A CHEEGER INEQUALITY FOR GRAPHS BASED ON A REFLECTION PRINCIPLE

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Abstract. Given a graph with a designated set of boundary vertices, we define a new notion of a Neumann Laplace operator on a graph using a reflection principle. We show that the first eigenvalue of this Neumann graph Laplacian satisfies a Cheeger inequality.

1. Introduction and Main Result

1.1. Introduction. Suppose that $G = (V, E)$ is a graph with vertices $V$ and edges $E$. Let $\partial V \subseteq V$ be a designated set of boundary vertices, and $\hat{V} := V \setminus \partial V$. We define the doubled graph $G'$ as follows. Let $\hat{G} = (U, F)$ be an isomorphic copy of the induced subgraph $G[\hat{V}]$, and let $f$ be an isomorphism from $\hat{V}$ to $U$. Set

$$F' := \{ \{u, v\} : u \in U, v \in \partial V, \{f^{-1}(u), v\} \in E \}.$$ 

Then, we define $G' := (V', E')$ where $V' := V \cup U$ and $E' := E \cup F \cup F'$. That is to say, $G'$ is defined by making an isomorphic copy of the interior of $G$ and attaching it to the boundary vertices $\partial V$ as in the original graph, see Figure 1.

![Figure 1. A graph $G$, and its doubled graph $G'$, where the black and white dots denote interior and boundary vertices, respectively.](image)

**Definition 1.1.** We say that a function $\varphi : V' \to \mathbb{R}$ is even with respect to $\partial V$ if

$$\varphi(v) = \varphi(f(v)) \text{ for } v \in \hat{V},$$

and we say that $\varphi$ is odd with respect to $\partial V$ if

$$\varphi(v) = -\varphi(f(v)) \text{ for } v \in \hat{V}, \text{ and } \varphi(v) = 0 \text{ for } v \in \partial V.$$ 

Let $L' := D - A$ denote the graph Laplacian of $G'$ where $D$ is the degree matrix of $G'$, and $A$ is the adjacency matrix of $G'$. The following proposition characterizes the eigenvectors of $L'$ as either even or odd.

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Proposition 1.1. The graph Laplacian $L'$ has $|V|$ eigenvectors that are even with respect to $\partial V$, and $|\bar{V}|$ eigenvectors that are odd with respect to $\partial V$; this accounts for all eigenvectors of $L'$.

Proof. The proof of this proposition is immediate from the block structure of the graph Laplacian $L'$. Indeed, let $L'(U,W)$ denote the submatrix of $L'$ whose rows and columns are indexed by $U \subseteq V$ and $W \subseteq V$, respectively. We can write

$$L' = \begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix},$$

where $X$ is the submatrix $L'(\bar{V},\bar{V})$, $Y$ is the submatrix $L'(\bar{V},\partial V)$, and $Z$ is the submatrix $L'(\partial V,\partial V)$. With this notation, the eigenvectors of $L'$ that are even with respect to $\partial V$ are solutions to the equation

$$\begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix} \begin{pmatrix} u \\ v \\ u \end{pmatrix} = \mu \begin{pmatrix} u \\ v \\ u \end{pmatrix}.$$

That is to say, the vectors $u$ and $v$ satisfy $Xu + Yv = \mu u$ and $2Y^\top u + Zv = \mu v$. Put differently, when concatenated, $u$ and $v$ form an eigenvector of the matrix $L_N := \begin{pmatrix} X & Y \\ 2Y^\top & Z \end{pmatrix}$.

Observe that $L_N$ is similar to a symmetric matrix

$$L_N = \begin{pmatrix} I & 0 \\ 0 & \sqrt{2}I \end{pmatrix} \begin{pmatrix} X & \sqrt{2}Y \\ \sqrt{2}Y^\top & Z \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sqrt{2}I \end{pmatrix}^{-1},$$

and thus by the Spectral Theorem $L_N$ has $|V|$ real eigenvectors, which give rise to $|\bar{V}|$ real eigenvectors of $L'$. The eigenvectors of $L'$ that are odd with respect to $\partial V$ are solutions to the equation

$$\begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix} \begin{pmatrix} u \\ 0 \\ -u \end{pmatrix} = \lambda \begin{pmatrix} u \\ 0 \\ -u \end{pmatrix}.$$

Thus, each vector $u$ such that $Xu = \lambda u$ gives rise to an odd eigenvector of $L'$. Let $L_D := X$.

