On the powers of the descent set statistic

Richard Ehrenborg and Alex Happ

Abstract

We study the sum of the $r$th powers of the descent set statistic and how many small prime factors occur in these numbers. Our results depend upon the base $p$ expansion of $n$ and $r$.

1 Introduction

It has always been interesting to study divisibility properties of sequences defined combinatorially. Three classical examples are Fibonacci numbers, the partition function, and binomial coefficients. The Fibonacci numbers satisfy $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. Ramanujan discovered that the partition function satisfies, among other relations, that $5$ divides $p(5n+4)$. The binomial coefficients are well-studied modulo a prime; see the theorems of Lucas and Kummer in Section 2. In this paper we consider divisibility properties of the sum of powers of the descent set statistic from permutation enumeration.

The descent set statistic was first studied by MacMahon [6]. For a permutation $\pi$ in the symmetric group $S_n$, the descent set of $\pi$ is the subset of $\{1, 2, \ldots, n-1\}$ given by $\text{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$. The descent set statistics $\beta_n(S)$ are defined for subsets $S$ of $[n-1]$ by

$$\beta_n(S) = |\{\pi \in S_n : \text{Des}(\pi) = S\}|.$$ 

Since there are $n!$ permutations, we directly have

$$n! = \sum_{S \subseteq [n-1]} \beta_n(S).$$

Define $A_n^r$ to be the sum of the $r$th powers of the descent set statistics, that is,

$$A_n^r = \sum_{S \subseteq [n-1]} \beta_n(S)^r.$$ 

This quantity occurs naturally as moments of the random variable $\text{Des}(S)$, where the set $S$ is chosen with a uniform distribution from all subsets of the set $[n-1]$.

In Section 3 we give two expressions, depending on the parity of $r$ for $A_n^r$; see Lemma 3.1. We continue by showing that for an odd prime $p$ and an even positive integer $r$, if $m$ and $n$ contain the same non-zero digits in base $p$, then the prime $p$ dividing $A_m^r$ is equivalent to $p$ dividing $A_n^r$. In Section 4 we give lower bounds for the number of prime factors in $A_n^r$. These bounds depend on the digit sum of $n$ in base $p$. Unfortunately, we do not obtain any bound when $p$ is an odd prime and $r$ is even. In Section 5 we sharpen the results by collecting terms together occurring in the expansion of Lemma 3.1. The method of collection is by considering orbits of a group action. First we use the cyclic group $\mathbb{Z}_{p^k}$, and then we use a group defined by the action on the balanced $p$-ary tree of cyclically
Proposition 2.1. Let $\alpha_s = \prod_{r < s} x \mod p$ for all integers $s \equiv 0 \mod p$. The lower bounds obtained in this section for the prime factors of $p$ in $A_n^r$ now also depend on the base $p$ expansion of $r$.

We end in the concluding remarks by presenting two tables obtained by computation to compare our bounds with the actual number of factors of 2 and 3 occurring in $A_n^r$.

2 Preliminaries

Define $\alpha_n(S)$ by the sum

$$\alpha_n(S) = \sum_{T \subseteq S} \beta_n(T).$$

Observe that $\alpha_n(S)$ enumerates the number of permutations in $\mathfrak{S}_n$ with descent set contained in the set $S$. Especially, we know that $A_n^1 = \alpha_n([n-1]) = n!$. For more on descents; see [8] Section 1.4.

Define a bijection $\co$ from subsets of the set $[n-1]$ to compositions of $n$ by sending the set $S = \{s_1 < s_2 < \cdots < s_k\}$ to the composition $\co(S) = (c_1, c_2, \ldots, c_k)$, where $c_i = s_i - s_{i-1}$ with $s_0 = 0$ and $s_k = n$. See, for instance, [1] or [7, Section 7.19]. It is now straightforward to observe that $\alpha_n(S)$ is given by the multinomial coefficient $\binom{n}{\co(S)}$.

Using elementary number theory we have three observations.

Proposition 2.1. Let $p$ be a prime. Assume that $r$ and $s$ are both greater than or equal to $k$ and $r \equiv s \mod p^{k-1} \cdot (p-1)$. Then the congruence $A_n^r \equiv A_n^s \mod p^k$ holds. Especially, the statement $p^k$ divides $A_n^r$ is equivalent to $p^k$ divides $A_n^s$.

Proof. We may assume that $r < s$, that is, $s-r = p^{k-1} \cdot (p-1) \cdot j$ for a positive integer $j$. For an integer $x$ which is relative prime to the prime $p$, Euler’s theorem implies that $x^s \equiv x^r \cdot (x^{p^{k-1}(p-1)})^j \equiv x^r \mod p^k$. For an integer $x$ which is divisible by the prime $p$, we have $x^s \equiv 0 \equiv x^r \mod p^k$ since $r, s \geq k$. Thus for all integers $x$ we have $x^s \equiv x^r \mod p^k$ and we conclude $A_n^s \equiv \sum_{S \subseteq [n-1]} \beta_n(S)^s \equiv \sum_{S \subseteq [n-1]} \beta_n(S)^r \equiv A_n^r \mod p^k$.

When the prime $p$ is 2 and $k \geq 3$, we have an improvement of a factor of 2.

Proposition 2.2. Assume that $r$ and $s$ are both greater than or equal to $k \geq 3$ and $r \equiv s \mod 2^{k-2}$. Then the congruence $A_n^r \equiv A_n^s \mod 2^k$ holds. Especially, the statement $2^k$ divides $A_n^r$ is equivalent to $2^k$ divides $A_n^s$.

Proof. For an odd integer $x$ we know that $x^{2^{k-2}} \equiv 1 \mod 2^k$, which yields the better bound using the same argument as in the proof of Proposition 2.1.

Proposition 2.3. Let $p$ be a prime and $r$ an integer such that $r \geq k \cdot p$. If $p^k$ divides the $k$ numbers $A_n^{r-(p-1)}, A_n^{r-2(p-1)}, \ldots, A_n^{r-k(p-1)}$, through $A_n^{r-k(p-1)}$, then $p^k$ divides $A_n^r$.

