Vortex Velocity Probability Distributions in Phase Ordering

Kinetics

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Abstract

The range of systems over which we can extract the point vortex velocity probability distribution function (vvpdf) is extended to include anisotropic models. The structure of the theory used to compute the vvpdf is explored.
I. INTRODUCTION

In recent work [1] it was shown that the theoretically predicted [2] velocity probability distribution for vortices in the case of a phase ordering system agrees very well with direct numerical simulations. In particular the predicted high velocity algebraic tail is found to be robust and the predicted exponent confirmed. In the original paper [2] describing the theory there were assumptions concerning the gaussian nature of an underlying auxiliary field. Here we clarify this result by showing that the assumption of an underlying gaussian field is consistent and does not imply in any way that the underlying order parameter field is gaussian. We only require that the order parameter field and the auxiliary field share the same zeros and symmetry. It is shown here how one can deal with the simplest types of anisotropy in the theory.

In ref. [2] it was shown that for annihilating point defects \( n = d \), where \( n \) is the number of components of the order parameter and \( d \) the spatial dimensionality of the system that the vortex velocity probability distribution function (vvpdf), as a function of time after a quench, is given by

\[
P(\mathbf{V}) = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{(\pi \bar{v}^2)^{n/2}} \frac{1}{\left(1 + \mathbf{V}^2/\bar{v}^2\right)^{(n+2)/2}}
\]

where the parameter \( \bar{v} \) is defined below and varies as \( t^{-1/2} \) for long times. Both the form of \( P(\mathbf{V}) \) and the time dependence of \( \bar{v} \) have been confirmed in ref. [1]. It is worth pointing out that the order parameter growth law, \( L(t) \), for the system studied in ref. [1] has a log correction \([3,4]\), \( L^2 \approx t/\ln t \). This log correction for \( L(t) \) is seen in our simulations but not in \( \bar{v}(t) \). Thus nonlinearities which influence \( L \) are not seen in \( \bar{v} \). We discuss in more detail here the range of validity of this result and it extension to systems with simple spatial anisotropy.

The set of problems of interest are driven by Langevin equations of the form

\[
\frac{\partial \psi_\alpha}{\partial t} = \Gamma \left[ -V'_\alpha(\psi) + \hat{O}_\alpha \psi \right]
\]
where \( \Gamma \) is a kinetic coefficient and \( \hat{O} \) is a gradient operator chosen to be \( \hat{O}(1) = c\nabla_1^2 \) in ref. [2]. We assume that \( V'_a(\psi = 0) = 0 \). The potential contribution must be such that system orders via annihilating point defects. We assume that the instantaneous positions of these defects are determined by the zeros of the order parameter field. Furthermore, it was pointed out in ref. [2], that the vortex charge density for this system can be written as

\[
\rho = \delta(\vec{\psi})D
\]  

(3)

where \( D \) is the Jacobian (determinant) for the change of variables from the set of vortex positions \( r_i(t) \) (where \( \vec{\psi} \) vanishes) to the field \( \vec{\psi} \):

\[
D = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \nabla_{\mu_1} \psi_{\nu_1} \nabla_{\mu_2} \psi_{\nu_2} \ldots \nabla_{\mu_n} \psi_{\nu_n}
\]  

(4)

where \( \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \) is the \( n \)-dimensional fully anti-symmetric tensor and summation over repeated indices here and below is implied. Furthermore, since topological charge is conserved, it was shown in ref. [2] that \( \rho \) satisfies a continuity equation:

\[
\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v})
\]  

(5)

where the vortex velocity is given by

\[
Dv_\beta = -\frac{1}{(n - 1)!} \epsilon_{\beta,\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \hat{O}(1) \nabla_{\mu_1} \psi_{\nu_1} \nabla_{\mu_2} \psi_{\nu_2} \ldots \nabla_{\mu_n} \psi_{\nu_n}.
\]  

(6)

It is assumed that the velocity field is used inside expressions multiplied by the vortex locating \( \delta \)-function. These results are rather general.

Notice that \( \rho \) and \( \vec{v} \) have certain important invariance properties. If we can write

\[
\psi_\nu(1) = (\alpha + \beta m^2(1) + \ldots) m_\nu(1)
\]  

(7)

for small \( m_\nu \) and where \( \alpha \) and \( \beta \) are constants, it is easy to see that

\[
\rho(\vec{\psi}) = \rho(\vec{m})
\]  

(8)

If we further assume, as \( \vec{m} \) and \( \vec{\psi} \) go to zero,
\[ \dot{\psi}_\nu(1) = \alpha \dot{m}_\nu(1) \quad , \]

then

\[ v_\mu(\vec{\psi}) = v_\mu(\vec{m}) \quad . \]

Thus the correlation function we compute in the next section, \( G(12) \), is for that set of fields \( m \), related to \( \psi \) by Eq.(7), for small values of \( m_\nu \) and \( \psi_\nu \), which is described by a gaussian distribution. Thus we assume there is a field \( m_\nu \) which is gaussian while the statistics of \( \psi_\nu \) are largely undetermined.

**II. CORRELATIONS FOR THE DEFECT SECTOR**

It was shown by Mazenko and Wickham [5] that the defect continuity equation can be used to determine the auxiliary field correlation function. Here we give a more general proof of this assertion. The idea is to look at the equation generated by multiplying the continuity equation by a source function and then averaging over \( m \). We have

\[ \langle \left[ \partial \rho(1) \partial t_1 + \vec{\nabla}(1) \cdot (\rho(1)v(1)) \right] S(H) \rangle = 0 \quad (11) \]

where

\[ S(H) = \exp[\int d\bar{H} \bar{H} \cdot m(\bar{1})] \quad . \]

The question is whether this equation can be satisfied for an underlying gaussian probability distribution for arbitrary external field \( H(1) \)?

To answer this question we evaluate first the quantity

\[ \langle \rho(1)S(H) \rangle = \langle S(H)\delta(1)D(1) \rangle \quad (13) \]

where we introduce the simplifying notation

\[ \delta(1) = \delta(m(1)) \quad (14) \]
and now \( D \) is a function of the field \( m \). When we talk about correlation in the defect sector we mean averages like in Eq.(13) where there is a vortex locating \( \delta \)-function inside the average.

By taking functional derivatives we are able to generate the correlations between fields \( m \) at arbitrary space-time points with the field at the space-time point 1. If we define

\[
Z_H(1) = \langle S(H)\delta(1) \rangle
\]

then

\[
\langle \delta(1)m_{\nu_2}(2)m_{\nu_3}(3)\ldots \rangle = Z_H^{-1}(1) \frac{\delta}{\delta H_{\nu_2}(2)} \frac{\delta}{\delta H_{\nu_3}(3)} \ldots Z_H(1) .
\]

In our development a key property of the underlying gaussian distribution function is

\[
\langle m_{\nu_1}(1)A \rangle = \sum_{\nu_1^\prime} \int d\bar{1} G_{\nu_1\nu_1^\prime}(1\bar{1}) \langle \frac{\delta}{\delta m_{\nu_1^\prime}(1)} A \rangle
\]

for arbitrary \( A \). For \( A = m_{\nu_2}(2) \) we obtain

\[
\langle m_{\nu_1}(1)m_{\nu_2}(2) \rangle = G_{\nu_1\nu_2}(12) .
\]

