Note on the super-extended Moyal formalism and its BBGKY hierarchy

Carlo Pagani

Institute of Physics, PRISMA & MITP,
Johannes Gutenberg University Mainz,
Staudingerweg 7, D–55099 Mainz, Germany

Abstract

We consider the path integral associated to the Moyal formalism for quantum mechanics extended to contain higher differential forms by means of Grassmann odd fields. After revisiting some properties of the functional integral associated to the (super-extended) Moyal formalism, we give a convenient functional derivation of the BBGKY hierarchy in this framework. In this case the distribution functions depend also on the Grassmann odd fields.
1 Introduction

Recently a lot of work has been devoted to study at a deeper level the formalism and the symmetries associated to the functional integral of the Schwinger-Keldysh (SK) formalism \cite{1,2}, which allows to deal with out of equilibrium phenomena in quantum field theory. It turns out that a super-extension of the SK formalism has a rich symmetry structure that allows to keep some features of the formalism manifest \cite{3–12}.

However, the Schwinger-Keldysh formalism is not the only possibility to deal with a time dependent density matrix. In particular, Moyal introduced a formalism which naturally evolves the density matrix and that allows to deal with quantum mechanics without considering operators but only \(c\)-functions \cite{13}. In this work we shall revisit the functional formulation of the super-extended Moyal formalism, that allowed to put forward geometric notions, such as exterior derivatives, in quantum mechanics, from the point of view of statistical mechanics. Besides its own interest, this approach is tightly connected with the recently proposed super-extension of the SK formalism. The main objective of this work is to discuss the properties of the super-extended Moyal formalism from the point of view of statistical mechanics and give a functional derivation of the super-extended Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of distribution functions.

We shall review the Moyal approach to quantum mechanics in section 2, for the time being it suffices to say that a special role is reserved to the density matrix, whose time dependence is given in terms of the quantum analogue of the Liouville operator, which, in the classical case, describes the time evolution of the classical phase-space density distribution. The Moyal approach to quantum mechanics has also the nice feature of displaying the classical limit in a very direct manner since all the standard quantum mechanical operations, like the commutator of two operators, are mapped to functions which are expressed as a series expansion in \(\hbar\).

The path integral associated to the Moyal formalism has been built in \cite{14,15}. In \cite{15} the action appearing in the path integral has been extended in order to contain new Grassmann odd fields, which we will call ghosts, that allowed to put forward a proposal for differential calculus in quantum mechanics. We will revisit this path integral and briefly discuss its relation with the SK formalism in section 2.3.

In section 3 we review the super-extension of the Moyal formalism and its classical limit. In doing so we discuss some properties of the associated functional integral. In particular, we show that the introduction of the ghosts does not modify the theory (i.e. the correlation functions), consider the associated topological properties and the relation with the super-extended SK formalism.

Finally, in section 4 we give a functional derivation of the super-extended BBGKY hierarchy. In particular, we shall derive the BBGKY hierarchy via a functional integral associated to the Moyal approach to quantum mechanics. More precisely, in section 4.1...
we first discuss the case of the path integral associated to the Moyal formalism and in section 4.2 we consider the extension containing the ghosts fields.

In section 5 we summarize our findings and comment on possible outlooks.

2 Review of the Moyal approach to quantum mechanics and its associated path integral

2.1 Moyal approach to quantum mechanics

In this section we review the approach to quantum mechanics (QM) put forward by Moyal [13], building on previous work [16, 17] by Weyl and Wigner (see also [18, 19]). The basic idea underlying Moyal formalism is that, ultimately, one can formulate QM without using operators, rather only \( \mathcal{C} \)-functions are present and they are multiplied via the so-called star-product that will be introduced shortly. (We employ the notation used in [15].)

It is useful, however, to keep track of the relation between the standard formalism and the Moyal one by considering the so-called “symbol calculus”. In particular, one sets up a one-to-one map between the operators \( \hat{O} \) acting on a Hilbert space \( \mathcal{V} \) and (complex valued) functions \( O \) on a suitable manifold \( \mathcal{M} \). The relation between the space of functions, indicated with \( \mathcal{F}(\mathcal{M}) \), and the operators is given by the following relation: given an operator \( \hat{O} \), the associated function is provided by the “symbol map” \( O = \text{symb} (\hat{O}) \).

Furthermore, the space of symbols \( \mathcal{F}(\mathcal{M}) \) is equipped with the star-product \( * \), which implements the multiplication of operators at the level of their representative in \( \mathcal{F}(\mathcal{M}) \):

\[
\text{symb} (\hat{O}_1 \hat{O}_2) = \text{symb} (\hat{O}_1) * \text{symb} (\hat{O}_2).
\]

This product is associative but non-commutative, thus keeping track of these crucial features of the operatorial formulation of QM.

We shall consider a particularly important realization of the ideas that we have just outlined by employing the Weyl symbol [16]. Essentially, Weyl introduced an association rule mapping \( \mathcal{C} \)-function in the phase-space to operators with a specific ordering prescription. Before introducing explicitly this correspondence, let us set up our notation for the phase-
space. We denote all the phase-space variables collectively as \( \varphi^a \equiv (q_1, \cdots q_n, p_1, \cdots, p_n) \). The symplectic two-form \( \omega_{ab} \) and its inverse \( \omega^{ab} \) can be written locally as
\[
\omega_{ab} \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\
\omega^{ab} \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]
The Hamilton equation of motion and the Poisson brackets can then be written respectively as
\[
\dot{\varphi}^a = \omega^{ab} \partial_b H, \\
\{f, g\}_{pb} = \partial_a f \omega^{ab} \partial_b g = \partial q f \partial p g - \partial p f \partial q g,
\]
where the subscript \( pb \) allows to distinguish the Poisson brackets from the Moyal brackets that will be introduced later on.

The Weyl symbol \( O = \text{symb} \left( \hat{O} \right) \) is defined as
\[
O (\varphi^a) \equiv \int \frac{d^{2n} \varphi_a}{(2 \pi \hbar)^n} \exp \left[ \frac{i}{\hbar} \varphi_0^a \omega_{ab} \varphi^b \right] \text{Tr} \left[ \hat{T} (\varphi_0) \hat{O} \right],
\]
with
\[
\hat{T} (\varphi_0) \equiv \exp \left[ \frac{i}{\hbar} \varphi_0^a \omega_{ab} \varphi^b \right] = \exp \left[ \frac{i}{\hbar} (p_0 q - q_0 p) \right].
\]
The inverse map is given by
\[
\hat{O} = \int \frac{d^{2n} \varphi_0 d^{2n} \varphi_a}{(2 \pi \hbar)^{2n}} O (\varphi^a) \exp \left[ \frac{i}{\hbar} \varphi_0^a \omega_{ab} \varphi^b \right] \hat{T} (\varphi_0) \\
= \int \frac{d^{2n} \varphi_0 d^{2n} \varphi_a}{(2 \pi \hbar)^{2n}} O (\varphi^a) \exp \left[ \frac{i}{\hbar} (p_0 (q - q_0) - q_0 (\hat{p} - p)) \right].
\]
A particularly important case is that of the density matrix. Considering a pure state density matrix \( \hat{\rho} = |\psi\rangle \langle \psi| \) we have
\[
\rho (\varphi^a) = \int \frac{d^{2n} \varphi_a}{(2 \pi \hbar)^n} \exp \left[ \frac{i}{\hbar} \varphi_0^a \omega_{ab} \varphi^b \right] \text{Tr} \left[ \hat{T} (\varphi_0) |\psi\rangle \langle \psi| \right] \\
= \int dq_0 e^{-i \frac{q_0}{2} p_0} \psi \left( q + \frac{q_0}{2} \right) \psi^\dagger \left( q - \frac{q_0}{2} \right).
\]

The so defined $ρ(q,p)$ is called the Wigner function (WF). While the WF does not have the meaning of a probability density as it can be also negative, standard quantum probability densities are retrieved via the following integrations:

$$|ψ(q)|^2 = \int \frac{dp}{(2\piℏ)^n} ρ(p,q),$$
$$|\tilde{ψ}(p)|^2 = \int \frac{dq}{(2\piℏ)^n} ρ(p,q).$$

The expectation value of an observable is given by

$$⟨ψ|\hat{O}|ψ⟩ = \int \frac{dq dp}{(2\piℏ)^n} ρ(p,q) O(p,q).$$

By construction, it is guaranteed that this formalism is fully equivalent to QM. Indeed, by introducing the star-product we have an isomorphism between operators and their multiplication, and the phase-space functions. Consider the operator $\hat{O}_3 = \hat{O}_1 \hat{O}_2$, its Weyl symbol $O_3$ satisfies

$$\hat{O}_3 = \int \frac{d^nϕ_0 d^nϕ_a}{(2\piℏ)^{2n}} O_3(ϕ_a) \exp \left[ \frac{i}{ℏ} ω_{ab} ϕ_b ϕ_a^0 \right] T(ϕ_0),$$

where

$$O_3(ϕ_a) \equiv (O_1 * O_2)(ϕ_a) = O_1(ϕ_a) \exp \left[ i \frac{ℏ}{2} \partial_a ω_{ab} ϕ_b^0 \right] O_2(ϕ_a),$$

where we introduced the so called *-product.