Since $L_D$ is symmetric, it follows from the Spectral Theorem that it has $|\bar{V}|$ real eigenvectors, and we conclude that $L'$ has $|\bar{V}|$ odd eigenvectors. $\square$

1.2. Motivation. We are motivated by the observation that the odd and even eigenvectors of $L'$ for the graph $G'$ seem like natural Dirichlet and Neumann Laplacian eigenvectors for the graph $G$. In fact, the matrix $L_D$ has been previously defined as the Dirichlet graph Laplacian by Chung [3], and inequalities involving its eigenvalues have been investigated [6]. However, the operator $L_N$ defined above, has not, to our knowledge been investigated. In [3], Chung defines the Neumann graph Laplacian by enforcing a condition that a discrete derivative vanishes on the boundary nodes of the graph. A Cheeger inequality for Chung’s definition has recently been established by Hua and Huang [7]. In this paper, we argue that the operator $L_N$ defined above is a more natural way to define a Neumann graph Laplacian for a graph with boundary. In Remark 1.1, we show that on the path
graph the eigenvectors of the Dirichlet and Neumann graph Laplacians $L_D$ and $L_N$ are the familiar discrete sine and cosine functions. Our main result Theorem 1.1 shows that the first eigenvalue of the normalized version of $L_N$ satisfies a Cheeger inequality for graphs with boundary. In Remark 1.2, we illustrate Theorem 1.1 with an example where the first eigenvector of the Neumann graph Laplacian $L_N$ suggests a drastically different cut than the first eigenvector of the standard graph Laplacian; we describe how the graph cut suggested by $L_N$ is consistent with the Cheeger inequality established in Theorem 1.1. It may be interesting to investigate the analog of other classical eigenvalue inequalities involving these definitions of $L_D$ and $L_N$ for graphs with boundary.

Remark 1.1. The operators $L_D$ and $L_N$ are particularly natural on the path graph. Let $P_n = (V,E)$ denote the path graph on $n$ vertices, where $V = \{1, \ldots, n\}$ and $\{u,v\} \in E$ if and only if $|u - v| = 1$. If $\partial V := \{1, n\}$, then the doubled graph $P'_n = C_{2n-2}$ is the cycle graph on $2n-2$ vertices, see Figure 2.

![Figure 2. A path graph and its doubled graph.](image)

Consider $L_D$ and $L_N$ of the path graph $P_n$. The Dirichlet eigenvectors $\varphi_k$ and eigenvalues $\lambda_k$, which satisfy $L_D \varphi_k = \lambda_k \varphi_k$ for $k = 1, \ldots, n-2$, are of the form

$$\lambda_k := 2 \left( 1 - \cos \left( \frac{\pi k}{n-1} \right) \right) \quad \text{and} \quad \varphi_k(j) = \sin \left( \frac{\pi jk}{n-1} \right),$$

for $j = 1, \ldots, n-2$, while the Neumann eigenvectors, $\psi_k$ and $\mu_k$, which satisfy $L_N \psi_k = \mu_k \psi_k$ for $k = 0, \ldots, n-1$, are of the form

$$\mu_k := 2 \left( 1 - \cos \left( \frac{\pi k}{n-1} \right) \right) \quad \text{and} \quad \psi_k(j) = \cos \left( \frac{\pi jk}{n-1} \right)$$

for $j = 0, \ldots, n-1$. Thus, the path graph doubling procedure defined in §1.1 gives the familiar sine and cosine functions, which are the Dirichlet and Neumann eigenfunctions of the Laplace operator of the unit interval.

1.3. Main result. Suppose that $G = (V,E)$ is a graph with vertices $V$ and edges $E$. Let $\partial V \subseteq V$ be a designated set of boundary vertices, and set $\bar{V} = V \setminus \partial V$. We can write the adjacency matrix $A$ of the graph $G$ as the block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix},$$

where $A_{11} = A(\bar{V}, \bar{V})$, $A_{12} = A(\bar{V}, \partial V)$, and $A_{22} = A(\partial V, \partial V)$. With this notation, it is straightforward to verify that the Neumann graph Laplacian $L_N$ defined in §1.1 can be written as

