Kahler-Einstein Structures of General Natural Lifted Type on the Cotangent Bundles

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Abstract. We study the conditions under which the cotangent bundle $T^*M$ of a Riemannian manifold $(M, g)$, endowed with a Kählerian structure $(G, J)$ of general natural lift type (see [1]), is Einstein. We first obtain a general natural Kähler-Einstein structure on the cotangent bundle $T^*M$. In this case, a certain parameter, $\lambda$ involved in the condition for $(T^*M, G, J)$ to be a Kählerian manifold, is expressed as a rational function of the other two, the value of the constant sectional curvature, $c$, of the base manifold $(M, g)$ and the constant $\rho$ involved in the condition for the structure of being Einstein. This expression of $\lambda$ is just that involved in the condition for the Kählerian manifold to have constant holomorphic sectional curvature (see [2]). In the second case, we obtain a general natural Kähler-Einstein structure only on $T^*_0M$, the bundle of nonzero cotangent vectors to $M$. For this structure, $\lambda$ is expressed as another function of the other two parameters, their derivatives, $c$ and $\rho$.

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Key words: cotangent bundle, Riemannian metric, general natural lift, Kähler-Einstein structure.

1 Introduction

A few natural lifted structures introduced on the cotangent bundle $T^*M$ of a Riemannian manifold $(M, g)$, have been studied in recent papers such as [1], [2], [8], [11], [12], [14]–[19]. The similitude between some results from the mentioned papers and results from the geometry of the tangent bundle $TM$ (e.g. [9], [10]), may be explained by the duality cotangent bundle-tangent bundle. The fundamental differences between the geometry of the cotangent bundle and that of the tangent bundle of a Riemannian manifold, are due to the different construction of lifts to $T^*M$, which cannot be defined just like in the case of $TM$ (see [21]).

The results from [1], [5], [6], [20], concerning the natural lifts, and the classification of the natural vector fields on the tangent bundle of a pseudo-Riemannian manifold, made by Janyška in [3], allowed the present author to introduce in the paper [1], a
general natural almost complex structure \( J \) of lifted type on the cotangent bundle \( T^* M \), and a general natural lifted metric \( G \) defined by the Riemannian metric \( g \) on \( T^* M \) (see the paper [9] by Oproiu, for the case of the tangent bundle). The main result from [1] is that the family of general natural Kähler structures on \( T^* M \) depends on three essential parameters (one is a certain proportionality factor obtained from the condition for the structure to be almost Hermitian and the other two are coefficients involved in the definition of the integrable almost complex structure \( J \) on \( T^* M \)).

In the present paper we are interested in finding the conditions under which the cotangent bundle \( T^* M \) of a Riemannian manifold \((M, g)\), endowed with a Kählerian structure \((G, J)\) of general natural lift type (see [1]), is an Einstein manifold. To this aim, we have to study the vanishing conditions for the components of the difference between the Ricci tensor of \((T^* M, G, J)\) and \(\rho G\), where \(\rho\) is a constant.

After some quite long computations with the RICCI package from the program Mathematica, we obtain two cases in which a general natural Kählerian manifold \((T^* M, G, J)\) is Einstein. In the first case, \((T^* M, G, J)\) is a Kähler-Einstein manifold if the proportionality factor \(\lambda\), involved in the condition for the manifold to be Kählerian, is expressed as a rational function of the first two essential parameters, their derivatives, the values of the constant sectional curvature of the base manifold \((M, g)\), and the constant \(\rho\), from the condition for the manifold to be Einstein. In this case the expression of \(\lambda\) leads to the condition obtained in [2] for \((T^* M, G, J)\) to have constant holomorphic sectional curvature. In the second case, \((G, J)\) is a Kähler-Einstein structure on the the bundle of nonzero cotangent vectors to \(M\), \(T^*_0 M\), if \(\lambda'\) is expressed as a certain function of \(\lambda\), the other two parameters, their first order derivatives, \(c\) and \(\rho\). The similar problem on tangent bundle \(T M\) was treated by Oproiu and Papaghiuc in the paper [13].

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class \(C^\infty\) (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices \(h, i, j, k, l, m, r\) being always \{1, \ldots, n\}.

## 2 Preliminary results

The cotangent bundle of a smooth \(n\)-dimensional Riemannian manifold may be endowed with a structure of a \(2n\)-dimensional smooth manifold, induced from the structure of the base manifold. If \((M, g)\) is a smooth Riemannian manifold of the dimension \(n\), we denote its cotangent bundle by \(\pi: T^* M \to M\). From every local chart \((U, \varphi) = (U, x^1, \ldots, x^n)\) on \(M\), it is induced a local chart \((\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)\), on \(T^* M\), as follows. For a cotangent vector \(p \in \pi^{-1}(U) \subset T^* M\), the first \(n\) local coordinates \(q^1, \ldots, q^n\) are the local coordinates of its base point \(x = \pi(p)\) in the local chart \((U, \varphi)\) (in fact we have \(q^i = \pi^* x^i = x^i \circ \pi, i = 1, \ldots, n\)). The last \(n\) local coordinates \(p_1, \ldots, p_n\) of \(p \in \pi^{-1}(U)\) are the vector space coordinates of \(p\) with respect to the natural basis \((dx_{\pi(p)}^1, \ldots, dx_{\pi(p)}^n)\), defined by the local chart \((U, \varphi)\), i.e. \(p = p_i dx_{\pi(p)}^i\).

The \(M\)-tensor fields on the cotangent bundle may be introduced in the same manner as the \(M\)-tensor fields were introduced in the paper [7] on the tangent bundle of a Riemannian manifold.
On $T^*M$, a few useful $M$-tensor fields may be obtained as follows. Let $v, w : [0, \infty) \to \mathbb{R}$ be smooth functions and let $\|p\|^2 = g_{\pi^{-1}(p), p}^{-1}$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ ($g^{-1}$ is the tensor field of type (2,0) having the components $(g^{kl}(x))$ which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of $g$ in the local chart $(U, \varphi)$). The components $vg_{ij}(\pi(p))$, $p_i$, $w(\|p\|^2)p_ip_j$ define respective $M$-tensor fields of types (0,2), (0,1), (0,2) on $T^*M$. Similarly, the components $\pi^{-1}(U)$ have the components of $\pi^{-1}(U)$ in the local chart $(U, \varphi)$. The components $vg^{kl}(\pi(p))$, $g^{0i} = p_h g^{hi}$, $w(\|p\|^2)g^{ik}g^{0l}$ define respective $M$-tensor fields of type (2,0), (1,0), (2,0) on $T^*M$. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

We recall the splitting of the tangent bundle to $T^*M$ into the vertical distribution $VT^*M = \text{Ker} \, \pi_*$ and the horizontal one determined by the Levi Civita connection $\nabla$ of $g$:

\begin{equation}
TT^*M = VT^*M \oplus HT^*M.
\end{equation}

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q_1, \ldots, q_n, p_1, \ldots, p_n)$ is a local chart on $T^*M$, induced from the local chart $(U, \varphi) = (U, x_1, \ldots, x^n)$, the local vector fields $\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}$ on $\pi^{-1}(U)$ define a local frame for $VT^*M$ over $\pi^{-1}(U)$ and the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ define a local frame for $HT^*M$ over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih} \frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k \Gamma^k_{ih},$$

and $\Gamma^k_{ih} (\pi(p))$ are the Christoffel symbols of $g$.

