ON TRAIN TRACK SPLITTING SEQUENCES

HOWARD MASUR, LEE MOSHER, AND SAUL SCHLEIMER

Abstract. We show that the subsurface projection of a train track splitting sequence is an unparameterized quasi-geodesic in the curve complex of the subsurface. For the proof we introduce induced tracks, efficient position, and wide curves.

This result is an important step in the proof that the disk complex is Gromov hyperbolic. As another application we show that train track sliding and splitting sequences give quasi-geodesics in the train track graph, generalizing a result of Hamenstädt [Invent. Math.].

1. Introduction

Train track splitting sequences are an integral part of the study of surface diffeomorphisms [19, 15, 14]. There are links between splitting sequences of tracks and the curves they carry, on the one hand, and Teichmüller geodesics and measured foliations on the other. Of particular importance, in light of the hierarchy machine [12] and the closely related control of Teichmüller geodesics [16, 17], is to understand how a splitting sequence interacts with subsurface projection. We give a detailed structure theorem (Theorem 5.3) that explains this interaction. Our main application is the following result, needed for Masur and Schleimer’s work on the disk complex.

Theorem 5.5. For any surface $S$ with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in $S$ and for any essential subsurface $X \subset S$ if $\pi_X(\tau_N) \neq \emptyset$ then the sequence $\{\pi_X(\tau_i)\}_{i=0}^N$ is a $Q$-unparameterized quasi-geodesic in the curve complex $C(X)$.

As another application of Theorem 5.3 we generalize, via a very different proof, a result of Hamenstädt [8, Corollary 3]:

Theorem 6.2. For any surface $S$ with $\xi(S) \geq 1$ there is a constant $Q = Q(S)$ with the following property: If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting...
sequence in the train track graph $T(S)$, injective on slide subsequences, then $\{\tau_i\}$ is a $Q$–quasi-geodesic.

For the proof of Theorem 5.3 we introduce induced tracks, efficient position, and wide curves. For any essential subsurface $X \subset S$ and track $\tau \subset S$ there is an induced track $\tau | X$. Induced tracks generalize the notion of subsurface projection of curves. Efficient position of a curve with respect to a track $\tau$ is a simultaneous generalization of curves carried by $\tau$ and curves dual to $\tau$ (called hitting $\tau$ efficiently in [15]). Efficient position of $\partial X$ allows us to pin down the location of the induced track $\tau | X$. Wide curves are our combinatorial analogue of curves of definite modulus in a Riemann surface. The structure theorem (5.3) then implies Theorem 5.5: this, together with subsurface projection, controls the motion of a splitting sequence through the complex of curves $C(X)$.

Theorem 6.2, our second application of the structure theorem, is a direct consequence of Theorem 6.1, stated in terms of the marking graph. Theorem 6.1 requires a delicate induction proof conceptually similar to the hierarchy machine developed in [12]. We do not deduce Theorem 6.1 directly from the results of [12]; in particular it is not known if splitting sequences fellow-travel resolutions of hierarchies.

Acknowledgments. We thank Yair Minsky for enlightening conversations.

2. Background

We provide the definitions needed for Theorem 5.3 and its corollaries.

2.1. Coarse geometry. Suppose that $Q \geq 1$ is a real number. For real numbers $r, s$ we write $r \leq_Q s$ if $r \leq Qs + Q$ and say that $r$ is quasi-bounded by $s$. We write $r =_Q s$ if $r \leq_Q s$ and $s \leq_Q r$; this is called a quasi-equality.

For a metric space $(X, d_X)$ and finite diameter subsets $A, B \subset X$ define $d_X(A, B) = \text{diam}_X(A \cup B)$. Following Gromov [6], a relation $f: X \to Y$ of metric spaces is a $Q$–quasi-isometric embedding if for all $x, y \in X$ we have $d_X(x, y) =_Q d_Y(f(x), f(y))$. (Here $f(x) \subset Y$ is the set of points related to $x$.) If, additionally, the $Q$–neighborhood of $f(X)$ equals $Y$ then $f$ is a $Q$–quasi-isometry and $X$ and $Y$ are quasi-isometric.

If $[m, n]$ is an interval in $\mathbb{Z}$ and $f: [m, n] \to Y$ is a quasi-isometric embedding then $f$ is a $Q$–quasi-geodesic. Now suppose that $Q > 1$ is a real number, $[m, n]$ and $[p, q]$ are intervals in $\mathbb{Z}$, and $f: [m, n] \to Y$ is a relation. Then $f$ is a $Q$–unparameterized quasi-geodesic if there is a
strictly increasing function \( \rho \colon [p,q] \to [m,n] \) so that \( f \circ \rho \) is a \( Q \)-quasi-geodesic and for all \( i \in [p,q-1] \) the diameter of \( f \left( \left[ \rho(i), \rho(i+1) \right] \right) \) is at most \( Q \).

2.2. Surfaces, arcs, and curves. Let \( S = S_{g,n} \) be a compact, connected, orientable surface of genus \( g \) with \( n \) boundary components. The complexity of \( S \) is \( \xi(S) = 3g - 3 + n \). A curve in \( S \) is an embedding of the circle into \( S \). An arc in \( S \) is a proper embedding of the interval \([0,1]\) into \( S \). A curve or arc \( \alpha \subset S \) is trivial if \( \alpha \) separates \( S \) and one component of \( S \setminus \alpha \) is a disk; otherwise \( \alpha \) is essential. A curve \( \alpha \) is peripheral if \( \alpha \) separates \( S \) and one component of \( S \setminus \alpha \) is an annulus; otherwise \( \alpha \) is non-peripheral. A connected subsurface \( X \subset S \) is essential if every component of \( \partial X \) is essential in \( S \) and \( X \) is neither a pair of pants \((S_{0,3})\) nor a peripheral annulus (the core curve is peripheral). Note that \( X \) inherits an orientation from \( S \). This, in turn, induces an orientation on \( \partial X \) so that \( X \) is to the left of \( \partial X \).

Define \( C(S) \) to be the set of isotopy classes of essential, non-peripheral curves in \( S \). Define \( A(S) \) to be the set of proper isotopy classes of essential arcs in \( S \). Let \( AC(S) = C(S) \cup A(S) \). If \( \alpha, \beta \in AC(S) \) then the geometric intersection number \([5]\) of \( \alpha \) and \( \beta \) is
\[
i(\alpha, \beta) = \min \{|a \cap b| : a \in \alpha, b \in \beta \}.
\]
A finite subset \( \Delta \subset AC(S) \) is a multicurve if \( i(\alpha, \beta) = 0 \) for all \( \alpha, \beta \in \Delta \).

If \( T \subset S \) is a subsurface with \( \partial T \) a union of smooth arcs, meeting perpendicularly at their endpoints, then define
\[
\text{index}(T) = \chi(T) - \frac{c^+(T)}{4} + \frac{c^-(T)}{4}
\]
where \( c^\pm(T) \) is the number of outward (inward) corners of \( \partial T \). Note that index is additive: \( \text{index}(T \cup T') = \text{index}(T) + \text{index}(T') \) as long as the interiors of \( T, T' \) are disjoint.

2.3. Train tracks. For detailed discussion of train tracks see \([19, 15, 14]\). A pretrack \( \tau \subset S \) is a properly embedded graph in \( S \) with additional structure. The vertices of \( \tau \) are called switches; every switch \( x \) is equipped with a tangent \( v_x \in T^1_x S \). We require every switch to have valence three; higher valence is dealt with in \([15]\). The edges of \( \tau \) are called branches. All branches are smoothly embedded in \( S \). All branches incident to a fixed switch \( x \) have derivative \( \pm v_x \) at \( x \).

An immersion \( \rho \colon \mathbb{R} \to S \) is a train-route (or simply a route) if
- \( \rho(\mathbb{R}) \subset \tau \) and
- \( \rho(n) \) is a switch if and only if \( n \in \mathbb{Z} \).
The restriction $\rho|[0, \infty)$ is a half-route. If $\rho$ factors through $\mathbb{R}/m\mathbb{Z}$ then $\rho$ is a train-loop. We require, for every branch $b$, a train-route travelling along $b$.

For each branch $b$ and point $p \in b$, a component $b'$ of $b \setminus \{p\}$ is a half-branch. Two half-branches $b', b'' \subset b$ are equivalent if $b' \cap b''$ is again a half-branch. Every switch divides the three incident half-branches into a pair of small half-branches on one side and a single large half-branch on the other. A branch $b$ is large (small) if both of its half-branches are large (small); if $b$ has one large and one small half-branch then $b$ is called mixed.

Let $B = B(\tau)$ be the set of branches of $\tau$. A function $w: B \rightarrow \mathbb{R}_{\geq 0}$ is a transverse measure on $\tau$ if $w$ satisfies the switch conditions: for every switch $x \in \tau$ we have $w(a) + w(b) = w(c)$, where $a', b'$ are the small half-branches and $c'$ is the large half-branch meeting $x$. Let $P(\tau)$ be the projectivization of the cone of transverse measures; define $V(\tau)$ to be the vertices of the polyhedron $P(\tau)$.

![Figure 1](image_url)

**Figure 1.** Top: A large branch admits a right, central, and left splitting. Bottom: a mixed branch admits a slide.

We may split a pretrack along a large branch or slide it along a mixed branch; see Figure 1. (Slides are called shifts in [15].) The inverse of a split or slide is called a fold. Note that the inverse of a slide may be obtained via a slide followed by an isotopy.

Suppose that $\tau \subset S$ is a pretrack. Let $N = N(\tau) \subset S$ be a tie neighborhood of $\tau$: so $N$ is a union of rectangles $\{R_b|b \in B\}$ foliated by
vertical intervals (the ties). At a switch, the upper and lower thirds of the vertical side of the large rectangle are identified with the vertical side of the small rectangles, as shown in Figure 2. Since $N$ is a union of rectangles it follows that $\text{index}(N) = 0$. The horizontal boundary $\partial_h N$ is the union of $\partial_h R_b$, for $b \in B$, while the vertical boundary is $\partial_v N = \partial N \setminus \partial_h N$.

Let $N = N(\tau)$ be a tie neighborhood. Let $T$ be a complementary region of $\tau$: a component of the closure of $S \setminus N$. Define the horizontal and vertical boundary of $T$ to be $\partial_h T = \partial T \cap \partial_h N$ and $\partial_v T = \partial T \cap \partial_v N$. Note that all corners of $T$ are outward, so $\text{index}(T) = \chi(T) - \frac{1}{4} |\partial \partial_h T|$.

Suppose $\tau \subset S$ is a pretrack. The subsurface filled by $\tau$ is the union of $N$ with all complementary regions $T$ of $\tau$ that are disks or peripheral annuli.

**Definition 2.1.** Suppose that $\tau \subset S$ is a pretrack and $N = N(\tau)$. We say that $\tau$ is a train track if $\tau$ is compact, every component of $\partial N$ has at least one corner and every complementary region $T$ of $\tau$ has negative index.

**Definition 2.2.** In a sliding and splitting sequence $\{\tau_i\}$ of train tracks each $\tau_{i+1}$ is obtained from $\tau_i$ by a slide or a split.

**2.4. Carrying, duality, and efficient position.** Suppose that $\tau \subset S$ is a train track. If $\sigma$ is also a track, contained in $N = N(\tau)$ and transverse to the ties, then we write $\sigma \prec \tau$ and say that $\sigma$ is carried by $\tau$. For example, if $\tau$ is a fold of $\sigma$ then $\sigma$ is carried by $\tau$.

A properly embedded arc or curve $\beta \subset N$ is carried by $\tau$ if $\beta$ is transverse to the ties and $\partial \beta \cap \partial_h N = \emptyset$. Thus if $\beta$ is carried then $\partial \beta \subset \partial_h N$. Again we write $\beta \prec \tau$ for carried arcs and curves.
Definition 2.3. Suppose that $\alpha \subset S$ is a properly embedded arc or curve. Then $\alpha$ is in efficient position with respect to $\tau$, denoted $\alpha \vdash \tau$, if

- every component of $\alpha \cap N$ is a tie or is carried by $\tau$ and
- every region $T \subset S \setminus (N \cup \alpha)$ has negative index or is a rectangle.

Suppose that $\alpha \vdash \tau$. If $\alpha \subset N$ then $\alpha$ is carried, $\alpha \prec \tau$. If no component of $\alpha \cap N$ is carried then $\alpha$ is dual to $\tau$ and we write $\alpha \blacktriangleleft \tau$.

If $\Delta \subset \mathcal{AC}(S)$ is a multicurve then we write $\Delta \prec \tau$, $\Delta \blacktriangleleft \tau$, or $\Delta \vdash \tau$ if all elements of $\Delta$ are disjointly and simultaneously carried, dual, or in efficient position.

Remark 2.4. Our notion of duality is called hitting efficiently by Penner and Harer [15, page 19]. Note that $\alpha \blacktriangleleft \tau$ if and only if $\alpha$ is carried by some extension of the dual track $\tau^*$, also defined in [15]. Likewise, if $\alpha \vdash \tau$ and $\alpha \cap N$ consists of carried arcs then $\alpha$ is carried by some extension of $\tau$.

An index argument proves:

Lemma 2.5. If $\alpha$ is a properly embedded curve or arc in efficient position with respect to a train track $\tau \subset S$ then $\alpha$ is essential and non-peripheral in $S$. $\square$

One of the goals of this paper is to prove the converse of Lemma 2.5; this is done in Theorem 4.1. Following Lemma 2.5 we may define $\mathcal{C}(\tau) = \{\alpha \mid \alpha \prec \tau\}$ and $\mathcal{C}^*(\tau) = \{\alpha \mid \alpha \blacktriangleleft \tau\}$. Notice that if $\sigma \prec \tau$ is a track then $\mathcal{C}(\sigma) \subset \mathcal{C}(\tau)$ and $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$.

A branch $b \in \mathcal{B}(\tau)$ is recurrent if there is some $\alpha \prec \tau$ that meets $R_b$. The track $\tau$ is recurrent if every branch is recurrent. Transverse recurrence is defined by replacing carrying by duality [15, page 20]. The track $\tau$ is birecurrent if $\tau$ is recurrent and transversely recurrent [15, Section 1.3]. In a slight departure from Penner and Harer’s terminology [15, page 27] we will call a birecurrent track $\tau$ complete if all complementary regions have index $-1/2$. (When $S = S_{1,1}$ there is, instead, a single complementary region with index $-1$.)

