Universidade do Estado do Rio de Janeiro
Instituto de Física

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Poincaré Sections of Hamiltonian Systems

E.S. Cheb-Terrab\[1] and H.P. de Oliveira\[1]

Abstract

A set of Maple R.3 software routines, for plotting 2D/3D projections of Poincaré surfaces-of-section of Hamiltonian dynamical systems, is presented. The package consists of a plotting-command plus a set of facility-commands for a quick setup of the Hamilton equations of motion, initial conditions for numerical experiments, and for the zooming of plots.

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\[1\]Departamento de Física Teórica, IF-UERJ. URL: [http://dft.if.uerj.br](http://dft.if.uerj.br)
E-mail: terrab@vmesa.uerj.br,
E-mail: henrique@vmesa.uerj.br
PROGRAM SUMMARY

Title of the software package: Poincaré.

Catalogue number: (supplied by Elsevier)

Software obtainable from: CPC Program Library, Queen’s University of Belfast, N. Ireland (see application form in this issue)

Licensing provisions: none

Operating systems under which the program has been tested: UNIX systems, Macintosh, DOS (AT 386, 486 and Pentium based) systems, DEC VMS, IBM CMS.

Programming language used: Maple V Release 3.

Memory required to execute with typical data: 8 Megabytes.

No. of lines in distributed program, including On-Line Help, etc.: 1381.

Keywords: Hamiltonian systems, surface-of-section method, symbolic computing.

Nature of mathematical problem
Computation and plotting of 2D/3D projections of Poincaré surfaces-of-section of Hamiltonian systems.

Methods of solution
A 4th order Runge-Kutta method with optional stepsize and number of iterations is used. However, it is possible to indicate any user-method to be used in the integration scheme.

Restrictions concerning the complexity of the problem
Besides the inherent restrictions of the Runge-Kutta method, this first version of the package does not makes use of adaptative stepsize control.

Typical running time
It depends strongly on the surface-of-section to be plotted. With a Pentium-90 PC (32 Mb. RAM), fast plots usually take from a few seconds to a few minutes. On the other extreme, in an example considered in this paper, a surface-of-section with 10,000 points and an energy threshold $\approx 10^{-8}$ took 35 minutes.

Unusual features of the program
This package provides easy-to-use software tools for plotings 2D/3D projections of Poincaré surfaces-of-section of Hamiltonians systems. The speed at which the returned plots are calculated is adjustable, in connection with their accuracy. This feature permits alternatively searching for, say, “first order” phenomena at remarkable high speed, or, say, “high order” detailed 2D/3D projections displaying “islands” and the inner structure of a surface-of-section, as desired. The 2D intersection plane over which the surface-of-section is plotted can be any one of the coordinate planes of the phase space, and can be shifted in the positive and negative directions. The package also provides routines for setting large sets of initial conditions for numerical experiments in seconds. The implementation in a symbolic computing environment allows for combined symbolic/numerical studies.
Introduction

The Poincaré surface-of-section method has become a popular technique for analyzing weakly perturbed Hamiltonian systems\(^1\). People working in correlated areas usually prepare numerical routines, for instance in Fortran or C, and use the computational environment to find the solution curves of the model, and to build the corresponding surfaces-of-section plots. In parallel, symbolic systems now present a satisfactory performance when computing with hardware floating-point numbers, while offering a comfortable computational environment for realizing symbolic mathematical studies and manipulating plots.

Related to these points, this paper presents a set of software routines, implemented in MapleV R.3, for plotting Poincaré surfaces-of-section, exploiting the MapleV routines for working with hardware floats and with plots. In preparing such a package, our intention was:

- to provide easy-to-use software tools for fast plottings of Poinc àre sections of Hamiltonian dynamical systems;
- to make these tools flexible enough to feature sections over any possible 2D/3D submanifold of the corresponding phase space (including the time as a possible variable);
- to allow for combined symbolic/numerical studies by implementing such software tools in a symbolic computing environment.

