THE EQUIVARIANT SPECTRAL FUNCTION OF AN INVARIANT ELLIPTIC OPERATOR. \(L^p\)-BOUNDS, CAUSTICS, AND CONCENTRATION OF EIGENFUNCTIONS

PABLO RAMACHER

Abstract. Let \(M\) be a compact boundaryless Riemannian manifold, carrying an effective and isometric action of a compact Lie group \(G\), and \(P_0\) an invariant elliptic classical pseudodifferential operator on \(M\). Using Fourier integral operator techniques, we prove a local Weyl law with remainder estimate for the equivariant (or reduced) spectral function of \(P_0\) for each isotypic component in the Peter-Weyl decomposition of \(L^2(M)\), generalizing work of Avacumović, Levitan, and Hörmander. From this we deduce a generalized Kuznecov sum formula for periods of \(G\)-orbits, and recover the local Weyl law for orbifolds shown by Stanhope and Uribe. Relying on recent results on singular equivariant asymptotics of oscillatory integrals, we further characterize the caustic behaviour of the reduced spectral function near singular orbits, which allows us to give corresponding point-wise bounds for clusters of eigenfunctions in specific isotypic components. In case that \(G\) acts on \(M\) without singular orbits, we are able to deduce hybrid \(L^p\)-bounds for \(2 \leq p \leq \infty\) in the eigenvalue and isotypic aspect that improve on the classical estimates of Seeger and Sogge for generic eigenfunctions. Our results are sharp in the eigenvalue aspect, but not in the isotypic aspect, and reduce to the classical ones in the case \(G = \{e\}\).

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1. Introduction

In this paper, we derive an asymptotic formula with remainder estimate for the equivariant (or reduced) spectral function of an invariant elliptic operator on a compact Riemannian manifold with an effective and isometric action of a compact Lie group \(G\), generalizing previous work of Avacumović [1], Levitan [19], Hörmander [13], and, more recently, Stanhope and Uribe [32]. If \(G\) acts on \(M\) with orbits of the same dimension, we obtain hybrid \(L^p\)-bounds for eigenfunctions in the eigenvalue and isotypic aspect that improve on the classical estimates for generic eigenfunctions proved by Seeger and Sogge [29, 26], but cannot hold when singular orbits are present. In the latter case, we are able to describe the caustic behaviour of the reduced spectral function as one approaches orbits of singular type, relying on our recent work [22] on singular equivariant asymptotics obtained via desingularization techniques. As an application, we are able to prove point-wise bounds for isotypic clusters of eigenfunctions, showing

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that they tend to concentrate on singular orbits. Since very little can be said about the shape of eigenfunctions in general, this result is rather striking. In particular, this gives a new interpretation of the classical bounds for spherical harmonics in terms of caustics of the equivariant spectral function, generalizing them to eigenfunctions on arbitrary compact manifolds with symmetries. The concentration of eigenfunctions along singular orbits was already observed in \[18\] for Schrödinger operators in the context of equivariant quantum ergodicity under the additional assumption that the reduced Hamiltonian flow is ergodic. Our results can be viewed as part of the more general problem of studying the eigenfunctions of a commuting family of differential operators on a general compact manifold that are independent in some sense \[21\].

To explain our results, consider a closed connected Riemannian manifold \(M\) of dimension \(n\), together with an elliptic classical pseudodifferential operator
\[
P_0 : C^\infty(M) \longrightarrow L^2(M)
\]
of degree \(m\), where \(C^\infty(M)\) denotes the space of smooth functions on \(M\) and \(L^2(M)\) the Hilbert space of square integrable functions with respect to the Riemannian volume density \(dM\) on \(M\). We assume that \(P_0\) is positive and symmetric, so that it has a unique self-adjoint extension \(P\). Furthermore, the compactness of \(M\) implies that \(P\) has discrete spectrum. Let \(\{E_\lambda\}\) be a spectral resolution of \(P\), and denote by \(e(x,y,\lambda)\) the Schwartz kernel of \(E_\lambda\), which is called the spectral function of \(P\). Within the theory of Fourier integral operators one can then show the following local Weyl formula \[1, 19, 13\]
\[
(1.1) \quad \left| e(x,x,\lambda) - \frac{\lambda^m}{(2\pi)^n} \int_{\{p(x,\xi) < 1\}} d\xi \right| \leq C \lambda^{-\frac{m-1}{2}}, \quad x \in M, \quad \lambda \to +\infty,
\]
for some constant \(C > 0\) independent of \(x\) and \(\lambda\), \(p\) being the principal symbol of \(P_0\). By integrating over \(M\) one deduces from this the spectral counting function \(N(\lambda) := \sum_{t \leq \lambda} \dim \mathcal{E}_t = \int_M e(x,x,\lambda) \, dM(x)\) the global Weyl formula
\[
N(\lambda) = \frac{\text{vol} S^* M}{n(2\pi)^n} \lambda^m + O(\lambda^{-\frac{m-1}{2}}),
\]
where \(\mathcal{E}_t\) denotes the eigenspace of \(P\) belonging to the eigenvalue \(t\) and \(S^* M\) the co-sphere bundle \(\{(x,\xi) \in T^* M \mid p(x,\xi) = 1\}\). In order to show the stronger point-wise formula (1.1) one first proves the estimate
\[
(1.2) \quad |e(x,x,\lambda + 1) - e(x,x,\lambda)| \leq C \cdot \lambda^{-\frac{m-1}{2}}, \quad x \in M,
\]
which describes the order of magnitude of the discontinuities of \(N(\lambda)\) or, more generally, the amount of eigenvalues in the interval \((\lambda, \lambda + 1)\) as \(\lambda \to +\infty\), yielding the asymptotics \(N(\lambda + 1) - N(\lambda) = O(\lambda^{-\frac{m-1}{2}})\). The bound (1.2) is equivalent to
\[
(1.3) \quad \sum_{\lambda_j \in (\lambda, \lambda + 1]} |e_j(x)|^2 \leq C \cdot \lambda^{-\frac{m-1}{2}}, \quad x \in M,
\]
where \(\{e_j\}_{j \geq 0}\) denotes an arbitrary orthonormal basis of eigenfunctions of \(P\) in \(L^2(M)\) corresponding eigenvalues \(\{\lambda_j\}_{j \geq 0}\), and actually implies the bound
\[
(1.4) \quad \|\chi_{\lambda} u\|_{L^\infty(M)} \leq C (1 + \lambda)^{-\frac{m-1}{2}} \|u\|_{L^2(M)}, \quad u \in L^2(M),
\]
where \(\chi_{\lambda}\) denotes the spectral projection onto the sum of eigenspaces with eigenvalues in the interval \((\lambda, \lambda + 1)\) with Schwartz kernel \(\chi_{\lambda}(x,y) = e(x,y,\lambda + 1) - e(x,y,\lambda)\), since \(\|\chi_{\lambda}\|_{L^2 \rightarrow L^\infty} = \text{sup}_{x \in M} \chi_{\lambda}(x,x)\). From this the estimate for \(N(\lambda + 1) - N(\lambda)\) immediately follows by taking the trace of \(\chi_{\lambda}\). In particular, one deduces from (1.4) the bound for eigenfunctions
\[
(1.5) \quad \|u\|_{L^\infty(M)} \leq C \lambda^{-\frac{m-1}{2}}, \quad u \in \mathcal{E}_\lambda, \quad \|u\|_{L^2} = 1.
\]
Under the additional assumption that the co-spheres \(S^*_t M\) are strictly convex, Seeger and Sogge \[20\] were also able to prove upper bounds for \(L^p\)-norms of eigenfunctions via analytic interpolation.
techniques, generalizing previous work of Sogge for second order elliptic differential operators \cite{29}. More precisely, let
\[
\delta_n(p) := \max \left( n \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right).
\]
Then, for \( u \in \mathcal{E}_\lambda \), \( \|u\|_{L^2} = 1 \) one has
\[
\|u\|_{L^p(M)} \leq \begin{cases} C\lambda^{\frac{\mu (n-1)}{n}}, & 2 \leq p \leq \frac{2(n+1)}{n-1}, \\
C\lambda^{\frac{n-1}{2p} - \frac{1}{2}}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \end{cases}
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

