A two-qubit algorithm involving quantum entanglement

Arvind\(^1\)∗ and N. Mukunda\(^2\)†

\(^1\)Department of Physics, Guru Nanak Dev University, Amritsar 143 005, India
\(^2\)Center for Theoretical Studies and Department of Physics, Indian Institute of Science, Bangalore 560012, India
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The phenomenon of quantum entanglement is fundamental to the implementation of quantum computation, and requires at least two qubits for its demonstration. However, both Deutsch algorithm and Grover’s search algorithm for two bits do not use entanglement. We develop a Deutsch-like problem, where we consider all possible binary functions for two bit inputs and distinguish their even or odd nature. The quantum algorithm to solve this problem requires entanglement at the level of two qubits. The final solution suggests that an NMR implementation of the problem would lead to interesting results.

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The fusion of ideas from classical information theory and the foundations of quantum mechanics has led to a quiet revolution in computer science today - namely, quantum computation [1]. One can now envisage building computing devices to physically implement fast quantum algorithms. Quantum algorithms rely on truly quantum phenomena like entanglement, interference and quantum superposition to achieve a significant computational advantage over classical algorithms [2,3]. The algorithms designed thus far to demonstrate the power of quantum computing range from the Deutsch-Jozsa algorithm to evaluate the constant or balanced nature of a function \(^2\) and Grover’s quantum search algorithm \(^3\) which achieve a polynomial speed-up, to Shor’s factorization algorithm which gains an exponential advantage over its best known classical counterpart \(^4\).

Quantum entanglement shows up qualitatively at the level of two qubits. It can be visualised as the non-separability of the state of a composite quantum system into the states of its parts \(^{10}\). For example, if two qubits are in a state such as \(\frac{1}{\sqrt{2}}(|01⟩ − |10⟩)\), which is not resolvable into the tensor product of the states of the individual qubits, they are entangled. Neither qubit by itself has a definite pure state, in contrast to a classical system which can be completely resolved into the states of each part of the system. The problem of quantum entanglement for general (pure or mixed) states has attracted a lot of attention \(^{11,12}\).

It is quantum entanglement which prevents the mapping or realisation of a quantum computation using just classical waves. Consider the polarisation states of a classical light beam. These states are in one-to-one correspondence with the states of a qubit. All possible states can be realised by using one half-wave and two quarter-wave plates. One can pass from any one chosen polarisation state to all others in this way, i.e. all \(U(2)\) transformations can be implemented using these gadgets \(^{13}\). Therefore a single qubit has a classical analogue. On the other hand, it is not possible to map the states of a two-qubit system onto the polarisation states of two light beams. The entangled states of the two qubits have no classical counterpart. Therefore at the level of two qubits itself the possibility of mapping a quantum computer onto classical optical fields breaks down \(^{14}\).

The Deutsch-Jozsa(DJ) algorithm was the first and the simplest example of an algorithm that demonstrated the appreciable advantage of quantum computing \(^{15}\). However, it was realised recently that this algorithm requires entangling transformations only for three or more input bits. For the case of two input qubits, it can be mapped onto an essentially classical optical problem \(^{16,17,18}\). Generalisations of the DJ problem are also being considered \(^{19}\). Our aim in this paper is to discuss a simple two qubit problem which requires manipulation of entangled states for its solution in an essential way. We thus evaluate the global property of a function (its even or odd nature) using fewer function calls than a classical algorithm. The algorithm is designed to exploit the entangled states of the two qubits. In the process of the formulation and solution of this problem we arrive at some interesting consequences for distinguishing non-orthogonal states through measurement. It turns out that this is an issue relevant to quantum computation using NMR which has been the most successful technique to implement quantum computation schemes, including the DJ problem \(^{19,24}\).

