MINIMAL SPACELIKE SURFACES AND THE GRAPHIC EQUATIONS IN $\mathbb{R}^4$

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Abstract. We will study an extension of the Bernstein Theorem for minimal spacelike surfaces of the four dimensional Minkowski vector space form. We will obtain the class of these surfaces for which has non zero Gauss curvature and that they are graphic surfaces. The class of entire solutions of a system of two elliptic non-linear equations, that is an extension of the equation of minimal graphic of the three dimensional Euclidean space.

Keywords: Minimal spacelike surface, graphic surfaces, Weiestrass representation.

1. Introduction and Notations and Basic Facts

First, we need to introduce our problem, which we will call of the Bernstein problem for the Minkowski space. To this end, we give an enunciate of the Bernstein Theorem.

Let $S$ be a regular surface of the Euclidean 3 dimensional space $\mathbb{E}^3$. Assume that there exists a fixed direction $[\partial_3]$ and a system of coordinates $(O, x, y, z)$ where $\partial_3 = (0, 0, 1)$ in this coordinates. The Bernstein Theorem, that can be found in [1], essentially establishes that “If $S$ is a minimal surface and the orthogonal projection in the coordinate plane $(O, x, y)$ is onto, then the surface is a plane.”

In this coordinates $(O, x, y)$ there exists a non-linear partial differential equation

$$(1 + p^2)r - 2pq + (1 + q^2)t = 0,$$

where in a non-parametric representation $z = \rho(x, y)$, we have $p(x, y) = \rho_x(x, y), q(x, y) = \rho_y(x, y), r(x, y) = \rho_{xx}(x, y), s(x, y) = \rho_{xy}(x, y)$ and $t(x, y) = \rho_{yy}(x, y)$. Moreover, it is needed that $d\rho = dx dy \sqrt{1 + p^2 + q^2} > 0$, which is the area function over $S$.

A nice proof of this theorem can be found in the work of J. C. C. Nitsche [7] and his equations (8) in [7] will be great influence in our work. Our work can be seen as an extension of the program developed by T. Radó in [9].

Furthermore, we can see the Bernstein Theorem as a restriction of the angle between the normal Gauss map and a fixed direction $[\partial_3]$ of the Euclidean space $\mathbb{E}^3$. There is a cone around this fixed direction that the normal Gauss map can not cross.
Therefore, we can consider as a variation of the Bernstein theorem, the questions related 
with number of points and numbers of directions omitted by complete minimal surfaces 
of \( E^3 \). On the work of R. Osserman [8] and B. Lawson [6], we can find a set of results 
related to these two questions.

The E. Calabi work in [2] is a transposition of the Bernstein Theorem for the 3 dimen-
sional Minkowski space, where the fixed direction is a timelike unitary vector.

We note that a set of examples of maximal surfaces in the 3-dimensional Minkowski 
space, can be found on the work of O. Kobayashi [4]. In particular, the new type of 
singularities are defined for those surfaces, which are points where as manifolds these 
surfaces are defined, but the metric vanishes. That means that in those points, the 
tangent plane of \( S \) is also tangent to the lightcone of the Minkowski space. The Helicoid 
is a beautiful example that we can found in [4].

On the work of K. Kommerell [5], which considers minimal surfaces in the Euclidean 
4-dimensional space, we have that graphic of entire holomorphic function, by example, 
\((x, y, \Re(f), \Im(f))\), establishes minimal surfaces where its projection in the plane \((O, x, y)\) 
are onto, and the Gauss curvature is not zero.

Our question is: There exists complex functions, not necessarily holomorphic, defined in 
all plane, which the spacelike graphic surface associated to orthogonal projection, directed 
by a timelike plane or a spacelike plane, in the Minkowski space \( \mathbb{R}^4_1 \) is onto with no zero 
Gauss curvature?

We say that this question is a generalization of the Bernstein and Kommerell Theorems.

To study this question we will need of an integral representation of those surfaces, and 
of an adaptation of [3] to the Minkowski space \( \mathbb{R}^4_1 \). Details of this adaptation can already 
be found in [M. P. Dussan and A. P. Franco Filho and P. A. Q. Simões] ([3]).

2. Basic Facts and Notations

The Minkowski space \( \mathbb{R}^4_1 \) will be the 4-dimensional real space \( \mathbb{R}^4 \) equipped with the 
bilinear form called of Lorentzian product, that is given by

\[
\langle (a, b, c, d), (t, x, y, z) \rangle = -at + bx + cy + dz.
\]

A spacelike plane \( V \subset \mathbb{R}^4_1 \) is a 2-dimensional vector subspace where \( \langle v, v \rangle > 0 \) for each 
\( v \neq 0 \) of the plane \( V \). A timelike plane \( T \subset \mathbb{R}^4_1 \) is a 2-dimensional vector subspace where 
there exists a timelike vector \( t \in T \) that means that \( \langle t, t \rangle < 0 \) and an other spacelike 
vector \( n \in T \) such that \( \langle n, n \rangle > 0 \) with \( \langle t, n \rangle = 0 \).
We say that timelike plane $T$ is the orthogonal complement of the spacelike plane $V$ denoted by the symbols $V = T^\perp$ and $T = V^\perp$ if
\[ \langle x, y \rangle = 0 \quad \text{for all} \quad x \in T \quad \text{and} \quad y \in V. \]

The following simple proposition is very usefully along this work, it establishes a special orthonormal base for each timelike plane. We denote by $\partial_0$ the vector $(1, 0, 0, 0)$.

**Proposition 2.1.** For each spacelike plane $V \not\subset E^3$ equipped with a orthonormal base $\{e_1, e_2\}$ the (unique) timelike plane $T = V^\perp$ has an orthonormal base $\{\tau, \nu\}$ satisfying the following conditions:
1. $\langle \tau, \tau \rangle = -1$ and $\langle \tau, \partial_0 \rangle < 0$. That means $\tau$ is timelike, unitary future directed vector of $T$.
2. $\langle \nu, \nu \rangle = 1$ with $\langle \nu, \partial_0 \rangle = 0$. That means that $\nu$ is a vector into the $3$-dimensional subspace $\{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4$, which we will identify with the Euclidean $3$-dimensional vector space $\mathbb{E}^3$.
3. $\langle \tau, \nu \rangle = 0$ and for all other orthonormal base $\{t, n\}$ of $T$ we have that $t^0 \leq |t^0|$.
4. The orthonormal base $\{\tau, e_1, e_2, \nu\}$ in this order is positive, relative to the Minkowski referential $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ that is the canonical base of $\mathbb{R}^4$.

**Proof.** We need to define the vector $\tau$, therefore all the statements of the proposition follow immediately. In fact, we take $\tau$ as being
\begin{equation}
\tau = \frac{1}{\sqrt{1 + (e_i^0)^2 + (e_j^0)^2}} (\partial_0 + e_i^0 e_1 + e_j^0 e_2),
\end{equation}
where $e_i^0 = -\langle \partial_0, e_i \rangle$ for $i = 1, 2$. It is trivial to see that $\langle \tau, \tau \rangle = -1$, $\langle \tau, e_i \rangle = 0$ for $i = 1, 2$, and that $\tau$ is future directed. Since by the assumption $V \not\subset \mathbb{E}^3$, we have the timelike plane generated by $\{\partial_0, \tau\}$, then we take $\nu$ to be the unique unitary vector of the line span$[\partial_0, \tau] \cap \mathbb{E}^3$ such that $\{\tau, e_1, e_2, \nu\}$ is a positive base.

Now, assuming that we have other orthonormal base $\{t, n\}$ for $T$ we can take the Lorentz transformation given by
\[ t = \cosh \varphi \tau + \sinh \varphi \nu \quad \text{and} \quad n = \sinh \varphi \tau + \cosh \varphi \nu, \]
assumed that $t^0 > 0$. Since $-\langle t, \partial_0 \rangle = -\cosh \varphi \langle \tau, \partial_0 \rangle$ it follows then that $t^0 > \tau^0$. \hfill $\Box$

### 2.1. A Semi-rigid frame.
Let $\mathbb{R}^4_1 = E \oplus T$ be given by directed sum of a spacelike plane $E$ and its orthogonal complement $T$, that is a timelike plane.

**Definition 2.2.** A semi-rigid referential of the the Minkowski space $\mathbb{R}^4_1$ associated to a directed sum $\mathbb{R}^4_1 = E \oplus T$ is a positive base $\{l_0, e_1, e_2, l_3\}$ of $\mathbb{R}^4_1$ satisfying the following conditions:
(1) \( E = \text{span}\{e_1, e_2\} \) and \( T = \text{span}\{l_0, l_3\} \).
(2) \( \{e_1, e_2\} \) is an orthonormal base for \( E \).
(3) \( \{l_0, l_3\} \) is a null or lightlike base for \( T \), such that \( l_0 = l_0 \) and \( l_3 = l_3 \).

**Proposition 2.3.** If we have two semi-rigid referentials \( \{l_0, e_1, e_2, l_3\} \) and \( \{\tilde{l}_0, \tilde{e}_1, \tilde{e}_2, \tilde{l}_3\} \) associated to the directed sum \( \mathbb{R}_4^1 = E \oplus T \) with \( T = E^\perp \), then \( l_0 = \tilde{l}_0 \) and \( l_3 = \tilde{l}_3 \).

Therefore the complex numbers given by
\[
a(l_3) = \frac{l_3^1 + il_3^2}{1 - l_3^3} \quad \text{and} \quad b(l_0) = \frac{l_0^1 + il_0^2}{1 + l_0^3},
\]
are univocally determined by the directed sum \( \mathbb{R}_4^1 = E \oplus T \).

**Proof.** In the lorentz plane \( T \) with induced orientation by \( \partial_0 \) there exists only two independent lightlike direction \( L_0 \) and \( L_3 \), therefore adding the condition
\[
\langle L_0, \partial_0 \rangle = -1 = \langle L_3, \partial_0 \rangle
\]
we obtain the unique base \( \{l_0, l_3\} \) for \( T \) given by (3) of the Definition 2.2. \( \square \)

**Corollary 2.4.** The matrix associated to the set of the semi-rigid referential of the directed sum \( \mathbb{R}_4^1 = E \oplus T \) is given by
\[
M(\vartheta) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \vartheta & \sin \vartheta & 0 \\
0 & -\sin \vartheta & \cos \vartheta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
for \( \vartheta \in \mathbb{R} \).

