Abstract. Generalizing the notion of Newton polytope, we define the Newton-
Okounkov body, respectively, for semigroups of integral points, graded alge-
bras, and linear series on varieties. We prove that any semigroup in the lattice
$\mathbb{Z}^n$ is asymptotically approximated by the semigroup of all the points in a sub-
lattice and lying in a convex cone. Applying this we obtain several results: we
show that for a large class of graded algebras, the Hilbert functions have poly-
nomial growth and their growth coefficients satisfy a Brunn-Minkowski type
inequality. We prove analogues of Fujita approximation theorem for semi-
groups of integral points and graded algebras, which implies a generalization
of this theorem for arbitrary linear series. Applications to intersection theory
include a far-reaching generalization of the Kusminenko theorem (from Newton
polytope theory) and a new version of the Hodge inequality. We also give
elementary proofs of the Alexandrov-Fenchel inequality (and its corollaries) in
convex geometry and their analogues in algebraic geometry.

Key words: semigroup of integral points, convex body, mixed volume, Alexan-
drov-Fenchel inequality, Hilbert function, graded algebra, Cartier divisor and linear se-
ries, Hodge index theorem, Bernstein-Kušnirenko theorem

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Introduction

This paper is dedicated to a generalization of the notion of Newton polytope (of a Laurent polynomial). We introduce the Newton-Okounkov body and prove a series of results related to it. It is a completely expanded and revised version of the second part of the preprint [Kaveh-Khovanskii08-1]. A revised and extended version of the first part can be found in [Kaveh-Khovanskii08-2]. Nevertheless, the present paper is totally independent and self-contained. Here we develop a geometric approach to semigroups in \( \mathbb{Z}^n \) and apply the results to graded algebras, intersection theory and convex geometry.

A generalization of the notion of Newton polytope was started by the pioneering works of Okounkov [Okounkov96, Okounkov03]. A systematic study of the Newton-Okounkov body was introduced about the same time in the papers [Lazarsfeld-Mustata08] and [Kaveh-Khovanskii08-1]. Recently the Newton-Okounkov body (which Lazarsfeld-Mustata call Okounkov body) has been explored and used in the papers [Yuan08], [Nystrom09] and [Jow09].

First, we briefly discuss the results we need from [Kaveh-Khovanskii08-2] and then we will explain the results of the present paper in more details. For the sake of simplicity throughout the introduction we may use slightly simplified notation compared to the rest of the paper.

The remarkable Bernstein-Kusnirenko theorem computes the number of solutions of a system of equations \( P_1 = \cdots = P_n = 0 \) in \( (\mathbb{C}^*)^n \), where each \( P_i \) is a generic Laurent polynomial taken from a finite dimensional subspace \( L_i \) spanned by Laurent monomials. The answer is given in terms of the mixed volumes of the Newton polytopes of the polynomials \( P_i \). (The Kusnirenko theorem deals with the case where the Newton polytopes of all the equations are the same; the Bernstein theorem concerns the general case.)

In [Kaveh-Khovanskii08-2] a much more general situation is addressed. Instead of \( (\mathbb{C}^*)^n \) one takes any irreducible \( n \)-dimensional algebraic variety \( X \), and instead of the finite dimensional subspaces \( L_i \) spanned by monomials one takes arbitrary non-zero finite dimensional subspaces of rational functions on \( X \). We denote the collection of all the non-zero finite dimensional subspaces of rational functions on \( X \) by \( K_{rat}(X) \). For an \( n \)-tuple \( L_1, \ldots, L_n \in K_{rat}(X) \), we define the intersection index \( [L_1, \ldots, L_n] \) as the number of solutions in \( X \) of a system of equations \( f_1 = \cdots = f_n = 0 \), where each \( f_i \) is a generic element in \( L_i \). In counting the number of solutions one neglects the solutions at which all the functions from a subspace \( L_i \), for some \( i \), are equal to 0, and the solutions at which at least one function in \( L_i \), for some \( i \), has a pole. One shows that this intersection index is well-defined and has all the properties of the intersection index of divisors on a complete variety. There is
a natural multiplication in the set $K_{rat}(X)$. For $L, M \in K_{rat}(X)$ the product $LM$ is the span of all the functions $fg$, where $f \in L$, $g \in M$. With this product, the set $K_{rat}(X)$ is a commutative semigroup. Moreover, the intersection index is multilinear with respect to this product and hence can be extended to the Grothendieck group of $K_{rat}(X)$, which we denote by $G_{rat}(X)$ (see Section 1.2). If $X$ is a normal projective variety, the group of (Cartier) divisors on $X$ can be embedded as a subgroup of the group $G_{rat}(X)$. Under this embedding, the intersection index in the group of divisors coincides with the intersection index in the group $G_{rat}(X)$. Thus the intersection index in $G_{rat}(X)$ can be considered as a generalization of the classical intersection index of divisors, which is birationally invariant and can be applied to non-complete varieties also (as discussed in [Khovanskii92]).

Now about the contents of the present paper: we begin with proving general (and not very hard) results regarding a large class of semigroups of integral points. The origin of our approach goes back to [Khovanskii92].

Let us start with a class of semigroups with a simple geometric construction: for an integer $0 \leq q < n$, let $L$ be a $(q + 1)$-dimensional rational subspace in $\mathbb{R}^n$, $C$ a $(q + 1)$-dimensional convex cone in $L$ with apex at the origin, and $G$ a subgroup of full rank $q + 1$ in $L \cap \mathbb{Z}^n$. The set $\tilde{S} = G \cap C$ is a semigroup with respect to addition. (After a linear change of coordinates, we can assume that the group $G$ coincides with $L \cap \mathbb{Z}^n$ and hence $\tilde{S} = C \cap \mathbb{Z}^n$.) In addition, assume that the cone $C$ is strongly convex, that is, $C$ does not contain any line. Let $M_0 \subset L$ be a rational $q$-dimensional linear subspace which intersects $C$ only at the origin. Consider the family of rational $q$-dimensional affine subspaces in $L$ parallel to $M_0$ such that they intersect the cone $C$ as well as the lattice $G$. Let $M_k$ denote the affine subspace in this family which has distance $k$ from the origin. Let us normalize the distance $k$ so that as values it takes all the non-negative integers. Then, this family of parallel affine subspaces can be enumerated as $M_0, M_1, M_2, \ldots$. It is not hard to estimate the number $H_\tilde{S}(k)$ of points in the set $\tilde{S}_k = M_k \cap \tilde{S}$. For sufficiently large $k$, $H_\tilde{S}(k)$ is approximately equal to the (normalized in the appropriate way) $q$-dimensional volume of the convex body $C \cap M_k$. This idea, which goes back to Minkowski, shows that $H_\tilde{S}(k)$ grows like $a_k k^q$ where the $q$-th growth coefficient $a_k$ is equal to the (normalized) $q$-dimensional volume of the convex body $\Delta(\tilde{S}) = C \cap M_1$.

We should point out that the class of semigroups $\tilde{S}$ above has already a rich and interesting geometry even when $C$ is just a simplicial cone. For example, it is related to a higher dimensional generalization of continuous fractions originating in the work of V. I. Arnold [Arnold98].

Now let us discuss the case of a general semigroup of integral points. Let $S \subset \mathbb{Z}^n$ be a semigroup. Let $G$ be the subgroup of $\mathbb{Z}^n$ generated by $S$, $L$ the subspace of $\mathbb{R}^n$ spanned by $S$, and $C$ the closure of the convex hull of $S \cup \{0\}$, that is, the smallest closed convex cone (with apex at the origin) containing $S$. Clearly, $G$ and $C$ are contained in the subspace $L$. We define the regularization $\tilde{S}$ of $S$ to

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1 A linear subspace of $\mathbb{R}^n$ is called \textit{rational} if it can be spanned by rational vectors (equivalently integral vectors). An affine subspace is said to be \textit{rational} if it is parallel to a rational subspace.

2 For a function $f$, we define the \textit{$q$-th growth coefficient} $a_q$ to be the limit $\lim_{k \to \infty} f(k)/k^q$ (whenever this limit exists).
be the semigroup $C \cap G$. From the definition $\hat{S}$ contains $S$. We prove that the regularization $\hat{S}$ asymptotically approximates the semigroup $S$. We call this the approximation theorem. More precisely:

**Theorem 1.** Let $C' \subset C$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L$) of the cone $C$ only at the origin. Then there exists a constant $N > 0$ (depending on $C'$) such that any point in the group $G$ which lies in $C'$ and whose distance from the origin is bigger than $N$ belongs to $S$.

Now, in addition, assume that the cone $C$ constructed from $S$ is strongly convex. Let $\dim L = q + 1$. Fix a rational $q$-dimensional subspace $M_0 \subset L$ intersecting $C$ only at the origin and as above let $M_k$, $k \in \mathbb{Z}_{\geq 0}$, be the family of $q$-dimensional affine subspaces parallel to $M_0$. That is, each $M_k$ intersects the cone $C$ as well as the group $G$. Let $H_S(k)$ and $H_{\hat{S}}(k)$ be the number of points in the levels $S_k = S \cap M_k$ and $\hat{S}_k = \hat{S} \cap M_k$ respectively. The function $H_S$ is called the Hilbert function of the semigroup $S$.

Let $\Delta(S) = C \cap M_1$. We call it the Newton-Okounkov body of the semigroup $S$. Note that $\dim \Delta(S) = q$. By the above discussion (Minkowski’s observation) the Hilbert function $H_S(k)$ grows like $a_q k^q$ where $a_q$ is the (normalized) $q$-dimensional volume of $\Delta(S)$. But, by the approximation theorem, the Hilbert functions $H_S(k)$ and $H_{\hat{S}}(k)$ have the same asymptotic, as $k$ goes to infinity. It thus follows that the volume of $\Delta(S)$ is responsible for the asymptotic of the Hilbert function $H_S$ as well, i.e.

**Theorem 2.** The function $H_S(k)$ grows like $a_q k^q$ where $q$ is the dimension of the convex body $\Delta(S)$, and the $q$-th growth coefficient $a_q$ is equal to the (normalized in the appropriate way) $q$-dimensional volume of $\Delta(S)$.

More generally, we extend the above theorem to the sum of values of a polynomial on the points in the semigroup $S$ (Theorem 1.13).

Next, we describe another result about the asymptotic behavior of a semigroup $S$. With each non-empty level $S_k = C \cap M_k$ we can associate a subsemigroup $\hat{S}_k \subset S$ generated by this level. It is non-empty only at the levels $kt$, $t \in \mathbb{N}$. Consider the Hilbert function $H_{\hat{S}_k}(kt)$ equal to the number of points in the level $kt$, of the semigroup $\hat{S}_k$. Then if $k$ is sufficiently large, $H_{\hat{S}_k}(kt)$, regarded as a function of $t \in \mathbb{N}$, grows like $a_{q,k} t^q$ where the $q$-th growth coefficient $a_{q,k}$ depends on $k$. We show that:

**Theorem 3.** The growth coefficient $a_{q,k}$ for the function $H_{\hat{S}_k}$, considered as a function of $k$, has the same asymptotic as the Hilbert function $H_S(k)$ of the original semigroup $S$.

Now we explain the results in the paper on graded algebras. Let $F$ be a finitely generated field of transcendence degree $n$ over a ground field $k$. Let $F[t]$ be the algebra of polynomials over $F$. We will be concerned with the graded $k$-subalgebras of $F[t]$ and their Hilbert functions. In order to apply the results about the semigroups to graded subalgebras of $F[t]$ one needs a $\mathbb{Z}^{n+1}$-valued valuation $v_t$ on the ring $F[t]$. Let $I$ be an ordered abelian group. An $I$-valued valuation on an algebra $A$ is a map from $A \setminus \{0\}$ to $I$ which respects the algebra operations (see Section 3). There is also the closely related notion of saturation of a semigroup $S$ which is the semigroup $C \cap \mathbb{Z}^n$. 
depends on \(k\) almost integral type. We prove it by reducing it, via the valuation prove an abstract analogue of the Fujita approximation theorem for algebras of almost integral type is new. We construct a valuation \(v_t\) on \(F[t]\) by extending a valuation \(v\) on \(F\). We will also need \(v\) to be faithful, i.e. it takes all the values in \(\mathbb{Z}^n\). It is well-known how to construct many such valuations for any finitely generated field \(F\) of transcendence degree \(n\) (over \(k\)). We present main examples of such valuations \(v\) (see Section \ref{22}).

The valuation \(v_t\) maps the non-zero elements of a graded subalgebra \(A \subset F[t]\) to a semigroup of integral points in \(\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}\). This gives a connection between the graded subalgebras of \(F[t]\) and semigroups in \(\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}\).

The following types of graded subalgebras in \(F[t]\) will play the main roles for us:

- The algebra \(A_L = \bigoplus_{k \geq 0} L^k t^k\), where \(L\) is a non-zero finite dimensional subspace of \(F\) over \(k\). Here \(L^0 = k\) and for \(k > 0\) the space \(L^k\) is the span of all the products \(f_1 \cdots f_k\) with \(f_1, \ldots, f_k \in L\). It is a graded algebra generated by \(k\) and finitely many degree 1 elements.
- An algebra of integral type is a graded subalgebra which is a finite module over some algebra \(A_L\), equivalently, a graded subalgebra which is finitely generated and integral over some algebra \(A_L\).
- An algebra of almost integral type is a graded subalgebra which is contained in an algebra of integral type, equivalently, a graded subalgebra which is contained in some algebra \(A_L\).

By the Hilbert-Serre theorem on finitely generated modules over a polynomial ring, it follows that the Hilbert function \(H_A(k)\) of an algebra \(A\) of almost integral type does not grow faster than \(k^n\). From this one can then show that the cone \(C\) associated to the semigroup \(S(A) = v_t(A \setminus \{0\})\) is strongly convex. Let \(\Delta(A)\) denote the Newton-Okounkov body of the semigroup \(S(A)\). We call \(\Delta(A)\) the Newton-Okounkov body of the algebra \(A\). Applying Theorem \ref{2} above we prove:

**Theorem 4.** 1) After appropriate rescaling of the argument \(k\), the Hilbert function \(H_A(k)\) grows like \(a_q k^q\), where \(q\) is an integer between 0 and \(n\). 2) Moreover, the degree \(q\) is equal to the dimension of \(\Delta(A)\), and \(a_q\) is the (normalized in the appropriate way) \(q\)-dimensional volume of \(\Delta(A)\).

When \(A\) is of integral type, again by the Hilbert-Serre theorem, the Hilbert function becomes a polynomial of degree \(q\) for large values of \(k\) and the number \(q! a_q\) is an integer. When \(A\) is of almost integral type, in general, the Hilbert function \(H_A\) is not a polynomial for large \(k\) and \(a_q\) can be transcendental. It seems to the authors that the result above on the polynomial growth of the Hilbert function of algebras of almost integral type is new.

The Fujita approximation theorem in the theory of divisors states that the so-called volume of a big divisor can be approximated by the self-intersection numbers of ample divisors (see \cite{Fujita94}, \cite{Lazarsfeld04} Section 11.4)). In this paper, we prove an abstract analogue of the Fujita approximation theorem for algebras of almost integral type. We prove it by reducing it, via the valuation \(v_t\), to the corresponding result on the semi groups (Theorem \ref{8} above). With each non-empty homogeneous component \(A_k\) of the algebra \(A\) one associates the graded subalgebra \(A_k\) generated by this component. For fixed large enough \(k\), the Hilbert function \(H_{A_k}(kt)\) of the algebra \(A_k\) grows like \(a_q,k t^q\) where the \(q\)-th growth coefficient \(a_{q,k}\) depends on \(k\).
Theorem 5. The q-th growth coefficient $a_{q,k}$ of the Hilbert function $H_{A_i}$, regarded as a function of $k$, has the same asymptotic as the Hilbert function $H_{A}(k)$ of the algebra $A$.

Hilbert’s theorem on the dimension and degree of a projective variety yields an algebro-geometric interpretation of the above facts. Let $X$ be an irreducible $n$-dimensional algebraic variety. To each subspace $L \in \mathbf{K}_{rat}(X)$ one can associate a rational Kodaira map $\Phi_L : X \rightarrow \mathbb{P}(L^*)$, where $\mathbb{P}(L^*)$ is the projectivization of the dual space to $L$. Take a point $x \in X$ such that all the $f \in L$ are defined at $x$ and not all are zero at $x$. To $x$ there corresponds a functional $\xi_x$ on $L$ given by $\xi_x(f) = f(x)$ for all $f \in L$. The Kodaira map sends $x$ to the image of this functional in the projective space $\mathbb{P}(L^*)$. Let $Y_L \subset \mathbb{P}(L^*)$ be the closure of the image of $X$ under the map $\Phi_L$. Consider the algebra $A_L$ associated to the subspace $L$. Then by Hilbert’s theorem we see that: the dimension of the variety $Y_L$ is equal to the dimension of the body $\Delta(A_L)$, and the degree of $Y_L$ (in the projective space $\mathbb{P}(L^*)$) is equal to $q!$ times the $q$-dimensional (normalized in the appropriate way) volume of $\Delta(A_L)$.

One naturally defines a componentwise product of graded subalgebras (see Definition 2.23). Consider the class of graded algebras of almost integral type such that, for large enough $k$, all their $k$-th homogeneous components are non-zero. Let $A_1$, $A_2$ be algebras of such kind and put $A_3 = A_1A_2$. It is easy to verify the inclusion

$$\Delta_0(A_1) + \Delta_0(A_2) \subset \Delta_0(A_3),$$

where $\Delta_0(A_1)$ is the Newton-Okounkov body for the algebra $A_1$ projected to $\mathbb{R}^n$ (via the projection on the first factor $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$). Using the previous result on the $n$-th growth coefficient $a_n(A_1)$ of the Hilbert function of the algebra $A_1$ and the classical Brunn-Minkowski inequality we then obtain the following inequality:

Theorem 6.

$$a_n(A_1) + a_n(A_2) \leq a_n(A_3).$$

The results about graded subalgebras of polynomials in particular apply to the algebra of sections of a divisor. In Section 3.2 we see that the algebra of sections of a divisor is an algebra of almost integral type. Applying the above results to this algebra we recover several well-known results regarding the asymptotic theory of divisors and linear series. Moreover, we obtain some new results about the case when the divisor is not a big divisor. As a corollary of our Theorem 5 we generalize the interesting Fujita approximation result in Lazarsfeld-Mustata08 Theorem 3.3. The result in Lazarsfeld-Mustata08 applies to the so-called big divisors (or more generally big graded linear series) on a projective variety. Our generalization holds for any divisor (more generally any graded linear series) on any complete variety (Corollary 3.13). The point is that beside following the ideas in Lazarsfeld-Mustata08, we use results which apply to arbitrary semigroups of integral points. Another difference between the approach in the present paper and that of Lazarsfeld-Mustata08 is that we use abstract valuations on algebras, as opposed to a valuation on the algebra of sections of a line bundle and coming from a flag of subvarieties. On the other hand, the use of special valuations with algebro-geometric nature is helpful to get more concrete information about the Newton-Okounkov bodies in special cases.

Let us now return back to the subspaces of rational functions on a variety $X$. Let $L \in \mathbf{K}_{rat}(X)$. If the Kodaira map $\Phi_L : X \rightarrow \mathbb{P}(L^*)$ is a birational isomorphism
between $X$ and its image $Y_L$ then the degree of $Y_L$ is equal to the self-intersection index $[L, \ldots, L]$ of the subspace $L$. We can then apply the results above to the intersection theory on $K_{rat}(X)$. Let us call a subspace $L$ a **big subspace** if for large $k$, $\Phi_{L,k}$ is a birational isomorphism between $X$ and $Y_{L,k}$.

With a space $L \in K_{rat}(X)$, we associate two graded algebras: the algebra $A_L$ and its integral closure $\overline{A_L}$ in the field of fractions of the polynomial algebra $\mathbb{C}(X)[t]$.

The algebra $A_L$ is easier to define and fits our purposes best when the subspace $L$ is big. On the other hand, the second algebra $\overline{A_L}$ is a little bit more complicated to define (it involves the integral closure) but leads to more convenient results for any $L \in K_{rat}(X)$ (Theorem 7 below). The algebraic construction of going from $A_L$ to its integral closure $\overline{A_L}$ can be considered as the analogue of the geometric operation of taking the convex hull of a set of points.

One can then associate to $L$ two convex bodies $\Delta(A_L)$ and $\Delta(\overline{A_L})$. In general $\Delta(A_L) \subseteq \Delta(\overline{A_L})$, while for a big subspace $L$ we have $\Delta(A_L) = \Delta(\overline{A_L})$.

