A FUNCTIONAL STABLE LIMIT THEOREM FOR GIBBS-MARKOV MAPS

DAVID KOCHERM, FABIAN PÜHRINGER, AND ROLAND ZWEIMÜLLER

Abstract. For a class of locally (but not necessarily uniformly) Lipschitz continuous $d$-dimensional observables over a Gibbs-Markov system, we show that convergence of (suitably normalized and centered) ergodic sums to a non-Gaussian stable vector is equivalent to the distribution belonging to the classical domain of attraction, and that it implies a weak invariance principle in the (strong) Skorohod $J_1$-topology on $D([0, \infty), \mathbb{R}^d)$. The argument uses the classical approach via finite-dimensional marginals and $J_1$-tightness. As applications, we record a Spitzer-type arcsine law for certain $Z$-extensions of Gibbs-Markov systems, and prove an asymptotic independence property of excursion processes of intermittent interval maps.

1. Introduction

The study of ergodic dynamical systems naturally leads to many questions about stationary sequences with nontrivial dependence structure. If $T$ is a map preserving a probability measure $\mu$ on $(X, \mathcal{A})$, and $f : X \to \mathbb{R}^d$ is some observable (that is, a measurable function), the stationary sequence $(f \circ T^k)_{k \geq 0}$ on $(X, \mathcal{A}, \mu)$ is expected to exhibit properties similar to those of iid sequences as soon as $T$ possesses sufficiently strong mixing properties, and $f$ is regular enough. In particular, the sequence of ergodic sums $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ should behave like a classical partial sum process, for which, among a multitude of other results, functional limit theorems are available. Indeed, there is a well developed theory clarifying the asymptotic behavior of such sequences in situations with a Gaussian limit, see for example [MPU].

In the present article we are interested in situations where $f$ has a heavy tail, and its distribution $\mu \circ f^{-1}$ is in the domain of attraction of some non-Gaussian stable random vector $S$. (We refer to [MS] and [ST] for background information on stable laws and processes, see also Section 2 below.) Then there exist two sequences of constants $A_n \in \mathbb{R}^d$ and $B_n > 0$, $n \geq 1$, such that distributional convergence

$$
\frac{1}{B_n} (S_n(f) - A_n) \Rightarrow S \quad \text{as } n \to \infty
$$

2010 Mathematics Subject Classification. Primary 28D05, 37A25, 37C30, 11K50.

Key words and phrases. Weak invariance principle, stable laws, stable Lévy processes, weakly dependent processes, stationary sequences.

Acknowledgement. F.P. gratefully acknowledges support through FWF-grant Y00782. R.Z. thanks Jon Aaronson, Alexey Korepanov and Ian Melbourne for inspiring discussions related to this topic. We are also grateful to the referee whose comments helped to improve the presentation.
takes place provided that \((f \circ T^k)_{k \geq 0}\) is an iid sequence. In the classical iid setup, the Stable Limit Theorem (SLT) (1.1) automatically entails a Functional Stable Limit Theorem (FSLT) (or weak invariance principle) in \((D([0, \infty), \mathbb{R}^d), \mathcal{J}_1)\), which asserts distributional convergence of the partial sum processes \(S[n]\) given by

\[
S[n] : X \rightarrow D([0, \infty), \mathbb{R}^d), \quad S[n] := \frac{1}{B_n} \left( S[t_n] (f) - \frac{[t_n]}{n} A_n \right),
\]

(1.2) to the \(\alpha\)-stable Lévy process \(S = (S_t)_{t \geq 0}\) with \(S_1\) distributed like \(S\) (see [S1], [S2]),

\[
S[n] \Rightarrow S \quad \text{in} \quad (D([0, \infty), \mathbb{R}^d), \mathcal{J}_1).
\]

(1.3)

Here, \(D([0, \infty), \mathbb{R}^d)\) is the Skorohod space of right-continuous functions \(x : [0, \infty) \rightarrow \mathbb{R}^d\) possessing left limits everywhere, and we always use the Skorohod \(\mathcal{J}_1\)-topology (details below).

It is known that for dependent stationary sequences the \(\mathcal{J}_1\)-FSLT does not, in general, follow from the SLT, not even if \(d = 1\) (see Examples 1.1 and 2.1 in [Ty2]).

The main result of the present article, Theorem 2.1 below, shows that a SLT (1.1) for sufficiently regular vector-valued observables over a Gibbs-Markov system implies the corresponding FSLT (1.3) in the \(\mathcal{J}_1\)-topology. The regularity condition we use is that the tail of the cylinderwise Lipschitz constant should not be heavier than the tail of \(f\) itself.

Gibbs-Markov maps form an important basic class of systems. In this context, SLTs have for example been established in [AD], [G1], [G2] for certain real-valued observables \(f\) which are Lipschitz on cylinders. Based on work in [Ty1], the article [Ty2] proves, for \(d = 1\), a \(\mathcal{J}_1\)-FSLT (1.3) in certain dynamical situations, including that of piecewise constant observables on a Gibbs-Markov system. In the case of stable laws of index \(\alpha \in (1, 2)\), a vector-valued FSLT similar to our Theorem 2.1 is given in Section 4 of [CFKM]. This, too, is based on [Ty1]. Our present result covers all \(\alpha \in (0, 2)\), and the proof is independent of [Ty1].

Our Theorem 2.2 (which rephrases the \(d = 1\) case of Theorem 2.1 in easily applicable form) gives a \(\mathcal{J}_1\)-FSLT for a large class of real observables. (More on the relation to the FSLT of [Ty2] in Remark 2.3 below.) The second special case of Theorem 2.1 we make explicit, Theorem 2.3, asserts that, for a vector-valued observable \(f = (f^{(1)}, \ldots, f^{(d)})\) with individual components \(f^{(j)}\) of said regularity and with asymptotically proportional tails in the domains of attraction of \(\alpha\)-stable laws, the partial sum processes converge, under the \(\mathcal{J}_1\)-topology, to an independent tuple of scalar stable Lévy motions, provided that the tails of the \(f^{(j)}\) are determined on non-overlapping sets.

In a final section, we illustrate the use of these results. First, we apply Theorem 2.2 in the setup of certain infinite measure preserving skew products with Gibbs-Markov base to obtain a Spitzer-type arcsine law (Theorem 6.1). Second, for prototypical interval maps on \([0, 1]\) with two indifferent fixed points of the same order, at \(x = 0\) and \(x = 1\), our Theorem 2.3 entails asymptotic independence between the two processes of excursions to small neighbourhoods of \(x = 0\) and \(x = 1\), respectively. Here, the maps may have finite or infinite invariant measure. We require the fixed points to be so strong that those excursion processes have non-Gaussian limits, see Theorem 6.2.
The approach used in [TY1], [TY2] was to study the asymptotic behaviour of the point processes which capture the occurrences of large individual observations, thus proving convergence to the limit process via Lévy-Itô-type representations. In contrast, the present article follows the classical “convergence of marginals plus tightness” approach.

2. Background and Main results

We begin by fixing notations and collecting the required background material. For functions \( \tau_1, \tau_2 \) and \( c \) a constant, we write \( \tau_1(t) \sim c\tau_2(t) \) as \( t \to \infty \) to indicate, as in [BGT], that \( \tau_2(t) > 0 \) for large \( t \) and \( \tau_1(t)/\tau_2(t) \to c \), even in case \( c = 0 \).

**Distributional convergence.** If \( R_l, l \geq 1 \), are Borel measurable maps of \((X,A)\) into some metric space \((\mathcal{E},d_\mathcal{E})\), while \( \nu_l, l \geq 1 \), are probability measures on \((X,A)\), and \( R \) is another random element of \( \mathcal{E} \) (defined on some \((\Omega,\mathcal{F},\mathbb{P})\)), then we write

\[
R_l \overset{\nu}{\Rightarrow} R \quad \text{as} \quad l \to \infty
\]

(2.1) to indicate that \( \nu_l \circ R_l^{-1} \Rightarrow \mathbb{P} \circ R^{-1} \) (the usual weak convergence of probability measures, [Br]). This is distributional convergence to \( R \) of the \( R_l \) when the latter functions are regarded as random variables on the probability spaces \((X,A,\nu_l)\), respectively. It includes the case of a single measure \( \nu \), where \( R_l \Rightarrow R \) means that the distributions \( \nu \circ R_l^{-1} \) of the \( R_l \) under \( \nu \) converge weakly to the law of \( R \). If \( \nu \) is understood, we may simply write \( R_l \Rightarrow R \).

The limit theorems we are to discuss are in fact instances of strong distributional convergence with respect to the invariant measure \( \mu \) (terminology taken from [A]) or mixing limit theorems, meaning that convergence \( \Rightarrow \) with respect to one probability measure \( \nu \ll \mu \) implies convergence w.r.t. all such measures, see Theorems 2.1 2.2 2.3 and Lemma 4.3 below.

**Stable random vectors.** A random vector \( S = (S^{(1)}, \ldots, S^{(d)}) \) (or its law) is stable if for any positive numbers \( a, b \) there are constants \( c > 0 \) and \( D \in \mathbb{R}^d \) such that \( aS_1 + bS_2 \overset{d}{=} cS + D \), where \( S_1, S_2 \) are independent copies of \( S \) and \( \overset{d}{=} \) indicates equality of distributions. This is equivalent to the assertion that there is some \( \alpha \in (0,2] \) such that for any \( n \geq 2 \) there is some constant \( D_n \in \mathbb{R}^d \) for which \( S_1 + \ldots + S_n \overset{d}{=} n^{1/\alpha}S + D_n \), whenever \( S_1, S_2, \ldots \) are independent copies of \( S \). In this case \( S \) (or its law) is said to be \( \alpha \)-stable, and \( \alpha \) is the index of \( S \) (see §2.1 of [ST]). \( S \) is a Gaussian iff \( \alpha = 2 \). Stable vectors are exactly those random elements \( S \) of \( \mathbb{R}^d \) which occur as limits

\[
\frac{1}{B_n} \left( \sum_{k=0}^{n-1} Z_k - A_n \right) \Rightarrow S \quad \text{as} \quad n \to \infty,
\]

(2.2) for iid sequences \((Z_k)_{k \geq 0}\) of random vectors and constants \( A_n \in \mathbb{R}^d \) and \( B_n > 0 \). In this case, \( Z_0 \) (or its law \( Q \)) is said to belong to the domain of attraction of \( S \), which we will indicate by writing \( Z_0 \) (or \( Q \)) \( \in \text{DOA}(S) \) (see §7.3 in [MS]).

We call a Borel measure on \( \mathbb{R}^d \) full if it is not supported on any proper affine subspace of \( \mathbb{R}^d \). A \( d \)-dimensional random vector is full if its law is. The collection of all full stable random vectors \( S \) with index \( \alpha \in (0,2] \) can be represented as a parametrized family \( \{S_\alpha(\Lambda,c)\}_{c \in \mathbb{R}^d, \Lambda \in \Sigma} \) where \( c \) is the center of \( S_\alpha(\Lambda,c) \) and \( \Lambda \) is
its spectral measure. Here $\Sigma$ is the family of all finite full Borel measures $\Lambda$ on $\mathbb{R}^d$ which are concentrated on the unit sphere $S^{d-1}$ (see Theorem 7.3.16 of [MS]). The constant vector $c$ is a simple location parameter such that $S_\alpha(A, c) \overset{d}{=} S_\alpha(A, 0) + c$, while $\Lambda$ encodes the tail behaviour of $S = S_\alpha(A, c)$ in that

$$\Pr \left[ \frac{S}{\|S\|} \in D \mid \|S\| > t \right] \to \frac{\Lambda(D)}{\Lambda(S^{d-1})} \quad \text{as } t \to \infty,$$

for Borel $D \subseteq S^{d-1}$ whose boundary in $S^{d-1}$ is a null set for $\Lambda$. Moreover, as shown in [R] (see also Theorem 8.2.18 of [MS]), the (law of a) random vector $Z$ belongs to DOA($S_\alpha(A, c)$) iff $V(t) := \Pr[\|Z\| > t]$, $t \geq 0$, is regularly varying of index $-\alpha$, and \[ \text{(2.3)} \]

\[ \mu(f > t) = (c_+ + o(1))t^{-\alpha}\ell(t) \quad \text{and} \quad \mu(f \leq -t) = (c_- + o(1))t^{-\alpha}\ell(t). \]

The law of a specific variable $S$ which partial sums are then attracted to is given by the Fourier transform

$$\mathbb{E}[e^{itS}] = e^{-c_\alpha \beta |t|^{1-i\beta \text{sgn}(t) \omega(\alpha, t))}, \quad t \in \mathbb{R},$$

where $c_\alpha := \Gamma(1-\alpha) \cos(\alpha \pi/2)$ if $\alpha \neq 1$ and $c_\alpha := \pi/2$ if $\alpha = 1$, while $\beta := c_+ + c_-$, $\bar{\beta} := c_+ - c_-$ and $\beta := \bar{\beta}/2$, while $\omega_\alpha(t) := \tan(\alpha \pi/2)$ if $\alpha \neq 1$ and $\omega_\alpha(t) := -(2/\pi) \log |t|$ if $\alpha = 1$. While the limit laws of partial sums are only unique up to type, we will take the limit to be this particular variable $S$. (This is the convention used in [AD], [G1], [G2].)

In this situation, partial sums of iid sequences $(Z_k)$ with the same distribution as $f$ satisfy an SLT \[ \text{(2.2)} \] with sequences $(A_n)$ and $(B_n)$ defined in terms of this distribution. Specifically, one can take $(B_n)$ so that

$$\text{(2.6)} \quad n \ell(B_n) = B_n^\alpha \quad \text{for } n \geq 1,$$

and $A_n = 0$ if $\alpha < 1$, $A_n = n \int f \, d\mu$ if $\alpha > 1$, while the definition of $A_n$ is more complicated in case $\alpha = 1$, see Section 6 of [AD]. In either case, \[ \text{(2.7)} \]

$$A_n = o(nB_n) \quad \text{as } n \to \infty.$$ We shall refer to this choice of $(A_n, B_n)_{n \geq 1}$ in $\mathbb{R} \times (0, \infty)$ as the canonical normalizing sequence for $[\alpha, c_+, c_-]$, and to $S$ as the canonical limit (law) for $[\alpha, c_+, c_-]$ (in the notation of §1.1 in [ST], we have $S \overset{d}{=} S([c_\alpha(c_+ - c_-)]^{1/\alpha}, (c_+ - c_-)/(c_+ + c_-), 0)$.)

