Mixtures of Mean-Preserving Contractions

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Abstract: Given a purely atomic probability measure with support on \( n \) points, 
\( P \), any mean-preserving contraction of \( P \), \( Q \), with support on \( m > n \) points is a 
mixture of mean-preserving contractions of \( P \), each with support on most \( n \) points. 
We illustrate an application of this result in economics.

Keywords: Fusion, Convex Domination, Mean-Preserving Spread, Information Design, Bayesian Persuasion

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1. Introduction

We prove a simple result pertaining to statistical experiments, as first introduced and 
promulgated in Blackwell (1951) \([4]\) and soon after in Blackwell (1953) \([5]\) and Blackwell 
and Girshick (1954) \([6]\). Other important related works include Hardy, Littlewood and 
Pólya (1959) \([11]\), Rothschild and Stiglitz (1970) \([18]\), and more recently Athey and Levin 
(2018) \([1]\). See also Rasmusen and Petrakis (1992) \([16]\) who establish a result in a similar 
spirit to this one; that any four-point mean-preserving spread can be constructed from 
two three-point mean-preserving spreads.

Throughout we use the language introduced in Elton and Hill (1992) \([7]\), who intro-
duce the idea of a fusion of a probability distribution. There, they define a fusion of a 
probability distribution \( P \) as a probability distribution that can be obtained by “fusing” 
together parts of \( P \). Indeed, the intuition they provide is physical: think of the probabil-

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ity measure as cups of liquid, and hence a fusion is any distribution that can be obtained by pouring parts of various cups together. $Q$ is a fusion of $P$ if and only if $P \succ_C Q$, which denotes that $P$ convexly dominates $Q$. Equivalently, $P$ is a mean-preserving spread of $Q$ (or $Q$ is a mean-preserving contraction of $P$).

The main result of this paper, Theorem 1.5, establishes that given a purely atomic probability measure $P$ on $\mathbb{R}$ with support on $n$ points, any fusion of $P$, $Q$, with support on $n + 1$ points is the convex combination of two fusions of $P$, $Q'$ and $Q''$, each of which have support on at most $n$ points. An important corollary of this result, Corollary 1.7, is that any fusion $Q$ of $P$ is a mixture of fusions with support on at most $n$ points.

In addition to the works mentioned in the first paragraph, several other related papers bear mention. Elton and Hill (1998) [8] provide a constructive proof of a seminal result of Hardy, Littlewood and Polya (1959) [11] on convex domination of measures. Similar results are obtained by Mirsky (1961) [15] and by Fischer and Holbrook (1980) [9].

In recent years a sub-field of economics, information design and Bayesian persuasion, has experienced a surge in interest.¹ Broadly, works in this area explore how a principal (or principals) can design information structures in various environments in order to achieve some objective. Corollary 1.7 has useful ramifications for competitive Bayesian persuasion problems²: instead of checking deviations to any mean preserving contraction of $P$, one need check only deviations to mean preserving contractions with support on $n$ points. In Subsection 1.2 we present a simple competitive persuasion problem in which this result is used. Jain and Whitmeyer (2019) [12] and Whitmeyer (2018) [19] both appeal to Corollary 1.7 in similar settings to this example.

1.1. The Main Result

Throughout, identify vectors with bold font. Let $X = \mathbb{R}$ and denote by $\mathcal{B}$ the Borel subsets of $X$. Moreover, let $\mathcal{P}$ denote the set of Borel probability measures on $(X, \mathcal{P})$.

¹See e.g. Rayo and Segal (2010) [17] and Kamenica and Gentzkow (2011) [14], which two papers first sparked the widespread interest in the topic; Bergemann and Morris (2018) [3], a more recent and relatively self-contained overview; and Kamenica (2018) [13], a recent survey paper.

²In which multiple principals compete by designing information structures.
Let $P \in \mathcal{P}$ be a purely atomic probability measure with support on $n$ points i.e. $\text{supp} P = a := \{a_1, a_2, \ldots, a_n\}$, with respective masses $p_1, p_2, \ldots, p_n$, where $p_i > 0$ for all $i = 1, \ldots, n$ and $\sum_i^n p_i = 1$. Denote $\mathbf{p} := (p_1, p_2, \ldots, p_n)$. Without loss of generality, $a_1 < a_2 < \cdots < a_n$. Likewise, let $Q \in \mathcal{P}$ be a purely atomic probability measure with support on $m$ points: $\text{supp} R = b := \{b_1, b_2, \ldots, b_m\}$, with respective masses $q_1, q_2, \ldots, q_m$, where $q_i > 0$ for all $i = 1, \ldots, m$ and $\sum_i^m q_i = 1$. Denote $\mathbf{q} := (q_1, q_2, \ldots, q_m)$. Finally, denote $\mathbf{p}_a := (p_1 a_1, p_2 a_2, \ldots, p_n a_n)$ and analogously for $\mathbf{q}_b$.

