The Large $N$ Limits of the Chiral Potts Model

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Abstract

In this paper we study the large-$N$ limits of the integrable $N$-state chiral Potts model. Three chiral solutions of the star-triangle equations are derived, with states taken from all integers, or from a finite or infinite real interval. These solutions are expected to be chiral-field lattice deformations of parafermionic conformal field theories. A new two-sided hypergeometric identity is derived as a corollary.

Key words: Chiral Potts Model; Star-Triangle Equations; $R$-matrix; Chiral Fields; Hypergeometric Functions

1 Introduction

When the integrable $N$-state chiral Potts model was introduced, it was the first example of an exactly solvable lattice model whose Boltzmann weights both require the use of higher-genus algebraic functions for their uniformization and do not have “the difference property” [1–6]. Since then, much has been written about many aspects of this model and we refer the reader to the recent review [7] for more information. In this paper we shall concentrate our attention on just one aspect, namely the large $N$ limit. We have written about this once before [8], but we can now present a much more complete and improved version containing several new results in addition.

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The chiral Potts model is a spin model on a two-dimensional lattice (or more generally a planar graph). At each site (or vertex) of the lattice (or graph), there is a state variable or “spin” that takes on \(N\) values \(a, b, \cdots = 1, 2, \cdots, N, (\text{mod } N)\). The Boltzmann weights are associated with pair interactions along edges. We assume that there are two types of such weights \(W\) and \(W\), which on a square lattice would correspond to horizontal and vertical interactions. We assume also that the weights only depend on the difference modulo \(N\) of the two spin states \(a\) and \(b\) at the two endpoints of each edge, which is the Potts property. The chiral character (handedness or breakdown of parity) is expressed by \(W(a - b) \neq W(b - a)\) and can only occur if \(N > 2\).

The integrable chiral Potts model [1–6] is a nontrivial generalization of the critical Fateev–Zamolodchikov model [9]. The fact that its rapidity variables lie on a higher-genus curve makes this model special among the many solvable lattice models. In spite of this several results exist for it. Therefore, its large-\(N\) limits should provide interesting generalizations of certain nonchiral \(\infty\)-state models of Fateev and Zamolodchikov [9–11], very different from the SOS-model of Baxter [12] and the few other \(\infty\)-state models [13–16] that have been introduced. From existing thermodynamic results for the finite-\(N\) case, we can infer corresponding results for the \(N = \infty\) cases that may be of interest in later studies. We expect, for example, a direct relation with new integrable chiral-field deformations of parafermionic conformal field theories.

This paper is organized as follows. In section 2 we present the Boltzmann weights of the integrable chiral Potts model and its dual model, also adding new details not given in [8]. In section 3 we give the three different large-\(N\) limits of the weights, while treating the more technical details in Appendix B. The three corresponding large-\(N\) limits of the star-triangle equations are given in detail in section 4. In section 5, it is shown that the results of the previous section 4 imply a new two-sided hypergeometric summation formula. Finally, a short discussion is given in section 6.

2 Integrable \(N\)-state Chiral Potts Model

In this section we shall review earlier results on our higher-genus solution of the star-triangle equations for the chiral Potts model [1–6] and present in more detail a reparametrization [8,17] that is particularly suitable for the large-\(N\) limit.

\(^2\) This chiral aspect allows us to mimic the effect of further-neighbor interactions within the context of a nearest-neighbor interaction model, see [7] and references quoted there.
2.1 Star-Triangle Equation for Chiral Potts Model

The $N$-state chiral Potts model can be defined on a general graph with spin states $a, b, \cdots$ taking values $1, \cdots, N$ on the vertices and Boltzmann weights $W(a, b) = W(a - b)$ associated with edges. $W(n)$ is periodic in $n \mod N$.

In the integrable model, one assumes that there are oriented straight lines (the rapidity lines) on the medial graph, which are dashed lines shown in Fig. 1 for the case of a square lattice. They are obtained by connecting the middles of all pairs of edges (solid lines in the figure) that are incident to a single site and share a common face. No more than two rapidity lines meet at any given point. These lines carry variables $p, q, \cdots$ and arrows specifying their orientations. In nearly all solvable models the weights depend on the differences of these rapidity variables. For our class of integrable spin-pair interaction models the weights can be graphically represented as in Fig. 2. These weights must satisfy the star-triangle equation

\[ \sum_{d=1}^{N} W_{qr}(b - d) W_{pr}(a - d) W_{pq}(d - c) = R_{pqr} W_{pq}(a - b) W_{pr}(b - c) W_{qr}(a - c). \] (2.1)
Fig. 2. Boltzmann weights $W_{pq}(a-b)$ and $\overline{W}_{pq}(a-b)$. We need to put an arrow on each edge to distinguish $W_{pq}(a-b)$ from $W_{pq}(b-a)$. Note the relative orientation of this arrow with respect to the orientations of the two rapidity lines in each case.

Here the factor $R_{pqr}$ can be determined as [5,6,18]

$$R_{pqr} = \frac{F_{pq}F_{qr}}{F_{pr}}, \quad F_{pq} = \left\{ \prod_{l=1}^{N} \sum_{j=1}^{N} \omega^{-jl} W_{pq}(j) \right\}^{1/N}.$$  \hspace{1cm} (2.2)

with

$$\omega \equiv e^{2\pi i/N} \equiv e^{2\pi \sqrt{-1/N}}.$$ \hspace{1cm} (2.3)

The easiest way to derive (2.2) is to set $a = 0$ in (2.1) and then to take the determinant with respect to the matrix indices $b$ and $c$, leading to determinants of products of diagonal and cyclic matrices; this argument first appeared in print in [18]. The star-triangle equation (2.1) can be symbolically represented as in Fig. 3.

Fig. 3. The star-triangle relations, which allow one to move a rapidity line $p$ through a vertex, which is the intersection of two other rapidity lines $q$ and $r$. 
2.2 Weights of Integrable Chiral Potts Model

In [1–6] one family of integrable chiral Potts models, with an arbitrary number of states per site \( N \geq 2 \), has been deduced with weights \( W_{pq}(a - b) \) and \( \overline{W}_{pq}(a - b) \) satisfying the star-triangle equation (2.1) for all \( a, b, c = 1, \ldots, N \). These weights are given by

\[
\frac{W_{pq}(n)}{W_{pq}(0)} = \left( \frac{\mu_p}{\mu_q} \right)^n \prod_{j=1}^{n} \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j},
\]

\[
\frac{W_{pq}(n)}{W_{pq}(0)} = \left( \mu_p \mu_q \right)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q \omega^j}{y_q - y_p \omega^j}.
\]

(2.4)

Here the parameters \( p \equiv (x_p, y_p, \mu_p) \) and \( q \equiv (x_q, y_q, \mu_q) \) are restricted by the two periodicity requirements

\[
W_{pq}(N + n) = W_{pq}(n), \quad \overline{W}_{pq}(N + n) = \overline{W}_{pq}(n),
\]

yielding

\[
\left( \frac{\mu_p}{\mu_q} \right)^N = \frac{y_p^N - x_q^N}{y_q^N - x_p^N}, \quad \left( \mu_p \mu_q \right)^N = \frac{y_q^N - y_p^N}{x_p^N - x_q^N},
\]

(2.5)

which can be recombined as

\[
\frac{\mu_p^N x_p^N \pm y_p^N}{1 \pm \mu_p^N} = \frac{\mu_q^N x_q^N \pm y_q^N}{1 \pm \mu_q^N} \equiv \lambda_{\pm},
\]

(2.6)

independent of \( p \) and \( q \). We write \( \lambda_{\pm} = \pm c(1 \mp k')/k \), with \( c \) a constant that can be absorbed by a trivial rescaling of all the \( x_p, y_p, x_q, y_q \) by a common factor that will drop out of (2.4), and with \( k \) and \( k' \) numbers related by \( k^2 + k'^2 = 1 \). Then the conditions (2.5) reduce to

\[
\mu_p^N = k'/(1 - k x_p^N) = (1 - k y_p^N)/k', \quad x_p^N + y_p^N = k(1 + x_p^N y_p^N).
\]

(2.7)

These equations describe a complex curve, which is the intersection of two “Fermat cylinders,” and the genus of this curve is \( g = N^2(N - 2) + 1 \). For each Boltzmann weight the two line (or rapidity) variables \( p \) and \( q \) are two points on this higher-genus algebraic curve, so that the usual difference-variable transformation cannot be carried out, except for special subcases where the genus degenerates to \( g \leq 1 \). Here the substitutions \( k = 0, k' = \pm 1 \) reduce the curve

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\( ^3 \) To connect with the original homogeneous notation [5,6], we must set \( x_p \equiv a_p/d_p, \ y_p \equiv b_p/c_p, \mu_p \equiv d_p/c_p, \) and similarly with \( p \) replaced by \( q, r, \cdots \). A proof that the star-triangle equation (2.1) is satisfied is given in the appendix of [6].
(2.7) to a product of genus-zero curves and the weights degenerate to those of the self-dual Fateev–Zamolodchikov model [9].

In general the model is not self-dual and the dual weights are found by Fourier transform. We first note that the periodic weights in (2.4) are of the form

\[
W(n) = \prod_{j=1}^{n} \frac{x_1 - x_2\omega^{i}}{x_4 - x_3\omega^{i}}, \quad \frac{x_1^{N} - x_2^{N}}{x_4^{N} - x_3^{N}} = 1, \tag{2.8}
\]

so that the linear recursion relation

\[
(x_4 - x_3\omega^n)W(n) = (x_1 - x_2\omega^n)W(n - 1) \tag{2.9}
\]

is satisfied. We can apply the Fourier (or duality) transformation\(^4\)

\[
W^{(f)}(m) \equiv N^{-1} \sum_{n=0}^{N-1} \omega^{-mn}W(n), \tag{2.10}
\]

leading to

\[
x_4W^{(f)}(m) - x_3W^{(f)}(m - 1) = \omega^{-m}x_4W^{(f)}(m) - \omega^{-m}x_2W^{(f)}(m - 1), \tag{2.11}
\]

or

\[
\frac{W^{(f)}(n)}{W^{(f)}(0)} = \prod_{j=1}^{n} \frac{\omega x_2 - x_3\omega^{j}}{x_1 - x_4\omega^{j}}. \tag{2.12}
\]

Therefore, the weights dual to (2.4) are

\[
\frac{W^{(f)}_{pq}(n)}{W^{(f)}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega \mu_p x_p - \mu_q x_q\omega^{j}}{\mu_p y_q - \mu_q y_p\omega^{j}}.
\]

\[
\frac{W^{(f)}_{pq}(n)}{W^{(f)}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega \mu_p \mu_q x_q - y_p\omega^{j}}{\omega \mu_p \mu_q x_p - y_q\omega^{j}}, \tag{2.13}
\]

which are again both of the form (2.8).