The exposition is organized as follows. In Sec.\(^1\), the basic ideas concerning the surface-of-section method are briefly reviewed. In Sec.\(^2\), a summary of the package’s commands is presented, followed by a detailed description\(^3\) of its most relevant commands, mainly \texttt{poincare}, for building the 2D/3D plots of Poincaré surface-of-section projections, and \texttt{gin}, for generating sets of initial conditions for numerical experiments. Sec.\(^3\) contains a brief illustration of how the new commands work in three selected examples that can also be regarded as simple input/output tests. They are: the Toda lattice\(^4\), the Hénon-Heiles Hamiltonian\(^5\), and a numerical study, in the context of general relativity, which appeared in a recent publication\(^6\). Finally, the Conclusions contain a brief discussion about this work and its possible extensions.

1 The Poincaré surface-of-section method

The Poincaré surface-of-section method has become popular in the last decades in connection with the KAM theorem for weakly perturbed (originally integrable) Hamiltonian systems.

An insight of the ideas underlying this method can be obtained by considering, for instance, a conservative system with two degrees of freedom. The physical trajectories

\(^2\)Aside from this, the package itself contains an \textit{On-Line} help in standard Maple format which can be viewed as the User’s manual for all the routines.
lie on the three-dimensional energy surface \( H(p_1, p_2, q_1, q_2) = H_0 \) and, if the motion is bounded, after a suitable time interval, the solution curves will repeatedly intersect any given embedded 2D-plane; for instance, the \( q_2 = \text{constant} \), \((p_1, q_1)\) plane. If another constant of motion involving \( p_1 \) and \( q_1 \) exists (then the system is integrable), it is possible to use it to express \( p_1 = p_1(q_1) \), i.e., the intersection points will lie on a smooth curve. The enclosed area will be an integral invariant and, as time flows, these smooth curves will draw surfaces in the phase space.

For weakly perturbed systems, these surfaces of solution curves of regular motion (KAM surfaces) continue to exist for most initial conditions, breaking their topology near resonances to form “islands” chains. Within these islands, the topology is broken again to other chains and so on. Generally speaking, the KAM surfaces isolate thin layers of stochasticity and, as the perturbation strength is increased, transitions between layers merge, and the KAM surfaces progressively disappear resulting in complete stochastic motion.

In this manner, the plotting of the intersection points of the solution curves with a given 2D-plane (the surface-of-section) permits the study of the existence of non-obvious constants of motion (isolating integrals in the context of Hamilton-Jacobi theory), local stability, transition from ordered to stochastic motion, and many other interesting properties. We recall that no general procedure for determining the integrability of an arbitrary Hamiltonian system, or even the number of such isolating integrals, has yet been found. As a consequence, the plotting of these surfaces-of-section plays an important role not only in numerical studies, but also in checking the consistency of analytical results obtained using perturbative methods.

In the multidimensional case, the Poincaré surface-of-section (PS) has dimensionality \( 2N-2 \). Although the case \( N \geq 3 \) presents some subtleties (Arnold diffusion etc.), for trajectories which are exactly separable in the \((p_i, q_i)\) coordinates, the intersection points of the solution curves with the corresponding \((p_i, q_i)\) plane also lie on a smooth curve. We will denominate two-dimensional surface-of-section (2PS) the plotting of these intersection points over a given 2D-plane of the phase space. In the general case the intersection points may not lie in such smooth curves, but they still fill an annulus of finite area, and the thickness can be related to the nearness concerning exact separability in the related \((p_i, q_i)\) coordinates.

It is possible to extend the concept of 2D surface-of-section to 3D space-of-section (3PS), which will contain 3D projections of the solution curves. The plotting of the 2PS embedded in a related 3PS may give interesting detailed information about the behavior of a given model.

