AN IMPROVED JULIA-CARATHEODORY THEOREM FOR SCHUR-AGLER MAPPINGS OF THE UNIT BALL

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Abstract. We adapt Sarason’s proof of the Julia-Caratheodory theorem to the class of Schur-Agler mappings of the unit ball, obtaining a strengthened form of this theorem. In particular those quantities which appear in the classical theorem and depend only on the component of the mapping in the complex normal direction have $K$-limits (not just restricted $K$-limits) at the boundary.

Let $\mathbb{B}^n$ denote the open unit ball in $n$-dimensional complex space. In this note we show that holomorphic mappings $\varphi : \mathbb{B}^n \to \mathbb{B}^m$ belonging to the Schur-Agler class (defined below) satisfy a strengthened form of the Julia-Caratheodory theorem (Theorem 1.9). While the Schur-Agler class has received much attention in the past several years from operator theorists, relatively little seems to be known about the function-theoretic behavior of this class.

For many operator theoretic applications, the Schur-Agler classes $\mathcal{S}(n,1)$ and $\mathcal{S}(n,n)$ are more appropriate analogues of the unit ball of $H^\infty(\mathbb{D})$ than are the larger classes $\text{Hol}(\mathbb{B}^n, \mathbb{D})$ and $\text{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. For example, the Schur-Agler class is a natural setting for multivariable versions of von Neumann’s inequality [5], the Sz.-Nagy dilation theorem [3], commutant lifting theorems [4] and the Nevanlinna-Pick interpolation theorem [1]. Additionally, every self-map of the ball belonging to the Schur-Agler class induces a bounded composition operator on the standard holomorphic function spaces $\mathcal{H}$, which is not true of general self-maps of the ball. This last fact suggests that mappings in the Schur-Agler class should also enjoy function-theoretic privileges over generic maps of the ball, and is the motivation for this paper.

Indeed there seems to be little known about the function theory of $\mathcal{S}(n,m)$ apart from what is true generically. Recently Anderson, Dritschel and Rovnyak [2] have established a family of inequalities for derivatives of Schur-Agler functions, though it is not known if these inequalities hold generically. In this paper we show that the Schur-Agler class satisfies a form of the Julia-Caratheodory theorem that is strictly

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stronger than what is true for general holomorphic functions on the unit ball. The result is proved by adapting Sarason’s Hilbert space proof of the classical Julia-Caratheodory theorem [9, Chapter VI] to the ball. In fact Sarason’s proof cannot prove the general Julia-Caratheodory theorem in higher dimensions, since it exploits the positivity of the de Branges-Rovnyak kernel. The analogous kernel in several variables need no longer be positive, but since the Schur-Agler class is precisely the class for which this kernel is positive, the proof goes through but in fact proves a stronger result.

Definition 1.1. The Schur-Agler class $S(n, m)$ is the set of all holomorphic mappings $\varphi : \mathbb{B}^n \to \mathbb{B}^m$ such that the Hermitian kernel

$$k^\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

The kernel $k^\varphi$ is called the de Branges-Rovnyak kernel associated to $\varphi$. We let $H^2_n$ denote the Hilbert space of holomorphic functions on $\mathbb{B}^n$ with reproducing kernel

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle}$$

When $n > 1$ the space $H^2_n$ is strictly smaller than the classical Hardy space $H^2$ (defined by spherical means); however in many ways it is the higher-variable analogue of $H^2(\mathbb{D})$ appropriate for multivariable operator theory, see e.g. [1, 3, 4]. In this context, as mentioned above, the Schur-Agler classes play the role of the unit ball of the algebra of bounded analytic functions in $\mathbb{D}$, though we stress that when $n > 1$ the inclusion $S(n, m) \subset \text{Hol}(\mathbb{B}^n, \mathbb{B}^m)$ is always proper.

