Error Bounds for the Method of Simultaneous Projections with Infinitely Many Subspaces

Rafał Zalas
Technion - Israel Institute of Technology
zalasrafal@gmail.com

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Introduction

$\mathcal{H}$ – real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

$r \in \{2, 3, \ldots\} \cup \{\infty\}$.

$M_i$ – closed and linear subspaces of $\mathcal{H}$, $i = 1, \ldots, r$.

$M := \bigcap_{i=1}^{r} M_i$.

$P_{M_i}, P_M$ – orthogonal projections onto $M_i$ and $M$, respectively.

Approximate $P_M$ by using $P_{M_1}, \ldots, P_{M_r}$.

Projection Methods

$x_0 := x \in \mathcal{H}, \quad x_k := T^k(x), \quad k = 1, 2, \ldots,$

$$\lim_{k \to \infty} \|x_k - P_{\text{Fix } T}(x_0)\| = 0,$$

$\text{Fix } T \approx M$. 

Rafał Zalas Technion – Israel Institute of Technology, Haifa, Israel zalasrafal@gmail.com
Alternating/Cyclic Projection Method

\[ T(x) := P_{M_r} \ldots P_{M_1}(x), \quad \text{Fix } T = M. \]

Simultaneous Projection Method

\[ T(x) := \sum_{i=1}^{r} \omega_i P_{M_i}(x), \quad \text{where } \omega_i > 0, \quad \sum_{i=1}^{r} \omega_i = 1, \quad \text{Fix } T = M. \]

Douglas-Rachford Projection Method \( (r = 2) \)

\[ T(x) := \frac{I + R_{M_2} R_{M_1}}{2}(x), \quad \text{where } R_{M_i} := 2P_{M_i} - I, \quad i = 1, 2, \]

\[ \text{Fix } T = M_1 \cap M_2 \oplus (M_1^\perp \cap M_2^\perp), \]

\[ P_{M_2} P_{\text{Fix } T} = P_{M_1} P_{\text{Fix } T} = P_M \]

\[ \ldots \]
Outline

- Known error bounds when $r < \infty$.
- New error bounds for the simultaneous projection method when $r = \infty$.

Keywords

linear convergence, arbitrarily slow convergence, super-polynomially fast convergence, polynomial convergence
Known Error Bounds

\[ r < \infty \]
Linear and Arbitrarily Slow Convergence

\[ T = P_{M_r} \ldots P_{M_1} \quad T = \frac{1}{r} \sum_{i=1}^{r} P_{M_i} \quad T = \frac{I + R_{M_2}R_{M_1}}{2} \]

**Theorem 1.** Exactly one of the following two statements holds:

(i) \( \sum_{i=1}^{r} M_i^\perp \) is closed. Then \( T^k \) converges linearly to \( P_{\text{Fix} \ T} \).

(ii) \( \sum_{i=1}^{r} M_i^\perp \) is not closed. Then \( T^k \) converges arbitrarily slowly to \( P_{\text{Fix} \ T} \).

**CPM and SPM** - Bauschke, Deutsch and Hundal, 2009 and 2010, Badea, Grivaux and Müller 2011; **DRPM** - Badea and Seifert 2017.

- \( \sum_{i=1}^{r} M_i^\perp = M^\perp \).
- \( T^k \) converges linearly to \( P_{\text{Fix} \ T} \) if there are \( c > 0 \) and \( q \in (0, 1) \) s.t.
  \[ \| T^k(x) - P_{\text{Fix} \ T}(x) \| \leq cq^k \| x \| \]
  holds for each \( k = 0, 1, 2, \ldots \) and \( x \in \mathcal{H} \).