Consider a Boolean function defined from a two-bit domain space to a one-bit range space: \(f(x) : \{0,1\}^2 \rightarrow \{0,1\}\). There are four possible input values \((00), (01), (10)\) and \((11)\) and the output for each of these could be either 0 or 1. There are thus 16 functions in all. For a given function, the output can have either: all ones, three ones and a zero, two ones and two zeros, three zeros and one one or all zeros. We can divide the function into classes \([0, 4]\), \([1, 3]\), \([2, 2]\), \([3, 1]\), and \([4, 0]\), the first...
entry indicating the number of ones and the second indicating the number of zeros in the output. The functions with an even number \((0, 2, 4)\) of ones (i.e. the functions \([0, 4], [2, 2]\) and \([4, 0]\)) are defined as “Even” functions while the functions with an odd number (1, 3) of ones in the output (i.e. the \([1, 3]\) and \([3, 1]\) functions) are defined to be “Odd” functions. Using this evaluation criterion, of the 16 possible functions for the two-bit case, eight are even and eight are odd.

The appropriate question to be asked is: given a function, how to decide whether it is even or odd. Classically, the classification of a given function would require computing the function at all input points, since even the last output can change the class of a function. We give here an algorithm which classifies a given function using fewer function calls. What we require in our analysis is a gate to call the function, and a judicious use of Hadamard transformations. A Hadamard transformation mixes the two eigenstates of a qubit maximally.

\[
|0\rangle \rightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle ) \\
|1\rangle \rightarrow \frac{H}{\sqrt{2}} (|0\rangle - |1\rangle )
\]

\(H = H^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

There are three different types of Hadamard transformations possible for the two qubit system. One can either apply the Hadamard transformation selectively on the first or the second qubit, or non-selectively on both the qubits. Explicitly,

\[
H^1 = H \otimes I \\
H^2 = I \otimes H \\
H^{1-2} = H \otimes H
\]

where 1 and 2 label the qubit involved.

The function call mechanism is similar to the one used by the Deutsch problem \([10, 11]\). Each function \(f\) can be encoded by a unitary transformation \(U_f\), with its action on the eigenstates of the two qubits being defined as

\[
|\text{x}\rangle_{2\text{-bit}} \xrightarrow{U_f} (-1)^{f(x)} |\text{x}\rangle_{2\text{-bit}}
\]

there are sixteen \(U_f\) matrices in all, with half of them being not separable. For example, consider the matrix with diagonal entries \([1, 1, -1, -1]\); it cannot be written as a tensor product of two matrices, one belonging to each qubit. Therefore the transformations \(U_f\) are in general entangling in character. The sub-class of functions that are either constant or balanced in the sense of Deutsch problem i.e. the functions \([4, 0], (0, 4)\) and \([2, 2]\) are all separable in character. For example consider the matrix for a balanced function with diagonal entries \([1, 1, -1, -1]\); is actually a tensor product of two \(2 \times 2\) matrices with diagonal entries \([1, -1]\) and \([1, 1]\). Therefore, the two-bit Deutsch problem, namely distinguishing between constant and balanced functions, can be implemented using non-entangling transformations alone. One can even conceive of using classical waves for its implementation as only the concepts of superposition and interference are required for its solution. The present problem of distinguishing between even and odd functions on the other hand requires entangling transformations for its implementation. These entangling transformations can produce entangled states which do not have any analogue in the classical world even given superposition and interference.

The computation proceeds with both qubits initially in the state \([00]\). The sequence of steps followed then is: Apply the transformation \(H^{-1} H_f\), call the function by applying \(U_f\), apply the selective Hadamard transformation \(H^2\) on second qubit alone, call the function a second time through \(U_f\), and finally again apply the two-bit Hadamard transformation \(H^{1-2}\). This leads to the result

\[
\begin{align*}
H^{1-2} U_f H^2 U_f H^{1-2} |00\rangle \\
&= \frac{1}{2 \sqrt{2}} \left[ \left((-1)^{f(00) \oplus f(01)} + (-1)^{f(10) \oplus f(11)} \right) |00\rangle \\
&+ 2 |01\rangle \\
&+ \left((-1)^{f(00) \oplus f(01)} - (-1)^{f(10) \oplus f(11)} \right) |10\rangle \right]
\end{align*}
\]