The following generalization of the Kušnirenko theorem gives a geometric interpretation of the self-intersection index of a subspace $L$:

**Theorem 7.** For any $n$-dimensional irreducible algebraic variety $X$ and for any $L \in K_{rat}(X)$ we have:

$$[L, \ldots, L] = n! \text{Vol}(\Delta(\overline{A_L})).$$

The Kušnirenko theorem is a special case of the formula (1). The Newton polytope of the product of two Laurent polynomials is equal to the sum of the corresponding Newton polytopes. This additivity property of the Newton polytope and multi-linearity of the intersection index in $K_{rat}(X)$ gives the Bernstein theorem as a corollary of the Kušnirenko theorem. Each of the bodies $\Delta(A_L)$ and $\Delta(\overline{A_L})$ satisfy a superadditivity property, that is, the convex body associated to the product of two subspaces, contains the sum of the convex bodies corresponding to the subspaces.

The formula (1) and the superadditivity of the Newton-Okounkov body $\Delta(\overline{A_L})$ together with the classical Brunn-Minkowski inequality for convex bodies, then imply an analogous inequality for the self-intersection index:

**Theorem 8.** Let $L_1, L_2 \in K_{rat}(X)$ and put $L_3 = L_1 L_2$. We have:

$$[L_1, \ldots, L_1]^{1/n} + [L_2, \ldots, L_2]^{1/n} \leq [L_3, \ldots, L_3]^{1/n}.$$

For an algebraic surface $X$, i.e. for $n = 2$, this inequality is equivalent to the following analogue of the Hodge inequality (from the Hodge index theorem):

$$[L_1, L_1][L_2, L_2] \leq [L_1, L_2]^2.$$  

The Hodge index theorem holds for smooth irreducible projective (or compact Kaehler) surfaces. Our inequality (2) holds for any irreducible surface, not necessarily smooth or complete, and hence is easier to apply. In contrast to the usual proofs of the Hodge inequality, our proof of the inequality (2) is completely elementary.

Using properties of the intersection index in $K_{rat}(X)$ and using the inequality (2) one can easily prove the algebraic analogue of Alexandrov-Fenchel inequality from convex geometry (and its corollaries). The classical Alexandrov-Fenchel inequality (and its corollaries) in convex geometry follow easily from their algebraic analogues via the Bernstein-Kušnirenko theorem. These inequalities from intersection theory and their application to deduce the corresponding inequalities in convex geometry,
have been known (see [Khovanskii88, Teissier79]). A contribution of the present paper is an elementary proof of the key inequality (2) which makes all the chain of arguments involved elementary and more natural.

This paper stems from an attempt to understand the right definition of the Newton polytope for representations of reductive groups. Unexpectedly, we found that one can define many convex bodies analogous to the Newton polytope and their definition, in general, is not related with the group action. We explore this convex bodies in this paper. It is unlikely that one can completely understand the shape of a Newton-Okounkov body in the general situation. In a paper in preparation, we return back to the reductive group actions. We consider the Newton-Okounkov bodies associated to invariant subspaces of rational functions on spherical varieties and constructed via special valuations. The Newton-Okounkov bodies in such cases can be described (in particular they are convex polytopes) and the results of the present paper become more concrete.

1. Part I: Semigroups of integral points

In this part we develop a geometric approach to the semigroups of integral points in \(\mathbb{R}^n\). The origin of this approach goes back to the paper [Khovanskii92]. We show that a semigroup of integral points is sufficiently close to the semigroup of all points in a sublattice and lying in a convex cone in \(\mathbb{R}^n\). We then introduce the notion of Newton-Okounkov body for a semigroup, which is responsible for the asymptotic of the number of points of the semigroup in a given (co)direction. Finally, we prove a theorem which compares the asymptotic of a semigroup and that of its subsemigroups. We regard this as an abstract version of the Fujita approximation theorem in the theory of divisors. Later in the paper, the results of this part will be applied to graded algebras and to intersection theory.

1.1. Semigroups of integral points and their regularizations. Let \( S \) be an additive semigroup in the lattice \( \mathbb{Z}^n \subset \mathbb{R}^n \). In this section we will define the regularization of \( S \), a simpler semigroup with more points constructed out of the semigroup \( S \). The main result is the approximation theorem (Theorem 1.3) which states that the regularization of \( S \) asymptotically approximates \( S \). Exact definitions and statement will be given below.

To a semigroup \( S \) we associate the following basic objects:

**Definition 1.1.**

(1) The *subspace generated by the semigroup* \( S \) is the real span \( L(S) \subset \mathbb{R}^n \) of the semigroup \( S \). By definition the linear space \( L(S) \) is spanned by integral vectors and thus the *rank of the lattice* \( L(S) \cap \mathbb{Z}^n \) is equal to \( \dim L(S) \).

(2) *Cone generated by the semigroup* \( S \) is the closed convex cone \( \text{Con}(S) \subset L(S) \) which is the closure of the set of all linear combinations \( \sum \lambda_i a_i \) for \( a_i \in S \) and \( \lambda_i \geq 0 \).

(3) *Group generated by the semigroup* \( S \) is the group \( G(S) \subset L(S) \) generated by all the elements in the semigroup \( S \). The group \( G(S) \) consists of all the linear combinations \( \sum k_i a_i \) where \( a_i \in S \) and \( k_i \in \mathbb{Z} \).

**Definition 1.2.** The *regularization of a semigroup* \( S \) is the semigroup

\[
\text{Reg}(S) = G(S) \cap \text{Con}(S).
\]

Clearly the semigroup \( S \) is contained in its regularization.
The **ridge** of a closed convex cone with apex at the origin is the biggest linear subspace contained in the cone. A cone is called **strictly convex** if its ridge contains only the origin. The **ridge** $L_0(S)$ of a semigroup $S$ is the ridge of the cone $\text{Con}(S)$.

First we consider the case of finitely generated semigroups. The following statement is obvious:

**Proposition 1.3.** Let $A \subset \mathbb{Z}^n$ be a finite set generating a semigroup $S$, and let $\Delta(A)$ be the convex hull of $A$. Then: 1) The space $L(S)$ is the smallest subspace containing the polytope $\Delta(A)$. 2) The cone $\text{Con}(S)$ is the cone with the apex at the origin over the polytope $\Delta(A)$. 3) If the origin $O$ belongs to $\Delta(A)$ then the ridge $L_0(S)$ is the space generated by the smallest face of the polytope $\Delta(A)$ containing $O$, otherwise $L_0(S) = \{0\}$.

The following statement is well-known in toric geometry (conductor ideal). For the sake of completeness we give a proof here.

**Theorem 1.4.** Let $S \subset \mathbb{Z}^n$ be a finitely generated semigroup. Then there is an element $g_0 \in S$ such that $\text{Reg}(S) + g_0 \subset S$, i.e. for any element $g \in \text{Reg}(S)$ we have $g + g_0 \in S$.

**Proof.** Let $A$ be a finite set generating $S$ and let $P \subset \mathbb{R}^n$ be the set of vectors $x$ which can be represented in the form $x = \sum \lambda_i a_i$, where $0 \leq \lambda_i < 1$ and $a_i \in A$. The set $P$ is bounded and hence $Q = P \cap G(S)$ is finite. For each $q \in Q$ fix a representation of $q$ in the form $q = \sum k_i(q) a_i$, where $k_i(q) \in \mathbb{Z}$ and $a_i \in A$. Let $g_0 = \sum_{a_i \in A} m_i a_i$, with $m_i = 1 - \min_{q \in Q} \{ k_i(q) \}$. Each vector $g \in \text{Reg}(S) \subset \text{Con}(S)$ can be represented in the form $g = \sum \lambda_i a_i$, where $\lambda_i \geq 0$ and $a_i \in A$. Let $g = x + y$, with $x = \sum [\lambda_i] a_i$ and $y = \sum (\lambda_i - [\lambda_i]) a_i$. Clearly $x \in S \cup \{0\}$ and $y \in P$. Let’s verify that $g + g_0 \in S$. In fact $g + g_0 = x + (y + g_0)$. Because $g \in \text{Reg}(S)$, we have $y \in Q$. Now $y + g_0 = \sum k_i(y) a_i + \sum m_i a_i = \sum (k_i(y) + m_i) a_i$. By definition $k_i(y) + m_i \geq 1$ and so $(y + g_0) \in S$. Thus $g + g_0 = x + (y + g_0) \in S$. This finishes the proof.

Fix any Euclidean metric in $L(S)$.

**Corollary 1.5.** Under the assumptions of Theorem 1.4 there is a constant $N > 0$ such that any point in $G(S)$ whose distance to the boundary of $\text{Con}(S)$ (as a subset of the topological space $L(S)$) is bigger than or equal to $N$, is in $S$.

**Proof.** It is enough to take $N$ to be the length of the vector $g_0$ from Theorem 1.4.

Now we consider the case where the semigroup $S$ is not necessarily finitely generated. Let $S \subset \mathbb{Z}^n$ be a semigroup and let $\text{Con}$ be any closed strictly convex cone inside $\text{Con}(S)$ which intersects the boundary of $\text{Con}(S)$ (as a subset of $L(S)$) only at the origin. We then have:

**Theorem 1.6 (Approximation of a semigroup by its regularization).** There is a constant $N > 0$ (depending on the choice of $\text{Con} \subset \text{Con}(S)$) such that each point in the group $G(S)$ which lies in $\text{Con}$ and whose distance from the origin is bigger than $N$ belongs to $S$.

**Proof.** Fix a Euclidean metric in $L(S)$ and equip $L(S)$ with the corresponding topology. We will only deal with $L(S)$ and the ambient space $\mathbb{R}^n$ will not be used in the proof below. Let us enumerate the points in the semigroup $S$ and let $A_i$ be...
the collection of the first \( i \) elements of \( S \). Denote by \( S_i \) the semigroup generated by \( A_i \). There is \( i_0 > 0 \) such that for \( i > i_0 \) the set \( A_i \) contains a set of generators for the group \( G(S) \). If \( i > i_0 \) then the group \( G(S_i) \) generated by the semigroup \( S_i \) coincides with \( G(S) \), and the space \( L(S_i) \) coincides with \( L(S) \).

Fix any linear function \( \ell : L(S) \to \mathbb{R} \) which is strictly positive on \( \text{Con} \setminus \{0\} \). Let \( \Delta_\ell(\text{Con}(S)) \) and \( \Delta_\ell(\text{Con}) \) be the closed convex sets obtained by intersecting \( \text{Con}(S) \) and \( \text{Con} \) by the hyperplane \( \ell = 1 \) respectively. By definition \( \Delta_\ell(\text{Con}) \) is bounded and is strictly inside \( \Delta_\ell(\text{Con}(S)) \).

The convex sets \( \Delta_\ell(\text{Con}(S_i)) \), obtained by intersecting \( \text{Con}(S_i) \) with the hyperplane \( \ell = 1 \), form an increasing sequence of closed convex sets in this hyperplane. The closure of the union of the sets \( \Delta_\ell(\text{Con}(S_i)) \) is, by construction, the convex set \( \Delta_\ell(\text{Con}(S)) \). So there is an integer \( i_1 \) such that for \( i > i_1 \) the set \( \Delta_\ell(\text{Con}) \) is strictly inside \( \Delta_\ell(\text{Con}(S_i)) \). Take any integer \( j \) bigger than \( i_0 \) and \( i_1 \). By Theorem 1.3 for the finitely generated semigroup \( S_j \) there is a vector \( g_0 \) such that any point in \( G(S) \cap (g_0 + \text{Con}(S_j)) \) belongs to \( S \). The convex cone \( \text{Con} \) is contained in \( \text{Con}(S_j) \) and their boundaries intersect only at the origin. Now it is elementary to verify that the shifted cone \( \text{Con}(S_j) + g_0 \) contains all the points of \( \text{Con} \) which are far enough from the origin. This finishes the proof of the theorem.

Example 1.7. In \( \mathbb{R}^2 \) with coordinates \( x \) and \( y \), consider the domain \( U \) defined by the inequality \( y \geq F(x) \) where \( F \) is an even function, i.e. \( F(x) = F(-x) \), such that \( F(0) = 0 \) and \( F \) is concave down and increasing on the ray \( x \geq 0 \). The set \( S = U \cap \mathbb{Z}^2 \) is a semigroup. The group \( G(S) \) associated to this semigroup is \( \mathbb{Z}^2 \). The cone \( \text{Con}(S) \) is defined by the inequality \( y \geq c|x| \) where \( c = \lim_{x \to -\infty} F(x)/x \) and the regularization \( \text{Reg}(S) \) is \( \text{Con}(S) \cap \mathbb{Z}^2 \). In particular, if \( F(x) = |x|^{\alpha} \) where \( 0 < \alpha < 1 \), then \( \text{Con}(S) \) is the half-plane \( y \geq 0 \) and \( \text{Reg}(S) \) is the set of integral points in this half-plane. Here the distance from the point \( (x,0) \in \text{Con}(S) \) to the semigroup \( S \) goes to infinity as \( x \) goes to infinity.

1.2. Rational half-spaces and admissible pairs. In this section we discuss admissible pairs consisting of a semigroup and a half-space. We define the Newton-Okounkov body and the Hilbert function for an admissible pair.

Let \( L \) be a linear subspace in \( \mathbb{R}^n \) and \( M \) a half-space in \( L \) with boundary \( \partial M \). A half-space \( M \subset L \) is rational if the spaces \( L \) and \( \partial M \) can be spanned by integral vectors.

With a rational half-space \( M \subset L \) one can associate \( \partial M_\mathbb{Z} = \partial M \cap \mathbb{Z}^n \) and \( L_\mathbb{Z} = L \cap \mathbb{Z}^n \). Take the linear map \( \pi_M : L \to \mathbb{R} \) such that \( \text{ker}(\pi_M) = \partial M \), \( \pi_M(L_\mathbb{Z}) = \mathbb{Z} \) and \( \pi_M(M \cap \mathbb{Z}^n) = \mathbb{Z}_{\geq 0} \), the set of all non-negative integers. The linear map \( \pi_M \) induces an isomorphism from \( L_\mathbb{Z}/\partial M_\mathbb{Z} \) to \( \mathbb{Z} \).

Now we define an admissible pair of a semigroup and a half-space.

Definition 1.8. A pair \((S,M)\) where \( S \) is a semigroup in \( \mathbb{Z}^n \) and \( M \) a rational half-space in \( L(S) \) is called admissible if \( S \subset M \). We call an admissible pair \((S,M)\) strongly admissible if the cone \( \text{Con}(S) \) is strictly convex and intersects the space \( \partial M \) only at the origin.

With an admissible pair \((S,M)\) we associate the following objects:
- \( \text{ind}(S,\partial M) \), the index of the subgroup \( G(S) \cap \partial M \) in the group \( \partial M_\mathbb{Z} \),
- \( \text{ind}(S,M) \), the index of the subgroup \( \pi_M(G(S)) \) in the group \( \mathbb{Z} \). (We will usually denote \( \text{ind}(S,M) \) by the letter \( m \).)
- $S_k$, the subset $S \cap \pi_M^{-1}(k)$ of the points of $S$ at level $k$.

**Definition 1.9.** The Newton-Okounkov convex set $\Delta(S, M)$ of an admissible pair $(S, M)$, is the convex set $\Delta(S, M) = \text{Con}(S) \cap \pi_M^{-1}(m)$, where $m = \text{ind}(S, M)$. It follows from the definition that the convex set $\Delta(S, M)$ is compact (i.e. is a convex body) if and only if the pair $(S, M)$ is strongly admissible. In this case we call $\Delta(S, M)$ the Newton-Okounkov body of $(S, M)$.

We now define the Hilbert function of an admissible pair $(S, M)$. It is convenient to define it in the following general situation. Let $T$ be a commutative semigroup and $\pi : T \to \mathbb{Z}_{\geq 0}$ a homomorphism of semigroups.

**Definition 1.10.** 1) The Hilbert function $H$ of $(T, \pi)$ is the function $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, defined by $H(k) = \#\pi^{-1}(k)$. The support $\text{supp}(H)$ of the Hilbert function is the set of $k \in \mathbb{Z}_{\geq 0}$ at which $H(k) \neq 0$. 2) The Hilbert function of an admissible pair $(S, M)$ is the Hilbert function of the semigroup $S$ and the homomorphism $\pi_M : S \to \mathbb{Z}_{\geq 0}$. That is, $H(k) = \#S_k$, for any $k \in \mathbb{Z}_{\geq 0}$.

The following is easy to verify.

**Proposition 1.11.** Let $T$ and $\pi$ be as above. 1) The support $\text{supp}(H)$ of the Hilbert function $H$ is a semigroup in $\mathbb{Z}_{\geq 0}$. 2) If the semigroup $T$ has the cancellation property then the set $H^{-1}(\infty)$ is an ideal in the semigroup $\text{supp}(H)$, i.e. if $x \in H^{-1}(\infty)$ and $y \in \text{supp}(H)$ then $x + y \in H^{-1}(\infty)$. 3) Let $m$ be the index of the subgroup generated by $\text{supp}(H) \subseteq \mathbb{Z}_{\geq 0}$ in $\mathbb{Z}$. Then $\text{supp}(H)$ is contained in $m\mathbb{Z}$ and there is a constant $N_1$ such that for $mk > N_1$ we have $mk \in \text{supp}(H)$. 4) If the semigroup $T$ has the cancellation property and $H^{-1}(\infty) \neq \emptyset$, then there is $N_2$ such that for $mk > N_2$ we have $H(mk) = \infty$.

**Proof.** 1) and 2) are obvious. 3) Follows from Theorem 1.9 applied to the semigroup $\text{supp}(H) \subseteq \mathbb{Z}$. Finally 4) Follows from 2) and 3). □

In particular, if the Hilbert function of an admissible pair $(S, M)$ is equal to infinity for at least one $k$, then for sufficiently large values of $k$, Proposition 1.11(4) describes this function completely. Thus in what follows we will assume that the Hilbert function always takes finite values.

1.3. **Hilbert function and volume of the Newton-Okounkov convex set.**

In this section we establish a connection between the asymptotic of the Hilbert function of an admissible pair and its Newton-Okounkov body.

First let us define the notion of integral volume in a rational affine subspace. We call a linear subspace of $\mathbb{R}^n$ *rational* if it is spanned by rational vectors (equivalently by integral vectors). An affine subspace of $\mathbb{R}^n$ is *rational* if it is parallel to a rational linear subspace.

**Definition 1.12 (Integral volume).** Let $L \subset \mathbb{R}^n$ be a rational linear subspace of dimension $q$. The integral measure in $L$ is the translation invariant Euclidean measure in $L$ normalized such that the smallest measure of a $q$-dimensional parallelepiped with vertices in $L \cap \mathbb{Z}^n$ is equal to 1. Let $E$ be a rational affine subspace of dimension $q$ and parallel to $L$. The integral measure on $E$ is the integral measure on $L$ shifted to $E$. The measure of a subset $\Delta \subset E$ will be called its integral volume and denoted by $\text{Vol}_q(\Delta)$.

\[\text{Vol}_q(\Delta)\]  

\[\text{By a convex body we mean a convex compact subset of } \mathbb{R}^n.\]
For the rest of the paper, unless otherwise stated, $\text{Vol}_q$ refers to the integral volume.

Now let $(S, M)$ be an admissible pair with $m = \text{ind}(S, M)$. Let $q = \dim \partial M$. We denote the integral measure in the affine space $\pi^{-1}_M(m)$ by $d\mu$. Take a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $d$ and let $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$ be its decomposition into homogeneous components.

**Theorem 1.13.** Let $(S, M)$ be a strongly admissible pair. Then

$$\lim_{k \to \infty} \frac{\sum_{x \in S \cap \mathbb{Z}^{q+1}} f(x)}{k^{q+d}} = \frac{\int_{\Delta(S, M)} f^{(d)}(x) d\mu}{\text{ind}(S, \partial M)}.$$

Let $M$ be the positive half-space $x_{q+1} \geq 0$ in $\mathbb{R}^{q+1}$. Take a $(q + 1)$-dimensional strongly convex cone $C \subset M$ which intersects $\partial M$ only at the origin. Let $S = C \cap \mathbb{Z}^{q+1}$ be the semigroup of all the integral points in $C$. Then $(S, M)$ is a strongly admissible pair. For such kind of a saturated semigroup $S$, Theorem 1.13 is relatively easy to show. We restate the above theorem in this case as it will be needed in the proof of the general case. Results of such kind have origins in the classical work of Minkowski.

**Theorem 1.14.** Let $S = C \cap \mathbb{Z}^{q+1}$ and $\Delta = C \cap \{x_{q+1} = 1\}$. Then:

$$\lim_{k \to \infty} \frac{\sum_{x \in S \cap \mathbb{Z}^{q+1}} f(x)}{k^{q+d}} = \int_{\Delta} f^{(d)}(x) d\mu.$$

Here $S_k$ is the set of all the integral points in $C \cap \{x_{q+1} = k\}$, and $d\mu$ is the Euclidean measure at the hyperplane $x_{q+1} = 1$.

We will not prove Theorem 1.14. It can be easily proved by considering the Riemann sums for the integral of the homogeneous component of $f$ over $\Delta$.