If we assume that the system is isotropic in the vector space then

\[
G_{\nu_1\nu_2}(12) = \delta_{\nu_1\nu_2}G(12)
\]

and

\[
\langle m_{\nu_1}(1)A \rangle = \int d\bar{1} G(1\bar{1}) \langle \frac{\delta}{\delta m_{\nu_1}(1)} A \rangle .
\]

We then need to work out

\[
\langle S(H)\rho(1) \rangle = \langle S(H)\delta(1) \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \nabla_{\mu_1} m_{\nu_1} \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle .
\]

Using Eq.(19) we have

\[
\langle S(H)\rho(1) \rangle = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \nabla_{\mu_1} G(1\bar{1}) \langle \frac{\delta}{\delta m_{\nu_1}(1)} \langle S(H)\delta(1) \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle \rangle .
\]
After using the results

$$\frac{\delta}{\delta m_{\nu_1}(1)} S(H) = H_{\nu_1}(1) S(H) \ ,$$

(22)

$$\frac{\delta}{\delta m_{\nu_1}(1)} \delta(1) = \delta_{\nu_1}(1) \delta(1) \ ,$$

(23)

where

$$\delta_{\nu_1}(1) = \frac{\partial}{\partial m_{\nu_1}(1)} \delta(m(1)) \ ,$$

(24)

we find

$$\langle S(H) \rho(1) \rangle = \frac{1}{n!} \epsilon_{\mu_1, \mu_2, \ldots, \mu_n} \epsilon_{\nu_1, \nu_2, \ldots, \nu_n} \times \left[ \nabla_{\mu_1} A_{\nu_1}(1) \langle S(H) \delta(1) \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle \right. \]$$

$$+ (\nabla_{\mu_1} G(1 \bar{1})) |_{1=1} \langle S(H) \delta_{\nu_1}(1) \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle$$

$$+ \nabla_{\mu_1} G(1 \bar{1}) \langle S(H) \delta(1) \frac{\delta}{\delta m_{\nu_1}(1)} (\nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n}) \rangle$$

(25)

where we define the key quantity

$$A_{\nu_1}(1) = \int d\bar{1} G(1 \bar{1}) H_{\nu_1}(\bar{1}) \ .$$

(26)

We assume, and check self-consistently, that

$$(\nabla_{\mu_1} G(1 \bar{1})) |_{1=1} = 0 \ .$$

(27)

The derivatives of the product $\nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n}$ with respect to $m_{\nu_1}(\bar{1})$ lead to contributions which all vanish because it picks out terms $\delta_{\nu_1 \nu_j}$ which multiplies $\epsilon_{\nu_1, \nu_2, \ldots, \nu_j, \nu_n}$ and $\epsilon_{\nu_1, \nu_2, \ldots, \nu_1, \nu_n} = 0$. We have then

$$\langle S(H) \rho(1) \rangle = \frac{1}{n!} \epsilon_{\mu_1, \mu_2, \ldots, \mu_n} \epsilon_{\nu_1, \nu_2, \ldots, \nu_n} \nabla_{\mu_1} A_{\nu_1}(1) \langle S(H) \delta(1) \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle \ .$$

(28)
Clearly we can go through this process $n - 1$ more times to obtain
\[
\langle S(H)\rho(1) \rangle = D_A(1)Z_H(1)
\]
(29)

where
\[
D_A(1) = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \nabla_{\mu_1} A_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1)
\]
(30)

Next we look at the current contributions to Eq.(11) in the form
\[
\langle S(H)\vec{\nabla} \cdot (\rho(1)v(1)) \rangle = \nabla^{(1)}_{\mu_1} \frac{1}{(n - 1)!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \langle S(H)\delta(1)m_{\nu_1} \nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle
\]
(31)

We assume that
\[
\delta(1)m_{\nu_1}(1) = -\delta(1)\hat{O}(1)m_{\nu_1}(1)
\]
(32)

where $\hat{O}(1)$ is a derivative operator. In ref. [2] one has the choice $\hat{O}(1) = -\Gamma c\nabla^2$. After inserting Eq.(32) back into Eq.(31) and using Eq.(19) we obtain
\[
\langle S(H)\vec{\nabla} \cdot (\rho(1)v(1)) \rangle = \nabla^{(1)}_{\mu_1} \frac{1}{(n - 1)!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n}
\]
\[
\times \left[ (-\hat{O}(1)A_{\nu_1}(1)) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1)Z_H(1)
\right.
\]
\[
+ \left( -\hat{O}(1)G(11) \right)_{1=1} \langle S(H)\delta_{\nu_1}(1)\nabla_{\mu_2} m_{\nu_2} \ldots \nabla_{\mu_n} m_{\nu_n} \rangle
\]
(33)

and the derivatives of the product of $m$’s vanish as in the previous case. Clearly the reduction of the term containing $\delta_{\nu_1}(1)$ follows just as for $\langle S(H)\rho(1) \rangle$ with the result:
\[
\langle S(H)\vec{\nabla} \cdot (\rho(1)v(1)) \rangle = \nabla^{(1)}_{\mu_1} \frac{1}{(n - 1)!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n}
\]
\[
\times \left[ (\hat{O}(1)A_{\nu_1}(1)) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1)Z_H(1)
\right.
\]
\[
+ \left( \hat{O}(1)G(11) \right)_{1=1} \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1)\langle S(H)\delta_{\nu_1}(1) \rangle
\]
(34)
We must next evaluate $Z_H(1)$ and the related quantity $\langle S(H)\delta_{\nu_1}(1) \rangle$. We have

$$Z_H(1) = \langle \delta(m(1))S(H) \rangle = \int \frac{d^d k}{(2\pi)^d} \langle e^{ik \cdot m(1)} e^{H(1) \cdot m(1)} \rangle$$

$$= \int \frac{d^d k}{(2\pi)^d} \langle e^{\tilde{H}(1) \cdot m(1)} \rangle \quad (35)$$

where

$$\tilde{H}(\bar{1}) = ik \bar{\delta}(11) + H(\bar{1}) \quad (36)$$

and

$$Z_H(1) = \int \frac{d^d k}{(2\pi)^d} e^{\frac{1}{2} \left[ \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_2}(2)G_{\nu_1\nu_2}(\bar{1}\bar{2}) \right]}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{\frac{1}{2} \left[ \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_1}(2)G(\bar{1}\bar{2}) \right]}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{\left[ -\frac{1}{2} k^2 G(11) + ik \cdot A(1) + \frac{1}{2} \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_1}(\bar{2})G(\bar{1}\bar{2}) \right]} \quad . \quad (37)$$

Letting $S_0(1) = G(11)$ we can then do the $k$ integration to obtain

$$Z_H(1) = \frac{e^{-\frac{1}{2} \frac{\delta^2(1)}{S_0(1)}}}{(2\pi S_0(1))^{n/2}} \left[ e^{\frac{1}{2} \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_1}(\bar{2})G(\bar{1}\bar{2})} \right] \quad . \quad (38)$$

