Improved Finite Element Approach for Modeling Three-Dimensional Linear-Elastic Bodies

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Abstract

**Objectives:** This paper proposes a general formula for computation of the stiffness matrix of three-dimensional linear-elastic bodies by the method of Moment Schema of Finite Elements (MSFE). **Methods:** Numerical methods based on Lagrange variational principle, such as Finite Element Method (FEM) and the method of Moment Schema of Finite Elements (MSFE). The paper focuses on the problem of obtaining the stiffness matrices of Finite Elements (FE), which ensure effectiveness of the developed version of FEM. Variational principle for calculation of energy functional of rectangular three-dimensional finite element is used. **Results:** Analyses of various problems of mechanics of deformable solids shows slow convergence of traditional FEM, especially for rigid bodies and surfaces of complex curved shapes. This paper proposes a scheme for inference for FEM ratios, which takes into account the basic properties of rigid displacements for isoparametric and curvilinear FE. The presented version of FEM allows us to obtain the stiffness matrix of FE, taking into account the effect of a false shift and low compressibility of elastomers. **Conclusion/Application:** The general method to find the potential energy of three-dimensional linear-elastic bodies is proposed, which allows us to solve the problems of mechanics of complex structures with higher accuracy.

**Keywords:** Energy Functional, Finite Element Method, Three-Dimensional Finite Element

1. Introduction

When using the traditional Finite Element Method (FEM) in the form of displacement method (based on the Lagrange variational principle for solution of problems with singularities, such as taking into account the low compressibility, calculation of plates and shells on the basis of three-dimensional finite elements), there are significant difficulties arose\textsuperscript{1}. It could be fixed by other variational principles - Castigliano (method of forces), Hellinger-Reissner, Hu-Vashitsu (mixed method)\textsuperscript{2}.

FEM in the form of the method of forces does not receive significant development due to its complexity in the approximation of the stress states. Greater use has the mixed FEM schemes. With positive features\textsuperscript{3}, they also have a number of shortcomings, such as the increasing order of system of solving equations in comparison with the FEM method in the form of displacement, impaired positive definiteness of the matrix equations. Thus, for such the problems, the development of hybrid schemes in the form of FEM displacement method based on the variational principle of Lagrange is preferable.

To increase effectiveness of computation of various types of structures, first the properties of stiffness matrices of Finite Element (FE) should be taken into account. This paper focuses on the problem of obtaining such the stiffness matrices of FE, which will ensure the effectiveness of the developed version of FEM. Standard FEM in the form of displacement method requires that the field of movements of points within FE...
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is approximated by polynomial functions and the contact on the boundaries of the elements meets the condition of continuity. As it follows from \(4,5\), this option of FEM has slow convergence due to use of polynomial functions. Approximating displacement fields do not include an element, describing the rigid FE movements. This effect is essential when using curved FE, and accounting for rigid FE displacements should not be seen as a necessary condition for convergence, but as an important method for improving the efficiency of FEM at computation of curved bodies.

Application of the standard FEM scheme in the form of displacement method has another negative feature, called “complex shear effect”\(6\); at bending thin plates and membranes the errors, caused by fictitious shear deformations, increase considerably.

To address these shortcomings, the Moment Schema of Finite Elements (MSFE) has been developed\(7\), which allow us to take into account the basic properties of rigid displacements for isoparametric and curved FE of isotropic elastic bodies. Its essence is a rejection of some members of the strain series expansion, reactive to rigid displacements and showing false shear strains. Here, the exact equations, linking deformations and displacements, are replaced by approximate.

\section{Moment Schema of Finite Elements}

\subsection{Basic Ideas of MSFE}

The classical FEM requires that the displacement field of points inside FE to be approximated by polynomial functions. This option has the slow convergence because approximating displacement fields functions include elements describing the rigid FE displacement but do not take into account the effects of “false shift”. These shortcomings we propose to eliminate by the method of moment schemes.

