NORMAL PROJECTIVE VARIETIES ADMITTING POLARIZED OR
INT-AMPLIFIED ENDOMORPHISMS

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Abstract. Let $X$ be a normal projective variety admitting a polarized or int-amplified
endomorphism $f$. First, we list up characteristic properties of such an endomorphism.
Second, we list up properties of such a variety from the aspects of its singularities,
anti-canonical divisor and Kodaira dimension. Third, we run the equivariant minimal
model program relative to not just the single $f$ but also the semigroup $\text{SEnd}(X)$ of
all surjective endomorphisms of $X$, up to finite-index. Several applications are given.
Finally, we give both algebraic and geometric characterizations of toric varieties via
polarized endomorphisms.

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1. Introduction

In this note, we report our recent results in studying non-isomorphic endomorphisms,
especially polarized endomorphisms and int-amplified endomorphisms of higher dimen-
sional algebraic varieties in arbitrary characteristics. The main focus is in setting up the
equivariant minimal model program (MMP) for such endomorphisms. We will outline
the ideas but refer to the original papers for the detailed proofs. Our approach is more

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geometric. We refer the reader to the survey paper [35] with more number theoretic
flavours and including many outstanding conjectures.

We work over an algebraically closed field $k$ of arbitrary characteristic. We will mention
the extra requirement on $k$ in each section or statement when needed.

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2. Polarized or int-amplified endomorphisms

Let $X$ be a projective variety over the field $k$. A Cartier divisor is assumed to be
integral, unless otherwise indicated. Denote by $\text{Pic}(X)$ the group of Cartier divisors
modulo linear equivalence and $\text{Pic}^0(X)$ the subgroup of the classes in $\text{Pic}(X)$ which are
algebraically equivalent to 0. Then

$$\text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X)$$

is the Néron-Severi group. Denote by

$$N^1(X) := \text{NS}(X) \otimes \mathbb{Z} \mathbb{R}$$

and

$$\text{NS}_K(X) := \text{NS}(X) \otimes \mathbb{Z} K$$

for $K := \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$. So $\text{NS}_\mathbb{R}(X) = N^1(X)$.

Let $n := \dim(X)$. We can regard $N^1(X)$ as the space of numerically equivalent classes
of $\mathbb{R}$-Cartier divisors. Two $\mathbb{R}$-Cartier divisors $D_1$ and $D_2$ are *numerically equivalent*,
denoted as

$$D_1 \equiv D_2$$

if their classes $[D_1]$ and $[D_2]$ in $N^1(X)$ are the same, i.e., if

$$(D_1 - D_2) \cdot C = 0$$

for any curve $C$ on $X$. Denote by

$$N_r(X)$$

the space of weakly numerically equivalent classes of $r$-cycles with $\mathbb{R}$-coefficients. Namely,
two $r$-cycles $D_1$ and $D_2$ are *weakly numerically equivalent*, if the intersection

$$(D_1 - D_2) \cdot L_1 \cdots L_{n-r} = 0$$
for all Cartier divisors $L_i$ (cf. [22, Definition 2.2]). When $X$ is normal, we also call $N_{n-1}(X)$ the space of weakly numerically equivalent classes of Weil $\mathbb{R}$-divisors. In this case, $N^1(X)$ can be regarded as a subspace of $N_{n-1}(X)$ (cf. [34, Lemma 3.2]). We recall the following cones:

- $\text{Amp}(X)$ is the cone of ample classes in $N^1(X)$.
- $\text{Nef}(X)$ is the cone of nef classes in $N^1(X)$.
- $\text{PEC}(X)$ is the cone of pseudo-effective $\mathbb{R}$-Cartier divisor classes in $N^1(X)$.
- $\text{PE}(X)$ is the cone of pseudo-effective Weil divisor classes in $N_{n-1}(X)$.

We refer to [22, §2] for more information.

**Definition 2.1.** Let $X$ be a projective variety.

1. Denote by $\text{SEnd}(X)$ the monoid of all surjective endomorphisms of $X$.

   In the following, let $f \in \text{SEnd}(X)$.

2. $f$ is *numerically polarized* if $f^*L \equiv qL$ for some ample Cartier divisor $L$ and integer $q > 1$.

3. $f$ is *numerically weakly polarized* if $f^*L \equiv qL$ for some big Cartier divisor $L$ and integer $q > 1$.

4. $f$ is *polarized* if $f^*L \sim qL$ for some ample Cartier divisor $L$ and integer $q > 1$.

5. $f$ is *amplified* if $f^*L - L = H$ for some (not necessarily ample) Cartier divisor $L$ and some ample Cartier divisor $H$.

6. $f$ is *int-amplified* if $f^*L - L = H$ for some ample Cartier divisors $L$ and $H$.

7. $f$ is *separable* if the induced field extension $f^*: k(X) \to k(X)$ is separable.

The following gives a norm criterion of numerically polarized endomorphisms.

**Proposition 2.2.** (cf. [22, Proposition 2.9], [9, Proposition 3.1]) Let $\varphi: V \to V$ be an invertible linear map of a positive dimensional real normed vector space $V$. Assume $\varphi^{\pm 1}(C) = C$ for a convex cone $C \subseteq V$ such that $C$ spans $V$ and its closure $\overline{C}$ contains no line. Let $q$ be a positive number. Then the conditions (i) and (ii) below are equivalent.

