The Discussion on Hulls of Cyclic Codes over the Ring $\mathbb{R} = \mathbb{Z}_4 + v\mathbb{Z}_4$, $v^2 = v$

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Abstract For odd length $n$, the cyclic codes construction over $\mathbb{R} = \mathbb{Z}_4[v]/(v^2 - v)$ is provided. The hulls of cyclic codes over $\mathbb{R}$ are studied. The average 2-dimension $E(n)$ of the hulls of cyclic codes over $\mathbb{R}$ is also conferred. Among these, the various examples of generators of hulls of cyclic codes over $\mathbb{R}$ are provided, whose $\mathbb{Z}_4$-images are good $\mathbb{Z}_4$-linear codes with good parameters.

Keywords Linear codes, Cyclic codes, Self-dual, Hulls.

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1 Introduction

The study of hulls of cyclic codes have crucial importance because of their applications. Therefore the hulls of linear and cyclic codes over finite fields have been well studied. Assmus et al. [14] introduced the theory of hull of linear codes to describe the finite projective planes. In this series, many authors discussed the properties of hulls to describe the complexity of some algorithms in [17,18,19,20,21,22]. Again, Sendrier [16] described the hulls of linear codes of length $n$ over square order field $\mathbb{F}_q$ and proved that the average dimension of the hulls depend on the order of $\mathbb{F}_q$. In [23], Skersys discussed the average dimension $E_q(n)$ of the hulls of cyclic codes over finite fields and studied the character of $\frac{E_q(n)}{n}$, as $n$ tends to infinity. Later, Sangwisut et al. [9]
described the generator polynomials and dimensions of the hulls of cyclic and negacyclic codes over finite field. Recently, Jitman and Sangwisut [7] provided the average dimension of the Hermitian hull of constacyclic codes over \( F_q \) and computed the upper and lower bounds. Furthermore, Jitman et al. [15] introduced the hulls of cyclic codes of odd length over \( Z_4 \) and gave an algorithm to determine types of hulls of cyclic codes over \( Z_4 \). In this article, utilizing the results provided by Jitman et al. [15], we study the construction of hulls for our purpose.

The cyclic codes are important family of linear codes. The study of cyclic codes over finite rings have become a crucial topic of research. The cyclic codes over finite rings have been studied in a series of papers [4,5,10,26,27]. In particular, Hammons et al. [27] done a brilliant work on codes over \( Z_4 \) and discussed \( Z_4 \)-linearity of different types of codes. After that many authors were attracted to do a lots of work over \( Z_4 \). The ring \( R = Z_4[v]/(v^2 - v) \) is an extension of ring \( Z_4 \). For the first time Bandi and Bhaintwal [12] introduced linear codes over finite ring \( R \). They also studied the MacWilliams identities for Lee weight and Gray enumerators for codes over \( R \). After that, Gao et al. [5] studied the different types of linear codes over \( R \) and also obtained many new \( Z_4 \)-linear codes with good parameters. Recently, Dinh et al. [10] introduced a new Gray map and they studied the all-constacyclic codes over \( R \). Moreover, they also obtained the various new \( Z_4 \)-linear codes with better parameters. Further, Kumar and Singh [11] studied the DNA computing by using cyclic codes over \( R \). They provided various examples of reversible cyclic codes over \( R \) and obtained many good \( Z_4 \)-linear codes with good parameters with the help of Gray map. These works encourage us to study the Hulls of cyclic codes over \( R = Z_4 + vZ_4, v^2 = v \).

The remaining part of the article is organised in such manner. In section 2, some basic properties and results related with cyclic codes over \( R \) are conferred. Using the cyclic codes structure, the construction of hulls of cyclic codes over \( R \) is studied in section 3. The types of hulls of cyclic codes over \( R \) are discussed in section 4. Section 5 is devoted to study of average 2-dimension of cyclic codes over \( R \). In section 6, various examples of hulls of cyclic codes over \( R \) are given and we obtain the \( Z_4 \)-linear codes with good parameters according to \( Z_4 \)-database [24].

2 Preliminaries

As a linear code \( C \) of length \( n \) over \( Z_4 \) can be viewed as a vector space over \( F_2 \) and the concept of 2-dimension of \( C \) was introduced in [28] and is given as \( \dim_2(C) = \log_2(|C|) \). Let \( h = (h_0, h_1, \ldots, h_{n-1}), k = (k_0, k_1, \ldots, k_{n-1}) \in R^n \), the inner product is given as

\[
h \cdot k = h_0k_0 + h_1k_1 + \cdots + h_{n-1}k_{n-1}.
\]
The dual code \( C^\perp \) is conferred as
\[
C^\perp = \{ h \mid h \cdot k = 0, \forall k \in C \}.
\]
Then \( C \) is called \textit{self-orthogonal} if \( C \subseteq C^\perp \) and \( C \) is \textit{self-dual} if \( C = C^\perp \). The \textit{hull} of \( C \) is defined as
\[
\text{Hull}(C) = C \cap C^\perp.
\]

Let \( C \) be a cyclic code over \( \mathbb{Z}_4 \) of odd length \( n \). Then generator of \( C \) is given by
\[
C = \langle f(x)g(x), 2f(x)h(x) \rangle = \langle f(x)g(x), 2f(x) \rangle,
\] (1)
where \( f(x), g(x), h(x) \) are unique monic polynomials over \( \mathbb{Z}_4 \) such that \( f(x)g(x)h(x) = x^n - 1 \) ([13], Theorem 12.3.13) and \( |C| = 4^{\deg h(x)}2^{\deg g(x)} \). Here, \( C \) is said to be of type \( 4^{\deg h(x)}2^{\deg g(x)}. \) In this case, the 2-dimension of \( C \) is \( \dim_2(C) = \log_2(|C|) = 2\deg h(x) + \deg g(x). \) A reciprocal polynomial of \( h(x) = a_0 + a_1x + a_2x^2 + \cdots + x^n \) is defined to be
\[
h^*(x) = a_0^{-1}x^{\deg h(x)}h\left(\frac{1}{x}\right).
\]
Clearly, \( (h^*)^*(x) = h(x). \) A polynomial \( h(x) \) is called self-reciprocal if \( h^*(x) = h(x). \) From ([13], Theorem 12.3.20), the generator of dual cyclic code \( C^\perp \) is given as
\[
\langle h^*(x)g^*(x), 2h^*(x)f^*(x) \rangle = \langle h^*(x)g^*(x), 2h^*(x) \rangle
\] (2)

