New Compactifications of Supergravities and Large $N$ QCD

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Abstract

We construct supergravity backgrounds representing non-homogeneous compactifications of $d = 10, 11$ supergravities to four dimensions, which cannot be written as a direct product. The geometries are regular and approach $AdS_7 \times S^4$ or $AdS_5 \times S^5$ at infinity; they are generically non-supersymmetric, except in a certain “extremal” limit, where a Bogomol’nyi bound is saturated and a naked singularity appears. By using these spaces, one can construct a model of QCD that generalizes by one (or two) extra parameters a recently proposed model of QCD based on the non-extremal D4 brane. This allows for some extra freedom to circumvent some (but not all) limitations of the simplest version.

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1. Introduction

In the last few months there have been a number of suggestions generalizing the
dualities proposed by Maldacena [1] to the case of four-dimensional gauge theories with
less supersymmetries (see e.g. refs. [2-10]). In most of the examples so far considered, the
supergravity backgrounds are of the form $AdS_n \times X$, where $X$ is some Einstein manifold,
and they are expected to be related to a conformal field theory “at the boundary” [1,11,2],
but, because of conformal invariance, not to a confining gauge theory (for a review of
supergravity compactifications on anti-de Sitter backgrounds, see [12]). The interesting
model introduced by Witten [5] and further investigated in [13-16] exhibits confinement,
but, as pointed out in these works, its applications to $3+1$ dimensional QCD are somewhat
limited by the fact that there is a single scale in the geometry. The model arises as
dimensional reduction of $4+1$ dimensional $SU(N)$ super Yang-Mills theory, which as
quantum field theory is non-renormalizable; in order to decouple Kaluza-Klein theory, which as
least two scales seem necessary: one representing $\Lambda_{\text{QCD}}$, and another one for the masses
of Kaluza-Klein particles $M_{\text{KK}}$, with $\Lambda_{\text{QCD}} \ll M_{\text{KK}}$. In addition, understanding certain
properties of the spectrum seems to require an understanding of string theory in Ramond-
Ramond backgrounds. Unfortunately, such string models are not solvable; to be able to
uncover the properties of large $N$ QCD, one would like to have a regime where there is a
supergravity description.

The present work is an attempt in this direction. We will assume that a supergravity
description of pure Yang-Mills theory at large ’t Hooft coupling exists and start with the
most general regular geometry that can plausibly describe a confining theory, in the sense
that will be explained below. The non-supersymmetric background of section 3 general-
izes the Witten QCD model in much the same way the Kerr black hole generalizes the
Schwarzschild black hole solution. It is, however, a static geometry – what is “time” in the
Kerr solution here plays the role of an angular coordinate –, which is regular everywhere,
and contains an extra parameter $a$ with respect to the Witten QCD model. When this
parameter is small, the model reduces to the Witten model, with $\Lambda_{\text{QCD}} \cong M_{\text{KK}}$. When
this parameter is large, one obtains the desired decoupling of unwanted Kaluza-Klein par-
ticles, $\Lambda_{\text{QCD}} \ll M_{\text{KK}}$, which guarantees that the gauge theory at the QCD scale is a $3+1$
(rather than $4+1$) dimensional one.
2. Large $N$ QCD from supergravity

In order to introduce our notation, in this section we review the approach to non-supersymmetric Yang-Mills theory of \cite{5}.

2.1. Construction of the metric from the black M5-brane

We start with the metric for the non-extremal M5-brane of eleven-dimensional supergravity, which is given by

$$ds^{2}_{11} = f^{-1/3}[-h\ dt^2 + dx_1^2 + ... + dx_5^2] + f^{2/3}\left[\frac{dr^2}{h} + r^2d\Omega_4^2\right], \quad (2.1)$$

where

$$f = 1 + \frac{2m \sinh^2 \alpha}{r^3}, \quad h = 1 - \frac{2m}{r^3}, \quad (2.2)$$

There is in addition non-zero flux of the four-form field strength in the four-sphere with quantized (magnetic) charge $N$ related to $m$ and $\alpha$ by

$$2m \cosh \alpha \sinh \alpha = \pi N l_P^3, \quad (2.3)$$

where $l_P$ is the Planck length in eleven dimensions (in terms of string theory parameters, defined as $l_P = g^{1/3/(\alpha')}$). There is an event horizon at $r = r_H$, $r_H = (2m)^{1/3}$. The Hawking temperature is given by

$$T_H = \frac{3r_H^2}{8\pi m \cosh \alpha}. \quad (2.4)$$

By dimensional reduction in $x_5$, one obtains the type IIA solution representing the non-extremal D4-brane \cite{17}:

$$ds^2_{IIA} = f^{-1/2}[-h\ dt^2 + dx_1^2 + ... + dx_4^2] + f^{1/2}\left[\frac{dr^2}{h} + r^2d\Omega_4^2\right], \quad (2.5)$$

$$e^{2(\phi(r) - \phi_{\infty})} = f^{-1/2}(r). \quad (2.6)$$

The supergravity solution that is related to a field theory is obtained by taking the ("decoupling") low-energy limit $l_P \to 0$ in (2.1). This is done by rescaling variables as follows $\cite{1,18}$:

$$r = U^2 l_P^3, \quad m = \frac{1}{2} U_6^6 l_P^3, \quad (2.7)$$

and taking the limit $l_P \to 0$ at fixed $U, U_0$, so that $2m \sinh^2 \alpha \to \pi N l_P^3$ (see (2.3)). We get

$$ds^2_{11} = l_P^2\left[\frac{U^2}{(\pi N)^{1/3}}[-(1 - \frac{U_6^6}{U^6})dt^2 + dx_1^2 + ... + dx_5^2] + \frac{4(\pi N)^{2/3}dU^2}{U^2(1 - \frac{U_6^6}{U^6})} + (\pi N)^{2/3}d\Omega_4^2\right]. \quad (2.8)$$

There is a horizon at $U = U_0$. At infinity, the solution approaches the geometry of $AdS_7 \times S^4$. It can also be obtained as a special limit of the Schwarzschild-de Sitter black hole $\cite{5}$. 

