LOWER AND UPPER BOUNDS OF LAPLACIAN EIGENVALUE PROBLEM BY WEAK GALERKIN METHOD ON TRIANGULAR MESHES

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(Communicated by Xiaoming He)

ABSTRACT. In this paper, we investigate the weak Galerkin method for the Laplacian eigenvalue problem. We use the weak Galerkin method to obtain lower bounds of Laplacian eigenvalues, and apply a postprocessing technique to get upper bounds. Thus, we can verify the accurate intervals which the exact eigenvalues lie in. This postprocessing technique is efficient and does not need to solve any auxiliary problem. Both theoretical analysis and numerical experiments are presented in this paper.

1. Introduction. The eigenvalue problems, especially the Laplacian eigenvalue problems, are of great concern in sciences and engineering fields. The Laplacian eigenvalue problem is closely related to the resonance phenomenon and plays an essential role in industrial designing. The eigenvalue problem is also very important in many other branches, such as plasma physics in fusion experiments and astrophysics, petroleum reservoir simulation, linear stability of flows in fluid mechanics, and electronic band structure calculations, which has been reviewed in [7]. Simulating the Laplacian eigenvalue numerically is of great importance and has a lot of applications in physics.

Many numerical methods have been proposed for the Laplacian eigenvalue problem, such as finite difference method [12] and finite element method [2, 3]. The finite element method can be applied on many types of region, and is also very efficient. However, due to the minimal-maximum principle, the conformal finite element method always produce upper bounds of the eigenvalues. As we know, eigenvalues of the Laplace operator are real numbers, and if we have both upper bounds and lower bounds, we can get accurate intervals of the eigenvalues. To this end, many efforts have been dedicated to getting lower bounds of Laplacian eigenvalues.

For the lower bounds approximation problem, there are mainly two types of methods. One type is the post-processing methods [13, 16]. Most of the postprocessing methods need to solve the Laplacian eigenvalue problem by conforming
finite element method, and then solve an auxiliary problem to get an a posterior estimate, which leads to lower bound approximation. However, in this way we need to solve an auxiliary problem in addition to the finite element eigenvalue problem. The other type is the non-conforming methods. For the non-conforming methods, the finite element space \( V_h \not\in \mathcal{H}_1^0(\Omega) \), so that they are not restricted by the minimum-maximum principle. Many kinds of non-conforming elements are proposed for the lower bounds of eigenvalue problems, such as Crouzeix-Raviart element [1, 5], Rotated \( Q_1 \) element and Wilson’s element [14, 15]. Some criterions [8, 9, 11] for non-conforming element methods are also developed.

Recently, a new numerical method, called the weak Galerkin finite element method, was developed for the partial differential equations (PDEs). The main idea of weak Galerkin method is to use totally discontinuous functions as basis and replace the classical derivative operators by specifically defined weak operators. The weak Galerkin method has been applied to many different types of PDEs, such as second order elliptic equation [6, 18, 23, 24], biharmonic equation [19, 20, 29], Stokes equation [22, 25, 26, 28], and Maxwell equation [21]. For the Laplacian eigenvalue problem, it has been shown in [27] that the weak Galerkin method produces asymptotic lower bounds of Laplacian eigenvalue problem, and the finite element space can be of any order without special construction.

In this paper, we are concerned with the problem of getting both upper and lower bounds of Laplacian eigenvalues. A direct idea is to solve the same problem twice by conforming finite element and non-conforming method or WG method. As the eigenvalue problem is indeed a non-linear problem, and solving the problem twice may be time-costing. Another idea is to solve the eigenvalue problem by conforming finite element method or non-conforming method, and then use some post-processing technique based on a posteriori estimate [4, 17]. However, it is still necessary to solve an extra linear problem. In [10], a post-processing technique based on interpolation was proposed. The main idea of [10] is to solve the Laplacian eigenvalue problem by non-conforming method to obtain a lower bound, then map the eigenvector to the conforming finite element space by interpolation, and calculate the Rayleigh quotient to get an upper bound. By this method, one only needs to solve the eigenvalue problem once and calculate the Rayleigh quotient, which can be very efficient. In this paper, we follow the idea in [10] and apply this method to the weak Galerkin method.