- \( T^k \) converges arbitrarily slowly to \( P_{\text{Fix} \ T} \) if \( T^k \to P_{\text{Fix} \ T} \) and for each sequence \( (a_k)_{k=0}^\infty \subset (0, \infty) \) satisfying \( a_k \to 0 \) as \( k \to \infty \), there is \( x \) s.t.
  \[ \| T^k(x) - P_{\text{Fix} \ T}(x) \| \geq a_k. \]
Linear and Arbitrarily Slow Convergence

\[ T = P_{M_r} \ldots P_{M_1} \quad T = \frac{1}{r} \sum_{i=1}^{r} P_{M_i} \quad T = \frac{I + R_{M_2} R_{M_1}}{2} \]

**Theorem 1.** Exactly one of the following two statements holds:

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**CPM** and **SPM** - Bauschke, Deutsch and Hundal, 2009 and 2010, Badea, Grivaux and Müller 2011; **DRPM** - Badea and Seifert 2017.

- What is the optimal error bound in (i)?

\[
\sup_{x \neq 0} \frac{\| T^k(x) - P_{\text{Fix } T}(x) \|}{\| x \|} = \| T^k - P_{\text{Fix } T} \|
\]

- Are there any “good” starting points in (ii)? What are the error bounds?
Error Bounds when \( r = 2 \)

\[
M_1^\perp + M_2^\perp \text{ is closed } \iff \cos(M_1, M_2) < 1
\]

\[
\cos(M_1, M_2) := \sup\{|\langle x_1, x_2 \rangle| : x_i \in M_i \cap (M_1 \cap M_2)^\perp, \|x_i\| \leq 1, i = 1, 2\}
\]

**Theorem 2.** \( \|(P_{M_2}P_{M_1})^k - P_M\| = \cos(M_1, M_2)^{2k-1}. \)

Aronszajn 1950 (inequality), Kayalar and Weinert 1988 (equality)

**Theorem 3.** \( \left\| \left( \frac{P_{M_1} + P_{M_2}}{2} \right)^k - P_M \right\| = \left( \frac{1}{2} + \frac{1}{2} \cos(M_1, M_2) \right)^k. \)

Reich, Z. 2017

**Theorem 4.** \( \left\| \left( \frac{I + R_{M_2}R_{M_1}}{2} \right)^k T - P_{\text{Fix } T} \right\| = \cos(M_1, M_2)^k. \)

Bauschke, Bello Cruz, Nghia, Phan, Wang 2014
Error Bounds when $2 < r < \infty$

$$\sum_{i=1}^{r} M_{i}^\perp \text{ is closed } \iff \cos(C, D) < 1$$

$C := M_1 \times \ldots \times M_r$ and $D := \{(x, \ldots, x) : x \in \mathcal{H}\}$

$$\mathcal{H} := \bigoplus_{i=1}^{r} \mathcal{H}, \quad \langle x, y \rangle = \sum_{i=1}^{r} \langle x_i, y_i \rangle, \quad \|x\| := \sqrt{\sum_{i=1}^{r} \|x_i\|^2}$$

**Theorem 5.** $\| (P_{M_r} \ldots P_{M_1})^k - P_M \| \leq \left(1 - \frac{1}{m^2} \left(1 - \cos(C, D)\right)^2 \right)^{k/2}.$

Badea, Grivaux, Müller 2011

**Theorem 6.** $\| \left(\frac{1}{r} \sum_{i=1}^{r} P_{M_i}\right)^k - P_M \| = \cos(C, D)^{2k}.$

Reich, Z. 2017
Super-Polynomially Fast Convergence

\[ T = P_{M_r} \ldots P_{M_1} \quad T = \frac{1}{r} \sum_{i=1}^{r} P_{M_i} \quad T = \frac{I + R_{M_2} R_{M_1}}{2} \]

**Theorem 7.** If \( \sum_{i=1}^{r} M_i^\perp \) is not closed, then \( T^k \) converges super-polynomially fast to \( P_{\text{Fix } T} \) on some dense linear subspace \( X \subseteq \mathcal{H} \).

