EFFICIENT ESTIMATION FOR DIMENSION REDUCTION WITH CENSORED DATA

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We propose a general index model for survival data, which generalizes many commonly used semiparametric survival models and belongs to the framework of dimension reduction. Using a combination of geometric approach in semiparametrics and martingale treatment in survival data analysis, we devise estimation procedures that are feasible and do not require covariate-independent censoring as assumed in many dimension reduction methods for censored survival data. We establish the root-$n$ consistency and asymptotic normality of the proposed estimators and derive the most efficient estimator in this class for the general index model. Numerical experiments are carried out to demonstrate the empirical performance of the proposed estimators and an application to an AIDS data further illustrates the usefulness of the work.

1. Introduction. Cox proportional hazards model (Cox, 1972) is probably the most widely used semiparametric model for analyzing survival data. In the Cox model, covariate effect is described by a single linear combination of covariates in an exponential function and is multiplicative in modeling the hazard function. Although this special way of modeling the hazard function permits a convenient estimation procedure, such as the maximum partial likelihood estimation (Cox, 1975), it has its limitations. As widely studied in the literature, there are many situations where the Cox model may not be proper. Due to the limitations of the Cox model, many other semiparametric survival models have been proposed in the literature, such as the accelerated failure time model (Buckley and James, 1979), proportional odds model (McCullagh, 1980) and linear transformation model (Dabrowska and Doksum, 1988), etc. Despite of all these efforts, the link between the summarized covariate effect, typically in the form of a linear combination of covariates, and the possibly transformed event time remains to have a predetermined form and hence can be restrictive sometimes.

The single index feature of the above mentioned semiparametric survival models is appealing since the covariates effect has a nice interpretation. It also naturally achieves dimension reduction when there is a large number of covariates. However, the specific model form to link the covariate index to the event time may be restrictive, and it is often difficult to check the goodness-of-fit of the specific link function form. To achieve a model that is flexible yet is feasible in practice, we borrow and extend the idea of linear summary of the covariate effects, while free up the specific functional relation between the event time and the linear summaries. Thus, we propose the following general index model

\[
\Pr(T \leq t \mid X) = \Pr(T \leq t \mid \beta_0^T X), \quad t > 0
\]

where $T$ is the survival time of interest, $X$ is the $p$-dimensional baseline covariates, and $\beta_0 \in \mathcal{R}^{p \times d}$ is the regression coefficient matrix, with $p > d$. Several properties of model (1) is worth mentioning.

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1) First of all, instead of a single linear summary, we allow $d$ linear summaries described by the $d$ columns of $\beta_0$. This increases the flexibility of how the covariate effects are combined. We can view this as a generalization from single index to multi index covariate summary. Imagine an extreme case when $d = p$, this model degenerates to the restriction free case where the dependence of $T$ on $X$ is arbitrary. Of course, in practice, when $d$ is large, the estimation will encounter difficulties and it is not feasible to carry out the analysis. However conceptually this provides a way of appreciating the flexibility of the model. In addition, we will see that in practice, when $d$ is often smaller than $p$, this model framework allows us to find and incorporate the suitable number of indices $d$. 2) Second, we do not specify any functional form of the conditional probability. Thus, the conditional probability in (1) is simply a function of both $t$ and $\beta_0^T X$. This relaxes both the exponential form of the covariate relation and the multiplicative form of the hazard function in the Cox model and is also much more flexible than other popular semiparametric survival models, such as the accelerated failure time and linear transformation models. Despite of the flexibility of the model in (1), we show that through properly incorporating semiparametric treatment and martingale techniques, estimation and inference is still possible. 3) In addition, the analysis can be carried out under the usual conditional independent censoring assumption, where the censoring time is allowed to depend on the covariates.

The proposed general index model and associated semiparametric estimation method naturally provide a dimension reduction tool for survival data. It has a few advantages over existing dimension reduction methods for survival data. 1) First, many existing dimension reduction methods for survival data require a stronger assumption on the censoring time, such as the covariate-independent censoring assumption (e.g. Li, Wang and Chen, 1999; Lu and Li, 2011), or requires nonparametric estimation of the conditional survival function of censored survival times (Xia, Zhang and Xu, 2010) or censoring times (Li, Wang and Chen, 1999) given all the covariates, which may suffer from the curse of dimensionality. All these drawbacks are avoided here. 2) Second, most of existing methods (Xia, Zhang and Xu, 2010; Li, Wang and Chen, 1999) are constructed based on general inverse probability weighted estimation techniques in one way or another, and are thus not efficient. In contrast, our proposed method is built on the semiparametric theory (Tsiatis, 2006) and achieves the optimal semiparametric efficient estimator.

The rest of the paper is organized as the following. In Section 2, we develop the estimation procedures for both the index parameters in $\beta$ and functional relation between event time and the multiple indices. In Section 3, we establish the large sample properties to enable inference. We perform extensive numerical experiments in Section 4, where both simulation and analysis of an AIDS data are included. We conclude the paper with a discussion in Section 5, while relegate all the technical details in an Appendix.

2. Methodology Development. Define $Z = \min(T, C)$ and $\Delta = I(T \leq C)$, where $C$ is the censoring time. Assume $C \perp T \mid X$ and the relation between $T$ and $X$ follows the model in (1). The observed data consist of $(X_i, Z_i, \Delta_i), i = 1, \ldots, n$, which are independent copies of $(X, Z, \Delta)$. Note that even without censoring, $\beta_0$ in (1) is not identifiable because for any $d \times d$ full rank matrix $A$, $\beta_0$ and $\beta_0 A$ suite the model (1) equally well. Thus, we fix a parameterization of $\beta_0$ by assuming the upper $d \times d$ block of $\beta_0$ to be the identity matrix $I_d$. This ensures the unique identification of $\beta_0$. Here we consider a fixed $d$, and our focus will be in estimating the lower block of $\beta_0$, which has dimension $(p - d) \times d$. We then proceed to estimate the conditional distribution function in (1). For convenience, write $X = (X^T_\alpha, X^T_\beta)^T$, where $X_\alpha \in \mathbb{R}^d$ and $X_\beta \in \mathbb{R}^{p-d}$. Note that under the assumption of $C \perp T \mid X$ and (1), we have

$$E\{f_1(C)f_2(T) \mid \beta_0^T X\} = E[E\{f_1(C)f_2(T) \mid X\} \mid \beta_0^T X] = E\{E\{f_1(C) \mid X\}E\{f_2(T) \mid \beta_0^T X\} \mid \beta_0^T X\}$$
Note that the functions causing confusion, we omit the last parameter for notationa l simplicity. Using these notation, the more precise notations are

\[ S_{\beta}, T_{\beta} \]

Note that following our notation, \( \lambda_{\beta} \) proof utilizes properties of martingale integration and the details are given in Appendix A.1. To this end, we first characterize the nuisance tangent space as described in Proposition 1. Here we find that NPMLE does not suit well without adaption due to the inseparable relation between the hazard function and the covariates. Martingale approach may enable us to obtain one specific estimator for \( \beta \), while we aim at obtaining a more comprehensive understanding of the estimation of \( \beta \). Thus we use a less conventional approach by adopting the geometrical treatment in semiparametrics. Similar practice has been performed in Tsiatis (2006) to rediscover the partial likelihood estimator for Cox proportional hazard model.

To this end, we first characterize the nuisance tangent space as described in Proposition 1. The proof utilizes properties of martingale integration and the details are given in Appendix A.1. Define \( M(t, \beta) \) and \( M_c(t, X) \) are mean-zero martingale processes.

**Proposition 1. The nuisance tangent space** \( \Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \), where

\[
\begin{align*}
\Lambda_1 & = \left\{ a(X) : E \{ a(X) \} = 0, a(X) \in \mathcal{R}^{(p-d)d} \right\}, \\
\Lambda_2 & = \left\{ \int_0^\infty h(s, \beta) dM(s, \beta) : \forall h(Z, \beta) \in \mathcal{R}^{(p-d)d} \right\}, \\
\Lambda_3 & = \left\{ \int_0^\infty h(s, X) dM_c(s, X) : \forall h(Z, X) \in \mathcal{R}^{(p-d)d} \right\}.
\end{align*}
\]

Having found the nuisance tangent space, we can now proceed to identify the efficient score function through projecting the score function onto \( \Lambda \) and calculating the residual. The score function is defined as \( S_{\beta}(\Delta, X) = \partial \log f_X, Z, \Delta(x, z, \delta, \beta, \lambda, \lambda_c, f_X) / \partial \beta \). Let \( \lambda_1(s, \beta) = \partial \lambda(s, \beta) / \partial (\beta) \) and \( \lambda_{10}(s, \beta) = \partial \lambda_0(s, \beta) / \partial (\beta) \). Straightforward calculation yields

\[
S_{\beta}(\Delta, Z, X) = \int_0^\infty \frac{\lambda_1(s, \beta) \otimes X_t dM(s, \beta)}{\lambda(s, \beta)}.
\]
We can verify that $S_{\beta_0}(\Delta, Z, X) \perp \Lambda_1$ and $S_{\beta_0}(\Delta, Z, X) \perp \Lambda_3$ due to the martingale properties. Thus to look for the efficient score, we only need to project $S_{\beta}(\Delta, Z, X)$ onto $\Lambda_2$ and calculate its residual. To this end, we search for $h^*(s, \beta_0^T X)$ so that

$$S_{\text{eff}}(\Delta, Z, X) = S_{\beta_0}(\Delta, Z, X) - \int_0^\infty h^*(s, \beta_0^T X) dM(s, \beta_0^T X)$$

$$= \int_0^\infty \left\{ \frac{\lambda_{10}(s, \beta_0^T X)}{\lambda_0(s, \beta_0^T X)} \otimes X_t - h^*(s, \beta_0^T X) \right\} dM(s, \beta_0^T X)$$

is orthogonal to $\Lambda_2$. This entails that for any $h(s, \beta_0^T X)$,

$$0 = E \left[ \int_0^\infty h^T(s, \beta_0^T X) dM(s, \beta_0^T X) \int_0^\infty \left\{ \frac{\lambda_{10}(s, \beta_0^T X)}{\lambda_0(s, \beta_0^T X)} \otimes X_t - h^*(s, \beta_0^T X) \right\} dM(s, \beta_0^T X) \right]$$

$$= E \left[ \int_0^\infty h^T(s, \beta_0^T X) \left\{ \frac{\lambda_{10}(s, \beta_0^T X)}{\lambda_0(s, \beta_0^T X)} \otimes X_t - h^*(s, \beta_0^T X) \right\} Y(s) \lambda_0(s, \beta_0^T X) ds \right].$$

By letting $h(s, \beta_0^T X) = I(s = t)a(\beta_0^T X)$ for any $a(\beta_0^T X)$, we obtain that

$$0 = E \left[ \left\{ \frac{\lambda_{10}(t, \beta_0^T X)}{\lambda_0(t, \beta_0^T X)} \otimes X_t - h^*(t, \beta_0^T X) \right\} Y(t) \lambda_0(t, \beta_0^T X) \right]$$

$$= E \left[ \left\{ \frac{\lambda_{10}(t, \beta_0^T X)}{\lambda_0(t, \beta_0^T X)} \otimes X_t - h^*(t, \beta_0^T X) \right\} Y(t) \right].$$

Note that

$$E \left\{ X_t Y(t) \mid \beta_0^T X \right\} = \frac{E \left\{ X_t S_c(t, X) \mid \beta_0^T X \right\}}{E \left\{ S_c(t, X) \mid \beta_0^T X \right\}}.$$ (3)

This leads to

$$h^*(t, \beta_0^T X) \frac{\lambda_{10}(t, \beta_0^T X)}{\lambda_0(t, \beta_0^T X)} \otimes \frac{E \left\{ X_t S_c(t, X) \mid \beta_0^T X \right\}}{E \left\{ S_c(t, X) \mid \beta_0^T X \right\}}.$$  

Thus the efficient score is

$$S_{\text{eff}}(\Delta, Z, X) = \int_0^\infty \frac{\lambda_{10}(s, \beta_0^T X)}{\lambda_0(s, \beta_0^T X)} \otimes \left[ X_t - \frac{E \left\{ X_t S_c(s, X) \mid \beta_0^T X \right\}}{E \left\{ S_c(s, X) \mid \beta_0^T X \right\}} \right] dM(s, \beta_0^T X).$$ (4)

Several properties of the efficient score are worth pointing out. First of all, $E\{S_{\text{eff}}(\Delta, Z, X) \mid X\} = 0$ is ensured by $E\{dM(t, \beta_0^T X) \mid X\} = 0$, hence to preserve the mean zero property, we can replace $\lambda_{10}(s, \beta_0^T X)/\lambda_0(s, \beta_0^T X)$ by any function of $s$ and $\beta_0^T X$, say $g(s, \beta_0^T X)$, and still obtain

$$E \int_0^\infty g(s, \beta_0^T X) \otimes \left[ X_t - \frac{E \left\{ X_t S_c(s, X) \mid \beta_0^T X \right\}}{E \left\{ S_c(s, X) \mid \beta_0^T X \right\}} \right] dM(s, \beta_0^T X) = 0.$$  

This implies that if we are only aiming at a consistent estimator, we can use an arbitrary function $g(s, \beta_0^T X)$ to replace $\lambda_{10}(s, \beta_0^T X)/\lambda_0(s, \beta_0^T X)$ in the efficient score to get a more general martingale integration.
This implies that we can construct estimating equations of the form
\[ E \int_0^\infty g(s, \beta_0^T X) \otimes \left[ X_t - \frac{E \{ X_t S_c(s, X) \mid \beta_0^T X \}}{E \{ S_c(s, X) \mid \beta_0^T X \}} \right] Y(s) \lambda_0(s, \beta_0^T X) \, ds \]
\[ = E \int_0^\infty g(s, \beta_0^T X) \otimes \left[ E \{ X_t Y(s) \mid \beta_0^T X \} - \frac{E \{ X_t S_c(s, X) \mid \beta_0^T X \}}{E \{ S_c(s, X) \mid \beta_0^T X \}} E \{ Y(s) \mid \beta_0^T X \} \right] \times \lambda_0(s, \beta_0^T X) \, ds \]
\[ = 0. \]

As a consequence,
\[ E \int_0^\infty g(s, \beta_0^T X) \otimes \left[ X_t - \frac{E \{ X_t Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] dN(s) = 0. \]

This implies that we can construct estimating equations of the form
\[ \sum_{i=1}^n \Delta_i g(Z_i, \beta^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) \mid \beta^T X_i \}}{E \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] = 0 \]

for any \( g \), where
\[ \hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^n K_h(\beta^T X_j - \beta^T X_i) I(Z_j \geq Z_i)}{\sum_{j=1}^n K_h(\beta^T X_j - \beta^T X_i)}, \]
\[ \hat{E} \{ X_i Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^n K_h(\beta^T X_j - \beta^T X_i) X_{ij} I(Z_j \geq Z_i)}{\sum_{j=1}^n K_h(\beta^T X_j - \beta^T X_i)}. \]

Here \( E \{ Y_i(Z_i) \mid \beta^T X_i \} = E \{ Y_i(t) \mid \beta^T X_i \} \mid_{t=Z_i} \) and similarly for other terms, \( K(\cdot) \) is a kernel function and \( K_h(\cdot) = K(\cdot/h)/h \).

Third, we can further relax the estimating equation form to
\[ \sum_{i=1}^n \Delta_i g(Z_i, \beta^T X_i) \otimes \left[ a(X_i) - \frac{\hat{E} \{ a(X_i) Y_i(Z_i) \mid \beta^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] = 0 \]

by taking advantage of the fact that
\[ E \Delta g(Z, \beta_0^T X) \otimes \left[ a(X) - \frac{E \{ a(X) Y(Z) \mid \beta_0^T X \}}{E \{ Y(Z) \mid \beta_0^T X \}} \right] = 0 \]

for any \( a(X) \).

Fourth, when we choose to estimate \( \lambda_0(s, \beta_0^T X)/\lambda_0(s, \beta_0^T X) \), using, for example, the smoothed Kaplan-Meier estimator, we can then obtain the efficient estimator from solving
\[ \sum_{i=1}^n \Delta_i \hat{\lambda}_i(Z_i, \beta^T X_i) \otimes \left[ X_i - \frac{\hat{E} \{ X_i Y_i(Z_i) \mid \beta^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] = 0. \]

Here we can use a local Nelson-Aalen estimator to estimator the cumulative hazard function
\[ \hat{\Lambda}(Z, \beta^T X) = \sum_{Z_i < Z} \frac{\Delta_i K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^n I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}. \]
The local Nelson-Aalen estimator of hazard function can be obtained from

$$\hat{\lambda}(Z, \beta^T X) = \int_0^\infty K_h(t - Z)d\hat{\Lambda}(t|\beta^T X)$$

(10)

$$= \sum_{i=1}^n K_h(Z_i - Z) \frac{\Delta_i K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^n I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}.$$

and we estimate the derivative

$$\hat{\lambda}_1(Z, \beta^T X) = \frac{\partial \hat{\lambda}(Z, \beta^T X)}{\partial (\beta^T X)}$$

(11)

Here $K_h'(v) = \frac{\partial K_h(v)}{\partial v}$ is the first derivative of $K_h$ with respect to its variables, which is a vector, and $b$ is a bandwidth. For any vector or matrix $a$, let $a^{\otimes 2} = aa^T$.

