GLOBAL WELL-POSEDNESS AND LIMIT BEHAVIOR FOR THE MODIFIED FINITE-DEPTH-FLUID EQUATION

ZHIHUA GUO AND BAOXIANG WANG

Abstract. Considering the Cauchy problem for the modified finite-depth-fluid equation
\[ \partial_t u - G_\delta (\partial_x^2 u) \mp u^2 u_x = 0, \quad u(0) = u_0, \]
where \( G_\delta f = -i \mathcal{F}^{-1} \left[ \coth(2\pi \delta \xi) - \frac{1}{2\pi \delta \xi} \right] \mathcal{F} f, \delta \gtrsim 1, \) and \( u \) is a real-valued function, we show that it is uniformly globally well-posed if \( u_0 \in H^s (s \geq 1/2) \) with \( \| u_0 \|_{L^2} \) sufficiently small for all \( \delta \gtrsim 1. \) Our result is sharp in the sense that the solution map fails to be \( C^3 \) in \( H^s (s < 1/2) \). Moreover, we prove that for any \( T > 0, \) its solution converges in \( C([0, T]; H^s) \) to that of the modified Benjamin-Ono equation if \( \delta \) tends to \( +\infty. \)

1. Introduction

In this paper, we study the Cauchy problem for the (defocusing) modified finite-depth-fluid (mFDF) equation (the focusing version with nonlinearity \( u^2 u_x \) can also be treated by our methods)
\[ \partial_t u - G_\delta (\partial_x^2 u) - u^2 u_x = 0, \quad u(x, 0) = u_0(x), \quad (1.1) \]
where \( u: \mathbb{R}^2 \to \mathbb{R} \) is a real-valued function of \( (x, t) \in \mathbb{R} \times \mathbb{R}, \)
\[ G_\delta f = -i \mathcal{F}^{-1} \left[ \coth(2\pi \delta \xi) - \frac{1}{2\pi \delta \xi} \right] \mathcal{F} f, \quad (1.2) \]
and \( \delta > 0 \) is a real number which characterizes the depth of the fluid layer. The equation (1.1) is a special one of the following so-called generalized finite-depth-fluid equations
\[ \partial_t u - G_\delta (\partial_x^2 u) + u^k u_x = 0, \quad u(x, 0) = u_0(x). \quad (1.3) \]

Eq. (1.3) with \( k = 1 \) was first derived by Joseph [9, 15, 18] to describe the propagation of internal waves in the stratified fluid of finite depth. From the physical point of view, if the depth \( \delta \) tends to infinity, then Eq. (1.1) reduces to the modified Benjamin-Ono equation
\[ \partial_t u - H (\partial_x^2 u) - u^2 u_x = 0, \quad u(x, 0) = u_0(x), \quad (1.4) \]
where \( H = -i \mathcal{F}^{-1} \text{sgn}(\xi) \mathcal{F} \) denotes the Hilbert transform. There is another form of the modified finite-depth-fluid equation which is
\[ \partial_t u - \frac{3}{2\pi \delta} G_\delta (\partial_x^2 u) - u^2 u_x = 0, \quad u(x, 0) = u_0(x). \quad (1.5) \]

2000 Mathematics Subject Classification. Primary: 35Q35; Secondary: 35Q53.
Key words and phrases. Global wellposedness, Modified finite-depth-fluid equation, Limit behavior.
It is easy to see that under the transformation
\[ u(t, x) \rightarrow \left( \frac{3}{2\pi \delta} \right)^{1/2} u\left( \frac{3}{2\pi \delta} t, x \right), \tag{1.6} \]
Eq. (1.1) turns into Eq. (1.5). If the depth \( \delta \) tends to 0, then Eq. (1.5) becomes the modified Korteweg-de Vries equation
\[ \partial_t u + \partial_x^3 u - u^2 u_x = 0. \tag{1.7} \]

There are a few literatures which are concerned with the well posedness for the Cauchy problem (1.3). For the case \( k = 1 \), using the energy methods, Abdelouhab, Bona, Folland and Saut [1] obtained global wellposedness in \( H^s \) with \( s > \frac{3}{2} \), and the limit behavior as \( \delta \rightarrow \infty \) and \( \delta \rightarrow 0 \) of the solutions of Eqs. (1.3) in \( C^k([0, T]; H^{s-2k}) \) (\( s > 3/2 \)) and \( C([0, T]; H^s) \) (\( s \geq 2 \)). For \( k \geq 4 \), Han and Wang [6] proved global wellposedness for the equation (1.3) with small initial data in the critical Besov spaces by using the smoothing effect estimates. To the authors’ knowledge, we are not aware of any other wellposedness results. On the other hand, the limit equations (1.4) and (1.7) have been extensively studied during the past decades. See [20] for a thorough review.

In the first part of this paper, we study the wellposedness for the Cauchy problem (1.1). Our methods are inspired by the important observation made by the first-named author [3] for the modified Benjamin-Ono equation. Precisely, for the modified Benjamin-Ono equation, one may use a direct contraction principle to prove wellposedness but without using a gauge transformation. We will adopt the same ideas for the mFDF equation. From the technical point of view, Eq. (1.1) is easier to handle than Eq. (1.5). Indeed, to prove wellposedness by iteration, the biggest enemy is the loss of derivative from the nonlinearity and the worst case is the high-low interaction. We will see from Lemma 3.1 that if \( \delta \gtrsim 1 \), the dispersion relation of Eq. (1.1) has uniform estimates in high frequency while that of Eq. (1.5) doesn’t. Our methods rely heavily on the symmetries of the mFDF equation (1.1). The first one is the scaling invariance which enables us to assume the initial data has small norm. It is easy to see that Eq. (1.1) is invariant under the following transformation
\[ u(x, t) \rightarrow u_\lambda = \frac{1}{\lambda^{1/2}} u\left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right), \quad u_0 \rightarrow u_{0, \lambda} = \frac{1}{\lambda^{1/2}} u_0\left( \frac{x}{\lambda} \right), \quad \delta \rightarrow \lambda \delta. \tag{1.8} \]
We will assume \( \lambda \gg 1 \), thus \( \lambda \delta \gtrsim 1 \) if \( \delta \gtrsim 1 \). There are at least the following three conservation laws preserved under the flow of (1.4)
\[ \frac{d}{dt} \int \mathbb{R} u(x, t) dx = 0, \tag{1.9} \]
\[ \frac{d}{dt} \int \mathbb{R} u(x, t)^2 dx = 0, \tag{1.10} \]
\[ \frac{d}{dt} \int \mathbb{R} \left[ \frac{1}{2} u G_0 u_x - \frac{1}{12} u(x, t)^4 \right] dx = 0. \tag{1.11} \]
These conservation laws provide a priori bounds on the solution. For example, we can get from Lemma 3.1, (1.10) and (1.11) that if \( u \) is a smooth solution to (1.1) (for the focusing case, we assume \( \|u_0\|_{L^2} \ll 1 \)) and \( \delta \gtrsim 1 \) then
\[ \|u\|_{H^{1/2}} \lesssim C(\|u_0\|_{H^{1/2}}). \tag{1.12} \]
There are several methods to compensate the loss of derivative from the non-linearity. Energy methods exploit the "energy cancelation", which usually requires high regularity of the initial data. Another approach is the smoothing effect estimate for the linear solution. On the other hand, Bourgain’s space \( X^{s,b} \) defined as a closure of the following space

\[
\{ f \in \mathcal{S}(\mathbb{R}^2) : \|f\|_{X^{s,b}} = \|\langle \xi \rangle^s (\tau - \omega_b(\xi))^b \hat{f}(\xi, \tau)\|_{L^2} \}
\]

is very useful in the study of the low regularity theory of the nonlinear dispersive equations [2, 13, 8]. One might try a direct perturbative approach in \( X^{s,b} \) space as we found for the mBO equation [3]. However, compared to the mFDF equation, there is a new difficulty caused by the component \( \xi/\delta \) in the dispersion relation. Fortunately, there is a cancelation we can use. Precisely, the resonance is almost the same as in the mBO equation. The spaces of these structures were first found and used by Ionescu and Kenig [7] to remove some logarithmic divergence. Now we state our main results:

**Theorem 1.1.** Fix \( 0 < c_0 < \infty \). Let \( s \geq 1/2 \) and \( \delta \geq c_0 \). Assume \( u_0 \in H^s \) and \( \|u_0\|_{L^2} \ll 1 \). Then

(a) Existence. There exists \( T = T(\|u_0\|_{H^{1/2}}, c_0) > 0 \) independent of \( \delta \) and a solution \( u \) to the mFDF equation (1.1) (or its focusing version) satisfying

\[
u \in F^s(T) \subset C([-T, T] : H^s), \tag{1.14}
\]

where the function space \( F^s(T) \) will be defined later (see section 2).

(b) Uniqueness. The solution mapping \( u_0 \to u \) is the unique extension of the mapping \( H^\infty \to C([-T, T] : H^\infty) \).

(c) Lipschitz continuity. For any \( R > 0 \), the mapping \( u_0 \to u \) is Lipschitz continuous from \( \{ u_0 \in H^s : \|u_0\|_{H^s} < R, \|u_0\|_{L^2} \ll 1 \} \) to \( C([-T, T] : H^s) \) uniformly for all \( \delta \geq c_0 \).

(d) Persistence of regularities. If in addition \( u_0 \in H^{s_1} \) for some \( s_1 > s \), then the solution \( u \) belongs to \( H^{s_1} \) uniformly for all \( \delta \geq c_0 \).

From the a-priori bound (1.12) and iterating Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** Fix \( 0 < c_0 < \infty \). The Cauchy problem for Eq. (1.1) (or its focusing version) is uniformly globally wellposed if \( \phi \) belongs to \( H^s \) for \( s \geq 1/2 \) with \( \|\phi\|_{L^2} \) sufficiently small for all \( \delta \geq c_0 \).

**Remark 1.3.** Our methods also work for the complex-valued mFDF equation (1.1). We can obtain local wellposedness but with some weaker uniqueness. See [3].

For the other mFDF equation (1.5), it is easy to see from (1.6) that local wellposedness also holds. However, we can not obtain uniform local (global) wellposedness for \( 0 < \delta \leq 1 \). This is the reason why we can not prove the limit behavior in \( C([0, T] : H^{1/2}) \) as \( \delta \to 0 \) for Eq. (1.5) which we conjecture holds. Our results are sharp in the following sense.
Theorem 1.4. Assume $\delta \geq 1$. If $s < 1/2$, then the solution map of Eq. (1.1) is not $C^3$ in $H^s$.

In the second part we study the limit behavior as $\delta \to \infty$ for Eq. (1.1). It is natural to conjecture that the solution of Eq. (1.1) converges to that of (1.4) as $\delta \to \infty$. Indeed, denote by $S^4_\delta$, $S_T$ the solution map of Eq. (1.1), Eq. (1.4) in [14, 3] and we proved the following

Theorem 1.5. Let $s \geq 1/2$. Assume $u_0 \in H^s(\mathbb{R})$ with $\|u_0\|_{L^2} \ll 1$. For any $T > 0$, then

$$\lim_{\delta \to \infty} \|S^4_\delta(u_0) - S_T(u_0)\|_{C([0,T],H^s)} = 0.$$  \hfill (1.15)

Remark 1.6. We are only concerned with the limit in the same regularity space. There seems no convergence rate. This can be seen from the linear solution,

$$\| \mathcal{F}^{-1} e^{i[t \coth(2\pi \xi) - \frac{1}{2\pi \xi}]\xi^4} \mathcal{F}u_0 - e^{i\theta \partial_x^2} u_0\|_{C([0,T],H^s)} \to 0, \quad \text{as } \delta \to \infty,$$

but without any convergence rate. If the initial data has higher regularity, then there is a convergence rate. For example, we prove that

$$\|u_\delta - v\|_{C([0,T],H^{1/2})} \lesssim \|\phi_1 - \phi_2\|_{H^{1/2}} + \frac{1}{\delta} C(T, \|\phi_1\|_{H^{1/2}}, \|\phi_2\|_{H^{1/2}}).$$

For the limit behavior for the other form Eq. (1.5) as $\delta \to 0$, we can’t prove the same results. One can obtain the similar results as in [1] using the energy methods.

In proving Theorem 1.5 we will adopt the same ideas as we did for the KdV-Burger equations [5]. Considering the difference equation, we first treat the difference term $(\mathcal{G}_\delta - \mathcal{H}) \partial_x^2 u$ as nonlinear term, then use the uniform global well-posedness.

The rest of the paper is organized as following. In Section 2 we present some notations and Banach function spaces. Some properties of the space are given in Section 3. In Section 4 we prove symmetric estimates that will be used to prove trilinear estimates in Section 5. Theorem 1.1, 1.4, 1.5 are proved in Section 6, 7, 8, respectively.

2. Notation and Definitions

Throughout this paper, we fix $0 < c_0 < \infty$. For $x, y > 0$, $x \lesssim y$ means that there exists $C > 0$ that may depend on $c_0$ such that $x \leq Cy$. By $x \sim y$ we mean $x \lesssim y$ and $y \lesssim x$. Similarly, we use $x \gtrsim y$, $x \ll y$ and $x \gg y$. For $f \in S'$ we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ for both spatial and time variables,

$$\hat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x, t) dx dt.$$

We denote by $\mathcal{F}_x$ ($\mathcal{F}_t$) the Fourier transform on spatial variable (time variable). If there is no confusion, we still write $\mathcal{F} = \mathcal{F}_x$. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers, respectively. Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. For $k \in \mathbb{Z}_+$ let $I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\}$ if $k \geq 1$ and $I_0 = [-2, 2]$.

Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$, $\chi_k$
By a slight abuse of notation we also define the operator 
\[ P \]
and
\[ \{ \}
Roughly speaking, \[ \eta_k = \chi_k \] if \( k \geq 1 \) and \( \eta_k = 0 \) if \( k \leq -1 \). Also, for \( k_1 \leq k_2 \in \mathbb{Z} \) let
\[ \eta_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \eta_k \]
and \( \eta_{\leq k_2} = \sum_{k=-\infty}^{k_2} \eta_k \).

