Weak type estimates of Marcinkiewicz integrals on the weighted Hardy and Herz-type Hardy spaces

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Abstract

The Marcinkiewicz integral is essentially a Littlewood-Paley \( g \)-function, which plays an important role in harmonic analysis. In this article, by using the atomic decomposition theory of weighted Hardy spaces and homogeneous weighted Herz-type Hardy spaces, we will obtain some weighted weak type estimates for Marcinkiewicz integrals on these spaces.

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1 Introduction

Suppose that \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n (n \geq 2) \) equipped with the normalized Lebesgue measure \( d\sigma \). Let \( \Omega \in L^1(S^{n-1}) \) be homogeneous of degree zero and satisfy the cancellation condition

\[
\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0,
\]

where \( x' = x/|x| \) for any \( x \neq 0 \). Then the Marcinkiewicz integral of higher dimension \( \mu_\Omega \) is defined by

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty \left| \frac{F_{\Omega,t}(x)}{t^{n/2}} \right|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy.
\]

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This operator $\mu_\Omega$ was first introduced by Stein in [14]. He proved that if $\Omega \in \text{Lip}_\alpha(S^{n-1})(0 < \alpha \leq 1)$, then $\mu_\Omega$ is of type $(p, p)$ for $1 < p \leq 2$ and of weak type $(1, 1)$. It is well known that the Littlewood-Paley $g$-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley $g$-function. Therefore, many authors have been interested in studying the boundedness properties of $\mu_\Omega$ on various function spaces, we refer the readers to see [1,2,3,7,9,16] for its developments and applications.

In 1990, Torchinsky and Wang [16] showed the following result.

**Theorem A.** Let $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$. If $w \in A_p$ (Muckenhoupt weight class), $1 < p < \infty$, then there exists a constant $C$ independent of $f$ such that

$$\|\mu_\Omega(f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$ 

Assume that $\Omega$ satisfies the same conditions as above, in [2] and [7], the authors proved the $H^p_w$ boundedness of Marcinkiewicz integrals provided that $\frac{n}{n+\beta} < p < 1$ and $w \in A_p(1+\beta/n)$, where $\beta = \min\{\alpha, 1/2\}$. The main purpose of this paper is to discuss the weak type estimate of $\mu_\Omega$ on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ when $p = \frac{n}{n+\alpha}$ and $w \in A_1$. In the meantime, the corresponding weak type estimate of $\mu_\Omega$ on the homogeneous weighted Herz-type Hardy spaces $H^\alpha_{q,p}(w_1, w_2)$ is also given. We now state our main results as follows.

**Theorem 1.** Let $0 < \alpha < 1$ and $\Omega \in \text{Lip}_\alpha(S^{n-1})$. If $p = \frac{n}{n+\alpha}$, $w \in A_1$, then there exists a constant $C > 0$ independent of $f$ such that

$$\|\mu_\Omega(f)\|_{W^p_{L^p_w}} \leq C\|f\|_{L^p_w}.$$ 

**Theorem 2.** Let $0 < \beta < 1$ and $\Omega \in \text{Lip}_\beta(S^{n-1})$. If $0 < p \leq 1$ and $1 < q < \infty$, $\alpha = n(1-1/q) + \beta$, $w_1, w_2 \in A_1$, then there exists a constant $C$ independent of $f$ such that

$$\|\mu_\Omega(f)\|_{W^\alpha_{q,p}(w_1, w_2)} \leq C\|f\|_{H^\alpha_{q,p}(w_1, w_2)}.$$ 

### 2 Notations and definitions

First, let’s recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [13]. Let $w$ be a nonnegative, locally integrable function defined on $\mathbb{R}^n$, all cubes are assumed
to have their sides parallel to the coordinate axes. We say that \( w \in A_p \), 
\( 1 < p < \infty \), if
\[
\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,
\]
where \( C \) is a positive constant which is independent of the choice of \( Q \).

