Two Non-Commutative Binomial Theorems

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Abstract

We derive two formulae for \((A + B)^n\), where \(A\) and \(B\) are elements in a non-commutative, associative algebra with identity.

1 Introduction

Let \(\mathfrak{A}\) be an associative algebra, not necessarily commutative, with identity. For two elements \(A\) and \(B\) in \(\mathfrak{A}\), that commute, i.e.

\[
AB = BA
\]

the well-known Binomial Theorem reads

\[
(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}
\]

If \(A\) and \(B\) do not commute, we find the first formula for \((A + B)^n\) that retains the binomial coefficient. It also gives a representation of \(e^{(A+B)}\) that is different from the Campell-Baker-Hausdorff representation \(3\). The first formula is then applied to a problem in non-commutative geometry. The second formula for \((A + B)^n\) complements the first one. We apply it to a problem in quantum mechanics.

2 The First Non-Commutative Binomial Theorem

Let \(\mathfrak{A}\) be an associative algebra, not necessarily commutative, with identity 1. \(L(\mathfrak{A})\) denotes the algebra of linear transformations from \(\mathfrak{A}\) to \(\mathfrak{A}\).

Definition 1

Let \(A\) and \(X\) be elements of \(\mathfrak{A}\).

1. \(A\) can be looked upon as an element in \(L(\mathfrak{A})\) by
A(X) = AX \hspace{1cm} (3)

i.e. leftmultiplication

2. The element $d_A$ in $L(\mathfrak{A})$ is defined by

$$d_A(X) = [A, X] = AX -XA \hspace{1cm} (4)$$

We now have the following trivial relations:

**Statements**

1. As elements in $L(\mathfrak{A})$, $A$ and $d_A$ commute, i.e.

$$Ad_A(X) = d_AA(X) \hspace{1cm} (5)$$

2. $d_A$ is a derivation on $\mathfrak{A}$, i.e.

$$d_A(XY) = (d_AX)Y + X(d_AY) \hspace{1cm} (6)$$

3. $$(A - d_A)X =XA \hspace{1cm} (7)$$

4. Jacobi identity

$$d_Ad_B(C) + d_Bd_C(A) + d_Cd_A(B) = 0 \hspace{1cm} (8)$$

These simple statements are sufficient to prove the following non-commutative Binomial Theorem [1], [2].

**Theorem 1**

For $A$ and $B$ elements in $\mathfrak{A}$, and $1$ being the identity in $\mathfrak{A}$

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} \{(A + d_B)^k 1\} B^{n-k} \hspace{1cm} (9)$$

**Proof.** The formula holds true for $n=1$. We now proceed by induction.

$$(A + B)^{n+1} = (A + B)(A + B)^n = (A + d_B + B - d_B)(A + B)^n$$

$$= (A + d_B + B - d_B) \sum_{k=0}^{n} \binom{n}{k} \{(A + d_B)^k 1\} B^{n-k}$$

Using the previous Statements, we get
\[(A + B)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} \left[ A \{(A + d_B)^{k+1}\} B^{n-k} + \{(A + d_B)^{k+1}\} B^{n-k} + \{(A + d_B)^{k+1}\} B^{n-k+1} \right] \]
\[= \sum_{k=0}^{n} \binom{n}{k} \left[ \{(A + d_B)^{k+1}\} B^{n-k} + \{(A + d_B)^{k+1}\} B^{n-k+1} \right] \]
\[= \sum_{k=1}^{n} \binom{n}{k} \{(A + d_B)^{k+1}\} B^{n-k+1} + B^{n+1} \]
\[+ \sum_{k=1}^{n} \binom{n}{k-1} \{(A + d_B)^{k+1}\} B^{n-k+1} + \{(A + d_B)^{k+1}\} B^{n+1+1} \]

From the identity
\[\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}\]
we then get
\[(A + B)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \{(A + d_B)^{k+1}\} B^{n+1-k} \]

\[\square\]

3 The Essential Non-Commutative Part

We write
\[(A + d_B)^n 1 = A^n + D_n (B, A) \quad (10)\]

For a commutative algebra, \(D_n (B, A)\) is identically zero. We thus call \(D_n (B, A)\) the essential non-commutative part.

\(D_n (B, A)\) satisfies the following recurrence relation
\[D_{n+1} (B, A) = d_B A^n + (A + d_B) D_n (B, A) \quad (11)\]
with
\[D_0 (B, A) = 0\]

Definition 2

1. \[M_n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \quad (12)\]
2. \[D_k (B, A) = D_k \quad (13)\]

We now have the following obvious corollary.
Corollary 1

\[(A + B)^n = M_n + \sum_{k=0}^{n} \binom{n}{k} D_k B^{n-k}\] (14)

4 Exponentials

We have as a consequence of the first non-commutative Binomial Theorem

Corollary 2

\[e^{A+B} = [e^{A+dB}]e^B\] (15)

Proof.

\[
e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \{ (A + dB)^k \} B^{n-k}
\]

\[
= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} \{ (A + dB)^k \} B^{n-k}
\]

\[
e^{A+B} = [e^{A+dB}]e^B
\]

By splitting of the essential non-commutative part we get

Corollary 3

\[e^{A+B} = e^A e^B + \sum_{n=0}^{\infty} \frac{1}{n!} D_n e^B\] (17)

This is different from the Campell-Baker-Hausdorff formula.