In this paper, we shall sharpen the bounds \( \|1\| \leq \|\| \) in the presence of symmetries. To explain our results, assume that \( M \) carries an effective and isometric action of a compact Lie group \( G \) with Lie algebra \( \mathfrak{g} \) and orbits of dimension less or equal \( n-1 \). The group \( G \) might be disconnected or even finite, though the case of interest is when \( G \) is continuous. Suppose that \( P \) commutes with the left-regular representation \( (\pi, L^2(M)) \) of \( G \) in \( L^2(M) \) given by
\[
\pi(g)u(x) = u(g^{-1} \cdot x), \quad u \in L^2(M),
\]
so that each eigenspace of \( P \) becomes a unitary \( G \)-module. If \( \mathcal{G} \) denotes the set of equivalence classes of irreducible unitary representations of \( G \), which we shall identify with the set of characters of \( G \), the Peter-Weyl theorem asserts that
\[
L^2(M) = \bigoplus_{\gamma \in \mathcal{G}} L^2_\gamma(M),
\]
a Hilbert sum decomposition, where \( L^2_\gamma(M) := \Pi_\gamma L^2(M) \) denotes the \( \gamma \)-isotypic component, and \( \Pi_\gamma \) the corresponding projection. Assume that the orthonormal basis \( \{e_j\}_{j \geq 0} \) has been chosen such that it is compatible with the decomposition \( \|1\| \leq \|\| \), and let \( e_\gamma(x, y, \lambda) \) be the spectral function of the operator \( P_\gamma := \Pi_\gamma \circ P \circ \Pi_\gamma = P \circ \Pi_\gamma = \Pi_\gamma \circ P \), which is also called the reduced spectral function of \( P \). Further, let \( \mathbb{J} : T^*M \to \mathfrak{g}^* \) denote the momentum map of the Hamiltonian \( G \)-action on \( T^*M \), induced by the action of \( G \) on \( M \), and write \( \Omega := \mathbb{J}^{-1}(\{0\}) \). As our first result, we show in Theorem 4.3 the equivariant local Weyl law
\[
e_\gamma(x, x, \lambda - \frac{n-\kappa_x}{n-1}, \frac{d\mathbb{J} \cap \Omega(x, \xi) < 1}{\text{vol}(\Omega(x, \xi))} \leq C_{x,\gamma} \lambda^{\frac{n-\kappa_x}{n-1}}, \quad x \in M,
\]
as \( \lambda \to +\infty \), where \( \kappa_x := \dim O_x \) is the dimension of the \( G \)-orbit through \( x \), \( d\mathbb{J} \) denotes the dimension of an irreducible \( G \)-representation \( \pi_\gamma \) belonging to \( \gamma \) and \( [\pi_\gamma|_{G_x} : 1] \) the multiplicity of the trivial representation in the restriction of \( \pi_\gamma \) to the isotropy group \( G_x \) of \( x \), while \( C_{x,\gamma} \) is a constant depending on \( x \) and \( \gamma \). It should be emphasized that \( \kappa_x \), and therefore also the leading term and the constant \( C_{x,\gamma} \), which are independent of \( \lambda \), will in general depend in a highly non-uniform way on \( x \in M \). In fact, the description of \( e_\gamma(x, y, \lambda) \) reduces in essence to the study of oscillatory integrals of the form
\[
I_{x,y}(\mu) := \int_G \int_{S^*_x Y} e^{\mu \Phi_{x,y}(\omega, g)} a(x, y, \omega, g) d(S^*_x Y)(\omega) \, dg, \quad \mu > 0,
\]
with phase function
\[
\Phi_{x,y}(\omega, g) := \langle \kappa(x) - \kappa(g \cdot y), \omega \rangle,
\]
where \( (Y, \kappa) \) is a local chart on \( M \) and \( a \in C^\infty_c \) an amplitude that might depend on \( \mu \) and is such that \( (x, y, \omega, g) \in \text{supp } a \) implies \( x, g \cdot y \in Y \), while \( d(S^*_Y) \) and \( dg \) denote Liouville and Haar measure, respectively. Now, when trying to describe the asymptotic behaviour of \( I_{x,y}(\mu) \) as \( \mu \to +\infty \) uniformly in \( x \) via the stationary phase principle, one encounters the phenomenon that the critical set of \( \Phi_{x,y} \) changes abruptly its dimension when \( x \) passes through points of singular orbits, leading to a drastic change in the asymptotics of \( I_{x,y}(\mu) \). Such points are called caustics \cite{31}, and are ultimately responsible for the qualitatively very different asymptotic behaviour of the reduced spectral function.
as \( x \) approaches such points. A precise description of the asymptotics of the integrals \( [1.9] \) is given in Theorems \( 3.3 \) and Proposition \( 3.6 \).

Though the leading coefficient in the asymptotic formula \( [1.8] \) for \( e_\gamma(x,x,\lambda) \) is explicit, and has a clear geometric meaning, it does not unveil the caustic nature of \( e_\gamma(x,x,\lambda) \) when singular orbits are present, and blows up in an unknown way as \( x \) approaches such orbits. To obtain a precise description of this caustic behaviour it is necessary to examine the integrals \( [1.9] \) more carefully. For this, we shall rely on our recent results \( [22] \) on singular equivariant asymptotics obtained via resolution of singularities, from which we will be able to deduce a uniform description of the integrals \( I_{x,x}(\mu) \) and the behaviour of \( e_\gamma(x,x,\lambda) \) near singular orbits. More precisely, consider the stratification \( M = M(H_1) \cup \ldots \cup M(H_L) \) of \( M \) into orbit types, arranged in such a way that \( (H_i) \leq (H_j) \) implies \( i \geq j \), and let \( \Lambda \) be the maximal length that a maximal totally ordered subset of isotropy types can have. Write \( M_{\text{prin}} := M(H_L), M_{\text{except}}, \) and \( M_{\text{sing}} \) for the union of all orbits of principal, exceptional, and singular type, respectively, so that

\[
M = M_{\text{prin}} \cup M_{\text{except}} \cup M_{\text{sing}},
\]

