PROPERTIES OF PARABOLA-INSERBED
PONCELET POLYGONS

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ABSTRACT. We investigate parabola-inscribed Poncelet $N$-gon families, enveloping a focus-centered circular caustic. For all $N$ studied ($N = 3, 4, 5, 6$), loci of vertex, perimeter, and area centroids are parabolas. We also study the hyperbola-inscribed family of polar polygons with respect to the parabola. We describe many curious loci of its triangle centers for $N = 3$. For all $N$s studied, we notice the locus of vertex and perimeter centroids of the polar family are straight lines while that of the perimeter centroid is a non-conic. Supported by experiments, we conjecture that the centroidal behavior seen in the examples studied herein can be generalized for all $N$.

1. INTRODUCTION

This is a continuation of our investigation of Euclidean phenomena of Poncelet families [10, 11, 13, 19]. Recall Poncelet’s porism: specially-chosen pairs of conics $C, C'$ admit a 1d family of polygons inscribed in $C$ while simultaneously circumscribing $C'$ [5, 7, 8].

Here we investigate a special kind of Poncelet family, where $C$ is a parabola $P$ while $C'$ is a circle centered on the focus of $P$. As shown in Figure 1, this can be regarded as the polar image of the well-studied bicentric family (interscribed between two circles) with respect to its outer circle, see Appendices A and B for construction details. We derive closure conditions for this new family for $N = 3, 4, 5, 6$, and describe many of its curious, almost degenerate, Euclidean phenomena. Also considered is its own polar image with respect to $P$.

Main results.

- The loci of vertex, perimeter, and area centroids are parabolas. Recall that in general, the locus of the perimeter centroid is not a conic [21]. We conjecture this to be true for all $N$.
- The loci of vertex and area centroids of polar polygons are straight lines, whereas that of the perimeter centroid is a quartic for $N = 3$ and a degree-10 algebraic curve for $N = 4$. We conjecture this to be true for all $N$.
- In the $N = 3$ case, the locus of the orthocenter is a straight line as are those of many triangle centers of the polar family. The Euler line of the polar family always passes through the parabola’s focus.
- Several centers of the $N = 3$ polar family are stationary and/or sweep circles. In the latter case they all belong to a single parabolic pencil.

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We prove the quantity $\sum \sin \theta_i/2$, where $\theta_i$ are directed angles of parabola-inscribed polygons, is conserved. In fact, this applies to any conic-inscribed polar image of the bicentric family.

Most of the above properties were first noticed via simulation [25], and later proved with a computer-algebra system (CAS) [16], using the explicit parametrizations given in Appendix A. For brevity, we omit any CAS-based proofs.

**Related work.** Some recent works close to this can be roughly divided into those which (i) study loci over certain triangle families [17, 18, 27], (ii) prove that loci of certain Poncelet triangle families are of a given shape [9, 12, 20, 23], to (iii) proving properties and invariants over $N \geq 3$ Poncelet families [2, 6, 4, 21].

**Article organization.** In Sections 2 and 3 we examine parabola-inscribed Poncelet triangles and their polar family, deriving closure conditions, and expressions for many of their triangle center loci. In Sections 4 to 6 we derive geometric closure conditions for $N = 4, 5, 6$ families, respectively, detecting the abovementioned pattern for the loci of their centroids (as well as in the polar family), followed by conjectured generalizations in Section 7. In Section 8 we describe a new quantity conserved by parabola-inscribed family (and variations thereof).
In Appendix A we provide explicit parametrizations for the vertices of both the \( N = 3 \) and \( N = 4 \) families, as well as their respective polar families. In Appendix B we explore the relation of parabola-inscribed families with the traditional bicentric family.

2. LOCI OF PARABOLA-INSERBED TRIANGLES

Referring to Figure 2, consider a Poncelet family \( T \) of triangles inscribed in a parabola \( \mathcal{P} \) and circumscribing a focus-centered circle. Let \( F = (-f, 0) \) and \( V = (0, 0) \) denote focus and vertex, respectively, where \( f \) is the focal distance. Consider a circle \( \mathcal{C} \) centered at \( F \) with radius \( r \).

**Proposition 1.** \( \mathcal{P} \) and \( \mathcal{C} \) will admit a Poncelet family of triangles iff \( r/f = 2(\sqrt{2} - 1) \).

