DIMENSION-INDEPENDENT HARNACK INEQUALITIES FOR SUBORDINATED SEMIGROUPS

MARIA GORDINA†, MICHAEL RÖCKNER‡, AND FENG-YU WANG∗.∗∗

Abstract. Dimension-independent Harnack inequalities are derived for a class of subordinate semigroups. In particular, for a diffusion satisfying the Bakry-Emery curvature condition, the subordinate semigroup with power \( \alpha \) satisfies a dimension-free Harnack inequality provided \( \alpha \in \left( \frac{1}{2}, 1 \right) \), and it satisfies the log-Harnack inequality for all \( \alpha \in (0, 1) \). Some infinite-dimensional examples are also presented.

CONTENTS

1. Introduction 1
2. Proofs 5
3. Some infinite-dimensional examples 8
  3.1. Stochastic porous medium equation 8
  3.2. Singular stochastic semi-linear equations 9
  3.3. The Ornstein–Uhlenbeck type semigroups with jumps 10
  3.4. Infinite-dimensional Heisenberg groups 11
References 12

1. INTRODUCTION

By using the gradient estimate for diffusion semigroups, the following dimension-free Harnack inequality was established in [19] for the diffusion semigroup \( P_t \) generated by \( L = \Delta + Z \) on a complete Riemannian manifold \( M \) with curvature \( \text{Ric} - \nabla Z \) bounded below by \( -K \in \mathbb{R} \)

\[
(P_t f(x))^p \leq \exp \left( \frac{pKp(x,y)^2}{2(p-1)(e^{2Kt} - 1)} \right) P_t f^p(y), \quad t > 0, x, y \in M, f \in \mathcal{B}_b^+(M),
\]

where \( p > 1, p \) is the Riemannian distance, and \( \mathcal{B}_b^+(M) \) is the class of all bounded positive measurable functions on \( M \). This inequality has been extended and applied in the study of contractivity properties, heat kernel bounds, strong Feller properties.

Date: April 15, 2010 File:12GRW.tex.
1991 Mathematics Subject Classification. Primary: 60G51, 58G32. Secondary: 58J65, 60J60.
Key words and phrases. Harnack inequality, log-Harnack inequality, subordination.
† Research was supported in part by NSF Grant DMS-0706784.
‡ Research was supported in part by the German Science Foundation (DFG) through CRC 701.
∗ Research was supported in part by WIMICS, NNSFC (10721091) and the 973-Project.
and cost-entropy properties for finite- and infinite-dimensional diffusions. In particular, using the coupling method and Girsanov transformations developed in [4], this inequality has been derived for diffusions without using curvature conditions, see e.g. [5, 6, 9, 13–15, 17, 20] and references therein. See also [1–3] for applications to the short time behavior of transition probabilities. On the other hand, however, due to absence of a chain rule for the “gradient estimate” argument and an explicit Girsanov theorem, this technique of proving a dimension independent Harnack inequalities is not applicable to pure jump processes. The main purpose of this paper is to establish such inequalities for a class of α-stable like jump processes by using subordination.

Let \((E, \rho)\) be a Polish space with the Borel \(\sigma\)-algebra \(\mathcal{B}(E)\), and \(P_t\) the semigroup for a time-homogeneous Markov process on \(E\). Let \(\{\mu_t\}_{t \geq 0}\) be a convolution semigroup of probability measures on \([0, \infty)\), i.e. one has \(\mu_{t+s} = \mu_t * \mu_s\) for \(s, t \geq 0\) and \(\mu_t \to \mu_0 := \delta_0\) weakly as \(t \to 0\). Thus, the Laplace transform for \(\mu_t\) has the form

\[
\int_0^\infty e^{-sx} \mu_t(ds) = e^{-tB(x)}, \quad \text{for any } x \geq 0, t \geq 0
\]

for some Bernstein function \(B\), see e.g. [12]. We shall study the Harnack inequality for the subordinated semigroup

\[
P_t^B := \int_0^\infty P_s \mu_t(ds), \quad t \geq 0.
\]

Obviously, if \(P_t\) is generated by a negatively definite self-adjoint operator \((L, \mathcal{D}(L))\) on \(L^2(\nu)\) for some \(\sigma\)-finite measure \(\nu\) on \(E\), then \(P_t^B\) is generated by \(-B(-L)\). In particular, if \(B(x) = x^\alpha\) for \(\alpha \in (0, 1]\), we shall denote the corresponding \(\mu_t\) by \(\mu_t^\alpha\), and \(P_t\) by \(P_t^\alpha\) respectively.

We shall use (1.3) and a known dimension independent Harnack inequality for \(P_t\) to establish the corresponding Harnack inequality for \(P_t^B\). For instance, suppose we know that

\[
(P_t f(x))^p \leq \exp(\Phi(p, t, x, y)) P_t f^p(y), \quad x, y \in E, \quad t > 0, \quad p > 1, \quad f \in \mathcal{B}_+^p(E)
\]

for some \(\Phi : (1, \infty) \times (0, \infty) \times E^2 \to [0, \infty)\). Then (1.3) implies

\[
(P_t^B f(x))^p = \left(\int_0^\infty P_s f(x) \mu_t(ds)\right)^p \leq \left(\int_0^\infty (P_s f^p(y))^{1/p} \exp \left(\frac{\Phi(p, s, x, y)}{p}\right) \mu_t(ds)\right)^p \leq (P_t^B f^p(y)) \left(\int_0^\infty \exp \left(\frac{\Phi(p, s, x, y)}{p - 1}\right) \mu_t(ds)\right)^{p-1}.
\]

