A general class of invariant diffusion processes in one dimension

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Abstract

This paper improves a previously established test involving only coefficients to decide a priori whether or not non-trivial symmetries of a large class of space-time dependent diffusion processes on the real line exist. When the existence of these symmetries are ensured, the transformation to canonical forms admitting either four- or six-dimensional symmetry groups and the full list of their infinitesimal generators are then immediately at our disposal without any cumbersome calculations that happen when at least one of the coefficients is arbitrarily chosen. We study in depth symmetry and reducibility properties and physically important solutions of six models arising in applications.

Keywords: Diffusion process, Brownian motion, Lie symmetry group, transformation formula to canonical forms, fundamental solution

1 Introduction

We reconsider a class of partial differential equations studied in an earlier paper [1] from the point of view of local equivalence and invariance

\[ u_t = a(x)u_{xx} + b(x)u_x + c(x)u, \quad a > 0, \quad x \in \mathbb{R}, \quad t > 0, \]

(1.1)

where \( a, b, c \) are arbitrary smooth functions. The coefficients \( a \) and \( b \) are called diffusion and drift functions. Partial differential equations (PDEs) from the class (1.1) are basic modelling equations for many applications in applied mathematics and science ranging from stochastic processes (Brownian motion), applied probability theory, financial mathematics to population genetics.

The purpose of this paper is to demonstrate that any PDE within the class (1.1) can be treated from group-theoretical point of view to find interesting solutions in a unified and systematic way. Here in particular we concentrate on the most general case when the coefficients depend on both space and time variables.
We observe that any PDE of the form (1.1) admits the time translation symmetry \( u(x, t) \rightarrow u(x, t + t_0) \) and the scaling symmetry \( u(x, t) \rightarrow \lambda u(x, t) \) for any constant \( \lambda \). This class has as well the linear superposition principle. These symmetries are the trivial ones. In this paper we shall be interested only in non-trivial symmetries (symmetries other than trivial ones).

The symmetry group of the standard heat equation where \( a = 1, b = c = 0 \) was first computed by Lie [2] and later recomputed by Appell [3]. A complete symmetry group classification of the superclass of (1.1) where the coefficients depend on \( x \) and \( t \) was also performed by Lie himself in [2]. The historical details have been discussed in [4]. For the most general class there are two canonical forms which admit non-trivial symmetry groups

\[
    u_t = u_{xx}, \quad u_t = u_{xx} + \frac{\mu}{x^2}u, \quad \mu \neq 0.
\]

(1.2)

Alternative derivations of this classification can be found in Refs. [5] or [1]. The first canonical form (the heat equation) has the Lie group structure \( G = \text{SL}(2, \mathbb{R}) \ltimes H(1) \), while the second canonical one has the Lie group structure \( G = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R} \), where \( \ltimes \) denotes a semi-direct product, \( H(1) \) is the 3-dimensional Heisenberg group.

The structure of this paper is as follows. In Section 2, we review the Lie point symmetry analysis of (1.1). Solutions of the invariance condition as a Riccati equation in the independent variable \( I(x) \) has been discussed. Section 3 is devoted to five examples of PDEs in connection with the study of Brownian motion and mathematical finance. The results of Section 2 are applied to them to construct solutions of physical interest.

## 2 Lie point symmetries

### 2.1 Semi-invariants

The group of equivalence transformations of the general PDE

\[
    u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u
\]

(2.1)

is known to be given by invertible point transformations of the form

\[
    \tilde{t} = T(t), \quad \tilde{x} = X(x, t), \quad \tilde{u} = \theta(x, t)u,
\]

(2.2)

where \( \theta(x, t) \neq 0 \). By an invariant of (2.1) we shall mean a differential function \( F(C, C^{(s)}) \) of the coefficients \( C = (a, b, c) \) and their space-time derivatives \( C^{(s)} \) up to some order \( s \), which is invariant under the change of dependent and independent variables so that \( F(\tilde{C}, \tilde{C}^{(s)}) = F(C, C^{(s)}) \) (2.2).

If only invariance under the rescaling of the dependent variable \( u \) by a nonvanishing function \( \theta(x, t) = e^{f(x, t)} \) (the so-called gauge factor) exists, then \( F \) will be called a semi-invariant of (2.1). Such a transformation changes the coefficients of the PDE (2.1) to

\[
    \tilde{a} = a, \quad \tilde{b} = b - 2af_x, \quad \tilde{c} = c + f_t - bf_x + a(f_x^2 - f_{xx}).
\]

(2.3)
The first relation implies that \( a \) is a semi-invariant.

We define the following two quantities in terms of the coefficients that will be used repeatedly throughout this paper

\[
I(x) = \int x \frac{ds}{\sqrt{a(s)}}, \quad J(x) = \frac{1}{\sqrt{a}} \left[ \frac{a'(x)}{2} - b(x) \right] = \frac{d}{dx}(\sqrt{a(x)}) - \frac{b(x)}{\sqrt{a(x)}}
\]

(2.4)

Another function that will play a key role in the identification of point symmetries and mapping of (1.1) to canonical forms is

\[
K(x) := F(a, b, c, a_x, a_{xx}, b_x) = \frac{1}{2} \sqrt{a} J'(x) - \frac{1}{4} J^2(x) + c(x).
\]

(2.5)

This function \( K(x) \) is a (second order) semi-invariant of (1.1) as it remains unaltered under the rescaling of the dependent variable: \( \tilde{u} = \theta(x,t) u \), where the gauge factor \( \theta \neq 0 \) (\( \theta_t = 0 \)) is arbitrary.

When we allow the coefficients to depend on both temporal and spatial variables

\[
a = a(x,t), \quad b = b(x,t), \quad c = c(x,t),
\]

(2.6)

we need to incorporate additional terms involving time derivatives of \( a \) and \( b \) into \( K(x) \) and replace (2.5) by a new semi-invariant

\[
K(x,t) := F(a, b, c, a_x, a_{xx}, a_{xxx}, b_x, b_{tt}) = \frac{1}{2} \sqrt{a} J'(x) - \frac{1}{4} J^2(x,t) + c + \frac{1}{2} \int \frac{\partial}{\partial t} \left( b \right) dx,
\]

(2.7)

where

\[
I(x, t) = \int x \frac{ds}{\sqrt{a(s,t)}}, \quad J(x, t) = \frac{\partial}{\partial x} \sqrt{a(x,t)} - \frac{b(x,t)}{\sqrt{a(x,t)}}.
\]

(2.8)

\( K(x,t) \) is again an invariant of the PDE (2.1) under the gauge transformation \( \tilde{u} = \theta(x,t) u \), \( \theta \neq 0 \). We note that that this invariant is not unique since we can add an arbitrary function of \( a \) and its derivatives.

On the other hand, two equations of the form (2.1) can be mapped to each other if and only if the (first-order) semi-invariant

\[
K_0(x, t) = c - b^2 4a + \frac{b a_x}{2a} - b \frac{3}{2} + \frac{1}{2} \int \frac{\partial}{\partial t} \left( b \right) dx
\]

(2.9)

has the same value. \( K(x,t) \) differs from \( K_0(x, t) \) by the invariant

\[
\frac{1}{2} \sqrt{a} (\sqrt{a})_{xx} - \frac{1}{4} (\sqrt{a})_{x}^2 = \frac{4aa_{xx} - 3a^2_x}{16a}.
\]

It is precisely \( K(x) \) or \( K(x,t) \) that will be needed in characterizing PDEs of the form (1.1) or (2.1) allowing symmetry algebras isomorphic to that of the heat equation.

Second order semi-invariants of the PDE (1.1) with general coefficients (2.6) using an infinitesimal approach was calculated in [6]. The spatial derivative of the directly calculated (nonlocal) first order semi-invariant (2.9) coincides with this second order invariant \( K_2 := K_{0,x} \). Later, in Ref. [7] a fifth order singular
invariant equation of the PDE invariant under the full equivalence group in terms of the second order invariant $K_2$ and the zeroth order invariant $a$ along with its higher derivatives was constructed with the same method. The authors of [7] demonstrated that a necessary and sufficient condition for the PDE to be transformable to the heat equation is that a fifth order singular invariant equation be satisfied.

An infinitesimal approach can be used to verify that the set \{ $a, K_2$ \} actually forms a basis of second order semi-invariants. Arbitrary function of this set is again a semi-invariant and any higher order ones can be generated by differentiating this basis with respect to time and space coordinates. In this approach, we look at the infinitesimal changes in the coefficients $a, b, c$ of the original PDE under the infinitesimal transformations corresponding to the rescaling of $u$

$$\tilde{x} = x, \quad \tilde{t} = t, \quad \tilde{u} = e^{\varepsilon \eta(x,t)} u = u + \varepsilon \eta(x,t) u + O(\varepsilon^2).$$

(2.10)

Putting $f = \varepsilon \eta$ in (2.3) and keeping terms linear in $\varepsilon$ we see that the coefficients transform infinitesimally as follows

$$\tilde{a} = a, \quad \tilde{b} = b - 2a \varepsilon \eta_x + O(\varepsilon^2), \quad \tilde{c} = c + \varepsilon(\eta_t - a \eta_{xx} - b \eta_x) + O(\varepsilon^2).$$

(2.11)

The associated infinitesimal generator of (2.11) in the space of coefficients is represented by

$$X = -2a \eta_x \partial_b + (\eta_t - a \eta_{xx} - b \eta_x) \partial_c.$$ 

(2.12)

We can infinitesimally verify that $K_2$, up to a factor of $2a^2$ multiplying $c_x$, 

$$K_2 = \frac{1}{2} b^2 a_x + (aa_{xx} - a_t - a_x^2) b + (aa_x - ab)b_x + ab_t - a^2 b_{xx} + 2a^2 c_x$$

(2.13)

is a second-order (local) semi-invariant. Indeed, the action of second prolongation of $X$ on $K_2$ vanishes for any choice of $\eta(x,t) \neq 0$ everywhere on the second jet space of the functions $a, b, c$. In other words, the semi-invariant $K_2$ is a differential invariant of the infinitesimal transformation (2.11). Any other semi-invariant is a function of $a, K_2$ and their derivatives with respect to $x$.