Next we need

$$\langle S(H)\delta_{\nu_1}(1) \rangle = \int \frac{d^d k}{(2\pi)^d} i k_{\nu_1} \langle e^{ik \cdot m(1)} e^{H(1) \cdot m(1)} \rangle$$

$$= \int \frac{d^d k}{(2\pi)^d} i k_{\nu_1} e^{\left[ -\frac{1}{2} k^2 G(11) + ik \cdot A(1) + \frac{1}{2} \tilde{H}_{\nu_2}(\bar{1}) \tilde{H}_{\nu_2}(2)G(\bar{1}\bar{2}) \right]}$$

$$= e^{\left[ \frac{1}{2} \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_1}(\bar{2})G(\bar{1}\bar{2}) \right]} \frac{\partial}{\partial A_{\nu_1}(1)} e^{\left[ -\frac{1}{2} \tilde{H}_{\nu_1}(\bar{1}) \tilde{H}_{\nu_1}(\bar{2})G(\bar{1}\bar{2}) \right]} Z_H(1)$$

$$= -A_{\nu_1}(1) \frac{\partial}{\partial S_0(1)} Z_H(1) \quad . \quad (39)$$
Putting Eqs.(38) and (39) back into Eq.(34)

\[
\langle S(H) \nabla \cdot (\rho(1)v(1)) \rangle = -\nabla^{(1)}_{\mu_1} \frac{1}{(n-1)!} \epsilon_{\mu_1, \mu_2, ..., \mu_n} \epsilon_{\nu_1, \nu_2, ..., \nu_n} 
\times (B_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) ... \nabla_{\mu_n} A_{\nu_n}(1)) Z_H(1)
\]

(40)

where

\[
B_{\nu_1}(1) = \left( -\hat{O}(1) + \Omega(1) \right) A_{\nu_1}(1)
\]

(41)

and

\[
\Omega(1) = \frac{1}{S_0(1)} \left( \hat{O}(1) G(12) \right)_{1=2}
\]

(42)

Allowing the gradient \(\nabla^{(1)}_{\mu_1}\) to act in Eq.(40) gives

\[
\langle S(H) \nabla \cdot (\rho(1)v(1)) \rangle = -D_B(1) Z_H(1) - D^{B}_{\mu_1}(1) \nabla^{(1)}_{\mu_1} Z_H(1)
\]

(43)

where

\[
D_B(1) = \frac{1}{(n-1)!} \epsilon_{\mu_1, \mu_2, ..., \mu_n} \epsilon_{\nu_1, \nu_2, ..., \nu_n} \nabla_{\mu_1} B_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) ... \nabla_{\mu_n} A_{\nu_n}(1)
\]

(44)

and

\[
D^{B}_{\mu_1}(1) = \frac{1}{(n-1)!} \epsilon_{\mu_1, \mu_2, ..., \mu_n} \epsilon_{\nu_1, \nu_2, ..., \nu_n} B_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) ... \nabla_{\mu_n} A_{\nu_n}(1)
\]

(45)

Putting the results together in Eq.(11) we obtain

\[
\frac{\partial}{\partial t_1} (D_A(1) Z_H(1)) = D_B(1) Z_H(1) + D^{B}_{\mu_1}(1) \nabla^{(1)}_{\mu_1} Z_H(1)
\]

(46)

We can write this in the form:

\[
\frac{\partial D_A(1)}{\partial t_1} + D_A(1) \frac{\partial}{\partial t_1} \ln Z_H(1) = D_B(1) + D^{B}_{\mu_1}(1) \nabla_{\mu_1} \ln Z_H(1)
\]

(47)

Taking the derivatives

\[
\frac{\partial}{\partial t_1} \ln Z_H(1) = \frac{\partial}{\partial t_1} \left( -\frac{1}{2} \frac{A^2(1)}{S_0(1)} - \frac{n}{2} \ln (2\pi S_0(1)) \right)
\]
\[ \frac{A(1)}{S_0(1)} \cdot \hat{A}(1) + \frac{1}{2} A^2(1) \frac{\dot{S}_0(1)}{S_0^2(1)} - \frac{n \dot{S}_0(1)}{2 S_0(1)} \]  

(48)

and

\[ \nabla_{\mu_1} \ln Z_H(1) = -\frac{A(1)}{S_0(1)} \cdot \nabla_{\mu_1} A(1) . \]  

(49)

We can then write Eq.(47) in the form

\[ W_2(1) = W_4(1) \]  

(50)

where

\[ W_2(1) = \frac{\partial D_A(1)}{\partial t_1} - D_A(1) \frac{n \dot{S}_0(1)}{2 S_0(1)} - D_B(1) \]  

(51)

and

\[ W_4(1) = D_A(1) \left( \frac{A(1)}{S_0(1)} \cdot \hat{A}(1) - \frac{1}{2} A^2(1) \frac{\dot{S}_0(1)}{S_0^2(1)} \right) - D_B^{\mu_1}(1) \frac{A(1)}{S_0(1)} \cdot \nabla_{\mu_1} A(1) . \]  

(52)

Look first at \( W_2(1) \) which can be written in the form:

\[ W_2(1) = \frac{1}{(n-1)!} \epsilon_{\mu_1,\mu_2,...,\mu_n} \epsilon_{\nu_1,\nu_2,...,\nu_n} \times \nabla_{\mu_1} \left( \hat{A}_{\nu_1}(1) - \frac{1}{2} \frac{\dot{S}_0(1)}{S_0(1)} A_{\nu_1}(1) - B_{\nu_1}(1) \right) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1) \]

\[ = \frac{1}{(n-1)!} \epsilon_{\mu_1,\mu_2,...,\mu_n} \epsilon_{\nu_1,\nu_2,...,\nu_n} \nabla_{\mu_1} g_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1) \]  

(53)

where

\[ g_{\nu_1}(1) = \hat{A}_{\nu_1}(1) - \frac{1}{2} \frac{\dot{S}_0(1)}{S_0(1)} A_{\nu_1}(1) - B_{\nu_1}(1) . \]  

(54)

In looking at \( W_4 \) we need to focus on the quantity

\[ D_B^{\mu_1}(1) A_{\nu}(1) \nabla_{\mu_1} A_{\nu}(1) \]

\[ = \frac{1}{(n-1)!} \epsilon_{\mu_1,\mu_2,...,\mu_n} \epsilon_{\nu_1,\nu_2,...,\nu_n} B_{\nu_1}(1) \nabla_{\mu_2} A_{\nu_2}(1) \ldots \nabla_{\mu_n} A_{\nu_n}(1) A_{\nu}(1) \nabla_{\mu_1} A_{\nu}(1) . \]  

(55)
Note that
\[\epsilon_{\mu_1, \mu_2, \ldots, \mu_n} \nabla_{\mu_1} A_\nu(1) \nabla_{\mu_2} A_\nu(2) \ldots \nabla_{\mu_n} A_\nu(n) = \epsilon_{\nu, \nu_2, \ldots, \nu_n} D_A(1). \tag{56}\]

Putting this back into Eq.(55)
\[D^{B}_{\mu_1}(1) A_\nu(1) \nabla_{\mu_1} A_\nu(1) = \frac{1}{(n-1)!} \epsilon_{\nu_1, \nu_2, \ldots, \nu_n} \epsilon_{\nu, \nu_2, \ldots, \nu_n} B_{\nu_1}(1) A_\nu(1) D_A(1) \]
\[= B_{\nu_1}(1) A_\nu(1) D_A(1) \tag{57}\]
and
\[W_4(1) = D_A(1) \frac{A_\nu(1)}{S_0(1)} \left( \dot{A}_\nu(1) - \frac{1}{2} \frac{\dot{S}_0(1)}{S_0(1)} A_\nu(1) - B_{\nu_1}(1) \right) \]
\[= D_A(1) \frac{A_\nu(1)}{S_0(1)} g_\nu(1). \tag{58}\]

