Correlation Functions in the Two-Dimensional Ising Model in a Magnetic Field at $T = T_c$

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Abstract

The one and two-particle form factors of the energy operator in the two-dimensional Ising model in a magnetic field at $T = T_c$ are exactly computed within the form factor bootstrap approach. Together with the matrix elements of the magnetisation operator already computed in ref. [1], they are used to write down the large distance expansion for the correlators of the two relevant fields of the model.
The last years have seen important progresses in the non-perturbative study of two-dimensional quantum field theories and related statistical mechanical models. If conformal symmetry provided us with an exact description of critical points and universality classes \[1, 2\], the study of off-critical models turned out to be better approached within the framework of relativistic scattering theory. In fact, if the off-critical model under consideration is integrable (i.e. admits infinite conservation laws), it can be usually solved exploiting very general bootstrap techniques \[3, 4, 5, 6\]. This circumstance appears to be particularly important in light of the fact that a large number of physically interesting two-dimensional systems can actually be described in terms of integrable models. A remarkable example is provided by the scaling limit of the two-dimensional Ising model in a magnetic field at \( T = T_c \) (IMMF in the sequel) \[4\]. It can be formally described by the action

\[
A = \mathcal{A}_{\text{CF}} + h \int d^2x \, \sigma(x),
\]

where \( \mathcal{A}_{\text{CF}} \) denotes the action of the conformal minimal model \( M_{3,4} \) and \( \sigma(x) \) the magnetisation operator of scaling dimension \( 2\Delta_\sigma = 1/8 \). The coupling constant \( h \) (magnetic field) has physical dimension \( h \sim m^{15/8} \), \( m \) being a mass scale. Apart form the magnetisation operator, the only other relevant scaling field in the Ising model is the energy density \( \varepsilon(x) \) with scaling dimension \( 2\Delta_\varepsilon = 1 \).

Zamolodchikov showed that the theory described by the action \(1\) possesses an infinite number of integrals of motion which can be used in order to determine the exact particle spectrum and \( S \)-matrix of the theory \(4\). He found that the spectrum consists of eight massive particles \( A_a \) \( (a = 1, 2, \ldots, 8) \) whose masses stay in the following ratios with the mass \( m_1 \) of the lightest particle

\[
m_2 = 2m_1 \cos \frac{\pi}{5} = (1.6180339887..) \, m_1, \\
m_3 = 2m_1 \cos \frac{\pi}{30} = (1.9890437907..) \, m_1, \\
m_4 = 2m_2 \cos \frac{7\pi}{30} = (2.4048671724..) \, m_1, \\
m_5 = 2m_2 \cos \frac{2\pi}{15} = (2.9562952015..) \, m_1, \\
m_6 = 2m_2 \cos \frac{\pi}{30} = (3.2183404585..) \, m_1, \\
m_7 = 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233..) \, m_1, \\
m_8 = 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168..) \, m_1.
\]

The interaction between these particles is described by a factorised, reflectionless \( S \)-matrix
characterised by the two–particle amplitudes
\[ S_{ab}(\theta) = \prod_{\alpha \in A_{ab}} \left[ \frac{\tanh \frac{1}{2} (\theta + i\pi \alpha)}{\tanh \frac{1}{2} (\theta - i\pi \alpha)} \right]^{\mu_\alpha}. \]  
(3)

The set of numbers \( A_{ab} \) and the multiplicity factors \( \mu_\alpha \) are given in Table 1. We use the standard rapidity parameterisation of the on–shell momenta \( p_\alpha^\mu = (m_\alpha \cosh \theta_\alpha, m_\alpha \sinh \theta_\alpha) \), so that \( \theta \equiv \theta_a - \theta_b \) in (3).

In ref. [7] the knowledge of the \( S \)–matrix (3) was exploited in order to approach the computation of the correlation functions of the model (1) within the form factor bootstrap method. The basic idea of this approach is to express the (euclidean) correlation functions (e.g. the two–point ones) as a spectral sum over a complete set of intermediate multiparticle states
\[ G_{\Phi_1 \Phi_2}(x) \equiv \langle \Phi_1(x)\Phi_2(0) \rangle \\
= \sum_{n=0}^{\infty} \int_{\theta_1 > \theta_2 > \ldots > \theta_n} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} e^{-|x|} \sum_{k=1}^{n} m_k \cosh \theta_k \]
\[ \times \langle 0|\Phi(0)|A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)\rangle \langle A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)|\Phi_2(0)|0 \rangle, \]  
(4)
and to exploit the fact that the form factors (FF)
\[ F_{a_1 \ldots a_n}^\Phi(\theta_1, \ldots, \theta_n) = \langle 0|\Phi(0)|A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)\rangle. \]  
(5)
are exactly computable in integrable models once the \( S \)–matrix is known [8, 6].

