On the Existence of certain Quantum Algorithms

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Abstract
We investigate the question if quantum algorithms exist that compute the maximum of a set of conjugated elements of a given number field in quantum polynomial time. We will relate the existence of these algorithms for a certain family of number fields to an open conjecture from elementary number theory.

Keywords: Quantum algorithm, Fermat quotient, Mirimanoff polynomial.

1 Introduction
Let \( \mathbb{Q}(\Gamma)/\mathbb{Q} \) be a Galois extension with Galois group \( G \) and define \( \Gamma_{\text{max}} := \max_{\sigma \in G} |\Gamma^\sigma| \), with \( \sigma \in G \). We ask if there exists an algorithm that, given some description of \( \Gamma \), efficiently computes a \( \varphi \in G \), such that \( |\Gamma^\varphi| = \Gamma_{\text{max}} \).

The first problem we encounter in this general setting is that two conjugated elements might not be efficiently distinguishable, i.e., for \( \sigma, \rho \in G \), the difference \( |\Gamma^\sigma| - |\Gamma^\rho| \) may become very small. We will avoid this problem by defining, for a positive integer \( t \in \mathbb{N} \), the set:

\[
\text{MAX}_{\Gamma_{\text{max}},t} := \{ \sigma \in G \mid |\Gamma^\sigma/\Gamma_{\text{max}}| > 1 - 1/(\log |G|)^t \},
\]

(1)
and ask for an efficient computation of an element \( \varphi \in \text{MAX}_{\Gamma_{\text{max}},t} \), say, the number of steps being a polynomial in \( \log |G| \).

In this article, we will relate the existence of these algorithms for a certain family of number fields to an open conjecture from elementary number theory in a sense that either algorithms of this kind exist or the conjecture is true or both.

To state the conjecture, we introduce the Fermat quotient \( q_p(k) \), where \( p \) is an odd prime and \( k \in \mathbb{Z} \) with \( (k, p) = 1 \), to be the smallest integer greater or equal to 0 that satisfies the equation

\[
k^{p-1} \equiv 1 + q_p(k)p \mod p^2.
\]

(2)
Here, we are interested in the number of the first consecutive zeros of this quotient, that is
\[ \kappa_p := \min \{ q \in \mathbb{N} | q_p(q) \neq 0 \} \tag{3} \]
For a long time this integer was closely related to first case of Fermats Last Theorem, but we will not go into this here (see [2] and the references given there).

It has been shown in [2] that
\[ \kappa_p \in \mathcal{O}(\sqrt{p}) \tag{4} \]
and it is still an open question, whether this bound is tight. The following Conjecture lowers this bound for infinitely many primes:

\textbf{Conjecture 1} \textit{For all } \epsilon > 0 \textit{ there exist infinitely many primes } p \textit{ with } \kappa_p < \epsilon \sqrt{p}.

Now, let \( p \) be again an odd prime and \( \zeta_{p^2} := e^{2\pi i/p^2} \) a primitive \( p^2 \)-th root of unity. Further, denote by \( \mathbb{Q}(\Gamma_p)/\mathbb{Q} \) the real subfield of \( \mathbb{Q}((\zeta_{p^2})) \) of degree \( p \), where \( \Gamma_p \) is given by
\[ \Gamma_p := \sum_{\sigma^{p^2-1} = 1} \zeta_{p^2}^{\sigma}, \tag{5} \]
with \( \sigma \in G(\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}) \simeq (\mathbb{Z}/p^2\mathbb{Z})^\times \).

The aim of this article is to prove the following theorem:

\textbf{Theorem 1} \textit{At least one of the following is true:}
\begin{enumerate}
\item For all positive integers \( t \in \mathbb{N} \), there exist a constant \( c_t \) and a quantum algorithm that, given an odd prime \( p \), computes in \((\log p)^{c_t}\) steps and with a probability close to 1 an element of the set \( \text{MAX}\Gamma_{p,t} \).
\item For all \( \epsilon > 0 \) there exist infinitely many primes \( p \) with \( \kappa_p < \epsilon \sqrt{p} \).
\end{enumerate}

The paper is organized as follows. In the next section, a quantum algorithm is presented that attempts to compute an element of the set \( \text{MAX}\Gamma_{p,t} \) in quantum polynomial time, at least if Conjecture \( \Pi \) is false. Then, after recalling some basic facts from number theory, we will state the proof of the Theorem.
2 The Algorithm

In the following let \( p \) be an odd prime. To present the algorithm, we define a polynomial time computable function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) by

\[
f(x) := \begin{cases} 
p, & \text{if } x \equiv 0 \mod p, \\
q_p(x), & \text{else}
\end{cases}
\]

where \( q_p(x) \) denotes the Fermat quotient of the integer \( x \), defined in the last section.