The main contribution of our method is two-fold. On the one hand, by this method we can get both upper and lower bounds of Laplacian eigenvalues, and determine rather accurate intervals which the exact eigenvalues lie in. The calculation cost is almost the same as one eigenvalue problem, which is very efficient. On the other hand, our method works with high order finite element spaces. We can get high order approximations without constructing finite element spaces specifically.

This paper is organized as follows. In Section 2, we introduce the weak Galerkin method for the Laplacian eigenvalue problem, and recall the estimates in [27]. Section 3 is devoted to the algorithm and theoretical analysis for the upper bounds. In Section 4, we present some numerical experiments to verify the analysis in this paper. Conclusion and future works are included in Section 5.

2. Lower bounds. In this section, we introduce the weak Galerkin method for the Laplacian eigenvalue problem. We recall the weak Galerkin algorithm and the asymptotic lower bound estimates in [27].
In this paper, we consider the Laplacian eigenvalue problem with Dirichlet boundary condition:
\begin{align}
-\Delta u &= \lambda u, \quad \text{in } \Omega, \\
\quad u &= 0, \quad \text{on } \partial\Omega, \\
\int_{\Omega} u^2 &= 1,
\end{align}
where $\Omega$ is a polygonal domain in $\mathbb{R}^2$. The classical variational form of (2.1)-(2.2) is: Find $\lambda \in \mathbb{R}$, $u \in H_0^1(\Omega)$, such that $b(u, u) = 1$ and $a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega)$,

where $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$, and $b(u, v) = \int_{\Omega} uv$.

First, we need to state some notations. We use the standard notations of Sobolev spaces. $||\cdot||_{m,D}$ and $|\cdot|_{m,D}$ stand for the norm and semi-norm of $H^m(D)$ space, and $(\cdot,\cdot)_{m,D}$ represents the inner-product on $H^m(D)$. When $D$ is an edge or boundary, we also use $(\cdot,\cdot)_{m,D}$ to represent the inner-product. When $D = \Omega$ or $m = 0$, we shall drop the subscript.

For a polygonal domain $\Omega$, suppose $T_h$ is a triangular mesh satisfying regular assumptions verified in [24], and we let $E_h$ be the union of edges in $T_h$. We denote $P_T = \{v|T \in P_k(T), \forall T \in T_h\}$ the union of all the piecewise polynomials on each $T$ whose degree is no more than $k$, and $P_E = \{v|e \in P_{k-1}(e), \forall e \in E_h\}$ the union of all the piecewise polynomials on each $e$ whose degree is no more than $k - 1$. Denote $Q_h$ the $L^2$ projection onto $P_E$. For any element $T \in T_h$, $h_T$ denotes the diameter of $T$, and the mesh size $h$ is given by $h = \max_{T \in T_h} h_T$. In this paper, $a \lesssim b$ means $a \leq Cb$ for positive numbers $a$ and $b$, where $C$ is a constant irrelevant to $h$.

Let $V_h \subset V$ be the $k$-th degree conforming finite element space whose set of nodes is $E_h$. For each $A \in E_h$, let $K(A)$ be the set of elements containing $A$, and $N_A$ the number of elements in $K(A)$. For any $T \in T_h$, define $K(T)$ the union of elements which have the same vertex as $T$.

For a given partition $T_h$, we introduce the weak Galerkin finite element space
\[ V_h = \{(v_0, v_b) : v_0 \in P_k(T), v_b \in P_{k-1}(e), \forall T \in T_h, e \in E_h, v_b = 0 \text{ on } \partial\Omega\} \]
It should be noticed that $v_b$ has a single value on each $e$, and is not necessary to be the trace of $v_0$. The weak Galerkin finite element space is made up of discontinuous functions, and we use the following weak gradient instead of the classical gradient operator in the weak Galerkin algorithm.

**Definition 2.1.** For any $v = \{v_0, v_b\} \in V_h$, define $\nabla_w v$ to be the piecewise polynomial such that, for every element $T$, $\nabla_w v|T$ is in $[P_{k-1}(T)]^2$ uniquely determined by
\[ (\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_{k-1}(T)]^2, \]
where $n$ is the outward unit normal vector of $T$.

In order to propose the weak Galerkin algorithm, we need to define three bilinear forms on $V_h$ for any $v, w \in V_h$,
\begin{align}
s(v, w) &= \sum_{T \in T_h} h^{-1+\epsilon} (Q_h v_0 - v_b, Q_h w_0 - w_b)_{\partial T}, \\
a_w(v, w) &= (\nabla_w v, \nabla_w w) + s(v, w), \\
b_w(v, w) &= (v_0, w_0),
\end{align}
where $0 < \varepsilon < 1$ is a small parameter to be determined. The selection of $\varepsilon$ will be discussed in Remark 2.1.

