MODELING AND ANALYSIS OF
REACTIVE MULTI-COMPONENT TWO-PHASE FLOWS
WITH MASS TRANSFER AND PHASE TRANSITION
– THE ISOTHERMAL INCOMPRESSIBLE CASE

DIETER BOTHE
Technische Universität Darmstadt
D-64287 Darmstadt, Germany

JAN PRÜSS*
Martin-Luther-Universität Halle-Wittenberg
Institut für Mathematik
Theodor-Lieser-Straße 5, D-06120 Halle, Germany

ABSTRACT. Isothermal incompressible multi-component two-phase flows with mass transfer, chemical reactions, and phase transition are modeled based on first principles. It is shown that the resulting system is thermodynamically consistent in the sense that the available energy is a strict Lyapunov functional, and the equilibria are identified. It is proved that the problem is well-posed in an $L_p$-setting, and generates a local semiflow in the proper state manifold. It is further shown that each non-degenerate equilibrium is dynamically stable in the natural state manifold. Finally, it is proved that a solution, which does not develop singularities, exists globally and converges to an equilibrium in the state manifold.

1. Introduction. In this paper, we continue our mathematical studies of Maxwell-Stefan diffusion begun in the paper by the first author [1], and continued by Herberg, Meyries, Prüss, and Wilke [7]. Here we are concerned with multi-component two-phase systems in the isothermal and incompressible case, where incompressibility is modeled in a simplified manner by assuming constant total densities in each phase. We include mass transfer, phase transition, and reactions of mass-action type, but exclude Marangoni forces. Still, in this generality, this paper seems to be first to tackle the problem in a mathematical rigorous way. For general background on multicomponent diffusion we refer to the monographs of Giovangigli [6] and Krishna and Taylor [9], as well as to the papers by Krishna and Wesselingh [10] and Boudin, Grec, and Salvarini [4]. For a rigorous derivation based on a fundamental mixture theory we refer to the paper Bothe and Dreyer [2]. Background information on the continuum physical modeling of two-phase flows can be found, e.g., in [8] and [17].

In Section 2 we present the underlying physics of the model to be investigated in this paper. We recommend [2] for more details and considerable extensions, which

2010 Mathematics Subject Classification. 35Q35, 76D27, 76E17, 35R37, 35K59.

Key words and phrases. Two-phase flows, phase transition, mass transfer, chemical reactions, available energy, quasilinear parabolic evolution equations, maximal regularity, generalized principle of linearized stability, convergence to equilibria.

* Corresponding author: Jan Prüss.
go far beyond the purposes of this article. Our model depends only on a few axioms, it is fairly general. We can show that the available energy is a strict Lyapunov functional for the system, hence the model is thermodynamically consistent, and we identify the non-degenerate equilibria of the system.

Based on methods, tools and results taken from the recent monograph Prüss and Simonett [14], we analyze the problem mathematically rigorously. We are able to prove \textit{short time existence} for arbitrarily \textit{large data}, i.e. local well-posedness, and to define the generated local semiflow in a natural state manifold. We prove that non-degenerate equilibria are \textit{dynamically stable}, i.e. stable for the semiflow in the state manifold, and we can show \textit{long time existence} for \textit{small data}, as well as convergence to equilibrium if a solutions does not develop singularities, in a sense to be specified.

2. \textbf{Modeling.} In this paper $\Omega \subset \mathbb{R}^n$ is an open bounded domain with boundary $\partial \Omega$ of class $C^3$ and outer normal field $\nu$. At time $t$, $\Omega$ decomposes into two parts, the disperse phase $\Omega_1(t)$ which is assumed not to be in contact with the outer boundary $\partial \Omega$, to avoid the contact angle problem, and the continuous phase $\Omega_2(t) = \Omega \setminus \bar{\Omega}_1(t)$. So $\Gamma(t) = \partial \Omega_1(t)$ is the moving free interface. A typical initial geometry is depicted in Figure 1.

![Figure 1. A typical geometry](image)

\textbf{2.1 Balance Equations}

Consider an isothermal mixture of $N \geq 2$ species $A_k$ with molar masses $M_k > 0$ and individual mass densities $\rho_k \geq 0$ filling the container $\Omega$. If there is no surface mass, individual mass balances read as

$$\partial_t \rho_k + \text{div}_x (\rho_k u_k) = M_k r_k \quad \text{in } \Omega \setminus \Gamma,$$

$$[\rho_k (u_k - u_{\Gamma}) \cdot \nu_{\Gamma}] = 0 \quad \text{on } \Gamma,$$

$$u_k \cdot \nu = 0 \quad \text{on } \partial \Omega,$$

where $u_k$ denotes the individual velocity of component $A_k$, with $k = 1, \ldots, N$. Here $\nu_{\Gamma}$ denotes the outer normal of $\Omega_1$, $u_{\Gamma}$ the velocity of the interface, and $[\phi]$ means the jump of a quantity $\phi$ across the interface.

Of interest are the mass densities $\rho_k$. To reduce the complexity of these balance laws, we introduce the \textit{total density} $\varrho = \sum_k \rho_k$, the \textit{barycentric velocity}
\[ u = \sum_k \varrho_k u_k / \varrho, \] 
the mass fractions \( y_k = \varrho_k / \varrho \), and the concentrations \( c_k = g_k / M_k \).
Then we obtain the overall mass balance
\[ \partial_t \varrho + \text{div}_x (\varrho u) = 0 \quad \text{in } \Omega, \]
\[ [\varrho (u - u_\Gamma) \cdot \nu_\Gamma] = 0 \quad \text{on } \Gamma, \]
and \( u \cdot \nu = 0 \) on \( \partial \Omega \). Next we introduce the phase flux by means of
\[ j_\Gamma = \varrho (u - u_\Gamma) \cdot \nu_\Gamma. \]

No phase transition means \( j_\Gamma \equiv 0 \), whence \([u \cdot \nu_\Gamma] = 0 \) and \( V_\Gamma = u \cdot \nu_\Gamma \), i.e. the interface is advected with the flow. On the other hand, if phase transition occurs, we have \( j_\Gamma \neq 0 \), and \([u \cdot \nu_\Gamma] = j_\Gamma [1/\varrho] \) as well as \( V_\Gamma = [\varrho u \cdot \nu_\Gamma] / [\varrho] \). Alternatively, we may write \( V_\Gamma = u \cdot \nu_\Gamma - j_\Gamma / \varrho \), which shows that \( \Gamma \) moves according to advection and to the phase flux.

The individual mass balances become
\[ \varrho (\partial_t y_k + u \cdot \nabla x y_k) + \text{div}_x J_k = M_k r_k \quad \text{in } \Omega, \]
\[ [y_k]_{\Gamma} + [J_k \cdot \nu_\Gamma] = 0 \quad \text{on } \Gamma, \]
where the diffusive fluxes \( J_k \) are defined by
\[ J_k = \varrho_k (u_k - u), \quad k = 1, \ldots, N. \]

Note that, by definition, \( y_k \geq 0, \sum_k y_k = 1 \) and \( \sum_k J_k = 0 \). The conditions \( u_k \cdot \nu = 0 \) at the outer boundary \( \partial \Omega \) then yield by definition of \( u \) and \( J_k \)
\[ u \cdot \nu = 0, \quad J_k \cdot \nu = 0, \quad k = 1, \ldots, N \quad \text{on } \partial \Omega. \]

In a similar way, we may define the individual phase fluxes \( j_k \) by means of
\[ j_k = \varrho_k (u_k - u_\Gamma) \cdot \nu_\Gamma = y_k j_\Gamma + J_k \nu_\Gamma. \]

Observe \( \sum_k j_k = j_\Gamma \), and note that \( j_k \equiv 0 \) means that there is no mass transfer of component \( k \) between the phases. For simplicity, we will here consider the case, where each species is present in each phase, i.e. \( y_k > 0 \), and also that mass transfer takes place for all species. Otherwise the notation would be more involved without changing modeling and analysis, substantially.

To obtain a closed \textit{Class-I-Model}, one has to prescribe laws for \( u, r_k \), and most importantly for the diffusive fluxes \( J_k \), to get rid of the individual velocities \( u_k \). For the barycentric velocity \( u \) we employ a Navier-Stokes equation. This assumption leads to the system
\[
\begin{align*}
\varrho (\partial_t y_k + u \cdot \nabla x y_k) + \text{div}_x J_k &= M_k r_k & \text{in } \Omega, \\
[y_k]_{\Gamma} + [J_k \cdot \nu_\Gamma] &= 0 & \text{on } \Gamma, \\
\varrho (\partial_t u + u \cdot \nabla x u) - \text{div}_x T &= 0 & \text{in } \Omega, \\
[u]_{\Gamma} - [T \nu_\Gamma] &= \sigma H_\Gamma \nu_\Gamma & \text{on } \Gamma, \\
V_\Gamma - u \cdot \nu_\Gamma + j_\Gamma / \varrho &= 0 & \text{on } \Gamma,
\end{align*}
\]
where \( k = 1, \ldots, N \). Here \( T \) denotes the stress tensor, \( \sigma H_\Gamma \) means surface tension, \( H_\Gamma = -\text{div}_x \nu_\Gamma \) designates \((n - 1)\text{-times}\) the mean curvature of the interface \( \Gamma \), and the constant \( \sigma > 0 \) the coefficient of surface tension. In particular, in this paper we do not allow for Marangoni forces, resulting from dependence of \( \sigma \) on the composition \( y \). This will be the subject of future contributions.
In this paper, for simplicity, the viscous stress tensor \( S = T + \pi I \) is defined by the Newtonian law
\[
S = \mu_s (\nabla_x u + [\nabla_x u]^T) + \mu_b \text{div}_x u I, \quad \mu_j = \mu_j (\varrho, y),
\]
where \( \pi \) denotes the pressure. The shear viscosity \( \mu_s \) as well as the bulk viscosity \( \mu_b \) are functions of density \( \varrho > 0 \) and composition \( y \), and these functions also depend on the phase.

2.2 Interface Conditions

For simplicity, we will assume below the no-slip condition for the velocity on the interface, which means
\[
[u] = \left[ \frac{1}{\varrho} j_T \nu_T, \right]
\]
or equivalently
\[
P_T [u] = 0, \quad j_T = [u \cdot \nu_T]/\left[ \frac{1}{\varrho} \right].
\]
Here \( P_T = I - \nu_T \otimes \nu_T \) means the orthogonal projection onto the tangent bundle \( TT \), and the second equation is used to eliminate the phase flux from all equations, showing that \( j_T \) is a dummy variable. For this we need of course \( [\varrho] \neq 0 \) on \( \Gamma \), for all times \( t \).

To derive the remaining interface conditions, we assume that the interface is ideal, i.e. it does neither carry mass nor energy and has no physical properties except for surface tension. Therefore, it is sensible to assume zero dissipation of energy on the interface. Moreover, without interface mass and without interfacial slip we are allowed to define \( P_T u_T \) as \( P_T u|_\Gamma \).

The mass specific density of available energy is defined by \( e_a = |u|^2/2 + \psi \), where \( \psi = \psi (\varrho, y) \) denotes the density of free energy which depends on the phase. We define the chemical potentials \( \mu_k \) by means of \( \mu_k = \partial_{y_k} \psi \), and \( P_0 = I - e \otimes e/N \) is the orthogonal projection onto the space \( E := \{ e \}^\perp \) with \( e = [1, \ldots, 1]^T \).

