The Minimum Shared Edges Problem on Planar Graphs

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Abstract

We study the Minimum Shared Edges problem introduced by Omran et al. [Journal of Combinatorial Optimization, 2015] on planar graphs: Planar MSE asks, given a planar graph $G = (V,E)$, two distinct vertices $s, t \in V$, and two integers $p, k \in \mathbb{N}$, whether there are $p$ $s$-$t$ paths in $G$ that share at most $k$ edges, where an edge is called shared if it appears in at least two of the $p$ $s$-$t$ paths. We show that Planar MSE is NP-hard by reduction from Vertex Cover. We make use of a grid-like structure, where the alignment (horizontal/vertical) of the edges in the grid correspond to selection and validation gadgets respectively.

Keywords: Grids, Reductions, NP-completeness.

1 Introduction

We study the following problem.

\textbf{Planar Minimum Shared Edges (Planar MSE)}

\textit{Input:} An undirected planar graph $G = (V,E)$ with distinct vertices $s, t \in V$, and two integers $p \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

\textit{Question:} Are there $p$ $s$-$t$ paths in $G$ that share at most $k$ edges?

Herein, an edge is called \textit{shared} if it appears in at least two $s$-$t$ paths of the solution. To clearly distinguish between paths in a solution and ordinary paths in a graph, we also call the paths in a solution \textit{s-t routes}.

Planar MSE is the special case of Minimum Shared Edges (MSE) in which the input graph is planar. MSE was introduced on directed graphs by Omran et al. \cite{OMN15}. Ye et al. \cite{YXZ18} proved that the problem is solvable in polynomial-time on graphs of bounded treewidth. Fluschnik et al. \cite{FJC19} proved that MSE is fixed-parameter tractable (FPT) with respect to the number $p$ of desired $s$-$t$ routes, that is, there is an algorithm solving MSE in $f(p) \cdot n^{O(1)}$ time, where $f$ is a computable function and $n$ denotes the number of vertices in the graph of the input instance of MSE. Moreover, Fluschnik et al. \cite{FJC19} showed that MSE parameterized by the treewidth $tw$ and the number $k$ of shared edges combined is $W[1]$-hard, that is, it is unlikely that MSE parameterized by $tw$ and $k$ combined admits an FPT algorithm. For more results on MSE and related work, we refer the reader to Fluschnik \cite{FJC19}.

In this paper, we prove the following result.

\textbf{Theorem 1.} Planar Minimum Shared Edges is NP-hard, even on planar graphs of degree at most four.

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constructed as follows: take two paths the relevant notions of parameterized complexity, we refer the reader to the literature [1, 3, 4, 8].

We also call the edge set \{\{(p_i, p_{i+1})\mid i \in [p]\}\} ⊆ \{\{(a, b)\mid a, b \in \mathbb{N}\}\} the subgraph \(G\). We call the family \{\{(a, b)\mid a, b \in \mathbb{N}\}\} horizontal and column \(i\) the vertical column beside the first one corresponds to an edge of \(G\). More precisely, row \(i\) corresponds to vertex \(v_i\) and column \(j + 1\) corresponds to edge \(e_j\). For each \(i \in [n]\), if edge \(e_j\) is not incident endpoints. We create a \((\ell, m)\)-bundle with a \((q, \ell, m)\)-feather the \((\ell, m)\)-rainbow is a graph constructed as follows: take two paths \(P_{\ell+1}^1, P_{\ell+1}^2\), represented as tuple of vertices \((p_1^1, \ldots, p_{\ell+1}^1)\) and \((p_1^2, \ldots, p_{\ell+1}^2)\), respectively, and, for all \(x \in [\ell]\), connect the pair \(p_x^1, p_x^2\) by an \((\ell, m)\)-chain.