Denote $V = H^1_0(\Omega)$, for any $v, w \in V + V_h$ define the following inner-product and norm on $V + V_h$ that

\[
(v, w)_V = (\nabla v_0, \nabla w_0) + \sum_{T \in T_h} h_T^{-1} (v_0 - v_b, w_0 - w_b)_{\partial T},
\]

\[
\|v\|_V^2 = \|\nabla v_0\|^2 + \sum_{T \in T_h} h_T^{-1} \|v_0 - v_b\|^2_{\partial T}.
\]

When $\|\cdot\|_V = 0$, we have $\|\nabla v_0\| = 0$ on each $T$ and $v_0 = v_b$ on each edge $e$, which implies $v$ is globally continuous and constant. From the vanishing boundary condition, it follows that $v = 0$, and $\|\cdot\|_V$ indeed defines a norm on $V + V_h$. In particular, when restricted on $V$, $\|\cdot\|_V$ coincides with $|\cdot|_1$.

Now, we can introduce the weak Galerkin scheme for the Laplacian eigenvalue problem as follows.

**Weak Galerkin Algorithm 1.** Find $\lambda_h \in \mathbb{R}$, $u_h \in V_h$ such that $b_{u}(u_h, u_h) = 1$ and

\[
a_{u}(u_h, v) = \lambda_h b_{u}(u_h, v), \quad \forall v \in V_h.
\]

For the weak Galerkin Algorithm 1, the following error estimate and asymptotic lower bound estimate hold true. The detailed proof can be found in Theorem 4.7 and Theorem 5.3 in [27].

**Theorem 2.1.** Assume the eigenfunctions of (2.1)-(2.3) have $H^{k+1}(\Omega)$-regularity. Suppose $\lambda_{j,h}$ is the $j-th$ eigenvalue of Algorithm 1 and $u_{j,h}$ is the corresponding eigenvector. There exist an exact eigenvalue $\lambda_j$ and the corresponding exact eigenfunction $u_j$ such that when $h$ is sufficiently small, the following error estimates hold

\[
0 \lesssim \lambda_j - \lambda_{j,h} \lesssim h^{2k - 2\varepsilon},
\]

\[
\|u_j - u_{j,h}\|_V \lesssim h^{k - \varepsilon},
\]

\[
\|u_j - u_{j,h}\| \lesssim h^{k+1 - \varepsilon}.
\]

For triangular meshes, define $I_h : C^0(\Omega) \to \tilde{V}_h$ the interpolation operator. From the standard interpolation theory we have the following estimate.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, we have the following estimate

\[
\|I_h u_j - u_{j,h}\|_V \lesssim h^{k-\varepsilon},
\]

\[
\|I_h u_j - u_{j,h}\| \lesssim h^{k+1 - \varepsilon}.
\]

**Remark 2.1.** The constant $C$ in the error estimates is irrelevant to $\varepsilon$. But when $\varepsilon > 0$, $\lambda_{j,h}$ is an asymptotic lower bound of $\lambda_j$, i.e. $\lambda_j - \lambda_{j,h} > 0$ for sufficiently small $h$.

3. **Upper bounds.** In this section, we shall propose an algorithm to get an upper bound of the Laplacian eigenvalue problem. The eigenvalue of Algorithm 1 is a lower bound of the exact eigenvalue, and we can calculate an upper bound by the following procedure. The idea of the algorithm comes from [10] but some modifications for the weak Galerkin method.
First, we need to define an interpolation operator \( \Pi_h : V_h \to \tilde{V}_h \). The definition of interpolation operator \( \Pi_h \) is as follows. For every \( v_h \in V_h \), we let \( \Pi_h v_h \) be the finite element function in \( \tilde{V}_h \) such that

\[
\Pi_h v_h(A) = \frac{1}{|K(A)|} \sum_{T \in K(A)} v_0|_T(A), \quad \forall A \in \tilde{E}_h.
\]

On the domain \( \Omega \), the value of \( \Pi_h v_h \) is determined by interpolation.

Now, we introduce the following algorithm to obtain upper bounds of Laplacian eigenvalue problem by the weak Galerkin method.