**Proof of Theorem 1.13** The theorem follows from Theorem 1.9 (approximation theorem) and Theorem 1.14. Firstly, one reduces to the case where $L(S) = \mathbb{R}^n$, $q + 1 = n$, $M$ is given by the inequality $x_{q+1} \geq 0$, $G(S) = \mathbb{Z}^{q+1}$, $\text{ind}(S, \partial M) = \text{ind}(S, M) = 1$ and $\Delta(S, M)$ is a $q$-dimensional convex body in the hyperplane $x_{q+1} = 1$, as follows: choose a basis $e_1, \ldots, e_q, e_{q+1}, \ldots, e_n$ in $\mathbb{R}^n$ such that $e_1, \ldots, e_q$ generate the group $G(S) \cap \partial M$ and the vectors $e_1, \ldots, e_{q+1}$ generate the group $G(S)$ (no condition on the rest of vectors in the basis). This choice of basis identifies the spaces $L(S)$ and $\partial M$ with $\mathbb{R}^{q+1}$ and $\mathbb{R}^q$ respectively. We will not deal with the vectors outside $\mathbb{R}^{q+1}$, and hence we can assume $q + 1 = n$. Under such choice of a basis the lattice $L(S)\mathbb{Z}$ identifies with a lattice $\Lambda \subset \mathbb{R}^{q+1}$ which may contain non-integral points. Also the lattice $\partial M\mathbb{Z}$ identifies with a lattice $\Lambda \cap \mathbb{R}^q$. The index of the subgroup $\mathbb{Z}^q$ in the group $\Lambda \cap \mathbb{R}^q$ is equal to $\text{ind}(S, \partial M)$. The coordinate $x_{q+1}$ of the points in the lattice $\Lambda \subset \mathbb{Z}^{q+1}$ is proportional to the number $1/m$ where $m = \text{ind}(S, M)$. The map $\pi_M : L(S)\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ then coincides with the restriction of the map $mx_{q+1}$ to the lattice $\Lambda$. The semigroup $S$ becomes a subsemigroup in the lattice $\mathbb{Z}^q$ and the level set $S_k$ is equal to $S \cap \{x_{q+1} = k\}$. Also the measure $d\mu$ is given by $d\mu = \rho d\mathbf{x} = \rho dx_1 \wedge \cdots \wedge x_q$, where $\rho = \text{ind}(S, \partial M)$. Thus with the above choice of basis the theorem is reduced to this particular case.

To prove that the limit exists and is equal to $\int_{\Delta(S, M)} f^{(d)}(x) dx$, it is enough to show that any limit point of the sequence $\{y_k\}$, $y_k = \sum_{x \in S_k} f(x)/k^{q+d}$, lies in arbitrarily small neighborhoods of $\int_{\Delta(S, M)} f^{(d)}(x) dx$. Take a convex body $\Delta$ in the
Consider the convex bodies $k\Delta(S, M)$ and $k\Delta$ in the hyperplane $x_{q+1} = k$. Let $S_k'$ and $S_k''$ be the sets $k\Delta \cap \mathbb{Z}^{q+1}$ and $k\Delta(S, M) \cap \mathbb{Z}^{q+1}$ respectively. By Theorem 1.16, for large values of $k$, we have $S_k' \subset S_k \subset S_k''$. Also by Theorem 1.14,

$$\lim_{k \to \infty} \frac{\sum_{x \in S_k'} f(x)}{k^{q+d}} = \int_{\Delta} f^{(d)}(x)dx,$$

$$\lim_{k \to \infty} \frac{\sum_{x \in S_k''} f(x)}{k^{q+d}} = \int_{\Delta(S, M)} f^{(d)}(x)dx,$$

$$\lim_{k \to \infty} \frac{\#(S_k'' \setminus S_k')}{k^q} = \text{Vol}_q(\Delta(S, M) \setminus \Delta).$$

Since $(S, M)$ is strongly admissible, one can find a constant $N > 0$ such that for any point $x \in \text{Con}(S)$ with $x_{q+1} \geq 1$ we have $|f(x)|/x_{q+1}^d < N$ and $|f^{(d)}(x)|/x_{q+1}^d < N$. This implies that for large values of $k$ we have:

$$\sum_{x \in S_k'' \setminus S_k'} |f(x)|/x_{q+1}^d \leq \tilde{N}\text{Vol}_q(\Delta(S, M) \setminus \Delta),$$

$$\int_{\Delta(S, M) \setminus \Delta} |f^{(d)}(x)|dx < \tilde{N}\text{Vol}_q(\Delta(S, M) \setminus \Delta),$$

where $\tilde{N}$ is any constant bigger than $N$. Thus

$$\left| \frac{\sum_{x \in S_k'} f(x)}{k^{q+d}} - \int_{\Delta(S, M)} f^{(d)}(x)dx \right| < 2\tilde{N}\text{Vol}_q(\Delta(S, M) \setminus \Delta).$$

For any given $\varepsilon > 0$ we may choose the convex body $\Delta > 0$ such that $\text{Vol}_q(\Delta(S, M) \setminus \Delta) < \varepsilon/2\tilde{N}$. This shows that for any $\varepsilon > 0$, all the limit points of the sequence $\{g_k\}$ belong to the $\varepsilon$-neighborhood of the number $\int_{\Delta(S, M)} f^{(d)}(x)dx$, which finishes the proof.

\[ \square \]

**Corollary 1.15.** With the assumptions as in Theorem 1.15, the following holds:

$$\lim_{k \to \infty} \frac{\#S_{mk}}{k^q} = \frac{\text{Vol}_q(\Delta(S, M))}{\text{ind}(S, \partial M)}.$$ 

**Proof.** Apply Theorem 1.13 to the polynomial $f = 1$. \[ \square \]

**Definition 1.16.** Let $(S, M)$ be an admissible pair with $m = \text{ind}(S, M)$ and $q = \dim \partial M$. We say that $S$ has bounded growth with respect to the half-space $M$ if there exists a sequence $k_i \to \infty$ of positive integers such that the sets $S_{mk_i}$ are finite and the sequence of numbers $\#S_{mk_i}/k_i^q$ is bounded.

**Theorem 1.17.** Let $(S, M)$ be an admissible pair. The semigroup $S$ has bounded growth with respect to $M$ if and only if the pair $(S, M)$ is strongly admissible. In fact, if $(S, M)$ is strongly admissible then $S$ not only has bounded growth but also has polynomial growth.

**Proof.** Let us show that if $S$ has bounded growth then $(S, M)$ is strongly admissible. Suppose the statement is false. Then the Newton-Okounkov convex set $\Delta(S, M)$ is an unbounded convex $q$-dimensional set and hence has infinite $q$-dimensional volume. Assume that $P$ is a constant such that for any $i$, $\#S_{mk_i}/k_i^q < P$. Choose a convex body $\Delta$ strictly inside $\Delta(S, M)$ in such a way that the $q$-dimensional volume of $\Delta$ is bigger than $mP$. Let $\text{Con}$ be the cone over the convex body $\Delta$ with
the apex at the origin. By Theorem 1.6, for large values of \( k_i \), the set \( S_{mk_i} \) contains the set \( S'_{mk_i} = \text{Con} \cap G(S) \cap \pi^{-1}_M(mk_i) \). Then by Corollary 1.15

\[
\lim_{k_i \to \infty} \frac{\#S_{mk_i}}{k^q_{i}} = \frac{\text{Vol}_q(\Delta)}{\text{ind}(S, \partial M)} > P.
\]

The contradiction proves the claim. The other direction, namely if \((S, M)\) is strongly admissible then it has polynomial growth (and hence bounded growth), follows immediately from Corollary 1.15.

\[\square\]

**Theorem 1.18.** Let \((S, M)\) be an admissible pair and assume that the sets \( S_k, k \in \mathbb{Z}_{\geq 0} \), are finite. Let \( H \) be the Hilbert function of \((S, M)\) and put \( \text{dim} \, \partial M = q \). Then

1. The limit

\[
\lim_{k \to \infty} \frac{H(mk)}{k^q}
\]

exists (possibly infinite), where \( m = \text{ind}(S, M) \).

2. This limit is equal to the volume (possibly infinite) of the Newton-Okounkov convex set \( \Delta(S, M) \) divided by the integer \( \text{ind}(S, \partial M) \).

**Proof.** First assume that \( H(mk)/k^q \) does not approach infinity (as \( k \) goes to infinity). Then there is a sequence \( k_i \to \infty \) with \( k_i \in \mathbb{Z}_{\geq 0} \) such that the sets \( S_{mk_i} \) are finite and the sequence \( \#S_{mk_i}/k^q_{i} \) is bounded. But this means that the semigroup \( S \) has bounded growth with respect to the half-space \( M \). Thus by Theorem 1.17 the cone \( \text{Con}(S) \) is strictly convex and intersects \( \partial M \) only at the origin. In this case the theorem follows from Corollary 1.15. Now if \( \lim_{k \to \infty} H(mk)/k^q = \infty \), then the conditions in Theorem 1.18 can not be satisfied. Hence the convex set \( \Delta(S, M) \) is unbounded and thus has infinite volume. This shows that Theorem 1.18 is true in this case as well.

\[\square\]

**Example 1.19.** Let \( S \) be the semigroup in Example 1.7 where \( F(x) = |x|^{1/n} \) for some natural number \( n > 1 \). Also let \( M \) be the half-space \( y \geq 0 \). Then the pair \((S, M)\) is admissible. Its Newton-Okounkov set \( \Delta(S, M) \) is the line \( y = 1 \) and its Hilbert function is given by \( H(k) = 2k^n + 1 \). Thus in spite of the fact that the dimension of the Newton-Okounkov convex set \( \Delta(S, M) \) is 1, the Hilbert function grows like \( k^n \). This effect is related to the fact that the pair \((S, M)\) is not strongly admissible.

1.4. **Non-negative semigroups and approximation theorem.** In \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \) there is a natural half-space \( \mathbb{R}^n \times \mathbb{R}_{\geq 0} \), consisting of the points whose last coordinate is non-negative. In this section we will deal with semigroups that are contained in this fixed half-space of full dimension. For such semigroups we refine the statements of theorems proved in the previous sections.

We start with definitions. A non-negative semigroup of integral points in \( \mathbb{R}^{n+1} \) is a semigroup \( S \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0} \) which is not contained in the hyperplane \( x_{n+1} = 0 \). With a non-negative semigroup \( S \) we can associate an admissible pair \((S, M(S))\) where \( M(S) = L(S) \cap (\mathbb{R}^n \times \mathbb{R}_{\geq 0}) \). We call a non-negative semigroup, strongly non-negative if the corresponding admissible pair is strongly admissible. Let \( \pi : \mathbb{R}^{n+1} \to \mathbb{R} \) be the projection on the \((n+1)\)-th coordinate. We can associate all the objects defined for an admissible pair to a non-negative semigroup:

- \( \text{Con}(S) \), the cone of the pair \((S, M(S))\).
- $G(S)$, the group generated by the semigroup $S$.
- $H_S$, the Hilbert function of the pair $(S, M(S))$.
- $\Delta(S)$, the Newton-Okounkov convex set of the pair $\Delta(S, M(S))$.
- $G_0(S) \subset G(S)$, the subgroup $\pi^{-1}(0) \cap G(S)$.
- $S_k$, the subset $S \cap \pi^{-1}(k)$ of points in $S$ at level $k$.
- ind$(S)$, the index of the subgroup $G_0(S)$ in $\mathbb{Z}^n \times \{0\}$, i.e. ind$(S, \partial M(S))$.
- $m(S)$, the index ind$(S, M(S))$.

We now give a more refined version of the approximation theorem for the non-negative semigroups. We will need the following elementary lemma.

**Lemma 1.20.** Let $B$ be a ball of radius $\sqrt{\pi}$ centered at a point $a$ in the Euclidean space $\mathbb{R}^n$ and let $A = B \cap \mathbb{Z}^n$. Then: 1) the point $a$ belongs to the convex hull of $A$. 2) The group generated by $x - y$ where $x, y \in A$ is $\mathbb{Z}^n$.

**Proof.** Let $K_a \subset B$ be the cube centered at $a$ and with the sides of length 2 parallel to the coordinate axes. If $a = (a_1, \ldots, a_n)$ then the cube $K_a$ is defined by the inequalities $(a_i - 1) \leq x_i \leq (a_i + 1)$, $i = 1, \ldots, n$. On the intervals defined by $(a_i - 1) \leq x_i < a_i$ and $a_i < x_i \leq (a_i + 1)$ there are integers $n_i^-$ and $n_i^+$ respectively. The set $A$ contains the subset $A' = K_a \cap \mathbb{Z}^n$ which contains the $2^n$ integral points $\mathbf{n} = (n_1^+, \ldots, n_n^+)$. The point $a$ belongs to the convex hull of $A'$. The differences of the points in $A'$ generates the group $\mathbb{Z}^n$. □

**Remark 1.21.** K. A. Matveev (an undergraduate student at the University of Toronto) has shown that the smallest radius for which the above proposition holds is $\sqrt{n + 3}/2$.

Let us now proceed with the refinement of the approximation theorem for non-negative semigroups. Let $\dim L(S) = q + 1$ and let $\text{Con} \subset \text{Con}(S)$ be a strongly convex $(q + 1)$-dimensional cone which intersects the boundary (in the topology of $L(S)$) of $\text{Con}(S)$ only at the origin $0$.

**Theorem 1.22.** There is a constant $N > 0$ (depending on the choice of $\text{Con}$) such that for any integer $p > N$ which is divisible by $m(S)$ we have: 1) The convex hull of the set $S_p$ contains the set $\Delta(p) = \text{Con} \cap \pi^{-1}(p)$. 2) The group generated by the differences $x - y$, $x, y \in S_p$ is independent of $p$ and coincides with the group $G_0(S)$.

**Proof.** By a linear change of variables we can assume that $L(S)$ is $\mathbb{R}^{q+1}$ (whose coordinates we denote by $x_1, \ldots, x_{q+1}$), $M(S)$ is the positive half-space $x_{q+1} \geq 0$, $G(S)$ is $\mathbb{Z}^{q+1}$ and the index $m(S)$ is $1$. To make the notation simpler denote $\text{Con}(S)$ by $\text{Con}_2$. Take any $(q + 1)$-dimensional convex cone $\text{Con}_1$ such that 1) $\text{Con} \subset \text{Con}_1 \subset \text{Con}_2$ and 2) $\text{Con}_1$ intersects the boundaries of the cones $\text{Con}$ and $\text{Con}_2$ only at the origin. Consider the sections $\Delta(p) \subset \Delta_1(p) \subset \Delta_2(p)$ of the cones $\text{Con} \subset \text{Con}_1 \subset \text{Con}_2$ (respectively) by the hyperplane $\pi^{-1}(p)$, for some positive integer $p$. Take $N_1 > 0$ large enough so that for any integer $p > N_1$ a ball of radius $\sqrt{q}$ centered at any point of the convex body $\Delta(p)$ is contained in $\Delta_1(p)$. Then by Lemma 1.20 the convex body $\Delta_1(p)$ is contained in the convex hull of the set of integral points in $\Delta_1(p)$. Also by Theorem 1.6 (approximation theorem) there is $N_2 > 0$ such that for $p > N_2$ the semigroup $S$ contains all the integral points in $\Delta_1(p)$. Thus if $p > N = \max\{N_1, N_2\}$, the convex hull of the set $S_p$ contains the convex body $\Delta(p)$. This proves Part 1). Moreover, since for $p > N \Delta_1(p)$ contains a ball of radius $\sqrt{q}$ and $S_p$ contains all the integral points in this ball, by Lemma 1.20
the differences of the integral points in $S_p$ generates the group $\mathbb{Z}^q = \mathbb{Z}^{q+1} \cap \pi^{-1}(0)$. This proves Part 2).

\section{Hilbert function of a semigroup $S$ and its subsemigroups $\widehat{S}_p$.}

Let $S$ be a strongly non-negative semigroup with the Hilbert function $H_S$. For an integer $p$ in the support of $H_S$ let $\widehat{S}_p$ denote the subsemigroup generated by $S_p = S \cap \pi^{-1}(p)$. In this section we compare the asymptotic of $H_S$ with the asymptotic, as $p \to \infty$, of the Hilbert functions of the semigroups $\widehat{S}_p$.

Later in Sections 2.4 and 3.2 we will apply the results here to prove a generalization of the Fujita approximation theorem (from the theory of divisors). Thus we consider the main result of this section (Theorem 1.26) as an analogue of the Fujita approximation theorem for semigroups.

We will follow the notation introduced in Section 1.2. In particular, $\Delta(S)$ is the Newton-Okounkov body of the semigroup $S$, $q = \dim \Delta(S)$ its dimension, and $m(S)$ and $\mathrm{ind}(S)$, the indices associated to $S$. Also $\mathrm{Con}(\widehat{S}_p)$, $\mathcal{G}(\widehat{S}_p)$, $H_{\widehat{S}_p}$, $\Delta(\widehat{S}_p)$, $\mathrm{ind}(\widehat{S}_p)$, $m(\widehat{S}_p)$, denote the corresponding objects for the semigroup $\widehat{S}_p$. If $S_p = \emptyset$ put $\widehat{S}_p = \Delta(\widehat{S}_p) = \mathcal{G}_0(\widehat{S}_p) = \emptyset$ and $H_{\widehat{S}_p} = 0$.

The next proposition is straightforward to verify:

**Proposition 1.23.** If the set $S_p$ is not empty then $m(\widehat{S}_p) = p$, $\Delta(\widehat{S}_p)$ is the convex hull of $S_p$, the cone $\mathrm{Con}(\widehat{S}_p)$ is the cone over $\Delta(\widehat{S}_p)$, $\mathcal{G}(\widehat{S}_p)$ is the group generated by the set $S_p$ and $\mathcal{G}_0(\widehat{S}_p) = G(\widehat{S}_p) \cap \pi^{-1}(0)$ is the group generated by the differences $a - b$, $a, b \in S_p$. Also $\mathrm{Con}(\widehat{S}_p) \subset \mathrm{Con}(S)$. If $p$ is not divisible by $m(S)$ then $S_p = \emptyset$.

Below we deal with functions defined on a non-negative semigroup $T \subset \mathbb{Z}_{\geq 0}$. A semigroup $T \subset \mathbb{Z}_{\geq 0}$ contains any large enough integer divisible by $m = m(T)$. Let $O_m : \mathbb{Z} \to \mathbb{Z}$ be the scaling map given by $O_m(k) = mk$. For any function $f : T \to \mathbb{R}$ and for sufficiently large $p$, the pull-back $O_m^*(f)$ is defined by $O_m^*(f)(k) = f(mk)$.

**Definition 1.24.** Let $\varphi$ be a function defined on a set of sufficiently large natural numbers. The $q$-th growth coefficient $a_q(\varphi)$ is the value of the limit $\lim_{k \to \infty} \varphi(k)/k^q$ (whenever this limit exists).

The following is a reformulation of Corollary 1.15.

**Theorem 1.25.** The $q$-th growth coefficient of the function $O_m^*(H_S)$, i.e.

$$a_q(\mathcal{G}_m^*(H_S)) = \lim_{k \to \infty} \frac{H_S(mk)}{k^q},$$

exists and is equal to $\mathrm{Vol}_q(\Delta(S))/\mathrm{ind}(S)$.

For large enough $p$ divisible by $m(S)$, $S_p \neq \emptyset$ and the subsemigroups $\widehat{S}_p$ are defined. The following theorem holds.

**Theorem 1.26.** For $p$ sufficiently large and divisible by $m = m(S)$ we have:

1. $\dim \Delta(\widehat{S}_p) = \dim \Delta(S) = q$.
2. $\mathrm{ind}(\widehat{S}_p) = \mathrm{ind}(S)$.
3. Let the function $\varphi$ be defined by

$$\varphi(p) = \lim_{t \to \infty} \frac{H_{\widehat{S}_p}(tp)}{t^q}.$$
That is, \( \varphi \) is the \( q \)-th growth coefficient of \( O^*_p(H_{\tilde{S}_p}) \). Then the \( q \)-th growth coefficient of the function \( O^*_m(\varphi) \), i.e.

\[
a_q(O^*_m(\varphi)) = \lim_{k \to \infty} \frac{\varphi(mk)}{k^q},
\]

exists and is equal to \( a_q(O^*_m(H_S)) = Vol_q(\Delta(S))/\text{ind}(S) \).