We use the following definition of a fusion, as introduced in Elton and Hill [7]:

**Definition 1.1.** $Q$ is a **Fusion** of $P$ if there exists a non-negative row-stochastic (partition) $n \times m$ matrix $F$ that satisfies

$$\mathbf{p} F = \mathbf{q}$$

and

$$\mathbf{p}_a F = \mathbf{q}_b$$

Elton and Hill denote the class of fusions of $P$ by $\mathcal{F}(P)$ and show (in Theorem 4.1. [7]) that $Q$ being a fusion of $P$ is equivalent to $P \succ_C Q$, which denotes that $P$ convexly dominates $Q$. Formally,

**Definition 1.2.** For probability measures $P, Q \in \mathcal{P}$, $P$ **Convexly Dominates** $Q$ if $\int \phi dP \geq \int \phi dQ$ for all nonnegative continuous convex functions $\phi$ for which both integrals exist.

Equivalently, $P$ is a mean-preserving spread of $Q$ (or $Q$ is a mean-preserving contraction of $P$). See Rothschild and Stiglitz (1970) [18], who formulate this idea in order to explore the preference of a risk-averse agent over lotteries of prizes.

From Definition 1.1, a fusion of an arbitrary $n$-atom probability measure $P$ with atoms, $\mathbf{a}$, into $n + 1$ atoms, $\mathbf{b}$, can be defined by the $n \times (n + 1)$ partition matrix $F$. Each row of $F$, $F_i$, sums to 1 and denotes the partition of the weight $P_i$ at atom $a_i$ across the atoms $\mathbf{b}$.

Next, we say that that $Q = \alpha Q' + (1 - \alpha) Q''$ for fusions $Q$, $Q'$, and $Q''$ if and only if

$$F = \alpha F' + (1 - \alpha) F''$$

(1)
for partition matrices $F$, $F'$, and $F''$, where 0-columns are inserted into the partition matrices corresponding to the zero-probability atoms in $Q'$ and $Q''$ but otherwise the ratios within columns in $F'$ and $F''$ are the same as in $F$.

Consider $F$ corresponding to fusion $Q$. Because $F$ is an $n \times (n + 1)$ matrix, there is some subset of column vectors that is linearly dependent, and because all atoms are distinct, this subset has a minimal size of three. Hence, any of these column vectors can be expressed as a linear sum of the others, which in turn means it can be zeroed with the other vectors modified correspondingly.

**Lemma 1.3.** Two different column vectors in the linearly dependent set can be chosen such that zeroing one of them gives entries in the remaining vectors from 0 to 1.

**Proof.** Let $0$ and $1$ denote the column vectors of all zeroes and ones, respectively. Without loss of generality, suppose the linearly dependent set consists of $n + 1$ column vectors $f_j$, where $0 \leq f_j \leq 1$ and $\sum_{j=1}^{n+1} f_{ij} = 1$ for all $i$ or $\sum_{j=1}^{n+1} f_j = 1$, the component-wise sum of the vectors $f_j$. Then there are scalars $c_j$ such that $\sum_{j=1}^{n+1} c_j f_j = 0$. They can be arranged and signs changed such that for some $k \leq n$

$$\sum_{j=1}^{k} c_j f_j = \sum_{j=k+1}^{n+1} c_j f_j$$

where $c_j > 0$ for all $j$. Define

$$c_j^* := \max_{1 \leq j \leq k} \{c_j\} \quad c_j^{**} := \max_{k+1 \leq j \leq n+1} \{c_j\}$$

Then, the corresponding $f_j^*$ and $f_j^{**}$ can be zeroed. To see that take one, say $f_j^*$:

$$f_j^* = -\sum_{j \leq k, j \neq j^*} \frac{c_j f_j}{c_j^*} + \sum_{j \geq k+1} \frac{c_i f_j}{c_{j^*}} = 0$$

In the new partition matrix for the new fusion,

$$f'_{ij} = \begin{cases} 
(1 - \frac{c_j}{c_{j^*}}) f_{ij}, & j \leq k, j \neq j^* \\
(1 + \frac{c_j}{c_{j^*}}) f_{ij}, & j \geq k + 1 
\end{cases}$$

