INVERSE SCATTERING TRANSFORM FOR N-WAVE INTERACTION PROBLEM WITH A DISPERSIVE TERM IN TWO SPATIAL DIMENSIONS

MANSUR ISMAILOV∗,**

Abstract. In this work, we introduce a dispersive $N$-wave interaction problem ($N = 2n, n \in \mathbb{N}$) involving $n$ velocities in two spatial dimensions and one temporal dimension. Exact solutions of the problem are exhibited. This is a generalization of the $N$-wave interaction problem and matrix Davey-Stewartson equation with 2+1 dimensions that examines the Benney-type model of interactions between short and long waves. Accordingly, associated with the solutions of two dimensional analog of the Manakov system, a Gelfand-Levitan-Marchenko (GLM)-type, or so-called inversion-like, equation is constructed. It is shown that the presence of the degenerate kernel reads exact soliton-like solutions of the dispersive $N$-wave interaction problem. We also mention the unique solution of the Cauchy problem on an arbitrary time interval for small initial data.

1. Introduction

The inverse scattering transform (IST) for nonlinear evolution equations with 2+1, i.e., two spatial and one temporal dimensions has been started with the papers by Zakharov and Shabat [18, 19]. For the general case of evolutive partial differential equations in 2+1 dimensions the IST requires a novel approach, namely either a nonlocal Riemann–Hilbert (RH) [3] or a $\partial$-bar formalism [1, 4], however for certain nonlinear two-dimensional equations, the classical approach of the IST via the GLM equation is still applicable [2, 11, 12]. The IST can be employed to the initial value problem for a variety of physically significant equations which are related to the inverse scattering problem for first order systems of partial differential equations. Concrete results with a wide class of the exact solutions for various forms of Davey–Stewartson, Kadomtsev–Petviashvili equations and the $N$-wave interaction in 2 + 1 dimension, has been obtained in [7, 13] on the basis of the analysis of GLM type integral equations, in [5, 17] on the basis of the nonlocal Riemann–Hilbert problem and in [6] via the $\partial$-bar method.

This paper considers the two-dimensional spatial dispersive 2$n$-wave interaction problem with $n$ velocities which is generalized the $N$–wave interaction problem of [7] and two component Davey–Stewartson equation of [13]. This nonlinear equation admits a Lax-type representation. Therefore we use the IST via the GLM equation for its integration. As the inverse problem we set the two dimensional inverse-scattering problem for the following Manakov system, studied in detail in [8]:

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\( \frac{\partial}{\partial y} \psi - \sigma \frac{\partial}{\partial x} \psi + Q(x, y) \psi = 0, \)

where \( \sigma = \begin{pmatrix} I_n \ 0_{n \times 1} \\ 0_{1 \times n} \ -1 \end{pmatrix} \) is constant \( n \times n \) diagonal matrix with the identity matrix \( I_n \) of the order \( n \) and \( n \times 1 \) column \( 0_{n \times 1} \) and \( 1 \times n \) row \( 0_{1 \times n} \) zero vectors;
\( Q = \begin{pmatrix} 0_n & q_{12} \\ q_{21} & 0 \end{pmatrix} \) is an off-diagonal matrix with the zero matrix entry \( 0_{n \times n} \) of the order \( n \) and \( n \times 1 \) column \( q_{12} \) and \( 1 \times n \) row \( q_{21} \) vector functions.

The case \( n = 1 \) makes this system two component nonstationary Dirac system [14] (and also two dimension analogue of Zakharov-Shabat (ZS) or AKNS system [15]). In the case \( n = 2 \) this system is two dimension analogue of Manakov system [10] and for arbitrary positive integer \( n \) is the two dimension analogue of Dirac-type syystem [16]. Inverse scattering theory for the system (1.1) is satisfactorily investigated in [14] for the case \( n = 1 \) and in [8] for arbitrary \( n > 1 \). This paper use this inverse problem to solve dispersive 2n-wave interaction problem with \( n \) velocities satisfied by additionally time dependent \( Q(x, y; t) \), which is generalized 2n-wave interaction with \( n \) velocities and \( n \times n \) matrix Davey-Stewartson equation in two spatial and one temporal dimensions.

This article is organized as follows: In Section 2, the dispersive 2n-wave interaction problem with \( n \) velocities and its Lax representation is introduced. It was clear that its spectral problem is the problem two dimensional analogue of Manakov system (1.1) that the inverse scattering problem is studied in detail in [8]. Section 3, deals with the inverse problem associated with the linear equation (1.1) and the corresponding multidimensional GLM equation with explicitly solvable degenerate kernel case. The aim of the Section 4 is to apply the results of [8] to the integration of the dispersive 2n-wave interaction problem with \( n \) velocities by using the IST method: The Cauchy problem is investigated the exact soliton-like solutions are derived.

2. Dispersive 2n-wave interaction problem with \( n \) velocities and its Lax representation

Consider the system of \( N \) wave equations with \( N = 2n \) in the following form:

\[
\begin{align*}
\frac{\partial}{\partial t} q_k + \alpha_k \frac{\partial}{\partial x} q_k + \beta_k \frac{\partial}{\partial y} q_k - i \gamma \frac{\partial^2}{\partial x \partial y} q_k &= \sum_{m=1}^{n} p_{km} q_m - pq_k, \\
\frac{\partial}{\partial t} q_{n+k} + \alpha_k \frac{\partial}{\partial x} q_{n+k} + \beta_k \frac{\partial}{\partial y} q_{n+k} + i \gamma \frac{\partial^2}{\partial x \partial y} q_{n+k} &= -\sum_{m=1}^{n} p_{mk} q_{n+m} + pq_{n+k},
\end{align*}
\]