Some results are taken from [13], which are given below. Let \( \text{ord}_j(i) \) denotes the multiplicative order of \( i \) modulo \( j \), where \( i \) and \( j \) are coprime integers. Let \( N_2 := \{ l \geq 1 : l \text{ divides } 2^l + 1 \text{ for some positive integer } l \}. \) The factorization of \( x^n - 1 \) is given as
\[
x^n - 1 = \prod_{j(n,j \in N_2} \left( \prod_{i=1}^{\gamma(j)} g_{ij}(x) \right) \prod_{j(n,j \in N_2} \left( \prod_{i=1}^{\beta(j)} f_{ij}(x)f_{ij}^*(x) \right) \text{ in } \mathbb{Z}_4[x]
\] (3)
where
\[
\gamma(j) = \frac{\phi(j)}{\text{ord}_j(2)}, \quad \beta(j) = \frac{\phi(j)}{2\text{ord}_j(2)}
\]
and \( f_{ij}(x), f_{ij}^*(x) \) form a monic basic irreducible reciprocal polynomial pair and \( g_{ij}(x) \) is a monic basic irreducible self-reciprocal self-reciprocal polynomial. Let \( B_n = \deg \prod_{j(n,j \in N_2} \left( \prod_{i=1}^{\gamma(j)} g_{ij}(x) \right) \). Then
\[
B_n = \deg \prod_{j(n,j \in N_2} \left( \prod_{i=1}^{\gamma(j)} g_{ij}(x) \right) = \sum_{j(n,j \in N_2} \frac{\phi(j)}{\text{ord}_j(2)}\text{ord}_j(2) = \sum_{j(n,j \in N_2} \phi(j). \quad (4)
\]

The Gray map \( \xi \) is taken from [10] that transfers the elements of \( \mathbb{R} \) to elements of \( \mathbb{Z}_4^3 \) such that
\[
\xi(a + bv) = (a, 2b + a).
\]
The Lee weight \( w_L(a) \) in \( \mathbb{Z}_4 \) is given as \( \min\{a, 4-a\} \). The Lee weight of any element of \( \mathbb{R} \) is defined as \( w_L(a + bv) = w_L(a) + w_L(2b + a) \).

**Theorem 1** \[10\] The Gray map \( \xi \) is \( \mathbb{Z}_4 \)-linear, and it is a distance-preserving map from \( \mathbb{R}^n \) (Lee distance) to \( \mathbb{Z}_4^{2n} \) (Lee distance).

### 3 Hulls of Cyclic Codes over \( \mathbb{R} \)

In present section, the theory of hulls of cyclic codes over \( \mathbb{R} \) is discussed. First, we review some useful results, which will be used to describe the hulls of cyclic codes over \( \mathbb{R} \).

**Theorem 3.1.** \[10\] Let \( C \) be a cyclic code of odd length \( n \) over \( \mathbb{Z}_4 \) generated by \( \langle f(x)g(x), 2f(x) \rangle \), where \( x^n - 1 = f(x)g(x)h(x) \) and \( f(x), g(x) \) and \( h(x) \) are pairwise coprime. Then \( \text{Hull}(C) \) is generated by

\[
\langle \text{lcm}(f(x)g(x), h^*(x)g^*(x)), 2 \text{lcm}(f(x), h^*(x)) \rangle
\]

Furthermore, \( \text{Hull}(C) \) is of type \( 4^{\deg H(x)}2^{\deg G(x)} \), where

\[
H(x) = \gcd (h(x), f^*(x)) \quad \text{and} \quad G(x) = \frac{x^n - 1}{\gcd(h(x), f^*(x)) \cdot \text{lcm} (f(x), h^*(x))}.
\]

Some important results of cyclic codes over \( \mathbb{R} \) are given as follows.

**Theorem 3.2.** \[10\] A linear code over \( \mathbb{R} \) is given as \( C = vC_1 \oplus (1 + 3v)C_2 \). Then, \( C \) is cyclic if and only if \( C_1 \) and \( C_2 \) are cyclic over \( \mathbb{Z}_4 \).

**Proposition 3.3.** \[10\] Let \( n \)-tuple \( k \in \mathbb{R}^n \), then \( \xi(\rho(k)) = \rho^2 \xi(k) \), where \( \rho \) is the cyclic shift on \( \mathbb{Z}_4^{2n} \).

**Proposition 3.4.** \[10\] Let \( C \) be a cyclic code of length \( n \) over \( \mathbb{R} \). Then its Gray image \( \xi(C) \) is a 2-quasi-cyclic code of length \( 2n \) over \( \mathbb{Z}_4 \).

Let \( C \) be cyclic code over \( \mathbb{R} \) of odd length \( n \). By utilizing the equation 1, the generator of cyclic codes is described in next result.

**Theorem 3.5.** Let \( C = vC_1 \oplus (1 - v)C_2 \) be a cyclic code of odd length \( n \) over \( \mathbb{R} \). Then

\[
C = \langle vp_1(x)q_1(x), 2vp_1(x), (1 - v)p_2(x)q_2(x), 2(1 - v)p_2(x) \rangle,
\]

where \( p_1(x)q_1(x) \) and \( p_2(x)q_2(x) \) are \( x^n - 1 \) and \( C_1 = \langle p_1(x)q_1(x), 2p_1(x) \rangle \), \( C_2 = \langle p_2(x)q_2(x), 2p_2(x) \rangle \) over \( \mathbb{Z}_4 \), respectively.

**Proof** Let \( \hat{C} = \langle vp_1(x)q_1(x), 2vp_1(x), (1 - v)p_2(x)q_2(x), 2(1 - v)p_2(x) \rangle \), then it is obvious that \( \hat{C} \subseteq \hat{C} \). We have \( vC_1 = vC \) since \( v^2 = v \) over \( \mathbb{Z}_4 \). Moreover, \( (1 - v)^2 = 1 - v \), then \( (1 - v)C_2 = (1 - v)\hat{C} \). Thus, \( vC_1 \oplus (1 - v)C_2 \subseteq \hat{C} \). Hence, \( C = \hat{C} \).
**Proposition 3.6.** [10] Let $C$ be a linear code of length $n$ over $\mathcal{R}$, then $C^\perp = vC^2_1 \oplus (1-v)C^2_2$.

By using the dual cyclic codes $C^\perp$ over $\mathcal{R}$ [given in equation 2], we describe the structure of dual of cyclic codes over $\mathcal{R}$ in next result.

**Proposition 3.7.** If $C = \langle vp_1(x)q_1(x), 2vp_1(x), (1-v)p_2(x)q_2(x), 2(1-v)p_2(x) \rangle$ is a cyclic code of odd length $n$ over $\mathcal{R}$. Then $C^\perp = \langle vr_1^*(x)q_1^*(x), 2vr_1^*(x), (1-v)r_2^*(x)q_2^*(x), 2(1-v)r_2^*(x) \rangle$, where $p^*(x) = x^{\deg p(x)} p(x^{-1})$.