Proof. By Fermat’s little theorem we know $x^{p-1} - 1 \equiv 0 \mod p$ for $x$ relative prime to $p$. Hence the $k$th power of this quantity is divisible by $p^k$, that is, $(x^{p-1} - 1)^k \equiv 0 \mod p^k$. Note that $x^k \equiv 0 \mod p^k$ for $x$ not relative prime to $p$. Multiplying these two statements we obtain

$$x^{kp} - \binom{k}{1} \cdot x^{kp-(p-1)} + \cdots + (-1)^k \cdot x^k \equiv 0 \mod p^k$$

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for all \( x \). Multiply this polynomial relation with \( x^{r-kp} \), substitute \( x \) to be \( \beta_n(S) \), and sum over all \( S \subseteq [n-1] \) to obtain the linear recursion

\[
A_n^r - \binom{k}{1} A_n^{r-(p-1)} + \cdots + (-1)^k \cdot A_n^{r-k(p-1)} \equiv 0 \pmod{p^k}.
\]

This relation yields the result. \( \square \)

**Example 2.4.** Note using Table 1 that for \( 8 \leq n \leq 20 \), the power \( 2^5 \) divides \( A_n^r \) when \( 5 \leq r \leq 9 \). Hence, Proposition 2.3 gives that \( 2^5 \) divides \( A_n^r \) for \( r \geq 5 \).

**Example 2.5.** Using Table 2 we know for \( n = 6 \) and \( 8 \leq n \leq 20 \) that \( 3^2 \) divides \( A_n^3 \) and \( A_n^5 \). Hence, Proposition 2.3 implies for \( r \) odd and \( r \geq 3 \) that \( 3^2 \) divides \( A_n^r \). Similarly, we know for \( n \in \{9, 10, 12, 13, 15, 16, 18, 19, 20\} \) that \( 3^3 \) divides \( A_n^3, A_n^5 \) and \( A_n^7 \). Therefore, for these same values of \( n \), \( 3^3 \) divides \( A_n^r \) for \( r \) odd and \( r \geq 3 \).

**Remark 2.6.** Note that Propositions 2.1 through 2.3 apply to any sequence of the form \( \sum_{i=1}^{N} c_i \cdot d_i^r \) where \( c_i \) and \( d_i \) are integers.

We end this section by reviewing Lucas’ theorem, see [5, Chapter XXIII, Section 228], and Kummer’s theorem, see [7], for multinomial coefficients.

**Theorem 2.7** (Lucas). Let \( p \) be a prime and \( \vec{c} = (c_1, c_2, \ldots, c_k) \) be a weak composition of \( n \), that is, \( 0 \) is allowed as an entry. Expand \( n \) and each \( c_i \) in base \( p \), that is, \( n = \sum_{j \geq 0} n_j \cdot p^j \) and \( c_i = \sum_{j \geq 0} c_{i,j} \cdot p^j \) where \( 0 \leq n_j, c_{i,j} \leq p - 1 \). Let \( \vec{c}_j \) be the weak composition \( \vec{c}_j = (c_{1,j}, c_{2,j}, \ldots, c_{k,j}) \). Then the multinomial coefficient \( \binom{n}{\vec{c}} \) modulo \( p \) is given by

\[
\binom{n}{\vec{c}} \equiv \prod_{j \geq 0} \binom{n_j}{c_{i,j}} \pmod{p}.
\]

Let \( \text{carries}_p(\vec{c}) \) denote the number of carries when adding \( c_1 + c_2 + \cdots + c_k \) in base \( p \).

**Theorem 2.8** (Kummer). For a prime \( p \) and a composition \( \vec{c} = (c_1, c_2, \ldots, c_k) \) of \( n \), the largest power \( d \) such that \( p^d \) divides the multinomial coefficient \( \binom{n}{\vec{c}} \) is given by \( \text{carries}_p(\vec{c}) \).

### 3 Divisibility by odd primes

First, we express the sum \( A_n^r = \sum_{S \subseteq [n-1]} \beta_n(S)^r \) in terms of \( \alpha_n(S) \).

**Lemma 3.1.** When \( r \) is even, \( A_n^r \) is given by

\[
A_n^r = \sum_{T_1 \cup T_2 \cup \cdots \cup T_r \subseteq [n-1]} (-1)^{\sum_{i=1}^{r} |T_i|} \cdot 2^{n-1-|\bigcup_{i=1}^{r} T_i|} \cdot \prod_{i=1}^{r} \alpha_n(T_i).
\]

When \( r \) is odd, we have

\[
A_n^r = \sum_{T_1 \cup T_2 \cup \cdots \cup T_r \subseteq [n-1]} (-1)^{n-1+\sum_{i=1}^{r} |T_i|} \cdot \prod_{i=1}^{r} \alpha_n(T_i).
\]
Proof. We begin by expanding $\beta_n(S)$ in terms of $\alpha_n(S)$:

$$A_n^r = \sum_{S \subseteq [n-1]} \beta_n(S)^r$$

$$= \sum_{S \subseteq [n-1]} \prod_{i=1}^{r} \left( \sum_{T_i \subseteq S} (-1)^{|S-T_i|} \cdot \alpha_n(T_i) \right)$$

$$= \sum_{T_1 \cup T_2 \cup \cdots \cup T_r \subseteq [n-1]} \sum_{T_i \subseteq S} (-1)^{|S-T_i|} \cdot (-1)^{\sum_{i=1}^{r} |T_i|} \cdot \prod_{i=1}^{r} \alpha_n(T_i).$$

When $r$ is even, we have $(-1)^{|S|} = 1$, and the inner sum has $2^{n-1-|\bigcup_{i=1}^{r} T_i|}$ terms. When $r$ is odd, the inner sum is zero unless the union $\bigcup_{i=1}^{r} T_i$ is the whole set $[n-1]$. \hfill \Box

**Theorem 3.2.** Let $p$ be an odd prime and $r$ an even positive integer. Assume that $m$ and $n$ contain the same non-zero digits in base $p$. Then the congruence $2^{-m} \cdot A_m^r \equiv 2^{-n} \cdot A_n^r \mod p$ holds. Especially, the prime $p$ divides $A_n^r$ if and only if $p$ divides $A_m^r$.