**CPM and DRPM** - Badea and Seifert 2016, 2017; **SPM** - Reich and Z. 2017.

- \( T^k \) converges super-polynomially fast to \( P_{\text{Fix } T} \) on \( X \) if

\[ k^n \| T^k(x) - P_{\text{Fix } T}(x) \| \rightarrow 0 \text{ as } k \rightarrow \infty \]

holds for all \( x \in X \) and for all \( n = 1, 2, \ldots \).
Polynomial Convergence

**Theorem 8.** $(r \geq 2)$ Assume $M = \{0\}$. For any $x \in \sum_{i=1}^{r} M_i^\perp$ there is $C(x) > 0$ s.t.

$$\| (P_{M_r} \ldots P_{M_1})^k (x) \| \leq C(x) \cdot k^{-1/(4r \sqrt{r} + 2)}, \quad k = 1, 2, \ldots .$$

**Theorem 9.** $(r = 2)$ Assume $M_1 \cap M_2 = \{0\}$. For any $x \in M_1^\perp + M_2^\perp$ there is $C(x) > 0$ s.t.

$$\| (P_{M_2} P_{M_1})^k (x) \| \leq \frac{C(x)}{\sqrt{k}}, \quad k = 1, 2, \ldots .$$

Moreover, $\sqrt{k}$ cannot be replaced by $n^{1/2 + \epsilon}$ for any $\epsilon > 0$.

Borodin and Kopecká 2020
New Error Bounds for the Simultaneous PM

\[ r = \infty \]

S. Reich, R. Zalas, Error bounds for the method of simultaneous projections with infinitely many subspaces, *J. Approx. Theory* 272 (2021), 105648.
Simultaneous Projection in $\mathcal{H}$

$$T_\omega(x) := \sum_{i=1}^{\infty} \omega_i P_{M_i}(x)$$

$$\omega := \{\omega_i\}_{i=1}^{\infty}, \quad \omega_i > 0, \ i = 1, 2, \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_i = 1$$

Fix $T_\omega = M$

Product Space Setup

$$\mathcal{H}_\omega := \left\{ x = \{x_i\}_{i=1}^{\infty} : x_i \in \mathcal{H}, i = 1, 2, \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \right\}$$

$$\langle x, y \rangle_\omega := \sum_{i=1}^{\infty} \omega_i \langle x_i, y_i \rangle \quad \text{and} \quad \|x\|_\omega = \sqrt{\langle x, x \rangle_\omega}$$

Fact 10. The elements of $\mathcal{H}_\omega$ can change if we change $\omega$. 
Alternating Projection in $\mathcal{H}_\omega$

$$C_\omega := \left\{ \{x_i\}_{i=1}^{\infty} : x_i \in M_i, i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \right\}$$

$$D_\omega := \{ \{x\}_{i=1}^{\infty} : x \in \mathcal{H} \}$$

**Theorem 11.** For each $x = \{x\}_{i=1}^{\infty} \in D_\omega$, we have

$$\left( P_{D_\omega} P_{C_\omega} \right)^k(x) = \{ T_\omega^k(x) \}_{i=1}^{\infty}$$

and

$$P_{C_\omega \cap D_\omega}(x) = \{ P_M(x) \}_{i=1}^{\infty}.$$ 

**Corollary 12.** For each $x \in \mathcal{H}$ and $x = \{x\}_{i=1}^{\infty}$, we have

$$\| T_\omega^k(x) - P_M(x) \| = \| (P_{D_\omega} P_{C_\omega})^k(x) - P_{C_\omega \cap D_\omega}(x) \|_{\omega} \to 0.$$
Towards the Dichotomy Theorem

How to define \( \sum_{i=1}^{\infty} M_i^\perp \)?

