Abstract

This paper proposes optimal mean squared error batch sizes for multivariate batch means and spectral variance estimators. We propose a novel estimation technique for the optimal batch sizes, which is computationally inexpensive and has low variability. Further, the asymptotic mean squared error for a family of spectral variance estimators is derived under conditions convenient to verify for Markov chain Monte Carlo simulations. Vector auto-regressive, Bayesian logistic regression, and Bayesian dynamic space-time examples illustrate the quality of the estimation procedure where optimal batch sizes proposed here outperform current batch size selection methods.

1 Introduction

In Markov chain Monte Carlo (MCMC) simulations, it is desirable to estimate the variability of ergodic averages to assess the quality of estimation (see e.g. Flegal et al., 2008; Geyer, 2011; Jones and Hobert, 2001). Estimation of this variability can be approached through a multivariate Markov chain central limit theorem (CLT). To this end, let $F$ be a probability distribution with support $X \in \mathbb{R}^d$ and $g : X \to \mathbb{R}^p$ be a $F$-integrable function. We are

*Research supported by National Science Foundation
interested in estimating the $p$-dimensional vector

$$\theta = \int_X g(x) dF$$

using draws from a Harris $F$-ergodic Markov chain, say $\{X_t\}$. Then, letting $Y_t = g(X_t)$ for $t \geq 1$, $\bar{Y}_n = n^{-1} \sum_{t=1}^{n} Y_t \rightarrow \theta$ with probability 1 as $n \rightarrow \infty$. The sampling distribution for $\bar{Y}_n - \theta$ is available via a Markov chain CLT if there exists a positive definite symmetric matrix $\Sigma$ such that

$$\sqrt{n}(\bar{Y}_n - \theta) \xrightarrow{d} N_p(0, \Sigma) \text{ as } n \rightarrow \infty,$$

where

$$\Sigma = \text{Var}_F(Y_1) + \sum_{k=1}^{\infty} \left[ \text{Cov}_F(Y_1, Y_{1+k}) + \text{Cov}_F(Y_1, Y_{1+k})^T \right].$$

We assume throughout that the CLT at (1) holds (see e.g. Jones, 2004) and consider estimation of $\Sigma$. Provided such an estimator is available, one can access variability of $\bar{Y}_n$ by constructing a $p$-dimensional confidence ellipsoid. Recent work of Vats et al. (2018a,b) obtains necessary conditions for strongly consistent estimation for multivariate spectral variance (SV) and batch means (BM) estimators. Liu and Flegal (2018) proposes computationally efficient weighted BM estimators that allow for flexible choice of window function similar to SV estimators. These broad family of estimators account for the serial correlation in the Markov chain up to a certain lag. This lag, denoted as $b$, is called the bandwidth and batch size in SV and BM estimators, respectively. We will refer to both as the batch size, unless otherwise stated. The choice of $b$ is crucial to finite sample performance, but estimating an optimal $b$ has not been carefully addressed in multivariate output analysis. Instead, suboptimal bandwidths and batch sizes inherited from univariate estimators are widely used by practitioners.

Optimal batch size selection has been studied carefully in other contexts such as non-parametric density and spectral density function estimation. For example, Politis (2003, 2011) and Politis and Romano (1999) discuss optimal bandwidth selection of SV estimators using the flat-top window function. Estimation of the time-average variance constant using a recursive estimator is proposed by Chan and Yau (2017) where an optimal batch size is suggested. An interested reader is further directed to Jones et al. (1996), Silverman (1999), Woodroofe (1970), Sheather (1983), Sheather and Jones (1991) for bandwidth selection in density estimation. Broadly speaking, these results are not computationally viable for high-dimensional MCMC where long run lengths are standard.

Until recently, most MCMC simulations only considered estimation of the diagonal entries of $\Sigma$. An incomplete list of appropriate univariate estimators includes BM and over-
lapping BM \cite{flegal2010approximation, jones2006monte, meketon1984}, SV methods including flat-top estimators \cite{anderson1994, politis1995, politis1996}, initial sequence estimators \cite{geyer1992}, and regenerative simulation \cite{hobert2002, mykland1995, seila1982}. For more general dependent processes, Song and Schmeiser \cite{song1995} and Damerdji \cite{damerdji1995} consider univariate BM estimators and obtain optimal batch sizes that minimize the asymptotic mean squared error (MSE). Flegal and Jones \cite{flegal2010approximation} also consider these estimators for MCMC simulations under weaker mixing and moment conditions. These papers show the optimal batch size is proportional to $n^{1/3}$, but there has been no work in MCMC settings with regard to estimating the proportionality constant. As a result, Flegal and Jones \cite{flegal2010approximation} suggest using the suboptimal batch size equal to $\lfloor n^{1/2} \rfloor$ to avoid estimation of the unknown proportionality constant.

This paper has two major contributions. First, we provide a fast and stable estimation procedure for the optimal batch size proportionality constant. In short, we use a stationary autoregressive process of order $m$ to approximate the marginals of $\{Y_t\}$, which yields a closed form expression for the unknown proportionality constant. This gives a different batch size for each entry of $\Sigma$ that is difficult to implement in practice. Hence we propose using the average of the univariate optimal batch sizes as an overall batch size.

Second, we obtain an optimal batch size expression for multivariate SV estimators. This is a substantial generalization since prior results only consider univariate overlapping BM estimators \cite{damerdji1995, flegal2010approximation}. That is, these results only consider SV estimators with a Bartlett lag window. As in the univariate case, the optimal batch size is proportional to $n^{1/3}$. As a comparison, we obtain the optimal batch size for BM estimators, which are also proportional to $n^{1/3}$.

A vector autoregressive process of order 1 is used to evaluate finite sample performance of our proposed method in comparison to the more common flat-top pilot estimators \cite{politis2003, politis2011, politis1999}. In this simple example, the optimal batch size can be calculated analytically. Next, we present a Bayesian logistic regression example with real data and compare the performance of the optimal batch size methods with the more commonly used batch sizes of $\lfloor n^{1/3} \rfloor$ and $\lfloor n^{1/2} \rfloor$. A similar analysis is done for a Bayesian dynamic space-time model with real data.

The rest of this paper is organized as follows. Section 2 presents the SV and BM estimators. Section 3 summarizes results on the optimal batch size and proposes a novel estimation technique. Section 4 presents the theoretical details on the derivation of the optimal batch size. Section 5 considers three examples to compare performances between suggested and more commonly used batch sizes. All proofs are relegated to the appendices.
2 Batch means and spectral variance estimators

The BM estimator of $\Sigma$ is constructed using the sample covariance matrix from batches of the Markov chain. For a Monte Carlo sample size $n$, let $n = ab$, where $a$ is the number of batches and $b$ is the batch size. For $l = 0, \ldots, a - 1$, let $\bar{Y}_l = b^{-1} \sum_{t=1}^{b} Y_{lb+t}$ denote the mean vector of the batch. The BM estimator is defined as,

$$\hat{\Sigma}_B = \frac{b}{a-1} \sum_{l=0}^{a-1} (\bar{Y}_l - \bar{Y}_n)(\bar{Y}_l - \bar{Y}_n)^T.$$  

(2)

Strong and mean square consistency requires both the batch size $b$ and the number of batches $a$ to increase with $n$. Large batch sizes capture more lag correlations with fewer batches, implying larger variance. Alternatively, small batch sizes yield higher bias. We present theoretical results on the optimal choice of $b$ in Section [4].

A similar trade-off is seen for SV estimators, which we present below. Consider estimating the lag $k$ autocovariance denoted by $R(k) = E_F (Y_t - \theta)(Y_{t+k} - \theta)^T$ with

$$\hat{R}(k) = \frac{1}{n} \sum_{t=1}^{n-k} (Y_t - \bar{Y}_n)(Y_{t+k} - \bar{Y}_n)^T.$$  

The SV estimator truncates and downweights the summed lag $k$ autocovariances. That is,

$$\hat{\Sigma}_s := \hat{R}(0) + \sum_{k=1}^{b} w_n(k) \left[ \hat{R}(k) + \hat{R}(k)^T \right],$$

where $b$ is the bandwidth or truncation point and $w_n(\cdot)$ is the lag window. The bandwidth serves the same role as the batch size in the BM estimator, hence we denote both by $b$.

We assume throughout that the lag window is an even function defined on $\mathbb{Z}$. In addition, we assume (i) $|w_n(k)| \leq 1$ for all $n$ and $k$, (ii) $w_n(0) = 1$ for all $n$, and (iii) $w_n(k) = 0$ for all $|k| \geq b$. Some examples of lag windows are Bartlett, Bartlett flat-top, and Tukey-Hanning lag windows defined as

$$w_n(k) = \left( 1 - |k|/b \right) I(|k| \leq b),$$

(3)

$$w_n(k) = I(|k| \leq b/2) + (2(1 - |k|/b)) I(b/2 < |k| \leq b),$$

(4)

$$w_n(k) = ((1 + \cos(\pi |k|/b))/2) I(|k| \leq b),$$

(5)

respectively. Figure [1] illustrates these lag windows. In MCMC, the Bartlett lag window is by far the most commonly used. However, the downweighting of lags induces significant
bias. The motivation behind flat-top lag windows is to reduce downweighting of small lag terms by letting \( w_n(k) = 1 \) for \( k \) near 0 (see e.g., Politis and Romano, 1995, 1996).

![Lag windows](image)

Figure 1: Plot of Bartlett, Tukey-Hanning, and Bartlett flat-top lag windows.

### 3 Estimating optimal batch sizes

As we show later for BM and SV estimators, the optimal \( b \) for estimating the \( ij \)th element of \( \Sigma \), \( \Sigma_{ij} \), is

\[
 b_{opt,ij} \propto \left( \frac{\Gamma_{ii}^2}{\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2} \right)^{1/3} \cdot n^{1/3},
\]

where

\[
 \Gamma = - \sum_{k=1}^{\infty} k \left[ R(k) + R(k)^T \right]
\]

with components \( \Gamma_{ij} \). Hence estimating \( b_{opt,ij} \) requires estimates of \( \Gamma_{ii} \) and \( \Sigma_{ii} \), which is the focus of this section.

Choosing a different \( b \) for each element of \( \Sigma \) requires substantial computational effort and it is unclear if it makes intuitive sense. Since \( b \) can be calculated for each univariate component, we define the overall optimal \( b \) by averaging the diagonals of \( b_{opt,ij} \). That is,

\[
 b_{opt} := \frac{1}{p} \sum_{i=1}^{p} b_{opt,ii} \propto \frac{1}{p} \sum_{i=1}^{p} \left( \frac{\Gamma_{ii}^2}{\Sigma_{ii}^2} \right)^{1/3} \cdot n^{1/3}.
\]
The proportionality constant is known and depends on the choice of variance estimator (see Section 4 for details).

To estimate \( b_{opt} \), we require estimates of \( \Gamma_{ii} \) and \( \Sigma_{ii} \), which is our initial goal. A common solution is to use pilot estimators, i.e. a pilot run of the process is obtained and used to estimate \( \Gamma_{ii} \) and \( \Sigma_{ii} \) (see e.g. Jones et al. 1996; Loader 1999; Politis 2003; Woodroofe 1970). Two such procedures are the flat-top pilot estimator and empirical rule (Politis 2003) and the iterative plug-in pilot estimator (Brockmann et al. 1993; Bühlmann 1996). In both settings, an SV estimator is constructed for each of the \( i \) components for the pilot run where the bandwidth is chosen by an empirical or iterative rule that monitors lag autocorrelations.

Unlike final estimators of \( \Sigma \), we do not require consistency for estimators of \( b_{opt} \). Thus, the pilot step need not be based on BM or SV estimators. The choice of the estimator for \( b_{opt} \) only impacts the finite sample properties of the estimator of \( \Sigma \), while maintaining the asymptotic results. That is, if the asymptotic results of strong consistency and mean square consistency hold for \( b = \lfloor n^{1/3} \rfloor \), then they also hold for \( b = c \lfloor n^{1/3} \rfloor \), for some constant \( c \).

We argue estimators of \( b_{opt} \) should be (i) computationally inexpensive and (ii) have low variability. Both the empirical rule and iterative plug-in estimators can be computationally involved (especially for slow mixing chains) and hence they fail our first criteria. These estimators can also yield highly variable estimates of \( b_{opt} \), which we illustrate in our examples. Low variability is particularly important since the user cannot be expected to run multiple pilot runs to obtain a stable estimate of \( b_{opt} \).

We now present pilot estimators of \( \Gamma_{ii} \) and \( \Sigma_{ii} \) satisfying our two criteria. In short, we propose using a stationary autoregressive process of order \( m \) (AR(\( m \))) approximation to the marginals of \( \{Y_t\} \). For \( t = 1, 2, \ldots, \) let \( W_t \in \mathbb{R} \) be such that

\[
W_t = \sum_{i=1}^{m} \phi_i W_{t-i} + \epsilon_t,
\]

where \( \epsilon_t \) has mean 0 and variance \( \sigma^2_\epsilon \), and \( \phi_1, \ldots, \phi_m \) are the autoregressive coefficients. Let \( \gamma(k) \) be the lag \( k \) autocovariance function for the process. Then by the Yule-Walker equations, it is known that for \( k > 0 \), \( \gamma(k) = \sum_{i=1}^{m} \phi_i \gamma(k-i) \), and \( \gamma(0) = \sum_{i=1}^{m} \phi_i \gamma(i) + \sigma^2_\epsilon \). The asymptotic variance in the CLT is known to be

\[
\sigma^2 = \sum_{k=-\infty}^{\infty} \gamma(k) = \frac{\sigma^2_\epsilon}{(1 - \sum_{i=1}^{m} \phi_i)^2}.
\]
Obtaining an expression for $\sum_{k=1}^{\infty} k\gamma(k)$ is more involved. Following [Taylor (2018)],

$$\sum_{k=1}^{\infty} k\gamma(k) = \sum_{k=1}^{\infty} k \sum_{i=1}^{m} \phi_i \gamma(k - i)$$

$$= \sum_{i=1}^{m} \phi_i \left( \sum_{k=1}^{\infty} k\gamma(k - i) \right)$$

$$= \left( \sum_{i=1}^{m} \phi_i \sum_{k=1}^{i} k\gamma(k - i) \right) + \left( \sum_{i=1}^{m} \phi_i \sum_{k=i+1}^{\infty} k\gamma(k - i) \right)$$

$$= \left( \sum_{i=1}^{m} \phi_i \sum_{k=1}^{i} k\gamma(k - i) \right) + \left( \sum_{i=1}^{m} \phi_i \sum_{s=1}^{\infty} (s + i)\gamma(s) \right)$$

$$= \left( \sum_{i=1}^{m} \phi_i \sum_{k=1}^{i} k\gamma(k - i) \right) + \left( \sum_{i=1}^{m} \phi_i \left[ \sum_{s=1}^{\infty} s\gamma(s) + i \sum_{s=1}^{\infty} \gamma(s) \right] \right)$$

$$= \left( \sum_{i=1}^{m} \phi_i \sum_{k=1}^{i} k\gamma(k - i) \right) + \left( \sum_{i=1}^{m} \phi_i \sum_{s=1}^{\infty} s\gamma(s) \right) + \left( \sum_{i=1}^{m} \phi_i \sum_{s=1}^{\infty} \gamma(s) \right)$$

$$\Rightarrow \sum_{k=1}^{\infty} k\gamma(k) = \left[ \left( \sum_{i=1}^{m} \phi_i \sum_{k=1}^{i} k\gamma(k - i) \right) + \frac{\sigma_e^2 - \gamma(0)}{2} \left( \sum_{i=1}^{m} i\phi_i \right) \right] \left( \frac{1}{1 - \sum_{i=1}^{m} \phi_i} \right).$$

We propose to fit an AR($m$) model for each marginal of the Markov chain, where $m$ is determined by Akaike information criterion. The autocovariances $\gamma(s)$ are estimated by the sample lag autocovariances, $\hat{\gamma}(s)$. Then $\sigma_e^2$ and $\phi_i$ are estimated by $\hat{\sigma}_e^2$ and $\hat{\phi}_i$, respectively, by solving the Yule-Walker equations. For the $i$th component of the Markov chain, the resulting pilot estimators are

$$\Sigma_{ii}^{(0)} = \frac{\hat{\sigma}_e^2}{(1 - \sum_{i=1}^{m} \hat{\phi}_i)^2},$$

and

$$\hat{\Gamma}_{ii}^{(0)} = -2 \left[ \left( \sum_{i=1}^{m} \hat{\phi}_i \sum_{k=1}^{i} k\hat{\gamma}(k - i) \right) + \frac{(\hat{\sigma}_e^2 - \hat{\gamma}(0))}{2} \left( \sum_{i=1}^{m} i\hat{\phi}_i \right) \right] \left( \frac{1}{1 - \sum_{i=1}^{m} \hat{\phi}_i} \right).$$

An AR($m$) approximation for MCMC has also been studied by [Thompson (2010)] who considers estimating the integrated autocorrelation time of a process. Further, the R package [coda (Plummer et al., 2006)] uses (8) as the final estimator of $\Sigma_{ii}$ when calculating univariate effective sample sizes.
4 Theoretical results

This section presents optimal bandwidth and batch size results for SV and BM estimators, respectively. We assume the following two assumptions for mean square consistency.