For the case \( p = 1 \), \( w \in A_1 \), if
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \cdot \text{ess inf}_{x \in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.
\]

A weight function \( w \) is said to belong to the reverse Hölder class \( RH_r \) if there exist two constants \( r > 1 \) and \( C > 0 \) such that the following reverse Hölder inequality holds
\[
\left( \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.
\]

It is well known that if \( w \in A_p \) with \( 1 < p < \infty \), then \( w \in A_r \) for all \( r > p \), and \( w \in A_q \) for some \( 1 < q < p \). We thus write \( q_w \equiv \inf \{ q > 1 : w \in A_q \} \) to denote the critical index of \( w \). If \( w \in A_p \) with \( 1 \leq p < \infty \), then there exists \( r > 1 \) such that \( w \in RH_r \).

Given a cube \( Q \) and \( \lambda > 0 \), \( \lambda Q \) denotes the cube with the same center as \( Q \) whose side length is \( \lambda \) times that of \( Q \). \( Q = Q(x_0, r_Q) \) denotes the cube centered at \( x_0 \) with side length \( r_Q \). For a weight function \( w \) and a measurable set \( E \), we set the weighted measure \( w(E) = \int_E w(x) \, dx \).

We shall need the following lemmas.

**Lemma B ([5]).** Let \( w \in A_p, p \geq 1 \). Then, for any cube \( Q \), there exists an absolute constant \( C > 0 \) such that
\[
w(2Q) \leq C \, w(Q).
\]
In general, for any \( \lambda > 1 \), we have
\[
w(\lambda Q) \leq C \cdot \lambda^{np} w(Q),
\]
where \( C \) does not depend on \( Q \) nor on \( \lambda \).

**Lemma C ([5,6]).** Let \( w \in A_p \cap RH_r, p \geq 1 \) and \( r > 1 \). Then there exist constants \( C_1, C_2 > 0 \) such that
\[
C_1 \left( \frac{|E|}{|Q|} \right)^p \leq \frac{w(E)}{w(Q)} \leq C_2 \left( \frac{|E|}{|Q|} \right)^{(r-1)/r}
\]
for any measurable subset \( E \) of a cube \( Q \).
It should be pointed out that the definition of \( A_p(1 \leq p < \infty) \) condition could have been given with balls \( B \) replacing the cubes \( Q \) and the conclusions of Lemmas B and C also hold.

Next we shall give the definitions of the weighted Hardy spaces \( H^p_w(\mathbb{R}^n) \) and homogeneous weighted Herz-type Hardy spaces \( \dot{K}^\alpha_{q}(w_1, w_2) \). Given a Muckenhoupt’s weight function \( w \) on \( \mathbb{R}^n \), for \( 0 < p < \infty \), we denote by \( L^p_w(\mathbb{R}^n) \) the space of all functions satisfying

\[
\|f\|_{L^p_w(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
\]

We also denote by \( WL^p_w(\mathbb{R}^n) \) the weak weighted \( L^p \) space which is formed by all functions satisfying

\[
\|f\|_{WL^p_w(\mathbb{R}^n)} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty.
\]

Let \( \mathcal{S}(\mathbb{R}^n) \) be the class of Schwartz functions and let \( \mathcal{S}'(\mathbb{R}^n) \) be its dual space. Suppose that \( \varphi \) is a function in \( \mathcal{S}(\mathbb{R}^n) \) satisfying \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \).

Set

\[
\varphi_t(x) = t^{-n} \varphi(x/t), \quad t > 0, \ x \in \mathbb{R}^n.
\]

For \( f \in \mathcal{S}'(\mathbb{R}^n) \), we will define the maximal function \( M_\varphi f(x) \) by

\[
M_\varphi f(x) = \sup_{t > 0} |f \ast \varphi_t(x)|.
\]

**Definition 1.** Let \( 0 < p < \infty \) and \( w \) be a weight function on \( \mathbb{R}^n \). Then the weighted Hardy space \( H^p_w(\mathbb{R}^n) \) is defined by

\[
H^p_w(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : M_\varphi f \in L^p_w(\mathbb{R}^n) \}
\]

and we define \( \|f\|_{H^p_w} = \|M_\varphi f\|_{L^p_w} \).