For simplicity of notation, let \( a \) and \( \text{thd}(\) integers,
\( N_1 \) be the dispersion relation associated to Eq. (1.1). The elementary properties of \( \omega \)
we define denote the solution of the free finite-depth-fluid evolution given by
\[ \hat{P}_k u(\xi) = \eta_k(\xi) \hat{u}(\xi). \]

By a slight abuse of notation we also define the operator \( P_k \) on \( L^2(\mathbb{R}) \) by formula \( \mathcal{F}(P_k u)(\xi,\tau) = \eta_k(\xi) \mathcal{F}(u)(\xi,\tau) \). For \( l \in \mathbb{Z} \) let
\[ P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k. \]

Let \( a_1, a_2, a_3, a_4 \in \mathbb{R} \). It will be convenient to define the quantities \( a_{\text{max}} \geq a_{\text{sub}} \geq a_{\text{thd}} \geq a_{\text{min}} \) to be the maximum, sub-maximum, third-maximum, and minimum of \( a_1, a_2, a_3, a_4 \) respectively. We also denote \( \text{sub}(a_1, a_2, a_3, a_4) = a_{\text{sub}} \) and \( \text{thd}(a_1, a_2, a_3, a_4) = a_{\text{thd}} \). Usually we use \( k_1, k_2, k_3, k_4 \) and \( j_1, j_2, j_3, j_4 \) to denote integers, \( N_i = 2^{k_i} \) and \( L_i = 2^{j_i} \) for \( i = 1, 2, 3, 4 \) to denote dyadic numbers.

For \( \xi \in \mathbb{R} \) let
\[ \omega_{\delta}(\xi) = [\coth(2\pi \delta \xi) - \frac{1}{2\pi \delta \xi}]\xi^2 \]
be the dispersion relation associated to Eq. (1.1). The elementary properties of the function \( \omega_{\delta}(\xi) \) are given in Lemma 3.1. For \( \phi \in L^2(\mathbb{R}) \) let \( U_{\delta}(t)\phi \in C(\mathbb{R} : L^2) \) denote the solution of the free finite-depth-fluid evolution given by
\[ \mathcal{F}_x[U_{\delta}(t)\phi](\xi,t) = \mathcal{E}^{i\delta \omega_{\delta}(\xi)}\tilde{\phi}(\xi), \]
where \( \omega_{\delta}(\xi) \) is defined in (2.1). For \( k, j \in \mathbb{Z}^+ \) let \( D_{k,j} = \{ (\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega_{\delta}(\xi) \in I_j \} \). We define first the Banach spaces \( X_k = X_k(\mathbb{R}^2) \). For \( k \in \mathbb{Z}^+ \) we define
\[ X_k = \{ f \in L^2(\mathbb{R}^2) : f(\xi,\tau) \text{ is supported in } I_k \times \mathbb{R} \text{ and} \]
\[ \| f \|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \| \eta_j(\tau - \omega_{\delta}(\xi)) \cdot f(\xi,\tau) \|_{L^2_{\xi,\tau}} < \infty \}, \]
where
\[ \beta_{k,j} = 1 + 2^{2j-2k}/5. \]

The precise choice of the coefficients \( \beta_{k,j} \) is important in order for all the trilinear estimates to hold. This factor is particularly important in controlling the high-low interaction.

The spaces \( X_k \) are not sufficient for our purpose, due to various logarithmic divergences involving the modulation variable. Fix \( M > 1 \) to be a large integer
which is dependent on $c_0$. For $k \geq M$ we also define the Banach spaces $Y_k = Y_k(\mathbb{R}^2)$. For $k \geq M$ we define
\[
Y_k = \{ f \in L^2(\mathbb{R}^2) : f(\xi, \tau) \text{ is supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and } \|f\|_{Y_k} := 2^{-k/2}\|\mathcal{F}^{-1}[\tau - \omega_\delta(\xi) + i] f(\xi, \tau)\|_{L^1 L^2_\tau} < \infty \}. \tag{2.5}
\]
Then for $k \in \mathbb{Z}_+$ we define
\[
Z_k := X_k \text{ if } k \leq M - 1 \text{ and } Z_k := X_k + Y_k \text{ if } k \geq M. \tag{2.6}
\]
The spaces $Z_k$ are our basic Banach spaces. For $s \geq 0$ we define the Banach spaces $F^s = F^s(\mathbb{R} \times \mathbb{R})$:
\[
F^s = \{ u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) : \|u\|_{F^s} = \sum_{k=0}^{\infty} 2^{sk}\|\eta_k(\xi)\mathcal{F}(u)\|_{Z_k}^2 < \infty \}, \tag{2.7}
\]
and $N^s = N^s(\mathbb{R} \times \mathbb{R})$ which is used to measure the nonlinear term and can be viewed as an analogue of $X^{s, b-1}$
\[
N^s = \{ u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) : \|u\|_{N^s}^2 = \sum_{k=0}^{\infty} 2^{sk}\|\eta_k(\xi)(\tau - \omega_\delta(\xi) + i)^{-1}\mathcal{F}(u)\|_{Z_k}^2 < \infty \}. \tag{2.8}
\]
We also define $F^s(T)$ and $N^s(T)$ to be the spaces that $F^s$ and $N^s$ restricted to the time interval $[-T, T]$, respectively.

These $I^1$-type $X^{s,b}$ structures $X_k$ were first introduced and used in [19, 7, 8]. It is also useful in the study of uniform global wellposedness and inviscid limit for the nonlinear dispersive equation with dissipative term [5]. The combination of $X^{s,b}$ structure and smoothing effect $Z_k$ were first used by Ionescu and Kenig [7].

3. Properties of the spaces $Z_k$

In this section we devote to study the properties of the spaces $Z_k$. We start with some elementary estimates on the dispersion relation $\omega_\delta(\xi)$ some of which were also proved in [6].

**Lemma 3.1.** If $\delta > 0$, then
\[
\begin{cases}
|\omega_\delta(\xi)| \sim |\xi|^2, & |\omega'_\delta(\xi)| \sim |\xi|, & |\omega''_\delta(\xi)| \sim 1: \text{ if } |\xi| \gtrsim 1/\delta. \\
|\omega_\delta(\xi)| \sim \delta|\xi|^3, & |\omega'_\delta(\xi)| \sim \delta|\xi|^2, & |\omega''_\delta(\xi)| \sim \delta|\xi|: \text{ if } |\xi| \lessgtr 1/\delta.
\end{cases} \tag{3.1}
\]

**Proof.** Since $\omega_\delta(\cdot)$ is odd, we may assume $\xi > 0$. Let $h(\xi) = [\coth(\xi) - \frac{1}{\xi}][e^\xi - e^{-\xi}]^2$, then we see that $\omega_\delta(\xi) = \frac{1}{2\pi\xi}h(2\pi\delta\xi)$. Using Taylor’s expansion, we get
\[
h(\xi) = \frac{\sum_{k=0}^{\infty} (\frac{\xi}{2k+1})^{2k+1}}{\xi(e^\xi - e^{-\xi})^2} \xi^2 > 0, \text{ if } \xi > 0.
\]
From $\lim_{\xi \to 0^+} \frac{h(\xi)}{\xi^2} > 0$ and $\lim_{\xi \to +\infty} \frac{h(\xi)}{\xi^2} > 0$, we get that $|h(\xi)| \sim |\xi|^2$ if $|\xi| \gtrsim 1$ and $|h(\xi)| \sim |\xi|^3$ if $|\xi| \lessgtr 1$. Direct computations show that
\[
\begin{align*}
h'(\xi) &= \left(\frac{1}{2\xi} - \frac{4}{(e^\xi - e^{-\xi})^2}\right)\xi^2 + [\coth(\xi) - \frac{1}{\xi}]2\xi;
\end{align*}
\]
\[
\begin{align*}
h''(\xi) &= \frac{8e^\xi(e^\xi + e^{-\xi}) - 16\xi(e^\xi - e^{-\xi}) + 2(e^\xi + e^{-\xi})(e^\xi - e^{-\xi})^2}{(e^\xi - e^{-\xi})^3}.
\end{align*}
\]
Using Taylor’s expansion, we easily see that if $\xi > 0$ then
\[
h''(\xi) = \frac{16\xi^2(e^\xi + e^{-\xi}) - 16\xi(e^\xi - e^{-\xi})}{(e^\xi - e^{-\xi})^3} > 0.
\]
From $\lim_{\xi \to 0^+} h''(\xi) > 0$ and $\lim_{\xi \to +\infty} h(\xi) > 0$, we get that $|h''(\xi)| \sim 1$ if $|\xi| \gtrsim 1$ and $|h''(\xi)| \sim |\xi|$ if $|\xi| \lesssim 1$. Similarly, we get $|h'(\xi)| \sim |\xi|$ if $|\xi| \gtrsim 1$ and $|h'(\xi)| \sim |\xi|^2$ if $|\xi| \lesssim 1$. Therefore, we complete the proof of the lemma.

For $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, let
\[
\Omega(\xi_1, \xi_2, \xi_3) = \omega_3(\xi_1) + \omega_3(\xi_2) + \omega_3(\xi_3) - \omega_3(\xi_1 + \xi_2 + \xi_3). \tag{3.2}
\]
This is the resonance function which plays crucial rule in the trilinear estimate. See [21] for more perspective discussion. We prove an estimate on the resonance in the following lemma.

**Lemma 3.2.** Let $\delta \geq 0$. Assume $|\xi_1| \leq |\xi_2| \leq |\xi_3|$, $|\xi_1| \ll |\xi_3|$, $|\xi_1 + \xi_2 + \xi_3| \sim |\xi_3|$ and $|\xi_3| \gg 1$. Then we have
\[
|\Omega(\xi_1, \xi_2, \xi_3)| \sim |\xi_1 + \xi_2| \cdot |\xi_3|. \tag{3.3}
\]

**Proof.** We consider first the case that $|\xi_2| \ll |\xi_3|$. From the mean value formula we see that
\[
|\omega_3(\xi_3) - \omega_3(\xi_1 + \xi_2 + \xi_3)| \sim |\xi_3| \cdot |\xi_1 + \xi_2|, \quad |\omega_3(\xi_1) + \omega_3(\xi_2)| \ll |\xi_3| \cdot |\xi_1 + \xi_2|,
\]
which immediately gives (3.3) in this case.

We consider now the case $|\xi_2| \gtrsim |\xi_3|$. Then it suffices to show that $|\Omega(\xi_1, \xi_2, \xi_3)| \sim |\xi_3|^2$. We get from the definition that
\[
\omega_3(\xi_2) + \omega_3(\xi_3) - \omega_3(\xi_2 + \xi_3) = \text{coth}(2\pi\delta(\xi_2 + \xi_3)) - \text{coth}(2\pi\delta\xi_2 + \xi_3) \lesssim |\xi_3|^2 + |\xi_3|^3 =: I + II + III.
\]
It is easy to see that $|I|, |III| \ll |\xi_3|^2$ and $|II| \sim |\xi_3|^3$, then (3.3) follows from the fact that $|\omega_3(\xi_1 + \xi_2 + \xi_3) - \omega_3(\xi_2 + \xi_3)| \ll |\xi_3|^2$.

From the definitions we see that if $k \in \mathbb{Z}_+$ and $f_k \in Z_k$ then $f_k$ can be written in the form
\[
\{ f_k = \sum_{j=0}^{k-1} f_{k,j} + g_k; \sum_{j=0}^{k-1} 2^{j/2} \beta_{k,j} \| f_{k,j} \|_{L^2} + \| g_k \|_{Y_k} \leq 2 \| f_k \|_{Z_k}; \tag{3.4}
\]
such that $f_{k,j}$ is supported in $D_{k,j}$ and $g_k$ is supported in $\cup_{j=0}^{k-1} D_{k,j}$ (if $k \leq M - 1$ then $g_k \equiv 0$). In analogy with Lemma 4.1 in [7] we have the following

**Lemma 3.3.** (a) If $m, m' : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}_+$, and $f_k \in Z_k$ then
\[
\{ \| m(\xi) f_k(\xi, \tau) \|_{Z_k} \leq C \| F^{-1}(m) \|_{L^1(\mathbb{R})} \| f_k \|_{Z_k}; \| m'(\tau) f_k(\xi, \tau) \|_{Z_k} \leq C \| m'(\tau) \|_{L^\infty(\mathbb{R})} \| f_k \|_{Z_k}. \tag{3.5}
\]
(b) If $\delta \geq 0$, $k \in \mathbb{Z}_+$, $j \geq 0$, and $f_k \in Z_k$ then
\[
\| \eta_j(\tau - \omega_3(\xi)) f_k(\xi, \tau) \|_{X_k} \lesssim \| f_k \|_{Z_k}. \tag{3.6}
\]
(c) If $\delta \geq 0$, $k \geq 1$, $j \in [0, k]$, and $f_k$ is supported in $I_k \times \mathbb{R}$ then
\[
\| F^{-1}[\eta_{\leq 1}(\tau - \omega_3(\xi)) f_k(\xi, \tau)] \|_{L^1_{\tau}L^2_{\xi}} \lesssim \| F^{-1}(f_k) \|_{L^1_{\tau}L^2_{\xi}}. \tag{3.7}
\]
Proof. It is easy to see that part (a) follows directly from Plancherel theorem and the definitions.