Assumption 1. The batch size $b$ is an integer sequence such that $b \to \infty$ and $n/b \to \infty$ as $n \to \infty$, where $b$ and $n/b$ are both monotonically non-decreasing.

Denote the Euclidean norm by $\| \cdot \|$ and let $\{ B(t), t \geq 0 \}$ be a $p$-dimensional Brownian motion. Our second assumption is that of a strong invariance principle.

Assumption 2. There exists a $p \times p$ lower triangular matrix $L$, a non-negative increasing function $\psi$ on the positive integers, a finite random variable $D$, and a sufficiently rich probability space $\Omega$ such that for almost all $\omega \in \Omega$ and for all $n > n_0$,

$$\left\| \sum_{t=1}^{n} Y_t - n\theta - LB(n) \right\| < D(\omega)\psi(n) \quad \text{with probability } 1. \quad (9)$$

Assumption 2 implies the CLT at (1) holds and is needed to obtain expressions for element-wise variance of BM and SV estimators. We now turn to deriving the bias and variance for the SV estimators.

Consider an alternative representation of the SV estimator, which differs on only some end effects. This expression has been used previously by Damerdji (1995) and Flegal and Jones (2010). First define $\Delta_1 w_n(k) = w_n(k-1) - w_n(k)$ and $\Delta_2 w_n(k) = w_n(k-1) - 2w_n(k) + w_n(k+1)$. Let $\bar{Y}_l(k) = k^{-1} \sum_{t=1}^{k} Y_{l+t}$ for $l = 0, ..., n - k$ and consider the estimator

$$\hat{\Sigma}_w = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_n(k)(\bar{Y}_l(k) - \bar{Y})(\bar{Y}_l(k) - \bar{Y})^T.$$

Suppose that $d$ is such that $\hat{\Sigma}_w = \bar{\Sigma}_w - d$ (the expression for $d$ is in Appendix B). When $b$ is fixed, (20) shows $d = o(n^{-1})$ hence the estimators are asymptotically equivalent. Our first result shows $d \to 0$ with probability 1 as $n \to \infty$. Define $h(X_t) = (g(X_t) - E F g)^2$, where the square is component-wise for $t = 1, 2, \ldots$. Preliminaries for the following result can be found in Appendix A followed by the proof in Appendix B.

Theorem 1. Assume Assumption 2 holds and Assumption 2 holds for both $g$ and $h$. If $n > 2b$, $b^{-1}\log n = O(1)$, $b^{-1}\psi(n) \to 0$, $b^{-1}\psi_h(n) \to 0$ and $b^{-1} \sum_{k=1}^{b} k |\Delta_1 w_n(k)| \to 0$ as $n \to \infty$, then $d \to 0$ with probability 1 as $n \to \infty$.

We now derive the asymptotic bias and variance for $\hat{\Sigma}_w$, which enables calculation of an optimal bandwidth by minimizing the MSE. We use this MSE to approximate that of
\[ \hat{\Sigma}_s, \text{ which is difficult to obtain directly. Let } \hat{\Sigma}_{w,ij} \text{ be the } ij \text{th entry of } \hat{\Sigma}_w, \text{ then we begin by calculating the asymptotic bias.} \]

**Theorem 2.** Suppose \( E_F \|g\|^{2+\delta} < \infty \) for \( \delta > 0 \) and the lag window satisfies \( \sum_{k=1}^{b} k \Delta_2 w_n(k) = 1. \) If \( \{X_t\} \) is a polynomially ergodic Markov chain of order \( \xi > (2 + \epsilon)(1 + 2/\delta) \) for some \( \epsilon > 0, \) then

\[
\text{Bias} (\hat{\Sigma}_{w,ij}) = \sum_{k=1}^{b} \Delta_2 w_n(k) \Gamma_{ij} + o \left( \frac{b}{n} \right) + o \left( \frac{1}{b} \right).
\]

**Proof.** By Vats and Flegal (2018, Theorem 2), \( |\Gamma_{ij}| < \infty. \) Then under Assumption 1, for all \( i \) and \( j, \)

\[
\text{Cov}[\overline{Y}^{(i)}(k), \overline{Y}^{(j)}(k)] - \text{Cov}[\overline{Y}^{(i)}_n, \overline{Y}^{(j)}_n] = \frac{n-k}{kn} \left( \Sigma_{ij} + \frac{n+k}{kn} \Gamma_{ij} + o \left( \frac{1}{k^2} \right) \right).
\]

Since \( \sum_{k=1}^{b} k \Delta_2 w_n(k) = 1, \) by (10),

\[
E \left( \hat{\Sigma}_{w,ij} \right) = \sum_{k=1}^{b} \frac{(n-k+1)(n-k)k \Delta_2 w_n(k)}{n^2} \cdot \Sigma_{ij}
\]

\[
+ \sum_{k=1}^{b} \frac{(n-k+1)(n^2-k^2) \Delta_2 w_n(k)}{n^3} \cdot \Gamma_{ij} + o \left( \frac{b}{n} \right) + o \left( \frac{1}{b} \right)
\]

\[
= \Sigma_{ij} + \sum_{k=1}^{b} \frac{(n-k+1)(n^2-k^2) \Delta_2 w_n(k)}{n^3} \cdot \Gamma_{ij} + o \left( \frac{b}{n} \right) + o \left( \frac{1}{b} \right)
\]

\[
= \Sigma_{ij} + \sum_{k=1}^{b} \Delta_2 w_n(k) \cdot \Gamma_{ij} + o \left( \frac{b}{n} \right) + o \left( \frac{1}{b} \right).
\]

Next, we present the element-wise variance of the SV estimator. Appendix C contains a number of preliminary results, followed by the proof of Theorem 3 in Appendix D.

**Theorem 3.** Suppose Assumption 2 holds for \( g \) with \( L, D, \psi \) and \( h \) with \( L_h, D_h, \psi_h. \) Further, suppose \( ED^4 < \infty, \) Assumption 1 holds and if

1. \( \sum_{k=1}^{b} (\Delta_2 w_k)^2 = O \left( 1/b^2 \right), \)
2. \( b \psi^2(n) \log n \left( \sum_{k=1}^{b} |\Delta_2 w_n(k)| \right)^2 \rightarrow 0, \) and
3. \( \psi^2(n) \sum_{k=1}^{b} |\Delta_2 w_n(k)| \rightarrow 0, \) then,
\[
\text{Var}(\hat{\Sigma}_{w,ij}) = [\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2]\left[\frac{2}{3} \sum_{k=1}^{b} (\Delta_2 w_k)^2 k^3 \frac{1}{n} + 2 \left(\sum_{t=1}^{b-1} \sum_{u=1}^{b-t} \Delta_2 w_u \Delta_2 w_{t+u} \left(\frac{2}{3} u^3 + u^2 t\right) \frac{1}{n}\right) + o \left(\frac{b}{n}\right)\right] := ([\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2]S + o(1)) \frac{b}{n}.
\]

Combining results from Theorem 2 and Theorem 3 yields \(\text{MSE}[\hat{\Sigma}_{w,ij}] \to 0\) as \(n \to \infty\).

If \(S \neq 0\) in Theorem 3 and for a constant \(C \neq 0\), if \(\sum_{k=1}^{b} \Delta_2 w_n(k) = C / b\), then

\[
\text{MSE} \left(\hat{\Sigma}_{w,ij}\right) = \frac{C^2 \Gamma_{ij}^2}{b^2} + [\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2]S \frac{b}{n} + o \left(\frac{b}{n}\right) + o \left(\frac{1}{b}\right).
\]

Further, the bandwidth \(b_{\text{opt},ij}\) that minimizes the asymptotic MSE is

\[
b_{\text{opt},ij} = \left(\frac{2C^2}{S \Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2}\right)^{1/3}.
\]

Thus, for the SV estimator, the proportionality constant in (6) is \(2C^2 / S\). Expressions of \(S\) and \(C\) can be derived for different lag windows.

**Example 1.** Consider the Bartlett window in (3). Here, \(\Delta_2 w_n(b) = b^{-1}\) and \(\Delta_2 w_n(k) = 0\) for \(k = 1, 2, \ldots, b - 1\). So, \(\sum_{k=1}^{b} k \Delta_2 w_n(k) = 1\), and since \(\sum_{k=1}^{b} \Delta_2 w_n(k) = 1 / b\), using Theorem 2, \(C = 1\). Condition 1 of Theorem 3 is satisfied since \(\sum_{k=1}^{b} (\Delta_2 w_k)^2 = 1 / b^2\), and since \(\sum_{k=1}^{b} (\Delta_2 w_k)^2 k^3 = b, S = 2 / 3\). Thus, using (7),

\[
b_{\text{opt}} = \frac{1}{p} \sum_{i=1}^{p} \left(\frac{3 \Gamma_{ij}^2}{2 \Sigma_{ii}^2}\right)^{1/3} n^{1/3}.
\]

We now present MSE results for the BM estimator. The result is a consequence of setting \(r = 1\) in Theorems 5 and 6 of [Vats and Flegal (2018)].

**Corollary 1.** Let \(E_F\|g\|^{2+\delta} < \infty\) for some \(\delta > 0\) and let \(\{X_t\}\) be a polynomially ergodic Markov chain of order \(\xi > (1 + \epsilon)(1 + 2/\delta)\) for \(\epsilon > 0\). Then Assumption 2 holds for \(\psi(n) = n^{1/2 - \lambda}\) for \(\lambda > 0\). If in addition, \(E_F\|g\|^4 < \infty, ED^4 < \infty,\) and \(b^{-1} n^{1-2\lambda} \log n \to 0\) as \(n \to \infty\), then

\[
\text{MSE} \left(\hat{\Sigma}_{B,ij}\right) = \left(\frac{\Gamma_{ij}}{b}\right)^2 + (\Sigma_{ij}^2 + \Sigma_{ii} \Sigma_{jj}) \frac{b}{n} + o \left(\frac{b}{n}\right) + o \left(\frac{1}{b}\right).
\]
As a consequence, the optimal batch size according to (7) is
\[ b_{\text{opt}} = \frac{1}{p} \sum_{i=1}^{p} \left( \frac{\Gamma_{ii}^2}{\Sigma_{ii}^2} \right)^{1/3} n^{1/3}. \]

5 Examples

5.1 Vector auto-regressive example

Consider the \( p \)-dimensional vector auto-regressive process of order 1 (VAR(1))
\[ X_t = \Phi X_{t-1} + \epsilon_t, \]
for \( t = 1, 2, \ldots \) where \( X_t \in \mathbb{R}^p, \epsilon_t \) are i.i.d. \( N_p(0, I_p) \) and \( \Phi \) is a \( p \times p \) matrix. The Markov chain is geometrically ergodic when the largest eigenvalue of \( \Phi \) in absolute value is less than 1 (Tjøstheim [1990]). In addition if \( \otimes \) denotes the Kronecker product, the invariant distribution is \( N_p(0, V) \), where \( \text{vec}(V) = (I_p^2 - \Phi \otimes \Phi)^{-1} \text{vec}(I_p) \). Consider estimating \( \theta = EX_1 = 0 \) with \( \bar{Y}_n = \bar{X}_n \). By the CLT at (1),
\[ \Sigma = \text{Var}[X_1] + \sum_{k=1}^{\infty} \left[ \text{Cov}(X_1, X_{1+k}) + \text{Cov}(X_1, X_{1+k})^T \right] = (I_p - \Phi)^{-1} V + V(I_p - \Phi)^{-1} - V. \]

With some algebra, it can be shown that
\[ \Gamma = - \left[ (I_p - \Phi)^{-2} \Phi V + V \Phi^T(I_p - \Phi^T)^{-2} \right]. \] (11)

Thus, the true coefficient of the optimal batch size can be obtained by using the diagonals of \( \Sigma \) and \( \Gamma \) above. To ensure geometric ergodicity, we generate the process as follows. Consider a \( p \times p \) matrix \( A \) with each entry generated from standard normal distribution, let \( B = AA^T \) be a symmetric matrix with the largest eigenvalue \( m \), then set \( \Phi_0 = B/(m + 0.001) \). We then evaluate a series of \( \Phi = \rho \Phi_0 \), where \( \rho = \{.80, .82, .84, \ldots, .90\} \), where larger \( \rho \) values imply stronger auto-covariance and cross-covariance of the chain.

We compare our proposed optimal batch size coefficient estimates with that of the flat-top pilot estimator and empirical rule of Politis (2003). For each \( \rho \), optimal coefficients using both methods are computed for the corresponding \( \Phi \) based on VAR(1) of length \( 1e4 \). MSE over 1000 replications are plotted in Figure 2 with 95% confidence intervals. Note, our proposed AR approximation method consistently provides better estimates of the
coefficients while yielding smaller variability.

For an estimator $\hat{\Sigma}$ of $\Sigma$, let $E = \hat{\Sigma} - \Sigma$. MSE across entries of $E$ can be reflected by

$$MSE = \frac{1}{p^2} \sum_i \sum_j e_{ij}^2,$$

which we use to evaluate various batch sizes. Figure 2 shows the log of the average $MSE$ over 1000 replications of Markov chain length $1e5$, illustrating the optimal batch size leads to smaller $MSE$ than batch sizes $\lfloor n^{1/3} \rfloor$ and $\lfloor n^{1/2} \rfloor$. The two estimation methods perform similarly in terms of $MSE$ of the matrix.

### 5.2 Bayesian logistic regression

We consider the *Anguilla australis* data from Elith et al. (2008) available in the dismo R package. The dataset records the presence or absence of the short-finned eel in 1000 sites over New Zealand. Following Leathwick et al. (2008), we choose six of the twelve covariates recorded in the data; SegSumT, DSDist, USNative, DMaxSlope and DSSlope are continuous and Method is categorical with five levels.