Set \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \) and \( C_k = B_k \setminus B_{k-1} \) for \( k \in \mathbb{Z} \). Denote \( \chi_k = \chi_{C_k} \) for \( k \in \mathbb{Z} \), \( \overline{\chi}_k = \chi_k \) if \( k \in \mathbb{N} \) and \( \overline{\chi}_0 = \chi_{B_0} \), where \( \chi_{C_k} \) is the characteristic function of \( C_k \). Let \( \alpha \in \mathbb{R} \), \( 0 < p, q < \infty \) and \( w_1, w_2 \) be two weight functions on \( \mathbb{R}^n \). The homogeneous weighted Herz space \( \dot{K}^\alpha_{q}(w_1, w_2) \) is defined by

\[
\dot{K}^\alpha_{q}(w_1, w_2) = \{ f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^\alpha_{q}(w_1, w_2)} < \infty \},
\]

where

\[
\|f\|_{\dot{K}^\alpha_{q}(w_1, w_2)} = \left( \sum_{k \in \mathbb{Z}} (w_1(B_k))^{\alpha p/n} \|f \chi_k\|_{L^p_{w_2}}^p \right)^{1/p}.
\]
For $k \in \mathbb{Z}$ and $\lambda > 0$, we set $E_k(\lambda, f) = |\{ x \in C_k : |f(x)| > \lambda \}|$. Let $\tilde{E}_k(\lambda, f) = E_k(\lambda, f)$ for $k \in \mathbb{N}$ and $E_0(\lambda, f) = |\{ x \in B(0, 1) : |f(x)| > \lambda \}|.$ A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak weighted Herz space $W\dot{K}^{\alpha,p}_q(w_1,w_2)$ if

$$\|f\|_{W\dot{K}^{\alpha,p}_q(w_1,w_2)} = \sup_{\lambda > 0} \lambda \left( \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(E_k(\lambda, f))^{p/q} \right)^{1/p} < \infty.$$ 

For $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of $f$ is defined by

$$G(f)(x) = \sup_{\varphi \in \mathcal{A}_N} \sup_{|y-x| < t} |\varphi_t * f(y)|,$$

where $N > n + 1$, $\mathcal{A}_N = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \varphi(x)| \leq 1 \}$.

**Definition 2.** Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $w_1$, $w_2$ be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz-type Hardy space $\mathcal{H}^{\alpha,p}_q(w_1,w_2)$ associated with the space $W\dot{K}^{\alpha,p}_q(w_1,w_2)$ is defined by

$$\mathcal{H}^{\alpha,p}_q(w_1,w_2) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G(f) \in \dot{K}^{s,p}_q(w_1,w_2) \}$$

and we define $\|f\|_{\mathcal{H}^{\alpha,p}_q(w_1,w_2)} = \|G(f)\|_{\dot{K}^{\alpha,p}_q(w_1,w_2)}$.

### 3 The atomic decomposition

In this section, we will give the atomic decomposition theorems for weighted Hardy spaces and homogeneous weighted Herz-type Hardy spaces. In [4], Garcia-Cuerva characterized weighted Hardy spaces in terms of atoms in the following way.

**Definition 3.** Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index $q_w$. Set $[\cdot]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \geq [n(q_w/p - 1)]$, a real-valued function $a(x)$ is called $(p,q,s)$-atom centered at $x_0$ with respect to $w$ or $w^-(p,q,s)$-atom centered at $x_0$ if the following conditions are satisfied:

(a) $a \in L^q_w(\mathbb{R}^n)$ and is supported in a cube $Q$ centered at $x_0$,

(b) $\|a\|_{L^q_w} \leq w(Q)^{1/q - 1/p},$

(c) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

**Theorem D.** Let $0 < p < 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index $q_w$. For each $f \in \mathcal{H}^p_w(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of $w^-(p,q,[n(q_w/p - 1)])$-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \leq C\|f\|_{\mathcal{H}^p_w}$ such that $f = \sum_j \lambda_j a_j$ both in the sense of distributions and in the $\mathcal{H}^p_w$ norm.
In [10] and [11], Lu and Yang characterized homogeneous weighted Herz-type Hardy spaces in terms of atoms as follows.

