Elastic gauge fields and zero-field 3D quantum Hall effect in hyperhoneycomb lattices

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Dirac materials respond to lattice deformations as if the electrons were coupled to gauge fields. We derive the elastic gauge fields in the hyperhoneycomb lattice, a three dimensional (3D) structure with trigonally connected sites. In its semimetallic form, this lattice is a nodal-line semimetal with a closed loop of Dirac nodes. Using strain engineering, we find a whole family of strain deformations that create uniform nearly flat Landau levels in 3D. We propose that those Landau levels can be created and tuned in metamaterials with the application of a simple uniaxial temperature gradient. In the 3D quantum anomalous Hall phase, which is topological, we show that the components of the elastic Hall viscosity tensor are multiples of \( \eta_H = \beta^2 \sqrt{3} / (8 \pi a^3) \), where \( \beta \) is an elastic parameter and \( a \) is the lattice constant.

**Introduction.** In honeycomb lattices such as graphene [1], strain deformations couple to electronic degrees of freedom as gauge fields and can induce Landau level (LL) quantization with very large effective pseudomagnetic fields [2, 3]. When the chemical potential is inside the gap of the LLs, the Hall conductivity per valley is quantized and the system is expected to show a zero-field quantum Hall effect (QHE). Due to the dispersion of the LLs, Hall conductivity quantization is not common in three dimensions (3D), and may occur only in extremely anisotropic systems such as Bechgaard salts [4, 5], Bernal graphite [6, 7], and in nodal-line semimetals [10–12]. Even in strongly anisotropic systems such as in nodal line semimetals, the physical implementation of the 3D QHE is challenging due to the unusual toroidal field geometry required [13]. With the help of strain engineering, one may in principle design 3D LLs with well defined gaps in between from real space configurations of magnetic field that would be otherwise impractical to realize.

In this Rapid communication, we derive the elastic gauge fields that follow from arbitrary lattice deformations in the hyperhoneycomb lattice, a natural 3D generalization of the honeycomb geometry where all sites are connected by coplanar trigonal bonds, as shown in Fig. 1a. In the semimetallic form, this lattice is an example of a nodal-line semimetal [10, 13–18]. We identify a whole family of lattice deformations that produce uniform nearly flat LLs in 3D, a prerequisite for the 3D zero-field QHE. We show that this family of non-trivial deformations can be physically implemented with the application of a simple temperature gradient along the axis perpendicular to the nodal line, leading to a tunable metal-insulator transition in the bulk. The strain deformations can be uniquely specified by the set of thermal expansion coefficients of the crystal. We propose that a tunable temperature controlled 3D zero-field QHE can be implemented in acoustic metamaterials [19].

In the presence of topological states, the topological invariants can manifest in the elastic response of the crystal through phonons. In the 3D quantum anomalous Hall (QAH) phase [20], which is the extension of the Haldane model [21] to the hyperhoneycomb lattice, we also calculate the elastic Hall viscosity tensor \( \eta_{\mu \nu \rho \sigma} \). Also known as the phonon Hall viscosity [23], this quantity is analogous to the dissipationless viscous response of electrons in the quantum Hall regime [24–27] and is topological in nature. We show that the components of the Hall viscosity tensor are \( \pm \eta_H \) or \( \pm 2 \eta_H \) (or zero), with \( \eta_H = \beta^2 \sqrt{3} / (8 \pi a^3) \), where \( \beta \) is an elastic parameter and \( a \) is the lattice constant.

**Hamiltonian.** The hyperhoneycomb lattice has four sites per unit cell \( \mu = 1, \ldots, 4 \) and is generated by the lattice vectors \( \mathbf{a}_1 = (\sqrt{3}, 0, 0), \mathbf{a}_2 = (0, \sqrt{3}, 0) \), and \( \mathbf{a}_3 = (-\sqrt{3}/2, \sqrt{3}/2, 3) \), in units of the lattice constant \( a \). In the momentum space, the reciprocal lattice is generated by the vectors \( \mathbf{b}_1 = (2\pi/\sqrt{3}, 0, -\pi/3) \), \( \mathbf{b}_2 = (0, -2\pi/\sqrt{3}, \pi/3) \) and \( \mathbf{b}_3 = (0, 0, 2\pi/3) \), shown in Fig. 1b. The tight-binding Hamiltonian is a \( 4 \times 4 \) matrix