and denote by \( \kappa := \dim G/H_L \) the dimension of an orbit of principal type. Then, by Theorem \( 7.7 \) one has for \( x \in M_{\text{prin}} \cup M_{\text{except}} \) and \( \lambda \to +\infty \) the singular equivariant local Weyl law

\[
\left| e_\gamma(x,x,\lambda) - \frac{d_\lambda \lambda^{n-\kappa}}{(2\pi)^{n-\kappa}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \ldots < i_N} \prod_{l=1}^N |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa} L^{0,0}_{i_1 \ldots i_N}(x,\gamma) \right|
\leq C_\gamma \lambda^{n-\kappa-1} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \ldots < i_N} \prod_{l=1}^N |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa-1},
\]

where the multiple sums run over all possible totally ordered subsets \( \{(H_{i_1}), \ldots, (H_{i_N})\} \) of singular isotropy types, the coefficients \( L^{0,0}_{i_1 \ldots i_N} \) are explicitly given and bounded functions in \( x \), and \( \tau_{i_l} = \tau_{i_l}(x) \in (-1,1) \) are desingularization parameters that arise in the resolution process satisfying \( |\tau_{i_l}| = \text{dist}(x, M(H_{i_l})) \), while \( C_\gamma > 0 \) is a constant independent of \( x \) and \( \lambda \). Thus, the combinatorial complexity of the underlying group action is reflected in the asymptotic shape of the equivariant spectral function. By integrating the asymptotic formul\( a [1.8] \) and \( [1.11] \) over \( x \in M \), one obtains for the equivariant counting function \( N_\gamma(\lambda) := \int_M e_\gamma(x,x,\lambda) \, dM(x) \) the equivariant Weyl law

\[
N_\gamma(\lambda) = \frac{d_\lambda |\pi_1| H_\infty}{(n-\kappa) \cdot (2\pi)^{n-\kappa}} \text{vol} \left[ \Omega \cap S^* M / G \right] \lambda^{\frac{n-\kappa-1}{m}} + O_\gamma(\lambda^{(n-\kappa-1)/m} \cdot (\log \lambda)^\Lambda).
\]

This was the main result of \( [22] \). Notice that in spite of the fact that the desingularization techniques developed there are necessary to establish the remainder estimate in \( [1.12] \), singular and exceptional orbits, being of measure zero, do not contribute to the equivariant Weyl law \( [1.12] \), and remain hidden. It is only in the stronger local Weyl laws \( [1.8] \) and \( [1.11] \) for the reduced spectral function that the whole orbit structure of the underlying group action becomes manifest.

As a major consequence, Theorems \( 4.3 \) and \( 7.7 \) lead to refined bounds for eigenfunctions. In the non-singular case, that is, when only principal and exceptional orbits are present, and consequently all \( G \)-orbits have the same dimension \( \kappa \), the obtained bounds are uniform in \( x \in M \), while in the singular case, they show that eigenfunctions tend to concentrate along lower dimensional orbits. Indeed, as in the non-equivariant case, the crucial bound for obtaining \( [1.8] \) is a bound for \( e_\gamma(x,x,\lambda+1) - e_\gamma(x,x,\lambda) \), which is equivalent to the non-uniform bound

\[
\sum_{\lambda_\gamma \in (\lambda,\lambda+1)} |e_\gamma(x)|^2 \leq C_{x,\gamma} \lambda^{\frac{n-\kappa-1}{m}}, \quad x \in M,
\]

see Corollary \( 4.6 \). From this one immediately deduces in the non-singular case the hybrid \( L^\infty \)-estimate in the eigenvalue and isotropic aspect

\[
\| (\chi_\lambda \circ \Pi_{\gamma}) u \|_{L^\infty(M)} \leq C_\gamma (1 + \lambda)^{\frac{n-\kappa-1}{2m}} \| u \|_{L^2(M)}, \quad u \in L^2(M),
\]
where $C_{\gamma} > 0$ is a constant independent of $\lambda$ satisfying the estimate
\begin{equation}
C_{\gamma} \ll \sqrt{d_{\gamma}} \sup_{l \leq [\kappa/2+1]} \|D^l \gamma\|_{\infty}, \tag{1.14}
\end{equation}
see Proposition~5.1 and (5.4). In particular, we obtain the hybrid equivariant bound for eigenfunctions
\[ \|u\|_{L^\infty(M)} \ll C_{\gamma} \lambda^{\frac{n-\kappa-1}{2m}}, \quad u \in \mathcal{E}_\lambda \cap L^2(M), \quad \|u\|_{L^2} = 1. \]
Note that if $n = \kappa + 1$, this bound reads $\|u\|_{\infty} \leq C_{\gamma}$. The proof of $L^p$-bounds is considerably more involved, since it no longer suffices to study the integrals $I_{x,q}(\mu)$ restricted to the diagonal. Instead, it is necessary to estimate their growth as $\mu \to +\infty$ in a neighborhood of the latter, for which we have to assume that the co-spheres $S^*_r M$ are strictly convex. Using complex interpolation techniques, we then prove in Theorem~5.4 the hybrid bounds in the eigenvalue and isotypic aspect
\[ \|(\chi_\lambda \circ \Pi_\gamma) u\|_{L^p(M)} \leq \begin{cases} C_{\gamma} \lambda^{\frac{\kappa-n-1}{m}} \|u\|_{L^2(M)}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C_{\gamma} \lambda^{\frac{n-\kappa}{m(2-q')}} \|u\|_{L^2(M)}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases} \]
where $\frac{1}{q} + \frac{1}{q'} = 1$, and $C_{\gamma}$ is as in (1.14). In particular, we have the hybrid equivariant bound
\[ \|u\|_{L^p(M)} \leq \begin{cases} C_{\gamma} \lambda^{\frac{\kappa-n-1}{m}}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C_{\gamma} \lambda^{\frac{n-\kappa}{m(2-q')}} & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases} \]
for any eigenfunction of $P$ belonging to $u \in \mathcal{E}_\lambda \cap L^2(M)$ and satisfying $\|u\|_{L^2} = 1$, provided that $G$ acts on $M$ with orbits of the same dimension $\kappa$. Nevertheless, the $L^p$-bounds above cannot hold when singular orbits are present, and the situation in this case is described by Corollary~7.9, by which one has the uniform bound
\begin{equation}
\sum_{\lambda \in (\Lambda, \Lambda+1], \ e_j \in L^2_M(M)} |e_j(x)|^2 \leq \begin{cases} C \lambda^{\frac{n-\kappa}{m}}, & x \in M_{\text{sing}}, \\ C_{\gamma} \lambda^{\frac{n-\kappa}{m} - \frac{1}{m} \sum_{N=1}^{\Lambda-1} \sum_{i_1 \cdots < i_N} N \prod_{l=1}^N |r_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}, & x \in M - M_{\text{sing}}, \end{cases} \tag{1.15}
\end{equation}
for a constant $C_{\gamma} > 0$ independent of $x$ and $\lambda$, and $C > 0$ even independent of $\gamma$. In comparison with the bound (1.13), where the dependency of the constant $C_{x,\gamma}$ on $x$ remains unspecified, the bound (1.15) gives a rather precise description of the growth of eigenfunctions near singular orbits.