**Gibbs-Markov systems.** A piecewise invertible probability preserving system is a tuple $(X, A, \mu, T, \xi)$ where $T : X \to X$ is a measure preserving map on the probability space $(X, A, \mu)$, and $\xi$ is a countable partition (mod $\mu$) of $X$ with $A = \sigma(T^{-k} \xi : k \geq 0)$ (mod $\mu$) and such that the restriction of $T$ to any cylinder $Z \in \xi$ is a measurably invertible map $T|_Z : Z \to TZ$ with inverse $v_Z : TZ \to Z$. The partition and the system are said to be Markov if for each $Z \in \xi$ the image $TZ$ is measurable $\xi$ (mod $\mu$). It has the big image property if $\inf_{Z \in \xi} \mu(TZ) > 0.$
Write $\xi_n := \bigvee_{k=0}^{n-1} T^{-k} \xi$ for the family of cylinders of rank $n \geq 1$. We denote the element of $\xi_n$ containing $x$ by $\xi_n(x) := \bigcap_{k=0}^{n-1} \xi(T^k x)$ (well defined for a.e. $x$).

The separation time of two points $x, y \in X$ is $s(x, y) := \inf \{ n \geq 1 : \xi_n(x) \neq \xi_n(y) \}$. For a parameter $\theta \in (0, 1)$ define a dynamical metric by letting $d_\theta(x, y) := \theta^{s(x, y)}$. Evidently, $T$ is uniformly expanding w.r.t. $d_\theta$ in that $d_\theta(x, y) = \theta d_\theta(T x, T y)$ a.e. For $f : X \to \mathfrak{X}$ (with $(\mathfrak{X}, d_\mathfrak{X})$ some metric space) and $W \subseteq X$ set $D_W(f) := \inf \{ L > 0 : d_\mathfrak{X}(f(x), f(y)) \leq L d_\theta(x, y) \text{ for } x, y \in W \}$, the least Lipschitz constant of $f$ on the set $W$, and $D_W(f) := \sup_{W \subseteq W} D_W(f)$ if $W$ is a collection of sets. Call $f$ uniformly piecewise Lipschitz in case $D_\mathfrak{X}(f) < \infty$.

Consider the Radon-Nikodym derivatives $v'_Z : TZ \to [0, \infty)$, $Z \in \xi$, with $v'_Z := d(\mu \circ v_Z)/d\mu$. A Markov system $(X, \mathcal{A}, \mu, T, \xi)$ with the big image property is said to be Gibbs-Markov if, in addition, $\sup_{Z \in \xi} D_T Z (\log \circ v'_Z) < \infty$ (for suitable versions of the a.e. defined functions $v'_Z$). In this case a routine argument shows that the system has bounded distortion in that there is some $R \in [0, \infty)$ such that

$$e^{-R} \frac{\mu(T^n Z \cap E)}{\mu(T^n Z)} \leq \frac{\mu(Z \cap T^{-n} E)}{\mu(Z)} \leq e^R \frac{\mu(T^n Z \cap E)}{\mu(T^n Z)}$$

whenever $n \geq 1$, $Z \in \xi$, and $E \in \mathcal{A}$.

For observables $f$ taking values in some normed space $(\mathfrak{X}, ||||)$, the ergodic sums will be denoted $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$. If $f$ is understood, we will simply abbreviate $S_n := S_n(f)$. To deal with observables $f : X \to \mathfrak{X}$ which are not necessarily uniformly piecewise Lipschitz, we consider the associated $\xi$-measurable function

$$\vartheta_f : X \to [0, \infty], \quad \vartheta_f := \sum_{Z \in \xi} D_Z(f) 1_Z$$

(defined almost everywhere), which collects the best Lipschitz constants on individual rank-one cylinders. If $\vartheta_f$ is unbounded, the decay rate of its tail $\mu(\vartheta_f > t)$ as $t \to \infty$ provides a meaningful way of quantifying the overall regularity of the function. The stable limit theorems for real-valued observables obtained in [AD], which assume boundedness of $\vartheta_f$, have been extended significantly in [G2], where it is shown that for $f : X \to \mathbb{R}$ in the domain of attraction of a stable random variable $S$, the assumption that $\int \vartheta_f^\eta d\mu < \infty$ for some $\eta \in (0, 1]$ is sufficient for the SLT (1.11).

**Link to the iid case.** The result just quoted is based on the insight (Theorem 1.5 of [G2]) that (excluding the square-integrable Gaussian case) for such $f$ the SLT for the dynamical system holds if it holds for the partial sums of an iid sequence of random variables with the same distribution as $f$. This easily extends from the scalar case studied in [G2] to the situation of $d$-dimensional random vectors.

**Proposition 2.1 (Nondegenerate distributional limits - GM versus iid).** Let $(X, \mathcal{A}, \mu, T, \xi)$ be a mixing probability preserving Gibbs-Markov system, and $f = (f^{(1)}, \ldots, f^{(d)}) : X \to \mathbb{R}^d$ an observable satisfying $\int \vartheta_f^\eta d\mu < \infty$ for some $\eta \in (0, 1]$. Suppose that $S$ is a full $d$-dimensional random vector, and $(A_n, B_n)_{n \geq 1}$ a sequence in $\mathbb{R}^d \times (0, \infty)$ such that $\sqrt{n} = o(B_n)$ as $n \to \infty$.

Let $(Z_k)_{k \geq 0}$ be an iid sequence with $Z_0 \overset{d}{=} f$. Then

$$\frac{1}{B_n} (S_n(f) - A_n) \Rightarrow S \quad \text{as } n \to \infty \quad (2.10)$$
iff

\[(2.11) \quad \frac{1}{B_n} \left( \sum_{k=0}^{n-1} Z_k - A_n \right) \Rightarrow S \quad \text{as } n \to \infty.\]

If \((2.10)\) and \((2.11)\) hold, then \(S\) is a stable random vector. Specifically, if \(S\) is \(\alpha\)-stable, \(\alpha \in (0, 2)\), then \((B_n)\) is regularly varying of index \(1/\alpha\), while each \(\tau_{f^{(i)}(s)} = \mu(|f^{(i)}| > s)\) is regularly varying of index \(-\alpha\) and satisfies

\[(2.12) \quad n \tau_{f^{(i)}(s)}(B_n) \to c^{(i)} \quad \text{as } n \to \infty\]

for some \(c^{(i)} \in (0, \infty)\). Moreover, the \(A_n = (A_n^{(1)}, \ldots, A_n^{(d)})\) satisfy

\[(2.13) \quad A_n^{(i)} = o(nB_n) \quad \text{as } n \to \infty.\]

Proof. (i) Take any non-zero linear form \(\psi : \mathbb{R}^d \to \mathbb{R}\). Due to \(\partial_{\psi \circ f} \leq \|\psi\| \partial_f\) we have \(\int \partial_{\psi \circ f} d\mu < \infty\). Hence the function \(\psi \circ f : X \to \mathbb{R}\) belongs to the family of scalar observables studied in \([G2]\).

Assume \((2.11)\). Then by linearity and continuity of \(\psi\),

\[(2.14) \quad \frac{1}{B_n} \left( \sum_{k=0}^{n-1} \psi(Z_k) - \psi(A_n) \right) \Rightarrow \psi(S) \quad \text{as } n \to \infty,\]

and the classical CLT shows that \(\int (\psi \circ f)^2 d\mu = \mathbb{E}[\psi(Z_k)^2] = \infty\) since \(\sqrt{n} = o(B_n)\) and \(\psi(S)\) is non-degenerate. On the other hand, if we assume \((2.10)\), then the \(L_2(\mu)\)-case of Theorem 1.5 in \([G2]\) shows that \(\int (\psi \circ f)^2 d\mu = \infty\). Consequently, either of \((2.10)\) and \((2.11)\) implies that the infinite variance case of Theorem 1.5 in \([G2]\) applies to \(\psi \circ f\). The latter states that \((2.14)\) holds iff

\[(2.15) \quad \frac{1}{B_n} (S_n(\psi \circ f) - \psi(A_n)) \Rightarrow \psi(S) \quad \text{as } n \to \infty.\]

(ii) By Cramér-Wold, \((2.10)\) is equivalent to the statement that \((2.15)\) holds for all \(\psi\), while \((2.11)\) is equivalent to the assertion that \((2.14)\) is valid for all \(\psi\). Therefore equivalence of \((2.10)\) and \((2.11)\) follows from step (i).

(iii) As mentioned before, \((2.11)\) implies stability of \(S\). Write \(S = (S^{(1)}, \ldots, S^{(d)})\), and fix any \(i \in \{1, \ldots, d\}\). As a special case of (i) we see that \(f^{(i)}\) is in the domain of attraction of \(S^{(i)}\), and in view of the explicit description \((2.4)\) of scalar domains of attraction above, we conclude that the tail \(\tau_{f^{(i)}}\) of \(f^{(i)}\) is regularly varying of index \(-\alpha\). Moreover, \((A_n^{(i)}, B_n)\) is a normalizing sequence for the partial sums of \(\psi\) of variables distributed like \(f^{(i)}\). By the standard one-dimensional convergence of types theorem (for example, Theorem II.10.2 of \([GK]\), \((B_n)\) here is asymptotically proportional to the canonical normalizing sequence in \((2.6)\), and hence regularly varying of index \(1/\alpha\). Combining \((2.4)\) and \((2.6)\) we get \((2.12)\). Regarding \((A_n^{(i)})\), convergence of types shows that (since \(B_n \to \infty\), statement \((2.13)\) follows from the corresponding statement \((2.7)\) for the canonical normalization. □

Skorohod spaces. Let \((\mathfrak{S}, d_\mathfrak{S})\) be a metric space. Recall that \(D([0, 1], \mathfrak{S})\) is the space of right-continuous real functions \(x : [0, 1] \to \mathfrak{S}\) possessing left limits everywhere (cadlag functions). Denote functions to be regarded as elements of the path space \(D([0, 1], \mathfrak{S})\) by \(x, y\) and their values at \(t\) by \(x_t, y_t\). Equip the space
$D([0,1],\mathfrak{F})$ with the standard Skorohod $J_1$-topology (see [Bi], [GS], [SI], or [W]). Two functions $x,y \in D([0,1],\mathfrak{F})$ are close in this topology if they are uniformly close after a small distortion of the domain. Formally, let $\Lambda$ be the set of increasing homeomorphisms $\lambda: [0,1] \to [0,1]$, and let $\lambda_{id} \in \Lambda$ denote the identity. Then $d_{J_1}(x,y) = d_{J_1,1}(x,y) := \inf_{\lambda \in \Lambda} \{ sup_{[0,1]} d_1(x \circ \lambda, y) / sup_{[0,1]} |\lambda - \lambda_{id}| \}$ defines a metric on $D([0,1],\mathfrak{F})$ which induces the $J_1$-topology. While its restriction to $C([0,1],\mathfrak{F})$ coincides with the uniform topology, discontinuous functions are $J_1$-close to each other only if they have similar jumps at nearby positions.

For any $s > 0$ the space $(D([0,s],\mathfrak{F}),J_1)$ of cadlag functions on $[0,s]$ is defined analogously, with metric $d_{J_1,s}$. To obtain the $J_1$-topology on $D([0,\infty),\mathfrak{F})$, the space of all cadlag functions $x: [0,\infty) \to \mathfrak{F}$, we use $d_{J_1,\infty} := \int_0^\infty e^{-s} (1 \wedge d_{J_1,s}) ds$. Then, convergence $x^{[n]} \to x$ as $n \to \infty$ in $(D([0,\infty),\mathfrak{F}),J_1)$ means $x^{[n]} \to x$ in $(D([0,s],\mathfrak{F}),J_1)$ for every continuity point $s > 0$ of $x$ (we simply denote the restriction of $x \in D([0,\infty),\mathfrak{F})$ to $[0,s]$ by $x$ again).

We will study situations in which $\mathfrak{F} = \mathbb{R}$ or $\mathbb{R}^d$. In the latter case, the standard $J_1$-topology on $D([0,\infty),\mathbb{R}^d)$ is sometimes called the strong $J_1$-topology, in order to distinguish it from the weak $J_1$-topology on $D([0,\infty),\mathbb{R}^d)$ given by componentwise convergence in $D([0,\infty),\mathbb{R})$, see Section 3.3 of [W]. The random elements of $D([0,1],\mathbb{R}^d)$ or $D([0,\infty),\mathbb{R}^d)$ of interest in this paper will be denoted by $S,S^{[n]}, \ldots$ with corresponding coordinate variables $S_t,S_t^{[n]}$ etc.

**SLT implies the $J_1$-FSLT.** The core result of the present paper shows that an SLT for $d$-dimensional observables of reasonable regularity over a Gibbs-Markov system automatically entails a functional version in the (strong) $J_1$-topology of $D([0,\infty),\mathbb{R}^d)$.

**Theorem 2.1 (SLT implies $J_1$-FSLT for Gibbs-Markov maps).** Let $(X,A,\mu,T,\xi)$ be a mixing probability preserving Gibbs-Markov system, and $f: X \to \mathbb{R}^d$ an observable satisfying $\mu(\partial_f > t) = O(\mu(\|f\| > t))$. Assume that $\mu \circ f^{-1} \in DOA(S)$ for some full $\alpha$-stable random vector $S$ in $\mathbb{R}^d$ with $\alpha \in (0,2)$, so that there are $(A_n,B_n) \in \mathbb{R}^d \times (0,\infty)$, $n \geq 1$, such that

$$\frac{1}{B_n} (S_n(f) - A_n) \overset{\mu}{\Rightarrow} S \quad \text{as } n \to \infty. \quad (2.16)$$

Then the partial sum processes $S^{[n]} = (S_t^{[n]})_{t \geq 0}$ given by

$$S^{[n]}: X \to D([0,\infty),\mathbb{R}^d), \quad n \geq 1, \quad S_t^{[n]} := \frac{1}{B_n} \left( S_{\lfloor nt \rfloor}(f) - \frac{\lfloor nt \rfloor}{n} A_n \right), \quad (2.17)$$

converge to the $\alpha$-stable Lévy process $S = (S_t)_{t \geq 0}$ with $S_t \overset{d}{=} S$ in that

$$S^{[n]} \overset{\nu}{\Rightarrow} S \quad \text{in } (D([0,\infty),\mathbb{R}^d),J_1) \quad (2.18)$$

for every probability measure $\nu \ll \mu$ on $(X,A)$.

**Remark 2.1.** The assumption that $\mu \circ f^{-1} \in DOA(S)$ entails regular variation with index $-\alpha$ of $t \to \mu(\|f\| > t)$. Therefore, for any $\eta \in (0,\alpha \wedge 1)$, we have $\int \|f\|^\eta \, d\mu < \infty$, and since $\mu(\partial_f > t) = O(\mu(\|f\| > t))$ the observable also satisfies
the condition $\int \vartheta^2 \, d\mu < \infty$ from Proposition 2.1. We do not know whether the latter alone is sufficient for the conclusion of Theorem 2.2.