Proof. Let $m$ and $n$ have the base $p$ expansions $m = \sum_{j \geq 0} m_j \cdot p^j$ and $n = \sum_{j \geq 0} n_j \cdot p^j$. Then there exists a permutation $\pi$ on the non-negative integers such that $m_j = n_{\pi(j)}$ for all $j \geq 0$. Essentially, the permutation $\pi$ permutes the powers of the prime $p$. Define a bijection $f$ on the non-negative integers by $f \left( \sum_{j \geq 0} a_j \cdot p^j \right) = \sum_{j \geq 0} a_j \cdot p^{\pi(j)}$, where $0 \leq a_j \leq p-1$. Note that

$$f(m) = \sum_{j \geq 0} m_j \cdot p^{\pi(j)} = \sum_{j \geq 0} n_{\pi(j)} \cdot p^{\pi(j)} = \sum_{j \geq 0} n_j \cdot p^j = n.$$

Furthermore, when there are no carries adding $x$ and $y$ in base $p$, this function is additive, that is, $f(x + y) = f(x) + f(y)$. Also note that the inverse function $f^{-1}$ is additive under the same condition. In terms of compositions, we have that if $\vec{c} = (c_1, c_2, \ldots, c_k)$ is a composition of $m$ such that $\text{carries}_p(\vec{c}) = 0$, then the composition $f(\vec{c}) = (f(c_1), f(c_2), \ldots, f(c_k))$ is a composition of $f(m) = n$.

Let the non-carry power set $\text{NCP}(m)$ be the collection of all subsets of $[m-1]$ whose associated composition has no carries when added in base $p$, that is,

$$\text{NCP}(m) = \{ T \subseteq [m-1] : \text{carries}_p(\text{co}(T)) = 0 \}.$$

Observe that $\text{NCP}(m)$ is closed under inclusion. Note that we can define a bijection $f : \text{NCP}(m) \rightarrow \text{NCP}(n)$ by composing the three maps

$$\text{NCP}(m) \xrightarrow{\text{co}} \{ \vec{c} \in \text{Comp}(m) : \text{carries}_p(\vec{c}) = 0 \} \xrightarrow{f} \{ \vec{d} \in \text{Comp}(n) : \text{carries}_p(\vec{d}) = 0 \} \xrightarrow{\text{co}^{-1}} \text{NCP}(n).$$

Since the compositions $\vec{c}$ and $f(\vec{c})$ have the same length, the function $f$ preserves cardinality. But there is a more direct description of the last map $f$ on sets. For $T = \{ t_1 < t_2 < \cdots < t_k \} \in \text{NCP}(m)$, we claim that $f(T) = \{ f(t_1), f(t_2), \ldots, f(t_k) \}$. Let $\vec{c}$ be the composition $\text{co}(T)$. By definition, the $i$th element of $f(T)$ is the initial partial sum of the $i$ first elements of $f(\vec{c})$, that is, $f(c_1) + \cdots + f(c_i)$. Since the whole sum $c_1 + \cdots + c_k$ has no carries, the partial sum also has no carries. Hence, the $i$th element of $f(T)$ is given by $f(c_1) + \cdots + f(c_i) = f(c_1 + \cdots + c_i) = f(t_i)$, proving the claim.

Also note that for a composition $\vec{c}$ without any carries, we have by Lucas’ Theorem that

$$\begin{pmatrix} m \atop \vec{c} \end{pmatrix} \equiv \begin{pmatrix} f(m) \atop f(\vec{c}) \end{pmatrix} \mod p,$$
since the factors of the product in Lucas’ Theorem are permuted by the permutation \( \pi \). Hence, for a set \( T \) in \( \text{NCP}(m) \) we know that \( \alpha_m(T) = \alpha_n(f(T)) \).

We now use the expansion in equation (3.1). Let \( \tilde{c}^i \) be the composition associated with the subset \( T_i \) of \( [m - 1] \). Similarly, let \( U_i \) be the subset of \( [n - 1] \) associated with the composition \( f(\tilde{c}^i) = \tilde{d}^i \). Next we study the two unions \( \bigcup_{i=1}^{r} T_i \) and \( \bigcup_{i=1}^{r} U_i \). However, they may not be in the collection \( \text{NCP}(m) \), respectively, \( \text{NCP}(n) \).

For \( I \) a non-empty subset of the index set \( [r] \) let \( T_I \) be the intersection \( \bigcap_{i \in I} T_i \). Note that \( T_I \) belongs to \( \text{NCP}(m) \) since this collection is closed under inclusion. Similarly let \( U_I \) be the intersection \( \bigcap_{i \in I} U_i \) which belongs to \( \text{NCP}(n) \). Note that \( f(T_I) = U_I \) so the two sets \( T_I \) and \( U_I \) have the same cardinality. By inclusion-exclusion we have

\[
\left| \bigcup_{i=1}^{r} T_i \right| = \sum_{\emptyset \neq I \subseteq [r]} (-1)^{|I|-1} \cdot |T_I| = \sum_{\emptyset \neq I \subseteq [r]} (-1)^{|I|-1} \cdot |U_I| = \left| \bigcup_{i=1}^{r} U_i \right|.
\]

Now observe that the non-zero terms in equation (3.1) modulo \( p \) are the terms where \( T_i \) belongs to \( \text{NCP}(m) \). Hence, modulo \( p \) we have that

\[
A_m^r \equiv \sum_{T_1, T_2, \ldots, T_r \in \text{NCP}(m)} (-1)^{\sum_{i=1}^{r} |T_i|} \cdot 2^{m-1 - |\bigcup_{i=1}^{r} T_i|} \cdot \prod_{i=1}^{r} \alpha_m(T_i)
\]

\[
\equiv 2^{m-n} \cdot \sum_{U_1, U_2, \ldots, U_r \in \text{NCP}(n)} (-1)^{\sum_{i=1}^{r} |U_i|} \cdot 2^{n-1 - |\bigcup_{i=1}^{r} U_i|} \cdot \prod_{i=1}^{r} \alpha_n(U_i)
\]

\[
\equiv 2^{m-n} \cdot A_n^r \mod p.
\]

This proves the identity. Finally, since 2 is invertible modulo \( p \), we obtain that \( A_m^r \) and \( A_n^r \) either both have a factor of \( p \) or none of them have a factor of \( p \).

**Corollary 3.3.** When \( r \) is even and \( p \) is an odd prime, the congruence \( A_m^r \equiv A_n^r \mod p \) holds.

**Proof.** Since \( n \) and \( p \cdot n \) have the same non-zero digits modulo \( p \), Theorem 3.2 applies. Hence, it is enough to observe that \( 2^{m-n} \equiv (2^n)^p \equiv 2^n \mod p \) using Fermat’s little theorem.