Balance of available energy then reads
\[
\varrho D_t e_a + \text{div}_x ([\mu_k (J_k - T u) + M_k \mu_k r_k, \nabla_x [\mu_k (J_k - T u)] + \mu_k J_k \cdot \nu_T - T \nu_T \cdot u] = \sigma H_T V_T,
\]
were we used sum convention, and \( D_t = \partial_t + u \cdot \nabla_x \) means the Lagrangian derivative. We discuss here the jump condition across \( \Gamma \) in more detail. Employing the definition of the phase flux \( j_T \) and employing the jump conditions from (1) yields
\[
0 = [\varrho e_a (u - u_T) \cdot \nu_T + \mu_k J_k \cdot \nu_T - T \nu_T \cdot u] - \sigma H_T V_T
\]
\[
= [e_a]_\Gamma + [\mu_k (J_k \cdot \nu_T + (y_k - 1/N) j_T)] - [\mu_k (y_k - 1/N)]_\Gamma
\]
\[
- [P_T T \nu_T \cdot P_T (u - u_T)] - [T \nu_T \nu_T (u - u_T) \cdot \nu_T] - [u \cdot \nu_T] V_T j_T
\]
\[
= [\psi + \frac{1}{2 \varrho^2} j_T^2 - \mu \cdot (y - e/N) - T \nu_T \cdot \nu_T / \varrho]_\Gamma + [P_0 \mu] (J \nu_T + y j_T),
\]
since \( [J_k \cdot \nu_T + y_k j_T] = 0 \) and \( \sum_j J_k = 0 \). As these terms are independent and do not have a sign, they must vanish simultaneously. Therefore we have either \( j_T \equiv 0 \), this is the case without phase transition, or the Gibbs-Thomson law
\[
[\psi - (y - e/N) \cdot \nabla_y \psi] + [1/2 \varrho^2] j_T^2 - [T \nu_T \cdot \nu_T / \varrho] = 0
\]
must hold. So, in general, the phase flux \( j_T \) can attain arbitrary values, which accounts for phase transition and enables so-called Ostwald-ripening. Secondly,
the simplest case where all components allow for change of phase, we also have

\[ P_0[\mu] = 0, \]

In more detail this condition reads

\[ \mathbb{I}[\mu_k] = \frac{1}{N} \sum_{j=1}^{N} [\mu_j], \quad k = 1, \ldots, N. \]

Observe that, in contrast to common textbook belief, the chemical potentials are not continuous across the interface, they may jump, but they all jump in the same way. The reason for this fact are the constraints \( y \cdot e = 1 \) and \( \sum_k J_k = 0 \).

Nevertheless, the Gibbs-Thomson law above together with the jump conditions for the chemical potentials are closely related to the so-called Henry law from mass transfer theory, stating that the ratio of the one-sided limits of the concentrations is prescribed, depending on the material and on the thermodynamic state. The connection becomes clear if \( \pi/\varrho \) is taken out of the last jump term in the Gibbs-Thomson law and is combined with the specific free energy in the first jump term there, to give the Gibbs free energy. The latter equals \( \sum_k y_k \mu_k \) by the Gibbs-Duhem equation, hence the Gibbs-Thomson law together with \( P_0[\mu] = 0 \) can be rewritten as

\[ \mathbb{I}[\mu_k] + \left[ \frac{1}{2} \varrho^2 j^2 \right] - \mathbb{I}[\nu_T \cdot \nu_T/\varrho] = 0 \quad \text{for } k = 1, \ldots, N. \]

This form of the jump condition for the chemical potentials can be found, e.g., in [5]; see also [3]. But the formulation above, which combines the Gibbs-Thomson law with the continuity of \( P_0[\mu] \), is new.

The latter form of the jump condition shows that in the quiescent case \( u \equiv 0 \), hence also \( j_\Gamma \equiv 0 \), the chemical potentials are in fact continuous across the interface \( \Gamma \). In general, a dimensionless form of the latter relation shows that for many technical processes, the first jump term is dominating the left-hand side by far. Hence \( \mathbb{I}[\mu_k] = 0 \) for all \( k \) is an excellent approximation, and the latter simplifies to the above mentioned Henry law under further assumptions; cf. [3] for more details.

To ensure conservation of mass and positivity of the concentrations, for the reactions we assume at least

\[ \sum_k M_k r_k = 0, \quad c \geq 0, \quad c_i = 0 \quad \Rightarrow \quad r_i(c) \geq 0. \quad (3) \]

We emphasize that, by definition, \( y \) must belong to the standard simplex

\[ \mathbb{D} = \{ y \in \mathbb{R}^N : y_k \geq 0, \ k = 1, \ldots, N, \ (y|e) = 1 \}, \]

where \( e = [1, \ldots, 1]^T \). For further reference, we denote by

\[ \mathbb{D}^0 = \{ y \in \mathbb{D} : y_k > 0, \ k = 1, \ldots, N \}, \]

the relative interior of \( \mathbb{D} \).

The complete system is rather complicated and we restrict here to the case of constant densities \( \varrho^1, \varrho^2 > 0 \) with \( \varrho^1 \neq \varrho^2 \), being aware that this is an idealization which does not strictly apply to real mixtures but, nevertheless, allows for a first
rigorous mathematical analysis of these complicated models.

\[
\rho \left( \partial_t y_k + u \cdot \nabla_x y_k \right) + \text{div}_x J_k = M_k r_k \quad \text{in } \Omega,
\]

\[
[y_k]_{\Gamma} + [J_k \cdot \nu_{\Gamma}] = 0 \quad \text{on } \Gamma,
\]

\[
P_0 [\mu_k] = 0 \quad \text{on } \Gamma,
\]

\[
\text{div}_x u = 0 \quad \text{in } \Omega,
\]

\[
\rho \left( \partial_t u + u \cdot \nabla_x u \right) - \text{div}_x T = 0 \quad \text{in } \Omega,
\]

\[
[u]_{\Gamma} - \left[ T \nu_{\Gamma} / \rho \right] = 0 \quad \text{on } \Gamma,
\]

\[
[y_k - e/N \cdot \mu_k]_{\Gamma} + \left[ (1/2\rho^2) [\nabla_x u]^2 - [T \nu_{\Gamma} \cdot \nu_{\Gamma} / \rho] \right]_{\Gamma} = 0 \quad \text{on } \Gamma,
\]

\[
V_{\Gamma} - u \cdot \nu_{\Gamma} + j_r / \rho = 0 \quad \text{on } \Gamma,
\]

for \( k = 1, \ldots, N \), completed by the outer boundary conditions

\[
u_k = 0 \quad \text{on } \partial \Omega, \quad k = 1, \ldots, N,
\]

and by the initial data

\[
y(0) = y_0 \in D^0, \quad u(0) = u_0, \quad \text{in } \Omega.
\]

2.3 The Total Available Energy

The total available energy is given by

\[
E_a = \int_{\Omega} \rho e_a \, dx + \sigma |\Gamma|;
\]

here the second term corresponds to surface energy due to surface tension. A short computation, employing the assumptions from Section 2.2, yields

\[
\frac{d}{dt} E_a = -\int_{\Omega} \left( S : \nabla_x u - \sum_k \nabla_x \mu_k \cdot J_k - \sum_k M_k \mu_k \cdot r_k \right) \, dx.
\]

Therefore, \( E_a \) is a Lyapunov functional for the problem, if the following well-known conditions hold.

\[
S : \nabla_x u \geq 0, \quad \sum_k \nabla_x \mu_k \cdot J_k \leq 0, \quad \sum_k M_k \mu_k \cdot r_k \leq 0.
\]

The same inequalities follow from the entropy principle, cf. [2]. This relationship between the decrease of the total available energy and the increase of total entropy holds in general for closed isothermal systems.

By the constitutive law for \( S \), the first term is nonnegative provided \( \mu_s > 0 \), which we will require below. Then we even have

\[
S : \nabla_x u = 0 \quad \Leftrightarrow \quad 2D := \nabla_x u + [\nabla_x u]^T = 0.
\]

The other two inequalities will be discussed below.

For the free energy density \( \psi \) we may use any function \( \psi \in C^4(D^0) \) which is strictly convex on \( E \) in the sense that \( \nabla^2 \psi(y) \) is positive definite on \( E \), for each \( y \in D^0 \), as long as no reactions are involved. If reactions are present, \( \psi \) is adapted.
to the reactions. For the class of reactions with mass-action kinetics, we will employ the classical free energies

\[ \psi(y) = \kappa \sum_{j=1}^{N} \frac{y_j}{M_j} \left( \log(y_j/y_j^0) - 1 \right), \]

where \( \kappa > 0 \) is constant, and \( y_j^0 = [y_j^0] \in \mathbb{D}^0 \) is a solution to \( r(c_\infty) = 0 \), \( c_\infty = qM^{-1}y_\infty \) with \( M = \text{diag}(M_j) \); both constants will depend on the phases, in general. Observe that in this case we have \( \mu_k = (\kappa/M_k) \log(y_k/y_k^0) \), hence \( \nabla^2 y \psi = \kappa \text{diag}(M_k^{-1}y_k^{-1}) \), which is obviously positive definite, for each \( y \in \mathbb{D}^0 \).

### 2.4 Modeling Diffusion: The Maxwell-Stefan Approach

To model the diffusion fluxes \( J_k \), following James Clerk Maxwell and Josef Stefan in their pioneering papers [13] and [18], a balance of so-called driving forces \( d_k \) and friction forces \( f_k \) is postulated, i.e., \( d_k = f_k \). The friction forces are given by

\[ f_k = \sum_{j \neq k} f_{kj} y_k y_j (u_j - u_k) = \sum_{j \neq k} f_{kj} (y_k J_j - y_j J_k), \]

with symmetric friction coefficients \( f_{kj} = f_{jk} > 0 \). These coefficients may depend on the composition and on the phase. Observe that \( \sum_k f_k = 0 \), so that the friction forces are internal forces which act only on the components but not on the mixture. In the sequel we assume **Condition (F)**

\[ 0 < f_{kj} = f_{jk} \in C^{2-}(\mathbb{D}^0), \quad k, j = 1, \ldots, N, \]

for the friction coefficients.

The driving forces \( d_k \) are related to the gradients in the chemical potentials \( \mu_k \) and the pressure. In the considered isothermal case and without body forces, \( d_k = y_k(\nabla x \mu_k - \frac{1}{\rho} \nabla x p) \); cf. [2]. In this case, \( \frac{1}{\rho} \nabla x p = \sum_j y_j \nabla x \mu_j \) as a consequence of the Gibbs-Duhem equation. We therefore let

\[ d_k = y_k(\nabla x \mu_k - \sum_{j=1}^{N} y_j \nabla x \mu_j); \]

note that \( \sum_k d_k = 0 \) as required. The force balances \( d_k = f_k \) lead to the **Maxwell-Stefan equations** for the columns \( J^\alpha \in \mathbb{R}^N \) of the flux matrix \( J = [J_1, \ldots, J_N]^T \in \mathbb{R}^{N \times n} \). These equations read as follows:

\[ B(y) J^\alpha = d^\alpha, \quad \alpha = 1, \ldots, n, \quad B(y) = [b_{ij}(y)], \]

\[ b_{ij}(y) = y_i f_{ij} \quad \text{for } i \neq j, \quad b_{ii}(y) = - \sum_{j \neq i} f_{ij} y_i, \quad i, j = 1, \ldots, N. \]

The Maxwell-Stefan equations have to be inverted to obtain the fluxes \( J_k \). For this observe that \( B(y) \) is not invertible, we have to take into account the side conditions

\[ (y|e) = 1, \quad (J^\alpha|e) = 0, \quad (d^\alpha|e) = 0. \]

For further use we set \( P(y) = I - y \otimes e = I - y(e|\cdot) \); note that \( P^T = I - e \otimes y \), as well as \( R(P(y)) = \mathbb{E} \) and \( N(P(y)) = \text{span}\{y\} \).

The **Perron-Frobenius theorem** yields the following result.

**Lemma 2.1.** For any \( y \in \mathbb{D}^0 \) we have

(i) the matrix \( B(y) = [b_{ij}(y)] \) is irreducible and quasi-positive;

(ii) the kernel of \( B(y) \) is \( N(B(y)) = \text{span}\{y\} \);

(iii) the range of \( B(y) \) is \( R(B(y)) = \mathbb{E} = \{e\}^\perp \).
In particular, \( B \) in the sequel its inverse is denoted by \( Y^{-1/2}B(y)Y^{1/2} \) is symmetric and negative definite on \( Y^{-1/2} \mathbb{E} \).