$$L_N = \begin{pmatrix} D_{11} - A_{11} & -A_{12} \\ -2A_{12}^\top & D_{22} - A_{22} \end{pmatrix},$$
where \( D_{11} = \text{diag}(A_{11} \bar{1} + A_{12} \bar{1}) \) and \( D_{22} = \text{diag}(2A_{12}^\top \bar{1} + A_{22} \bar{1}) \). Here, \( \bar{1} \) denotes a vector whose entries are all 1, and whose dimensions are such that the matrix-vector multiplication is well defined. Let
\[
D := \begin{pmatrix}
D_{11} & 0 \\
0 & D_{22}
\end{pmatrix}, \quad \text{and} \quad Q := \begin{pmatrix}
I_{|\bar{V}|} & 0 \\
0 & \frac{1}{2} I_{|\partial V|}
\end{pmatrix},
\]
where \( I_n \) is an identity matrix of size \( n \times n \). We define the normalized Neumann graph Laplacian \( \mathcal{L}_N \) by
\[
\mathcal{L}_N := D^{-1/2} Q^{1/2} \mathcal{L}_N Q^{-1/2} D^{-1/2}.
\]
Since \( \mathcal{L}_N \) is symmetric, positive semi-definite, and has the eigenvector \( D_{11}^{-1/2} Q^{1/2} \bar{1} \) of eigenvalue 0, it follows that the first nontrivial eigenvalue \( \lambda_N \) of \( \mathcal{L}_N \) satisfies
\[
\lambda_N := \inf_{x \top D^{1/2} Q^{1/2} \bar{1} = 0} \frac{x \top \mathcal{L}_N x}{x \top x}.
\]
Let \( E(U, W) := \{ \{u, w\} \in E : u \in U, w \in W \} \), that is, \( E(U, W) \) is the set of edges between \( U \) and \( W \). We define a measure \( m(U, W) \) on this set of edges by
\[
m(U, W) = |E(U, W)| - \frac{1}{2} |E(U \cap \partial V, W \cap \partial V)|,
\]
and we define the volume \( \text{vol}(U) \) of \( U \subseteq V \) by
\[
\text{vol}(U) := \sum_{u \in U} m(\{u\}, V).
\]
The following theorem is our main result.

**Theorem 1.1.** Suppose that \( G = (V, E) \) is a graph with a designated set of boundary vertices \( \partial V \subseteq V \), and define the Cheeger constant \( h_N \) by
\[
h_N := \min_{S \subseteq V} \frac{m(S, V \setminus S)}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}.
\]
Then,
\[
\sqrt{2\lambda_N} \geq h_N \geq \frac{\lambda_N}{2},
\]
where \( \lambda_N \) is the first nontrivial eigenvalue of \( \mathcal{L}_N \).

Recall that the standard Cheeger inequality is constructive in the sense that a cut that achieves the upper bound on the Cheeger constant can be determined from the eigenfunction corresponding to the first eigenvalue of the Laplacian \([1, 2]\). Specifically, the partition that achieves the upper bound can be determined by dividing vertices into two groups based on if the value of the first eigenvector is more or less than some threshold; for a detailed exposition see for example \([3, 4]\). Similarly, the result of Theorem 1.1 is constructive in the sense that a cut which achieves the upper bound on \( h_N \) can be determined from the eigenvector \( \psi_N \) of \( \mathcal{L}_N \) that corresponds to \( \lambda_N \). In the following remark, we present an example where the cut arising from \( \psi_N \) actually minimizes \( h_N \). Moreover, in this remark we demonstrate that cuts arising from the normalized Neumann graph Laplacian \( \mathcal{L}_N \) can differ significantly from cuts arising from the standard normalized graph Laplacian \( \mathcal{L} \).
Figure 3. A graph with 16 vertices, where the black and white dots denote interior and boundary vertices, respectively.

Remark 1.2. Let $G = (V, E)$ be the graph illustrated in Figure 3. Suppose that $S^* \cup (V \setminus S^*)$ is the partition of the vertices of $G$ that minimizes

$$S^* = \arg\min_{S \subseteq V} \frac{m(S, V \setminus S)}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}.$$

By inspection, this minimizing partition $S^* \cup (V \setminus S^*)$ is given by cutting the graph into two equal pieces with a horizontal line; this can be verified by exhaustive calculation since the graph is small. Let $\psi_N$ denote the first eigenvector of the normalized Neumann Laplacian $L_N$ for this graph, and let $\psi$ denote the first eigenvector of the standard normalized Laplacian $L$, which results from treating all vertices of $G$ as interior vertices. We illustrate $\psi_N$ and $\psi$ in Figure 4.

