Bargmann Representation of Quantum Absorption Refrigerators

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Abstract

In this work, we solve the quantum absorption refrigerator analytically in the space of holomorphic functions with Gaussian measure. Our approach simplifies the calculations since for a given quantum system the coordinate representation of any quantum state is always more complicated than its corresponding expression written with respect to the phase-space coordinate $z_i$. We finally discuss the computational complexity of the holomorphic representation and compare it with the computational complexity of the standard operator method and prove the efficiency of the holomorphic representation in computing some tasks. Our treatment is applicable to all quantum heat engines and refrigerators.

Keywords: Bargmann representation, Quantum Absorption Refrigerators, Computational Complexity, Open Quantum Systems

1 Introduction

Thermodynamics is one of the oldest and well-established branches of physics that sets boundaries to what can or cannot be achieved at macroscopic level\textsuperscript{1}. It also has deep connection with other branches of science such as philosophy and computing theory\textsuperscript{2,3}. It was realized that large quantum devices, such as masers and lasers, could be treated under the scope of thermodynamics. The pioneering work of Scovil et al. has shown the equivalence of the Carnot engine with three-level Maser\textsuperscript{4-6}. This opened the road for flourishing progress in the study of the connection between the laws of thermodynamics and quantum devices\textsuperscript{7-13,14}.

The interaction between any quantum mechanical system with environment may lead to dissipation or loss of information contained in the system to its
environment. To obtain a complete description of any quantum mechanical system, one needs to incorporate the effect of environment or baths to the original system Hamiltonian. This was the philosophy behind investigating open quantum systems [15, 16, 17]. One of the main tasks in this theory is to solve the so-called Master equation of open quantum systems. Generally solving such equation can be very complicated and requires many tedious calculations especially in the non-Markovian case [18, 19].

In [20], Fock noticed the fact that one in principle express the raising and lowering operators analytically in the complex $z$-plane as $z$ and $\frac{\partial}{\partial z}$ respectively. In this case the wavefunction should be written in term of the complex variable $z$ in a closed analytical form not as a vector in the Hilbert space. For example, the wavefunctions of the quantum harmonic oscillator in coordinate representation are given by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!} (m \omega \pi \hbar)^{1/4}} e^{-m\omega x^2/2\hbar} H_n \left( \sqrt{m\omega/\hbar} x \right)$$

where $H_n(y) = (-1)^n e^{y^2/2} \frac{d^n}{dy^n} \left( e^{-y^2} \right)$ is the Hermite polynomials while in the Bargmann representation it is simply $\psi_n(z) = \frac{z^n}{\sqrt{n!}}$ written with respect to the phase space coordinate $z$. The rigorous treatment of the holomorphic function spaces described here was done by Segal and Bargmann in 60s. They also provided a transform formula where one could obtain the coordinate representation of wavefunctions from the monomials $\psi_n(z) = \frac{z^n}{\sqrt{n!}}$ [21, 22]. The Bargmann representation of some quantum mechanical system has been studied before such as for Jaynes-Cummings model, thermal coherent states and WKB approximation [23, 24, 25]. Very interestingly it was shown that Bargmann representation gives better results in the case of WKB approximation comparing with other known methods [25]. In [26], a generalization of the Segal-Bargmann transform has been used in the context of canonical gravity and loop quantum gravity.

For sake of simplicity we systematically discuss the theory of open quantum systems in term of holomorphic functions. Moreover we solve the quantum absorption refrigerators driven by a Gaussian noise as a model example in term of analytical functions in the complex $z$-plane [27, 28]. Recently the quantum absorption refrigerator was realized experimentally in the case of trapped ions [29].

2 Bargmann representation of open quantum systems

Definition 1: The Bargmann space also known as the Fock-Bargmann or Segal-Bargmann space, denoted by $\mathcal{H} L^2(\mathbb{C}^n, \mu)$, is the space of holomorphic functions with $\mu(z) = (\pi)^{-n} e^{-|z|^2}$ and $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. Any entire analytic function $f(z)$ in this space obeys the following square-integrability condition
\( \|f\|^2 := (f|f)_\mu = (\pi)^{-n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} \, dz < \infty, \)  

(1)

where \( dz \) is the \( 2n \)-dimensional Lebesgue measure on \( \mathbb{C}^n \).

**Remark 1:** The inner-product of any two analytic functions \( f(z) \) and \( g(z) \) satisfying the condition (1) is

\[
(f|g)_\mu = (\pi)^{-n} \int_{\mathbb{C}^n} \overline{f}(z) g(z) e^{-|z|^2} \, dz.
\]

(2)

Using the inner product defined in (2) we can prove that both \( \overline{z} \) and \( \frac{\partial}{\partial z} \) have the same effect i.e. \( (\frac{\partial}{\partial z}, g)_\mu = (f, z g)_\mu \).

**Lemma 1:** The Bargmann space \( \mathcal{H}L^2(\mathbb{C}^n, \mu) \) is a Hilbert space.

This observation was due to Bargmann in [21]. More generally, \( L^2 \) is the only Hilbert space among the function spaces \( L^p \) [35].

