Abstract. We give a combinatorial proof of an elementary property of generalized Lucas Polynomials, inspired by [1]. These polynomials in $s$ and $t$ are defined by the recurrence relation $\langle n \rangle = s\langle n-1 \rangle + t\langle n-2 \rangle$ for $n \geq 2$. The initial values are $\langle 0 \rangle = 2, \langle 1 \rangle = s$ respectively.

1. Introduction and Motivation

In this paper, we shall focus on giving a combinatorial proof of a result on the generalized Lucas polynomials. But first we give some introductory remarks and motivation. The famous Fibonacci numbers, $F_n$ are defined by $F_0 = 0, F_1 = 1$ and, for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$ 

The Lucas numbers $L_n$ are defined by the same recurrence, with the initial conditions $L_0 = 2$ and $L_1 = 1$.

One generalization of these numbers which has received much attention is the sequence of Fibonacci polynomials

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2,$$

with initial conditions $F_0(x) = 0, F_1(x) = 1$. The generalized Fibonacci polynomials depend on two variables $s, t$ and are defined by $\{0\}_{s,t} = 0$,
\{1\}_{s,t} = 1 \text{ and, for } n \geq 2, \\
\{n\}_{s,t} = s\{n-1\}_{s,t} + t\{n-2\}_{s,t}.

Here and with other quantities depending on \(s\) and \(t\), we will drop the subscripts as they will be clear from context. For example, we have

\[\{2\} = s, \quad \{3\} = s^2 + t, \quad \{4\} = s^3 + 2st, \quad \{5\} = s^4 + 3s^2t + t^2.\]

For some historical remarks and relations of these polynomials we refer the reader to [1], [2] and [3].

The main focus of our paper are the generalized Lucas polynomials defined by

\[\langle n \rangle_{s,t} = s\langle n-1 \rangle_{s,t} + t\langle n-2 \rangle_{s,t}, \quad n \geq 2\]

together with the initial conditions \(\langle 0 \rangle_{s,t} = 2\) and \(\langle 1 \rangle_{s,t} = s\). The first few polynomials are

\[\langle 2 \rangle_{s,t} = s^2 + 2t, \quad \langle 3 \rangle_{s,t} = s^3 + 3st, \quad \langle 4 \rangle_{s,t} = s^4 + 4s^2t + 2t^2, \quad \langle 5 \rangle_{s,t} = s^5 + 5s^3t + 5st^2.\]

When \(s = t = 1\) these reduce to the ordinary Lucas numbers.

2. Combinatorial Interpretations of \(\{n\}\) and \(\langle n \rangle\)

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet - \bullet & \bullet & \\
\bullet & \bullet - \bullet \\
\end{array}
\]

Figure 1: Linear and circular tilings

In addition to the algebraic approach to our polynomials, there is a combinatorial interpretation derived from the standard interpretation of \(F_n\) via tiling, given in [3]. A \textit{linear tiling}, \(T\), of a row of squares is a covering of the squares with dominos (which cover two squares) and monominos (which cover one square). We let,

\[\mathcal{L}_n = \{T \ : \ T \text{ a linear tiling of a row of } n \text{ squares}\}.\]
The three tilings in the first row of Figure 1 are the elements of $L_3$. We will also consider circular tilings where the (deformed) squares are arranged in a circle. We will use the notation $C_n$ for the set of circular tilings of $n$ squares. So the set of tilings in the bottom row of Figure 1 is $C_3$. For any type of nonempty tiling, $T$, we define its weight to be

$$\text{wt } T = s\# \text{ of monominos in } T + t\# \text{ of dominos in } T.$$ 

We give the empty tiling $\epsilon$ of zero boxes the weight $\text{wt } \epsilon = 1$, if it is being considered as a linear tiling or $\text{wt } \epsilon = 2$, if it is being considered as a circular tiling. The following proposition is immediate from the definitions of weight and of our generalized polynomials.

**Proposition 2.1** (Sagan and Savage, [3]). For $n \geq 0$, we have

$$\{n + 1\} = \sum_{T \in L_n} \text{wt } T$$

and

$$\langle n \rangle = \sum_{T \in C_n} \text{wt } T.$$ 

From the above discussions on the combinatorial interpretations of $\{n\}$ and $\langle n \rangle$ we get the following.