Moreover, \( M(\vartheta) \) is a 1 parameter sub-group of the Minkowski group of isometry of \( \mathbb{R}_4^1 \), and all geometric facts that we will see in this work is invariant by this sub-group. Indeed, we will see that the complex functions \( a(l_3) \) and \( b(l_0) \) determine the geometric properties of minimal spacelike surfaces of \( \mathbb{R}_4^1 \).

**Proposition 2.5.** The frame associated to the vector subspace \( E \) can be taken in terms of \( a(p) \) and \( b(p) \), namely,
\[
e_1(p) = \frac{W(p) + \overline{W(p)}}{2|1 - a(p)b(p)|} \quad \text{and} \quad e_2(p) = \frac{W(p) - \overline{W(p)}}{2i||1 - a(p)b(p)||},
\]
where
\[
W(p) = (a(p) + b(p), 1 + a(p)b(p), i(1 - a(p)b(p)), a(p) - b(p))
\]
with \( \langle W(p), \overline{W(p)} \rangle = 2|1 - a(p)b(p)|^2 \)
2.2. Spacelike Surfaces in $\mathbb{R}^4_1$.

**Definition 2.6.** A spacelike surface $S \subset \mathbb{R}^4_1$ is a smooth 2-dimensional sub-manifold of the topological real vector space $\mathbb{R}^4$ that at each point $p \in S$ its tangent plane $T_p S$ relative to the lorentz product of $\mathbb{R}^4_1$ is a spacelike plane.

A spacelike parametric surface of $\mathbb{R}^4_1$ is a two parameters map $(U, X)$ from a connected open subset $U \subset \mathbb{R}^2$ into $\mathbb{R}^4_1$ such that the topological subspace $X(U)$ is a spacelike surface.

We will assume, always, that $(X(U), X^{-1})$ is a chart of a complete atlas for a spacelike surface $S$ of $\mathbb{R}^4_1$.

Let $((x, y), U)$ be a connected and simply connected open subset of the Euclidean plane $\mathbb{R}^2$. If $X(x, y) = (X^0(x, y), X^1(x, y), X^2(x, y), X^3(x, y))$ is a spacelike parametric surface of $\mathbb{R}^4_1$ then, we have a metric tensor induced by the lorentzian semi-metric of $\mathbb{R}^4_1$ given by

$$g = \sum \langle D_i X, D_j X \rangle dx^i \otimes dx^j,$$

and the second quadratic form of $S = X(U)$ is a quadratic symmetric 2-form

$$B = \sum \Psi_{ij} dx^i \otimes dx^j,$$

that is given by covariant partial derivative by the formula

$$D_{ij}X - \sum \Gamma^k_{ij} D_k X = \Psi_{ij}.$$

From the definition of the Christoffel symbols $\Gamma^k_{ij}$ it follows that $\langle \Psi_{ij}, D_k X \rangle \equiv 0$. Setting a pair of pointwise orthonormal base for the normal bundle $NS$ given by $\tau(x, y)$ and $\nu(x, y)$, where $\tau(x, y)$ is an unitary future directed timelike vector and $\nu(x, y)$ is an unitary spacelike vector, we can assume that $\langle \nu(x, y), (1, 0, 0, 0) \rangle \equiv 0$. Therefore we have

$$\Psi_{ij} = h_{ij} \tau + n_{ij} \nu$$

where by definition

$$h_{ij} = -\langle D_{ij} X, \tau \rangle \quad \text{and} \quad n_{ij} = \langle D_{ij} X, \nu \rangle.$$

Since $\dim(N_p S) = 2$ we need to define the normal connection for $S$, that is given by a covariant vector $\gamma = \sum \gamma_k dx^k$ where

$$\gamma_k = \langle D_k \tau, \nu \rangle = \langle D_k \nu, \tau \rangle.$$
Next we will display these set of structural equations for \( S = X(U) \), equation (2) being the Gauss equation, (3) and (4) corresponding to Weingarten equations for \( S \). Namely,

\[
D_{ij}X - \sum \Gamma^k_{ij} D_k X = h_{ij} \tau + n_{ij} \nu \tag{2}
\]

\[
D_k \tau = \sum h^k_m D_m X + \gamma_k \nu \tag{3}
\]

\[
D_k \nu = - \sum n^k_m D_m X + \gamma_k \tau. \tag{4}
\]

**Definition 2.7.** We say that the surface \( S = X(U) \) is a minimal surface if and only if

\[ H_S = \frac{1}{2} \sum \Psi_{ij} g^{ij} = 0 \]

The vector field \( H_S \) is called the mean curvature vector of \( S \).

Follows from equations (2) that an equivalent definition for minimal surfaces is the condition

\[ 2H_S = \sum g^{ij}(D_{ij}X - \sum \Gamma^k_{ij} D_k X) = (\Delta g X^0, \Delta g X^1, \Delta g X^2, \Delta g X^3) = 0, \]

where \( \Delta g \) is the Laplace-Beltrami operator over \( S = (U, g) \).

Next we observe that one can associate a Riemann surface to \( S \). In fact, from the well know theorem that say that all spacelike surface admits a isothermic coordinate atlas, that means that there exists parametrization

\[ f(w) = (f^0(w), f^1(w), f^2(w), f^3(w)) \quad w = u + iv \in U' \subset \mathbb{C}, \]

such that \( f(U') \subset S = X(U) \), and the induced metric tensor now, is \( g = \lambda^2 dw dw \), we have that

\[ \langle f_u, f_u \rangle = \lambda^2 = \langle f_v, f_v \rangle \quad \text{and} \quad \langle f_u, f_v \rangle = 0. \]

Since \( f_w = \frac{1}{2}(f_u - if_v) \) we extend the bilinear form of \( \mathbb{R}^4 \) to a complex bilinear form over \( \mathbb{C}^4 \equiv \mathbb{R}^4 + i\mathbb{R}^4 \), as follows

\[ \langle X + iY, A + iB \rangle = \langle X, A \rangle - \langle Y, B \rangle + i(\langle X, B \rangle - \langle Y, A \rangle), \]

that imply that

\[ \langle f_w, f_w \rangle = 0 \quad \text{and} \quad \langle f_w, \bar{f}_w \rangle = \langle f_w, f_\bar{w} \rangle = \lambda^2 / 2. \tag{5} \]

Now, if we have two isothermic charts \((U', f)\) and \((V, h)\) for \( S \) then, when makes sense, the overlapping map is a holomorphic function and therefore we can see \( M = (S, A) \) as a Riemann surface equipped conformal atlas \( A \), and the induced metric tensor or line element \( ds^2 = \lambda^2(w)|dw|^2 \) is a compatible metric for this Riemann surface \( M \).
Finally we note that does not exist compact spacelike surfaces in $\mathbb{R}^4_1$, therefore from now $M$ will be either the disk

$$D = \{ z \in \mathbb{C} : z\bar{z} < 1 \}$$

that is a hyperbolic Riemann surface or the complex plane $\mathbb{C}$ which is a parabolic Riemann surface, since we are assuming that $M$ is a connected and simply-connected Riemann surface. Moreover, if $h(z(w)) = f(w)$ then from chain rule it follows

$$f_w(w) = h_z(z(w)) \frac{dz}{dw}(w)$$

and

$$\langle f_w, f_w \rangle = \langle h_z, h_z \rangle \left| \frac{dz}{dw} \right|^2.$$

2.3. A solution for the equations (5). Expanding in its coordinates we have that the equations (5) become

$$-(f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0$$

and if we denote the complex derivate of the components $f^i_w$ by $Z^i$ and we assume that $Z^1 - iZ^2 \neq 0$, we have

$$\frac{Z^0 - Z^3}{Z^1 - iZ^2}, \frac{Z^0 + Z^3}{Z^1 - iZ^2} = \frac{Z^1 + iZ^2}{Z^1 - iZ^2}.$$  

Defining

$$a = \frac{Z^0 + Z^3}{Z^1 - iZ^2} \quad \text{and} \quad b = \frac{Z^0 - Z^3}{Z^1 - iZ^2} \quad \text{and} \quad \mu = \frac{Z^1 - iZ^2}{2},$$

we obtain the following map

$$f_w = \mu W(a, b) \quad \text{where} \quad W(a, b) = (a + b, 1 + ab, i(1 - ab), a - b),$$

from $(a, b) \in \mathcal{F}(M, \mathbb{C}) \times \mathcal{F}(M, \mathbb{C})$ in $\mathbb{C}^4$.

Moreover, we have that $\lambda^2 = 2\langle f_w, f_w \rangle = 4\mu\bar{\mu}(1 - ab)(1 - \bar{b})$. Therefore, $\mu \neq 0$ and $1 - ab \neq 0$ are the conditions to obtain a surface without singularities in its metric.

Now, since we can write $W(a, b) = (a, 1, i, a) + b(1, a, -ia, -1)$ we obtain the cases where $(Z^1 + iZ^2)(Z^1 - iZ^2) = 0$, with the expression $f_w = \eta(a, 1, i, a)$ and $f_w = \xi(1, a, -ia, -1)$. Moreover when $Z^0 = 0 = Z^3$ we obtain $f_w = \eta(0, 1, i, 0)$ that is the plane $\{0\} \times \mathbb{R}^2 \times \{0\}$.

The following lemma is an extension to $\mathbb{R}^4_1$ of a theorem obtained by Monge:

**Lemma 2.8.** For a $\lambda$-isothermic spacelike parametric surface $(U, f)$ the following statement are equivalent:

(i) The surface $f(U)$ is minimal, $H_f(w) \equiv 0$.

(ii) The maps $\mu, a, b$ are holomorphic functions from $U$ into $\mathbb{C}$. 
Proof. Follows from the Laplace-Beltrami operator: \( \Delta_M f_i(w) = \frac{2}{\bar{w}}(f_i(w))_{w\bar{w}} = 0 \) for \( i = 0, 1, 2, 3 \). □

2.4. An Integral Representation. Let \((U, X)\) be a spacelike parametric surface of \( \mathbb{R}^4_1 \) where \( X(x, y) = (X^0(x, y), X^1(x, y), X^2(x, y), X^3(x, y)) \) and \( U \subset \mathbb{R}^2 \) is a simply connected domain. Then, the vector 1-form given by \( dX = \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy \) is exact, therefore it is closed. The integral equation associated to \((U, X)\) is

\[
X(x, y) = X(x_0, y_0) + \int_{(x_0, y_0)}^{(x, y)} \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy.
\]

Moreover, each solution of equation (6) is a spacelike parametric surface \((U, X)\) if it holds

\[
E = \langle X_x, X_x \rangle > 0, \quad G = \langle X_y, X_y \rangle > 0, \quad F = \langle X_x, X_y \rangle \quad \text{and} \quad EG - F^2 > 0.
\]

From Definition 2.7 and Lemma 2.8 we obtain:

Corollary 2.9. Let \( U \subset \mathbb{R}^2 \) a simply connected domain. If \((U, X)\) is a minimal spacelike parametric surface which is solution of the integral equation (6), then each coordinate function of \( X(x, y) \) is a harmonic real-valued function on \( U \).