For part (b), we may assume $k \geq M$, $f_k = g_k \in Y_k$, and $j \leq k$. From the definition we see that if $g_k \in Y_k$ then $g_k$ can be written in the form
\[
\begin{cases}
g_k(\xi, \tau) = 2^{k/2} \chi_{[k-1, k+1]}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} \eta_{\leq k}(\tau - \omega_\delta(\xi)) \mathcal{F}_x(h)(\xi, \tau); \\
\|g_k\|_{Y_k} = C\|h\|_{L^1_t L^2_x}.
\end{cases}
\]
Thus from Plancherel’s equality we get
\[
\|\eta_j(\tau - \omega_\delta(\xi))g_k(\xi, \tau)\|_{X_k} \lesssim \sup_{0 \leq j \leq k} \|\chi_k(\xi)\eta_j(\tau - \omega_\delta(\xi))\mathcal{F}_x(h)(\xi, \tau)\|_{L^2_t L^2_x},
\]
which suffices to prove (3.9).

On the other hand, by changing of variable $\mu = \omega_\delta(\xi)$ we get from Lemma 3.1 and $\delta \geq \delta_0$ that
\[
\|\chi_k(\xi)\eta_j(\tau - \omega_\delta(\xi))\|_{L^\infty_t L^2_x} \lesssim 2^{j-k},
\]
which completes the proof of part (b).

For part (c), from Plancherel’s equality, it suffices to prove that
\[
\left\|\int_{\mathbb{R}} e^{ix\xi} \chi_k(\xi) \eta_{\leq j}(\tau - \omega_\delta(\xi)) d\xi\right\|_{L^1_t L^p_x} \leq C.
\]
We may assume $k \geq M$ in proving (3.9). By the change of variable $\tau - \omega_\delta(\xi) = \alpha$, integration by parts and Lemma 3.1 we obtain that
\[
\left|\int_{\mathbb{R}} e^{ix\xi} \chi_k(\xi) \eta_{\leq j}(\tau - \omega_\delta(\xi)) d\xi\right| \lesssim \frac{2^{j-k}}{1 + (2^{j-k}x)^2},
\]
which suffices to prove (3.9).

We study now the embedding properties of the spaces $Z_k$ which is important in the trilinear estimates.

Lemma 3.4. Assume $\delta \geq \delta_0$. Let $k \in \mathbb{Z}_+$, $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let $Y$ be $L^p_t L^q_x$ or $L^p_t L^q_x$ for some $1 \leq p < \infty$ with the property that
\[
\|U_\delta(t)f\|_Y \lesssim 2^k \|f\|_{L^1(\mathbb{R})}
\]
for all $f \in L^2(\mathbb{R})$ with $\hat{f}$ supported in $I_k$. Then we have that if $f_k \in Z_k$
\[
\|\mathcal{F}^{-1}(f_k)\|_Y \lesssim 2^k \|f_k\|_{Z_k}.
\]

Proof. We assume first that $f_k = f_{k,j}$ with $\|f_k\|_{X_k} = 2^{j/2} \beta_{k,j} \|f_{k,j}\|_{L^2}$ and $f_{k,j}$ is supported in $D_{k,j}$ for some $j \geq 0$. Then we have
\[
\mathcal{F}^{-1}(f_k)(x,t) = \int f_{k,j}(\xi, \tau)e^{ix\xi} e^{it\tau} d\xi d\tau = \int f_{k,j}(\xi, \tau + \omega_\delta(\xi)) e^{ix\xi} e^{it\omega_\delta(\xi)} d\xi d\tau.
\]
From the hypothesis on $Y$, we obtain
\[
\|\mathcal{F}^{-1}(f_k)(x,t)\|_Y \lesssim \int \eta_j(\tau) \left\|e^{it\tau} \int f_{k,j}(\xi, \tau + \omega_\delta(\xi)) e^{ix\xi} e^{it\omega_\delta(\xi)} d\xi\right\|_Y d\tau \lesssim 2^{k/2} \|f_{k,j}\|_{L^2}.
\]
which completes the proof in this case.

We assume now that \( k \geq M \) and \( f_k = g_k \in Y_k \). From definitions and (3.8), it suffices to prove that if

\[
f(\xi, \tau) = 2^{k/2} \chi_{[k-1, k+1]}(\xi)(\tau - \omega_\delta(\xi) + i)\eta_{\leq k}(\tau - \omega_\delta(\xi)) \cdot h(\tau) \tag{3.11}
\]

then

\[
\left\| \int_{\mathbb{R}^2} f(\xi, \tau)e^{ix\xi}e^{i\tau \xi} d\xi d\tau \right\|_Y \lesssim 2^{ks} \|h\|_{L^2}.
\tag{3.12}
\]

Since \( k \geq 100 \) and \( |\xi| \in [2^{k-2}, 2^{k+2}] \), we may assume that the function \( h \) in (3.11) is supported in the set \( \{ t : |\tau| \in [2^{2k-C}, 2^{2k+C}] \} \). Let \( h_+ = h \cdot 1_{[0, \infty)} \), \( h_- = h \cdot 1_{(-\infty, 0]} \), and define the corresponding functions \( f_+ \) and \( f_- \) as in (3.11). By symmetry, it suffices to prove the bounds (3.12) for the function \( f_+ \), which is supported in the set \( \{ (\xi, \tau) : \xi \in [2^{k-2}, 2^{k+2}], \tau \in [2^{2k-C}, 2^{2k+C}] \} \). From the fact that \( \omega_\delta(\xi) \) is strictly increasing, we have an inverse function \( \varphi(\alpha) = \omega_\delta^{-1}(\alpha) \). It follows from Lemma 3.1 that \( \varphi(\tau) \sim 2^k \) for \( \tau \in [2^{k-C}, 2^{k+C}] \). Thus \( f_+(\xi, \tau) \equiv 0 \) unless \( |\varphi(\tau) - \xi| \leq C \).

Let

\[
f_+(\xi, \tau) = 2^{k/2} \chi_{[k-1, k+1]}(\varphi(\tau))(\varphi(\tau)^2 - \xi^2 + (\varphi(\tau) - \xi)^2 + i\varphi(\tau)2^{-k})^{-1} \cdot \eta_0(\varphi(\tau) - \xi) \cdot h(\tau) \tag{3.13}
\]

Since for \( \varphi(\tau) \sim 2^k, \tau \in [2^{k-C}, 2^{k+C}] \) and \( |\varphi(\tau) - \xi| \leq C \) we have

\[
|\tau - \omega_\delta(\xi) - \varphi(\tau)^2 + \xi^2| = |\omega_\delta(\varphi(\tau)) - \omega_\delta(\xi) - \varphi(\tau)^2 + \xi^2| \leq C,
\]

it is easy to see that

\[
\|f_- - f_+\|_{X_k} \leq C \|h_+\|_{L^2}.
\]

Thus using the estimate for \( f \in X_k \), we get

\[
\|\mathcal{F}^{-1}(f_+ - f_-)\|_Y \leq C 2^{ks} \|h_+\|_{L^2}.
\]

It remains to estimate \( \|\mathcal{F}^{-1}(f_+^\ast)\|_Y \). We make the change of variables \( \xi = \varphi(\tau) - \mu \), then

\[
\mathcal{F}^{-1}(f_+^\ast)(x, t) = 2^{k/2} \int_{\mathbb{R}} h_+(\tau)(\varphi(\tau)^{-1} \chi_{[k-1, k+1]}(\varphi(\tau))e^{ix\tau}e^{i\xi\varphi(\tau)}d\tau
\]

\[
\cdot \int_{\mathbb{R}} \eta_0(\mu)(\mu + i/2^{k+1})^{-1}e^{ix\mu}d\mu.
\tag{3.14}
\]

The second integral is bounded by \( C \). We make the change of variable \( \xi = \varphi(\tau) \) in the first integral, then by the hypothesis of \( Y \) we get

\[
\|\mathcal{F}^{-1}(f_+^\ast)\|_Y \lesssim 2^{ks} 2^{k/2} \|h_+(\omega_\delta(\xi))\chi_k(\xi)\|_{L^2} \lesssim 2^{ks} \|h_+\|_{L^2}.
\]

Therefore, we complete the proof of the lemma.

In order to obtain the more specific embedding properties of the spaces \( Z_k \), we need the estimates for the free finite-depth-fluid equation. We prove the Strichartz estimates, smoothing effects, and maximal function estimates for the free solutions in the following lemma.

Lemma 3.5. Assume \( \delta \geq c_0 \). Let \( k \in \mathbb{Z}_+ \) and \( I \subset \mathbb{R} \) be an interval with \( |I| \leq 1 \). Then for all \( \phi \in L^2(\mathbb{R}) \) with \( \tilde{\phi} \) supported in \( I_k \),

(a) Strichartz estimates: if \( k \geq 1 \) then

\[
\|U_\delta(t)\phi\|_{L^q_tL^r_x} \lesssim \|\phi\|_{L^2(\mathbb{R})},
\tag{3.15}
\]

where \( (q, r) \) is admissible, namely \( 2 \leq q, r \leq \infty \) and \( 2/q = 1/2 - 1/r \).
(b) Smoothing effect: if \( k \geq 1 \) then
\[
\|U_s(t)\phi\|_{L^\infty_t L^2_x} \lesssim 2^{-k/2}\|\phi\|_{L^2(\mathbb{R})}.
\] (3.16)

(c) Maximal function estimate: if \( k \geq 0 \) then
\[
\|U_s(t)\phi\|_{L^\infty_t L^\infty_x} \lesssim 2^{k/2}\|\phi\|_{L^2(\mathbb{R})},
\] (3.17)
\[
\|U_s(t)\phi\|_{L^\infty_t L^\infty_x} \lesssim 2^{k/4}\|\phi\|_{L^2(\mathbb{R})}.
\] (3.18)

**Proof.** For part (a), we use the results in [4] and Lemma 3.1. We easily see that the Strichartz estimates (3.15) holds if \((q,r)\) is admissible pairs as for Schrödinger equation.

Part (b) and the second inequality in part (c) follow from Lemma 3.1 and the results in [11]. They were also proved in [6]. The first inequality in part (c) follows from slightly modified argument in the proof of theorem 3.1 in [12]. We omit the details. \(\blacksquare\)

From Lemma 3.4 and Lemma 3.5, we immediately get the following

**Lemma 3.6.** Assume \( \delta \geq c_0 \). Let \( k \in \mathbb{Z}_+ \) and \( I \subset \mathbb{R} \) be an interval with \(|I| \lesssim 1\). Assume \((q,r)\) is admissible and \( f_k \in L_k \). Then
(a) If \( k \geq 1 \), then
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^\infty_t L^2_x} \lesssim 2^{-k/2}\|f_k\|_{L^\infty_t L^2_x}, \quad \|\mathcal{F}^{-1}(f_k)\|_{L^2_t L^\infty_x} \lesssim \|f_k\|_{L^\infty_t L^\infty_x}.
\]
(b) For all \( k \in \mathbb{Z}_+ \),
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^\infty_t L^\infty_x} \lesssim 2^{k/4}\|f_k\|_{L^\infty_t L^\infty_x}, \quad \|\mathcal{F}^{-1}(f_k)\|_{L^2_t L^\infty_x} \lesssim 2^{k/2}\|f_k\|_{L^\infty_t L^\infty_x}.
\]
As a consequence, for any \( s \geq 0 \) we have \( F^s \subseteq C(\mathbb{R}; H^s) \) with \( \|u\|_{L^\infty_t H^s} \lesssim \|u\|_{F^s} \).

4. A Symmetric Estimate

Following the standard fixed point argument, we will need to prove a trilinear estimate. We start with a symmetric estimate for nonnegative functions. For \( \xi_1, \xi_2, \xi_3 \in \mathbb{R} \) and \( \Omega : \mathbb{R}^3 \to \mathbb{R} \) as in (3.2), and for compactly supported nonnegative functions \( f, g, h, u \in L^2(\mathbb{R} \times \mathbb{R}) \) let
\[
J(f, g, h, u) = \int_{\mathbb{R}^6} f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_3, \mu_3)
\]
\[
u(\xi_1 + \xi_2 + \xi_3, \mu_1 + \mu_2 + \mu_3 + \Omega(\xi_1, \xi_2, \xi_3))d\xi_1 d\xi_2 d\xi_3 d\mu_1 d\mu_2 d\mu_3.
\]

**Lemma 4.1.** Assume \( \delta \geq c_0 \). Assume \( k_i, j_i \in \mathbb{Z}_+ \) and \( N_i = 2^{k_i}, L_i = 2^{j_i} \), and \( f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in \( I_{k_i} \times J_{j_i} \), \( i = 1, 2, 3, 4 \). For simplicity we write \( J = |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \).
(a) For any \( k_i, j_i \in \mathbb{Z}_+, i = 1, 2, 3, 4 \),
\[
J \lesssim 2^{(j_{\min} + j_{\max})/2 + 2g(k_{\min} + k_{\max})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (4.1)

(b) If \( N_{\text{sub}} \ll N_{\text{sub}}, N_{\text{max}} \gg 1 \), and \( (k_i, j_i) \neq (k_{\text{thd}}, j_{\text{thd}}) \) for \( i = 1, 2, 3, 4 \),
\[
J \gtrsim 2^{(j_1 + j_2 + j_3 + j_4)/2} - 2^{j_{\max}/2} - 2^{j_{\max}/2} 2^{k_{\min}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2};
\] (4.2)
if \( N_{\text{thd}} \ll N_{\text{sub}}, \ N_{\text{max}} \gg 1 \), and \((k_i, j_i) = (k_{\text{thd}}, j_{\text{max}})\) for some \( i \in \{1, 2, 3, 4\}\),

\[
J \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2 - j_{\text{max}}/2}2^{-k_{\text{max}}/2}2^{k_{\text{thd}}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.3}
\]

(c) For any \( k_i, j_i \in \mathbb{Z}_+, \ i = 1, 2, 3, 4 \), with \( N_{\text{max}} \gg 1 \),

\[
J \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2 - j_{\text{max}}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.4}
\]

(d) If \( N_{\text{min}} \ll N_{\text{max}} \) and \( N_{\text{max}} \gg 1 \), then

\[
J \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2 - k_{\text{max}}} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.5}
\]

**Proof.** Let

\[
A_k(\xi) = \int_{\mathbb{R}} |f_{k,j}(\xi, \mu)|^2 d\mu|^{1/2}, \ i = 1, 2, 3, 4.
\]

Using the Cauchy-Schwarz inequality and the support properties of the functions \( f_{k,j} \),

\[
|J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4})| \lesssim \int_{\mathbb{R}^3} A_{k_1}(\xi_1)A_{k_2}(\xi_1)A_{k_3}(\xi_1)A_{k_4}(\xi_1 + \xi_2 + \xi_3)d\xi_1d\xi_2d\xi_3
\]

which is part (a), as desired.