Let us describe the procedure for deriving the coefficients for the stiffness matrix by considering arbitrary curvilinear cube. We assume that the region, filled by the element, is mapped into the cube with unit edges. In the center of isoparametric FE we put the origin of the local coordinate system \(O'x'x'x'\), directing axis along the edges of the cube (Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{curvilinear_element.png}
\caption{Curvilinear finite element.}
\end{figure}

To derive the basic FEM relations let’s use the principle of virtual work\(8\), according to which to obtain the coefficients of stiffness matrix of FE is necessary to construct an expression for the variation of the elastic strain energy. It has the following form:

\begin{equation}
\delta W = \int \int \int \sigma_{ij} \delta \varepsilon_{ij} dV
\end{equation}

We give the formula (1) the following form:

\begin{equation}
\delta W = \int \int \int (2\mu \varepsilon_{ik} \varepsilon_{jl} - \lambda \delta \varepsilon_{ij} + \lambda \delta \theta) dV
\end{equation}

For the construction of the stiffness matrix of the FE we apply triple approximation applicable in the fields of displacements \(u_j\), the components of the strain tensor \(\varepsilon_{ij}\) and the function of volume change \(\theta\).

In general, the approximation of the displacement fields with respect to the base coordinate system is expressed by the formula:

\begin{equation}
u_k = \sum_{pqrs} \omega_k^{(pqrs)} \psi_k^{(pqrs)}
\end{equation}

where \(\omega_k^{(pqrs)}\) - the coefficients of the expansion; \(\Psi^{(pqrs)}\) - the set of power coordinate functions of the form:

\begin{equation}
\psi_k^{(pqrs)} = \frac{(x^1)^p (x^2)^q (x^3)^r}{p! q! r!}
\end{equation}
\( p = 0,1,\ldots; \quad q = 0,1,\ldots; \quad m; \quad r = 0,1,\ldots; \quad n \) - the powers of approximating polynomial for corresponding coordinate directions;

\[
\sum_{pqrs}^{lmn} = \sum_{p=0}^{l} \sum_{q=0}^{m} \sum_{r=0}^{n} \sum_{s=0}^{n}
\]

(5)

where (5) is the sign of summation; a trace over the index \( k \) indicates the basic system of coordinates.

The components of the strain tensor is approximated by the expansion of the components \( \varepsilon_{ij} \) in a Maclaurin series in the neighborhood of the origin:

\[
\varepsilon_{ij} = \sum_{stg}^{(ij)} \varepsilon_{ij}^{(stg)} \psi_{stg}^{(ij)}
\]

(6)

Where,

\[
\sum_{stg}^{(ij)} = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} I_{ij} = \begin{cases} l - n p u i, j \neq 1, \\ l - 1; \end{cases}
\]

\[
M_{ij} = \begin{cases} m - n p u i, j \neq 2, \\ m - 1; \end{cases} \quad N_{ij} = \begin{cases} n - n p u i, j \neq 3, \\ n - 1, \end{cases}
\]

In the expansion (6) are only those members, that do not change with increasing order of approximation of displacements.

The expansion coefficients \( \varepsilon_{ij}^{(stg)} \) are calculated by the formulas:

\[
\varepsilon_{11}^{(pq)} = \sum_{pqrs}^{(i)} \omega_{k}^{(p+q)} b_{k}^{(i)} \varepsilon_{ij}^{(pq)} b_{k}^{(i)}
\]

\[
\varepsilon_{33}^{(pq)} = \sum_{pqrs}^{(3)} \omega_{k}^{(p+q)} b_{k}^{(3)} \varepsilon_{ij}^{(pq)} b_{k}^{(3)}
\]

\[
\varepsilon_{12}^{(pq)} = \frac{1}{2} \sum_{pqrs}^{(12)} \left( \omega_{k}^{(p+q)} b_{k}^{(12)} \varepsilon_{ij}^{(pq)} b_{k}^{(12)} + \omega_{k}^{(p+q)} b_{k}^{(i)} \varepsilon_{ij}^{(pq)} b_{k}^{(i)} \right)
\]