**Proof.** Consider the Poncelet triangle with two parallel sides shown in Figure 3, inscribed in the parabola \( y = x^2/(4f) \), where \( f \) is the focal length. At \( x = r \) the parabola must be at \( y = f - r \), i.e., \( f - r = r^2/(4f) \), and the result follows. \( \square \)
2.1. **Straight-line orthocenter locus.** Let \( T \) denote our parabola-inscribed triangle family and \( D \) the directrix of \( \mathcal{P} \). Henceforth we shall adopt Kimberling’s notation \( X_k \) to refer to triangle centers [15].

**Proposition 2.** Over \( T \), the locus of the orthocenter \( X_4 \) is a line parallel to \( D \), and given explicitly by:

\[
X_4 : \left[ (5 - 2\sqrt{2})f, \frac{(3\sqrt{2} + 2)(2\sqrt{2}y^2 - 28f^2 + y^2)}{7(y\sqrt{2} - 2f + y)(y\sqrt{2} + 2f + y)} \right]
\]

The proof below was kindly contributed by Alexey Zaslavsky [26]:

*Proof.* Let \( C \) be the unit circle in the complex plane and \( A, B, C \) the touching points with the sides of the parabola-inscribed triangles. The polar transformation with center \( F \) maps the parabola to a circle with center \( I \) passing through \( F \) and touching \( AB, BC, CA \). Using Euler’s formula \( |FI|^2 = r^2 = R(R - 2r) \) [24], with \( R = 1 \) its radius \( r = \sqrt{2} - 1 \). Consider the line \( FI \) as the real axis. Since \( I \) is self-conjugated with respect to \( ABC \), we have \( a + b + c = 2\sqrt{2} - 2 + (3 - 2\sqrt{2})abc \), \( ab + bc + ca = 3 - \sqrt{2} + (2\sqrt{2} - 2)abc \). The polar images of the altitudes of the original triangle are the common points of \( BC, CA, AB \) with the lines passing through \( F \), and perpendicular to \( FA, FB, FC \) respectively. We have to calculate the common point of the line passing through these three points and the real axis. This point is a symmetric function in \( a, b, c \), so we can express it through \( abc \) and see that it is constant. \( \square \)

In Appendix B we describe how the parabola-inscribed family is the polar image of the bicentric family with respect to the bicentric circumcircle. Referring to Figure 4, Proposition 2 is actually a special case of:

**Proposition 3.** The locus of \( X_4 \) of an family which is the polar image of \( N = 3 \) bicentrics with respect to its outer circle is an ellipse, straight line, or hyperbola if the bicentric circumcenter lies in the interior, on top, or outside the bicentric incircle.

2.2. **Three parabolic loci.** Referring to Figure 2, we show below that over \( T \), the loci of the barycenter, circumcenter, and Spieker centers are all parabolas. The first and last correspond to the vertex and perimeter centroids of a triangle. This is curious since in general the perimeter centroid of a Poncelet family is not a conic [22].
Figure 4. Consider slight perturbations of the parabola-inscribed family (bottom left) such that the outer conic is an ellipse (top left and right) or a hyperbola (bottom right). In each case, the locus of $X_4$ is a conic (ellipse in the first two cases), and straight line and hyperbola in the remaining.

**Proposition 4.** Over $T$, the locus of the barycenter $X_2$ is a parabola coaxial with $P$, with focus $F_2 = [-f/3, 0]$, and vertex $V_2 = [2f(1 - 2\sqrt{2})/3, 0]$.

**Proposition 5.** Over $T$, the locus of the circumcenter $X_3$ of $T$ is a parabola coaxial with $P$, with focus $F_3 = [-f(2\sqrt{2} - 3)/2, 0]$, and vertex $V_3 = [-f(2\sqrt{2} + 3)/2, 0]$.

**Proposition 6.** Over $T$, the locus of the Spieker center $X_{10}$ is a parabola coaxial with $P$, with focus $F_{10} = [(1 - 2\sqrt{2})f, 0]$ and vertex $V_{10} = [f(3/2 - 2\sqrt{2}), 0]$. In particular, $F_{10} = [-f - r, 0]$, i.e., it lies on the left extreme of $C$.

3. **The polar $N = 3$ family**

Referring to Figure 5, let $T'$ denote the polar triangle of a triangle $T$ in $T$, i.e., whose interior is bounded by the polars of $T$ with respect to $P$. Since $T$ is inscribed in $P$ these are simply the tangents.
Recall some known properties of the polar triangle with respect to any parabola [3]: (i) the circumcircle of $T'$ passes through the focus $F$; (ii) the orthocenter of $T'$ is on the directrix; (iii) its area is half that of the reference triangle.

