Interaction of light with gravitational waves

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The physical properties of electromagnetic waves in the presence of a gravitational plane wave are analyzed. Formulas for the Stokes parameters describing the polarization of light are obtained in closed form. The particular case of a constant amplitude gravitational wave is worked out explicitly and it is shown that it produces a linear polarization of light.

Keywords: Gravitational waves; electromagnetic waves; Stokes parameters.

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1. Introduction

The propagation of electromagnetic waves in a gravitational field is an interesting problem in general, and it is particularly relevant to the detection of gravitational waves by interferometric methods [1] or by the polarization of the cosmic microwave background [2, 3]. Previous works on the subject started with Plebanski’s article on the scattering of electromagnetic waves by weak gravitational fields [4]. Electromagnetic waves in the field of a gravitational wave were analyzed by Mashhoon and Grishchuk [5] in a general context. Exact but purely formal solutions of Einstein’s equations for electromagnetic waves by weak gravitational fields [4]. Electromagnetic waves in the background of a gravitational field [4]. The relations between electric and magnetic field vectors \( \mathbf{E} \) and \( \mathbf{H} \), and electric displacement and magnetic induction \( \mathbf{D} \) and \( \mathbf{B} \) are the usual ones,

\[
\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M},
\]

and the Maxwell equations imply

\[
\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}, \quad \mathbf{B} = \nabla \times \mathbf{A},
\]

where the scalar and vector potentials, \( \Phi \) and \( \mathbf{A} \), satisfy the equations

\[
\Box\Phi = -4\pi \nabla \cdot \mathbf{P},
\]

\[
\Box\mathbf{A} = 4\pi(\dot{\mathbf{P}} + \nabla \times \mathbf{M}),
\]

with the Lorentz gauge condition \( \dot{\Phi} + \nabla \cdot \mathbf{A} = 0 \).

Now, for the metric (1) in particular, it follows that

\[
4\pi\mathbf{P} = \mathbf{G} \cdot \mathbf{E},
\]

\[
4\pi\mathbf{M} = \mathbf{G} \cdot \mathbf{B},
\]

where \( \mathbf{G} \) is a dyad with components:

\[
G_{ab} = \begin{pmatrix} f & g & 0 \\ g & -f & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

In flat space-time, an electromagnetic plane wave is given by \( \mathbf{E}^{(0)} = \mathbf{E} e^{-i\omega t + \mathbf{k} \cdot \mathbf{r}} \) and \( \mathbf{B}^{(0)} = \mathbf{B} e^{-i\omega t + \mathbf{k} \cdot \mathbf{r}} \), where \( \mathbf{E} \) and \( \mathbf{B} \) are constant vectors such that

\[
\omega \mathbf{B} = \mathbf{k} \times \mathbf{E}, \quad \omega \mathbf{E} = -\mathbf{k} \times \mathbf{B},
\]

\( \mathbf{k} \) is the wave vector and \( \omega = |\mathbf{k}| \) the frequency of the wave. The important point is that, if terms of second order in \( G_{ab} \) are neglected, \( \mathbf{P} \) and \( \mathbf{M} \) depend only on the unperturbed electric and magnetic fields, \( \mathbf{E}^{(0)} \) and \( \mathbf{B}^{(0)} \), and, accordingly, we can set

\[
4\pi\mathbf{P} = \mathbf{G} \cdot \mathbf{E} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}},
\]

\[
4\pi\mathbf{M} = \mathbf{G} \cdot \mathbf{B} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}}.
\]
It is now convenient to define
\[ h_{\pm}(u) = f(u) \pm ig(u), \]
so that
\begin{align*}
G \cdot E &= h_+(u)E_+ + h_-(u)E_- \\
G \cdot B &= h_+(u)B_+ + h_-(u)B_-. 
\end{align*}
(12)
(13)
where
\[ e_{\pm} = e_x \pm ie_y \]
and
\[ E_{\pm} = \frac{1}{2}(E_x \pm iE_y) \]
\[ B_{\pm} = \frac{1}{2}(B_x \pm iB_y). \]

Setting the first order corrections to the potentials in the forms
\begin{align*}
\Phi^{(1)} &= \phi(u) e^{-i\omega t + ik \cdot r}, \\
A^{(1)} &= A(u) e^{-i\omega t + ik \cdot r},
\end{align*}
it follows that
\begin{align*}
\Box \Phi^{(1)} &= -2i(\omega - k_z)\phi^\prime(u) e^{-i\omega t + k \cdot r} \\
&= -4\pi \nabla \cdot P, \quad \Box A^{(1)} = -2i(\omega - k_z)A^\prime(u) e^{-i\omega t + k \cdot r} \\
&= 4\pi(\dot{P} + \nabla \times \mathbf{M}),
\end{align*}
(14)
(15)
where the primes denote derivation with respect to \( u \). These last equations can be integrated separating + and - components:
\begin{align*}
\phi^{(1)} &= \phi_{\pm}^{(1)} + \phi_{\mp}^{(1)}, \\
A^{(1)} &= A_{\mp}^{(1)} + A_{\pm}^{(1)}.
\end{align*}