The spectral series (4) is manifestly a large distance expansion. Nevertheless, it has been observed in several models [9–13] that it is characterised by a fast rate of convergence also at intermediate and short distance scales (see in particular ref. [11] for a theoretical justification of this property), so that a truncation of the series including few lowest states turns out to be sufficient for most practical purposes. In ref. [7] the one and two–particle FF of the magnetisation operator in the IMMF were exactly computed and shown to be sufficient to reproduce with remarkable accuracy the numerical data for \( G_{\sigma\sigma}(x) \) available from Monte Carlo simulations [14]. It is the purpose of this letter to carry out a similar program for the FF of the energy operator \( \varepsilon(x) \) and to write down the large distance expansion for the correlators \( G_{\varepsilon\varepsilon}(x) \) and \( G_{\sigma\varepsilon}(x) \).

Let’s briefly recall the basic strategy for the computation of FF in the IMMF referring the reader to ref. [7] for details. Form factors can be generally computed in an integrable model using a recursive procedure based on a set of residue equations relating matrix elements with different particle content. For instance, if the scattering amplitude \( S_{ab}(\theta) \) has a simple pole with positive residue \( (\Gamma_{ab}^c)^2 \) at \( \theta = iv_{ab}^c \) corresponding to the particle \( A_c \) with mass \( m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos v_{ab}^c \) appearing as a bound state in the direct \( ab \)
channel, then we can write

\[ F_{ab}^\Phi(\theta \simeq iu_{ab}^c) \simeq \frac{i\Gamma_{ab}^c}{\theta - iu_{ab}^c} F_{c}^\Phi, \]

as well as similar relations among higher matrix elements containing spectator particles. Other recursive equations are associated to the higher poles in the \( S \)–matrix and to the “kinematical” poles in the matrix elements. Together with the equations ruling the monodromy properties of FF, these residue equations provide a system of linear relations whose general solution amounts to a complete classification of the operator content of the theory \[15\].

In theories with diagonal scattering, the basic information about the structure of FF is already encoded in the two–particle matrix elements. In the IMMF they can be generally parameterised as \[7\]

\[ F_{ab}^\Phi(\theta) = \frac{Q_{ab}^\Phi(\theta)}{D_{ab}(\theta)} F_{min}^{ab}(\theta), \] (7)

where

\[ F_{min}^{ab}(\theta) = \left(-i \sinh \frac{\theta}{2}\right) \delta_{ab} \prod_{\alpha \in A_{ab}} (G_{\alpha}(\theta))^{\mu_{\alpha}}, \] (8)

\[ G_{\alpha}(\theta) = \exp \left\{ \int_{0}^{\infty} \frac{dt}{t} \cosh \left( \alpha - \frac{1}{2} \right) t \sinh t \sin^2 \left( i \pi - \theta \right) t \right\}, \] (9)

and

\[ D_{ab}(\theta) = \prod_{\alpha \in A_{ab}} (P_{\alpha}(\theta))^{i_{\alpha}} (P_{1-\alpha}(\theta))^{j_{\alpha}}, \] (10)

\[ i_{\alpha} = n + 1, \quad j_{\alpha} = n, \quad \text{if} \quad \mu_{\alpha} = 2n + 1; \]

\[ i_{\alpha} = n, \quad j_{\alpha} = n, \quad \text{if} \quad \mu_{\alpha} = 2n, \] (11)

\[ P_{\alpha}(\theta) \equiv \frac{\cos \pi \alpha - \cosh \theta}{2 \cos^2 \frac{\pi \alpha}{2}}. \] (12)

The terms \( F_{min}^{ab}(\theta) \) and \( D_{ab}(\theta) \) in eq. (7) take into account the monodromy properties and the singularity structure of the matrix elements, respectively, and are both uniquely determined by the knowledge of the \( S \)–matrix. The whole information about the operator \( \Phi(x) \) is then contained in the polynomial

\[ Q_{ab}^\Phi(\theta) = \sum_{k=0}^{N_{ab}} c_{ab}^k \cosh^k \theta, \] (13)

whose (operator dependent) coefficients \( c_{ab}^k \) are the only remaining unknowns.