**Corollary 3.4.** When \( r \) is even and \( p \) is an odd prime, \( A_m^r \) is not divisible by \( p \).

**Proof.** It is enough to check that \( A_m^1 = 1 \) is not divisible by \( p \).

**Example 3.5.** We can compute \( A_4^3 \) to observe that this number has a factor of 3. Hence by Proposition 2.1 we know that for all even \( r \), the prime 3 divides \( A_4^r \). Furthermore, 14 in base 3 consists of two 1’s and one 2. Hence Theorem 3.2 implies for \( n = 16, 22, 32, 34, 38, 42, 46, 48, 58, 64, 66, 86, 88, \ldots \) that 3 divides \( A_n^r \) as well.

**Example 3.6.** Note that 5 divides \( A_3^2 = 10 \). Hence, we know that 5 divides \( A_n^{4i+2} \) for \( n \) of the form \( 3 \cdot 5^k \). One may compute that 5 also divides \( A_2^2 \) and \( A_3^2 \). This implies that 5 divides \( A_n^{4i+2} \) for \( n \) belonging to the following two sequences: 12, 52, 60, 252, 260, 300, 1252, 1260, 1300, \ldots and 13, 17, 53, 65, 77, 85, 253, 265, 325, 377, 385, \ldots.
4 On the number of prime factors

For a positive integer $n$, let $u_p(n)$ be the sum of the digits when $n$ is written in base $p$. More formally, for $n = \sum_{i \geq 0} n_i \cdot p^i$, where $0 \leq n_i \leq p-1$, the function $u_p(n)$ is given by the sum $\sum_{i \geq 0} n_i$. Furthermore, for a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$, define $u_p(\vec{c})$ to be the sum of the digits when all the parts of $\vec{c}$ are written in base $p$, that is, $u_p(\vec{c}) = \sum_{i=1}^{k} u_p(c_i)$. Finally, recall that $\text{carries}_p(\vec{c})$ denotes the number of carries when adding $c_1 + c_2 + \cdots + c_k$ in base $p$.

Lemma 4.1. For a composition $\vec{c}$ of $n$, the sum of its digits in base $p$ satisfies

$$u_p(\vec{c}) = (p-1) \cdot \text{carries}_p(\vec{c}) + u_p(n).$$

Proof. If one lines up the parts $c_1, c_2, \ldots, c_k$ of $\vec{c}$ in base $p$, note that any one of the $u_p(\vec{c})$ units in any of these addends has only two options: It may either contribute to a carry along with another $p-1$ units in its column, or it can directly become one of the $u_p(n)$ units in $n$. \hfill \Box

Corollary 4.2. Let $p$ be a prime. Then the number of factors of $p$ in $A_n^1$ is $(n - u_p(n))/(p-1)$.

Proof. Note that $A_n^1 = n! = \binom{n}{1,1,\ldots,1}$. Hence by Kummer’s theorem the number of factors of $p$ is $\text{carries}_p(1,1,\ldots,1) = (u_p(1,1,\ldots,1) - u_p(n))/(p-1)$. \hfill \Box

Similarly, define the depth of $n$ to be $d_p(n) = u_p(n) - 1$, that is, the sum of the digits of $n$ in base $p$ beyond the requisite digit greater than zero in its first position. Further, define the depth of a composition $\vec{c}$ to be the sum of the depth of each of its parts, that is, $d_p(\vec{c}) = \sum_{i=1}^{k} d_p(c_i)$. The next lemma is direct.

Lemma 4.3. For a composition $\vec{c}$ into $k$ parts, $u_p(\vec{c}) = d_p(\vec{c}) + k$.

Recall according to the map $\text{co}$ from subsets $S \subseteq [n - 1]$ to compositions $\vec{c}$ of $n$ that the number of parts $k$ of $\vec{c}$ is one more than the cardinality of $S$. Combining this observation with the previous two lemmas yields the next result.

Proposition 4.4. For a set $S \subseteq [n - 1]$ and its associated composition $\text{co}(S) = \vec{c}$ of $n$, the number of carries $\text{carries}_p(\vec{c})$ is given by $(d_p(\vec{c}) + |S| - d_p(n))/(p-1)$.

This gives way to the main result in this section.

Theorem 4.5. When $r$ is odd and $p$ is prime, the sum $A_n^r$ has at least $(n - 1 - r \cdot d_p(n))/(p-1)$ factors of $p$.

Proof. Consider a term in equation (32), where we let $\text{co}(T_i) = \vec{c}^i$. The number of factors of $p$ in this term is given by

$$\sum_{i=1}^{r} \text{carries}_p(\vec{c}^i) = \sum_{i=1}^{r} \frac{d_p(\vec{c}^i) + |T_i| - d_p(n)}{p-1} \geq \sum_{i=1}^{r} \frac{|T_i| - d_p(n)}{p-1} \geq \frac{n - 1 - r \cdot d_p(n)}{p-1},$$

since $d_p(\vec{c}^i) \geq 0$ for all $i$ and $\sum_{i=1}^{r} |T_i| \geq |\bigcup_{i=1}^{r} T_i| = n - 1$. \hfill \Box

We can say something stronger when the prime $p$ is 2.

Theorem 4.6. The sum $A_n^r$ is divisible by $2^{n-1-r \cdot d_2(n)}$. 

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Proof. The case when $r$ is odd follows from Theorem 4.5. Now suppose $r$ is even, and consider a term in equation (3.1). The number of factors of 2 in this term is given by

\[
\begin{align*}
\left| \bigcup_{i=1}^{r} T_i \right| + \sum_{i=1}^{r} \text{carries}_2(\hat{c}^i) &\geq n - 1 - \sum_{i=1}^{r} |T_i| + \sum_{i=1}^{r} \text{carries}_2(\hat{c}^i) \\
&= n - 1 - \sum_{i=1}^{r} |T_i| + \sum_{i=1}^{r} (d_2(\hat{c}^i) + |T_i| - d_2(n)) \\
&\geq n - 1 - r \cdot d_2(n).
\end{align*}
\]

Since $d_2(2^k) = 0$, we have the following corollary.

**Corollary 4.7.** When $n$ is a power of 2, then $A_n^r$ is divisible by $2^{n-1}$.

In this case, we actually have equality.