**Definition 4.** Let \( 1 < q < \infty, n(1 - 1/q) \leq \alpha < \infty \) and \( s \geq \lceil \alpha + n(1/q - 1) \rceil \).

A real-valued function \( a(x) \) is called a central \((\alpha, q, s)\)-atom with respect to \((w_1, w_2)\)(or a central \((\alpha, q, s; w_1, w_2)\)-atom), if it satisfies

(a) \( \text{supp} \ a \subseteq B(0, R) = \{ x \in \mathbb{R}^n : |x| < R \} \),

(b) \( \| a \|_{L^q_{w_2}} \leq w_1(B(0, R))^{-\alpha/n} \),

(c) \( \int_{\mathbb{R}^n} a(x)x^\beta \, dx = 0 \) for every multi-index \( \beta \) with \( |\beta| \leq s \).

**Theorem E.** Let \( w_1, w_2 \in A_1, 0 < p < 1, 1 < q < \infty \) and \( n(1 - 1/q) \leq \alpha < \infty \). Then we have that \( f \in H^{\alpha,p}_q(w_1, w_2) \) if and only if

\[
 f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x), \quad \text{in the sense of} \ \mathcal{S}'(\mathbb{R}^n),
\]

where \( \sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty \), each \( a_k \) is a central \((\alpha, q, s; w_1, w_2)\)-atom. Moreover,

\[
 \| f \|_{H^{\alpha,p}_q(w_1, w_2)} \approx \inf \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p},
\]

where the infimum is taken over all the above decompositions of \( f \).

For the properties and applications of the above two spaces, we refer the readers to the books [12] and [15] for further details. Throughout this article, we will use \( C \) to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By \( A \sim B \), we mean that there exists a constant \( C > 1 \) such that \( \frac{1}{C} \leq \frac{A}{B} \leq C \). Moreover, we will denote the conjugate exponent of \( q > 1 \) by \( q' = q/(q - 1) \).

## 4 Proof of Theorem 1

In order to prove our main result, we shall need the following superposition principle on the weighted weak type estimates.

**Theorem 4.1.** Let \( w \in A_1 \) and \( 0 < p < 1 \). If a sequence of measurable functions \( \{f_j\} \) satisfy

\[
 \| f_j \|_{WL^p_w} \leq 1 \quad \text{for all} \ j \in \mathbb{Z}
\]

and

\[
 \sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq 1,
\]
then we have

\[ \left\| \sum_{j \in \mathbb{Z}} \lambda_j f_j \right\|_{W^{L_w}} \leq \left( \frac{2 - p}{1 - p} \right)^{1/p}. \]

**Proof.** The proof of this lemma is similar to the corresponding result for the unweighted case. See [8, page 123].

We are now in a position to give the proof of Theorem 1.

**Proof.** We note that when \( w \in A_1 \) and \( p = n/(n + \alpha) \), then \([n(q_w/p - 1)] = [\alpha] = 0\). By Lemma 4.1 and Theorem D, it suffices to show that for any \( w- (p, q, 0)\)-atom \( a \), there exists a constant \( C > 0 \) independent of \( a \) such that

\[ \| \mu_{\Omega} \|_{W^{L_w}} \leq C. \]

Let \( a \) be a \( w- (p, q, 0)\)-atom with \( \text{supp} \, a \subseteq Q = Q(x_0, rQ) \), \( 1 < q < \infty \) and let \( Q^* = 2\sqrt{n}Q \). For any given \( \lambda > 0 \), we can write

\[ \lambda^p \cdot w(\{ x \in \mathbb{R}^n : |\mu_{\Omega}(a)(x)| > \lambda \}) \leq \lambda^p \cdot w(\{ x \in Q^* : |\mu_{\Omega}(a)(x)| > \lambda \}) \]

\[ = I_1 + I_2. \]