**Weak Galerkin Algorithm 2.**

**Step 1.** Find \( \lambda_h \in \mathbb{R}, \ u_h \in V_h \) such that

\[
a_w(u_h, v) = \lambda_h b_w(u_h, v), \quad \forall v \in V_h.
\]

**Step 2.** Calculate \( \tilde{u}_h = \Pi_h u_h \).

**Step 3.** Calculate the Rayleigh quotient

\[
\tilde{\lambda}_h = \frac{a(\tilde{u}_h, \tilde{u}_h)}{b(\tilde{u}_h, \tilde{u}_h)}.
\]

In order to get the upper bound estimate, we still need some technical results. The relationship of \( \Pi_h v_h \) and \( v_h \) are revealed in the following lemma.

**Lemma 3.1.** For any \( v_h \in V_h \), we have the following estimates

\[
|\Pi_h v_h|_1 \lesssim \|v_h\|_V, \\
\|\Pi_h v_h\| \lesssim \|v_h\|.
\]

**Proof.** For any \( T \in \mathcal{T}_h \), define \( K(T) \) the union of elements which have the same vertex as \( T \). We only need to prove that

\[
|\Pi_h v_h|_{1,T} \lesssim \|v_h\|_{V,K(T)}.
\]

Summing (3.1) over \( T \in \mathcal{T}_h \) and we get the conclusion.

Define \( \tilde{T} \) the reference element, and \( P : T \to \tilde{T} \) the affine isomorphism. Denote \( K(\tilde{T}) = P(K(T)) \). It follows from the regularity assumption of mesh that \( K(\tilde{T}) \) is also of unit size.

Consider the Banach space \( \tilde{V}_h = V_h|_{K(\tilde{T})} \) and \( M = \{ \tilde{v}_h = (\tilde{v}_0, \tilde{v}_h) : \tilde{v}_h \in \tilde{V}_h, \int_{\tilde{T}} \tilde{v}_0 = 0 \} \). Next we prove that the complement of \( M \) in \( \tilde{V}_h \) is \( M^\perp = \{ \tilde{v}_h : \tilde{v}_h \in \tilde{V}_h, \tilde{v}_h \) is constant on \( K(\tilde{T}) \} \).

Define \( N = \{ \tilde{v}_h : \tilde{v}_h \in \tilde{V}_h, \tilde{v}_h \) is constant on \( K(\tilde{T}) \} \). For any \( \tilde{v}_h \in \tilde{V}_h \), denote \( \tilde{v}_h = \int_{\tilde{T}} \tilde{v}_h \), i.e., on each element both \( \tilde{v}_0 \) and \( \tilde{v}_h \) equal to \( \int_{\tilde{T}} \tilde{v}_h \). Denote \( \tilde{v}_h = \tilde{v}_h - \tilde{v}_h \). Since \( \tilde{v}_h \) is a number, and \( \int_{\tilde{T}} \tilde{v}_h = 0 \), so we have \( \tilde{v}_h \in M \) and \( \tilde{v}_h \in N \). Thus, for any \( \tilde{v}_h \in \tilde{V}_h \), one can decompose \( \tilde{v}_h \) as

\[
\tilde{v}_h = \tilde{v}_h + \tilde{v}_h, \quad \tilde{v}_h \in M, \tilde{v}_h \in N.
\]

From the definition of \( \langle \cdot, \cdot \rangle_V \) we have

\[
(\tilde{v}_h, \tilde{v}_h)_{V,K(\tilde{T})} = (\nabla \tilde{v}_0, \nabla \tilde{v}_0) + \sum_{T \in K(\tilde{T})} \langle \tilde{v}_0 - \tilde{v}_h, \tilde{v}_0 - \tilde{v}_h \rangle_{\partial T} = 0,
\]

