BOUNDED NEGATIVITY, MIYAOKA-SAKAI INEQUALITY AND
ELLIPTIC CURVE CONFIGURATIONS

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Abstract. Using the Miyaoka-Sakai inequality for curves with normal crossings, we find lower bounds for $H$-constants of curves involving their geometric genus and the number of their singularities. Similarly to the linear $H$-constant introduced in [1], we study the elliptic $H$-constants of $\mathbb{P}^2$ and Abelian surfaces and we find some lower bounds. We exhibit configurations of smooth plane cubic curves whose $H$-constant is arbitrarily close to $-4$. As a Corollary, we obtain that the global $H$-constant of any surface $X$ is less or equal to $-4$. Related to these problems, we moreover give a new inequality for the number and type of singularities of elliptic curves arrangements on Abelian surfaces, inequality which has a close similarity to the one of Hirzebruch for arrangements of lines in the plane.

1. Introduction

The $H$-constant of a surface $X$ and some related variants of it have been recently introduced in [1], bringing new problems and open questions on curves on surfaces, and their singularities. As explained there, these constants measure the local negativity of curves on surfaces, in analogy with the local positivity measured by Seshadri constants. These constants come from the context of the Bounded Negativity Conjecture (BNC):

**Conjecture.** Let $S/\mathbb{C}$ be a smooth projective surface. There exists an integer $b(X)$ such that $C^2 \geq -b(X)$.

This ancient and now intensively studied conjecture ([1], [2], [3], [4], [10], [12]) is trivially true for the plane, but we do not know the behavior of the problem if one takes blow-ups of it. The $H$-constant was introduced to handle that question. For a blow-up $X' \to X$ at a set $\mathcal{P}$ of $s > 0$ distinct points and $C \to X$ a curve, we denote by $\bar{C}$ the strict transform in $X$ of $C$. The $H$-constant of $X$ is defined by

$$H_X := \min_{C, \mathcal{P}} H(C, \mathcal{P})$$

where

$$H(C, \mathcal{P}) = \frac{(\bar{C}^2)}{s}$$

for $C \to X$ varying over all reduced curves on $S$, and the strict transform of $C$ varying over the blow-ups at set $\mathcal{P}$ of $s$ distinct points on $X$ ($s > 0$ any). If finite, the $H$-constant has the interesting property that whenever BNC holds for $X$, then it holds for any blow-ups of it.

Variation of the $H$-constants are proposed in [1], among them the linear $H$-constant $H_L$ for the plane. There $X = \mathbb{P}^2$ and the minimum $H_{L, \mathbb{P}^2} = \min_{C, \mathcal{P}} H(C, \mathcal{P})$ is taken over every curves $C$ that are union of lines in $\mathbb{P}^2$. Using Hirzebruch bounds on the singularities of lines configurations, the authors proves that $H_{L, \mathbb{P}^2} \geq -4$. They moreover found an example of union of lines $C$ such that $H(C, \mathcal{P})$ is very negative ($H(C, \mathcal{P}) = -225/67$) for some particular blow-ups of $\mathbb{P}^2$ over some set of points $\mathcal{P}$, and therefore we know that $-4 \leq H_{L, \mathbb{P}^2} \leq -225/67$, $H_{\mathbb{P}^2} \leq H_{L, \mathbb{P}^2}$.

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In this paper, we similarly propose to study elliptic $H$-constant $H_{E_1,X}$ defined by

$$H_{E_1,X} = \min_{C,P} H(C,P)$$

for $C$ the unions of smooth elliptic curves on $X$. We obtain:

**Proposition 1.** There exists configurations $C_n$ of smooth cubic curves in $\mathbb{P}^2$ such that for $P_n$ the set of singular points of $C_n$, we have

$$\lim_n H(C_n,P_n) = -4,$$

and therefore $H_{E_1,\mathbb{P}^2} \leq -4$.

The configurations $C_n$ are described in [14]. Note that for configurations $C$ of smooth cubic curves with $k$-points singularities only (ie singularities locally like $\{x^k = y^k\}$, $k \geq 2$), we obtain in Corollary 8 the reverse inequality $H(C,P) \geq 4$.

From Proposition 1, we get the following general result:

**Corollary 2.** Let $X$ be a smooth surface. Then $H_X \leq H_{E_1,\mathbb{P}^2} \leq -4$.

Then, it is natural to study the elliptic $H$-constant of Abelian surfaces. Let $C$ be a configuration of elliptic curves on an Abelian surface $A$ and let $k_t$ be the number of $k$-points singularities ($k \geq 2$) on $C$. Let be $f_0 = \sum k_t$ and $f_1 = \sum k t_k$.

**Theorem 3.** For any set $P$ of points, we have

$$H(C,P) \geq -\frac{f_1}{f_0} = \frac{t_2 + \frac{1}{2}t_3}{f_0} - 4 \geq -4$$

(1.1)

The elliptic $H$-constant satisfies $H_{E_1,A} \geq -4$. If we have equality $H(C,P) = -4$ for some $P$, then the only singularities on $C$ are 4-points. The $H$-constant and elliptic $H$-constant are isogeny invariants : if $A$ and $B$ are isogeneous, then $H_A = H_B$ and $H_{E_1,A} = H_{E_1,B}$.