It follows that
\[ \phi_{\pm}^{(1)} = \frac{1}{(\omega - k_z)}k_z E_{\pm} H_{\pm} e^{-i\omega t + ik \cdot r}, \]
(16)
and
\[ A_{\pm}^{(1)} = \frac{1}{(\omega - k_z)} \left\{ \frac{i}{2} \left( E_{\pm} (H_{\pm}' - i\omega H_{\pm}) \mp iB_{\pm} \right) \right\} e^{-i\omega t + ik \cdot r}, \]
(17)
where we have defined
\[ H_{\pm}'(u) = h_{\pm}(u) \]
and
\[ k_{\pm} = \frac{1}{2}(k_x \pm ik_y). \]

Accordingly, the first order correction to the electric field vector can be written as the sum of two terms, \( \mathbf{E}_{\pm}^{(1)} \), such that
\[ \mathbf{E}_{\pm}^{(1)} = (E_{\pm} M_{\pm} \pm iB_{\pm} N_{\pm}) e^{-i\omega t + ik \cdot r}, \]
(18)
where
\[ M_{\pm} \equiv M_{\pm} e_{\pm} + M_{\pm z} e_z - \frac{ik_{\pm}}{\omega - k_z} H_{\pm}, \]
\[ N_{\pm} \equiv N_{\pm} e_{\pm} + N_{\pm z} e_z, \]
with
\[ M_{\mp} = -\frac{i}{2(\omega - k_z)}(H_{\mp}' - 2i\omega H_{\mp} - \omega^2 H_{\mp}) , \]
(19)
\[ M_{\pm z} = \frac{k_{\pm}}{\omega - k_z} H_{\pm}' , \]
(20)
and
\[ N_{\mp} = -\frac{i}{2(\omega - k_z)}(H_{\mp}' - i(\omega + k_z)H_{\mp}' - \omega k_z H_{\mp}) , \]
(21)
\[ N_{\pm z} = \frac{k_{\pm}}{\omega - k_z} (H_{\pm}' - i\omega H_{\pm}). \]
(22)

Define now two orthonormal vectors perpendicular to \( k \):
\begin{align*}
\epsilon_1 &= \frac{1}{k_{\perp}} e_z \times k , \\
\epsilon_2 &= \frac{1}{\omega k_{\perp}} (\omega^2 e_z - k_z k) ,
\end{align*}
(23)
where \( k_{\perp} = (k_x^2 + k_y^2)^{1/2} \), and also a circular polarization basis, which is conveniently chosen as
\[ \epsilon_{\pm} = \epsilon_2 \pm i\epsilon_1. \]
(24)

The matrix of the Stokes parameters, as defined in general in the Appendix, can be written in the form \( \mathbf{S} + \Delta \mathbf{S} \), where \( \mathbf{S} \) is the corresponding matrix in flat space-time and \( \Delta \mathbf{S} \) is the first order correction produced by the gravitational wave. Explicitly:
\[ \Delta \mathbf{S} = \left( \begin{array}{c} e_+ \cdot \mathbf{E}^{(1)} \\
\epsilon_- \cdot \mathbf{E}^{(1)} \end{array} \right) \times \left( \begin{array}{c} (e_+ \cdot \mathbf{E}^{(0)*})^*, \\
(e_- \cdot \mathbf{E}^{(0)*})^* \end{array} \right) + \text{h. c.} \]
(25)
Setting \( \Delta \mathbf{S} \equiv \Delta \mathbf{S}_+ + \Delta \mathbf{S}_- \) and using Eqs. (A.4) and (A.6) in the Appendix, it follows with some straightforward matrix algebra that
\[ \Delta \mathbf{S}_{\pm} = \frac{1}{2} \left( \begin{array}{ccc} e_+ \cdot e_\pm & e_+ \cdot e_z \\
\epsilon_- \cdot e_\pm & \epsilon_- \cdot e_z \end{array} \right) \left( \begin{array}{c} M_{\mp} \\
N_{\pm z} \end{array} \right) \]
\[ \times \left( \begin{array}{c} \epsilon_+ \cdot e_\pm \\
\epsilon_- \cdot e_\pm \end{array} \right) + \text{h. c.} \]
(26)
where, according to our previous definitions (23) and (24),
\[ e_+ \cdot e_\pm = \frac{1}{\omega k_{\perp}} (\omega \mp k_z) k_{\pm}, \]
\[ e_- \cdot e_\pm = \frac{1}{\omega k_{\perp}} (\omega \pm k_z) k_{\pm}, \]
\[ \epsilon_{\pm} \cdot e_z = \frac{k_{\perp}}{\omega}, \]
(27)
In particular, we can choose without loss of generality the coordinates system such that the vector $k$ lies in the $(x, z)$ plane, that is $k_y = 0$ and $k = \frac{1}{2}k_x$. In this case, Eq. (26) takes the simpler form:

$$\Delta S = \frac{1}{2\omega^2} \left[ \begin{pmatrix} \mp \omega - k_x & k_x \\ \pm \omega - k_x & k_x \end{pmatrix} \begin{pmatrix} M_{\pm} \\ N_{\pm} \end{pmatrix} \right] S + \text{h.c.}$$

$$\Delta S = \Delta S_+ + \Delta S_- = -\frac{i}{4(\omega - k_z)} \begin{pmatrix} 3k_z^2(H_+ + H_-) \\ -(\omega - k_z)^2 H_+ - (\omega + k_z)^2 H_- \end{pmatrix}$$