According to Riesz representation theorem between any given Hilbert space and its dual space there must exist a unique \( f_\alpha \) such that

\[
f(\alpha) = (f_\alpha(z)|f(z))_\mu, \]

(3)

where the function \( f_\alpha \) is usually called a coherent state with parameter \( \alpha \) whereas the quantity \( f_\alpha \) is known as the reproducing kernel [37, 38].

**Lemma 2:** The reproducing kernel in the Bargmann space is

\[
K(z, w) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \frac{w^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{1}{n!} (zw)^n = e^{zw}.
\]

(4)

**Lemma 3:** The monomials \( z^n/\sqrt{n!} \) or its alternate expression \( z^n/\sqrt{\Gamma(n+1)} \) form an orthonormal basis

\[
\int \frac{dz \, d\overline{z}}{\pi} \exp[-z\overline{z}] z^n z^m = n! \delta_{mn}
\]

(5)

Thus the bosonic wave functions can be written as a uniformly convergent series \( f(z) = \sum_n c_n \frac{z^n}{\sqrt{n!}} \). The convergence of this series in any compact domain of the complex \( z \)-plane is fixed by the condition \( \sum_n |c_n|^2 = 1 \) [30, 36].

**Lemma 4:** In Bargmann representation, the raising and lowering operators can be defined as \( a_i^\dagger = z_i \) and \( a_i = \frac{\partial}{\partial z_i} \) or \( \overline{\partial}_i \).
This can be proven by straightforward calculations of the commutators,

\[
\left[ \frac{\partial}{\partial z_i}, z_j \right] = \delta_{ij}, \quad (6)
\]

\[
\left[ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right] = [z_i, z_j] = [z_i, z_j] = 0. \quad (7)
\]

With this construction we define the position and momentum operators in natural units i.e. \( \hbar = 1 \) as

\[
x_j = \frac{1}{2} \left( \frac{\partial}{\partial z_j} + z_j \right), \quad (8)
\]

\[
p_j = \frac{1}{2i} \left( \frac{\partial}{\partial z_j} - z_j \right). \quad (9)
\]

Since the operators \( x_j \) and \( p_j \) satisfy the Weyl relations and act irreducibly in the Bargmann space, one could in principle as consequence of the Stone-von-Neumann theorem maps each position dependent quantity in the Hilbert space \( L^2(\mathbb{R}^{2n}) \) to its corresponding holomorphic expression in the Bargmann space \( \mathcal{H}L^2(\mathbb{C}^n, \mu) \). This can be done in virtue of the Segal-Bargmann transform given by

\[
(Af)(z) = \int_{\mathbb{R}^n} \exp \left[ -\left( z \cdot z - 2\sqrt{2}z \cdot x + x \cdot x \right) \right] f(x) \, dx, \quad (10)
\]

As an example the harmonic oscillator Hamiltonian (up to a constant term) \( H_0 = \hbar \omega a^\dagger a \) assumes the following form in the Bargmann representation \( H_0 = \hbar \omega z \frac{d}{dz} \) \cite{39}. Acting by \( H_0 \) to the energy eigenstates we get

\[
H_0|n\rangle = \hbar \omega \frac{d}{dz} \frac{z^n}{\sqrt{n!}} = n\hbar \omega \frac{z^n}{\sqrt{n!}} = n\hbar \omega |n\rangle. \quad (11)
\]

Another interesting example from quantum optics is the cat-states which are defined as the quantum superposition of two opposite-phase coherent states of a single mode \cite{4, 40, 41}. The even and odd cat states can be defined respectively as

\[
|\alpha_+\rangle = |\alpha\rangle + | - \alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{2n!}} |2n\rangle, \quad (12)
\]

\[
|\alpha_-\rangle = |\alpha\rangle - | - \alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle, \quad (13)
\]

where \( |\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle \) in the Bosonic Fock number basis \cite{41}.  

4
The even and odd cat states defined in 12 and 13 can be rewritten in a closed analytic form using the Bargmann representation as

\[
|\alpha_+\rangle = |\alpha\rangle + |-\alpha\rangle = 2e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(az)^{2n}}{2n!} \sinh (az),
\]

\[
|\alpha_-\rangle = |\alpha\rangle - |-\alpha\rangle = 2e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(az)^{2n+1}}{(2n+1)!} \cosh (az),
\]

since

\[
|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(az)^n}{n!} = e^{-\frac{1}{2}|\alpha|^2} e^{az},
\]

and \(I_{1/2,-1/2}\) are the modified Bessel functions. Interestingly the equations 14 and 15 are written with respect to complex variables \(\alpha\) and \(z\). As far as we are aware, these equations has not been given explicitly before in the literature. It would be interesting to see if they will reproduce all physical results found previously for the even and odd cat states analytically.