Proof. Indeed, the Laplace-Beltrami operator \( \Delta_M \) is a tensorial operator, defined by contraction of the Gauss equation (2), as follows in Definition 2.7. □

We note that with local isothermic coordinates the integral representation (6) is usually called of Weierstrass integral equation, namely,

\[
f(w) = p_0 + 2\Re \int_{w_0}^{w} \mu(\xi)W(a(\xi), b(\xi))d\xi,
\]

where \( f_w(w) \) is the solution of equation (5) in Subsection 2.2.

2.5. The structural equations with isothermic parameters. Let \((U, f)\) be a parametric sub-surface of \( X(M) \) given with isothermic parameters \( w = u + iv \) and \( \langle f_w, f_w \rangle = 0 \) and \( \langle f_w, f_{\bar{w}} \rangle = \lambda^2/2 \). We have the following version of equations (2),(3) and (4).
Lemma 2.10. Let \((U, f)\) be a \(\lambda^2\)-isothermic coordinate system for a minimal surface \((M, X)\) of \(\mathbb{R}^4_1\). We have the following structural equations in \(w = u + iv \in U\):

\[
\begin{align*}
\tau_w &= \sigma f_w + \Gamma \nu \quad \text{and} \quad \nu_w = \chi f_w + \Gamma \tau \\
2f_{ww} &= 2\frac{\lambda w}{\lambda} f_w + \sigma \frac{\lambda^2}{2} \tau - \frac{\chi \lambda^2}{2} \nu \quad \text{and} \quad f_{w\bar{w}} = 0
\end{align*}
\]

\[
\Gamma(w) = \langle \tau_w, \nu \rangle = -\langle \nu_w, \tau \rangle = \frac{\sigma_1(w) - i\sigma_2(w)}{2}
\]

Proof. We start showing equation (8). For that we take \(f_{ww} = Af_w + Bf_{\bar{w}} + C\tau + D\nu\), and assume that equations (7) and (9) are the definition of the functions associated to the normal connection for \((U, f)\).

From \(\langle f_w, f_w \rangle = 0\) it follows that \(\langle f_{ww}, f_w \rangle = 0\), therefore \(B = 0\). From \(\langle f_w, f_{\bar{w}} \rangle = \lambda^2/2\) it follows that \(\langle f_{ww}, f_w \rangle + \langle f_w, f_{w\bar{w}} \rangle = \lambda w \lambda\), and, since \(f_{w\bar{w}} = 0\) we obtain \(A = 2\frac{\lambda w}{\lambda}\).

Now, from \(\langle f_w, \tau \rangle = 0\) we have that \(\langle f_{ww}, \tau \rangle + \langle f_w, \tau_w \rangle = 0\), therefore we obtain \(C = \sigma \frac{\lambda^2}{2}\). Analogously one has \(D = -\chi \frac{\lambda^2}{2}\). So we have showed equation (8).

The definition of the functions \(\sigma\) and \(\chi\) is obtained by equations (7), that from \(\langle f_{\bar{w}}, \tau \rangle = 0\) and from the minimal condition for \((M, f)\) it follows that \(\langle \tau_w, f_{\bar{w}} \rangle + \langle \tau, f_{w\bar{w}} \rangle = 0\), therefore the tangent component of \(\tau_w\) is \(\sigma f_{\bar{w}}\). Then, we take the equations (7) as a definition of the functions associated to the normal connection of \((M, f)\). Equation (9) defines the function \(\Gamma\). \(\square\)

3. Two Types of Graphics for Minimal Surfaces of \(\mathbb{R}^4_1\)

First, let us recall that \(\mathbb{R}^4_1\) has topological structure and differential structure of the Euclidean space \(\mathbb{R}^4\).

If \(T(u, v) = (\varphi(u, v), \psi(u, v))\) is a function from \(U \subset \mathbb{R}^2\) in \(\mathbb{R}^2\) we can see as a graphic of \(T\) the set of point of \(\mathbb{R}^4\) such that

\[
\text{graphic}(T) = \{((u, v), (\varphi(u, v), \psi(u, v))) \in \mathbb{R}^4 : (u, v) \in U \subset \mathbb{R}^2\}.
\]

Then we can choice four equivalent positions for the timelike axis in \(\mathbb{R}^4_1\). But we only need to choice two of those possible positions to have all possibilities of type graphic surfaces. In fact:

Fixing the signature of \(\mathbb{R}^4_1\) by \((-1, +1, +1, +1)\) we take by definition:

1. The first type of graphic surfaces are given by

\[
X(x, y) = (A(x, y), x, y, B(x, y))\quad \text{where} \quad (x, y) \in U \subset \mathbb{R}^2.
\]
The second type of graphic surfaces are given by
\[ X(x, y) = (x, y, A(x, y), B(x, y)) \text{ where } (x, y) \in U \subset \mathbb{R}^2. \]

We will always assume that the functions \( A \) and \( B \) are \( C^\infty(U) \), \( U \) is a connected and simply connected open subset of \( \mathbb{R}^2 \) and that the surface \( X(U) \) is a spacelike surfaces of \( \mathbb{R}_4^1 \).

**Proposition 3.1.** A minimal graphic surface (first or second type) of \( \mathbb{R}_4^1 \) satisfies the following system of equations

\[
\begin{align*}
 g_{22} D_{11} A - 2 g_{11} D_{12} A + g_{11} D_{22} A &= 0 \\
 g_{22} D_{11} B - 2 g_{11} D_{12} B + g_{11} D_{22} B &= 0
\end{align*}
\]

where \( g = \sum_{ij} g_{ij} du^i du^j \) is the metric tensor associated to the surface \( S = X(U) \). The system of equations (10) only says that \( A \) and \( B \) are harmonic functions of the Riemann surface \( (U, X) \).

**Proof.** Taking the matrix representation of metric tensor and its inverse tensor
\[
[g_{ij}] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \quad [g^{ij}] = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}
\]

one has that by Definition 2.7 the mean curvature vector is given by
\[
H_X = \frac{1}{EG - F^2} (G \Psi_{11} - 2F \Psi_{12} + E \Psi_{22}).
\]

For each type of surface we take a pointwise base \( \{N_1, N_2\} \) for its normal bundle, as follows. If \( X(x, y) = (A, x, y, B) \) we take the orthogonal vectors
\[
N_1 = (1, A_x, A_y, 0) \quad \text{and} \quad N_2 = (0, B_x, B_y, -1).
\]

So in this case \( D_{ij} X = (D_{ij} A, 0, 0, D_{ij} B) \).

If \( X(x, y) = (x, A, B, y) \) we take the orthogonal vectors
\[
N_1 = (A_x, 1, 0, -A_y) \quad \text{and} \quad N_2 = (B_x, 0, 1, -B_y).
\]

Then in this case \( D_{ij} X = (0, D_{ij} A, D_{ij} B, 0) \). Now, the system (10) follows immediately. \( \square \)

Our first example corresponds to a minimal spacelike surface, which represents a first type graphic surface, defined in all plane \( \mathbb{R}^2 \).
**Example 1.** For each harmonic function \( \theta : \mathbb{R}^2 \rightarrow \mathbb{R} \) the maps

\[
X(x, y) = (\theta(x, y), x, y, \theta(x, y)) \quad \text{or} \quad X(x, y) = (\theta(x, y), x, y, -\theta(x, y))
\]

are both minimal spacelike parametric surfaces, locally isometric to the Euclidean plane \( \mathbb{R}^2 \), therefore it is a flat surface.

In fact, assuming the first expression of \( X(x, y) \), since \( X_x = (\theta x, 1, 0, \theta x) \) and \( X_y = (\theta y, 0, 1, \theta y) \) it follows \( \langle X_x, X_x \rangle = 1 = \langle X_y, X_y \rangle \) with \( \langle X_x, X_y \rangle = 0 \). Now, by assumption \( \Delta \theta = \theta_{xx} + \theta_{yy} = 0 \) it follows that \( H_X(x, y) = (0, 0, 0, 0) \).

We also observe that, according the notation of Subsection 2.3, this class of surfaces correspond to \( Z_1 + iZ_2 = 0 \), with \( Z_0 - Z_3 = 0 \) and \( Z_0 \neq 0 \), where \( Z_i \) are the components in the representation \( X_w(w) = (\theta_w, \frac{1}{2}, i, \theta_w) \). Moreover we can write these parametric surfaces as follows: For \( A = B = \theta(x, y) \) we have that \( X(x, y) = (0, x, y, 0) + \theta(x, y)(\partial_0 + \partial_3) \), therefore \( X(\mathbb{R}^2) \) is a subset of a degenerated hyperplane, and this shows that its normal curvature vanishes identically.

The Example 1 shows that we need a formula of the second quadratic form in terms of functions \( \mu, a \) and \( b \). That formula was already obtained in Theorem 3.3 from [3], so we rewrite next.

**Lemma 3.2.** If \( f_w = \mu W(a, b) \) where \( a \) and \( b \) are holomorphic functions from \( M \) into \( \mathbb{C} \). The second quadratic form with complex notation is given by

\[
(f_w)_{1} = \frac{\mu a_w}{1 - ab} L_0(b) + \frac{\mu b_w}{1 - ba} L_3(a),
\]

where \( L_0(b) \) and \( L_3(a) \) are future directed lightlike vectors given by

\[
L_0(b) = (1 + b\overline{b}, b + \overline{b}, -i(b - \overline{b}), 1 - b\overline{b}) \quad \text{and} \quad L_3(a) = (1 + a\overline{a}, a + \overline{a}, -i(a - \overline{a}), -1 + a\overline{a}).
\]

**Corollary 3.3.** The second quadratic form of a minimal spacelike surface \( (U, f) \) is lightlike type if and only if \( a_w = 0 \) or \( b_w = 0 \). Therefore in this case, the Gauss curvature \( K(f) = 0 \) and the surface is contained in a degenerated hyperplane.

