ON THE VIRTUAL LEVEL OF TWO-BODY INTERACTIONS AND APPLICATIONS TO THREE-BODY SYSTEMS IN HIGHER DIMENSIONS

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Abstract. We consider a system of three particles in dimension $d \geq 4$ interacting via short-range potentials, where the two-body Hamiltonians have a virtual level at the bottom of the essential spectrum. In dimensions $d = 2$ (in case of fermions) and $d = 3$ the corresponding three-body Hamiltonian admits an infinite number of bound states, which is known as the Efimov effect. In this work we prove that this is not the case in higher dimensions. We investigate how the dimension and symmetries of the system influence this effect and prove the finiteness of the discrete spectrum of the corresponding three-body Hamiltonian.

1. Introduction

In the early seventies a large amount of literature was developed around the investigation of the discrete spectrum of many-body operators, such as [8] by D.R. Yafaev and [28] by G. Zhislin. Especially the three-body Hamiltonian has attracted a lot of interest in mathematics and physics since then. It is well known that if the two-body Hamiltonian has negative spectrum, then the three-body Hamiltonian has at most a finite number of negative eigenvalues [8]. Therefore, from a mathematical point of view it was surprising what the physicists V. Efimov has predicted [2]; he claimed that the three-body system in dimension three exhibits an infinite number of bound states, provided every two-body subsystem admits a positive spectrum and at least two of the three two-body subsystems possess a resonance at zero.

From a physical point of view for a long time it was unclear whether this so called Efimov effect could actually be observed experimentally. It became an outstanding challenge to observe this phenomenon. Thirty-five years after Efimov’s prediction a group of experimental physicists observed these quantum states for the first time in an ultracold gas of caesium atoms [11], where temperartures around 10 nK were necessary to make the desired observation. Other observations of the Efimov states in different experiments followed later on, such as in the year 2009, where researchers reported experimental verification of the effect in bosonic quantum gases [27].

The first rigorous mathematical proof of the Efimov effect was given by D.R. Yafaev [6] where he used a symmetrized form of Faddeev equations for the three-body Hamiltonian. He also verified that if at least two of the three two-body Hamiltonians do not have a resonance, then the Efimov effect does not occur [8], which was also predicted in the original work of V. Efimov. The first proof of this fact based on variational arguments is due to S. Vugalter and G. Zhislin [20]. Later, this result was generalized in different directions, see [21],[24],[25],[26]. The first variational proof of the Efimov effect is due to Yu. N. Ovchinnikov and I. M. Sigal [16], which is based on the Born-Oppenheimer approximation. The proof has been improved later by H. Tamura [18]. The technique of A. Sobolev [17], which uses similar methods as Yafaev’s proof, combined with the low-energy behaviour of the resolvent [10] and the calculation of the distribution of a Toeplitz...
operator is of great importance. Using this technique he established the low energy asymptotics
\[
\lim_{z \to 0^-} \frac{N(z)}{\ln |z|} = \mathcal{U}_0 > 0,
\]
where \( N(z) \) is the counting function of the eigenvalues of the three-body Hamiltonian below \( z < 0 \). Later, H. Tamura improved this result by considering less restrictive pair potentials [19].

The original work of V. Efimov also discussed whether the effect is possible in dimensions one and two or in three-dimensional subspaces with fixed symmetries. The absence of the Efimov effect in such symmetry subspaces was proved by S. Vugalter and G. Zhislin [22]. They showed that the discrete spectrum of the Hamiltonian is always finite if one restricts its domain to functions underlying symmetries of irreducible representations of the group \( S_3 \) of permutation of particles. These subspaces are associated with nonzero angular momentum states or two or three identical fermions. The main reason of the finiteness of the discrete spectrum is due to the fact that in such subspaces a resonance state is always an eigenfunction corresponding to the eigenvalue zero. The existence and non-existence of the Efimov effect in lower dimensions was studied by the same authors in [23]. Under restrictive assumptions on the pair potentials and considered without symmetries they proved that the three-body system can admit at most a finite number of bound states in dimension two. In contrast to that, D.K. Gridnev has recently proved the existence of the so called super-Efimov effect [4] which at first was predicted in the physical work [14], where he considered a system of three spinless fermions in dimension two, each interacting through spherically-symmetric pair potentials. It turns out that such systems have two infinite series of bound states, each corresponding to the orbital angular momentum \( l = \pm 1 \) and
\[
\lim_{z \to 0^-} \frac{N(z)}{\ln |\ln z^2|} = \frac{8}{3\pi}.
\]

We want to emphasize that in the case of two-dimensional fermions, the two-body resonance behaves like \( c|x|^{-1} \) as \( |x| \to \infty \), which is the borderline case of not being square-integrable. A similar situation occurs in dimension four, where the Hamiltonian is considered without symmetry restrictions. The corresponding resonance function decays as \( c|x|^{-2} \), which is also on the edge of being in \( L^2 \). However, the physicists predicted that there is no Efimov effect in this case [15]. Our goal is to give rigorous mathematical proofs of these statements for any dimension greater than three. To the best of our knowledge it has not been studied in full detail so far. We highlight the differences between the behaviour of the corresponding counting functions and give a precise mathematical proof why the Efimov effect is absent in the four-dimensional case by using Sobolev’s technique [17] of a symmetrized form of the Faddeev equations. We study a more general concept of resonances called virtual levels and prove some of their fundamental characteristics, such as the impact on the behaviour of the resolvent of the two-body Hamiltonians and its consequences for the three-body system. This also leads to the finiteness of the discrete spectrum of the corresponding Hamiltonian in any dimension equal or greater than five.

The paper is organized as follows. In Section 2 we introduce the notation for the three-body system and the corresponding two-body subsystems, and formulate the main results. In Section 3 we consider two-body Hamiltonians with a virtual level at the bottom of the essential spectrum. We describe the relation between virtual levels, resonances and actual bound states at zero energy, which depend not only on the dimension of the particles, but also on the corresponding symmetry. Section 4 is devoted to the four-dimensional case. We prove resolvent-related properties of resonances, which will be used in the proof of the main result. In Section 5 we prove the main results by combining the auxiliary statements from the previous sections.
2. Notation and main result

We consider a system of three particles with masses $m_1, m_2, m_3 > 0$ and corresponding position vectors $x_1, x_2, x_3 \in \mathbb{R}^d$, $d \geq 4$. The Hamiltonian of such a system in coordinate representation is given by

$$-\frac{1}{2m_1} \Delta_{x_1} - \frac{1}{2m_2} \Delta_{x_2} - \frac{1}{2m_3} \Delta_{x_3} + v_{12}(x_{12}) + v_{23}(x_{23}) + v_{31}(x_{31}),$$

where $x_{ij} = x_i - x_j$, $i, j = 1, 2, 3$, and $\Delta_{x_i}$ denotes the Laplacian with respect to coordinate $x_i = (x_{1i}, \ldots, x_{id}) \in \mathbb{R}^d$ of the i-th particle. The real valued potential $v_{\alpha}$, $\alpha \in \{12, 23, 31\}$, describes the interaction of the corresponding particles with masses $m_1$ and $m_2$ or $m_2$ and $m_3$ or $m_3$ and $m_1$, respectively. We assume that the potentials $v_{\alpha}$ satisfy

$$v_{\alpha} \in L^d_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad |v_{\alpha}(x)| \leq C|x|^{-b}, \quad \text{if} \quad |x| \geq \gamma \quad (2.1)$$

for some constant $\gamma > 0$ and $b > 2$. After separation of the center of mass the Hamiltonian of relative motion can be written as

$$H = H_0 + \sum_{\alpha} v_{\alpha}, \quad (2.2)$$

where $H_0$ denotes the free Hamiltonian of the system. The corresponding configuration space $R_0$ is a $3(d-1)$-dimensional subspace of $\mathbb{R}^{3d}$. Under assumptions (2.1) on the potentials $v_{\alpha}$ the operator $H$ is essentially self-adjoint. Every two-body subsystem corresponding to the subscript $\alpha \in \{12, 23, 31\}$ is described in the center of mass frame by the Hamiltonian

$$h_{\alpha} = -\frac{1}{2 m_{\alpha}} \Delta + v_{\alpha} \quad (2.3)$$

in $L^2(\mathbb{R}^d)$, where $m_{\alpha}$ is the reduced mass. Denote

$$\mu = \min_{\alpha} \inf \sigma(h_{\alpha}), \quad (2.4)$$

then by the HVZ-Theorem one has

$$\sigma_{\text{ess}}(H) = [\mu, \infty). \quad (2.5)$$

The case $\mu < 0$ in dimension three was studied earlier [28] and can be adapted to dimension $d \geq 4$. Hence, we only consider the case $\mu = 0$. Our main results are the following two Theorems.

**Theorem 2.1.** For $d = 4$ let $v_{\alpha}(x) \leq 0$ as well as

$$|v_{\alpha}(x)| \leq C(1 + |x|)^{-b}, \quad b > 4,$$

and for $d \geq 5$ let $v_{\alpha}$ satisfy (2.1). Then $\sigma_{\text{disc}}(H)$ is finite.

**Remark.** In dimension $d = 4$ we analyse the spectrum of the three-body Hamiltonian by the use of Faddeev equations following [17], which require more restrictions on the potentials. In case of $d \geq 5$ we make use of a variational type of argument following [22], which allows us to have more general assumptions on the potentials, namely (2.1). In case of three identical particles the potentials $v_{\alpha}$ satisfy $v_{\alpha}(x_{ij}) = v_{\alpha}(-x_{ij})$ and the operator $H$ is invariant under the action of the group $S_3$ of permutatation of particles. Denote by $\sigma_1, \sigma_2$ and $\sigma_3$ the three irreducible representations of $S_3$, where $\sigma_1$ is the symmetric representation, $\sigma_2$ the antisymmetric representation and $\sigma_3$ the two-dimensional irreducible representation, respectively. Denote by $P^{\sigma_i}$, $i = 1, 2, 3$ the corresponding projection. Symmetries of types $\sigma_1$ and $\sigma_3$ do not put any restrictions on the symmetry of the two-particle subsystems. In case of $\sigma_2$ we denote the two-body Hamiltonians on $P^{\sigma_2} L^2$ by $h_{\sigma_2}^{\alpha}$ and the corresponding three-body Hamiltonian by $H^{\alpha_2}$. In dimensions $d \in \{2, 3, 4\}$ the operator $h_{\sigma_2}^{\alpha}$ has different properties of so called virtual levels (see later for a precise definition), which itself determines the existence or non-existence of the Efimov effect.
Theorem 2.2. For \( d \geq 4 \) let the potentials \( v_\alpha \) satisfy (2.1) and \( v_\alpha(x) = v_\alpha(-x) \). Then \( \sigma_{\text{disc}}(H^{\text{as}}) \) is finite.

Remark. Consequently, the super-Efimov effect does not exist in dimension \( d \geq 4 \).

3. Virtual levels of two-body subsystems

The finiteness of the discrete spectrum of the three-body Hamiltonian depends on the existence and properties of resonances in the two-body subsystems, which appear in the frame of critical potentials and are sometimes called virtual levels of the Hamiltonian. We introduce the concept of virtual levels following [7]. For the sake of brevity we omit the subscript \( \alpha \) in this section and add it for every corresponding expression only to distinguish between different subsystems, i.e. we consider the Schrödinger operator

\[
h = -\frac{1}{2m} \Delta + v.
\]

This operator is given in the center of mass frame, acting in \( L^2(\mathbb{R}^d) \), \( d \in \mathbb{N} \), where \( m > 0 \) is the reduced mass and \( v \) is the multiplication by a real valued function satisfying (2.1).