For a proof we refer to Bothe [1] or Giovangigli [6] or to the more recent paper [7].

Therefore the restriction of \( B(y) \) to \( \mathbb{E} = \{ e \}^+ \) is invertible, for all \( y \in \mathbb{D}^0 \), and in the sequel its inverse is denoted by \( B(y) = (B(y)|\mathbb{E})^{-1} \); then \( J^\alpha = A(y)\alpha^\alpha, \alpha = 1, \ldots, n \).

We have with \( Y := \operatorname{diag}(y) \) and \( B_s(y) = Y^{-1/2}B(y)Y^{1/2} \):

\[
\sum_{k=1}^N (\nabla x \mu_k | J_k) = \sum_{\alpha=1}^n (\partial x_\alpha \mu | J^\alpha) = \sum_{\alpha=1}^n (Y^{-1}d^\alpha | J^\alpha) = \sum_{\alpha=1}^n (B(y)J^\alpha | Y^{-1}J^\alpha) = \sum_{\alpha=1}^n (B_s(y)Y^{-1/2}J^\alpha | Y^{-1/2}J^\alpha) \leq 0,
\]

as \( B_s(y) \) is symmetric and negative semi-definite; see e.g. [1]. Moreover, equality holds if and only if \( J^\alpha = 0 \) and equivalently \( d^\alpha = 0 \) for all \( \alpha \). Further, with

\[
d^\alpha = Y(\partial x_\alpha \mu - e(y)\partial x_\alpha \mu) = YP_\alpha(y)^2 \psi(y)\partial x_\alpha y = P(y)Y \nabla_y^2 \psi(y)P(y)\partial x_\alpha y,
\]

we see that \( d^\alpha = 0 \) for all \( \alpha \) is equivalent to \( \nabla x y = 0 \), provided

\[
\nabla_y^2 \psi(y) \quad \text{is positive definite on} \quad \mathbb{E}, \quad \text{for all} \quad y \in \mathbb{D}^0.
\]

Then we have

\[
\sum_{k=1}^N (\nabla x \mu_k | J_k) \leq 0,
\]

with equality if and only if, equivalently, \( \nabla x \mu = 0 \) or \( J = 0 \) or \( \nabla x y = 0 \).

### 2.5 Modeling Chemistry: Mass-Action Kinetics

Next we consider the chemistry more closely. For \textit{reversible mass-action kinetics} the reaction rate \( r \) is given by

\[
r(y) = \nu \psi(c) = \sum_{l=1}^m \nu_l r_l(c), \tag{9}
\]

where \( c = \varrho M^{-1} y, \) and

\[
\nu = [\nu_1, \ldots, \nu_m] = [\nu_{j+}, \nu_{j-}] = [\nu_{jl}^+, \nu_{jl}^-] \in \mathbb{Z}^{N \times m}, \quad \nu_{jl}^+, \nu_{jl}^- \in \mathbb{N}_0
\]

is the \textit{stoechiometry} of the reactions. The individual reaction rates in terms of the concentrations \( c \) are given by

\[
r_l(c) = -k^l_+ c_l^+ + k^-_l c_l^- , \quad l = 1, \ldots, m,
\]

where \( c^\alpha := \prod_{l=1}^N c_{l}^\alpha \) for \( c \in \mathbb{R}_+^N, \alpha \in \mathbb{N}_0^N \). The \textit{stoechiometric subspace} \( \mathbb{S} \subset \mathbb{R}^N \) is defined by

\[
\mathbb{S} := R(\nu) = \text{span}\{\nu_1, \ldots, \nu_m\}, \quad s := \text{dim} \mathbb{S}.
\]
Throughout we make the following Assumption (R).

\[ Me \in S^+, \quad k^+_l, k^-_l > 0, \]
\[ r(c_*) = 0, \quad c_* = \rho M^{-1} y_*, \quad \text{for some } y_* \in D^0, \quad (10) \]
\[ \psi(y) = \kappa \sum_{j=1}^{N} \frac{y_j}{M_j} \left( \log\left( \frac{y_j}{y_j^\circ} \right) - 1 \right). \]

Let us note in passing that the first condition is always fulfilled for a physical system with conservation of atoms. Note also that \( \kappa \) and \( k^{\pm}_l \) in general will depend on the phases, and we could allow for \( k^{\pm}_l = 0 \) in one of the phases. But for simplicity in presentation, we will not do this here. On the other hand, the stoechiometric vectors \( \nu_l \) are independent of the phase. But observe that we only consider reversible reactions, as \( k^{\pm}_l > 0 \). The second hypothesis in (R) implies that the set of chemical equilibria

\[ \mathcal{E}_c = \{ y_* \in D^0 : r(c_*) = 0, \quad c_* = \rho M^{-1} y_* \} \]

is nonempty, in each of the phases. This assumption is physically reasonable, it is discussed in more detail in [7]. In the sequel, we fix an arbitrary chemical equilibrium \( y_\circ = [y^\circ_k] \in \mathcal{E}_c \). Then, with the above choice of the free energy \( \psi \), the chemical potentials are given by

\[ \mu_k(y) = \frac{\kappa}{M_k} \log\left( \frac{y_k}{y^\circ_k} \right), \quad y \in D^0, \quad k = 1, \ldots, N, \]

where the constant \( \kappa > 0 \) may depend on the phase. Then we have

**Lemma 2.2. Assume (R). Then the set \( \mathcal{E}_c \) of chemical equilibria forms an \((N - s - 1)\)-dimensional real analytic surface contained in \( D^0 \). At \( y \in \mathcal{E}_c \) the tangent space of \( \mathcal{E}_c \) is given by \( T_y \mathcal{E}_c = \mathbb{N}(\nu^T Y^{-1}) \cap \mathbb{E} \). If \( s = N - 1 \) then \( \mathcal{E}_c \subset D^0 \) is discrete.**

For a proof we refer to [7].

A short computation shows that for mass action kinetics

\[ r_l(c) = 0 \quad \Leftrightarrow \quad c^{\nu_l} = K_l := k^-_l / k^+_l. \]

This yields

\[ \sum_{k=1}^{N} \mu_k M_k r_k = \kappa \sum_{k=1}^{N} \sum_{l=1}^{m} \log(c_k/c^\circ_k) \nu_l k^-_l c^{\nu_l} - k^+_l c^{\nu_l} \]
\[ = \kappa \sum_{l=1}^{m} k^+_l c^{\nu_l} \left( \log(c^{\nu_l}) - \log(c^\circ_{\circ}) \right) \left( c^\circ_{\circ} - c^{\nu_l} \right) \leq 0, \]

and equality implies \( c^{\nu_l} = c^\circ_{\circ} = K_l \) for each \( l = 1, \ldots m \) as \( \log \) is strictly monotone. This shows that \( y \in \mathcal{E}_c \). So we see that

\[ \sum_{k=1}^{N} \mu_k M_k r_k \leq 0, \]

with equality if and only if \( y \in \mathcal{E}_c \).

**2.6 The Available Energy and Equilibria**

We have seen above that the total available energy

\[ E_a = \int_{\Omega} \rho e_a dx + \sigma |\Gamma| \]
is a Lyapunov functional for the system, as
\[
\frac{d}{dt} E_a = - \int_{\Omega} (S : \nabla u - \sum_k \nabla x \mu_k \cdot J_k - \sum_k M_k \mu_k \cdot r_k) dx
\]
and, by the constitutive assumptions,
\[
S : \nabla u \geq 0, \quad \sum_{k=1}^N \nabla x \mu_k \cdot J_k \leq 0, \quad \sum_{k=1}^N M_k \mu_k r_k \leq 0. \tag{11}
\]
We want to show that $E_a$ is even a strict Lyapunov functional. For this purpose, suppose that along a smooth solution, $E_a(t)$ is not strictly decreasing, but constant on some time interval $(t_1, t_2)$, hence $dE_a/dt = 0$ on $(t_1, t_2)$. Note that by a smooth solution we mean in particular that it is non-degenerate, i.e. at each time instant $\Gamma(t)$ is a $C^2$-manifold which does not touch the outer boundary. Then we know from before $D = 0$, $\nabla_x y = d = J = 0$, and $r(c) = 0$ in $(t_1, t_2) \times \Omega$. By a variant of Korn’s inequality, see Lemma 1.2.1 in [14], this yields $u = 0$ for $t \in (t_1, t_2)$, hence $\partial_t u = 0$ on $(t_1, t_2)$, and so $\nabla \pi = 0$, as well as $j_\Gamma = 0$, and $V_\Gamma = 0$. Therefore, the pressure $\pi$ is constant in the components of the phases, and the curvature $H_{\Gamma}$ is constant on the components of $\Gamma$, as well. As $\Omega$ is bounded, this implies that $\Gamma$ consists of finitely many spheres.

Moreover, the jump condition $[P_0[\mu]] = 0$ on $\Gamma$ shows that the values of $y$ on either side of $\Gamma$ determine each other, hence the compositions $y_k$ are constant even throughout the phases. The Gibbs-Thomson condition then shows that the same is valid for the pressures, and so $H_{\Gamma}$ is constant all over $\Gamma$, which implies that the spheres all have the same radius. In particular, the continuous phase is connected.

Therefore, we are at an equilibrium, and we also found all non-degenerate equilibria. These are given by the manifold $\mathcal{E}$, defined by
\[
z = (y, u, \Gamma) \in \mathcal{E} \quad \Leftrightarrow \quad u = 0, \quad y \in \mathcal{E}_c \text{ constant in } \Omega, \quad P_0[\mu] = 0, \quad \Gamma = \bigcup_i S_R(x_i). \tag{12}
\]
The interfacial pressure jumps are determined by the equations
\[
[\pi] = \sigma H_{\Gamma} = -\frac{\sigma(n-1)}{R}, \quad [\pi/\rho] = -[\psi - (y-e/N) \cdot \mu].
\]
We emphasize again that we exclude equilibria which have boundary contact, as these would involve the contact angle problem which is not a subject in this paper.

Summarizing, we have shown the following result.

**Theorem 2.3.** Assume that condition (R) holds. Then
(a) the total mass is preserved along smooth solutions;
(b) the total available energy $E_a$ is a strict Lyapunov functional;
(c) the non-degenerate equilibria are zero velocity $u = 0$, constant mass fractions $y \in \mathcal{E}_c$ in the phases, satisfying $P_0[\mu] = 0$, and the interface is a finite union of non-intersecting spheres of equal size, which do not touch the outer boundary $\partial \Omega$;
(d) the continuous phase $\Omega_2$ is connected.
In particular, the model (4) is thermodynamically consistent.

Observe that in the case of no phase transition, i.e. $j_\Gamma \equiv 0$, there is no Ostwald ripening, the spheres may have arbitrary radii, the pressures may depend on the components of the phases, and $\Omega_2$ may be disconnected. Physically, this would be unrealistic, and so we take into account phase transition throughout this paper.
3. **Analysis.** Employing all modeling assumptions, and using sum convention, system (4) can be written as

\[
\varrho(\partial_t y + u \cdot \nabla_x y) - \partial_{x_a}(\mathcal{A}(y)\partial_{x_a} y) = M\nu_t r_t(c) \quad \text{in } \Omega \setminus \Gamma,
\]

\[
[y]_{\Gamma} - [\mathcal{A}(y)\partial_{x_a} y] = 0 \quad \text{on } \Gamma,
\]

\[
P_0[\mu(y)] = 0 \quad \text{on } \Gamma,
\]

\[
div_x u = 0 \quad \text{in } \Omega \setminus \Gamma,
\]

\[
\varrho(\partial_t u + u \cdot \nabla_x u) - \div_x S + \nabla_x \pi = 0 \quad \text{in } \Omega \setminus \Gamma,
\]

(13)

with \(S = \mu_s(y)(\nabla_x u + [\nabla_x u]^T)\) and \(\mu = \nabla_y \psi(y)\). The matrix \(\mathcal{A}(y)\) is given by

\[
\mathcal{A}(y) = -\kappa \mathcal{A}(y)YP_T(y)B(y), \quad B(y) = \nabla^2 \psi(y), \quad \mathcal{A}(y) = [B(y)]^{-1}
\]

with \(\kappa\) from the definition of \(\psi\). This is a quasilinear evolution problem with moving boundary, which is of parabolic type. Therefore, we can apply the techniques and tools presented in the monograph Prüss and Simonett [14] for its analysis.

