ON SOME UPPER BOUNDS FOR THE ZETA-FUNCTION
AND THE DIRICHLET DIVISOR PROBLEM

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ABSTRACT. Let $d(n)$ be the number of divisors of $n$, let

$$
\Delta(x) := \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1)
$$

denote the error term in the classical Dirichlet divisor problem, and let $\zeta(s)$ denote the Riemann zeta-function. Several upper bounds for integrals of the type

$$
\int_0^T \Delta^k(t)|\zeta(\frac{1}{2} + it)|^{2m} \, dt \quad (k, m \in \mathbb{N})
$$

are given. This complements the results of the paper Ivić-Zhai [9], where asymptotic formulas for $2 \leq k \leq 8, m = 1$ were established for the above integral.

1. INTRODUCTION

As usual, let

$$
\Delta(x) := \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \quad (x \geq 2)
$$

denote the error term in the classical Dirichlet divisor problem (see e.g., Chapter 3 of [5]). Also let

$$
E(T) := \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - T \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right) \quad (T \geq 2)
$$

denote the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|$. Here $d(n)$ is the number of all positive divisors of $n$, $\zeta(s)$ is the Riemann zeta-function, and

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\( \gamma = -\Gamma'(1) = 0.577215 \ldots \) is Euler’s constant. In [6] the author proved several results involving the mean values of \( \Delta(x), E(t) \) and

\[
\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)
\]

which is the “modified” divisor function, introduced and studied by M. Jutila [10], [11].

In [9] the author and W. Zhai studied the moments

\[
\int_0^T \Delta^*(t) \zeta(\frac{1}{2} + it)^2 \, dt \ll T (\log T)^4,
\]

and if \( k \) is a fixed integer for which \( 2 \leq k \leq 8 \), then we have

\[
\int_1^T \Delta^k(t) \zeta(\frac{1}{2} + it)^2 \, dt = c_1(k) T^{1+\frac{k}{4}} \log T + c_2(k) T^{1+\frac{k}{4}} + O_\varepsilon(T^{1+\frac{k}{4}-\eta_k + \varepsilon}),
\]

where \( c_1(k) \) and \( c_2(k) \) are explicit constants, and where

\( \eta_2 = \eta_3 = \eta_4 = 1/10, \eta_5 = 3/80, \eta_6 = 35/4742, \eta_7 = 17/6312, \eta_8 = 8/9433. \)

It was also shown how the value of \( \eta_2 \) can be improved to \( \eta_2 = 3/20 \). It may be well conjectured that the asymptotic formula (1.5) holds for integers \( k \geq 9 \) as well (with some \( \eta_k > 0 \)), although this is beyond reach at present. This is in tune with (1.2) and the classical conjecture that

\[
\int_0^T \Delta^k(t) \, dt = C_k T^{1+k/4} + O_\varepsilon(T^{1+k/4-c(k)+\varepsilon})
\]

holds with an explicit constant \( C_k \) and some \( c(k) > 0 \), when \( k > 1 \) is a given natural number. We note that (1.6) is at present known to hold for \( 2 \leq k \leq 9 \) (see W. Zhai [14]). In what concerns (1.4) it was conjectured in [9] that one has

\[
\int_1^T \Delta(t) \zeta(\frac{1}{2} + it)^2 \, dt = \frac{T}{4} \left( \log \frac{T}{2\pi} + 2\gamma - 1 \right) + O_\varepsilon(T^{3/4+\varepsilon}),
\]

however obtaining any asymptotic formula for the integral in (1.4) is difficult. Here and later \( \varepsilon \) denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence, while \( f \ll a, b, \ldots \) \( g \) (same as \( f = O_{a, b, \ldots}(g) \)) means that the implied constant depends on \( a, b, \ldots \).
2. Statement of results

A natural continuation of the previous investigations related to the integral in (1.5) is the estimation of the more general integral

\[ \int_0^T \Delta^k(t) |\zeta(\frac{1}{2} + it)|^{2m} \, dt \quad (k, m \in \mathbb{N}), \]

where \( k \geq 1, m > 1 \). One would naturally want to obtain non-trivial upper bounds, where by trivial we mean bounds coming from the use of the currently best known upper bounds

\[ \Delta(x) \ll x^{\theta + \epsilon}, \quad \theta = \frac{131}{416} = 0.3149 \ldots \]

and

\[ \zeta(\frac{1}{2} + it) \ll |t|^{\frac{32}{205} + \epsilon}, \quad \frac{32}{205} = 0.15609 \ldots \]

We note that the exponent \( \frac{32}{205} \) has been recently improved to \( \frac{53}{342} \approx 0.15497 \ldots \) in a forthcoming paper by J. Bourgain [2].

In the case of \( m = 2 \) in (2.1) one would naturally wish to use results on the fourth moment of \( |\zeta(\frac{1}{2} + it)| \).