**Proposition 4.8.** When $n$ is a power of 2, then $2^{n-1}$ is the highest power dividing $A_n^r$.

**Proof.** For $x$ an $r$-tuple $(T_1, T_2, \ldots, T_r)$, let $h(x)$ denote the associated term in Lemma 3.4. Note that the expression for $h(x)$ depends on the parity of $r$. Observe in the proofs of Theorems 4.5 and 4.6 that we have equality in the bound for those terms where the sets $T_i$ are disjoint and $d_2(\hat{c}^i) = 0$ for all $i$. Note that the latter condition requires all the parts of $\hat{c}^i$ to be powers of 2. Let $X$ be the collection of all such $r$-tuples.

Consider an $r$-tuple $x = (T_1, T_2, \ldots, T_r) \in X$, and choose the smallest $1 \leq k \leq |r/2|$ such that $T_{2k-1} \neq T_{2k}$, if one exists. Let $x'$ be obtained by switching the $(2k-1)$st and $2k$th subsets, that is, $x' = (T_1, \ldots, T_{2k-2}, T_{2k}, T_{2k-1}, T_{2k+1}, \ldots, T_r)$. Observe that $h(x') = h(x)$. Since the subsets $T_i$ are all disjoint, the only case where such a $k$ does not exist is when $T_1, T_2, \ldots, T_{|r/2|}$ are all empty. This occurs in a single $r$-tuple $x_0$, where $x_0 = (\emptyset, \emptyset, \ldots, \emptyset)$ if $r$ is even, and $x_0 = (\emptyset, \emptyset, \ldots, \emptyset, [n-1])$ if $r$ is odd. When $r$ is even, we directly observe that $h(x_0) \equiv 2^{n-1} \mod 2^n$. When $r$ is odd we have $h(x_0) \equiv n! \equiv 2^{n-1} \mod 2^n$, using that $n$ is a power of 2. Now, after pairing up all these terms except the term $h(x_0)$, the result follows by

\[
A_n^r \equiv \sum_{x \in X} h(x) \equiv h(x_0) \equiv 2^{n-1} \mod 2^n. \qed
\]

## 5 Improving the bound

We now improve upon the bounds of Theorems 4.5 and 4.6.

**Proposition 5.1.** When $p$ is an odd prime and $n \geq 2$, the sum $A_n^{p^k}$ has at least

\[
\frac{n - 1 - p^k \cdot d_p(n)}{p - 1} + k
\]

factors of $p$.

**Proof.** Consider the term indexed by the $p^k$-tuple $(T_1, T_2, \ldots, T_{p^k})$ in equation (3.2), and consider the action by the shift $(T_2, T_3, \ldots, T_{p^k}, T_1)$. Note that the size of the orbit of this action is $p^i$, for some $0 \leq i \leq k$. Grouping these $p^i$ identical terms gives $i$ factors of $p$. However, this means that
our tuple \((T_1, T_2, \ldots, T_{p^k})\) can be written as \((T_1, T_2, \ldots, T_{p^r}, T_1, T_2, \ldots, T_{p^i}, \ldots, T_1, T_2, \ldots, T_{p^j})\) up to a cyclic shift, and further, \(\bigcup_{j=1}^{p^i} T_j = [n - 1]\). Hence, the number of factors of \(p\) in these terms is

\[
i + \sum_{j=1}^{p^k} \text{carries}_p(\overline{c^j}) = i + \sum_{j=1}^{p^k} \frac{d_p(\overline{c^j}) + |T_j| - d_p(n)}{p - 1}
\]

\[
\geq i + \frac{\left(\sum_{j=1}^{p^k} |T_j|\right) - p^k \cdot d_p(n)}{p - 1}
\]

\[
\geq i + \frac{p^{k-i} \cdot (n - 1) - p^k \cdot d_p(n)}{p - 1}
\]

\[
= i + \frac{(p^{k-i} - 1) \cdot (n - 1)}{p - 1} + \frac{n - 1 - p^k \cdot d_p(n)}{p - 1}
\]

\[
\geq k + \frac{n - 1 - p^k \cdot d_p(n)}{p - 1},
\]

where in the last step we used \((p^{k-i} - 1)/(p - 1) = 1 + p + \cdots + p^{k-i-1} \geq k - i\) and \(n - 1 \geq 1\). \(\square\)

The above proof uses the action of the cyclic group \(\mathbb{Z}_{p^k}\) to collect terms together. We can improve the bound of Proposition 5.1 in some cases by using a larger group acting on the \(r\)-tuples.

Let \(q\) be the prime power \(p^k\). We define the group \(G_q\) acting on the set \([q]\). The generators are indexed by pairs \((a, b)\) where \(1 \leq a \leq k\) and \(0 \leq b \leq p^k - a - 1\). The generator \(\sigma_{a,b}\) is given by the following product of \(p\)-cycles,

\[
\sigma_{a,b} = \prod_{i=1}^{p^a-1} (i + bp^a, i + bp^a + p^{a-1}, \ldots, i + bp^a + (p - 1)p^{a-1}).
\]

To give a geometric picture of the action of this group, consider a balanced \(p\)-ary tree of depth \(k\). This tree has \(q\) leaves, which we label 1 through \(q\). Furthermore, the tree has \(\frac{q^k}{p-1}\) internal nodes, which are indexed by the pairs \((a, b)\). The \(a\) coordinate states that the internal node is at depth \(k - a\). The \(b\) coordinate indicates which node at that depth, reading from left to right. The generator \(\sigma_{a,b}\) then cyclically shifts the \(p\) children of this node. See Figure I for an example.

With this geometric picture, it is straightforward to observe that the group has order \(p^{\frac{q^k}{p-1}}\). Given a \(q\)-tuple of sets \(x = (T_1, T_2, \ldots, T_q)\), let the group \(G_q\) act on \(x\) by permuting the indices. Let \(\text{Orb}_x\) be the orbit of the \(q\)-tuple \(x\), that is, \(\text{Orb}_x = \{g \cdot x : g \in G_q\}\). Note that the cardinality of the orbit \(\text{Orb}_x\) is a power of \(p\).

Additionally, for an \(r\)-tuple \(x = (T_1, \ldots, T_r)\) let \(f_n^r(x) = (-1)^{\sum_{i=1}^{r} |T_i|} \cdot \prod_{i=1}^{r} \alpha_n(T_i)\).