Lemma 2.6. Suppose that $\sigma \subset S$ is a birecurrent track. Then $\mathcal{C}^*(\sigma)$ has infinite diameter inside of $\mathcal{C}(S)$. $\square$

Proof. Let $\tau$ be a complete track extending $\sigma$ [15, Corollary 1.4.2]. Section 3.4 of [15] and a dimension count gives a lamination $\lambda \blacktriangleleft \tau$ so that $i(\lambda, \alpha) \neq 0$ for all $\alpha \in \mathcal{C}(S)$. Now an argument of Kobayashi [10], refined by Luo [11, page 124], implies that $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$ has infinite diameter. $\square$
2.5. Vertex cycles. When $\alpha \prec \tau$ is a curve there is a transverse measure $w_\alpha$ defined by taking $w_\alpha(b) = |\alpha \cap t|$ where $t$ is any tie of the rectangle $R_b$. Conversely, for any integral transverse measure $w$ there is a multicurve $\alpha_w$ — take $w(b)$-many horizontal arcs in $R_b$ and glue endpoints as dictated by the switch conditions.

Note that if $v \in V(\tau)$ then there is a minimal integral measure $w$ projecting to $v$. Since $v$ is an extreme point of $P(\tau)$ deduce that $\alpha_w$ is an embedded curve. We call $\alpha_w$ a vertex cycle of $\tau$ and henceforth use $V(\tau)$ to denote the set of vertex cycles.

2.6. Wide curves. Let $N = N(\tau)$ be a tie neighborhood.

**Definition 2.7.** A multicurve $\Delta \dashv \tau$ is wide if there is an orientation of the components of $\Delta$ so that

- for every $b \in B(\tau)$, all arcs of $\Delta \cap R_b$ are to the right of each other (see the top of Figure 3) and
- for every complementary region $T$ of $\tau$, all arcs of $\Delta \cap T$ are to the right of each other (see the bottom of Figure 3).

![Figure 3](image)

**Figure 3.** Above: Arcs of $\Delta \cap R_b$ to the right of each other. The vertical dotted line are ties; the heavy horizontal lines are arcs of $\partial_b N$. Below: Arcs meeting the complementary region $T$, all to the right of each other.

It follows from the definition that if $\Delta \dashv \tau$ is wide then for any branch $b \in B(\tau)$ the intersection $\Delta \cap R_b$ has at most two components.

**Lemma 2.8.** Every vertex cycle $\alpha \in V(\tau)$ is wide.
For an even more precise characterization of vertex cycles see Lemma 3.11.3 of [14]. The proof below recalls a surgery technique used in the sequel.

Proof of Lemma 2.8. We prove the contrapositive. Suppose that $\alpha$ is not wide. Orient $\alpha$. There are three cases.

Suppose there is a branch $b \subset \tau$ and an oriented tie $t \subset R_b$ where $x$ and $y$ are consecutive (along $t$) points of $\alpha \cap t$ so that the signs of intersection at $x$ and $y$ are equal. Let $[x, y]$ be the subarc of $t$ bounded by $x$ and $y$. Surger $\alpha$ along $[x, y]$ to form curves $\beta, \gamma \prec \tau$. See Figure 4. Thus $w_\alpha = w_\beta + w_\gamma$ and $\alpha$ is not a vertex cycle.

![Figure 4](image)

**Figure 4.** Surgery when adjacent intersections have the same sign.

Suppose instead that $x, y, z$ are consecutive (along $t$) points of $\alpha \cap t$ with alternating sign. In this case there is again a surgery along $[x, z]$ producing curves $\beta$ and $\gamma$. See Figure 5 for one of the possible arrangements of $\alpha, \beta$ and $\gamma$. Again $w_\alpha = w_\beta + w_\gamma$ is a non-trivial sum and $\alpha$ is not a vertex cycle.

In the remaining case $w_\alpha(a) \leq 2$ for all $a \in \mathcal{B}$ and there are branches $b, c \in \mathcal{B}$ where the arcs of $\alpha \cap R_b$ are to the right of each other while the arcs of $\alpha \cap R_c$ are to the left of each other. See Figure 6 for the two ways $\alpha$ may be carried by $\tau$.

If $\alpha$ is carried as in the top line of Figure 6 then surger $\alpha$ in both rectangles $R_b$ and $R_c$ as was done in Figure 4. This shows that $w_\alpha$ is a non-trivial sum and so $\alpha$ is not a vertex cycle. Now suppose that $\alpha$ is carried as in the bottom line of Figure 6. Note that the closure of $\alpha \setminus (R_b \cup R_c)$ is a union of four arcs. Two of these, $\beta'$ and $\gamma'$, meet both $R_b$ and $R_c$. Since $S$ is orientable no tie-preserving isotopy of $N$ throws $\beta'$ onto $\gamma'$. Let $\alpha' \cup \alpha'' = \alpha \setminus (\beta' \cup \gamma')$. Create an embedded curve $\beta$ by taking two parallel copies of $\beta'$ and joining them to $\alpha'$ and $\alpha''$. Similarly create $\gamma$ by joining two parallel copies of $\gamma'$ to the arcs.
Figure 5. Surgery when the three intersections have alternating sign.

Figure 6. The curve $\alpha$ meets $R_b$ and $R_c$ twice.

$\alpha'$ and $\alpha''$. It follows that $w_\beta \neq w_\gamma$. Since $2w_\alpha = w_\beta + w_\gamma$ again $\alpha$ is not a vertex cycle. $\Box$

2.7. Combing. Suppose that $\alpha \prec \tau$ has $w_\alpha(b) \leq 1$ for every branch $b \subset \tau$. Orient and transversely orient $\alpha$ to agree with the orientation of the surface $S$. We think of the orientation as pointing in the $x$-direction and the transverse orientation pointing in the $y$-direction. A half-branch $b \subset \tau \prec \alpha$, sharing a switch with $\alpha$, twists to the right if any train-route through $b$ locally has positive slope. Otherwise $b$ twists to the left. If all branches on one side of $\alpha$ twist to the right then that side of $\alpha$ has a right combing, and similarly for a left combing. See Figure 8 for an example where both sides are combed to the left.
2.8. Curve complexes and subsurface projection. For more information on the curve complex see [11, 12]. Impose a simplicial structure on $\mathcal{AC}(S)$ where $\Delta \subset \mathcal{AC}(S)$ is a simplex if and only if $\Delta$ is a multicurve. The complex of curves $\mathcal{C}(S)$ and the arc complex $\mathcal{A}(S)$ are the subcomplexes spanned by curves and arcs, respectively. Note that if the complexity $\xi(S)$ is at least two then $\mathcal{C}(S)$ is connected [9, Proposition 2]. For surfaces of lower complexity we alter the simplicial structure on $\mathcal{C}(S)$.

Define the Farey tessellation $\mathcal{F}$ to have vertex set $\mathbb{Q} \cup \{\infty\}$. A collection of slopes $\Delta \subset \mathcal{F}$ spans a simplex if $ps - rq = \pm 1$ for all $p/q, r/s \in \Delta$. If $S = S_{1,1}$ or $S_{0,4}$ we take $\mathcal{C}(S) = \mathcal{F}$; that is, there is an edge between curves that intersect in exactly one point (for $S_{1,1}$) or two points (for $S_{0,4}$). Note that for surfaces with $\xi(S) \geq 1$ the inclusion of $\mathcal{C}(S)$ into $\mathcal{AC}(S)$ is a quasi-isometry.

Suppose now that $X \cong S_{0,2}$ is an annulus. Define $\mathcal{A}(X)$ to be the set of all essential arcs in $X$, up to isotopy fixing $\partial X$ pointwise. For $\alpha, \beta \in \mathcal{A}(X)$ define

$$i(\alpha, \beta) = \min \{|(a \cap b) \setminus \partial X| : a \in \alpha, b \in \beta\}.$$ 

As usual, multicurves give simplices for $\mathcal{A}(X)$.

If $\alpha, \beta$ are vertices of $\mathcal{C}(S)$, $\mathcal{A}(S)$, or $\mathcal{AC}(S)$ then define $d_S(\alpha, \beta)$ to be the minimal number of edges in a path, in the one-skeleton, connecting $\alpha$ to $\beta$; the containing complex will be clear from context. Note that if $\alpha \beta$ are distinct arcs of $\mathcal{A}(X)$, when $X$ is an annulus, then $d_X(\alpha, \beta) = 1 + i(\alpha, \beta)$ [12, Equation 2.3].

As usual, suppose that $\xi(X) \geq 1$. Fix an essential subsurface $X \subset S$ with $\xi(X) < \xi(S)$. We suppose that $X$ is either a non-peripheral annulus or a surface of complexity at least one. (The case of an essential annulus inside of $S_1$ is not relevant here.) Following [12], we will define the subsurface projection relation $\pi_X : \mathcal{AC}(S) \to \mathcal{AC}(X)$. Let $S^X$ be the cover of $S$ corresponding to the inclusion $\pi_1(X) < \pi_1(S)$. The surface $S^X$ is not compact; however, there is a canonical (up to isotopy) homeomorphism between $X$ and the Gromov compactification of $S^X$. This identifies the arc and curve complexes of $X$ and $S^X$. Fix $\alpha \in \mathcal{AC}(S)$. Let $\alpha^X$ be the preimage of $\alpha$ in $S^X$.

If $\alpha^X$ contains a non-peripheral curve in $S^X$ then $\pi_X(\alpha) = \{\alpha\}$. Otherwise, place every essential arc of $\alpha^X$ into the set $\pi_X(\alpha)$. If neither obtains then $\pi_X(\alpha) = \emptyset$. If $\pi_X(\alpha) = \emptyset$ then we say that $\alpha$ misses $X$. If $\pi_X(\alpha) \neq \emptyset$ then $\alpha$ cuts $X$.

Suppose that $\alpha, \beta \in \mathcal{C}(S)$. If $\pi_X(\alpha)$ and $\pi_X(\beta)$ are nonempty define

$$d_X(\alpha, \beta) = \text{diam}_X (\pi_X(\alpha) \cup \pi_X(\beta)).$$
Likewise define the distance $d_X(A, B)$ between finite sets $A, B \subset C(S)$. When $\tau$ is a track, we use the shorthand $\pi_X(\tau)$ for the set $\pi_X(V(\tau))$. If $\sigma$ is also a track, we write $d_X(\pi_X(\tau), \pi_X(\sigma))$.

We end with a lemma connecting the subsurface projection of carried (or dual) curves to the behavior of wide curves.

**Lemma 2.9.** Suppose that $X \subset S$ is an essential surface and $\tau$ is a track. If some $\alpha \prec \tau (\alpha \cap \tau)$ cuts $X$ then there is a vertex cycle $\beta \prec \tau$ (wide dual $\beta \cap \tau$) cutting $X$.

**Proof.** Some multiple of $\alpha \prec \tau$ is a sum of vertices: $m \cdot w_\alpha = \sum n_i w_i$ where $w_i$ is the integral transverse measure for the vertex $\beta_i \in V(\tau)$. Via a sequence of tie-preserving isotopies of $N$ we may arrange for all of the $\beta_i$ to realize their geometric intersection with each other. Note that there is an isotopy representative of $\alpha$ contained inside of a small neighborhood of the union $B = \bigcup \beta_i$.

To prove the contrapositive, suppose that none of the $\beta_i$ cut $X$. It follows that $X$ may be isotoped in $S$ to be disjoint from $B$. Thus $\alpha$ misses $X$, as desired. A similar discussion applies when $\alpha \cap \tau$. □

### 3. Induced Tracks

Suppose that $\tau \subset S$ is a train track. Suppose that $X \subset S$ is an essential subsurface with $\xi(X) < \xi(S)$. Let $S^X$ be the corresponding cover of $S$. Let $\tau^X$ be the preimage of $\tau$ in $S^X$; note that the pretrack $\tau^X$ satisfies all of the axioms of a train track except compactness.

Define $\mathcal{AC}(\tau^X)$ to be the set of essential arcs and essential, non-peripheral curves properly embedded in the Gromov compactification of $S^X$ with interior a train-route or train-loop carried by $\tau^X$. A bit of caution is required here — inessential arcs and peripheral curves may be carried by $\tau^X$ but these are not admitted into $\mathcal{AC}(\tau^X)$. Define $\mathcal{A}(\tau^X), \mathcal{C}(\tau^X) \subset \mathcal{AC}(\tau^X)$ to be the subsets of arcs and curves respectively. Define $\mathcal{AC}^*(\tau^X)$ to be the set of dual essential arcs and dual essential, non-peripheral curves, up to isotopies fixing $\tau^X$ setwise.

#### 3.1. Induced tracks for non-annuli

If $X$ is not an annulus define $\tau|X$, the induced track, to be the union of the branches of $\tau^X$ crossed by an element of $\mathcal{C}(\tau^X)$.

**Lemma 3.1.** If $X$ is not an annulus then the induced track $\tau|X$ is compact.

**Proof.** Note train-routes in $\tau^X$ that are mapped properly to $S^X$ are uniform quasi-geodesics in $S^X$ [14, Proposition 3.3.3]. Thus there is a compact core $X' \subset S^X$, homeomorphic to $X$, so that any route meeting
$S^X \setminus X'$ has one endpoint on the Gromov boundary of $S^X$. It follows that $\tau|X \subset X'$.

\[\Box\]

Note that $\tau|X$ may not be a train track: $N = N(\tau|X)$ may have smooth boundary components and complementary regions with non-negative index. However, since all complementary regions of $\tau^X$ have negative index it follows that if a complementary region $T$ of $\tau|X$ has non-negative index then $T$ is a peripheral annulus meeting a smooth component of $\partial N$.

The definition of $\tau|X$ implies that $\tau|X$ is recurrent. Carrying, duality, efficient position and wideness with respect to an induced track are defined as in Section 2.4. Define $\mathcal{C}(\tau|X) \subset \mathcal{C}(X)$, the subset of curves carried by $\tau|X$. Note that $\mathcal{C}(\tau|X) = \mathcal{C}(\tau^X)$. Define $\mathcal{AC}^*(\tau|X) \subset \mathcal{AC}(X)$ to be the subset of arcs and curves dual to $\tau|X$. Note that $\mathcal{AC}^*(\tau|X) \supset \mathcal{AC}^*(\tau^X)$.

Now, $\tau|X$ fails to be transversely recurrent exactly when it carries a peripheral curve. We say that a branch $b \subset \tau|X$ is transversely recurrent with respect to arcs and curves if there is $\alpha \in \mathcal{AC}^*(\tau|X)$ meeting $b$. Then $\tau|X$ is transversely recurrent with respect to arcs and curves if every branch $b$ is.

**Lemma 3.2.** Suppose that $\tau$ is transversely recurrent in $S$. Then $\tau|X$ is transversely recurrent with respect to arcs and curves in $X$. Furthermore: suppose that $\tau|X$ is transversely recurrent with respect to arcs and curves in $X$. If $\sigma \subset \tau|X$ is a train track then $\sigma$ is transversely recurrent in $X$.