Let $A$ be either the surface $(\mathbb{C}/\mathbb{Z}[\alpha])^2$, where $\alpha^2 + \alpha + 1 = 0$ or $(\mathbb{C}/\mathbb{Z}[i])^2$, $i^2 = -1$. By the constructions of Hirzebruch [6] and Holzapfel [8, Example 5.4], there exists configurations $C$ of elliptic curves on $A$ and points $P$ such that $H_{A,E_1} = H(C,P) = -4$. We thus see that the bound $H_{E_1,A} \geq -4$ is optimal. Actually, by Kobayashi’s results on hyperbolicity [9], the equality $H(C,P) = -4$ is attained if and only if the complement of $C$ in $A$ is birational to an open ball quotient surface. Inequality (1.1) in Theorem 3 is a corollary of a more general result (Theorem 15) proved by using $(\mathbb{Z}/n\mathbb{Z})^d$-covers of Abelian surfaces. In particular we obtain:

**Proposition 4.** For a configuration of elliptic curves on an Abelian surface, we have

$$t_2 + \frac{3}{4}t_3 \geq \sum_{k \geq 5} (2k - 9)t_k.$$

This is the very analog of the inequality $t_2 + \frac{3}{4}t_3 \geq d + \sum_{k \geq 5}(2k - 9)t_k$ of Hirzebruch for a configuration of $d$ lines in $\mathbb{P}^2$.

In the same vein of ideas, we also obtain the following general result on the $H$-constant for a configuration $D = \sum_{i=1}^d C_i$ of smooth curves on a surface $X$, with only $k$-points singularities ($k \geq 2$). Let $k_t$ be the number of $k$-points on $D$ and let be as above $f_0 = \sum t_k$. We denote by $g(C_i)$ the genus of a smooth curve $C_i$.

**Theorem 5.** Suppose that $K_X + D$ is nef. Then

$$-t_2 + 4f_0 - f_1 \geq K_X^2 - 3c_2(X) - 2\sum C_i^2 - \sum K_XC_i.$$  

(1.2)
Let \( \mathcal{P} = \mathcal{P}(D) \) be the set of singularities on \( D \). Then

\[
H(D, \mathcal{P}) \geq \frac{t_3 + K^2_X - 3c_2(X) - \sum (2g(C_i) - 2)}{f_0} - 4. \quad (1.3)
\]

In [11, Problem 3.10], is discussed the question whether the global \( H \)-constant of \( \mathbb{P}^2 \) equals \(-4\). Because of the part \(- \sum (2g(C_i) - 2) \) in equation (1.3) we see that for configurations of curves of genus \( \geq 2 \), the bound \(-4\) (obtained for \( g = 0 \) in [11]) is less visible.

The proof of Theorem 5 uses the logarithmic Bogomolov Miyaoka Yau inequality, also called Miyaoka-Sakai inequality, proved by Miyaoka in [11] and generalizing a result of Sakai [15].

2. **Miyaoka-Sakai inequality and refinements for \( \mathbb{P}^2 \).**

A word of explanation on the motivation of Theorem 5. In view of the Bounded Negativity Conjecture, it would be interesting to know irreducible curves \( C \) with low \( H(C, \mathcal{P}) \): by taking the blow-up one could get very negative curves. By example, if one consider the rational curves of degree \( d \) in \( \mathbb{P}^2 \) with the maximal number of nodes, and \( \mathcal{P} \) the set of nodes, one get \( H(C, \mathcal{P}) \) close to \(-2\) (on that subject, see also Introduction of [1] and Remark 10 below). If one is willing to go down, we must impose singularities with higher multiplicities on \( C \), but it forces \( C \) to have “negative genus” ie to be union of (at least) two curves. That explains the idea to consider union of curves rather than irreducible curves. Moreover considering reducible curves gives more functorialities to the \( H \)-constants, e.g. when one wish to compare these constants through a dominant map \( f : X \to Y \).

Let \( C = \sum D_i \) be a smooth normal crossing divisor on a surface \( X \). The main tool for the proof of Theorem 5 is the Miyaoka-Sakai inequality, also known as the logarithmic Bogomolov Miyaoka Yau inequality:

**Theorem 6.** [11] Suppose that \( K_X + C \) is nef. Then:

\[
(K_X + C)^2 \leq 3e(X \setminus C).
\]

Let \( X \) be a smooth projective complex surface and \( C \) a curve on \( X \) (reduced but maybe not irreducible). Let us define inductively a finite sequence of surfaces \( X_j \) and curves \( C_j \hookrightarrow X_j \).

In the first step, \( X_0 = X \), \( C_0 = C \) and at step \( j \), the surface \( X_j \) is the blow-up of \( X_{j-1} \) at the singular points of \( C_{j-1} \), the curve \( C_j \) is the strict transform on \( X_j \) of \( C_{j-1} \).

