Cosmological Solution in M-theory on $S^1/Z_2$

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**Abstract**

We provide the first example of a cosmological solution of the Horava-Witten supergravity. This solution is obtained by exchanging the role of time with the radial coordinate of the transverse space to the five-brane soliton. On the boundary this corresponds to rotating an instanton solution into a tunneling process in a space with Lorentzian signature, leading to an expanding universe. Due to the freedom to choose different non-trivial Yang-Mills backgrounds on the boundaries, the two walls of the universe (visible and hidden worlds) expand differently. However at late times the anisotropy is washed away by gravitational interactions.

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Recent years have lead to surprising new developments in our attempt to build a fundamental theory. In particular it has been conjectured that some (yet unknown) $M$-theory unifies all the known vacua of string theory as well as eleven dimensional supergravity. These vacua are believed to be interconnected through a network of dualities. Many pieces of evidence have been gathered in this direction.

One of the simplest vacua to analyze, for providing new phenomenological or cosmological scenarios, is the strongly coupled limit of heterotic string theory. Horava and Witten have identified it with a $Z_2$ orbifold \([1]\). In the infrared limit, it has the description of an eleven dimensional supergravity on a manifold with boundaries. Gravitational anomalies are canceled by coupling the supergravity in each boundary with an $E_8$ super–Yang–Mills theory.

Unlike usual orbifold compactifications of string theory, in the Horava-Witten orbifold the communication between the two boundaries is due only to gravitational interactions. There is no gauge bosons propagating in the bulk. This leads to the picture of two parallel worlds (the observable and the hidden world) communicating through gravity.

It has been pointed out recently that cosmological solutions can be generated from p-brane solutions \([2–4]\). The process consists in the exchange the role of time with the radial coordinate of the transverse space\([4]\). In this note we will use a similar approach to obtain a cosmological solution to supergravity theory on an eleven dimensional manifold with boundaries. A large number of cosmological models have been derived in the past from a variety of vacua of M-theory. These include solutions of type II strings with both R-R and NS-NS background fields as well as weakly coupled heterotic string compactifications \([2–21]\). We will promote the discussion to the level of the strongly coupled limit of heterotic string theory.

In Horava-Witten model the eleven dimensional bulk theory is described by a graviton supermultiplet. Its degrees of freedom are the metric $g_{IJ}$, the three–form gauge potential $C_{IJK}$ with the associated field strength components $G_{IJKL}$ and the gravitino. Here $I, J...$ take values $0, 1, ..., 9, 11$. In addition on each boundary lives a super–Yang–Mills ten–dimensional supermultiplet with a gauge field $A_\alpha^A$ and the corresponding gauginos ($A, B... = 0, 1, ..., 9$). The gauge field strength is denoted by $F^{a(1,2)}_{AB}$. The bosonic part of the corresponding lagrangian (up to two derivatives) takes the form:
\[ S = \int d^{11}x \frac{1}{2\kappa^2} \sqrt{-g} \left[ -R - \frac{1}{24} G_{IJKL} G^{IJKL} \right] \\
- \int d^{11}x \frac{1}{\kappa^2} \sqrt{-g} \epsilon_{I_{11}...I_1} C_{I_1...I_2} G_{I_2...I_7} G_{I_8...I_{11}} \\
- \int d^{11}x \delta(x^{11}) \frac{1}{4\pi} \sqrt{-g_{10}} \text{tr}(F_{AB}^{(1)} F^{(1)AB}) \\
- \int d^{11}x \delta(x^{11} - \pi \rho) \frac{1}{4\pi} \sqrt{-g_{10}} \text{tr}(F_{AB}^{(2)} F^{(2)AB}) \]  \hspace{1cm} (1)

where \( \lambda^2 = 4\pi (4\pi \kappa^2)^{2/3} \) [1,24]. The orbifold segment is parametrized by \( x^{11} \in [0, \pi \rho] \). The metric \( g_{10} \) is the restriction of the eleven dimensional metric to the boundaries.