\(^4\) Here we added a normalization factor \(N^{-1}\) that is needed in the \(N \to \infty\) limit.
2.3 Reparametrization

In order to proceed, we introduce new parameters to describe the higher-genus curve of rapidity variables. These parameters are real when the Boltzmann weights \( W(p|a-b) \) and \( \bar{W}(p|a-b) \) are real and positive. We begin with the substitutions

\[
\begin{align*}
    x_p &= e^{i\phi_p/N}, \quad y_p = \omega^{1/2} e^{i\theta_p/N}, \quad x_q &= e^{i\phi_q/N}, \quad y_q = \omega^{1/2} e^{i\theta_q/N},
\end{align*}
\]

so that from the last identity in (2.7) we find

\[
k = \sin \frac{1}{2}(\theta_p - \phi_p) = \sin \frac{1}{2}(\theta_q - \phi_q).
\]

This is equivalent to

\[
e^{i\phi_p} = \frac{e^{i\theta_p} + k}{1 + k e^{i\theta_p}},
\]

and similarly with \( p \) replaced by \( q \). From (2.16) we have

\[
\begin{align*}
    \cos \phi_p &= \frac{2k + (1 + k^2) \cos \theta_p}{1 + k^2 + 2k \cos \theta_p}, & \sin \phi_p &= \frac{(1 - k^2) \sin \theta_p}{1 + k^2 + 2k \cos \theta_p}.
\end{align*}
\]

We will also need two parameters that will describe the dual model, see e.g. (2.29). The first one is given by

\[
\lambda_p \equiv \frac{\theta_p + \phi_p}{2\pi} = \frac{1}{\pi} \arctan \frac{\sin \theta_p}{\cos \theta_p + k},
\]

where the last step follows from (2.16). Also using (2.15) we find

\[
\begin{align*}
    \theta_p - \phi_p &= 2 \arcsin(k \sin \pi \lambda_p), & \theta_p &= \pi \lambda_p + \arcsin(k \sin \pi \lambda_p), \quad \phi_p &= \pi \lambda_p - \arcsin(k \sin \pi \lambda_p),
\end{align*}
\]

which expresses \( \theta_p \) and \( \phi_p \) in terms of \( \lambda_p \). The other parameter \( \gamma_p \) is defined by

\[
e^{\gamma_p \pm \pi i \lambda_p} \equiv \frac{e^{\pm i\theta_p} + k}{\sqrt{1 - k^2}}.
\]

Our definitions of \( \theta_p \) and \( \phi_p \) differ by a factor \( N \) from Baxter’s [17]. This change of normalization will be necessary in the large \( N \) limit.
These two expressions are equivalent in view of (2.16) and (2.18). Multiplying them and using the second equality in (2.17) we find

\[ e^{2\gamma_p} = \frac{1 + k^2 + 2k \cos \theta_p}{1 - k^2} = \frac{\sin \theta_p}{\sin \phi_p}. \]  

(2.22)

Because of (2.20) this \( \gamma_p \) is also a function of \( \lambda_p \), i.e.

\[ e^{\pm \gamma_p} = \sqrt{1 - k^2 \sin^2 \pi \lambda_p} \pm k \cos \pi \lambda_p \sqrt{1 - k^2}. \]  

(2.23)

From (2.7) and (2.14) we have \( \mu_p^N = (1 + k e^{\theta_p})/k' \) so that

\[ \mu_p = \left( \frac{e^{i \theta_p} \sin \theta_p}{e^{i \phi_p} \sin \phi_p} \right)^{1/2N}, \]  

(2.24)

after using (2.18) and (2.21).

With the help of (2.14) and (2.24) we can now rewrite the results (2.4) as [17]

\[
\frac{W_{pq}(n)}{W_{pq}(0)} = \left( \frac{\sin \theta_p \sin \phi_q}{\sin \theta_q \sin \phi_p} \right)^{n/2N} \prod_{j=1}^{n} \frac{\sin[\pi(j - \frac{1}{2})/N - (\theta_q - \phi_p)/2N]}{\sin[\pi(j - \frac{1}{2})/N + (\phi_q - \theta_p)/2N]},
\]

(2.25)

\[
\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \left( \frac{\sin \theta_p \sin \theta_q}{\sin \phi_p \sin \phi_q} \right)^{n/2N} \prod_{j=1}^{n} \frac{\sin[\pi(j - 1)/N + (\phi_q - \phi_p)/2N]}{\sin[\pi j/N - (\theta_q - \theta_p)/2N]}.
\]

(2.26)

Similarly, using (2.14) and (2.24), their Fourier transforms (2.10) become [17]

\[
\frac{W_{pq}^{(f)}(n)}{W_{pq}^{(f)}(0)} = e^{in(\phi_p - \theta_p + \phi_q - \theta_q)/2N} \prod_{j=1}^{n} \frac{\sin[\pi(j - 1)/N + (\phi_q - \phi_p)/2N]}{\sin[\pi j/N - (\theta_q - \theta_p)/2N]},
\]

(2.27)

\[
\frac{\overline{W}_{pq}^{(f)}(n)}{\overline{W}_{pq}^{(f)}(0)} = e^{in(\phi_p - \theta_p + \phi_q)/2N} \prod_{j=1}^{n} \frac{\sin[\pi(j - \frac{1}{2})/N - (\phi_q - \phi_p)/2N]}{\sin[\pi(j - \frac{1}{2})/N + (\theta_q - \phi_p)/2N]}.
\]

(2.28)

\[\text{Eq. (7) and (8) of [8] have misprints, which can be corrected by replacing } n \text{ by } N - n \text{ in their left-hand sides.}\]
where

\[ \tilde{\phi}_p = \pi \lambda_p - i \gamma_p = \frac{1}{2} (\theta_p + \phi_p) - \frac{1}{2} i \log \frac{\sin \theta_p}{\sin \phi_p}, \]

\[ \tilde{\theta}_p = \pi \lambda_p + i \gamma_p = \frac{1}{2} (\theta_p + \phi_p) + \frac{1}{2} i \log \frac{\sin \theta_p}{\sin \phi_p}. \]  

(2.29)

By direct substitution we can show that if the weights satisfy the star-triangle equation (2.1) then their Fourier transforms satisfy the star-triangle equation

\[ \frac{N}{R_{pq}} \mathcal{W}^{(f)}_{qr}(a) \mathcal{W}^{(f)}_{pr}(b) \mathcal{W}^{(f)}_{pq}(a + b) \]

\[ = \sum_{d=0}^{N-1} \mathcal{W}^{(f)}_{pq}(b - d) \mathcal{W}^{(f)}_{pr}(a + b - d) \mathcal{W}^{(f)}_{qr}(d). \]  

(2.30)

This equation has the exact same form as equation (2.1), as can be seen replacing \( a \rightarrow a - b, \ b \rightarrow b - c, \ a + b \rightarrow a - c, \) and \( c + d \rightarrow d. \) Therefore, from the proof [6] that the weights (2.25), (2.26) satisfy (2.1) we conclude that the weights (2.27), (2.28) satisfy (2.30).

For \( \theta_p = \phi_p, \ \theta_q = \phi_q \) we recover the self-dual Fateev and Zamolodchikov [9] solution with

\[ \frac{W^{(f)}(n)}{W^{(f)}(0)} = \frac{\mathcal{W}(n)}{\mathcal{W}(0)} = \frac{\mathcal{W}(N - n)}{\mathcal{W}(0)}, \]

\[ \frac{\mathcal{W}^{(f)}(n)}{\mathcal{W}^{(f)}(0)} = \frac{W(n)}{W(0)} = \frac{W(N - n)}{W(0)}. \]  

(2.31)

which are trigonometric expressions \( (g = 0) \) of the difference variable \( \theta_q - \theta_p. \)

In this nonchiral special case the Boltzmann weights depend only on this one parameter, which is the difference of two rapidity variables. The more general chiral weights depend on the two rapidity variables separately, living on a higher-genus curve.

### 3 The \( N \rightarrow \infty \) Limit of the Boltzmann Weights

In this section we shall obtain the \( N \rightarrow \infty \) limit of the Boltzmann weights of the previous section. We shall give explicit formulae for all three regimes.
3.1 General form of the Boltzmann weights

Note that the Boltzmann weights (2.25) and (2.26) or their dual weights (2.27) and (2.28) all have the product form

\[ W(n) = A^{n/N} \prod_{j=1}^{n} \frac{\sin(\pi(j + \alpha - 1)/N)}{\sin(\pi(j + \beta - 1)/N)}, \]  

(3.1)

where

\[ A = \frac{\sin\pi\beta}{\sin\pi\alpha}, \]  

(3.2)

with \( \alpha \) and \( \beta \) given constants depending on parameters \( \theta_p, \theta_q, \phi_p, \) and \( \phi_q \) satisfying (2.15). Also, the condition on \( A \) guarantees that \( W(n+N) = W(n) \), using a trivial exercise on complex exponentials.

More precisely, we have to use in case of (2.25) and (2.26)

\[ \alpha_{pq} = \frac{1}{2} + \frac{\phi_p - \theta_q}{2\pi}, \quad \beta_{pq} = \frac{1}{2} + \frac{\phi_q - \theta_p}{2\pi}, \]  

(3.3)

\[ \overline{\alpha}_{pq} = \frac{\phi_q - \phi_p}{2\pi}, \quad \overline{\beta}_{pq} = 1 + \frac{\theta_p - \theta_q}{2\pi}, \]  

(3.4)

which all four satisfy equations of the form \( \xi_{pq} + \xi_{qr} - \xi_{pr} = \xi_{qq} \), or

\[ \alpha_{pq} + \alpha_{qr} - \alpha_{pr} = \beta_{pq} + \beta_{qr} - \beta_{pr}, \]  

(3.5)

\[ \overline{\alpha}_{pq} + \overline{\alpha}_{qr} = \overline{\alpha}_{pr}, \quad \overline{\beta}_{pq} + \overline{\beta}_{qr} = 1 + \overline{\beta}_{pr}. \]  

(3.6)

Similarly, we have to use in case of (2.27) and (2.28)

\[ \alpha_{pq}^{(f)} = \frac{\tilde{\phi}_q - \tilde{\phi}_p}{2\pi}, \quad \beta_{pq}^{(f)} = 1 + \frac{\tilde{\theta}_p - \tilde{\theta}_q}{2\pi}, \]  

(3.7)

\[ \overline{\alpha}_{pq}^{(f)} = \frac{1}{2} + \frac{\tilde{\theta}_q - \tilde{\phi}_p}{2\pi}, \quad \overline{\beta}_{pq}^{(f)} = \frac{1}{2} + \frac{\tilde{\theta}_q - \tilde{\phi}_p}{2\pi}, \]  

(3.8)

satisfying

\[ \alpha_{pq}^{(f)} + \alpha_{qr}^{(f)} = \alpha_{pr}^{(f)}, \quad \beta_{pq}^{(f)} + \beta_{qr}^{(f)} = 1 + \beta_{pr}^{(f)}, \]  

(3.9)

\[ \overline{\alpha}_{pq}^{(f)} + \overline{\alpha}_{qr}^{(f)} - \overline{\alpha}_{pr}^{(f)} = \overline{\beta}_{pq}^{(f)} + \overline{\beta}_{qr}^{(f)} - \overline{\beta}_{pr}^{(f)}. \]  

(3.10)
The four corresponding constants $A, \bar{A}, A^{(f)}, \bar{A}^{(f)}$, as given in (3.2) with the corresponding $\alpha$ and $\beta$ substituted, are worked out in Appendix A and they agree with (2.25) through (2.28), as was to be expected.

Important symmetries of weight (3.1) are

$$W(n|\alpha, \beta) = W(0|\alpha, \beta),$$

$$W(n|\beta, \alpha) = W(n|\alpha, \beta),$$

$$W(n|\beta, \alpha) = W(n+\pm N|\alpha, \beta),$$

which is easily verified from (3.1). This allows us to restrict ourselves to study $W(n)$ only for $0 \leq n \leq \frac{1}{2}N$, while reducing the other case $\frac{1}{2}N \leq n \leq N$ or equivalently $-\frac{1}{2}N \leq n \leq 0$ to this case. This symmetry shows up explicitly in the following, particularly in (3.17), (3.19), (3.21), and (B.4).

We can conclude from (3.12) that $W(N - n) = W(n)$ for $\alpha + \beta = 1$. Then the chirality disappears and the model reduces to the model of Fateev and Zamolodchikov [9].

