Research Article

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Applications of some operators on supra topological spaces

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Abstract: In this paper, the notion of an operator $y$ on a supra topological space $(X, \mu)$ is studied and then utilized to analyze supra $y$-open sets. The notions of $\mu_y$-$g$-closed sets on the subspace are introduced and investigated. Furthermore, some new $\mu_y$-separation axioms are formulated and the relationships between them are shown. Moreover, some characterizations of the new functions via operator $y$ on $\mu$ are presented and investigated. Finally, we give some properties of $S_{y,g}$-closed graph and strongly $S_{y,g}$-closed graph.

Keywords: supra topology, operator $y$ on $\mu$, $\mu_y$-$g$-closed sets, operator on subspace ST, $\mu_y$-separation axioms, $S_{y,g}$-continuous functions, $S_{y,g}$-closed and strongly $S_{y,g}$-closed graphs

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1 Introduction

Kasahara [1] defined an operator associated with a topology, namely, an $\alpha$ operator. He initiated some concepts that are equivalent to those given in topological spaces when the operator is the identity operator. Also, he studied $\alpha$-closed graphs of $\alpha$-continuous functions and $\alpha$-compact spaces. Then, Jankovic [2] used $\alpha$ operator to introduce $\alpha$-closure of a set and give some characterizations on $\alpha$-closed graph of functions. Later, Ogata [3] defined the notion of $y$-open sets to study operator-functions and operator-separation. Rosas and Vielma [4] investigated some features of operator-compact spaces and defined the concept of operator-connected spaces. Kalaivani and Krishnan [5] formulated the concept of $\alpha$-$y$-open sets in a topological space and studied their corresponding closure and interior operators. Quite recently, many notions of operators have been investigated on different classes of open sets and generalizations of open sets; see [6–15]. In 1983, Mashhour et al. [16] introduced supra topological spaces (STSs) by neglecting an intersection condition of topology. This makes supra topology (ST) more flexible to describe some real-life problems (see, [17]) and construct easily some examples that show the relationships between certain topological concepts. Al-shami [18] investigated the classical topological notions such as limit points of a set, compactness, and separation axioms on the STSs. Investigation of several notions of compactness and Lindelöfness was the goal of some papers such as [19–22]. Al-shami [23] introduced the concept of paracompactness on STSs and explored main properties. Recently, the authors of [24–27] have employed some generalizations of supra open sets given in [28–31] to study limit points and separation axioms on STSs. They have provided
various interesting examples to explain the given relationships and results. In [32–34], some new concepts and notions were introduced using supra b-open sets and supra D-open sets.

This paper is organized as follows: after this introduction, we recall some basic definitions that are necessary to understand this work. In Section 3, an operator γ depending on supra open sets is studied and then employed to analyze supra γ-open sets. In Section 4, we introduce and discuss μγ-g-closed sets on subspace ST. In Section 5, some new μγ-separation axioms are formulated using the operator γ on μ and the relationships between them are elucidated. In Section 6, some new classes of functions are defined and some characterizations of these functions are given. In Section 7, two new classes of closed graphs are studied and some relations and properties are obtained. Section 8 concludes the paper with summary.

2 Preliminaries

Let X be a non-empty set and P(X) be the power set of X.

**Definition 2.1.** [16] A subfamily μ of P(X) is called an ST if it is closed under arbitrary union and X is a member of μ.

Then the pair (X, μ) is called an STS. Terminologically, a member of μ is called a supra open set and its complement is called a supra closed set.

**Definition 2.2.** [16] For A ⊆ (X, μ), intμ(A) is the union of all supra open sets that are contained in A and clμ(A) is the intersection of all supra closed sets containing A.

**Remark 2.3.** μ is called an associated ST with a topology τ if τ ⊆ μ.

**Definition 2.4.** [16,18] An STS (X, μ) is said to be:

(i) supra T0 (briefly, S-T0) if ∀x ≠ y ∈ X, ∃U ∈ μ such that either x ∈ U and y /∈ U or y ∈ U and x /∈ U;
(ii) supra T1 (briefly, S-T1) if ∀x ≠ y ∈ X, ∃U, V ∈ μ with x ∈ U but y /∈ U and y ∈ V but x /∈ V;
(iii) supra T2 (briefly, S-T2) if ∀x ≠ y ∈ X, ∃U, V ∈ μ with x ∈ U, y ∈ V and U ∩ V = φ;
(iv) supra regular if for every supra closed set F and every a /∈ F, there exist disjoint supra open sets U and V containing F and a, respectively;
(v) supra normal if for every disjoint supra open sets F and H, there exist disjoint supra open sets U and V containing F and H, respectively;
(vi) S-T3 (resp. S-T4) if it is both supra regular (resp. supra normal) and S-T2.

**Definition 2.5.** [16] The graph G(f) of a function f : X → Y is called S-closed if ∀(x, y) ∈ (X × Y) \ G(f), there exist two supra open sets U in X and V in Y with x ∈ U, y ∈ V and (U × V) ∩ G(f) = φ.

3 Supra γ-open sets and operators

In this section, we introduce and study the concept of γ operator on an ST. Then, we define supra γ-regular and supra open operators and investigate main properties. We construct some examples to show the obtained results.

**Definition 3.1.** Let (X, μ) be an STS. An operator γ on an ST μ is a mapping from μ to P(X) such that U ⊆ γ(U) ∀U ∈ μ, where γ(U) denotes the value of γ at U. This operator will be denoted by γ : μ ↦ P(X).
Definition 3.2. Let \((X, \mu)\) be an STS and \(\gamma : \mu \to P(X)\) be an operator on \(\mu\). A non-empty set \(A\) of \(X\) is called supra \(\gamma\)-open if \(\forall x \in A, \exists U \in \mu\) with \(x \in U \subseteq \gamma(U) \subseteq A\).

Suppose that the empty set \(\phi\) is also supra \(\gamma\)-open for any operator \(\gamma : \mu \to P(X)\).

We denote the class of all supra \(\gamma\)-open subsets of an STS \((X, \mu)\) by \(\mu_{\gamma}\).

The identity operator \(id\) on \(\mu\) is a mapping \(id : \mu \to P(X)\) such that \(V^{id} = V\) for every \(V \in \mu\). This leads to that a subset \(A\) is \(\mu_{id}\)-open of \(X\) iff \(A\) is supra open in \(X\). In other words, \(\mu_{id} = \mu\).

Remark 3.3. In fact, if \(A \in \mu_{\gamma}\), then \(\forall x \in A, \exists U \in \mu\) such that \(x \in U \subseteq \gamma(U) \subseteq A\). Thus, \(x \in int^{\mu}(A)\). This implies that \(A \in \mu\). That is, every supra \(\gamma\)-open set is supra open. Hence, \(\mu_{\gamma} \subseteq \mu\). But the converse of this relation is not true as illustrated in the following example.

Example 3.4. Let \(X = \{a, b, c\}\) and \(\mu = \{\phi, X, \{a, b\}, \{a, c\}\}\). Then \((X, \mu)\) is an STS. Define an operator \(\gamma : \mu \to P(X)\) as follows: \(\forall A \in \mu\)

\[
\gamma(A) = \begin{cases} 
A & \text{if } A = \{a, b\}; \\
cl(A) & \text{if } A \neq \{a, b\}.
\end{cases}
\]

Clearly, \(\mu_{\gamma} = \{\phi, X, \{a, b\}\}\). Then it can be easy to check that the set \(\{a, c\}\) is supra open, but not supra \(\gamma\)-open. So, \(\mu \notin \mu_{\gamma}\).

Lemma 3.5. Arbitrary union of supra \(\gamma\)-open sets are also supra \(\gamma\)-open.

Proof. Suppose \(\{A_{\lambda} : \lambda \in \Lambda\}\) is a class of supra \(\gamma\)-open sets in \(X\). We have to show that \(\bigcup_{\lambda \in \Lambda}A_{\lambda} \in \mu_{\gamma}\). For this, let \(x \in \bigcup_{\lambda \in \Lambda}A_{\lambda}\). Then \(x \in A_{\lambda_{0}}\) for some \(\lambda_{0} \in \Lambda\). Hence, \(\exists U \in \mu\) such that \(x \in U \subseteq \gamma(U) \subseteq A_{\lambda_{0}} \subseteq \bigcup_{\lambda \in \Lambda}A_{\lambda}\). Therefore, \(\bigcup_{\lambda \in \Lambda}A_{\lambda} \in \mu_{\gamma}\).

Example 3.6. Let \(X = \{a, b, c\}\) and \(\mu = \{\phi, X, \{b\}, \{b, c\}, \{a, b\}, \{a, c\}\}\). Define an operator \(\gamma : \mu \to P(X)\) by \(\gamma(A) = A, \forall A \in \mu\). Thus, \(\mu_{\gamma} = \mu\). So, \(\{a, b\} \in \mu_{\gamma}\) and \(\{a, c\} \in \mu_{\gamma}\), but \(\{a, b\} \cap \{a, c\} = \{a\} \notin \mu_{\gamma}\).

Remark 3.7. Lemma 3.5 demonstrates that \(\mu_{\gamma}\) is an ST on \(X\), and Example 3.6 shows that \(\mu_{\gamma}\) is not always a topology.

Definition 3.8. Let \((X, \mu)\) be any STS. An operator \(\gamma\) on \(\mu\) is said to be supra regular if \(\forall x \in X \land \forall U, V \in \mu\) both containing \(x\), \(\exists W \in \mu\) containing \(x\) such that \(\gamma(W) \subseteq \gamma(U) \cap \gamma(V)\).

Example 3.9. Let \(X = \{a, b, c\}\) and \(\mu = \{\phi, X, \{a\}, \{b, c\}\}\). Let \(\gamma : \mu \to P(X)\) be the mapping defined by: \(\forall A \in \mu\)

\[
\gamma(A) = \begin{cases} 
A & \text{if } a \in A; \\
X & \text{if } a \notin A.
\end{cases}
\]

Thus, it can easily check that \(\gamma : \mu \to P(X)\) is a supra regular operator.

Theorem 3.10. Let \(\gamma : \mu \to P(X)\) be supra regular operator on \(\mu\). If \(A, B \in \mu_{\gamma}\), then \(A \cap B \in \mu_{\gamma}\).