2 The Poincaré package

Basically, the Poincaré package consists of a plotting-command, \texttt{poincare}, for the 2D/3D plotting of the corresponding projections of surfaces-of-section, plus a set of facility-commands for a quick setup of the Hamilton equations of motion, initial conditions for numerical experiments, and for the zooming of plots. All plots are built by numerically integrating Hamilton’s equations for a given set of initial conditions. Once the plot is realized, all the Maple facilities for handling plots are available. Complementary symbolic
studies in the same environment can be developed with the commands of the PDEtools package\[3\] for the analytical solving of systems of ODE’s, scalar PDE’s (the corresponding Hamilton-Jacobi equation), and for changing variables.

**Summary**
A brief review of the commands of the package is as follows:\[4\]:

- **hameqs** receives a Hamiltonian and returns Hamilton’s equations and a list with the p’s and q’s involved;
- **poincare** receives a Hamiltonian, a time range and initial conditions, and returns a 2D plot of a 2PS; or a 3D plot of 2PS embedded in a related 3PS. In both the 2D and 3D plots, the 2D plane over which the 2PS is plotted can be any of the coordinate planes of the phase space, and can optionally be shifted in the positive or negative directions;
- **gin** generates a set of lists of initial conditions for plotting a 2PS/3PS, from a given incomplete set of fixed values and/or ranges for the p’s, q’s and the energy. This command speeds-up the time-expensive task of setting appropriate initial conditions for performing numerical experiments;
- **zoom** allows for changing the ranges of the display of a given 2D/3D plot without having to recalculate it again, thus saving time and memory resources.

**Description**
A complete description of the Poincaré package’s commands is found in the On-Line help. Therefore, a detailed description will be given here only for the most relevant routines, namely poincare and gin. Some commented input/output examples can be found in Sec.3.

### 2.1 Command name: poincare

*Feature:* plot 2D/3D projections of the Poincaré surface-of-section of a given Hamiltonian system.

*Calling sequence:*

```maple
> poincare(H, t=a..b, ics, optional_parameters);
```

*Parameters:*

- `H` - any algebraic expression representing the Hamiltonian.
- `t=a..b` - `t` represents the time and `a..b` is a numerical range.
- `ics` - a set of initial conditions for the p’s and q’s in the form:
  ```plaintext
  {[n1,n2,n3,...], [...],...}, where [n1,n2,n3,...] is a list of numbers representing the initial values for [t, p1, p2,...pk, q1,q2,...,qk].
  ```

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\[3\]This subsection and the next one may contain some information already presented in the previous sections; this was necessary to produce a complete-cut description of the package.

\[4\]In what follows, the input can be recognized by the Maple prompt >.
**Optional Parameters:**

- **stepsize=n** - indication of a positive number representing the size of the step used in the integration scheme.

- **iterations=N** - indication of a positive integer representing the number of iterations used in the integration scheme.

- **scene=[xi,xj]** - indication of the variables constituting the 2D-plane of the phase space where the 2PS is plotted; it is possible to specify ranges for the plot as in: scene=[xi=a..b,xj=c..d]

- **scene=[xi,xj,xk]** - indication of the 2D-plane and a third variable, xk, to be used as cross-variable or third axis in plots of 2PS/3PS’s, respectively

- **shift=s** - indication of a number representing a positive or negative shift of the intersection plane in the plots of 2Ps/3Ps

- **method=procedure** - indication of a user procedure to be used as integration method.

**Synopsis:**

The `poincare` command was designed to build either fast but not so accurate, or slower, as accurate as desired, 2D/3D projection plots of Poincare sections (see optional arguments below). Instead of analytically enforcing the Hamiltonian constraint, the value of the energy of each plotted point is checked against the corresponding $H_0$. As a complement, another routine, `gin` (see subsection 3.2), was programmed in order to speed-up the preparation of initial conditions for the numerical experiments.

The returned plots can be manipulated using all the standard Maple facilities (icon tool-bar) for 2D/3D plots such as reorientation, perspective, etc., and using the `zoom` command, also part of this package, for zooming in/out the interesting regions. These facilities usually permit a detailed visual distinction between the KAM surfaces and the layers of stochasticity for, say, typical weakly perturbed systems.