In case of fermions, we can construct analytical representation on the fermionic phase space \(\mathbb{R}^{2d}\) by realizing the raising and lowering operators in term of the Grassmann variable \(\theta\) as

\[
b_j = \frac{\partial}{\partial \theta_j},
\]

\[
b_j^\dagger = \theta_j,
\]

where \(\theta_j = \frac{1}{2} (\xi_j - i \xi_j)\), \(\overline{\theta}_j = \frac{1}{2} (\xi_j - i \xi_j)\) with \(j = 1, \ldots, d\), \(\{\xi_i, \xi_k\} = 2i\epsilon_{ijk}\xi_k\) and \(\{\xi_i, \xi_k\} = 0\) for \(i \neq k\) and 2 for \(i = k\). It is important to note that this specific construction is when the fermionic phase space is for spin-1/2 particles or in general any two-level systems and this explains the 2 in front of the commutation relation between any two different components of the vector \(\xi\). For higher spins the generalization is not difficult and one only need to generalize the Grassmann algebra for higher spins and write the exact commutation and anti-commutation relations for each case. The fermionic phase space is even since the complex structure should be preserved. With this construction, the anticommutation relations are

\[
\{\theta_i, \theta_j\} = \{\overline{\theta}_i, \overline{\theta}_j\} = 0, \quad \{\theta_i, \overline{\theta}_j\} = \delta_{ij}.
\]

The Gaussian integral over complex Grassmann (anti-commuting) variables is

\[
\int d\overline{\theta}d\theta \ e^{-\overline{\theta}\theta} = b,
\]
From this result we find
\[ \int d\theta d\theta e^{-\theta^2} = 1. \]  
(21)
The basic integral rules for a single Grassmann variable \( \theta \) are given by the Berezin integrals defined as \[ 43\, 44\]
\[ \int \theta d\theta = 1, \quad \int d\theta = 1. \]  
(22)
Applying Berezin integrals over the most general one-variable Grassmann polynomial \( f(\theta) = a\theta + b \) gives
\[ \int f(\theta) d\theta = \int (a\theta + b) d\theta = a, \]  
(23)
where \( a \) and \( b \) are elements from \( C \). The orthogonal basis for a single fermion are simply \( 1 = |0\rangle \) and \( \theta = |1\rangle \) since \( \theta^2 = 0 \).

The total fermionic number operator for many-body fermionic system is
\[ N_F = \sum_i \bar{\theta}_i \theta_i, \]  
(24)
and satisfies \( N^2_F = N_F \).

By direct calculation we find for each \( i \)
\[ N^2_{F_i} = \theta_i \bar{\theta}_i \theta_i \bar{\theta}_i = \theta_i (1 - \theta_i \bar{\theta}_i) \bar{\theta}_i = \theta_i \bar{\theta}_i = N_{F_i}, \]  
(25)
since according to \[ 19\] we have \( \theta^2_i = \bar{\theta}^2_i = 0 \).

The total Hamiltonian should be written as a function of \( z_i, \theta_i \) and their complex conjugates i.e. \( H(z_i, \bar{z}_i, \theta_i, \bar{\theta}_i) \) in general. Moreover the corresponding bosonic eigenstates are written as a polynomial of complex variable \( z \)
\[ |n_1 \ldots, n_k\rangle = \prod_{i=1}^{i=k} \frac{z^n_k}{\sqrt{n_k!}}, \]  
(26)
and the fermionic eigenstates for each single fermion are 1 for the ground state and \( \theta \) for the excited state.

As a model example we consider the Jaynes-Cummings model that describes the interaction between two-level atom and quantized field in an optical cavity. The Jaynes-Cummings Hamiltonian up to a constant term in operator language reads \[ 4, 45\]
\[ \hat{H}_{JC} = \hbar \omega_c \hat{a}^\dagger \hat{a} + \hbar \omega_{eg} |e\rangle \langle e| + \hbar g_c (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger), \]  
(27)
where \( \omega_c \) is the cavity frequency and \( \omega_{eg} \) is the resonance frequency of the transition between atomic sub-levels. \( \hat{a}, \hat{a}^\dagger \) are the bosonic raising and lowering
operators of the cavity, $\hat{\sigma}_+ = |e\rangle \langle g|$, $\hat{\sigma}_- = |g\rangle \langle e|$ are the raising and lowering operators of the atom. Moreover we can define the atomic inversion operator as $\hat{\sigma}_z = |e\rangle \langle e| - |g\rangle \langle g|$. We re-write the analytical version of the Jaynes-Cummings Hamiltonian as

$$H_{JC} = \hbar \omega_c z \bar{\sigma} + \hbar \omega_c \theta \sigma + \hbar g_c (\theta \sigma + \bar{\sigma} z),$$  

(28)

or equivalently

$$H_{JC} = \hbar \omega_c z \frac{\partial}{\partial z} + \hbar \omega_c \theta \frac{\partial}{\partial \theta} + \hbar g_c \left( \theta \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta} z \right),$$  

(29)

where we identified the excited and ground states of the atom as $|e\rangle = \theta$ and $|g\rangle = 1$. We many define the number operator as

$$N = \theta \bar{\sigma} + z \sigma = \theta \frac{\partial}{\partial \theta} + z \frac{\partial}{\partial z}. $$  