5 Application of Theorem 1 for

\[d_B A = hA^2\] (18)

Definition 3

For \(h\) a scalar and \(n\) an integer we introduce

\[
\gamma_n(h) = [1 + h][1 + 2h] \cdots [1 + (n-1)h], \quad \gamma_0(h) = 1
\] (19)
Lemma 1
The following properties hold

1. \( \gamma_1(h) = 1, \gamma_n(0) = 1, \gamma_n(1) = n! \)

2. \( \gamma_{k+1}(h) = (1 + kh)\gamma_k(h) \)

Proof. Direct verification

Now, from Corollary 1 (14)

\[(A + B)^n = M_n + \sum_{k=2}^{n} \binom{n}{k} D_k B^{n-k} \]

\[D_k = d_B A^{k-1} + (A + d_B)D_{k-1}, \quad D_2 = d_B A \]

we find

Lemma 2

1. \( d_B A^k = khA^{k+1} \)

2. \( D_k = \{\gamma_k(h) - 1\} A^k \)

Proof.

1. \( d_B A = hA^2 \)

   Since \( d_B \) is a derivation we have by induction

   \[ d_B A^k = (d_B A^{k-1})A + A^{k-1}(d_B A) \]
   \[ = (k-1)hA^{k+1} + A^{k-1}hA^2 = khA^{k+1} \]

2. By induction and \( D_2 = hA^2 \), we find

\[ D_k = d_B A^{k-1} + (A + d_B)\{\gamma_{k-1}(h) - 1\} A^{k-1} \]
\[ = d_B A^{k-1} + \{\gamma_{k-1}(h) - 1\} A^k + \gamma_{k-1}(h)d_B A^{k-1} - d_B A^{k-1} \]
\[ = \{\gamma_{k-1}(h) - 1\} A^k + \gamma_{k-1}(h)(k - 1)hA^k \]
\[ = \{[1 + (k - 1)h]\gamma_{k-1}(h) - 1\} A^k \]
\[ D_k = \{\gamma_k(h) - 1\} A^k \]
Now

\[(A + B)^n = M_n + \sum_{k=2}^{n} \binom{n}{k} D_k B^{n-k}\]

\[= \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} + \sum_{k=2}^{n} \binom{n}{k} \{\gamma_k(h) - 1\} A^k B^{n-k}\]

\[= B^n + \binom{n}{1} A B^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \gamma_k(h) A^k B^{n-k}\]

Finally,

\[(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(h) A^k B^{n-k}\]

The result can also be found in [4]

Note: For \(h = 1\), i.e. \(d_B A = A^2\), we find

\[(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} k! A^k B^{n-k}\]

\[(A + B)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} A^k B^{n-k}\]

Also, if on the vector space of infinitely often differentiable function on \(\mathbb{R}\) we introduce the operators

\[A = x, \quad B = x^2 \frac{d}{dx}\]

we have \(d_B A = A^2\). Thus the representation (21) applies.

### 6 The Second Non-Commutative Binomial Theorem

Let \(A\) and \(B\) be in \(\mathfrak{A}\). With

\[M_n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \in \mathfrak{A}\]

we have
Lemma 3

1. \[ M_0 = 1, M_1 = A + B \]  \hspace{2cm} (24)

2. \[ M_1 M_n = M_{n+1} + d_B M_n \]  \hspace{2cm} (25)

Proof.

1. Obvious

2.

\[
M_1 M_n = (A + B) \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} A^{k+1} B^{n-k} + \sum_{k=0}^{n} \binom{n}{k} B A^k B^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} A^{k+1} B^{n-k} + \sum_{k=0}^{n} \binom{n}{k} \{d_B A^k + A^k B\} B^{n-k}
\]

\[
= \sum_{s=1}^{n+1} \binom{n}{s-1} A^s B^{n+1-s} + \sum_{k=0}^{n} \binom{n}{k} A^k B^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} \{d_B A^k\} B^{n-k}
\]

\[
= A^{n+1} + B^{n+1} + \sum_{k=1}^{n} \left( \binom{n}{k-1} + \binom{n}{k} \right) A^k B^{n+1-k} + d_B \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}
\]

\[
= A^{n+1} + B^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} A^k B^{n+1-k} + d_B M_n
\]

\[ M_1 M_n = M_{n+1} + d_B M_n \]

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\square
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Lemma 4

\[
M_1^n = M_n + \sum_{k=0}^{n-2} M_1^k d_B M_{n-1-k}
\]  \hspace{2cm} (26)

Proof. This is true for \( n = 2 \),

\[ M_1^2 = M_1 M_1 = M_2 + d_B M_1 \]
Now by induction

\[ M_1^{n-1} = M_{n-1} + \sum_{k=0}^{n-3} M_1^k d_B M_{n-2-k} \]

\[ M_1^n = M_1 M_1^{n-1} = M_1 M_{n-1} + \sum_{k=0}^{n-3} M_1^{k+1} d_B M_{n-2-k} \]

\[ = M_n + d_B M_{n-1} + \sum_{s=1}^{n-2} (s) d_B M_{n-1-s} \]

\[ M_1^n = M_n + \sum_{k=0}^{n-2} M_1^k d_B M_{n-1-k} \]

\[ \square \]

