Counter-examples to the high-order version and strong version of the generalized Eshelby conjecture for anisotropic media

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Abstract

In this work, we prove that in anisotropic media possessing cubic, transversely isotropic, orthotropic, and monoclinic symmetries, there exist non-ellipsoidal inclusions that can transform particular quadratic eigenstrains into quadratic elastic strain fields in them. Further, we prove that in these anisotropic media, there exist non-ellipsoidal inclusions that can transform particular polynomial eigenstrains of even degrees into polynomial elastic strain fields of the same even degrees in them. A sufficient condition for the existence of those counter-examples is provided. These results constitute counter-examples, in the strong sense, to the generalized high-order Eshelby conjecture (inverse problem of Eshelby’s polynomial conservation theorem) for polynomial eigenstrains in both anisotropic media and the isotropic medium (quadratic eigenstrain only). In addition, we also show that there are counter-examples to the strong version of the generalized Eshelby conjecture for uniform eigenstrains in these anisotropic media. These findings reveal striking richness of the uniformity between the eigenstrains and the correspondingly induced elastic strains in inclusions in anisotropic media beyond the canonical ellipsoidal inclusion.

Subject Areas: solid mechanics, applied mathematics, engineering

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1 Introduction

In a recent paper [1], we prove that the weak version of the generalized Eshelby conjecture for anisotropic media is true, which is associated with the combination of the elastic parameters and uniform eigenstrains in the context of anisotropic elasticity for materials possessing cubic, transversely isotropic, orthotropic, and monoclinic symmetries. That the weak conjecture is true does not exclude the possibility that some non-ellipsoidal inclusions can transform some uniform eigenstrain(s) into a uniform elastic strain field in the inclusion in these anisotropic media. However, the generalized strong conjecture excludes such cases, but whether the strong conjecture is true or false is yet to be verified. In this paper, we explore the high-order Eshelby conjecture concerning polynomial eigenstrains, and thus, a uniform eigenstrain is considered as a particular polynomial eigenstrain of degree zero. By searching for counter-examples, the generalized strong versions of both the high-order Eshelby conjecture and the conventional Eshelby conjecture are studied. As a matter of fact, compared with the uniform eigenstrain, polynomial eigenstrains are of more practical implications, besides theoretical significance. For instance, Mura [2] pointed out that the equivalent inclusion method can be extended to nonuniform stress fields in inclusion problems by expanding the equivalent eigenstrains into polynomials of the coordinates. Moreover, as the strain fields in embedded quantum dots, which are anisotropic crystals, can be used to tune the behaviour of the emitted photons by the dots [1, 3, 4, 5, 6], the solutions of the inclusion problems in the context of anisotropy can help to correlate the mismatch strain (eigenstrain) to the final strain fields in the quantum dot structures, and thus have particular implications in the technology of quantum information.

In 1961, in addition to setting forth the conventional conjecture regarding the uniform eigenstrain, Eshelby [7] also verified that such an extraordinary uniformity property could be extended to the case when the eigenstrain is a polynomial of the coordinates of the interior points of the ellipsoidal inclusion. Specifically, Eshelby stated that if the eigenstrain within an ellipsoidal inclusion is a polynomial of the coordinates of the points with degree \( n \), resultantly, the induced elastic strain must be a polynomial with the same degree \( n \), which is called Eshelby’s polynomial conservation property [8] or Eshelby’s polynomial conservation theorem [9] in the subsequent research. The explicit expression of the induced elastic strain inside an ellipsoidal inclusion, when subjected to an eigenstrain of a polynomial form, is explicitly formulated by Sendeckyj [10] in the context of isotropy with the utilization of Ferrers and Dyson’s theorem [11, 12] that is concerned with the Newtonian potential with a polynomial mass density, which reveals that there exist formulas connecting the elastic problem to the potential problem. Thus, Eshelby and the following researchers validated the Eshelby’s polynomial conservation theorem for the isotropic elastic medium. In 1975, Asaro and Barnett [13] firstly studied the interior strain field of an anisotropic ellipsoidal inclusion subjected to an eigenstrain of a polynomial form, and substantiated the validity of Eshelby’s polynomial conservation theorem for the anisotropic elastic medium, and the same result is obtained by Mura and Kinoshita [14] with the exterior strain field additionally derived. The solutions derived by Asaro and Barnett [13], and Mura and Kinoshita [14] are not entirely explicit; thus other researchers derived the explicit closed-form results for a spherical inhomogeneity [15] and a cylindrical inhomogeneity [16], which can form the corresponding results for spherical and cylindrical inclusions via the equivalent inclusion method (EIM) [17, 18], noting that the solutions to the Eshelby conjecture in the sense of the inhomogeneity problem is equivalent to that in the sense of the inclusion problem. Moreover, Rahman [9] reported an explicit closed-form strain field inside an isotropic ellipsoid for a particular polynomial eigenstrain, which is characterized by the equation of the surface of the ellipsoid, and Nie and co-workers [19, 20] derived the strain field of an elliptic inhomogeneity embedded in an orthotropic medium.
under eigenstrains of linear and quadratic polynomial forms.

Recently, Liu [21] proposed a mathematically rigorous proof of Eshelby’s polynomial conservation theorem for an ellipsoidal inclusion in an anisotropic medium via solving particular p-harmonic problems in arbitrary dimensions. Calvo-Jurado and Parnell presented a new scheme to evaluate the field inside an isolated elliptical inhomogeneity and further verified Eshelby’s polynomial conservation theorem in two dimensions [8], via the approximation method [22] firstly established to deal with the Eshelby problem in the sense of Newtonian potentials. Rashidinejad and Shodja [23] proved that Eshelby’s polynomial conservation theorem remains valid even when multi-field effects are considered. They found that magneto-electro-elastic ellipsoidal inclusions retain Eshelby’s polynomial property, but pointed a limitation of this striking property, which requires that the anisotropy is rectilinear.

Therefore, Eshelby’s polynomial conservation theorem has been proved for the ellipsoidal inclusion problem in the context of linearly elastic isotropy and rectilinear anisotropy. Conversely, the inverse problem, namely, whether the ellipsoid is the only shape that possesses Eshelby’s polynomial property for any single polynomial eigenstrain, is not explored. The answer to this question depends on the proof or disproof of the conjecture that no inclusion other than an ellipsoid transforms a polynomial eigenstrain into a polynomial elastic strain field of the same degree in it. This conjecture is the counterpart of the strong version of the conventional Eshelby conjecture for the uniform eigenstrain [24, 25, 26], and thus, when including the material symmetry, the generalized strong conjecture in the recent paper [1] is a special case when the degree of the polynomial is zero.

There are studies dealing with the non-ellipsoidal or non-elliptical inclusions like polygons [27, 28, 29] with polynomial eigenstrains prescribed, but the results only show that the considered non-ellipsoidal or non-elliptical inclusions do not exhibit Eshelby’s polynomial property, which does not lead to either falsification or substantiation of the high-order Eshelby conjecture. In this work, with the help of the variational method proposed by Liu [25], we firstly prove the existence of a three-dimensional non-ellipsoidal inclusion that possesses Eshelby’s polynomial property in linearly elastic anisotropic media of cubic, transversely isotropic, orthotropic, and monoclinic symmetries, when the eigenstrain is expressed in the form of a polynomial of degree two. We also find that such a result is also applicable to isotropic materials, which implies the disproof of the validity of the generalized strong version of the high-order Eshelby conjecture for the case when the eigenstrain is in the form of quadratic polynomials in both anisotropic and isotropic media. Secondly, more counter-examples are constructed to extend the proof of the invalidity of the generalized strong version of the high-order Eshelby conjecture for the polynomial eigenstrain of degree two to that for polynomial eigenstrains of any non-negative even degree. We also show that the counter-example given by Liu [25] concerning the strong version of the Eshelby conjecture in the isotropic medium can be utilized to disprove the generalized strong version of the Eshelby conjecture for uniform eigenstrains in the anisotropic media listed above. A sufficient condition for the existence of counter-examples for polynomial eigenstrains of any non-negative even degree in anisotropic media and the proof of its validity are provided as a lemma.

2 Polynomial eigenstrain of degree two

We start with the equilibrium equation in the expression of the displacement for the Eshelby inclusion problem in the context of linear elasticity, i.e.,

$$\nabla \cdot \left[ C : (\nabla \otimes u(x) - \chi_{\Omega}(x) e^*(x)) \right] = 0 \text{ in } \mathbb{R}^3,$$

(1)
where the repeated indexes obey the summation convention; \( C \) is the fourth-order elastic tensor; \( u(x) \) is the displacement vector; \( \varepsilon^e(x) \) is the eigenstrain which is a second-order tensor; \( \Omega \subset \mathbb{R}^3 \) is a simply connected and bounded inclusion domain with a Lipschitz boundary; and \( \chi_\Omega(x) \) is the characteristic function satisfying

\[
\begin{aligned}
\chi_\Omega(x) &= 1 & x &\in \Omega, \\
\chi_\Omega(x) &= 0 & x &\notin \Omega.
\end{aligned}
\] (2)

In the context of anisotropy, it turns to be hard to capture an explicit solution to \( u \) in (1), owing to the complexity of the elastic tensor.

By using the Fourier analysis as what we did in [1], we note that \( u \) can be given as

\[
u_m(x) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} T^{-1}_{mn}(\eta)C_{nspq}\varepsilon^s_{pq}(y)\eta_s \int_\Omega e^{-i\eta(x-y)}dyd\eta,
\] (3)

and thus the strain field is

\[
\varepsilon_{ij}(x) = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{2} \left( T^{-1}_{in}(\eta)\eta_j + T^{-1}_{jn}(\eta)\eta_i \right) C_{nspq}\varepsilon^s_{pq}(y)\eta_s \int_\Omega e^{-i\eta(x-y)}dyd\eta,
\] (4)

where the right-hand sides of (3) and (4) are integrals over \( \Omega \times \mathbb{R}^3 \) with respect to \( (y, \eta) \), and

\[
T_{mn}(\eta) = C_{mknl}\eta_k\eta_l \quad (m, n, k, l = 1, 2, 3)
\] (5)
is a symmetric tensor, with \( i \) the imaginary unit.

Note that the Newtonian potential \( N_{\Omega}[\rho] \) induced by \( \Omega \) with mass density \( \rho \) is the solution to

\[
\Delta N_{\Omega}[\rho] = \chi_\Omega\rho,
\] (6)

whose Hessian matrix \( H_{ij}(N_{\Omega}[\rho]) \), by the Fourier analysis, can be expressed as

\[
H_{ij}(N_{\Omega}[\rho]) = \frac{\partial^2 N_{\Omega}[\rho]}{\partial x_i \partial x_j} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_i\eta_j}{\eta_1^2 + \eta_2^2 + \eta_3^2} \int_\Omega \rho(y)e^{-i\eta(x-y)}dyd\eta.
\] (7)

Previously, Liu [25] pointed out that there is correlation between \( H_{ij} \) in (7) and \( \varepsilon_{ij} \) in (4) when \( C_{nspq} \) is isotropic, and \( \varepsilon^s_{pq}(y) \), which is appropriately chosen, and \( \rho(y) \) are both constant function of \( y \). In this work, we find that there still exists a relationship between \( H_{ij} \) in (7) and \( \varepsilon_{ij} \) in (4) even when \( C_{nspq} \) is anisotropic, and \( \varepsilon^s_{pq}(y) \) and \( \rho(y) \) are in some particular polynomial forms.

To continue our analysis, we will follow the Voigt notation of the anisotropic elastic tensor \( C \), that is, \( C_{ijkl} \) \((i, j, k, l = 1, 2, 3)\) is represented by \( C_{mn} \) \((m, n = 1, 2, 3, 4, 5, 6)\) via the following contraction of the indexes:

\[
m (\text{or} n) = 1 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 11; \quad m (\text{or} n) = 2 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 22;
\]
\[
m (\text{or} n) = 3 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 33; \quad m (\text{or} n) = 4 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 23;
\]
\[
m (\text{or} n) = 5 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 13; \quad m (\text{or} n) = 6 \quad \text{when} \quad ij \quad (\text{or} \quad kl) = 12.
\] (8)

Then we consider different material symmetries separately.
2.1 Cubic material

Let the three axes of the Cartesian coordinate system \( x = (x_1, x_2, x_3) \) coincide with the 4-fold rotation axes of the cubic material. Then there are three mutually independent elastic parameters \( C_{11}, C_{12} \) and \( C_{44} \) satisfying

\[
C_{11} > 0, \quad C_{44} > 0, \quad C_{11} > C_{12} > -\frac{1}{3} C_{11},
\]

which guarantee the positive definiteness of \( C \).

Then we will present and prove a theorem, which provides a counter-example to the generalized strong version of the high-order Eshelby conjecture for cubic materials.

**Theorem 2.1** For cubic materials whose elastic parameters satisfy \( C_{12} + C_{44} = 0 \), there exists a non-ellipsoidal inclusion \( \Omega \) that possesses Eshelby’s polynomial property for a particular polynomial eigenstrain of degree two.

Note that under \( C_{12} + C_{44} = 0 \), the cubic material will not degenerate into an isotropic material, since for an isotropic medium where \( C_{44} = \frac{C_{11} - C_{12}}{2} \), \( C_{12} + C_{44} = 0 \) implies \( C_{11} + C_{12} = 0 \), which contradicts (9).

The particular eigenstrain in the form of a quadratic polynomial is chosen such that the corresponding eigenstress \( \sigma_{ij}^* = C_{ijmn} \varepsilon_{mn}^* \) takes the form

\[
\sigma_{ij}^* = \rho(x) P_{ij},
\]

where \( \rho(x) \) is a quadratic polynomial, and \( P_{ij} \) denotes the uniaxial stress state. Here, we just consider the case where

\[
P := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

and the other two cases can be analysed in the same way.