\[
\mathcal{H}_{\mu \nu} (\mathbf{k}) = -t_0 \sum_{\delta_{\mu \nu}} e^{i \mathbf{k} \cdot \delta_{\mu \nu}},
\]

where \( t_0 \) is the hopping amplitude, \( \delta_{\mu \nu} \) are the nearest neighbor (NN) vectors between sites of species \( \mu \) and \( \nu \) and \( \mathbf{k} \) is the momentum measured from the center of

![Figure 1](https://example.com/figure1.png)

**Figure 1.** a) Hyperhoneycomb lattice, with four atoms per unit cell. All sites are linked by coplanar trigonal bonds spaced by 120°. b) Brillouin zone (BZ) of the hyperhoneycomb lattice, with the nodal line shown in red. The arrows show the reciprocal lattice vectors.
the Brillouin zone (BZ). In total, there are six NN vectors
\[ \delta_{12} = \pm \sqrt{3} a / 2, 0, a / 2 \], \[ \delta_{14} = (0, \pm \sqrt{3} a / 2, a / 2) \],
\[ \delta_{14} = (0, 0, -a) \] and \[ \delta_{23} = (0, 0, a) \]. The low
energy bands of this lattice have a line of Dirac nodes
\[ h_0 = [k_x(s), k_y(s), 0] \] in the \( k_z = 0 \) plane, which can be
written in terms of some parameter \( s \) that satisfies the
equation \[ 4 \cos(3k_x(s)/2) \cos(3k_y(s)/2) = 1 \]. The low
energy projected Hamiltonian is described by a \( 2 \times 2 \) matrix
expanded around the nodal line
\[ \mathcal{H}_{0,p}(q) = [v_x(s)q_x + v_y(s)q_y] \sigma_1 + v_z(s)q_z \sigma_2 \] (2)

where \( q \equiv k - k_0(s) \) is the relative momentum, \( \sigma_1, \sigma_2 \)
are the two off-diagonal Pauli matrices and
\[ v_x(s) = \frac{\sqrt{3}}{1 + \alpha^2} \sin \left( \frac{\sqrt{3}}{2} k_x(s) \right) t_0 \]
\[ v_y(s) = \frac{\alpha^2 \sqrt{3}}{1 + \alpha^2} \sin \left( \frac{\sqrt{3}}{2} k_y(s) \right) t_0 \] (3)
\[ v_z(s) = -\frac{3\alpha}{1 + \alpha^2} t_0, \]

are the velocities of the quasiparticles, with \( \alpha(s) \equiv 2 \cos(\sqrt{3} k_x(s)a/2) \) [20]. The energy spectrum of the
quasiparticles is \( E_0(q) = \pm \sqrt{(v_x q_x + v_y q_y)^2 + v_z^2 q_z^2} \). The
wavefunctions have a \( \pi \) Berry phase for closed line trajec-
tories that encircle the nodal loop.

**Elastic gauge fields.** The inclusion of lattice deformations
can be done by locally changing the distance be-
tween lattice sites, which affect value of the hopping
constant. Expanding it to lowest order in the displacement
of the lattice,
\[ t \left( \tilde{\delta}^{(n)} + \delta r \right) \approx t_0 + \frac{\beta}{a^2} \tilde{\delta}^{(n)} \delta_{ij} u_{ij} + \mathcal{O} (\delta r^2), \] (4)

with \( n = 1, \ldots, 6 \) indexing the 6 NN lattice vectors
\( \delta^{(n)} \), \( u_{ij} = \frac{1}{2} (\partial_i u_{j} + \partial_j u_{i}) \) is the strain tensor defined
in terms of the displacement field \( u \) of the lattice and
\( \beta = a^2 \frac{\partial \sigma}{\partial \log L} = \frac{\partial \log \rho}{\partial \log L} \) is the Grüneisen parameter of the
model [21]. Including the lattice distortions in Hamiltonian
\[ \mathcal{H}_p = \mathcal{H}_{0,p} + \mathcal{H}_{el}, \]