Among the different constructions of consistent estimators, the estimator obtained from (9) will be shown to achieve the smallest possible variability, hence this estimator is efficient and is what we recommend. The efficient estimator will be the focus of our study. We provide the detailed algorithm of the efficient estimation procedure below.

1. Obtain an initial estimator of $\beta$ through, for example, lmmave (Xia, Zhang and Xu, 2010). Denote the result $\hat{\beta}$.
2. Replacing $E\{Y(Z) \mid \beta^T X\}$, $E\{X_iY(Z) \mid \beta^T X\}$, $\lambda(Z, \beta^T X)$ and $\lambda_1(Z, \beta^T X)$ with their nonparametric estimated versions given in (6), (7), (10) and (11) respectively. Write the resulting estimators as $\hat{E}\{X_iY(Z) \mid \beta^T X\}$, $\hat{E}\{Y(Z) \mid \beta^T X\}$, $\hat{\lambda}(Z, \beta^T X)$ and $\hat{\lambda}_1(Z, \beta^T X)$.
3. Plug $\hat{E}\{X_iY(Z) \mid \beta^T X\}$, $\hat{E}\{Y(Z) \mid \beta^T X\}$, $\hat{\lambda}(Z, \beta^T X)$ and $\hat{\lambda}_1(Z, \beta^T X)$ into (9) and solve the estimating equation to obtain the efficient estimator $\hat{\beta}$, using $\hat{\beta}$ as starting value.

In performing the nonparametric estimations, bandwidths need to be selected. Because the final estimator is insensitive to the bandwidths, as indicated by condition C2, Lemma 1, Theorems 1 and 2, where a range of different bandwidths all lead to the same asymptotic property, we suggest that one should select the corresponding bandwidths by taking the sample size $n$ to its suitable power to satisfy C2, and then multiply a constant to scale it. For example, when $d = 1$, we let $h$ be $n^{-1/4-1/32}$ multiplying the standard deviation of $\hat{\beta}^T X_i$, let $b$ be $n^{-1/4+1/8}$ multiplying the standard deviation of $Z_i$.

3. Asymptotics. We will show that the efficient estimator described in Section 2 is root-$n$ consistent, asymptotically normally distributed and achieves the optimal efficiency. Let the parameter space of $\beta$ be $\mathcal{B}$. We first list some regularity conditions.

C1 (The kernel function.) The univariate kernel function $K(x)$ is symmetric, monotonically decreasing when $x > 0$ and differentiable, with bounded derivative. In addition, $\int x^j K(x)dx = 0$, for $1 \leq j < \nu$, $0 < \int x^\nu K(x)dx < \infty$, $\int K^2(x)dx < \infty$, $\int x^2 K^2(x)dx < \infty$, $\int K''(x)dx < \infty$, $\int x^2 K''(x)dx < \infty$, $\int x^2 K''(x)dx < \infty$. The $d$-dimension kernel function is a product of $d$ univariate kernel functions, that is $K(u) = \prod_{j=1}^d K(u_j)$ for $u = (u_1, ..., u_d)^T$. For simplicity, we use the same $K$ for both univariate and multivariate kernel functions.
C2 (The bandwidths.) The bandwidths satisfy $h \to 0$, $b \to 0$, $nh^{d+2}b \to \infty$ and $nh^{2\nu} \to 0$, where $2\nu > d + 1$.

C3 (The boundedness.) The parameter space $\mathcal{B}$ is bounded.

C4 (The density functions of covariates.) For all $\beta \in \mathcal{B}$, the probability density function of $\beta^\top X$, $f_{\beta^\top X}(\beta^\top x)$, has a compact support and has four derivatives. The function $f_{\beta^\top X}(\beta^\top x)$ is bounded away from zero and infinity on the support, and its first four derivatives are bounded uniformly on the support.

C5 (The smoothness.) For all $\beta \in \mathcal{B}$, the absolute value of $E\{X_j I(Z_j \geq Z) \mid \beta^\top x\}$, $E\{I(Z_j \geq Z) \mid \beta^\top x\}$, and their first four derivatives are bounded uniformly component wise. The absolute value of $E\{X_j X_j^\top I(Z_j \geq Z) \mid \beta^\top x\}$ and its first two derivatives are bounded uniformly component wise.

C6 (The survival function.) For all $\beta \in \mathcal{B}$, the survival function of the event process $S(t, \beta^\top x)$, the conditional expectation of the survival function of the censoring processes $E\{S_c(t, X) \mid \beta^\top x\}$ and the probability density function of the survival process $f(t, \beta^\top x)$ satisfy $\partial^{i+j} S(t, \beta^\top x)/\partial t^i \partial \beta^j(\beta^\top x)^j$, $\partial^{i+j} E\{S_c(t, X) \mid \beta^\top x\}/\partial t^i \partial \beta^j(\beta^\top x)^j$, $\partial^{i+j} f(t, \beta^\top x)/\partial t^i \partial \beta^j(\beta^\top x)^j$ exist and are bounded and bounded away from zero, for all $i \geq 0, j \geq 0, i + j \leq 4$.

C7 (The uniqueness.) The equation

$$E \left( \frac{\Delta \lambda_1(Z, \beta^\top X)}{\lambda(Z, \beta^\top X)} \otimes \left[ X_i - \frac{E\{X_i Y(Z) \mid \beta^\top X\}}{E\{Y(Z) \mid \beta^\top X\}} \right] \right) = 0$$

has a unique solution on $\mathcal{B}$. Because the true parameter $\beta_0$ satisfies the equation, hence the unique solution is $\beta_0$.

These conditions are quite commonly imposed in nonparametrics, survival analysis and estimating equations and are generally mild. Conditions C1 and C2 contain some basic requirements on the kernel function and the bandwidths, which are common in kernel related works. The boundedness of the parameter space $\mathcal{B}$ in C3 is satisfied in most practical problems. Condition C4-C6 impose certain boundedness condition of event time, censoring time, covariates, their expectations and corresponding derivatives. The unique solution requirement in C7 is needed to ensure the convergence of the estimator, which could be further relaxed to local uniqueness if needed.

Before presenting the main results, we summarize several preliminary results first. These results highlight the theoretical properties of the kernel based estimators of several conditional expectations, as well as the estimation properties of the hazard function and its derivative, hence are of their own interest. These properties also play an important role in the proof of the Theorems 1 and 2. The proofs of Lemma 1 and both theorems are respectively in the Appendix A.2, A.3 and A.4.

**Lemma 1.** Under the regularity conditions C1-C7 listed above,

(12) $\hat{E}\{Y(Z) \mid \beta^\top X\} = E\{Y(Z) \mid \beta^\top X\} + O_p\{(nh)^{-1/2} + h^2\}$,

(13) $\hat{E}\{XY(Z) \mid \beta^\top X\} = E\{XY(Z) \mid \beta^\top X\} + O_p\{(nh)^{-1/2} + h^2\}$,

(14) $\frac{\partial}{\partial \beta^\top X} \hat{E}\{Y(Z) \mid \beta^\top X\} = \frac{\partial}{\partial \beta^\top X} E\{Y(Z) \mid \beta^\top X\} + O_p\{(nh^3)^{-1/2} + h^2\}$

(15) $\frac{\partial}{\partial \beta^\top X} \hat{E}\{XY(Z) \mid \beta^\top X\} = \frac{\partial}{\partial \beta^\top X} E\{XY(Z) \mid \beta^\top X\} + O_p\{(nh^3)^{-1/2} + h^2\}$

(16) $\hat{\lambda}(z, \beta^\top X) = \lambda(z, \beta^\top X) + O_p\{(nhb)^{-1/2} + h^2 + b^2\}$

(17) $\hat{\lambda}_1(z, \beta^\top X) = \lambda_1(z, \beta^\top X) + O_p\{(nhb)^{-1/2} + h^2 + b^2\}$
uniformly for all $Z, \beta^T X$.

**Theorem 1.** The estimator obtained from solving (9) is consistent, i.e. $\hat{\beta} - \beta_0 \to 0$ in probability when $n \to \infty$.

**Theorem 2.** The estimator obtained from solving (9) satisfies

$$\sqrt{n}(\hat{\beta} - \beta_0) \to N(0, [E\{S_{\text{eff}}^2(\Delta, Z, X)\}]^{-1})$$

in distribution when $n \to \infty$. Here $S_{\text{eff}}(\Delta, Z, X)$ is given in (4). Thus, the estimator is efficient.

Note that because $S_{\text{eff}}$ is a martingale, we have

$$E\{S_{\text{eff}}^2(\Delta, Z, X)\} = E \int_0^\infty \left( \frac{\lambda_{10}(s, \beta_0^T X)}{\lambda_0(s, \beta_0^T X)} \otimes \left[ X_i - \frac{E \{X_i S_c(s, X) | \beta_0^T X\}}{E \{S_c(s, X) | \beta_0^T X\}} \right] \right)^2 Y(s)ds$$

Therefore, a natural estimator of the estimation variance is the inverse of

$$\frac{1}{n} \sum_{i=1}^n \delta_i \left( \frac{\hat{\lambda}_1(z_i, \hat{\beta}^T x_i)}{\hat{\lambda}(z_i, \hat{\beta}^T x_i)} \otimes \left[ x_{it} - \frac{\hat{E} \{X_i S_c(z_i, X) | \hat{\beta}^T x_i\}}{\hat{E} \{S_c(z_i, X) | \hat{\beta}^T x_i\}} \right] \right)^2.$$

**4. Numerical Experiments.**

4.1. Simulation. To evaluate the finite sample performance of our method, we perform four simulation studies. In the first study, we generate event times from $Z$, uniformly for all $X$.

$$T = \Phi \left[ 5\epsilon \{ \exp(3\beta^T X) + 1 \} - 2 \right]$$

where $\Phi$ is the cumulative distribution function (cdf) of the standard normal distribution, $\epsilon$ has an exponential distribution with parameter 1, and $X$ follows a standard normal distribution independent with $\epsilon$. We consider $d = 1, p = 7$ and the true parameter values are taken to be $\beta = (1, 0, -1, 0, 1, 0, -1)^T$. We further generate the covariate dependent censoring times using $C = \Phi(2X_2 + 2X_3) + U$ where $U$ denotes a random variable uniformly distributed on $(0, c_1)$, where $c_1$ is a constant controlling the proportion of censoring.

In the second study, we generate the event times from

$$T = \exp(\beta^T X + \epsilon)$$

where $\epsilon$ follows an exponential distribution with parameter 1 and each component in $X$ follows independent uniform distribution on $(0, 1)$. We consider $d = 1, p = 7$ and set the true parameter value to be $\beta = (1, 1.3, -1.3, 1, -0.5, 0.5, -0.5)^T$. We generate the censoring time from a uniform distribution on $(0, c_2)$, where different values of $c_2$ are used to achieve various censoring rate.

In the third study, we generate the event times from

$$T = \exp \left\{ 5 - 10(1 - \beta^T X)^2 + \epsilon \right\}$$
where \( \epsilon \sim \text{Normal}(0, 1) \), and each component of \( \mathbf{X} \) is independently distributed with uniform distribution on \((0, 1)\). We consider \( d = 1, p = 10 \) and set the true parameter value to be \( \beta = (1, -0.6, 0, -0.3, -0.1, 0, 0.1, 0.3, 0, 0.6)^T \). The censoring time is generated from \( C = U \beta^T \mathbf{X} \) where \( \beta_c = (0, 0, 0, 1, 1, 0, 0, 0, 0, 0)^T \) and \( U \) is uniformly distributed on \((0, c_3)\), and \( c_3 \) is a constant controlling the censoring proportion.

In the last simulation study, we increase \( d \) to 2 to further evaluate the performance of the proposed method. We set the event times

\[
T = \exp \left\{ 5 - 10 \sum_{j=1}^{2} (1 - \beta_j^T \mathbf{X})^2 + \epsilon \right\}
\]

where \( \epsilon \sim \text{Normal}(0, 1) \) and each component of \( \mathbf{X} \) is independently distributed with uniform distribution on \((0, 1)\), \( \beta_j, j = 1, 2 \), denotes the \( j \)th column of \( \beta \) with \( p = 6 \). The censoring time is generated from a uniform distribution on \((0, c_4)\), where \( c_4 \) controls the censoring rate.

These studies are designed to resemble and extend the simulation studies considered in Xia, Zhang and Xu (2010), which proposed \( \text{hmave} \), the best method so far in the literature for dimension reduction under censored data. In fact, we compared our results with those from \( \text{hmave} \).

The results of the first simulation study are given in Table 1 and Figure 1, where we considered three different censoring rates, 0%, 20% and 40% separately. From these results, we can see that the semiparametric method we proposed generally performs better, and often is much better than \( \text{hmave} \), in that it has much smaller absolute biases as well as smaller sample standard errors. The semiparametric method also yields smaller difference between the estimated projection matrix \( \hat{\mathbf{P}} \equiv \hat{\beta}(\hat{\beta}^T \hat{\beta})^{-1} \hat{\beta}^T \) and the true projection matrix \( \mathbf{P} \equiv \beta(\beta^T \beta)^{-1} \beta^T \), in that both the mean and variance of the largest singular value of \( \hat{\mathbf{P}} - \mathbf{P} \) are much smaller based on the semiparametric method than based on \( \text{hmave} \). The same results are also presented in Figure 1 to provide a quick visual inspection.

**Table 1**

Results of study 1, based on 1000 simulations with sample size 100. “bias” is the absolute value of mean(\( \hat{\beta} \)) \(-\beta \) of each component in \( \beta \), “sd” is the sample standard errors of the corresponding estimation. The last column is the mean and standard errors of the largest singular value of \( \hat{\mathbf{P}} - \mathbf{P} \).

|                | \( \beta_2 \) | \( \beta_3 \) | \( \beta_4 \) | \( \beta_5 \) | \( \beta_6 \) | \( \beta_7 \) | \( \lambda_{\text{max}} \) |
|----------------|--------------|--------------|--------------|--------------|--------------|--------------|----------------|
| **true**       |              |              |              |              |              |              |                |
| No censoring   | 0.0010       | 0.0256       | 0.0099       | 0.0058       | 0.0052       | 0.0182       | 0.2208         |
| \( \text{hmave} \) bias | 0.1700       | 0.2344       | 0.1710       | 0.2320       | 0.1643       | 0.2264       | 0.1578         |
| \( \text{semi} \) bias | 0.0025       | 0.0129       | 0.0071       | 0.0059       | 0.0071       | 0.0033       | 0.0903         |
| \( \text{semi} \) sd | 0.1298       | 0.1333       | 0.1314       | 0.1277       | 0.1337       | 0.1335       | 0.0622         |
| **20% censoring** |              |              |              |              |              |              |                |
| \( \text{hmave} \) bias | 0.0747       | 0.0994       | 0.0095       | 0.0042       | 0.0099       | 0.0228       | 0.2256         |
| \( \text{semi} \) bias | 0.1688       | 0.2236       | 0.1663       | 0.2281       | 0.1612       | 0.2217       | 0.1560         |
| \( \text{semi} \) sd | 0.0003       | 0.0143       | 0.0064       | 0.0079       | 0.0055       | 0.0054       | 0.0928         |
| \( \text{semi} \) sd | 0.1301       | 0.1339       | 0.1300       | 0.1268       | 0.1320       | 0.1294       | 0.0574         |
| **40% censoring** |              |              |              |              |              |              |                |
| \( \text{hmave} \) bias | 0.0056       | 0.0261       | 0.0078       | 0.0079       | 0.0169       | 0.0189       | 0.2314         |
| \( \text{semi} \) bias | 0.1812       | 0.2502       | 0.1784       | 0.2462       | 0.1707       | 0.2416       | 0.1604         |
| \( \text{semi} \) sd | 0.0012       | 0.0130       | 0.0064       | 0.0090       | 0.0103       | 0.0056       | 0.0948         |
| \( \text{semi} \) sd | 0.1345       | 0.1352       | 0.1353       | 0.1305       | 0.1351       | 0.1354       | 0.0694         |
The results of the second study are presented in Tables 2, and Figures 2, where the same conclusion can be drawn as in the first study. The superiority of the semiparametric method to hmave is even more prominent in the third study, as reflected in Table 3 and Figure 3. Here, the semiparametric method is substantially more accurate in estimating each component in $\beta$, yielding smaller biases and variances. The largest singular value of the difference between the estimated and true projection matrices is also much smaller for the semiparametric method in comparison with hmave. When we increased $d$ to 2 in the last simulation, the semiparametric method continues to generate satisfactory results, see Table 4 and Figure 4. In this case, the performance of hmave is rather concerning, possibly caused by the difficulties associated with multiple indices.

Fig 1. Boxplot of hmave and the semiparametric methods of study 1. First row: no censoring; Second row: 20% censoring rate; Third row: 40% censoring rate. Dashed line: True $\beta$. 