(i) For the quantum part of the algorithm, we start with the state

\[
\frac{1}{p} \sum_{x=0}^{p^2-1} |x\rangle |f(x)\rangle
\]

and (ii) apply the Quantum Fourier Transform (QFT) to the first register, which leads to

\[
\frac{1}{p^2} \sum_{a,x=0}^{p^2-1} \xi_{p^2}^{ax} |a\rangle |f(x)\rangle.
\]

(iii) We now measure the system and obtain the state \( |a\rangle |s\rangle \) with probability

\[
\frac{1}{p^4} \left| \sum_{f(x)=s} \xi_{p^2}^{ax} \right|^2.
\]

(iv) If \( a \not\equiv 0 \mod p \) and \( s \not= p \), we note down the smallest nonnegative integer \( \sigma' \) that satisfies the equation

\[
\sigma' \equiv q_p(a) + s \mod p.
\]

For a constant \( c \), to be specified later, we repeat the whole process \((\log p)^c \) times and output the integer \( \sigma' \) that has occurred most frequently (in case of a tie we choose one of the “leaders” by random).

In order to obtain the desired element \( \sigma \in \mathbb{G}_p := G(\mathbb{Q}(\Gamma_p)/\mathbb{Q}) \), we define the group homomorphism \( j_p : \mathbb{Z} \rightarrow G(\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}) \) such that \( j_p(a)(\zeta_{p^2}) := \zeta_{p^2}^{1-\alpha_p} \) and set

\[
\sigma := j_p(\sigma')|_{|Q\Gamma_r}.
\]

3 Analysis

For the analysis of the algorithm, we will begin with two Lemmata stating the probabilities of the possible outcomes of step (iv) of the quantum subroutine from the last section:
Lemma 1  The probability that step (iv) produces some integer equals \(1 - \frac{1}{p} \left(2 - \frac{1}{p}\right)\).

Proof. Let \(|a⟩|s⟩\) be the state given after the measurement in step (iii) of the procedure. We will collect the probabilities for the event \(s = p\) or \(a = kp\), with \(k \in \{0, 1, \ldots, p - 1\}:

\(s = p\) and \(a \neq kp\): The probability for this event equals 0, since by definition of the function \(f\) the inner sum of equation (9) equals \(\sum_{r=0}^{p-1} r^a = 0\).

\(s = p\) and \(a = kp\): Here, the inner sum of equation (9) is equal to \(p\) and since there are also \(p\) possibilities for the integer \(k\), the probability for this event equals \(\frac{1}{p}\).

\(s \neq p\) and \(a = 0\): In this case, the value of the inner sum equals \(p - 1\), and since there are \(p\) possible values for the integer \(s\) the probability here is \(\frac{p(p-1)^2}{p^4}\).

\(s \neq p\) and \(0 \neq a = kp\): In this setting, the inner sum equals the trace of the element \(\zeta_p^k\) and is therefore equal to \(-1\). Again, there are \(p\) possible values for \(s\) and \(p - 1\) values for the integer \(k\), so that in summary this probability equals \(\frac{p(p-1)}{p^4}\).

Finally, the sum of these probabilities leads to the statement of the Lemma. \(\square\)

Lemma 2  The probability \(\Pr(σ')\), that in step (iv) the integer \(σ'\) is recorded, satisfies

\[
\Pr(σ') = \left(1 - \frac{1}{p}\right) \left(\frac{Γ_{p}(σ')}{p}\right)^2,
\]

where \(σ := j_p(σ')|Q(Γ_p)\).