\[ \mathcal{H}_{el} = \frac{3}{4} \frac{\beta}{a} v_z (u_{xx} + u_{yy} - 2u_{zz}) \sigma_1 - \frac{\beta}{a} (v_x u_{xz} + v_y u_{yz}) \sigma_2 \] (5)

is the elastic contribution. As in the 2D case (graphene),
the deformation of the lattice couples to the Dirac
fermions as an elastic gauge field \( \mathbf{A} \). It is convenient
to rewrite the Hamiltonian in the more familiar form
\[ \mathcal{H}_p(q) = [v_x(q_x + A_x) + v_y(q_y + A_y)] \sigma_1 + v_z(q_z + A_z) \sigma_2, \] (6)

\[ \begin{align*}
\text{a)} & \\
\text{b)} & \\
\end{align*} \]

Figure 2. Pseudomagnetic field \( \mathbf{B} \) along the nodal line for two
different strain field configurations. a) \( \mathbf{u} = (2xz, 2yz, z^2) \) and
b) \( \mathbf{u} = (2yz, 2zx, 0) \). Both configurations lead to uniform \( \mathbf{B} \)
fields in real space, but only the former produces nearly flat
LLs.

\[ \begin{align*}
A_x(s) &= \frac{v_z}{v_x^2} \frac{3\beta}{4a} \left( u_{xx} + u_{yy} - 2u_{zz} \right) \\
A_y(s) &= \frac{v_y}{v_x^2} \frac{3\beta}{4a} \left( u_{xx} + u_{yy} - 2u_{zz} \right) \quad (7) \\
A_z(s) &= -\frac{\beta}{a} \left( \frac{v_x}{v_z} u_{xz} + \frac{v_y}{v_z} u_{yz} \right)
\end{align*} \]

are the components of the elastic gauge field along the
nodal line, with \( v_x^2(s) = v_y^2(s) + v_z^2(s) \). The definition of the
\( A_x \) and \( A_y \) components is to a degree arbitrary. In [7]
we chose the most symmetric combination, although this
choice has no effect in physical observables.

Those gauge fields can be associated to a pseudomag-
netic field \( \mathbf{B} = \nabla \times \mathbf{A} \), which follows from lattice de-
formations and hence must preserve time reversal symme-
try (TRS). While pseudo magnetic fields couple to the
Dirac fermions similarly to conventional magnetic fields
and can produce Landau level (LL) quantization, they
create a zero net magnetic flux at each lattice site. There-
fore, electrons sitting at opposite points in the nodal line
are related by TRS and must necessarily couple to op-
posite \( \mathbf{B} \) fields. In order to produce zero-field quantum
Hall effect, one needs to create 3D LL quantization with
well defined gaps in between. In 2D, the conventional
Hall conductivity \( \sigma_{xy} \) is a dimensionless and quantized
in units of \( e^2/h \). In 3D, it has an extra unit of inverse
length. According to Halperin [28], the Hall conductivity
tensor is \( \sigma_{ij} = e^2/(2\pi h) \epsilon_{ijk} G_k \), where \( G \) is a reciproc-
ical lattice vector (and could be zero). In general, a finite Hall
conductivity in 2D (3D) is allowed whenever the chem-
ical potential is in the gap between different LLs, and
implies in the existence of chiral edge (surface) states.
At zero field, the Hall conductivity tensor due to pseudo-
magnetic fields does not create chiral charge currents
as in the conventional quantum Hall effect, but rather a
valley current.