Below it will be clear that the use of our semi-invariant (2.7) calculated through a direct approach rather than far too complicated singular invariant equation of the full equivalence group calculated in [7] is much simpler for checking a priori reducibility to the first and second canonical forms. On the other hand, the singular invariant equation of [7] will impose conditions as fifth, fourth and third order nonlinear ODEs, respectively when we let one of the coefficients $a, b, c$ remain free. We refer to Example 3.5 to see how our criterion in comparison to the singular invariant condition simply reveals the required conditions for the PDE under study to possess heat or Schrödinger algebras as Lie point symmetry algebras.

### 2.2 Symmetry algebra

Now we can turn to discuss the infinitesimal symmetries of (1.1). In order to make this article self-contained we shall recall some relevant results of [1]. We take this opportunity to correct a number of misprints therein.
A general element of the symmetry algebra of (1.1) will be a vector field of the form
\[ v = \tau(t)\partial_t + \xi(x,t)\partial_x + \phi(x,t)u\partial_u, \]  
(2.14)
where the coefficients \(\tau, \xi, \phi\) are determined by applying Lie’s infinitesimal invariance algorithm [1] (see [4] for the details of the general theory). The coefficients \(\xi, \phi\) of \(v\) are given by
\[ \xi(x,t) = \sqrt{a} \left( \frac{1}{2} \dot{\tau}(x) + \rho(t) \right), \]
(2.15)
and
\[ \phi(x,t) = -\frac{1}{8} \ddot{\tau} + \frac{1}{4} \dot{\rho} I + \frac{1}{2} \dot{\tau} I + \frac{1}{4} \rho J + \sigma(t). \]
(2.16)
The undetermined functions \(\tau, \rho, \sigma\) must satisfy the classifying (determining) ODE
\[ \frac{1}{8} \dddot{\tau} I^2 - \frac{1}{4} \dddot{\tau} + \dot{\tau} \left( K(x) + \frac{1}{2} \sqrt{a} K'(x) I(x) \right) + \frac{1}{2} \ddot{\rho} I + \rho \dot{\sqrt{a}} K'(x) - \dot{\sigma} = 0. \]
(2.17)
Analysis of (2.17) reveals two possibilities for the invariant \(K\) depending on \(\rho\):
1) \(\rho = 0\) : \(K(x) = f_4(I) = \mu I^{-2} + c_2 I^2 + c_0, \quad \mu \neq 0, \)
(2.18)
2) \(\rho \neq 0\) : \(K(x) = f_0(I) = c_2 I^2 + c_1 I + c_0. \)
(2.19)
Substituting the above forms of \(K\) into (2.17) and splitting with respect to the powers of \(I\) result in the system of ODEs for functions \(\tau, \rho, \) and \(\sigma\)
\[ \dddot{\tau} + 16c_2 \dot{\tau} = 0, \quad \dddot{\rho} + 4c_2 \rho = -3c_1 \dot{\tau}, \quad \dddot{\sigma} = -\frac{1}{4} \dddot{\tau} + c_0 \dot{\tau} + c_1 \rho. \]
(2.20)

### 2.3 Four-dimensional symmetry algebra

In the case \(\rho = 0\), the general solution of this system will depend on 4 arbitrary independent constants which lead to the following bases for the corresponding algebras.

Depending on the sign of \(c_2\) we have three possible cases for the infinitesimal symmetries. The symmetry vector fields in some convenient basis together with the commutation relations among them are given as follows:

1. \(c_2 = 0\).
\[ v_1 = \partial_t, \]
\[ v_2 = t\partial_t + \frac{1}{2} \sqrt{a} I \partial_x + (c_0 t + \frac{1}{4} I J) u \partial_u, \]
\[ v_3 = t^2 \partial_t + t \sqrt{a} I \partial_x + \frac{1}{4} [2(2c_0 t - 1)t - I^2 + 2t I J] u \partial_u, \]
\[ v_4 = u \partial_u. \]
(2.21)
The non-zero commutation relations are
\[ [v_1, v_2] = v_1 + c_0 v_4, \]
\[ [v_1, v_3] = 2v_2 - \frac{1}{2} v_4, \quad [v_2, v_3] = v_3. \]
(2.22)
2. \( c_2 = -\kappa^2, \kappa > 0. \)

\[
\begin{align*}
v_1 &= \partial_t, \\
v_2 &= e^{4\kappa t} \left[ \partial_t + 2\kappa \sqrt{a} I \partial_x - (-c_0 + \kappa + 2\kappa^2 I^2 - \kappa I J) u \partial_u \right], \\
v_3 &= e^{-4\kappa t} \left[ \partial_t - 2\kappa \sqrt{a} I \partial_x + (c_0 + \kappa - 2\kappa^2 I^2 - \kappa I J) u \partial_u \right], \\
v_4 &= u \partial_u.
\end{align*}
\] (2.23)

The non-zero commutation relations are

\[
\begin{align*}
[v_1, v_2] &= 4\kappa v_2, \\
[v_1, v_3] &= -4\kappa v_3, \\
[v_2, v_3] &= -8\kappa v_1 - 8c_0 \kappa v_4.
\end{align*}
\] (2.24)

The basis elements \( v_2 \) and \( v_3 \) of (2.23) can be expressed in terms of hyperbolic functions \( \{\cosh 4\kappa t, \sinh 4\kappa t\} \) by taking the combination

\[
v_2 \rightarrow \frac{1}{2} (v_2 + v_3), \quad v_3 \rightarrow \frac{1}{2} (v_2 - v_3).
\]

3. \( c_2 = \kappa^2, \kappa > 0. \)

\[
\begin{align*}
v_1 &= \partial_t, \\
v_2 &= \cos 4\kappa t \partial_t - 2\kappa \sin 4\kappa t \sqrt{a} I \partial_x + [\cos 4\kappa t (c_0 + 2\kappa^2 I^2) \\
&\quad + \kappa \sin 4\kappa t (1 - I J)] u \partial_u, \\
v_3 &= \sin 4\kappa t \partial_t + 2\kappa \cos 4\kappa t \sqrt{a} I \partial_x + [\sin 4\kappa t (c_0 + 2\kappa^2 I^2) \\
&\quad - \kappa \cos 4\kappa t (1 - I J)] u \partial_u, \\
v_4 &= u \partial_u.
\end{align*}
\] (2.25)

The non-zero commutation relations are

\[
\begin{align*}
[v_1, v_2] &= -4\kappa v_3, \\
[v_1, v_3] &= 4\kappa v_2, \\
[v_2, v_3] &= 4\kappa (v_1 + c_0 v_4).
\end{align*}
\] (2.26)

**Remark 2.1.** The vector fields \( v_1, v_2, v_3 \) above span a Lie algebra isomorphic to \( \text{sl}(2, \mathbb{R}) \). For example, this is the case for the algebra with basis (2.22), which is easily seen by the change of basis

\[
v_1 \rightarrow v_1 + c_0 v_4, \quad v_2 \rightarrow 2v_2 - \frac{1}{2} v_4, \quad v_3 \rightarrow v_3.
\]

The entire algebra is isomorphic to the direct sum \( \mathfrak{g}_4 = \text{sl}(2, \mathbb{R}) \oplus \langle u \partial_u \rangle \) The same isomorphism is also true for the algebras spanned by (2.23) and (2.25).

### 2.4 Six-dimensional symmetry algebra

In the case \( \rho \neq 0 \), the general solution of the system (2.20) will depend on 6 arbitrary independent constants which lead to the following bases for the corresponding algebras.
1. $c_2 = 0$

\[ v_1 = \partial_t, \]
\[ v_2 = \frac{1}{2}\sqrt{a}(I - 3c_1t^2)\partial_x + t(2c_0 - 3c_1t)\partial_x, \]
\[ v_3 = t\sqrt{a}(I - c_1t^2)\partial_x - 4t^2(I - 2c_0t + c_1t^2)\partial_x, \]
\[ v_4 = t\sqrt{a}\partial_x + \frac{1}{2}[-I + t(c_1t + J)]u\partial_u, \]
\[ v_5 = \sqrt{a}\partial_x + \frac{1}{2}[2c_1t + J]u\partial_u, \]
\[ v_6 = u\partial_u. \]  

The non-zero commutation relations are

\[
[v_1, v_2] = v_1 - 3c_1v_4 + c_0v_6, \quad [v_1, v_3] = 2v_2 - \frac{v_6}{3}, \quad [v_1, v_4] = v_5, \\
[v_1, v_5] = c_1v_6, \quad [v_2, v_3] = v_3, \quad [v_2, v_4] = \frac{1}{2}v_4, \quad [v_2, v_5] = -\frac{1}{2}v_5, \\
[v_3, v_5] = -v_4, \quad [v_4, v_5] = \frac{v_6}{2}. \]

2. $c_2 = -\kappa^2, \kappa > 0$

\[ v_1 = \partial_t, \]
\[ v_2 = e^{4\kappa t}\partial_t + \frac{1}{\kappa}e^{4\kappa t}\sqrt{a}(-c_1 + 2\kappa^2 I)\partial_x - \frac{1}{4\kappa^2}e^{4\kappa t}[c_1^2 - 4c_0\kappa^2 + 4\kappa^3 + 8\kappa^4 I^2 + 2c_1\kappa J - 4\kappa^2 I(2c_1 + \kappa J)]u\partial_u, \]
\[ v_3 = e^{-4\kappa t}\partial_t + \frac{1}{\kappa}e^{-4\kappa t}\sqrt{a}(c_1 - 2\kappa^2 I)\partial_x - \frac{1}{4\kappa^2}e^{-4\kappa t}[c_1^2 - 4c_0\kappa^2 - 4\kappa^3 + 8\kappa^4 I^2 - 2c_1\kappa J + 4\kappa^2 I(-2c_1 + \kappa J)]u\partial_u, \]
\[ v_4 = e^{2\kappa t}\sqrt{a}\partial_x + \frac{1}{2\kappa}e^{2\kappa t}[c_1 - 2\kappa^2 I + \kappa J]u\partial_u, \]
\[ v_5 = e^{-2\kappa t}\sqrt{a}\partial_x + \frac{1}{2\kappa}e^{-2\kappa t}[-c_1 + 2\kappa^2 I + \kappa J]u\partial_u, \]
\[ v_6 = u\partial_u. \]  

The non-zero commutation relations are

\[
[v_1, v_2] = 4\kappa v_2, \quad [v_1, v_3] = -4\kappa v_3, \quad [v_1, v_4] = 2\kappa v_4, \\
[v_1, v_5] = -2\kappa v_5, \quad [v_2, v_3] = -2\kappa(4v_1 + rv_6), \quad [v_2, v_5] = -4\kappa v_4, \\
[v_3, v_4] = 4\kappa v_5, \quad [v_4, v_5] = 2\kappa v_6, \]

where $r = 4c_0 + (c_1/\kappa)^2$.  