**Theorem 2.2** (Sagan and Savage, [3]). For $m \geq 1$ and $n \geq 0$ we have

$$\{m + n\} = \{m\}\{n + 1\} + t\{m - 1\}\{n\}.$$ 

**Proposition 2.3** (Sagan and Savage, [3]). For $n \geq 1$ we have

$$\langle n \rangle = \{n + 1\} + t\{n - 1\}.$$ 

And for $m, n \geq 0$ we have

$$\{m + n\} = \frac{\langle m \rangle\{n\} + \langle m \rangle\langle n \rangle}{2}.$$ 

For some more interesting combinatorial interpretations, we refer the reader to [1] and [3].

3. Main Result

The main aim of this paper is to give a combinatorial proof of the following result, inspired by [1].
**Theorem 3.1.** For $s, t \in \mathbb{N}$ such that $\frac{1}{s + t} < \min \left\{ \frac{1}{|X|}, \frac{1}{|Y|} \right\}$, we have for $X, Y \neq 0$

$$\sum_{n=0}^{\infty} \frac{\langle n \rangle_{s,t}}{(s + t)^{n+1}} = \frac{s + 2t}{t(s + t - 1)}.$$  

**Proof.** We consider an infinite row of squares which extends to both directions. A random square is marked as the $0^{th}$ place. The squares are numbered from left to right of 0 by the positive integers and from the right to left of 0 by the negative integers.

We now suppose that each square can be coloured with one of $s$ shades of white and $t$ shades of black. Let $Z$ be the random variable which returns the box number at the end of the first odd-length block of boxes of the same black shade starting from the right of 0, and let $Z'$ be the random variable which returns the box number at the end of the first odd-length block of boxes of the same black shade from the left of 0. And let $W$ be the event be the combination of both $Z$ and $Z'$.

For any integer $n$, the event $W = n$ is the combination of $Z = n$ and $Z' = -n$. Here $Z = n$ is equivalent to having box $n$ painted with one of the shades of black among the first $n$ squares being of even length including 0 to the right of 0. So there are $t$ choices for the colour of box $n$ and $s + t - 1$ choices for the colour of box $n + 1$. Similarly $Z' = -n$ is equivalent to having box $-n$ painted with one of the shades of black among the first $n$ squares being of even length including 0 to the left of 0. So, there are $t$ choices for the colour of box $-n$ and $s + t - 1$ choices for the colour of box $-(n + 1)$.

Each colouring of the first $n$ squares gives a tiling where each white box is replaced by a monomino and a block of $2k$ boxes of the same shade of black is replaced by $k$ dominoes. Also, the weight of the tiling is just the number of colourings attached to it. Thus, the number of the colourings of the first $n$ boxes is $\langle n \rangle$ since the $n$ boxes both to the right and left of 0 will give rise to a circular tiling in this case. Indeed, the number of the colourings of the first $n$ boxes to the right of 0, including the box 0 is $\{n + 1\}$. Moreover, if the number of black shades boxes to the left of 0, not including the box 0, is even, then the number of the colourings of the first $n - 1$ boxes to the left of 0, not including the box 0 is $\{n\}$. By convention, we fix the shade of the box...
0 to be white. Since there are $s$ possible white shades for the box 0, there are $s\{n\}$ colourings of the first $n$ boxes to the left of 0, including the box 0. So, there are $(n + 1) - s\{n\} = t\{n - 1\}$ colourings of the first $n - 1$ boxes to the left of 0, not including the box 0. This implies that the number of the colourings of the first $n$ boxes is $(n + 1) + t\{n - 1\} = \langle n \rangle$.

Notice that if the shade of the box 0 is white, then the box 0 contributes by a factor $s$ to the total number of circular tilings for which $W = n$ whereas if the shade of the box 0 is black, then the box 0 contributes by a factor $2t$ to the total number of circular tilings since for each black shade, there are two possibilities (namely the two neighbours of box 0 in a circular tiling). It gives rise to a multiplicative factor $s + 2t$ in the expression of the total number of circular tilings for which $W = n$. Notice that once we count $s + 2t$ for the box 0, the other boxes (including the box $n + 1$) contributes by a multiplicative factor $s + t$ to the total number of circular tiling. Thus, the total number of circular tilings for which $W = n$ is given by $(s + 2t)(s + t)^{n+1}$.

Hence we have,

$$P(W = n) = \frac{t(s + t - 1)\langle n \rangle_{s,t}}{(s + 2t)(s + t)^{n+1}}.$$ 

Summing these will give us the desired result. "

Acknowledgements

The second author was supported by DST INSPIRE Scholarship 422/2009 from the Department of Science and Technology, Government of India when this work was done.
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