We can then define the Moyal brackets (indicated with the subscript $mb$) \[13\]

$$\{O_1, O_2\}_{mb} \equiv \frac{1}{iℏ} [O_1 * O_2 - O_2 * O_1] = \text{symb} \left( \frac{1}{iℏ} [\hat{O}_1, \hat{O}_2] \right).$$

Interestingly, the $ℏ \to 0$ limit of the Moyal brackets gives back the standard Poisson brackets of classical mechanics:

$$\lim_{ℏ \to 0} \{O_1, O_2\}_{mb} = \{O_1, O_2\}_{pb}.$$

Thus, already at this stage, it is evident that the classical limit is particularly clear in this formalism.
Furthermore, we wish to consider the time evolution of the density matrix, which is given by

\[ \partial_t \hat{\rho} = -\frac{1}{i\hbar} \left[ \hat{\rho}, \hat{H} \right]. \]

By applying the symbol map we have the evolution equation of the WF \( \rho(q,p;t) \):

\[ \partial_t \rho = -\{\rho, H\}_{mb}. \]

It is clear that in the classical limit \( \hbar \to 0 \) we obtain

\[ \partial_t \rho = -\{\rho, H\}_{pb} = -\left( \partial_a \rho \omega^{ab} \partial_b H \right) = \omega^{ab} \partial_a H \partial_b \rho \equiv -\hat{L}_{\rho}, \]

where \( \hat{L} = \omega^{ab} \partial_b H \partial_a \) is the Liouville operator. On the other hand, the fully quantum evolution of the WF is given by:

\[ \partial_t \rho = -\{\rho, H\}_{mb} = -\frac{1}{i\hbar} \left( \exp \left[ i\frac{\hbar}{2} \omega^{ab} \partial_a (\rho) \partial_b (H) \right] \rho H - \exp \left[ i\frac{\hbar}{2} \omega^{ab} \partial_a (H) \partial_b (\rho) \right] \rho H \right). \]

Writing

\[ \exp \left[ i\frac{\hbar}{2} \omega^{ab} \partial_a (\rho) \partial_b (H) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left( i\frac{\hbar}{2} \omega^{ab} \partial_a (\rho) \partial_b (H) \right)^n, \]

we can re-express the evolution equation for the WF as follows:

\[ \partial_t \rho = -\frac{2}{\hbar} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} \left( i\frac{\hbar}{2} \omega^{ab} \partial_a (\rho) \partial_b (H) \right)^{2n+1} H \rho \right] \]

\[ = -\frac{2}{\hbar} \sin \left( i\frac{\hbar}{2} \omega^{ab} \partial_a (H) \partial_b (\rho) \right) H \rho \equiv -\hat{L}_{\rho}. \]

Finally, one can write the (generic) Moyal brackets as

\[ \{A, B\}_{mb} = A \frac{2}{\hbar} \sin \left( i\frac{\hbar}{2} \partial_a \omega^{ab} \partial_b \right) B. \]

A generalization of the Moyal brackets including Grassmann odd variables can be introduced using the star-product between Grassmann odd variables \( [20] \), see \( [15] \).
2.2 Path integral for the Moyal formalism

In this section we present a simple derivation of the path integral associated to the Moyal formalism following ref. [21]. For a more detailed description and more careful analysis we refer the reader to [14, 15]. Let us note that the Schrödinger equation,

\[ i\hbar \partial_t \psi = \hat{H} \psi, \]  

(2.1)

is formally very similar to the evolution equation for the Wigner function, which indeed can be written as:

\[ i\partial_t \rho = -\frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \omega_{ab} \partial_b \partial_a \right) \hat{H} \rho = -i\hat{L}_h \rho, \]  

(2.2)

where \(-i\hat{L}_h\) is a sort of “Hamiltonian” operator and the Planck constant in (2.1) takes the value \(\hbar = 1\) in the LHS of (2.2). Moreover, in the present case the phase-space variables \(\varphi^a\) play the role of the position operator \(\hat{x}\) of standard QM. Therefore we need to introduce a conjugate variable to \(\varphi^a\), such conjugate variable is indicated with \(\lambda^a\). Now, invoking the QM commutators between conjugate variables, i.e. \([\hat{x}, \hat{p}] = i\hbar\), we define, in full analogy, \([\hat{\varphi}^a, \hat{\lambda}_b] = i\delta^a_b\).

In the \(\varphi\)-representation, \(\hat{\varphi}^a = \varphi^a\), the conjugate operator reads \(\hat{\lambda}_a = -i \frac{\partial}{\partial \varphi^a} = -i \partial_a\). Thus, we can rewrite

\[ i\partial_t \rho = -\frac{2}{\hbar} H \sin \left( \frac{\hbar}{2} \omega_{ab} \partial_b \partial_a \right) \rho = -\frac{2}{\hbar} H \sin \left( \frac{i\hbar}{2} \omega_{ab} \partial_b \partial_a \right) \rho \equiv \hat{\mathcal{H}}_B^h \rho, \]

where we defined the new “Hamiltonian” \(\hat{\mathcal{H}}_B^h\). As a function of \(\varphi^a\) and \(\lambda^a\) the Hamiltonian \(\hat{\mathcal{H}}_B^h\) reads

\[ \hat{\mathcal{H}}_B^h = -\frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \lambda^a \omega_{ab} \partial_b \right) H \]

\[ = \frac{2}{\hbar} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{\hbar}{2} \lambda^a \omega_{ab} \partial_b \right)^{2n+1} H. \]

It is straightforward to check that \(\hat{\mathcal{H}}_B^h\) can be rewritten as

\[ \hat{\mathcal{H}}_B^h = \frac{1}{\hbar} \left[ H \left( \varphi^a - \frac{\hbar}{2} \omega_{ab} \lambda_b \right) - H \left( \varphi^a + \frac{\hbar}{2} \omega_{ab} \lambda_b \right) \right] \]
or via the following expression

\[ \tilde{H}_B^h = \frac{2}{\hbar} \sinh \left( \frac{\hbar}{2} \omega^{ab} \lambda_a \partial_b \right) H. \]

For later purposes, let us check the classical limit of \( \tilde{H}_B^h \):

\[ \lim_{\hbar \to 0} \frac{1}{\hbar} \left[ H \left( \varphi^a - \frac{\hbar}{2} \omega^{ab} \lambda_b \right) - H \left( \varphi^a + \frac{\hbar}{2} \omega^{ab} \lambda_b \right) \right] = -\partial_aH\omega^{ab}\lambda_b = \lambda_a\omega^{ab}\partial_bH. \]

In QM we know that, given an Hamiltonian \( H \), the associated path integral weight is characterized by an action \( S = \int dt \left[ p \dot{q} - H \right] \). Therefore, the action associated to \( \tilde{H}_B^h \) is given by

\[ S_M = \int dt \left[ \lambda_a \varphi^a - \tilde{H}_B^h \right], \]

where the subscript M reminds of Moyal. In the classical limit, we obtain the action

\[ S_{h \to 0}^M = \int dt \left[ \lambda_a \varphi^a - \lambda_a\omega^{ab}\partial_bH \right], \]

which is indeed known to be the action for the path integral formulation of classical mechanics [22]. We will study in more detail the so called classical path integral (CPI) in section 3.1.1.

For later purposes, let us come back to the Hamiltonian \( \tilde{H}_B^h \) and write explicitly the combinations \( \varphi^a - \frac{\hbar}{2} \omega^{ab} \lambda_b \) and \( \varphi^a + \frac{\hbar}{2} \omega^{ab} \lambda_b \) in the case of a single particle, i.e. \( \varphi^a = q, p \) and \( \lambda_a = \lambda_q, \lambda_p \). We obtain

\[ \tilde{H}_B^h = \frac{1}{\hbar} \left[ H \left( \varphi^a - \frac{\hbar}{2} \omega^{ab} \lambda_b \right) - H \left( \varphi^a + \frac{\hbar}{2} \omega^{ab} \lambda_b \right) \right] = \frac{1}{\hbar} \left[ H \left( p + \frac{\hbar}{2} \lambda_q, q - \frac{\hbar}{2} \lambda_p \right) - H \left( p - \frac{\hbar}{2} \lambda_q, q + \frac{\hbar}{2} \lambda_p \right) \right]. \]

2.3 Comparison with the Schwinger-Keldysh formalism and its phase-space path integral

In this section we briefly review the SK formalism and make contact with the formalism of section 2.2. For the sake of clarity we shall consider a simple one-dimensional description.
Given the density matrix $\rho_0$ at an initial time $t_0$, the time evolution of the density matrix is given by

$$\hat{\rho}(t) = U(t,t_0) \hat{\rho}_0 U^\dagger(t,t_0).$$

The matrix element $\langle x'|\rho(t)|x \rangle$ can be written as

$$\langle x'|\rho(t)|x \rangle = \int dx_0 dx_0' \langle x'|U(t,t_0)|x_0 \rangle \langle x_0|\hat{\rho}_0|x_0' \rangle \langle x_0'|U^\dagger(t,t_0)|x \rangle = \int dx_0 dx_0' \langle x_0|\hat{\rho}_0|x_0' \rangle \int_{x_1(t)=x'} Dx_1 e^{iS[x_1]} \int_{x_2(t)=x} Dx_2 e^{-iS[x_2]}.$$

The trace is obtained via

$$\text{Tr} [\hat{\rho}(t)] = \int dx \langle x|\hat{\rho}(t)|x \rangle = \int dx dx_0 dx_0' \langle x_0|\hat{\rho}_0|x_0' \rangle \int_{x_1(t)=x} Dx_1 \int_{x_2(t)=x} Dx_2 e^{iS[x_1]} e^{-iS[x_2]}.$$

The path integrals over $x_1$ and $x_2$ can be re-expressed as path integrals in phase-space where one integrates also over the momenta, which is not subject to any boundary condition. Ignoring for a moment the factor $\langle x_0|\hat{\rho}_0|x_0' \rangle$, which carries the information on the initial state, we rewrite

$$\int Dx_1 Dx_2 e^{\frac{i}{\hbar}S[x_1]} e^{-\frac{i}{\hbar}S[x_2]} = \int Dx_1 Dx_2 D\pi_1 D\pi_2 e^{\frac{i}{\hbar} \int (\pi_1 \dot{x}_1 - H(\pi_1,x_1)) - \frac{1}{i} \int (\pi_2 \dot{x}_2 - H(\pi_2,x_2))}.$$