In general, \(\Phi(p, s, x, y) \to \infty\) as \(s \to 0\), so we have to verify that \(\exp(\Phi(p, s, x, y)/(p-1))\) is integrable w.r.t. \(\mu_t(ds)\). Similarly to (1.1), for many specific models the singularity of \(\Phi(p, s, x, y)\) at \(s = 0\) behaves like \(e^{\delta/s^\kappa}\) for some \(\delta = \delta(p, x, y) > 0, \kappa \geq 1\) (see Section 3 below for specific examples). In this case, the following results say that the Harnack inequality provided by (1.4) is valid for \(P_t^\alpha\) with \(\alpha > \kappa/(\kappa + 1)\).
Theorem 1.1. Let $p > 1, \kappa > 0$ and $\alpha \in \left(\frac{\kappa}{\kappa + 1}, 1\right)$ be fixed. Suppose that $P_t$ satisfies the Harnack inequality

$$\left(P_t f(x)\right)^p \leq \exp \left(H(x, y) (\varepsilon + t^{-\kappa})\right) P_t f^p(y), \quad x, y \in E, f \in B^+_b(E), t > 0,$$

for some positive measurable function $H$ on $E \times E$ and a constant $\varepsilon \geq 0$. Then there exists a constant $c > 0$ depending on $\alpha$ and $\kappa$ such that

$$\left(P_t f(x)\right)^p \leq e^{cH(x,y)} \left(1 + \left[\exp \left(\frac{cH(x,y)}{(p-1)\kappa/\alpha}\right)^{1/(1-(\alpha^{-1}-1)\kappa)} - 1\right]^{(1-(\alpha^{-1}-1)\kappa)}\right)^{p-1} P_t^\alpha f^p(y)$$

holds for all $f \in B^+_b(E)$, where

$$C_{p,\kappa,\alpha} = \left(1 - (\alpha^{-1}-1)\kappa\right)c^{1/(1-(\alpha^{-1}-1)\kappa)}\left(p-1\right)^{1/(1-(\alpha^{-1}-1)\kappa)}.$$

Consequently, if $P_t$ has an invariant probability measure $\mu$, we have that

(i) for any $p, q > 1$,

$$\frac{\|P_t^\alpha\|_{p \to q}}{2^{(p-1)/p}} \leq \left(\int_E \frac{\mu(dx)}{\exp \left[\varepsilon H(x,y) - C_{p,\kappa,\alpha} \left(H(x,y) \frac{\kappa/\alpha}{\kappa/\alpha}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right] \mu(dy)}^{q/p}\right)^{1/q};$$

(ii) if $P_t^\alpha$ has a transition density $p_t^\alpha(x,y)$ w.r.t. $\mu$ such that for any $x \in \operatorname{supp}(\mu)$

$$\int_E p_t^\alpha(x,y)^2 \mu(dy) \leq 2 \left(\int_E \exp \left[-\varepsilon H(x,y) - C_{p,\kappa,\alpha} \left(H(x,y) \frac{\kappa/\alpha}{\kappa/\alpha}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right] \mu(dy)\right)^{-1}.$$

As an application of Theorem 1.1 (ii), we have the following explicit heat kernel upper bounds for stable like processes.

Example 1.2. Let $P_t$ be generated by $L = \Delta + Z$ on a complete Riemannian manifold such that $\text{Ric} - \nabla Z \geq -K$. By (1.1), (1.5) holds for $H(x,y) = \rho(x,y)^2$ and $\kappa = 1$. So, for $\alpha \in (1/2, 1]$, Theorem 1.1 (ii) implies

$$p_t^\alpha(x,x) \leq \frac{c}{\mu\left(\{y : \rho(x,y) \leq t^{1/2\alpha}\}\right)} \quad x \in M, t > 0$$

for some constant $c > 0$. In particular, for $L = \Delta$ on $\mathbb{R}^d$, $\mu(dx) = dx$ and $K = 0$, we have

$$\sup_{x,y \in \mathbb{R}^d} p_t^\alpha(x,y) = \sup_{x \in \mathbb{R}^d} p_t^\alpha(x,x) \leq ct^{-d/2\alpha}, t > 0,$$
for some constant $c > 0$. This is sharp due to the well known explicit bounds of heat kernels for the classical stable processes on $\mathbb{R}^d$.

Theorem 1.3 does not apply to $\alpha \in (0, \frac{\kappa}{\kappa+1}]$, since in this case $\int_0^\infty e^{\delta/\kappa} \mu_t^\kappa(ds) = \infty$ for large $\delta > 0$. A more careful analysis allows us to treat the case $\alpha = \frac{\kappa}{\kappa+1}$ under certain restrictions on $x, y, t$. Thus results of this type apply also to the Cauchy process.

**Proposition 1.3** (The case $\alpha = \frac{\kappa}{\kappa+1}$). Suppose that $P_t$ satisfies the Harnack inequality (1.5) for some positive measurable function $H$ on $E \times E$ and a constant $\varepsilon \geq 0$. Then there exists a constant $C > 0$ depending on $\kappa$ such that

$$
(P_t^\frac{\kappa}{\kappa+1} f(x))^p 
\leq e^{\varepsilon H(x,y)} \left( 1 + \frac{C}{\varepsilon^{(p-1)/\kappa}} \right) P_t^\frac{\kappa}{\kappa+1} f^p(y), \quad f \in B^+_0(E)
$$

holds for all $t > 0, x, y \in E$ such that $e(\varepsilon - 1)(t\kappa)^{\kappa+1} < \kappa(1)^{\kappa+1} H(x,y)$.