**Proposition 7.** The $T'$ family is Ponceletian. It circumscribes $\mathcal{P}$ and is inscribed in a hyperbola $\mathcal{H}$ centered at $(f, 0)$. Its axes are the axis and directrix of $\mathcal{P}$. Implicitly:

$$\mathcal{H} : \left(\sqrt{2} + \frac{3}{2}\right)(x - f)^2 - \frac{y^2}{2} - 2f^2 = 0$$

3.1. **Straight and nearly-straight loci.** An enduring conjecture has been that the locus of the incenter $X_1$ of a Poncelet triangle family can only be a conic if the pair is confocal [14].

As shown in Figure 5, over the polars, the locus of the incenter is, to the naked eye, a straight line. However, upon an algebraic investigation:
Proposition 8. The locus of the incenter $X'_1$ of $T'$ is one of four branches of the following quartic:

$$X'_1 : (-5\sqrt{2} - 6)x^2y^2 + (4\sqrt{2} + 2)f^2x^2 + (10\sqrt{2} + 12)fxy^2 + (8\sqrt{2} + 4)f^3x + (3\sqrt{2} - 16)f^2y^2 - 14f^4 = 0.$$  

Specifically, this branch:

$$X'_1 : \left[ \frac{\sqrt{2}y^2 + 2 + 2y^2 - \sqrt{-4y^2 + 4y^4 + 8\sqrt{2} + 8y^2} - 2\sqrt{2}y^4}{\sqrt{2}y^2 - 2 + 2y^2}, y \right]$$

In particular, said locus $X'_1$ is bounded by two lines parallel to the directrix and approximately $f/850$ apart, see Figure 6.

Still referring to Figure 5:

Proposition 9. The locus of the barycenter $X'_2$ of $T'$ is a line parallel to $\mathcal{D}$ and parametrized by:

$$X'_2 : \frac{1}{3} \left[ (2\sqrt{2} - 1)f \left( \frac{4 - 8\sqrt{2}}{(8\sqrt{2} - 12)f^2 + y^2} \right) \right].$$

Proposition 10. The locus of the circumcenter $X'_3$ of $T'$ is a line parallel to $\mathcal{D}$ and parametrized by:

$$X'_3 : \left[ (\sqrt{2} - 1)f \frac{(3\sqrt{2} + 2)(2\sqrt{2}y^2 - 28f^2 + y^2)y}{14(y\sqrt{2} - 2f + y)(y\sqrt{2} + 2f + y)} \right].$$
Referring to Figure 7, the above expressions for $X'_2$ and $X'_3$ yield:

**Corollary 1.** The (varying) Euler line $X'_2X'_3$ of the polar family passes through the focus $F = (-f, 0)$ of $\mathcal{P}$.

Still referring to Figure 7:

**Proposition 11.** The locus of the symmedian point $X'_6$ of $T'$ is a line parallel to $D$ and parametrized by:

$$X'_6 : \left[ \frac{(5 - 3\sqrt{2})f, (3\sqrt{2} + 4)(2\sqrt{2}y^2 - 28f^2 + y^2)y}{14(y\sqrt{2} - 2f + y)(y\sqrt{2} + 2f + y)} \right]$$

**Proposition 12.** The locus of $X'_{10}$ of the polar family is an algebraic curve of degree four given by:

$$X'_{10} : 4 \left( 11\sqrt{2} + 16 \right) x^4 - 4 \left( 3\sqrt{2} + 5 \right) x^2 y^2 - 4 \left( 37\sqrt{2} + 50 \right) f x^3 + 8 \left( 2\sqrt{2} + 1 \right) f xy^2 + 21 \left( 5\sqrt{2} + 8 \right) f^2 x^2 - 4 \left( 9\sqrt{2} + 8 \right) f^3 x - \left( \sqrt{2} + 4 \right) f^2 y^2 + 7 f^4 = 0$$

This locus is tightly bound by the following two lines parallel to the directrix:

$$x = \left( \sqrt{2} - 1 + \frac{\sqrt{10} - 7\sqrt{2}}{2} \right) f \quad \text{and} \quad x = \left( \sqrt{2} - 2^{-1/4} \right) f.$$  

The distance between these lines is approx. $f/1700$.

3.2. **Stationary points.** The circumcenter of the tangential triangle appears as $X_{26}$ on [15].

**Proposition 13.** Point $X'_{26}$ of $T'$ is stationary at the focus $F$ of $\mathcal{P}$.

Note $X_{26}$ does not lie in general on the circumcircle of a reference triangle. In our case it does since as mentioned above, the circumcircle of the polar contains the focus.