Furthermore, let us introduce the following symmetric and anti-symmetric combinations, which we call $s$-fields and $a$-fields respectively:

$$x_a \equiv \frac{1}{\hbar} (x_1 - x_2)$$

$$x_s \equiv \frac{x_1 + x_2}{2}$$

$$\pi_a \equiv \frac{1}{\hbar} (\pi_1 - \pi_2)$$

$$\pi_s \equiv \frac{\pi_1 + \pi_2}{2}.$$
Now, rewriting our path integral in terms of the new fields we have
\[
\int D\mathbf{x}_1 D\mathbf{x}_2 \exp\left[ i \int \left( \pi_s \dot{x}_a + \pi_a \dot{x}_s + \frac{1}{\hbar} H \left( \pi_s - \hbar \frac{\pi_a}{2}, x_s - \hbar \frac{x_a}{2} \right) - H \left( \pi_s + \hbar \frac{\pi_a}{2}, x_s + \hbar \frac{x_a}{2} \right) \right) \right],
\]
so that final SK action reads
\[
S_{\text{SK}} = \int dt \left[ \pi_s \dot{x}_a + \pi_a \dot{x}_s + \frac{1}{\hbar} H \left( \pi_s - \hbar \frac{\pi_a}{2}, x_s - \hbar \frac{x_a}{2} \right) - \frac{1}{\hbar} H \left( \pi_s + \hbar \frac{\pi_a}{2}, x_s + \hbar \frac{x_a}{2} \right) \right].
\]
By performing an expansion in \( \hbar \) we obtain, at the first non-trivial order, the following expression
\[
S_{\text{SK}}^{\hbar \to 0} = \int dt \left[ -x_a \left( \pi_s + \partial_{x_a} H (\pi_s, x_s) \right) + \pi_a \left( \dot{x}_s - \partial_{x_s} H (\pi_s, x_s) \right) \right] + O(\hbar).
\]
Interestingly enough, the terms isolated in the round brackets are exactly the Hamilton equation of motion in terms of the \( s \)-fields. On the other hand, the \( a \)-fields appear only linearly after the \( \hbar \) expansion, thus acquiring the status of Lagrange multipliers for the Hamilton’s equation of motion.

However, not only the classical limit coincides with its counterpart found in the path integral for the Moyal formalism. Actually the two path integrals are fully equivalent. To see this explicitly let us first compare the classical limit of the actions involved, i.e. \( S_{\text{SK}}^{\hbar \to 0} \) and \( S_{\text{M}}^{\hbar \to 0} \). We see that the two actions are the same provided that we make the following identifications:
\[
\begin{align*}
\pi_s & \equiv p \\
x_s & \equiv q \\
x_a & \equiv -\lambda_p \\
\pi_a & \equiv \lambda_q.
\end{align*}
\]
If we implement this identification in the still exact \( S_{\text{SK}} \) we obtain
\[
\begin{align*}
S_{\text{SK}} &= \int dt \left[ \pi_s \dot{x}_a + \pi_a \dot{x}_s + \frac{1}{\hbar} H \left( \pi_s - \hbar \frac{\pi_a}{2}, x_s - \hbar \frac{x_a}{2} \right) - \frac{1}{\hbar} H \left( \pi_s + \hbar \frac{\pi_a}{2}, x_s + \hbar \frac{x_a}{2} \right) \right] \\
&= \int dt \left[ -p \dot{\lambda}_p + \lambda_q \dot{q} + \frac{1}{\hbar} H \left( p - \hbar \frac{\lambda_p}{2}, q + \hbar \frac{\lambda_q}{2} \right) - \frac{1}{\hbar} H \left( p + \hbar \frac{\lambda_q}{2}, q - \hbar \frac{\lambda_p}{2} \right) \right] \\
&= \int dt \left[ \lambda_p \dot{p} + \lambda_q \dot{q} - \tilde{H}_B \right] + \text{surface term}.
\end{align*}
\]
This makes it clear that the functional integrals of the Moyal and Schwinger-Keldysh approach are actually identical and are related by simple identifications: the s-fields correspond to the Moyal’s phase-space coordinates \( \varphi = (q, p) \) while the a-fields play the role of the conjugate variables \( \lambda = (\lambda_q, \lambda_p) \).

Let us note that the integration by parts performed in (2.3) leaves a surface term which is somewhat unpleasant. This surface term is however not present after a more careful analysis. The kernel of propagation associated to the SK action \( S_{SK} \) has the form \( \langle x_{1,t}, x_{2,t}; t| x_{1,0}, x_{2,0}; 0 \rangle \), where \( x_{i,0} \) and \( x_{i,t} \) are eigenvalues of the position operators \( \hat{x}_1 \) and \( \hat{x}_2 \). Equivalently, a state \( | x_{1,t}, x_{2,t}; t \rangle \) can also be labelled as an eigenstate of the operators \( \hat{q} = \frac{1}{\hbar} (\hat{x}_1 + \hat{x}_2) \) and \( \hat{\lambda}_p = (\hat{x}_2 - \hat{x}_1) / \hbar \) so that we can rewrite \( \langle x_{1,t}, x_{2,t}; t| x_{1,0}, x_{2,0}; 0 \rangle = \langle q_t, \lambda_{p,t}; t| q_0, \lambda_{p,0}; 0 \rangle \). On the other hand, when we consider the path integral associated to the Moyal action \( S_M \) we are actually considering the kernel of propagation \( \langle q_t, p_t; t| q_0, p_0; 0 \rangle \). The two kernels are related as follows

\[
\langle q_t, p_t; t| q_0, p_0; 0 \rangle = \int d\lambda_{p,t} d\lambda_{p,0} \frac{1}{(\sqrt{2\pi})^2} e^{i\lambda_{p,t} p_t - i\lambda_{p,0} p_0} \langle q_t, \lambda_{p,t}; t| q_0, \lambda_{p,0}; 0 \rangle. \tag{2.4}
\]

We observe that, using the path integral expression for the kernels of propagation, the surface terms appearing in (2.3) cancel against the exponential appearing in (2.4).

Before concluding this section let us point out some generic features of the Schwinger-Keldysh formalism for later purposes. Let us define the generating functional of connected Green’s function via

\[
e^{W[J_1; J_2]} = \text{Tr} \left[ U_1 (+\infty, -\infty; J_1) \rho_0 U_2^\dagger (+\infty, -\infty; J_2) \right], \tag{2.5}
\]

where \( J_i \) are sources which are associated to operator insertions in the forward and backward evolutions. Clearly, if we set \( J_1 = J_2 = J \) the cyclicity of the trace implies that (see for instance \([3]\))

\[
e^{W[J; J]} = 1. \tag{2.6}
\]

This condition is sometimes referred to as the unitarity condition.

Moreover, if the initial density matrix is thermal, namely \( \rho_0 \sim e^{-\beta H} \), then a further condition, known as the KMS condition, applies. Indeed, in this case \( \rho_0 \) can be thought of as an evolution operator for imaginary times. For an initial thermal density matrix, manipulating the trace (2.5), one can connect \( W[J_1, J_2] \) to the generating functional associated to the time reversed process, see e.g. \([3, 23]\). By combining the KMS condition with time reversal symmetries\(^1\) one obtains a constraint directly on the functional \( W[J_1, J_2]: \)

\[
W[J_1(t_1), J_2(t_2)] = W[J_1(-t_1), J_2(-t_2 - i\beta)]. \tag{2.7}
\]

\(^1\) \( PT \) and \( CPT \) symmetries have also been used for the same purpose \([3, 12]\).
If the microscopic action $S$ is time reversal (and time translation) invariant, it is possible to check that relation (2.7) follows from a symmetry of the action in the forward and backward branches [3, 5, 12].

3 Super-extended Moyal formalism

In this section we consider the super-extended Moyal formalism put forward in [15] as a proposal to introduce an exterior differential calculus in QM. This formalism naturally evolves not only the Wigner function $\rho(\varphi; t)$ but also its generalization including additional Grassmann odd fields.

In section 3.1 we show that the super-extended Moyal formalism yields the same correlation functions as the standard one (provided one considers the very same observable).

In section 3.2 we compare the super-extended Moyal formalism with the super-extended SK framework. We pay particular attention to the topological sector of the theory and show that by changing the boundary condition of the path integral one can construct topological invariants of the phase-space manifold.

3.1 Super-extended Moyal formalism

We have seen that the path integral description of the Moyal formalism allows easily to make contact with the path integral description of classical mechanics [22]. It turns out that in this limiting case a supersymmetric extension is very useful and it has been studied thoroughly giving light to many interesting results related to ergodicity, the symplectic geometry of classical mechanics, quantization, and so on [24–30]. In section 3.1.1 we give a brief overview of the CPI formalism (the reader already familiar with it may skip this section). In section 3.1.2 we review the super-extended Moyal formalism pointing out that the introduction of the ghosts does not modify the correlation functions.

3.1.1 Classical path integral

In the 1930s Koopman and von Neumann gave an operatorial formulation of classical mechanics [31, 32]. Let us consider the Liouville equation for a probability density in the phase-space, $\rho(q, p; t)$:

$$\partial_t \rho(q, p) = \left[ \partial_q H \partial_p - \partial_p H \partial_q \right] \rho(q, p; t).$$

Koopman and von Neumann, inspired by the formalism of quantum mechanics, introduced an Hilbert space of the square integrable functions on the phase-space and constructed an operatorial approach to CM [31, 32]. Given this operatorial formalism for CM, we expect that there should be a path integral counterpart of it and, of course, it should be different
from the one associated to QM. This path integral approach was studied in [22], whose main properties we now review.