3. \( c_2 = \kappa^2, \kappa > 0. \)

\[
\begin{align*}
    v_1 &= \partial_t, \\
    v_2 &= \cos 4\kappa t \partial_t - \frac{1}{\kappa} \sin 4\kappa t \sqrt{a(c_1 + 2\kappa^2 I)} \partial_x + \\
    &\quad \left[ \cos 4\kappa t (c_0 + \frac{c_1^2}{4\kappa^2} + 2(c_1 + 2\kappa^2 I)I) + \sin 4\kappa t (\kappa - \frac{1}{2\kappa}(c_1 + 2\kappa^2 I)J) \right] u \partial_u, \\
    v_3 &= \sin 4\kappa t \partial_t + \frac{1}{\kappa} e^{-4\kappa t} \sqrt{a(c_1 - 2\kappa^2 I)} \partial_x + \\
    &\quad \left[ \cos 4\kappa t (-\kappa + \frac{1}{2\kappa}(c_1 + 2\kappa^2 I)J) + \sin 4\kappa t (c_0 + \frac{c_1^2}{4\kappa^2} + 2(c_1 + 2\kappa^2 I)I) \right] u \partial_u, \\
    v_4 &= \cos 2\kappa t \sqrt{a} \partial_x + \frac{1}{2\kappa} \left[ \sin 2\kappa t (c_1 + 2\kappa^2 I) + \kappa \cos 2\kappa t J \right] u \partial_u, \\
    v_5 &= \sin 2\kappa t \sqrt{a} \partial_x + \frac{1}{2\kappa} \left[ -\cos 2\kappa t (c_1 + 2\kappa^2 I) + \kappa \sin 2\kappa t J \right] u \partial_u, \\
    v_6 &= u \partial_u. 
\end{align*}
\]

The non-zero commutation relations are

\[
\begin{align*}
    [v_1, v_2] &= -4\kappa v_3, & [v_1, v_3] &= 4\kappa v_2, & [v_1, v_4] &= -2\kappa v_5, \\
    [v_1, v_5] &= 2\kappa v_4, & [v_2, v_3] &= \kappa (4v_1 + sv_6), & [v_2, v_4] &= 2\kappa v_5, & [v_2, v_5] &= 2\kappa v_4, \\
    [v_3, v_4] &= -2\kappa v_4, & [v_3, v_5] &= 2\kappa v_5, & [v_4, v_5] &= -\kappa v_6, 
\end{align*}
\]

where \( s = 4c_0 - (c_1/\kappa)^2. \)

The symmetry algebras in cases (2.27), (2.29) and (2.31) has a Levi-decomposition structure in the form of semi-direct sum \( \mathfrak{g}_6 = \text{sl}(2,\mathbb{R}) \oplus \mathfrak{h}(1), \) where \( \mathfrak{h}(1) \) is the three-dimensional nilradical (Heisenberg algebra) with \( v_6 \) being the center element. \( \mathfrak{g}_6 \) is isomorphic to the heat algebra. A compete study of the symmetry properties of the heat equation has recently been discussed in [8]. A general treatment of Kolmogorov and Fokker–Planck type PDEs from a new perspective can be found in [9].

We note that the vector field \( v_3 \) has a sign error in the two exponential terms of [1] and the basis elements (2.31) appears mistakenly as a copy of (2.29) in the same reference.

We summarise the above results of Sections 2.3 and 2.4 as two theorems.

**Theorem 2.1.** The dimension of the Lie symmetry algebra of Eq. (1.1) is either four or six. A four-dimensional symmetry algebra occurs if and only if

\[
K(x) = f_4(I) := \frac{\mu}{I^2} + c_2 I^2 + c_0, \quad \mu \neq 0, \tag{2.33}
\]

a six-dimensional one (maximal) if and only if

\[
K(x) = f_6(I) := c_2 I^2 + c_1 I + c_0 \tag{2.34}
\]

for some constants \( c_2, c_1, c_0, \mu. \) The first algebra is locally isomorphic to \( \mathfrak{g}_4 = \text{sl}(2,\mathbb{R}) \oplus \mathbb{R}, \) the second one isomorphic to the Schrödinger (or heat) algebra.
Theorem 2.2. An equation from the class (1.1) can be transformed to the heat equation if and only if the semi-invariant $K$ equals a quadratic polynomial in $I$

$$K(x) := \frac{1}{2} \sqrt{a} J' - \frac{1}{4} J^2 + c = c_2 I^2 + c_1 I + c_0,$$  

where $c_2$, $c_1$ and $c_0$ are certain constants. Furthermore, the transformation mapping (1.1) to the standard heat equation $\tilde{\partial}_t \tilde{u} = \tilde{\partial}_x^2 \tilde{u}$ is given by

$$\tilde{t} = T(t), \quad \tilde{x} = \sqrt{T(I + \omega(t))}, \quad \dot{T} > 0,$$

$$u = C(Ta)^{1/4} \exp \left[ - \int \frac{b}{2a} dx + \frac{T}{8T}(I + \omega)^2 + \frac{\dot{\omega} I}{2} + \int \left( c_0 + \frac{\dot{\omega}^2}{4} - c_2 \omega^2 \right) dt \right] \tilde{u},$$

where $C$ is an arbitrary nonzero constant and $T$, $\omega$ are solutions to the set of ODEs

$$\{T, t\} = 8c_2, \quad \dot{\omega} + 4c_2 \omega = 2c_1,$$  

where $\{T, t\}$ denotes the Schwarzian derivative of $T$. This transformation in general depends on six arbitrary constants.

When the most general dependence on coefficients is imposed the transformation formula (2.36) will still hold true. But in this case we replace $I(x)$ and $J(x)$ with $I(x, t)$ and $J(x, t)$ of (2.8), $K(x)$ with $K(x, t)$ of (2.7) and then the appropriately specified constants $c_2, c_1, c_0$ will be functions of $t$ (see Example 3.3 for an application).

We comment that the same transformation can be used to map to the second canonical form (1.2) of the heat equation which in this case we must have $c_1 = 0$ and $\omega = 0$.

We recall that the ratio $\varrho = \dot{T}/\dot{T}$ satisfies the Riccati equation

$$\dot{\varrho} - \frac{1}{2} \varrho^2 = 8c_2.$$  

There are three possible cases for the solution of the Schwarzian equation $\{T, t\} = 8c_2$ depending on the sign of $c_2$:

1. $c_2 = 0$:

$$T(t) = M(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \Delta = \alpha \delta - \beta \gamma > 0, \quad \varrho = \tilde{T} = \frac{-2\gamma}{\gamma t + \delta}.$$  

2. $c_2 = -\lambda^2$, $\lambda > 0$:

$$T(t) = \frac{\alpha \cosh 2\lambda t + \beta \sinh 2\lambda t}{\gamma \cosh 2\lambda t + \delta \sinh 2\lambda t}, \quad \dot{T} = -2\Delta \lambda (\gamma \cosh 2\lambda t + \delta \sinh 2\lambda t)^{-2}.$$  

$$\varrho = \frac{\ddot{T}}{T} = -4\lambda \tanh(2\lambda t + \varepsilon), \quad \tanh \varepsilon = \frac{\delta}{\gamma}, \quad \Delta < 0.$$  

9
3. \( c_2 = \lambda^2, \lambda > 0:\)

\[
T(t) = \frac{\alpha \cos 2\lambda t + \beta \sin 2\lambda t}{\gamma \cos 2\lambda t + \delta \sin 2\lambda t}, \quad \dot{T} = -2\Delta \lambda (\gamma \cos 2\lambda t + \delta \sin 2\lambda t)^{-2}.
\]

\[
\vartheta = \frac{\dot{T}}{T} = 4\lambda \tan(2\lambda t + \varepsilon), \quad \tan \varepsilon = \frac{\delta}{\gamma}, \quad \Delta < 0.
\]

We point out that as \( \alpha, \beta, \gamma, \delta \) are defined up to nonzero constant multiplicative and \( \dot{T} > 0 \), we can always put \( \Delta = \alpha \delta - \beta \gamma = \pm 1 \).

Formula (2.36) means that if \( u(x,t) \) satisfies (1.1), then \( \tilde{u}(\tilde{x},\tilde{t}) \) will satisfy the heat equation \( \tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} \). In some cases, this transformation can map the trivial solutions (a constant one) to interesting solutions like fundamental solutions (see Section 3).

If, in particular, we require \( u \) to be solution of the heat equation, then Theorem 2.2 provides us with the symmetry group of this PDE (see Theorem 2 of [8] for an alternative derivation of this result). It is sufficient to choose \( a = 1, b = c = 0 \), which implies \( K = 0 \) and \( c_2 = c_1 = 0 \). So we have to solve \{\( T, t \)\} = 0 and \( \ddot{\omega} = 0 \). \( T(t) \) are Möbius transformations of \( t \) and \( \omega(t) = \frac{vt}{2} + x_0 \).

**Corollary 2.1.** The complete symmetry group of the heat equation, up to adding an arbitrary solution to \( u \), is given by

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{x + vt + x_0}{\gamma t + \delta},
\]

\[
\tilde{u} = C_0|\gamma t + \delta|^{1/2} \exp \left[ \frac{\gamma(x + vt + x_0)^2}{4(\gamma t + \delta)} - \frac{v}{2}t - \frac{v^2}{4}t \right] u,
\]  \hspace{1cm} (2.39)

where \( C_0, v, x_0 \) and \( \alpha, \beta, \gamma, \delta \) such that \( \alpha \delta - \beta \gamma = 1 \) are arbitrary parameters.