**Proof.** The first claim follows from the definitions. An index argument proves the second claim. \[\Box\]

Here is our second surgery argument.

**Lemma 3.3.** Suppose that $\tau$ is a track and $X \subset S$ is an essential subsurface, yet not an annulus. For every $\alpha \in \mathcal{A}(\tau^X)$ at least one of the following holds:

- There is an arc $\beta \in \mathcal{A}(\tau^X)$ so that $\beta$ is wide and $i(\alpha, \beta) = 0$.
- There is a curve $\gamma \in \mathcal{C}(\tau|X)$ so that $i(\alpha, \gamma) \leq 2$.

The statement also holds replacing $\mathcal{A}, \mathcal{C}$ by $\mathcal{A}^*, \mathcal{C}^*$.

**Proof.** The proof is modelled on that of Lemma 2.8. If $\alpha < \tau^X$ is wide we are done. If not, as $\alpha$ is a quasi-geodesic [14, Proposition 3.3.3], orient $\alpha$ so that $\alpha$ is wide outside of a compact core for $S^X$. Now we induct on the total number of arcs of intersection between $\alpha$ and rectangles $R_b \subset N(\tau^X)$ meeting the compact core.
Let $t$ be a tie of $R_b$. Orient $t$. Suppose that $x, y$ are consecutive (along $t$) points of $\alpha \cap t$. Suppose that the sign of intersection at $x$ equals the sign at $y$. Let $[x, y]$ be the subarc of $t$ bounded by $x$ and $y$. As in Lemma 2.8 surger $\alpha$ along $[x, y]$ to form an arc $\beta'$ and a curve $\gamma$. See Figure 4 with $\beta'$ substituted for $\beta$.

Note that $\gamma$ is essential in $S^X$, by an index argument. If $\gamma$ is non-peripheral then the second conclusion holds. So suppose that $\gamma$ is peripheral. Then $\alpha$ is obtained by Dehn twisting $\beta'$ about $\gamma$. So $\beta'$ is properly isotopic to $\alpha$ and has smaller intersection with $R_b$; thus we are done by induction.

Suppose instead that $x, y, z$ are consecutive (along $t$) points of $\alpha \cap t$, with alternating sign. Surger $\alpha$ along $[x, z]$ to form an arc $\beta'$ and a curve $\gamma$. See Figure 5, with $\beta'$ substituted for $\beta$, for one of the possible arrangements of $\alpha, \beta'$, and $\gamma$. Again $\gamma$ is essential. If $\gamma$ is non-peripheral then the second conclusion holds and we are done. If $\gamma$ is peripheral then, as $\alpha$ and $\beta'$ differ by a half-twist about $\gamma$, we find that $\beta'$ is properly isotopic to $\alpha$. Since $\beta'$ has smaller intersection with $R_b$ we are done by induction.

All that remains is the case that $\alpha$ meets every rectangle $R_b$ in most a pair of arcs of opposite orientation. For every branch $b$ where $\alpha$ meets $R_b$ twice, choose a subarc $t_b$ of a tie in $R_b$ so that $\alpha \cap t_b = \partial t_b$. We call $t_b$ a chord for $\alpha$. For every $t_b$ there is a subarc $\alpha_b \subset \alpha$ so that $\partial t_b = \partial \alpha_b$. A chord $t_b$ is innermost if there is no chord $t_c$ with $\alpha_c$ strictly contained in $\alpha_b$. Let $t_b$ be the first innermost chord. Let $\alpha'$ be the component of $\alpha \setminus \alpha_b$ before $\alpha_b$. Build a route $\beta$ by taking two copies of $\alpha'$ and joining them to $\alpha_b$. Note that $\beta \cap R_c$ is a single arc or a pair of arcs exactly as $\alpha_b$ or $\alpha'$ meets $R_c$. Thus $\beta$ is wide. Also, $\beta$ is essential: otherwise $t_b \cup \alpha_b$ bounds a disk with index one-half, a contradiction. By construction $i(\alpha, \beta) = 0$ and Lemma 3.3 is proved. $\square$

3.2. Induced tracks for annuli. Suppose that $X \subset S$ is an annulus. Define $\tau|X$ to be the union of branches $b \subset \tau^X$ so that some element of $A(\tau^X)$ travels along $b$. (Note that $\tau|X$, if nonempty, is not compact.) Define $A(\tau|X) = A(\tau^X)$ and also the duals $A^*(\tau|X) \supset A^*(\tau^X)$.

Define $V(\tau|X)$ in $A(\tau|X)$ to be the set of wide carried arcs. Define $V^*(\tau|X)$ dually.

Lemma 3.4. Suppose that $X \subset S$ is an essential annulus. If $A^*(\tau|X)$ is nonempty then $V^*(\tau|X)$ is nonempty. Let $N = N(\tau|X)$. If $\gamma \perp \tau|X$ is a wide essential arc then $\gamma$ meets each rectangle of $N$ and each region of $S^X \setminus N$ in at most a single arc.
For example, if \( \gamma \prec \tau|X \) is a wide essential arc then \( \gamma \) embeds into \( \tau|X \).

**Proof of Lemma 3.4.** We prove the second conclusion; the first is similar. Suppose that \( R \) is either a rectangle or region so that \( \gamma \cap R \) is a pair of arcs to the right of each other. Let \( \delta \) be an arc properly embedded in \( R \setminus \gamma \) so that \( \delta \cap \gamma = \partial \delta \). Let \( \gamma' \) be the component of \( \gamma \setminus \partial \delta \) so that \( \partial \gamma' = \partial \delta \). If \( \gamma' \cup \delta \) bounds a disk in \( S^X \) then this disk has index one-half and we contradict efficient position. If \( \gamma' \cup \delta \) bounds an annulus then \( \gamma \) was not essential, another contradiction. \( \Box \)

Suppose that \( \alpha \) is the core curve of the annulus \( X \).

**Lemma 3.5.** If \( \alpha \) is not carried by \( \tau|X \) then \( V(\tau|X) = A(\tau|X) \). If \( \alpha \) is not dual to \( \tau|X \) then \( V^*(\tau|X) = A^*(\tau|X) \). \( \Box \)

**Lemma 3.6.** Suppose that \( \alpha \prec \tau|X \). One side of \( \alpha \) is combed if and only if both sides are combed in the same direction if and only if some isotopy representative of \( \alpha \) is dual to \( \tau|X \). \( \Box \)

### 4. Finding efficient position

After discussing the various sources of non-uniqueness we prove in Theorem 4.1 that efficient position exists.

Let \( N = N(\tau) \); suppose that \( \alpha \vdash \tau \). A rectangle \( T \subset S \setminus (N \cup \alpha) \) is *vertical* if \( \partial T \) has a pair of opposite sides meeting \( \alpha \) and \( \partial_v N \) respectively. Define *horizontal* rectangles similarly. Figure 7 depicts the two kinds of *rectangle swap*.

**Figure 7.** Pieces of \( N \) are shown, with vertical and horizontal boundary in the correct orientation; the dotted lines are ties. The left and right pictures show a vertical and horizontal rectangle swap, respectively.

Now suppose that \( \alpha \prec \tau \), every rectangle \( R_b \subset N \) meets \( \alpha \) in at most a single arc, and one side of \( \alpha \) is combed. Let \( A \) be a small
regular neighborhood of $\alpha$. Then an annulus swap interchanges $\alpha$ and the component of $\partial A$ on the combed side. See Figure 8.

![Figure 8](image-url)

**Figure 8.** Both sides of $\alpha$ are combed (to the left). Thus both boundary components of the annulus shown are dual to $\tau$ and both differ from the carried core curve by an annulus swap.

**Theorem 4.1.** Suppose that $\xi(S) \geq 1$ and $\tau \subset S$ is a birecurrent train track. Suppose that $\Delta \subset AC(S)$ is a multicurve. Then efficient position for $\Delta$ with respect to $\tau$ exists and is unique up to rectangle swaps, annulus swaps, and isotopies of $S$ preserving the foliation of $N(\tau)$ by ties.

**Remark 4.2.** When $S = S_1$ is a torus, Lemma 14 of [7] proves the existence of efficient position for curves with respect to Reebless bigon tracks. Uniqueness of efficient position follows from a slight generalization of Section 4.1 using bigon swaps.

4.1. **Uniqueness of efficient position.** Suppose that $\alpha$ and $\beta$ are isotopic curves and in efficient position with respect to $\tau$. We induct on $i(\alpha, \beta)$. For the base case suppose that $|\alpha \cap \beta| = 0$. Then $\alpha$ and $\beta$ cobound an annulus $A \subset S$ so that $\partial A$ has no corners [3, Lemma 2.4].

Since $N = N(\tau)$ is a union of rectangles the intersection $A \cap N$ is also a union of rectangles. Thus $\text{index}(N \cap A) = 0$. By the hypothesis of efficient position any region $T \subset A \setminus N$ has non-positive index and has all corners outwards. By the additivity of index it follows that $\text{index}(T) = 0$. It follows that each region $T$ is either an annulus without corners or a rectangle.

Suppose that some region $T$ is an annulus without corners. Then we must have $T = A$. For if $\partial T$ meets $\partial N$ then $\partial N$ has a component without corners, contrary to assumption. Since $T = A$ it follows that $\alpha$ and $\beta$ are isotopic in the complement of $N$ and we are done.
So we may assume that all regions of $A \setminus N$ are rectangles. (In particular, $A \cap N \neq \emptyset$.) Note that if a region $R$ is a horizontal rectangle then there is no obstruction to doing a rectangle swap across $R$. After doing all such swaps we may assume that $A \setminus N$ contains no horizontal rectangles.

We now abuse terminology slightly by assuming that the position of $N$ determines that of $\tau$. So if $A$ contains vertical rectangles then there are switches of $\tau$ contained in $A$. This implies that $A$ contains half-branches of $\tau$. Let $b'$ be a half-branch in $A$, meeting $\partial A$. If $b'$ is large then there is a vertical rectangle swap removing three half-branches from $A$. After doing all such swaps we may assume that any such $b'$ is small. If $R$ is a vertical rectangle meeting $\partial A$ then $R$ has two horizontal sides. If neither of these meets a switch on its interior then there is a swap removing three half-branches from $A$.

After doing all such swaps if there are still vertical rectangles in $A$ then we proceed as follows: every vertical rectangle must have a horizontal side that properly contains the horizontal side of another vertical rectangle. (For example, in Figure 8 number the rectangles above the core curve $R_0, R_1, R_2$ from left to right. Note that the left horizontal side of $R_i$ strictly contains the right horizontal side of $R_{i-1}$.) It follows that the union of these vertical rectangles gives an annulus swap which we perform. Thus, we are reduced to the situation where $A$ contains no horizontal or vertical rectangles.

If $A \subset N$ then $\alpha$ and $\beta$ are both carried. For any tie $t \subset N$, any component $t' \subset t \cap A$ is an essential arc in $A$. (To see this, suppose that $t'$ is inessential. Let $B \subset A$ be the bigon cobounded by $t'$ and $\alpha' \subset \alpha$, say. Since $\alpha$ is carried, $\alpha'$ is transverse to the ties. We define a continuous involution on $\alpha'$; for every tie $s$ and for every component $s' \subset s \cap B$ transpose the endpoints of $s'$. As this involution is fixed point free, we have reached a contradiction.) It follows that $A$ is foliated by subarcs of ties and we are done.

There is one remaining possibility in the base case of our induction: $A \cap N \neq \emptyset$, $A \not\subset N$, and $A$ contains no switches of $\tau$. Thus every region of $A \cap N$ and of $A \setminus N$ is a rectangle meeting both $\alpha$ and $\beta$. Any region $R$ of $A \cap N$ is foliated by (subarcs of) ties and, as above, all ties meet $R$ essentially. Thus $R$ gives a parallelism between (carried arcs) ties of $\alpha$ and $\beta$. It follows that $A$ gives an isotopy between $\alpha$ and $\beta$, sending ties to ties. This completes the proof of uniqueness when $|\alpha \cap \beta| = 0$.

For the induction step assume $|\alpha \cap \beta| > 0$. Since $\alpha$ is isotopic to $\beta$ the Bigon Criterion [3, Lemma 2.5], [4, Proposition 1.3] implies that there is a disk $B \subset S$ with exactly two outward corners $x$ and $y$ so that
Suppose that $x$ is a dual intersection: an intersection of a tie of $\alpha$ and a carried arc of $\beta$. See Figure 9.

**Figure 9.** The left corner is a dual intersection, between a tie of $\alpha$ and a carried arc of $\beta$. If the half-route $\rho$ exits through $\alpha$ or $\beta$ then a bigon or non-trivial trigon is created.

Let $\alpha' = \alpha \cap B$ and $\beta' = \beta \cap B$. Orient $\beta'$ away from $x$. Let $z \in \alpha'$ be immediately adjacent to $x$. Without loss of generality we may assume that $z$ is to the left of $\beta'$, near $x$. Let $\rho$ be the half-route starting at $z$, initially agreeing with $\beta$, and turning left at every switch. If $\rho \subset B$ then eventually $\rho$ repeats a branch $b$ in the same direction; it follows that there is a curve $\gamma \prec \tau$ contained in $B$ contradicting Lemma 2.5. However, if $\rho$ exits $B$ through $\alpha'$ ($\beta'$) then we contradict efficient position of $\alpha$ ($\beta$).

It follows that the corner $x$ either lies in $S \setminus N$ or is the intersection of carried arcs of $\alpha$ and $\beta$. The same holds for $y$. In either case we cut off of $B$ a small neighborhood of $x$ and of $y$: when the corner lies in $N$ we use a subarc of a tie to do the cutting. The result $B'$ is a rectangle with the components of $\partial_h B$ contained in $\alpha$ and $\beta$ respectively. As $\text{index}(B') = 0$ the argument given in the case of an annulus gives a sequence of rectangle swaps moving $\alpha$ across $B$. This reduces $|\alpha \cap \beta|$ by two and so completes the induction step.

The proof when $\alpha$ and $\beta$ are arcs follows the above but omitting any mention of annulus swaps.

Finally, suppose that $\Delta, \Gamma$ are isotopic multicurves, both in efficient position. We may isotope $\Gamma$ to $\Delta$, as above, being careful to always use innermost bigons. This completes the proof that efficient position is unique.

We end this subsection with a useful corollary:
Corollary 4.3. Suppose that $\Gamma \subset \mathcal{AC}(S)$ is a finite collection of arcs and curves in efficient position. Then we may perform a sequence of rectangle swaps to realize the pairwise geometric intersection numbers.