Proof. Assume that \(A, B \in \mu_{\gamma}\) and let \(x \in A \cap B\). Then \(x \in A \land x \in B\). So, \(\exists U, V \in \mu\) such that \(U \subseteq A\) and \(V \subseteq B\). Since \(\gamma : \mu \to P(X)\) is a supra regular operator on \(\mu\), then \(\exists W \in \mu\) containing \(x\) such that \(\gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B\). Hence, \(A \cap B \in \mu_{\gamma}\).}

Remark 3.11. If \(\gamma\) is a supra regular operator on \(\mu\), then \(\mu_{\gamma}\) is a topology on \(X\).
Example 3.12. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$. Let $\gamma : \mu \rightarrow P(X)$ be the mapping defined by:

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\}; \\ A & \text{if } A \neq \{a\}. \end{cases}$$

Clearly, $\gamma$ is not a supra regular operator on $\mu$. Thus, $\mu_{\gamma} = \{\phi, X, \{a, c\}, \{b, c\}\}$ is not a topology on $X$.

Definition 3.13. An STS $(X, \mu)$ with an operator $\gamma$ on $\mu$ is said to be supra $\gamma$-regular if $\forall x \in X$ and $\forall U \in \mu$ with $x \in U$, $\exists W \in \mu$ with $x \in W$ and $\gamma(W) \subseteq U$.

Theorem 3.14. Let $(X, \mu)$ be an STS and $\gamma : \mu \rightarrow P(X)$ be an operator on $\mu$. Then the following statements are equivalent:

1. $\mu = \mu_{\gamma}$.
2. $(X, \mu)$ is supra $\gamma$-regular.
3. $\forall x \in X$ and $\forall U \in \mu$ with $x \in U$, $\exists W \in \mu_{\gamma}$ with $x \in W$ and $W \subseteq U$.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $U \in \mu$ with $x \in U$. It follows from assumption that $U \in \mu_{\gamma}$. This implies that $\exists W \in \mu$ with $x \in W$ and $\gamma(W) \subseteq U$. Thus, $(X, \mu)$ is a supra $\gamma$-regular space.

(2) $\Rightarrow$ (3) Let $x \in X$ and $U \in \mu$ with $x \in U$. Then by (2), $\exists W \in \mu$ such that $x \in W \subseteq \gamma(W) \subseteq U$. Again, by using (2) for the set $W$, we obtain $W \in \mu_{\gamma}$ such that $x \in W$ and $W \subseteq U$.

(3) $\Rightarrow$ (1) By using (3) and Lemma 3.5, we obtain $U \in \mu_{\gamma}$. That is, $\mu \subseteq \mu_{\gamma}$. Since $\mu_{\gamma} \subseteq \mu$ in general, Thus, $\mu = \mu_{\gamma}$.

Definition 3.15. A subset $B$ of an STS $(X, \mu)$ is called supra $\gamma$-closed if $X \setminus B$ is supra $\gamma$-open in $(X, \mu)$.

Definition 3.16. Let $A$ be any subset of an STS $(X, \mu)$ and $\gamma$ be an operator on $\mu$. Then

1. $\forall x \in X$, $x \in Cl_{\mu}(A)$ if $\gamma(U) \cap A \neq \phi \forall U \in \mu$ with $x \in U$.
2. The supra $\gamma$-closure of $A$ is denoted by $\mu_{\gamma}\cdot Cl_{\mu}(A)$ and is defined as $\mu_{\gamma}\cdot Cl_{\mu}(A) = \bigcap \{F : F \text{ is a supra } \gamma \text{-closed set in } X \text{ and } A \subseteq F\}$.

Theorem 3.17. Let $A$ be any subset of an STS $(X, \mu)$ and $\gamma$ be an operator on $\mu$. Then $x \in \mu_{\gamma}\cdot Cl_{\mu}(A)$ iff $A \cap U \neq \phi \forall U \in \mu_{\gamma}$ with $x \in U$.

Proof. Let $x \in \mu_{\gamma}\cdot Cl_{\mu}(A)$ and $A \cap U = \phi$ for some $U \in \mu_{\gamma}$ with $x \in U$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is a supra $\gamma$-closed set in $X$. Hence, $\mu_{\gamma}\cdot Cl_{\mu}(A) \subseteq X \setminus U$. Thus, $x \in X \setminus U$. This is a contradiction. Hence, the proof is complete.

Conversely, let $x \notin \mu_{\gamma}\cdot Cl_{\mu}(A)$. So $\exists$ a supra $\gamma$-closed set $F$ containing $A$ with $x \notin F$. Thus, $X \setminus F \in \mu_{\gamma}$ with $x \in X \setminus F$ and $(X \setminus F) \cap A = \phi$. This is a contradiction. Therefore, $x \notin \mu_{\gamma}\cdot Cl_{\mu}(A)$.

Lemma 3.18. Let $(X, \mu)$ be an STS and $\gamma$ be an operator on $\mu$. Then the following statements are true for any subsets $A, B \subseteq X$:

1. $\mu_{\gamma}\cdot Cl_{\mu}(A)$ is supra $\gamma$-closed in $X$ and $Cl_{\mu}(A)$ is supra closed in $X$.
2. $A \subseteq Cl_{\mu}(A) \subseteq \mu_{\gamma}\cdot Cl_{\mu}(A)$.
3. (a) $A$ is supra $\gamma$-closed iff $\mu_{\gamma}\cdot Cl_{\mu}(A) = A$ and
   (b) $A$ is supra $\gamma$-closed iff $Cl_{\mu}(A) = A$.
4. If $A \subseteq B$, then $\mu_{\gamma}\cdot Cl_{\mu}(A) \subseteq \mu_{\gamma}\cdot Cl_{\mu}(B)$ and $Cl_{\mu}(A) \subseteq Cl_{\mu}(B)$.
5. (a) $\mu_{\gamma}\cdot Cl_{\mu}(A \cap B) \subseteq \mu_{\gamma}\cdot Cl_{\mu}(A) \cap \mu_{\gamma}\cdot Cl_{\mu}(B)$ and
   (b) $Cl_{\mu}(A \cap B) \subseteq Cl_{\mu}(A) \cap Cl_{\mu}(B)$.
6. (a) \( \mu_x - \text{cl}^p(A) \cup \mu_y - \text{cl}^p(B) \subseteq \mu_x - \text{cl}^p(A \cup B) \) and 
(b) \( \text{cl}^p_x(A) \cup \text{cl}^p_y(B) \subseteq \text{cl}^p_x(A \cup B) \).
7. \( \mu_y - \text{cl}^p(\mu_x - \text{cl}^p(A)) = \mu_y - \text{cl}^p(A) \).

**Proof.** Straightforward. \(\square\)

**Lemma 3.19.** Let \( A, B \) be subsets of an STS \( (X, \mu) \) and \( y \) be a supra regular operator on \( \mu \). Then
1. \( \mu_x - \text{cl}^p(A) \cup \mu_y - \text{cl}^p(B) = \mu_x - \text{cl}^p(A \cup B) \).
2. \( \text{cl}^p_x(A) \cup \text{cl}^p_y(B) = \text{cl}^p_x(A \cup B) \).

**Proof.**
(1) It follows directly from Lemma 3.18 (6) that \( \mu_y - \text{cl}^p(A \cup B) \subseteq \mu_y - \text{cl}^p(A \cup B) \). Then it is enough to prove that \( \mu_x - \text{cl}^p(A \cup B) \subseteq \mu_y - \text{cl}^p(A \cup B) \). Let \( x \in \mu_x - \text{cl}^p(A \cup B) \). Then \( \exists U, V \in \mu_x \) with \( x \in U, x \in V, A \cap U = \phi \) and \( B \cap V = \phi \). Since \( y \) is a supra regular operator on \( \mu \), then by Theorem 3.10, \( U \cap V \in \mu_x \) such that 

\[
(U \cap V) \cap (A \cup B) = \phi.
\]

This means that \( x \notin \mu_y - \text{cl}^p(A \cup B) \). Hence, 

\[
\mu_x - \text{cl}^p(A \cup B) \subseteq \mu_y - \text{cl}^p(A \cup B).
\]

(2) Let \( x \notin \text{cl}^p_x(A) \cup \text{cl}^p_y(B) \). Then \( \exists U_1 \in \mu \) and \( U_2 \in \mu \) with \( x \in U_1, x \in U_2, A \cap \gamma(U_1) = \phi \) and \( A \cap \gamma(U_2) = \phi \).

Since \( y \) is a supra regular operator on \( \mu \), then \( \exists W \in \mu \) with \( x \in W \) and \( \gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \). Thus, we have 

\[
(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).
\]

The disjoint of \( (A \cup B) \) and \( (\gamma(U_1) \cap \gamma(U_2)) \) leads to \( (A \cup B) \cap \gamma(W) = \phi \). This means that \( x \notin \text{cl}^p_x(A \cup B) \). Therefore, \( \text{cl}^p_x(A \cup B) \subseteq \text{cl}^p_x(A) \cup \text{cl}^p_y(B) \). From Lemma 3.18 (6), we obtain the equality. \(\square\)

**Lemma 3.20.** Let \( (X, \mu) \) be an STS and \( y \) be a supra regular operator on \( \mu \). Then \( \mu_x - \text{cl}^p(A \cap U \subseteq \mu_y - \text{cl}^p(A \cup U) \) holds \( \forall U \in \mu_x \) and \( \forall A \subseteq X \).

**Proof.** Suppose that \( x \in \mu_x - \text{cl}^p(A \cap U \forall U \in \mu_x \), then \( x \in \mu_x - \text{cl}^p(A) \) and \( x \in U \). Let \( V \in \mu_x \) with \( x \in V \). Since \( y \) is supra regular on \( \mu \). So by Theorem 3.10, \( U \cap V \in \mu_x \) with \( x \in U \cap V \). Since \( x \in \mu_x - \text{cl}^p(A) \), then by Theorem 3.17, we find that \( A \cap (U \cap V) \neq \phi \). Therefore, \( (A \cup U) \cap V \neq \phi \). Thus, by Theorem 3.17, we have that \( x \in \mu_x - \text{cl}^p(A \cup U) \). Hence, \( \mu_x - \text{cl}^p(A \cup U) \). \(\square\)

**Theorem 3.21.** If \( A \subseteq (X, \mu) \) and \( y \) is an operator on \( \mu \), then the next four properties are equivalent:
1. \( A \in \mu_x \).
2. \( \text{cl}^p_x(X \! \setminus \! A) = X \! \setminus \! A \).
3. \( \mu_y - \text{cl}^p(X \! \setminus \! A) = X \! \setminus \! A \).
4. \( X \! \setminus \! A \) is supra \( y \)-closed.

**Definition 3.22.** Let \( (X, \mu) \) be any STS. An operator \( y \) on \( \mu \) is said to be supra open if \( \forall x \in X \) and \( \forall U \in \mu \) with \( x \in U, \exists W \in \mu_y \) with \( x \in W \) and \( W \subseteq \gamma(U) \).