Proof. 1) Follows from Theorem 1.22 (1). 2) Follows from Theorem 1.22 (2). 3) By Theorem 1.25 applied to the semigroup \( \tilde{S}_p \) we have:

\[
\varphi(p) = \frac{Vol_q(\Delta(\tilde{S}_p))}{\text{ind}(S)}.
\]

Now we use Theorem 1.22 to estimate the quantity \( Vol_q(\Delta(\tilde{S}_p)) \). Let \( Con_0 \) be a \((q + 1)\)-dimensional cone contained in \( Con(S) \) which intersects its boundary (in the topology of the space \( L(S) \)) only at the origin. Then, for sufficiently large \( p \) and divisible by \( m \), the volume \( Vol_q(\Delta(\tilde{S}_p)) \) satisfies the inequalities

\[
Vol_q(\text{Con}_0 \cap \pi^{-1}(p)) < Vol_q(\Delta(\tilde{S}_p)) < Vol_q(\text{Con}(S) \cap \pi^{-1}(p)).
\]

Let \( p = km \). Dividing the inequalities above by \( k^q \text{ind}(S) \) we obtain

\[
\frac{Vol_q(\text{Con}_0 \cap \pi^{-1}(m))}{\text{ind}(S)} < \frac{\varphi(mk)}{k^q} < \frac{Vol_q(\text{Con}(S) \cap \pi^{-1}(m))}{\text{ind}(S)} = a_q(O^*_m(H_S)).
\]

Since we can choose \( Con_0 \) as close as we want to \( Con(S) \) this proves Part 3). \( \square \)

1.6. Levelwise addition of semigroups. In this section we define the levelwise addition of non-negative semigroups, and we consider a subclass of semigroups for which the \( n \)-th growth coefficient of the Hilbert function depends on the semigroup in a polynomial way.

Let \( \pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( \pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be the projections on the first and second factors respectively. Define the operation of levelwise addition \( \oplus_t \) on the pairs of points with the same last coordinate by:

\[
(x_1, h) \oplus_t (x_2, h) = (x_1 + x_2, h),
\]

where \( x_1, x_2 \in \mathbb{R}^n \), \( h \in \mathbb{R} \). In other words, if \( e \) is the \((n + 1)\)-th standard basis vector in \( \mathbb{R}^n \times \mathbb{R} \) and \( y_1 = (x_1, h), \ y_2 = (x_2, h) \in \mathbb{R}^n \times \mathbb{R} \), then we have \( y_1 \oplus_t y_2 = y_1 + y_2 - h e \).

Next we define the operation of levelwise addition between any two subsets. Let \( X, Y \subset \mathbb{R}^n \times \mathbb{R} \). Then \( X \oplus_t Y = Z \) where \( Z \) is the set such that for any \( h \in \mathbb{R} \) we have:

\[
\pi_1(Z \cap \pi^{-1}(h)) = \pi_1(X \cap \pi^{-1}(h)) + \pi_1(Y \cap \pi^{-1}(h)).
\]

(By convention the sum of the empty set with any other set is the empty set.)

The following proposition can be easily verified.

**Proposition 1.27.** For any two non-negative semigroups \( S_1, S_2 \), the set \( S = S_1 \oplus_t S_2 \) is a non-negative semigroup and the following holds:

1. \( L(S) = L(S_1) \oplus_t L(S_2) \).
2. \( M(S) = M(S_1) \oplus_t M(S_2) \).
3. \( \partial M(S) = \partial M(S_1) \oplus_t \partial M(S_2) \).
4. \( G(S) = G(S_1) \oplus_t G(S_2) \).
5. \( G_0(S) = G_0(S_1) + G_0(S_2) \).
Let us say that a non-negative semigroup has \textit{almost all levels} if \(m(S) = 1\). Also for a non-negative semigroup \(S\), let \(\Delta_0(S)\) denote its Newton-Okounkov convex set shifted to level 0, i.e. \(\Delta_0(S) = \pi_1(\Delta(S))\).

\textbf{Proposition 1.28.} For non-negative semigroups \(S_1, S_2\) and \(S = S_1 \oplus_t S_2\), the following relations holds: 1) The cone \(\text{Con}(S)\) is the closure of the levelwise addition \(\text{Con}(S_1) \oplus_t \text{Con}(S_2)\) of the cones \(\text{Con}(S_1)\) and \(\text{Con}(S_2)\). 2) If the semigroups \(S_1, S_2\) have almost all levels then the Newton-Okounkov set \(\Delta(S)\) is the closure of the levelwise addition \(\Delta(S_1) \oplus_t \Delta(S_2)\) of the Newton-Okounkov sets \(\Delta(S_1)\) and \(\Delta(S_2)\). (In fact, since in this case the Newton-Okounkov convex sets live in the level 1, we have \(\Delta_0(S)\) is the closure of the Minkowski sum \(\Delta_0(S_1) + \Delta_0(S_2)\).)

\textbf{Proof.} 1) It is easy to see that \(S_1 \oplus_t S_2 \subset \text{Con}(S_1) \oplus_t \text{Con}(S_2) \subset \text{Con}(S)\) and the set \(\text{Con}(S_1) \oplus_t \text{Con}(S_2)\) is dense in \(\text{Con}(S)\). Note that the set \(\text{Con}(S_1) \oplus_t \text{Con}(S_2)\) may not be closed (see Example 1.29 below). 2) Follows from Part 1). Note that the Minkowski sum of closed convex subsets may not be closed (see Example 1.29). \(\Box\)

\textbf{Example 1.29.} Let \(\Delta_1, \Delta_2\) be closed convex sets in \(\mathbb{R}^2\) with coordinates \((x, y)\) defined by \(\{(x, y) \mid xy \geq 1, x > 0\}\) and \(\{(x, y) \mid -xy \geq 1, x > 0\}\) respectively. Then the Minkowski sum \(\Delta_1 + \Delta_2\) is the open upper half-plane \(\{(x, y) \mid y > 0\}\).

\textbf{Example 1.30.} Let \(\Delta_1, \Delta_2\) be the sets from Example 1.29 and let \(\Delta_1 \times \{1\}, \Delta_2 \times \{1\}\) in \(\mathbb{R}^2 \times \mathbb{R}\) (with coordinates \((x, y, z)\)) be the shifted copies of these sets to the plane \(z = 1\). Let \(\text{Con}_1\) and \(\text{Con}_2\) be the closures of the cones over these sets. Then \(\text{Con}_1 \oplus \text{Con}_2\) is a non-closed cone which is the union of the set \(\{(x, y, z) \mid 0 \leq z, 0 < y\}\) and the line \(\{(x, y, z) \mid y = z = 0\}\).

\textbf{Proposition 1.31.} Let \(S_1, S_2\) be non-negative semigroups. Moreover assume that \(S_1\) is a strongly non-negative semigroup. Let \(S = S_1 \oplus S_2\). Then \(\text{Con}(S) = \text{Con}(S_1) \oplus \text{Con}(S_2)\) and \(\text{Reg}(S) = \text{Reg}(S_1) \oplus \text{Reg}(S_2)\). If in addition, \(S_1, S_2\) have almost all levels, then \(\Delta(S) = \Delta(S_1) \oplus \Delta(S_2)\). (In other words, \(\Delta_0(S) = \Delta_0(S_1) + \Delta_0(S_2)\).)

\textbf{Proof.} Let \(D\) be the set of pairs \((y_1, y_2) \in \text{Con}(S_1) \times \text{Con}(S_2)\) defined by the condition \(\pi(y_1) = \pi(y_2)\). Let us show that the map \(F : D \to \mathbb{R}^n \times \mathbb{R}\) given by the function \(F(y_1, y_2) = y_1 \oplus_t y_2\) is proper. Consider a compact set \(K \subset \mathbb{R}^n \times \mathbb{R}\). The function \(x_{n+1}\) is bounded on the compact set \(K\), i.e. there are constants \(N_1, N_2\) such that \(N_1 \leq x_{n+1} \leq N_2\). The subset \(K_1\) in the cone \(\text{Con}(S_1)\) defined by the inequalities \(N_1 \leq x_{n+1} \leq N_2\) is compact. Consider the set \(K_2\) consisting of the points \(y_2 \in \text{Con}(S_2)\) for which there is \(y_1 \in K_1\) such that \(y_1 \oplus_t y_2 \in K\). The compactness of \(K\) and \(K_1\) implies that \(K_2\) is also compact and hence the map \(F\) is proper. The properness of \(F\) implies that the sum \(\text{Con}(S_1) \oplus \text{Con}(S_2)\) is closed which proves \(\text{Con}(S) = \text{Con}(S_1) \oplus \text{Con}(S_2)\). The other statements follow from this and Proposition 1.28. \(\Box\)

Finally, let us define \(S(n)\) to be the collection of all strongly non-negative semigroups \(S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}\) with almost all levels, i.e. \(m(S) = 1\), and \(\text{ind}(S) = 1\). The set \(S(n)\) is a (commutative) semigroup with respect to the levelwise addition.

Let \(f : S \to \mathbb{R}\) be a function defined on a (commutative) semigroup \(S\). We say that \(f\) is a \textit{homogeneous polynomial of degree} \(d\) if for any choice of the elements \(a_1, \ldots, a_r \in S\), the function \(F(k_1, \ldots, k_r) = f(k_1 a_1 + \cdots + k_r a_r)\), where \(k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}\), is a homogeneous polynomial of degree \(d\) in the \(k_i\).
Theorem 1.32. The function on \( S(n) \) which associates to a semigroup \( S \in S(n) \), the \( n \)-th growth coefficient of its Hilbert function, is a homogeneous polynomial of degree \( n \). The value of the polarization of this polynomial on an \( n \)-tuple \((S_1, \ldots, S_n)\) is equal to the mixed volume of the Newton-Okounkov bodies \( \Delta(S_1), \ldots, \Delta(S_n) \).

Proof. According to Theorem 1.25, the \( n \)-th growth coefficient of a semigroup \( S \in S(n) \) exists and is equal to the \( n \)-dimensional volume of the convex body \( \Delta(S) \). By Proposition 1.31 under the levelwise addition of semigroups the Newton-Okounkov bodies are added. Thus the \( n \)-th growth coefficient is a homogeneous polynomial of degree \( n \) and the value of its polarization is the mixed volume (see Section 4.1 for a review of the mixed volume). \( \square \)

2. Part II: Valuations and graded algebras

In this part we consider the graded subalgebras of a polynomial ring in one variable with coefficients in a field \( F \) of transcendence degree \( n \) over a ground field \( k \). For a large class of graded subalgebras (which are not necessarily finitely generated), we prove the polynomial growth of the Hilbert function, a Brunn-Minkowski inequality for their growth coefficients and an abstract version of the Fujita approximation theorem. We obtain all these from the analogous results for the semigroups of integral points. The conversion of problems about algebras into problems about semigroups is made possible via a faithful \( \mathbb{Z}^n \)-valued valuation on the field \( F \). Two sections of this part are devoted to valuations.

2.1. Prevaluation on a vector space. In this section we define a prevaluation and discuss its basic properties. A prevaluation is a weaker version of a valuation which is defined for a vector space (while a valuation is defined for an algebra).

Let \( V \) be a vector space over a field \( k \) and \( I \) a totally ordered set with respect to some ordering \( < \).

Definition 2.1. A prevaluation on \( V \) with values in \( I \) is a function \( v : V \setminus \{0\} \to I \) satisfying the following:

1. For all \( f, g \in V \) with \( f, g, f + g \neq 0 \), we have \( v(f + g) \geq \min(v(f), v(g)) \).
2. For all \( 0 \neq f \in V \) and \( 0 \neq \lambda \in k \), \( v(\lambda f) = v(f) \).

Example 2.2. Let \( V \) be a finite dimensional vector space with a basis \( \{e_1, \ldots, e_n\} \) and \( I = \{1, \ldots, n\} \), ordered with the usual ordering of numbers. For \( f = \sum_i \lambda_i e_i \) define

\[
    v(f) = \min\{i \mid \lambda_i \neq 0\}.
\]

Then \( v \) is a prevaluation on \( V \) with values in \( I \).

Let \( v : V \setminus \{0\} \to I \) be a prevaluation. For \( \alpha \in I \), let \( V_\alpha = \{ f \in V \mid v(f) \geq \alpha \text{ or } f = 0 \} \). It follows immediately from the definition of a prevaluation that \( V_\alpha \) is a subspace of \( V \). The leaf \( \hat{V}_\alpha \) above the point \( \alpha \in I \) is the quotient vector space \( V_\alpha / \bigcup_{\beta \leq \alpha} V_\beta \).

Proposition 2.3. Let \( P \subset V \) be a set of vectors. If the prevaluation \( v \) sends different vectors in \( P \) to different points in \( I \) then the vectors in \( P \) are linearly independent.

Proof. Let \( \sum_{i=1}^s \lambda_i w_i = 0, \lambda_i \neq 0 \), be a non-trivial linear relation between the vectors in \( P \). Let \( \alpha_i = v(w_i), i = 1, \ldots, s \), and without loss of generality assume...
\[ \alpha_1 < \cdots < \alpha_s. \] We can rewrite the linear relation in the form \[ \lambda_1 w_1 = - \sum_{i=2}^s \lambda_i w_i. \] But this cannot hold since \( \lambda_1 w_1 \not\in \mathcal{V}_{\alpha_2} \) while \( \sum_{i>1} \lambda_i w_i \in \mathcal{V}_{\alpha_2}. \]

**Proposition 2.4.** Let \( V \) be finite dimensional. Then for all but a finite set of \( \alpha \in I \), the leaf \( \hat{V}_\alpha \) is zero, and we have:

\[ \sum_{\alpha \in I} \dim \hat{V}_\alpha = \dim V. \]

**Proof.** From Proposition 2.3 it follows that \( \varphi(V \setminus \{0\}) \) contains no more than \( \dim V \) points. Let \( \varphi(V \setminus \{0\}) = \{\alpha_1, \ldots, \alpha_s\} \) where \( \alpha_1 < \cdots < \alpha_s \). We have a filtration \( V = \mathcal{V}_{\alpha_1} \supset \mathcal{V}_{\alpha_2} \supset \cdots \supset \mathcal{V}_{\alpha_s} \) and \( \dim V \) is equal to \( \sum_{k=1}^{s-1} \dim \mathcal{V}_{\alpha_k}/\mathcal{V}_{\alpha_{k+1}} = \sum_{k=1}^{s-1} \dim \hat{V}_{\alpha_k}. \)

Let \( W \subset V \) be a non-zero subspace. Let \( J \subset I \) be the image of \( W \setminus \{0\} \) under the prevaluation \( \varphi \). The set \( J \) inherits a total ordering from \( I \). The following is clear:

**Proposition 2.5.** The restriction \( \varphi|_W : W \setminus \{0\} \to J \) is a prevaluation on \( W \). For each \( \alpha \in J \) we have \( \dim \hat{V}_\alpha \geq \dim \hat{W}_\alpha \).

A prevaluation \( \varphi \) is said to have one-dimensional leaves if for every \( \alpha \in I \) the dimension of the leaf \( \hat{V}_\alpha \) is at most 1.

**Proposition 2.6.** Let \( V \) be equipped with an \( I \)-valued prevaluation \( \varphi \) with one-dimensional leaves. Let \( W \subset V \) be a non-zero subspace. Then the number of elements in \( \varphi(W \setminus \{0\}) \) is equal to \( \dim W \).

**Proof.** Let \( J = \varphi(W \setminus \{0\}) \). From Proposition 2.5 \( \varphi \) induces a \( J \)-valued prevaluation with one-dimensional leaves on the space \( W \). The proposition now follows from Proposition 2.4 applied to \( W \).

**Example 2.7** (Schubert cells in Grassmannian). Let \( \text{Gr}(n, k) \) be the Grassmannian of \( k \)-dimensional planes in \( \mathbb{C}^n \). Take the prevaluation \( \varphi \) in Example 2.2 for \( V = \mathbb{C}^n \) and the standard basis. Under this prevaluation each \( k \)-dimensional subspace \( L \subset \mathbb{C}^n \) goes to a subset \( J \subset I \) with \( k \) elements. The set of all \( k \)-dimensional subspaces which are mapped onto \( J \) forms the Schubert cell \( X_J \) in the Grassmannian \( \text{Gr}(n, k) \).

In a similar fashion to Example 2.7 the Schubert cells in the variety of complete flags can also be recovered from the above prevaluation \( \varphi \) on \( \mathbb{C}^n \).

### 2.2. Valuations on algebras

In this section we define a valuation on an algebra and describe its basic properties. It will allow us to reduce the properties of the Hilbert functions of graded algebras to the corresponding properties of semigroups. We will present several examples of valuations.

An **ordered abelian group** \( \Gamma \) equipped with a total order \( < \) which respects the group operation, i.e. for \( a, b, c \in \Gamma \), \( a < b \) implies \( a + c < b + c \).

**Definition 2.8.** Let \( A \) be an algebra over a field \( k \) and \( \Gamma \) an ordered abelian group. A prevaluation \( \varphi : A \setminus \{0\} \to \Gamma \) is a **valuation** if, in addition, it satisfies the following: for any \( f, g \in A \) with \( f, g \neq 0 \), we have \( \varphi(fg) = \varphi(f) + \varphi(g) \). The valuation \( \varphi \) is called **faithful** if its image is the whole \( \Gamma \).
We will only deal with faithful valuations with one-dimensional leaves. For the rest of the paper, without explicitly mentioning, we will assume that all the valuations are faithful and have one-dimensional leaves.

**Example 2.9.** Let $X$ be an irreducible curve. As the algebra take the field of rational functions $\mathbb{C}(X)$ and $\Gamma = \mathbb{Z}$. Let $a \in X$ be a smooth point. Then the map

$$v(f) = \text{ord}_a(f)$$

defines a faithful $\mathbb{Z}$-valued valuation (with one-dimensional leaves) on $\mathbb{C}(X)$.

The following proposition is straightforward.

**Proposition 2.10.** Let $A$ be an algebra over $k$ together with a $\Gamma$-valued valuation $v: A \setminus \{0\} \to \Gamma$. 1) For each subalgebra $B \subset A$, the set $v(B \setminus \{0\})$ is a subsemigroup of $\Gamma$. 2) For two linear subspaces $L_1, L_2 \subset A$, let the product $L_1L_2$ denote the span (over $k$) of all the products $fg$ with $f \in L_1$, $g \in L_2$. Put $D_1 = v(L_1 \setminus \{0\})$, $D_2 = v(L_2 \setminus \{0\})$ and $D = v(L_1L_2 \setminus \{0\})$. Then we have $D_1 + D_2 \subset D$.

In general it is not true that $D = D_1 + D_2$ as the following example shows.

**Example 2.11.** Let $F = \mathbb{C}(t)$ be the field of rational polynomials in one variable, $\Gamma = \mathbb{Z}$ (with the usual ordering of numbers) and $v$ the valuation which associates to a polynomial its order of vanishing at the origin. Let $L_1 = \text{span}\{1, t\}$ and $L_2 = \text{span}\{t, 1 + t^2\}$. Then $D_1 = D_2 = \{0,1\}$. The space $L_1L_2$ is spanned by the polynomials $t, 1 + t^2, t^2 + t^3$. We have $D = \{0,1,2,3\}$, while $D_1 + D_2 = \{0,1,2\}$.

We will work with valuations with values in the group $\mathbb{Z}^n$ (equipped with some total ordering). One can define orderings on $\mathbb{Z}^n$ as follows. Take $n$ independent linear functions $\ell_1, \ldots, \ell_n$ on $\mathbb{R}^n$. For $p, q \in \mathbb{Z}^n$ we say $p > q$ if for some $1 \leq r < n$ we have $\ell_r(p) = \ell_r(q)$, $i = 1, \ldots, r$, and $\ell_{r+1}(p) > \ell_{r+1}(q)$. This is a total ordering on $\mathbb{Z}^n$ which respects the addition.

We are essentially interested in orderings on $\mathbb{Z}^n$ whose restriction to the semigroup $\mathbb{Z}_{\geq 0}^n$ is a well-ordering. This holds for the above ordering if the following properness condition is satisfied: there is $1 \leq k \leq n$ such that $\ell_1, \ldots, \ell_k$ are non-negative on $\mathbb{Z}_{\geq 0}^n$ and the map $\ell = (\ell_1, \ldots, \ell_k)$ is a proper map from $\mathbb{Z}_{\geq 0}^n$ to $\mathbb{R}^k$.

Let us now define the Gröbner valuation on the algebra $A = k[[x_1, \ldots, x_n]]$ of formal power series in the variables $x_1, \ldots, x_n$ and with the coefficients in a field $k$. Fix a total ordering on $\mathbb{Z}^n$ (respecting the addition) which restricts to a well-ordering on $\mathbb{Z}_{\geq 0}^n$. For $f \in A$ let $cx_1^{a_1} \cdots x_n^{a_n}$ be the term in $f$ with the smallest exponent $(a_1, \ldots, a_n)$ with respect to this ordering. It exists since $\mathbb{Z}_{\geq 0}^n$ is well-ordered. Define $v(f) = (a_1, \ldots, a_n)$. We extend $v$ to the field of fractions $K$ of $A$ by defining $v(f/g) = v(f) - v(g)$, for any $f, g \in A$, $g \neq 0$. One verifies that $v$ is a faithful $\mathbb{Z}^n$-valued valuation with one-dimensional leaves on the field $K$.

Most of the examples of valuations that we will need can be realized as restrictions of the above Gröbner valuation to subfields of $K$.

