Block extensions, local categories, and basic Morita equivalences

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Abstract

Let \( (k, \mathcal{O}, k) \) be a \( p \)-modular system with \( k \) algebraically closed, let \( b \) be a block of the normal subgroup \( H \) of \( G \) having defect pointed group \( Q_\delta \) in \( H \) and \( P_\gamma \) in \( G \), and consider the block extension \( b \sigma G \). One may attach to \( b \) an extended local category \( \mathcal{E}(b, H, G) \), a group extension \( L \) of \( Z(Q) \) by \( N_G(Q_\delta)/C_H(Q) \) having \( P \) as a Sylow \( p \)-subgroup, and a cohomology class \( [\alpha] \in H^2(N_G(Q_\delta)/QC_H(Q), k^\times) \). We prove that these objects are invariant under the \( G/H \)-graded basic Morita equivalences introduced in [4]. Along the way, we give alternative proofs of the results of [11] and [23] on extensions of nilpotent blocks, and of [27] on \( p' \)-extensions of inertial blocks.

Keywords: Group algebras, block extensions, defect groups, group graded algebras, local categories, nilpotent block, inertial block, basic Morita equivalence.

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1. Introduction

This paper is an effort to unify and simplify results involving nilpotent or inertial blocks of normal subgroups and the basic Morita equivalences introduced by L. Puig [18]. Recall that a block algebra is called inertial in [21] if it is basic Morita equivalent to its Brauer correspondent. Nilpotent blocks are the particular case when the inertial quotient is trivial. The structure of the source algebra of an extension of a nilpotent block has been determined by B. Külshammer and L. Puig [11, Theorem 1.12], together with a strong statement of uniqueness of a group controlling the fusion [11, Theorem 1.8]. Their results significantly extend the result of [13], and both theorems have been given simplified proofs in [23, Theorems 3.14 and 3.5]. In a similar fashion, Y. Zhou [27] has determined the source algebra of a \( p' \)-extension of an inertial block.

We treat here these results in the framework of group graded basic Morita equivalences introduced in [4], and further investigated in [5]. To summarize our results, let us introduce some notation. Let \((K, O, k)\) be a \( p \)-modular system, with \( k = O/J(O) \) algebraically closed (although this assumption is not needed everywhere). Let \( H \) be a normal subgroup of a finite group \( G \), let \( \bar{G} = G/H \), and let \( b \) be a block of \( O_H \), which may be assumed to be \( G \)-invariant. Let \( Q_\delta \) be a defect pointed subgroup of \( H \setminus \{ b \} \), and \( P_\gamma \) a defect pointed subgroup of \( G \setminus \{ b \} \), such that \( Q_\delta \leq P_\gamma \) and \( Q_\delta = P_\gamma \cap H \).

Let \( b_\delta \) be the block of \( O_C H(Q) \) with defect group \( Z(Q) \) determined by \( Q_\delta \), so \( N_G(Q_\delta) \) is the stabilizer of \( b_\delta \) in \( N_G(Q) \). We denote \( E = E_G(Q_\delta) = N_G(Q_\delta)/C_H(Q) \) and \( \bar{E} = N_G(Q_\delta)/QC_H(Q) \). Recall that \( b_\delta \) is also a block of \( O_C H(Q) \) with defect group \( Q \), and it has a unique simple module \( \bar{V} \). Since \( QC_H(Q) \leq N_G(Q_\delta) \), the Clifford extension of \( \bar{V} \) with respect to this situation gives a 2-cocycle \( \alpha \in Z^2(\bar{E}, k) \).

As in [11, Theorem 1.8] and [23, Theorem 3.5], the block \( b_\delta \) determines a group extension \( L \) of \( Z(Q) \) by \( E \), having \( P \) as a Sylow \( p \)-subgroup, such that the conjugation actions of \( N_G(Q_\delta) \) on \( Q \) and of \( L \) on \( Q \) are strongly related. We may therefore consider (with some abuse of notation) the twisted group algebra \( O_\alpha L \). Note that we do not assume here that the block \( b \) is nilpotent.

**Theorem 1.1.** Consider the block extension \( A = bO_G \), and assume that \( B = bO_H \) is an inertial block. Denote \( A_\delta = jA j \), where \( j \in \delta \). Then the following statement hold.

a) \( A_\delta \) is an \( \bar{E} \)-graded algebra with identity component \( C_\delta \) Morita equivalent to \( O \bar{Q} \), and the Clifford extension of the unique simple \( C_\delta \)-module is isomorphic to the Clifford extension of \( \bar{V} \).

b) The bimodule inducing the Morita equivalence between \( B \) and \( O_\alpha(Q \rtimes E_H(Q_\delta)) \) is \( \bar{G} \)-invariant.

c) ([11, Theorem 1.12] and [23, Theorem 3.14]) If \( b \) is nilpotent (hence \( \bar{E} \simeq \bar{G} \)), then there is a \( \bar{G} \)-graded basic Morita equivalence between \( A \) and \( O_\alpha L \).
d) ([27, Theorem]) If \( p \) does not divide the order of \( \tilde{G} \) (hence \( Q_\delta = P_\gamma \)), then there is a \( \tilde{G} \)-graded basic Morita equivalence between \( A \) and \( \mathcal{O}_\alpha (Q \rtimes E_G(Q_\delta)) \).

We obtain the Morita equivalences in c) and d) by extending a bimodule inducing the given Morita equivalence to a certain \( \tilde{G} \)-graded diagonal subalgebra, as in [14, Theorem 5.1.2]. Note that in d), the original bimodule extends directly, while in c), the original bimodule does not extend in general, it must be replaced by another one.

We do not know whether there is a common generalization of c) and d). Zhou [26] considered a particular case of a \( p \)-extension of an inertial block, but the Morita equivalence obtained there is not basic.

The extended local category \( \mathcal{E}_{(b,H,G)} \) of \( (G, \tilde{G}) \)-fusions was introduced in [23, 3.4]. The objects are pointed subgroups of \( P_\gamma \), and the morphisms are conjugations by elements \( x \in G \), also taking into account the class \( \bar{x} \in \tilde{G} \).

Now let \( b' \mathcal{O}_{\tilde{G}}' \) be another block extension, where \( H' \subseteq G', b' \in \mathcal{O}(\mathcal{O}H') \) is a \( G' \)-invariant block, and \( G'/H' \cong G/H = \tilde{G} \). We use “′” to denote the objects associated with \( b' \).

**Theorem 1.2.** Assume that there is a \( \tilde{G} \)-graded basic Morita equivalence between \( A = b\mathcal{O}_{\tilde{G}} \) and \( A' = b'\mathcal{O}_{\tilde{G}}' \). Then, by identifying \( P \) and \( P' \), we have:

a) The categories \( \mathcal{E}_{(b,H,G)} \) and \( \mathcal{E}_{(b',H',G')} \) are equivalent; in particular, \( E \simeq E' \).

b) The group extensions \( L \) and \( L' \) of \( \mathcal{Z}(Q) \) by \( E \) are isomorphic.

c) \([\alpha] = [\alpha'] \) in \( H^2(\tilde{E}, k^*) \).

In particular, [11, Theorem 1.8] and [23, Theorem 3.5] follow from Theorem 1.2 and Theorem 1.1d). Note that in these papers, the fact that \( L \) controls the fusion was first established, and then used to get the Morita equivalence.

The paper is organized as follows. In Section 2 we recall the main concepts and facts needed for the proofs of the main results. In Section 3 we give the construction of of the group extension \( L \) of \( \mathcal{Z}(Q) \) by \( E \), and of the twisted group algebra \( \mathcal{O}_\alpha L \), by using the properties of the block extension \( \mathcal{O}N_G(Q_\delta)b_\delta \). The next section contains a variant of the Fong-Reynolds reduction, which reduces us to the case when \( b_\delta \) is \( N_G(Q) \)-invariant. In Section 5 we show that under a certain condition, there is an injective algebra map from the source algebra of the block extension \( \mathcal{O}N_G(Q_\delta)b_\delta \) to the source algebra of \( \mathcal{O}Gb \). In Section 6 we prove a useful lemma which allows to construct group graded Morita equivalences, based on some results of E.C. Dade on extendibility of modules. This is applied in the next section, where we prove the statements of Theorem 1.1. We discuss \((A, \tilde{G})\)-fusions and \((G, \tilde{G})\)-fusions in Section 8 and the last section contains the proof of Theorem 1.2.
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2. Preliminaries and quoted results

In this paper, all algebras and modules, considered as left modules, are finitely generated. We introduce in this section our notation and recall some basic concepts and facts. Our assumptions are standard, and we refer to [25] and [14] for general results on block theory and group graded algebras, respectively.

2.1. Throughout this paper \( p \) is a prime, and let \( k \) be the residue field, with characteristic \( p \), of a discrete valuation ring \( O \). For the moment, we do not assume that \( k \) is algebraically closed, some of the results in the next sections will require this assumption.

Let \( H \) be a normal subgroup of a finite group \( G \). Then the group algebra \( O_H \) is a \( p \)-permutation \( G \)-algebra. Let \( b \) be a \( G \)-invariant block of \( O_H \), hence, in particular, \( b \) remains a primitive idempotent in \( (O_H)^G \). Set the notations

\[
\bar{G} := G/H, \quad A := OGb, \quad B := OHb,
\]

and we usually regard \( A \) as a \( \bar{G} \)-graded \( O \)-algebra with identity component \( B \).

2.2. Take a defect pointed group \( P_\gamma \) of \( G_{\langle b \rangle} \). Then, by [11, Proposition 5.3], there is a defect pointed group \( Q_\delta \) of \( H_{\langle b \rangle} \) on the \( H \)-interior algebra \( O_H \) such that \( Q_\delta \leq P_\gamma \); in this case we may assume that \( Q_\delta = P_\cap H \).

The Frattini argument implies that \( G = HN_G(Q_\delta) \), hence

\[
\bar{G} \simeq N_G(Q_\delta)/N_H(Q_\delta).
\]

Let \( i \in \gamma \) and \( j \in \delta \) be source idempotents such that \( j = ij = ji \). Set

\[
A_\gamma = iAi, \quad A_\delta = jAj, \quad B_\delta = jBj, \quad B_\gamma = iBi.
\]

By [15, Proposition 3.2], both \( A_\delta \) and \( A_\gamma \) are strongly \( \bar{G} \)-graded \( O \)-algebras, and we have \( \bar{G} \)-graded Morita equivalences between \( A, A_\delta \) and \( A_\gamma \).

2.3. Recall (see [25, § 40]) that there is a maximal \( (H,b) \)-Brauer pair, denoted \( (Q,b_\delta) \), associated with \( Q_\delta \). Here \( b_\delta \) is a block of \( kC_H(Q) \) with defect group \( Z(Q) \). It is well known that \( b_\delta \) lifts uniquely to a block (still denoted by \( b_\delta \)) of \( \bar{O}C_H(Q) \) with defect group \( Z(Q) \), and that we have the equality \( N_H(Q_\delta) = N_H(Q,b_\delta) \). Moreover, \( b_\delta \) is also a block of \( \bar{O}QC_H(Q) \) with defect group \( Q \). As in [5, Section 4], we also denote

\[
E := E^\bar{G}_G(Q_\delta) = N_G(Q_\delta)/C_H(Q), \quad \bar{E} := \bar{E}^G_G(Q_\delta) = N_G(Q_\delta)/QC_H(Q),
\]
\[ E_H(Q_\delta) = N_H(Q_\delta)/C_H(Q), \quad \tilde{E}_H(Q_\delta) = N_H(Q_\delta)/QCH(Q). \]

These groups will be regarded in Section 8 as automorphism groups in a certain extended local category.