**Theorem 3.23.** Let \( A \) be any subset of an STS \( (X, \mu) \). If \( y \) is a supra open operator on \( \mu \), then
1. \( \text{cl}^p_x(A) = \mu_x - \text{cl}^p(A) \),
2. \( \text{cl}^p_x(\text{cl}^p_y(A)) = \text{cl}^p_y(A) \),
3. \( \text{cl}^p_x(A) \) is supra \( y \)-closed in \( X \).
Proof.
(1) First we need to show that $\mu_f \cdot \text{cl}_\mu(A) \subseteq \text{cl}_\mu(A)$. By Lemma 3.18 (2), we have $\text{cl}_\mu(A) \subseteq \mu_f \cdot \text{cl}_\mu(A)$. Now let $x \notin \text{cl}_\mu(A)$, then $\exists U \in \mu$ with $x \in U$ and $A \cap U = \phi$. Since $\gamma$ is a supra open on $\mu$, then $\exists W \in \mu$ with $x \in W$ and $W \subseteq \gamma(U)$. So $A \cap W = \phi$ and hence by Theorem 3.17, $x \notin \mu_f \cdot \text{cl}_\mu(A)$. Therefore, $\mu_f \cdot \text{cl}_\mu(A) \subseteq \text{cl}_\mu(A)$.

Hence, $\text{cl}_\mu(A) = \mu_f \cdot \text{cl}_\mu(A)$.

(2) By (1) and Lemma 3.18 (7), we have $\text{cl}_\mu(A) = \text{cl}_\mu(A)$.

(3) By (2) and Lemma 3.18 (3b), we get $\text{cl}_\mu(A)$ is supra $\gamma$-closed in $X$. \hfill \Box

4 $\mu_f \cdot g$-closed sets and operator on subspace ST

Through this section, we present the concept of $\mu_f \cdot g$-generalized closed and give some characterizations.

**Definition 4.1.** A subset $A$ of an STS $(X, \mu)$ with an operator $\gamma$ on $\mu$ is called $\mu_f \cdot g$-generalized closed (briefly, $\mu_f \cdot g$-closed) if $\text{cl}_\mu(A) \subseteq U \forall U \in \mu_f \gamma$, satisfies that $A \subseteq U$.

**Lemma 4.2.** Let $(X, \mu)$ be an STS and $\gamma$ be an operator on $\mu$. A set $A$ in $(X, \mu)$ is $\mu_f \cdot g$-closed iff $A \cap \mu_f \cdot \text{cl}_\mu(A) \neq \phi \forall x \in \text{cl}_\mu(A)$.

**Proof.** Let $A$ be a $\mu_f \cdot g$-closed set in $X$ and suppose (if possible) that $\exists x \in \text{cl}_\mu(A)$ such that $A \cap \mu_f \cdot \text{cl}_\mu(A) = \phi$. This follows that $A \subseteq X \setminus \mu_f \cdot \text{cl}_\mu(A)$. Since $\mu_f \cdot \text{cl}_\mu(A)$ is supra $\gamma$-closed and hence $X \setminus \mu_f \cdot \text{cl}_\mu(A) \in \mu_f$. Now, $\mu_f \cdot g$-closedness of $A$ in $X$ implies that $\text{cl}_\mu(A) \subseteq X \setminus \mu_f \cdot \text{cl}_\mu(A)$. Therefore, $x \notin \text{cl}_\mu(A)$, which is a contradiction.

Thus, $A \cap \mu_f \cdot \text{cl}_\mu(A) \neq \phi$.

Conversely, let $U \in \mu_f$ with $A \subseteq U$. To show that $\text{cl}_\mu(A) \subseteq U$. Let $x \in \text{cl}_\mu(A)$. Then by hypothesis, $A \cap \mu_f \cdot \text{cl}_\mu(A) \neq \phi$. So, $\exists y \in A \cap \mu_f \cdot \text{cl}_\mu(A)$. Thus, $y \in A \subseteq U$ and $y \in \mu_f \cdot \text{cl}_\mu(A)$. By Theorem 3.17, $\exists x \in \text{cl}_\mu(A)$, which is a contradiction.

Therefore, $x \in \text{cl}_\mu(A) \subseteq U$. Hence, $A$ is $\mu_f \cdot g$-closed.

**Theorem 4.3.** Let $\gamma$ be an operator on $\mu$. If $A$ is $\mu_f \cdot g$-closed subset of $(X, \mu)$, then $\text{cl}_\mu(A) \setminus A$ does not contain any non-empty supra $\gamma$-closed set in $(X, \mu)$.

**Proof.** Suppose that $F \neq \emptyset$ is a supra $\gamma$-closed set in $X$ with $F \subseteq \text{cl}_\mu(A) \setminus A$. Then $F \subseteq X \setminus A$. Obviously, $A \subseteq X \setminus F$. Since $X \setminus F \in \mu_f$ and $A$ is $\mu_f \cdot g$-closed, then $\text{cl}_\mu(A) \subseteq X \setminus F$. That is, $F \subseteq X \setminus \text{cl}_\mu(A)$. Therefore, $F \subseteq X \setminus \text{cl}_\mu(A)$ and $X \setminus \text{cl}_\mu(A) \subseteq X \setminus \text{cl}_\mu(A) \cap \text{cl}_\mu(A) = \phi$. Thus, $F = \phi$. But this is a contradiction. Hence, $F \notin \text{cl}_\mu(A) \setminus A$. \hfill \Box

**Theorem 4.4.** The converse of Theorem 4.3 is true when the operator $\gamma : \mu \rightarrow P(X)$ is supra open.

**Proof.** Let $U \in \mu_f$ with $A \subseteq U$. Since $\gamma : \mu \rightarrow P(X)$ is a supra open operator, then by Theorem 3.23 (3), $\text{cl}_\mu(A)$ is supra $\gamma$-closed set in $X$. Hence, we have $\text{cl}_\mu(A) \cap X \setminus U$ is a supra $\gamma$-closed set in $(X, \mu)$. Since $X \setminus U \subseteq X \setminus A$, $\text{cl}_\mu(A) \cap X \setminus U \subseteq \text{cl}_\mu(A) \setminus A$. By using the assumption of the converse of Theorem 4.3, $\text{cl}_\mu(A) \subseteq U$. Thus, $A$ is $\mu_f \cdot g$-closed set in $(X, \mu)$.

**Corollary 4.5.** Let $A$ be a $\mu_f \cdot g$-closed subset of STS $(X, \mu)$ and let $\gamma$ be an operator on $\mu$. Then $A$ is $\gamma$-closed iff $\text{cl}_\mu(A) \setminus A$ is supra $\gamma$-closed set in $(X, \mu)$.

**Proof.** (Necessity) Let $A$ be a supra $\gamma$-closed set in $(X, \mu)$. It follows from Lemma 3.18 (3b) that $\text{cl}_\mu(A) = A$ and hence $\text{cl}_\mu(A) \setminus A = \phi$ which is supra $\gamma$-closed.
(Sufficiency) Suppose \( cl^\mu(A) \setminus A \) is supra \( \gamma \)-closed and \( A \) is \( \mu_\gamma \)-g.closed. It follows from Theorem 4.3 that \( cl^\mu(A) \setminus A \) does not contain any non-empty supra \( \gamma \)-closed set in \((X, \mu)\). Since \( cl^\mu(A) \setminus A \) is supra \( \gamma \)-closed subset of itself, then \( cl^\mu(A) \setminus A = \emptyset \) implies \( cl^\mu(A) \cap X \setminus A = \emptyset \). Hence, \( cl^\mu(A) = A \). By Lemma 3.18 (3b), we obtain \( A \) is a supra \( \gamma \)-closed set in \((X, \mu)\).

\begin{theorem} \label{thm:4.6}
Let \((X, \mu)\) be an STS and \( \gamma \) be an operator on \( \mu \). If \( A \) is \( \mu_\gamma \)-g.closed and supra \( \gamma \)-open subset of \( X \), then \( A \) is supra \( \gamma \)-closed.
\end{theorem}

\begin{proof}
Since \( A \) is \( \mu_\gamma \)-g.closed and supra \( \gamma \)-open set in \( X \), then \( cl^\mu(A) \subseteq A \) and hence by Lemma 3.18 (3b), \( A \) is supra \( \gamma \)-closed.
\end{proof}

\begin{theorem} \label{thm:4.7}
For any STS \((X, \mu)\) with an operator \( \gamma \) on \( \mu \), \( X \setminus \{x\} \) is \( \mu_\gamma \)-g.closed or supra \( \gamma \)-open \( \forall x \in X \).
\end{theorem}

\begin{proof}
Let \( X \setminus \{x\} \notin \mu_\gamma \). Then the only supra \( \gamma \)-open set containing \( X \setminus \{x\} \) is \( X \). Automatically, we have \( cl^\mu(X \setminus \{x\}) \subseteq X \). This ends the proof that \( X \setminus \{x\} \) is a supra \( \gamma \)-open set in \( X \).
\end{proof}

\begin{corollary} \label{cor:4.8}
For any STS \((X, \mu)\) with an operator \( \gamma \) on \( \mu \), \( \{x\} \) is a supra \( \gamma \)-closed set or \( \{x\} \) is a supra \( \gamma \)-open set \( \forall x \in X \).
\end{corollary}

\begin{proof}
Let \( \{x\} \) be not supra \( \gamma \)-open. Then \( X \setminus \{x\} \) is not supra \( \gamma \)-open. Therefore, it follows from Theorem 4.7 that \( X \setminus \{x\} \) is \( \mu_\gamma \)-g.closed.
\end{proof}

\begin{definition} \label{def:4.9}
Let \( A \subseteq (X, \mu) \) and \( \gamma \) be an operator on \( \mu \). Then the \( \mu_\gamma \)-kernel of \( A \), denoted by \( \mu_\gamma \text{-ker}(A) \), is defined as follows:
\[
\mu_\gamma \text{-ker}(A) = \cap\{U : A \subseteq U \text{ and } U \in \mu_\gamma\},
\]
i.e., \( \mu_\gamma \text{-ker}(A) \) is the intersection of all supra \( \gamma \)-open sets of \((X, \mu)\) containing \( A \).
\end{definition}

\begin{theorem} \label{thm:4.10}
Let \( A \subseteq (X, \mu) \) and \( \gamma \) be an operator on \( \mu \). Then \( A \) is \( \mu_\gamma \)-g.closed iff \( cl^\mu(A) \subseteq \mu_\gamma \text{-ker}(A) \).
\end{theorem}

\begin{proof}
Suppose that \( A \) is \( \mu_\gamma \)-g.closed. Then \( cl^\mu(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \in \mu_\gamma \). Let \( x \in cl^\mu(A) \). Then by Lemma 4.2, \( A \cap \mu_\gamma \text{-cl}(\{x\}) \neq \emptyset \). So \( \exists \) a point \( z \in X \) such that \( z \in A \cap \mu_\gamma \text{-cl}(\{x\}) \) implies \( z \in A \subseteq U \) and \( z \in \mu_\gamma \text{-cl}(\{x\}) \). By Theorem 3.17, \( \{x\} \cap U \neq \emptyset \). Hence, we show that \( x \in \mu_\gamma \text{-ker}(A) \). Thus, \( cl^\mu(A) \subseteq \mu_\gamma \text{-ker}(A) \).