The process stops after a finite number \( t \) of steps, when \( C_t \) becomes smooth. On each step \( X_j \), we denote by \( E_j \), the exceptional divisor (union of \((-1)-\)curves) of the map \( X_j \to X_{j-1} \) and we denote by \( \mathcal{P}_j \) the set of singular points of \( C_j \). For \( k \geq 2 \), we denote by \( t_k^j \) the number of points of \( C_j \) that have multiplicity \( k \). Then \( f_k^j = \sum_{k \geq 2} t_k^j \) is the number of singularities of \( C_j \), and we define the quantities

\[
f_1^j = \sum k t_k^j, \quad f_2^j = \sum k^2 t_k^j.
\]

We observe that when \( \mathcal{P}_0 \neq \emptyset \), i.e. when \( C \) is singular, then

\[
H(C_0, \mathcal{P}_0) = \frac{(C_0)^2 - f_0^j}{f_0^j}.
\]

We denote by \( F_0 = \sum_{j=0}^{t-1} f_0^j \) the number of blow-ups and we define:

\[
F_1 = \sum_{j=0}^{t-1} f_1^{j-1}, \quad F_2 = \sum_{j=0}^{t-1} f_2^{j-1}.
\]

Let \( g \) be the genus of \( C \) i.e.

\[
g - 1 = \sum g_i - 1
\]
where the $g_i$‘s are the geometric genus of the irreducible components of $C$. We define the smooth \textit{H-constant of a singular curve} $C$ by $H_{\text{smooth}}(C) = \frac{(C_1)^2}{F_0}$.

\textbf{Theorem 7.} Suppose that $C$ is singular and $K_X + C$ is nef. Then:

$$H_{\text{smooth}}(C) = \frac{(C)^2 - F_2}{F_0} \geq -4 + \frac{1}{F_0} (K_X^2 - 3c_2(X) - (2g - 2))$$

If $t = 1$, i.e. the curve has only $k$-points ($k \geq 2$), then

$$H(C, \mathcal{P}(C)) \geq -4 + \frac{t_2 + K_X^2 - 3c_2(X) - 2g + 2}{F_0}.$$ \hspace{1cm} (2.1)

where $\mathcal{P}(C)$ is the set of singular points of $C$.

\textbf{Proof.} Let $K_t$ be the canonical divisor of $X_t$. The curve $C_t$ is smooth and $K_t C_t + (C_t)^2 = 2g - 2$. We have $(C_1)^2 = (C_0)^2 - f_2^0$ and

$$(C_j)^2 = (C_{j-1})^2 - f_2^{j-1} = (C_0)^2 - \sum_{u=0}^{j-1} f_2^{u-1}.$$  

The Chern numbers of $X_t$ are $K_t^2 = K_X^2 - F_0$, $c_2(X_t) = c_2(X) + F_0$. Using the fact that $K_X + C$ is nef, it is easy to check that $K_t + C_t$ is nef, we can apply the Miyaoka-Sakai inequality

$$(K_t + C_t)^2 \leq 3e(X_t - C_t)$$

and we have

$$(K_t + C_t)^2 = (K_t)^2 + 2(2g - 2) - C_t^2 \leq 3(c_2(X) + F_0 + (2g - 2)).$$

Using $(C_t)^2 = (C_0)^2 - F_2$, we get

$$F_2 - (C_0)^2 - (2g - 2) \leq 4F_0 + 3c_2(X) - K_X^2.$$  

Thus

$$-4 + \frac{1}{F_0} (K_X^2 - 3c_2(X) - (2g - 2)) \leq \frac{(C_0)^2 - F_2}{F_0}.$$  

Let us suppose that $t = 1$, i.e. all singularities are $k$-points ($k \geq 2$), then instead of blowing up every point, one can blow-up at $k$-points for $k \geq 3$ only. The Miyaoka Sakai inequality apply to the strict transform of $C$ and one find inequality 2.1. \hfill $\square$

We obtain the following applications:

\textbf{Corollary 8.} A) If $X = \mathbb{P}^2$ and the $C_i$, $i = 1, \ldots, d$ are lines with $d \geq 3$ then

$$H(D, \mathcal{P}) \geq \frac{t_2 + 2d}{s} - 4 \geq -4.$$  

B) If $X$ is $\mathbb{P}^2$ and the $C_i$ are smooth genus 1 curves such that $D = \sum C_i$ has only $k$-points ($k \geq 2$) then

$$H(D, \mathcal{P}) \geq \frac{t_2}{s} - 4 \geq -4.$$ \hspace{1cm} (2.2)

The first inequality in part A) of Corollary 8 is a weaker version of the inequality:

$$H(C, \mathcal{P}) \geq B_1 := \frac{1}{s} (2d + t_2 + \frac{1}{4} t_3) - 4$$ \hspace{1cm} (2.3)

obtained in [1] that uses a refinement of Hirzebruch [5] 140]

$$t_2 + \frac{3}{4} t_3 \geq d + \sum_{k \geq 5} (k - 4) t_k$$ \hspace{1cm} (2.4)
for the case of an arrangement of \( d \) lines in \( \mathbb{P}^2 \), which is better than the Miyaoka-Sakai inequality. In fact an even better inequality is obtain in [7 eq. (9)], which is
\[
t_2 + \frac{3}{4} t_3 \geq d + \sum_{k \geq 5} (2k - 9)t_k. \quad (2.5)
\]

**Proposition 9.** Using inequality \((2.5)\), we obtain:
\[
H(C, \mathcal{P}) \geq B_2 := \frac{1}{s} \left( \frac{3}{2}d + 2t_2 + \frac{9}{8} t_3 + \frac{1}{2}t_4 \right) - \frac{9}{2}
\]
and moreover that inequality \((2.6)\) is sharper than \((2.3)\): \( B_2 \geq B_1 \).