In addition to the returned plot, some relevant information related to each solution curve is displayed on the screen during the calculations. This information is:

- the initial value of the Hamiltonian, p’s and q’s for the curve;

- the number of intersection points found in the given time interval (2D plots);

- the maximum percentile “energy-deviation” of the intersection points.

It is useful to know the number of intersection points since it may be an indication of how appropriate is the indicated time interval. Concerning the percentile energy-deviation, it is calculated as $\frac{H_0 - H(\text{point})}{H_0} \times 100$. We note that all numerical algorithms will lead to values of $H$ different from the initial $H_0$, specially in the case of optional fast plottings. Percentile deviations below $10^{-8}$ are displayed as $0$. Also, the deviation is calculated for each intersection point, but only the greatest value is displayed. This information will give an idea of the accuracy of the plot, and the user will have the option of either reentering the instruction looking for a more accurate/slower plot or using his/her own numerical integration algorithm (see below). In typical situations, the smooth curves can be recognized even with energy-deviations of the order of $H_0/10$.

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5 When $H_0 = 0$, just the maximum absolute deviation is displayed.
The arguments

The first argument of a poincare calling is the Hamiltonian. Some handy conventions were adopted to represent the q's and p's. These are: all p's and q's must appear as \(p_n\) or \(q_n\) where n is a positive integer, as in \(p_1, p_2, \text{etc.}\), and the time dependence need not be explicit, as in \(p_n\) or \(q_n\) instead of \(q_n(t)\) or \(p_n(t)\). The Hamiltonian is assumed to be time independent and the number of degrees of freedom is expected to be \(\leq 10\).

The solution curves are calculated within a given time interval specified in the second argument as \(t=a..b\). When this time range leads to no intersection points, one can use the 3 option, see how far the intersection plane is from the trajectories, and reenter the instruction shifting the intersection plane using the shift option (see below).

The third argument, a set, may have any number of lists of initial conditions for the time, p's and q's, corresponding or not to the same initial value of \(H\); they can be generated by the user or by the gin command, as explained in subsection 2.2. The initial conditions must be inside a set structure as in \([n_1,n_2,n_3,\ldots], \ldots,\ldots\), where \([n_1,n_2,n_3,\ldots]\) is a list of numbers representing the initial values for \([t, p_1, \ldots p_k, q_1, \ldots q_k]\).

The optional arguments

The optional arguments can be given alone or in conjunction and in any order.

By default, the step interval is \((b-a)/20\), where \(a..b\) is the range for \(t\). This can be changed by giving the extra argument \textit{stepsize} = \(n\), where \(n\) is a positive number. As the stepsize is decreased, the accuracy and the smoothness of the integral curves (as well as the time consumed in the calculations) will increase.

The default numerical algorithm used in the integration scheme is basically the 4\(^{th}\) order Runge-Kutta of the \textit{DEtools} package, but this can be changed by giving the extra argument \textit{method} = \textit{usermethod}, where \textit{usermethod} should be a numerical integration algorithm (see the help pages of the \textit{DEtools} package). Also, when requesting a plot, the numerical algorithm can be iterated, as many times as desired, by giving the extra argument \textit{iterations} = \(N\) (default: \(\text{iterations} = 1\)).

The default scene for the plots is: the \((p_1,q_1)\) plane, at \(q_2 = 0\) for the 2PS, or the \((p_1,q_1,q_2)\) 3D-submanifold, for the plot of a 2PS embedded in a 3PS, when the 3 option is indicated. The intersection points constituting the 2PS are obtained by looking for the sign change of a “third” coordinate, here denominated the cross-variable, by default \(q_2\). The default ranges for the display of a plot are such as to include all the calculated intersection points in the case of 2D plots; or all calculated pieces of projections of solution curves plus the intersection 2D-plane, in the case of 3D plots.