Figure 4. The graph $G$ with vertices colored by grayscale values proportional to $\psi_N$ and $\psi$, respectively.

We claim that the vector $\psi_N$ pictured in Figure 4 illustrates the result of Theorem 1.1. Indeed, observe that there exists a threshold such that partitioning with $\psi_N$ actually yields the optimal partition $S^* \cup (V \setminus S^*)$ resulting from a horizontal cut. In contrast, the first eigenvector $\psi$ of the standard normalized Laplacian suggests a vertical cut. In fact, if all vertices are treated as interior vertices it is straightforward to verify that the optimal partition of $G$ is given by dividing the vertices into two equal groups by a vertical line. Thus, this example demonstrates that the eigenvectors of $L_N$ and $L$ can differ in a meaningful way.
2. Proof of Main Result

2.1. Summary. The proof of Theorem 1.1 is divided into two lemmas: first, in Lemma 2.1 we show that $\lambda_N \leq 2h_N$, and second, in Lemma 2.2 we show that $h_N^2/2 \leq \lambda_N$. The structure of our argument is similar to classical Cheeger inequality proofs, see [3, 5].

2.2. Proof of Theorem 1.1

Lemma 2.1 (Trivial direction). We have

$$\lambda_N \leq 2h_N.$$  

Proof of Lemma 2.1. Recall that

$$L_N := D^{-1/2}Q^{1/2}L_NQ^{-1/2}D^{-1/2}.$$  

First, we observe that $QL_N$ can be written as

$$QL_N = L - \frac{1}{2}L_\partial,$$

where

$$L = \begin{pmatrix} \text{diag}(A_{11}\vec{1} + A_{12}\vec{1}) - A_{11} & -A_{12} \\ -A_{12}^\top & \text{diag}(A_{12}\vec{1} + A_{22}\vec{1}) - A_{22} \end{pmatrix},$$

and

$$L_\partial := \begin{pmatrix} 0 & \text{diag}(A_{22}\vec{1}) - A_{22} \\ 0 & \text{diag}(A_{22}\vec{1}) - A_{22} \end{pmatrix}.$$  

Observe that $L$ is the standard graph Laplacian of $G$, while $L_\partial$ is the graph Laplacian of the vertex induced subgraph $G[\partial V]$. Fix a subset $S \subseteq V$, and let $\chi_S$ be the indicator function for $S$. Define

$$x := Q^{1/2}D^{1/2}\chi_S - \frac{\chi_S^\top DQ\chi_S}{\chi_S^\top DQ\chi_S}D^{1/2}Q^{1/2}\vec{1}.$$  

By construction, we have $x^\top D^{1/2}Q^{1/2}\vec{1} = 0$, and it follows that

$$\lambda_N \leq \frac{x^\top D^{-1/2}Q^{1/2}L_NQ^{-1/2}D^{-1/2}x}{x^\top x}.$$  

$$= \frac{\chi_S^\top QL_N\chi_S}{\chi_S^\top DQ\chi_S} \left( \frac{\chi_S^\top DQ\chi_S}{\chi_S^\top DQ\chi_S} \right)$$  

$$= \chi_S^\top L - \frac{1}{2}L_\partial \chi_S \left( \frac{\chi_S^\top DQ\chi_S}{\chi_S^\top DQ\chi_S} \right)$$  

$$= \chi_S^\top \left( \frac{2}{\chi_S^\top DQ\chi_S} \right) \begin{pmatrix} \min\{\chi_S^\top D\chi_S, \chi_S^\top D\chi_{V \setminus S}\} \\ \min\{\text{vol}(S), \text{vol}(V \setminus S)\} \end{pmatrix}.$$  

Since this inequality holds for all subsets $S \subseteq V$, we conclude that $\lambda_N \leq 2h_N$, as was to be shown. $\square$
Lemma 2.2 (Nontrivial direction). We have
\[ \lambda_N \geq \frac{h_N^2}{2} \]

Proof of Lemma 2.2. Recall that
\[ \lambda_N = \inf_{x^\top D^{1/2}Q^{1/2}x = 1} \frac{x^\top L_N x}{x^\top x} = \inf_{y^\top Q\mathbf{1} = 0} \frac{y^\top QL_N y}{y^\top QD\mathbf{1}}. \]