Definition 3.1. The operator \( h = -\frac{1}{2m} \Delta + v \) has a virtual level at the bottom of its spectrum, if

\[
h \geq 0 \quad \text{and} \quad \sigma_{\text{disc}} \left( -\frac{1}{2m} \Delta + (1 + \varepsilon) v \right) \neq \emptyset
\]

holds for every \( \varepsilon > 0 \).

The existence of a virtual level of \( h \) is connected to the following homogeneous Sobolev-space

\[
\dot{H}^1(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d), \quad \|f\|_{(1)} = \left( \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

Theorem 3.2. Let \( d \geq 4 \). If the operator \( h \) has a virtual level, then there exists a positive function \( f \in \dot{H}^1(\mathbb{R}^d) \), \( f \neq 0 \), such that

\[
\left( -\frac{1}{2m} \Delta + v \right) f = 0 \quad (3.1)
\]

in the distributional sense.

Remark. If \( f \notin L^2(\mathbb{R}^d) \), then \( \lambda = 0 \) is called resonance of \( h \) with the corresponding resonance state \( f \).

Proof. Let \( k, n \in \mathbb{N} \), \( B_k = \{ x \in \mathbb{R}^d : |x| \leq k \} \) and

\[
h_0^k = -\frac{1}{2m} \Delta + v, \quad h_n^k = -\frac{1}{2m} \Delta + \left(1 + \frac{1}{n}\right) v,
\]

considered as operators in \( L^2(B_k) \) with form domains \( H^1_0(B_k) \). Denote by \( \lambda_0^k \) and \( \lambda_n^k \) the smallest Dirichlet-eigenvalue of the operator \( h_0^k \) and \( h_n^k \), respectively. Since every domain \( B_k \) is bounded we have \( \lambda_0^k > 0 \) and \( \lambda_n^k < 0 \) for every \( n \in \mathbb{N} \) and sufficiently large \( k \in \mathbb{N} \) due to the existence of the virtual level. For every sufficiently large \( n \in \mathbb{N} \) we can find a \( k_n \in \mathbb{N} \), such that \( \lambda_n^{k_n} \geq 0 \) and \( \lambda_n^{k_n+1} < 0 \), which implies the existence of a region \( A_n \subset \mathbb{R}^d \) and zero-eigenfunctions \( u_n \in H^1_0(A_n) \).

We normalize \( u_n \) by

\[
\|u_n\|_{(1)} = \int_{A_n} |\nabla u_n(x)|^2 \, dx = 1. \quad (3.2)
\]

Let

\[
f_n : \mathbb{R}^d \to \mathbb{R}, \quad f_n(x) = \begin{cases} u_n(x), & x \in A_n \\ 0, & x \in \mathbb{R}^d \setminus A_n \end{cases}
\]
Then we have \( f_n \in H^1(\mathbb{R}^d) \) and \( \|f_n\|_{(1)} = 1 \). Due to (3.2) there exists a subsequence (also denoted by \( f_n \)) and a function \( f \), such that \( f_n \rightharpoonup f \) in \( H^1(\mathbb{R}^d) \), i.e.

\[
\int_{\mathbb{R}^d} \nabla f_n \cdot \nabla \phi \, dx \xrightarrow{n \to \infty} \int_{\mathbb{R}^d} \nabla f \cdot \nabla \phi \, dx
\]

holds for every \( \phi \in C_0^\infty(\mathbb{R}^d) \). Note that by assumptions (2.1) the following operator is well defined.

\[ T_v : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad (T_v u)(x) = |v(x)|^{\frac{1}{2}} u(x). \]

\( T_v \) is compact, since for every \( R > 0 \) the operator

\[ T_v^R : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad (T_v^R u)(x) = |v(x)|^{\frac{1}{2}} u(x) \chi_R(x) \]

is obviously compact and for sufficiently large \( R > 0 \) we have

\[
\|(T_v - T_v^R)u\|_{L^2(\mathbb{R}^d)}^2 = \int_{\{|x| > R\}} |v(x)||u(x)|^2 \, dx = \int_{\{|x| > R\}} |v(x)||x|^2 \frac{|u(x)|^2}{|x|^2} \, dx
\]

\[
\leq C \sup_{|y| > R} |y|^{2-b} \int_{\{|x| > R\}} |\nabla u(x)|^2 \, dx \xrightarrow{R \to \infty} 0.
\]

Due to the compactness of \( T_v \), the sequence \((T_v f_n)\) converges in \( L^2(\mathbb{R}^d) \) and therefore

\[
\left(1 + \frac{1}{n}\right) \int_{\mathbb{R}^d} v(x)f_n(x)\bar{\phi}(x) \, dx \longrightarrow \int_{\mathbb{R}^d} v(x)f(x)\bar{\phi}(x) \, dx
\]

(3.3)

holds for every \( \phi \in C_0^\infty(\mathbb{R}^d) \). Note that

\[
\int_{\mathbb{R}^d} vf(f - \phi) \, dx - \int_{\mathbb{R}^d} v|f|^2 \, dx = -\int_{\mathbb{R}^d} vf \phi \, dx
\]

\[
= \int_{\mathbb{R}^d} \nabla f \cdot \nabla \phi \, dx = \int_{\mathbb{R}^d} |\nabla f|^2 \, dx - \int_{\mathbb{R}^d} \nabla f \cdot (\nabla f - \nabla \phi) \, dx.
\]

Hence, by approximating \( f \) with functions \( \phi_k \in C_0^\infty(\mathbb{R}^d) \), together with \( \|v|f|\| \leq C\|\nabla f\| \) and \( \|f\|_{(1)} = 1 \) we have \( \int |v||f|^2 \, dx = 1 \), which implies \( f \neq 0 \). Therefore \( f \in H^1(\mathbb{R}^d) \) is a positive solution of

\[
\left(-\Delta + \frac{1}{2m} \right) f = 0.
\]

(3.4)

This completes the proof. \( \square \)

**Remark.** If the operator \( h \) is considered without symmetry restrictions, then the solution \( f \) is non-degenerate. Therefore, the proof of Theorem 3.2 yields the following Corollary.

**Corollary 3.3.** There exists a constant \( \mu > 0 \), such that for every function \( g \in H^1(\mathbb{R}^d) \setminus \{0\} \) with \( \langle \nabla g, \nabla f \rangle = 0 \) one has

\[
\langle (-\Delta + v)g, g \rangle \geq \mu \|\nabla g\|^2.
\]

(3.5)

**Lemma 3.4.** Let \( d \geq 4 \) and let \( v \) satisfy assumptions (2.1). Then for every \( \psi \in H^1(\mathbb{R}^d) \) one has \( v\psi \in L^2(\mathbb{R}^d) \).

**Proof.** We consider

\[
\int_{\mathbb{R}^d} |v(x)\psi(x)|^2 \, dx = \int_{\{|x| < \gamma\}} |v(x)\psi(x)|^2 \, dx + \int_{\{|x| \geq \gamma\}} |v(x)\psi(x)|^2 \, dx,
\]
where $\gamma > 0$ is the constant in (2.1). Since $v \in L_{\text{loc}}^q(\mathbb{R}^d)$ and $\psi \in \dot{H}^1(\mathbb{R}^d)$ we obtain by the use of the Sobolev inequality

$$\int_{\{|x|<\gamma\}} |v(x)\psi(x)|^2 \, dx \leq \left( \int_{\{|x|<\gamma\}} |v(x)|^q \, dx \right)^{\frac{2}{q}} \left( \int_{\{|x|<\gamma\}} |\psi(x)|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \leq c||\psi||_{(1)}^{\frac{2(d-2)}{d}}.$$  

By the second assumption of (2.1) we can make use of Hardy’s inequality to conclude

$$\int_{\{|x|\geq\gamma\}} |v(x)\psi(x)|^2 \, dx \leq c\int_{\{|x|\geq\gamma\}} \frac{|\psi(x)|^2}{|x|^2} \, dx \leq C||\psi||_{(1)}^2.$$  

Hence, $v\psi \in L^2(\mathbb{R}^d)$, which completes proof. \qed

**Lemma 3.5.** If $d = 4$, then the solution $f$ of (3.1) belongs to the space

$L^2_{-\sigma}(\mathbb{R}^4) = \{ \varphi : \mathbb{R}^4 \to \mathbb{R} \mid (1 + |\cdot|)^{-\sigma} \varphi \in L^2(\mathbb{R}^4) \}$

for every $\sigma > 0$. If $d \geq 5$, then $f \in L^2(\mathbb{R}^d)$.

**Proof.** At first let $\gamma_0 \in (1, 2)$. Then we have

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{(1 + |x|)^{2\gamma_0}} \, dx = \int_{\mathbb{R}^d} \frac{|x|^2}{(1 + |x|)^{2\gamma_0}} \left| f(x) \right|^2 \, dx \leq C \int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx = C||f||_{(1)}^2,$$  

which implies $f \in L^2_{-\gamma_0}(\mathbb{R}^d)$. By Lemma 3.4 and [12] we have

$$f(x) = G \ast (v \cdot f)(x) = -\frac{2m}{(d-2)w_d} \int_{\mathbb{R}^d} \frac{v(y)f(y)}{|x-y|^{d-2}} \, dy,$$  

where $G$ is the fundamental solution of $\frac{\partial}{\partial t} \Delta$, which is given by

$$G(x) = -\frac{2m}{(d-2)w_d} \frac{1}{|x|^{d-2}}$$

and $w_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. We write

$$-\frac{(d-2)w_d}{2m} f = f_1 + f_2,$$  

where

$$f_1(x) = \int_{\{|x-y|<2\}} \frac{v(y)f(y)}{|x-y|^{d-2}} \, dy \quad \text{and} \quad f_2(x) = \int_{\{|x-y|\geq2\}} \frac{v(y)f(y)}{|x-y|^{d-2}} \, dy.$$  

Since the function $x \mapsto |x|^{-(d-2)} \chi_{\{|x|<2\}}(x)$ belongs to $L^1(\mathbb{R}^d)$ and by Lemma 3.4 we have $vf \in L^2(\mathbb{R}^d)$, Young’s inequality implies $f_1 \in L^2(\mathbb{R}^d)$. Let us show that $f_2 \in L^2_{-\sigma}(\mathbb{R}^d)$ for every $\sigma > 0$. We rewrite assumption (2.1) as

$$|v(x)| \leq C|x|^{-(2+\theta)},$$

where $|x| \geq \gamma$ and $\theta > 0$. Note that we can always assume $\theta < 1$. Let $1 \leq q < 2$, then by Lemma 3.4 we have $vf \in L^q_{\text{loc}}(\mathbb{R}^d)$. Hence, by the use of Hölder’s inequality with $p_1 = \frac{2}{q}$ and $p_2 = \frac{2}{2-q}$ we obtain

$$\|vf\|_{L^q_{\text{loc}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(y)v(y)|^q \, dy = \int_{\mathbb{R}^d} \frac{|f(y)|^q}{(1 + |y|)^{2\gamma_0}} |v(y)|^q(1 + |y|)^{\gamma_0} \, dy$$

$$\leq C \left( \int_{\mathbb{R}^d} \frac{|f(y)|^2}{(1 + |y|)^{2\gamma_0}} \, dy \right)^{\frac{q}{2}} \left( \int_{\mathbb{R}^d} (1 + |y|)^{-p_2q(2+\theta-\gamma_0)} \, dy \right)^{\frac{1}{q}}.$$  