Under the condition

\[
C_{12} + C_{44} = 0,
\]

by substituting (10) and (11) into (4), we can obtain

\[
\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0,
\]

\[
\varepsilon_{13}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_1 \eta_3}{C_{11} \eta_3^2 + C_{44} (\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta,
\]

\[
\varepsilon_{23}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_2 \eta_3}{C_{11} \eta_3^2 + C_{44} (\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta,
\]

\[
\varepsilon_{33}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_3^2}{C_{11} \eta_3^2 + C_{44} (\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta.
\]

By transformations of coordinates

\[
x' := \tilde{Q} \cdot x, \quad y' := \tilde{Q} \cdot y, \quad \eta' := \tilde{Q}^{-1} \cdot \eta
\]
with

\[
\tilde{Q} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}
\]

and then substitution of (14) into (13) with \(s = \sqrt{\frac{C_{33}}{C_{11}}}\), we obtain

\[ \varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0, \]

\[
\varepsilon_{13}(x') = -\frac{1}{2\sqrt{C_{11}C_{44}}(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta^2_1 + \eta^2_2 + \eta^2_3}{\eta^2_1 + \eta^2_2 + \eta^2_3} \int_{\partial \Omega} \rho(y') e^{-in' \cdot (x'-y')} dy' d\eta',
\]

\[
\varepsilon_{23}(x') = -\frac{1}{2\sqrt{C_{11}C_{44}}(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta^2_2 + \eta^2_3}{\eta^2_1 + \eta^2_2 + \eta^2_3} \int_{\partial \Omega} \rho(y') e^{-in' \cdot (x'-y')} dy' d\eta',
\]

\[
\varepsilon_{33}(x') = -\frac{1}{\sqrt{C_{11}}(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\eta^2_1 + \eta^2_2 + \eta^2_3} \int_{\partial \Omega} \rho(y') e^{-in' \cdot (x'-y')} dy' d\eta',
\]

with

\[
\Omega' := \{ y' \mid \tilde{Q}^{-1} \cdot y' \in \Omega \}.
\]

Through comparison of (17) with (16), we see

\[
\varepsilon_{ij}(x') = \frac{1}{2C_{44}} \left( \tilde{Q}_{il} P_{mj} \tilde{Q}_{mq} \frac{\partial^2 N_{\Omega'}[\rho](x')}{\partial x_i \partial x_j} + \tilde{Q}_{jm} P_{il} \tilde{Q}_{is} \frac{\partial^2 N_{\Omega'}[\rho](x')}{\partial x_m \partial x_s} \right),
\]

where

\[
N_{\Omega'}[\rho](x') = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\eta^2_1 + \eta^2_2 + \eta^2_3} \int_{\partial \Omega} \rho(y') e^{-in' \cdot (x'-y')} dy' d\eta'
\]

is the Newtonian potential induced by \(\Omega'\) with the mass density \(\rho\).

We note that if we can find \(\Omega'\) that leads to \(\frac{\partial^2 N_{\Omega'}[\rho](x')}{\partial x_i \partial x_j}\) being a polynomial of degree two, which implies \(N_{\Omega'}[\rho](x')\) is a polynomial of degree four, then \(\Omega'\) must be the inclusion that possesses Eschelby’s polynomial property due to (18).

Given this, the aim to prove Theorem 2.1 is achieved by proving the following theorem:

**Theorem 2.2** For \(\psi(y) := -(y_1^2 + y_2^2 + y_3^2)\), there exists at least one simply-connected bounded Lipschitz domain \(\Omega\) of non-ellipsoidal shape which leads to

\[
N_{\Omega}[\psi](x) := -\int_{\Omega} \frac{\psi(y)}{4\pi |x-y|} dy = \text{quartic, } x \in \Omega.
\]

The theorem will be verified by proving two lemmas, i.e.,

**Lemma 2.1** Let \(\varphi(x) := C - \frac{1}{12} (x_1^4 + x_2^4 + x_3^4)\) with \(C\) a positive real constant. There exists a simply-connected bounded Lipschitz domain \(\Omega \subset \mathbb{R}^3\) which leads to

\[
N_{\Omega}[\psi](x) = -\int_{\Omega} \frac{\psi(y)}{4\pi |x-y|} dy = \varphi(x), \quad x \in \Omega.
\]

**Lemma 2.2** Let \(E\) denote an ellipsoid arbitrarily oriented and placed in \(\mathbb{R}^3\).

\[
\forall E \subset \mathbb{R}^3, \quad N_{E}[\psi](x) := -\int_{E} \frac{\psi(y)}{4\pi |x-y|} dy \neq \varphi(x), \quad x \in \Omega.
\]
Proof of Lemma 2.1

To handle the free boundary problem when the boundary is undetermined, Friedman [30] has set up a variational inequality to analyze a series of potential problems. Further, the variational method has been extended by Liu [25], achieving the construction of non-ellipsoidal extremal structures that possess the Eshelby uniformity property in a medium with a fourth-order isotropic elastic tensor of three elastic constants by solving a particular over-determined problem concerning the Newtonian potential with a constant mass density.

We note that the variational scheme proposed by Liu [25] can also be applied to proving the existence of non-ellipsoidal inclusions that possess Eshelby’s polynomial property in anisotropic media by solving a corresponding Newtonian potential problem but with a quadratic mass density, as is shown in Lemma 2.1 and Lemma 2.2.

Firstly, let us recall the variational method given in [25]. According to [25], we know that for an obstacle function \( \phi \) satisfying

1. \( \phi \in C^{0,1}(\mathbb{R}^3) \), which implies \( \phi \) possesses Lipschitz continuity with a norm
   \[
   \|\phi\|_{0,1} = \sup_{x \in \mathbb{R}^3} |\phi(x)| + \sup_{x, y \in \mathbb{R}^3} \left| \frac{\phi(x) - \phi(y)}{|x - y|} \right|
   \] (23)

2. there exists \( r_0 > 0 \), such that \( \forall |x| \geq r_0, \phi(x) \leq 0 \);

3. \( |\Delta \phi| \) is bounded in \( B_{r_0} \setminus U^* \), with \( B_{r_0} = \{x||x| \leq r_0, x \in \mathbb{R}^3\} \) and \( U^* \) the set of the singular points where \( |\nabla \otimes \nabla \phi| \), which denotes the norm of the second-order tensor \( \nabla \otimes \nabla \phi \), is unbounded, in the sense of distribution;

4. \( \exists C^\phi \in \mathbb{R} \), such that \( \forall \zeta \in \mathbb{R}^3 \) with \( |\zeta| = 1 \),
   \[
   \int_{U^*} \frac{\partial^2 \xi}{\partial \zeta^2} \left( \phi + \frac{1}{2} C^\phi |x|^2 \right) dx \geq 0,
   \] (24)

for any smooth function \( \xi \in C^\infty_c(\mathbb{R}^3) \) with a compact support, where \( \frac{\partial}{\partial \zeta} \) denotes the directional derivative,

the variational inequality

\[
\Pi(V_\phi) = \inf_{\nu \in K_\phi} \left\{ \Pi(\nu) \equiv \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \nu|^2 \right\},
\] (25)

where \( K_\phi = \{ \nu \in W^{1,2}_0(\mathbb{R}^3) : \nu \geq \phi \} \), admits a unique minimizer \( V_\phi \in W^{2,\infty}_{loc}(\mathbb{R}^3) \cap K_\phi \) satisfying

\[
\Delta V_\phi \leq 0, \quad V_\phi \geq \phi, \quad (V_\phi - \phi) \Delta V_\phi = 0 \quad \text{in} \quad \mathbb{R}^3,
\] (26)

and there exists a coincident set \( \Omega = \{ x | V_\phi(x) = \phi(x), x \in \mathbb{R}^3 \} \) with \( \Omega \subseteq B_{r_0} \).

In the above expressions, \( W^{1,2}_0(\mathbb{R}^3) \) denotes the class of functions in \( L^2(\mathbb{R}^3) \) with a zero boundary value, and the first derivatives of the functions in \( W^{1,2}_0(\mathbb{R}^3) \) also belong to \( L^2(\mathbb{R}^3) \) in the sense of distribution. \( W^{2,\infty}_{loc}(\mathbb{R}^3) \) denotes the class of functions in \( L^\infty(\mathbb{R}^3) \), whose first and second derivatives also belong to \( L^\infty(\mathbb{R}^3) \) with the norm

\[
||V_\phi(x)||_{2,\infty} = \sum_{m \leq 2} \operatorname{ess sup}_{x \in \mathbb{R}^3} |\partial^m V_\phi(x)|
\] (27)
where \( m \geq 0; \partial^m V_\phi(x) = \frac{\partial^m v(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} \) with \( k_i \geq 0, k_i \in \mathbb{Z} \) \( (i = 1, 2, 3) \) and \( \sum_{i=1}^{3} k_i = m \) denote weak derivatives; and ‘ess sup’ denotes the essential supremum. Besides, the subscript ‘loc’ implies the norm in \( (27) \) must be bounded with \( \mathbb{R}^3 \) changed by any bounded strictly interior subdomain of it.

Based on the obstacle function \( \phi \), the following over-determined problem
\[
\begin{align*}
&\Delta v_{od} = \chi_{\Omega}^\alpha \Delta \phi &\quad \text{in } \mathbb{R}^3 \\
&\nabla v_{od} = \nabla \phi &\quad \text{for } \mathbf{x} \in \Omega \\
&|v_{od}| \leq \frac{C}{|\mathbf{x}|} &\quad \text{for } \mathbf{x} \geq r_0
\end{align*}
\]

admits a solution \( v_{od} = V_\phi \). Here \( C \) is a constant. The details of the above formulation can be found in \( [25] \).

Let \( \Gamma(x - y) := -\frac{1}{4\pi |x - y|} \). Then owing (28), and \( \phi \), for any \( v_{od} \in W^2_{loc}(\mathbb{R}^3) \), we can get
\[
N_\Omega[\Delta \phi](x) = \int_{\mathbb{R}^3} \chi_{\Omega}(y) \Delta \phi(y) \Gamma(x - y) dy
\]
\[
= \int_{\mathbb{R}^3} \Delta v_{od}(y) \Gamma(x - y) dy
\]
\[
= \sum_{i = 1}^{3} \int_{\partial B_{\infty}} \frac{\partial v_{od}(y)}{\partial y_i} \Gamma(x - y) n_i dy - \sum_{i = 1}^{3} \int_{\mathbb{R}^3} \frac{\partial v_{od}(y)}{\partial y_i} \frac{\partial \Gamma(x - y)}{\partial y_i} dy
\]
\[
= -\sum_{i = 1}^{3} \int_{\partial B_{\infty}} v_{od}(y) \frac{\partial \Gamma(x - y)}{\partial y_i} n_i dy + \int_{\mathbb{R}^3} v_{od}(y) \Delta \Gamma(x - y) dy
\]
\[
= \int_{\mathbb{R}^3} v_{od}(y) \delta(x - y) dy = v_{od}(x),
\]
where \( B_{\infty} = \lim_{r \to \infty} \{x | |x| \leq r, x \in \mathbb{R}^3 \} \), and \( \mathbf{n} = (n_1, n_2, n_3) \) is the unit outward normal to \( \partial B_{\infty} \). Hence we conclude that any solution \( v_{od} \in W^2_{loc}(\mathbb{R}^3) \) of (28) must be the Newtonian potential induced by \( \Omega \) with the mass density \( \Delta \phi \).

Secondly, based on the result above, we will introduce a particular \( \phi^* \) and prove that such a \( \phi^* \) satisfies all of the properties of the obstacle function listed above. We specify the particular \( \phi^*(x) \) as
\[
\phi^*(x) := \begin{cases} 
\phi(x), & \mathbf{x} \in U \\
-3C, & \mathbf{x} \in \mathbb{R}^3 \setminus U
\end{cases}
\]
with \( U := \{ x | x^4 + x^2 + x^4 \leq 48C, x \in \mathbb{R}^3 \} \). Let \( U' := \{ x | |x_i| \leq (48C)^{\frac{1}{i}} (i = 1, 2, 3), x \in \mathbb{R}^3 \} \). It is easily seen that \( U \subset \tilde{U} \); hence \( U \) is a bounded domain with \( \partial U \) defined by the surface \( x^4 + x^2 + x^4 = 48C = 0 \). In addition,
\[
\phi^*|_{\partial U^-} = \phi^*|_{\partial U^+} = -3C,
\]

where \( \partial U^- \) means the limiting value approached from the interior of \( U \), and \( \partial U^+ \) means the limiting value approached from the exterior of \( U \). Thus \( \phi^* \in C^0(\mathbb{R}^3) \). Given this, we can further calculate
\[
||\phi^*||_{0, 1} = \max \left\{ \sup_{x \in U} |\phi(x)|, -3C \right\}
+ \max \left\{ \sup_{x, y \in U} \frac{|\phi(x) - \phi(y)|}{|x - y|}, \sup_{x \in U, y \in \mathbb{R}^3} \frac{|\phi(x) + 3C|}{|x - y|} \right\}.
\]
Substituting $\phi(x) = C - \frac{1}{12} (x_1^4 + x_2^4 + x_3^4)$ into (32) yields

$$||\phi^*||_{0,1} = \sup_{x \in U} |\phi(x)| + \sup_{x \in U} |\nabla \phi(x)|.$$  \hspace{1cm} (33)

Since $|\phi(x)|$ and $|\nabla \phi(x)|$ are bounded in $U$, we verify that $\phi^* \in C^{0,1}(\mathbb{R}^3)$.

Let $r_0 = 6\sqrt{C}$. Because

$$\forall x \in B_{r_0}, \ x_1^4 + x_2^4 + x_3^4 \leq (x_1^2 + x_2^2 + x_3^2)^2 \leq 36C < 48C,$$  \hspace{1cm} (34)

we see that $B_{r_0} \subset U$.