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2.2. Witten QCD model

The low-energy theory on the non-extremal D4 brane is 4+1 dimensional \( SU(N) \) Yang-Mills theory at finite temperature \( T_H \). In the path integral formulation, the gauge theory at finite temperature is described by going to Euclidean time \( t \to -i\tau \), and identifying \( \tau \) periodically with period \( 1/T_H \). Zero-temperature 3+1 Yang-Mills theory can be described by making \( x_4 \to -ix_0 \), and viewing \( \tau \) as parametrizing a space-like circle with radius \( R_0 = (2\pi T_H)^{-1} \), where – as in the thermal ensemble – fermions obey antiperiodic boundary conditions. At energies much lower than \( 1/R_0 \), the theory is effectively 3+1 dimensional. Because of the boundary conditions, fermions and scalar particles get masses proportional to the inverse radius, so that, as \( R_0 \to 0 \), they should decouple from the low-energy physics. The low-energy theory is thus pure Yang-Mills theory.

The gauge coupling \( g_4^2 \) in the 3+1 dimensional Yang-Mills theory is given by the ratio between the periods of the eleven-dimensional coordinate \( x_5 \) and \( \tau \). It is convenient to introduce ordinary angular coordinates \( \theta_1, \theta_2 \) which are \( 2\pi \)-periodic by

\[
\tau = R_0 \theta_2 \ , \quad x_5 = g_4^2 R_0 \theta_1 = \frac{\lambda}{N} R_0 \theta_1 ,
\]

where we have introduced the ‘t Hooft coupling

\[
\lambda \equiv g_4^2 N .
\]

The eleven-dimensional metric takes the form

\[
ds_{11}^2 = \frac{U^2}{(\pi N)^{1/3}} \left[ -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + \frac{4U^2}{9U_0^2} (\pi N)^{2/3} \left( 1 - \frac{U_0^6}{U^6} \right) d\theta_2^2
\]

\[
+ \frac{4U^2 \lambda^2}{9U_0^2 N^2} (\pi N)^{2/3} d\theta_1^2 + \frac{4(\pi N)^{2/3} dU^2}{U^2 (1 - \frac{U_0^6}{U^6})} + (\pi N)^{2/3} d\Omega_4^2 .
\]

In the large \( N \) limit at fixed \( \lambda \), the circle parametrized by \( \theta_1 \) is much smaller than that parametrized by \( \theta_2 \), so that we can dimensionally reduce in \( \theta_1 \). We obtain \( (U = 2(\pi N)^{1/2}u)\)

\[
ds_{10A}^2 = \frac{8\pi \lambda u_3}{3u_0} \left[ -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + \frac{8\pi \lambda u_3}{27} \frac{u_3}{u_0} (1 - \frac{u_0^6}{u^6}) d\theta_2^2
\]

\[ \]
\[ + \frac{8\pi\lambda}{3} \frac{du^2}{u_0u(1 - \frac{u_0^6}{u^6})} + \frac{2\pi\lambda}{3u_0} u d\Omega_4^2, \]  

(2.12)

\[ e^{2\phi} = \frac{8\pi \lambda^3 u^3}{27} \frac{1}{u_0^3 N^2}, \quad R_0^{-1} = 3u_0. \]  

(2.13)

Wilson loops in the model exhibit a confining area-law behavior [3,13,19], where the string tension \( T_{YM} \) is proportional to the coefficient of \( \sum_{i=0}^3 dx_i^2 \) at the horizon,

\[ T_{YM} = \frac{4}{3} \lambda u_0^2. \]  

(2.14)

The supergravity fields can be classified in terms of representations of the isometry group of the internal space \( SO(5) \times U(1) \). Glueballs are expected to be associated with singlets of \( SO(5) \) carrying vanishing \( U(1) \) charge [13]. The squared mass of glueballs should be proportional to the string tension, i.e.

\[ M_{\text{glueballs}} \approx \Lambda_{\text{QCD}} = u_0. \]  

(2.15)

The Kaluza-Klein states associated with the circle parametrized by \( \theta_2 \) are of the form \( \Phi = \chi(u)e^{ik\cdot x}e^{in\theta_2} \). These states will have mass of order \( R_0^{-1} \), i.e. in our units,

\[ M_{\text{KK}} \approx R_0^{-1} = 3u_0. \]  

(2.16)

Comparing eqs. (2.15) and (2.16) one sees that Kaluza-Klein states cannot be decoupled while keeping the glueballs in the spectrum. In other words, as long as \( \Lambda_{\text{QCD}} \) is finite, the theory is always 4 + 1 dimensional. In sect. 3 we explore a generalization of the Witten model which overcomes this problem.