2.4. As we deal with Morita equivalences, we consider another finite group \( G' \), and assume that \( \omega : G \to \hat{G}, \omega' : G' \to \hat{G} \) are group epimorphisms with \( H = \text{Ker} \omega \) and \( H' = \text{Ker} \omega' \).

Let \( b' \) be a \( G' \)-invariant block of \( \mathcal{O}_Hb \) and let \( A' := \mathcal{O}_G'b' \) be the associated \( \hat{G} \)-graded, \( G' \)-interior algebra with 1-component \( B' := A'_1 = \mathcal{O}_Hb' \). We will use notations similar to the above (and self-explanatory) for the objects associated with \( b' \).

Set \( \hat{\mathcal{G}} := (\omega \times \omega')^{-1}(\Delta(\hat{G})) \) and \( \hat{\Delta} := (b \otimes b')\mathcal{O}_\hat{G} \).

2.5. Basic Morita equivalences have been introduced by Puig \cite{18}. Recall that if the indecomposable \( \mathcal{O}(H \times H') \)-module \( M \) such that \( bMb' = M \) induces a Morita equivalence between \( \mathcal{O}Hb \) and \( \mathcal{O}H'b' \), then by \cite{18, Theorem 6.9} there is a \( \hat{\mathcal{Q}} \)-interior algebra embedding \( B_\delta \to T \otimes B'_\gamma \), where \( T = \text{End}_{\mathcal{O}}(N) \) is a \( \hat{\mathcal{Q}} \)-interior algebra, with \( \hat{\mathcal{Q}} \leq H \times H' \) a vertex of \( M \), and \( N \) a source \( \mathcal{O}\hat{\mathcal{Q}} \)-module of \( M \). The equivalence is called basic if \( T \) is a Dade \( \hat{\mathcal{Q}} \)-algebra (see \cite{18, Chapter 7}). In this case, the projections from \( \hat{G} \) to \( G \) and \( G' \) give an identification of the vertex \( \hat{\mathcal{Q}} \) with the defect groups \( Q \) and \( Q' \).

The block \( b \) is called inertial (see \cite{21, 2.16}) if it is basic Morita equivalent to the block \( \mathcal{O}_N(H(Q_\delta))b_\delta \) (hence to its Brauer correspondent as well).

2.6. Basic Morita equivalences have been generalized to block extensions in \cite{4}. According to \cite{4, Definition 4.2}, \( A \) is \( \hat{\mathcal{G}} \)-graded basic Morita equivalent to \( A' \) if and only if there is an indecomposable \( \mathcal{O}[H \times H'] \)-module \( M \) associated with \( b \otimes b' \) that extends to \( \hat{\Delta} \) such that there is an embedding \( f : A_\gamma \to S \otimes A'_\gamma \) of \( \hat{\mathcal{G}} \)-graded \( P \)-interior algebras, where \( S := \text{End}_{\mathcal{O}}(\bar{N}) \) is a Dade \( \hat{\mathcal{P}} \)-algebra, with \( \bar{P} \) a vertex of \( M \), and \( \bar{N} \) a source \( \mathcal{O}\bar{P} \)-module of \( M \).

In this case, we can again identify
\[ P \simeq P' \simeq \bar{P} \quad (1) \]

In particular, it follows that the embedding \( f \) restricts to a \( Q \)-interior algebra embedding
\[ f : B_\gamma \to S \otimes B'_\gamma \quad (2) \]

Let \( R \leq P \) and \( R' \leq P' \) such that \( R' \) corresponds to \( R \) via (1). By applying the Brauer construction to (2) we obtain the \( N_P(R) \)-algebra embedding
\[ \bar{f} : B_\gamma(R) \to S(R) \otimes B'_\gamma(R') \quad (3) \]
If \( R_e \) is a local pointed subgroup of \( P_\gamma \) then, by (3), there is a unique local pointed group \( R'_e \) such that \( R'_e \leq P'_\gamma \). This construction yields a bijection

\[
R_e \leftrightarrow R'_e.
\]

between the local pointed groups included in \( P_\gamma \) and local pointed groups included in \( P'_\gamma \).

2.7. The following remarks hold, more generally, when \( A \) is a crossed product of the \( \mathcal{O} \)-algebra \( B := A_1 \) and any finite group \( \tilde{G} \). This means that the group \( hU(A) = \bigcup_{g \in \tilde{G}} (A^\times \cap A_g) \) of homogeneous units of \( A \) is a group extension of \( B^\times \) by \( \tilde{G} \).

Let \( V \) be an \( A_1 \)-module. For \( g \in \tilde{G} \), the \( A_1 \)-module \( A_g \otimes_B V \) is called the \( g \)-conjugate of \( V \), and \( V \) is called \( \tilde{G} \)-invariant, if \( V \cong A_g \otimes_B V \) as \( B \)-modules for all \( g \in \tilde{G} \). Consider the \( \tilde{G} \)-graded \( A \)-module \( U = A \otimes_B V \). Then the endomorphism algebra \( A' := \text{End}_A(U)^{\text{op}} \) is a \( \tilde{G} \)-graded algebra with \( g \)-component given by

\[
A'_g \cong \text{Hom}_{A_1}(V, A_g \otimes_B V)
\]

for any \( g \in \tilde{G} \). In this way, \( U \) becomes a \( \tilde{G} \)-graded \( (A, A') \)-bimodule.

Note that \( A' \) is a crossed product of \( B' \cong \text{End}_B(V)^{\text{op}} \) and \( \tilde{G} \) if and only if \( V \) is a \( \tilde{G} \)-invariant \( B \)-module. Then the graded Jacobson radical \( J_{\text{gr}}(A') \) equals \( J(B')A' = A'J(B') \), and \( \tilde{A}' := A'/J_{\text{gr}}(A') \) is still a crossed product of \( \tilde{B}' := B'/J(B') \) and \( \tilde{G} \). In this case, the group extension \( hU(A') \) is called the Clifford extension of \( V \), while \( hU(\tilde{A}') \) is the residual Clifford extension of \( V \).

If, in addition, \( k \) is algebraically closed, then \( \tilde{A}' \) is a twisted group algebra of the form \( k_\alpha \tilde{G} \) for some 2-cocycle \( \alpha \in Z^2(\tilde{G}, k^\times) \).

3. The block \( b_\delta \) and the extension \( L \) of \( Q \)

3.1. We keep the setting of Section 1. We consider the \( N_G(Q_\delta) \)-invariant nilpotent block \( b_\delta \) of \( \mathcal{O}QC_H(Q) \). In this case \( (Q, b_\delta) \) remains a maximal \( (b_\delta, QC_H(Q)) \)-Brauer pair. We obtain as a particular case of the next proposition that \( P \) is also a defect of \( b_\delta \) in \( N_G(Q_\delta) \). For this, note that associated with \( P_\gamma \), there is a so-called generalized maximal \( (G, H, b) \)-Brauer pair denoted \( (P, b_\gamma) \) such that \( (Q, b_\delta) \leq (P, b_\gamma) \). In particular, if \( G/H \) is a \( p' \)-group, then \( Q = P \) and \( b_\gamma = b_\delta \). We refer to [9] for more details regarding generalized Brauer pairs.

The next proposition is a generalization of [24, Proposition 3.1], but we will use it here only in the case \( R = Q \) and \( b_e = b_\delta \).

**Proposition 3.2.** Let \( R_e \leq P_\gamma \), where \( R \) is a normal subgroup of \( P \) such that \( R \leq H \) and \( (R, b_e) \leq (P, b_\gamma) \). Then the pair \( (P, b_\gamma) \) is a maximal \( (N_G(R_e), N_H(R_e), b_e) \)-Brauer pair.
Proof. Let $X = N_G(P) \cap N_G(Q_e)$. The following restriction of the Brauer homomorphism

$$\text{Br}_P : (kC_H(R))^N_{G(Q_e)} \rightarrow (kC_H(P))^X$$

is an epimorphism of $N_G(P)$-algebras. Indeed, we have

$$\text{Br}_P \left( \text{Tr}_P^{N_G(Q_e)}(a) \right) = \text{Br}_P \left( \sum_{x \in [X \setminus N_G(Q_e)/P]} \text{Tr}_X^P(a^x) \right) = \sum_{x \in [X/P]} \text{Br}_P(a^x) = \text{Tr}_P^X(\text{Br}_P(a)),$$

where $a \in (kC_H(R))^P$. Note that $x \in X$ if and only if $X \cap P^x = P$, and then, if $x \notin X$, we get

$$\text{Tr}_X^P(a^x) = \sum_{y \in [P \setminus (X \cap P^x)]} \text{Tr}_P^Y(a^y),$$

where $P \cap (X \cap P^x)^Y$ is always a proper subgroup of $P$. By using the inclusion

$$(kC_H(P))^X \subseteq (kC_H(P))^N_{G(Q_e)} = (kC_H(P))^N_{G(Q_e)}$$

and the fact that $b_\gamma$ is a primitive idempotent of the algebra $(kC_H(P))^N_{G(Q_e)}$, we can find a primitive idempotent $b_X \in (kC_H(P))^X$ that verifies

$$b_X b_\gamma = b_\gamma b_X = b_\gamma. $$

By lifting it, we obtain the equality $\text{Br}_P(\bar{b}_e) = b_X$, for a primitive idempotent $\bar{b}_e$ with defect group $P$ in the algebra $(kC_H(R))^N_{G(Q_e)}$.

Further, we have

$$b_\gamma b_\gamma \text{Br}_P(b_e) = b_X b_\gamma \text{Br}_P(b_e) = b_\gamma b_\gamma \text{Br}_P(\bar{b}_e) \text{Br}_P(b_e),$$

hence $\bar{b}_e b_e \neq 0$. This forces

$$\bar{b}_e b_e = b_e = b_e,$$

since both idempotents are primitive in $(kC_H(R))^N_{G(Q_e)}$. 

\[ \square \]

3.3. By [11, Proposition 5.3] we have that $P_{C_H(Q)} / C_H(Q) \simeq P / Z(Q)$ is a Sylow $p$-subgroup of $E = N_G(Q_\delta) / C_H(Q)$. For now, we only need to show the existence of an extension $L$ of $Q$ by $\tilde{E} = N_G(Q_\delta) / Q_{CH}(Q)$ containing $P$ as a Sylow $p$-subgroup. It turns out that when $b$ is nilpotent, $L$ controls the fusions in the block extension $b \mathcal{O} G$ ([11, Theorem 1.8]), but this will be obtained as a consequence of a more general result in Section 8. The group $L$ exists without any assumption on $b$. For convenience, we include the proof, which follows the first part of the proof of [23, Theorem 3.5].
Proposition 3.4. With the above notations there is a group extension

\[
1 \rightarrow Q \xrightarrow{\tau} L \xrightarrow{\bar{\pi}} \tilde{E} \rightarrow 1
\]

and an injective group homomorphism \( \tau : P \rightarrow L \) such that \( \tau(P) \) a Sylow \( p \)-subgroup in \( L \), \( \ker \bar{\pi} = \tau(Q) \) and \( (\pi \circ \tau)(u) = \bar{\pi}(u) \) for any \( u \in P \).