Let \( g \) be a vector satisfying
\[ \lambda_N = \frac{g^\top QL_N g}{g^\top QDg}, \quad \text{and} \quad g^\top QD\mathbf{1} = 0. \]
Let \( \{v_1, \ldots, v_n\} \) be an enumeration of the vertices \( V \) so that \( g_{v_1} \leq \ldots \leq g_{v_n} \), and set \( S_j := \{v_1, \ldots, v_j\} \), for \( j = 1, \ldots, n \). Let \( p \) be the largest integer such that \( \text{vol}(S_p) \leq \text{vol}(V)/2 \), that is,
\[ p := \max \{ j \in \{1, \ldots, n\} : \text{vol}(S_j) \leq \text{vol}(V)/2 \}. \]
Let \( g^+ \) and \( g^- \) denote the positive and negative parts of \( g - g_{v_p} \), respectively. That is, \( g^+_v := \max\{g_v - g_{v_p}, 0\} \) and \( g^-_v := \max\{g_{v_p} - g_v, 0\} \). Let \( u \sim v \) denote \( \{u, v\} \in E \) and \( q = \text{diag}(Q) \). Then
\[ \lambda_N = \frac{g^\top (L - \frac{1}{2}L_0)g}{g^\top QDg} \]
\[ = \frac{\sum_{u \sim v} (g_u - g_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (g_u - g_v)^2}{\sum_v g_v^2 d_v q_v} \]
\[ \geq \frac{\sum_{u \sim v} (g_u - g_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (g_u - g_v)^2}{\sum_v q(v) d_v q_v} \]
\[ \geq \frac{\sum_{u \sim v} ((g^+_u - g^+_v)^2 + (g^-_u - g^-_v)^2) - \frac{1}{2} \sum_{u,v \in \partial V} ((g^+_u - g^+_v)^2 + (g^-_u - g^-_v)^2)}{\sum_v ((g^+_v)^2 + (g^-_v)^2) d_v q_v}, \]
where the second to last inequality holds because we have increased the denominator, and where the last inequality holds for each fixed \( u, v \) in the numerator. Recall that
\[ \frac{a+b}{c+d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\}, \]
for any \( a, b, c, d > 0 \). Without loss of generality, we will assume that
\[ \lambda_N \geq \frac{\sum_{u \sim v} (g^+_u - g^-_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (g^+_u - g^+_v)^2}{\sum_v (g^+_v)^2 d_v q_v}. \]
To simplify notation in the following, let \( f = g^+ \). We begin by setting
\[ \lambda := \frac{\sum_{u \sim v} (f_u - f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u - f_v)^2}{\sum_v f^2_v d_v q_v}. \]
Multiplying the numerator and denominator by the same term gives
\[ \lambda = \frac{\left( \sum_{u \sim v} (f_u - f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u - f_v)^2 \right) \left( \sum_{u \sim v} (f_u + f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u + f_v)^2 \right)}{\left( \sum_v f^2_v d_v q_v \right) \left( \sum_{u \sim v} (f_u + f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u + f_v)^2 \right)}. \]
Applying the Cauchy-Schwarz inequality in the numerator gives

\[ \lambda \geq \frac{\left( \sum_{u \sim v} |f_u^2 - f_v^2| - \frac{1}{2} \sum_{u,v \in \partial V} |f_u^2 - f_v^2| \right)^2}{\left( \sum_v f_v^2 d_v q_v \right) \left( \sum_{u \sim v} (f_u + f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u + f_v)^2 \right)}. \]

Next, we observe that

\[ \sum_{u \sim v} (f_u + f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u + f_v)^2 = \sum_v f_v^2 d_v q_v - \left( \sum_{u \sim v} (f_u - f_v)^2 - \frac{1}{2} \sum_{u,v \in \partial V} (f_u - f_v)^2 \right), \]

and thus it follows that

\[ \lambda \geq \frac{\left( \sum_{u \sim v} |f_u^2 - f_v^2| - \frac{1}{2} \sum_{u,v \in \partial V} |f_u^2 - f_v^2| \right)^2}{\left( \sum_v f_v^2 d_v q_v \right)^2 (2 - \lambda)}. \]

We want to show that

\[ \sum_{u \sim v} |f_u^2 - f_v^2| - \frac{1}{2} \sum_{u,v \in \partial V} |f_u^2 - f_v^2| \geq \sum_{i=1}^n \left| f_{v_i}^2 - f_{v_{i+1}}^2 \right| \cdot m(S_i, V \setminus S_i). \]