The Kiepert parabola of a triangle is an inscribed parabola with focus on $X_{110}$. Its directrix is the Euler line [24]. Referring to Figure 7:

**Proposition 14.** The focus $X'_{110}$ of the Kiepert parabola (resp. the Prasolov point $X'_{168}$) of the polar family is stationary at the right (resp. left) vertex of $\mathcal{H}$. Furthermore, $X'_{161}$ is stationary at the leftmost end of the incircle of the original family, i.e., at $[(1 - 2\sqrt{2})f, 0]$.

**Observation 1.** Over the polar family, the vertex of the polar Kiepert sweeps a circle.

3.3. **Linear loci galore.** Referring to Figure 8:

**Observation 2.** Over the first 1000 triangle centers in [15], the following triangle centers of $T'$ sweep linear loci parallel to $D$: $X'_{1}, k = 2, 3, 4, 5, 6, 20, 22, 23, 24, 25, 49, 51, 52, 54, 64, 66, 67, 69, 74, 113, 125, 140, 141, 143, 146, 154, 155, 156, 159, 182, 184, 185, 186, 193, 195, 206, 235, 265, 323, 343, 368, 370, 373, 376, 378, 381, 382, 389, 391, 399, 403, 427, 428, 468, 546, 547, 549, 549, 550, 567, 568, 569, 575, 576, 578, 597, 599, 631, 632, 858, 895, 973, 974.
3.4. A pencil of circular loci. Referring to Figure 7:

**Proposition 15.** The locus of the Steiner point $X'_{99}$ is a circle whose center $O'_{99}$ lies on the axis of $P$ of radius $R'_{99}$ such that at its right endpoint it touches $X'_{110}$. Explicitly:

$$O'_{99} = \left[(6\sqrt{2} - 7)f, 0\right], \quad R'_{99} = 2f\sqrt{17 - 12\sqrt{2}}$$

Referring to Figure 9:

**Observation 3.** Over the first 1000 triangle centers in [15], the following triangle centers of $T'$ sweep circular loci with centers on the axis of $P$ and passing through $X'_{110}$: $X'_k, k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935$.

This gives credence to:

**Conjecture 1.** If the locus of $X'_k$ is a circle with nonzero radius, it is in the parabolic pencil with $X'_{110}$ as a common point.

4. PARABOLA-INSCRIBED QUADRILATERALS

Referring to Figure 10, consider a Poncelet family $Q$ of quadrilaterals inscribed in a parabola $P$ and circumscribing a focus-centered circle $C$ of radius $r$. As before,
let $f$ denote the parabola’s focal distance, and $V = (0,0)$, $F = (-f, 0)$, its vertex and focus, respectively.

**Proposition 16.** $\mathcal{P}$ and $\mathcal{C}$ will admit a Poncelet family of quadrilaterals iff $r/f = 2\sqrt{\sqrt{5} - 2}$.

**Proof.** Referring to Figure 11, consider the symmetric Poncelet quadrilateral $P_i = (x_i, y_i), i = 1, \ldots , 4$, inscribed in the parabola $y = x^2/(4f)$, i.e., $x = 2\sqrt{fy}$. Clearly, $y_1 = f - r$, and $y_2 = f + r$. Requiring that $P_1P_2$ be tangent to $\mathcal{C}$ yields the quartic $r^2 + 4f\sqrt{f^2 - r^2} = 0$. The claim is the one positive root of this quartic. $\square$

**Proposition 17.** Over $\mathbb{Q}$, the two major diagonals $P_1P_3$ and $P_2P_4$ intersect at a stationary point $W = [(2 - \sqrt{5})f, 0]$.

4.1. **The three centroids.** Referring to Figure 10, let $C_0$, $C_1$, and $C_2$ denote the vertex, perimeter, and area centroids of the quadrilaterals in $\mathcal{Q}$, respectively.

**Proposition 18.** Over the family, $C_0$, $C_2$, and $W$ are collinear.

**Proposition 19.** Over the Poncelet family, the loci of $C_0, C_1, C_2$ are parabolas coaxial with $\mathcal{P}$, with foci and vertices given by:
Figure 9. Over the polar family we find that if a certain triangle center sweeps a circular locus, said locus will be an element of a parabolic pencil with $X_{110}$ as their common point (not labeled). In the figure the circular loci of $X_k$, $k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935$ are shown. Notice all lie on the dynamically-moving circumcircle (dashed red) except for $X_{249}$.

| centroid (N=4) | focal dist. | vertex $x/f$ | vtx. $x/f$ (num.) |
|---------------|-------------|--------------|-------------------|
| $C_0$         | $f/4$       | $-1$         | -1                |
| $C_1$         | $f/2$       | $(\sqrt{5} - 5)/2$ | -1.381966       |
| $C_2$         | $f/3$       | $\sqrt{5}/3 - 2$ | -1.25464         |

4.2. The polar quadrilateral. Referring to Figure 10, consider the polar quadrilateral bounded by the tangents to $P$ at the vertices of the original family. Let $P'_i$, $i = 1 \ldots 4$ denote its vertices and $C'_0$, $C'_1$, and $C'_2$ denote its vertex, perimeter, and area centroids.