Conversely, let \( cl^\mu(A) \subseteq \mu_\gamma \text{-ker}(A) \). Let \( U \in \mu_\gamma \) with \( A \subseteq U \). Let \( x \) be a point in \( X \) such that \( x \in cl^\mu(A) \). Then \( x \in \mu_\gamma \text{-ker}(A) \). We have \( x \in U \), because \( A \subseteq U \) and \( U \in \mu_\gamma \). That is, \( cl^\mu(A) \subseteq \mu_\gamma \text{-ker}(A) \subseteq U \). Thus, \( A \) is \( \mu_\gamma \)-g.closed in \( X \).
\end{proof}

Now we define an operator on subspace ST as follows:

\begin{definition} \label{def:4.11}
Let \((A, \mu_\gamma)\) be a subspace of an STS \((X, \mu)\) and \( \gamma : \mu \to P(X) \) be an operator on \( \mu \). We define the restriction of \( \gamma \) to \( \mu_\gamma \), denoted by \( \gamma_\mu \), to be the mapping from \( \mu_\gamma \) into \( P(X) \) such that \( \forall U \in \mu_\gamma \),
\[
\gamma_\mu(U) = \gamma(V) \cap A \text{ for some } V \in \mu \text{ with } U = V \cap A.
\]
\end{definition}

\begin{lemma} \label{lem:4.12}
Let \((A, \mu_\gamma)\) be a subspace of an STS \((X, \mu)\) and \( \mu_\gamma \) be the restriction of \( \gamma \) to \( \mu_\gamma \). If \( B \in \mu_\gamma \) in \( X \), then \( B \cap A \) is supra \( \gamma_\mu \)-open in \( A \).
\end{lemma}

\begin{proof}
Let \( x \in B \cap A \). Since \( B \in \mu_\gamma \) in \( X \), then \( \exists U \in \mu \) with \( x \in U \) and \( \gamma(U) \subseteq B \). So, \( U \cap A \) is supra \( \gamma_\mu \)-open set with \( x \in U \cap A \) and
\[
\gamma_\mu(U \cap A) = \gamma(U) \cap A \subseteq B \cap A.
\]
Thus, \( B \cap A \) is supra \( \gamma_\mu \)-open in \( A \).
\end{proof}
Lemma 4.13. Let \( (A, \mu_A) \) be a subspace of an STS \((X, \mu)\) and \( \mu_A \) be the restriction of \( \gamma \) to \( \mu_A \). If the mapping \( \gamma \) is supra open and the set \( B \) is supra \( \gamma_A \)-open in \( A \), then \( \exists C \in \mu_A \) with \( B = C \cap A \).

**Proof.** Since \( B \) is supra \( \gamma_A \)-open in \( A \), then \( \forall x \in B, \exists U_x \in \mu \) with \( x \in U_x \) and

\[
\gamma_A(U_x \cap A) = \gamma(U_x) \cap A \subseteq B.
\]

Since \( \gamma \) is supra open, then \( \exists W_x \in \mu \) with \( x \in W_x \) and \( W_x \subseteq \gamma(U_x) \). Put \( C = \bigcup_{x \in B} W_x \). So, \( C \in \mu_A \) in \( X \) and

\[
B \subseteq C \cap A = \left( \bigcup_{x \in B} W_x \right) \cap A \subseteq \left( \bigcup_{x \in B} \gamma(U_x) \cap A \right) \subseteq B.
\]

This completes the proof. \( \square \)

Theorem 4.14. Let \( (A, \mu_A) \) be a subspace of an STS \((X, \mu)\) and \( B \subseteq A \subseteq X \). If \( \mu_A \) is the restriction of \( \gamma \) to \( \mu_A \), then \( \text{cl}_A^\mu(B) = \text{cl}_A^\gamma(B) \cap A \).

**Proof.** Let \( x \in \text{cl}_A^\mu(B) \) and \( U \in \mu \) with \( x \in U \). Then \( \gamma_A(U \cap A) \cap B = \gamma(U) \cap B \neq \phi \) and hence \( x \in (\text{cl}_A^\mu(B) \cap A) \). On the other hand, let \( x \in (\text{cl}_A^\mu(B) \cap A) \) and \( V \in \mu_A \) with \( x \in V \). Then \( V = U \cap A \) for some \( U \in \mu \) with \( x \in U \). Since \( x \in \text{cl}_A^\mu(B) \),

\[
\gamma_A(V) \cap A = (\gamma(U) \cap A) \cap B = (\gamma(U) \cap B) \neq \phi.
\]

Thus, \( x \in \text{cl}_A^\mu(B) \). \( \square \)

Theorem 4.15. Let \( (A, \mu_A) \) be a subspace of an STS \((X, \mu)\) and \( \mu_A \) be the restriction of \( \gamma \) to \( \mu_A \). If the mapping \( \gamma \) is supra open, the set \( B \) is \( \mu_A \)-closed in \( A \) and \( A \) is \( \mu_A \)-closed in \( X \), then \( B \) is \( \mu_A \)-closed in \( X \).

**Proof.** Let \( U \in \mu_A \) in \( X \) with \( B \subseteq U \). Then by Lemma 4.12, \( U \cap A \) is supra \( \gamma_A \)-open in \( A \) and \( B \subseteq U \cap A \). By hypothesis, \( \text{cl}_A^\mu(B) = \text{cl}_A^\gamma(B) \cap A \subseteq U \cap A \). Hence, \( A \subseteq U \cup (X \setminus \text{cl}_A^\mu(B)) \). Since \( A \) is \( \mu_A \)-closed in \( X \), \( \gamma \) is supra open and \( X \setminus \text{cl}_A^\mu(B) \) is \( \mu_A \)-open. So \( \text{cl}_A^\mu(A) \subseteq U \cup (X \setminus \text{cl}_A^\mu(B)) \). Hence,

\[
\text{cl}_A^\mu(B) \subseteq \text{cl}_A^\mu(A) \subseteq U \cup (X \setminus \text{cl}_A^\mu(B)).
\]

Thus, \( \text{cl}_A^\mu(B) \subseteq U \). Therefore, \( B \) is \( \mu_A \)-closed in \( X \). \( \square \)

Corollary 4.16. If \( \gamma \) is supra open, \( A \) is \( \mu_A \)-closed in \( X \) and \( F \) is supra \( \gamma \)-closed in \( X \), then \( A \cap F \) is \( \mu_A \)-closed in \( X \).

**Proof.** By Lemma 4.12, \( A \cap F \) is supra \( \gamma_A \)-closed in \( A \) and hence \( A \cap F \) is \( \mu_A \)-closed in \( A \). Thus, by Theorem 4.15, \( A \cap F \) is \( \mu_A \)-closed in \( X \). \( \square \)

Theorem 4.17. If \( \gamma \) is supra open, \( A \) is \( \mu_A \)-closed in \( X \) and \( A \subseteq B \subseteq \text{cl}_A^\mu(A) \), then \( B \) is \( \mu_A \)-closed in \( X \).

**Proof.** Since \( \text{cl}_A^\mu(B) \setminus B \subseteq \text{cl}_A^\mu(A) \setminus A \) and \( \text{cl}_A^\mu(A) \setminus A \) has no non-empty supra \( \gamma \)-closed set in \( X \), neither does \( \text{cl}_A^\mu(B) \setminus B \). So, by Theorem 4.4, \( B \) is \( \mu_A \)-closed in \( X \). \( \square \)

Theorem 4.18. Let \( (A, \mu_A) \) be a subspace of an STS \((X, \mu)\), \( \mu_A \) be the restriction of \( \gamma \) to \( \mu_A \) and \( B \subseteq A \subseteq X \). If \( \gamma \) is supra open, \( B \) is \( \mu_A \)-closed in \( X \), then \( B \) is \( \mu_A \)-closed in \( A \).

**Proof.** Let \( U \) be supra \( \gamma_A \)-open in \( A \) and \( B \subseteq U \). Then \( \exists V \in \mu_A \) in \( X \) with \( U = A \cap V \) and hence \( B \subseteq V \). Thus, by hypothesis, \( \text{cl}_A^\mu(B) \subseteq V \). By Theorem 4.14, \( \text{cl}_A^\mu(B) = \text{cl}_A^\mu(B) \cap A \subseteq V \cap A = U \). Therefore, \( B \) is \( \mu_A \)-closed in \( A \). \( \square \)
5 $\mu_y$-Separation axioms

In this section, we investigate some types of $\mu_y$-separation axioms. Some results and examples of these spaces are studied.

**Definition 5.1.** An STS $(X, \mu)$ with an operator $y$ on $\mu$ is said to be:

(i) $\mu_y-T_0$ if $\forall x, y \in X$ with $x \neq y$, $\exists U \in \mu$ such that either $x \in U$ and $y \notin y(U)$ or $y \in U$ and $x \notin y(U)$.

(ii) $\mu_y-T_0^0$ if $\forall x, y \in X$ with $x \neq y$, $\exists U \in \mu_y$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

(iii) $\mu_y-T_1^i$ if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu$ with $x \in U$ but $y \notin y(U)$ and $y \in V$ but $x \notin y(V)$.

(iv) $\mu_y-T_1^0$ if $\forall x, y \in X$ with $x \neq y$, $\exists U \in \mu_y$ with $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

(v) $\mu_y-T_2$ if $\forall x, y \in X$ with $x \neq y$, $\exists U, V \in \mu_y$ with $x \in U$, $y \in V$ and $y(U) \cap y(V) = \emptyset$.

(vi) $\mu_y-T_2^i$ if $\exists x \in X$ with $x \neq y$, $\exists U \in \mu_y$ with $x \in U$ and $y \notin y(U)$.

(vii) $\mu_y-T_2^0$ if every $\mu_y$-closed set in $X$ is supra $y$-closed.