**Strain engineering.** In all possible strain configu-
rations, the effective Hamiltonian [6] has the form
\[ \mathcal{H}_p(q) = h_1 \sigma_1 + h_2 \sigma_2. \] In specific, for configuration
Energy of the Landau levels (LLs) around the nodal line vs. polar angle $\theta$, $(0 \leq \theta < \pi)$ for three strain configurations: (a) $\mathbf{u} = (2xz, 2yz, z^2)$, (b) $\mathbf{u} = (2yz, 2xz, 0)$ and (c) $\mathbf{u} = (2xz, 2yz, 0)$. The former and the latter configurations belong to a broader family of deformations $\mathbf{u} = (\alpha_x x, \alpha_y y, \alpha_z z^2)$ that produce nearly flat LLs in 3D. In (b), the LLs collapse at discrete points of the nodal line, preventing the zero-field QHE. Application of a uniform temperature gradient, $\Delta T \propto z$, creates strain fields in that family. (d) Phase space for $\alpha_x = \alpha_y$ and $\alpha_z$ with a zero-field 3D QHE (light red regions).

In general, one can define families of strain deformations that lead to a 3D zero-field QHE. While the energy spectrum is generically defined by Eq. (11), in those families $\omega(s) = \sqrt{2|h_2, h_1|}$ can be non zero for all points along the nodal line. For instance, one can build a family of strain deformations

$$\mathbf{u} = (\alpha_x x, \alpha_y y, \alpha_z z^2),$$

where the constants $\alpha_i$ ($i = x, y, z$) are such that $\omega(s) = \sqrt{(|3|/a)|\alpha_x v_x^2(s) + \alpha_y v_y^2(s) + 2(\alpha_x + \alpha_y - 4\alpha_z)v_z^2(s)|}$ is non-zero for all $s$. The anisotropic case $\alpha_x = \alpha_y \gg \alpha_z$ is shown in Figure 3c. The phase space of parameters with $\alpha_x = \alpha_y$ that leads to a zero-field QHE is shown in the light red areas of Fig. 3d.

The deformation pattern $\mathbf{u} = (2xz, 2yz, 0)$ can be created with the strain forces indicated by the arrows in Fig. 4a. Interestingly, the physical implementation of the family of deformations (1) can be achieved with the application of a uniform temperature gradient along the $z$ axis of the crystal (see Fig. 4b). Since $\mathbf{u}$ describes the displacement of the lattice sites from their equilibrium position, the thermal expansion is represented as $u_i = \Delta x_i = x_i \gamma_i \Delta T \propto \gamma_i x_i z$, where $\gamma_i = \alpha x_i / \Delta x$ is the linear thermal expansion coefficient in the $i = x, y, z$ direction and $\Delta T(z) = T - T_0 \propto z$ is the temperature variation from equilibrium. This tunable pattern of deformations could be created with temperature gradients in crystals and acoustic metamaterials [19].

Elastic Hall viscosity. In quantum Hall systems, the Hall viscosity follows from the linear response of the system to gravitational fluctuations, which manifest through local changes in the metric of space $\xi_{ij} = 1/2(\partial_i \xi_j + \partial_j \xi_i)$, where $\xi_i$ has the physical meaning of a strain field. The so called gravitational Hall viscosity is defined as the variation of the stress tensor $T_{\mu \nu} = \partial \mathcal{H} / \partial \xi_{\mu \nu}$ to time variations of the strain tensor $\xi_{ij}$. By analogy, the elastic (phonon) Hall viscosity can be derived using linear re-
Figure 4. (a) Elastic deformation of a cylinder under the strain configuration $u = (2xz, 2yz, 0)$. The arrows indicate the strain forces that create uniform nearly flat LLs in a 3D material (see fig. 3c). (b) Temperature gradient along the $z$ axis that implements the strain field $u = (\alpha_x z, \alpha_y y, \alpha_z z^2)$, with $\alpha_i (i = x, y, z)$ proportional to the thermal expansion coefficients. Red: hot region. Blue: cold.

The effective action

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} d^3k \sum_{m} \sigma_{ij} (1)$$

where $\sigma_{ij}$ is the antisymmetric tensor, $u_{ij}$ the strain-rate tensor and $\eta_{\mu\nu\rho\gamma}$ the elastic Hall viscosity tensor. The first term is the elastic response of a charge neutral fluid and the second one describes a non-dissipative response as \[\tau_{ij}(1)\] proportional to the thermal expansion coefficients.