To illustrate our results, consider the classical case where $M = S^2$, and $G = SO(2)$ acts on $M$ by rotations around the symmetry axis through the poles. The eigenfunctions of the Laplace-Beltrami operator on $M = S^2$ are given by the spherical functions
\[ Y_{k,m}(\phi, \theta) = \sqrt{\frac{2k+1}{4\pi}} \frac{(k-m)!}{(k+m)!} P_{k,m}(\cos \theta) e^{im\phi}, \quad 0 \leq \phi < 2\pi, 0 \leq \theta < \pi, \]
with corresponding eigenvalues $k(k+1)$, where $k \in \mathbb{N}$, $|m| \leq k$, and $P_{k,m}$ are the associated Legendre polynomials. Furthermore, with the identification $SO(2) \simeq \mathbb{Z}$ the spherical function $Y_{k,m}$ belongs to the isotypic component $L^2_M(S^2)$. The Legendre polynomials $P_k(\cos \theta) := P_{k,0}(\cos \theta)$ satisfy $P_k(1) = 1$, and for $k \sin \theta > 1$ obey the classical asymptotics
\[ P_k(\cos \theta) = \frac{2}{\pi k \sin \theta} \cos \left( \left( k + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right) + O \left( \frac{1}{(k \sin \theta)^{3/2}} \right), \quad \theta \in (0, \pi), \]
where the remainder is uniform in $\theta$ on any interval $[\varepsilon, \pi - \varepsilon]$ with $0 < \varepsilon$ small, see [12, p. 303]. From this one concludes in the limit $k \to \infty$ that
\begin{equation}
|Y_{k,0}(\phi, \theta)|^2 = \frac{2k+1}{4\pi} |P_k(\cos \theta)|^2 \approx \begin{cases} k, & \theta = 0, \pi, \\ \frac{1}{\sin \theta}, & \theta \in (0, \pi), \end{cases} \tag{1.16}
\end{equation}
Thus, as $k \to \infty$ the eigenfunctions $Y_{k,0}$ concentrate on the poles, which are precisely the fixed points of the SO(2)-action on $S^2$, and maximize the bound (1.5). The bounds (1.15) are precisely of the type (1.16), and provide an interpretation of the latter in terms of the caustic behaviour of the equivariant spectral function, compare also Example 7.10. Furthermore, as discussed in Section 8 the bounds (1.16) show that the point-wise bounds (1.15) are sharp in the spectral parameter.

Collecting everything, the main conclusions to be drawn from this work are that

- asymptotics for the equivariant spectral function of an invariant elliptic operator are determined by the orbit structure of the underlying group action;
- symmetries lead to refined $L^p$-estimates for eigenfunctions of invariant elliptic operators, provided that all orbits of the underlying group action have the same dimension;
- lower dimensional orbits are responsible for concentration of eigenfunctions, and this concentration is due to the caustic behaviour of the equivariant spectral function. In other words, the orbit structure is reflected in the shape of eigenfunctions.

We would like to close this introduction by making some final comments. In the particular case that $\gamma = \gamma_{\text{triv}}$ is the trivial representation, (1.8) actually implies in passing a generalized Kuznecov sum formula for periods of $G$-orbits, see Corollary 4.7 which generalizes previous results of Zelditch [36] on periods of closed geodesics. In case that $G$ acts with finite isotropy groups on $M$, that is, when $\tilde{M} := M/G$ is an orbifold, an asymptotic formula for the spectral function of an elliptic operator on $\tilde{M}$ was given by Stanhope and Uribe in [32], and we recover their result in Corollary 4.8. If $G = \{e\}$, our results just reduce to the classical ones. Finally, let us mention that one can deduce also bounds for the spectral function $e(x, y, \lambda)$ of an elliptic operator of the form

$$|e(x, y, \lambda)| \leq C \cdot \lambda^{\frac{n}{m}}, \quad x, y \in M,$$

by using heat-equation or, equivalently, zeta-function methods. Nevertheless, bounds of the form (1.2), which are necessary for proving the local Weyl law (1.1), are not accessible via these techniques, and can only be obtained within the theory of Fourier integral operators, see [13] and [27] Sections 15 and 21, in particular Problem 15.1 and Lemma 21.4. In the equivariant case, bounds of the form

$$|e_\gamma(x, y, \lambda)| \leq C_\gamma \cdot \lambda^{\frac{n-k}{m}}, \quad x, y \in M,$$

could in principle be deduced from work of Donnelly [7] and Br"uning-Heintze [3], at least when $G$ acts on $M$ with orbits of the same dimension $\kappa$. But they would not be sufficient to imply our results, and the desingularization techniques developed in [22] are necessary in order to describe the precise nature of the reduced spectral function of an invariant elliptic operator.

$L^p$-bounds for spectral clusters for elliptic second-order differential operators on 2-dimensional compact manifolds with boundary and either Dirichlet or Neumann conditions were shown in [28], while manifolds with maximal eigenfunction growth were studied in [31]. For locally symmetric spaces of higher rank, improved $L^p$-bounds have been shown by Sarnak and Marshall in [25, 21]. They also derived corresponding subconvex $L^\infty$-bounds based on the presence of an additional family of commuting operators given by the Hecke algebra [16, 20]. In a forthcoming article [24] we shall extend their results to compact arithmetic quotients of semisimple algebraic groups relying on the asymptotic description of the integrals (1.9) given in Theorems 3.3 and 3.4. For a general overview on eigenfunctions on Riemannian manifolds, we refer to the survey articles [38, 37].

Through the whole document, the notation $O(\mu^k), k \in \mathbb{R} \cup \{\pm \infty\}$, will mean an upper bound of the form $C\mu^k$ with a constant $C > 0$ that is uniform in all relevant variables, while $O_N(\mu^k)$ will denote an upper bound of the form $C_N \mu^k$ with a constant $C_N > 0$ that depends on the indicated variable $N$. In the same way, we shall write $a \ll b$ for two real numbers $a$ and $b$, if there exists a constant $C_N > 0$ depending only on $N$ such that $|a| \leq C_N b$, and similarly $a \ll b$, if the bound is uniform in all relevant
variables. Finally, \( \mathbb{N} \) will denote the set of natural numbers 0, 1, 2, 3, \ldots.

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2. The reduced spectral function of an invariant elliptic operator

Let \( M \) be a closed connected Riemannian manifold of dimension \( n \) with Riemannian volume density \( dM \), and \( P_0 \) an elliptic classical pseudodifferential operator on \( M \) of degree \( m \) which is positive and symmetric. Let further \( T^*M \) be the cotangent bundle of \( M \). The principal symbol \( p(x, \xi) \) of \( P_0 \) constitutes a strictly positive function on \( T^*M \setminus \{0\} \), and is homogeneous in \( \xi \) of degree \( m \). Denote by \( P \) the unique self-adjoint extension of \( P_0 \), its domain being the \( m \)-th Sobolev space \( H^m(M) \), and let \( \{e_j\}_{j \geq 0} \) be an orthonormal basis of \( L^2(M) \) consisting of eigenfunctions of \( P \) with eigenvalues \( \{\lambda_j\}_{j \geq 0} \) repeated according to their multiplicity. In order to deal with a hyperbolic problem, consider the \( m \)-th root \( Q := \sqrt[2m]{P} \) of \( P \) given by the spectral theorem. It is well known that \( Q \) is a classical pseudodifferential operator of order \( 1 \) with principal symbol \( q(x, \xi) := \sqrt[2m]{p(x, \xi)} \) and domain \( H^1(M) \).