Table 2
Results of study 2, based on 1000 simulations with sample size 200. “bias” is the absolute value of mean($\hat{\beta}$) – $\beta$ of each component in $\beta$, “sd” is the sample standard errors of the corresponding estimation. The last column is the mean and standard errors of the largest singular value of $P^\top P$.

|          | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ | $\lambda_{max}$ |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------------|----------------|
| true     |           |           |           |           |           |           |           |           |             |                |
| hmax bias| 0.1958    | 0.3428    | 0.1644    | 0.0137    | 0.3699    | 0.1672    | 0.4247    |            |             |                |
|          | 7.0919    | 7.6671    | 9.8179    | 6.2261    | 10.976    | 5.6038    | 3.011     |            |             |                |
| semi bias| 0.2483    | 0.0617    | 0.0929    | 0.2354    | 0.1273    | 0.0646    | 0.2915    |            |             |                |
|          | 5.3549    | 4.3952    | 2.5835    | 5.9618    | 3.2538    | 1.9439    | 0.1133    |            |             |                |
|          |           |           |           |           |           |           |           |            |             |                |
|          |           |           |           |           |           |           |           |            |             |                |
| 20% censoring |     |     |     |     |     |     |     |     |     |     |
| hmax bias| 0.5650    | 0.5997    | 0.3841    | 0.1947    | 0.2155    | 0.2656    | 0.3212    |            |             |                |
|          | 4.2947    | 3.4980    | 2.4377    | 1.5136    | 2.7222    | 2.4448    | 0.1356    |            |             |                |
| semi bias| 0.0864    | 0.0289    | 0.0167    | 0.0109    | 0.0537    | 0.1300    | 0.1448    |            |             |                |
|          | 1.9872    | 0.5161    | 1.2467    | 0.8233    | 0.9729    | 3.9002    | 0.0909    |            |             |                |
|          |           |           |           |           |           |           |           |            |             |                |
| 40% censoring |     |     |     |     |     |     |     |     |     |     |
| hmax bias| 0.0998    | 0.1024    | 0.0566    | 0.0381    | 0.0386    | 0.0371    | 0.1991    |            |             |                |
|          | 0.4603    | 0.4678    | 0.3877    | 0.3085    | 0.2939    | 0.3085    | 0.1303    |            |             |                |
| semi bias| 0.0158    | 0.0107    | 0.0053    | 0.0128    | 0.0063    | 0.0062    | 0.0781    |            |             |                |
|          | 0.3976    | 0.4678    | 0.3492    | 0.5517    | 0.2718    | 0.2873    | 0.0720    |            |             |                |

Table 3
Results of study 3, based on 1000 simulations with sample size 200. “bias” is the absolute value of mean($\hat{\beta}$) – $\beta$ of each component in $\beta$, “sd” is the sample standard errors of the corresponding estimation. The last column is the mean and standard errors of the largest singular value of $P^\top P$.

|          | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ | $\lambda_{max}$ |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------------|----------------|
| true     |           |           |           |           |           |           |           |           |             |                |
| hmax bias| 0.3711    | 0.3643    | 1.3056    | 0.5394    | 0.0913    | 0.1954    | 0.4499    | 0.2092    | 0.3660      | 0.8706        |
|          | 17.267    | 10.472    | 35.110    | 16.246    | 9.449     | 16.931    | 26.510    | 12.342    | 17.304      | 0.2953        |
| semi bias| 0.0124    | 0.0044    | 0.0175    | 0.0035    | 0.0026    | 0.0111    | 0.0323    | 0.0013    | 0.0180      | 0.2337        |
|          | 0.1639    | 0.1538    | 0.1523    | 0.1563    | 0.1585    | 0.1535    | 0.1590    | 0.1543    | 0.1631      | 0.0637        |
|          |           |           |           |           |           |           |           |            |             |                |
| 20% censoring |     |     |     |     |     |     |     |     |     |     |
| hmax bias| 1.4974    | 2.3355    | 1.6021    | 0.4699    | 2.3620    | 1.6553    | 0.5311    | 1.6506    | 2.9421      | 0.8822        |
|          | 41.451    | 44.735    | 66.302    | 49.280    | 40.673    | 47.488    | 58.485    | 57.025    | 72.228      | 0.2952        |
| semi bias| 0.0035    | 0.0003    | 0.0239    | 0.0063    | 0.0023    | 0.0072    | 0.0184    | 0.0035    | 0.0177      | 0.2148        |
|          | 0.1600    | 0.1584    | 0.1722    | 0.1716    | 0.1555    | 0.1615    | 0.1531    | 0.1633    | 0.1595      | 0.0691        |
|          |           |           |           |           |           |           |           |            |             |                |
| 40% censoring |     |     |     |     |     |     |     |     |     |     |
| hmax bias| 0.5909    | 5.5304    | 3.2835    | 0.8370    | 1.775     | 5.6482    | 4.8272    | 1.2442    | 0.6951      | 0.8382        |
|          | 20.946    | 146.58    | 68.000    | 25.442    | 38.877    | 145.82    | 90.530    | 58.178    | 23.032      | 0.2900        |
| semi bias| 0.0209    | 0.0004    | 0.0198    | 0.0062    | 0.0041    | 0.0138    | 0.0244    | 0.0021    | 0.0166      | 0.2656        |
|          | 0.1500    | 0.1514    | 0.1518    | 0.1506    | 0.1492    | 0.1560    | 0.1453    | 0.1503    | 0.1579      | 0.0556        |
Table 4
Results of study 4, based on 1000 simulations with sample size 200. “bias” is the absolute value of $\text{mean}(\hat{\beta}) - \beta$ of each component in $\beta$, “sd” is the sample standard errors of the corresponding estimation. The last column is the mean and standard errors of the largest singular value of $\hat{P} - P$.

|        | $\beta_{3.1}$ | $\beta_{4.1}$ | $\beta_{5.1}$ | $\beta_{6.1}$ | $\beta_{3.2}$ | $\beta_{4.2}$ | $\beta_{5.2}$ | $\beta_{6.2}$ | $\lambda_{\text{max}}$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| true   |                |                |                |                |                |                |                |                |                |
|        | 2.75           | -0.75          | -1             | 2.0            | -3.125         | -1.125         | 1.0            | -2.0           |                |
|        |                |                |                |                |                |                |                |                |                |
| No censoring |                |                |                |                |                |                |                |                |                |
| hmxave bias | 5.8883         | 3.4252         | 0.4704         | 2.9062         | 43.257         | 38.079         | 8.0696         | 33.551         | 0.8351         |
| sd     | 130.6          | 109.3          | 24.214         | 101.81         | 1091.9         | 939.06         | 174.9          | 855.8          | 0.1116         |
| semi bias | 0.1609         | 0.0914         | 0.0609         | 0.1388         | 0.1257         | 0.1002         | 0.0372         | 0.0887         | 0.1791         |
| sd     | 0.3393         | 0.2143         | 0.2545         | 0.2837         | 0.3674         | 0.2619         | 0.2559         | 0.3121         | 0.0788         |
| 20% censoring |                |                |                |                |                |                |                |                |                |
| hmxave bias | 4.2011         | 2.8472         | 1.7685         | 3.4267         | 3.5963         | 0.4313         | 0.6443         | 0.9892         | 0.9273         |
| sd     | 64.499         | 39.786         | 56.389         | 41.570         | 20.496         | 24.564         | 23.323         | 22.339         | 0.1038         |
| semi bias | 0.0846         | 0.0358         | 0.0311         | 0.0596         | 0.1133         | 0.0726         | 0.0305         | 0.0724         | 0.0971         |
| sd     | 0.3433         | 0.2068         | 0.2018         | 0.2646         | 0.4051         | 0.2690         | 0.2374         | 0.2952         | 0.0777         |
| 40% censoring |                |                |                |                |                |                |                |                |                |
| hmxave bias | 3.2328         | 0.5529         | 2.0826         | 1.0118         | 4.2734         | 3.6817         | 2.0000         | 2.9417         | 0.9363         |
| sd     | 14.712         | 19.789         | 17.661         | 19.938         | 26.892         | 73.550         | 29.838         | 31.115         | 0.1071         |
| semi bias | 0.0986         | 0.0555         | 0.0246         | 0.0808         | 0.1420         | 0.0868         | 0.0451         | 0.0950         | 0.0915         |
| sd     | 0.3604         | 0.2173         | 0.2168         | 0.2991         | 0.4864         | 0.2627         | 0.2645         | 0.3099         | 0.0898         |
Fig 2. Boxplot of $hmave$ and the semiparametric methods of study 2. First row: no censoring; Second row: 20% censoring rate; Third row: 40% censoring rate. Dashed line: True $\beta$. 

$\beta_2 = 1.3$ 
$\beta_3 = -1.3$ 
$\beta_4 = 1$ 
$\beta_5 = -0.5$ 
$\beta_6 = 0.5$ 
$\beta_7 = -0.5$
Fig 3. Boxplot of hmave and the semiparametric methods of study 3. First row: no censoring; Second row: 20% censoring rate; Third row: 40% censoring rate. Dashed line: True $\beta$. 
Figure 4. Boxplot of $h_{\text{ave}}$ and the semiparametric methods of study 4. First row: no censoring; Second row: 20% censoring rate; Third row: 40% censoring rate. Dashed line: True $\beta$.

We also performed an additional experiment to further assess the finite sample performance of the asymptotic results established in Section 3. To this end, we generate covariates $X$ from a standard normal distribution and event times $T$ from a distribution with hazard function

$$
\lambda(t|X) = \lambda_0(t) \left\{ \sum_{j=1}^{2} \exp (\beta_j^T X) \right\},
$$

where the baseline hazard $\lambda_0(t) = t$ and the dimension of $\beta$ is $d = 2, p = 6$. We use the parameter values $\beta = \{(2.75, -0.75, -1, 2)^T; (-3.125, -1.125, 1, -2)^T\}^T$, and adopt the same censoring process as in the second study to yield 40% censoring rate. We carry out 1000 simulations and consider sample sizes $n = 100, 500$ and $1000$. The estimation results, together with sample standard errors, average of the estimated standard deviations and coverage probabilities of the 95% confidence intervals are given in Table 5. These results indicate that the large sample properties of the estimator requires more sample size than 1000. However, the general trend is that when sample size increases, the results are approaching what we expect based on the asymptotic results, in that the sample standard errors and their estimated versions are becoming closer to each other, and the 95% confidence probabilities are getting closer to the nominal level. The phenomenon that
asymptotic result requires very large sample size to illustrate itself is quite common in survival data analysis and is not unique to our semiparametric method. Due to the limited sample size in practice, we recommend to use bootstrap to assess estimation variability.

Table 5
Results of study 5, based on 1000 simulations with sample size 100, 500, 1000 respectively. “bias” is the absolute value of mean(\(\hat{\beta}\)) - \(\beta\) of each component in \(\beta\). “sd” is the sample standard errors of the corresponding estimation, “\(\hat{\sigma}\)” is the mean of the estimated standard errors of \(\hat{\beta}\) component, “95%” is the sample coverage of the 95% confidence intervals.

|       | \(\beta_{31}\) | \(\beta_{41}\) | \(\beta_{51}\) | \(\beta_{61}\) | \(\beta_{32}\) | \(\beta_{42}\) | \(\beta_{52}\) | \(\beta_{62}\) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(n = 100\) | 2.75 | -0.75 | -1 | 2.0 | -3.125 | -1.125 | 1.0 | -2.0 |
| bias | 0.3995 | 0.5031 | 0.2066 | 0.3799 | 0.5515 | 0.5349 | 0.1757 | 0.3395 |
| sd | 0.5760 | 0.4236 | 0.4673 | 0.5608 | 0.6163 | 0.4376 | 0.4772 | 0.5377 |
| \(\hat{\sigma}\) | 0.3868 | 0.3188 | 0.3312 | 0.3427 | 0.3956 | 0.3131 | 0.3331 | 0.3602 |
| 95% | 0.7100 | 0.6577 | 0.8051 | 0.7034 | 0.7414 | 0.7414 | 0.8089 | 0.7414 |
| \(n = 500\) | | | | | | | | |
| bias | 0.1790 | 0.1258 | 0.0714 | 0.1338 | 0.2100 | 0.1489 | 0.07386 | 0.1340 |
| sd | 0.2741 | 0.1714 | 0.2177 | 0.2380 | 0.2979 | 0.1897 | 0.2202 | 0.2244 |
| \(\hat{\sigma}\) | 0.1585 | 0.1371 | 0.1644 | 0.1659 | 0.2683 | 0.2179 | 0.2538 | 0.2558 |
| 95% | 0.6663 | 0.8022 | 0.8298 | 0.7566 | 0.8127 | 0.8773 | 0.9125 | 0.8764 |
| \(n = 1000\) | | | | | | | | |
| bias | 0.0611 | 0.0492 | 0.0188 | 0.0423 | 0.0695 | 0.0467 | 0.0209 | 0.0448 |
| sd | 0.1951 | 0.1451 | 0.1555 | 0.1538 | 0.1867 | 0.1433 | 0.1650 | 0.1711 |
| \(\hat{\sigma}\) | 0.1062 | 0.1113 | 0.1134 | 0.1190 | 0.1823 | 0.1712 | 0.1749 | 0.1740 |
| 95% | 0.8060 | 0.8830 | 0.8783 | 0.8621 | 0.9268 | 0.9705 | 0.9515 | 0.9287 |

4.2. AIDS Application. We apply the proposed method to analyze the HIV data from AIDS Clinical Trials Group Protocol 175 (ACTG175) (Hammer et al. (1996)). In this study, 2137 HIV-infected subjects were randomized to receive one of four treatments: zidovudine (ZDV) monotherapy, ZDV plus didanosine, ZDV plus zalcitabine and ddI monotherapy. As in Geng, Lu and Zhang (2015) and Jiang et al. (2017), the survival time of interest was chosen as the time to having a larger than 50% decline in the CD4 count, or progressing to AIDS or death, whichever comes first. Besides the treatments, there were 12 covariates included in our study, specifically, patient age in years at baseline (\(X_1\)), patient weight in kilogram at baseline (\(X_2\)), hemophilia indicator (\(X_3\)), homosexual activity (\(X_4\)), history of IV drug use (\(X_5\)), Karnofsky score on a scale of 0-100 (\(X_6\)), race (\(X_7\)), gender (\(X_8\)), antiretroviral history (\(X_9\)), symptomatic indicator (\(X_{10}\)), number of CD4 at baseline (\(X_{11}\)), number of CD8 at baseline (\(X_{12}\)), treatment indicator (\(X_{13}\)), where we coded \(X_{13} = 0\) for treatment ZDV+ddl and \(X_{13} = 1\) for treatment ZDV+Zal. As in Jiang et al. (2017), we only analyzed data from the two composite treatments: ZDV plus didanosine and ZDV plus zalcitabine, which had been shown to have significantly better survival than the other two treatments (Geng, Lu and Zhang, 2015). This subset of data contains 1046 subjects with the censoring rate around 75%. In addition, each covariate is standardized respectively with no obvious outliers and no missing values.

To determine the proper reduced space dimension \(d\), we employ the Validated Information Criterion (VIC) (Ma et al., 2015), where the \(d\) corresponding to the smallest VIC value is selected. In the example, the VIC value at \(d = 1\) is 90.38. Further, when \(d \geq 2\), the VIC values are all greater than 180.7 which result from the penalty term alone. Hence we choose \(d = 1\) as the final model.

Table 6 contains the estimated coefficient \(\hat{\beta}\)'s under the selected model, with the corresponding estimation standard errors and \(p\)-values. Here, we implemented the semiparametric estimator to obtain these results due to its superior theoretical and numerical performance.

The results in Table 6 indicate that in forming the index described by \(\hat{\beta}_{1,1}\), all covariates are
significant except hemophilia indicator ($X_3$), gender ($X_8$) and number of CD4 at baseline ($X_{11}$). The estimated cumulative hazard functions are also reported in Figure 5, where it is plotted as a function of time (upper left panel), a function of the covariate index $\beta^T x$ (upper right panel) and as a function of both (bottom panel). Specifically, in plotting the cumulative hazard as a function of time $t$, we fixed the covariate index at three different sets of covariate values, respectively $X_{1:12} = (40,60,1,0,0,80,0,0,1,200,800)^T$, $X_{1:12} = (20,70,1,0,0,80,0,1,0,1,200,800)^T$, $X_{1:12} = (60,70,1,0,20,0,0,1,1,200,200)^T$, in combination with the treatment indicator of both $X_{13} = 0$ and $X_{13} = 1$. Based on the plots, the estimated cumulative hazard of treatment ZDV+ddl is slightly larger than that of treatment ZDV+Zal in all scenarios. In plotting the estimated cumulative hazard $\hat{\Lambda}$ as a function of the index $\hat{\beta}^T x$, we fixed the time at $t = 100,500$ and $1000$. Finally, we also plotted the cumulative hazard as a function of two variables $t$ and $\beta^T x$ using the contour plot, where the hazard values are explicitly written out on each contour.