**Theorem 5.2.** An STS $(X, \mu)$ with an operator $y$ on $\mu$ is $\mu_y-T_2$ iff the set $\{x\}$ is supra $y$-closed or supra $y$-open $\forall x \in X$.

**Proof.** Suppose that $\{x\}$ is not supra $y$-closed set in a $\mu_y-T_2$ space $(X, \mu)$. Then Corollary 4.8 implies that $X \setminus \{x\}$ is a $\mu_y$-g. closed set. Since $(X, \mu)$ is $\mu_y-T_2$, then $\{x\}$ is a supra $y$-open set.

Conversely, let $F$ be any $\mu_y$-g. closed set in the STS $(X, \mu)$. We have to show that $F$ is supra $y$-closed (i.e., $cl^p y(F) = F$ by Lemma 3.18 (3b)). It is sufficient to show that $cl^p y(F) \subseteq F$. Let $x \in cl^p y(F)$. By hypothesis $\{x\}$ is supra $y$-closed or supra $y$-open $\forall x \in X$. So there are two cases:

Case 1: If $\{x\}$ is supra $y$-closed. Let $x \notin F$, then $x \in cl^p y(F) \setminus F$ contains a non-empty supra $y$-closed set $\{x\}$. Since $F$ is $\mu_y$-g. closed and according to Theorem 4.3, we obtain a contradiction. Hence, it must be that $x \in F$ and $F \subseteq cl^p y(F)$. Thus, Lemma 3.18 (3b) $F$ is supra $y$-closed in $(X, \mu)$. Therefore, $(X, \mu)$ is $\mu_y-T_2$ space.

Case 2: If $\{x\}$ is supra $y$-open. Then by Theorem 3.17, $F \cap \{x\} \neq \emptyset$ which implies that $x \in F$. So $cl^p y(F) \subseteq F$.

Thus, by Lemma 3.18 (3b), $F$ is supra $y$-closed. Thus, $(X, \mu)$ is $\mu_y-T_2$ space.

**Theorem 5.3.** Suppose that $y$ is a supra open operator on $\mu$. An STS $(X, \mu)$ is a $\mu_y-T_0^0$ iff $cl^p y(\{x\}) \neq cl^p y(\{y\})$, $\forall x, y \in X$ with $x \neq y$.

**Proof.** (Necessity) Let $x, y \in X$ with $x \neq y$, where $(X, \mu)$ be a $\mu_y-T_0^0$ space. Thus, $\exists U \in \mu_x$ with $x \in U$ and $y \notin y(U)$. Since $y$ is a supra open operator on $\mu$, then $\exists W \in \mu_y$ with $x \in W$ and $W \subseteq y(U)$. So, $y \in X \setminus y(U) \subseteq X \setminus W$. Since $X \setminus W$ is a supra $y$-closed set in $(X, \mu)$. Therefore, we obtain that $cl^p y(\{y\}) \subseteq X \setminus W$ and hence $cl^p y(\{x\}) \neq cl^p y(\{y\})$.

(Sufficiency) Suppose that $cl^p y(\{x\}) \neq cl^p y(\{y\})$, $\forall x, y \in X$ with $x \neq y$. Now, we assume that $\exists z \in X$ such that $z \notin cl^p y(\{x\})$, but $z \notin cl^p y(\{y\})$. If $x \in cl^p y(\{y\})$, then $\exists x \subseteq cl^p y(\{y\})$, which implies that $cl^p y(\{x\}) \subseteq cl^p y(\{y\})$ (by Lemma 3.18 (4)). Therefore, $\exists z \in cl^p y(\{y\})$. This contradiction shows that $x \notin cl^p y(\{y\})$. Thus, $\exists U \in \mu_y$ such that $x \in U$ and $y(U) \setminus \{y\} = \emptyset$. Hence, we obtain $x \in U$ and $y \notin y(U)$. It gives that the STS $(X, \mu)$ is $\mu_y-T_0^0$.

**Theorem 5.4.** An STS $(X, \mu)$ is $\mu_y-T_0$ iff $\mu_y-cl^p y(\{x\}) \neq \mu_y-cl^p y(\{y\})$, $\forall x, y \in X$ with $x \neq y$.

**Proof.** (Necessity) Let $X$ be a $\mu_y-T_0$ space and $x, y \in X$ with $x \neq y$. Then $\exists U \in \mu_y$ (say $x \in U$, but $y \notin U$). So $X \setminus U$ is a supra $y$-closed set, which does not contain $x$, but contains $y$. Since $\mu_y-cl^p y(\{y\})$ is the smallest supra $y$-closed set containing $y$, $\mu_y-cl^p y(\{y\}) \subseteq X \setminus U$, and so $x \notin \mu_y-cl^p y(\{y\})$. Therefore, $\mu_y-cl^p y(\{x\}) \neq \mu_y-cl^p y(\{y\})$.
(Sufficiency) Let $\mu_\gamma \text{-cl}^\mu(\{x\}) \neq \mu_\gamma \text{-cl}^\mu(\{y\}) \forall x, y \in X$ with $x \neq y$. Now, let $z \in X$ such that $z \in \mu_\gamma \text{-cl}^\mu(\{x\})$, but $z \notin \mu_\gamma \text{-cl}^\mu(\{y\})$. Now, we claim that $x \in \mu_\gamma \text{-cl}^\mu(\{y\})$. For, if $x \in \mu_\gamma \text{-cl}^\mu(\{y\})$, then $\{x\} \subseteq \mu_\gamma \text{-cl}^\mu(\{y\})$, which implies that $\mu_\gamma \text{-cl}^\mu(\{x\}) \subseteq \mu_\gamma \text{-cl}^\mu(\{y\})$. This is a contradiction to the fact that $z \notin \mu_\gamma \text{-cl}^\mu(\{y\})$. Hence, $x$ belongs to the supra $\gamma$-open set $X \setminus \mu_\gamma \text{-cl}^\mu(\{y\})$ to which $y$ does not belong. It gives that $X$ is $\mu_\gamma$-$T_0$ space.

From Theorem 5.3, Theorem 5.4 and the fact that $\text{cl}^\mu(A) = \mu_\gamma \text{-cl}^\mu(A) \forall A \subseteq X$ holds under Theorem 3.23 (1) that $y$ is a supra open operator on $\mu$, so we have the following corollary.

**Corollary 5.5.** Suppose that $y$ is a supra open operator on $\mu$. An STS $(X, \mu)$ is $\mu_\gamma$-$T_0^*$ if $(X, \mu)$ is $\mu_\gamma$-$T_0$.

**Theorem 5.6.** For an STS $(X, \mu)$ with an operator $y$ on $\mu$. Then the following conditions are true:
1. $(X, \mu)$ is $\mu_\gamma$-$T_1^*$.
2. The set $\{x\}$ is supra $\gamma$-closed $\forall x \in X$.
3. $(X, \mu)$ is $\mu_\gamma$-$T_0$.

**Proof.** (1) $\Rightarrow$ (2) Let $x$ be a point of a $\mu_\gamma$-$T_1^*$ space. Then $\forall y \in X \setminus \{x\}$, $\exists V_y \in \mu$ such that $y \in V_y$ but $x \notin y(V_y)$. Thus, $y \in y(V_y) \subseteq X \setminus \{x\}$. This implies that

$$X \setminus \{x\} = \bigcup \{y(V_y) : y \in X \setminus \{x\}\}.$$ 

It is shown that $X \setminus \{x\} \in \mu_\gamma$. Hence, $\{x\}$ is supra $\gamma$-closed in $(X, \mu)$.

(2) $\Rightarrow$ (3) Suppose every singleton set in $X$ is supra $\gamma$-closed. Let $x, y \in X$ with $x \neq y$. Then by (2), the sets $X \setminus \{x\} \in \mu_\gamma$ and $X \setminus \{y\} \in \mu_\gamma$ with $y \in X \setminus \{x\}$ but $x \notin X \setminus \{x\}$ and $x \in X \setminus \{y\}$ but $y \notin X \setminus \{y\}$. Thus, $(X, \mu)$ is $\mu_\gamma$-$T_0$.

(3) $\Rightarrow$ (1) It is shown that if $x \in U$, where $U \in \mu_\gamma$, then $\exists V \in \mu$ with $x \in V \subseteq y(V) \subseteq U$. Hence, by using (3), we have that $(X, \mu)$ is $\mu_\gamma$-$T_1^*$.

**Theorem 5.7.** Let $(X, \mu)$ be an STS and $y$ be an operator on $\mu$. Then the following statements are equivalent:
1. $X$ is $\mu_\gamma$-$T_3$.
2. If $x \in X$, then $\exists U \in \mu_\gamma$ with $x \in U$ and $y \notin \mu_\gamma \text{-cl}^\mu(U) \forall y \in X$ with $x \neq y$.
3. $\bigcap \{\mu_\gamma \text{-cl}^\mu(U) : U \in \mu_\gamma\} = \{x\} \forall x \in X$.

**Proof.** (1) $\Rightarrow$ (2) Let $X$ be any $\mu_\gamma$-$T_3$ space and $\forall x, y \in X$ with $x \neq y$, then $\exists U, V \in \mu_\gamma$ with $x \in U$, $y \in V$ and $U \cap \gamma V = \emptyset$. This implies that $U \subseteq X \setminus \gamma V$ and hence $\mu_\gamma \text{-cl}^\mu(U) \subseteq X \setminus \gamma V$ since $X \setminus \gamma V$ is supra $\gamma$-closed in $X$ and $y \notin X \setminus \gamma V$. Thus, $y \notin \mu_\gamma \text{-cl}^\mu(U)$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $x, y \in X$ with $x \neq y$. By (3), $\exists U \in \mu_\gamma$ with $x \in U$ and $y \notin \mu_\gamma \text{-cl}^\mu(U)$. Then $y \in X \setminus \mu_\gamma \text{-cl}^\mu(U)$ and $X \setminus \mu_\gamma \text{-cl}^\mu(U) \in \mu_\gamma$. Thus,

$$U \cap X \setminus \mu_\gamma \text{-cl}^\mu(U) = \emptyset.$$ 

Hence, $X$ is $\mu_\gamma$-$T_3$.