Again, \( Q \) has discrete spectrum, and its eigenvalues are given by \( \mu_j := \sqrt[2m]{\lambda_j} \). The spectral function \( e(x, y, \lambda) \) of \( P \) can then be described by studying the spectral function of \( Q \), which in terms of the basis \( \{e_j\} \) is given by

\[
e(x, y, \mu) := \sum_{\mu_j \leq \mu} e_j(x) \overline{e_j(y)}, \quad \mu \in \mathbb{R},
\]

and belongs to \( C^\infty(M \times M) \) as a function of \( x \) and \( y \). Let \( \chi_\mu \) be the spectral projection onto the sum of eigenspaces of \( Q \) with eigenvalues in the interval \( (\mu, \mu + 1] \), and denote its Schwartz kernel by \( \chi_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu) \). To obtain an asymptotic description of the spectral function of \( Q \), one first derives a description of \( \chi_\mu(x, y) \) by approximating \( \chi_\mu \) by Fourier integral operators. To do so, let \( \varphi \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+) \) be such that \( \varphi(0) = 1 \) and \( \text{supp } \hat{\varphi} \in (-\delta/2, \delta/2) \) for a given \( \delta > 0 \), and define the approximate spectral projection operator

\[
\overline{\chi}_\mu u := \sum_{j=0}^\infty \varphi(\mu - \mu_j) E_j u, \quad u \in L^2(M),
\]

where \( E_j \) denotes the orthogonal projection onto the subspace spanned by \( e_j \). Clearly, \( K_{\overline{\chi}_\mu}(x, y) := \sum_{j=0}^\infty \varphi(\mu - \mu_j) e_j(x) \overline{e_j(y)} \in C^\infty(M \times M) \) constitutes the kernel of \( \overline{\chi}_\mu \). Now, notice that for \( \mu, \tau \in \mathbb{R} \) one has

\[
\varphi(\mu - \tau) = \frac{1}{2\pi} \int \hat{\varphi}(t) e^{-it\tau} e^{it\mu} \, dt,
\]

where \( \hat{\varphi}(t) \) denotes the Fourier transform of \( \varphi \), so that for \( u \in L^2(M) \) we obtain

\[
\overline{\chi}_\mu u = \frac{1}{2\pi} \sum_{j=0}^\infty \int \hat{\varphi}(t) e^{it\mu_j} e^{-it\mu_j} \, dt \, E_j u = \frac{1}{2\pi} \int \hat{\varphi}(t) e^{it\mu} U(t) u \, dt,
\]

where \( U(t) \) denotes the one-parameter group of unitary operators in \( L^2(M) \)

\[
U(t) = \int e^{-it\mu} dE^Q_\mu = e^{-itQ}, \quad t \in \mathbb{R},
\]

given by the Fourier transform of the spectral measure, \( \{E^Q_\mu\} \) being a spectral resolution of \( Q \). The central result of Hörmander [13] then says that \( U(t) = e^{-itQ} : L^2(M) \to L^2(M) \) can be approximated by Fourier integral operators, yielding an asymptotic formula for the kernels of \( \overline{\chi}_\mu \) and \( \chi_\mu \), and finally for the spectral function of \( Q \) and \( P \).

Let us now come back to the problem described in the introduction, and assume that \( M \) carries an effective and isometric action of a compact Lie group \( G \). Let \( P \) commute with the left-regular
A reducible representation \((\pi, L^2(M))\) of \(G\). Consider the Peter-Weyl decomposition (1.7) of \(L^2(M)\), and let \(\Pi_\gamma\) be the projection onto the isotypic component belonging to \(\gamma \in \hat{G}\), which is given by the Bochner integral

\[
\Pi_\gamma = d_\gamma \int_G \overline{\gamma(g)} \pi(g) \, dG(g),
\]

where \(d_\gamma\) is the dimension of an unitary irreducible representation of class \(\gamma\), and \(d_G(g) \equiv dg\) Haar measure on \(G\), which we assume to be normalized such that \(\text{vol} \, G = 1\). If \(G\) is finite, \(d_G\) is simply the counting measure. In addition, let us suppose that the orthonormal basis \(\{e_j\}_{j \geq 0}\) is compatible with the decomposition (1.7) in the sense that each vector \(e_j\) is contained in some isotypic component \(L^2_\gamma(M)\). In order to describe the spectral function of the operator \(Q_\gamma := \Pi_\gamma \circ Q \circ \Pi_\gamma = Q \circ \Pi_\gamma = \Pi_\gamma \circ Q\) given by

\[
e_{\gamma}(x, y, \mu) := \sum_{\mu_j \leq \mu, \, e_j \in L^2_\gamma(M)} e_j(x) e_j(y),
\]

we consider the composition

\[
(\chi_{\mu} \circ \Pi_\gamma) u = \sum_{\mu_j \in (\mu, \mu+1)} (E_j \circ \Pi_\gamma) u = \sum_{\mu_j \in (\mu, \mu+1), \, e_j \in L^2_\gamma(M)} E_j u, \quad u \in L^2(M),
\]

with kernel \(K_{\chi_{\mu} \circ \Pi_\gamma}(x, y) = e_\gamma(x, y, \lambda + 1) - e_\gamma(x, y, \lambda)\), together with the corresponding equivariant approximate spectral projection

\[
(\tilde{\chi}_{\mu} \circ \Pi_\gamma) u = \sum_{j \geq 0, \, e_j \in L^2_\gamma(M)} \phi(\mu - \mu_j) E_j u = \frac{d_\gamma}{2\pi} \int G \int \phi(t)e^{i\mu_\gamma \overline{\gamma(g)} (U(t) \circ \pi(g))} u \, dt \, dg.
\]

Its kernel can be written as

\[
K_{\tilde{\chi}_{\mu} \circ \Pi_\gamma}(x, y) := \sum_{j \geq 0, \, e_j \in L^2_\gamma(M)} \phi(\mu - \mu_j) e_j(x) e_j(y) \in C^\infty(M \times M).
\]

Put \(m_\gamma(\mu_j) := d_\gamma \cdot \text{mult}_\gamma(\mu_j) / \dim \mathcal{E}_{\mu_j}\), where \(\text{mult}_\gamma(\mu_j)\) denotes the multiplicity of an unitary irreducible representation of class \(\gamma\) in the eigenspace \(\mathcal{E}_{\mu_j}\). In (22), an asymptotic formula for

\[
\text{tr} (\tilde{\chi}_{\mu} \circ \Pi_\gamma) = \int_M K_{\tilde{\chi}_{\mu} \circ \Pi_\gamma}(x, x) \, dM(x) = \sum_{j=0}^\infty m_\gamma(\mu_j) \phi(\mu - \mu_j)
\]

was given in order to describe the behaviour of the equivariant counting function as the eigenvalues become large, while now we are interested in the spectral function itself, which makes it necessary to derive asymptotics for the restriction of \(K_{\tilde{\chi}_{\mu} \circ \Pi_\gamma}\) to the diagonal, or even to a neighborhood of it, and is therefore considerably more subtle than computing the trace.