Proof. Consider the element \( \bar{h} \in H^2(N_G(Q_\delta)/C_H(Q), Z(Q)) \) corresponding to the extension

\[
1 \rightarrow Z(Q) \rightarrow P \xrightarrow{\tau} P/Z(Q) \rightarrow 1.
\]

We show that \( \bar{h} \) is in the image of the restriction map \( H^2(N_G(Q_\delta)/C_H(Q), Z(Q)) \rightarrow H^2(PC_H(Q)/C_H(Q), Z(Q)) \). By [6, Chapter XII, Theorem 10.1], it is enough to prove that for any subgroup \( R \) of \( P \) containing \( Q \), and any \( \bar{x} \in N_G(Q_\delta)/C_H(Q) \) such that \( RC_H(Q) \leq P^rC_H(Q) \), the restriction to \( RC_H(Q)/C_H(Q) \) of the class from \( H^2(P^rC_H(Q)/C_H(Q), Z(Q)) \) determined by \( P^r \) coincides with the class determined by \( R \).

Indeed, by Proposition 3.2 applied for \( Q \), we get that \( b_\delta \) is a block idempotent of \( \partial PC_H(Q) \), of \( \partial P^rC_H(Q) \) and of \( \partial RC_H(Q) \) having defect groups \( P \), \( P^r \) and \( R \), respectively. Since \( RC_H(Q) \leq P^rC_H(Q) \), \( R \) must be contained in a conjugate of \( P^r \), so there is \( z \in C_H(Q) \) such that \( R \leq P^rz \). This implies that the extension

\[
1 \rightarrow Z(Q) \rightarrow R \rightarrow R/Z(Q) \rightarrow 1
\]

is a subextension of

\[
1 \rightarrow Z(Q) \rightarrow P^r \rightarrow P^r/Z(Q) \rightarrow 1,
\]

and the claim follows.

Therefore, we obtain a group extension

\[
1 \rightarrow Z(Q) \xrightarrow{\tau} L \xrightarrow{\bar{\pi}} \tilde{E} \rightarrow 1
\]

corresponding to the element in \( H^2(N_G(Q_\delta)/C_H(Q), Z(Q)) \) whose image by restriction in \( H^2(PC_H(Q)/C_H(Q), Z(Q)) \) is \( \bar{h} \), and also the injective group extension map \( \tau : P \rightarrow L \).

Remark 3.5. In the situation of the above proposition, it is clear by the construction of the group \( L \) that for any \( x \in N_G(Q_\delta) \) there is \( y \in L \) such that

\[
y\tau(u)y^{-1} = \tau(xux^{-1})
\]

for all \( u \in Q \).
3.6. We consider here the case when $\tilde{G}$ is a $p'$-group, which is the assumption in Subsection 7.2 below. It is known that when $k$ is algebraically closed, $\tilde{E}_H(Q_\delta) = N_H(Q_\delta)/QCH(Q)$ is a $p'$-group, and therefore $\tilde{E}$ is still a $p'$-group, since $\tilde{E}/\tilde{E}_H(Q_\delta) \simeq G/H$. We have the commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & Q/Z(Q) & \rightarrow & N_G(Q_\delta)/CH(Q) & \rightarrow \tilde{E} & \rightarrow 1 \\
\downarrow & & \downarrow & & \downarrow & \theta & \\
1 & \rightarrow & \text{Int}Q & \rightarrow & \text{Aut}Q & \rightarrow \text{Out}Q & \rightarrow 1.
\end{array}
\]

Since $\tilde{E}$ is a $p'$-group we obtain an action of $\tilde{E}$ on $\mathcal{O}Q$. In this case, we have that $L \simeq Q \ltimes _{\theta} \tilde{E}$. Note also that we have the group isomorphism $\tilde{E} \simeq L/\mathcal{O}$.

3.7. Assume that $k$ is algebraically closed. Let $\tilde{V}$ be the unique simple $\mathcal{O}QC_H(Q)b_\delta$-module. The Clifford extension of $\tilde{V}$ (see 2.7) gives a 2-cocycle $\alpha \in Z^2(\tilde{E}, k^\times)$. Since the group extension \[1 \rightarrow 1 + J(O) \rightarrow \tilde{E} \rightarrow k^\times \rightarrow 1\] splits uniquely, we obtain a 2-cocycle in $Z^2(\tilde{E}, k^\times)$. From the isomorphism $L/Q \simeq \tilde{E}$, we obtain by inflation a 2-cocycle, also denoted by $\alpha$, in $Z^2(L, \tilde{E}^\times)$. Using this, we construct the twisted group algebra $\mathcal{O}_\alpha L$, which will be regarded as an $\tilde{E}$-graded algebra with 1-component $\mathcal{O}Q$, or even as an $E$-graded algebra with 1-component $\mathcal{O}Z(Q)$. The restrictions of $\alpha$ to subgroups of $L$ will be denoted by the same $\alpha$.

4. The Fong-Reynolds correspondence

4.1. It is clear that the idempotent $b_\delta$ remains a primitive idempotent in $(\mathcal{O}N_H(Q_\delta))_{N(G(Q_\delta))}$, and the induced block $b' := b_{N_H(Q_\delta)}$ is the Brauer correspondent of $b$. Since $b$ is a $G$-invariant block of $\mathcal{O}H$ with defect $Q$, it is well known that $b'$ is an $N_G(Q)$-invariant block of $\mathcal{O}N_H(Q)$ with defect group $Q$ in $N_H(Q)$. In this section, set

\[G' := N_G(Q), \quad H' := N_H(Q), \quad A' := \mathcal{O}N_G(Q)b', \quad B' := \mathcal{O}N_H(Q)b'.\]

Since

\[G/H \simeq N_G(Q_\delta)/N_H(Q_\delta) = G'/H',\]

it follows that $A'$ is also a $\tilde{G}$-graded algebra, with identity component $B'$. 
The Fong-Reynolds theorem says that there is a Morita equivalence between the block algebras \( B = \mathcal{O}N_H(Q_\delta)b_\delta \) and \( B' = \mathcal{O}N_H(Q)b' \). This equivalence is actually basic. Although the argument is well known, for completeness, we give an explicit proof of the fact that Fong-Reynolds equivalence extends to group-graded Morita equivalences.

We start with a useful lemma, whose proof is actually hidden in the proof of [24, Proposition 3.2], see [24, Remarks 3.3, 3.4].

**Lemma 4.2.** With the above notations, the Brauer correspondent \( b' \) of \( b \) in \( \mathcal{O}N_H(Q) \) is a primitive idempotent of \( (\mathcal{O}C_H(Q))_{N_G(Q)} \), and moreover, \( b' = \text{Tr}_{N_G(Q)}^N(b_\delta) \).

**Proof.** Since \( b' \) is the induced block of \( b_\delta \) from \( N_H(Q_\delta) \) to \( N_H(Q) \), we have that

\[
\text{Br}_Q(c)b_\delta = b_\delta.
\]  

(6)

By [24, Lemma 2.1] it is clear that \( c \) is a primitive idempotent of \( (\mathcal{O}C_H(Q))_{N_G(Q)} \) and then from [2, Proposition 3.10, Part IV] we deduce that \( c = \text{Tr}_{N_G(Q)}^{N_G(Q,b_1)}(b_1) \), where \( b_1 \) is a block of \( \mathcal{O}C_H(Q) \). It is enough to show that

\[
b_\delta = g b_1
\]

(7)

for some \( g \in G \). By using (6), we deduce the equalities

\[
\text{Br}_Q(c)b_\delta = \sum_{g \in [N_G(Q)/N_G(Q,b_1)]} \text{Br}_Q(g b_1)b_\delta = b_\delta.
\]

Since for all elements \( g \in [N_G(Q)/N_G(Q,b_1)] \) the idempotents \( g b_1 \) are blocks of \( \mathcal{O}C_H(Q) \), it follows that

\[
b_\delta = \sum_{g \in [N_G(Q)/N_G(Q,b_1)]} g b_1 b_\delta,
\]

hence there is a unique \( g \in [N_G(Q)/N_G(Q,b_1)] \) such that \( b_\delta = g b_1 \). \( \square \)

The main result of this section is the following refinement of the Fong-Reynolds correspondence.

**Proposition 4.3.** With the above notations, there is a \( G \)-graded basic Morita equivalence between \( \mathcal{O}N_G(Q_\delta)b_\delta \) and \( \mathcal{O}N_G(Q)b' \).

**Proof.** First notice that by Lemma [4.2] we have \( b' = \text{Tr}_{N_G(Q_\delta)}^{N_G(Q)}(b_\delta) \). Next, by [14, Lemma 2.3.16] we know that there is a \( G \)-graded Morita equivalence between \( b_\delta \mathcal{O}N_G(Q)b_\delta \) and

10
\(\mathcal{O}N_G(Q)b_\delta \mathcal{O}N_G(Q)\) induced by \(\mathcal{O}N_G(Q)b_\delta\) and its \(\mathcal{O}\)-dual \(b_\delta \mathcal{O}N_G(Q)\), viewed as \(\tilde{G}\)-graded bimodules.

The identity component \(\mathcal{O}N_H(Q)b_\delta\) of \(\mathcal{O}N_G(Q)b_\delta\) determines a Morita equivalence between \(\mathcal{O}N_H(Q)b_\delta\) and \(\mathcal{O}N_H(Q_\delta b_\delta\). Moreover, we have that \(\mathcal{O}N_H(Q)b_\delta\) is an indecomposable \(\mathcal{O}(N_H(Q) \times N_H(Q_\delta))\)-module that extends to the diagonal subalgebra, and we claim that this extension has vertex \(\Delta(P \times P)\). Indeed, if \(\hat{P} \leq G \times G\) denotes such a vertex of \(\mathcal{O}N_H(Q)b_\delta\) we know that the projection \(G \times G' \to G\) restricts to the epimorphism \(\hat{P} \to P\), since \(P\) is a defect group in \(N_G(Q_\delta)\) of \(b_\delta\). Since \(b_\delta\) is projective relative to \(P\) it follows that \(\mathcal{O}N_H(Q)b_\delta\) is relatively \(\Delta(P \times P)\)-projective. Assuming that \(\hat{P} \leq \Delta(P \times P)\) would be a contradiction to the previous statement. This proves that the group-graded Morita equivalence is basic.

5. Source algebras of block extensions

5.1. Let \(G' = N_G(Q)\) and \(H' = N_H(Q)\) as in the previous section. Similarly to \(2.2\) let \(Q_\delta' \leq H_\delta', P_\delta' \leq G_\delta'\) be defect pointed groups with source idempotents \(i' \in \delta', j' \in \delta'\) such that \(j' = i'j' = j'i\). Set

\[
A_{i'}' = i'A_{i'}, \quad A_{j'}' = j'A_{j'}, \quad B_{i}j' = j' B_{i}j', \quad B_{j}i' = i' B_{j}i'.
\]

It is well known (see [1, Theorem 5 and Corollary 7]) that there is an injective algebra map \(B_{i}j' \to B_{j}i'\). We may adapt the argument to obtain a graded version this property, with an additional condition. Note that this condition obviously holds when \(\hat{G}\) is a \(p'\)-group, or when \(G = PH\) and \(P\) is abelian, as in [26].