This means that for local well-posedness as well as for stability of equilibria we use the direct mapping method, reducing the problem by means of a Hanzawa transform to a quasilinear parabolic system on a fixed domain. The generated semiflow will be defined in the original variables, and the global results are proved applying dynamical system methods involving the available energy as a strict Lyapunov functional and the local stability result.

### 3.1 Transformation to a Fixed Domain

A basic idea is to transform the problem to a domain with a fixed interface \(\Sigma\), where \(\Gamma(t)\) is parameterized over \(\Sigma\) by means of a height function \(h(t)\). For this purpose one may employ the so-called **Hanzawa transform** which will be explained below. For the necessary geometric background and more details we refer to the monograph by Prüss and Simonett [14], Chapter 2.

#### (a) The Hanzawa Transform

Assume that \(\Omega \subset \mathbb{R}^n\) is a bounded domain with boundary \(\partial \Omega\) of class \(C^2\) and that \(\Gamma \subset \Omega\) is a hypersurface of class \(C^2\), i.e. a \(C^2\)-manifold which is the boundary of a bounded domain \(\Omega_1 \subset \Omega\). As above, one sets \(\Omega_2 = \Omega \setminus \Omega_1\). Note that \(\Omega_2\) typically is connected, while \(\Omega_1\) may be disconnected. In the later case, \(\Omega_1\) consists of finitely many components, since \(\partial \Omega_1 = \Gamma \subset \Omega\) by assumption is a manifold, at least of class \(C^2\). Recall that the second normal bundle \(\mathcal{N}^2\Gamma\) is defined by

\[
\mathcal{N}^2\Gamma = \{ (p, \nu_T(p), L_T(p)) : p \in \Gamma \}.
\]

Here \(L_T = -\nabla_T \nu_T\) denotes the Weingarten tensor of the manifold \(\Gamma\). The hypersurface \(\Gamma\) can be approximated by a real analytic hypersurface \(\Sigma\), in the sense that the Hausdorff distance of the second order normal bundles is as small as we please. More precisely, given \(\eta > 0\), there exists an analytic hypersurface \(\Sigma\) such that \(d_H(\mathcal{N}^2\Sigma, \mathcal{N}^2\Gamma) \leq \eta\). If \(\eta > 0\) is small enough, then \(\Sigma\) bounds a domain \(\Omega^\Sigma_1\).
with $\Omega^0_1 \subset \Omega$ and then we set $\Omega^0_2 = \Omega \setminus \Omega^0_1 \subset \Omega$. The $C^2$-hypersurface $\Sigma$ admits a tubular neighborhood, which means that there is $a_0 > 0$ such that the map
\[ \Lambda : \Sigma \times (-a_0, a_0) \to \mathbb{R}^n, \quad \Lambda(p, r) := p + r\nu_{\Sigma}(p) \]
is a diffeomorphism from $\Sigma \times (-a_0, a_0)$ onto $\text{im}(\Lambda)$, the image of $\Lambda$. The inverse
\[ \Lambda^{-1} : \text{im}(\Lambda) \to \Sigma \times (-a_0, a_0) \]
of this map is conveniently decomposed as
\[ \Lambda^{-1}(x) = (\Pi_{\Sigma}(x), d_{\Sigma}(x)), \quad x \in \text{im}(\Lambda). \]
Here $\Pi_{\Sigma}(x)$ means the metric projection of $x$ onto $\Sigma$ and $d_{\Sigma}(x)$ the signed distance from $x$ to $\Sigma$: so $|d_{\Sigma}(x)| = \text{dist}(x, \Sigma)$ and $d_{\Sigma}(x) < 0$ if and only if $x \in \Omega^0_1$. In particular, $\text{im}(\Lambda) = \{ x \in \mathbb{R}^n : \text{dist}(x, \Sigma) < a_0 \}$. The maximal number $a_0$ is given by the radius $r_\Sigma > 0$, defined as the largest number $r$ such the exterior and interior ball conditions for $\Sigma$ in $\Omega$ holds. In the following, let
\[ a_0 = r_\Sigma/2 \quad \text{and} \quad a = a_0/3. \]
The derivatives of $\Pi_{\Sigma}(x)$ and $d_{\Sigma}(x)$ are given by
\[ \nabla_x d_{\Sigma}(x) = \nu_{\Sigma}(\Pi_{\Sigma}(x)), \quad \Pi'_x \Sigma(x) = M_0(d_{\Sigma}(x))\mathcal{P}_\Sigma(\Pi_{\Sigma}(x)), \]
where, as before, $\mathcal{P}_\Sigma(p) = I - \nu_{\Sigma}(p) \otimes \nu_{\Sigma}(p)$ denotes the orthogonal projection onto the tangent space $T_p\Sigma$ of $\Sigma$ at $p \in \Sigma$, and $M_0(r) = (I - rL_\Sigma)^{-1}$, with $L_\Sigma$ the Weingarten tensor. Then
\[ |M_0(r)| \leq 1/(1 - r|L_\Sigma|) \leq 3 \quad \text{for all} \quad |r| \leq 2r_\Sigma/3. \]
If $\text{dist}(\Gamma, \Sigma)$ is small enough, the map $\Lambda$ can be used to parameterize the unknown free boundary $\Gamma(t)$ over $\Sigma$ by means of a height function $h(t)$ via
\[ \Gamma(t) = \{ p + h(t, p)\nu_{\Sigma}(p) : p \in \Sigma \}, \quad t \geq 0, \]
for small $t \geq 0$, at least. Extend this diffeomorphism to all of $\Omega$ by means of
\[ \Xi_h(t, x) = x + \chi(d_{\Sigma}(x)/a)h(t, \Pi_{\Sigma}(x))\nu_{\Sigma}(\Pi_{\Sigma}(x)) =: x + \xi_h(t, x). \]
Here $\chi$ denotes a suitable cut-off function. More precisely, let $\chi \in \mathcal{D}(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(r) = 1$ for $|r| < 1$, and $\chi(r) = 0$ for $|r| > 2$; $\chi$ may be chosen in such a way that $1 < |\chi'|_{\infty} \leq 3$. Note that $\Xi_h(t, x) = x$ for $|d_{\Sigma}(x)| > 2a$, and
\[ \Pi_{\Sigma}(\Xi_h(t, x)) = \Pi_{\Sigma}(x), \quad |d_{\Sigma}(x)| < a, \]
as well as
\[ d_{\Sigma}(\Xi_h(t, x)) = d_{\Sigma}(x) + \chi(d_{\Sigma}(x)/a)h(t, \Pi_{\Sigma}(x)), \quad |d_{\Sigma}(x)| < 2a. \]
This yields
\[ \Xi^{-1}_h(t, x) = x - h(t, \Pi_{\Sigma}(x))\nu_{\Sigma}(\Pi_{\Sigma}(x)) \quad \text{for} \quad |d_{\Sigma}(x)| < a, \]
in particular
\[ \Xi^{-1}_h(t, x) = x - h(t, x)\nu_{\Sigma}(x) \quad \text{for} \quad x \in \Sigma. \]
Furthermore,
\[ \partial \Xi_h = I + \partial \xi_h, \quad (\partial \Xi_h)^{-1} = I - [I + \partial \xi_h]^\top \partial \xi_h =: I - M'^T_1(h), \]
where $\partial := \partial_x$ denotes the derivative with respect to $x \in \mathbb{R}^n$, and
\[ \partial \xi_h(t, x) =
\nu_{\Sigma}(\Pi_{\Sigma}(x)) \otimes M_0(d_{\Sigma}(x))\nabla_{\Sigma} h(t, \Pi_{\Sigma}(x)) - h(t, \Pi_{\Sigma}(x))M_0(d_{\Sigma}(x))L_{\Sigma}(\Pi_{\Sigma}(x)) \]
\[ \otimes M_0(d_{\Sigma}(x))\nabla_{\Sigma} h(t, \Pi_{\Sigma}(x)) - h(t, \Pi_{\Sigma}(x))M_0(d_{\Sigma}(x))L_{\Sigma}(\Pi_{\Sigma}(x)) \]
for \(|d_\Sigma(x)| < a\), \(\xi'_h(t, x) = 0\) for \(|d_\Sigma(x)| > 2a\), and in general

\[
\partial \xi_h(t, x) = \frac{1}{a} \chi'(d_\Sigma(x)/a) h(t, \Pi_\Sigma(x)) \nu_\Sigma(\Pi_\Sigma(x)) \otimes \nu_\Sigma(\Pi_\Sigma(x)) \\
+ \chi(d_\Sigma(x)/a) \nu_\Sigma(\Pi_\Sigma(x)) \otimes M_0(d_\Sigma(x)) \nabla \Sigma h(t, \Pi_\Sigma(x)) \\
- \chi(d_\Sigma(x)/a) h(t, \Pi_\Sigma(x)) M_0(d_\Sigma(x)) L_\Sigma(\Pi_\Sigma(x)).
\]

It is a matter of simple algebra to determine the inverse of \(\partial \Xi_h\), to the result

\[
(\partial \Xi_h(t, x))^{-1} = I - \left(\chi h L_\Sigma - \frac{\chi'h/a}{1 + \chi'h/a} \nu_\Sigma \otimes \nu_\Sigma - \frac{\chi}{1 + \chi'h/a} \nu_\Sigma \otimes \nabla \Sigma h\right) M_0(d_\Sigma + ch).
\]

where obvious variables have been dropped. This implies

\[
M_1(h) = \chi M_0(d_\Sigma + ch) \left(\frac{\nabla \Sigma h \otimes \nu_\Sigma}{1 + \chi'h/a} - h L_\Sigma\right) + \frac{\chi'h/a}{1 + \chi'h/a} \nu_\Sigma \otimes \nu_\Sigma.
\]

Note that \(M_1(h)\) depends linearly on \(\nabla \Sigma h\). On the interface we then have

\[
M_1(h) = M_0(h)(\nabla \Sigma h \otimes \nu_\Sigma - h L_\Sigma).
\]

In particular, \(\partial \Xi_h\) is invertible, provided \(M_0(d_\Sigma + ch) = (I - (d_\Sigma + ch)L_\Sigma)^{-1}\) exists, and \(1 + \chi'h/a > 0\). This certainly holds if

\[
|d_\Sigma + ch| L_\Sigma \leq 2/3 \quad \text{and} \quad |\chi'|_\infty h/a < 1/2,
\]

which leads to the restriction \(|h|_\infty \leq h_\infty := a/|\chi'|_\infty\); note that \(|\chi'|_\infty > 1\). Observe that at this place no restrictions on \(\nabla \Sigma h\) are required.