We note that in the case of four-dimensional symmetry algebra, \( \omega = 0 \) \((c_1 = 0)\). So we only have to solve \( \{T, t\} = 8c_2 \). Now we can use formula (2.36) to obtain a point transformation mapping to the second canonical form (heat equation with inverse square potential). This gives us the symmetry group of this PDE for the choice

\[
a = 1, \quad b = 0, \quad c = \frac{\mu}{x^2}.
\]  \hspace{1cm} (2.40)

Then, \( J = 0, K = c \) and form the condition (2.33), it follows that \( c_2 = 0 \) \((c_1 \text{ is already zero})\).

**Corollary 2.2.** The complete symmetry group of the second canonical form consists of point transformations of the form

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{x}{\gamma t + \delta},
\]

\[
\tilde{u} = C_0|\gamma t + \delta|^{1/2} \exp \left[ \frac{\gamma x^2}{4(\gamma t + \delta)} \right] u,
\]  \hspace{1cm} (2.41)

where \( C_0 \) and \( \alpha, \beta, \gamma, \delta \) such that \( \alpha \delta - \beta \gamma = 1 \) are arbitrary parameters.

See Theorem 9 of [11] for an alternative construction of (2.41).
2.5 Discussion of solutions of the symmetry criteria

Regarding $I$ as the independent variable in the definition of $K$ we can write the invariance condition $K(x) = f(I)$, where $f \in \{f_4, f_6\}$ in the form

$$\frac{1}{2} \frac{d\Omega}{dI} - \frac{1}{4} \Omega^2 + \frac{q}{a} (a' - b + q) = f(I), \quad q' = c, \quad \Omega = J + \frac{2q}{\sqrt{a}}. \tag{2.42}$$

This ODE can be integrated as a Riccati equation if the last term on the left hand side of (2.42) is expressed in terms of a function of $I$ alone. In the simpler case where Eq. (1.1) is in conserved form ($a'' - b' + c = 0$), it becomes tractable in terms of special functions. The conserved PDE has the form

$$u_t = \partial_x [au_x + (b - a')u]. \tag{2.43}$$

Letting $a = p$, $q = b - a'$ leads to the Fokker–Planck equation

$$u_t = \partial_x (pu_x + qu) \tag{2.44}$$

with $J = -[(\sqrt{p})' + q/\sqrt{p}]$. In this case, in (2.42) we have $a' - b + q = \text{const.}$ which we pick zero for simplicity. Thus $\Omega(I)$ satisfies the special Riccati equation

$$\frac{1}{2} \frac{d\Omega}{dI} - \frac{1}{4} \Omega^2 = f(I), \tag{2.45}$$

where

$$\Omega = - (\sqrt{p})' + \frac{q}{\sqrt{p}}. \tag{2.46}$$

From (2.46) we find that for given coefficient $p$ solving (2.45) we can determine $q$ such that the main PDE admits a non-trivial symmetry group

$$q = \frac{p'}{2} + \sqrt{p} \Omega(I). \tag{2.47}$$

The solutions of (2.45) can be expressed in terms of the confluent hypergeometric functions. The Cole–Hopf transformation $\Omega = -2v'/v$ linearizes (2.45) to the second order ordinary differential equation (ODE)

$$4v'' - fv = 0. \tag{2.48}$$

Here we have replaced $f$ by $-(1/4)f$ for convenience.

When $f = f_6$ the transformation of both the independent and dependent variables

$$v = e^{-z/2}w(z), \quad z = \frac{c_2}{2} \left( I + \frac{c_1}{2c_2} \right)^2 \tag{2.49}$$

maps this ODE to confluent hypergeometric (Kummer) equation

$$zw'' + \left( \frac{1}{2} - z \right) - aw = 0, \tag{2.50}$$

where

$$a = \frac{1}{4} \left[ 1 + \frac{4c_0 c_2 - c_1^2}{8c_2^{3/2}} \right].$$
For the case $f = f_4$ we apply the transformation
\[ v = z^s e^{-z/2} w(z), \quad z = \frac{\sqrt{c_2}}{2} f^2, \quad s = \frac{1}{2} (1 \pm \sqrt{1 + \mu}) \] (2.51)
to transform (2.48) to another confluent hypergeometric (Kummer) differential equation
\[ zw'' + (2s - z)w' - \left( \frac{c_0}{2\sqrt{c_2}} + \frac{1}{4} + \frac{s}{2} \right) w = 0. \] (2.52)

### 3 Applications

In this section we illustrate the ideas developed in Section 2 with several PDEs with varying diffusion and drift coefficients arising in a wide range of applications. Our intention is to show how they can serve to effectively simplify the symmetry group analysis, which otherwise might be rather tricky, specifically when at least one coefficient, say the drift term, is allowed to remain arbitrary (see Example 3.5).

**Example 3.1.** A PDE that arises in a study of Brownian motion is
\[ u_t = (1 + k^2 x^2)^2 u_{xx}. \] (3.1)
The coefficients are $a(x) = (1 + k^2 x^2)^2$, $b(x) = c(x) = 0$ and we find
\[ I(x) = \frac{1}{k} \arctan(kx), \quad J(x) = 2k^2 x, \quad K(x) = k^2. \]

We check that condition (2.35) for the PDE (3.1) to have maximal symmetry is satisfied for $c_1 = c_2 = 0$ and $c_0 = k^2$. It follows that a reduction to the first canonical form (heat equation) is possible by the point transformation (2.36) (for the simple choice $\omega = 0$) given by
\[ \tilde{t} = M(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{1}{k} \sqrt{\Delta} (\gamma t + \delta)^{-1} \arctan kx, \]
\[ u = C(\gamma t + \delta)^{-1/2} (1 + k^2 x^2)^{1/2} \exp \left[ -\frac{\gamma \arctan kx}{4k^2(\gamma t + \delta)} + k^2 t \right] \tilde{u}, \] (3.2)
where $C$ is a nonzero constant. As a bonus, the transformation formula (3.2) in the limit $k \to 0$ produces the well-known Appell’s transformation mapping the heat equation into each other [3]
\[ \tilde{t} = M(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \sqrt{\Delta} (\gamma t + \delta)^{-1} x, \]
\[ u = C|\gamma t + \delta|^{-1/2} \exp \left[ -\frac{\gamma x^2}{4(\gamma t + \delta)} \right] \tilde{u}(\tilde{x}, M(t)). \] (3.3)

A well-known fact is that transformation (3.3) maps the constant solution $\tilde{u} = 1$ to the heat kernel (fundamental solution of the heat equation) up to a multiplicative constant. This argument has substantially been used in the present work to produce physically significant solutions.
From (2.27) we find that the corresponding six-dimensional symmetry algebra is spanned by the vector fields

\[ v_1 = \partial_t, \]
\[ v_2 = t\partial_t + \frac{1}{2k}(1 + k^2x^2)\arctan{kx}\partial_x + \frac{k}{2}(2kt + x\arctan{kx})u\partial_u, \]
\[ v_3 = t^2\partial_t + \frac{t\arctan{kx}}{k}(1 + k^2x^2)\partial_x - \frac{1}{4}[2t - 4k^2t^2 - 4ktx\arctan{kx} + (\arctan{kx})^2]u\partial_u, \]
\[ v_4 = t(1 + k^2x^2)\partial_x + \frac{t\arctan{kx}}{2k}u\partial_u, \]
\[ v_5 = (1 + k^2x^2)\partial_x + k^2xu\partial_u, \]
\[ v_6 = u\partial_u. \]

The non-zero commutation relations are

\[ [v_1, v_2] = v_1 + k^2v_6, \quad [v_1, v_3] = 2v_2 - \frac{v_6}{2}, \quad [v_1, v_4] = v_5, \]
\[ [v_2, v_3] = v_3, \quad [v_2, v_4] = \frac{1}{2}v_4, \quad [v_2, v_5] = -\frac{1}{2}v_5, \]
\[ [v_3, v_5] = -v_4, \quad [v_4, v_5] = \frac{v_6}{2}. \]

In the limit \( k \to 0 \), the symmetry algebra spanned by (2.27) reduces to that of the standard heat equation by a slight change of the basis \( v_2 \to 2v_2 - v_6/2 \).

**Example 3.2.** We discuss the special Fokker–Planck PDE

\[ u_t = u_{xx} + \partial_x(q(x)u). \] (3.6)

This PDE was first studied in [12]. We use our results for an alternative treatment of it. We shall require that \( q(x) \) be specified in such a manner that the PDE is left invariant under the full symmetry group of the heat equation. Here we have \( p = 1, I = x, J = -q \). Then, Riccati equation (2.45) with \( \Omega(x) = q(x) \) or condition (2.34) now takes the form

\[ \frac{1}{2}q' - \frac{1}{4}q^2 = f_6(x) = c_2x^2 + c_1x + c_0. \] (3.7)

We, in particular, choose the constants \( c_2, c_1, c_0 \) as

\[ c_2 = -\frac{\ell^2}{4}, \quad c_1 = 0, \quad c_0 = \frac{\ell}{2}. \] (3.8)

With these values of \( c_2, c_1 \) and \( c_0 \), it is straightforward to check that \( q = \ell x \) is a particular solution of (3.7). The general solution of course can be found. This enables to constructs the most general PDE of the form (3.6) equivalent to the heat equation by a point transformation. The PDE corresponding to the special solution is the celebrated Orstein–Uhlenbeck model. We conclude that
for this PDE there exists a transformation (not unique) mapping (3.6) with \( q = \ell x \) to the first canonical form

\[
\begin{align*}
\tilde{t} &= -\coth \ell t, \quad \tilde{x} = \frac{\sqrt{\ell}}{\sinh \ell t} x, \\
u &= C (\sinh \ell t)^{-1/2} \exp \left[ -\frac{\ell x^2}{4} (1 + \coth \ell t) + \frac{\ell t}{2} \right] \tilde{u}(\tilde{x}, \tilde{t}) \\
&= C (1 - e^{-2\ell t})^{-1/2} \exp \left[ -\frac{\ell}{2} \frac{x^2}{1 - e^{-2\ell t}} \right] \tilde{u}(\tilde{x}, \tilde{t}).
\end{align*}
\]

(3.9)

It is remarkable to observe that the constant solution \( \tilde{u} = 1 \) is mapped to the fundamental solution of (3.6) at the singularity \((0,0)\)

\[
u(x,t) = K(x,t;0) = \sqrt{\frac{\ell}{2\pi}} (1 - e^{-2\ell t})^{-1/2} \exp \left[ -\frac{\ell}{2} \frac{x^2}{1 - e^{-2\ell t}} \right]
\]

(3.10)

with the normalization constant \( C = (\ell/2\pi)^{1/2} \). We note that in the limit \( t \to \infty \), this solution approaches the stationary solution \( u_0(x) \) of (3.6) (a Gaussian distribution)

\[
u_0 = \sqrt{\frac{\ell}{2\pi}} e^{-\frac{\ell}{2} x^2}.
\]

Alternatively, the basis elements of the corresponding Lie symmetry algebra which are read off from (2.29) can be used to construct group invariant solution satisfying the condition \( u(x,0) = \delta(x) \). This solution coincides with the fundamental solution or probability density function (3.10) possessing the property \( \int u(x,t) dx = 1 \) (See [14] for this derivation and other symmetry group methods).