1. $\varphi(u) = qu$ for some $u \in C^\circ$ (the interior part of $C$).
2. There exists a constant $N > 0$, such that

$$\frac{||\varphi^i||}{q^i} < N$$

for all $i \in \mathbb{Z}$. 
Assume further the equivalent conditions (i) and (ii). Then the following are true.

1. \( \varphi \) is a diagonalizable linear map with all eigenvalues of modulus \( q \).
2. Suppose \( q > 1 \). Then, for any \( v \in V \) such that \( \varphi(v) - v \in C \), we have \( v \in C \).

Applying the above criterion to the cones \( \text{Nef}(X) \) and \( \text{PEC}(X) \), we now can say the equivalence of “numerically weakly polarized” and “numerically polarized”; see [22, Proposition 3.6]. In the case of characteristic 0, “numerically polarized” is equivalent to “polarized” by [26, Lemma 2.3]. In the case of arbitrary characteristic, they are also equivalent if we further require the endomorphism is separable and the variety is normal; see [9, Theorem 5.1]. Together, we have the following result.

**Theorem 2.3.** Let \( f : X \to X \) be a numerically weakly polarized separable endomorphism of a normal projective variety over the field \( k \) of arbitrary characteristic. Then \( f \) is polarized.

The following gives useful criteria of int-amplified endomorphisms.

**Proposition 2.4.** (cf. [21, Proposition 3.3]) Let \( f : X \to X \) be a surjective endomorphism of a projective variety \( X \). Then the following are equivalent.

1. The endomorphism \( f \) is int-amplified.
2. All the eigenvalues of
   \[
   \varphi := f^*|_{\text{N}^1(X)}
   \]
   are of modulus greater than 1.
3. There exists some big \( \mathbb{R} \)-Cartier divisor \( B \) such that \( f^*B - B \) is big.
4. If \( C \) is a \( \varphi^\pm \)-invariant convex cone in \( \text{N}^1(X) \), then
   \[
   \emptyset \neq (\varphi - \text{id}_{\text{N}^1(X)})^{-1}(C) \subseteq C.
   \]

Considering the action \( f^*|_{\text{N}^1_{\text{an}}(X)} \) and the cone \( \text{PE}(X) \), we have similar criteria as follows.

**Proposition 2.5.** (cf. [21, Proposition 3.4]) Let \( f : X \to X \) be a surjective endomorphism of an \( n \)-dimensional normal projective variety \( X \). Then the following are equivalent.

1. The endomorphism \( f \) is int-amplified.
2. All the eigenvalues of
   \[
   \varphi := f^*|_{\text{N}^1_{\text{an}}(X)}
   \]
   are of modulus greater than 1.
3. There exists some big Weil \( \mathbb{R} \)-divisor \( B \) such that \( f^*B - B \) is a big Weil \( \mathbb{R} \)-divisor.
(4) If $C$ is a $\varphi^\pm$-invariant convex cone in $N_{n-1}(X)$, then
$$\emptyset \neq (\varphi - \text{id}_{N_{n-1}(X)})^{-1}(C) \subseteq C.$$ 

By the above criteria, we may easily claim that the property of “numerically polarized” and “int-amplified” are preserved via every equivariant descending.

**Lemma 2.6.** (cf. [22, Theorem 3.11], [21, Lemmas 3.5, 3.6], [9, Lemma 2.5]) Let $\pi : X \to Y$ be a dominant map of projective varieties. Let $f : X \to X$ and $g : Y \to Y$ be two surjective endomorphisms such that
$$g \circ \pi = \pi \circ f.$$ 
Suppose $f$ is numerically polarized (resp. int-amplified, separable). Then $g$ is numerically polarized (resp. int-amplified, separable).

### 3. Singularities

We first introduce the ramification divisor formula for a separable finite surjective morphism.

**Proposition 3.1.** (cf. [27, Lemma 4.4]) Let $f : X' \to X$ be a separable finite surjective morphism of normal varieties over the field $k$ of characteristic $p$. Let $P'$ be a prime divisor on $X'$ and let $r$ be the ramification index of $f$ along $P'$. Then there exists a non-negative integer $b \geq r - 1$ such that
$$K_{X'} = f^*K_X + bP'$$ 
holds around the generic point of $P'$ and $b > r - 1$ holds exactly when $p \mid r$. In particular, we have the ramification divisor formula:
$$K_{X'} = f^*K_X + R_f,$$
where
$$R_f := \sum_{P'} bP'$$
is the ramification divisor of $f$.

We refer to [20, Chapters 2 and 5] for the definitions and the properties of log canonical (lc), Kawamata log terminal (klt), canonical and terminal singularities. Let $f : X \to X$ be a separable surjective endomorphism of a normal projective variety $X$. In the case of characteristic 0, one can apply [20, Proposition 5.20] to control the singularities of $X$. Wahl [31, Theorem 2.8] showed that $X$ has at worst lc singularities when $\dim(X) = 2$. Broustet and Höring [7, Corollary 1.5] generalized this result to higher dimensional case with additional assumptions that $f$ is polarized and $X$ is $\mathbb{Q}$-Gorenstein. However, in
the case of positive characteristic, one requires an additional “tame” assumption on \( f \) to apply [20, Proposition 5.20]; see [27, Proposition 4.6, Remark 4.7, Example 4.8].

In characteristic 0, we may weaken the condition to the int-amplified case.