For \(a, b \in \mathbb{N}\), we denote by \(\boxtimes_{a,b}\) the graph with vertex set \(\{(i, j) \mid i \in [a], j \in [b]\}\) and edge set \(\{(i, j), (k, \ell)\} \mid |i - k| + |j - \ell| = 1\}\). We call \(\boxtimes_{a,b}\) the \(a \times b\)-grid. For \(i \in [a]\), we call the subgraph \(R_i\) in graph \(\boxtimes_{a,b}\) induced by the vertex set \(\{(i, j) \mid j \in [b]\}\) the row \(i\). For \(j \in [b]\), we call the subgraph \(C_j\) in graph \(\boxtimes_{a,b}\) induced by the vertex set \(\{(i, j) \mid i \in [a]\}\) the column \(j\). We call the family \(\{R_1, \ldots, R_a\}\) the rows of \(\boxtimes_{a,b}\), and we call the family \(\{C_1, \ldots, C_b\}\) the columns of \(\boxtimes_{a,b}\). We denote the edges in the rows by horizontal edges and the edges in the columns by vertical edges.

We now prove \(1\). Given an instance of VC we first describe how to obtain an equivalent instance of Planar MSE containing a planar graph of arbitrary maximum degree. We then show how to reduce the maximum degree to at most 4.

Let \((G = (V, E), k)\) be an instance of VC and let \(n := |V|\) and \(m := |E|\). We denote \(V = \{v_1, \ldots, v_n\}\) and \(E = \{e_1, \ldots, e_m\}\). We construct an instance \((G', s, t, p, k')\) of Planar MSE as follows. First, define \(M := 2 \cdot (m + 1) + 2\), and \(k' := k \cdot (m^3 + m + 1)\). Below we also call \(k'\) the budget. The graph \(G'\) is constructed as follows. Refer to Figure 2 for an illustration.

Initially, we set \(G'\) to \(\mathbb{H}_{n,m+1}\). Each row in \(\mathbb{H}_{n,m+1}\) corresponds to a vertex of \(G\) and each column beside the first one corresponds to an edge of \(G\). More precisely, row \(i\) corresponds to vertex \(v_i\) and column \(j + 1\) corresponds to edge \(e_j\). For each \(i \in [n]\), if edge \(e_j\) is not incident
with vertex $v_i$, then we replace the horizontal edge in row $i$ connecting column $j$ and $j+1$ by a $(1, M, k'+1)$-feather. We replace all vertical edges by $(k'+1)$-chains. We call the resulting graph $\Gamma'$. By row $i$ of $\Gamma'$ we refer to the subgraph obtained from row $i$ in $\boxplus_{n,m+1}$ by the previously described modifications. Similarly, by the column $j$ of $\Gamma'$ we refer to the subgraph obtained from column $j$ in $\boxplus_{n,m+1}$ by the previously described modifications. Instead of horizontal and vertical edges, we talk about horizontal and vertical connections in $\Gamma'$, meaning the graphs that replaced the corresponding edges.

We now add the new vertices $s$ and $t$ to $G'$. We connect all vertices in column 1 with $s$ via $(m^3, M, k'+1)$-feathers and all vertices in column $m+1$ with $t$ via $(1, M, k'+1)$-feathers. Herein, we merge $s$ with the end points of the shafts of the feathers and analogously for $t$. Last, we add two $(k'+1)$-chains, one connecting $s$ with vertex $(1,1)$, and the other connecting vertex $(n,m+1)$ with $t$. Below, we call these paths validation paths. We denote the finally obtained graph by $G'$. Further, we set the number $p$ of desired $s$-$t$ routes to $k \cdot M + (n - k) + 1$. This concludes the construction of the instance $(G', s, t, p, k')$.

**Planarity** The $\boxplus_{n,m+1}$ is planar, feathers and chains are planar as well. Replacing (vertical/horizontal) edges in $\boxplus_{n,m+1}$ preserves planarity. Connecting $s$ with all vertices in column 1 can be done preserving the planarity, and by symmetry, the same holds for connecting $t$ with all vertices in the column $m + 1$.

**Correctness** We claim that $(G', s, t, p, k')$ is a yes-instance of PLANAR MSE if and only if $(G, k)$ is a yes-instance of VC.