(3.9)
By (3.6) we have \( f \in L^2_{s_0}(\mathbb{R}^d) \). Therefore, (3.9) is finite for
\[
p_{2q}(2 + \theta - s_0) > d \quad \Leftrightarrow \quad q > \frac{d}{2 + \frac{d}{2} - (s_0 - \theta)},
\]
which implies that \( v f \in L^q(\mathbb{R}^d) \) for
\[
\frac{1}{q} = \frac{2}{d} + \frac{d}{2} - (s_0 - \theta).
\]
Since \( x \mapsto |x|^{-(d-2)} \chi_{\{|x| \geq 2\}}(x) \) belongs to \( L^p(\mathbb{R}^d) \), where \( p > \frac{d}{(d-2)} \), i.e. \( \frac{1}{p} < 1 - \frac{2}{d} \), we conclude by Young’s inequality that \( f_2 \in L^r(\mathbb{R}^d) \), where
\[
r > \frac{d}{2} - (s_0 - \theta).
\]
By the use of Hölder’s inequality it follows \( f_2 \in L^2_{s_1}(\mathbb{R}^d) \), where \( s_0 - \theta < s_1 < 1 \). Subsequently applying this type of argument for \( d = 4 \) yields \( f \in L^2_{s_0}(\mathbb{R}^4) \) for all \( s > 0 \). Let us show that in the case of \( d \geq 5 \) we have \( f \in L^2(\mathbb{R}^d) \). Note that \( 2^{-1} < 2^{-1} + 2d^{-1} < 1 \) holds if and only if \( d \geq 5 \). Since \( v f \in L^q(\mathbb{R}^d) \) for any \( 1 \leq q \leq 2 \), we can choose
\[
2^{-1} + 2d^{-1} < q^{-1} < 1 \quad \text{and} \quad 2^{-1} < p^{-1} < 1 - 2d^{-1},
\]
such that \( q^{-1} + p^{-1} = 1 + 2^{-1} \). Applying Young’s inequality yields \( f \in L^2(\mathbb{R}^d) \).
\( \square \)

**Remark.** In dimension \( d = 3 \) (see [7]) the corresponding resonance state belongs to the weighted Sobolev space
\[
L^2_{s_0}(\mathbb{R}^3) = \left\{ \varphi : \mathbb{R}^3 \to \mathbb{R} : (1 + |\cdot|)^{-s} \varphi \in L^2(\mathbb{R}^3) \right\},
\]
where \( s > \frac{1}{2} \). Our proof can be adapted to the case \( d = 3 \), where in view of (3.10) one has \( s_0 - \theta > \frac{1}{2} \), which leads to \( f \in L^2_{s_0}(\mathbb{R}^3) \), \( s > \frac{1}{2} \).

**Lemma 3.6.** If \( d = 4 \), then the solution \( f \) of (3.1) satisfies
\[
f(x) = -\frac{2\pi^2}{m^2} \frac{\langle v, f \rangle}{|x|^2} + \tilde{f}(x)
\]
as \( |x| \to \infty \), where \( \tilde{f} \in L^2(\mathbb{R}^4) \).

**Proof.** For \( d = 4 \) we have \( w_4 = 4\pi^2 \) and in view of (3.8)
\[
-\frac{2\pi^2}{m} f = f_1 + f_2,
\]
where \( f_1 \in L^2(\mathbb{R}^4) \) and \( f_2 = g_2 + h_2 \), such that
\[
g_2(x) = \int_{\{|x-y| \geq 2 \wedge |y| \geq |x|^2\}} \frac{v(y)f(y)}{|x-y|^2} \, dy, \quad h_2(x) = \int_{\{|x-y| \geq 2 \wedge |y| \leq |x|^2\}} \frac{v(y)f(y)}{|x-y|^2} \, dy.
\]
Since we consider \( |x| \to \infty \), we can set \( |x| > \max\{4, \gamma^2\} \). By the use of \( |v(y)| \leq C(1 + |y|)^{-2-\theta} \) for \( |y| > |x|^\frac{1}{2} \geq \gamma \) and \( \theta > 0 \) we obtain
\[
|g_2(x)| \leq C \left(1 + |x|^{\frac{1}{2}}\right)^{-\frac{\theta}{2}} \int_{\{|x-y| \geq 2\}} \frac{|f(y)|}{(1 + |y|)^{\frac{\theta}{2} + \frac{\theta}{2}|x-y|^2}} \, dy.
\]
Now since \( f \in L^2_{\text{loc}}(\mathbb{R}^4) \) holds for all \( s > 0 \), we have \((1 + |\cdot|)^{-(2+\frac{s}{2})} f \in L^1(\mathbb{R}^4)\) and therefore \( g_2 \in L^2(\mathbb{R}^4)\). For \(|y| \leq |x|^\frac{1}{2}\) we have

\[
|x-y|^{-2} = |x|^{-2} - \frac{x}{|x|} \cdot \frac{y}{|y|} \geq |x|^{-2} \left(1 + |x|^{-\frac{1}{2}} \right)^{-2} = |x|^{-2} \frac{(1 - |x|^{-\frac{1}{2}})^2}{(1 - |x|^{-1})^2} \geq |x|^{-2} \left(1 - |x|^{-\frac{1}{2}} \right)^2.
\]

On the other hand we can estimate

\[
|x-y|^{-2} \leq (|x| - |y|)^{-2} = |x|^{-2} \left(\sum_{k=0}^{\infty} \left|\frac{y}{|x|}\right|^{k}\right)^2 \leq |x|^{-2} \left(1 + 2|x|^{-\frac{1}{2}} \right)^2.
\]

This implies

\[
\frac{\langle v, f \rangle}{|x|^2} \left(1 - |x|^{-\frac{1}{2}} \right)^2 \leq |h_2(x)| \leq \frac{\langle v, f \rangle}{|x|^2} \left(1 + 2|x|^{-\frac{1}{2}} \right)^2,
\]

which completes the proof. \(\square\)

4. Resonance Interaction in Dimension Four

In this section we consider \( d = 4 \) and we further assume that the pair potentials \( \alpha \) satisfy

\[
v_\alpha(x) \leq 0 \quad \text{and} \quad |v_\alpha(x)| \leq C(1 + |x|)^{-b}, \quad b > 4. \quad (4.1)
\]

We will adapt the technique of [17] to simplify the representation of \( H \) and carry out the computations in the momentum space. By using this method we can also highlight the main reason why the Efimov effect is absent in this case.

Following [3] we denote by \( k_i \) the conjugate variable of \( x_i \) and introduce the set of variables \((k_\alpha, p_\alpha)\), conjugate with respect to the Jacobi-coordinates \((x_\alpha, y_\alpha)\), see (A.1)-(A.3) in the Appendix. The shift from \( x_\alpha \) to \( k_\alpha \) is carried out by the partial Fourier transform

\[
(\Phi_\alpha f)(k_\alpha, \cdot) = (2\pi)^{-2} \int_{\mathbb{R}^4} e^{-i(k_\alpha x_\alpha)} f(x_\alpha, \cdot) \, dx_\alpha.
\]

Every pair of variables \( k_\alpha, p_\alpha \) can be expressed by means of every other pair (see (A.7)-(A.9)). In this setting the three-particle Schrödinger operator (by abuse of notation) has the form

\[
H = H_0 + \sum_\alpha V_\alpha,
\]

where the kinetic energy is given by the multiplication of the function

\[
H_0 f(k, p) = H^0(k, p) \cdot f(k, p),
\]

where

\[
H^0(k, p) = \frac{k_\alpha^2}{2m_\alpha} + \frac{p_\alpha^2}{2n_\alpha} = \frac{k_\beta^2}{2m_\beta} + \frac{p_\beta^2}{2n_\beta} = \frac{k_\gamma^2}{2m_\gamma} + \frac{p_\gamma^2}{2n_\gamma}
\]

and the interactions are given by

\[
V_\alpha = \Phi_\alpha v_\alpha \Phi_\alpha^*.
\]

Often it is useful to work with coordinates \((p_\alpha, p_\beta)\) instead of \((k_\alpha, p_\alpha)\). The relations are given by

\[
k_\alpha = d_{\alpha\beta} p_\alpha + e_{\alpha\beta} p_\beta,
\]

where the constants \( d_{\alpha\beta} \) and \( e_{\alpha\beta} \) depend only on the masses \( m_1, m_2 \) and \( m_3 \) (see (A.7)-(A.9)). We denote by \( H_{\alpha\beta}^0 \) the function \( H^0 \) expressed in terms of \( p_\alpha, p_\beta \), which then takes the form

\[
H_{\alpha\beta}^0(p_\alpha, p_\beta) = \frac{p_\alpha^2}{2m_\alpha} + \frac{(p_\alpha, p_\beta)}{l_{\gamma}} + \frac{p_\beta^2}{2m_\beta}.
\]
where \( l_1 \in \{ m_1, m_2, m_3 \} \) (see (A.10)). By virtue of (A.4)-(A.6) it follows that

\[
H^0_{\alpha\beta}(p_\alpha, p_\beta) \geq \frac{p_\alpha^2}{2l_\alpha} + \frac{p_\beta^2}{2l_\beta}.
\]  

(4.6)

Hence, by the use of the elementary Young’s inequality one obtains

\[
H^0_{\alpha\beta}(p_\alpha, p_\beta) \geq c|p_\alpha|^{2\kappa}|p_\beta|^{2\kappa'}, \quad \kappa + \kappa' = 1.
\]  

(4.7)

Following [6] and [17] we will use a symmetrized form of Faddeev equations to study the discrete spectrum of \( H \). See in the Appendix for a detailed derivation of the Faddeev equations.

**Definition 4.1.** Let \( z < 0 \) and

\[
A(z) = W^{\frac{1}{2}}(z)K(z)W^{\frac{1}{2}}(z),
\]

where

\[
W(z) = \begin{pmatrix}
W_{12}(z) & 0 & 0 \\
0 & W_{23}(z) & 0 \\
0 & 0 & W_{31}(z)
\end{pmatrix},
\]

such that

\[
W_\alpha(z) = I + |V_\alpha|^\frac{1}{2}R_\alpha(z)|V_\alpha|^{\frac{1}{2}}, \quad R_\alpha(z) = (H_0 + V_\alpha - z)^{-1}
\]  

(4.8)

and

\[
K(z) = \begin{pmatrix}
0 & K_{12|23}(z) & K_{12|31}(z) \\
K_{23|12}(z) & 0 & K_{23|31}(z) \\
K_{31|12}(z) & K_{32|23}(z) & 0
\end{pmatrix},
\]

(4.9)

such that

\[
K_{\alpha\beta}(z) = |V_\alpha|^{\frac{1}{2}}R_\alpha(z)|V_\beta|^{\frac{1}{2}}, \quad R_\alpha(z) = (H_0 - z)^{-1}.
\]

The main property of \( A(z) \) is the Birman-Schwinger-type characteristic. The proof of the following statement is given in [17].

**Theorem 4.2.** [17, Theorem 4.1] Let \( N(z) \) be the number of eigenvalues of the operator \( H \) below \( z < 0 \) and let \( n(1, A(z)) \) be the number of eigenvalues of the operator \( A(z) \) greater than one. Then

\[
N(z) = n(1, A(z)).
\]

**Remark.** In view of Theorem 4.2, to prove Theorem 2.1 in the case \( d = 4 \), it is sufficient to prove the compactness of \( A(z) \) for \( z \to 0 \). In the presence of resonances in the two-body subsystems the corresponding operator \( A_{R^3}(z) \) in dimension three is not compact up to \( z = 0 \), due to a singularity of \( W_{R^3}(z) \). To this end, we need to study the the operator \( W_\alpha(z) \) in the two-body subsystems.