Table 6
The estimated coefficients, standard errors and p-value in AIDS data.

|     | $\hat{\beta}_{2,1}$ | $\hat{\beta}_{3,1}$ | $\hat{\beta}_{4,1}$ | $\hat{\beta}_{5,1}$ | $\hat{\beta}_{6,1}$ | $\hat{\beta}_{7,1}$ | $\hat{\beta}_{8,1}$ | $\hat{\beta}_{9,1}$ | $\hat{\beta}_{10,1}$ | $\hat{\beta}_{11,1}$ | $\hat{\beta}_{12,1}$ | $\hat{\beta}_{13,1}$ |
|-----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| est | 0.115                | -0.002               | 0.093                | 0.088                | -0.090               | 0.231                | -0.003               | -0.178               | 0.058                | -0.031               | 0.201                | 0.156                |
| std | 0.039                | 0.039                | 0.039                | 0.037                | 0.043                | 0.046                | 0.036                | 0.046                | 0.035                | 0.042                | 0.033                | 0.038                |
| p-value | 0.003               | 0.965               | 0.017               | 0.017               | 0.036               | 0.001               | 0.928               | 0.001               | 0.100               | 0.457               | 0.001               | 0.001               |

5. Discussion. We have considered a very general model for analyzing time to event data subject to censoring. The model allows the event times to link to the covariate indices in an unspecified fashion. Because both the number of indices and the functional form of the linkage to the indices are data determined, conceptually the model is maximally flexible. In practice, relatively low number of indices are expected to avoid curse of dimensionality. The work is conducted without requiring covariate independent censoring. Instead, it only requires event independent censoring conditional on covariates, which is the minimum requirement for identification. We derived a class of estimators which are consistent and asymptotically normal. We also proposed a procedure to construct the semiparametric efficient estimator that achieves the optimal estimation variability among all possible consistent estimators.
Fig 5. Estimated cumulative hazard function $\hat{\Lambda}$ in AIDS data. (a). Comparisons of $\hat{\Lambda}$ as a function of $t$ between treatments ZDV+ddI and ZDV+Zal when other covariates are fixed at three indices. (b). $\hat{\Lambda}$ as a function of $\beta^T X$ at $T = 100, 500, 1000$. (c). Contour plot of $\hat{\Lambda}$ as a function of $T$ and $\beta^T X$.

Appendix.

A.1. Proof of Proposition 1. The result of $\Lambda_1$ is obvious. To obtain $\Lambda_2$, let $h(t, \beta_0^T X, \gamma) = \partial \log \lambda(t, \beta_0^T X, \gamma)/\partial \gamma$, where $\lambda(t, \beta_0^T X, \gamma)$ is a submodel of $\lambda(t, \beta_0^T X)$. Hence

$$\frac{\partial \log f(X, Z, \Delta)}{\partial \gamma} = \frac{\Delta \partial \log \lambda(Z, \beta_0^T X, \gamma)}{\gamma} - \int_0^Z \frac{\partial \lambda(s, \beta_0^T X, \gamma)}{\partial \gamma} ds$$

$$= \Delta h(Z, \beta_0^T X, \gamma) - \int_0^Z h(s, \beta_0^T X, \gamma) \lambda(s, \beta_0^T X) ds$$
Because $\lambda_0(t, \beta_0^T X)$ can be any positive function, $h(s, \beta_0^T X, \gamma)$ can be any function. We denote it $h(s, \beta_0^T X)$. This leads to the form of $\Lambda_2$.

Similar derivation leads to $\Lambda_3$. It is easy to verify that $\Lambda_1 \perp \Lambda_2$ and $\Lambda_1 \perp \Lambda_3$. Because $C_1 T \mid X$, the martingale integrations associated with $M(t, \beta_0^T X)$ and $M_C(z, X)$ are also independent conditional on $X$, hence $\Lambda_2 \perp \Lambda_3$. This completes the proof. \hfill \Box

A.2. Proof of Lemma 1. For notational convenience, we prove the results for $d = 1$ and assume the first component of $\beta$ is 1. The first four results are obtained from the uniform convergence property of the kernel estimation (Mack and Silverman, 1982) under conditions C1-C2. Specifically, to derive the first four results, we first establish the following preliminary conclusion for any $Z$ and $\beta^T X$,

\begin{equation}
\frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T x) = f_{\beta^T x}(\beta^T x) + O_p(n^{-1/2} h^{-1/2} + h^2).
\end{equation}

To see this, we compute the absolute bias of the left hand side of (A.1) as

\begin{align*}
\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T x) \right\} - f_{\beta^T x}(\beta^T x) \right| & = \left| EK_h(\beta^T X_j - \beta^T x) - f_{\beta^T x}(\beta^T x) \right| \\
& = \int \frac{1}{h} K \left( \frac{\beta^T x_j - \beta^T x}{h} \right) f_{\beta^T x}(\beta^T x_j) d\beta^T x_j - f_{\beta^T x}(\beta^T x) \\
& = \int K(u) f_{\beta^T x}(\beta^T x + hu) du - f_{\beta^T x}(\beta^T x) \\
& = \int K(u) \left\{ f_{\beta^T x}(\beta^T x) + \left[ f_{\beta^T x}(\beta^T x) hu + \frac{1}{2} f_{\beta^T x}(\beta^T x^*) h^2 u^2 \right] \right\} du - f_{\beta^T x}(\beta^T x) \\
& \leq \frac{h^2}{2} \sup_{\beta^T x} |f''_{\beta^T x}(\beta^T x)| \int u^2 K(u) du,
\end{align*}

where throughout the text, $\beta^T x^*$ is on the line connecting $\beta^T x$ and $\beta^T x + hu$, and the variance to be

\begin{align*}
\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T x) \right\} & = \frac{1}{n} \text{var} K_h(\beta^T X_j - \beta^T x) \\
& = \frac{1}{n} \left[ EK_h^2(\beta^T X_j - \beta^T x) - \{ EK_h(\beta^T X_j - \beta^T x) \}^2 \right] \\
& = \frac{1}{n} \left[ \int \frac{1}{h^2} K^2(\beta^T x_j - \beta^T x) f_{\beta^T x}(\beta^T x_j) d\beta^T x_j - f_{\beta^T x}(\beta^T x) d\beta^T x_j - f_{\beta^T x}(\beta^T x) \right] \\
& = \frac{1}{nh} \int K^2(u) f_{\beta^T x}(\beta^T x + hu) du - \frac{1}{n} f_{\beta^T x}(\beta^T x) + O(h^2/n)
\end{align*}
\[ \leq \frac{1}{nh} f_{\beta^T}(\beta^T x) \int K^2(u)du + \frac{h}{2n} \sup_{\beta^T x} |f''_{\beta^T}(\beta^T x)| \int u^2 K^2(u)du + \frac{1}{n} |f_{\beta^T}(\beta^T x)| + O(h^2/n). \]

Therefore, we have

\[ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T x) = f_{\beta^T X}(\beta^T x) + O_p(n^{-1/2}h^{-1/2} + h^2) \]

uniformly under conditions C1, C2 and C4.

Following similar derivations, we have

\[ E \left\{ -\frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq Z) K'_h(\beta^T X_j - \beta^T x) \right\} \]

\[ = - \int \frac{1}{h^2} x_j K'(\beta^T x_j - \beta^T x) I(c \geq Z) I(t \geq Z) K' \left\{ (\beta^T x_j - \beta^T x)/h \right\} \\
\times f_{\beta^T X}(\beta^T X_j) dx_j d\beta^T x_j \\
= - \frac{1}{h} \int (\beta^T x + hu, x_j^T)^T K'(u) S_c(Z, x_j, \beta^T x + hu) S(Z, \beta^T x) f_{\beta^T}(x_j^T | \beta^T x + hu) \\
\times f_{\beta^T X}(\beta^T x + hu) du dx_j du \\
= - \frac{1}{h} \int (\beta^T x + hu, x_j^T)^T S_c(Z, x_j, \beta^T x + hu) S(Z, \beta^T x) f_{\beta^T}(x_j^T | \beta^T x + hu) \\
\times f_{\beta^T X}(\beta^T x + hu) du dx_j du
\[ \begin{align*}
&= -\frac{\partial}{\partial \beta^T x} f_{\beta^T x}(\beta^T x)E\{X_j I(Z_j \geq Z) \mid \beta^T x\} \\
&\quad - \frac{h^2}{6} \int \frac{\partial^3}{\partial (\beta^T x)^3} f_{\beta^T x}(\beta^T x) E\{X_j I(Z_j \geq Z) \mid \beta^T x^*\} u^3 K'(u) du.
\end{align*} \]

Hence the absolute bias is
\[\begin{align*}
& \leq \frac{h^2}{6} \sup_{\beta^T x} \left| \frac{\partial^3}{\partial (\beta^T x)^3} f_{\beta^T x}(\beta^T x) E\{X_j I(Z_j \geq Z) \mid \beta^T x\} \right| \left( \int u^2 K(u) du \right)
\end{align*} \]

and
\[\begin{align*}
&\text{var} \left\{ \frac{1}{n} \sum_{j=1}^n X_j I(Z_j \geq Z) K'_1(\beta^T x_j - \beta^T x) \right\} \\
&\leq \frac{1}{n} \left[ \int h_3^2(\beta^T x_h + hu)^T(\beta^T x + hu, x_{j1}) S_c(Z, x_{j1}, \beta^T x + hu) \right. \\
&\quad \times \left. S(Z, \beta^T x + hu) f_{\beta^T x}(x_{j1} \mid \beta^T x + hu) f_{\beta^T x}(\beta^T x + hu) dx_{j1} K'^2(u) du \right] + O(1/n)
\end{align*} \]
So

\[
\frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq Z) K'_h(\beta^T X_j - \beta^T x)
\]

(A.2)

\[-\frac{c}{\partial \beta^T x} f_{\beta^T X} (\beta^T x) E \{X_j I(Z_j \geq Z) \mid \beta^T x\} + O_p(n^{-1/2} h^{-3/2} + h^2)\]

uniformly under conditions C1-C2.

We next show

\[
-\frac{1}{n} \sum_{j=1}^{n} K'_h(\beta^T X_j - \beta^T x) = f_{\beta^T X} (\beta^T x) + O_p(n^{-1/2} h^{-3/2} + h^2).
\]

(A.3)

To see this, we compute the absolute bias of the left hand side as

\[
\begin{align*}
&\left| E \left\{ -\frac{1}{n} \sum_{j=1}^{n} K'_h(\beta^T X_j - \beta^T x) \right\} - f_{\beta^T x}(\beta^T x) \right| \\
&= \left| E \left\{ -K'_h(\beta^T X - \beta^T x) \right\} - f_{\beta^T x}(\beta^T x) \right| \\
&= \left| \int -\frac{1}{h^2} K'(\beta^T X_j - \beta^T x) f_{\beta^T x}(\beta^T x) d\beta^T x_j - f_{\beta^T x}(\beta^T x) \right| \\
&= \left| \int -\frac{1}{h} K'(u) f_{\beta^T x}(\beta^T x + hu) du - f_{\beta^T x}(\beta^T x) \right| \\
&= \left| \int -\frac{1}{h} K'(u) \left\{ f_{\beta^T x}(\beta^T x) + f''_{\beta^T x}(\beta^T x) hu + \frac{1}{2} f'''_{\beta^T x}(\beta^T x) h^2 u^2 + \frac{1}{6} f^{(3)}_{\beta^T x}(\beta^T x) h^3 u^3 \right\} du \\
&\quad - f_{\beta^T x}(\beta^T x) \right| \\
&= \left| \frac{h^2}{2} \sup_{\beta^T x} \left| f^{(3)}_{\beta^T x}(\beta^T x) \right| \int u^2 K(u) du, \right| \\
\end{align*}
\]

and the variance to be

\[
\begin{align*}
\text{var} \left\{ -\frac{1}{n} \sum_{j=1}^{n} K'_h(\beta^T X_j - \beta^T x) \right\} \\
&= \frac{1}{n} \text{var} K'_h(\beta^T X - \beta^T x) \\
&= \frac{1}{n} \left[ E K'^2_h(\beta^T X_j - \beta^T x) - \{EK'_h(\beta^T X_j - \beta^T x)\}^2 \right] \\
&= \frac{1}{n} \left[ \int \frac{1}{h^4} K'^2(\beta^T x_j - \beta^T x)/h f_{\beta^T x}(\beta^T x_j) d\beta^T x_j - f_{\beta^T x}(\beta^T x) + O(h^2) \right] \\
&= \frac{1}{n h^3} \int K'^2(\beta^T x + hu) du - \frac{1}{n} f_{\beta^T x}(\beta^T x) + O(h^2/n) \\
&\leq \frac{1}{n h^3} f_{\beta^T x}(\beta^T x) \int K'^2(u) du + \frac{1}{2n} \sup_{\beta^T x} \left| f''_{\beta^T x}(\beta^T x) \right| \int u^2 K'^2(u) du + O(1/n).
\end{align*}
\]

Therefore, (A.3) holds uniformly under conditions C1, C2 and C4.
We compute the expectation

\[
E \left\{ -\frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K'_h(\beta^T x_j - \beta^T x) \right\}
\]

\[
= E \left\{ -I(Z_j \geq Z) K'_h(\beta^T x_j - \beta^T x) \right\}
\]

\[
= -\frac{1}{h^2} I(c \geq Z) I(t \geq Z) K' \{(\beta^T x_j - \beta^T x)/h \}
\times f_r(c, x_j) f(t, \beta^T x_j) f_{X \mid \beta^T x}(x_j | \beta^T x_j) f_{\beta^T x}(\beta^T x_j) dcdtdx_j d\beta^T x_j
\]

\[
= -\frac{1}{h^2} K'(u) S_c(Z, x_{jl}, \beta^T x + hu) S(Z, \beta^T x + hu) f_{X \mid \beta^T x}(x_{jl} | \beta^T x + hu)
\times f_{\beta^T x}(\beta^T x + hu) dx_j du
\]

\[
= -\frac{1}{h} \int S_c(Z, x_{jl}, \beta^T x) S(Z, \beta^T x) f_{X \mid \beta^T x}(x_{jl} | \beta^T x) f_{\beta^T x}(\beta^T x) dx_j du
\]

\[
= -\frac{1}{h} \int \frac{\partial}{\partial \beta^T x} \left\{ S_c(Z, x_{jl}, \beta^T x) S(Z, \beta^T x) f_{X \mid \beta^T x}(x_{jl} | \beta^T x) f_{\beta^T x}(\beta^T x) \right\} dx_j
\times \frac{\partial^2}{\partial(\beta^T x)^2} \{ S_c(Z, x_{jl}, \beta^T x) S(Z, \beta^T x) f_{X \mid \beta^T x}(x_{jl} | \beta^T x) f_{\beta^T x}(\beta^T x) \} dx_j
\]

\[
= \frac{\partial}{\partial \beta^T x} S(Z, \beta^T x) f_{\beta^T x}(\beta^T x) E\{ S_c(Z, X_j) | \beta^T x \}
\]

\[
= \frac{\partial}{\partial \beta^T x} f_{\beta^T x}(\beta^T x) E\{ I(Z_j \geq Z) | \beta^T x \}
\]

Hence the absolute bias is

\[
\left| E \left\{ -\frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K'_h(\beta^T x_j - \beta^T x) \right\} - \frac{\partial}{\partial \beta^T x} f_{\beta^T x}(\beta^T x) E\{ I(Z_j \geq Z) | \beta^T x \} \right|
\]
\[
\begin{align*}
&= \left| -\frac{h^2}{6} \int \frac{\varepsilon^3}{\partial (\beta^T \mathbf{x})^3} f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}^*) E \left\{ I(Z_j \geq Z) \mid \beta^T \mathbf{x} \right\} u^3 K'(u) du \right| \\
&\leq \frac{h^2}{2} \sup_{\beta^T \mathbf{x}} \left| \frac{\varepsilon^3}{\partial (\beta^T \mathbf{x})^3} f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}) E \left\{ I(Z_j \geq Z) \mid \beta^T \mathbf{x} \right\} \right| \left\{ \int u^2 K(u) du \right\}
\end{align*}
\]

and

\[
\text{var} \left\{ -\frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \right\} = \frac{1}{n} \left[ E \left\{ I(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \right\} E \left\{ I(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \right\}^T - \{EI(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \} \{EI(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \}^T \right]
\]

\[
= \frac{1}{n} \left( \int \frac{1}{h^2} E \left\{ I(Z_j \geq Z) K^2 \left\{ (\beta^T \mathbf{x}_j - \beta^T \mathbf{X}) / \sqrt{h} \right\} \right\} f_t(c, \mathbf{x}_j) f_t(\beta^T \mathbf{x}) f_{X_t j} \beta^T \mathbf{x}(\beta^T \mathbf{x}_j) f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}_j) df_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}) df_{X_t j} \beta^T \mathbf{x} \right) + O(h^2)
\]

\[
= \frac{1}{n} \int \frac{1}{h^3} S_c(Z, \mathbf{x}_j, \beta^T \mathbf{x} + hu) f_{X_t j} \beta^T \mathbf{x}(\mathbf{x}_j \mid \beta^T \mathbf{x} + hu) f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x} + hu) du + O(1/n)
\]

\[
= \frac{1}{nh^3} \int \frac{1}{h^3} S_c(Z, \mathbf{x}_j, \beta^T \mathbf{x}) S(Z, \beta^T \mathbf{x}) f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}) f_{X_t j} \beta^T \mathbf{x}(\mathbf{x}_j \mid \beta^T \mathbf{x}) du + \frac{1}{nh^3} \int \frac{1}{h^3} S_c(Z, \mathbf{x}_j, \beta^T \mathbf{x}) S(Z, \beta^T \mathbf{x}) du + \frac{1}{nh^3} \int \frac{1}{h^3} S_c(Z, \mathbf{x}_j, \beta^T \mathbf{x}) S(Z, \beta^T \mathbf{x}) du + O(1/n)
\]

\[
\leq \frac{1}{nh^3} \sup_{\beta^T \mathbf{x}} \left| f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}) E \left\{ I(Z_j \geq Z) \mid \beta^T \mathbf{x} \right\} \right| \left\{ \int K^2(u) du \right\}
\]

So

\[
-\frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K'_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) = \frac{\partial}{\partial \beta^T \mathbf{x}} f_{\beta^T \mathbf{x}}(\beta^T \mathbf{x}) E \left\{ I(Z_j \geq Z) \mid \beta^T \mathbf{x} \right\} + O_p(n^{-1/2}h^{-3/2} + h^2)
\]

(A.4)

uniformly under conditions C1-C2.