The presence of a four-dimensional symmetry group is guaranteed if we take \( q \) as any solution of the Riccati equation

\[
\frac{1}{2} q' - \frac{1}{4} q^2 = f_4(x) = \frac{\mu}{x^2} + c_2 x^2 + c_0
\]

(3.11)

for some suitable choice of the parameters. For \( \mu = -\alpha(\alpha + 2), \alpha \neq -2, \ c_2 = c_0 = 0 \), it admits the solution \( q = \alpha/x \). The corresponding PDE is equivalent to the second canonical form of (1.2).

**Example 3.3.** We consider the Cox–Ingersoll–Ross (CIR) PDE

\[
u_t = \sigma x u_{xx} + (m + nx) u_x, \quad \sigma, m, n > 0, \quad x \in \mathbb{R}^+.
\]

(3.12)

We have \( a = \sigma x, \ b = m + nx, \ c = 0 \). We start calculating the following quantities

\[
I = 2 \sqrt{\frac{x}{\sigma}}, \quad J = \frac{1}{\sqrt{\sigma x}} \left[ \frac{\sigma}{2} - (m + nx) \right], \quad K = -\frac{1}{2\sigma x} \left[ \frac{1}{2} (m + nx)^2 - \sigma m \right].
\]

We establish the condition ensuring that a non-trivial symmetry algebra exists

\[
K(x) = \frac{\mu}{I^2} + c_2 I^2 + c_1 I + c_0.
\]

(3.13)
After some manipulation, this implies that

\[ c_1 = 0, \quad c_2 = \frac{n^2}{16}, \quad c_0 = -\frac{mn}{2\sigma}, \quad \mu = -\frac{\Phi}{4\sigma^2}, \]

where \( \Phi = (2m - \sigma)(2m - 3\sigma) \). If \( \mu \neq 0 \), then the corresponding non-trivial symmetry algebra becomes four-dimensional. Otherwise, if we let \( \mu = 0 \) (\( \Phi \equiv 0 \)) then we find that we must have \( m \in \{ \sigma/2, (3\sigma)/2 \} \) for the presence of a six-dimensional symmetry algebra. In the latter case, the corresponding CIR equation is isomorphic to the heat equation \( u_t = u_{xx} \) by the action of the transformation formula (2.36). We can write such transformations for both of the values \( m \) in a unified manner for the special choice \( \omega = 0 \) using (2.36):

\[
\dot{t} = T(t) = \frac{\alpha \cosh \frac{nt}{2} + \beta \sinh \frac{nt}{2}}{\gamma \cosh \frac{nt}{2} + \delta \sinh \frac{nt}{2}}, \quad \dot{x} = \sqrt{-\frac{2n\Delta}{\sigma}} \left( \gamma \cosh \frac{nt}{2} + \delta \sinh \frac{nt}{2} \right)^{-1} \sqrt{x},
\]

\[ u = C \left( \gamma \cosh \frac{nt}{2} + \delta \sinh \frac{nt}{2} \right)^{-1/2} g(x) \exp \left[ -\frac{n}{2\sigma} \left( 1 + \tanh \left( \frac{nt}{2} + \varepsilon \right) \right) x + c_0 t \right] \hat{u}, \quad (3.14)
\]

where \( C \) is a nonzero arbitrary constant, \( \tanh \varepsilon = \sigma/\gamma \), \( c_0 \in \{ -n/4, -3n/4 \} \) and

\[ g(x) = \begin{cases} 1, & m = \frac{\sigma}{2}, \\ \frac{1}{\sqrt{x}}, & m = \frac{3\sigma}{2}. \end{cases} \]

An interesting solution to (3.12) is obtained if we choose \( \hat{u} = 1 \). We first choose the pair \((m, c_0) = (\sigma/2, -n/4)\), \( \alpha = \delta = 0 \), \( \beta = \gamma = 1 \) (\( \Delta = -1 \)) and obtain the solution

\[ u = C(1 + e^{nt})^{-1/2} \exp \left[ -\frac{n}{\sigma} \frac{x}{(1 + e^{-nt})} \right]. \quad (3.16) \]

For the other possibility \((m, c_0) = (3\sigma/2, -3n/4)\), the solution produced from mapping to \( \hat{u} = 1 \) is

\[ u = C(e^{2nt} + e^{nt})^{-1/2} x^{-1/2} \exp \left[ -\frac{n}{\sigma} \frac{x}{(1 + e^{-nt})} \right]. \quad (3.17) \]

If we have \( m \not\in \{ \sigma/2, (3\sigma)/2 \} \), then transformation (3.14) with \( g(x) = x^{(\sigma-2m)/(4\sigma)} \) maps the equation to the second canonical form

\[ \ddot{u}_t = \ddot{u}_{xx} + \frac{\mu}{x^2}, \quad \mu = -\frac{(2m - \sigma)(2m - 3\sigma)}{4\sigma^2}. \]

We refer the interested reader to Example 5.3 of [15] for a detailed study of the non-trivial Lie point symmetries, relevant transformations and their use for construction of the fundamental solution \( u = K(x, t; x_0) \) satisfying the boundary condition \( K(x, 0; x_0) = \delta(x - x_0) \) in the special case \( n = 0 \).

**Example 3.4.** We now consider a PDE whose drift term not only has \( x \) dependence but also \( t \) dependence

\[ a = 1, \quad b = m(t) + n(t)x, \quad c = q(t) + r(t)x. \quad (3.18) \]
We first let $n \neq 0$. We impose the condition for the PDE to be transformable to the heat equation

$$K(x, t) = c_2(t)I^2 + c_1(t)I + c_0(t)$$

with $K$ as given by (2.7), more explicitly

$$\frac{1}{2} J_x - \frac{1}{4} J^2 + c + \frac{1}{2} \int (\dot{n} + \dot{n}x) dx = c_2(t)x^2 + c_1(t)x + c_0(t), \quad J(x, t) = -b(x, t)$$

for some functions $c_2, c_1, c_0$ that need to be fixed conveniently. This occurs if and only if

$$\dot{n} - n^2 = 4c_2(t), \quad \dot{m} - mn + 2r = 2c_1(t), \quad q - \frac{n}{2} - \frac{m^2}{4} = c_0(t). \quad (3.19)$$

With this choice the symmetry algebra of the corresponding PDE should be isomorphic to that of the heat equation.

Infinitesimal generators $\xi, \phi$ of the general symmetry vector field (2.14) are given by

$$\xi(x, t) = \frac{1}{2} \dot{\tau}x + \rho(t), \quad (3.20)$$

and

$$\phi(x, t) = -\frac{1}{8} \dot{\tau}x^2 - \frac{1}{2} \dot{\rho}x - \left(\frac{1}{4} \dot{\tau}x + \frac{1}{2} \rho\right) b(x, t) - \frac{\tau}{2} \int b_t dx + \sigma(t), \quad (3.21)$$

where $\tau, \rho, \sigma$ satisfy a classifying equation splitting into the set of ODEs (compare with (2.20))

$$\ddot{\tau} + 16c_2 \dot{\tau} + 8c_2 \tau = 0, \quad \ddot{\rho} + 4c_2 \rho + 3c_1 \dot{\tau} + 2c_1 \tau = 0$$

$$\ddot{\sigma} = \frac{d}{dt}(c_0 \tau) + c_1 \rho - \frac{\tau}{4}. \quad (3.22)$$

The function $\tau(t)$ has the fundamental system of solutions $\{\psi_1^2, \psi_1 \psi_2, \psi_2^2\}$, where $\psi_1, \psi_2$ are two independent solutions of the ODE $\psi'' + 4c_2\psi = 0$.

For fixed values of $c_2(t)$ in the first condition of (3.19), we can specify $n(t)$ by solving a Riccati equation. Otherwise, for given $n(t)$, $c_2(t)$ remains fixed. Hence, fixing $c_2(t) = \varepsilon \lambda^2/4$, $\lambda = \text{const.}$ leads to the Riccati equation

$$\dot{\tau} - n^2 = \varepsilon \lambda^2, \quad \varepsilon \in \{0, 1, -1\}$$

with solutions in terms of elementary functions. $T$ is found as a solution of the Schwarzian equation $\{T, t\} = 2\varepsilon \lambda^2$. Once $n$ has been determined, then any solution $\omega$ of the ODE

$$\ddot{\omega} + \varepsilon \lambda^2 \omega = 2c_1 = \dot{m} - mn + 2r \quad (3.23)$$

will complete the required transformation (2.36).

Moreover, setting $c_2(t) = (A/4)t^\alpha$ for $\alpha = -2$ and $\alpha = -4k/(2k + 1), k \in \mathbb{Z}$ gives an explicitly solvable Riccati equation of which solutions will produce many time-dependent drift coefficients $n(t)$ in terms of elementary functions. Solutions of $\{T, t\} = 2At^\alpha$ will also be elementary. Both differential equations will require solving the second order linear ODE $\ddot{\varepsilon} + At^\alpha \varepsilon = 0$. 