Remark 2.2. This result remains valid if $S^{[n]}$ is replaced by $\overline{S}^{[n]} = (\overline{S}^t_{[n]})_{t \geq 0}$ with
\[
\overline{S}^n_t : X \rightarrow \mathcal{D}([0, \infty), \mathbb{R}^d), \quad n \geq 1, \quad \overline{S}^n_t \coloneqq \frac{1}{B_n} (S^{|tn|}_t(f) - tA_n),
\]
a variant of the partial sum process which some authors prefer (see [Ty1], [Ty2], [W]). This is clear since $\sup_{t \geq 0} \|S^t_t^n - \overline{S}^n_t\| \leq A_n/(nB_n) \rightarrow 0$ as $n \rightarrow \infty$, see (2.13) in Proposition 2.1.

To facilitate the application of our FSLT, we now provide explicit versions of two important special cases in Theorems 2.2 and 2.3 below. The first deals with scalar observables, while the second gives an easy sufficient condition for a $d$-dimensional observable to lead to an independent tuple of scalar Lévy processes in the limit.

Special case: $J_1$-FSLT for real-valued observables. In the scalar case we obtain

**Theorem 2.2 (Scalar Functional Stable Limit Theorem for GM-maps).** Let $(X, A, \mu, T, \xi)$ be a mixing probability preserving Gibbs-Markov system, and $f : X \rightarrow \mathbb{R}$ an observable in the domain of attraction of some $\alpha$-stable random variable with $\alpha \in (0, 2)$, as in (2.17). Assume also that
\[
\mu(\partial f > t) = O(\mu(\{|f| > t\})) \quad \text{as} \quad t \rightarrow \infty.
\]
Then, for the canonical normalizing sequence $(A_n, B_n)$ for $[\alpha, c_+, c_-]$, the partial sum processes $S^{[n]} = (S^t_{[n]})_{t \geq 0}$ given by (2.17) converge to the real-valued $\alpha$-stable Lévy process $S = (S_t)_{t \geq 0}$ with $S_1 \overset{d}{=} S$, the canonical limit law for $[\alpha, c_+, c_-]$ characterized by (2.19), so that
\[
S^{[n]} \xrightarrow{\nu} S \quad \text{in} \quad \mathcal{D}([0, \infty), \mathbb{R}), J_1
\]
for every probability measure $\nu \ll \mu$ on $(X, A)$.

**Remark 2.3.** As mentioned in the introduction, scalar FSLTs for similar situations have been obtained in [Ty2]. We briefly outline how our Theorem 2.2 relates to the results of §4 in [Ty2] (in the case of a mixing probability preserving Gibbs-Markov system $(X, A, \mu, T, \xi)$ and an observable $f : X \rightarrow \mathbb{R}$ in the domain of attraction of some $\alpha$-stable variable).

a) If $f$ is constant on cylinders, then [Ty2] shows that it satisfies the $J_1$-FSLT (2.21). Theorem 2.2 above generalizes this to all observables which fulfil (2.20).

b) Example 1.2 of [Ty2] shows that (2.20) is not necessary for the validity of the $J_1$-FSLT for a specific $f$. However, the idea that the regularity of the particular $f$ used in this example might give a more general condition sufficient for all $f$, which could replace (2.20) in Theorem 2.2 is misleading. Indeed, a different example in [Ty2] (Example 2.1 there) illustrates the fact that an observable of the same regularity and tail can fail the $J_1$-FSLT.

**Special case: $J_1$-FSLT for tuples with disjoint tail supports.** Suppose that $f^{(1)}, \ldots, f^{(d)} : X \rightarrow \mathbb{R}$ are observables, each attracted to an $\alpha$-stable law, with
asymptotically proportional absolute tails determined on non-overlapping sets, then the vector-valued observable \( f := (f^{(1)}, \ldots, f^{(d)}) : X \to \mathbb{R}^d \) also satisfies a FSLT, the limit being a tuple of independent \( \alpha \)-stable Lévy motions:

**Theorem 2.3 (\( \mathcal{F}_t \)-convergence to an independent tuple of Lévy processes).**
Let \((X, \mathcal{A}, \mu, T, \xi)\) be a mixing probability preserving Gibbs-Markov system, and let \( f^{(1)}, \ldots, f^{(d)} : X \to \mathbb{R} \) be observables, each satisfying the assumptions of Theorem 2.2 and such that there are \( \alpha \in (0, 2) \), a slowly varying function \( \ell \) and, for each \( j \in \{1, \ldots, d\} \), constants \( c^+_j, c^-_j \geq 0 \) with \( c^+_j + c^-_j > 0 \) such that, as \( t \to \infty \),

\[
\mu(f^{(j)} > t) \sim c^+_j t^{-\alpha} \ell(t) \quad \text{and} \quad \mu(f^{(j)} < -t) \sim c^-_j t^{-\alpha} \ell(t).
\]

Suppose in addition that there is some \( M > 0 \) such that

\[
\{ f^{(i)} \mid M \} \cap \{ f^{(j)} \mid M \} = \emptyset \quad \text{whenever} \ i \neq j.
\]

Then the partial sum processes \( S^{[n]} = \{(S_t^{[n]})_{t \geq 0} \mid f := (f^{(1)}, \ldots, f^{(d)}) : X \to \mathbb{R}^d \text{ given by (2.17)} \) converge to a full \( d \)-dimensional \( \alpha \)-stable Lévy process \( S = (S_t)_{t \geq 0} \), so that

\[
S^{[n]} \xrightarrow{d} S \quad \text{in} \quad (\mathcal{D}([0, \infty), \mathbb{R}^d), \mathcal{F}_t)
\]

for every probability measure \( \nu \ll \mu \) on \((X, \mathcal{A})\).

Here, \((A_n^{(1)}, B_n^{(1)})_{n \geq 1} \subseteq \mathbb{R} \times (0, \infty)\) is the canonical normalizing sequence for \([\alpha, c^+_j, c^-_j] \), and we let \( S^{(j)} \) be the corresponding \( \alpha \)-stable variable as in (2,5), while \( B_n := B_n^{(1)} \) and \( A_n := (A_n^{(1)}, \ldots, A_n^{(d)}) \).

The limit process \( S \) is a tuple \((S^{(1)}, \ldots, S^{(d)}) \) of \( d \) independent scalar \( \alpha \)-stable Lévy processes \((S_t^{(j)})_{t \geq 0} \) determined by \( S_t^{(j)} = S^{(j)} \).

### 3. Preparations and Convergence of Marginals

**Regularity control.** We next record that due to bounded distortion (2.8) and the big image property, \( a := \inf_{Z \in \xi} \mu(TZ) > 0 \), a Gibbs-Markov system satisfies

\[
\mu(Z \cap T^{-n} E) \leq \frac{e^R}{a} \mu(Z) \mu(E) \quad \text{whenever} \ n \geq 1, \ 
Z \in \xi_n, \text{ and } E \in \mathcal{A}.
\]

This enables good control of conditional probabilities on cylinders. Let \((\mathfrak{F}, ||\cdot||)\) be a normed space, and \( f : X \to \mathfrak{F} \). For our argument it will be crucial to keep track of the oscillations of ergodic sums on cylinders. We do so using the functions

\[
\vartheta_{f,n} : X \to [0, \infty], \quad \vartheta_{f,n} := \sum_{k=0}^{n-1} \vartheta^{n-k}(\vartheta_f \circ T^k) \quad \text{for} \ n \geq 1.
\]

The value which the \( \xi_n \)-measurable function \( \vartheta_{f,n} \) takes on some rank-\( n \) cylinder \( Z \in \xi_n \) will be denoted \( \vartheta_{f,n}(Z) \). This controls the oscillations of \( S_n(f) \):

**Lemma 3.1 (Oscillation of ergodic sums on cylinders).** Let \((\mathfrak{F}, ||\cdot||)\) be a normed space, \((X, \mathcal{A}, \mu, T, \xi)\) a probability preserving Gibbs-Markov system, and \( f : X \to \mathfrak{F} \) an observable with \( \vartheta_f < \infty \) a.e. on \( X \). Then, for any \( n \geq 1 \) and \( Z \in \xi_n \), we have

\[
\sup_{x, y \in Z} ||S_n(f)(x) - S_n(f)(y)|| \leq \vartheta_{f,n}(Z).
\]
Proof. If \( x, y \in Z \in \xi_n \) and \( k \in \{0, \ldots, n-1\} \), then \( \xi(T^kx) = \xi(T^ky) \). Therefore,

\[
\|S_n(f(x) - S_n(f(y))\| \leq \sum_{k=0}^{n-1} \|f(T^kx) - f(T^ky)\| \leq \sum_{k=0}^{n-1} D_{\xi(T^kx)}(f) d_\theta(T^kx, T^ky)
\]

\[
= \sum_{k=0}^{n-1} (\partial_f \circ T^k)(x) \theta^{n-k} d_\theta(T^nx, T^ny) \leq \partial_{f,n}(x),
\]

since \( \text{diam}(X) = 1 \). \( \square \)

This easy estimate will be exploited through the following general observation.

Lemma 3.2 (Tails of exponentially weighted ergodic sums). Let \((\mathfrak{F}, \|\|)\) be a normed space, \( T \) a measure-preserving map on the probability space \((X, \mathcal{A}, \mu)\), assume that \( g : X \to \mathfrak{F} \) is a measurable function, and \( \rho \in (0, 1) \). Let

\[
G_n := \sum_{k=0}^{n-1} \rho^{n-k} (g \circ T^k) \quad \text{for } n \geq 1.
\]

If the tail of \( \|g\| \) satisfies \( \mu(\|g\| > t) = O(\tau(t)) \) as \( t \to \infty \) for some regularly varying function \( \tau \) of order \(-\alpha\) (some \( \alpha > 0 \)), then there are constants \( \zeta, s_* > 0 \) such that

\[
\mu(\|G_n\| > s) \leq \zeta \tau(s) \quad \text{for } n \geq 1 \text{ and } s \geq s_*.
\]

Proof. (i) Write \( \tau(t) = t^{-\alpha} \ell(t) \), \( t > 0 \), with \( \ell \) slowly varying. Fix some \( q \in (\rho, 1) \). According to Potter’s Theorem for slowly varying functions (Theorem 1.5.6 of [BGT]), there is some \( s_0 > 0 \) such that

\[
\frac{\ell((1-q)(q/\rho)^{j}s)}{\ell(s)} \leq \kappa \left(\frac{q}{\rho}\right)^{j\alpha/2} \quad \text{whenever } j \geq 1 \text{ and } s \geq s_0.
\]

As a consequence,

\[
\sum_{j=1}^{n} \left(\frac{q}{\rho}\right)^{j\alpha} \frac{\ell((1-q)(q/\rho)^{j}s)}{\ell(s)} \leq \kappa \frac{1}{1-(\rho/q)^{\alpha/2}} \quad \text{for } s \geq s_0.
\]

(ii) Now observe, using \( T \)-invariance of \( \mu \), that for \( 1 \leq m < n \) and \( t > 0 \),

\[
\mu \left( \sum_{k=0}^{m} \rho^{n-k}(\|g\| \circ T^k) > t \right) \leq \mu(\rho^{n-m} \|g\| > (1-q)t) + \mu \left( \sum_{k=0}^{m-1} \rho^{n-k}(\|g\| \circ T^k) > qt \right).
\]
Iterating this we obtain, for every $n \geq 1$ and $s > 0$,

$$
\mu(\|G_n\| > s) \leq \mu\left(\sum_{k=0}^{n-1} \rho^{n-k}(\|g\| \circ T^k) > s\right)
$$

$$
\leq \mu(\rho \|g\| > (1-q)s) + \mu\left(\sum_{k=0}^{n-2} \rho^{n-k}(\|g\| \circ T^k) > qs\right)
$$

$$
\vdots
$$

$$
\leq \sum_{j=0}^{n-2} \mu(\rho^{j+1} \|g\| > (1-q)q^j s) + \mu(\rho^n \|g\| > q^{n-1} s)
$$

$$
\leq \sum_{j=0}^{n-1} \mu(\|g\| > (1-q)(q/\rho)^j s).
$$

(In the last step we can drop the surplus $\rho$ since $\rho < 1$.) With $c > 0$ a constant such that $\mu(\|g\| > t) \leq c \tau(t)$ for $t \geq t_0$, we then obtain, recalling (3.6),

$$
\mu(\|G_n\| > s) \leq \tau(s) \sum_{j=1}^{n} \frac{\tau((1-q)(q/\rho)^j s)}{\tau(s)}
$$

$$
= \tau(s) c(1-q)^{-\alpha} \sum_{j=1}^{n} \left(\frac{\rho}{q}\right)^{j} \frac{\ell((1-q)(q/\rho)^j s)}{\ell(s)}
$$

$$
\leq \tau(s) c \kappa (1-q)^{-\alpha} \frac{1}{1-(\rho/q)^{\alpha/2}} \text{ for } s \geq \max\left(s_0, \frac{\rho t_0}{(1-q)q}\right),
$$

which establishes our claim (3.5). \(\square\)

This leads to

**Lemma 3.3 (Uniform tail estimate for the $\vartheta_{f,k}$).** Under the assumptions of Theorem 2.1, there is some constant $\zeta_f > 0$ such that

$$
\text{lim}_{n \to \infty} \sup_{k \geq 1} \mu\left(\frac{\vartheta_{f,k}}{B_n} > \varepsilon\right) \leq \zeta_f \varepsilon^{-\alpha} \text{ for } \varepsilon > 0.
$$

**Proof.** Take any $i \in \{1, \ldots, d\}$, and consider the tail $\tau_{f(i)}$ of $|f^{(i)}|$. Then regular variation of $(B_n)$ and (2.12) of Proposition 2.1 show that for any $\varepsilon > 0$ there is some $n_0^{(i)}(\varepsilon)$ such that

$$
\tau_{f(i)}\left(\frac{\varepsilon}{d} B_n\right) \leq 2c^{(i)} d^\alpha \frac{\varepsilon^{-\alpha}}{n} \text{ for } n \geq n_0^{(i)}(\varepsilon).
$$

Our regularity assumption allows us to apply Lemma 3.2 to $g := \vartheta_{f(i)}$ and $\tau := \tau_{f(i)}$ to obtain $\zeta^{(i)}, s^{(i)}_k > 0$ such that for all $\varepsilon > 0$ we have $\mu(\vartheta_{f(i),k}/B_n > \varepsilon) \leq \zeta^{(i)} \tau_{f(i)}(\varepsilon B_n)$ whenever $k \geq 1$ and $n \geq n_0^{(i)}(\varepsilon)$ (so large that $\varepsilon B_n \geq s^{(i)}_k$). Now $D_Z(f) \leq \sum_{i=1}^{d} D_Z(f^{(i)})$, and hence $\vartheta_{f,k} \leq \sum_{i=1}^{d} \vartheta_{f(i),k}$, so that for any $\varepsilon > 0$,

$$
\mu\left(\frac{\vartheta_{f,k}}{B_n} > \varepsilon\right) \leq \sum_{i=1}^{d} \mu\left(\frac{\vartheta_{f(i),k}}{B_n} > \varepsilon\right) \leq \sum_{i=1}^{d} \zeta^{(i)} \tau_{f(i)}\left(\frac{\varepsilon}{d} B_n\right) \text{ for } k \geq 1 \text{ and } n \geq \max_{1 \leq i \leq d} n_0^{(i)}(\varepsilon).
$$
Combined with (3.8) this gives (3.7) with \( \zeta_f := 2d^n \sum_{i=1}^{d} \zeta^{(i)} c^{(i)}. \)

**Uniform control under conditional measures.** By standard arguments, the good distortion properties of \( T \) can also be expressed in terms of the transfer operator \( \tilde{T} : L_1(\mu) \to L_1(\mu) \) of \( T \), which is characterized by \( \int f \circ T \cdot u \, d\mu = \int f \cdot \tilde{T} u \, d\mu \) for \( f \in L_\infty(\mu) \) and \( u \in L_1(\mu) \). We are going to use this via

**Lemma 3.4 (An \( L_1 \)-compact invariant set for \( \tilde{T} \)).** Let \((X, A, \mu, T, \xi)\) be a probability preserving Gibbs-Markov map. There is some strongly compact convex set \( \mathcal{H} \subseteq L_1(\mu) \) such that \( \mathcal{H} \mathcal{T} \subseteq \mathcal{H} \), while for every \( n \geq 1 \) and \( W \in \xi_n \), the normalized density \( \mu(W)^{-1} \mathcal{T}^n W \) belongs to \( \mathcal{H} \).

