A TRANSVERSE LINK INVARIANT FROM \( \mathbb{Z}_2 \)-EQUIVARIANT HEEGAARD FLOER COHOMOLOGY

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Abstract. We define an invariant of based transverse links, as a well-defined element inside the equivariant Heegaard Floer cohomology of its branched double cover, defined by Lipschitz, Hendricks, and Sarkar. We prove the naturality and functoriality of equivariant Heegaard Floer cohomology for branched double covers of \( S^3 \) along based knots, and then prove that our transverse link invariant \( c_{\mathbb{Z}_2}(\xi_K) \) is an well-defined element which is always nonvanishing and functorial under certain classes of symplectic cobordisms, and describe its behavior under negative stabilization. It follows that we can use properties of \( c_{\mathbb{Z}_2}(\xi_K) \) to give a condition on transverse knots \( K \) which implies the vanishing/nonvanishing of the contact class \( c(\xi_K) \).

1. Introduction

In the paper [OS2], Ozsvath and Szabo defined an element \( c(\xi) \in \hat{HF}(M) \) associated to a contact 3-manifold \( (M, \xi) \), which is an invariant of the isotopy class of the given contact structure \( \xi \) on \( M \). In particular, they defined an element in \( \hat{HF}(M) \) associated to an open book decomposition of \( M \) which supports \( \xi \), and then proved its invariance under isotopy and positive stabilization. Later, in the paper [HKM], Honda, Kazez, and Matic provided a new way to define the element \( c(\xi) \), by working with Heegaard diagrams induced by arc diagrams on open books.

In the paper [HLS], Lipshitz, Hendricks, and Sarkar defined an \( \mathbb{F}_2[\theta] \)-module \( HF_{\mathbb{Z}_2}(L_1, L_2) \) associated to a pair of Lagrangian submanifolds \( L_1, L_2 \) in a symplectic manifold \( M \), where the group \( \mathbb{Z}_2 \) acts on \( M \) by symplectomorphisms and leaves \( L_1, L_2 \) invariant as sets. The equivariant Floer cohomology \( HF_{\mathbb{Z}_2}(L_1, L_2) \) turned out to be invariant under \( \mathbb{Z}_2 \)-invariant Hamiltonian isotopies, and in some special cases, noninvariant Hamiltonian isotopies. This construction was applied to construct an \( \mathbb{F}_2[\theta] \)-module

\[ \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \in \text{Mod}_{\mathbb{F}_2[\theta]} \]

associated to a bridge diagram of a based link \((L, p)\) on a sphere, whose isomorphism type is an invariant of the isotopy class of \((L, p)\).

In this paper, we construct an element \( c_{\mathbb{Z}_2}(\xi_L) \in \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \), where \( L \) is a transverse knot in the standard contact 3-sphere \((S^3, \xi_{\text{std}})\), by considering the contact branched double cover \((\Sigma(L), \xi_L)\) of \((S^3, \xi_{\text{std}})\), branched along \( L \), as defined by Plamenevskaya [PI]. We prove that this element is indeed a well-defined element inside the equivariant Floer cohomology, and becomes an invariant of the transverse (based) isotopy class of \((L, p)\).

Then, before discussing the functoriality of \( c_{\mathbb{Z}_2}(\xi_{\text{std}}) \), we first prove the functoriality of \( \hat{HF}_{\mathbb{Z}_2} \). Using the techniques introduced by Juhasz and Thurston [JT], we prove that the \( \mathbb{F}_2[\theta] \)-module \( \hat{HF}_{\mathbb{Z}_2}(\Sigma(K), p) \) is natural in the sense that it admits an action of \( MCG(S^3, K, p) \), when \((K, p)\) is a based knot. Then, using the naturality of \( \hat{HF}_{\mathbb{Z}_2} \) for based knots, we prove that a based cobordism \( S = (S_0, s) \) between two based knots \((K_1, p_1)\) and \((K_2, p_2)\) in \( S^3 \) defines a map

\[ \hat{f}_S : \hat{HF}_{\mathbb{Z}_2}(\Sigma(K_2), p_2) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(K_1), p_1), \]

which is an invariant of the isotopy class of \( S \) rel \( \partial S \). Here, a based cobordism is a cobordism together with a curve from \( p_1 \) to \( p_2 \).

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After establishing the functoriality of $\widehat{HF}_{\mathbb{Z}_2}$, we discuss the functoriality of the element $c_{\mathbb{Z}_2}(\xi_{\text{std}})$ inside $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$, when $(K, p)$ is a based transverse knot in $(S^3, p)$. Recall that any smooth cobordism between knots (or links) is a composition of isotopies, births, saddles, and deaths. We define their analogues (except for deaths) in the symplectic setting, which turn out to be well-defined up to a weaker version of symplectic isotopy, and restrict our attention to symplectically constructible symplectic cobordisms, which can roughly be defined to be based symplectic cobordisms which can be weakly symplectically isotoped to a composition of symplectic versions of isotopies, births, and saddles. Then we prove that $c_{\mathbb{Z}_2}(\xi_{\text{std}})$ is preserved under the maps associated to symplectically constructible based cobordisms.

To summarize, we prove the following theorem.

**Theorem.** For a based link $(K, p)$ in $S^3$, the $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ is natural. Given a based knot cobordism $(S, s)$ in $S^3 \times I$ with $\partial S = K_1 \times \{0\} \cup K_2 \times \{1\}$ and $\partial S = \{(p_1, 0), (p_2, 1)\}$, we have an $\mathbb{F}_2[\theta]$-module homomorphism

$$\hat{f}_{(S, s)} : \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_2), p_2) \to \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K_1), p_1),$$

which is an invariant of the isotopy class of $(S, s)$. When $(K, p)$ is a based transverse knot inside $(S^3, \xi_{\text{std}})$, we have an element

$$c_{\mathbb{Z}_2}(\xi_K) \in \widehat{HF}(\Sigma(K), p)$$

which is an invariant of the transverse isotopy class of $(K, p)$. Furthermore, if $(S, s)$ is a symplectically constructible symplectic cobordism in the $(S^3 \times I, \text{d}(\xi^{\alpha_{\text{std}}}))$ between based transverse knots, say, $\partial S = K_1 \times \{0\} \cup K_2 \times \{1\}$ and $\partial S = \{(p_1, 0), (p_2, 1)\}$, then we have

$$\hat{f}_{(S, s)}(c_{\mathbb{Z}_2}(\xi_{K_2})) = c_{\mathbb{Z}_2}(\xi_{K_1}).$$

Note that, since moving the basepoint by an isotopy preserves the element $c_{\mathbb{Z}_2}(\xi_K)$, we can drop the choice of a basepoint on $K$ and say that $c_{\mathbb{Z}_2}(\xi_K)$ is an invariant of the transverse isotopy class of a given transverse knot $K$ in $(S^3, \xi_{\text{std}})$.

Surprisingly, basic properties of $c_{\mathbb{Z}_2}(\xi_K)$ allows us to give a condition on the self-linking number of $K$ which ensures the vanishing/nonvanishing of the (non-equivariant) contact class $c(\xi_K)$. This can be seen as a Heegaard Floer analogue of a similar condition, for the Plamenevskaya $\psi$-invariant, proven in [P3]. The two results are shown to be the same for transverse representatives of quasi-alternating knots.

**Theorem 1.1.** Let $K$ be a knot in $S^3$ and $T$ be a transverse representative of $K$. Then $c(\xi) \neq 0$ if $d_3(\xi_K) = \frac{q_2(K)-1}{2}$ and $c(\xi) = 0$ if $d_3(\xi_K) > \frac{q_2(K)-1}{2} + v_r(K)$.

It is natural to ask whether the transverse knot invariant $c_{\mathbb{Z}_2}(\xi_K)$ is effective, in the sense that it can distinguish between two knots with the same self-linking number and topological knot type. There are several transverse knot invariants defined previously. For example, the Plamenevskaya invariant of transverse knots is an element of Khovanov homology, as shown in [P2]; Wu [W] defined an $sl_n$ invariant of transverse knots, as an element of $sl_n$ homology. Lisca, Ozsvath, Stipsicz, and Szabo [LOSS2] defined the LOSS invariant of Legendrian and transverse knots, as an element of knot Floer homology, which was proved to be the same (up to automorphisms of $HF(K)$) as the HFK grid invariant by Baldwin and Vela-Vick [BVV]. Also, Ekholm, Etnyre, Ng, and Sullivan [ENS2] defined transverse homology, which is a filtered version of knot contact homology. Baldwin and Sivek [BS] defined a monopole version of the LOSS invariant, as an element of the sutured monopole homology, which is functorial under Lagrangian concordances and maps to the LOSS invariant via isomorphism between KHM and HFK [BS2]. Among them, the LOSS invariant, its monopole version, and the transverse homology are proven to be effective. It is still not known whether the others are effective invariants of transverse knots. We do not know whether the invariant $c_{\mathbb{Z}_2}$ is effective.

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2. Branched double covers along transverse links

Recall that, given a link $L$ in $S^3$, we can remove its neighborhood, take the double cover with respect to the meridian of its boundary, and then reglue a solid torus along the boundary to get the branched double cover $\Sigma(L)$ of $S^3$ along $L$. The covering transformation, i.e. natural $\mathbb{Z}_2$-action on $\Sigma(L)$ is an orientation-preserving homeomorphism.

Now suppose that we are working with the standard contact sphere $(S^3, \xi_{std})$, and a transverse link $L$ inside it. It has a standard contact neighborhood:

$$N(L) \simeq (S^1 \times D^2, \ker(\alpha = d\phi + r^2d\theta)),$$

where $\phi$ parametrizes $S^1$ and $(r, \theta)$ are the polar coordinates on $D^2$. The pullback of $\alpha$ along the branched covering $\pi : (z, r, \theta) \mapsto (z, r^2, 2\theta)$ is given by

$$p^*\alpha = dz + 2r^4d\theta,$$

which satisfies the contact condition away from the fixed locus $L = \{r = 0\}$. However, the branched double cover construction of Plamenevskaya\cite{Pl} tells us that we may consider the interpolated 1-forms

$$\alpha_f = dz + f(r^2)d\theta,$$

for smooth increasing functions $f$ which satisfy $f(r^2) = r^2$ near $r = 0$ and $f(r^2) = 2r^4$ away from $r = 0$. For such a function $f$, the form $\alpha_f$ is always contact. Also, for any two such functions $f$ and $g$, the forms $\alpha_f$ and $\alpha_g$ are obviously isotopic by a radial isotopy. Hence, by gluing the solid torus to the double cover of $S^3 \setminus N(L)$, we see that the contact branched double cover

$$(\Sigma(L), \xi_L) = (S^3 - \overline{N(L)}, \xi_{std}) \cup (S^1 \times D^2, \alpha_f)$$

is well-defined up to isotopy supported near $L$.

Now we consider the case when $L$ is braided along the $z$-axis in $S^3$. This notion can be made precise as follows:

**Definition 2.1.** Consider the genus 0 open book of $S^3$, as follows:

$$\pi : S^3 \setminus \{z\text{-axis}\} \to S^1.$$

Then a link $L$ in $S^3$ is braided if it does not intersect the $z$-axis and the map $\pi|_L : L \to S^1$ is a regular covering map.

Clearly, when $L$ is braided along the $z$-axis, it is a closed braid. The corresponding braid word is unique up to positive/negative stabilizations and conjugations inside the braid group.

When $L$ is transverse in $(S^3, \xi_{std})$, we do not have positive stabilizations, since they increase the self-linking number of $L$ by 2. However, we have the following theorems.

**Theorem 2.2.** (Bennequin \cite{B}) For any transverse link $L$ in $(S^3, \xi_{std})$, there exists a transverse braid $B$ around the $z$-axis, such that $L$ is transversely isotopic to $B$.

**Theorem 2.3.** (Orevkov-Shevchishin \cite{OS}) Two (closed) transverse braids around the $z$-axis are transversely isotopic as transverse links if and only if they are related by braid isotopies, conjugations in the braid group, and positive braid stabilizations.

We adopt the usual notation for braids. If we consider braids with $n$ strings, the corresponding braid group is generated by the standard generators $\sigma_1, \ldots, \sigma_{n-1}$, where $\sigma_i$ creates a positive crossing between the $i$th and the $(i + 1)$th strands. This notation can also be applied to transverse braids around the $z$-axis.

Now we recall the way to construct the contact branched double cover from a braid representative of a given transverse link, due to Plamenevskaya\cite{Pl}. Suppose that the transverse link $L$ is represented by a transverse braid $B$ with a braid word $w = \sigma_{i_1}^{\pm 1} \cdots \sigma_{i_k}^{\pm 1}$. Consider the disk $D$ with $n$ points $p_1, \ldots, p_n$ in its interior and pairwise (interior-)disjoint simple arcs $c_i$ connecting $p_i$ and $p_{i+1}$. Then we define the following self-diffeomorphism of $D$:

$$h(w) = T_{i_1}^{\pm 1} \cdots T_{i_k}^{\pm 1} \in \text{Diff}^+(D, \partial D, \{p_1, \ldots, p_n\}).$$
Here, \( T_i \) denotes the Dehn half-twist along the arc \( c_i \). Clearly this definition depends on the choice of points \( p_i \) and arcs \( c_i \), but since any two choices of such data are related by a self-diffeomorphism of \( D \) (due to the fact that any two such data are isotopic to each other), we see that \( h(w) \) is well-defined up to conjugation. Since the self-diffeomorphism \( h(w) \) is orientation-preserving and fixes the boundary \( \partial D \) pointwise, the pair \((D, h)\) becomes an abstract open book of \( S^3 \).

We then consider the branched double cover \( \Sigma \) of \( D \), branched along the points \( p_1, \ldots, p_n \). The arcs \( c_i \) smoothly lift to smooth simple closed curves \( C_i \) and the self-diffeomorphism \( h \) of \( D \) lifts smoothly to a self-diffeomorphism \( \tilde{h} \) of \( S \), which is now given by products of positive/negative Dehn twists along the curves \( C_i \). By the argument used above, the conjugacy class of \( \tilde{h} \) is uniquely determined. Hence the abstract open book \( \mathcal{P}_B = (\Sigma, \tilde{h}) \) is defined up to diffeomorphism. Since this abstract open book has a natural \( \mathbb{Z}_2 \)-action (i.e. covering transformation) and the monodromy \( \tilde{h} \) is equivariant with respect to that action, \( \mathcal{P}_B \) represents a uniquely determined closed contact 3-manifold \((M_B, \xi_{M_B})\) with a contact \( \mathbb{Z}_2 \)-action, which turns out to be the contact branched double cover of \((S^3, \xi_{std})\) along \( L \). In other words, we get an open book description of \((\Sigma(L), \xi_L)\).

**Theorem 2.4.** [HLS] We have a \( \mathbb{Z}_2 \)-equivariant contactomorphism 
\[
(M_B, \xi_{M_B}) \simeq (\Sigma(L), \xi_L).
\]

**Remark.** The construction of \((M_B, \xi_B)\) depends on the braid isotopy class of \( B \), not only on its link isotopy class. However, by the definition of contact branched double covers, we know that if \( L, L' \) are transversely isotopic, then the double covers \((\Sigma(L), \xi_L)\) and \((\Sigma(L'), \xi_{L'})\) are \((\mathbb{Z}_2 \text{-equivariantly})\) contactomorphic. Hence we see that, if two transverse braids \( B, B' \) are related by braid isotopies and positive stabilizations (conjugations and positive stabilizations in terms of braid group elements), then \((M_B, \xi_B) \simeq (M_{B'}, \xi_{B'})\).

3. \( \widehat{HF}_{\mathbb{Z}_2} \) of Branched Double Covers of \( S^3 \) Along a Based Link

Hendricks, Lipshitz, and Sarkar [HLS] constructed the \( \mathbb{F}_2[\theta] \)-module \( \widehat{HF}_{\mathbb{Z}_2}(L_0, L_1) \) when \( \mathbb{Z}_2 \) acts symplectically on a symplectic manifold \((M, \omega)\) and \( L_0, L_1 \subset M \) are transversely intersecting Lagrangians which are fixed by the \( \mathbb{Z}_2 \)-action and satisfy Hypothesis 3.2 of [HLS], and use it to define \( \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), z) \) for the branched double cover \( \Sigma(L) \), where \((L, z)\) is a based link in \( S^3 \). In this section, we will briefly review their construction to make the whole paper more self-contained.

**Homotopy coherent diagrams of almost complex structures.** By a cylindrical complex structure on a symplectic manifold \((M, \omega)\), we mean a smooth 1-parameter family \( J = J(t) \), \( 0 \leq t \leq 1 \), of almost complex structures on \( M \), compatible with \( \omega \). By an eventually cylindrical almost complex structure on \((M, \omega)\), we mean a smooth 1-parameter family \( \hat{J}(s) \), \( s \in \mathbb{R} \), of cylindrical complex structures, which is constant outside a compact subset of \( \mathbb{R} \), modulo translation by \( \mathbb{R} \). Denote the set of eventually cylindrical almost complex structures by \( \mathcal{J} \). Then it carries a natural topology by declaring that a sequence \( \{\hat{J}_i\} \) converges if and only if we can replace \( \hat{J}_i \) by their representatives so that every \( \hat{J}_i \) is constant outside a fixed compact set \( C \) and \( \{\hat{J}_i|_C\} \) is convergent in the \( C^\infty \) topology. Given an element \( J \in \mathcal{J} \), its limit at \( -\infty \) and \( \infty \) are well-defined cylindrical complex structures, which we will denote as \( J_{-\infty} \) and \( J_{+\infty} \). Let \( \mathcal{J}(J_{-\infty}, J_{+\infty}) \) be the subspace of \( \mathcal{J} \) consisting of \( J' \in \mathcal{J} \) with \( J'_{-\infty} = J_{-\infty} \) and \( J'_{+\infty} = J_{+\infty} \).

We now define a category \( \overline{\mathcal{J}} \) as follows. Its objects are cylindrical complex structures. For any cylindrical complex structures \( J \) and \( J' \), the morphism set \( \overline{\mathcal{J}}(J, J') \) consists of finite nonempty sequences
\[
(\tilde{J}^1, \ldots, \tilde{J}^n) \in \mathcal{J}(J, J_1) \times \cdots \times \mathcal{J}(J_{n-1}, J'),
\]
where \( J_1, \ldots, J_{n-1} \) are cylindrical complex structures, modulo the equivalence relation
\[
(\tilde{J}^1, \ldots, \tilde{J}^{i-1}, \tilde{J}^i, \tilde{J}^{i+1}, \ldots, \tilde{J}^n) \sim (\tilde{J}^1, \ldots, \tilde{J}^{i-1}, \tilde{J}^{i+1}, \ldots, \tilde{J}^n),
\]
whenever \( \tilde{J}^i \) is a constant path. Then we have a natural composition map
\[
\circ : \overline{\mathcal{J}}(J, J') \times \overline{\mathcal{J}}(J', J'') \rightarrow \overline{\mathcal{J}}(J, J''),
\]
defined by concatenation which makes \( \overline{\mathcal{J}} \) into a category. This category can be given a natural topology, which turns it into a topological category; see Section 3.2 of [HLS] for details.
Note that, since the space \( \mathcal{J}(J,J') \) are weakly contractible, we can also consider continuous multi-parameter families of elements in \( \mathcal{J}(J,J') \) and consider them as higher morphisms in the category \( \mathcal{J} \). Thus we can define the notion of homotopy coherent diagrams in \( \mathcal{J} \) as follows.

**Definition 3.1.** Given a small category \( \mathcal{C} \), a homotopy coherent \( \mathcal{C} \)-diagram \( F \) in \( \mathcal{J} \) consists of the following data.

(i) For each \( x \in \text{ob}(\mathcal{C}) \), an object \( F(x) \) of \( \mathcal{J} \),

(ii) For each integer \( n \geq 1 \) and each composable sequence of morphisms \( f_1, \ldots, f_n \) in \( \mathcal{C} \), i.e. \( f_{i+1} \circ f_i \) is defined for each \( i \), a continuous family

\[
F(f_n, \ldots, f_1) : [0,1]^{n-1} \to \mathcal{J}(F(x_0), F(x_1)),
\]

so that the conditions in Definition 3.3 of [HLN] are satisfied.