Using Eqs.(53) and (58) in Eq.(50) we find a solution for general \(\mathbf{H}\) if
\[g_\nu(1) = \dot{A}_\nu(1) - \frac{1}{2} \frac{\dot{S}_0(1)}{S_0(1)} A_\nu(1) - B_{\nu_1}(1) = 0. \tag{59}\]

This will hold for all source fields if
\[\frac{\partial}{\partial t} G_{(12)} - \frac{1}{2} \frac{\dot{S}_0(1)}{S_0(1)} G_{(12)} = \left( -\dot{\mathbf{O}}(1) + \Omega(1) \right) G_{(12)}. \tag{60}\]

III. SOLUTION FOR \(G_{(12)}\)

We can solve Eq.(60) in some generality. The first step is to write
\[G_{(12)} = \sqrt{S_0(1) S_0(2)} f_{(12)}. \tag{61}\]

Inserting this form into Eq.(60) gives
\[\frac{\partial}{\partial t} f_{(12)} = \left( -\dot{\mathbf{O}}(1) + \Omega(1) \right) f_{(12)}. \tag{62}\]
We assume that the system is translationally invariant and on Fourier transformation the operator $\hat{O}(1)$ is *diagonalized* and time independent. We have then

$$\frac{\partial}{\partial t_1} f(q, t_1, t_2) = (-O(q) + \Omega(1)) f(q, t_1, t_2). \quad (63)$$

We see that $\Omega(1)$ is determined by the constraint

$$\Omega(1) = \int \frac{dq}{(2\pi)^d} O(q) f(q, t_1, t_1). \quad (64)$$

We also have the equation

$$\frac{\partial}{\partial t_2} f(q, t_1, t_2) = (-O(q) + \Omega(2)) f(q, t_1, t_2). \quad (65)$$

Adding Eqs.(63) and (65) and setting $t_1 = t_2 = t$ we obtain for the equal-time correlation function $f(q, t) \equiv f(q, t, t)$:

$$\frac{\partial}{\partial t} f(q, t) = 2 (-O(q) + \Omega(2)) f(q, t). \quad (66)$$

For equal times we have from Eq.(61) that $f(11) = 1$ or

$$1 = \int \frac{dq}{(2\pi)^d} f(q, t). \quad (67)$$

The partial solution for Eq.(66) is given by

$$f(q, t) = \exp \left( 2 \int_{t_0}^{t} d\tau (\Omega(\tau) - O(q)) \right) f(q, t_0)$$

$$= R^2(t, t_0) e^{-2O(q)(t-t_0)} f(q, t_0) \quad (68)$$

where

$$R(t_1, t_2) = \exp \left( \int_{t_2}^{t_1} d\tau \Omega(\tau) \right). \quad (69)$$

We then need to solve for $\Omega(t)$. Inserting Eq.(68) into Eq.(67) gives

$$1 = R^2(t, t_0) I(t, t_0) \quad (70)$$

where
\[
I(t, t_0) = \int \frac{d^d q}{(2\pi)^d} e^{-2O(q) (t-t_0)} f(q, t_0) .
\]  

(71)

We then have

\[
R^2(t, t_0) = I^{-1}(t, t_0) .
\]  

(72)

The constraint condition, Eq.(64), is given by

\[
\Omega(t) = R^2(t, t_0) \int \frac{d^d q}{(2\pi)^d} O(q) e^{-2O(q) (t-t_0)} f(q, t_0)
\]

\[
= -\frac{1}{2} \frac{\dot{I}(t, t_0)}{I(t, t_0)} .
\]

(73)

Thus the determination of \( \Omega(1) \) is reduced to evaluation of some integrals. The equal time correlation function is given then by

\[
f(q, t) = I^{-1}(t, t_0) e^{-2O(q) (t-t_0)} f(q, t_0) .
\]

(75)

Going back to the unequal time correlation function we can integrate Eq.(63).

\[
f(q, t_1, t_2) = \exp \left( \int_{t_2}^{t_1} d\tau (\Omega(\tau) - O(q)) \right) f(q, t_2)
\]

\[
= R(t_1, t_2) e^{-O(q) (t_1-t_2)} f(q, t_2)
\]

\[
= R(t_1, t_2) e^{-O(q) (t_1-t_2)} R^2(t_2, t_0) e^{-2O(q) (t_2-t_0)} f(q, t_0)
\]

\[
= R(t_1, t_0) R(t_2, t_0) e^{-O(q) (t_1+t_2-2t_0)} f(q, t_0)
\]

(76)

where we have used

\[
R(t_1, t_2) = \frac{R(t_1, t_0)}{R(t_2, t_0)} .
\]

(77)

We obtain a complete solution once one does the integral for \( I(t, t_0) \). Notice that these results are independent of the specific form for \( S_0(t) \).
IV. EVALUATION OF VORTEX VELOCITY PROBABILITY DISTRIBUTION

The vortex velocity probability distribution function defined by

\[ n_0 P(V) \equiv \langle n(1) \delta(V - v(1)) \rangle \] (78)

where \( v \) as a function of the order parameter is given by Eq.(6) with \( \tilde{\psi} \) replaced by \( \tilde{m} \), \( n(1) = \delta(m(1))|\mathcal{D}(m(1))| \) is the unsigned defect density, and \( n_0(1) = \langle n(1) \rangle \). We notice in evaluating \( P(V) \) that it is of the defect sector form, thus there is a defect locating \( \delta \)-function in the average via the factor of \( n(1) \). Our results from section 2 suggest that in this sector we can treat the field \( m \) as gaussian with variance given by \( G(12) \) calculated in section 3.

In carrying out the average we need the auxiliary quantity

\[ W(\xi, b) = \langle \delta(m) \prod_{\mu,\nu} \delta(\xi^\nu - \nabla_\mu m_\nu) \delta(b - K) \rangle \] (79)

where

\[ K(1) = \hat{O}(1)m(1) \] (80)

Then

\[ n_0 P(V) = \int d^nb \prod_{\mu,\nu} d\xi^\nu |\mathcal{D}(\xi)| \delta(V - v(b, \xi)) W(\xi, b) \] (81)

where

\[ v(b, \xi) = \frac{J(b, \xi)}{\mathcal{D}(\xi)} \] (82)

with

\[ \mathcal{D}(\xi) = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \xi^{\nu_1}_{\mu_1} \xi^{\nu_2}_{\mu_2} \cdots \xi^{\nu_n}_{\mu_n} \] (83)

and

\[ J_\alpha(b, \xi) = \frac{1}{(n - 1)!} \epsilon_{\alpha,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} b_{\nu_1} \xi^{\nu_2}_{\mu_2} \cdots \xi^{\nu_n}_{\mu_n} \] (84)
Let us turn to the gaussian average determining $W(\xi, b)$. Following the analysis used to evaluate $Z_H$ in section 2 we introduce the integral representation for the $\delta$-function to obtain:

$$ W(\xi, b) = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \left( \prod_{\mu, \nu} \int \frac{dt^\nu}{2\pi} \right) e^{-ib \cdot q} e^{-i\xi_{\mu} t_{\mu}} \Gamma(k, q, t) \label{eq:85} $$

where

$$ \Gamma(k, q, t) = \langle e^{i k \cdot m(1)} e^{i q \cdot K(1)} e^{it_{\mu} \nabla_{\mu} m_{\mu}(1)} \rangle. \label{eq:86} $$