Given a Schur-Agler mapping $\varphi \in S(n, m)$, we can define another Hilbert function space $\mathcal{H}(\varphi)$ to be the space of holomorphic functions on $\mathbb{B}^n$ with reproducing kernel $k^\varphi$. This space is always contractively contained in $H^2_n$:

Lemma 1.2. If $\varphi \in S(n, m)$ and $f \in \mathcal{H}(\varphi)$ then $f \in H^2_n$ and

$$\|f\|_{H^2_n} \leq \|f\|_{\mathcal{H}(\varphi)}$$

Proof. The positivity of $k^\varphi$ implies that the operator

$$T : (f_1, \ldots, f_m) \to \sum_{k=1}^m \varphi_k f_k$$
is contractive from the direct sum of $m$ copies of $H^2_n$ to $H^2_n$. The de Branges-Rovnyak kernel may then be written as
\[
k^\varphi(z, w) = \langle (I - TT^*)^{1/2}k_w, (I - TT^*)^{1/2}k_z \rangle_{H^2_n}\]

Now let $f \in \mathcal{H}(\varphi)$. It follows from the standard de Branges-Rovnyak construction applied to $T$ \cite{9}*{Chapter 1} that there exists $g \in H^2_n$ such that $f = (I - TT^*)^{1/2}g$ and $\|f\|_{\mathcal{H}(\varphi)} = \|g\|_{H^2_n}$. Thus $f \in H^2_n$ and $\|f\|_{\mathcal{H}(\varphi)} \geq \|f\|_{H^2_n}$. \hfill \Box

We will be examining the boundary behavior of Schur-Agler mappings and to a lesser extent the behavior of functions in $\mathcal{H}(\varphi)$. We recall here some basic notions in the study of boundary behavior of holomorphic functions on the unit ball, and refer to Rudin \cite{8}*{Chapter 8} (or Krantz \cite{7}*{Section 8.6}) for details.

Given a point $\zeta \in \partial \mathbb{B}^n$ and a real number $\alpha > 0$, the Koranyi region $D_\alpha(\zeta)$ is the set
\[
D_\alpha(\zeta) = \{ z \in \mathbb{B}^n : |1 - \langle z, \zeta \rangle| \leq \frac{\alpha}{2}(1 - |z|^2) \}
\]

A function $f : \mathbb{B}^n \to \mathbb{C}$ has $K$-limit $L$ at $\zeta$ if $\lim_{z \to \zeta} f(z) = L$ whenever $z$ tends to $\zeta$ within a Koranyi region. Note that when $n = 1$, a $K$-limit is just a nontangential limit; however for $n > 1$ $K$-limits allow for parabolic approach in directions orthogonal to $\zeta$. We shall also require the notion of a restricted $K$-limit: to define this, fix a point $\zeta \in \partial \mathbb{B}^n$ and consider a curve $\Gamma : [0, 1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of $\Gamma$ onto the complex line through $\zeta$. The curve $\Gamma$ is called special if
\[
(1) \quad \lim_{t \to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0
\]

and restricted if it is special and in addition
\[
(2) \quad \frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A
\]

for some constant $A > 0$. We say that a function $f : \mathbb{B}^n \to \mathbb{C}$ has restricted $K$-limit $L$ at $\zeta$ if $\lim_{z \to \zeta} f(z) = L$ along every restricted curve.

Lemma 1.3. If $f \in H^2_n$ then
\[
|f(z)| = o((1 - |z|^2)^{-1/2})
\]
as $|z| \to 1$. 

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Proof. The Hilbert space norm of the reproducing kernel $k_z$ is
\[ \sqrt{k(z, z)} = (1 - |z|^2)^{-1/2}. \]
The statement of the lemma is thus equivalent to the statement that the normalized kernel functions \( \tilde{k}_z = k_z/\|k_z\| \) tend weakly to 0 as \( |z| \to 1 \). That this is the case follows readily from two observations: 1) if \( f \in H^2_\phi \) is bounded, then \( \langle f, \tilde{k}_z \rangle \to 0 \) since \( \|k_z\| \to \infty \), and 2) the bounded functions belonging to \( H^2_\phi \) (e.g. the polynomials) are norm dense in \( H^2_\phi \).