$(\Rightarrow)$: Suppose that $(G', s, t, p, k')$ is a yes-instance of PLANAR MSE and consider a solution to $(G', s, t, p, k')$. We show that we can construct a vertex cover of size $k$ in $G$.

First we state observations about a solution to $(G', s, t, p, k')$. The first observation is about $s$, its incident chain and feathers, and how $k'$ $s$-$t$ routes determine a $k$-vertex subset of $G$. Note that the degree of $s$ is exactly $(n + 1)$. At most one $s$-$t$ route contains the validation paths, otherwise there are $k'+1$ shared edges, contradicting the fact that $(G', s, t, p, k')$ is a yes-instance. For the same reason, each $(m^3, M, k'+1)$-feather appears in at most $M$ routes. If a $(m^3, M, k'+1)$-feather appears in at least two routes, then $m^3$ edges are shared. Since the budget allows for
can "leave" a row via vertical connections. This observation together with \( s \) yield the following.

**Observation 1.** In any solution to the instance \((G', s, t, p, k')\), there are exactly \( k \) feathers connecting \( s \) with the vertices in the first column that contain \( M \) \( s-t \) routes each. All the other \( n-k \) feathers contain exactly one route each. Moreover, the \((k' + 1)\)-chain incident with \( s \) appears in exactly one \( s-t \) route.

We say that the row \( i \) is **selected** if the feather connecting \( s \) with vertex \((i,1)\) is contained in \( M \) routes.

The second observation is about the number of shared edges in a selected row. Note that each vertical connection in \( E' \) is a \((k' + 1)\)-chain and, thus, none of them appears in at least two routes. Recall that if row \( i \) is selected, then vertex \((i,1)\) appears in at least \( M = 2 \cdot (m + 1) + 2 \) routes. Since each vertical connection appears in at most one route, there are at most \( 2 \cdot (m + 1) \) routes that can "leave" a row via vertical connections. This observation together with \( s \) yield the following.

**Observation 2.** In any solution to the instance \((G', s, t, p, k')\), each selected row \( i \) induces \((m + 1)\) shared edges. These shared edges appear only in the horizontal connections in the selected row \( i \) and in the feather connecting \((i, m + 1)\) with \((i, t)\).

By \( s \) and \( t \), we know that in any solution to \((G', s, t, p, k')\), there are exactly \( k \) selected rows in \( G' \) and all shared edges appear in the selected rows, in the feathers connecting \( s \) with the selected rows, and in the feathers connecting \( t \) with the selected rows. Let rows \( i_1, \ldots, i_k \) be the selected rows and let \( w_1, \ldots, w_k \) be the vertices in \( G \) corresponding to the selected rows. Recall that by \( s \) and \( t \), no budget is left. We claim that \( W := \{ w_1, \ldots, w_k \} \) is a vertex cover in \( G \).

Suppose that \( W \) is not a vertex cover in \( G \), that is, there is an edge \( e_j \) such that \( v \cap e_j = \emptyset \) for all \( v \in W \). We show that this induces at least one additional shared edge, contradicting the fact that \((G', s, t, p, k')\) is a yes-instance of Planar MSE. If \( W \) is not a vertex cover in \( G \), then \((i, e_j)\) and \((i, e_j + 1)\) are connected by a \((1, M, k' + 1)\)-feather for each \( \ell \in [k] \). Observe that there are at most \( M \cdot k \) routes crossing column \( j = j + 1 \) over the feathers connecting \((i, e_j)\) and \((i, e_j + 1)\) with \( \ell \in [k] \). There are \( n-k \) remaining horizontal connections to cross column \( j = j + 1 \). Furthermore, all \( p \) \( s-t \) routes appear in each column. Hence there are at least \( n-k+1 \) routes that cross column \( j = j + 1 \) over the \( n-k \) remaining horizontal connections (recall that \( p = k \cdot M + (n-k) + 1 \)). By the pigeon-hole principle, at least one of these horizontal connections appears in at least two routes. Since each horizontal connection is either a single edge or a \((1, M, k' + 1)\)-feather, the two routes induce at least one further shared edge. Thus, there at least \( k' + 1 \) edges shared by the \( p \) \( s-t \) routes, which contradicts the fact that \((G', s, t, p, k')\) is a yes-instance of Planar MSE. It follows that \( W \) is a vertex cover in \( G \).