\[
\sum_{i=1}^{\infty} \omega_i M_i^\perp := \left\{ \sum_{i=1}^{\infty} \omega_i x_i : x_i \in M_i^\perp \text{ and } \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \right\}
\]

\[
\sum_{i=1}^{\infty} \omega_i \|x_i\|^2 < \infty \implies \sum_{i=1}^{\infty} \omega_i \|x_i\| < \infty \implies \sum_{i=1}^{\infty} \omega_i x_i \in \mathcal{H}
\]

The elements of \( \sum_{i=1}^{\infty} \omega_i M_i^\perp \) can change if we change \( \omega \).

\[
\sum_{i=1}^{\infty} \omega_i M_i^\perp = M^\perp.
\]
\[ \sum_{i=1}^{\infty} \omega_i M_i^\perp \text{ is closed} \iff \cos_{\omega}(C_\omega, D_\omega) < 1 \]

\[
\cos_{\omega}(C_\omega, D_\omega) = \sup \left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M_i^\perp, \sum_{i=1}^{\infty} \omega_i \|x_i\|^2 \leq 1 \right\}
\]

\[
\cos_{\omega}(C_\omega, D_\omega)^2 \leq \sup \left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M_i^\perp, \|x_i\| \leq 1 \right\} \leq \cos_{\omega}(C_\omega, D_\omega)
\]

**Theorem 13.** The following conditions are equivalent:

(i) \( \cos_{\omega}(C_\omega, D_\omega) = 1 \) for some sequence of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \).

(ii) \( \cos_{\omega}(C_\omega, D_\omega) = 1 \) for all sequences of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \).
\[ \sum_{i=1}^{\infty} \omega_i M_i^\perp \text{ is closed } \iff \cos_\omega(C_\omega, D_\omega) < 1 \]

\[ \cos_\omega(C_\omega, D_\omega) = \sup \left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M_i^\perp, \sum_{i=1}^{\infty} \omega_i \| x_i \| ^2 \leq 1 \right\} \]

\[ \cos_\omega(C_\omega, D_\omega)^2 \leq \sup \left\{ \left\| \sum_{i=1}^{\infty} \omega_i x_i \right\| : x_i \in M_i \cap M_i^\perp, \| x_i \| \leq 1 \right\} \leq \cos_\omega(C_\omega, D_\omega) \]

**Theorem 13.** The following conditions are equivalent:

(i) \( \sum_{i=1}^{\infty} \omega_i M_i^\perp \) is closed for some sequence of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \).

(ii) \( \sum_{i=1}^{\infty} \omega_i M_i^\perp \) is closed for all sequences of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \).
Dichotomy

**Theorem 14.** Exactly one of the following two statements holds:

(i) \( \sum_{i=1}^{\infty} \omega_i M_i^\perp \) is closed for all sequences of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \). Then \( T_{\omega}^k \) converges linearly to \( P_M \).

(ii) \( \sum_{i=1}^{\infty} \omega_i M_i^\perp \) is not closed for all sequences of weights \( \omega = \{\omega_i\}_{i=1}^{\infty} \). Then \( T_{\omega}^k \) converges arbitrarily slowly to \( P_M \).

Optimal error bound in (i)? / “Good” starting points in (ii)?
Optimal error bound

**Theorem 15.** \( \| T^k_\omega - P_M \| = \cos(\omega, D_\omega)^{2k} \).

Super-Polynomially Fast Convergence

**Theorem 16.** If \( \sum_{i=1}^{\infty} \omega_i M_i \perp \) is not closed, then \( T^k_\omega \) converges super-polynomially fast to \( P_M \) on some dense linear subspace \( X \subseteq \mathcal{H} \).

Polynomial Convergence

**Theorem 17.** For any \( x \in \sum_{i=1}^{\infty} \omega_i M_i \perp \) there is \( C(x) > 0 \) s.t.

\[
\| T^k_\omega(x) - P_M(x) \| \leq \frac{C(x)}{\sqrt{k}}, \quad k = 1, 2, \ldots.
\]
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Thank you for your attention!