**Proof.** Using the combinatorics, we have
\[
H(C, \mathcal{P}) = \frac{d^2 - \sum_{k \geq 2} k^2 t_k}{s} = \frac{d - \sum_{k \geq 2} kt_k}{s}.
\]
By \((2.5)\) we have \( t_2 + \frac{3}{4} t_3 \geq d + \sum_{k \geq 4} (2k - 9)t_k \), thus
\[
- \sum_{k \geq 2} kt_k \geq \frac{1}{2} \left( d + 4t_2 + \frac{9}{4} t_3 + t_4 - 9s \right) \quad (2.7)
\]
and we obtain inequality \((2.6)\). The fact that this new inequality is sharper comes by construction; it is simple to check it. \( \square \)

For the Klein configuration of lines (see [11 § 4.1]), inequality \((2.6)\) is even an equality. By considering inequality
\[
t_2 + \frac{3}{4} t_3 \geq d + \sum_{k \geq 5} (ak + b)t_k
\]
holding for \((a, b)\) with \( a > 0, ak + b \leq 2k - 9 \) for all \( k \geq 5 \), one obtain:
\[
H(C, \mathcal{P}) \geq \frac{1}{as} \left( d - (2a + 1 + b)t_2 - (3a + \frac{3}{4} + b)t_3 - (4a + b)t_4 \right) + \frac{b}{a}.
\]
If one is looking to an inequality
\[
H(C, \mathcal{P}) \geq \frac{1}{s} (g_1 d + g_2 t_2 + g_3 t_3 + g_4 t_4) + g_5
\]
with positive coefficients \( g_i, i = 1 \ldots 4 \) then \( g_5 \leq -4 \) and this gives no improvement on the bound \(-4\). For \( a = \frac{7}{8} \) and \( b = -\frac{27}{7} \), we have
\[
H(C, \mathcal{P}) \geq \frac{16d + 5t_2 - t_4}{7s} - \frac{27}{7},
\]
therefore if a configuration has no 4-points, then \( H(C, \mathcal{P}) \geq -\frac{27}{7} > -4 \).

**Remark 10.** The singularities of a totally geodesic curve on a Shimura or a Ball quotient surface are at most \( k \)-points \((k \geq 2)\), and so are the singularities of a union of such curves. Let \( X \) be a Shimura surface and \( \mathcal{C} \) be a totally geodesic curve on it. Let \( \delta \in \mathbb{N} \) be defined by \( K_X C + C^2 = 2g - 2 + 2\delta \). We have \( 2\delta = f_2 - f_1 \), moreover using [4 Cor. 4.2] for totally geodesic curves on \( X \), we know that \( C^2 \sim 2\delta \) when the geometric genus \( g \) of \( \mathcal{C} \) goes to \( \infty \), therefore
\[
H(C, \mathcal{P}(C)) \sim -\frac{f_1}{f_0}
\]
when \( g \to \infty \) and since always \( f_1 \geq 2f_0 \), we obtain \( \liminf_C H(C, \mathcal{P}(C)) \leq -2 \). In view of inequality \((2.1)\) in Theorem \([7]\) it would be interesting to know if there is a bound for \( \frac{f_1}{f_0} \) for such curves.
3. Negative elliptic curve configurations

Let \( p : Z \rightarrow \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at the 12 singular points \( \mathcal{P}_{HE} = \{ p_1, \ldots, p_{12} \} \) of the dual Hesse configuration. In [14], is described some elliptic curve configurations \( \mathcal{H}_n \) \( (n \in 3 \mathbb{N}, n > 0) \) on \( Z \) with the following properties:

i) \( \mathcal{H}_n \) is the union of \( \frac{1}{3}(n^2 - 3) \) elliptic curves which are fibers of some elliptic fibrations of \( Z \),

ii) Each elliptic curve of \( \mathcal{H}_n \) has 9 3-points and \( n^2 - 9 \) 4-points,

iii) There are \( \frac{1}{3}(n^2 - 3)(n^2 - 9) \) 4-points and \( 4(n^2 - 3) \) 3-points on \( \mathcal{H}_n \).

Let be \( \mathcal{L}_n \) be the image in \( \mathbb{P}^2 \) of \( \mathcal{H}_n \) by the blow-down map \( p \). Then:

i) \( \mathcal{L}_n \) is the union of \( \frac{1}{3}(n^2 - 3) \) smooth degree 3 curves.

ii) Each elliptic curve contains \( n^2 - 9 \) 4-points and goes through 9 of the 12 singular points of the dual Hesse configuration.

iii) On \( \mathcal{L}_n \), there are \( \frac{1}{3}(n^2 - 3)(n^2 - 9) \) 4-points and the 12 points in \( \mathcal{P}_{HE} \) have multiplicity \( (n^2 - 3) \) after one blow up in \( p_i \), the strict transform of the curve \( \mathcal{L}_n \) has \( \frac{1}{3}(n^2 - 3) \) 3-points on the exceptional divisor.

**Proposition 11.** The configurations \( \mathcal{L}_n \) of smooth cubic curves on \( \mathbb{P}^2 \) satisfy

\[
\lim_n H(\mathcal{L}_n, \mathcal{P}_n) = -4,
\]

where \( \mathcal{P}_n \) is the set of singular points of \( \mathcal{L}_n \).