**Proof.** Let $\Gamma = \{\gamma_i\}_{i=1}^k$. By induction, the curves of $\Gamma' = \Gamma \setminus \{\gamma_k\}$ realize their pairwise geometric intersection numbers. If $\gamma_k$ meets some $\gamma_i \in \Gamma'$ non-minimally then by the Bigon Criterion [5, page 46] there is an innermost bigon between $\gamma_k$ and some $\gamma_j \in \Gamma'$. We now may reduce the intersection number following the proof of uniqueness of efficient position. □

4.2. Existence of efficient position. Our hypotheses are weaker, and thus our discussion is more detailed, but the heart of the matter is inspired by [11, pages 122-123].

We may assume that $\tau$ fills $S$; for if not we replace $S$ by the subsurface filled by $\tau$. Since $\tau$ is transversely recurrent for any $\epsilon, L > 0$ there is a finite area hyperbolic metric on the interior of $S$ and an isotopy of $\tau$ so that: every branch of $\tau$ has length at least $L$ and every train-route $\rho \prec \tau$ has geodesic curvature less than $\epsilon$ at every point [15, Theorem 1.4.3].

Let $\tau^H$ be the lift of $\tau$ to $\mathbb{H} = \mathbb{H}^2$, the universal cover of $S$. Every train-route $\rho \prec \tau^H$ cuts $\mathbb{H}$ into a pair $H^\pm(\rho)$ of open half-planes. Fix a route $\rho \prec \tau^H$ and a half-branch $b' \subset \tau^H$ so that there is some $n \in \mathbb{Z}$ with $b' \cap \rho = \rho(n)$. We say the branch $b$ is rising or falling with respect to $\rho$ as the large half-branch at the switch $\rho(n)$ is contained in $\rho|[-\infty, n]$ or contained in $\rho|[n, \infty)$.  

**Claim 4.4.** For any route $\rho \prec \tau^H$ one side of $\rho$ has infinitely many rising branches while the other side has infinitely many falling branches.

**Proof.** Note that there are infinitely many half-branches on both sides of $\rho$: if not then $\partial_h N(\tau)$ would have a component without corners, contrary to assumption. Suppose that there are only finitely many rising branches along $\rho$. Then there is a curve $\gamma \prec \tau$ so that $w_\gamma(b) \leq 1$ for every branch $b$ and so that the two sides of $\gamma$ are combed in opposite directions. Thus $\tau$ is not recurrent, a contradiction. The same contradiction is obtained if there are only finitely many falling branches along $\rho$. □

**Claim 4.5.** For any route $\rho \prec \tau^H$ and for any family of half-routes $\{\beta_n\}$ if $\beta_n \cap \rho = \rho(n)$ then $\lim_{n \to \infty} \beta_n(\infty) = \rho(\infty)$.

**Proof.** Let $x = \rho(\infty) \in \partial_\infty \mathbb{H}$. Consider the subsequence $\{\beta_n\}$ where the first branch of each $\beta_n$ is falling. Let $P_n = \rho|[-n, n] \cup \beta_n$, oriented away from $\rho(-\infty)$. Note that $P_n(\infty) = \beta_n(\infty)$. Recall that $\rho$ and $P_n$
are both uniformly close to geodesics [15, pages 61–62]. Thus \( P_n(\infty) \rightarrow x \) as \( n \rightarrow \infty \).

Now consider the subsequence \( \{ \beta_n \} \) where the first branch of each \( \beta_n \) is rising. Let \( P_n = \beta_n \cup \rho \mid [n, \infty) \) oriented towards \( x \); so \( P_n(\infty) = x \) for all \( n \). Note that \( P(-\infty) = \beta_n(\infty) \). Since all complementary regions of \( \tau \) have negative index none of the \( P_n \) may cross each other. It follows that either the \( P_n \) exit compact subsets of \( H \), and we are done, or the \( P_n \) converge [15, Theorem 1.5.4] to \( P \), a train-route with \( P(\infty) = x \). Since \( P \) does not cross any \( P_n \) deduce that \( P \) and \( \rho \) are disjoint. But this contradicts [14, Corollary 3.3.4]: train-routes that share an endpoint must share a half-route. \( \square \)

Given distinct points \( x, y, z \in S^1 = \partial_\infty \mathbb{H} \), arranged counterclockwise, let \( (y, z) \) be the component of \( S^1 \setminus \{y, z\} \) that does not contain \( x \). Let \( [y, z] \) be the closure of \( (y, z) \). Thus \( x \in (z, y) \), \( (y, z) \cap [z, y] = \emptyset \), and \( (y, z) \cup [z, y] = S^1 \).

**Claim 4.6.** For any distinct \( z, y \in S^1 \) there is a train-route \( \rho \) so that one of the intervals \( \partial_\infty H^\pm(\rho) \) is contained in \((z, y)\).

**Proof.** The endpoints of train-routes are dense in \( S^1 = \partial_\infty \mathbb{H} \). Fix \( x \in (z, y) \) so that \( x \) is the endpoint of a train-route \( \gamma \). Since there are infinitely many rising branches along \( \gamma \) (Claim 4.4) the claim follows from the rising case of Claim 4.5. \( \square \)

Let \( H_{x,y} \subset \mathbb{H} \) be the convex hull of \( (x, y) \in \partial_\infty \mathbb{H} \). Let \( \mathcal{H}_{x,y} \) be the union of all open half-planes \( H(\rho) \) so that \( \partial_\infty H(\rho) \subset (x, y) \). Since train-routes have geodesic curvature less than \( \epsilon \) at every point:

**Claim 4.7.** The union \( \mathcal{H}_{x,y} \) is contained in an \( \delta \)-neighborhood of \( H_{x,y} \), where \( \delta \) may be taken as small as desired by choosing appropriate \( \epsilon, L \).

A set \( X \subset \mathbb{H} \) is \( \epsilon' \)-convex if every pair of points in \( X \) can be connected by a path in \( X \) which has geodesic curvature less than \( \epsilon' \) at every point.

**Claim 4.8.** \( \mathbb{H} \setminus \mathcal{H}_{x,y} \) is closed and \( \epsilon' \)-convex, where \( \epsilon' \) may be taken as small as desired by choosing appropriate \( \epsilon, L \).

**Proof.** This is proved in detail on pages 122-123 of [11]. \( \square \)

**Claim 4.9.** The point \( x \) is an accumulation point of \( \partial(\mathbb{H} \setminus \mathcal{H}_{x,y}) \).

**Proof.** Pick a sequence of subintervals \( (x_n, y_n) \subset (x, y) \) so that \( x_n, y_n \rightarrow x \) as \( n \rightarrow \infty \). By Claim 4.6 for every \( n \) there is a route \( \rho_n \) and a half-plane \( H_n = H(\rho_n) \) so that \( \partial_\infty H_n \subset (x_n, y_n) \). It follows that \( H_n \subset \mathcal{H}_{x,y} \). Let \( r_n \) be any bi-infinite geodesic perpendicular to \( \partial H_{x,y} \) and
meeting $H_n$. Thus $r_n \to x$ as $n \to \infty$. By Claim 4.7 the intersection $r_n \cap \partial(\mathbb{H} \setminus \mathcal{H}_{z,x})$ is nonempty, and we are done. \hfill \Box

The next lemma is not needed for the proof of Theorem 4.1: we state it and give the proof in order to introduce necessary techniques and terminology.

**Lemma 4.10.** For any non-parabolic point $x \subset S^1$ there is a sequence of train-routes $\{\rho_n\}$ with associated half-planes $\{H(\rho_n)\}$ forming a neighborhood basis for $x$.

**Proof.** Let $y, z$ be arbitrary points of $S^1$ so that $x, y, z$ are ordered counterclockwise. It suffices to construct a train-route separating $x$ from $(y, z)$.

First assume that $x$ is the endpoint of a route $\rho$. Claim 4.4 implies that there are infinitely many rising branches $\{a_m\}$ on one side of $\rho$ and infinitely many falling branches $\{c_n\}$ on the other side. Run half-routes $\alpha_m$ and $\gamma_n$ through $a_m$ and $c_n$; so each half-route meets $\rho$ in a single switch. By Claim 4.5 the endpoints converge $\alpha_m(\infty), \gamma_n(\infty) \to x$. Thus sufficiently large $m, n$ give a train-route

$$\alpha_m \cup \rho|_{[m,n]} \cup \gamma_n$$

that separates $x$ from $(y, z)$, as desired.

For the general case consider

$$K = \mathbb{H} \setminus (\mathcal{H}_{z,x} \cup \mathcal{H}_{x,y})$$

Note that $x$ is an accumulation point of $K$ (by Claim 4.9 and because $\mathcal{H}_{z,x}$ cannot contain points of $\partial(\mathbb{H} \setminus \mathcal{H}_{x,y})$). Fix any basepoint $w \in K$. By Claim 4.8 $K$ is $\epsilon'$-convex. Thus there is a path $r \subset K$ from $w$ to $x$ which has geodesic curvature less than $\epsilon'$ at every point. Since $x$ is not a parabolic point the projection of $r$ to $S$ recurs to the thick part of $S$; thus $r$ meets infinitely many branches $\{b_n\}$ of $\tau^\mathbb{H}$.

Suppose that $b$ is a branch of $\tau^\mathbb{H}$ lying in $K$. If the two sides of $b$ meet $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$ then $b$ is a bridge of $K$. If the sides of $b$ meet neither $\mathcal{H}_{z,x}$ nor $\mathcal{H}_{x,y}$ then $b$ is an interior branch of $K$. If exactly one side of $b$ lies in $K$ then $b$ is a boundary branch. If both sides lie in $\mathcal{H}_{z,x}$ (or both sides lie in $\mathcal{H}_{x,y}$) then $b$ is an exterior branch. See Figure 10 below.

By convexity the path $r$ is disjoint from the exterior branches. After a small isotopy the path $r$ is also disjoint from the boundary branches, meets the interior branches transversely, and still has geodesic curvature less than $\epsilon'$ at every point.

Now, if $r$ travels along a bridge then there are routes $\rho^\pm$ cutting off half-planes $H^\pm$ lying in $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$ respectively. Then either $x$ is
the endpoint of a train-route or a cut and paste of $\rho^\pm$ gives the desired route $\Gamma$ separating $x$ from $(y, z)$. In either case we are done.

So suppose that $r$ only meets interior branches $\{b_n\}$ of $K$. Let $\gamma_n$ be any train-route travelling along $b_n$. If any of the $\gamma_n$ land at $x$ we are done, as above. Supposing not: Fixing orientations and passing to a subsequence we may assume that $\gamma_n(\infty) \to x$ as $n \to \infty$. There are now two cases: suppose that for infinitely many $n$ we find that $\gamma_n(-\infty) \in [y, z]$. Then passing to a further subsequence we have that $\gamma_n \to \Gamma$ where $\Gamma(\infty) = x$ [15, Theorem 1.5.4]; thus $x$ is the endpoint of a train-route and we are done as above. The other possibility is that for some sufficiently large $n$ both endpoints $\gamma_n(\pm \infty)$ lie in $(z, y)$. Since $b_n$ is an interior branch $\gamma_n$ separates $x$ from $(y, z)$ and Lemma 4.10 is proved.

4.3. Finding invariant efficient position. Fix $\alpha \in C(S)$. (The case where $\alpha \in A(S)$ is dealt with at the end.) Let $\alpha'$ be a component of the lift of $\alpha$ to the universal cover $\mathbb{H}$. Let $\pi_1(\alpha)$ be the cyclic subgroup (of the deck group) preserving $\alpha'$. Let $\{x, y\} = \partial_{\infty} \alpha' \subset S^1$. We take

$$K = \mathbb{H} \setminus (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y}).$$

By construction $K$ is $\pi_1(\alpha)$–invariant. By Claims 4.8 and 4.9 the set $K$ is closed, $\epsilon'$–convex, and has $\{x, y\} \subset \partial_{\infty} K$. By Lemma 4.10 the only non-parabolic points of $\partial_{\infty} K$ are $x$ and $y$. As in the proof of Lemma 4.10 we find a bi-infinite path $r \subset K$ connecting $y$ to $x$, with geodesic curvature less than $\epsilon'$ at every point. See Figure 10.

Let $H(r)$ be the open half-plane to the right of $r$. If we remove the union

$$\bigcup_{g \in \pi_1(\alpha)} g \cdot H(r)$$

from $\mathbb{H}$ then, as with Claim 4.8, what remains is closed and $\epsilon''$–convex for some small $\epsilon''$. It follows that we may homotope the path $r$ to become a $\pi_1(\alpha)$–invariant smooth path, contained in $K$ and transverse to the interior branches, and avoiding the exterior branches of $K$. A further equivariant isotopy ensures that $r$ also avoids the boundary branches of $K$. Orient $r$ from $y$ to $x$.

Remark 4.11. Suppose that $\gamma \prec \tau^\mathbb{H}$ is a train-route that separates $x$ from $y$. Note that if there exists a non-identity element $g \in \pi_1(\alpha)$ so that $\gamma$ and $g \cdot \gamma$ meet then $r$ is carried by $\tau^\mathbb{H}$, thus $\alpha \prec \tau$, and we are done. We will henceforth assume that train-routes separating $x$ from $y$ are disjoint from their non-trivial translates.
Figure 10. The path $r$ runs from $y$ to $x$. To simplify the figure, no interior branches are shown.

Let $b$ be any interior branch of $\mathcal{K}$ and let $\gamma$ be any train-route travelling along $b$. Since $b$ is interior, $\gamma$ must separate $x$ from $y$. Orient $\gamma$ from $\mathcal{H}_{y,x}$ to $\mathcal{H}_{x,y}$. (Thus if $\gamma$ and $r$ meet once then the tangent vectors to $r$ and $\gamma$, in that order, form a positive frame.) The orientation of $\gamma$ gives an orientation to $b$. Moreover, as $b$ is an interior branch a cut and paste argument shows that the orientation on $b$ is independent of our choice of $\gamma$. Orient all interior branches in this fashion and note that these orientations agree at interior switches.

We say that $p \in r \cap b$ has positive or negative sign as the tangent vectors to $r$ and $b$ (in that order) form a positive or negative frame. Suppose that there are $N \in \mathbb{N}$ orbits of points of negative sign, under the action of $\pi_1(\alpha)$. We now induct on $N$.