We have (see Ivić - Motohashi [7],[8] and Y. Motohashi [12])

\[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = TQ_4(\log T) + E_2(T), \quad E_2(T) = O(T^{2/3} \log^8 T), \]

where \( Q_4(x) \) is an explicit polynomial of degree four in \( x \) with leading coefficient \( 1/(2\pi^2) \). We also have (here and later \( C \) denotes positive generic constants)

\[ \int_0^T E_2^2(t) \, dt \ll T^2 \log^C T \]

but neither (2.3) nor (2.4)-(2.5) are sufficiently strong to obtain non-trivial results regarding (2.1) when \( m = 2 \).

Our results are contained in the following

**THEOREM 1.** We have

\[ \int_0^T \Delta(t) |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T^{41/32} \log^C T \quad (41/32 = 1.28125), \]

\[ \int_0^T \Delta^2(t) |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T^{25/16} \log^C T \quad (25/16 = 1.56250), \]

\[ \int_0^T \Delta^3(t) |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T^{59/32} \log^C T \quad (59/32 = 1.84375), \]

\[ \int_0^T \Delta^4(t) |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T^{17/8} \log^C T \quad (17/8 = 2.125). \]
THEOREM 2. We have

\begin{align}
\int_0^T \Delta(t) |\zeta(\frac{1}{2} + it)|^6 \, dt &\ll T^{49/32} \log^C T \quad (49/32 = 1.53125), \\
\int_0^T \Delta^2(t) |\zeta(\frac{1}{2} + it)|^6 \, dt &\ll T^{29/16} \log^C T \quad (29/16 = 1.8125). 
\end{align}

Remark 1. The values of the constants $C$ in (2.6) and (2.7) are not important, since the exponents are certainly not the best possible ones, as will be discussed later. Indeed, if one assumes the classical conjecture

\begin{equation}
\Delta(x) \ll \epsilon x^{1/4 + \epsilon}
\end{equation}

and the famous Lindelöf Hypothesis

\begin{equation}
\zeta(\frac{1}{2} + it) \ll \epsilon |t|^{\epsilon},
\end{equation}

then trivially we have, for natural numbers $k > 1, m \geq 1$

\begin{equation}
\int_0^T \Delta^k(t) |\zeta(\frac{1}{2} + it)|^{2m} \, dt \ll_{\epsilon,k,m} T^{1+k/4+\epsilon}.
\end{equation}

Proving (2.10) in full generality is not possible nowadays, since neither (2.8) nor (2.9) is yet known to be true. It is classical that the Lindelöf Hypothesis follows from the Riemann Hypothesis (that all complex zeros of $\zeta(s)$ have real parts 1/2); see e.g. [5, Chapter 1]. However, (2.8) does not seem to follow from any known hypotheses, and the best known exponent $\theta = 131/416 = 0.3149\ldots$ in (2.2) is very far from the conjectural exponent $1/4 + \epsilon$.

The upper bound in (2.10) can be probably sharpened, at least for some values of $k$ and $m$, to an asymptotic formula of the form

\begin{equation}
\int_0^T \Delta^k(t) |\zeta(\frac{1}{2} + it)|^{2m} \, dt = T^{1+k/4} Q_m^2(\log T) + O_{\epsilon,k,m}(T^{1+k/4+\epsilon})
\end{equation}

for some constant $\rho_{k,m} > 0$, where $Q_m^2(x)$ is a polynomial of degree $m^2$, whose coefficients depend on $k$ and $m$.

Remark 2. The methods of proofs of the results allow one to carry over the results of Theorem 1 and Theorem 2 to the integrals where $\Delta(t)$ is replaced by $\Delta(\alpha t)$ or $\Delta^*(\alpha t)$ for any given $\alpha > 0$. Here

\begin{equation}
\Delta^*(x) : = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)
= \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1),
\end{equation}
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which is the “modified” function in the divisor problem. In view of F.V. Atkinson’s classical explicit formula (see [1] and Chapter 15 of [5]) for $E(T)$, which shows analogies between $\Delta(x)$ and $E(T)$, it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$ itself.

**Remark 3.** Finally, as in [9], we indicate two possible generalizations of our results. Namely the results can be generalized if $\Delta(x)$ is replaced either by $P(x) := \sum_{n \leq x} r(n) - \pi x$, or $A^*(t) := \sum_{n \leq t} a(n)n^{\frac{1}{2}+it}$. As usual, $r(n) = \sum_{a^2 + b^2 = n}$ denotes the number of ways $n$ may be represented as a sum of two integer squares, and $a(n)$ the $n$-th Fourier coefficient of $\varphi(z)$, a normalized eigenfunction of weight $\kappa$ for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$.