**Two-body subsystems in dimension four.**

**Definition 4.3.** Let \( r_\alpha(z), z < 0 \), be the resolvent of \( h_\alpha = -\frac{1}{2m_\alpha}\Delta + v_\alpha \) and let

\[
w_\alpha(z) = I + |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}}.
\]

(4.10)

Note that \( w_\alpha(z) \) is uniformly bounded in \( L^2(\mathbb{R}^d) \) for every \( z \leq z_0 < 0 \), where \( |z_0| \) can be chosen arbitrarily small. Using the resolvent identity

\[
r_\alpha(z) = r_0(z) - r_\alpha(z)v_\alpha r(z) = r_0(z) - r_\alpha(z)v_\alpha r_0(z),
\]

we have

\[
I = \left( I - |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right) \left( I + |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right)
\]

\[
= \left( I + |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right) \left( I - |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right),
\]

\[
I = \left( I - |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right) \left( I + |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right)
\]

\[
= \left( I + |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right) \left( I - |v_\alpha|^\frac{1}{2}r_\alpha(z)|v_\alpha|^{\frac{1}{2}} \right),
\]
which implies
\[ w_\alpha(z) = I + |v_\alpha|^2 r_\alpha(z)|v_\alpha|^2 = \left( I - |v_\alpha|^2 r_0(z)|v_\alpha|^2 \right)^{-1}. \] (4.11)

Note that in accordance with Definition 4.1 we have
\[ W_\alpha(z) = I + |V_\alpha|^2 R_\alpha(z)|V_\alpha|^2 = \Phi^* \alpha w_\alpha \left( z - \frac{p^2}{2m_\alpha} \right) \Phi^* \alpha, \] (4.12)
where \( \Phi^* \alpha \) is the partial Fourier transform defined in (4.2). The existence of a resonance of the two-body Hamiltonian \( h_\alpha \) affects the behaviour of \( w_\alpha(z) \) for \( z \to 0 \) (see [9]). It produces a singularity of \( w_\alpha(z) \) at \( z = 0 \), which leads in dimension \( d = 3 \) to the Efimov effect (see [17]). We will see that in dimension four the singularity is not strong enough to break the compactness of \( A(z) \) for \( z \to 0 \).

**Lemma 4.4.** Let \( G_\alpha \) be the integral operator with the kernel
\[ G_\alpha(x,y) = \frac{m_\alpha |v_\alpha(x)|^2 |v_\alpha(y)|^2}{|x - y|^2}, \] (4.13)
acting in \( L^2(\mathbb{R}^4) \). If \( \lambda = 0 \) is a resonance of \( h_\alpha \), then \( \mu = 1 \) is a simple eigenvalue of \( G_\alpha \).

**Proof.** By Lemma 3.6 the resonance is non-degenerate. Let \( f \) be a resonance state of \( h_\alpha \) and let \( \varphi = |v_\alpha|^2 f \). Then by Lemma 3.5 we have \( \varphi \in L^2(\mathbb{R}^4) \) and
\[ (G_\alpha \varphi)(x) = \frac{m_\alpha}{2\pi^2} \int_{\mathbb{R}^4} \frac{|v_\alpha(x)|^2 |v_\alpha(y)|^2}{|x - y|^2} \varphi(y) dy = |v_\alpha(x)|^2 \left( -\frac{m_\alpha}{2\pi^2} \int_{\mathbb{R}^4} \frac{v_\alpha(y)f(y)}{|x - y|^2} dy \right) = |v_\alpha(x)|^2 f(x) = \varphi(x). \]

**Lemma 4.5.** Let \( G_\alpha \) be the operator defined by the kernel (4.13). For \( z < 0 \), \( |z| \) sufficiently small, there exist compact operators \( G_1, G_2 \) and a constant \( \delta > 0 \), such that
\[ |v_\alpha|^2 r_0(z)|v_\alpha|^2 = G_\alpha + zG_1 + z \ln |z|G_2 + |z|^{1+\delta}G_\alpha^{(\delta)}(z), \]
where \( G_\alpha^{(\delta)}(z) \) is an operator, such that \( \|G_\alpha^{(\delta)}(z)\|_{HS} \leq C_\delta |z|^\delta \), where \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm.

**Proof.** In the following we consider \( |z| < 1 \). The kernel of \( (-\Delta - z)^{-1} \) is given by
\[ (-\Delta - z)^{-1}(|x - y|) = \frac{i \sqrt{z}}{8\pi|x - y|} H_1^{(1)}(\sqrt{|x - y|}), \quad x, y, \in \mathbb{R}^4, \]
where \( H_1^{(1)} \) is the first Hankel function (see [1]). Hence,
\[ r_0(z,|x - y|) = \left( -\frac{1}{2m_\alpha} \Delta - z \right)^{-1}(|x - y|) = 2m_\alpha (-\Delta - 2m_\alpha z)^{-1}(|x - y|) = \frac{m_\alpha \sqrt{2m_\alpha z}}{4\pi|x - y|} H_1^{(1)}(\sqrt{2m_\alpha z}|x - y|). \] (4.14)

According to [1], p.360, one has \( H_1^{(1)}(\zeta) = J_1(\zeta) + iY_1(\zeta) \) and
\[ J_1(\zeta) = \frac{\zeta}{\pi} \sum_{k=0}^{\infty} \frac{(-\zeta^2)^k}{k!(k + 1)!}, \]
\[ Y_1(\zeta) = -\frac{2}{\pi \zeta} + \frac{2}{\pi} \ln \left( \frac{\zeta}{2} \right) J_1(\zeta) - \frac{\zeta}{2\pi} \sum_{k=0}^{\infty} (\psi(k + 1) + \psi(k + 2)) \frac{(-\zeta^2)^k}{k!(k + 1)!}. \]
We will show that the remainder \( G \) Schmidt norm is of order 

\[ \| G \| = \tfrac{1}{4\pi^{2}} \sum_{k=0}^{\infty} a_{k} (\xi^{2})^{k}, \]

(4.15)

where 

\[ a_{k} = \frac{(-1)^{k}}{k!} \quad \text{and} \quad b_{k} = (\psi(k + 1) + \psi(k + 2)) a_{k}. \]

Note that both series in (4.15) converge for every \( \xi \in \mathbb{C} \). By (4.14) and (4.15) we get

\[
|v_{n}|^{\frac{1}{2}}(x) v_{0}(z, |x - y|)|v_{n}|^{\frac{1}{2}}(y) = |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y) m_{\omega} \sqrt{2m_{\omega}} H_{1}^{(1)} \left( \frac{|2m_{\omega}|}{|x - y|} \right) \]

\[
= G_{\alpha}(x, y) + zG_{1}(x, y) + z \ln |z| G_{2}(x, y) + G(x, y, z), \]

(4.16)

where

\[
G_{\alpha}(x, y) = \frac{m_{\omega}}{2\pi^{2}} \frac{|v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y)}{|x - y|^{2}}, \quad (4.17)
\]

\[
G_{1}(x, y) = \frac{m_{\omega}^{2}}{4\pi^{2}} |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y) \left( \psi(1) + \psi(2) - \ln(2m_{\omega}) - 2 \ln \left( \frac{|x - y|}{2} \right) \right), \quad (4.18)
\]

\[
G_{2}(x, y) = - \frac{m_{\omega}^{2}}{4\pi^{2}} |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y). \quad (4.19)
\]

We will show that the remainder \( G(x, y, z) \) is a Hilbert-Schmidt kernel and that the Hilbert-Schmidt norm is of order \( O \left( |z|^{-1} \right) \) as \( z \to 0 \), where \( \delta > 0 \) is sufficiently small.

Let \( \sqrt{2m_{\omega}|z||x - y|} \) be \( 1 \). By [1], p.364, we have

\[
\left| H_{1}^{(1)}(\xi) \right| \leq c|\xi|^{-\frac{1}{2}}, \quad |\xi| \geq 1. \quad (4.20)
\]

Relations (4.14) and (4.16) imply

\[
|G(x, y, z)| \chi_{\left\{ \sqrt{2m_{\omega}|z||x - y|} > 1 \right\}} \leq c \left| \xi \right|^{\frac{1}{2}} \frac{|v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y)}{|x - y|} \left| H_{1}^{(1)} \left( \frac{|2m_{\omega}|}{|x - y|} \right) \right| + |G_{\alpha}| + |z||G_{1}| + |z \ln |z||G_{2}|.
\]

Hence, by definition of the kernels (4.17)-(4.19) together with (4.20) we have

\[
|G(x, y, z)| \chi_{\left\{ \sqrt{2m_{\omega}|z||x - y|} > 1 \right\}} \leq c \left| \xi \right| \ln |z| |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y) \left( c_{1} + c_{2} \left| \ln \left( \frac{|x - y|}{2} \right) \right| \right)
\]

\[
\leq |z| \ln |z| |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y) \left( \frac{1 + |x|}{1 + |x - y|} \right)^{4\delta} \left( c_{1} + c_{2} \left| \ln \left( \frac{|x - y|}{2} \right) \right| \right)
\]

\[
\leq c_{1} \left| \xi \right|^{4\delta} \ln |z| |v_{n}|^{\frac{1}{2}}(x)|v_{n}|^{\frac{1}{2}}(y) \left( 1 + |z| \right)^{4\delta} \left( c_{1} + c_{2} \ln \left( \frac{|x - y|}{2} \right) \right). \quad (4.21)
\]

Now let \( \sqrt{2m_{\omega}|z||x - y|} \leq 1 \). Note that in view of (4.15) we have

\[
G(x, y, z) = z|v_{n}|^{\frac{1}{2}}(x) \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (\ln |z|)^{k} G_{j}^{k}(x, y) |v_{n}|^{\frac{1}{2}}(y). \quad (4.22)
\]
where the kernels $G_j^k$ are defined by

$$G_j^k(x, y) = -(2m_\alpha)^j \alpha_j |x-y|^{2j}, \quad G_j^0(x, y) = (2m_\alpha)^j \alpha_j |x-y|^{2j} \left( \beta_j - 2 \ln \left( \frac{|x-y|}{2} \right) \right)$$

(4.23)

and the constants $\alpha_j, \beta_j$ are given by

$$\alpha_j = \frac{(-1)^j m_\alpha^2}{4^{j+1} \pi^j (j+1)!}, \quad \beta_j = \psi(j+1) + \psi(j+2) - \ln(2m_\alpha).$$

By definition of the kernels $G_j^k$, we have

$$G(x, y, z) = z|v_\alpha|^{\frac{1}{2}}(x) (\sigma_1(x, y, z) + \sigma_2(x, y, z) + \sigma_3(x, y, z)) |v_\alpha|^{\frac{1}{2}}(y),$$

(4.24)

where

$$\sigma_1(x, y, z) = \sum_{j=1}^{\infty} \alpha_j \beta_j \left( \sqrt{2m_\alpha} |x-y| \right)^{2j},$$

$$\sigma_2(x, y, z) = -2 \ln \left( \frac{|x-y|}{2} \right) \sum_{j=1}^{\infty} \alpha_j \left( \sqrt{2m_\alpha} |x-y| \right)^{2j},$$

$$\sigma_3(x, y, z) = -\ln |z| \sum_{j=1}^{\infty} \alpha_j \left( \sqrt{2m_\alpha} |x-y| \right)^{2j}.$$