The set of vector fields $\left\{ \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q_1}, \ldots, \frac{\delta}{\delta q_n} \right\}$ defines a local frame on $T^*M$, adapted to the direct sum decomposition (2.1).

We consider

$$t = \frac{1}{2} \|p\|^2 = \frac{1}{2} g_{\pi^{-1}(p), p}^{-1} = \frac{1}{2} g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U)$$

the energy density defined by $g$ in the cotangent vector $p$. We have $t \in [0, \infty)$ for all $p \in T^*M$.

The computations will be done in local coordinates, using a local chart $(U, \varphi)$ on $M$ and the induced local chart $(\pi^{-1}(U), \Phi)$ on $T^*M$.

We shall use the following lemma, which may be proved easily.

**Lemma 2.1.** If $n > 1$ and $u, v$ are smooth functions on $T^*M$ such that

\begin{equation}
ug_{ij} + vp_ip_j = 0, \quad ug^{ij} + vg^{0i}g^{0j} = 0, \quad \text{or} \quad u\delta^i_j + vg^{0i}p_j = 0, \quad \forall i, j = 1, \ldots, n,
\end{equation}

on the domain of any induced local chart on $T^*M$, then $u = 0, \ v = 0$.

In the paper [1], the present author considered the real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ on $[0, \infty) \subset \mathbb{R}$ and studied a general natural tensor of
type \((1,1)\) on \(T^*M\), defined by the relations

\[
\begin{align*}
JX^H_p &= a_1(t)g_X^V p + b_1(t)p(X)p^V_p + a_4(t)X^H_p + b_4(t)p(X)(p^V)^H_p, \\
J\theta^V_p &= a_3(t)\theta^V_p + b_3(t)g^{-1}_{\pi(p)}(p, \theta)p^V_p - a_2(t)(\theta^V)^H_p - b_2(t)g^{-1}_{\pi(p)}(p, \theta)(p^V)^H_p,
\end{align*}
\]

(2.2)

in every point \(p\) of the induced local card \((\pi^{-1}(U), \Phi)\) on \(T^*M, \forall X \in \mathcal{X}(M), \forall \theta \in \Lambda^1(M)\), where \(g_X\) is the 1-form on \(M\) defined by \(g_X(Y) = g(X, Y), \forall Y \in \mathcal{X}(M)\), \(\theta^2 = g^{\circ-1}_b\) is a vector field on \(M\) defined by \(g(\theta^2, Y) = \theta(Y), \forall Y \in \mathcal{X}(M)\), the vector \(p^2\) is tangent to \(M\) in \(\pi(p)\), \(p^V\) is the Liouville vector field on \(T^*M\), and \((p^V)^H\) is the similar horizontal vector field on \(T^*M\).

The definition of the general natural lift given by (2.2), is based on the Janyška’s classification of the natural vector fields on the tangent bundle, but the construction is different, being specific for the cotangent bundle.

**Theorem 2.1.** ([1]) A natural tensor field \(J\) of type \((1,1)\) on \(T^*M\), given by (2.2), defines an almost complex structure on \(T^*M\), if and only if \(a_4 = -a_3, b_4 = -b_3\) and the coefficients \(a_1, a_2, a_3, b_1, b_2\) and \(b_3\) are related by

\[
a_1a_2 = 1 + a_3^2, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.
\]

Studying the vanishing conditions for the Nijenhuis tensor field \(N_J\), we may state:

**Theorem 2.2.** ([1]) Let \((M, g)\) be an \(n(>2)\)-dimensional connected Riemannian manifold. The almost complex structure \(J\) defined by (2.2) on \(T^*M\) is integrable if and only if \((M, g)\) has constant sectional curvature \(c\) and the coefficients \(b_1, b_2, b_3\) are given by:

\[
\begin{align*}
b_1 &= \frac{2c_1^2a_2^2 + 2cta_1a_3' + a_1a_3' - c + 3ca_2^2}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_1^2}, \\
b_2 &= \frac{2a_1^2 - 2a_1a_3' + ca_2^2 + 2cta_2a_3' + a_1a_3'}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_1^2}, \\
b_3 &= \frac{a_1^2 + 2a_1a_3' + 4cta_2a_3' - 2cta_2a_3'}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_1^2},
\end{align*}
\]

(2.4)

**Remark 2.3.** The integrability conditions (2.2) for the almost complex structure \(J\) on \(T^*M\), may be expressed in the equivalent form

\[
a_1' = \frac{1}{a_1 + 2tb_1}(a_1b_1 + c - 3ca_2^2 - 4cta_3b_3), \\
a_2' = \frac{1}{a_1 + 2tb_1}(2a_3b_3 - a_2b_1 - ca_2^2), \\
a_3' = \frac{1}{a_1 + 2tb_1}(a_1b_3 - 2ca_2a_3 - 2cta_3b_3).
\]

In the paper [1], the author defined a Riemannian metric \(G\) of general natural lift type, given by the relations

\[
\begin{align*}
G_p(X^H, Y^H) &= c_1(t)g_{\pi(p)}(X, Y) + d_1(t)p(X)p(Y), \\
G_p(\theta^V, \omega^V) &= c_2(t)g_{\pi(p)}^{-1}(\theta, \omega) + d_2(t)g_{\pi(p)}^{-1}(p, \theta)g_{\pi(p)}^{-1}(p, \omega), \\
G_p(X^H, \theta^V) &= G_p(\theta^V, X^H) = c_3(t)\theta(X) + d_3(t)p(X)g_{\pi(p)}^{-1}(p, \theta),
\end{align*}
\]

(2.5)

\(\forall X, Y \in \mathcal{X}(M), \forall \theta, \omega \in \Lambda^1(M), \forall p \in T^*M\).
Using the adapted frame \( \{ \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j} \}_{i,j=1,\ldots,n} \) on \( T^*M \), we may write the expression in the next form

\[
G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j}) = c_1(t)g_{ij} + d_1(t)p_ip_j = G_{(1)}^{ij},
\]

\[
G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j}) = c_2(t)g^{ij} + d_2(t)g^{0i}g^{0j} = G_{(2)}^{ij},
\]

\[
G(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) = c_3(t)\delta^i_j + d_3(t)p_i g^{0j} = G_{33}^{ij},
\]

where \( c_1, c_2, c_3, d_1, d_2, d_3 \) are six smooth functions of the density energy on \( T^*M \).