If we introduce

$$ H_\alpha(\bar{1}) = i \left[ k_\alpha + q_\alpha \hat{O}(1) + t^\alpha_{\mu} \nabla_{\mu} \right] \delta(\bar{1}1) \label{eq:87} $$

then we can write

$$ \Gamma(k, q, t) = \langle e^{H_\alpha(1)m_{\alpha}(1)} \rangle = \exp \left[ \frac{1}{2} H_\alpha(\bar{1}) H_\alpha(2) G(12) \right] \label{eq:88} $$

where $G(12)$ was determined in section 3. In the argument of the exponential we have

$$ -H_\alpha(\bar{1}) H_\alpha(2) G(12) $$

$$ = \left[ \left[ k_\alpha + q_\alpha \hat{O}(1) + t^\alpha_{\mu} \nabla_{\mu} \right] \left[ k_\alpha + q_\alpha \hat{O}(2) + t^\alpha_{\mu} \nabla_{\mu}^{(2)} \right] G(12) \right]_{1=2} $$

$$ = k^2 S_0(1) + 2k \cdot q S_c(1) + q^2 S_{O2}(1) + t_{\mu} t^{\alpha}_{\mu} S_{\mu \nu}(1) \label{eq:89} $$

where

$$ S_0(1) = G(11) \label{eq:90} $$

$$ \frac{S_c(1)}{S_0(1)} = \int \frac{d^n q}{(2\pi)^n} O(q) f(q, t) = \Omega(1) = -\frac{1}{2} \frac{\tilde{I}(1)}{I(1)} \label{eq:91} $$

$$ \frac{S_{O2}(1)}{S_0(1)} = \int \frac{d^n q}{(2\pi)^n} O^2(q) f(q, t) = \frac{1}{4} \frac{\tilde{I}(1)}{I(1)} \label{eq:92} $$
and

$$\frac{S^{(2)}_{\mu\nu}(1)}{S_0(1)} = \int \frac{d^nq}{(2\pi)^n} q_\mu q_{\nu'} f(q, t_1) = \delta_{\mu\nu'} \frac{S^{(2)}_{\mu}(1)}{S_0(1)} . \tag{93}$$

We assume that the cross terms involving an odd number of gradients vanishes. We have then

$$\Gamma(k, q, t) = \exp \left[-\frac{1}{2} (k^2 S_0(1) + 2k \cdot q S_c(1) + q^2 S_{O2}(1) + t_\mu t_{\nu} S^{(2)}_{\mu\nu}(1)) \right] . \tag{94}$$

The next step in extracting $W$ is to integrate over $k$. This involves the integral

$$J_1 = \int \frac{d^n k}{(2\pi)^n} e^{-\frac{1}{2} (k^2 S_0(1) + 2k \cdot q S_c)} . \tag{95}$$

This integral can be evaluated by completing the square:

$$k^2 S_0 + 2k \cdot q S_c = S_0 \left( k + \frac{S_c}{S_0} q \right)^2 - q^2 S_c^2 S_0^2 . \tag{96}$$

and

$$J_1 = \frac{1}{(2\pi S_0)^{n/2}} e^{-\frac{1}{2} q^2 S_c^2 S_0} . \tag{97}$$

Then

$$\Gamma(q, t) = \int \frac{d^n k}{(2\pi)^n} \Gamma(k, q, t) = \frac{1}{(2\pi S_0)^{n/2}} e^{-\frac{1}{2} q^2 S} e^{-\frac{1}{2} t_\mu t_{\nu} S^{(2)}_{\mu\nu}} \tag{98}$$

where

$$\bar{S} = S_{O2} - \frac{S_c^2}{S_0} . \tag{99}$$

Using Eq.(98) back in Eq.(85) we obtain

$$W(\xi, b) = \int \frac{d^n q}{(2\pi)^n} \left( \prod_{\mu, \nu} \right) e^{-ib \cdot q} e^{-i\xi_{\mu} t_{\nu}^\mu} \Gamma(q, t)$$

$$= \int \frac{d^n q}{(2\pi)^n} \left( \prod_{\mu, \nu} \right) e^{-ib \cdot q} e^{-i\xi_{\mu} t_{\nu}^\mu} \frac{1}{(2\pi S_0)^{n/2}} e^{-\frac{1}{2} q^2 S} e^{-\frac{1}{2} t_\mu t_{\nu} S^{(2)}_{\mu\nu}} \tag{100}$$

This factorizes into a product of three natural parts.
\[ W(\xi, b) = \frac{1}{(2\pi S_0)^{n/2}} W(\xi)W(b) \quad (101) \]

where

\[ W(\xi) = \left( \prod_{\mu, \nu} \int \frac{dt^{\nu}}{2\pi} \right) e^{-i\xi^{\nu}_{\mu}\nu} e^{-\frac{1}{2}S_{\mu}^{(2)}} \quad (102) \]

and

\[ W(b) = \int \frac{dnq}{(2\pi)^n} e^{-ib \cdot q} e^{-\frac{1}{2}q^2 S} \quad . \quad (103) \]

Using the basic integral

\[ \int \frac{dx}{2\pi} e^{-iyx} e^{-\frac{x^2}{2a}} = \frac{1}{\sqrt{2\pi a}} e^{-\frac{y^2}{2a^2}} \quad (104) \]

we can evaluate both factors:

\[ W(\xi) = \left( \prod_{\mu} \frac{1}{2\pi S_{\mu}^{(2)}} \right)^{n/2} e^{-\frac{(v^{\nu}_{\mu})^2}{2S_{\mu}^{(2)}}} \quad , \quad (105) \]

\[ W(b) = \frac{1}{(2\pi S)^{n/2}} e^{-\frac{i^2}{2S}} \quad . \quad (106) \]

Turning to \( n_0 P(V) \) given by Eq.(81), we see that we have the integral over \( b \) of the form

\[ J_b = \int d^n b \delta(V - v(b, \xi)) W(b) \]

\[ = \int d^n b \int \frac{dnz}{(2\pi)^n} e^{-iV \cdot z} e^{iv(b, \xi) \cdot z} W(b) \quad . \quad (107) \]

We can then write

\[ z \cdot v(b, \xi) = a \cdot b \quad (108) \]

where

\[ a_{\nu} = \frac{1}{D(n-1)!} \epsilon_{\alpha, \mu_1 \ldots \mu_n} \epsilon_{\nu_1, \nu_2 \ldots \nu_n} \epsilon_{\mu_1}^{\nu_2} \ldots \epsilon_{\mu_n}^{\nu_n} \quad (109) \]