**Theorem 3.2.** (cf. [21, Theorem 1.6]) Let \( X \) be a \( \mathbb{Q} \)-Gorenstein normal projective variety over the field \( k \) of characteristic 0 admitting an int-amplified endomorphism. Then \( X \) has at worst lc singularities.

In the case of positive characteristic, with additional assumptions and some extra work, we can deal with the surface case, extending [31] to positive characteristics.

**Theorem 3.3.** (cf. [9, Theorem 10.2]) Let \( X \) be a normal algebraic surface over the field \( k \) of characteristic \( p > 0 \). Let \( f : X \to X \) be a non-isomorphic surjective endomorphism of \( X \) with the degree \( \deg f^{\text{Gal}} \) of its Galois closure co-prime to \( p \). Then \( X \) is lc.

4. Anti-canonical divisor and Kodaira dimension

Let \( f : X \to X \) be a separable surjective endomorphism of a normal projective variety \( X \). When \( f \) is polarized and \( X \) is smooth, Boucksom, de Fernex and Favre [5, Theorem C] showed that \( -K_X \) is pseudo-effective. Cascini, Meng and Zhang [9, Theorem 1.1 and Remark 3.2] used a different method to show further that \( -K_X \) is weakly numerically equivalent to an effective Weil \( \mathbb{Q} \)-divisor without the assumption of \( X \) being smooth. By applying Propositions 2.4 and 2.5 to the ramification divisor formula, we have the following result for the int-amplified case.

**Theorem 4.1.** (cf. [21, Theorem 1.5]) Let \( X \) be a normal projective variety admitting an int-amplified separable endomorphism. Then \( -K_X \) is weakly numerically equivalent to some effective Weil \( \mathbb{Q} \)-divisor. If \( X \) is further assumed to be \( \mathbb{Q} \)-Gorenstein, then \( -K_X \) is numerically equivalent to some effective \( \mathbb{Q} \)-Cartier divisor.

Let \( f : X \to X \) be an amplified endomorphism of a projective variety \( X \). Then Fakhruddin [14, Theorem 5.1] showed that the set of \( f \)-periodic points is Zariski dense in \( X \). In characteristic 0, by taking the equivariant Iitaka fibration (cf. [25, Theorem A]), we then can control the Kodaira dimension of \( X \).

**Lemma 4.2.** (cf. [21, Lemma 2.5]) Let \( f : X \to X \) be an amplified endomorphism of a projective variety \( X \) over the field \( k \) of characteristic 0. Then the Kodaira dimension \( \kappa(X) \leq 0 \).

5. Equivariant MMP relative to \( f \)

This section generalises and extends results in [33] to higher dimensions.
Let 

\[(*) : X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_r \]

be a finite sequence of dominant rational maps of projective varieties. Let \( f : X_1 \to X_1 \) be a surjective endomorphism. We say the sequence \((*)\) is \(f\)-equivariant if the following diagram commutes

\[
\begin{array}{ccc}
X_1 & \rightarrow & X_2 \\
f_1 & \downarrow & \downarrow f_2 \\
X_1 & \rightarrow & X_2 \\
& \vdots & \\
& \rightarrow & \\
f_r & \downarrow & f_r \\
X_1 & \rightarrow & X_r
\end{array}
\]

where \( f_1 = f \) and all \( f_i \) are surjective endomorphisms. Let \( G \) be a subset of \( \text{SEnd}(X) \). We say the sequence \((*)\) is \( G \)-equivariant if \((*)\) is \( g \)-equivariant for any \( g \in G \).

Following the proof of [22, Lemma 6.1], [9, Lemma 6.2] and [21, Lemma 8.1], we have the following key lemma. We also refer to [9, Remark 6.3] to show that the following condition (2) is necessary.

**Lemma 5.1.** (cf. [24, Lemmas 3.4 and 3.5]) Let \( f : X \to X \) be an int-amplified separable endomorphism of a projective variety \( X \) over the field \( k \) of characteristic \( p \geq 0 \). Assume \( A \subseteq X \) is a closed subvariety with

\[ f^{-i}f^i(A) = A \]

for all \( i \geq 0 \). Assume further either one of the following conditions.

1. \( A \) is a prime divisor of \( X \).
2. \( p \) and \( \deg f \) are co-prime.
3. \( p = 0 \).

Then

\[ M(A) := \{ f^i(A) \mid i \in \mathbb{Z} \} \]

is a finite set.

Applying the same argument and proofs of [22, Lemma 6.2 to Lemma 6.6], we may state the following theorem. Note that Lemma 2.6 and Theorem 2.3 are used to show the following \( g \) is again polarized or int-amplified.

**Theorem 5.2.** (cf. [21, Theorem 8.2]) Let \( f : X \to X \) be a polarized (resp. int-amplified) endomorphism of a \( \mathbb{Q} \)-factorial lc projective variety \( X \) over the field \( k \) of characteristic \( p \) such that either \( p = 0 \) or \( p \) and \( \deg f \) are co-prime. Let \( \pi : X \dashrightarrow Y \) be a dominant rational map which is

1. either a divisorial contraction, or
2. a Fano contraction, or
(iii) a flipping contraction, or
(iv) a flip
induced by a $K_X$-negative extremal ray. Then there exists a polarized (resp. int-amplified) endomorphism $g : Y \to Y$ such that

$$g \circ \pi = \pi \circ f$$

after replacing $f$ by a positive power.