**Example 2.12.** Let $X$ be an irreducible $n$-dimensional variety over a field $k$, and $p \in X$ a smooth point over $k$. (When $k$ is algebraically closed almost all the points are smooth.) Let $x_1, \ldots, x_n$ be regular functions at $p$ which form a system of local coordinates, i.e. $x_1, \ldots, x_n$ are in the maximal ideal $\mathfrak{m}_p$ of $p$ and their images in the tangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a vector space basis over $k$. Then each regular function at $p$ can be represented by a formal power series in the $x_i$. This gives
an injective homomorphism from the algebra $\mathcal{O}_p$ of regular functions at $p$ to the algebra of formal power series $A = k[[x_1, \ldots, x_n]]$. The restriction of $v$ to $\mathcal{O}_p$ gives a $\mathbb{Z}^n$-valued valuation. Since $\mathcal{O}_p$ contains $x_1, \ldots, x_n$ this valuation is faithful. The valuation $v$ extends to a valuation on the field of rational functions on $X$ by defining $v(f/g) = v(f) - v(g)$. It is a faithful valuation with one-dimensional leaves.

The existence of faithful $\mathbb{Z}^n$-valued valuations on any field of transcendence degree $n$ is well-known (cf. [Jacobson80, Chapter 9]).

Birationally isomorphic varieties have isomorphic fields of rational functions. It allows one to modify the above example: let $Y$ be an irreducible variety birationally isomorphic to $X$. Then a valuation $v$ on the field of rational functions on $Y$ (e.g. the faithful $\mathbb{Z}^n$-valued valuation in Example 2.12) automatically gives a valuation on the field of rational functions on $X$. The following is an example of this kind of valuation defined in terms of information on the variety $X$, although it indeed corresponds to a system of parameters at a smooth point on some birational model $Y$ of $X$.

**Example 2.13** (Valuation constructed from a Parshin point on $X$). Let $X$ be an irreducible normal $n$-dimensional variety. Consider a sequence of maps

$$X_i \导向 \{a\} = X_0 \pi_0 \rightarrow X_1 \pi_1 \rightarrow \cdots \rightarrow X_{n-1} \pi_{n-1} \rightarrow X_n = X,$$

where each $X_i$, $i = 0, \ldots, n - 1$, is a normal irreducible variety of dimension $i$ and the map $X_i \rightarrow X_{i+1}$ is the normalization map for the image $\pi_i(X_i) \subset X_{i+1}$. We call such a sequence $X_i$ a Parshin point on the variety $X$. We say that a collection of rational functions $f_1, \ldots, f_n$ is a system of parameters about $X_i$, if for each $i$, the function $\pi_i \circ \cdots \circ \pi_{n-1}(f_i)$ vanishes at order 1 along the hypersurface $\pi_{i-1}(X_{i-1})$ in the normal variety $X_i$. Given a Parshin point $X_i$, together with a system of parameters, one can associate an iterated Laurent series to each rational function $g$ on $X$ (see [Parshin83, Okounkov03]). An iterated Laurent series is defined inductively (on the number of parameters). It is a usual Laurent series $\sum c_k f_k^n$ with a finite number of terms with negative degrees in the variable $f_n$ and every coefficient $c_k$ is an iterated Laurent series in the variables $f_1, \ldots, f_{n-1}$. An iterated Laurent series has a monominal $c f_1^{k_1} \cdots f_n^{k_n}$ of the smallest degree with respect to the lexicographic order in the degrees $(k_1, \ldots, k_n)$ (where we order the parameters by $f_n > f_{n-1} > \cdots > f_1$). The map which assigns to a Laurent series its smallest monominal defines a faithful valuation (with one-dimensional leaves) on the field of rational functions on $X$.

Finally let us give an example of a faithful $\mathbb{Z}^n$-valued valuation on a field of transcendence degree $n$ (over the ground field $\mathbb{C}$) which is not finitely generated over $\mathbb{C}$.

**Example 2.14.** As above let $K$ be the field of fractions of the algebra of formal power series $\mathbb{C}[[x_1, \ldots, x_n]]$ and let $K'$ be the subfield consisting of the elements which are algebraic over the field of rational functions $\mathbb{C}(x_1, \ldots, x_n)$. This has transcendence degree $n$ over $\mathbb{C}$ but is not finitely generated over $\mathbb{C}$. The restriction of the Gröbner valuation above on $K$ to $K'$ gives a faithful valuation on $K'$.

### 2.3. Graded subalgebras of the polynomial ring $F[t]$

In this section we introduce certain large classes of graded algebras and discuss their basic properties.

Let $F$ be a field containing a field $k$ which we take as the ground field. The main example of $k$ will be $\mathbb{C}$, the field of complex numbers. Nevertheless, here $k$ can be
taken to be any field, not necessarily algebraically closed and it can have positive characteristic.

A homogeneous element of degree $m \geq 0$ in $F[t]$ is an element $a_m t^m$ where $a_m \in F$. (For any $m$ the element $0 \in F$ is a homogeneous element of degree $m$.) Let $M$ be a linear subspace of $F[t]$. For any $k \geq 0$ the collection $M_k$ of homogeneous elements of degree $k$ in $M$ is a linear subspace over $k$ called the $k$-th homogeneous component of $M$. Similarly the linear subspace $L_k \subset F$ consisting of those $a$ such that $at^k \in M_k$ is called the $k$-th subspace of $M$. A linear subspace $M \subset F[t]$ over $k$ is called a graded space if it is the direct sum of its homogeneous components. A subalgebra $A \subset F[t]$ is called graded if it is graded as a linear subspace of $F[t]$.

We now define three classes of graded subalgebras which will play main roles later:

(1) To each non-zero finite dimensional linear subspace $L \subset F$ over $k$ we associate the graded algebra $A_L$ defined as follows: its zero-th homogeneous component is $k$ and for each $k > 0$ its $k$-th subspace is $L^k$, the subspace spanned by all the products $f_1 \cdots f_k$ with $f_i \in L$. That is,

$$A_L = \bigoplus_{k \geq 0} L^k.$$

The algebra $A_L$ is a graded algebra generated by $k$ and finitely many elements of degree 1.

(2) We call a graded subalgebra $A \subset F[t]$, an algebra of integral type, if there is an algebra $A_L$, for some non-zero finite dimensional subspace $L$ over $k$, such that $A$ is a finitely generated $A_L$-module. (Equivalently, if $A$ is finitely generated over $k$ and is integral over some $A_L$.)

(3) We call a graded subalgebra $A \subset F[t]$, an algebra of almost integral type, if there is an algebra $A' \subset F[t]$ of integral type such that $A \subset A'$. (Equivalently if $A \subset A_L$ for some finite dimensional subspace $L \subset F$.)

As the following shows, the class of algebras of almost integral type already contains the class of finitely generated graded subalgebras. Although, in general, an algebra of almost integral type may not be finitely generated.

**Proposition 2.15.** Let $A$ be a finitely generated graded subalgebra of $F[t]$ (over $k$). Then $A$ is an algebra of almost integral type.

**Proof.** Let $f_1^{d_1}, \ldots, f_r^{d_r}$ be a set of homogeneous generators for $A$. Let $L$ be the subspace spanned by 1 and all the $f_i$. Then $A$ is contained in the algebra $A_L$ and hence is of almost integral type. \(\square\)

The following proposition is easy to show.

**Proposition 2.16.** Let $M \subset F[t]$ be a graded subspace and write $M = \bigoplus_{k \geq 0} L_k t^k$, where $L_k$ is the $k$-th subspace of $M$. Then $M$ is a finitely generated module over an algebra $A_L$ if and only if there exists $N > 0$ such that for any $m \geq N$ and $\ell > 0$ we have $L_{m+\ell} = L_m L^{\ell}$.\footnote{Let $A \subset B$ be commutative rings. An element $f \in B$ is called integral over $A$ if $f$ satisfies an equation $f^m + a_1 f^{m-1} + \cdots + a_m = 0$, for $m > 0$ and $a_i \in A$, $i = 1, \ldots, m$. The integral}

Let $A$ be a graded subalgebra of $F[t]$. Let us denote the integral closure of $A$ in the field of fractions $F(t)$ by $\overline{A}$.\footnote{It is a standard result that $\overline{A}$ is contained in $F[t]$ and is graded (see [Eisenbud95] Ex. 4.21).}
The following is a corollary of the classical theorem of Noether on finiteness of integral closure.

**Theorem 2.17.** Let \( F \) be a finitely generated field over a field \( k \) and \( A \) a graded subalgebra of \( F[t] \). 1) If \( A \) is of integral type then \( \overline{A} \) is also of integral type. 2) If \( A \) is of almost integral type then \( \overline{A} \) is also of almost integral type.

Let \( L \subset F \) be a linear subspace over \( k \). Let \( P(L) \subset F \) denote the field consisting of all the elements \( f/g \) where \( f, g \in L^k \) for some \( k > 0 \) and \( g \neq 0 \). We call \( P(L) \), the subfield associated to \( L \), and its transcendence degree over \( k \), the projective transcendence degree of the subspace \( L \).

**Definition 2.18.** The Hilbert function of a graded space \( M \subset F[t] \) is the function \( H_M \) defined by \( H_M(k) = \dim M_k \) (over \( k \)), where \( M_k \) is the \( k \)-th homogeneous component of \( M \). We put \( H_M(k) = \infty \) if \( M_k \) is infinite dimensional.

The theorem below is a corollary of the so-called Hilbert-Serre theorem on Hilbert function of a finitely generated module over a polynomial ring. Algebraic and combinatorial proofs of this theorem can be found in [Samuel-Zariski60, Chap. VII, §12], [Khovanskii95] and [Chulkov-Khovanskii06].

**Theorem 2.19.** Let \( L \subset F \) be a finite dimensional subspace over \( k \) and let \( q \) be its projective transcendence degree. Let \( M \subset F[t] \) be a finitely generated graded module over \( A_L \). Then for sufficiently large values of \( k \), the Hilbert function \( H_M(k) \) of \( M \) coincides with a polynomial \( \tilde{H}_M(k) \) of degree \( q \). The leading coefficient of this polynomial multiplied by \( q! \) is a positive integer.

**Definition 2.20.** The polynomial \( \tilde{H}_M \) in Theorem 2.19 is called the Hilbert polynomial of the graded module \( M \).

Two numbers appear in Theorem 2.19: the degree \( q \) of the Hilbert polynomial and its leading coefficient multiplied by \( q! \). When \( M = A_L \), both of these numbers have geometric meanings (see Section 3.1).

Assume that a graded algebra \( A \subset F[t] \) has at least one non-zero homogeneous component of positive degree. Then the set of \( k \) for which the homogeneous component \( A_k \) is not 0 forms a non-trivial semigroup \( T \subset \mathbb{Z}_{\geq 0} \). Let \( m(A) \) be the index of the group \( G(T) \) in \( \mathbb{Z} \). When \( k \) is sufficiently large, the homogeneous component \( A_k \) is non-zero (and hence \( H_A(k) \) is non-zero) if and only if \( k \) is divisible by \( m(A) \).

**Corollary 2.21.** For an algebra \( A \subset F[t] \) of integral type we have \( m(A) = 1 \).

**Proof.** The Hilbert polynomial \( \tilde{H}_A \) of an algebra \( A \) of integral type is not identically zero. So any large enough integer belongs to the support of \( \tilde{H}_A \) and hence \( m(A) = 1 \).

Next we define the componentwise product of graded spaces. First we define the product of two subspaces of \( F \).

**Definition 2.22.** Let \( L_1, L_2 \subset F \) be two finite dimensional subspaces. Define \( L_1 L_2 \) to be the \( k \)-linear subspace spanned by all the products \( fg \) where \( f \in L_1 \) and \( g \in L_2 \). The collection of all the non-zero finite dimensional subspaces of \( F \) is a (commutative) semigroup with respect to this product. We will denote it by \( \text{K}(F) \).

---

closure \( \overline{A} \) of \( A \) in \( B \) is the collection of all the elements of \( B \) which are integral over \( A \). It is a ring containing \( A \).
Definition 2.23. Let $M', M''$ be graded spaces with $k$-th subspaces $L'_k$, $L''_k$ respectively. The componentwise product of spaces $M'$ and $M''$ is the graded space $M = M'M''$ whose $k$-th subspace $L_k$ is $L'_kL''_k$.

In particular, the componentwise product can be applied to graded subalgebras of $F[t]$. The following can be easily verified:

Proposition 2.24. 1) The componentwise product of graded algebras is a graded algebra. 2) Let $L', L'' \subset F$ be two non-zero finite dimensional subspaces over $k$ and let $L = L'L''$. Then $A_L = A_LA_{L''}$. 3) Let $M'$, $M''$ be two finitely generated modules over $A_L$ and $A_{L''}$ respectively. Then $M = M'M''$ is a finitely generated module over $A_L$, where $L = L'L''$. 4) If $A'$, $A''$ are algebras of integral type (respectively of almost integral type) then $A = A'A''$ is also of integral type (respectively of almost integral type).

Corollary 2.25. 1) The map $L \mapsto A_L$ is an isomorphism between the semigroup $K(F)$ of non-zero finite dimensional subspaces in $F$, and the semigroup of subalgebras $A_L$ with respect to the componentwise product. 2) The collection of algebras of almost integral type in $F[t]$ is a semigroup with respect to the componentwise product of subalgebras.

2.4. Valuations on graded algebras and semigroups. In this section given a valuation on the field $F$ we construct a valuation on the ring $F[t]$. Using this valuation we will deduce results about the graded algebras of almost integral type from the analogous results for the strongly non-negative semigroups.

It will be easier to prove the statements in this section if in addition $F$ is assumed to be finitely generated over $k$. One knows that a field extension $F/k$ is finitely generated if and only if it is the field of rational functions of an irreducible algebraic variety over $k$. Moreover, the transcendence degree of $F/k$ is the dimension of the variety $X$. The following simple proposition justifies that the general case can be reduced to the case where $F$ is finitely generated over $k$.

Proposition 2.26. Let $A_1, \ldots, A_k \subset F[t]$ be algebras of almost integral type over $k$. Then there exists a field $F_0 \subset F$ which is finitely generated over $k$ such that $A_1, \ldots, A_k \subset F_0[t]$. If $F$ has finite transcendence degree over $k$ then the field $F_0$ can be chosen to have the same transcendence degree.

Thus To prove a statement about a finite collection of subalgebras $A_1, \ldots, A_k \subset F[t]$ of almost integral type over $k$, it is enough to prove it for the case where $F$ is finitely generated over $k$.

To carry out our constructions we require a faithful $\mathbb{Z}^n$-valued valuation on the field $F$ (where $n$ is the transcendence degree of $F/k$). When $F$ is the field of rational functions on an irreducible algebraic variety there are many examples of such a valuation (see Example 2.12). Also if $F$ is not finitely generated over $k$ such a valuation still may exist (see Example 2.13). Nevertheless, by the above proposition, it is enough to prove all the statements in this section for the case when $F$ is finitely generated over $k$, and hence is the field of rational functions of some algebraic variety. So without loss of generality we assume that there is a faithful $\mathbb{Z}^n$-valued valuation on $F$. Fix one such valuation $v : F \setminus \{0\} \to \mathbb{Z}^n$ (where it is understood that $\mathbb{Z}^n$ is equipped with a total order $< \text{respecting the addition}$). Using $v$ on $F$ we define a $\mathbb{Z}^n \times \mathbb{Z}$-valued valuation $v_t$ on the algebra $F[t]$.
Consider the total ordering \( \prec \) on the group \( \mathbb{Z}^n \times \mathbb{Z} \) given by the following: let \((\alpha, n), (\beta, m) \in \mathbb{Z}^n \times \mathbb{Z}\).

1. If \( n > m \) then \((\alpha, n) \prec (\beta, m)\).
2. If \( n = m \) and \( \alpha < \beta \) then \((\alpha, n) \prec (\beta, m)\).

**Definition 2.27.** Define \( v_t : F[t] \setminus \{0\} \to \mathbb{Z}^n \times \mathbb{Z} \) as follows: Let \( P(t) = a_n t^n + \cdots + a_0, a_n \neq 0 \), be a polynomial in \( F[t] \). Then
\[
v_t(P) = (v(a_n), n).
\]

It is easy to verify that \( v_t \) is a valuation (extending \( v \) on \( F \)) where \( \mathbb{Z}^n \times \mathbb{Z} \) is equipped with the total ordering \( \prec \). The extension of \( v_t \) to the field of fractions \( F(t) \) is faithful and has one-dimensional leaves.

Let \( A \subset F[t] \) be a graded subalgebra. Then
\[
S(A) = v_t(A \setminus \{0\}),
\]
is a non-negative semigroup (see Proposition 2.10). We will use the following notations:
- \( \text{Con}(A) \), the cone of the semigroup \( S(A) \),
- \( G(A) \), the group generated by the semigroup \( S(A) \),
- \( G_0(A) \), the subgroup \( G_0(S(A)) \).
- \( H_A \), the Hilbert function of the graded algebra \( A \),
- \( \Delta(A) \), the Newton-Okounkov convex set of the semigroup \( S(A) \),
- \( m(A) \), \( \text{ind}(A) \), the indices \( m(S(A)) \), \( \text{ind}(S(A)) \) for the semigroup \( S(A) \) respectively.

**Proposition 2.28.** The Hilbert function \( H_{S(A)} \) of the non-negative semigroup \( S(A) \) coincides with the Hilbert function \( H_A \) of the algebra \( A \).

**Proof.** Follows from Proposition 2.6.

Now we show that when \( A \) is an algebra of almost integral type then the semigroup \( S(A) \) is strongly non-negative.

**Lemma 2.29.** Let \( A \) be an algebra of integral type. Assume that the rank of \( G(A) \subset \mathbb{Z}^n \times \mathbb{Z} \) is equal to \( n + 1 \). Then the semigroup \( S(A) \) is strongly non-negative.

**Proof.** It is obvious that the semigroup \( S(A) \) is non-negative. Let \( A \) be a finitely generated module over some algebra \( A_L \). Since \( P(L) \subset F \), the projective transcendence degree of \( L \) cannot be bigger than \( n \). By Theorem 2.19 (Hilbert-Serre theorem), for large values of \( k \), the Hilbert function of the algebra \( A \) is a polynomial in \( k \) of degree \( \leq n \). Thus by Theorem 1.17 the semigroup \( S(A) \) is strongly non-negative.

**Lemma 2.30.** Let \( A \) be an algebra of integral type. Then there exists an algebra of integral type \( B \) containing \( A \) such that the group \( G(B) \) is the whole \( \mathbb{Z}^n \times \mathbb{Z} \).

**Proof.** By assumption \( v \) is faithful and hence we can find elements \( f_1, \ldots, f_n \in F \) such that \( v(f_1), \ldots, v(f_n) \) is the standard basis for \( \mathbb{Z}^n \). Consider the space \( L \) spanned by \( 1 \) and \( f_1, \ldots, f_n \) and take its associated graded algebra \( A_L \). The semigroup \( S(A_L) \) contains the basis \( \{ e_{n+1}, e_1 + e_{n+1}, \ldots, e_n + e_{n+1} \} \), where \( \{ e_1, \ldots, e_{n+1} \} \) is the standard basis in \( \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \). Hence \( G(A_L) = \mathbb{Z}^n \times \mathbb{Z} \). Let \( B = A_L A \) be the componentwise product of \( A \) and \( A_L \). One sees that \( G(B) = \mathbb{Z}^n \times \mathbb{Z} \). Since \( 1 \in L \) we have \( A \subset B \).
Theorem 2.31. Let $A \subset F[t]$ be an algebra of almost integral type. Then $S(A)$ is a strongly non-negative semigroup, and hence its Newton-Okounkov convex set $\Delta(A)$ is a convex body.

Proof. By definition the algebra $A$ is contained in some algebra of integral type, and moreover by Lemma 2.30 it is contained in an algebra $B$ of integral type such that $G(B) = \mathbb{Z}^n \times \mathbb{Z}$. By Lemma 2.29 $S(B)$ is strongly non-negative. Since $A \subset B$ we have $S(A) \subset S(B)$ which shows that $S(A)$ is also strongly non-negative. 

Using Theorem 2.31 we can translate the results in Part I about the Hilbert function of strongly non-negative semigroups to results about the Hilbert function of algebras of almost integral type.

Let $A$ be an algebra of almost integral type with the Newton-Okounkov body $\Delta(A)$. Put $m = m(A)$ and $q = \dim \Delta(A)$. For large values of $p$, the Hilbert function $H_A$ vanishes at those $p$ which are not divisible by $m$. Recall that $O_m$ denotes the scaling map $O_m(k) = mk$. For a function $f$, $O_m^*(f)$ is the pull-back of $f$ defined by $O_m^*(f)(k) = f(mk)$, for all $k$. Also $\text{Vol}_q$ denotes the integral volume (Definition 1.12).

Theorem 2.32. The $q$-th growth coefficient of the function $O_m^*(H_A)$, i.e.

$$a_q(O_m^*(H_A)) = \lim_{k \to \infty} \frac{H_A(mk)}{k^q},$$

exists and is equal to $\text{Vol}_q(\Delta(A))/\text{ind}(A)$.