The source category \( \mathcal{C} \) which we will use frequently is the groupoid \( \mathcal{E}Z_2 \), which has two objects \( a \) and \( b \), and four morphisms, as follows.

- \( \text{Hom}_\mathcal{C}(x,x) = \{ \text{id}_x \} \) for \( x = a, b \),
- \( \text{Hom}_\mathcal{C}(a,b) = \{ \alpha \} \) and \( \text{Hom}_\mathcal{C}(b,a) = \{ \beta \} \).

**The freed Floer complex.**

**Definition 3.2.** Given an morphism \( \tilde{J} = (\tilde{J}^1, \ldots, \tilde{J}^n) \in \mathcal{J} \), where each \( \tilde{J}^i \in \mathcal{J}(J_{i-1}, J_i) \) is nonconstant, and points \( x, y \in L_0 \cap L_1 \), a \( \tilde{J} \)-holomorphic disk from \( x \) to \( y \) is a sequence

\[
(x^{i,j-1}, \ldots, x^{i,1}, x^i, \ldots, x^{i,0}, y) \in \mathcal{T},
\]

where \( m_0, \ldots, m_n \) are nonnegative integers and the following conditions are satisfied.

- Each \( x^{i,j} = x^{i,j} \) is a \( J_i \)-holomorphic Whitney disk with boundary on \( L_0 \) and \( L_1 \), connecting some points \( x^{i,j-1} \) and \( x^{i,j} \) in \( L_0 \cap L_1 \).
- Each \( x^i = x^i \) is a \( J_i \)-holomorphic Whitney disk with boundary on \( L_0 \) and \( L_1 \), connecting some points \( x^{i,j-1} \) and \( x^i \) in \( L_0 \cap L_1 \).
- \( x^{i,0} = x^i \) for \( i \geq 1 \), \( x^{i,m_i} = x^{i+1} \) for all \( i \), \( x^{0,0} = x \), and \( x^{n,m_n} = y \).

Given a map \( \tilde{J} : [0,1]^k \to \mathcal{J} \), points \( x, y \in L_0 \cap L_1 \), and a homotopy class \( \phi \in \pi_2(x,y) \), let \( \mathcal{M}(x,y;\tilde{J}) \) denote the moduli space of pairs \((p,u)\) where \( p \in [0,1]^k \) and \( u \) is a \( \tilde{J}(p) \)-holomorphic disk from \( x \) to \( y \), and \( \mathcal{M}(\phi;\tilde{J}) \) denotes the subspace of disks representing the class \( \phi \). Then we have a splitting, as follows.

\[
\mathcal{M}(x,y;\tilde{J}) = \coprod_{\phi \in \pi_2(x,y)} \mathcal{M}(\phi;\tilde{J})
\]

Since \( \tilde{J} \) is a \( k \)-parameter family, the expected dimension of \( \mathcal{M}(\phi;\tilde{J}) \) is given by \( \mu(\phi) + k \). When a homotopy coherent diagram \( F : \mathcal{C} \to \mathcal{J} \) is sufficiently generic, i.e. for any points \( x, y \in L_0 \cap L_1 \), a homotopy class \( \phi \in \pi_2(x,y) \), and a composable sequence of morphisms \( f_1, \ldots, f_n \) in \( \mathcal{C} \), the space \( \mathcal{M}(\phi;F(f_n, \ldots, f_1)) \) is transversely cut out, and its dimension is given by \( \mu(\phi) + n - 1 \).

Now, given a sufficiently generic homotopy coherent diagram \( F : \mathcal{C} \to \mathcal{J} \), for each \( a \in \text{ob}(\mathcal{C}) \), let \( G(a) \) be the Floer chain complex \( (CF(L_0, L_1), \partial_{F(a)}) \) with respect to the cylindrical complex structure \( F(a) \). For a composable sequence of morphisms \( f_1, \ldots, f_n \) in \( \mathcal{C} \), and a \( k \)-dimensional face \( \sigma \) of \( [0,1]^{n-1} \), we have a \( k \)-dimensional subfamily:

\[
F(f_n, \ldots, f_1)|_\sigma : [0,1]^k \to \mathcal{J}.
\]

So we define:

\[
G(f_n, \ldots, f_1)(\sigma \otimes x) = \sum_{y \in L_0 \cap L_1} \sum_{\phi \in \pi_2(x,y), \mu(\phi) = 1-k} |\mathcal{M}(\phi;F(f_n, \ldots, f_1)|_\sigma) \cdot y.
\]

Then \( G : \mathcal{C} \to \text{Kom}_{\mathcal{F}_2} \) turns out to be a homotopy coherent \( \mathcal{C} \)-diagram in the \((\infty-)\)category of complexes of \( \mathbb{F}_2 \)-vector spaces, in the following sense.

**Definition 3.3.** Given a small category \( \mathcal{C} \), a homotopy coherent \( \mathcal{C} \)-diagram \( F \) in \( \text{Kom}_{\mathcal{F}_2} \) consists of:

- For each \( x \in \text{ob}(\mathcal{C}) \), a chain complex \( F(x) \in \text{ob}(\text{Kom}_{\mathcal{F}_2}) \).
For each $n \geq 1$ and each composable sequence $f_1, \ldots, f_n$ of morphisms in $C$, a chain map

$$G(f_n, \ldots, f_1) : I^*_{i-1} \otimes G(x_0) \to G(x_n),$$

where $I_* = C_*^{\text{simplicial}}([0,1])$, such that $G(f_n, \ldots, f_1)(t_1 \otimes \cdots \otimes t_{n-1})$, where $t_i \in I_*$, is equal to:

- $G(f_n, \ldots, f_2)(\pi(t_1) \otimes t_2 \otimes \cdots \otimes t_{n-1})$ if $f_1 = [0,1]$, where $\pi : I_* \to F_2$ is the map induced the projection $[0,1] \to \{pt\}$;
- $G(f_n, \ldots, f_i+1, f_i-1, \ldots, f_1)(t_1 \otimes \cdots \otimes m(t_{i-1} \otimes t_i) \otimes \cdots \otimes t_{n-1})$ if $f_i = [0,1]$ and $1 < i < n$, where $m : I_* \otimes I_* \to I_*$ is the map induced by the multiplication $[0,1] \times [0,1] \to [0,1]$;
- $G(f_{n-1}, \ldots, f_1)(t_1 \otimes \cdots \otimes t_{n-2} \otimes \pi(t_{n-1}))$ if $f_n = [0,1]$;
- $G(f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1)(t_1 \otimes \cdots \otimes t_{i-1} \otimes t_{i+1} \otimes \cdots \otimes t_{n-1})$ if $t_i = \{1\}$;
- $G(f_n, \ldots, f_{i+1})(t_{i+1} \otimes \cdots \otimes t_{n-1}) \circ G(f_i, \ldots, f_1)(t_1 \otimes \cdots \otimes f_{i-1})$ if $t_i = \{0\}$.

Since $G$ is homotopy coherent, we can consider its homotopy colimit $\text{hocolim} G$, which is a single chain complex of $\mathbb{F}_2$-vector spaces.

**Definition 3.4.** When $C = \mathcal{E} \mathbb{Z}_2$ and the homotopy coherent diagram $F : \mathcal{E} \mathbb{Z}_2 \to \mathcal{J}$ is $\mathbb{Z}_2$-equivariant and sufficiently generic, the chain complex $\text{hocolim} G$ is defined as the freed Floer complex, and denoted as $\text{CF}_{\mathbb{Z}_2}(L_0, L_1)$. Since we have a natural $\mathbb{Z}_2$-action on the freed Floer complex, the complex

$$\text{CF}_{\mathbb{Z}_2}(L_0, L_1) = \text{Hom}_{\mathbb{F}_2[\mathbb{Z}_2]}(\text{CF}_{\mathbb{Z}_2}(L_0, L_1), \mathbb{F}_2)$$

is defined as the equivariant Floer cochain complex, and its cohomology

$$H\text{F}_{\mathbb{Z}_2}(L_0, L_1) = H^*(\text{CF}_{\mathbb{Z}_2}(L_0, L_1))$$

is defined as the equivariant Floer cohomology. In the category $\mathcal{E} \mathbb{Z}_2$, any composable sequence of morphisms is either of the form $\alpha_n = (\alpha, \beta, \alpha, \cdots)$ or of the form $\beta_n = (\beta, \alpha, \beta, \cdots)$. Thus the elements of the freed Floer chain complex $\text{CF}_{\mathbb{Z}_2}(L_0, L_1)$ are of the form $\alpha_n \otimes x$ or $\beta_n \otimes x$ for $x \in L_0 \cap L_1$. The differential is of the following form.

$$\partial(\alpha_n \otimes x) = \alpha_n \otimes (\partial x) + \beta_{n-1} \otimes x + \sum_{i=1}^{n} \alpha_{n-i} \otimes \begin{cases} G_{\alpha,\alpha,\cdots}(x) & \text{if } n-i \text{ is odd} \\ G_{\alpha,\beta,\cdots}(x) & \text{if } n-i \text{ is even} \end{cases}$$

$$\partial(\beta_n \otimes x) = \beta_n \otimes (\partial x) + \alpha_{n-1} \otimes x + \sum_{i=1}^{n} \beta_{n-i} \otimes \begin{cases} G_{\alpha,\beta,\cdots}(x) & \text{if } n-i \text{ is odd} \\ G_{\beta,\beta,\cdots}(x) & \text{if } n-i \text{ is even} \end{cases}$$

Here, $G_{\alpha,\alpha,\cdots}(x)$ and $G_{\beta,\alpha,\cdots}(x)$ can be evaluated by counting holomorphic disks of Maslov index $2 + i - n$ from $x$. Note that the holomorphic disks which we are counting here are the ones defined in $32$.

The $\mathbb{Z}_2$-action on the freed Floer chain complex is given by

$$\tau(\alpha_n \otimes x) = \beta_n \otimes \tau x,$$

where $\mathbb{Z}_2 = \langle \tau \rangle$. Hence the elements of the equivariant Floer cochain complex $C_{\mathbb{Z}_2}(L_0, L_1)$ are of the form $\theta^n \otimes x^*$, where $\theta$ is a formal variable and $x^*$ is the dual of a given Floer generator $x$, and its differential is given by

$$d(\theta^n \otimes x^*) = \theta^n \otimes dx^* + \theta^{n+1} \otimes \tau x^* + \sum_{i=1}^{\infty} \theta^{n+i} \otimes (x^* \circ G_{\alpha,\beta,\cdots}).$$

We have an action of $\mathbb{F}_2[\theta]$ on $C_{\mathbb{Z}_2}(L_0, L_1)$, given as follows.

$$\theta \cdot (\theta^n \otimes x^*) = \theta^{n+1} \otimes \tau x^*$$

Since the differential of $C_{\mathbb{Z}_2}(L_0, L_1)$ is $\theta$-equivariant, we get a natural $\mathbb{F}_2[\theta]$-module structure on $H\text{F}_{\mathbb{Z}_2}(L_0, L_1)$.

The quasi-isomorphism type of $C_{\mathbb{Z}_2}(L_0, L_1)$, and thus the $\mathbb{F}_2[\theta]$-module isomorphism type of $H\text{F}_{\mathbb{Z}_2}(L_0, L_1)$, turns out to be invariant under the choice of sufficiently generic $\mathbb{Z}_2$-equivariant diagrams in $\mathcal{J}$, $\mathbb{Z}_2$-equivariant Hamiltonian isotopies, and non-$\mathbb{Z}_2$-equivariant Hamiltonian isotopies which satisfy the conditions in Proposition 3.25 of [HLS]. The proof is given in Proposition 3.23, 3.24, and 3.25 of [HLS].
When a generic $\mathbb{Z}_2$-equivariant cylindrical complex structure achieves transversality for all Whitney disks of Maslov index at most 1, then we have the following isomorphism.

$$CF_{\mathbb{Z}_2}(L_0, L_1) \cong CF(L_0, L_1) \otimes F_2^1 \mathbb{Z}_2$$

**Diffeomorphism maps.** Given a $\mathbb{Z}_2$-equivariant symplectomorphism $\phi : M \to M'$ which sends $\mathbb{Z}_2$-invariant Lagrangians $L_0, L_1 \subset M'$ to $L_0' = \phi(L_0)$ and $L_1' = \phi(L_1)$, we have a naturally defined chain isomorphism

$$\phi_* : CF_{\mathbb{Z}_2}(L_0, L_1; J) \to CF_{\mathbb{Z}_2}(L_0', L_1'; \phi(J)),$$

for homotopy coherent diagrams $J$, when $(M, L_0, L_1)$ satisfies Hypothesis 3.2 of [HLS]. Hence we get a natural map between equivariant Floer cohomology.

$$\phi^* : HF_{\mathbb{Z}_2}(L_0', L_1', \phi(J)) \xrightarrow{\sim} HF_{\mathbb{Z}_2}(L_0, L_1, J)$$

**Equivariant triangle maps.** Suppose that $L_0$, $L_0'$, and $L_1$ are $\mathbb{Z}_2$-invariant Lagrangians, which are pairwise transverse, and there is a $\tau$-invariant $\omega$-compatible almost complex structure $J$ on $M$ which achieves transversality for all moduli spaces of holomorphic disks with boundary on $(L_0, L_0')$ of Maslov index at most 1. Fix a cocycle $\Theta \in CF(L_0, L_0')$, which is $\mathbb{Z}_2$-invariant. As in the proof of Proposition 3.25 of [HLS], when $L_0 \cap L_0' \cap L_1 = \emptyset$, define a category $\mathcal{D}$ as follows. (The case when $L_0 \cap L_0' \cap L_1$ is nonempty can be done by extending $\mathcal{D}$ to include all possible Hamiltonian perturbations of $(L_0, L_0', L_1)$)

- $\text{ob}(\mathcal{D}) = \{0, 1\} \times \text{ob}(\overline{J})$.
- For $(i, J), (i', J') \in \{0, 1\} \times \text{ob}(\overline{J})$, $\text{Hom}_\mathcal{D}(i, J), (i', J') = \text{Hom}_{\overline{J}}(J, J') = \mathcal{J}(J, J')$.
- $\text{Hom}_\mathcal{D}(0, J), (1, J')$ is the space of sequences $(\overline{J}_i, \overline{J}_j, J_0, J_1, J_2, \ldots, J_j)$ where:
  - For $k \neq 0, J_k \in \mathcal{J}(J_k, J_{k+1})$ for some sequence $J_{i-1}, \ldots, J_{j+1}$ of cylindrical complex structures.
  - $J_{i-1} = J$ and $J_{j+1} = J'$.
  - $J_0 \in \mathcal{J}_\Delta$ agrees with $J_0$ on some cylindrical neighborhood $[n, \infty) \times [0, 1]$ of $p_2$, $J_1$ on some cylindrical neighborhood of $[n, \infty) \times [0, 1]$ of $p_3$, and $J$ on some cylindrical neighborhood $[n, \infty) \times [0, 1]$ of $p_1$.

Here, $\mathcal{J}_\Delta$ is the space of almost complex structures parametrized by $\Delta$, where $\Delta$ is a disk with three boundary punctures $p_1, p_2, p_3$, together with identifications of a small closed neighborhood of $p_i$ with $[n, \infty) \times [0, 1]$. Then, like $\overline{J}$, the category $\mathcal{D}$ also becomes a topological category.

For families $\tilde{J} : [0, 1]^\ell \to \mathcal{J}_\Delta$ and a homotopy class $\phi$ of triangles in $(L_0, L_0', L_1)$, we can consider the moduli space

$$\mathcal{M}(\phi; \tilde{J}) = \cup_{t \in [0, 1]^{\ell}} \mathcal{M}(\phi; \tilde{J}(t)).$$

For generic $\tilde{J}$, the moduli space $\mathcal{M}(\phi; \tilde{J})$ is transversely cut out, and so we can define a map

$$G(\tilde{J}) : CF(L_0, L_1; J_0) \to CF(L_0', L_1; J_1)$$

by the following equation. Here, $p \in \Theta$ means that $p$ runs over all Floer generators appearing in the given cocycle $\Theta$.

$$G(x) = \sum_{\tilde{J} : \mathcal{J}_\Delta} \sum_{\phi \in \pi_2(x, y, p, \mu(\phi) = -\ell)} \left| \mathcal{M}(\phi; \tilde{J}) \right| \cdot y.$$
This gives a map $G'' : CF_{Z_2}(L_0, L_1; F) \rightarrow CF_{Z_2}(L_0', L_1', F')$. Since it is $\mathbb{Z}_2$-equivariant, we get the equivariant triangle map between equivariant Floer cochain complexes:

$$F : CF_{Z_2}(L_0, L_1; F) \rightarrow CF_{Z_2}(L_0', L_1', F').$$

### Equivariant Floer cohomology of branched double covers of $S^3$.

A based link is a pair $(L, p)$ where $L$ is a link in $S^4$ and $p \in L$ is a choice of a basepoint. Given a genus 0 Heegaard surface $\Sigma \subset S^3$, a based link $(L, p)$ is in a bridge position with respect to $\Sigma$ if, for a Heegaard splitting $S^3 = H_a \cup_\Sigma H_b$, the connected arcs $\{a_i\}$ and $\{b_j\}$, $1 \leq i \leq n$, given by

$$\cup a_i = L \cap H_a, \cup b_i = L \cap H_b,$$

satisfy the following conditions.

- There exist disks $D_{a_i}$ and $D_{b_j}$ such that $a_i \subset \partial D_{a_i} \subset a_i \cup \Sigma$ and $b_j \subset \partial D_{b_j} \subset b_j \cup \Sigma$. 

- The disks $D_{a_i}$ and $D_{b_j}$ can be chosen to have pairwise disjoint interiors.

- $p \in L \cap \Sigma$.

If $\partial D_{a_i} = a_i \cup A_i$ and $\partial D_{b_j} = b_j \cup B_j$ where $A_i$ and $B_j$ are simple arcs on $\Sigma$, we say that $(\{A_i\}, \{B_j\})$ is the bridge diagram for the based link $(L, p)$. Given a bridge diagram, by taking the branched double cover of the whole diagram, with the branching locus given by $L \cap \Sigma$, and removing the curves which contain the basepoint $p$, gives a Heegaard diagram $(\tilde{\Sigma}, \alpha, \beta, p)$ together with the covering $\mathbb{Z}_2$-action, where the alpha(beta)-curves are given by the inverse images of the arcs $A_i(B_j)$.

The $\mathbb{Z}_2$-equivariant Floer cohomology theory can be applied to the symplectic $\mathbb{Z}_2$-action on the symmetrized power $(\text{Sym}^3(\tilde{\Sigma} - \{p\}), T_\alpha, T_\beta)$. It turns out that the $F_2[\theta]$-isomorphism class of the equivariant Floer cohomology

$$HF_{Z_2}(\Sigma(L), p) = HF_{Z_2}(\tilde{T}_\alpha, \tilde{T}_\beta)$$

is an invariant of the isotopy class of $(L, p)$.

The proof of the invariance uses the fact that any two bridge diagram of a based link are related by three types of moves: isotopy, handleslide, and stabilization. An isotopy of bridge diagram $(\{A_i\}, \{B_j\})$ is an isotopy of each arc $A_i$ and $B_j$, while fixing their endpoints. A handleslide is a move which replaces $A_i$ (or $B_j$) by $A_i'$ (or $B_j'$) with the same endpoints, when there exists another arc $A_k$ such that $A_i$ and $A_k$ bound a disk $D$ which contains $A_k$, and $D$ does not intersect any other A-arcs(or B-arcs). Finally, a stabilization is a move which adds an A-arc and an B-arc to an arc $A_i$ (or $B_j$) near one of its endpoint; see Figure 13 of [HLS] for details.

An isotopy of an arc induces a Hamiltonian isotopy of an alpha(beta)-curve on the branched double cover, which induces an isomorphism of the equivariant Floer cohomology $HF_{Z_2}(\Sigma(L), p)$. A handleslide also induces an isomorphism of the equivariant Floer cohomology, by Proposition 3.25 of [HLS]. A stabilization can be seen as a combination of a creation of an unknot and a saddle move, which can be translated as a composition of a stabilization map, i.e. performing a connected sum with a genus 1 Heegaard diagram of $S^3$, followed by an equivariant triangle map. The proof that this also induces an isomorphism of equivariant cohomology is given in the proof of Theorem 1.24 in [HLS].