\[ ^1 \text{In this letter we consider only scalar operators.} \]
The identification of specific operators out of the general solution of the recursive
equations is a nontrivial task, especially in theories lacking any internal symmetry, as is
the case for the IMMF. A progress with respect to this problem was made in ref. [7] where
it was shown that in a unitary theory the FF of an operator $\Phi(x)$ satisfy the asymptotic
bound

$$
\lim_{|\theta_i| \to \infty} F_{a_1,\ldots,a_n}^\Phi(\theta_1,\ldots,\theta_n) \leq \text{const.} e^{\Delta \Phi |\theta_i|},
$$

where $2\Delta \Phi$ being the scaling dimension. The consequences of this result are easily illustrated
using the parameterisation (7). Consider the simplest two-particle FF in the IMMF,

$$
F_{11}^\Phi(\theta) \sim \exp(|\theta|/2) \quad \text{as} \quad |\theta| \to \infty,
$$

the constraint (14) implies that, for any relevant scalar field $\varphi(x)$ ($\Delta \varphi < 1$), the total degree $N_{11}$ of the polynomial $Q_{11}^\varphi(\theta)$ must be less than 2. Since it can be checked that no solution of the residue equations exists if $N_{11} = 0$, one concludes

$$
Q_{11}^\varphi(\theta) = c_{11}^1 \cosh \theta + c_{11}^0, \quad c_{11}^1 \neq 0.
$$

It turns out that, once the FF $F_{11}^\varphi(\theta)$ corresponding to a specific relevant operator $\varphi(x)$ has been fixed assigning the two coefficients $c_{11}^1$ and $c_{11}^0$, all the FF of $\varphi(x)$ can be uniquely determined using the residue equations. This amounts to say that the solutions of the FF bootstrap for the relevant scalar operators of the IMMF form a two–dimensional linear space, which is what expected from the fact that such operators can only correspond to linear combinations of $\sigma(x)$ and $\varepsilon(x)$. Solutions corresponding to operators not just differing for an inessential normalisation constant can be labelled by the ratio

$$
z_\varphi \equiv \frac{c_{11}^0}{c_{11}^1}.
$$

It is of particular physical interest to determine the values of this ratio which select the two scaling fields $\sigma(x)$ and $\varepsilon(x)$. The problem for $\sigma(x)$ was solved in ref. [7] using the relation

$$
\Theta(x) = 2\pi h (2 - 2\Delta \sigma) \sigma(x),
$$

and exploiting the constraints imposed on the matrix elements of $\Theta(x)$ by energy–momentum
conservation. On the other hand, no similar method can be used for $\varepsilon(x)$ and an alternative
way must be found in order to characterise the FF of this operator.

The fields $\sigma(x)$ and $\varepsilon(x)$ are uniquely identified by the short distance behaviour of their correlation functions predicted by the conformal operator product expansion [1]

$$
G_{\sigma\sigma}(x) \sim |x|^{-4\Delta_\sigma} = |x|^{-1/4}, \quad |x| \to 0
$$

$$
G_{\sigma\varepsilon}(x) \sim |x|^{-2\Delta_\varepsilon} = |x|^{-1}, \quad |x| \to 0
$$

$$
G_{\varepsilon\varepsilon}(x) \sim |x|^{-4\Delta_\varepsilon} = |x|^{-2}, \quad |x| \to 0.
$$

4
Since in the FF approach the determination of the exact ultraviolet behaviour of the correlators requires in principle the resummation of the spectral series (4), it seems quite difficult to make a direct use of eqs. (18) for the determination of $z_\sigma$ and $z_\varepsilon$. Nevertheless, eqs. (18) suggest that, if a property characterising the FF of the scaling fields $\sigma(x)$ and $\varepsilon(x)$ exists, it is probably related to the high energy asymptotics of the matrix elements. Moreover, if such property should be able to select the scaling fields among their linear combinations, it must be non-linear in the operator. Interestingly enough, a property with these features is known in the FF literature. It was noticed in the context of the sine–Gordon model that the FF of the operators exponential of the elementary field satisfy the \textit{cluster property} [6, 16, 17]

$$\lim_{\alpha \to \infty} \tilde{F}_\Phi^{a_1...a_k a_{k+1}...a_n}(\theta_1 + \alpha, \ldots, \theta_k + \alpha, \theta_{k+1}, \ldots, \theta_n) = \tilde{F}_\Phi^{a_1...a_k}(\theta_1, \ldots, \theta_k) \tilde{F}_\Phi^{a_{k+1}...a_n}(\theta_{k+1}, \ldots, \theta_n),$$