(2.1)

where \( \alpha_k, \beta_k \) and \( \gamma \) are real numbers with \( \beta_k - \alpha_k \neq \beta_j - \alpha_j \) and \( \beta_k + \alpha_k = \beta_j + \alpha_j \) when \( k \neq j \). The functions \( p_{km} \) and \( p \) are solutions of the equations

\[
\frac{\partial}{\partial \xi} p = -i \gamma \sum_{m=1}^{n} \frac{\partial}{\partial x} (q_m q_{n+m}), \quad \frac{\partial}{\partial \eta} p_k = -i \gamma \frac{\partial}{\partial x} (q_k q_{n+k}), \quad k = 1, \ldots, n,
\]

\[
\frac{\partial}{\partial \eta} p_{km} = (\beta_m - \beta_k) q_k q_{n+m} - i \gamma \frac{\partial}{\partial x} (q_k q_{n+m}), \quad m, k = 1, \ldots, n; \quad m \neq k.
\]

(2.2)
where \( \frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \frac{\partial}{\partial x}, \frac{\partial}{\partial y} = \frac{\partial}{\partial y} - \frac{\partial}{\partial x}. \)

This system (2.1) is the 2+1 dimensional \( N \)-wave interaction problem with the dispersive term \( i\gamma \frac{\partial^2}{\partial x \partial y} q_k \) and also with the quasi-potential (2.2). The aim of this paper is the integrability of this system by using the suitable method of inverse scattering transform. The case when the terms \( \alpha_k \frac{\partial}{\partial x} q_k + \beta_k \frac{\partial}{\partial y} q_k \) absence the equation (2.1) becomes to 2+1 dimensional nonlinear Schrodinger equation and the IST method for its integration is realized in [13] for \( n = 1 \) by using the inverse scattering problem (ISP) for two component nonstationary Dirac equation, [14]. The undispersive system (2.1) when \( \gamma = 0 \), can be integrate by using IST method in [7, 9] which the integrate ISP is matrix nonstationary Dirac system of \( n + 1 \) components with \( n > 1 \), [8].

Generally, in physical systems, waves with different length scales appear. They are examine the interactions between waves for certain model partial differential equations. In Benney model [20], the equations (2.1), (2.2) examines the interactions between short and long waves, where \( p \) is the long wave profile and \( q \) is, to leading order, the short wave envelope. The constants \( \alpha_k \) and \( \beta_k \) are the group velocities of the short waves, \( \gamma \) is due to the linear dispersion.

Let us denote \( q_{12} = \text{col} \{ q_1, \ldots, q_n \}, q_{21} = \text{col} \{ q_{n+1}, \ldots, q_{2n} \}^T, P = (p_{km})_{k,m=1}^n \). Then, the equation (2.1) and (2.2) are reduced to the matrix system

\[
\partial_t q_{12} - B \partial_x q_{12} + b \partial_y q_{12} - i\gamma(q_{12})_{xy} = P q_{12} - \rho q_{12},
\]

\[
\partial_t q_{21} - \delta_k q_{21} B + b \partial_y q_{21} + i\gamma(q_{21})_{xy} = \rho q_{21} - q_{21} P,
\]

and

\[
\partial_y P = [q_{12} q_{21}, B] - i\gamma q_{12} q_{21},
\]

respectively, where \( \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \partial_y = \frac{\partial}{\partial y} - \frac{\partial}{\partial x}, B = \text{diag}(b_1, \ldots, b_n) \) with \( b_k = \frac{\beta_k - \alpha_k}{2} \) and \( b = -\frac{\beta_1 + \alpha_2}{2} \).

Let \( M \) and \( A \) be first order matrix operators:

\[
M = \sigma \frac{\partial}{\partial x} + Q, \quad A = \delta \frac{\partial}{\partial x} + i\gamma I_n \frac{\partial^2}{\partial x^2} + \Gamma.
\]

Then the equation (2.3), (2.4) admits the following Lax representation:

\[
\left[ \frac{\partial}{\partial t} - M, \frac{\partial}{\partial y} - A \right] = 0.
\]

Here \( \delta \) and \( \Gamma \) are \( n + 1 \)-th (\( n \geq 2 \)) order square matrices. The matrix \( \delta \) be a real and diagonal: \( \delta = \begin{pmatrix} 2B & 0_{n \times 1} \\ 0_{1 \times n} & 2b \end{pmatrix} \), where \( B = \text{diag}(b_1, \ldots, b_n) \) with \( b_k \neq b_j \neq b \)

when \( k \neq j \) and \( \Gamma \) is the following form \( \Gamma = \begin{pmatrix} P & (B - b I_n) q_{12} \\ q_{21} (B - b I_n) & P \end{pmatrix} \)

\( \) that obey the relation \( [\sigma, \Gamma] = [\delta, Q] + 2i\gamma Q_x. \)

Let us denote \( L = \frac{\partial}{\partial y} - M \) and \( D = \frac{\partial}{\partial \eta} - A. \)

**Lemma 1.** Let \( \psi \) be a solution of the system (1.1), whose the coefficients \( q_{12} \) and \( q_{21} \) satisfy system (2.3). Then the function \( \phi = D \psi \) also satisfy the system (1.1).
Proof. From (2.5) we obtain:

$$(LD - DL)\psi = L(D\psi) - D(L\psi) = 0.$$

Since $L\psi = 0$, then

$$L(D\psi) = 0.$$

It means that $D\psi$ is solution of system (1.1). □

3. Manakov-type systems on the plane

Let us consider the system (1.1) on the plane $-\infty < x, y < +\infty$ with the matrix function $q_{12}$ and $q_{21}$ has measurable complex-valued rapidly decreasing (Schwartz) entries. Notice that if the potential is independent on $y$, then by taking $\psi(x, y) = \psi(x) \exp(i\lambda y)$, we can convert equation (1.1) into the Manakov system given by

$$-\sigma \frac{d}{dx} \psi(x) + Q(x)\psi(x) = i\lambda \psi(x)$$

which is considered in [10].

Throughout this chapter the following notations will be used:

◮ We partition $(n + 1) \times (n + 1)$ matrix $A$ as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11}$ is $n \times n$ matrix $A_{12}$ is and $n \times 1$ column vector, $A_{21}$ is $1 \times n$ row vector and $A_{22}$ is scalar.