Then we can evaluate $\phi^*(x)$ for $|x| \geq r_0$. When $x \in U \setminus B_{r_0}$,

$$\phi^*(x) = C - \frac{1}{12} (x_1^4 + x_2^4 + x_3^4) \leq C - \frac{1}{36} (x_1^2 + x_2^2 + x_3^2)^2 \leq 0.$$  \hspace{1cm} (35)

And when $x \in \mathbb{R}^3 \setminus U$,

$$\phi^*(x) = -3C < 0.$$  \hspace{1cm} (36)

It can be concluded from (35) and (36) that $\forall |x| \geq r_0, \ \phi^*(x) \leq 0$. And it is easy to see that $|\Delta \phi^*|$ is bounded in $B_{r_0} \setminus U^*$, since $U^* = \emptyset$ and $\phi^* \in C^\infty(B_{r_0})$, which results from $B_{r_0} \subset U$.

Finally, let $U^\xi$ denote the compact support of a smooth function $\xi \in C^\infty(\mathbb{R}^3)$, on which $\xi \geq 0$. By definition, we know that $\xi = 0$ in $\mathbb{R}^3 \setminus U^\xi$, and

$$\forall \ n = i + j + k \ with \ i, j, k \geq 0, \ \frac{\partial^n \xi}{\partial x_1^i \partial x_2^j \partial x_3^k} \bigg|_\partial U = 0.$$  \hspace{1cm} (37)

Then by substituting (30) into the left hand side of (24), we obtain

$$\int_{U^\xi} \frac{\partial^2 \xi}{\partial \xi^2} \left( \phi^* + \frac{1}{2} C^\phi |x|^2 \right) \ dx = \int_{U^\xi} \frac{\partial^2 \xi}{\partial \xi^2} \phi^* \ dx + \int_{U^\xi} \frac{1}{2} C^\phi |x|^2 \frac{\partial^2 \xi}{\partial \xi^2} \ dx$$

$$= \int_{U^\xi \cap U} \frac{\partial^2 \xi}{\partial \xi^2} \phi \ dx + \int_{U^\xi \cap (\mathbb{R}^3 \setminus U)} (-3C) \frac{\partial^2 \xi}{\partial \xi^2} \ dx + \int_{U^\xi} \xi C^\phi \ dx$$

$$= \int_{\partial U \cap U^\xi} \frac{\partial \xi}{\partial \xi} \phi \cdot dS - \int_{U^\xi \cap (\mathbb{R}^3 \setminus U)} \frac{\partial \xi}{\partial \xi} \frac{\partial \phi}{\partial \xi} \ dx + \int_{\partial U \cap U^\xi} \frac{\partial \xi}{\partial \xi} (-3C) \cdot (-dS)$$

$$- \int_{U^\xi \cap (\mathbb{R}^3 \setminus U)} \frac{\partial \xi}{\partial \xi} \cdot dS + \int_{U^\xi} \xi C^\phi \ dx$$  \hspace{1cm} (38)

$$= -\int_{\partial U \cap U^\xi} \xi \frac{\partial \phi}{\partial \xi} \cdot dS + \int_{U^\xi \cap U} \xi \frac{\partial^2 \phi}{\partial \xi^2} \ dx + \int_{U^\xi} \xi C^\phi \ dx$$

$$\geq -\int_{\partial U \cap U} \xi \left( C^\phi + \sup_{x \in \partial U} \left| \frac{\partial \phi}{\partial \xi} \right| + \sup_{x \in U} \left| \frac{\partial^2 \phi}{\partial \xi^2} \right| \right) \ dx + \int_{U^\xi} \xi C^\phi \ dx$$

Based on (38), we know that $\exists C^\phi \in \mathbb{R}$ satisfying

$$C^\phi \geq \sup_{x \in U} \left| \frac{\partial \phi}{\partial \xi} \right| + \sup_{x \in U} \left| \frac{\partial^2 \phi}{\partial \xi^2} \right|,$$  \hspace{1cm} (39)
such that
\[ \int \frac{\partial^2 \xi}{\partial \zeta^2} \left( \phi^* + \frac{1}{2} C^\phi |x|^2 \right) dx \geq \int_{U^\xi} \xi \left( C^\phi - \sup_{x \in U} \left| \frac{\partial \phi}{\partial \zeta} \right| - \sup_{x \in U} \left| \frac{\partial^2 \phi}{\partial \zeta^2} \right| \right) dx \geq 0. \] (40)

Therefore, we have proved that \( \phi^* \) satisfies all of the properties that an obstacle function \( \phi \) must possess. Then for \( \phi^* \), the over-determined problem (28) with \( \phi \) replaced by \( \phi^* \) admits a corresponding solution \( v_{od} = V_{\phi^*} \in W_{\text{loc}}^{2,\infty}(\mathbb{R}^3) \), and there is a coincident set \( \Omega \subseteq B_{\varrho_0} \subseteq U \), satisfying \( V_{\phi^*} = \phi^* = \phi \) for \( x \in \Omega \).

According to (29), \( V_{\phi^*}(x) \) is actually the Newtonian potential induced by \( \Omega \) with the mass density \( \Delta \phi^* \). Considering \( \Delta \phi^* = \Delta \phi = \psi \), since \( \Omega \subseteq U \), we see that \( V_\phi(x) = N_\Omega[\psi](x) = \phi(x) \) for \( x \in \Omega \). Therefore, we have substantiated the existence of a domain \( \Omega \) that leads to \( N_\Omega[\psi](x) = \phi(x) \) for \( x \in \Omega \), and thus the proof of Lemma 2.2 is achieved.

**Proof of Lemma 2.2**

Firstly, we know that the Newtonian potential of an ellipsoid should rely on the orientation of the ellipsoid, and it is also dependent on the position, since the mass density is not homogeneous, which varies with the position in the coordinate system \( x = (x_1, x_2, x_3) \). Here we let \( z = (z_1, z_2, z_3) \) be the Cartesian coordinate whose origin is at the center of the ellipsoid and its axes are along the axes of the ellipsoid so that the ellipsoid is expressed as \( E = \{ z \mid \frac{z_1^2}{a_1^2} + \frac{z_2^2}{a_2^2} + \frac{z_3^2}{a_3^2} \leq 1 \} \).

Introduce a coordinate transformation
\[ x = \mathbf{Q} \cdot z + d, \] (41)

where \( \mathbf{Q} \) is an second-degree orthogonal tensor denoting rotation, and \( d \in \mathbb{R}^3 \) denotes the translation. Then, the Newtonian potential induced by \( E \) with the mass density \( \psi \) given in the coordinates \( x = (x_1, x_2, x_3) \) of the global frame is expressed in the coordinates \( z = (z_1, z_2, z_3) \) of the body frame of the ellipsoid as
\[ N_E[\psi](z) = -\frac{1}{4\pi} \int_E \frac{|\mathbf{Q} \cdot z' + d|^2}{|z - z'|} dz' = \int_E \frac{|z'|^2 + 2(\mathbf{d} \cdot \mathbf{Q}) \cdot z' + |d|^2}{4\pi |z - z'|} dz'. \] (42)

Let \( f = 2(\mathbf{d} \cdot \mathbf{Q}) \) and define
\[ I = \Pi_{j=1}^3 k \frac{a_k}{2} \int_0^{+\infty} \frac{ds}{\sqrt{\Pi_{q=1}^3 (a_q^2 + s)}}; \]
\[ I_i = \Pi_{k=1}^3 k \frac{a_k}{2} \int_0^{+\infty} \frac{ds}{(a_i^2 + s) \sqrt{\Pi_{q=1}^3 (a_q^2 + s)}}; \]
\[ I_{ij} = \Pi_{k=1}^3 k \frac{a_k}{2} \int_0^{+\infty} \frac{ds}{(a_i^2 + s)(a_j^2 + s) \sqrt{\Pi_{q=1}^3 (a_q^2 + s)}}; \]
\[ I_{ijn} = \frac{\Pi_{k=1}^3 k}{2} \int_0^{+\infty} \frac{ds}{(a_i^2 + s)(a_j^2 + s)(a_n^2 + s) \sqrt{\Pi_{q=1}^3 (a_q^2 + s)}}. \] (43)

According to [2], we can directly calculate (42), i.e.,
\[ N_E[\psi](z) = C_E + A_1 z_1 + A_2 z_2 + A_3 z_3 + B_1 z_1^2 + B_2 z_2^2 + B_3 z_3^2 + H_1 z_1^4 + H_2 z_2^4 + H_3 z_3^4 + H_4 z_1^4 + H_5 z_2^4 + H_6 z_3^4 + H_7 z_1 z_2 z_3^2 + H_8 z_2 z_3 z_1^2 + H_9 z_3 z_1 z_2^2 \]
\[ + J_1 z_1^4 + J_2 z_2^4 + J_3 z_3^4 + J_4 z_1^2 z_2^2 + J_5 z_2^2 z_3^2 + J_6 z_3^2 z_1^2, \] (44)
with

\[ C_E := \frac{1}{8} \left( (a_1^2 + a_2^2 + a_3^2) I - (a_1^4 I_1 + a_2^4 I_2 + a_3^4 I_3) \right) + \frac{1}{2} |d|^2 I; \]

\[ A_1 := \frac{1}{2} a_1^2 I_1 f_1; \quad A_2 := \frac{1}{2} a_2^2 I_2 f_2; \quad A_3 := \frac{1}{2} a_3^2 I_3 f_3; \]

\[ B_1 := \frac{3}{4} I_{11} a_1^4 + \frac{1}{4} I_{12} a_2^4 + \frac{1}{4} I_{13} a_3^4 - \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_1 - \frac{1}{2} |d|^2 I; \]

\[ B_2 := \frac{3}{4} I_{22} a_2^4 + \frac{1}{4} I_{21} a_1^4 + \frac{1}{4} I_{23} a_3^4 - \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_2 - \frac{1}{2} |d|^2 I_2; \]

\[ B_3 := \frac{3}{4} I_{33} a_3^4 + \frac{1}{4} I_{31} a_1^4 + \frac{1}{4} I_{32} a_2^4 - \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_3 - \frac{1}{2} |d|^2 I_3; \]

\[ H_1 := - \frac{1}{2} a_1^2 I_{11} f_1; \quad H_2 := - \frac{1}{2} a_2^2 I_{22} f_2; \quad H_3 := - \frac{1}{2} a_3^2 I_{33} f_3; \]

\[ H_4 := - \frac{1}{2} a_2^2 I_{21} f_1; \quad H_5 := - \frac{1}{2} a_2^2 I_{31} f_1; \quad H_6 := - \frac{1}{2} a_2^2 I_{21} f_2; \]

\[ H_7 := - \frac{1}{2} a_2^2 I_{23} f_2; \quad H_8 := - \frac{1}{2} a_2^2 I_{13} f_3; \quad H_9 := - \frac{1}{2} a_2^2 I_{23} f_3; \quad (45) \]

\[ J_1 := \frac{1}{8} I_{11} (a_1^2 + a_2^2 + a_3^2) - \frac{5}{8} I_{111} a_1^4 - \frac{1}{8} a_1^2 I_{112} - \frac{1}{8} a_1^2 I_{113}; \]

\[ J_2 := \frac{1}{8} I_{22} (a_1^2 + a_2^2 + a_3^2) - \frac{5}{8} I_{222} a_2^4 - \frac{1}{8} a_1^2 I_{221} - \frac{1}{8} a_2^4 I_{223}; \]

\[ J_3 := \frac{1}{8} I_{33} (a_1^2 + a_2^2 + a_3^2) - \frac{5}{8} I_{333} a_3^4 - \frac{1}{8} a_1^2 I_{331} - \frac{1}{8} a_2^4 I_{332}; \]

\[ J_4 := \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_{12} - \frac{3}{4} (a_1^4 I_{1211} + a_2^4 I_{1212}) - \frac{1}{4} a_3^4 I_{1221}; \]

\[ J_5 := \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_{23} - \frac{3}{4} (a_1^4 I_{2322} + a_3^4 I_{2323}) - \frac{1}{4} a_2^4 I_{2312}; \]

\[ J_6 := \frac{1}{4} (a_1^2 + a_2^2 + a_3^2) I_{31} - \frac{3}{4} (a_1^4 I_{3133} + a_3^4 I_{3131}) - \frac{1}{4} a_3^4 I_{3121}. \]

Substituting (41) into \( \varphi(x) \) yields

\[ \varphi(z) = C - \frac{1}{12} (Q_{11} z_1 + Q_{12} z_2 + Q_{13} z_3 + d_1)^4 \]

\[ - \frac{1}{12} (Q_{21} z_1 + Q_{22} z_2 + Q_{23} z_3 + d_2)^4 \]

\[ - \frac{1}{12} (Q_{31} z_1 + Q_{32} z_2 + Q_{33} z_3 + d_3)^4. \] (46)

Then to prove Lemma 2.2, we are going to prove \( N_E \{ \psi \} (z) \neq \varphi(z) \), and we will achieve the proof by contradiction.