By construction, for $j \leq k, j \neq j^*$:

$$0 \leq \frac{c_j}{c_{j^*}} \leq 1$$
and so

\[ 0 \leq \left( 1 - \frac{c_j}{c_j^*} \right) f_{ij} \leq 1 \]

Also,

\[ 0 \leq \sum_{j \leq k \atop j \neq j^*} \left( 1 - \frac{c_j}{c_j^*} \right) f_j < \sum_{j \leq k \atop j \neq j^*} f_j < 1 \]

Then

\[ 1 = \sum_{j=1}^{n+1} f_j = \sum_{j \neq j^*} f_j' = \sum_{j \leq k \atop j \neq j^*} \left( 1 - \frac{c_j}{c_j^*} \right) f_j + \sum_{j \geq k+1} \left( 1 + \frac{c_j}{c_j^*} \right) f_j \]

This means

\[ \sum_{j \geq k+1} \left( 1 + \frac{c_j}{c_j^*} \right) f_j \leq 1 \]

i.e. for all \( i \),

\[ \sum_{j \geq k+1} \left( 1 + \frac{c_j}{c_j^*} \right) f_{ij} \leq 1 \]

and since

\[ \left( 1 + \frac{c_j}{c_j^*} \right) \geq 1 \]

for all \( j \), we have

\[ 0 \leq \left( 1 + \frac{c_j}{c_j^*} \right) f_{ij} \leq 1 \]

\[ \blacksquare \]

**Lemma 1.4.** The partition matrices \( F' \) and \( F'' \) given in Equation 1 are unique.

**Proof.** Suppose for the sake of contradiction that \( f_{j^*} \) and \( f_{j^*} \) are not unique, i.e. that there is a third \( f_{j^*} \) that can be zeroed, where \( j^* \neq j^\dagger \neq j^{**} \). Without loss of generality suppose \( 1 \leq j^\dagger \leq k \). Then by definition \( c_{j^\dagger} \leq c_{j^*} \). Without loss of generality let \( c_{j^\dagger} < c_{j^*} \) (in the case of an equality zeroing \( f_{j^*} \) would also zero \( f_{j^\dagger} \)). Thus,

\[ f_{j^\dagger} = -\sum_{j \leq k \atop j \neq j^\dagger} \frac{c_j}{c_{j^\dagger}} f_j + \sum_{j \geq k+1} \frac{c_j}{c_{j^\dagger}} f_j = 0 \]

Then, because

\[ \frac{c_j}{c_{j^\dagger}} > 1 \]
the other zeroing produces the following component,

\[ f''_{ij} = \left(1 - \frac{c_j}{c_j^*}\right) f_{ij} < 0 \]

which is not permitted. ■

Since this zeroing uniquely determines \( F' \) and \( F'' \), \( \alpha \) in Equation 1 follows as well and can be calculated from the coefficients in the linear dependence.

Combining Lemmata 1.3 and 1.4 we obtain the following theorem.

**Theorem 1.5.** Let \( Q \in \mathcal{F}(P) \) be any fusion of \( P \) with support on \( n + 1 \) points, \( n \geq 2 \). Then \( Q \) is the convex combination of two purely atomic probability measures \( Q' \) and \( Q'' \); \( Q', Q'' \in \mathcal{F}(P) \), each with support on at most \( n \) points. \( Q' \) and \( Q'' \) are unique.

**Example 1.6.** Here we provide an example. Let the purely atomic measure \( P \) be given by

\[ P = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{3}{10} & \frac{3}{10} & \frac{2}{5} \end{pmatrix} \]

and \( F \) be given by

\[ F = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \]

Hence,

\[ Q = \begin{pmatrix} \frac{5}{6} & \frac{3}{4} & \frac{1}{2} & \frac{1}{6} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \end{pmatrix} \]

Then, \( F' \) and \( F'' \) are

\[ F' = \begin{pmatrix} 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{7}{12} & 0 & 0 & \frac{5}{12} \\ \frac{7}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix} \quad F'' = \begin{pmatrix} 0 & 0 & \frac{5}{9} & \frac{4}{9} \\ \frac{7}{9} & 0 & 0 & \frac{2}{9} \\ 0 & \frac{7}{12} & \frac{5}{12} & 0 \end{pmatrix} \]

Thus,

\[ Q' = \begin{pmatrix} \frac{5}{6} & \frac{1}{2} & \frac{1}{6} \\ \frac{21}{40} & \frac{1}{10} & \frac{3}{8} \end{pmatrix}, \quad Q'' = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{6} \\ \frac{7}{15} & \frac{1}{3} & \frac{1}{5} \end{pmatrix} \]

and \( \alpha = 4/7 \).
Corollary 1.7. Any fusion $Q \in \mathcal{F}(P)$ supported on $m > n$ points is a mixture of fusions with support on at most $n$ points.