(19)

where

$$\tilde{F}_\Phi^{a_1...a_n}(\theta_1, \ldots, \theta_n) \equiv \frac{1}{\langle \Phi \rangle} F_{a_1...a_n}^{\Phi}(\theta_1, \ldots, \theta_n).$$

(20)

It is known that massive deformations of minimal models can be obtained from sine–Gordon through a suitable restriction of the Hilbert space and that some exponential operators are mapped into the scaling fields of the restricted models. In refs. [16, 18], it was shown for some specific cases that the factorisation property (19) survives the reduction procedure and is satisfied by the FF of the scaling fields in the reduced models. Here we simply assume that the cluster property (19) characterises the FF of the operators $\sigma(x)$ and $\varepsilon(x)$ in the IMMF and provide what we think is strong evidence that this is indeed the case.

Before proceeding further, notice that the asymptotic relation (19) involves the vev $\langle \Phi \rangle$. In the general case, the computation of this quantity is a nontrivial problem (the thermodynamic Bethe ansatz only provides the vev of the field which perturbs the conformal point [19]). It is then remarkable that, if the FF of $\Phi(x)$ satisfy the cluster property, the vev can be obtained, for instance, as

$$\langle \Phi \rangle = \frac{F_{\Phi}^{\varphi} F_{b}^{\Phi}}{\lim_{\theta \to \infty} F_{ab}^{\Phi}(\theta)}.$$ 

(21)

Going back to the determination of $z_\sigma$ and $z_\varepsilon$, consider the two–particle FF $F_{12}^c(\theta)$. Following the same arguments used above for $F_{11}^{c}(\theta)$ one concludes that $Q_{12}^{c}(\theta)$ is a polynomial of degree 2 in $\cosh \theta$. Since the amplitudes $S_{11}(\theta)$ and $S_{12}(\theta)$ have common poles corresponding to the particles $A_1$, $A_2$ and $A_3$ (see Table 1), eq. (8) provides the linear system

$$\frac{1}{\Gamma_{11}} \text{Res}_{\theta = iu_{11}} F_{11}^{c}(\theta) = \frac{1}{\Gamma_{12}} \text{Res}_{\theta = iu_{12}} F_{12}^{c}(\theta), \quad c = 1, 2, 3.$$ 

(22)
Once an overall normalisation of the operator $\varphi(x)$ has been fixed, these equations uniquely determine the coefficients $c_{12}^0$, $c_{12}^1$ and $c_{12}^0$ in terms of $z_{\varphi}$. Finally, in order to search for solutions satisfying the cluster property, we use eq. (21) and require

$$\frac{F_1^{\varphi}}{\lim_{\theta \to \infty} F_1^{\varphi}(\theta)} = \frac{F_2^{\varphi}}{\lim_{\theta \to \infty} F_2^{\varphi}(\theta)},$$

with $F_1^{\varphi}$ and $F_2^{\varphi}$ also determined in function of $z_{\varphi}$ using eq. (24).

There exist only two values of the parameter $z_{\varphi}$ satisfying the last equation. One of them exactly coincides with the value of $z_{\sigma}$ which had been determined in ref. [7] without any reference to the cluster property.

$$z_{\sigma} = \frac{2m_1^2 + m_3 m_7}{2m_1^2} = 4.869840...$$

As we said above, once this initial condition has been fixed, all the matrix elements of $\sigma(x)$ can in principle be determined using the residue equations only, without further use of the cluster property. All the one-particle and several two-particle FF of $\sigma(x)$ are given in tables 2 and 4, respectively. It must be stressed that all the two-particle FF of $\sigma(x)$ computed in this way automatically satisfy the cluster property which then should be regarded as characteristic of the whole solution selected by the initial condition (24). The same pattern is observed for the solution arising from the other value of $z_{\varphi}$ satisfying eq. (23), which we identify with $z_{\varepsilon}$

$$z_{\varepsilon} = 1.255585...$$

The one and two-particle FF corresponding to this initial condition are contained in tables 3 and 5; they were used in ref. [20] in order to compute the corrections the energy spectrum of the theory (1) undergoes under a small thermal perturbation induced by the energy operator $\varepsilon(x)$. The remarkable agreement observed in ref. [20] between the theoretical predictions and the data coming from a numerical diagonalisation of the Hamiltonian strongly supports the conclusion that the cluster property correctly selects the matrix elements of both relevant scaling operators in the IMMF.