3. Generalizations

In the holographic picture [2], \( SU(N) \) gauge theory on \( R^4 \) is expected to be described in terms of a sum over geometries that have \( R^4 \) isometry and asymptotically approach \( AdS_7 \times S^4 \) (or \( AdS_5 \times S^5 \)). A natural question is whether there are other metrics (beside the non-extremal D4 brane) that can be relevant for a description of QCD within the framework of supergravity. We first look for regular geometries that approach \( AdS_7 \times S^4 \) at infinity, i.e. something of the form

\[ ds_{11}^2 = U^2 h_1(U, \varphi_m) \sum_{i=0}^3 dx_i^2 + \frac{R^2_{AdS} dU^2}{U^2 h_2(U, \varphi_m)} + \text{angular part}, \]
where $h_1, h_2 \to 1$ at $U \to \infty$. In order to have an area law for Wilson loops within a supergravity description, it is necessary that the coefficient of $\sum dx_i^2$ be bounded above zero \cite{5}. This suggests that there must be a horizon at some $U = U_H > 0$. No-hair theorems imply that the most general stationary M5 metric with a regular horizon is given by a Kerr-type metric with angular momentum components in two planes, and they preclude any other possibility. In the present case, since we want full Poincaré symmetry in the four-dimensional space $x_i$, we look for a static geometry, so what plays the role of “time” in the Kerr metric here will be an angular coordinate describing an internal circle (just as in the model of the preceding section based on the non-extremal M5 brane). Alternatively, one may try to look for geometries with a regular horizon and $\mathbb{R}^4$ isometry that approach $AdS_5 \times S^5$ at infinity. The no-hair theorem implies that there are none (black D3 branes do not have $\mathbb{R}^4$ isometry). On the face of it, there are no other options to describe a $3+1$ dimensional confining gauge theory other than a Kerr-type generalization of the M5 brane. Standing possibilities are of course singular metrics, but these require an understanding of the corresponding string theory.

3.1. A static $D4$-brane

We start with the metric for the rotating M5 brane of eleven-dimensional supergravity with angular component in one plane. This solution can be obtained by uplifting the magnetically charged rotating black hole in $D = 6$ \cite{20} and it was explicitly constructed in \cite{21}. It is given by

$$ds_{11}^2 = f^{-1/3} \left[ -h dt^2 + dx_1^2 + ... + dx_5^2 \right] + f^{2/3} \left[ \frac{dr^2}{h} + r^2 (\Delta d\theta^2 + \tilde{\Delta} \sin^2 \theta d\varphi^2 \right. \right.$$  

$$+ \left. \cos^2 \theta d\Omega_2^2 \right] - \frac{4lm \cosh \alpha}{r^3 \Delta f} \sin^2 \theta dt d\varphi \right), \tag{3.1}$$

where

$$f = 1 + \frac{2m \sinh^2 \alpha}{r^3 \Delta}, \quad \Delta = 1 + \frac{l^2 \cos^2 \theta}{r^2}, \quad \tilde{\Delta} = 1 + \frac{l^2}{r^2} + \frac{2ml^2 \sin^2 \theta}{r^5 \Delta f}, \quad \tag{3.2}$$

$$h = 1 - \frac{2m}{r^3 \Delta}, \quad \tilde{h} = \frac{1}{\Delta} \left( 1 + \frac{l^2}{r^2} - \frac{2m}{r^3} \right), \quad \tag{3.3}$$

$$d\Omega_2^2 = d\psi_1^2 + \sin^2 \psi_1 d\psi_2^2.$$
There are in addition non-zero components for the three-form field
\[ C_{\varphi \psi_1 \psi_2} = 2m\Delta^{-1}(1 + \frac{l^2}{r^2}) \cosh \alpha \sinh \alpha \cos^2 \theta , \quad C_{t \psi_1 \psi_2} = -\frac{2ml}{r^2 \Delta} \sinh \alpha \cos^2 \theta . \]

The quantized (magnetic) charge \( N \) is related to \( \alpha \) and \( m \) as in (2.3). There is an event horizon at \( r = r_H \), where \( r_H \) is the real solution of \( r^3 + l^2r - 2m = 0 \), which exists for all values of \( m > 0 \) and \( l \). The Killing vector \( \partial/\partial t \) diverges at \( r = 0 \) and becomes space-like in the ergosphere region outside the horizon where \( r^2 + l^2 \cos^2 \theta - 2m/r < 0 \). There is no inner horizon, just a space-like singularity at \( r = 0 \). In the limit \( m \to 0 \), horizon and singularity coalesce, and the solution saturates the BPS bound [20]. The Hawking temperature is given by
\[ T_H = \frac{3r_H^2 + l^2}{8\pi m \cosh \alpha} . \] (3.4)

By dimensional reduction in \( x_5 \), one obtains the following type IIA solution, representing a rotating black D4-brane:
\[ ds_{\text{IIA}}^2 = f^{-1/2}[-h \, dt^2 + dx_1^2 + \ldots + dx_4^2] + f^{1/2}\left[\frac{dr^2}{h} + r^2(\Delta d\theta^2 + \tilde{\Delta} \sin^2 \theta d\varphi^2 \right. \\
+ \cos^2 \theta \, d\Omega_2^2 - \frac{4lm \cosh \alpha}{r^3 \Delta f} \sin^2 \theta dt d\varphi \bigg], \] (3.5)
\[ e^{2(\phi(r) - \phi_\infty)} = f^{-1/2} . \] (3.6)