**Proof** If $C$ is a cyclic code, then dual $C^\perp$ is also cyclic code. Moreover, $C^\perp = vC^2_1 \oplus (1-v)C^2_2$ by Proposition 3.6. Thus, equation 2 and Proposition 3.5 imply our result.

**Lemma 3.8.** Let $a = (a_0,a_1,\ldots,a_{n-1})$ and $b = (b_0,b_1,\ldots,b_{n-1})$ be vectors in $\mathbb{R}_n$ with corresponding polynomials $a(x)$ and $b(x)$, respectively. Then $a$ is orthogonal to $b$ and all its shifts if and only if $a(x)b'(x) = 0$ in $\mathbb{R}[x]/(x^n-1)$.

**Proof** Proof is similar to ([13], Theorem 12.3.18).

Next result provide the generator of the hulls of the cyclic codes over $\mathcal{R}$.

**Theorem 3.9.** Let $C$ be cyclic code of odd length $n$ over $\mathcal{R}$ such that $C = vC^1_1 \oplus (1-v)C^1_2$ and $C$ is generated by

$$C = \langle vp_1(x)q_1(x), 2vp_1(x), (1-v)p_2(x)q_2(x), 2(1-v)p_2(x) \rangle,$$

where $p_1(x)q_1(x) = p_2(x)q_2(x) = x^n-1$ and $C_1 = \langle p_1(x)q_1(x), 2p_1(x) \rangle$, $C_2 = \langle p_2(x)q_2(x), 2p_2(x) \rangle$ over $\mathcal{R}$, respectively. Then Hull($C$) has generator of the form $\langle v \text{lcm} (p_1(x)q_1(x), r_1^*(x)q_1^*(x)), 2v \text{lcm} (p_1(x), r_1^*(x)), 2(1-v) \text{lcm} (p_2(x), r_2^*(x)) \rangle$.

In addition, Hull($C$) is of the type $4^{\deg R_1(x)+\deg R_2(x)}2^{\deg Q_1(x)+\deg Q_2(x)}$, where

$$R_1(x) = \gcd (r_1(x), p_1^*(x)), R_2(x) = \gcd (r_2(x), p_2^*(x))$$

and

$$Q_1(x) = \frac{x^n-1}{\gcd (r_1(x), p_1^*(x)) \text{lcm} (p_1(x), r_1^*(x))}, Q_2(x) = \frac{x^n-1}{\gcd (r_2(x), p_2^*(x)) \text{lcm} (p_2(x), r_2^*(x))}.$$

**Proof** From Theorem 3.5, the generator of a cyclic code $C$ over $\mathcal{R}$ is conferred as

$$C = \langle vp_1(x)q_1(x), 2vp_1(x), (1-v)p_2(x)q_2(x), 2(1-v)p_2(x) \rangle,$$

and from Proposition 3.7, the dual of cyclic codes over $\mathcal{R}$ is given as

$$C^\perp = \langle vr_1^*(x)q_1^*(x), 2vr_1^*(x), (1-v)r_2^*(x)q_2^*(x), 2(1-v)r_2^*(x) \rangle.$$
We consider $\bar{C}$ is a cyclic code over $\mathbb{R}$, which has generator of the form

$$\bar{C} = \langle vP_1(x)Q_1(x), 2vP_1(x), (1-v)P_2(x)Q_2(x), 2(1-v)P_2(x) \rangle,$$

where

$$P_1(x) = \text{lcm} \left( p_1(x), r_1^2(x) \right), P_2(x) = \text{lcm} \left( p_2(x), r_2^2(x) \right)$$

and

$$Q_1(x) = \frac{x^n - 1}{\gcd \left( r_1(x), p_1^2(x) \right). \text{lcm} \left( p_1(x), r_1^2(x) \right)}, Q_2(x) = \frac{x^n - 1}{\gcd \left( r_2(x), p_2^2(x) \right). \text{lcm} \left( p_2(x), r_2^2(x) \right)}.$$

$$R_1(x) = \frac{x^n - 1}{\text{lcm} \left( p_1(x)q_1(x), r_1^2(x)q_1^2(x) \right)} = \gcd \left( r_1(x), p_1^2(x) \right),$$

$$R_2(x) = \frac{x^n - 1}{\text{lcm} \left( p_2(x)q_2(x), r_2^2(x)q_2^2(x) \right)} = \gcd \left( r_2(x), p_2^2(x) \right)$$

such that $x^n - 1 = P_1(x)Q_1(x)R_1(x) = P_2(x)Q_2(x)R_2(x)$ and $P_1(x), Q_1(x), R_1(x)$, $P_2(x), Q_2(x), R_2(x)$ are pairwise coprime polynomials. Note that

$$\langle vP_1(x)Q_1(x), 2vP_1(x), (1-v)P_2(x)Q_2(x), 2(1-v)P_2(x) \rangle \subseteq \langle vp_1(x)q_1(x), 2vp_1(x), (1-v)p_2(x)q_2(x), 2(1-v)p_2(x) \rangle$$

and also

$$\langle vP_1(x)Q_1(x), 2vP_1(x), (1-v)P_2(x)Q_2(x), 2(1-v)P_2(x) \rangle \subseteq \langle vr_1^2(x)q_1^2(x), 2vr_1^2(x), (1-v)r_2^2(x)q_2^2(x), 2(1-v)r_2^2(x) \rangle,$$

therefore we get $\bar{C} \subseteq \text{Hull}(C)$.

Now, we have to show that $\text{Hull}(C) \subseteq \bar{C}$. Note that $\text{Hull}(C)$ is a cyclic code over $\mathbb{R}$ and generated by

$$\langle vL_1(x)M_1(x), 2vL_1(x), (1-v)L_2(x)M_2(x), 2(1-v)L_2(x) \rangle,$$

where the polynomials $L_1(x), M_1(x), N_1(x), M_2(x), L_2(x), N_2(x)$ are pairwise coprime such that

$$x^n - 1 = L_1(x)M_1(x)N_1(x) = M_2(x)L_2(x)N_2(x).$$

As the $\text{Hull}(C) \subseteq C^+$ is orthogonal to $C$ and from Lemma 3.8, we have

$$L_1(x)M_1(x)2p_1^2(x) = 0, \quad L_2(x)M_2(x)2p_2^2(x) = 0$$

that means $r_1^2(x)q_1^2(x)\|L_1(x)M_1(x)$, $r_2^2(x)q_2^2(x)\|L_2(x)M_2(x)$ and also

$$2L_1(x)p_1^2(x)q_1^2(x) = 0, \quad 2L_2(x)p_2^2(x)q_2^2(x) = 0$$

which provided that $r_1^2(x)\|L_1(x), r_2^2(x)\|L_2(x)$.