Since \( w \in A_1 \), then \( w \in A_q \) for \( 1 < q < \infty \). Applying Chebyshev’s inequality, Hölder’s inequality, Lemma B and Theorem A, we thus have

\[ I_1 \leq \int_{Q^*} |\mu_{\Omega}(a)(x)|^p w(x) \, dx \]

\[ \leq \left( \int_{Q^*} |\mu_{\Omega}(a)(x)|^q w(x) \, dx \right)^{p/q} \left( \int_{Q^*} w(x) \, dx \right)^{1-p/q} \]

\[ \leq \| \mu_{\Omega}(a) \|_{L_w}^p w(Q)^{1-p/q} \]

\[ \leq C \cdot \| a \|_{L_w}^p w(Q)^{1-p/q} \]

\[ \leq C. \]

We now turn to estimate \( I_2 \). If we set \( \varphi(x) = \Omega(x)|x|^{-n+1} \chi_{\{|x| \leq 1\}}(x) \), then

\[ \mu_{\Omega}(f)(x) = \left( \int_{0}^{\infty} |\varphi_t \ast f(x)|^2 \frac{dt}{t} \right)^{1/2}. \]
By the vanishing moment condition of atom $a$, we have
\[ |\varphi_t * a(x)| = \frac{1}{t} \left| \int_Q \left( \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right) a(y) \, dy \right| \]
\[ \leq C \cdot \frac{1}{t} \int_Q \left| \frac{1}{|x-y|^{n-1}} - \frac{1}{|x-x_0|^{n-1}} \right| |a(y)| \, dy \]
\[ + \frac{1}{t} \int_Q \frac{|\Omega(x-y) - \Omega(x-x_0)|}{|x-x_0|^{n-1}} |a(y)| \, dy \]
\[ = I + II. \]

Observe that when $y \in Q$, $x \in (Q^*)^c$, then $|x-y| \sim |x-x_0|$. This together with the mean value theorem gives
\[ I \leq C \cdot \frac{r_Q}{t|x-x_0|^n} \int_Q |a(y)| \, dy. \quad (1) \]

On the other hand, since $\Omega \in Lip_\alpha(S^{n-1})$, $0 < \alpha < 1$, then we can get
\[ II \leq C \cdot \frac{1}{t|x-x_0|^{n-1}} \int_Q \left( \frac{|x-y|}{|x-y|} - \frac{x-x_0}{|x-x_0|} \right)^\alpha |a(y)| \, dy \]
\[ \leq C \cdot \frac{1}{t|x-x_0|^{n-1}} \int_Q \left( \frac{|y-x_0|}{|x-x_0|} \right)^\alpha |a(y)| \, dy \]
\[ \leq C \cdot \frac{(r_Q)^\alpha}{t|x-x_0|^{n-1+\alpha}} \int_Q |a(y)| \, dy. \quad (2) \]

By using Hölder’s inequality and the $A_q$ condition, we thus obtain
\[ \int_Q |a(y)| \, dy \leq \left( \int_Q |a(y)|^q w(y) \, dy \right)^{1/q} \left( \int_Q w(y)^{-q'/q} \, dy \right)^{1/q'} \]
\[ \leq C \cdot \|a\|_{L^q_w} \left( \frac{|Q|^q}{w(Q)} \right)^{1/q} \]
\[ \leq C \cdot \frac{|Q|}{w(Q)^{1/p}}. \quad (3) \]