All these defaults for the scene can also be changed. First of all, the intersection plane over which the 2PS is plotted can be shifted in the positive or negative directions by indicating \textit{shift} = s (a real number) as an additional argument. Concerning the 2D/3D submanifolds or the cross-variable, they can be changed by giving the extra argument \textit{scene} = \([x_1,x_2]\) or \textit{scene} = \([x_1,x_2,x_3]\), with or without extra ranges (for displaying a plot of a particular region), as in \textit{scene} = \([x_1=a..b,\ldots]\). The 2PS will then be plotted over the plane formed by the first two variables appearing in the right-hand-side, and \(x_3\), when given, will be the cross-variable; or the third axis when the 3 option is given as

\(^{6}\)To avoid useless large code this number was restricted to 10 (20-dimensional phase space), but this restriction can easily be removed.
2.2 Command name: gin

Feature: generates a set of lists of initial conditions satisfying the Hamiltonian constraint.

Calling sequence:

\[ \texttt{gin}(\text{H}, \text{ic}, N) \];

Parameters:

\texttt{H} - any algebraic expression representing the Hamiltonian.

\texttt{ic} - a set of single initial conditions for the time \( t \), \( p \)'s, \( q \)'s and the energy, in the form:
\[
\{ t=X_1, p_1=X_2, \ldots, q_n=X_k, \text{energy}=X_{2n+2} \},
\]
where the \( X_i \) are numbers or ranges of numbers, representing initial values for the time, \( p \)'s, \( q \)'s, and the energy.

\texttt{N} - a positive integer representing the number of desired lists of initial conditions.

Synopsis:

The numerical study of a given Hamiltonian system usually requires a lot of suitable lists of initial conditions (IC's) satisfying the Hamiltonian constraint (see examples in Sec.3). The \texttt{gin} command was thus proposed as a tool for the fast generation of a set of such lists, with the appropriate syntax as required by the \texttt{poincare} command explained above.

The first argument received by \texttt{gin} is the Hamiltonian. As second argument, a single specification of IC's for all but one of \( (t, p \)'s, \( q \)'s, \( H_0 \)) must be given. If IC's are specified for all these variables, the command just checks the values against \( H \) for consistency. The IC for a given variable may be a fixed number or a numerical range.

The third argument is the number of requested lists of IC's. Face the request of say \( N \) IC's giving ranges or fixed numbers for all variables but one, say \( x \), the routine proceeds as follows. The received IC's are separated into numerical ranges and fixed numbers. Each range is then uniformly divided into \( N \) numbers, and \( N \) lists are built by taking one number, from each divided range, together with the received fixed numbers for the other variables. These lists are sequentially introduced in \( H \), resulting in \( N \) algebraic equations for \( x \). Each equation is then numerically solved, and a set of \( N \) lists of complete IC's satisfying the Hamiltonian constraint, with values inside the given ranges, is returned. Remarkably, though \texttt{gin} is a very simple command, it plays a fundamental role in speeding-up the numerical studies.

3 Examples

This section contains a brief illustration of how the routines here presented work in some selected examples: the Toda lattice, the Hénon-Heiles Hamiltonian, and a numerical study.

\footnote{It is possible to indicate the time "t" as the third variable, in which case a convenient mouse-manipulation of the 3D-plot can display the projection of the curves over each \((q_i, q_j)\) plane. This may be useful to study the bounded/unbounded properties of a given potential. Furthermore, in the case of a system with 3 degrees of freedom, the use of the "3" option with scene=\([q_1,q_2,q_3]\) will render the 3D-plot of the physical trajectory.}
in the context of general relativity, which appeared in a recent publication[3]. Although
the idea is just to test the obtainment of some of the related PSs using the package’s
commands, we also included, for completeness, brief introductions to each example as
well as short comments about the resulting PS plots.