Following similar derivations, we have

\[
E \left\{ \frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_j I(Z_j \geq Z) K_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \right\} = E \left\{ \mathbf{X}_j I(Z_j \geq Z) K_h(\beta^T \mathbf{X}_j - \beta^T \mathbf{x}) \right\}
\]
\[
\begin{align*}
&= \int \frac{1}{h} x_j I(c \geq Z)I(t \geq Z)K \{ (\beta^T x_j - \beta^T x)/h \} \\
&\quad \times f(c, x_j) f(t, \beta^T x_j) f X | \beta^T x(X_j | \beta^T x_j) f \beta^T x(\beta^T x_j) dcdtx_j d\beta^T x_j \\
&= \int \frac{1}{h} x_j K \{ (\beta^T x_j - \beta^T x)/h \} S(c, x_j, \beta^T x_j) S(Z, \beta^T x_j) f X | \beta^T x(X_j | \beta^T x_j) \\
&\quad \times f \beta^T x(\beta^T x_j) dx_j d\beta^T x_j \\
&= \int (\beta^T x + hu, x_j)^T K(u) S(c, x_j, \beta^T x + hu) S(Z, \beta^T x + hu) f X | \beta^T x(X_j | \beta^T x + hu) \\
&\quad \times f \beta^T x(\beta^T x + hu) dx_j du \\
&= \int (\beta^T x, x_j)^T S(c, x_j, \beta^T x) S(Z, \beta^T x) f X | \beta^T x(X_j | \beta^T x) f \beta^T x(\beta^T x) dx_j \\
&\quad \times \int K(u) du \\
&\quad + \frac{1}{2} \int \frac{\partial^2}{\partial (\beta^T x)^2} \left\{ (\beta^T x^*, x_j)^T S(c, x_j, \beta^T x^*) S(Z, \beta^T x^*) f X | \beta^T x(X_j | \beta^T x) f \beta^T x(\beta^T x) \right\} dx_j \\
&\quad \times h^2 u^2 K(u) du \\
&= \int (\beta^T x, x_j)^T S(c, x_j, \beta^T x) S(Z, \beta^T x) f X | \beta^T x(X_j | \beta^T x) f \beta^T x(\beta^T x) dx_j \\
&\quad + \frac{h^2}{2} \int \frac{\partial^2}{\partial (\beta^T x)^2} S(c, x_j, \beta^T x) S(Z, \beta^T x) f X | \beta^T x(X_j | \beta^T x) f \beta^T x(\beta^T x) \right\} dx_j \\
&\quad \times u^2 K(u) du \\
&= f \beta^T x(\beta^T x) E \{ X_j S(c, x_j) | \beta^T x \} \\
&\quad + \frac{h^2}{2} \int \frac{\partial^2}{\partial (\beta^T x)^2} f \beta^T x(\beta^T x) E \{ X_j S(c, x_j) | \beta^T x \} u^2 K(u) du \\
&= f \beta^T x(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \\
&\quad + \frac{h^2}{2} \sup_{\beta^T x} \left| \frac{\partial^2}{\partial (\beta^T x)^2} f \beta^T x(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \right| u^2 K(u) du.
\end{align*}
\]

Hence the absolute bias is
\[
\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \right\} - f \beta^T x(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \right| \leq \frac{h^2}{2} \sup_{\beta^T x} \left| \frac{\partial^2}{\partial (\beta^T x)^2} f \beta^T x(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \right| \int u^2 K(u) du
\]

and
\[
\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \right\}
\]
\[
\begin{align*}
&= \frac{1}{n} \left[ E \left\{ X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \right\} \{ X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \}^T \\
&\quad - \{ E X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \} \{ E X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) \}^T \right] \\
&= \frac{1}{n} \left( \int \frac{1}{h^2} x_j x_j^T I(Z_j \geq Z) K^2 \left\{ (\beta^T x_j - \beta^T x)/h \right\} \\
&\quad \times f_{c}(c, x_j) f(t, \beta^T x_j) f_{X_j|\beta^T X}(x_{jt} | \beta^T x_j) f_{\beta^T X}(\beta^T x_j) dcdtdx_j d\beta^T x_j \\
&\quad - \left[ f_{\beta^T X}(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \right] \left[ f_{\beta^T X}(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} \right]^T \\
&\quad + O(h^2) \right) \\
&= \frac{1}{n} \int \frac{1}{h} (\beta^T x + hu, x_{jt})^T (\beta^T x + hu, x_{jt}) \times S(Z, \beta^T x + hu) \times f_{X_j|\beta^T X}(x_{jt} | \beta^T x + hu) f_{\beta^T X}(\beta^T x + hu) dx_j K^2(u) du \\
&\quad + O(1/n) \\
&= \frac{1}{nh} \int (\beta^T x, x_{jt})^T (\beta^T x, x_{jt}) \times S(Z, \beta^T x) S(Z, \beta^T x) f_{\beta^T X}(\beta^T x) f_{X_j|\beta^T X}(x_{jt} | \beta^T x) dx_{jt} \\
&\quad \times \int K^2(u) du \\
&\quad + \frac{1}{2nh} \frac{\partial^2}{\partial (\beta^T x)^2} \int (\beta^T x^*, x_{jt}^*)^T (\beta^T x^*, x_{jt}^*) \times S(Z, \beta^T x^*) S(Z, \beta^T x^*) \\
&\quad \times f_{\beta^T X}(\beta^T x^*) f_{X_j|\beta^T X}(x_{jt} | \beta^T x^*) dx_{jt} h^2 u^2 K^2(u) du + O(1/n) \\
&\leq \frac{1}{nh} \sup_{\beta^T x} \left\| f_{\beta^T X}(\beta^T x) E \{ X_j X_j^T I(Z_j \geq Z) | \beta^T x \} \right\| \int K^2(u) du \\
&\quad + \frac{h}{2n} \sup_{\beta^T x^*} \left\| \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x^*) E \{ X_j X_j^T I(Z_j \geq Z) | \beta^T x^* \} \right\| \int u^2 K^2(u) du + O(1/n).
\end{align*}
\]

So
\[
\frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T x) = f_{\beta^T X}(\beta^T x) E \{ X_j I(Z_j \geq Z) | \beta^T x \} + O_p(n^{-1/2} h^{-1/2} + h^2)
\]
(A.5)

uniformly under conditions C1-C2.
\[
\begin{align*}
= & \int K(u)S_c(Z, x_{jl}, \beta^T x + hu)S(Z, \beta^T x + hu)f_{X_i|\beta^T x}(x_{jl} | \beta^T x + hu) \\
& \times f_{\beta^T X}(\beta^T x + hu)dx_{jl}du \\
= & \int S_c(Z, x_{jl}, \beta^T x)S(Z, \beta^T x)f_{X_i|\beta^T x}(x_{jl} | \beta^T x)f_{\beta^T X}(\beta^T x)dx_{jl} \int K(u)du \\
& + \int \frac{\partial}{\partial \beta^T x} \left\{ S_c(Z, x_{jl}, \beta^T x)S(Z, \beta^T x)f_{X_i|\beta^T x}(x_{jl} | \beta^T x)f_{\beta^T X}(\beta^T x) \right\} dx_{jl} \\
& \times h^2 u^2 K(u)du \\
& + \frac{1}{2} \int \int \frac{\partial^2}{\partial (\beta^T x)^2} \left\{ S_c(Z, x_{jl}, \beta^T x^*)S(Z, \beta^T x^*)f_{X_i|\beta^T x}(x_{jl} | \beta^T x^*)f_{\beta^T X}(\beta^T x^*) \right\} dx_{jl} \\
& \times h^2 u^2 K(u)du \\
= & S(Z, \beta^T x)f_{\beta^T X}(\beta^T x)E \{ S_c(Z, X_j) | \beta^T x \} \\
& + \frac{h^2}{2} \int \int \frac{\partial^2}{\partial (\beta^T x)^2} S(Z, \beta^T x^*)f_{\beta^T X}(\beta^T x^*)E \{ S_c(Z, X_j) | \beta^T x^* \} u^2 K(u)du \\
= & f_{\beta^T X}(\beta^T x)E \{ I(Z_j \geq Z) | \beta^T x \} \\
& + \frac{h^2}{2} \int \int \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x^*)E \{ I(Z_j \geq Z) | \beta^T x^* \} u^2 K(u)du \\
\] \\
Hence the absolute bias is \\
\[
\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \right\} - f_{\beta^T X}(\beta^T x)E \{ I(Z_j \geq Z) | \beta^T x \} \right| \\
\leq \frac{h^2}{2} \sup_{\beta^T x} \left| \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x)E \{ I(Z_j \geq Z) | \beta^T x \} \right| \left\{ \int u^2 K(u)du \right\} \\
\] \\
and \\
\[
\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \right\} \\
\leq \frac{1}{n} \left[ E \{ I(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \} \{ X_j I(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \}^T \\
- \{ EI(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \} \{ EI(Z_j \geq Z)K_h(\beta^T X_j - \beta^T x) \}^T \right] \\
\leq \frac{1}{n} \left( \int \frac{1}{h^2} I(Z_j \geq Z)K^2 \{ (\beta^T x_j - \beta^T x)/h \} \\
\times f_c(c, x_j)f(t, \beta^T x_j)f_{X_i|\beta^T x}(x_{jl} | \beta^T x_j)f_{\beta^T X}(\beta^T x_j)dc dtdx_{jl}d\beta^T x_j \right) \\
\]
\[
- \left[ f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z) \mid \beta^T X \} \right] \left[ f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z) \mid \beta^T X \} \right]^T + O(h^2)
\]
\[
= \frac{1}{n} \int \frac{1}{h} S_c(Z, x_{jl}, \beta^T X + hu) \\
\times S(Z, \beta^T X + hu) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X + hu) f_{\beta^T X}(\beta^T X + hu) dx_j K^2(u) du + O(1/n)
\]
\[
= \frac{1}{nh} \int S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T X) f_{X_l|\beta^T X}(\beta^T X) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X) dx_j \\
\times f_{\beta^T X}(\beta^T X) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X) dx_j h^2 u^2 K^2(u) du + O(1/n)
\]
\[
\leq \frac{1}{nh} \sup_{\beta^T X} \left| f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z) \mid \beta^T X \} \right| \int K^2(u) du
\]
\[
+ \frac{h}{2n} \sup_{\beta^T X^*} \left| \frac{\partial^2}{\partial (\beta^T X)^2} f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z) \mid \beta^T X^* \} \right| \int u^2 K^2(u) du + O(1/n).
\]

So
\[
\frac{1}{n} \sum_{j=1}^n I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X) = f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z) \mid \beta^T X \} + O_p(n^{-1/2} h^{-1/2} + h^2)
\]
(A.6)

uniformly under conditions C1-C2.

\[
E \left\{ \frac{1}{n} \sum_{j=1}^n I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T X) \right\}
\]
\[
= E \{ I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T X) \}
\]
\[
= \int \frac{1}{h^2} I(c \geq Z) I(t \geq Z) K'' \left\{ (\beta^T x_j - \beta^T X)/h \right\}
\times f_c(c, x_j) f(t, \beta^T x_j) f_{X_l|\beta^T X}(x_j \mid \beta^T x_j) f_{\beta^T X}(\beta^T x_j) dce dx_j d\beta^T X_j
\]
\[
= \int \frac{1}{h^2} K'' \left\{ (\beta^T x_j - \beta^T X)/h \right\} S_c(Z, x_{jl}, \beta^T x_j) S(Z, \beta^T x_j) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T x_j)
\times f_{\beta^T X}(\beta^T x_j) dx_{jl} d\beta^T X_j
\]
\[
= \int \frac{1}{h^2} K''(u) S_c(Z, x_{jl}, \beta^T X + hu) S(Z, \beta^T X + hu) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X + hu)
\times f_{\beta^T X}(\beta^T X + hu) dx_{jl} du
\]
\[
= \frac{1}{2h^2} \int \frac{\partial^2}{\partial (\beta^T X)^2} \left\{ S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T X) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X) f_{\beta^T X}(\beta^T X) \right\} dx_{jl}
\]
\[
+ \frac{h^2}{24} \int \frac{\partial^4}{\partial (\beta^T X)^4} \left\{ S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T X) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X) f_{\beta^T X}(\beta^T X) \right\} dx_{jl}
\]
\[
= \int \frac{\partial^2}{\partial (\beta^T X)^2} \left\{ S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T X) f_{X_l|\beta^T X}(x_{jl} \mid \beta^T X) f_{\beta^T X}(\beta^T X) \right\} dx_{jl}
\]
\[
\frac{h^2}{24} \int \frac{\partial^4}{\partial (\beta^T x)^4} \left\{ S_c(Z, x_{jl}, \beta^T x^*) S(Z, \beta^T x^*) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x^*) f_{\beta^T X}(\beta^T x^*) \right\} dx_{jl} \\
\times u^4 K''(u) du \\
= \frac{\partial^2}{\partial (\beta^T x)^2} S(Z, \beta^T x) f_{\beta^T X}(\beta^T x) E\{S_c(Z, X_j) \mid \beta^T x\} \\
+ \frac{h^2}{24} \int \frac{\partial^4}{\partial (\beta^T x)^4} \left\{ S_c(Z, x_{jl}, \beta^T x^*) S(Z, \beta^T x^*) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x^*) f_{\beta^T X}(\beta^T x^*) \right\} dx_{jl} \\
\times u^4 K''(u) du \\
= \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x) E\{I(Z_j \geq Z) \mid \beta^T x\} \\
+ \frac{h^2}{24} \int \frac{\partial^4}{\partial (\beta^T x)^4} \left\{ S_c(Z, x_{jl}, \beta^T x^*) S(Z, \beta^T x^*) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x^*) f_{\beta^T X}(\beta^T x^*) \right\} dx_{jl} \\
\times u^4 K''(u) du.
\]

Hence the absolute bias is
\[
\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \right\} - \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x) E\{I(Z_j \geq Z) \mid \beta^T x\} \right| \\
= \frac{h^2}{24} \sup_{\beta^T x^*} \left| \frac{\partial^4}{\partial (\beta^T x)^4} f_{\beta^T X}(\beta^T x) E\{I(Z_j \geq Z) \mid \beta^T x\} \right| \left\{ \int u^2 K(u) du \right\}.
\]

To analyze the variance,
\[
\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \right\} \\
= \frac{1}{n} \left[ E \left\{ I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \right\} \{ X_j I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \}^T \right. \\
- \{ E I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \} \{ E I(Z_j \geq Z) K''_h(\beta^T X_j - \beta^T x) \}^T \left] \\
= \frac{1}{n} \left( \int \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x) E\{I(Z_j \geq Z) \mid \beta^T x\} \right) \left[ \left( \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T X}(\beta^T x) E\{I(Z_j \geq Z) \mid \beta^T x\} \right)^T + O(h^2) \right) \\
= \frac{1}{n} \int \frac{\partial^2}{\partial (\beta^T x)^2} S_c(Z, x_{jl}, \beta^T X + hu) \\
\times S(Z, \beta^T x + hu) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x + hu) f_{\beta^T X}(\beta^T x + hu) dx_{jl} K''(u) du + O(1/n) \\
= \frac{1}{nh^2} \int S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T x) f_{\beta^T X}(\beta^T x) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x) dx_{jl} K''(u) du \\
+ \frac{1}{2nh^5} \frac{\partial^2}{\partial (\beta^T x)^2} \int S_c(Z, x_{jl}, \beta^T X) S(Z, \beta^T x^*) \\
\times f_{\beta^T X}(\beta^T x^*) f_{\chi, |\beta^T X}(x_{jl} \mid \beta^T x^*) dx_{jl} h^2 u^2 K''(u) du + O(1/n)
\]
\[ \begin{align*}
&\leq \frac{1}{nh^3}\sup_{\beta^T x}\left| f_{\beta^T x}(\beta^T x)E\{I(Z_j \geq Z) \mid \beta^T x\}\right| \int K^{n2}(u)du \\
&\quad + \frac{1}{2nh^3}\sup_{\beta^T x^*}\left| \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T x}(\beta^T x^*)E\{I(Z_j \geq Z) \mid \beta^T x^*\}\right| \int u^2 K^{n2}(u)du + O(1/n).
\end{align*} \]

So
\[ \frac{1}{n} \sum_{j=1}^{n} I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X) = \frac{\partial^2}{\partial (\beta^T x)^2} f_{\beta^T x}(\beta^T x)E\{I(Z_j \geq Z) \mid \beta^T x\} + O_p(n^{-1/2}h^{-5/2} + h^2) \]
(A.7)

uniformly under conditions C1-C2.