### 3.2 General $N \to \infty$ formula

Naively, in the limit $N \to \infty$, we can drop the sin symbols in (3.1). This leads us to introduce the function

$$P(n|\alpha, \beta) \equiv \frac{\Gamma(\alpha+n)\Gamma(\beta)}{\Gamma(\beta+n)\Gamma(\alpha)} = \prod_{j=1}^{n} \frac{j + \alpha - 1}{j + \beta - 1} = \frac{(\alpha)_n}{(\beta)_n}, \quad \text{if } n \geq 0,$$

$$= \prod_{j=1}^{-n} \frac{j - \beta}{j - \alpha} = \frac{(1 - \beta)_{-n}}{(1 - \alpha)_{-n}}, \quad \text{if } n \leq 0,$$

where $\Gamma(x)$ is the Gamma function and $(x)_n = \Gamma(x+n)/\Gamma(x)$ the Pochhammer symbol [19,20]. The finite-$N$ corrections are described by the function

$$S_n(\alpha) \equiv \log \prod_{j=1}^{n} \frac{\sin[\pi(j+\alpha-1)/N]}{\pi(j+\alpha-1)/N},$$

which has an asymptotic expansion derived in Appendix B. Using (B.2) there, we immediately have an asymptotic expansion formula for (3.1) in terms of
powers of \(1/N\), i.e.\[\log \frac{W(n)}{W(0)} = \log \left[ A^{n/N} P(n|\alpha, \beta) \right] + S_n(\alpha) - S_n(\beta) \]

\[= \log \left[ A^{n/N} P(n|\alpha, \beta) \right] + \sum_{l=0}^{\infty} \frac{B_{l+1}(\alpha) - B_{l+1}(\beta)}{(l + 1)!} \left( \frac{\pi}{N} \right)^l \times \left[ \left( \frac{d}{dz} \right)^l \log \left( \frac{\sin z}{z} \right) \right]_{z = \pi n/N} - \left( \frac{d}{dz} \right)^l \log \left( \frac{\sin z}{z} \right) \right]_{z = 0}, \quad (3.15)\]

where the \(B_m(x)\) are Bernoulli polynomials \([19,20]\). Only the term \(l = 0\) will be relevant in the limit \(N \to \infty\) and the terms \(l \geq 1\) are finite-\(N\) corrections, for which bounds are derived in Appendix B. Using \(B_1(x) = x - \frac{1}{2}\) and \(B_2(x) = x^2 - x + \frac{1}{6}\) \([19,20]\) and restricting ourselves to \(l \leq 1\) we can rewrite (3.15) as

\[
\frac{W(n)}{W(0)} = A^{n/N} P(n|\alpha, \beta) \left( \frac{\sin(\pi n/N)}{\pi n/N} \right)^{\alpha - \beta} \times \exp \left[ \frac{\pi(\alpha - \beta)(\alpha + \beta - 1)}{2N} \left( \cot \frac{\pi n}{N} - \frac{N}{\pi n} \right) + O(N^{-2}) \right]. \quad (3.16)\]

The last line of (3.16) gives the leading correction for large \(N\) and can be ignored in the limit. We can use (3.16) to study three regimes for the large \(N\) limit. We shall work this out in the following three subsections.

3.3 The regime I: \(N \to \infty\), \(n\) finite

First we study the limit \(N \to \infty\), while \(n\) remains finite. In this case, (3.16) results in

\[
\frac{W(n)}{W(0)} = \frac{\Gamma(\alpha + n)\Gamma(\beta)}{\Gamma(\beta + n)\Gamma(\alpha)} = \frac{\Gamma(1 - \beta - n)\Gamma(1 - \alpha)}{\Gamma(1 - \alpha - n)\Gamma(1 - \beta)}, \quad -\infty < n < \infty, \quad (3.17)\]

which is just the naive limit \(P(n|\alpha, \beta)\) given in (3.13). Using the well-known asymptotic expansion formula of the Gamma function \([19,20]\), we have

\[
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} \left( 1 + \frac{(\alpha - \beta)(\alpha + \beta - 1)}{2z} + O(z^{-2}) \right), \quad (3.18)\]

\[\text{Formula (12) in [8] has the higher orders misprinted and is only correct to the order needed in the actual } N \to \infty \text{ limits, which are presented correctly in [8].}\]
for \( z \to +\infty \), which is also equation 1.18(4) of [19]. Therefore,

\[
\frac{W(n)}{W(0)} = n^{\alpha-\beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \left( 1 + O(n^{-1}) \right), \quad \text{for } n \to +\infty,
\]

\[
= |n|^{\alpha-\beta} \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} \left( 1 + O(|n|^{-1}) \right), \quad \text{for } n \to -\infty. \tag{3.19}
\]

This shows that the Boltzmann weights vanish in the limit whenever \( \Re \alpha < \Re \beta \), where \( \Re z \) is the real part of \( z \).

### 3.4 The regime II: \( N, n \to \infty, \ n/N \) finite

For the second regime we study \( N, n \to \infty \) such that

\[
x \equiv \frac{2\pi n}{N} \tag{3.20}
\]

remains finite. Consequently, the weights \( W(n) \) in (3.1), which originally took \( N \) different values and which were periodic modulo \( N \), now depend on the continuous spin values \( x \) and they are periodic modulo \( 2\pi \).

We can now substitute the asymptotic formula (3.19) for \( P(n|\alpha, \beta) \) into (3.16), while assuming without loss of generality \( -\frac{1}{2} N \leq n \leq \frac{1}{2} N \) and rearranging the resulting expression. We immediately arrive at

\[
W(x) = W(0) A \frac{\pi}{2\pi} \left( \frac{N}{\pi} \sin \frac{1}{2} x \right)^{\alpha-\beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)}, \quad \text{if } 0 < x \leq \frac{1}{2} \pi,
\]

\[
= W(0) A \frac{\pi}{2\pi} \left( \frac{N}{\pi} \sin \frac{1}{2} |x| \right)^{\alpha-\beta} \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)}, \quad \text{if } -\frac{1}{2} \pi \leq x < 0. \tag{3.21}
\]

This can be summarized as a function periodic modulo \( 2\pi \), i.e.

\[
W(x) = C A \frac{x}{2\pi} - \left\lfloor \frac{x}{2\pi} \right\rfloor \left| \sin \frac{1}{2} x \right|^{\alpha-\beta}, \tag{3.22}
\]

where \( \lfloor x \rfloor \) stands for the largest integer \( \leq x \) and

\[
C = W(0) \left( \frac{N}{\pi} \right)^{\alpha-\beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)}, \quad A = \frac{\sin \pi \beta}{\sin \pi \alpha} = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\beta)}. \tag{3.23}
\]

In this regime II, we have to rescale \( W(0) \) with a power of \( N \) as \( N \to \infty \) in order to keep the constant \( C \) finite.
For the special case $\alpha + \beta = 1$, we have $A = 1$ and the chirality vanishes. This special limit has been mentioned first by Fateev and Zamolodchikov in [9] and is generalized above to $\alpha + \beta \neq 1$.

### 3.5 The regime III: $N, n \to \infty$, $n/N \to 0$

A crossover regime intermediate between regimes I and II appears when both $N, n \to \infty$ such that $n/\varphi(N) \to x$ for some function $\varphi(N)$ with $\varphi(N) \to \infty$ and $\varphi(N)/N \to 0$. We have

$$W(x) = D A^{\frac{1}{2} \text{sign}(x)} |x|^{\alpha - \beta}, \quad -\infty < x < \infty, \quad (3.24)$$

which is a chiral generalization of the Boltzmann weight in Zamolodchikov’s Fishnet Model [10]. Here,

$$D = W(0) \varphi(N)^{\alpha - \beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)} A^{\frac{1}{2}}, \quad (3.25)$$

implying again that $W(0)$ need be suitably rescaled in the limit $N \to \infty$. We note that (3.24) is also the asymptotic large-$n$ behavior (3.19) for regime I and the small-$x$ limiting behavior of (3.22) in regime II. The sign function in (3.24) arises as coefficients in (3.19) and (3.21) differ by a factor $A$ for $n$ positive or negative, see also (3.23).

We note that we can reproduce the previously known cases [9,10] by setting $\alpha + \beta = 1$ and $A = 1$. Now we have only one condition (3.2) on $A$, i.e. $A = \sin \frac{\pi \beta}{\sin \pi \alpha}$. This provides us with the deformations (3.17), (3.22), and (3.24), which define integrable field theories with chirality.

### 3.6 Duality of regime I and regime II

The limiting Boltzmann weights in Regimes I and II are each other’s dual under Fourier duality transformation. More precisely, if the limiting weights are in Regime II, their Fourier transforms are in Regime I, and vice versa. This follows from the way that we have constructed the limits. However, there is also a direct way to show this, as the infinite Fourier sum can be performed using a transformation formula of the Gauss hypergeometric function $F(a, b; c; x)$ [21]. Thus we obtain a formula for the double-sided hypergeometric function $1\ _2H_1$.
as defined for example by Slater [22], i.e.

\[ 1 \text{H}_1 \left[ \frac{\alpha}{\beta} \right] e^{ix} = \sum_{n=-\infty}^{\infty} \frac{(\alpha)_n}{(\beta)_n} e^{inx} = 1 \text{H}_1 \left[ \frac{1-\beta}{1-\alpha} \right] e^{-ix} \]

\[ = F(\alpha, 1; \beta; e^{ix}) + F(1 - \beta, 1; 1 - \alpha; e^{-ix}) - 1 \]

\[ = \frac{2^{\beta-\alpha-1} \Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{i(1-\alpha-\beta)(x-\pi)/2} (\sin \frac{1}{2}x)^{\beta-\alpha-1}, \tag{3.26} \]

for \( 0 < x < 2\pi \) (and periodically extended mod \( 2\pi \)). The inverse Fourier transform of (3.26) corresponds to integral 3.892.1 of [20], where we need to replace \( \nu \mapsto \beta - \alpha, \beta \mapsto \alpha + \beta - 1 + 2n, x \mapsto \pi - \frac{1}{2}z \), resulting in

\[ \frac{1}{2\pi} \int_0^{2\pi} dz \, e^{-inz+i(1-\alpha-\beta)z/2} (\sin \frac{1}{2}z)^{\beta-\alpha-1} = \frac{2^{1-\beta+\alpha} e^{i\pi(1-\alpha-\beta)/2} (-1)^n}{(\beta-\alpha)\Gamma(\beta+n, 1-\alpha-n)} \]

\[ = \frac{2^{1-\beta+\alpha} e^{i\pi(1-\alpha-\beta)/2} \Gamma(\beta-\alpha) (-1)^n}{\Gamma(\beta+n)\Gamma(1-\alpha-n)}, \tag{3.27} \]

in agreement with (3.26) as

\[ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)} = \frac{(-1)^n}{(1-x)_n}. \tag{3.28} \]

It is easily verified that, up to a possible overall constant factor, (3.26) and (3.27) relate the regime I result (3.17) with the regime II result (3.22), provided an appropriate transformation of the \( \alpha \) and \( \beta \) parameters is made. From (3.3), (A.1), (2.29), and (A.3), we find for this duality transformation

\[ \beta - \alpha - 1 = \alpha^{(f)} - \beta^{(f)}, \]

\[ \alpha + \beta - 1 = -\frac{i}{\pi} \log A^{(f)}, \quad A^{(f)} = \frac{\sin \pi \beta^{(f)}}{\sin \pi \alpha^{(f)}}, \tag{3.29} \]

whereas, from (3.3), (A.1), (2.29), and (3.7), its inverse is given by

\[ \beta^{(f)} - \alpha^{(f)} - 1 = \alpha - \beta, \]

\[ \alpha^{(f)} + \beta^{(f)} - 1 = \frac{i}{\pi} \log A, \quad A = \frac{\sin \pi \beta}{\sin \pi \alpha}. \tag{3.30} \]

\[ ^8 \text{Start from 2.9 (27) of [19], with } a = \alpha, b = 1, c = \beta, z = e^{ix}, \text{ substituting the definitions 2.9 (1), (13), (22). Next use eqs. 2.1.2 (6) and } F(a, 0; c; z) = 1. \]
The equations (3.29) and (3.30) differ only by one minus sign in front of the i, coming from the corresponding difference of the finite Fourier transform and its inverse.

4 The $N \to \infty$ Limit of the Star-Triangle Equation

In this section we shall examine the various limits of the star-triangle equation (2.1), now we have obtained explicit prescriptions on how to take the $N \to \infty$ limit of the Boltzmann weights of the $N$-state chiral Potts model.

4.1 Principal domain

There are several parameter domains that require separate treatment. But from now on, we shall assume that all $W(n)$ encountered become vanishingly small whenever $n, N - n \gg 0$. In view of (3.19) and (3.21), this means that all $\Re(\beta - \alpha) > 0$.