3.1 The Toda lattice

Our first example is the three-particle Toda lattice[1]. This model describes the motion of
three particles of equal mass moving on a ring with exponentially decreasing interactions.
Early numerical experiments[5], followed by analytical results[6], showed that the related
dynamical system is integrable. After some manipulations, the original Hamiltonian can
be written as:

\[ H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{24} \left( e^{(2q_2 + 2\sqrt{3} q_1)} + e^{(2q_2 - 2\sqrt{3} q_1)} + e^{(-4q_2)} \right) - \frac{1}{8} \]  (1)

A fast plot of the surfaces-of-section, projected over the \((p,q)\) planes, with just one initial
condition, can be obtained via:

\[
\text{> H, t=-150..150, \{0,.1,1.4,.1,0\}; \# ics: t=0,p1=.1,p2=1.4,q1=.1,q2=0
> poincare("", stepsize=.05,iterations=5);
> poincare("",stepsize=.05,iterations=5,scene=[p2,q2]);}
\]

Fig.1.a. 2PS over the \(q_2=0\) plane, with 127 intersection points lying on smooth curves. \(H_0=0.99, \text{H-deviation}=5 \times 10^{-6}\%\). Time: 73s.

Fig.1.b. 2PS over the \(q_1=0\) plane with 146 intersection points. Time: 76s.

Generally speaking, it is interesting to have the 2PS projected over all the \((p_i,q_i)\) planes
of the phase space, since the existence of smooth patterns in all of them is an indication
of the possible integrability of the system.

A Poincaré space-of-section corresponding to Fig.1.a can be manipulated using the
mouse to obtain the following illustrative perspectives:

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\[^8\text{In what follows, the values of } \theta \text{ and } \phi \text{ mentioned in the figures correspond to the Maple values for the plots.}\]
Another indication of the integrability of the system is that regular curves exist whatever the value of $H_0$. As an example of this, a surface-of-section (one solution curve) and a related 3PS, at $H_0 = 256$, can be built as follows:

```plaintext
> ics2 := gin(H,\{t=0,p2=22,q1=0,q2=0,energy=256\},1):
> poincare(H,t=-50..50,ics2,stepsize=.005,iterations=4,scene=[p2,q2]);
> poincare(H,t=0..20,ics2,stepsize=.01,iterations=4,scene=[p2,q2,q1],3);
```

Another indication of the integrability of the system is that regular curves exist whatever the value of $H_0$. As an example of this, a surface-of-section (one solution curve) and a related 3PS, at $H_0 = 256$, can be built as follows:

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> ics2 := gin(H,\{t=0,p2=22,q1=0,q2=0,energy=256\},1):
> poincare(H,t=-50..50,ics2,stepsize=.005,iterations=4,scene=[p2,q2]);
> poincare(H,t=0..20,ics2,stepsize=.01,iterations=4,scene=[p2,q2,q1],3);
```

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```plaintext
> ics2 := gin(H,\{t=0,p2=22,q1=0,q2=0,energy=256\},1):
> poincare(H,t=-50..50,ics2,stepsize=.005,iterations=4,scene=[p2,q2]);
> poincare(H,t=0..20,ics2,stepsize=.01,iterations=4,scene=[p2,q2,q1],3);
```

**3.2 Hénon-Heiles Hamiltonian**

The Hénon-Heiles Hamiltonian produces one of the most famous and studied surfaces-of-section. The corresponding Hamiltonian can be obtained by expanding the Toda Hamil-
tonian to cubic terms in $q_1$ and $q_2$, and is given by:

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 + q_1^2 + q_2^2 \right) + q_1^2 q_2 - \frac{1}{3} q_2^3.$$  \hspace{1cm} (2)

This model can be related to the motion of a star in a cylindrically symmetric gravitational galactic potential, is not integrable, and the phase space is bounded only if the energy is less than $1/6$. Well-studied surfaces-of-section, presented in several treatises of chaos, with $H_0$ equal to $1/24, 1/18, 1/12, 1/8, 1/7, \text{ and } 1/6$, are obtained here using the \texttt{gin} and \texttt{poincare} commands as follows. To start with, six sets, related to each value of $H_0$ respectively, with three different initial conditions each, are generated via:

> for h in [1/24,1/18,1/12,1/8,1/7,1/6] do
>  
>  ics[h] := gin(H,{t=0,p2=0.1,q2=-0.2..0.2,q1=-0.2..-0.1,energy=h},3)
>  od:

After that, surfaces-of-section with around 550 points, calculated in approximately 5 minutes each, with percentile $H$-deviations $\approx 10^{-7}\%$, can be obtained via:

> for h in [1/24,1/18,1/12,1/8,1/7,1/6] do
>  
>  poincare(H,t=-300..300,ics[h],
>  
>  stepsize=.05,iterations=3,scene=[p2=-.5..0.5,q2=-.5..0.5]);
>  od:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Fig.4.a. $H_0 = 1/24$ \hspace{1cm} Fig.4.b. $H_0 = 1/18$ \hspace{1cm} Fig.4.c. $H_0 = 1/12$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Fig.4.d. $H_0 = 1/8$ \hspace{1cm} Fig.4.e. $H_0 = 1/7$ \hspace{1cm} Fig.4.f. $H_0 = 1/6$}
\end{figure}
The figures above reflect the progressive disintegration of the KAM surfaces, occurring with the increase of $H_0$ up to $1/6$. In the plots for $H_0$ equal to $1/24$ and $1/12$, invariant curves apparently exist everywhere, but this is not strictly correct. In fact, the model is not integrable, as is reflected by the sequence of figures, and thin resonance layers of stochasticity are densely distributed throughout the 2PS, even for small $H_0$.

Another interesting parameter is given by the relevant number of calculations involved in the building of each of Figs.4.(a,b,c,d,e,f): $600/0.05=12,000$ points, iterated $4.3=12$ times each; that is, $144,000$ calculations in five minutes, or $480$ calculations per second.

Concluding this example, an instructive test is to compare the regularity of the physical trajectories of Fig.4.a ($H_0=1/24$) with that of the trajectories of Fig.4.f ($H_0=1/6$). These six displays are obtained by plotting “3PS’s” over $(q_1,q_2,t)$ and changing the perspective, appropriately, using the mouse:

```maple
> for ic in [ics[1/24][1..3],ics[1/6][1..3]] do
> poincare(H,t=-100..100,{ic},scene=[q1,q2,t],stepsize=.05,iterations=3,3):
> od:
```

| Fig.5.a. $H_0=1/24$, ics[1/24][1], $H$-dev.$=2\times10^{-7}$ %, time: 124s. |
| Fig.5.b. $H_0=1/24$, ics[1/24][2], $H$-dev.$=2\times10^{-7}$ %, time: 149s. |
| Fig.5.c. $H_0=1/24$, ics[1/24][3], $H$-dev.$=3\times10^{-7}$ %, time: 159s. |
| Fig.5.d. $H_0=1/6$, ics[1/6][1], $H$-dev.$=4\times10^{-7}$ %, time: 169s. |
| Fig.5.e. $H_0=1/6$, ics[1/6][2], $H$-dev.$=5\times10^{-7}$ %, time: 178s. |
| Fig.5.e. $H_0=1/6$, ics[1/6][2], $H$-dev.$=6\times10^{-7}$ %, time: 191s. |

$^9$The number of digits carried in floats is 16; also, the true calculations per second is greater than 480: part of the five minutes were dedicated to determine the intersection points and the maximum percentile deviation.
The contrast between Figs. 5.(a,b,c) and Figs. 5.(d,e,f) emphasizes the connection between the smoothness of the curves over a surface-of-section and the regularity of the corresponding physical trajectories.

3.3 An example from general relativity

Chaos in General Relativity is an issue of rich debate. Since the pioneering work of Belinski, Khalatnikov and Lifshitz, who discussed chaotic behavior in anisotropic Bianchi IX models, much advance has been made. Francisco and Matsas, studying that model, originally noted that the Liapunov exponents tend to zero in the numerical experiments, due to the choice made for the time variable. This result has been confirmed by several authors. In this context, the Poincaré surfaces-of-section play a crucial role, since the destruction of KAM surfaces does not depend on the time variable and constitutes the signature of chaos.