First of all, we can consider that the transition amplitude for CM should give weight “one” to the classical trajectories and zero to all other paths. So the classical “transition amplitude” is:

$$K(\varphi^a, t; \varphi^a_0, t_0) = \delta(\varphi^a - \Phi^a_{cl}(t; \varphi^a_0, t_0))$$  \hspace{1cm} (3.1)

where $$\Phi^a_{cl}(t; \varphi^a_0, t_0)$$ is the classical solution of the Hamiltonian equation of motion,

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b},$$

with initial conditions $$\Phi^a_{cl}(t_0; \varphi^a_0, t_0) = \varphi^a_0$$. The functional Dirac delta in (3.1) can be written as:

$$\delta(\varphi^a - \Phi^a_{cl}) = \delta(\dot{\varphi} - \omega^{ab} \partial_b H) \det \left( \delta^a_b \partial_t - \omega^{ac} \partial_c \partial_b H \right),$$

where $$\det [\cdots]$$ is a functional determinant. By expressing the Dirac delta as a Fourier transform and exponentiating the determinant via Grassmann odd fields, to which we refer to as ghosts, we finally obtain the representation:

$$K(\varphi^a, t; \varphi^a_0, t_0) \propto \int D\varphi^a D\lambda^a D\bar{c}^a D\bar{\bar{c}}^a \exp \left[ i \int_{t_0}^t d\tau \tilde{L} \right],$$

where the integration $$D\varphi^a$$ indicate the integration over $$\varphi^a$$ with fixed initial and final configuration. The Lagrangean $$\tilde{L}$$ which appears in the formula is:

$$\tilde{L} = \lambda^a(\dot{\varphi}^a - \omega^{ab} \partial_b H) + i\bar{c}^a \dot{c}^a - i\bar{\bar{c}}^a \omega^{ac} \partial_c \partial_b H c^b$$  \hspace{1cm} (3.2)

and its associated Hamiltonian reads:

$$\tilde{H} = \lambda^a \omega^{ab} \partial_b H + i\bar{c}^a \omega^{ac} \partial_c \partial_b H c^b.$$  

The action associated to $$\tilde{L}$$ enjoys a set of universal symmetries. In particular, the action is invariant under the group $$ISp(2)$$ and enjoys a $$N = 2$$ supersymmetry [22]. The reader may wonder about the role of the ghosts $$c^a$$ and $$\bar{c}^a$$. It turns out that these ghosts have been crucial to make contact with many interesting topics like Cartan calculus, ergodicity and so on (we refer the reader to [21] for an overview).

Before concluding, let us recall one of the universal symmetries present in the CPI and consider the symmetry transformation

$$\delta \varphi^a = \epsilon c^a, \quad \delta \bar{c}_a = i\epsilon \lambda_a, \quad \delta c^a = \delta \lambda_a = 0.$$  \hspace{1cm} (3.3)
Remarkably, the conserved charge associated to this symmetry, denoted \( Q_{BRS} = ic^a \lambda_a \) in [22], can be understood as an operator implementing the exterior derivative on the phase-space (the ghosts play the role of differentials \( c^a \leftrightarrow d \varphi^a \)). The full set of symmetries gives birth to a set of charges whose algebra is that of \( ISp(2) \). All these charges can be given a nice geometrical meaning in the framework of Cartan calculus [22].

### 3.1.2 Super-extension of the Moyal formalism

Inspired by the classical case, we study the super-extension of the Moyal formalism, which has been put forward in [15] in order to investigate how to possibly introduce geometric operators, like the exterior derivative, in QM. Quite generally, given the Hamiltonian

\[
\tilde{H} = \mathcal{F} \left( \lambda_a \omega^{ab} \partial_b \right) \mathcal{H} + i \bar{c}^a c^b \partial_a \partial_b \mathcal{H} + \mathcal{F} \left( \lambda_a \omega^{ab} \partial_b \right) \mathcal{H},
\]

where

\[
\mathcal{F} \left( \lambda_a \omega^{ab} \partial_b \right) = \frac{2}{\hbar} \left[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{\hbar}{2} \lambda_a \omega^{ab} \partial_b \right)^{2n+1} \right] = \frac{2}{\hbar} \sinh \left( \frac{\hbar}{2} \lambda_a \omega^{ab} \partial_b \right),
\]

and \( \bar{c}^a \) and \( c^b \) are Grassmann odd fields, which we call ghosts.

The classical case is recovered via the identification \( \mathcal{F} \left( \lambda_a \omega^{ab} \partial_b \right) \rightarrow \lambda_a \omega^{ab} \partial_b \), consistently with the limit \( \hbar \rightarrow 0 \). As shown in [15], the super-extended Hamiltonian \( \tilde{H}^h \) is determined by the following two requirements:

(i) the complete Hamiltonian \( \tilde{H}^h \) is assumed to be invariant under the BRST transformation (3.3);

(ii) the ghost part of the Hamiltonian, \( \tilde{H}^h_F \), is at most bilinear in the ghosts.

The action associated to the Hamiltonian \( \tilde{H}^h \) is

\[
\int dt \tilde{\mathcal{E}}^h = \int dt \left[ \lambda_a \dot{\varphi}^a - \tilde{H}^h_B + i \bar{c}^a c^a - \tilde{H}^h_F \right]. \tag{3.4}
\]

This action characterizes the path integral weight associated to the super-extended Moyal formalism, whose partition function is given by

\[
Z = \int \mathcal{D} \varphi^a \mathcal{D} \lambda_a \mathcal{D} c^a \mathcal{D} \bar{c}^a e^{i \int dt \tilde{\mathcal{E}}^h}. \tag{3.5}
\]
One can readily check that the new action (3.4) is BRS invariant. Moreover, it is interesting to note that not only the BRS symmetry survives the “quantization process”, namely to go from $\tilde{\mathcal{H}}^{\hbar=0}$ to $\tilde{\mathcal{H}}^{\hbar}$. Rather, all the $\text{ISp}(2)$ symmetry group is still present \cite{15}. However, the action is no longer invariant under the $N = 2$ supersymmetry present in the classical case.\footnote{As suggested in \cite{15} and \cite{3} it would be interesting to investigate whether a deformed version of the SUSY survives instead.}

To the super-extended path integral (3.5) one can associate an operatorial formalism, analogous to the one presented in section 2.2 (see \cite{15} for more details). In particular, we may consider the representation

$$\hat{c}^a = c^a \quad \hat{c}_a = \frac{\partial}{\partial c^a}.$$  

The path integral can then be thought of as representing the kernel of propagation associated to the following Schrödinger-like equation

$$i\partial_t \rho(\varphi^a, c^a; t) - \tilde{\mathcal{H}}^{\hbar} \rho(\varphi^a, c^a; t) = 0,$$

where now the variables in $\tilde{\mathcal{H}}^{\hbar}$ are interpreted as operators. We have already detailed the operatorial form of $\tilde{\mathcal{H}}^{\hbar}$ in section 2.2 its ghost extension acts as follows:

$$\tilde{\mathcal{H}}^{\hbar}_F \rho(\varphi^a, c^a; t) = \left[ i\frac{\partial}{\partial c^a} \omega^{ac} \partial_c \partial_b F(\lambda^a \omega_{ab} \partial_b) \right] \rho(\varphi^a, c^a; t).$$

Next, let us show that the ghosts that we just added are harmless as their contribution corresponds to a unit determinant. Indeed, the full ghost Lagrangean is given by

$$\tilde{\mathcal{L}}_F = i\hat{c}_a \hat{c}^b - i\hat{c}_a \omega^{ac} \partial_c \partial_b F(\lambda^a \omega_{ab} \partial_b) H c^b.$$
Integrating over the ghosts, the Lagrangean $\tilde{L}_F$ gives rise to a determinant in the path integral. This determinant reads

$$\int Dc^a D\bar{c}_a e^{i \tilde{L}_F} = \det \left( \partial_t - \omega^{ac} \partial_c \partial_b \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H \right) \delta (t - t')$$

$$= \exp \text{Tr} \log \left( \partial_t \left( 1 + \omega^{ac} \partial_c \partial_b \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H \right) \delta (t - t') \right)$$

$$= \exp \text{tr} \left[ \log \left( 1 + \omega^{ac} \partial_c \partial_b \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H \right) \delta (t - t') \right]$$

$$= \exp \text{tr} \left[ \log \left( 1 + \omega^{ac} \partial_c \partial_b \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H \right) \delta (t - t') \right] = 1,$$

where in the third line we dropped a field independent term. (Note that we are considering retarded boundary condition for the Green’s function.)

Therefore, if we consider the insertion of operators that do not depend on the ghosts, like any operator present in the standard Moyal formalism, we would obtain the same correlation function as in the standard case without ghosts. This is due to the fact that the integration over the ghosts produces no terms that depend on the dynamical variables.

Thus, the super-extension of the Moyal formalism is in some sense a redundant description. However, from the CPI case, we know that, despite being not strictly needed, the ghost sector considerably enlightens many aspects of the theory. Indeed, the super-extended version of the Moyal formalism has been used in [15] to investigate a possible extension of Cartan calculus to QM.

Finally let us recall that, as noted in [15], the full Lagrangean can be written as a BRS-variation, in particular

$$\tilde{L}^h = \delta_{\text{BRS}} \left( \bar{c}_a \left( \phi^a - \omega^{ab} \partial_b \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H \right) \right).$$

Concluding, let us also recall the following rewriting for the Hamiltonian $\tilde{H}^h$ found in [15]. We define

$$H^h \equiv \frac{F(\lambda^a \omega^{ab} \partial_b)}{\lambda^a \omega^{ab} \partial_b} H$$

$$= \frac{1}{2} \int_{-1}^1 ds \exp \left[ -s \left( \frac{\hbar}{2} \lambda^a \omega^{ab} \partial_b \right) \right] H$$

$$= \frac{1}{2} \int_{-1}^1 ds H \left( \phi^a - s \frac{\hbar}{2} \lambda^a \omega^{ba} \right).$$

15
Using this and introducing $h^a_h \equiv \omega^{ab} \partial_b H_h$ we have

$$\tilde{\mathcal{H}}^h = \lambda^a h^a_h + i \bar{c}^a \partial_b h^a_h H^b.$$  \hfill (3.8)

At this point we note that the super-extended Hamiltonian of the Moyal formalism, $\tilde{\mathcal{H}}^h$, can be obtained from its classical limit by replacing $H(\varphi)$ with $H_h(\varphi, \lambda)$.