The results contained in tables 2-5 can be used in the spectral representation (4) in order to write down the large distance expansion of the correlators $G_{\sigma\sigma}(x)$, $G_{\sigma\varepsilon}(x)$ and

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The leading infrared contributions are simply

\[ G_{\Phi_1\Phi_2}(x) = \langle \Phi_1 \rangle \langle \Phi_2 \rangle + \frac{1}{\pi} \sum_{a=1}^{3} F_{a}^{\Phi_1} F_{a}^{\Phi_2} K_0(m_a|x|) + O\left(e^{-2m_1|x|}\right) \]  

(26)

where \( K_0(x) \) is the modified Bessel function. In ref. [7], the fast convergence of the FF series for \( G_{\sigma\sigma}(x) \) was tested against the Monte Carlo data available for that correlator; as far as we know, no similar data exist for \( G_{\sigma\varepsilon}(x) \) and \( G_{\varepsilon\varepsilon}(x) \) but it seems natural to expect a similar convergence pattern. An integral check is provided by the following sum rule for the scaling dimension [21]

\[ \Delta^\Phi = -\frac{1}{4\pi \langle \Phi \rangle} \int d^2x \langle \Theta(x)\Phi(0)\rangle_c . \]  

(27)

The results obtained for \( \Delta^\sigma \) and \( \Delta^\varepsilon \) using in the last formula the spectral representation of the correlators \( G_{\sigma\sigma}(x) \) and \( G_{\sigma\varepsilon}(x) \) are contained in tables 6 and 7, respectively (\( \Delta_{a_1...a_n}^\Phi \) denotes the contribution coming from the intermediate state containing the particles \( A_{a_1} ... A_{a_n} \)). We recall that the exact results are \( \Delta^\sigma = 0.0625 \) and \( \Delta^\varepsilon = 0.5 \). The reason for the slower convergence of the series for \( \Delta^\varepsilon \) appears quite clear. Indeed, according to (18), \( G_{\sigma\varepsilon}(x) \) is much more singular than \( G_{\sigma\sigma}(x) \) as \( x \to 0 \). As a consequence, the integral in eq. (27) receives a larger contribution from short distances for \( \Phi = \varepsilon \) than for \( \Phi = \sigma \). On the other hand more and more terms in the spectral series are needed in order to approximate precisely the correlators at small \( x \).

In conclusion, it is clear that it would be highly desirable to reach a satisfactory physical understanding of one of the basic ingredients we used in this letter, namely the cluster property (19). It is very tempting to argue that this property (or better, a suitable generalisation applying also to theories with internal symmetries) characterises the matrix elements of the scaling fields in two-dimensional quantum field theories. The Smirnov’s observation that, if the FF of an operator \( \Phi(x) \) factorise asymptotically as in (19), then it is particularly simple to show that the correlator \( G_{\Phi\Phi}(x) \) behaves as a power law at short distances [16], seems to go in this direction. We hope that the results of this letter will stimulate further investigations on this point.

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Table Caption

Table 1. Two-particle scattering amplitudes of the IMMF. Each factor $(\gamma)^\mu$ stays for \[ \left[ \tanh \frac{\theta}{2} \left( \theta + i \frac{\pi}{30} \right) / \tanh \frac{\theta}{2} \left( \theta - i \frac{\pi}{30} \right) \right]^\mu. \] The indeces $i$ placed above the functions $(\gamma)$ correspond to the particles $A_i$ appearing as bound states in the $ab$ channel.

Table 2. One-particle Form Factors of the operator $\sigma(x)$. The results are given in units of $m_1^{1/8}$ and refer to the normalisation of the operator in which $\langle \sigma \rangle = m_1^{1/8}$.

Table 3. One-particle Form Factors of the operator $\varepsilon(x)$. The results are given in units of $m_1$ and refer to the normalisation of the operator in which $\langle \varepsilon \rangle = m_1$.