In other cases we can still prove the log-Harnack inequality. For diffusion semigroups, the known log-Harnack inequality looks like

$$(1.6) \quad P_t \log f(x) \leq \log P_t f(y) + H(x,y)(\varepsilon + t^{-\kappa}), \quad x, y \in E, t > 0, f \geq 1,$$

for some positive measurable function $H$ on $E \times E$ and some constants $\varepsilon \geq 0, \kappa \geq 1$.

In many cases, one has $H(x,y) = c\rho(x,y)^2$ for a constant $c > 0$ and the intrinsic distance $\rho$ induced by the diffusion (see e.g. [18]).

**Theorem 1.4.** If (1.6) holds, then for any $\alpha \in (0, 1],$

$$P_t^\alpha \log f(x) \leq \log P_t^\alpha f(y) + H(x,y) \left( \varepsilon + \frac{\Gamma(\frac{\alpha}{\kappa})}{\alpha t^{\frac{\alpha}{\kappa}} \Gamma(\kappa)} \right),$$

$t > 0, x, y \in E, f \geq 1$.

As observed in [6] and [18], the log-Harnack inequality implies an entropy-cost inequality for the semigroup and an entropy inequality for the corresponding transition density. Let $W_H$ be the Wasserstein distance induced by $H$, i.e.

$$W_H(\mu_1, \mu_2) = \inf_{\pi \in C(\mu_1, \mu_2)} \int_{E \times E} H(x,y) \pi(dx, dy),$$

where $\mu_1, \mu_2$ are probability measures on $E$ and $C(\mu_1, \mu_2)$ is the set of all couplings for $\mu_1$ and $\mu_2$.

**Corollary 1.5.** Assume that (1.6) holds and let $P_t$ have an invariant probability measure $\mu$. Then for any $\alpha \in (0, 1],$

1. The entropy-cost inequality

$$\mu((P_t^\alpha f)^*) \log (P_t^\alpha f) \leq W_H(f \mu, \mu) \left( \varepsilon + \frac{1}{\alpha t^{\frac{1}{\kappa}} \Gamma(\kappa)} \right),$$

$t > 0, f \geq 0, \mu(f) = 1$.
If \( (P^\alpha_t)^* \) is the adjoint of \( P^\alpha_t \) in \( L^2(E;\mu) \).

2. Proofs

Proof of Theorem 1.1. The consequences of the desired Harnack inequality are straightforward. Indeed, (i) follows by noting that the claimed Harnack inequality implies

\[
(P^\alpha f(x))^p \int_E \exp \left[ -\varepsilon H(x, y) - C_{p, \kappa, \alpha} \left( \frac{H(x, y)}{\kappa/p} \right)^{1/(\alpha^{-1} - 1)} \right] \mu(dy) 
\leq \mu(P^\alpha f)^p = \mu(f)^p,
\]

which also implies (ii) by taking \( p = 2 \) and \( f(z) = p^\alpha_t(x, z), z \in E \). Indeed, with \( f = 1_A \) for a \( \mu \)-null set \( A \), this inequality implies that the associated transition probability \( P^\alpha_t(x, \cdot) \) is absolutely continuous w.r.t. \( \mu \) and hence, has a density \( p^\alpha_t(x, \cdot) \) for every \( x \in E \). Then the desired upper bound for \( \int_E P^\alpha_t(x, y)^2 \mu(dy) \) follows by first applying the above inequality with \( p = 2 \) and \( f(z) = p^\alpha_t(x, z) \) and then letting \( n \to \infty \). So, it remains to prove the first assertion.

By (1.5), (1.4) holds for \( \Phi(p, s, x, y) = H(x, y)(\varepsilon + s^{-\kappa}) \), i.e.

\[
(2.1) \quad (P^\alpha_t f(x))^p \leq e^{\varepsilon H(x, y)}(P^\alpha_t f^p(y)) \left( \int_0^\infty \exp \left[ \frac{H(x, y)}{(p - 1)s^\kappa} \right] \mu_t(ds) \right)^{p-1}.
\]

So it suffices to estimate the integral \( \int_0^\infty e^{\delta/s^\kappa} \mu_t(ds) \) for \( \delta := \frac{H(x, y)}{(p - 1)} > 0 \).

We use the formula

\[
s^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-xs} dx, \quad r > 0.
\]

to obtain

\[
\int_0^\infty \frac{\mu_t^\alpha(ds)}{s^r} = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-xs} dx \mu_t(ds) = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-x\alpha} dx \mu_t(ds) dx = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-tB(x)} dx.
\]

In particular, for \( B(x) = x^\alpha \) we have

\[
(2.2) \quad \int_0^\infty \frac{\mu_t^\alpha(ds)}{s^r} = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-tx^\alpha} dx = \frac{1}{\alpha \Gamma(r)} \int_0^\infty y^{\frac{\alpha}{\alpha - 1}} e^{-ty} dy = \frac{\Gamma \left( \frac{\alpha}{\alpha - 1} \right)}{\alpha \Gamma(r)} t^{\frac{\alpha}{\alpha - 1}}.
\]