The conditions for \( G \) to be positive definite are assured if

\[
c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 > 0.
\]

The author proved the following result

**Theorem 2.3.** ([1]) The family of Riemannian metrics \( G \) of general natural lifted type on \( T^*M \) such that \( (T^*M, G, J) \) is an almost Hermitian manifold, is given by (2.6), provided that the coefficients \( c_1, c_2, c_3, d_1, d_2, d_3 \) are related to the coefficients \( a_1, a_2, a_3, b_1, b_2, b_3 \) by the following proportionality relations

\[
\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,
\]

where the proportionality coefficients \( \lambda > 0 \) and \( \lambda + 2t\mu > 0 \) are functions depending on \( t \).

Considering the two-form \( \Omega \) defined by the almost Hermitian structure \( (G, J) \) on \( T^*M \), given by \( \Omega(X, Y) = G(X, JY) \), for any vector fields \( X, Y \) on \( T^*M \), we may formulate the main results from [1]:

**Theorem 2.4.** ([1]) The almost Hermitian structure \( (T^*M, G, J) \) is almost Kählerian if and only if

\[
\mu = \lambda'.
\]

**Theorem 2.5.** A general natural lifted almost Hermitian structure \( (G, J) \) on \( T^*M \) is Kählerian if and only if the almost complex structure \( J \) is integrable (see Theorem 2.2) and \( \mu = \lambda' \).

Examples of such structures may be found in [15], [17].

### 3 General natural Kähler-Einstein structures on cotangent bundles

The Levi-Civita connection \( \nabla \) of the Riemannian manifold \( (T^*M, G) \) is obtained from the Koszul formula, and it is characterized by the conditions

\[
\nabla G = 0, \quad T = 0,
\]
where $T$ is the torsion tensor of $\nabla$.

In the case of the cotangent bundle $T^*M$ we may obtain the explicit expression of $\nabla$.

The symmetric $2n \times 2n$ matrix

$$
\begin{pmatrix}
G_{ij}^{(1)} & G_{ij}^{(2)} \\
G_{ij}^{(2)} & G_{ij}^{(3)}
\end{pmatrix}
$$

associated to the metric $G$ in the base $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\ldots,n}$ has the inverse

$$
\begin{pmatrix}
H_{ij}^{(1)} & H_{ij}^{(3)} \\
H_{ij}^{(3)} & H_{ij}^{(2)}
\end{pmatrix}
$$

where the entries are the blocks

$$
(3.1) \quad H_{ij}^{(1)} = e_1 g_{ij} + f_1 g_{ik} g_{jl}, \quad H_{ij}^{(2)} = e_2 g_{ij} + f_2 p_k p_l, \quad H_{ij}^{(3)} = e_3 d_i + f_3 g_{ij} p_l.
$$

Here $g^{kl}$ are the components of the inverse of the matrix $(g_{ij})$, $g^{0k} = p_l g_{ik}$, and $e_1, f_1, e_2, f_2, e_3, f_3 : [0, \infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained by solving the system:

$$
\begin{align*}
G_{ih}^{(1)} H_{jk}^{(1)} + G_{ih}^{(3)} H_{jk}^{(3)} &= \delta_i^j, \\
G_{ih}^{(1)} H_{jk}^{(2)} + G_{ih}^{(3)} H_{jk}^{(3)} &= 0, \\
G_{ih}^{(1)} H_{jk}^{(3)} + G_{ih}^{(2)} H_{jk}^{(3)} &= 0, \\
G_{ih}^{(1)} H_{jk}^{(3)} + G_{ih}^{(2)} H_{jk}^{(2)} &= \delta_i^j,
\end{align*}
$$

in which we substitute the relations (2.6) and (3.1). By using Lemma 2.1, we get $e_1, e_2, e_3$ as functions of $c_1, c_2, c_3$

$$
(3.2) \quad e_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \quad e_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \quad e_3 = -\frac{c_3}{c_1 c_2 - c_3^2},
$$

and $f_1, f_2, f_3$ as functions of $c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3$

$$
(3.3) \quad f_1 = -\frac{c_2 d_1 c_1 - c_2 d_2 c_1 - c_2 d_3 c_1 + c_2 d_4 c_1 + 2 d_1 d_2 c_1 - 2 d_1^2 c_1 t}{c_1 c_2 - c_3^2 + 2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_3 t + 4 d_1 d_2 t - 4 d_1^2 t^2},
$$

$$
(3.3) \quad f_2 = \frac{(c_3 + 2 d_3 t)(c_1 + 2 d_3 t) - (c_1 + 2 d_1 t)(c_1 + 2 d_3 t)}{(c_1 + 2 d_3 t)(c_1 + 2 d_1 t) - (c_1 + 2 d_3 t)^2} - \frac{d_2 c_2 + d_4 c_4}{c_3 + 2 d_3 t},
$$

$$
(3.3) \quad f_3 = \frac{(c_3 + 2 d_3 t)(c_1 + 2 d_1 t) - (c_1 + 2 d_3 t)(c_1 + 2 d_3 t)}{(c_1 + 2 d_1 t)(c_1 + 2 d_3 t) - (c_1 + 2 d_3 t)^2}.
$$

Next we may obtain the expression of the Levi Civita connection of the Riemannian metric $G$ on $T^*M$. 