**Proposition 5.2.** With the above notations assume that \(N_G(Q_\delta) \setminus N_G(P_\gamma) \subseteq H\). Then there is a unital injective homomorphism

\[
A_{i'}' \to A_{j'}, \quad a \mapsto fa'
\]

of \(G/H\)-graded algebras, where \(f\) is a primitive idempotent in \((\mathcal{O}H)^{G'}\) with defect \(P\), which verifies \(bb' f = f = fbb'\) and \(i' f = f i' = i\).

**Proof.** Note that the assumption implies that

\[
G/H \cong G'/H' \simeq N_G(Q_\delta)/N_H(Q_\delta) = N_G(P_\gamma)/N_H(P_\gamma).
\]

Further, let \(K_H, K\) and \(K_{H'}\) be the inverse image in \(G \times G\) of \(\Delta(G/H \times G/H)\), the inverse image in \(G \times G'\) of \(\Delta(G/H \times G'/H')\) and the inverse image in \(G' \times G'\) of \(\Delta(G'/H' \times G'/H')\), respectively. Then the inclusions

\[
\Delta(P \times P) \leq N_{K_H}(\Delta(P \times P)) \leq K' \leq K \leq K_H
\]
of groups hold.

Now, the indecomposable $\mathcal{O}K_H$-module $\mathcal{O}Hb$ admitting $\Delta(P \times P)$ as vertex is the Green correspondent of the indecomposable $\mathcal{O}K_H$-module $\mathcal{O}H'b'$, which also has vertex $\Delta(P \times P)$. Again, using the same correspondence we determine a unique indecomposable $\mathcal{O}K$-module with vertex $\Delta(P \times P)$, say $X$, that lies in $\text{Res}^K_{KH}(\mathcal{O}Hb)$ and in $\text{Ind}_{K_{H'}}^K(\mathcal{O}H'b')$. Explicitly, we have that $X = \mathcal{O}Hf$ for a primitive idempotent $f$ lying in $(\mathcal{O}H)^G$ such that $\text{Br}_P(f) \neq 0$. Also note that $X \mid \text{Ind}_{K_{H}}^{K}(\mathcal{O}H\Delta(P \times P))(Z)$ for an indecomposable $\mathcal{O}H\Delta(P \times P)$-module $Z$ that has vertex $\Delta(P \times P)$.

We have that $\mathcal{O}H'i'$ is an indecomposable $\mathcal{O}H'\Delta(P \times P)$-module with vertex $\Delta(P \times P)$ that is also a source module of $\mathcal{O}H'b'$, where recall that $i'$ is a primitive idempotent of $(\mathcal{O}H')^P$. We consider the $\mathcal{O}H\Delta(P \times P)$-module

$$M := X \otimes_{\mathcal{O}H'} \mathcal{O}H'i',$$

and we claim that $M$ is an indecomposable module with vertex $\Delta(P \times P)$, and that it is also a source module of $\mathcal{O}Hb$. Indeed, we have that $M$ is a direct summand of $\mathcal{O}H'i'$, and the isomorphism

$$\mathcal{O}H'i' \simeq \text{Ind}_{H'\Delta(P \times P)}^{H\Delta(P \times P)}(\mathcal{O}H'i')$$

of $\mathcal{O}H\Delta(P \times P)$-modules shows that

$$M \mid \text{Ind}_{H'\Delta(P \times P)}^{H\Delta(P \times P)}(\mathcal{O}H'i').$$

Let us emphasise that we have the inclusion

$$N_{H\Delta(P \times P)}(\Delta(P \times P)) \leq H'\Delta(P \times P),$$

and this forces the Green correspondent of $\mathcal{O}H'i'$ to lie in $M$. It follows that $M = M' \oplus M''$ where $M'$ is an indecomposable $\mathcal{O}H\Delta(P \times P)$-module with vertex $\Delta(P \times P)$, while $M''$ is a direct sum of indecomposable modules with vertices strictly smaller than $\Delta(P \times P)$.

We have, by our hypothesis,

$$M \mid \text{Res}_{H\Delta(P \times P)}^{K}(X) \mid \text{Res}_{H\Delta(P \times P)}^{K}(\text{Ind}_{H\Delta(P \times P)}^{K}(Z)) =$$

$$= \sum_{(x,x') \in [H\Delta(P \times P)/K \cdot H\Delta(P \times P)]} \text{Ind}_{H\Delta(P \times P)}^{H\Delta(P \times P)}(x,x') \left(\text{Res}_{H\Delta(P \times P)}^{K}(Z)\right) =$$

$$= \sum_{(x,x') \in [H\Delta(P \times P)/K \cdot H\Delta(P \times P)]} Z^{(x,x')}.$$
Consequently $M'' = 0$ and then $M = M'$ is the Green correspondent of $\mathcal{O}H'i'$, this forces $fi' = i'f$ to be primitive idempotent of $(\mathcal{O}H)^P$ with $bfi' = f'i' = f'i'b$, hence $M$ is a source module of $\mathcal{O}Hb$. We may therefore assume that $fi' = i$ and that $M = \mathcal{O}Hi$.

With these notations we get the isomorphism

$$\mathcal{O}H'i \simeq \mathcal{O}H'f \otimes_{\mathcal{O}H'} \mathcal{O}H' \simeq \mathcal{O}H'i'$$

of $\mathcal{O}H\Delta(P \times P)$-modules, hence $\mathcal{O}H'i' \mid \text{Res}_{H'\times 1}^H(\mathcal{O}Hi)$. We obtain a $P$-algebra homomorphism

$$B'_\gamma \simeq \text{End}_{\mathcal{O}H'}(\mathcal{O}H'i') \rightarrow \text{End}_{\mathcal{O}H}(\mathcal{O}Hi) \simeq B_\gamma,$$

which is given by $a' \mapsto a'f = f'a'$ for any $a' \in B'_\gamma$. Finally, this map extends in an obvious way to a homomorphism

$$A'_\gamma \rightarrow A_\gamma,$$

of $P$-interior $\tilde{G}$-graded algebras which, since

$$\mathcal{O}G'i' \simeq \mathcal{O}G' \otimes_{\mathcal{O}H'} \mathcal{O}H'i' \mid \mathcal{O}G' \otimes_{\mathcal{O}H'} \mathcal{O}Hi \mid \mathcal{O}Gi,$$

it is also injective.

6. Lifting Morita equivalences

In this section we give a general technical lemma which is useful to lift Morita equivalences between 1-components to $G$-graded Morita equivalences. The notation below will be used only in this section.

6.1. Let $G$ be a finite group, and let $A$ and $A'$ be strongly $G$-graded $\mathcal{O}$-algebras with 1-components $B := A_1$ and $B' := A'_1$. Denote $\tilde{A} := A / J_{gr}(A)$ and $\tilde{A}' := A' / J_{gr}(A')$.

6.2. We consider the diagonal subalgebra

$$\Delta := \Delta(A \otimes A'^{\text{op}}) = \sum_{g \in G} A_g \otimes A'_{g^{-1}}$$

of $A \otimes A'^{\text{op}}$, and let

$$\tilde{\Delta} := \Delta / J_{gr}(\Delta) \cong \Delta(\tilde{A} \otimes_k \tilde{A}'^{\text{op}}).$$
6.3. Let $M$ be a $G$-invariant $\Delta_1$-module (that is, a $G$-invariant $(B, B')$-bimodule), let $\bar{M} := M/J(\Delta_1)M$, and consider the $G$-graded endomorphism algebras

$$D := \text{End}_\Delta(\Delta \otimes_{\Delta_1} M)^{\text{op}}, \quad \bar{D} := \text{End}_{\bar{\Delta}}(\bar{\Delta} \otimes_{\bar{\Delta}_1} \bar{M})^{\text{op}}.$$  

As in [2.7] the group extension

$$1 \longrightarrow \mathcal{D}_1 \longrightarrow \text{hU}(\mathcal{D}) \longrightarrow G \longrightarrow 1$$

is the Clifford extension of the $\Delta_1$-module $M$, and the group extension $\text{hU}(\mathcal{D}/J_{\text{gr}}(\mathcal{D}))$ is the residual Clifford extension of $M$.

**Lemma 6.4.** With the above notations, assume that the following conditions hold.

1. $M$ induces a Morita equivalence between $B$ and $B'$.
2. $\bar{M}$ is a simple $\bar{\Delta}_1$-module.
3. The algebra $\mathcal{D}_1$ is commutative.
4. $\text{End}_{\bar{\Delta}}(\bar{\Delta} \otimes_{\bar{B}} \bar{M})^{\text{op}} \simeq \bar{A}'$.
5. For a Sylow $p$-subgroup $P$ of $G$, $M$ extends to a $\Delta_P$-module.

Then $A \otimes_B M$ induces a $G$-graded Morita equivalence between $A$ and $A'$.

**Proof.** Since $\bar{M}$ is a simple $\bar{\Delta}_1$-module, we have that $\bar{\mathcal{D}} \simeq \bar{\mathcal{D}}/J_{\text{gr}}(\bar{\mathcal{D}})$ (see [16, Lemma 2.4]), which means that the Clifford extension $\text{hU}(\bar{\mathcal{D}})$ of $\bar{M}$ is also the residual Clifford extension of $M$.

Condition (4) implies that $\bar{\Delta} \otimes_{\bar{B}} \bar{M}$ is a $G$-graded $(\bar{\Delta}, \bar{A}')$-bimodule, so by [14, Lemma 1.6.3], $\bar{M}$ extends to a $\bar{\Delta}$-module. By [8, (1.7)] it follows that the Clifford extension $\text{hU}(\mathcal{D})$ of $M$ splits. Consequently, by [8, Theorem 2.8], we deduce that for any Sylow $q$-subgroup $Q$ of $G$, where $q \neq p$, the Clifford extension

$$1 \longrightarrow \mathcal{D}_1 \longrightarrow \text{hU}(\mathcal{D}_Q) \longrightarrow Q \longrightarrow 1$$

of $M$ splits. By assumption (5) we have that the extension

$$1 \longrightarrow \mathcal{D}_1 \longrightarrow \text{hU}(\mathcal{D}_P) \longrightarrow P \longrightarrow 1$$

also splits.

Since $\mathcal{D}_1$ is commutative, we deduce that the Clifford extension $\text{hU}(\mathcal{D})$ splits (see, for instance [8, Theorem 7.2]), hence $M$ extends to a $\Delta$-module. It follows by [14, Theorem 5.1.2] that

$$A \otimes_B M \cong M \otimes_{B'} A' \cong (A \otimes (A')^{\text{op}}) \otimes_\Delta M$$

induces a $G$-graded Morita equivalence between $A$ and $A'$. \qed
7. Extensions of nilpotent and inertial blocks

7.1. Let \( b \) be a \( G \)-invariant block of \( O_H \) which is with defect pointed group \( Q_\delta \) in \( H(b) \), as in the Introduction, and let \( j \in \delta \). Because of the Fong-Reynolds reduction from Section 4, the notations in this section and in the last one are as follows:

\[
A' := \mathcal{O}N_G(Q_\delta)b_\delta, \quad B' := \mathcal{O}N_H(Q_\delta)b_\delta; \quad C' := \mathcal{O}QC_H(Q)b_\delta.
\]

and we regard \( A' \) as an \( \tilde{E} \)-graded algebra with 1-component \( C' \). Let \( A'_{\gamma} = i' \mathcal{O}N_G(Q_\delta)i' \) be the source algebra of the block extension \( A' \).