(30)

The eigenstates of the number operator commutes with the atom-field Hamiltonian $[N, H_{JC}] = 0$. Thus it can be used as a basis of the tensor product states $|e,n\rangle, |g,n\rangle, |e,n-1\rangle \ldots$ However in the analytical formulation we express these states as a product of monomials not as a tensor product of state vectors. For example, the matrix element

$$\langle g,n|H_{JC}|e,n-1\rangle = h g_c \langle g,n|\hat{a}^\dagger \hat{\sigma}_-|e,n-1\rangle + h g_c \langle g,n|\hat{a} \hat{\sigma}_+|e,n-1\rangle = \hbar \sqrt{n} g_c, $$  

(31)

can be calculated analytically as

$$\langle g,n|H_{JC}|e,n-1\rangle = h g_c \langle g,n|z \bar{\sigma}|e,n-1\rangle + h g_c \langle g,n|\bar{\sigma}|e,n-1\rangle$$  

(32)

$$= h g_c \langle g|\bar{\sigma}|e\rangle \langle n|z|n-1\rangle \mu + h g_c \langle g|\bar{\sigma}|e\rangle \langle n|\bar{\sigma}|n-1\rangle \mu$$  

(33)

$$= h g_c \langle g|\bar{\sigma}|e\rangle \langle n|z|n-1\rangle \mu = h g_c \langle g|\bar{\sigma}|e\rangle \int \frac{dz d\bar{z}}{\pi^2} \frac{z^{-|z|^2}}{\sqrt{n^!}} \sqrt{n-1^!}$$  

(34)

$$= h g_c \int \frac{dz d\bar{z}}{\pi^2} \frac{z^{-|z|^2}}{\sqrt{n^!}} \sqrt{n-1^!}$$  

(35)

$$= \sqrt{n} h g_c \int \frac{dz d\bar{z}}{\pi^2} \frac{z^{-|z|^2}}{\sqrt{n^!}} \sqrt{n-1^!} = \sqrt{n} h g_c, $$  

(36)

since $|e\rangle = \theta$, thus $\theta \langle e| = \theta^2 = 0$, also $\frac{\partial}{\partial \theta}|n-1\rangle = \sqrt{n-1}|n-2\rangle$ thus the quantity $\langle g,n|\bar{\sigma}|e,n-1\rangle = \langle g,n|\bar{\sigma}|e,n-1\rangle = \langle g|\bar{\sigma}|e\rangle = 1$ since $\bar{\sigma}|e\rangle = \bar{\sigma} \theta = 1 = |g\rangle$ and $\langle g|g\rangle = 1$ in an orthonormal basis. The previous calculations show that one can separate the cavity states from the atomic level states safely and express them as a product of analytical quantities so no need to worry about the matrix dimensionality of the constructed space from tensor product of the cavity and atomic states and other related issues. Finally we can
express the eigenstates of Jaynes-Cummings Hamiltonian as

\[ |n, + \rangle = \cos \left( \frac{\phi_n}{2} \right) |e, n - 1 \rangle + \sin \left( \frac{\phi_n}{2} \right) |g, n \rangle \] (37)

\[ = \cos \left( \frac{\phi_n}{2} \right) \theta \frac{z^{n-1}}{\sqrt{(n-1)!}} + \sin \left( \frac{\phi_n}{2} \right) \frac{1}{\sqrt{n!}} \]

\[ |n, - \rangle = \cos \left( \frac{\phi_n}{2} \right) |g, n \rangle - \sin \left( \frac{\phi_n}{2} \right) |e, n - 1 \rangle \] (38)

\[ = \cos \left( \frac{\phi_n}{2} \right) \frac{1}{\sqrt{n!}} \theta z^{n-1} - \sin \left( \frac{\phi_n}{2} \right) \frac{1}{\sqrt{(n-1)!}} \]

where 1 is the fermionic ground state and obeys the Grassmann algebra rules.

For a given open quantum systems, we connect the system with one or more reservoirs (baths). In the case of system coupled to a single bath we have the Hamiltonian

\[ H = H_s + H_b + H_{sb}, \] (39)

where \( H_s \) is the system Hamiltonian, \( H_b \) is the bath Hamiltonian and \( H_{sb} \) is the system-bath interaction term. In the Markovian approximation in which the time derivative of the operator depends on the operator itself not its histories, the evolution of any observer \( O \) such as the density matrix \( \rho \) is given by the Gorini–Kossakowski–Sudarshan–Lindblad (GKS-L) equation

\[ \frac{d}{dt} O = -\frac{i}{\hbar} [H_s, O] + \mathcal{L}_D (O), \] (40)

where \( \mathcal{L}_D \) is the dissipative part or dissipator given by

\[ \mathcal{L}_D (O) = \sum_n \gamma_n \left( L_n O \overline{T_n} - \frac{1}{2} \overline{T_n} L_n, O \right), \quad \gamma_n \geq 0, \] (41)

and \( L_n \) are the Lindblad jump operators and \( \overline{T_n} \) are their complex conjugates.