**Proposition 5.2.** Let \(q = p^k\) and \(d_p(n) > 0\). For a \(q\)-tuple \(x = (T_1, T_2, \ldots, T_q)\), the sum \(\sum_{y \in \text{Orb}_x} f_n^q(y)\) has at least

\[
\frac{q - 1 + \left|\bigcup_{j=1}^{p^i} T_j\right| - q \cdot d_p(n)}{p - 1}
\]

factors of \(p\).

**Proof.** The proof is by induction on \(k\). The induction basis is \(k = 0\), that is, \(q = 1\). Here \(\text{Orb}_x\) consists only of \((T)\). The number of \(p\)-factors are

\[
\text{carries}_p(\overline{c}) = \frac{d_p(\overline{c}) + |T| - d_p(n)}{p - 1} \geq \frac{|T| - d_p(n)}{p - 1}.
\]
Figure 1: A balanced ternary tree of depth 3 with the action of $\sigma_{2,1}$ shown.

Figure 1: A balanced ternary tree of depth 3 with the action of $\sigma_{2,1}$ shown.

since $d_p(\vec{c}) \geq 0$, which completes the basis of the induction.

Now assume that the statement is true for all $p$-powers strictly less than $q$. Notice that $f_n(y) = f_n(x)$ for all $y \in \text{Orb}_x$. Hence,

$$
\sum_{y \in \text{Orb}_x} f_n(y) = |\text{Orb}_x| \cdot f_n^q(x) = |\text{Orb}_x| \cdot (-1)^{\sum_{i=1}^{q} |T_i|} \prod_{i=1}^{q} \alpha_n(T_i).
$$

Furthermore, the number of factors of $p$ in the last expression is

$$
\log_p(|\text{Orb}_x|) + \sum_{i=1}^{q} \text{carries}_p(\vec{c}^i) = \log_p(|\text{Orb}_x|) + \sum_{i=1}^{q} \left( d_p(\vec{c}^i) + |T_i| - d_p(n) \right).
$$

For $0 \leq b \leq p-1$ let $x_b$ denote the $q/p$-tuple $(T_{b,q/p+1}, \ldots, T_{(b+1)q/p})$, that is, the $q/p$-tuple of sets below the node $(k-1,b)$ in the tree.

First, assume that the stabilizer of $x$ contains an element involving the permutation $\sigma_{k,0}$. That is, the stabilizer contains a rotation centered at the root $(k,0)$ of the tree. Then the leaves below the nodes $(k-1,0)$ are the same as the leaves below $(k-1,b)$. Then the cardinality of the orbit $\text{Orb}_x$ is the same as the size of the orbit $\text{Orb}_{x_0}$. Hence we can apply the induction hypotheses to the node $(k-1,0)$ of the tree:

$$
\log_p(|\text{Orb}_x|) + \sum_{i=1}^{q} \text{carries}_p(\vec{c}^i) = \log_p(|\text{Orb}_{x_0}|) + \sum_{i=1}^{q/p} \text{carries}_p(\vec{c}^i) + \sum_{i=q/p+1}^{q} \text{carries}_p(\vec{c}^i)
$$

$$
\geq \frac{q/p - 1 + \left| \bigcup_{i=1}^{q/p} T_i \right| - q/p \cdot d_p(n)}{p-1} + \sum_{i=q/p+1}^{q} \frac{d_p(\vec{c}^i) + |T_i| - d_p(n)}{p-1}
$$

$$
= \frac{q/p - 1 + \left| \bigcup_{i=1}^{q/p} T_i \right| - q \cdot d_p(n)}{p-1} + \sum_{i=q/p+1}^{q} \frac{d_p(\vec{c}^i) + |T_i|}{p-1}.
$$

If $T_i$ is non-empty, then $|T_i| \geq 1$. If $T_i$ is empty, then $\vec{c}^i$ is the composition $n$, so $d_p(\vec{c}^i) = d_p(n) \geq 1$ by our assumption. In both cases we have $d_p(\vec{c}^i) + |T_i| \geq 1$ for all $q/p + 1 \leq i \leq q$. Thus, we can
apply this inequality
\[
\log_p(|\text{Orb}_x|) + \sum_{i=1}^{q} \text{carries}_p(c^i) \geq \frac{q/p - 1 + |\bigcup_{i=1}^{q} T_i| - q \cdot d_p(n)}{p - 1} + \frac{q - q/p}{p - 1},
\]
which yields the bound.

It remains to consider the case when the stabilizer of \(x\) does not contain a rotation centered at the root \((k, 0)\). Now the cardinality of the orbit of \(x\) is given by the product
\[
|\text{Orb}_x| = \prod_{b=0}^{p-1} |\text{Orb}_{x_b}|.
\]
Hence we apply the induction hypotheses to each child of the root
\[
\log_p(|\text{Orb}_x|) + \sum_{i=1}^{q} \text{carries}_p(c^i) = \sum_{b=0}^{p-1} \left( \log_p(|\text{Orb}_{x_b}|) + \sum_{i=1}^{q/p} \text{carries}_p(c^{bq/p+i}) \right)
\geq \sum_{b=0}^{p-1} \frac{q/p - 1 + |\bigcup_{i=1}^{q/p} T_{bq/p+i}| - q/p \cdot d_p(n)}{p - 1} \geq \frac{q - p + |\bigcup_{i=1}^{q} T_i| - q \cdot d_p(n)}{p - 1},
\]
which yields the bound. This completes the second case and the induction. ⊓⊔

**Theorem 5.3.** For \(r\) odd and \(d_p(n) > 0\), the sum \(A_n^r\) contains at least
\[
\left\lfloor \frac{r - u_p(r) + n - 1 - r \cdot d_p(n)}{p - 1} \right\rfloor
\]
factors of \(p\).

**Proof.** Let \(r = \sum_{i=1}^{u_p(r)} q_i\) where \(q_i\) is a power of \(p\). Note that a power \(p^j\) occurs at most \(p - 1\) times in this sum. Now define the group \(G\) to be the Cartesian product \(G = \prod_{i=1}^{u_p(r)} G_{q_i}\). Furthermore, let \(G\) act on the set \([r]\) by letting the \(G_{q_i}\) act on the interval \([q_1 + \cdots + q_{i-1} + 1, q_1 + \cdots + q_{i-1} + q_i]\). The action of the group \(G\) can be viewed as forest consisting of \(u_p(r)\) trees. Finally, let \(G\) act on a \(r\)-tuple by acting on the indices of the tuple.