### 3. Proofs of the Theorems

The ingredients in the proof are the asymptotic formula (1.5) (with $k = 8$), results on upper bounds for the moments of $|\zeta(\frac{1}{2} + it)|$, and Hölder’s classical inequality for integrals in the form

\begin{equation}
\int_a^b f_1(x) \ldots f_r(x) \, dx \leq \left( \int_a^b f_1^{p_1}(x) \, dx \right)^{1/p_1} \cdots \left( \int_a^b f_r^{p_r}(x) \, dx \right)^{1/p_r},
\end{equation}

where $p_1, p_2, \ldots, p_r > 0$ and $f_1(x), f_2(x), \ldots, f_r(x) \geq 0$ are integrable functions in $[a, b]$ ($a < b$), and

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} = 1.$$

The case $r = 2, p_1 = p_2 = 2$ is the standard Cauchy-Schwarz inequality for integrals.

For the moments of $|\zeta(\frac{1}{2} + it)|$ we use the bound (this is [5, Theorem 8.3])

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^A \, dt \ll T^{1+(A-4)/8} \log^{C(A)} T \quad (4 \leq A \leq 12).
\end{equation}

The value of the constant $C(A)$ in (3.2) can be given explicitly, but as mentioned, its value is not important for our applications.
To prove the first upper bound in (2.6) note that

$$\int_0^T \Delta(t) |\zeta(1/2 + it)|^4 \, dt = \int_0^T \Delta(t) |\zeta(1/2 + it)|^{1/4} |\zeta(1/2 + it)|^{15/4} \, dt$$

\begin{equation}
\ll \left( \int_0^T \Delta^8(t) |\zeta(1/2 + it)|^2 \, dt \right)^{1/8} \left( \int_0^T |\zeta(1/2 + it)|^{30/7} \, dt \right)^{7/8}
\end{equation}

$$\ll (T^3 \log T)^{1/8} \left( \int_0^T |\zeta(1/2 + it)|^{4+\frac{7}{8}} \, dt \right)^{7/8}$$

$$\ll T^{41/32} \log C T,$$

where $C$ denotes positive, generic constants as already mentioned. Here we used Hölder’s inequality (3.1), (1.5) with $k = 8$ and (3.2).

**Remark 4.** It is readily checked that the exponent (3.3) cannot be improved by using trivial estimation coming from the bounds in (2.2) and (2.3).

**Remark 5.** The idea in proving (3.3), and other upper bounds of Theorem 1 as well, is to use (1.5) with $k = 8$. However, not the full asymptotic formula implied by (1.5) is used, but just the upper bound $T^3 \log T$. There is a possibility to obtain small improvements on all exponents in Theorem 1 and Theorem 2 as follows. First, recall (see (2.2)) that there exists a constant $\theta$ such that $1/4 \leq \theta < 1/3$ and

\begin{equation}
\Delta(x) \ll_{\varepsilon} x^{\theta + \varepsilon}, \quad E(t) \ll_{\varepsilon} t^{\theta + \varepsilon}.
\end{equation}

In particular, we can take $\theta = 131/416 = 0.3149 \cdots$. The proofs of the bounds in (3.4) are due to M.N. Huxley [4] and N. Watt [13], respectively, and they are the sharpest ones known. Then for any $A$ satisfying $0 \leq A \leq 11$ we have

\begin{equation}
\int_1^T |\Delta(x)|^A \, dx \ll_{\varepsilon} T^{1+M(A)+\varepsilon}
\end{equation}

and

\begin{equation}
\int_1^T |E(t)|^A \, dt \ll_{\varepsilon} T^{1+M(A)+\varepsilon},
\end{equation}

where

\begin{equation}
M(A) := \max \left( \frac{A}{4}, \theta(A - 2) \right).
\end{equation}

This follows by the discussion given in the author’s monograph [5, Chapter 13].
For completeness, we also note that, for real \( k \in [0, 9] \), the limits

\[
E_k := \lim_{T \to \infty} T^{-1-k/4} \int_0^T |E(t)|^k \, dt
\]

exist. The analogous result holds also for the moments of \( \Delta(t) \). This was proved by D.R. Heath-Brown [3], who used (3.5) and (3.6) in his proof. He also showed that the limits of moments (both of \( \Delta(t) \) and \( E(t) \)) without absolute values also exist when \( k = 1, 3, 5, 7 \) or 9. For the asymptotic formulas for the moments of \( \Delta(t), E(t) \) see W. Zhai [14], [15]. The merit of (3.8) that it gets rid of “\( \varepsilon \)” and establishes the existence of the limit (but without an error term). Note that, with \( \theta = 131/416 = 0.3149 \cdots \), in (3.7) we have \( M(A) = A/4 \) for \( A \leq 262/27 = 9.703 \). Using the method of [9] we can find a constant \( 8 < A_0 < A = 262/27 \) for which the bound