Bearing the above remarks in mind, we examine here a numerical study appearing in a recent work by Calzetta and El Hasi. The authors developed a perturbative study of the influence of the scalar radiation field on the expansion of the universe in the early stages of inflation. They performed numerical experiments to exhibit chaotic behavior indicated by the destruction of tori structures, formation of cantori, and Arnold diffusion. The Hamiltonian for the model is given by:

$$H = \frac{1}{2} \left( -p_1^2 - q_1^2 + 2 \Lambda q_1^4 + p_2^2 + q_2^2 + m^2 q_2^2 q_3^2 \right) = 0$$

Such an expression describes closed homogeneous and isotropic universes with a cosmological constant, $\Lambda$, playing the role of the inflaton. The degrees of freedom of the model are the “radius of the universe”, $q_1$, and the conformally scaled radiation field, $q_2$; $m$ represents its mass, and $p_1, p_2$ are the conjugated momenta.

Taking convenient values $m = 0.65$ and $\Lambda = 0.125$ and choosing the $q_2 = 0, (q_1, p_1)$ plane as in [3], it is possible to reproduce the relevant 2PS there displayed as Fig. 2 as follows. First, an appropriate set of one hundred lists of initial conditions, satisfying $H_0 = 0$ and in accordance with the initial values for $p_1, q_1$ indicated in that paper, is generated via:

```maple
> ic1 := gin(H,{t=0,p1=-.2..-0.7,q1=0,q2=0,energy=0},15):
> ic2 := gin(H,{t=0,p1=-.7..-0.812,q1=0,q2=0,energy=0},85):
> ics:= ic1 union ic2:
> poincare(H,t=0..300,ics,
>        stepsize=.1,iterations=3,scene=[q1=-1.5..1.5,p1=-1..1,q2=-1..1]);
```
The presence of smooth curves related to KAM surfaces, as well as a region of broken tori. More than 10,000 intersection points, absolute $H$-dev. $\approx 10^{-8}$. Time: 35 minutes.

Another good test is given by the plotting of a 3PS over \((p_1, q_1, q_2)\), for initial conditions very close to the critical point \(P_c\), where \(p_1 = p_2 = q_1 = q_2 = 0\):

\[
> \text{ics} := \text{gin}(H, \{t=0, p_1=0..-0.31, q_1=0, q_2=0, \text{energy}=0\}, 40);
> \text{poincare}(H, t=0..20, \text{ics},
  > \quad \text{stepsize}=0.1, \text{scene}=[q_1=-0.3..0.3, p_1=-0.3..0.3, q_2=-0.1..0.1], 3);
\]

\[\text{Fig.7.a. } \theta=-160, \phi=50. \text{ Regular circles close to } P_c, \text{ in agreement with complex eigenvalues for the linearized system. Time: 61s.}\]

\[\text{Fig.7.b. } \theta=-20, \phi=18. \text{ The action of higher order terms in splitting-up the KAM surfaces. Absolute } H\text{-deviation } < 10^{-8}.\]
4 Conclusions

This work presented a set of software-tools for the plotting of Poincaré surfaces-of-section of Hamiltonian systems. This method is a valuable tool for studying dynamical systems, since it conveys relevant information on the dynamics, in a simple manner. Due to the characteristic flexibility of symbolic programming languages, the main result was an easy-to-use package of commands permitting reasonably fast and varied numerical studies of Hamiltonian systems, in a general purpose symbolic computing environment.

An important remark, taking into account that this software was written for realizing intensive numerical studies, is that great emphasis was put in the interactive character of the package. That is, the user is given the possibility of alternatively searching for “first order” phenomena at remarkably high speed, obtaining draft Poincaré sections in just a few seconds; or “high order” detailed 2D/3D projections, displaying “islands” and the inner structure of a PS, as desired.

On the other hand, this is a first version of the package and, as such, it does not make use of the theory of Liapunov coefficients, does not discuss the determination of the analytical Poincaré mappings, does not feature options related to the structure of Arnold’s web, and is not designed to study the interesting case of dissipative Hamiltonians and correlated phenomena (strange attractors etc.). All these topics are possible extensions of this work and we expect to report related work in the near future.

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