Table 4. Coefficients of the polynomials $Q_\sigma^{ab}(\theta)$. The results are given in units of $m_1^{1/8}$ and refer to the normalisation of the operator in which $\langle \sigma \rangle = m_1^{1/8}$.

Table 5. Coefficients of the polynomials $Q_\varepsilon^{ab}(\theta)$. The results are given in units of $m_1$ and refer to the normalisation of the operator in which $\langle \varepsilon \rangle = m_1$.

Table 6. The first eight contributions to the sum rule for $\Delta^\sigma$.

Table 7. The first eight contributions to the sum rule for $\Delta^\varepsilon$. 
| $a$ | $b$ | $S_{ab}$ |
|-----|-----|---------|
| 1   | 1   | (20) (12) (2) |
| 1   | 2   | (24) (18) (14) (8) |
| 1   | 3   | (29) (21) (13) (3) (11)² |
| 1   | 4   | (25) (21) (17) (11) (7) (15) |
| 1   | 5   | (28) (22) (14) (4) (10)² (12)² |
| 1   | 6   | (25) (19) (9) (7)² (13)² (15) |
| 1   | 7   | (27) (23) (5) (9)² (11)² (13)² (15) |
| 1   | 8   | (26) (16)³ (6)² (8)² (10)² (12)² |
| 2   | 2   | (24) (20) (14) (8) (2) (12)² |
| 2   | 3   | (25) (19) (9) (7)² (13)² (15) |
| 2   | 4   | (27) (23) (5) (9)² (11)² (13)² (15) |
| 2   | 5   | (26) (16)³ (6)² (8)² (10)² (12)² |
| 2   | 6   | (29) (25) (19)³ (13)³ (3) (7)² (9)² (15) |
| 2   | 7   | (27) (21)³ (17)³ (11)³ (5)² (7)² (15)² |
| 2   | 8   | (28) (22)³ (4)² (6)² (10)⁴ (12)⁴ (16)⁴ |
| 3   | 3   | (22) (20)³ (14) (12)³ (4) (2)² |
| 3   | 4   | (26) (16)³ (6)² (8)² (10)² (12)² |
| 3   | 5   | (29) (23) (21)³ (13)³ (5) (3)² (11)⁴ (15) |
| 3   | 6   | (26) (24)³ (18)³ (8)³ (10)² (16)⁴ |
| 3   | 7   | (28) (22)³ (4)² (6)² (10)⁴ (12)⁴ (16)⁴ |
| 3   | 8   | (27) (25)³ (17)⁵ (7)⁴ (9)⁴ (11)² (15)³ |

Continued
| $a$ | $b$ | $S_{ab}$                  |
|-----|-----|---------------------------|
| 4   | 4   | (26) (20)$^3$ (16)$^3$ (12)$^3$ (2) (6)$^2$ (8)$^2$ |
| 4   | 5   | (27) (23)$^3$ (19)$^3$ (9)$^3$ (5)$^2$ (13)$^4$ (15)$^2$ |
| 4   | 6   | (28) (22)$^3$ (4)$^2$ (6)$^2$ (10)$^4$ (12)$^4$ (16)$^4$ |
| 4   | 7   | (28) (24)$^3$ (18)$^5$ (14)$^5$ (4)$^2$ (8)$^4$ (10)$^4$ |
| 4   | 8   | (29) (25)$^3$ (21)$^5$ (3)$^2$ (7)$^4$ (11)$^6$ (13)$^6$ (15)$^3$ |
| 5   | 5   | (22)$^3$ (20)$^5$ (12)$^5$ (2)$^2$ (4)$^2$ (6)$^2$ (16)$^4$ |
| 5   | 6   | (27) (25)$^3$ (17)$^5$ (7)$^4$ (9)$^4$ (11)$^4$ (15)$^3$ |
| 5   | 7   | (29) (25)$^3$ (21)$^5$ (3)$^2$ (7)$^4$ (11)$^6$ (13)$^6$ (15)$^3$ |
| 5   | 8   | (28) (26)$^3$ (24)$^5$ (18)$^7$ (8)$^6$ (10)$^6$ (16)$^8$ |
| 6   | 6   | (24)$^3$ (20)$^5$ (14)$^5$ (2)$^2$ (4)$^2$ (8)$^4$ (12)$^6$ |
| 6   | 7   | (28) (26)$^3$ (22)$^5$ (16)$^7$ (6)$^4$ (10)$^6$ (12)$^6$ |
| 6   | 8   | (29) (27)$^3$ (23)$^5$ (21)$^7$ (5)$^4$ (11)$^8$ (13)$^8$ (15)$^4$ |
| 7   | 7   | (26)$^3$ (24)$^5$ (20)$^7$ (2)$^2$ (8)$^6$ (12)$^8$ (16)$^8$ |
| 7   | 8   | (29) (27)$^3$ (25)$^5$ (23)$^7$ (19)$^9$ (9)$^8$ (13)$^{10}$ (15)$^5$ |
| 8   | 8   | (28)$^3$ (26)$^5$ (24)$^7$ (22)$^9$ (20)$^{11}$ (12)$^{12}$ (16)$^{12}$ |