**Theorem 2**

\[ (A + B)^n = M_n + \sum_{k=0}^{n-2} (A + B)^k d_B M_{n-1-k} \] (27)

**Proof.** This is lemma 4 with \( M_1 = A + B \)

**7 Application of Theorem 2 for the case**

\[ d_B A = d_B M_1 = C, \quad \text{and} \quad d_A C = d_B C = 0 \] (28)

Then

\[ d_B A^k = kCA^{k-1}, \quad d_B M_n = nCM_{n-1} \] (29)

and

\[ (A + B)^n = M_n + \sum_{k=0}^{n-2} (n - 1 - k) CM_1^k M_{n-2-k} \]

Ansatz

\[ (A + B)^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} A_{n,k} \] (30)

with

\[ A_{n,0} = 1 \] (31)

and \( A_{n,k} \) commuting with \( A \) and \( B \).

\([\frac{a}{b}] \) denotes the greatest integer less than \( \frac{a}{b} \).
From
\[(A + B)^{n+1} = M_1(A + B)^n\]
we have
\[\sum_{k=0}^{\left\lceil \frac{n+1}{2} \right\rceil} M_{n+1-2k}A_{n+1,k} = M_1 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} M_{n-2k}A_{n,k}\]
or
\[M_{n+1} + \sum_{k=1}^{\left\lceil \frac{n+1}{2} \right\rceil} M_{n+1-2k}A_{n+1,k} = M_1\{M_n + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} M_{n-2k}A_{n,k}\}\]
From (25) and (23) we find
\[M_1M_n = M_{n+1} + nCM_{n-1}\]
resulting in
\[\sum_{k=1}^{\left\lceil \frac{n+1}{2} \right\rceil} M_{n+1-2k}A_{n+1,k} = nCM_{n-1} + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} M_{n+1-2k}A_{n,k} + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (n-2k)CM_{n-1-2k}A_{n,k}\]
(32)
For \(n\) even, \(n = 2N\), (32) reads
\[\sum_{k=1}^{N} M_{2N+1-2k}A_{2N+1,k} = 2NCM_{2N-1} + \sum_{k=1}^{N} M_{2N+1-2k}A_{2N,k} + \sum_{k=1}^{N-1} (2N-2k)CM_{2N-1-2k}A_{2N,k}\]
or
\[M_{2N-1}A_{2N+1,1} + \sum_{k=2}^{N} M_{2N+1-2k}A_{2N+1,k} = 2NCM_{2N-1} + M_{2N-1}A_{2N,1} + \sum_{k=2}^{N} M_{2N+1-2k}A_{2N,k} + \sum_{k=2}^{N} M_{2N+1-2k}(2N+2-2k)A_{2N,k-1}\]
Comparing coefficients gives the recurrence relation
\[A_{2N+1,k} = A_{2N,k} + (2N+2-2k)CA_{2N,k-1}\]
or
\[A_{n+1,k} = A_{n,k} + (n+2-2k)CA_{n,k-1}, k \geq 1\]
(33)
Note, that for \(n\) odd, \(n = 2N + 1\), we get the same relation
Lemma 5

The recurrence relation (33) with $A_{n,0} = 1$ has the solution

$$A_{n,k} = \frac{n!}{(n-2k)!k!2^k} C^k$$  \hspace{1cm} (34)

and (30) becomes

$$(A + B)^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} \frac{n!}{(n-2k)!k!2^k} C^k$$  \hspace{1cm} (35)

Proof. by direct verification

This result can also be found in [5]

Note: On the vector space of infinitely often differentiable function on $\mathbb{R}$ we introduce the operators

$A = x, B = \lambda \frac{d}{dx}$, where $\lambda$ is a scalar.  \hspace{1cm} (36)

Then $d_B A = \lambda$, or $C = \lambda I$. Thus the above representation (35) applies.

In particular

$$(x + \lambda \frac{d}{dx})^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} \frac{n!}{(n-2k)!k!2^k} \lambda^k$$

where

$$M_n = \sum_{r=0}^{n} \binom{n}{r} x^r \frac{d^{n-r}}{dx^{n-r}}, \quad M_n 1 = x^n$$

resulting in

$$(x + \lambda \frac{d}{dx})^n 1 = \sum_{k=0}^{[\frac{n}{2}]} x^{n-2k} \frac{n!}{(n-2k)!k!2^k} \lambda^k$$  \hspace{1cm} (37)

For $\lambda = -1$, we get

$$(x - \frac{d}{dx})^n 1 = n! \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{x^{n-2k}}{(n-2k)!k!2^k}$$  \hspace{1cm} (38)
The right-hand side are the Hermite polynomials.

Thus

\[ H_n(x) = (x - \frac{d}{dx})^n 1 \]  \hspace{1cm} (39)

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