Note that the function \(f^r_n\) is multiplicative in the following meaning. For an \(r\)-tuple \(x = (T_1, \ldots, T_r)\) define \(x_i\) to be the \(q_i\)-tuple \((T_{q_1+\cdots+q_{i-1}+1}, \ldots, T_{q_1+\cdots+q_{i-1}+q_i})\). Then we have
\[
f^r_n(T_1, \ldots, T_r) = \prod_{i=1}^{u_p(r)} f^q_n(x_i).
\]
Now the sum over an orbit of the \(r\)-tuple \(x = (T_1, \ldots, T_r)\) factors as
\[
\sum_{y \in \text{Orb}_x} f^r_n(y) = \prod_{i=1}^{u_p(r)} \sum_{y_i \in \text{Orb}_{x_i}} f^q_n(y_i).
\]
Hence we can apply Proposition 5.2 to each factor, and the sum over the orbit has at least
\[
\sum_{i=1}^{u_p(r)} \frac{1}{p-1} \left( q_i - 1 + \left| \bigcup_{j=q_i+\cdots+q_{i-1}+1}^{q_i+\cdots+q_i} T_j \right| - q_i \cdot d_p(n) \right) \\
\geq \frac{1}{p-1} \left( r - u_p(r) + \bigcup_{j=1}^{r} T_j \right) - r \cdot d_p(n) \\
= \frac{1}{p-1} \cdot (r - u_p(r) + n - 1 - r \cdot d_p(n)),
\]
where the last equality comes from the assumption \( T_1 \cup T_2 \cup \cdots \cup T_r = [n-1] \) in equation (3.2). \( \square \)

Again, we can make a stronger statement when \( p = 2 \).

**Theorem 5.4.** For \( d_2(n) > 0 \), the sum \( A_n^r \) contains at least \( r - u_2(r) + n - 1 - r \cdot d_2(n) \) factors of 2.

**Proof.** The case where \( r \) is odd follows from Theorem 5.3. We retain the notation of the proof of Theorem 5.3. Note that in that proof, we did not use the parity of \( r \) until the very end. Now assume that \( r \) is even. For an \( r \)-tuple \( x = (T_1, \ldots, T_r) \) define the function \( g_n^r(x) = 2^{n-1-|\bigcup_{i=1}^{r} T_i|} \cdot f_n^r(x) \), which is the expression in equation (3.1). Hence the number of factors of 2 in the sum over the orbit
\[
\sum_{y \in \text{Orb}_x} g_n^r(y) = 2^{n-1-|\bigcup_{i=1}^{r} T_i|} \cdot \sum_{y \in \text{Orb}_x} f_n^r(y)
\]
is bounded from below by the sum of \( n - 1 - |\bigcup_{i=1}^{r} T_i| \) and the expression (5.1). That is,
\[
n - 1 - \bigg| \bigcup_{i=1}^{r} T_i \bigg| + r - u_2(r) + \bigg| \bigcup_{j=1}^{r} T_j \bigg| - r \cdot d_2(n) = n - 1 + r - u_2(r) - r \cdot d_2(n). \]
\( \square \)

Theorem 5.4 improves upon Theorem 4.6 by at least 1 when \( n \) is not a 2-power.

**Corollary 5.5.** When \( n \) is not a power of 2 and \( r \geq 2 \), then \( A_n^r \) has at least \( n - r \cdot d_2(n) \) factors of 2.

**Proof.** Note that \( r \geq 2 \) implies that \( r > u_2(r) \), that is, \( r - u_2(r) - 1 \geq 0 \). Hence by Theorem 5.4 we have \( r - u_2(r) + n - 1 - r \cdot d_2(n) \geq n - r \cdot d_2(n) \).

**Corollary 5.6.** Let \( n \) satisfy the inequality \( 2^k \leq n \leq 2^{k+1} - 1 \). Then \( A_n^2 \) is divisible by \( 2^{2^k-1} \).

**Proof.** Write \( n \) as the sum \( 2^k + a \). When \( a = 0 \) there is nothing to prove by Corollary 4.7. When \( a \geq 1 \) we have \( d_2(n) = u_2(a) \). Furthermore, since for each 2-power \( 2^j \), where \( j \geq 1 \), we have \( 2^j - 2 \cdot u_2(2^j) \geq 0 \). But for \( j = 0 \) we have \( 2^j - 2 \cdot u_2(2^j) = -1 \). Hence for all non-negative \( a \) we have \( a - 2 \cdot u_2(a) \geq -1 \). Hence the bound by Corollary 5.5 yields \( n - 2 \cdot d_2(n) = 2^k + a - 2 \cdot u_2(a) \geq 2^k - 1 \).

6 Concluding remarks

Some of the results in this paper are reminiscent of results in the papers [1] [2] [3], where there are results which depend on the binary expansion of the parameters. However, as the reader can see from Tables 1 and 2, where we present computational results for the numbers of factors of the primes 2 and 3 in \( A_n^r \), a lot of work remains in order to understand these numbers.