then
\[
J_b = \int d^n b \int \frac{d^n z}{(2\pi)^n} e^{-iV \cdot z} e^{i a \cdot b} \frac{1}{(2\pi S)^{n/2}} e^{-\frac{\xi^2}{2 S}}
\]
\[
= \int \frac{d^n z}{(2\pi)^n} e^{-iV \cdot z} e^{-\frac{\xi^2}{2 S}}
\]
where
\[
a^2 = z_\alpha M_{\alpha\beta} z_\beta
\]
where the matrix \( M \) is given by
\[
M_{\alpha\beta} = \frac{1}{D^2[(n-1)!]^2} \epsilon_{\alpha\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \ldots
\]
Doing the remaining gaussian \( z \) integration in Eq.(110) we obtain
\[
J_b = \frac{1}{(2\pi S)^{n/2}} \frac{1}{\sqrt{\det M}} \exp \left[ -\frac{1}{2 S} \sum_{\mu,\nu} V^\mu [M^{-1}]_{\mu\nu} V^\nu \right]
\]
and
\[
n_0 P(V) = \int \prod_{\mu,\nu} d\xi^\nu \sqrt{D(\xi)} \frac{1}{(4\pi^2 S_0 S)^{n/2}} \frac{1}{\sqrt{\det M}} \exp \left[ -\frac{1}{2 S} \sum_{\mu,\nu} V^\mu [M^{-1}]_{\mu\nu} V^\nu \right]
\]
We must look at the matrix \( M \) and its inverse. First multiply \( M_{\alpha\beta} \) by \( \xi^\nu \) to obtain
\[
\xi^\nu \xi^\mu_{\alpha\beta} = \frac{1}{D^2[(n-1)!]^2} \epsilon_{\alpha,\mu_2,\ldots,\mu_n} \epsilon_{\nu,\mu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\mu_n} \epsilon_{\nu,\mu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\mu_n} \epsilon_{\nu,\mu_2,\ldots,\nu_n} \ldots
\]
However
\[
\xi^\nu_{\mu_1\mu_2\ldots\mu_n} \xi^\mu_{\mu_2\ldots\mu_n} = \sqrt{D(\xi)} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n}
\]
then
\[
\xi^\nu_{\alpha\beta} = \frac{1}{D^2[(n-1)!]^2} \sqrt{D(\xi)} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\nu_n} \epsilon_{\mu_2,\ldots,\nu_n} \ldots
\]
where we have used
\[ \epsilon_{\nu_1, \nu_2, \ldots, \nu_n} \epsilon_{\nu_1', \nu_2', \ldots, \nu_n'} = \delta_{\nu, \nu_1}(n - 1)! \]  

(119)

Multiply Eq.(118) by \( \xi^\nu_\gamma \) to obtain

\[
\xi^\nu_\gamma \xi^\gamma_\alpha M_{\alpha\beta} = \frac{1}{D(n-1)!} \epsilon^{\nu_\gamma}_{\beta_2, \ldots, \beta_n} \epsilon^{\nu_\gamma}_{\nu_1, \nu_2, \ldots, \nu_n} \xi^\nu_{\nu_1, \nu_2, \ldots, \nu_n} \frac{\mathcal{D}(\xi)}{\mathcal{D}(\xi)} \epsilon^{\nu_\gamma}_{\beta_2, \ldots, \beta_n}
\]

(120)

Thus we have the beautiful result:

\[
(M^{-1})_{\alpha\beta} = \sum_\nu \xi^{\nu}_\alpha \xi^{\nu}_\beta
\]

(121)

We need \( \det M = 1/\det M^{-1} \). We have

\[
\det M^{-1} = \frac{1}{n!} \epsilon^{\alpha_1, \alpha_2, \ldots, \alpha_n} \epsilon^{\beta_1, \beta_2, \ldots, \beta_n} \xi^\nu_{\alpha_1} \xi^\nu_{\beta_1} \xi^\nu_{\alpha_2} \xi^\nu_{\beta_2} \cdots \xi^\nu_{\alpha_n} \xi^\nu_{\beta_n}
\]

(122)

Using the clean result

\[
\det(M) = \frac{1}{(D)^2}
\]

(123)

and Eq.(121), we have

\[
n_0 P(V) = \int \left( \prod_{\mu, \nu} \frac{d\xi^\nu_{\mu}}{\sqrt{2\pi S^{(2)}_{\mu}}} \right) \exp \left[ -\sum_{\mu, \nu} \frac{1}{2S^{(2)}_{\mu}} (\xi^\nu_{\mu})^2 \right] \frac{\mathcal{D}^2(\xi)}{(4\pi^2 S_0 S)^n/2} \exp \left[ -\frac{1}{2S} \sum_{\alpha, \beta, \nu} V^\alpha \xi^\nu_{\alpha} \xi^\nu_{\beta} \right] V^\beta
\]

\[
= \int \left( \prod_{\mu, \nu} \frac{d\xi^\nu_{\mu}}{\sqrt{2\pi S^{(2)}_{\mu}}} \right) \frac{\mathcal{D}^2(\xi)}{(4\pi^2 S_0 S)^n/2} e^{-\frac{1}{2} \lambda(\xi)}
\]

(124)

where

\[ V^\alpha = \sum_{\nu} \epsilon^{\nu}_{\alpha, \nu'} V^{\nu'} \]

\[ \lambda(\xi) = \sum_{\alpha, \beta} \xi^{\nu}_{\alpha} \xi^{\nu}_{\beta} V^{\nu} \]

and

\[ \beta(\xi) = \sum_{\mu, \nu} (\xi^\nu_{\mu})^2 \]

\[ \alpha(\xi) = \sum_{\alpha, \beta} V^\alpha \xi^{\nu}_{\alpha} \xi^{\nu}_{\beta} \]
\[ A(\xi) = \sum_{\mu, \nu} \frac{1}{S^{(2)}_\mu} (\xi^{\nu})^2 + \frac{1}{S} \sum_{\alpha, \beta, \nu} V^\alpha \xi^{\nu} \xi^\nu V^\beta \]  

(125)

Net make the change of variables

\[ \xi^{\nu} = \sqrt{S^{(2)}_\mu} \tilde{\xi}^{\nu} \]  

(126)

to obtain,

\[ D(\xi) = \left( \Pi_\mu \sqrt{S^{(2)}_\mu} \right) D(\tilde{\xi}), \]

\[ n_0 P(V) = \prod_\mu \left( \frac{S^{(2)}_\mu}{2\pi \sqrt{S_0 S}} \right) \int \left( \prod_{\mu, \nu} \frac{d\tilde{\xi}^{\nu}}{(2\pi)} \right) D^2(\tilde{\xi}) e^{-\frac{1}{2} A(\tilde{\xi})} \]  

(127)

where

\[ A(\tilde{\xi}) = \sum_{\mu, \nu} (\tilde{\xi}^{\nu})^2 + \sum_{\alpha, \beta, \nu} \check{V}^\alpha \check{\xi}^{\nu} \check{\xi}^\nu \check{V}^\beta \]  

(128)

where

\[ \check{V}^\alpha = \sqrt{\frac{S^{(2)}_\alpha}{S}} V^\alpha. \]  

(129)

Next we make the transformation from \( \tilde{\xi}^{\nu} \) to \( \chi^\nu \) via

\[ \tilde{\xi}^{\nu} = N_{\alpha, \beta} \chi^\nu \]  

(130)

such that

\[ A(\tilde{\xi}) = \sum_{\alpha, \nu} (\chi^\nu)^2 \]  

(131)

\[ = \sum_{\mu, \nu} N_{\mu, \bar{\mu}_1} N_{\mu, \bar{\mu}_2} \chi^\nu_{\bar{\mu}_1} \chi^\nu_{\bar{\mu}_2} + \sum_{\alpha, \beta, \nu} \check{V}^\alpha N_{\alpha, \bar{\mu}_1} \chi^\nu_{\bar{\mu}_1} N_{\beta, \bar{\mu}_2} \chi^\nu_{\bar{\mu}_2} \check{V}^\beta \]  

(132)