Let \( \{ \Lambda_s | s \geq 0 \} \) be a quantum dynamical semigroup and let \( \rho^0 \in T (HL^2 (\mathbb{C}^n, \mu)) \) be an \( \Lambda_s \)-invariant state where \( T (HL^2 (\mathbb{C}^n, \mu)) \) is the Banach space of trace class operators, then the associated entropy production \( \sigma \) relative to \( \rho^0 \) is

\[ \sigma (\rho) = -\frac{d}{dt} S (\Lambda_s \rho | \rho^0 \rangle \langle \rho^0 |) \big|_{s=0}. \] (42)

As a final comment in this section, the holomorphic representation works also in the non-Markovian approximation and in any quantum mechanical system indeed.
3 Bargmann representation solution of quantum absorption refrigerator

The Hamiltonian of quantum absorption refrigerator is composed of three interacting oscillators \[12\]

\[ H = H_0 + H_{\text{int}}, \] (43)

\[ H_0 = \hbar \omega_h z_h \frac{\partial}{\partial z_h} + \hbar \omega_c z_c \frac{\partial}{\partial z_c} + \hbar \omega_w z_w \frac{\partial}{\partial z_w} = \sum_i \hbar \omega_i z_i \frac{\partial}{\partial z_i}. \] (44)

\[ H_{\text{int}} = \hbar \omega_{\text{int}} \left( z_h \frac{\partial}{\partial z_c} \frac{\partial}{\partial z_w} + z_c z_w \frac{\partial}{\partial z_h} \right), \] (45)

where \( i = h, c, w \) denotes the hot, cold and work reservoirs. The unperturbed energy eigenstates for the quantum absorption refrigerator in the Bargmann representation are

\[ |n_h, n_c, n_w \rangle (0) = \frac{1}{\sqrt{n_h! n_c! n_w!}} z_h^{n_h} z_c^{n_c} z_w^{n_w}. \]

Under steady-state conditions, the first and second laws of thermodynamics should be satisfied, namely the following relations hold

\[ \mathcal{J}_h + \mathcal{J}_c + \mathcal{P} = 0, \] (46)

\[ -\frac{\mathcal{J}_h}{T_h} - \frac{\mathcal{J}_c}{T_c} - \frac{\mathcal{P}}{T_w} \geq 0, \] (47)

where the first relation represents the conservation of energy (first law of thermodynamics) while the second inequality states that the sum of entropies is equal or greater than 0 (second law of thermodynamics, positive production of entropy in the universe).

Quantum mechanically, we may re-write the Hamiltonian as

\[ H = H_0 + \lambda H_{\text{int}}. \] (48)

Assuming the energy spectrum to be non-degenerate, we apply the holomorphic perturbation theory where we expand the energy eigenstates in terms of phase-space coordinates \( \{ z_i \} \) not \( \{ \psi(z_i) \} \). The energy levels and eigenstates of the perturbed Hamiltonian are given by the time-independent Schrödinger equation

\[ (H_0 + \lambda H_{\text{int}}) |n \rangle = E_n |n \rangle. \] (49)

If we assume the perturbation to be extremely small, we may expand the energy levels and eigenstates as power series

\[ E_n |n \rangle = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots \] (50)

\[ |n \rangle = |n^{(0)} \rangle + \lambda |n^{(1)} \rangle + \lambda^2 |n^{(2)} \rangle + \ldots \] (51)

The first-order energy shift in the quantum absorption refrigerator is

\[ E_n^{(1)} = \frac{\langle m^{(0)} | H_{\text{int}} | n^{(0)} \rangle}{|m^{(0)} \rangle}, \] (52)

\[ = \hbar \omega_{\text{int}} \left( \delta_{m_h, n_h+1} \delta_{m_c, n_c-1} \delta_{m_w, n_w-1} \sqrt{(n_h + 1)n_c n_w} \\
+ \delta_{m_h, n_h-1} \delta_{m_c, n_c+1} \delta_{m_w, n_w+1} \sqrt{n_h (n_c + 1)(n_w + 1)} \right) \] (53)
and the first-order energy eigenstates are
\[ |n^{(1)} \rangle = \sum_{m \neq n} \frac{\langle m^{(0)} | H_{\text{int}} | n^{(0)} \rangle |m^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \]
(54)
where \( |m^{(0)} \rangle = \frac{1}{\sqrt{m_h!m_c!m_w!}} z_h^{m_h} z_c^{m_c} z_w^{m_w} \) is the energy eigenstates of index \( m = (m_h, m_c, m_w) \) and \( E_n^{(0)} - E_m^{(0)} = \hbar \omega_{\text{int}} (n - m) = \hbar \omega_{\text{int}} (n_h + n_c + n_w - m_h - m_c - m_w) \). By plugging the expressions of \( \langle m^{(0)} | H_{\text{int}} | n^{(0)} \rangle | \) and \( E_n^{(0)} - E_m^{(0)} \) in 54, we found the first-order correction of the energy eigenstates to be independent of \( \hbar \omega_{\text{int}} \).