From (3.3), (3.4), (3.7), (3.8), (3.29), and (3.30) we find

$$\beta - \alpha = 1 - \beta + \alpha = 1 - \beta^{(f)} + \alpha^{(f)} = \beta^{(f)} - \alpha^{(f)}, \quad (4.1)$$

independent of the choice of rapidity variables $p, q$, which we have suppressed in (4.1). Therefore, we define our principal domain by the condition

$$0 < \Re(\beta - \alpha) < 1, \quad (4.2)$$

for all occurring $(\alpha, \beta)$ pairs. This condition is easily satisfied, even with $\alpha_{pq}, \beta_{pq}, \alpha_{pr}, \beta_{pr}, \alpha_{qr}, \beta_{qr}$, and their barred versions, all being real.

The summation over $d$ in (2.1) has to be split in several pieces as we must choose to which of the three $N \to \infty$ regimes each of the three weights in the summand belongs. We shall see that under condition (4.2) all pieces but one can be ignored and that the three types of large-$N$ behavior I, II, or III do not mix: If we take the three spin states $a, b$, and $c$ in (2.1) such that all three weights in the right-hand side of (2.1) are in the same regime, the dominant part of the sum over spin state $d$ comes from the piece with all three weights in the left-hand side of (2.1) being in the identical regime.
4.2 The constant \( R_{pq} \)

The next calculation to be done is the evaluation of the large-\( N \) limit of the constants \( R_{pq} \) or \( F_{pq} \) given in (2.2). We break this up in several steps.

First, we can rewrite the \( F_{pq} \) in (2.2) as

\[
F_{pq} = N \frac{\overline{W}^{(f)}_{pq}(0)}{W_{pq}(0)} \exp(\overline{L}_{pq}^{(f)} - L_{pq}), \tag{4.3}
\]

where

\[
L_{pq} \equiv \frac{1}{N} \sum_{l=1}^{N} \log \frac{W_{pq}(l)}{W_{pq}(0)}, \quad \overline{L}_{pq}^{(f)} \equiv \frac{1}{N} \sum_{l=1}^{N} \log \left( \frac{\overline{W}^{(f)}_{pq}(l)}{\overline{W}^{(f)}_{pq}(0)} \right). \tag{4.4}
\]

Here both the \( L_{pq} \) and the \( \overline{L}_{pq}^{(f)} \) can be evaluated in an identical fashion under condition (4.2), substituting the regime II asymptotic form (3.22) and (3.23), while replacing the sum by an integral. Therefore, in an obvious simplification of notation suppressing the rapidity subscripts, \( L_{pq} \) becomes

\[
L = \log \left[ \left( \frac{N}{\pi} \right)^{\alpha - \beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \right] + \frac{1}{N} \sum_{l=1}^{N} \left[ \left( \frac{n}{N} - \lfloor \frac{n}{N} \rfloor \right) \log A + (\alpha - \beta) \log \sin |\pi n/N| \right] \approx \log \left[ \left( \frac{N}{\pi} \right)^{\alpha - \beta} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \right] + \frac{1}{2\pi} \int_{0}^{2\pi} dx \left[ \left( \frac{x}{2\pi} - \lfloor \frac{x}{2\pi} \rfloor \right) \log A + (\alpha - \beta) \log \sin |\frac{1}{2}x| \right] = \log \left[ \left( \frac{N}{2\pi} \right)^{\alpha - \beta} A^{1/2} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \right], \quad \text{for } N \to \infty, \tag{4.5}
\]

where the elementary integral 4.224.3 of [20] has been used. Similarly,

\[
\overline{L}^{(f)} \approx \log \left[ \left( \frac{N}{2\pi} \right)^{\overline{\alpha}(f) - \overline{\beta}(f)} \frac{\overline{A}^{(f)}(1/2)}{\Gamma(\overline{\beta}(f))} \right], \quad \text{for } N \to \infty. \tag{4.6}
\]

It is easily checked that the corrections to (4.5) and (4.6) are irrelevant in the large-\( N \) limit.

Similarly, \( \overline{W}^{(f)}(0)/\overline{W}(0) \) is also dominated by the regime II contribution given by (3.22) and (3.23), i.e.
\[
\frac{W(t)(0)}{W(0)} = \frac{1}{N} \sum_{l=1}^{N} \left[ \frac{(N/\pi)^{\frac{n}{N} - \bar{\beta}}}{\Gamma(\bar{\beta})/\Gamma(\bar{\alpha})} \right] \bar{A}^{(n/N) - \left\lfloor \frac{n}{N} \right\rfloor} |\sin \pi n/N|^{\frac{n}{N} - \bar{\beta}}
\]
\[
\approx \left[ \frac{(N/\pi)^{\frac{n}{N} - \bar{\beta}}}{\Gamma(\bar{\beta})/\Gamma(\bar{\alpha})} \right] \frac{1}{2\pi} \int_{0}^{2\pi} dx \bar{A}^{x/2\pi} (\sin \frac{x}{2})^{\frac{n}{N} - \bar{\beta}}
\]
\[
= (N/2\pi)^{\frac{n}{N} - \bar{\beta}} \bar{A}^{1/2} \frac{\Gamma(\bar{\beta}) \Gamma(1 - \bar{\beta} + \bar{\alpha})}{\Gamma(\bar{\alpha}) \Gamma(\bar{\beta}(f)) \Gamma(1 - \bar{\alpha}(f))}, \quad \text{for } N \to \infty. \tag{4.7}
\]

We have used the integral (3.27) for \( n = 0 \), after substituting the barred version of (3.30), i.e.
\[
\bar{\alpha} - \bar{\beta} = \bar{\beta}(f) - \bar{\alpha}(f) - 1, \quad \log \bar{A} = \pi i (1 - \bar{\alpha}(f) - \bar{\beta}(f)). \tag{4.8}
\]

Then the last line of (4.7) is obtained using (4.8) once more. As before in (4.5) and (4.6), (4.7) gives the coefficient of the leading \( N \)-power as \( N \to \infty \). Correction terms can be obtained, but they will not be needed.

We can now substitute (4.4)–(4.8) into (4.3) and simplify the result. This is worked out in Appendix C and the result is
\[
F_{pq} = \left[ (N/2\pi)^{\beta_{pq} - \alpha_{pq}} A_{pq}^{-1/2} \right] \frac{W_{pq}(0) W_{qr}(0) W_{qr}(0)}{W_{pq}(0) W_{pr}(0) W_{qr}(0)} \Gamma(\alpha_{pq}) \Gamma(\beta_{pq}) \Gamma(1 - \bar{\alpha}_{pq}) \Gamma(\bar{\beta}_{pq} - \bar{\alpha}_{pq}), \tag{4.9}
\]
giving us the desired expression for \( R_{pqr} = F_{pq} F_{qr}/F_{pr} \). We note that the factor in square brackets in (4.9) cancels out in view of (3.5) and (A.1). Hence,
\[
\lim_{N \to \infty} R_{pqr} = \frac{W_{pq}(0) W_{pq}(0) W_{qr}(0)}{W_{pq}(0) W_{qr}(0) W_{qr}(0)} F_{pq}, \tag{4.10}
\]

where
\[
r_{pqr}^\infty = \frac{f_{pq} f_{qr}}{f_{pr}}, \quad f_{pq} = \frac{\Gamma(\alpha_{pq}) \Gamma(\beta_{pq}) \Gamma(1 - \bar{\alpha}_{pq})}{\Gamma(\beta_{pq}) \Gamma(\beta_{pq} - \bar{\alpha}_{pq})}. \tag{4.11}
\]

We see that \( R_{pqr} \) is independent of \( N \) in leading order in the large-\( N \) limit. Finite-\( N \) corrections can be worked out but are not needed here, as we shall only consider the actual \( N \to \infty \) limit.

Note that \( f_{pq} \) and \( r_{pqr}^\infty \) are invariant under \( \alpha_{pq} \mapsto 1 - \beta_{pq}, \beta_{pq} \mapsto 1 - \bar{\alpha}_{pq} \), whereas \( F_{pq} \) and \( r_{pqr}^\infty \) are invariant under \( \alpha_{pq} \mapsto 1 - \beta_{pq}, \beta_{pq} \mapsto 1 - \alpha_{pq} \), but \( f_{pq} \) is not.
4.3 Regime I

We can first consider the large-$N$ limit of (2.1), while keeping $a - b$, $a - c$, and thus also $b - c$ finite. Without loss of generality, we can then restrict ourselves to considering the case $a$, $b$, and $c$ finite. The three Boltzmann weights in the right-hand side of (2.1) take the regime-I form (3.17). The sum over $d$ in (2.1) needs to be split up in the limit. In one part $|d|$ remains finite but can become arbitrarily large and the three weights in the left-hand side of (2.1) also take the regime-I form. The summand of (2.1) then decays as $|d|^{-\kappa}$ for $|d| \to \infty$. From (3.17), and using (3.6) and (4.1), we see that

$$
\kappa = \alpha_{qr} - \beta_{qr} + \alpha_{pr} - \beta_{pr} + \alpha_{pq} - \beta_{pq} \\
= \alpha_{qr} - \beta_{qr} + \beta_{pr} - \alpha_{pr} - 1 + \alpha_{pq} - \beta_{pq} = -2,
$$

(4.12)

so that the sum over $d$ converges as $\sum |d|^{-2}$.

In the part of the sum for which $|d|/N$ does not tend to zero, the three weights in the left-hand side of (2.1) belong to regime II and now we can use (3.21) to show that the summand scales as $N^\kappa = N^{-2}$. As the total sum has $N$ terms, this contribution vanishes in the limit.

The contribution of the crossover regime III connecting regimes I and II also vanishes. To show this in more detail, we can split the sum over $d$ in a piece $-N^{2/3} < d \leq N^{2/3}$, and a piece $N^{2/3} < d \leq N - N^{2/3}$. In the first piece, the summand is bounded by $|d|^{-2}$-behavior as shown above and in (3.24) and (3.25); therefore, the error made by replacing the sum with $-N^{2/3} < d \leq N^{2/3}$ by a sum $-\infty < d \leq \infty$ vanishes as $O(N^{-2/3})$. For the second piece we can use (3.21), (3.24) and (3.25) to show that the summand is $O(N^{2\kappa/3})$ and with less than $N$ terms of order $O(N^{-4/3})$ its contribution vanishes as $O(N^{-1/3})$.

To summarize, all six Boltzmann weights must take the regime-I form (3.17)

$$
W_{pq}(n) = \frac{(\alpha_{pq})_n}{(\beta_{pq})_n}, \quad \overline{W}_{pq}(n) = \frac{(\overline{\alpha}_{pq})_n}{(\overline{\beta}_{pq})_n},
$$

(4.13)

solving the star-triangle equation

$$
\sum_{d=-\infty}^{\infty} \overline{W}_{qr}(b-d) W_{pr}(a-d) \overline{W}_{pq}(d-c) \\
= r_{pqr} W_{pq}(a-b) \overline{W}_{pr}(b-c) W_{qr}(a-c),
$$

(4.14)

provided the $\alpha$ and $\beta$ parameters satisfy (3.3), (3.4), and (2.16). In (4.13) and
we have chosen the normalization $W_{pq}(0) = \overline{W}_{pq}(0) = 1$.

Equation (4.14) is related to the Dougall–Ramanujan identity [23]. We shall return to this in the next section.

4.4 Regime II

We can next consider the large-$N$ limit of (2.1), while keeping $(a - b)/N$, $(a - c)/N$, and $(b - c)/N$ fixed and nonzero. The three Boltzmann weights in the right-hand side of (2.1) now take the regime-II form (3.21). Again, the sum over $d$ in (2.1) needs to be split up in this limit. Now the dominant part is the one with all three weights in the left-hand side of (2.1) belonging to regime II. Only when $d$ is close to $a$, $b$, or $c$, one of the weights can be of the form of regime I or III. Because of the principal domain condition (4.2) these contributions can be ignored in the large-$N$ limit and the sum can be replaced by an integral as is done twice in subsection 4.2.

