Note on Entanglement Temperature for Low Thermal Excited States in Higher Derivative Gravity

Wu-zhong Guo, a,b,c Song He, a,b Jun Tao d,a,b

a State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Science, Beijing 100190, People’s Republic of China
b Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China
c University of the Chinese Academy of Sciences, Beijing 100049, China
d Center for Theoretical Physics, College of Physical Science and Technology, Sichuan University, Chengdu, 610064, PR China

E-mail: wuzhong@itp.ac.cn, hesong@itp.ac.cn, taojun@scu.edu.cn

Abstract: We investigate the entanglement temperature of a small scale subsystem in low excited states by using holographic method. Especially, we study the entanglement entropy and entanglement temperature in higher derivative gravities which are considered as low thermal excitation of pure AdS gravity. We find that the entanglement entropy are related to the central charges of CFT living on the boundary. The relation between the variance of entanglement entropy and energy of a small scale subsystem has been also obtained. Furthermore, the relation is consistent with the first law-like relation that is proposed by Phys. Rev. Lett. 110, 091602 (2013). Finally, we derive the formula of the variance of entanglement entropy in general excited states in gravity background with the Fefferman-Graham coordinates and the entanglement temperature can be figured out in special case.

Keywords: Higher derivative gravity, Holographic entanglement entropy, gauge/gravity duality
1 Introduction

The AdS/CFT correspondence [1][2][3][4](also called by gauge/gravity duality) is a very important and fundamental relation which connects gravitational theories and quantum field theories. As an application in [5] Ryu and Takayanagi proposed a framework for calculation of entanglement entropy of conformal field theory through AdS/CFT correspondence. Their approach is simple and elegant. The main point is that the entanglement entropy in
field theory side can be mapped to an area of minimal surface in gravity side. The holographic entanglement entropy (HEE) have been proved in [6][7]. And there are so many evidences [8][9][10] to confirm the correspondence within $AdS_3/CFT_2$. As application of HEE, there are intensive studies [11–23] recently. More recently, in [24], a free falling particle in an AdS space mimic the holographic dual of local quenches and the HEE has been computed to show the evolution of quantum entanglement. In [25], an analytical framework for holographic counterpart of global quantum quenches. In [28], authors studied how a small perturbation of HEE evolves dynamically through solving the Einstein equation in AdS spaces.

In vacuum state entanglement entropy (EE) is proportional to the surface area (in many models)[26][27] in the leading divergent term, which is the original motivation for relating EE with black hole entropy. The EE is also an useful quantity to describe the quantum correlations between the in and out side of a subsystem in QFT. The behavior of EE in low excited states is also important to understand the quantum entanglement nature of the system. This topic has been studied by many authors, for example [30][31]. The elegant method of HEE could also be used to study the property of EE in low excited states of CFT, which may be related with the background perturbation of the bulk.

In [32] the authors go on this study about the low thermal excited state in the holographic view, and more, they find an interesting relation between the variance of energy and EE for the subsystem in low excited states of CFT, which is similar as the first law of thermodynamics, i.e., $\Delta E = T_{eff} \Delta S$, where they also introduce a universal quantity $T_{eff}$ called entanglement temperature, which is only related with shape of the subsystem.

In this paper we will study the property of EE after low excitation in the holographic view. Considering that the HEE formula should have quantum correction when the bulk theory has higher curvature terms[14][15][33], we expect new property of EE would appear in low excited states. The general formula of HEE with the bulk theory containing arbitrary higher curvature terms is still a open question needed to further study. But the HEE formula for Lovelock gravity has been studied in [14][15] by comparing the logarithm term with the CFT prediction. In terms of [15][14][20][34], one can study the HEE with higher derivative gravity and see what will happen for the variance of EE of a small scale subsystem. Furthermore, the energy momentum tensor will also include contribution from higher derivative gravity. So it would also be interesting to investigate the first law-like relation and entanglement temperature that are proposed in [32] then.

This paper is organized as follows. In section 2 we calculate the entanglement temperature with roll ball profile and infinite stripe profile with a small scale in the 5-dimensional Lovelock gravity and study correction of the first law-like relation for these subsystems. In section 3, we extend to the holographic entanglement entropy and entanglement temperature for 7-dimensional Lovelock gravity. In section 4, we analyze the holographic entanglement entropy and entanglement temperature in asymptotical AdS with Fefferman-Graham gauged background in diverse dimension. Section 5 is devoted to conclusions and discussions. We put some details of the computations in this paper in the Appendix.

While proceeding with this project, we learned that the small overlap topic in section 4 was also studied in recent papers [28][29] which are reminded by Tadashi Takayanagi and
2 Entanglement temperature in D=4 CFT with dual 5-dimensional Lovelock gravity

In this section, we will calculate the entanglement temperature in the 4-dimensional CFT with the dual 5-dimensional Lovelock gravity. To obtain the novel quantities, we should calculate the variance of entanglement entropy after low thermal excitation in two examples, i.e., the entangling surfaces are sphere and infinite stripe, and also the variance of the energy within the subsystem.

2.1 HEE for 4-dimensional CFT

The 5-dimensional Lovelock gravity can be realized by adding the Gauss-Bonnet term\[35\] to pure Einstein gravity theory. The theory can be described by the following action

\[I = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_5 L^2}{2} L_4 \right], \tag{2.1}\]

with

\[L_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \tag{2.2}\]

and \(\lambda_5\) denotes the coupling of Gauss-Bonnet gravity and \(L\) stands for the Radius of AdS background. In order to obtain a well-defined vacuum for the gravity theory, one has to have \(\lambda_5 \leq 1/4\). Additional constraint on the Gauss-Bonnet parameter \(\lambda_5\) by considering the positivity of the energy flux and causality on the boundary theory is \(-7/36 \leq \lambda_5 \leq 9/100\). The AdS Gauss-Bonnet (GB) gravity would admit a pure AdS solution\[15\],

\[ds^2 = \tilde{L}^2 \left( -dt^2 + dz^2 + dx_1^2 + dx_2^2 + dx_3^2 \right), \tag{2.3}\]

\(\tilde{L}\) is the effective AdS radius in Gauss-Bonnet gravity and is defined by \(\tilde{L}^2 = \frac{L^2}{f_\infty}\) with

\[f_\infty = \frac{1 - \sqrt{1 - 4\lambda_5}}{2\lambda_5}. \tag{2.4}\]

The holographic entanglement entropy formula for the GB case is discussed in \[14\]|\[15\]|\[33\]. The subsystem in a time slice of boundary is \(A\), we recall that the HEE formula should be

\[S_A = \frac{2\pi}{\ell_p^3} \int_M d^3x \sqrt{h} \left[ 1 + \lambda_5 L^2 \mathcal{R} \right] + \frac{4\pi}{\ell_p^3} \int_{\partial M} d^2x \sqrt{h} \lambda_5 L^2 \mathcal{K}, \tag{2.5}\]

where the first integral is evaluated on the bulk surface \(M\), the second one is on \(\partial M\), which is the boundary of \(M\) regularized at \(z = \epsilon\), \(\mathcal{R}\) is the Ricci scalar for the intrinsic geometry of \(M\), and \(\mathcal{K}\) is the trace of the extrinsic curvature of the boundary of \(M\), \(h\) is the determinant of the induced metric on \(M\). The second term in the first integral is present due to higher derivative gravity in the background. The minimal value of the functional (2.5) would give the entanglement entropy of the subsystem \(A\).
2.2 Variance of EE after low excitation

Our purpose is to discuss the variance of the entanglement entropy for the low excited state of CFT from the holographic point of view. Following the logic given by [32], the low exciting state of the CFT is dual to the asymptotical AdS background and the pure AdS solution can be considered as the ground state of AdS GB gravity. Here we will assume the low excited state with the dual gravity background as\(^1\)[32]

\[
ds^2 = \frac{\tilde{L}^2}{z^2}(-f(z)dt^2 + \frac{dz^2}{g(z)} + dx_1^2 + dx_2^2 + dx_3^2),
\]

(2.6)

where \(f(z) \simeq g(z) = 1 - mz^4\), \(m\) reflects the asymptotical behavior of gravity background near the boundary. We identify the excitation as low thermal excitation in this paper.

In the next two subsections, we would like to discuss two special examples to see the behavior of the variance of the entanglement entropy after the low thermal excitation (2.6).

2.2.1 Subsystem with a round ball configuration

In this subsection, we choose the subsystem \(A\) as a two-sphere with a radius \(R_0\) in a time slice of the CFT living on the boundary. The bulk surface \(M\) can be parameterized as \(r = r(u)\) and \(z = z(u)\), where \(r\) is the radial direction of the ball, with \(u_{\text{min}} \leq u \leq u_{\text{max}}\).