We are going to estimate $\sigma_1, \sigma_2$ and $\sigma_3$ separately. Let $0 < \delta < 2^{-1}$. Since $\sqrt{2m_\alpha} |z||x-y| \leq 1$, we have

$$|\sigma_1(x, y, z)| \leq \left( \sqrt{2m_\alpha} |z||x-y| \right)^{2\delta} \sum_{j=1}^{\infty} |\alpha_j \beta_j| \left( \sqrt{2m_\alpha} |z||x-y| \right)^{2(j-2\delta)},$$

$$\leq C |z|^{2\delta} |x-y|^{2\delta} \sum_{j=1}^{\infty} |\alpha_j \beta_j| \leq C_1 |z|^{2\delta} (1 + |x|)^{4\delta} (1 + |y|)^{4\delta}. \quad (4.25)$$

In the last inequality we used the fact that $\sum_{j=1}^{\infty} |\alpha_j \beta_j| < \infty$. Analogously we obtain

$$|\sigma_2(x, y, z)| \leq 2 \left| \ln \left( \frac{|x-y|}{2} \right) \right| \sum_{j=1}^{\infty} |\alpha_j| \left( \sqrt{2m_\alpha} |z||x-y| \right)^{2j},$$

$$\leq C_2 |z|^{2\delta} \left| \ln \left( \frac{|x-y|}{2} \right) \right| (1 + |x|)^{4\delta} (1 + |y|)^{4\delta} \quad (4.26)$$

and also

$$|\sigma_3(x, y, z)| \leq C_3 |z|^{2\delta} \ln |z|(1 + |x|)^{4\delta} (1 + |y|)^{4\delta}. \quad (4.27)$$

Hence, by collecting estimates (4.25)-(4.27) together with (4.24) we get for $|z| < 1$ sufficiently small

$$|G(x, y, z)| \chi_{\{ \sqrt{2m_\alpha} |z||x-y| \leq 1 \}}$$

$$\leq |z|^{1+2\delta} \ln |z| |v_\alpha|^\frac{1}{2}(x) |v_\alpha|^\frac{1}{2}(y) (1 + |x|)^{4\delta} (1 + |y|)^{4\delta} \left( c_3 + c_4 \ln \left( \frac{|x-y|}{2} \right) \right). \quad (4.28)$$

By combining estimates (4.28) and (4.21) we get

$$|G(x, y, z)| \leq C |z|^{1+2\delta} \ln |z| |v_\alpha|^\frac{1}{2}(x) |v_\alpha|^\frac{1}{2}(y) (1 + |x|)^{4\delta} (1 + |y|)^{4\delta} \left( 1 + \ln \left( \frac{|x-y|}{2} \right) \right).$$
Since
\[ \left| \ln \left( \frac{|x - y|}{2} \right) \right| \leq C \max \left\{ |x - y|^\varepsilon, |x - y|^{-\varepsilon} \right\}, \varepsilon > 0 \]
and
\[ |v_\alpha(x)| \leq C(1 + |x|)^{-b}, \ b > 4, \]
we can choose \( \varepsilon, \delta > 0 \), such that \( 0 < \delta < \frac{b - \varepsilon - 2\varepsilon}{8} \), which implies that the remainder \( G(z) \) is Hilbert-Schmidt and the operator norm is of order \( \mathcal{O}(\!|z|^{1+2\varepsilon} \ln |z|) \). Hence, the operator \( G_\alpha^{(\delta)}(z) = |z|^{-1-\delta}G(z) \) is bounded up to \( z \leq 0 \). Further, we have
\[ |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}} = G_\alpha + zG_1 + z\ln |z|G_2 + |z|^{1+\delta}G_\alpha^{(\delta)}(z), \ \delta > 0, \]
which completes the proof. \( \square \)

Remark. We used similar arguments as in [9], where it was shown that for
\[ |v_\alpha(x)| \leq C(1 + |x|)^{-b}, \ b > 8, \]
\( G(z) \) is of order \( \mathcal{O}(|z|^2 \ln |z|) \). We allow weaker assumptions on the potential and obtain a weaker estimate as a result. By using arguments from [9] we can prove the following Lemma.

Lemma 4.6. If \( \lambda = 0 \) is a resonance of \( h_a \), then for \( z < 0, |z| \) sufficiently small, the operator \( w_\alpha(z) \) has the representation
\[ w_\alpha(z) = (z(\ln |z| - \tau_\alpha))^{-1}\langle \cdot, \varphi \rangle \varphi + (z(\ln |z| - \tau_\alpha))^{-1+\delta}w_\alpha^{(\delta)}(z), \]
where \( \delta > 0 \) is sufficiently small, \( \varphi \) is an eigenfunction of the operator \( G_\alpha \) corresponding to the eigenvalue \( \mu = 1 \) and \( \tau_\alpha \in \mathbb{R} \) is a constant, which depends on the potential \( v_\alpha \) and the mass \( m_\alpha \).

In addition, the operator \( w_\alpha^{(\delta)}(z) \) is bounded for \( z \leq 0 \).

Proof. Let
\[ s_\alpha(z) = I - |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}}. \]

We will use the expansion of Lemma 4.5 in order to compute the inverse
\[ s_\alpha^{-1}(z) = w_\alpha(z). \]

Let \( P_0 \) be the one-dimensional projection on the subspace associated with the eigenfunction \( \varphi \) of the operator \( G_\alpha \) corresponding to the eigenvalue \( \mu = 1 \) (c.f. Lemma 4.4) and denote by \( P_1 \) the projection onto the orthogonal complement of the eigenspace of \( \mu \) in \( L^2(\mathbb{R}^4) \). Following [9], for every \( \psi \in L^2(\mathbb{R}^3) \) we have the unique decomposition \( \psi = P_0\psi + P_1\psi \), which allows us to write \( s_\alpha(z) \psi \) as \( S(z)(P_0\psi, P_1\psi) \), where
\[ S(z) = \begin{pmatrix} P_0s_\alpha(z)P_0 & P_0s_\alpha(z)P_1 \\ P_1s_\alpha(z)P_0 & P_1s_\alpha(z)P_1 \end{pmatrix}. \]

Further, let
\[ P(z) = \begin{pmatrix} |z|^{\frac{\delta}{2}} & 0 \\ 0 & P_1 \end{pmatrix} \]
and
\[ B(z) = P(z)S(z)P(z). \]
The entries of \( B(z) \) are given by
\[ b_{00}(z) = |z|^{-\frac{\delta}{2}}P_0(I - |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}})P_0, \]
\[ b_{01}(z) = |z|^{-\frac{\delta}{2}}P_0(I - |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}})P_1, \]
\[ b_{10}(z) = |z|^{-\frac{\delta}{2}}P_1(I - |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}})P_0, \]
\[ b_{11}(z) = P_1(I - |v_\alpha|^{\frac{\delta}{2}}r_0(z)|v_\alpha|^{\frac{\delta}{2}})P_1. \]

By the use of Lemma 4.5 we have \( B(z) = C(z) + D(z) \), where
\[ C(z) = \begin{pmatrix} P_0 (G_1 + \ln |z|G_2) P_0 & 0 \\ 0 & P_1 (I - G_\alpha) P_1 \end{pmatrix} \]
and
\[ D(z) = \begin{pmatrix} d_{00}(z) & d_{01}(z) \\ d_{10}(z) & d_{11}(z) \end{pmatrix}. \]
The entries of $D(z)$ are given by
\begin{align*}
  d_{00}(z) &= -|z|^6 p_0 g^{(6)}(z) P_0, \\
  d_{01}(z) &= |z|^5 p_0 \left( G_1 + \ln|z| G_2 - |z|^6 g^{(6)}(z) \right) P_1, \\
  d_{10}(z) &= |z|^5 p_1 \left( G_1 + \ln|z| G_2 - |z|^6 g^{(6)}(z) \right) P_0, \\
  d_{11}(z) &= |z| p_1 \left( G_1 + \ln|z| G_2 - |z|^6 g^{(6)}(z) \right) P_0.
\end{align*}

By abuse of notation we write
\begin{equation}
D(z) = O\left(|z|^6\right). \tag{4.32}
\end{equation}

Since $P_1$ projects onto the subspace of functions orthogonal to $\varphi$, the operator $P_1(I - G_\alpha)P_1$ is invertible. Note that by Lemma 4.4 we have $\langle |v|^\frac{1}{2}, \varphi \rangle \neq 0$. Hence, we can normalize $\varphi$, such that
\begin{equation}
\langle |v|^\frac{1}{2}, \varphi \rangle = \frac{2\pi}{m_\alpha}. \tag{4.33}
\end{equation}

Then we have
\begin{equation}
\langle G_2 \varphi, \varphi \rangle = -1 \quad \text{and} \quad \langle G_1 \varphi, \varphi \rangle = \tau_\alpha,
\end{equation}
where due to (4.18)
\begin{equation}
\tau_\alpha = \frac{m^2}{4\pi^2} \int \int \left( \psi(1) + \psi(2) - \ln(2m_\alpha) - 2 \ln\left( \frac{|x - y|}{2} \right) \right) |\varphi_\alpha(x)|^2 |\varphi_\alpha(y)|^2 \varphi(x) \varphi(y) \, dxdy.
\end{equation}

Using $P_0 = ||\varphi||^{-2}(\cdot, \varphi)\varphi$ we obtain
\begin{equation}
P_0 (G_1 + \ln|z| G_2) P_0 = \frac{(\tau_\alpha - \ln|z|)}{||\varphi||^2} P_0
\end{equation}
and therefore
\begin{equation}
C^{-1}(z) = \begin{pmatrix}
\frac{(\cdot, \varphi)^2}{(\tau_\alpha - \ln|z|)} & 0 \\
0 & K
\end{pmatrix},
\end{equation}
where $K = (P_1(I - G_\alpha)P_1)^{-1}$. Now we can write
\begin{equation}
B(z) = C(\cdot) + D(\cdot) = (I + D(z) C^{-1}(z)) C(z).
\end{equation}

By (4.32) we have
\begin{equation}
||D(z) C^{-1}(z)|| \xrightarrow{z \to 0} 0.
\end{equation}

Therefore, we obtain the inverse of $B(z)$ by the Neumann series
\begin{equation}
B^{-1}(z) = C^{-1}(z) \left( I - (-D(z) C^{-1}(z)) \right)^{-1} = C^{-1}(z) + C^{-1}(z) \sum_{n=1}^{\infty} (-D(z) C^{-1}(z))^n.
\end{equation}

Note that
\begin{equation}
\sum_{n=1}^{\infty} ||D(z) C^{-1}(z)||^n \leq \frac{||D(z) C^{-1}(z)||}{1 - ||D(z) C^{-1}(z)||},
\end{equation}
which together with (4.32) yields
\begin{equation}
B^{-1}(z) = \begin{pmatrix}
\frac{(\cdot, \varphi)^2}{(\tau_\alpha - \ln|z|)} & 0 \\
0 & K
\end{pmatrix} + O\left(|z|^6\right).
\end{equation}

Note that
\begin{equation}
S^{-1}(z) = P(z) B^{-1}(z) P(z)
\end{equation}
and $|z|(|\tau_\alpha - \ln|z|| = z(\ln|z| - \tau_\alpha)$ for sufficiently small $|z|$. This completes the proof. □
The proof of the next Lemma follows from similar arguments as in [17]. We adapt the proof to our case.

**Lemma 4.7.** For $z < 0$, $|z|$ sufficiently small, the operator $w_\alpha(z)$ is positive and we have

$$w_\alpha(z) = \frac{\langle \cdot, \varphi \rangle \varphi}{\| \varphi \| \sqrt{z(\ln |z| - \tau_\alpha)}} + (z(\ln |z| - \tau_\alpha))^{-\frac{1}{2}} w_\alpha^{(z)}(z),$$

where $w_\alpha^{(z)}(z)$ is bounded for $z \leq 0$.