Assume \( N_E \{ \psi \} (z) = \varphi(z) \), and hence the right-hand side of (46) equals the right-hand side of (44). By comparison of the coefficients of \( z_1^4 z_2, z_1 z_2^3, z_2 z_3, z_1 z_3, z_1 z_3^2, z_1 z_3^2, z_1 z_3^2, z_1 z_3^2, z_1 z_3^2, z_1 z_3^2 \) in (46) with those in (44), we obtain

\[ Q_{11}^3 Q_{12} + Q_{21}^3 Q_{22} + Q_{31}^3 Q_{32} = 0, \quad Q_{11} Q_{12}^3 + Q_{21} Q_{22}^3 + Q_{31} Q_{32}^3 = 0, \]

\[ Q_{12}^3 Q_{13} + Q_{22}^3 Q_{23} + Q_{32}^3 Q_{33} = 0, \quad Q_{12} Q_{13}^3 + Q_{22} Q_{23}^3 + Q_{32} Q_{33}^3 = 0, \]

\[ Q_{13}^3 Q_{11} + Q_{23}^3 Q_{21} + Q_{33}^3 Q_{31} = 0, \quad Q_{13} Q_{11}^3 + Q_{23} Q_{21}^3 + Q_{33} Q_{31}^3 = 0. \] (47)
In addition, since $Q$ is orthogonal, we see

\[
Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0, \quad Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0,
\]
\[
Q_{13}Q_{11} + Q_{23}Q_{21} + Q_{33}Q_{31} = 0, \quad Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1,
\]
\[
Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1, \quad Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1.
\] (48)

By combining (47) and (47)5 with (48)1 and (48)3, we obtain

\[
\begin{bmatrix}
Q_{12} & Q_{22} & Q_{32} \\
Q_{13} & Q_{23} & Q_{33} \\
Q_{12}^3 & Q_{22}^3 & Q_{32}^3 \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix}
\begin{bmatrix}
Q_{11} \\
Q_{21} \\
Q_{31}
\end{bmatrix} = 0.
\] (49)

We regard (49) as a homogenous linear system of equations with respect to $(Q_{11}, Q_{21}, Q_{31})$, so $(Q_{12}, Q_{22}, Q_{32}), (Q_{13}, Q_{23}, Q_{33}), (Q_{12}^3, Q_{22}^3, Q_{32}^3)$ and $(Q_{13}^3, Q_{23}^3, Q_{33}^3)$ denote four corresponding coefficients for four different linear equations in this system.

If there are more than 2 independent linear equations in (49), the solution will be trivial. However, (49) only admits non-trivial solutions owing to (48)4. Thus, there are at most two independent equations in the homogenous linear system shown in (49). Then, let us choose the first two equations in (49) as two independent equations, and the independence between them can be proved by (48)2.

Given this, we have

\[
(Q_{12}^3, Q_{22}^3, Q_{32}^3) = k_1(Q_{12}, Q_{22}, Q_{32}) + m_1(Q_{13}, Q_{23}, Q_{33}),
\]
\[
(Q_{13}^3, Q_{23}^3, Q_{33}^3) = k_2(Q_{13}, Q_{23}, Q_{33}) + m_2(Q_{12}, Q_{22}, Q_{32}),
\] (50)

with $k_i, m_i$ $(i = 1, 2)$ four real constants.

Then by substituting (50)1 and (48)2 into (47)2 and substituting (50)2 and (48)2 into (47)5, we get

\[
m_1 = m_2 = 0,
\] (51)

which means

\[
(Q_{12}^3, Q_{22}^3, Q_{32}^3) = k_1(Q_{12}, Q_{22}, Q_{32}),
\]
\[
(Q_{13}^3, Q_{23}^3, Q_{33}^3) = k_2(Q_{13}, Q_{23}, Q_{33}),
\] (52)

with $k_1, k_2 \neq 0$ due to (48)5,6.

Then we take three cases concerning $(Q_{12}, Q_{22}, Q_{32})$ into consideration.

1. Only one component of $(Q_{12}, Q_{22}, Q_{32})$ is nonzero.

Without loss of generality, we take $Q_{12} \neq 0$ so that $Q_{22} = Q_{32} = 0$.

Since $|(Q_{12}, Q_{22}, Q_{32})| = 1$, we have $Q_{12} = \pm 1$. Then by substituting $Q_{12} = 1$ into (48) yields $Q_{13} = 0$. Then we consider three cases concerning $(Q_{13}, Q_{23}, Q_{33})$.

(a) $Q_{23} = 0, Q_{33} \neq 0$ or $Q_{33} = 0, Q_{23} \neq 0$.

Without loss of generality, we take $Q_{23} = 0, Q_{33} \neq 0$. Likewise, $|(Q_{13}, Q_{23}, Q_{33})| = 1$ so that $Q_{33} = \pm 1$. Based on (48), since $(Q_{12}, Q_{22}, Q_{32}) = (\pm 1, 0, 0)$ and $(Q_{13}, Q_{23}, Q_{33}) = (0, 0, \pm 1)$, we know that $(Q_{11}, Q_{21}, Q_{31}) = (0, \pm 1, 0)$.

\[
Q = \begin{bmatrix}
0 & \pm 1 & 0 \\
\pm 1 & 0 & 0 \\
0 & 0 & \pm 1
\end{bmatrix}.
\] (53)
By following the same procedure, we can construct more \( Q_s \) that only possesses three \( \pm 1 \) components. Such \( Q_s \) denotes the rotation of the coordinate system \( z = (z_1, z_2, z_3) \) around any basis of it by \( \pm \frac{\pi}{4} \) or symmetric transformations with respect to any plane spanned by two axes of the coordinate system \( z = (z_1, z_2, z_3) \) or the superposition of them. There are 48 \( Q_s \) in total.

Let \( \varphi^{(4)}(z) \) denote the summation of terms in \( \varphi \) with degree 4. In this case,

\[
\varphi^{(4)}(z) = -\frac{1}{12}(z_1^4 + z_2^4 + z_3^4).
\]  

(54)

(b) \( Q_{23} \neq 0, Q_{33} \neq 0 \).

According to (52) and (48), we have four cases:

\[
Q_{23} = \pm \sqrt{k_2}, \quad Q_{33} = \pm \sqrt{k_2}, \quad k_2 = \frac{1}{2}.
\]  

(55)

We discuss \( Q_{23} = Q_{33} = \frac{\sqrt{k_2}}{2} \), and other cases can be discussed in the same way. When \( Q_{23} = Q_{33} = \frac{\sqrt{k_2}}{2} \), by resorting to (48), we gain \( (Q_{11}, Q_{21}, Q_{31}) = (0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \) or \( (Q_{11}, Q_{21}, Q_{31}) = (0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \); we just consider the former case,

\[
Q = \begin{bmatrix}
0 & \pm 1 & 0 \\
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}
\end{bmatrix}.
\]  

(56)

Likewise, we could construct more \( Q_s' \) in a similar form, which means the rotations of the coordinate system \( z = (z_1, z_2, z_3) \) around any basis of it by \( \pm \frac{\pi}{4} \) or further superposition of such rotations on the coordinates transformation represented by \( Q_s \). There are 72 \( Q_s' \) in total. In this case,

\[
\varphi^{(4)}(z) = -\frac{1}{12}\left(\frac{1}{2}z_1^4 + \frac{1}{2}z_2^4 + z_3^4 + 3z_1^2z_2^2\right)
\]

or \( \varphi^{(4)}(z) = -\frac{1}{12}\left(\frac{1}{2}z_1^4 + \frac{1}{2}z_2^4 + z_3^4 + 3z_1^2z_2^2\right) \)  

or \( \varphi^{(4)}(z) = -\frac{1}{12}\left(\frac{1}{2}z_1^4 + \frac{1}{2}z_3^4 + z_1^4 + 3z_2^2z_3^2\right) \).

(57)

2. Two components of \( (Q_{12}, Q_{22}, Q_{32}) \) are nonzero and the other is zero.

Without loss of generality, we take \( Q_{12}, Q_{22} \neq 0 \) so that \( Q_{32} = 0 \). Due to (52) and (48), we have

\[
Q_{12} = \pm \sqrt{k_1}, \quad Q_{22} = \pm \sqrt{k_1}, \quad k_1 = \frac{1}{2}.
\]  

(58)

Then we consider two cases concerning \( (Q_{13}, Q_{23}, Q_{33}) \).

(a) At least one component of \( (Q_{13}, Q_{23}, Q_{33}) \) is zero.

This situation is the same as that discussed in (b) of (ii), since \( (Q_{13}, Q_{23}, Q_{33}) \), \( (Q_{12}, Q_{22}, Q_{32}) \) and \( (Q_{11}, Q_{21}, Q_{31}) \) are equivalent, which can be replaced by each other. For example,
when \((Q_{12}, Q_{22}, Q_{32}) = (-\sqrt{2}/2, \sqrt{2}/2, 0)\) is fixed, and if there is only one nonzero component in \((Q_{13}, Q_{23}, Q_{33})\), we have \((Q_{13}, Q_{23}, Q_{33}) = (0, 0, \pm 1)\) and \((Q_{11}, Q_{21}, Q_{31}) = (\sqrt{3}/2, \sqrt{3}/2, 0)\); thus

\[
Q = \begin{bmatrix}
-\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \\
0 & 0 & \pm 1
\end{bmatrix}.
\]  

(59)

If there are two nonzero components in \((Q_{13}, Q_{23}, Q_{33})\), we have \((Q_{13}, Q_{23}, Q_{33}) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0)\) and \((Q_{11}, Q_{21}, Q_{31}) = (0, 0, \pm 1)\); thus

\[
Q = \begin{bmatrix}
0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
\pm 1 & 0 & 0
\end{bmatrix}.
\]  

(60)

(b) Three components of \((Q_{13}, Q_{23}, Q_{33})\) are all nonzero.

In this case, according to \((52)\), we have

\[
Q_{13} = \pm \sqrt{k_2}, \quad Q_{23} = \pm \sqrt{k_2}, \quad Q_{33} = \pm \sqrt{k_2}, \quad k_3 = \frac{1}{3}.
\]

(61)

We fix \((Q_{12}, Q_{22}, Q_{32}) = (\sqrt{3}/2, \sqrt{3}/2, 0)\) and \((Q_{13}, Q_{23}, Q_{33}) = (\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)\). Other situations when the sign of any component of \((Q_{12}, Q_{22}, Q_{32})\) and \((Q_{13}, Q_{23}, Q_{33})\) changes can be discussed via the same method. We can easily calculate

\[
(Q_{11}, Q_{21}, Q_{31}) = \pm (Q_{12}, Q_{22}, Q_{32}) \times (Q_{13}, Q_{23}, Q_{33}) = \pm (\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}),
\]  

(62)

which contradicts \((47)\). Hence this case is invalid.

3. Three components of \((Q_{12}, Q_{22}, Q_{32})\) are all nonzero.

In this case, it is obvious that \((Q_{13}, Q_{23}, Q_{33})\) cannot have one nonzero component or three nonzero components due to \((45)\). Given this, there must be two nonzero components of \((Q_{13}, Q_{23}, Q_{33})\). However, when \((Q_{13}, Q_{23}, Q_{33})\) has two nonzero components, the situation is the same as that discussed in (b) of (ii), since \((Q_{13}, Q_{23}, Q_{33})\) and \((Q_{12}, Q_{22}, Q_{32})\) can be exchanged, which does not influence the discussion.

Then we draw the conclusion that if there exists an ellipsoid whose Newtonian potential is \(N_E[\psi](\mathbf{z}) = \varphi(\mathbf{z})\), then the sum of the forth-degree terms \(\varphi^{(4)}(\mathbf{z})\) in the polynomial \(\varphi(\mathbf{z})\) can only be expressed as either \((54)\) or \((57)\).

According to \((44)\), we know that

\[
\varphi^{(4)}(\mathbf{z}) = J_1 z_1^4 + J_2 z_2^4 + J_3 z_3^4 + J_4 z_1^2 z_2^2 + J_5 z_2^2 z_3^2 + J_6 z_3^2 z_1^2.
\]

(63)

Then based on \((45)\), we are going to prove that the coefficients in \((63)\), which result from the Newtonian potential of an ellipsoid, cannot be in the form given in \((54)\) or \((57)\).
We will achieve the proof by determining the range of $J_i$ ($i = 1, 2, 3, 4, 5, 6$) that relies on the shape of the ellipsoid. By Mura \[2\], we know that there are relationships among $I_i, I_{ij}, I_{ijk}$ $(i, j, k = 1, 2, 3)$, i.e.,

\[
I_1 + I_2 + I_3 = 1; \quad I_{ij} = \frac{I_j - I_i}{a_i^2 - a_j^2}, \text{ for } i \neq j; \quad I_{ii} = \frac{1}{3} \left( \frac{1}{a_i^2} - \sum_{q=1,q \neq i}^{3} I_{iq} \right); \\
I_{ijk} = \frac{I_{ik} - I_{ij}}{a_i^2 - a_j^2}, \text{ for } i \neq j \neq k; \quad I_{ijj} = \frac{I_{ij} - I_{ii}}{a_i^2 - a_j^2}, \text{ for } i \neq j; \quad I_{iii} = \frac{1}{5} \left( \frac{1}{a_i^4} - \sum_{q=1,q \neq i}^{3} I_{iiq} \right),
\]

where the summation convention is not utilized. Then substituting (64) and (43) into (45) yields

\[
J_1 = -\frac{1}{6} + \frac{1}{24} (4a_1^2 + 3a_2^2) I_{12} + \frac{1}{24} (4a_1^2 + 3a_3^2) I_{13}; \\
J_2 = -\frac{1}{6} + \frac{1}{24} (4a_2^2 + 3a_3^2) I_{21} + \frac{1}{24} (4a_2^2 + 3a_3^2) I_{23}; \\
J_3 = -\frac{1}{6} + \frac{1}{24} (4a_3^2 + 3a_1^2) I_{31} + \frac{1}{24} (4a_3^2 + 3a_1^2) I_{32}; \\
J_4 = \frac{1}{4} (I_1 + I_2) - \frac{3}{4} (a_1^2 + a_2^2) I_{12} + \frac{a_1 a_2 a_3}{8} \int_0^{+\infty} \frac{s}{(a_1^2 + s)(a_2^2 + s) \sqrt{\prod_{q=1}^{3} (a_q^2 + s)}} ds; \\
J_5 = \frac{1}{4} (I_2 + I_3) - \frac{3}{4} (a_2^2 + a_3^2) I_{23} + \frac{a_1 a_2 a_3}{8} \int_0^{+\infty} \frac{s}{(a_2^2 + s)(a_3^2 + s) \sqrt{\prod_{q=1}^{3} (a_q^2 + s)}} ds; \\
J_6 = \frac{1}{4} (I_3 + I_1) - \frac{3}{4} (a_3^2 + a_1^2) I_{31} + \frac{a_1 a_2 a_3}{8} \int_0^{+\infty} \frac{s}{(a_3^2 + s)(a_1^2 + s) \sqrt{\prod_{q=1}^{3} (a_q^2 + s)}} ds,
\]

which are always valid even when two of $a_1, a_2, a_3$ are equal. Then we consider three cases concerning the shape of the ellipsoid.