Although string theory on this space cannot be dual to Lorentz-invariant quantum field theory, a metric with \( \mathbf{R}^4 \) Poincaré symmetry can nevertheless be obtained by the formal substitution:
\[ t \to -i\tau , \quad x_4 \to ix_0 , \quad l \to ib \]
where \( \tau \) is periodic of period \( 1/T_H \). We obtain
\[ ds_{\text{IIA}}^2 = f^{-1/2}[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2] + f^{-1/2}h \, d\tau^2 + f^{1/2}\left[\frac{dr^2}{\tilde{h}} \\
+ r^2(\Delta d\theta^2 + \tilde{\Delta} \sin^2 \theta d\varphi^2 + \cos^2 \theta d\Omega_2^2) - \frac{4bm \cosh \alpha}{r^3 \Delta f} \sin^2 \theta d\tau d\varphi \bigg] , \] (3.7)
with \( \phi \) and \( \Delta, \tilde{\Delta}, \tilde{h} \) as before with \( l \to ib \), i.e.
\[ \Delta = 1 - \frac{b^2 \cos^2 \theta}{r^2} , \quad \tilde{\Delta} = 1 - \frac{b^2}{r^2} - \frac{2mb^2 \sin^2 \theta}{r^5 \Delta f} , \quad \tilde{h} = \frac{1}{\Delta} \left( 1 - \frac{b^2}{r^2} - \frac{2m}{r^3} \right) . \] (3.8)
The geometry is regular and geodesically complete (being essentially the Euclidean Kerr metric) with the space-time restricted to the region $r > r_H$, where $r_H$ is the real solution of $r^3 - b^2r - 2m = 0$. Since $m > 0$, one has $r_H > b$ (there is a singularity at the surface $r^2 = b^2 \cos^2 \theta$, but this lies inside the horizon). It should be noted that the interpretation as rotating D4-brane is no longer valid. The geometry now represents a static 4-brane configuration with Ramond-Ramond charge, where one of the world-volume directions $\tau$ is compactified on a circle and mixed with the angular directions.

Let us now consider the eleven-dimensional metric that is obtained in the “decoupling” limit. As in (2.7), we redefine variables as follows:

$$r = U^2 l_P^3, \quad m = \frac{1}{2} U_0^6 l_P^0, \quad b = a^2 l_P^0,$$

and take the limit $l_P \to 0$ at fixed $U, U_0$ and $a$. We get

$$ds^2_{11} = l_P^2 \Delta^{1/3} \left[ \frac{U^2}{(\pi N)^{1/3}} \left( -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_5^2 + (1 - \frac{U_0^6}{U^6 \Delta}) d\tau^2 \right) + \frac{4(\pi N)^{2/3} dU^2}{U^2(1 - \frac{a^4}{U^4} - \frac{U_0^6}{U^6})} \right.$$

$$+ (\pi N)^{2/3} \left( \frac{d\theta^2 + \Delta \sin^2 \theta d\varphi^2 + \frac{1}{\Delta} \cos^2 \theta d\Omega_2^2 - \frac{2a^2 U_0^3}{U^4 \Delta (\pi N)^{1/2}} \sin^2 \theta d\tau d\varphi} \right) \left], \quad (3.10) \right.$$

where

$$\Delta = 1 - \frac{a^4 \cos^2 \theta}{U^4}, \quad \tilde{\Delta} = 1 - \frac{a^4}{U^4}.$$

The background thus represents a smooth compactification of $d = 11$ supergravity which is not of “direct product” form. Near infinity, it looks like $AdS_7 \times S^4$.

Similarly, one can construct the more general solution with parameters $\{U_0, N, a_1, a_2\}$ by starting with the M5-brane with angular momentum in two planes [21], with parameters $\{m, N, l_1, l_2\}$, and making $t \to -it$, $l_{1,2} = ia_{1,2}^2 l_P^3$, $2m = U_0^6 l_P^0$. In the decoupling limit, it is of the form

$$ds^2_{11} = l_P^2 \Delta^{1/3} \left[ \frac{U^2}{(\pi N)^{1/3}} \left( -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_5^2 + (1 - \frac{U_0^6}{U^6 \Delta}) d\tau^2 \right) \right.$$

$$+ \frac{4(\pi N)^{2/3} \Delta^{1/3} dU^2}{U^2(1 - \frac{a_1^4}{U^4} - \frac{a_2^4}{U^4} + \frac{a_1^4 a_2^4}{U^8} - \frac{U_0^6}{U^6})} + \text{angular part}, \quad (3.11)$$

$$\Delta = 1 - \frac{a^4}{U^4} \cos^2 \theta - \frac{a_1^4}{U^4} (\cos^2 \theta \cos^2 \psi_1 + \sin^2 \theta) + \frac{a_2^4}{U^8} \cos^2 \theta \cos^2 \psi_1.$$

(3.12)
3.2. Type IIB vacua

Analogous spaces containing 3-branes of type IIB theory, which are of the form \( AdS_5 \times S^5 \) at infinity, can be constructed by starting with a “stack” of rotating M2 branes in eleven dimensions with two extra translational isometries. This is obtained by a slight generalization of the solutions of [20,21]. The metric is given by

\[
 ds^2_{11} = f_0^{-2/3} \left[ -h_0 \, dx_0^2 + dx_1^2 + dx_2^2 \right] + f_0^{1/3} \left[ \frac{dr^2}{h_0} + d\tau_1^2 + d\tau_2^2 + r^2 (\Delta_0 d\theta^2 + \tilde{\Delta}_0 \sin^2 \theta d\varphi^2 \\
+ \cos^2 \theta \, d\Omega_3^2) - \frac{4lm \cosh \alpha}{r^4 \Delta_0 f_0} \sin^2 \theta \, dx_0 d\varphi \right]. \tag{3.13}
\]

The functions \( f_0, \Delta_0, \) etc. are as before with the replacement \( m/r^3 \rightarrow m/r^4 \), that is,

\[
 f_0 = 1 + \frac{2m \sinh^2 \alpha}{r^4 \Delta_0}, \quad \Delta_0 = 1 + \frac{l^2 \cos^2 \theta}{r^2}, \quad \tilde{\Delta}_0 = 1 + \frac{l^2}{r^2} + \frac{2ml^2 \sin^2 \theta}{r^6 \Delta_0 f_0}, \tag{3.14}
\]

\[
 h_0 = 1 - \frac{2m}{r^4 \Delta_0}, \quad \tilde{h}_0 = \frac{1}{\Delta_0} \left( 1 + \frac{l^2}{r^2} - \frac{2m}{r^4} \right). \tag{3.15}
\]