◮ $F_x$ denotes the $(n + 1) \times (n + 1)$ diagonal matrix shift operator, such that for a $n + 1$ dimensional vector function $h(t)$

$$F_x h(y) = \begin{pmatrix} h_1(y + x) \\ h_2(y - x) \end{pmatrix}$$

where $h_1(y)$ is a vector function that consists of the first $n$ component of vector $h(y)$, $h_2(y)$ is a scalar function.

◮ We denote

$$A_\pm (x) h(x, y) = \mp \int_y^{+\infty} A_\pm (x, y, \tau) h(x, \tau) d\tau$$

by the upper and lower Volterra integral operators.

For any $a_\pm(y) \in L_2(\mathbb{R}, \mathbb{C}^{n+1})$ there exist unique solutions in $L_2(\mathbb{R}^2, \mathbb{C}^{n+1})$ of the systems (1.1) with the conditions $\psi(x, y) = F_x a_\pm(y) + o(1), \ y \rightarrow \pm \infty$ and these solutions admit the representations

$$(3.1)\quad \psi(x, y) = (I + A_\pm(x)) F_x a_\pm(y),$$

where $I$ is identity operator and the kernels $A_\pm(x, y, \tau) = \begin{pmatrix} A_{11}^\pm(x, y, \tau) & A_{12}^\pm(x, y, \tau) \\ A_{21}^\pm(x, y, \tau) & A_{22}^\pm(x, y, \tau) \end{pmatrix}$

of the integral operators $A_\pm(x)$ are uniquely determined by the coefficients of the system (1.1), and for the fixed $x$, these kernels are the Hilbert-Schmidt kernels. In addition, these kernels are connected with the potential by the following equalities
where matrix elements of the kernel of matrix integral operator $K_{B\mined by A}(x)$, $A(x,y,y) = \pm 1/2 q_{12}(x,y), A_{21}(x,y,y) = \mp 1/2 q_{21}(x,y)$.

An operator $S$ transforming the given incident waves $a_{-}(y) \in L_{2}(\mathbb{R}, \mathbb{C}^{n+1})$ into the scattered waves $a_{+}(y) \in L_{2}(\mathbb{R}, \mathbb{C}^{n+1})$ is called the scattering operator for the system (1.1) on the plane:

$$a_{+}(y) = S a_{-}(y)$$

where $a_{+}(y) = a_{-}(y) + \int_{-\infty}^{+\infty} f_{y-x-s}(Q\psi)(x,s) ds$. Operator $S$ is $(n+1) \times (n+1)$ matrix linear operator on the space $L_{2}(\mathbb{R}, \mathbb{C}^{n+1})$.

From the representations (3.1) it follows that the next factorization results for $S$. For every $x$, the operator $F_{x}Sf_{-x}$ admits factorizations

$$F_{x}Sf_{-x} = (I + A_{-}(x))^{-1}(I + A_{+}(x)),$$

We can analogously introduce the next representations corresponding to asymptotics $\psi(x,y) = F_{x}b_{y}(y) + o(1), x \rightarrow -\infty$ and $\psi(x,y) = F_{x}b_{y}(y) + o(1), x \rightarrow +\infty$.

For any $b_{y}(y) \in L_{2}(\mathbb{R}, \mathbb{C}^{n})$ there exist unique solutions in $L_{2}(\mathbb{R}^{2}, \mathbb{C}^{n})$ of the systems (\ref{system}) and these solutions admit the representations

$$\psi(x,y) = (I + B_{y}(y)) F_{x}b_{y}(y),$$

where the kernels $B_{\pm}(x,y,\tau) = \left(\begin{array}{cc} B_{11}^{\pm}(x,y,\tau) & B_{12}^{\pm}(x,y,\tau) \\ B_{21}^{\pm}(x,y,\tau) & B_{22}^{\pm}(x,y,\tau) \end{array}\right)$ of the integral operators $B_{\pm}(x)$ are uniquely determined by the coefficients of the system (1.1) and these kernels are the Hilbert-Schmidt kernels for fixed $x$. In addition, these kernels are connected with the potential by the following equalities

$$B_{12}^{\pm}(x,y,x) = \pm 1/2 q_{12}(x,y), B_{21}^{\pm}(x,y,x) = \mp 1/2 q_{21}(x,y).$$

From the representation (3.4) follows the next factorization results for $S$. For every $x$, the operator $F_{x}Sf_{-x}$ admits factorizations

$$F_{x}Sf_{-x} = (I + K_{-}(x))^{-1}(I + K_{+}(x))$$

where matrix elements of the kernel of matrix integral operator $K_{\pm}(x)$ are determined by $B_{\pm}(y)$ as follows

$$K_{11}^{\pm}(x,y,\tau) = B_{11}^{\pm}(x,y,x-y+\tau), \quad K_{12}^{\pm}(x,y,\tau) = B_{12}^{\pm}(x,y,x+y-\tau), i = 1,2.$$  

It is possible the unique restoration of the potential by scattering operator.

Let $S$ be the scattering operator for the system (1.1) on the plane with the potential $Q(x,y)$ belonging to the Schwartz class. Then the potential $Q(x,y)$ is uniquely determined by the known scattering operator $S$. The ISP is solved with the following steps:

1) Construct the operator $F_{x}Sf_{-x}$;

2) Find the factorization factors $A_{-}(x)$ and $A_{+}(x)$ from the (3.3), since $F_{x}Sf_{-x}$ admits left factorization;

3) Find the matrix coefficients of the system (1.1) with respect to the kernels $A_{\pm}(x,y,\tau)$ of the operators $A_{\pm}(x)$, by formula (3.2).
Let us denote the kernels of the matrix integral operators \( S - I \) and \( S^{-1} - I \) as \( F(y, \tau) \) and \( G(y, \tau) \), respectively. Let \( F(y, \tau) = \begin{pmatrix} F_{11}(y, \tau) & F_{12}(y, \tau) \\ F_{21}(y, \tau) & F_{22}(y, \tau) \end{pmatrix} \) and \( G(t, \tau) = \begin{pmatrix} G_{11}(y, \tau) & G_{12}(y, \tau) \\ G_{21}(y, \tau) & G_{22}(y, \tau) \end{pmatrix} \). We will call the collection of functions \( \{ F_{12}(y, \tau), G_{21}(y, \tau) \} \) as the scattering data for the system (1.1).