1. When the ellipsoid is spherical.

In this case, $a_1 = a_2 = a_3$, and it is easy to calculate that

\[
J_1 = J_2 = J_3 = -\frac{1}{20}, \quad J_4 = J_5 = J_6 = -\frac{1}{10}.
\]

Thus the summation of the forth-degree terms $\varphi_{[\text{sph}]}^{(4)}(z)$ for a sphere is

\[
\varphi_{[\text{sph}]}^{(4)}(z) = -\frac{1}{20} (z_1^4 + z_2^4 + z_3^4) - \frac{1}{10} (z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2),
\]

which does not satisfy either (54) or (57). Hence we conclude that $\varphi$ cannot be the Newtonian potential induced by a sphere with the mass density $\Delta \varphi$.

2. When the ellipsoid is spheroidal.

In this case, without loss of the generality, we take $a_1 = a_2 = \frac{a_3}{e}$, and $e > 0$. 

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(a) Oblate spheroid: \( e < 1 \).

When \( e < 1 \), we can directly calculate

\[
J_1 = J_2 = -\frac{(2e^2 - 23)e^2\sqrt{1 - e^2} + 3e(3 + 4e^2) \arccos e}{64(1 - e^2)^{\frac{5}{2}}};
\]

\[
J_3 = -\frac{(2 + 19e^2)\sqrt{1 - e^2} + 3e(3 + 4e^2) \arccos e}{24(1 - e^2)^{\frac{5}{2}}};
\]

\[
J_4 = -\frac{(2e^2 - 23)e^2\sqrt{1 - e^2} + 3e(3 + 4e^2) \arccos e}{32(1 - e^2)^{\frac{5}{2}}};
\]

\[
J_5 = J_6 = -\frac{(2e^4 + 15e^2 + 4)\sqrt{1 - e^2} - 3e(3 + 4e^2) \arccos e}{8(1 - e^2)^{\frac{5}{2}}},
\]

where there is

\[
J_1 = J_2 = \frac{J_4}{2}.
\]

If \( \phi_{\text{obl}}^{(4)} \) is expressed as (54), we have \( J_1, J_2, J_3 \neq 0, J_4 = J_5 = J_6 = 0 \), which contradicts (69). Thus, \( \phi_{\text{obl}}^{(4)} \) cannot be expressed as (54).

If \( \phi_{\text{obl}}^{(4)} \) is expressed as (57), we obtain \( J_5 = J_6 = 0 \) and \( J_1 = J_2 = \frac{1}{3}J_3 = \frac{1}{6}J_4 \neq 0 \), which also contradicts (69). Hence \( \phi_{\text{obl}}^{(4)} \) cannot be expressed as (57), either.

Through the same method, we can discuss the case when \( a = c = \frac{b}{e} \) and \( b = c = \frac{e}{e} \). Hence we conclude that \( \phi \) cannot be the Newtonian potential induced by an oblate spheroid with the mass density \( \Delta \phi \).

(b) Prolate spheroid: \( e > 1 \).

When \( e > 1 \), we can directly calculate

\[
J_1 = J_2 = -\frac{(2e^2 - 23)e^2\sqrt{e^2 - 1} + 3e(3 + 4e^2) \cosh^{-1} e}{64(e^2 - 1)^{\frac{5}{2}}};
\]

\[
J_3 = -\frac{(2 + 19e^2)\sqrt{e^2 - 1} + 3e(3 + 4e^2) \cosh^{-1} e}{24(e^2 - 1)^{\frac{5}{2}}};
\]

\[
J_4 = -\frac{(2e^2 - 23)e^2\sqrt{e^2 - 1} + 3e(3 + 4e^2) \cosh^{-1} e}{32(e^2 - 1)^{\frac{5}{2}}};
\]

\[
J_5 = J_6 = -\frac{(2e^4 + 15e^2 + 4)\sqrt{e^2 - 1} - 3e(3 + 4e^2) \cosh^{-1} e}{8(e^2 - 1)^{\frac{5}{2}}},
\]

where there is also a result in (69), and the only difference between (68) and (70) is the replacement of \( \arccos e \) with \( \cosh^{-1} e \). By a similar analysis to that for the oblate spheroid, we can reach the conclusion that \( \phi \) cannot be the Newtonian potential induced by a prolate spheroid with the mass density \( \Delta \phi \).

3. When the ellipsoid is in a general shape: \( a_1 \neq a_2 \neq a_3 \).

We divide our analysis into two parts:
(a) Firstly, we are going to prove that (54) cannot be the summation of the forth-degree terms of the polynomial Newtonian potential \( \varphi_{[\text{gen}]}^{(4)} \) induced by a general ellipsoid with \( a_1 \neq a_2 \neq a_3 \) and the mass density \( \Delta \varphi \).

When \( a_1 \neq a_2 \neq a_3 \), we can simplify the expression of \( J_i \) \((i = 1, 2, 3, 4, 5, 6) \) in (65), i.e.,

\[
\begin{align*}
J_1 &= -\frac{1}{6} + \frac{(I_2 - I_1)(4a_1^2 + 3a_2^2)}{24(a_1^2 - a_2^2)} + \frac{(I_3 - I_1)(4a_1^2 + 3a_3^2)}{24(a_1^2 - a_3^2)}, \\
J_2 &= -\frac{1}{6} + \frac{(I_2 - I_1)(4a_2^2 + 3a_1^2)}{24(a_1^2 - a_2^2)} + \frac{(I_3 - I_2)(4a_2^2 + 3a_3^2)}{24(a_2^2 - a_3^2)}, \\
J_3 &= -\frac{1}{6} + \frac{(I_3 - I_1)(4a_3^2 + 3a_1^2)}{24(a_1^2 - a_3^2)} + \frac{(I_3 - I_2)(4a_3^2 + 3a_2^2)}{24(a_2^2 - a_3^2)}, \\
J_4 &= \frac{(5a_1^2 + 2a_3^2)I_1 - (5a_1^2 + 2a_2^2)I_2}{4(a_1^2 - a_2^2)}, \\
J_5 &= \frac{(5a_2^2 + 2a_3^2)I_2 - (5a_2^2 + 2a_3^2)I_3}{4(a_2^2 - a_3^2)}, \\
J_6 &= \frac{(5a_3^2 + 2a_1^2)I_3 - (5a_1^2 + 2a_3^2)I_1}{4(a_3^2 - a_1^2)}. \\
\end{align*}
\]

If \( \varphi_{[\text{gen}]}^{(4)} \) is given as (54), we see that \( J_4 = J_5 = J_6 = 0 \), which indicates that

\[
\begin{bmatrix}
5a_1^2 + 2a_2^2 & -(5a_1^2 + 2a_2^2) & 0 \\
0 & 5a_2^2 + 2a_3^2 & -(5a_2^2 + 2a_3^2) \\
5a_1^2 + 2a_3^2 & 0 & -(5a_2^2 + 2a_3^2)
\end{bmatrix}
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Since

\[
\begin{vmatrix}
5a_1^2 + 2a_2^2 & -(5a_1^2 + 2a_2^2) & 0 \\
0 & 5a_2^2 + 2a_3^2 & -(5a_2^2 + 2a_3^2) \\
5a_1^2 + 2a_3^2 & 0 & -(5a_2^2 + 2a_3^2)
\end{vmatrix}
= 30(a_1^2 - a_2^2)(a_2^2 - a_3^2)(a_3^2 - a_1^2) \neq 0,
\]

we know that the linear system of equations with respect \( (I_1, I_2, I_3) \) in (72) only admits a trivial solution. However, according to (43), we know that \( I_i > 0 \) \((i = 1, 2, 3) \), which forms a contradiction. Hence we conclude that \( \varphi_{[\text{gen}]}^{(4)} \) cannot be in the form of (54).

(b) Secondly, we are going to prove that (57) cannot be the summation of the forth-degree terms of the polynomial Newtonian potential \( \varphi_{[\text{gen}]}^{(4)} \) induced by a general ellipsoid with \( a_1 \neq a_2 \neq a_3 \) and the mass density \( \Delta \varphi \).

Based on (57), we take \( J_4 = J_5 = 0, J_6 \neq 0 \), and the situations when \( J_4 = J_6 = 0, J_5 \neq 0 \) and \( J_5 = J_6 = 0, J_4 \neq 0 \) can be analysed through the same procedure.
When \( J_4 = J_5 = 0, J_6 \neq 0 \), comparing \((71)\) with \((57)\) yields
\[
I_1 = \frac{5a_1^2 + 2a_2^2}{5a_1^2 + 2a_2^2} 5a_3^2 + 2a_3^2 I_3; \quad I_2 = \frac{5a_2^2 + 2a_2^2}{5a_2^2 + 2a_2^2} I_3;
\]
\[
J_1 = -\frac{1}{6} + \frac{2a_1^2(9a_2^2 + 5a_3^2) + 3(a_2^2 + 6a_2^2a_3^2)}{4(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = -\frac{1}{24};
\]
\[
J_2 = -\frac{1}{6} + \frac{48a_1^2 + 78a_2^2a_3^2 + a_1^2(78a_2^2 + 90a_3^2)}{24(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = -\frac{1}{12};
\]
\[
J_3 = -\frac{1}{6} + \frac{2a_1^2(9a_2^2 + 5a_3^2) + 3(a_2^2 + 6a_2^2a_3^2)}{4(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = -\frac{1}{24};
\]
\[
J_4 = J_5 = 0; \quad J_6 = -\frac{15(a_1^2 - a_2^2)(a_2^2 - a_3^2)}{2(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = -\frac{1}{4},
\]
which are valid only when
\[
\frac{2a_1^2(9a_2^2 + 5a_3^2) + 3(a_2^4 + 6a_2^2a_3^2)}{(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = \frac{15(a_1^2 - a_2^2)(a_2^2 - a_3^2)}{(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = \frac{1}{2},
\]
\[
\frac{48a_1^4 + 78a_2^2a_3^2 + a_1^2(78a_2^2 + 90a_3^2)}{(5a_1^2 + 2a_2^2)(5a_2^2 + 2a_3^2)} I_3 = 2.
\]
It follows from \((75)\) that
\[
2a_1^2(9a_2^2 + 5a_3^2) + 3(a_2^4 + 6a_2^2a_3^2) = 15(a_1^2 - a_2^2)(a_2^2 - a_3^2);
\]
\[
\Rightarrow \quad a_2^2(6a_2^2 + a_3^2) + a_1^2(3a_1^2 + 25a_3^2) = 0.
\]
However, \( a_2^2(6a_2^2 + a_3^2) + a_1^2(3a_1^2 + 25a_3^2) > 0 \), which indicates that \((76)\) is impossible. Hence we conclude that \( \varphi \) cannot be the Newtonian potential induced by a general ellipsoid with \( a_1 \neq a_2 \neq a_3 \) and the mass density \( \Delta \varphi \).

In conclusion, by reduction, we have proved that \( \varphi \) is not equal to the Newtonian potential induced by spheres, spheroids, and general ellipsoids with any orientation, any position, and the fixed mass density \( \Delta \varphi \), which implies the completion of the proof that \( \varphi \) cannot be the Newtonian potential induced by any ellipsoidal inclusion which possesses the mass density \( \Delta \varphi \). Hence Lemma 2.2 is proved. The proofs of Lemma 2.1 and Lemma 2.2 imply Theorem 2.2.

Based on Theorem 2.2 there must be a non-ellipsoidal \( \Omega' \) that leads to the quadratic polynomial form of \( N_{\Omega'}[\rho](\mathbf{x}') \) by letting \( \rho = \psi \). Thus, owing to \((18)\), such a non-ellipsoid \( \Omega' \) will generate polynomial \( e(\mathbf{x}') \) of degree two so that \( e(\mathbf{x}) \) is also in the form of polynomial of degree two due to \((14)\). Given this, we have proved the existence of an \( \Omega \), which can be transformed from \( \Omega' \) by inverse of \((14)\), and possesses Eshelby’s polynomial property for the particular polynomial eigenstrain \( \varepsilon^* \) whose corresponding polynomial eigenstress is given in \((10)\) when \( \rho(\mathbf{x}) = -(x_1^2 + x_2^2 + x_3^2) \). Thus Theorem 2.1 has been proved, which substantiates the invalidity of the generalized strong version of the high-order Eshelby conjecture for the polynomial eigenstrain of degree two in cubic materials. The shape of a counter-example inclusion is shown as \( \Omega^{(1)} \) in Fig. 1 in the Appendix.

It is worth mentioning that if the eigenstrain is dilational \( \varepsilon^*_{ij} = \psi \delta_{ij} \) and the medium is isotropic, which means
\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (i, j, k, l = 1, 2, 3),
\]
with \( \lambda \) and \( \mu \) Lamé constants, then there is a linear relationship between the second derivatives of the Newtonian potential \( N_\Omega[\psi](x) \) induced by the domain \( \Omega \) of the inclusion with mass density \( \psi \) and the strain field \( \varepsilon_{ij}(x) \) inside, i.e.,
\[
\varepsilon_{ij}(x) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{3\lambda + 2\mu}{\lambda + 2\mu} \frac{\partial^2 N_\Omega[\psi](x)}{\partial x_i \partial x_j}, \quad x \in \Omega. \tag{78}
\]

Substituting (78) into Theorem 2.2 also yields the verification of the invalidity of the generalized strong version of the high-order Eshelby conjecture for a polynomial eigenstrain of degree two in isotropic materials. We note that the polynomial form of \( N_\Omega[\psi] \) in (78) is the necessary condition for the existence of a non-ellipsoidal \( \Omega \) that has Eshelby’s polynomial property for the polynomial eigenstrain that is dilatational in an isotropic medium. However, the polynomial form of \( N_\Omega[\psi](x') \) in (18) is not a necessity for the existence of a non-ellipsoidal \( \Omega' \) that possesses Eshelby’s polynomial property for a particular polynomial eigenstrain in a medium of cubic symmetry – more non-ellipsoidal \( \Omega' \) that possess Eshelby’s polynomial property could be found according to (18) by letting \( \rho = \psi \) and then requiring
\[
N_\Omega[\psi](x') = \varphi(x') + \omega(x'_1, x'_2), \quad x' \in \Omega', \tag{79}
\]
where
\[
\omega(x'_1, x'_2) := -\beta \log \frac{(x'_1 - 12\sqrt{C})^2 + (x'_2 - 12\sqrt{C})^2}{36C}, \tag{80}
\]
and \( \beta \) is a positive real constant. It is easily seen that \( \Delta x' \omega = 0 \) so that \( \omega \) is harmonic. Then the right-hand side of (79) satisfies (6), and substitution of (79) into (18) yields the quadratic polynomial form of the strain field, which signifies that Eshelby’s polynomial conservation theorem holds for \( \Omega' \) that generates the Newtonian potential in (79).