In the above theorem, it is easy to see that either $p = 0$ or $p$ and $\deg g$ are still co-prime. Now assuming the existence of MMP, we have the following result of equivariant MMP by successively applying Theorem 5.2. We will also give a stronger version of the following theorem in the next section (cf. Theorem 6.3).

**Theorem 5.3.** Let $f : X \to X$ be an int-amplified endomorphism of a $\mathbb{Q}$-factorial lc projective variety $X$ over the field $k$ of characteristic $p$ such that either $p = 0$ or $p$ and $\deg f$ are co-prime. Then any finite sequence of MMP starting from $X$ is $f^s$-equivariant for some $s > 0$.

We refer to [20], [3], [15], [4], [17] and [30] for things about MMP. In the case of characteristic 0, the existence of a polarized or int-amplified endomorphism on a variety exerts strong constraints on the geometry of the variety. Recall that a normal projective variety $X$ is said to be $\mathbb{Q}$-abelian if there is a finite surjective morphism $\pi : A \to X$ étale in codimension 1 with $A$ being an abelian variety.

**Theorem 5.4.** (cf. [26, Theorem 3.4], [16, Theorem 1.21], [22, Lemma 6.9], [21, Theorem 1.9]) Let $f : X \to X$ be an int-amplified endomorphism of a normal projective variety $X$ over the field $k$ of characteristic 0. Suppose either $X$ is klt and $K_X$ is pseudo-effective or $X$ is non-uniruled. Then $X$ is $\mathbb{Q}$-abelian.

The following theorem further characterizes step by step the equivariant MMP and the action $f^*|_{\Omega^1(X)}$.

**Theorem 5.5.** (cf. [22, Theorem 1.8], [21, Theorem 1.10]) Let $f : X \to X$ be an int-amplified endomorphism of a $\mathbb{Q}$-factorial klt projective variety $X$ over the field $k$ of characteristic 0. Then after replacing $f$ by a positive power, there exists a finite sequence of $f$-equivariant MMP $(\ast)$ which ends up with a $\mathbb{Q}$-abelian variety $Y = X_r$ such that

1. If $K_X$ is pseudo-effective, then $X = Y$ and it is $\mathbb{Q}$-abelian.
2. If $K_X$ is not pseudo-effective, then for each $i$, $X_i \to Y$ is equi-dimensional holomorphic with every fibre irreducible (and rationally connected if $k$ is further uncountable) and $f_i$ is int-amplified. The $X_{r-1} \to X_r = Y$ is a Fano contraction.
(3) \( f^*|_{N^1(X)} \) is diagonalizable over \( \mathbb{C} \) if and only if so is \( f^*_r|_{N^1(Y)} \).

(4) If \( f \) is polarized, then \( f^*|_{N^1(X)} \) is a scalar multiplication:
\[
f^*|_{N^1(X)} = q \text{id},
\]
if and only if so is \( f^*_r|_{N^1(Y)} \).

**Definition 5.6.** Let \( X \) be a normal projective variety.

1. \( q(X) := h^1(X, O_X) = \dim H^1(X, O_X) \) (the irregularity).
2. \( \tilde{q}(X) := q(\tilde{X}) \) with \( \tilde{X} \) a smooth projective model of \( X \).
3. \( \tilde{q}^e(X) := \sup \{ \tilde{q}(X') \mid X' \to X \text{ is finite surjective and étale in codimension one} \} \).
4. \( \pi_{1_{\text{alg}}}(X_{\text{reg}}) \) is the algebraic fundamental group of the smooth locus \( X_{\text{reg}} \) of \( X \).

The following result is an application of Theorem 5.5. Note that when \( f : X \to X \) is a polarized endomorphism of a projective variety \( X \), the action \( f^*|_{N^1(X)} \) is always diagonalizable over \( \mathbb{C} \) and all the eigenvalues are of the same modulus (cf. [22, Proposition 2.9]). However, Theorem 5.7 fails in general due to Example 5.8 given by Najmuddin Fakhruddin.

**Theorem 5.7.** (cf. [22, Lemma 9.1, Theorem 1.10], [21, Theorem 1.11]) Let \( f : X \to X \) be an int-amplified endomorphism of a \( \mathbb{Q} \)-factorial klt projective variety \( X \) over the field \( k \) of characteristic 0. Suppose either \( \tilde{q}^e(X) = 0 \) or \( \pi_{1_{\text{alg}}}(X_{\text{reg}}) \) is finite (eg. \( X \) is smooth rationally connected). Then we have:

1. There exists a finite sequence of MMP which ends up with a point.
2. There exists some \( s > 0 \), such that \( (f^*)^s|_{N^1(X)} \) is diagonalizable over \( \mathbb{Q} \) with all the eigenvalues being positive integers greater than 1.
3. If \( f \) is further \( q \)-polarized, then \( (f^*)^s|_{N^1(X)} = q^s \text{id} \).