The Einstein equation are given by

\[ R_{MN} - \frac{1}{2} g_{MN} R = -\frac{1}{6} \left( G_{MJKL} G^{JKL} - \frac{1}{8} g_{MNG} G_{JIKL} G^{JIKL} \right) \\
- \delta(x^{11}) \frac{\lambda^2}{\kappa^2} \left( g_{11,11} \right)^{-\frac{1}{2}} \left[ \text{tr}(F_{AC}^{(1)} F_B^{(1)C}) - \frac{1}{4} g_{AB} \text{tr}(F_{CD}^{(1)} F^{(1)CD}) \right] \\
- \delta(x^{11} - \pi \rho) \frac{\lambda^2}{\kappa^2} \left( g_{11,11} \right)^{-\frac{1}{2}} \left[ \text{tr}(F_{AC}^{(2)} F_B^{(2)C}) - \frac{1}{4} g_{AB} \text{tr}(F_{CD}^{(2)} F^{(2)CD}) \right] . \]  \hspace{1cm} (2)

The equations of motion of the three-form \( C_{IJK} \) take the form:

\[ \partial_M \left( \sqrt{-g} G^{MNKL} \right) = \frac{\sqrt{2}}{1152} \epsilon^{NKL} G_{I_{11}...I_1} G_{I_1...I_2} G_{I_2...I_{11}} \\
- \frac{\kappa^2}{\lambda^2} \delta(x^{11}) \epsilon^{11NKL} G_{I_{11}...I_1} \omega_{I_1...I_7} G_{I_8...I_{11}} , \]  \hspace{1cm} (3)

where \( \omega_{I_5...I_7} \) is the Yang–Mills Chern–Simons form. The four-form field strength \( G_{ABCD} \) is also subject to the Bianchi identity [\(^{[1]}\)]:

\[ (dG)_{11ABCD} = -3\sqrt{2} \frac{\lambda^2}{\kappa^2} \delta(x^{11}) \left( \text{tr}(F_{[AB} F_{CD]} - \frac{1}{2} \text{tr}(R_{[AB} R_{CD]})) \right) . \]  \hspace{1cm} (4)

The boundary gauge fields satisfy

\[ D_A F^{iAB} = \frac{1}{\sqrt{2}} \left( g_{11,11} \right)^{-\frac{1}{2}} F^{i}_{DE} G^{11BDE} + \frac{1}{1152} \frac{1}{\sqrt{-g_{10}}} \epsilon^{B_{I_2...I_{10}} \hat{i}} G_{I_3...I_6} G_{I_7...I_{10}} , \]  \hspace{1cm} (5)

with \( D_A \) the space–time and gauge covariant derivative.

We want to exhibit a solution with a metric in eleven dimension of the form (in polar coordinates):

\[ ds^2 = e^{2A} \left( -q(dx^0)^2 + dx^\mu dx^\nu \delta_{\mu\nu} \right) + e^{2B} \left( \frac{dr^2}{q} + \tau^2 d\Omega^2 \right) + e^{2B} (dx^{11})^2 , \]  \hspace{1cm} (6)

where \( q \) will later be taken equal to \( -1 \) later. The unit spherical volume element is given by:

\[ d\Omega^2 = d\hat{r}^2 + \sin^2 \left( \frac{\sqrt{k}\hat{r}}{\kappa} \right) \left( d\theta^2 + \sin^2 \theta d\omega^2 \right) . \]  \hspace{1cm} (7)
Taking \( k = q \) the metric (3) can be transformed to:

\[
\begin{align*}
\text{ds}^2 &= e^{2A} (-dt^2 + dx^\mu dx^\nu \delta_{\mu\nu}) + e^{2B} \left( dr^2 + r^2 \left( d\phi^2 + \sin^2(\phi) \left( d\theta^2 + \sin^2\theta d\omega^2 \right) \right) \right) + e^{2B} (dx^{11})^2,
\end{align*}
\]

through the changement of variables:

\[
\begin{align*}
t &= \sqrt{q}x^0, & \phi &= \sqrt{q}\chi, & r &= \frac{\tau}{\sqrt{q}},
\end{align*}
\]