The $\text{Hull}(C) \subseteq C$ is orthogonal to $C^+$ and from Lemma 3.8, we have

$$L_1(x)M_1(x)2r_1(x) = 0$$

that means $p_1(x)q_1(x)\|L_1(x)M_1(x), p_2(x)q_2(x)\|L_2(x)M_2(x)$ and also

$$2L_1(x)r_1(x)q_1(x) = 0, \quad 2L_2(x)r_2(x)q_2(x) = 0,$$

that gives $p_1(x)\|L_1(x)$ and $p_2(x)\|L_2(x)$.

Therefore, $\text{lcm}(p_1(x)q_1(x), r_1^2(x)q_1^2(x)\|L_1(x)M_1(x), \text{lcm}(p_2(x)q_2(x), r_2^2(x)q_2^2(x)\|L_2(x)M_2(x)$ and

$$\text{lcm}(r_1^2(x), p_1^2(x)\|L_1(x), \text{lcm}(r_2^2(x), p_2^2(x)\|L_2(x).$$

That means $P_1(x)Q_1(x)\|L_1(x)M_1(x), P_2(x)Q_2(x)\|L_2(x)M_2(x)$ and $P_1(x)\|L_1(x), P_2(x)\|L_2(x)$.

Hence, $\text{Hull}(C) \subseteq \bar{C}$. Thus, $\text{Hull}(C) = \bar{C}$. 
4 Types of Hulls of Cyclic Codes over \( \mathcal{R} \)

Present section devotes to study the types of hulls of cyclic codes of odd length \( n \) over \( \mathcal{R} \). For this, first we discuss some results.

**Lemma 4.1** [15] Let \( \beta \) be a positive integer. For \( 1 \leq i \leq \beta \), let \( (v_i, z_i), (w_i, d_i) \) and \( (u_i, b_i) \) be elements in \( \{(0,0), (1,0), (0,1)\} \). Let \( a_i = \min \{1 - v_i - z_i, w_i\} + \min \{1 - w_i - d_i, v_i\} \). Then \( a_i \in \{0,1\} \). Moreover, the following statements hold.

1. \( 2 - \min \{1 - v_i - z_i, w_i\} - \max \{v_i, 1 - w_i - d_i\} - \min \{1 - w_i - d_i, v_i\} - \max \{w_i, 1 - v_i - z_i\} = z_i + d_i \).
2. If \( a_i = 0 \), then \( z_i + d_i \in \{0, 1, 2\} \).
3. If \( a_i = 1 \), then \( z_i + d_i = 0 \).
4. Let \( a = \sum_{i=1}^{n} a_{ij} \), then \( \sum_{i=1}^{n} (z_i + d_i) = c \) for some \( 0 \leq c \leq 2(\beta - a) \).

**Theorem 4.2** The types of the hull of a cyclic code of odd length \( n \) over \( \mathcal{R} \) are \( 4^{k_1} \cdot 2^{k_2} \), where

\[
k_1 = \sum_{j \in n,j \notin N_2} \text{ord}_j(2) a_{1j} + \sum_{j \in n,j \notin N_2} \text{ord}_j(2) a_{2j}
\]

\[
k_2 = \sum_{j \in n,j \notin N_2} \text{ord}_j(2) b_{1j} + \sum_{j \in n,j \notin N_2} \text{ord}_j(2) b_{2j} + \sum_{j \in n,j \notin N_2} \text{ord}_j(2) c_{1j} + \sum_{j \in n,j \notin N_2} \text{ord}_j(2) c_{2j},
\]

\( 0 \leq a_{1j}, a_{2j} \leq \beta(j), 0 \leq b_{1j}, b_{2j} \leq \gamma(j) \) and \( 0 \leq c_{1j} \leq 2(\beta(j) - a_{1j}), 0 \leq c_{2j} \leq 2(\beta(j) - a_{2j}) \).

**Proof** From Theorem 3.9, Hull(C) has type

\( q^{\deg R_1(x) + \deg R_2(x)} 2^{\deg Q_1(x) + \deg Q_2(x)} \).

From equation (3), we get

\[
p_k(x) = \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\gamma(j)} q_{ij}(x)^{w_{k_{ij}}} \right) \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\beta(j)} p_{k_{ij}}(x)^{v_{k_{ij}}} p_{k_{ij}^*}(x)^{w_{k_{ij}}} \right)
\]

\[
q_k(x) = \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\gamma(j)} q_{ij}(x)^{b_{k_{ij}}} \right) \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\beta(j)} p_{k_{ij}}(x)^{z_{k_{ij}}} p_{k_{ij}^*}(x)^{d_{k_{ij}}} \right)
\]

\[
r_k(x) = \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\gamma(j)} q_{ij}(x)^{1-w_{k_{ij}}-b_{k_{ij}}} \right) \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\beta(j)} p_{k_{ij}}(x)^{1-v_{k_{ij}}-z_{k_{ij}}} p_{k_{ij}^*}(x)^{1-w_{k_{ij}}-d_{k_{ij}}} \right),
\]

and

\[
p_k^*(x) = \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\gamma(j)} q_{ij}(x)^{w_{k_{ij}}} \right) \prod_{j \in n,j \notin N_2} \left( \prod_{i=1}^{\beta(j)} p_{k_{ij}}(x)^{w_{k_{ij}}} p_{k_{ij}^*}(x)^{v_{k_{ij}}} \right)
\]
\[ q_k^*(x) = \prod_{j | n, j \in N_2} \gamma(j) q_{kij}(x)^{b_{kij}} \prod_{j | n, j \notin N_2} \beta(j) p_{kij}(x)^{d_{kij}^* f_{kij}(x)^{z_{kij}}} \]

\[ r_k^*(x) = \prod_{j | n, j \in N_2} \gamma(j) q_{kij}(x)^{1-u_{kij}-b_{kij}} \prod_{j | n, j \notin N_2} \beta(j) p_{kij}(x)^{1-w_{kij}-d_{kij}^* f_{kij}(x)^{1-v_{kij}-z_{kij}}} \]

where \( k = 1, 2 \) and \((u_{kij}, b_{kij}), (v_{kij}, z_{kij}), (w_{kij}, d_{kij}) \in \{(0, 0), (1, 0)(0, 1)\}.