Reciprocally, if the Gauss curvature \( K(f) = 0 \) then the second quadratic form is lightlike type or it is zero, \((f_{ww})_{1} = 0\).

Next, we will apply equations (10) for type graphic minimal surfaces in \( \mathbb{E}^3 \) and \( \mathbb{L}^3 \). We will give an explicit equation for each case.

We start with the first type:

1. When we have \( A(x, y) \equiv 0 \) we obtain the graphics in \( \mathbb{E}^3 \) given by an unique function \( B(x, y) \):

\[
f(x, y) = (0, x, y, B(x, y)) \in \mathbb{E}^3,
\]
with the induced metric tensor over $f(U)$ as a spacelike surface of $\mathbb{R}_1^4$. Then system (10) becomes to the equation

$$\left(1 + B_y^2\right)B_{xx} - 2B_xB_yB_{xy} + (1 + B_y^2)B_{yy} = 0,$$

which is called the equation of minimal graphic for smooth surface of the Euclidean space $\mathbb{R}^3 \equiv \mathbb{E}^3$. In this case Bernstein showed that if $U = \mathbb{R}^2$ then the solution of equation (12) is a plane.

(2) When we have $B(x, y) \equiv 0$ we obtain the graphics in $\mathbb{E}^3$ given by an unique function $A(x, y)$:

$$f(x, y) = (A(x, y), x, y, 0) \in \mathbb{L}^3,$$

with the induced metric tensor over $f(U)$ as a spacelike surface of $\mathbb{R}_1^4$. Then system (10) becomes to the equation

$$\left(1 - A_y^2\right)A_{xx} + 2A_xA_yA_{xy} + (1 - A_x^2)A_{yy} = 0, \quad \text{with } (A_x^2 < 1 \text{ and } A_x^2 + A_y^2 < 1),$$

which is called the equation of minimal graphic for smooth surface of the Lorentzian space $\mathbb{L}^3$. For this case, Calabi showed that if $U = \mathbb{R}^2$ then the solution of equation (13) is a plane.

Of the second type are the minimal spacelike surfaces given by the representation $f(x, y) = (x, y, A(x, y), B(x, y))$. In this case,

(3) When we have $B(x, y) \equiv 0$ we obtain the graphics given by an unique function $A(x, y)$:

$$f(x, y) = (x, y, A(x, y), 0) \in \mathbb{L}^3,$$

with the induced metric tensor over $f(U)$ as a spacelike surface of $\mathbb{R}_1^4$. Then system (10) becomes to the equation

$$\left(1 + A_y^2\right)A_{xx} - 2A_xA_yA_{xy} + (-1 + A_x^2)A_{yy} = 0, \quad \text{with } (A_x^2 > A_y^2 + 1),$$

and, we will say that this equation is the equation of second type of minimal graphic for smooth surface of the $\mathbb{L}^3$.

4. About the Extension of Local Solutions of the Graphic Equations

In this section we study if it is possible to extend to all the complex plane $\mathbb{C}$ the local solutions for the graphic equations given in system (10).

We start identifying a formula for the Gauss curvature of the surface. In fact, for $f_w = \mu W(a, b)$ where $(U, f)$ is a minimal spacelike surface of $\mathbb{R}_1^4$, with holomorphic functions
A(w), b(w), µ(w), we know that the expression for the Gauss curvature is given by

\[ K(f) = -\frac{\Delta \ln \lambda^2}{2\lambda^2} = -\frac{1}{\lambda^2} \Delta \ln \lambda. \]

Now, since \( \lambda^2 = 4\mu \overline{\mu}(1 - ab)(1 - \overline{ab}) \) and \( \Delta = 4\partial_w \overline{\partial_w} \), we obtain

\[ K(f) = -\frac{(\ln(1 - ab)(1 - \overline{ab}))_{w\overline{w}}}{2\mu \overline{\mu}(1 - ab)(1 - \overline{ab})}. \]

Since

\[ (\ln(1 - ab)(1 - \overline{ab}))_{w\overline{w}} = -a_w \left( \frac{b}{1 - ab} \right)_{\overline{w}} - b_w \left( \frac{a}{1 - ab} \right)_{\overline{w}}, \]

it follows that

\[ K(f) = -\frac{\Re(a_w \overline{b}(1 - \overline{ab})^2)}{\mu \overline{\mu}(1 - ab)^3(1 - \overline{ab})^3}. \]

**First case.** We will focus our attention to find surfaces given by

\[ X(x, y) = (A(x, y), x, y, B(x, y)) \]

for all \((x, y) \in \mathbb{R}^2\), satisfying the equations (10), that means that \( X(\mathbb{R}^2) = S \) is a minimal surface of \( \mathbb{R}^4 \).

The question is: There exists non flat solution for this problem?

### 4.1. A pointwise base for the normal bundle.

Let us to take the vector fields along \( S = X(\mathbb{R}^2) \)

\[ N_1 = (1, A_x, A_y, 0) \quad \text{and} \quad N_2 = (0, -B_x, -B_y, 1) \]

used in the proof of Proposition \[3.1\]

**Proposition 4.1.** The spacelike Gauss map \( \nu(x, y) \) for the minimal surface \( S \subset \mathbb{R}^4 \) is given by

\[ \nu(x, y) = \frac{1}{\sqrt{1 + (B_x)^2 + (B_y)^2}} (0, -B_x, -B_y, 1). \]

**Proof.** We only need see if the orientation of \( \{ N_1, N_2 \} \) and the orientation of \( \{ \partial_0, \partial_3 \} \)

are compatible each other. The compatibly orientations follows from the projected vectors:

\( (N_1^0, 0, 0, N_2^0) = \partial_0 \) and \( (N_2^0, 0, 0, N_2^0) = \partial_3. \)

**Corollary 4.2.** The Gauss map \( \nu : S \rightarrow S^2 \subset \mathbb{E}^3 \) is such that

\[ \nu^3 = \frac{1}{\sqrt{1 + (B_x)^2 + (B_y)^2}} > 0. \]

In other words, \( \nu(S) \) is the (open) north hemisphere of the Riemann sphere \( S^2 \).
Now we assume that we have a local representation \((U,f)\) such that \(f(U) \subset S\) and
\[
f_w = \mu(a + b, 1 + ab, i(1 - ab), a - b),
\]
where \(a, b, \mu\) are holomorphic functions from \(U\) into \(\mathbb{C}\). Moreover, we assume that \(U\) is a connected and simply connected open subset of \(\mathbb{C}\).

The normal bundle has a pointwise base of lightlike vectors given by \(\{L_3(a), L_0(b)\}\), which allows, in easier form, to compute the component \(\nu^3\) of the spacelike Gauss map \(\nu(a, b)\).

**Lemma 4.3.** For an isothermic local representation \((U,f)\) such that \(f(U) \subset S\) we have
\[
\nu^3(a, b) = \frac{1}{|1 - \bar{\mu}b| \sqrt{1 + |a|^2} \sqrt{1 + |b|^2}} (1 - |ab|^2).
\]

The maximal extension of holomorphic functions \(a, b\), is forced by the inequalities:
\[
|1 - \bar{\mu}b| \neq 0 \quad \text{and} \quad |ab|^2 \neq 1.
\]

**Proof.** Taking a normalization of the vector
\[
N_3(a, b) = \frac{1}{1 + bb} L_0(b) - \frac{1}{1 + aa} L_3(a)
\]
we obtain \(\nu(a, b)\) because \(N_3^0(a, b) = 0\). Therefore, we obtain the component \(\nu^3(a, b)\) and the equations (23). \(\Box\)

The first equation of (23) is the functional area \(\sqrt{EG - F^2} = |\mu| |1 - \bar{\mu}b|\). Then, we will find of a necessary and sufficient condition to obtain a maximal extension for \(\sqrt{EG - F^2}\). Next corollary follows from Liouville theorem.

**Corollary 4.4.** 1. If \(a\) and \(b\) can be extended for all plane \(\mathbb{C}\), there exists a constant \(c \in \mathbb{C}\) such that \(a(w)b(w) = c\).

Hence it follows as direct consequence of Corollary 4.3, that if \(a(w) = b(w)\) or \(a(w) = -b(w)\) for all \(w \in \mathbb{C}\) then \(a(w) = \sqrt{c}\). That means that \((\mathbb{C}, f)\) is a spacelike plane of \(\mathbb{R}^4\).

From the Corollary 4.4, we can also construct an example of a minimal surface \((\mathbb{C}, f)\), which is a graphic with Gauss curvature \(K(f) \neq 0\). That means a set of points \(p \in S\) such that the condition \(K(p) = 0\) is not satisfied on all the plane \(\mathbb{C}\). Much more, now we are abled to prove our next result which establishes a general class of examples of entire graphic surfaces of first type such that the Gauss curvature \(K(f) \neq 0\).
THEOREM 4.5. Let $a = a(w)$ be a holomorphic function defined in all plane $\mathbb{C}$ such that $a(w) \neq 0$ for each $w \in \mathbb{C}$. Let $c = \alpha + i\beta \in \mathbb{C} \setminus \{0, 1, -1\}$ such that $\alpha^2 + \beta^2 \neq 1$, and take the holomorphic function $b(w) = \frac{c}{a(w)}$ from $\mathbb{C}$ in $\mathbb{C}$. Then the surface given by

$$f(w) = X_0 + 2\Re \int_0^w \left( a(\xi) + \frac{c}{a(\xi)} \right) d\xi,$$

is minimal surface of $\mathbb{R}^4$ which is a graphic of type $X(x, y) = (A(x, y), x, y, B(x, y))$, where the transformation of coordinates are given by $x_w = (1 + c)$ and $y_w = i(1 - c)$.

Moreover, assuming that $a(w)$ is not a constant function then, there exists a point $p \in S$ such that $K(p) \neq 0$. Hence the surface can not be contained in hyperplanes of $\mathbb{R}^4$.

**Proof.** Taking $x(u, v) = 2[(1 + \alpha)u - \beta v]$ and $y(u, v) = 2[\beta u + (\alpha - 1)v]$ we get the equation of the coordinates change, namely,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2(\alpha^2 + \beta^2 - 1)} \begin{bmatrix} \alpha - 1 & \beta \\ -\beta & 1 + \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, since $a$ and $b$ are holomorphic functions and $\alpha^2 + \beta^2 \neq 1$, we obtain that equation (17) represents a graphic minimal surfaces of first type.