Proposition 1.4. Suppose \( \varphi \in S(n, m) \) and \( \zeta \in \partial B^m \). If
\[
h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}
\]
belongs to \( \mathcal{H}(\varphi) \) for some \( \xi \in \mathbb{C}^m \), then \( |\xi| = 1 \) and \( \varphi \) has K-limit \( \xi \) at \( \zeta \).

Proof. If \( h \in \mathcal{H}(\varphi) \) then by growth lemma \( |h(z)| = o((1 - |z|^2)^{-1/2}) \). So
\[
|1 - \langle \varphi(z), \xi \rangle| = o\left( \frac{|1 - \langle z, \zeta \rangle|}{1 - |z|^2} (1 - |z|^2)^{1/2} \right)
\]
which goes to 0 as \( z \to \zeta \) within a Koranyi region; this establishes the claim.

We are interested in Schur-Agler mappings satisfying the following condition, which we call condition (C) following Sarason:

\[
(C) \quad L = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty
\]

The following is then the analogue, for the Schur-Agler class, of Sarason’s Hilbert space formulation of the Julia-Caratheodory theorem [9, Theorem VI-4]:

Theorem 1.5. Let \( \varphi \in S(n, m) \) and \( \zeta \in \partial B^n \). Then the following are equivalent:

1. Condition (C).
2. There exists \( \xi \in \partial B^m \) such that the function
\[
h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}
\]
belongs to \( \mathcal{H}(\varphi) \).
3. Every \( f \in \mathcal{H}(\varphi) \) has a finite K-limit at \( \zeta \).
Proof. First, suppose condition (C) holds. Then there exists a sequence $z_n \to \zeta$ such that

$$L = \lim \|k_{z_n}^\varphi\|^2_\varphi$$

and by passing to a subsequence we may assume that $\varphi(z_n) \to \xi$ for some $\xi$ (necessarily $|\xi| = 1$). By weak compactness of the closed unit ball in $H(\varphi)$ (passing to a further subsequence if necessary) we have $k_{z_n}^\varphi \to h$ weakly for some $h \in H(\varphi)$. Thus for all $z \in \mathbb{B}^n$,

$$h(z) = \langle h, k_{z_n}^\varphi \rangle_\varphi = \lim_{n \to \infty} \frac{1 - \langle \varphi(z), \varphi(z_n) \rangle}{1 - \langle z, z_n \rangle} = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}$$

which proves (2).

Now assume (2). By the lemma, $\varphi$ has $K$-limit $\xi$ at $\zeta$; we will write $\alpha = \varphi(\zeta)$ and $k_{\alpha}^\varphi$ for the function $h$ in (2). To prove (3) it suffices to prove that $k_z^\varphi \to k_{\alpha}^\varphi$ weakly as $z \to \zeta$ within a Koranyi region. By taking inner products with the kernel functions $k_{\alpha}^\varphi$ it is clear that $k_z^\varphi \to k_{\alpha}^\varphi$ pointwise on $\mathbb{B}^n$ as $z \to \zeta$ in a Koranyi region. Since the kernel functions $k_{\alpha}^\varphi$ span $H(\varphi)$, it suffices to prove that the norms $\|k_z^\varphi\|_\varphi$ remain bounded as $z \to \zeta$ in a Koranyi region. For each $z \in \mathbb{B}^n$ we have

$$\langle k_{\alpha}^\varphi, k_z^\varphi \rangle = \frac{1 - \langle \varphi(z), \varphi(\zeta) \rangle}{1 - \langle z, \zeta \rangle}$$

so by the Cauchy-Schwarz inequality

$$\left| \frac{1 - \langle \varphi(z), \varphi(\zeta) \rangle}{1 - \langle z, \zeta \rangle} \right|^2 \leq \|k_{\alpha}^\varphi\|^2_\varphi \|k_z^\varphi\|^2_\varphi = \|k_z^\varphi\|^2_\varphi \left( \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right)$$