**Example 5.8.** (Fakhruddin) Let \( E \) be an elliptic curve admitting a complex multiplication. Let
\[
S = E \times E.
\]
Then \( \dim(N^1(S)) = 4 \). Let \( \sigma : S \to S \) be an automorphism given as:
\[
(x, y) \mapsto (x, x + y).
\]
Then \( \sigma^*|_{N^1(S)} \) is not diagonalizable over \( \mathbb{C} \). Let \( m_S \) be the multiplication endomorphism of \( S \). Note that
\[
m_S^*|_{N^1(S)} = m^2 \text{id}_{N^1(S)}.
\]
So \( f := \sigma \circ m_S \) is int-amplified for \( m > 1 \), by Proposition 2.4. Clearly,
\[
f^*|_{N^1(S)}
is not diagonalizable over $\mathbb{C}$.

In the case of positive characteristic, the theory of MMP is still far from being established (only known for lc 3-fold with characteristic $> 5$, cf. [30] and the references therein). For the lower dimensional cases, we refer to [9, Theorem 1.6] for a similar version of Theorem 5.5 and [9, Theorem 1.8] for a similar version of Theorem 5.7.

6. Equivariant MMP relative to $\text{SEnd}(X)$

We use Proposition 6.1 below in proving the results in the rest of this section. As kindly informed by Professors Dinh and Sibony, when $k = \mathbb{C}$, this kind of result (with a complete proof) first appeared in [12, Section 3.4]; [11, Theorem 3.2] is a more general form including Proposition 6.1 below, requiring a weaker condition and dealing with also dominant meromorphic self-maps of Kähler manifolds; see comments in [13, page 615] for the history of these results; see also [6]. The following assumption that $p = \text{char} k$ and $\deg f$ are co-prime is necessary; see [24, Example 3.7].

Proposition 6.1. (cf. [24, Proposition 3.6]) Let $f : X \to X$ be an int-amplified endo-
morphism of a projective variety $X$ over the field $k$ of characteristic $p \geq 0$. Suppose either $p = 0$, or $p$ and $\deg f$ are co-prime. Then there are only finitely many $f^{-1}$-periodic Zariski closed subsets.

Let $X$ be a projective variety and let $C$ be a curve. Denote by

$$R_C := \mathbb{R}_{\geq 0}[C]$$

the ray generated by $[C]$ in $\overline{\text{NE}}(X)$. Denote by

$$\Sigma_C$$

the union of curves whose classes are in $R_C$.

Definition 6.2. Let $X$ be a projective variety. Let $C$ be a curve such that $R_C$ is an extremal ray in $\overline{\text{NE}}(X)$. We say $C$ or $R_C$ is contractible if there is a surjective morphism $\pi : X \to Y$ to a projective variety $Y$ such that the following hold.

1. $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.
2. Let $C'$ be a curve in $X$. Then $\pi(C')$ is a point if and only if $[C'] \in R_C$.
3. Let $D$ be a $\mathbb{Q}$-Cartier divisor of $X$. Then $D \cdot C = 0$ if and only if $D \equiv \pi^* D_Y$ (numerical equivalence) for some $\mathbb{Q}$-Cartier divisor $D_Y$ of $Y$.

A submonoid $G$ of a monoid $\Gamma$ is said to be of finite-index in $\Gamma$ if there is a chain

$$G = G_0 \leq G_1 \leq \cdots \leq G_r = \Gamma$$
of submonoids and homomorphisms $\rho_i : G_i \to F_i$ such that $\text{Ker}(\rho_i) = G_{i-1}$ and all $F_i$ are finite groups.

**Theorem 6.3.** (cf. [24, Theorem 1.1]) Let $X$ be a (not necessarily normal or $\mathbb{Q}$-Gorenstein) projective variety with a polarized (or int-amplified) endomorphism. Then:

1. $X$ has only finitely many (not necessarily $K_X$-negative) contractible extremal rays in the sense of Definition 6.2.
2. Suppose $X$ is $\mathbb{Q}$-factorial normal. Then any finite sequence of MMP starting from $X$ is $G$-equivariant for some finite-index submonoid $G$ of $\text{SEnd}(X)$.

In the rest of this section, we work over the field $k$ of characteristic 0.

**Theorem 6.4.** (cf. [24, Theorem 1.2]) Let $f : X \to X$ be an int-amplified endomorphism of a $\mathbb{Q}$-factorial klt projective variety $X$ over the field $k$ of characteristic 0. Then there exist a finite-index submonoid $G$ of $\text{SEnd}(X)$, a $\mathbb{Q}$-abelian variety $Y$, and a $G$-equivariant relative MMP over $Y$

$$X = X_0 \longrightarrow \cdots \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_r = Y$$

(i.e. $g \in G = G_0$ descends to $g_i \in G_i$ on each $X_i$), such that:

1. There is a finite quasi-étale Galois cover $A \to Y$ from an abelian variety $A$ such that $G_Y := G_r$ lifts to a submonoid $G_{A}$ of $\text{SEnd}(A)$.
2. If $g$ in $G$ is amplified and its descending $g_i$ on $X_i$ is int-amplified for some $i$, then $g$ is int-amplified. The $X_{r-1} \to X_r = Y$ is a Fano contraction.
3. For any subset $H \subseteq G$ and its descending

$$H_Y \subseteq \text{SEnd}(Y)$$

$H$ acts via pullback on $\text{NS}_\mathbb{Q}(X)$ or $\text{NS}_\mathbb{C}(X)$ as commutative diagonal matrices with respect to a suitable basis if and only if so does $H_Y$.