We also observe that $supp \varphi \subseteq \{ x \in \mathbb{R}^n : |x| \leq 1 \}$, then for any $y \in Q$, $x \in (Q^*)^c$, we have $t \geq |x-y| \geq |x-x_0| - |y-x_0| \geq \frac{|x-x_0|}{2}$. Substituting
the above inequality (3) into (1) and (2), we can deduce

\[ |\mu_\Omega(a)(x)|^2 \leq C \left( \frac{r_Q^{n+1}}{|x-x_0|^n w(Q)^{1/p}} + \frac{r_Q^{n+\alpha}}{|x-x_0|^{n-1+\alpha} w(Q)^{1/p}} \right)^2 \left( \int_{-\infty}^{\infty} \frac{dt}{t^3} \right) \]

\[ \leq C \left( \frac{r_Q^{n+1}}{|x-x_0|^n w(Q)^{1/p}} + \frac{r_Q^{n+\alpha}}{|x-x_0|^{n+\alpha} w(Q)^{1/p}} \right)^2 \]

\[ \leq C \left( \frac{1}{w(Q)^{1/p}} \right)^2. \]

Set \( Q_0^* = Q, Q_1^* = Q^* \) and \( Q_k^* = (Q_{k-1}^*)^*, k = 2, 3, \ldots \). Following along the same lines as above, we can also show that for any \( x \in (Q_k^*)^c \), then

\[ |\mu_\Omega(a)(x)| \leq C \cdot \frac{1}{w(Q_{k-1}^*)^{1/p}} \]

\[ k = 1, 2, \ldots. \]

We shall consider the following two cases:

If \( \{ x \in (Q^*)^c : |\mu_\Omega(a)(x)| > \lambda \} = \emptyset \), then the inequality

\[ I_2 \leq C \]

holds trivially.

If \( \{ x \in (Q^*)^c : |\mu_\Omega(a)(x)| > \lambda \} \neq \emptyset \), then for \( p = n/(n + \alpha) \), it is easy to check that

\[ \lim_{k \to \infty} \frac{1}{w(Q_k^*)^{1/p}} = 0. \]

Consequently, for any fixed \( \lambda > 0 \), we are able to find a maximal positive integer \( N \) such that

\[ \lambda < C \cdot \frac{1}{w(Q_N^*)^{1/p}}. \]

Therefore

\[ I_2 \leq \lambda^p \cdot \sum_{k=1}^{N} w(\{ x \in Q_{k+1}^* \setminus Q_k^* : |\mu_\Omega(a)(x)| > \lambda \}) \]

\[ \leq C \cdot \frac{1}{w(Q_N^*)} \sum_{k=1}^{N} w(Q_k^*) \]

\[ \leq C. \]

Combining the above estimates for \( I_1, I_2 \) and taking the supremum over all \( \lambda > 0 \), we complete the proof of Theorem 1. \( \square \)
5 Proof of Theorem 2

Proof. We note that our assumption $\alpha = n(1 - 1/q) + \beta$ implies that $s = \lfloor \alpha + n(1/q - 1) \rfloor = \lfloor \beta \rfloor = 0$. For every $f \in H K^\alpha_p (w_1, w_2)$, then by Theorem E, we have the decomposition $f = \sum_{j \in \mathbb{Z}} \lambda_j a_j$, where $\sum_{j \in \mathbb{Z}} |\lambda_j|^p < \infty$ and each $a_j$ is a central $(\alpha, q, 0; w_1, w_2)$-atom. Without loss of generality, we may assume that $\text{supp } a_j \subseteq B(0, R_j)$ and $R_j = 2^j$. For any given $\sigma > 0$, we write

\begin{align*}
\sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : |\mu_\Omega(f)(x)| > \sigma\})^{p/q} \\
\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : \sum_{j=k-1}^{\infty} |\lambda_j| |\mu_\Omega(a_j)(x)| > \sigma/2\})^{p/q} \\
+ \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j| |\mu_\Omega(a_j)(x)| > \sigma/2\})^{p/q} \\
= J_1 + J_2.
\end{align*}

Since $w_2 \in A_1$, then $w_2 \in A_q$ for any $1 < q < \infty$. Note that $0 < p \leq 1$, then by using Chebyshev’s inequality and Theorem A, we can get

\begin{align*}
J_1 &\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left(\sum_{j=k-1}^{\infty} |\lambda_j| |\mu_\Omega(a_j)\chi_k|_{L^q_{w_2}}\right)^p \\
&\leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p |\mu_\Omega(a_j)|_{L^p_{w_2}}^p\right) \\
&\leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p |a_j|_{L^p_{w_2}}^p\right).
\end{align*}