The numerator on the left hand side dominates $(1 - |\varphi(z)|)^2$, so

$$\frac{(1 - |\varphi(z)|)^2}{|1 - \langle z, \zeta \rangle|^2} \leq \|k_z^\varphi\|^2_\varphi \left( \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right)$$

which implies

$$\|k_z^\varphi\|^2_\varphi = \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \leq \|k_z^\varphi\|^2_\varphi \left( \frac{1 + |\varphi(z)|^2}{1 + |z|^2} \right) \left( \frac{|1 - \langle z, \zeta \rangle|^2}{1 - |z|^2} \right)^2$$

The right hand side remains bounded as $z \to \zeta$ in a Koranyi region, which proves (3).

The proof that (3) implies (1) is immediate, since by the principle of uniform boundedness the norms $\|k_z^\varphi\|_\varphi$ stay bounded as $z \to \zeta$ in a Koranyi region, which implies condition (C).
Theorem 1.6. Suppose $\varphi \in \mathcal{S}(n,m)$ and satisfies condition (C). Then the function
$$\frac{1 - |(\varphi(z), \xi)|^2}{1 - |(z, \zeta)|^2}$$
has $K$-limit $L$ at $\zeta$.

Proof. By pre- and post-composing with unitary rotations, we may assume without loss of generality that (in the nomenclature of previous theorem) $\xi = e_1$ and $\zeta = e_1$. (We are using $e_1$ to refer to vectors in two different spaces, but this should cause no confusion.)

Starting with the identity
$$1 - \varphi(z) = (1 - z_1)\langle k_{e_1}^\varphi, k_z^\varphi \rangle$$
we find
$$|\varphi(z)|^2 = 1 - 2\text{Re}[(1 - z_1)\langle k_{e_1}^\varphi, k_z^\varphi \rangle] + |1 - z_1|^2|\langle k_{e_1}^\varphi, k_z^\varphi \rangle|^2$$
From what has already been proved, the last term is $o(1 - |z_1|^2)$ as $z \to e_1$ within a Koranyi region. Thus
$$K\text{-lim}_{z \to e_1} \frac{1 - |\varphi(z)|^2}{1 - |z_1|^2} = K\text{-lim}_{z \to e_1} \frac{2\text{Re}[(1 - z_1)\langle k_{e_1}^\varphi, k_z^\varphi \rangle]}{1 - |z_1|^2}$$
As $z \to e_1$ in a Koranyi region, the real part of
$$\frac{1 - z_1}{1 - |z_1|^2}$$
tends to $1/2$ and its imaginary part remains bounded. The real part of $\langle k_{e_1}^\varphi, k_z^\varphi \rangle$ tends to $\|k_{e_1}^\varphi\|^2$ and its imaginary part tends to $0$. Thus
$$K\text{-lim}_{z \to e_1} \frac{2\text{Re}[(1 - z_1)\langle k_{e_1}^\varphi, k_z^\varphi \rangle]}{1 - |z_1|^2} = \|k_{e_1}^\varphi\|^2 = L$$
which completes the proof.

Combining statements (2) and (3) of Theorem 1.5 we obtain our first strengthened conclusion, namely that the function $h$ has finite $K$-limit at $\zeta$. For general $\varphi$ this will exist only as a restricted $K$-limit. The same is true for the expression in Theorem 1.6. These facts will allow us to strengthen the convergence results for directional derivatives of the component of $\varphi$ in the $\zeta$ direction.