Let us denote the kernels of the integral operators \( B \) and \( S \). By the right factorization (3.6) of \( F_x S F_{-x} - I \) and \( F_x S^{-1} F_{-x} - I \) as \( F(x, y, \tau) \) and \( G(x, y, \tau) \), respectively. It is clear that \( F(0, y, \tau) = F(y, \tau) \) and \( G(0, y, \tau) = G(y, \tau) \).

Therefore, we obtain the following Gelfand- Levitan-Marchenko type matrix integral equations from (3.6).

\[
K_{12}^-(x, y, \tau) - \int_{-\infty}^{y} \left[ \int_{y}^{+\infty} K_{12}^-(x, y, z) G_{21}(z - x, s + x) dz \right] F_{12}(s + x, \tau - x) ds = F_{12}(y + x, \tau - x), \quad \tau \geq y,
\]

(3.8)

\[
K_{21}^+(x, y, \tau) - \int_{y}^{+\infty} \left[ \int_{-\infty}^{y} K_{21}^+(x, y, z) F_{12}(z + x, s - x) dz \right] G_{21}(s - x, \tau + x) ds = G_{21}(y - x, \tau + x), \quad \tau \leq y,
\]

By the right factorization (3.6) of \( F_x S F_{-x} \) there exist unique solutions of these equations.

Considering the relationships (3.5) between the potential \( Q(x, t) \) and the operators \( B_{\pm}(y) \), and also (3.7) between the operators \( B_{\pm}(y) \) and \( K_{\pm}(x) \) we obtain the following results for the ISP in the plane:

Let \( S = I + F \) be the scattering operator for the system (1.1) on the whole plane. Then there exists \( S^2 = I + G \), where \( F \) and \( G \) are the Hilbert-Schmidt matrix integral operators. Let us partition \( F = (F_{ij})_{i,j=1}^2 \), \( G = (G_{ij})_{i,j=1}^2 \) and let the kernels of the operators \( F_{12} \) and \( G_{21} \) be given. Then there exists a unique solution of the system of integral equations (3.8) and the solution of this system determines the potential by formulae

\[
q_{12}(x, y) = -2K_{12}^-(x, y, y), \quad q_{21}(x, y) = -2K_{21}^+(x, y, y).
\]

(3.9)

Thus, for the system (1.1) with the coefficient \( Q(x, y) \) there is scattering operator \( S \) with the scattering data \( F_{12} \) and \( G_{21} \) which are the Hilbert-Schmidt integral operators with the kernels \( F_{12}(x, y) \) and \( G_{21}(x, y) \) that decrease quite fast with respect to variables at infinity. This defines the mapping of the scattering data \( \{ q_{12}(x, y), \sigma_{21}^T(x, y) \} \) to \( \{ F_{12}(x, y), G_{21}^T(x, y) \} \).

This operator mapping coefficients of the system (1.1) into the scattering data is continuous in \( L_2 \) and its inverse \( \Pi^{-1} \) exists and is continuous and its action can be constructively described by means of the uniquely solvable of systems (3.8).
4. Inverse scattering method

To integrate the Cauchy problem for the system (2.1) with the initial condition
\begin{equation}
q_k(x, y, t)|_{t=0} = q_k^0(x, y), \quad k = 1, \ldots, 2n
\end{equation}
by the inverse scattering method. We use the ISP for the system (1.1) with the potential \( Q^0(x, y) = \begin{pmatrix} 0_n & q_{12}^0(x, y) \\ q_{21}^0(x, y) & 0 \end{pmatrix} \), where \( q_{12}^0 = \text{col} \{q_{11}^0, \ldots, q_{n}^0\} \), \( q_{21}^0 = \text{col} \{q_{n+1}^0, \ldots, q_{2n}^0\} \) in the whole plane, that is given in Chapter 3. Let \( F_{12}^0(x, y) \) and \( G_{21}^0(x, y) \) are the scattering data for the system (1.1) with the coefficient \( Q^0(x, y) \) which decrease quite fast with respect to variables at infinity. This defines the mapping of the scattering data
\[
q^0 = \left[ \begin{array}{c} q_{12}^0 \\ q_{21}^0 \end{array} \right]^T \mapsto \left[ \begin{array}{c} F_{12}^0 \\ G_{21}^0 \end{array} \right]^T.
\]

Let us investigate the evolution of this scattering data, when the coefficients of the operator \( L \) satisfy the equations (2.3).

The pair \( \{F_{12}, G_{21}\} \) is denoted the scattering data correspond to operator \( L \) with the coefficients \( q_{12}(x, y; t) \) and \( q_{21}(x, y; t) \) which are satisfy the system of equation (2.3).

**Theorem 1.** Let the coefficients \( q_{12} \) and \( q_{21} \) of the system (1.1) depend on \( t \) as a parameter and satisfy the system of equation (2.3). Besides that \( P(x, +\infty) = 0, \quad p(x, -\infty) = 0^- \).

Then the kernels \( F_{12}(y, \tau; t), G_{21}(y, \tau; t) \) of the integral operators \( F_{12}, G_{21} \) corresponding to the scattering operator \( S \) for the system (1.1) on the plane satisfy the system of equations (4.5) and (4.6).