For $i = 1, \ldots, 1000$, let $Y_i$ record the presence ($Y_i = 1$) or absence of *Anguilla australis*. Let $x_i$ denote the vector of covariates for observation $i$. We fit a model with intercept so
that the regression coefficient $\beta \in \mathbb{R}^9$. Let

$$Y_i \mid x_i, \beta \sim \text{Bernoulli}\left(\frac{1}{1 + \exp(x_i^T \beta)}\right) \quad \text{and} \quad \beta \sim N(0, \sigma_\beta^2 I_9).$$

We set $\sigma_\beta^2 = 100$ as in [Boone et al. (2014)]. The posterior distribution is intractable and we use the `MCMClogit` function in the R package `MCMCpack` to obtain posterior samples; this random walk Metropolis-Hastings sampler is geometrically ergodic (Vats et al., 2018a).

In 1000 replications, we ran a pilot run of length 1e4 to estimate the optimal batch size using our AR approximation methods and the flat-top pilot estimators. We then reran the chain to estimate $\Sigma$ using the optimal batch size estimates. This was repeated for three Monte Carlo sample sizes, $n = 1e4, 1e5$, and $1e5$. Figure 3 presents the variability in the estimates of the coefficient of the optimal batch size. Note that our AR estimator has significantly lower variability compared to the flat-top estimators. This is particularly useful since a pilot estimator is only run once by a user. In addition, since $n^{1/6}$ is relatively close to the estimated coefficients of the optimal batch size, we expect a batch size of $\lfloor n^{1/2} \rfloor$ to perform relatively well for $n = 5e4$ and $n = 1e5$.

Coverage probabilities over the 1000 replications are provided in Table 1. Given that the estimated coefficient is significantly larger than 1, it is not surprising that $\lfloor n^{1/3} \rfloor$ performs poorly. For small sample sizes, both the optimal methods have better coverage probabilities.

Figure 3: Boxplot of estimated coefficient of optimal batch size using our proposed method and the flat-top pilot estimates. The three horizontal lines correspond to $(1e4)^{1/6}, (5e4)^{1/6}$, and $(1e5)^{1/6}$.
| Sample Size / b | Batch Means | Spectral Variance |
|----------------|-------------|------------------|
|                | \(n^{1/3}\) | \(n^{1/2}\) | AR | FT | \(n^{1/3}\) | \(n^{1/2}\) | AR | FT |
| 1e4            | 0.279       | 0.722           | 0.725 | 0.716 | 0.276 | 0.727 | 0.733 | 0.733 |
| 5e4            | 0.499       | 0.826           | 0.832 | 0.826 | 0.497 | 0.834 | 0.838 | 0.839 |
| 1e5            | 0.615       | 0.861           | 0.857 | 0.853 | 0.615 | 0.864 | 0.864 | 0.863 |

Table 1: Coverage probabilities for 90% confidence regions over 1000 replications.

For Monte Carlo sample size 1e5, as expected, \(\lfloor n^{1/2} \rfloor\) fares marginally better. Almost universally, our AR approximation method outperforms the flat-top pilot estimators.

### 5.3 Bayesian dynamic space-time model

This example considers the Bayesian dynamic model of [Finley et al. (2012)] to model monthly temperature data collected at 10 nearby stations in northeastern United States in 2000. A data description can be found in the `spBayes` R package [Finley and Banerjee (2013)].

Suppose \(y_t(s)\) denotes the temperature observed at location \(s\) and time \(t\) for \(s = 1, 2, \ldots, N_s\) and \(t = 1, 2, \ldots, N_t\). Let \(x_t(s)\) be a \(m \times 1\) vector of predictors and \(\beta_t\) be a \(m \times 1\) coefficient vector, which is a purely time component and \(u_t(s)\) be a space-time component. The model is

\[
y_t(s) = x_t(s)^T \beta_t + u_t(s) + \epsilon_t(s), \quad \epsilon_t(s) \sim N(0, \tau^2_t),
\]

\[
\beta_t = \beta_{t-1} + \eta_t; \quad \eta_t \sim N_p(0, \Sigma_t),
\]

\[
\tau_t = \nu_t + \omega_t; \quad \nu_t \sim \nu_t(s), \quad \omega_t \sim \nu_t(s),
\]

where \(GP(0, C_t(\cdot, \sigma_t^2, \phi_t))\) is a spatial Gaussian process with \(C_t(s_1, s_2; \sigma_t^2, \phi_t) = \sigma_t^2 \rho(s_1, s_2; \phi_t)\), \(\rho(\cdot; \phi)\) is an exponential correlation function with \(\phi\) controlling the correlation decay, and \(\sigma_t^2\) represents the spatial variance components. The Gaussian spatial process allows closer locations to have higher correlations. Time effect for both \(\beta_t\) and \(u_t(s)\) is characterized by transition equations to achieve reasonable dependence structure. We are interested in estimating posterior expectation of 185 parameters \(\theta = (\beta_t, u_t(s), \sigma^2_t, \Sigma_t, \tau^2_t, \phi_t)\), their prior follows the `spDyn1M` function in the `spBayes` package.

The only predictor in our analysis is elevation, hence \(\beta_t = (\beta_{t}^{(0)}, \beta_{t}^{(1)})^T\) for \(t = 1, 2, \ldots, 12\), where \(\beta_{t}^{(0)}\) is the intercept and \(\beta_{t}^{(1)}\) is the coefficient for elevation. Consider estimating the coefficient of the covariate for the last two months, \(\beta_{11}^{(1)}\) and \(\beta_{12}^{(1)}\). We obtain the true posterior mean of these two components by averaging over 1000 chains of length 1e6. Our simulation setup is similar to that in Section 5.2. In Figure 4, we present boxplots of the estimated coefficient of the optimal batch size; here again the variability in the flat-top
estimator is high. Also note that the average estimated coefficient of 25.8, immediately indicates the inappropriateness of using batch size $\lfloor n^{1/3} \rfloor$ and $\lfloor n^{1/2} \rfloor$. We also present the 90% confidence regions created in one run of length $1e5$ using the BM estimators for varying batch sizes. Given the dependence in time of the two components, we expect such thin ellipses, due to high posterior correlation, assisted by high Markov chain lag cross-correlation.

Figure 4: Confidence regions for $(\hat{\beta}_{11}^{(1)}, \hat{\beta}_{12}^{(1)})$ based on $\hat{\Sigma}_{bt}$ and a chain length of $1e5$.

Coverage probabilities over 1000 replications are shown in Table 2. Unsurprisingly, the $\lfloor n^{1/3} \rfloor$ and $\lfloor n^{1/2} \rfloor$ do not perform well for both the BM and SV estimators. The AR approximation method and the flat-top pilot estimators are comparable, with our AR approximation method being yielding marginally better coverage probabilities.

| Sample Size / b | Batch Means | Spectral Variance |
|-----------------|-------------|-------------------|
|                 | $|n^{1/3}|$ | $|n^{1/2}|$ | AR | FT | $|n^{1/3}|$ | $|n^{1/2}|$ | AR | FT |
| $1e4$           | 0.274       | 0.571             | 0.771 | 0.758 | 0.274 | 0.565 | 0.765 | 0.761 |
| $1e5$           | 0.426       | 0.764             | 0.869 | 0.864 | 0.424 | 0.756 | 0.872 | 0.865 |
| $2e5$           | 0.445       | 0.789             | 0.856 | 0.857 | 0.446 | 0.787 | 0.865 | 0.859 |

Table 2: Coverage probabilities for 90% confidence regions over 1000 replications.

6 Discussion

This paper provides theoretical evidence and practical guidance for optimal batch size selection in MCMC simulations. Estimators with the proposed optimal batch sizes are shown
to have superior performance versus conventional batch sizes. Optimal batch size has not been addressed previously in multivariate MCMC setting although sampling multivariate posteriors is ubiquitous in Bayesian analyses. Optimal batch size in univariate MCMC simulations has been discussed previously, where sub-optimal batch sizes that ignore the proportionality constant are conventionally used.

To reduce computational effort, we used $1e4$ MCMC samples to obtain pilot estimates regardless of the total chain length. As a result, performance of the estimator can be improved without significant additional computation, which is crucial for multivariate problems in practice. Our choice was a compromise between computation effort and accuracy.

Optimal batch sizes with estimated coefficients can also be expanded to other estimators such as weighted BM estimators [Liu and Flegal 2018]. These estimators are based on fewer batches compared with mSV estimators. Hence they are more computationally efficient with a larger variance. Nevertheless, obtaining optimal batch sizes should follow results discussed in this paper and is a direction of future work.

Appendix

A Preliminaries

First, we introduce some notation and propositions. Recall $B = \{B(t), t \geq 0\}$ is a $p$-dimensional standard Brownian motion. Let $B^{(i)}(t)$ be the $i$th component of vector $B(t)$. Denote $\bar{B} = n^{-1}B(n)$, $\bar{B}(k) = k^{-1}[B(l + k) - B(l)]$. The Brownian motion counterpart of $\hat{\Sigma}_w$ is

$$\bar{\Sigma}_w = \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_n(k)(\bar{B}_l(k) - \bar{B})(\bar{B}_l(k) - \bar{B})^T.$$ 

Proposition 1. (Csörgő and Révész [1981]) Suppose Assumption 1 holds, then for all $\epsilon > 0$ and for almost all sample paths, there exists $n_0(\epsilon)$ such that for all $n \geq n_0$ and all $i = 1, ..., p$

$$\sup_{0 \leq t \leq n-b} \sup_{0 \leq s \leq b} |B^{(i)}(t + s) - B^{(i)}(t)| < (1 + \epsilon) \left(2b \left(\log \frac{n}{b} + \log \log n\right)\right)^{1/2},$$

$$\sup_{0 \leq s \leq b} |B^{(i)}(n) - B^{(i)}(n - s)| < (1 + \epsilon) \left(2b \left(\log \frac{n}{b} + \log \log n\right)\right)^{1/2},$$

$$|B^{(i)}(n)| < (1 + \epsilon)\sqrt{2n \log \log n}.$$

Let $\Sigma = LL^T$, where $L$ is a lower triangular matrix. Let $C(t) = LB(t)$ and $C^{(i)}(t)$ be the $i$th component of $C(t)$. Suppose $\bar{C}^{(i)}_l(k) = k^{-1}(C^{(i)}(l + k) - C^{(i)}(l))$, and $\bar{C}^{(i)} = n^{-1}C^{(i)}(n)$.
Proposition 2. 
\cite{Vats et al, 2018b} For all $\epsilon > 0$ and for almost all sample paths, there exists $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$ and all $i = 1, \ldots, p$

$$|C^{(i)}(n)| < (1 + \epsilon) (2n\Sigma_{ii} \log \log n)^{1/2},$$

where $\Sigma_{ii}$ is the $i$th diagonal entry of $\Sigma$.

Proposition 3. \cite{Vats et al, 2018b} If Assumption \cite{Vats et al, 2018b} holds, then for all $\epsilon > 0$ and for almost all sample paths, there exists $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$ and all $i = 1, \ldots, p$

$$|\tilde{C}_t^{(i)}(k)| \leq \frac{1}{k} \sup_{0 \leq l \leq n - b} \sup_{0 \leq s \leq b} |C^{(i)}(l + s) - C^{(i)}(l)| < \frac{1}{k} (1 + \epsilon) (b\Sigma_{ii} \log n)^{1/2},$$

where $\Sigma_{ii}$ is the $i$th diagonal entry of $\Sigma$.

Proposition 4. If variable $X$ and $Y$ are jointly normally distributed with

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix} \right),$$

then $E[X^2Y^2] = 2l_{12}^2 + l_{11}l_{22}$.

Proposition 5. \cite{Janssen and Stoica, 1987} If $X_1$, $X_2$, $X_3$, and $X_4$ are jointly normally distributed with mean 0, then

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3].$$

**B Proof of Theorem \cite{Vats et al, 2018b}**

Throughout, for any matrix $A$, let $A_{ij}$ denote the $i, j$th element of the matrix. Denote $w_n(k)$ by $w_k$ for simplification. Let $V_t = Y_t - \bar{Y}$ and $T_t = U_t - \bar{B}$ where $U_t \overset{iid}{\sim} N_p(0, I_p)$ for $t = 1, 2, \ldots, n$. Define

$$d = \frac{1}{n} \left[ \sum_{h=1}^{b} \Delta_1 w_h \sum_{r=1}^{h-1} V_r V_r^T + \sum_{s=1}^{b-1} \sum_{h=1}^{b-s} \Delta_1 w_{s+h} \left( \sum_{r=1}^{h-1} (V_r V_r^T + V_r V_r^T) \right) \right]$$

$$+ \frac{1}{n} \left[ \sum_{h=1}^{b} \sum_{r=n-b+h+1}^{n} \Delta_1 w_{n-r+h+1} V_r V_r^T + \sum_{s=1}^{b-1} \sum_{h=1}^{b-s} \sum_{r=n-b+h+1}^{n-s} \Delta_1 w_{n-r+s+h} \left( V_r V_r^T + V_r V_r^T \right) \right],$$

and

$$\tilde{d} = \frac{1}{n} \left[ \sum_{h=1}^{b} \Delta_1 w_h \sum_{r=1}^{h-1} T_r T_r^T + \sum_{s=1}^{b-1} \sum_{h=1}^{b-s} \Delta_1 w_{s+h} \left( \sum_{r=1}^{h-1} (T_r T_r^T + T_r T_r^T) \right) \right].$$
+ \frac{1}{n} \left[ \sum_{h=1}^{b} \sum_{r=n-b+h+1}^{n} \Delta_1 w_{n-r+h+1} T_r T_r^T + \sum_{s=1}^{b-1} \sum_{h=1}^{b-s} \sum_{r=n-b+h+1}^{n-s} \Delta_1 w_{n-r+h+1} (T_r T_{r+s}^T + T_{r+s} T_r^T) \right].