So the induced line element on the bulk surface is

\[
ds_D^2 = \frac{\tilde{L}^2}{z^2} \left( (1 + mz^4) \dot{z}^2 + \dot{r}^2 \right) du^2 + r^2 d\Omega^2,
\]

(2.7)

where the dot denotes the derivative with respect to \(u\). Following the notation of [15], we define

\[
h_{uu} = h_{uu}^{-1} = \frac{\tilde{L}^2}{z^2} \left( (1 + mz^4) \dot{z}^2 + \dot{r}^2 \right), \quad F = \ln \left( \frac{r}{z} \right).
\]

(2.8)

We can also get the intrinsic Ricci scalar of \(M\)

\[
\mathcal{R}_D = e^{-2F} \frac{2}{L^2} - 4\Delta_u F - 6h^{uu} \dot{F}^2,
\]

(2.9)

where \(\Delta_u F = \frac{1}{\sqrt{h_{uu}}} \partial_u \left( \frac{1}{\sqrt{h_{uu}}} \dot{F} \right)\). The trace of extrinsic curvature \(\mathcal{K}\) of the boundary \(\partial M\) is

\[
\mathcal{K} = -\frac{2}{\sqrt{h_{uu}}} \dot{F} \bigg|_{u=u_{\text{min}}}
\]

(2.10)

\(^1\)We recall that the 5-dimensional Gauss-Bonnet-AdS black brane solution with a Ricci flat horizon is given by [40]

\[
ds_{BB}^2 = \frac{L^2}{z^2} \left( -f(z)dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{dz^2}{f(z)} \right),
\]

\[
f(z) = \frac{1}{2\lambda_5} \left( 1 - \sqrt{1 - 4\lambda_5 \left( 1 - \frac{4}{z_h^4} \right)} \right),
\]

where \(z_h\) is the horizon of the black brane. The asymptotical AdS background (2.6) is actually the above solution with the horizon \(z_h\) far away from the boundary. So we may call this kind of excited state of the CFT living on the boundary as the low thermal excitation.
Plug (2.10) and (2.9) into (2.5), one can obtain
\[
S_A = \frac{8\pi^2 \tilde{L}^3}{\ell_p^3} \int du \sqrt{R_{uu}} e^{2F}(1 + L^2 \lambda_5 (e^{-2F} \frac{2}{L^2} + 2h^{uu}(\tilde{F})^2)).
\]  
(2.11)

It is convenient to parameterize \( r \) and \( z \) as following form
\[
r(u) = f \left( \frac{u}{R_0} \right) \cos \left( \frac{u}{R_0} \right), \quad z(u) = f \left( \frac{u}{R_0} \right) \sin \left( \frac{u}{R_0} \right), \quad \text{with} \quad \epsilon \leq u \leq \frac{\pi}{2} R_0.
\]
(2.12)

Finally, we get \( S_A \) as a functional of \( f \left( \frac{u}{R_0} \right) \). The minimal value of functional \( S_A \) corresponds to the holographic entanglement entropy. We first need to get the surface that minimizes the functional when the perturbation parameter \( m \) is turned off. \( S_A \) is functional of \( f(x) \) as
\[
S_A = \frac{8\pi^2 \tilde{L}^3}{\ell_p^3} \int_{\epsilon/R_0}^{\frac{\pi}{2}} dx \frac{\cos^2(x)}{\sin^3(x)} \sqrt{1 + \left( \frac{d\log f}{dx} \right)^2} \left( 1 + 2 \left( \frac{L}{\tilde{L}} \right)^2 \lambda_5 (\tan^2(x) + \frac{\cos^{-2}(x)}{1 + (d\log f/dx)^2}) \right),
\]
(2.13)

where we denote \( \frac{\epsilon}{R_0} \). One should note that we should introduce the regulation procedure to make the \( S_A \) to be finite. The regulation scale is denoted by the \( \epsilon \) in (2.13). Using Euler-Lagrange equation, we get
\[
\frac{d}{dx} \left[ \frac{d\log f \cos^2(x)}{\sin^3(x)} \left( 1 + 2 \frac{L}{\tilde{L}} \lambda_5 \tan^2(x) - \frac{f_{\infty} \lambda_5 \cos^{-2}(x)}{1 + (d\log f/dx)^2} \right) \right] = 0.
\]
(2.14)

One solution is that \( f(x) \) is a constant. We fix the boundary condition as \( f \left( \frac{\pi}{2} R_0 \right) = R_0 \). The minimal surface can be parameterized as
\[
r(u) = R_0 \cos(u/R_0), \quad z(u) = R_0 \sin(u/R_0), \quad \text{with} \quad \epsilon \leq u \leq \frac{\pi}{2} R_0.
\]
(2.15)

The holographic entanglement entropy for pure AdS state is
\[
S_A = \frac{8\pi^2 \tilde{L}^3}{\ell_p^3} \int_{\epsilon/R_0}^{\frac{\pi}{2}} dx \frac{\cos^2(x)}{\sin^3(x)} \left( 1 + 2 \left( \frac{L}{\tilde{L}} \right)^2 \lambda_5 (\tan^2(x) + \cos^{-2}(x)) \right).
\]
(2.16)

We formally turn on \( m \neq 0 \) (2.6) and consider it as a perturbation of the pure AdS background. Finally, we can consider the AdS-Gauss-Bonnet gravity with \( m \) as an excitation state with comparing with pure AdS with radius \( \tilde{L} \). Then the holographic entanglement entropy after the excitation is the minimal value of the functional
\[
\tilde{S}_A = \frac{8\pi^2 \tilde{L}^3}{\ell_p^3} \int_{\epsilon/R_0}^{\frac{\pi}{2}} dx \frac{\cos^2(x)}{\sin^3(x)} \sqrt{1 + mz^4 \lambda^2 \left( 1 + 2 \left( \frac{L}{\tilde{L}} \right)^2 \lambda_5 (\tan^2(x) + \cos^{-2}(x)) \right)}.
\]
(2.17)

We can calculate the variance of the holographic entanglement entropy up to \( O(m R_0^4) \) as follows
\[
\Delta S_A = \tilde{S}_A - S_A
\]
(2.18)
Considering the limit $mR_0^4 \ll 1$, we expand (2.17) to first order of $mR_0^4 \ll 1$ and use (2.15) and then the variance of holographic entanglement entropy up to $O(mR_0^4)$ is

$$\Delta S_A = \frac{8\pi^2 \tilde{L}^3}{\ell_p^3} m_0 \int dx \left( \frac{1}{2} \sin x \cos^4 x - \lambda_5 \frac{\tilde{L}^2}{L^2} \sin x \cos^4 x \right).$$

In the procedure of getting (2.19), we assume that the bulk surface that corresponds to the minimal functional does not change with turning on excitation. As appendix A shows, the configuration of bulk surface is protected after the low excitation up to $O(mR_0^4)$. As a consistent check, we reproduce the result given in [32] with the Gauss-Bonnet parameter $\lambda_5 = 0$. According to AdS/CFT correspondence, the Gauss-Bonnet gravity parameters $\lambda_5$ would related with the two kinds of central charges of the CFT side. The A-type and B-type central charges [41]

$$c = \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 - 2\lambda_5 f_\infty), \quad a = \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 - 6\lambda_5 f_\infty).$$

According to (2.20), we can rewrite $\Delta S_A$ as

$$\Delta S_A = \frac{4c}{5} m R_0^4.$$  

One can see that the variance of the entanglement entropy after the low thermal excitation is related to the central charges of CFT. One will also see this property in entanglement entropy of subsystem with an infinite stripe profile.

### 2.2.2 Subsystem with stripe configuration

In this subsection, we consider the subsystem with a stripe profile which is defined by $-\frac{l_0}{2} < x_1 \equiv x < \frac{l_0}{2}$ and $\frac{l_0}{2} < x_2, x_3 < \frac{l_0}{2}$ where $l_0$ is infinite. We also consider the AdS Gauss-Bonnet solution (2.6) as a excitation of pure AdS gravity. Then the induced metric $h_{\mu\nu}$ of the bulk surface after perturbation (or excitation) is

$$ds_S^2 = \tilde{L}^2 \left[ \frac{1}{z^2} \left( x^2 + 1 + mz^4 \right) dz^2 + dx_2^2 + dx_3^2 \right].$$

Where $x$ is parameterized as $x = x(z)$ and the dot stands for the derivative with respect to $z$ in this subsection. We denote

$$h_{zz} = \frac{\tilde{L}^2}{z^2} (x^2 + 1 + mz^4), \quad e^{2F} = \frac{\tilde{L}^2}{z^2}.$$  

$\text{Here we have used convention}$

$$\langle T^i_i \rangle = \frac{c}{16\pi^2} C_{ijkl} C^{ijkl} - \frac{a}{16\pi^2} (R_{ijkl} R^{ijkl} - 4R_{kl} R^{kl} + R^2).$$

Where the $\langle T^i_i \rangle$ stands for the trace anomaly which have been well studied and $a, c$ stands for different kinds of central charges. The $a$ and $c$ are equal at the limit $\lambda_5 \to 0$ [15].
The intrinsic Ricci curvature $R_S$ is
\[ R_S = -\frac{4}{\sqrt{h_{zz}}} \partial_z (\sqrt{h_{zz}} h^{zz} \dot{F}) - 6 h^{zz} \dot{F}^2. \tag{2.24} \]