In the absorption refrigerator, the noise substitutes the work bath and its contact leading to the following interaction term
\[ H_{\text{int}} = g(t) \left( z_h \frac{\partial}{\partial z_c} + z_c \frac{\partial}{\partial z_h} \right) = g(t) X_1, \]
(55)
where \( g(t) \) is the stochastic noise field with zero mean \( \langle g(t) \rangle = 0 \) and delta time correlation \( \langle g(t) g(t') \rangle = 2 \eta \delta (t - t') \) and \( X_1 \) is the generator of swap operation between the two oscillators and belongs to the SU(2) Lie algebra together with the following operators
\[ X_2 = i \left( z_h \frac{\partial}{\partial z_c} - z_c \frac{\partial}{\partial z_h} \right), \]
(56)
\[ X_3 = \left( z_h \frac{\partial}{\partial z_h} - z_c \frac{\partial}{\partial z_c} \right), \]
(57)
\[ N = \left( z_h \frac{\partial}{\partial z_h} + z_c \frac{\partial}{\partial z_c} \right), \]
(58)
forming a closed set of equations. The Heisenberg equation for an arbitrary time-independent analytic function \( O(z_i, \bar{z}_i) \) reads
\[ \frac{d}{dt} O = -\frac{i}{\hbar} [H_s, O] + \mathcal{L}_n (O) + \mathcal{L}_h (O) + \mathcal{L}_c (O), \]
(59)
Where the noise dissipator for Gaussian noise is \( \mathcal{L}_n (O) = -\eta [X_1, [X_1, O]] \) 27. For simplicity we assume the heat baths to be uncorrelated between themselves and also uncorrelated with the driving noise. In the Bargmann representation, the hot and cold dissipators are
\[ \mathcal{L}_h (O) = \gamma_h (\overline{\pi}_h + 1) \left( z_h O \overline{\pi}_h - \frac{1}{2} \{ z_h \overline{\pi}_h, O \} \right) \]
(60)
\[ + \gamma_h \overline{\pi}_h \left( \overline{\pi}_h O z_h - \frac{1}{2} \{ \overline{\pi}_h z_h, O \} \right), \]
\[ \mathcal{L}_c (O) = \gamma_c (\overline{\pi}_c + 1) \left( z_c O \overline{\pi}_c - \frac{1}{2} \{ z_c \overline{\pi}_c, O \} \right) \]
(61)
\[ + \gamma_c \overline{\pi}_c \left( \overline{\pi}_c O z_c - \frac{1}{2} \{ \overline{\pi}_c z_c, O \} \right), \]
For a vanishing stochastic driving field, these equations guide separately the oscillators of hot and cold baths to thermal equilibrium provided that

\[
\overline{n}_i = \left[ e^{-\frac{\hbar \omega_i}{kB_i}} - 1 \right]^{-1}
\]

where \( i = h, c \). To obtain the cooling current \( J_c = \langle L_c (h \omega_c \frac{\partial}{\partial z}) \rangle \), we search for the stationary solutions of \( X_3 \) and \( N \) we find

\[
J_c = \hbar \omega_c \frac{n_c - n_h}{(2\eta)^{-1} + \gamma_c^{-1} + \gamma_h^{-1}}. \tag{62}
\]

Obviously cooling occurs when \( n_c - n_h > 0 \). The coefficient of performance (COP) is

\[
\text{COP} = \frac{J_c}{J_n} = \frac{\omega_c}{\omega_h - \omega_c} \leq \frac{T_c}{T_c - T_h}. \tag{63}
\]

In this section, we have considered Levy and Kosloff model driven by Gaussian noise for the quantum refrigerator, however it is possible to consider the Poisson noise case [27] or more enhanced models presented in [51].

4 The computational complexity

Bargmann representation of quantum absorption refrigerators and in principle any quantum heat engine has some advantages comparing with the standard treatment based on the operators \( a \) and \( a^\dagger \) that act in Hilbert space \( L^2(\mathbb{R}^d) \). To put this into context we compute the computational complexity of both pictures for a specific example. By definition, the computational complexity is the time consumed by a multitape Turing machine in performing computational tasks [52]. The coordinate representation of any wavefunction in the bosonic hot or cold reservoir is given by a Hermite polynomial of degree \( n \) multiplied by exponential function of Gaussian signature up to some numerical constants proportional with the ground-state length scale \( \xi = \sqrt{\hbar/2m\omega} \). However in Bargmann representation the wavefunctions of the bosonic hot or cold reservoirs are simply monomials of \( z \) with power \( n \). As an example consider the ground-state of one specific state in hot or cold reservoir, it has simply the formula \( \psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-m\omega x^2/2\hbar} \) with computational cost of order \( O(N^{5/2}) \) using schoolbook multiplication algorithms and of order \( O(N^{2.085}) \) using Karatsuba multiplication algorithm where \( N \) denotes the input’s digits number. This complexity is simplification of the problem since we considered the quantities such as \( m, \omega, h, x^2 \) and other possible combinations to be given previously to calculations and thus have complexity of order \( O(1) \) [52]. However it is very large comparing with the computational cost of the ground-state in Bargmann representation which is of order \( O(1) \). The situation becomes very drastically complicated considering higher excited states.