Denote

$$\tilde{R}(k) = \frac{1}{n} \sum_{t=1}^{n-k} (U_t - \bar{B})(U_{t+k} - \bar{B})^T = \frac{1}{n} \sum_{t=1}^{n-k} T_t T_{t+1}^T. \quad (12)$$

Then the following is the Brownian motion equivalent of $\tilde{\Sigma}_s$:

$$\tilde{\Sigma}_s = \tilde{R}(0) + \sum_{s=1}^{b-1} w_s \tilde{R}(s) + \sum_{s=1}^{b-1} w_s \tilde{R}(s). \quad (13)$$

**Lemma 1.** $\tilde{\Sigma}_w = \tilde{\Sigma}_s - \tilde{d}$.

**Proof.** The proof is similar to that in [Damerdji (1991)](Damerdji1991) Theorem 3.1. Note

$$\Delta_1 w_l = \sum_{k=1}^{b} \Delta_2 w_k, \quad \sum_{l=s+1}^{b} \Delta_1 w_l = w_s, \quad \text{and} \quad \sum_{l=1}^{b} \Delta_1 w_l = 1. \quad (14)$$

We will prove that for $i, j = 1, \ldots, p$, $\tilde{\Sigma}_{w,ij} = \tilde{\Sigma}_{s,ij} - \tilde{d}_{ij}$.

$$\tilde{\Sigma}_{w,ij} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_k \left( \tilde{B}^{(i)}_l(k) - B^{(i)} \right) \left( \tilde{B}^{(j)}_l(k) - B^{(j)} \right)$$

$$= \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_k \left( \frac{1}{k} \left[ \left( U_{l+1}^{(i)} + U_{l+2}^{(i)} + \cdots + U_{l+k}^{(i)} \right) - kB^{(i)} \right] \right)$$

$$\times \left( \frac{1}{k} \left[ \left( U_{l+1}^{(j)} + U_{l+2}^{(j)} + \cdots + U_{l+k}^{(j)} \right) - kB^{(j)} \right] \right)$$

$$= \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_k \left( \frac{1}{k^2} \left( \sum_{h=1}^{k} T_{l+h}^{(i)} \right) \left( \sum_{h=1}^{k} T_{l+h}^{(j)} \right) \right)$$

$$= \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \left( \sum_{h=1}^{k} T_{l+h}^{(i)} T_{l+h}^{(j)} + \sum_{h=1}^{k-1} \sum_{s=1}^{k-s} T_{l+h}^{(i)} T_{l+h+s}^{(j)} + \sum_{s=1}^{k-1} \sum_{h=1}^{k-s} T_{l+h}^{(i)} T_{l+h+s}^{(j)} \right)$$

$$= \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \sum_{h=1}^{k} T_{l+h}^{(i)} T_{l+h}^{(j)} + \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \sum_{h=1}^{k-1} \sum_{s=1}^{k-s} \left( T_{l+h}^{(i)} T_{l+h+s}^{(j)} + T_{l+h}^{(j)} T_{l+h+s}^{(i)} \right)$$

$$:= I + \Pi_{ij} + \Pi_{ji},$$

where

$$I = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \sum_{h=1}^{k} T_{l+h}^{(i)} T_{l+h}^{(j)}, \quad (15)$$
\[
\Pi_{ij} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \sum_{s=1}^{k-1} \sum_{h=1}^{k-s} T_{l+h}^{(i)} T_{l+h+s}^{(j)}, \quad \text{and} \quad \Pi_{ji} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \Delta_2 w_k \sum_{s=1}^{k-1} \sum_{h=1}^{k-s} T_{l+h+s}^{(i)} T_{l+h}^{(j)}.
\]

(16)

Change the order of sums in (15), (16) and (17) while applying (14) and (12),

\[
I = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \sum_{h=1}^{k-l} \Delta_2 w_k T_{l+h}^{(i)} T_{l+h}^{(j)} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \sum_{h=1}^{k-l} \Delta_2 w_k T_{l+h}^{(i)} T_{l+h}^{(j)} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \sum_{h=1}^{k-l} \Delta_2 w_k T_{l+h}^{(i)} T_{l+h}^{(j)}
\]

(17)

Next,

\[
\Pi_{ij} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \sum_{s=1}^{k-1} \sum_{h=1}^{k-s} \Delta_2 w_k \sum_{r=1}^{h-1} T_{r+h}^{(i)} T_{r+h+s}^{(j)} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \sum_{s=1}^{k-1} \sum_{h=1}^{k-s} \Delta_2 w_k \sum_{r=1}^{h-1} T_{r+h}^{(i)} T_{r+h+s}^{(j)}
\]

(18)
\[
\begin{align*}
\tilde{\Sigma}_{w,ij} &= I + \Pi_{ij} + \Pi_{ji} \\
&= \tilde{R}_{ij}(0) + \sum_{s=1}^{b-1} w_s \tilde{R}_{ij}(s) + \sum_{s=1}^{b-1} w_s \tilde{R}_{ji}(s) \\
&\quad - \frac{1}{n} \sum_{h=1}^{b-1} \sum_{s=1}^{b-1} T_{r}^{(i)} T_{r+s}^{(j)} + \sum_{r=n-b+h+1}^{n-s} \Delta_1 w_{n-r+h+1} T_{r}^{(i)} T_{r+s}^{(j)}
\end{align*}
\]
By (18) and (19), and

\[
\sum_{r=1}^{h-1} T_r^{(i)} T_r^{(j)} | \leq \frac{1}{2} \sum_{r=1}^{h-1} \left[ (T_r^{(i)})^2 + (T_r^{(j)})^2 \right] \leq \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(i)})^2 + \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(j)})^2,
\]
and

\[
\sum_{r=1}^{h-1} |T_r^{(i)} T_r^{(j)}| + |T_r^{(i)} T_{r+s}^{(i)}| \leq \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(i)})^2 + \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(j)})^2 + \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(s)})^2.
\]

**Lemma 2.** Suppose Assumption holds and \( n > 2b \). If \( bn^{-1} \sum_{k=1}^{b} k |\Delta_1 w_n(k)| \to 0 \) as \( n \to \infty \), then \( \tilde{d} \to 0 \) w.p. 1.

**Proof.** For \( i, j = 1, \ldots, p \), we prove that \( \tilde{d}_{ij} \to 0 \) w.p. 1. Using the inequality \(|ab| \leq (a^2 + b^2)/2\),

\[
|\tilde{d}_{ij}| = \left| \frac{1}{n} \left( \sum_{h=1}^{b} \Delta_1 w_n(h) \sum_{r=1}^{h-1} T_r^{(i)} T_r^{(j)} + \sum_{h=1}^{b-1} \sum_{h=1}^{b-s} \Delta_1 w_n(s+h) \sum_{r=1}^{h-1} T_r^{(i)} T_{r+s}^{(i)} \right) \right|
\]

\[
\leq \frac{1}{n} \left( \sum_{h=1}^{b} |\Delta_1 w_n(h)| \cdot \sum_{h=1}^{b} |T_r^{(i)} T_r^{(j)}| + \sum_{h=1}^{b-1} \sum_{h=1}^{b-s} |\Delta_1 w_n(s+h)| \cdot \sum_{h=1}^{h-1} (|T_r^{(i)} T_r^{(j)}| + |T_r^{(i)} T_{r+s}^{(i)}|) \right)
\]

\[
\leq \frac{1}{n} \sum_{h=1}^{b} b |\Delta_1 w_n(h)| \cdot \left( \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(i)})^2 + \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(j)})^2 \right)
\]

\[
+ \frac{1}{n} \sum_{h=1}^{b-1} \sum_{h=1}^{b-s} |\Delta_1 w_n(s+h)| \cdot \left( \sum_{h=1}^{b} (T_r^{(i)})^2 + \sum_{r=1}^{2b} (T_r^{(j)})^2 \right)
\]

\[
= \frac{1}{b} \left( \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(i)})^2 + \frac{1}{2} \sum_{r=1}^{2b} (T_r^{(j)})^2 \right) \cdot \frac{b}{n} \left( \sum_{h=1}^{b} |\Delta_1 w_n(h)| + \sum_{h=1}^{b-1} \sum_{h=1}^{b-s} |\Delta_1 w_n(s+h)| \right)
\]

\[
(20)
\]
\[ H = \frac{1}{b} \left( \frac{1}{2} \sum_{r=1}^{2b} \left( T^{(i)}_r \right)^2 + \frac{1}{2} \sum_{r=1}^{2b} \left( T^{(j)}_r \right)^2 \right) \]

and
\[ K = \frac{b}{n} \left( \sum_{h=1}^{b} |\Delta_1 w_n(h)| + 2 \sum_{s=1}^{b-1} \sum_{h=1}^{b-s} |\Delta_1 w_n(s+h)| \right) \]

We show that \( H \) stays bounded w.p. 1. Recall that \( T_r = U_r - \bar{B} \) and apply Proposition 1.

\[ \frac{1}{2b} \sum_{r=1}^{2b} (T^{(i)}_r)^2 = \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r - \bar{B}^{(i)})^2 \]
\[ \leq \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r)^2 - \frac{1}{2b} 2\bar{B}^{(i)} \cdot \sum_{r=1}^{2b} U^{(i)}_r + (\bar{B}^{(i)})^2 \]
\[ < \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r)^2 \left( \frac{2}{n} (1 + \epsilon)(2n \log \log n)^{1/2} \right) \cdot \left( \left| \frac{1}{2b} \sum_{r=1}^{2b} U^{(i)}_r \right| \right)^2 \]
\[ = \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r)^2 + \left( \frac{1}{n} (1 + \epsilon)(2n \log \log n)^{1/2} \right)^2 \cdot \left( \frac{1}{2b} \sum_{r=1}^{2b} U^{(i)}_r \right)^2 \]
\[ \leq \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r)^2 + \frac{1}{2b} \sum_{r=1}^{2b} \left( U^{(i)}_r \right)^2 \cdot O \left( \left( n^{-1} \log n \right)^{1/2} \right) + O(n^{-1} \log n). \quad (21) \]

For \( r = 1, ..., 2b \), since \( U^{(i)}_r \overset{iid}{\sim} N(0, 1) \), by classical strong law of large numbers, both
\[ \frac{1}{2b} \sum_{r=1}^{2b} U^{(i)}_r \quad \text{and} \quad \frac{1}{2b} \sum_{r=1}^{2b} (U^{(i)}_r)^2 \]
in (21) stay bounded w.p. 1 hence \( H \) stays bounded. Since \( bn^{-1} \sum_{k=1}^{b} k |\Delta_1 w_n(k)| \rightarrow 0 \) as \( n \rightarrow \infty \), by Vats et al. (2018b, Lemma 6), \( K \rightarrow 0 \) as \( n \rightarrow \infty \) and thus \( \tilde{d}_{ij} \rightarrow 0 \) as \( n \rightarrow \infty \).

Combining Lemmas 1 and 2, the Brownian motion equivalent of Theorem 1 holds. To prove Theorem 1, the following Lemma is needed.

**Lemma 3.** (Vats et al. 2018b, Lemma 8) Set \( h(X_t) = [g(X_t) - E_F g]^2 \) for \( t = 1, 2, 3, ..., \) were the square is element-wise; assume \( \|E_F h\| < \infty \). Let Assumption 2 hold for \( h \) so that
there exists a non-negative increasing function $\psi_h$ on the positive integers, a lower triangular matrix $L_h$, a finite random variable $D_h$ and an $n_0 \in N$ such that w.p. 1, for $n \geq n_0$,
\[
\left\| \sum_{k=1}^{n} h(X_k) - nE_F h - L_h B(n) \right\| < D_h \psi_h(n).
\]

Also let Assumption 1 hold and as $n \to \infty$, let $b^{-1} \psi_h(n) \to 0$ and $b^{-1} \log n = O(1)$, then $b^{-1} \sum_{k=1}^{b} h(X_k)$ stays bounded w.p. 1 as $n \to \infty$.

By Lemmas 2 and 3 and following on the lines of Vats et al. (2018b, Lemma 9), $d \to 0$.

**C Preliminaries for Theorem 3**

Denote $\lim_{n \to \infty} f(n)/g(n) = 0$ by $f(n) = o(g(n))$. For $0 < c_2 < c_1 < 1$, let
\[
A_2 = \frac{(c_1 b)^2}{n^2} E \left[ \sum_{l=0}^{n-c_1 b} \left( \bar{C}_l^{(i)} (c_1 b) - C_l^{(i)} \right) \left( \bar{C}_l^{(j)} (c_1 b) - C_l^{(j)} \right) \right]^2, \text{ and }
\]
\[
A_3 = \frac{-c_1 c_2 b^2}{n^2} E \left[ \sum_{l=0}^{n-c_1 b} \left( \bar{C}_l^{(i)} (c_1 b) - C_l^{(i)} \right) \left( \bar{C}_l^{(j)} (c_1 b) - C_l^{(j)} \right) \right] \left[ \sum_{l=1}^{n-c_2 b} \left( \bar{C}_l^{(i)} (c_2 b) - C_l^{(i)} \right) \left( \bar{C}_l^{(j)} (c_2 b) - C_l^{(j)} \right) \right].
\]

**Lemma 4.** For $0 < c_2 < c_1 < 1$,
\[
A_2 = \left[ \frac{2}{3} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{c_1 b}{n} + \Sigma_{ij}^2 \cdot \frac{c_1 b}{n} - 4 \Sigma_{ij}^2 \cdot \frac{b}{n} \right] + o \left( \frac{b}{n} \right) \quad \text{ and }, \quad (22)
\]
\[
A_3 = \frac{(c_2 - 3c_1) c_2}{3c_1} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{b}{n} + 2 (c_1 + c_2) \cdot \Sigma_{ij}^2 \cdot \frac{b}{n} - \Sigma_{ij}^2 + o \left( \frac{b}{n} \right). \quad (23)
\]

**Proof.** Denote
\[
a_1 = \sum_{l=0}^{n-c_1 b} \left( \bar{C}_l^{(i)} (c_1 b) - C_l^{(i)} \right)^2 \left( \bar{C}_l^{(j)} (c_1 b) - C_l^{(j)} \right)^2, \quad (24)
\]
\[
a_2 = \sum_{s=1}^{c_1 b - 1} \sum_{l=0}^{n-c_1 b-s} \left( \bar{C}_l^{(i)} (c_1 b) - C_l^{(i)} \right) \left( \bar{C}_l^{(j)} (c_1 b) - C_l^{(j)} \right) \left( \bar{C}_{l+s}^{(i)} (c_1 b) - C_{l+s}^{(i)} \right) \left( \bar{C}_{l+s}^{(j)} (c_1 b) - C_{l+s}^{(j)} \right), \quad (25)
\]
\[
a_3 = \sum_{s=b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b-s} \left( \bar{C}_l^{(i)} (c_1 b) - C_l^{(i)} \right) \left( \bar{C}_l^{(j)} (c_1 b) - C_l^{(j)} \right) \left( \bar{C}_{l+s}^{(i)} (c_1 b) - C_{l+s}^{(i)} \right) \left( \bar{C}_{l+s}^{(j)} (c_1 b) - C_{l+s}^{(j)} \right). \quad (26)
\]
Then
\[
A_1 = \frac{c_1 b^2}{n^2} E \left[ \sum_{l=0}^{n-c_1 b} \left( \bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)} \right)^2 \left( \bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)} \right)^2 \right]
+ 2 \sum_{s=1}^{c_1 b-1} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)}) (\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)}) (\bar{C}_{l+s}^{(i)}(c_1 b) - \bar{C}^{(i)}) (\bar{C}_{l+s}^{(j)}(c_1 b) - \bar{C}^{(j)})
+ 2 \sum_{s=b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)}) (\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)}) (\bar{C}_{l+s}^{(i)}(c_1 b) - \bar{C}^{(i)}) (\bar{C}_{l+s}^{(j)}(c_1 b) - \bar{C}^{(j)})
= \frac{c_1 b^2}{n^2} E[a_1 + 2a_2 + 2a_3].
\]

First we calculate \(E[a_1]\) at (24). Let \(U_t^{(i)} = B^{(i)}(t) - B^{(i)}(t-1)\), then \(U_t^{(i)} \overset{iid}{\sim} N(0,1)\) for \(t = 1, 2, ..., n\) and
\[
\bar{B}_t^{(i)}(c_1 b) - \bar{B}^{(i)} = \frac{n-c_1 b}{nc_1 b} \sum_{t=1}^{l+c_1 b} U_t^{(i)} - \frac{1}{n} \sum_{t=1}^{l} U_t^{(i)} - \frac{1}{n} \sum_{t=l+c_1 b+1}^{n} U_t^{(i)}.
\]

Notice that \(E[\bar{B}_t^{(i)}(c_1 b) - \bar{B}^{(i)}] = 0\) for \(l = 0, ..., (n-c_1 b)\) and
\[
\text{Var}[\bar{B}_t^{(i)}(c_1 b) - \bar{B}^{(i)}] = \left( \frac{n-c_1 b}{nc_1 b} \right)^2 c_1 b + \frac{n-c_1 b}{n^2} c_1 b = \frac{n-c_1 b}{c_1 bn},
\]
therefore
\[
\bar{B}_t^{(i)}(c_1 b) - \bar{B}^{(i)} \sim N \left( 0, \frac{n-c_1 b}{c_1 bn} \right)
\]
and
\[
\bar{B}_t(c_1 b) - \bar{B}_n \sim N \left( 0, \frac{n-c_1 b}{c_1 bn} I_p \right),
\]

hence
\[
\bar{C}_t(c_1 b) - \bar{C}_n = L(\bar{B}_t - \bar{B}_n) \sim N \left( 0, \frac{n-c_1 b}{c_1 bn} L L^T \right).
\]