Table 1
\[
\begin{array}{c}
F_1^\sigma = -0.64090211 \\
F_2^\sigma = 0.33867436 \\
F_3^\sigma = -0.18662854 \\
F_4^\sigma = 0.14277176 \\
F_5^\sigma = 0.06032607 \\
F_6^\sigma = -0.04338937 \\
F_7^\sigma = 0.01642569 \\
F_8^\sigma = -0.00303607 \\
\end{array}
\]

Table 2

\[
\begin{array}{c}
F_1^e = -3.70658437 \\
F_2^e = 3.42228876 \\
F_3^e = -2.38433446 \\
F_4^e = 2.26840624 \\
F_5^e = 1.21338371 \\
F_6^e = -0.96176431 \\
F_7^e = 0.45230320 \\
F_8^e = -0.10584899 \\
\end{array}
\]

Table 3
\[ \begin{align*}
  c^1_{11} &= -2.093102832 \\
  c^0_{11} &= -10.19307727 \\
  c^1_{12} &= -7.979022182 \\
  c^1_{12} &= -71.79206351 \\
  c^0_{12} &= -70.29218939 \\
  c^3_{13} &= -582.2557366 \\
  c^2_{13} &= -6944.416956 \\
  c^1_{13} &= -13406.48877 \\
  c^0_{13} &= -7049.622303 \\
  c^3_{22} &= -21.48559881 \\
  c^2_{22} &= -333.8125724 \\
  c^1_{22} &= -791.3745549 \\
  c^0_{22} &= -500.2535896 \\
  c^3_{14} &= 22.57778351 \\
  c^2_{14} &= 318.7122159 \\
  c^1_{14} &= 672.2210098 \\
  c^0_{14} &= 377.4586311 \\
  c^1_{15} &= -260.7643072 \\
  c^0_{15} &= -4719.877128 \\
  c^1_{15} &= -15172.07643 \\
  c^0_{15} &= -17428.22924 \\
  c^0_{15} &= -6716.787925 \\
  c^3_{23} &= -92.73452350 \\
  c^2_{23} &= -1846.579035 \\
  c^2_{23} &= -6618.297073 \\
  c^1_{23} &= -8436.850082 \\
  c^0_{23} &= -3579.556465 \\
  c^3_{33} &= -1197.056497 \\
  c^2_{33} &= -30166.99117 \\
  c^2_{33} &= -150512.4122 \\
  c^2_{33} &= -301093.9432 \\
  c^1_{33} &= -267341.1276 \\
  c^0_{33} &= -87821.70785 \\
\end{align*}\]
\begin{table}
\begin{tabular}{|c|c|}
\hline
$c^6_{25}$ & 1425.995027 \\
$c^5_{25}$ & 44219.03877 \\
$c^4_{25}$ & 286184.1535 \\
$c^3_{25}$ & 788413.2178 \\
$c^2_{25}$ & 1078996.488 \\
$c^1_{25}$ & 723566.4417 \\
$c^0_{25}$ & 191383.5734 \\
\hline
$c^5_{17}$ & 190.8548023 \\
$c^4_{17}$ & 4633.706068 \\
$c^3_{17}$ & 21406.72691 \\
$c^2_{17}$ & 39514.82959 \\
$c^1_{17}$ & 32456.91939 \\
$c^0_{17}$ & 9906.265607 \\
\hline
$c^6_{14}$ & -7249.785565 \\
$c^5_{14}$ & -276406.7236 \\
$c^4_{14}$ & -2299573.212 \\
$c^3_{14}$ & -849276.3526 \\
$c^2_{14}$ & -16615618.39 \\
$c^1_{14}$ & -17950817.11 \\
$c^0_{14}$ & -10139089.36 \\
$c^0_{14}$ & -2341590.241 \\
\hline
\end{tabular}
\end{table}