**Proof.** This follows from bounded distortion, see for example \[AD\]. One can choose \( \mathcal{H} := \{ f \in L_1(\mu) : f \geq 0, \int f \, d\mu = 1, \text{ and } f \text{ has a version with } D_\xi(f) < K \} \) for a suitable constant \( K > 0 \).

We provide one more abstract lemma which is useful for proving convergence of finite-dimensional marginals.

**Lemma 3.5 (Uniform changes of measures).** Let \((X, A, \mu, T)\) be an ergodic probability preserving system, and \( (G_n)_{n \geq 0} \) a uniformly bounded sequence of measurable functions \( G_n : X \to [0, \infty) \) satisfying

\[
G_n \circ T - G_n \xrightarrow{\mu} 0.
\]

Suppose that \( \mathcal{H} \) is a family of probability densities, strongly compact in \( L_1(\mu) \). Then

\[
\int_X G_n \cdot v \, d\mu - \int_X G_n \cdot u^* \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty, \quad \text{uniformly in } u, u^* \in \mathcal{H}.
\]

**Proof.** This follows from a classical companion (see \[Y\] or Theorem 2 of \[Z3\]) to the mean ergodic theorem. In fact, it is contained in Proposition 3.1 of \[Z2\].

**Convergence of marginals.** We can then establish

**Proposition 3.1 (Convergence of finite-dimensional marginals).** Under the assumptions of Theorem (2.7) we have, for all \( m \geq 1 \),

\[
(\mathcal{S}_t^{[n]}, \ldots, \mathcal{S}_{t_m}^{[n]}) \xrightarrow{\mu} (\mathcal{S}_t, \ldots, \mathcal{S}_{t_m}) \quad \text{as } n \to \infty, \quad \text{for } 0 \leq t_1 < \ldots < t_m \leq 1.
\]

**Proof.** (i) Assumption (2.16) implies that (3.11) is satisfied for \( m = 1 \). For the inductive step, we fix any \( m \geq 1 \) and assume validity of (3.11). To prove (3.11) with \( m \) replaced by \( m + 1 \) we fix any tuple \( 0 < t < t_1 < \ldots < t_m \leq 1 \) (the case \( t = 0 \) being trivial), and any \( s = (s^{(1)}, \ldots, s^{(d)}) \in \mathbb{R}^d \) for which \( \Pr[\mathcal{S}_t \leq s] > 0 \), where \( (r^{(1)}, \ldots, r^{(d)}) \leq (s^{(1)}, \ldots, s^{(d)}) \) means that \( r^{(i)} \leq s^{(i)} \) for all \( i \in \{1, \ldots, d\} \).

We are going to show that

\[
(\mathcal{S}_t^{[n]} - \mathcal{S}_t^{[n]}, \ldots, \mathcal{S}_{t_m}^{[n]} - \mathcal{S}_{t}^{[n]}) \xrightarrow{\mu_E h} (\mathcal{S}_t_{t-t}, \ldots, \mathcal{S}_{t_m-t}) \quad \text{as } n \to \infty,
\]

where \( E_n := \{ \mathcal{S}_t^{[n]} \leq s \} \). This suffices since \( \mathcal{S} \) is a stable Lévy motion, and the \( m = 1 \) case of (3.11) guarantees that \( \mu(E_n) \to \Pr[\mathcal{S}_t \leq s] > 0 \).

(ii) We will work with conditioning events \( F_n \) more convenient than the \( E_n \). Define \( E_n' := \{ \mathcal{S}_t^{[n]} - B_n^{-1}\partial f_{ \{t \}} (1, \ldots, 1) \leq s \} \) and \( F_n := \bigcup_{Z \in \xi(t)} \mathcal{S}_t^{[n]} \leq s \) somewhere on \( Z^d \),
$n \geq 1$. Obviously, $E_n \subseteq E_n'$. Now $\vartheta_{f_{(s)}} \leq \vartheta_{f_{s}}$ if $f = (f^{(1)}, \ldots, f^{(d)})$, and Lemma 3.3 shows that $F_n \subseteq E_n'$. Next, Lemma 3.3 gives

$$
\vartheta_{f_{[t]n}}/B_n \xrightarrow{\mu} 0 \quad \text{as } n \to \infty.
$$

Since $S_t$ has a continuous distribution, $(\mu(E_n'))$ has the same limit as $(\mu(E_n))$, and hence we also have $\mu(F_n) \to \Pr[S_t \leq s]$. Therefore, (3.12) is equivalent to

$$
(S_{t_{1}^n} - S_{t}^n, \ldots, S_{t_{m}^n} - S_{t}^n) \xrightarrow{\mu_F} (S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t}) \quad \text{as } n \to \infty.
$$

But $S_{t_{i}^n} - S_{t}^n = B_n^{-1}(S_{t_{i}^n-t} - t_{i}^n)(f) - (\lfloor t'_{i}^n \rfloor - \lfloor t_{i}^n \rfloor)A_n/n) \circ T^{\lfloor t'_{i}^n \rfloor}$ for any $t' < t''$. Also, since $\lfloor t''_{i}^n \rfloor - \lfloor t_{i}^n \rfloor - \lfloor t'_{i}^n - t_{i}^n \rfloor \leq 2$, we get

$$
\|S_{t_{i}^n} - S_{t}^n - S_{t_{i}^n-t} \circ T^{\lfloor t'_{i}^n \rfloor} \| \xrightarrow{\mu} 0 \quad \text{as } n \to \infty.
$$

(Use (2.13) and the fact that $(f \circ T^k)/B_k \xrightarrow{\mu} 0$, which is immediate from (2.4) and (2.6).) Therefore (3.14) is equivalent to

$$
(S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t}) \xrightarrow{\mu_F \circ T^{-[tn]}} (S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t}) \quad \text{as } n \to \infty.
$$

Note that $F_n$ is a $\xi_{[tn]}$-measurable set. Therefore the density of $\mu_{F_n} \circ T^{-[tn]}$, that is, $u_n := \hat{T}^{[tn]}(\mu(F_n)^{-1}1_{F_n})$ belongs to the closed convex set $\mathcal{F}$ of Lemma 3.4.

(iii) The desired convergence (3.15) can be established by checking that for every $G : \mathbb{R}^{md} \to \mathbb{R}$ of the form $G(x_1, \ldots, x_m) = g_1(x_1) \cdots g_m(x_m)$ with bounded Lipschitz functions $g_j : \mathbb{R}^d \to \mathbb{R}$ we have

$$
\int G(S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t}) u_n \, d\mu \longrightarrow \mathbb{E}[G(S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t})] \quad \text{as } n \to \infty.
$$

Due to assumption (3.11), (3.15) is valid if the $\mu_{F_n} \circ T^{-[tn]}$ are replaced by the single measure $\mu$, and hence (3.16) is valid if the $u_n$ are replaced by the density $1_X$ of $\mu$. Therefore, (3.10) follows once we prove that

$$
\int G(S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t}) (u_n - 1_X) \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty.
$$

But letting $G_n := G(S_{t_{1}^n-t}, \ldots, S_{t_{m}^n-t})$ it is easy to see that

$$
|G_n \circ T - G_n| \leq \frac{L \Gamma_{m-1}}{B_n} \sum_{j=1}^{m} \left( \|f\| + \|T^{\lfloor t_{j}^n-t_{j-1}^n \rfloor}f\| \right),
$$

with $L$ a common Lipschitz constant for the $g_j$, and $\Gamma := \max_{1 \leq j \leq m} \sup \|g_j\|$. Since $T$ preserves $\mu$ and $B_n \to \infty$, the asymptotic invariance property (3.9) follows. Lemma 3.5 then gives (3.17) since $u_n \in \mathcal{F}$ for all $n \geq 1$.

4. Maximal Inequalities and Tightness

Maximal inequalities. The proof of tightness will depend on the following maximal inequalities, which constitute the main technical tool of the present paper. (An easier version of these arguments has been used in [Z1].) Let $(\mathcal{F}, \|\cdot\|)$ be a normed space.
Lemma 4.1 (Maximal inequalities for ergodic sums). Let \((X, \mathcal{A}, \mu, T, \xi)\) be a probability preserving Gibbs-Markov map, and \(g : X \to \mathbb{R}\) an observable with \(\vartheta_g < \infty\) a.e. on \(X\). Denote \(S_n = S_n(g)\), \(n \geq 0\). Then, for any \(n \geq 1\) and \(\kappa > 0\),

\[
\mu \left( \max_{1 \leq k \leq n} \| S_k \| > \kappa \right) \leq \frac{2e^R}{a} \max_{1 \leq k \leq n} \mu \left( \| S_k \| > \frac{\kappa}{4} \right) + n \max_{1 \leq k \leq n} \mu \left( \vartheta_{g,k} > \frac{\kappa}{4} \right),
\]

while

\[
\mu \left( \max_{1 \leq k \leq n} \| S_n - S_k \| > \kappa \right) \leq \mu \left( \max_{1 \leq k \leq n} \| S_k \| > \frac{\kappa}{2} \right).
\]

Moreover, for any \(n \geq 1\) and \(\kappa > 0\),

\[
\mu \left( \max_{1 \leq j < l \leq n} (\| S_j - S_l \| \wedge \| S_l - S_j \|) > \kappa \right)
\]

\[
\leq \frac{e^R}{a} \mu \left( \max_{1 \leq k \leq n} \| S_k \| > \frac{\kappa}{4} \right) \left[ \mu \left( \max_{1 \leq k \leq n} \| S_k \| > \frac{\kappa}{4} \right) + n \max_{1 \leq k \leq n} \mu \left( \vartheta_{g,k} > \frac{\kappa}{4} \right) \right].
\]

Proof. (1) We fix \(\kappa > 0\) and define families of cylinders by

\[
\gamma_1(\kappa) := \left\{ Z \in \xi_1 : \sup_Z \| S_t \| > \kappa \right\} \quad \text{and, for} \quad k \geq 1,
\]

\[
\gamma_{k+1}(\kappa) := \left\{ Z \in \left( \bigcup_{j=1}^k \bigcup_{W \in \gamma_j(\kappa)} W \right)^c \cap \xi_{k+1} : \sup_Z \| S_{k+1} \| > \kappa \right\}
\]

(where, for \(E\) measurable w.r.t. a partition \(\eta\), \(E \cap \eta := \{T \in \eta : T \subseteq E\}\)). Then,

\[
\left\{ \max_{1 \leq k < n} \| S_k \| > \kappa \right\} \subseteq \bigcup_{k=1}^{n-1} \bigcup_{Z \in \gamma_k(\kappa)} Z \quad \text{(disjoint),}
\]

and, since \(\inf_Z \| S_k \| > \kappa - \vartheta_{g,k}(Z)\) for \(Z \in \gamma_k(\kappa)\) by Lemma 3.1, we get

\[
\left\{ \max_{1 \leq k < n} \| S_k \| > \kappa \right\} \cap \left\{ \| S_n \| \leq \frac{\kappa}{2} \right\} \subseteq \bigcup_{k=1}^{n-1} \bigcup_{Z \in \gamma_k(\kappa)} Z \cap \left\{ \| S_n \| \leq \frac{\kappa}{2} \right\}
\]

\[
\subseteq \bigcup_{k=1}^{n-1} \bigcup_{Z \in \gamma_k(\kappa)} Z \cap \left( \left\{ \| S_n - S_k \| > \frac{\kappa}{4} \right\} \cup \left\{ \vartheta_{g,k} > \frac{\kappa}{4} \right\} \right).
\]

According to 3.1 we see that for any \(Z \in \gamma_k(\kappa)\), \(1 \leq k < n\),

\[
\mu \left( Z \cap \left\{ \| S_n - S_k \| > \frac{\kappa}{4} \right\} \right) = \mu \left( Z \cap T^{-k} \left\{ \| S_{n-k} \| > \frac{\kappa}{4} \right\} \right)
\]

\[
\leq \frac{e^R}{a} \mu(Z) \mu \left( \| S_{n-k} \| > \frac{\kappa}{4} \right).
\]
Combining these observations we find that

\[
\mu \left( \max_{1 \leq k < n} \|S_k\| > \kappa \right) \cap \left\{ \|S_n\| \leq \frac{\kappa}{2} \right\} \\
\leq \frac{e^R}{a} \sum_{k=1}^{n-1} \sum_{Z \in \gamma_k(\kappa)} \mu(Z) \mu \left( \|S_{n-k}\| > \frac{\kappa}{4} \right) + \sum_{k=1}^{n-1} \sum_{Z \in \gamma_k(\kappa)} \mu \left( Z \cap \left\{ \vartheta_{g,k} > \frac{\kappa}{4} \right\} \right) \\
\leq \frac{e^R}{a} \max_{1 \leq k < n} \mu \left( \|S_k\| > \frac{\kappa}{4} \right) + n \max_{1 \leq k < n} \mu \left( \vartheta_{g,k} > \frac{\kappa}{4} \right),
\]

where the last step again uses that \( \{Z \in \gamma_k(\kappa) : k \geq 1\} \) is a family of pairwise disjoint sets. This implies our maximal inequality \((4.1)\) if we also note that

\[
\mu \left( \max_{1 \leq k < n} \|S_k\| > \kappa \right) \leq \mu \left( \|S_n\| > \frac{\kappa}{2} \right) + \mu \left( \left\{ \max_{1 \leq k < n} \|S_k\| > \kappa \right\} \cap \left\{ \|S_n\| \leq \frac{\kappa}{2} \right\} \right). \]