**Proof.** Let be \( X_n \rightarrow \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at \( \mathcal{P}_n \), the \( s_n = 12 + \frac{1}{3}(n^2 - 3)(n^2 - 9) = \frac{1}{3}n^4 + o(n^3) \) singularities of \( \mathcal{L}_n \). Let \( L_n \) be the strict transform of \( \mathcal{L}_n \). The blow down map \( X_n \rightarrow \mathbb{P}^2 \) factorizes through \( Z \). Then

\[
L_n^2 = \sum C_i^2 + 2 \sum_{i < j} C_i C_j
\]

where \( C_i \) is the strict transform of a curve \( C_i \) in \( \mathcal{H}_n \). Since \( C_i^2 = -(n^2 - 9) \) and since from the configuration one has:

\[
\sum_{i < j} C_i C_j = 4(n^2 - 3),
\]

we get

\[
L_n^2 = -(n^2 - 9) \times \frac{4}{3}(n^2 - 3) + 8(n^2 - 3) = -\frac{4}{3}n^4 + o(n^3).
\]

As \( s_n = \frac{1}{3}n^4 + o(n^3) \), we obtain the result. We could also use the formulas for \( t_4 = \frac{n^4}{3} + o(n^3) \), \( t_{n^2 - 3} = 12 \) and a degree \( 4(n^2 - 3) \) curve. \( \square \)

**Remark 12.** A) By example, for \( n = 7 \), we obtain \( H(\mathcal{L}_n, \mathcal{P}_n) = -\frac{20148}{5027} \approx -3.83 \).

B) The above configurations \( C_n \) are strongly linked to some compactifications \( X_n \) of some open ball quotient surfaces, for which \( \lim \frac{ds}{C^2}(X_n) = 3 \), ie one is close to the upper bound in the Miyaoka-Yau inequality.

Let \( f : X \rightarrow Y \) be a dominant morphism between two smooth surfaces. Let \( C \) be a reduced curve on \( Y \). Suppose that \( C \) do not contain components in the branch locus \( B \) and let \( \mathcal{P} \) be a set of \( s \) point in \( Y \) disjoint with \( B \) (so that \( f^*C \) and \( f^*\mathcal{P} \) are reduced of pure dimension 1 and 0 respectively).

**Lemma 13.** Then \( H_X(f^*C, f^*\mathcal{P}) = H_Y(C, \mathcal{P}) \).

**Proof.** Let \( d \) be the degree of \( f \). Let \( p \) be a point of \( \mathcal{P} \). Then the \( d \) points above \( p \) have the same multiplicity \( m_p \) inside \( C' = f^*C \) than \( p \) inside \( C \) and we have:

\[
H_X(f^*C, f^*\mathcal{P}) = \frac{C^2}{ds} = \frac{C^2 - d \sum m_p^2}{ds} = \frac{dC^2 - d \sum m_p^2}{ds} = \frac{C^2}{s} = H_Y(C, \mathcal{P}).
\]

\( \square \)
Another consequence of Proposition [11] is:

**Corollary 14.** Let $X$ be a smooth surface. Then $H_X \leq H_{\mathbb{P}^2} \leq -4$.

*Proof.* Let $f : X \to \mathbb{P}^2$ be a generic projection of $X$ onto the plane. For any curve $C$ in $\mathbb{P}^2$ and set of point $P \subset \mathbb{P}^2$, we can choose an automorphism $g$ such that $g^*C$ do not contain any components of the branch divisor $B$ and $g^*P$ is disjoint with $B$. We then apply Lemma [13] and results of Proposition [11] to conclude that $H_X \leq H_{\mathbb{P}^2} \leq -4$. $\square$

### 4. Abelian Surfaces

#### 4.1. Arrangement of curves on Abelian surfaces and $(\mathbb{Z}/n\mathbb{Z})^k$-covers.

Let $A$ be an abelian surface and let $C = \sum_{i=1}^d C_i$ be a reduced divisor with only $k$-points singularities ($k \geq 2$ any), a union of $d \geq 2$ smooth divisors $C_i$. Let $g$ be the geometric genus of $C$ ie

$$g - 1 = \sum g_j - 1,$$

where the $g_j$’s are the geometric genus of the irreducible components of $C$.

**Theorem 15.** We have

$$-\frac{f_1}{f_0} \geq \frac{2t_2 + \frac{9}{2}t_3 + \frac{1}{2}t_4 + 10 - 10g}{2} - \frac{9}{2}. \quad (4.1)$$

If $C$ is a configuration of elliptic curves, then

$$H(C) \geq \frac{t_2 + \frac{1}{2}t_3}{f_0} - 4 \geq -4 \quad (4.2)$$

where $H(C) = \min \tau H(C, \mathcal{P})$. In particular, if $H(C) = -4$, then the only singularities on $C$ are 4-points.