6
Theorem 3.1. The Levi-Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame \( \{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \}_{i,j=1,\ldots,n} \)

\[
\begin{align*}
\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} & = Q^{ij}_{\;\;h} \frac{\partial}{\partial y^h} + \tilde{Q}^{ijh} \frac{\delta}{\delta y^h}, \\
\nabla_{\frac{\delta}{\delta y^j}} \frac{\delta}{\delta x^i} & = P^{ih}_{j} \frac{\delta}{\delta x^h} + \tilde{P}^{i}_{j} \frac{\delta}{\delta x^h}, \\
\nabla_{\frac{\delta}{\delta y^j}} \frac{\delta}{\delta y^i} & = \Gamma^{i}_{jh} \frac{\delta}{\delta y^h} + S_{ijh} \frac{\delta}{\delta y^h},
\end{align*}
\]

where $\Gamma^{i}_{jh}$ are the Christoffel symbols of the connection $\nabla$ and $M$-tensor fields appearing as coefficients in the above expressions are given as

\begin{align*}
Q^{ij}_{\;\;h} & = \frac{1}{2} (\partial^i G^{jk}_{(2)} + \partial^j G^{ik}_{(2)} - \partial^k G^{ij}_{(2)}) H^{(2)}_{kh} + \frac{1}{2} (\partial^i G^{jk}_{(2)} + \partial^j G^{ik}_{(2)}) H^{(1)}_{kh}, \\
\tilde{Q}^{ijh} & = \frac{1}{2} (\partial^i G^{jk}_{(2)} - \partial^k G^{ij}_{(2)}) H^{(2)}_{kh} + \frac{1}{2} (\partial^i G^{jk}_{(2)} - \partial^j G^{ik}_{(2)}) H^{(1)}_{kh}, \\
P^{ih}_{j} & = \frac{1}{2} (\partial^i G^{jk}_{(2)} - \partial^k G^{ij}_{(2)}) H^{(2)}_{kh} + \frac{1}{2} (\partial^i G^{jk}_{(2)} - \partial^j G^{ik}_{(2)}) H^{(1)}_{kh}, \\
S_{ijh} & = \frac{1}{2} (c_2 R^0_{lij} - \partial^k G^{ij}_{(1)}) H^{(2)}_{kh} - c_3 R^0_{lij} H^{(2)}_{kh}, \\
\tilde{S}_{ijh} & = \frac{1}{2} (c_2 R^0_{lij} - \partial^k G^{ij}_{(1)}) H^{(2)}_{kh} - c_3 R^0_{lij} H^{(2)}_{kh},
\end{align*}

where $R^0_{lij}$ are the components of the curvature tensor field of the Levi-Civita connection $\nabla$ of the base manifold $(M, g)$.

If we replace in (3.4) the relations (2.6), which define the metric $G$, the expressions (3.1) for the inverse matrix $H$ of $G$, and the formulas (3.2), (3.3) we obtain the detailed expressions of $P^{ih}_{j}$, $Q^{ij}_{\;\;h}$, $S_{ijh}$, $\tilde{P}^{i}_{j}$, $\tilde{Q}^{ijh}$, $\tilde{S}_{ijh}$.

The curvature tensor field $K$ of the connection $\nabla$ is defined by

\[
K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X,Y,Z \in \Gamma(TM).
\]

By using the local adapted frame \( \{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \}_{i,j=1,\ldots,n} = \{ \delta_i, \partial^j \}_{i,j=1,\ldots,n} \) we obtain the horizontal and vertical components of the curvature tensor field:

\[
\begin{align*}
K(\delta_i, \partial^j) \delta_k &= QQ QQ Q^i_{\;\;j}^k \delta_h + QQ Q P^i_{\;\;j}^k \delta_h, \\
K(\partial^i, \partial^j) \partial^k &= QQ PP Q^i_{\;\;j}^k \delta_h + QQ PP P^i_{\;\;j}^k \partial^h, \\
K(\partial^i, \partial^j) \delta_k &= PP QQ Q^i_{\;\;j}^k \delta_h + PP QQ P^i_{\;\;j}^k \delta_h, \\
K(\partial^i, \partial^j) \partial^k &= PP PP Q^i_{\;\;j}^k \delta_h + PP PP P^i_{\;\;j}^k \delta_h, \\
K(\partial^i, \delta_j) \delta_k &= QQ QQ Q^i_{\;\;j}^k \delta_h + QQ QQ P^i_{\;\;j}^k \delta_h, \\
K(\partial^i, \delta_j) \partial^k &= QQ QQ Q^i_{\;\;j}^k \delta_h + QQ QQ P^i_{\;\;j}^k \delta_h,
\end{align*}
\]

where the coefficients are the $M$-tensor fields given by

\[
QQQ Q^i_{\;\;j}^k = \tilde{S}_{jk} \tilde{S}_{il}^h + P^i_{\;\;l} S_{jkl} - \tilde{S}_{jl}^i \tilde{S}_{ikl} - P^i_{\;\;l} S_{jkl} - R^0_{lij} P^i_{\;\;l} + R^i_{lij},
\]
\[ QQ P_{ij} = \tilde{S}_{ik} S_{ilh} + P_{i}^{l} P_{j}^{l} S_{kl} - \tilde{S}_{ik} S_{jih} - P_{j}^{l} S_{ikl} - P_{k}^{l} R_{ij}^{0}, \]

\[ QQ Q_{ij}^{kh} = \tilde{P}_{i}^{k} P_{j}^{l} + P_{j}^{kl} S_{i} - \tilde{P}_{i}^{k} + P_{i}^{kl} S_{j} - P_{j}^{kl} h - R_{ij}^{0} Q_{ih}^{l}, \]

\[ QQ P_{ij}^{kh} = \tilde{P}_{i}^{k} P_{j}^{l} + P_{j}^{kl} S_{i} - \tilde{P}_{i}^{k} + P_{i}^{kl} S_{j} - P_{j}^{kl} h - R_{ij}^{0} Q_{ih}^{l}, \]

\[ PP Q_{ij}^{kh} = \delta_{ij} \tilde{P}_{i}^{l} + P_{i}^{kl} S_{j} - P_{j}^{kl} h - P_{j}^{kl} S_{j} - P_{j}^{kl} h, \]

\[ PP P_{ij}^{kh} = \delta_{ij} \tilde{P}_{i}^{l} + P_{j}^{kl} S_{j} - P_{j}^{kl} h - P_{j}^{kl} S_{j} - P_{j}^{kl} h, \]

\[ PP P_{ij}^{kh} = \delta_{ij} \tilde{P}_{i}^{l} + P_{j}^{kl} S_{j} - P_{j}^{kl} h - P_{j}^{kl} S_{j} - P_{j}^{kl} h, \]

\[ PP P_{ij}^{kh} = \delta_{ij} \tilde{P}_{i}^{l} + P_{j}^{kl} S_{j} - P_{j}^{kl} h - P_{j}^{kl} S_{j} - P_{j}^{kl} h, \]

In order to get the final expressions of the above M-tensor fields, we have to compute the first and second order partial derivatives with respect to the cotangential coordinates, \( p_i \) of the usual tensor fields involved in the definition of the Riemannian metric \( G \).