Note that the proof of the existence of an \( \Omega' \) that yields (79) also inspires us to construct more counter-examples to deal with the high-order Eshelby conjecture for polynomial eigenstrains of any non-negative even degree in the next section.

The existence of an \( \Omega' \) that yields (79) can be briefly verified via the same method in the proof of Lemma 2.1

We introduce
\[
\phi' := \phi^* + \omega^*, \tag{81}
\]
where
\[
\omega^*(x'_1, x'_2) := \begin{cases} 
0 & x' \in U^{(1)}, \\
-\beta \log 9 & x' \in U^{(2)}, \\
\omega & x' \in U^\omega,
\end{cases} \tag{82}
\]
with \( U^{(1)} := \{ x' | (x'_1 - 12\sqrt{C})^2 + (x'_2 - 12\sqrt{C})^2 \leq 36C, \ x' \in \mathbb{R}^3 \}, \)
\( U^{(2)} := \{ x' | (x'_1 - 12\sqrt{C})^2 + (x'_2 - 12\sqrt{C})^2 \geq 324C, \ x' \in \mathbb{R}^3 \} \) and \( U^\omega := \{ x' | 324C \geq (x'_1 - 12\sqrt{C})^2 + (x'_2 - 12\sqrt{C})^2 \geq 36C, \ x' \in \mathbb{R}^3 \} \).

Then we will achieve our goal by confirming that \( \phi' \) meets all of the conditions in regard to an obstacle function listed above.

Firstly, we see \( \phi' \in C^0(\mathbb{R}^3) \), and
\[
||\phi'||_{0,1} \leq ||\phi^*||_{0,1} + ||\omega^*||_{0,1}, \tag{83}
\]
where
\[ \| \omega^* \|_{0,1} = \sup_{\omega(x')} |\omega(x')| + \sup_{\omega(x')} |\nabla \omega(x')|. \] (84)

Since \(|\omega(x')|\) and \(|\nabla \omega(x')|\) are bounded in \(U^\omega\), substituting (84) into (83) and considering that \(\|\phi^*\|_{0,1}\) is bounded lead to the result that \(\|\phi^\ast\|_{0,1}\) is bounded, and \(\phi' \in C^{0,1}([\mathbb{R}^3])\).

We still let \(r_0 = 6\sqrt{C}\). It has been proved that \(\forall |x'| \geq r_0, \phi^*(x') \leq 0\). Since (82) signifies \(\omega^* \leq 0\), we conclude that \(\forall |x'| \geq r_0, \phi'(x') = \phi^*(x') + \omega^*(x') \leq 0\). \(|\Delta \phi^\ast|\) is bounded in \(B_{r_0}\), since \(\phi' \in C^\omega(B_{r_0})\).

Finally, by introducing the smooth function \(\xi\) that possesses the property in (37), we see
\[ \int_{U^\xi} \frac{\partial^2 \xi}{\partial \zeta^2} \left( \phi' + \frac{1}{2} C^\phi |x'|^2 \right) \, dx' \geq \int_{U^\xi} \xi (C^\phi - \sup_{x' \in U} \frac{\partial \phi}{\partial \zeta} - \sup_{x' \in U} \left| \frac{\partial^2 \phi}{\partial \zeta^2} \right|) \, dx' + \int_{U^\xi} \frac{\partial^2 \xi}{\partial \zeta^2} \omega^* \, dx', \] (85)
whose second term satisfies
\[ \int_{U^\xi} \frac{\partial^2 \xi}{\partial \zeta^2} \omega^* \, dx' = \int_{U^\xi \cap U^\omega} \frac{\partial^2 \xi}{\partial \zeta^2} \omega \, dx' + \int_{U^\xi \cap U^{(2)}} (-\beta \log 9) \frac{\partial^2 \xi}{\partial \zeta^2} \, dx' \]
\[ = \int_{U^\xi \cap \partial U^\omega} \frac{\partial \xi}{\partial \zeta} \omega \cdot dS - \int_{U^\xi \cap U^\omega} \frac{\partial \xi}{\partial \zeta} \frac{\partial \omega}{\partial \zeta} \, dx' \]
\[ - \int_{U^\xi \cap \partial U^{(2)}} (\beta \log 9) \frac{\partial \xi}{\partial \zeta} \omega \cdot dS + \int_{U^\xi \cap U^{(2)}} \frac{\partial \omega}{\partial \zeta} \frac{\partial (\beta \log 9 \log 9)}{\partial \zeta} \omega \, dS \]
\[ = - \int_{U^\xi \cap \partial U^\omega} \xi \frac{\partial \omega}{\partial \zeta} \cdot dS + \int_{U^\xi \cap U^\omega} \xi \frac{\partial^2 \omega}{\partial \zeta^2} \, dx' \]
\[ \geq \int_{U^\xi} \sup_{x' \in U^\omega} \left| \frac{\partial \omega}{\partial \zeta} \right| - \sup_{x' \in U^\omega} \left| \frac{\partial^2 \omega}{\partial \zeta^2} \right| \, dx'. \]

Then substitution of (86) back into (85) yields
\[ \exists C^\phi \geq \sup_{x' \in U} \frac{\partial \phi}{\partial \zeta} + \sup_{x' \in U} \left| \frac{\partial^2 \phi}{\partial \zeta^2} \right| + \sup_{x' \in U^\omega} \frac{\partial \omega}{\partial \zeta} + \sup_{x' \in U^\omega} \left| \frac{\partial^2 \omega}{\partial \zeta^2} \right|, \] (87)
s.t.
\[ \int_{U^\xi} \frac{\partial^2 \xi}{\partial \zeta^2} \left( \phi' + \frac{1}{2} C^\phi |x'|^2 \right) \, dx' \geq 0. \]

Therefore, we have shown that \(\phi'\) is also capable of being an obstacle function for the over-determined problem in (28), which results in the existence of a coincident set \(\Omega' \subseteq B_{r_0}\), where \(N_{\Omega'}[\psi](x') = \phi'(x')\) for \(x' \in \Omega'\). Further, since \(\Omega' \subseteq B_{r_0} \subseteq (U \cap U^\omega)\), we conclude that there is an \(\Omega'\), within which (79) is satisfied.

Finally, if \(\phi\) is updated to \(\phi + \omega\) in Lemma 2.2, Lemma 2.2 is automatically satisfied by comparison of (79) with (44), due to the non-polynomial term \(\omega\) in (79). Given this, we have proved the existence of another kind of non-ellipsoidal inclusion that possesses Eshelby’s polynomial property.

### 2.2 Transversely isotropic material

For a transversely isotropic material, we reset the Cartesian coordinate system \(x = (x_1, x_2, x_3)\) to make the \(x_3\)-axis perpendicular to the plane of isotropy of the transversely isotropic medium. Then there
are five independent elastic parameters $C_{11}$, $C_{12}$, $C_{13}$, $C_{33}$ and $C_{44}$, and the positive definiteness of $C$ requires

$$C_{11} > 0, \quad C_{11} > C_{12} > -C_{11}, \quad -2C_{13}^2 + (C_{11} + C_{12})C_{33} > 0, \quad C_{44} > 0.$$ \hspace{1cm} (88)

Then we present the following theorem:

**Theorem 2.3** For any transversely isotropic material, there exists a non-ellipsoidal inclusion $\Omega$ that possesses Eshelby’s polynomial property for a particular polynomial eigenstrain of degree two.

Here, we do not impose any additional constraint on the elastic parameters of the transversely isotropic material except those in (88). The proof of Theorem 2.3 is divided into two parts concerning the elastic parameters.

1. $C_{13} + C_{44} = 0$

   In this case, we still choose the eigentress $\sigma^*_i$ in (10). Likewise, substitution of (10) into (4) generates

   $$\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0,$$

   $$\varepsilon_{13}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_1 \eta_3}{C_{33} \eta_3^2 + C_{44}(\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta,$$

   $$\varepsilon_{23}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_2 \eta_3}{C_{33} \eta_3^2 + C_{44}(\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta,$$

   $$\varepsilon_{33}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\eta_3^2}{C_{33} \eta_3^2 + C_{44}(\eta_1^2 + \eta_2^2)} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta.$$ \hspace{1cm} (89)

   By comparison of (89) with (13) and introduction of the same transformation as (14) with the magnitude of $s$ replaced by $s:=\sqrt{\frac{C_{44}}{C_{33}}}$, we can obtain the same results in (18). Therefore, by following the same discussion from (18) to (86), we verify the existence of a non-ellipsoidal inclusion that possesses Eshelby’s polynomial property for a polynomial eigenstrain of degree two, when the elastic parameters of the transversely isotropic medium satisfy $C_{13} + C_{44} = 0$, which sustains Theorem 2.3 under the condition $C_{13} + C_{44} = 0$.

2. $C_{13} + C_{44} \neq 0$

   In this case, we need to choose appropriate eigenstrain to achieve our goal. We specify the eigentrain that makes the corresponding eigenstress belong to the transversely isotropic category, i.e.,

   $$\sigma^*(x) \in \{ \sigma^*(x) | \sigma^*(x) = \rho(x)(\sigma_{11}^* \hat{\alpha} + \sigma_{33}^* \hat{\beta}), \quad \sigma_{11}^*, \sigma_{33}^* \in \mathbb{R}, \quad x \in \mathbb{R}^3 \},$$ \hspace{1cm} (90)

   with $\hat{\alpha}:=\vec{I} - \hat{\beta}$ and $\hat{\beta}:=\vec{n} \otimes \vec{n}$ utilized for the convenience for describing transverse isotropy [31] where $\vec{n}$ is the unit vector normal to the plane of isotropy of the transversely isotropic material.

   In addition, we require

   $$\sigma_{33}^* = \frac{C_{11} C_{33} - C_{33} C_{44} \eta^2}{(C_{13} + C_{44}) C_{11}} \sigma_{11}^*,$$ \hspace{1cm} (91)
with \( v \) is the root of

\[
C_{33}C_{44}v^4 - (C_{11}C_{33} + C_{22}^2 - (C_{13} + C_{44})^2)v^2 + C_{11}C_{44} = 0. \tag{92}
\]

Then by substituting (90)-(92) into (4), we can get a concise expression of the strain field, i.e.,

\[
\varepsilon(x) = \frac{1}{2} \left[ \nabla \otimes (K^* \cdot \nabla u^*) + (K^* \cdot \nabla u^*) \otimes \nabla \right], \tag{93}
\]

with

\[
u^*(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\eta_1^2 + \eta_2^2 + \eta_3^2} \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} dy d\eta, \tag{94}
\]

and

\[
K^* := \begin{bmatrix}
\sigma_{11} & 0 & 0 \\
0 & \sigma_{11} & 0 \\
0 & 0 & \frac{(C_{13} - C_{44}v^2)\sigma_{11}}{v^2(C_{13} + C_{44})}
\end{bmatrix}. \tag{95}
\]

Then substituting (14) into (93) along with letting \( \rho = \psi \) yields

\[
\varepsilon_{ij}(x') = \frac{1}{2} \left( \tilde{Q}_{ip}K^*_{jl}\tilde{Q}_{lm} \frac{\partial^2 N_{\Omega'}[\psi](x')}{\partial x'_p \partial x'_m} + \tilde{Q}_{jl}K^*_{ip}\tilde{Q}_{pq} \frac{\partial^2 N_{\Omega'}[\psi](x')}{\partial x'_l \partial x'_q} \right). \tag{96}
\]

Based on (96) and Theorem 2.2, we draw the conclusion that under the condition \( C_{13} + C_{44} \neq 0 \), Eshelby's polynomial conservation theorem holds for a non-ellipsoidal inclusion in the transversely isotropic medium when subjected to a polynomial eigenstrain of degree two, which sustains Theorem 2.3 when \( C_{13} + C_{44} \neq 0 \).

Up to now, the proof of Theorem 2.3 is fulfilled. We note that, when \( C_{13} + C_{44} \neq 0 \), the non-ellipsoidal \( \Omega \) that is a counter-example to the generalized strong version of the high-order Eshelby conjecture for the eigenstrain of degree two can only be constructed by (21), just like the isotropic material.

### 2.3 Orthotropic material

For an orthotropic material, let the axes of the Cartesian coordinate system \( x = (x_1, x_2, x_3) \) be oriented along the three 2-fold axes of rotational symmetry of the material. Now, there are nine independent elastic parameters \( C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}, C_{12}, C_{23} \) and \( C_{13} \), which satisfy

\[
C_{11} > 0, C_{11}C_{22} - C_{12}^2 > 0, C_{44} > 0, C_{55} > 0, C_{66} > 0,
(C_{11}C_{22} - C_{12}^2)C_{33} + 2C_{12}C_{23}C_{13} - (C_{11}C_{23} + C_{22}C_{13}) > 0,
\tag{97}
\]

owing to the positive definiteness of \( C \).