A final question is to understand the asymptotic behavior of \( A_n^r \) as \( n \) tends to infinity. How similar is this behavior to Stirling’s formula?
| \( n \) | \( n_2 \) | \( d_2(n) \) | \( r = 1 \) | \( r = 2 \) | \( r = 3 \) | \( r = 4 \) | \( r = 5 \) | \( r = 6 \) | \( r = 7 \) | \( r = 8 \) | \( r = 9 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 10 | 0 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 |
| 3 | 11 | 1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 |
| 4 | 100 | 0 | 3,3 | 3,3 | 3,3 | 3,3 | 3,3 | 3,3 | 3,3 | 3,3 |
| 5 | 101 | 1 | 3,3 | 3,6 | 3,2 | 3,3 | 3,2 | 3,2 | 3,3 | 2,2 |
| 6 | 110 | 1 | 4,4 | 4,4 | 3,3 | 4,4 | 3,3 | 3,5 | 3,3 | 4,4 |
| 7 | 111 | 2 | 4,4 | 3,3 | 1,2 | 2,5 | 2,2 | 3,4 | 2,2 | 4,5 |
| 8 | 1000 | 0 | 7,7 | 7,7 | 7,7 | 7,7 | 7,7 | 7,7 | 7,7 | 7,7 |
| 9 | 1001 | 1 | 7,7 | 7,7 | 6,7 | 7,8 | 6,6 | 6,6 | 5,7 | 6,6 |
| 10 | 1010 | 1 | 8,8 | 8,8 | 7,7 | 8,8 | 7,7 | 7,8 | 6,10 | 8,8 |
| 11 | 1011 | 2 | 8,8 | 7,8 | 5,5 | 5,6 | 3,6 | 3,5 | 3,5 | 4,7 |
| 12 | 1100 | 1 | 10,10 | 10,10 | 9,9 | 10,10 | 9,9 | 9,11 | 8,12 | 10,10 |
| 13 | 1101 | 2 | 10,10 | 9,9 | 7,7 | 7,10 | 5,7 | 4,7 | 3,6 | 4,11 |
| 14 | 1110 | 2 | 11,11 | 10,11 | 8,10 | 8,13 | 6,8 | 5,9 | 3,7 | 4,11 |
| 15 | 1111 | 3 | 11,11 | 9,9 | 6,7 | 5,8 | 3,7 | 3,8 | 3,6 | 4,8 |
| 16 | 10000 | 0 | 15,15 | 15,15 | 15,15 | 15,15 | 15,15 | 15,15 | 15,15 | 15,15 |
| 17 | 10001 | 1 | 15,15 | 15,15 | 14,15 | 15,17 | 14,14 | 14,14 | 13,14 | 15,15 |
| 18 | 10010 | 1 | 16,16 | 16,16 | 15,15 | 16,16 | 15,15 | 15,17 | 14,15 | 16,16 |
| 19 | 10011 | 2 | 16,16 | 15,15 | 13,14 | 13,13 | 11,12 | 10,16 | 8,11 | 9,14 |
| 20 | 10100 | 1 | 18,18 | 18,18 | 17,17 | 18,18 | 17,17 | 17,19 | 16,18 | 18,18 |

Table 1: A comparison of our best prediction of the number of factors of 2 in \( A_n^r \) with the actual number. Predictions are given first, colored according to whether the result is given by Proposition 2.1, Proposition 2.2, Theorem 4.6 or Theorem 5.4 and the actual value is given second.
Theorem 5.3 and the actual value is given  

| n  | n₃ | d₃(n) | r = 1 | r = 2 | r = 3 | r = 4 | r = 5 | r = 6 | r = 7 | r = 8 | r = 9 |
|-----|----|-------|------|------|------|------|------|------|------|------|------|
| 2   |    | 1     | 0,0  | 0,0  | 0,0  | 0,0  | 0,0  | 0,0  | 0,0  | 0,0  | 0,0  |
| 3   | 10 | 0     | 1,1  | 0,0  | 2,2  | 0,0  | 1,1  | 0,0  | 1,1  | 0,0  | 3,3  |
| 4   | 11 | 1     | 1,1  | 0,0  | 1,2  | 0,0  | 1,1  | 0,0  | 1,1  | 0,0  | 2,3  |
| 5   | 12 | 2     | 1,1  | 0,0  | 1,1  | 0,0  | 1,1  | 0,0  | 1,2  | 0,0  | 1,1  |
| 6   | 20 | 1     | 2,2  | 0,0  | 2,4  | 0,0  | 1,2  | 0,0  | 2,2  | 0,0  | 2,5  |
| 7   | 21 | 2     | 2,2  | 0,0  | 1,2  | 0,0  | 1,1  | 0,0  | 1,1  | 0,0  | 2,3  |
| 8   | 22 | 3     | 2,2  | 0,0  | 1,2  | 0,0  | 1,2  | 0,0  | 2,2  | 0,0  | 2,2  |
| 9   | 100| 0     | 4,4  | 0,0  | 5,6  | 0,0  | 4,4  | 0,0  | 4,4  | 0,0  | 6,7  |
| 10  | 101| 1     | 4,4  | 0,0  | 4,6  | 0,0  | 3,4  | 0,0  | 3,4  | 0,0  | 4,6  |
| 11  | 102| 2     | 4,4  | 0,0  | 3,3  | 0,0  | 1,2  | 0,0  | 2,2  | 0,0  | 2,4  |
| 12  | 110| 1     | 5,5  | 0,0  | 5,6  | 0,0  | 4,5  | 0,0  | 4,5  | 0,0  | 5,7  |
| 13  | 111| 2     | 5,5  | 0,0  | 4,4  | 0,0  | 2,3  | 0,0  | 2,3  | 0,0  | 3,5  |
| 14  | 112| 3     | 5,5  | 0,1  | 3,4  | 1,2  | 1,1  | 1,1  | 2,2  | 1,1  | 2,5  |
| 15  | 120| 2     | 6,6  | 0,0  | 5,5  | 0,0  | 3,5  | 0,0  | 2,5  | 0,0  | 3,6  |
| 16  | 121| 3     | 6,6  | 1,1  | 4,5  | 1,1  | 1,3  | 1,1  | 2,3  | 1,1  | 3,7  |
| 17  | 122| 4     | 6,6  | 0,0  | 3,4  | 0,0  | 1,2  | 0,0  | 2,2  | 0,0  | 2,3  |
| 18  | 200| 1     | 8,8  | 0,0  | 8,11 | 0,0  | 7,8  | 0,0  | 7,8  | 0,0  | 8,12 |
| 19  | 201| 2     | 8,8  | 0,0  | 7,8  | 0,0  | 5,7  | 0,0  | 4,7  | 0,0  | 4,10 |
| 20  | 202| 3     | 8,8  | 0,0  | 6,8  | 0,0  | 3,5  | 0,0  | 2,6  | 0,0  | 3,7  |

Table 2: A comparison of our best prediction of the number of factors of 3 in $A_n^r$ with the actual number. Predictions are given first, colored according to whether the result is given by Proposition 2.1, Example 2.5, Example 3.5, Theorem 4.5, Proposition 5.1, or Theorem 5.3 and the actual value is given second.
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R. Ehrenborg, A. Happ. Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, richard.ehrenborg@uky.edu, alex.happ@uky.edu