7.1. Extensions of nilpotent blocks

We assume in this subsection that \( B = \mathcal{O}Hb \) is a nilpotent block, and we refer to [25, Chapter 7] for a comprehensive presentation of Puig’s theorem on the source algebras of nilpotent blocks.

7.2. First, note that \( N_H(Q_\delta) = QC_H(Q) \), hence \( \tilde{E} = \bar{G} \) and \( C' = B' \). Moreover, the source algebra \( B_\delta = jBj \) has an \( \mathcal{O} \)-simple, \( Q \)-stable subalgebra \( S_\delta \) such that

\[
jBj \simeq S_\delta \otimes \mathcal{O}Q; \quad S_\delta \simeq \text{End}_\mathcal{O}(V_\delta)
\]

where \( V_\delta \) is the unique (up to isomorphism) \( \mathcal{O} \)-simple \( jBj \)-module, and \( p \nmid \text{rank}_\mathcal{O}(S_\delta) \), that is, \( S_\delta \) is a Dade \( Q \)-algebra; denoting \( \bar{V}_\delta = k \otimes_\mathcal{O} V_\delta \) and \( \bar{S} = k \otimes_\mathcal{O} S \), we have \( \bar{S}_\delta \simeq jBj/J(jBj) \) is a simple \( k \)-algebra, and \( \bar{V}_\delta \) the unique simple \( \bar{S}_\delta \)-module. The Morita equivalence between \( jBj \) and \( \mathcal{O}Q \) is given by the functor

\[
V_\delta \otimes_\mathcal{O} - : \mathcal{O}Q\text{-Mod} \to jBj\text{-Mod}.
\]

Since \( B \) is Morita equivalent to \( \mathcal{O}Q \), there is a unique \( \mathcal{O} \)-simple \( B \)-module \( U \), and to \( U \) it corresponds a unique \( \mathcal{O} \)-simple \( jBj \)-module \( V_\delta \) such that \( U = Bj \otimes jBjV_\delta \).

The following Lemma should be compared to [11, 1.11 and 1.15]

**Lemma 7.3.** The residual Clifford extensions of \( U \) and \( V_\delta \) are isomorphic to \( k_\alpha \tilde{E} \) as \( \tilde{E} \)-graded algebras, where \( \alpha \in \tilde{Z}^2(\tilde{E}, k^\times) \) is defined in 3.7.

**Proof.** From the \( \tilde{E} \)-graded Morita equivalence between \( A \) and \( jAj \) we obtain the isomorphism

\[
\text{End}_A(A \otimes_B U)^{\text{op}} \simeq \text{End}_{jAj}(jA)jBjV_\delta)^{\text{op}}
\]

15
of $\tilde{E}$-graded algebras, see [14, Corollary 5.1.4]. In particular, $U$ and $V_\delta$ have isomorphic residual Clifford extensions.

Observe that $B^Q$ and $N_G(Q_\delta)$ generate a $\tilde{G}$-graded subalgebra of $A$, isomorphic to $B^Q \otimes_{C_H(Q)} N_G(Q_\delta)$, with 1-component $B^Q \otimes_{C_H(Q)} Q$. We have that $B^Q \otimes_{C_H(Q)} Q$ is an $N_G(Q_\delta)$-invariant indecomposable projective $B^Q \otimes_{C_H(Q)} Q$-module. By [5, Proposition 6.2], its residual Clifford extension is isomorphic to $k\alpha\tilde{E}$ as $\tilde{E}$-graded algebras, since $\tilde{E} \simeq \tilde{G}$ in our situation. But we have a unital injective map

$$jB^Q \otimes_{C_H(Q)} N_G(Q_\delta) j \to jA j$$

of $\tilde{G}$-graded algebras, sending $b \otimes x$ to $bx$ for all $b \in B^Q$ and $x \in N_G(Q_\delta)$. By taking quotients modulo the graded Jacobson radicals, we obtain a homomorphism

$$k\alpha\tilde{E} \to A_\delta := A_\delta / J_{gr}(A_\delta)$$

of $\tilde{E}$-graded algebras. In particular, $\tilde{E}$ acts on $S_\delta$ and $\tilde{A}_\delta$ is a crossed product of the form $\tilde{A}_\delta \simeq S_\delta \otimes k\alpha\tilde{E}$.

To prove the Lemma, we need to show the isomorphism

$$\text{End}_{\tilde{A}_\delta}(\tilde{A}_\delta \otimes \tilde{S}_\delta \tilde{V}_\delta) \simeq k\alpha\tilde{E}$$

of $\tilde{E}$-graded algebras. But this is the same to show that the $\tilde{S}_\delta$-module structure of $\tilde{V}_\delta$ extends to a module structure over the diagonal subalgebra

$$\Delta(\tilde{A}_\delta \otimes (k\alpha\tilde{E})^\text{op}) \simeq \tilde{S}_\delta \otimes k\tilde{E}.$$

This condition, in turn, is equivalent to the splitting of the Clifford extension $\text{End}_{\tilde{S}_\delta \otimes k\tilde{E}}((\tilde{S}_\delta \otimes k\tilde{E}) \otimes_{S_\delta} \tilde{V}_\delta)$.

Since $\tilde{V}_\delta$ is an $\tilde{E}$-invariant indecomposable endopermutation $kQ$-module, its residual Clifford extension splits, by Dade’s theorem. Therefore, it is enough to show that there is an $\tilde{E}$-graded algebra homomorphism

$$\text{End}_{\tilde{S}_\delta \otimes k\tilde{E}}((\tilde{S}_\delta \otimes k\tilde{E}) \otimes_{S_\delta} \tilde{V}_\delta) \to \text{End}_{k\ell}(k\ell \otimes_{kQ} \tilde{V}_\delta).$$

Indeed, such a homomorphism exists, and it is defined (in a way similar to [5, Proposition 5.4]) as follows. Let $\tilde{x} \in \tilde{E}$, where $x \in N_G(Q_\delta)$, and let $y \in L$ such that $\pi(y) = \tilde{x}$. An element of degree $\tilde{x}$ from the domain is a $k$-linear map $f: \tilde{V}_\delta \to \tilde{V}_\delta$ satisfying $f \circ s = s^x \circ f$ for all $s \in S_\delta$, while an element of degree $\tilde{y} \in L / Q$ from the codomain is a $k$-linear map $f': V_\delta \to V_\delta$ such that $f'(uv) = u^x f'(v)$ for all $v \in V$ and $u \in Q$. It is straightforward to check, by using Remark [5,5], that the restriction of scalars $f \to f$, via $kQ \to S$, gives the required $\tilde{E}$-graded algebra homomorphism. \qed
7.4. We know by [11, Proposition 6.5], or by [3, Theorem 2] that $BP = b\mathcal{O}PH$ is also a nilpotent block of $\mathcal{O}PH$. Then, by Puig’s results on nilpotent blocks, we have:

- the $\bar{P} \simeq P/Q$-graded source algebra $iBP$ has an $\mathcal{O}$-simple, $P$-stable subalgebra $S_\gamma$ such that

  $$iBP \simeq S_\gamma \otimes \mathcal{O}P; \quad S_\gamma \simeq \text{End}_\mathcal{O}(V_\gamma)$$

  where $V_\gamma$ is the unique $\mathcal{O}$-simple $iBP$-module, and $p \nmid \text{rank}_\mathcal{O}(S_\gamma)$, so $S_\gamma$ is a Dade $P$-algebra;

- denoting $\bar{V}_\gamma = k \otimes_\mathcal{O} V_\gamma$ and $\bar{S}_\gamma = k \otimes_\mathcal{O} S$, we have $\bar{S}_\gamma \simeq iBP/J(iBP)$ is a simple $k$-algebra, and $\bar{V}_\gamma$ the unique simple $\bar{S}_\gamma$-module.

Now Theorem 1.1 c) is a consequence of the following more precise statement.

**Theorem 7.5.** Assume that $B = \mathcal{O}Hb$ is a nilpotent block. There is an $\bar{G}$-graded Morita equivalence between $A_\gamma$ and $\mathcal{O}_\alpha L$ induced by $V_\gamma$, or equivalently, an isomorphism

$$A_\gamma \simeq S_\gamma \otimes \mathcal{O}_\alpha L.$$  

**Proof.** The $\bar{P}$-graded Morita equivalence between $A_\gamma$ and $\mathcal{O}P$ restricts to the Morita equivalence $V_\gamma \otimes_\mathcal{O} - : \mathcal{O}Q\text{-Mod} \to B_\gamma\text{-Mod}$. We aim to use Lemma 6.4 to lift this to a Morita equivalence $V_\gamma \otimes_\mathcal{O} - : \mathcal{O}_\alpha L\text{-Mod} \to A_\gamma\text{-Mod}$.

We have that $\bar{A}_\gamma := A_\gamma/J_{gr}(A_\gamma)$ is an $\bar{G}$-graded crossed product with 1-component $\bar{B}_\gamma := B_\gamma/J(B_\gamma) \simeq S_\gamma$. The algebra $\bar{R} := \mathcal{O}_\alpha L$ is $\bar{G} \simeq L/Q$-graded with 1-component $\mathcal{O}Q$, and $\bar{R} := R/J_{gr}(R) \simeq k_\alpha \bar{G}$. Let

$$\Delta := \Delta(A_\gamma \otimes_\mathcal{O} R^{\text{op}}), \quad \Delta_1 = B_\gamma \otimes_\mathcal{O} (\mathcal{O}Q)^{\text{op}}, \quad \bar{\Delta} = \Delta/J_{gr}(\Delta).$$

We have $\Delta_1 \simeq (S_\gamma \otimes \mathcal{O}Q) \otimes (\mathcal{O}Q)^{\text{op}}, \Delta_1/J(\Delta_1) \simeq S_\gamma$. From $V_\gamma$ we get the $(iBi, \mathcal{O}Q)$-bimodule

$$M = M_\gamma := V_\gamma \otimes_\mathcal{O} Q$$

inducing a Morita equivalence between $iBi$ and $\mathcal{O}Q$, hence $M$ satisfies condition (1) of Lemma 6.4. Let $\tilde{M} = M/J(\Delta_1)M$; then $\tilde{M} \simeq \bar{V}_\gamma$ is a simple $\bar{S}_\gamma$-module, hence $M$ satisfies condition (2) of Lemma 6.4. Moreover,

$$\text{End}_{\Delta_1}(M) \simeq \text{End}_{S_\gamma \otimes \mathcal{O}Q}(V_\gamma \otimes \mathcal{O}Q) \simeq Z(\mathcal{O}Q)$$
is commutative, therefore condition (3) of Lemma 6.4 also holds. For the Clifford extensions, observe that
\[
\text{End}_{\tilde{A}_{\gamma}}(\tilde{A}_{\gamma} \otimes \tilde{B}_{\gamma})_{\gamma} \simeq \text{End}_{\tilde{A}_{\delta}}(\tilde{A}_{\delta} \otimes \tilde{B}_{\delta})_{\delta} \simeq k_{\alpha} G,
\]
as $G$-graded $k$-algebras, where the first isomorphism is a consequence of the $G$-graded Morita equivalence between $A_{\delta}$ and $A_{\gamma}$ (see [15, Proposition 3.2]), while the second isomorphism follows from Lemma 7.3. This gives condition (4) of Lemma 6.4. Finally, $M = V_{\gamma} \otimes \delta Q$ has a diagonal action of $P$, because $\delta Q$ is a $P$-algebra and $V_{\gamma}$ is an $\delta P$-module, so condition (5) of Lemma 6.4 also holds.

