1101.5145 [hep-th].

[16] A. Faraggi, W. Mueck and L.A. Pando Zayas, “One-loop Effective Action of the
Holographic Antisymmetric Wilson Loop,” Phys. Rev. D 85 (2012) 106015,
arXiv:1112.5028 [hep-th].

[17] M. Ammon, J. Erdmenger, R. Meyer, A. O’Bannon and T. Wrase, “Adding Flavor to
AdS(4)/CFT(3),” JHEP 0911, 125 (2009) arXiv:0909.3845 [hep-th].

[18] B.E. W. Nilsson and C.N. Pope, “Hopf Fibration Of Eleven-dimensional Supergravity,”
Class. Quant. Grav. 1 (1984) 499.

[19] M.A. Awada, M.J. Duff and C.N. Pope, “N = 8 Supergravity Breaks Down to N = 1,”
Phys. Rev. Lett. 50 (1983) 294.

[20] H. Ooguri and C.S. Park, “Superconformal Chern–Simons Theories and the Squashed
Seven Sphere,” JHEP 0811 (2008) 082, arXiv:0808.0500 [hep-th].

[21] C.G. Callan, A. Guijosa and K.G. Savvidy, “Baryons and string creation from the fivebrane worldvolume action,” Nucl. Phys. B547 (1999) 127, hep-th/9810092.

[22] C.G. Callan, A. Guijosa, K.G. Savvidy and O. Tafjord, “Baryons and flux tubes in confining gauge theories from brane actions,” Nucl. Phys. B555 (1999) 183, hep-th/9902197.

[23] N. Drukker, J. Plefka and D. Young, “Wilson loops in 3-dimensional $N = 6$ supersymmetric Chern–Simons Theory and their string theory duals,” JHEP 0811 (2008) 019, arXiv:0809.2787 [hep-th].

[24] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[25] M. Kruczenski, D. Mateos, R. C. Myers and D.J. Winters, “Meson spectroscopy in AdS/CFT with flavor,” JHEP 0307 (2003) 049, hep-th/0304032.

[26] N. Jokela, J. Mas, A. V. Ramallo and D. Zoakos, “Thermodynamics of the brane in Chern-Simons matter theories with flavor,” arXiv:1211.0630 [hep-th].