### 3.2 Comparison with the Schwinger-Keldysh formalism

We wish to briefly discuss the super-extension of the SK formalism and its link with the Moyal super-extension. The SK extension has been studied in detail recently in [3–12].

In [3] the authors construct an effective field theory (EFT) for hydrodynamics by expressing the low energy path integral in terms of Stuckelberg fields which allow to keep manifest the symmetries of the original action when equipped with sources (background gauge fields and the metric). In order to maintain the unitarity property (2.6) in the low energy EFT, the authors note that introducing certain Grassmann odd fields (ghosts), and imposing a BRS symmetry analogous to the one we considered in section 3.1.1 allow to keep the property (2.6) manifest. Moreover, it is explicitly checked that this ghost sector is required in the classical limit.

On the other hand, in [9] the authors introduce the ghost sector by considering a generic field redefinition and carrying out a gauge-fixing like procedure. This procedure does not change the content of the theory but allows once again to make manifest the unitarity condition (2.6). The relation between these two latter frameworks is analyzed in [11].

In this section we observe that the super-extension of the path integral for the Moyal approach to QM carries many similarities with the various super-extensions proposed for the SK formalism. In particular, in section 3.1.2 we have seen that the action (3.4) has been tailored precisely to inherit the BRS symmetry present in the classical case. A similar reasoning underlies the extension proposed in [3]. However, in [15] the ghosts were added with the purpose of putting forward a path integral implementation of geometric operations, like the exterior derivative, but essentially the ghost action has been selected without any strong need behind. From the SK formalism point of view, we see that there is a further reason to add the ghost fields, namely to preserve the condition (2.6) in a manifest way.

The theory obtained by setting equal the sources on the forward and backward branches, so that condition (2.6) is realized, is sometimes referred to as the topological sector of the SK formalism. We wish to show here that also in the Moyal approach certain topological properties are present. This has been fully investigated in the classical limit $\hbar \to 0$ in [33]. We will prove that similar observations apply also in the quantum case. Before discussing
the quantum case let us recall the classical result. In [33] it is shown that the following functional integral describes a topological field theory:

\[
Z_{\text{pbc}} \equiv \int_{\text{pbc}} \mathcal{D}\varphi^{a} \mathcal{D}\lambda_{a} \mathcal{D}c^{a} \mathcal{D}\bar{c}^{a} e^{i \int_{0}^{T} L_{\hbar} dt} = \int d\varphi_{0}^{a} dc_{0}^{a} K (\varphi_{0}^{a}, c_{0}^{a}, T; \varphi_{0}^{a}, c_{0}^{a}, 0),
\]

(3.9)

where \(L_{\hbar} = 0\) is the Lagrangean given in (3.2), and periodic boundary conditions (pbc) have been imposed on both the phase-space variables \(\varphi^{a}\) and the ghosts \(c^{a}\). The fact of having periodic boundary conditions also for the Grassmann odd fields \(c^{a}\) is due to the fact that only in this case the boundary conditions are also BRS invariant, as is the integrand see [33]. The next crucial fact proven in [33] is that \(Z_{\text{pbc}}\) is actually independent of the dynamics, i.e. it does not depend on \(H\), and

\[
Z_{\text{pbc}} = \text{Tr} \left[ (-1)^{F} e^{-i \tilde{L}_{\hbar=0} T} \right] = \text{Tr} \left[ (-1)^{F} \right].
\]

(3.10)

In (3.10) the trace is performed over functions \(\rho(\varphi^{a}, c^{a})\) or, equivalently, a set of forms \(\rho_{1,\ldots,p}(\varphi^{a})\), with \(p \in [0, 2n]\). \(F\) indicates the “fermion” number operator which counts the degree of the respective differential form, i.e. \((-1)^{F} = +1\) for even \(p\) and \((-1)^{F} = -1\) otherwise. Thus, \(Z_{\text{pbc}}\) evaluates the Witten index for the CPI.

In order to generalize these results to the quantum case \(\hbar \neq 0\) we consider

\[
Z_{\text{pbc}} \equiv \int_{\text{pbc}} \mathcal{D}\varphi^{a} \mathcal{D}\lambda_{a} \mathcal{D}c^{a} \mathcal{D}\bar{c}^{a} e^{i \int_{0}^{T} L_{\hbar} dt} \quad (3.11)
\]

and ask if the value of \(Z_{\text{pbc}}\) in the super-extended Moyal approach depends on \(\hbar\):

\[
\frac{d}{d\hbar} Z_{\text{pbc}} = \frac{d}{d\hbar} \int_{\text{pbc}} \mathcal{D}\varphi^{a} \mathcal{D}\lambda_{a} \mathcal{D}c^{a} \mathcal{D}\bar{c}^{a} e^{i \int_{0}^{T} L_{\hbar} dt}.
\]

It is convenient to re-express the path integral via the following rescaled fields: \(\lambda_{a}' \equiv \hbar \lambda_{a}\) and \(\bar{c}_{a}' \equiv \hbar \bar{c}_{a}\) so that, dropping the prime, the action reads

\[
\int dt \, \tilde{L}_{\hbar} = \frac{1}{\hbar} \int dt \left[ \lambda_{a} \dot{\varphi}^{a} - \tilde{H}_{B}^{\hbar=1} + i \bar{c}_{a} \dot{c}^{a} - \tilde{H}_{F}^{\hbar=1} \right],
\]

where the \(\hbar\) is now present only as an overall factor. Let us point out that the path integral measure is invariant under this rescaling due to the properties of the Jacobian of Grassmann even and odd variables. In particular \(d\lambda_{a} = d\lambda_{a}' / \hbar\) and \(d\bar{c}_{a} = d\bar{c}_{a}' / \hbar\), owning to the fact that the Jacobian for Grassmann variables is just the inverse of the standard one.
As a consequence, when differentiating \( Z_{\text{pbc}} \) with respect to \( \hbar \) we find

\[
\frac{d}{d\hbar} Z_{\text{pbc}} = \int_{\text{pbc}} \mathcal{D} \varphi^a \mathcal{D} \lambda_a \mathcal{D} c^a \mathcal{D} \bar{c}_a \, e^{\frac{i}{\hbar} \int_0^T dt \mathcal{L}^{\hbar = 1}} \left( -\frac{i}{\hbar^2} \int_0^T dt' L^{\hbar = 1} \right).
\]

Since the Lagrangian can be re-expressed as BRS variation we have that

\[
\frac{d}{d\hbar} Z_{\text{pbc}} = -\frac{i}{\hbar^2} \int_0^T dt \int_{\text{pbc}} \mathcal{D} \varphi^a \mathcal{D} \lambda_a \mathcal{D} c^a \mathcal{D} \bar{c}_a \, e^{\frac{i}{\hbar} \int_0^T dt \mathcal{L}^{\hbar = 1}} \left[ \delta_{\text{BRS}} (\bar{c}_a (\dot{\varphi}^a - \omega^{ab} \partial_b \mathcal{L}^{\hbar = 1})) \right],
\]

which, upon integrating by parts in field space, yields a vanishing result due to the nilpotent character of the BRS transformation, \((\delta_{\text{BRS}})^2 = 0\).

One can also give a further argument that suggests that \( Z_{\text{pbc}} \) is actually independent of \( \hbar \). The topological nature of \( Z_{\text{pbc}} \) for the CPI is tied to its invariance with respect to arbitrary deformations of the Hamiltonian, \( H(\varphi^a) \rightarrow H(\varphi^a) + \delta H(\varphi^a) \). However, it is straightforward to check that the super-extended Moyal Lagrangian \( \mathcal{L}^{\hbar} \) can be obtained from the CPI one, i.e. \( \bar{\mathcal{L}} \), by replacing \( H \) with \( H^{\hbar} \), introduced in equation (3.7). Since, the classical \( Z_{\text{pbc}} \) is invariant under deformation of \( H \) it is also invariant under the deformation \( H \rightarrow H^{\hbar} \). Note, however, that \( H^{\hbar} \) depends also on \( \lambda_a \), and not only on \( \varphi^a \), and so, strictly speaking, the deformation \( H \rightarrow H^{\hbar} \) does not belong to class \( H(\varphi^a) \rightarrow H(\varphi^a) + \delta H(\varphi^a) \) that we are considering. Therefore, this latter observation is just an argument, the proof of our previous statements is actually based on the BRS invariance.

To make the discussion self-contained, let us mention how to obtain \( Z_{\text{pbc}} \). All the details can be found in ref. [33]. Since \( Z_{\text{pbc}} \) is independent from \( \hbar \), we set \( \hbar = 0 \) and work with the CPI Lagrangian. Due to equation (3.10), \( Z_{\text{pbc}} \) is actually independent of the time \( T \) and we can compute it via the associated path integral (3.9) in the limit \( T \rightarrow 0 \).

In the CPI Lagrangian the fields \( \lambda_a \) and \( \bar{c}_a \) appear linearly and functionally integrating them out simply produces a delta function. Hence, using the linearized solution of the equation of motion in the limit \( T \rightarrow 0 \) we have [33]

\[
Z_{\text{pbc}} = \int d\varphi^a_0 dc^a_0 \delta \left( \varphi^a_0 - \left( \varphi^a_0 + \omega^{ab} \partial_b H \right) \right) \delta \left( c^a_0 - \left( c^a_0 + \omega^{ac} \partial_c \partial_b H \right) c^b_0 \right)
\]

\[
= \sum_p \int d\varphi^a_0 dc^a_0 \frac{\det (\omega^{ac} \partial_c \partial_b H (\varphi_p))}{\det (\omega^{ac} \partial_c \partial_b H (\varphi_p))} \delta \left( \varphi_0 - \varphi_p \right) \delta (c^a_0)
\]

\[
= \sum_p (-1)^{i_p} = \chi (\mathcal{M}_{2n}).
\]

In the above lines \( \varphi_p \) are the critical points of \( H \), i.e. those points in which the Hamiltonian vector field vanishes, while \( i_p \) is the number of negative eigenvalues of the Hessian of \( H \) at the critical point. It follows that \( Z_{\text{pbc}} \) is actually the Morse theory representation...
of the Euler number of the manifold on which the “Morse function” $H$ is defined, i.e. the phase-space manifold associated to the mechanical system $[33]$. Along the lines of ref. $[33]$, it is also possible to study other topological observables.