Effective action. In the QAH state, the Hamiltonian for the hyperhoneycomb lattice, described in detail in [23–26], is given by

$$\mathcal{H}_{\text{QAH}} = \mathcal{H}_p(q) + m(s)\sigma_3,$$

where $m(s)$ is the mass of the charge, and $m(s) = 0$, are Weyl points with a well defined helicity \[\mathbf{20}\]. Weyl points with opposite helicities are connected by surface states in the form of topological Fermi arcs \[\mathbf{33}\].

Effective action. In the QAH state, the Hamiltonian for the hyperhoneycomb lattice, described in detail in [23–26], is given by

$$\mathcal{H}_{\text{QAH}} = \mathcal{H}_p(q) + m(s)\sigma_3,$$

The effective action in terms of the strain tensor $u_{ij}$ can be derived by integrating out the fermions. That results in the effective action $S_{\text{ef}}(u) = \mathcal{T}_x \ln \left( G^{-1} \right)$, where $G^{-1}(q) = ig_0 - \mathcal{H}_{\text{QAH}}(q) = G_0^{-1}(q) - \Sigma_{\text{el}}$ is the Green's function

$$\Sigma_{\text{el}}(u) = v_z A_1 \sigma_1 + (v_x A_2 + v_y A'_2)\sigma_2$$

is the self-energy due to elastic terms. For convenience, we define the elastic gauge fields in \[\mathbf{20}\] as

$$A_1 = \frac{a_0}{\mathbf{4}} (u_{xx} + u_{yy} - 2u_{zz})$$

Expanding the action in powers of the elastic gauge fields, namely $S_{\text{ef}} = \mathcal{T}_x \ln G_0^{-1} - \mathcal{T}_x \ln G_0 + \mathcal{T}_x \ln G_\Sigma$, the lowest order contribution to the Hall viscosity comes from

$$S_{\text{el}}^{(2)} = \frac{1}{2} \mathcal{T}_x \ln \left( G_0 \Sigma G_0 \Sigma \right).$$

More explicitly,

$$\delta S_{\text{el}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ v_x v_x A_1(1) \Pi_1(2) (k) A_2(k) + v_y v_y A_1(1) \Pi_1(2) (k) A_2(k) + (1 \leftrightarrow 2) \right],$$

where $\Pi_1(2) = -\Pi_2(1)$ is the standard polarization tensor, with antisymmetric off-diagonal terms, $\Pi_1(2) = -\Pi_2(1)$. Integration can be done by slicing the BZ into planes intersecting the nodal line at two points. Integrating over a slice in the $xz$ plane for the first term,

$$v_x v_x \Pi_1(2) = \frac{k_0}{2\pi} \int \frac{d^2q_y}{(2\pi)^2} \nu_q (k_0) = \frac{k_0}{2\pi} \lambda_y,$$

where $\nu_q (k_0) = \frac{1}{2} \text{sign} [v_x (v_x v_y (m(s))] = \pm \frac{1}{2}$ is the topological charge of 2D massive Dirac fermions confined to an $xz$ plane crossing the nodal line at $k_0$. Integration along the nodal loop $C$ gives the $y$ component of the Chern vector $\lambda_y = (\lambda_x, \lambda_y, \lambda_z)$, which is belongs to the reciprocal lattice $G$ and sets the 3D quantum Hall conductivity of the system, $\sigma_{ij} = e^2/(2\pi \hbar) \epsilon_{ijk} \lambda_k$. From a similar argument, $v_y v_y \Pi_2(1) = -v_y v_y \Pi_2(1) = k_0 \lambda_x/2\pi$. Hence,

$$\delta S_{\text{ef}} = \frac{1}{16\pi^2} \int \frac{d^3k}{(2\pi)^3} \left[ \left[ -\lambda_x A_1(1) - k_0 A'_2(1) \right] + \lambda_y A_1(1) - k_0 A_2(k) - (1 \leftrightarrow 2) \right].$$