16
For example, for $A = 1, \alpha = -4 \ (k = -1), c_2(t) = t^{-4}/4$ we find

$$n(t) = t^{-2}(\cot(t^{-1} + C) - t), \quad T(t) = \frac{\alpha \cos(\frac{1}{t}) + \beta \sin(\frac{1}{t})}{\gamma \cos(\frac{1}{t}) + \delta \sin(\frac{1}{t})}, \quad \alpha \delta - \beta \gamma = 1.$$ 

For the particular choice $m = 0, q = 0$ and $r = r_0 t^{-3},$ we have $c_1(t) = r(t), c_0(t) = -n(t)/2$ we can solve the system (3.22)

$$\tau(t) = t^2 \left[ \tau_1 \cos^2 \left( \frac{1}{t} \right) + \tau_2 \sin \left( \frac{2}{t} \right) + \tau_3 \sin^2 \left( \frac{1}{t} \right) \right],$$

$$\rho(t) = t \left[ \rho_1 \cos \left( \frac{1}{t} \right) + \rho_2 \sin \left( \frac{1}{t} \right) \right] + r_0 t \left[ (\tau_1 - \tau_3) \sin \left( \frac{2}{t} \right) - 2\tau_2 \cos \left( \frac{2}{t} \right) \right].$$

Finally, we obtain

$$\sigma = c_0 \tau + r_0 \int t^{-3} \rho(t) - \frac{\dot{r}}{4} + \sigma_0,$$

whose explicit form will not be produced here. Six arbitrary integration constants enter into the infinitesimal symmetries.

In summary, we have constructed a PDE of the form

$$u_t = u_{xx} + \frac{x}{t^2} \left[ \cot \left( \frac{1}{t} + C \right) - t \right] u_x + r_0 t^{-3} xu \quad (3.24)$$

admitting a symmetry algebra isomorphic to that of the heat equation with a basis given as discussed above. The related point transformation is again easy to write down from the formula (2.36) in which $\omega(t)$ can be taken as a private solution $\omega = 2r_0 t \ \text{of} \ \ddot{\omega} + t^{-4} \omega = 2r_0 t^{-3}.$

The case $n = 0 \quad (c_2 = 0)$ is easy to handle and we find the corresponding simplest form of the transformation as

$$\dot{t} = t, \quad \dot{x} = x + \int (m + 2s) dt, \quad u = C \exp \left[ sx + \int [s(s + m) + q] dt \right] \tilde{u}, \quad (3.25)$$

where $r = \dot{s}.$

Another simple case occurs when $m = 0, n = -1/t \quad (\varepsilon = 0)$ and $r = 0$ then we have $c_2 = c_1 = 0$ and $c_0(t) = q(t) + 1/(2t)$ and the corresponding transformation depending on four arbitrary parameters becomes

$$\dot{t} = M(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \dot{x} = \frac{\sqrt{\Delta}}{\gamma t + \delta} (x + t), \quad \Delta = \alpha \delta - \beta \gamma,$$

$$u = C(\gamma t + \delta)^{-1/2} \sqrt{t} \exp \left[ \frac{(x + t)^2}{4t} - \frac{\gamma (x + t)^2}{4(\gamma t + \delta)} + \int q(t) dt \right] \tilde{u}(\tilde{x}, \tilde{t}), \quad (3.26)$$

where we have chosen $\omega(t) = t$ in (2.36). From this group action on solutions, using the trivial solution $\tilde{u} = 1$ we obtain the non-trivial solution of the diffusion equation

$$u_t = u_{xx} - \frac{x}{t} u_x + q(t) u = 0 \quad (3.27)$$

as

$$u = C(\gamma t + \delta)^{-1/2} \sqrt{t} \exp \left[ \frac{\delta (x + t)^2}{4t(\gamma t + \delta)} + \int q(t) dt \right]. \quad (3.28)$$
If $r \neq 0$, $\omega$ should be taken as any solution of $\ddot{\omega} = 2r$.

By solving the system (3.22) it can be shown that the finite dimensional part of the Lie symmetry algebra of (3.27) for $q = 0$ is spanned by

$$
\begin{align*}
    v_1 &= \partial_t + \left( \frac{1}{2t} - \frac{x^2}{4t^2} \right) u \partial_u, \\
v_2 &= t \partial_t + \frac{x}{2} \partial_x, \\
v_3 &= t^2 \partial_t + xt \partial_x, \\
v_4 &= t \partial_x, \\
v_5 &= t^2 \partial_t + (1 + xt) \partial_x + \frac{x}{2t} u \partial_u, \\
v_6 &= u \partial_u.
\end{align*}
$$

(3.29)

The case $\varepsilon = -1$: We have $n(t) = -\lambda \tanh \lambda t$. We can single out from (3.19) another time-space dependent invariant PDE of the form (3.18) with $m, r$ fixed further

$$
m(t) = t \sech \lambda t, \quad r(t) = -\frac{1}{2} \sech \lambda t.
$$

Subsequently, functions $c_2(t), c_1(t), c_0(t)$ are also fixed as

$$
c_2 = -\frac{\lambda^2}{4}, \quad c_1 = 0, \quad c_0 = q(t) + \frac{\lambda}{2} \tanh \lambda t - \frac{1}{4} t^2 \sech^2 \lambda t.
$$

For $q = 0$, a basis for the symmetry algebra is immediately obtained from (3.20)-(3.21) by solving the system (3.22) in the form

$$
\begin{align*}
    \tau &= \tau_0 + \tau_1 \cosh 2\lambda t + \tau_2 \sinh 2\lambda t, \\
    \rho &= \rho_1 \cosh \lambda t + \rho_2 \sinh \lambda t, \\
    \sigma &= -\frac{1}{4} \sech^2 \lambda t[(\tau_0 + \tau_1 \cosh 2\lambda t + \tau_2 \sinh 2\lambda t) t^2 + (\tau_1 - \tau_0)\lambda \sinh 2\lambda t] + \sigma_0.
\end{align*}
$$

(3.30)

The arbitrary constants $\tau_0, \tau_1, \tau_2$ correspond to $\mathfrak{sl}(2, \mathbb{R})$ part and $\rho_1, \rho_2, \sigma_0$ to $\mathfrak{h}(1)$ Heisenberg part of the maximal symmetry algebra. We already know that this algebra is isomorphic to that of the heat equation. Of course, the transformation mapping (3.18) to the heat equation is given by (2.36).

The other possibility $\varepsilon = 1$ can be dealt with in the same manner.

**Remark 3.1.** We comment that the complete characterization of the drift functions for the existence of a non-trivial Lie point algebra of symmetries of a class of Fokker–Planck equations associated to Ito integrable equations analyzed in the recent paper [16] is readily recovered from the results of our above discussion of Example 3.4 in the special case $q(t) = n(t), r(t) = 0$.

The following transformation mapping this PDE in conserved form to the heat equation given in [7] is surely covered in our general transformation formula (2.36)

$$
\tilde{t} = T(t), \quad \tilde{x} = e^{\alpha(t)} x + \beta(t), \quad u = e^{\alpha(t)} \tilde{u},
$$

(3.31)

where $\alpha, \beta$ satisfy

$$
\dot{\alpha} = n, \quad \dot{\beta} = e^{\alpha(t)} m.
$$

(3.32)
In order to achieve this we choose $\beta = e^{\alpha} \omega$ and observe that

$$\{T, t\} = 2(\ddot{\alpha} - \dot{\alpha}^2) = 2(\dot{n} - n^2), \quad \dot{\omega} + n \omega = m, \quad \ddot{\omega} + (\dot{n} - n^2) \omega = m - mn. \quad (3.33)$$

This means then the pair of ODEs (2.37) is exactly the set of constraints (3.19) with $r = 0$ and if we further choose $c_0 = (2n - m^2)/4$, the whole term multiplying $\tilde{u}$ in the formula (2.36) is independent of $x$ and the rest boils down to $e^{\alpha}$. So we arrive at (3.31)-(3.32).

We note that in the general case (3.18), a transformation similar to (3.31) exists in the form

$$\tilde{t} = \int e^{2\alpha(t)} dt, \quad \tilde{x} = e^{\alpha(t)} x + \beta(t), \quad u = \exp \left[ \int (q + e^{2\alpha} \gamma^2 + me^\alpha \gamma) dt \right] \tilde{u},$$

where

$$\dot{\alpha} = n, \quad \dot{\beta} = e^{\alpha}(m + 2e^{\alpha} \gamma), \quad \gamma(t) = \int e^{-\alpha(t)} r(t) dt. \quad (3.35)$$

Moreover, $\beta$ satisfies $\ddot{\beta} - 2n \dot{\beta} = 2e^{\alpha}c_1$. The Fokker–Planck case means we have $q = n = \dot{\alpha}$ and $\gamma = 0$ and the transformation (3.34)-(3.35) simplifies to (3.31)-(3.32).

For the special PDE (3.24) with the help of the following functions

$$e^{2\alpha} = t^{-2} \csc^2(t^{-1}), \quad \beta = 2r_0 \csc(t^{-1}), \quad \gamma = r_0 \cos(t^{-1})$$

the transformation (3.34) specializes to

$$\tilde{t} = \cot(t^{-1}), \quad \tilde{x} = \csc(t^{-1})(t^{-1} x + 2r_0), \quad u = \exp \left[ r_0^2 (t^{-1} + \cot(t^{-1})) \right] \tilde{u},$$

which provides us with a mapping of (3.24) to the heat equation.

A final note is that if we additionally let the coefficient $c(x, t)$ to be a quadratic polynomial in $x$ as $c = q(t) + r(t)x + s(t)x^2$, then the corresponding PDE still remains transformable to the heat equation. However, there is no such simple formula like (3.34)-(3.35). In this case, only the first relation of (3.19) changes as $\dot{n} - n^2 + 4s = 4c_2$.