The trace of extrinsic curvature $K$ of the boundary $z = \epsilon$ is
\[ K = -\frac{2}{\sqrt{h_{zz}}} \dot{F} \big|_{z=\epsilon}. \tag{2.25} \]

Plug (2.24) and (2.25) into (2.5)
\[ S_A = \frac{2\pi l_0^2}{\ell_p^3} \int dz \sqrt{h_{zz}} \frac{\tilde{L}^2}{z^3} (1 + 2 h^{zz} \dot{F}^2). \tag{2.26} \]

Firstly let’s turn off the perturbation, then the bulk is the pure AdS space. $S_A$ is a functional of $x(z)$ as follows
\[ S_A(m = 0) = \frac{4\pi l_0^2}{\ell_p^3} \int_{\epsilon}^{z_*} dz \frac{\tilde{L}^3}{z^3} (\sqrt{1 + \dot{x}^2} + 2 \lambda_5 \frac{L^2}{L^2} \frac{1}{\sqrt{1 + \dot{x}^2}}), \tag{2.27} \]
where $z_*$ is the maximal value of $z$ on the surface in the bulk, which is controlled by the following constraint
\[ \frac{l}{2} = \int_{0}^{z_*} dz \dot{x}, \tag{2.28} \]
with minimizing the functional (2.27), we get the following equation of motion
\[ \dot{x} \frac{1 - 2 \lambda_5 f_\infty + \dot{x}^2}{(1 + \dot{x}^2)^{3/2}} = \frac{z^3}{z_*^3} \tag{2.29} \]
This is a cubic equation for $\dot{x}$, there are three solutions for $\dot{x}$. We are only interested in the solution of this equation with $\lambda_5$ is near 0. In this case, we can continuously connect AdS Gauss-Bonnet gravity with the solution of $\lambda_5 = 0$, i.e., the pure Einstein gravity. Here we consider the pure AdS gravity as a ground state of AdS GB gravity. Assuming that $\lambda_5$ is close to 0 and $\lambda_5 f_\infty \ll 1$, we can solve the equation (2.29) with expanding as small parameter $\lambda_5 f_\infty$. Up to the $O(\lambda_5 f_\infty)$, we get
\[ \dot{x} = (1 + 2 \lambda_5 f_\infty) \frac{\frac{z^3}{z_*^3}}{\sqrt{1 - \frac{z^6}{z_*^6}}}. \tag{2.30} \]

Using the constraint (2.28), we can get $z_*$
\[ z_* = \frac{\Gamma(\frac{1}{2})}{2(1 + 2 \lambda_5 f_\infty) \Gamma(\frac{2}{3}) \sqrt{\pi}}. \tag{2.31} \]

With turning on the excitation (2.6), the functional of the entangling surface is
\[ S_A(m) = \frac{4\pi l_0^2}{\ell_p^3} \int_{\epsilon}^{z_*} dz \frac{\tilde{L}^3}{z^3} (\sqrt{1 + mz^4 + \dot{x}^2} + 2 \lambda_5 \frac{L^2}{L^2} \frac{1}{\sqrt{1 + mz^4 + \dot{x}^2}}). \tag{2.32} \]
Using the conclusion in Appendix A we get the variance of the HEE is
\[
\Delta S_A = S_A(m) - S_A(m = 0) = \frac{2mL^3\pi l_0^2}{\ell_p^3} \int_\epsilon^{\infty} dzz \frac{1 - 2\lambda_5 f_\infty + x^2}{(1 + x^2)^{3/2}}, \tag{2.33}
\]

Use \( ml^4 \ll 1 \) and plug the solution of \( x \) (2.30) into (2.34):
\[
\Delta S_A = \frac{2mL^3\pi l_0^2}{\ell_p^3} \int_\epsilon^{\infty} dzu \sqrt{1 - u^6}
= \frac{mL^3 \sqrt{\pi l_0^2}^2}{20(1 + 2\lambda_5 f_\infty)^{3/2}} \frac{\Gamma(\frac{1}{5}) \Gamma(\frac{1}{10})}{\Gamma(\frac{2}{5})^2 \Gamma(\frac{3}{5})}.
\tag{2.34}
\]
where we use (2.20) in the last step. \( \Delta S \) depends on the A-type central charge of CFT living on the boundary with the assumption that \( \lambda_5 f_\infty \ll 1 \). (2.34) will reproduce result in [32] with \( \lambda_5 \to 0 \).

### 2.3 Variance of energy and EE

In [32] they have proposed a universal relation between the variance of the energy and the entanglement entropy for a small subsystem A on the boundary theory. The universal relation induce a novel concept called by entanglement temperature. We would like to obtain the entanglement temperature in higher derivative gravity. We apply to calculate the boundary energy-stress tensor in the low excitation AdS Gauss-Bonnet background in Appendix C. The \( t - t \) component of the energy-stress tensor [42][43] corresponds to the energy density (C.8)
\[
T_{tt} = \frac{3mL^3(1 - 2\lambda_5 f_\infty)}{2\ell_p^3} \tag{2.35}
\]

For the sphere case, the variance of energy in the subsystem is
\[
\Delta E = \frac{2\pi m(1 - 2f_\infty \lambda_5) L^3 R_0^3}{\ell_p^3} \tag{2.36}
\]

The entanglement temperature defined by [32] is
\[
\frac{1}{T_{ent}} = \frac{\Delta S}{\Delta E} = \frac{2\pi}{5} R_0 \tag{2.37}
\]

We find that the relation is same as the result of [32] for the 2-sphere in the 4-dimensional CFT. The dependence of the central charges of variance of entanglement entropy do not appear for entanglement temperature. The entanglement temperature is proportional to the inverse of \( R_0 \). As we will see in d=7 dimensional Lovelock gravity the sphere would also admit this property in the next section.
For the infinite stripe, the variance of energy in the subsystem is

\[ \Delta E = \frac{3(1 - 2f_\infty \lambda) ml \tilde{L}^3 l_0^2}{2\ell_p^3}. \] (2.38)

The entanglement temperature is

\[ \frac{1}{T_{\text{ent}}} = \frac{\Delta S}{\Delta E} = \frac{a \sqrt{\pi} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{6}\right)^2}{c \cdot 30 \Gamma\left(\frac{5}{3}\right) \Gamma\left(\frac{3}{8}\right)} l. \] (2.39)

Where we have used (2.20). The entanglement temperature in infinite stripe subsystem is similar to the sphere case, but entanglement temperature depends on the two kinds of central charges of CFT living on the boundary.

In limit \( ml^4 \to 0 \), the entanglement entropy in CFTs, as shown in (2.39), satisfies a universal relation analogous to the first law of thermodynamics proposed by [32]. Comparing with the exact result given by [32], one can find that the coefficient of \( l \) have included the contribution from higher derivative gravity. With taking the \( \lambda = 0 \), one can reproduce the coefficient of \( l \) in [32]. We find that the coefficient of \( l \) is no longer universal constant that is only related to the shape of the entangling surface, but may be related with the central charges of the CFT.

3 Entanglement temperature in D=6 CFT with dual 7-dimensional Lovelock gravity

In this section, we will continue to study entanglement temperature in the D=6 CFT with dual 7-dimensional Lovelock gravity.

3.1 HEE for 6-dimensional CFT

The cubic gravity interaction would appear in Lovelock gravity in 7 dimensional Lovelock gravity[35]. The 7-dimensional Lovelock gravity action is

\[ I = \frac{1}{2\ell_p^3} \int d^7x \sqrt{-g} \left( \frac{30}{L^2} + R + \frac{L^2}{12} \gamma_4 \mathcal{L}_4(R) - \frac{L^4}{24} \gamma_6 \mathcal{L}_6(R) \right), \] (3.1)

where \( \mathcal{L}_4(R) \) is the curvature-squared term (i.e., the Gauss-Bonnet term) and \( \mathcal{L}_6(R) \) is the cubic term, which is as follows [15]

\[ \mathcal{L}_6(R) = \frac{1}{23} \delta^{\nu_1 \nu_2 \ldots \nu_6}_{\mu_1 \mu_2 \ldots \mu_6} R^{\mu_1 \mu_2 \ldots \mu_6}_{\nu_1 \nu_2 \ldots \nu_6}, \] (3.2)

where \( \delta^{\nu_1 \nu_2 \ldots \nu_6}_{\mu_1 \mu_2 \ldots \mu_6} \) is the totally antisymmetric product of Kronecker delta symbols.

The 7 dimensional Lovelock gravity also admit the pure AdS solution with the effective radius \( \tilde{L}^2 = f_\infty^2 \), \( f_\infty \) is one special root of following equation

\[ 1 = f_\infty - f_\infty^2 \lambda_7 - f_\infty^3 \mu_7, \] (3.3)
where \( f_\infty \) can go to \( f_\infty = 1 \) with \( \lambda_7, \mu_7 \to 0 \) continuously. One can turn off the coupling of cubic term to obtain the \( f_\infty \) which is defined previous section (2.4).

