FORMAL MEROMORPHIC FUNCTIONS ON MANIFOLDS OF FINITE TYPE

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Dedicated to Prof. J.J. Kohn on the occasion of his 75th birthday.

Abstract. It is shown that a real-valued formal meromorphic function on a formal generic submanifold of finite Kohn-Bloom-Graham type is necessarily constant.

1. Introduction

It is easy to see (and known, see [1]) that if \( M \subset \mathbb{C}^N \) is a connected generic real-analytic CR manifold which is of finite type in the sense of Kohn [2] and Bloom-Graham [3] at some point \( p \in M \), then any meromorphic map \( H: U \to \mathbb{C}^m \) defined on a connected neighbourhood of \( M \) which satisfies \( H(M) \subset E \), where \( E \subset \mathbb{C}^m \) is a totally real analytic submanifold, is necessarily constant.

Let us give a short proof of this fact. First, we recall the definition of the Segre sets \( S_p^1 \). These are defined inductively. First, we define the Segre variety \( S_p^0 = S_p^1 \) for \( p \in M \). Let \( \rho(Z, \bar{Z}) = (\rho_1(Z, \bar{Z}), \ldots, \rho_d(Z, \bar{Z})) \) be a (vector-valued) defining function for \( H \) defined in a neighbourhood \( U \times \tilde{U} \) of \( (p, \tilde{p}) \), i.e.

\[
M \cap U = \{ Z \in U : \rho(Z, \bar{Z}) = 0 \}, \quad d\rho_1 \wedge \cdots d\rho_d \neq 0 \text{ on } U, \quad \rho(Z, \bar{Z}) = \bar{\rho}(\tilde{Z}, Z).
\]

With this notation, \( S_p^1 \) is defined by

\[
S_p^1 = \{ Z \in U : \rho(Z, q) = 0 \}, \quad q \in U,
\]

and the \( j \)-th Segre set \( S_p^j \), \( j > 1 \), is defined inductively by

\[
S_p^j = \bigcup_{q \in S_p^{j-1}} S_p^1.
\]

For consistency, we also put \( S_p^0 = \{ p \} \).

We are using the following Theorem, which characterizes finite type in terms of properties of the Segre sets:

Theorem 1 (Baouendi, Ebenfelt and Rothschild [1]). Let \( M \subset \mathbb{C}^N \) be a generic real-analytic CR manifold. Then \( M \) is of finite type at \( p \in M \) if and only if there exists an open set \( V \subset \mathbb{C}^N \) with \( V \subset S_p^{d+1} \).

Now assume that \( H: U \to \mathbb{C}^m \) is a meromorphic map which satisfies \( H(M) \subset E \), where \( E \) is totally real. First note that since \( M \) is of finite type at some point \( p \), it is of finite type on the complement of a proper real-analytic subvariety \( F \subset M \). So there exists a point \( p \in M \) with the property that \( M \) is of finite type at \( p \) and \( H \) is holomorphic in some neighbourhood of \( p \) (because \( M \) is generic, it is a set of uniqueness for holomorphic functions). We shall prove that in this situation, \( H \) is constant on an open set in \( \mathbb{C}^N \), and thus constant.

We can find coordinates \( \eta \) in \( \mathbb{C}^m \) such that near \( H(p) \), \( E \) is given by an equation of the form \( \eta = \varphi(\bar{\eta}) \). Thus, \( H(Z) = \varphi(H(Z)) \), whenever \( Z \in M \), and from this we have that \( H(Z) = \varphi(H(\bar{\zeta})) \) whenever \( Z \in S_\zeta \) (restricting to a suitable neighbourhood \( U \) of \( p \)). Thus, \( H(Z) = \varphi(H(\bar{p})) \) for \( Z \in S_p \); since \( p \in S_p \), \( H(Z) = H(p) \) for \( Z \in S_p \). Now we consider \( Z \in S_p^2 \). For each such \( Z \), there is \( \zeta \in S_p^1 \) with \( Z \in S_\zeta \). Our equation tells us that \( H(Z) = \varphi(H(\bar{\zeta})) = \varphi(H(\bar{p})) \), and again, since \( p \in S_p^2 \), \( H(Z) = H(p) \) for \( Z \in S_p^2 \).