Let $\text{Pol}(X)$ be the set of all polarized endomorphisms on $X$, and let $\text{IAmp}(X)$ be the set of all int-amplified endomorphisms on $X$. In general, they are not semigroups, i.e., they may not be closed under composition; see [21, Example 10.4]. When $X$ is rationally connected and smooth, Theorem 6.5 below gives the assertion that if $g$ and $h$ are in $\text{Pol}(X)$ (resp. $\text{IAmp}(X)$) then $g^M \circ h^M$ remains in $\text{Pol}(X)$ (resp. $\text{IAmp}(X)$) for some $M > 0$ depending only on $X$. In particular, it answers affirmatively [32, Question 4.15], “up to finite-index”, when $X$ is rationally connected and smooth. By [24, Example 1.7], this extra “up to finite-index” assumption is necessary. For a subset $S$ of a semigroup $H$ and an integer $M \geq 1$, denote by

$$\langle S^M \rangle := \{ s_1^M \cdots s_r^M | r \geq 1, s_i \in S \}.$$
Theorem 6.5. (cf. [24, Theorems 1.4, 6.2]) Let $X$ be a rationally connected smooth projective variety over the field $k$ of characteristic 0. Suppose $X$ admits a polarized (or int-amplified) endomorphism $f$. We use the notation

$$X = X_0 \rightarrow \cdots \rightarrow X_r = Y$$

and the finite-index submonoid $G \leq \text{SEnd}(X)$ as in Theorem 6.4. Then there is an integer $M \geq 1$ depending only on $X$ such that:

1. The $Y$ in Theorem 6.4 is a point.
2. $G^*|_{\text{NS}_Q(X)}$ is a commutative diagonal monoid with respect to a suitable $Q$-basis $B$ of $\text{NS}_Q(X)$. Further, for every $g$ in $G$, the representation matrix $[g^*|_{\text{NS}_Q(X)}]_B$ relative to $B$, is equal to

$$\text{diag}[q_1, q_2, \ldots]$$

with integers $q_i \geq 1$.
3. $G \cap \text{Pol}(X)$ is a subsemigroup of $G$, and consists exactly of those $g$ in $G$ such that

$$[g^*|_{\text{NS}_Q(X)}]_B = \text{diag}[q, \ldots, q]$$

for some integer $q \geq 2$. Further,

$$G \cap \text{Pol}(X) \supseteq \langle \text{Pol}(X)^{[M]} \rangle.$$  

4. $G \cap \text{IAmp}(X)$ is a subsemigroup of $G$, and consists exactly of those $g$ in $G$ such that

$$[g^*|_{\text{NS}_Q(X)}]_B = \text{diag}[q_1, q_2, \ldots]$$

with integers $q_i \geq 2$. Further,

$$G (G \cap \text{IAmp}(X)) = G \cap \text{IAmp}(X) \supseteq \langle \text{IAmp}(X)^{[M]} \rangle;$$

any $h$ in $\text{SEnd}(X)$ has

$$(h^M)^* = (g_1^*)^{-1} g_2^*$$

on $\text{NS}_Q(X)$ for some $g_i$ in $G \cap \text{IAmp}(X)$.
5. We have $h^M \in G$ and that $h^*|_{\text{NS}_C(X)}$ is diagonalizable for every $h \in \text{SEnd}(X)$.

Let $\text{Aut}(X)$ be the group of all automorphisms of $X$, and $\text{Aut}_0(X)$ its neutral connected component. By applying Theorem 6.5, we have the following result.

Theorem 6.6. (cf. [24, Theorems 1.5, 6.3]) Let $X$ be a rationally connected smooth projective variety over the field $k$ of characteristic 0. Suppose $X$ admits a polarized (or int-amplified) endomorphism. Then we have:

1. $\text{Aut}(X)/\text{Aut}_0(X)$ is a finite group. More precisely, $\text{Aut}(X)$ is a linear algebraic group (with only finitely many connected components).
(2) Every amplified endomorphism of $X$ is int-amplified.
(3) $X$ has no automorphism of positive entropy (nor amplified automorphism).

Even when $X$ has Picard number one, the following question is still open when $n \geq 4$.

**Question 6.7.** Let $X$ be a rationally connected smooth projective variety of dimension $n \geq 1$ which admits a polarized endomorphism. Is $X$ (close to be) a toric variety?

### 7. Characterizations of toric varieties

Throughout this section, we always work over the field $k$ of characteristic 0.

A normal projective variety $X$ is said to be *toric* or a *toric variety* if $X$ contains an algebraic torus $T = (k^*)^n$ as an (affine) open dense subset such that the natural multiplication action of $T$ on itself extends to an action on the whole variety $X$. In this case, let

$$D := X \setminus T$$

which is a divisor; the pair $(X, D)$ is said to be a *toric pair*.

Let $G$ be a linear algebraic group acting on a normal projective variety $X$. We say $X$ is $G$-almost homogeneous if there exists an open dense $G$-orbit in $X$. Note that $X$ is toric if and only if $X$ is $T$-almost homogeneous for some algebraic torus $T$. The following result gives a sufficient condition, in terms of a polarized endomorphism, for an almost homogeneous variety to be toric.

**Theorem 7.1.** (cf. [23, Theorem 1.1]) Let $f : X \to X$ be a polarized endomorphism of a $G$-almost homogeneous normal projective variety $X$ with $G$ being a linear algebraic group. Assume further the following conditions.

(i) Let $U$ be the open dense $G$-orbit in $X$ and $D$ the codimension-1 part of $X \setminus U$.