**Proof.** For $|z|$ sufficiently small one has $z(\ln |z| - \tau_\alpha) > 0$ Hence, by $P_0 = P_0^2$ we have

$$(\frac{\langle \cdot, \varphi \rangle \varphi}{z(\ln |z| - \tau_\alpha)})^\frac{1}{2} = \frac{\| \varphi \|^2 P_0^2}{z(\ln |z| - \tau_\alpha)} = \frac{\langle \cdot, \varphi \rangle \varphi}{\| \varphi \| \sqrt{z(\ln |z| - \tau_\alpha)}}$$

Since $r_\alpha(z) \geq 0$ for $h_\alpha \geq 0$ we have $w_\alpha(z) \geq I \geq 0$. By the use of

$$\| A^\frac{1}{2} - B^\frac{1}{2} \| \leq \| A - B \|$$

for positive operators $A$, $B$ we obtain from Lemma 4.6

$$\left\| w_\alpha^{(z)}(z) - \frac{\langle \cdot, \varphi \rangle \varphi}{\| \varphi \| \sqrt{z(\ln |z| - \tau_\alpha)}} \right\| \leq C(z(\ln |z| - \tau_\alpha))^{-\frac{1}{2}},$$

which completes the proof. 

**Three-body system in dimension four.** Now we move to the three-body system. In this section we will prove that every entry $A_{\alpha\beta}(z)$ of the matrix $A(z)$ is a compact operator for every $z \leq 0$. By Definition 4.1 we have

$$A_{\alpha\beta}(z) = W_\alpha^\frac{1}{2}(z) K_{\alpha\beta}(z) W_\beta^\frac{1}{2}(z).$$

Due to the partial Fourier transform $\Phi_\alpha, \Phi_\beta$, defined by (4.2), and the structure of the operator $A_{\alpha\beta}(z)$, we will make use of the mixed coordinates $(x_\alpha, p_\alpha)$, $(x_\beta, p_\beta)$ and various relations such as (4.4) and (4.5). We start with the proof of the compactness of $K_{\alpha\beta}(z)$ by adapting the proof of [17] to our case.

**Lemma 4.8.** The operator $K_{\alpha\beta}(z)$ is compact for every $z \leq 0$.

**Proof.** In view of (4.9) the operator $K_{\alpha\beta}(z)$ is given by

$$K_{\alpha\beta}(z) = |V_\alpha|^\frac{1}{2} R_0(z) |V_\beta|^\frac{1}{2}.$$

It is sufficient to consider the operator

$$\tilde{K}_{\alpha\beta}(z) = \Phi_\alpha^* K_{\alpha\beta}(z) \Phi_\beta.$$

For $R \geq 1$ let

$$\chi_R : \mathbb{R}^4 \to \mathbb{R}, \chi_R(p) = \begin{cases} 1, & |p| \leq R \\ 0, & |p| > R \end{cases}$$

We decompose $\tilde{K}_{\alpha\beta}(z)$ as

$$\tilde{K}_{\alpha\beta}(z) = Z_{\alpha\beta}^R(z) + Y_{\alpha\beta}^R(z),$$

where

$$Z_{\alpha\beta}^R(z) = \chi_R(p_\alpha) \tilde{K}_{\alpha\beta} \chi_R(p_\beta),$$

$$Y_{\alpha\beta}^R(z) = (I - \chi_R(p_\alpha)) \tilde{K}_{\alpha\beta} \chi_R(p_\beta) + \chi_R(p_\alpha) \tilde{K}_{\alpha\beta}(I - \chi_R(p_\beta)) + (I - \chi_R(p_\alpha)) \tilde{K}_{\alpha\beta}(I - \chi_R(p_\beta)).$$
we have the behaviour of the operator $w_i$.

The critical case is the existence of a resonance of the two-body Hamiltonian $H$.

The kernel of the operator $\tilde{K}_{\alpha \beta}(z)\, f(x_{\alpha}, p_{\alpha})$

$$= \int_{\mathbb{R}^4} \frac{d\kappa}{H^{0}(\kappa, p_{\alpha}) - z} \int_{\mathbb{R}^4} dx_{\beta} e^{-i\kappa z_{\beta}} |v_{\beta}(x_{\beta})|^\frac{1}{2} f(x_{\beta}, p_{\beta})$$

$$= c \int_{\mathbb{R}^4} dp_{\beta} \frac{e^{ix_{\alpha}(d_{\alpha \beta}p_{\alpha} + c_{\alpha \beta}p_{\beta})} |v_{\alpha}(z_{\alpha})|^\frac{1}{2}}{H^{0}_{\alpha \beta}(p_{\alpha}, p_{\beta}) - z} \int_{\mathbb{R}^4} dx_{\beta} e^{-ix_{\beta}(d_{\alpha \beta}p_{\alpha} + c_{\alpha \beta}p_{\beta})} |v_{\beta}(x_{\beta})|^\frac{1}{2} f(x_{\beta}, p_{\beta}),$$

i.e. the kernel of $\tilde{K}_{\alpha \beta}(z)$ is of the form

$$\tilde{K}_{\alpha \beta}(z)((x, p), (x', p')) = ce^{ix_{\alpha}p_{\alpha}} \frac{|v_{\alpha}(x)|^\frac{1}{2} e^{i\kappa' z_{\alpha}} e^{-ix'_{\beta}c_{\alpha \beta}} |v_{\beta}(x')|^\frac{1}{2}}{H^{0}_{\alpha \beta}(p, p') - z} e^{-ix'_{\beta}c_{\alpha \beta}}, \quad (4.34)$$

where the constants $d_{\alpha \beta}, c_{\alpha \beta}$ are given by (4.5). By the use of (4.7) with $\kappa = 1 = \frac{1}{2}$ it follows that $Z^{R}_{\alpha \beta}(z)$ belongs to the Hilbert-Schmidt class for every $z \leq 0$. Using estimate (4.6) one can see that the norm of the operator $Y^{R}_{\alpha \beta}(z)$ is bounded by $CR^{-2}$ for every $z \leq 0$, where $C$ does not depend on $z$. Hence, we have

$$\|\tilde{K}_{\alpha \beta}(z) - Z^{R}_{\alpha \beta}(z)\| = \|Y^{R}_{\alpha \beta}(z)\| \to 0$$

as $R \to \infty$, which completes the proof. \hfill \square

By Lemma 4.8 we have $N(z) < \infty$ for every $z < 0$, since $W_{\alpha}(z)$ is bounded for such $z$. Recall relation (4.12)

$$W_{\alpha}(z) = \Phi_{\alpha}w_{\alpha}(z - \frac{p_{\alpha}}{2\tau_{\alpha}}) \Phi_{\alpha}^*. \quad (4.35)$$

The critical case is the existence of a resonance of the two-body Hamiltonian $h_{\alpha}$, which affects the behaviour of the operator $w_{\alpha}(z)$ as $z \to 0$. If $h_{\alpha}$ has a resonance at zero, then according to Lemma 4.7 the operator $w_{\alpha}^{\frac{1}{2}}(z)$ has the representation

$$w_{\alpha}^{\frac{1}{2}}(z) = \frac{\langle \cdot, \varphi \rangle \varphi}{\|\varphi\| \sqrt{2(\ln |z| - \tau_{\alpha})}} + (z(\ln |z| - \tau_{\alpha}))^{-\frac{1}{2} + \frac{i}{2}} \tilde{w}_{\alpha}^{(\delta)}(z), \quad (4.36)$$

where $|z| < 1$ can be chosen sufficiently small, such that $\ln |z| - \tau_{\alpha} < 0$. We only have the representation (4.36) where $z - \frac{p_{\alpha}^2}{2\tau_{\alpha}}$ is sufficiently small and $\ln \left(z - \frac{p_{\alpha}^2}{2\tau_{\alpha}}\right) - \tau_{\alpha} < 0$. Therefore, for every $\alpha$ we introduce the following auxiliary function $\zeta_{\alpha} : (-\infty, 0) \to \mathbb{R}$, where $\zeta_{\alpha} \in C^{\infty}(\mathbb{R}_-)$, $\zeta_{\alpha}(t) > 0$ for all $t < 0$ and

$$\zeta_{\alpha}(t) = \begin{cases} \sqrt{t(\ln |t| - \tau_{\alpha})}, & t \in (\mu_{\alpha}, 0) \\ 1, & t \leq -1 \end{cases} \quad (4.37)$$

The constant $\mu_{\alpha} \in (-1, 0)$ is chosen such that $\ln |t| - \tau_{\alpha} < 0$ holds for all $t \in [\mu_{\alpha}, 0)$. This allows us to represent $w_{\alpha}^{\frac{1}{2}}(z)$ as (4.36) not only for small $z$ but for every $z < 0$ by defining the operator

$$\tilde{w}_{\alpha}^{(\delta)}(z) = \begin{cases} w_{\alpha}^{\frac{1}{2}}(z), & z \in (\mu_{\alpha}, 0) \\ \zeta_{\alpha}(z)w_{\alpha}^{\frac{1}{2}}(z) - \|\varphi\|^{-1}\langle \cdot, \varphi \rangle \varphi, & z \in (-\infty, \mu_{\alpha}] \end{cases} \quad (4.38)$$

Since $w_{\alpha}^{\frac{1}{2}}(z)$ is uniformly bounded for $z \leq \mu_{\alpha} < 0$ and $w_{\alpha}^{(\delta)}(z)$ is continuous up to $z = 0$, it follows that

$$w_{\alpha}^{\frac{1}{2}}(z) = \zeta_{\alpha}(z)^{-1}\|\varphi\|^{-1}\langle \cdot, \varphi \rangle \varphi + \zeta_{\alpha}(z)^{-1+\delta} \tilde{w}_{\alpha}^{(\delta)}(z) \quad (4.39)$$
holds true for every $z < 0$ and the operator $\tilde{u}_\alpha^{(s)}(z)$ is continuous up to $z = 0$. By relation (4.12) it is evident that for $z = 0$ the kernel of $A_{\alpha\beta}(z)$ admits a singularity in $p_\alpha = 0$ and $p_\beta = 0$. In the following we will decompose the kernel of $A_{\alpha\beta}(z)$ into four kernels. Simply put, we will cut the region in the variables $p_\alpha, p_\beta$ where both $|p_\alpha|, |p_\beta|$ are small, both $|p_\alpha|, |p_\beta|$ are large and the other two cases where $|p_\alpha|$ is small and $|p_\beta|$ is large, and vice versa.

Here, it should be noted that in dimension four the mixed cases of one of the variables $|p_\alpha|, |p_\beta|$ being small and the other one being large is more complicated compared to the three-dimensional case [17]. After squaring the kernel the resolvent provides in both cases a decay like $|p_\alpha|^{-4}$ and $|p_\beta|^{-4}$, which in dimension three yields the Hilbert-Schmidt property. This argument cannot be adapted to the four-dimensional case.

**Lemma 4.9.** Let $\Gamma_\alpha(z)$ be the operator of multiplication by $\zeta_\alpha \left( z - \frac{p_\alpha^2}{2n_\alpha} \right)$, i.e.
\[
(\Gamma_\alpha(z) f)(k_\alpha, p_\alpha) = \zeta_\alpha \left( z - \frac{p_\alpha^2}{2n_\alpha} \right) \cdot f(k_\alpha, p_\alpha).
\]
Then the operator
\[
M_{\alpha\beta}(z) = \Gamma_\alpha(z)^{-1} K_{\alpha\beta}(z) \Gamma_\beta(z)^{-1}
\]
is compact for every $z \leq 0$.