**Proof.** By virtue of definition of the scattering operator \( S \) we get
\begin{equation}
\varphi_+ = S \varphi_-,
\end{equation}
where \( \varphi_\pm = P_\pm a_\pm \), \( P_\pm = \frac{\partial}{\partial x} - \delta \sigma \frac{\partial}{\partial y} - i\gamma I_\sigma \frac{\partial}{\partial y} \), \( P(x, \pm\infty) = 0, \quad p(x, \pm\infty) = 0. \)

Since \( a_+ = Sa_- \), \( S \) from (4.2) we obtain:
\begin{equation}
P_+ S = SP_-.
\end{equation}

Analogously,
\begin{equation}
P_- S^{-1} = S^{-1} P_+.
\end{equation}

Since \( S = I + F \) and \( S^{-1} = I + G \), where \( Ff(y) = \int_{-\infty}^{+\infty} F(y, \tau; t)f(\tau)d\tau, \quad F(y, \tau; t) = (F_{ij}(y, \tau; t) + 2)_{i,j=1} \)
\begin{equation}
\left( F_{ij}(y, \tau; t) + 2 \right)_{i,j=1} \quad \text{and} \quad Gf(y) = \int_{-\infty}^{+\infty} G(y, \tau; t)f(\tau)d\tau, \quad G(y, \tau; t) = (G_{ij}(y, \tau; t) + 2)_{i,j=1},
\end{equation}
from the matrix operator equation (4.3) it follows that the kernels of the integral operator \( F_{12} \) satisfy the equation
\begin{equation}
\frac{\partial}{\partial t} F_{12} - 2 \left( B \frac{\partial}{\partial y} F_{12} - b \frac{\partial}{\partial r} F_{12} \right) - i\gamma \left( \frac{\partial^2}{\partial y^2} F_{12} - \frac{\partial^2}{\partial r^2} F_{12} \right) = 0,
\end{equation}
The similarly equation for the kernels of the integral operator $G_{21}$ follows from the matrix operator equation (4.4):

\[
\left(\frac{\partial}{\partial t}G_{21} + 2 \left( b \frac{\partial}{\partial y}G_{21} - \frac{\partial}{\partial \tau}G_{21}B \right) \right) + i\gamma \left( \frac{\partial^2}{\partial y^2}G_{21} - \frac{\partial^2}{\partial \tau^2}G_{21} \right) = 0.
\]

\[
\square
\]

Now, let us give a procedure for the solution of the system (2.3) by inverse scattering method.

**Theorem 2.** Let functions $F_{12}(y, \tau; t)$ and $G_{21}(y, \tau; t)$ satisfy the equations (4.5) and (4.6) and these functions together with their derivatives with respect to $t$ and their first and second derivatives with respect to $y$ and $\tau$ belong to $L_2(R^2)$. Then the equations (4.7) are uniquely solvable and the functions

\[
q_{12}(x, y; t) = -2K^-(x, y, y; t), \quad q_{21}(x, y; t) = -2K^+(x, y; t),
\]

is the solution of the non-linear equation (2.3).

**Proof.** If the coefficients of the system (1.1) depend on $t$ as a parameter and satisfy the system of equation (2.3), then the kernels $F_{12}(y, \tau; t)$, $G_{21}(y, \tau; t)$ of the integral operators $F_{12}$, $G_{21}$ satisfy the system of equations (4.5), (4.6). In addition, the coefficients $q_{12}$, $q_{21}$ of the system (1.1) is uniquely determined by (3.9) and the analogues of the equations (3.8)

\[
K^-(x, y, \tau; t) - \int_{-\infty}^{y} K^-(x, y, z; t) \left( \int_{y}^{+\infty} G_{21}(z - x, s + x; t) F_{12}(s + x, \tau - x; t) ds \right) dz = F_{12}(y + x, \tau - x; t), \quad \tau \geq y,
\]

\[
K^+(x, y, \tau; t) - \int_{y}^{+\infty} K^+(x, y, z; t) \left( \int_{-\infty}^{y} F_{12}(z + x, s - x; t) G_{21}(s - x, \tau + x; t) ds \right) dz = G_{21}(y - x, \tau + x; t), \quad \tau \leq y,
\]

which are constructed by kernels $F_{12}(y, \tau; t)$, $G_{21}(y, \tau; t)$ are uniquely solved by $K^-(x, y, \tau)$, $K^+(x, y, \tau)$

The statement of this theorem is equivalent to the assertion that the function

\[
q = \left[ \begin{array}{cc} q_{12}^T \\ (q_{21})^T \end{array} \right] = \Pi^{-1}e^{-iA^T}\Pi q^0
\]

is the solution of equation (2.3) with the initial condition (4.1) if it is determined, where

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A_1 = 2i \begin{bmatrix} B_1 \frac{\partial}{\partial y} - b_1 \frac{\partial}{\partial \tau} \end{bmatrix} - \gamma I_n \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \tau^2} \right),
\]

\[
A_2 = 2i \begin{bmatrix} B_2 \frac{\partial}{\partial y} - b_2 \frac{\partial}{\partial \tau} \end{bmatrix} + \gamma I_n \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \tau^2} \right).
\]

It is well known that $\Pi q^0 = \left[ \begin{array}{c} F_{12}^0 \\ (G_{21}^0)^T \end{array} \right]$ is direct problem of determining scattering data, $e^{-iA^T}\Pi q^0$ is the evolution of scattering data and $\Pi^{-1}e^{-iA^T}\Pi q^0$ is the inverse scattering problem of finding $q$.

The function $q(x, y; t)$ these functions together with their derivatives with respect to $t$ and their first and second derivatives with respect to $x$ and $y$ belong to $L_2(R^2)$ is called the solution of Cauchy problem (1.1), (4.1) if at $t = 0$ it coincides with the initial data $q(x, y, 0) = q^0(x, y)$. 

\[\square\]
Theorem 3. The solution of Cauchy problem (1.1), (1.4) is unique. The solution of this Cauchy problem exists on an arbitrary interval of time for small initial data.