\[
\varepsilon_{13}^{(pq)} = \frac{1}{2} \sum_{pqrs}^{(13)} \left( \omega_{k}^{(p+q)} b_{k}^{(13)} \varepsilon_{ij}^{(pq)} b_{k}^{(13)} + \omega_{k}^{(p+q)} b_{k}^{(i)} \varepsilon_{ij}^{(pq)} b_{k}^{(i)} \right)
\]

\[
\varepsilon_{23}^{(pq)} = \frac{1}{2} \sum_{pqrs}^{(23)} \left( \omega_{k}^{(p+q)} b_{k}^{(23)} \varepsilon_{ij}^{(pq)} b_{k}^{(23)} + \omega_{k}^{(p+q)} b_{k}^{(i)} \varepsilon_{ij}^{(pq)} b_{k}^{(i)} \right)
\]

(8)

where

\[
b_{k}^{(pq)} = \frac{\partial^{(p+q)} \varepsilon_{ij}^{(k)}}{(\partial x^1)^{p}(\partial x^2)^{q}(\partial x^3)^{r}}|_{x^1 = x^2 = x^3 = 0}
\]

(9)

Relations (8) and (9) can be represented in the matrix form:

\[
\varepsilon_{ij} = [e_{ij}]^T \{\psi_{ij}\} + \{e_{ij}\} = [F_{ij}^{k}] \{\omega_{k}\}
\]

(10)

Matrices \( F_{ij}^{k} \) are constructed on the base of (3).

Let us approximate the function of volume change, defining the weak compressibility of a material:

\[
\theta = \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} \sum_{\gamma=0}^{l-1} \xi^{(\alpha \beta \gamma)} \psi_{(\alpha \beta \gamma)}
\]

(11)

where \( \xi^{(\alpha \beta \gamma)} \) - coefficient of expansion, determined by the relation:

\[
\xi^{(\alpha \beta \gamma)} = \frac{\partial^{(\alpha + \beta + \gamma)} \varepsilon_{ij}^{(g)}}{(\partial x^1)^{\alpha}(\partial x^2)^{\beta}(\partial x^3)^{\gamma}} |_{x^1 = x^2 = x^3 = 0}
\]

(12)

In the matrix form, expressions (11) and (12) have the form:

\[
\theta = [\xi]^T \{\psi_{0}\}
\]

(13)

\[
[\xi] = [F_{ij}^{k}] \{\omega_{k}\}
\]

(14)

Substituting (10) and (13) into (2), we obtain:

\[
\delta W = \int \int \sum_{ij} \left[ \delta \varepsilon_{ij}^{(g)} \right] 2\mu \varepsilon_{ij}^{(g)} \varepsilon_{ij}^{(g)} \sqrt{g} dV d^2 x d^3 x.
\]

(16)

\[
[H^{gij}] = \int \int \int \sum_{ij} \left[ \varepsilon_{ij}^{(g)} \right] 2\mu \varepsilon_{ij}^{(g)} \varepsilon_{ij}^{(g)} \sqrt{g} dV d^2 x d^3 x.
\]

(17)

Taking into account the accepted notation (16), (17) and (10), (14) the expression (15) takes the form:

\[
\delta W = \delta \left[ \omega_{ij}^{(g)} \right] \left[ F_{ij}^{(g)} \right] \left[ H^{gij} \right] \left[ F_{ij}^{(g)} \right] \left[ \omega_{ij}^{(g)} \right]
\]

(18)

For the construction of the stiffness matrix FE let's move in the expression (16) from the coefficients of the expansion \( \{\omega_{ij}^{(g)}\} \) to coefficients of the expansion for the displacements through the Lagrange interpolation polynomials of the form:

\[
u_{i} = \sum_{p=0}^{M} \sum_{q=0}^{N} \sum_{r=0}^{L} u_{i}^{(pqr)} \varphi_{(pqr)} = u_{i}^{(pqr)} N_{(pqr)},
\]

(19)
where \( N_{(pqrs)} \) - the shape function; \( u^{(pqrs)} \) - the values of displacements FE; \( \Psi^{(pqrs)} \) - approximating function, defined by one-dimensional Lagrange polynomials in the form:

\[
N_{(pqrs)} = \varphi^{(pqrs)} = R^{M}_{1p} R^{N}_{2q} R^{L}_{3r} \]