According to above relations, we have

\[ R_k(x) = \gcd(r_k(x), p_k^*(x)) \]

\[ = \prod_{j | n, j \in N_2} \gamma(j) q_{kij}(x)^{\min\{1-u_{kij}-b_{kij}, w_{kij}\}} \prod_{j | n, j \notin N_2} \beta(j) p_{kij}(x)^{\min\{1-v_{kij}-z_{kij}, w_{kij}\}} \]

\[ \times p_{kij}^*(x)^{\min\{1-w_{kij}-d_{kij}, v_{kij}\}} \]

\[ = \prod_{j | n, j \notin N_2} \beta(j) q_{kij}(x)^{\min\{1-v_{kij}-z_{kij}, w_{kij}\}} p_{kij}^*(x)^{\min\{1-w_{kij}-d_{kij}, v_{kij}\}} \]

that means

\[ \deg R_k(x) = \deg \gcd(r_k(x), p_k^*(x)) \]

\[ = \deg \prod_{j | n, j \notin N_2} \beta(j) q_{kij}(x)^{\min\{1-v_{kij}-z_{kij}, w_{kij}\}} p_{kij}^*(x)^{\min\{1-w_{kij}-d_{kij}, v_{kij}\}} \]

\[ = \sum_{j | n, j \notin N_2} \operatorname{ord}_j (2) \sum_{i=1}^{\beta(j)} \left( \min\{1-v_{kij}-z_{kij}, w_{kij}\} + \min\{1-w_{kij}-d_{kij}, v_{kij}\} \right) \]

\[ = \sum_{j | n, j \notin N_2} \operatorname{ord}_j (2) \sum_{i=1}^{\beta(j)} \min\{1-v_{kij}-z_{kij}, w_{kij}\} \]

\[ = \sum_{j | n, j \notin N_2} \operatorname{ord}_j (2) \cdot a_{kij}, \text{ where } 0 \leq a_{kij} \leq 1. \]

\[ = \sum_{j | n, j \notin N_2} \operatorname{ord}_j (2) \cdot a_{kij}, \text{ where } 0 \leq a_{kj} \leq \beta(j). \]
The Discussion on Hulls of Cyclic Codes over the Ring $\mathbb{R} = \mathbb{Z}_4 + v\mathbb{Z}_4$, $v^2 = v$

In the similar way, we determine the $\deg Q_k(x)$, where $k = 1, 2$.

$$Q_K(x) = \frac{x^n - 1}{\gcd (R_k(x), p_k^*(x)) \cdot \text{lcm} (p_k(x), R_k^*(x))}$$

$$= \prod_{j|n,j \in N_2} \gamma(j) \prod_{i=1}^q q_{kij}(x)^{1 - \max\{u_{kij}, 1 - u_{kij} - b_{kij}\}}$$

$$\times \prod_{j|n,j \notin N_2} \beta(j) \prod_{i=1}^p p_{kij}(x)^{\min\{1 - v_{kij} - z_{kij}, w_{kij} - \max\{v_{kij}, 1 - w_{kij} - d_{kij}\}\}}$$

$$\times p_{kij}^*(x) \left[1 - \min\{1 - w_{kij} - d_{kij} - v_{kij}\} - \max\{w_{kij}, 1 - v_{kij} - z_{kij}\}\right]$$

Thus,

$$\deg Q_k(x) = \deg \prod_{j|n,j \in N_2} \gamma(j) \prod_{i=1}^q q_{kij}(x)^{1 - \max\{u_{kij}, 1 - u_{kij} - b_{kij}\}}$$

$$\times \prod_{j|n,j \notin N_2} \beta(j) \prod_{i=1}^p p_{kij}(x)^{\min\{1 - v_{kij} - z_{kij}, w_{kij} - \max\{v_{kij}, 1 - w_{kij} - d_{kij}\}\}}$$

$$\times p_{kij}^*(x) \left[1 - \min\{1 - w_{kij} - d_{kij} - v_{kij}\} - \max\{w_{kij}, 1 - v_{kij} - z_{kij}\}\right]$$

$$= \sum_{j|n,j \in N_2} \text{ord}_j(2) \sum_{i=1}^q (1 - \max\{u_{kij}, 1 - u_{kij} - b_{kij}\})$$

$$+ \sum_{j|n,j \notin N_2} \text{ord}_j(2) \sum_{i=1}^p (2 - \min\{1 - v_{kij} - z_{kij}, w_{kij}\}$$

$$- \max\{w_{kij}, 1 - v_{kij} - z_{kij}\} - \min\{1 - w_{kij} - d_{kij}, v_{kij}\}$$

$$- \max\{v_{kij}, 1 - w_{kij} - d_{kij}\})$$

$$= \sum_{j|n,j \in N_2} \text{ord}_j(2) \sum_{i=1}^q (1 - \max\{u_{kij}, 1 - u_{kij} - b_{kij}\})$$

$$+ \sum_{j|n,j \notin N_2} \text{ord}_j(2) \sum_{i=1}^p (z_{kij} + d_{kij})$$

$$= \sum_{j|n,j \in N_2} \text{ord}_j(2) \cdot b_{kj} + \sum_{j|n,j \notin N_2} \text{ord}_j(2) \cdot c_{kj}$$. 
From Theorem 3.9, Hull(C) is of the type $4^{\deg R_1(x) + \deg R_2(x)}2^{\deg Q_1(x) + \deg Q_2(x)}$, where

\[ k_1 = \deg R_1(x) + \deg R_2(x) \]
\[ = \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot a_{1j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot a_{2j} \]
and
\[ k_2 = \deg Q_1(x) + \deg Q_2(x) \]
\[ = \sum_{j|n,j \in \mathbb{N}_2} \ord_j(2) \cdot b_{1j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot b_{2j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot c_{2j}, \]

where $0 \leq a_{1j}, a_{2j} \leq \beta(j), 0 \leq b_{1j}, b_{2j} \leq \gamma(j)$ and $0 \leq c_{1j} \leq 2(\beta(j) - a_{1j}), 0 \leq c_{2j} \leq 2(\beta(j) - a_{2j})$.

**Corollary 4.3** For odd length $n$ such that $n \in \mathbb{N}_2$. Then the types of the hull of a cyclic codes over $\mathbb{R}$ are of the form $4^b2^k$, where

\[ k_2 = \sum_{j|n,j \in \mathbb{N}_2} \ord_j(2) \cdot b_{1j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot b_{2j}, \]
\[ 0 \leq b_{1j}, b_{2j} \leq \gamma(j) \]

**Proof** From the Theorem 4.2, if $n \in \mathbb{N}_2$ and $j|n,j \in \mathbb{N}_2$, we get the required result.

**Algorithm:** The types of the hull of a cyclic code of odd length $n$ over $\mathbb{R}$.

1. For each $j|n$, consider the following cases.
   (a) If $j \in \mathbb{N}_2$, then compute $\ord_j(2)$ and $\gamma(j)$.
   (b) If $j \not\in \mathbb{N}_2$, then compute $\ord_j(2)$ and $\beta(j)$.
2. Compute
\[ k_1 = \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot a_{1j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot a_{2j}, \]
where $0 \leq a_{1j}, a_{2j} \leq \beta(j)$.
3. Next, we determine $k_2$ for fixed $a_{1j}, a_{2j}$ in 2,
\[ k_2 = \sum_{j|n,j \in \mathbb{N}_2} \ord_j(2) \cdot b_{1j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot b_{2j} + \sum_{j|n,j \not\in \mathbb{N}_2} \ord_j(2) \cdot c_{2j}, \]
where $0 \leq b_{1j}, b_{2j} \leq \gamma(j)$ and $0 \leq c_{1j} \leq 2(\beta(j) - a_{1j}), 0 \leq c_{2j} \leq 2(\beta(j) - a_{2j})$.

