A SHORT PROOF THAT THE FREE ASSOCIATIVE ALGEBRA IS HOPFIAN

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Abstract. A short proof is given of the fact that for various classes of algebras including the free associative algebra are Hopfian, i.e., every epimorphism is an automorphism. This further simplifies the Dicks-Lewin solution of the Jacobian conjecture for the free associative algebra. It also relates to the Jacobian conjecture for meta-abelian algebras.

1. Introduction and main results

The main object of this note is to use PI-theory to simplify the results of Dicks and Lewin [6] on the automorphisms of the free algebra $F\{X\}$, namely that if the Jacobian is invertible, then every endomorphism is an automorphism. Their proof is in two parts: Every epimorphism $F\{X\} \to F\{X\}$ is an isomorphism, and if the Jacobian is invertible, then every endomorphism is an epimorphism. We follow this route, and see how these arguments apply to wider classes of rings. In the appendix, we tie this in with the Jacobian conjecture.

2. Hopfian rings

Definition 2.1. An algebra $R$ is Hopfian if every epimorphism (i.e., onto algebra homomorphism) $R \to R$ is an isomorphism.

Dicks and Lewin reduced the Jacobian conjecture for $F\{X\}$ to the question of whether $F\{X\}$ is Hopfian, and proved it (using some deep
theory) for the free algebra in two variables. In fact, this had already been resolved for any finite set of variables by Orzech and Ribes [9, 10], with a rather direct proof given in [5].

In this note, relying on considerations of growth, we give a quick and easy proof of a more general result (based on facts about the Gelfand-Kirillov dimension) which implies at once that the free associative algebra $F\{X\}$ is Hopfian.

Recall that the Gelfand-Kirillov dimension $\text{GKdim}(A)$ of an affine algebra $A = F\{a_1, \ldots, a_\ell\}$ is

$$(1) \quad \text{GKdim}(A) := \lim_{n \to \infty} \log_n d_n,$$

where $A_n = \sum F a_{i_1} \cdots a_{i_n}$ and $d_n = \dim_F A_n$.

The standard reference on Gelfand-Kirillov dimension is [8]. Although the $d_n$ depend on the choice of the generating set $a_1, \ldots, a_\ell$, $\text{GKdim}(A)$ is independent of the choice of the generating set. Let us tighten this fact a bit: Suppose that $A' = F\{a_1', \ldots, a_\ell'\}$ and $d'_n = \dim_F A'_n$. We say that the growth rate of the $d_n$ is less than or equal to the growth rate of the $d'_n$ if there are constants $c, k$ such that $d'_n \leq cd^k n$. This defines an equivalence, and it is easy to see that the growth rate of $A$ with respect to any two sets of generators is the same.

**Lemma 2.2.** Suppose $R$ is an affine algebra in which the growth of $R/I$ is less than the growth of $R$, for each ideal $I$ of $R$. Then $R$ is Hopfian.

In particular, if $\text{GKdim}(R/I) < \text{GKdim}(R)$ for all ideals $I$ of $R$, then $R$ is Hopfian.

**Proof.** For any epimorphism $\varphi : R \to R$, one has $\varphi(R) \cong R / \ker \varphi$, but then $\varphi(R)$ and $R$ have the same growth rates, implying $\ker \varphi = 0$. □

The hypothesis of Lemma 2.2 holds for prime PI-algebras, cf. [4, Theorem 11.2.12], so we have:

**Corollary 2.3.** Any prime affine PI-algebra is Hopfian.

On the other hand, [1] provides an affine PI-algebra that is not Hopfian.
Example 2.4. $R$ and $R/I$ could have different growth rates even if $\text{GKdim}(R/I) = \text{GKdim}(R)$. For example, let $R$ be the subalgebra of the free associative algebra generated by all subwords of $u_n$ for any $n$, where $u_1 = xyx$ and $u_{n+1} = x^{10^n}u_n^{10^n}yx^{10^n}u_n^{10^n}$, a prime algebra, of $\text{GKdim} 2$, and $I$ be the ideal generated by all words of degree 2 in $y$. Then $\text{GKdim}(R/I) = 2$, although the growth rate of $R/I$ is less than that of $R$. This example is not a PI-algebra.

A $T$-ideal of an ideal $R$ is an ideal invariant under all ring endomorphisms.

Lemma 2.5. If $\mathcal{I}$ is a $T$-ideal of $R$, then any endomorphism $\varphi$ of $R$ induces an endomorphism of $R/\mathcal{I}$.

Proof. Define $\varphi : R/\mathcal{I} \to R/\mathcal{I}$ by $\varphi(a + \mathcal{I}) = \varphi(a) + \mathcal{I}$. This is well-defined since $\varphi(\mathcal{I}) \subseteq \mathcal{I}$ by hypothesis. □

We say that $R$ is $T$-residually Hopfian if the intersection of those $T$-ideals $I$ of $R$ for which $R/I$ is Hopfian is 0.

The following result, whose proof follows that of [3, Theorem 7], attributed to Markov, unifies instances of Hopfian algebras.

Proposition 2.6. Any $T$-residually Hopfian algebra is Hopfian.

Proof. Let $\varphi : R \to R$ be an epimorphism, with some nonzero element $r \in \ker(\varphi)$. By hypothesis there is some $T$-ideal $\mathcal{I}$ not containing $r$, for which $R/\mathcal{I}$ is Hopfian, but Lemma 2.5 implies that $R/\mathcal{I}$ is not Hopfian, a contradiction. □

We are ready to treat the free algebra.

Theorem 2.7 ([9]). When $X$ is a finite set of noncommuting indeterminates, the free associative algebra $F\{X\}$ is Hopfian.

Proof. Let $\varphi : F\{X\} \to F\{X\}$ be an epimorphism, with some nonzero polynomial $f \in \ker(\varphi)$. Let $n = \deg(f)$. Let $\mathcal{I}_n$ be the $T$-ideal of identities of the algebra of generic $n \times n$ matrices. Then $\cap \mathcal{I}_n = 0$, so $F\{X\}$ is $T$-residually Hopfian, and we apply Proposition 2.6. □
3. Appendix: The Jacobian Conjecture and the Free Metabelean Algebra

Dicks and Lewin [6] Proposition 3.1] proved that an endomorphism of the free associative algebra \( F\{X\} \) is an epimorphism iff its Jacobian matrix is invertible. Their proof was utilized by Umirbaev to obtain the following result:

**Theorem 3.1** ([12]). If any endomorphism \( \varphi \) of the polynomial ring with Jacobian 1 could be lifted to some endomorphism of the free metabelean algebra with invertible Jacobi matrix, then the Jacobian Conjecture would hold.

**Remark 3.2.** The free meta-abelian algebra was used by Umiibaev [13], and Drensky and Yu [7] to prove the Anick and strong Anick conjectures.

Also see [11] for a treatment of the Jacobian conjecture over a free algebra, using his deep theory of skew fields, and [2] for an overview of Yagzev’s method to attack the Jacobian conjecture.

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