**Remark.** The equivariant triangle map can also be used to construct a cobordism map in equivariant Floer cohomology, using the construction given in Lemma 6.10 of [HLS]. It is constructed by slicing a given cobordism into basic pieces, i.e. cylinders, births, deaths, and saddles. Saddles correspond to equivariant triangle maps, births/deaths correspond to the stabilization/destabilization of the surface $\tilde{\Sigma}$, and cylinders correspond to isotopy maps. After isotoping a given bridge diagram as drawn in Figure 4 of [HLS], this saddle map becomes the map induced by the surgery cobordism map between the ordinary Heegaard Floer homology. We will show in this paper that this cobordism map is independent of all auxiliary choices, and thus is well-defined.

4. **Weak admissibility and naturality of $\widehat{HF}_{Z_2}$**

Recall that, to define hat-versions of HF-groups and the maps between them, we need to assure that all diagrams we consider satisfy weak admissibility. In particular, to define the Floer chain complex, we need weak admissibility for Heegaard diagrams. We also need admissibilities for higher polygons; the triangle maps
need weakly admissible triple-diagrams and the proof of its associativity needs weakly admissible quadruple-diagrams. Counting of higher polygons is not needed.

In this section, we will prove that all Heegaard (double, triple, quadruple)-diagrams that we will use in this paper will be weakly admissible, thus allowing us to freely use all hat-flavored aspects of Heegaard Floer theory. The Heegaard diagrams that we will use are involutive ones, which we will define as follows.

**Definition 4.1.** A Heegaard (double-)diagram $(\Sigma, \alpha, \beta, z)$ is involutive (with respect to a orientation-preserving involution $\tau$) if $\tau$ fixes $z$ and the alpha- and beta-curves setwise and $\tau|_{\alpha_i}, \tau|_{\beta_j}$ is orientation-reversing with two distinct fixed points.

A Heegaard triple-diagram $(\Sigma, \alpha, \beta, \gamma, z)$ is involutive with respect to $\tau$ if it is a small perturbation of a diagram $(\Sigma, \alpha_0, \beta_0, \gamma_0, z)$ such that the tuples $(\Sigma, \alpha_0, \beta_0, z), (\Sigma, \beta_0, \gamma_0, z), (\Sigma, \alpha_0, \gamma_0, z)$ are involutive with respect to $\tau$.

A Heegaard quadruple-diagram is involutive if it is a small perturbation of a diagram such that the triple-diagrams given by choosing any three of the curve systems among the given four are involutive with respect to $\tau$.

For simplicity, we usually do not specify the action of an involution $\tau$ in figures, unless necessary. An example of possible local pictures of Heegaard triple-diagrams and quadruple-diagrams around the intersection points $\alpha_0 \cap \beta_0 \cap \gamma_0 (\cap \delta_0)$ are drawn in Figure 4.1. Note that any diagram given by a small perturbation can be obtained by permuting the labels of $\alpha, \beta, \cdots$ in the figures.

**Figure 4.1.** Possible local pictures of nice Heegaard diagrams. A choice of a perturbation changes the figures on the left to the figures of nice diagrams, shown on the right.

**Lemma 4.2.** Every involutive Heegaard diagram is weakly admissible.
Proof. Suppose that a nontrivial periodic domain $D$ with nonnegative coefficients is given. If we denote the involution by $\tau$, the domain $\tilde{D} = D + \tau_* D$ is nontrivial, periodic, has nonnegative coefficients and is $\tau$-invariant. Thus, in a neighborhood of any $\tau$-invariant point $p \in \alpha_i \cap \beta_j$, the domain $\tilde{D}$ should have coefficients as described in Figure 4.2.

By periodicity, $2a = 2b$, i.e. $a = b$. But then, near an adjacent intersection point $q \in \alpha_i \cap \beta_k$, the domain $\tilde{D}$ is given as in Figure 4.3. By periodicity, $a + c = a + d$, i.e. $c = d$. Continuing in this manner, we see that if two components of $\Sigma - \cup \alpha_i - \cup \beta_j$ share a segment of a beta-curve, then the coefficients of $\tilde{D}$ for those components are the same. But by the same argument, we can prove the same for components sharing a segment of a alpha-curve. This implies that all coefficients of $\tilde{D}$ should be the same. Since $n_z(\tilde{D}) = 0$, we deduce that $\tilde{D} = 0$. Contradiction. 

\[\begin{array}{c}
\alpha \\
Z_2 \\
\beta \\
b \\
\end{array}\]

Figure 4.2. The periodic domain $\tilde{D}$ near an invariant intersection point.

\[\begin{array}{c}
\alpha \\
a \\
\beta \\
c \\
d \\
b \\
\end{array}\]

Figure 4.3. The periodic domain $\tilde{D}$ near a non-invariant intersection point.

Lemma 4.3. Every involutive Heegaard triple-diagram $\mathcal{D} = (\Sigma, \alpha, \beta, \gamma, z)$ is weakly admissible.

Proof. Suppose that a nontrivial triply-periodic domain $D$ with nonnegative coefficients in $\mathcal{D}$ is given. If we denote the involution by $\tau$, $\tau_* D$ would be well-defined away from the triple intersections. A typical local picture near a triple intersection point is drawn in Figure 4.4.

We denote the coefficients by $a, b, c, d, e, f, g$ as in Figure 4.4 and claim that there exists an integer $g'$ which makes the domain described in Figure 4.5 achieve periodicity. Define $g'$ as $g' = a + c - b$. Then we
have
\[ c + e - d = b + g' - a + e - d = g' + g - f + f - g = g', \]
and similarly \( a + c - f = g' \), so our choice of \( g' \) makes the domain periodic; denote the resulting periodic domain by \( \tau_* D \). Then \( \tau_* D \) may not have nonnegative coefficients, since we do not know whether \( g' \geq 0 \) holds. However, we have
\[
\begin{align*}
g + g' &= g + a + c - b \\
&= g + f + b - g + d + b - g - b \\
&= f + d + b - g \\
&= e + b \geq 0.
\end{align*}
\]
Hence \( \tilde{D} = D + \tau_* D \) is a periodic domain with nonnegative coefficients. Also, since \( a, \cdots, f = 0 \) implies \( g = 0 \), we see that \( \tilde{D} \neq 0 \). Now, the local picture of \( D \) near triple intersections is given as in Figure 4.6. By periodicity, we get
\[
\begin{align*}
\tilde{a} + \tilde{b} &= \tilde{c} + \tilde{d}, \\
\tilde{a} + \tilde{c} &= \tilde{b} + \tilde{d}, \\
\tilde{a} + \tilde{d} &= \tilde{b} + \tilde{c}.
\end{align*}
\]
Thus \( \tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} \). Now, by the argument used to prove the previous lemma, we see that all coefficients of \( \tilde{D} \) are the same. Since \( n_z(\tilde{D}) = 0 \), we get \( \tilde{D} = 0 \), which is a contradiction. \( \Box \)

**Figure 4.4.** The periodic domain \( D \) near a non-invariant intersection point.

**Figure 4.5.** The periodic domain \( \tau_* D \) near a non-invariant intersection point.

**Lemma 4.4.** Every involutive Heegaard quadruple-diagram is weakly admissible.
Figure 4.6. The periodic domain $\tilde{D}$ near a non-invariant intersection point.

Proof. We continue to use the above approach and start from Figure 4.7 for a nontrivial quadruply-periodic domain $D$ with nonnegative coefficients. Suppose that we are given $a, b, c, d, e, f, g, h$ and try to find $x, y, z$ so that the resulting domain becomes periodic. If such $x, y, z$ exists, then we must have

\[
\begin{align*}
  y &= b + h - f, \\
  z &= b + g - d, \\
  x &= c + e - a.
\end{align*}
\]

The remaining three equations are

\[
\begin{align*}
  c + y - h - x &= 0, \\
  x + g - e - z &= 0, \\
  b + x - y - z &= 0.
\end{align*}
\]

Using the first three set of equations, we get the following.

\[
\begin{align*}
  0 &= c + y - h - x = c + b + h - f - h - c - e + a \\
  &= b - f - e + a, \\
  0 &= x + g - e - z = c + e - a + g - e - b - g + d \\
  &= c + d - a - b, \\
  0 &= b + x - y - z = b + c + e - a - b - h + f - b - g + d \\
  &= c + d + e + f - a - b - g - h.
\end{align*}
\]

Hence we get

\[
a + b = c + d = e + f = g + h,
\]

which we will call as the filling condition. Now, if a given $a, b, c, d, e, f, g, h$ satisfies the filling condition, we can reverse the above argument to deduce that there exists a unique choice of $x, y, z$ which makes the resulting domain periodic.

Here we notice that the filling condition is invariant with respect to the change

\[
a \leftrightarrow b, c \leftrightarrow d, e \leftrightarrow f, g \leftrightarrow h.
\]

This implies that there exists a unique choice of integers $x', y', z'$ making the domain described in Figure 4.8 periodic, where coefficients for all other domains are transformed by the involution $\tau$, which makes the given quadruple-diagram nice. Denote the resulting domain by $\tau_\ast D$. Then $\tilde{D} = D + \tau_\ast D$ has the local picture as in Figure 4.9 near any $\tau$-invariant intersection point, by the filling condition. Note that we obviously have $\tilde{D} \neq 0$. Now, by the argument used for Heegaard (double) diagrams, we deduce that all coefficients of $\tilde{D}$ are the same. Since $n_z(\tilde{D}) = 0$, we must have $\tilde{D} = 0$, which is a contradiction.

By the above results, we see that we can now freely talk about counting holomorphic disks, triangles, and squares in involutive diagrams. Proposition 3.25 of [HLS] proves that counting triangles of Maslov indices at most zero gives a map between equivariant Floer (co)homologies.
Choose a system of pairwise disjoint simple arcs $a_i^0$ in $D$, where $a_i^0$ connects $p_i$ to a boundary point. Pick a point $p_1$ among $p_1, \ldots, p_n$ and regard it as a basepoint; $z := p_1$ (such a system is called a **half-arc basis**). The arcs $a_i^0$ lift to nonseparating smooth simple arcs $a_i$ in $S$, which pass through $p_i$ and connect two points in $\partial S$. We claim that $\{a_i\}_{i \neq 1}$ is an arc basis on the surface $S$, so that the data $(S, \{a_i\}_{i \neq 1}, h, z = p_1)$ defines an open book diagram of $(M_B, \xi)$ in the sense of [HKM], which is invariant under the covering transformation. To prove this, recall that a pairwise disjoint system of simple arcs $\{a_i\} \subset S$ is called an arc basis if the two
endpoints of each $a_i$ lie on $\partial S$ and $S - \cup a_i$ is a disk. Since each $a_i$ is a lift of $a_0^i$ and $D - \cup_{i \neq 1} a_0^i$ is a disk with one branching point $p_i$ in its interior, its inverse image $S - \cup_{i \neq 1} a_i$ is the double cover of a disk branched along an interior point, which is a disk. This proves our claim.

**Example.** Suppose that we are given a disk with four marked points $p_1, \cdots, p_4$ in its interior and a half-arc basis $\{a_2^0, a_3^0, a_4^0\}$ given as in Figure 5.1.

![Figure 5.1. A disk with four marked points and a half-arc basis](image)

After taking the branched double cover, we get the twice-puncture torus with three arcs $a_2, a_3, a_4$. From Figure 5.2 We see that $S - (a_2 \cup a_3 \cup a_4)$ is a disk, i.e. $\{a_2, a_3, a_4\}$ gives an arc basis on $S$.

![Figure 5.2. The branched double cover.](image)

Thus, if we define the following $\alpha$- and $\beta$-curves on the Heegaard surface

$$
\Sigma = (S \times \{0, 1\}) / \partial S \times \{0, 1\},
$$
we get a $\mathbb{Z}_2$-invariant Heegaard diagram of $M_B$:
\[
\alpha_i = a_i \cup a_i,
\beta_i = b_i \cup \hat{h}(b_i),
\]
where $b_i$ is a slight perturbation of $a_i$ in positive direction along an orientation of $\partial S$ and $|a_i \cap b_i| = 1$. For simplicity, we denote the sets of $\alpha_i$ and $\beta_i$ as $\alpha$ and $\beta$. If we denote the half-arc basis which we have started with by $A$, then the contact element $EH(\xi_K, A)$ is defined as the element $\{p_1, \cdots, p_n\}$ in the Heegaard Floer cochain complex of $M_B$, as in section 3.1 of [HLS].

\[
EH(\xi_K, A) \in CF^*(\Sigma, \alpha, \beta, z) \simeq CF^*(M_B),
\]
which is a $\mathbb{Z}_2$-invariant cocycle. This element induces the following element in the equivariant Floer cochain complex
\[
EH_{\mathbb{Z}_2}(\xi_K, A) = \theta^0 \otimes EH(\xi_K) \in \hat{CF}_{\mathbb{Z}_2}(\Sigma, \alpha, \beta, z).
\]
When the choice of a half-arc basis $A$ is not important, we will drop $A$ and write $EH_{\mathbb{Z}_2}(\xi_K)$ for simplicity.

Remark. As we have seen above, given a half-arc basis $\{a_i^0\}$ and a monodromy $h$ of $D^2$ which fixes the points $p_i$, we can take its branched double cover along the points $p_i$ to get an arc basis $\{a_i\}$ in the open book diagram $(S, \hat{h})$, and applying the Honda-Kazez-Matić construction to it gives a $\mathbb{Z}_2$-invariant Heegaard diagram. Now, if we apply the Honda-Kazez-Matić construction directly to the given half-arc basis, what we get is a bridge diagram of a link in $S^3$, drawn on a genus 0 Heegaard surface, as follows.

\[
S^2 \simeq \Sigma = (D^2 \times \{0,1\})/(\partial D^2 \times \{0,1\}),
\]
\[
A_i = a_i^0 \cup a_i^0,
B_i = b_i^0 \cup \hat{h}(b_i^0).
\]
Here, $b_i^0$ is a slight perturbation of $a_i^0$ along the positive direction of $\partial D^2$, so that $a_i^0$ and $b_i^0$ intersect only at the endpoint of $a_i^0$ which lies in the interior of $D^2$. Then, taking its branched double cover along the set $\{p_1, p_2, \cdots\} \times \{0,1\}$ also gives a $\mathbb{Z}_2$-invariant Heegaard diagram. The two Heegaard diagrams we get are identical. To summarize, we have a following commutative diagram of objects which we consider in this paper.

\[
\text{half-arc diagram} \xrightarrow{\text{branched double cover}} \text{arc diagram}
\]

\[
\text{HKM construction} \quad \text{bridge diagram} \xrightarrow{\text{branched double cover}} \text{Heegaard diagram} \quad \text{HKM construction}
\]

Now we argue that, for a generic almost complex structure $J$ on $S^2$, the symmetric product of the lifted structure $\hat{J}$ on $\Sigma$ achieves equivariant transversality.

**Theorem 5.1.** For a generic 1-parameter family of almost complex structures $J$ on $S^2$, the $\mathbb{Z}_2$-equivariant cylindrical complex structure $\text{Sym}^g(\hat{J})$ on $\text{Sym}^g(\Sigma)$, where $g$ is the genus of $\Sigma$, achieves equivariant transversality, in the sense of [HLS].

**Proof.** Since the $\mathbb{Z}_2$-invariant locus $(T_a \cap T_b)^{inv}$ consists of 0-dimensional components of $(\text{Sym}^g(\Sigma))^{inv}$ (see Section 6.1 of [HLS] for details), for any choice of an almost complex structure $J$ on $\text{Sym}^g(\Sigma)$, any $J$-holomorphic disk connecting two points in $T_a \cap T_b$ is not completely contained in $(\text{Sym}^g(\Sigma))^{inv}$. Thus, as in the proof of Proposition 5.13 in [KS], the argument used in the proof of Corollary 1.12 in [HLS] actually gives transversality for all homotopy classes of Whitney disks in this case. $\square$

The above theorem tells us that we only have to work with $\mathbb{Z}_2$-invariant almost complex structures of the form $\text{Sym}^g(\hat{J})$. For such almost complex structures, the argument in Section 3.1 of [HJM] tells us that there are no holomorphic disks going towards $EH(\xi_K)$. Thus, if we denote the generator of $\mathbb{Z}_2$ as $\tau$, the higher degree terms in the formula 3.1 vanishes, except for the term $\theta^1 \otimes EH(\xi_K)$ which comes from the constant disk of Maslov index 0. So the following equality holds.

\[
d_{\mathbb{Z}_2}(EH_{\mathbb{Z}_2}(\xi_K)) = \theta^0 \otimes d(EH(\xi_K)) + \theta^1 \otimes (EH(\xi_K) + \tau^* EH(\xi_K)) = 0.
\]
Hence $EH_{Z_2}(\xi_K)$ is a cocycle, i.e. defines a cohomology class
$$c_{Z_2}(\xi_K) := [EH_{Z_2}(\xi_K)] \in \widehat{HF}_{Z_2}(\Sigma(K)).$$

**Definition 5.2.** Given a half-arc basis $\{a^0_i\}_{i \neq 1}$ on a disk $D$, suppose that an (half-)arc $b$ starting at $p_i$ and ending in $\partial D$ satisfies the property that there exists a unique $j \neq i, 1$ such that $p_j$ is contained in the region bounded by $\partial D$, $a^0_j$, and $b$. If $a^0_j$ is also contained in that region, we say that $\{a^0_k\}_{k \neq 1, i} \cup \{b\}$ is obtained by performing a half-arc slide of $a^0_j$ along $a^0_j$.

![Figure 5.3. A picture describing a half-arc slide.](image)

**Proposition 5.3.** Let $\{a_i\}_{1 < i \leq n}, \{b_i\}_{1 < i \leq n}$ on a disk $D$ be two half-arc bases, where $a_i$ and $b_i$ connect an interior point $p_i \in \text{int}(D)$ with $\partial D$. Then they are related by a sequence of isotopies and half-arc slides.

**Proof.** We can isotope the arcs so that $\partial a_i = \partial b_i$ for all $1 < i \leq n$. Then, for each $k$, the closed curve $\gamma$ given by the concatenation of $a_k$ and $b_k$ gives an element of $\pi_1(D - \{p_1\}, p_k) \simeq \mathbb{Z}$. If the homotopy class of $[\gamma]$ is $n \cdot s$ times the generator, then we can apply $n|s|$ half-arc slides on $b_k$ so that $a_k$ is homotopic to $b_k$ in $D - \{p_1\}$. Thus we can apply this process for each $1 < k \leq n$, so that $a_k$ is homotopic to $b_k$ rel endpoints in $D - \{p_1\}$.

Now assume that, for any $i, j$ with $1 < i, j \leq n$, $a_i$ and $b_i$ intersect transversely, and denote the total number of intersection points between a-half-arcs and b-half-arcs by $N$. We can find a disk $D$ which is innermost, i.e. no half-arcs intersect its interior. Then, we can apply an isotopy along the disk to remove a pair of intersection points; the number of remaining intersection points in $N - 2$. Therefore, by induction on $N$, we see that, after a sequence of isotopies, we may assume that $a_i$ and $b_i$ cobound a disk $D_i$ for each $1 < i \leq n$, and the disks $D_i$ are pairwise disjoint. Then, we can isotope the half-arcs along the disks $D_i$ to isotope $a_i$ to $b_i$.

Now we will prove invariance of $c_{Z_2}(\xi_K)$ with respect to half-arc slides. Lifting the whole picture to $S$ shows that, in the branched double cover, a half-arc slide corresponds to an arcslide in the sense of [HKM], which then corresponds to an $\alpha$-handleslide followed by a $\beta$-handleslide. Hence, if $\{a_i\}$ is obtained by performing an arcslide to $\{a_i\}$, and $\alpha, \beta, \hat{\alpha}, \hat{\beta}$ are the associated $\alpha$- and $\beta$-curves on the invariant Heegaard surface $\Sigma$, then we have the following quasi-isomorphism, which is induced by a composition of an equivariant triangle map for an $\alpha$-handleslide followed by an equivariant triangle map for a $\beta$-handleslide:

$$\widetilde{CF}_{Z_2}(\Sigma, \hat{\alpha}, \hat{\beta}) \sim \widetilde{CF}_{Z_2}(\Sigma, \alpha, \beta).$$

Since this quasi-isomorphism is clearly $Z_2$-equivariant, we get the following induced quasi-isomorphism between equivariant Floer cochain complexes.

$$\widetilde{CF}_{Z_2}(\Sigma, \hat{\alpha}, \hat{\beta}) \sim \widetilde{CF}_{Z_2}(\Sigma, \alpha, \beta).$$
Theorem 5.4. The map (5.1) sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$.