Next,

\[
\partial_\nu \Xi_h(t, x) = \chi(d_\Sigma(x)/a) \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \Omega,
\]

hence the relation \(\Xi_h^{-1}(t, \Xi_h(t, x)) = x\) implies

\[
\partial_\nu \Xi_h^{-1}(t, \Xi_h(t, x)) = -m_0(h) \partial_\nu h(t, \Pi_\Sigma(x)) \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \Omega,
\]

where

\[
m_0(h)(t, x) = \frac{\chi(d_\Sigma(x)/a)}{(1 + h(t, \Pi_\Sigma(x)) \chi'(d_\Sigma(x)/a)}.
\]

With the Weingarten tensor \(L_\Sigma\) and the surface gradient \(\nabla \Sigma\), we have

\[
\nu_\nu(h) = \beta(h)(\nu_\Sigma - a(h)), \quad a(h) = M_0(h) \nabla \Sigma h, \\
M_0(h) = (I - h L_\Sigma)^{-1}, \quad \beta(h) = (1 + |a(h)|^2)^{-1/2},
\]

and

\[
V_\nu = \partial_\nu \Xi_h \cdot \nu_\nu = (\nu_\Sigma \cdot \nu_\nu) \partial_\nu h = \beta(h) \partial_\nu h.
\]

Finally, the mean curvature (times \((n - 1)\)) \(H_\nu(h)\) is given by

\[
H_\nu(h) = \beta(h) \{\text{tr}[M_0(h)(L_\Sigma + \nabla \Sigma a(h))] - \beta^2(h)(M_0(h)a(h)[\nabla \Sigma a(h)]a(h))\},
\]

a differential expression involving second order derivatives of \(h\) only linearly. Hence

\[
H_\nu(h) = C_0(h) : \nabla_\Sigma^2 h + C_1(h),
\]

where \(C_0(h)\) and \(C_1(h)\) depend on \(h\) and \(\nabla \Sigma h\), provided \(|h| \leq h_\infty\) holds. The linearization of \(H_\nu(h)\) at \(h = 0\) is given by

\[
H_\nu'(0) = \text{tr} L_\Sigma^2 + \Delta \Sigma.
\]

Here \(\Delta \Sigma\) denotes the Laplace-Beltrami operator on \(\Sigma\).
(b) The Transformed Problem

We define the transformed quantities by

\[ \tilde{y}(t, x) = y(t, \Xi_h(t, x)), \quad \tilde{u}(t, x) = u(t, \Xi_h(t, x)) \quad \text{in} \quad \Omega \setminus \Sigma, \]

\[ \tilde{\pi}(t, x) = \pi(t, \Xi_h(t, x)), \quad \tilde{\mu}(t, x) = \mu(\tilde{y}(t, x)) \quad \text{in} \quad \Omega \setminus \Sigma, \quad (14) \]

\[ j_{\Sigma}(t, p) = j_{\Gamma}(t, \Xi_h(t, p)) \quad \text{on} \quad \Sigma, \]

the pull back of \((y, u, \pi, j_{\Gamma})\). This way, the time varying regions \(\Omega \setminus \Gamma(t)\) are transformed to the fixed region \(\Omega \setminus \Sigma\). This transforms the general problem (13) to the following problem for \((\tilde{y}, \tilde{u}, \tilde{\pi}, j_{\Sigma}, h)\).

\[ \partial_t \tilde{y} - \mathcal{G}(h) \cdot \Lambda(\tilde{y}) \mathcal{G}(h) \tilde{y} = \mathcal{R}_y(\tilde{u}, \tilde{y}, h) \quad \text{in} \quad \Omega \setminus \Sigma, \]

\[ [\tilde{y}]_{\Sigma} - [\Lambda(\tilde{y}) \nu_T \cdot \mathcal{G}(h) \tilde{y}] = 0 \quad \text{on} \quad \Sigma, \]

\[ P_h[\tilde{\mu}(\tilde{y})] = 0 \quad \text{on} \quad \Sigma, \]

\[ \tilde{\theta} - \mathcal{G}(h) \cdot \tilde{\bar{S}} + \mathcal{G}(h) \tilde{\pi} = \tilde{\theta} \mathcal{R}_u(\tilde{u}, \tilde{\theta}, h) \quad \text{in} \quad \Omega \setminus \Sigma, \]

\[ \mathcal{G}(h) \cdot \tilde{\nu} = 0 \quad \text{in} \quad \Omega, \quad (15) \]

\[ \tilde{u} = 0 \quad \text{on} \quad \partial \Omega, \]

\[ \left[1/\tilde{\theta}\right]_{\Sigma}^2 \nu_T(h) - \left[\tilde{S} \nu_T(h)\right] + \left[\tilde{\pi}\right] \nu_T(h) = \sigma H_\Sigma(h) \nu_T(h) \quad \text{on} \quad \Sigma, \]

\[ \left[\tilde{u}\right] - \left[1/\tilde{\theta}\right]_{\Sigma}^2 \nu_T(h) = 0 \quad \text{on} \quad \Sigma, \]

\[ \tilde{y}(0) = \tilde{\theta}_0, \quad \tilde{u}(0) = \tilde{u}_0, \]

where

\[ \tilde{S} = \mu_s(\tilde{y})(\mathcal{G}(h) \tilde{u} + [\mathcal{G}(h) \tilde{u}]^T). \]

Here \(\mathcal{G}(h)\) denotes the transformed gradient. More precisely, the relations

\[ \nabla_x \pi \circ \Xi_h = \mathcal{G}(h) \tilde{\pi} = [(\partial \Xi_h^{-1})^T \circ \Xi_h] \nabla_x \tilde{\pi} = (I - \mathcal{M}_1(h)) \nabla_x \tilde{\pi} \]

and

\[ \nabla_x y_k \circ \Xi_h = (I - \mathcal{M}_1(h)) \nabla_x \tilde{y}_k, \]

as well as

\[ (\nabla_x \cdot u) \circ \Xi_h = (\mathcal{G}(h) \nu \tilde{u} = ((I - \mathcal{M}_1(h)) \nabla_x \tilde{u}), \]

are valid, and

\[ \partial_x u \circ \Xi_h = \partial_x \tilde{u} + \partial_x [\partial_x \Xi_h^{-1} \circ \Xi_h] = \partial_x \tilde{u} - \mathcal{M}_0(h) \partial_x h(\nu_\Sigma \cdot \nabla_x) \tilde{u}. \]

Therefore,

\[ \mathcal{R}_u(\tilde{u}, \tilde{y}, h) = -\tilde{u} \cdot \mathcal{G}(h) \tilde{u} + \mathcal{M}_0(h) \partial_x h(\nu_\Sigma \cdot \nabla_x) \tilde{u} \]

and, similarly,

\[ \partial_x y_k \circ \Xi_h = \partial_x \tilde{y}_k - \mathcal{M}_0(h) \partial_x h(\nu_\Sigma \cdot \nabla_x) \tilde{y}_k, \]

and so

\[ \mathcal{R}_u(\tilde{u}, \tilde{y}, h) = -\tilde{u} \cdot \mathcal{G}(h) \tilde{y} + \mathcal{M}_0(h) \partial_x h(\nu_\Sigma \cdot \nabla_x) \tilde{y} + M \mathcal{N}_l r_l(\tilde{y}). \]

3.2 Local Well-Posedness

To obtain local well-posedness, the transformed problem is rewritten in the abstract form

\[ \mathcal{L}z = (N(z), z_0). \]

Here \(\mathcal{L}\) is the principal linear part of the problem in question, and \(N\) is the remaining nonlinear part which is small in the sense that \(N\) collects all lower-order terms and contains only highest-order terms which carry a factor \(|\nabla_\Sigma h|\) which is small on small...
time intervals due to the choice of the Hanzawa transform. The variable \( z \) with initial value \( z_0 \) collects all essential variables of the problem under consideration.

The first step is to find function spaces \( \mathcal{E}(J) \) and \( \mathcal{F}(J) \), \( J = (0, a) \) or \( J = \mathbb{R}_+ \), such that \( L : \mathcal{E}(J) \to \mathcal{F}(J) \times \mathcal{E}_x \) is an isomorphism. Here \( \mathcal{E}_x \) denotes the time-trace space of \( \mathcal{E}(J) \) where the initial value \( z_0 \) should belong to. This is the question of maximal regularity.

The second step then employs the contraction mapping principle to obtain local solutions. For this, estimates of the nonlinearity \( N \) are needed, eventually showing that \( N : \mathcal{E}(J) \to \mathcal{F}(J) \) is continuously Fréchet-differentiable, at least. This requires some smoothness of the coefficient functions in the constitutive laws. If these are real analytic, the interface will become instantly real analytic, which shows the strong regularizing effect, characteristic for parabolic problems.

The solution space \( \mathcal{E}(J) \) with \( J = [0, a] \) for the transformed problem is given by

\[
\begin{align*}
\tilde{u} &\in H^1_p(J; L_p(\Omega)^n) \cap L_p(J; H^2_p(\Omega \setminus \Sigma)^n), \\
\tilde{\pi} &\in L_p(J; H^1_p(\Omega \setminus \Sigma)) \\
\tilde{y} &\in H^1_p(J; L_p(\Omega)^N) \cap L_p(J; H^2_p(\Omega \setminus \Sigma)^N),
\end{align*}
\]  

(16)

with compatibility conditions

\[
\begin{align*}
\mathcal{G} \cdot \tilde{u} &= 0, \quad \tilde{y} \in \mathcal{D}^0 \quad \text{in} \ \Omega \setminus \Sigma, \\
\tilde{u} &= 0, \quad \partial_\nu \tilde{y} = 0 \quad \text{on} \ \partial \Omega, \\
P_0[\mu(\tilde{y})] &= \tilde{g}[(\tilde{u} \cdot \nu_T)/\|\tilde{g}\| - \|\mathcal{A}(\tilde{y})\nu_T \cdot \mathcal{G}(\tilde{h})\|] = 0 \quad \text{on} \ \Sigma, \\
P_T[\tilde{u}] &= P_T[\tilde{S}n_T] = 0 \quad \text{on} \ \Sigma.
\end{align*}
\]

Observe that these conditions are highly nonlinear. For smoothness of the nonlinear part \( N \) we employ the regularity of the coefficients \( \mu, f_{ij} \) and of \( \psi \), and we require \( p > n + 2 \) to have appropriate Sobolev embeddings at our disposal, which ensure that \( \tilde{y}, \tilde{u}, \nabla_x v, \nabla_x u \) are uniformly continuous in \( J \times (\Omega \setminus \Sigma) \).

According to the strategy in [14], Chapter 9, we extract the principal linearization of the problem at a given initial state, referring to a reference hypersurface \( \Sigma \), i.e. from now on we drop the bars. This problem reads as follows.

\[
\begin{align*}
\rho \partial_t v - \partial_{x_\omega} (\mathcal{A}(x) \partial_{x_\omega} v) &= f_v \quad \text{in} \ \Omega \setminus \Sigma, \\
\|\mathcal{A}(x) \partial_{x_\omega} v\| &= g_v^1 \quad \text{on} \ \Sigma, \\
P_0[\mathcal{B}(x) v] &= g_v^0 \quad \text{on} \ \Sigma, \\
\text{div}_x w &= g_\omega \quad \text{in} \ \Omega \setminus \Sigma, \\
\rho \partial_t w - \text{div}_x (\mu(x)(\nabla_x w + [\nabla_x w]^T)) + \nabla_x p &= f_w \quad \text{in} \ \Omega \setminus \Sigma, \\
-\mu(x)(\nabla_x w + [\nabla_x w]^T) \nu_{\Sigma} + [p] \nu_{\Sigma} - \sigma H^1_t h \nu_{\Sigma} &= g_w^1 \quad \text{on} \ \Sigma, \\
P_{\Sigma}[w] &= g_w^0 \quad \text{on} \ \Gamma, \\
[p/\rho] - 2\mu(x)[\nabla_x w \nu_{\Sigma}/\rho] &= g_w^2 \quad \text{on} \ \Sigma, \\
\partial_t h - [g_\omega \nu_{\Sigma}/\rho] &= f_h \quad \text{on} \ \Sigma, \\
w &= \partial_x v = 0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]

(18)

Here we have set \( \mathcal{A}(x) = \mathcal{A}(y_0(x)), \mathcal{B}(x) = \mathcal{B}(y_0(x)), \mu(x) = \mu_\omega(u_0(x)), \) and the right-hand sides \( f_v, \ldots \) denote generic inhomogeneities. Evidently, the diffusion problem decouples completely from the Navier-Stokes problem. The latter has been studied thoroughly in [14], Section 7.4. For the diffusion problem we also may refer
to the monograph by Prüss and Simonett [14], Section 6.5, where such transmission problems are studied in detail. To apply these results, we have to verify normal ellipticity of $A(x)$ and the Lopatinskii-Shapiro condition at the interface.