Then we present the following theorem:

**Theorem 2.4** For orthotropic materials whose elastic parameters satisfy \( C_{12} + C_{66} = 0 \) and \( C_{13} + C_{55} \), there exists a non-ellipsoidal inclusion \( \Omega \) that possesses Eshelby’s polynomial property for a particular polynomial eigenstrain of degree two.
We consider the eigentress $\sigma_{ij}'$ in (10). Under the conditions $C_{12} + C_{66} = 0$ and $C_{13} + C_{55}$, substituting (10) into (4) leads to

$$
\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0,
$$

$$
\varepsilon_{13}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} C_{55} \eta_1 \eta_3 \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} \, dy \, d\eta,
$$

$$
\varepsilon_{23}(x) = -\frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} C_{55} \eta_2 \eta_3 \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} \, dy \, d\eta,
$$

$$
\varepsilon_{33}(x) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} C_{55} \eta_3^2 \int_{\Omega} \rho(y) e^{-i\eta \cdot (x-y)} \, dy \, d\eta.
$$

(98)

By transformations of coordinates

$$
x'' := \hat{Q} \cdot x, \quad y'' := \hat{Q} \cdot y, \quad \eta'' := \hat{Q}^{-1} \cdot \eta
$$

(99)

with

$$
\hat{Q} := \begin{bmatrix}
  s_1 & 0 & 0 \\
  0 & s_2 & 0 \\
  0 & 0 & 1
\end{bmatrix}
$$

(100)

and then substitution of (100) into (98) with $s_1 := \sqrt{\frac{C_{33}}{C_{55}}}, s_2 := \sqrt{\frac{C_{33}}{C_{44}}}$, it follows that

$$
\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0,
$$

$$
\varepsilon_{13}(x'') = -\frac{1}{2C_{33}C_{55}} \int_{\mathbb{R}^3} \eta_1^2 + \eta_2^2 + \eta_3^2 \int_{\Omega''} \rho(y'') e^{-i\eta'' \cdot (x''-y'')} \, dy' \, d\eta'',
$$

$$
\varepsilon_{23}(x'') = -\frac{1}{2C_{33}C_{44}} \int_{\mathbb{R}^3} \eta_1^2 + \eta_2^2 + \eta_3^2 \int_{\Omega''} \rho(y'') e^{-i\eta'' \cdot (x''-y'')} \, dy' \, d\eta'',
$$

$$
\varepsilon_{33}(x'') = -\frac{1}{C_{33}} \int_{\mathbb{R}^3} \eta_3^2 \int_{\Omega''} \rho(y'') e^{-i\eta'' \cdot (x''-y'')} \, dy' \, d\eta''
$$

(101)

with

$$
\Omega'' := \{ y'' | \hat{Q}^{-1} \cdot y'' \in \Omega \}.
$$

(102)

Let $\rho = \psi$, and by comparing (7) with (101), we find

$$
\varepsilon_{ij}(x'') = \frac{1}{2C_{33}} \left( \hat{Q}_{ij} P_{jm} \hat{Q}_{m} \frac{\partial^2 N_{Q} \rho(x'')}{\partial x_j'' \partial x_i''} + \hat{Q}_{jm} P_{il} \hat{Q}_{l} \frac{\partial^2 N_{Q} \rho(x'')}{\partial x_i'' \partial x_j''} \right).
$$

(103)

Resorting to Theorem 2.2, there exists a non-ellipsoid $\Omega''$ leading to the quadratic polynomial form of the right-hand side of (103), which implies that the strain field within $\Omega''$ should be a quadratic function of $x''$ and thus $x$ via the transformation (99).

Therefore, the existence of a non-ellipsoidal $\Omega$ constructed by stretching $\Omega''$ along the $x''_1$-axis and $x''_2$-axis by proportions $s_1$ and $s_2$, respectively, has been proved. Thus the proof of Theorem 2.4 is completed, which constitutes the counter-example to the generalized strong version of the high-order Eshelby conjecture for a polynomial eigenstrain of degree two in orthotropic materials.
2.4 Monoclinic material

For a monoclinic material, let the $x_3$-axis of the Cartesian coordinate system $x = (x_1, x_2, x_3)$ be the 2-fold axis of rotational symmetry of the material. There are thirteen independent elastic parameters, i.e., $C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}, C_{12}, C_{13}, C_{23}, C_{16}, C_{26}, C_{36}$ and $C_{45}$. The positive definiteness of $C$ requires

$$C_{11} > 0, \ C_{11}C_{22} - C_{12}^2 > 0, \ C_{44}C_{55} > C_{45}^2, \ C_{55} > 0, \ C_{66} > 0,$$

$$(C_{11}C_{22} - C_{12}^2)C_{33} + 2C_{12}C_{23}C_{13} - (C_{11}C_{23} + C_{22}C_{13}) > 0.$$  \text{(104)}

Then we present the following theorem:

**Theorem 2.5** For monoclinic materials whose elastic parameters satisfy $C_{45} = 0$, $C_{36} = 0$, $C_{23} + C_{44} = 0$ and $C_{13} + C_{55} = 0$, there exists a non-ellipsoidal inclusion $\Omega$ that possesses Eshelby’s polynomial property for a particular polynomial eigenstrain of degree two.

If we select the eigenstrain that results in (10), then substitution of (10) into (4) still yields (98). Likewise, through the same discussion as before, (103) is derived from (98), comparison of which with Theorem 2.2 implies the existence of a non-ellipsoidal $\Omega''$ that generates the polynomial form of the strain field within it. By (99), the existence of the corresponding non-ellipsoidal $\Omega$ is also obtained, which sustains Theorem 2.5 and thus forms the counter-example to the generalized strong version of the high-order Eshelby conjecture for a polynomial eigenstrain of degree two in monoclinic materials.

3 Polynomial eigenstrain of even degrees

In this section, we prove the invalidity of the generalized strong version of the high-order Eshelby conjecture in the case of polynomial eigenstrains of even degrees. For any polynomial $\rho$, if we define the eigenstress as (10), then (18) and (103) can always be derived from (4), which implies

$$\varepsilon_{ij}[\rho](x') \propto \left( Q'_{il}P_{jm}Q'_{mq} \frac{\partial^2 N_{\Omega'}[\rho](x')}{\partial x'_l \partial x'_q} + Q'_{jm}P_{il}Q'_{ls} \frac{\partial^2 N_{\Omega'}[\rho](x')}{\partial x'_m \partial x'_s} \right),$$  \text{(105)}

where $\varepsilon_{ij}[\rho](x')$ denotes the strain field owing to the polynomial eigenstrain that is linear with respect to the polynomial $\rho$; $Q'$ denotes a constant diagonal matrix with all diagonal elements being positive; and $N_{\Omega'}(x')$ denotes the Newtonian potential induced by $\Omega'$ with the polynomial mass density $\rho$. Here $\Omega'$ is transformed from the original domain $\Omega$ through stretching transformations defined by the inverse of $Q'$.

Based on (105), we present the following theorem:

**Theorem 3.1** There exists a non-ellipsoidal inclusion $\Omega$ that possesses Eshelby’s polynomial property for a polynomial eigenstrain of any non-negative even degree in the anisotropic media, if the elastic parameters satisfy

1. $C_{12} + C_{44} = 0$ for cubic materials;
2. $C_{13} + C_{44} = 0$ for transversely isotropic materials;
3. $C_{23} + C_{44} = 0$ and $C_{13} + C_{55} = 0$ for orthotropic materials;
4. $C_{45} = 0, C_{36} = 0, C_{23} + C_{44} = 0$ and $C_{13} + C_{55} = 0$ for monoclinic materials.

It is emphasized that all of the constraints on the elastic parameters listed above are the necessary conditions for (105). Theorem 3.1 substantiates the invalidity of the generalized strong version of the high-order Eshelby conjecture for polynomial eigenstrains of any non-negative even degree in the anisotropic media mentioned above.

Inspired by the derivation from (79) to (86), we present a lemma to prove Theorem 3.1, i.e.,

**Lemma 3.1** $\forall n \in \{n \mid n=2k, k \geq 0, k \in \mathbb{Z}\}$, there exists at least one simply-connected bounded Lipschitz domain $\Omega' \subseteq \mathbb{R}^3$ of non-ellipsoidal shape which leads to

\[
N_{\Omega}[\hat{\rho}](\mathbf{x}') = -\int_{\Omega'} \frac{\hat{\rho}(\mathbf{y}', n)}{4\pi |\mathbf{x}' - \mathbf{y}'|} d\mathbf{y}' = \hat{\varphi}(\mathbf{x}', n) + \hat{\omega}(x_1', x_2'), \; \mathbf{x}' \in \Omega',
\]

with $\hat{\rho}(\mathbf{x}', n) = - (x_1'' + x_2'' + x_3'')$, $\hat{\varphi}(\mathbf{x}', n)$ an undetermined polynomial function of $\mathbf{x}'$ with degree $n + 2$ that satisfies $\Delta_n \hat{\varphi}(\mathbf{x}', n) = \hat{\rho}(\mathbf{x}', n)$, and $\hat{\omega}(x_1', x_2')$ an undetermined harmonic function that is non-polynomial and Lipschitz continuous.

We see that owing to the non-polynomial form of $\hat{\omega}(x_1', x_2')$, Lemma 2.2 is automatically satisfied with $\Psi$ and $\Phi$ replaced by $\hat{\varphi}(\mathbf{x}')$ and $\hat{\rho}(\mathbf{x}', n) + \hat{\omega}(x_1', x_2')$, respectively, since the interior Newtonian potential induced by an ellipsoid with an arbitrary polynomial mass density must be polynomial [12, 11]. Given this, once Lemma 3.1 is proved, we can prove Theorem 3.1 by substituting (106) into (105).

Therefore, we see that (106) is actually a sufficient condition for the existence of counter-examples $\Omega'$ for polynomial eigenstrains of any non-negative even degree in the anisotropic media, which is in terms of the Newtonian potential induced by $\Omega'$. In other words, by appropriately choosing $\hat{\varphi}(\mathbf{x}', n)$ and $\hat{\omega}(x_1', x_2')$ at the right-hand side of (106) to make $\hat{\varphi}(\mathbf{x}', n) + \hat{\omega}(x_1', x_2')$ meet with the property of an obstacle function defined previously, we can construct numerous counter-examples $\Omega'$ to the strong version of the high-order Eshelby conjecture for polynomial eigenstrains of any non-negative even degree in the anisotropic media. The validity of such a sufficient condition is stated in Lemma 3.1

**Proof of Lemma 3.1**

Lemma 3.1 has been proved for $n = 2$ from (79) to (86). To complete the proof of Lemma 3.1, we consider the remaining cases concerning the degree $n$ of $\hat{\rho}$.

1. $n = 0$.

By resorting to [25], we see that there is an $\Omega'$, leading to

\[
N_{\Omega'}(\mathbf{x}') = -\int_{\Omega'} \frac{1}{4\pi |\mathbf{x}' - \mathbf{y}'|} d\mathbf{y}' = q'(\mathbf{x}') + \omega'(x_1', x_2'), \; \mathbf{x}' \in \Omega',
\]

where $q'$ denotes a quadratic function and $\omega'$ is a harmonic function whose structure is the same as the harmonic function $\omega^*$ in (82) with Lipschitz continuity. We note that (107) is exactly (106) when $\hat{\rho} \equiv 1$. Hence the proof of Lemma 3.1 is obtained for $n = 0$. Note that the non-ellipsoidal structure $\Omega'$ that leads to (107) in [25] is initially constructed as a counter-example that possesses Eshelby’s uniformity property in the isotropic medium. Here we find that even if the medium is anisotropic, substituting (107) into (105) yields the Eshelby’s uniformity property of $\Omega'$, which verifies the invalidity of the generalized strong version of the Eshelby conjecture for uniform eigenstrains in the anisotropic media shown in Theorem 3.1.
2. \( n > 2 \).

We take \( \hat{\omega} = \omega^* \) and

\[
\hat{\phi}(x', n) := \hat{\phi}(x') = \hat{C} - \frac{1}{(n+2)(n+1)} \left( x_1^{n+2} + x_2^{n+2} + x_3^{n+2} \right)
\]

where \( \hat{C} \) is a positive real constant. It can be testified that \( \hat{\phi}(x') + \omega^*(x_1', x_2') \) satisfies (6), which is necessary for a function to be Newtonian potential. Then we are going to prove the existence of an \( \Omega' \) that generates (106) under the conditions \( \hat{\omega} = \omega^* \) and (108).

We note that if we can substantiate

\[
\hat{\phi}(x') := \begin{cases} 
\hat{\phi}(x'), & x' \in \hat{U} \\
-C, & x' \in \mathbb{R}^3 \setminus \hat{U}
\end{cases}
\]

where \( \hat{U} := \{ x' \mid x_1^{n+2} + x_2^{n+2} + x_3^{n+2} \leq 2(n+1)(n+2)\hat{C}, x' \in \mathbb{R}^3 \} \) possesses all of the properties of an obstacle function introduced above, then by following the same procedure from (79) to (86) with \( \phi^* \) replaced by \( \hat{\phi}^* \), we can prove Lemma 3.1 for \( n > 2 \).

We will evaluate \( \hat{\phi}^* \) in the same way as that we evaluate \( \phi^* \) from (31) to (39). Since

\[
\hat{\phi}^*|_{\partial \hat{U}^-} = \hat{\phi}^*|_{\partial \hat{U}^+} = -\hat{C},
\]

where \( \hat{U} \) is bounded due to \( \hat{U} \) being contained in another bounded domain \( \{ x' \mid |x_1'| \leq (2(n+1)(n+2)\hat{C})^{\frac{1}{2}}, x' \in \mathbb{R}^3 \} \), we see \( \hat{\phi}^* \in C^0(\mathbb{R}^3) \).