For part (b), we observe that \( J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \equiv 0 \) unless \( N_{\text{max}} \sim N_{\text{sub}} \).

Simple changes of variables in the integration and the fact that the function \( \omega \) is odd show that

\[
|J(f, g, h, u)| = |J(g, f, h, u)| = |J(f, h, g, u)| = |J(f, g, u, h)|,
\]

where \( \tilde{f}(\xi, \mu) = f(-\xi, -\mu) \). Thus we may assume \( k_1 \leq k_2 \leq k_3 \leq k_4 \). We assume first that \( j_3 \neq j_{\text{max}} \). Then we have several cases: if \( j_4 = j_{\text{max}} \), then we will prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) functions supported in \( I_{k_i}, \ i = 1, 2, 3 \), and \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) is an \( L^2 \) function supported in \( I_{k_4} \times J_{j_4} \), then

\[
\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3), d\xi_1d\xi_2d\xi_3
\]

This suffices for (4.2).

To prove (4.7), we first observe that since \( N_{\text{thd}} \ll N_{\text{sub}} \) then \( |\xi_3 + \xi_2| \sim |\xi_3| \). By change of variable \( \xi' = \xi_1, \xi'' = \xi_2, \xi''' = \xi_3 + \xi_2 \), we get that the left side of (4.7) is bounded by

\[
\int_{|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}, |\xi_3| \sim 2^{k_3}} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3 - \xi_2)g(\xi_1 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3 - \xi_2), d\xi_1d\xi_2d\xi_3.
\]

Note that in the integration area we have

\[
|\frac{\partial}{\partial \xi_2} [\Omega(\xi_1, \xi_2, \xi_3 - \xi_2)]| = \mid \omega'_b'(\xi_2) - \omega'_b(\xi_3 - \xi_2) \mid \sim 2^{k_3},
\]
where we use by Lemma 3.1 that $\omega_\delta'(\xi) \sim |\xi|$ and $N_2 \ll N_3$. By change of variable $\mu_2 = \Omega(\xi_1, \xi_2, \xi_3 - \xi_2)$, we get that (4.8) is bounded by

$$2^{-k_3/2} \int_{|\xi_1| \sim 2^{k_1}} g_1(\xi_1) \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2} d\xi_1$$

$$\lesssim 2^{-k_{\max}/2} 2^{k_{\min}/2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}.$$  \hspace{1cm} (4.9)

If $j_3 = j_{\max}$, this case is identical to the case $j_4 = j_{\max}$ in view of (4.6). If $j_1 = j_{\max}$ it suffices to prove that if $g_i : \mathbb{R} \to \mathbb{R}_+$ are $L^2$ functions supported in $I_k$, $i = 2, 3, 4$, and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an $L^2$ function supported in $I_k \times I_j$, then

$$\int_{\mathbb{R}^3} g_2(\xi_2) g_3(\xi_3) g_4(\xi_4) g(\xi_2 + \xi_3 + \xi_4, \Omega(\xi_2, \xi_3, \xi_4)) d\xi_2 d\xi_3 d\xi_4$$

$$\lesssim 2^{-k_{\max}/2} 2^{k_{\min}/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}.$$  \hspace{1cm} (4.10)

Indeed, by change of variables $\xi_2' = \xi_2, \xi_3' = \xi_3, \xi_4' = \xi_2 + \xi_3 + \xi_4$ and noting that in the area $|\xi_\ell| \sim 2^{k_2}, |\xi_\ell'| \sim 2^{k_3}, |\xi_\ell''| \sim 2^{k_1}$,

$$|\frac{\partial}{\partial \xi_\ell'} \Omega(\xi_2', \xi_3', \xi_4' - \xi_2' - \xi_3')| = |\omega_\delta'(\xi_2') - \omega_\delta'(\xi_2' - \xi_2'' - \xi_3'')| \sim 2^{k_3},$$

we get from Cauchy-Schwarz inequality that

$$\int_{\mathbb{R}^3} g_2(\xi_2) g_3(\xi_3) g_4(\xi_4) g(\xi_2 + \xi_3 + \xi_4, \Omega(\xi_2, \xi_3, \xi_4)) d\xi_2 d\xi_3 d\xi_4$$

$$\lesssim \int_{|\xi_\ell| \sim 2^{k_2}, |\xi_\ell'| \sim 2^{k_3}, |\xi_\ell''| \sim 2^{k_1}} g_2(\xi_2') g_3(\xi_3') g(\xi_4', \Omega(\xi_2', \xi_3', \xi_4' - \xi_2' - \xi_3') d\xi_2' d\xi_3' d\xi_4'$$

$$\lesssim 2^{-k_3/2} \int_{|\xi_\ell| \sim 2^{k_2}, |\xi_\ell'| \sim 2^{k_3}, |\xi_\ell''| \sim 2^{k_1}} g_3(\xi_3') \|g_2(\xi_2') g_4(\xi_4' - \xi_2' - \xi_3')\|_{L^2} \|g(\xi_4', \cdot)\|_{L^2} d\xi_2' d\xi_3' d\xi_4'$$

$$\lesssim 2^{-k_{\max}/2} 2^{k_{\min}/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}.$$  \hspace{1cm} (4.11)

We assume now that $j_2 = j_{\max}$. The proof is identical to the case $j_1 = j_{\max}$. We note that we actually prove that if $N_{\text{thd}} \ll N_{\text{sub}}$ then

$$J \leq C^2(2^{j_1+j_2+j_3+j_4})^{2/2-2j_1/2-2j_2/2-2j_3/2-2j_4/2} \prod_{i=1}^4 \|f_{k_1,j_i}\|_{L^2}.$$  \hspace{1cm} (4.12)

Therefore, we complete the proof for part (b).

For part (c), setting $f_{k_1,j_i} = f_{k_1,j_i}(\xi, \tau + \omega_\delta(\xi)), i = 1, 2, 3$, then we get

$$|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})|$$

$$= \int f_{k_1,j_1}^* f_{k_2,j_2}^* f_{k_3,j_3}^* f_{k_4,j_4}^*$$

$$\lesssim \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2} \|f_{k_3,j_3}\|_{L^2} \|f_{k_4,j_4}\|_{L^2}$$

$$\lesssim \|f_{k_1,j_1}\|_{L^2} \|f_{k_2,j_2}\|_{L^2} \|f_{k_3,j_3}\|_{L^2} \|f_{k_4,j_4}\|_{L^2}.$$  \hspace{1cm} (4.13)

On the other hand,

$$F^{-1}(f_{k_1,j_1}^*) = \int_{\mathbb{R}^2} f_{k_1,j_1}(\xi, \tau + \omega_\delta(\xi)) e^{ix\xi} e^{it\tau} d\xi d\tau$$

$$= \int_{\mathbb{R}^2} f_{k_1,j_1}(\xi, \tau) e^{ix\xi} e^{itw_\delta(\xi)} e^{it\tau} d\xi d\tau,$$
Indeed in the other cases we get from the fact implies that (4.14) is bounded by from (b) and (c) in these cases. We assume now similarly we can bound the other terms. Thus part (c) follows form the symmetry.

By changing variable of integration we get that

\[
\int_{\mathbb{R}^3 \cap \{ \xi_1, \xi_2 < 0 \}} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1 d\xi_2 d\xi_3
\]

is bounded by

\[
\lesssim 2^{3j/2 - k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}.
\]

(4.13)

By localizing \(|\xi_1 + \xi_2| \sim 2^l\) for \(l \in \mathbb{Z}\), we get that the right-hand side of (4.13) is bounded by

\[
\sum_l \int_{\mathbb{R}^3} \chi_l(\xi_1 + \xi_2)g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1 d\xi_2 d\xi_3.
\]

(4.14)

From the support properties of the functions \(g_i\), \(g\) and Lemma 3.2 that in the integration area

\[
|\Omega(\xi_1, \xi_2, \xi_3)| \sim |(\xi_1 + \xi_2)(\xi_1 + \xi_3)| \sim 2^{l+k_3},
\]

we get that

\[
L_{\max} \lesssim 2^{l+k_3}.
\]

(4.15)

By changing variable of integration \(\xi_1' = \xi_1 + \xi_2, \xi_2' = \xi_2, \xi_3' = \xi_1 + \xi_3\), we obtain that (4.14) is bounded by

\[
\sum_l \int_{|\xi_1'| \sim 2^l, |\xi_2'| \sim 2^k, |\xi_3'| \sim 2^{k_3}} \chi_l(\xi_1')g_1(\xi_1' - \xi_2')g_2(\xi_2')g_3(\xi_2' + \xi_3' - \xi_1')
\]

\[
g(\xi_2' + \xi_3', \Omega(\xi_1' - \xi_2', \xi_2' + \xi_3' - \xi_1'))d\xi_1' d\xi_2' d\xi_3'.
\]

(4.16)

Since in the integration area

\[
|\frac{\partial}{\partial \xi_1}|[\Omega(\xi_1' - \xi_2', \xi_2' + \xi_3' - \xi_1')] = |\omega_1'(\xi_1' - \xi_2') - \omega_3'(\xi_2' + \xi_3' - \xi_1')| \sim 2^{k_3},
\]

(4.17)

then we get from (4.17) that (4.16) is bounded by

\[
\sum_l \int_{|\xi_1'| \sim 2^l} \chi_l(\xi_1') \|g_1\|_{L^2} \|g_3\|_{L^2} \\
\|g_2(\xi_2')g(\xi_2' + \xi_3', \Omega(\xi_1' - \xi_2', \xi_2' + \xi_3' - \xi_1'))\|_{L^2} \|g\|_{L^2} d\xi_1'
\]

\[
\lesssim \sum_l 2^{l/2} 2^{-k_3/2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}
\]

\[
\lesssim 2^{l_{\max}/2} 2^{-k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2},
\]

(4.18)

where we used (4.15) in the last inequality.

From symmetry we know the case \(j_3 = j_{\max}\) is identical to the case \(j_4 = j_{\max}\), and the case \(j_1 = j_{\max}\) is identical to the case \(j_2 = j_{\max}\), thus it reduces to prove the
case \( j_2 = \max_i \). It suffices to prove that if \( g_i \) is \( L^2 \) nonnegative functions supported in \( I_{k_i} \), \( i = 1, 3, 4 \), and \( g \) is a \( L^2 \) nonnegative function supported in \( I_{k_3} \times I_{j_2} \), then
\[
\int_{\mathbb{R}^3 \cap \{\xi_1 \leq 0\}} g_1(\xi_1)g_3(\xi_3)g_4(\xi_4)g(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1 d\xi_3 d\xi_4 \lesssim 2^{j_2/2 - k_3/2} \|g_1\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}. \tag{4.19}
\]
As the case \( j_4 = \max_i \), we get that the right-hand side of (4.19) is bounded by
\[
\sum I \int_{\mathbb{R}^3} \chi_i(\xi_3 + \xi_4)g_1(\xi_1)g_4(\xi_4)g_3(\xi_3)g(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1 d\xi_4 d\xi_3. \tag{4.20}
\]
From the support properties of the functions \( g_i \), \( g \) and Lemma 3.2 that in the integration area
\[
|\Omega(\xi_1, \xi_3, \xi_4)| \sim |(\xi_1 + \xi_4)(\xi_1 + \xi_3)| \sim 2^{l+k_3},
\]
we get that
\[
L_{\max} \gtrsim 2^{l+k_3}. \tag{4.21}
\]
By changing variable of integration \( \xi'_1 = \xi_1 + \xi_3, \xi'_3 = \xi_3 + \xi_4, \xi'_4 = \xi_1 + \xi_3 + \xi_4 \), we obtain that (4.20) is bounded by
\[
\sum I \int_{|\xi'_1| - 2^l, |\xi'_3| - 2^k, |\xi'_4| - 2^k} \chi_i(\xi'_3)g_1(\xi'_1 - \xi'_3)g_3(\xi'_1 + \xi'_3 - \xi'_4)g_4(\xi'_1 - \xi'_4)
\times g(\xi'_1, \Omega(\xi'_1 - \xi'_3, \xi'_1 + \xi'_3 - \xi'_4, \xi'_1 - \xi'_4))d\xi'_1 d\xi'_3 d\xi'_4. \tag{4.22}
\]
Since in the integration area,
\[
\left| \frac{\partial}{\partial \xi'_3} [\Omega(\xi'_1 - \xi'_3, \xi'_1 + \xi'_3 - \xi'_4, \xi'_1 - \xi'_4)] \right| = | - \omega^l(\xi'_1 - \xi'_3) + \omega^l(\xi'_1 + \xi'_3 - \xi'_4)| \sim 2^{k_3}, \tag{4.23}
\]
then we get from (4.23) that (4.22) is bounded by
\[
\sum I \int_{|\xi'_1| - 2^l} \chi_i(\xi'_3) \|g_1\|_{L^2} \|g_3\|_{L^2} \\
\|g_4(\xi'_1 - \xi'_3)g(\xi'_1, \Omega(\xi'_1 - \xi'_3, \xi'_1 + \xi'_3 - \xi'_4, \xi'_1 - \xi'_4))\|_{L^2_{\xi'_1, \xi'_4}} d\xi'_3 \lesssim \sum I 2^{l/2 - k_3/2} \|g_1\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2} \lesssim 2^{j_{\max}/2 - k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}, \tag{4.24}
\]
where we used (4.21) in the last inequality. Therefore, we complete the proof for part (d).

We restate now Lemma 4.1 in a form that is suitable for the trilinear estimates in the next sections.