Proof. Let \(w\) be an integer, with \((w - 1, p) = 1\) and \(w^{p-1} \equiv 1 \mod p^2\). Then any integer \(k \not\equiv 0 \mod p\) can be written in the form

\[
k \equiv w^{d_k}(1 - q_p(k)p) \mod p^2,
\]

where...
for some integer $d_k$. Now suppose that at the end of step (iii), we obtain a state $|a\rangle|s\rangle$, with $a \not\equiv 0 \mod p$ and $s \not= p$. It then follows that the inner sum of equation (9) equals

$$\sum_{j=1}^{p-1} \zeta_p j^d a (1-q_p(a)p) j^d (1-sp) = \sum_{j=1}^{p-1} \zeta_p j^d a (1-(q_p(a)+s)p) = \Gamma_p^\sigma,$$

with $\sigma := j_p(q_p(a)+s)|Q_p|$, by definition of the element $\Gamma_p$. Since there are $p(p-1)$ ways which lead to the same $\sigma$, the statement of the Lemma follows.

\section{Proof of the Main Theorem}

To state the proof of Theorem \ref{main} we first define the Mirimanoff polynomial

$$\gamma_p(t) := \sum_{j=1}^{p-1} \frac{t^j}{j}. \quad (14)$$

This polynomial is closely related to the Fermat quotient, since

$$\gamma_p(t) \equiv \frac{1-t^p-(1-t)^p}{p} \equiv (t-1)q_p(t-1) - t q_p(t) \mod p, \quad (15)$$

and therefore

$$\kappa_p = \min\{n > 0 | \gamma_p(n) \not\equiv 0 \mod p\}. \quad (16)$$

For an introduction Mirimanoff polynomials and their basic properties, we refer to \cite{2}.

If we denote the zeros of $\gamma_p$ modulo $p$ by $\eta_p$ it can be shown that:

\textbf{Theorem 2} \textit{There exist positive constants $c_1$ and $c_2$ such that for all primes $p$}

$$\kappa_p^2 < c_1 \eta_p < c_2 \Gamma_{\max,p}. \quad (17)$$

\textbf{Proof.} The first inequality is given by Theorem 1 in \cite{2}, while the second is shown in \cite{3}, Prop. 3.16. \hfill \Box

Now, in order to prove Theorem \ref{main} we look at the following statement:

\textbf{Statement 1} \textit{There exist positive integers $s, p_0 \in \mathbb{N}$ such that, for all primes $p > p_0$,}

$$\Gamma_{\max,p} > \frac{p}{(\log p)^{\pi}}. \quad (18)$$
It now follows immediately from Theorem [2] that Conjecture [1] is true, if Statement [1] is false. So, from now on, we will assume that Statement [1] is true. From this, Lemma [2] gives us

\[
\Pr(\sigma_{\max}') = \left(1 - \frac{1}{p}\right) \left(\frac{\Gamma_{\text{max},p}}{p}\right)^2 > \left(1 - \frac{1}{p}\right) \frac{1}{(\log p)^{2s}}.
\]

(19)

Further, we define for each \(\varphi \in G(Q(\Gamma_p)/Q)\) the real number \(\alpha_{\varphi}\) by \(|\Gamma_p^{\varphi}| = \alpha_{\varphi} \Gamma_{\text{max},p}\) and (again) \(\varphi'\) by \(\varphi = j_p(\varphi')|Q(\Gamma_p)|\).

Finally, calling the algorithm \((\log p)^k\) times, where \(k > \max\{t + 10, 12 + 4s\}\), and demanding that the difference of the expected values of \(\sigma_{\max}'\) and \(\varphi'\) lie above a certain bound,

\[
(\log p)^k \left(1 - \frac{1}{p}\right)^2 \left(\frac{\Gamma_{\text{max},p}}{p}\right)^2 - (\log p)^k \left(1 - \frac{1}{p}\right) \alpha_{\varphi}^2 \left(\frac{\Gamma_{\text{max},p}}{p}\right)^2 > (\log p)^{5+k/2},
\]

(20)

it follows that

\[
\alpha_{\varphi} < 1 - \frac{1}{(\log p)^{k-10}}.
\]

(21)

This completes the proof of Theorem [1]

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