Proof. Note that performing a half-arc slide to a half-arc basis corresponds to performing an arcslide to the induced arc-basis in the branched double cover. Thus the Heegaard triple-diagrams we get are the same as the diagrams which arise in the proof of the invariance of contact classes under arcslides, as in [HKM]. Since all holomorphic triangles involved are small (see the proof of Lemma 3.5 in [HKM]), all holomorphic triangles we count here have Maslov index 0, and their moduli spaces consist of a single point by Riemann mapping theorem. Therefore we deduce that $EH_{Z_2}(\xi_K)$ is mapped to $EH_{Z_2}(\xi_K)$. □

The invariance under perturbations of almost complex structures is proved similarly.

Theorem 5.5. The map induced by changing the choice of almost complex structures sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$.

Proof. The induced map, which is defined in the proof of Proposition 3.23 of [HLS], counts holomorphic disks going towards $EH$ with Maslov index at most 0. By Theorem 5.1 we can choose cylindrical complex structures of the form $\text{Sym}^n(\mathfrak{j})$ to compute equivariant Floer cohomology. However, by the argument of Section 3.1 in [HKM], such disks must intersect the basepoint. This completes the proof. □

Corollary 5.6. The map induced by an isotopy, from a half-arc basis $A_0$ to another basis $A$, sends $EH_{Z_2}(\xi_K,A)$ to $EH_{Z_2}(\xi_K,A_0)$.

Proof. An isotopy of half-arc basis can be replaced by a 1-parameter family of self-diffeomorphisms of $D^2$, starting from the identity. Such a family induces a 1-parameter family of cylindrical complex structures which we use to compute equivariant Floer cohomology. By Theorem 5.5 we see that the the induced isomorphism maps $EH_{Z_2}(\xi_K,A)$ to $EH_{Z_2}(\xi_K,A_0)$. □

Theorem 5.7. The map induced by an isotopy of the monodromy $h$ sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$.

Proof. Let $\{h_t\}$ be an isotopy of monodromy functions. As in Theorem 7.3 of [OS2], we can reduce to the case where the isotopy $\{h_t\}$ of self-diffeomorphisms of $\Sigma$ is a $Z_2$-equivariant Hamiltonian isotopy. Then we can apply the proof of Lemma 3.3 in [HKM] to deduce that the equivariant isotopy map sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$. □

Thus we proved the invariance under the choice of Floer-theoretic auxiliary data, so it remains to prove the invariance under a basepoint change and positive braid stabilization. Before proving the invariance under positive braid stabilization, we prove the invariance under the choice of a basepoint and stabilizations.

Theorem 5.8. The map induced by changing the (invariant) basepoint sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$.

Proof. According to the definition in the previous section, we choose the basepoint $z$ to be one of the points $p_1, \ldots, p_n$, which form the fixed locus of the $Z_2$-action. If we choose another such basepoint $z'$, then since our transverse braid forms a knot, there exists a positive integer $k$ satisfying $h^k(z) = z'$. Now applying the self-diffeomorphism $h^k$ to the open book diagram $(\Sigma, \{a_i\}, \hat{h}, z)$ gives $(\Sigma, \{h^k(a_i)\}, \hat{h}^k \hat{h}^{-k} = h, h^k(z) = z')$. But since we can always change the half-arc basis $\{h^k(a_i)\}$ back to $\{a_i\}$ via half-arc slides, we can change the arc basis $\{h^k(a_i)\}$ back to $\{a_i\}$ via arcslides, in the same manner. Since the maps induced by arcslides and diffeomorphisms preserve $EH^*_{Z_2}$, the theorem follows. □

Theorem 5.9. The map induced by a positive braid stabilization sends $EH_{Z_2}(\xi_K)$ to $EH_{Z_2}(\xi_K)$.

Proof. The induced map between $CF^*_{Z_2}$ is induced by taking $\text{RHom}$ at the following $Z_2$-equivariant quasi-isomorphism of chain complexes (see the proof of Theorem 1.24 in [HLS] for details):

$$\overline{CF}(\Sigma(B)) \rightarrow \overline{CF}(\Sigma(B \amalg \text{unknot})) \rightarrow \overline{CF}(\Sigma(B_+)),$$

where $B$ is the original braid and $B_+$ is its positive stabilization. Dualizing this gives

$$\overline{CF}^*(\Sigma(B_+)) \rightarrow \overline{CF}^*(\Sigma(B \amalg \text{unknot})) \rightarrow \overline{CF}^*(\Sigma(B)).$$

The second map preserves $EH$ by its definition. The first map is the saddle map induced by a Legendrian $(-1)$-surgery along a lift $c$ of a small Legendrian arc connecting $B$ with a trivial braid. The Heegaard triple
diagram for the saddle is drawn in Figure 5.4. Then, by the convenient placement of the basepoint, we see that all holomorphic triangles connecting \( x \) and \( \Theta \) are small. Therefore the induced isomorphism between \( \text{CF}^*_Z \) preserves \( \text{EH}_Z \). □

![Figure 5.4. The induced Heegaard triple-diagram.](image)

Combining these invariance theorems, we get the complete invariance of \( \text{EH}_Z(\xi K) \) and its cohomology class \( c_Z(\xi K) \).

**Theorem 5.10.** The cohomology class \( c_Z(\xi K) \in \widehat{HF}_Z(\Sigma(K)) \) depends only on the transverse isotopy class of the transverse knot \( K \).

**Proof.** By Theorem 5.5, the class \( c_Z(\xi K) \) is independent of the choice of almost complex structures. From Theorem 5.4, Theorem 5.7 and Corollary 5.6, we see that \( c_Z(\xi K) \) is invariant under isotopy and half-arc slide. Thus \( c_Z(\xi K) \) does not depend on the choice of half-arc basis by Proposition 5.3, which means that it only depends on the choice of a transverse braid representative of the given transverse knot \( K \). However, Theorem 5.9 tells us that \( c_Z(\xi K) \) is also invariant under positive braid stabilizations. Therefore, by Theorem 2.3, we deduce that \( c_Z(\xi K) \) is an invariant of the transverse isotopy class of \( K \). □

**Definition 5.11.** The class \( c_Z(\xi K) \), which is an invariant of the transverse isotopy class of \( K \) in \( (S^3, \xi_{std}) \), is called the equivariant contact class of \( (\Sigma(K), \xi_K) \).

**Equivariant contact classes of transverse links.** When we work with a multi-component transverse link \( L \), the same argument can be applied to establish the existence and the invariance of equivariant contact classes. However, we have a small issue with the choice of a basepoint; the equivariant contact class still depends on the component of \( L \) in which the basepoint lies. Hence, what we get is a cohomology class

\[
c_Z(\xi_L, z) \in \widehat{HF}_Z(\Sigma(L), z),
\]

which depends on the component of \( L \) in which \( z \) lies. Writing the basepoint \( z \) explicitly will be useful in the next section, where we deal with symplectic functoriality.
6. Naturality and Functoriality of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ when $K$ is a knot

In this section, we will prove that the equivariant Floer cohomology $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), p)$ is well-defined up to natural isomorphism, in the sense of [JT], so that it admits a natural action of the mapping class group $\text{MCG}(S^3, L) = \pi_0 \text{Diff}^+(S^3, L)$.

Recall that any two bridge diagrams of a given based link are related by isotopies, handleslides, and (de)stabilizations, applied to arcs which do not contain the basepoint.

**Definition 6.1.** Let $\{A_i\}, \{B_i\}$ denote the A- and B-arcs of a bridge diagram of a based link $(L, p)$. A basic move on $L$ is an isotopy, a handleslide, or a (de)stabilization applied to either a single A-arc or a single B-arc, which does not contain $p$. An A(B)-equivalence is an isotopy or a handleslide applied to a single A(B)-arc which does not contain $p$.

We can easily point out some naturally arising commutative triangles, squares, and hexagons, consisting of basic moves. Those diagrams are described below. Please note that, by an A(B)-arc, we mean an A(B)-arc of a given bridge diagram of a given based link, which does not contain the basepoint.

(a) *A-equivalences and B-equivalences commute with each other.* Given a bridge diagram $D = (\{A_i\}, \{B_i\})$ of a based link $(L, p)$, we can consider applying an A-equivalence on $A_i$ and a B-equivalence on $B_j$. Suppose that applying an A-equivalence on $A_i$ of $D$ transforms it into $D_a$ and applying a B-equivalence on $B_j$ of $D$ transforms it into $D_b$. Denote the result of applying both equivalences on $D$ gives $D_{ab}$. Then we have a following distinguished square.

\[
\begin{array}{ccc}
D & \rightarrow & D_b \\
A & \downarrow & A \\
D_a & \rightarrow & D_{ab}
\end{array}
\]

(b) *Commutative triangles of A(B)-equivalences.* Suppose that applying an A-equivalence on a bridge diagram $D$ gives $D_1$, applying another A-equivalence on $D_1$ gives $D_2$, and there exists an A-equivalence which transforms $D$ into $D_2$. Suppose further that, if two of the three A-equivalences are handleslides, then they are handleslides along the same A-arc. Then we get a distinguished triangle. The same thing also holds for B-equivalences.

\[
\begin{array}{ccc}
D & \rightarrow & D_1 \\
A & \downarrow & A \\
D_2
\end{array}
\]

(c) *Handleslides on different arcs commute.* Suppose that applying a handleslide on an A-arc $A_i$ of a bridge diagram $D$ gives $D_i$, applying a handleslide on an A-arc $A_j$ gives $D_j$, and applying both on $D$ gives $D_{ij}$. If $i \neq j$, then we have a distinguished square. The same thing also holds for B-equivalences.

\[
\begin{array}{ccc}
D & \rightarrow & D_i \\
A & \downarrow & A \\
D_j & \rightarrow & D_{ij}
\end{array}
\]

(d) *Commutative hexagon of handleslides.* Suppose that there are three A-arcs $A_i, A_j, A_k$ lying close to each other in a bridge diagram $D$, as in Figure 6.1. Then we have two choices when handlesliding $A_i$ and $A_j$ over $A_k$ to reach Figure 6.2; we can either move $A_j$ over $A_k$ first and then move $A_i$ over $A_k$ and $A_j$, or move $A_i$ over $A_j$ and $A_k$ first and then move $A_j$ over $A_k$.  


This gives a distinguished hexagon, and the same thing holds for B-arcs. Here, $A_i/A_j$ denotes the handleslide of $A_i$ over $A_j$.

(e) $A(B)$-equivalences commute with stabilizations. Suppose that applying an $A(B)$-equivalence on an $A(B)$-arc $A_i(B_j)$ in the bridge diagram $D$ gives $D_1$, and applying a stabilization on an arc of $D$, which does not contain the basepoint and is different from $A_i$, gives $D'$, and applying a stabilization on the corresponding arc of $D_1$ gives $D'_1$. Then $D'_1$ can be obtained from $D'$ by an $A(B)$-equivalence, and so we get a distinguished square.
(f) Stabilizations applied on different arcs commute. Consider applying stabilization on two different arcs of a bridge diagram $D$. There are two possible orders, which give us a distinguished square.

\[
\begin{array}{ccc}
D & \longrightarrow & D_2 \\
\downarrow s & & \downarrow s \\
D_1 & \longrightarrow & D_3
\end{array}
\]

(g) Commutative triangle of two stabilizations and an isotopy. Given a point $x \in \partial A_i \cap \partial B_j$ of a bridge diagram $D = (\{A_i\}, \{B_j\})$, suppose that applying a stabilization on $A_i$ near $x$ gives $D_1$ and applying a stabilization on $B_j$ near $x$ gives $D_2$. Then $D_1$ and $D_2$ differ by an isotopy. So we get a distinguished triangle.

\[
\begin{array}{ccc}
D & \longrightarrow & D_2 \\
\downarrow s & & \downarrow \text{isotopy} \\
D_1 & \longrightarrow & D
\end{array}
\]

Given a bridge diagram $D$ of a based link $(L, p)$, whose set of endpoints of arcs is given by $S \subset \Sigma$, we can also define maps on the equivariant Floer cohomology, associated to diffeomorphisms $\phi \in \text{Diff}^+(\Sigma)$ which fix $p$ and $S$ pointwise in a natural way. Then we get few more types of distinguished diagrams.

(h) Diffeomorphism and basic moves commute.

\[
\begin{array}{ccc}
D & \longrightarrow & \phi(D) \\
\downarrow \text{basic} & & \downarrow \text{basic} \\
D_1 & \longrightarrow & \phi(D')
\end{array}
\]

Here, we use diffeomorphisms $\phi \in \text{Diff}^+(\Sigma, S, p)$.

(i) Diffeomorphism can be undone by basic moves. Suppose that a diffeomorphism $\phi \in \text{Diff}^+(\Sigma, S, p)$ maps a bridge diagram $D$ of $(L, p)$ to $\phi(D)$. Since they represent the same based link, we can obtain $\phi(D)$ from $D$ by a sequence of basic moves. So we get a distinguished diagram.

\[
\begin{array}{ccc}
D & \longrightarrow & \phi(D) \\
\downarrow \text{diffeo} & & \downarrow \text{basic} \\
D_1 & \longrightarrow & \ldots & \longrightarrow & D_k
\end{array}
\]

Remark. In the paper [HLS], the authors define stabilizations on an arc only when the stabilization is applied near an endpoint of an arc and that endpoint is very close to the basepoint. However, we can similarly define stabilization maps even when the point where stabilizations occur is not close to the basepoint and the endpoints of arcs, by taking a family of almost complex structures of type $\text{Sym}^0(j)$ as in Theorem 5.1, where the 1-parameter family $j$ is split and has long neck near the point at which a stabilization is performed. Once we prove that the equivariant Floer cohomology satisfies the commutative squares of type (e), we can immediately deduce that such maps are indeed isomorphisms.

Lemma 6.2. Let $(S, \alpha, \beta)$ and $(S, \beta, \gamma)$ be $\mathbb{Z}_2$-Heegaard diagrams given by taking branched double covers of bridge diagrams of links drawn on a sphere $\Sigma$, and suppose that $\{\alpha_t\}_{t \in [0, 1]}$ is a $\mathbb{Z}_2$-invariant isotopy so that $\alpha_1 = \alpha$. Suppose further that $\alpha_t, \beta, \gamma$ are pairwise transverse for $t = 0$ and $t = 1$. Then, for each $\mathbb{Z}_2$-invariant cycle $\theta_{\beta, \gamma} \in \widehat{CF}(S, \beta, \gamma)$ and each element $x_{\alpha_0, \beta} \in H_s(\widehat{CF}_{\mathbb{Z}_2}(S, \alpha_0, \beta) \otimes \mathbb{F}_{\mathbb{Z}_2}[\mathbb{F}_2])$, we have

\[
\Gamma_{\alpha_1, \gamma}(\hat{f}_{\alpha_0, \beta, \gamma}(x_{\alpha_0, \beta} \otimes \theta_{\beta, \gamma})) = \hat{f}_{\alpha, \beta, \gamma}(\Gamma_{\alpha_1, \gamma}(x_{\alpha_0, \beta} \otimes \theta_{\beta, \gamma})),
\]

where $\hat{f}$ denote the equivariant triangle maps, as defined in the proof of Proposition 3.25 of [HLS], and $\Gamma$ denote the equivariant isotopy map.
Proof. Recall that we used a topological category $\mathcal{D}$ when constructing equivariant triangle map. Denote the three edges of the triangle $\Delta$ by $e_\alpha, e_\beta, e_\gamma$, and parametrize the edge $e_\alpha$ by a function $E_\alpha : [0,1] \to \Delta$. Given a parametrized family $J : [0,1]^t \to J_\Delta$, consider the moduli spaces

$$\mathcal{M}_t(J) = \bigcup_{t \in [0,1]^t} \left\{ u : \Delta \to \text{Sym}^2(\Sigma) \mid u \circ e_\alpha(t) \in T_{\mathcal{A}_{t,x+}}, u(e_\beta) \subset T_\beta, u(e_\gamma) \subset T_\gamma \right\},$$

$$\mathcal{M}(J) = \bigcup_{t \in \mathbb{R}} \mathcal{M}_t(J),$$

and split them into homotopy classes $\phi \in \pi_2^{T_{\mathcal{A}_{t,x}}} T_\beta, T_\gamma(x,y,z)$ for $x \in T_{\mathcal{A}_{t,x}} \cap T_\beta, y \in T_\beta \cap T_\gamma, z \in T_\gamma \cap T_{\mathcal{A}_{t,x}}$, as follows.

$$\mathcal{M}_t(J) = \bigcup_{\phi} \mathcal{M}_t(\phi; J), \quad \mathcal{M}(J) = \bigcup_{\phi} \mathcal{M}(\phi; J)$$

Then we define the map $G(J) : \widehat{CF}(S, \alpha_0, \beta) \to \widehat{CF}(S, \alpha, \gamma)$ as follows:

$$G(J)(x) = \sum_{x \in \mathcal{A}_t \cap \mathcal{A}_y} \sum_{y \in \mathcal{A}_\beta \cap \mathcal{A}_\gamma} \sum_{\phi \in \pi_2^{T_{\mathcal{A}_{t,x}}} T_\beta, T_\gamma(x,y,z), \mu(\phi) = -1 - \ell} \left| \mathcal{M}(\phi; J) \right| \cdot z.$$

As in the proof of Proposition 3.25 of [HLS], the function $G$ induces a map $F_G : \widehat{CF}_{\mathbb{Z}_2}(S, \alpha_0, \beta) \to \widehat{CF}_{\mathbb{Z}_2}(S, \alpha, \gamma)$.

There are three types of ends in the moduli space $\mathcal{M}(\phi; J)$ when $\mu(\phi) = -\ell$. The first type is the degeneration of the almost complex structure to the boundary $\tilde{J}[\partial[0,1]^t]$, which does not contribute to the total count of ends; since we are using $\mathbb{Z}_2$-equivariant diagrams of almost complex structures, the count of such ends must be even and hence zero in $\mathbb{F}_2$. The second and third types are the ones in the proof of Proposition 8.14 in [OSZ], which contribute to $\Gamma_{\{a_1\},, \gamma}(f_{a_0, \beta, \gamma}(x \otimes \theta_{\beta, \gamma})) + \hat{f}_{a_0, \beta, \gamma}(\Gamma_{\{a_1\},, \beta}(x) \otimes \theta_{\beta, \gamma})$ and $\partial F_G(x) + F_G(\partial x)$, respectively. Therefore we deduce that

$$\Gamma_{\{a_1\},, \gamma}(f_{a_0, \beta, \gamma}(x_{a_0, \beta} \otimes \theta_{\beta, \gamma})) + \hat{f}_{a_0, \beta, \gamma}(\Gamma_{\{a_1\},, \beta}(x_{a_0, \beta}) \otimes \theta_{\beta, \gamma}) = \partial F_G(x_{a_0, \beta}) + F_G(\partial x_{a_0, \beta}).$$

Since $\theta_{\beta, \gamma}$ is $\mathbb{Z}_2$-invariant, the map $F_G$ is also $\mathbb{Z}_2$-invariant. Therefore we get the desired result.

Lemma 6.3. Let $(S, \alpha, \beta)$ be a $\mathbb{Z}_2$-Heegaard diagram given by taking branched double cover of a bridge diagram of a link drawn on a sphere $\Sigma$, and suppose that $\{\alpha_t\}_{t \in [0,1]}$ is a $\mathbb{Z}_2$-invariant isotopy so that $\alpha_1 = \alpha$. Suppose further that $\alpha_1, \beta, \gamma$ are pairwise transverse for $t = 0, \frac{1}{2}, 1$. Then, for each element $x_{a_0, \beta} \in H_s(\widetilde{CF}_{\mathbb{Z}_2}(S, \alpha_0, \beta) \otimes_{\mathbb{Z}_2} \mathbb{F}_2)$, we have

$$\Gamma_{\{a_1\},, \gamma}(\Gamma_{\{a_1\},, \beta}(x_{a_0, \beta})) = \Gamma_{\{a_1\},, \gamma}(x_{a_0, \beta}),$$

where $\Gamma$ denotes the equivariant isotopy map.