Suppose that $N$ is zero. Any bigon between $r$ and a train-route is contained in $\mathcal{K}$ and so contributes one point of positive and one point of negative sign. So if there are no points of negative sign then there are no bigons and $r$ is in efficient position with respect to $\tau^\mathbb{R}$. Recall that $r$ is $\pi_1(\alpha)$–invariant. So $\beta \subset S$, the image of $r$ under the universal covering map, is an immersed curve in $S$ homotopic to $\alpha$. If $\beta$ is embedded then we are done. If not then the Bigon Criterion for immersed curves [18] implies that $\beta$ must have either a monogon or a bigon of self-intersection. If $\beta$ has a monogon $B$ of self-intersection then, since index is additive, $\tau$ must be disjoint from $B$. Thus we can
homotope $\beta$ to remove $B$ while fixing $\tau$ pointwise. If $\beta$ has a bigon $B$ of self-intersection then, as in the proof of uniqueness in Section 4.1, we may remove $B$ via a sequence of rectangle swaps. After removing all monogons and bigons of self-intersection the curve $\beta$ is embedded and in efficient position.

![Figure 11](image)

Figure 11. Left: The lowest point shown has negative sign. The paths $r$ and $\gamma_L$ form a bigon. Right: The corresponding figure for $\gamma_R$.

Suppose that $N$ is positive. Let $b$ be a branch with a point $p \in r \cap b$ of negative sign. Let $\gamma_R$ ($\gamma_L$) be the half-route starting at $p$, travelling in the direction of $b$, and thereafter turning only right (left). Each of $\gamma_R$ and $\gamma_L$ must have at least one bigon with $r$, as their points at infinity lie in $(x, y)$. There are now two (essentially identical) cases:

- There is a bigon $B$ between $r$ and $\gamma_R$, to the right of $r$, so that the corners of $B$ appear in the same order along $r$ and $\gamma_R$.
- There is a bigon $B$ between $r$ and $\gamma_L$, to the right of $r$, so that the corners of $B$ appear in opposite order along $r$ and $\gamma_L$.

See Figure 11. If neither case holds then any half-route ending at $p$ must originate in $(x, y)$, contradicting the fact that $p$ has negative sign. Note that, by Remark 4.11, $\gamma_R$ ($\gamma_L$) is disjoint from its non-trivial $\pi_1(\alpha)$ translates. It follows that the bigon $B$ is also disjoint from its non-trivial translates. Finally, we may equivariantly isotope $r$ across $\pi_1(\alpha) \cdot B$. Since the arc of $\gamma_R \cap \partial B$ (respectively $\gamma_L \cap \partial B$) is combed outside of $B$ this isotopy reduces $N$ by at least one. (Again, see Figure 11.) This completes the proof of Theorem 4.1 when $\alpha$ is a curve.

4.4. Efficient position for arcs and multicurves. Now suppose that $\alpha \subset S$ is an essential arc. Let $\alpha'$ be a lift of $\alpha$ to $\mathbb{H}$, the universal cover of $S$. Note that $\{x, y\} = \partial_\infty \alpha'$ is a pair of parabolic points.
Construct $\mathcal{K} = \mathbb{H} \setminus (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y})$ as before. The proof now proceeds as above, omitting any mention of $\pi_1(\alpha)$, equivariance, or annulus swaps.

Finally suppose that $\Delta$ is a multicurve. As shown in Section 4.2 we may isotope, individually, every $\alpha \in \Delta$ into efficient position. By Corollary 4.3 all $\alpha \in \Delta$ may be realized disjointly in efficient position. This completes the proof of Theorem 4.1. \hfill \Box

5. The structure theorem

5.1. Bounding diameter. We now bound the diameters of the sets of wide arcs and curves carried by the induced track.

**Lemma 5.1.** Suppose that $\tau$ is a birecurrent track and $X \subset S$ is an essential annulus. If $\tau|X \neq \emptyset$ then the diameter of $V(\tau|X) \cup V^*(\tau|X)$ inside of $\mathcal{A}(X)$ is at most eight.

**Proof.** In the proof we use $V, V^*$ to represent $V(\tau|X)$ and $V^*(\tau|X)$. Since $\tau|X \neq \emptyset$ it follows that $\mathcal{A}(\tau|X)$ is nonempty. The first conclusion of Lemma 3.4 now implies that $V^*$ is nonempty.

**Claim.** $V^* \neq \emptyset$.

**Proof.** By Lemma 2.6 there is a dual curve $\beta \in \mathcal{C}^*(\tau)$ so that $i(\alpha, \beta) > 0$. Thus there is a lift $\beta' \subset S^X$ with closure an essential arc. Since $\tau|X \subset \tau^X$ it follows that $\beta' \in \mathcal{A}^*(\tau|X)$. The first conclusion of Lemma 3.4 now implies that $V^*$ is nonempty. \hfill \Box

**Claim.** If $\beta \in V$ and $\gamma \in V^*$ then $i(\beta, \gamma) \leq 3$.

**Proof.** Suppose that $i(\beta, \gamma) = n \geq 4$. Let $\{\gamma_i\}_{i=1}^{n-1}$ be the components of $\gamma \setminus \beta$ with both endpoints on $\beta$. Let $R_i$ be the components of $S^X \setminus (\beta \cup \gamma)$ with compact closure. We arrange matters so that opposite sides of $R_i$ are on $\gamma_i$ and $\gamma_{i+1}$. Let $R$ be the union of the $R_i$. Since index$(R) = 0$ every region $T$ of the closure of $R \setminus N(\tau|X)$ also has index zero and so is a rectangle. If $T$ meets both $\gamma_i$ and $\gamma_{i+1}$ then $\gamma$ was not wide, a contradiction. As $n - 1 \geq 3$ any region $T$ meeting $\gamma_2$ is a compact rectangle component of the closure of $S^X \setminus N(\tau|X)$. An index argument implies that $\tau^X$ and thus $\tau \subset S$ has a complementary region with non-negative index, a contradiction. \hfill \Box

Since $V, V^*$ are nonempty it follows that diam$(V \cup V^*) \leq 8$. \hfill \Box

Now suppose that $X$ is not an annulus. Prompted by Lemma 2.8 we define

$$W(\tau^X) = \{\alpha \in \mathcal{A}(\tau^X) \mid \alpha \text{ is wide}\}.$$ 

Define $W^*(\tau^X)$ similarly, replacing $\mathcal{A}(\tau^X)$ by $\mathcal{A}^*(\tau^X)$. 
Lemma 5.2. There is a constant $K_1 = K_1(S)$ with the following property. Suppose that $\tau$ is a track and $X \subset S$ is an essential subsurface (not an annulus) with $\pi_X(\tau) \neq \emptyset$. Then the diameter of $W(\tau^X) \cup W^*(\tau^X)$ inside of $\mathcal{AC}(X)$ is at most $K_1$. Furthermore if, after isotoping $\partial X$ into efficient position, the induced orientation on $\partial X$ is not wide then either $C(\tau|X)$ or $C^*(\tau|X)$ has diameter at most two in $C(X)$.

Proof. In the proof we use $W, W^*, AC, AC^*$ to represent $W(\tau^X)$ and so on. Since $\pi_X(\tau) \neq \emptyset$ there is some vertex cycle $\alpha \in V(\tau)$ so that $\alpha$ cuts $X$. Since $\alpha$ is wide (Lemma 2.8) there is a lift $\alpha' \prec \tau^X$ which is also wide; deduce that $W$ is nonempty.

Claim. $W^* \neq \emptyset$.

Proof. By Lemma 2.6 there is a dual curve $\alpha \in C^*(\tau)$ cutting $X$. By Lemma 2.9 there is a wide dual $\beta$ that also cuts $X$. Thus there is a lift $\beta' \subset S^X$ with closure an essential wide arc or wide essential, non-peripheral curve. So $\beta' \in W^*$ as desired. □

Now isotope $\partial X$ into efficient position. Let $X'$ be the compact component of the preimage of $X$ under the covering map $S^X \rightarrow S$. Note that $\partial X'$ is in efficient position with respect to $\tau^X$. Note that the covering map $S^X \rightarrow S$ induces a homeomorphism between $X'$ and $X$. Let $N^X = N(\tau^X) \subset S^X$ be the preimage of $N = N(\tau)$. Let $N' = X' \cap N^X$. Again, the covering map induces a homeomorphism between $N'$ and $N \cap X$.

Suppose that $\partial X$, with its induced orientation, is not wide. If $\partial X$ fails to be wide in $S^X \smallsetminus N(\tau)$ then there is a properly embedded, essential arc $\gamma \subset X$ disjoint from $N(\tau)$. Lift $\gamma$ to $\gamma' \subset X'$. Adjoin to $\gamma'$ geodesic rays in $S^X \smallsetminus X'$ to obtain an essential, properly embedded arc $\gamma'' \subset S^X$. Note that $i(\gamma'', \alpha) = 0$ for every $\alpha \in AC$; only intersections in $X'$ contribute to geometric intersection number as computed in $S^X$. This implies that $\text{diam}_X(C(\tau|X)) \leq 2$. Furthermore, $i(\gamma'', \beta) \leq 2$ for every $\beta \in W^*$. This gives the desired diameter bound for $W \cup W^*$.

If, instead, $\partial X$ fails to be wide in $N(\tau)$ then there is a properly embedded, essential arc $\gamma \subset X$ that is a subarc of a tie. Again, lift and extend to an essential arc $\gamma'' \subset S^X$ so that $i(\gamma'', \beta) = 0$ for any $\beta \in AC^*$. This implies that $\text{diam}_X(C^*(\tau|X)) \leq 2$. We also have $i(\gamma'', \alpha) \leq 2$ for any $\alpha \in W$. Again the diameter is bounded.

Now suppose that $\partial X$ is wide. Thus, for every $b \in B(\tau)$, the rectangle $R_b$ meets $\partial X$ in at most a pair of arcs. It follows that $N \cap X$, and thus $N'$, is a union of at most $2|B(\tau)|$ subrectangles of the form $R'$, $R'' \subset R_b$. Suppose that $\alpha \in W$ and $\beta \in W^*$. Then $\alpha$ and $\beta$ each meet a subrectangle $R'$ in at most two arcs. Thus $\alpha$ and $\beta$ intersect in at
most four points inside of $R'$. Thus $i(\alpha, \beta) \leq 8|\mathcal{B}(\tau)|$ and Lemma 5.2 is proved.

5.2. Accessible intervals. Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks. Suppose $X \subset S$ is an essential subsurface, yet not an annulus, with $\xi(X) < \xi(S)$. Define

$$m_X = \min \{ i \in [0, N] \mid \text{diam}_X(C^*(\tau_i|X)) \geq 3 \}$$

and

$$n_X = \max \{ i \in [0, N] \mid \text{diam}_X(C(\tau_i|X)) \geq 3 \}.$$ 

If either $m_X$ or $n_X$ is undefined or if $n_X < m_X$ then $I_X$, the accessible interval is empty. Otherwise, $I_X = [m_X, n_X]$.

If $X \subset S$ is an annulus, then $I_X$ is defined by replacing $C$ by $A$ and increasing the lower bound on diameter from 3 to 9. We may now state the structure theorem:

**Theorem 5.3.** For any surface $S$ with $\xi(S) \geq 1$ there is a constant $K_0 = K_0(S)$ with the following property: Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks in $S$ and suppose that $X \subset S$ is an essential subsurface.

- For every $[a, b] \subset [0, N]$ if $[a, b] \cap I_X = \emptyset$ and $\pi_X(\tau_b) \neq \emptyset$ then $d_X(\tau_a, \tau_b) \leq K_0$.

Suppose $i \in I_X$. If $X$ is an annulus:

- The core curve $\alpha$ is carried by and wide in $\tau_i$.
- Both sides of $\alpha$ are combed in the induced track $\tau_i|X$.
- If $i + 1 \in I_X$ then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.

If $X$ is not an annulus:

- When in efficient position $\partial X$ is wide with respect to $\tau_i$.
- The track $\tau_i|X$ is birecurrent and fills $X$.
- If $i + 1 \in I_X$ then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.

**Proof.** Fix an interval $[a, b] \subset [0, N]$. Note that $\tau_b \prec \tau_a$ and so $\tau_b^X \prec \tau_a^X$. Thus $\mathcal{AC}(\tau_b^X) \subset \mathcal{AC}(\tau_a^X)$ while $\mathcal{AC}^*(\tau_b^X) \subset \mathcal{AC}^*(\tau_a^X)$.

**Claim.** If $[a, b] \cap I_X = \emptyset$ and $\pi_X(\tau_b) \neq \emptyset$ then $d_X(\tau_a, \tau_b) \leq K_0$.

**Proof.** Fix, for the duration of the claim, a vertex cycle $\beta \in V(\tau_b)$ so that $\beta$ cuts $X$. Since $\beta$ is also carried by $\tau_a$ there is, by Lemma 2.9, a vertex cycle $\alpha \in V(\tau_a)$ cutting $X$. Pick $\alpha' \in \pi_X(\alpha)$ and $\beta' \in \pi_X(\beta)$. Note Lemma 2.8 implies that $\alpha'$ is wide in $\tau_a^X$ while $\beta'$ is wide in $\tau_b^X$. The proof divides into cases depending on the relative positions of $a, b, m_X$ and $n_X$. 


Case I. Suppose \( n_X < a \) or \( n_X \) is undefined.

Note that \( \beta' \prec \tau^X_a \). If \( X \) is an annulus then since \( a \notin I_X \) the diameter of \( \mathcal{A}(\tau_a|X) \) is at most eight; thus \( d_X(\alpha, \beta) \leq 8 \) and we are done.

Suppose that \( X \) is not an annulus. If \( \beta' \) is an arc then Lemma 3.3 gives two cases: we may replace \( \beta' \) by \( \gamma \) which is either a wide arc in \( \tau^X_a \) or is an essential non-peripheral curve in \( \tau_a|X \). (If \( \beta' \) is a curve then let \( \gamma = \beta' \).) In either case Lemma 3.3 ensures that \( i(\gamma, \beta') \leq 2 \) and so \( d_X(\gamma, \beta') \leq 4 \). If \( \gamma \) is an arc then both \( \alpha' \) and \( \gamma \) are wide so Lemma 5.2 gives \( d_X(\alpha, \beta) \leq K_1 + 4 \). If \( \gamma \) is a curve pick any \( \delta \in V(\tau_a|X) \). Then Lemma 5.2 implies that \( d_X(\alpha', \delta) \leq K_1 \). Also, \( a \notin I_X \) implies that \( d_X(\delta, \gamma) \leq 2 \). Thus \( d_X(\alpha, \beta) \leq K_1 + 6 \).

Case II. Suppose \( b < m_X \) or \( m_X \) is undefined.