**Theorem 5.8.** For any STS $(X, \mu)$ and any operator $y$ on $\mu$, the following properties hold:
1. Each $\mu_\gamma$-$T_1^*$ space is $\mu_\gamma$-$T_{i-1}$, where $j \in \{2, 1\}$.
2. Each $\mu_\gamma$-$T_2^*$ space is $\mu_\gamma$-$T_{i-1}$, where $j \in \{2, 0\}$.
3. Each $\mu_\gamma$-$T_2$ space is $\mu_\gamma$-$T_{i-1}^*$.
4. Each $\mu_\gamma$-$T_3^*$ space is $\mu_\gamma$-$T_{i-1}^*$.
5. Each $\mu_\gamma$-$T_{i-1}^*$ space is $\mu_\gamma$-$T_0$.
6. Each $\mu_\gamma$-$T_{i-1}$ space is $\mu_\gamma$-$T_0$.
7. Each $\mu_\gamma$-$T_{i-1}$ space is $S$-$T_i$, where $j \in \{2, 1, 0\}$.
Proof. The proofs are obvious from their definitions and hence they are omitted. □

Observe that the converse of each part of Theorem 5.8 is not true as shown by the following examples.

Example 5.9. Suppose $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator $\gamma$ on $\mu$ as follows:

$$\forall A \in \mu, \gamma(A) = \begin{cases} A & \text{if } c \in A; \\ X & \text{if } c \notin A. \end{cases}$$

Thus, the space $(X, \mu)$ is $\mu_{\gamma}\cdot T_0$, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_1$.

Example 5.10. Let $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator $\gamma$ on $\mu$ as follows:

$$\forall A \in \mu, \gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

(i) Thus, STS $(X, \mu)$ is $\mu_{\gamma}\cdot T_1$ space, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_2$.

(ii) Thus, STS $(X, \mu)$ is $\mu_{\gamma}\cdot T_1$ space, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_2$.

Example 5.11. The STS $(X, \mu)$ in Example 3.12 is both $\mu_{\gamma}\cdot T_0$ and $\mu_{\gamma}\cdot T_0^*$, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_2^*$.  

Example 5.12. Let $X = \{a, b, c\}$ and $\mu = P(X)$. Let $\gamma : \mu \to P(X)$ be an operator on $\mu$ defined as follows:

$$\forall A \in \mu, \gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\} \text{ or } \{a, b\} \text{ or } \{b, c\}; \\ \{b, c\} & \text{if } A = \{c\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

Obviously, $\mu_{\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Thus, the STS $(X, \mu)$ is $\mu_{\gamma}\cdot T_1^*$, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_1^*$. 

Example 5.13. Consider $X = \{a, b, c\}$ and $\mu = P(X)$. Define an operator $\gamma$ on $\mu$ as follows:

$$\forall A \in \mu, \gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{b\}; \\ \{b, c\} & \text{if } A = \{c\} \text{ or } \{b, c\}; \\ X & \text{otherwise.} \end{cases}$$

Thus, the STS $(X, \mu)$ is $\mu_{\gamma}\cdot T_0^*$, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_0$.

Example 5.14. Let $X = \{a, b, c\}$ and $\mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Define an operator $\gamma : \mu \to P(X)$ by $\gamma(A) = X, \forall A \in \mu$. Thus, the STS $(X, \mu)$ is $S\cdot T_j$, but $(X, \mu)$ is not $\mu_{\gamma}\cdot T_j^*$ for $j \in \{2, 1, 0\}$.

Remark 5.15. By Theorem 5.6 and Theorem 5.8, we obtain the following diagram of implications.

![Diagram of Implications](image_url)
6 \( S_{(Y,\beta)} \)-Continuous functions

Throughout this section and Section 7, let \((X, \mu)\) and \((Y, \nu)\) be STS and let \(\gamma : \mu \to \mathcal{P}(X)\) and \(\beta : \nu \to \mathcal{P}(Y)\) be operators on ST \(\mu\) and ST \(\nu\), respectively.

**Definition 6.1.** A function \(f : (X, \mu) \to (Y, \nu)\) is said to be \(S_{(Y,\beta)}\)-continuous if \(\forall x \in X\) and \(\forall V \in \nu\) in \(Y\) with \(f(x) \in Y\), \(\exists U \in \mu\) in \(X\) with \(x \in U\) and \(f(\gamma(U)) \subseteq \beta(V)\).

**Theorem 6.2.** Let \(f : (X, \mu) \to (Y, \nu)\) be an \(S_{(Y,\beta)}\)-continuous function, then,
1. \(f(\text{cl}_\mu^Y(A)) \subseteq \text{cl}_\nu^Y(f(A)), \forall A \subseteq (X, \mu)\).
2. \(f^{-1}(F)\) is supra \(\gamma\)-closed set in \((X, \mu)\), \(\forall\) supra \(\beta\)-closed set \(F\) of \((Y, \nu)\).

**Proof.**
(1) Let \(y \in f(\text{cl}_\mu^Y(A))\) and \(V \in \nu\) in \(Y\) with \(y \in V\). Then by hypothesis, \(\exists x \in X\) and \(\exists U \in \mu\) in \(X\) with \(x \in U\) and \(f(x) = y\) and \(f(\gamma(U)) \subseteq \beta(V)\). Since \(x \in \text{cl}_\mu^Y(A)\), then \(\gamma(U) \cap A \neq \emptyset\). Hence, \(\emptyset \neq f(\gamma(U) \cap A) \subseteq f(\gamma(U)) \cap f(A) \subseteq \beta(V) \cap f(A)\). This implies that \(y \in \text{cl}_\nu^Y(f(A))\). Therefore, \(f(\text{cl}_\mu^Y(A)) \subseteq \text{cl}_\nu^Y(f(A))\).

(2) Let \(F\) be any supra \(\beta\)-closed set of \((Y, \nu)\). So, \(f^{-1}(F) \subseteq (X, \mu)\). Then by using (1), we have \(f(\text{cl}_\mu^Y(f^{-1}(F))) \subseteq \text{cl}_\nu^Y(f^{-1}(F)) = F\).

Hence, \(\text{cl}_\nu^Y(f^{-1}(F)) = f^{-1}(F)\). Therefore, \(f^{-1}(F)\) is supra \(\gamma\)-closed in \((X, \mu)\). \(\square\)

**Theorem 6.3.** Items (1) and (2) in the aforementioned theorem are equivalent to each other if either the space \((Y, \nu)\) is supra \(\beta\)-regular or the operator \(\beta\) is supra open.

**Proof.** The implications: “\(S_{(Y,\beta)}\)-continuity of \(f^{-1}\) ⇒ (1) ⇒ (2)” follow from the proof of Theorem 6.2. Then, when the space \((Y, \nu)\) is supra \(\beta\)-regular, we prove the implication: (2) ⇒ “\(S_{(Y,\beta)}\)-continuity of \(f\). Let \(x \in X\) and let \(V \in \nu\) in \(Y\) with \(f(x) \in V\). Since \((Y, \nu)\) is a supra \(\beta\)-regular space, then by Theorem 3.14, \(V \in \nu_\gamma\) in \(Y\). By using (2) of Theorem 6.2, \(f^{-1}(V) \in \mu_\gamma\) in \(X\) with \(x \in f^{-1}(V)\). So \(\exists U \in \mu\) in \(X\) with \(x \in U\) and \(\gamma(U) \subseteq f^{-1}(V)\). This implies that \(f(\gamma(U)) \subseteq V \subseteq \beta(V)\). Therefore, \(f \in S_{(Y,\beta)}\)-continuous.

Now, when \(\beta\) is a supra open operator, we show the implication: (2) ⇒ “\(S_{(Y,\beta)}\)-continuity of \(f\). Let \(x \in X\) and let \(V \in \nu\) in \(Y\) with \(f(x) \in V\). Since \(\beta\) is a supra open operator, then \(\exists W \in \nu_\gamma\) in \(Y\) with \(f(x) \in W\) and \(W \subseteq \beta(V)\). By using (2) of Theorem 6.2, \(f^{-1}(W) \in \mu_\gamma\) in \(X\) with \(x \in f^{-1}(W)\). So \(\exists U \in \mu\) in \(X\) with \(x \in U\) and \(\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))\). This implies that \(f(\gamma(U)) \subseteq \beta(V)\). Hence, \(f \in S_{(Y,\beta)}\)-continuous. \(\square\)

**Definition 6.4.** A function \(f : (X, \mu) \to (Y, \nu)\) is said to be
1. \(\nu_\gamma\)-closed if the image of each supra \(\gamma\)-closed set of \(X\) is supra \(\beta\)-closed in \(Y\).
2. \(S_{(id,\beta)}\)-closed if the image of each supra closed set of \(X\) is supra \(\beta\)-closed in \(Y\).

**Theorem 6.5.** Suppose that a function \(f : (X, \mu) \to (Y, \nu)\) is both \(S_{(Y,\beta)}\)-continuous and \(S_{(id,\beta)}\)-closed, then:
1. The image \(f(A)\) is \(\nu_\gamma\)-closed in \((Y, \nu)\), \(\forall \mu_\gamma\)-closed set \(A\) of \((X, \mu)\).
2. The inverse set \(f^{-1}(B)\) is \(\mu_\gamma\)-closed in \((X, \mu)\), \(\forall \nu_\beta\)-closed set \(B\) of \((Y, \nu)\).

**Proof.**
(1) Let \(U \in \nu_\gamma\) with \(f(A) \subseteq U\). Since \(f\) is \(S_{(Y,\beta)}\)-continuous function, then by using Theorem 6.2 (2), \(f^{-1}(U) \subseteq \mu_\gamma\) in \(X\). Since \(A\) is \(\mu_\gamma\)-closed and \(A \subseteq f^{-1}(U)\), then we have \(\text{cl}_\mu^Y(A) \subseteq f^{-1}(U)\), and hence \(f(\text{cl}_\mu^Y(A)) \subseteq U\). Thus, by Lemma 3.18 (1), \(\text{cl}_\nu^Y(f(A))\) is a supra closed set and since \(f\) is \(S_{(Y,\beta)}\)-closed, then \(f(\text{cl}_\mu^Y(A))\) is supra \(\beta\)-closed set in \(Y\). Therefore, \(\text{cl}_\nu^Y(f(A)) \subseteq \text{cl}_\nu^Y(f(\text{cl}_\mu^Y(A))) = f(\text{cl}_\mu^Y(A)) \subseteq U\). This implies that \(f(A)\) is \(\nu_\gamma\)-closed in \((Y, \nu)\).
Let $V \in \mu_2$ in $X$ with $f^{-1}(B) \subseteq V$. Let $C = cl_{\scriptscriptstyle V}(f^{-1}(B)) \cap (X \setminus V)$, then by Lemma 3.18 (1), $C$ is a supra closed set in $(X, \mu)$. Since $f$ is the $S_{id,\beta}$-closed function, then $f(C)$ is supra $\beta$-closed in $(Y, \nu)$. Since $f$ is the $S_{\nu,\beta}$-continuous function, then by using Theorem 6.2 (1), we have $f(C) = f(cl_{\scriptscriptstyle V}(f^{-1}(B))) \cap f(X \setminus V) \subseteq cl_{\scriptscriptstyle Y}(f(B)) \cap f(X \setminus V) \subseteq f(cl_{\scriptscriptstyle V}(B)) \cap (Y \setminus B) = cl_{\scriptscriptstyle Y}(B) \setminus B$. This implies from Theorem 4.3 that $f(C) = \phi$, and hence $C = \phi$. So, $cl_{\scriptscriptstyle Y}(f^{-1}(B)) \subseteq V$. Thus, $f^{-1}(B)$ is $\mu_2$-g closed in $(X, \mu)$.