**Proof.** We consider the operator
\[
\tilde{M}_{\alpha\beta}(z) = \Phi^*_\alpha M_{\alpha\beta}(z) \Phi_\beta.
\]
The compactness for $z < 0$ follows from Lemma 4.8. We only need to consider the case $z = 0$. Similar to (4.34) the kernel of $\tilde{M}_{\alpha\beta}(0)$ is given by
\[
\tilde{M}_{\alpha\beta}((x, p), (x', p')) = e^{ipd_\alpha} e^{ip'd_\alpha} e^{-iz'd \epsilon_\alpha} e^{-iz \epsilon_\beta} \zeta_\alpha \left( -\frac{p_\alpha^2}{2n_\alpha} \right) H^0_{\alpha\beta}(p, p') \zeta_\beta \left( -\frac{p_\beta^2}{2n_\beta} \right).
\]
Let $\mu_\alpha, \mu_\beta < 0$ be in accordance with (4.37) and $0 < r < \min(|\mu_\alpha|, |\mu_\beta|)$. Denote by $\chi_r(p)$ the multiplication by the characteristic function of $\{ p \in \mathbb{R}^4 : \frac{p^2}{2n} < r \}$, where $n = \min\{n_\alpha, n_\beta\}$. We decompose
\[
\tilde{M}_{\alpha\beta} = \tilde{M}_{\alpha\beta}^1 + \tilde{M}_{\alpha\beta}^2 + \tilde{M}_{\alpha\beta}^3,
\]
where
\[
\tilde{M}_{\alpha\beta}^1 = \chi_r(p) \tilde{M}_{\alpha\beta} \chi_r(p'),
\]
\[
\tilde{M}_{\alpha\beta}^2 = \chi_r(p) M_{\alpha\beta} (I - \chi_r(p')) + (I - \chi_r(p)) \tilde{M}_{\alpha\beta} \chi_r(p'),
\]
\[
\tilde{M}_{\alpha\beta}^3 = (I - \chi_r(p)) \tilde{M}_{\alpha\beta} (I - \chi_r(p')).
\]
The compactness of $\tilde{M}_{\alpha\beta}^3$ follows from Lemma 4.8. Let us prove that $\tilde{M}_{\alpha\beta}^2$ is compact. We consider only the first summand, the second one can be treated analogously. Let $R > r > 0$ be fixed. Then the first summand of $\tilde{M}_{\alpha\beta}^2$ can be written as
\[
\chi_r(p) M_{\alpha\beta} (I - \chi_r(p')) = X_{\alpha\beta} + Y_{\alpha\beta},
\]
where
\[
X_{\alpha\beta} = \chi_r(p) M_{\alpha\beta} (\chi_R(p') - \chi_r(p')),
\]
\[
Y_{\alpha\beta} = \chi_r(p) M_{\alpha\beta} (I - \chi_R(p')).
\]
By the use of $H^0_{\alpha\beta}(p, p') \geq cp^2$, the absolute value of the kernel of $X_{\alpha\beta}$ can be estimated from above by
\[
\left| c \chi_r(p) \frac{|v_\alpha(x)|^2 |v_\beta(x')|^2}{|p||p'|^2} (\chi_R(p') - \chi_r(p')) \right|
\]
which is square-integrable with respect to the arguments $x, x', p, p'$. The kernel of $Y_{\alpha\beta}$ is given by

$$Y_{\alpha\beta}(x, p, (x', p')) = c\chi_r(p)(1 - \chi_R(p'))e^{ixp/\alpha}e^{-ix'p'/\alpha}v_{\alpha}(x)|\frac{x'}{\alpha}|^{\frac{1}{2}}v_{\beta}(x')|\frac{x'}{\alpha}|^{\frac{1}{2}}e^{-ixp/\alpha}e^{-ix'p'/\alpha}.$$ 

We will show that $Y^*_{\alpha\beta}Y_{\alpha\beta}$ is continuous and the operator norm tends to zero as $R \to \infty$. The kernel of $Y^*_{\alpha\beta}Y_{\alpha\beta}$ is given by

$$Y^*_{\alpha\beta}Y_{\alpha\beta}((x'', p''), (x', p')) = \int \int Y_{\alpha\beta}((x, p), (x', p'))Y_{\alpha\beta}((x, p), (x'', p''))\, dx\, dp.$$ 

Hence, we have

$$|Y^*_{\alpha\beta}Y_{\alpha\beta}((x'', p''), (x', p'))| \leq c|v_{\alpha}(p' - p'')||v_{\beta}(x')|\frac{1}{2}|v_{\beta}(x'')|\frac{1}{2}J(p', p'')(1 - \chi_R(p'))(1 - \chi_R(p')).$$

(4.47)

where

$$\overline{v}_{\alpha}(p' - p'') = \int_{\mathbb{R}^4} |v_{\alpha}(x)|e^{-i\alpha x(p' - p'')}\, dx,$$

$$J(p', p'') = \int_{\{p < \sqrt{2}\mu\}} \frac{1}{p^2(p^2 + p'^2)(p^2 + p''^2)}\, dp.$$ 

In view of the characteristic functions $(1 - \chi_R(p'))$ and $(1 - \chi_R(p''))$ we have $|p'|, |p''| \geq c > 0$ and therefore

$$J(p', p'') \leq \frac{C}{p'^2p''^2}$$

for such $p'$ and $p''$, which implies

$$|Y^*_{\alpha\beta}Y_{\alpha\beta}((x'', p''), (x', p'))| \leq C\frac{|\overline{v}_{\alpha}(p' - p'')|}{p'^2p''^2}|v_{\beta}(x')|\frac{1}{2}|v_{\beta}(x'')|\frac{1}{2}(1 - \chi_R(p'))(1 - \chi_R(p')).$$ 

(4.48)

For $\xi_1, \xi_2 \in \mathbb{R}^4 \setminus \{0\}$ and $b > 4$ we define the function

$$y(\xi_1, \xi_2) = (1 + |\xi_1|)^{-\frac{1}{2}}|\xi_2|^{-2}.$$ 

By assumption (4.1) we have $v_{\alpha} \in L^1(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$ and $\overline{v}_{\alpha} \in L^2(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$. Hence, by the use of (4.48) and $(1 + |\cdot|)^{-\frac{1}{2}}|v_{\beta}(\cdot)|\frac{1}{2} \in L^1(\mathbb{R}^4)$ we have

$$\int |Y^*_{\alpha\beta}Y_{\alpha\beta}((x'', p''), (x', p'))| y(x', p')\, dx'\, dp' \leq \frac{|v_{\beta}(x'')|\frac{1}{2}}{|p''|^2}C_R \leq y(x'', p'')C_R,$$

where $C_R \to 0$ as $R \to \infty$. By symmetry we also have

$$\int |Y^*_{\alpha\beta}Y_{\alpha\beta}((x'', p''), (x', p'))| y(x'', p')\, dx''\, dp'' \leq \frac{|v_{\beta}(x')|\frac{1}{2}}{|p'|^2}C_R \leq y(x', p')C_R.$$ 

Hence, we can apply the Schur test (see [5]) to conclude that $Y_{\alpha\beta}$ is a bounded operator on $L^2(\mathbb{R}^4)$, where the operator norm tends to zero as $R \to \infty$. By applying the same arguments to the second kernel of (4.43) we conclude that $M^1_{\alpha\beta}$ is compact.

It remains to show that $\tilde{M}^1_{\alpha\beta}$ is compact. By definition of the function (4.37) and in view of the characteristic functions $\chi_r(p)$, $\chi_r(p')$, where $0 < r < \mu < \min\{\mu_\alpha, |\mu_\beta|\}$, it is sufficient to show that the integral

$$\int_{|p| < \mu} \int_{|p'| < \mu} K(p, p')\, dp'dp$$

(4.49)

is finite, where $\mu > 0$ is sufficiently small and the kernel $K$ is given by

$$K(p, p') = \frac{1}{|p|^2|\ln|p||} \left| H^0_{\alpha\beta}(p, p') \right|^2 |p'|^2|\ln|p'||.$$
Note that
\[
(H_{\alpha\beta}^0(p, p'))^2 \geq c|p|^{4\kappa}|p'|^{4\kappa'}, \quad \kappa + \kappa' = 1.
\]
We set \( \kappa = 0 \) and use spherical coordinates \( p = (\omega, \rho), \ p' = (\omega', \rho') \) to obtain
\[
\int_{|p| < \mu} K(p, p') \, dp' \, dp = \int_{|p| < \mu} \left( \int_{|p'| \leq |p|} K(p, p') \, dp' + \int_{|p'| < |p|} K(p, p') \, dp' \right) \, dp \\
\leq C \int_{|p| < \mu} \frac{1}{p^2 \ln |p|} \left( \int_{|p'| \leq |p|} \frac{1}{p'^6 \ln |p'|} \, dp' \right) \, dp \\
\leq C' \int_0^\mu \frac{\rho}{\ln |\rho|} \left( \int_\rho^{\mu \rho_0} \frac{1}{p'^3 \ln |p'|} \, dp' \right) \, d\rho \\
= C' \int_0^\mu \frac{\rho}{\ln |\rho|} F(\mu, \rho) \, d\rho, \tag{4.50}
\]
where the function \( F(\mu, \rho) \) is given by
\[
F(\mu, \rho) = \int_\rho^{\mu \rho_0} \frac{1}{p^2 \ln |p'|} \, dp' = -\frac{1}{2\rho^2 \ln |\rho|} - \frac{1}{2} \frac{1}{2\rho^3 \ln |\rho'|^2}.
\]
For \( \mu > 0 \) sufficiently small we have \( |\ln |p'|| \geq 1 \) and therefore
\[
|F(\mu, \rho)| \leq C(\mu) + \frac{1}{2\rho^2 \ln |\rho|} + \frac{1}{2} |F(\mu, \rho)|. \tag{4.51}
\]
Hence, by inserting (4.51) into (4.50) we get
\[
\int_{|p| < \mu} K(p, p') \, dp' \, dp \leq C' \int_0^\mu \frac{\rho}{\ln |\rho|} F(\mu, \rho) \, d\rho \leq C_1 + C_2 \int_0^\mu \frac{1}{\rho(\ln \rho)^2} \, d\rho < \infty, \tag{4.52}
\]
which completes the proof. \( \square \)

5. PROOF OF THE MAIN RESULTS

We are ready to prove Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. Let \( d = 4 \). At first we assume that every two-body Hamiltonian \( h_\alpha, \ \alpha \in \{12, 23, 31\} \) has a virtual level at zero. According to Lemma 3.6, \( \lambda = 0 \) is not an eigenvalue of \( h_\alpha \). In the case of resonances in every subsystem, every entry \( A_{\alpha\beta}(z) \) of \( A(z) \) can be represented as
\[
A_{\alpha\beta}(z) = \Pi_\alpha (\Gamma_\alpha(z))^{-1} K_{\alpha\beta}(z) \Gamma_\beta(z)^{-1} \Pi_\beta + \tilde{U}_\alpha(\beta)(\Gamma_\beta(z))^{-1+\delta} K_{\alpha\beta}(z) (\Gamma_\beta(z))^{-1} \Pi_\beta \\
+ \Pi_\alpha \Gamma_\alpha(z)^{-1} K_{\alpha\beta}(z) (\Gamma_\beta(z))^{-1+\delta} \tilde{U}_\alpha(\beta)(\Gamma_\beta(z))^{-1+\delta} K_{\alpha\beta}(z) (\Gamma_\beta(z))^{-1+\delta} \tilde{U}_\beta(z),
\]
where the operator \( \Pi_\alpha \) is defined by
\[
(\Pi_\alpha f)(k_\alpha, p_\alpha) = ||\varphi||^{-1}(\Phi_\alpha \varphi)(k_\alpha) \int f(k_\alpha', p_\alpha)(\Phi_\alpha \varphi)(k_\alpha') \, dk_\alpha'
\]
and \( \tilde{U}_\alpha(\beta)(z) \) is given by
\[
\tilde{U}_\alpha(\beta)(z) = \Phi_\alpha \varphi(\beta) \left( z - \frac{p_\alpha^2}{2\bar{n}_\alpha} \right) \Phi_\beta^*.
\]
where \( \tilde{w}_\alpha^{(\delta)}(z) \) is defined by (4.38). Since \( \Pi_\alpha, \Pi_\beta, \tilde{U}_\alpha^{(\delta)}(z), \tilde{U}_\beta^{(\delta)}(z) \) are bounded operators for \( z \leq 0 \), the finiteness of \( \sigma_{\text{disc}}(H) \) follows from Lemma 4.9 and Theorem 4.2. Now assume that one subsystem, say \( \alpha \), does not have a resonance. In case of \( \lambda = 0 \) being a regular point of \( h_\alpha \), the operator \( w_\alpha(z) \) is continuous up to \( z = 0 \). Indeed, one can easily see that \( \mu = 1 \) is not an eigenvalue of the operator with the kernel

\[
G_\alpha(x, y) = \frac{m_\alpha}{2\pi^2} \frac{|v_\alpha(x)|^2 |\tilde{v}_\alpha(y)|}{|x - y|^2}.
\]