i.e., the decomposition is an orthogonal decomposition. Then, we have \( N = M^\perp \).
Notice that \(|\Pi_h \hat{v}_h|_{1,T}\) defines a seminorm on \(M\) and \(\|\hat{v}_h\|_{V,K(\hat{T})}\) defines a norm on \(M\), from the equivalence of norms on finite dimensional Banach spaces we can obtain
\[
|\Pi_h \hat{v}_h|_{1,T} \lesssim \|\hat{v}_h\|_{V,K(\hat{T})}, \quad \forall \hat{v}_h \in M.
\]
Furthermore, since \(|\Pi_h \hat{v}_h|_{1,T} = \|\hat{v}_h\|_{V,K(\hat{T})} = 0\) for all \(\hat{v}_h \in \hat{M}\), we have
\[
|\Pi_h \hat{v}_h|_{1,T} \lesssim \|\hat{v}_h\|_{V,K(\hat{T})}, \quad \forall \hat{v}_h \in \hat{M}.
\]
From the property of affine isomorphism we can obtain
\[
|\Pi_h v_h|_{1,T} \lesssim |\Pi_h \hat{v}_h|_{1,T} \lesssim \|\hat{v}_h\|_{V,K(\hat{T})} \lesssim \|v_h\|_{V,K(T)},
\]
which completes the proof. The proof for \(L^2\) norm is similar. \(\square\)

We also need some approximation property of interpolation operator \(\Pi_h\), which is verified in the following lemma.

**Lemma 3.2.** For any \(v_h \in V_h\), we have the following estimate
\[
\|v_h - \Pi_h v_h\| \lesssim h\|v_h\|_V.
\]

**Proof.** We only need to prove that
\[
\|v_h - \Pi_h v_h\|_T \lesssim h\|v_h\|_{V,K(T)}. \tag{3.2}
\]

On an element \(T\) denote by \(A_0\) the interior interpolation points, and \(A_b = \{A_{b,1}, \ldots, A_{b,N_b}\}\) the boundary interpolation points. Notice that for any \(A \in A_0\), \(v_h(A) = \Pi_h v_h(A)\), so we have
\[
\|v_h - \Pi_h v_h\|^2_T \lesssim \int_T \sum_{i=1}^{N_b} |v_0(A_{b,i}) - \Pi_h v_0(A_{b,i})|^2 \varphi_{i,T}^2 dT
\]
\[
\lesssim h^2 \sum_{i=1}^{N_b} |v_0(A_{b,i}) - \Pi_h v_0(A_{b,i})|^2,
\]
where \(\varphi_{i}\) is the Lagrange basis function corresponding to \(A_{b,i}\).

For a given \(A_{b,i}\), denote \(\{T_1, T_2, \ldots, T_{N_i}\}\) the elements adjoint to \(A_{b,i}\) in counter-clock order. From the definition of \(\Pi_h v_h\) we can obtain
\[
|v_0(A_{b,i}) - \Pi_h v_0(A_{b,i})| \leq \frac{1}{N_i} \sum_{j=1}^{N_i} |v_0|_{T_j} (A_{b,i}) - v_0|_{T_j} (A_{b,i})|
\]
\[
\leq \frac{1}{N_i} \sum_{j=1}^{N_i} \sum_{k=1}^{j} |v_0|_{T_k} (A_{b,i}) - v_0|_{T_{k-1}} (A_{b,i})|.
\]
From the \(L^{\infty} - L^2\) inverse inequality, it follows that
\[
|v_0|_{T_k} (A_{b,i}) - v_0|_{T_{k-1}} (A_{b,i})| \lesssim h^{-\frac{3}{2}} \|[v_0]\|_e.
\]
where $e$ is the edge between $T_k$ and $T_{k-1}$. Thus, we have
\[
\|v_h - \Pi_h v_h\|_T^2 \\
\lesssim h \sum_{e \in K(T)} \|[v_0]\|_e^2 \\
\lesssim h \sum_{T \in K(T)} \|v_0 - v_b\|_{\partial T}^2 \\
\lesssim h^2 \|v_h\|_V^2,
\]
which completes the proof.

For the eigenvalue approximation problem, the following identity plays an essential role in the estimate, which was proposed in [3].

**Lemma 3.3.** Suppose $(\lambda, u)$ satisfies
\[
a(u, v) = \lambda b(u, v), \quad \forall v \in V.
\]
Then for any $w \in V$, $b(w, w) \neq 0$ we have
\[
a(w, w) - \lambda = \frac{a(w - u, w - u)}{b(w, w)} - \lambda \frac{b(w - u, w - u)}{b(w, w)}.
\]

With these preparations, we finally reach the following upper bound estimate.