For an “even” function the final state becomes

\[
\frac{1}{\sqrt{2}} (|00\rangle + |01\rangle)
\]

and for an “odd” function it becomes

\[
\frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)
\]

These final states are clearly different and hence can be used to classify the function as “even” or “odd”. However, unlike the case of the DJ algorithm, these states

| Class   | Number | Nature | \(U_f\)          | DJ Class |
|---------|--------|--------|------------------|----------|
| [0,4]   | 1      | Even   | Separable       | Constant |
| [1,3]   | 4      | Odd    | Entangling      | ———     |
| [2,2]   | 6      | Even   | Separable       | Balanced |
| [3,1]   | 4      | Odd    | Entangling      | ———     |
| [4,0]   | 1      | Even   | Separable       | Constant |

**TABLE I.** Characteristics of different classes of functions. In each class we give number of functions, their even or odd nature, the entangling or separable nature of \(U_f\) and their status in DJ problem.
are not orthogonal. Can one hence unambiguously conclude from a single measurement the character of the function? In conventional quantum measurement theory one expects that, to distinguish between such states the experiment has to be repeated at least a few times. Recently there have been refinements where a single measurement can unambiguously distinguish between such states, though the experiment may not work all the time \cite{23}.

The most successful method of implementing quantum algorithms to date has been NMR. As far as NMR is concerned the measurements obtained from the above two states leads to very different spectra \cite{26}. To see it more clearly we compute the density matrices corresponding to the two non-orthogonal final states which are

\[
\rho_{\text{even}} = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 & 0 & 0 \\
\pm 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \rho_{\text{odd}} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & \pm 1 & 0 \\
0 & \pm 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(5)

The density matrix \(\rho_{\text{even}}\) has off-diagonal terms corresponding to single quantum coherences and therefore will give rise to a line in the NMR spectrum. On the other hand \(\rho_{\text{odd}}\) has off-diagonal terms corresponding to zero quantum coherence which does not give rise to any observable NMR signal \cite{24}.

To further elaborate this point we calculate the reduced density matrix corresponding to the second spin for both the cases.

\[
\rho^{(2)}_{\text{even}} = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 \\
\pm 1 & 1
\end{pmatrix} = \frac{1}{2} I + \frac{1}{2} \begin{pmatrix}
0 & \pm 1 \\
\pm 1 & 0
\end{pmatrix}
\]

\[
\rho^{(2)}_{\text{odd}} = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \frac{1}{2} I + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

(6)

Multiples of identity do not give rise to any NMR signal at all. After a multiple of identity is taken out, the two reduced density matrices are clearly very different. The first one is zero and has single quantum coherences, therefore it will give rise to a line in the NMR spectrum while the second one is zero and hence will lead to no measurable NMR signal. Therefore a clear demarcation of even and odd functions is possible using the NMR quantum computer. It might be interesting to do experiments in this direction! We can argue similarly for the first spin by calculating reduced density matrices corresponding to it. It turns out that measurements on the first spin alone do not distinguish between the even or odd nature of the function.

The reduced density matrices (6) also show that for odd functions we have quantum entanglement. It follows from the fact that \(\rho^{(2)}_{\text{odd}} \not\equiv \rho^{(2)}_{\text{even}}\). Therefore, the second qubit alone, after partial trace is in a mixed state (in fact the reduced density matrix for the second qubit for odd functions is a multiple of identity and is therefore maximally mixed) indicating clearly, that the original two-qubit pure density matrix has entanglement for odd functions.

We have described an algorithm for two qubits which requires the implementation of entangling transformations for its execution. As explained by Schr"{o}dinger, entanglement is not just one way but the way in which quantum mechanics differs from classical physics \cite{17}. The problem we have addressed is a natural generalisation of the Deutsch problem. In our solution we require two function calls as opposed to four for the classical solution. Since in this problem the manipulation of entanglement is essential it will be very interesting to implement this simple algorithm experimentally and track down the amount of entanglement. This is being currently pursued and will be reported elsewhere.

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