Proof. From Theorem 1.5, given a purely atomic probability measure $P$ with support on $n$ points, any fusion $Q \in \mathcal{F}(P)$ with support on $n + 1$ points is a convex combination of two fusions $Q', Q'' \in \mathcal{F}(P)$ with support on at most $n$ points. If $m = n + 1$ then the result is simply Theorem 1.5. If not, simply iterate backward until the desired mixture of fusions of $P$ with support on $n$ points is obtained.

Taking weak limits, this result holds for fusions of $P$ that are not purely atomic. ■

1.2. Simplifying Competitive Persuasion

Here we illustrate the usefulness of Corollary 1.7 in persuasion problems, and look at a simple competitive persuasion problem. General results about this problem have been established in Au and Kawai (2018) [2]. We show how our result can be used to easily show candidate vectors of strategies are (or are not) equilibria.

There are two competing sellers who wish a single buyer to purchase their good. The sellers sell an identical product with random quality: each seller’s product is an independent identically distributed random variable $X$ and $Y$, distributed according to the purely atomic measure $P$:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

Each seller, without knowing the quality of her (or the other seller’s) good, simultaneously chooses a Blackwell experiment conditioned on her type. The buyer observes the realization of the two experiments and naturally buys from the seller whose product has highest expected value for the product, conditional on the observed signals. The buyer randomizes fairly over the two sellers if they have the same expected quality. A seller gets a payoff normalized to one if the buyer buys from her and zero if the buyer buys from the other seller.

From Gentzkow and Kamenica (2016) [10] and others, the choice of experiment by a
seller is one of choosing a feasible distribution of posterior means. The set of feasible
distributions of posterior means is the set of distributions that are mean-preserving
contractions of the prior; each seller may choose any \( Q \in \mathcal{F}(P) \).

**Proposition 1.8.** There is a Nash Equilibrium in which both sellers choose the cdf
\( F(x) := \mathbb{P}(X \leq x) \), where

\[
F(x) = \begin{cases} 
\frac{2}{3}x, & 0 \leq x \leq \frac{1}{2} \\
\frac{8}{3}x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4}
\end{cases}
\]

**Proof.** From Corollary 1.7 it suffices to check that there is no profitable deviation to a
fusion with support on three points. To that end, suppose that seller 2 chooses such a
distribution, distribution \( Q \):

\[
Q = \begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}
\]

where \( Q \in \mathcal{F}(P) \), \( pa + qb + rc = 1/2 \), and \( p + q + r = 1 \). Without loss of generality
\( a \leq 1/2 \) and \( c \geq 1/2 \). First suppose that \( b \leq 1/2 \). Then, seller 2’s payoff from deviating is

\[
u_2 = pF(a) + qF(b) + rF(c)
\]

\[
= \frac{2}{3}pa + \frac{2}{3}qb + r\left(\frac{8}{3}c - 1\right)
\]

\[
= \frac{2}{3}(pa + qb + rc) + r(2c - 1) = \frac{1}{3} + r(2c - 1)
\]

However, note that we must have \( c \leq 1/2 + 1/(12r) \). Hence,

\[
u_2 \leq \frac{1}{3} + r\left(\frac{1}{2} + \frac{1}{6r} - 1\right) = \frac{1}{2}
\]

In a similar manner, for \( b \geq 1/2 \) we have

\[
u_2 = pF(a) + qF(b) + rF(c)
\]

\[
= \frac{2}{3}pa + q\left(\frac{8}{3}b - 1\right) + r\left(\frac{8}{3}c - 1\right)
\]

\[
= \frac{2}{3}(pa + qb + rc) + 2(qb + rc) - (q + r) = \frac{1}{3} + p(1 - 2a)
\]
where we used the fact that $p + q + r = 1$ and that $pa + qb + rc = 1/2$. Then, since $a \geq (1 - 1/(6p))/2$ we have

$$u_2 \leq \frac{1}{3} + p \left(1 + \frac{1}{6p} - 1\right) = \frac{1}{2}$$

Thus, there is no profitable deviation to any $Q \in \mathcal{F}(P)$ with support on three points, and so Proposition 1.8 is proved.

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