**Proposition 20.** The locus of the polar quadrilateral’s vertices is the hyperbola $\mathcal{H}$ given by:

$$\mathcal{H} : \frac{(x - f)^2}{4(\sqrt{5} - 2)f^2} - \frac{y^2}{4f^2} - 1 = 0$$

with center at $[f, 0]$ and foci $[f(1 \pm 2\sqrt{5} - 1), 0]$.

Let $W$ be defined as in Proposition 17.
Figure 10. A Poncelet quadrilateral (green) is shown inscribed in a parabola \( P \) (gold) and circumcribing a focus-centered circle (brown). Over the family, (i) the intersection \( W \) of its diagonals (dashed green) is stationary; (ii) the loci of vertex \( C_0 \), perimeter \( C_1 \), and area \( C_2 \), centroids sweep 3 distinct parabolas (blue) coaxial with \( P \) with foci on \( F_0, F_1 \), and \( F_2 \). Notice the vertex of \( C_0 \) is \( F \) and that of \( C_1 \) is \( F_0 \). (iii) \( C_0, C_2, W \) are collinear (dashed blue). Also shown is the polar quadrilateral \( Q' \) (red) with respect to \( P \), inscribed in a hyperbola (dashed, red) centered at \([f,0]\). One observes that: (i) its diagonals (dashed red) also intersect at \( W \); (ii) the loci of its vertex \( C_0' \) and area \( C_2' \) centroids are lines (dashed orange) perpendicular to the axis of \( P \), (iii) \( C_0', C_2', W \) are collinear (dashed red); (iv) the locus of the polar perimeter centroid \( C'_1 \) is non-conic.

Figure 11. Construction used to derive \( r/f \) in for parabola-inscribed quadrilaterals in Proposition 16.

Proposition 21. The two major diagonals of the polar quadrilateral intersect at \( W \).

Proposition 22. Over the polar quadrilateral family, \( C'_0, C'_2, \) and \( W \) are collinear.
Proposition 23. Over \( Q \), the loci of \( C_0' \) and \( C_2' \) are lines perpendicular to the parabola’s axis intersecting it at:

\[
C_0' : x = (3 - \sqrt{5})f/2
\]

\[
C_2' : x = (4 - \sqrt{5})f/3
\]

Proposition 24. Over \( Q \) the locus of \( C_1' \) is one connected component of an algebraic curve of degree ten, given by the following equation:

\[
C_1' : - (1457008 \sqrt{5} + 3257968 + 122156 \sqrt{5} + 273148 + 465164 \sqrt{5} + 1040132) f^7 x^2 y^2 + (96506 \sqrt{5} + 215698) f^6 x^2 y^2 - (119256 \sqrt{5} + 266664) f^5 x y^4 + (505052 \sqrt{5} + 1129268) f^5 x^2 y^2 + 8564 \sqrt{5} + 192894 f^4 x y^2 - (881712 \sqrt{5} + 1971568) x^{10} + (43955 \sqrt{5} + 98289) f^3 x^2 y^4 + (24568 \sqrt{5} + 54936) f x^5 - (7250 \sqrt{5} + 16210) f^6 x y^4 - (1274930 \sqrt{5} + 2850838) f^4 x y^2 + (1235568 \sqrt{5} + 2762832) f^3 x^2 y^2 + (4457696 \sqrt{5} + 9967712) f x^5 - (7787152 \sqrt{5} + 17412608) f^2 x^5 + (5470456 \sqrt{5} + 12232344) f^2 x^7 - (1690535 + 755997 \sqrt{5}) f^4 x^6 - (812098 \sqrt{5} + 1815898) f^3 x^5 + (330222 \sqrt{5} + 738968) f^5 x^4 + (1002 + 448 \sqrt{5}) f^4 y^4 - (228 \sqrt{5} + 672) f^3 y^2 - (7300 \sqrt{5} + 169956) f^3 x^3 + (2750 \sqrt{5} + 7150) f^3 x - (16145 \sqrt{5} + 36103) f^2 x^2 - (84196 \sqrt{5} + 188268) x^6 y^4 + (544928 \sqrt{5} + 1218496) x^5 y^2 - 726 f^{10} = 0
\]

Furthermore, \( C_1' \) is bound by the following two lines parallel to the directrix and approximately \( f/25 \) apart: \( x = \frac{(5 + \sqrt{3} - \sqrt{3} - \sqrt{3} - \sqrt{3} + 3)}{2} f \) and \( x = \frac{(\sqrt{5} - \sqrt{3} - \sqrt{3} - \sqrt{3} + 3)}{2} f \).