We can use the generalization of Stirling’s formula giving the asymptotic behavior of the Gamma function for large \( r \)

\[
\Gamma (r) = \sqrt{2\pi r^{r-\frac{1}{2}}} e^{-r + \gamma(r)},
\]

where
\[
\eta(r) = \sum_{n=0}^{\infty} \left( r + n + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{r + n} \right) - 1 = \frac{\theta}{12r}, 0 < \theta < 1.
\]

We apply this estimate to \( \Gamma (\kappa n) \), \( \Gamma \left( \frac{\kappa n}{\alpha} \right) \) and \( n! \). Thus

\[
\int_0^\infty e^{-s} \mu_t^\alpha (ds) = 1 + \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \Gamma (\kappa n) t^{-\frac{\alpha n}{2}} = 1 + \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} (\kappa n)^{\kappa n \left( \frac{1}{\alpha} - 1 \right)} e^{-\kappa n \left( \frac{1}{\alpha} - 1 \right) t^{-\frac{\alpha}{2}}} e^{\frac{\alpha \alpha n - \theta \alpha n}{12\pi}} t^{-\frac{\alpha n}{2}} \leq \left( \begin{array}{c}
\end{array} \right)
\]

This series converges for \( \alpha > \frac{\kappa}{\kappa + 1} \), moreover, there is a constant \( c \) depending only on \( \kappa \) such that

\[
\frac{1}{\sqrt{2\pi n}} \left( \frac{K}{e} \right)^{\kappa \left( \frac{1}{\alpha} - 1 \right)} \alpha^{-\frac{\alpha}{2} t^{-\frac{\alpha}{2}}} e^{\frac{\alpha}{12\pi}} \leq c^n.
\]

Denote

\[
c(\delta, \alpha, \kappa) := 1 + \sum_{n=1}^{\infty} n^{\kappa \left( \frac{1}{\alpha} - 1 \right)} (\delta t^{-\frac{\alpha}{2}})^n,
\]

then

\[
(P_t^\alpha f(x))^p \leq e^{c H(x,y)} \left( c \left( \frac{H(x,y)}{p-1}, \alpha, \kappa \right) \right)^{p-1} P_t^\alpha f^p(y).
\]

Note that for \( a > 0, 1 \geq b > 0 \) we have the following estimate

\[
\sum_{n=1}^{\infty} \frac{a^n}{n^{bn}} = \sum_{n=1}^{\infty} \frac{(2a)^n}{n^{bn}} \leq \left( \sum_{n=1}^{\infty} \frac{(2a)^n}{n^n} \right)^{\frac{1}{2n}} \leq \left( \sum_{n=1}^{\infty} \frac{(2a)^n}{n^n} \right)^{\frac{1}{2n}} = \left( e^{\frac{(2a)^{1/b}}{2} - 1} \right)^b,
\]

where we used Jensen’s inequality. Thus for any \( \alpha \in \left( \frac{\kappa}{\kappa + 1}, 1 \right) \) we use the above estimate with \( b := \kappa \left( 1 - \frac{1}{\alpha} \right) + 1 \leq 1 \) to see that
\[ c(\delta, \alpha, \kappa) = 1 + \sum_{n=1}^{\infty} n^{\kappa(\frac{1}{\alpha}-1)} (c\delta t^{-\frac{2}{\alpha}})^n \leq \]
\[ 1 + \left( \exp \left( \frac{(2c\delta t^{-\frac{2}{\alpha}})^{\frac{1}{\kappa(1-\frac{2}{\alpha})+1}}}{2} \right) - 1 \right)^{\kappa(1-\frac{1}{\alpha})+1}. \]

Thus we can say that there is \( c > 0 \) depending on \( \alpha \) and \( \kappa \) such that

\[ \int_{0}^{\infty} e^{\frac{H(x,y)}{p-1}} \mu_{t}^p (ds) \leq 1 + \left( \exp \left( \frac{cH(x,y)}{(p-1)t^\frac{\alpha}{\pi}} \right)^{\frac{1}{\kappa(1-\frac{2}{\alpha})+1}} - 1 \right)^{\kappa(1-\frac{1}{\alpha})+1} \]

Using the inequality

\[ 1 + (x-1)^a \leq 2x^a \]

for any \( x \geq 1 \) and \( 0 \leq a \leq 1 \) we see that

\[ \int_{0}^{\infty} e^{\frac{H(x,y)}{p-1}} \mu_{t}^p (ds) \leq 2 \exp \left( \kappa \left( 1 - \frac{1}{\alpha} \right) + 1 \right) \left( \frac{cH(x,y)}{(p-1)t^\frac{\alpha}{\pi}} \right)^{\frac{1}{\kappa(1-\frac{2}{\alpha})+1}} \]

which completes the proof.

\[ \square \]

**Proof of Proposition 1.3.** In the case \( \alpha = \frac{\kappa}{\kappa+1} \) the series in (2.3) converges for \( t > 0 \) and \( x, y \in E \) such that

(2.4) \[ e(p-1)(tk)^{\kappa+1} > \kappa(\kappa+1)^{\kappa+1} H(x,y). \]

Note that for \( \delta := \frac{H(x,y)}{p-1} \) the last line of (2.3) reduces to

\[ 1 + \sqrt{\frac{\kappa+1}{2\pi\kappa}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \frac{\delta\kappa}{e} \left( \frac{\kappa+1}{kt} \right)^{\kappa+1} n^{\frac{1}{2(\kappa+1)n}} \]

\[ \leq 1 + C \sum_{n=1}^{\infty} \left( \frac{\delta\kappa}{e} \left( \frac{\kappa+1}{kt} \right) \right)^{\kappa+1} n \]

\[ = 1 + \frac{C}{\delta\kappa} \left( \frac{\kappa+1}{\kappa+1} \right)^{\kappa+1} - 1. \]