(a) Normal Ellipticity

Fix any point $x_0 \in \Omega \setminus \Sigma$, and let $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^n$, $v \in \mathbb{E}$, $v \neq 0$, such that

$$\lambda v + |\xi|^2 A_0 v = 0, \quad A_0 = A(x_0), \quad B_0 = B(x_0).$$

Multiplying with $B_0 v$ and observing $A_0 = A(y_0(x_0))P(y_0(x_0))Y_0(x_0)B_0$ yields

$$0 = \lambda (v|B_0 v) + |\xi|^2 (A(y_0(x_0))Y_0(x_0)P^T(y_0(x_0))B_0 v|B_0 v)$$

$$= \lambda (v|B_0 v) + |\xi|^2 (Y_0^{-1/2}(x_0)A(y_0(x_0))Y_0^{1/2}(x_0)w|w),$$

where $w = Y_0^{-1/2}(x_0)P^T(y_0(x_0))B_0 v$. This implies that $\lambda$ is real, and as $B_0$ and $Y_0^{-1/2}(x_0)A(y_0(x_0))Y_0^{1/2}(x_0)$ are positive definite on $\mathbb{E}$ by the assumptions and by Lemma 2.1 (v), we obtain $\lambda \leq 0$, and if $\lambda = 0$ then $w = 0$, hence

$$0 = (w|Y_0^{-1/2}(x_0)v) = (B_0 v|v),$$

which implies $v = 0$, a contradiction. This shows that the differential operator $-\partial_{x_0}(A(x)\partial_{x_0})$ is normally elliptic for each $x \in \Omega \setminus \Sigma$.

(b) Lopatinskii-Shapiro Condition

Fix any point $x_0 \in \Sigma$ and let $A_0$, $B_0$ as before. Suppose

$$\lambda v(\tau) + A_0(|\xi|^2 - \partial_\tau^2)v(\tau) = 0, \quad \tau \neq 0,$$

for some $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^{n-1}$ and $v \in L_2(\mathbb{R}; \mathbb{E})$, $v \neq 0$. Moreover, let the jump conditions

$$[A_0 \partial_\tau v] = P_0[B_0 v] = 0,$$

be satisfied. Multiplying the equation by $B_0 v(\tau)$ and integrating over $\tau \in \mathbb{R}$, we obtain

$$0 = \lambda (v|B_0 v) + |\xi|^2 (A_0 v|B_0 v) = \int \left( A_0 \partial_\tau^2 v|B_0 v \right) d\tau$$

$$= \lambda (v|B_0 v) + |\xi|^2 (A_0 v|B_0 v) + (A_0 \partial_\tau v|B_0 \partial_\tau v) + [(A_0 \partial_\tau v|B_0 v)]$$

$$= \lambda (v|B_0 v) + |\xi|^2 (A_0 v|B_0 v) + (A_0 \partial_\tau v|B_0 \partial_\tau v),$$

as by the jump conditions

$$[(A_0 \partial_\tau v|B_0 v)] = [(P_0 A_0 \partial_\tau v|B_0 v)] = (A_0 \partial_\tau v|P_0 B_0 v) = 0.$$

This shows that $\lambda$ is real and negative, as $B_0$ and $Y_0^{-1/2}(x_0)A(y_0(x_0))Y_0^{1/2}(x_0)$ are positive definite on $\mathbb{E}$. Therefore the Lopatinskii-Shapiro condition is fulfilled.

As a result, from Prüss and Simonett [14], Section 6.5, we may conclude that the principal linearization has the property of maximal $L_p$-regularity. Then we may follow the route in [14], Chapter 9, to obtain local well-posedness of the Hanzawa transformed problem.

3.3 The State Manifold

The state manifold $\mathcal{SM}$ for the problem is defined by taking time traces of the solutions of (15), inverting the Hanzawa transform and employing the compatibility conditions.

$z := (y, u, \Gamma) \in \mathcal{SM}$ if and only if

- $(y, u) \in W_p^{2-2/p}(\Omega \setminus \Gamma)^{N \times n}$, \quad $\Gamma \in W_p^{3-2/p}$;
Theorem 3.1. Let general kinetics the local existence result from the previous subsection implies 
Here \( j \) assume \( 0 \) and generalized principle finitely many. Therefore, one may employ what is called the \( L \) the linearized problem enjoys maximal \( \) this is the \( \) normally stable case of linearized stability principle of linearized stability ural to study these eigenvalues and to apply the \( C \) a compact analytic \( \) again the Hanzawa transform, where the reference manifold \( \Sigma \) now is the equilib-
rium interface \( \Gamma \), constructed from \( ( \) again the \( 1 \) to the tangent space of \( E \) to the standard principle of linearized stability is not applicable. Fortunately, \( 0 \) is the \( L \) is nontrivial, i.e. the imaginary axis is not in the resolvent set of \( L \), and so the standard principle of linearized stability is not applicable. Fortunately, \( 0 \) is the only eigenvalue of \( L \) on \( i \mathbb{R} \) and it is nicely behaved: the kernel \( N(L) \) is isomorphic to the tangent space of \( E \) at this equilibrium, and \( 0 \) is semi-simple. This shows that \( 0 \) is normally stable if the remaining eigenvalues of \( L \) have positive real parts, and normally hyperbolic if some of them have negative real parts; these are only finitely many. Therefore, one may employ what is called the generalized principle of linearized stability, a method which is adapted to such a situation and has been worked out recently for quasi-linear parabolic evolution equations; see e.g. Prüss and Simonett [14], Chapter 5. So the stability analysis of equilibria proceeds in two steps.

In the first step, the eigenvalues of \( L \) are analyzed and one finds necessary and sufficient conditions which ensure that all eigenvalues of \( L \) except \( 0 \) have positive real parts; this is the normally stable case. In the normally hyperbolic case the
dimension of the unstable subspace of $L$ is determined. It also turns out that 0 is semi-simple, the kernel of $L$ is determined and one can show that $N(L)$ is isomorphic to the tangent space of $E$. In the second step, the generalized principle of linearized stability is applied to the nonlinear problem. This method yields the following result.

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^n$ be bounded with boundary $\partial \Omega \in C^{3-}$, $p > n+2$, assume $0 < \mu_\ast \in C^2(\partial \Omega)$, $\psi \in C^1(\partial \Omega)$, as well as condition (F) and assumption (R).

Then, in the topology of the state manifold $SM$, we have

(i) Any solution starting in a neighborhood of a stable equilibrium converges to another stable equilibrium.

(ii) Any solution starting and staying in a neighborhood of an unstable equilibrium converges to another unstable equilibrium.

(iii) Any solution starting in a neighborhood of a stable equilibrium converges to another stable equilibrium.

**Proof.** For the central first step, we investigate normal stability of a given non-degenerate equilibrium $z_\ast = (y_\ast, u_\ast, \Gamma_\ast)$. Setting $\Sigma = \Gamma_\ast$, the full linearization of the problem becomes

$$
\begin{align*}
\partial_t v - A_\ast \Delta_x v + a_1 M \mathbf{v}_i (v | Y^{-1}_\ast \mathbf{v}_i) &= f_v \quad \text{in } \Omega \setminus \Sigma, \\
([P_0 y_\ast] / [1/\partial]) [w \cdot \nu_\Sigma] - [A_\ast \partial_v v] &= g_v^1 \quad \text{on } \Sigma, \\
P_\Sigma [B_\ast v] &= g^0_v \quad \text{on } \Sigma, \\
\text{div}_x w &= g_d \quad \text{in } \Omega \setminus \Sigma, \\
\partial_t w - \mu_\ast \Delta_x w + \nabla_x p &= f_w \quad \text{in } \Omega \setminus \Sigma, \\
- [\mu_\ast (\nabla_x w + [\nabla_x w]^2) \nu_\Sigma] + [p] \nu_\Sigma + \sigma A_\Sigma \nu_\Sigma &= g^1_w \quad \text{on } \Sigma, \\
P_\Sigma [w] &= g^0_w \quad \text{on } \Sigma, \\
- [P_0 y_\ast \cdot B_\ast v + p / \partial] - 2 [\mu_\ast \nu_\Sigma \cdot \nabla_x w (w \nu_\Sigma) / \partial] &= g^2_w \quad \text{on } \Sigma, \\
\partial_t h - [g w \cdot \nu_\Sigma] / \partial &= f_h \quad \text{on } \Sigma, \\
w = \partial_v v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

Here we have set $A_\ast = \mathbb{A}(y_\ast)$, $B_\ast = \mathbb{B}(y_\ast) = \kappa M^{-1} Y^{-1}_\ast$, $\mu_\ast = \mu_\ast(y_\ast) > 0$, $a_1 = k \mu^{-1}_\ast 0 > 0$, and $A_\Sigma = -H_\Sigma$ is the curvature operator (times $(n-1)$). Again, the right hand sides $f_v, \ldots$ denote generic inhomogeneities. By maximal $L_p$-regularity of the principal part of the linearization proved in Section 3.1 and perturbation, this problem has maximal $L_p$-regularity as well, hence the underlying linear operator $L$ is the negative generator of an analytic $C_0$-semigroup. More precisely, $L$ is defined as follows. As a base space we take $X_0 = L_p(\Omega; \mathbb{E}) \times L_{p,\sigma}(\Omega) \times W^{2-1/p}_p(\Sigma)$, where the index $\sigma$ indicates solenoidal. Then we define $L$ by means of

$$
Lz = (- A_\ast \Delta_x v / \partial + (a_1 / \partial) M \mathbf{v}_i (v | Y^{-1}_\ast \mathbf{v}_i) - (\mu_\ast / \partial) \Delta_x w + \nabla_x p / \partial, - [g w \cdot \nu_\Sigma] / \partial)
$$

with domain

$$
X_1 = D(L) = \{ z = (v, w, h) \in H^2_p(\Omega \setminus \Sigma) \times \mathbb{R}^{N^2} \times W^{3-1/p}_p(\Sigma), z \in X_{0\ast} (C) \text{ holds} \},
$$

where the compatibilities (C) are given by

$$
\begin{align*}
([P_0 y_\ast] / [1/\partial]) [w \cdot \nu_\Sigma] - [A_\ast \partial_v v] &= P_0 [B_\ast v] = 0 \quad \text{on } \Sigma, \\
P_\Sigma [\mathbf{v}_\ast \nu_\Sigma] &= P_\Sigma [w] = 0 \quad \text{on } \Sigma, \\
w = \partial_v v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$
As $\Omega$ is bounded, the resolvent of $L$ is compact, hence its spectrum consists only of eigenvalues of finite algebraic multiplicity, which by ellipticity are independent of $p$. So it is enough to look at the eigenvalue problem for $L$ with $p = 2$, i.e.
\[
\begin{align*}
\varrho \lambda v - \mathcal{A}_v \Delta x v &= a_1 M\nu_1(v)Y^{-1}_*\nu_1 = 0 \quad \text{in } \Omega \setminus \Sigma, \\
\{P_0 y_0\} - [a_1 \nabla \nu \cdot \mathcal{B}_v] &= 0 \quad \text{on } \Sigma, \\
\nabla x w &= 0 \quad \text{in } \Omega \setminus \Sigma,
\end{align*}
\]
and if \(\lambda\) is not real we may divide by \(\text{Im}\ \lambda\) and insert into (22) to the result
\[
0 = 2\text{Re } \lambda (aw|w)_2 + (\mathcal{A}_v \mathcal{B}_v + \sigma(\mathcal{A}_v h|h)_2) + \int_{\Omega} \kappa a_1 |(v|Y^{-1}_*\nu_1|^2\,dx.
\]
This identity together with (21) implies that $\lambda$ is real. In fact, it implies $D = 0$ and $\nabla v = 0$, hence $w = 0$ by Poincaré's inequality, and $v$ constant, and so also $v = 0$ by the first equation in (21), if eigenvalue $\lambda$ is not real.