**Corollary 4.2.** Assume \( \delta \geq c_0 \). Let \( k_i, j_i \in \mathbb{Z}_+ \) and \( N_i = 2^{k_i}, L_i = 2^{j_i} \) for \( i = 1, 2, 3, 4 \). Assume \( f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) are functions supported in \( D_{k_i, j_i}, i = 1, 2, 3 \).
Then Corollary 4.2 follows from Lemma 4.1.

Let $f$ be a function such that $\|f\|_L^2 = 1$. We have

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_L^2 \leq C2^{(k_{min} + k_{thd})/2j(M_{min} + j_{thd})/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_L^2. \quad (4.25)$$

(b) For any $k_i, j_i \in \mathbb{Z}_+, i = 1, 2, 3, 4$ with $N_{thd} \ll N_{sub}$. If for some $i \in \{1, 2, 3, 4\}$ such that $(k_i, j_i) = (k_{thd}, j_{max})$, then

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_L^2 \leq C2^{(-k_{max} + k_{thd})/2j_1 + j_2 + j_3 + j_4)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_L^2. \quad (4.26)$$

else we have

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_L^2 \leq C2^{(-k_{max} + k_{min})/2j_1 + j_2 + j_3 + j_4)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_L^2. \quad (4.27)$$

(c) For any $k_i, j_i \in \mathbb{Z}_+, i = 1, 2, 3, 4$, with $N_{max} \gg 1$,

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_L^2 \leq C2^{(j_1 + j_2 + j_3 + j_4)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_L^2. \quad (4.28)$$

(d) If $N_{min} \ll N_{max}$ and $N_{max} \gg 1$, then

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_L^2 \leq C2^{(j_1 + j_2 + j_3 + j_4)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_L^2. \quad (4.29)$$

**Proof.** Clearly, we have

$$\|1D_{k_i,j_i}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})(\xi, \tau)\|_L^2 = \sup_{\|f\|_L^2 = 1} \left| \int_{D_{k_i,j_i}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \, d\xi d\tau \right|. \quad (4.30)$$

Let $f_{k_{4},j_{4}} = 1_{D_{k_{4},j_{4}}} \cdot f$, and then $f_{k_{i},j_{i}}^\sharp(\xi, \mu) = f_{k_{i},j_{i}}(\xi, \mu + \omega(\xi))$, $i = 1, 2, 3, 4$. The functions $f_{k_{i},j_{i}}^\sharp$ are supported in $f_{k_{i}} \times \cup_{|m| \leq 3} I_{j_{i} + m}$, $\|f_{k_{i},j_{i}}^\sharp\|_L^2 = \|f_{k_{i},j_{i}}\|_L^2$. Using simple changes of variables, we get

$$\int_{D_{k_{4},j_{4}}} f \cdot f_{k_{1},j_{1}} * f_{k_{2},j_{2}} * f_{k_{3},j_{3}} \, d\xi d\tau = J(f_{k_{1},j_{1}}^\sharp, f_{k_{2},j_{2}}^\sharp, f_{k_{3},j_{3}}^\sharp, f_{k_{4},j_{4}}^\sharp).$$

Then Corollary 4.2 follows from Lemma 4.1. □
5. Trilinear Estimate

In this section we devote to prove the trilinear estimates. We divide it into several cases. The first case is low \times high \rightarrow high interactions.

**Proposition 5.1.** Assume $\delta \geq \epsilon_0$. Let $k_i \in \mathbb{Z}_+, N_i = 2^{k_i}, i = 1, 2, 3, 4$. Assume $N_3 \gg 1, N_4 \sim N_3, N_1 \sim N_2 \ll N_3$, and $f_{k_i} \in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in $I$ with $|I| \lesssim 1, i = 1, 2, 3$. Then

$$2^{k_1} \| \chi_{k_1}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_1}} \lesssim 2^{(k_1 + k_2)/2} \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}}. \quad (5.1)$$

**Proof.** We first divide it into three parts. Fix $(\delta \gg 3)_{X_i \sim 2^{X_i+M}}$, Finally we consider the contribution of III. Let

$$I \leq \sum_{k_4 \leq 2k_4 + M} 2^{k_4} \| \mathcal{F}^{-1}(f_{k_1}) \|_{L^2} \| \mathcal{F}^{-1}(f_{k_2}) \|_{L^2} \| \mathcal{F}^{-1}(f_{k_3}) \|_{L^2} \| \mathcal{F}^{-1}(f_{k_4}) \|_{L^2} \lesssim 2^{(k_1 + k_2)/2} \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}}.$$

which is (5.1) as desired.

For the contribution of II, we use $X_k$ norm. Then we get from Lemma 3.6 that

$$II \leq \sum_{j_1 \geq 2k_4 + M} \sum_{j_2, j_3 \geq 0} \| 1_{D_{k_1, j_1}}(\xi, \tau) f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2} \lesssim 2^{(k_1 + k_2)/2} \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}}.$$

Finally we consider the contribution of III. Let $f_{k_{i,j_i}}(\xi, \tau) = f_{k_i}(\xi, \tau) \eta_{j_i}(\tau - \omega_\delta(\xi)), j_i \geq 0, i = 1, 2, 3, \text{ using } X_k \text{ norm, we get}$

$$III \leq \sum_{j_1 \geq 2k_4 + M+1} \sum_{j_2, j_3 \geq 0} \| 1_{D_{k_1, j_1}}(\xi, \tau) f_{k_{1,j_1}} * f_{k_2, j_2} * f_{k_3, j_3} \|_{L^2}.$$

Since in the area $\{|\xi_i| \in I_{k_i}, i = 1, 2, 3\}$, we have $|\Omega(\xi_1, \xi_2, \xi_3)| \ll 2^{2k_4 + M}$. By checking the support properties of $f_{k_{i,j_i}}$, we get $L_{\max} \sim L_{sub}$. We consider only
the worst case \( L_4 \sim L_3 \geq L_1, L_2 \), since the other cases are better. It follows from Corollary 4.2 and Lemma 3.1 (b) that

\[
III \lesssim \sum_{j_3 \geq 2k_4 + M + 1} 2^{j_3} \sum_{j_1, j_2 \geq 0} 2^{(j_1 + j_2)/2} 2^{(k_1 + k_2)/2} \|f_{k_1, j_1}\|_L^2 \|f_{k_2, j_2}\|_L^2 \|f_{k_3, j_3}\|_L^2 \\
\lesssim \sum_{j_3 \geq 2k_4 + M + 1} 2^{j_3} 2^{k_3 - j_3} 2^{(k_1 + k_2)/2} \|f_{k_1}\|_L \|f_{k_2}\|_L \|f_{k_3}\|_L \\
\lesssim \sum_{j_3 \geq 2k_4 + M + 1} 2^{k_3 - (j_3 - k_3)/2} \|f_{k_1}\|_L \|f_{k_2}\|_L \|f_{k_3}\|_L \|f_{k_3}\|_L \\
\lesssim 2^{(k_1 + k_2)/2} \|f_{k_1}\|_L \|f_{k_2}\|_L \|f_{k_3}\|_L \|f_{k_3}\|_L.
\]

Therefore, we complete the proof of the proposition. 

This proposition suffices to control high \times low interaction in the case that the two low frequencies is comparable. However, for the case that the two low frequencies is not comparable, we will need an improvement.

**Proposition 5.2.** Assume \( \delta \geq c_0 \). Let \( k_i \in \mathbb{Z}, N_i = 2^k_i, i = 1, 2, 3, 4 \). Assume \( N_3 \gg 1 \), \( N_4 \sim N_3, N_1 \ll N_2 \ll N_3 \), and \( f_{k_i} \in Z_{k_i} \) with \( \mathcal{F}^{-1}(f_{k_i}) \) compactly supported (in time) in \( I \) with \( |I| \leq 1 \), \( i = 1, 2, 3 \). Then

\[
2^{k_4} \|\chi_{k_4}(\xi) (\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2} \lesssim 2^{(k_1 + k_2)/2} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}.
\]

**Proof.** We first observe that in this case we get from Lemma 3.2 that

\[
|\Omega(\xi_1, \xi_2, \xi_3) | \sim 2^{k_4 + k_2}.
\]

Dividing it into three parts and fixing an integer \( M \) such that \( N_2 \geq M \gg 1 \), we obtain

\[
2^{k_4} \|\chi_{k_4}(\xi) (\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}}
\]

\[
\leq 2^{k_4} \|\chi_{k_4}(\xi) \eta_{\leq k_4 - 1} (\tau - \omega_5(\xi))(\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
+ 2^{k_4} \|\chi_{k_4}(\xi) \eta_{\leq 2k_4 + M} (\tau - \omega_5(\xi))(\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
+ 2^{k_4} \|\chi_{k_4}(\xi) \eta_{\leq 2k_4 + M + 1} (\tau - \omega_5(\xi))(\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}}
\]

\[
= I + II + III.
\]

For the last two terms II, III, we can use the same argument as for II, III in the proof of Proposition 5.1. We consider now the first term I.

\[
I \leq 2^{k_4} \|\chi_{k_4}(\xi) \eta_{\leq k_4 - 1} (\tau - \omega_5(\xi))(\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
+ 2^{k_4} \|\chi_{k_4}(\xi) \eta_{\leq k_4 - 1} (\tau - \omega_5(\xi))(\tau - \omega_5(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}}
\]

\[
= I_1 + I_2,
\]

where

\[
f_{k_3}^h = f_{k_3}(\xi, \tau) \eta_{\leq k_3 + k_2 - M} (\tau - \omega_5(\xi)), \quad f_{k_3}^l = f_{k_3}(\xi, \tau) \eta_{\leq k_3 + k_2 - M + 1} (\tau - \omega_5(\xi)).
\]

For the contribution of \( I_1 \), we observe first that from the support of \( f_{k_3}^h \) and the definition of \( Y_{k_3} \), one easily get that

\[
\|f_{k_3}^h \| \chi_{k_3} \lesssim \| f_{k_3} \|_{Z_{k_3}}.
\]
Thus from the definition of $Y_k$, and from Hölder’s inequality, Lemma 3.3, Lemma 3.6 (b), we get

$$I_1 \lesssim 2^{k_1} \|\chi_{k_1}(\xi)\eta_{k_1-1}(\tau - \omega_3(\xi))(\tau - \omega_3(\xi) + i)^{-1} f_{k_1} \ast f_{k_2} \ast f_{k_3}^h\|_{Y_{k_1}}$$

$$\lesssim 2^{k_1/2} \|F^{-1} [f_{k_1} \ast f_{k_2} \ast f_{k_3}^h]\|_{L^1_x L^2_t}$$

$$\lesssim 2^{k_1/2} \|F^{-1} (f_{k_1}^h)\|_{L^2_x L^2_t} \|F^{-1} (f_{k_2})\|_{L^2_x L^\infty_t} \|F^{-1} (f_{k_3})\|_{L^2_x L^\infty_t}$$

$$\lesssim 2^{k_1/2} \|f_{k_1}^h\|_{L^2} \|f_{k_1}\|_{Z_{k_1}} \|f_{k_2}\|_{Z_{k_2}} \|f_{k_3}\|_{Z_{k_3}}.$$ 

Then from the fact that

$$2^{k_1/2} \|f_{k_3}^h\|_{L^2} \lesssim \sum_{j \geq k_1 + k_2 - 10} 2^{k_1/2} \|f_{k_3}^h\|_{L^2} \|f_{k_1}\|_{Z_{k_1}} \|f_{k_2}\|_{Z_{k_2}}$$

we conclude the proof for $I_1$.

We consider now the contribution of $I_2$ where $\beta_{k,j}$ plays crucial roles. Let $f_{k_i,j}((\xi, \tau) = f_{k_i,j}(\tau - \omega_3(\xi)))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, we get

$$I_2 \leq \sum_{j_3 \leq k_1} \sum_{j_2 \leq k_2 - M, j_3 \geq 0} 2^{k_1 - j_3/2} \|1_{D_{k_1,j_3}} f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3}\|_{L^2}.$$ 

From the support properties, we get that $1_{D_{k_1,j_3}}(\xi, \tau)f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3} \equiv 0$ unless

$$\begin{cases} L_1 \sim L_2 \geq N_3 N_2; & \text{or} \\
L_2 \ll L_1 \sim N_3 N_2; & \text{or} \\
L_1 \ll L_2 \sim N_3 N_2.
\end{cases}$$

If $L_1 \sim L_2 \geq N_3 N_2$, it follows from Corollary 4.2 (b) and Lemma 3.1 (b) that

$$I_2 \lesssim \sum_{j_3 \leq k_1 \geq 0} 2^{k_1 - j_3/2} \|f_{k_1,j_1}\|_{Z_{k_1}} \|f_{k_2,j_2}\|_{Z_{k_2}} \|f_{k_3,j_3}\|_{Z_{k_3}}$$

$$\lesssim \sum_{j_3 \geq 0} \sum_{j_2 \geq 0} 2^{k_1 - j_3/2} \|f_{k_1,j_1}\|_{Z_{k_1}} \|f_{k_2,j_2}\|_{Z_{k_2}} \|f_{k_3,j_3}\|_{Z_{k_3}}$$

$$\lesssim \sum_{j_3 \geq 0} \sum_{j_2 \geq 0} 2^{k_1 - j_3/2} \|f_{k_1,j_1}\|_{Z_{k_1}} \|f_{k_2,j_2}\|_{Z_{k_2}} \|f_{k_3,j_3}\|_{Z_{k_3}}$$

which is acceptable. If $L_1 \ll L_2 \sim N_3 N_2$, it follows from Corollary 4.2 (b) that

$$I_2 \lesssim \sum_{j_1 \geq 0} \sum_{j_3 \leq 2 k_1} \sum_{j_4 \leq 2 k_1} 2^{k_1 - j_1/2} \|f_{k_1,j_1}\|_{Z_{k_1}} \|f_{k_2,j_2}\|_{Z_{k_2}}$$

$$\lesssim \sum_{j_1 \geq 0} \sum_{j_2 \geq 0} 2^{k_1/2} \|f_{k_1,j_1}\|_{Z_{k_1}} \|f_{k_2,j_2}\|_{Z_{k_2}}$$