This completes the proof.

\[ \square \]

**Proof of Theorem 1.4.** By (2.2) with \( r = \kappa \), we have

\[ \int_{0}^{\infty} \frac{\mu_{t}^p (ds)}{s^\kappa} = \frac{\Gamma \left( \frac{\kappa}{\alpha} \right)}{\alpha t^\frac{\alpha}{\pi} \Gamma (\kappa)}. \]

Using (1.2), (1.6) we obtain
\[ P_t^\alpha \log f(x) = \int_0^\infty P_s \log f(x) \mu_t^\alpha(ds) \leq \int_0^\infty \left( \log P_s f(y) + H(x,y)(\varepsilon + s^{-\kappa}) \right) \mu_t^\alpha(ds) \]
\[ = \log P_t^\alpha f(y) + H(x,y) \left( \varepsilon + \frac{\Gamma \left( \frac{\alpha}{\gamma} \right)}{\alpha \tau \Gamma(\kappa)} \right). \]

This completes the proof. \hfill \Box

**Proof of Corollary 1.5.** (1) It suffices to prove for \( f \in B_b^+(E) \) such that \( \inf f > 0 \) and \( \mu(f) = 1 \). In this case, there exists a constant \( c > 0 \) such that \( cf \geq 1 \). By Theorem 1.4 for \( cP_t^\alpha f \) in place of \( f \), we obtain

\[ P_t^\alpha \log(P_t^\alpha)^* f(x) \leq \log P_t^\alpha (P_t^\alpha)^* f(y) + H(x,y) \left( \varepsilon + \frac{\Gamma \left( \frac{\alpha}{\gamma} \right)}{\alpha \tau \Gamma(\kappa)} \right) \]

Since \( \mu \) is invariant for \( P_t^\alpha \) and \( (P_t^\alpha)^* \), taking the integral for both sides w.r.t. \( \pi \in (f\mu, \mu) \) and minimizing in \( \pi \), we prove the first assertion.

(2) The strong Feller property follows from Theorem 1.4 according to [18, Proposition 2.3], while by [18, Proposition 2.4] the desired entropy inequality for the transition density is equivalent to the log-Harnack inequality for \( P_t^\alpha \) provided by Theorem 1.3. \hfill \Box

3. SOME INFINITE-DIMENSIONAL EXAMPLES

As explained in Section 1, Theorems 1.1 and 1.4 hold for \( \kappa = 1 \) if \( P_t \) is a diffusion semigroup on a Riemannian manifold with the Ricci curvature bounded below. In this section we present some infinite dimensional examples where these theorems can be used.

3.1. **Stochastic porous medium equation.** Let \( \Delta \) be the Dirichlet Laplace operator on a bounded interval \( (a, b) \) and \( W_t \) the cylindrical Brownian motion on \( L^2((a, b); dx) \). Since the eigenvalues \( \{\lambda_i\} \) of \( -\Delta \) satisfies \( \sum_{i=1}^{\infty} \lambda_i^{-1} < \infty \), \( W_t \) is a continuous process on \( H \), the completion of \( L^2((a, b); dx) \) under the inner product

\[ \langle x, y \rangle := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle x, e_i \rangle \langle y, e_i \rangle, \]

where \( e_i \) is the unit eigenfunction corresponding to \( \lambda_i \) for each \( i \geq 1 \). Let \( \| \cdot \| \) denote the norm on \( H \), and suppose \( r > 1 \). Then the following stochastic porous medium equation has a unique strong solution on \( H \) for any \( X_0 \in H \) (see e.g. [7]):

\[ dX_t = \Delta X_t^r dt + dW_t. \]

Let \( P_t \) be the corresponding Markov semigroup. According to [20, Remark 1.1 and Theorem 1.2], Theorem 1.1 in [20] holds for \( \theta = r-1 \) and some constant \( \gamma, \delta, \xi > 0 \). Thus, there exist two constants \( c_1, c_2 > 0 \) depending on \( r \) such that

\[ (P_t f)^p(x) \leq (P_t f^p(y)) \exp \left[ \frac{c_1 p \|x - y\|^{4/(1+r)}}{(p-1)(1- e^{-c_2 t})^{(3+r)/(1+r)}} \right], \quad p > 1, t > 0, x, y \in H \]
holds for all \( f \in B^+_H(\mathbb{H}). \) By [18, Proposition 2.2] for \( \rho(x, y)^2 = \|x - y\|^{2/(1 + r)} \), this implies the log-Harnack inequality
\[
P_t \log f(x) \leq \log P_t f(x) + \frac{c_1 \|x - y\|^{4/(1 + r)}}{(1 - e^{-c_2 t})^{(3 + r)/(1 + r)}}, \quad x, y \in \mathbb{H}, f \geq 1.
\]
Therefore, Theorems 1.1 and 1.2 apply to \( P_t^\alpha \) for
\[
\kappa = \frac{r + r}{1 + r}
\]
and some constant \( \varepsilon \) depending on \( r \).