Combining the results of (A.1) and (A.6), we have
\[ \hat{E}\{Y(Z) \mid \beta^T X\} = \frac{\sum_{j=1}^{n} I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} = \frac{f_{\beta^T x}(\beta^T X)E\{I(Z_j \geq Z) \mid \beta^T X\} + O_p(n^{-1/2}h^{-1/2} + h^2)}{f_{\beta^T x}(\beta^T X) + O_p(n^{-1/2}h^{-1/2} + h^2)} \]
uniformly under conditions C1-C2. Combining the results of (A.1) and (A.5), we have
\[ \hat{E}\{XY(Z) \mid \beta^T X\} = \frac{\sum_{j=1}^{n} X_j I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} = \frac{f_{\beta^T x}(\beta^T X)E\{X_j I(Z_j \geq Z) \mid \beta^T X\} + O_p(n^{-1/2}h^{-1/2} + h^2)}{f_{\beta^T x}(\beta^T X) + O_p(n^{-1/2}h^{-1/2} + h^2)} \]
uniformly under conditions C1-C2. Combining the results of (A.1), (A.3), (A.6) and (A.4), we have
\[ \frac{\partial}{\partial \beta^T X} \hat{E}\{Y(Z) \mid \beta^T X\} \]
\[ = \frac{\sum_{j=1}^{n} I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} \]
\[ + \frac{\{\sum_{j=1}^{n} I(Z_j \geq Z) K_h(\beta^T X_j - \beta^T X)\}\{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)\}}{\left(\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)\right)^2} \]
\[ \frac{\partial f_{\beta^T x}(\beta^T X)E\{I(Z_j \geq Z) \mid \beta^T X\}/\partial \beta^T X + O_p(n^{-1/2}h^{-3/2} + h^2)}{f_{\beta^T x}(\beta^T X) + O_p(n^{-1/2}h^{-1/2} + h^2)} \]
uniformly under conditions C1-C2. Finally, combining the results of (A.1), (A.3), (A.5) and (A.2), we have

\[
\frac{\partial}{\partial \beta^T X} \hat{E}\{XY(Z) \mid \beta^T X\} = \frac{f_{\beta^T X}(\beta^T X)E\{X_i(Z_i \geq Z) \mid \beta^T X\}}{f_{\beta^T X}(\beta^T X) + O_p(n^{-1/2}h^{-3/2} + h^2)} \frac{\partial E\{I(Z_i \geq Z) \mid \beta^T X\}}{\partial \beta^T X} - \frac{f'_{\beta^T X}(\beta^T X)E\{I(Z_i \geq Z) \mid \beta^T X\}}{f_{\beta^T X}(\beta^T X) + O_p(n^{-1/2}h^{-3/2} + h^2)} \]

uniformly under conditions C1-C2.

Now we inspect the consistency of the Kaplan Meier estimator on the hazard function and its derivatives. Let \( A = n^{-1} \sum_{i=1}^{n} I(Z_i \geq Z_i)K_h(\beta^T X_i - \beta^T X) - f_{\beta^T X}(\beta^T X)E\{I(Z \geq Z_i) \mid \beta^T X\}. \)

\[
\hat{\lambda}(z, \beta^T x) = \int_0^\infty K_h(t - z) d\hat{\Lambda}(t \mid \beta^T x) = \sum_{i=1}^{n} K_h(Z_i - z) \sum_{j=1}^{n} I(Z_j \geq Z_i)K_h(\beta^T X_j - \beta^T X) = 1 \sum_{i=1}^{n} K_h(Z_i - z) \frac{\Delta_i K_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X)E\{I(Z \geq Z_i) \mid \beta^T X\} + A} = \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i \frac{K_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T x)E\{I(Z \geq Z_i) \mid \beta^T X\}} + O_p(A),
\]
We first inspect
\[
\frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x).
\]
First,
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \right] = \int \int \frac{K_b(z_i - z) K_h(\beta^T x_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(z_i, \beta^T x)} \frac{f(z_i, \beta^T X_i) S_c(z_i, X_i)}{f_{\beta^T X}(\beta^T x_i)} f(z_i, \beta^T x_i) dz_i dx_i d\beta^T x_i.
\]
where throughout the text, \( z^* \) is on the line connecting \( z \) and \( z + bv \). Thus, the absolute bias is
\[
\left| E \left[ \frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \right] - \lambda(z, \beta^T x) \right|
\leq b^2 \sup_{z^*, \beta^T x} \left| \frac{\partial^2}{2 \hat{v}^2} K(v) \right| \int v^2 K(v) dv
+ h^2 \sup_{z^*, \beta^T x, \beta^T x^*} \left| \frac{\partial^2}{2 \hat{v}^2} E \left[ S_c(z, X_i) | \beta^T x^* \right] f(z_i, \beta^T x) f_{\beta^T X}(\beta^T x) \right| \int u^2 K(u) dv,
\]
under conditions C1-C6. Following the same procedure, noting that \( A = O_p((nh)^{-1/2} + h^2) \) uniformly, we can show that
\[
\frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \sim O_p(h^2 + (nh)^{-1/2}).
\]
hence the bias of $\hat{\lambda}(z, \beta^T x)$ is of order $O_p\{(nh)^{-1/2} + h^2 + b^2\}$ uniformly. On the other hand, the variance of $\hat{\lambda}(z, \beta^T x)$ is

\[
\text{var}\left\{ \hat{\lambda}(z, \beta^T x) \right\} = \text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] = \text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] (1 + O_p(A)) \]

\[
\leq 2\text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] + 2\text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] O_p(A). \]

We inspect the first term first.

\[
2\text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] = 2\text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right]
\]

\[
\leq 2\text{var}\left[ \frac{1}{n} \sum_{i=1}^{n} K_i(z_i - z) \Delta_i K_i(\beta^T X_i - \beta^T x) \right] + O(1/n)
\]

\[
= \frac{2}{b^2 h \text{hn}} \int \frac{K^2((z_i - z)/b)K^2((\beta^T x_i - \beta^T x)/h)}{f_{\beta^T X}(\beta^T x)S^2(z_i, \beta^T x)E^2\{S^c(z_i, X_i) | \beta^T x\}} \times f(z, \beta^T x)S_c(z, x_i) f_{\beta^T X}(x_i, \beta^T x_i) f_{\beta^T X}(\beta^T x_i) dz_i dx_i d\beta^T x_i + O(1/n)
\]

\[
= \frac{2}{b \text{hn}} \int \frac{K^2(\beta^T x)K^2(v)}{f_{\beta^T X}(\beta^T x)S^2(z + bv, \beta^T x)E^2\{S^c(z + bv, X_i) | \beta^T x\}} \times f(z + bv, \beta^T x + hu)E\{S^c(z + bv, X_i) | \beta^T x + hu\} f_{\beta^T X}(\beta^T x + hu) dv du + O(1/n)
\]

\[
= \frac{2}{b \text{hn}} \int \frac{f(z, \beta^T x)K^2(v)K^2(u)}{f_{\beta^T X}(\beta^T x)S^2(z, \beta^T x)E\{S^c(z, X_i) | \beta^T x\}} dv du
\]

\[
+ \frac{b c^2}{nh} \int \frac{f(z^*, \beta^T x)K^2(v)K^2(u)}{f_{\beta^T X}(\beta^T x)S^2(z^*, \beta^T x)E\{S^c(z^*, X_i) | \beta^T x\}} v^2 dv du
\]

\[
+ \frac{h c^2}{nb c^2(\beta^T x)^2} \int \frac{f(z, \beta^T x)K^2(v)K^2(u)}{f_{\beta^T X}(\beta^T x)S^2(z, \beta^T x)E\{S^c(z, X_i) | \beta^T x\}} u^2 dv du + O(1/n)
\]

\[
\leq \frac{2}{b \text{hn}} \int \frac{f(z, \beta^T x)}{f_{\beta^T X}(\beta^T x)S^2(z, \beta^T x)E\{S^c(z, X_i) | \beta^T x\}} \left\{ \int K^2(u) du \right\}^2
\]

\[
+ \frac{b}{nh} \sup_{z^*, \beta^T x} \frac{c^2}{\beta^T x} \int \frac{f(z^*, \beta^T x)}{f_{\beta^T X}(\beta^T x)S^2(z^*, \beta^T x)E\{S^c(z^*, X_i) | \beta^T x\}} \left\{ \int K^2(u) du \right\}^2
\]

\[
\times \left\{ \int K^2(u) du \right\} \left\{ \int K^2(u) du \right\}
\]
\[ + \frac{h}{nb} \sup_{z, \beta^T x, \beta^T x} \left| \frac{\partial^2}{\partial(\beta^T x)^2} f_{\beta^T x}(\beta^T x) f(z, \beta^T x^*) E\{S_c(z, X_i) \mid \beta^T x^*\} \right| \]

\[ \times \left\{ \int K^2(u)u^2 du \right\} \left\{ \int K^2(u)du \right\} + O(1/n) \]

\[ = O\left\{ \frac{1}{nhb} \right\} + h/(nb) + b/(nh) + 1/n \]

\[ = O\left\{ \frac{1}{nhb} \right\} \]

uniformly under conditions C1-C6 and \( \beta^T x^* \) is on the line connecting \( \beta^T x \) and \( \beta^T x + hu \). For the second term

\[ 2 \text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) O_p(1) \right] \]

\[ \leq 2E \left[ \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) O_p(1) \right]^2 \sup \{ O_p^2 (A) \} \]

\[ = 2E \left( \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \right)^2 \sup \{ O_p^2 (A) \} \]

\[ = 2E \left( \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \right)^2 \sup \{ O_p^2 ((nh)^{-1} + h^4) \} \]

\[ = 2E \left( \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \right)^2 \sup \{ O_p^2 ((nh)^{-1} + h^4) \} \]

\[ = 2 \left( \frac{f(z, \beta^T x)}{bhn f_{\beta^T x}(\beta^T x) S^2(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\}} \right) \left( \int K^2(u)du \right) + O(n^{-1}b^{-1}h + n^{-1}h^{-1}b) \]

\[ = O\left\{ (nh)^{-1} + h^4 \right\} \]

under conditions C1-C6 uniformly. Summarizing the above results, the variance of \( \hat{\lambda}(z, \beta^T x) \) is of order \( 1/(nhb) \) uniformly. Hence we have the consistency of estimator \( \hat{\lambda}(z, \beta^T x) \), specifically

\[ \hat{\lambda}(z, \beta^T x) = \lambda(z, \beta^T x) + O_p\left( (nhb)^{-1/2} + h^2 + b^2 \right) \]

uniformly under condition C1-C6. Next we inspect the estimator for the first derivative of hazard function \( \lambda(z, \beta^T x) \). Let

\[ \hat{\lambda}_{11} = -\sum_{i=1}^{n} K_h(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{\sum_j I(Z_j \geq z) K_h(\beta^T X_j - \beta^T x)} \]

\[ \hat{\lambda}_{12} = \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\sum_{j=1}^{n} I(Z_j \geq z) K'_h(\beta^T X_j - \beta^T x)}{\sum_{j=1}^{n} I(Z_j \geq z) K_h(\beta^T X_j - \beta^T x)^2} \]

Then \( \hat{\lambda}(x, \beta^T x) = \hat{\lambda}_{11} + \hat{\lambda}_{12} \). To analyze \( \hat{\lambda}_{11} \),

\[ E\hat{\lambda}_{11} = E \left\{ -\sum_{i=1}^{n} K_h(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{\sum_{j=1}^{n} I(Z_j \geq z) K_h(\beta^T X_j - \beta^T x)} \right\} \]
\[
\begin{align*}
&= E \left[ \frac{1}{n} \sum_{i=1}^{n} -K_b(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) E(I(Z \geq Z_i) \mid \beta^T x) + A} \right] \\
&= E \left[ \frac{1}{n} \sum_{i=1}^{n} -K_b(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} \right] \\
&\quad + E \left[ \frac{1}{n} \sum_{i=1}^{n} -K_b(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} O_p(A) \right] \\
&= E \left[ \frac{1}{n} \sum_{i=1}^{n} -K_b(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} \right] \\
&\quad + E \left[ \frac{1}{n} \sum_{i=1}^{n} -K_b(Z_i - z) \frac{\Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} O_p(A) \right].
\end{align*}
\]

We inspect the first term to obtain
\[
E \left[ \frac{-K_b(Z_i - z) \Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} \right]
\]
\[
= - \iint \frac{K_b(z_i - z) K'_h(\beta^T x_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(z_i, \beta^T x) E\{S_c(z_i, X_i) \mid \beta^T x\}} f(z_i, \beta^T x_i) S_c(z_i, x_i) f_{X_i}(z_i, \beta^T x_i)
\times f_{\beta^T X}(\beta^T x_i) dz_i dx_i d\beta^T x_i
\]
\[
= - \iint \frac{K_b(z_i - z) K'_h(\beta^T x_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(z_i, \beta^T x) E\{S_c(z_i, X_i) \mid \beta^T x\}} f(z_i, \beta^T x_i) S_c(z_i, x_i) f_{X_i}(z_i, \beta^T x_i) d\beta^T x_i
\]
\[
= - \iint \frac{\partial}{\partial \beta^T x} \left[ f(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) \right] / \partial \beta^T x \]
\[
\times \frac{K(v) u K'(u)}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\}}
\]
\[
= - \iint \frac{h^2 \beta^3}{2 \partial^2 \beta^T x} \left[ f_{\beta^T X}(\beta^T x) f(z^*, \beta^T x) E\{S_c(z^*, X_i) \mid \beta^T x\} \right] / \partial \beta^T x
\]
\[
\times \frac{-2 \hat{v}^2 K(v) u K'(u)}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\}}
\]
\[
= - \iint \frac{h^2 \beta^3}{6 \partial^3 (\beta^T x)} \left[ f_{\beta^T X}(\beta^T x) f(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\} \right]
\times \frac{K(v) u^3 K'(u)}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\}}
\]

Hence the absolute bias is given by
\[
\left| E \left[ \frac{-K_b(Z_i - z) \Delta_i K'_h(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x) S(Z_i, \beta^T x) E\{S_c(Z_i, X_i) \mid \beta^T x\}} \right] \right|
\]
\[
\left| \frac{\partial}{\partial \beta^T x} \left[ f(z, \beta^T x) E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) \right] / \partial \beta^T x \right|
\]
\[
\begin{align*}
& \leq b^2 \sup_{z^*, \beta^T x, \beta^T x^*} \left| \frac{\partial^3}{\partial \beta^T x^3} f_{\beta^T x} (\beta^T x^*) f(z^*, \beta^T x^*) E\{S_c(z^*, x_i) \mid \beta^T x^*\} \right| \int v^2 K(v) dv \\
& + h^2 \sup_{z, \beta^T x, \beta^T x^*} \left| \frac{\partial^3}{\partial (\beta^T x)^3} f_{\beta^T x} (\beta^T x^*) f(z, \beta^T x^*) E\{S_c(z, x_i) \mid \beta^T x^*\} \right| \int v^2 K(u) du \\
& = O(b^2 + h^2)
\end{align*}
\]

uniformly under condition C1-C6.