The other case can be handled in the same way. Therefore, we complete the proof of the proposition.
Proposition 5.3. Assume $\delta \geq c_0$. Let $k_i \in \mathbb{Z}^+, N_i = 2^{k_i}, i = 1, 2, 3, 4$. Assume $N_3 \gg 1, N_1 \ll N_2 \sim N_3 \sim N_4$, and $f_{k_i} \in\mathcal{F}^{-1}(f_i)$ with $\mathcal{F}^{-1}(f_i)$ compactly supported (in time) in $I$ with $|I| \leq 1, i = 1, 2, 3$. Then

$$2^k \|\chi_{k_i}(\xi - \omega_3(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \lesssim 2^{(k_1 + k_2) / 4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}.$$ \(\square\)

Proof. We first observe that this case corresponds to an integration in the area \(\{\xi_i \in I_{k_i}, i = 1, 2, 3\} \cap \{\xi_1 + \xi_2 + \xi_3 \in I_{k_4}\}\), where we have from Lemma 3.2 that

$$|\Omega(\xi_1, \xi_2, \xi_3)| = 2^{2k_3}.$$  \(5.3\)

Let \(f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau)\eta_{j_i}(\tau - \omega_3(\xi))\), \(j_i \geq 0, i = 1, 2, 3\). Using \(X_k\) norm, we get

$$2^{k_i} \|\chi_{k_i}(\xi - \omega_3(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \lesssim \sum_{j_i \geq 0} 2^{k_i} 2^{-j_i/2} (1 + 2^{(j_4 - 2k_4)/2}) \|1_{D_{k_4,j_4}}(\xi, \tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}.$$ \(5.4\)

From the support properties of the functions \(f_{k_i,j_i}, i = 1, 2, 3\), it is easy to see that \(1_{D_{k_4,j_4}}(\xi, \tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \equiv 0\) unless

$$\begin{cases} L_{\text{max}} \sim L_{\text{sub}} \gtrsim N_3^2; \\
L_{\text{sub}} \ll L_{\text{max}} \sim N_3^2. \end{cases}$$

If \(L_{\text{max}} \sim j_{\text{sub}} \gtrsim N_3^2\), it follows from Corollary 4.2 (a) that the right-hand side of \(5.4\) is bounded by

$$\sum_{j_i \geq 0} 2^{k_i} 2^{(j_1 + j_2 + j_3)/2} (1 + 2^{(j_4 - 2k_4)/2}) 2^{-(j_{\text{max}} + j_{\text{sub}})/2} 2^{(k_1 + k_2)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \quad (5.5)$$

It suffices to consider the worst case \(j_3 = j_{\text{max}}, j_4 = j_{\text{sub}}\). We get from Lemma 3.3 (b) that \(5.5\) is bounded by

$$\sum_{j_i \geq 2k_3 - 10} 2^{k_i} 2^{4j_3} 2^{(k_1 + k_2)/2} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}} \lesssim 2^{(k_1 + k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}.$$ \(5.6\)

If \(L_{\text{sub}} \ll L_{\text{max}} \sim N_3^2\), then from Corollary 4.2 (c) we get that the right side of \(5.4\) is bounded by

$$\sum_{j_i \geq 0} 2^{k_i} 2^{(j_1 + j_2 + j_3)/2} 2^{-j_{\text{max}}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_2 \lesssim 2^{(k_1 + k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}},$$

where we used Lemma 3.3 (b). Thus, we complete the proof of the proposition. \(\square\)

Proposition 5.4. Assume $\delta \geq c_0$. Let $k_i \in \mathbb{Z}^+, N_i = 2^{k_i}, i = 1, 2, 3, 4$. Assume $N_3 \gg 1, N_1 \sim N_2 \sim N_3 \sim N_4$, and $f_{k_i} \in\mathcal{F}^{-1}(f_i)$ with $\mathcal{F}^{-1}(f_i)$ compactly supported (in time) in $I$ with $|I| \leq 1, i = 1, 2, 3$. Then

$$2^{k_4} \|\chi_{k_i}(\xi - \omega_3(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \lesssim 2^{k_4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}.$$
Proof. First we divide it into two parts. Fixing $M \gg 1$, then we have

$$2^{k_i} \| \chi_{k_i}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2}$$

$$\lesssim 2^{k_i} \| \chi_{k_i}(\xi) \eta \leq 2^{k_i} + M(\tau - \omega_\delta(\xi))(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2}$$

$$+ 2^{k_i} \| \chi_{k_i}(\xi) \eta \leq 2^{k_i} + M(\tau - \omega_\delta(\xi))(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2}$$

$$= I + II.$$  

We consider first the contribution of the first term $I$. Using the $X_k$ norm and Lemma 3.6 (a), then we get

$$I \lesssim 2^{k_i} \sum_{j_k \geq 0} 2^{-j_k/2} \| 1_{D_{k_1,j_1}}(\xi, \tau) f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2}$$

$$\lesssim 2^{k_i} \prod_{i=1}^n \| F^{-1}(f_{k_i}) \|_{L^2} \lesssim 2^{k_i} \prod_{i=1}^n \| f_{k_i} \|_{Z_{k_i}}.$$  

We consider now the contribution of the second term $II$. Let $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau) n_{j_i}(\tau - \omega_\delta(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using the $X_k$ norm, we get

$$II \lesssim \sum_{j_k \geq 2^{k_i} + 20} \sum_{j_1, j_2, j_3 \geq 0} \| 1_{D_{k_1,j_1}}(\xi, \tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \|_{L^2}.$$  

Since in the area $\{ |\xi| \leq L_{k_i}, i = 1, 2, 3 \}$ we have $|\Omega(\xi_1, \xi_2, \xi_3)| \leq 2^{k_3} < 2^{k_i}$, by checking the support properties of the functions $f_{k_i,j_i}$, $i = 1, 2, 3$, we get $L_{\text{max}} \sim L_{\text{sub}} \gg N_3^2$. From symmetry, we assume $j_3, j_4 = j_{\text{max}}, j_{\text{sub}}$, then we get

$$II \lesssim \sum_{j_k \geq 2^{k_i} + 20} \sum_{j_1, j_2, j_3 \geq 0} 2^{j_1+j_2/2} 2^{k_3} 2^{j_3} \prod_{i=1}^3 \| f_{k_i,j_i} \|_{L^2}$$

$$\lesssim \sum_{j_k \geq 2^{k_i}} 2^{k_i} 2^{-j_3/2} \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}} \lesssim 2^{k_i} \prod_{i=1}^n \| f_{k_i} \|_{Z_{k_i}}.$$  

Therefore we complete the proof of the proposition.  

We consider now the case which corresponds to high × high interactions. This case is better than high × low interaction case.

**Proposition 5.5.** Assume $\delta \geq c_0$. Let $k_i \in \mathbb{Z}_+$, $N_i = 2^{k_i}$, $i = 1, 2, 3, 4$. Assume $N_1 \gg 1$, $N_4 \ll N_1$, $N_3 \ll N_1 \sim N_2$, and $f_{k_i} \in Z_{k_i}$ compactly supported (in time) in $I$ with $|I| \sim 1$, $i = 1, 2, 3$. Then

$$2^{k_i} \| \eta_{k_i}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2} \lesssim 2^{k_i} \prod_{i=1}^n \| f_{k_i} \|_{Z_{k_i}}.$$  

**Proof.** Let $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau) n_{j_i}(\tau - \omega_\delta(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, then we get

$$2^{k_i} \| \chi_{k_i}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2} \lesssim \sum_{j_k \geq 0} 2^{k_i} 2^{-j_3/2} \| 1_{D_{k_1,j_1}}(\xi, \tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \|_{L^2}. (5.8)$$
If $L_{\text{max}} \lesssim N_1^2$, then it follows from Corollary 4.2 (d) that the right side of (5.8) is bounded by

$$\sum_{j_i \geq 0} 2^{k_i}(1 + 2^{(j_i - 2k_i)/2})2^{(j_1 + j_2 + j_3)/2} 2^{-k_i} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2} \lesssim k_1 2^{3} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}},$$

where we used Lemma 3.3 (b).

If $L_{\text{max}} \gg N_1^2$, then by checking the support properties, we get $L_{\text{max}} \sim L_{\text{sub}}$. We consider only the worst case $j_1, j_4 = j_{\text{max}}, j_{\text{sub}}$. It follows from Corollary 4.2 (a) and Lemma 3.3 (b) that the right side of (5.8) is bounded by

$$\sum_{j_i \geq 0} 2^{k_i} 2^{-j_i/2}(1 + 2^{(j_i - 2k_i)/2})2^{(j_2 + j_3)/2} 2^{k_i} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2} \lesssim k_1 2^{3} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}}.$$ 

Therefore, we complete the proof of the proposition.

The next proposition is used to control low $\times$ low interactions. This interaction is easy to control.

**Proposition 5.6.** Assume $\delta \geq c_0$. Let $k_i \in \mathbb{Z}_+$, $N_i = 2^{k_i}$, $i = 1, 2, 3, 4$. Assume $N_{\text{max}} \lesssim 1$, and $f_{k_i} \in Z_{k_i}$, $i = 1, 2, 3$. Then

$$2^{k_i} \| \eta_k(\xi) (\tau - \omega_k(\xi) + i) \| f_{k_i} \| f_{k_2} \| f_{k_3} \| f_{k_4} \| Z_{k_4} \lesssim \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}}, \quad (5.9)$$

**Proof.** Let $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau) \eta_{j_i}(\tau - \omega_{j_i}(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, Corollary 4.2 (a) and Lemma 3.1 (b), then we get

$$2^{k_i} \| \eta_k(\xi) (\tau - \omega_k(\xi) + i) \| f_{k_i} \| f_{k_2} \| f_{k_3} \| f_{k_4} \| Z_{k_4} \lesssim \sum_{j_i \geq 0} \sum_{j} \| f_{k_i,j_i} \|_{L^2} \lesssim 2^{(k_{\text{max}} + k_{\text{max}})/2} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}},$$

since for the case $j_{\text{max}} \gg 1$ we have $L_{\text{max}} \sim L_{\text{sub}}$ by checking the support properties of the functions $f_{k_i,j_i}$, $i = 1, 2, 3$.

Finally we present two counterexamples as in [3]. The first one shows why we use a $l^1$-type $X^{*,b}$ structure. The other one shows a logarithmic divergence if we only use $X_k$ which is the reason for us applying $Y_k$ structure.

**Proposition 5.7.** Let $\delta \geq c_0$. Assume $k \geq M$. Then there exist $f_1 \in X_1$, $f_k \in X_k$ such that

$$2^{k} \| \eta_k(\xi) (\tau - \omega_k(\xi) + i) \| f_{k} \| X_k \gtrsim k \| f_1 \|_{X_1} \| f_1 \|_{X_1} \| f_k \|_{X_k}. \quad (5.10)$$

**Proof.** From the proof of Proposition 5.1, we easily see that the worst interaction comes from the case that largest frequency component has a largest modulation. So we construct this case explicitly. Let $I = [1/2, 1]$, and take

$$f_1(\xi, \tau) = \chi_I(\xi) \eta_1(\tau - \omega_1(\xi)), \quad f_k(\xi, \tau) = \chi_{k}(\xi) \eta_k(\tau - \omega_k(\xi)).$$

From definition, we easily get $\| f_1 \|_{X_1} \sim 1$ and $\| f_k \|_{X_k} \sim 2^{k/2}$ and

$$2^{k} \| \eta_k(\xi) (\tau - \omega_k(\xi) + i) \| f_{k} \| X_k \gtrsim 2^{k/2} \sum_{j=0}^{k/2} 2^{-j/2} \| D_{k, j} \cdot f_1 \|_{L^2} \| f_k \|_{L^2}. \quad (5.10)$$
On the other hand, we have for \( j \leq k/2 \)
\[
1_{D_{k,j}}(\xi, \tau) \cdot f_1 \ast f_1 \ast f_k
\]
\[
= \int f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_k(\xi - \xi_1, \xi - \xi_2, \tau - \tau_1 - \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi_1(\xi_1) \chi_1(\xi_2) \eta_1(\tau_1) \eta_2(\tau_2) \chi_n(\xi - \xi_1 - \xi_2)
\]
\[
\cdot \eta_k(\tau - \tau_1 - \tau_2 - \omega_\delta(\xi_1) - \omega_\delta(\xi_2) - \omega_\delta(\xi - \xi_1 - \xi_2)) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
\gtrsim \chi_{2^{10} + 2} \frac{1}{2^{10} + 2} \cdot (\xi) \eta_j(\tau - \omega_\delta(\xi)).
\]
Therefore, we get
\[
2^{k/2} \sum_{j=0}^{k/2} 2^{-j/2} \| 1_{D_{k,j}} \cdot f_1 \ast f_1 \ast f_k \|_{L^2_{\xi, \tau}} \gtrsim \kappa 2^{3k/2}, \quad \text{(5.11)}
\]
which completes the proof of the proposition.