Changing the order of summation yields

\begin{align*}
J_1 &\leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n}\right).
\end{align*}

When $k \leq j + 1$, then $B_k \subseteq B_{j+1}$. Since $w_1 \in A_1$, then we know $w \in RH_r$ for some $r > 1$. It follows directly from Lemma C that

\begin{align*}
w_1(B_k) &\leq C \cdot w_1(B_{j+1}) |B_k|^{\delta} |B_{j+1}|^{-\delta},
\end{align*}

(4)
where $\delta = (r - 1)/r > 0$. By Lemma B and the above inequality (4), we get

$$
\sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n} \leq C \sum_{k=-\infty}^{j+1} \left( \frac{w_1(B_{j+1})}{w_1(B_j)} \right)^{\alpha p/n} \left( \frac{|B_k|}{|B_{j+1}|} \right)^{\alpha \delta p/n} \leq C \sum_{k=-\infty}^{j+1} \left( \frac{w_1(B_{j+1})}{w_1(B_j)} \right)^{\alpha p/n} \left( \frac{|B_k|}{|B_{j+1}|} \right)^{\alpha \delta p/n} \leq C \sum_{k=-\infty}^{j+1} \left( \frac{w_1(B_{j+1})}{w_1(B_j)} \right)^{\alpha p/n} \left( \frac{|B_k|}{|B_{j+1}|} \right)^{\alpha \delta p/n} \leq \sum_{k=0}^{\infty} 2^{(k-j-1)\alpha \delta p}
$$

where the last series is convergent since $\alpha \delta p > 0$. Furthermore, it is bounded by a constant which is independent of $j \in \mathbb{Z}$. Hence

$$J_1 \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq C \|f\|_{HK_{w_1}(w_2)}^p.
$$

We turn to deal with $J_2$. As in the proof of Theorem 1, we can also write

$$
|\varphi_t \ast a_j(x)| = \frac{1}{t} \left| \int_{B_j} \left( \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \right) a_j(y) \, dy \right| 
\leq C \cdot \frac{1}{t} \int_{B_j} \left| \frac{1}{|x-y|^{n-1}} - \frac{1}{|x|^{n-1}} \right| |a_j(y)| \, dy 
+ \frac{1}{t} \int_{B_j} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \right| |a_j(y)| \, dy 
= \text{III+IV}.
$$

Observe that when $j \leq k-2$, then for any $y \in B_j$ and $x \in C_k = B_k \backslash B_{k-1}$, we have $|x| \geq 2|y|$, which implies $|x-y| \sim |x|$. We also observe that $\text{supp} \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$, then we can get $t \geq |x-y| \geq \frac{|x|}{2}$. Hence, by using the same arguments as that of Theorem 1, we obtain

$$\text{III} \leq C \cdot \frac{R_j}{t|x|^n} \int_{B_j} |a_j(y)| \, dy \quad (5)
$$

and

$$\text{IV} \leq C \cdot \frac{(R_j)^{\beta}}{t|x|^{n-1+\beta}} \int_{B_j} |a_j(y)| \, dy. \quad (6)
$$
Similarly, it follows from Hölder’s inequality and the $A_q$ condition that

$$\int_{B_j} |a_j(y)| \, dy \leq \left( \int_{B_j} |a_j(y)|^q w_2(y) \, dy \right)^{1/q} \left( \int_{B_j} w_2(y)^{q'/q} \, dy \right)^{1/q'} \leq C \cdot |B_j| w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q}. \tag{7}$$

Substituting the above inequality (7) into (5) and (6), we can deduce

$$|\mu_\Omega(a_j)(x)|^2 \leq C \left( \frac{2^{j(n+1)}}{|x|^n w_1(B_j)^{\alpha/n} w_2(B_j)^{1/q}} + \frac{2^{j(n+\beta)}}{|x|^{n-1+\beta} w_1(B_j)^{\alpha/n} w_2(B_j)^{1/q}} \right)^2 \left( \int_{|x|}^\infty \frac{dt}{t^\beta} \right). \tag{8}$$