5. Parabola-inscribed pentagons

Referring to Figure 12, consider a family of pentagons inscribed in a parabola \( \mathcal{P} \) of focal distance \( f \), and circumscribing a focus-centered circle \( \mathcal{C} \) of radius \( r \).

**Proposition 25.** The pair \( \mathcal{P}, \mathcal{C} \) will admit a Poncelet family of pentagons iff \( r/f \) is the only positive root of the following sextic polynomial (\( r/f \approx 0.995219 \)):

\[
x^6 + 12x^5 - 28x^4 + 32x^3 + 112x^2 - 64x - 64 = 0
\]

**Proof.** Referring to Figure 13, without loss of generality, let \( \mathcal{P} \) be the unit parabola \( y = x^2 \) with focus \( F = (0,1/4) \) and let \( \mathcal{C} \) be a circle of radius \( r \) centered at \( F \). Consider the Poncelet pentagon \( P_i \), \( i = 1 \ldots 5 \) with \( P_i \) at infinity, and \( P_2 \) horizontal and tangent to \( \mathcal{C} \) at \( 0,1/4 - r \). Compute the next Poncelet vertex \( P_3 = (x_3, y_3) \) as the intersection of a tangent to \( \mathcal{C} \) from \( P_2 \) with \( \mathcal{P} \). By requiring that \( x_3 = r \) we obtain the sextic in the claim. \( \square \)

Referring to Figure 12:

**Conjecture 2.** Over the parabola-inscribed pentagon family, the loci of vertex, perimeter, and area centroids are parabolas coaxial with \( \mathcal{P} \).

**Conjecture 3.** Over the family of polar polygons to parabola-inscribed pentagons, the locus of vertex and area vertices are lines perpendicular to the directrix while that of the perimeter centroid is an algebraic curve of degree at least four.
6. PARABOLA-INSERVED HEXAGONS AND SUMMARY

6.1. Hexagons and summary. Referring to Figure 14, we can also consider a family of parabola-inscribed hexagons.

A similar approach can be used to compute the $r/f$ required for $N = 6$. A summary of all $r/f$ thus obtained appears in Table 1.
7. Generalizing Centroidal Loci

Let \( \mathcal{R} \) be a Poncelet family of \( N \)-gons inscribed to a parabola \( \mathcal{P} \) circumscribing a focus-centered circle \( \mathcal{C} \). Given evidence for the cases above, namely, \( N = 3, 4, 5, 6 \), we propose the following generalizations:

### Table 1. Table of \( r/f \) for closure of parabola-inscribed families of various \( N \). Algebraic expressions (2nd column) are only possible for \( N = 3, 4 \).

| \( N \) | \( r/f \) | \( r/f \) (num.)  |
|--------|----------|-------------------|
| 3      | \( 2(\sqrt{2} - 1) \) | 0.828427          |
| 4      | \( 2\sqrt{\sqrt{5} - 2} \) | 0.971737          |
| 5      | n/a      | 0.995219          |
| 6      | n/a      | 0.999183          |

Figure 13. Construction used to derive \( r/f \) in Proposition 25. Top (resp. bottom) shows the complete picture (resp. a detailed view near the vertex).
Conjecture 4. Over $\mathcal{R}$, for arbitrary $N$, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $\mathcal{P}$.

Conjecture 5. Over the polar polygons of $\mathcal{R}$ with respect to $\mathcal{P}$, for arbitrary $N$, the loci of their vertex and area centroids are straight lines parallel to the directrix of $\mathcal{P}$.

Experimental evidence allows us to further generalize the previous conjectures. Let $\mathcal{B}'$ be the conic-inscribed polar image of a generic bicentric family of $N$-gons with respect to the bicentric circumcircle (see Appendix B).

Recall that the locus of vertex and area centroids are conics over any Poncelet family, while that of the perimeter centroid is not, in general, a conic [21]. Sufficient experimental evidence suggests that:

Conjecture 6. Over $\mathcal{B}'$, the locus of the perimeter centroid is a conic.

Let $\mathcal{P}'$ be the conic to which $\mathcal{B}'$ is inscribed.
Conjecture 7. Over the polar polygons of $\mathcal{B}'$ with respect to $\mathcal{P}'$, the locus of the perimeter centroid is never a conic.