The second inequality \((12.2)\) of our lemma is immediate from

\[
\left\{ \max_{1 \leq k < n} \|S_n - S_k\| > \kappa \right\} \subseteq \left\{ \|S_n\| > \frac{\kappa}{2} \right\} \cup \left\{ \max_{1 \leq k < n} \|S_k\| > \frac{\kappa}{2} \right\}.
\]

(ii) Fixing \( \kappa > 0 \), we first note that by \((4.5)\),

\[
\left\{ \max_{1 \leq i < j < n} \|S_j - S_i\| > \kappa \right\} \subseteq \left\{ \max_{1 \leq k < n} \|S_k\| > \frac{\kappa}{2} \right\} \subseteq \bigcup_{k=1}^{n-1} \bigcup_{Z \in \gamma_k(\kappa/2)} Z \text{ (disjoint)}.
\]

Assume that \( x \in \{ \max_{1 \leq i < j < l \leq n} (\|S_j - S_i\| \wedge \|S_l - S_j\|) > \kappa \} \). Then \( \|S_m(x)\| > \kappa/2 \) for some \( m \leq n \), and therefore there are \( k \in \{1, \ldots, n\} \) and \( Z \in \gamma_k(\kappa/2) \) such that \( x \in Z \) (use \((15)\) again). Choose \( 1 \leq i < j < l \leq n \) such that \( \|S_j(x) - S_i(x)\| > \kappa \) and \( \|S_l(x) - S_j(x)\| > \kappa \). According to the definition of \( \gamma_k(\kappa/2) \) above, we have \( \|S_h(x)\| \leq \kappa/2 \) for all \( h < k \), so that (due to \( \|S_j(x) - S_i(x)\| > \kappa \)) necessarily \( j \geq k \). We claim that

\[
\max_{k < h \leq n} \|S_h(x) - S_k(x)\| > \kappa/2.
\]

In case \( j = k \) this is clear for \( h = l \), by our choice of \( j \) and \( l \). On the other hand, if \( j > k \), then \( \|S_l(x) - S_j(x)\| > \kappa \) ensures that \( \|S_j(x) - S_k(x)\| > \kappa/2 \) or \( \|S_l(x) - S_k(x)\| > \kappa/2 \), and we can take \( h = j \) or \( h = l \), proving our claim.

We therefore see that for any \( k \in \{1, \ldots, n\} \) and \( Z \in \gamma_k(\kappa/2) \),

\[
Z \cap \left\{ \max_{1 \leq i < j < l \leq n} (\|S_j - S_i\| \wedge \|S_l - S_j\|) > \kappa \right\} \subseteq Z \cap \left\{ \max_{k < h \leq n} \|S_h - S_k\| > \frac{\kappa}{2} \right\} \\
= Z \cap T^{-k} \left\{ \max_{1 \leq l \leq n-k} \|S_l\| > \frac{\kappa}{2} \right\},
\]
and hence
\[
\mu \left( \max_{1 \leq i < j \leq n} (\|S_j - S_i\| \wedge \|S_l - S_j\|) > \kappa \right) \\
\leq \sum_{k=1}^{n} \sum_{Z \in \gamma_k(\kappa/2)} \mu \left( Z \cap T^{-k} \left\{ \max_{1 \leq i \leq n-k} \|S_i\| > \frac{\kappa}{2} \right\} \right) \\
\leq \frac{R}{a} \sum_{k=1}^{n} \sum_{Z \in \gamma_k(\kappa/2)} \mu(Z) \mu \left( \max_{1 \leq i \leq n-k} \|S_i\| > \frac{\kappa}{2} \right) \\
\leq \frac{e^{R}}{a} \mu \left( \max_{1 \leq i \leq n} \|S_i\| > \frac{\kappa}{2} \right) \mu \left( \bigcup_{k=1}^{n} \bigcup_{Z \in \gamma_k(\kappa/2)} Z \right).
\]

But as \( \inf_{Z} \|S_k\| > \kappa/2 - \vartheta_{g,k}(Z) \) for \( Z \in \gamma_k(\kappa/2) \), we have \( Z \subseteq \{ \|S_k\| > \kappa/4 \} \cup \{ \vartheta_{g,k} > \kappa/4 \} \) for such \( Z \), and therefore
\[
\mu \left( \bigcup_{k=1}^{n} \bigcup_{Z \in \gamma_k(\kappa/2)} Z \right) \leq \mu \left( \max_{1 \leq i \leq n} \|S_i\| > \frac{\kappa}{4} \right) + n \max_{1 \leq k \leq n} \mu \left( \vartheta_{g,k} > \frac{\kappa}{4} \right).
\]
Combining the last two estimates yields inequality (4.3).

\[ \Box \]

**Control of translation sequence.** We shall also use the following observation regarding the sequence \((A_n)\). Note that the assertion of the lemma below is trivial in many cases (when \( A_n = 0 \) or \( A_n = \text{const} \cdot n \)). The \( d \)-dimensional \( \alpha \)-stable Lévy process \( S = (S_t)_{t \geq 0} \) with \( S_1 \overset{d}{=} S \) satisfies \( S_t \overset{d}{=} t^{1/\alpha} S + a_t \), \( t \geq 0 \), for some continuous function \( t \mapsto a_t = (a_t^{(1)}, \ldots, a_t^{(d)}) \), \( t \geq 0 \), which appears in the following statement.

**Lemma 4.2 (Uniform control of translation vectors \( A_n \)).** Under the assumptions of Theorem 2.1, the normalizing sequence \((A_n, B_n)\) satisfies
\[
(4.6) \quad \max_{1 \leq k \leq n} \left\| \frac{1}{B_{n_j}} \left( A^{(i)}_{k} - \frac{k_j}{n_j} A^{(i)}_{n_j} \right) - \frac{a^{(i)}_{s_j}}{\sigma_j} \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

**Proof.** (i) Assume the contrary, and write \( A_n = (A_n^{(1)}, \ldots, A_n^{(d)}) \). Then there are \( i \in \{1, \ldots, d\} \), \( \eta > 0 \) and sequences \((n_j)_{j \geq 1}\) and \((k_j)_{j \geq 1}\) in \( \mathbb{N} \) such that \( k_j \leq n_j \) and
\[
(4.7) \quad \left\| \frac{1}{B_{n_j}} \left( A^{(i)}_{k_j} - \frac{k_j}{n_j} A^{(i)}_{n_j} \right) - \frac{a^{(i)}_{s_j}}{\sigma_j} \right\| > \eta \quad \text{for } j \geq 1,
\]
while \( s_j := k_j/n_j \rightarrow s \) as \( j \rightarrow \infty \) for some \( s \in [0,1] \). But we are going to show that for any \( s \) and \((s_n)_{n \geq 1} \in (0,1] \) with \( s_n \rightarrow s \),
\[
(4.8) \quad \frac{1}{B_{n_j}} \left( A^{(i)}_{s_{n_j}} - s_{n_j} A^{(i)}_{n_j} \right) - a^{(i)}_{s_{n_j}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]
which contradicts (4.7) and therefore proves (4.6).

(ii) To validate (4.8), recall first that we have observed (Proposition 2.1) that if \((Z_k)_{k \geq 1}\) is an iid sequence of random variables on some probability space \((\Omega, \mathcal{F}, \Pr)\) with the same distribution as \( f^{(i)} \), and \( S_n := \sum_{k=0}^{n-1} Z_k \), then \( B_{n}^{-1}(S_n - A^{(i)}_{n}) \rightarrow S^{(i)} \).
Define \( S^{[n]} = (S_t^{[n]})_{t \geq 0} \) by \( S_t^{[n]} := B_n^{-1}(\hat{S}_{tn} - |tn|A_n^{(i)}/n) \). Now use Skorokhod’s classical \( J_1 \)-FSLT (\([S2]\)) to see that \( \hat{S}^{[n]} \Rightarrow \hat{S} \) in the \( J_1 \)-topology, with \( \hat{S} \) the one-dimensional stable Lévy process with \( \hat{S}_1 \overset{d}{=} S^{(i)} \).

In particular, \( \hat{S}^{[n]}_t \Rightarrow \hat{S}_t \), and due to regular variation of \( (B_n) \),

\[
V_n := \frac{B_{|tn|}}{B_n} (|tn| - A_n^{(i)}) \Rightarrow s^{1/\alpha} \hat{S}_1 \quad \text{as } n \to \infty.
\]

Now consider homeomorphisms \( \lambda_n \in \Lambda \) of \([0, 1]\) with \( \lambda_n(s) = s_n \), and affine on \([0, s]\) and \([s, 1]\), then \( \sup_{[0,1]} |\lambda_n - \lambda| \to 0 \) as \( n \to \infty \). Define time-changed processes \( \tilde{S}^{[n]} = (\tilde{S}_t^{[n]})_{t \in [0,1]} \) by \( \tilde{S}_t^{[n]} := \hat{S}_{\lambda_n(t)} \), and note that

\[
d_{J_1}(\tilde{S}^{[n]}, \tilde{S}^{[n]}) \leq \sup_{[0,1]} |\lambda_n - \lambda| \quad \text{on } \Omega \quad \text{for } n \geq 1.
\]

(For any \( n \geq 1 \) and \( \omega \in \Omega \), the paths \( x = (x_t) = (\tilde{S}_t^{[n]}(\omega)) \) and \( y = (y_t) = (\tilde{S}_t^{[n]}(\omega)) \) are related by \( y = x \circ \lambda_n \). Now recall the definition of \( d_{J_1} \).) This shows that \( d_{J_1}(\tilde{S}^{[n]}, \tilde{S}^{[n]}) \to 0 \) uniformly on \( \Omega \) as \( n \to \infty \), and therefore convergence of \( (\tilde{S}^{[n]}) \) entails \( \tilde{S}^{[n]} \Rightarrow \tilde{S} \). In particular,

\[
V_n + \frac{1}{B_n} (A_n^{(i)} - |s_n|A_n^{(i)}) = \tilde{S}_n^{[n]} \Rightarrow \tilde{S}_n \overset{d}{=} \tilde{S}_1 + A_n^{(i)} \quad \text{as } n \to \infty.
\]

But \((4.9)\) and \((4.11)\), together with continuity of \( t \to A_n^{(i)}(t) \), give \((4.10)\). \( \square \)

**Tightness in** \((D([0,1], \mathbb{R}^d), J_1)\). We can now tackle the crucial tightness condition.

**Proposition 4.1 (Tightness)**. Under the assumptions of Theorem [2.7], the sequence \((\hat{S}^{[n]})_{n \geq 1}\) on \((X, \mathcal{A}, \mu)\) is tight for the (strong) \( J_1 \)-topology on \( D([0,1], \mathbb{R}^d) \).

**Proof.** (i) We check that the distributions under \( \mu \) of the \( S^{[n]} \) are uniformly tight as a sequence of Borel probability measures on \( D([0,1], \mathbb{R}^d) \) equipped with the (strong) \( J_1 \)-topology. For \( x \in D([0,1], \mathbb{R}^d) \) and \( \delta \in (0,1) \) define

\[
\Delta_\delta^{(1)}(x) := \sup_{0 \leq t \leq \delta} \|x_t - x_0\| \quad \text{and} \quad \Delta_\delta^{(2)}(x) := \sup_{1/\delta \leq t \leq 1} \|x_1 - x_t\|,
\]

while

\[
\Delta_\delta^{(3)}(x) := \sup_{0 \leq \omega(t) \leq \delta} \|x_\omega - x_\omega\| \wedge \|x_{\omega'} - x_{\omega'}\|.
\]

A set \( K \subseteq D([0,1], \mathbb{R}^d) \) with \( x_0 = 0 \) for all \( x \in K \) is relatively compact in the \( J_1 \)-topology iff for every \( j \in \{1, 2, 3\} \),

\[
\lim_{\delta \to 0} \sup_{x \in K} \Delta_\delta^{(j)}(x) = 0,
\]

see statements 2.7.2 and 2.7.3 of \([S1]\). As a consequence, uniform tightness of \((S^{[n]})_{n \geq 1}\) in \( (D([0,1], \mathbb{R}^d), J_1) \) can be verified by showing that for every \( j \in \{1, 2, 3\} \),

\[
\lim_{\delta \to 0} \mu \left( \Delta_\delta^{(j)}(S^{[n]}) > \varepsilon \right) = 0 \quad \text{for } \varepsilon > 0.
\]

(ii) To efficiently deal with the \( A_n \) we define new observables \( f_n : X \to \mathbb{R}^d \) with \( f_n := f - A_n/n \), \( n \geq 1 \), so that \( S_k(f_n) = S_k(f) - (k/n)A_n \) whenever \( k, n \geq 1 \). Consequently,

\[
S_t^{[n]} = \frac{1}{B_n} S_{tn}(f_n) \quad \text{for } n \geq 1 \text{ and } t \in [0,1],
\]
whereas

\begin{equation}
S'_t - S'_{t'} = \frac{1}{B_n} S_{[t_n]-[t'_n]} (f_n) \circ T_{[t']n} \quad \text{for } n \geq 1 \text{ and } 0 \leq t' \leq t \leq 1.
\end{equation}

For later use, pick a constant \( C > 0 \) such that \( \Pr[\|S\| > t] \leq C \cdot t^{-\alpha} \) for \( t \geq 1/24 \).

(iii) We first establish (4.13) for \( j = 1 \). To this end, we fix \( \varepsilon > 0 \) and observe that (4.14) yields \( \Delta_k^{(1)}(S^{[n]}) = \max_{1 \leq k \leq \delta n} \|B_n^{-1}S_k(f_n)\| \), which allows us to apply (4.1) of Lemma 4.11 to \( g := f_n \). This gives

\begin{equation}
\mu \left( \Delta_k^{(1)}(S^{[n]}) > \varepsilon \right) \leq \frac{2e^R}{a} \max_{1 \leq k \leq \delta n} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{4} \right)
+ \delta n \max_{1 \leq k \leq \delta n} \mu \left( \frac{\partial f_n,k}{B_n} > \frac{\varepsilon}{4} \right)
\end{equation}

for \( \delta \in (0, 1) \) and \( n \geq 1 \).