Therefore, we have shown that there exists a topological sector also in the super-extended Moyal formalism. Despite being seemingly close, the topological sector that we have just discussed and that usually considered in the SK formalism are not the same. To see this we recall that the topological sector in the SK formalism is related to the unitarity condition $[2.6]$, which can be rewritten as

$$\text{Tr} \left[ U(t, 0; J) \rho_0 U^\dagger(t, 0; J) \right] = \text{Tr} [\rho_0],$$

(3.12)

whose RHS is explicitly independent from the Hamiltonian evolving the system. Therefore, also the path integral representation of (3.12) is independent from the dynamics of the system. However, as we shall check in a moment, the path integral expression associated to (3.12) is different from the one concerning $Z_{\text{pbc}}$. We have

$$\text{Tr} \left[ U(t, 0; J) \rho_0 U^\dagger(t, 0; J) \right] = \text{Tr} [\rho_0]$$

$$= \int d\varphi^a dc^a \rho (\varphi^a, c^a; t)$$

$$= \int d\varphi^a dc^a d\varphi^a_0 dc^a_0 K (\varphi^a_0, c^a_0; t; \varphi^a_0, c^a_0, 0) \rho (\varphi^a_0, c^a_0; 0),$$

where the kernel of propagation $K (\varphi^a_0, c^a_0; t; \varphi^a_0, c^a_0, 0)$ is given in terms of the super-extended Moyal path integral. A particularly simple choice for the initial density $\rho (\varphi^a_0, c^a_0; 0)$ is

$$\rho (\varphi^a_0, c^a_0; 0) = \delta (\varphi^a_0 - \varphi^a) \delta (c^a_0 - c^a).$$

With this choice we have the following path integral representation

$$\text{Tr} \left[ U(t, 0; J) \rho_0 U^\dagger(t, 0; J) \right] = \int d\varphi^a dc^a K (\varphi^a_0, c^a_0; t; \varphi^a, c^a, 0)$$

$$= \int d\varphi^a dc^a \int_{\varphi, c} D\varphi^a D\lambda_a Dc^a D\bar{c}^a e^{i \int_0^t \mathcal{L}}.$$

This latter expression is clearly different from the one concerning $Z_{\text{pbc}}$ due to the different boundary conditions. To see this more explicitly we can mimic the reasoning that we did for $Z_{\text{pbc}}$. Since we know that actually the topological SK sector is independent of $\hbar$ due to equation (3.12), at least whenever $\hbar$ does not enter in the initial density matrix as in
our case, we can set $\hbar = 0$ and work once again with the CPI. Integrating over $\lambda_a$ and $\bar{c}_a$ produces again a delta function, and in the limit $t \to 0$

$$
\text{Tr} \left[ U (t, 0; J) \rho_0 U^\dagger (t, 0; J) \right] = \int d\varphi^a_i d\bar{c}_i \delta \left( \varphi^i_a - (\varphi^i_a + \omega^{ab} \partial_b Ht) \right) 
\times \delta \left( c^i_a - (c^i_a + \omega^{ac} \partial_c \partial_b H T c^b) \right)
\times 1,$$

where we essentially found the normalization condition $\text{Tr} [\hat{\rho}_t] = \text{Tr} [\hat{\rho}_0] = 1$, as it should be.

Next, let us discuss a further aspect concerning the super-extended Moyal approach of section 3.1.2 and the super-extended SK formalism. In section 2.3 we have seen that the actions of the Moyal path integral and the SK one coincide after a suitable redefinition of the fields. It follows that any symmetry present in the SK formulation can be translated into an associated symmetry transformation in the Moyal formalism. As we mentioned at the end of section 2.3, the KMS condition (2.7) can be seen as a consequence of the time reversal symmetry on the forward branch and the time reversal plus an imaginary time translation in the backward branch (provided that the action is time reversal invariant, see e.g. [3, 12]). This symmetry can be translated also in the Moyal language, for instance $q(t) = \frac{1}{2} (x_1 (t) + x_2 (t))$ is mapped to $\frac{1}{2} (x_1 (-t) + x_2 (-t - i\beta)) = \frac{1}{4} (-\lambda q (-t) + \lambda q (-t - i\beta) + 2q (-t) + 2q (-t - i\beta))$. The crucial fact that allows to translate a SK symmetry in the Moyal framework is that the Moyal action $S_M$ and the SK action $S_{SK}$ are mapped one onto the other via the mapping discussed in section 2.3. In particular, via a suitable redefinition of the field, we have seen that the action $S_M$ can be “split” into two parts, which correspond to those associated to the backward and forward branches in the SK formalism. It is natural to ask whether this splitting property is enjoyed also by the ghost sector. Seemingly, this is not the case as the part of the action containing $\tilde{H}_F\hbar$ can not be split along the lines used for $\tilde{H}_B\hbar$, at least straightforwardly.

The reader may worry that the KMS condition is lost even at the level of correlation functions. This is not the case for observables which do not depend on the ghosts. Indeed we have shown that the ghost integration is actually harmless and produces just a factor of unity in the path integral. Because of this, the KMS condition is still satisfied at the level of correlation functions.

With regard to the KMS condition, a very interesting observation applies to the classical limit $\hbar \to 0$. In [24] it was shown that actually the classical limit of the KMS condition can be derived from the CPI Lagrangean as a Ward identity associated to the supersymmetry present in the CPI. However, the supersymmetry present in the CPI is no longer present in the quantum case and it has been speculated that a quantum version
of it might be present \[8, 12, 15\]. Recently, a recipe to write down supersymmetric out of equilibrium field theories has been put forward \[5\], it would be interesting to investigate whether or not this is applicable to the Moyal case. If this were the case, one would have a new proposal for the super-extension of the theory and possibly have a new way to define geometric operations such as exterior derivatives in QM.

Finally, it would be interesting to look in more detail at the relation between the KMS condition as arising in the path integral formulation of the Moyal formalism and that developed in deformation quantization, which introduces the notion of conformal symplectic structure \[34\].

## 4 BBGKY hierarchy

So far we have been discussing the path integral associated to the evolution of the phase-space density distribution \(\rho(\varphi)\). However, \(\rho(\varphi)\) often carries much more information than what one is interested in. In kinetic theory the reduced distribution functions (defined below) play a very important role. The reduced distribution functions \(f_n\) for a system with \(N\) degrees of freedom are defined by

\[
 f_n(\varphi_1, \cdots \varphi_n; t) \equiv \frac{N!}{(N-n)!} \int d\varphi_{n+1} \cdots d\varphi_N \rho(\varphi^n; t),
\]

where \(d\varphi_i \equiv dq_i dp_i\). Eventually, we will denote \(d\Gamma\) the integration element over the full phase-space, while the integration element over the phase-space of \(n\) particles only will be indicated with \(d\Gamma_n\).

In classical mechanics the functions \(f_n\) satisfy an hierarchy of equations, the BBGKY hierarchy, that describes the temporal evolution of \(f_n\) via a partial differential equation involving \(f_n\) and \(f_{n+1}\), giving rise to an infinite hierarchy of equations. In the following we deduce the BBGKY hierarchy via path integral methods connected to the Moyal approach to QM. We shall first deal with the bosonic, i.e. without ghost, case and then consider the full super-extended framework.

### 4.1 Bosonic BBGKY hierarchy

In order to derive the BBGKY hierarchy in our framework, we need first to consider a suitable path integral description. For formal manipulations it turns out convenient to consider the path integral implementing the kernel of propagation directly on \(\rho\). The evolution of \(\rho\) is determined by equation (2.2) and its kernel of propagation can be given a path integral representation with the techniques reviewed in section 3.1.1. This program has been implemented successfully in the classical case in \[35, 37\], in this section we extend this program to the quantum case.
In practice, $\rho$ becomes a field and the phase-space variables are integrated over in the associated action. Thus, we consider the following functional integral:

$$\tilde{Z} = \int D\Lambda D\rho \exp\left[i \int dt d\Gamma \left(\partial_t - \hat{L}_h\right) \rho\right].$$

(4.1)

Since equation (2.2), which determines the evolution of $\rho$, is linear in $\rho$ then its kernel of propagation is simply quadratic in $\rho$ and its associate “conjugate” field $\Lambda$. Furthermore, following [35–37], we can generically parametrize the field $\Lambda$ as follows:

$$\Lambda (\varphi^a, t) = \Lambda_0 (t) + \left(\sum_{i=1}^N \Lambda_1 (q_i, p_i, t)\right) + \left(\sum_{i<j} \Lambda_2 (q_i, p_i, q_j, p_j, t)\right) + \cdots.
$$

Note that since the particles are indistinguishable the functions $\rho$, $f_n$ and the fields $\Lambda_i$ are totally symmetric. The action is now characterized by the following structure:

$$\int \Delta \left(\partial_t - \hat{L}_h\right) \rho = \int dt d\Gamma \left(\partial_t + \hat{L}_h\right) \rho$$

$$+ \int dt d\Gamma \left[\sum_{i=1}^N \Lambda_1 (q_i, p_i, t) \left(\partial_t + \hat{L}_h\right) \rho\right]$$

$$+ \int dt d\Gamma \left[\sum_{i<j} \Lambda_2 (q_i, p_i, q_j, p_j, t) \left(\partial_t + \hat{L}_h\right) \rho\right] + \cdots. \quad (4.2)$$

The operator $\hat{L}_h$, introduced in equation (2.2), contains derivatives with respect to the phase-space variables acting on $\rho$. Now we wish to show that the part of the action identified by the field $\Lambda_n (q_{i_1}, p_{i_1}, \ldots, q_{i_n}, p_{i_n}, t)$ reproduces the quantum version of the equation for $\rho_n$ in the BBGKY hierarchy.