We can now use the fact that if some function \( A \) and \( B \), as well as a background configuration of gauge fields \( \mathfrak{g} \) solves the equations of motion and satisfies all the constraints, then the new metric (8) obtained through a simple coordinate transformation will also do. A particular non-trivial solution with a metric of the form (8) is the five-brane soliton. This solution was obtained in the weakly coupled regime by [23] and elevated to a solution of the strongly coupled theory by [24], then successively revisited by [25] and [26]. Here we will closely follow the discussion of [25]. For the five-brane the functions \( A, B \) and the four-form \( G \) are related by \( A = -B/2 = -C/6 \) and

\[
G_{mnrs} = \pm \frac{1}{\sqrt{2}} e^{-\frac{8C}{q}} \epsilon_{mnrs} t \partial_t e^C
\]

with \( m, n, \ldots = 6, \ldots, 9, 11 \). The function \( C \) depends only on \( x^{11} \) and \( r \). We first solve the equations of motion in the metric we will extract a cosmological solution by exchanging the role of time with the radial coordinate of the transverse space through (9).

In our ansatz (10) the only non-vanishing components of the four-field strength are \( G_{abcd} \) and \( G_{11abc} \) with \( a, b, \ldots = 6, \ldots, 9 \). For such configuration \( G \wedge G \) vanishes and the equation of motion becomes \( d*G = 0 \). In (10) \( *G \) is a closed form so the equation of motion are automatically satisfied.

The Bianchi identity involves the gauge fields at the boundaries that need to be specified. A non-trivial Yang-Mills background is obtained with a field strength self-dual with respect to the metric:

\[
ds_4^2 = \frac{d\tau^2}{q} + \tau^2 d\Omega_3^2,
\]

We choose a configuration which in the case \( q = 1 \) corresponds to an \( SU(2) \) instanton. For \( q = -1 \), the gauge vectors take value in a subset of \( SL(2, \mathbb{C}) \) and the background considered can be interpreted as mapping of a pseudo-sphere in the target space on a a pseudo-sphere of the gauge group space.
It is useful to present the expressions for the gauge fields for a generic value of $q$ as they were derived in \cite{4}. The gauge vector potential takes the form:

$$ A(r) = \gamma(\tau) \sqrt{q}g^{-1}dg, \quad (12) $$

$g^{-1}dg$ is the Cartan-Maurer form where

$$ g = \cos \sqrt{q} \chi - i \cos \omega \sin \theta \sin \sqrt{q} \chi \sigma_1 - i \sin \omega \sin \theta \sin \sqrt{q} \chi \sigma_2 - i \cos \theta \sin \sqrt{q} \chi \sigma_3, \quad (13) $$

with

$$ \gamma(r) = \frac{\tau^2}{\sqrt{q} \sqrt{q} \sigma^2 + 1}, \quad (14) $$

where $\sqrt{q} \sigma^2$ is an integration constant. The associated field strength is given by:

$$ F^a = \frac{2\sqrt{q} \sigma^2}{(\tau^2 + \sigma^2)^2} \eta^a_{mn} dy^m \wedge dy^n, \quad (15) $$

where $a$ is an adjoint index of $SU(2)$ for $q = 1$ or its analytical continuation in $SL(2, \mathbb{C})$ for $q = -1$, $\eta^a_{mn}$ are the corresponding 't Hooft symbols. Thus,

$$ \text{tr}(F_{[AB}F_{CD]} = -\sqrt{q} \epsilon_{ABCD} \frac{16q \sigma^4}{(\sigma^2 + \tau^2)^4}. \quad (16) $$