Proof. As in the proof of 6.2 we can mimic the proof of Theorem 7.3 of [OSZ] to make it work in the equivariant setting.

Proposition 6.4. The equivariant Floer cohomology of based links in $S^3$ makes the distinguished diagrams of type (a)-(h) commutative.

Proof. By equivariant transversality and the above lemma, we only have to prove that the corresponding commutative diagrams of $\widetilde{CF}$ groups are satisfied up to $\mathbb{Z}_2$-equivariant chain homotopies.

For the distinguished diagrams of type (c), we already know that it is satisfied on the $\widetilde{CF}$ level, up to a chain homotopy. The chain homotopy is given by counting holomorphic squares, after a perturbation as in Figure 4 of [HLS]. Here, we can always perturb a given bridge diagram by an isotopy using Lemma 6.2. After making such a perturbation, we have no constant triangle of negative Maslov index, and thus the equivariant triangle map is induced by the ordinary triangle map with respect to generic 1-parameter families of almost
complex structures. Since holomorphic squares contained in the $\mathbb{Z}_2$-fixed locus are constant squares, and such squares have Maslov index 0, they are not counted in the square map. This implies, by the arguments of the proof of Proposition 5.13 in [KS], that a generic $\mathbb{Z}_2$-invariant 1-parameter family of almost complex structures achieves transversality for squares of Maslov index $-1$, which tells us that the holomorphic square map is well-defined for generic $\mathbb{Z}_2$-invariant families. Hence the given chain homotopy is $\mathbb{Z}_2$-equivariant, i.e. the given square diagram commutes up to $\mathbb{Z}_2$-equivariant chain homotopy on the $\hat{CF}$ level. Therefore we get a commuting square diagram of corresponding $\hat{HF}_{\mathbb{Z}_2}$ groups. The same argument can be used to prove the commutativity for distinguished squares of type (a), (f), and (h). Also, by Theorem 2.14 of [OSz], we can follow the proof of Lemma 2.15 in [OSz] to show that distinguished squares of type (e) also commute.

For the distinguished diagrams of type (b), the A-equivalence from $D$ to $D_1$ is given by evaluating the triangle map using a $\mathbb{Z}_2$-invariant cocycle $\Theta_{D_1,D_2}$, which represent the top class, and similarly consider $\mathbb{Z}_2$-invariant cocycles $\Theta_{D_1,D_2}$ and $\Theta_{D_2,D_2}$. By the technique used to prove the commutativity of distinguished diagrams of type (c), it suffices to prove that the image of $\Theta_{D_1,D_1} \otimes \Theta_{D_1,D_2}$ under the triangle map is the same as the cocycle $\Theta_{D_1,D_2}$. Since the image of $\Theta_{D_1,D_1} \otimes \Theta_{D_1,D_2}$ must also be a $\mathbb{Z}_2$-invariant cocycle which represents the top class, the proof will be finished if $\Theta_{D_1,D_2}$ is the only $\mathbb{Z}_2$-invariant cocycle which represents the top class. Since the Heegaard diagram for an isotopy or a saddle obviously admits a unique representative of its top class, we are done. The same argument can be used to prove the commutativity of distinguished squares of type (d) and (g).

Lemma 6.6. Given a bridge diagram $\{(A_i), (B_i)\}$ of a based link $(L,p)$ in $S^3$, drawn on a sphere $\Sigma = S^2$, let $\{p_1, \ldots, p_n\}$ be the set of endpoints of arcs $A_i$ and $B_i$, which are not equal to the basepoint $p$. Given a 1-parameter family of self-diffeomorphisms $\{\phi_t\}_{t \in [0,1]}$ of $\Sigma$, such that $\phi_0 = \text{id}_\Sigma$, each $\phi_t$ fixes $p$ pointwise and $\{p_1, \ldots, p_n\}$ setwise, and the images of $A$- and $B$-curves under $\phi_t$ intersect transversely with the original $A$- and $B$-curves, consider the following two maps. First, the isotopy map induced by the isotopy $\{\phi_t(A_i)\}$ and $\{\phi_t(B_i)\}$ of $A$- and $B$-arcs:

$$\Gamma : \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p).$$

Next, the diffeomorphism map

$$\phi^* : \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p).$$

Then we have $\Gamma = \phi^*$.

Proof. The argument used in the proof of Lemma 9.5 of [MT] and Proposition 9.8 of [OSz] directly generalizes to the equivariant setting. Hence we see that the lemma holds when the given isotopy $\{\phi_t\}$ is sufficiently small. Hence, by Lemma 6.3, we deduce that the lemma holds for any isotopy.

Lemma 6.6. Given a bridge diagram $\{(A_i), (B_i)\}$ of a based link $(L,p)$ in $S^3$, drawn on a sphere $\Sigma = S^2$, let $\{p_1, \ldots, p_n\}$ be the set of endpoints of arcs $A_i$ and $B_i$, which are not equal to the basepoint $p$. Given a diffeomorphism $\phi$ of $\Sigma$ which fix $p$ pointwise and $\{p_1, \ldots, p_n\}$ setwise, choose a sequence of basic moves from $\{(A_i), (B_i)\}$ to $\{(\phi(A_i)), (\phi(B_i))\}$, and denote the induced map between $\hat{HF}_{\mathbb{Z}_2}$ as follows:

$$T_\phi : \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(L), p).$$

Similarly construct a map $T_{\phi^{-1}}$ by choosing a sequence of basic moves from $\{(\phi(A_i)), (\phi(B_i))\}$ to $\{(A_i), (B_i)\}$. Then we have $T_\phi \circ T_{\phi^{-1}} = \text{id}$.

Proof. By Lemma 6.3, we can assume, without losing generality, that the given sequences of basic moves do not contain isotopies. Since the maps induced by A-equivalences and B-equivalences commute, it suffices to prove that the composition of all A-equivalence maps arising in $T_\phi$ and $T_{\phi^{-1}}$ is the identity, since it will also imply the same thing for B-equivalence maps.

The Heegaard diagram given by taking the branched double cover of $\Sigma \cup \{A_i\}$, $\{A'_i\}$ has unique $\mathbb{Z}_2$-invariant cocycle which represents the top class, where $A'_i$ is a slight perturbation of $A_i$ so that $A_i \cap A'_i = \partial A_i$. Therefore we can use the arguments in the proof of Proposition 6.4 to conclude that the composition of all A-equivalence maps in $T_\phi \circ T_{\phi^{-1}}$ is the identity, and so $T_\phi \circ T_{\phi^{-1}} = \text{id}$.

Proposition 6.7. The equivariant Floer cohomology of based knots in $S^3$ satisfies the commutative diagrams of type (i)
Proof. For any positive integer \( n \), the pure mapping class group of a disk with \( n \) punctures is given by the pure braid group on \( n \) strands, and taking its quotient by the Dehn twists along the boundary gives the mapping class group of a sphere with \( n + 1 \) punctures. Thus, if we consider the standard generating set \( \{ T_i \} \) of the pure braid group \( B_n \), where \( T_i \) denotes a positive twist of the \( i \)th and the \((i+1)\)th strand, the set \( \{ T_i^2 \} \) normally generates the pure braid group \( PB_n \), and thus generates the group \( \text{PMod}(S_{0,n+1}) \). Hence, by Lemma 6.6, given a bridge diagram of a based knot \((K,p)\) on a sphere \( \Sigma \), we only have to prove that the commutativity diagrams of type (i) holds for \( \hat{HF}_{\mathbb{Z}_2}(\Sigma(K),p) \) only for the full Dehn twists along an A-arc or a B-arc, which does not contain \( p \), and any choice of a sequence of basic moves.

Choose such an A-arc \( A_i \). The bridge diagram given by applying a Dehn twist along \( A_i \) is drawn in Figure 6.3. Its branched double cover admits a unique \( \mathbb{Z}_2 \)-invariant cycle which represents the top generator in homology. Thus we can compute the composition of the maps associated to our choice of basic moves by computing the equivariant triangle map for the Heegaard triple-diagram, which is given by taking the branched double cover of the diagram drawn in Figure 6.4. Note that \( A_i \) is assumed to not intersect any B-arcs other than the two B-arcs adjacent to it, and the basepoint is placed near the leftmost point in Figure 6.4. We claim that the only nonconstant triangles, each of which consists of a green arc, a blue arc, and a constant red arc, and involves at least one of the two endpoints of \( A_i \), are those shown in Figure 6.5. Let \( T \) be such a triangle. Then, without loss of generality, we may assume that it uses the blue arc and the green arc connected to the leftmost point in Figure 6.4. If \( T \) uses the leftmost red arc instead, then by the assumption on the placement of the basepoint, \( T \) must intersect the basepoint, so this case is impossible. Hence \( T \) must use the constant red arc at that point. Then, the triangle we get is the one shown in Figure 6.5.

Now, the other triangles are exactly the ones which arise when calculating the triangle map for the triple-diagram in Figure 6.6 except for the two shaded regions. Thus, the triangle map for the triple-diagram in Figure 6.4 is the composition of the triangle map for the triple-diagram in Figure 6.6 followed by the diffeomorphism map induced by the Dehn twist along \( A_i \).

However, using the argument of Proposition 9.8 in [OS], we see that the triangle map for Figure 6.6 agrees with the diffeomorphism map, induced by a diffeomorphism which is isotopic to the identity. Therefore we see that the triangle map must give the same result as the diffeomorphism map.

\[ \square \]

Figure 6.3. The diagram after applying a Dehn twist along (the boundary of a neighborhood of) \( A_i \)

For a based link \((L,p)\) in \( S^3 \) and a genus 0 Heegaard surface \( \Sigma \subset S^3 \), let \( B_{L,\Sigma,p} \) be the 2-dimensional cell complex defined as follows.

- The 0-cells are bridge diagrams of \((L,p)\) on \( \Sigma \),
- The 1-cells are basic moves and diffeomorphism maps,
- The 2-cells are commutative diagrams of type (a)-(i).

If we denote the space of parametrized based links in \( S^3 \) which are isotopic to \((L,p)\) as \( \text{Emb}_L(\coprod S^1, S^3) \), we have a canonical map

\[ R : B_{L,\Sigma,p} \to \text{Emb}_L(\coprod S^1, S^3). \]
Lemma 6.8. The map $\mathcal{R}$ is 1-connected.

Proof. Let $x, y$ be the north and south pole of $S^3$, and choose a projection function $p_\Sigma : S^3 - \{x, y\} \to \Sigma$, together with a height function $h_\Sigma : S^3 - \{x, y\} \to \mathbb{R}$, so that $p_\Sigma \times h_\Sigma$ is a diffeomorphism, $p_\Sigma|_{\Sigma} = \text{id}_\Sigma$, and $h_\Sigma|_{\Sigma} = 0$. A generic point in $\text{Emb}_L(\coprod S^1, S^3)$ is a based links in $S^3$, isotopic to $(L, p)$, which do not pass through $x$ nor $y$, intersects $\Sigma$ transversely, no two points $(x_1, x_2)$ on the link have the same image under $p_\Sigma$ if $x_1 \in \Sigma$ or $x_2 \in \Sigma$, and no three points on the link have the same image under $p_\Sigma$. Such a link can be canonically isotoped to a bridge position by moving it across $\Sigma$ so that, for each pair of points $(x_1, x_2)$ on the link which satisfies $p_\Sigma(x_1) = p_\Sigma(x_2)$ and $h_\Sigma(x_1) < h_\Sigma(x_2)$, we have

$$h_\Sigma(x_1) < 0 < h_\Sigma(x_2),$$
while fixing the points in \(L \cap \Sigma\). The codimension 1 points are the links which satisfy one of the following cases.

(1a) The link projects to the cusp \(y^2 = x^3\)

(1b) Exactly two generic points \(x_1, x_2\) on the link have the same image under \(p\Sigma\), and the projected image of the segments of the link near \(x_1\) and \(x_2\) are tangent to each other, where the order of tangency is 1

(1c) Exactly three generic points \(x_1, x_2, x_3\) on the link have the same image under \(p\Sigma\)

(1d) The link is tangent to \(\Sigma\) at a generic point \(x_{\Sigma}\), where the order of tangency is 1

The case (1a) corresponds to isotopies, (1b) and (1c) corresponds to handleslides, and (1d) corresponds to stabilizations. Note that a path of generic points, which does not pass through codimension 1 singularities, corresponds to isotopy maps or diffeomorphism maps. The choice of a diffeomorphism map is unique up to distinguished squares of type (i).

Now the codimension 2 points are given as follows.

(2a) Exactly four generic points \(x_1, x_2, x_3, x_4\) on the link have the same image under \(p\Sigma\)

(2b) Exactly three points \(x_1, x_2, x_3\) on the link have the same image under \(p\Sigma\), where the segments of the link near \(x_1, x_2\) are tangent to each other, where the order of tangency is 1

(2c) Exactly two points \(x_1, x_2\) on the link have the same image under \(p\Sigma\), where the segments of the link near \(x_1, x_2\) are tangent to each other, and the order of tangency is 2

(2d) Two codimension 1 states of type (1c) occur at two different points of \(\Sigma\)

(2e) Exactly two points \(x_1, x_2\) on the link have the same image under \(p\Sigma\), where the segments of the link near \(x_1\) projects to the cusp \(y^2 = x^3\)

(2f) The link projects to \(\Sigma\) as the degenerate cusp \(y^2 = x^5\)

(2g) A codimension 1 state of type (1d) and a state of type (1a) occur at two different points of \(\Sigma\)

(2h) A codimension 1 state of type (1d) and a state of type (1b) or (1c) occur at two different points of \(\Sigma\)

(2i) Two codimension 1 states of type (1d) occur at two different points of \(\Sigma\)

(2j) The link is tangent to \(\Sigma\), where the order of tangency is 2

The monodromies of codimension 2 points are given in Table 1. Note that (none) means a monodromy along a boundary of a 2-cell which does not contain codimension 2 points. Therefore, using the triangulation technique of [JT], we deduce that the map \(R\) induces isomorphisms of \(\pi_0\) and \(\pi_1\).

| Codimension 2 points | (2a) | (2b) | (2c) | (2d) | (2e) | (2f) | (2g) | (2h) | (2i) | (2j) | (none) |
|-----------------------|------|------|------|------|------|------|------|------|------|------|--------|
| Monodromy             | (d)  | (b)  | (c)  | (e)  | (a)  | (i)  | (g)  | (e)  | (f)  | (g)  | (b),(h) |

**Table 1.** Monodromies of codimension 2 points in terms of distinguished diagrams

Now we stick to the case when \(L = K\) is a knot. By the above lemma, the map

\[ R : B_{K,\Sigma,p} \to \text{Emb}_K(S^1, S^3) \]

is 1-connected. Consider the space \(E_{K,p}\) of unparametrized based knots isotopic to \((K,p)\). Since we have a fibration

\[ \text{Diff}^+(S^1, p) \to \text{Emb}_K(S^1, S^3) \to E_{K,p} \]

and the group \(\text{Diff}^+(S^1, p)\) is contractible, the map \(\text{Emb}_K(S^1, S^3) \to E_{K,p}\) is a homotopy equivalence. Hence we have an isomorphism \(\pi_1(B_{K,\Sigma,p}) \simeq \pi_1(E_{K,p})\).

The natural action of the diffeomorphism group on \(E_{K,p}\) gives a fibration

\[ \text{Diff}^+(S^3, K, p) \to \text{Diff}^+(S^3) \to E_{K,p}. \]
Since $\text{Diff}^+(S^3)$ is path-connected, $\pi_1(\text{Diff}^+(S^3)) \cong \mathbb{Z}_2$ with the generator given by rotation, and $\pi_0(\text{Diff}^+(S^3, K, p))$ is the mapping class group $\text{MCG}(S^3, K, p)$, we get an exact sequence

$$\mathbb{Z}_2 \rightarrow \pi_1(E_{K,p}) \rightarrow \text{MCG}(S^3, K, p) \rightarrow 1.$$ 

However, since we can place the genus 0 Heegaard surface of $S^3$ so that the generator of $\mathbb{Z}_2$ acts on it by a full rotation, and such a rotation induces the identity map of $\mathbb{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$ by Lemma 6.5, the monodromy representation

$$\pi_1(\mathcal{B}_{K,\Sigma,p}) \rightarrow \text{GL}(\mathbb{HF}_{\mathbb{Z}_2}(\Sigma(K), p))$$

factors through $\text{MCG}(S^3, K, p)$. Therefore we have a natural action of the mapping class group on the equivariant Floer cohomology $\mathbb{HF}_{\mathbb{Z}_2}(\Sigma(K), p)$.

**Theorem 6.9.** Let $\text{Knot}_*$ be the category whose objects are based knots in $S^3$ and morphisms are self-diffeomorphisms of $S^3$ which preserve the knot (as a set) and the basepoint. Then we have a functor

$$\mathbb{HF}_{\mathbb{Z}_2} : \text{Knot}_* \rightarrow \text{Vect}_{\mathbb{Z}_2},$$

agreeing up to isomorphism with the invariants $\mathbb{HF}_{\mathbb{Z}_2}(\Sigma(K))$ defined in [HLS].

**Remark.** A similar argument shows that the same statement holds for links, when each component has a basepoint. However, since we were working with based links, where only one component has a basepoint, we cannot say that $\mathbb{HF}_{\mathbb{Z}_2}(\Sigma(L), p)$ is natural with respect to the based link $(L, p)$.

This causes a slight problem when proving functoriality, so we will only consider cobordisms where both ends are knots. In this case, we can “push off” the excessive monodromies coming from absence of basepoints toward an end and then cancel them out.

Now we will consider cobordisms between based links in $S^3$.

**Definition 6.10.** A based cobordism $(S, s)$ between based links $(L_1, p_1)$ and $(L_2, p_2)$ is an oriented cobordism $S \subset S^3 \times I$ between $L_1$ and $L_2$, together with a smooth curve $s : I \rightarrow S$ such that $s(0) = p_1$ and $s(1) = p_2$.

We first consider the case when the based cobordism $(S, s)$ is very simple. There are three possible cases of such cobordisms, which we will call as basic pieces. The basic pieces can be defined as the follows.

1. cylindrical pieces
2. birth/death of a component without a basepoint
3. saddle along an arc joining two points on the link

To define a map associated to a based cobordism, the most natural strategy is to chop it into simple pieces. Given a based cobordism $(S, s)$, consider the projection map $p : S^3 \times I \rightarrow I$. Then we can isotope the pair $(S, s)$ so that it satisfies the following conditions.

- $p|_S$ is Morse on $S$,
- $p \circ s$ is regular.
- No two critical points of $p|_S$ have the same value under $p$.

Once such conditions are satisfied, we can cut $S$ horizontally to reduce it into basic pieces, and thus we can represent $(S, s)$ as a composition of basic pieces. Note that the third condition can be satisfied because, if there are two critical points of $p$ with the same value, then we can perturb $p$ slightly to make them have different values, and if the perturbation is sufficiently $C^1$-small, then the first and second conditions remain hold.

In [HLS], it is proved that when $S$ is a basic piece from $(L_1, p_1)$ to $(L_2, p_2)$, there exists a map

$$\hat{f}_S : \mathbb{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \rightarrow \mathbb{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1),$$
which is compatible with the cobordism map between \( \widehat{HF}^*(\Sigma(L_i)) \), via the naturally defined spectral sequence

\[
E_1 = \widehat{HF}^*(\Sigma(L)) \otimes \mathbb{F}_2[\theta] \Rightarrow \widehat{HF}_{\mathbb{Z}}^*(\Sigma(L)).
\]

However, the construction of this map depends on choices of auxiliary data, and we should prove that our maps are invariant under such choices.

Suppose that \( S \) is a saddle along an arc \( a \) from \( x_1 \in L_1 \) to \( x_2 \in L_2 \), which is indeed a based link cobordism from \( (L_1, p_1) \) to \( (L_2, p_2) \), provided \( x_i \neq p_i \). Choose a genus zero Heegaard surface \( \Sigma \subset S^3 \). To define a map \( \widehat{f}_S \) associated to \( S \), we need to draw bridge diagrams of \( (L_i, p_i) \) on the surface \( \Sigma \) and then project the arc \( a \) on \( \Sigma \).