If \( X \) is an annulus, then Lemma 5.1 gives wide duals \( \alpha^* \in V^*(\tau_a|X) \) and \( \beta^* \in V^*(\tau_b|X) \) so that \( d_X(\alpha', \alpha^*), d_X(\beta', \beta^*) \leq 8 \). It follows that the arc \( \alpha^* \) also lies in \( \mathcal{A}^*(\tau_v|X) \). Since \( b \notin I_X \) we have \( d_X(\alpha^*, \beta^*) \leq 8 \). Thus \( d_X(\alpha, \beta) \leq 24 \), as desired.

If \( X \) is not an annulus, then by Lemma 5.2 there is a wide dual \( \alpha^* \in W^*(\tau^X_a) \) so that \( d_X(\alpha', \alpha^*) \leq K_1 \). Again, \( \alpha^* \) is also an element of \( \mathcal{A}C^*(\tau_b^X) \) but may not be wide there. If \( \alpha^* \) is an arc then Lemma 3.3 gives two cases: we may replace \( \alpha^* \) by \( \gamma^* \) which is either a wide dual arc to \( \tau_b^X \) or is an essential non-peripheral dual curve to \( \tau_b^X \). (If \( \alpha^* \) is a curve then let \( \gamma^* = \alpha^* \).) So \( i(\alpha^*, \gamma^*) \leq 2 \) and thus \( d_X(\alpha^*, \gamma^*) \leq 4 \). If \( \gamma^* \) is a wide dual arc then Lemma 5.2 implies that \( d_X(\gamma^*, \beta') \leq K_1 \) and so \( d_X(\alpha, \beta) \leq 2K_1 + 4 \). If \( \gamma^* \) is a dual curve then, as \( b \notin I_X \), any dual wide curve \( \delta^* \in V^*(\tau_v|X) \) has \( d_X(\gamma^*, \delta^*) \leq 2 \). Again, Lemma 5.2 implies that \( d_X(\delta^*, \beta') \leq K_1 \) and so \( d_X(\alpha, \beta) \leq 2K_1 + 6 \).

Case III. Suppose \( a \leq n_X < c < m_X \leq b \).

The first two cases bound \( d_X(\tau_c, \tau_b) \) and \( d_X(\tau_a, \tau_c) \); thus we are done by the triangle inequality.

Case IV. Suppose \( a \leq n_X < m_X \leq b \) and \( m_X = n_X + 1 \).

Let \( c = n_X \) and \( d = m_X \). The first two cases bound \( d_X(\tau_d, \tau_b) \) and \( d_X(\tau_a, \tau_c) \). Since \( V(\tau_c) \) and \( V(\tau_d) \) have bounded intersection \( d_X(\tau_c, \tau_d) \) is also bounded and the claim is proved.

Now fix \( i \in I_X \).

Claim. If \( X \) is an annulus:

- The core curve \( \alpha \) is carried by and is wide in \( \tau_i \).
- Both sides of \( \alpha \) are combed in the induced track \( \tau_i|X \).
• If $i + 1 \in I_X$ then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.

**Proof.** Since $i \in I_X$, both $\mathcal{A}(\tau_i|X)$ and $\mathcal{A}^*(\tau_i|X)$ have diameter at least nine. From Lemma 5.1 deduce that the inclusions $V \subset \mathcal{A}$ and $V^* \subset \mathcal{A}^*$ are strict. Thus by Lemma 3.5 the core curve $\alpha$ is both carried by and dual to $\tau_i|X$. The second statement now follows from Lemma 3.6. Thus at least one side of $\alpha$ is combed in $\tau_i^X$. Projecting from $S^X$ to $S$ we find that $\alpha \prec \tau_i$. If $\alpha$ is not wide in $\tau_i$ then we deduce that neither side of $\alpha$ is combed in $\tau_i^X$, a contradiction.

Suppose that $\tau_i$ slides to $\tau_{i+1}$. Then, up to isotopy, $\tau_{i+1}$ slides to $\tau_i$. Since slides do not kill essential arcs it follows that $\tau_{i+1}|X$ is obtained from $\tau_i|X$ by an at most countable collection of slides.

Now suppose that $\tau_{i+1}$ is obtained by splitting $\tau_i$ along a large branch $b$. Thus $\tau_i^X$ is obtained from $\tau_i^X$ by splitting all of the countably many lifts of $b$. Every essential arc carried by $\tau_i+1|X$ is also carried by $\tau_i^X$. Let $\tau' \subset \tau_i^X$ be the union of these essential routes. It follows that $\tau_{i+1}|X$ is obtained from $\tau'$ by splitting along lifts of $b$ that are also large branches of $\tau'$. Since both sides of $\alpha$ are combed in $\tau_i|X$ the same is true in $\tau'$ and so any component of $\tau' \setminus \alpha$ is a tree without large branches. The track $\tau'$ therefore has only finitely many large branches, all contained in $\alpha$. Since $\alpha$ is wide in $\tau_i$ there are at most two preimages of the large branch $b$ contained in $\alpha \subset S^X$. Thus $\tau_{i+1}|X$ is obtained from $\tau'$ by at most two splittings. This proves the claim. (See Figure 12 for pictures of how $\alpha$ may be carried by $\tau_i$ and how splitting $b$ affects $\tau_i|X$.)

**Claim.** Suppose $X$ is not an annulus.

• When in efficient position $\partial X$ is wide with respect to $\tau_i$.

• The track $\tau_i|X$ is birecurrent and fills $X$.

• If $i + 1 \in I_X$ then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.

**Proof.** Since $i \in I_X$ the second conclusion of Lemma 5.2 implies that $\partial X$ is wide. The induced track $\tau_i|X$ carries a pair of curves at distance at least three, so fills $X$. Also, $\tau_i|X$ is recurrent by definition. For any branch $b' \in \tau_i|X \subset S^X$, let $b \subset \tau$ be the image in $S$. Since $\tau$ is transversely recurrent there is a dual curve $\beta$ meeting $b$. Lifting $\beta$ to a curve or arc $\beta' \subset S^X$ gives a dual to $\tau|X$ meeting $b'$. Thus $\tau|X$ is transversely recurrent with respect to arcs and curves, as defined in Section 3.1.

Now, if $\tau_i$ slides to $\tau_{i+1}$ then, as in the annulus case, $\tau_i|X$ slides to $\tau_{i+1}|X$. Suppose instead that $\tau_i$ splits to $\tau_{i+1}$ along the branch $b \in \mathcal{B}(\tau_i)$. Thus $\tau_i|X$ splits (or isotopes) to a track $\tau'$ so that $\tau_{i+1}|X$ is a subtrack.
Figure 12. Four of the possible ways for an oriented, carried, wide curve $\alpha$ to meet a large rectangle $R_b$ of $N(\tau)$. Note that when $X$ is an annulus and $\alpha$ is the core curve the upper left picture implies that neither side of $\alpha$ is combed in $\tau|X$. Splitting the upper right deletes zero or two components of $\tau_i|X \setminus \alpha$. In the bottom row, only the left splitting is possible when $i, i+1 \in I_X$. On the bottom left one component of $\tau_i|X \setminus \alpha$ is deleted and $\tau_i|X$ is split once. On the bottom right $\tau_i|X$ is split twice.

Let $R_b \subset N(\tau_i)$ be the rectangle corresponding to the branch $b$. Isotope $\partial X$ into efficient position with respect to $N(\tau_i)$ and recall that $X$ is to the left of $\partial X$. Note that, by an isotopy, we may arrange for all curves in $\mathcal{C}(\tau_i|X)$ to be disjoint from $\partial X$. Let $\beta \subset R_b$ be a central splitting arc: a carried arc completely contained in $R_b$.

If $\beta \cap X$ is empty then $\tau_i|X$ is identical to $\tau_{i+1}|X$. If $\beta \subset X$ then $\tau_{i+1}|X$ is either a subtrack or a splitting of $\tau_i|X$, depending how the carried curves of $\tau_i|X$ meet the lift of $R_b$. In all other cases $\tau_{i+1}|X$ is a subtrack of $\tau_i|X$. See Figure 12 for some of the ways carried subarcs of $\partial X$ may meet $R_b$. If $\partial X \cap R_b$ contains a tie then $\tau_i|X$ is identical to $\tau_{i+1}|X$. This completes the proof of the claim.

Thus Theorem 5.3 is proved.

We now rephrase a result of Masur and Minsky using the refinement procedure of Penner and Harer [15, page 122].
Theorem 5.4. [13, Theorem 1.3] For any surface \( S \) with \( \xi(S) \geq 1 \) there is a constant \( Q = Q(S) \) with the following property: For any sliding and splitting sequence \( \{ \tau_i \}_{i=0}^N \) of birecurrent train tracks in \( S \) the sequence \( \{ V(\tau_i) \}_{i=0}^N \) forms a \( Q \)-unparameterized quasi-geodesic in \( \mathcal{C}(S) \).

Theorem 5.3 implies that the same result holds after subsurface projection.

Theorem 5.5. For any surface \( S \) with \( \xi(S) \geq 1 \) there is a constant \( Q = Q(S) \) with the following property: For any sliding and splitting sequence \( \{ \tau_i \}_{i=0}^N \) of birecurrent train tracks in \( S \) and for any essential subsurface \( X \subset S \) if \( \pi_X(\tau_N) \neq \emptyset \) then the sequence \( \{ \pi_X(\tau_i) \}_{i=0}^N \) is a \( Q \)-unparameterized quasi-geodesic in \( \mathcal{C}(X) \).

Proof. By the first conclusion of Theorem 5.3 we may restrict attention to the subinterval \([p, q] = I_X \subset [0, N]\).

Fix any vertex \( \alpha \in V(\tau_q|X) \). Note \( \alpha \) is carried by \( \tau_i|X \) for all \( i \leq q \). So define \( \sigma_i \subset \tau_i|X \) to be the minimal pretrack carrying \( \alpha \). Since \( \sigma_i \) does not carry any peripheral curves \( \sigma_i \) is a train track. Note that \( \sigma_i \) is recurrent by definition and transversely recurrent by Lemma 3.2. Applying Theorem 5.3, for all \( i \in [p, q - 1] \) the track \( \sigma_{i+1} \) is a slide, a split, or identical to the track \( \sigma_i \).

Theorem 5.4 implies the sequence \( \{ V(\sigma_i) \} \) is a \( Q \)-unparameterized quasi-geodesic in \( \mathcal{C}(X) \). Note that \( d_X(\sigma_i, \tau_i|X) \) is uniformly bounded because \( \sigma_i \) is a subtrack.

Since \( \alpha \prec \tau_i|X \) the curve \( \alpha \) is also carried by \( \tau_i \). By Lemma 2.9 there is a vertex cycle \( \beta_i \prec \tau_i \) that cuts \( X \). Since \( \beta_i \) is wide (Lemma 2.8) any element \( \beta'_i \in \pi_X(\beta) \) is carried by and wide in \( \tau_i^X \). It follows that \( \beta'_i \) and the vertex cycles of \( \tau_i|X \) have bounded intersection. Thus \( d_X(\tau_i, \tau_i|X) \) is uniformly bounded and we are done. \( \square \)

6. Further applications of the structure theorem

We now turn to Theorems 6.1 and 6.2; both are slight generalizations of a result of Hamenstädt [8, Corollary 3]. Our proofs, however, rely on Theorem 5.3 and are quite different from the proof found in [8].

6.1. The marking and train track graphs. Suppose that \( S \) is not an annulus. A finite subset \( \mu \subset \mathcal{AC}(S) \) fills \( S \) if for all \( \beta \in \mathcal{C}(S) \) there is a \( \gamma \in \mu \) so that \( i(\beta, \gamma) \neq 0 \). If \( \mu, \nu \subset \mathcal{AC}(S) \) then we define

\[
i(\mu, \nu) = \sum_{\alpha \in \mu, \beta \in \nu} i(\alpha, \beta).
\]
Also, let \( i(\mu) = i(\mu, \mu) \) be the self-intersection number. A set \( \mu \) is a \( k \)-marking if \( \mu \) fills \( S \) and \( i(\mu) \leq k \). Two sets \( \mu, \nu \) are \( \ell \)-close if \( i(\mu, \nu) \leq \ell \).

Define \( k_0 = \max_{\tau} i(V(\tau)) \), where \( \tau \) ranges over tracks with vertex cycles \( V(\tau) \) filling \( S \). Define \( \ell_0 = \max_{\tau, \sigma} i(V(\tau), V(\sigma)) \), where \( \sigma \) ranges over tracks obtained from \( \tau \) by a single splitting. Referring to [12] for the necessary definitions, we define \( k_1 = \max_{\mu} i(\mu) \), where \( \mu \) ranges over complete clean markings of \( S \). Define \( \ell_1 = \max_{\mu, \nu} i(\mu, \nu) \), where \( \nu \) ranges over markings obtained from \( \mu \) by a single elementary move.

Define \( \ell_2 = \max_{\tau} \min_{\mu} i(V(\tau), \mu) \).

Note that there are only finitely many tracks \( \tau \) and finitely many complete clean markings \( \mu \), up to the action of \( \mathrm{MCG}(S) \). An index argument bounds \( |B(\tau)| \) and so bounds the number of splittings of \( \tau \). Lemma 2.4 of [12] bounds the number of elementary moves for \( \mu \). Thus the quantities \( k_0, k_1, \ell_0, \ell_1 \) are well-defined. An upper bound for \( \ell_2 \) can be obtained by surgering \( V(\tau) \) to obtain a complete clean marking: see the discussion preceding Lemma 6.1 in [1]. Now define \( k = \max\{k_0, k_1\} \) and \( \ell = \max\{\ell_0, \ell_1, \ell_2\} \). Define \( \mathcal{M}(S) \) to be the marking graph: the vertices are \( k \)-markings and the edges are given by \( \ell \)-closeness. (When \( S \) is an annulus we take \( \mathcal{M}(S) = \mathcal{A}(S) \). Recall that \( \mathcal{A}(S) \) is quasi-isometric to \( \mathrm{MCG}(S, \partial) \simeq \mathbb{Z} \).

That \( \mathcal{M}(S) \) is connected now follows from the discussion at the beginning of [12, Section 6.4]. Accordingly, define \( d_{\mathcal{M}(S)}(\mu, \nu) \) to be the length of the shortest edge-path between the markings \( \mu \) and \( \nu \).

Since the above definitions are stated in terms of geometric intersection number, the mapping class group \( \mathrm{MCG}(S) \) acts via isometry on \( \mathcal{M}(S) \). Counting the appropriate set of ribbon graphs proves that the action has finitely many orbits of vertices and edges. The Alexander method [4, Section 2.4] proves that vertex stabilizers are finite and hence the action is proper. It now follows from the Milnor-Švarc Lemma [2, Proposition I.8.19] that any Cayley graph for \( \mathrm{MCG}(S) \) is quasi-isometric to \( \mathcal{M}(S) \).