Proof. The uniqueness of the solution follows from the possibility of representing it in the form (4.8) that the assumption of the existence of solution requires the representation (4.8) which is expressed by initial data. If the initial data \( q^0 \) are sufficiently small in (4.8) then this formula has a sense by virtue of continuity of \( \Pi \) and unitarity of \( e^{-i\mathbf{A}t} \) that \( \|e^{-i\mathbf{A}t}\Pi q\| \) is less than 1. \( \square \)

5. Exact soliton-like solutions of the dispersive 4-wave interaction problem

Let \( n = 2 \) in (1.1) and

\[
Q = \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}, \quad q_{12} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad q_{21} = \begin{pmatrix} q_3 & q_4 \end{pmatrix}.
\]

It is easy to see that the scattering data corresponding to the potential (5.1) is in the form of

\[
F_{12}(y, \tau; t) = \begin{pmatrix} f_{11}(y, \tau; t) \\ f_{21}(y, \tau; t) \end{pmatrix}, \quad G_{21}(y, \tau; t) = \begin{pmatrix} g_{11}(y, \tau; t) & g_{12}(y, \tau; t) \end{pmatrix}.
\]

The nonlinear system of equations (2.1) becomes to the form

\[
\begin{align*}
\partial_t q_1 + \alpha_1 \partial_x q_1 + \beta_1 \partial_y q_1 - i\gamma \partial^2_{xy} q_1 &= (p_{11} - p) q_1 + p_{12} q_2, \\
\partial_t q_2 + \alpha_2 \partial_x q_2 + \beta_2 \partial_y q_2 - i\gamma \partial^2_{xy} q_2 &= p_{21} q_1 + (p_{22} - p) q_2, \\
\partial_t q_3 + \alpha_1 \partial_x q_3 + \beta_1 \partial_y q_3 + i\gamma \partial^2_{xy} q_3 &= (p - p_{11}) q_3 - p_{21} q_4, \\
\partial_t q_4 + \alpha_2 \partial_x q_4 + \beta_2 \partial_y q_4 + i\gamma \partial^2_{xy} q_4 &= (p - p_{22}) q_4 - p_{12} q_3,
\end{align*}
\]

where

\[
\begin{align*}
\frac{\partial}{\partial x} p &= -i\gamma \frac{\partial}{\partial x} (q_1 q_3) - i\gamma \frac{\partial}{\partial x} (q_2 q_4), \\
\frac{\partial}{\partial \eta} p_{11} &= -i\gamma \frac{\partial}{\partial \eta} (q_1 q_3), \quad \frac{\partial}{\partial \eta} p_{22} = -i\gamma \frac{\partial}{\partial \eta} (q_2 q_4), \\
\frac{\partial}{\partial \eta} p_{km} &= (\beta_m - \beta_k) q_k q_{2m} - i\gamma \frac{\partial}{\partial x} (q_k q_{2m}), \quad m, k = 1, 2; \ m \neq k.
\end{align*}
\]

In the case \( P^+(x) = p^-(x) = 0 \), these derivatives comes to form

\[
\begin{align*}
p &= i\frac{\gamma}{2} (q_1 q_3) + \frac{\gamma}{2} (q_2 q_4) - i\gamma \frac{1}{2} \int_{-\infty}^{\xi} \left[ \frac{\partial}{\partial \eta} (q_1 q_3) + \frac{\partial}{\partial \eta} (q_2 q_4) \right] ds, \\
p_{11} &= i\frac{\gamma}{2} (q_1 q_3) - \frac{\gamma}{2} \int_{\eta}^{+\infty} \frac{\partial}{\partial \xi} (q_1 q_3) d\tau, \quad p_{22} = i\frac{\gamma}{2} (q_2 q_4) - \frac{\gamma}{2} \int_{\eta}^{+\infty} \frac{\partial}{\partial \xi} (q_2 q_4) d\tau, \\
p_{km} &= \frac{\beta_m - \beta_k}{2} q_k q_{2m} + i\frac{\gamma}{2} (q_k q_{2m}) - i\gamma \frac{1}{2} \int_{\eta}^{+\infty} \frac{\partial}{\partial \xi} (q_k q_{2m}) d\tau, \quad m, k = 1, 2; \ m \neq k.
\end{align*}
\]
and after the elimination of $p$ and $p_{km}$, the system (5.2) represents a system of integro-differential equations.

The evolution of the scattering data are in the following form according to (4.5) and (4.6):

\[
\begin{align*}
\partial_t f_{11} - 2 b_1 \partial_y f_{11} + 2 b \partial_x f_{11} - i \gamma (\partial_y^2 f_{11} - \partial_x^2 f_{11}) &= 0, \\
\partial_t f_{21} + 2 b_2 \partial_y f_{21} - 2 b \partial_x f_{21} - i \gamma (\partial_y^2 f_{21} - \partial_x^2 f_{21}) &= 0, \\
\partial_t g_{11} + 2 b \partial_y g_{11} - 2 b_1 \partial_x g_{11} + i \gamma (\partial_y^2 g_{11} - \partial_x^2 g_{11}) &= 0, \\
\partial_t g_{12} + 2 b \partial_y g_{12} - 2 b_2 \partial_x g_{12} + i \gamma (\partial_y^2 g_{12} - \partial_x^2 g_{12}) &= 0.
\end{align*}
\]

(5.5)