In the previous section we have seen that the summand in the left-hand side of (2.1) scales as $N^{-2}$ in the large-$N$ limit, when the three weights belong to regime II. The right-hand side of (2.1) now scales as $N^{\kappa} = N^{-1}$, since

\[
\kappa = \alpha_{pq} - \beta_{pq} + \alpha_{pr} - \beta_{pr} + \alpha_{qr} - \beta_{qr} = \alpha_{pq} - \beta_{pq} + \beta_{pr} - \alpha_{pr} - 1 + \alpha_{qr} - \beta_{qr} = -1.
\] (4.15)

Therefore, it is natural to multiply (2.1) by $N$ and to replace $N^{-1}\sum_d$ by $(2\pi)^{-1}\int dx$, so that the star-triangle equation becomes

\[
\frac{1}{2\pi} \int_0^{2\pi} dw \ W_{qr}(y-w) W_{pr}(x-w) W_{pq}(w-z) = R_{pqr}^\infty W_{pq}(x-y) \overline{W}_{pr}(y-z) \overline{W}_{qr}(x-z),
\] (4.16)

as also follows after suitable rescalings of the Boltzmann weights and $R_{pqr}$, i.e.

\[
R_{pqr}^\infty \equiv \lim_{N \to \infty} N^{-1} R_{pqr}
\] (4.17)

and the two equations (4.20) below. Equation (4.16) has the solution

\[
W_{pq}(x) = e^{(\gamma_p - \gamma_q)(\frac{x}{2\pi} - \frac{1}{2\pi})} \sin^{\frac{1}{2}x} |\lambda_p - \lambda_q|,
\]

\[
\overline{W}_{pq}(x) = e^{(\gamma_p + \gamma_q)(\frac{x}{2\pi} - \frac{1}{2\pi})} \sin^{\frac{1}{2}x} |\lambda_q - \lambda_p - 1|,
\] (4.18)
using the normalization $C_{pq} = \overline{C_{pq}} = 1$ in (3.22), together with (3.3), (3.4), (2.18), (A.1), and (A.2), which imply

\[ A_{pq} = e^{\gamma_p - \gamma_q}, \quad \alpha_{pq} - \beta_{pq} = \lambda_p - \lambda_q, \]
\[ A_{pq} = e^{\gamma_p + \gamma_q}, \quad \overline{\alpha_{pq}} - \overline{\beta_{pq}} = \lambda_q - \lambda_p - 1. \]  

(4.19)

Here $\gamma_p$ and $\lambda_p$ are related by (2.23). If $\lambda_p < \lambda_q < \lambda_r < 1 + \lambda_p$ all six Boltzmann weights in (4.16) are real and positive and the parameters are in the principal domain.

According to (3.23), the condition $C_{pq} = \overline{C_{pq}} = 1$ implies

\[ W_{pq}(0) = (\frac{N}{\pi})^{\beta_{pq} - \alpha_{pq}} \frac{\Gamma(\alpha_{pq})}{\Gamma(\beta_{pq})}, \quad \overline{W_{pq}}(0) = (\frac{N}{\pi})^{\overline{\beta_{pq}} - \overline{\alpha_{pq}}} \frac{\Gamma(\overline{\alpha_{pq}})}{\Gamma(\overline{\beta_{pq}})}. \]  

(4.20)

Substituting this in (4.17), while using (4.10) and (4.11), we arrive at

\[ R_\infty^{pq} = \frac{\tilde{f}_{pq} \tilde{f}_{qr}}{f_{pr}}, \quad \tilde{f}_{pq} = \frac{1}{\pi} \frac{\Gamma(\beta_{pq})\Gamma(\overline{\alpha_{pq}})}{\Gamma(\alpha_{pq})\Gamma(\beta_{pq})} f_{pq} = \frac{\Gamma(\overline{\alpha_{pq}})\Gamma(1 - \overline{\alpha_{pq}})}{\pi \Gamma(\overline{\beta_{pq}} - \overline{\alpha_{pq}})}, \]  

(4.21)

where the $\pi$ and $N$ factors have been redistributed using (3.5) and (3.6).

It can be shown that the Boltzmann weights obtained by dropping the integral part $\left\lfloor \frac{x}{2\pi} \right\rfloor$ in (4.18) also satisfy the same star-triangle equation (4.16). The resulting chiral solution can be viewed as the nonchiral Fateev–Zamolodchikov large-$N$ solution [9] with a site-dependent gauge transformation.

Since $W_{pq}(x)$ and $\overline{W_{pq}}(x)$ as given in (4.18) are now functions of $x$ periodic modulo $2\pi$, their Fourier transforms

\[ W_{pq}^{(f)}(j) = \frac{1}{2\pi} \int_0^{2\pi} dx \ e^{-ijx} W_{pq}(x), \quad \overline{W_{pq}}^{(f)}(j) = \frac{1}{2\pi} \int_0^{2\pi} dx \ e^{-ijx} \overline{W_{pq}}(x), \]  

(4.22)

are over all integer value $j$, ranging from $-\infty$ to $\infty$. Substituting (4.22) into (4.16), we find that these Fourier transforms satisfy the star-triangle equation

\[ \frac{1}{R_\infty^{pq}} \overline{W_{qr}}^{(f)}(a - b) W_{pr}^{(f)}(b - c) \overline{W_{pq}}^{(f)}(a - c) \]
\[ = \sum_{d=-\infty}^{\infty} W_{pq}^{(f)}(b - d) \overline{W_{pr}}^{(f)}(a - d) \overline{W_{qr}}^{(f)}(d - c), \]  

(4.23)
in which the sum is over all integer values of \( d \).

### 4.5 Regime III

As the final case, we can consider the large-\( N \) limit of (2.1), while keeping \((a - b)/\varphi(N), (a - c)/\varphi(N), \) and \((b - c)/\varphi(N)\) fixed at finite and nonzero values. The three Boltzmann weights in the right-hand side of (2.1) now take the regime-III form (3.24). Also in this case, the sum over \( d \) in (2.1) needs to be split up. Now the dominant part is the one with all three weights in the left-hand side of (2.1) belonging to regime III. Only when \( d \) is close to \( a, b, \) or \( c \), one of these weights can be of the form of regime I, while the other two weights are of the form of regime III. Because of the principal domain condition (4.2) these regime-I contributions can be ignored in the large-\( N \) limit and the sum can be replaced by an integral as is done twice already in subsections 4.2 and 4.4. Also, the three weights in the summand of (2.1) can only simultaneously take the form (3.21) of regime II. This contribution is \( O(N^{-1}) \) as in subsection 4.3 and can be ignored.

Hence, we can prove a star-triangle equation of the form (4.16), but with integration over \((-\infty, +\infty)\), i.e.

\[
\int_{-\infty}^{+\infty} dw \, W_{qr}(y - w) W_{pr}(x - w) W_{pq}(w - z) = \hat{R}_{pq}^{\infty} W_{pq}(x - y) W_{pr}(y - z) W_{qr}(x - z),
\]

(4.24)

where

\[
\hat{R}_{pq}^{\infty} \equiv \lim_{N \to \infty} \varphi(N)^{-1} R_{pq}.
\]

(4.25)

Equation (4.24) has the solution

\[
W_{pq}(x) = e^{-\frac{1}{2}(\gamma_p - \gamma_q)\text{sign}(x)} |x|^{|\lambda_p - \lambda_q|},
\]

\[
W_{pq}(x) = e^{-\frac{1}{2}(\gamma_p + \gamma_q)\text{sign}(x)} |x|^{|\lambda_q - \lambda_p - 1|},
\]

(4.26)

using the normalization \( D_{pq} = \overline{D}_{pq} = 1 \) in (3.24), together with (4.19). Again, \( \gamma_p \) and \( \lambda_p \) are related by (2.23). If \( \lambda_p < \lambda_q < \lambda_r < 1 + \lambda_p \) all six Boltzmann weights in (4.24) are real and positive.
According to (3.25), the condition $D_{pq} = \overline{D}_{pq} = 1$ implies

$$W_{pq}(0) = \varphi(N)^{\beta_{pq} - \alpha_{pq}} \frac{\Gamma(\alpha_{pq})}{\Gamma(\beta_{pq})}, \quad \overline{W}_{pq}(0) = \varphi(N)^{\overline{\beta}_{pq} - \overline{\alpha}_{pq}} \frac{\Gamma(\overline{\alpha}_{pq})}{\Gamma(\overline{\beta}_{pq})}, \quad (4.27)$$

Substituting this in (4.25), while using (4.10) and (4.11), we arrive at

$$\hat{R}_{pqr} = \frac{\hat{f}_{pq} \hat{f}_{qr}}{\hat{f}_{pr}}, \quad \hat{f}_{pq} = \frac{\Gamma(\beta_{pq}) \Gamma(\overline{\alpha}_{pq})}{\Gamma(\alpha_{pq}) \Gamma(\overline{\beta}_{pq})} f_{pq} = \frac{\Gamma(\overline{\alpha}_{pq}) \Gamma(1 - \overline{\alpha}_{pq})}{\Gamma(\beta_{pq} - \overline{\alpha}_{pq})}, \quad (4.28)$$

where the $\varphi(N)$ factors have been redistributed using (3.5) and (3.6).

It can be shown that the nonchiral Boltzmann weights obtained by setting $\gamma_p = \gamma_q = 0$ in (4.26) satisfy the same star-triangle equation (4.24). The resulting solution can be viewed as the Fishnet Model of Zamolodchikov [10]. It also generalizes Symanzik’s conformal integral [11], which has been used by Zamolodchikov to prove the star-triangle equation for the Fishnet Model and which has also provided the proof for the $\hat{N} = \infty$ Fateev-Zamolodchikov model [9] via a conformal transformation, $\xi = \tan \frac{1}{2} w$.

4.6 Remark on $R$-matrix

Following our joint work with Baxter [5], we can make an $R$-matrix by taking the product of four weights in any of the above regimes I, II, or III. Taking four weights of type I, we have

$$R(a, b, c, d) = \overline{W}_{p_1 q_1} (a - c) W_{p_1 q_2} (c - b) \overline{W}_{p_2 q_2} (d - b) W_{p_2 q_1} (a - d). \quad (4.29)$$

Similarly, we can also take four weights of type II or III. Then any such infinite-dimensional $R$-matrix satisfies the usual Yang-Baxter equation. But these solutions are very different from those of [12–16].

5 Two-Sided Hypergeometric Sum

In this section we shall rewrite the star-triangle equation (4.14) with solution (4.13) as a new double-sided hypergeometric identity.
5.1 More symmetric star-triangle equation

The star-triangle equation (4.14) can be written in a more symmetric form with permutation symmetry among the Boltzmann weights in each of the two sides.