3.2. Singular stochastic semi-linear equations. Let \( \mathbb{H} \) be a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \), and \( W_t \) the cylindrical Brownian motion on \( \mathbb{H} \). Consider the stochastic equation
\[
dX_t = (AX_t + F(X_t))dt + \sigma dW_t, \quad X_0 \in H.
\]
Let \( A, F, \sigma \) satisfy the following hypotheses:

(H1) \((A, D(A))\) is the generator of a \( C_0\)-semigroup, \( T_t = e^{tA}, t \geq 0, \) on \( \mathbb{H} \) and for some \( \omega \in \mathbb{R}\)
\[
\langle Ax, x \rangle \leq \omega\|x\|^2, \quad \forall x \in D(A).
\]

(H2) \( \sigma \) is a bounded positively definite, self-adjoint operator on \( \mathbb{H} \) such that \( \sigma^{-1} \) is bounded and \( \int_0^\infty \|T_t\sigma\|_{HS}^2 dt < \infty \), where \( \| \cdot \|_{HS} \) denotes the norm on the space of all Hilbert–Schmidt operators on \( \mathbb{H} \).

(H3) \( F : D(F) \subset \mathbb{H} \to \mathbb{H} \) is an \( m\)-dissipative map, i.e.,
\[
\langle F(x) - F(y), x - y \rangle \leq 0, \quad x, y \in D(F), \quad u \in F(x), \quad v \in F(y),
\]
\( \text{“dissipativity”} \) and
\[
\text{Range} \ (I - F) := \bigcup_{x \in D(F)} (x - F(x)) = \mathbb{H}.
\]
Furthermore, \( F_0(x) \in F(x), \ x \in D(F), \) is such that
\[
\|F_0(x)\| = \min_{y \in F(x)} \|y\|.
\]

Here we recall that for \( F \) as in (H3) we have that \( F(x) \) is closed, non-empty and convex.

The corresponding Kolmogorov operator is then given as follows: Let \( \mathcal{E}_A(H) \) denote the linear span of all real parts of functions of the form \( \varphi = e^{i(h \cdot \cdot \cdot)}, \ h \in D(A^*), \) where \( A^* \) denotes the adjoint operator of \( A, \) and define for any \( x \in D(F), \)
\[
L_0 \varphi(x) = \frac{1}{2} \text{Tr} (\sigma^2 D^2 \varphi(x)) + \langle x, A^* D \varphi(x) \rangle + \langle F_0(x), D \varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H).
\]
Additionally, we assume:

(H4) There exists a probability measure \( \mu \) on \( H \) (equipped with its Borel \( \sigma\)-algebra \( \mathcal{B}(H) \)) such that
\begin{enumerate}
  \item \( \mu(D(F)) = 1, \)
  \item \( \int_H (1 + \|x\|^2)(1 + \|F_0(x)\|)\mu(dx) < \infty, \)
  \item \( \int_H L_0 \varphi d\mu = 0 \) for all \( \varphi \in \mathcal{E}_A(H). \)
\end{enumerate}
By [8], the closure of \((L_0, \mathcal{E}_A(\mathbb{H}))\) in \(L^1(\mathbb{H}; \mu)\) generates a Markov semigroup \(P_t\) with \(\mu\) as an invariant probability measure, which is point-wise determined on \(\mathbb{H}_0 := \text{supp}\mu\). If moreover the following hypotheses holds:

(H5) (i) \((1 + \omega - A, \mathcal{D}(A))\) satisfies the weak sector condition: there exists a constant \(K > 0\) such that

\[
(3.3) \quad \langle (1 + \omega - A)x, y \rangle \leq K \langle (1 + \omega - A)x, x \rangle^{1/2} \langle (1 + \omega - A)y, y \rangle^{1/2}, \quad \forall x, y \in \mathcal{D}(A).
\]

(ii) There exists a sequence of \(A\)-invariant finite dimensional subspaces \(\mathbb{H}_n \subset \mathcal{D}(A)\) such that \(\bigcup_{n=1}^{\infty} \mathbb{H}_n\) is dense in \(\mathbb{H}\).

Then (see [9, Theorem 1.6])

\[
(P_t f(x))^p \leq P_t f^p(y) \exp \left[ \frac{\|\sigma\|^2}{2} \frac{p \omega \|x - y\|^2}{(p - 1)(1 - e^{-2\omega t})} \right], \quad t > 0, \ x, y \in \mathbb{H}_0.
\]

As mentioned above, according to [18, Proposition 2.2] this implies the corresponding log-Harnack inequality. Therefore, our Theorems 1.1 and 1.4 apply to \(P_t^p\) for \(\kappa = 1\).

3.3. The Ornstein–Uhlenbeck type semigroups with jumps. Consider the following stochastic differential equation driven by a Lévy process

\[
(3.4) \quad dX_t = AX_t dt + dZ_t, \quad X_0 = x \in \mathbb{H},
\]

where \(A\) is the infinitesimal generator of a strongly continuous semigroup \((T_t)_{t \geq 0}\) on \(\mathbb{H}\), \(Z_t := \{Z_t^u, u \in \mathbb{H}\}\) is a cylindrical Lévy process with characteristic triplet \((a, R, M)\) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), that is, for every \(u \in \mathbb{H}\) and \(t \geq 0\)

\[
\mathbb{E} \exp(i\langle Z_t, u \rangle) = \exp (it\langle a, u \rangle - \frac{t}{2} (Ru, u) - \int_{\mathbb{H}} [1 - \exp(i\langle x, u \rangle) + i\langle x, u \rangle 1_{\{\|x\| \leq 1\}}(x)] , M(dx)),
\]

where \(a \in \mathbb{H}\), \(R\) is a symmetric linear operator on \(\mathbb{H}\) such that

\[
R_t := \int_0^t T_s RT_s^* ds
\]

is a trace class operator for each \(t > 0\), and \(M\) is a Lévy measure on \(\mathbb{H}\). (For simplicity, we shall write \(Z_t^u := \langle Z_t, u \rangle\) for every \(u \in \mathbb{H}\).) In this case, (3.3) has a unique mild solution

\[
X_t = T_t x + \int_0^t T_{t-s} dZ_s, t \geq 0.
\]