Upon dimensional reduction in \( \tau_2 \) and T-duality in \( \tau_1 \) one finds

\[
 ds^2_{\text{IIB}} = f_0^{-1/2} \left[ -h_0 \, dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + f_0^{1/2} \left[ \frac{dr^2}{h_0} + r^2 (\Delta_0 d\theta^2 + \tilde{\Delta}_0 \sin^2 \theta d\varphi^2 \\
+ \cos^2 \theta \, d\Omega_3^2) - \frac{4lm \cosh \alpha}{r^4 \Delta_0 f_0} \sin^2 \theta \, dx_0 d\varphi \right], \tag{3.16}
\]

\[
 e^{\phi_B} = g_s = \text{const}.
\]

where we have replaced the compact coordinate T-dual to \( \tau_1 \) by an uncompact coordinate \( x_3 \). The solution (3.16) represents a rotating black D3-brane with charge \( N \), related to \( m, \alpha \) by

\[
 4\pi g_s N \alpha'{}^2 = 2m \cosh \alpha \sinh \alpha. \tag{3.17}
\]

There is an event horizon at \( r^4 + l^2 r^2 - 2m = 0 \), or \( r_H^2 = \frac{1}{2} (\sqrt{l^4 + 8m} - l^2) \). For any \( m > 0 \), there is a space-like singularity at \( r = 0 \). The ADM mass per unit volume of the three brane is given by

\[
 \frac{M_{\text{ADM}}}{V_3} = \frac{cm}{2\pi g_s \alpha'{}^2} \left( \cosh^2 \alpha + \frac{1}{4} \right). 
\]
Let us now consider the extremal limit \(m \to 0\) at fixed charge \(N\). In this limit the solution saturates the Bogomol’nyi bound \(\frac{M_{\text{ADM}}}{\Delta_3} = cN\) (there should be 16 unbroken supersymmetries, i.e. 1/2 of the supersymmetries of \(AdS_5 \times S^5\)). We obtain

\[
ds_{\text{IIB}}^2 = f_0^{-1/2} \left[ - dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + f_0^{1/2} \left[ \frac{(r^2 + l^2 \cos^2 \theta) dr^2}{r^2 + l^2} \right. \\
+ \left. (r^2 + l^2 \cos^2 \theta) d\theta^2 + (r^2 + l^2) \sin^2 \theta d\varphi^2 + r^2 \cos^2 \theta d\Omega_3^2 \right],
\]

\(f_0 = 1 + \frac{4\pi g s N \alpha'}{r^4 \Delta_0}\). (3.18)

We now rescale the variables

\(r = U \alpha', \quad l = a \alpha'\),

and take the limit \(\alpha' \to 0\), with \(U, a\) fixed. We obtain the following type IIB background:

\[
ds_{\text{IIB}}^2 = \alpha' \Delta_0^{1/2} \left[ \frac{U^2}{\sqrt{4\pi g s N}} \left[ - dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + \sqrt{4\pi g s N} \frac{dU^2}{(U^2 + a^2)} \right. \\
+ \left. \sqrt{4\pi g s N} \left[ d\theta^2 + \Delta_0^{-1} (1 + \frac{a^2}{U^2}) \sin^2 \theta d\varphi^2 + \Delta_0^{-1} \cos^2 \theta d\Omega_3^2 \right] \right],
\]

\(\Delta_0 = 1 + \frac{a^2 \cos^2 \theta}{U^2}\). (3.20)

The background asymptotically approaches \(AdS_5 \times S^5\). The scalar curvature vanishes identically, but there is a naked singularity at \(U = 0\), as can be seen from the invariant

\[
R_{\mu\nu} R^{\mu\nu} = \frac{160}{\Delta_0^3} \left( 1 + \frac{a^2}{U^2} + \frac{a^4}{4U^4} \cos^2 \theta \right)^2.
\]

The metric (3.18) takes a simple form in spheroidal coordinates:

\[
y_1 = \sqrt{r^2 + l^2} \sin \theta \cos \varphi, \quad y_2 = \sqrt{r^2 + l^2} \sin \theta \sin \varphi, \\
y_3 = r \cos \theta \cos \psi_1, \quad y_4 = r \cos \theta \sin \psi_1 \cos \psi_2, \\
y_5 = r \cos \theta \sin \psi_1 \sin \psi_2 \cos \psi_3, \quad y_6 = r \cos \theta \sin \psi_1 \sin \psi_2 \sin \psi_3.
\]

One finds

\[
ds_{\text{IIB}}^2 = f_0^{-1/2} \left[ - dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + f_0^{1/2} \sum_{m=1}^{6} dy_m^2,
\]

(3.22)
where
\[ f_0 = 1 + \frac{4\pi g_s N \alpha'^2}{r^2 (r^2 + l^2 \cos^2 \theta)}, \]
\[ r^2 = \frac{1}{2}(\rho_1^2 - l^2) + \frac{1}{2}\sqrt{(\rho_1^2 - l^2)^2 + 4l^2 \rho_2^2}, \quad r^2 + l^2 \cos^2 \theta = \sqrt{(\rho_1^2 - l^2)^2 + 4l^2 \rho_2^2}, \]
\[ \rho_1^2 \equiv y_1^2 + \ldots + y_6^2, \quad \rho_2^2 \equiv y_3^2 + \ldots + y_6^2. \]

It is easy to check that \( f_0 = f_0(y_m) \) is a harmonic function in the space \( y_m \). It would be interesting to find an \( \mathcal{N} = 2 \) four-dimensional field theory associated with the supersymmetric compactification (3.22).