\[ \partial^{i} G^{(1)}_{kl} = c_{i}^{j} g^{ij} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk}, \]

\[ \partial^{i} G^{(2)}_{kl} = c_{i}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk}, \]

\[ \partial^{i} G^{(3)}_{kl} = c_{i}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk}, \]

\[ \partial^{i} \partial^{i} G^{(1)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk}}, \]

\[ \partial^{i} \partial^{i} G^{(2)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk}}, \]

\[ \partial^{i} \partial^{i} G^{(3)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk}}, \]

\[ \partial^{i} \partial^{i} G^{(1)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk} + d_{1}^{j} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk}}, \]

\[ \partial^{i} \partial^{i} G^{(2)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk} + d_{2}^{j} g^{ij} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk}}, \]

\[ \partial^{i} \partial^{i} G^{(3)}_{kl} = \frac{c_{i}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk} + d_{3}^{j} g^{ij} \delta_{jk}}{c_{i}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk} + d_{4}^{j} g^{ij} \delta_{jk}}, \]
\[ + d_4 g^{0i} g^{0k} \delta^i_j + d_3 g^{ik} \delta^i_j, \]
\[ \partial^i H^{jk}_{(1)} = e_1' g^{0i} g^{jk} + f_1 g^{0i} g^{0j} g^{0k} + f_1 g^{ij} g^{0k} + f_1 g^{0j} g^{ik}, \]
\[ \partial^i H^{jk}_{(2)} = e_2 g^{0i} g_{jk} + f_2 g^{0i} p_j p_k + f_2 \delta^i_j \delta_k, \]
\[ \partial^0 H^{ij}_{k} = e_3 g^{0i} \delta^0_j + f_3 g^{0i} g^{0j} p_k + f_3 g^{ij} p_k + f_3 g^{0j} \delta^i_k. \]

We get the first order partial derivatives of the M-tensor fields \(P^j_{i h}, Q^j_{i h}, S_{ijh}, \bar{P}^j_{i h}, \bar{Q}^j_{i h}, \bar{S}_{ijh}\) with respect to the cotangential coordinates \(p_i\) and we replace these derivatives, and the expressions (3.2), (3.3) of the functions \(e_1, e_2, e_3, f_1, f_2, f_3\) and of their derivatives in order to obtain the components of the curvature tensor as functions of \(a_1, a_2, a_3\) and their derivatives of first, second and third order only. The expressions are obtained by using the Mathematica package RICCI.

\[ \partial^i Q^j_{kh} = \frac{1}{2} \partial^i H^{kh}_{jl} (\partial^j G^{0l}_{(2)} + \partial^0 \partial^j G^{kl}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ + \frac{1}{2} \partial^i H^{kl}_{jl} (\partial^j G^{0l}_{(2)} + \partial^0 \partial^j G^{kl}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ \partial^i \bar{Q}^{jkh} = \frac{1}{2} \partial^i H^{jkl}_{lh} (\partial^0 G^{0l}_{(2)} + \partial^0 \partial^0 G^{ij}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ + \frac{1}{2} \partial^i H^{jkl}_{lh} (\partial^0 G^{0l}_{(2)} + \partial^0 \partial^0 G^{ij}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ \partial^i \bar{P}^j_{kh} = \frac{1}{2} \partial^i H^{jih}_{kl} (\partial^0 G^{0l}_{(2)} + \partial^0 \partial^0 G^{ij}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ + \frac{1}{2} \partial^i H^{jih}_{kl} (\partial^0 G^{0l}_{(2)} + \partial^0 \partial^0 G^{ij}_{(2)} - \partial^0 \partial^l G^{jk}_{(2)}) + \]
\[ \partial^i S_{jkh} = \frac{1}{2} [(e_1' d_4 R_{mjk} + c_2 R_{mjk} - \partial^i \partial^j G^{0j}_{(2)} H^{ih}_{kl} + (c_2 R_{mjk} - \partial^0 G^{0j}_{(2)} H^{ih}_{kl}) - \]
\[ - c_3 \partial^j R_{jkl} H^{0h}_{kl} - c_3 (R_{jkl} H^{0i}_{kl} + R_{jkl} \partial^i H^{0l}_{kl})], \]
\[ \partial^i \bar{S}^{j}_{ik} = \frac{1}{2} [(e_1' d_4 R_{mjk} + c_2 R_{mjk} - \partial^i \partial^j G^{0j}_{(2)} H^{ih}_{kl} + (c_2 R_{mjk} - \partial^0 G^{0j}_{(2)} H^{ih}_{kl}) - \]
\[ - c_3 \partial^j R_{jkl} H^{0h}_{kl} - c_3 (R_{jkl} H^{0i}_{kl} + R_{jkl} \partial^i H^{0l}_{kl})]. \]
In the following, we shall obtain the conditions under which the general natural Kählerian manifold \((T^* M, G, J)\) is an Einstein manifold. The components of the Ricci tensor \(\text{Ric}(Y, Z) = \text{trace}(X \rightarrow K(X, Y)Z)\) of the Kählerian manifold \((T^* M, G, J)\) are given by the formulas:

\[
\text{Ric}_{QQ}^{jk} = \text{Ric}(\delta_j, \delta_k) = QQ_{hjk}^h + PQ^h P_{jkh}^h, \\
\text{Ric}_{PP}^{jk} = \text{Ric}(\delta_j, \delta_k) = PPP_{hjk}^h - PQPQ_{hkh}^h, \\
\text{Ric}_{P}^{k} = \text{Ric}(\delta_j, \delta_k) = \text{Ric}(\delta_k, \delta_j) = PQP^h P_{jkh}^h + QQ_{hjk}^h.
\]

The conditions for the general natural Kählerian manifold \((T^* M, G, J)\) to be Einstein, are

\[
\begin{cases}
\text{Ric}_{QQ}^{jk} - \rho G_{jk}^{(1)} = 0, \\
\text{Ric}_{PP}^{jk} - \rho G_{jk}^{(2)} = 0, \\
\text{Ric}_{P}^{k} - \rho G_{k}^{3} = 0,
\end{cases}
\]

where \(\rho\) is a constant.

After a straightforward computation, using the RICCI package from Mathematica, the three differences which we have to study, become of the next forms:

\[
\begin{align*}
\text{Ric}_{QQ}^{jk} - \rho G_{jk}^{(1)} &= (\lambda + 2\lambda t)^2 \left[\lambda(\lambda + 2\lambda t)\alpha_1 g_{jk} + \beta_1 p_{jk}p_k\right], \\
\text{Ric}_{PP}^{jk} - \rho G_{jk}^{(2)} &= \lambda(\lambda + 2\lambda t)^2 \left[(\lambda + 2\lambda t)\alpha_2 g^{jk} + 2\lambda\beta_2 g^{0i} g^{0k}\right], \\
\text{Ric}_{P}^{k} - \rho G_{k}^{3} &= (\lambda + 2\lambda t)^2 \left[\lambda(\lambda + 2\lambda t)\alpha_3 g^{jk} + \beta_3 p_{jk}p_k\right],
\end{align*}
\]

where \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\) are rational functions depending on \(a_1, a_3, \lambda\), their derivatives of the first two orders, and \(\rho\). We do not present here the explicit expressions of the functions, since they are quite long.