The behavior of holographic entanglement entropy has been studied in [15] for the Lovelock gravity. In this section, we are interested in the variation of holographic entanglement entropy after low excitation of pure AdS. For 7 dimensional Lovelock gravity, the holographic entanglement entropy should be the minimal value of the following functional\(^3\)

\[
S_A = \frac{2\pi}{\ell_p^5} \int_M d^5x \sqrt{h} \left( 1 + \frac{\lambda_7 L^2}{6} R - \frac{\mu_7 L^4}{8} \left( R_{\mu\nu\kappa\sigma} R^{\mu\nu\kappa\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right) + S_{\text{surfaceterm}},
\]

(3.4)

where \( h \) is the determinant of the induced metric on the bulk surface \( M \), \( R_{\mu\nu\kappa\sigma} \) is the Riemann tensor of \( M \). The surface term is added to functional to make the variational problem work well. Following [44], the surface term should be

\[
S_{\text{surfaceterm}} = \frac{2\pi}{\ell_p^5} \int_{\partial M} d^4x \sqrt{h_{\partial M}} \left( \frac{\lambda_7 L^2}{3} K - \frac{\mu_7 L^4}{8} \left( 4 R^B K - 8 R^B_{ij} K^{ij} - \frac{4}{3} K^3 + 4 K_{ij} K^{ij} - \frac{8}{3} K_{ij} K^{jk} K^{ik} \right) \right),
\]

(3.5)

where \( \partial M \) is the boundary of \( M \), \( K_{ij} \) and \( K \) are the extrinsic curvature and the trace of extrinsic curvature on boundary \( \partial M \), \( R^B_{ij} \) and \( R^B \) are the intrinsic Ricci tensor and Ricci scalar of the boundary \( \partial M \) respectively. In the following part, we will discuss the case that the entangling surface in the 6-dimensional CFT is a 4-dimensional sphere with radius \( R_0 \).

### 3.2 Variance of EE after low thermal excitation

We continue to consider low thermal excitation for 6-dimensional CFT. For simplicity, we choose \( \mu_7 = -\frac{\lambda_7^3}{3} \).

The low thermal excitation of pure AdS in 7D Lovelock gravity can be expressed by

\[
ds^2 = \frac{\tilde{L}^2}{z^2} (-f(z) dt^2 + \frac{dz^2}{g(z)} + dr^2 + r^2 d\Omega_2^2),
\]

(3.7)

\( ^3 \)There is no general formula for HEE for higher derivative gravity as shown in [15]. For the entangling surfaces with rotation symmetry, logarithm divergent term of HEE from the (3.4) is consistent with calculation from pure CFT side. In this paper, we will only focus on the entangling surface is 4-dimensional sphere in (3.4). Therefore, it is reasonable to use (3.4) as the formula for HEE. We refer readers to [15] for more general discussion.

\( ^4 \)Here we consider the pure AdS as a ground state of Lovelock gravity. Generally speaking, the third order Lovelock gravity with general parameters [45] can not admit the black hole solution(3.6) without the constrain condition \( \mu_7 = -\frac{\lambda_7^3}{3} \). With this condition, we can get a simple black brane solution[46]

\[
ds^2 = \frac{L^2}{z^2} \left[ -f(z) dt^2 + \frac{dz^2}{f(z)} \right] + \sum_{i=1}^5 dx_i^2,
\]

(3.6)

\[
f(z) = \frac{1}{\lambda_7} \left( 1 - (1 - 3 \lambda_7 \left( 1 - \frac{6}{z_h^6} \right)^{1/3} \right),
\]

when \( z_h \) is far away from boundary, we get (3.7). Here we get \( f_\infty = \frac{1 - (1 - 3 \lambda_7 \lambda_7^3)^{1/3}}{\lambda_7} \) by taking \( \mu_7 = -\frac{\lambda_7^3}{3} \) into (3.3).
with \( f(z) \simeq g(z) = 1 - mz^6 \), \( m \) is a parameter which corresponds to the asymptotical behavior of the gravity background. The induced metric of bulk surface \( M \) is
\[
\frac{ds^2_M}{z^2} = \frac{\tilde{L}^2}{z^2} \left( (1 + mz^6) dz^2 + dr^2 + r^2 d\Omega^2 \right).
\] (3.8)

We parameterize the surface \( M \) as \( r = r(u) = f(u/R_0) \cos(u/R_0) \) and \( z = z(u) = f(u/R_0) \sin(u/R_0) \), with \( \epsilon \leq u \leq \frac{\pi}{2} R_0 \), and define
\[
h_{uu} = \frac{\tilde{L}^2}{z^2} \left( (1 + mz^6) \dot{z}^2 + \dot{r}^2 \right), \quad F = \log \left( \frac{r}{z} \right). \] (3.9)

Taking the parametrization of the bulk surface \( M \) into (3.4), we would get a functional of \( f(x) \) as follows
\[
S_A = \frac{2\pi \tilde{L}^5 S_4}{\ell_5 p} \int_{R_0}^{\frac{\pi}{2} R_0} dx \cos^4(x) \sqrt{1 + \left( \frac{d \log f}{dx} \right)^2 \left[ 1 + \frac{\lambda \tau f_\infty}{6} (12 \tan^2(x) + 12 \frac{1}{\cos^2(x)} + \frac{1}{1 + \left( \frac{d \log f}{dx} \right)^2}) \right]}.
\] (3.10)

The details of the calculation is given in Appendix B. With turning off the low thermal excitation, the minimal value of functional \( S_A \) corresponds to the surface \( M \):
\[
r(u) = R_0 \cos(u/R_0), \quad z(u) = R_0 \sin(u/R_0), \quad \text{with} \quad \epsilon \leq u \leq \frac{\pi}{2} R_0. \] (3.11)

As shown in Appendix A, the low thermal excitation do not affect the configuration of bulk surface up to \( O(m R_0^6) \), we can calculate the variance of \( S_A \) after the perturbation is turned on (3.7).
\[
\Delta S_A = \frac{2m R_0^6 \pi \tilde{L}^5 S_4}{\ell_5 p} \int_{R_0}^{\frac{\pi}{2} R_0} dx \left( \frac{1}{2} \sin(x) \cos^6(x) + \frac{\lambda \tau f_\infty}{6} \sin^3(x) \cos^4(x) - \frac{\lambda \tau f_\infty}{6} \sin^3(x) \cos^4(x) \right.
\] - \[
\frac{3}{2} \frac{f_\infty^2 \mu_7}{f_\infty^2} \sin^5(x) \cos^2(x) - \frac{3}{2} \frac{f_\infty^2 \mu_7}{f_\infty^2} \sin(x) \cos^2(x) + \frac{3}{2} f_\infty^2 \mu_7 \sin^3(x) \cos^2(x) \right)
\[
= \frac{m R_0^6 \pi \tilde{L}^5 S_4}{\ell_5 p} \left( 1 - 2\lambda \tau f_\infty - 3f_\infty^2 \mu_7 \right)
= \frac{m R_0^6 \pi S_4}{7} \left( - \frac{56 B_1}{9} + 128 B_2 \right), \] (3.12)

where \( S_4 \) is the volume of the 4-dimensional unit sphere. We have expressed the formula in terms of the central charges of 6-dimensional CFT in the last step (3.12). Where we have quoted the holographic expressions for the four types of central charges denoted by
\[ A, B_1, B_2, B_3 \] \[ \frac{47}{5} \]

\[ B_1 = \frac{\hat{L}^5 - 9 + 26f_\infty \lambda_7 + 51f_\infty^2 \mu_7}{\ell_p^5 \ell_p^5} \]

\[ B_2 = \frac{\hat{L}^5 - 9 + 34f_\infty \lambda_7 + 75f_\infty^2 \mu_7}{1152} \]

\[ B_3 = \frac{\hat{L}^5 - 2f_\infty \lambda_7 - 3f_\infty^2 \mu_7}{384} \]

\[ A = \frac{\pi^3 \hat{L}^5 3 - 10f_\infty \lambda_7 - 45f_\infty^2 \mu_7}{6}. \] (3.13)

In Appendix C we get the energy-stress tensor of CFT with the dual bulk background (3.6). The variance of the energy is easy to obtain

\[ \Delta E = \left( 1 - 2f_\infty \lambda_7 - 3f_\infty^2 \right) \frac{mR_0^5 \hat{L}^5 S_4}{2\ell_p^5} \]

\[ = \frac{mR_0^5 S_4}{2} \left( -\frac{56B_1}{9} + 128B_2 \right). \] (3.14)

Both (3.12) and (3.14) show that the higher derivative gravity will make contribution to the variation of entanglement entropy and energy-stress tensor, as we expected. The entanglement temperature is

\[ \frac{1}{T_{ent}} = \frac{2\pi}{l} R_0 \] (3.15)

(3.12)(3.14) show that the variation of entanglement entropy and energy momentum tensor of theory living on boundary should be related to CFT data. The behavior of entanglement temperature (3.15) for the round ball shaped subsystem calculated in both 5-dimensional and 7-dimensional Lovelock gravity cases do not depends on CFT data.