We can write

\[ \sum_{u \sim v} |f_u^2 - f_v^2| - \frac{1}{2} \sum_{u,v \in \partial V} |f_u^2 - f_v^2| = \sum_{i=2}^n \sum_{j=1}^{i-1} \left( \chi_{E_{i,j}} - \frac{1}{2} \chi_{\partial_i} \chi_{\partial_j} \right) (f_{v_i}^2 - f_{v_j}^2), \]

where

\[ \chi_{E_{i,j}} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases} \]

is the indicator function for \( \{v_i, v_j\} \in E \), and

\[ \chi_{\partial_i} = \begin{cases} 1 & \text{if } i \in \partial V \\ 0 & \text{otherwise} \end{cases} \]

is the indicator function for \( v_i \in \partial V \). Note that we are justified in dropping the absolute value signs because \( f_{v_i}^2 \) is an increasing function of \( i \). Next we write \( f_{v_i}^2 - f_{v_j}^2 \) as a telescoping series

\[ f_{v_i}^2 - f_{v_j}^2 = (f_{v_i}^2 - f_{v_{i-1}}^2) + (f_{v_{i-1}}^2 - f_{v_{i-2}}^2) + \ldots + (f_{v_{j+1}}^2 - f_{v_j}^2), \]

and rearrange terms in the summation to conclude that

\[ \sum_{i=2}^n \sum_{j=1}^{i-1} \left( \chi_{E_{i,j}} - \frac{1}{2} \chi_{\partial_i} \chi_{\partial_j} \right) (f_{v_i}^2 - f_{v_j}^2) = \sum_{i=1}^n \sum_{k=1}^{n} \sum_{j=1}^{n} \left( \chi_{E_{j,k+1}} - \frac{1}{2} \chi_{\partial_j} \chi_{\partial_{k+1}} \right) \chi_{j \leq l} (f_{v_{k+1}}^2 - f_{v_j}^2), \]

where

\[ \chi_{j \leq l} = \begin{cases} 1 & \text{if } j \leq l \\ 0 & \text{otherwise.} \end{cases} \]
Then, to complete this step, we note that

\[
\sum_{k=1}^{n} \sum_{j=1}^{n} \left( \chi_{E_{j,k+1}} - \frac{1}{2} \chi_{\partial E_{k+1}} \right) \chi_{j \leq t} = m(S_t, V \setminus S_t).
\]

Returning to our main sequence of inequalities for \( \lambda \), we have

\[
\lambda \geq \frac{\left( \sum_{i=1}^{n} |f_{v_i}^2 - f_{v_{i+1}}^2| \cdot m(S_i, V \setminus S_i) \right)^2}{2(\sum_v f_v^2 d_v q_v)^2} \geq \left( \sum_{i=1}^{n} |f_{v_i}^2 - f_{v_{i+1}}^2| \cdot \min \{ \text{vol}(S_i), \text{vol}(V \setminus S_i) \} \right)^2 \frac{m(S_i, V \setminus S_i)}{2(\sum_v f_v^2 d_v q_v)^2},
\]

where

\[
\alpha := \min_{1 \leq i \leq n} \frac{m(S_i, V \setminus S_i)}{\min \{ \text{vol}(S_i), \text{vol}(V \setminus S_i) \}}.
\]

Since \( f_{v_i}^2 \) is nondecreasing, a rearrangement of the numerator of the previous expression gives

\[
\lambda \geq \frac{\alpha^2 \left( \sum_v (f_v^2 \cdot \min \{ \text{vol}(S_i), \text{vol}(V \setminus S_i) \} - \min \{ \text{vol}(S_{i+1}), \text{vol}(V \setminus S_{i+1}) \}) \right)^2}{2(\sum_u f_u^2 d_u q_u)^2}.
\]

It follows that

\[
\lambda_N \geq \lambda \geq \frac{\alpha^2 \left( \sum_v (f_v^2 \cdot \text{vol}(S_i, V \setminus S_i) - \min \{ \text{vol}(S_{i+1}), \text{vol}(V \setminus S_{i+1}) \}) \right)^2}{2(\sum_u f_u^2 d_u q_u)^2} = \frac{\alpha^2}{2} \geq \frac{h_N^2}{2},
\]

which completes the proof. \( \square \)

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