Using lemma 2.1, and taking into account that \(\lambda \neq 0, \lambda + 2\lambda t \neq 0\), we obtain that \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\) must vanish.

Solving the equations \(\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0\) with respect to \(\rho\) we get the same value of \(\rho\), which is quite long and we shall not write here.

Next, from \(\beta_1 = 0, \beta_2 = 0,\) and \(\beta_3 = 0\), we obtain another three values for \(\rho\), which we denote respectively by \(\rho_1, \rho_2,\) and \(\rho_3\). This values must coincide with \(\rho\).

When we impose the conditions \(\rho_2 - \rho = 0, \rho_3 - \rho = 0\), we obtain two equations:

\[
(3.5) \quad (a_1^2 + a_1^2 a_3^2 - 4a_1 a'_1 t - 4a_1 a'_1 a_3^2 t + 4a_1^2 t^2 + 4a_1^2 a_3^2 t^2 - 8a_1 a_3 a'_3 t^2 + 4a_1^2 a_3^2 t^2)(A_1 + B)/N_1 = 0
\]

\[
(3.6) \quad (a_3^2 a_3 - 2a_1 a'_1 a_3 t + 2a_1 a'_3 t + 2a_1 a_3 ct + 2a_1 a_3 ct - 4a_1 a_3 ct - 4a_1 a_3 ct^2 - 4a_1 a_3 ct^2 - 4a_1 a_3 ct^2 + 4a_1 a_3 ct^2)(A_1 + B)/N_2 = 0
\]

where the expressions of \(A, B, N_1, N_2\) are quite long, depending on \(a_1, a_3, \lambda,\) and their derivatives.
Let us study the first parenthesis from (3.5) and (3.6), namely
\[
E = a_1^2 + a_2^2a_3^2 - 4a_1a_1' a_3^2 - 4a_1a_1'a_3't + 4a_1^2a_3a_3't +
+ 4a_1^2t^2 + 4a_1^2a_3^2t^2 - 8a_1a_1'a_3a_3't^2 + 4a_1^2a_3^2t^2,
\]
\[
F = a_1^3a_3 - 2a_1^2a_1'a_3t + 2a_1^3a_3t + 2a_1a_3ct + 2a_1a_3'ct -
- 4a_1a_3't^2 - 4a_1^2a_3't^2 - 4a_1a_3ct^2 + 4a_1^2a_3'ct^2.
\]
The sign of \(E\) may be studied thinking it as a second degree function of the variable \(a_3'\). The associated equation has the discriminant \(\Delta = -(a_1^2t^2(a_1 - 2a_1't)^2) < 0, \forall t > 0\) and the coefficient of \(a_3'^2\), \(4a_1'^2t^2 > 0, \forall t > 0\). Thus, \(E > 0\) for every \(t > 0\). If \(t = 0\), the expression becomes \(a_1^2(1 + a_3'^2) > 0\). Hence we obtained that \(E\) is always positive.

Taking into account of the values of \(a_3'\) and \(a_3''\) from (2.5) and then multiplying by \(\frac{a_1+2a_1't}{a_1+2a_1't} > 0\), \(F = 0\) becomes an equation of the second order with respect to \(a_3'^2\)
\[
(3.7) \quad (a_3'^2)^2 - 4a_1(1-a_3'^2)ct + 4c^2t^2(1+a_3'^2) = 0,
\]
with the discriminant \(\Delta = -16a_1^2(1-a_3'^2)c^2t^2 < 0, \forall t > 0\). Thus \(F > 0, \forall t > 0\) and if \(t = 0, F = a_3'^2 > 0\).

Since \(E\) and \(F\) are always positive, the relations (3.5) and (3.6) are fulfilled if and only if \(An + B = 0\). The obtained equations do not depend on the dimension \(n\) of the base manifold, so we get that both \(A\) and \(B\) must vanish.