Then since \( |\hat{\phi}(x')| \) and \( |\nabla x' \hat{\phi}(x)| \) are bounded in \( \hat{U} \),

\[
|\hat{\phi}^*|_{0,1} = \sup_{x' \in \hat{U}} |\hat{\phi}(x')| + \sup_{x' \in \hat{U}} |\nabla x' \hat{\phi}(x)|
\]

is bounded, which implies \( \hat{\phi}^* \in C^{0,1}(\mathbb{R}^3) \). By taking the ball \( B_{\hat{r}} \supset \hat{U} \), we see that \( \forall |x'| \geq \hat{r}, \hat{\phi}^* = -\hat{C} < 0 \).

Likewise, by introducing the smooth function \( \xi \) defined in (37), (38) can be deduced with \( \phi^* \) replaced by \( \hat{\phi}^* \), which implies the existence of \( \hat{\tilde{C}} \) that satisfies

\[
\hat{\tilde{C}} \geq \sup_{x' \in \hat{U}} \left| \frac{\partial \hat{\phi}}{\partial \xi} \right| + \sup_{x' \in \hat{U}} \left| \frac{\partial^2 \hat{\phi}}{\partial \xi^2} \right|
\]

leading to

\[
\int_{\hat{U}} \frac{\partial^2 \xi}{\partial \xi^2} \left( \hat{\phi}^* + \frac{1}{2} \hat{\tilde{C}} |x'|^2 \right) \, dx' \geq 0.
\]

Up to now, \( \hat{\phi}^* \) has been testified to possess all of the properties of an obstacle function. As is mentioned above, replacing the old obstacle function \( \phi^* \) by \( \hat{\phi}^* \) and following the same procedure from (79) to (86) yield the existence of an \( \Omega' \) that leads to

\[
N_{\Omega'}[\hat{\phi}](x') = -\int_{\Omega'} \frac{-x_1^n + x_2^n + x_3^n}{4\pi|x'| - y'} \, dy' = \hat{\tilde{C}} - \frac{1}{(n+2)(n+1)} \left( x_1^{n+2} + x_2^{n+2} + x_3^{n+2} \right) + \omega^*(x_1', x_2'), \quad x' \in \Omega'.
\]
Hence the proof of Lemma 3.1 is fulfilled, and thus Theorem 3.1 is proved due to the substantiation of Lemma 3.1.

As an example, the shape of a particular counter-example, which sustains Theorem 3.1 when the degree of the polynomial eigenstrain is four is shown as $\Omega^{(2)}$ in Fig. 2 in the Appendix. $\Omega^{(2)}$ is one of the structures that sustain Theorem 3.1 and more counter-examples that sustain Theorem 3.1 but correspond to polynomial eigenstrains of higher even degrees can be constructed via the same procedure.

4 Conclusions and Discussion

In this work, we prove that in anisotropic media possessing cubic, transversely isotropic, orthotropic, and monoclinic symmetries, there exist non-ellipsoidal inclusions that can transform particular polynomial eigenstrains of even degrees into polynomial elastic strain fields of the same even degrees in them, and also in the isotropic medium, there exist non-ellipsoidal inclusions that can transform particular quadratic eigenstrains into quadratic elastic strain fields in them. A sufficient condition for the existence of non-ellipsoidal inclusions that retain Eshelby’s polynomial conservation theorem in anisotropic media is also presented, which can help construct more counter-examples to the strong version of the high-order Eshelby conjecture in anisotropic media. These findings reveal that in anisotropic media, a striking rich class of inclusions beyond ellipsoids can exhibit the uniformity between the eigenstrains and the induced elastic strains. They may have implications in the quantum dots technology where the strain fields can be used to modulate the properties of the photons emitted by the dots of anisotropic materials.

When the eigenstrain is in the expression of the polynomial of odd degrees, the variational method is incapable of dealing with the high-order Eshelby conjecture, since the obstacle function $\phi$, in this case, shall be chosen in the form of polynomials whose highest degree is odd so that $\not\exists r \in \mathbb{R}^+ , \forall |x| \geq r , \phi(x) < 0$, which contradicts the property of an obstacle function.

In contrast to the generalized strong version, the problem concerning the generalized weak version of the high-order Eshelby conjecture is hard to solve. The strategy for the verification of the generalized weak version in the recent paper [1] cannot be utilized here due to the fact that the ellipsoidal shape will not be necessity for the domain to generate a polynomial Newtonian potential with a polynomial mass density of a degree larger than zero. Therefore, the generalized weak version of the high-order Eshelby conjecture remains to be solved.

Ethics. This work does not involve ethical issues.

Data Accessibility. This work has no supplementary data.

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Appendix

A  The shapes of counter-example non-ellipsoidal inclusions

The counter-example is constructed by (25), which implies that each counter-example corresponds to a specific obstacle function \( \phi \). In other words, once \( \phi \) is given, (25) admits a unique solution \( V_\phi \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^3) \cap K_\phi \), and thus the shape of the counter-example \( \Omega \) can be constructed by assembling the points where \( V_\phi(x) = \phi(x) \), since \( \Omega = \{ x \mid V_\phi(x) = \phi(x), \ x \in \mathbb{R}^3 \} \).

According to [25], (25) can be discretized into

\[
\min \left\{ \prod(\tilde{v}) = -\frac{1}{2} \tilde{V}_\phi \cdot \hat{K} \cdot \tilde{V}_\phi, \ \tilde{V}_\phi \geq \hat{\phi} \right\},
\]

(A.1)

where \( \tilde{V}_\phi \) denotes the vector whose components are the values of \( V_\phi \) at the nodal points when using discretization of the finite element method; \( \hat{\phi} \) similarly denotes the vector whose components are the values of \( \phi \) at the nodal points; and \( \hat{K} \) is the stiffness tensor corresponding to the Laplacian equation \( \nabla^2 V_\phi = 0 \) discretized via the finite element method. Note that (A.1) is a standard quadratic programming problem that can be easily solved.

The first counter-example \( \Omega^{(1)} \) to the generalized strong version of the high-order Eshelby conjecture for the quadratic eigenstrain is constructed by letting \( \phi \) be given as (30) when \( C = \frac{1}{36} \). The shape of \( \Omega^{(1)} \) is shown in Fig. 1 below. The notation \( \{x, y, z\} \) denoting the axes of the coordinates in the figure is corresponding to \( \{x_1, x_2, x_3\} \) (or \( \{x'_1, x'_2, x'_3\} \)) in the main text.

![Image of the first counter-example](image1)

(a) Nodal points where \( |\{\tilde{V}_\phi\}_k - \{\hat{\phi}\}_k| \leq 1 \times 10^{-4} \) with \( k \) the sequence of the nodes. 

(b) The configuration of \( \Omega^{(1)} \) assembled by the nodal points.

(c) The view of \( \Omega^{(1)} \) in x-y plane. 
(d) The view of \( \Omega^{(1)} \) in x-z plane. 
(e) The view of \( \Omega^{(1)} \) in y-z plane.

Figure 1: The counter-example \( \Omega^{(1)} \) for a quadratic eigenstrain.

The second counter-example \( \Omega^{(2)} \) to the generalized strong version of the high-order Eshelby
conjecture for the quartic eigenstrain is constructed by letting $\phi$ equal $\{109\}$ plus $\{82\}$ along with $\tilde{\phi}(x')$ in $\{109\}$ being given by $\{108\}$, when $n = 4$, $C = \hat{C} = \frac{1}{36}$ and $\beta = \frac{1}{600}$. The shape of $\Omega^{(2)}$ is shown in Fig. 2 below.

(a) Nodal points where $|\{\hat{V}_\phi\}_k - \{\hat{\phi}\}_k| \leq 1 \times 10^{-4}$ with $k$ the sequence of the nodes.

(b) The configuration of $\Omega^{(2)}$ assembled by the nodal points.

(c) The view of $\Omega^{(2)}$ in x-y plane.

(d) The view of $\Omega^{(2)}$ in x-z plane.

(e) The view of $\Omega^{(2)}$ in y-z plane.

Figure 2: The counter-example $\Omega^{(2)}$ for a quartic eigenstrains.

References

[1] Yuan T, Huang K, Wang J. Proofs of the weak version of the generalized Eshelby conjecture for anisotropic media. arXiv:2105.08295

[2] Mura T. 1987 *Micromechanics of Defects in Solids*. Leiden: Springer Netherlands.

[3] Chen Y, Zhang J, Zopf M, Jung K, Zhang Y, Keil R, Ding F, Schmidt OG. 2016 Wavelength-tunable entangled photons from silicon-integrated III-V quantum dots. *Nature Communications* **7**, 10387.

[4] Stepanov P, Elzo-Aizarna M, Bleuse J, Malik NS, Curé Y, Gautier E, Favre-Nicolin V, Gérard JM, Claudon J. 2016 Large and uniform optical emission shifts in quantum dots strained along their growth axis. *Nano Letters* **16**, 3215–3220.
[5] Trotta R, Martín-Sánchez J, Daruka I, Ortix C, Rastelli A. 2015 Energy-tunable sources of entangled photons: a viable concept for solid-state-based quantum relays. Physical Review Letters 114, 150502.

[6] Trotta R, Martín-Sánchez J, Wildmann JS, Piredda G, Reindl M, Schimpf C, Zallo E, Stroj S, Edlinger J, Rastelli A. 2016 Wavelength-tunable sources of entangled photons interfaced with atomic vapours. Nature Communications 7, 10375.

[7] Eshelby JD. 1961 Elastic inclusions and inhomogeneities. In Progress in solid mechanics II (eds I. N. Sneddon, R. Hill) pp. 89–140 Amsterdam, The Netherlands. North-Holland Publishing Company.

[8] Calvo-Jurado C, Parnell WJ. 2020 Induced fields in isolated elliptical inhomogeneities due to imposed polynomial fields at infinity. International Journal of Computer Mathematics 97, 18–29.

[9] Rahman M. 2002 The isotropic ellipsoidal inclusion with a polynomial distribution of eigenstrain. Journal of Applied Mechanics-Transactions of the ASME-Transactions of the Asme 69, 593–601.

[10] Sendeckyj G. 1967 Ellipsoidal inhomogeneity problem (Ph.D. Dissertation). Evanston: Northwestern University.

[11] Dyson F. 1891 The potentials of ellipsoids of variable densities. The Quarterly journal of pure and applied mathematics 25, 259–288.

[12] Ferrers NM. 1877 On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae and elliptic rings of variable densities. The Quarterly journal of pure and applied mathematics 14, 1–22.

[13] Asaro R, Barnett D. 1975 The non-uniform transformation strain problem for an anisotropic ellipsoidal inclusion. Journal of the Mechanics and Physics of Solids 23, 77 – 83.

[14] Mura T, Kinoshita N. 1978 The polygonal eigenstrain problem for an anisotropic ellipsoidal inclusion. Physica Status Solidi A 48, 447–450.

[15] Monchiet V, Bonnet G. 2011 Inversion of higher order isotropic tensors with minor symmetries and solution of higher order heterogeneity problems. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 467, 314–332.

[16] Monchiet V, Bonnet G. 2013 Algebra of transversely isotropic sixth order tensors and solution to higher order inhomogeneity problems. Journal of Elasticity 110, 159–183.

[17] Eshelby JD. 1957 The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 241, 376–396.

[18] Eshelby JD. 1959 The elastic field outside an ellipsoidal inclusion. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 252, 561–569.

[19] Nie G, Guo L, Chan C, Shin F. 2007 Non-uniform eigenstrain induced stress field in an elliptic inhomogeneity embedded in orthotropic media with complex roots. International Journal of Solids and Structures 44, 3575 – 3593.
[20] Guo L, Nie GH, Chan CK. 2011 Elliptical inhomogeneity with polynomial eigenstrains embedded in orthotropic materials. *Archive of Applied Mechanics* **81**, 157–170.

[21] Liu LP. 2013 Polynomial eigenstress inducing polynomial strain of the same degree in an ellipsoidal inclusion and its applications. *Mathematics and Mechanics of Solids* **18**, 168–180.

[22] Joyce D, Parnell WJ. 2017 The Newtonian potential inhomogeneity problem: non-uniform eigenstrains in cylinders of non-elliptical cross section. *Journal of Engineering Mathematics* **107**, 283–303.

[23] Rashidinejad E, Shodja HM. 2019 On the exact nature of the coupled-fields of magneto-electro-elastic ellipsoidal inclusions with non-uniform eigenfields and general anisotropy. *Mechanics of Materials* **128**, 89–104.

[24] Kang H, Milton GW. 2008 Solutions to the Pólya-Szegő conjecture and the weak Eshelby conjecture. *Archive for Rational Mechanics and Analysis* **188**, 93–116.

[25] Liu LP. 2008 Solutions to the Eshelby conjectures. *Proceedings of the Royal Society a-Mathematical Physical and Engineering Sciences* **464**, 573–594.

[26] Xu BX, Zhao YT, Gross D, Wang MZ. 2009 Proof of the strong Eshelby conjecture for plane and anti-plane anisotropic inclusion problems. *Journal of Elasticity* **97**, 173–188.

[27] Duong CN, Wang JJ, Yu J. 2001 An approximate algorithmic solution for the elastic fields in bonded patched sheets. *International Journal of Solids and Structures* **38**, 4685–4699.

[28] Lee YG, Zou WN, Pan E. 2015 Eshelby’s problem of polygonal inclusions with polynomial eigenstrains in an anisotropic magneto-electro-elastic full plane. *Proceedings of the Royal Society a-Mathematical Physical and Engineering Sciences* **471**, 20140827.

[29] Yue YM, Xu KY, Chen QD, Pan E. 2015 Eshelby problem of an arbitrary polygonal inclusion in anisotropic piezoelectric media with quadratic eigenstrains. *Acta Mechanica* **226**, 2365–2378.

[30] Friedman A. 1982 *Variational Principles and Free Boundary Problems*. New York: Wiley.

[31] Walpole LJ. 1981 Elastic behavior of composite materials: theoretical foundations. *Advances in Applied Mechanics* **21**, 169–242.