As mentioned before, one can approximate \(U(t)\) by means of Fourier integral operators. More precisely, let \(\{(\kappa_s, Y_s)\}_{s \in \mathbb{R}}, \, \kappa_s : Y_s \xrightarrow{\sim} \tilde{Y}_s \subset \mathbb{R}^n\), be an atlas for \(M\), \(\{f_s\}\) a corresponding partition of unity and \(\hat{v}(\eta) := \mathcal{F}(v)(\eta) = \int_{\mathbb{R}^n} e^{-i(s - \eta) \cdot \nu(\tilde{y})} \tilde{v}(\tilde{y}) \, d\tilde{y}\) the Fourier transform of \(v \in C^\infty_c(\tilde{Y}_s)\). Write \(d\eta := d\eta/(2\pi)^n\), and introduce the operator

\[
[\tilde{U}_s(t)v](\tilde{x}) := \int_{\mathbb{R}^n} e^{i\psi_s(t, x, \eta)} a_s(t, \tilde{x}, \eta) \hat{v}(\eta) \, d\eta
\]

on \(\tilde{Y}_s\), where \(a_s \in S^0_{ph}\) is a classical polyhomogeneous symbol satisfying \(a_s(0, \tilde{x}, \eta) = 1\) and \(\psi_s\) the defining phase function given as the solution of the Hamilton-Jacobi equation

\[
\frac{\partial \psi_s}{\partial t} + q(x, \frac{\partial \psi_s}{\partial \tilde{x}}) = 0, \quad \psi_s(0, \tilde{x}, \eta) = (\tilde{x}, \eta),
\]

see [15, p. 254]. Let us remark that \(\psi_s\) is homogeneous in \(\eta\) of degree 1, so that Taylor expansion for small \(t\) gives

\[
\psi_s(t, \tilde{x}, \eta) = \psi_s(0, \tilde{x}, \eta) + t \frac{\partial \psi_s}{\partial t}(0, \tilde{x}, \eta) + O(t^2 |\eta|) = (\tilde{x}, \eta) - tq_s(\tilde{x}, \eta) + O(t^2 |\eta|),
\]
where we wrote \( q(\tilde{x}, \eta) := q(\kappa^{-1}_i(\tilde{x}), \eta) \). In other words, there exists a smooth function \( \zeta_\eta \) which is homogeneous in \( \eta \) of degree 1 and satisfies

\[
\psi_i(t, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle - t\zeta_\eta(t, \tilde{x}, \eta), \quad \zeta_\eta(0, \tilde{x}, \eta) = q(\tilde{x}, \eta).
\]

Let now \( \bar{U}_i(t) u := [\bar{U}_i(t)(u \circ \kappa_i^{-1})] \circ \kappa_i, \ u \in \mathcal{C}_c^\infty(Y_i) \). Consider further test functions \( \bar{f}_i \in \mathcal{C}_c^\infty(Y_i) \) satisfying \( \bar{f}_i \equiv 1 \) on \( \text{supp} \ f_i \), and define

\[
\bar{U}(t) := \sum_i F_i \bar{U}_i(t) \bar{f}_i,
\]

where \( F_i, \bar{f}_i \) denote the multiplication operators corresponding to \( f_i \) and \( \bar{f}_i \), respectively. Then Hörmander showed that for small \(|t|\)

\[
R(t) := U(t) - \bar{U}(t) \text{ is an operator with smooth kernel},
\]

compare [10, p. 134] and [27, Theorem 20.1]; in particular, the kernel \( R_i(x,y) \) of \( R(t) \) is smooth in \( t \).

We now have the following

**Proposition 2.1.** Let \( \delta > 0 \) be sufficiently small and \( x, y \in M \). Then, as \( \mu \to +\infty \),

\[
K_{\tilde{\chi}_\mu \circ \Pi_y}(x, y) = \frac{\mu^n d_\gamma}{(2\pi)^{n+1}} \int_0^{+\infty} \int_G \int_{\mathbb{R}^n} e^{it\mu[1-\zeta_\mu(t, \kappa_i(x), \eta)]} e^{it\mu[(\kappa_i(x) - \kappa_i(g \cdot y), \eta)]} \hat{\phi}(t) \gamma(g) f_i(x) \cdot a_i(t, \kappa_i(x, \eta)) J_i(g, y) d\eta d\mu dt,
\]

up to terms of order \( O(\mu^{-\infty}) \) which are uniform in \( x \) and \( y \), where \( 0 \leq \alpha \in \mathbb{C}_c^\infty(1/2, 3/2) \) is a test function such that \( \alpha \equiv 1 \) in a neighborhood of 1, \( J_i(g, y) \) is a Jacobian, and \( d\mu \) denotes Lebesgue measure on \( \mathbb{R}^n \). On the other hand, \( K_{\tilde{\chi}_\mu \circ \Pi_y}(x, y) \) is rapidly decaying as \( \mu \to -\infty \).

**Proof.** To obtain an explicit expression for the kernel of \( \tilde{\chi}_\mu \circ \Pi_y \) let \( u \in \mathcal{C}_c^\infty(M) \), and notice that

\[
F_i \bar{U}_i(t) \bar{f}_i u(x) = f_i(x) [\bar{U}_i(t)(\bar{f}_i u \circ \kappa_i^{-1})] \circ \kappa_i(x)
\]

\[
= f_i(x) \int_{\mathbb{R}^n} e^{i\psi_i(t, \kappa_i(x), \eta)} a_i(t, \kappa_i(x), \eta)(\bar{f}_i u \circ \kappa_i^{-1})(\eta) d\eta
\]

\[
= \int_{Y_i} \int_{\mathbb{R}^n} f_i(x) e^{i\psi_i(t, \kappa_i(x), \eta) - \langle g, \eta \rangle} a_i(t, \kappa_i(x), \eta) \bar{f}_i u(\kappa_i^{-1}(\eta)) d\eta d\eta
\]

\[
= \int_{Y_i} \left[ \int_{\mathbb{R}^n} e^{i\psi_i(t, \kappa_i(x), \eta) - \langle \kappa_i(y), \eta \rangle} a_i(t, \kappa_i(x), \eta) d\eta f_i(x) \bar{f}_i(y) (\beta^{-1}_i \circ \kappa_i)(y) \right] u(y) dM(y),
\]

where we wrote \( (\kappa_i^{-1})^* dM = \beta_i d\tilde{\mu} \). The last two expressions are oscillatory integrals with suitable regularizations. With \( (2.3) \) and \( (2.6) \) we therefore obtain for \( K_{\tilde{\chi}_\mu \circ \Pi_y}(x, y) \) the expression

\[
\frac{d_\gamma}{(2\pi)^{n+1}} \sum_i \int_0^{+\infty} \int_G \int_{\mathbb{R}^n} \tilde{\phi}(t) e^{it\mu \gamma(g)} f_i(x) e^{i\psi_i(t, \kappa_i(x), \eta) - \langle \kappa_i(y), \eta \rangle} a_i(t, \kappa_i(x), \eta) \cdot \bar{f}_i(g \cdot y) J_i(g, y) d\eta dt + O(\mu^{-\infty}),
\]