**Example 3.5.** The following PDE as local volatility model in mathematical finance has been studied from the symmetry point of view (see [17] for the details):

$$u_t = \frac{1}{2} \sigma^2(x)u_{xx} - g(x)u, \quad \sigma > 0, \quad (3.37)$$

where $\sigma$ and $g$ are functions defined in some subinterval of $\mathbb{R}$ and the drift coefficient is zero.

All the symmetry calculation results of this paper can immediately be obtained from our general result for the given coefficients of PDE (3.37). All that is needed is the calculation of the semi-invariant $K(x)$ defined by (2.5). Then, the condition for the equation to have a six-dimensional symmetry algebra is simply obtained by forming the condition (see formula (2.19))

$$K(x) = c_2 I^2(x) + c_1 I(x) + c_0, \quad (3.38)$$
where \( c_i, i = 0, 1, 2 \) are some constants to be determined later. The coefficients of the symmetry vector field are given by (2.15) and (2.16) under the condition (2.17). Here there is a slight change of the factor \( \rho(t): \rho \to \rho/\sqrt{2} \). All the infinitesimal generators of the symmetry algebra can be produced using the formulae of Subsection (2.4).

In this case,

\[
a(x) = \frac{1}{2} \sigma(x)^2, \quad b(x) = 0, \quad c(x) = -g(x)
\]

and it is straightforward to calculate

\[
I(x) = \sqrt{2} \psi(x), \quad J(x) = \frac{\sigma'(x)}{\sqrt{2}}, \quad K(x) = \frac{1}{4} \sigma'' - \frac{1}{8} \sigma'^2 - g,
\]

where \( \psi(x) \) denotes the Lamperti transformation implicitly defined by \( \psi' = 1/\sigma(x) \). Condition (3.38) now gives rise to

\[
\frac{1}{8}(2\sigma'' - \sigma'^2) - g = 2c_2\psi^2 + c_1\sqrt{2}\psi + c_0.
\]

This condition, when differentiated once,

\[
K'(x) = \frac{1}{4} \sigma''' - g' = \frac{4c_2\psi}{\sigma} + \frac{\sqrt{2}c_1}{\sigma}
\]

coincides with the result of Theorem (3.4) of [17] for \( c_1 = A/\sqrt{2}, c_2 = B/4 \). Also, Theorems 3.2, 3.4, 3.5 of the same work contain calculations that can be easily obtained in terms of \( I, J \) without having to start from scratch with applying Lie’s method for different coefficients of the equation. For the Lie algebra given in Theorem 3.5, it is sufficient to choose \( c_0 = 0 \) in (2.27). We note a misprint in \( v_2 \): the factor 1/8 of the last term of the \( u \)-coefficient should be 1/4.

An interesting observation here is that condition (3.40), when the set of functions \((c_2, c_1, c_0)\) are eliminated by three successive differentiations with respect to \( x \), turns into a nonlinear ODE of fifth order in \( \sigma(x) \) and fully agrees with the singular invariant equation \( \lambda = 0 \) given in [7].

By (3.40), if we allow \( \sigma \) to be a fixed quadratic normal volatility model

\[
\sigma(x) = \alpha + \beta x + \gamma x^2, \quad \gamma \neq 0,
\]

then \( g(x) \) can be determined as a quadratic polynomial in \( \psi(x) \)

\[
g = g_2\psi^2 + g_1\psi + g_0,
\]

where \( g_2 = -2c_2, g_1 = -\sqrt{2}c_1, g_0 = -(c_0 + 1/8(\beta^2 - 4\alpha\gamma)) \).

For the choice \( c_1 = c_2 = 0, c_0 = -1/8(\beta^2 - 4\alpha\gamma) \) we have \( g = 0 \). This implies that if the quadratic (3.42) has distinct roots \((c_0 \neq 0)\), then the vector fields \( v_2 \) and \( v_3 \) of (2.27) \((v_2 \text{ and } v_1 \text{ with the notation of [17]}\)) will also depend on \( c_0 \). So in that case \( v_1 \text{ and } v_2 \) of [17] are no longer correct and should be rectified. If we choose \( \sigma = \sqrt{2}(1 + k^2 x^2) \) \((c_0 = k^2)\) then this process coincides with Example 3.1. If \( \sigma \) has two equal roots then \( c_0 = 0 \).
When the quadratic $\sigma$ (3.42) has two distinct and multiple real roots two different fundamental solutions have been obtained in [17] by using the action of the symmetry $v_5$ (free of $c_0$) ($v_4$ with the notation of [17]) on the constant solution $u = 1$.

Finally, we remark that with $c_1 = c_2 = 0$ and $g = -h(x)\sigma^2/2$ for some function $h$, our PDE is

$$u_t = \frac{\sigma^2}{2}(u_{xx} + hu) \quad (3.43)$$

and (3.40) has the form

$$\frac{1}{4}(2\sigma\sigma'' - \sigma'^2) + h\sigma^2 = 2c_0. \quad (3.44)$$

General solution of (3.44) is given by

$$\sigma = \alpha\varphi_1^2 + \beta\varphi_1\varphi_2 + \gamma\varphi_2^2, \quad (4\alpha\gamma - \beta^2)W^2 = 8c_0 \quad (3.45)$$

where the set $\{\varphi_1, \varphi_2\}$ is the fundamental set of solutions of the linear ODE $\varphi'' + h\varphi = 0$ and $W$ is its Wronskian [18]. Notice that $\varphi$ coincides with the stationary solution of (3.43).

We recall that Eq. (3.44) for given $p(x)$ and $c_0$ is connected with the Er- makov equation

$$\chi'' + h\chi = 2c_0\chi^{-3} \quad (3.46)$$

through the transformation $\sigma = \chi^2$. We refer to [18] for more details of this equation and its extensions.

In particular, if we set $h = 0$ ($g = 0$), then the fundamental set is $\{1, x\}$ with $W = 1$ and we recover the quadratic volatility (3.42) under the condition $(\alpha\gamma - 4\beta^2) = 2c_0$ as obtained above.

Using one-dimensional subalgebras of (2.27) interesting group invariant solutions of (3.43) with $h = 0$ can be obtained. We recall that the vector fields $(v_1, v_2, v_3)$ span an algebra $\mathfrak{o}(2, 1)$ of pseudo-rotations, up to change of basis, isomorphic to the $\mathfrak{sl}(2, \mathbb{R})$ algebra. For example, solution invariant under the subalgebra $v_4$ (up to a multiplicative constant) of (2.27)

$$v_4 = 2t\sigma\partial_x + (t\sigma' - 2\psi)u\partial_u$$

should have the form

$$u = R(t)\sigma^{1/2}\exp\left[-\frac{\psi^2}{2t}\right]. \quad (3.47)$$

Substituting (3.47) into (3.43) and using the relation (3.44) we find

$$\frac{\dot{R}}{R} = c_0 - \frac{1}{2t}$$

leading to the solution

$$u(x, t) = C \sqrt{\frac{\sigma}{t}} \exp\left[c_0t - \frac{\psi^2}{2t}\right]. \quad (3.48)$$
This solution includes the heat kernel for \( \sigma = \sqrt{2} \) and \( \psi = x/\sqrt{2} \) \((c_0 = 0)\). A case where \( c_0 \neq 0 \) arises, among many others, is \( \sigma = \sin \nu x \cos \nu x \) \((h = \nu^2 \neq 0, c_0 = -\nu^2/8)\) and \( \psi = \nu^{-1} \ln(\tan(\nu x)) \). Then solution (3.48) specializes to

\[
\begin{align*}
  u &= Ct^{-1/2} (\sin 2\nu x)^{1/2} \exp \left[ -\frac{\nu^2 t}{8} - \frac{\ln(\tan(\nu x))^2}{2\nu^2 t} \right].
\end{align*}
\]

In a similar way, invariance under the symmetry

\[
v_5 = \sigma \partial_x + \frac{1}{2} \sigma' u \partial_u
\]

produces the solution

\[
u(x, t) = C \sqrt{\sigma} e^{\cot}. \tag{3.49}
\]

Both invariant solutions (3.48) and (3.49) could be obtained from (2.36) by transforming the constant solution \( \tilde{u} = 1 \) into \( u(x, t) \) by choosing \( T(t) = -1/t \) and \( T(t) = t \), respectively. Moreover, in the special case \( \sigma = \gamma(x-r)^2 \) \((c_0 = 0)\), formula (2.36) gives an SL(2, \( \mathbb{R} \)) group action on a solution \( U(x, t) \) of the heat equation

\[
\begin{align*}
  u &= C(ct + d)^{-1/2}(x-r) \exp \left[ -\frac{d}{2\gamma^2(x-r)^2(ct+d)} \right] U(\tilde{x}, \tilde{t}), \\
  \tilde{x} &= -\frac{\sqrt{2(ad-bc)}}{\gamma(ct+d)(x-r)}, \quad \tilde{t} = M(t) = \frac{at+b}{ct+d}, \quad ad - bc > 0.
\end{align*}
\tag{3.50}
\]

Solution invariant under the dilatational type vector field \( v_2 \) of (2.27)

\[
v_2 = t \partial_t + \frac{1}{2} \psi \partial_x + \frac{1}{4}(4c_0 t + \psi \sigma') u \partial_u
\]

assumes the form

\[
u(x, t) = R(\zeta) \sqrt{\sigma} e^{\cot}, \quad \zeta = \frac{\psi}{\sqrt{t}}. \tag{3.51}
\]

Substitution into (3.43) and taking into account (3.44) we find that \( R \) should satisfy \( R'' + \zeta R' = 0 \) with solution

\[
R = C_1 \text{erf} \left( \frac{\zeta}{\sqrt{2}} \right) + C_2,
\]

where \( \text{erf}(z) \) is the error function defined by

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2} d\eta.
\]

Solution (3.51) includes (3.49) for \( C_1 = 0 \). Taking \( C_2 = 0 \) we obtain the solution

\[
u(x, t) = C \text{erf} \left( \frac{\psi}{\sqrt{2t}} \right) \sqrt{\sigma} e^{\cot} \tag{3.52}
\]

with the initial condition \( u(x, 0^+) = C \sqrt{\sigma} \). Obviously, the error function solution of the heat equation is included in (3.52).
Invariance under the projective vector field

\[ v_3 = t^2 \partial_t + t \sigma \psi \partial_x + \frac{1}{2} [t(2c_0 t - 1) - \psi^2 + t \psi \sigma'] u \partial_u \]

leads to the invariant solution

\[ u = R(\zeta) \sqrt{\frac{\sigma}{\psi}} \exp \left[ c_0 t - \frac{\psi^2}{2t} \right], \quad \zeta = \frac{\psi}{t}. \]

Substituting this into (3.43) and using the relation \(8c_0 + \sigma'^2 - 2\sigma\sigma'' = 0\) we find that \(R\) satisfies the second order linear ODE

\[ 4\zeta^2 R'' - 4\zeta R' + 3R = 0 \]

with general solution \(R = C_1 \zeta^{1/2} + C_2 \zeta^{3/2}\). Finally, the corresponding solution is given by

\[ u = \left\{ C_1 t^{-1/2} + C_2 t^{-3/2} \psi \right\} \sqrt{\frac{\sigma}{\psi}} \exp \left[ c_0 t - \frac{\psi^2}{2t} \right]. \quad (3.53) \]

Obviously, the solution (3.48) is included in this solution.