We start by applying (3.12) to the \( \mathbf{W}_{pq} \) in (4.14), followed by substituting \( d = -n \) and and replacing all six weights by (4.13). We arrive at

\[
\sum_{n=-\infty}^{\infty} \frac{\alpha_{qr} \alpha_{pr} (1 - \beta_{pq})}{\beta_{pr} \alpha_{pq} (1 - \alpha_{pr})} = \frac{\alpha_{pq}}{\beta_{pq}} \frac{\alpha_{pr}}{\beta_{pr}} \frac{\alpha_{qr}}{\beta_{qr}} \frac{\alpha_{pq}}{\beta_{pq}} \frac{\alpha_{pr}}{\beta_{pr}} \frac{\alpha_{qr}}{\beta_{qr}},
\]

(5.1)

This result seems to invite us to introduce a more symmetric notation. For the three external spin states in the star-triangle equation we write

\[
m_1 \equiv a, \quad m_2 \equiv b, \quad m_3 \equiv c.
\]

(5.2)

It is logical to associate \( pr \) with 1, \( qr \) with 2, and \( pq \) with 3. The other quantities in (5.1) are then rewritten as

\[
x_1 \equiv \alpha_{pr}, \quad x_2 \equiv \alpha_{qr}, \quad x_3 \equiv 1 - \beta_{pq};
\]

\[
y_1 \equiv \beta_{pr}, \quad y_2 \equiv \beta_{qr}, \quad y_3 \equiv 1 - \alpha_{pq};
\]

\[
u_1 \equiv \alpha_{pr}, \quad v_2 \equiv \alpha_{qr}, \quad v_3 \equiv \alpha_{pq};
\]

\[
u_1 \equiv \beta_{pr}, \quad v_2 \equiv \beta_{qr}, \quad v_3 \equiv \beta_{pq};
\]

(5.3)

\[
f_1 \equiv f_{pr}, \quad f_2 \equiv f_{qr}, \quad f_3 \equiv f_{pq};
\]

(5.4)

where the \( f_j \) for \( j = 1, 2, 3 \) are given by (4.11), i.e.

\[
f_1 = \frac{\Gamma(x_1) \Gamma(v_1) \Gamma(1 - u_1)}{\Gamma(y_1) \Gamma(v_1 - u_1)}, \quad f_2 = \frac{\Gamma(u_2) \Gamma(y_2) \Gamma(1 - x_2)}{\Gamma(v_2) \Gamma(y_2 - x_2)}, \quad f_3 = \frac{\Gamma(u_3) \Gamma(y_3) \Gamma(1 - x_3)}{\Gamma(v_3) \Gamma(y_3 - x_3)},
\]

(5.5)

and \( r_{pq} = f_2 f_3 / f_1 \). Therefore, (5.1) takes the much more symmetric form

---

9 The parameters \( x_j \) and \( y_j \), for \( j = 1, 2, 3 \), should not be confused with the rapidity variables \( x_p \) and \( y_p \) of (2.4).
The quantities (5.3) are not independent. From (3.3) and (3.4) we find that the six variables \( u_j \) and \( v_j \) depend in a linear and symmetric fashion on the six variables \( x_j \) and \( y_j \), i.e.

\[
\begin{align*}
  u_1 &= 1 + x_2 - y_3, & v_1 &= y_2 - x_3, \\
  u_2 &= 1 + x_1 - y_3, & v_2 &= y_1 - x_3, \\
  u_3 &= 1 + x_1 - y_2, & v_3 &= y_1 - x_2,
\end{align*}
\]

so that (5.6) simplifies further to

\[
\sum_{n=-\infty}^{\infty} \frac{(x_j)_{m_j+n}(y_j)_{m_j+n}}{f_1(x_1)_{m_1+n}(x_2)_{m_2+n}(x_3)_{m_3+n}} = \frac{f_2 f_3}{f_1} \prod_{1 \leq i < j \leq 3} \frac{(1 + x_i - y_j)_{m_i-m_j}}{(y_i - x_j)_{m_j-m_i}}. \tag{5.8}
\]

Since (3.28) implies

\[
\frac{(1 + x_i - y_j)_{m_i-m_j}}{(y_i - x_j)_{m_j-m_i}} = \frac{(1 + x_j - y_i)_{m_j-m_i}}{(y_j - x_i)_{m_i-m_i}}, \tag{5.9}
\]

the symmetry in (5.8) is even larger than is manifested there, namely the full permutation symmetry group \( S_3 \).

\subsection*{5.2 The two conditions to be satisfied}

The integrable chiral Potts model solution (2.4) of the star-triangle equation (2.1) has not six free parameters, but only four independent ones, namely the rapidity variables \( x_p, x_q, x_r \), and the modulus parameter \( k \). Therefore, two conditions must be imposed on the new variables \( x_j \) and \( y_j \) with \( j = 1, 2, 3 \).

The first relation is a linear relation, which is a direct consequence of (4.12) and (5.3). It reads

\[
x_1 + x_2 + x_3 + 2 = y_1 + y_2 + y_3. \tag{5.10}
\]

This is in fact the Saalschütz condition, which plays such an important role in the theory of hypergeometric functions [19,22].
The second relation can be found from (4.19), which is derived in Appendix A and holds for all \(N\). From (4.19) we can derive

\[
A_{pr}A_{qr} = \bar{A}_{pq} = A_{qr}A_{pr}, \quad A_{pq}A_{qr} = A_{pr}, \quad A_{pq}A_{qp} = 1. \tag{5.11}
\]

From the first equality in (5.11) and (5.3), we then find

\[
\sin \pi x_1 \sin \pi x_2 \sin \pi x_3 = \sin \pi y_1 \sin \pi y_2 \sin \pi y_3, \tag{5.12}
\]

and we may call this the “periodicity condition” due to its relation with the periodicity mod \(N\) property.

Since (5.12) is a nonlinear relation, we may ask ourselves what the ambiguity is in solving \(x_3\) and \(y_3\) from them. Let us use the abbreviations

\[
S \equiv \frac{\sin \pi x_1 \sin \pi x_2}{\sin \pi y_1 \sin \pi y_2}, \quad T \equiv x_1 + x_2 - y_1 - y_2, \tag{5.13}
\]

which are single-valued functions of \(x_1, x_2, y_1, y_2\). We then must solve

\[
y_3 = x_3 + T + 2, \quad \sin \pi y_3 = S \sin \pi x_3. \tag{5.14}
\]

This has the solution

\[
x_3 = \frac{1}{2\pi i} \log \frac{S - e^{-i\pi T}}{S - e^{i\pi T}}, \quad y_3 = x_3 + T + 2, \tag{5.15}
\]

so that the only ambiguity is a translation \(x_3 \mapsto x_3 + M, y_3 \mapsto y_3 + M\), shifting \(x_3\) and \(y_3\) by a common integer \(M\). We will use this freedom below and we conclude that we have indeed found the required two conditions for (5.8) to hold.

### 5.3 A double-sided hypergeometric identity

Equation (5.8) can be further simplified after we work out \(f_2f_3/f_1\). We can do this using (5.5) and \(v_j - u_j = 1 + x_j - y_j\), for \(j = 1, 2, 3\), which follows from (5.7). The result is

\[
\frac{f_2f_3}{f_1} = \frac{\pi^2 \sin \pi u_1}{\sin \pi x_2 \sin \pi x_3 \sin \pi (y_1 - x_1)} \prod_{j=1}^{3} \frac{\Gamma(y_j)\Gamma(u_j)}{\Gamma(x_j)\Gamma(v_j)\Gamma(y_j - x_j)}. \tag{5.16}
\]
Here the functional equation of the Gamma function \( \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z \) has been used. Substituting (5.16) and (5.7) in (5.8), we obtain

\[
\sum_{n=-\infty}^{\infty} \prod_{j=1}^{3} \frac{\Gamma(x_j + m_j + n)}{\Gamma(y_j + m_j + n)} = \frac{G(x_1, x_2, x_3 | y_1, y_2, y_3)}{\prod_{i=1}^{3} \prod_{j=1}^{3} \Gamma(y_i - x_j + m_i - m_j)},
\]

(5.17)

where the function \( G(x_1, x_2, x_3 | y_1, y_2, y_3) \) is given by

\[
G(x_1, x_2, x_3 | y_1, y_2, y_3) \equiv \frac{\pi^5}{\sin \pi x_2 \sin \pi x_3 \prod_{i=1}^{3} \sin \pi (y_i - x_1)},
\]

(5.18)

and can also be expressed as a product of ten Gamma functions.

We can simplify (5.17) further by absorbing the integers \( m_j \) in the \( x_j \) and the \( y_j \), i.e. \( x_j + m_j \mapsto x_j, y_j + m_j \mapsto y_j \). The function \( G(x_1, x_2, x_3 | y_1, y_2, y_3) \) is invariant under this translation. We also note that in this way we utilize the freedom in solving the two conditions (5.10) and (5.12) in subsection 5.2.

As the main conclusion of this section, we find from the large-\( N \) limit of the chiral Potts model that the following double-sided hypergeometric identity holds:

\[
\sum_{n=-\infty}^{\infty} \prod_{i=1}^{3} \frac{\Gamma(x_i + n)}{\Gamma(y_i + n)} = \frac{G(x_1, x_2, x_3 | y_1, y_2, y_3)}{\prod_{i=1}^{3} \prod_{j=1}^{3} \Gamma(y_i - x_j)},
\]

(5.19)

where

\[
G(x_1, x_2, x_3 | y_1, y_2, y_3) = \prod_{j=2}^{3} \Gamma(x_j) \Gamma(1-x_j) \prod_{i=1}^{3} \Gamma(y_i - x_1) \Gamma(1-y_i + x_1),
\]

(5.20)

or equivalently (5.18). The identity (5.19) holds provided both the Saalschütz condition and the periodicity condition of subsection 5.2 hold, i.e.

\[
x_1 + x_2 + x_3 + 2 = y_1 + y_2 + y_3,
\]

\[
\sin \pi x_1 \sin \pi x_2 \sin \pi x_3 = \sin \pi y_1 \sin \pi y_2 \sin \pi y_3.
\]

(5.21)

Equation (5.19) is clearly also an identity for

\[
\begin{aligned}
\mathbf{3}_H \left[ x_1, x_2, x_3 \left| y_1, y_2, y_3, 1 \right. \right] &= \prod_{i=1}^{3} \frac{\Gamma(y_i)}{\Gamma(x_i)} \sum_{n=-\infty}^{\infty} \prod_{i=1}^{3} \frac{\Gamma(x_i + n)}{\Gamma(y_i + n)},
\end{aligned}
\]

(5.22)
which involves the double-sided hypergeometric function \( _3H_3 \) as defined e.g. by Slater [22].

We should emphasize that (5.19) holds whenever (5.21) is satisfied and none of the \( x_j \) is an integer. The summand and the right-hand side of (5.19) are meromorphic functions of their six variables and the sum converges absolutely like \(|n|^{-2}\) due to (5.21), see also subsection 4.3. From (5.15) we see that all solutions of (5.21) are connected by analytic continuation, and the ambiguities in solving (5.21) relate to the way we go around the logarithmic branchpoints in (5.15).

More generally, the \( N \to \infty \) limit treated in this paper corresponds to a \( q \equiv \omega \to 1 \) limit of the cyclic (basic) hypergeometric functions of [7,24]. Looking back, the theory of these cyclic hypergeometric functions seems to be nearly synonymous with the theory of the integrable chiral Potts model. The fact that we have found yet another new hypergeometric identity confirms this point of view.

No direct proof of (5.19) has been given here. Such a proof should exist and the various symmetries of (5.19),

- Permutations of \( x_1, x_2, x_3 \),
- Permutations of \( y_1, y_2, y_3 \),
- Reflections \( x_j \mapsto 1 - y_j, y_j \mapsto 1 - x_j \), for \( j = 1, 2, 3 \) simultaneously,
- Translations \( x_j \mapsto x_j + M, y_j \mapsto y_j + M \), shifting \( x_j \) and \( y_j \) by a common integer \( M \), for \( j = 1, 2 \) or 3,
- Shifts \( y_i \mapsto y_i + M, y_j \mapsto y_j - M, i \neq j, M \) an integer,

should be helpful in such a proof. The left-hand side and the double product in the denominator of the right-hand side of (5.19) are both separately invariant under the first three of these symmetries, where for the reflection symmetry it has been assumed that the conditions (5.21) hold and the functional equation of the Gamma function has to be used. The fourth symmetry extends (5.19) to a star-triangle equation, as seen above in (5.17), transforming one solution of (5.21) into another one. Similarly, the fifth symmetry also maintains the validity of (5.19).

Therefore, the function \( G(x_1, x_2, x_3|y_1, y_2, y_3) \) given by (5.18) or (5.20) should also exhibit the same symmetries, i.e.

\[
G(x_1, x_2, x_3|y_1, y_2, y_3) = G(1-y_1, 1-y_2, 1-y_3|1-x_1, 1-x_2, 1-x_3)
= G(x_2, x_1, x_3|y_1, y_2, y_3) = G(x_3, x_2, x_1|y_1, y_2, y_3)
= G(x_1, x_2, x_3|y_2, y_1, y_3) = G(x_1, x_2, x_3|y_3, y_2, y_1). \tag{5.23}
\]

This is verified in Appendix D, to which we refer for some details.
5.4 Relation with the Dougall–Ramanujan formula

It is probably now a good idea to ask ourselves how equation (5.19) fits in the existing mathematical literature. There are not many identities available for double-sided hypergeometric functions\(^{10}\) and one obvious source for them is the textbook of Slater [22]. It is easily seen that the double-sided hypergeometric sum \(p\bar{H}_p\{\{a\};\{b\}; z\}\) converges only on parts of the unit circle in the \(z\)-plane, and there are a few identities for \(z = 1\) available.