Consider the first expression on the right-hand side of (4.16). Since \( t \mapsto a_t \) is continuous, we can find \( \delta' > 0 \) such that \( \|a_{k/n}\| < \varepsilon/16 \) whenever \( 1 \leq k \leq \delta'n \). By Lemma 1.2, there is some \( n^* = n^*(\varepsilon) \) s.t. \( \max_{1 \leq k \leq n} \|B_n^{-1}(A_k - \frac{k}{n}A_n) - a_{k/n}\| < \varepsilon/16 \) for \( n \geq n^* \). Now

\begin{equation}
\frac{1}{B_n} S_k(f_n) = \frac{1}{B_n} (S_k(f) - A_k) + a_{k/n} + \frac{1}{B_n} \left( A_k - \frac{k}{n}A_n \right) - a_{k/n},
\end{equation}

so that

\begin{equation}
\left\{ \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{4} \right\} \subseteq \left\{ \left\| \frac{1}{B_n} (S_k(f) - A_k) \right\| > \frac{\varepsilon}{8} \right\}
\text{ for } n \geq n^* \text{ and } 1 \leq k \leq \delta'n.
\end{equation}

Due to (4.16), we see that for every \( \delta \in (0, \varepsilon^\alpha) \) there is some \( k^*(\delta) \) such that

\begin{equation}
\mu \left( \left\| \frac{1}{B_n} (S_k(f) - A_k) \right\| > \frac{\varepsilon}{24} \delta^{-\frac{1}{\alpha}} \right) \leq \Pr \left[ \|S\| > \frac{\varepsilon}{24} \delta^{-\frac{1}{\alpha}} \right] + \delta
\end{equation}

\begin{equation}
\leq \left[ C \left( \frac{\varepsilon}{24} \right)^{-\alpha} + 1 \right] \delta \quad \text{ for } k \geq k^*(\delta).
\end{equation}

Since \( (B_n) \in \mathcal{R}_{1/\alpha} \), there are \( k_* \) and, for each \( \delta \in (0, \varepsilon^\alpha) \), some \( n_*(\delta) \) for which

\begin{equation}
\frac{B_n}{B_k} \geq \frac{1}{4} \delta^{-\frac{1}{\alpha}} \quad \text{ for } n \geq n_*(\delta) \text{ and } k_* \leq k \leq \delta n.
\end{equation}

(Indeed, there is some non-decreasing sequence \( (B_n^*) \in \mathcal{R}_{1/\alpha} \) for which \( B_n^* \sim B_n \) as \( n \to \infty \), see Theorem 1.5.3 of [BGT]). Choose \( k_* \) for which \( B_n^*/B_k \in (1/\sqrt{2}, \sqrt{2}) \) whenever \( k \geq k_* \). Given \( \delta > 0 \) there is some \( n_*(\delta) \) such that \( B_n^*/B_{\delta n} \geq \delta^{-\frac{1}{\alpha}}/2 \) for \( n \geq n_*(\delta) \). Now (4.20) follows if we write \( B_n/B_k = (B_n/B_n^*)(B_n^*/B_k)(B_n^*/B_k^*) \) and use \( B_n^*/B_k^* \geq B_n^*/B_{\delta n} \) for \( k \leq \delta n \). Combining (4.19) and (4.20) we find that

\begin{equation}
\mu \left( \left\| \frac{1}{B_n} (S_k(f) - A_k) \right\| > \frac{\varepsilon}{8} \right) = \mu \left( \left\| \frac{1}{B_n} (S_k(f) - A_k) \right\| > \frac{\varepsilon}{24} \delta^{-\frac{1}{\alpha}} \right)
\leq \left[ C \left( \frac{\varepsilon}{24} \right)^{-\alpha} + 1 \right] \delta 
\text{ for } n \geq n^* \lor n_*(\delta) \text{ and } k \geq k_* \lor k^*(\delta).
\end{equation}

Recalling (4.15) we thus obtain, for all \( \delta \in (0, \delta' \lor \varepsilon^\alpha) \),

\begin{equation}
\lim_{n \to \infty} \max_{k_*, \lor k^*(\delta) \leq k \leq \delta n} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{4} \right) \leq \left[ C \left( \frac{\varepsilon}{24} \right)^{-\alpha} + 1 \right] \delta.
\end{equation}
On the other hand, (4.18) shows that for every given \( k \geq 1 \),

\[
\left\| \frac{S_k(f_n)}{B_n} \right\| \xrightarrow{n \to \infty} 0
\]

Therefore, whatever \( \delta \in (0, \delta' \wedge \varepsilon') \), we have

\[
\lim_{n \to \infty} \max_{1 \leq k < k_n \vee k_n^*} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{4} \right) = 0.
\]

Together with (4.22) this yields, for every \( \delta \in (0, \delta' \wedge \varepsilon') \),

\[
\lim_{n \to \infty} \max_{1 \leq k \leq \delta n} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{4} \right) \leq \left[ C \left( \frac{\varepsilon}{24} \right)^{-\alpha} + 1 \right] \delta =: C_\ast(\varepsilon) \delta.
\]

Turning to the second term on the right-hand side of (4.16), observe first that trivially \( \vartheta_{f_n} = \vartheta_f \) for every \( n \geq 1 \), and hence \( \vartheta_{f_n,k} = \vartheta_{f,k} \) for all \( k, n \geq 1 \). But then Lemma 3.3 immediately shows that

\[
\lim_{n \to \infty} \delta n \max_{1 \leq k \leq \delta n} \mu \left( \vartheta_{f_n,k} > \frac{\varepsilon}{4} \right) \leq \zeta_j \left( \frac{\varepsilon}{4} \right)^{-\alpha} \delta =: C_\ast(\varepsilon) \delta.
\]

When combined with (4.24) this implies (4.13) for \( j = 1 \).

(iv) To deal with (4.13) for \( j = 2 \), write

\[
\Delta_\delta^{(2)}(S^n] = \max_{\lfloor (1-\delta)n \rfloor \leq j \leq n} \left\| \frac{S_n(f_n) - S_j(f_n)}{B_n} \right\| = \left( \max_{0 \leq k \leq n-\lfloor (1-\delta)n \rfloor} \left\| \frac{S_k(f_n)}{B_n} \right\| \right) \circ T^{\lfloor (1-\delta)n \rfloor}.
\]

Using \( T \)-invariance of \( \mu \), and the inequalities (4.2) and (4.1) (again for \( g := f_n \)), we then find that

\[
\mu \left( \Delta_\delta^{(2)}(S^n] > \varepsilon \right) \leq \mu \left( \max_{0 \leq k \leq n-\lfloor (1-\delta)n \rfloor} \left\| \frac{S_k(f_n)}{B_n} \right\| > \varepsilon \right) \leq \frac{2eR}{a} \max_{0 \leq k \leq n-\lfloor (1-\delta)n \rfloor} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{2} \right) \leq \frac{2eR}{a} \max_{0 \leq k \leq n-\lfloor (1-\delta)n \rfloor} \mu \left( \left\| \frac{S_k(f_n)}{B_n} \right\| > \frac{\varepsilon}{8} \right) + (n - \lfloor (1-\delta)n \rfloor) \max_{0 \leq k \leq n-\lfloor (1-\delta)n \rfloor} \mu \left( \vartheta_{f_n,k} > \frac{\varepsilon}{8} \right).
\]

Since \( n - \lfloor (1-\delta)n \rfloor \sim \delta n \) as \( n \to \infty \), assertion (4.13) for \( j = 2 \) follows from (4.24) and (4.25).

(v) We finally turn to (4.13) for \( j = 3 \). For any \( \delta \in (0, 1) \), if \( n \geq 1/\delta \) then \( \lfloor 2\delta n \rfloor \geq \delta n \), and thus for any triple \( t' < t < t'' \) as in the definition of \( \Delta_\delta^{(3)} \), the points \( \lfloor t'n \rfloor \leq \lfloor tn \rfloor \leq \lfloor t''n \rfloor \) are contained in an interval of the form \( [k \lfloor 2\delta n \rfloor, \lfloor k+3 \lfloor 2\delta n \rfloor \rfloor ] \). Consequently, \( \Delta_\delta^{(3)}(S^n] \) cannot exceed

\[
\max_{0 \leq k \leq n/2\delta n} \left( \max_{0 \leq i < j < l \leq 4\lfloor 2\delta n \rfloor} \left( \left\| \frac{S_j(f_n) - S_i(f_n)}{B_n} \right\| \wedge \left\| \frac{S_i(f_n) - S_j(f_n)}{B_n} \right\| \right) \right) \circ T^{k \lfloor 2\delta n \rfloor}.
\]
Therefore, \( \{ \Delta_\delta^{(3)}(S^{[n]}) > \varepsilon \} \) is contained in
\[
\bigcup_{0 \leq k \leq n/2\delta n} T^{-k[2\delta n]} \left\{ \max_{0 \leq i < j < l \leq 4[2\delta n]} \left( \frac{\|S_j(f_n) - S_i(f_n)\|}{B_n} \wedge \frac{\|S_i(f_n) - S_j(f_n)\|}{B_n} \right) > \varepsilon \right\},
\]
and due to \( T \)-invariance of \( \mu \), we find that for \( \delta \in (0, 1/2) \) and \( n \) so large that \( 1 + n/2\delta n \leq 1/\delta \),
\[
\mu \left( \Delta_\delta^{(3)}(S^{[n]}) > \varepsilon \right) \leq \frac{1}{\delta} \mu \left( \max_{0 \leq i < j < l \leq 4[2\delta n]} \left( \frac{\|S_j(f_n) - S_i(f_n)\|}{B_n} \wedge \frac{\|S_i(f_n) - S_j(f_n)\|}{B_n} \right) > \varepsilon \right).
\]
Hence, applying (4.24) to \( g := f_n \),
\[
\mu \left( \Delta_\delta^{(3)}(S^{[n]}) > \varepsilon \right) \leq \frac{e^R}{\delta a} \mu \left( \max_{1 \leq k \leq 8\delta n} \left\| S_k(f_n) \right\| \frac{\|S_k(f_n)\|}{B_n} > \frac{\varepsilon}{4} \right)
\cdot \left[ \mu \left( \max_{1 \leq k \leq 8\delta n} \left\| S_k(f_n) \right\| \frac{\|S_k(f_n)\|}{B_n} > \frac{\varepsilon}{4} \right) + 8\delta n \max_{1 \leq k \leq 8\delta n} \mu \left\{ \frac{\|f_{n,k}\|}{\|f_n\|} > \frac{\varepsilon}{4} \right\} \right].
\]
In view of (4.24) and (4.25) this shows that for every \( \delta \in (0, \varepsilon \wedge 1/2) \),
\[
\liminf_{n \to \infty} \mu \left( \Delta_\delta^{(3)}(S^{[n]}) > \varepsilon \right) \leq \frac{e^R}{\delta a} C_* (\varepsilon) [C_* (\varepsilon) + 8C_* (\varepsilon)] \delta,
\]
and (4.13) with \( j = 3 \) follows as \( \delta \searrow 0 \). \( \Box \)

We finally provide a \( d \)-dimensional version of Corollary 3 of [Z3].

**Lemma 4.3 (Strong distributional convergence of \( S^{[n]} \) in \( (D([0, \infty), \mathbb{R}^d), J_1) \)).**

Let \( T \) be an ergodic measure preserving map on the probability space \( (X, \mathcal{A}, \mu) \), \( f : X \to \mathbb{R}^d \) a measurable function, and \( S^{[n]} \) defined as in (2.17) with \( (A_n, B_n) \in \mathbb{R}^d \times (0, \infty) \) satisfying \( B_n \to \infty \) and \( \|A_n\| = o(nB_n) \) as \( n \to \infty \). Then
\[
d_{J_1, \infty}(S^{[n]}, S^{[n]} \circ T) \xrightarrow{\mu} 0 \quad \text{as} \quad n \to \infty.
\]
Consequently, whenever \( S \) is a random element of \( (D([0, \infty), \mathbb{R}^d), J_1) \), then
\[
S^{[n]} \xrightarrow{\nu} S \quad \text{in} \quad (D([0, \infty), \mathbb{R}^d), J_1)
\]
holds for all probabilities \( \nu \ll \mu \) as soon as it holds for some \( \nu \).

**Proof.** (i) Below we shall show that for every \( \varepsilon > 0 \) there are constants \( g_\varepsilon(n) \) such that \( g_\varepsilon(n) \to 0 \) as \( n \to \infty \) while
\[
\mu \left( d_{J_1, s}(S^{[n]}, S^{[n]} \circ T) > \varepsilon \right) \leq g_\varepsilon(n) \quad \text{for} \quad n \geq 1 \text{ and } s \in [0, \infty).
\]
Therefore, whatever \( \varepsilon > 0 \),
\[
\int_{0}^{\infty} \int_{X} e^{-s} (1 \wedge d_{J_1, s}(S^{[n]}, S^{[n]} \circ T)) \, d\mu \, ds
\leq \int_{0}^{\infty} e^{-s} \left[ \mu \left( d_{J_1, s}(S^{[n]}, S^{[n]} \circ T) > \varepsilon \right) + \varepsilon \right] \, ds
\leq g_\varepsilon(n) + \varepsilon < 2\varepsilon \quad \text{for} \quad n \geq n_1(\varepsilon),
\]
proving that \( \int X d_{J_1, \infty}(S^{[n]}, S^{[n]} \circ T) \, d\mu \to 0 \), which implies (4.20). Now Theorem 1 of [Z3] immediately yields the second assertion of our lemma, about strong distributional convergence of \( (S^{[n]}) \).
(ii) For $n \geq 4$ define a time-change $\lambda_n \in \Lambda$ by affinely interpolating between $\lambda_n(0) = 0$, $\lambda_n(1/n) = 2/n$, $\lambda_n((n-2)/n) = (n-1)/n$, and $\lambda_n(1) = 1$. By straightforward elementary considerations,

$$d_{\mathcal{F},1}(S^{[n]}, S^{[n]} \circ T) \leq \sup_{t \in [0,1]} \left\| S^{[n]}_{\lambda_n(t)} - (S^{[n]} \circ T)_t \right\|$$

$$\leq \frac{2}{n} + \frac{2 \|A_n\|}{nB_n} + \frac{\|f\| + \|f \circ T^{n-1}\| + \|f \circ T^n\|}{B_n}.$$  

Taking $n_0(\varepsilon)$ so large that $n \geq n_0(\varepsilon)$ implies $2(B_n + \|A_n\|)/(nB_n) < \varepsilon/2$,

$$\mu \left( d_{\mathcal{F},1}(S^{[n]}, S^{[n]} \circ T) > \varepsilon \right) \leq \mu \left( \frac{\|f\| + \|f \circ T^{n-1}\| + \|f \circ T^n\|}{B_n} > \varepsilon B_n/2 \right)$$

$$\leq \mu \left( \frac{\|f\|}{B_n} > \frac{\varepsilon}{6} \right) + \mu \left( \frac{\|f \circ T^{n-1}\|}{B_n} > \frac{\varepsilon}{6} \right) + \mu \left( \frac{\|f \circ T^n\|}{B_n} > \frac{\varepsilon}{6} \right)$$

$$\leq 3 \mu \left( \frac{\|f\|}{B_n} > \frac{\varepsilon}{6} \right)$$

for $n \geq n_0(\varepsilon)$. But $\mu(\|f\|/B_n > \varepsilon/6) \to 0$ since $\|f\|/B_n \to 0$ and $\mu$ is finite.