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Continuing the iteration process like this, we see that $H(Z) = H(p)$ for $Z \in S_p^j$ for $j \in \mathbb{N}$. Since $S_p^d$ contains an open subset of $\mathbb{C}^N$ by Theorem 1 the identity principle implies that $H(Z) = H(p)$ on $U$. This proves the constancy of such an $H$.

Our main point in this paper is the extension of this result to the formal category. Here we cannot “move to a good point”. Let us be a bit more specific about the notions which we are going to use (and refer the reader to Baouendi, Mir and Rothschild [3] for more information). A generic real formal submanifold $(M, 0) \subset (\mathbb{C}^N, 0)$ of codimension $d$ is given by its manifold ideal

$$I(M, 0) \subset \mathbb{C}[[Z, \zeta]],$$

which satisfies that $I(M, 0)$ can be generated by $d$ functions $\rho_1, \ldots, \rho_d$, where $\rho_1, \ldots, \rho_d$ have the following properties:

1. $\overline{\rho_j}(\zeta, Z) = \rho_j(Z, \zeta)$ (the $\rho_j$ are real)
2. $\rho_j, Z(0) \wedge \cdots \wedge \rho_d, Z(0) \neq 0$, where $\rho_j, Z = \left(\frac{\partial \rho_j}{\partial Z_1}, \ldots, \frac{\partial \rho_j}{\partial Z_N}\right)$.

A formal meromorphic map $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ is given by $H = \frac{N}{D}$, where $D$ is a formal power series which is not identically zero, and $N: (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ is a formal holomorphic map (i.e., $N = (N_1, \ldots, N_m)$ where $N_j \in \mathbb{C}[[Z]]$). In the present context, $(E, 0) \subset (\mathbb{C}^m, 0)$ is a formal totally real manifold if it is a formal real submanifold which in suitable (formal) holomorphic coordinates $\eta \in \mathbb{C}^m$, $(E, 0)$ is given by $\Im \eta = 0$—by this we mean that $I(E, 0) \subset \mathbb{C}[[\eta, \nu]]$ can be generated by the functions $\frac{1}{2i} (\nu_j - \eta_j)$.

**Remark 1.** Usually, a totally real CR-manifold is defined as a CR-manifold of CR-dimension 0; what we refer to as “totally real” here is usually referred to as “maximally totally real”. However, in the formal category, every “totally real” submanifold is automatically equivalent to a “maximally totally real” submanifold; this justifies the chosen terminology.

**Definition 1.** We say that a formal meromorphic map $H = N/D: (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ maps $(M, 0) \subset (\mathbb{C}^N, 0)$ into the totally real submanifold $(E, 0) \subset (\mathbb{C}^m, 0)$ (with coordinates as above) if for any formal holomorphic map $\gamma(t) = (\gamma_1(t), \gamma_2(t)): (\mathbb{C}^{2N-d}, 0) \to (\mathbb{C}^{2N})$ satisfying $\rho(\gamma_1(t), \gamma_2(t)) = 0$ for every $\rho \in I(M, 0)$ it holds that

$$N_j(\gamma_1(t)) \dot{D}(\gamma_2(t)) - \dot{N}_j(\gamma_2(t)) D(\gamma_1(t)) = 0$$

for every $j = 1, \ldots, m$.

We also recall that a formal generic manifold $(M, 0) \subset (\mathbb{C}^N, 0)$ is of finite type at 0 if the Lie algebra generated by the formal $(1,0)$- and $(0,1)$-vector fields tangent to $(M, 0)$, evaluated at 0, spans $\mathbb{C}^N$. We can now state our main result.

**Theorem 2.** Let $(M, 0) \subset (\mathbb{C}^N, 0)$ be a formal generic manifold of finite type, $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ a formal meromorphic map which maps $(M, 0)$ into $(E, 0)$, where $(E, 0)$ is a formal totally real manifold. Then $H$ is constant.