This requires that \( N \) satisfy

\[ N_{\mu, \bar{\mu}_1} N_{\mu, \bar{\mu}_2} + \check{V}^\alpha N_{\alpha, \bar{\mu}_1} \check{V}^\beta N_{\beta, \bar{\mu}_2} = \delta_{\mu_1, \mu_2}. \]  

(133)

We look for a solution of the form

\[ N_{\mu_1, \mu_2} = N_0 \delta_{\mu_1, \mu_2} + N_1 \check{V}^{\mu_1} \check{V}^{\mu_2}. \]  

(134)
We then find
\[ N_{\mu_1, \mu_2} N_{\mu, \mu_2} = N_0^2 \delta_{\mu_1, \mu_2} + (2N_0 N_1 + N^2_1 (\tilde{V})^2)\tilde{V}^{\mu_1} \tilde{V}^{\mu_2} \] (135)
\[ \tilde{V}^{\alpha} N_{\alpha, \mu_1} = (N_0 + N_1 (\tilde{V})^2)\tilde{V}^{\mu_1} . \] (136)

Eq.(133) then takes the form
\[ N_0^2 \delta_{\mu_1, \mu_2} + (2N_0 N_1 + N^2_1 (\tilde{V})^2)\tilde{V}^{\mu_1} \tilde{V}^{\mu_2} = \delta_{\mu_1, \mu_2} . \] (137)

Setting the coefficients of \( \delta_{\mu_1, \mu_2} \) and \( \tilde{V}^{\mu_1} \tilde{V}^{\mu_2} \) to zero:
\[ N_0^2 = 1 \] (138)
and
\[ 2N_0 N_1 + N^2_1 (\tilde{V})^2 + (N_0 + N_1 (\tilde{V})^2)^2 = 0 . \] (139)

This is a quadratic equation for \( N_1 \) with the solution
\[ N_1 = \frac{1}{V^2} \left[ \frac{1}{\sqrt{1 + V^2}} - 1 \right] \] (140)
and
\[ N_{\alpha \beta} = \delta_{\alpha \beta} + \left[ \frac{1}{\sqrt{1 + V^2}} - 1 \right] \tilde{V}^\alpha \tilde{V}^\beta \] (141)

We then need the Jacobian of the transformation \( d\xi_\mu^\nu \rightarrow d\chi_\mu^\nu \) and \( D(\tilde{\xi}) \) evaluated in terms of \( \chi \). Look first at \( D(\tilde{\xi}) \).
\[ D(\tilde{\xi}) = \frac{1}{n!} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon_{\nu_1 \nu_2 \ldots \nu_n} \]
\[ \times \left[ \chi^\nu_1 + N_1 \tilde{V}^{\alpha_1} \chi^{\nu_1}_{\mu_1} \tilde{V}^{\mu_1} \right] \left[ \chi^\nu_2 + N_1 \tilde{V}^{\alpha_2} \chi^{\nu_2}_{\mu_2} \tilde{V}^{\mu_2} \right] \ldots \left[ \chi^\nu_n + N_1 \tilde{V}^{\alpha_n} \chi^{\nu_n}_{\mu_n} \tilde{V}^{\mu_n} \right] . \] (142)

If we multiply this out in powers of \( \tilde{V} \) we see that if we have more than one factor of \( \tilde{V}^{\alpha_i} \) then the contribution vanishes due to antisymmetry, therefore
\[ D(\tilde{\xi}) = D(\chi) + \frac{1}{n!} n \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon_{\nu_1 \nu_2 \ldots \nu_n} N_1 \tilde{V}^{\alpha_1} \chi_{\bar{\mu}_1} \tilde{V}^{\bar{\mu}_1} \chi_{\alpha_2} \ldots \chi_{\alpha_n} \]

\[ = D(\chi) + \frac{1}{n!} n N_1 \tilde{V}^{\alpha_1} \tilde{V}_{\bar{\mu}_1} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon_{\bar{\mu}_1 \alpha_2 \ldots \alpha_n} D(\chi) \]

\[ = D(\chi) \left[ 1 + \frac{1}{n!} n N_1 \tilde{V}^{\alpha_1} \tilde{V}_{\bar{\mu}_1} (n-1)! \delta_{\bar{\mu}_1 \alpha_1} \right] \]

\[ = D(\chi) \left[ 1 + N_1 \tilde{V}^2 \right] . \quad (143) \]

We have then

\[ 1 + N_1 \tilde{V}^2 = 1 - 1 + \frac{1}{\sqrt{1 + \tilde{V}^2}} = \frac{1}{\sqrt{1 + \tilde{V}^2}} \quad (144) \]

and

\[ D(\tilde{\xi}) = \frac{D(\chi)}{\sqrt{1 + \tilde{V}^2}} . \quad (145) \]

Next we need the Jacobian

\[ J = \prod_{\nu} \det \left( \frac{\partial \tilde{\xi}_\nu}{\partial \chi_{\mu'}} \right) \]

\[ = \left[ \det \left( \delta_{\mu, \mu'} + N_1 \tilde{V}^\mu \tilde{V}_{\mu'} \right) \right]^n = (J_0)^n \quad (147) \]

where

\[ J_0 = \det \left( \delta_{\mu, \mu'} + N_1 \tilde{V}^\mu \tilde{V}_{\mu'} \right) \]

\[ = \frac{1}{n!} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon_{\beta_1 \beta_2 \ldots \beta_n} \left[ \delta_{\alpha_1, \beta_1} + N_1 \tilde{V}^{\alpha_1} \tilde{V}_{\beta_1} \right] \left[ \delta_{\alpha_2, \beta_2} + N_1 \tilde{V}^{\alpha_2} \tilde{V}_{\beta_2} \right] \ldots \left[ \delta_{\alpha_3, \beta_3} + N_1 \tilde{V}^{\alpha_3} \tilde{V}_{\beta_3} \right] . \quad (148) \]

Again, expanding this out in powers of \( \tilde{V} \), only the first two terms contribute due to symmetry and we have

\[ J_0 = \frac{1}{n!} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n}^2 + n N_1 \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon_{\beta_1 \alpha_2 \ldots \alpha_n} \tilde{V}^{\alpha_1} \tilde{V}_{\beta_1} \]
\[= 1 + N_1 \tilde{V}^2 = \frac{1}{\sqrt{1 + \tilde{V}^2}}. \quad (150)\]

Going back to Eq. (127) we have

\[
n_0 P[V] = \prod_\mu \left( \frac{S^{(2)}_\mu}{2\pi \sqrt{S_0 S}} \right) \frac{1}{(1 + \tilde{V}^2)^{(n+2)/2}} J_F \quad (151)\]

where we have the final integral

\[
J_F = \int \prod_{\mu, \nu} \frac{d\chi^\nu}{\sqrt{2\pi}} D^2(\chi) e^{-\frac{1}{2} A(\chi)}. \quad (152)\]

We can evaluate \(J_F\) directly. The first step is to write:

\[
J_F = \int \prod_{\mu} \frac{d\chi^\mu}{\sqrt{2\pi}} \epsilon_{\mu_1 \mu_2 \ldots \mu_n} \epsilon_{\mu'_1 \mu'_2 \ldots \mu'_n} \chi^{(1)}_{\mu_1} \chi^{(2)}_{\mu_2} \ldots \chi^{(n)}_{\mu_n} e^{-\frac{1}{2} \sum_{\mu} (\chi^\mu)^2}. \quad (153)\]