Finally, we construct a fundamental solution for a multidimensional PDE invariant under rotation.

**Example 3.6.** We wish to solve the Cauchy problem for the PDE

\[ u_t = \Delta u + \left( \frac{\mu}{|x|^2} + \omega^2 |x|^2 \right) u = 0, \quad x \in \mathbb{R}^n, \quad \mu > 0 \quad (3.54) \]

with the condition

\[ u(x, 0; y) = \delta(x - y), \quad y \in \mathbb{R}^n, \]

where \(\Delta(x - y)\) is the usual Laplacian in \(\mathbb{R}^n\) and \(\delta\) is the Dirac distribution with source at \(y\). The case \(n = 1\) was already treated in [19].

Fundamental solution should be invariant under rotations so that we look for a radial solution as fundamental solution in the form \(u(x, t) = u(|x|, t) = u(r, t)\), where \(r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}\) is the radial variable. We require to solve the one-dimensional diffusion PDE

\[ u_t = u_{rr} + \frac{n - 1}{r} u_r + \left( \frac{\mu}{r^2} + \omega^2 r^2 \right) u = 0 \quad (3.55) \]

with the condition

\[ u(r, 0; \rho) = \delta(r - \rho), \quad \rho = |y|. \quad (3.56) \]

We are dealing with a PDE of the form (1.1) with coefficients

\[ a(r) = 1, \quad b(r) = \frac{n - 1}{r}, \quad c(r) = \frac{\mu}{r^2} + \omega^2 r^2. \]

We need the following quantities

\[ I = r, \quad J = -b, \quad IJ = 1 - n, \]
and
\[ K = \left[ \mu - \frac{(n-1)(n-3)}{4} \right] r^{-2} + \omega^2 r^2. \]

The last relation implies that provided \( \mu \neq (n-1)(n-3)/4 \), the symmetry algebra must be four-dimensional.

Comparing this with (2.18) we have
\[ c_0 = c_1 = 0, \quad c_2 = \omega^2, \]

and from (2.25), we find the basis
\[ v_1 = \partial_t, \]
\[ v_2 = \cos(4\omega t)\partial_t - 2\omega \sin(4\omega t)r\partial_r + [n\omega \sin(4\omega t) + 2\omega^2 \cos(4\omega t)r^2]u\partial_u, \]
\[ v_3 = \sin(4\omega t)\partial_t + 2\omega \cos(4\omega t)r\partial_r + [-n\omega \cos(4\omega t) + 2\omega^2 \sin(4\omega t)r^2]u\partial_u, \]
\[ v_4 = u\partial_u \]

with nonzero commutation relations
\[ [v_1, v_2] = -4\omega v_3, \quad [v_1, v_3] = 4\omega v_2, \quad [v_2, v_3] = 4\omega v_1. \]

The subalgebra \( v_1, v_2, v_3 \) is an so(2,1) algebra, which is isomorphic to the sl(2, \( \mathbb{R} \)) algebra. This is seen by making the change of basis
\[ v'_1 = 4\omega(v_1 + v_2), \quad v'_2 = 4\omega v_3 \quad v'_3 = 4\omega(v_1 - v_2) \]

The vector fields \( v'_1, v'_2, v'_3 \) thus form an sl(2, \( \mathbb{R} \)) algebra with the commutation relations
\[ [v'_1, v'_2] = v'_1, \quad [v'_1, v'_3] = 2v'_2, \quad [v'_2, v'_3] = v'_3. \]

The symmetry algebra has the direct-sum structure
\[ \mathfrak{g} = \text{sl}(2, \mathbb{R}) \oplus \{ u\partial_u \}. \]

When \( \mu = (n-1)(n-3)/4 \), the PDE (3.55) has six-dimensional symmetry algebra isomorphic to the heat algebra. In this case, two additional symmetries exist. They are vector fields \( v_4, v_5 \) in the notation of (2.31) given by
\[ v_4 = \cos(2\omega t)\partial_r + \left[ \omega \sin(2\omega t)r - \frac{(n-1)}{2r} \cos(2\omega t) \right]u\partial_u, \]
\[ v_5 = \sin(2\omega t)\partial_r - \left[ \omega \cos(2\omega t)r + \frac{(n-1)}{2r} \sin(2\omega t) \right]u\partial_u. \]

We now pass to find the group invariant solution preserving the boundary conditions \( t = 0, r = \rho \) and (3.56). With these conditions applied to the vector field
\[ v = \sum_{i=1}^{4} k_i v_i, \]
we get the conditions
\[ k_1 + k_2 = 0, \quad k_3 = 0, \quad k_4 - 2k_1 \omega^2 \rho^2 = 0. \]
This leads to the subalgebra generated by the vector field
\[ v = 2 \sin^2(2\omega t) \partial_t + 2 \omega \sin(4\omega t) r \partial_r + \omega [2\omega \rho^2 - 2\omega \cos(4\omega t) r^2 - n \sin(4\omega t)] u \partial_u. \] 
(3.60)
We find two functionally independent invariants of \( v \) by solving the linear PDE \( v(F) = 0 \) by the method of characteristics as
\[ \eta = \frac{r}{\sin(2\omega t)}, \quad \zeta = u^{-1} \sin(2\omega t)^{-n/2} \exp \left[ -\frac{\omega (r^2 + \rho^2)}{\tan(2\omega t)} \right]. \] 
(3.61)
We make the ansatz
\[ u(r, t) = F(\eta) \sin(2\omega t)^{-n/2} \exp \left[ -\frac{\omega (r^2 + \rho^2)}{\tan(2\omega t)} \right]. \] 
(3.62)
If \( u \) solves the original PDE, \( F \) satisfies an ODE of Bessel type
\[ \eta^2 F'' + (n - 1) F' + \left( \mu - \omega^2 \rho^2 \eta^2 \right) F = 0. \] 
(3.63)
Hence we have the solution
\[ F(\eta) = \eta^{(2-n)/2} \left( c_1(\rho) J_\nu(\omega r \rho) + c_2(\rho) J_{-\nu}(\omega r \rho) \right), \quad \nu = \sqrt{\mu - \frac{(n - 2)^2}{4}}, \]
where \( J_\nu(z) \) is the Bessel function of the first kind of order \( \nu \) and we interpret \( J_{-\nu}(z) \) to be \( Y_\nu(z) \) if \( \nu \) is an integer. Since \( J_{-\nu}(z) \) is not integrable near zero for \( \nu \geq 1 \) then we will have to impose \( c_2(\rho) = 0 \). With the choice \( c_1(\rho) = \rho^{(2-n)/2} \) we obtain the fundamental solution
\[ u(r, t; \rho) = \frac{(r \rho)^{(2-n)/2}}{\sin(2\omega t)} \exp \left[ -\frac{\omega (r^2 + \rho^2)}{\tan(2\omega t)} \right] J_\nu \left( \frac{\omega r \rho}{\sin 2\omega t} \right). \] 
(3.64)
In particular, the two-dimensional PDE
\[ u_t = u_{x_1 x_1} + u_{x_2 x_2} + \left( \frac{\mu}{x_1^2 + x_2^2} + \omega^2 (x_1^2 + x_2^2) \right) u, \quad (x_1, x_2) \in \mathbb{R}^2 \] 
(3.65)
has the fundamental solution
\[ u(x_1, x_2, t; y_1, y_2) = \frac{1}{\sin(2\omega t)} \exp \left[ -\frac{\omega (x_1^2 + x_2^2 + y_1^2 + y_2^2)}{\tan(2\omega t)} \right] J_\nu \left( \frac{\omega r \rho}{\sin 2\omega t} \right), \] 
(3.66)
where \( r = \sqrt{x_1^2 + x_2^2} \) and \( \rho = \sqrt{y_1^2 + y_2^2} \).
We note that when \( \omega = 0 \) (\( c_2 = 0 \)), formula (2.21) gives the symmetry algebra
\[ v_1 = \partial_t, \]
\[ v_2 = t \partial_t + \frac{r}{2} \partial_r + \frac{(1 - n)}{4} u \partial_u, \]
\[ v_3 = t^2 \partial_t + t r \partial_r - \frac{1}{4} (r^2 + 2nt) u \partial_u, \]
\[ v_4 = u \partial_u. \] 
(3.67)
The vector fields \( v_1, v_2 - v_4/4, v_3 \) span the algebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) algebra. The corresponding fundamental solution for the one-dimensional version of (3.55) can be shown to be (see [14] for a derivation)

\[
u(x,t;y) = \sqrt{xy} \exp \left\{ -\frac{x^2 + y^2}{4t} \right\} I_\nu \left( \frac{xy}{2t} \right), \quad x > 0, \ y > 0, \tag{3.68}
\]

where \( I_\nu \) denotes the modified Bessel function of the first kind of order \( \nu = \sqrt{1/4 - \mu} \).

Conflict of interest statement
The author declares that there is no conflict of interest.

Data availability statement
The data that support the findings of this work are available within the article.

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