**Theorem 6.6.** Let $f : (X, \mu) \to (Y, \nu)$ be a surjective, $S_{\nu,\beta}$-continuous and $S_{id,\beta}$-closed function. If $(X, \mu)$ is $\mu_2$-$T_{T_2}$, then $(Y, \nu)$ is $\nu_2$-$T_{T_2}$. □

**Proof.** Let $V$ be a $\nu_2$-$g$ closed set of $(Y, \nu)$. Since $f$ is $S_{\nu,\beta}$-continuous and $S_{id,\beta}$-closed function. Then by Theorem 6.5 (2), $f^{-1}(V)$ is $\mu_2$-$g$-closed in $(X, \mu)$. Since $(X, \mu)$ is $\mu_2$-$T_{T_2}$, then we have $f^{-1}(V)$ is supra $\gamma$-closed set in $X$. Again, since $f$ is the $S_{id,\beta}$-closed function, then $f(f^{-1}(V))$ is supra $\beta$-closed in $Y$. Therefore, $V$ is supra $\beta$-closed in $Y$ since $f$ is surjective. Hence, $(Y, \nu)$ is $\nu_2$-$T_{T_2}$ space.

**Theorem 6.7.** Let $f : (X, \mu) \to (Y, \nu)$ be an injective, $S_{\nu,\beta}$-continuous and $S_{id,\beta}$-closed function. If $(Y, \nu)$ is $\nu_2$-$T_{T_2}$ space, then $(X, \mu)$ is $\mu_2$-$T_{T_2}$ space.

**Proof.** Let $U$ be any $\mu_2$-$g$-closed set of $(X, \mu)$. Since $f$ is $S_{\nu,\beta}$-continuous and $S_{id,\beta}$-closed function. Then by Theorem 6.5 (1), $f(U)$ is $\nu_2$-$g$-closed in $(Y, \nu)$. Since $(Y, \nu)$ is $\nu_2$-$T_{T_2}$, then $f(U)$ is supra $\beta$-closed in $Y$. Again, since $f$ is $S_{\nu,\beta}$-continuous, so by Theorem 6.2 (2), $f^{-1}(f(U))$ is supra $\gamma$-closed in $(X, \mu)$. Thus, $U$ is supra $\gamma$-closed in $(X, \mu)$ because $f$ is injective. Thus, the space $(X, \mu)$ is $\mu_2$-$T_{T_2}$.

**Theorem 6.8.** If a function $f : (X, \mu) \to (Y, \nu)$ is injective $S_{\nu,\beta}$-continuous and the space $(Y, \nu)$ is $\nu_2$-$T_{T_2}$, then the STS $(X, \mu)$ is $\mu_2$-$T_{T_2}$.

**Proof.** Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $f$ is an injective function and $(Y, \nu)$ is a $\nu_2$-$T_{T_2}$ space. Then $\exists U_1 \in \nu$ and $U_2 \in \nu$ in $Y$ with $f(x_1) \in U_1, f(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since $f$ is $S_{\nu,\beta}$-continuous, $\exists V_1 \in \mu$ and $V_2 \in \mu$ in $X$ with $x_1 \in V_1, x_2 \in V_2, f(y(V_1)) \subseteq \beta(U_1)$ and $f(y(V_2)) \subseteq \beta(U_2)$. Hence, $\beta(U_1) \cap \beta(U_2) = \phi$. Thus, $(X, \mu)$ is $\mu_2$-$T_{T_2}$.

**Theorem 6.9.** If a function $f : (X, \mu) \to (Y, \nu)$ is injective $S_{\nu,\beta}$-continuous and the space $(Y, \nu)$ is $\nu_2$-$T_{T_2}$, then the space $(X, \mu)$ is $\mu_2$-$T_{T_2}$ for $f \in [0, 1]$.

**Proof.** The proof is similar to Theorem 6.8. □

**Definition 6.10.** A function $f : (X, \mu) \to (Y, \nu)$ is said to be $S_{\nu,\beta}$-homeomorphism if $f$ is bijective, $S_{\nu,\beta}$-continuous and $f^{-1}$ is $\mu_{\beta,\nu}$-continuous.

**Theorem 6.11.** Suppose that a function $f : (X, \mu) \to (Y, \nu)$ is an $S_{\nu,\beta}$-homeomorphism. If $(X, \mu)$ is $\mu_2$-$T_{T_2}$, then $(Y, \nu)$ is $\nu_2$-$T_{T_2}$.

**Proof.** Let $\{y\}$ be any singleton set of $(Y, \nu)$. Then $\exists x \in X$ with $y = f(x)$. Then by hypothesis and Theorem 5.2, we get $\{x\}$ is supra $\gamma$-closed or supra $\gamma$-open in $(X, \mu)$. By using Theorem 6.2, we obtain $\{y\}$ is supra $\beta$-closed or supra $\beta$-open. Thus, by Theorem 5.2, the space $(Y, \nu)$ is $\nu_2$-$T_{T_2}$.

### 7 $S_{(\nu,\beta)}$-closed graphs and strongly $S_{(\nu,\beta)}$-closed graphs

In this section, we further investigate general operator approaches of closed graphs of functions. Let $(X \times Y, \tau \times \sigma)$ be the product space of the STS $(X, \mu)$ and $(Y, \nu)$, and let $\rho : \mu \times \nu \to P(X \times Y)$ be an operator on $\mu \times \nu$. 
Definition 7.1. The graph \( G(f) \) of a function \( f : (X, \mu) \to (Y, \nu) \) is called \( S_{Y, \beta} \)-closed if \( \forall (x, y) \in (X \times Y) \backslash G(f) \), \( \exists U \in \mu \) in \( X \) and \( V \in \nu \) in \( Y \) with \( x \in U \), \( y \in V \) and \( (\gamma(U) \times \beta(V)) \backslash G(f) = \phi \).

Lemma 7.2. A function \( f : (X, \mu) \to (Y, \nu) \) has \( S_{Y, \beta} \)-closed graph iff \( \forall (x, y) \in (X \times Y) \backslash G(f) \), \( \exists U \in \mu \) in \( X \) and \( V \in \nu \) in \( Y \) with \( x \in U \), \( y \in V \) and \( f(\gamma(U)) \cap \beta(V) = \phi \).

Proof. The proof is directly from the above definition.

Theorem 7.3. If \( f : (X, \mu) \to (Y, \nu) \) is an \( S_{Y, \beta} \)-continuous function and \( (Y, \nu) \) is a \( \nu_{T_2} \)-space, then \( f \) has an \( S_{Y, \beta} \)-closed graph.

Proof. Let \( (x, y) \in (X \times Y) \backslash G(f) \). Then \( y \neq f(x) \), and since \( (Y, \nu) \) is \( \nu_{T_2} \), \( \exists U, V \in \nu \) in \( Y \) with \( f(x) \in U \), \( y \in V \) and \( \beta(U) \cap \beta(V) = \phi \). Since \( f \) is \( S_{Y, \beta} \)-continuous, then \( \exists W \in \mu \) in \( X \) with \( x \in W \) and \( f(\gamma(W)) \subseteq \beta(U) \). Thus, \( f(\gamma(W)) \cap \beta(V) = \phi \). Therefore, by using Lemma 7.2, \( f \) has an \( S_{Y, \beta} \)-closed graph.

Theorem 7.4. If \( f : (X, \mu) \to (Y, \nu) \) is an \( S_{Y, \beta} \)-continuous injective function with an \( S_{Y, \beta} \)-closed graph, then \( (X, \mu) \) is a \( \mu_{T_2} \) space.

Proof. Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Then \( f(x_1) \neq f(x_2) \). This implies that \( (x_1, f(x_2)) \in (X \times Y) \backslash G(f) \). Since \( f \) has an \( S_{Y, \beta} \)-closed graph, then by using Lemma 7.2, \( \exists U \in \mu \) in \( X \) and \( V \in \nu \) in \( Y \) with \( x_1 \in U \), \( f(x_2) \in V \) and \( f(\gamma(U)) \cap \beta(V) = \phi \). Therefore, \( f \) has an \( S_{Y, \beta} \)-closed graph.

Definition 7.5. The graph \( G(f) \) of a function \( f : (X, \mu) \to (Y, \nu) \) is called strongly \( S_{Y, \beta} \)-closed if \( \forall (x, y) \in (X \times Y) \backslash G(f) \), \( \exists U \in \mu_2 \) in \( X \) and \( V \in \nu_{T_2} \) in \( Y \) with \( x \in U \), \( y \in V \) and \( (U \times V) \backslash G(f) = \phi \).

Lemma 7.6. A function \( f : (X, \mu) \to (Y, \nu) \) has strongly \( S_{Y, \beta} \)-closed graph iff \( \forall (x, y) \in (X \times Y) \backslash G(f) \), \( \exists U \in \mu_2 \) in \( X \) and \( V \in \nu_{T_2} \) in \( Y \) with \( x \in U \), \( y \in V \) and \( f(U) \cap V = \phi \).

Proof. Obvious.

Definition 7.7. A function \( f : (X, \mu) \to (Y, \nu) \) is said to be \( S_{Y, \beta} \)-irresolute if \( \forall x \in X \) and \( \forall V \in \nu_{T_2} \) with \( f(x) \in V \), \( \exists U \in \mu_2 \) with \( x \in U \) and \( f(U) \subseteq V \).

Theorem 7.8. If \( f : (X, \mu) \to (Y, \nu) \) is an \( S_{Y, \beta} \)-irresolute function and \( (Y, \nu) \) is a \( \nu_{T_2} \)-space, then \( f \) has a strongly \( S_{Y, \beta} \)-closed graph.

