Symmetry properties of a nonlinear acoustics model

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Abstract We give a classification into conjugacy classes of subalgebras of the symmetry algebra generated by the Zabolotskaya–Khokhlov equation, and obtain all similarity reductions of this equation into \((1+1)\)-dimensional equations. We thus show that the Lie classical reduction approach may also give rise to more general reduced equations as those expected from the direct method of Clarkson and Kruskal. By transforming the determining system for the similarity variables into the equivalent adjoint system of total differential equations, similarity reductions to \(\text{ODEs}\) which are independent of the three arbitrary functions defining the symmetries are also obtained. These results are again compared with those obtained by the direct method of Clarkson and Kruskal, by finding in particular equivalence transformations mapping some of the reduced equations to each other. Various families of new exact solutions are also derived.

Keywords Lie algebra classification · Comparison of reduction techniques · Adjoint system of total differential equations · Equivalence transformations · Exact solutions

1 Introduction

The Zabolotskaya–Khokhlov (ZK) equation is a nonlinear model of sound wave propagation derived from the incompressible Navier–Stokes equation \([1–3]\). The \((2+1)\)-dimensional version of this equation has the form

\[
\Delta(t, x, y, u) \equiv u_{tt} - (uu_x)_x - u_{yy} = 0, \tag{1}
\]

and it has been studied from the Lie group approach in a number of papers \([4–7]\). Chowdhury and Naser \([4]\) attempted the determination of the symmetry algebra of this equation and calculated some of their conservation laws. However, Schwarz \([5]\) and Hereman \([6]\) were certainly the first to provide independently the correct generators of the Lie symmetry algebra for this popular acoustics model. Although the calculation of symmetry algebras for almost all systems of differential equations has long been reduced to a mere function on a number of modern computing systems, many symmetry properties of this equation are still to be uncovered.

In \([7]\), some similarity reductions of \((1)\) to \((1+1)\)-dimensional models were obtained, based on the direct method of Clarkson and Kruskal \([8]\). More specifically, it was shown that if a similarity solution of \((1)\) of the form

\[
u = U(t, x, y, W(\xi, \eta)),
\]

\[
\xi = \xi(t, x, y), \quad \eta = \eta(t, x, y) \tag{2}
\]
can be found by solving a reduced \((1 + 1)\)-dimensional equation, then when \(\xi_1 \neq 0\), such an equation must be one of three nonequivalent \((1 + 1)\)-dimensional equations found in that paper. However, these reduced equations are either determined only implicitly in terms of solutions of certain partial differential equations (PDEs), or they depend on up to four arbitrary functions, and the same can be said about their solutions. In the same paper, using again the direct method of Clarkson and Kruskal and a restrictive ansatz, the most general ODE that every similarity solution of (1) obtained by solving an ODE must satisfy is shown to be of the form:

\[ w'^2 + w'' + (Az + B)w' + 2A = \frac{1}{3} (Az + B)^2 \tag{3} \]

Some of the unanswered questions raised in [7] were how to find all the nonequivalent similarity reductions of the ZK equation to an ODE by the classical Lie symmetry approach, and whether there is any connection between these two types of reduction techniques. The first of these two questions stems from the fact that the symmetries of (1) depend on three arbitrary functions of time, and so its similarity reductions are usually achieved by restricting these functions to some specific types of elementary functions, such as exponential or simple polynomial functions of time [4, 5]. It also stems from the fact that no classification of low-dimensional subalgebras of the ZK symmetry algebra into conjugacy classes is available.

In this paper, we obtain all canonical forms of nonequivalent one- and two-dimensional subalgebras of the symmetry algebra \(L\) of (1), under the adjoint representation of the symmetry group. We then apply them to obtain all similarity reductions of the ZK equation to \((1 + 1)\)-dimensional equations, using the classical Lie approach. The same reductions are also obtained by direct case analysis. We thus show that in addition to its simpler algorithm and other properties, Lie classical reduction method gives rise not only to simpler equations, but it may also yield more unified and general reduced equations as those expected from the direct method (of Clarkson and Kruskal). Next, by transforming the determining system for the similarity variables into the equivalent adjoint system of total differential equations, similarity reductions to large classes of ODEs which are independent of the three arbitrary functions defining the symmetries are also obtained. The latter system of total differential equations allows for an easier determination of the invariant functions defining the similarity coordinates. Large classes of similarity solutions depending on much less arbitrary functions than those obtained in [7] are also derived in this way.

Finally, we find equivalence transformations mapping some of the equations that we’ve obtained by the Lie classical method to some subequations of the reduced equation (3) obtained by the direct method of Clarkson and Kruskal. Our discussions also show that in principle any reduced equation achievable with the direct method can also be achieved by Lie classical method although the converse is totally out of question, as far as the properties of the reduced equations are concerned.

This paper is organized as follows. In the next section, we discuss the symmetry algebra of the ZK equation and determine its algebraic structure as well as its connection with Kac–Moody–Virasoro (KMV) algebras. Section 3 is devoted to the classification of low-dimensional subalgebras of \(L\) and Sect. 4 to the similarity reductions of the ZK equation. We investigate the connection between the two types of reductions invoked above in Sect. 5. Some concluding remarks are given in the last section.

2 Symmetry group of the ZK equation

2.1 Structure of the symmetry algebra

The Lie algebra of the ZK equation is well known [5, 6]. This is the Lie algebra defined by the infinitesimal generators of the point symmetry group \(G\) of the equation, that is, the Lie group of point transformations that map every solution of the equation to another solution of the same equation [9, 10]. These infinitesimal generators are vector fields of the form

\[ \mathbf{v} = \xi_1(t, x, y, u) \partial_t + \xi_2(t, x, y, u) \partial_x + \xi_3(t, x, y, u) \partial_y + \phi(t, x, y, u) \partial_u \tag{4} \]

acting on the space of independent variables coordinatized by \((t, x, y)\) and the space of the dependent variable coordinatized by \(u\), and such that the second prolongation \(\Pr^{(2)} \mathbf{v}\) of \(\mathbf{v}\) satisfies

\[ \Pr^{(2)} \mathbf{v} \Delta(t, x, y, u) \big|_{\Delta(t, x, y, u) = 0} = 0. \tag{5} \]