The Weil divisor $K_X + D$ is $\mathbb{Q}$-Cartier.

(ii) The endomorphism $f$ is $G$-equivariant in the sense: there is a surjective homomorphism $\varphi : G \to G$ such that

$$f \circ g = \varphi(g) \circ f$$

for all $g$ in $G$.

Then $(X, D)$ is a toric pair.

Let $(X, \Delta)$ be a log pair. The *complexity* $c = c(X, \Delta)$ of $(X, \Delta)$ is defined as

$$c := \inf\{n + \dim_{\mathbb{R}}(\sum \mathbb{R}[S_i]) - \sum a_i | \sum a_i S_i \leq \Delta, a_i \geq 0, S_i \geq 0\}.$$
Brown, M’Kernan, Svaldi and Zong recently gave a geometric characterization of toric varieties involving the complexity; see [8, Theorem 1.2]. Their result is a special case of a conjecture of Shokurov, which is stated in the relative case (cf. [29]). A simple version of their result shows that if \((X, \Delta)\) is a log canonical pair such that \(\Delta\) is reduced, \(- (K_X + \Delta)\) is nef, and \(c(X, \Delta)\) is non-positive, then \((X, \Delta)\) is a toric pair; see [23, Remark 4.4].

Let \(X\) be an \(n\)-dimensional smooth Fano variety of Picard number one and \(D \subset X\) a reduced divisor. Assume the existence of a non-isomorphic surjective endomorphism \(f : X \rightarrow X\) such that \(D\) is \(f^{-1}\)-invariant and \(f|_{X\setminus D}\) is étale. Hwang and Nakayama show that \(X\) is isomorphic to \(\mathbb{P}^n\) and \(D\) is a simple normal crossing divisor consisting of \(n + 1\) hyperplanes; see [19, Theorem 2.1]. In particular, \((X, D)\) is a toric pair. Indeed, their argument shows that the complexity \(c(X, D)\) is non-positive. Our Theorem 7.2 follows their idea and tries to generalize their result to the singular case. A key step of the following is to verify that 
\[
\hat{\Omega}^1_X(\log D)
\]
is free, i.e., isomorphic to \(\mathcal{O}_X^{\oplus m}\); see [19, Proposition 2.3] and [23, Theorem 5.4].

**Theorem 7.2. (cf. [23, Theorem 1.2])** Let \(X\) be a normal projective variety which is smooth in codimension 2, and \(D \subset X\) a reduced divisor such that

(i) there is a Weil \(\mathbb{Q}\)-divisor \(\Gamma\) such that the pair \((X, \Gamma)\) has only klt singularities;
(ii) there is a polarized endomorphism \(f : X \rightarrow X\) such that \(D\) is \(f^{-1}\)-invariant and \(f|_{X\setminus D}\) is quasi-étale;
(iii) the algebraic fundamental group \(\pi^\alg_1(X_{\text{reg}})\) of the smooth locus \(X_{\text{reg}}\) of \(X\) is trivial (this holds when \(X\) is smooth and rationally connected); and
(iv) the irregularity \(q(X) := h^1(X, \mathcal{O}_X)\) is zero (this holds when \(X\) is rationally connected).

Then the complexity \(c(X, D)\) is non-positive.

An immediate corollary is the following.

**Corollary 7.3. (cf. [23, Corollary 1.4])** Let \(X\) be a rationally connected smooth projective variety and \(D \subset X\) a reduced divisor. Suppose \(f : X \rightarrow X\) is a polarized endomorphism such that \(D\) is \(f^{-1}\)-invariant and \(f|_{X\setminus D}\) is étale. Then \((X, D)\) is a toric pair.

We say that a normal projective variety \(X\) is of **Fano type** if there is a Weil \(\mathbb{Q}\)-divisor \(\Delta\) such that the pair \((X, \Delta)\) has only klt singularities and \(- (K_X + \Delta)\) is an ample \(\mathbb{Q}\)-Cartier divisor. The assumption below of \(X\) being of Fano type is necessary, since a normal projective toric variety is known to be of Fano type. The lifting in the following corollary is necessary; see [23, Remark 1.7].
Corollary 7.4. (cf. [23, Corollary 1.5]) Let $f: X \to X$ be a polarized endomorphism of a normal projective variety $X$ of Fano type which is smooth in codimension 2. Let $D \subset X$ be an $f^{-1}$-invariant reduced divisor such that $f|_{X \setminus D}$ is quasi-étale and $K_X + D$ is $\mathbb{Q}$-Cartier. Then there exist a quasi-étale cover
\[ \pi: \tilde{X} \to X \]
and a polarized endomorphism
\[ \tilde{f}: \tilde{X} \to \tilde{X} \]
such that
1. the endomorphism $f$ lifts to $\tilde{f}$, i.e.,
\[ \pi \circ \tilde{f} = f \circ \pi, \]
and
2. the pair $(\tilde{X}, \tilde{D})$ is toric, where $\tilde{D} = \pi^{-1}(D)$.

The following well known conjecture is still open (see also Question 6.7). It has been affirmatively solved in dimension $\leq 3$. We refer to [1], [2], [10], [18], [19], [26] and [28] for details.

Conjecture 7.5. Let $X$ be a Fano manifold of Picard number one different from the projective space. Then a surjective endomorphism $X \to X$ must be bijective.