Now consider \(E[(\bar{C}_t^{(i)}(c_1 b) - \bar{C}^{(i)})^2 (\bar{C}_t^{(j)}(c_1 b) - \bar{C}^{(j)})^2] := E[Z_i^2 Z_j^2]\) where \(Z_i = \bar{C}_t^{(i)}(c_1 b) - \bar{C}^{(i)}\) and \(Z_j = \bar{C}_t^{(j)}(c_1 b) - \bar{C}^{(j)}\). Recall \(\Sigma = L L^T\), then
\[
\begin{bmatrix} Z_i \\ Z_j \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{n-c_1 b}{c_1 bn} \begin{bmatrix} \Sigma_{ii} & \Sigma_{ij} \\ \Sigma_{ij} & \Sigma_{jj} \end{bmatrix} \right).
\]
Apply Proposition 4

\[
E \left[ \left( \bar{C}_l^{(i)}(c_1b) - \bar{C}^{(i)} \right) \left( \bar{C}_l^{(j)}(c_1b) - \bar{C}^{(j)} \right) \right] = 2 \left( \frac{n - c_1b}{c_1bn} \right)^2 \left( \frac{n - c_1b}{c_1bn} \sum_{ij} \right) \left( \frac{n - c_1b}{c_1bn} \sum_{jj} \right) = \left( \frac{n - c_1b}{c_1bn} \right)^2 \left( \sum_{ij} + \sum_{ii} \right) + \left( \frac{n - c_1b}{c_1bn} \right)^2 \Sigma_{ij}.
\] (28)

Replace (28) in (24)

\[
E[a_1] = \sum_{l=0}^{n-c_1b} E \left[ \left( \bar{C}_l^{(i)}(c_1b) - \bar{C}^{(i)} \right) \left( \bar{C}_l^{(j)}(c_1b) - \bar{C}^{(j)} \right) \right] = (n - c_1b + 1) \left( \frac{n - c_1b}{c_1bn} \right)^2 \left( \sum_{ij} + \sum_{ii} \right) + \sum_{l=0}^{n-c_1b} \left( \frac{n - c_1b}{c_1bn} \right)^2 \Sigma_{ij}
\]

= \sum_{l=0}^{n-c_1b} \left( \frac{n - c_1b}{c_1bn} \right)^2 \Sigma_{ij} + o \left( \frac{n}{b} \right). \quad (29)

To calculate \( E[a_2] \) for \( s = 1, 2, ..., (c_1b - 1) \), we calculate

\[
E \left[ \left( \bar{C}_l^{(i)}(c_1b) - \bar{C}^{(i)} \right) \left( \bar{C}_l^{(j)}(c_1b) - \bar{C}^{(j)} \right) \right] \left( \bar{C}_{l+s}^{(i)}(c_1b) - \bar{C}^{(i)} \right) \left( \bar{C}_{l+s}^{(j)}(c_1b) - \bar{C}^{(j)} \right)
\].

Notice that

\[
\text{Cov}(\bar{C}_t^{(i)}(c_1b) - \bar{C}_n, \bar{C}_{t+s}^{(i)}(c_1b) - \bar{C}_n) = E[(\bar{C}_t^{(i)}(c_1b) - \bar{C}_n)(\bar{C}_{t+s}^{(i)}(c_1b) - \bar{C}_n)^T] = L \cdot E[\bar{B}_t(c_1b) - \bar{B}_n](\bar{B}_{t+s}(c_1b) - \bar{B}_n)^T \cdot L^T.
\]

Consider each entry of \( E[(\bar{B}_t(c_1b) - \bar{B})(\bar{B}_{t+s}(c_1b) - \bar{B})^T] \). For \( i \neq j \),

\[
E \left[ B_l^{(i)}(c_1b) - B^{(i)} \right] \left[ B_{l+s}^{(j)}(c_1b) - B^{(j)} \right] = E \left[ B_l^{(i)}(c_1b) - B^{(i)} \right] \cdot E \left[ B_{l+s}^{(j)}(c_1b) - B^{(j)} \right] = 0.
\] (30)

For \( i = j \), we need to calculate \( E[B_l^{(i)}(c_1b) - B^{(i)}][B_{l+s}(c_1b) - B^{(j)}] \).

\[
E \left[ B_l^{(i)}(c_1b) - B^{(i)} \right] \left[ B_{l+s}(c_1b) - B^{(j)} \right] = E \left[ B_l^{(i)}(c_1b)B_{l+s}(c_1b) \right] - E \left[ B^{(i)} \right] - E \left[ B_l^{(i)}B_{l+s}(c_1b) \right] - E \left[ B_l^{(i)}B^{(i)} \right] \left[ B_{l+s}(c_1b) \right]
\]

= \frac{1}{c_1b^2} E \left( B^{(i)}(l + c_1b) - B^{(i)}(l) \right) \left( B^{(i)}(l + s + c_1b) - B^{(i)}(l + s) \right)

+ \frac{1}{n^2} E \left( B^{(i)}(n) \right)^2 - \frac{1}{nc_1b} E \left[ B^{(i)}(n) \left( B^{(i)}(l + c_1b + s) - B^{(i)}(l + s) \right) \right]
\]
Proposition 5, Only the upper triangle entries are considered due to symmetry of the matrix. Apply \( E \) Plug (33) in (25),

\[
- \frac{1}{nc_{1}b} \mathbb{E} \left[ B^{(i)}(n) \left( B^{(i)}(l + c_{1}b) - B^{(i)}(l) \right) \right] = \frac{c_{1}b - s}{c_{1}^{2}b^{2}} + \frac{1}{n} - \frac{2}{n} = \frac{n - c_{1}b}{c_{1}bn} - \frac{s}{n^2}.
\]  

(31)

Combine (31) and (30),

\[
\text{Cov} (\hat{C}_{l}(c_{1}b) - \hat{C}_n, \hat{C}_{l+s}(c_{1}b) - \hat{C}_n) = L \cdot \left( \frac{n - c_{1}b}{c_{1}bn} - \frac{s}{c_{1}^{2}b^{2}} \right) I_p \cdot L^T = \left( \frac{n - c_{1}b}{c_{1}bn} - \frac{s}{c_{1}^{2}b^{2}} \right) \cdot \Sigma.
\]  

(32)

Given (27), (30) and (31) and let \( Z_1 = \hat{C}_l(c_{1}b)^{(i)} - \hat{C}_n^{(i)}, Z_2 = \hat{C}_l(c_{1}b)^{(j)} - \hat{C}_n^{(j)}, Z_3 = \hat{C}_{l+s}(c_{1}b)^{(i)} - \hat{C}_n^{(i)}, Z_4 = \hat{C}_{l+s}(c_{1}b)^{(j)} - \hat{C}_n^{(j)}, (Z_1, Z_2, Z_3, Z_4)^T \) has a 4-dimensional Normal distribution with mean 0, and covariance matrix,

\[
\begin{bmatrix}
\left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ii} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ij} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ji} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} \\
\left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ji} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ij} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} \\
\left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ji} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ij} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} \\
\left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{ij} & \left( \frac{n - c_{1}b}{c_{1}bn} \right) \Sigma_{jj}
\end{bmatrix}.
\]

Only the upper triangle entries are considered due to symmetry of the matrix. Apply Proposition 3,

\[
E[Z_1Z_2Z_3Z_4]
\]  

(33)

\[
= E[Z_1Z_2] \cdot E[Z_3Z_4] + E[Z_1Z_3] \cdot E[Z_2Z_4] + E[Z_1Z_4] \cdot E[Z_2Z_3]
\]  

(34)

Plug (33) in (25),

\[
E[a_2] = \sum_{s=1}^{c_{1}b} \sum_{l=0}^{n-c_{1}b-s} E[(\hat{C}_{l}^{(i)}(c_{1}b) - \hat{C}_n^{(i)})(\hat{C}_{l}^{(j)}(c_{1}b) - \hat{C}_n^{(j)})(\hat{C}_{l+s}^{(i)}(c_{1}b) - \hat{C}_n^{(i)})(\hat{C}_{l+s}^{(j)}(c_{1}b) - \hat{C}_n^{(j)})]
\]

\[
= \sum_{s=1}^{c_{1}b} \sum_{l=0}^{n-c_{1}b-s} E[Z_1Z_2Z_3Z_4]
\]
Notice that

$$= \sum_{s=1}^{c_1 b - 1} \sum_{l=0}^{n-c_1 b - s} \left[ \left( \frac{n-c_1 b}{c_1 bn} \right)^2 \Sigma_{ij}^2 + \left( \frac{n-c_1 b}{c_1 bn} - \frac{s}{c_1^2 b^2} \right)^2 \Sigma_{ii} \Sigma_{jj} + \left( \frac{n-c_1 b}{c_1 bn} - \frac{s}{c_1^2 b^2} \right)^2 \Sigma_{ij}^2 \right].$$

(35)

Notice that

$$= \sum_{s=1}^{c_1 b - 1} \sum_{l=0}^{n-c_1 b - s} \left( \frac{n-c_1 b}{c_1 bn} - \frac{s}{c_1^2 b^2} \right)^2$$

$$= \sum_{s=1}^{c_1 b - 1} \sum_{l=0}^{n-c_1 b - s} \left[ \frac{s^2}{c_1^4 b^4} + \left( \frac{n}{c_1^4 b^4} + \frac{1}{c_1^2 b^4} - \frac{2}{c_1^2 b^2 n} \right) s^2 + \left( \frac{3}{c_1^2 b^2 n} - \frac{2}{c_1^4 b^2 n} - \frac{2}{c_1^2 b^2} - \frac{1}{n^2} \right) s + \left( \frac{n}{c_1^4 b^4} - \frac{3}{c_1 b} + \frac{1}{c_1^2 b^2} + \frac{1}{n^2} - \frac{2}{c_1 b n} - \frac{c_1 b}{n^2} \right) \right]$$

$$= -\frac{1}{c_1^4 b^4} \left( \frac{c_1^4 b^4}{4} - \frac{c_1^2 b^2}{2} + \frac{c_1^2 b^2}{4} \right) + \left( \frac{n}{c_1^4 b^4} + \frac{1}{c_1^2 b^4} - \frac{2}{c_1^2 b^2 n} \right) \left( \frac{c_1^3 b^3}{3} - \frac{c_1^2 b^2}{2} + \frac{c_1 b}{6} \right)$$

$$+ \left( \frac{3}{b^2} - \frac{2}{b^3} + \frac{2}{b^2 n} - \frac{2}{b^3} - \frac{1}{n^2} \right) \left( \frac{b^2}{2} - \frac{b}{2} \right)$$

$$+ \left( \frac{n}{c_1^2 b^2} + \frac{3}{c_1 b} - \frac{3}{c_1 b^2} + \frac{1}{c_1^2 b^2} + \frac{1}{n^2} - \frac{2}{c_1 b n} - \frac{c_1 b}{n^2} \right) (c_1 b - 1)$$

$$= \frac{n}{c_1^4 b^4} \cdot \frac{c_1^3 b^3}{3} - \frac{2n}{c_1^4 b^4} \cdot \frac{c_1^2 b^2}{2} + \frac{n}{c_1^4 b^4} \cdot \frac{c_1^2 b^2}{2} \cdot c_1 b$$

$$= \frac{1}{3} \frac{n}{c_1 b} + o \left( \frac{n}{b} \right).$$

(36)

Plug (36) in (35)

$$E[a_2] = \Sigma_{ij}^2 \sum_{s=1}^{c_1 b - 1} \sum_{l=0}^{n-c_1 b - s} \left( \frac{n-c_1 b}{c_1 bn} \right)^2 + \left( \Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2 \right) \left[ \frac{1}{3} \frac{n}{c_1 b} + o \left( \frac{n}{b} \right) \right].$$

(37)

Similarly as $E[a_2]$, we calculate $E[a_3]$ by first calculating

$$E \left[ \left( \tilde{C}_l^{(i)} (c_1 b) - \tilde{C}^{(i)} \right) \left( \tilde{C}_l^{(j)} (c_1 b) - \tilde{C}^{(j)} \right) \left( \tilde{C}_{l+s}^{(i)} (c_1 b) - \tilde{C}^{(i)} \right) \left( \tilde{C}_{l+s}^{(j)} (c_1 b) - \tilde{C}^{(j)} \right) \right]$$

27
for $s = c_1 b, \ldots, (n - c_1 b)$. We will show that

$$\begin{align*}
\begin{bmatrix}
C_l(b) - \bar{C} \\
\tilde{C}_{l+s}(b) - \bar{C}
\end{bmatrix}
\sim N
\left(
\begin{bmatrix}
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\frac{n-b}{bn} \Sigma & -\frac{1}{n} \Sigma \\
-\frac{1}{n} \Sigma & \frac{n-b}{bn} \Sigma
\end{bmatrix}
\right).
\end{align*}
$$

Continuing as in (30)

$$\text{Cov}(\tilde{C}_l(c_1 b) - \bar{C}, \tilde{C}_{l+s}(c_1 b) - \bar{C}) = L \cdot \left( -\frac{1}{n} \right) I_p \cdot L^T = -\frac{1}{n} \cdot \Sigma. \quad (39)$$

The joint distribution at (38) follows (39) and (27). Denote $Z_1 = \tilde{C}_l(c_1 b)^{(i)} - \bar{C}^{(i)}$, $Z_2 = \tilde{C}_l(c_1 b)^{(j)} - \bar{C}^{(j)}$, $Z_3 = \tilde{C}_{l+s}(c_1 b)^{(i)} - \bar{C}^{(i)}$, $Z_4 = \tilde{C}_{l+s}(c_1 b)^{(j)} - \bar{C}^{(j)}$. Apply Proposition 5 on (38),

$$E[a_3] = \sum_{s=c_1 b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b - s} E[(\tilde{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\tilde{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)})(\tilde{C}_{l+s}^{(i)}(c_1 b) - \bar{C}^{(i)})(\tilde{C}_{l+s}^{(j)}(c_1 b) - \bar{C}^{(j)})]$$

$$= \sum_{s=b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b - s} E[Z_1 Z_2 Z_3 Z_4]$$

$$= \sum_{s=b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b - s} \left[ \left( \frac{n-c_1 b}{c_1 bn} \right)^2 \Sigma_{ij} + \frac{1}{n^2} \Sigma_{ii} \Sigma_{jj} + \frac{1}{n^2} \Sigma_{ij}^2 \right]. \quad (40)$$

Notice

$$\sum_{s=c_1 b}^{n-c_1 b+1} \sum_{l=1}^{n-c_1 b} \frac{1}{n^2} = \frac{1}{n^2} \cdot \sum_{s=c_1 b}^{n-c_1 b} (n - c_1 b - s + 1)$$

$$= -\frac{1}{n^2} \left( \frac{n^2}{2} - c_1 bn + \frac{n}{2} \right) + \left( \frac{1}{n} - \frac{c_1 b}{n^2} + \frac{1}{n^2} \right) (n - c_1 b + 1)$$

$$= o \left( \frac{n}{b} \right). \quad (41)$$

Plug (41) in (40),

$$E[a_3] = \Sigma_{ij}^2 \sum_{s=b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b - s} \left( \frac{n-c_1 b}{c_1 bn} \right)^2 + o \left( \frac{n}{b} \right). \quad (42)$$