Since the metric is given by $\lambda^2 = 4|1 - \bar{c}c/a|^2$ follows that $\Delta \ln \lambda \neq 0$ in point where $a_w(w) \neq 0$. Then, since $K(f) = -\frac{1}{\lambda^2} \Delta \ln \lambda$, it follows that in those points $K(f) \neq 0$. Next, by integration we can obtain the components functions $A(w) = f^0(w)$ and $B(w) = f^3(w)$, and through of the coordinate transformation given by the equation (18) we obtain the explicit representation as graphic surface.

Now we see the real spacial property of $S$. In fact, it supposes that there is a vector $v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4$ such that $\langle v, f_w \rangle = 0$. From $-v^0(a + b) + v^1(1 + ab) + iv^2(1-ab) + v^3(a - b) = 0$ we obtain

$$(v^3 - v^0)a - (v^3 + v^0)b + (v^1 + iv^2) + ab(v^1 - iv^2) = 0.$$

Defining $T = v^3 - v^0$, $S = v^3 + v^0$, $Z = v^1 + iv^2$ we obtain $(Ta + Z) + b(aZ - S) = 0$ which implies that

$$b = \frac{Ta + Z}{S - aZ} = \frac{c}{a} \quad \text{if and only if} \quad T = 0 = S \quad \text{and} \quad c = -\frac{Z}{Z}.$$

Hence, from $\frac{Z}{Z} = -c$ and $v^0 - v^3 = 0 = v^0 + v^3$ it follows that $v \notin \mathbb{R}^4$. Contradiction. □

**Example 2.** By a simple example, we take $a = e^w$ and $c = 2$. Then according to Theorem 5.7 we can take $b = \frac{c}{a} = 2e^{-w}$ and $X_0 = (0, 0, 0, 0)$, to have the parametrization

$$f(w) = 2((e^u - 2e^{-u}) \cos v, 3u, v, (e^u + 2e^{-u}) \cos v).$$
Therefore taking the coordinates transformation given by \( x = 6u, y = 2v \), we get the graphic parametrization given by

\[
X(x, y) = (2(e^{i\phi} - 2e^{-i\phi}) \cos(\frac{H}{2}), x, y, 2(e^{i\phi} + 2e^{-i\phi}) \cos(\frac{H}{2})),
\]

for which there exists points such that the Gaussian curvature is not zero. In fact, it is just to take \( \alpha = \cos(2y) \) and \( \beta = \sin(2y) \), that means, \( c \neq e^{2iy} \).

**Example 3.** In this example we use Theorem 5.7 to construct a first type of minimal graphic surfaces. We start assuming \( a(w) = e^w \) and \( b(w) = \frac{2e^{i\theta}}{a(w)} \) for \( \theta \in [0, \pi] \). Since \( |c| = |2e^{i\theta}| = 2 \), the condition \( \alpha^2 + \beta^2 \neq 1 \) is hold. Then \( W(a, b) \) is given by

\[
W(a, b) = (e^w + 2e^{i\theta}e^{-w}, 1 + 2e^{i\theta}, i(1 - 2e^{i\theta}), e^w - 2e^{i\theta}e^{-w}).
\]

Now we take the factor of integration \( \mu = 1 \), to obtain the integral representation (17) given by

\[
f(w) = 2\Re \int_0^w (e^\xi + 2e^{i\theta}e^{-\xi}, 1 + 2e^{i\theta}, i(1 - 2e^{i\theta}), e^\xi - 2e^{i\theta}e^{-\xi})d\xi,
\]

more explicitly

\[
(19) \quad f(u, v) = 2(e^u \cos v - 2e^{-u}(\cos v \cos \theta + \sin v \sin \theta), (1 + 2 \cos \theta)u - 2v \sin \theta,
\]

\[
(-1 + 2 \cos \theta)v + 2u \sin \theta, e^u \cos v + 2e^{-u}(\cos v \cos \theta + \sin v \sin \theta)).
\]

Hence making the coordinates transformation \( x_w = 1 + 2e^{i\theta} \) and \( y_w = i(1 - 2e^{i\theta}) \), we get

\[
x = 2[(1 + 2 \cos \theta)u - 2v \sin \theta] \quad \text{and} \quad y = 2[(-1 + 2 \cos \theta)v + 2u \sin \theta],
\]

Therefore the graphic minimal surface is given by \( X(x, y) = (A(x, y), x, y, B(x, y)) \), where the functions \( A(x, y), B(x, y) \) are given by the first and four components of formula (19) with

\[
u = \frac{1}{6}(-2x \sin \theta + (1 + 2 \cos \theta)y).
\]

We observe that since \( a_w = e^w \) never vanishes, all the points of the graphic surface are such that \( K(p) \neq 0 \).

**Second case.** We will focus our attention to find surfaces given by

\[
X(x, y) = (x, A(x, y), B(x, y), y) \quad \text{for all} \quad (x, y) \in \mathbb{R}^2,
\]

satisfying the equations (10). That means that \( X(\mathbb{R}^2) = S \) is a graphic minimal surface of \( \mathbb{R}^4 \), of second type.

**The question is:** There exists non flat solution for this problem?
4.2. A pointwise base for the normal bundle. Let us take the attitude matrix of \(dX\):

\[
[dX]^t = \begin{bmatrix}
1 & A_x & B_x & 0 \\
0 & A_y & B_y & 1
\end{bmatrix}.
\]

The unitary spacelike Gauss map \(\nu = \nu(x, y)\) is given by

\[
\nu(x, y) = \frac{1}{\sqrt{J^2 + (B_x)^2 + (A_x)^2}}(0, B_x, -A_x, J)
\]

for \(J = \frac{\partial(A, B)}{\partial(x, y)} = A_xB_y - A_yB_x\).

Since we can not control the functions \(\nu^i\) for \(i = 1, 2, 3\), we will work with the Weierstrass form

\[
f_w = \mu(a + b, 1 + ab, i(1 - ab), a - b)
\]

and the transformation of coordinates

\[x_w, y_w = \mu(a + b) \text{ and } y_w = \mu(a - b)\]

where \(x_wy_w^* - x_w^*y_w = 2|\mu|^2 (\overline{a} - \overline{b})\).

**Lemma 4.6.** It considers the coordinates transformation given by equations \((20)\). Then Jacobian function \(x_wy_w^* - x_w^*y_w = 2|\mu|^2 (\overline{a} - \overline{b})\) does not vanish in a domain \(U \subset M\) if and only if for each \(w \in U\)

\[a(w) \neq 0 \neq b(w) \text{ and } \Im\left(\frac{a(w)}{b(w)}\right) \neq 0.
\]

A maximal extension of holomorphic functions \(a, b\) is restricted by the conditions \((21)\) and by \(|1 - \overline{ab}| \neq 0\).

**Proof.** First we see that \(a(w) \neq 0 \neq b(w)\) is a necessary condition. Moreover, for each \(w \in U\),

\[-2i\Im\left(\frac{a(w)}{b(w)}\right) = \frac{\overline{a(w)} - a(w)}{b(w)} = \frac{\overline{a(w)b(w)} - a(w)b(w)}{b(w)b(w)}.
\]

Hence, since the Jacobian function does not vanish, it follows that \(\Im\left(\frac{a(w)}{b(w)}\right) \neq 0\). The conversely follows immediately. \(\square\)

From Lemma 4.6 and from Picard Theorem, it follows the next corollary.

**Corollary 4.7.** It assumes that the holomorphic functions \(a(w)\) and \(b(w)\) can be extended for all plane \(\mathbb{C}\). Then there exists a constant \(c \in \mathbb{C} \setminus \{0, 1, -1\}\) such that \(b(w) = ca(w)\).
Moreover, as consequence, if \( f_w \) is such that \( f_w = \mu(a(1+c), 1+ca^2, i(1-ca^2), a(1-c)) \) then
\[
x(w) = 2\Re \left( (1 + c) \int_0^w \mu(\xi)a(\xi)d\xi \right) \quad \text{and} \quad y(w) = 2\Re \left( (1 - c) \int_0^w \mu(\xi)a(\xi)d\xi \right).
\]

Taking \( P(w) + iQ(w) = \int_0^w \mu(\xi)a(\xi)d\xi \) and \( c = \alpha + i\beta \) we obtain
\[
x(u,v) = 2[(1 + \alpha)P(w) - \beta Q(w)] \quad \text{and} \quad y(u,v) = 2[(1 - \alpha)P(w) + \beta Q(w)].
\]

**Remark 1.** We observe that Corollary 4.7 has a weakness because while in Theorem 5.7 the equation (18) gives us the inversion function which is linear, and which we can use to construct the graphic over all the complex plane \( \mathbb{C} \), Corollary 4.7 cannot guarantee that we have a graphic over all complex plane, since it could exist ramifications. For instance, taking \( a(w) = e^w \) and \( \mu = 1 \), we obtain \( P(u,v) = e^u \cos v \) and \( Q(u,v) = e^u \sin v \). So, \( x(u,v) = 2[(1 + \alpha)e^u \cos v - \beta e^u \sin v] \) and \( y(u,v) = 2[(1 - \alpha)e^u \cos v + \beta e^u \sin v] \), which are periodic functions in the variable \( v \).

In the next theorem we answer the question if there exist non-flat solution which are entire graphic surfaces of second type. In fact, we argue that if \( a = a(w) \) is a given holomorphic function defined in all \( \mathbb{C} \) and such that \( a(w) \neq 0 \), then we can take the holomorphic function \( \mu(w) = \frac{1}{a(w)} \) and take also \( f_w = \mu W(a(w), ca(w)) \) with constant \( c \in \mathbb{C} \setminus \{0,1,-1\} \). Then we will show that in this case, it can exist point in the surface such that its Gauss curvature is not zero.

**Theorem 4.8.** Let \( a = a(w) \) be a holomorphic function defined in all plane \( \mathbb{C} \) such that \( a(w) \neq 0 \) for each \( w \in \mathbb{C} \). For \( c = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R} \) we take \( b(w) = ca(w) \) and \( \mu(w) = \frac{1}{a(w)} \).