Proof. This follows from Theorem 2.31 and Theorem 1.25. 

The semigroup associated to an algebra of almost integral type has the following superadditivity property with respect to the componentwise product:

Proposition 2.33. Let $A', A''$ be algebras of almost integral type and $A = A' A''$. Put $S = v_t(A \setminus \{0\})$, $S' = v_t(A' \setminus \{0\})$ and $S'' = v_t(A'' \setminus \{0\})$. Then $S' \oplus_t S'' \subset S$. Moreover, if $m(A') = m(A'') = 1$ then

$$\Delta(A') \oplus_t \Delta(A'') \subset \Delta(A).$$

(In other words, $\Delta_0(A') + \Delta_0(A'') \subset \Delta_0(A)$, where $\Delta_0$ is the Newton-Okounkov body projected to the level 0 and + is the Minkowski sum.)

Proof. If $L_k$, $L'_k$ and $L''_k$ are the $k$-th subspaces corresponding to $A, A'$ and $A''$ respectively, then by definition $L_k = L'_k L''_k$. According to Proposition 2.10 we have $v(L'_k \setminus \{0\}) + v(L''_k \setminus \{0\}) \subset v(L_k \setminus \{0\})$. The proposition follows from this inclusion. 

Next we prove a Brunn-Minkowski type inequality for the $n$-th growth coefficients of Hilbert functions of algebras of almost integral type, where as usual $n$ is the transcendence degree of $F$ over $k$. This is a generalization of the corresponding inequality for the volume of big divisors (see Corollary 3.13 and Remark 3.14).

Theorem 2.34. Let $A_1, A_2$ be algebras of almost integral type and let $A_3 = A_1 A_2$ be their componentwise product. Moreover assume $m(A_1) = m(A_2) = 1$. Then the $n$-growth coefficients $\rho_1, \rho_2$ and $\rho_3$ of the Hilbert functions of the algebras $A_1, A_2, A_3$ respectively, satisfy the following Brunn-Minkowski type inequality:

$$\rho_1^{1/n} + \rho_2^{1/n} \leq \rho_3^{1/n}. $$
Proof. By Proposition [2.33] applied to the valuation $v_i$, we have $S(A_1) \oplus_t S(A_2) \subset S(A_3)$, and $\Delta(A_1) \oplus_t \Delta(A_2) \subset \Delta(A_3)$. From the classical Brunn-Minkowski inequality (Theorem [1.13]) we then get

$$\text{Vol}_{n}^{1/n}(\Delta(A_1)) + \text{Vol}_{n}^{1/n}(\Delta(A_2)) \leq \text{Vol}_{n}^{1/n}(\Delta(A_3)).$$

For $i = 1, 2, 3$, we have $p_i = \text{Vol}_n(\Delta(A_i))/\text{ind}(A_i)$ (Theorem [2.32]). Since $S(A_1) \oplus_t S(A_2) \subset S(A_3)$, the index $\text{ind}(A_3)$ is less than or equal to both of the indices $\text{ind}(A_1)$ and $\text{ind}(A_2)$. From this and (3) the required inequality (3) follows. \hfill \Box

Let $A \subset F[t]$ be an algebra of almost integral type. For an integer $p$ in the support of the Hilbert function $H_A$ let $\widehat{A}_p$ be the graded subalgebra generated by the $p$-th homogeneous component $A_p$ of $A$. We wish to compare the asymptotic of $H_A$ with the asymptotic, as $p$ tends to infinity, of the growth coefficients of the Hilbert functions of the algebras $\widehat{A}_p$.

For every $p$ in the support of $H_A$ we associate two semigroups: 1) the semigroup $\widehat{S}_p(A)$ generated by the set $S_p(A)$ of points at level $p$ in $S(A)$, and 2) The semigroup $S(\widehat{A}_p)$ associated to the algebra $\widehat{A}_p$.

**Theorem 2.35.** Let $A$ be an algebra of almost integral type and $p$ any integer in the support of $H_A$. Then the semigroup $S(\widehat{A}_p)$ satisfies the inclusions:

$$\widehat{S}_p(A) \subset S(\widehat{A}_p) \subset S(A).$$

Proof. The inclusion $S(\widehat{A}_p) \subset S(A)$ follows from $\widehat{A}_p \subset A$. By definition, the set of points at level $p$ in the semigroups $\widehat{S}_p(A)$ and $S(\widehat{A}_p)$ coincide. Denote this set by $S_p$. For any $k > 0$, the set of points in $\widehat{S}_p(A)$ at the level $kp$ is equal to $k \ast S_p = S_p + \cdots + S_p$ ($k$-times), and the set $S_{kp}(\widehat{A}_p)$ is equal to $v_1(A_p^k \setminus \{0\})$. By Proposition [2.33] we get $k \ast S_p \subset v_1(A_p^k \setminus \{0\})$, i.e. $k \ast S_p \subset S_{kp}(\widehat{A}_p)$, which implies the required inclusion. \hfill \Box

Let $A$ be an algebra of almost integral type with index $m = m(A)$. Any positive integer $p$ which is sufficiently large and is divisible by $m$ lies in the support of the Hilbert function $H_A$ and hence the subalgebra $\widehat{A}_p$ is defined. To this subalgebra there corresponds its Hilbert function $H_{\widehat{A}_p}$, the semigroup $S(\widehat{A}_p)$, the Newton-Okounkov body $\Delta(\widehat{A}_p)$, and the indices $m(\widehat{A}_p), \text{ind}(\widehat{A}_p)$.

The following can be considered as a generalization of the Fujita approximation theorem (regarding the volume of big divisors) to algebras of almost integral type.

**Theorem 2.36.** For $p$ sufficiently large and divisible by $m = m(A)$ we have:

1. $\dim(\widehat{A}_p) = \dim(\Delta(A)) = q$.
2. $\text{ind}(\widehat{A}_p) = \text{ind}(A)$.
3. Let the function $\varphi$ be defined by

$$\varphi(p) = \lim_{t \to \infty} \frac{H_{\widehat{A}_p}(tp)}{t^q}.$$

That is, $\varphi$ is the $q$-th growth coefficient of $O_p^*(H_{\widehat{A}_p})$. Then the $q$-th growth coefficient of the function $O_m^*(\varphi)$, i.e.

$$a_q(O_m^*(\varphi)) = \lim_{k \to \infty} \frac{\varphi(mk)}{k^q},$$

exists and is equal to $a_q(O_m^*(H_A)) = \text{Vol}_q(\Delta(A))/\text{ind}(A)$. 

Proof. It follows from Theorem 2.35 and Theorem 2.32. □

When \( A \) is an algebra of integral type, Theorem 2.36 can be refined using the Hilbert-Serre theorem (Theorem 2.19). Note that by Corollary 2.21 when \( A \) is of integral type have \( m(A) = 1 \).

**Theorem 2.37.** Let \( A \) be an algebra of integral type and, as in Theorem 2.36, let \( \varphi(p) \) be the \( q \)-th growth coefficient of \( O_p^*(H_{\hat{A}}) \). Then for sufficiently large \( p \), the number \( \frac{\varphi(p)}{p^q} \) is independent of \( p \) and we have:

\[
\frac{\varphi(p)}{p^q} = \frac{\text{Vol}_q(\Delta(A))}{\text{ind}(A)} = a_q(H_A).
\]

Proof. This follows from Theorem 2.19. Let \( \tilde{H}_A(k) = a_qk^q + \cdots + a_0 \) be the Hilbert polynomial of the algebra \( A \). From Proposition 2.16 it follows that if \( p \) is sufficiently large then, for any \( k > 0 \), the \( (kp) \)-th homogeneous component of the algebra \( \hat{A}_p \) coincides with the \( (kp) \)-th homogeneous component of the algebra \( A \), and hence the dimension of the \( (kp) \)-th homogeneous component of \( \hat{A}_p \) is equal to \( H_A(kp) \). Thus the \( q \)-th growth coefficient of the function \( O_p^*(H_{\hat{A}}) \) equals \( p^q a_q \) which proves the theorem. □

3. Part III: Projective varieties and algebras of almost integral type

The famous Hilbert theorem computes the dimension and degree of a projective subvariety of projective space by means of the asymptotic growth of its Hilbert function. The constructions and results in the previous parts relate the asymptotic of Hilbert function with the Newton-Okounkov body. In this part we use Hilbert’s theorem to give geometric interpretations of these results. We will take the ground field to be \( \mathbb{C} \), the field of complex numbers, although it can be replaced with any algebraically closed field of characteristic 0.

3.1. Dimension and degree of projective varieties. In this section we give a geometric interpretation of the dimension and degree of (the closure of) the image of an irreducible variety under a rational map to projective space.

Let \( X \) be an irreducible algebraic variety over \( \mathbb{C} \) of dimension \( n \), and let \( F = \mathbb{C}(X) \) denote the field of rational functions on \( X \). To each finite dimensional subspace \( L \subset F \) we can associate a rational map \( \Phi_L : X \dashrightarrow \mathbb{P}(L^*) \), the projectivization of the dual space of \( L \), as follows:

**Definition 3.1.** Let \( x \in X \) be such that \( f(x) \) is defined for all \( f \in L \). To \( x \) there corresponds a functional in \( L^* \) which evaluates any \( f \in L \) at \( x \). The map \( \Phi_L \) sends \( x \) to the image of this functional in \( \mathbb{P}(L^*) \). We call \( \Phi_L \) the Kodaira map of \( L \).

Let \( Y_L \) denote the closure of the image of \( X \) under the Kodaira map in \( \mathbb{P}(L^*) \) (more precisely, the image of a Zariski open subset of \( X \) where \( \Phi_L \) is defined).

Consider the algebra \( A_L \) associated to \( L \). For large values of \( k \), the Hilbert function \( H_{A_L}(k) \) coincides with the Hilbert polynomial \( \tilde{H}_{A_L}(k) = a_qk^q + \cdots + a_0 \). The following is a version of the celebrated Hilbert theorem on the dimension and degree of a projective subvariety customized for the purposes of this paper.

**Theorem 3.2 (Hilbert).** The degree \( q \) of the Hilbert polynomial \( \tilde{H}_{A_L} \) is equal to the dimension of the variety \( Y_L \), and its leading coefficient \( a_q \) multiplied by \( q! \) is equal to the degree of the subvariety \( Y_L \) in the projective space \( \mathbb{P}(L^*) \).
Fix a faithful \( \mathbb{Z}^n \)-valued valuation \( v \) on the field of rational functions \( F = \mathbb{C}(X) \). The extension \( v_t \) of \( v \) to \( F[t] \), associates to any algebra \( A \) of almost integral type the strongly non-negative semigroup \( S(A) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0} \). Comparing Theorem \( \ref{thm:main-theorem} \) and Theorem \( \ref{thm:Hilbert-theorem} \) (Hilbert’s theorem) we obtain that for \( A = A_L \), the Newton-Okounkov body \( \Delta(A_L) \) is responsible for the dimension and degree of the variety \( Y_L \).

**Corollary 3.3.** The dimension \( q \) of the Newton-Okounkov body \( \Delta(A_L) \) is equal to the dimension of the variety \( Y_L \), and its \( q \)-dimensional integral volume \( \text{Vol}_q(\Delta(A_L)) \) multiplied by \( q! / \text{ind}(A_L) \) is equal to the degree of \( Y_L \).

Let \( A \) be an algebra of almost integral type in \( F[t] \). Let \( L_k \) be the \( k \)-th subspace of the algebra \( A \). To each non-zero subspace \( L \) we can associate the following objects: the Kodaira map \( \Phi_L : X \to \mathbb{P}(L^*_k) \), the variety \( Y_L \subset \mathbb{P}(L^*_k) \) (i.e. the closure of the image of \( \Phi_L \)) and its dimension and degree. Recall that for a sufficiently large integer \( p \) divisible by \( m = m(A) \), the space \( L_p \) is non-zero. As before let \( O_m \) be the scaling map \( O_m(k) = mk \) and \( O_m^\ast \) the pull-back given by \( O_m(f)(k) = f(mk) \). We have the following:

**Theorem 3.4.** If \( p \) is sufficiently large and divisible by \( m = m(A) \), the dimension of the variety \( Y_{L_p} \) is independent of \( p \) and is equal to the dimension \( q \) of the Newton-Okounkov body \( \Delta(A) \). Let \( \text{deg} \) be the function given by \( \text{deg}(p) = \text{deg} Y_{L_p} \). Then the \( q \)-th growth coefficient of the function \( O_m^\ast(\text{deg}) \), i.e.

\[
a_q(O_m^\ast(\text{deg})) = \lim_{k \to \infty} \frac{\text{deg} Y_{Lmk}}{k^q},
\]

exists and is equal to \( q! a_q(O^\ast_m(H_A)) \), which in turn is equal to \( q! \text{Vol}_q(\Delta(A))/\text{ind}(A) \).

**Proof.** Follows from Theorem \( \ref{thm:main-theorem} \) and Theorem \( \ref{thm:Hilbert-theorem} \).

When \( A \) is an algebra of integral type Theorem \( \ref{thm:integral-theorem} \) can be refined. Recall again that by Corollary \( \ref{cor:integral-corollary} \), \( m(A) = 1 \).

**Theorem 3.5.** Let \( A \) be an algebra of integral type. Then for sufficiently large \( p \), the dimension \( q \) of the variety \( Y_{L_p} \), as well as the degree of the variety \( Y_{L_p} \) divided by \( p^3 \), are independent of \( p \). Moreover, the dimension of \( Y_{L_p} \) is equal to the dimension of the Newton-Okounkov body \( \Delta(A) \) and its degree is given by:

\[
\text{deg} Y_{L_p} = q! p^3 a_q(O^\ast_m(H_A)) = \frac{q! p^3 \text{Vol}_q(\Delta(A))}{\text{ind}(A)}.
\]

**Proof.** Follows from Theorem \( \ref{thm:main-theorem} \) (Hilbert’s theorem (Theorem \( \ref{thm:Hilbert-theorem} \) and Theorem \( \ref{thm:integral-theorem} \)).

### 3.2. Algebras of almost integral type associated to linear series.

In this section we apply the results on graded algebras to the ring of sections of divisors and more generally linear series. One of the main results is a generalization of the Fujita approximation theorem (for a big divisor) to any divisor on a complete variety.

Let \( X \) be an irreducible variety over \( \mathbb{C} \) of dimension \( n \) and let \( D \) be a Cartier divisor on \( X \). To \( D \) one associates the subspace \( \mathcal{L}(D) \) of rational functions defined by

\[
\mathcal{L}(D) = \{ f \in \mathbb{C}(X) \mid (f) + D \geq 0 \}.
\]

Let \( \mathcal{O}(D) \) denote the line bundle corresponding to \( D \). The elements of the subspace \( \mathcal{L}(D) \) are in one-to-one correspondence with the sections in \( H^0(X, \mathcal{O}(D)) \).
The following is well-known (see [Hartshorne77, Theorem 5.19]).

**Theorem 3.6.** Let $X$ be a complete variety and $D$ a Cartier divisor on $X$. Then $\mathcal{L}(D)$ is finite dimensional.

Let $D, E$ be divisors and let $f \in \mathcal{L}(D)$, $g \in \mathcal{L}(E)$. From definition it is clear that $fg \in \mathcal{L}(D + E)$. Thus multiplication of functions gives a map

\[
\mathcal{L}(D) \times \mathcal{L}(E) \to \mathcal{L}(D + E).
\]

In general this map may not be surjective.

To a divisor $D$ we associate a graded subalgebra $\mathcal{R}(D)$ of the ring $F[t]$ of polynomials in $t$ with coefficients in the field of rational functions $F = \mathbb{C}(X)$ as follows.

**Definition 3.7.** Define $\mathcal{R}(D)$ to be the collection of all the polynomials $f(t) = \sum_k f_k t^k$ with $f_k \in \mathcal{L}(kD)$, for all $k$. In other words,

\[
\mathcal{R}(D) = \bigoplus_{k=0}^{\infty} \mathcal{L}(kD)t^k.
\]

From (5) it follows that $\mathcal{R}(D)$ is a graded subalgebra of $F[t]$.

**Remark 3.8.** One can find example of a divisor $D$ such that the algebra $\mathcal{R}(D)$ is not finitely generated. See for example [Lazarsfeld04, Section 2.3].

The basic result of this section is that the graded algebra $\mathcal{R}(D)$ is of almost integral type. It allows us to apply the results of Section 2.4 to the algebra $\mathcal{R}(D)$ in order to recover several important results about the asymptotic behavior of divisors (and line bundles).

**Theorem 3.9.** For any Cartier divisor $D$ on a complete variety $X$, the algebra $\mathcal{R}(D)$ is of almost integral type.

To prove Theorem 3.9 we need some preliminaries which we recall here.

When $D$ is a very ample divisor, the following well-known result describes $\mathcal{R}(D)$ (see [Hartshorne77, Ex. 5.14]).

**Theorem 3.10.** Let $X$ be a normal projective variety and $D$ a very ample divisor. Let $L = \mathcal{L}(D)$ be the finite dimensional subspace of rational functions associated to $D$, and let $A_L = \bigoplus_{k\geq 0} L^k t^k$ be the algebra corresponding to $L$. Then 1) $\mathcal{R}(D)$ is the integral closure of $A_L$ in its field of fractions. 2) $\mathcal{R}(D)$ is a graded subalgebra of integral type.

It is well-known that very ample divisors generate the group of all Cartier divisors (see [Lazarsfeld04, Example 1.2.10]). More precisely,

**Theorem 3.11.** Let $X$ be a projective variety. Let $D$ be a Cartier divisor and $E$ a very ample divisor. Then for large enough $k$, the divisor $D + kE$ is very ample. In particular, $D$ can be written as the difference of two very ample divisors $D + kE$ and $kE$.

Finally we need the following statement which is an immediate corollary of Chow’s lemma and the normalization theorem.

**Lemma 3.12.** Let $X$ be any complete variety. Then there exists a normal projective variety $X'$ and a morphism $\pi : X' \to X$ which is a birational isomorphism.
Proof of Theorem 7.9. Let \( \pi : X' \to X \) be as in Lemma 3.12. Let \( D' = \pi^*(D) \) be the pull-back of \( D \) to \( X' \). Then \( \pi^*(\mathcal{R}(D)) \subset \mathcal{R}(D') \). Thus replacing \( X \) with \( X' \), it is enough to prove the statement when \( X \) is normal and projective. Now by Theorem 3.11 we can find very ample divisors \( D_1 \) and \( D_2 \) with \( D = D_1 - D_2 \), moreover, we can take \( D_2 \) to be an effective divisor. It follows that \( \mathcal{R}(D) \subset \mathcal{R}(D_1) \). By Theorem 3.10 \( \mathcal{R}(D_1) \) is of integral type and hence \( \mathcal{R}(D) \) is of almost integral type.

We can now apply the results of Section 2.4 to the graded algebra \( \mathcal{R}(D) \) and derive some results on the asymptotic of the dimensions of the spaces \( \mathcal{L}(kD) \).

Let us recall some terminology from the theory of divisors and linear series (see [Lazarsfeld04, Chapter 2]). These are special cases of the corresponding general definitions for graded algebras in Part II of the paper.

A graded subalgebra \( W \) of \( \mathcal{R}(D) \) is usually called a graded linear series for \( D \). Since \( \mathcal{R}(D) \) is of almost integral type, then any graded linear series \( W \) for \( D \) is also an algebra of almost integral type. Let us write \( W = \bigoplus_{k \geq 0} W_k = \bigoplus_{k \geq 0} L_k t^k \), where \( W_k \) (respectively \( L_k \)) is the \( k \)-th homogeneous component (respectively \( k \)-th subspace) of the graded subalgebra \( W \).

1) The \( n \)-th growth coefficient of the algebra \( W \) multiplied with \( n! \), is called the volume of the graded linear series \( W \) and denoted by \( \text{Vol}(W) \). When \( W = \mathcal{R}(D) \), the volume of \( W \) is denoted by \( \text{Vol}(D) \). In the classical case when \( D \) is ample, \( \text{Vol}(D) \) is equal to its self-intersection number. Moreover, if \( D \) is very ample it induces an embedding of \( X \) into a projective space and \( \text{Vol}(D) \) is the (symplectic) volume of the image of \( X \) under this embedding and hence the term volume.

2) The index \( m = m(W) \) of the algebra \( W \) is usually called the exponent of the graded linear series \( W \). Recall that for large enough \( p \) and divisible by \( m \), the homogeneous component \( W_p \) is non-zero.

3) The growth degree \( q \) of the Hilbert function of the algebra \( W \) is called the Kodaira-Iitaka dimension of \( W \). This term is used when \( X \) is normal (see [Lazarsfeld04, Section 2.1.A]), although we will use it even when \( X \) is not necessarily normal.