Let

\[
P_t f(x) = \mathbb{E} f(X_t), \quad x \in \mathbb{H}, \ f \in \mathcal{B}_b(\mathbb{H}).
\]

If

\[
\|R^{-1/2} R_t x\| \leq \sqrt{h(t)} \|R^{1/2} x\|, \quad x \in \mathbb{H}, \ t \geq 0
\]

holds for some positive function \(h \in C([0, \infty))\). Then by [16, Theorem 1.2] (see also [17] for the diffusion case),
\[(P_t f)^\alpha(x) \leq \exp \left[ \frac{\alpha \| R^{-1/2}(x - y) \|^2}{2(\alpha - 1)} \int_0^t h(s)^{-1} \, ds \right] P_t f^\alpha(y), \quad t > 0, x - y \in R^{1/2} \mathbb{H} \]

holds for all \( f \in B^+_0(\mathbb{H}) \). By this and [18, Proposition 2.2] which implies the corresponding log-Harnack inequality, Theorems 1.1 and 1.4 apply to some \( \varepsilon \geq 0 \) and \( \kappa \geq 1 \) if

\[
\lim_{t \to 0} \frac{1}{t^{\kappa}} \int_0^t \frac{1}{h(s)} \, ds > 0.
\]

3.4. Infinite-dimensional Heisenberg groups. In [10] an integrated Harnack inequality similar to (1.1) has been established for a Brownian motion on infinite-dimensional Heisenberg groups modeled on an abstract Wiener space. The inequality is the consequence of the Ricci curvature bounds for both finite-dimensional approximations to these groups and the group itself, and the results established for inductive limits of finite-dimensional Lie groups in [11]. Even though the methods described in that paper are applicable to inductive and projective limits of finite-dimensional Lie groups, the infinite-dimensional Heisenberg groups provide a very concrete setting. We follow the exposition in [10].

Let \((W, H, \mu)\) be an abstract Wiener space over \(\mathbb{R}(C)\), \(C\) be a real (complex) finite dimensional inner product space, and \(\omega: W \times W \to C\) be a continuous skew symmetric bilinear quadratic form on \(W\). Further, let

(3.5) \[
\|\omega\|_0 := \sup \{ \|\omega(w_1, w_2)\|_C : w_1, w_2 \in W \text{ with } \|w_1\|_W = \|w_2\|_W = 1 \}
\]

be the uniform norm on \(\omega\) which is finite since \(\omega\) is assumed to be continuous. We will need the Hilbert-Schmidt norm of \(\omega\) which is defined as

\[
\|\omega\|_2 = \|\omega\|_{H^* \otimes H^* \otimes C} := \sum_{i,j=1}^{\infty} \|\omega(e_i, e_j)\|_C^2,
\]

which is finite by Proposition 3.14 in [10].

**Definition 3.1.** Let \(\mathfrak{g}\) denote \(W \times C\) when thought of as a Lie algebra with the Lie bracket operation given by

(3.6) \[
\left[(A, a), (B, b)\right] := (0, \omega(A, B))
\]

Let \(G := G(\omega)\) denote \(W \times C\) when thought of as a group with the multiplication law given by

(3.7) \[
g_1 g_2 = g_1 + g_2 + \frac{1}{2} [g_1, g_2] \quad \text{for any } g_1, g_2 \in G.
\]

It is easily verified that \(\mathfrak{g}\) is a Lie algebra and \(G\) is a group. The identity of \(G\) is the zero element, \(e := (0, 0)\).

**Notation 3.2.** Let \(\mathfrak{g}_{CM}\) denote \(H \times C\) when viewed as a Lie subalgebra of \(\mathfrak{g}\) and \(G_{CM}\) denote \(H \times C\) when viewed as a subgroup of \(G = G(\omega)\). We will refer to \(\mathfrak{g}_{CM} / G_{CM}\) as the *Cameron–Martin subalgebra (subgroup)* of \(\mathfrak{g} (G)\). (For explicit examples of such \((W, H, C, \omega)\), see [10].)
We equip $G = \mathfrak{g} = W \times C$ with the Banach space norm
\begin{equation}
\|(w, c)\|_\mathfrak{g} := \|w\|_W + \|c\|_C
\end{equation}
and $G_{CM} = \mathfrak{g}_{CM} = H \times C$ with the Hilbert space inner product,
\begin{equation}
\langle (A, a), (B, b) \rangle_{\mathfrak{g}_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_C.
\end{equation}
The associate Hilbertian norm is given by
\begin{equation}
\|(A, \delta)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|\delta\|_C^2}.
\end{equation}
As was shown in [10, Lemma 3.3], these Banach space topologies on $W \times C$ and $H \times C$ make $G$ and $G_{CM}$ into topological groups.
Then we can define a Brownian motion on $G$ starting at $e = (0, 0) \in G$ to be the process
\begin{equation}
g(t) = \left( B(t), B_0(t) + \frac{1}{2} \int_0^t \omega(B(\tau), dB(\tau)) \right).
\end{equation}
We denote by $\nu_t$ the corresponding heat kernel measure on $G$. The following estimate was used in the proof of Theorem 8.1 in [10]. For any $h \in G_{CM}$, $1 < p < \infty$
\begin{equation}
\int_G |f(xh)| \, d\nu_t(x) \leq \|f\|_{L^p(G, \nu_t)} \exp\left( \frac{c(-k(\omega)t)(p-1)}{2t} d_{G_{CM}}^2(e, h) \right),
\end{equation}
where
\begin{equation}
c(t) = \frac{t}{e^t - 1}
\end{equation}
for all $t \in \mathbb{R}$
with the convention that $c(0) = 1$ and $k(\omega) := \frac{1}{2} \sup_{\|A\|_H = 1} \|\omega(\cdot, A)\|_{H \otimes C}^2 \leq \frac{1}{2} \|\omega\|_C^2 < \infty.$
Equation (3.12) implies the corresponding $L^p$-estimates of Radon-Nikodym derivatives of $\nu_t$ relative to the left and right multiplication by elements in $G_{CM}$. This in turn is equivalent to the Harnack inequality (1.1) following an argument similar to Lemma D.1 in [11]
\begin{equation}
[(P_tf)(x)]^p \leq C_p (P_t f^p)(y)
\end{equation}
for all $f \geq 0$.
Thus we are in position to apply our results to the heat kernel measure $\nu_t$ subordinated as described in Section 1.