In the disk, Theorem 1.6 says that $\|k_z^\varphi\|_{\varphi} \to \|k_\zeta^\varphi\|_{\varphi}$ as $z \to \zeta$ nontangentially; together with the weak convergence of $k_z^\varphi$ to $k_\zeta^\varphi$ this shows
that in fact $k_z^\varphi \to k_\zeta^\varphi$ in norm. In the ball we would like to establish $\|k_z^\varphi\|_\varphi \to \|k_\zeta^\varphi\|_\varphi$ or equivalently
\[
\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \to L
\]
in as general a sense as possible. For generic self-maps $\varphi$, this limit exists restrictedly but not as a $K$-limit in general. Unlike the previous results, however, this cannot be improved for Schur-Agler mappings; in fact for the Schur-Agler mapping $\varphi(z) = z$ the above expression does not have a $K$-limit at $e_1$. Thus in the ball we only have $\|k_z^\varphi\|_\varphi \to \|k_\zeta^\varphi\|_\varphi$ (and hence $k_z^\varphi \to k_\zeta^\varphi$ in norm) when $z \to \zeta$ restrictedly.

The following is Rudin’s version of the Caratheodory theorem on the ball:

**Theorem 1.7.** Suppose $\varphi = (\varphi_1, \ldots, \varphi_m)$ is a holomorphic mapping from $B^n$ to $B^m$ satisfying condition (C) at $e_1$. Suppose $2 \leq j \leq m$ and $2 \leq k \leq n$. The following functions are then bounded in every Koranyi region $D_\alpha(e_1)$:

(i) $(1 - \varphi_1(z))/(1 - z_1)$
(ii) $(D_{1\varphi_1})(z)$
(iii) $\varphi_j(z)/(1 - z_1)^{1/2}$
(iv) $(1 - z_1)^{1/2}(D_{1\varphi_j})(z)$
(v) $(D_{k\varphi_1})(z)/(1 - z_1)^{1/2}$
(vi) $(D_{k\varphi_j})(z)$

Moreover, the functions (i), (ii) have restricted $K$-limit $L$ at $e_1$, and the functions (iii), (iv), (v) have restricted $K$-limit $0$ at $e_1$.

We next show that for $\varphi \in S(n, m)$, the restricted $K$-limits in (i), (ii) and (v) can be improved to $K$-limits. Note that these are precisely the expressions that involve only the $e_1$ component of $\varphi$. This is to be expected, since the improvement derives from the fact that the kernel $k_\zeta^\varphi$ has a $K$-limit at $\zeta$, and this kernel depends only on the component of $\varphi$ in the $\zeta$ (that is, the complex normal) direction. Indeed, the limits of (iii) and (iv) cannot be improved to $K$-limits, since the counterexamples given in [S] are in fact Schur-Agler mappings; this will be shown after proving the next theorem. Before beginning we recall Lemma 8.5.5 of [S] which will be used in the proof.

**Lemma 1.8.** Suppose $1 < \alpha < \beta$, $\delta = \frac{1}{3}(1/\alpha - 1/\beta)$, and $z = (z_1, z') \in D_\alpha$.

(i) If $|\lambda| \leq \delta|1 - z_1|$ then $(z_1 + \lambda, z') \in D_\beta$.
(ii) If $|w| \leq \delta|1 - z_1|^{1/2}$ then $(z_1, z' + w') \in D_\beta$. 
Theorem 1.9. Suppose that $\varphi \in S(n, m)$ and satisfies condition (C). Then in (i), (ii) and (v) of Theorem 1.7, restricted $K$-limit can be improved to $K$-limit.

Proof. Since we are assuming condition (C), we know from statement (2) of Theorem 1.5 that the function

$$k_{e_1}^\varphi(z) = \frac{1 - \varphi_1(z)}{1 - z_1}$$

belongs to $\mathcal{H}(\varphi)$ and hence by statement (3) has a $K$-limit at $e_1$; this limit must of course equal $L$.