Following the same procedure, we conclude that
\[
\frac{1}{n} \sum_{i=1}^{n} -K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{1}{\sum_{i=1}^{n} I(Z_i \geq z)} K_h(\beta^T x_i - \beta^T x) O_p(A) = O_p(h^2 + (nh)^{-1/2})
\]

uniformly under conditions C1-C6 due to \( A = O_p(h^2 + (nh)^{-1/2}) \). Therefore, we have
\[
E\hat{\lambda}_{11} = \frac{\partial}{\partial \beta^T x} \left[ \frac{f(z, \beta^T x) E\{S_c(z, x_i) \mid \beta^T x\} f_{\beta^T x} (\beta^T x)}{f_{\beta^T x} (\beta^T x) E\{S_c(z, x_i) \mid \beta^T x\}} \right] + O\{(nh)^{-1/2} + b^2 + h^2\}
\]

For \( \hat{\lambda}_{12} \), let \( B = -1/n \sum_{j=1}^{n} I(Z_j \geq z) K_h' (\beta^T x_j - \beta^T x) - \partial f_{\beta^T x} (\beta^T x) E\{I(Z_j \geq z) \mid \beta^T x\} / \partial \beta^T x \), then
\[
\hat{\lambda}_{12} = K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\sum_{j=1}^{n} I(Z_j \geq z) K_h' (\beta^T x_j - \beta^T x)}{(\sum_{j=1}^{n} I(Z_j \geq z))^2 K_h(\beta^T x_j - \beta^T x)}
\]

\[
= - \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\} + A} \right]
\]

\[
= - \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}}{\sum_{i=1}^{n} I(Z \geq Z_i)^2} \right]
\]

\[
\times \{1 + O_p(B) + O_p(A)\}
\]

We have
\[
E \left[ - \frac{1}{n} \sum_{i=1}^{n} K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}} \right] \right]
\]

\[
= - E \left[ K_h(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{I(Z \geq Z_i) \mid \beta^T x\}} \right] \right]
\]

\[
= - \int K_h(z_i - z) K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{S_c(z, \beta^T x) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{S_c(z, \beta^T x) \mid \beta^T x\}} \right] \frac{f_2(z_i, \beta^T x_i) S_2(z_i, \beta^T x_i) E\{S_c(z_i, x_i) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{S_c(z_i, x_i) \mid \beta^T x\}}
\]

\[
\times f(z_i, \beta^T x_i) S_c(z_i, x_i) f_{\beta^T x} (\beta^T x) E\{S_c(z_i, x_i) \mid \beta^T x\} \frac{d\beta^T x_i}{d\beta^T x_i}
\]

\[
= - \int K_h(z_i - z) K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ \frac{f_{\beta^T x} (\beta^T x) E\{S_c(z, \beta^T x) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{S_c(z, \beta^T x) \mid \beta^T x\}} \right] \frac{f_2(z_i, \beta^T x_i) S_2(z_i, \beta^T x_i) E\{S_c(z_i, x_i) \mid \beta^T x\}}{f_{\beta^T x} (\beta^T x) E\{S_c(z_i, x_i) \mid \beta^T x\}}
\]

\[
\times f(z_i, \beta^T x_i) S_c(z_i, x_i) f_{\beta^T x} (\beta^T x) E\{S_c(z_i, x_i) \mid \beta^T x\} \frac{d\beta^T x_i}{d\beta^T x_i}
\]
\[
\begin{align*}
&= - \int K(v)K(u) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T X}(\beta^T x)S(z + bv, \beta^T x)E\{S_c(z + bv, X_i) \mid \beta^T x\} \right] f(z, \beta^T x) dvdu \\
&\times f(z + bv, \beta^T x + hu)E\{S_c(z + bv, X_i) \mid \beta^T x + hu\} f_{\beta^T X}(\beta^T x + hu) dvdu \\
&= - \int K(v)K(u) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T X}(\beta^T x)S(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} \right] f(z, \beta^T x) dvdu \\
&\times f(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) u^2 dvdu \\
&= - \frac{\partial^2}{\partial \beta^T x^2} \int K(v)K(u) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T X}(\beta^T x)S(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} \right] f(z, \beta^T x) dvdu \\
&\times f(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) u^2 dvdu \\
&= - \frac{\partial^2}{\partial \beta^T x^2} \int K(v)K(u) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T X}(\beta^T x)S(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} \right] f(z, \beta^T x) dvdu \\
&\times f(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) u^2 dvdu \\
&\times f(z, \beta^T x)E\{S_c(z, X_i) \mid \beta^T x\} f_{\beta^T X}(\beta^T x) u^2 dvdu.
\end{align*}
\]

Therefore

\[
\begin{align*}
E \left[ - \frac{1}{n} \sum_{i=1}^n K_h(z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T X}(\beta^T z)E\{I(Z \geq Z_i) \mid \beta^T x\} \right] f_{\beta^T X}(\beta^T x) \right] \\
&\leq b^2 \sup_{z, \beta^T x} \left\{ \int u^2 K(v) dv \right\} \\
&\leq h^2 \sup_{z, \beta^T x, \beta^T x^*} \left\{ \int u^2 K(v) dv \right\} \\
&= O(b^2 + h^2)
\end{align*}
\]

uniformly under conditions C1-C6. Noting that \( B = O_p(n^{-1/2}h^{-3/2} + h^2) \), based on similar proce-
We exam each term separately.

dure, we have

\[
-\frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} \frac{f_{\beta^T X}(\beta^T z) E\{I(Z \geq Z_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x) E^2\{I(Z \geq Z_i) | \beta^T x\}} O_p(B)
\]

\[
= O_p(n^{-1/2} h^{-3/2} + h^2),
\]

\[
-\frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\partial}{\partial \beta} \frac{f_{\beta^T X}(\beta^T z) E\{I(Z \geq Z_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x) E^2\{I(Z \geq Z_i) | \beta^T x\}} O_p(A)
\]

\[
= O_p(n^{-1/2} h^{-1/2} + h^2)
\]

uniformly under condition C1-C6. Therefore, we can conclude that

\[
E\hat{\lambda}_{12} = -\frac{\partial}{\partial \beta} \frac{f_{\beta^T X}(\beta^T x) S(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x) S^2(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}} f(z, \beta^T x) + O(n^{-1/2} h^{-3/2} + b^2 + h^2)
\]

In addition, we have

\[
\frac{\partial}{\partial \beta} \frac{f(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}} f(z, \beta^T x)
\]

\[
= \frac{\partial f(z, \beta^T x)}{S(z, \beta^T x)} + \frac{f(z, \beta^T x) \partial E\{S_c(z, X_i) | \beta^T x\}}{S(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}} + \frac{f_{\beta^T X}(\beta^T x) f(z, \beta^T x)}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x)}
\]

\[
-\frac{f_{\beta^T X}(\beta^T x) f(z, \beta^T x)}{f_{\beta^T X}(\beta^T x) S(z, \beta^T x)} - \frac{f(z, \beta^T x) \partial S(z, \beta^T x)}{S(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}} - \frac{f(z, \beta^T x) \partial E\{S_c(z, X_i) | \beta^T x\}}{S(z, \beta^T x) E\{S_c(z, X_i) | \beta^T x\}}
\]

\[
= \frac{\partial f(z, \beta^T x)}{S(z, \beta^T x)} - \frac{f(z, \beta^T x) \partial S(z, \beta^T x)}{S(z, \beta^T x)}
\]

\[
= \lambda_1(z, \beta^T x).
\]

Combining \(E\hat{\lambda}_{11}\) and \(E\hat{\lambda}_{12}\), we readily obtain

\[
\left| E\hat{\lambda}_1(z, \beta^T x) - \frac{\partial}{\partial \beta} \lambda_1(z, \beta^T x) \right| = \left| E\hat{\lambda}_{11} + E\hat{\lambda}_{12} - \frac{\partial}{\partial \beta} \lambda_1(z, \beta^T x) \right|
\]

\[
= O(n^{-1/2} h^{-3/2} + b^2 + h^2)
\]

uniformly under conditions C1-C6.

The variance of \(\hat{\lambda}_1(z, \beta^T x)\) is given by

\[
var\left\{\hat{\lambda}_1(z, \beta^T x)\right\} = var\left\{\hat{\lambda}_{11} + \hat{\lambda}_{12}\right\} \leq 2var(\hat{\lambda}_{11}) + 2var(\hat{\lambda}_{12}).
\]

We exam each term separately.

\[
var(\hat{\lambda}_{11})
\]
\[
\begin{align*}
\text{The first part is given by} & \\
& = \frac{2}{n} \var \left[ \frac{-K_{b}(Z_i - z) \Delta_i K'_{b}(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) | \beta^T x\}} \right] \\
& \leq \frac{2}{n} \left( E \left[ \frac{-K_{b}(Z_i - z) \Delta_i K'_{b}(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) | \beta^T x\}} \right] \right)^2 \\
& = \frac{2}{n} \left( E \left[ \frac{-K_{b}(Z_i - z) \Delta_i K'_{b}(\beta^T X_i - \beta^T x)}{f_{\beta^T X}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) | \beta^T x\}} \right] \right)^2 \\
& = 2 \frac{n}{nbh^3} \int_0^1 \int_0^1 K^2(v)K^2(u)f(z, \beta^T x)dv \right. \\
& + \left. b \var \left[ \int \frac{f(z, \beta^T x)S^2(z, \beta^T x)E\{S_c(z, X_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x)S^2(z, \beta^T x)E^2\{S_c(z, X_i) | \beta^T x\}} d^2 K^2(v)K^2(u)dvdu \right] \\
& + \left. \frac{1}{nbh} \int_0^1 \int_0^1 \int \frac{f(z, \beta^T x)S^2(z, \beta^T x)E\{S_c(z, X_i) | \beta^T x\}}{f_{\beta^T X}(\beta^T x)S^2(z, \beta^T x)E^2\{S_c(z, X_i) | \beta^T x\}} d^2 K^2(v)u^2 K^2(u)dvdu + O(1/n) \right)
\end{align*}
\]
uniformly under conditions C1-C6. Noting that $A = O_p(n^{-1/2}h^{-1/2} + h^2)$, the second part is

$$2\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{-K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) \mid \beta^T x\}} O_p(A) \right]$$

$$\leq 2E \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{-K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) \mid \beta^T x\}} O_p(A) \right]^2$$

$$\leq 2E \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{-K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X_i) \mid \beta^T x\}} \right]^2 \right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \frac{f(z, \beta^T x)}{\beta^T x} S(2z, \beta^T x) E\{S_c(z, X) \mid \beta^T x\} \left\{ \int K^2(v)dv \right\} \left\{ \int K^2(u)du \right\} + O\left( \frac{b}{nh^3} + \frac{1}{nh} \right)$$

uniformly under conditions C1-C6. Therefore

$$\text{var}(\hat{\lambda}_{11}) = O(1/(nhb^3))$$

uniformly under conditions C1-C6.

For $\lambda_{12}$,

$$\text{var}(\hat{\lambda}_{12}) = \left[ \frac{1}{n^2} \sum_{i=1}^{n} \frac{K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X) \mid \beta^T x\}} \right]$$

$$\leq \left[ \frac{1}{n^2} \sum_{i=1}^{n} \frac{K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X) \mid \beta^T x\}} \right]$$

$$\leq \frac{2}{n} \text{var} \left[ \frac{K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X) \mid \beta^T x\}} \right]$$

The first part is given by

$$\frac{2}{n} \text{var} \left[ \frac{K_h(Z_i - z)\Delta_i K_h'(\beta^T X_i - \beta^T x)}{f_{\beta^T x}(\beta^T x)S(Z_i, \beta^T x)E\{S_c(Z_i, X) \mid \beta^T x\}} \right]$$
\[\begin{aligned}
&= \frac{2}{n} E \left[ K_h(z_i - Z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ I(Z \geq z_i) \mid \beta^T x \} \right)^2 \right] \\
&\quad + \frac{2}{n} \left( E \left[ K_h(Z_i - z) \Delta_i K_h(\beta^T X_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ I(Z \geq z_i) \mid \beta^T x \} \right)^2 \right] \right) \\
&= \frac{2}{n} \int \int K^2(v)K^2(u) \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z + v, \beta^T x) \mid \beta^T x \} \right)^2 \right) f(z, \beta^T x) dv du \\
&\quad + \frac{b}{nh} \sup_{\beta^T x} \left| \frac{\partial^2}{\partial z^2} f(z, \beta^T x) \right| \left| \frac{\partial^2}{\partial z^2} \right| \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z + v, \beta^T x) \mid \beta^T x \} \right)^2 \right) f(z, \beta^T x) dv du \\
&\quad + \frac{h}{nb} \sup_{\beta^T x} \left| \frac{\partial^2}{\partial z^2} \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z + v, \beta^T x) \mid \beta^T x \} \right)^2 \right) \right| \left( \frac{\partial^2}{\partial z^2} \right) f(z, \beta^T x) dv du \\
&\quad \leq \frac{2}{nbh} \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z, \beta^T x) \mid \beta^T x \} \right)^2 \right) \left( \frac{\partial^2}{\partial z^2} \right) f(z, \beta^T x) dv du \\
&\quad + \frac{b}{nh} \sup_{z^*, \beta^T x} \left| \frac{\partial^2}{\partial z^2} \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z, \beta^T x) \mid \beta^T x \} \right)^2 \right) \right| \left( \frac{\partial^2}{\partial z^2} \right) f(z, \beta^T x) dv du \\
&\quad + \frac{h}{nb} \sup_{z^*, \beta^T x} \left| \frac{\partial^2}{\partial z^2} \left( \frac{\partial}{\partial \beta^T x} \left( f_{\beta^T i} X(\beta^T x) E \{ S_c(z, \beta^T x) \mid \beta^T x \} \right)^2 \right) \right| \left( \frac{\partial^2}{\partial z^2} \right) f(z, \beta^T x) dv du \\
&\quad \times f(z, \beta^T x) E \{ S_c(z, \beta^T x) \mid \beta^T x \} \left( \frac{\partial}{\partial \beta^T x} \right) f_{\beta^T i} X(\beta^T x) dv du + O(1/n)
\end{aligned}\]
uniformly under conditions C1-C6. Noting that \( A = O_p(n^{-1/2} h^{-1/2} + h^2) \) and \( B = O_p(n^{-1/2} h^{-3/2} + h^2) \), the second part is

\[
\begin{align*}
4\text{var} & \left[ \frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) E \{ I(Z \geq Z_i) \mid \beta^T x \} \right] / \partial \beta^T x \right] \bigg| \bigg| O_p(B) \\
\leq & \quad 4E \left( \frac{1}{n} \sum_{i=1}^{n} \left[ K_b(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) E \{ I(Z \geq Z_i) \mid \beta^T x \} \right] / \partial \beta^T x \right]^2 \right) \\
& \times O(1/(nh) + h^4) \\
= & \quad \frac{4}{nbh} f(z, \beta^T x) \left( \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) S(z, \beta^T x) E \{ S_c(z, X_i) \mid \beta^T x \} \right] / \partial \beta^T x \right)^2 \\
& \times \left\{ \int K^2(v) dv \right\} \left\{ \int K^2(u) du \right\} + O(1/(nh) + h/(nb)) \quad O(n^{-1}h^{-3} + h^4) \\
= & \quad O(n^{-2}b^{-1}h^{-4} + (nb)^{-1}h^3)
\end{align*}
\]

under conditions C1-C6 uniformly. The last part is

\[
\begin{align*}
4\text{var} & \left[ \frac{1}{n} \sum_{i=1}^{n} K_b(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) E \{ I(Z \geq Z_i) \mid \beta^T x \} \right] / \partial \beta^T x \right] \bigg| \bigg| O_p(A) \\
\leq & \quad 4E \left( \frac{1}{n} \sum_{i=1}^{n} \left[ K_b(Z_i - z) \Delta_i K_h(\beta^T x_i - \beta^T x) \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) E \{ I(Z \geq Z_i) \mid \beta^T x \} \right] / \partial \beta^T x \right]^2 \right) \\
& \times O((nh)^{-1} + h^4) \\
= & \quad \frac{4}{nbh} f(z, \beta^T x) \left( \frac{\partial}{\partial \beta^T x} \left[ f_{\beta^T x}(\beta^T z) S(z, \beta^T x) E \{ S_c(z, X_i) \mid \beta^T x \} \right] / \partial \beta^T x \right)^2 \\
& \times \left\{ \int K^2(v) dv \right\} \left\{ \int K^2(u) du \right\} + O(1/(nh) + h/(nb)) \quad O(n^{-1}h^{-1} + h^4) \\
= & \quad O((nh)^{-2}b^{-1} + h^3(nb)^{-1})
\end{align*}
\]

under conditions C1-C6 uniformly. Therefore,

\[
\text{var}(\hat{\lambda}_{12}) = O\{1/(nbh)\}
\]

uniformly under conditions C1-C6.

Summarizing the results above, \( \text{var}\{\hat{\lambda}_1(z, \beta^T x)\} = O\{1/(nbh^3)\} \) uniformly. Hence the estimator \( \hat{\lambda}_1(z, \beta^T x) \) satisfies

\[
\hat{\lambda}_1(x, \beta^T x) = \lambda_1(x, \beta^T x) + O_p\{((nbh^3)^{-1/2} + h^2 + b^2)
\]

uniformly under conditions C1-C6.
A.3. Proof of Theorem 1. For each \( n \), let \( \hat{\beta}_n \) satisfy

\[
\frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{n}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] = 0.
\]

Under condition C3, there exists a subsequence of \( \hat{\beta}_n, n = 1, 2, \ldots \), that converges. For notational simplicity, we still write \( \beta_n, n = 1, 2, \ldots \), as the subsequence that converges and let the limit be \( \beta^* \).