The general theorems proved in Section 2.4 about algebras of almost integral type, applied to a graded linear series \( W \) gives the following results:

**Corollary 3.13.** Let \( X \) be a complete irreducible \( n \)-dimensional complex variety. Let \( D \) be a Cartier divisor on \( X \) and \( W \subset \mathcal{R}(D) \) a graded linear series. Then: 1) The \( q \)-th growth coefficient of the function \( O^*_m(H_W) \), i.e.

\[
a_q(O^*_m(H_W)) = \lim_{k \to \infty} \frac{\dim W_{mk}}{k^q},
\]

exists. Fix a faithful \( \mathbb{Z}^n \)-valued valuation for the field \( \mathbb{C}(X) \). Then the Kodaira-Iitaka dimension \( q \) of \( W \) is equal to the dimension of the convex body \( \Delta(W) \) and the growth coefficient \( a_q(O^*_m(H_W)) \) is equal to \( \text{Vol}_q(\Delta(W)) \). Following the notation for the volume of a divisor we denote the quantity \( q! a_q(O^*_m(H_W)) \) by \( \text{Vol}_q(W) \).

2) (A generalized version of Fujita approximation) For \( p \) sufficiently large and divisible by \( m \), let \( \varphi(p) \) be the \( p \)-th growth coefficient of the graded algebra \( A_{L_p} = \bigoplus_k L_k t^k \) associated to the \( q \)-th subspace \( L_p \) of \( W \), i.e. \( \varphi(p) = \lim_{t \to \infty} \dim L'_p/t^q \). Then the \( q \)-th growth coefficient of the function \( O^*_m(\varphi) \), i.e.

\[
a_q(O^*_m(\varphi)) = \lim_{k \to \infty} \frac{\varphi(mk)}{k^q},
\]
exists and is equal to $\Vol_q(\Delta(W))/\ind(W) = \Vol_q(W)/q!\ind(W)$.

3) (Brunn-Minkowski for volume of graded linear series) Suppose $W_1$ and $W_2$ are two graded linear series for divisors $D_1$ and $D_2$ respectively. Also assume $m(W_1) = m(W_2) = 1$, then we have:

$$\Vol^{1/n}(W_1) + \Vol^{1/n}(W_2) \leq \Vol^{1/n}(W_1W_2),$$

where $W_1W_2$ denotes the componentwise product of $W_1$ and $W_2$. In particular, if $W_1 = R(D_1)$ and $W_2 = R(D_2)$ then $W_1W_2 \subset R(D_1 + D_2)$ and hence

$$\Vol^{1/n}(D_1) + \Vol^{1/n}(D_2) \leq \Vol^{1/n}(D_1 + D_2).$$

**Remark 3.14.** The existence of the limit in 1) has been known for the graded algebra $R(D)$, where $D$ is a so-called big divisor (see [Lazarsfeld04]). A divisor $D$ is big if its volume $\Vol(D)$ is strictly positive. Equivalently, $D$ is big if for some $k > 0$, the Kodaira map of the subspace $L(kD)$ is a birational isomorphism onto its image.

It seems that for a general graded linear series (and in particular the algebra $R(D)$ of a general divisor $D$) the existence of the limit in 1) has not previously been known (see [Lazarsfeld04, Remark 2.1.39]).

Part 2) above is in fact a generalization of the Fujita approximation result of [Lazarsfeld-Mustata08, Theorem 3.3]. Using similar methods, for certain graded linear series of big divisors, Lazarsfeld and Mustata prove a statement very close to the statement 2) above.

In [Lazarsfeld-Mustata08] and [Kaveh-Khovanskii08-1, Theorem 5.13] the Brunn-Minkowski inequality in 3) is proved with similar methods.

4. **Part IV: Applications to intersection theory and mixed volume**

In this part we associate a convex body to any non-zero finite dimensional subspace of rational functions on an $n$-dimensional irreducible variety such that: 1) the volume of the body multiplied by $n!$ is equal to the self-intersection index of the subspace, 2) the body which corresponds to the product of subspaces contains the sum of the bodies corresponding to the factors. This construction allows us to prove that the intersection index enjoys all the main inequalities concerning the mixed volume, and also to prove these inequalities for the mixed volume itself.

4.1. **Mixed volume.** In this section we recall the notion of mixed volume of convex bodies and list its main properties (without proofs).

There are two operations of addition and scalar multiplication for convex bodies: let $\Delta_1$, $\Delta_2$ be convex bodies, then their sum

$$\Delta_1 + \Delta_2 = \{x + y \mid x \in \Delta_1, y \in \Delta_2\},$$

is also a convex body called the Minkowski sum of $\Delta_1$, $\Delta_2$. Also for a convex body $\Delta$ and a scalar $\lambda \geq 0$,

$$\lambda \Delta = \{\lambda x \mid x \in \Delta\},$$

is a convex body.

Let $\Vol$ denotes the $n$-dimensional volume in $\mathbb{R}^n$ with respect to the standard Euclidean metric. Function $\Vol$ is a homogeneous polynomial of degree $n$ on the cone of convex bodies, i.e. its restriction to each finite dimensional section of the cone is a homogeneous polynomial of degree $n$. More precisely: for any $k > 0$ let $\mathbb{R}_+^k$ be the positive octant in $\mathbb{R}^k$ consisting of all $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq$
Mixed volume is monotone, that is, for two $n$-tuples of convex bodies $\Delta_1, \ldots, \Delta_k$, the function $P_{\Delta_1, \ldots, \Delta_k}$ defined on $\mathbb{R}_+^k$ by

$$P_{\Delta_1, \ldots, \Delta_k}(\lambda_1, \ldots, \lambda_k) = \text{Vol}(\lambda_1 \Delta_1 + \cdots + \lambda_k \Delta_k),$$

is a homogeneous polynomial of degree $n$.

By definition the mixed volume $V(\Delta_1, \ldots, \Delta_n)$ of an $n$-tuple $(\Delta_1, \ldots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \cdot \cdots \cdot \lambda_n$ in the polynomial $P_{\Delta_1, \ldots, \Delta_n}$ divided by $n!$.

This definition implies that the mixed volume is the polarization of the volume polynomial, that is, it is the function on the $n$-tuples of convex bodies satisfying the following:

(i) Symmetry $V$ is symmetric with respect to permuting the bodies $\Delta_1, \ldots, \Delta_n$.

(ii) Multi-linearity It is linear in each argument with respect to the Minkowski sum. Linearity in the first argument means that for convex bodies $\Delta'_1, \Delta''_1$ and $\Delta_2, \ldots, \Delta_n$ we have:

$$V(\Delta'_1 + \Delta''_1, \ldots, \Delta_n) = V(\Delta'_1, \ldots, \Delta_n) + V(\Delta''_1, \ldots, \Delta_n).$$

(iii) Relation with volume On the diagonal it coincides with the volume, i.e. if $\Delta_1 = \cdots = \Delta_n = \Delta$, then $V(\Delta_1, \ldots, \Delta_n) = \text{Vol}(\Delta)$.

The above three properties characterize the mixed volume: it is the unique function satisfying (i)-(iii).

The next two inequalities are easy to verify:

1) Mixed volume is non-negative, that is, for any $n$-tuple of convex bodies $\Delta_1, \ldots, \Delta_n$ we have:

$$V(\Delta_1, \ldots, \Delta_n) \geq 0.$$

2) Mixed volume is monotone, that is, for two $n$-tuples of convex bodies $\Delta'_1 \subset \Delta_1, \ldots, \Delta'_n \subset \Delta_n$ we have:

$$V(\Delta'_1, \ldots, \Delta'_n) \leq V(\Delta_1, \ldots, \Delta_n).$$

The following inequality attributed to Alexandrov and Fenchel is important and very useful in convex geometry. All its previously known proofs are rather complicated (see [Burago-Zalgaller88]).

**Theorem 4.1** (Alexandrov-Fenchel). Let $\Delta_1, \ldots, \Delta_n$ be convex bodies in $\mathbb{R}^n$. Then

$$V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n)V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n) \leq V^2(\Delta_1, \Delta_2, \ldots, \Delta_n).$$

Below we mention a formal corollary of the Alexandrov-Fenchel inequality. First we need to introduce a notation for repetition of convex bodies in the mixed volume. Let $2 \leq m \leq n$ be an integer and $k_1 + \cdots + k_r = m$ a partition of $m$ with $k_i \in \mathbb{N}$. Denote by $V(k_1 * \Delta_1, \ldots, k_r * \Delta_r, \Delta_{m+1}, \ldots, \Delta_n)$ the mixed volume of the $\Delta$ where $\Delta_1$ is repeated $k_1$ times, $\Delta_2$ is repeated $k_2$ times, etc. and $\Delta_{m+1}, \ldots, \Delta_n$ appear once.

**Corollary 4.2.** With the notation as above, the following inequality holds:

$$\prod_{1 \leq j \leq r} V^{k_j}(m * \Delta_j, \Delta_{m+1}, \ldots, \Delta_n) \leq V^m(k_1 * \Delta_1, \ldots, k_r * \Delta_r, \Delta_{m+1}, \ldots, \Delta_n).$$

The celebrated Brunn-Minkowski inequality concerns volume of convex bodies in $\mathbb{R}^n$. 

Theorem 4.3 (Brunn-Minkowski). Let $\Delta_1$, $\Delta_2$ be convex bodies in $\mathbb{R}^n$. Then
\[
\text{Vol}^{1/n}(\Delta_1) + \text{Vol}^{1/n}(\Delta_2) \leq \text{Vol}^{1/n}(\Delta_1 + \Delta_2).
\]

The following generalization of the Brunn-Minkowski inequality is a corollary of the Alexandrov-Fenchel inequality.

Corollary 4.4 (Generalized Brunn-Minkowski inequality). For any $0 < m \leq n$ and for any fixed convex bodies $\Delta_{m+1}, \ldots, \Delta_n$, the function $F$ which assigns to a body $\Delta$, the number $F(\Delta) = V^{1/m}(m \ast \Delta, \Delta_{m+1}, \ldots, \Delta_n)$, is concave, i.e. for any two convex bodies $\Delta_1, \Delta_2$ we have:
\[
F(\Delta_1) + F(\Delta_2) \leq F(\Delta_1 + \Delta_2).
\]

On the other hand, all the classical proofs of the Alexandrov-Fenchel inequality deduce it from the Brunn-Minkowski inequality. But these deductions are the main and most complicated part of the proofs ([Burago-Zalgaller88]). Interestingly, the main construction in the present paper (using algebraic geometry) allows us to obtain the Alexandrov-Fenchel inequality as an immediate corollary of the simplest case of the Brunn-Minkowski, i.e. when $n = 2$. The Brunn-Minkowski inequality for $n = 2$ is elementary and its proof can be understood by high-school mathematics background. The Alexandrov-Fenchel inequality implies many immediate corollaries and in particular the Brunn-Minkowski and its generalization for any $n$ (Corollary 4.4).

Let us discuss the relation between the 2-dimensional versions of the Brunn-Minkowski and the Alexander-Fenchel. We recall the classical isoperimetric inequality whose origins date back to the antiquity. According to this inequality if $P$ is the perimeter of a simple closed curve in the plane and $A$ is the area enclosed by the curve then
\[
4\pi A \leq P^2.
\]

The equality is obtained when the curve is a circle. To prove (6) it is enough to prove it for convex regions. The Alexandrov-Fenchel inequality for $n = 2$ implies the isoperimetric inequality (6) as a particular case and hence has inherited the name.

Theorem 4.5 (Isoperimetric inequality). If $\Delta_1$ and $\Delta_2$ are convex regions in the plane then
\[
\text{Area}(\Delta_1)\text{Area}(\Delta_2) \leq A(\Delta_1, \Delta_2)^2,
\]
where $A(\Delta_1, \Delta_2)$ is the mixed area.

When $\Delta_2$ is the unit disc in the plane, $A(\Delta_1, \Delta_2)$ is $1/2$ times the perimeter of $\Delta_1$. Thus the classical form (6) of the inequality (for convex regions) follows from Theorem 4.5. It is easy to verify that Theorem 4.5 is equivalent to the Brunn-Minkowski for $n=2$.

4.2. Semigroup of subspaces and intersection index. In this section we briefly review some concepts and results from [Kaveh-Khovanskii08-2]. That is, we discuss the semigroup of subspaces of rational functions, its Grothendieck group and the intersection index on the Grothendieck group. We also recall the key notion of the completion of a subspace.

Let $F$ be a field finitely generated over a ground field $k$. Later we will deal with the case where $F = \mathbb{C}(X)$ is the field of rational functions on a variety $X$ over
Recall (Definition 2.22) that \( K(F) \) denotes the collection of all non-zero finite dimensional subspaces of \( F \) over \( k \). Moreover, for \( L_1, L_2 \in K(F) \), the product \( L_1L_2 \) is the \( k \)-linear subspace spanned by all the products \( fg \) where \( f \in L_1 \) and \( g \in L_2 \). With respect to this product \( K(F) \) is a (commutative) semigroup.

In general the semigroup \( K(F) \) does not have the cancellation property. that is, the equality \( L_1M = L_2M \) if \( L_1, L_2, M \in K(F) \), does not imply \( L_1 = L_2 \). Let us say that \( L_1 \) and \( L_2 \) are equivalent and write \( L_1 \sim L_2 \), if there is \( M \in K(F) \) with \( L_1M = L_2M \). Naturally the quotient \( K(F)/\sim \) is a semigroup with the cancellation property and hence can be extended to a group. The Grothendieck group has the following universal property:

\[ \text{for any group } G \text{ and a homomorphism } \phi : K(F) \to G, \]

there exists a unique homomorphism \( \psi : G(F) \to G \) such that \( \phi = \psi \circ \phi \).

Similar to the notion of integrality of an element over a ring, one defines the integrality of an element over a linear subspace.

**Definition 4.6.** Let \( L \) be a \( k \)-linear subspace in \( F \). An element \( f \in F \) is integral over \( L \) if it satisfies an equation

\[ f^m + a_1 f^{m-1} + \cdots + a_m = 0, \]

where \( m > 0 \) and \( a_i \in L^i, i = 1, \ldots, m \). The completion or integral closure \( \overline{L} \) of \( L \) in \( F \) is the collection of all \( f \in F \) which are integral over \( L \).

The facts below about the completion of a subspace can be found, for example, in [Samuel-Zariski60, Appendix 4]. One shows that \( f \in F \) is integral over a subspace \( L \) if and only if \( L \sim L + \langle f \rangle \). Moreover, the completion \( \overline{L} \) is a subspace containing \( L \), and if \( L \) is finite dimensional then \( \overline{L} \) is also finite dimensional.

The completion \( \overline{L} \) of a subspace \( L \in K(F) \) can be characterized in terms of the notion of equivalence of subspaces: take \( L \in K(F) \). Then \( \overline{L} \) is the largest subspace which is equivalent to \( L \), that is, 1) \( L \sim \overline{L} \) and, 2) If \( L \sim M \) then \( M \subset \overline{L} \).

The following standard result shows the connection between the completion of subspaces and integral closure of algebras.

**Theorem 4.7.** Let \( L \) be a finite dimensional \( k \)-subspace and let \( A_L = \bigoplus_k L^k t^k \) be the corresponding graded subalgebra of \( F[t] \). Then the \( k \)-th subspace of the integral closure \( \overline{A_L} \) is \( \overline{L^k} \), the completion of the \( k \)-th subspace of \( A_L \). That is,

\[ \overline{A_L} = \bigoplus_k \overline{L^k t^k}. \]
Definition 4.8. Let us say that for an \( n \)-tuple of subspaces \((L_1, \ldots, L_n)\), the intersection index is defined and equal to \([L_1, \ldots, L_n]\) if there is a proper algebraic subvariety \( R \subset \mathbb{L} = L_1 \times \cdots \times L_n \) such that for each \( n \)-tuple \((f_1, \ldots, f_n) \in \mathbb{L} \setminus R\) the following holds:

1) The number of solutions of the system \( f_1 = \cdots = f_n = 0 \) in the set \( U_L \setminus Z_L \) is independent of the choice of \((f_1, \ldots, f_n)\) and is equal to \([L_1, \ldots, L_n]\).
2) Each solution \( a \in U_L \setminus Z_L \) of the system \( f_1 = \cdots = f_n = 0 \) is non-degenerate, i.e. the form \( df_1 \wedge \cdots \wedge df_n \) does not vanish at \( a \).

The following is proved in [Kaveh-Khovanskii08-2 Proposition 5.7].

Theorem 4.9. For any \( n \)-tuple \((L_1, \ldots, L_n)\) of subspaces \( L_i \in \mathbb{K}_{rat}(X) \) the intersection index \([L_1, \ldots, L_n]\) is defined.

The following are immediate corollaries of the definition of the intersection index:
1) \([L_1, \ldots, L_n]\) is a symmetric function of \( L_1, \ldots, L_n \in \mathbb{K}_{rat}(X) \), 2) The intersection index is monotone, (i.e. if \( L_1 \subseteq L_1', L_2, \ldots, L_n \subseteq L_n' \), then \([L_1, \ldots, L_n]\) \( \leq \) \([L_1, \ldots, L_n] \), and 3) The intersection index is non-negative.

The next theorem contains the main properties of the intersection index (see [Kaveh-Khovanskii08-2 Section 5]).

Theorem 4.10. 1) (Multi-linearity) Let \( L_1', L_2', \ldots, L_n' \in \mathbb{K}_{rat}(X) \) and put \( L_1 = L_1' L_1'' \). Then
\[
[L_1, \ldots, L_n] = [L_1', L_2, \ldots, L_n] + [L_1'', L_2, \ldots, L_n].
\]
2) (Invariance under the completion) Let \( L_1 \in \mathbb{K}_{rat}(X) \) and let \( \overline{L_1} \) be its completion. Then for any \((n-1)\)-tuple \( L_2, \ldots, L_n \in \mathbb{K}_{rat}(X) \) we have:
\[
[L_1, L_2, \ldots, L_n] = [\overline{L_1}, L_2, \ldots, L_n].
\]

Because of the multi-linearity, the intersection index can be extended to the Grothendieck group \( \mathbb{G}_{rat}(X) \) of the semigroup \( \mathbb{K}_{rat}(X) \). The Grothendieck group of \( \mathbb{K}_{rat}(X) \) can be considered as an analogue (for a typically non-complete variety \( X \)) of the group of Cartier divisors on a complete variety, and the intersection index on the Grothendieck group \( \mathbb{G}_{rat}(X) \) as an analogue of the intersection index of Cartier divisors.

The next proposition relates the self-intersection index of a subspace with the degree of the image of the Kodaira map. It easily follows from the definition of the intersection index.

Proposition 4.11 (self-intersection index and degree). Let \( L \in \mathbb{K}_{rat}(X) \) be a subspace and \( \Phi_L : X \to Y_L \subset \mathbb{P}(L^*) \) its Kodaira map. 1) If \( \dim X = \dim Y_L \) then \( \Phi_L \) has finite mapping degree \( d \) and \([L, \ldots, L]\) is equal to the degree of the subvariety \( Y_L \) (in \( \mathbb{P}(L^*) \)) multiplied with \( d \). 2) If \( \dim X > \dim Y_L \) then \([L, \ldots, L] = 0\).

4.3. **Newton-Okounkov body and intersection index.** We now discuss the relation between the self-intersection index of a subspace of rational functions and the volume of the Newton-Okounkov body.

Let \( X \) be an irreducible \( n \)-dimensional variety (over \( \mathbb{C} \)) and \( L \in \mathbb{K}_{rat}(X) \) a non-zero finite dimensional subspace of rational functions. We can naturally associate two algebras of integral type to \( L \): the algebra \( A_L \) and its integral closure \( \overline{A_L} \).

(Note that by Theorem 2.47 \( A_L \) is an algebra of integral type.)
Let $F = \mathbb{C}(X)$. As in Section 2.3 let $v : F \setminus \{0\} \to \mathbb{Z}^n$ be a faithful valuation with one-dimensional leaves, and $v_t$ its extension to the polynomial ring $F[t]$. Then $v_t$ associates two convex bodies to the space $L$, namely $\Delta(A_L)$ and $\Delta(\overline{A_L})$.

Since $A_L \subset \overline{A_L}$ then $\Delta(A_L) \subset \Delta(\overline{A_L})$. In general $\Delta(\overline{A_L})$ can be strictly bigger than $\Delta(A_L)$.

The following two theorems can be considered as far generalizations of the Kušnirek theorem in Newton polytope theory and toric geometry. Below $\text{Vol}_n$ denotes the standard Euclidean measure in $\mathbb{R}^n$.