4 Entanglement temperature in aAdS

To extend our analysis about the entanglement temperature, we can consider background in Fefferman-Graham coordinates corresponds to a special excitation of the dual CFT.

\[ T^i_i = \sum_{m=1}^3 B_i I_i + A E_6, \]

where

\[ I_1 = C_{ijkl} C^{i mnk} C_{m \; n}, \quad I_2 = C_{ijkl} C_{kl \; mn} C_{mn \; ij}, \]

\[ I_3 = C_{ijkl} (\delta_m \nabla_2 + 4R^i - \frac{6}{5} \delta^i_m) C_{m j k l}, \quad E_6 = \frac{1}{192\pi^3} \mathcal{L}_6, \]

where \( C_{ijkl} \) is the Weyl tensor, \( \mathcal{L}_6(R) \) is defined as (3.2).
4.1 Asymptotical AdS in Fefferman-Graham gauge

We assume the CFT is living on the \( d \)-dimensional flat space. The entangling surface is a codimension 2 surface which is parameterized by the coordinate \( \{ x_a \} \), with \( a = 1, 2, \ldots, d-2 \).

And we use the Gauss coordinate, the other space coordinate is \( y \), the metric is

\[
d s_{\text{CFT}}^2 = -dt^2 + dy^2 + h_{ab}(y, x_a)dx^a dx^b. \tag{4.1}
\]

To study the asymptotical AdS space, it is convenient to use the Fefferman-Graham coordinates in background,

\[
d s_{\text{FG}}^2 = \frac{L^2}{z^2} \left( dz^2 + g_{ij}(z, x)dx^i dx^j \right). \tag{4.2}
\]

It has been show that \( g_{ij}(z, x) \) allow an expanding as follows

\[
g_{ij}(z, x) = g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x)z^2 + \ldots + g_{ij}^{(d)}(x)z^d + \mathcal{H} z^d \log z \ldots \tag{4.3}
\]

Where the \( d \) is even. Via the conformal symmetry and Einstein equation the term \( g_{ij}^{(2k)} \) up to order \( d-2 \) and \( \mathcal{H} \) can be computable from \( g_{ij}^{(0)} \) which is the dual boundary metric. \( g_{ij}^{(d)} \) is undetermined by boundary metric. Our interesting case is the dual CFT lives on the flat spacetime, i.e., \( g_{ij}^{(0)} = \eta_{ij} \). We consider the case that the term \( g_{ij}^{(2k)} \) up to order \( d-2 \) and \( \mathcal{H} \) are all vanishing. The asymptotical AdS is

\[
d s_{\text{bulk}}^2 = \frac{L^2}{z^2} \left[ dz^2 + \left( \eta_{ij} + \delta \eta_{ij}(x)z^d \right)dx^i dx^j \right]. \tag{4.4}
\]

Where we have denoted \( \delta \eta_{ij} = g_{ij}^{(d)} \). We can use (4.1) and write in an explicit form as

\[
d s_{\text{bulk}}^2 = \frac{L^2}{z^2} \left[ dz^2 - dt^2 + \delta \eta_i z^d dt dx^i + dy^2 + \delta \eta_{pi} z^d dy dx^i + (h_{ab} + \delta \eta_{ab} z^d)dx^a dx^b \right]. \tag{4.5}
\]

We consider general low excitation of the pure AdS gravity would correspond to (4.5). We can use the holographic method to analyze the variance of EE and relation with variance of the energy of the subsystem.

4.2 Variance of EE and energy

When the excitation is turned off, i.e., the background should go back to pure AdS. We can get the entanglement entropy of the subsystem \( A \) by minimizing the functional of bulk surface \( M \), which is parameterized as \( z = z(y) \). Here we have assumed the entangling surface has the symmetry to make the parametrization possible. The induced metric \( g_{\mu\nu} \) on \( M \) is

\[
d s^2_M = \frac{L^2}{z^2} \left( (1 + z'^2) dy^2 + h_{ab} dx^a dx^b \right), \tag{4.6}
\]

where \( z' \equiv \frac{dz}{dy} \).

\[
S = \frac{2\pi}{\ell_p} \int_M dy d^{d-2} x \sqrt{g} = \frac{2\pi L^{d-1}}{\ell_p} \int_M dy d^{d-2} x \frac{\sqrt{1 + z'^2}}{z^{d-1}} \sqrt{h}, \tag{4.7}
\]
with $h \equiv \det h_{ab}$. We can find the minimal surface $M_0$ with a solution $z = z_0(y)$. When the excitation is turned on, the entanglement entropy would also make a change and the induced metric on $M$ would become $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$,

$$dz_M^2 = \frac{L^2}{z_0^2} \left( 1 + z'^2 + \delta \eta_{yy} z^d \right) dz^2 + (h_{ab} + \delta \eta_{ab} z^d) dx^a dx^b + \delta \eta_{ya} z^d dy dx^a,$$  \hfill (4.8)

We use statement in Appendix A to get the variance of holographic entanglement entropy. The perturbation does not change the shape of $M$ up to $O(\delta g_{\mu\nu})$. So we just need expand the functional

$$\tilde{S} = \frac{2\pi}{\ell_p^{d-1}} \int_M dy d^{d-2}x \sqrt{\tilde{g}}.$$  \hfill (4.9)

to $O(\delta g_{\mu\nu})$ and keep the minimal surface to be $M_0$ with $z = z_0(y)$.

$$\tilde{S} = \frac{2\pi}{\ell_p^{d-1}} \int_{M_0} dy d^{d-2}x \left( \sqrt{g} \left( 1 + \frac{1}{2} g_{\mu\nu} \delta g_{\mu\nu} + O(\delta g_{\mu\nu}^2) \right) \right).$$  \hfill (4.10)

Then we get

$$\Delta S = \tilde{S} - S = \frac{2\pi}{\ell_p^{d-1}} \int_{M_0} dy d^{d-2}x \left( \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} + O(\delta g_{\mu\nu}^2) \right).$$  \hfill (4.11)

We can read $\delta g_{\mu\nu}$ by comparing (4.8) with (4.6). The result is

$$\Delta S = \frac{\pi L^{d-1}}{\ell_p^{d-1}} \int_{M_0} dy d^{d-2}x \sqrt{h} \left( 1 + z'^2 + \frac{\delta \eta_{yy}}{1 + z'^2} \right).$$  \hfill (4.12)

Now we turn to calculate the variance of energy of the subsystem after the low excitation of the ground state (pure AdS). The energy-stress tensor for the CFT living on the boundary is

$$T_{ij} = \frac{dL^{d-1}}{\ell_p^{d-1}} \delta \eta_{ij}.$$  \hfill (4.13)

The conformal symmetry would restrict the energy-stress tensor to be traceless, i.e., $T_{ij} \eta^{ij} = 0$. According to (4.13) $\delta \eta_{ij}$ would also restrict to be traceless, in the coordinate (4.1) we get

$$- \delta \eta_{tt} + \delta \eta_{yy} + h^{ab} \delta \eta_{ab} = 0.$$  \hfill (4.14)

The variance of energy of the subsystem A is

$$\Delta E = \int_A dy d^{d-2}x \sqrt{h} T_{tt} = \frac{dL^{d-1}}{\ell_p^{d-1}} \int_A dy d^{d-2}x \sqrt{h} \delta \eta_{tt}.$$  \hfill (4.15)

Using (4.13)(4.14), we rewritten (4.12) as

$$\Delta S = \frac{\pi}{d} \int_A dy d^{d-2}x \sqrt{h} \sqrt{1 + \frac{z'^2}{z_0^2}} \left( T_{tt} - \frac{z'^2}{1 + z'^2} T_{yy} \right).$$  \hfill (4.16)

The variance of holographic entanglement entropy (4.16) of the subsystem A is only related to $T_{tt}$ and $T_{yy}$, i.e., the variance energy density and the strength of pressure in the direction normal to the entangling surface.
4.3 Entanglement temperature with sphere profile

In this subsection, we would like to consider the subsystem $A$ to be an $(d - 2)$-dimensional sphere with radius $R_0$ as an example to investigate the entanglement temperature. We get $h_{ab}dx^a dx^b = y^2 d\Omega_{d-2}$. It is easy to get the minimal surface $M_0$, $z_0 = \sqrt{R_0^2 - y^2}$, with $0 \leq y \leq R_0$. Taking this solution into (4.16), we get

$$\Delta S = \frac{\pi}{d} \int_A dy d^{d-2}x \sqrt{h} \left( R_0 T_{tt} - \frac{y^2}{R_0} T_{yy} \right)$$  \hspace{1cm} (4.17)

We can rewrite the above formula in a more compact form by using (4.15),

$$\Delta S = \frac{\pi R_0}{d} \Delta E - \pi \int_A dy d^{d-2}x \sqrt{h} \frac{y^2}{R_0} T_{yy}.$$  \hspace{1cm} (4.18)

We find that the second term on the right hand side of (4.18) is not linear dependence on $\Delta E$ generally. There is possibility to break first law-like relation in some special cases. We do not consider the dynamic constrain [28] of the perturbation of the bulk background in this paper. At this stage, we have no exact statement about whether the first law-like relation will be hold or not. The (4.18) shows that $\Delta S$ highly depend on the configurations of subsystem and dynamic constrain of the perturbation.