Another interesting example which shows the advantage of the developed formalism in this work appears in the calculations of quasiprobability distributions in phase space [4, 54]. More concretely, the Husimi-Kano \( Q \) representation has a simpler form in the Bargmann representation comparing with its form in
the phase space $\mathbb{R}^d$, and can be written formally in natural units for normalized state $||\psi|| = 1$ as $H_\psi = (2\pi)^{-d}|A\psi|^2 \left(\frac{x-\mu}{\sqrt{2}}\right)e^{-|z|^2/2}$ \cite{34,53}. This simplification is mainly because instead of representing $Q$ in term of the conjugate coordinates $X$ and $P$ in the phase space, we simply unify the treatment using the phase space coordinate $z$ only. Thus Bargmann space can be regarded as a natural home for the Husimi-Kano $Q$-representation and other quasi-probability distributions in the phase space \cite{54}. In the context of heat engines and refrigerators, the Husimi-Kano $Q$-representation has been used in the calculation of the quantum synchronization since the synchronization measure $S$ is the integral of $Q$ up to a numerical constant \cite{55,56,57}. Thus it is legitimate to claim that quantum synchronization formalism simplifies using Bargmann representation and this in turn reduces the computational complexity of the problem.

5 Conclusion

In this work, we discussed the analytical theory of open quantum systems using Bargmann representation of the bosonic raising and lowering operators systematically. We also provided similar procedure for fermions in term of the anti-commuting Grassmann variables $\{\theta_i\}$ and their partial derivatives. This construction is useful in two counts first it allows us to exploit the whole theory of analytical functions and all its techniques throughout the computation of open system characterizations such as the dissipators. Moreover it appears to be conceptually easier to understand than the standard canonical approach based on the raising and lowering operators $a$ and $a^\dagger$ especially for the bosonic case where there is no upper bound on the number of excited states a particle can take.

More precisely, we have considered the quantum absorption refrigerator driven by a Gaussian noise as a model example. However, the holomorphic representation is applicable in all heat engines and refrigerators. We discussed the computational complexity, the time required by a multitape Turing machine to perform specific tasks, associated with both standard and Bargmann representation for excitations in bosonic heat baths. We found that working within Bargmann representation has less computational complexity comparing with the standard algebraic or analytical methods in coordinate representation. Another advantage comes from the fact that in holomorphic picture, we use a phase coordinate $z$ instead of the canonical variables $X$ and $P$. This fact simplifies the computation of quasiprobability distributions defined normally in phase space such as the Husimi-Kano $Q$ distributions and this might have impact on the numerical studies of quantum synchronization in heat engines and in principle for any quantum system.

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References

[1] L. D. Landau, E. M. Lifshitz: *Statistical Physics: part 1*, 3rd edition, Pergamon Press, Oxford (1980).

[2] J. Uffink: *Stud. Hist. Philo. MP.* 32(3), Pages 305-394 (2001).

[3] D. J. C. MacKay: *Information Theory, Inference and Learning Algorithms*, Cambridge University Press, Cambridge 2003.

[4] Z. Ficek, M. R. B. Wahiddin: *Quantum Optics for Beginners*, Jenny Stanford Publishing 2014.

[5] H. E. D. Scovil, E. O. Schulz-DuBois: *Phys. Rev. Lett.* 2, 262–263 (1959).

[6] J. E. Geusic, E. O. Schulz-DuBios, H. E. D. Scovil: *Phys. Rev.* 156, 343–351 (1967).

[7] R. Alicki: *J. Phys. A* 12, L103 (1979).

[8] R. Kosloff: *J. Chem. Phys.* 80, 1625 (1984).

[9] E. Geva, R. Kosloff: *J. Chem. Phys.* 96,(4) 3054–3067 (1992).

[10] H. T. Quan, Yu-xi Liu, C. P. Sun, F. Nori: *Phys. Rev. E* 76, 031105 (2007).

[11] U. Harbola, S. Rahav, S. Mukamel: *EPL* 99 50005 (2012).

[12] R. Kosloff, A. Levy: *Annu. Rev. Phys. Chem.* 65:365–393 (2014).

[13] S. Vinjanampathy, J. Anders: *Contemp. Phys.* 57, 545 (2016).

[14] A. Friedenberger, E. Lutz: *EPL* 120(1), 10002 (2017).

[15] E. B. Davies: *Quantum theory of open systems*, Academic Press, Massachusetts 1976.

[16] H. Breuer, F. Petruccione: *The Theory of Open Quantum Systems*, Oxford University Press, Oxford 2007.