since

\[
\frac{1}{2\pi} \int_G \int_{-\infty}^{+\infty} \tilde{\phi}(t) e^{it\mu} R_i(x, g \cdot y) dt \gamma(g) J_i(g, y) d\mu = \int_G F^{-1}(\tilde{\phi}(\bullet) R_\bullet(x, g \cdot y))(\mu) \gamma(g) J_i(g, y) d\mu,
\]

and \( F^{-1}(\tilde{\phi}(\bullet) R_\bullet(x, g \cdot y)) \) is rapidly falling in \( \mu \); in particular, \( O(\mu^{-\infty}) \) is uniform in \( x, y \). We are interested in the asymptotic behaviour of \( K_{\tilde{\chi}_\mu \circ \Pi_y}(x, y) \) as \( \mu \to \pm\infty \). In order to study it by means of the stationary phase theorem, we define

\[
\mathcal{G}(\tau, \tilde{x}, \eta) := \int_{-\infty}^{+\infty} e^{it\tau} \bar{\phi}(t) a_i(t, \tilde{x}, \eta) e^{O(t^2|\eta|)} dt,
\]
where \(O(t^2|\eta|)\) denotes the remainder in \(t^4\). Clearly, \(G(\tau, \tilde{x}, \eta)\) is rapidly decaying as a function in \(\tau\). On the other hand, there must exist a constant \(C > 0\) such that
\[
C|\eta| \geq q_i(\tilde{x}, \eta) \geq \frac{1}{C}|\eta|
\]
\(\forall \tilde{x} \in \tilde{Y}_s, \eta \in \mathbb{R}^n\),
which implies that for fixed \(\mu\), \(G(\mu - q_i(\tilde{x}, \eta), \tilde{x}, \eta)\) is rapidly decaying in \(\eta\). This yields a regularization of the oscillatory integral above, and we obtain
\[
K_{\tilde{x}, \eta}(x, y) = \frac{d}{(2\pi)^{n+1}} \sum_i \int_{G} \int_{\mathbb{R}^n} e^{i(\kappa_i(x) - \kappa_i(y, \eta))} \gamma(g) f_i(x)
\]
\[
\cdot G(\mu - q(x, \eta), \kappa_i(x), \eta) f_i(g \cdot y) J_i(g, y) d\eta d\gamma + O(|\mu|^{-\infty}).
\]
But even more is true. \(K_{\tilde{x}, \eta}(x, y)\) is rapidly decreasing as \(\mu \to -\infty\), reflecting the positivity of the spectrum. Furthermore, assume that \(|1 - q_i(\tilde{x}, \eta/\mu)| \geq \text{const} > 0\). Then
\[
|G(\mu - q_i(\tilde{x}, \eta), \tilde{x}, \eta)| \leq C_{N+M} \frac{1}{|\mu|^N} \frac{1}{|1 - q_i(\tilde{x}, \eta/\mu)|^N} \frac{1}{|\mu - q_i(\tilde{x}, \eta)|^M}
\]
\[
\leq C'_{N+M} \frac{1}{|\mu|^N} \frac{1}{|\mu - q_i(\tilde{x}, \eta)|^M}
\]
for arbitrary \(N, M \in \mathbb{N}\) and suitable constants. Let therefore \(\alpha \in C^\infty_c(1/2, 3/2)\) be as indicated, so that
\[
1 - \alpha(q_i(\tilde{x}, \eta/\mu)) \neq 0 \implies |1 - q_i(\tilde{x}, \eta/\mu)| \geq C > 0
\]
for a constant depending only on \(\alpha\). Substituting \(\eta = \mu \gamma\), we can re-write \(K_{\tilde{x}, \eta}(x, y)\) as
\[
K_{\tilde{x}, \eta}(x, y) = \frac{|\mu|^n d \gamma}{(2\pi)^{n+1}} \sum_i \int_{G} \int_{\mathbb{R}^n} e^{i\mu [\psi_i(t, \kappa_i(x), \eta) - (\kappa_i(g \cdot \eta))]} \gamma(t) \gamma(g) f_i(x)
\]
\[
\cdot a_i(t, \kappa_i(x), \mu \gamma) f_i(g \cdot y) \alpha(q(x, \eta)) J_i(g, y) d\eta d\gamma + O(|\mu|^{-\infty}),
\]
where all integrals are absolutely convergent, and the remainder is uniform in \(x, y\). The proposition now follows with \(2.5\).

Since \(\zeta_i(0, \tilde{x}, \eta) = q_i(\tilde{x}, \eta)\), there exists a constant \(C > 0\) such that for sufficiently small \(t \in (-\delta/2, \delta/2)\)
\[
C|\eta| \geq \zeta_i(t, \tilde{x}, \eta) \geq \frac{1}{C}|\eta|
\]
\(\forall \tilde{x} \in \tilde{Y}_s, \eta \in \mathbb{R}^n\).

We can therefore introduce in \(\mathbb{R}^n \setminus \{0\}\) the coordinates
\[
\eta = R \omega, \quad R > 0, \quad \zeta_i(t, \kappa_i(x), \omega_1) = 1.
\]
Indeed, since \(\zeta_i(t, \kappa_i(x), \eta)\) is homogeneous of degree 1 in \(\eta\), its derivative in radial direction reads
\[
\lim_{s \to 0} s^{-1}(R + s - R)\zeta_i(t, \kappa_i(x), \omega_1) = 1,
\]
so that for all \(\eta = R \omega_1\) we have
\[
(2.7) \quad \langle \text{grad}_\eta \zeta_i(t, \tilde{x}, \eta), \eta \rangle = R > 0.
\]

Consequently, the Jacobian of the coordinate change \(\eta = R \omega_1\) does not vanish. Re-writing the expression for the kernel of \(X_{\mu} \circ \Pi_s\) in Proposition \(2.1\) in terms of the coordinates \(\eta = R \omega_1\) we obtain
\[
K_{\tilde{x}, \eta}(x, y) = \frac{|\mu|^n d \gamma}{(2\pi)^{n+1}} \sum_i \int_{G} \int_{\mathbb{R}^n} e^{i\mu [\psi_i(t-R\omega)]} \int_{\Sigma_{i,x}} e^{i\mu [\kappa_i(x) - \kappa_i(g \cdot \omega)]} \gamma(t) \gamma(g) f_i(x)
\]
\[
\cdot a_i(t, \kappa_i(x), \mu \omega) f_i(g \cdot y) \alpha(q(x, \omega)) J_i(g, y) d\gamma dR d\omega dR dt
\]
up to terms of order \(O(\mu^{-\infty})\) which are uniform in \(x, y\) and \(\omega\), where we set
\[
(2.9) \quad \Sigma_{i,x} := \{ \omega \in \mathbb{R}^n \mid \zeta_i(t, \kappa_i(x), \omega) = R \}.
\]
Here \(d\Sigma_{i,x}(\omega)\) denotes the quotient of Lebesgue measure in \(\mathbb{R}^n\) by Lebesgue measure in \(\mathbb{R}\) with respect to \(\zeta_i(t, \tilde{x}, \omega)\). Note that for sufficiently small \(\delta > 0\) we can assume that the \(R\)-integration is over a
compact set. Furthermore, $R$ and $t$ are close to 1 and 0, respectively. To describe the asymptotic behaviour of $K_{\mu,\omega,\nu}(x, y)$ as $\mu \to +\infty$, we shall now first apply the stationary phase theorem to the integral over $R$ and $t$, and then to the integral over $G \times \Sigma_x$.