\((\Leftarrow):\) Suppose that \((G,k)\) is a yes-instance of VC, and let \( W \subseteq V \) be a vertex cover in \( G \) with \(|W| = k \). We show that we can construct \( p \) \( s-t \) routes in \( G' \) that share \( k' \) edges.

We lead \( M \) routes from \( s \) to each vertex \((w,1)\) in \( G' \) with \( w \in W \) and one route to each vertex \((x,1)\) with \( x \in V \setminus W \). Note that these are exactly \( k \cdot M + (n-k) = p-1 \) routes. Moreover, by the construction of the routes so far, \( k \cdot m^3 \) edges are shared. These shared edges appear in the \( m^3 \)-shafts of the \( k \) feathers connecting \( s \) with the vertices \((w,1)\) in \( G' \) with \( w \in W \).

For each row \( i \in [n] \), we lead all the routes containing vertex \((i,1)\) from vertex \((i,1)\) to vertex \((i, m + 1)\) using only the connections in row \( i \). Note that this construction of the routes induce \( k \cdot (m + 1) \) further shared edges: In feathers, only the shafts need to be shared, since we define \( M \) routes and the bundle in each feather contain \( M \) edge-disjoint paths. Finally, for each row \( i \in [n] \), we lead all routes containing \((i, m+1)\) via the feather incident with vertex \((i, m+1)\) to \( t \). This construction yields \( k \) further shared edges, namely those in the \( 1 \)-shafts of the \( k \) feathers that connect column \( m + 1 \) with \( t \), each appearing in \( M \) \( s-t \) routes. Observe that, so far, \( k' \) edges are shared and, thus, no budget for sharing any further edge is left.

So far, \( p-1 \) routes are constructed connecting \( s \) with \( t \). Thus, one \( s-t \) route remains, that we call the **validation route** and which we construct as follows. First, we lead the validation route to \((1,1)\) over the \((k'+1)\)-chain connecting \( s \) with vertex \((1,1)\). Next, we route the validation route through \( E' \) as follows. Since \( W \) is a vertex cover in \( G \), for each edge \( e_j \) there exists a vertex \( v_j \in W \) such that \( v_j \in e_j \). Thus, by construction of \( G' \) it holds that \((i,j)\) and \((i,j+1)\) are connected by an edge. By the construction of the \( p-1 \) \( s-t \) routes before, it holds that the edge connecting \((i,j)\) with \((i,j+1)\) is shared by exactly \( M \) paths. Thus, we can find in every column \( j \in [m] \) exactly
one index \(i_j\) corresponding to row \(i_j\) such that \((i_j, j)\) and \((i_j, j + 1)\) are connected by an edge that is shared by \(M\) \(s\)-\(t\) routes. We lead the validation route in each column \(j \in [m]\) to the row \(i_j\) using the vertical connections, and then over the shared edge \(\{(i_j, j), (i_j, j + 1)\}\) to column \(j + 1\). In column \(m + 1\), we lead the validation route via the vertical connections to \((n, m + 1)\). Finally, we lead the validation route over the \((k' + 1)\)-chain connecting \((n, m + 1)\) with \(t\) to \(t\). Note that in the construction of the validation route, we do not share any additional edge.

We constructed \(p\) \(s\)-\(t\) routes sharing \(k'\) edges in \(G'\) and, thus, \((G', s, t, p, k')\) is a yes-instance of Planar MSE.

**Maximum degree at most four** We now make the following modifications to the graph \(G'\) and the budget \(k'\) in the instance of Planar MSE constructed above. We first replace each bundle in the graph \(G'\) by a rainbow. (For this to yield an equivalent instance, we need to subdivide edges in \(G'\) before.) Then, we replace the high-degree vertices \(s\) and \(t\) by binary trees, in a similar fashion as done by Fluschnik [5, Theorem 5.2].