References

[1] E. Amerik, M. Rovinsky and A. Van de Ven, A boundedness theorem for morphisms between threefolds, Ann. Inst. Fourier (Grenoble) 49 (1999), 405-415.
[2] A. Beauville, Endomorphisms of hypersurfaces and other manifolds, Intern. Math. Res. Notices 2001, no. 1 (2001): 53-58.
[3] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type. J. Amer. Math. Soc., 23(2):405-468, 2010.
[4] C. Birkar and J. Waldron, Existence of Mori fibre spaces for 3-folds in char $p$, Adv. Math. 313 (2017), 62-101.
[5] S. Boucksom, T. de Fernex and C. Favre, The volume of an isolated singularity, Duke Math. J. 161 (2012), no. 8, 1455-1520.
[6] J. -Y. Briand and J. Duval, Erratum: Deux caractérisations de la mesure d’équilibre d’un endomorphisme de $P^n(\mathbb{C})$, Publ. Math. Inst. Hautes Études Sci. No. 109 (2009), 295-296.
[7] A. Broustet and A. Höring, Singularities of varieties admitting an endomorphism, Math. Ann. 360 (2014), no. 1-2, 439-456.
[8] M. Brown, J. McKernan, R. Svaldi and H. Zong, A geometric characterisation of toric varieties, Duke Math. J. Volume 167, Number 5 (2018), 923-968.
[9] P. Cascini, S. Meng and D.-Q. Zhang, Polarized endomorphisms of normal projective threefolds in arbitrary characteristic, arxiv:1710.01903
[10] D. Cerveau and A. Lins Neto, Hypersurfaces exceptionnelles des endomorphismes de CP(n), Bol. Soc. Brasil. Mat. (N.S.) 31 (2000), no. 2, 155-161.

[11] T. -C. Dinh, Analytic multiplicative cocycles over holomorphic dynamical systems, Complex Var. Elliptic Equ. 54 (2009), no. 3-4, 243-251.

[12] T. -C. Dinh and N. Sibony, Dynamique des applications d’allure polynomiale, J. Math. Pures Appl. 82 (2003), pp. 367-423.

[13] T. -C. Dinh and N. Sibony, Equidistribution speed for endomorphisms of projective spaces, Math. Ann. 347 (2010), no. 3, 613-626.

[14] N. Fakhruddin, Questions on self-maps of algebraic varieties, J. Ramanujan Math. Soc., 18(2):109-122, 2003.

[15] O. Fujino, Some remarks on the minimal model program for log canonical pairs, J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, 149-192.

[16] D. Greb, S. Kebekus and T. Peternell, Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of Abelian varieties, Duke Math. J. 165 (2016), no. 10, 1965-2004.

[17] C. Hacon and C. Xu, On the three dimensional minimal model program in positive characteristic, J. Amer. Math. Soc. 28 (2015), no. 3, 711-744.

[18] J.-M. Hwang and N. Mok, Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles, J. Alg. Geom. 12 (2003), 627-651.

[19] J. M. Hwang and N. Nakayama, On endomorphisms of Fano manifolds of Picard number one, Pure Appl. Math. Q. 7(4), pp.1407-1426, 2011.

[20] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math., 134 Cambridge Univ. Press, 1998.

[21] S. Meng, Building blocks of amplified endomorphisms of normal projective varieties, arXiv:1712.08995

[22] S. Meng and D. -Q. Zhang, Building blocks of polarized endomorphisms of normal projective varieties, Adv. Math. 325 (2018), 243-273.

[23] S. Meng and D. -Q. Zhang, Characterizations of toric varieties via polarized endomorphisms, arXiv:1702.07883

[24] S. Meng and D. -Q. Zhang, Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism, arXiv:1806.05828

[25] N. Nakayama and D.-Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds, Proc. Lond. Math. Soc. (3) 99 (2009), no. 3, 725-756.

[26] N. Nakayama and D.-Q. Zhang, Polarized endomorphisms of complex normal varieties, Math. Ann. 346 (2010), no. 4, 991-1018.

[27] S. Okawa, Extensions of two Chow stability criteria to positive characteristics, Michigan Math. J. 60 (2011), no. 3, 687-703.

[28] K. H. Paranjape and V. Srinivas, Self maps of homogeneous spaces, Invent. Math. 98 (1989), 425-444.

[29] V. V. Shokurov, Complements on surfaces, J. Math. Sci. (New York) 102 (2000), no. 2, 3876-3932, Algebraic geometry, 10.

[30] J. Waldron, The LMMP for log canonical 3-folds in char p, arXiv:1603.02967

[31] J. Wahl, A characteristic number for links of surface singularities, J. Amer. Math. Soc. 3 (1990), no. 3, 625-637.
[32] X. Yuan and S. Zhang, The arithmetic Hodge index theorem for adelic line bundles, Math. Ann. 367 (2017), no. 3-4, 1123-1171.

[33] D.-Q. Zhang, Polarized endomorphisms of uniruled varieties. Compos. Math. 146 (2010), no. 1, 145-168.

[34] D. -Q. Zhang, n-dimensional projective varieties with the action of an abelian group of rank n − 1, Trans. Amer. Math. Soc. 368 (2016), no. 12, 8849-8872.

[35] S. W. Zhang, Distributions in algebraic dynamics, 381-430, Survey in Differential Geometry 10, Somerville, MA, International Press, 2006.

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