4. Model of QCD

The backgrounds (3.10), (3.11), generalize the Witten model (2.11) by two extra parameters \( a_1, a_2 \). Based on the no-hair theorem, in the previous section we have argued that there are no other smooth manifolds which can describe a confining gauge theory in four dimensions. It is therefore of interest to explore whether there is a sense in which these compactifications can be related to non-supersymmetric gauge theories.

Let us first consider the case (3.10), in which \( a_2 = 0 \). It is convenient to work with coordinates \( \theta_1, \theta_2 \) which are \( 2\pi \)-periodic, defined as in (2.9), with (see (3.4), (2.3))
\[ R_0 = (2\pi T_H)^{-1} = \frac{A}{3u_0}, \quad A \equiv \frac{u_0^4}{u_H^4 - \frac{2}{3} a_4}, \]
where we have introduced the coordinate \( u \) by \( U = 2(\pi N)^{1/2} u \), and redefined \( a \to 2(\pi N)^{1/2} a \). By dimensional reduction in the \( x_5 \) direction, we obtain
\[ ds^2_{IIA} = \frac{8\pi \lambda A u^3}{3u_0} \Delta^{1/2} \left[ -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + \frac{8\pi \lambda A^3 u^3}{27 u_0^3} \Delta^{1/2} (1 - \frac{u_0^6}{u^6}) d\theta_2^2 \]
\[ + \frac{8\pi \lambda A}{3u_0} d\theta_1^2 \Delta^{1/2} \left( d\theta^2 + \frac{\tilde{\Delta}}{\Delta} \sin^2 \theta d\varphi^2 + \frac{1}{\Delta} \cos^2 \theta d\Omega_2^2 - \frac{4a^2 A u_0^2}{3u^4 \Delta} \sin^2 \theta d\theta_2 d\varphi \right), \]
\[ e^{2\phi} = \frac{8\pi A^3 \lambda^3 u^3 \Delta^{1/2}}{27 u_0^3} \frac{1}{N^2}, \quad \Delta = 1 - \frac{a^4}{u^4} \cos^2 \theta, \quad \tilde{\Delta} = 1 - \frac{a^4}{u^4}. \]
With this normalization, the metric reduces to eq. (4.8) of ref. [5] after setting $a = 0$ (cf. eqs. (2.12), (2.13)). The string coupling $e^\phi$ is of order $1/N$, and the metric has become independent of $N$, which is consistent with the expectation that in the large $N$ limit the string spectrum should be independent of $N$.

The location of the horizon is at

$$u_H^6 - a^4 u_H^2 - u_0^6 = 0,$$

i.e.

$$u_H^2 = \frac{a^4}{\gamma u_0^2} + \frac{1}{3} \gamma u_0^2, \quad \gamma = 3 \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{27} \left( \frac{a}{u_0} \right)^{12}} \right)^{1/3}. \quad (4.5)$$

For all $a^4, u_0^6 > 0$, $u_H^2$ is a positive real number, $u_H > a > 0$.

The spectrum of the Laplacian corresponding to the space of this model, eq. (3.10), has a mass gap. This is characteristic of smooth geometries with a cutoff at some $u = u_0$ that approach $AdS$ at infinity: there are no oscillatory solutions to the Laplace equation at infinity, so the spectrum of normalizable eigenfunctions that are regular at $u = u_0$ must therefore be discrete [5]. The isometry group of the internal space is $SO(3) \times U(1) \times U(1)$. The supergravity fields on this space can be classified in terms of representations of $SO(3)$, given by the standard spherical harmonics. Glueballs of QCD are presumably related to $SO(3)$ singlets, with vanishing $U(1) \times U(1)$ charges.

Let us consider the following Laplace operators corresponding to the background (4.2), (4.3):

$$\nabla^2 \equiv \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu \nu} \partial_\nu, \quad \nabla^2 \equiv \frac{e^{2\phi}}{\sqrt{g}} \partial_\mu \sqrt{g} e^{-2\phi} g^{\mu \nu} \partial_\nu,$$

$$\sqrt{g} = C \ u^9 \Delta \cos^2 \theta \sin \theta \sin \psi_1, \quad C = \frac{1}{2\pi \lambda} \left( \frac{4\pi \lambda A}{3 u_0} \right)^6.$$

It is easy to see that there are no solutions to the equation $\nabla^2 \Phi = 0$ of the form $\Phi = \chi(u) e^{ik \cdot x}$, one needs $\Phi = \chi(u, \theta) e^{ik \cdot x}$. Interestingly, there are however $\theta$-independent solutions $\Phi = \chi(u) e^{ik \cdot x}$ to the equation $\nabla^2 \Phi = 0$. One obtains

$$\frac{1}{u^3} \partial_u \left( u^6 - a^4 u^2 - u_0^6 \right) \partial_u \chi(u) = -M^2 \chi(u), \quad M^2 = -k^2. \quad (4.6)$$

This equation can be treated, for instance, by using the WKB method. Glueballs are associated with the massless scalar that couples to the operator $F_{\mu \nu} F^{\mu \nu}$. This obeys the
Laplace equation in the Einstein frame (cf. ref. [17]), which in the string frame corresponds to $\hat{\nabla}^2 \Phi = 0$. Therefore glueball masses should be determined by eq. (4.6).