From the uniform convergence in (12), (13), (16), (17) given in Lemma 1, we obtain

\[
\frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \hat{\beta}_n^T X_i)}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right]
= \frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \hat{\beta}_n^T X_i)}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] + O_p\{(nbh)^{-1/2} + h^2 + b^2\}

\[
\otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] + O_p\{(nbh)^{-1/2} + h^2 + b^2\}
\]

\[
\frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \hat{\beta}_n^T X_i)}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] + o_p(1).
\]

Thus, for sufficiently large \( n \), we have

\[
\frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \beta^* X_i)}{\lambda(Z_i, \beta^* X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \beta^* X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^* X_i \right\}} \right]
= \frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \beta^* X_i)}{\lambda(Z_i, \beta^* X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \beta^* X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^* X_i \right\}} \right] + \frac{1}{n} \sum_{i = 1}^{n} \Delta_i \left[ \frac{\lambda_1(Z_i, \beta^* X_i)}{\lambda(Z_i, \beta^* X_i)} \right] \otimes \left[ \frac{E \left\{ X_{i} Y_i(Z_i) \mid \beta^* X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^* X_i \right\}} \right] + o_p(1),
\]

where the first equality is because the first derivative of the summation with respect to \( \beta \) is bounded uniformly under conditions C1-C2 by Lemma 1, and the last equality is because \( \hat{\beta}_n \) converges to \( \beta^* \). In addition,
under conditions C1-C2. Thus, for sufficient large \( n \) we have

\[
0 = \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\lambda}_i(Z_i, \hat{\beta}_n^T X_i)}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \otimes \left[ X_{il} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] + o_p(1)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{il} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \beta_0^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \beta_0^T X_i \right\}} \right] + o_p(1)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\lambda}_i(Z_i, \beta^* T X_i)}{\lambda(Z_i, \beta^* T X_i)} \otimes \left[ X_{il} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \beta^* T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \beta^* T X_i \right\}} \right] + o_p(1)
\]

under conditions C1-C2 and C3. Note that

\[
E \left( \frac{\Delta \lambda_i(Z, \beta^T X)}{\lambda(Z, \beta^T X)} \right) \otimes \left[ X_{il} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z) \mid \beta^* T X_i \right\}}{\hat{E} \left\{ Y(Z) \mid \beta^* T X_i \right\}} \right] + o_p(1)
\]

is a nonrandom quantity that does not depend on \( n \), hence it is zero. Thus the uniqueness requirement in Condition C7 ensures that \( \beta^* = \beta_0 \).

We now show that the subsequence that converges includes all but a finite number of \( n \)'s. Assume this is not the case, then we can obtain an infinite sequence of \( \hat{\beta}_n \)'s that do not converge to \( \beta^* \). As an infinite sequence in a compact set \( \mathcal{B} \), we can thus obtain another subsequence that converges, say to \( \beta^{**} \neq \beta^* \). Identical derivation as before then leads to \( \beta^{**} = \beta_0 \), which is a contradiction to \( \beta^{**} \neq \beta^* \). Thus we conclude \( \hat{\beta} - \beta_0 \rightarrow 0 \) in probability when \( n \rightarrow \infty \) under condition C1-C7. \( \square \)

A.4. Proof of Theorem 2. We first expand (9) as

\[
0 = n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\hat{\lambda}_i(Z_i, \hat{\beta}_0^T X_i)}{\lambda(Z_i, \hat{\beta}_0^T X_i)} \otimes \left[ X_{ili} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \hat{\beta}_0^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \hat{\beta}_0^T X_i \right\}} \right]
\]

(A.8) \[
= n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{ili} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \beta_0^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \beta_0^T X_i \right\}} \right]
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_{ili})} \left( \Delta_i \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{ili} - \frac{\hat{E} \left\{ X_{ili} Y_i(Z_i) \mid \beta_0^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \beta_0^T X_i \right\}} \right] \right) \right\} \bigg|_{\beta = \hat{\beta}}
\]

(A.9)

\[
\times \sqrt{n} (\hat{\beta} - \beta_0),
\]

where \( \hat{\beta} \) is on the line connecting \( \beta_0 \) and \( \hat{\beta} \).
We first consider (A.9). Because of Theorem 1 and Lemma 1, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_i^T \beta)} \left( \Delta \frac{\partial^2 \lambda_i(Z_i, \beta^T X_i)}{\partial (Y_i)^2} \right) \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] \right\} \bigg|_{\beta=\tilde{\beta}} \otimes X_{li}^T \right) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_i^T \beta_0)} \left( \Delta \frac{\partial^2 \lambda_i(Z_i, \beta_0^T X_i)}{\partial (Y_i)^2} \right) \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right\} \otimes X_{li}^T + o_p(1)
\]

(A.10)

\[- \frac{1}{n} \sum_{i=1}^{n} \left\{ \Delta \frac{\partial^2 \lambda_i(Z_i, \beta_0^T X_i)}{\partial (X_i^T \beta_0)} \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right\} \otimes X_{li}^T + o_p(1)
\]

(A.11)

Because of Lemma 1, (A.10) converges uniformly in probability to

\[-E \left( \int_0^\infty \frac{\lambda_i^{\otimes 2}(s, \beta_0^T X)}{\lambda^2(s, \beta_0^T X)} \otimes \left[ X_l - \frac{E \{ X_l Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes X_l^T dN(s) \right) = -E \left( \int_0^\infty \frac{\lambda_i^{\otimes 2}(s, \beta_0^T X)}{\lambda(s, \beta_0^T X)} \otimes \left[ X_l - \frac{E \{ X_l Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes \frac{E \{ X_l Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} Y(s) ds \right)
\]

where the last equality is because the second term above is zero by first taking expectation conditional on \(\beta_0^T X\).

Similarly, from Lemma 1, the term in (A.12) converges uniformly in probability to the limit of

\[E \left( \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_1(Z_i, \beta_0^T X_i) \otimes \left[ X_{li} - \frac{E \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right) \otimes X_{li}^T \right) \]

Now let \(\hat{\lambda}_{1,-i}(Z, \beta_0^T X)\) be the leave-one-out version of \(\hat{\lambda}_1(Z, \beta_0^T X)\), i.e. it is constructed the same as \(\hat{\lambda}_1(Z, \beta_0^T X)\) except that the \(i\)th observation is not used. Obviously,

\[
\frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_1(Z_i, \beta_0^T X_i) \otimes \left[ X_{li} - \frac{E \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right) \otimes X_{li}^T
\]

\[
- \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1,-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_{li} - \frac{E \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right) \otimes X_{li}^T = o_p(1).
\]
Now let $E_i$ mean taking expectation with respect to the $i$th observation conditional on all other observations, then

$$
E_i \left\{ \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right) \otimes X_i^T \right\} = E_i \left\{ \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(s, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(s) | \beta_0^T X_i \}}{E \{ Y_i(s) | \beta_0^T X_i \}} \right] \right) \otimes X_i^T dN_i(s) \right\}
$$

$$
= \frac{\partial}{\partial \beta_0} E_i \left\{ \hat{\lambda}_{1-i}(s, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(s) | \beta_0^T X_i \}}{E \{ Y_i(s) | \beta_0^T X_i \}} \right] \right\} = 0.
$$

Here, the last equality is because the integrand has expectation zero conditional on $\beta_0^T X_i$ and all other observations, and the third last equality is because the expectation is with respect to $X_i$ and does not involve $\beta_0$. Therefore, the term in (A.12) converges in probability uniformly to

$$
E \left\{ \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right) \otimes X_i^T \right\} = 0.
$$

Combining the results concerning (A.10) and (A.12), we thus have obtained that the expression in (A.9) is $-E \{s_{\text{eff}}(\Delta, Z, X) \otimes \alpha \} + o_p(1)$.

Next we decompose (A.8) into

$$
(A.12)^{1/2} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right) = T_1 + T_2 + T_3 + T_4,
$$

where

$$
T_1 = n^{-1/2} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right),
$$

$$
T_2 = n^{-1/2} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right),
$$

$$
T_3 = n^{-1/2} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right),
$$

$$
T_4 = n^{-1/2} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_i - \frac{E \{ X_i Y_i(Z_i) | \beta_0^T X_i \}}{E \{ Y_i(Z_i) | \beta_0^T X_i \}} \right] \right).
$$
First, note that

\[
T_2 = n^{-1/2} \sum_{i=1}^{n} \int \left\{ \frac{\hat{\lambda}_1(s, \beta_0^T X_i)}{\lambda(s, \beta_0^T X_i)} - \frac{\lambda_1(s, \beta_0^T X_i)}{\lambda(s, \beta_0^T X_i)} \right\} \otimes \left[ X_{ti} - \frac{E \left\{ Y_{ti} Y_i(s) \mid \beta_0^T X_i \right\}}{E \{ Y_i(s) \mid \beta_0^T X_i \}} \right] dN_i(s)
\]

\[
= o_p \left( n^{-1/2} \sum_{i=1}^{n} \int \left[ X_{ti} - \frac{E \left\{ Y_{ti} Y_i(s) \mid \beta_0^T X_i \right\}}{E \{ Y_i(s) \mid \beta_0^T X_i \}} \right] Y_i(s) \lambda(s, \beta_0^T X_{ti}) ds \right)
\]

= o_p(1),

where the last equality above is because the quantity inside the parenthesis is a mean zero normal random quantity of order \(O_p(1)\). Further,

\[
T_3 = n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( -\frac{\hat{E} \{ X_{ti} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right) + o_p(1)
\]

\[
= n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( \frac{n^{-1} \sum_{j=1}^{n} K_h(\beta_0^T X_j - \beta_0^T X_i) X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i) E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right) + o_p(1)
\]

\[
= n^{-3/2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_i \frac{\lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( \frac{E \{ X_{ti} Y_i(Z_i) \mid \beta_0^T X_i \} K_h(\beta_0^T X_j - \beta_0^T X_i) I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i) [E \{ Y_i(Z_i) \mid \beta_0^T X_i \}]^2} \right) + o_p(1)
\]

\[
= T_{31} + T_{32} + T_{33} + o_p(1),
\]

where

\[
T_{31} = n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( -\frac{K_h(\beta_0^T X_j - \beta_0^T X_i) X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i) E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right) \mid \Delta_i, Z_i, X_i
\]

\[
T_{32} = n^{-1/2} \sum_{j=1}^{n} \Delta_j \frac{\lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( -\frac{K_h(\beta_0^T X_j - \beta_0^T X_i) X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i) E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right) \mid \Delta_j, Z_j, X_j
\]
\[ T_{33} = -n^{1/2} E \left( \frac{\Delta_i \lambda_1(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \frac{-K_h(\beta_0^T X_i - \beta_0^T X_i)X_{ij}I(Z_j \geq Z_i)}{f_{\beta_0}^T X(\beta_0^T X_i)E \{ I(Z_j \geq Z_i) \mid \beta_0^T X_i \} } \right. \\
\left. \quad + \frac{E \{ X_iY_i(Z_i) \mid \beta_0^T X_i \} K_h(\beta_0^T X_j - \beta_0^T X_i)I(Z_j \geq Z_i)}{f_{\beta_0}^T X(\beta_0^T X_j)E \{ I(Z_j \geq Z_i) \mid \beta_0^T X_j \}^2} \right) \].

Here we used U-statistic property in the last equality above. Now when \( nh^2 \to 0 \),

\[ T_{31} = -n^{-1/2} \sum_{i=1}^{n} \Delta_i \lambda_1(Z_i, \beta_0^T X_i) \otimes \left[ -\frac{E \{ X_iY_i(Z_i) \mid \beta_0^T X_i \} }{E \{ Y_i(Z_i) \mid \beta_0^T X_i \} } \right. \\
\left. \quad + \frac{E \{ X_iY_i(Z_i) \mid \beta_0^T X_i \} E \{ Y_i(Z_i) \mid \beta_0^T X_i \} }{[E \{ Y_i(Z_i) \mid \beta_0^T X_i \} ]^2} \right] + O(n^{1/2}h^2) \]

Thus, \( T_{33} = o_p(1) \) as well. To analyze \( T_{32} \),

\[ T_{32} = -n^{-1/2} \sum_{j=1}^{n} E \left( \Delta_i \lambda_1(Z_i, \beta_0^T X_i) \otimes \left[ \frac{-K_h(\beta_0^T X_j - \beta_0^T X_i)X_{ij}I(Z_j \geq Z_i)}{f_{\beta_0}^T X(\beta_0^T X_i)E \{ I(Z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \} } \right. \\
\left. \quad + \frac{E \{ X_iI(Z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \} K_h(\beta_0^T X_j - \beta_0^T X_i)I(Z_j \geq Z_i)}{f_{\beta_0}^T X(\beta_0^T X_j)E \{ I(Z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \}^2} \right] \mid \Delta_j, Z_j, X_j \right) \]

\[ = -n^{-1/2} \sum_{j=1}^{n} E \left( \Delta_i \lambda_1(Z_i, \beta_0^T X_i) \otimes \left[ \frac{-x_{ij}I(z_j \geq Z_i)}{f_{\beta_0}^T X(\beta_0^T X_i)E \{ I(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \} } \right. \\
\left. \quad + \frac{E \{ X_iI(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \} }{f_{\beta_0}^T X(\beta_0^T X_i)E \{ I(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T X_i, Z_i \}^2} \right] \mid \beta_0^T X_i \right) \\
K_h(\beta_0^T x_j - \beta_0^T X_i) \}
\]

\[ = -n^{-1/2} \sum_{j=1}^{n} E \left( \Delta_i \lambda_1(Z_i, \beta_0^T x_j) \otimes \left[ \frac{-x_{ij}I(z_j \geq Z_i)}{E \{ I(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T x_j, Z_i \} } \right. \\
\left. \quad + \frac{E \{ X_iI(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T x_j, Z_i \} }{[E \{ I(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T x_j, Z_i \} ]^2} \right] \mid \beta_0^T x_i = \beta_0^T x_j \right) + O_p(n^{1/2}h^2) \]

\[ = -n^{-1/2} \sum_{j=1}^{n} E \left( \Delta_i \lambda_1(Z_i, \beta_0^T x_j) \right. \\
\left. \quad \otimes \left[ \frac{-x_{ij}I(z_j \geq Z_i)}{E \{ I(z_j \geq Z_i) \mid \beta_0^T X = \beta_0^T x_j, Z_i \} } \right. \\
\left. \quad + \frac{E \{ X_iS_c(Z_i, X) \mid \beta_0^T X = \beta_0^T x_j, Z_i \} }{E \{ S_c(Z_i, X) \mid \beta_0^T X = \beta_0^T x_j, Z_i \}^2} \right] \mid \beta_0^T x_i = \beta_0^T x_j \right) + O_p(n^{1/2}h^2) \]

\[ = -n^{-1/2} \sum_{j=1}^{n} \left( \int_{0}^{\infty} \frac{\Delta_i \lambda_1(s, \beta_0^T x_j)}{E \{ S_c(s, X) \mid \beta_0^T X = \beta_0^T x_j \} } \right. \\
\left. \quad \otimes \left[ \frac{E \{ X_iS_c(s, X) \mid \beta_0^T X = \beta_0^T x_j \} }{E \{ S_c(s, X) \mid \beta_0^T X = \beta_0^T x_j \}^2} \right] \mid \beta_0^T x_i = \beta_0^T x_j \right) + O_p(n^{1/2}h^2) \]
\[ \begin{align*}
  &= n^{-1/2} \sum_{j=1}^n \int_0^{z_j} \lambda_1(s, \beta_0^T X_j) \otimes \left[ \frac{E \{ X_{lj} S_c(s, X_j) \mid \beta_0^T X_j \} - x_{lj}}{E \{ S_c(s, X_j) \mid \beta_0^T X_j \}} - x_{lj} \right] ds + O_p(n^{1/2}h^2) \\
  &= n^{-1/2} \sum_{j=1}^n \int Y_j(s) \lambda(s, \beta_0^T X_j) \otimes \left[ \frac{E \{ X_{lj} Y_j(s) \mid \beta_0^T X_j \} - \frac{E \{ X_{lj} Y_j(s) \mid \beta_0^T X_j \}}{E \{ Y_j(s) \mid \beta_0^T X_j \}} - x_{lj} \right] ds + O_p(n^{1/2}h^2).
\end{align*} \]

When \( nh^4 \to 0 \), plugging the results of \( \mathbf{T}_1 \) and \( \mathbf{T}_{32} \) to (A.12), we obtain that the expression in (A.8) is

\[ n^{-1/2} \sum_{i=1}^n \Delta_i \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes X_{li} - \frac{\widehat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \] 

\[ = n^{-1/2} \sum_{i=1}^n \int \Delta_i \frac{\lambda_i(t, \beta_0^T X_i)}{\lambda(t, \beta_0^T X_i)} \otimes X_{li} - \frac{E \{ X_{li} Y_i(t) \mid \beta_0^T X_i \}}{E \{ Y_i(t) \mid \beta_0^T X_i \}} \] 

\[ = n^{-1/2} \sum_{i=1}^n \mathbf{S}_{\text{eff}}(\Delta_i, Z_i, X_i) + o_p(1). \]

Finally, note that

\[ \mathbf{T}_4 = n^{-1/2} \sum_{i=1}^n \Delta_i \left( \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} - \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \right) \] 

\[ \times \left[ \frac{E \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} - \frac{\widehat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \] 

\[ = o_p \left( n^{-1/2} \sum_{i=1}^n \Delta_i \left[ \frac{E \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} - \frac{\widehat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right) \] 

\[ = o_p \left( n^{-1/2} \sum_{i=1}^n \int Y_i(s) \lambda(s, \beta_0^T X_i) \otimes \left[ \frac{E \{ X_{li} Y_i(s) \mid \beta_0^T X_i \}}{E \{ Y_i(s) \mid \beta_0^T X_i \}} - x_{li} \right] ds \right) + o_p(n^{1/2}h^2) \]

\[ = o_p(1), \]

where the last equality is because the integrands have mean zero conditional on \( \beta_0^T X \), and the second last equality is obtained following the same derivation of \( \mathbf{T}_3 \). Using these results in (A.8), combined with the results on (A.9), it is now clear that the theorem holds. \( \square \)

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