In the low thermal excitation, $\delta \eta_{tt}$ and $\delta \eta_{yy}$ are constant. Here we take them to be function of the coordinate $y$ which is normal to the sphere. Generally speaking, we assume

$$\delta \eta_{tt} = \sum_i m(i) \left( \frac{y}{R_0} \right)^i,$$  \hspace{1cm} (4.19)

$$\delta \eta_{yy} = \sum_i n(i) \left( \frac{y}{R_0} \right)^i.$$  \hspace{1cm} (4.20)

The parameters $m(i), n(i) \ll 1$. Taking (4.19) into (4.15), we get

$$\Delta E = \frac{d L^{d-1} R_0^{d-1} S_{d-2}}{2 \ell_p^{d-1}} \sum_i \frac{m(i)}{d + i - 1}.$$  \hspace{1cm} (4.21)

Taking (4.19) and (4.20) into (4.12), we get $\Delta S$,

$$\Delta S = \frac{2\pi R_0}{d} \Delta E - \frac{\pi L^{d-1} R_0^{d-1} S_{d-2}}{\ell_p^{d-1}} \sum_i \frac{n(i)}{d + i + 1}.$$  \hspace{1cm} (4.22)

Or by using (4.21),

$$\Delta S = \frac{2\pi R_0}{d} \left( 1 - \frac{A}{B} \right) \Delta E,$$  \hspace{1cm} (4.23)

with

$$A = \sum_i \frac{n(i)}{d + i + 1}, \quad B = \sum_i \frac{m(i)}{d + i - 1}.$$  \hspace{1cm} (4.24)

In this example, the $\Delta S$ is also proportional to the variance $\Delta E$ and the entanglement temperature is related to $m(i)$ and $n(i)$, which correspond to the low excitation mode.
When \( n_i = m_i(0) = 0 \), for \( i \geq 1 \), and \( n(0) = \frac{2}{t}, m(0) = \frac{(d-1)m}{d} \), (4.23) recover the result of [32], i.e., \( \Delta S = \frac{2\pi R}{d} \Delta E \).

This example shows that the first law-like relation and entanglement temperature is not so obvious that it will need to be more carefully study. It is worth to study entanglement temperature in subsystem with general configurations and excitations.

5 Conclusion and Discussion

The entanglement entropy is a useful quantity to describe the entanglement structure of the CFT in vacuum state. It is also interesting to study the property of the entanglement entropy after low excitation of the vacuum state. In this paper, we mainly focus on the entanglement entropy property of the low excited state in higher derivative gravity by using the holographic method. The variance of the HEE would contain more information of the CFT after the low thermal excitation. The entanglement temperature will absorb the contribution from higher derivative gravity. When assuming the bulk theories are Lovelock gravity, one considers these solutions are the low excited state of vacuum of CFT. We just turn on the low thermal excitation and we get the variance of the entanglement entropy for sphere and infinite stripe profile. Combing the variation of energy of these subsystem on the boundary, one can find a novel relation which is first law-like theorem for entanglement entropy. Finally, we also obtain the entanglement temperature which is similar to [32] and the coefficient absorbs the contribution of higher derivative gravity. We find that the entanglement temperature should depend on the central charges of corresponding CFT in strip shaped subsystem in 5-dimensional Lovelock gravity. The behavior of entanglement temperature for the round ball shaped subsystem calculated in both 5-dimensional and 7-dimensional Lovelock gravity cases do not depends on CFT data.

Finally, we study more general low excited states in the Fefferman-Graham coordinates. We get the general result of variance of entanglement entropy (4.16), which is related with the \( t - t \) and \( y - y \) component of energy-stress tensor. The variance of the entanglement entropy with the energy density and pressure which is normal to the entangling surface in general excited states. At this stage, we can not conclude whether the first law-like relation will be hold or not. We have show (4.18) shows that \( \Delta S \) highly depend on the configurations of subsystem and dynamic constrain of the perturbation. We also take a subsystem A with a \( (d-2) \)-dimensional sphere as an example to show the entanglement temperature and the first law-like relation. This may give us more clues to study the quantum entanglement structure of the low excited CFT and more insight to the probable first law relation of entanglement entropy.

In the future, we would like to study the entanglement temperature and entanglement density in general higher derivative gravity. Although there is no universal formula to count the entanglement entropy in general higher derivative gravities, it is worth to try and check whether the first and second law like theorem are correct. One more project is to study the HEE and entanglement temperature of subsystem with other general configurations. Finally, authors [48][49][50][51] have used potential construction approach to generate some
gravity solutions analytically. These solutions are good place to study the dynamics of entanglement entropy [28].

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APPENDIX

A Variation of HEE

In this appendix, we will give a brief proof that the method to calculate the variance of HEE with turning on low excitation is reasonable. The basic logic is that the low excitation of the CFT corresponds to turning on small perturbation term related to $m_i$ in the pure AdS background. The perturbation is described by some dimensionless parameters $m_1...m_i...m_k$, which are quite small, i.e., $m_i \ll 1 (1 \leq i \leq k)$. We only care about the leading effect of $m_i$ in changing the classical configuration of entangling surface. The holographic entanglement entropy is given by minimizing the functional $S_A$ of entanglement surface $M$, which is parameterized by $z = z(x, ..., y)$, where $x, ..., y$ are the boundary coordinates. When the perturbation is turn off, the background goes back to pure AdS, we assume that the minimal $S_A(z_0, m_i = 0)$ corresponds to the surface $m z = z_0(x, ..., y)$. Then

$$\frac{\delta S_A}{\delta z}|_{z=z_0} = 0 \quad (A.1)$$

with turning on the perturbation, $S_A$ is still functional of the bulk surface with the parameters $m_i (1 \leq i \leq k)$, i.e., $\tilde{S}_A \equiv S_A(z; m_1, ..., m_k)$. In principle, we can variance $S_A$ and get the minimal value with a solution $\tilde{z} \equiv z(x, ..., y; m_1, ..., m_k)$. With assuming that the perturbation is quite small, we can expand $z = \tilde{z}(x, ..., y; m_1, ..., m_k)$ up to first order of the parameters $m_i (1 \leq i \leq k)$.

$$z(x, ..., y; m_1, ..., m_k) = z_0(x, ..., y) + \delta z(x, ..., y; m_1, ..., m_k). \quad (A.2)$$
We have denoted \( \delta z(x, ..., y; m_1, ..., m_k) = m_i z^{(i)}(x, ..., y) \). We can also expand \( \tilde{S}_A \) as
\[
\tilde{S}_A \equiv S_A(z; m_1, ..., m_k) = S_A(z, m_i = 0) + m_i S^{(i)}(z).
\] (A.3)

After low thermal excitation, the holographic entanglement entropy should be
\[
S_A(z_0(x, ..., y)) + \delta z(x, ..., y; m_1, ..., m_k) = S_A(z_0(x, ..., y)) + \frac{\delta S_A}{\delta z}|_{z=z_0} \delta z(x, ..., y; m_1, ..., m_k) + O(m_i^2)
\]
\[
= S_A(z_0(x, ..., y)) + m_i S^{(i)}(z_0) + O(m_i^2).
\] (A.4)

Where we have used (A.1). Finally, the variance of the holographic entanglement entropy \( \Delta S_A = m_i S^{(i)}(z_0) + O(m_i^2) \). In other word, the variance of the shape of the bulk surface do not give contribution to variance of the functional \( S_A \) up to \( O(m_i) \).

### B Surface term

In this subsection, we will list some details about the surface term which make the variation to be well defined. The bulk surface \( M \) is parameterized as \( r = r(u) \) and \( z = z(u) \), with \( u_i \leq u \leq u_f \). So the boundary \( \partial M \) is the hypersurface \( u = u_i \). Using the induced metric on \( M \), we can get
\[
R_{\mu
u\kappa\sigma}R^{\mu
u\kappa\sigma} = \frac{1}{6} (R^B - 12h^{uu} \dot{F}^2)^2 + 16(|\Delta u F + h^{uu} \dot{F}^2|^2),
\]
\[
R_{\mu
u}R^{\mu
u} = \frac{1}{4} (R^B - 4(|\Delta u F + 4h^{uu} \dot{F}^2)|^2) + 16(|\Delta u F + h^{uu} \dot{F}^2|^2),
\]
\[
R^2 = (R^B - 8|\Delta u F + 20h^{uu} \dot{F}^2|^2).
\] (B.1)

Where \( \Delta u F \equiv \frac{1}{\sqrt{h_{uu}}} \partial_u \frac{\dot{F}}{\sqrt{h_{uu}}} \). The boundary is a 4-dimensional sphere with radius \( e^{2F} L^2 \), we get \( R^B_{ij} = \frac{3h_{ij}}{e^{2F} L^2} \) and \( R^B = \frac{12}{e^{2F} L^2} \), where \( h_{ij} \) is the induced metric on the boundary \( \partial M \). The normal outward unit vector to \( \partial M \) is \( n_u = -\sqrt{h_{uu}} \partial_u \). It is easy to get the extrinsic curvature \( K_{\nu\mu} = \nabla_{\nu} n_{\mu}|_{u=u_i} \) and its trace, where \( \nabla \) is defined on the bulk surface \( M \) can be obtain as follows
\[
K_{uu} = K_{ui} = K_{iu} = 0, \quad K_{ij} = \frac{\dot{F} h_{ij}}{\sqrt{h_{uu}}}|_{u=u_i}.
\] (B.2)