References
1. Shigeki Aida, Uniform positivity improving property, Sobolev inequalities, and spectral gaps, J. Funct. Anal. 158 (1998), no. 1, 152–185. MR MR1641566 (2000d:60125)
2. Shigeki Aida and Hiroshi Kawabi, Short time asymptotics of a certain infinite dimensional diffusion process, Stochastic analysis and related topics, VII (Kusadasi, 1998), Progr. Probab., vol. 48, Birkhäuser Boston, Boston, MA, 2001, pp. 77–124. MR MR1915450 (2003m:60219)
3. Shigeki Aida and Tusheng Zhang, On the small time asymptotics of diffusion processes on path groups, Potential Anal. 16 (2002), no. 1, 67–78. MR MR1880348 (2003e:58052)
4. Marc Arnaudon, Anton Thalmaier, and Feng-Yu Wang, Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below, Bull. Sci. Math. 130 (2006), no. 3, 223–233. MR MR2215664 (2007i:58032)
5. , Gradient estimates and harnack inequalities on non-compact riemannian manifolds, to appear x (2009), x.
6. Sergey G. Bobkov, Ivan Gentil, and Michel Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) **80** (2001), no. 7, 669–696. MR MR1846020 (2003b:47073)
7. G. Da Prato, Michael Röckner, B. L. Rozovskii, and Feng-Yu Wang, *Strong solutions of stochastic generalized porous media equations: existence, uniqueness, and ergodicity*, Comm. Partial Differential Equations **31** (2006), no. 1-3, 277–291. MR MR2209754 (2007b:60153)
8. Giuseppe Da Prato and Michael Röckner, *Singular dissipative stochastic equations in Hilbert spaces*, Probab. Theory Related Fields **124** (2002), no. 2, 261–303. MR MR1936019 (2003k:47051)
9. Giuseppe Da Prato, Michael Röckner, and Feng-Yu Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. **257** (2009), no. 4, 992–1017. MR MR2535460
10. Bruce K. Driver and Maria Gordina, *Heat kernel analysis on infinite-dimensional Heisenberg groups*, J. Funct. Anal. **255** (2008), no. 9, 2395–2461. MR MR2473262
11. , *Integrated Harnack inequalities on Lie groups*, Journal of Differential Geometry **3** (2009), 501–550.
12. N. Jacob, *Pseudo differential operators and Markov processes. Vol. I*, Imperial College Press, London, 2001, Fourier analysis and semigroups. MR MR1873235 (2003a:47104)
13. Wei Liu, *Fine properties of stochastic evolution equations and their applications*, Ph.D. thesis, Bielefeld University, 2009.
14. Wei Liu and Feng-Yu Wang, *Harnack inequality and strong Feller property for stochastic fast-diffusion equations*, J. Math. Anal. Appl. **342** (2008), no. 1, 651–662. MR MR2440828 (2009k:60137)
15. Shun-Xiang Ouyang, *Harnack inequalities and applications for stochastic equations*, Ph.D. thesis, Bielefeld University, 2009.
16. Shun-Xiang Ouyang, Michael Röckner, and Feng-Yu Wang, *Harnack inequalities and applications for Ornstein-Uhlenbeck semigroups with jumps*, preprint.
17. Michael Röckner and Feng-Yu Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. **203** (2003), no. 1, 237–261. MR MR1996872 (2005d:47077)
18. Feng-Yu Wang, *Heat kernel inequalities for curvature and second fundamental form*, preprint.
19. , *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields **109** (1997), no. 3, 417–424. MR MR1481127 (98i:58253)
20. , *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. **35** (2007), no. 4, 1333–1350. MR MR2330974 (2008e:60192)

† Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A. E-mail address: gordina@math.uconn.edu
‡ Department of Mathematics, Bielefeld University, D-33501 Bielefeld, Germany
‡ Departments of Mathematics and Statistics, Purdue University, 150 N. University St, West Lafayette, IN 47907-2067 USA
E-mail address: roeckner@math.uni-bielefeld.de
∗ Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK
E-mail address: wangfy@bnu.edu.cn
∗∗ School of Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China
E-mail address: wangfy@bnu.edu.cn