We aim to replace each bundle in \(G'\) by a rainbow. Since replacing a bundle by a rainbow may introduce additional shared edges in a solution, we have to increase the budget \(k'\) as well. However, increasing the budget may allow to share new edges outside of rainbows which we did not intend to be shareable. To circumvent this issue, we first subdivide each edge in \(G'\) several times and, only after the subdivision, replace bundles by rainbows. Subdividing edges has the effect that the number of shared edges in any solution is a multiple of the number \(b\) of subdivisions. Hence, if \(b\) is larger than the increase of the budget when replacing the bundles by rainbows, no further edges other than the ones in the rainbows can be shared. We now formalize this approach.

We introduce the following notation. Let \(H\) be a graph. A proper chain in \(H\) is an induced path \(P\) in \(H\) such that all inner vertices of \(P\) have degree exactly two in \(H\). If the endpoints of \(P\) each have degree different from two in \(H\), then we call \(P\) a maximal proper chain.

To replace bundles by rainbows, we use the following claim.

**Claim 1.** Let \(H\) be a graph such that each maximal proper chain in \(H\) has length at least \(b\). Let \(c\) be the number of \((a, d)\)-bundles in \(H\), such that \(2ac < b\), and let \(p, k\) be integers such that \(d > k\).

Let \(H'\) be the graph obtained from \(H\) by replacing each \((a, d)\)-bundle by a \((a, d + 2ac)\)-rainbow. Graph \(H\) admits \(p\) \(s\)-\(t\) routes with at most \(k\) shared edges if and only if \(H'\) admits \(p\) \(s\)-\(t\) routes with at most \(k + 2ac\) shared edges.

**Proof.** \((\Rightarrow):\) Assume that \(H\) admits \(p\) \(s\)-\(t\) routes with at most \(k\) shared edges. Note that, since \(d > k\), there are no two routes that share one of the \(d\)-chains in any bundle. Transform this set of routes into a set of routes in \(H'\) as follows. Outside of any rainbow, each route in \(H'\) equals one route in \(H\). Inside a rainbow, lead each route over a distinct \(d\)-chain. In this way, at most \(c \cdot 2a\) more edges are shared. That is, \(H'\) admits \(p\) \(s\)-\(t\) routes with at most \(k + 2ac\) shared edges.

\((\Leftarrow):\) Assume that \(H'\) admits \(p\) \(s\)-\(t\) routes with at most \(k + 2ac\) shared edges. Note that, since \(d + 2ac > k + 2ac\), there are no two routes that share one of the \(d\)-chains in any rainbow. Transform a corresponding set of routes in \(H'\) into a set of routes in \(H\) as follows. Outside of any bundle, each route in \(H\) equals one route in \(H'\). Inside a bundle, lead each route over a distinct \(d\)-chain. This yields a set of \(s\)-\(t\) routes in \(H\) with at most \(k + 2ac\) shared edges. Since each maximal proper chain in \(H\) has length at least \(b > 2ac\) and since each maximal proper chain is either shared completely or not at all, indeed, there are at most \(b[(k + 2ac)/b]\) \(k\) shared edges.

Now consider \(G'\) and \(k'\) from the instance of Planar MSE. Note that each bundle in \(G'\) is an \((M, k' + 1)\)-bundle. We now to replace each of these bundles by rainbows using [1]. To satisfy the precondition of [1], we need that each maximal proper chain in \(G'\) has length at least \(b' > 2Mc'\), where \(c'\) is the number of \((M, k' + 1)\)-bundles in \(G'\). For this, we perform the operation “subdivide each edge in \(G\) and multiply \(k'\) by two” sufficiently often. Note that this operation yields an equivalent instance, because each \(P_i\) resulting from subdividing an edge has to be traversed either completely or not at all by each route. Furthermore, each application of this operation doubles the minimum length \(b\) of a maximal proper chain in \(G'\), meaning that, after \(O(\log(Mc'))\) applications, we have \(2Mc' < b\). Note that subdividing and multiplying \(k'\) by two does not invalidate the property that each bundle is a \((M', d')\)-bundle for some \(d' > k'\). Thus, we can replace all bundles in \(G'\) by rainbows and \(k'\) by \(k' + 2Mc'\), yielding an equivalent instance. Clearly, none of the above
operations increases the degree of any vertex. On the contrary, after all operations have been applied, each vertex except $s$ and $t$ has degree at most 4.