Define \( \mathcal{T}(S) \), the train track graph as follows: vertices are isotopy classes of birecurrent train tracks \( \tau \subset S \) so that \( V(\tau) \) fills \( S \). Connect vertices \( \tau \) and \( \sigma \) by an edge exactly when \( \sigma \) is a slide or split of \( \tau \). Let \( d_{\mathcal{T}(S)}(\tau, \nu) \) be the minimal number of edges in a path in \( \mathcal{T}(S) \) connecting \( \tau \) to \( \nu \), if such a path exists. Note that the map \( \tau \mapsto V(\tau) \) from \( \mathcal{T}(S) \) to \( \mathcal{M}(S) \) sends edges to edges (or to vertices) and thus is distance non-increasing. For further discussion of graphs tightly related to \( \mathcal{T}(S) \) see [8].
We adopt the following notations. If \( \{\tau_i\}_{i=0}^N \) is a sliding and splitting sequence in \( \mathcal{T}(S) \) and \( I = [p, q] \subset [0, N] \) is a subinterval then \( |I| = q - p \) and \( d_{\mathcal{T}(S)}(I) = d_{\mathcal{T}(S)}(\tau_p, \tau_q) \). If \( \tau, \sigma \in \mathcal{T}(S) \) then define

\[
d_{\mathcal{M}(X)}(\tau, \sigma) = d_{\mathcal{M}(X)}(\tau|X), V(\sigma|X)).
\]

Also take \( d_{\mathcal{M}(X)}(I) = d_{\mathcal{M}(X)}(\tau_p, \tau_q) \).

**Theorem 6.1.** For any surface \( S \) with \( \xi(S) \geq 1 \) there is a constant \( Q = Q(S) \) with the following property: Suppose that \( \{\tau_i\}_{i=0}^N \) is a sliding and splitting sequence in \( \mathcal{T}(S) \). Then the sequence \( \{V(\tau_i)\}_{i=0}^N \) as parameterized by splittings, is a \( Q \)-quasi-geodesic in the marking graph.

Our final generalization of [8, Corollary 3] follows from Theorem 6.1:

**Theorem 6.2.** For any surface \( S \) with \( \xi(S) \geq 1 \) there is a constant \( Q = Q(S) \) with the following property: If \( \{\tau_i\}_{i=0}^N \) is a sliding and splitting sequence in \( \mathcal{T}(S) \), injective on slide subsequences, then \( \{\tau_i\} \) is a \( Q \)-quasi-geodesic.

Notice that here, unlike Theorem 6.1, the parameterization is by index. We require:

**Lemma 6.3.** There is a constant \( A = A(S) \) so that if \( \{\tau_i\}_{i=1}^N \) is an injective sliding sequence in \( \mathcal{T}(S) \) then \( N + 1 \leq A \).

**Proof of Theorem 6.2.** Let \( \{\tau_i\} \) be the given sliding and splitting sequence in \( \mathcal{T}(S) \). Let \( I = [p, q] \subset [0, N] \) be a subinterval. Note that \( d_{\mathcal{T}(S)}(I) \leq |I| \) because \( \{\tau_i\} \) is an edge-path in \( \mathcal{T}(S) \).

Define \( S(I) \) to be the set of indices \( r \in I \) where \( r + 1 \in I \) and \( \tau_{r+1} \) is a splitting of \( \tau_r \). Thus \( |I| \leq A |S(I)| \), where \( A \) is the constant of Lemma 6.3. Now, Theorem 6.1 implies \( |S(I)| \leq Q d_{\mathcal{M}(S)}(I) \). Finally, since \( \tau \mapsto V(\tau) \) is distance non-increasing we have \( d_{\mathcal{M}(S)}(I) \leq d_{\mathcal{T}(S)}(I) \). Deduce that \( |I| \leq Q d_{\mathcal{M}(S)}(I) \), for a somewhat larger value of \( Q \).

**Remark 6.4.** Note that we have not used the connectedness of \( \mathcal{T}(S) \), an issue that appears to be difficult to approach combinatorially. For a proof of connectedness see [8, Corollary 3.7].

6.2. **Hyperbolicity and the distance estimate.** To prove Theorem 6.1 we will need:

**Theorem 6.5.** [11, Theorem 1.1] For every connected compact orientable surface \( X \) there is a constant \( \delta_X \) so that \( \mathcal{C}(X) \) is \( \delta_X \)-hyperbolic.

An important consequence of the Morse Lemma [2, Theorem III.H.1.7] is a reverse triangle inequality.
Lemma 6.6. For every $\delta$ and $Q$, there is a constant $R_0 = R_0(\delta, Q)$ with the following property: For any $\delta$–hyperbolic space $X$, for any $Q$–unparameterized quasi-geodesic $f: [m, n] \to X$, and for any $a, b, c \in [m, n]$, if $a \leq b \leq c$ then

$$d_X(\alpha, \beta) + d_X(\beta, \gamma) \leq d_X(\alpha, \gamma) + R_0$$

where $\alpha, \beta, \gamma = f(a), f(b), f(c)$. □

We now take $R_0 = R_0(\delta, Q(S))$ where $\delta = \max\{\delta_X | X \subset S\}$, as provided by Theorem 6.5, and $Q(S)$ is the constant of Theorem 5.5.

The next central result needed is the distance estimate for $M(S)$. Let $[\cdot]_C$ be the cut-off function:

$$[x]_C = \begin{cases} 0, & \text{if } x < C \\ x, & \text{if } x \geq C \end{cases}.$$

We may now state the distance estimate:

Theorem 6.7. [12, Theorem 6.12] For any surface $S$, there is a constant $C(S)$ so that for every $C \geq C(S)$ there is an $E \geq 1$ so that for all $\mu, \nu \in M(S)$

$$d_{M(S)}(\mu, \nu) = E \sum [d_X(\mu, \nu)]_C$$

where the sum ranges over essential subsurfaces $X \subset S$.

6.3. Marking distance. Suppose that $\{\tau_i\}_{i=0}^N \subset T(S)$ is a sliding and splitting sequence. Let $V_i = V(\tau_i)$ be the set of vertex cycles of $\tau_i$. As $i(V_i) \leq k_0$ and $i(V_i, V_{i+1}) \leq \ell_0$ the map $i \mapsto V_i$ gives rise to an edge-path in $M(S)$.

Suppose that $[p, q] \subset [0, N]$. Let $S_X(p, q)$ be the set of indices $r \in [p, q - 1]$ so that $\tau_{r+1}|X$ is a splitting of $\tau_r|X$. (When $X$ is an annulus $\tau_{r+1}|X$ may also differ from $\tau_r|X$ by a pair of splits.) Since slides do not effect $P(\tau)$ [15, Proposition 2.2.2] the distance in $M(S)$ between $V_p$ and $V_q$ is at most $|S_S(p, q)|$.

Remark 6.8. We do not place indices $r$ onto $S_X$ where $\tau_{r+1}|X$ is a subtrack of $\tau_r|X$; the number of such indices is bounded by a constant depending only on $X$.

In the other direction, we must bound the number of splittings between $\tau_p$ and $\tau_q$ in terms of the marking distance between $V_p$ and $V_q$. This will be done inductively. As a notational matter set $I_S = [0, N]$. When $X \subset S$ is essential take $V(\tau|X)$ to be the vertex cycles of the induced track. Recall that $I_X \subset I_S$, defined in Section 5.2, is the accessible interval for $X \subset S$. If $I = [m, n] \subset [0, N]$ is an interval then define $S_X(I) = S_X(m, n)$, $d_X(I) = d_X(\tau_m, \tau_n)$, and so on.
Proposition 6.9. Suppose that $X \subset S$ is an essential subsurface and $J_X \subset I_X$ is a subinterval. There is a constant $A = A(X)$, independent of the sequence $\{\tau_i\}$, so that $|S_X(J_X)| \leq_A d_{\mathcal{M}(X)}(J_X)$.

The rest of this section is devoted to the proof of Proposition 6.9, from which Theorem 6.1 follows.

6.4. Inductive and straight intervals. We fix two thresholds $T_0, T_1$ so that:

$$\max \left\{ 6N_1 + 2N_2 + 2K_0(X) + 2, 2R_0, M_2(X), C(X) \right\} \leq T_0(X)$$
$$\max \left\{ T_0(X) + 2R_0, B_0N_2 \right\} \leq T_1(X)$$

Here $N_1$ is an upper bound for $d_Y(\alpha, \beta)$ where $Y \subset S$ is any essential subsurface, $\tau$ is a track, and $\alpha$ and $\beta$ are wide with respect to $\tau$. The constant $B_0$ is an upper bound for the number of branches in any induced track. The constant $N_2$ is an upper bound for the distance (in any subsurface projection) between the vertices of $\tau$ (or $\tau|X$) and the vertices of a single splitting or subtrack of $\tau$. Also, $M_2(X)$ is the constant provided by Lemma 6.1 of [12].

Recall that the interval $J_X \subset I_X$ is given in Proposition 6.9.

Definition 6.10. Suppose that $Y \subset X$ is an essential subsurface with $\xi(Y) < \xi(X)$. If $d_Y(J_X) \geq T_0(X)$ then we call $Y$ an inductive subsurface of $X$ and take $J_Y = I_Y \cap J_X$ as the associated inductive subinterval of $J_X$. If $d_Y(J_X) < T_0(X)$ then we set $J_Y = \emptyset$.

Suppose $I$ is a subinterval of $J_X$. Define $diam_Y(I)$ to be the diameter, in $\mathcal{AC}(Y)$ of the union $\cup_{i \in I} \pi_Y(\tau_i)$.

Definition 6.11. A subinterval $I \subset J_X$ is a straight subinterval for $X$ if for all essential subsurfaces $Y \subset X$, with $\xi(Y) < \xi(X)$, we have $diam_Y(I) \leq T_1(X)$.

Lemma 6.12. If $I \subset J_X$ is disjoint from all inductive subintervals of $J_X$ then $I$ is straight for $X$.

Proof. Fix an essential $Y \subset X$ with $\xi(Y) < \xi(X)$. It suffices to show, for every subinterval $J \subset I$, that $d_Y(J) \leq T_1(X)$.

If $J \cap I_Y = \emptyset$ then Theorem 5.3 implies $d_Y(J) \leq K_0$. Suppose that $J$ meets $I_Y$; thus $J_Y = \emptyset$ by hypothesis and so $Y$ is not inductive. It follows that $d_Y(J) < T_0(X)$. By Lemma 6.6 we have $d_Y(J) < T_0(X) + 2R_0$. \qed

Lemma 6.13. There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that if $I \subset J_X$ is straight then $|S_X(I)| \leq_A d_X(I)$.
Proof. If \(X\) is an annulus then, by Theorem 5.3, for every \(r \in I\) the core curve \(\alpha \subset X\) is carried by and wide in \(\tau_r\). It follows that the number of switches in \(\alpha \subset \tau_r|X\) is bounded by some constant \(K = K(S)\). Let \(q = \max I\) and pick any \(\beta \in V(\tau_q|X)\). As in the proof of Theorem 5.5 let \(\sigma_r \subset \tau_r|X\) be the minimal subtrack carrying \(\beta\). Thus \(\sigma_r\) has either exactly four branches and two switches, or is an embedded arc. It follows that every \(K^2/4\) consecutive splittings in \(S_X(I)\) induces at least one splitting in the sequence of tracks \(\{\sigma_r\}\). Therefore the singleton sets \(V(\sigma_r)\) form a quasi-geodesic in \(A(X)\). Since \(V(\sigma_r) \subset V(\tau_r|X)\) the proof is complete when \(X\) is an annulus.

We assume for the rest of the proof that \(X\) is not an annulus. The map \(i \mapsto V(\tau_i|X)\), taking tracks to their vertex cycles, is generally not injective. (For example, see [15, Proposition 2.2.2].) However:

Claim 6.14. There is a constant \(N_0 = N_0(X)\), independent of \(\{\tau_i\}\), so that if \(V(\tau_r|X) = V(\tau_s|X)\) then \(|S_X(r, s)| \leq N_0\).

Proof. Let \(\mu = V(\tau_r|X)\). Our hypothesis on \(\tau_s|X\) and induction proves that \(V(\tau_t|X) = \mu\) for all \(t \in [r, s]\). Recurrence and uniqueness of carrying [14, Proposition 3.7.3] implies that \(\tau_{t+1}|X\) is a split or a slide of \(\tau_t|X\), and not a subtrack, for all \(t \in [r, s - 1]\).

If \(t \in [r, s]\) and \(b \in B(\tau_t|X)\) then define \(w_\mu(b) = \sum_{\alpha \in \mu} w_\alpha(b)\). Let

\[
M(t) = (w_\mu(b) : b \text{ is a large branch of } \tau_t|X)
\]

be the sequence of given numbers, arranged in non-increasing order. Note that if \(\tau_{t+1}|X\) is a slide of \(\tau_t|X\) then \(M(t + 1) = M(t)\). However, if \(t \in S_X(r, s)\) then the recurrence of \(\tau_t|X\) implies that \(M(t+1) < M(t)\), in lexicographic order. Since there are only finitely many possibilities for an induced track \(\tau|X\), up to the action of \(\text{MCG}(X)\), the claim follows.

Notice that if \(V(\tau_{t+1}|X) \neq V(\tau_t|X)\) then \(V(\tau_{t+1}|X) \neq V(\tau_j|X)\) for \(j \leq i\). This is because \(P(\tau_{k+1}|X) \subset P(\tau_k|X)\) for all \(k\). Using \(C = 1 + \max\{C(X), T_1(X)\}\) as the cut-off in Theorem 6.7 gives some constant of quasi-equality, say \(E\). Define \(R_1 = E + 1\).

Suppose that \([p, q] = I\), the straight subinterval of \(J_X\) given by Lemma 6.13. We define a function \(\rho: [0, M] \to I\) as follows: let \(\rho(0) = p\) and let \(\rho(n + 1)\) be the smallest element in \([\rho(n), q]\) with \(d_{\text{M}(X)}(\tau_{\rho(n)}, \tau_{\rho(n+1)}) = R_1\). (If \(\rho(n + 1)\) is undefined then take \(M = n+1\) and \(\rho(M) = q\).) Let \(B(\mu)\) be the ball of radius \(R_1\) about the marking \(\mu \in \text{M}(X)\). Define

\[
V = \max \{|B(\mu)| : \mu \in \text{M}(X)\}.
\]
Deduce from Claim 6.14 and the remark immediately following that, for all \( n \in [0, M - 1] \),

\[
|S_X(\rho(n), \rho(n + 1))| \leq N_0 V.
\]

Thus

\[
|S_X(I)| \leq N_0 V \cdot M.
\]

So to prove Lemma 6.13 it suffices to bound \( M \) from above in terms of \( d_X(I) \).