**Theorem 3.1.** Suppose the exact eigenfunctions of problem (2.1)-(2.3) $u_j \in H^{k+1}(\Omega)$ are not polynomials of degree $k$. If $\lambda_{j,h}$ is the $j$-th eigenvalue of Algorithm 2, then there exists an exact eigenvalue $\lambda_j$ such that when $h$ is small enough, the following error estimate holds
\[
0 \leq \lambda_{j,h} - \lambda_j \lesssim h^{2k-2\varepsilon}.
\]

*Proof.* First, we need to estimate the eigenfunction $\tilde{u}_h$. From Corollary 2.1 we know that there exists an exact eigenfunction $u$ such that
\[
\|u_h - I_h u\|_V \lesssim h^{k-\varepsilon},
\]
\[
\|u_h - I_h u\| \lesssim h^{k+1-\varepsilon}.
\]

From the triangle inequality and Lemma 3.1 we have
\[
|u - \tilde{u}_h|_1 \lesssim h^{k-\varepsilon}, \quad (3.3)
\]
\[
|u - I_h u|_1 + |I_h u - \Pi_h u_h|_1 \\
= |u - I_h u|_1 + |\Pi_h I_h u - \Pi_h u_h|_1 \\
\lesssim |u - I_h u|_1 + \|I_h u - u_h\|_V \\
\lesssim h^{k-\varepsilon}.
\]

From Theorem 2.1 in [15], we also have when $u \not\in P_k(T)$ on each element $T$,
\[
|u - \tilde{u}_h|_1 \gtrsim h^k. \quad (3.4)
\]

Similarly, for $L^2$ norm we have
\[
\|u - \tilde{u}_h\|_2 \lesssim \|u - I_h u\| + \|I_h u - \Pi_h u_h\| \\
= \|u - I_h u\| + \|\Pi_h I_h u - \Pi_h u_h\| \\
\lesssim \|u - I_h u\| + \|I_h u - u_h\|_V.
\]

[15]
\[ h^{k+1-\varepsilon}. \]

When \( h \) is sufficiently small, from Lemma 3.2 we also have

\[
\| \tilde{u}_h \| \geq \| u_h \| - \| u_h - \tilde{u}_h \| \geq \| u_h \| - C \| u_h - \tilde{u}_h \| \geq \| u_h \| - Ch \| u_h \|_V \geq 1/2, \]

and

\[
\| \tilde{u}_h \| \leq \| u_h \| + \| u_h - \tilde{u}_h \| \leq \| u_h \| + Ch \| u_h \|_V \leq 2. \]

Notice that \( \tilde{u}_h \in V \), and \( a_w(\tilde{u}_h, \tilde{u}_h) = a(\tilde{u}_h, \tilde{u}_h) \), it follows Lemma 3.3 that

\[
\hat{\lambda}_h - \lambda = a(\tilde{u}_h, \tilde{u}_h) - \lambda = \frac{a(\tilde{u}_h - u, \tilde{u}_h - u)}{b(\tilde{u}_h, \tilde{u}_h)} - \lambda \frac{b(\tilde{u}_h - u, \tilde{u}_h - u)}{b(\tilde{u}_h, \tilde{u}_h)} = \frac{|\tilde{u}_h - u|^2 - \lambda |\tilde{u}_h - u|^2}{\| \tilde{u}_h \|^2}. \]

When \( h \) is sufficiently small, from (3.3)-(3.7) we can obtain

\[ 0 \leq \hat{\lambda}_h - \lambda \lesssim h^{2k-2\varepsilon}, \]

which completes the proof. \( \Box \)

**Remark 3.1.** If the exact solution \( u \) is polynomial of degree \( k \) on \( \Omega \), then \( u \) lies in the finite element space \( V_h \). In this case, the numerical solution coincides with the exact solution and the conclusion of Theorem 3.1 still holds true.

4. Numerical experiments. In this section, we shall present some numerical examples to verify the validity of the theoretical analysis in the previous sections.

4.1. Example 1. In the first example, we shall solve problems (2.1)-(2.3) on a square domain \( \Omega = (0, \pi) \times (0, \pi) \). The uniform triangular mesh is employed, and we select the parameter \( \varepsilon = 0.1 \). The first four eigenvalues of problems (2.1)-(2.3) is known as 2, 5, 5, 8, respectively. The numerical results of Algorithm 1 and Algorithm 2 are listed in Tables 1-2. In Tables 1, the degree of polynomial \( k = 1 \), and \( k = 2 \) in Tables 2. The numerical results coincide the theoretical predicts.