The same argument, with the same upper bound, works if the time interval $[0, 1]$ is replaced by any $[0, s]$ with $s > 0$. This proves (4.28).

We can now wrap up the

**Proof of Theorem 2.7.** According to Lemma 4.3 it suffices to prove $S^{[n]} \xrightarrow{\nu} S$ in $(D([0, \infty), \mathbb{R}^d), \mathcal{F})$ with respect to $\nu = \mu$, distributional convergence $S^{[n]} \xrightarrow{\mu} S$ for arbitrary $\nu \ll \mu$ then being automatic.

Due to stochastic continuity of $S$ we see that $S^{[n]} \xrightarrow{\nu} S$ in $(D([0, \infty), \mathbb{R}^d), \mathcal{F})$ follows as soon as $S^{[n]} \xrightarrow{\mu} S$ in $(D([0, s], \mathbb{R}^d), \mathcal{F})$ for every $s > 0$. Now Propositions 3.1 and 4.1 immediately show, via Prohorov’s Theorem, that $S^{[n]} \xrightarrow{\mu} S$ in $(D([0, 1], \mathbb{R}^d), \mathcal{F})$, and the argument for $s \neq 1$ is the same.

5. **Proof of the concrete limit theorems**

The one-dimensional case does not present any difficulties.

**Proof of Theorem 2.2.** The assumption that $f$ be in the domain of attraction of $S$ and the condition (2.20) on the tail of $\vartheta_f$ together imply that

$$(5.1) \quad \int \vartheta^n_f \, d\mu < \infty \quad \text{for } \eta \in (0, 1 \wedge \alpha).$$

Finiteness of $\int \vartheta^n_f \, d\mu$ for some $\eta \in (0, 1]$, however, is the fundamental regularity assumption of (G2). Theorem 1.5 of (G2) asserts that if such an observable $f$ is in the domain of attraction of $S$, then its ergodic sums satisfy a stable limit theorem as in (1.1), with constants $(A_n)$ and $(B_n)$ obtained from the law of $f$ in exactly the same way as in the iid case. In particular, the assumptions of Theorem 2.1 are fulfilled, and (2.21) follows. \hfill $\square$
Turning to the specific vector-valued scenario of Theorem 2.3, we recall (2.3). It is then easy to see (Example 2.3.5 of [ST]) that if a full stable vector is of the form \( S = (S^{(1)}, \ldots, S^{(d)}) \) with independent scalar \( \alpha \)-stable variables \( S^{(j)} \) which satisfy \( \mu(S^{(j)}) = t \sim c_+^{(j)} t^{-\alpha} \ell(t) \) and \( \mu(S^{(j)}) = -t \sim c_-^{(j)} t^{-\alpha} \ell(t) \) as \( t \to \infty \), then the spectral measure \( \Lambda \) of \( S \) coincides with

\[
\Lambda_{\{c_+^{(j)}, c_-^{(j)}\}_{j=1}^d} = \sum_{j=1}^d \left( c_+^{(j)} \delta_{e_j} + c_-^{(j)} \delta_{-e_j} \right),
\]

where \( e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1) \) are the standard basis of \( \mathbb{R}^d \).

**Proof of Theorem 2.3.**

(i) Because of Theorem 2.1, the main point is to prove

\[
R_n := \frac{1}{B_n} (S_n(f) - A_n) \xrightarrow{\mu} S \quad \text{as } n \to \infty
\]

with \( S = (S^{(1)}, \ldots, S^{(d)}) \) an independent tuple as in the statement of the present theorem. In view of (2.6), we have \( (B_n^{(j)}) = (B_n) \) for each \( j \in \{1, \ldots, d\} \), and Theorem 2.2 immediately gives

\[
\frac{1}{B_n} (S_n(f^{(j)}) - A_n^{(j)}) \xrightarrow{\mu} S^{(j)} \quad \text{as } n \to \infty.
\]

As a consequence, the sequence \( (R_n)_{n \geq 1} \) of random vectors is tight, and every distributional limit point is a full random vector. Therefore (5.3) follows once we show that for every sequence of indices \( n_k \to \infty \) such that \( R_{n_k} \mu \Rightarrow R \) for some random vector \( R \), that limit necessarily satisfies \( R \sim S \).

Below we prove that \( \mu \circ f^{-1} \in \text{DOA}(S) \). By Proposition 2.1, this implies that

\[
\frac{1}{B_n} (S_n(f) - A_n) \xrightarrow{\mu} S \quad \text{as } n \to \infty
\]

for a suitable sequence \( (A_n, B_n)_{n \geq 1} \in \mathbb{R}^d \times (0, \infty) \). By the \( d \)-dimensional convergence of types principle (Theorem 2.3.17 of [MS]) and (5.4), this ensures that \( R \sim S \) as required.

(ii) In view of (5.2) proving our earlier claim \( \mu \circ f^{-1} \in \text{DOA}(S) \) only requires us to check that

\[
\mu(\|f\| > t) \left( \frac{f}{\|f\|} \in D \right) \xrightarrow{\mu} \frac{\Lambda_{\{c_+^{(j)}, c_-^{(j)}\}_{j=1}^d}(D)}{\sum_{i=1}^d (c_+^{(i)} + c_-^{(i)})} \quad \text{as } t \to \infty,
\]

for Borel \( D \subseteq \mathbb{S}^{d-1} \) whose boundary in \( \mathbb{S}^{d-1} \) is disjoint from \( \{\pm e_j\}_{j=1}^d \). This follows as soon as we prove that for every \( j \in \{1, \ldots, d\} \) and \( \delta > 0 \),

\[
\mu(\|f\| > t) \left( \text{sgn} f^{(j)} = \pm 1 \text{ and } |f^{(i)}| \leq \delta |f^{(j)}| \text{ for } i \neq j \right) \xrightarrow{\mu} \frac{c_+^{(j)}}{\sum_{i=1}^d (c_+^{(i)} + c_-^{(i)})}.
\]

To this end, fix \( j \) and observe that for \( t > \sqrt{dM} \) our assumption (2.23) ensures that

\[
\{f^{(j)} > t\} \subseteq \{\|f\| > t \text{ and } f^{(j)} > M\} \subseteq \{f^{(j)} > \sqrt{t^2 - dM^2}\}.
\]
Since $\sqrt{t^2 - dM^2} \sim t$ the tail assumption (2.22) yields
\begin{equation}
\mu(\|f\| > t \text{ and } f^{(i)} > M) \sim c^{(i)}_+ t^{-\alpha} \ell(t) \quad \text{as } t \to \infty,
\end{equation}
and in the same manner we obtain
\begin{equation}
\mu(\|f\| > t \text{ and } f^{(j)} < -M) \sim c^{(j)}_- t^{-\alpha} \ell(t) \quad \text{as } t \to \infty.
\end{equation}
On the other hand, (2.23) also guarantees that for $t > \sqrt{dM}$,
\begin{equation}
\{\|f\| > t\} = \bigcup_{\sigma = \pm 1} \bigcup_{j=1}^{d} \{\|f\| > t \text{ and } \left|f^{(j)}\right| > M \text{ with } \text{sgn} f^{(j)} = \sigma\},
\end{equation}
which is a disjoint union. Hence,
\begin{equation}
\mu(\|f\| > t) \sim \left(\sum_{i=1}^{d} (c^{(i)}_+ + c^{(i)}_-)\right) t^{-\alpha} \ell(t) \quad \text{as } t \to \infty.
\end{equation}
But (5.9) also implies that for any $\delta > 0$ we have
\begin{equation}
\{\|f\| > t \text{ and } \left|f^{(i)}\right| \leq \delta \left|f^{(j)}\right| \text{ for } i \neq j \text{ while } \text{sgn} f^{(j)} = \pm 1\} = \{\|f\| > t \text{ and } \left|f^{(j)}\right| > M \text{ with } \text{sgn} f^{(j)} = \pm 1\} \quad \text{for } t \geq t_0(\delta).
\end{equation}
In view of (5.7), (5.8) and (5.10) this validates (5.6).

6. Two applications

We illustrate the use of our general results in two specific situations.

An arcsine law for some $\mathbb{Z}$-extensions of Gibbs-Markov systems. We first mention an application of Theorem 2.2 to the infinite measure preserving dynamical system given by the dynamically defined random walk $(S_n(f))_{n \geq 0}$. More precisely, let $(X, \mathcal{A}, \mu, T, \xi)$ and $f$ be as in Theorem 2.2 where we assume for simplicity that $f$ is integer-valued. Define the skew product transformation
\begin{equation}
T_f : X \times \mathbb{Z} \to X \times \mathbb{Z}, \quad T_f(x, m) := (T(x), m + f(x)).
\end{equation}
The system $(X \times \mathbb{Z}, \mathcal{A} \otimes \mathcal{P}(\mathbb{Z}), \mu \otimes \nu, T_f)$ is the $\mathbb{Z}$-extension of $(X, \mathcal{A}, \mu, T)$ by $f$ (see Chapter 8 of [A]). The infinite but $\sigma$-finite measure $\mu \otimes \nu$ (with $\nu$ denoting counting measure on $\mathbb{Z}$) is invariant under $T_f$.

Various interesting properties of classical random walks hold for more general infinite measure preserving systems. In the case of the simplest symmetric random walk on $\mathbb{Z}$ several relevant quantities converge to the arcsine law with distribution function $t \mapsto (2/\pi) \arcsin \sqrt{t}$, $t \in [0, 1]$. These well-known results (Chapter 3 of [F]) can be generalized in different ways. Classical probability theory provides, for example, arcsine-type limit theorems for the time of the last visit to a reference set of finite measure ([D] and [L1]), for occupation times of infinite-measure sets separated from their complement by some finite-measure set ([L2]), and for occupation times of a half-line under a random walk ([Sp]). The first two types have been extended to more general infinite measure preserving systems, see [Th3], [Th4], [TZ], [Z2], [KZ] and [SY], while it seems that the last (Spitzer’s arcsine law, which requires a random walk or skew-product structure) has not. We therefore take the opportunity to point out that the FSLT above entails a result of this flavour: For $\rho \in (0, 1)$ let
A_\rho \) denote a \([0,1]\)-valued random variable which has the \textit{generalized arcsine law with parameter } \rho, \textit{that is,}

\begin{equation}
\Pr[0 \leq A_\rho \leq t] = \frac{\sin \rho \pi}{\pi} \int_0^t s^{\rho - 1} (1 - s)^{-\rho} \, ds \quad \text{for } t \in [0,1].
\end{equation}

We then obtain

\textbf{Theorem 6.1 (Arcsine Law for } \mathbb{Z}\text{-extensions of Gibbs-Markov systems).}

Let \((X,A,\mu,T,\xi)\) be a mixing probability preserving Gibbs-Markov system, and \(f : X \to \mathbb{Z}\) an observable in the domain of attraction of some \(\alpha\)-stable random variable \(S\), \(\alpha \in (0,2)\). Assume also that \(\mu(\vartheta_f > t) = O(\mu(|f| > t))\) and that \(f\) is centered, \(\int f \, d\mu = 0\), in case \(\alpha > 1\), and symmetrically distributed, \(\mu(f > t) = \mu(f < -t)\), in case \(\alpha = 1\). Then

\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} 1_{X \times \mathbb{N}} \circ T_f^k \xrightarrow{\nu} \eta \quad \text{as } n \to \infty
\end{equation}

for all probability measures \(\eta \ll \mu \otimes \iota\), where \(\rho := \Pr[S > 0]\).

\textbf{Proof.} (i) An arbitrary probability measure \(\eta \ll \mu \otimes \iota\) can be represented as \(\eta = \sum_{m \in \mathbb{Z}} p_m \nu_m \otimes \delta_m\) (with \(\delta_m\) denoting unit point mass at \(m\)) for probabilities \(\nu_m \ll \mu\) and weights \(p_m \geq 0\) with \(\sum_{m \in \mathbb{Z}} p_m = 1\). It is straightforward that (6.3) follows once we prove for that all for \(m \in \mathbb{Z}\) and \(\nu_m \ll \mu\),

\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} 1_{X \times \mathbb{N}} \circ T_f^k \nu_m \otimes \delta_m \xrightarrow{\rho} A_\rho \quad \text{as } n \to \infty.
\end{equation}

Fix \(m \in \mathbb{Z}\) and any probability \(\nu_m \ll \mu\). Clearly, \(T_f^n(x,m) = (T^n(x), m + S_n(f)(x))\) for \(n \geq 0\). As \(f\) is centered in case \(\alpha > 1\), and symmetrically distributed in case \(\alpha = 1\), the canonical normalizing sequence \((A_n, B_n)\) for \([\alpha, c^+, c^-]\) is of the form \((0, B_n)\). Therefore we see that \(T_f^n(x,m) \in X \times \mathbb{N}\) iff \(S_{[k/n]}^n(x) + m/B_n > 0\), where \(S^n\) is as in (2.11). Defining \(\psi : \mathcal{D}(0,\infty) \to \mathbb{R}\) by \(\psi(x) := \int_0^1 1_{(0,\infty)}(x_s) \, ds\), we get

\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} 1_{X \times \mathbb{N}} \circ T_f^k(x,m) = \psi(S^n(x) + m/B_n),
\end{equation}

so that (6.4) is equivalent to

\begin{equation}
\psi(S^n + m/B_n) \xrightarrow{\nu} A_\rho \quad \text{as } n \to \infty.
\end{equation}

(ii) Now Theorem 2.7 applies to \(f\), and since \(B_n \to \infty\), we have

\begin{equation}
S^n + m/B_n \xrightarrow{\nu} S \quad \text{in } (\mathcal{D}(0,\infty), \mathcal{J}_1) \quad \text{as } n \to \infty,
\end{equation}

where \(S = (S_t)_{t \geq 0}\) is the \(\alpha\)-stable motion with \(S_1 \overset{d}{=} S\). Observe then that while the map \(\psi\) is not continuous (for example, it is discontinuous at \(x := 0\)), we have that

\begin{equation}
\psi \text{ is } \mathcal{J}_1\text{-continuous at almost every path of } S.
\end{equation}

This follows by the argument presented in Appendix M15 of [3], because \(\Pr[S_s = 0] = \Pr[S = 0] = 0\) for all \(s > 0\). In view of (6.6) the continuous mapping theorem (Theorem 2.7 in [3]) immediately shows that

\begin{equation}
S^n + m/B_n \xrightarrow{\nu} S \quad \text{implies} \quad \psi(S^n + m/B_n) \xrightarrow{\nu} \psi(S).
\end{equation}

By Theorem VI.13 of [3], however, \(\psi(S) \overset{d}{=} A_\rho\), which completes the proof. \(\square\)
Remark 6.1. Note that (6.3) holds for all probabilities $\eta \ll \mu \otimes \ell$, even if $T_f$ is not ergodic w.r.t. $\mu \otimes \ell$. (Which happens, for example, if $f$ takes its values in the set $2\mathbb{Z}$ of even integers.) This is in contrast to the other types of arcsine laws for infinite measure preserving systems mentioned before, where convergence to the same limit law under all probabilities absolutely continuous w.r.t. the invariant measure depends on ergodicity of the system (via the device discussed in [Z3]).