8. A conserved quantity

As in Appendix B, let $\mathcal{B}$ denote a bicentric family of $N$-gons (inscribed to a circle $\mathcal{C} = (O, R)$ and circumscribing a second, nested circle. Let $d_i$ denote the perpendicular distance from the bicentric circumcenter $O$ to side $P_iP_{i+1}$.

Referring to Figure 15:

Lemma 1. Over $\mathcal{B}$, the quantity $\sum d_i$ is invariant.

The argument below was kindly provided by A. Akopyan [1].

Proof. The above statement is equivalent to stating that (i) the sum of unit vectors from a point $P$ in the direction perpendicular to bicentric sides is constant, and that (ii) the latter is equivalent to the well-known fact that the center of mass of the points of tangency of the bicentric polygon with the incircle is stationary. □

Proposition 26. For all $N$, the family of polygons polar to $\mathcal{B}$ with respect to its circumcenter conserves $\sum \sin \theta_i/2 = (1/R) \sum d_i$.

Proof. The vertices of the tangential polygon are the poles of each side of $\mathcal{B}$ with respect to the circumcircle. Therefore, said vertices are at a distance $D_i = R^2/d_i$ from the $O$. Since $\sin \theta_i/2 = R/D_i = d_i/R$, per Lemma 1, the claim follows. □

Note that in general, $\theta_i$ is the directed angle $P_{i-1}P_iP_{i+1}$. In the case when $r < d$, the tangential polygon will be inscribed in two branches of a hyperbola. There are only two cases: either (i) all vertices lie on a first proximal branch of the hyperbola, or (ii) all but one vertex $P_k$ will lie on said branch, with $P_k$ lying on the distal branch. In case (i) all $\theta_i$ are positive whereas in (ii) all are positive except for $\theta_k$. Furthermore, in this case, the supplement of angles $\theta_{i-1}$ and $\theta_{i+1}$ need to be used in the sum. So the invariant sum becomes:
\sin \theta_1/2 + \ldots + \sin \pi \theta_{k-1}/2 - \sin \theta_k/2 + \sin \pi \theta_{k+1}/2 + \ldots + \sin \theta_N/2 = \\
\sin \theta_1/2 + \ldots + \cos \theta_{k-1}/2 - \sin \theta_k/2 + \cos \theta_{k+1}/2 + \ldots + \sin \theta_N/2

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Appendix A. Vertex Parametrizations

A.1. Parabola-inscribed triangles. A 3-periodic orbit \( P_i = [x_i, y_i] = [-y_i^2/(4f), y_i] \) is such that:

\[
y_2 = \frac{2 \left(1 - \sqrt{2}\right) (4fy_1 + \Delta)f}{8f^2\sqrt{2} - 12f^2 + y_i^2},
\]

\[
y_3 = \frac{2 \left(\sqrt{2} - 1\right)f\Delta}{8f^2\sqrt{2} - 12f^2 + y_i^2},
\]

\[
\Delta = \sqrt{16(8\sqrt{2} - 11)f^4 + 8f^2y_i^2 + y_i^4}
\]

A.2. Hyperbola-inscribed polar triangles. A 3-periodic orbit \( Q_i = [q_{1,i}, q_{2,i}] \) is such that:

\[
Q_1 = \left(1 + \sqrt{2}\right) \left[ \frac{(4fy_1 + \Delta)y_1}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} : \frac{(1 + \sqrt{2})y_i^2 - 4(1 + \sqrt{2})f^2y_1 - 2\Delta f}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} \right]
\]

\[
Q_2 = \left(1 + \sqrt{2}\right) \left[ \frac{(4fy_1 - \Delta)y_1}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} : \frac{(1 + \sqrt{2})y_i^2 - 4(1 + \sqrt{2})f^2y_1 + 2\Delta f}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} \right]
\]

\[
Q_3 = \left(1 + \sqrt{2}\right) \left[ \frac{(5 - 3\sqrt{2})((1 + 2\sqrt{2})y_1^2 - 28f^2)f}{7((3 + 2\sqrt{2})y_i^2 - 4f^2)} , -\frac{8f^2y_1}{(3 + 2\sqrt{2})y_i^2 - 4f^2} \right]
\]

\[
\Delta = \sqrt{y_i^4 + 8f^2y_i^2 + 16(8\sqrt{2} - 11)f^4}
\]