4.2. Example 2. In this example, we shall solve problems (2.1)-(2.3) on a L-shape domain \( \Omega = (-1,1)^2 \setminus (0,1)^2 \). The uniform triangular mesh is employed. We select the parameter \( \varepsilon = 0.1 \). The analytic solution on L-shape domain is unknown, so we just list the first four eigenvalues of Algorithm 1 and Algorithm 2 in Tables 3-4 to show the lower bound and upper bound. The numerical results for \( k = 1 \) and \( k = 2 \) are listed in Table 3 and Table 4, respectively. On the L-shape domain, the exact eigenfunctions does not have \( H^2(\Omega) \)-regularity, but from Tables 3-4 we still get both the lower and upper bounds of the eigenvalues.
Table 1. Numerical results for the eigenvalues with $k = 1$.

| $h$  | $1/4$   | $1/8$   | $1/16$  | $1/32$  | $1/64$  | $1/128$ |
|------|---------|---------|---------|---------|---------|---------|
| $\lambda_{1,h}$ | 1.540722 | 1.851769 | 1.958022 | 1.988587 | 1.996933 | 1.999178 |
| rate | 1.6315  | 1.8202  | 1.8790  | 1.9055  | 1.9001  | 2       |
| $\tilde{\lambda}_{1,h}$ | 2.223673 | 2.068547 | 2.018267 | 2.004693 | 2.001189 | 2.000299 |
| rate | 1.7062  | 1.9079  | 1.9606  | 1.9809  | 1.9905  | 2       |
| $\lambda_{2,h}$ | 2.821831 | 4.135190 | 4.734921 | 4.926297 | 4.980094 | 4.994667 |
| rate | 1.3327  | 1.7060  | 1.8466  | 1.8885  | 1.9001  | 5       |
| $\tilde{\lambda}_{2,h}$ | 6.588245 | 5.484282 | 5.125546 | 5.031824 | 5.008012 | 5.002010 |
| rate | 1.6315  | 1.8202  | 1.8790  | 1.9055  | 1.9001  | 2       |
| $\lambda_{3,h}$ | 2.836209 | 4.142792 | 4.737301 | 4.926955 | 4.980270 | 4.994713 |
| rate | 1.3358  | 1.7062  | 1.8465  | 1.8884  | 1.8999  | 5       |
| $\tilde{\lambda}_{3,h}$ | 5.909723 | 5.304362 | 5.079723 | 5.020284 | 5.005121 | 5.001287 |
| rate | 1.7062  | 1.9476  | 1.9800  | 1.9884  | 1.9905  | 2       |
| $\lambda_{4,h}$ | 3.681649 | 4.973361 | 4.998298 | 4.999888 | 4.999992 | 4.999994 |
| rate | 1.3327  | 1.7062  | 1.8465  | 1.8884  | 1.8999  | 5       |
| $\tilde{\lambda}_{4,h}$ | 6.352000 | 5.023431 | 5.000784 | 5.000034 | 5.000008 | 5.0000008 |
| rate | 1.6315  | 1.8202  | 1.8790  | 1.9055  | 1.9001  | 2       |

Table 2. Numerical results for the eigenvalues with $k = 2$.

| $h$  | $1/4$   | $1/8$   | $1/16$  | $1/32$  | $1/64$  | $1/128$ |
|------|---------|---------|---------|---------|---------|---------|
| $\lambda_{1,h}$ | 1.977623 | 1.998594 | 1.99907 | 1.999994 | 1.999999 | 1.99999997 |
| rate | 3.9922  | 3.9260  | 3.9060  | 3.9006  | 3.8999  | 2       |
| $\tilde{\lambda}_{1,h}$ | 2.047797 | 2.001416 | 2.000060 | 2.000003 | 2.000000000000 | 2       |
| rate | 5.0796  | 4.5640  | 4.3668  | 4.2495  | 4.1640  | 2       |
| $\lambda_{2,h}$ | 4.562639 | 4.973361 | 4.998298 | 4.999888 | 4.999992 | 4.9999994 |
| rate | 4.0372  | 3.9683  | 3.9193  | 3.9049  | 3.9011  | 5       |
| $\tilde{\lambda}_{2,h}$ | 5.965094 | 5.205528 | 5.000095 | 5.000048 | 5.00003 | 5.000000000000 |
| rate | 5.2405  | 4.6807  | 4.3769  | 4.2375  | 4.1526  | 5       |
| $\lambda_{3,h}$ | 4.656697 | 4.980159 | 4.998739 | 4.999917 | 4.999994 | 4.999996 |
| rate | 4.1130  | 3.9757  | 3.9208  | 3.9056  | 3.9016  | 5       |
| $\tilde{\lambda}_{3,h}$ | 6.352000 | 5.023431 | 5.000784 | 5.000034 | 5.000008 | 5.000000000000 |
| rate | 5.8505  | 4.9017  | 4.5228  | 4.3518  | 4.2405  | 5       |
| $\lambda_{4,h}$ | 6.423791 | 7.902345 | 7.993940 | 7.999603 | 7.999974 | 7.999998 |
| rate | 4.0126  | 4.0103  | 3.9302  | 3.9073  | 3.9011  | 8       |
| $\tilde{\lambda}_{4,h}$ | 14.174453 | 8.132512 | 8.004208 | 8.000190 | 8.000010 | 8.000001 |
| rate | 5.5421  | 4.9768  | 4.4660  | 4.2751  | 4.1726  | 8       |