We deduce explicit solutions of the system (5.2) by using the formulas for the exactly solvable case of the inverse-scattering problem for the system (1.1). We get an elementary example for $F_{12}(y, \tau) = \left( f_1 (y; t) f_2 (\tau; t), f_3 (y; t) f_4 (\tau; t) \right)$, $G_{21}(y, \tau) = \left( g_1 (y; t) g_2 (\tau; t), g_3 (y; t) g_4 (\tau; t) \right)$, where the functions $f_k$ and $g_k$ satisfy the equations

\[
\begin{align*}
\partial_t f_1 - 2 b_1 \partial_y f_1 - i \gamma \partial_y^2 f_1 &= 0, & \partial_t f_2 + 2 b \partial_x f_2 + i \gamma \partial_x^2 f_2 &= 0, \\
\partial_t f_3 - 2 b_2 \partial_y f_3 - i \gamma \partial_y^2 f_3 &= 0, & \partial_t f_4 + 2 b \partial_x f_4 + i \gamma \partial_x^2 f_4 &= 0, \\
\partial_t g_1 + 2 b_1 \partial_y g_1 + i \gamma \partial_y^2 g_1 &= 0, & \partial_t g_2 + 2 b \partial_x g_2 - i \gamma \partial_x^2 g_2 &= 0, \\
\partial_t g_3 + 2 b_2 \partial_y g_3 + i \gamma \partial_y^2 g_3 &= 0, & \partial_t g_4 - 2 b \partial_x g_4 - i \gamma \partial_x^2 g_4 &= 0.
\end{align*}
\]

(5.6)

Let $K_{12}^{-} = \left[ \begin{array}{c} K_1^{-} \\ K_2^{-} \end{array} \right]$, $K_{21}^{+} = \left[ \begin{array}{cc} K_1^{+} & K_2^{+} \end{array} \right]$ in (4.7). Then

\[
\begin{align*}
K_1^{-} (x, y, \tau; t) &= a_{11}(x, y; t) f_2 (\tau - x; t) f_1 (y + x; t) + a_{12}(x, y; t) f_4 (\tau - x; t) f_1 (y + x; t), \\
K_2^{-} (x, y, \tau; t) &= a_{11}(x, y; t) f_3 (y + x; t) f_2 (\tau - x; t) + a_{22}(x, y; t) f_3 (y + x) f_4 (\tau - x),
\end{align*}
\]

(5.7)

where

\[
\begin{align*}
a_{11}(x, y; t) &= \frac{1 - \alpha_{34} \alpha_{43}}{1 - \alpha_{12} \alpha_{21} - \alpha_{34} \alpha_{43} + \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{34} \alpha_{41} \alpha_{12} \alpha_{23}}, \\
a_{12}(x, y; t) &= \frac{\alpha_{34} \alpha_{23}}{1 - \alpha_{12} \alpha_{21} - \alpha_{34} \alpha_{43} + \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{34} \alpha_{41} \alpha_{12} \alpha_{23}}, \\
a_{21}(x, y; t) &= \frac{\alpha_{12} \alpha_{41}}{1 - \alpha_{12} \alpha_{21} - \alpha_{34} \alpha_{43} + \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{34} \alpha_{41} \alpha_{12} \alpha_{23}}, \\
a_{22}(x, y; t) &= \frac{1 - \alpha_{12} \alpha_{21}}{1 - \alpha_{12} \alpha_{21} - \alpha_{34} \alpha_{43} + \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{34} \alpha_{41} \alpha_{12} \alpha_{23}}.
\end{align*}
\]

with

\[
\begin{align*}
\alpha_{21} &= \int_{y}^{+\infty} f_2 (s - x; t) g_1 (s - x; t) \, ds, & \alpha_{41} &= \int_{y}^{+\infty} f_4 (s - x; t) g_1 (s - x; t) \, ds, \\
\alpha_{23} &= \int_{y}^{+\infty} f_2 (s - x; t) g_3 (s - x; t) \, ds, & \alpha_{43} &= \int_{y}^{+\infty} f_4 (s - x; t) g_3 (s - x; t) \, ds,
\end{align*}
\]
the scattering data in the degenerate form of (5.6), exists and it is in the following

$$\alpha_{12} = \int_{-\infty}^{y} g_2 (s + x) f_1 (s + x) ds, \quad \alpha_{34} = \int_{-\infty}^{y} g_4 (s + x; t) f_3 (s + x; t) ds,$$

$$\alpha_{12} = \int_{-\infty}^{y} g_2 (s + x; t) f_1 (s + x; t) ds, \quad \alpha_{34} = \int_{-\infty}^{y} g_4 (s + x; t) f_3 (s + x; t) ds$$

and

$$K_1^+(x, y, \tau; t) = b_{11}(x, y; t) g_2(\tau + x; t) g_1(y - x; t) + b_{12}(x, y; t) g_2(\tau + x; t) g_3(y - x; t),$$

$$K_2^+(x, y, \tau; t) = b_{21}(x, y; t) g_4(\tau + x; t) g_1(y - x; t) + b_{22}(x, y; t) g_4(\tau + x; t) g_3(y - x; t),$$

where

$$b_{11}(x, y; t) = \frac{1 - \beta_{34} \beta_{43} + \beta_{12} \beta_{23} \beta_{34} \beta_{41}}{1 - \beta_{12} \beta_{21} - \beta_{34} \beta_{43} + \beta_{12} \beta_{21} \beta_{34} \beta_{43}},$$

$$b_{12}(x, y; t) = \frac{\beta_{34} \beta_{41}}{1 - \beta_{12} \beta_{21} - \beta_{34} \beta_{43} + \beta_{12} \beta_{21} \beta_{34} \beta_{43}},$$

$$b_{21}(x, y; t) = \frac{1 - \beta_{12} \beta_{21} - \beta_{34} \beta_{43} + \beta_{12} \beta_{21} \beta_{34} \beta_{43}}{1 - \beta_{12} \beta_{21} - \beta_{34} \beta_{43} + \beta_{12} \beta_{21} \beta_{34} \beta_{43}},$$

$$b_{22}(x, y; t) = \frac{1 - \beta_{12} \beta_{21} + \beta_{12} \beta_{23} \beta_{34} \beta_{41}}{1 - \beta_{12} \beta_{21} - \beta_{34} \beta_{43} + \beta_{12} \beta_{21} \beta_{34} \beta_{43}}$$

with

$$\beta_{12} = \int_{-\infty}^{y} g_2(s + x; t) f_1(s + x; t) ds, \quad \beta_{32} = \int_{-\infty}^{y} g_2(s + x; t) f_3(s + x; t) ds,$$

$$\beta_{14} = \int_{-\infty}^{y} g_4(s + x; t) f_1(s + x; t) ds, \quad \beta_{34} = \int_{-\infty}^{y} g_4(s + x; t) f_3(s + x; t) ds,$$

$$\beta_{21} = \int_{y}^{+\infty} f_2(s - x; t) g_1(s - x; t) ds, \quad \beta_{41} = \int_{y}^{+\infty} f_4(s - x; t) g_1(s - x; t) ds,$$

$$\beta_{23} = \int_{y}^{+\infty} f_2(s - x; t) g_3(s - x; t) ds, \quad \beta_{43} = \int_{y}^{+\infty} f_4(s - x; t) g_3(s - x; t) ds.$$