**Theorem 4.12.** Let $L \in K_{\text{rat}}(X)$ with the Kodaira map $\Phi_L$. 1) If $\Phi_L$ has finite mapping degree then

$$[L, \ldots, L] = \frac{n! \deg \Phi_L}{\text{ind}(A_L)} \text{Vol}_n(\Delta(A_L)).$$

Otherwise, both $[L, \ldots, L]$ and $\text{Vol}_n(\Delta(A_L))$ are equal to 0. 2) In particular, if $\Phi_L$ is a birational isomorphism between $X$ and $Y_L$ then $\deg \Phi_L = \text{ind}(A_L) = 1$ and we obtain

$$[L, \ldots, L] = n! \text{Vol}_n(\Delta(A_L)).$$

3) The correspondence $L \mapsto \Delta(A_L)$ is superadditive, i.e. if $L_1, L_2$ are finite dimensional subspaces of rational functions. Then

$$\Delta(A_{L_1}) \oplus_t \Delta(A_{L_2}) \subset \Delta(A_{L_1L_2}).$$

(In other words, $\Delta_0(A_{L_1}) + \Delta_0(A_{L_2}) \subset \Delta_0(A_{L_1L_2})$, where $\Delta_0$ is the Newton-Kouchnikov body projected to the level 0 and + is the Minkowski sum.)

**Proof.** 1) Follows from Proposition 4.1 and Corollary 4.2. 2) If $\Phi_L$ is a birational isomorphism then it has degree 1. On the other hand, from the birational isomorphism of $\Phi_L$ it follows that the subfield $P(L)$ associated to $L$ coincides with the whole field $\mathbb{C}(X)$. Since the valuation $v$ is faithful we then conclude that the subgroup $G_0(A_L)$ coincides with the whole $\mathbb{Z}^n$ and hence $\text{ind}(A_L) = 1$. Part 2) then follows from 1). 3) We know that $A_{L_1L_2} = A_{L_1}A_{L_2}$ and $m(A_{L_1}) = m(A_{L_2}) = m(A_{L_1L_2}) = 1$. Proposition 2.3 now gives the required result.

**Theorem 4.13.** 1) We have:

$$[L, \ldots, L] = n! \text{Vol}_n(\Delta(\overline{A_L})).$$

2) The correspondence $L \mapsto \Delta(\overline{A_L})$ is superadditive, i.e. if $L_1, L_2$ are finite dimensional subspaces of rational functions. Then

$$\Delta(\overline{A_{L_1}}) \oplus_t \Delta(\overline{A_{L_2}}) \subset \Delta(\overline{A_{L_1L_2}}).$$

(In other words, $\Delta_0(\overline{A_{L_1}}) + \Delta_0(\overline{A_{L_2}}) \subset \Delta_0(\overline{A_{L_1L_2}}).$)

We need the following lemma.

**Lemma 4.14.** Let $L$ be a subspace of rational functions. Suppose $\dim Y_L = n$ i.e. the Kodaira map $\Phi_L$ has finite mapping degree. Then there exists $N > 0$ such that the following holds: for any $p > N$ the subfield associated to the completion $\overline{P^L}$ coincides with the whole $\mathbb{C}(X)$.

**Proof.** Let $E = P(L) \cong \mathbb{C}(Y_L)$ and $F = \mathbb{C}(X)$. The extension $F/E$ is a finite extension because $\dim Y_L = n$. Clearly for any $p > 0$, $P(\overline{P^L}) \subset F$. We will show
that there is $N > 0$ such that for $p > N$ we have $F \subset P(L^p)$. Let $f_1, \ldots, f_r$ be a basis for $F/E$. Let $f \in \{f_1, \ldots, f_r\}$. Then $f$ satisfies an equation
\begin{equation}
\tag{8}
 a_0 f^m + \cdots + a_m = 0,
\end{equation}
where $a_i = P_i/Q_i$ with $P_i, Q_i \in L^{d_i}$ for some $d_i > 0$. Let $N_f = \sum_{i=0}^{m} d_i$ and put $Q = Q_0 \cdots Q_m$. Then $Q \in L^{N_f}$. Multiplying (8) with $Q$ we have $b_0 f^m + \cdots + b_m = 0$, where $b_i = P_i Q_i / Q_i \in L^{N_f}$. Then multiplying with $b_0^{-1}$ gives
\[(b_0 f)^m + b_1 (b_0 f)^{m-1} + \cdots + (b_0^{-1}) b_m = 0,
\]
which shows that $b_0 f$ is integral over $L^{N_f}$. Now $f \in P(L^{N_f})$ because $f = b_0 f / b_0$ and $b_0 \in L^{N_f}$. Let $N$ be the maximum of the $N_f$ for $f \in \{f_1, \ldots, f_r\}$. It follows that $F \subset P(L^N)$. It is easy to see that for $p > N$ we have $L^N L^p - N \subset L^p$ and hence $P(L^N) \subset P(L^p)$. Thus $F = P(L^p)$ as required.

Proof of Theorem 4.13.
1) Suppose $\dim Y_L < n$. Then from the definition of the self-intersection index it follows that $[L, \ldots, L] = 0$. But we know that the $n$-dimensional volume of $\Delta(\overline{A_L})$ is $0$ because $\dim \Delta(\overline{A_L})$ equals the dimension of $Y_L$ and hence is less than $n$. This proves the theorem in this case. Now suppose $\dim Y_L = n$. Then the Kodaira map $\Phi_L$ has finite mapping degree. By Lemma 4.14 we know there is $N$ such that if $p > N$ then the field $P(L^p)$ coincides with $\mathbb{C}(X)$ and thus the Kodaira map $\Phi_{L^p}$ is a birational isomorphism onto its image. Thus the self-intersection index of $L^p$ is equal to the degree of the variety $Y_{L^p}$. By the main properties of intersection index (Theorem 4.11) we have:
\[
[L^p, \ldots, L^p] = [L^p, \ldots, L^p] = p^n [L, \ldots, L].
\]

On the other hand, by Theorem 4.14,
\[
\deg Y_{L^p} = \frac{p^n}{\text{ind}(\overline{A_L})} \text{Vol}_n(\Delta(\overline{A_L})).
\]
But since the field $P(L^p)$ coincides with $\mathbb{C}(X)$ we have: $\text{ind}(\overline{A_L}) = 1$ which finishes the proof of 1). To prove 2) first note that we have the inclusion:
\[
\overline{A_L}_1 \cap \overline{A_L}_2 \subset \overline{A_{L_1}} \cap \overline{A_{L_2}},
\]
(this follows from the fact that for any two subspaces $L, M$ we have $\overline{L} \cap \overline{M} \subset \overline{L \cap M}$). Secondly, since $m(\overline{A_L}_1) = m(\overline{A_L}_2) = 1$ by Proposition 2.33 we know:
\[
\Delta(\overline{A_L}_1) \cap \Delta(\overline{A_L}_2) \subset \Delta(\overline{A_{L_1}}) \cap \Delta(\overline{A_{L_2}}).
\]
The theorem is proved. \hfill \Box

Next, let us see that the well-known Bernstein-Kušnirenko theorem follows from Theorem 4.12. For this, we take the variety $X$ to be $(\mathbb{C}^*)^n$ and the subspace $L$ a subspace spanned by Laurent monomials.

We identify the lattice $\mathbb{Z}^n$ with the Laurent monomials in $(\mathbb{C}^*)^n$: to each integral point $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we associate the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ where $x = (x_1, \ldots, x_n)$. A Laurent polynomial $P(x) = \sum c_a x^a$ is a finite linear combination of Laurent monomials with complex coefficients. The support $\text{supp}(P)$ of a Laurent polynomial $P$, is the set of exponents $a$ for which $c_a \neq 0$. We denote the convex hull of a finite set $I \subset \mathbb{Z}^n$ by $\Delta_I \subset \mathbb{R}^n$. The Newton polytope $\Delta(P)$ of a Laurent polynomial $P$ is the convex hull $\Delta_{\text{supp}(P)}$ of its support. With each finite set $I \subset \mathbb{Z}^n$ one associates the linear space $L(I)$ of Laurent polynomials $P$ with $\text{supp}(P) \subset I$. 

Theorem 4.15 (Kušnirenko). The number of solutions in \((\mathbb{C}^*)^n\) of a generic system of Laurent polynomial equations \(P_1 = \cdots = P_n = 0\), with \(P_1, \ldots, P_n \in L(I)\) is equal to \(n! \text{Vol}(\Delta_I)\), i.e.

\[ [L(I), \ldots, L(I)] = n! \text{Vol}(\Delta_I). \]

Proof. If \(I, J\) are finite subsets in \(\mathbb{Z}^n\) then \(L_I L_J = L_{I+J}\). Consider the graded algebra \(A_{L(I)} \subset F[t]\) where \(F\) is the field of rational functions on \(\mathbb{C}^n\). Take any valuation \(v\) on \(F\) coming from the Gröbner valuation on the field of fractions of the algebra of formal power series \(\mathbb{C}[[x_1, \ldots, x_n]]\) (see the paragraph before Example 2.12). From the definition it is easy to see that \(v(L(I) \setminus \{0\}) = I\) and more generally \(v(L(I)^k \setminus \{0\}) = k \cdot I\), where \(k \cdot I\) is the sum of \(k\) copies of the set \(I\). Let \(S = S(A_{L(I)})\) be the semigroup associated to the algebra \(A_{L(I)}\). Then the cone \(\text{Con}(S)\) is the cone in \(\mathbb{R}^{n+1}\) over \(\Delta_I \times \{1\}\) and the group \(G_0(S)\) is generated by the differences \(a-b, a, b \in I\). Let \(I = \{\alpha_1, \ldots, \alpha_r\}\). Then \(\{x^{\alpha_1}, \ldots, x^{\alpha_r}\}\) is a basis for \(L(I)\) and one verifies that, in the dual basis for \(L(I)^*\), the Kodaira map is given by \(\Phi_{L(I)}(x) = (x^{\alpha_1} : \cdots : x^{\alpha_r})\). From this it follows that the mapping degree of \(\Phi_{L(I)}\) is equal to the index of the subgroup \(G_0(S)\), i.e. \(\text{ind}(A_{L(I)})\). By Theorem 4.12 we then have:

\[ [L(I), \ldots, L(I)] = n! \frac{\text{deg} \Phi_{L(I)}}{\text{ind}(A_{L(I)})} \text{Vol}(\Delta_I) = n! \text{Vol}(\Delta_I), \]

which proves the theorem. \(\square\)

The Bernstein theorem computes the intersection index of an \(n\)-tuple of subspaces of Laurent polynomials in terms of the mixed volume of their Newton polytopes.

Theorem 4.16 (Bernstein). Let \(I_1, \ldots, I_n \subset \mathbb{Z}^n\) be finite subsets. The number of solutions in \((\mathbb{C}^*)^n\) of a generic system of Laurent polynomial equations \(P_1 = \cdots = P_n = 0\), where \(P_i \in L(I_i)\), is equal to \(n! V(\Delta_{I_1}, \ldots, \Delta_{I_n})\), i.e.

\[ [L(I_1), \ldots, L(I_n)] = n! V(\Delta_{I_1}, \ldots, \Delta_{I_n}), \]

(where, as before, \(V\) denotes the mixed volume of convex bodies in \(\mathbb{R}^n\)).

Proof. The Bernstein theorem readily follows from the multi-linearity of the intersection index, Theorem 4.15 (the Kušnirenko theorem) and the observation that for any two finite subsets \(I, J \subset \mathbb{Z}^n\) we have \(L(I + J) = L(I)L(J)\) and \(\Delta_I + \Delta_J = \Delta_{I+J}\). \(\square\)

The above proofs of the Bernstein and Kušnirenko theorems are in fact very close to the ones in [Khovanskii92].

4.4. Proof of the Alexandrov-Fenchel inequality and its algebraic analogues. Finally in this section, using the notion of Newton-Okounkov body, we prove algebraic analogues of the Alexandrov-Fenchel inequality (and its corollaries). From this we deduce the classical Alexandrov-Fenchel inequality (and its corollaries) in convex geometry.

As before \(X\) is an \(n\)-dimensional irreducible complex algebraic variety. The self-intersection index enjoys the following analogue of the Brunn-Minkowski inequality.
Corollary 4.17. Let $L_1, L_2$ be finite dimensional subspaces in $\mathbb{C}(X)$ and $L_3 = L_1 L_2$. Then

$$[L_1, \ldots, L_1]^{1/n} + [L_2, \ldots, L_2]^{1/n} \leq [L_3, \ldots, L_3]^{1/n}.$$ 

Proof. By definition the Newton-Okounkov body $\Delta(A)$ of an algebra $A$ lives at level 1. For $i = 1, 2, 3$, let $\Delta_i$ be the Newton-Okounkov body of the algebra $\hat{A}_{L_i}$ projected to the level 0. Then by Theorem 4.13 we have $[L_i, \ldots, L_i] = n! \text{Vol}_n(\Delta_i)$, and $\Delta_1 + \Delta_2 \subset \Delta_3$. Now by the classical Brunn-Minkowski inequality $\text{Vol}_n(\Delta_1) + \text{Vol}_n(\Delta_2) \leq \text{Vol}_n(\Delta_3)$ which proves the corollary (cf. Theorem 2.34). □

Surprisingly the most important case of the above inequality is the $n=2$ case, i.e. when $X$ is an algebraic surface. As we show next, the general case of the above inequality and many other inequalities for the intersection index follow from this $n=2$ case and the basic properties of the intersection index.

Corollary 4.18 (A version of the Hodge inequality). Let $X$ be an irreducible algebraic surface and let $L, M$ be non-zero finite dimensional subspaces of $\mathbb{C}(X)$. Then

$$[L, L][M, M] \leq [L, M]^2.$$ 

Proof. From Corollary 4.17 for $n = 2$, we have:

$$[L, L] + 2[L, M] + [M, M] = [LM, LM] \geq ([L, L]^{1/2} + [M, M]^{1/2})^2 \geq [L, L] + 2[L, L]^{1/2}[M, M]^{1/2} + [M, M],$$

which readily implies the claim. □

In other words, Theorem 4.13 allowed us to easily reduce the Hodge inequality above to the isoperimetric inequality. We can now give an easy proof of the Alexandrov-Fenchel inequality and its corollaries for the intersection index.

Let us call a subspace $L \in \mathbb{K}_{rat}(X)$ a very big subspace if the Kodaira rational map of $L$ is a birational isomorphism between $X$ and its image. Also we call a subspace big if for some $m > 0$, the subspace $L^m$ is very big. It is not hard to show that the product of two big subspaces is again a big subspace and thus the big subspaces form a subsemigroup of $\mathbb{K}_{rat}(X)$.

Theorem 4.19 (A version of the Bertini-Lefschetz theorem). Let $X$ be a smooth irreducible $n$-dimensional variety and let $L_1, \ldots, L_k \in \mathbb{K}_{rat}(X)$, $k < n$, be very big subspaces. Then there is a Zariski open set $U$ in $L = L_1 \times \cdots \times L_k$ such that for each point $f = (f_1, \ldots, f_k) \in U$ the variety $X_f$ defined in $X$ by the system of equations $f_1 = \cdots = f_k = 0$ is smooth and irreducible.

A proof of the Bertini-Lefschetz theorem can be found in [Hartshorne77, Theorem 8.18]

One can slightly extend Theorem 4.19. Assume that we are given $k$ very big spaces $L_1, \ldots, L_k \in \mathbb{K}_{rat}(X)$ and $(n-k)$ arbitrary subspaces $L_{k+1}, \ldots, L_n$. We denote by $[L_{k+1}, \ldots, L_n]_{X_f}$ the intersection index of the restriction of the subspaces $L_{k+1}, \ldots, L_n$ to the subvariety $X_f$. It is easy to verify the following reduction theorem.
Theorem 4.20. There is a Zariski open subset $U$ in $L_1 \times \cdots \times L_k$ such that for $f = (f_1, \ldots, f_k) \in U$, the system $f_1 = \cdots = f_k = 0$ defines a smooth irreducible subvariety $X_f$ in $X$ and the identity

$$[L_1, \ldots, L_n]_X = [L_{k+1}, \ldots, L_n]_{X_f},$$

holds.

Theorem 4.21 (Algebraic analogue of the Alexandrov-Fenchel inequality). Let $X$ be an irreducible $n$-dimensional variety and let $L_1, \ldots, L_n \in K_{rat}(X)$. Also assume that $L_3, \ldots, L_n$ are big subspaces. Then the following inequality holds:

$$[L_1, L_1, L_3, \ldots, L_n][L_2, L_2, L_3, \ldots, L_n] \leq [L_1, L_2, L_3, \ldots, L_n]^2.$$  

Proof. Because of the multi-linearity of the intersection index, if the inequality holds for the spaces $L_i$ replaced with $L_i^N$, for some $N$, then it holds for the original spaces $L_i$. So without loss of generality we can assume that $L_3, \ldots, L_n$ are very big. By Theorem 4.20 for almost all the $(f_3, \ldots, f_n) \in L_3 \times \cdots \times L_n$ and the variety $Y$ defined by the system $f_3 = \cdots = f_n = 0$ we have:

$$[L_1, L_1, L_3, \ldots, L_n] = [L_1, L_2]_Y,$$

$$[L_1, L_1, L_3, \ldots, L_n] = [L_1, L_1]_Y,$$

$$[L_2, L_2, L_3, \ldots, L_n] = [L_2, L_2]_Y.$$  

Now applying Corollary 4.18 (the Hodge inequality) for surface $Y$ we have:

$$[L_1, L_1]_Y[L_2, L_2]_Y \leq [L_1, L_2]^2_Y,$$

which proves the theorem. \(\square\)

Similar to Section 4.1, let us introduce a notation for repetition of subspaces in the intersection index. Let $2 \leq m \leq n$ be an integer and $k_1 + \cdots + k_r = m$ be a partition of $m$ with $k_i \in \mathbb{N}$. Consider the subspaces $L_1, \ldots, L_n \in K_{rat}(X)$. Denote by $[k_1 * L_1, \ldots, k_r * L_r, L_{m+1}, \ldots, L_n]$ the intersection index of $L_1, \ldots, L_n$ where $L_1$ is repeated $k_1$ times, $L_2$ is repeated $k_2$ times, etc. and $L_{m+1}, \ldots, L_n$ appear once.

Corollary 4.22 (Corollaries of the algebraic analogue of the Alexandrov-Fenchel inequality). Let $X$ be an $n$-dimensional irreducible variety. 1) Let $2 \leq m \leq n$ and $k_1 + \cdots + k_r = m$ with $k_i \in \mathbb{N}$. Take big subspaces of rational functions $L_1, \ldots, L_n \in K_{rat}(X)$. Then

$$\prod_{1 \leq j \leq r} [m * L_j, L_{m+1}, \ldots, L_n]^{k_j} \leq [k_1 * L_1, \ldots, k_r * L_r, L_{m+1}, \ldots, L_n]^m.$$  

2) (Generalized Brunn-Minkowski inequality) For any fixed big subspaces $L_{m+1}, \ldots, L_n \in K_{rat}(X)$, the function

$$F : L \mapsto [m * L, L_{m+1}, \ldots, L_n]^{1/m},$$

is a concave function on the semigroup $K_{rat}(X)$.

1) follows formally from the algebraic analogue of the Alexandrov-Fenchel, the same way that the corresponding inequalities follow from the classical Alexandrov-Fenchel in convex geometry. 2) can be easily deduced from Corollary 4.17 and Theorem 4.20.

We now prove the classical Alexandrov-Fenchel inequality in convex geometry (Theorem 4.1).
Proof of Theorem 4.1. As we saw above, the Bernstein-Kušnirenko theorem follows from Theorem 4.12. Applying the algebraic analogue of the Alexandrov-Fenchel inequality to the situation considered in the Bernstein-Kušnirenko theorem one proves the Alexandrov-Fenchel inequality for convex polytopes with integral vertices. The homogeneity then implies the Alexandrov-Fenchel inequality for convex polytopes with rational vertices. But since any convex body can be approximated by convex polytopes with rational vertices, by continuity we obtain the Alexandrov-Fenchel inequality in complete generality.

At the end of 1970s and the beginning of 1980s, Teissier [Teissier79] and the second author (see [Khovanskii88] for a survey) proved the algebraic analogue of the Alexandrov-Fenchel inequality (using the Hodge inequality) and out of it deduced the original Alexandrov-Fenchel in convex geometry. This was done in a similar manner as discussed above. The new contribution of the present paper is that we do not use the classical Hodge inequality as in [Teissier79] and [Khovanskii88], but rather we get a version of this inequality along the way in our chain of arguments. Our arguments rely on Hilbert’s theorem on dimension and degree of a projective variety, and the elementary $n = 2$ case of the Brunn-Minkowski inequality (which can be proved with high-school mathematics background). We use a birational version of intersection theory constructed in our previous paper [Kaveh-Khovanskii08-2]. Then via a valuation, we reduce the required results to general statements on semigroups of integral points. These theorems on the semigroups of integral points are among the main results of the present paper. We should note that while in the classical Hodge inequality the surface needs to be smooth and projective (or compact Kaehler), our version of the Hodge inequality (Corollary 4.18) holds for any irreducible surface (not necessarily smooth and projective or compact Kaehler) and hence is much easier to apply.

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A. G. KHOVANSKI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, CANADA; MOSCOW INDEPENDENT UNIVERSITY; INSTITUTE FOR SYSTEMS ANALYSIS, RUSSIAN ACADEMY OF SCIENCES.