5. Conclusion. In this paper, we study the weak Galerkin method for the Laplacian eigenvalue problem. The weak Galerkin method is applied to obtain lower bounds, and we use a post-processing method based on interpolation to get upper bounds. The method can be applied to high order elements and some numerical experiments are presented. In the future works, we shall combine the weak Galerkin method with other tools, such as two-grid method or PPR, and solve more types of eigenvalue problems.
Table 3. Numerical results for the Lshape domain with $k = 1$.

| $h$   | 1/4  | 1/8  | 1/16 | 1/32 | 1/64 | 1/128 |
|-------|------|------|------|------|------|-------|
| $\lambda_{1,h}$ | 5.907113 | 8.154417 | 9.155869 | 9.490441 | 9.593250 | 9.624793 |
| $\tilde{\lambda}_{1,h}$ | 11.656280 | 10.498323 | 9.908178 | 9.722418 | 9.666084 | 9.648513 |
| $\lambda_{2,h}$ | 8.201188 | 12.308821 | 14.293314 | 14.943779 | 15.128548 | 15.178811 |
| $\tilde{\lambda}_{2,h}$ | 17.974323 | 16.396266 | 15.533543 | 15.284908 | 15.219633 | 15.201931 |
| $\lambda_{3,h}$ | 9.444494 | 15.205676 | 18.276194 | 19.324905 | 19.626571 | 19.709394 |
| $\tilde{\lambda}_{3,h}$ | 26.262083 | 22.353621 | 20.453712 | 19.923453 | 19.785982 | 19.750997 |
| $\lambda_{4,h}$ | 11.124165 | 20.258292 | 26.287193 | 28.580159 | 29.263739 | 29.452134 |
| $\tilde{\lambda}_{4,h}$ | 44.270371 | 34.588644 | 30.891333 | 29.870772 | 29.609766 | 29.543696 |

Table 4. Numerical results for the Lshape domain with $k = 2$.

| $h$   | 1/4  | 1/8  | 1/16 | 1/32 | 1/64 | 1/128 |
|-------|------|------|------|------|------|-------|
| $\lambda_{1,h}$ | 9.076541 | 9.556156 | 9.615812 | 9.630832 | 9.636223 | 9.638331 |
| $\tilde{\lambda}_{1,h}$ | 12.345634 | 10.075057 | 9.790403 | 9.698357 | 9.662955 | 9.648961 |
| $\lambda_{2,h}$ | 13.584241 | 15.097385 | 15.190417 | 15.196722 | 15.197204 | 15.197247 |
| $\tilde{\lambda}_{2,h}$ | 22.666873 | 15.346520 | 15.202918 | 15.197605 | 15.197285 | 15.197256 |
| $\lambda_{3,h}$ | 16.280287 | 19.517968 | 19.725325 | 19.738296 | 19.739148 | 19.739205 |
| $\tilde{\lambda}_{3,h}$ | 35.044030 | 20.076880 | 19.750149 | 19.739700 | 19.739234 | 19.739210 |
| $\lambda_{4,h}$ | 19.966348 | 28.777512 | 29.476177 | 29.518504 | 29.521278 | 29.521467 |
| $\tilde{\lambda}_{4,h}$ | 74.081123 | 30.746432 | 29.554518 | 29.522929 | 29.521560 | 29.521486 |

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Received November 2016; revised October 2017.

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