**Example 1:**
For \( n = 7 \), we discuss the types of hulls of cyclic codes over \( R \). If \( 1 \in N_2 \), then \( \text{ord}_1(2) = 1 \), and \( \gamma(1) = 1 \) and if \( 7 \notin N_2 \), then \( \text{ord}_7(2) = 3 \) and \( \beta(7) = 1 \).

Which implies \( k_1 = 3a_{17} + 3a_{27} \), with \( 0 \leq (a_{17}, a_{27}) \leq 1 \). Then we have following cases.

1. Let \( (a_{17}, a_{27}) = (0, 0) \), i.e. \( k_1 = 0 \) and
   \[
   k_2 = b_{11} + b_{21} + 3b_{17} + 3c_{27}, \text{ with } 0 \leq (b_{11}, b_{21}) \leq 1 \text{ and } 0 \leq c_{17}, c_{27} \leq 2.
   \]
   Then \( k_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) \)

2. Let \( (a_{17}, a_{27}) = (1, 0) \), i.e. \( k_1 = 3 \) and
   \[
   k_2 = b_{11} + b_{21} + 3b_{17} + 3c_{27}, \text{ with } 0 \leq (b_{11}, b_{21}) \leq 1 \text{ and } c_{17} = 0, 0 \leq c_{27} \leq 2.
   \]
   Then \( k_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8) \)

3. Let \( (a_{17}, a_{27}) = (0, 1) \), i.e. \( k_1 = 3 \) and
   \[
   k_2 = b_{11} + b_{21} + 3b_{17} + 3c_{27}, \text{ with } 0 \leq (b_{11}, b_{21}) \leq 1 \text{ and } c_{27} = 0, 0 \leq c_{17} \leq 2.
   \]
   Then \( k_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8) \)

4. Let \( (a_{17}, a_{27}) = (1, 1) \), i.e. \( k_1 = 6 \) and
   \[
   k_2 = b_{11} + b_{21} + 3b_{17} + 3c_{27}, \text{ with } 0 \leq (b_{11}, b_{21}) \leq 1 \text{ and } c_{17}, c_{27} = 0.
   \]
   Then \( k_2 = (0, 1, 2) \)

**Example 2:**

For \( n = 15 \), we discuss the types of hulls of cyclic codes over \( R \). Let \( 1, 3, 5 \in N_2 \), we get \( \text{ord}_3(2) = 1, \text{ord}_5(2) = 2, \text{ord}_5(2) = 4, \) and \( \gamma(1) = \gamma(3) = \gamma(5) = 1 \) and \( 15 \notin N_2 \), then \( \text{ord}_{15}(2) = 4 \) and \( \beta(15) = 1 \).

Which implies \( k_1 = 4a_{115} + 4a_{215} \), with \( 0 \leq (a_{115}, a_{215}) \leq 1 \). Then, we have following cases.

1. Let \( (a_{115}, a_{215}) = (0, 0) \), i.e. \( k_1 = 0 \) and
   \[
   k_2 = b_{11} + b_{21} + 2b_{13} + 2b_{23} + 4b_{15} + 4b_{25} + 4c_{115} + 4c_{215}, \text{ with } 0 \leq (b_{11}, b_{21}, b_{13}, b_{23}, b_{15}, b_{25}) \leq 1 \text{ and } 0 \leq c_{115}, c_{215} \leq 2.
   \]
   Then \( k_2 = (0, 1, 2, \ldots, 30) \)

2. Let \( (a_{115}, a_{215}) = (1, 0) \), i.e. \( k_1 = 4 \) and
   \[
   k_2 = b_{11} + b_{21} + 2b_{13} + 2b_{23} + 4b_{15} + 4b_{25} + 4c_{115} + 4c_{215}, \text{ with } 0 \leq (b_{11}, b_{21}, b_{13}, b_{23}, b_{15}, b_{25}) \leq 1 \text{ and } c_{115} = 0, 0 \leq c_{215} \leq 2.
   \]
   Then \( k_2 = (0, 1, 2, \ldots, 22) \)

3. Let \( (a_{115}, a_{215}) = (0, 1) \), i.e. \( k_1 = 4 \) and
   \[
   k_2 = b_{11} + b_{21} + 2b_{13} + 2b_{23} + 4b_{15} + 4b_{25} + 4c_{115} + 4c_{215}, \text{ with } 0 \leq (b_{11}, b_{21}, b_{13}, b_{23}, b_{15}, b_{25}) \leq 1 \text{ and } 0 \leq c_{115} \leq 2, c_{215} = 0.
   \]
   Then \( k_2 = (0, 1, 2, \ldots, 22) \)

4. Let \( (a_{115}, a_{215}) = (1, 1) \), i.e. \( k_1 = 8 \) and
   \[
   k_2 = b_{11} + b_{21} + 2b_{13} + 2b_{23} + 4b_{15} + 4b_{25} + 4c_{115} + 4c_{215}, \text{ with } 0 \leq (b_{11}, b_{21}, b_{13}, b_{23}, b_{15}, b_{25}) \leq 1 \text{ and } c_{115} = c_{215} = 0.
   \]
   Then \( k_2 = (0, 1, 2, \ldots, 14) \)
5 The Average 2-Dimension $E(n)$

In this section, the average 2-dimension of the cyclic codes of length $n$ over $\mathbb{R}$ is discussed. The average 2-dimension of the cyclic codes over $\mathbb{Z}_4$ is given as

$$E(n) = \sum_{C \in C(n, 4)} \frac{\text{deg}_2(\text{Hull}(C))}{|C(n, 4)|},$$

(5)

where $C(n, 4)$ denotes the set of all cyclic codes over $\mathbb{Z}_4$.