The remaining of this section is the proof of Theorem [15]. Let us recall a Theorem of Namba on branched covers. Let $M$ be a manifold, $D_1, \ldots, D_s$ be irreducible reduced divisors on $M$ and $n_1, \ldots, n_s$ be positive integers. We denote by $D$ the divisor $D = \sum n_i D_i$. Let $\text{Div}(M, D)$ be the sub-group of the $\mathbb{Q}$-divisors generated by the entire divisors

$$\frac{1}{n_1}D_1, \ldots, \frac{1}{n_s}D_s.$$

Let $\sim$ be the linear equivalence in $\text{Div}(M, D)$, where $G \sim G'$ if and only if $G - G'$ is a principal divisor. Let $\text{Div}(M, D)/\sim$ the quotient and let $\text{Div}^0(M, D)/\sim$ be the kernel of the Chern class map

$$\text{Div}^0(M, D)/\sim \to H^{1,1}(M, \mathbb{R})$$

$$G \to c_1(G).$$

**Theorem 16.** (Namba, [13] Thm. 2.3.20). There exists a finite Abelian cover which branches at $D$ with index $n_i$ over $D_i$ for all $i = 1, \ldots, s$ if and only if for every $j = 1, \ldots, s$ there exists an element $v_j = \sum \frac{n_i}{n_j}D_i + E_j$ of $\text{Div}^0(M, D)/\sim$ $(E_j$ an entire divisor) such that $a_{i,j}$ is coprime to $n_j$. Then the group in $\text{Div}^0(M, D)/\sim$ generated by the $v_j$ is isomorphic to the Galois group of one of such Abelian cover.

We find the inequalities among the $t_k$ in Theorem [15] using $(\mathbb{Z}/n\mathbb{Z})^d$-covers of $A$ ramified above curves (related to the) $C_i$. These inequalities involve quantities that are “linear” under isogenies, by which we mean that if $\phi : B \to A$ is an isogeny of degree $m$, then the number of $k$-points on $\phi^*C$ (a reduced curve), the intersections between the $\phi^*C_i$’s and $\phi^*C$, the geometric genus of $\phi^*C$ are the one of $C, C_i$ etc... multiplied by $m$. Thus inequalities involving linear terms in the $t_k$’s and $C_i^2$ proved on abelian surface $B$ are then inequalities for $A$. Recall that if $\phi = [m]$ is the multiplication by $m \in \mathbb{N}$ map, then
\( \phi^* D \sim \frac{m(m+1)}{2} D + \frac{m(m-1)}{2} [ -1]^* D \) for any divisor \( D \). By taking \( m = 2n \), one can therefore suppose that the divisors \( C_i \) are \( n \)-divisible if there exists divisors \( L_i \) such that \( C_i \sim nL_i \).
The divisor \( \frac{1}{n} C_i - L_i \) is in \( \text{Div}^0(A, nC) / \sim \) and the multiplicity of an irreducible component \( C_i' \) in \( \frac{1}{n} C_i - L_i \) is \( \frac{1}{n} \). The group generated by divisors \( \frac{1}{n} C_i - L_i \) is isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^d \) and there exists a \( (\mathbb{Z}/n\mathbb{Z})^d \)-cover of \( A \) branched with index \( n \) over \( C \). For the computation of the Chern numbers of the resolution \( X_n \) of the cover will use the local analysis of the \( (\mathbb{Z}/n\mathbb{Z})^d \)-branched covers of the plane constructed by Hirzebruch in [6].
The following quantities \( f_0, f_1, f_2 \) are linear under \([2n]\):
\[
 f_0 = \sum_{k \geq 2} t_k, \quad f_1 = \sum_{k \geq 2} kt_k, \quad f_2 = \sum_{k \geq 2} k^2 t_k.
\]
Let \( Z \to A \) be the blow-up at the \( f_0 - t_2 = \sum_{k \geq 3} t_k \) singularities of \( C \) of multiplicities \( k \geq 3 \) and let \( \bar{C} = \sum \bar{C}_i \) be the strict transform of \( C \) in \( Z \). For a singularity \( p \) of \( C \) of multiplicity \( k_p \geq 3 \), we denote by \( E_p \to Z \) the exceptional curve over \( p \). There exists a map
\[
 f : X_n \to Z
\]
ramified with index \( n \) above the curve \( \bar{C} \). As in [3], above \( E_p \) lies in \( X_n \) \( n^{d-r} \) \((r = k_p)\) copies of a smooth curve \( F_p \) which is a \( (\mathbb{Z}/n\mathbb{Z})^{r-1} \)-cover of \( E_p \) ramified with index \( n \) at \( r \) points, thus
\[
e(C(F_p)) = n^{r-1}(2 - r) + rn^{r-2} = n^{r-2}(2n + r(1 - n)).
\]
Since the Galois group permutes these curves, we have \( (F_p)^2 = -n^{r-2} \). If a singularity \( p \) of \( C \) is a node, then \( X_n \) is smooth over \( p \) and the fiber of \( f \) at \( p \) has only \( n^{d-2} \) points.
We have
\[
e(C) = 2 - 2g + f_0 - f_1, \quad e(C \setminus \text{sing}(C)) = 2 - 2g - f_1, \quad e(A \setminus C) = -e(C) = 2g - 2 + f_1 - f_0,
\]
and
\[
 C^2 = \sum C_i^2 + f_2 - f_1 = 2g - 2 + f_2 - f_1.
\]
Therefore we get
\[
e(X_n \setminus f^{-1} E_p) = n^d e(A \setminus C) + n^{d-1} e(C \setminus \text{sing}(C)) + n^{d-2} t_2
\]
and
\[
e(X_n \setminus f^{-1} E_p)/n^{d-2} = n^2(2g - 2 + f_1 - f_0) + n(2 - 2g - f_1) + t_2.
\]
Since above each exceptional divisor \( E_p \) in \( Z \), we have \( n^{d-k} \) curves with Euler number \( e(F_p) \), we get
\[
e(X_n)/n^{d-2} = e(X_n \setminus f^{-1} E_p)/n^{d-2} + \sum_{k \geq 3} n^{d-3} t_k (2n + k(1 - n))
\]
thus
\[
e(X_n)/n^{d-2} = (2g - 2 + f_1 - f_0)n^2 + 2(1 - g + f_0 - f_1)n - t_2.
\]
Let us now compute the canonical divisor: \( K_{X_n} \) is numerically equivalent to the pullback of
\[
 K = \sum E_p + \frac{n - 1}{n} (\sum E_p + f^* C - \sum k_p E_p)
\]
which is
\[
 K = \sum \frac{2n - 1 + k(1 - n)}{n} E_p + \frac{n - 1}{n} f^* C
\]
thus
\[
 K^2 = \sum \frac{(2n - 1 + k(1 - n))^2}{n^2} + \frac{(n - 1)^2}{n} C^2
\]
Since \( C^2 = \sum C_i^2 + 2 \sum_{i < j} C_i C_j = 2g - 2 + f_2 - f_1 \), we obtain
\[
 K_{X_n}^2/n^{d-2} = (2g - 2 + 3f_1 - 4f_0)n^2 + 4(f_0 - f_1 - g + 1)n - f_0 + f_1 + t_2 + 2g - 2.
\]
One can check that we always have $K_X^2 > 0$ and therefore $X_n$ has general type, and it is minimal since it covers an abelian surface and there are no $(-1)$ curves above $E_p \hookrightarrow Z$. Then we get by using the Miyaoka-Yau inequality