This factorizes into a product of integrals for fixed \(\nu\)

\[
J_F = \epsilon_{\mu_1 \mu_2 \ldots \mu_n} \epsilon_{\mu'_1 \mu'_2 \ldots \mu'_n} \int \prod_{\mu} \frac{d\chi^{(1)}_{\mu}}{\sqrt{2\pi}} \chi^{(1)}_{\mu_1} e^{-\frac{1}{2} \sum_{\mu} (\chi^{(1)}_{\mu})^2} \int \prod_{\mu} \frac{d\chi^{(2)}_{\mu}}{\sqrt{2\pi}} \chi^{(2)}_{\mu_1} e^{-\frac{1}{2} \sum_{\mu} (\chi^{(2)}_{\mu})^2} \ldots

\times \int \prod_{\mu} \frac{d\chi^{(n)}_{\mu}}{\sqrt{2\pi}} \chi^{(n)}_{\mu_1} e^{-\frac{1}{2} \sum_{\mu} (\chi^{(n)}_{\mu})^2}. \quad (153)\]

Each integral in the product is equal to 1 except for those giving a \(\delta\)-function with unit coefficient:

\[
J_F = \epsilon_{\mu_1 \mu_2 \ldots \mu_n} \epsilon_{\mu'_1 \mu'_2 \ldots \mu'_n} \delta_{\mu_1, \mu'_1} \delta_{\mu_2, \mu'_2} \ldots \delta_{\mu_n, \mu'_n}

= \epsilon^2_{\mu_1 \mu_2 \ldots \mu_n} = n!. \quad (154)\]

We have then

\[
n_0 P[V] = n! \prod_\mu \left( \frac{S^{(2)}_\mu}{2\pi \sqrt{S_0 S}} \right) \frac{1}{(1 + \tilde{V}^2)^{(n+2)/2}}. \quad (155)\]

Since \(P(V)\) is normalized to one, we find on integration over \(V\), the result

\[
n_0 = \frac{n!}{2^{n/2} \Gamma\left(\frac{n}{2} + 1\right)} \prod_\mu \sqrt{\frac{S^{(2)}_\mu}{2\pi S_0}} \quad (156)\]
which agrees with previous results in the isotropic limit. Eliminating $n_0$ in Eq.(155) we obtain

$$P[V] = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}} \left( \prod_{\mu} \frac{1}{\bar{v}_{\mu}} \right) \frac{1}{(1 + \tilde{V}^2)^{(n+2)/2}}$$  \hspace{1cm} (157)

where

$$\bar{v}_{\mu} = \sqrt{\frac{S}{S^{(2)}_{\mu}}}$$  \hspace{1cm} (158)

and $\tilde{V}_{\alpha}$ is given by Eq.(129). This result basically says that the probability of finding a large velocity decreases with time. However, since this distribution falls off only as $V^{-(n+2)}$ for large $V$ only the first moment beyond the normalization integral exists. This seems to imply the existence of a source of large velocities.

**V. ANISOTROPIC CASE**

As a particular example, suppose that we have the choice in the governing Langevin equation

$$\hat{O}(1)\psi_\nu(1) = -\sum_\alpha c_\alpha \nabla^2_\alpha \psi_\nu(1),$$  \hspace{1cm} (159)

or, in terms of Fourier transforms,

$$O(q) = \sum_\alpha c_\alpha q^2_\alpha.$$  \hspace{1cm} (160)

What is the associated vortex-velocity probability distribution? Clearly this is given by Eq.(157) with $\tilde{V}_{\alpha}$ given by Eq.(129). We then need to use the results of section III to compute $S_c(1)/S(1)$, $S_{O2}(1)/S(1)$, $\tilde{S}(1)/S(1)$ and $S^{(2)}_{\mu}(1)/S(1)$. We easily find

$$\frac{S_c(1)}{S(1)} = \Omega(1) = -\frac{1}{2} \frac{\dot{I}(1)}{I(1)}$$  \hspace{1cm} (161)

$$\frac{S_{O2}(1)}{S(1)} = \Omega_4(1) = \int \frac{d^nq}{(2\pi)^n} O^2(q)f(q,t) = \frac{1}{4} \frac{\dot{I}(1)}{I(1)}$$  \hspace{1cm} (162)
\begin{equation}
\frac{S(1)}{S_0(1)} = \Omega_4(1) - \Omega^2(1) \tag{163}
\end{equation}

and

\begin{equation}
\frac{S^{(2)}(1)}{S_0(1)} = \int \frac{d^nq}{(2\pi)^n} q^2_\mu f(q, t_1) \tag{164}
\end{equation}

Assuming the initial condition

\begin{equation}
f(q, 0) = \left( \prod_\alpha (2\pi h_\alpha)^{1/2} \right) e^{-\frac{1}{2} h_\mu q^2_\mu}, \tag{165}
\end{equation}

we have using Eq.(71)

\begin{equation}I(t) = \int \frac{d^nq}{(2\pi)^n} \left( \prod_\alpha (2\pi h_\alpha)^{1/2} \right) e^{-\frac{1}{2} h_\mu(t) q^2_\mu} = \left( \prod_\alpha \left( \frac{h_\alpha}{h_\alpha(t)} \right)^{1/2} \right) \tag{166}
\end{equation}

where

\begin{equation}h_\alpha(t) = h_\alpha + 4c_\alpha(t - t_0) \tag{167}
\end{equation}

Using this result we easily obtain

\begin{equation}\dot{I}(t) = I(t) \left(-\frac{1}{2} \sum_\mu \frac{4c_\mu}{h_\mu(t)} \right) \tag{168}
\end{equation}

and

\begin{equation}\Omega(1) = \sum_\mu \frac{c_\mu}{h_\mu(t)} \tag{169}
\end{equation}

Similarly

\begin{equation}\Omega_4(1) = \Omega^2(1) + 2 \sum_\mu \left( \frac{c_\mu}{h_\mu(t)} \right)^2 \tag{170}
\end{equation}

and

\begin{equation}\frac{\bar{S}}{S_0} = 2 \sum_\mu \left( \frac{c_\mu}{h_\mu(t)} \right)^2 \tag{171}
\end{equation}
\[
\frac{S^{(2)}_{\mu}}{S_{\mu}(1)} = \int \frac{d^nq}{(2\pi)^n} q^2_f(q, t_1) = \frac{1}{h_\mu(t)} .
\] (172)

Putting these results back into Eq.(158) we find the general final result for the scaling velocity for a simple anisotropic system

\[
\bar{v}_\mu^2 = 2h_\mu(t) \sum_\alpha \left( \frac{c_\alpha}{h_\alpha(t)} \right)^2
\] (173)

\[
= 2(h_\mu + 4c_\mu(t - t_0)) \sum_\alpha \left( \frac{c_\alpha}{h_\alpha + 4c_\alpha(t - t_0)} \right)^2 .
\] (174)

In the large time limit we have

\[
\bar{v}_\mu^2 = \frac{d c_\mu}{2 t}
\] (175)

and the final form is a simple generalization of the isotropic result.

VI. CONCLUSIONS

We have presented here the complete calculation of the vvpdf including the time dependent vortex scaling velocity \( \bar{v}_\mu \). We see that there is self-consistent confirmation that in dealing with vortex velocities one can organize things in terms of averages over an auxiliary gaussian field. We require self-consistently that this field and the order parameter field share the same zeros. A similar development holds for string defects \([5,6]\).

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