From the condition \(A = 0\) we get an expression of \(\lambda''\), given by
\[
\lambda'' = (a_3'^2(-\lambda^2(2a_1^2a_3'^2 + a_1^3a_3'^2 + 4a_1a_1'a_3^2 + 8a_1^2a_3a_3'^2 + 8a_3^2a_3'^2) -
- 2a_1^2\lambda\lambda'(a_1a_1' + 2a_3c) + 2a_1^2\lambda^2) - 2a_1^2t(\lambda^2(-a_1^2a_3'^2 - 6a_1a_1'a_3'^2 - 6a_1a_1'a_3'^2 + 2a_1^3a_3^2c + 2a_1^3a_3'^2c - 20a_1'^2c^2 -
- 20a_1'a_3c + 8a_1a_3a_3'^2 + 32a_1a_3a_3'^2c) + a_1\lambda\lambda'(-a_1^2a_3'^2 + a_1a_3'^2 - 6a_1a_3'^2c - 6a_1a_3'^2c + 12a_1a_3a_3'^2 + 4a_1^3c^2 - 4a_1^3c^2) + 2a_1^3\lambda^2(a_1a_3' + 3c - a_3^2c')c -
- 4a_1^2c\lambda(\lambda^2(-a_1^2a_3'^2 - 3a_1a_3'^2 + 6a_1a_3'^2c - 6a_1c - 3a_1a_3'^2) -
- 20a_1a_1'a_3'^2 - 14a_1a_3'^2c + a_1a_3'^2c + 16a_1a_3a_3'^2c + 32a_1a_1'a_3'^2c +
+ 4a_1^2a_3'^2c - 12a_1a_3a_3'^2c + 4a_1^2a_3a_3'^2c - 4a_1^2a_3a_3'^2c) + a_1\lambda\lambda'(3a_1a_3'^2 - 3a_1a_3'^2 +
+ 3a_1a_3a_3'^2 - 3a_1a_3a_3'^2 - 8a_1a_3a_3'^2 + 2a_1a_3^2c + 2a_1a_3^2c + 6a_1c - 4a_1a_3^2c -
- 10a_1a_3^2c - 16a_1a_3a_3'^2 + 16a_1a_3a_3'^2c + 2a_1^2\lambda^2(-3a_1a_3' + a_1a_3'^2 + 2a_1a_3a_3'^2 -
- 3c - 2a_3^2c + a_3^2c)) - 8a_1c\lambda^2(-\lambda^2(3a_1a_3'^2 + 6a_1a_3'^2c + 3a_1a_3a_3'^2 -
- 12a_1a_1'^2a_3'^2 - 12a_1a_1'^2a_3^2a_3'^2 + 12a_1a_1'^2a_3'^2c + 2a_1^3c + a_1a_3'^2 + 6a_1c^2 +
+ 3a_1a_1'^2a_3'^2 + 6a_1^2a_3'^2 + 3a_1a_1'^2a_3'^2 + 2a_1^3a_3'^2c + a_1a_3'^2c - 12a_1a_1'a_3'^2c -
- 24a_1a_1'a_3^2a_3'^2 - 12a_1a_1'a_3^2a_3'^2c - 12a_1a_1'a_3a_3'^2c + 14a_1^2a_3'^2c -
- 2a_1^2a_3a_3'^2 + 4a_1a_3'^2c - a_1\lambda\lambda'(3a_1a_3'^2 + 3a_1a_3^2c + 2a_1a_3^2c +
+ 2a_1a_3^2c + 5a_1a_3^2c - 7a_1a_1'a_3'^2 + 8a_1a_3a_3'^2 - 8a_1a_3a_3'^2 - 4a_1^2a_3^2c -
- 4a_1a_3a_3'^2 + 4a_1a_3a_3'^2 - 2a_1'c - 6a_1a_3'^2c - 6a_1a_3'^2c + 4a_1a_3a_3'^2 +
+ 8a_1a_3a_3'^2 + 4a_1a_1'a_3'^2c + 2a_1^2\lambda^2(3a_1a_3'^2 + a_1a_3'^2 + 2a_1a_3'^2c -
+ 4a_1^2a_3'^2c + c + 3a_1^2c + 3a_1c + a_3^2c)) - 16a_1^2t^4(\lambda^2(a_1^2a_3'^2 - 2a_1a_1'a_3'a_3'^2) +
+ a_1\lambda\lambda'(-1 + a_3'^2)(a_1' - a_1a_3'^2 + 2a_1^2a_3'^2c - 2a_1a_1'^2a_3^2 + 2a_1^2a_3'a_3'^2 - a_1a_1'a_3'^2 + 2a_1^2a_3'^2 -
- 2a_1a_1'a_3^2a_3'^2 + 2a_1a_1'a_3a_3'^2 + 2a_1^2\lambda^2(1 + a_3^2)(-a_1' - a_1a_3'^2 +
+ 2a_1a_3')c)/(a_1\lambda(a_1' - 2a_1a_3't - 2a_1a_3't - a_1a_3'ct - 4a_1'^2t^2 + 4a_1^2a_3'^2t^2 -
- 8a_1a_3a_3'^2t^2)(a_1' - 4a_1ct + 4a_1^2a_3't + 4c^2t^2 + 8a_1t^2c^2t^2 + 4a_1^2t^2c^2t)^2))
\]
From $B = 0$ we may get the expression of $\lambda'''$, which is quite long and we shall not present it here.

By doing some quite long computations with RICCI, we prove that the differences $\rho_1 - \rho$, $\rho_2 - \rho$ and $\rho_3 - \rho$ vanish when we replace the obtained values for $\lambda''$ and $\lambda'''$. Hence all the expressions obtained for the constant $\rho$ coincide.

Next we have to find the conditions under which the derivative of $\lambda'''$ is equal to $\lambda'''$:

\[(\lambda''')' = \lambda'''.\]

Computing the above difference, we obtain that its numerator must vanish:

\[(a_1^2a_1'\lambda + 2a_1c\lambda + 2a_1a_3^2c\lambda + a_1^3\lambda' - 2a_1'c\lambda t - 2a_1'a_3^2c\lambda t + 4a_1a_3a_3'c\lambda t + 2a_1c\lambda t + 2a_1a_3^2c\lambda t) \]
\[= (a_1^2a_1'\lambda^2 + 2a_1^2a_3^2c\lambda^2 + a_1^4\lambda \lambda' - a_1^4a_1'\lambda^2 t - 4a_1^3a_1'c\lambda^2 t - 4a_1^3a_1'a_3^2c\lambda^2 t + + 4a_1a_1a_3a_3'c\lambda^2 t + 4a_1^2a_3^2c\lambda^2 t + a_1^6\lambda^2 t + 4a_1^2a_1'a_3^2c\lambda^2 t + + 4a_1^2a_1a_3a_3'c\lambda^2 t - 8a_1^3a_1a_3a_3'c\lambda^2 t^2 + 4a_1a_1'a_3^2c^2\lambda^2 t^2 + 8a_1a_1'a_3^2c\lambda^2 t^2 + + 4a_1a_1'a_3a_3'c^2\lambda^2 t^2 - 8a_1^2a_3a_3'c^2\lambda^2 t^2 - 8a_1^2a_3a_3'c^2\lambda^2 t^2 + + 4a_1^3c^2\lambda^2 t^2 - 4a_1^2c^2\lambda^2 t^2 + 4a_1a_3a_3'c^2\lambda^2 t^2 - + 4a_1a_3^2c^2\lambda^2 t^2 - 8a_1^2a_3^2c^2\lambda^2 t^2 + 4a_1^2c^2\lambda^2 t^2)
\[= (a_1^2a_1'\lambda^2 - 2a_1a_3^2c\lambda t + 4a_1a_3^2c\lambda t^2 + 4a_1a_3a_3'c\lambda t + 2a_1a_3^2c\lambda t) = 0.\]

If we replace in the last parenthesis the values of $a_1'$ and $a_3'$ given by $[2.5]$ and then we multiply by the denominator $(a_1 + 2b_1t) > 0$, we obtain the expression $[3.7]$

\[(a_1^2)^2 - 4a_1^2(1 + a_3^2)ct + 4a_1^2(1 + a_3^2)^2,\]

about which we proved that it is always positive.

Hence the cases which must be studied are the next two:

$I)$ $a_1^2a_1'\lambda + 2a_1c\lambda + 2a_1a_3^2c\lambda + a_1^3\lambda' - 2a_1'c\lambda t - 2a_1'a_3^2c\lambda t + + 2a_1a_3a_3'c\lambda t + a_1a_3a_3'c\lambda t = 0,