So let $\lambda > 0$. Then $\nabla x w = 0$ implies that $h$ has vanishing mean value over $\Sigma$. As $\mathcal{A}_v$ is positive semi-definite in $L_{2,0}(\Sigma)$, i.e. $L_2$-functions with mean zero, if and
Therefore, only if $\Sigma$ is connected, we see that all terms in (22) are nonnegative, hence must vanish simultaneously, which implies $z = (v, w, h) = 0$, as $\mathcal{B}_s$ is positive definite. Therefore, $L$ has no negative eigenvalues if $\Sigma$ is connected.

(b) Next let us look at $\lambda = 0$. In this case (22) implies

$$\langle \mu, D|D \rangle_2 = (\mathcal{A}_s \partial_{x,v}|\mathcal{B}_s \partial_{x,v})_2 = \int_{\Omega} \kappa \alpha_l |v(Y^{-1}_s v)|^2 \, dx = 0,$$

hence $D = 0$ on $\Omega$, as well as $\nabla_x v = 0$ and $(v_l Y^{-1}_s v) = 0$ for all $l = 1, \ldots, m$.

Lemma 1.2.1 in Prüss and Simonett [14] implies $w = 0$, and by Lemma 2.2 above, $v \in T_{\nu} \mathcal{E}_c$ is constant in the components of the phases. Then the jump condition

$$P_0[\mathcal{B}_s v] = 0$$

shows that $v$ is even constant in the phases, which induces $N - s - 1$ dimensions in the kernel of $L$. Furthermore, $\nabla_x p = 0$ in $\Omega \setminus \Sigma$, hence $p$ is constant in the components of the phases, but due to the linearized Gibbs-Thomson relation, $p$ is even constant in the phases, hence by the normal stress jump condition we have

$$[p] + \sigma A\Sigma h = 0, \quad [p/\rho] = [(P_0 Y_s |\mathcal{B}_s v)].$$

These relations add $nr + 1$ dimensions to the kernel of $L$, in case $\Sigma$ consists of $r \in \mathbb{N}$ components. This shows that the dimension of the kernel of $L$ equals the dimension of the tangent space $T_z \mathcal{E}$ of the equilibrium manifold at $z_s \in \mathcal{E}$. Moreover, the kernel of $L$ is given by

$$N(L) = \text{span}\{(v, 0, 0)^T, [0, 0, Y_j^k]^T : v \in T_{y_s} \mathcal{E}_c, P_0[\mathcal{B}_s v] = 0\},$$

where $Y_j^k = \chi_{\Sigma_k}$, and $Y_j^k$, $j = 1, \ldots, n$, denote the spherical harmonics of degree 1 on $\Sigma_k$, $k = 1, \ldots, r$.

Next we prove that 0 is a semi-simple eigenvalue of $L$. More precisely, let $(\tilde{v}, \tilde{w}, \tilde{h})$ be a solution of (21) with $\lambda = 0$, and suppose $(v, w, h)$ satisfies

$$-\mathcal{A}_s \Delta_x v + a_l \mathcal{M} \nu_l (v(Y^{-1}_s v)) = \varrho \tilde{v} \quad \text{in } \Omega \setminus \Sigma,$n

$$([(P_0 Y_s)/[1/\rho] \nu_1 [w \cdot \nu_S] - [\mathcal{A}_s \partial_{x,v}]] = 0 \quad \text{on } \Sigma,$n

$$P_0[\mathcal{B}_s v] = 0 \quad \text{on } \Sigma,$n

$$\text{div}_x w = 0 \quad \text{in } \Omega \setminus \Sigma,$n

$$-\mu_s \Delta_x w + \nabla_x p = \varrho \tilde{w} = 0 \quad \text{in } \Omega \setminus \Sigma, \quad (23)$$

$$-\mu_s (\nabla_x w + [\nabla_x w]^T) \nu_S + \nu_1 \nu_2 + \sigma A\Sigma h \nu_2 = 0 \quad \text{on } \Sigma,$n

$$P_3[w] = 0 \quad \text{on } \Gamma,$n

$$-P_0 Y_s \cdot \mathcal{B}_s v + p/\rho = 2[\mu_s \nu_1 \cdot \nabla_x w \cdot \nu_S / \rho] = 0 \quad \text{on } \Sigma,$n

$$-\nu \nu_1 / \rho = \tilde{h} \quad \text{on } \Sigma,$n

$$w = \partial_x v = 0 \quad \text{on } \partial \Omega.$$

Then, taking the inner product of the equation for $v$ with $\mathcal{B}_s \tilde{v}$ and integrating by parts, we obtain

$$\langle \varrho \tilde{v}|\mathcal{B}_s \tilde{v}\rangle_2 = \int_{\Sigma} [\|P_0 Y_s |\mathcal{B}_s \tilde{v}]] d\Sigma,$n

as $\tilde{v}$ is constant in the components of $\Omega \setminus \Sigma$ and $(Y^{-1}_s v)|\tilde{v}) = 0$ for all $l = 1, \ldots, r$.

Next we take the inner product of the equation for $\tilde{w}$ with $w$ which, after an integration by parts, yields

$$\int_{\Sigma} \sigma(A\Sigma \tilde{h}) d\Sigma + \int_{\Sigma} \|[(\mathcal{B}_s \tilde{v}|P_0 Y_s)] d\Sigma = 0,$n

as $\tilde{v}$ is constant in the components of $\Omega \setminus \Sigma$ and $(Y^{-1}_s v)|\tilde{v}) = 0$ for all $l = 1, \ldots, r$. Then we integrate by parts,

$$\int_{\Sigma} \sigma(A\Sigma \tilde{h}) d\Sigma + \int_{\Sigma} \|[(\mathcal{B}_s \tilde{v}|P_0 Y_s)] d\Sigma = 0,$n

as $\tilde{v}$ is constant in the components of $\Omega \setminus \Sigma$ and $(Y^{-1}_s v)|\tilde{v}) = 0$ for all $l = 1, \ldots, r$. Then we integrate by parts,
as \( \tilde{w} = 0 \). Combining these two identities we obtain

\[
(\varrho \tilde{v}|\mathcal{B}_s \tilde{v})_2 + \int_{\Sigma} \sigma (A_\Sigma \tilde{h}|\tilde{h}) d\Sigma = 0.
\]

Using the structure of \( N(L) \), this implies \( \tilde{v} = 0 \) as well as \( A_\Sigma \tilde{h} = 0 \). Finally, multiplying the equation for \( v \) with \( B_s v \) and that for \( w \) with \( w \), integrating by parts, using the symmetry

\[
(A_\Sigma h|\tilde{h})_{\Sigma} = (h|A_\Sigma \tilde{h}) = 0
\]

and addition yields the relation

\[
(\mu_s D|D)_2 + (A_s \partial_{x_{\alpha}} v|\mathcal{B}_s \partial_{x_{\alpha}} v)_2 + \int_{\Omega} \kappa a_l|(v|Y^{-1}_s v_l)|^2 dx = 0,
\]

which shows as above \( w = 0 \), hence \( \tilde{h} = 0 \). This shows that the eigenvalue 0 of \( L \) is semi-simple.

(c) We want to prove that \( L \) has precisely \( r - 1 \) negative eigenvalues in case \( \Sigma \) has \( r \geq 2 \) components. To prove this we proceed as in Prüss and Simonett [14], Section 10.7. We first introduce an operator \( T_\lambda \) by solving the problem

\[
\varrho \lambda v - A_s \Delta_x v + \alpha_l M v_l(v|Y^{-1}_s v_l) = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
[A_s \partial_{x_{\alpha}} v] = [P_0 y_s], \quad \text{on} \quad \Sigma,
\]

\[
P_0[B_s v] = 0 \quad \text{on} \quad \Sigma,
\]

\[
div_x w = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
\varrho \lambda w - \mu_s \Delta_x w + \nabla_x p = 0 \quad \text{in} \quad \Omega \setminus \Sigma,
\]

\[
P_2: [\mu_s (\nabla_x w + [\nabla_x w]^T) \nu_\Sigma] = 0 \quad \text{on} \quad \Sigma,
\]

\[
P_2: [w] = 0 \quad \text{on} \quad \Gamma,
\]

\[
[p/\varrho] - 2[\mu_s \nu_\Sigma \cdot \nabla_x w \nu_\Sigma/\varrho] = g \quad \text{on} \quad \Sigma,
\]

\[
- \frac{[\varrho w \cdot \nu_\Sigma]}{[\varrho]} = h \quad \text{on} \quad \Sigma,
\]

\[
w = \partial_{x_{\alpha}} v = 0 \quad \text{on} \quad \partial \Omega.
\]

Here we have set

\[
j = [w \cdot \nu_\Sigma]/[1/\varrho], \quad g = [(\mathcal{B}_s v|P_s y_s)].
\]

Given the input \( h \), the output is then \( T_\lambda h = -2[\mu_s \partial_{x_{\alpha}} w \cdot \nu_\Sigma] + [p] \), and \( \lambda > 0 \) is an eigenvalue of \( -L \) if and only if

\[
B_\lambda h := \lambda T_\lambda h + \sigma A_\Sigma h = 0,
\]

for some nontrivial \( h \in H^2_\Sigma(\Sigma) \). The operator \( T_\lambda \) is a complicated Dirichlet to Neumann operator, its existence will be discussed below. A simple computation shows that for \( \lambda > 0 \),

\[
(T_\lambda g|g)_\Sigma = \lambda ((\varrho v|\mathcal{B}_s v)_2 + (\varrho w|w)_2)
\]

\[
+ (\mu_s D|D)_2 + (A_s \partial_{x_{\alpha}} v|\mathcal{B}_s \partial_{x_{\alpha}} v)_2 + \int_{\Omega} \kappa a_l|(v|Y^{-1}_s v_l)|^2 dx \geq 0.
\]

In a similar way one shows that \( T_\lambda \) is symmetric, it is even self-adjoint and positive definite and maps \( H^1_{1,0}(\Sigma) \) into \( L^2_{2,0}(\Sigma) \). Therefore, \( \lambda T_\lambda \) is a relatively compact perturbation of \( A_\Sigma \), hence the domain of \( B_\lambda \) is \( H^2_{2,0}(\Sigma) \), as well. Here we have used the notation \( H^2_{2,0}(\Sigma) = H^2_{2}(\Sigma) \cap L^2_{2,0}(\Sigma) \).
(d) To show that $T_\lambda$ is well defined, we define operators
\[
S^0_\lambda: (g,0) \mapsto -2[\mu, \partial \nu \cdot \nu_\Sigma] + [p], \quad S^0_\lambda: H^2_\Sigma(\Sigma) \to H^2_\Sigma(\Sigma),
\]
\[
S^1: (0,h) \mapsto -2[\mu, \partial \nu \cdot \nu_\Sigma] + [p], \quad S^1: H^2_\Sigma(\Sigma) \to H^2_{-1}(\Sigma),
\]
\[
N^S_\lambda: (g,0) \mapsto \|w \cdot \nu_\Sigma\|/[1/\rho] = j, \quad N^S_\lambda: H^2_\Sigma(\Sigma) \to H^{s+1}_\Sigma(\Sigma),
\]
\[
R^S_\lambda: (0,h) \mapsto \|w \cdot \nu_\Sigma\|/[1/\rho] = j, \quad R^S_\lambda: H^2_\Sigma(\Sigma) \to H^2_\Sigma(\Sigma),
\]
\[
N^D_\lambda: j \mapsto \|(P_0 y_\lambda[B, v])\| = g, \quad N^D_\lambda: H^2_\Sigma(\Sigma) \to H^{s+1}_\Sigma(\Sigma).
\]
These operators are well-defined for all $\lambda > 0$, and uniformly bounded for $\lambda \geq \lambda_0 > 0$, as they refer to the uncoupled systems, i.e. the asymmetric two-phase Stokes problem (superscripts $S, 1, 2$), or the diffusion problem (superscript $D$). Given $h \in H^2_{0,0}(\Sigma)$, we obtain the equations
\[
g = N^D_\lambda(N^S_\lambda g + R^S_\lambda h), \quad T_\lambda h = S^1_\lambda h + S^0_\lambda g,
\]
hence
\[
T_\lambda h = S^1_\lambda h + S^0_\lambda (I - N^D_\lambda N^S_\lambda)^{-1} N^D_\lambda R^S_\lambda h,
\]
as soon as $I - N^D_\lambda N^S_\lambda$ is invertible. Observe that $N^D_\lambda N^S_\lambda$ gains two orders, hence it is compact in $H^2_\Sigma(\Sigma)$, and
\[
|N^D_\lambda|_{B(H^2_\Sigma(\Sigma))} \leq c/\lambda^\beta,
\]
for some constants $c > 0$ and $\beta > 0$.