**Definition 6.11.** A saddle diagram of \( S \) consists of bridge diagrams \( (\{A^i_j\}, \{B^i_j\}) \) of \( (L_i, p_i) \) on \( \Sigma \) for \( i = 0, 1 \) and an arc \( a_\Sigma \subset \Sigma \) from an \( X \)-arc \( A^0_0(B^0_k) \) to an \( X \)-arc \( A^1_1(B^1_k) \), where \( X \) is either \( A \) or \( B \), so that the 1-subcomplex of \( S^3 \) given by pushing the \( A \)-arcs inside \( \Sigma \) and the \( B \)-arcs outside \( \Sigma \) is isotopic to \( L_1 \cup a \cup L_2 \), and \( \text{int}(a_\Sigma \cap A^i_j(B^i_k)) = \emptyset \) for all \( i \) and \( j(k) \).

There are several choice of saddle diagrams which represent \( S \). However, given two bridge diagrams of \( (L_i, p_i) \) for \( i = 0, 1 \), which coincide outside the saddle region, we only have to choose the placement of the arc \( a_\Sigma \). Any two choice of \( a_\Sigma \) are related by isotopies and handleslides(over \( A(B) \)-arcs). If \( a_\Sigma \) and \( a'_\Sigma \) are related by a single handleslide, then the bridge diagrams representing the saddle moves along \( a_\Sigma \) and \( a'_\Sigma \) are related by a composition of two handleslides, as shown in Figure 6.7. This can be represented as a distinguished diagram of bridge diagrams as follows. Note that we are using saddle diagrams in a perturbed form, as in [HLS].

![Figure 6.7. Taking saddles along two arcs \( a_\Sigma \) and \( a'_\Sigma \), which differ by a handleslide](image)

**Lemma 6.12.** The above diagram induces a commutative diagram of corresponding \( \widehat{HF}_{\mathbb{Z}_2} \) groups:

\[
\begin{align*}
\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \xrightarrow{\text{saddle along } a'_\Sigma} & \rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1). \\
\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \xrightarrow{\text{handleslides}} & \rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \xrightarrow{\text{saddle along } a_\Sigma} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1).
\end{align*}
\]
Proof. The Heegaard diagram for a saddle along $a'_\Sigma$ admits a unique $\mathbb{Z}_2$-invariant top generator. \hfill \Box

The above lemma means that the saddle map is invariant under the choice of $a'_\Sigma$. However, it remains to show the invariant under the choice of bridge diagrams of $L_1$ and $L_2$. For that, we need to show that the following commutative diagrams induce commutative diagrams of $\hat{HF}_{\mathbb{Z}_2}$ groups.

- Basic moves and saddles commute.
- Diffeomorphisms and saddles commute.

\begin{equation}
D \xrightarrow{\text{saddle}} D_2 \xrightarrow{\text{basic moves/diffeomorphisms}} D_1 \xrightarrow{\text{saddle}} D_3
\end{equation}

Lemma 6.13. The diagram (6.1) induces commutative diagrams of $\hat{HF}_{\mathbb{Z}_2}$ groups:

$\hat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \xrightarrow{\text{saddle}} \hat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1)$

\begin{equation}
\xrightarrow{\text{basic moves/diffeomorphisms}} \xrightarrow{\text{basic moves/diffeomorphisms}} \xrightarrow{\text{basic moves/diffeomorphisms}}
\end{equation}

$\hat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \xrightarrow{\text{saddle}} \hat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1)$

Proof. The lemma is obvious for diffeomorphisms, and Lemma 6.2 implies that the lemma holds for isotopies. The proof for handleslides and stabilizations are the same as the proof of Proposition 6.4 for distinguished diagrams of type (c) and (e), respectively. \hfill \Box

Using the above lemma, we see that the saddles give well-defined saddle maps.

Theorem 6.14. The basic pieces give well-defined maps between $\hat{HF}_{\mathbb{Z}_2}$ groups.

Proof. The saddle maps are well-defined by the above lemma. The birth map and the death map correspond to taking/untaking a connecting sum with an invariant Heegaard diagram of $S^1 \times S^2$ on the branched double cover. Since they commute with basic moves (and, obviously, diffeomorphisms) when the connected sum neck is very long, the theorem follows. \hfill \Box

Now consider the case when we are given a based cobordism $S$ from a based knot $(K_1, p_1)$ to a based knot $(K_2, p_2)$. Then we can isotope $S$, slice it into basic pieces, convert them into maps between $\hat{HF}_{\mathbb{Z}_2}$ groups, and then compose them to get a cobordism map

$\hat{f}_S : \hat{HF}_{\mathbb{Z}_2}(\Sigma(K_2), p_2) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(K_1), p_1)$

We claim that $\hat{f}_S$ does not depend on the process of slicing $S$ into basic pieces.

Lemma 6.15. $\hat{f}_S$ depends only on the isotopy class of based cobordisms $S = (S_0, s) \ rel \ s \cup \partial S_0$.

Proof. Isotoping $S$ and then slicing it into basic pieces is equivalent to representing the cobordism $S$ as a movie from $(K_1, p_1)$ to $(K_2, p_2)$. It is known that any two movies of $S$ are related through a sequence of 15 possible types of movie moves, which are defined by Carter and Saito in the paper \cite{CS}. Note that, although the result of Carter and Saito is about non-based cobordisms, it can also be applied directly to based cobordisms, by taking saddles moves to occur away from the basepoint.

Among the 15 types of movie moves, the only type that needs a proof for functoriality of equivariant Floer cohomology is the one drawn in Fig.30 of \cite{CS}, which corresponds to the following commutative triangles.

\begin{center}
\begin{tikzpicture}
\node (L) at (0,0) {$L$};
\node (LU) at (0,-1.5) {$L \sqcup U$};
\node (L1) at (2,0) {$L_1$};
\node (L2) at (2,-1.5) {$L_2 \sqcup U$};
\node (L3) at (4,0) {$L_3$};
\node (L4) at (4,-1.5) {$L_4 \sqcup U$};
\node (L5) at (6,0) {$L_5$};
\node (L6) at (6,-1.5) {$L_6 \sqcup U$};
\draw[->] (L) -- (LU) node[midway,above] {destabilization} node[midway,below] {birth};
\draw[->] (LU) -- (L2) node[midway,above] {saddle};
\draw[->] (L2) -- (L3) node[midway,above] {saddle};
\draw[->] (L3) -- (L4) node[midway,above] {saddle};
\draw[->] (L4) -- (L5) node[midway,above] {saddle};
\draw[->] (L5) -- (L6) node[midway,above] {saddle};
\end{tikzpicture}
\end{center}
But the composition of a birth map followed by a saddle map is precisely the definition of the stabilization map, which proves that the triangle on the left induces a commutative triangle of $\hat{HF}_{\mathbb{Z}}$ groups; see the proof of Theorem 1.24 of [HLS] for the definition.

The triangle on the right induces a commutative triangle of $\hat{HF}_{\mathbb{Z}}$ groups if the composite cobordism

$S : L \xrightarrow{\text{saddle}} L \coprod U \xrightarrow{\text{death}} L \coprod U \xrightarrow{\text{saddle}} L$

induces identity on equivariant Heegaard Floer cohomology. First, if $L$ is a union of two unlinked unknots, then this is obvious. Next, if $L = L_0 \coprod U_0$, where $U_0$ is an unlinked unknot component, the basepoint lies in $L_0$, and the saddle moves are taken on the component $U_0$, then the question reduces to the previous case by making a long neck between the bridge diagrams of $L_0$ and $U_0$, as drawn in 6.9.

For general link $L$, consider the saddle move on $L$, as in Figure 6.8 (the green dashed line is the saddle arc) so that the resulting link $L'$ is isotopic to $L \coprod$ unknot where the basepoint does not lie on the unknot component. Since the saddle map $\hat{HF}_{\mathbb{Z}}(\Sigma(L), p) \xrightarrow{f_{S_a}} \hat{HF}_{\mathbb{Z}}(\Sigma(L'), p)$ postcomposed with the birth map $\hat{HF}_{\mathbb{Z}}(\Sigma(L'), p) \rightarrow \hat{HF}_{\mathbb{Z}}(\Sigma(L), p)$ is the stabilization map, which is an isomorphism, we see that $f_{S_a}$ is injective. Now, by the proof of 6.4 for distinguished diagrams of type (c), the following diagram commutes.

But we already know that $f_{S \coprod (L \times I)}$ is the identity. Therefore, since $f_{S_a}$ is injective, the map $f_S$, too, is the identity. 

\[ \text{Figure 6.8. Saddle along a small arc near } L \]

**Theorem 6.16.** $f_S$ depends only on the isotopy class of based cobordisms $S = (S_0, s) \text{ rel } \partial S_0$. 

defined functor. We first define a notion of isotopies of based cobordisms, as follows.

Theorem 6.17. Then what we proved up to now can be rephrased as follows.

depend on the orientation of \( S \), which is well-defined up to monodromy. In other words, if \( S \) and \( S' \), we have

\[ \hat{f}_{(S_0,s)} = \hat{f}_{(S_0,s')} \]

Isotope \( S_0 \) rel \( s \cup s' \), so that \( p|_{S_0} \) is Morse and the critical points of \( p|_{S_0} \) have distinct images under \( p \). We can isotope \( s \) slightly and horizontally so that it intersects \( s' \) transversely. We know that \( s \) and \( s' \) cobound a disk \( D \subset S \), which is innermost in the sense that \( D \cap (s \cup s') = \partial D \). Using Lemma 6.15 we can isotope \( S \) so that \( p|_D \) has no critical points. Then we get an isotopy \( \{s_t\}_{t \in [0,1]} \) from \( s \) to \( s' \) rel \( \partial s \), such that \( h \) is regular on \( s_t \) for each \( t \in [0,1] \). Since isotopy maps (on \( \Sigma \)) commute with all other types of maps, we deduce that the based cobordisms \( (S, s_t) \) induce the same map for all \( t \). Therefore, by induction on \( |s \cap s'| \), we get \( \hat{f}_{(S_0,s)} = \hat{f}_{(S_0,s')} \).

\[ \square \]

Thus we can now rewrite the equivariant Heegaard Floer theory \( \hat{HF}_{\mathbb{Z}_2}(\Sigma(L)) \) of based links as a well-defined functor. We first define a notion of isotopies of based cobordisms, as follows.

We can now define a category \( \mathbf{bCob} \) as follows.

- The objects of \( \mathbf{bCob} \) are based knots in \( S^3 \).
- The morphisms of \( \mathbf{bCob} \) between based knots \( K_1, K_2 \) are isotopy classes of based cobordisms from \( K_1 \) to \( K_2 \).

Then what we proved up to now can be rephrased as follows.

**Theorem 6.17.** \( \hat{HF}_{\mathbb{Z}_2} : \mathbf{bCob} \to \text{Mod}_{\mathbb{Z}_2[\theta]} \) is a functor.

Note that the equivariant HF theory of links is an unoriented theory. The group \( \hat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \) does not depend on the orientation of \( K \), and a knot cobordism induces maps between \( \hat{HF}_{\mathbb{Z}_2} \) in both ways.

**Remark.** Using the same argument, we can prove that a based cobordism of links induces a cobordism map, which is well-defined up to monodromy. In other words, if \( S \) is a based cobordism from \( (L_1, p_1) \) to \( (L_2, p_2) \) and we denote the monodromy of \( \hat{HF}_{\mathbb{Z}_2}(\Sigma(L_i), p_i) \) by

\[ \rho_i : \pi_1(\mathcal{B}_{L_i, \Sigma, p_i}) \to \text{GL}(\hat{HF}_{\mathbb{Z}_2}(\Sigma(L_i), p_i)) \]

then the cobordism map

\[ \hat{f}_S : \hat{HF}_{\mathbb{Z}_2}(\Sigma(L_2), p_2) \to \hat{HF}_{\mathbb{Z}_2}(\Sigma(L_1), p_1) \]

is well-defined up to composition with elements in the images of \( \rho_1 \) (and \( \rho_2 \)).

7. Functoriality of \( c_{\mathbb{Z}_2} \) under certain symplectic cobordisms

Now we restrict to the case when the links are transverse, with respect to the standard contact structure \( \xi_{\text{std}} \) on \( S^3 \). Then the cobordisms we should use in this case are symplectic cobordisms. Recall that symplectic cobordism of links is defined as follows.

**Definition 7.1.** Let \( L_0, L_1 \) be transverse links in \( (S^3, \xi_{\text{std}}) \). A symplectic cobordism from \( L_0 \) to \( L_1 \) is an embedded symplectic surface \( S \subset (S^3 \times [0, R], d(-e^t\xi_{\text{std}})) \) where \( R \) is a positive real, \( S \cap (S^3 \times \{i\}) = L_i \) for \( i = 0, 1 \), and \( S \) is cylindrical near both ends.
Example 7.2. Suppose that we are given an isotopy $rels \cup s^t \{L_t\}$ of transverse links, $0 \leq t \leq 1$. Given a symplectization

$$(S^3 \times [0, 1], \omega_R = d(e^R \alpha_{std})) \simeq (S^3 \times [0, R], d(e^t \alpha_{std}))$$

for $R > 0$, where $\alpha_{std} = dz + xdy(on \mathbb{R}^3)$, consider the cobordism

$S_R = \bigcup_{0 \leq t \leq 1} L_t \times \{Rt\}.$

Choose a point $(p, t) \in S^3 \times [0, R]$. Then the tangent plane $T_{(p, Rt)}S_R$ is spanned by the vector $\partial_t + w_{p,t}$ and the line $T_pL_t$, for some vector $w_{p,t} \in T_pS^3$. Let $v_p$ be a tangent vector spanning $T_pL_t$, which is chosen to vary smoothly. Then we have

$$(\omega_R)_{(p, Rt)}(\partial_t + w_{p,t}, v_p) = e^{Rt}(Rdt \wedge \xi_{std} + d\alpha_{std})(\partial_t + w_{p,t}, v_p)$$

$$= Re^{Rt}(\alpha_{std}(v_p) + \frac{1}{R}d\alpha_{std}(\partial_t + w_{p,t}, v_p)).$$

Since $L_t$ are transverse to $\xi_{std}$, by compactness, the value of $\alpha_{std}(v_p)$ is bounded away from zero. Hence, for sufficiently large $R$, the cobordism $S$ is symplectic. In other words, $S_R$ is symplectic for sufficiently large $R > 0$; we will call such surfaces as isotopy cylinders.

Now suppose that we are given a cobordism $S \subset (S^3 \times [0, R], d(-e^t \alpha_{std}))$ which is symplectic and cylindrical near both ends. Let $L = S \cap (S^3 \times \{0\}) \subset S^3$ and choose a generator $v_p \in T_pL$ for $p \in S^3$. Since $S$ is symplectic, by assumption, we must have

$$0 \neq d(e^t \alpha_{std})(\partial_t, v_p) = e^t(dt \wedge \alpha_{std} + d\alpha_{std})(\partial_t, v_p) = e^t \alpha_{std}(v_p).$$

This implies $\alpha_{std}(v_p) \neq 0$, i.e. $L$ is transverse to $\xi_{std}$. Therefore we see that symplectic cobordisms form a good notion of cobordisms between transverse links.

Before we explicitly construct “basic pieces” of based symplectic cobordisms which achieve the functoriality for equivariant contact classes, we need to define the notion of weak symplectic isotopies between symplectic cobordisms. They are defined as follows.

Definition 7.3. Two symplectic cobordisms $S_1, S_2$ are weakly symplectically isotopic if they are symplectically isotopic after concatenating with (trivial) cylindrical cobordisms on both ends.

A based symplectic cobordism is a based cobordism $(S, s)$ where $S$ is symplectic. Two based symplectic cobordisms $(S, s)$ and $(S', s')$ are weakly symplectically isotopic if $S$ and $S'$ are weakly symplectically isotopic and, after performing the isotopy, the homotopy class $[s] - [s']$ is contained in the image of the map $\pi_1(\partial S) \to \pi_1(S)$.

Thus we are led to define a category $s\text{Cob}_w$ as follows:
- objects of $s\text{Cob}_w$ are based transverse knots;
- morphisms of $s\text{Cob}_w$ are weak symplectic based isotopy classes of based symplectic cobordisms.

Then we have a well-defined natural functor

$$s\text{Cob}_w \to b\text{Cob},$$

which forgets all the contact and symplectic structures involved and reverses the directions of cobordisms. In other words, a weak symplectic based isotopy class of based symplectic cobordisms can be seen as a based isotopy class of based cobordisms.

Now we construct the “basic pieces” of symplectic cobordisms, which will only be defined up to weak symplectic based isotopy. The idea is to mimic the symplectic handle constructions on the branched double cover side, so we can construct the symplectic models of “birth” and “saddle” cobordisms. Note that a birth corresponds to a 1-handle and a saddle corresponds to a 2-handle, while a death corresponds to a 3-handle; 4-dimensional symplectic 3-handles do not exist.

The birth case is easy:
**Definition 7.4.** Given a transverse based link $L$ in $(S^3, \xi_{std})$, choose a point $p \in S^3$ which lies outside $L$. Consider a 1-parameter family $\{U_t\}_{t \in (0,1)}$ which converges to $p$, so that in the front projection $(x, y, z) \mapsto (x, z)$, the family is represented by Figure 7.1. Then the symplectic birth cobordism from $L$ is defined as

$$B(L) = L \times [0, R] \cup \left( \bigcup_{t \in [0,1]} U_{1-t} \times \{Rt\} \right) \subset S^3 \times [0, R],$$

where $R > 0$ is chosen so that $B(L)$ becomes symplectic.

![Figure 7.1.](image)

**Lemma 7.5.** There exists $R > 0$ such that $B(L)$ in Definition 7.4 becomes symplectic, and the weak symplectic based isotopy type of $B(L)$ depends only on $L$.

**Proof.** The proof is very similar to the proof of Theorem 7.8, so we do not write it down here. □

We will now define symplectic saddle cobordisms. In the case of ordinary links and cobordisms, a saddle is defined by the following topological constructions.

- (C1) Given a link $L$, choose an arc $a$ which intersects transversely with $L$ at $\partial a$, such that $a \cap L = \partial a$.
- (C2) Remove a small neighborhood of $\partial a$ in $L$ and replace it by two arcs parallel to $a$. The saddle between those two links is the desired saddle cobordism of $L$ along $a$.

In our case, the link $L$ is transverse, and the arcs $a_i$ should be Legendrian. The reason is explained in Example 7.6.

**Example 7.6.** Let $L$ be a transverse link in $(S^3, \xi_{std})$. As we have seen previously, $\xi_{std}$ lifts to a contact structure $\xi_L$ on the branched double cover $\Sigma(L)$, which is determined uniquely up to isotopy (actually, $\mathbb{Z}_2$-equivariant isotopy). As a smooth manifold, $\Sigma(L)$ is defined by removing a standard neighborhood

$$\left(N(L), \xi_{std}\right|_{N(L)} \simeq (S^1 \times D^2, \ker(dz + f(r)d\theta)),$$

taking double cover with respect to the meridian $\partial D^2$, and then regluing a copy of $S^1 \times D^2$ where the covering transformation acts by $(r, \theta) \mapsto (r^2, 2\theta)$.

Suppose that a small smooth curve $\gamma \subset \Sigma(L)$ defined near a point $p \in L$ is invariant under $\mathbb{Z}_2$. Then, in the parametrization $N(L) \simeq S^1 \times D^2$, the tangent line $T_p\gamma$ must be spanned by the vector $\partial_r$. Since the contact structure $\xi_L$ on $\Sigma(L)$ is defined near $L$ by $\ker(\alpha_L)$ where $\alpha_L = dz + f(r)d\theta$ for a good increasing $f$, we deduce that, for any $v \in T_p\gamma$,

$$(\alpha_L)_p(v) = dz(v) + f(r)d\theta(v) = 0.$$
Now let $L$ be a based transverse link with basepoint $z \in L$, and suppose that we are given a Legendrian arc $a$ satisfying the conditions (C1) and (C2), which does not contain the basepoint $z$. Then the set $L \cup a$ is an embedded graph in $S^3$, such that it consists of transverse edge-cycles together with one Legendrian edge connecting points on the transverse cycles, and has a basepoint in some transverse edge. We shall impose a further condition on $a$ to ensure that the lift $\gamma$ of $a$ becomes a smooth Legendrian knot in $(\Sigma(L), (L, a, z))$:

- There exists a standard neighborhood $(N(L), \xi_{std}(L)) \simeq \coprod(S^1 \times D^2, \ker(dz + r^2 d\theta))$ such that the curve segment $a \cap N(L)$ is a straight radial line (i.e. point towards the $r$-axis).