Plug (29), (37) and (42) in (26),

$$A_2 = E \left[ \frac{c_1^2 b^2}{n^2} \sum_{l=0}^{n-c_1 b} \left( \tilde{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)} \right) \left( \tilde{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)} \right) \right]^2$$
\[
\begin{align*}
&= \frac{c_1 b^2}{n^2} \cdot [Ea_1 + 2Ea_2 + 2Ea_3] \\
&= \frac{c_1 b^2}{n^2} \cdot \left[ \frac{2}{3} (\Sigma_{ii} - \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{n}{c_1 b} + o\left(\frac{n}{b}\right) \right] \\
&\quad + \Sigma_{ij}^2 \left( \sum_{l=0}^{n-c_1 b} \left( \frac{n - c_1 b}{c_1 b n} \right)^2 + 2 \sum_{s=1}^{c_1 b-1} \sum_{l=0}^{n-c_1 b-s} \left( \frac{n - c_1 b}{c_1 b n} \right)^2 \right) \\
&\quad + 2 \sum_{s=c_1 b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b-s} \left( \frac{n - c_1 b}{c_1 b n} \right)^2 \right) \\
&= \frac{c_1 b^2}{n^2} \cdot \left[ \frac{2}{3} (\Sigma_{ii} - \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{n}{c_1 b} + o\left(\frac{n}{b}\right) + \Sigma_{ij}^2 \left( \frac{n - c_1 b}{c_1 b n} \right)^2 (n - c_1 b + 1)^2 \right] \\
&= \frac{c_1 b^2}{n^2} \cdot \left[ \frac{2}{3} (\Sigma_{ii} - \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{n}{c_1 b} + o\left(\frac{n}{b}\right) + \Sigma_{ij}^2 \left( \frac{n^2}{c_1 b^2} - \frac{4n}{c_1 b} \right) \right] \\
&= \frac{2}{3} (\Sigma_{ii} - \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{c_1 b}{n} + \Sigma_{ij}^2 - 4\Sigma_{ij}^2 \cdot \frac{c_1 b}{n} + o\left(\frac{b}{n}\right). \quad (43)
\end{align*}
\]

That proves the first part of the lemma. We now move on to term \(A_3\). Let
\[
OL^{(i)} = (\bar{C}_p^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_q^{(i)}(c_2 b) - \bar{C}^{(i)}),
\]
and
\[
OL^{(j)} = (\bar{C}_p^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_q^{(j)}(c_2 b) - \bar{C}^{(j)}),
\]
for \(p, q\) satisfying \(q \geq p\) and \(q + c_1 b \leq p + c_2 b\). Then
\[
A_3 \quad (44)
\]
\[
A_3 = -\frac{c_2 b^2}{n^2} \cdot \left[ \sum_{l=0}^{n-c_1 b} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)}) \right] \left[ \sum_{l=0}^{n-c_2 b} (\bar{C}_l^{(i)}(c_2 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_2 b) - \bar{C}^{(j)}) \right] \\
+ 2 \sum_{s=1}^{c_2 b-1} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_l^{(i)}(c_2 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_2 b) - \bar{C}^{(j)}) \\
+ 2 \sum_{s=c_2 b}^{n-c_2 b} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_l^{(i)}(c_2 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_2 b) - \bar{C}^{(j)}) \right], \quad (45)
\]

Denote the two double sums in \((44)\) by:
\[
a_4 = \sum_{s=1}^{c_2 b-1} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_l^{(i)}(c_2 b) - \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_2 b) - \bar{C}^{(j)}) \right),
\]
\[ a_5 = \sum_{s=c_2 b}^{n-c_1 b} \sum_{l=0}^{n-c_1 b-s} (\bar{C}_l(c_1 b)^{(i)} - \bar{C}^{(i)}) (\bar{C}_l(c_1 b)^{(j)} - \bar{C}^{(j)}) (\bar{C}_l^{(i)}(c_1 b - 2) + \bar{C}^{(i)})(\bar{C}_l^{(j)}(c_1 b - 2) + \bar{C}^{(j)})]. \]

First consider \( E[OL^{(i)}OL^{(j)}] \) at [44]. We will show that

\[
\begin{bmatrix}
\bar{C}_p(c_1 b) - \bar{C}_n \\
\bar{C}_q(c_2 b) - \bar{C}_n
\end{bmatrix}
\sim N\left([0, 0], \begin{bmatrix}
\sum \left(\frac{n-c_1 b}{c_1 b n} \right) & \sum \left(\frac{n-c_1 b}{c_2 b n} \right) \\
\sum \left(\frac{n-c_1 b}{c_1 b n} \right) & \sum \left(\frac{n-c_2 b}{c_2 b n} \right)
\end{bmatrix}\right). \tag{46}
\]

For \( i \neq j \),

\[
E[\bar{B}_p^{(i)}(c_1 b) - \bar{B}^{(i)}][\bar{B}_q^{(j)}(c_2 b) - \bar{B}^{(j)}] = E[\bar{B}_p^{(i)}(c_1 b) - \bar{B}^{(i)}] \cdot E[\bar{B}_q^{(j)}(c_2 b) - \bar{B}^{(j)}] = 0. \tag{47}
\]

For \( i = j \) and \( q \) satisfying \( q \geq p \) and \( q + c_2 b \leq p + c_1 b \), following steps similar to [31],

\[
E[\bar{B}_p^{(i)}(c_1 b) - \bar{B}^{(i)}][\bar{B}_q^{(j)}(c_2 b) - \bar{B}^{(j)}] = \frac{n-c_1 b}{nc_1 b}. \tag{48}
\]

By [47] and [48]

\[
\text{Cov}(\bar{C}_p(c_1 b) - \bar{C}_n, \bar{C}_q(c_2 b) - \bar{C}_n) = L \cdot \left(\frac{n-c_1 b}{c_1 b n}\right) I_p \cdot L^T = \frac{n-c_1 b}{c_1 b n} \cdot \Sigma. \tag{49}
\]

Equation [49] yields the joint distribution at [46]. Denote \( Z_1 = \bar{C}_p(c_1 b)^{(i)} - \bar{C}^{(i)} \), \( Z_2 = \bar{C}_p(c_1 b)^{(j)} - \bar{C}^{(j)} \), \( Z_3 = \bar{C}_q(c_2 b)^{(i)} - \bar{C}^{(i)} \), \( Z_4 = \bar{C}_q(c_2 b)^{(j)} - \bar{C}^{(j)} \). Then

\[
E[((c_1 - c_2) b + 1)(n-c_1 b + 1)OL^{(i)}OL^{(j)}]
\]

\[ = ((c_1 - c_2) b + 1)(n-c_1 b + 1) \times E[(\bar{C}_p^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_p^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_q^{(i)}(c_2 b) - \bar{C}^{(i)})(\bar{C}_q^{(j)}(c_2 b) - \bar{C}^{(j)})]
\]

\[ = ((c_1 - c_2) b + 1)(n-c_1 b + 1) \cdot E[Z_1 Z_2 Z_3 Z_4]
\]

\[ = ((c_1 - c_2) b + 1)(n-c_1 b + 1) \cdot \left(\frac{n-c_1 b}{c_1 b n}\right)^2 (\Sigma_{ij}^2 + \Sigma_{ii} \Sigma_{jj})
\]

\[ + ((c_1 - c_2) b + 1)(n-c_1 b + 1) \cdot \left(\frac{n-c_1 b}{c_1 b n}\right) \left(\frac{n-c_2 b}{c_2 b n}\right) \Sigma_{ij}^2. \tag{51}
\]

Notice that

\[
((c_1 - c_2) b + 1)(n-c_1 b + 1) \cdot \left(\frac{n-c_1 b}{c_1 b n}\right)^2
\]

\[ = ((c_1 - c_2) b + 1)(n-c_1 b + 1) \cdot \left(\frac{1}{c_1 b^2} + \frac{1}{n^2} - \frac{2}{c_1 b n}\right)
\]

30
\[ \begin{align*}
&= (c_1 - c_2)bn \frac{1}{c_1 b^2} + o \left( \frac{n}{b} \right) \\
&= (c_1 - c_2) \frac{n}{c_1 b} + o \left( \frac{n}{b} \right),
\end{align*} \]  

(52)

and
\[ ((c_1 - c_2)b + 1)(n - c_1 b + 1) \cdot \left( \frac{n - c_1 b}{c_1 b n} \right) \left( \frac{n - c_2 b}{c_2 b n} \right) \]
\[ = ((c_1 - c_2)b + 1)(n - c_1 b + 1) \cdot \left( \frac{1}{c_1 c_2 b^2} + \frac{1}{n^2} - \frac{(c_1 + c_1)}{c_1 c_2 b n} \right) \\
= (c_1 - c_2)bn \frac{1}{c_1 c_2 b^2} + o \left( \frac{n}{b} \right) \\
= \frac{c_1 - c_2 n}{c_1 c_2} \frac{n}{b} + o \left( \frac{n}{b} \right). \]  

(53)

Plug (52) and (53) in (50),
\[ E[[(c_1 - c_2)b + 1)(n - c_1 b + 1)OL^{(i)}OL^{(j)}] \\
= \left[ \frac{(c_1 - c_2)}{c_1^2} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) + \frac{c_1 - c_2}{c_1 c_2} \Sigma_{ij}^2 \right] \cdot \frac{n}{b} + o \left( \frac{n}{b} \right). \]  

(54)

We calculate \( E[a_4] \) by first deriving
\[ \begin{bmatrix}
\tilde{C}_l(c_1 b) - \bar{C} \\
\tilde{C}_{l+(c_1-c_2)b+s}(c_2 b) - \bar{C}
\end{bmatrix} \sim N \left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, \begin{bmatrix}
\left( \frac{n - c_1 b}{c_1 b n} \right) \Sigma & \left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_2 b^2} \right) \Sigma \\
\left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_2 b^2} \right) \Sigma & \left( \frac{n - c_2 b}{c_2 b n} \right) \Sigma
\end{bmatrix} \right). \]  

(55)

All we need to obtain is the covariance matrix. Continuing as before in (30), For \( i \neq j \),
\[ E[\tilde{B}_l^{(i)}(c_1 b) - \bar{B}^{(i)}][\tilde{B}_{l+(c_1-c_2)b+s}(c_2 b) - \bar{B}^{(j)}] = 0. \]  

(56)

For \( i = j \), we need to calculate \( E[\tilde{B}_l^{(i)}(c_1 b) - \bar{B}^{(i)}][\tilde{B}_{l+(c_1-c_2)b+s}(c_1 b) - \bar{B}^{(i)}] \) for \( s = 1, \ldots, (c_2 b - 1) \). Continuing as before in (31),
\[ E[\tilde{B}_l^{(i)}(c_1 b) - \bar{B}^{(i)}][\tilde{B}_{l+(c_1-c_2)b+s}(c_1 b) - \bar{B}^{(i)}] \\
= \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_1 c_2 b^2}. \]  

(57)

By (56) and (57),
\[ \text{Cov}(\tilde{C}_l(b) - \bar{C}, \tilde{C}_{l+s}(b) - \bar{C}) = \left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_1 c_2 b^2} \right) \cdot \Sigma. \]  

(58)
Therefore (55) follows from (27) and (58). Again denote \( Z_1 = \bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)}, Z_2 = C_t^{(j)}(c_1 b) - \bar{C}^{(j)}, Z_3 = \bar{C}_t^{(i)}(c_1 + c_2 b) + s (c_2 b) - \bar{C}^{(i)}, Z_4 = \bar{C}_t^{(j)}(c_1 + c_2 b) + s (c_2 b) - \bar{C}^{(j)}, \)

\[
E[a_1] = \sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} E[(\bar{C}_l^{(i)}(c_1 b) - \bar{C}^{(i)})(\bar{C}_t^{(j)}(c_1 b) - \bar{C}^{(j)})(\bar{C}_t^{(i)}(c_1 + c_2 b) + s (c_2 b) - \bar{C}^{(i)})(\bar{C}_t^{(j)}(c_1 + c_2 b) + s (c_2 b) - \bar{C}^{(j)})] \\
= \sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} E[Z_1 Z_2 Z_3 Z_4] \\
= \sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} \left[ \left( \frac{n-c_1 b}{c_1 b m} \right) \left( \frac{n-c_2 b}{c_2 b n} \right) \right] \Sigma_{ij}^2 + \left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_1 c_2 b^2} \right)^2 \Sigma_{ii} \Sigma_{jj} + \left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_1 c_2 b^2} \right)^2 \Sigma_{ij}^2 \right].
\]

Notice

\[
\sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} \left( \frac{n-c_1 b}{c_1 b m} \right) \left( \frac{n-c_2 b}{c_2 b n} \right) \\
= \sum_{s=1}^{c_2 b - 1} \left[ \left( \frac{1}{c_1 c_2 b^2} - \frac{c_2 + c_1}{c_1 c_2} \frac{1}{b m} + \frac{1}{n^2} \right) (n-c_1 b + 1) - \left( \frac{1}{c_1 c_2 b^2} - \frac{c_1 + c_2}{c_1 c_2} \frac{1}{b m} + \frac{1}{n^2} \right) s \right] \\
= \left( \frac{1}{c_1 c_2 b^2} - \frac{c_2 + c_1}{c_1 c_2} \frac{1}{b m} + \frac{1}{n^2} \right) (n-c_1 b + 1)(c_2 b - 1) - \left( \frac{1}{c_1 c_2 b^2} - \frac{c_1 + c_2}{c_1 c_2} \frac{1}{b m} + \frac{1}{n^2} \right) \left( \frac{c_2 b^2}{2} - \frac{c b}{2} \right) \\
= \frac{1}{c_1 c_2 b^2} \cdot n \cdot c_2 b + o \left( \frac{n}{b} \right) \\
= \frac{n}{c_1 b} + o \left( \frac{n}{b} \right),
\]

and

\[
\sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} \left( \frac{1}{c_1 b} - \frac{1}{n} - \frac{s}{c_1 c_2 b^2} \right)^2 \\
= \sum_{s=1}^{c_2 b - 1} \sum_{l=0}^{n-c_1 b-s} \left[ \frac{s^2}{c_1 c_2 b^4} + \left( \frac{2}{c_1 c_2 b^2 n} - \frac{2}{c_1 c_2 b^3} \right) s + \left( \frac{1}{c_1 b^2} + \frac{1}{n^2} - \frac{2}{c_1 b n} \right) \right] \\
= \sum_{s=1}^{c_2 b - 1} \left[ -\frac{s^3}{c_1 c_2 b^4} + \left( \frac{n}{c_1 c_2 b^2} - \left( \frac{2}{c_1 c_2} + \frac{1}{c_1 c_2} \right) \frac{1}{b^3} + \frac{1}{c_1 c_2 b^4} + \frac{2}{c_1 c_2 b^2 n} \right) s^2 \\
+ \left[ \left( \frac{4}{c_1 c_2} - \frac{1}{c_1^2} \right) \frac{1}{b^2} - \frac{2}{c_1 c_2 b^3} + \left( \frac{2}{c_1} - \frac{1}{c_2} \right) \frac{1}{b n} + \frac{2}{c_1 c_2 b^2 n} - \frac{2}{c_1 c_2} \frac{1}{b^3} + \frac{1}{n^2} \right] \right] s \\
+ \left( \frac{4}{c_1 b^2} + \frac{3}{c_1 b} - \frac{3}{c_1 b^2 n} + \frac{1}{c_1 c_2 b^2} + \frac{1}{n^2} - \frac{2}{c_1 b n} \right) \right].
\]

32
Therefore (65) follows from (27) and (68). Again denote

Finally, we calculate

Plug (61) and (62) in (59)

For \(i \neq j\),

For \(i = j\), we need to calculate \(E[B_i^{(i)}(b) - B^{(i)}] [B_i^{(j)}(b) - B^{(j)}] \) for \(s = c_b, \ldots, (n-b)\).