Then the surfaces given by
\[
f(w) = X_0 + 2\Re \int_0^w \left( 1 + c, \frac{1}{a(\xi)} + ca(\xi), i \left( \frac{1}{a(\xi)} - ca(\xi) \right), 1 - c \right) d\xi,
\]
are minimal surfaces of \( \mathbb{R}^4 \), which represent graphic of type \( X(x,y) = (x, A(x,y), B(x,y), y) \), where the coordinates transformation is given by \( x_w = (1 + c) \) and \( y_w = (1 - c) \).

Moreover, in this case, the Gauss curvature \( K(f)(w) = 0 \) if and only if \( a_w(w) = 0 \). Therefore, assuming that \( a = a(w) \) is not a constant function, there exists \( p \in S \) such that \( K(p) \neq 0 \). Again, there is not a hyperplane containing the surface \( S \).

**Proof.** By integration we obtain \( x = 2\Re((1 + \alpha) + i\beta)(u + iv)) = 2[(1 + \alpha)u - \beta v] \) and \( y = 2[(1 - \alpha)u + \beta v] \).

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = \frac{1}{4\beta} \begin{bmatrix}
  \beta & \beta \\
  \alpha - 1 & \alpha + 1
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}.
\]
Since $a, b$ and $\mu$ are holomorphic functions, formula (22) in the $(x, y)$ coordinates, represents a graphic minimal surface of second type. Moreover, since the Gauss curvature is given by $K(f) = -\frac{1}{\lambda^2} \Delta \ln \lambda$ where $\lambda^2 = 4|a|^2 - c|2$, it follows that $\Delta \ln \lambda \neq 0$ in point where $a_w(w) \neq 0$. Hence in those points $K(f) \neq 0$.

Next, by integration we can obtain the components functions $A(w) = f^1(w)$ and $B(w) = f^2(w)$, and through of the coordinate transformation we obtain the explicit representation as graphic surface.

Finally we note that it is needed to assume $c \not\in \mathbb{R}$, since we can not have $x_w y_w - x y_w = 0$, and it also is impossible to obtain a timelike vector $v \in \mathbb{R}^4$ such that $\langle v, f_w \rangle = 0$. □

**Example 4.** In this example we use Theorem 4.8 to construct second type of minimal graphic surfaces. Let $a = e^w$ and $b = e^{i\theta}a$ for $\theta \in [0, \pi]$. Then the expression of $W(a, b)$ is

$$W(a, b) = ((1 + e^{i\theta})e^w, 1 + e^{i\theta}e^{2w}, i(1 - e^{i\theta}e^{2w}), (1 - e^{i\theta})e^w),$$

and we take the factor of integration $\mu(w) = e^{-w}$. Therefore, the integral representation (22) is given by

$$f(w) = 2\Re \int_0^w (1 + e^{i\theta}, e^{-\xi} + e^{i\theta}e^{\xi}, i(e^{-\xi} - e^{i\theta}e^{\xi}), 1 - e^{i\theta})d\xi,$$

or more explicitly

$$f(u, v) = 2((1 + \cos \theta)u - v \sin \theta, -e^{-u} \cos v + e^u(\cos v \cos \theta - \sin v \sin \theta),$$

$$-e^{-u} \sin v + e^u(\sin v \cos \theta + \cos v \sin \theta), (1 - \cos \theta)u + v \sin \theta)).$$

Now making the coordinates transformation $x_w = 1 + e^{i\theta}$ and $y_w = 1 - e^{i\theta}$, we get

$$x = 2[(1 + \cos \theta)u - v \sin \theta] \quad \text{and} \quad y = 2[(1 - \cos \theta)u + v \sin \theta],$$

and hence the graphic minimal surface is given by $X(x, y) = (x, A(x, y), B(x, y), y)$ where the functions $A(x, y)$ and $B(x, y)$ are given by the second and third component of formula (24) with

$$u = \frac{x + y}{4} \quad \text{and} \quad v = \frac{1}{4 \sin \theta}[-(1 + \cos \theta)x + (1 + \cos \theta)y].$$

Finally we observe that since $a_w = e^w$ never vanishes, for all the points of the surface one gets that $K(p) \neq 0$. 

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5. The construction of the conjugated surface \((M,Y)\)

We dedicate this last section for looking the explicit expression of the conjugated surface to a minimal spacelike surface \((M,X)\) of \(\mathbb{R}^4_1\), using the Weierstrass notation. We start defining an operator on tangent bundle \(TS\) to a surface, as follows.

**Definition 5.1.** Let \((M,X)\) be a spacelike surface with line element \(ds^2(X) = Edu^2 + 2Fdx dy + Gdy^2\), and \(TS\) be its tangent bundle, where, pointwise, \(\{X_x(p), X_y(p)\}\) is a base of \(T_pS\). Let \(J : TS \rightarrow TS\) be the function given by

\[
J(V) = \frac{1}{\sqrt{EG - F^2}} \left( \langle X_x, V \rangle X_y - \langle X_y, V \rangle X_x \right).
\]

**Proposition 5.2.** Let \(J : TS \rightarrow TS\) be the function given by the equation (21). Then \(\forall V \in TS\), the following equations are satisfied:

\[
\langle V, J(V) \rangle = 0, \quad \langle J(V), J(V) \rangle = \langle V, V \rangle \quad \text{and} \quad J(J(V)) = -V.
\]

**Proof.** The first equation follows from \(\sqrt{EG - F^2} \langle V, J(V) \rangle = \langle X_x, V \rangle \langle X_y, V \rangle - \langle X_y, V \rangle \langle X_x, V \rangle = 0\). For getting second equation we take the values of \(J\) in the basis, namely,

\[
J(X_x) = \frac{1}{\sqrt{EG - F^2}} (EX_y - FX_x) \quad \text{and} \quad J(X_y) = \frac{1}{\sqrt{EG - F^2}} (FX_y - GX_x).
\]

It follows

\[
\langle J(X_x), J(X_x) \rangle = E, \quad \langle J(X_y), J(X_y) \rangle = G \quad \text{and} \quad \langle J(X_x), J(X_y) \rangle = F.
\]

Now we note that from the pointwise bi-linearity of \(\langle, \rangle\), it follows the pointwise linearity of \(J\). Therefore if \(V = aX_x + bX_y\) then it follows the second equation of the proposition.

For the third equation follows directly from the linearity and from the following facts:

\[
J(J(X_x)) = -X_x \quad \text{and} \quad J(J(X_y)) = -X_y.
\]

We observe that if \(S = (M,X)\) be a spacelike surface of \(\mathbb{R}^4_1\), the vector 1-form associated to \(S\) is given by \(\beta = X_x dx + X_y dy\). Therefore, by definition \(J(\beta)\) is 1-form given by

\[
J(\beta) = J(X_x)dx + J(X_y)dy.
\]

Next we related the operator \(J\) with the special normal frame \(\{\tau, \nu\}\) in \(\mathbb{R}^4_1\).
Let \( \vec{u} = \mathfrak{X}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \) be the exterior product in \( \mathbb{R}^4_1 \) of a set of 3 vectors. By definition, since \( \Omega(\mathbb{R}^4_1) = (-dx^0) \wedge dx^1 \wedge dx^2 \wedge dx^3 \) is the volume form, then \( \vec{u} = \mathfrak{X}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \) is defined by
\[
\langle \vec{u}, \vec{w} \rangle = \Omega(\mathbb{R}^4_1)(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}), \quad \forall \vec{w} \in \mathbb{R}^4_1.
\]
Then the \( J \) operator is equivalent to \( J(V) = \mathfrak{X}(\tau, \nu, V) \).

**Theorem 5.3.** Let \( S = (M, X) \) be a spacelike surface of \( \mathbb{R}^4_1 \) and let the vector 1-form associated to \( S \) be given by \( \beta = X_x dx + X_y dy \). Then
\[
(28) \quad J(\beta) = -\frac{F dx - G dy}{\sqrt{EG - F^2}} X_x + \frac{E dx + F dy}{\sqrt{EG - F^2}} X_y.
\]
The 1-form \( J(\beta) \) is closed if and only if \( (M, X) \) is a minimal spacelike surface.

**Proof.** The equation (28) follows from equations (26) and (27).

For the second statement, we use the representation of the operator \( J \) as a exterior product, to obtain
\[
J(\beta) = \mathfrak{X}(\tau, \nu, X_x dx + X_y dy) = \tau \times \nu \times \beta.
\]
Now, since \( d\beta = 0 \), we get the exterior derivative \( dJ(\beta) = ((d\tau) \times \nu \times \beta) + (\tau \times (d\nu) \times \beta) \).

Next we will calculate explicitly \( dJ(\beta) \). For that we use \( d\tau = \tau_x dx + \tau_y dy \), \( d\nu = \nu_x dx + \nu_y dy \), the Weingarten formulas (2.8), (6) and the anti-commutative properties of the exterior product in \( \mathbb{R}^4_1 \) and of the exterior product of 1-forms, to obtain
\[
d(J\beta) = (h_1^1 + h_2^3)(X_x \times \nu \times X_y) dx \wedge dy + (n_1^1 + n_2^3)(\tau \times X_x \times X_y) dx \wedge dy.
\]
Since \( X_x \times \nu \times X_y = -\sqrt{EG - F^2} \tau \) and \( \tau \times X_x \times X_y = \sqrt{EG - F^2} \nu \) it follows
\[
dJ(\beta) = -2H_X \sqrt{EG - F^2} dx \wedge dy.
\]
Hence it follows from equation (29) that \( dJ(\beta) = 0 \) if and only if \( (M, X) \) is minimal. \( \square \)

Theorem 5.3 allows us to establish the next corollary which shows the explicit expression of the minimal conjugate spacelike surface \( (M, Y) \) in \( \mathbb{R}^4_1 \). It comes from the fact that since \( J(\beta) \) is a closed 1-form in a connected simply-connected open subset of \( \mathbb{C} \) then it is exact.

**Corollary 5.4.** Let \( M \) be a connected and simple connected open subset of the plane \( \mathbb{C} \), and let \( (M, X) \) be a solution of the graphic equations. The integral representation equation can be extended to \( Z = X + iY \in \mathbb{C}^4 \) by
\[
Z(x, y) = Z(x_0, y_0) + \int_{x_0}^{x} \beta + iJ(\beta),
\]
(30)
where

\[ (31) \quad Y(x, y) = Y(x_0, y_0) + \int_{x_0}^{x} \frac{-Fdx - Gdy}{\sqrt{EG - F^2}} X_x + \frac{Edx + Fdy}{\sqrt{EG - F^2}} X_y, \]

give us the conjugated minimal spacelike surface \((M, Y)\) of \(\mathbb{R}^4_1\).