The surface term (3.5) are listed
\[
K = -\frac{4\dot{F}}{\sqrt{h_{uu}}}|_{u=u_i}, \quad R^B K = -e^{-2F} \frac{4\dot{F}}{L^2 \sqrt{h_{uu}}}|_{u=u_i}, \quad R^B_{ij} K^{ij} = -e^{-2F} \frac{12F}{L^2 \sqrt{h_{uu}}}|_{u=u_i},
\]
\[
KK_{ij} K^{ij} = -16 \left( \frac{\dot{F}}{\sqrt{h_{uu}}} \right)^3|_{u=u_i}, \quad K_{ij} K^{jk} K_k = -4 \left( \frac{\dot{F}}{\sqrt{h_{uu}}} \right)^3|_{u=u_i}.
\] (B.3)
The surface term can be expressed as

\[ S_{\text{surfaceterm}} = \frac{2\pi}{\ell_p^4} \int d\Omega_A \hat{L}^4 e^{4F} \left( - \frac{4\lambda_L L^2 \hat{F}}{3 \sqrt{h_{uu}}} \right. \]

\[ \left. - \frac{\mu_L L^4}{8} (e^{-2F} \frac{96\hat{F}}{L^2 \sqrt{h_{uu}}} + 32(\frac{\hat{F}}{\sqrt{h_{uu}}})^3) \right)_{u=u_i} \]

\[ = 2\pi S_4 \hat{L}^4 e^{4F} \left( - \frac{4\lambda_L L^2 \hat{F}}{3 \sqrt{h_{uu}}} \right. \]

\[ \left. - \frac{\mu_L L^4}{8} (e^{-2F} \frac{96\hat{F}}{L^2 \sqrt{h_{uu}}} + 32(\frac{\hat{F}}{\sqrt{h_{uu}}})^3) \right)_{u=u_i} \cdot \quad \text{(B.4)} \]

Where \( S_4 \) is the volume of the 4-dimensional unit sphere. Using \((B.1)\), we get \((3.4)\) as

\[ S_A = \frac{2\pi \hat{L}^4 S_4}{\ell_p^4} \int_{u_i}^{u_f} du \sqrt{h_{uu}} e^{4F} \left[ 1 + \frac{\lambda L^2}{6} (e^{-2F} \frac{12}{L^2} + 12 h_{uu} \hat{F}^2) \right] \]

\[ - \frac{\mu L^4}{8} \left( e^{-4F} \frac{3}{L^4} - (h_{uu} \hat{F}^2)^2 + e^{-2F} \frac{6}{L^2} h_{uu} \hat{F}^2 \right) \]

\[ + S_{\text{surfaceterm}}. \quad \text{(B.5)} \]

By integrating by parts, we get

\[ S_A = \frac{2\pi \hat{L}^4 S_4}{\ell_p^4} \int_{u_i}^{u_f} du \sqrt{h_{uu}} e^{4F} \left[ 1 + \frac{\lambda L^2}{6} (e^{-2F} \frac{12}{L^2} + 12 h_{uu} \hat{F}^2) \right] \]

\[ - L^4 \mu_T \left( e^{-4F} \frac{3}{L^4} - (h_{uu} \hat{F}^2)^2 + e^{-2F} \frac{6}{L^2} h_{uu} \hat{F}^2 \right) \]

\[ = \frac{8\pi \lambda L^2 \hat{L}^4 S_4}{\ell_p^4} e^{4F} \left( \frac{\hat{F}}{\sqrt{h_{uu}}} \right)_{u=u_i} \]

\[ - \frac{2\pi \mu_T L^4 \hat{L}^4 S_4}{\ell_p^4} \left( e^{-4F} \frac{3}{L^4} - (h_{uu} \hat{F}^2)^2 + e^{-2F} \frac{6}{L^2} h_{uu} \hat{F}^2 \right) \]

\[ + S_{\text{surfaceterm}}. \quad \text{(B.6)} \]

As we can see the surface term exactly cancel the boundary contribution from the integration by parts. \( S_A \) is

\[ S_A = \frac{2\pi \hat{L}^4 S_4}{\ell_p^4} \int_{u_i}^{u_f} du \sqrt{h_{uu}} e^{4F} \left[ 1 + \frac{\lambda L^2}{6} (e^{-2F} \frac{12}{L^2} + 12 h_{uu} \hat{F}^2) \right] \]

\[ - L^4 \mu_T \left( e^{-4F} \frac{3}{L^4} - (h_{uu} \hat{F}^2)^2 + e^{-2F} \frac{6}{L^2} h_{uu} \hat{F}^2 \right). \quad \text{(B.7)} \]

To minimal functional, one can obtain the following configuration which can be parameterized \( r \) and \( z \) as

\[ r(u) = f(u/R_0) \cos(u/R_0), \quad z(u) = f(u/R_0) \sin(u/R_0), \quad \text{with} \quad \epsilon \leq u \leq \frac{\pi}{2} R_0. \quad \text{(B.8)} \]

One can find that \( S_A \) is a functional of \( f(x) \):

\[ S_A = \frac{2\pi \hat{L}^4 S_4}{\ell_p^4} \int_{R_0}^{\pi} dx \cos^4(x) \sin^3(x) \sqrt{1 + \left( \frac{d \log f}{dx} \right)^2} \left[ 1 + \frac{\lambda f_{\infty}}{6} (12 \tan^2(x) + \frac{1}{\cos^2(x)} - 1 + \left( \frac{d \log f}{dx} \right)^2) \right] \]

\[ - f_{\infty}^2 \mu_T \left( 3 \tan^4(x) - \frac{1}{\cos^4(x)} \left( \frac{1}{1 + \left( \frac{d \log f}{dx} \right)^2} \right)^2 + 6 \tan^2(x) \frac{1}{\cos^2(x)} - 1 + \left( \frac{d \log f}{dx} \right)^2 \right). \quad \text{(B.9)} \]
From the above action, the equation of motion can be obtain

\[ 0 = \cot^4(x) \csc(x) G''(x) \left( 2\lambda f_\infty + 3\mu_7 f_\infty^2 - 1 \right) + \cot(x) \csc(x) G'(x)^2 \left( f_\infty \left( 4\lambda \csc^2(x) - 3\mu_7 f_\infty \right) + 2 \cot^2(x) \left( \lambda f_\infty + 2\csc^2(x) \right) \right) + \frac{1}{2} \left( \cos(2x) + 9 \right) \cot^3(x) \csc^3(x) G'(x) \left( -2\lambda f_\infty - 3\mu_7 f_\infty^2 + 1 \right) + \frac{1}{8} \cot(x) \csc^5(x) G'(x)^5 \left( -2\lambda f_\infty + 45\mu_7 f_\infty^2 - 12\cos(2x) \left( 6\lambda f_\infty + 3\mu_7 f_\infty^2 - 5 \right) \right) - \cos(2x) \left( 6\lambda f_\infty + 9\mu_7 f_\infty^2 - 3 \right) - 57 \right) + \frac{1}{8} \cot(x) \csc^5(x) G'(x)^3 \left( -58 \lambda f_\infty - 3\mu_7 f_\infty^2 - \cos(4x) \left( 6\lambda f_\infty + 9\mu_7 f_\infty^2 - 3 \right) \right) - 12 \cos(2x) \left( 8\lambda f_\infty + 9\mu_7 f_\infty^2 - 5 \right) + 57 \right) + \cot^2(x) \csc^3(x) G'(x)^2 G'''(x) \left( -4\lambda f_\infty - 15\mu_7 f_\infty^2 + \cos(2x) \left( 2\lambda f_\infty + 3\mu_7 f_\infty^2 - 1 \right) - 1 \right) + \cot(x) G'(x)^4 G''(x) \left( \lambda f_\infty \left( \cos(2x) - 5 \right) \cot(x) \csc^3(x) G''(x) \right) + 3\mu_7 f_\infty^2 \left( 4 \csc^2(x) + 1 \right) \sec(x) + \cot^3(x)(-\csc(x)) \right) \]

with \( G(x) = \log f(x) \). Where the prime is derivative with respect to \( x \). The Euler-Lagrange equation allow a solution that \( f = \) Constant. The minimal value of functional \( S_A \) corresponds to the surface \( M \) classically.

\[ r(u) = R_0 \cos(u/R_0), \quad z(u) = R_0 \sin(u/R_0), \quad \text{with} \quad \epsilon \leq u \leq \frac{\pi}{2} R_0. \]

Where we have used boundary condition \( f(u/R_0) = R_0 \).