Proof. Let \( (x, y) \in (X \times Y) \backslash G(f) \). Then \( y \neq f(x) \), and since \( (Y, \nu) \) is \( \nu_{T_2} \), \( \exists U, V \in \nu_{T_2} \) in \( Y \) with \( f(x) \in U \), \( y \in V \) and \( \beta(U) \cap \beta(V) = \phi \). Since \( f \) is \( S_{Y, \beta} \)-irresolute, then \( \exists W \in \mu_2 \) in \( X \) with \( x \in W \) and \( f(W) \subseteq U \). Thus, \( f(W) \cap V = \phi \). Therefore, by using Lemma 7.6, \( f \) has a strongly \( S_{Y, \beta} \)-closed graph.

Theorem 7.9. If \( f : (X, \mu) \to (Y, \nu) \) is an \( S_{Y, \beta} \)-irresolute injective function with a strongly \( S_{Y, \beta} \)-closed graph, then \( (X, \mu) \) is a \( \mu_{T_2} \) space.

Proof. Let \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Then \( f(x_1) \neq f(x_2) \). This implies that \( (x_1, f(x_2)) \in (X \times Y) \backslash G(f) \). Since \( f \) has a strongly \( S_{Y, \beta} \)-closed graph, then by using Lemma 7.6, \( \exists U \in \mu_2 \) in \( X \) and \( V \in \nu_{T_2} \) in \( Y \) with \( x_1 \in U \), \( f(x_2) \in V \) and \( f(U) \cap V = \phi \). Since \( f \) is \( S_{Y, \beta} \)-irresolute, then \( \exists W \in \mu_2 \) in \( X \) with \( x_2 \in W \) and \( f(W) \subseteq V \). Thus, \( f(U) \cap f(W) = \phi \). Therefore, \( U \cap W = \phi \). Hence, \( (X, \mu) \) is a \( \mu_{T_2} \) space.
Lemma 7.10. Suppose $y : \mu \to P(X)$ and $\beta : v \to P(Y)$ are operators on $\mu$ and $v$ respectively. Then we have
1. If $f : (X, \mu) \to (Y, v)$ has a strongly $S_{y,\beta}$-closed graph, then it has an $S_{y,\beta}$-closed graph.
2. If $f : (X, \mu) \to (Y, v)$ has an $S_{y,\beta}$-closed graph, then it has an $S$-closed graph.

Proof. (1) Let $x \in X$, $y \in Y$ with $f(x) \neq y$. Since $f$ has a strongly $S_{y,\beta}$-closed graph, then by Lemma 7.6, $\exists U \in \mu_y$ in $X$ and $V \in v_y$ in $Y$ with $x \in U$, $f(y) \in V$ and $f(U) \cap V = \emptyset$. Thus, $\exists E \in \mu_x$ in $X$ with $x \in E \subseteq y(E) \subseteq U_x$ and $F \in v_y$ in $Y$ with $y \in F \subseteq \beta(F) \subseteq V_y$. Therefore, $f(y(E)) \cap \beta(F) = \emptyset$. Hence, by Lemma 7.2, $f$ has an $S_{y,\beta}$-closed graph.

(2) Let $x \in X$, $y \in Y$ with $f(x) \neq y$. Since $f$ has an $S_{y,\beta}$-closed graph, then by Lemma 7.2, $\exists U \in \mu_x$ in $X$ and $V \in v_y$ in $Y$ with $x \in U$, $f(y) \in V$ and $f(y(U)) \cap \beta(V) = \emptyset$. Thus, $f(U) \cap V = \emptyset$. Hence, $f$ has an $S$-closed graph. □

Remark 7.11. From Lemma 7.10, we obtain the following diagram.

$$\text{strongly } S_{(y,\beta)}\text{-closed graph} \rightarrow S_{(y,\beta)}\text{-closed graph} \rightarrow S\text{-closed graph}$$

Theorem 7.12. Suppose that $y$ and $\beta$ are operators on $\mu$ and $v$ respectively. If $STS (X, \mu)$ and $STS (Y, v)$ are supra $y$-regular and supra $\beta$-regular spaces, then the following are equivalent for any function $f : (X, \mu) \to (Y, v)$:
1. $f$ has strongly $S_{y,\beta}$-closed graph.
2. $f$ has $S_{y,\beta}$-closed graph.
3. $f$ has $S$-closed graph.

Proof. Follows directly from Theorem 3.14 and the above diagram. □

Definition 7.13. An operator $\rho : \mu \times v \to P(X \times Y)$ is said to be supra associated with $y$ and $\beta$ if $\rho(U \times V) = y(U) \times \beta(V)$ holds $\forall U \in \mu$ and $\forall V \in v$.

Definition 7.14. The operator $\rho : \mu \times v \to P(X \times Y)$ is said to be supra regular with respect to $y$ and $\beta$ if $\forall (x, y) \in X \times Y$ and $\forall W \in (\mu \times v)$ in $X \times Y$ with $(x, y) \in W$, $\exists U \in \mu$ in $X$ and $V \in v$ in $Y$ with $x \in U$, $y \in V$ and $y(U) \times \beta(V) \subseteq \rho(W)$.

Theorem 7.15. Let $\rho : \mu \times \mu \to P(X \times X)$ be a supra associated operator with $y$ and $\beta$. If $f : (X, \mu) \to (Y, v)$ is an $S_{y,\beta}$-continuous function and $(Y, v)$ is a $v$-$T^*_2$ space, then the set $A = \{ (x, y) \in X \times X : f(x) = f(y) \}$ is supra $\rho$-closed of $(X \times X, \mu \times \mu)$.

Proof. We have to show that $cD^\rho_{v}(A) \subseteq A$. Let $(x, y) \in (X \times X) \setminus A$. Since $(y, v)$ is $v$-$T^*_2$. Then $\exists U, V \in v$ in $Y$ with $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \emptyset$. Furthermore, for $U$ and $V$, $\exists G, H \in \mu$ in $X$ with $x \in G$, $y \in H$ and $f(y(G)) \subseteq \beta(U)$ and $f(y(H)) \subseteq \beta(V)$ because $f$ is $S_{y,\beta}$-continuous. Thus, we obtain $(x, y) \in y(G) \times y(H) = \rho(G \times H) \cap A = \emptyset$ since $G \times H \in \mu \times \mu$. This gives that $(x, y) \notin cD^\rho_{v}(A)$. Hence, the proof is complete. □

Corollary 7.16. Suppose $\rho : \mu \times \mu \to P(X \times X)$ is supra associated operator with $y$ and $\beta$, and it is supra regular with $y$ and $\beta$. An $STS (X, \mu)$ is $\mu$, $T^*_2$ iff the diagonal set $\Delta = \{ (x, x) : x \in X \}$ is supra $\rho$-closed of $(X \times X, \mu \times \mu)$. 

Theorem 7.17. Let \( \rho : \mu \times \nu \to P(X \times Y) \) be a supra associated operator with \( y \) and \( \beta \). If \( f : (X, \mu) \to (Y, \nu) \) is \( S_{y,\beta}\)-continuous and \( (Y, \nu) \) is \( \nu_f\-T_2 \), then the graph of \( f \), \( G(f) = \{(x, f(x)) \in X \times Y \} \) is a supra \( \rho \)-closed set of \( (X \times Y, \mu \times \nu) \).

Proof. Similar to Theorem 7.15. \( \square \)

Definition 7.18. Let \( (X, \mu) \) be an STS and \( y \) be an operator on \( \mu \). A subset \( S \) of \( X \) is said to be \( \mu_{y,\beta}\)-compact if for any supra open cover \( \{U_i, i \in \mathbb{N}\} \) of \( S \), \( \exists \) a finite subfamily \( \{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\} \) with \( S \subseteq \gamma(U_{i_1}) \cup \gamma(U_{i_2}) \cup \ldots \cup \gamma(U_{i_n}) \).

Theorem 7.19. Suppose that \( y \) is supra regular and \( \rho : \mu \times \nu \to P(X \times Y) \) is supra regular with respect to \( y \) and \( \beta \). Let \( f : (X, \mu) \to (Y, \nu) \) be a function whose graph \( G(f) \) is supra \( \rho \)-closed in \( (X \times Y, \mu \times \nu) \). If a subset \( S \) is \( S_{y,\beta}\)-compact in \( (Y, \nu) \), then \( f^{-1}(S) \) is supra \( y \)-closed in \( (X, \mu) \).

Proof. Suppose that \( f^{-1}(S) \) is not supra \( y \)-closed, then \( \exists \) a point \( x \) with \( x \in \text{cl}^\rho_y(f^{-1}(S)) \) and \( x \notin f^{-1}(S) \). Since \((x, s) \notin G(f) \) and \( \forall s \in S \) and \( \text{cl}^\rho_y(G(f)) \subseteq G(f) \), \( \exists W \in (\mu \times \nu) \) in \( X \times Y \) with \((x, s) \in W \) and \( \beta(W) \cap G(f) = \phi \).

By supra regularity of \( \rho \), \( \forall s \in S \) we can take \( U(s) \in \nu \) and \( V(s) \in \nu \) in \( Y \) with \( x \notin U(s) \) and \( s \in V(s) \) and \( \gamma(U(s)) \times \beta(V(s)) \subseteq \rho(W) \). Then we have \( f(\gamma(U(s))) \cap \beta(V(s)) = \phi \). Since \( \{V(s) : s \in S\} \) is supra open cover of \( S \), then by \( \mu_{y,\beta}\)-compactness \( \exists \) a finite number \( s_1, s_2, \ldots, s_n \) with \( S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup \ldots \cup \beta(V(s_n)) \).

By the supra regularity of \( y \), \( \exists \) a supra open set \( U \in \mu \) in \( X \) with \( x \in U \) and \( \gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap \ldots \cap \gamma(U(s_n)) \). Therefore, we have \( \gamma(U) \cap f^{-1}(S) \subseteq U(s_1) \cap f^{-1}(\beta(V(s_1))) = \phi \). This shows that \( x \notin \text{cl}^\rho_y(f^{-1}(S)) \). This is a contradiction. Thus, \( f^{-1}(S) \) is supra \( y \)-closed in \( X \). \( \square \)

8 Conclusion

In this paper, the notion of an operator \( y \) on supra open sets has been studied and the notion of supra \( y \)-open sets of an STS \((X, \mu)\) has been defined. The notions of \( \mu y\-g \)-closed sets and operator on subspace ST have been presented and investigated. New \( \mu y\)-separation axioms have been introduced and explored. Some characterizations of \( S_{y,\beta}\)-continuous functions have been studied and some properties of \( S_{y,\beta}\)-closed graph and strongly \( S_{y,\beta}\)-closed graph have been given. Several examples have been exhibited to validate the discussed results.

In the upcoming works, we plan to study these concepts on the contents of supra soft topological spaces [35] and binary STS [36].

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