**Proof.** Since \(J(dY) = J(J(dX) = -dX = -(X_x dx + X_y dy)\) is a closed vector 1-form, from Theorem 5.3 it follows that \(H_Y(p) = 0\) for each \(p \in M\). □

**Example 5.** Let \(X(x, y) = (0, x \cos y, x \sin y, y)\) be a parametric Helicoid of \(\mathbb{E}^3\). The conjugated minimal spacelike surface, given by equation (31) with \(Y(0, 0) = (0, 0, 1, 0)\), is the Catenoid given in coordinates by

\[ Y(x, y) = (0, -\sqrt{1 + x^2} \sin y, \sqrt{1 + x^2} \cos y, \ln(x + \sqrt{1 + x^2})). \]

In fact, from \(X_x = (0, \cos y, \sin y, 0)\) and \(X_y = (0, -x \sin y, x \cos y, 1)\) it follows that \(E = 1, F = 0\) and \(G = 1 + x^2\). Now from the line integral equation (31) we obtain

\[ dY = \frac{1}{\sqrt{1 + x^2}}(0, -x \sin y, x \cos y, 1) dx - \sqrt{1 + x^2}(0, \cos y, \sin y, 0) dy. \]

Hence by integrating \(Y_x = \frac{1}{\sqrt{1 + x^2}}(0, -x \sin y, x \cos y, 1)\) and \(Y_y = -\sqrt{1 + x^2}(0, \cos y, \sin y, 0)\), we get the Catenoid surface \((\mathbb{R}^2, Y(x, y))\).

Moreover, if \(x \geq 0\) we have the part that correspond to \(Y^3 \geq 0\) and, if \(x \leq 0\) we have the part that correspond to \(Y^3 \leq 0\). Moreover, both surfaces \((\mathbb{R}^2, X)\) and its conjugated \((\mathbb{R}^2, Y(x, y))\) are ramified.

Finally, if we make \(x = \sinh u\) and \(y = v\), we obtain

\[ \tilde{X}(u, v) = (0, \sinh u \cos v, \sinh u \sin v, v) \quad \text{and} \quad \tilde{Y}(u, v) = (0, -\cosh u \sin v, \cosh u \cos v, u), \]

in the isothermic coordinates \((u, v)\). As it is expected it follows \(\tilde{X}_u = -\tilde{Y}_v\) and \(\tilde{X}_v = \tilde{Y}_u\).

### 5.1. Cauchy-Riemann equations over \((M, X)\)

In this subsection we identify the Cauchy-Riemann type equations over the surface \((M, X)\) when the parameters are not isothermic, and we identify the needed conditions to extend in continua way any local solution of those equations.

For starting, we observe that if we have a sub-surface \(f(U) \subset X(M)\) with isothermic parameters \(w = (u, v) \in U\) and that \(X(x, y) = f(u(x, y), v(u, y))\), then

\[ \frac{\partial X}{\partial x} = \frac{\partial u}{\partial x} f_u + \frac{\partial v}{\partial x} f_v \quad \text{and} \quad \frac{\partial X}{\partial y} = \frac{\partial u}{\partial y} f_u + \frac{\partial v}{\partial y} f_v. \]
**Lemma 5.5.** For each local solution of the equations

\[ \frac{\partial w}{\partial y} = \alpha(x, y) \frac{\partial w}{\partial x} \]

where \( \alpha(x, y) = \frac{F(x, y) + i\sqrt{E(x, y)G(x, y) - F^2(x, y)}}{E(x, y)} \),

in a neighborhood \( U \subset M \) of a point \( p \in M \), there exists a parametric isothermic sub-surface \((U, f)\) of \((M, X)\) such that \( X(x, y) = f(u(x, y), v(u, y)) \). Moreover, \( \alpha\overline{\alpha} = \frac{G}{E} \).

**Proof.** Let \( W = \sqrt{EG - F^2} \) be the area function in coordinates \( z = x + iy \in U \). Applying the operator \( J \) and since \( J(f_u) = f_v, \ J(f_v) = -f_u \), it follows

\[ J(X_x) = u_x J(f_u) + v_x J(f_v) = u_x f_v - v_x f_u. \]

Hence by equation (26) one gets

\[ u_x f_v - v_x f_u = \frac{E}{W} (u_y f_u + v_y f_v) - \frac{F}{W} (u_x f_u + v_x f_v). \]

From this last equations, we obtain the following equations with matrix representation

\[ \begin{bmatrix} u_y \\ v_y \end{bmatrix} = \begin{bmatrix} F/E & -W/E \\ W/E & F/E \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} \]

and

\[ \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} E \\ G \end{bmatrix} \begin{bmatrix} F/E & W/E \\ -W/E & F/E \end{bmatrix} \begin{bmatrix} u_y \\ v_y \end{bmatrix}. \]

Now we observe that the square matrices of order \( 2 \times 2 \) of these equations are the matrix representation of a complex number. Therefore we can write

\[ u_y + iv_y = \frac{F + iW}{E} (u_x + iv_x), \]

that is equation (32). \( \square \)

Our first corollary we describe the equations, which we identify as the Cauchy-Riemann type equation for \((M, X)\), namely equation (34):

**Corollary 5.6.** If a function \( h = \varphi + i\psi : S \rightarrow \mathbb{C} \) is holomorphic over the Riemann surface \( S = X(M) \) then

\[ \begin{bmatrix} \varphi_y \\ \psi_y \end{bmatrix} = \begin{bmatrix} F/E & -W/E \\ W/E & F/E \end{bmatrix} \begin{bmatrix} \varphi_x \\ \psi_x \end{bmatrix} \]

or

\[ \frac{\partial h}{\partial y} = \alpha \frac{\partial h}{\partial x}. \]

**Proof.** Since in an isothermic neighborhood \((U, \tilde{h})\) the function \( \tilde{h}(u, v) \) is holomorphic in the sense of complex variable if and only if \( h(x, y) \) is holomorphic over \( S \), we have

\[ \frac{\partial h}{\partial x} = \frac{\partial \tilde{h}}{\partial u} (u_x + iv_x) \quad \text{and} \quad \frac{\partial h}{\partial y} = \frac{\partial \tilde{h}}{\partial u} (u_y + iv_y), \]
because $i\tilde{h}_u = \tilde{h}_v$ holds for $\mathbb{C}$-holomorphic functions. Therefore $h_y = \alpha h_x$ follows from the definition of the function $\alpha(x,y)$.

Next we are interested in relating the isothermic neighborhood $(U, \tilde{f})$ with the Weierstrass datas $a(w)$ and $b(w)$ for graphic spacelike surfaces in $\mathbb{R}^4_1$.

In fact, fixing the semi-rigid referential associated to $(M,X)$ given by

$$M_0 = \{l_0(b(p)), e_1(p), e_2(p), l_3(a(p))\}$$

where

$$e_1(p) = \frac{1}{\sqrt{E}} \frac{\partial X}{\partial x}$$

and

$$e_2 = J(e_1) = \frac{1}{\sqrt{E}} J(X_x),$$

we obtain the next result.

**Proposition 5.7.** Let $S = (M,X)$ be a solution of the graphic equation for $\mathbb{R}^4_1$ and $(U,f)$ be a given locally isothermic sub-surface of $S$. Let $r(u,v)$ be a real-valued function and $\mathcal{M}(\vartheta) = \{l_0(b), e_1, e_2, l_3(a)\}_{u,v}$ be the semi-rigid referential associated to $f_w(w) = \mu(w)W(a(w), b(w))$ with $f_w(w) = r(w)(\hat{e}_1(w) - i\hat{e}_2(w))$. Then the following relation is hold:

$$\hat{e}_1(w) - i\hat{e}_2(w) = (\cos \vartheta e_1 + \sin \vartheta e_2) - i(-\sin \vartheta e_1 + \cos \vartheta e_2)) = e^{i\vartheta}(e_1 - ie_2).$$

From Proposition 5.7 it follows that if the coordinates $(M,X), (U,f)$ and $(\tilde{U}, \tilde{f})$ around a point $p \in f(U) \cap \tilde{f}(\tilde{U})$, are related by the equations

$$X(x,y) = f(u(x,y), v(x,y)) = f \circ \Phi(x,y), \quad X(x,y) = \tilde{f}(\tilde{u}(x,y), \tilde{v}(x,y)) = \tilde{f} \circ \tilde{\Phi}(x,y),$$

then the transition function are given by

$$f \circ \Phi(x,y) = \tilde{f} \circ \tilde{\Phi} \quad \text{therefore} \quad \Psi = \tilde{\Phi} \circ \Phi^{-1} = \tilde{f}^{-1} \circ f.\quad (35)$$

Now, applying the Proposition 4.8 to $f_w$ and $\tilde{f}_w$ we obtain:

$$\frac{1}{r} f_w = e^{i\phi}(\hat{e}_1 - i\hat{e}_2) \quad \text{with} \quad \frac{1}{r} \tilde{f}_w = e^{i\tilde{\phi}}(\tilde{e}_1 - i\tilde{e}_2)$$

which imply that the set of angle functions, are related each other by the equation:

$$\hat{\phi}(u,v) - \tilde{\phi} \circ \Psi(u,v) = \tilde{\theta}(u,v) - \hat{\theta} \circ \Psi(u,v).\quad (36)$$

Now we have the following facts, which come from equation (36).

1. If two holomorphic functions agree each other along a Jordan arc, then they agree each other along all connected component of this arc.

From (1) we obtain.
(2) If \((U, f)\) and \((\tilde{U}, \tilde{f})\) agree each other along an Jordan arc in \(S\), they agree each other along the open subset \(f(U) \cap \tilde{f}(\tilde{U})\).

(3) The overlapping or transition map between two isothermic coordinates system for a spacelike surface of \(\mathbb{R}^4\) are holomorphic function in sense of complex analysis.

(4) Each holomorphic function \(h : U \subset \mathbb{C} \rightarrow V \subset \mathbb{C}\) can be seen as a pointwise \(\mathbb{C}\)-linear transformation \(dh_{z_0} : T_{z_0}\mathbb{C} \rightarrow T_{h(z_0)}\mathbb{C}\) that preserve oriented angles.

**Lemma 5.8.** The angle function \(\tilde{\vartheta} - \vartheta\) determines the transition map of \((U, f)\) and \((\tilde{U}, \tilde{f})\), for two conformal parametrization of the neighborhood \(f(U) \cap \tilde{f}(\tilde{U}) \subset S\) around \(p \in S\).