Following \cite{25}, we denote $e^C$ by $\Phi$ and write the solution as:

$$ \Phi = \Phi_0 + \phi, \quad (17) $$

where $\Phi_0$ is a function of $r$ only, while $\phi$ is a function of both $r$ and $x^{11}$ with vanishing average on $x^{11}$. $\phi$ contains the corrections of the strong coupling limit of the heterotic string theory. The Bianchi identities can be written as an homogeneous condition $dG = 0$ in the bulk supplemented with the boundary conditions \cite{3}

$$ G_{ABCD}|_{x^{11}=0} = -\frac{3}{2} \frac{\kappa^2}{\Lambda^2} \text{tr}(F^{(1)}_{[AB}F^{(1)}_{CD]}), $$

$$ G_{ABCD}|_{x^{11}=\pi\rho} = \frac{3}{2} \frac{\kappa^2}{\Lambda^2} \text{tr}(F^{(2)}_{[AB}F^{(2)}_{CD]}). \quad (18) $$

\footnote{Note that after wick rotation these quantities become imaginary, as expected because they represent a tunneling effect.}
They lead to the differential equations (we assume \( q = \pm 1 \)):

\[
\frac{1}{r^3} \partial_r \left( r^3 \partial_r \Phi_0 \right) = \frac{q \kappa^2}{4 \pi \rho \lambda^2} \left( \frac{16 \sigma_1^4}{(\frac{\sigma_1^2}{q} + r^2)^4} + \frac{16 \sigma_2^4}{(\frac{\sigma_2^2}{q} + r^2)^4} \right)
\]  
(19)

and

\[
\frac{1}{r^3} \partial_r \left( r^3 \partial_r \phi \right) + \partial_{11}^2 \phi = 0
\]  
(20)

The function \( \phi \) is subject to the boundary conditions:

\[
\partial_{11} \phi \big|_{x^{11} = 0} = -q \frac{\kappa^2 \sigma_1^4}{4 \lambda^2 (\frac{\sigma_1^2}{q} + r^2)^4}, \quad \partial_{11} \phi \big|_{x^{11} = \pi \rho} = q \frac{\kappa^2 \sigma_2^4}{4 \lambda^2 (\frac{\sigma_2^2}{q} + r^2)^4}.
\]  
(21)

We do not reproduce here the different steps of the solution and refer for details to the original work by [25]. After making the change of variables (9), we obtain:

\[
\Phi_0 = 1 + \frac{2 \kappa^2}{\pi \rho \lambda^2} \left( \frac{2 \sigma_1^2 + \tau^2}{(\sigma_1^2 + \tau^2)^2} + \frac{2 \sigma_2^2 + \tau^2}{(\sigma_2^2 + \tau^2)^2} \right).
\]  
(22)

and \( \phi \) is given as an expansion in \((\pi \rho)^2/\sigma^2\):

\[
\phi = \sum_{n=0}^{\infty} \phi_n
\]  
(23)

The first terms of the expansion are given by:

\[
\phi_0 = 48 \frac{\kappa^2}{\lambda^2} \left( P_0(x^{11}) \frac{\sigma_1^4}{(\sigma_1^2 + \tau^2)^4} + Q_0(x^{11}) \frac{\sigma_2^4}{(\sigma_2^2 + \tau^2)^4} \right)
\]  
(24)

and

\[
\phi_1 = 768 \frac{\kappa^2}{\lambda^2} \left( P_1(x^{11}) \frac{3 \tau^2 - 2 \sigma_1^2}{(\sigma_1^2 + \tau^2)^6} + Q_1(x^{11}) \frac{3 \tau^2 - 2 \sigma_2^2}{(\sigma_2^2 + \tau^2)^6} \right).
\]  
(25)

with:

\[
P_0 = -\frac{(x^{11})^2}{4 \pi \rho} + \frac{x^{11}}{2} - \frac{\pi \rho}{6}, \quad Q_0 = -\frac{(x^{11})^2}{4 \pi \rho} + \frac{\pi \rho}{12}
\]  
(26)

and

\[
P_1 = \frac{(x^{11})^4}{48 \pi \rho} - \frac{(x^{11})^3}{12} + \frac{\pi \rho (x^{11})^2}{12} - \frac{\pi^3 \rho^3}{90}, \quad Q_1 = \frac{(x^{11})^4}{48 \pi \rho} - \frac{\pi \rho (x^{11})^2}{24} + \frac{7 \pi^3 \rho^3}{720}
\]  
(27)