Slater gives a general \(3H_3\)-identity following from a more general \(5H_5\)-identity of Dougall. The identity reads

\[
3H_3 \left[ \begin{array}{ccc} x_1, & x_2, & x_3 \\ 1 + a - x_1, & 1 + a - x_2, & 1 + a - x_3 \end{array} \right] = \frac{\Gamma(1 - \frac{1}{2}a)\Gamma(1 + \frac{1}{2}a)\Gamma(1 + \frac{3}{2}a - x_1 - x_2 - x_3)}{\Gamma(1 - a)\Gamma(1 + a)} \times \prod_{j=1}^{3} \frac{\Gamma(1 - x_j)\Gamma(1 + a - x_j)}{\Gamma(1 + a - x_1 - x_2 - x_3 + x_j)\Gamma(1 + \frac{1}{2}a - x_j)}. \tag{5.24}
\]

see (6.1.2.6) of [22], with \(b = x_1, c = x_2, d = x_3\). Note that this has four free parameters, just like in (5.19) where there are two relations (5.21) among the six parameters. It is easily checked that (5.19) and (5.24) are different, and one may say that (5.24) also has six parameters \(x_j\) and \(y_j = 1 + a - x_j\), but with two linear relations, whereas one of the relations in (5.21) is nonlinear.

When \(a = 0\), (5.24) reduces to the Dougall–Ramanujan formula,

\[
\sum_{n=-\infty}^{\infty} \prod_{i=1}^{3} \frac{\Gamma(x_i + n)}{\Gamma(1 - x_i + n)} = \frac{\Gamma(1 - x_1 - x_2 - x_3) \prod_{i=1}^{3} \Gamma(x_i)}{\prod_{1 \leq i < j \leq 3} \Gamma(1 - x_i - x_j)}, \tag{5.25}
\]

with \(x_1 = -x, x_2 = -y, x_3 = -z\) in [23] and valid for \(\Re(x_1 + x_2 + x_3) < 1\).

Note, while comparing with (5.19), that we must set \(y_i = 1 - x_i\) and the nonlinear periodicity condition (5.12) is automatically satisfied. Imposing the Saalschütz condition (5.10) implies

\[
\sum_{i=1}^{3} x_i = \frac{1}{2}, \quad \sum_{i=1}^{3} y_i = \frac{5}{2}, \quad y_i = 1 - x_i, \tag{5.26}
\]

\(^{10}\)We are grateful to Professor G.E. Andrews for his comments on this section.
so that (5.25) reduces to

$$\sum_{n=-\infty}^{\infty} \prod_{i=1}^{3} \frac{\Gamma(x_i + n)}{\Gamma(1 - x_i + n)} = \Gamma(\frac{1}{2}) \prod_{i=1}^{3} \frac{\Gamma(x_i)}{\Gamma(x_i + \frac{1}{2})}.$$  \hspace{1cm} (5.27)$$

It is a simple exercise to show that the special case \(x_1 + x_2 + x_3 = \frac{1}{2}, y_i = 1 - x_i\) of (5.19) coincides with (5.27).\(^{11}\)

Some other interesting hypergeometric sums have been evaluated in [25], now under certain nonlinear conditions, but (5.19) appears still to be new.

6 Discussion

There are several reasons why the above results may be of interest, although much of this will have to be deferred to future publications.

First, the integrable chiral Potts model for finite \(N\) is intimately related with integrable deformations of a series of parafermionic conformal field theories [26]. It was originally proposed in [26] that the Fateev–Zamolodchikov model [9] constitutes a series of critical lattice models in the same universality classes for each \(N\). This has been numerically checked by Alcaraz [27] for \(N \leq 8\). The more general chiral Potts model provides therefore lattice deformations of the conformal theory in the chiral-field direction, see e.g. [28,29].

This picture should extend also to the large-\(N\) limit. In section 3 we have constructed three different \(N \to \infty\) limits of the integrable chiral Potts model. These lead to three solutions of the star-triangle equations, with an infinite sum or finite or infinite integral, as is shown in section 4. Such exact solutions with an infinite state space per spin ought to be of interest as they should relate to new chiral integrable deformations of parafermionic conformal field theories.

There are not many nontrivial exact results for nearest-neighbor systems with infinite spin dimensionality. However, for the chiral Potts model with finite \(N\), exact results exist for the free energy [17,30–32], order parameter [33], groundstate energy and excitation spectra [34–36] of the associated quantum chain, and surface tensions [37–40]. Many of these results have been obtained using systems of functional equations for transfer matrices [41,42].

Large-\(N\) limits of these quantities can be constructed. For example, we can use

\(^{11}\) Only the functional relation \(\Gamma(z)\Gamma(1 - z) = \pi / \sin \pi z\) and the duplication formula \(\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2})/\Gamma(\frac{1}{2})\) of the Gamma function [19,20] are required.
a conjecture for the order parameters \([33]\), i.e. for each \(n\) with \(1 \leq n \leq N - 1\) an order parameter is given by the expectation value

\[
\langle \omega^{n a_0} \rangle = (1 - k'^2)^n(N-n)/2N^2,
\]

(6.1)

where \(a_0 = 0, \cdots, N - 1\) is the random value of a given bulk spin, say at the origin. In the limiting regime I, where this spin \(a_0\) can run through all positive and negative integers, (6.1) tends to

\[
\langle e^{i x a_0} \rangle = (1 - k'^2)^{x(2\pi - x)/8\pi^2},
\]

(6.2)

where \(x\) is a real number \(0 < x < 2\pi\) resulting from the limit \(2\pi n/N \to x\). Similar limits can be constructed from existing exact results for some other thermodynamic quantities.

The precise status of (6.2) must still be determined as the above construction presumes the interchange of the \(N \to \infty\) limit with the thermodynamic limit followed by the field limit defining the order parameter. Furthermore, it would be interesting to study all these thermodynamic quantities further within a larger \(\infty\)-state model containing the integrable manifold and we hope to return to this in the future.

Finally, from a mathematical point of view, the \(N \to \infty\) limit corresponds to a \(q \equiv \omega \to 1\) limit of the cyclic (basic) hypergeometric functions of [7,24], which are intimately related with the integrable chiral Potts model. However, the connection of the double-sided series (3.26) and (5.19) with more general cyclic hypergeometric series will be treated elsewhere.

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A  The Constant $A$

In this appendix we verify that the formula for constant $A$ given in (3.2) works out in all four cases.

Substituting (3.3) and (3.4) we get, using (2.17) and (2.22),

$$\begin{align*}
(A_{pq})^2 &= \frac{\cos^2 \frac{1}{2}(\theta_p - \phi_q)}{\cos^2 \frac{1}{2}(\theta_q - \phi_p)} = \frac{1 + \cos \theta_p \cos \phi_q + \sin \theta_p \sin \phi_q}{1 + \cos \theta_q \cos \phi_p + \sin \theta_q \sin \phi_p} \\
&= \frac{1 + k^2 + 2k \cos \theta_p}{1 + k^2 + 2k \cos \theta_q} \frac{\sin \theta_p \sin \phi_q}{\sin \phi_p \sin \theta_q} = e^{2\gamma_p - 2\gamma_q}, \quad (A.1)
\end{align*}$$

and

$$\begin{align*}
(\bar{A}_{pq})^2 &= \frac{\sin^2 \frac{1}{2}(\theta_q - \theta_p)}{\sin^2 \frac{1}{2}(\phi_q - \phi_p)} = \frac{1 - \cos \theta_p \cos \theta_q - \sin \theta_p \sin \theta_q}{1 - \cos \phi_q \cos \phi_p - \sin \phi_q \sin \phi_p} \\
&= \frac{(1 + k^2 + 2k \cos \theta_p)(1 + k^2 + 2k \cos \theta_q)}{1 - k^2} \frac{\sin \theta_p \sin \theta_q}{\sin \phi_p \sin \phi_q} = e^{2\gamma_p + 2\gamma_q}, \quad (A.2)
\end{align*}$$

so that we receive agreement with (2.25) and (2.26). The signs of the square roots are easily verified for the limit $\theta_q \to \theta_p$ and $\phi_q \to \phi_p$ in case of (A.1), or $\phi_p \to \theta_p$ and $\phi_q \to \theta_q$ for case (A.2).

Similarly, substituting (3.7) and (3.8) into the constants $A$ of (3.2) and using (2.21) we have

$$\begin{align*}
A^{(f)}_{pq} &= \frac{\sin \frac{1}{2}(\tilde{\theta}_p - \tilde{\theta}_q)}{\sin \frac{1}{2}(\tilde{\phi}_q - \tilde{\phi}_p)} = e^{i\pi(\lambda_p + \lambda_q)} \frac{e^{\gamma_p - i\pi \lambda_q} - e^{\gamma_p - i\pi \lambda_p}}{e^{\gamma_q + i\pi \lambda_q} - e^{\gamma_p + i\pi \lambda_p}} \\
&= e^{i(\phi_p + \phi_q - \theta_p - \theta_q)/2}, \quad (A.3)
\end{align*}$$

$$\begin{align*}
\bar{A}^{(f)}_{pq} &= \frac{\cos \frac{1}{2}(\tilde{\phi}_p - \tilde{\theta}_q)}{\cos \frac{1}{2}(\tilde{\theta}_p - \tilde{\phi}_q)} = e^{i\pi(\lambda_q - \lambda_p)} \frac{e^{\gamma_p + i\pi \lambda_p} e^{\gamma_q - i\pi \lambda_q} + 1}{e^{\gamma_p - i\pi \lambda_p} e^{\gamma_q + i\pi \lambda_q} + 1} \\
&= e^{i(\phi_q - \theta_q - \phi_p + \theta_p)/2}, \quad (A.4)
\end{align*}$$

in agreement with (2.27) and (2.28).
B Mathematical Details of Large-\(N\) Limit

In this appendix we prove that

\[
S_n(\alpha) \equiv \sum_{j=1}^{n} \log \frac{\sin[\pi(j+\alpha-1)/N]}{\pi(j+\alpha-1)/N}
\]  

(B.1)

is asymptotically for large \(N\) given by

\[
S_n(\alpha) \approx \sum_{j=0}^{\infty} \frac{B_j(\alpha)}{j!} \left( \frac{\pi}{N} \right)^{j-1} \left[ \phi^{(j)}(\pi n/N) - \phi^{(j)}(0) \right],
\]  

(B.2)

where

\[
\phi(z) \equiv \int_{0}^{z} d\zeta \log \frac{\sin \zeta}{\zeta}
\]  

(B.3)

and \(\phi^{(j)}(z)\) is its \(j\)th derivative.

The expression (B.1), valid for \(0 \leq n < N\), must be replaced by

\[
S_n(\alpha) = -\sum_{j=1}^{-n} \log \frac{\sin[\pi(j-\alpha)/N]}{\pi(j-\alpha)/N} = -S_{-n-1}(1-\alpha),
\]  

(B.4)

for \(-N < n \leq 0\), which has the identical asymptotic expansion (B.2). This can be easily verified using

\[
B_j(1-\alpha) = (-1)^j B_j(\alpha)
\]  

(B.5)

and

\[
\phi^{(j)}(-z) - \phi^{(j)}(0) = (-1)^{j+1} [\phi^{(j)}(z) - \phi^{(j)}(0)].
\]  

(B.6)

From elementary calculus we note that the derivatives

\[
\frac{d\phi(z)}{dz} = \log \frac{\sin z}{z}, \quad \frac{d^2\phi(z)}{dz^2} = \cot z - \frac{1}{z},
\]  

(B.7)

have Taylor expansions involving the Bernoulli numbers [19,20,43]. We have
\[ \phi(z) = \sum_{l=2}^{\infty} \frac{(2i)^l B_l}{l(l+1)!} z^{l+1} = \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{2k(2k+1)!} z^{2k+1} \]

\[ = -\sum_{k=1}^{\infty} \frac{\zeta(2k) z^{2k+1}}{k(2k+1)\pi^{2k}}. \tag{B.8} \]

convergent for \(|z| < \pi\). Here \(\zeta(s)\) is the Riemann zeta function.