\[
\int_0^T |\Delta(t)|^{A_0} |\zeta((1/2 + \varepsilon t)|^2 \, dt \ll_{\varepsilon} T^{1+A_0/4+\varepsilon}
\]

will hold. Hence an improvement of (3.3) will consist by using Hölder’s inequality in such a way that instead of the integral of \( \Delta^8(t)|\zeta((1/2 + \varepsilon t)|^2 \) we have the integral of \( |\Delta(t)|^{A_0} |\zeta((1/2 + \varepsilon t)|^2 \) with \( A_0 \) as in (3.9). However, this would entail unwieldy exponents and the improvement would not be large, so we worked out explicitly the results using only the integral of \( \Delta^8(t)|\zeta((1/2 + \varepsilon t)|^2 \).

We continue with the proof of the remaining bounds in (2.6). We have

\[
\int_0^T \Delta^2(t)|\zeta((1/2 + \varepsilon t)|^4 \, dt = \int_0^T \Delta^2(t)|\zeta((1/2 + \varepsilon t)|^{1/2}|\zeta((1/2 + \varepsilon t)|^{7/2} \, dt
\]

\[
\leq \left( \int_0^T \Delta^8(t)|\zeta((1/2 + \varepsilon t)|^2 \, dt \right)^{1/4} \left( \int_0^T |\zeta((1/2 + \varepsilon t)|^{4+2/3} \, dt \right)^{3/4}
\]

\[
\ll (T^3 \log T)^{1/4} (T^{1+1/12})^{3/4} \log^C T \ll T^{25/16} \log^C T,
\]

\[
\int_0^T \Delta^3(t)|\zeta((1/2 + \varepsilon t)|^4 \, dt = \int_0^T \Delta^3(t)|\zeta((1/2 + \varepsilon t)|^{3/4}|\zeta((1/2 + \varepsilon t)|^{13/4} \, dt
\]

\[
\ll \left( \int_0^T \Delta^8(t)|\zeta((1/2 + \varepsilon t)|^2 \, dt \right)^{3/8} \left( \int_0^T |\zeta((1/2 + \varepsilon t)|^{4+6/5} \, dt \right)^{5/8}
\]

\[
\ll (T^3 \log T)^{3/8} (T^{1+3/20})^{5/8} \log^C T \ll T^{59/32} \log^C T,
\]

\[
\int_0^T \Delta^4(t)|\zeta((1/2 + \varepsilon t)|^4 \, dt = \int_0^T \Delta^2(t)|\zeta((1/2 + \varepsilon t)|^3 \, dt
\]

\[
\leq \left( \int_0^T \Delta^8(t)|\zeta((1/2 + \varepsilon t)|^2 \, dt \right)^{1/2} \left( \int_0^T |\zeta((1/2 + \varepsilon t)|^6 \, dt \right)^{1/2}
\]

\[
\ll (T^3 \log T)^{1/2} (T^{1+1/4})^{1/2} \log^C T \ll T^{17/8} \log^C T.
\]
This proves Theorem 1. The proof of Theorem 2 is on similar lines. Namely

\[
\int_0^T \Delta(t) |\zeta(\frac{1}{2} + it)|^6 \, dt = \int_0^T \Delta(t) |\zeta(\frac{1}{2} + it)|^{1/4} |\zeta(\frac{1}{2} + it)|^{23/4} \, dt
\]

\[
\ll \left( \int_0^T \Delta^8(t) |\zeta(\frac{1}{2} + it)|^2 \, dt \right)^{1/8} \left( \int_0^T |\zeta(\frac{1}{2} + it)|^{46/7} \, dt \right)^{7/8}
\]

\[
\ll (T^3 \log T)^{1/8} \left( \int_0^T |\zeta(\frac{1}{2} + it)|^{4 + \frac{48}{7}} \, dt \right)^{7/8}
\]

\[
\ll T^{49/32} \log^C T,
\]

where again we used (1.5) with \(k = 8\), (3.1) and (3.2). Finally

\[
\int_0^T \Delta^2(t) |\zeta(\frac{1}{2} + it)|^6 \, dt = \int_0^T \Delta^2(t) |\zeta(\frac{1}{2} + it)|^{1/2} |\zeta(\frac{1}{2} + it)|^{11/2} \, dt
\]

\[
\leq \left( \int_0^T \Delta^8(t) |\zeta(\frac{1}{2} + it)|^2 \, dt \right)^{1/4} \left( \int_0^T |\zeta(\frac{1}{2} + it)|^{22/3} \, dt \right)^{3/4}
\]

\[
\ll (T^3 \log T)^{1/4} (T^{1 + 10/24})^{3/4} \log^C T \ll T^{29/16} \log^C T,
\]

as asserted.
On some upper bounds for $|\zeta\left(\frac{1}{2} + it\right)|$ and a divisor problem

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