(20)

\[
R^{k}_{(i,j)} = \frac{\prod_{m=1}^{k}(x' - x^{(m)}_{i})}{\prod_{m=1}^{k}(x' - x^{(m)}_{i} + \delta^{(m)}_{i})} \]

(21)

\[ M, N, L \] - maximum power of approximating polynomials in (19) with respect to the coordinate axes \( x', x'', x''' \), correspondingly. Combining (3) and (17), written in the matrix form.

\[ u^{k} = \{\omega^{T}\{\psi\}, u = \{\omega^{T}\{N\} \]

Note, that in this case the relationship between approximating Lagrange functions (18) and the powers (4) has the form:

\[
\{\omega^{k}\} = [A]\{u^{k}\}. \quad (22)
\]

or

\[
\{N\} = [A]^T\{\psi\}, \quad (23)
\]

where \([A]\) - the transformation matrix to be defined for a particular type of approximating functions.

Substituting (20) into (16) yields:

\[
\delta W = \delta\{u^{k}\}^T[\varepsilon^{T}][\sigma^{T}][\varepsilon^{T}][\sigma^{T}][A]\{u^{k}\} + \delta\{u^{k}\}^T[A]\{F^{T}H^{T}F^{T}][A]\{u^{k}\} = \delta\{u^{k}\}^T[G^{T}]\{u^{k}\} + \delta\{u^{k}\}^T[G_{\psi}^{T}]\{u^{k}\}, \quad (24)
\]

where \([G^{T}]\) and \([G_{\psi}^{T}]\) - matrices, are defined by the expressions:

\[
[G^{T}] = [A]^T[F^{T}][H^{T}[F^{T}][A]]] \quad (25)
\]

Stiffness matrix for FE is finally calculated by the formula

\[
[K^{T}] = [A]^T[F^{T}][H^{T}[F^{T}][A]]] + [A]^T[F^{T}][H^{T}[F^{T}][A]] \quad (26)
\]

The presented version of FEM allows us to obtain the stiffness matrix of FE, taking into account the rigid displacement, the effect of a false shift and low compressibility of elastomers. In the next section of the paper, we will derive the stiffness matrix for an arbitrary curvilinear cube.

2.2 Inference of Variational Relations in Statics

The basic principle of the method of moment schemes is the expansion of the approximating function in a Taylor series, followed by discarding \( n \)-members of the series.

For the spatial hexagonal FE, this function will have the form:

\[
u_{k} = w_{k}^{100} + w_{k}^{110} + w_{k}^{010} + w_{k}^{001} + w_{k}^{111} \quad (27)
\]

where \( w_{k}^{pqrs} \) - the coefficients of the expansion, \( \Psi^{(pqrs)} \) - the set of coordinate functions, defined by the formula:

\[
\psi^{pqrs} = \frac{x^{p}y^{q}z^{r}}{p!q!r!}(p=0,1; q=0,1; r=0,1) \quad (28)
\]

On the base of the formulas (27) and (28) we obtain the expressions for the derivatives of displacement function inside FE:

\[
u_{k,1} = w_{k}^{100} + w_{k}^{110} + w_{k}^{010} + w_{k}^{001} + w_{k}^{111} \quad (29)
\]

Components of the strain tensor can be expanded into Maclaurin series in the neighborhood of the origin

\[
\varepsilon_{ij}^{k} = \sum_{n=0}^{N} \frac{\partial\psi_{ij}^{(nk)}}{\partial u^{k}}(\varepsilon_{ij}^{(nk)} \psi_{ij}^{(nk)}) \quad (30)
\]