Now, in the particularly important case of unpolarized light, the averaged Stokes parameters are

$$\langle s_i \rangle = 0, \quad i = 1, 2, 3$$

and $\langle s_0 \rangle$ is just the intensity of the wave. In this case, it follows from Eq. (38) that

$$\Delta \langle s_0 \rangle = 0, \quad \Delta \langle s_1 \rangle = 0,$$

and

$$\langle s_1 \rangle + i\langle s_2 \rangle = \langle s_0 \rangle \frac{h_0}{2(\omega - k_z)} \left[ 2\omega k_z \sin \theta + i(\omega^2 + k_z^2) \cos \theta \right],$$

where $\theta = \Omega u + \alpha$. These are precisely the conditions for a light beam to be linearly polarized (as can be seen, for instance, from the definition of the Poincaré sphere; see, e.g., Born and Wolf [12]).

3. Constant amplitude gravitational wave

As an example of application of the general formula given above, consider a constant amplitude sinusoidal gravitational wave, such as one generated by a periodically varying configuration of massive bodies (see, e.g., Landau and Lifshitz [10]). Accordingly we set

$$H_\pm = h_0 e^{\mp i\Omega u + ia},$$

where $h_0$ is a real valued constant, $\Omega$ is the frequency of the wave and $\alpha$ is a constant phase. In this particular case:

$$M_\mp = \frac{i\omega^2}{2(\omega - k_z)}(\Omega \pm \omega)^2 H_\pm,$$

and

$$N_\mp = \frac{i}{2(\omega - k_z)}(\Omega \mp \omega)(\omega \pm k_z)H_\pm,$$

$$N_\pm = -ik_\pm \frac{\omega \pm \Omega}{\omega - k_z} H_\pm.$$
s_1 + is_2 = (\epsilon_+ \cdot \mathbf{E})^* (\epsilon_- \cdot \mathbf{E})
\nonumber
s_3 = \frac{1}{2} (|\epsilon_+ \cdot \mathbf{E}|^2 - |\epsilon_- \cdot \mathbf{E}|^2), \quad (A.1)

following the notation of Jackson [11] (except for a factor \sqrt{2} in the definition of \epsilon_{\pm}). This can be written in matrix form as
\nonumber
S \equiv \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix}
\nonumber
= \begin{pmatrix} \epsilon_+ \cdot \mathbf{E} \\ \epsilon_- \cdot \mathbf{E} \end{pmatrix} \begin{pmatrix} (\epsilon_+ \cdot \mathbf{E})^* \\ (\epsilon_- \cdot \mathbf{E})^* \end{pmatrix}. \quad (A.2)

Using the relations \omega \mathbf{B} = \mathbf{k} \times \mathbf{E} and \omega \mathbf{E} = -\mathbf{k} \times \mathbf{B} in combination with (23) and (24), it follows that
\nonumber
\epsilon_\pm \cdot \mathbf{E} = \frac{\omega}{k} (E_z \pm iB_z), \quad (A.3)

and therefore
\nonumber
S = \frac{2\omega^2}{k^2} \begin{pmatrix} (E_z + iB_z) \\ (E_z - iB_z) \end{pmatrix}
\nonumber\times \begin{pmatrix} (E_z + iB_z)^* \\ (E_z - iB_z)^* \end{pmatrix}. \quad (A.4)

Also
\nonumber
\mathbf{E} = \frac{\omega}{2k} \begin{pmatrix} (E_z - iB_z)\epsilon_+ + (E_z + iB_z)\epsilon_- \\ (E_z - iB_z)\epsilon_+ + (E_z + iB_z)\epsilon_- \end{pmatrix}, \quad (A.5)

and since
\nonumber
\mathbf{B} = \frac{\omega}{2k} \begin{pmatrix} - (E_z - iB_z)\epsilon_+ + (E_z + iB_z)\epsilon_- \\ - (E_z - iB_z)\epsilon_+ + (E_z + iB_z)\epsilon_- \end{pmatrix},

with a similar expression for \mathbf{B}, it follows that
\nonumber
\begin{pmatrix} E_\pm \\ \pm iB_\pm \end{pmatrix} = \frac{\omega}{4k} \begin{pmatrix} \epsilon_- \cdot \mathbf{e}_\pm \\ \pm \epsilon_- \cdot \mathbf{e}_\pm + \epsilon_+ \cdot \mathbf{e}_\pm \end{pmatrix} \begin{pmatrix} E_z + iB_z \\ E_z - iB_z \end{pmatrix}. \quad (A.6)

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