Similar to the steps in (31), we get

By (66) and (67),

Therefore (65) follows from (27) and (68). Again denote \(Z_1 = \tilde{C}_t(c_1b) - \tilde{C}, Z_2 = \tilde{C}_t(c_1b) - C \), \(Z_3 = \tilde{C}_t(c_1b) - \tilde{C}, Z_4 = \tilde{C}_t(c_1b) - C \)

\[ E[a_5] \]
\[
= \sum_{s=c_2b}^{n-c_1b-n-c_1b-s} \sum_{l=0}^{n-c_1b-n-c_1b-s} E[(\bar{C}_i^{(i)}(c_1b) - \bar{C}_i^{(i)})(\bar{C}_i^{(j)}(c_1b) - \bar{C}_i^{(j)})(\bar{C}_{l+(c_1-c_2)b+s}^{(i)}(c_2b) - \bar{C}_i^{(i)})(\bar{C}_{l+(c_1-c_2)b+s}^{(j)}(c_2b) - \bar{C}_i^{(j)})]
\]

\[
= \sum_{s=c_2b}^{n-c_1b-n-c_1b-s} E[Z_1Z_2Z_3Z_4]
\]

\[
= \sum_{s=c_2b}^{n-c_1b-n-c_1b-s} \sum_{l=0}^{n-c_1b-n-c_1b-s} \left[ \left( \frac{n - c_1b}{c_1bn} \right) \left( \frac{n - c_2b}{c_2bn} \right) \Sigma_{ii}^2 + \frac{1}{n^2} \Sigma_{ii} \Sigma_{jj} + \frac{1}{n^2} \Sigma_{ii}^2 \right] .
\] (70)

Notice

\[
\sum_{s=c_2b}^{n-c_1b-n-c_1b-s} \sum_{l=0}^{n-c_1b-n-c_1b-s} \left( \frac{n - c_1b}{c_1bn} \right) \left( \frac{n - c_2b}{c_2bn} \right) 
\]

\[
= \sum_{s=c_2b}^{n-c_1b-n-c_1b-s} \left( \frac{1}{c_1c_2b^2} - \frac{c_1 + c_2}{c_1c_2} \frac{1}{bn} + \frac{1}{n^2} \right) (n - c_1b + 1) - \left( \frac{1}{c_1c_2b^2} - \frac{c_1 + c_2}{c_1c_2} \frac{1}{bn} + \frac{1}{n^2} \right) s
\]

\[
= \left( \frac{1}{c_1c_2b^2} - \frac{c_1 + c_2}{c_1c_2} \frac{1}{bn} + \frac{1}{n^2} \right) (n - c_1b + 1)[n - (c_1 + c_2)b + 1]
\]

\[
- \left( \frac{1}{c_1c_2b^2} - \frac{c_1 + c_2}{c_1c_2} \frac{1}{bn} + \frac{1}{n^2} \right) \frac{[(c_2 - c_1)b + n][n - (c_1 + c_2)b + 1]}{2}
\]

\[
= \left( \frac{1}{c_1c_2b^2} - \frac{c_1 + c_2}{c_1c_2} \frac{1}{bn} + \frac{1}{n^2} \right) \frac{n^2 - (2c_1 + c_2)b + 2n + (c_1^2 + c_1)b^2 - (2c_1 + c_2)b + 1}{2}
\]

\[
= \left( \frac{1}{c_1c_2b^2} \cdot \frac{n^2}{2} - \frac{c_1 + c_2}{c_1c_2} \cdot bn - \frac{c_1 + c_2}{c_1c_2} \cdot bn \cdot \frac{n^2}{2} \right) + o\left( \frac{n}{b} \right)
\]

\[
= \frac{1}{2c_1c_2b^2} \cdot \frac{n^2}{2} - \left( \frac{3}{2c_1} + \frac{3}{2c_2} \right) \frac{n^2}{b} + o\left( \frac{n}{b} \right),
\] (71)

and

\[
\sum_{s=c_2b}^{n-c_1b-n-c_1b-s} \sum_{l=0}^{n-c_1b-n-c_1b-s} \frac{1}{n^2}
\]

\[
= \sum_{s=c_2b}^{n-c_1b} \frac{s}{n^2} + \left( \frac{1}{n} - \frac{c_1b}{n^2} + \frac{1}{n^2} \right)
\]

\[
= - \frac{1}{n^2} \left( \frac{(n - c_1b - c_2b + 1)(n - (c_1 - c_2))}{2} \right) + \left( \frac{1}{n} - \frac{c_1b}{n^2} + \frac{1}{n^2} \right) \cdot [n - (c_1 + c)2b + 1]
\]

\[
= o\left( \frac{n}{b} \right).
\] (72)
Lemma 5. If Assumption 1 holds and

\[ E[a_5] = \sum_{ij} \left( \frac{1}{2c_1c_2} n^2 b^2 - \left( \frac{3}{2c_1} + \frac{3}{2c_2} \right) \frac{n}{b} \right). \]  

(73)

Replace (54), (64) and (73) in (44)

\[
A_3 = E \left[ -\frac{c_1c_2b^2}{n^2} \left[ \sum_{i=0}^{n-b} \left( \bar{C}_l^{(i)}(b) - \bar{C'}_l^{(i)}(b) - \bar{C'}_l^{(j)}(b) - \bar{C}_l^{(j)}(b) \right) \right] \left[ \sum_{i=0}^{n-c_b} \left( \bar{C}_l^{(i)}(cb) - \bar{C'}_l^{(i)}(cb) - \bar{C'}_l^{(j)}(cb) - \bar{C}_l^{(j)}(cb) \right) \right] \right]
\]

\[
= -\frac{c_1c_2b^2}{n^2} \cdot E \left[ ((c_1 - c_2)b + 1)(n - b + 1) \cdot O(L)OL + 2a_4 + 2a_5 \right]
\]

\[
= -\frac{c_1c_2b}{n} \cdot \left[ \frac{(c_1 - c_2)}{c_1} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}) + \frac{c_1 - c_2}{c_1c_2} \Sigma_{ij} \right]
\]

\[
+ \frac{2c_2}{3c_1} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}) + \Sigma_{ij} \left( \frac{1}{c_1c_2} \frac{n}{b} - \left( \frac{3}{c_1} + \frac{3}{c_2} \right) \right) + o \left( \frac{b}{n} \right)
\]

\[
= -\frac{c_1c_2b}{n} \left[ \frac{3c_1 - c_2}{3c_1} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}) \cdot -2 \left( \frac{c_1 + c_2}{c_1c_2} \right) \cdot \Sigma_{ij} + \frac{1}{c_1c_2} \frac{n}{b} \cdot \Sigma_{ij} \right] + o \left( \frac{b}{n} \right)
\]

\[
= \frac{(c_2 - 3c_1)c_2}{3c_1} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}) \cdot \frac{b}{n} + 2 (c_1 + c_2) \cdot \Sigma_{ij} \cdot \frac{b}{n} - \Sigma_{ij} + o \left( \frac{b}{n} \right). \]  

(74)

Let \( \hat{\Sigma}_w = L\hat{\Sigma}_w L^T \) where \( L \) is the lower triangular matrix such that \( \Sigma = LL^T \). Then

\[
\hat{\Sigma}_w = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_n(k)[\bar{C}_l(k) - \bar{C}][\bar{C}'_l(k) - \bar{C}].
\]

**Lemma 5.** If Assumption 4 holds and

\[
\sum_{k=1}^{b} (\Delta_2 w_k)^2 \leq O \left( \frac{1}{b^2} \right),
\]

(75)

then

\[
Var[\hat{\Sigma}_{wL,ij}] = (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}) \left[ \frac{2}{3} \sum_{k=1}^{b} (\Delta_2 w_k)^2 k^3 \cdot \frac{1}{n} + 2 \sum_{t=1}^{b-t} \Delta_2 w_u \Delta_2 w_{u+} \left( \frac{2}{3} u^2 + u^2 t \right) \frac{1}{n} \right] + o \left( \frac{b}{n} \right). \]  

(76)

**Proof.** Note

\[
(\Delta_2 w_k)^2 \leq \sum_{k=1}^{b} (\Delta_2 w_k)^2 \leq O \left( \frac{1}{b^2} \right),
\]
\[ a_k = b \cdot \Delta_2 w_k \leq O(1). \]

Consider

\[
\tilde{\Sigma}_{wL,ij} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} k^2 \Delta_2 w_n(k) [\tilde{C}_l^{(i)}(k) - \tilde{C}^{(i)}] [\tilde{C}_l^{(j)}(k) - \tilde{C}^{(j)}].
\]

Let \( c_k = k/b \) for \( k = 1, \ldots, b \), also denote \( a_k = b \cdot \Delta_2 w_k \) for simplicity. Hence

\[
\tilde{\Sigma}_{wL,ij} = \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} \left( c_k b^2 \Delta_2 w_n(k) [\tilde{C}_l^{(i)}(k) - \tilde{C}^{(i)}] [\tilde{C}_l^{(j)}(k) - \tilde{C}^{(j)}] \right)
\]

\[
= \frac{1}{n} \sum_{k=1}^{b} \sum_{l=0}^{n-k} c_k b \left( \sum_{i,j} \left( \sum_{p=0}^{n-c_k b} (\tilde{C}_l^{(i)}(c_k b) - \tilde{C}^{(i)}) (\tilde{C}_l^{(j)}(c_k b) - \tilde{C}^{(j)}) \right) \right)
\]

\[
= \sum_{k=1}^{b} c_k a_k \left( \frac{c_k b}{n} \sum_{i,j} \left( \sum_{p=0}^{n-c_k b} (\tilde{C}_l^{(i)}(c_k b) - \tilde{C}^{(i)}) (\tilde{C}_l^{(j)}(c_k b) - \tilde{C}^{(j)}) \right) \right).
\]

Define \( A_{1,ij}^{(k)} \) and \( A_{2,ij}^{(ut)} \) below and apply Lemma 4

\[
A_{1,ij}^{(k)} = E \left[ \left( c_k b \right)^2 \left( \sum_{i,j} \left( \sum_{p=0}^{n-c_k b} (\tilde{C}_l^{(i)}(c_k b) - \tilde{C}^{(i)}) (\tilde{C}_l^{(j)}(c_k b) - \tilde{C}^{(j)}) \right) \right)^2 \right]
\]

\[
= \left( \frac{2}{3} \Sigma_{ii} \Sigma_{jj} + 4 \Sigma_{ij}^2 \right) \cdot \frac{c_k b}{n} + \Sigma_{ij}^2 + O \left( \frac{b}{n} \right), \quad (77)
\]

and

\[
A_{2,ij}^{(ut)} = E \left[ \left( c_u + t \right)^2 \left( \sum_{i,j} \left( \sum_{p=0}^{n-c_u + t} (\tilde{C}_l^{(i)}(c_t + u b) - \tilde{C}^{(i)}) (\tilde{C}_l^{(j)}(c_t + u b) - \tilde{C}^{(j)}) \right) \right) \right]
\]

\[
= \left( \frac{c_u + t}{3} \right) \left( \Sigma_{ii} \Sigma_{jj} + 3 \Sigma_{ij}^2 \right) - \left( 2 c_u + t + \frac{2 c_u + t}{c_u} \right) \Sigma_{ij} \left( \frac{b}{n} + \frac{c_u + t}{c_u} \Sigma_{ij}^2 + O \left( \frac{b}{n} \right) \right). \quad (78)
\]

To calculate \( Var[\tilde{\Sigma}_{wL,ij}] \), we will calculate \( E[\tilde{\Sigma}_{wL,ij}^2] \) and \( (E[\tilde{\Sigma}_{wL,ij}])^2 \). Plugging (77) and (78) in the expression of \( E[\tilde{\Sigma}_{wL,ij}^2] \) results in

\[
E[\tilde{\Sigma}_{wL,ij}^2] = E \left[ \left( \sum_{k=1}^{b} c_k a_k \cdot \left( \frac{c_k b}{n} \sum_{i,j} \left( \sum_{p=0}^{n-c_k b} (\tilde{C}_l^{(i)}(k) - \tilde{C}^{(i)}) (\tilde{C}_l^{(j)}(k) - \tilde{C}^{(j)}) \right) \right) \right)^2 \right]
\]

36
\[ E \left[ \sum_{k=1}^{b} \left( c_{k}a_{k} \cdot \frac{c_{k}b}{n} \sum_{l=0}^{n-c_{l}b} (C_{l}^{(i)}(k) - \bar{C}^{(i)})(\bar{C}_{l}^{(j)}(k) - \bar{C}^{(j)}) \right)^{2} \right] \]

\[ + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u}a_{t+u} \cdot E \left[ \frac{(c_{u+t}b)^{2}}{n^2} \cdot \left( \sum_{l=0}^{n-c_{l}b} (C_{l}^{(i)}(k) - \bar{C}^{(i)})(\bar{C}_{l}^{(j)}(k) - \bar{C}^{(j)}) \right)^{2} \right] \]

\[ = \sum_{k=1}^{b} c_{k}^{2}a_{k}^{2}A_{1,ij}^{(k)} + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u}a_{u+t}A_{2,ij}^{(ut)} \]

\[ = o \left( \frac{b}{n} \right) + \sum_{k=1}^{b} c_{k}^{2}a_{k}^{2} \left[ \left( \frac{2}{3} \Sigma_{i} \Sigma_{j} + \Sigma_{ij}^{2} - 4\Sigma_{ij}^{2} \right) - \frac{c_{k}b}{n} + \Sigma_{ij}^{2} \right] \]

\[ + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u+u} \left[ \left( c_{u+t} - \frac{c_{u}}{3} \right) \left( \Sigma_{i} \Sigma_{j} + \Sigma_{ij}^{2} \right) - \left( 2c_{u+t} + \frac{2c_{u+t}^{2}}{c_{u}} \right) \Sigma_{ij}^{2} \right] \cdot \frac{b}{n} + \frac{c_{u+t}\Sigma_{ij}^{2}}{c_{u}} \]

\[ = \sum_{k=1}^{b} c_{k}^{2}a_{k}^{2} \cdot \Sigma_{ij}^{2} + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u}a_{u+t} \cdot \frac{c_{u+t}}{c_{u}} \cdot \Sigma_{ij}^{2} \]

\[ + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u}a_{u+t} \left[ \left( c_{u+t} - \frac{c_{u}}{3} \right) \left( \Sigma_{i} \Sigma_{j} + \Sigma_{ij}^{2} \right) - \left( 2c_{u+t} + \frac{2c_{u+t}^{2}}{c_{u}} \right) \Sigma_{ij}^{2} \right] \cdot \frac{b}{n} + o \left( \frac{b}{n} \right) \]

\[ = \sum_{k=1}^{b} c_{k}^{2}a_{k}^{2} \Sigma_{ij}^{2} + 2 \sum_{k=1}^{b} c_{k}^{3}a_{k}^{2} \left( \frac{2}{3} \Sigma_{i} \Sigma_{j} + \Sigma_{ij}^{2} - 4\Sigma_{ij}^{2} \right) \cdot \frac{b}{n} \]

\[ + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} c_{u}a_{u}a_{u+t} \left[ \left( c_{u+t} - \frac{c_{u}}{3} \right) \left( \Sigma_{i} \Sigma_{j} + \Sigma_{ij}^{2} \right) - \left( 2c_{u+t} + \frac{2c_{u+t}^{2}}{c_{u}} \right) \Sigma_{ij}^{2} \right] \cdot \frac{b}{n} + o \left( \frac{b}{n} \right) . \]