When this condition is also satisfied, we shall call such a graph $(L, a, z)$ a nice graph.

Suppose that a nice graph $G = (L, a, z)$ is given. By the definition of nice graphs, there exists a standard neighborhood $(N(L), \xi_{std}(L)) \simeq \coprod(S^1 \times D^2, \ker(dz + r^2 d\theta))$ such that the curve segment $a \cap N(L)$ is a straight radial line (i.e. towards the $r$-axis). So we have a contact 1-form representative $\alpha$ of $\xi_{std}$ such that

$$\alpha|_{N(L)} = dz + r^2 d\theta,$$

in the given neighborhood parametrization. Now, by the standard neighborhood theorem for Legendrian arcs, we have a neighborhood parametrization, which is uniquely determined up to isotopy:

$$(N(a), \alpha|_{N(a)}) \simeq ([0, \pi/2] \times D^2, \ker(\cos(z)dx + \sin(z)dy)).$$

Here, by a (closed) neighborhood of the arc $a$, we mean an embedded closed disk-bundle over $a$. Sometimes we will choose a slight extension of the above parametrization, so that we have a contact embedding of the cylinder

$$\left([\epsilon, \frac{\pi}{2}] \times D^2, \ker(\cos(z)dx + \sin(z)dy)\right).$$

**Example 7.7.** Consider the contact manifold

$$U = ([0, \pi/2] \times D^2, \alpha = \cos(z)dx + \sin(z)dy),$$

which is the standard neighborhood of the Legendrian arc $[0, \pi] \times \{0\}$. If a curve $\gamma = (\gamma_x, \gamma_y, \gamma_z)$ satisfies

$$\frac{\gamma'_y}{\gamma'_z} = 2 \tan(z),$$

Then $\alpha(\gamma') = (1 + \sin^2 z) \cdot v \neq 0$ where $v = \frac{\gamma'_y}{\cos(z)} = \frac{\gamma'_y}{2 \sin(z)}$. So, given a regular curve $a(t) = (a_x(t), a_y(t)) \in D^2$,

the induced curve

$$V_a = \left(a_x, a_y, \arctan\left(\frac{a'_y}{2a'_z}\right)\right) \in D \times [0, \pi/2],$$

if defined, is always transverse. Also note that we have

$$d\alpha_U = dz \wedge (-\sin(z)dx + \cos(z)dy).$$

**Remark.** In this section, when we draw a regular curve $a$ on $D^2$, we will mean the induced transverse curve $V_a$ in a slightly extended cylinder $[\epsilon, \frac{\pi}{2} + \epsilon] \times D^2$ for a very small $\epsilon > 0$.

Since $L$ is a Reeb orbit of $\alpha$ and the Reeb orbits of $\cos \theta dx + \sin \theta dy$ passing through a point in $I \times \{0\}$ are radial lines, we deduce that the curve segments $L \cap N(a_i)$ are given by the $y$-axes on $D^2 \times \{\pm \pi/2\}$. Then we consider a family of curves $\{L_t\}_{t \in [0, 1]}$ defined as in Figure 7.2 embedded in the cylinder $S^3 \times [0, R]$.

However, since the diagram in the middle gives two curves which are tangent at a point with tangent line $\mathbb{R} \partial_z$, we see that taking the union $\cup_{t \in [0, 1]} L_t \times \{Rt\}$ of such curves does not give us a surface in $S^3 \times [0, R]$. To resolve this problem, we perform a slight perturbation to give a new family $\{\tilde{L}_t\}$, where the projection of $\tilde{L}_{1/2} \subset D^2 \times I$ to $D^2$ is given as in Figure 7.3.

If the perturbation is small enough, $\tilde{L}_t$ is transverse for $t \neq \frac{1}{2}$ and $\tilde{L}_t \setminus \{p\}$ is transverse for $t = \frac{1}{2}$. Also, the two tangent vectors to $\tilde{L}_{1/2}$ at $p$ define a positive orientation of $D^2$. We can now form a symplectic cobordism from such a family.
Theorem 7.8. When $R$ is sufficiently large, the cobordism $S_R = \bigcup_{t \in [0,1]} \tilde{L}_t \times \{Rt\}$ is symplectic, and its weak symplectic isotopy class depends only on the given transverse link $L$ and a Legendrian arc $a$.

Proof. For simplicity, fix $R = 1$ and let the symplectic structures on $S^3 \times I$ vary:

$$\omega_R = d(e^{Rt} \alpha_{std}) = e^{Rt} (Rdt \wedge \alpha_{std} + d\alpha_{std}).$$

We are asking whether the given cobordism, which we will denote by $S$, is symplectic with respect to the symplectic form

$$\frac{1}{Re^{Rt}} \omega_R = dt \wedge \alpha_{std} + \frac{1}{R} d\alpha_{std}.$$

Choose a smooth nonvanishing tangent bivector field $\sigma$ on $S$. By the transversality assumption, the term $dt \wedge \alpha_{std}(\sigma)$ is everywhere nonnegative, and it vanishes only at the saddle point, which we will call $p$. On the other hand, the term $d\alpha_{std}(\sigma)$ may not be everywhere nonnegative, but it takes a positive value at $p$. So, suppose that $d\alpha_{std}(\sigma) > 0$ in the $r$-ball centered at the saddle point, and let

$$\max_S |d\alpha_{std}(\sigma)| < M, \quad \min_{S-B_r(p)} dt \wedge \alpha_{std}(\sigma) > m.$$

Then, if $R > \frac{M}{m}$, we get

$$\left| \left( dt \wedge \alpha_{std} + \frac{1}{R} d\alpha_{std} \right)(\sigma) \right| \geq m - \frac{M}{R} > 0$$
denote such a cobordism, with the curve \( \{ z \} \times [0, R] \) on it, by \( S = S(L, a, z) \). Then \( S \) has \((L, z)\) as a concave end and a based transverse link \( L' = (C(L, a), z) \) as a convex end. The (transverse isotopy class of) based transverse link \((C(L, a), z)\) is called the transverse surgery of \((L, z)\) along \( a \). The (weak symplectic isotopy class of) based symplectic cobordism \( S(L, a, z) \) is called the symplectic saddle of \((L, z)\) along \( a \).

With those constructions in mind, we will say that the based symplectic cobordisms which can be constructed from the symplectic births and saddles are symplectically constructible. More precisely, we have the following definition.

**Definition 7.10.** The weak symplectic isotopy class of based symplectic cobordisms, obtained from gluing the classes of symplectic births and saddles, are called constructible classes. The cobordisms contained in constructible classes are called symplectically constructible based cobordisms.

Now we prove that the maps between \( \hat{HF}_{Z_2} \), associated to symplectically constructible based cobordisms, maps the equivariant contact class of the concave end to the equivariant contact class of the convex end. We start from the simplest case, when the based link \( L \) is already braided along the \( z \)-axis and the Legendrian arc \( a \) is in a very nice position.

**Definition 7.11.** Let \( L \) be a transverse link in \((S^3, \xi_{std})\), which is braided along the \( z \)-axis. A simple Legendrian arc \( a \) is basic if there exists a genus-zero open book supporting \((S^3, \xi_{std})\), whose binding is the \( z \)-axis, such that \( a \) is contained in a single page.

**Lemma 7.12.** Let \((L, a, z)\) be a nice graph, where \( L \) is braided along the \( z \)-axis, and suppose that \( a \) is basic. Let

\[
\hat{f}_{S(L, a, z)} : \hat{HF}_{Z_2}(\Sigma(C(L, a)), z) \to \hat{HF}_{Z_2}(\Sigma(L), z)
\]

be the map associated to the based cobordism class \( S(L, a, z) \). Then we have

\[
\hat{f}_{S(L, a, z)}(c_{Z_2}(\xi_{C(L, a)}, z)) = c_{Z_2}(\xi_L, z).
\]

**Proof.** Since \( a \) is basic, we know that the contact branched double cover of \((S^3, \xi_{std})\) along \( C(L, a) \) is the contact (-1)-surgery of \((\Sigma(L), \xi_L)\) along the lift \( \gamma \) of \( a \). The map \( \hat{f}_{S(L, a, z)} \) is given by the equivariant triangle map. The Heegaard triple-diagram we get is described in Figure 7.4.

As in the proof of invariance under positive stabilization, by the convenient placement of a basepoint, all triangles connecting \( x \) and \( \Theta \) in the diagram are small. So we deduce that the higher order terms in the equivariant triangle map vanish, since they count triangles of Maslov index at most 0. Thus we get the desired equality:

\[
\hat{f}_{S(L, a, z)}(\xi_{C(L, a)}, z) = c_{Z_2}(\xi_L, z) + \text{higher order terms} = c_{Z_2}(\xi_L, z).
\]

In the general case when \( L \) is not in a braid position and \( a \) is arbitrary, we argue that we can always isotope the whole situation to the above case, where \( L \) is braided and \( a \) is basic. For that we will have to deform our nice graph into a 4-valent transverse graph.

**Definition 7.13.** A based 4-valent transverse graph embedded in \((S^3, \xi_{std})\) is a 4-valent directed graph \( \Gamma \) with exactly one 4-valent vertex and several 2-valent vertices, together with a basepoint \( z \) on an edge of \( \Gamma \) and an embedding

\[
\Gamma \hookrightarrow S^3.
\]
such that each edge is transverse, the two adjacent edges of 2-valent vertices glue smoothly, and the 4-valent vertex has an labelling $l_1^\pm, l_2^\pm$ of its adjacent edges so that, as directed smooth curves, $l_1^\pm$ glue smoothly with $l_2^\pm$.

Recall that, given a nice graph $(L, a, z)$, we have defined the surgered link $C(L, a)$ and the symplectic saddle $S(L, a, z)$ by a sequence of diagrams drawn on $D^2$. By considering the intermediate slice (before applying a perturbation to make it a well-defined symplectic surface), which is defined as in Figure 7.5, we can see that $L$ transforms to $C(L, a)$ through a based 4-valent transverse graph $G(L, a, z)$ in $(S^3, \xi_{std})$.

Now, given a nice graph $(L, a, z)$, we want to isotope it, through a 1-parameter family of nice graphs, to another nice graph $(L', a', z')$, where $L'$ is braided along the $z$-axis and $a'$ is basic. To do that, we first isotope
the based 4-valent transverse graph \( G(L, a, z) \), so that its edges are in braid position. This is always possible due to the following (variant of) theorem of Bennequin.

**Theorem 7.14.** [BM] Any smooth transverse graph can be transversely isotooped into a braided position along the \( z \)-axis.

**Lemma 7.15.** Every nice graph \((L, a, z)\) can be isotooped to another nice graph \((L', a', z)\) through nice graphs, so that \( L' \) is braided and \( a' \) is basic.

**Proof.** By Theorem 7.14, we can transversely isotope the based 4-valent transverse graph \( G(L, a, z) \), so that the isotoopy is supported away from its unique 4-valent vertex, say \( v \). Since \((L, a, z)\) can be isotooped so that it agrees with \( G(L, a, z) \) away from the vertex \( v \) by the definition of \( G(L, a, z) \), we see that the transverse isotoopy of \( G(L, a, z) \) can also be applied to \((L, a, z)\) so that \( a \) is very short and \( L \) is braided. Therefore, by isotooping the pages of the open book of \( S^3 \) in a small neighborhood of \( v \), we see that \( a \) can be made basic, while remaining \( L \) braided. \( \Box \)

Now we can prove the functoriality in the most general setting.

**Lemma 7.16.** Given a nice graph \((L, a, z)\), let

\[
\hat{f}_{S(L, a, z)} : \widetilde{HF}_{Z_2}(\Sigma(C(L, a), z)) \to \widetilde{HF}_{Z_2}(\Sigma(L), z)
\]

be the map associated to the based cobordism class \( S(L, a, z) \). Then we have

\[
\hat{f}_{S(L, a, z)}(c_{Z_2}(\xi_{C(L, a)}, z)) = c_{Z_2}(\xi_L, z).
\]

**Proof.** Using Bennequin’s theorem, we can isotope the decorated transverse 4-valent graph

\[
(\Gamma, V, z) = G(L, a, z)
\]

into a braided position along the \( z \)-axis, with respect to some genus-0 open book having the \( z \)-axis as its binding. Furthermore, we can isotope the vector field \( V \) so that it is tangent to the pages of the open book. Then its detachment is a nice graph \((L', a', z')\), which is isotoopic to the original nice graph \((L, a, z)\), such that \( L' \) is braided along the \( z \)-axis and \( a \) is basic. Therefore, by the above lemma and the invariance of equivariant contact classes under isotoopies, we must have

\[
\hat{f}_{S(L, a, z)}(c_{Z_2}(\xi_{C(L, a)}, z)) = c_{Z_2}(\xi_L, z).
\]

\( \Box \)

**Theorem 7.17.** Given any symplectically constructible (weak symplectic isotoopy) class \( S \) of based symplectic cobordisms, with its concave and convex ends given by transverse isotoopy classes \((L_1, z_1)\) and \((L_2, z_2)\) of based transverse links, we have

\[
\hat{f}_S(c_{Z_2}(\xi_{L_2}, z_2)) = c_{Z_2}(\xi_{L_1}, z_1),
\]

where \( \hat{f}_S : \widetilde{HF}_{Z_2}(\Sigma(L_2), z_2) \to \widetilde{HF}_{Z_2}(\Sigma(L_1), z_1) \) is the cobordism map, induced by \( S \).

**Proof.** We already have the functoriality for both symplectic birth cobordisms and symplectic saddle cobordisms, and the functoriality for cylindrical cobordisms is obvious. Therefore, by composition, we get the functoriality for all symplectically constructible cobordisms. \( \Box \)

**Corollary 7.18.** We have a functor

\[
(\widetilde{HF}_{Z_2}, c_{Z_2}) : s\text{Cob}^c_w \to (\mathbb{F}_2[\theta] \downarrow \text{Mod}_{\mathbb{F}_2[\theta]}),
\]

where \( s\text{Cob}^c_w \) is the wide subcategory of \( s\text{Cob}_w \) spanned by symplectically constructible cobordism classes and \( \mathbb{F}_2[\theta] \downarrow \text{Mod}_{\mathbb{F}_2[\theta]} \) is the category of modules over \( \mathbb{F}_2[\theta] \) with a \( \theta \)-tower generated by a distinguished element.

**Proof.** This is just a category-theoretic statement for the theorem above. \( \Box \)
8. Properties

The first property of \( c_2(\xi_L, z) \) is that it contains the information about the ordinary contact class \( c(\xi_L) \).

**Theorem 8.1.** The natural map
\[
\widehat{HF}_{\mathbb{Z}_2}(\Sigma(L), z) \to \widehat{HF}^*(\Sigma(L))
\]
sends \( c_2(\xi_L, z) \) to \( c(\xi_L) \).

*Proof.* The chain level map is given by truncating all terms with nontrivial \( \theta \)-degree, so it sends \( EH^*_{\mathbb{Z}_2}(\xi_L) = EH^*_{\mathbb{Z}_2}(\xi_L) \otimes \theta^0 \) to \( EH^*_{\mathbb{Z}_2}(\xi_L) \). Hence, on the cohomology level, \( c_2(\xi_L) \) is sent to \( c(\xi_L) \). \( \square \)

Also, as in Section 6.1 of [HLS], we have a localization isomorphism
\[
\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \cong \widehat{HF}^*(S^3) \otimes \mathbb{F}_2[\theta, \theta^{-1}],
\]
where \( K \) is a knot and \( \theta^{-1} \) means that we are formally inverting \( \theta \), i.e., we define
\[
\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) = \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), z) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}].
\]
In turns out that the image of the equivariant contact class under the localization isomorphism takes a very simple form.

**Theorem 8.2.** Let \( K \) be a transverse knot in \((S^3, \xi_{\text{std}})\). Then the localization map
\[
\theta^{-1}\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \twoheadrightarrow \widehat{HF}(S^3) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \cong \mathbb{F}_2[\theta, \theta^{-1}],
\]
is defined up to multiplication by powers of \( \theta \), sends \( c_2(\xi_K) \) to a power of \( \theta \).

*Proof.* By the construction of the bare localization map in [SS], the localization map can be written as follows:
\[
c_2(\xi_K) \mapsto (c(\xi_{\text{std}}) + \text{higher order terms}) \otimes \theta^d,
\]
for some \( d \). But since there are no holomorphic disks going towards \( EH^*(\xi_K) \), the higher order terms vanish. Therefore the localization map sends \( c_2(\xi_K) \) to \( c(\xi_{\text{std}}) \otimes \theta^d = \theta^d \). \( \square \)

This theorem gives us a lower bound for the \( d_3 \)-invariants of the branched double covers along transverse knots in \((S^3, \xi_{\text{std}})\). Recall that \( q_r(K) \) is defined as:
\[
q_r(K) = 2 \cdot \min \{ \text{gr}(x) \mid x \in \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)), \theta^k x \neq 0 \text{ for all } k \geq 0 \}.
\]

**Corollary 8.3.** For a knot \( K \) in \( S^3 \), denote the set of all transverse representatives of \( K \) as \( T_K \). Then we have
\[
\frac{q_r(K) - 1}{2} \leq \min_{T \in T_K} d_3(T_K).
\]

*Proof.* For any transverse representative \( T \in T_K \), the absolute \( \mathbb{Q} \)-grading \( \text{gr}(EH(\xi_T)) \) of the contact element of \((\Sigma(K), \xi_T)\), which is the same as \( \text{gr}(c_2(\xi_T)) \), is given by \( \frac{1}{2} + d_3(\xi_T) \); see Proposition 4.6 of [OSZ2]. By the above theorem, \( c_2(\xi_T) \) cannot be annihilated by a power of \( \theta \), since the localization map is an isomorphism of \( \mathbb{F}_2[\theta] \)-modules. Therefore we have \( \frac{q_r(K) - 1}{2} \leq \min_{T \in T_K} d_3(T_K) \) for all such \( T \). \( \square \)

The functoriality of equivariant contact classes for symplectically constructible cobordisms also gives us some results about symplectic representatives of link cobordisms.

**Theorem 8.4.** Let \((L_1, z_1)\) and \((L_2, z_2)\) be two based transverse links in \((S^3, \xi_{\text{std}})\). If the based isotopy class of a based cobordism \( S \) from \((L_1, z_1)\) to \((L_2, z_2)\) has a symplectically constructible representative, then we must have
\[
f_S(c_2(L_2, z_2)) = c_2(L_1, z_1),
\]
where \( f_S \) is the cobordism map induced by \( S \).

*Proof.* This follows directly from the functoriality and the fact that \( f_S \) depends only on the based isotopy class of \( S \). \( \square \)

We can also explicitly calculate the equivariant contact class for some very simple transverse knots.
Example 8.5. Consider the trivial transverse braid $U$ and its positive/negative stabilizations $P$, $N$, respectively. We will see from the proof of the next theorem that $c_{Z_2}^*(\xi_U) = c_{Z_2}^*(\xi_P) = 1$ but $c_{Z_2}^*(\xi_N) = \theta$. This reflects the fact that, while the contact elements of $\xi_U$ and $\xi_P$ have Maslov degree zero, the contact element of $\xi_N$ has Maslov degree one, in the Floer chain complex of $S^3$.

Note that, while performing a positive stabilization to a transverse link (on any of its components) does not change its transverse isotopy class, performing a negative stabilization does change its transverse isotopy class. However, the topological isotopy class does not change under negative stabilizations, so the equivariant contact class of a transverse link and its positive stabilization lie in the same group. It turns out that the behavior of the equivariant contact class under a negative stabilization is very simple.