An estimate of the glueball masses can be given from the Yang-Mills string tension. The string tension can be determined by computing the Wilson loop as in [22]. We look for string configurations that minimize the Nambu-Goto action. The new feature is that now the metric components depend on $\cos^2 \theta$. Let us first consider configurations with constant $\theta$. This is a solution provided

$$\frac{\delta \cos^2 \theta}{\delta \theta} = 0,$$

i.e. $\theta = 0, \frac{\pi}{2}, \pi$. The string tension is then given by $\frac{1}{2\pi}$ times the coefficient of $\sum dx_i^2$, evaluated at the horizon $u = u_H$, that is (cf. eq. (2.14))

$$T_{YM}(\theta = 0, \pi) = \frac{4}{3} \lambda A u_H^3 \frac{u_0^3}{u_0} \sqrt{1 - \frac{a^4}{u_H^4}}.$$  

$$T_{YM}(\theta = \frac{\pi}{2}) = \frac{4}{3} \lambda A u_H^3 \frac{u_0^3}{u_0}.$$  

(4.7)

(4.8)

The string configuration with minimal area is thus at $\theta = 0, \pi$, with the tension given by (4.7). For configurations with non-constant $\theta = \theta(\sigma)$ one needs to find the tension by explicitly computing the Nambu-Goto action on the solution $U(\sigma), \theta(\sigma)$. The system of equations is highly non-linear, but we expect that the absolute minimum is at $\theta = 0, \pi$, for which $\Delta^{1/2}$ (and hence $T_{YM} \propto \Delta^{1/2}$) takes the minimum value that it can have. Using (4.4), the tension (4.7) becomes

$$T_{YM} = \frac{4}{3} \lambda A u_0^2 = \frac{4}{3} \lambda \frac{u_0^6}{u_H^4 - \frac{1}{3} a^4}.$$  

(4.9)

Thus masses of glueballs associated with excitations of the string with tension $T_{YM}$ should be of the order $(T_{YM})^{1/2}$. The masses of the supergravity glueballs – associated with the zero modes of the string – are determined from the Laplace equation. Since $\lambda$ appears only as an overall factor in the metric (4.2), these masses are independent of $\lambda$ (they may however receive corrections suppressed by factors $1/\lambda$), and they are of the form $M_{\text{glueballs}} = u_0 f(a/u_0)$. The mass scale should be of order

$$M_{\text{glueballs}} \approx \Lambda_{\text{QCD}} \approx \frac{u_0^3}{\sqrt{u_H^4 - \frac{1}{3} a^4}}.$$  

(4.10)
On the other hand, Kaluza-Klein states associated with the circle parametrized by \( \theta_2 \) (of the form \( \Phi = \chi(u, \theta) e^{ik \cdot x} e^{i n \theta_2} \)) will have masses

\[
M_{\text{KK}} \simeq R_0^{-1} = \frac{3}{u_0^3} \left( u_H^4 - \frac{1}{3} a^4 \right) .
\]

This means

\[
\frac{M_{\text{glueballs}}}{M_{\text{KK}}} \simeq A^{3/2} = \frac{u_0^6}{(u_H^4 - \frac{1}{3} a^4)^{3/2}} .
\]

This is a function of \( a/u_0 \) (see (4.13)), and the behavior at small and large \( a/u_0 \) is as follows:

\[
M_{\text{glueballs}} \simeq M_{\text{KK}} + O(a^4) , \quad u_0 \gg a ,
\]

\[
\frac{M_{\text{glueballs}}}{M_{\text{KK}}} \simeq \frac{u_0^6}{a^6} \ll 1 , \quad u_0 \ll a .
\]

For the glueballs related to string excitations, the ratio of masses will be

\[
\frac{(T_{YM})^{1/2}}{M_{\text{KK}}} \simeq \frac{\lambda u_0^6}{a^6} , \quad u_0 \ll a .
\]

In order for the Yang-Mills tension \( T_{YM} \simeq \lambda u_0^6/a^4 \) to remain finite at large \( a \) (and fixed \( u_0 \)) one has to take \( \lambda \sim a^4/u_0^4 \). Thus for \( a \gg u_0 \) the low-energy theory will be effectively 3 + 1 dimensional with a finite value for the string tension, and hence a finite value for glueball masses. In this sense \( a \) (which has dimension of mass) acts as a momentum cutoff. For \( u_0 \gg a \), one recovers the relation (4.13) of the Witten model.

Having a regime \( u_0 \ll a \) where \( M_{\text{glueballs}} \ll M_{\text{KK}} \), the next question is where the supergravity approximation applies. For this it is necessary that curvature invariants are small in the whole space. The maximum curvature is attained at \( u = u_H \) and at the poles, \( \theta = 0, \pi \). For \( u_0 \ll a \) one has \( \alpha' R \sim u_0/(u \lambda A \Delta^{1/6}) \). Using eqs. (4.1), (4.3) and (4.4) we find

\[
\alpha' R < O \left( \frac{a^4}{\lambda u_0^3} \right) , \quad u_0 \ll a .
\]

Therefore the regime where we can use supergravity and at the same time decouple the Kaluza-Klein particles with \( U(1) \) charge is large \( \lambda \) and large \( a/u_0 \), with \( \lambda \gg a^4/u_0^4 \). [In general, for any given \( a \) and \( u_0 \) one can pick a sufficiently large \( \lambda \) such that all curvature invariants are small]. This is precisely the region where the string excitations of mass \( (T_{YM})^{1/2} \) become heavy. The supergravity approximation only incorporates particles which have spin \( \leq 2 \). In order to incorporate the full Regge trajectories of spin \( J \)-glueballs
with squared masses of order $T_{YM}|J|$, one needs to use string theory. Although the string spectrum in this background is a function of $\lambda$, $u_0$ and $a$, which can be quite complicated, it is nonetheless possible to anticipate a hierarchy of scales as follows. Equation (4.9) indicates that the masses of excitations of the string increase with $\lambda$: for $\lambda \gg 1$ there will be a gap between the lowest glueball states of supergravity and the glueballs represented by string excitations. The analysis in eqs. (4.13)-(4.15) shows that at $a/u_0 \gg 1$ there will be another gap separating the scale of the string excitations from the scale of Kaluza-Klein particles with $U(1)$ charge.