For (ii), suppose $1 < \alpha < \beta$, choose $\delta$ as in the lemma, let $z \in D_\alpha$ and put

$$r = r(z) = \delta |1 - z_1|$$

As in [8], express $D_1\varphi_1$ using the Cauchy formula; after some manipulation we obtain

$$(D_1\varphi_1)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \varphi_1(z_1 + re^{i\theta}, z')}{1 - (z_1 + re^{i\theta})} \cdot \left\{1 - \frac{1 - z_1}{re^{i\theta}}\right\} d\theta \tag{3}$$

We must show that the above expression tends to $L$ along any sequence converging to $e_1$ within $D_\alpha$; in fact it suffices to show that given any such sequence, $D_1\varphi_1$ converges to $L$ along some subsequence. In particular we may assume that we have chosen a sequence $(z_n)$ such that

$$\lim_{n \to \infty} \frac{1 - z_{n,1}}{r(z_n)e^{i\theta}} = \frac{1}{\delta e^{i\theta}} \lim_{n \to \infty} \frac{1 - z_{n,1}}{|1 - z_{n,1}|}$$

exists, and is equal to some complex number $\lambda$. Then as $z_n \to e_1$ in $D_\alpha$, we have

$$z_n + r(z_n)e^{i\theta}e_1 \to e_1$$

in $D_\beta$, so the integrand in (3) converges to

$$L \cdot (1 - \frac{\lambda}{\delta e^{i\theta}})$$

for every $\theta \in [0, 2\pi]$. Since this integrates to $L$, and the integrands are uniformly bounded, we conclude $D_1\varphi_1(z_n) \to L$ by the dominated convergence theorem.

The $K$-limit of (v) is established similarly: we let $\alpha, \beta, \delta$ be as before, and for $z \in D_\alpha(e_1)$ we define

$$\rho = \rho(z) = \delta |1 - z_1|^{1/2}$$

Then by the lemma, $(z_1, z' + w') \in D_\beta(e_1)$ for all $w'$ with $|w'| \leq \rho$. Assuming $k = 2$ (without loss of generality), we apply the Cauchy
formula to obtain for every $z \in D_\alpha$

$$\frac{(D_2\varphi_1)(z)}{(1-z_1)^{1/2}} = -\frac{(1-z_1)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\varphi_1(z_1, z_2 + \rho e^{i\theta}, \ldots)}{1-z_1} e^{i\theta} \, d\theta$$

The factor outside the integral is bounded. As $z \to e_1$ within $D_\alpha$, $z + \rho(z)e^{i\theta}e_2 \to e_1$ within $D_\beta$, so the integrand tends to $Le^{i\theta}$ for every $\theta$. Thus

$$\frac{(D_2\varphi_1)(z)}{(1-z_1)^{1/2}} \to 0$$

by the dominated convergence theorem. \[\Box\]

Rudin \[8\] gives counterexamples to show that “restricted $K$-limit” cannot be improved to “$K$-limit” in Theorem 1.7 in the case of (iii) and (iv), the example is a map $\varphi : \mathbb{B}^2 \to \mathbb{B}^2$ of the form

$$\varphi(z_1, z_2) = (z_1, z_2g(z_1))$$

for a suitably chosen holomorphic function $g : \mathbb{D} \to \mathbb{D}$. It is not hard to show that any map of the form (4) belongs to $S(2, 2)$. To see this, first observe that because $g : \mathbb{D} \to \mathbb{D}$, the kernel

$$\frac{1 - g(z)g(w)}{1 - zw}$$

is positive. We may then write

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} = \frac{1 - \langle z, w \rangle + z_2\overline{w_2} - z_2\overline{w_2}g(z_1)g(w_1)}{1 - \langle z, w \rangle}$$

$$= 1 + z_2\overline{w_2} \frac{1 - g(z_1)g(w_1)}{1 - z_1\overline{w_1}} \cdot \frac{1 - z_1\overline{w_1}}{1 - \langle z, w \rangle}$$

which is positive.

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