**Lemma 5.1** [15] Let $(v, z), (w, d), (u, b) \in \{(0, 0), (1, 0), (0, 1)\}$. Then

1. $E(1 - \max\{u, 1 - u + b\}) = \frac{1}{3}$
2. $E(2 + \min\{1 - v - z, w\} - \max\{v, 1 - w - d\} + \min\{1 - w - d, v\} - \max\{w, 1 - v - z\}) = \frac{10}{9}$

Next, we study the formula of average 2-dimension of cyclic codes over $\mathbb{R}$ by utilizing above result and the expectation $E(Y)$, where $Y$ is the random variable of the 2-dimension $\text{dim}_2(\text{Hull}(C))$ and $C$ is chosen randomly from $C(n, 4)$ with uniform probability.

**Theorem 5.2** The value of $E(n)$ of hulls of cyclic codes of odd length $n$ over $\mathbb{R}$ is conferred as

$$E(n) = \frac{10n}{9} - \frac{4B_n}{9},$$

where $B_n$ is given in Equation (4).

**Proof** The structure of cyclic codes $C$ of odd length $n$ over $\mathbb{R}$ is determined as

$$C = \langle vp_1(x)q_1(x), 2vp_1(x), (1 - v)p_2(x)q_2(x), 2(1 - v)p_2(x) \rangle,$$

where $p_1(x)q_1(x)r_1(x) = p_2(x)q_2(x)r_2(x) = x^n - 1$, where for $i = 1, 2$, $p_i(x), q_i(x)$ and $r_i(x)$ are pairwise coprime polynomials over $\mathbb{Z}_4$. The Hull($C$) has type

$$4^{\deg R_1(x)} + \deg R_2(x) + 2^{\deg Q_1(x)} + \deg Q_2(x)$$

and 2-dimension of Hull($C$) is

$$2(\deg R_1(x) + \deg R_2(x)) + \deg Q_1(x) + \deg Q_2(x)$$
\[ \dim_2(\text{Hull}(C)) = 2(\deg R_1(x) + \deg R_2(x)) + \deg Q_1(x) + \deg Q_2(x) \]

\[
= 2\left( \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \left( \min\{1 - v_{1ij} - z_{1ij}, w_{1ij}\} + \min\{1 - w_{1ij} - d_{1ij}, v_{1ij}\} \right) \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \left( \min\{1 - v_{2ij} - z_{2ij}, w_{2ij}\} + \min\{1 - w_{2ij} - d_{2ij}, v_{2ij}\} \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \left( 1 - \max\{u_{1ij}, 1 - u_{1ij} - b_{1ij}\} \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \left( 2 - \min\{1 - v_{1ij} - z_{1ij}, w_{1ij}\} - \max\{w_{1ij}, 1 - v_{1ij} - z_{1ij}\} \right) \\
- \min\{1 - w_{1ij} - d_{1ij}, v_{1ij}\} - \max\{v_{1ij}, 1 - w_{1ij} - d_{1ij}\} \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \left( 2 - \min\{1 - v_{2ij} - z_{2ij}, w_{2ij}\} - \max\{w_{2ij}, 1 - v_{2ij} - z_{2ij}\} \right) \\
- \min\{1 - w_{2ij} - d_{2ij}, v_{2ij}\} - \max\{v_{2ij}, 1 - w_{2ij} - d_{2ij}\} \right) \\
= \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \left( 2 + \min\{1 - v_{1ij} - z_{1ij}, w_{1ij}\} - \max\{v_{1ij}, 1 - w_{1ij} - d_{1ij}\} \right) \\
+ \min\{1 - w_{1ij} - d_{1ij}, v_{1ij}\} - \max\{w_{1ij}, 1 - v_{1ij} - z_{1ij}\} \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \left( 1 - \max\{u_{2ij}, 1 - u_{2ij} - b_{2ij}\} \right) \\
+ \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \left( 2 + \min\{1 - v_{2ij} - z_{2ij}, w_{2ij}\} - \max\{v_{2ij}, 1 - w_{2ij} - d_{2ij}\} \right) \\
+ \min\{1 - w_{2ij} - d_{2ij}, v_{2ij}\} - \max\{w_{2ij}, 1 - v_{2ij} - z_{2ij}\} \right) \]
Next, utilizing the Lemma 5.1, we have

\[
E(n) = E(Y) \\
= E(2k_1 + k_2) \\
= E(2(\deg R_3(x) + \deg R_2(x)) + \deg Q_1(x) + \deg Q_2(x)) \\
= E\left(\sum_{j,n,j \in N_2} \gamma(j) \sum_{i=1}^{\deg j} \left(1 - \max\{u_{1ij}, 1 - u_{1ij} - b_{1ij}\}\right)\right) \\
+ E\left(\sum_{j,n,j \notin N_2} \delta(j) \sum_{i=1}^{\deg j} \left(2 + \min\{1 - v_{1ij} - z_{1ij}, w_{1ij}\} - \max\{v_{1ij}, 1 - w_{1ij} - d_{1ij}\} + \min\{1 - w_{1ij} - d_{1ij}, v_{1ij}\} - \max\{w_{1ij}, 1 - v_{1ij} - z_{1ij}\}\right)\right) \\
+ E\left(\sum_{j,n,j \notin N_2} \gamma(j) \sum_{i=1}^{\deg j} \left(1 - \max\{u_{2ij}, 1 - u_{2ij} - b_{2ij}\}\right)\right) \\
+ E\left(\sum_{j,n,j \notin N_2} \delta(j) \sum_{i=1}^{\deg j} \left(2 + \min\{1 - v_{2ij} - z_{2ij}, w_{2ij}\} - \max\{v_{2ij}, 1 - w_{2ij} - d_{2ij}\} + \min\{1 - w_{2ij} - d_{2ij}, v_{2ij}\} - \max\{w_{2ij}, 1 - v_{2ij} - z_{2ij}\}\right)\right) \\
= \sum_{j,n,j \in N_2} \ord_j(2) \cdot \gamma(j) \cdot E\left(1 - \max\{u_{1ij}, 1 - u_{1ij} - b_{1ij}\}\right) \\
+ \sum_{j,n,j \notin N_2} \ord_j(2) \cdot \beta(j) \cdot \left(2 + \min\{1 - v_{1ij} - z_{1ij}, w_{1ij}\} - \max\{v_{1ij}, 1 - w_{1ij} - d_{1ij}\} + \min\{1 - w_{1ij} - d_{1ij}, v_{1ij}\} - \max\{w_{1ij}, 1 - v_{1ij} - z_{1ij}\}\right) \\
+ \sum_{j,n,j \notin N_2} \ord_j(2) \cdot \gamma(j) \cdot E\left(1 - \max\{u_{2ij}, 1 - u_{2ij} - b_{2ij}\}\right) \\
+ \sum_{j,n,j \notin N_2} \ord_j(2) \cdot \beta(j) \cdot \left(2 + \min\{1 - v_{2ij} - z_{2ij}, w_{2ij}\} - \max\{v_{2ij}, 1 - w_{2ij} - d_{2ij}\} + \min\{1 - w_{2ij} - d_{2ij}, v_{2ij}\} - \max\{w_{2ij}, 1 - v_{2ij} - z_{2ij}\}\right) \\
= \sum_{j,n,j \in N_2} \phi(j) \cdot \frac{1}{3} + \sum_{j,n,j \notin N_2} \phi(j) \cdot \frac{10}{9} + \sum_{j,n,j \notin N_2} \phi(j) \cdot \frac{1}{3} + \sum_{j,n,j \notin N_2} \phi(j) \cdot \frac{10}{9} \quad \text{using Lemma 5.1} \\
= \frac{B_n}{3} + \frac{5(n - B_n)}{9} + \frac{B_n}{3} + \frac{5(n - B_n)}{9} \quad \text{from equation 4} \\
= \frac{10n}{9} - \frac{4B_n}{9}
\]
Corollary 5.3 In the Theorem 5.2, we have $E(n) < \frac{10n}{9}$.