The relation between confinement and monopole condensation was recently discussed in this context by Gross and Ooguri [13]. The magnetic monopole is represented by a D2 brane wrapped around the $\theta_2$ circle that ends on a D4 brane. The potential between a monopole $m$ and an anti-monopole $\bar{m}$ can then be computed by considering a D2 brane bounded by $S^1$ times the trajectories of $m$ and $\bar{m}$. Let us consider a configuration with $\theta = 0$ (or $\theta = \pi$), with the monopoles travelling along $x_1$ and separated in the $x_2$ direction by a distance $L$. Using eq. (4.2) we find for the D2 brane action

$$E_{m\bar{m}} = \int_0^L d\theta_2 dx_2 \ e^{-\phi} \sqrt{g_{\theta_2 \theta_2} g_{11} g_{22}}$$

$$= \frac{1}{2g_s T_H} \int_0^L dx_2 \left\{ \frac{d\bar{u}}{dx_2} \right\}^2 + \frac{1}{g_s N} \left( \bar{u}^3 - \bar{a}^2 \bar{u} - \bar{u}_0^3 \right), \quad (4.16)$$

where $\bar{u} = 4g_s Nu^2$, $\bar{a} = 4g_s Na^2$. When $\bar{a} = 0$, this reduces to eq. (12) of [13]. By minimizing the action (4.16), one can see that the phenomenon of complete screening of the magnetic monopole observed in [13] persists for any value of $\bar{a}, \bar{u}_0 > 0$. This happens because when $L$ goes beyond some critical distance $L_{\text{crit}}$ there is no connected $D2$ brane configuration minimizing the action. For a connected minimal surface, the distance $L$ between $m$ and $\bar{m}$ can be expressed in terms of the minimum value of $\bar{u} = \bar{u}_m$ reached by the D2 brane as follows

$$L = \frac{2\sqrt{g_s N}}{\bar{u}_m^2} \sqrt{\bar{u}_m^3 - \bar{a}^2 \bar{u}_m - \bar{u}_0^3} \int_1^\infty \frac{dy}{\sqrt{(y^3 - \bar{a}^2 \bar{y} - \bar{u}_0^3) (y^3 - 1 - \bar{a}^2 (y - 1))}}.$$  

1 We thank J. Maldacena for comments on this point.
This integral can be evaluated numerically as a function of \( \bar{u}_m, \bar{u}_H < \bar{u}_m < \infty \), showing that there is indeed a maximum distance \( L_{\text{crit}} \) for any real \( \bar{a}, \bar{u}_0 > 0 \). As a result, the potential between monopoles becomes constant for \( L > L_{\text{crit}} \).

Finally, let us consider the case \((3.11)\), where there are two non-vanishing components of the “angular momentum” parameters \( a_1, a_2 \). The dilaton is now given by \((4.3)\) with

\[
A^{-1} = \frac{1}{u_0^4} \left[ u_H^4 - \frac{1}{3}(a_1^4 + a_2^4) - \frac{a_1^4 a_2^4}{3u_H^4} \right],
\]

\[
\Delta = 1 - \frac{a_1^4}{u_0^4} \cos^2 \theta - \frac{a_2^4}{u_0^4} \left( \cos^2 \theta \cos^2 \psi_1 + \sin^2 \theta \right) + \frac{a_1^4 a_2^4}{u_0^8} \cos^2 \theta \cos^2 \psi_1 , \quad (4.18)
\]

and \( u_H \) is the largest solution of the equation

\[
(1 - \frac{a_1^4}{u_H^4})(1 - \frac{a_2^4}{u_H^4}) = \frac{u_0^6}{u_H^6}.
\]

Note that for any \( u_0^6 > 0 \) one has \( u_H > a_1, a_2 \).

To compute the Wilson loop we consider a configuration with constant \( \theta, \psi_1 \), which minimizes the Nambu-Goto action provided \( \delta \cos^2 \theta \delta \theta = 0 \), \( \delta \cos^2 \psi_1 \delta \psi_1 = 0 \), i.e. \( \theta, \psi_1 = 0, \frac{1}{2} \pi, \pi \). The string tension is thus given by

\[
T_{\text{YM}} = 4 \lambda R_0 u_0^3 \Delta_H^{1/2}, \quad R_0 = \frac{A}{3u_0}
\]

The tension takes the minimum value when \( \theta \) and \( \psi_1 \) are equal to 0 or \( \pi \), so that

\[
\Delta_H = (1 - \frac{a_1^4}{u_H^4})(1 - \frac{a_2^4}{u_H^4}) = \frac{u_0^6}{u_H^6}, \quad (4.20)
\]

which leads to

\[
T_{\text{YM}} = 4 \lambda R_0 u_0^3.
\]

Thus

\[
\frac{M_{\text{glueballs}}}{M_{\text{KK}}} \approx (R_0 u_0)^{3/2} \approx A^{3/2}.
\]

As in the previous \( a_2 = 0 \) case, a \( 3 + 1 \) dimensional theory with a finite scale \( \Lambda_{QCD} \) seems to arise in some regime with large \( a_1 \) or large \( a_2 \).

The mass scale corresponding to non-singlet representations of \( SO(3) \) cannot be estimated in a simple way. To establish what are the precise mass scales of the different particles of the spectrum a thorough analysis is obviously needed. This may require a numerical treatment of the Laplace equation as the one carried out in \([14,16]\).

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