First let us specify the Hamiltonian explicitly:

$$H \equiv H_n + H_{N-n} + H_m,$$

where we distinguished the following three contributions

$$H_n = \sum_{i=1}^n \frac{p_i^2}{2m} + U (q_i) + \frac{1}{2} \sum_{i \neq j=1}^n V (q_i - q_j)$$

$$H_{N-n} = \sum_{i=n+1}^N \frac{p_i^2}{2m} + U (q_i) + \frac{1}{2} \sum_{i \neq j=n+1}^N V (q_i - q_j)$$

$$H_m = \sum_{i<j} V (q_i - q_j).$$
The Hamiltonians $H_n$ and $H_{N-n}$ describes respectively the interaction of two subsets of $n$ and $N - n$ particles with themselves. The term $H_m$, where the subscript $m$ stays for mixed, contains the interaction between the $N - n$ particles, that we wish to integrate over in $\rho$, and the remaining $n$ particles (the sum in $H_m$ is ordered for practical purposes).

Let us consider that the operator $\hat{L}_B^h \equiv \hat{L}_B$ is linear in $H$ (the subscript $B$ is introduced for later purposes). This implies that we can easily separate the three contributions and distinguish the three operators $\hat{L}_B^h (H_n)$, $\hat{L}_B^h (H_{N-n})$ and $\hat{L}_B^h (H_m)$. We shall focus on the term containing the field $\Lambda_n$ in the action (4.2) and show that it yields a suitable quantum version of the BBKGY hierarchy. In practice, we focus on the term

$$\int \mathcal{L}_n \equiv \int dtd\Gamma \frac{1}{n!} \left[ \frac{N!}{(N-n)!} \Lambda_n (q_1, \ldots, q_n, p_1, \ldots, p_n; t) \left( \partial_t + \hat{L}_B^h (H_n) \right. \right.$$  

$$+ \hat{L}_B^h (H_{N-n}) + \hat{L}_B^h (H_m) \left. \right] \rho,$$

where we took advantage of the totally symmetric character of the terms in the action. We note that a functional integration over the field $\Lambda_n$ produces a delta function in the path integral. In the following we will show that the expression contained in this delta function yields precisely the equation for $f_n$ in the BBGKY hierarchy.

Let us consider each of the terms appearing in $\mathcal{L}_n$ of (4.3). We have

$$\int dtd\Gamma \frac{1}{n!} \left[ \frac{N!}{(N-n)!} \Lambda_n \partial_t \rho \right] = \int dtd\Gamma_n \frac{1}{n!} \left[ \frac{N!}{(N-n)!} \Lambda_n \partial_t \left( \int d\Gamma_{N-n} \rho \right) \right]$$

$$= \int dtd\Gamma_n \frac{1}{n!} \left[ \Lambda_n \partial_t f_n \right],$$

where we exploited the fact that the field $\Lambda_n$ does not depend on the phase-space variables of the $N - n$ particle set. Thus, we note the natural appearance of the reduced distribution $f_n$.

The second term in (4.3) can be dealt with in full analogy. Indeed, the Hamiltonian $H_n$ contains by definition only phase-space variables associated to the set having $n$ particles. We then have

$$\int dtd\Gamma \frac{1}{n!} \left[ \frac{N!}{(N-n)!} \Lambda_n \hat{L}_B^h (H_n) \right] \rho = \int dtd\Gamma_n \frac{1}{n!} \left[ \Lambda_n \hat{L}_B^h (H_n) f_n \right].$$
Let us turn now to the third term in equation (4.3). The operator $\hat{L}_B^h (H_{N-n})$ reads

$$\hat{L}_B^h (H_{N-n}) = -\frac{2}{\hbar} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left( \frac{\hbar}{2} \right)^{2m+1} \omega^{a_1 b_1} \cdots \omega^{a_{2m+1} b_{2m+1}} \partial_{a_1} \cdots \partial_{a_{2m+1}} H_{N-n} \times \partial_{b_1} \cdots \partial_{b_{2m+1}}.$$

In $\mathcal{L}_n$, the derivatives contained in $\hat{L}_B^h (H_{N-n})$ act on $\rho$. Given that $H_{N-n}$ depends only on the $N-n$ particles, we note that the “effective” derivatives acting on $H_{N-n}$ are those with respect to these $N-n$ particles. This determines which derivatives act on $\rho$. For instance, let $a_i = p_i$ then in $\hat{L}_B^h (H_{N-n})$ we have a term proportional to $-\partial_{p_i} H_{N-n} \partial_q$.

Now we note that we can integrate by parts in $\mathcal{L}_n$ the derivatives acting on $\rho$ so that the term $-\partial_{p_i} H_{N-n} \partial_q$ becomes $\partial_q \partial_{p_i} H_{N-n} = 0$. Therefore, we have shown that the term coming from $\hat{L}_B^h (H_{N-n})$ actually vanishes.

Finally, we turn to the term in (4.3) due to $\hat{L}_B^h (H_m)$, which reads

$$\hat{L}_B^h (H_m) = -\frac{2}{\hbar} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left( \frac{\hbar}{2} \right)^{2m+1} \omega^{a_1 b_1} \cdots \omega^{a_{2m+1} b_{2m+1}} \partial_{a_1} \cdots \partial_{a_{2m+1}} H_m \times \partial_{b_1} \cdots \partial_{b_{2m+1}}.$$

Let us focus on a single term of the sum. If one of the derivatives acting on $H_m$ is performed with respect to one of the phase-space variables of the $N-n$ particles set then, by the same reasoning used for $H_{N-n}$, the contribution vanishes. Therefore, the only non-vanishing contributions are those in which all the derivative acting on $H_m$ are performed with respect to the $n$ phase-space variables of the $n$ particles set. Such contributions are proportional to

$$\partial_{a_1} \cdots \partial_{a_{2m+1}} H_m = \partial_{a_1} \cdots \partial_{a_{2m+1}} \left( \sum_{i=1}^{n} \sum_{j=n+1}^{N} V (q_i - q_j) \right),$$

where the derivatives acts only on the $q_i$ dependence. Given the totally symmetric character of $\rho$ and $\Lambda_n$ we can relabel each term in $\sum_{i=1}^{n} \sum_{j=n+1}^{N} V (q_i - q_j)$ in the action via $\sum_{j=n+1}^{N} V (q_i - q_j) = (N-n) V (q_i - q_{n+1})$. Once rewritten in this way, we note that the dependence on the phase-space variables $\{q_i, p_i\}_{i=n+2, \ldots, N}$ in the terms of the action due to $\hat{L}_B^h (H_m)$ is contained only in $\rho$. Thus, we can already forecast that by integrating
over these latter variables we will determine a contributions proportional to $f_{n+1}$. More precisely:

$$
\int d\Gamma \frac{1}{n!} \left[ \frac{N!}{(N-n)!} \Lambda_n \hat{L}^h (H_m) \right] \rho = \int d\Gamma \frac{\Lambda_n}{n!} \left[ -\frac{2}{\hbar} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left( \frac{\hbar}{2} \right)^{2m+1} \times \omega^{a_1 b_1} \cdots \omega^{a_{2m+1} b_{2m+1}} \times \partial^{a_1} \cdots \partial^{a_{2m+1}} \left( \sum_{i=1}^{n} V (q_i - q_{n+1}) \right) \times \partial_{b_1} \cdots \partial_{b_{2m+1}} f_{n+1} \right].
$$

As we anticipated, by performing a functional integration over $\Lambda_n$ a (functional) Dirac delta is generated implying

$$
\partial_t f_n + \hat{L}^h (H_n) f_n = \sum_{i=1}^{n} \int dq_{n+1} dp_{n+1} \left[ -\hat{L}^h (V (q_i - q_{n+1})) f_{n+1} \right]. \quad (4.4)
$$

The above expression tells us that the evolution of the reduced density $f_n$ depends on the reduced density $f_{n+1}$, thus giving rise to an infinite hierarchy. Equation (4.4) can be rewritten in term of Moyal brackets as

$$
\partial_t f_n - \{H_n, f_n\}_m = \sum_{i=1}^{n} \int dq_{n+1} dp_{n+1} \{V (q_i - q_{n+1}), f_{n+1}\}_m. \quad (4.5)
$$

By using the isomorphism between the Moyal approach and the operatorial approach to QM we can further write

$$
i\hbar \partial_t \hat{f}_n - \left[ H_n, \hat{f}_n \right] = \sum_{i=1}^{n} \text{Tr}_{n+1} \left[ \hat{V} (q_i - q_{n+1}), \hat{f}_{n+1} \right],
$$

which is the standard form of the quantum version of the BBGKY hierarchy [38].

Furthermore, equation (4.5) is seen to go into its classical limit simply by replacing the Moyal brackets with the Poisson brackets or, equivalently, setting $\hbar \to 0$. Clearly, it is simple to keep the first non-trivial quantum corrections by truncating at some order in $\hbar$ the expression for the Moyal brackets.

Finally we point out that, besides deriving the BBGKY hierarchy in a novel way, our formalism gives an handy framework to be used in kinetic theory. Indeed, the classical limit of the field theory (4.1) has been used in [35] to derive in an easy way the Balescu-
Lenard collision operator of a high temperature plasma. Further applications have been investigated in [39, 40].