To decrease the degree of $s$ and $t$, we replace $s$ and $t$ by a complete binary trees as follows. Recall that the number of neighbors of $s$ and $t$ is $n + 1$ each. Assume that the number $n$ of vertices in the instance of VC is such that $n + 1$ is a power of two. Otherwise, add degree-zero vertices until this is the case. Replace $s$ with a complete binary tree with root $s$ and $(n + 1)/2$ leaves. Make incident each previous neighbor of $s$ in $G'$ with one of the leaves of the complete binary tree in such a way that each leaf has degree exactly three. Replace $t$ with a complete binary tree in the same way, and replace $k'$ by $k' + 2\ell$, where $\ell$ is the number of edges in a complete binary tree with $(n + 1)/2$ leaves. To see that the resulting instance is yes if and only if the original instance is yes, note that in any solution for $G'$, each row of $\sqcap$ receives at least one $s$-$t$ route from $s$ and sends at least one $s$-$t$ route to $t$. Thus, in any solution all $2\ell$ edges in the complete binary trees are shared. Finally, it is clear that each of the above operations can be performed while maintaining planarity.

### 3.1 Directed graphs

Let $G''$ be the planar graph of maximum degree four constructed in section 3. We now sketch how to modify graph $G''$ in such a way that an equivalent instance of **Planar Minimum Shared Edges** on directed planar graphs is obtained. To this end, we direct all edges except those in vertical connections from “left to right” with respect to a drawing as in Figure 2. Herein, we direct the edges in the binary tree containing vertex $s$ from $s$ to the leaves, and the edges in the binary tree containing vertex $t$ from the leaves to $t$. We replace each vertical connection by a directed graph gadget as follows (refer to Figure 3 in the following).

Consider a vertical connection between two vertices $v$ and $w$ (recall that $v$ and $w$ are in consecutive rows). Remove the $(k' + 1)$-chain, and add two vertices $a_{vw}$ and $b_{vw}$. Connect $a_{vw}$ with $b_{vw}$ via a $(k' + 1)$-chain, and direct all edges towards $b_{vw}$. Finally, add the arcs $(v, a_{vw})$ and $(w, a_{vw})$, as well as the arcs $(b_{vw}, v)$ and $(b_{vw}, w)$. We apply this to each vertical connection in $G''$.

Observe that no two routes can traverse a gadget without sharing at least $k' + 1$ arcs. Moreover, any route in row $i$, $1 \leq i < n$, can traverse through each gadget in each column to row $i + 1$, and vice versa. Thus, the introduced gadgets work like the vertical $(k' + 1)$-chains in $G''$.

Analogously to the proof of 1, we obtain the following.

**Corollary 1.** **Planar Minimum Shared Edges** is **NP-hard** on directed planar graphs of maximum out- and indegree three.

### 4 Conclusion

We proved that **Planar Minimum Shared Edges** is **NP-hard**, even on planar graphs of maximum degree four, leading to the natural question, whether the problem remains **NP-hard** on planar
graphs of maximum degree three. Furthermore, we showed that Planar Minimum Shared Edges is NP-hard on directed planar graphs of maximum out- and indegree three.

From a parameterized complexity perspective, since our reduction is not a parameterized reduction with respect to the number $k$ of shared edges, the question about the parameterized complexity of Planar MSE parameterized by $k$ remains open. Moreover, it would be interesting to see whether the running times of known FPT-algorithms \[6, 10\] for Minimum Shared Edges can be improved for Planar Minimum Shared Edges.

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