Therefore, we can write

\[ S_n(\alpha) = \sum_{j=1}^{n} \sum_{l=2}^{\infty} \left(\frac{2\pi i}{N}\right)^l \frac{B_l}{l!} (j+\alpha-1)^l. \tag{B.9} \]

For

\[ -N < \alpha, n + \alpha - 1 < N, \tag{B.10} \]

this is absolutely convergent, allowing us to perform the sum over \(l\) in terms of Bernoulli polynomials [19,20],

\[ B_n(x) \equiv \left(x + B\right)^n \equiv \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{B.11} \]

with the properties

\[ B_n(x + 1) = B_n(x) + nx^{n-1}, \tag{B.12} \]

\[ \sum_{j=1}^{l} (j+\alpha-1)^l = \frac{B_{l+1}(\alpha+n) - B_{l+1}(\alpha)}{l+1}, \tag{B.13} \]

\[ B_{l+1}(\alpha+n) \equiv \left(\alpha + n + B\right)^{l+1} = \sum_{k=0}^{l+1} \binom{l+1}{k} B_k(\alpha) x^{l+1-k}. \tag{B.14} \]

We find therefore

\[ S_n(\alpha) = \sum_{l=2}^{\infty} \sum_{k=0}^{l} \left(\frac{2\pi i}{N}\right)^l \frac{B_l}{l!} \frac{B_k(\alpha)}{k!} \left(\frac{d}{dx}\right)^k \left| \frac{x^{l+1}}{(l+1)!} \right|_{x=n}. \tag{B.15} \]

Here the sum over \(l\) converges absolutely within range (B.10).

However, the double sum converges only relatively, as one can also verify numerically. In order to obtain the result (B.2) we need to interchange the
two sums, leading to an asymptotic expansion as a consequence. We find

\[ S_n(\alpha) = \sum_{k=0}^{\infty} \frac{B_k(\alpha)}{k!} \frac{N}{\pi} \left( \frac{d}{dx} \right)^k \left( \frac{\pi x}{N} \right)^{l+1} \bigg|_{x=n}, \quad (B.16) \]

from which (B.2) immediately follows, or equivalently

\[ S_n(\alpha) \approx N \phi(\pi n/N) + (\alpha - \frac{1}{2})\phi^{(1)}(\pi n/N) \]

\[ + \sum_{l=1}^{\infty} \frac{B_{l+1}(\alpha)}{(l+1)!} \left( \frac{\pi}{N} \right)^l \left[ \phi^{(l+1)}(\pi n/N) - \phi^{(l+1)}(0) \right], \quad (B.17) \]

after using \( B_0(\alpha) = 1, B_1(\alpha) = \alpha - \frac{1}{2}, \phi(0) = \phi^{(1)}(0) = 0. \)

The first line of (B.17) is sufficient for our purposes, as we need to keep only terms of order \( O(1) \) as \( N \to \infty. \) Therefore, we conclude this appendix with estimating the second line. For \( n \geq 2, 0 \leq x \leq 1, \) the Bernoulli polynomial can be expressed as [19,20]

\[ B_n(x) = -\frac{2 \cdot n!}{(2\pi)^n} \sum_{k=1}^{\infty} \cos(2\pi kx - \frac{1}{2} \pi n) \frac{k^n}{k^n}, \quad (B.18) \]

so that

\[ |B_n(x)| \leq \frac{2 \cdot n!}{(2\pi)^n} \zeta(n), \quad \text{for } n \geq 2, \quad 0 \leq x \leq 1, \quad (B.19) \]

with \( 1 < \zeta(n) \leq \zeta(2) = \pi^2/6. \) Also, for \( m = 2, 3, \cdots, \) and \( |z| < \pi, \) we have from (B.8)

\[ \phi^{(m)}(z) - \phi^{(m)}(0) = -\sum_{k=\lceil m/2 \rceil}^{\infty} \frac{2 \cdot (2k-1)! \zeta(2k) z^{2k+1-m}}{(2k+1-m)! \pi^{2k}}, \quad (B.20) \]

where \( \lceil x \rceil \) is the smallest integer \( \geq x. \) Since all the coefficients in (B.20) are negative, we have

\[ \phi^{(m)}(z) - \phi^{(m)}(0) = (-1)^{m+1} \left[ \phi^{(m)}(-z) - \phi^{(m)}(0) \right] \leq 0 \quad (B.21) \]

and monotonically decreasing for \( 0 \leq z \leq \pi. \) Therefore, we can estimate (B.20) by replacing all \( \zeta(2k) \) by its minimum 1 or its maximum \( \zeta(2\lceil \frac{1}{2}m \rceil), \)
which makes the sum a binomial-type expansion. We find

\[ \chi^{(m)}(z) \leq -\left(\text{sign } z\right)^{m+1} \left[ \phi^{(m)}(z) - \phi^{(m)}(0) \right] \leq \zeta(2\left\lfloor \frac{1}{2}m \right\rfloor) \chi^{(m)}(z), \quad (B.22) \]

where

\[ \chi^{(2n)}(z) \equiv (2n - 2)! \left[ \frac{1}{(\pi - |z|)^{2n-1}} - \frac{1}{(\pi + |z|)^{2n-1}} \right], \]

\[ \chi^{(2n+1)}(z) \equiv (2n - 1)! \left[ \frac{1}{(\pi - |z|)^{2n}} + \frac{1}{(\pi + |z|)^{2n}} - \frac{2}{\pi^{2n}} \right], \quad (B.23) \]

which are both positive for \( n = 1, 2, \cdots \).

Bounds (B.19) and (B.22) are quite sharp, as can also be numerically verified. The bounds show that (B.2) is an asymptotic expansion, with absolute value of terms roughly bounded by \( \frac{1!}{(\pi N)^l} \) and error less than the bound on the first ignored term. This is true for \( N \geq 2 \) and \( |n| < N \). But the expansion becomes particularly useful for large \( N \) and the full range of \( n \) taken as \( |n| \leq \frac{1}{2}N \).

C Derivation of (4.8)

The factor \( F \equiv F_{pq} \) is obtained substituting (4.4)–(4.8) into (4.3). We can simplify it using (4.1) and the functional equation of the Gamma function \( \Gamma(x)\Gamma(1-x) = \pi / \sin \pi x \). This gives

\[ F = 2 \left( \frac{N}{2\pi} \right)^{\beta-\alpha} \frac{\bar{A}^{1/2} \bar{A}^{(f)}(1/2) \Gamma(\bar{\alpha})\Gamma(\bar{\beta})\Gamma(1-\bar{\beta}+\bar{\alpha}) \sin \pi \bar{\alpha}^{(f)}}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})}. \quad (C.1) \]

Part of this expression can be simplified further using \( A \equiv \sin \pi \beta / \sin \pi \alpha \), (4.8) and (4.1). More precisely,

\[ 4\bar{A} \bar{A}^{(f)} \sin^2 \pi \bar{\alpha}^{(f)} = 4\bar{A} \sin \pi \bar{\alpha}^{(f)} \sin \pi \bar{\beta}^{(f)} \]

\[ = 2\bar{A} \cos \pi (\bar{\alpha}^{(f)} - \bar{\beta}^{(f)}) - 2\bar{A} \cos \pi (\bar{\alpha}^{(f)} + \bar{\beta}^{(f)}) \]

\[ = -2\bar{A} \cos \pi (\bar{\alpha} - \bar{\beta}) + 2\bar{A} \cosh \log \bar{A} \]

\[ = \bar{A}^2 - 2\bar{A} \cos \pi (\bar{\alpha} - \bar{\beta}) + 1 = \frac{\sin^2 \pi (\bar{\beta} - \bar{\alpha})}{\sin^2 \pi \bar{\alpha}}. \quad (C.2) \]
Taking the square root of this in domain (4.2) and applying it to (C.1) we find

\[ F = \left( \frac{N}{2\pi} \right)^{\beta-\alpha} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\beta+\alpha)\sin \pi(\beta-\alpha)}{A^{1/2}\Gamma(\pi)\Gamma(\beta)\sin \pi\alpha}. \]  

(C.3)

This can be further reduced to (4.8) using the functional equation of the Gamma function again.

## D Verification of (5.23)

In this appendix we show that under the conditions (5.21), or

\[ 2 + \sum_{j=1}^{3} x_j = \sum_{j=1}^{3} y_j, \quad \prod_{j=1}^{3} \sin \pi x_j = \prod_{j=1}^{3} \sin \pi y_j \equiv \tau/(2i)^3. \]  

(D.1)

the symmetry relations (5.23) hold or, equivalently,

\[ \frac{\sin \pi(y_1 - x_1) \sin \pi(y_2 - x_1) \sin \pi(y_3 - x_1)}{\sin \pi x_1} \equiv \sigma/(2i)^2 \]  

(D.2)

is fully symmetric both in \( \{x_1, x_2, x_3\} \) and in \( \{y_1, y_2, y_3\} \) and it is also invariant under \( x_j \mapsto 1 - y_j, \ y_j \mapsto 1 - x_j, \) for \( j = 1, 2, 3 \) simultaneously. Indeed this gives (5.23), as \( G(x_1, x_2, x_3|y_1, y_2, y_3) = (2\pi i)^5 \tau/\sigma. \)

With the definitions

\[ \xi_i \equiv e^{i\pi x_i}, \quad \eta_i \equiv e^{i\pi y_i}, \quad \text{for} \ i = 1, 2, 3, \]  

(D.3)

the conditions (D.1) become

\[ \xi_1\xi_2\xi_3 = \eta_1\eta_2\eta_3 \equiv \rho, \quad \prod_{i=1}^{3}(\xi_i - \xi_i^{-1}) = \prod_{i=1}^{3}(\eta_i - \eta_i^{-1}) = \tau. \]  

(D.4)

We can now expand \( \rho \tau \) and rearrange terms. We obtain

\[ \rho^2 \sum_{i=1}^{3}(\xi_i^{-2} - \eta_i^{-2}) = \sum_{i=1}^{3}(\xi_i^2 - \eta_i^2). \]  

(D.5)

We use this to expand \( \sigma. \) Successively, we find
\[
\sigma = \frac{\rho^2 - \xi_1^2 \rho^2 \sum_{i=1}^{3} \eta_i^{-2} + \xi_1^4 \sum_{i=1}^{3} \eta_i^2 - \xi_1^6}{\rho \xi_1^2 (\xi_1^2 - 1)}
\]

\[
= \frac{\xi_2 \xi_3^2 - \rho^2 \sum_{i=1}^{3} \xi_i^{-2} + \sum_{i=1}^{3} \xi_i^2 - \sum_{i=1}^{3} \eta_i^2 + \xi_1^2 \sum_{i=1}^{3} \eta_i^2 - \xi_1^4}{\rho (\xi_1^2 - 1)}
\]

\[
= \sum_{i=1}^{3} (\eta_i^2 - \xi_i^2)/\rho. \quad (D.6)
\]

This shows that \( \sigma \) has the required permutation symmetries. The invariance of \( \sigma \) under the reflection symmetry

\[
\xi_i \mapsto -\eta_i^{-1}, \quad \eta_i \mapsto -\xi_i^{-1}, \quad (i = 1, 2, 3), \quad \text{and} \quad \rho \mapsto -\rho^{-1}, \quad (D.7)
\]

follows from (D.5) and (D.6). As \( \tau \) obeys these symmetries trivially, we can now complete the proof of (5.23).

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