Performing the substitution $A_1 = d_z$, $A_2 = d_x$ and $A'_2 = d_y$, the effective action can be written in a more compact form,

$$\delta S_{\text{ef}} = \frac{1}{16\pi^2} \int d^3x \epsilon_{\mu\nu\rho} \lambda_\mu d_\nu d_\rho.$$
where
\[\begin{align*}
d_x &= -\frac{\beta}{a} u_{xx} , \\
d_y &= -\frac{\beta}{a} u_{yz} , \\
d_z &= -\frac{3}{a} \left( u_{xx} + u_{yy} - 2u_{zz} \right).
\end{align*}\]

For the hyperhoneycomb lattice, the Chern vector is \( \lambda = b_1 + b_2 = (2\pi/\sqrt{3}, -2\pi/\sqrt{3}, 0) a^{-1} \). Writing the action in a more explicit form,
\[
\delta S_{\text{ef}} = \frac{1}{2} \int d^4 x \nabla \left[ (u_{xx} + u_{yy} - 2u_{zz}) (\dot{u}_{yz} + \dot{u}_{xz}) - (u_{yz} + u_{xz}) (\dot{u}_{xx} + \dot{u}_{yy} - 2\dot{u}_{zz}) \right],
\]
with \( \eta_H = \beta^2 \sqrt{3}/8\pi a^3 \). The action can be cast in the form of \([13]\), where the elastic Hall viscosity tensor is \( \eta_{xxxz} = \eta_{xyyz} = \eta_{yyxz} = \eta_{xyyx} = \eta_H \), and \( \eta_{zzzz} = -2\eta_H \). The elastic Hall viscosity tensor is anisotropic, as expected in 3D \([20]\), and reflects the topological nature of the QAH state \([34]\). In nodal-line semimetals, the Chern vector is related to the arclength separating two Weyls points \textit{along} the nodal line. Hence, the shape of the nodal line contains information about the lattice and can be used even in effective low energy models to determine the \textit{exact} elastic Hall viscosity in terms of the elastic parameter and lattice constant \( a \).

\textbf{Experimental observation.} Although there are no known examples of semimetallic hyperhoneycomb crystals \([35]\), this lattice may be artificially created in optical lattices \([30]\), and also in photonic \([4, 37]\) and acoustic metamaterials \([19]\). In twisted graphene bilayers, elastic gauge fields can be created with electric field effects \([38]\). In synthetic lattices, strain deformations can be readily implemented with local displacements of the lattice sites, without the need to apply pressure. While local probes such as scanning tunneling spectroscopy can fully characterize the LLs in 2D \([2, 3]\), this method can be used to characterize the surface states of the LLs in the 3D case.

In quantum Hall systems, the measurement of the Hall viscosity is typically challenging \([39]\), as it involves probing the response of the stress tensor under changes of the space metric \([20]\). In Galilean invariant systems in the hydrodynamic regime, the Hall viscosity can be determined solely in terms of the electromagnetic response due to a non-homogeneous electric field \([40, 41]\). The elastic Hall viscosity nevertheless can be measured in terms of the dispersion of sound waves. When \( \eta_H = \eta_{xxxz} \) is zero, the longitudinal and transverse modes are decoupled at long wavelengths. In the topological phase, where \( \eta_H \) is finite, the transverse and longitudinal modes are expected to mix, allowing one to measure the elastic Hall viscosity through the corrections to the dispersion of the phonons \([22]\). The quantum simulation of Chern insulating phases has been done in honeycomb lattices of cold atoms \([29]\), in quantum circuits \([12]\) and acoustic metamaterials \([19]\). We conjecture that the QAH state in 3D may be experimentally realized in synthetic lattices as well.

\textbf{Conclusions.} We have derived the elastic gauge fields that are created due to lattice deformations in the hyperhoneycomb lattice. We proposed a family of strain configurations that lead to uniform nearly flat LLs in 3D. The strain fields can be created with the application of uniform temperature gradient, driving a controllable reconstruction of the bulk states into nearly flat LLs. That raises the prospect of engineering tunable zero-field 3D QHE in metamaterials. In the topological phase, we have also shown that the components of the elastic Hall viscosity tensor in the 3D QAH state for this lattice are \( \pm \eta_H \) or \( \pm 2\eta_H \) (or zero), with \( \eta_H = \beta^2 \sqrt{3}/(8\pi a^3) \).

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