Since $B_j \subseteq B_{k-2}$, then by using Lemma C, we get

$$w_i(B_j) \geq C \cdot w_i(B_{k-2}) |B_j||B_{k-2}|^{-1} \text{ for } i = 1 \text{ or } 2.$$

From our assumption $\alpha = n(1 - 1/q) + \beta$ and (8), it follows that

$$|\mu_\Omega(a_j)(x)| \leq C \cdot \left( \frac{2^j}{2^{k-2}} \right)^{n+\beta-\alpha-n/q} \frac{1}{w_1(B_{k-2})^{\alpha/n} w_2(B_{k-2})^{1/q}}.$$ \tag{9}

We now set $A_k = w_1(B_{k-2})^{-\alpha/n} w_2(B_{k-2})^{-1/q}$. Once again, let us consider the following two cases:

If $\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||\mu_\Omega(a_j)(x)| > \sigma/2\} = \emptyset$, then the inequality

$$J_2 \leq C \|f\|_{H^{\alpha,p}_q(w_1, w_2)}^p$$

holds trivially.

If $\{x \in C_k : \sum_{j=-\infty}^{k-2} |\lambda_j||\mu_\Omega(a_j)(x)| > \sigma/2\} \neq \emptyset$, then by the inequality (9), we have

$$\sigma < C \cdot A_k \left( \sum_{j \in \mathbb{Z}} |\lambda_j| \right) \leq C \cdot A_k \left( \sum_{j \in \mathbb{Z}} |\lambda_j|^p \right)^{1/p} \leq C \cdot A_k \|f\|_{H^{\alpha,p}_q(w_1, w_2)}^p.$$
In addition, it is easy to verify that \( \lim_{k \to \infty} A_k = 0 \). Then for any given \( \sigma > 0 \), we are able to find a maximal positive integer \( k_\sigma \) such that

\[
\sigma < C \cdot A_{k_\sigma} \|f\|_{H^\alpha_{q,p}(w_1,w_2)}.
\]

From the above discussion, we have that \( B_{k-2} \subseteq B_{k_\sigma-2} \). As (4), by using Lemma C again, we obtain

\[
\frac{w_i(B_{k-2})}{w_i(B_{k_\sigma-2})} \leq C\left(\frac{|B_{k-2}|}{|B_{k_\sigma-2}|}\right)^{\delta} \text{ for } i = 1 \text{ or } 2.
\]

Furthermore, it follows immediately from Lemma B that

\[
\frac{w_i(B_k)}{w_i(B_{k_\sigma-2})} \leq C\left(\frac{|B_{k-2}|}{|B_{k_\sigma-2}|}\right)^{\delta} \text{ for } i = 1 \text{ or } 2.
\]

Therefore

\[
J_2 \leq \sigma^p \cdot \sum_{k=-\infty}^{k_\sigma} w_1(B_k)^{\alpha p/n} w_2(B_k)^{p/q}
\]

\[
\leq C\|f\|_{H^\alpha_{q,p}(w_1,w_2)}^p \sum_{k=-\infty}^{k_\sigma} \left(\frac{w_1(B_k)}{w_1(B_{k_\sigma-2})}\right)^{\alpha p/n} \left(\frac{w_2(B_k)}{w_2(B_{k_\sigma-2})}\right)^{p/q}
\]

\[
\leq C\|f\|_{H^\alpha_{q,p}(w_1,w_2)}^p \sum_{k=-\infty}^{k_\sigma} \frac{1}{2^{(k_\sigma-k)n\delta}}
\]

\[
\leq C\|f\|_{H^\alpha_{q,p}(w_1,w_2)}^p.
\]

Finally, by combining the above estimates for \( J_1, J_2 \) and taking the supremum over all \( \sigma > 0 \), we conclude the proof of Theorem 2.

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