**Proposition 5.8.** For any \( s \in \mathbb{R} \), there doesn't exist \( b \in \mathbb{R} \) such that
\[
\| \partial_x(uvw) \|_{X^{s,b}_{\mathbb{R}}} \lesssim \| u \|_{X^{s,b}_{\mathbb{R}}} \| v \|_{X^{s,b}_{\mathbb{R}}} \| w \|_{X^{s,b}_{\mathbb{R}}}.
\]

**Proof.** It is easy to see that the counterexample in the proof of Proposition 5.7 shows that (5.12) doesn't hold for \( b = 1/2 \) with a \( k^{1/2} \) divergence in (5.11). We assume now \( b \neq 1/2 \). Using Plancherel’s equality, we get that (5.12) is equivalent to
\[
\left\| \frac{\langle \eta \rangle^s}{(\tau - \omega_\delta(\xi))^b} \right\|_{L^2_{\xi, \tau}} \lesssim \| u \|_{L^2_{\xi, \tau}} \| v \|_{L^2_{\xi, \tau}} \| w \|_{L^2_{\xi, \tau}}.
\]

Fix any dyadic number \( N \gg 1 \). Let
\[
A = \{ 1/2 \leq \xi \leq 10, \ |\tau| \leq 1 \} \quad \text{and} \quad B = \{ N/2 \leq \xi \leq 2N, \ |\tau| \leq 2^{10} \}.
\]
Take
\[
u(\xi, \tau) = v(\xi, \tau) = \chi_A(\xi, \tau - \omega_\delta(\xi)), \quad w(\xi, \tau) = \chi_B(\xi, \tau - \omega_\delta(\xi)).
\]

We easily see that \( \| u \|_{L^2_{\xi, \tau}} \sim 1 \) and \( \| v \|_{L^2_{\xi, \tau}} \sim N^{1/2} \). Denote \( f(\xi, \tau) = u * v * w(\xi, \tau + \omega_\delta(\xi)) \). Then we have
\[
f(\xi, \tau)
\]
\[
= \int u(\xi_1, \tau_1)v(\xi_2, \tau_2)w(\xi - \xi_1 - \xi_2, \tau + \omega_\delta(\xi) - \tau_1 - \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi_{2^{10}}(\tau - \tau_1 - \tau_2 + \omega_\delta(\xi) - \omega_\delta(\xi - \xi_1 - \xi_2)) \cdot \chi_A(\xi_1, \tau_1) \chi_B(\xi_2, \tau_2)\chi_{\mathbb{R}/2^{N/2}N}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi_{2^{10}}(\tau - \tau_1 - \tau_2 + 2(\xi_1 + \xi_2) \xi + (\xi_1 - \xi_2)^2 - \omega_\delta(\xi_1) - \omega_\delta(\xi_2) + o(1)) \cdot \chi_A(\xi_1, \tau_1) \chi_B(\xi_2, \tau_2)\chi_{\mathbb{R}/2^{N/2}N}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\]
Therefore, fixing $M \gg 1$, we get for any $(\xi, \tau) \in [(M - 1)N/M, (M + 1)N/M] \times [-8N, -4N]$, then $\tau = -C_0\xi$ for some $2 \leq C_0 \leq 9$ and

$$f(\xi, \tau) \gtrsim \int \chi_{\Lambda}(\xi_1, \tau_1) \chi_{\Lambda}(\xi_2, \tau_2) \chi_{\xi_1 + \xi_2 - C_0|\xi| \leq N - 1} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \gtrsim N^{-1}.$$ 

Thus we see that the left-hand side of (5.13) is larger than $N^b$, while the right-hand side is $N^{1/2}$, which implies $b < 1/2$.

Similarly, by taking $B' = \{N/2 \leq \xi \leq 2N, N \leq |\tau| \leq N\}$ as before, we obtain that $b > 1/2$. Therefore we complete the proof of the proposition.

6. Proof of Theorem 1.1

In this section we devote to prove Theorem 1.1 by using the standard fixed-point machinery. From Duhamel’s principle, we get that the equation (1.1) is equivalent to the following integral equation:

$$u = U_\delta(t)\phi + \int_0^t U_\delta(t - t')(\partial_x(u^3)(t'))dt'. \quad (6.1)$$

We will mainly work on the following truncated version

$$u = \psi(t)U_\delta(t)\phi + \psi(t)\int_0^t U_\delta(t - t')\partial_x[\psi(t')u^3]dt', \quad (6.2)$$

where $\psi(t) = \eta_0(t)$ is a smooth cut-off function. Then we easily see that if $u$ is a solution to (6.2) on $\mathbb{R}$, then $u$ solves (6.1) on $t \in [-1, 1]$. Our first lemma is on the estimate for the linear solution.

**Lemma 6.1.** If $\delta \geq c_0$, $s \geq 0$ and $\phi \in H^s$ then

$$\|\psi(t) \cdot (U_\delta(t)\phi)\|_{F^s} \lesssim \|\phi\|_{H^s}. \quad (6.3)$$

**Proof.** A direct computation shows that

$$\mathcal{F}[\psi(t) \cdot (U_\delta(t)\phi)](\xi, \tau) = \hat{\psi}(\xi)\hat{\psi}(\tau - \omega_\delta(\xi)).$$

In view of definition, it suffices to prove that if $k \in \mathbb{Z}_+$ then

$$\|\eta_k(\xi)\hat{\psi}(\tau - \omega_\delta(\xi))\|_{Z_k} \leq C\|\eta_k(\xi)\hat{\psi}(\xi)\|_{L^2}. \quad (6.4)$$

Indeed, from definition we have

$$\|\eta_k(\xi)\hat{\psi}(\tau - \omega_\delta(\xi))\|_{Z_k} \leq \|\eta_k(\xi)\hat{\psi}(\tau - \omega_\delta(\xi))\|_{X_k} \leq C\sum_{j=0}^{\infty} 2^j \|\eta_k(\xi)\hat{\psi}(\xi)\|_{L^2} \|\eta_j(\tau)\hat{\psi}(\tau)\|_{L^2} \leq C\|\eta_k(\xi)\hat{\psi}(\xi)\|_{L^2},$$

which is (6.4) as desired.

Next lemma is on the estimate for the retarded linear term. We will follow the method in [7] to prove it.

**Lemma 6.2.** If $\delta \geq c_0$, $s \geq 0$ and $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ then

$$\left\|\psi(t) \cdot \int_0^t U_\delta(t - s)u(s)ds\right\|_{F^s} \leq C\|u\|_{N^s}. \quad (6.5)$$
Proof. A straightforward computation shows that
\[ F \left[ \psi(t) \cdot \int_0^t U_5(t-s)(u(s))ds \right] (x,\tau) = c \int \mathcal{F}(u)(\xi,\tau') \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(\tau - \omega_\delta(\xi))}{\tau' - \omega_\delta(\xi)} d\tau'. \]

For \( k \in \mathbb{Z}_+ \) let \( f_k(\xi,\tau') = \mathcal{F}(u)(\xi,\tau') \eta_k(\xi)(\tau' - \omega_\delta(\xi) + i)^{-1} \). For \( f_k \in \mathbb{Z}_k \) let
\[ T(f_k)(\xi,\tau) = \int \mathcal{F}(f_k)(\xi,\tau') \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(\tau - \omega_\delta(\xi))}{\tau' - \omega_\delta(\xi)} (\tau' - \omega_\delta(\xi) + i) d\tau'. \]

In view of the definitions, it suffices to prove that
\[ \| T \|_{\mathbb{Z}_k \to \mathbb{Z}_k} \leq C \text{ uniformly in } k \in \mathbb{Z}_+, \tag{6.6} \]
which follows from the slightly modified proof of Lemma 5.2 in [7]. We omit the details.

We prove a trilinear estimate in the following proposition which is an important component for using fixed-point argument.

**Proposition 6.3.** Assume \( \delta \geq c_0 \). Let \( s \geq 1/2 \). Then
\[ \| \partial_x (\psi(t)^3 uvw) \|_{N_x} \lesssim \| u \|_{F^s} \| v \|_{F^{1/2}} \| w \|_{F^{1/2}} + \| u \|_{F^{1/2}} \| v \|_{F^s} \| w \|_{F^{1/2}} + \| u \|_{F^{1/2}} \| v \|_{F^{1/2}} \| w \|_{F^s}. \]

**Proof.** In view of definition, we get
\[ \| \partial_x (\psi(t)^3 uvw) \|_{N_x}^2 = \sum_{k_4=0}^{\infty} 2^{2k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} \mathcal{F}(\partial_x (\psi(t)^3 uvw)) \|_{\mathbb{Z}_{k_4}}^2. \]

For \( k_1, k_2, k_3 \in \mathbb{Z}_+ \), setting \( f_{k_1} = \eta_{k_1}(\xi) \mathcal{F}(\psi(t)u)(\xi,\tau) \), \( f_{k_2} = \eta_{k_2}(\xi) \mathcal{F}(\psi(t)v)(\xi,\tau) \), and \( f_{k_3} = \eta_{k_3}(\xi) \mathcal{F}(\psi(t)w)(\xi,\tau) \), then we get
\[ 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} \mathcal{F}(\psi(t)^3 uvw) \|_{\mathbb{Z}_{k_4}} \lesssim \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{\mathbb{Z}_{k_4}}. \]

From symmetry it suffices to bound
\[ \sum_{0 \leq k_1 \leq k_2 \leq k_3} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{\mathbb{Z}_{k_4}}. \]

Setting \( N_i = 2^{k_i}, i = 1, 2, 3, 4 \), we get
\[ \sum_{k_1 \leq k_2 \leq k_3} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{\mathbb{Z}_{k_4}} \]
\[ \leq \sum_{j=1}^{6} \sum_{(k_1, k_2, k_3, k_4) \in A_j} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega_\delta(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{\mathbb{Z}_{k_4}}. \tag{6.7} \]
where we denote
\[ A_1 = \{ 0 \leq N_1 \leq N_2 \ll N_3, N_3 \gg 1, N_4 \sim N_3, N_1 \sim N_2 \}; \]
\[ A_2 = \{ 0 \leq N_1 \leq N_2 \ll N_3, N_3 \gg 1, N_4 \sim N_3, N_1 \ll N_2 \}; \]
\[ A_3 = \{ 0 \leq N_1 \leq N_2 \leq N_3, N_3 \gg 1, N_4 \sim N_3, N_1 \ll N_2 \}; \]
\[ A_4 = \{ N_1 \sim N_2 \sim N_3, N_3 \gg 1 \}; \]
\[ A_5 = \{ 0 \leq N_1 \leq N_2 \leq N_3, N_4 \ll N_3, N_3 \gg 1, N_2 \sim N_3 \}; \]
\[ A_6 = \{ \max(N_3, N_4) \leq 1 \}. \]

We will apply Proposition 5.1-5.6 obtained in the last section to bound the six terms in (7.3). For example, for the first term, from Proposition 5.1, we have
\[
\| 2^{sk_4} \sum_{i \in A_1} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} \ast f_{k_2} \ast f_{k_3} \|_{L^2}^2 \leq C 2^{sk_4} \sum_{i \in A_1} 2^{(k_1+k_2)/2} \| f_{k_1} \|_{L^2} \| f_{k_2} \|_{L^2} \| f_{k_3} \|_{L^2} \|_{L^2}^2 \leq \| u \|_{F^{1/2}} \| \| u \|_{F^{1/2}} \| \| u \|_{F^s}^2.
\]

For the other terms we can handle them in the similar ways. Therefore we complete the proof of the proposition.  

Now we prove Theorem 1.1. To begin with, we renormalize the data a bit via scaling. By the scaling (1.8), we see that if \( s \geq 1/2 \)
\[
\| \phi \|_{L^2} = \| \phi \|_{L^2},
\]
\[
\| \phi \|_{H^s} = \lambda^{-s} \| \phi \|_{H^s}.
\]

From the assumption \( \| \phi \|_{L^2} \ll 1 \), thus we can first restrict ourselves to considering (1.4) with data \( \phi \) satisfying
\[
\| \phi \|_{H^s} = r \ll 1. \tag{6.8}
\]

This indicates the reason why we assume that \( \| \phi \|_{L^2} \ll 1 \).

Define the operator
\[
\Phi_\phi(u) = \psi(t)U_\delta(t)\phi + \psi(t) \int_0^t U_\delta(t-t')\partial_x((\psi(t')u^3)(t'))dt',
\]
and we will prove that \( \Phi_\phi(\cdot) \) is a contraction mapping from
\[
\mathcal{B} = \{ w \in F^s : \| w \|_{F^s} \leq 2cr \} \tag{6.9}
\]
into itself. From Lemma 6.1, 6.2 and Proposition 6.3 we get if \( w \in \mathcal{B} \), then
\[
\| \Phi_\phi(w) \|_{F^s} \leq c \| \phi \|_{H^s} + \| \partial_x(\psi(t^3w^3(\cdot, t))) \|_{N^s} \leq cr + c\| w \|_{F^s} \leq cr + c(2cr)^3 \leq 2cr, \tag{6.10}
\]
provided that \( r \) satisfies \( 8c^3r^2 \leq 1/2 \). Similarly, for \( w, h \in \mathcal{B} \)
\[
\| \Phi_\phi(w) - \Phi_\phi(h) \|_{F^s} \leq c \| \psi(t) \int_0^t \partial_x[\psi^3(\tau)(w^3(\tau) - h^3(\tau))]d\tau \|_{F^s} \leq c(\| w \|_{F^s}^2 + \| h \|_{F^s}^2)\| w - h \|_{F^s} \leq 8c^3r^2\| w - h \|_{F^s} \leq \frac{1}{2}\| w - h \|_{F^s}. \tag{6.11}
\]
Thus $\Phi_\delta(\cdot)$ is a contraction. Therefore, there exists a unique $u \in B$ such that
\[
u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x[(\psi(t')u)^3](t'))dt'.
\]
Hence $u$ solves the integral equation (6.1) in the time interval $[-1,1]$.

Part (c) of Theorem 1.1 follows from the scaling (1.8), Lemma 3.6 and Proposition 6.3. Part (d) follows from the standard argument. We prove now part (b). It is easy to see that the energy methods as in [1] show local well-posedness for Eq. (1.1) in $H^s$ for $s > 3/2$. One may improve this to $H^3$, using the methods in [10]. According to Theorem 1.2 in [10], it suffices to prove that if $s > 1$ then
\[
\partial_x u \in L^1_{t \in [0,T]}L^\infty_x.
\]
Indeed, this follows from the fact that $u \in F^s(T)$ and $(4,\infty)$ is an admissible pair and Lemma 3.6. Therefore, we complete the proof of Theorem 1.1.