In Table 1, we are given the average 2-dimension $E(n)$ of the hull of cyclic codes of odd length $n$ from 55 upto 151. The (†) denotes the condition the $n \in N_2$ and otherwise $n \notin N_2$ in the Table 1. The Table 1 is given after the references.

Table 1.

6 Computational work

In this section, we have provided the various examples of hulls of cyclic codes of odd length over $\mathbb{R}$. Among these, the generator polynomials of hulls of cyclic codes over $\mathbb{R}$ are obtained. We know that cyclic codes over $\mathbb{R}$ are mapped via Gray map into $\mathbb{Z}_4^4$ and these Gray images are $\mathbb{Z}_4$-linear codes. Thus, the Lee weights of the $\mathbb{Z}_4$-images are obtained. Various good $\mathbb{Z}_4$-linear codes with good parameters are obtained according to data-base[24]. Here, the monic polynomials are considered in ascending order such that $x^4 + 3x^3 + 2x^2 + 1$ as 11231 in Tables 2, 3. The generator of hulls of cyclic codes over $\mathbb{R}$ are obtained in following manner such that $(3 + 2v) + (1 + 2v)x + vx^2 + (2 + v)x^3 + 2vx^4 = (31020) + v(22112)$ in Tables 2, 3. The (*) denotes good $\mathbb{Z}_4$-parameter.

Example 3. Let $n = 15$ and

$$x^{15} - 1 = (x-1)(x^2+x+1)(x^4+x^3-x^2+x+1)(x^4+3x^3+2x^2+1)(x^4+2x^2+3x+1)$$

over $\mathbb{Z}_4$. In Table 2, the generator of hulls of cyclic codes over $\mathbb{R}$ are obtained.

Table 2.
Example 4. Let \( n = 21 \) and \( x^{21} - 1 = (x - 1)(x^2 + x + 1)(x^6 + 2x^5 + 3x^4 + 3x^2 + x + 1)(x^6 + x^3 + 3x^4 + 3x^2 + 2x + 1)(x^3 + 2x^2 + x + 3)(x^3 + 3x^2 + 2x + 3) \) over \( \mathbb{Z}_4 \). In Table 3, the generator of hulls of cyclic codes over \( \mathbb{R} \) are obtained.
In this article, the construction of cyclic codes of odd length over $\mathbb{R} = \mathbb{Z}_4 + v\mathbb{Z}_4$, $v^2 = v$ is conferred. The $v^{th}$ powers of cyclic codes over $\mathbb{R}$ are studied. Moreover, the types of cyclic codes of even length over $\mathbb{R}$ will be another open problem.

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| n   | $B(n)$ | $E(n) = \frac{10n - 4B_n}{9}$ | n   | $B(n)$ | $E(n) = \frac{10n - 4B_n}{9}$ |
|-----|--------|-------------------------------|-----|--------|-------------------------------|
| 55  | 15     | $\frac{455}{9}$              | 105 | 13     | $\frac{938}{9}$              |
| 57† | 21     | 54                            | 107†| 107    | $\frac{642}{9}$              |
| 59† | 59     | $\frac{554}{9}$              | 109†| 109    | $\frac{644}{9}$              |
| 61† | 61     | $\frac{596}{9}$              | 111 | 39     | 106                           |
| 63  | 7      | $\frac{692}{9}$              | 113†| 113    | $\frac{678}{9}$              |
| 65† | 17     | $\frac{582}{9}$              | 115 | 5      | $\frac{1130}{9}$             |
| 67† | 67     | $\frac{1022}{9}$             | 117 | 19     | $\frac{1094}{9}$             |
| 69  | 3      | $\frac{678}{9}$              | 119 | 17     | $\frac{1122}{9}$             |
| 71  | 1      | $\frac{706}{9}$              | 121†| 111    | $\frac{706}{9}$              |
| 73  | 1      | $\frac{126}{9}$              | 123 | 43     | $\frac{1158}{9}$             |
| 75  | 23     | $\frac{658}{9}$              | 125†| 101    | $\frac{846}{9}$              |
| 77  | 11     | $\frac{126}{9}$              | 127 | 1      | $\frac{1266}{9}$             |
| 79  | 1      | $\frac{126}{9}$              | 129†| 45     | $\frac{1110}{9}$             |
| 81† | 55     | $\frac{204}{9}$              | 131†| 131    | $\frac{786}{9}$              |
| 83† | 83     | $\frac{198}{9}$              | 133 | 19     | $\frac{1259}{9}$             |
| 85  | 21     | $\frac{166}{9}$              | 135 | 23     | $\frac{1258}{9}$             |
| 87  | 31     | $\frac{46}{9}$               | 137†| 137    | $\frac{822}{9}$              |
| 89  | 1      | $\frac{586}{9}$              | 139†| 139    | $\frac{834}{9}$              |
| 91  | 13     | $\frac{308}{9}$              | 141 | 3      | $\frac{1398}{9}$             |
| 93  | 3      | 102                            | 143 | 23     | $\frac{1538}{9}$             |
| 95  | 23     | $\frac{588}{9}$              | 145†| 33     | $\frac{1348}{9}$             |
| 97† | 97     | $\frac{582}{9}$              | 147 | 45     | $\frac{1290}{9}$             |
| 99† | 17     | $\frac{722}{9}$              | 149†| 149    | $\frac{894}{9}$              |
| 101†| 101    | $\frac{906}{9}$              | 151 | 1      | $\frac{1306}{9}$             |
| 103 | 1      | 114                            | 153 | 23     | $\frac{1438}{9}$             |