$$(3c_2 - K_Y^2)/n^{d-2} = (f_0 + 4g - 4)n^2 + 2(f_0 - f_1 - g + 1)n + 2f_1 + f_0 - 4t_2 - 2g + 2 \geq 0. \quad (4.3)$$

As in [7], we will use refinements of the Miyaoka-Yau inequality for surfaces that contain smooth rational curves and elliptic curves, ie for $n = 2$ or $3$. Let $Y$ be a surface of non negative Kodaira dimension. Suppose that there exists on $Y$ configurations $ADE$ of $(-2)$-curves and let $a_n$ (resp. $d_n, e_6, e_7, e_8$) denote the number of $A_n$ curves (resp. $D_n, E_n$). Suppose moreover that $Y$ contains smooth disjoint elliptic curves $D_j$ with no intersections with the $(-2)$-curves, then:

**Theorem 17.** (Hirzebruch [7] Theorem 3, Miyaoka [11]). We have

$$3c_2(Y) - K_Y^2 \geq 3 \sum_{n \geq 1} (n+1)(a_n + d_n + e_n) - \sum \frac{3a_n}{n+1} + \frac{3d_n}{4(n-2)} + \frac{3e_6}{24} + \frac{3e_7}{48} + \frac{3e_8}{120} - \sum (D_j)^2. \quad (4.3)$$

For $n = 3$, we get from equation (4.3)

$$14g - 14 + 8f_0 \geq 2f_1 + 2t_2.$$ 

Taking into account the fact that the surface contains $3^{d-3}t_3$ elliptic curves of self-intersection $-3$, we can refine that inequality into

$$14g - 14 + 8f_0 \geq 2f_1 + 2t_2 + t_3,$$

which is $-\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{2}t_3 - 7g + 7}{f_0} - 4$. For $g = 1$, ie a configuration $C$ of elliptic curves, let $P(C)$ be the singularity set of $C$, then:

$$H(C, P(C)) = -\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{2}t_3}{f_0} - 4.$$

For $P$ be arbitrary one can use the same demonstration as in [11] thm 3.3 for the linear $H$-constant of $P^2$ to conclude that

$$H(C) \geq \frac{t_2 + \frac{1}{2}t_3}{f_0} - 4.$$

Suppose that the bound $-4$ is attained, then $4f_0 = f_1$, $t_2 = t_3 = 0$ and from equality $\sum 7(k - 4)t_k = 2t_2 + t_3$, we see that $t_k = 0 \forall k \geq 5$. The only possibility is $t_4 \neq 0$, which indeed exists (see below).

For $n = 2$, we have from equation (4.3):

$$10g - 10 + 9f_0 \geq 2f_1 + 4t_2.$$ 

Moreover the surface $X_n$ contains $2^{d-3}t_3$ disjoint rational curves of self intersection $(-2)$, each one contributing for $\frac{9}{4}$ and it contains $t_42^{d-4}$ elliptic curves of self-intersection $-4$, therefore

$$3c_2 - K_Y^2)/2^{d-2} \geq \frac{9}{4}t_3 + t_4$$

and

$$10g - 10 + 9f_0 \geq 2f_1 + 4t_2 + \frac{9}{4}t_3 + t_4.$$ 

We obtain:

$$10g - 10 + t_2 + \frac{3}{4}t_3 \geq \sum_{k \geq 5} (2k - 9)t_k.$$ 

This is also

$$-\frac{f_1}{f_0} \geq \frac{2t_2 + \frac{9}{4}t_3 + \frac{1}{2}t_4 + 10 - 10g}{f_0} - \frac{9}{2}.$$
4.2. The Elliptic $H$-constants for Abelian surfaces. Let $X$ be a smooth projective surface and $\mathcal{L}$ be a configuration of smooth disjoint elliptic curves on $X$. Let us recall the following result:

**Theorem 18.** (Kobayashi) There exists a torsion free lattice $\Gamma \subset PU(2, 1)$ such that $(X, \mathcal{L})$ is a smooth compactification of the ball quotient surface $\mathbb{B}_2/\Gamma$ if and only if $(K_X + \mathcal{L})^2 = 3e(X \setminus \mathcal{L})$.