From Lemma 5.8 it follows the next extension’s result of the local solutions.

**Proposition 5.9.** Let \(w, \tilde{w}\) two local solutions of equation (32), around a point \(p \in S\), with \(w_y = \alpha w_x\) and \(\tilde{w}_y = \alpha \tilde{w}_x\). If \(w_x = \tilde{w}_x\) then \(w_y = \tilde{w}_y\).

Therefore, all local solution of the equation (32) can be continuously extended whenever \(E(x, y) > 0\) and \(\sqrt{EG - F^2}(x, y) > 0\).

**Proof.** The conclusions are immediated. \(\square\)

Now we prove that the solutions of equations (33) are Nitsche type equations (equation (8), page 23 of [7]).

**Theorem 5.10.** The solution for equations (33) are given by Nitsche type equations (28), that is

\[
\begin{align*}
  u &= u(x, y) = x + \int_{z_0}^{z} \frac{Edx + Fdy}{W} \\
  v &= v(x, y) = y + \int_{z_0}^{z} \frac{Fdx + Gdy}{W}.
\end{align*}
\]

Moreover, from equations (37), it is possible to obtain global isothermic coordinates \((U, f)\) for the surface \(S = X(M)\).

**Proof.** In fact, since

\[
\frac{\partial u}{\partial x} = \frac{W + E}{W}, \quad \frac{\partial u}{\partial y} = \frac{F}{W}, \quad \frac{\partial v}{\partial x} = \frac{F}{W}, \quad \frac{\partial v}{\partial y} = \frac{W + G}{W}
\]

the matrix equation (33) is satisfied. In fact, remembering that \(W^2 + F^2 = EG\), we obtain

\[
\begin{bmatrix}
  F/W \\
  (W + G)/W
\end{bmatrix} = \begin{bmatrix}
  W/E & -W/E \\
  F/E & F/W
\end{bmatrix} \begin{bmatrix}
  (E + W)/W \\
  F/W
\end{bmatrix}.
\]

\(\square\)
We note that equations (37) are related to the equation describing the parametrization of conjugated minimal spacelike surface \((M, Y)\) given by \(J(\beta)\) or more explicitly by formula (31).

Finally we have the following corollary for equations of minimal graphic surfaces in \(\mathbb{R}^4_1\).

**Corollary 5.11.** If \(S = (\mathbb{R}^2, X)\) is a solution of the minimal graphic equation (10) then, for all \(p \in S\), the functions \(a(w)\) and \(b(w)\) satisfy either \(b(p) = ca(p)\) with \(c \notin \{-1, 1\}\) or \(a(p)b(p) = c\) with \(\Im(c) \neq 0\) for some constant \(c \in \mathbb{C}\).

The Bernstein Theorem and the Calabi Theorem follows from that \(c \neq 1\) and \(c \neq -1\) and for the second type of surfaces from \(\Im(c) \neq 0\).

Finally, if as a submanifold of the topological vector space \(\mathbb{R}^4\) there exists \(S = (\mathbb{R}^2, X)\) such that with the induced metric of \(\mathbb{R}^4_1\), is a spacelike graphic solution in connected and simply connected open subset \(M \subset \mathbb{C}\), with the condition that in some point \(p \in S\) the following statement fails:

"either \(b(p) = ca(p)\) with \(c \notin \{-1, 1\}\) or \(a(p)b(p) = c\) with \(\Im(c) \neq 0\) and for some constant \(c \in \mathbb{C}\),"

then the points \(X(x, y)\) where \(EG - F^2 = 0\), are points such that the tangent planes of \(X(\mathbb{R}^2)\) are tangent to the lightcone of \(\mathbb{R}^4_1\).

We observe that the points where happen the last condition of Corollary 5.11 are known as singularities lightlike as defined in [5].

### 6. A Particular Family of Minimal Surfaces of \(\mathbb{R}^4_1\)

In this section we construct examples of minimal spacelike surfaces in \(\mathbb{R}^4_1\) close related to surfaces in \(\mathbb{E}^3\) and \(\mathbb{L}^3\).

From the representation \(f_w = \mu(a + b, 1 + ab, 1 - ab, a - b)\) for holomorphic functions \(\mu, a, b\) from \(M\) into \(\mathbb{C}\), for a connected and simply connected open subset \(M\) of the complex plane, we take the relation

\[
b = ae^{i\theta}
\]

for a parameter \(\theta \in \mathbb{R}\).

**Definition 6.1.** A \(\theta\)-family is a set of minimal surfaces linking each other by a parameter \(\theta \in \mathbb{R}\) given by the following equation

\[
(38) \quad F(\theta; w) = P_0 + 2\Re \int_{w_0}^w \mu(\xi)[(1 + e^{i\theta})a(\xi), 1 + e^{i\theta}a^2(\xi), i(1 - e^{i\theta}a^2(\xi)), (1 + e^{i\theta})a(\xi)]d\xi,
\]

on a connected and simply connected domain \(M \subset \mathbb{C}\).
When \( \theta = 0 \) we say that the surface of \( \mathbb{L}^3 \), given by \( X(w) = F(0; w) \), is the initial surface of the family, and when \( \theta = \pi \) we say that the surface of \( \mathbb{E}^3 \), given by \( Y(w) = F(\pi; w) \), is the associated surface of the initial surface of the family.

**Lemma 6.2.** For a \( \theta \)-family \( (M, F(\theta; \ldots)) \) of minimal spacelike isothermic parametric surfaces in \( \mathbb{R}^4_1 \) the equations that related the initial surface \( (M, X) \) and the associated surface \( (M, Y) \), are given by:

\[
\frac{\partial Y^3}{\partial w} = \frac{\partial X^0}{\partial w}, \quad \frac{\partial Y^1}{\partial w} = -i \frac{\partial X^2}{\partial w}, \quad \frac{\partial Y^2}{\partial w} = i \frac{\partial X^1}{\partial w}.
\]

**Proof.** The equations of lemma follows from \( X_w = \mu(2a, 1 + a^2, i(1 - a^2), 0) \) and \( Y_w = \mu(0, 1 - a^2, i(1 + a^2), 2a) \). \( \square \)

Now we construct an example for these equations:

**Example 6.** Let \( (M, X) \) be the minimal spacelike surface of \( \mathbb{L}^3 \) given, in isothermic parameters, by

\[
X(u, v) = (u, \sinh u \cos v, \sinh u \sin v, 0).
\]

Since \( X_u = (1, \cosh u \cos v, \cosh u \sin v, 0) \) and \( X_v = (0, -\sinh u \sin v, \sinh u \cos v, 0) \) we obtain \( \lambda^2(X) = \sinh^2 u \). We assume that \( (u, v) \in M \) for \( u > 0 \).

Therefore, it follows \( X_w = \frac{1}{2}(1, \cosh w, -i \sinh w, 0) \). To obtain the associated surface we find the functions \( a(w) \) and \( \mu(w) \). In fact, since

\[
2\mu a = \frac{1}{2}, \quad \mu(1 + a^2) = \frac{\cosh w}{2}, \quad i\mu(1 - a^2) = -\frac{\sinh w}{2},
\]

it follows that \( 4\mu(w) = e^{-w} \) and \( a(w) = e^{w} \).

For obtaining the associated surface \( (M, Y) \), we use \( Y_w = \mu(0, 1 - a^2, i(1 + a^2), 2a) \), and so the surface is such \( Y_w = \frac{1}{2}(0, -\sinh w, i \cosh w, 1) \). Hence the holomorphic integral curve is given by

\[
\tilde{Y}(w) = \frac{1}{2}(0, -\cosh w, i \sinh w, w).
\]

Thus, the real part of \( \tilde{Y} \) gives us a Catenoid of \( \mathbb{E}^3 \) parametrized by

\[
Y(u, v) = (0, -\cosh u \cos v, -\cosh u \sin v, u) \quad \text{com} \quad \lambda^2(Y) = \cosh^2 u.
\]

Now, we looking for the non parametric representation (first type graphic equations) for these two associated surfaces.

For \( (M, X) \) and the representation \( P(x, y) = (A(x, y), x, y, 0) \) and for \( (M, Y) \) and the representation \( Q(p, q) = (0, p, q, B(p, q)) \) we have:
Taking $x = \sinh u \cos v \ e \ y = \sinh u \sin v$ therefore $\sinh u = \sqrt{x^2 + y^2}$ for $(M, X)$ we obtain the function
$$A(x, y) = \ln(\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 + 1}).$$

Taking $p = -\cosh u \cos v \ e \ q = -\cosh u \sin v$ therefore $\cosh u = \sqrt{p^2 + q^2}$ for $(M, Y)$ we obtain the function
$$B(p, q) = \ln(\sqrt{p^2 + q^2} + \sqrt{p^2 + q^2 - 1}).$$

Example 6 and equations linking the initial surface $(M, X)$ and its associated surface $(M, Y)$ in the $\theta$-family, suggest the following result.

**Lemma 6.3.** For the associated $\theta$-family surfaces given by
$$X(w) = (A(x(w), y(w)), x(w), y(w), 0) \quad \text{and} \quad Y(w) = (0, p(w), q(w), B(p(w), q(w))),$$
the Jacobian functions for the transformation of coordinates are related by
$$\frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial (p, q)}{\partial (u, v)}. \quad \Box$$

**Proof.** From equations of associated surfaces (39) it follows that $p_w = -iy_w$ and $q_w = ix_w$. Then $p_w q_w - p_w q_w = x_w y - x_w y w$, which implies the relation $\frac{i}{2}[p_u q_v - p_v q_u] = \frac{1}{2}[y_u x_v - y_v x_u]$. \quad \Box

Now, from Lemma 6.3 and from our version of the Nitsche equations for transformation of coordinates (37), we obtain the following result.

**Theorem 6.4.** The $\theta$-family transports minimal first type graphic solutions $P(x, y) = (A(x, y), x, y, 0)$ to minimal associated graphic solutions $Q(p, q) = (0, p, q, B(p, q))$ preserving the domain $\text{dom}(A) = \text{dom}(B) = M$.

If $M = \mathbb{C}$ then $P(\mathbb{C})$ and $Q(\mathbb{C})$ are spacelike planes of $\mathbb{R}^4_1$.

We can say that “the Bernstein theorem holds if and only if the Calabi theorem holds”.

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