Remark 6.2. The regularity condition $\mu(\vartheta_f > t) = O(\mu(|f| > t))$ allows for integer-valued observables $f$ which need not be constant on cylinders of any fixed rank.

**Excursions to cusps of intermittent maps.** Interval maps with indifferent fixed points constitute a basic class of dynamical systems at the edge of hyperbolicity which has been studied extensively in the last decades. Their dynamics can be viewed as being driven by a uniformly hyperbolic induced map, with delays caused by long excursions to the vicinity of the neutral points. The results of the present paper can be used to clarify questions about the asymptotic (in)dependence between excursion processes to individual indifferent fixed points.

To limit technicalities, we focus on the prototypical situation of maps $T$ on $X := [0, 1]$ with two full branches and neutral fixed points at $x = 0$ and $x = 1$. Specifically, assume that there is some $c \in (0, 1)$, defining cylinders $Z_1 := (0, c)$ and $Z_2 := (c, 1)$, such that (with $\lambda$ denoting Lebesgue measure)

- a) $T|_{Z_0}$ is an increasing homeomorphism of $(0, c)$ onto $(0, 1)$,
- b) $T|_{Z_0}$ extends to a $C^2$-map of $(0, c)$ onto $(0, 1)$, with $T' > 1$ on $(0, c)$, while $T'x \rightarrow 1$ as $x \searrow 0$, and $T''$ is increasing on some neighbourhood of 0,
- c) there is some decreasing function $\gamma : (0, c] \rightarrow [0, \infty)$ for which
  \[ \int_0^\gamma d\lambda < \infty \quad \text{and} \quad |T''| \leq \gamma \text{ on } (0, c), \]
- d) the map $\widetilde{T}x := 1 - T(1 - x)$ on $\widetilde{Z}_0 := (0, \overline{c})$ with $\overline{c} := 1 - c$ satisfies all the conditions which a)-c) impose on $T|_{Z_0}$.

Define $Y = Y(T) := \{y_0, y_1\} \subseteq X$ where $y_0$ is the unique point of period 2 in $Z_0$ and $y_1 := T y_0$. Set $\varphi_Y(x) := \inf\{n \geq 1 : T^n x \in Y\}$, the first return time of $Y$, and let $T_Y x := T^{\varphi_Y(x)} x$ define the first return map $T_Y : Y \rightarrow Y$ of $Y$, which comes with a natural partition $\xi_Y := \{Y \cap T^{-1}Z_j \cap \{\varphi_Y = k\} : j \in \{0, 1\}, k \geq 1\}$. The function $\varphi_Y$ is obviously constant on elements of $\xi_Y$. Its distribution depends on the details of $T$ near the indifferent fixed points, which can be expressed in terms of $r_0(x) := Tx - x$, $x \in (0, c)$ and $r_1(x) := \widetilde{T}x - x$, $x \in (0, \overline{c})$.

The following collects some basic facts about such maps.

**Proposition 6.1 (Basic ergodic properties of intermittent maps).** a) Any map $T$ satisfying a)-d) is conservative ergodic and exact with respect to Lebesgue measure $\lambda$ and preserves a $\sigma$-finite Borel measure $\mu \ll \lambda$ with a strictly positive density $h$ continuous on $(0, 1)$. Moreover, the first return map $(Y, \mathcal{B}_Y, \mu_Y, T_Y, \xi_Y)$ is a probability preserving Gibbs-Markov system.

b) Assume, in addition, that $\ell$ is regularly varying at $0^+$ and that $\kappa_0, \kappa_1, p \in (0, \infty)$ are constants such that

\[ r_j(x) \sim \kappa_j x^{1+p}\ell(x) \quad \text{as } x \searrow 0, \]
for $j \in \{0, 1\}$. Then

$$
\mu(X) < \infty \quad \text{iff} \quad \int_0^\infty \frac{x \, dx}{r_0(x)} < \infty,
$$

and the functions $\varphi_Y^{(0)} := 1_{Y \cap T^{-1}Z_0} \varphi_Y$ and $\varphi_Y^{(1)} := 1_{Y \cap T^{-1}Z_1} \varphi_Y$ on $Y$ which record the durations of excursions from $Y$ to $Z_0$ and $Z_1$, respectively, satisfy

$$
\mu_Y \left( \varphi_Y^{(j)} > m \right) \sim Cc_+^{(j)} A^{-1}(m) \quad \text{as } m \to \infty,
$$

where $A^{-1}$ is regularly varying of index $-1/p$ and asymptotically inverse to $A(t) := t/r_0(t)$, the constant $C := h(c)/[p^{1/p} \int Y \, du] \in (0, \infty)$ does not depend on $j$, and $c_+^{(0)} := \kappa_0^p/T'(e^+)\text{ while } c_+^{(1)} := \kappa_1^p/T'(e^-)$.

Proof. These are well known facts, see \[\text{A}, \text{C1}, \text{Th1}, \text{Th5}\] and \[\text{Z1}\]. \qed

It is the joint behaviour of consecutive excursions from $Y$ which determines interesting aspects of the long-term behaviour of $T$. For example, in the infinite measure case, the Arcsine Law for occupation times of neighbourhoods of the neutral fixed points (first established in \[\text{Th4}\], see also \[\text{TZ}\] and \[\text{Z2}\]) is best viewed as a consequence of the asymptotic independence between the excursion processes to $x = 0$ and $x = 1$, respectively. This is made explicit in \[\text{Se}\], which contains the $\alpha \in (0, 1)$ case of Theorem 6.2 below. The latter shows that this type of asymptotic independence also holds in finite measure situations as soon as the limit is not Gaussian.

In the situation of Proposition 6.1 b) set $\alpha := 1/p$ and let $(A_n^{(j)}, B_n^{(j)})_{n \geq 1}$ be the canonical normalizing sequence for $[\alpha, c_+^{(j)}]$, and let $S^{(j)}$ be the corresponding $\alpha$-stable variable as in (2.5), while $B_n := B_n^{(1)}$. Consider the processes $S^{(j)[n]} = (S^{(j)[n]}_t)_{t \geq 0}$ of excursions to $Z_j$ given by

$$
S^{(j)[n]} : X \to D([0, \infty), \mathbb{R}), \quad S^{(j)[n]}_t := \frac{1}{B_n} \left( \sum_{k=0}^{\lfloor tn \rfloor - 1} \varphi_Y^{(j)} \circ T_Y - \frac{|tn|}{n} A_n^{(j)} \right).
$$

**Theorem 6.2 (Asymptotically independent excursion processes).** Suppose that $T$ satisfies a)-d), and (6.3) with $p > 2$. Then,

$$(S^{(0)[n]}, S^{(1)[n]}) \overset{D}{\longrightarrow} (S^{(0)}, S^{(1)}) \quad \text{in } D([0, \infty), \mathbb{R}^2), \mathcal{F}_1
$$

for any probability $\nu \ll \lambda$, where $S^{(0)}, S^{(1)}$ are independent scalar $\alpha$-stable Lévy processes with $\alpha := 1/p \in (0, 2)$ and $S^{(j)} \overset{D}{=} S^{(j)}$ characterized by

$$
E[e^{i t S^{(j)}}] = e^{-c_\alpha^{(j)} |t|^{\alpha}(1-\text{sgn}(t)\omega(\alpha,t))}, \quad t \in \mathbb{R}.
$$

Proof. By the tail estimate (6.10) in Proposition 6.1 the function $\varphi_Y^{(j)} \geq 0$ is in the domain of attraction of the $\alpha$-stable variable $S^{(j)}$, as

$$
\Pr[S^{(j)} > t] = (c_+^{(j)} + o(1)) t^{-\alpha} \ell^*(t) \quad \text{and} \quad \Pr[S^{(j)} < -t] = O(t^{-\alpha} \ell^*(t)),
$$

with $\ell^*(t) := C t^{1/p} A^{-1}(t)$, which corresponds to the case $c_- = 0$ in (2.4), meaning that $\beta = c_+^{(j)}$ and $\beta = 1$ in (2.5).

On the other hand, $\{\varphi_Y^{(0)} > 0\} \cap \{\varphi_Y^{(1)} > 0\} = \emptyset$, and Theorem 2.3 applies. \qed
Remark 6.3. By routine arguments this theorem extends to more general Markovian interval maps with finitely many neutral fixed (or periodic) points at which the map satisfies conditions analogous to b) and c) above.

References

[A] J. Aaronson: An introduction to infinite ergodic theory. AMS 1997.
[AD] J. Aaronson, M. Denker: Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. Stoch. Dyn. 1 (2001), 193-237.
[ADSZ] J. Aaronson, M. Denker, O. Sarig, R. Zweimüller: Aperiodicity of cocycles and conditional local limit theorems. Stoch. Dyn. 4 (2004), 31-62.
[Be] J. Bertoin: Lévy Processes. Cambridge University Press 1996.
[Bi] P. Billingsley: Convergence of Probability Measures. 2nd ed., Wiley 1999.
[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels: Regular Variation. Cambridge University Press 1989.
[CFKM] I. Chevyrev, P.K. Friz, A. Korepanov, I. Melbourne: Superdiffusive limits for deterministic fast-slow dynamical systems. Probab. Theory Relat. Fields 178 (2020), 735-770.
[D] E.B. Dynkin: Some limit theorems for sums of independent random variables with infinite mathematical expectation. Selected Transl. in Math. Statist. and Probability 1 (1961), 171-189.
[F] W. Feller: An introduction to probability theory and its applications, Vol. I. 3rd ed., Wiley, New York, 1968.
[GS] I.I. Gikhman, A.V. Skorohod: Introduction to the theory of random processes. W.B. Saunders, Philadelphia, 1969. reprint Dover 1996.
[GK] B.V. Gnedenko, A.N. Kolmogorov: Limit Distributions for Sums of Independent Random Variables. Addison-Wesley 1968.
[G1] S. Gouëzel: Central limit theorem and stable laws for intermittent maps. Probab. Theory Relat. Fields 128 (2004), 82-122.
[G2] S. Gouëzel: Characterization of weak convergence of Birkhoff sums for Gibbs-Markov maps. Israel J. Math. 180 (2010), 1-41.
[KZ] D. Kocheim, R. Zweimüller: A joint limit theorem for compactly regenerative ergodic transformations. Studia Math. 203 (2011), 33-45.
[L1] J. Lamperti: Some limit theorems for stochastic processes. J. Math. Mech. 7 (1958), 433-448.
[L2] J. Lamperti: An occupation time theorem for a class of stochastic processes. Trans. Amer. Math. Soc. 88 (1958), 380-387.
[MS] M.M. Meerschaert, H.-P. Scheffler: Limit Distributions for Sums of Independent Random Vectors. Wiley, New York, 2001.
[MZ] I. Melbourne, R. Zweimüller: Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems. Ann. Inst. H. Poincare 51 (2015), 545-556.
[MPU] F. Merlevède, M. Peligrad, S. Utev: Recent advances in invariance principles for stationary sequences. Probability Surveys 3 (2006), 1-36.
[R] E.L. Rvaceva: On domains of attraction of multi-dimensional distributions. in: Select. Transl. Math. Statist. and Probability, AMS, Vol. 2 (1962), 183-205.
[ST] G. Samorodnitsky, M.S. Taqqu: Stable non-Gaussian Random Processes. Chapman & Hall 1994.
[SY] T. Sera, K. Yano: Multiray generalization of the arcsine laws for occupation times of infinite ergodic transformations. Trans. Amer. Math. Soc. 372 (2019), 3191-3209.
[Se] T. Sera: Functional limit theorem for occupation time processes of intermittent maps. Preprint, arXiv:1810.04571.
[S1] A.V. Skorohod: Limit theorems for stochastic processes. Theor. Probability Appl. 1 (1956), 261-290.
[S2] A.V. Skorohod: Limit theorems for stochastic processes with independent increments. Trans. Amer. Math. Soc. 82 (1956), 323-339.
[Sp] F. Spitzer: A combinatorial lemma and its application to probability theory. Trans. Amer. Math. Soc. 82 (1956), 323-339.
[Th1] M. Thaler: Estimates of the invariant densities of endomorphisms with indifferent fixed points. Isr. J. Math. 37 (1980), 393-314.
[Th2] M. Thaler: Transformations on $[0,1]$ with infinite invariant measures. Isr. J. Math. 46 (1983), 67-96.

[Th3] M. Thaler: The Dynkin-Lamperti Arc-Sine Laws for Measure Preserving Transformations. Trans. Amer. Math. Soc. 350 (1998), 4593-4607.

[Th4] M. Thaler: A limit theorem for sojourns near indifferent fixed points of one-dimensional maps. Ergod. Th. & Dynam. Sys. 22 (2002), 1289-1312.

[Th5] M. Thaler: Asymptotic distributions and large deviations for iterated maps with an indifferent fixed point. Stoch. Dyn. 5 (2005), 425-440.

[TZ] M. Thaler, R. Zweimüller: Distributional limit theorems in infinite ergodic theory. Probab. Theory Relat. Fields 135 (2006), 15-52.

[Ty1] M. Tyran-Kaminska: Convergence to Lévy stable processes under some weak dependence conditions. Stoch. Proc. Appl. 120 (2010), 1629-1650.

[Ty2] M. Tyran-Kaminska: Weak convergence to Lévy stable processes in dynamical systems. Stoch. Dyn. 10 (2010), 263-289.

[W] W. Whitt: Stochastic-Process limits. Springer 2002.

[Y] K. Yosida: Mean ergodic theorem in Banach spaces. Proc. Imp. Acad. Tokyo 14 (1938), 292-294.

[Z1] R. Zweimüller: Stable limits for probability preserving maps with indifferent fixed points. Stoch. Dyn. 3 (2003), 83-99.

[Z2] R. Zweimüller: Infinite measure preserving transformations with compact first regeneration. Journal d’Analyse Mathematique 103 (2007), 93-131.

[Z3] R. Zweimüller: Mixing limit theorems for ergodic transformations. Journal of Theoretical Probability 20 (2007), 1059-1071.