The expansion coefficients \( \varepsilon_{ij}^{(nk)} \) are calculated by the formulas:

\[
\varepsilon_{11}^{(pqrs)} = \sum_{m=0}^{N} w_{k}^{(pqrs)}b_{k}^{(pqrs)}(\varepsilon_{11}^{(pqrs)} \psi_{11}^{(pqrs)}), \quad \varepsilon_{12}^{(pqrs)} = \sum_{m=0}^{N} w_{k}^{(pqrs)}b_{k}^{(pqrs)}(\varepsilon_{12}^{(pqrs)} \psi_{12}^{(pqrs)}), \quad (31)
\]
where $b_{(\nu\mu)}^k = \frac{\partial^{(\mu+\nu)} z^k}{(\partial x)^\nu (\partial y)^\mu} |_{x=0, y=0, z=0}$ (32)

In the expansion of the strain components, along with the strain coefficients there are expansion coefficients of the rigid body rotation. This fact causes the slow convergence of FEM. To eliminate it, we discard these members in the series. After converting for a given finite element, using (30)-(32), strain tensors will be as follows:

\[
\begin{align*}
\varepsilon_{11} &= e_{11}^{000} + e_{11}^{001} \psi + e_{11}^{010} \psi + e_{11}^{111} \psi^2; \\
\varepsilon_{22} &= e_{22}^{000} + e_{22}^{001} \psi + e_{22}^{010} \psi^2 + e_{22}^{111} \psi^2; \\
\varepsilon_{33} &= e_{33}^{000} + e_{33}^{001} \psi + e_{33}^{010} \psi^2 + e_{33}^{111} \psi^2; \\
\varepsilon_{12} &= e_{12}^{000} + e_{12}^{001} \psi; \\
\varepsilon_{13} &= e_{13}^{000} + e_{13}^{001} \psi; \\
\varepsilon_{23} &= e_{23}^{000} + e_{23}^{001} \psi^2.
\end{align*}
\]

(33)

To find the stress components by the method of moment schemes, let's use the formulas of Hooke for linear approximation of displacements. After transformation, we obtain the following expressions:

\[
\begin{align*}
\sigma_{11} &= (2\mu + \lambda) \varepsilon_{11} + \lambda \varepsilon_{22} + \lambda \varepsilon_{33}; \\
\sigma_{22} &= \lambda \varepsilon_{11} + (2\mu + \lambda) \varepsilon_{22} + \lambda \varepsilon_{33}; \\
\sigma_{33} &= \lambda \varepsilon_{11} + \lambda \varepsilon_{22} + (2\mu + \lambda) \varepsilon_{33}; \\
\sigma_{12} &= \mu \varepsilon_{12}; \\
\sigma_{13} &= \mu \varepsilon_{13}; \\
\sigma_{23} &= \mu \varepsilon_{23}.
\end{align*}
\]

(34)

To find the strain energy of the system we use the formula (1). Initially, we transform it to the form:

\[
\delta W = \frac{1}{2} \int \int \left( a + a_x + a_y + a_z + a_{xy} + a_{xz} + a_{yz} + a_{x^2} + a_{y^2} + a_{z^2} + a_{x^2y} + a_{x^2z} + a_{y^2z} + a_{x^2y^2} + a_{x^2z^2} + a_{y^2z^2} \right) dV.
\]

(35)

For a cubic FE, having unit sides and given in the natural coordinate system, it is possible to determine the coefficients $b_{(\nu\mu)}^k$. Thus,

\[
\begin{align*}
b_{100}^k &= b_{010}^k = b_{001}^k = \frac{1}{2} a b_{110}^k = b_{101}^k = b_{011}^k = b_{111}^k = 0.
\end{align*}
\]

(36)

After substitution and applicable transformations we obtain the formula for finding the potential energy of the system, expressed only through the displacements and variables $x, y, z$. 

\[
\delta W = \frac{1}{8} \mu \int \int \int (a + a_x + a_y + a_z + a_{xy} + a_{xz} + a_{yz} + a_{x^2} + a_{y^2} + a_{z^2} + a_{x^2y} + a_{x^2z} + a_{y^2z} + a_{x^2y^2} + a_{x^2z^2} + a_{y^2z^2}) dV,
\]

(37)
3. Example of Use of MSFE

Let's consider the simple problem of the bending console having a narrow rectangular cross-section, one end of which is firmly embedded in the wall and other is loaded with the force \( P \) (Figure 2). If the thickness of the console is small compared with the height \( h \), than this problem can be considered as a plane stress state. Let's solve this problem under the following conditions: \( L = 10 \), \( h = 2c = 0.5 \), \( P = 1 \). The coefficient of elasticity of the material is \( E = 10^6 \), the Poisson coefficient \( \nu = 0.25 \). We will neglect gravity for simplicity.