For $A$ an Abelian surface we obtain:

**Corollary 19.** An elliptic curve configuration $\mathcal{L}$ on $A$ is such that $H_{El}(P, \mathcal{L}) = -4$ if and only if $(X, \mathcal{L})$ is a smooth compactification of a ball quotient surface for $P = P(\mathcal{L})$ and $X \rightarrow A$ the blow-up at $P(\mathcal{L})$. Such configuration $\mathcal{L}$ has only 4-points and in that case, there exists a covering $X_3 \rightarrow A$ branched with order 3 over points in $[3]^4 \mathcal{L}$ such that $X_3$ is a smooth compact ball quotient surface $c_1^2(X_3) = 3c_2(X_3)$.

**Proof.** We have equality $(K + \mathcal{L})^2 = 3e(X \setminus \mathcal{L})$ in the Miyaoka-Sakai inequality if and only if $\sum (4 - k)t_k = 0$. Since for any $P$,

$$H_{El}(P, \mathcal{L}) = -\frac{\sum kt_k}{s} \leq H_{El}(P, \mathcal{L})$$

(with $s = \sum t_k$), we have $H_{El}(P, \mathcal{L}) = -4$ if and only if $\sum 4t_k = \sum kt_k$. By Theorem 18, $H(\mathcal{L}) \geq \frac{4t_1 + t_2}{4} - 4$, therefore we must have $t_2 = t_3 = 0 = t_k$, $\forall k \geq 5$. The configuration $\mathcal{L}$ has then only 4-points and $f_1 = 4f_0$. By equation [13] we know that the value of $(3c_2(X_n) - K_{X_n}^2)/n^{d-2}$ is $f_0(n - 3)^2$, therefore when $n = 3$, we obtain a compact ball quotient surface.

Let be $j = \frac{-1 + \sqrt{5}}{2}$ with $i^2 = -1$, we have:

**Proposition 20.** The elliptic $H$-constants of $(\mathbb{C}/\mathbb{Z}[i])^2$ and $(\mathbb{C}/\mathbb{Z}[j])^2$ are equal to $-4$.

**Proof.** Hirzebruch and Holzapfel found elliptic curves arrangements on $(\mathbb{C}/\mathbb{Z}[i])^2$ and $(\mathbb{C}/\mathbb{Z}[j])^2$ respectively that satisfies Kobayashi’s criteria on suitable blow-up. In the first example, one has 4 elliptic curves with $t_4 = 1$ and $t_k = 0$, $k \neq 4$, on the second one has 6 elliptic curves with $t_4 = 3$ and $t_k = 0$, $k \neq 4$.

**Remark 21.** Hirzebruch and Holzapfel examples are elliptic curve configurations with only 4-points on Abelian surfaces. That situation has to be compared with the plane where there do not exist configurations of $d > 1$ lines with $t_2 = t_3 = 0$ (by inequality [23]).

Let $E, E'$ be two elliptic curves and let $A$ be an Abelian surface.

**Proposition 22.** The $H$-constant and elliptic $H$-constant of an Abelian surface are invariants under isogeny. Suppose $A$ is isogeneous to $E \times E'$. If $E$ and $E'$ are not isogeneous, then $H_{El,A} = -2$. If $E$ and $E'$ are isogeneous, then $H_{El,A} \leq -3$.

**Proof.** Let us prove that $H_{El}$ is an invariant of the isogeny class of $A$. Let $A, B$ be two Abelian surfaces and let $\phi : A \rightarrow B$ be an isogeny; it is an étale map. Let $C$ be a (reduced) curve on $B$, and let $P$ be a set of points on $B$. By Lemma 13, $H_A(\phi^*P, \phi^*C) = H_B(C, P)$, thus

$$\min_{C,P} H_A(C, P) \leq \min_{C,P} H_B(C, P).$$

Since there exists an isogeny $\psi : B \rightarrow A$ too, we have the reverse inequality. That holds also for the elliptic $H$-constant, since the pull-back by an isogeny of a genus one curve is a union of genus one curves.
Suppose that \( E \) and \( E' \) are not isogeneous. Then a configuration \( C \) of elliptic curves on \( A \) has a decomposition
\[
C = \sum_{k=1}^{m} F_k + \sum_{k=1}^{n} F'_k
\]
where the \( F_k \) (resp. \( F'_k \)) are fibers of the fibrations of \( A \) onto \( E \) (resp. \( E' \)). Then one sees that \( \overline{C^2/s} \) is minimal and equals \(-2\) when the blow-up is taken over the \( mn \) intersection points of \( F_i \) and \( F'_j \).

Suppose that \( E \) and \( E' \) are isogeneous. Since the elliptic \( H \)-constant is isogeny invariant, we can suppose that \( E = E' \). Let \( \Delta \) be the diagonal in \( E \times E \) and let be \( F = \{ y = 0 \} \), \( F' = \{ x = 0 \} \), where \( x, y \) are the coordinates. Then \( C = \Delta + F + F' \) has one 3-point in 0, and in the blow-up at 0, we get \( \overline{C^2} = -3 \). \( \square \)

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