II)$ $a_1^2a_1'\lambda^2 + 2a_1^2a_3^2c\lambda^2 + a_1^4\lambda \lambda' - a_1^4a_1'\lambda^2 t - 4a_1^3a_1'c\lambda^2 t - 4a_1^3a_1'a_3^2c\lambda^2 t + + 4a_1a_1a_3a_3'c\lambda^2 t - 4a_1^2a_3^2c\lambda^2 t + a_1^6\lambda^2 t + 4a_1^2a_1'a_3^2c\lambda^2 t + + 4a_1^2a_1a_3a_3'c\lambda^2 t - 8a_1^3a_1a_3a_3'c\lambda^2 t^2 + 4a_1a_1'a_3^2c^2\lambda^2 t^2 + 8a_1a_1'a_3^2c\lambda^2 t^2 + + 4a_1a_1'a_3a_3'c^2\lambda^2 t^2 - 8a_1^2a_3a_3'c^2\lambda^2 t^2 - 8a_1^2a_3a_3'c^2\lambda^2 t^2 + + 4a_1^3c^2\lambda^2 t^2 - 4a_1^2c^2\lambda^2 t^2 + 4a_1a_3a_3'c^2\lambda^2 t^2 - + 4a_1a_3^2c^2\lambda^2 t^2 + 4a_1^2a_3^2c^2\lambda^2 t^2 - 4a_1^2c^2\lambda^2 t^2 + 4a_1^2a_3^2c^2\lambda^2 t^2 -
In the case I, we may obtain the following expression of $\lambda'$

$$
\lambda' = -\lambda \frac{a_1 (a_1 a'_1 + 2 c (1 + a_3^2)) - 2 c t (a_1^t + 2 a_1 a_3^2 - 4 a_1 a_3 a_3')}{a_1 (a_1^t + 2 c t (1 + a_3^2))},
$$

Replacing this expression of $\lambda'$ in the first value obtained for $\rho$, we get a rational function of the density of energy, $\rho$, the coefficients $a_1$ and $a_3$, the proportionality factor $\lambda$, and the constant sectional curvature, $c$, of the base manifold $M$

$$
\rho = \frac{2 a_1 c (n + 1)}{\lambda (a_1^t + 2 c t (1 + a_3^2))},
$$

from which we get the value of $\lambda$

$$
(3.8) \quad \lambda = \frac{2 a_1 c (n + 1)}{\rho (a_1^t + 2 c t (1 + a_3^2))}.
$$

Now we may state:

**Theorem 3.2.** Let $(M, g)$ be a smooth $n$-dimensional Riemannian manifold. If $(G, J)$ is a general natural Kählerian structure on the cotangent bundle $T^* M$ and the parameter $\lambda$ is expressed by (3.8), where $\rho$ is a nonzero real constant, then $(T^* M, G, J)$ is a Kähler-Einstein manifold, i.e. $\text{Ric} = \rho G$.

**Remark 3.1.** Taking into account of a theorem from [2], the expression (9.8) of $\lambda$ implies that $(T^* M, G, J)$ is a Kählerian manifold of constant holomorphic sectional curvature $k = \frac{2 \rho}{n + 1}$.

**Example 3.1.** The Kähler-Einstein structure on $T^* M$, from the paper [15] by Oproiu and Poroșniuc, may be obtained from the theorem 3.2, as a particular case. If in the expression (3.8) we impose the condition $a_3 = 0$, we get the same expression of $\lambda$ obtained in [15], in the case of the natural structure of diagonal lifted type on the cotangent bundle $T^* M$ of a Riemannian manifold $(M, g)$.

In the case II, we obtain

$$
\begin{align*}
-a_1^2 t (a_1^t - 4 a_1^2 c t^2 + 4 a_1^2 a_3^2 c^2 t^2 + 4 c t^2 + 8 a_3^2 c^2 t^2 + 4 a_3^2 c^2 t^2) \lambda t^2 + a_1^2 (a_1^t - 4 a_1^2 c t + \\
+ 4 a_1^2 a_3^2 c t^2 + 4 c^2 t^2 + 8 a_3^2 c^2 t^2 + 4 a_3^2 c^2 t^2) \lambda' \lambda + (a_1^2 a_1 + 2 a_1 a_3^2 - a_1^4 a_1^t - 4 a_1^3 a_1 c t - \\
- 4 a_1^3 a_3^2 c t + 4 a_1^2 a_3 a_3^2 c t + 4 a_1^2 a_3^2 c t^2 + 4 a_1^2 a_3^2 c t^2 - 8 a_1^3 a_3 a_3 a_3^2 c^2 t^2 + \\
+ 4 a_1 a_1^t t^2 + 8 a_1 a_1^t a_3^2 c^2 t^2 + 4 a_1 a_1^t a_3^2 c^2 t^2 - 8 a_1^2 a_3 a_3^2 c^2 t^2 - 8 a_1^2 a_3 a_3^2 c^2 t^2 -
\end{align*}
$$
\[-4a_1^4c^2t^3 - 8a_1^2a_3^2c^3t^3 - 4a_1^2a_3^4c^2t^3 + 16a_1a_3a_3^2c^2t^3 + 16a_1a_3^4a_3^2c^2t^3 - \]
\[ -16a_1^2a_3^2a_3^4c^2t^3)\lambda^2 = 0.\]

This is a homogeneous equation of second order in \( \lambda' \) and \( \lambda \) and it may be solved with respect to \( \frac{\lambda'}{\lambda} \). Then we obtain two expressions for \( \lambda' \)

\[
\lambda' = \lambda(\pm \frac{1}{2t} + \frac{a_3^2 - 2a_1a_3^2t - 2a_1a_3^2t - 2a_1a_3^2ct + 4a_1^2a_3^2ct^2 + 4a_1^2a_3^2ct^2 + 8a_1a_3a_3^2ct^2}{2a_3^2t^2 + 4a_1^2a_3^2t^2 + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^2c^2t^2}).
\]

When we replace this expression of \( \lambda' \) and its derivative \( \lambda'' \) in the first value of \( \rho \), we obtain

\[
\rho = \frac{n(a_3^2 + 2ct + 2a_3^2ct \pm \sqrt{a_3^2 - 4a_1^2a_3^2t^2 + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^2c^2t^2})}{4a_1^2a_3^2t^2 + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^2c^2t^2}.
\]

In this case \( \rho \) is defined on the set \( T_0M \subset TM \) of the nonzero cotangent vectors to \( M \), and the value of \( \lambda \) is given by

\[
(3.9) \quad \lambda = \frac{n(a_3^2 + 2ct + 2a_3^2ct \pm \sqrt{a_3^2 - 4a_1^2a_3^2t^2 + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^2c^2t^2})}{4a_1^2a_3^2t^2 + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^2c^2t^2}.
\]

Now we may formulate the next theorem:

**Theorem 3.3.** Let \( (G, J) \) be a general natural Kählerian structure on the cotangent bundle \( T^*M \) of a smooth \( n \)-dimensional Riemannian manifold. If the parameter \( \lambda \) is expressed by [3.9], where \( \rho \) is a nonzero real constant, then \( (G, J) \) is a Kähler-Einstein structure on the bundle \( T^*_0M \), of nonzero cotangent vectors to \( M \), i.e. \( \text{Ric} = \rho G \).

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