Theorem 8.6. Let $L$ be a transverse link in $(S^3, \xi_{std})$ and denote its negative stabilization (i.e. transverse stabilization), applied to any of its components, by $L^-$. Then we have

$$c_{Z_2}^*(\xi_{L^-}) = \theta \cdot c_{Z_2}^*(\xi_L).$$

Proof. Put $L$ in a braided position. Then the negative stabilization $L^-$ is given by adding a negative twist to the last two strands in $L \coprod U$ where $U$ is the trivial braid. So the equivariant Heegaard diagram for $\Sigma(L^-)$ near the last strand is given by Figure 8.1. Put $x = EH(\xi_L) \otimes q$. Then, in the dual of the freed Floer complex, we have

$$dz_2 x = EH(\xi_L) \otimes p \otimes \theta^0 + EH(\xi_L) \otimes (q + \tau q) \otimes \theta^1 = EH_{Z_2}(\xi_L) + EH(\xi_L) \otimes (q + \tau q) \otimes \theta^1.$$ 

Now consider the equivariant triple Heegaard diagram in Figure 8.2 which describes the negative stabilization $L^-$ and the positive stabilization $L^+$ of $L$: The two shaded triangles are the only holomorphic triangles in the above diagram, so by the associativity of equivariant triangle maps, working with equivariant contact classes in equivariant HF$\bar{s}$ should be the same as working with contact classes in ordinary HF$\bar{s}$. In other words, we have

$$c_{Z_2}^*(\xi_{L^+}) = [EH_{Z_2}(\xi_{L^+})] = [EH(\xi_L) \otimes (q + \tau q) \otimes \theta^0] \in \widehat{HF}_{Z_2}(\Sigma(L)).$$

Therefore we get

$$c_{Z_2}^*(\xi_{L^-}) = \theta \cdot c_{Z_2}^*(\xi_{L^+}) = \theta \cdot c_{Z_2}^*(\xi_L).$$

□

![Figure 8.1](image-url)
A TRANSVERSE LINK INVARIANT FROM $\mathbb{Z}_2$-EQUIVARIANT HEegaRD FLOER COHOMOLOGY

Figure 8.2. Relevant triangles in the associated equivariant Heegaard triple-diagram, each of which has zero Maslov index.

Finally, using the naturality and functoriality of equivariant Heegaard Floer cohomology, we can construct an isotopy invariant of slice disks of a given slice knot in $S^3$, as follows. Note that a similar invariant can be constructed using functoriality of knot Floer homology under decorated cobordisms, as the $t_{S,P}$ invariant in [JM].

**Theorem 8.7.** Let $K \subset S^3$ be a (smoothly) slice knot, and $D \subset B^4$ be a slice disk which bounds $K$. Choose any point $p \in D$, its neighborhood $B(p) \subset B^4$, and draw a smooth simple arc $s$ on the annulus $D^2 - B(p)$, so that $s \cap K$ is a point and $(D^2 - B(p), s)$ is a based cobordism between $(K, s \cap K)$ and the based unknot. Consider the induced cobordism map:

$$
\hat{f}_{(D^2 - B(p), s)} : \widehat{HF}_{Z_2}(\Sigma(\text{unknot}), pt) \to \widehat{HF}_{Z_2}(\Sigma(K), s \cap K).
$$

Then the element $\hat{f}_{(D^2 - B(p), s)}(x)$, where $x \in \widehat{HF}_{Z_2}(\Sigma(\text{unknot}), pt)$, is nonvanishing and depends only on the isotopy class of $D$ rel $K$.

**Proof.** The fact that $\hat{f}_{(D^2 - B(p), s)}(x)$ depends only on the isotopy class of $D$ rel $K$ following from Theorem 6.17 since $D^2 - B(p)$ is an annulus and thus the choice of $s$ is unique up to homotopy. The fact that $\hat{f}_{(D^2 - B(p), s)}(x)$ is nonvanishing follows from Theorem 8.2 and Lemma 6.11 of [HLS].

9. VANISHING AND NONVANISHING OF $c_\xi(K)$

Recall that, for any knot $K \subset S^3$, we have a spectral sequence

$$
E_1 = \widehat{HF}^*(\Sigma(K)) \otimes_{Z_2} F_2[\theta] \Rightarrow \widehat{HF}_{Z_2}(\Sigma(K)),
$$

constructed in Section 6.1 of [HLS], whose pages depend only on the isotopy class of $K$. This spectral sequence is induced by the $\theta$-filtration on $\mathcal{C}\bar{F}_{Z_2}(\Sigma(K))$, which can be written up to quasi-isomorphism as

$$
\mathcal{C}\bar{F}_{Z_2}(\Sigma(K)) = (\mathcal{C}\bar{F}(\Sigma(K)) \otimes_{Z_2} F_2[\theta], d_{Z_2}),
$$

$$
d_{Z_2}(x \otimes \theta^i) = dx \otimes \theta^i + (x + \tau x) \otimes \theta^i,
$$

where $d$ denotes the differential on the cochain complex $\mathcal{C}\bar{F}^*(\Sigma(K))$ and $\tau$ denotes the generator of the $Z_2$-action.

From the construction on the transverse knot invariant $c_{Z_2}(\xi_K)$, we know that, given a transverse braid representation of $K$ along the z-axis, we have an element

$$
EH_{Z_2}(\xi_K) = EH(\xi_K) \otimes \theta^0 \in \mathcal{C}\bar{F}_{Z_2}(\Sigma(K)),
$$
which is a $d_{\mathbb{Z}_2}$-cocycle. The same element represents $c(\xi_K) \otimes \theta^0$ in the $E_1$ page, so we see that the element $c(\xi_K) \otimes \theta^0$ in the $E_1$ page of our spectral sequence induces an element on each page.

However, we have to be careful here: the limit of $c(\xi_K) \otimes \theta^0$ on the $E^\infty$ page is not $c_{\mathbb{Z}_2}(\xi_K)$. This is because our spectral sequence actually does not converge directly to $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, but instead converges to its associated graded module

$$\text{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) = \bigoplus_{i=0}^\infty \theta^i \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta^{i+1} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

Thus, considering the bigrading on each page of the sequence, we see that the limit of $c(\xi_K) \otimes \theta^0$ is the image $c_{\mathbb{Z}_2}(\xi_K)$ under the following map:

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \rightarrow \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \hookrightarrow \text{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

Similarly, for every nonnegative integer $n$, the limit of $c(\xi_K) \otimes \theta^n$ is the image of $\theta^n \cdot c_{\mathbb{Z}_2}(\xi_K)$ under the map

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \rightarrow \theta^n \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta^{n+1} \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \hookrightarrow \text{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)).$$

For simplicity, we will denote that image by $c^n_{\mathbb{Z}_2}(\xi_K)$.

Now suppose that the transverse knot $K$ achieves equality in the inequality of Corollary 8.3, i.e. we have

$$\text{gr}(c_{\mathbb{Z}_2}(\xi_K)) = d_3(\xi_K) + \frac{1}{2} = \frac{q_\tau(K) - 1}{2}.$$

Then $c_{\mathbb{Z}_2}(\xi_K)$ lies in the smallest possible grading among all elements of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ not annihilated by any powers of $\theta$. Under this setting, $c_{\mathbb{Z}_2}(\xi_K)$ cannot be written as a multiple of $\theta$ by the grading minimality condition, so we have $c^n_{\mathbb{Z}_2}(\xi_K) \neq 0$ in $\text{gr}_\theta \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, and similarly we have $c^n_{\mathbb{Z}_2}(\xi_K)$ for every nonnegative integer $n$. This simple fact can now be used to prove the following nonvanishing condition for $c(\xi_K)$.

**Theorem 9.1.** Let $K$ be a knot in $S^3$, and suppose that a transverse representative $T$ of $K$ satisfies $d_3(\xi_K) = \frac{q_\tau(K) - 1}{2}$. Then the following statements hold.

1. $c(\xi_T) \neq a + \tau^a$ for every $a \in \widehat{HF}(\Sigma(K))$, where $\tau$ denotes the deck transformation of the branched covering map $\Sigma(K) \rightarrow S^3$.

2. The cardinality of the set

$$\left\{ c(\xi_T) \right\}$$

is at most half of the cardinality of the $d_3(\xi_T) + \frac{1}{2}$-graded component of $\widehat{HF}(\Sigma(K), s_0^K)$, where $s_0^K$ is the Spin$^c$-structure on $\Sigma(K)$ induced by the unique spin structure on $\Sigma(K)$.

**Proof.** The statement $c(\xi_T) \neq a + \tau^a$ for every $a \in \widehat{HF}(\Sigma(K))$ means that the element $c(\xi_K) \otimes \theta^1$ in the $E_1$ page also survives in the $E_2$ page. By the condition we imposed on $K$, we have $c^n_{\mathbb{Z}_2}(\xi_K) \neq 0$ for every nonnegative integer $n$, which means that the element $c(\xi_K) \otimes \theta^n$ in the $E_1$ page must survive on every page. This proves (1).

Also, statement (3) follows from the fact that the rank of the $(q_\tau(K) + N)$-graded component of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K), s_0^K)$ converges to 1 under the limit $N \rightarrow \infty$ (and that $c(\xi_K) \otimes \theta^N$ must survive in every page). Note that this a direct corollary of the existence of the localization isomorphism

$$\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2[\theta, \theta^{-1}] \rightarrow \mathbb{F}_2[\theta, \theta^{-1}].$$

On the other hand, by analyzing the structure of $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$, we can find a condition which implies the vanishing of $c(\xi_T)$ for a transverse representative $T$ of $K$. We start by observing that taking a quotient of $\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K))$ by the $\theta$-action gives the ordinary Floer cochain complex $\widehat{CF}^*(\Sigma(K))$:}

$$\widehat{CF}_{\mathbb{Z}_2}(\Sigma(K)) \otimes_{\mathbb{F}_2[\theta]} \mathbb{F}_2 \simeq \widehat{CF}^*(\Sigma(K)).$$


We already know from Theorem 8.1 that the induced map
\[ \tilde{HF}_{\mathbb{Z}_2}(\Sigma(K)) \to \tilde{HF}^*(\Sigma(K)) \]
maps \( c_{\mathbb{Z}_2}(\xi_K) \) to \( c(\xi_K) \). Now, since the above map is an \( \mathbb{F}_2[\theta] \)-module homomorphism and the \( \theta \)-action on its codomain \( \tilde{HF}^*(\Sigma(K)) \) is trivial, we get a map
\[ \tilde{HF}_{\mathbb{Z}_2}(\Sigma(K))/\theta \tilde{HF}_{\mathbb{Z}_2}(\Sigma(K)) \to \tilde{HF}^*(\Sigma(K)), \]
which then maps \( c_0(\xi_K) \) to \( c(\xi_K) \). Hence, if \( c_{\mathbb{Z}_2}(\xi_K) \) is divisible by \( \theta \), then \( c(\xi_K) = 0 \). This can be used to prove a vanishing property of \( c(\xi_K) \).

**Definition 9.2.** Given a knot \( K \) in \( S^3 \), we define \( v_r(K) \) to be the biggest nonnegative integer \( N \) such that there exists an element in the \((q_r(K) + N)\)-graded piece of \( \tilde{HF}_{\mathbb{Z}_2}(\Sigma(K), s^K_0) \) which is not annihilated by any powers of \( \theta \) and not divisible by \( \theta \); such an integer always exists by the existence of the localization isomorphism. Here, \( s^K_0 \) is defined as in the statement (2) of Theorem 9.1.

**Theorem 9.3.** Let \( K \) be a knot in \( S^3 \) and \( T \) be a transverse representative of \( K \). Then \( c(\xi) \neq 0 \) if \( d_3(\xi_K) = \frac{q_r(\xi_k)}{2} \) and \( c(\xi) = 0 \) if \( d_3(\xi_K) > \frac{q_r(\xi_k)}{2} + v_r(K) \).

**Proof.** If \( d_3(\xi_K) > v_r(K) - \frac{1}{2} \), then \( c_{\mathbb{Z}_2}(\xi_K) \) is divisible by \( \theta \), so \( c(\xi_K) = 0 \). On the other hand, if \( d_3(\xi_K) = \frac{q_r(\xi_k)}{2} \), then \( c(\xi_K) \neq a + \tau^*a \) for any \( a \in \tilde{HF}_{\mathbb{Z}_2}(\Sigma(K)) \) by Theorem 9.1, so \( c(\xi_K) \neq 0 \) as \( 0 + \tau^*0 = 0 \).

Theorem 9.3 can be seen as a Heegaard Floer analogue of the following theorem of Plamenevskaya.

**Theorem 9.4 (Theorem 1.2, [P]).** If \( K \) is a transverse knot such that \( \text{s}(K) = \text{s}(K) - 1 \), then \( \psi(K) \neq 0 \), where \( \text{s}(K) \) stands for the Rasmussen invariant [R], and \( \psi(K) \) stands for the Plamenevskaya invariant [P2]. The converse holds if \( K \) is Kh\( \mathbb{Z}_2 \)-thin.

As its corollary, Plamenevskaya gives a vanishing/nonvanishing property of \( c(\xi_K) \) when \( K \) is a transverse representative of a quasi-alternating knot.

**Corollary 9.5 (Corollary 1.3, [P]).** Let \( K \) be a transverse representative of a quasi-alternating knot. Then \( c(\xi_K) \neq 0 \) if and only if \( \text{s}(K) = \sigma(K) - 1 \).

We will now show that Corollary 9.5 is also a direct consequence of Theorem 9.3 by giving another proof of it.

**Proof.** When \( K \) is a quasi-alternating knot and \( T \) is a transverse representative of \( K \) which satisfies \( \text{s}(T) = \text{s}(K) - 1 \), then by Proposition 6 of [P], we have
\[ d_3(\xi_T) = 3 \sigma(X) - \frac{1}{2} \text{s}(T) = -\frac{3}{4} \sigma(X) - \frac{1}{2} \sigma(K) - \frac{1}{2}, \]
where \( X \) is a 4-manifold consisting only of 2-handles, as defined in section 3.1 of [P]. Note that the original formula of Plamenevskaya is wrong by an additive factor of \( \frac{1}{2} \); it is easy to check it by testing it with the transverse unknot, which has self-linking number \(-1\).}

Now, by the construction of \( X \), it is a branched double cover of the 4-ball \( B^4 \) along a smooth surface which bounds \( K \), but with the opposite orientation, so by Theorem 3.1 of [KT], we have \( \sigma(X) = -\sigma(K) \). Hence we get
\[ d_3(\xi_T) = \frac{3}{4} \sigma(K) - \frac{1}{2} \sigma(K) - \frac{1}{2} = \frac{1}{4} \sigma(K) - \frac{1}{2}. \]
On the other hand, since \( \Sigma(K) \) is an L-space, we have
\[ \frac{q_r(K)}{2} = d(K, s^K_0), \]
which is then equal to \( \frac{\sigma(K)}{4} \) by Theorem 1 of [LO]; note that we are using the sign convention which makes the right handed trefoil have signature 2. Also, \( v_r(K) = 0 \) by the same reason. Hence we finally get
\[ d_3(\xi_T) + \frac{1}{2} = \frac{q_r(K)}{2} = \frac{q_r(K)}{4} + v_r(K). \]
Therefore, by Theorem 10.3 $c(\xi_T) \neq 0$. Similarly, we can prove that $c(\xi_T) = 0$ if $\text{sl}(T) > s(K) - 1$ by replacing equalities by inequalities.

10. Conclusion

Given a based link $(L, p)$ in $S^3$, the isomorphism type of $\tilde{HF}_{Z_2}(\Sigma(L), p)$ is well-defined. When $L = K$ is a knot, then $\tilde{HF}_{Z_2}(\Sigma(K), p)$ satisfies naturality, and an isotopy class of a based cobordism between two based knots induces a uniquely defined map between $\tilde{HF}_{Z_2}$.

Given a transverse based link $L$ in $(S^3, \xi_{std})$, we have constructed a distinguished element

$$c_{Z_2}(\xi_L) \in \tilde{HF}_{Z_2}(\Sigma(L), z),$$

which is invariant under transverse isotopies and thus is a transverse based link invariant. When $L = K$ is a transverse knot, then we can talk about its image under cobordism maps; it satisfies functoriality under symplectically constructible based cobordisms. When we work with a transverse knot $K$, we can forget the choice of a basepoint, so we get an element

$$c_{Z_2}(\xi_K) \in \tilde{HF}_{Z_2}(\Sigma(K)),$$

which may not be functorial under unbased cobordisms. The most natural question to ask about it would be about the effectivity of $c_{Z_2}$ in distinguishing topologically isotopic transverse knots.

Recall that we have two important classical invariants of transverse knot invariants:

- the topological knot type;
- the self-linking number.

Thus a transverse knot invariant is said to be effective if it can distinguish two transverse knots, which are topologically isotopic and have the same self-linking number, but are not transversely isotopic, i.e. isotopic through transverse knots.

The LOSS invariant, defined in $\text{LOSS}_{Z_2}$, is an example of an effective transverse knot invariant, lying in the knot Floer homology of a given transverse knot:

$$\hat{c}(K) \in \tilde{HFK}(-Y,K),$$

$$c^-(K) \in HFK^-(Y,K).$$

The invariant $c^-$ has some basic properties which similar to some properties of equivariant contact classes. First of all, it lies in the knot Floer homology of $(Y,K)$, which means that it is a cohomological invariant in $(Y,K)$. Also, if $K^-$ is the transverse(negative) stabilization of $K$, then we have

$$c^-(K^-) = U \cdot c^-(K),$$

which is very similar to the property $c_{Z_2}(\xi_{K^-}) = \theta \cdot c_{Z_2}(\xi_K)$. So, it is natural to ask the following question.

**Question.** Is there a way to calculate $c^-(K)$ or $\hat{c}(K)$ using $c_{Z_2}(\xi_K)$? If not, then is there a way to calculate $c_{Z_2}(\xi_K)$ using $c^-(K)$?

Of course, if we can recover either $c^-(K)$ or $\hat{c}(K)$ from $c_{Z_2}(\xi_K)$, then we immediately see that $c_{Z_2}$ is an effective transverse link invariant. However, even if we cannot, we can still ask whether the equivariant contact class is an effective invariant:

**Question.** Is there a knot $K$ in $S^3$ such that there exist two transverse knots $K_1, K_2$ in $(S^3, \xi_{std})$, topologically isotopic to $K$, with $\text{sl}(K_1) = \text{sl}(K_2)$, but

$$c_{Z_2}(K_1) \neq c_{Z_2}(K_2)$$

as elements in $\tilde{HF}_{Z_2}(\Sigma(K))$?

If the answer to the above question is yes, then we get a new effective transverse invariant. However, if the answer is no, we also have an interesting consequence. Given a knot $K$ in $S^3$, let $K_1, K_2$ be two transverse knots, topologically isotopic to $K$. Then, we apply the following well-known theorem.

**Theorem.** Any two topologically isotopic transverse knots are related by a sequence of (de)stabilizations.
Using the above theorem, suppose that the $n$th and $m$th positive stabilizations of $K_1$ and $K_2$ are transversely isotopic, and assume that $n \geq m$. Then the $n - m$th positive stabilization $K_1^{n-m}$ of $K_1$ satisfies $\text{sl}(K_1^{n-m}) = \text{sl}(K_2)$. Hence, by the assumed non-effectiveness of $c_{Z^2}$ and its behavior under positive stabilizations, we get

$$c_{Z^2}(\xi_{K_2}) = c_{Z^2}(\xi_{K_1^{n-m}}) = c_{Z^2}(\xi_{K_1}) \cdot \theta^{n-m}.$$ 

Hence, if $T_K$ is the transverse representative of $K$ with the minimal self-linking number, then for any other transverse representative $T$ of $K$, we must have

$$c_{Z^2}(\xi_T) = c_{Z^2}(\xi_{T_K}) \cdot \theta^{nt}$$

for some $nt \geq 0$. Therefore we deduce that the subset

$$\{c_{Z^2}(\xi_T) | T \text{ is a transverse representative of } K\} \subset \widehat{HF}_{Z^2}(\Sigma(K))$$

is a single $\theta$-tower; its minimal-order element $c_{Z^2}(\xi_{T_K}) \in \widehat{HF}_{Z^2}(\Sigma(K))$ becomes a knot invariant.

**Question.** Is there a non-constructible symplectic cobordism between two transverse knots in the standard contact $S^3$?

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