It follows that $M$ extends to $\Delta$, and consequently, $A_{\gamma} \otimes_{B_{\gamma}} M$ induces an $G$-graded Morita equivalence between $A_{\gamma}$ and $\delta Q$.  

7.2. Extensions of blocks with normal defect groups

7.6. Consider the block extension $A' = \delta N_{G}(Q_{\delta}) b_{\delta}$, regarded as an $\tilde{E}$-graded algebra with identity component $C' := \delta QC_H(Q) b_{\delta}$. The block $b_{\delta}$ has defect pointed group $Q_{\delta'}$ in $N_{H}(Q_{\delta})$ and, by [3.2] applied for $R = Q$, it has defect pointed group $P_{\delta'}$ in $N_{G}(Q_{\delta})$.

We know that the source algebra of $C'$ is $j' \delta QC_H(Q) j' \simeq \delta Q$, where $j' \in \delta'$, and the bimodule which gives the Morita equivalence is $\delta QC_H(Q) j'$, so here we have a particular case of extensions of nilpotent blocks. By Külshammer [10, Theorem A], (see also [1, Theorem 13]), we have that the source algebra of $B' = \delta N_{H}(Q_{\delta}) b_{\delta}$ is $B'_{\delta'} = j' B' j' \simeq \delta \alpha(Q \rtimes E_{H}(Q_{\delta}))$.

Let $A'_{\gamma} = i' A' i'$ be the source algebra of $A'$, where $i' \in (\delta QC_H(Q))^{P}$. As above, $P C' = \delta PC_H(Q) b_{\delta}$ is a nilpotent block. Let $V_{\gamma'}$ be the unique, up to isomorphism, simple $i' \delta PC_H(Q) i'$-module, and let $S_{\gamma'} = \text{End}_{\delta}(V_{\gamma'})$, which is a Dade $P$-algebra. The source algebra of $\delta PC_H(Q) b_{\delta}$ is isomorphic to $S_{\gamma'} \otimes \delta P$, and by Theorem 7.5 and 3.6 we get:

**Corollary 7.7.** There is an $E$-graded Morita equivalence between $A'_{\gamma}$ and $\delta_{\alpha} L$ induced by $V_{\gamma}$. More precisely, there is an isomorphism

\[
A'_{\gamma} \simeq S_{\gamma'} \otimes \delta_{\alpha} L
\]
of $P$-interior $E$-graded algebras, where $L$ is defined in Section 3.

7.3. $p'$-extensions of inertial blocks

In this subsection we assume that the block $B$ is inertial, and we prove the statements a), b) and d) of Theorem 1.1. By Puig [21, 2.16], the assumption means that

\[
B_{\delta} \simeq S \otimes B'_{\delta'}.
\]
where $S$ is a Dade $Q$-algebra, unique up to isomorphism. Statement 2) of the next theorem is due to Y. Zhou [27, Proposition 3.3]. Here we give an alternative proof based on Lemma 6.4 (see also the Remark following [27, Theorem]).

**Theorem 7.8.** Assume that the block $B$ is inertial.

1) Then $A_\delta$ is an $\tilde{E}$-graded crossed product.

2) If, in addition, $p$ does not divide the order of $\tilde{G}$, then there is an isomorphism

$$A_\delta \simeq S \otimes A'_\delta,$$

of $\tilde{E}$-graded algebras.

**Proof.** 1) Since $B'_{\delta'} \simeq \mathcal{O}_\alpha(Q \rtimes E_H(Q_\delta))$, by our assumption we have an isomorphism $B_\delta \simeq S \otimes \mathcal{O}_\alpha(Q \rtimes E_H(Q_\delta))$, which gives an $E_H(Q_\delta)$-grading on $B_\delta$, with 1-component denoted $C_\delta \simeq S_\delta \otimes \mathcal{O}Q$. To show that the $\tilde{G}$-grading of $A_\delta$ can be refined to an $\tilde{E}$-grading, it is enough to prove that $C_\delta$ is an $hU(A_\delta)$-invariant subalgebra of $B_\delta$.

We regard $S$ as a subalgebra of $C_\delta$, and let $W$ be the unique, up to isomorphism $\mathcal{O}$-simple $C_\delta$-module, so $W$ is also the unique $\mathcal{O}$-simple $S$-module. We have that the Morita equivalence between $B_\delta$ and $B'_{\delta'}$ is given by the functor

$$W \otimes \mathcal{O} - : (B'_{\delta'})\text{-mod} \to (B_\delta)\text{-mod},$$

that is, is induced by the $E_H(Q_\delta)$-graded $(B_\delta, B'_{\delta'})$-bimodule $W \otimes B'_{\delta'}$.

Let $\bar{x} \in \tilde{G}$; since $\tilde{G} \cong N_G(Q_\delta)/N_H(Q_\delta)$, we may assume that $\bar{x}$ is the coset of an element $x \in N_G(Q_\delta)$. Then there is $b_x \in (B^{Q})^x$ such that $xjx^{-1} = b_xjb_x^{-1}$, hence $b_x^{-1}x$ commutes with $j$, and let $a_x := (b_x^{-1}x)j \in N_{hU(A_\delta)}(Q)$. The $Q$-interior algebra $S$ given by the hypothesis is unique up to isomorphism, and this implies that $a_xSa_x^{-1} \simeq S$ as $Q$-interior algebra. Since $a_x(uv)a_x^{-1} = x(uv)x^{-1}$ for all $u \in Q$, we deduce that $W$ is an $N_G(Q_\delta)$-invariant $\mathcal{O}Q$-module. It follows that the subalgebra $C_\delta \simeq \text{End}_{\mathcal{O}Q}(W \otimes \mathcal{O}Q)$ of $B_\delta$ is $N_G(Q_\delta)$-stable, and the statement is proved.

2) We consider the $\tilde{E}$-graded algebras $A_\delta$ with 1-component $C_\delta$, $A'_{\delta'}$, with 1-component $C'_{\delta'} = \mathcal{O}Q$, and the diagonal subalgebra $\Delta := \Delta(A_{\delta} \otimes A'_{\delta'})$ with 1-component $\Delta_1 = C_\delta \otimes (\mathcal{O}Q)^{op}$. Let $M = W \otimes \mathcal{O}Q$. Then $M$ is an $\tilde{E}$-invariant $(C_\delta, \mathcal{O}Q)$-bimodule (that is, $\Delta \otimes \Delta_1$-modules for all $x \in \tilde{E}$), because $\mathcal{O}Q$ is $\tilde{E}$-invariant and $W$ is an $\tilde{E}$-invariant $\mathcal{O}Q$-module. (Note that we don’t need for this the assumption that $G$ is a $p'$-group, and Theorem 1.1 b) follows immediately.)

We are going to verify the conditions of Lemma 6.4. By assumption, $M$ induces a Morita equivalence between $C_\delta$ and $\mathcal{O}Q$, so condition (1) holds, while condition (5) is trivially true, since $\tilde{E}$ is a $p'$-group. For condition (2), note that $C_\delta/J(C_\delta) \simeq S$, $\tilde{\Lambda}_1 = \ldots$
\[ \Delta_1/J(\Delta_1) \simeq \bar{S} = k \otimes \delta S \text{ and } \bar{M} = M/J(\Delta_1)M \simeq \bar{W}, \]  
so \( \bar{M} \) is a simple \( \bar{\Delta}_1 \)-module. Since \( C_\delta \simeq S \otimes \mathcal{O}Q \), we have that 
\[ \mathcal{D}_1 = \text{End}_{\Delta_1}(M)^{\text{op}} = \text{End}_{C_\delta}\otimes(\mathcal{O}Q)^{\text{op}}(W \otimes \mathcal{O}Q)^{\text{op}} \simeq Z(\mathcal{O}Q) \]
is commutative, so condition (3) also holds.

Finally, for condition (4), observe that \( \bar{A}_\delta = A_\delta/J_{\text{gr}}(A_\delta) \) is a crossed product of \( \bar{S} \) and \( \bar{E} \), while \( \bar{A}_\delta' = A_\delta/J_{\text{gr}}(A_\delta') \simeq k_\alpha \bar{E} \), where \( \alpha \in Z^2(\bar{E}, k^\times) \) is defined in [3.7]. By Proposition [5.2] there are injective maps \( A_\delta' \to A_\delta \) and \( \bar{A}_\delta' \to \bar{A}_\delta \) of \( \bar{E} \)-graded algebras, hence \( \bar{E} \) acts on \( \bar{S} \), and \( \bar{A}_\delta \) is a crossed product of the form 
\[ \bar{A}_\delta \simeq \bar{S} \otimes k_\alpha \bar{E}. \]

To prove that 
\[ \text{End}_{\bar{A}_\delta}(\bar{A}_\delta \otimes \delta \bar{W})^{\text{op}} \simeq k_\alpha \bar{E}, \]
it is enough to show that the simple \( \bar{S} \)-module \( \bar{W} \) extends to the diagonal subalgebra \( \Delta(\bar{A}_\delta \otimes (\bar{A}_\delta')^{\text{op}}) \). Observe that 
\[ \Delta(\bar{A}_\delta \otimes (\bar{A}_\delta')^{\text{op}}) \simeq \Delta((\bar{S} \otimes k_\alpha \bar{E}) \otimes (k_\alpha \bar{E})^{\text{op}}) \simeq \bar{S} \otimes k\bar{E}, \]

hence it is enough to show that \( \bar{W} \) has a structure of a \( k\bar{E} \)-module. We now use the fact that \( W \) is an \( \bar{E} \)-invariant indecomposable endopermutation \( \mathcal{O}Q \)-module, where we regard \( Q \) as a normal subgroup of \( \bar{Q} \times \bar{E} \). By Dade’s theorem [7, (12)], the residual Clifford extension of \( W \) splits, and since \( \bar{E} \) is a \( p' \)-group, the Clifford extension of \( W \) also splits, hence the \( \mathcal{O}Q \)-module structure of \( W \) extends to an \( \mathcal{O}(\bar{Q} \times \bar{E}) \)-module structure (see [8, Theorem 6.7]). In particular, \( W \) is an \( \mathcal{O}\bar{E} \)-module and hence \( \bar{W} \) is an \( k\bar{E} \)-module. \( \square \)

**Remark 7.9.** Note that in the above proof, condition (4) of Lemma [6.4] holds for \( \bar{M} = \bar{W} \) without the assumption that \( \bar{G} \) is a \( p' \)-group or that \( \bar{G} = \bar{E} \). Indeed, we may argue as in Lemma [7.3]. We have the \( \bar{G} \)-graded algebra homomorphism \( BQ \otimes C_{\text{Ch}}(Q)N_{\bar{G}}(Q_\delta) \), and by the assumption that \( B \) is an inertial block, we already know that \( \bar{B}_\delta \simeq \bar{S} \otimes k_\alpha \bar{E}_H(Q_\delta) \). We deduce that \( \bar{E} \) acts on \( \bar{S} \) and that \( \bar{B}_\delta \) is a crossed product of the form \( \bar{B}_\delta \simeq \bar{S} \otimes k_\alpha \bar{E} \). Then we use again Dade’s theorem for the endopermutation \( kQ \)-module \( \bar{W} \) to get the isomorphism \( \text{End}_{\bar{A}_\delta}(\bar{A}_\delta \otimes \delta \bar{W})^{\text{op}} \simeq k_\alpha \bar{E} \) of \( \bar{E} \)-graded algebras. This proves the second part of Theorem [11.1a].