In Prüss and Simonett [14], Section 10.7, it has been shown that there exists a constant $c_0 > 0$ such that
\[
(S^1_\lambda h[h]_{L^2(\Sigma)} \geq c_0 |h|^2_{L^2(\Sigma)}, \quad \lambda \geq 0.
\]
This implies
\[
(T_\lambda h[h]_{L^2(\Sigma)} \geq c_0 |h|^2_{L^2(\Sigma)} - C\lambda^{-\beta}|h|^2_{L^2(\Sigma)} \geq (c_0/2)|h|^2_{L^2(\Sigma)}, \quad \lambda \geq \lambda_1,
\]
for some sufficiently large $\lambda_1$. As
\[
(A_\Sigma h[h]_{L^2(\Sigma)} \geq -\frac{n-1}{R^2_*} |h|^2_{L^2(\Sigma)},
\]
this shows that
\[
(B_\lambda h[h]_{L^2(\Sigma)} \geq (\lambda c_0/2 - \sigma(n-1)/R^2_*)|h|^2_{L^2(\Sigma)},
\]
hence $B_\lambda$ is positive definite for large $\lambda$, it has only positive eigenvalues.

(e) Next we consider small $\lambda$. For this purpose, we write (24) abstractly as
\[
(\lambda + A_1)z = 0, \quad B_1 z = (0,h),
\]
in $L^2(\Omega; E) \times L^2(\sigma(\Omega))$, with $z = [v, w]^T$. As we have shown in (d), this problem has a unique solution $z_0$ if $\lambda = \lambda_0$ is large enough. The operator $A_1$ with homogeneous boundary conditions $B_1 z = 0$ will be denoted by $A_0$, and we know from previous computations that $A_0$ is accretive in $L^2$. Then the solution of (24) is given by
\[
z = z_0 + z_1, \quad \lambda z_1 + A_0 z_1 = (\lambda_0 - \lambda)z_0,
\]
which yields
\[
z = z(\lambda) = z_0 + (\lambda_0 - \lambda)(\lambda + A_0)^{-1} z_0 \in H^2_\Sigma(\Omega \setminus \Sigma)^{N \times n}.
\]
This shows that $T_\lambda$ is well-defined for all $\lambda > 0$. To compute the limit $R_0 := \lim_{\lambda \to 0} \lambda T_\lambda$, we recall

$$
\lim_{\lambda \to 0} \lambda (\lambda + A_0)^{-1} = P_{A_0} \quad \text{in } H^2_2(\Omega \setminus \Sigma)^{N \times n},
$$

where $P_{A_0}$ denotes the projection onto $N(A_0)$ along the range $R(A_0)$. This implies

$$
\lim_{\lambda \to 0} \lambda z(\lambda) = \lambda_0 P_{A_0} z_0 \quad \text{in } H^2_2(\Omega \setminus \Sigma)^{N \times n},
$$

As the $w$-component in the kernel of $A_0$ vanishes, this shows that $R_0 = 0$, hence $B_0 = \sigma A_0$. If $\Sigma$ consists of $r \geq 2$ components, then $-\sigma(n-1)/R^2_2$ is an eigenvalue of $B_0$ in $L^2_2(\Sigma)$ of multiplicity $r-1$. In fact, functions $h = \sum_k h_k \chi_{\Sigma_k}$, with constants $h_k$ such that $\sum_k h_k = 0$, form the corresponding eigenspace. Recall that the mean of $h$ is necessarily 0, as we have seen before.

As $B_\lambda$ is selfadjoint for each $\lambda \geq 0$, it has only real, semi-simple eigenvalues, $r-1$ negative ones for $\lambda = 0$ and no negative one for $\lambda$ sufficiently large. So by continuous dependence, as $\lambda$ runs from 0 to $\infty$, $r-1$ eigenvalues of $B_\lambda$, counted with multiplicity, must cross the imaginary axis through 0, thereby producing $r-1$ negative eigenvalues of $L$.

Concluding, we have shown that an equilibrium $z_* \in \mathcal{E}$ is

- normally stable if and only if $\Gamma_*$ is connected;
- normally hyperbolic if and only if $\Gamma_*$ is disconnected.

Now we may follow the route explained in Prüss and Simonett [14], Chapter 11, to implement the generalized principle of linearized stability for a proof of Theorem 3.2.

3.5 Long-Time Behaviour

In general, solutions in $\mathcal{SM}$ will exist on a maximal time interval $[0, t_+(z_0))$ which typically will be finite, due to several obstructions, such as missing a priori bounds, loss of well-posedness, or topological changes in the moving interface. However, if a solution does not develop singularities in a sense to be specified, then the solution exists globally, i.e. $t_+(z_0) = \infty$, and it converges in the topology of $\mathcal{SM}$ to an equilibrium. This essentially relies on a method using time weights to improve regularity and on compact Sobolev embeddings. Actually, one can characterize solutions which exist globally and converge as $t \to \infty$. This only involves general properties of semiflows, relative compactness of bounded orbits, the existence of a strict Lyapunov functional (the total available energy), and the results on stability of equilibria.

In the situation considered here, the obstructions to global existence are:

- regularity: the norms of either $y(t)$, $u(t)$, $\Gamma(t)$ may become unbounded;
- positivity: one of the mass fractions $y_k(t)$ may vanish;
- geometry: the topology of the interface may change;
  or the interface may touch the boundary of $\Omega$;
  or a part of the interface may contract to a point.

Note that the relevant compatibility conditions and the regularity of the solutions are preserved by the semiflow.

Let $z$ be a solution in the state manifold $\mathcal{SM}$. By the uniform ball condition we mean the existence of a radius $r_0 > 0$ such that $B_{r_0}(x + (-1)^i r_0 \nu \Gamma(x)) \subset \Omega_i$, $i = 1, 2$, for each $t \in [0, t_+]$ and each $x \in \Gamma(t)$. Note that this condition bounds
the curvature of $\Gamma(t)$, prevents parts of it to shrink to points, to touch the outer boundary $\partial \Omega$, and to undergo topological changes.

We conclude this paper with the following result on long-time behaviour of the system.

**Theorem 3.3.** Let $\Omega \subset \mathbb{R}^n$ be bounded with boundary $\partial \Omega \in C^3$, $p > n+2$, assume $0 < \mu_s \in C^2(\mathbb{B})$, $\psi \in C^4(\mathbb{B})$, as well as condition (F) and assumption (R). Let $z$ be the solution of (4) with initial value $z_0 \in \mathcal{SM}$. Suppose on $[0,t_+)$

(i) $z(t)$ stays bounded in $\mathcal{SM}$;

(ii) there is a constant $c_0 > 0$ such that $\psi_k(t) \geq c_0$, for all $k$;

(iii) $\Gamma(t)$ is connected and satisfies the uniform ball condition.

Then $z$ is a global solution which, as $t \to \infty$, converges exponentially in $\mathcal{SM}$ to an equilibrium $z_\infty \in \mathcal{E}$ of (4). The converse of this result is also valid.

This result follows from abstract dynamical system arguments; see Köhne, Prüss, Wilke [1], Prüss, Shimizu, Wilke [16], and Prüss, Simonett [14].

**REFERENCES**

[1] D. Bothe, On the Maxwell-Stefan approach to multicomponent diffusion, *Progress in Nonlinear Differential Equation and Their Applications*, 80 (2011), 81–93.

[2] D. Bothe and W. Dreyer, Continuum thermodynamics of chemically reacting fluid mixtures, *Acta Mechanica*, 226 (2015), 1757–1805.

[3] D. Bothe and S. Fleckenstein, A Volume-of-Fluid-based method for mass transfer processes at fluid particles, *Chem. Engin. Sci.*, 101 (2013), 283–302.

[4] L. Boudin, B. Grec and F. Salvarini, A mathematical and numerical analysis of the Maxwell-Stefan diffusion equations, *Discrete Contin. Dyn. Syst. Ser. B*, 17 (2012), 1427–1440.

[5] W. Dreyer, On jump conditions at phase boundaries for ordered and disordered phases, WIAS preprint 869 (2003).

[6] V. Giovangigli, *Multicomponent Flow Modeling*, Birkhäuser, Basel, 1999.

[7] M. Herberg, M. Meyries, J. Prüss and M. Wilke, Reaction-diffusion systems of Maxwell-Stefan type with reversible mass-action kinetics, *Nonlinear Analysis*, in press.

[8] M. Ishii, *Thermo-Fluid Dynamic Theory of Two-Phase Flow*, Eyrolles, Paris 1975.

[9] R. Krishna and R. Taylor, Multicomponent mass transfer theory and applications, in *Handbook for Heat and Mass Transfer*, Vol. 2, Chapter 7, Gulf, Houston, 1986.

[10] R. Krishna and J. Wesselingh, The Maxwell-Stefan approach to mass transfer, *Chem. Engin. Sci.*, 52 (1997), 861–911.

[11] M. Köhne, J. Prüss and M. Wilke, On quasi-linear parabolic evolution equations in weighted $L_p$-spaces, *Journal of Evolution Equations*, 10 (2010), 443–463.

[12] J. LeCrone, J. Prüss and M. Wilke, On quasi-linear parabolic evolution equations in weighted $L_p$-spaces II, *Journal of Evolution Equations*, 14 (2014), 509–533.

[13] J. C. Maxwell, On the dynamical theory of gases. *Phil. Trans. Roy. Soc. London*, 157 (1866), 49–88.

[14] J. Prüss and G. Simonett, *Moving Interfaces and Quasilinear Parabolic Problems*, Monographs in Mathematics, 105, Birkhäuser, Basel, 2016.

[15] J. Prüss, G. Simonett and R. Zacher, On convergence of solutions to equilibria for quasi-linear parabolic problems, *Journal of Differential Equations*, 246 (2009), 3902–3931.

[16] J. Prüss, S. Shimizu and M. Wilke, Qualitative behaviour of incompressible two-phase flows with phase transitions: The case of non-equal densities. *Comm. Partial Differential Equations*, 39 (2014), 1236–1283.

[17] J. C. Slattery, *Advanced Transport Phenomena*, Cambridge Univ. Press, 1999.

[18] J. Stefan, Über das Gleichgewicht und Bewegung, insbesondere die Diffusion von Gemischen. *Sitzungsberichte Kaiserl. Akad. Wiss. Wien*, 63 (1871), 63–124.

Received June 2016; revised October 2016.

E-mail address: bothe@mma.tu-darmstadt.de
E-mail address: jan.pruess@mathematik.uni-halle.de