**Claim 6.15.** Fix \( n \in [0, M - 2] \). Let \( \tau, \sigma = \tau_{\rho(n)}, \tau_{\rho(n+1)} \). Then

\[
d_X(\tau, \sigma) \geq R_0 + 1.
\]

**Proof.** We use Theorem 6.7. Note that \( d_M(X)(\tau, \sigma) = R_1 \). Since \( R_1 \) is greater than the additive error there is at least one non-vanishing term in the sum \( \sum_{Y \subset X} [d_Y(\tau, \sigma)]_C \).

However, since \( [\rho(n), \rho(n + 1)] \subset [p, q] \) and \( [p, q] = I \) is straight we have \( d_Y(\tau, \sigma) \leq T_1(X) \) for all \( Y \subset X \) with \( \xi(Y) < \xi(X) \). Thus \( d_X(\tau, \sigma) \) is the only term of the sum greater than the cut-off \( C \). Since \( C > T_1(X) \geq R_0 \), we have \( d_X(\tau, \sigma) \geq R_0 + 1 \) and the claim is proved.

Thus we have

\[
d_X(I) \geq -(M - 1) \cdot R_0 + \sum_{n=0}^{M-1} d_X(\tau_{\rho(n)}, \tau_{\rho(n+1)})
\]

\[
\geq M - 1 + d_X(\tau_{\rho(M-1)}, \tau_{\rho(M)})
\]

\[
\geq M - 1
\]

where the first and second lines follow from Lemma 6.6 and Claim 6.15 respectively. This completes the proof of Lemma 6.13.

**Lemma 6.16.** There is a constant \( A = A(X) \) with the following property: Suppose that \( J_Y \subset J_X \) is an inductive interval. Suppose that \( I \subset J_Y \) is a straight subinterval for \( X \). Then \( |S_X(I)| \leq A \).

**Proof.** Let \( [p, q] = I \). Applying Theorem 5.3, as \( p \in J_Y \subset J_Y \), the multicurve \( \partial Y \) is wide with respect to \( \tau_p \). It follows that \( \partial Y \) is also wide with respect to \( \tau^X_p \). Note that the curves of \( V(\tau^Y_p, X) \) are also wide with respect to \( \tau^X_p \). Repeating this discussion for \( q \), and then applying Lemma 5.2 and the triangle inequality gives a uniform bound for \( d_X(\tau_p, \tau_q) \). The lemma now follows from Lemma 6.13.

**6.5. Long and short intervals.**

**Definition 6.17.** A straight subinterval \( I \) for \( X \) is short if \( d_X(I) \leq 4R_0 \). Otherwise \( I \) is long.
By Lemma 6.13, if $I$ is a short straight interval then $|S_X(I)|$ is uniformly bounded by a constant depending only on $X$.

Let $\text{Ind}$ be the set of inductive subsurfaces $Y \subset X$. Define $\text{Ind}' = \text{Ind} \cup \{X\}$. Note that every maximal subinterval of $J_X \setminus \bigcup_{Y \in \text{Ind}} J_Y$ is straight, by Lemma 6.12. We partition these maximal subintervals into the sets $\text{Long}$ and $\text{Short}$ as the given interval is long or short respectively.

**Lemma 6.18.** There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that

$$\sum_{I \in \text{Long}} |S_X(I)| \leq A d_X(J_X).$$

**Proof.** From Lemma 6.13 we deduce

$$\sum_{I \in \text{Long}} |S_X(I)| \leq A |\text{Long}| + \sum_{I \in \text{Long}} d_X(I)$$

where the first term on the right arises from addition of additive errors. By the definition of a long straight interval and from Lemma 6.6 deduce

$$4R_0|\text{Long}| \leq \sum_{I \in \text{Long}} d_X(I) \leq d_X(J_X) + 2R_0|\text{Long}|.$$

Thus $2R_0|\text{Long}| \leq d_X(J_X)$. These inequalities combine to prove the lemma, for a somewhat larger value of $A = A(X)$.

**Lemma 6.19.** There is a constant $A = A(X)$, independent of $\{\tau_i\}$, so that

$$\sum_{I \in \text{Short}} |S_X(I)| \leq A |\text{Ind}'|.$$

**Proof.** By Lemma 6.13 the number of splittings in any short straight interval is a priori bounded (depending only on $X$). Since $|\text{Short}| \leq |\text{Ind}'|$ the lemma follows.

**Lemma 6.20.** If $Z \in \text{Ind}$ then

$$\text{card} \{Y \in \text{Ind} \mid Z \subset Y, \xi(Z) < \xi(Y)\} \leq 2(\xi(X) - \xi(Z) - 1).$$

This follows from and is strictly weaker than Theorem 4.7 and Lemma 6.1 of [12]. We give a proof, using our structure theorem, to extract the necessary lower bound for $T_0(X)$.

**Proof of Lemma 6.20.** Suppose that $U \in \text{Ind}$ contains $Z$. Suppose $J_Z = [p, q]$ and $J_X = [m, n]$. Thus $\partial Z$ is wide with respect to $\tau_p$. So $d_U(\tau_p, \partial Z) \leq N_1$, by the definition of $N_1$, and the same holds at the index $q$. Thus $d_U(\tau_p, \tau_q) \leq 2N_1$. The subsurface $U$ precedes or succeeds $Z$ if $d_U(\tau_m, \tau_p)$ or $d_U(\tau_q, \tau_n)$, respectively, is greater than or equal to $2N_1 + N_2 + K_0(X) + 1$. Note that $U$ must precede or succeed $Z$ (or
both) as otherwise \(d_U(\tau_m, \tau_n) < 6N_1 + 2N_2 + 2K_0(X) + 2 \leq T_0(X)\), a contradiction.

It now suffices to consider subsurfaces \(U\) and \(V\) that both succeed and both contain \(Z\). If \(\max J_U \leq \max J_V\) then \(U \subset V\). For, if not, \(\partial V\) cuts \(U\) while missing \(Z\). Since \(\partial V\) is wide at the index \(r = \max J_V\) we deduce that

\[
d_U(\tau_q, \tau_n) \leq d_U(\tau_q, \partial Z) + d_U(\partial Z, \partial V) + d_U(\partial V, \tau_r) + \\
+ d_U(\tau_r, \tau_{r+1}) + d_U(\tau_{r+1}, \tau_n)
\]

\[
\leq 2N_1 + N_2 + K_0(X) + 1
\]

and this is a contradiction. Thus the surfaces in \(\text{Ind}\) that strictly contain \(Z\), and succeed \(Z\), are nested. \(\square\)

**Definition 6.21.** Assign an index \(r \in S_X(J_X)\) to a subsurface \(Y \subset X\) if \(Y \in \text{Ind}'\), \(r \in J_Y\), \(\tau_{r+1}|Y\) is a splitting of \(\tau_r|Y\) and there is no subsurface \(Z \subset Y, \xi(Z) < \xi(Y)\) with those three properties.

**Lemma 6.22.** There is a constant \(A = A(X)\), independent of \(\{\tau_i\}\), so that the number of splittings contained in inductive intervals is quasi-bounded by \(|\text{Ind}| + \sum_{Y \in \text{Ind}} d_Y(J_X)\).

**Proof.** Fix \(Y \in \text{Ind}\). Consider an index \(r \in J_Y\) that is assigned to \(X\). Let \(I \subset J_Y\) be the maximal interval containing \(r\) so that all indices in \(S_X(I)\) are assigned to \(X\). We now show that \(I\) is straight. Let \(Z\) be any essential subsurface of \(X\) with \(\xi(Z) < \xi(X)\) and let \([r, s] = J \subset I\) be any subinterval. If \(J \cap I_Z = \emptyset\) then Theorem 5.3 implies that \(d_Z(J) \leq K_0\). If \(J\) meets \(I_Z\) then, as no splittings of \(J\) are assigned to \(Z\) we deduce that \(\tau_s|Z\) is obtained from \(\tau_r|Z\) by sliding and taking subtracks only. Thus \(d_Z(J) \leq B_0N_2 \leq T_1(X)\), as desired.

By Lemma 6.16 we find that \(|S_X(I)|\) is bounded. It follows that the number of splittings in the inductive intervals is quasi-bounded by \(\sum_{Y \in \text{Ind}} |S_Y(J_Y)|\).

By induction, Proposition 6.9 gives

\[
|S_Y(J_Y)| \leq A d_{M(Y)}(J_Y).
\]

Taking a cutoff of \(C = 1 + \max\{C(Y), T_0(X) + 2R_0\}\) and applying the distance estimate Theorem 6.7 we have a quasi-inequality

\[
d_{M(Y)}(J_Y) \leq E \sum_{Z \subset Y} [d_Z(J_Y)]_C.
\]

Since \(d_Z(J_Y) \leq d_Z(J_X) + 2R_0\) for all \(Z \subset Y\), it follows that non-zero terms in the sum only arise for subsurfaces in \(\text{Ind}'(Y) = \{Z \in \text{Ind}'| Z \subset Y\}\).
Since \( 2R_0 \leq T_0(X) \) we have \( [d_Z(J_Y)]_C \leq 2 \cdot d_Z(J_X) \). Making \( A = A(X) \) larger if necessary we have

\[
|S_Y(J_Y)| \leq_A \sum_{Z \in \text{Ind}'(Y)} d_Z(J_X).
\]

Thus

\[
\sum_{Y \in \text{Ind}} |S_Y(J_Y)| \leq_A |\text{Ind}| + \sum_{Y \in \text{Ind}} \sum_{Z \in \text{Ind}(Y)} d_Z(J_X)
\]

\[
\leq_A |\text{Ind}| + \sum_{Y \in \text{Ind}} d_Y(J_X)
\]

where the final quasi-inequality follows from Lemma 6.20, taking \( A \) larger as necessary. Note that the term \( |\text{Ind}| \) on the middle line arises by adding additive errors. This proves Lemma 6.22.

Since every index in \( S_X(J_X) \) is either in a long or short straight interval or in an inductive interval, from Lemmas 6.18, 6.19, and 6.22 and increasing \( A \) slightly, we have:

\[
|S_X(J_X)| \leq_A d_X(J_X) + |\text{Ind}'| + \sum_{Y \in \text{Ind}} d_Y(J_X).
\]

Note that \( |\text{Ind}'| \leq_A d_{M(X)}(J_X) \); this follows from the hierarchy machine (in particular Lemma 6.2 and Theorem 6.10 of [12]) and because \( T_0(X) \geq M_2(X) \), the constant of Lemma 6.1 in [12]. Finally,

\[
\sum_{Y \in \text{Ind}'} d_Y(J_X) \leq_A d_{M(X)}(J_X)
\]

follows from the distance estimate (Theorem 6.7) and because \( T_0(X) \geq C(X) \). This completes the proof of Proposition 6.9 and thus the proof of Theorem 6.1. □

References

[1] Jason A. Behrstock. Asymptotic geometry of the mapping class group and Teichmüller space. \textit{Geom. Topol.}, 10:1523–1578 (electronic), 2006. arXiv:math/0502367. [31]

[2] Martin R. Bridson and André Haefliger. \textit{Metric spaces of non-positive curvature}. Springer-Verlag, Berlin, 1999. [31, 32]

[3] D. B. A. Epstein. Curves on 2-manifolds and isotopies. \textit{Acta Math.}, 115:83–107, 1966. [15, 16]

[4] Benson Farb and Dan Margalit. A primer on mapping class groups, 2010. \url{http://www.math.utah.edu/~margalit/primer/}. [16, 31]

[5] Fathi, Laudenbach, and Poénaru, editors. \textit{Travaux de Thurston sur les surfaces}. Société Mathématique de France, Paris, 1991. Séminaire Orsay. Reprint of \textit{Travaux de Thurston sur les surfaces}, Soc. Math. France, Paris, 1979 [MR 82m:57003], Astérisque No. 66-67 (1991). [3, 18]
[6] Mikhail Gromov. Hyperbolic groups. In Essays in group theory, pages 75–263. Springer, New York, 1987. [2]
[7] François Guéritaud. Deforming ideal solid tori, 2009. arXiv:0911.3067. [15]
[8] Ursula Hamenstädt. Geometry of the mapping class groups. I. Boundary amenability. Invent. Math., 175(3):545–609, 2009. [1, 30, 31, 32]
[9] William J. Harvey. Boundary structure of the modular group. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pages 245–251, Princeton, N.J., 1981. Princeton Univ. Press. [10]
[10] Tsuyoshi Kobayashi. Heights of simple loops and pseudo-Anosov homeomorphisms. In Braids (Santa Cruz, CA, 1986), pages 327–338. Amer. Math. Soc., Providence, RI, 1988. [6]
[11] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. Invent. Math., 138(1):103–149, 1999. arXiv:math/9804098. [6, 10, 18, 19, 32]
[12] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. Geom. Funct. Anal., 10(4):902–974, 2000. arXiv:math/9807150. [1, 2, 10, 31, 33, 34, 37, 39]
[13] Howard A. Masur and Yair N. Minsky. Quasiconvexity in the curve complex. In In the tradition of Ahlfors and Bers, III, volume 355 of Contemp. Math., pages 309–320. Amer. Math. Soc., Providence, RI, 2004. arXiv:math/0307083. [30]
[14] Lee Mosher. Train track expansions of measured foliations. 2003. http://newark.rutgers.edu/~mosher/. [1, 3, 8, 11, 12, 19, 35]
[15] R. C. Penner and J. L. Harer. Combinatorics of train tracks, volume 125 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1992. [1, 2, 3, 4, 6, 18, 19, 21, 29, 33, 35]
[16] Kasra Rafi. A characterization of short curves of a Teichmüller geodesic. Geom. Topol., 9:179–202 (electronic), 2005. arXiv:math/0404227. [1]
[17] Kasra Rafi. A combinatorial model for the Teichmüller metric. Geom. Funct. Anal., 17(3):936–959, 2007. arXiv:math/0509584. [1]
[18] Dylan Thurston. Geometric intersection of curves on surfaces. 2010. http://www.math.columbia.edu/~dpt/DehnCoordinates.pdf. [22]
[19] William P. Thurston. The Geometry and Topology of Three-Manifolds. MSRI, 2002. http://www.msri.org/publications/books/gt3m/. [1, 3]

Department of Mathematics, University of Chicago, Chicago, IL 60607, USA
E-mail address: masur@math.uchicago.edu

Department of Mathematics, Rutgers-Newark, State University of New Jersey, Newark, NJ 07102, USA
E-mail address: mosher@rutgers.edu

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK
E-mail address: s.schleimer@warwick.ac.uk