**8. Extended local categories**

In this section we introduce fusions in the general context of \( \bar{G} \)-graded \( G \)-interior algebras, and study their properties. We extend here some notions and results from [5], where only automorphisms of \( P \)-groups are considered.
8.1. Recall that there is an extended local category denoted $\mathcal{E}_{(b,H,G)}$ introduced by Puig and Zhou in [23]. The objects are the local pointed subgroups included in $P_\gamma$ and the set of morphisms $\text{Hom}_{\mathcal{E}_{(b,H,G)}}(R_\epsilon, T_\rho)$ from $R_\epsilon$ to $T_\rho$ (here $T,R \leq P$ and $\rho \subseteq B^T, \epsilon \subseteq B^R$ are local points) is formed by the pairs $(c_x, \bar{x})$ with $\bar{x} \in \bar{G}$, such that

$$c_x : R \to T \quad c_x(u) = xu$$

for any $u \in R$; here $^\epsilon R_\epsilon \leq T_\rho$, with $\epsilon$ and $\rho$ local points such that $T_\rho \leq P_\gamma$ and $R_\epsilon \leq P_\gamma$. If $R = T$ the automorphism group is

$$\text{Aut}_{\mathcal{E}_{(b,H,G)}}(R_\epsilon) := E_G^G(R_\epsilon) \cong N_G(R_\epsilon)/C_H(R).$$

For brevity, we denote this category by $\mathcal{E}$, and if $P_\gamma'$ is some defect pointed group in $G'_{\{b\}'}$ on $B'$ (which is a $G'$-algebra), we have a similar extended local category $\mathcal{E}'$.

8.2. We introduce $(A, \bar{G})$-fusions and $(G, \bar{G})$-fusions in a more general context. Let $\bar{G} = G/H$, let $A$ be a $\bar{G}$-graded $G$-interior algebra such that $B := A_1$ is an $H$-interior $G$-algebra. Let $P$ be a $p$-subgroup of $G$, and let $P_\gamma$ be a local pointed group on the $G$-algebra $B$ with $i \in \gamma$. We will assume that the structural map $P \to A^\times$ is injective. In particular, $Ai$ becomes a $\bar{G}$-graded $(A, \mathcal{O}P)$-bimodule, with $(\text{End}_A(Ai))^{op} \cong iAi$ as $\bar{G}$-graded $P$-interior algebras.

For two pointed subgroups $R_\epsilon \leq P_\gamma$ and $T_\rho \leq P_\gamma$ we denote

$$N_p(R,T) = \{u \in P \mid ^uR \leq T\}$$

$$N_G(R_\epsilon, T_\rho) = \{g \in G \mid ^gR_\epsilon \leq T_\rho\} = \{g \in N_G(R,T) \mid \text{ for } l \in \rho \text{ there is } ^gj \in ^g\epsilon \text{ such that } ^gj = l \cdot ^gj = ^gj \cdot l\}.$$

For $i \in \gamma$ choose $j \in \epsilon$ and $l \in \rho$ such that

$$j = ij = ji, \quad l = il = li.$$

In particular, to $R_\epsilon$ corresponds the $\bar{G}$-graded $(A, \mathcal{O}R)$-bimodule $Aj$, while to $T_\rho$ corresponds the $\bar{G}$-graded $(A, \mathcal{O}T)$-bimodule $Al$. Denote by $\text{Inj}(R,T)$ the set of injective group homomorphisms from $R$ to $T$.

**Definition 8.3.**

a) The set of $\bar{G}$-fusions from $R$ to $T$ is

$$\text{Hom}^{\bar{G}}(R,T) = \{(\varphi, \bar{g}) \mid \varphi \in \text{Inj}(R,T), \bar{g} \in \bar{G}, \varphi(u) = \bar{g}u, \forall u \in R\}.$$

b) The set of interior $\bar{G}$-fusions from $R$ to $T$ is

$$\text{Int}^{\bar{G}}(R,T) = \{(c_v, \bar{g}) \mid v \in N_p(R,T), \bar{g} \in \bar{G}, c_v(u) = \bar{g}u, \forall u \in R\}.$$
Remark 8.4. a) We have the inclusions
\[
\text{Int}^\hat{G}(R, T) \subseteq \text{Hom}^\hat{G}(R, T) \subseteq \text{Inj}(R, T) \times \hat{G}.
\]
If \((\varphi, \bar{g}) \in \text{Hom}^\hat{G}(R, T)\), then \(\bar{g} \in N_G(\bar{R}, \bar{T})\).

b) If \(R = T = P\), then \(\text{Hom}^\hat{G}(P, P) = \text{Aut}^\hat{G}(P)\), and \(\text{Int}^\hat{G}(P, P) = \text{Int}^\hat{G}(P)\), see [5, Definition 3.2].

Definition 8.5. With the above notations, assume \(R \cong T\). We define the set
\[
N_{hU}(A)(R_j, T_l) := \{a \in A^x \cap A_{\bar{g}} \mid \bar{g} \in \hat{G}, \ a(R_j)a^{-1} = T_l\}.
\]
If \(R_j = T_l = Pi\), then \(N_{hU}(A)(Pi)\) is a group, defined in [5].

Definition 8.6. a) The set of \((A, \hat{G})\)-fusions from \(R_\epsilon\) to \(T_\rho\) is
\[
F^\hat{G}_A(R_\epsilon, T_\rho) = \{(\varphi, \bar{g}) \in \text{Hom}^\hat{G}(R, T) \mid A_j \text{ is a summand of } (Al)(\bar{g}^{-1})_\varphi
\]
as \(\hat{G}\)-graded \((A, \mathcal{O}R)\)-bimodules, \}
where \((Al)(\bar{g}^{-1})_\bar{x} = Al_{\bar{g}^{-1}}\bar{x}\) for all \(\bar{x} \in \hat{G}\), see [5, 2.2].

b) The set of \((G, \hat{G})\)-fusions from \(R_\epsilon\) to \(T_\rho\) is
\[
E^\hat{G}_G(R_\epsilon, T_\rho) = \{(c_x, \bar{x}) \mid x \in N_G(R_\epsilon, T_\rho), \overline{c_x(u)} = \bar{x}u, \forall u \in R\}.
\]
Note that in the case of a block extension \(A = \mathcal{O}Gb\), this set is actually \(\text{Hom}_G(R_\epsilon, T_\rho)\).

Remark 8.7. If \(R_\epsilon = T_\rho = P_\gamma\) then \(F^\hat{G}_A(P_\gamma, P_\gamma) = F^\hat{G}_A(P_\gamma)\) and
\[
E^\hat{G}_G(P_\gamma, P_\gamma) = E^\hat{G}_G(P_\gamma) \cong N_G(P_\gamma)/C_H(P),
\]
see [5, Definition 3.3].

Lemma 8.8. With the above assumptions, if \(R \cong T\), then there is a surjective map
\[
\Phi : N_{hU}(A)(R_j, T_l) \to F^\hat{G}_A(R_\epsilon, T_\rho),
\]
which induces a bijective map
\[
\overline{\Phi} : N_{hU}(A)(R_j, T_l)/\ker \Phi \to F^\hat{G}_A(R_\epsilon, T_\rho).
\]
The proof is an adaptation of [5, Proposition 3.5]. Let \( \Phi \) be defined by

\[
\Phi(a) = (\varphi_a, \tilde{g})
\]

for any \( a \in A^\times \cap A_{\tilde{g}} \) such that \( a(Rj)a^{-1} = Tl \), where \( \varphi_a : R \rightarrow T \) is a bijection given by the conjugation with \( a \). We can do this, since \( Rj \cong R \cong T \cong Tl \). It is clear that \( Aj \) is isomorphic to \( (Al)(\tilde{g}^{-1})_\varphi \) as \( \hat{G} \)-graded \( (A, \partial R) \)-bimodules, hence \( \Phi \) is a well-defined map.

Next we verify that \( \Phi \) is surjective. For this, let \( (\varphi, \tilde{g}) \in E^G_A(R_\varepsilon, T_\rho) \) such that there is an isomorphism \( f : Aj \rightarrow (Al)(\tilde{g}^{-1})_\varphi \) of \( \hat{G} \)-graded \( (A, \partial R) \)-bimodules. Since \( Aj \) and \( (Al)(\tilde{g}^{-1})_\varphi \) are direct summands of \( A \) as left \( A \)-modules, it follows by the Krull-Schmidt theorem that there is \( \overline{f} \in \text{Aut}_A(A) \) such that \( \overline{f}|_{Aj} = f \). Then there is \( a \in A^\times \) such that \( \overline{f}(b) = ba \) for any \( b \in A \). In particular,

\[
f : Aj \rightarrow (Aj)(\tilde{g}^{-1})_\varphi \quad f(bj) = bja
\]

for any \( b \in A \). Since \( f \) is a homomorphism of \( \hat{G} \)-graded \( (A, \partial R) \)-bimodules, we obtain that \( a \) is an homogeneous unit; that is, \( a \in A^\times \cap A_{\tilde{g}} \) for some \( \tilde{g} \in \hat{G} \). We obtain that \( aRja^{-1} = Tl \) and \( \Phi(a) = (\varphi_a, \tilde{g}) = (\varphi, \tilde{g}) \).

We return to the case of the block extension \( A = b\partial G \).

**Proposition 8.9.** Let \( T_\rho \leq P_\gamma \) and \( R_\varepsilon \leq P_\delta \) be local pointed groups on \( B = \partial Hb \) such that \( R \cong T \). Then there is a bijection

\[
E^G_A(R_\varepsilon, T_\rho) \rightarrow F^G_A(R_\varepsilon, T_\rho).
\]

**Proof.** We will show the existence of two bijections

\[
E^\hat{G}_A(R_\varepsilon, T_\rho) \rightarrow N_{hU(A)}(Rj, Tl) / \ker \Phi \rightarrow F^\hat{G}_A(R_\varepsilon, T_\rho).
\]

The second bijection exists since our algebra satisfies the hypotheses of Lemma 8.8.