We note that the finite type assumption is necessary. Indeed, every manifold of the form $M = \tilde{M} \times E$, where $\tilde{M}$ is some CR manifold and $E$ is totally real, has nonconstant CR maps onto a totally real manifold (the projection onto its second coordinate). On the other hand, here is another example, due to J. Lebl:

**Example 1.** Let $M \subset \mathbb{C}^3$ be given by

$$w_1 = w_1 e^{ip|z|^2}, \quad w_2 = w_2 e^{iq|z|^2},$$

for some integers $p$ and $q$. Then the function

$$H(z, w_1, w_2) = \frac{w_1^q}{w_2^p}$$

maps $M$ into $\mathbb{R}$ and is not the restriction of a holomorphic function. Also note that this function is not even continuous on $M$. Our results imply that no nonconstant holomorphic choice of projection onto $\mathbb{R}$ can be made.
2. Reflected Identities and Consequences

We shall first show that we can simplify our situation somewhat by choosing "normal" coordinates. Recall that normal coordinates for a formal generic submanifold $(M,0) \subset (\mathbb{C}^N,0)$ means a choice of coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ (d being the real codimension of $(M,0)$) together with formal functions $Q_j(z,\chi,\tau) \in \mathbb{C}[[z,\chi,\tau]], j = 1, \ldots, d$, satisfying

$$Q_j(z,0,\tau) = Q_j(0,\chi,\tau) = \tau_j, \quad j = 1, \ldots, d,$$

such that $w_j - Q_j(z,\chi,\tau)$ generate the manifold ideal associated to $(M,0)$ in $\mathbb{C}[[z,w,\chi,\tau]]$. We will write $Q = (Q_1, \ldots, Q_d)$, and abbreviate the generating set with $w - Q(z,\chi,\tau)$.

We will show that in normal coordinates, a formal meromorphic function $H$ which maps $(M,0)$ into $(\mathbb{R},0)$ actually only depends on the transverse variables $w$. To do this, we first give a reflection identity which we will use.

**Proposition 2.** If $(M,0) \subset (\mathbb{C}^N,0)$ is a formal generic submanifold, and $(z, w)$ are normal coordinates for $(M,0)$ with corresponding generators $w - Q(z,\chi,\tau)$. If $H = \frac{N}{D} : (M,0) \to (\mathbb{R},0)$ is formal meromorphic, and $N$ and $D$ do not have any common factors, then there exists a formal holomorphic function $a(z,\chi, z^1, w)$, with $a(0,0,0,0) = 1$, such that

$$N(z, Q(z, \chi, \bar{Q}(z^1, w))) = a(z, \chi, z^1, w)N(z^1, w),$$

$$D(z, Q(z, \chi, \bar{Q}(z^1, w))) = a(z, \chi, z^1, w)D(z^1, w).$$

**Proof.** From the definition, we have

$$\bar{D}(\chi, \tau)N(z, Q(z, \chi, \tau)) = \bar{N}(\chi, \tau)D(z, Q(z, \chi, \tau)).$$

Taking the complex conjugate and replacing $z$ by $z^1$ in this equation, we also have that

$$D(z^1, w)\bar{N}(\chi, \bar{Q}(z^1, w)) = N(z^1, w)\bar{D}(\chi, \bar{Q}(z^1, w)).$$

We now substitute $\tau = \bar{Q}(\chi, z^1, w)$ into (2) to obtain

$$\bar{D}(\chi, \bar{Q}(\chi, z^1, w))N(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = \bar{N}(\chi, \bar{Q}(\chi, z^1, w))D(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))).$$

We now multiply the left (and right, respectively) hand sides of (3) and (4) with each other, and after cancelling a common factor of $\bar{N}(\chi, \bar{Q}(\chi, z^1, w))\bar{D}(\chi, \bar{Q}(\chi, z^1, w))$ we obtain

$$D(z^1, w)N(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = D(z, Q(z, \chi, \bar{Q}(\chi, z^1, w)))N(z^1, w).$$

Now, using the fact that $N$ and $D$ do not have any common factors, unique factorization in the ring $\mathbb{C}[[z, \chi, z^1, w]]$ implies that there exists a unit $a(z, \chi, z^1, w)$ such that (1) holds. By evaluating (1) at $z = z^1$, and using the reality property $Q(z, \chi, \bar{Q}(\chi, z, w)) = w$, we have that $a(z, \chi, z, w) = 1$, so in particular, $a(0,0,0,0) = 1$. \hfill \Box

**Lemma 3.** Let $(M,0) \subset (\mathbb{C}^N,0)$ be a formal generic submanifold. Assume that $H(Z) = \frac{N(Z)}{D(Z)}$ is a formal meromorphic map sending $(M,0)$ into $(\mathbb{R},0)$. Then for any choice of normal coordinates $(z,w)$ for $(M,0)$, we have that $H(z,w) = H(0,w)$; in particular, there exist formal functions $\bar{N}(w)$ and $\bar{D}(w)$ such that $H(z,w) = \frac{\bar{N}(w)}{\bar{D}(w)}$.