A.3. Parabola-inscribed quadrilaterals. A 4-periodic orbit \( P_i = [x_i, y_i] = [-\frac{1}{4f}y_i^2, y_i] \) is such that:

\[
y_2 = \frac{\left(2 \sqrt{5} - 2 \Delta_i + 4f y_1(3 - \sqrt{5})\right)f}{4f^2\sqrt{5} - 8f^2 - y_i^2},
\]

\[
y_3 = \frac{4(2 - \sqrt{5})f^2}{y_i^1},
\]

\[
y_4 = -\frac{\left(2 \sqrt{5} - 2 \Delta_i + 4f y_1(\sqrt{5} - 3)\right)f}{4f^2\sqrt{5} - 8f^2 - y_i^2},
\]

\[
\Delta_i = \sqrt{y_i^4 + 8f^2y_i^2 + 16\left(9 - 4\sqrt{5}\right)f^4}
\]
A.4. Hyperbola-inscribed polar quadrilaterals. A 4-periodic orbit \( P_i = [p_i, q_i] \) is such that:

\[
\begin{align*}
    p_1 &= \frac{\sqrt{5} - 2 \left( \Delta_1 + 6 f y_1 \sqrt{5} \sqrt{\sqrt{5} - 2} + 14 f y_1 \sqrt{\sqrt{5} - 2} \right) y_1}{4 y_1^2 + 2 y_1^2 \sqrt{5} - 8 f^2} \\
    q_1 &= \frac{2 \sqrt{5} - 2 (\Delta_1 + 2 \sqrt{2} (\sqrt{5} - 1) y_1) (\sqrt{5} - 2 y_1 + 4 f^2)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    p_2 &= \frac{\sqrt{5} + 2 \left( \Delta_1 - 2 \sqrt{5} + 2 f y_1^2 - 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right) \left( \sqrt{5} - 2 y_1 + 4 f^2 \right)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    q_2 &= \frac{-2 \sqrt{5} + 2 \left( \Delta_1 - 2 \sqrt{5} + 2 f y_1^2 - 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right) \left( \sqrt{5} - 2 y_1 + 4 f^2 \right)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    p_3 &= \frac{\sqrt{5} + 2 \left( \Delta_1 - 2 \sqrt{5} + 2 f y_1^2 - 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right) \left( \sqrt{5} - 2 y_1 + 4 f^2 \right)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    q_3 &= \frac{-2 \Delta_1 \sqrt{\sqrt{5} - 2 + 4 f^2 y_1 - y_1^2} (4 f^2 \sqrt{5} + 8 f^2 + y_1^2)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    p_4 &= \frac{\sqrt{5} - 2 \left( \Delta_1 - 2 \sqrt{5} + 2 f y_1^2 - 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right) \left( \sqrt{5} - 2 y_1 + 4 f^2 \right)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)} \\
    q_4 &= \frac{-2 \Delta_1 \sqrt{\sqrt{5} - 2 + 4 f^2 y_1 - y_1^2} (4 f^2 \sqrt{5} + 8 f^2 + y_1^2)}{y_1 (16 f^4 - 16 f^2 y_1^2 - y_1^4)}
\end{align*}
\]

Appendix B. Relation to the bicentric family

Referring to Figure 16, the bicentric family \( B \) of \( N \)-gons is a family of Poncelet \( N \)-gons inscribed in a circle \( C = (O, R) \) and circumscribing another circle \( C' = (O', r_b) \). Let \( d = |O - O'| \). Relations between \( d, R, r_b \) are known for many “low \( N \)” and are listed in [24, Poncelet’s porism].

**Definition 1** (Polar polygon). Given a polygon \( P \), its polar polygon \( P' \) with respect to a conic \( C \) is bounded by the tangents to \( C \) at the vertices of \( P \).

Referring to Figure 17:
Proposition 27. The polar family $B'$ of $B$ with respect to $C$ is an ellipse, parabola, or hyperbola-inscribed if $d$ is smaller, equal, or greater than $R'$, respectively ($O$ is interior, on the boundary, or exterior to $C'$, respectively). Furthermore, one of the foci coincides with $O'$.

In the hyperbolic case, the polar polygon may lie with either all vertices on a first branch or with all but a single one in the first branch, see Figure 17(right).

Proposition 28. The parabola $P$ which is the polar image of $B$ with $d = r_b$, has focal distance $f = R_b^2/(2r_b)$.

Proof. Let $O = (0, 0)$. Consider a polygon in $B$ with a vertical side $P_1P_2$ tangent to the incircle at $(2r_b, 0)$. The vertex $V$ of $P$ is the pole of said side which can be obtained as an inversion of point $(2r_b, 0)$ with respect to the circumcircle. This yields the result. 

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