7. Ill-posedness Result

In this section we will prove that the solution map of Eq. (1.1) is not $C^3$ differentiable at origin in $H^s$ if $s < 1/2$, closely following the method in [17, 16]. Thus we see $H^{1/2}$ is the critical regularity for which one can get well-posedness by fixed point argument. Following standard fixed point argument, one need to find the Banach space $X^s \subset C([0,T];H^s)$ such that it verifies
\[
\|U_\delta(t)u_0\|_{X^s} \lesssim \|u_0\|_{H^s}, \quad (7.1)
\]
\[
\left\| \int_0^t U_\delta(t - \tau)\partial_x(u_1u_2u_3)(\tau)d\tau \right\|_{X^s} \lesssim \|u_1\|_{X^s}\|u_2\|_{X^s}\|u_3\|_{X^s}. \quad (7.2)
\]
In particular, if we set $u_i = U_\delta(t)\phi_i$, $i = 1, 2, 3$, then we can obtain from (7.1) and (7.2) that for $0 < t < T$,
\[
\left\| \int_0^t U_\delta(t - \tau)\partial_x(\prod_{i=1}^3 U_\delta(\tau)\phi_i)\right\|_{H^s} \lesssim \prod_{i=1}^3 \|\phi_i\|_{H^s}. \quad (7.3)
\]
We will construct concrete functions $\phi_i$, $i = 1, 2, 3$ such that (7.3) fails if $s < 1/2$ for any $t > 0$.

As in [16], we fix $t \neq 0$ and define the real valued function $\phi_N$ by:
\[
\hat{\phi}_N(\xi) = N^{-s}\gamma^{-1/2}\left(\chi_{[-N,-N]}(\xi) + \chi_{[N,N]}(\xi)\right),
\]
with $\gamma = o(t^{-1})$. Then $\|\phi_N\|_{H^s} \sim 1$. Let
\[
u(x,t) = \int_0^t U_\delta(t - \tau)\partial_x(\prod_{i=1}^3 U_\delta(\tau)\phi_i)d\tau,
\]
them by straightforward calculating we have
\[
F_x(u)(\xi,t) = i\xi e^{it\omega_2(\xi)} \int_{\mathbb{R} \times \mathbb{R}} \frac{e^{iP(\xi,\xi_1,\xi_2)}}{iP(\xi,\xi_1,\xi_2)} \hat{\phi}_N(\xi_1)\hat{\phi}_N(\xi_2)\hat{\phi}_N(\xi - \xi_1 - \xi_2)d\xi_1d\xi_2,
\]
where
\[
P(\xi,\xi_1,\xi_2) = \omega_3(\xi_1) + \omega_3(\xi_2) + \omega_3(\xi - \xi_1 - \xi_2) - \omega_3(\xi).
\]
Lemma 8.1. which follows immediately from the definition. Since the other case can be treated in the same ways. We need the following lemma:

\[ \int_{\mathbb{R}^2} e^{itP(\xi_1, \xi_2)} - 1 \frac{iP(\xi_1, \xi_2)}{iP(\xi_1, \xi_2)} \chi_{[N,N+\gamma]}(\xi_1) \chi_{[N,N+\gamma]}(\xi_2) \chi_{[N,N+\gamma]}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2. \]

Since \( N \gg 1 \) then due to the localization (Note that there is a cancelation which is crucial)

\[ |P(\xi, \xi_1, \xi_2)| = |\coth(\xi_1) + \coth(\xi_2) + \coth(\xi - \xi_1 - \xi_2) - \coth(\xi)| \approx |\xi_1^2 + \xi_2^2 - (\xi - \xi_1 - \xi_2)^2 - \xi^2| \approx \gamma^2. \]

Therefore,

\[ \|u\|_{H^s} \lesssim |t|^{-2s} N \lesssim N^{-2s} N, \tag{7.4} \]

which implies \( s \geq 1/2 \).

Considering the solution map of Eq. (1.1) \( \phi \rightarrow u(t) \), then by computing the Frechet derivatives, we get

\[ \frac{\partial^3 u}{\partial \phi^3}|_{\phi=0} (h_N, h_N, h_N) = \int_0^t U_\delta(t - \tau) \partial_x [(U_\delta(\tau) h_N)^3] d\tau. \]

So, if \( \phi \rightarrow u \) is of class \( C^3 \) at the origin, then we have

\[ \left\| \int_0^t U_\delta(t - \tau) \partial_x [(U_\delta(\tau) h_N)^3] d\tau \right\|_{H^s} \lesssim \|h_N\|_{H^s}^3, \tag{7.5} \]

which fails as we have showed.

8. Limit Behavior

In this section we prove Theorem 1.5. We only prove the theorem for \( s = 1/2 \) since the other case can be treated in the same ways. We need the following lemma which follows immediately from the definition.

Lemma 8.1. Assume \( \delta > 0 \). If \( s \in \mathbb{R} \) and \( u \in L^2_t H^s_x \), then

\[ \|u\|_{H^s} \lesssim \|u\|_{L^2_t H^s_x}. \tag{8.1} \]

Assume \( u_\delta \) is a \( H^{1/2} \)-strong solution to (1.1) obtained in the last section and \( v \) is a \( H^{1/2} \)-strong solution to (1.4) in [3], with initial data \( \phi_1, \phi_2 \in H^{1/2} \) satisfying \( \|\phi_i\|_{L^2} \ll 1, i = 1, 2, \) respectively. From the scaling (1.8), we may assume first that \( \|\phi_1\|_{H^{1/2}}, \|\phi_2\|_{H^{1/2}} \ll 1 \). We still denote by \( u_\varepsilon, v \) the extension of \( u_\varepsilon, v \). Let \( w = u_\delta - v \) and \( \phi = \phi_1 - \phi_2 \), then \( w \) solves

\[ \begin{cases} \partial_t w - G_\delta(\partial_x^2 w) + (G_\delta - \mathcal{H}) \partial_x^2 v + \left( \frac{w(u^2 + 3u_\varepsilon v)}{3} \right)_x = 0, & t \in \mathbb{R}_+, \ x \in \mathbb{R}, \\ v(0) = \phi. \end{cases} \tag{8.2} \]

We first view \( (G_\delta - \mathcal{H}) \partial_x^2 v \) as a perturbation to the difference equation, and consider the integral equation of (8.2)

\[ w(x, t) = U_\delta(t) \phi - \int_0^t U_\delta(t - \tau) [(G_\delta - \mathcal{H}) \partial_x^2 v + \left( \frac{w(u^2 + 3u_\varepsilon v)}{3} \right)_x] d\tau. \]
Then \( w \) solves the following integral equation on \( t \in [0, 1] \),
\[
    w(x, t) = \psi(t) [U_\delta(t)\phi - \int_0^t U_\delta(t - \tau) \chi_{b_+} (\tau) \psi(\tau)(G_\delta - H) \partial_x^2 v(\tau)d\tau] - \int_0^t U_\delta(t - \tau) \partial_x [\psi^3(\tau)w(w^2 + 3u_4)\phi(\tau)]d\tau. 
\]
(8.3)
From Lemma 6.1 and Lemma 6.2, 8.1 and Proposition 6.3, we get
\[
    \|w\|_{L_{[0,2]}^2} + \frac{1}{\delta}\|u_\delta\|_{L_{[0,2]}^2} \leq \|w\|_{L_{[0,2]}^2} + \|v\|_{L_{[0,2]}^2}^2 + \|u_\delta\|_{L_{[0,2]}^2}^2.
\]
Since from the proof of Theorem 1.1 we have
\[
    \|v\|_{L_{[0,2]}^2} \lesssim \|\phi_2\|_{H^{1/2}} \ll 1, \quad \|u_\delta\|_{L_{[0,2]}^2} \lesssim \|\phi_1\|_{H^{1/2}} \ll 1,
\]
then we get that
\[
    \|w\|_{L_{[0,2]}^2} \lesssim \|\phi\|_{H^{1/2}} + \frac{1}{\delta}\|u_\delta\|_{L_{[0,2]}^2}^2. 
\]
(8.4)
From Lemma 3.6 and Theorem 1.1 (d) we get
\[
    \|u_\delta - v\|_{C([0, T]; H^{1/2})} \lesssim \|\phi_1 - \phi_2\|_{H^{1/2}} + \frac{1}{\delta}C(\|\phi_1\|_{H^{1/2}}, \|\phi_2\|_{H^{1/2}}).
\]
For general \( \phi_1, \phi_2 \in H^{1/2} \) satisfying \( \|\phi_i\|_{L^2} \ll 1, \; i = 1, 2 \), using the scaling (1.8), then we immediately get that there exists \( T = T(\|\phi_1\|_{H^{1/2}}, \|\phi_2\|_{H^{1/2}}) > 0 \) such that
\[
    \|u_\delta - v\|_{C([0, T]; H^{1/2})} \lesssim \|\phi_1 - \phi_2\|_{H^{1/2}} + \frac{1}{\delta}C(T, \|\phi_1\|_{H^{1/2}}, \|\phi_2\|_{H^{1/2}}). 
\]
(8.5)
Therefore, it follows that (8.5) automatically holds for any \( T > 0 \) due to (1.12) and Theorem 1.1 (d).

**Proof of Theorem 1.5.** For fixed \( T > 0 \), we need to prove that \( \forall \; \eta > 0 \), there exists \( N > 0 \) such that if \( \delta > N \) then
\[
    \|S_T^K(\varphi) - S_T(\varphi)\|_{C([0, T]; H^{1/2})} < \eta. 
\]
(8.6)
We denote \( \varphi_K = P_{\leq K}\varphi \). Then we get
\[
    \|S_T^K(\varphi) - S_T(\varphi)\|_{C([0, T]; H^{1/2})} 
\]
\[
    \leq \|S_T^K(\varphi - \varphi_K)\|_{C([0, T]; H^{1/2})} 
\]
\[
    + \|S_T^K(\varphi_K) - S_T(\varphi_K)\|_{C([0, T]; H^{1/2})} + \|S_T(\varphi_K) - S_T(\varphi)\|_{C([0, T]; H^{1/2})}. 
\]
From Theorem 1.1 (d) and (8.5) and the results in [14, 3] that the solution map of the modified Benjamin-Ono equation is Lipschitz continuous, we get
\[
    \|S_T^K(\varphi) - S_T(\varphi)\|_{C([0, T]; H^{1/2})} \lesssim \|\varphi_K - \varphi\|_{H^{1/2}} + \frac{1}{\delta}C(T, K, \|\varphi\|_{H^{1/2}}). 
\]
(8.7)
We first fix \( K \) large enough, then let \( \delta \) go to infinity, therefore (8.6) holds. \( \blacksquare \)

**Acknowledgment.** This work is supported in part by the National Science Foundation of China, grant 10571004; and the 973 Project Foundation of China, grant 2006CB805902, and the Innovation Group Foundation of NSFC, grant 10621061.
References

[1] L. Abdelouhab, J. L. Bona, M. Felland, J. C. Saut, Nonlocal models for nonlinear dispersive waves, Physica D, 40 (1989), 360-392.
[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II, Geom. Funct. Anal., 3:107-156, 209-262, 1993.
[3] Z. Guo, Local well-posedness and a priori bounds for the modified Benjamin-Ono equation without using a gauge transformation, arXiv:0807.3764v1
[4] Z. Guo, L. Peng, B. Wang, Decay estimates for a class of wave equations, Journal of Functional Analysis, 254/6 (2008) 1642-1660.
[5] Z. Guo, B. Wang, Global well posedness and inviscid limit for the Korteweg-de Vries-Burgers equation, arXiv:0803.2450v2
[6] L. Han, B. Wang, Global wellposedness and limit behavior for the generalized finite-depth-fluid equation with small critical data, J. Differential Equations 245 (2008), 2103-2144.
[7] A. D. Ionescu, C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces, J. Amer. Math. Soc., 20 (2007), no. 3, 753-798.
[8] A. D. Ionescu, C. E. Kenig, D. Tataru, Global well-posedness of KP-I initial-value problem in the energy space, arXiv:0705.4239v1.
[9] R.I. Joseph, Solitary waves in a finite depth fluid, J. Phys. A, 10 (1977), L225-L227.
[10] C. E. Kenig and K. D. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Mathematical Research Letters 10 (2003), no. 5-6, 879-895.
[11] C. E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., 40 (1991), 32-69.
[12] C. E. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. Henri. Poincaré, Vol. 10, No. 3, 1993, 255-2288.
[13] C. E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation. J. Amer. Math. Soc., 9:573-603, 1996. MR 96k:35159.
[14] C. E. Kenig, H. Takaoka, Global wellposedness of the Modified Benjamin-Ono equation with initial data in $H^{1/2}$, International Mathematics Research Notices 2006 (2006), no. 1, 1-44.
[15] Y. Kodama, J. Satsuma and M. J. Ablowitz, Nonlinear intermediate long-wave equation: analysis and method of solution, Phys. Rev. Lett., 46 (1981), 687-690.
[16] L. Molinet, F. Ribaud, Well-posedness results for the generalized BenjaminCOono equation with small initial data, J. Math. Pures Appl. 83 (2004) 277C311.
[17] L. Molinet, J.C. Saut, N. Tzvetkov, Ill-posedness issues for the BenjaminCOono and related equations, SIAM J. Math. Anal. 33 (4) (2001) 982C988.
[18] J. Satsuma, M. J. Ablowitz, Y. Kodama, On an internal Wave equation describing a stratified uid weth nite depth, Phys. Lett. A, 73 (1979), 283-286.
[19] D. Tataru, Local and global results for wave maps I, Comm. Partial Differential Equations 23 (1998), 1781-1793.
[20] Terence Tao’s home page, http://www.math.ucla.edu/~tao/Dispersive/
[21] T. Tao, Multiplinl weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations. Amer. J. Math., 123(5):839-908, 2001. MR 2002k:35283

LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, CHINA
E-mail address: zihuaguo, wbx@math.pku.edu.cn
URL: http://guo.5188.org