![Figure 2. Console with a rigid fixed end.](image)

Let's denote by \( U(x,y) \) the horizontal component of displacement, and by \( V(x,y) \) the vertical one.

Tables 1 and 2 show a comparison of the obtained numerical solution with known analytical solution of this problem. Table 1 compares the calculated values of the vertical component of displacement \( v \) with analytically calculated by the formula:

\[
\varphi_{l=0} = \frac{P x^3}{6EI} + \frac{P L x}{2EI} + \frac{P L^3}{3EI} + \frac{P c^3}{2IG} (L - x)
\]  

(38)

**Table 1.** Relative error for the vertical component of displacement  

| \( x, y \) | \( v \) | \( \varphi \) | \( \varepsilon,\% \) |
|-----------|------|---------|--------|
| 0,0       | -0.03199585 | -0.0320750 | 0.25   |
| 1,0       | -0.02721236  | -0.0272835 | 0.26   |
| 2,0       | -0.02252476  | -0.0225880 | 0.28   |
| 3,0       | -0.01802911  | -0.0180845 | 0.31   |
| 4,0       | -0.01382151  | -0.0138690 | 0.34   |
| 5,0       | -0.00999778  | -0.0100375 | 0.39   |
| 6,0       | -0.00665408  | -0.0066860 | 0.47   |
| 7,0       | -0.00388638  | -0.0039105 | 0.61   |
| 8,0       | -0.00179084  | -0.0018070 | 0.89   |
| 9,0       | -0.00046314  | -0.0004715 | 1.77   |
| 10,0      | 0              | 0         | 0.00   |

Note, that the relative error of the obtained results does not exceed 2%.

Table 2 shows the relative error obtained for the horizontal stress component \( \sigma_{xx} \), which is compared with the analytically calculated by the formula:

\[
\sigma_{xx} = -\frac{3P}{2c^3}x y
\]  

(39)

**Table 2.** Relative error for the horizontal component of the stress  

| \( x, y \) | \( \sigma_{xx} \) | \( \sigma_{yy} \) | \( \varepsilon,\% \) |
|-----------|-------------|-------------|--------|
| 0,0,25    | -2.46878589 | 0.0         | -      |
| 1,0,25    | -25.08465381| -24.0       | 4.2    |
| 2,0,25    | -46.90945003| -48.0       | 2.3    |
| 3,0,25    | -72.003141  | -72.0       | 0.0    |
| 4,0,25    | -94.9268793 | -96.0       | 1.2    |
| 5,0,25    | -121.10594595| -120.0     | 0.9    |
| 6,0,25    | -145.10994279| -144.0    | 0.8    |
| 7,0,25    | -169.10453595| -168.0    | 0.7    |
| 8,0,25    | -190.89481302| -192.0    | 0.6    |
| 9,0,25    | -217.07616722 | -216.0   | 0.5    |
| 10,0,25   | -241.81762206 | -240.0  | 0.7    |

Note, that for \( \sigma_{xx} \), the maximum relative error is about 4%, which gives the possibility to say about the high accuracy of the proposed solution.

4. Conclusion

The analysis of various problems of mechanics of deformable solids shows that the traditional FEM often has a slow convergence, especially for rigid bodies and surfaces of complex curved shapes.

This paper proposes a scheme of inference for FEM ratios, which allows us to take into account the basic properties of hard displacements for isoparametric and curvilinear finite elements.

We present the general method to find the potential energy of the system, based on the moment scheme of finite elements, which allows us to solve the problems of the mechanics of complex structures with high accuracy.

In our future work the developed model of numerical calculations will be implemented in computer-aided design system FORTU-FEM\(^{9,10}\) and will be used for the computations of structures, being in the stressed and the deformed states.
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