C Energy momentum tensor on the boundary theory

In this section, we would like to deal with the energy momentum tensor of CFT when the dual gravity are 5-dimensional AdS GB gravity and 7-dimensional Lovelock gravity respectively. To define the proper boundary stress-energy tensor, the total gravitational action should have following contribution from three parts\[42][43]

\[ I = I_{\text{bulk}}(g_{ij}) + I_{\text{surf}}(g_{ij}) + I_{\text{ct}}(\gamma_{ij}). \]

\( I_{\text{bulk}}(g_{ij}) \) denotes for the bulk dynamics, \( I_{\text{surf}}(g_{ij}) \) denotes for surface terms contribution and \( I_{\text{ct}}(\gamma_{ij}) \) denotes the terms to make the total action to be finite.

C.1 5-dimensional Lovelock gravity

The 5-dimensional gravity action with the Gauss-Bonnet term in the bulk \( M \) with a boundary \( \partial M \) is\[52][53][54]

\[ I_{\text{ren}} = \frac{1}{2\ell_p^4} \int_M d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + \frac{\lambda_5 L^2}{2} L_4 \right] - \frac{1}{2\ell_p^4} \int_{\partial M} d^4x \sqrt{-\gamma} \left[ 2K - \frac{3}{L} + 2\lambda_5 L^2 \left( J - (R_{ij} - \frac{1}{2} R_{\gamma ij}) K^{ij} \right) \right] + I_{\text{ct}}, \]

\[ (C.2) \]
with
\[ J_{ij} = \frac{1}{3}(2KK_{ik}K^k_j + K_{kl}K^{kl}K_{ij} - 2K_{ik}K^{kl}K_{lj} + K^2K_{ij}), \quad J = J_{ij}\gamma^{ij}. \] (C.3)

Where \( K_{ij} \) and \( K \) are respective the extrinsic curvature and its trace of the boundary \( \partial M \), \( \gamma_{ij} \) is the induced metric on the boundary \( \partial M \), \( R_{ij} \) and \( R \) are respective the induced Ricci tensor and Ricci scalar of the boundary \( \partial M \). The metric of the bulk theory is \( (2.6) \). To regulate the theory, we restrict to the region \( z \geq \epsilon \) and the surface term is evaluated at \( z = \epsilon \). As the boundary is flat the term \( R_{ij} \) and \( R \) do not contribute. The induced metric
\[ \gamma_{ij} = \tilde{L}_2^2\epsilon^2g_{ij}(x, \epsilon), \]
expanded as \( \epsilon \) is the flat metric \( g_{ij}(0) \).

Then the one point function of stress-energy tensor of the dual CFT is given by
\[ T_{ij} = \frac{2}{\sqrt{-\det g(0)}} \frac{\delta I_{\text{ren}}}{\delta g_{ij}(0)} = \lim_{\epsilon \to 0} \left( \tilde{L}_2^2 \frac{2}{\epsilon^2} \frac{\delta I_{\text{ren}}}{\sqrt{-\gamma} \delta \gamma^{ij}} \right). \] (C.4)

Two terms will contribute to the finite part of boundary energy-stress tensor according to \( (C.2) \), one is from the \( O(\epsilon^2) \) of the Brown-York tensor \( T_{ij}^{BY} \) on the boundary \( z = \epsilon \), with
\[ T_{ij}^{BY} = -\frac{1}{\ell_p^3} \left[ (K_{ij} - K\gamma_{ij}) + \lambda_5 L^2(3J_{ij} - J\gamma_{ij}) \right], \] (C.5)
other one is from the counter term. In our case, the counter term is very simply for the boundary is flat. There is only one necessary counter term
\[ I_{ct} = -\frac{1}{\ell_p^3} \int_{\partial M} d^d x \frac{(d-1)\sqrt{-\gamma}}{L'}, \] (C.6)
where \( L' \) depends on \( L \) and Gauss-Bonnet parameter \( \lambda_5 \). The explicit formula about \( L' \) for \( \lambda_5 \) \( (C.2) \) is
\[ L' = \frac{3\tilde{L}^3L}{-3\tilde{L}^3 + 3\tilde{L}^2L + 2L^3\lambda_5}. \] (C.7)

Directly evaluate \( (C.5) \) using \( (2.6) \), we get
\[ T_{tt} = \frac{3m\tilde{L}^3(1 - 2\lambda_5 f_\infty)}{2\ell_p^3}. \] (C.8)

We can see that the contribution tensor from Gauss bonet gravity is related to the term which is proportional to \( \lambda \). If one turns off the Gauss-Bonnet correction, the energy density can be reduced to that of CFT with the dual pure Einstein gravity[32].

**C.2 7-dimensional Lovelock gravity**
In this subsection, we will deal with energy momentum tensor in the 6-dimensional CFT with the dual 7-dimensional Lovelock gravity. We recall that the gravity action should
The Brown-York tensor is the boundary is flat, we obtain
\[ \mathcal{E}_{ij} = -\frac{1}{\ell_p^2} \int d^3 x \sqrt{-g} \left( \frac{30}{L^2} + R + \frac{L^2}{12} \lambda_7 \mathcal{L}_4(R) - \frac{L^4}{24 \mu_7 \mathcal{L}_6(R)} \right) \]
\[ - \frac{1}{2 \ell_p^2} \int d^3 x \sqrt{-g} \left[ 2K - \frac{5}{L} + \frac{\lambda_7 L^2}{3} \left( J - 2(R_{ij} - \frac{1}{2} R \gamma_{ij}) K_{ij} \right) \right] \]
\[ - \frac{\mu_7 L^4}{4} \left( P - 2\tilde{G}_{ij} K^{ij} - 12R_{ij} J^{ij} + 2RJ - 4K R_{ijkl} K^{ij} K^{jl} - 8R_{ijkl} K^{ik} K^j_m K^{nl} \right) \]
\[ + I_{ct}. \]
(C.9)

with

\[ P_{ij} = \frac{1}{5} \left( \left( K^4 - 6K^2 K_{mn} K^{mn} + 8K K_{mn} K^n_k K^{km} - 6K_{mn} K^{nk} K_k l K^{lm} + 3(K_{mn} K^{mn})^2 \right) K_{ij} \right. \]
\[ - (4K^3 - 12K K_{kl} K^{kl} + 8K_{mn} K^n_l K^{lm}) K_{ik} K^j_k - 24K K_{kl} K^{lk} K_{km} K_j^m \]
\[ + 12(K^2 - K_{mn} K^{mn}) K_{il} K^{kj} K_{jk} + 24K K_{ik} K^{kl} K_{lm} K^{mn} K_{nj} \right) \]
\[ \left. P = \gamma_{ij} P^{ij} \right) \]
(C.10)

and

\[ \tilde{G}_{ij} = 2(R_{ikmn} R_j^{kmn} - 2R_{ikjl} R^{kl} - 2R_{ik} R_{lj} + RR_{ij}) \]
\[ - \frac{1}{2} (R_{ijkl} R^{ijkl} - 4R_{ij} R^{ij} + R^2) \gamma_{ij}. \]
(C.11)

we can get the dual CFT stress-energy tensor using the same procedure mentioned in above section, (C.4) should be

\[ T_{ij} = \frac{2}{\sqrt{-\det g_{(0)}}} \frac{\delta I_{ren}}{\delta g^{ij}_{(0)}} = \lim_{\epsilon \to 0} \left( \frac{L^4}{\epsilon^4} \frac{2}{\sqrt{-\gamma}} \frac{\delta I_{ren}}{\delta \gamma^{ij}} \right). \]
(C.12)

The Brown-York tensor is

\[ T^{BY}_{ij} = -\frac{1}{\ell_p^2} \left[ (K_{ij} - K \gamma_{ij}) + \frac{\lambda_7 L^2}{6} (3J_{ij} - J \gamma_{ij}) - \frac{\mu_7 L^4}{8} (5P_{ij} - P \gamma_{ij}) \right]. \]
(C.13)

The counter term is still as the form (C.6), but \( L' \) should also depend on the new parameters \( \lambda_7 \) and \( \mu_7 \). The explicit formula about \( L' \) for (C.9) is

\[ L' = \frac{15L^5 L}{-15L^5 + 15L^4 L + 10L^2 L^3 \lambda_7 + 9L^5 \mu_7}. \]
(C.14)

In our special asymptotically \( AdS_7 \) background (3.7), the boundary is regularized at \( z = \epsilon \), the boundary is flat, we obtain \( t - t \) component of the energy-stress tensor,

\[ T_{tt} = \frac{5mL}{\ell_p^2} \left( \frac{L^4 - 2L^2 L^2 \lambda_7 - 3L^4 \mu_7}{2L^2} \right) \]
\[ = \frac{5mL^5 (1 - f_\infty \lambda_7)^2}{2\ell_p^2}. \]
(C.15)

Where we have used the constraint condition \( \mu_7 = -\frac{\lambda_7^2}{3} \) and \( \tilde{L}^2 = \frac{L^2}{\ell_p^2} \) in the last step. As a consistent check, if one turns off the Lovelock gravity, one can reproduce the energy density of CFT which is dual to pure Einstein gravity.
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