Similar to Lemma 4.6 one has

\[
w_\alpha(z) = (I - G_\alpha + o(1))^{-1} = (I - G_\alpha)^{-1} + o(1), \quad z \to 0.
\]

This implies the finiteness of \( \sigma_{\text{disc}}(H) \) in this case as well. If \( \lambda = 0 \) is an eigenvalue of \( h_\alpha \), then we do not need to distinguish between dimensions \( d = 4 \) and \( d \geq 5 \), since the finiteness of \( \sigma_{\text{disc}}(H) \) follows from similar arguments as in [22]. Indeed, by Lemma 3.5 the virtual level is always an eigenvalue for \( d \geq 5 \). By Corollary 3.3 there exists a constant \( \mu > 0 \), such that for every function \( g \in \dot{H}^1(\mathbb{R}^d) \) with \( \langle \nabla g, \nabla f \rangle = 0 \) one has

\[
\langle (-\Delta + v) g, g \rangle \geq \mu \| \nabla g \|^2.
\]

Now we can repeat the same arguments as in the proof of [22, Theorem 2.1] to prove the existence of a finite-dimensional subspace \( M \subset \mathcal{D}(H) \), such that

\[
\langle H\psi, \psi \rangle \geq 0
\]

holds for every \( \psi \perp M \). This concludes the proof. \( \square \)

The absence of the Efimov effect for the antisymmetric case follows from Theorem 2.1 by a slight modification.

**Proof of Theorem 2.2.** We have \( \langle v_\alpha, \varphi \rangle = 0 \) for every antisymmetric function \( \varphi \), since the potentials \( v_\alpha \) satisfy \( v_\alpha(x_{ij}) = v_\alpha(-x_{ij}) \). Hence, by Lemma 3.6 a virtual level of \( h_\alpha^{as} \) is always an eigenvalue for \( d \geq 4 \). It is easy to see that this eigenvalue has always finite multiplicity. Let \( E_d \) be the corresponding eigenspace. Similar to Corollary 3.3, by considering the orthogonal complement of \( E_d \) with respect to \( \| \cdot \|_{(1)} \), there exists a constant \( \mu_d > 0 \) such that for every function \( g \perp E_d \) in \( \dot{H}^1(\mathbb{R}^d) \) one has

\[
\langle h_\alpha^{as} g, g \rangle \geq \mu_d \| \nabla g \|^2.
\]

The finiteness of \( \sigma_{\text{disc}}(H^{as}) \) now follows from [22, Theorem 2.1]. \( \square \)

**Appendix A Coordinate system**

Let \( x_i \) be the coordinate of the particle with mass \( m_i, \ i \in \{1, 2, 3\} \) and denote by \( k_i \) the conjugate variable of \( x_i \). Let \( (x_\alpha, y_\alpha), \ \alpha \in \{12, 23, 31\} \) be any pair of Jacobi coordinates and introduce the set of variables conjugate with respect to the Jacobi-coordinates

\[
k_{12} = \frac{m_2 k_1 - m_1 k_2}{m_1 + m_2}, \quad p_{12} = \frac{m_3 (k_1 + k_3) - (m_1 + m_2) k_3}{m_1 + m_2 + m_3}, \quad (A.1)
\]

\[
k_{23} = \frac{m_3 k_2 - m_2 k_3}{m_2 + m_3}, \quad p_{23} = \frac{m_1 (k_2 + k_3) - (m_2 + m_3) k_1}{m_1 + m_2 + m_3}, \quad (A.2)
\]

\[
k_{31} = \frac{m_1 k_3 - m_3 k_1}{m_3 + m_1}, \quad p_{31} = \frac{m_2 (k_3 + k_1) - (m_3 + m_1) k_2}{m_1 + m_2 + m_3}. \quad (A.3)
\]
The reduced masses \(m_\alpha, n_\alpha\) in (4.4) are given by
\[
m_{12} = \frac{m_1 m_2}{m_1 + m_2}, \quad n_{12} = \frac{m_3 (m_1 + m_2)}{m_1 + m_2 + m_3}, \quad (A.4)
\]
\[
m_{23} = \frac{m_2 m_3}{m_2 + m_3}, \quad n_{23} = \frac{m_1 (m_2 + m_3)}{m_2 + m_2 + m_3}, \quad (A.5)
\]
\[
m_{31} = \frac{m_3 m_1}{m_3 + m_1}, \quad n_{31} = \frac{m_2 (m_3 + m_1)}{m_2 + m_2 + m_3}. \quad (A.6)
\]
The relations (4.5) of coordinates \((p_\alpha, p_\beta)\) and \((k_\alpha, p_\alpha)\) are given by
\[
k_{12} = -p_{23} - \frac{m_1}{m_1 + m_2} p_{12} = p_{31} + \frac{m_2}{m_1 + m_2} p_{12}, \quad (A.7)
\]
\[
k_{23} = -p_{31} - \frac{m_2}{m_2 + m_3} p_{23} = p_{12} + \frac{m_1}{m_2 + m_3} p_{23}, \quad (A.8)
\]
\[
k_{31} = -p_{12} - \frac{m_3}{m_1 + m_2} p_{31} = p_{23} + \frac{m_1}{m_1 + m_2} p_{31}. \quad (A.9)
\]
The kinetic energy \(H^0\) can now be expressed by any pair \((p_\alpha, p_\beta)\), i.e. \(H^0\) takes the form
\[
\frac{p_{12}^2}{2m_{23}} + \frac{p_{23}^2}{2m_{12}} \frac{p_{12}^2}{2m_{23}} + \frac{p_{23}^2}{2m_{12}} + \frac{p_{31}^2}{2m_{31}} + \frac{p_{12}^2}{2m_{31}}. \quad (A.10)
\]
For more details see [3].

**Appendix Appendix B Faddeev equations**

We only sketch a brief derivation of the Faddeev equations (see [13] for more details). Consider the bound-state equation for the three-body Hamiltonian \(H\).
\[
\left( H_0 + \sum_\alpha V_\alpha \right) u = z u, \quad (B.1)
\]
where \(z < 0\) and \(\alpha \in \{12, 23, 31\}\). Denote \(R_0(z) = (H_0 - z)^{-1}\), then
\[
u = -R_0(z) \sum_\alpha V_\alpha u.
\]
By decomposing \(u\) in the so called Faddeev components
\[
u = \sum_\alpha u_\alpha, \quad u_\alpha = -R_0 V_\alpha u
\]
and defining
\[
H_\alpha = H_0 + V_\alpha \quad \text{and} \quad R_\alpha(z) = (H_\alpha - z)^{-1},
\]
one has
\[
u_\alpha = -R_0(z) V_\alpha \sum_\alpha u_\alpha \quad \Leftrightarrow \quad u_\alpha + R_0(z) V_\alpha u_\alpha = -R_0(z) V_\alpha (u_\beta + u_\gamma)
\]
\[
\Leftrightarrow \quad R_\alpha(z) (H_\alpha - z) (I + R_0(z) V_\alpha) u_\alpha = -R_0(z) V_\alpha (u_\beta + u_\gamma)
\]
\[
\Leftrightarrow \quad R_\alpha(z) (H_0 + V_\alpha - z) u_\alpha = -R_0(z) V_\alpha (u_\beta + u_\gamma)
\]
\[
\Leftrightarrow \quad u_\alpha = -R_\alpha(z) V_\alpha (u_\beta + u_\gamma).
\]
By the assumption \(V_\alpha \leq 0\) and the resolvent identity
\[
R_\alpha(z) = R_0(z) - R_0(z) V_\alpha R_\alpha(z),
\]
one arrives at
\[
u_\alpha = R_0(z) (I + |V_\alpha| R_\alpha) V_\alpha (u_\beta + u_\gamma). \quad (B.2)
Now we define
\[ W_\alpha(z) = I + |V_\alpha|^{1/2} R_\alpha(z)|V_\alpha|^{1/2}, \]
then equation (B.2) becomes
\[ u_\alpha = R_0(z)|V_\alpha|^{1/2} W_\alpha(z)|V_\alpha|^{1/2}(u_\beta + u_\gamma). \]
By making the substitution
\[ f_\alpha = W_\alpha^\frac{1}{2}(z)|V_\alpha|^{1/2}(u_\beta + u_\gamma), \]
\[ f_\beta = W_\alpha^\frac{1}{2}(z)|V_\beta|^{1/2}(u_\alpha + u_\gamma), \]
\[ f_\gamma = W_\alpha^\frac{1}{2}(z)|V_\gamma|^{1/2}(u_\alpha + u_\beta), \]
the system of equations is now given by the Faddeev equations
\[ f_{12} = W_{12}^\frac{1}{2}(z)|V_{12}|^{1/2} R_0(z)|V_{23}|^{1/2} W_{23}^\frac{1}{2}(z)f_{23} + W_{12}^\frac{1}{2}(z)|V_{12}|^{1/2} R_0(z)|V_{31}|^{1/2} W_{31}^\frac{1}{2}(z)f_{31}, \]
\[ f_{23} = W_{23}^\frac{1}{2}(z)|V_{23}|^{1/2} R_0(z)|V_{12}|^{1/2} W_{12}^\frac{1}{2}(z)f_{12} + W_{23}^\frac{1}{2}(z)|V_{23}|^{1/2} R_0(z)|V_{31}|^{1/2} W_{31}^\frac{1}{2}(z)f_{31}, \]
\[ f_{31} = W_{31}^\frac{1}{2}(z)|V_{31}|^{1/2} R_0(z)|V_{12}|^{1/2} W_{12}^\frac{1}{2}(z)f_{12} + W_{31}^\frac{1}{2}(z)|V_{31}|^{1/2} R_0(z)|V_{23}|^{1/2} W_{23}^\frac{1}{2}(z)f_{23}. \]
In other words, the eigenvalue equation (B.1) is now formulated as
\[ A(z)F = F, \quad F = (f_{12}, f_{23}, f_{31}), \]
where \( A(z) \) is a \( 3 \times 3 \)-matrix with entries
\[ A_{\alpha\beta}(z) = W_{\alpha}^\frac{1}{2}(z)|V_{\alpha}|^{1/2} R_0(z)|V_{\beta}|^{1/2} W_{\beta}^\frac{1}{2}(z). \]

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