[17] A. Rivas, S. F. Huelga: *Open Quantum Systems. An Introduction*, Springer Briefs in Physics, New York 2012.

[18] A. Rivas, S. F. Huelga, M. B. Plenio: *Rep. Prog. Phys.* 77, 094001 (2014).

[19] I. de Vega, D. Alonso: *Rev. Mod. Phys.* 89, 15001 (2017).

[20] V. A. Fock: *Z. Phys.* 49, 339 (1928).

[21] V. Bargmann: *Commun. Pure. Appl. Math.* 14 (3): 187 (1961).

[22] I. E. Segal: *Mathematical problems of relativistic physics*, in Kac, M. (ed.), Proceedings of the Summer Seminar, Boulder, Colorado, 1960, Vol. II, Lectures in Applied Mathematics, American Mathematical Society (1963).
[23] S. Stenholm: *Opt. Commun.* 36, Pages 75-78 (1981).

[24] A. Vourdas, R. F. Bishop: *Phys. Rev. A* 50, 3331 (1994).

[25] A. Voros: *Phys. Rev. A* 40, 6814 (1989).

[26] T. Thiemann: *Class. Quant. Grav.* 13, 1383–1404 (1996).

[27] A. Levy, R. Kosloff: *Phys. Rev. Lett.* 108, 070604 (2012).

[28] M. T. Mitchison, P. P. Potts: In: Binder F., Correa L., Gogolin C., Anders J., Adesso G. (eds) *Thermodynamics in the Quantum Regime. Fundamental Theories of Physics*, 195, Springer, New York 2018.

[29] G. Maslennikov, S. Ding, R. Hablützel et al. *Nat. Commun.* 10, 202 (2019).

[30] A. Perelomov: *Generalized Coherent States and Their Applications*, Springer-Verlag Berlin Heidelberg 1986.

[31] G. Folland: *Harmonic Analysis in Phase Space*, Princeton University Press, New Jersey 1989.

[32] B. C. Hall: *Contemp. Math.* 260, 1–59 (2000).

[33] S. Twareque Ali, M. Enlisy: *Rev. Math. Phys.* 17, 391–490, (2005).

[34] B. C. Hall: *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics 267, Springer Verlag, New York 2013.

[35] G. Folland: *Real Analysis: Modern Techniques and Their Applications*, Wiley; 2nd edition, New Jersey 2007.

[36] R. P. Boas: *Entire Functions*, Academic Press, Massachusetts 1954.

[37] N. Aronszajn: *Trans. Amer. Math. Soc.* 68, pp. 337–404 (1950).

[38] S. Bergman: *The kernel function and conformal mapping*, Amer. Math. Soc., Rhode Island 1950.

[39] J. Zinn-Justin: *Path Integrals in Quantum Mechanics*, Oxford University Press, Oxford 2010.

[40] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouri, P. Grangier: *Nature* 448, 784–786 (2007).

[41] R. J. Glauber: *Phys. Rev.* 131, 2766 (1963).

[42] V. V. Dodonov, I. A. Malkin, V. I. Man’ko: *Physica* 72(3), 597–615 (1974).

[43] P. Cartier, Cecile DeWitt-Morette: *Functional Integration: Action and Symmetries*, Cambridge University Press, Cambridge 2007.

[44] L. A. Takhtajan: *Quantum Mechanics for Mathematicians*, Graduate Studies in Mathematics 95, American Mathematical Society, Rhode Island 2008.
[45] W. Vogel, D-G. Welsch: *Quantum Optics*, Wiley-VCH., Weinheim 2006.

[46] A. Kossakowski: *Rep. Math. Phys.* 3 (4): 247 (1972).

[47] V. Gorini, A. Kossakowski, E.C.G Sudarshan: *J. Math. Phys.* 17(5): 821 (1976).

[48] H. Spohn: *Rep. Math. Phys.* 10(2),189–194 (1976).

[49] G. Lindblad: *Commun. Math. Phys.* 48, 119–130 (1976).

[50] H. Spohn: *J. Math. Phys.*19, 1227 (1978).

[51] L. A. Correa, J. P. Palao, D. Alonso, G. Adesso: *Sci. Rep.* 4, 3949 (2014).

[52] R. P. Brent, P. Zimmermann: *Modern Computer Arithmetic*, Cambridge University Press, Cambridge 2010.

[53] K. Husimi: *Proc. Phys. Math. Soc. Jpn.* 22: 264–314 (1940).

[54] D. F. Walls, G. J. Milburn, *Quantum Optics*, Springer-Verlag Berlin Heidelberg, 2nd edition 2008.

[55] C. Davis-Tilley, A. D. Armour: *Phys. Rev. A* 94, 063819 (2016).

[56] H. Eneriz, D. Z. Rossatto, F. A. Cárdenas-López, et al.: *Sci. Rep.* 9, 19933 (2019).

[57] N. Jaseem, M. Hajdušek, V. Vedral, Rosario Fazio, Leong-Chuan Kwek, and S. Vinjanampathy: *Phys. Rev. E* 101, 020201(R) (2020).