4.2 Super BBGKY hierarachy

In this section we generalize our findings to the super-extended framework. In this case \( \rho \) also depends on the ghost fields and has the following structure:

\[
\rho (\varphi^a, c^a) = \rho_0 (\varphi^a) + \rho_d (\varphi^a) c^d + \cdots + \frac{1}{(2N)!} \rho_{d_1 \cdots d_{2N}} (\varphi^a) c^{d_1} \cdots c^{d_{2N}}.
\]

The zero form density \( \rho_0 (\varphi^a) \) corresponds to the standard density \( \rho (\varphi) \) that we were considering for the bosonic theory. Indeed, it can be checked that \( \rho_0 (\varphi^a) \) is actually determined by (2.2) given that the terms in (3.6) due to \( \mathcal{H}_F \) vanish automatically on zero-forms. However, also the 2\( N \)-form can be identified with the ordinary density \( \rho (\varphi) \) once the ghosts are integrated out. In fact, the operator \( \mathcal{H}_F \) vanishes automatically also on 2\( N \)-forms so that its coefficient is simply determined by the bosonic sector, recovering thus the standard density via

\[
\frac{1}{(2N)!} \rho_{d_1 \cdots d_{2N}} (\varphi^a) c^{d_1} \cdots c^{d_{2N}} = \rho (\varphi) c^{q_1} \cdots c^{p_N}.
\]

We define the super reduced density distributions as follows:

\[
f_n (\varphi^a, c^a; t) \equiv \frac{N!}{(N - s)!} \int d\Gamma_{N - n} d\Gamma_{N - n}^g \rho (\varphi^a, c^a; t),
\]

where \( d\Gamma_{N - n} \) denotes the integration element over the ghost associated to the particles in the \( N - n \) set. Equivalently, we can also introduce the super-coordinate \( \chi_i \equiv (q_i, p_i, c^0_i \equiv \xi_i, c^0_i \equiv \pi_i) \) and write

\[
f_n (\chi_1, \cdots, \chi_n, t) = \frac{N!}{(N - n)!} \int d\chi_{n+1} \cdots d\chi_N \rho (\chi_1, \cdots, \chi_n; t).
\]

The higher forms of \( \rho \) are defined with an unrestricted index summation intended, hence the factorial in the denominator. Alternatively an ordering must be chosen, for instance

\[
\frac{1}{(2N)!} \rho_{d_1 \cdots d_{2N}} (\varphi^a) c^{d_1} \cdots c^{d_{2N}} = \rho_{q_1 p_1 \cdots q_N p_N} (\varphi^a) c^{q_1} c^{p_1} \cdots c^{q_N} c^{p_N}.
\]
The first observation is that also the super-Liouville operator encoded in $\tilde{H}^{h}$ is linear in the Hamiltonian $H$. Thus, we can again separate this operator into three terms associated to $H_n$, $H_{N-n}$, and $H_m$, respectively. In particular, we consider

$$L_F^{h}\rho(\chi; t) = i\tilde{H}^{h}_F\rho(\chi; t) = \left[\omega^{ac}\partial_c\partial_b\frac{2}{\hbar} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\hbar}{2}\right)^{2n+1} \left(\omega^{ab}\partial_b(H)\partial_a(\rho)\right)^{2n} H_m^b\frac{\partial}{\partial c^a}\right)\right] \rho(\chi; t).$$

Having already discussed the properties of the operator $L_B^{h}$ in section 4.1, let us move to those of $L_F^{h}$. It is straightforward to check that $L_F^{h}(H_n)$ and $L_F^{h}(H_{N-n})$ depend only on the $n$ and $N-n$ $\chi$-variables respectively, and so the derivatives appearing in these operators are effectively acting only on such variables. Because of this, in the very same way as we discussed in section 4.1, the operator $L_F^{h}(H_{-n})$ disappears from the part of the action proportional to $\Lambda_n$. Looking at the part of the action where $L_F^{h}(H_n)$ appear, we note that the variables $\chi_{n+1}, \ldots, \chi_N$ only appear in the density $\rho$. Thus, we can integrate over them and conclude that $L_F^{h}(H_n)$ acts on the reduced density $f_n$.

Next, let us consider $L_F^{h}(H_m)$. We have

$$L_F^{h}(H_m) = \left[\omega^{ac}\partial_c\partial_b\frac{2}{\hbar} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\hbar}{2}\right)^{2n+1} \left(\omega^{ab}\partial_b(H)\partial_a(\rho)\right)^{2n} H_m^b\frac{\partial}{\partial c^a}\right)\right] = \left[\frac{2}{h} \tilde{O}\omega^{ac}\partial_c\partial_b H_m^b\frac{\partial}{\partial c^a}\right],$$

where $\tilde{O}$ abbreviates the sum in the round brackets of the first line. We note that

$$\omega^{ac}\partial_c\partial_b H_m^b\frac{\partial}{\partial c^a} = -\sum_{i \neq j}^{N} \xi^i\partial_i \partial_q H_m \frac{\partial}{\partial \pi_j}. \quad (4.6)$$

As far as the bosonic part $L_B^{h}(H_m)$ is concerned, we can follow the same logic as in section 4.1, which applies also for the generalized density $\rho(\varphi^a, \pi^a)$. On the other hand, for $L_F^{h}(H_m)$, whenever one of the indices $i, j$ in (4.6) belongs to the $N-n$ particles set, we see that the differentiation with respect to $\pi_j$ is actually a total derivative in the action and therefore such contributions vanish. Thus effectively

$$\omega^{ac}\partial_c\partial_b H_m^b\frac{\partial}{\partial c^a} = -\sum_{i \neq j=1}^{n} \xi^i\partial_i \partial_q H_m \frac{\partial}{\partial \pi_j}. $$
By following exactly the same steps as in section 4.1 and exploiting the symmetry under particle permutations we can rewrite

\[
\omega^{ac} \partial_c \partial_b H \frac{\partial}{\partial c^a} = - \sum_{i \neq j = 1}^{n} \xi^i \partial_q \partial_{\chi_j} \left( \sum_{k=1}^{n} \sum_{l=n+1}^{N} V(q_k - q_l) \right) \frac{\partial}{\partial \chi_j} \\
= - \sum_{i \neq j = 1}^{n} \xi^i \partial_q \partial_{\chi_j} \left( (N - n) \sum_{k=1}^{n} V(q_k - q_{n+1}) \right) \frac{\partial}{\partial \chi_j} \\
= (N - n) \sum_{k=1}^{n} \omega^{ac} \partial_c \partial_b V(q_k - q_{n+1}) c^b \frac{\partial}{\partial c^a}.
\]

This implies that, in the action, we can integrate over the variables \(\chi_{n+2}, \ldots \chi_N\) and conclude that also \(L^b\) acts on the reduced density \(f_{n+1}\). Note that, on top of the terms considered up to now, we also need to take into account the derivatives present in \(\tilde{O}\). This does not modify the picture since the very same consideration goes through applying the reasoning already detailed in section 4.1.

Our final result then reads:

\[
\partial_t f_n + L^b_B (H_n) f_n + L^b_F (H_n) f_n = - \int d\chi_{n+1} \left[ \sum_{k=1}^{n} L^b_B (V(q_k - q_{n+1})) f_{n+1} \\
+ L^b_F (V(q_k - q_{n+1})) f_{n+1} \right].
\]

This is our main result. It reproduces equation (4.4) for zero form densities and generalizes the standard BBGKY hierarchy by including higher order forms for the reduced densities \(f_n\).

The reader may wonder why one should consider an extended BBGKY hierarchy that includes also higher density forms. The main reason is that the ghosts are naturally present in the path integral formulation of out of equilibrium statistical mechanics [3–12], and as such one may wish to keep track of them also when considering BBKGY. Moreover, as observed in the classical limit, i.e. the CPI reviewed in section 3.1.1, the ghosts have proved to carry relevant physical information [24, 25], in particular in relation to the evolution of nearby trajectories (the so called Jacobi fields).

Finally, further studies might lead to new insights. In fact let us suppose that it is possible to supersymmetrize the Moyal formalism via the recipe provided in [5], then we would be lead to a field theory on which we could exploit the powerful methods of supersymmetry.
5 Summary and outlook

In this work we considered the functional formulation of the super-extended Moyal formalism. Originally, this formalism was put forward to introduce a proposal for differential calculus in quantum mechanics \[15\]. It turns out that this formalism is also related to the recently proposed super-extension of the Schwinger-Keldysh formalism \[3–12\]. In some sense, by viewing the super-extended Moyal formalism from the point of view of the Schwinger-Keldysh functional integral, one could say that there are further arguments which lead to introduce the ghost sector of the theory, in particular in relation to the unitarity condition \(2.6\).

In section 3.2 we considered the topological properties of the path integral associated to the Moyal formalism. In particular, we showed that, besides the topological sector usually considered in the Schwinger-Keldysh formulation, one can obtain a further topological sector by changing the boundary condition of the path integral. Building on the CPI result \[33\], we showed that the functional \(Z_{pbc}\) introduced in \(3.11\) yields the Euler characteristic of the phase-space manifold.

In section 4 we gave a functional derivation of the quantum, super-extended BBGKY hierarchy. To the best of our knowledge, a similar derivation was available only in the classical limit \[35–37\], where it proved useful to derive in a simple way the Balescu-Lenard collision operator \[35\], see also \[39, 40\]. The reason for considering such an extended hierarchy is the following. As we said, in the functional description of out of equilibrium phenomena a ghost sector naturally arises in order to preserve the unitarity condition \(2.6\) in a manifest way \[3–12\]. It is then natural to keep track explicitly of this ghost sector and, possibly, a consistent truncation of the BBGKY hierarchy has to contain also these further ghost fields. On top of this, the ghost fields have the meaning of Jacobi fields (i.e. infinitesimal displacement of nearby trajectories) in the classical limit and their evolution contains information regarding the chaoticity of the system \[25\]. Since in certain regimes the evolution of non equilibrium quantum fields can be well approximated by classical systems \[41\] and the universal behaviour of far from equilibrium phenomena can be sometimes described by nonthermal fixed points whose universality class encompass both quantum and classical systems \[42\], we believe that the introduction of these ghost fields carries useful information. We hope to come back to these topics in the future.

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