By (27),

\[ E[(C_{l}^{(i)}(c_{k}b) - \bar{C}^{(i)})(\bar{C}_{l}^{(j)}(c_{k}b) - \bar{C}^{(j)})] = \frac{n - c_{k}b}{c_{k}bm} \Sigma_{ij}. \]
Plug (80) in \((E[\hat{\Sigma}_{wL,ij}])^2\),

\[
(E[\hat{\Sigma}_{wL,ij}])^2 = \left( \frac{1}{n} \sum_{k=1}^{b} \sum_{t=0}^{n-k} k^2 \Delta_2 w_k E[(C_t^{(i)}(c_k b) - C^{(i)})(C_t^{(j)}(c_k b) - C^{(j)})] \right)^2
\]

\[
= \left( \sum_{k=1}^{b} c_k a_k \left[ \frac{c_k b}{n} \sum_{t=0}^{n-c_k b} E[(C_t^{(i)}(c_k b) - C^{(i)})(C_t^{(j)}(c_k b) - C^{(j)})] \right] \right)^2
\]

\[
= \left( \sum_{k=1}^{b} c_k a_k \left[ \frac{c_k b}{n} \cdot (n - c_k b + 1) \cdot \frac{n - c_k b}{c_k b n} \cdot \Sigma_{ij} \right] \right)^2 \text{ apply (1.4.3)}
\]

\[
= \Sigma_{ij}^2 \left[ \sum_{k=1}^{b} a_k c_k \right]^2 - \sum_{k=1}^{b} 4a_k^2 c_k^3 \cdot \frac{b}{n} - 2 \sum_{t=1}^{b-t} \sum_{u=1}^{b-1} a_u a_{u+t}(2c_u^2 c_{u+t} + 2c_u c_{u+t}^2) \cdot \frac{b}{n} + o \left( \frac{b}{n} \right) .
\]

Combine (79) and (81),

\[
\text{Var}[\hat{\Sigma}_{wL,ij}] = E[\hat{\Sigma}_{wL,ij}]^2 - (E[\hat{\Sigma}_{wL,ij}])^2
\]

\[
= \sum_{k=1}^{b} \frac{3c_k^2 a_k^2}{n} \left[ \frac{2}{3} (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) - 4 \Sigma_{ij}^2 \right] + 4 \Sigma_{ij}^2 \cdot \frac{b}{n}
\]

\[
+ 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} \left( c_u^2 a_u a_{u+t} \left[ (c_{u+t} - \frac{c_u}{3}) (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) - (2c_{u+t} + \frac{2c_{u+t}}{c_u}) \Sigma_{ij}^2 \right] 
\]

\[
+ a_u a_{u+t}(2c_u^2 c_{u+t} + 2c_u c_{u+t}^2) \Sigma_{ij}^2 \right) \cdot \frac{b}{n} + o \left( \frac{b}{n} \right)
\]

\[
= \sum_{k=1}^{b} \frac{2}{3} \left( \frac{k}{b} \right)^3 (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{b}{n}
\]

\[
+ 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} \left( \left( \frac{u}{b} \right)^2 \frac{u + t}{b} - \frac{1}{3} \left( \frac{u}{b} \right)^3 \right) b \Delta_2 w_u \cdot b \Delta_2 w_{u+t} \right) (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) \cdot \frac{b}{n} + o \left( \frac{b}{n} \right)
\]

\[
= (\Sigma_{ii} \Sigma_{jj} + \Sigma_{ij}^2) \cdot \left[ \sum_{k=1}^{b} \frac{2}{3} (\Delta_2 w_k)^2 k^3 \cdot \frac{1}{n} + 2 \sum_{t=1}^{b-1} \sum_{u=1}^{b-t} \Delta_2 w_u \cdot \Delta_2 w_{u+t} \left( \frac{2}{3} u^3 + u^2 t \right) \cdot \frac{1}{n} \right] + o \left( \frac{b}{n} \right).
\]

Lemma 5 is proved. \qed
Lemma 6. (Lemma 2 [Liu and Flegal, 2018]) Let Assumptions 2 and 1 hold. If as \( n \to \infty \),

\[
 b\psi(n)^2 \log n \left( \sum_{k=1}^{b} |\Delta_2 w_n(k)| \right)^2 \to 0,
\]

and

\[
 \psi(n)^2 b \sum_{k=1}^{b} |\Delta_2 w_n(k)| \to 0,
\]

then \( \hat{\Sigma}_w \to L \tilde{\Sigma}_w L^T \) as \( n \to \infty \) w.p. 1.

Lemma 7. Assume Assumption 1 holds. Further suppose Assumption 2 holds with \( ED^4 < \infty \), \( E_F \|g^4\| < \infty \) and as \( n \to \infty \), let \( \psi^2(n)b^{-1}\log n \to 0 \). Then

\[
 E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}]^2 \to 0 \text{ as } n \to \infty.
\]

Proof. An observation of Lemma B.4 of Jones et al. (2006), Lemmas 12, 13 and 14 of Flegal and Jones (2010) and Lemma 5 of Liu and Flegal (2018) show that Lemma 7 hold. \( \square \)

D Proof of Theorem 3

Define

\[
 \eta = \text{Var}[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}] + 2E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}](\hat{\Sigma}_{w,L,ij} - E\tilde{\Sigma}_{w,L,ij}],
\]

we first show that \( \eta \to 0 \) as \( n \to \infty \). Apply Lemma 7 by Cauchy-Schwarz inequality and \( \text{Var}[X] \leq EX^2 \).

\[
 |\eta| = |\text{Var}[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}] + 2E[(\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij})(\hat{\Sigma}_{w,L,ij} - E\tilde{\Sigma}_{w,L,ij})]| \\
 \leq E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}]^2 + 2\sqrt{E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}]^2 \cdot E[\hat{\Sigma}_{w,L,ij} - E\tilde{\Sigma}_{w,L,ij}]^2} \\
 = E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}]^2 + 2{(E[\hat{\Sigma}_{w,ij} - \tilde{\Sigma}_{w,L,ij}]^2)^{1/2} \cdot (\text{Var}[\hat{\Sigma}_{w,L,ij}])^{1/2}} \\
 \]

By (75),

\[
 \frac{1}{n} \sum_{k=1}^{b} (\Delta_2 w_k)^2 k^3 \leq \frac{b^3}{n} \sum_{k=1}^{b} (\Delta_2 w_k)^2 \leq O \left( \frac{b}{n} \right). 
\]

Hence (76) can be written as

\[
 \text{Var}[\hat{\Sigma}_{w,L,ij}] = ((\Sigma_{ii}^2 + \Sigma_{ij}^2)S + o(1)) \cdot \frac{b}{n}.
\]
By Lemma 7, $E[\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}]^2 = o(1)$, therefore

$$|\eta| \leq E[\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}]^2 + 2(E[\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}]^2)^{1/2} \cdot (\text{Var}[\tilde{\Sigma}_{wL,ij}])^{1/2}$$

$$= o(1) + 2 \sqrt{o(1) \cdot \left[\left((\Sigma_{ii} + \Sigma_{jj}^2)S + o(1)\right) \cdot \frac{b}{n}\right]}$$

$$= o(1) + 2 \left(\frac{b}{n}\right)^{1/2} \left[o(1) \cdot \left((\Sigma_{ii} + \Sigma_{jj}^2)S + o(1)\right)^{1/2} = o(1). \quad (82)\right]$$

Since $b/n \to 0$ as $n \to \infty$, plug in (82)

$$\text{Var}[\tilde{\Sigma}_{w,ij}] = E[\tilde{\Sigma}_{w,ij} - E\tilde{\Sigma}_{w,ij}]^2$$

$$= E[\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij} + \tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{wL,ij} + E\tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{w,ij}]^2$$

$$= E[(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}) + (\tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{wL,ij}) - (E\tilde{\Sigma}_{w,ij} - E\tilde{\Sigma}_{wL,ij})]^2$$

$$= E[(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}) - E(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij})]^2 + E[\tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{wL,ij}]^2$$

$$+ 2E[(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}) - E(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij})] \cdot [\tilde{\Sigma}_{w,ij} - E\tilde{\Sigma}_{w,ij}]$$

$$= E[(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}) - E(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij})]^2 + E[\tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{wL,ij}]^2$$

$$+ 2E[(\tilde{\Sigma}_{w,ij} - \tilde{\Sigma}_{wL,ij}) \cdot (\tilde{\Sigma}_{w,ij} - E\tilde{\Sigma}_{w,ij})]$$

$$= E[\tilde{\Sigma}_{wL,ij} - E\tilde{\Sigma}_{wL,ij}]^2 + \eta$$

$$= (\Sigma_{ii} + \Sigma_{jj}^2)S \cdot \frac{b}{n} + o\left(\frac{b}{n}\right) + o(1).$$

References

Anderson, T. W. (1994). *The Statistical Analysis of Time Series*. Wiley-Interscience.

Boone, E. L., Merrick, J. R., and Krachey, M. J. (2014). A hellinger distance approach to mcmc diagnostics. *Journal of Statistical Computation and Simulation*, 84(4):833–849.

Brockmann, M., Gasser, T., and Herrmann, E. (1993). Locally adaptive bandwidth choice for kernel regression estimators. *Journal of the American Statistical Association*, 88(424):1302–1309.

Bühlmann, P. (1996). Locally adaptive lag-window spectral estimation. *Journal of Time Series Analysis*, 17(3):247–270.

Chan, K. W. and Yau, C. Y. (2017). Automatic optimal batch size selection for recursive estimators of time-average covariance matrix. *Journal of the American Statistical Association*, 112(519):1076–1089.

Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press.
Damerdji, H. (1991). Strong consistency and other properties of the spectral variance estimator. Management Science, 37:1424–1440.

Damerdji, H. (1995). Mean-square consistency of the variance estimator in steady-state simulation output analysis. Operations Research, 43:282–291.

Elith, J., Leathwick, J., and Hastie, T. (2008). A working guide to boosted regression trees. Journal of Animal Ecology, 77(4):802–813.

Finley, A. O. and Banerjee, S. (2013). spBayes: Univariate and multivariate spatial modeling R package version 0.3-7. http://CRAN.R-project.org/package=spBayes.

Finley, A. O., Banerjee, S., and Gelfand, A. E. (2012). Bayesian dynamic modeling for large space-time datasets using gaussian predictive processes. Journal of Geographical Systems, 14(1):29–47.

Flegal, J. M., Haran, M., and Jones, G. L. (2008). Markov chain Monte Carlo: Can we trust the third significant figure? Statistical Science, 23:250–260.

Flegal, J. M. and Jones, G. L. (2010). Batch means and spectral variance estimators in Markov chain Monte Carlo. The Annals of Statistics, 38:1034–1070.

Geyer, C. J. (1992). Practical Markov chain Monte Carlo. Statistical science, pages 473–483.

Geyer, C. J. (2011). Introduction to Markov chain Monte Carlo. In Handbook of Markov Chain Monte Carlo. CRC, London.

Hobert, J. P., Jones, G. L., Presnell, B., and Rosenthal, J. S. (2002). On the applicability of regenerative simulation in Markov chain Monte Carlo. Biometrika, 89:731–743.

Janssen, P. H. and Stoica, P. (1987). On the expectation of the product of four matrix-valued Gaussian random variables. Eindhoven University of Technology.

Jones, G. L. (2004). On the Markov chain central limit theorem. Probability Surveys, 1:299–320.

Jones, G. L., Haran, M., Caffo, B. S., and Neath, R. (2006). Fixed-width output analysis for Markov chain Monte Carlo. Journal of the American Statistical Association, 101:1537–1547.

Jones, G. L. and Hobert, J. P. (2001). Honest exploration of intractable probability distributions via Markov chain Monte Carlo. Statistical Science, 16:312–334.

Jones, M. C., Marron, J. S., and Sheather, S. J. (1996). A brief survey of bandwidth selection for density estimation. Journal of the American Statistical Association, 91(433):401–407.

Leathwick, J., Elith, J., Chadderton, W., Rowe, D., and Hastie, T. (2008). Dispersal, disturbance and the contrasting biogeographies of New Zealand’s diadromous and non-diadromous fish species. Journal of Biogeography, 35(8):1481–1497.
Liu, Y. and Flegal, J. (2018). Weighted batch means estimators in Markov chain Monte Carlo. Electronic Journal of Statistics, 12:3397–3442.

Loader, C. R. (1999). Bandwidth selection: classical or plug-in? Annals of Statistics, 27:415–438.

Meketon, M. S. and Schmeiser, B. (1984). Overlapping batch means: Something for nothing? In WSC ’84: Proceedings of the 16th conference on Winter simulation, pages 226–230, Piscataway, NJ, USA. IEEE Press.

Mykland, P., Tierney, L., and Yu, B. (1995). Regeneration in Markov chain samplers. Journal of the American Statistical Association, 90:233–241.

Plummer, M., Best, N., Cowles, K., and Vines, K. (2006). CODA: convergence diagnosis and output analysis for MCMC. R news, 6(1):7–11.

Politis, D. N. (2003). Adaptive bandwidth choice. Journal of Nonparametric Statistics, 15(4-5):517–533.

Politis, D. N. (2011). Higher-order accurate, positive semidefinite estimation of large-sample covariance and spectral density matrices. Econometric Theory, 27(4):703–744.

Politis, D. N. and Romano, J. P. (1995). Bias-corrected nonparametric spectral estimation. Journal of Time Series Analysis, 16(1):67–103.

Politis, D. N. and Romano, J. P. (1996). On flat-top kernel spectral density estimators for homogeneous random fields. Journal of Statistical Planning and Inference, 51(1):41–53.

Politis, D. N. and Romano, J. P. (1999). Multivariate density estimation with general flat-top kernels of infinite order. Journal of Multivariate Analysis, 68(1):1–25.

Seila, A. F. (1982). Multivariate estimation in regenerative simulation. Operations Research Letters, 1:153–156.

Sheather, S. (1983). A data-based algorithm for choosing the window width when estimating the density at a point. Computational Statistics & Data Analysis, 1:229–238.

Sheather, S. J. and Jones, M. C. (1991). A reliable data-based bandwidth selection method for kernel density estimation. Journal of the Royal Statistical Society. Series B (Methodological), pages 683–690.

Silverman, B. W. (1999). Density Estimation for Statistics and Data Analysis. Chapman & Hall Ltd.

Song, W. T. and Schmeiser, B. W. (1995). Optimal mean-squared-error batch sizes. Management Science, 41:110–123.

Taylor (2018). Sum of autocovariances for AR(p) model. Cross Validated. URL:https://stats.stackexchange.com/q/372006 (version: 2018-10-17).
Thompson, M. B. (2010). A comparison of methods for computing autocorrelation time. *arXiv preprint arXiv:1011.0175*.

Tjøstheim, D. (1990). Non-linear time series and Markov chains. *Advances in Applied Probability*, pages 587–611.

Vats, D. and Flegal, J. M. (2018). Lugsail lag windows and their application to MCMC. *arXiv preprint arXiv:1809.04541*.

Vats, D., Flegal, J. M., and Jones, G. L. (2018a). Multivariate output analysis for Markov chain Monte Carlo. *Biometrika, to appear*.

Vats, D., Flegal, J. M., and Jones, G. L. (2018b). Strong consistency of multivariate spectral variance estimators in Markov chain Monte Carlo. *Bernoulli*, 24:1860–1909.

Woodroofe, M. (1970). On choosing a delta-sequence. *The Annals of Mathematical Statistics*, 41(5):1665–1671.