We thus found a cosmological solution of the form:

\[
ds^2 = \Phi^{-1/3} \left( (dx^0)^2 + dx^\mu dx^\nu \delta_{\mu \nu} \right) + \Phi^{2/3} \left( d\tau^2 + \tau^2 d\Omega^2_{3,-1} \right) + \Phi^{2/3} (dx^{11})^2
\]  
(28)
with $\Phi$ given by (22) to (27). To have a time coordinate which goes from $-\infty$ to $+\infty$ one makes the rescaling $\tau = e^\eta$. The choice $\tau = e^{-\eta}$ lead to a similar branch disconnected from this one.

We have thus transformed the gauge five-brane solution (in the form given by [25]) to a non-singular cosmological solution[4]. We would like to discuss now some interesting features of our solution. A four dimensional solution is obtained by considering the six-dimensional transverse space parametrized by $x^0, \ldots, x^5$ as compact.

For $t \to 0$ or equivalently $\eta \to -\infty$ the size of the compact internal space is finite. However the radius of the pseudo-sphere of the universe goes to zero. Through tunneling effect a bubble with a new expectation value of the field $\Phi$ (the dual of the four-form $G$) is created at $\eta \to -\infty$ and expands. This event is at finite proper time from any point in the future. The initial conditions on the two boundaries may be different (if $\sigma_1 \neq \sigma_2$). Thus the tunneling on the two boundaries follows different time dependence. On the $x^{11} = 0$ boundary:

$$\Phi = 1 + \frac{2\kappa^2}{\pi \rho \lambda^2} \left( \frac{2\sigma_1^2 + \tau^2}{(\sigma_1^2 + \tau^2)^2} + \frac{2\sigma_2^2 + \tau^2}{(\sigma_2^2 + \tau^2)^2} \right) + \frac{8\kappa^2 \pi \rho}{\lambda^2} \left( -\frac{\sigma_1^4}{(\sigma_1^2 + \tau^2)^4} + \frac{\sigma_2^4}{2(\sigma_2^2 + \tau^2)^4} \right) + \ldots, \quad (29)$$

while on the other edge $x^{11} = \pi \rho$:

$$\Phi = 1 + \frac{2\kappa^2}{\pi \rho \lambda^2} \left( \frac{2\sigma_1^2 + \tau^2}{(\sigma_1^2 + \tau^2)^2} + \frac{2\sigma_2^2 + \tau^2}{(\sigma_2^2 + \tau^2)^2} \right) + \frac{8\kappa^2 \pi \rho}{\lambda^2} \left( \frac{\sigma_1^4}{2(\sigma_1^2 + \tau^2)^4} - \frac{\sigma_2^4}{(\sigma_2^2 + \tau^2)^4} \right) + \ldots, \quad (30)$$

where the dots are for higher orders of the expansion in $(\pi \rho)^2/\sigma^2$.

At later times as $t \to +\infty$ or equivalently $\eta \to +\infty$, we find $\Phi \to 1$. Thus the solution describes an expanding four dimensional flat universe. The expansion anisotropy between the two boundaries is washed away by gravitational interactions between them. It is also interesting to notice that the volume of the internal space and the size of the segment remain finite (small) and thus we have a mechanism of spontaneous compactification.

It is tempting to find a relation between our toy example and the recent work of Hawking and Turok [28]. Our field $\Phi$ (obtained through dualization of the four form) or more precisely $\Phi_0$ would be identified with the scalar field considered in [28].

We have illustrated the possibility of having different initial conditions on the two boundaries of the universe which might lead to different expansions at early time. This might also be extended to the case of neutral five-brane [27] as a trivial issue and leads to a singular cosmology.
have implication for dark matter physics at early times if as suggested in [29] it lives on the hidden wall of the universe.

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