**Proof.** We use Proposition 2. Setting $\chi = z^1 = 0$, we see that

$$N(z,w) = a(z,0,0,w)N(0,w), \quad D(z,w) = a(z,0,0,w)D(0,w).$$

The Lemma follows. \hfill \Box
3. Prolongation of the reflection along Segre maps and proof of Theorem 2

We will denote by
\[ v^1(z, \chi, z^1; w) = Q(z, \chi, Q(z, z^1, w)); \]
in the usual Segre-map terminology, \( v^1(z, \chi, z^1; 0) \) is the transversal component of the second Segre map of \( (M, 0) \). Since we shall only have use for the Segre-maps of even order, we introduce the notation adapted to our setting. We define \( S^{(0)} = z \), and for \( j \geq 1 \)
\[ S^{(j)} = (z, \chi, z^1, \chi^1, \ldots, z^j), \]
and write \( S^{(j)}_k = (z^k, \chi^k, \ldots, z^j) \) for \( k \leq j \). By Lemma 4, \( H \) does not depend on \( z \) and we can assume that
\[ H(z, w) = \frac{N(w)}{D(w)}. \]

With that notation and our simplification from Lemma 3, our reflection identity now reads
\[
N \left( v^1(S^{(1)}; w) \right) = a(S^{(1)}, w) N(w), \\
D \left( v^1(S^{(1)}; w) \right) = a(S^{(1)}, w) D(w).
\]

For \( j \geq 1 \), we define inductively
\[
v^j \left( S^{(j)}; w \right) = v^1(z, \chi, z^1; v^{j-1}(S_1^{(j)}; w)).
\]

We can now state the finite type criterion of Baouendi, Ebenfelt and Rothschild for formal submanifolds, for later reference, as follows:

**Theorem 3.** If \((M, 0)\) is of finite type in the sense of Kohn-Bloom-Graham, then there exists a \( j \geq 1 \) such that
\[ S^{(j)} \to v^j \left( S^{(j)}; 0 \right), \quad (\mathbb{C}^{(2j-1)n}, 0) \to (\mathbb{C}^d, 0), \]
is of generic full rank \( d \).

Thus, if we replace \( w \) by \( v^{j-1}(S_1^{(j)}; w) \) in (6), we obtain
\[
N \left( v^j(S^{(j)}; w) \right) = N \left( v^1(S^{(1)}; v^{j-1}(S_1^{(j)}; w)) \right) = a(S^{(1)}; v^{j-1}(S_1^{(j)}; w)) N \left( v^{j-1}(S_1^{(j)}; w) \right).
\]

Applying induction, we see that the following holds:

**Lemma 4.** For every \( j \geq 1 \), there exists a unit \( a_j(S^{(j)}, w) \) such that
\[
N \left( v^j(S^{(j)}; w) \right) = a_j(S^{(j)}, w) N(w), \quad D \left( v^j(S^{(j)}; w) \right) = a_j(S^{(j)}, w) D(w).
\]

We can now prove Theorem 2. By Theorem 3 there exists a \( j \) such that \( v^j(S^{(j)}; 0) \) is of generic full rank. Assuming that \( D(0) = 0 \), we see that \( D(v^j(S^{(j)}; 0)) = 0 \). Since \( v^j \) is of generic full rank, this implies that \( D(w) = 0 \); this contradiction shows that \( D(0) \neq 0 \). Hence, we can assume that \( H(z, w) = N(w) \) is formal holomorphic, and without loss of generality, \( N(0) = 0 \). The same argument shows that \( N(w) = 0 \), and so, \( H \) is constant.

**Remark 2.** More generally, if we do not assume that \((M, 0)\) is of finite type, then we can define the formal variety
\[ V_j = \overline{\text{image}(v^j(S^{(j)}; 0))} \cong \{ f \in \mathbb{C}[[w]] : f \circ v^j(S^{(j)}; 0) = 0 \}, \]
and \( V = \bigcup_j V_j \) (which is again a formal variety). The same arguments as above show that \( D \), as well as \( N \), are constant on \( V \). This corresponds to the statement that a real-valued CR meromorphic function is constant along the CR-orbits of \( M \).
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