Table 4
\[
\begin{array}{l}
c_{11}^1 = -70.00917205 \\
c_{11}^0 = -87.90247670 \\
c_{12}^1 = -466.3008246 \\
c_{12}^0 = -1307.331521 \\
c_{13}^1 = -853.2803886 \\
c_{13}^0 = -102574.1349 \\
c_{22}^1 = -2193.896354 \\
c_{22}^0 = -10870.05277 \\
c_{22}^1 = -16161.44508 \\
c_{22}^0 = -7510.235388 \\
c_{14}^1 = 2074.636471 \\
c_{14}^0 = 9881.413381 \\
c_{14}^1 = 14357.04570 \\
c_{14}^0 = 6568.762583 \\
c_{15}^1 = -30333.56619 \\
c_{15}^0 = -198757.2340 \\
c_{15}^1 = -447504.5720 \\
c_{15}^0 = -422808.9295 \\
c_{15}^1 = -143743.2050 \\
c_{23}^1 = -11971.94909 \\
c_{23}^0 = -81253.72269 \\
c_{23}^1 = -186593.8661 \\
c_{23}^0 = -178494.3378 \\
c_{23}^1 = -61194.62416 \\
c_{33}^5 = -195385.7662 \\
c_{33}^4 = -1743171.802 \\
c_{33}^5 = -5603957.324 \\
c_{33}^2 = -8422606.859 \\
c_{33}^1 = -6035102.896 \\
c_{33}^0 = -1668721.004 \\
\end{array}
\]
Continued
| $c_0^{10}$ | 289831.4882 |
|-----------|-------------|
| $c_0^{11}$ | 3275586.983 |
| $c_0^{12}$ | 13872077.63 |
| $c_0^{13}$ | 29236961.96 |
| $c_0^{14}$ | 32979257.31 |
| $c_0^{15}$ | 19100224.04 |
| $c_0^{16}$ | 4471623.121 |

| $c_1^{10}$ | 30394.23374 |
|-----------|-------------|
| $c_1^{11}$ | 274294.8033 |
| $c_1^{12}$ | 897781.3229 |
| $c_1^{13}$ | 1.375919456 |
| $c_1^{14}$ | 1.004969466 |
| $c_1^{15}$ | 282938.1974 |

| $c_2^{10}$ | −1830120.693 |
|-----------|-------------|
| $c_2^{11}$ | −25699492.93 |
| $c_2^{12}$ | −138411873.8 |
| $c_2^{13}$ | −384776478.8 |
| $c_2^{14}$ | −608371427.1 |
| $c_2^{15}$ | −553818699.0 |
| $c_2^{16}$ | −270964337.7 |
| $c_2^{17}$ | −55283137.91 |

**Table 5**
| $\Delta_1^\sigma$ | 0.0507107 |
| $\Delta_2^\sigma$ | 0.0054088 |
| $\Delta_3^\sigma$ | 0.0010868 |
| $\Delta_{11}^\sigma$ | 0.0025274 |
| $\Delta_4^\sigma$ | 0.0004351 |
| $\Delta_{12}^\sigma$ | 0.0010446 |
| $\Delta_5^\sigma$ | 0.0000514 |
| $\Delta_{13}^\sigma$ | 0.0002283 |
| $\Delta_{\text{partial}}^\sigma$ | 0.0614934 |

**Table 6**

| $\Delta_1^\varepsilon$ | 0.2932796 |
| $\Delta_2^\varepsilon$ | 0.0546562 |
| $\Delta_3^\varepsilon$ | 0.0138858 |
| $\Delta_{11}^\varepsilon$ | 0.0425125 |
| $\Delta_4^\varepsilon$ | 0.0069134 |
| $\Delta_{12}^\varepsilon$ | 0.0245129 |
| $\Delta_5^\varepsilon$ | 0.0010340 |
| $\Delta_{13}^\varepsilon$ | 0.0065067 |
| $\Delta_{\text{partial}}^\varepsilon$ | 0.4433015 |

**Table 7**