For the first bijection let \( (\varphi, \tilde{g}) \in E^G_A(R_\varepsilon, T_\rho) \). Since \( R \cong T \), we obtain that \( \varepsilon R_\varepsilon = T_\rho \), hence \( \varepsilon = \rho \) and \( \varepsilon R = T \). It follows that for \( j \in \varepsilon \) and \( l \in \delta \), there is \( b_1 \in (B^T)^\times \) such that \( gjg^{-1} = b_1lb_1^{-1} \), hence \( l = b_1^{-1}gjg^{-1}b_1 \). We denote

\[
a = b_1^{-1}g
\]

(8)
which is a homogeneous invertible element. Then
\[ aRja^{-1} = b_1^{-1} gRjg^{-1}b_1 = b_1^{-1}(gR)(^gj)b_1 = b_1^{-1}T(^gj)b_1 = Tl. \]

Let\[
\Theta : E_G(R_\varepsilon, \rho) \to N_{hU(A)}(Rj, Tl) / \ker \Phi, \quad \Theta(\phi, \bar{g}) = [a]_{\ker \Phi},
\]
where \(a\) is obtained in (8) and \(\Phi\) is the map from Lemma 8.8; here \([a]_{\ker \Phi}\) is the set
\[
\{a' \in N_{hU(A)}(Rj, Tl) \mid \Phi(a) = \Phi(a')\}.
\]

Our definition of \(\Theta\) does not depend on the choice of \(b_1\), and the injectivity of \(\Theta\) follows by straightforward verification.

To show that \(\Theta\) is surjective, let \(\bar{a} \in N_{hU(A)}(Rj, Tl) / \ker \Phi\). By Lemma 8.8 we have \((\phi_{\bar{a}e}, \bar{g}) \in F_A^G(R_\varepsilon, \rho)\), and denote by \(\varphi\) the map \(\phi_{\bar{a}e}\). By Definition 8.6, since \(\varphi\) is an isomorphism, we obtain
\[
A j \cong (Al)(\bar{g}^{-1})_{\varphi}
\]
as \(\bar{G}\)-graded \((A, \varnothing R)\)-bimodules. Next we mimic the proof of [12, 7.2, 7.3] in our graded context. It follows
\[
ja j \cong (ja l)(\bar{g}^{-1})_{\varphi}
\]
as \(\bar{G}\)-graded \((\varnothing R, \varnothing R)\)-bimodules. Since \(Br_R(j) \neq 0\) and \(A = \varnothing Gb\), it follows that \(ja j\) has a direct summand isomorphic to \(\varnothing R\) as \((\varnothing R, \varnothing R)\)-bimodules, hence \(ja l\) has a direct summand isomorphic to \((\varnothing R)(\bar{g})_{\varphi^{-1}}\) as \(\bar{G}\)-graded \((\varnothing R, \varnothing T)\)-bimodules. Thus, in particular,
\[
(\varnothing R)(\bar{g})_{\varphi^{-1}} \cong \varnothing[Rx^{-1}] \cong \varnothing[x^{-1}T]
\]
for some \(x \in G\) such that \(\varphi(u) = xux^{-1}\) for any \(u \in R\). But then \(\bar{x} = \bar{g}\), and moreover, since
\[
A j \cong (Al)(\bar{g}^{-1})_{\varphi} \cong Al(\bar{g}^{-1})x
\]
as \(\bar{G}\)-graded \((A, \varnothing R)\)-bimodules, we get
\[
A jx^{-1} \cong (Al)(\bar{g}^{-1}),
\]
and thus we obtain \(xR_\varepsilon = T\rho\). In conclusion, there is \(x \in N_G(R_\varepsilon, \rho)\) such that \(\bar{x} = \bar{g}\) and \(\varphi = \varphi_x\). \(\square\)
9. Group-graded basic Morita equivalences

In the final section we prove the statements of Theorem 1.2. We show that if there is a group-graded basic Morita equivalence between two block extensions, then their extended local categories are preserved. As a consequence, we deduce the important uniqueness statement [11, Theorem 1.8] and [23, Theorem 3.5] concerning extensions of nilpotent blocks.

We start with the following two properties of fusions, motivated by the fact that a basic Morita equivalence is the composition of an embedding of $P$-interior algebras and an equivalence given by tensoring with a Dade $P$-algebra. We state them without proof, since [11, Lemma 1.17] and [17, Proposition 2.14] can be easily adapted to our graded context.

Proposition 9.1. Let $A$ be a $\tilde{G}$-graded $P$-interior algebra having an $\mathcal{O}$-basis with homogeneous elements such that $P XP = P$ and $|P \cdot x| = |P| = |x \cdot P|$ for any $x \in X$. Let $S$ be a Dade $P$-interior algebra.

Let $R_\varepsilon, T_\rho$ be local pointed groups on the 1-component $B$ of $A$, let $\varepsilon''$, $\rho''$ be the unique local points of $R$, $T$ on $S$, and let $\varepsilon'$, $\rho'$ be the unique local points of $R$, $T$ on $S \otimes B$ such that $Br^S_R(\varepsilon'') \otimes Br^B_R(\varepsilon') \subset Br^S_R(\varepsilon) \otimes Br^B_R(\rho') \subset Br^S_R(\varepsilon') \otimes Br^B_R(\rho'')$. Then we have the equality

$$F^\tilde{G}_S \otimes A(R_\varepsilon', T_\rho') = F^\tilde{G}_A(R_\varepsilon, T_\rho) \cap (F^\mathcal{O}_S(R_\varepsilon'', T_\rho'') \times \tilde{G})$$

Proposition 9.2. Let $f : A \to A'$ be an embedding of $\tilde{G}$-graded $P \simeq P'$-interior algebras. Let $R_\varepsilon, T_\rho$ be two local pointed groups on the first component $A_1$ and let $R_\varepsilon', T_\rho'$ be local pointed groups on $B'$ which correspond under the embedding $f$ such that $f(\varepsilon) \subset \varepsilon'$ and $f(\rho) \subset \rho'$. Then there exist bijections

$$E^\tilde{G}_A(R_\varepsilon, T_\rho) \to E^\tilde{G}_A(R_\varepsilon', T_\rho') \quad \text{and} \quad F^\tilde{G}_A(R_\varepsilon, T_\rho) \to F^\tilde{G}_A(R_\varepsilon', T_\rho').$$

We can now state our main result.

Theorem 9.3. With the notations of 8.1 we assume that $A$ is $\tilde{G}$-graded basic Morita equivalent to $A'$. Then $\mathcal{E}$ is equivalent to $\mathcal{E}'$.

Proof. By [2,6,4] there is a bijection between the objects of $\mathcal{E}$ and the objects of $\mathcal{E}'$ such that $R_\varepsilon$ (which is included in $P_\gamma$) is mapped into $R_\varepsilon'$ (which is included in $P'_{\gamma'}$), with $R \simeq R'$. Let $R_\varepsilon, T_\rho \leq P_\gamma$, respectively $R_\varepsilon', T_\rho' \leq P'_{\gamma'}$ be local pointed groups which correspond under the above bijection.

Since the morphisms in $\mathcal{E}, \mathcal{E}'$ are pairs given by compositions of “inclusions” and isomorphisms it is enough to prove that we have a bijection between $E^\tilde{G}_A(R_\varepsilon, T_\rho)$ and
\(E_G^G(R,v',T'_p)\) when \(R \cong T\). By using Proposition 9.1 and 9.2, the same arguments as in [18, 7.6.3] assure us that there is a bijection between \(F_A^G(R_e,T_p)\) and \(F_A^G(R'_e,T'_p)\) for any \(R \cong T\). Next, by Proposition 8.9, for any \(R \cong T\) there is a bijection from \(F_A^G(R_e,T_p)\) to \(E_G^G(R_e,T_p)\) and the same bijection exists for the pointed groups on \(B'\) from \(F_A^G(R'_e,T'_p)\) to \(E_G^G(R'_e,T'_p)\), hence our categories are equivalent.

We have already mentioned that we may consider \((A,G)\)-fusions on \(G\)-graded \(O\)-interior algebras. It is easy to see that \((G,G)\)-fusions and the category \(\mathcal{E}\) generalize to twisted group algebras of the form \(G_\alpha L\), where \(\alpha \in Z^2(L,k^x)\) (that is, to \(k^x\)-groups in Puig’s terminology), since such algebras are still \(P\)-interior. Thus, we obtain the next corollary, which is [11, Theorem 1.8] and [23, Theorem 3.5], as a consequence of Theorems 7.5 and 9.3.

**Corollary 9.4.** Assume that the block \(B\) is nilpotent. Then the categories \(\mathcal{E}_{(b,H,G)}\) and \(\mathcal{E}_{(1,\tau(Q),L)}\) are equivalent.

Finally, we show that the twisted group algebras \(G_\alpha L\) defined in Section 3 is invariant under graded basic Morita equivalences, thus proving statements b) and c) of Theorem 1.2. We denote by \(G_\alpha L'\) the twisted group algebra obtained from \(A'\).

**Corollary 9.5.** Assume that \(A\) is \(G\)-graded basic Morita equivalent to \(A'\). Then there is a isomorphism \(L \cong L'\) as extensions of \(Z(Q) \cong Z(Q')\) by \(E = N_G(Q_\delta)/C_H(Q) \cong N_G(Q'_\delta)/C_{H'}(Q')\), and with this identification, \([\alpha] = [\alpha']\) in \(H^2(E,k^x)\).

**Proof.** By Theorem 9.3, we have defect pointed groups \(Q_\delta \leq P_\epsilon\) corresponding to \(Q_\delta \leq P_\gamma\) such that \(P \cong P'\). By [5, Theorem 1.2], we have the isomorphism \(E = E_G^G(Q_\delta) \cong E_G^G(Q'_\delta)\), and there is an \(E\)-graded basic Morita equivalence between \(kN_G(Q_\delta)b_\delta\) and \(kN_G(Q'_\delta)b'_\delta\). Under this equivalence, the unique simple \(kQC_H(Q)b_\delta\)-module \(\bar{V}\) corresponds to the unique simple \(kQC_{H'}(Q')b'_\delta\)-module \(\bar{V}'\) (see 3.7). By [14, Theorem 5.1.8], the Clifford extensions of \(\bar{V}\) and \(\bar{V}'\) are isomorphic, hence \([\alpha] = [\alpha']\) in \(H^2(E,k^x)\).

By Theorem 9.3 and Corollary 9.4, we have the equivalences
\[
\mathcal{E}_{(b_\delta,QC_H(Q),N_G(Q_\delta))} \simeq \mathcal{E}_{(b'_\delta,QC_{H'}(Q'),N_G(Q'_\delta))}
\quad \text{and} \quad
\mathcal{E}_{(1,\tau(Q),L)} \simeq \mathcal{E}_{(1,\tau'(Q'),L')}
\]
of categories. Since in particular,
\[
L/\tau(Z(Q)) \cong N_G^G(Q_\delta)/C_H(Q) \simeq N_G^G(Q'_\delta)/C_{H'}(Q) \simeq L'/\tau(Z(Q')),
\]
we may now use the the argument in the final part of the proof of [23, Theorem 3.5, p. 820] and [23, Lemma 3.6] to deduce the isomorphism \(L \cong L'\) of group extensions. \(\square\)
Remark 9.6. Under the assumptions of Theorem 9.3, from the \( \tilde{E} \)-graded basic Morita equivalence between \( kN_G(Q_\delta)b_{\delta} \simeq kN_G(Q'_{\delta'})b_{\delta'} \), [4, Theorem 3.1] and Corollary 7.7, we immediately get an embedding \( k_{\alpha'}L' \to T \otimes k\alpha L \) of \( \tilde{E} \)-graded \( P \)-interior algebras, for some Dade \( P \)-algebra \( T \). Then, by [20, Lemma 4.5], we have that \( T \) is similar to \( k \).

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