Thus, the explicit solution of the equation (5.2) with the potential (5.1) having the scattering data in the degenerate form of (5.6), exists and it is in the following form:

$$q_1(x, y, t) = -2K_1^-(x, y, t), \quad q_2(x, y, t) = -2K_2^-(x, y, t),$$

$$q_3(x, y, t) = -2K_1^+(x, y, t), \quad q_4(x, y, t) = -2K_2^+(x, y, t),$$

where $K_1^-(x, y, \tau; t)$ and $K_2^-(x, y, \tau; t)$ are determined by (5.7), $K_1^+(x, y, \tau; t)$ and $K_2^+(x, y, \tau; t)$ are determined by (5.8).

6. Conclusion

In this paper, we consider the inverse scattering method for a generalization of both systems, the Davey-Stewartson system in two spatial dimensions and 2n-wave interaction problem with n velocities that examines the Benney-type model of interactions between short and long waves. We show the existence of a unique solution for Cauchy problem on an arbitrary interval of time for small initial data and also the multi-solitons corresponding to degenerate kernels of GLM-type equation that is associated with solutions of the two dimensional analogue of Manakov...
system. The construction of multi-solitons within inverse scattering techniques for a more generalized two dimensional nonlinear evolutional wave equations will be considered as a future work.

References

[1] R. Beals and R. R. Coifman, Linear spectral problem, nonlinear equations and the \( \partial \)-bar method, Inverse Problems 5 (1989), 87–130.

[2] H. Cornille, Solutions of the generalized nonlinear Schrödinger equation in two spatial dimensions, J. Math. Phys. 20 (1978), no. 1, 199-209.

[3] Fokas A S and Ablowitz M J., On the inverse scattering of the time dependent Schrödinger equation and the associated KPI equation, Stud. Appl. Math. 69 (1983), 211–28.

[4] A. S. Fokas and M. J. Ablowitz, Methods of solution for a class of multi-dimensional nonlinear evolution equations, Phys. Rev. Lett. 51 (1983), 7–10.

[5] A. S. Fokas and L. Y. Sung, On the solvability of the \( N \)-Wave, Davey-Stewartson and KdVetsev-Petviashvili equations, Inverse Problems 8 (1992), 673-708.

[6] A. S. Fokas and M. J. Ablowitz, On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane, J. Math. Phys. 25 (1984) no. 8, 2494-2505.

[7] N. Sh. Iskenderov and M. I. Ismailov, On the inverse scattering transform of a nonlinear evolution equation with 2+1 dimensions related to nonstrict hyperbolic systems. Nonlinearity 25 (2012), no. 7, 1967–1979.

[8] M. I. Ismailov, Inverse scattering problem for nonstationary Dirac-type systems on the plane, Journal of Mathematical Analysis and Applications, 365 (2010), 498-509.

[9] M. I. Ismailov, Integration of nonlinear system of 4-waves with two velocities in 2+1 dimensions by the inverse scattering transform method, Journal of Mathematical Physics 52 (2011), 033504.

[10] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys. JETP 38 (1974), 248-253.

[11] L. P. Nizhnik, Integration of multidimensional nonlinear equations by the inverse problem method. Soviet Phys. Dokl. 25 (1980), no. 9, 706–708.

[12] L. P. Nizhnik, M. D. Pochinaiko, Integration of a spatially two-dimensional nonlinear Schrödinger equation by the inverse problem method, Functional Anal. Appl. 16 (1982), no. 1, 66–69.

[13] L. P. Nizhnik, The inverse scattering problem for hyperbolic equations and their application to nonlinear integrable systems. Reports on Math Phys. 26 (1988), no.2, 261-283.

[14] L. P. Nizhnik, An inverse problem of nonstationary scattering for the Dirac equations, Ukr. Mat. Zh., 24 (1972), no. 1, 112-115.

[15] R. G. Novikov, Inverse scattering up to smooth functions for the Dirac-ZS-AKNS system, Selecta Math. (N.S.) 3 (1997), no. 2, 245–302.

[16] A. L. Sakhnovich, Dirac type system on the axis: explicit formulae for matrix potentials with singularities and soliton-position interactions, Inverse Problems 19 (2003), no. 4, 845-854.

[17] L. Y. Sung, and A. S. Fokas, Inverse problem for \( N \times N \) hyperbolic systems on the plane and the \( N \)-wave interactions. Comm. Pure Appl. Math. 64 (1991), 535-571.

[18] V. E. Zakharov, A. B. Shabat, The scheme of integration of nonlinear equations of mathematical physics by inverse scattering method. I, Funct. Anal. Appl. 8 (1974), no. 3, 226-235;

[19] V. E. Zakharov, A. B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II Funct. Anal. Appl. 13 (1979), no. 3, 166-174.

[20] D. J. Benney, A general theory for interactions between short and long waves, Stud. Appl. Math.. 56 (1977), 81 - 94.

*Department of Mathematics, Gebze Technical University, Gebze-Kocaeli 41400, Turkey. **Institute of Mathematics and Mechanics, Azerbaijan National Academy of Science, 1141 Baku, Azerbaijan

E-mail address: mismailov@gtu.edu.tr