**Corollary 2.2.** Let $\mu \geq 1$, $x, y \in M$, and with the notation of Proposition 2.1 set

$$I^\gamma_\mu(R, t, x, y) := \int_G \int_{\Sigma_x} e^{i\mu \Phi_x(y,\omega, g)} \varphi(t) \gamma(g) f_i(x),$$

(3.1)

as in (2.8) and the classical stationary phase theorem [10, Proposition 2.3].

**Proof.** Since $(R, t) = (1, 0)$ is the only critical point of $t - R t$, the assertion follows immediately from (2.8) and the classical stationary phase theorem [10, Proposition 2.3].

Thus, we are left with the task of describing the asymptotics of the oscillatory integrals $I^\gamma_\mu(R, t, x, y)$ as $\mu \to +\infty$, which will occupy us in the next sections.

3. Equivariant asymptotics of oscillatory integrals

Let the notation be as in the previous section. As we have seen there, the question of describing the spectral function in the equivariant setting reduces to the study of oscillatory integrals of the form

$$I_{x,y}(\mu) := \int_G \int_{\Sigma_x} e^{i\mu \Phi_{x,y}(\omega, g)} a(x, y, \omega, g) d\Sigma_x(\omega) d\mu,$$

(3.2)

with $\Sigma_x$ as in (2.9) and phase function

$$\Phi_{x,y}(\omega, g) := \langle \kappa(x) - \kappa(g \cdot y), \omega \rangle,$$

where we have skipped the index $\iota$ for simplicity of notation, and $a \in C_0^\infty$ is an amplitude that might depend on $\mu$ and other parameters such that $(x, y, \omega, g) \in \text{supp } a$ implies $x, g \cdot y \in Y$. In what follows, we shall also write

$$I_x(\mu) := I_{x,x}(\mu), \quad \Phi_x := \Phi_{x,x}.$$

The asymptotic behaviour of these integrals is related to that of oscillatory integrals of the form

$$I(\mu) := \int_G \int_{T^* Y} e^{i\mu \Phi(x,\eta, g)} a(x, \eta, g) d(T^* Y)(x, \eta) d\mu,$$

(3.3)

with phase function

$$\Phi(x,\eta, g) := \langle \kappa(x) - \kappa(g \cdot x), \eta \rangle,$$

(3.4)

and suitable amplitude $a$. Let us assume in the following that $G$ is a continuous group. Asymptotics for the integrals (3.3) were given in [22] using the stationary phase principle, and we will rely on these results in parts to perform a similar analysis for the integrals $I_{x,y}(\mu)$. Write $\kappa(x) = (\vec{x}_1, \ldots, \vec{x}_n)$ so that the canonical local trivialization of $T^* Y$ reads

$$Y \times \mathbb{R}^n \ni (x, \eta) \equiv \sum_{k=1}^n \eta_k (d\vec{x}_k)_x \in T^*_x Y.$$
With respect to this trivialization, we shall identify \( \Sigma^{R,t}_x \) with a subset in \( T^*_xY \) for eventually different \( x \) and \( x', \) if convenient. Let \( \Omega := J^{-1}(\{0\}) \) be the zero level set of the momentum map \( J : T^*M \to \mathfrak{g}^* \) of the underlying Hamiltonian \( G \)-action on \( T^*M. \) Since
\[
(x, \eta) \in \Omega \cap T^*_xM \iff (x, \eta) \in \text{Ann}(T_x(G \cdot x)),
\]
where \( \text{Ann}(V_x) \subset T^*_xM \) denotes the annihilator of a vector subspace \( V_x \subset T_xM, \) a simple computation shows that the critical set of \( \Phi \) is given by
\[
\text{Crit} \Phi = \{(x, \eta, g) \in T^*Y \times G \mid d(\Phi)_{(x,\eta,g)} = 0\} = \{(x, \eta, g) \in (\Omega \cap T^*Y) \times G \mid g \in G(x,\eta)\}.
\]
In what follows, we shall compute the critical set of the phase function \( \Phi \), which is much more involved. Let \( \mathcal{O}_x := G \cdot x \) denote the \( G \)-orbit and \( G_x := \{g \in G \mid g \cdot x = x\} \) the stabilizer or isotropy group of a point \( x \in M. \) Throughout the paper, we shall assume that
\[
\dim \mathcal{O}_x \leq n - 1 \quad \text{for all } x \in M.
\]
Let further \( N_g \mathcal{O}_x \) be the normal space to the orbit \( \mathcal{O}_x \) at a point \( y \in \mathcal{O}_x, \) which can be identified with \( \text{Ann}(T_y \mathcal{O}_x) \) via the underlying Riemannian metric. With \( M_{\text{prin}}, M_{\text{except}}, \) and \( M_{\text{sing}} \) as in (1.10) we now have the following

**Lemma 3.1.** Let \( x \in Y, \mathcal{O}_y \cap Y \neq \emptyset, \) and
\[
\text{Crit} \Phi_{x,y} := \left\{ (\omega, g) \in \Sigma^{R,t}_x \times \{g \in G \mid g \cdot y \in Y\} \mid d(\Phi_{x,y})_{(\omega,g)} = 0 \right\}
\]
be the critical set of \( \Phi_{x,y}. \)

(a) If \( y \in \mathcal{O}_x, \) the set \( \text{Crit} \Phi_{x,y} \) is clean and given by the smooth submanifold
\[
\mathcal{C}_{x,y} := \left\{ (\omega, g) \mid (g \cdot y, \omega) \in \Omega, x = g \cdot y \right\}
\]
of codimension \( 2 \dim \mathcal{O}_x. \)

(b) If \( y \notin \mathcal{O}_x, \)
\[
\text{Crit} \Phi_{x,y} = \left\{ (\omega, g) \mid (g \cdot y, \omega) \in \Omega, x = g \cdot y \in N_{\omega,\Sigma^{R,t}_x}\right\};
\]
Furthermore, assume that \( G \) acts on \( M \) with orbits of the same dimension \( \kappa, \) that is, \( M = M_{\text{prin}} \cup M_{\text{except}}, \) and that the co-spheres \( S^*_xM \) are strictly convex. Then, either \( \text{Crit} \Phi_{x,y} \) is empty, or, choosing \( Y \) sufficiently small, \( \text{Crit} \Phi_{x,y} \) is locally diffeomorphic to \( G_y, \) clean, and of codimension \( n - 1 - \kappa. \)

(c) In case that \( x \in Y \cap M_{\text{prin}} \) one has
\[
\mathcal{C}_{x,x} = \text{Crit} \Phi \cap (\Sigma^{R,t}_x \times G),
\]
a transversal intersection. In particular \( \mathcal{C}_{x,x} \) is a smooth submanifold of codimension \( 2\kappa. \)

**Proof.** Consider a local parametrization
\[
F : \mathbb{R}^{n-1} \supset W \longrightarrow \Sigma^{R,t}_x \subset \mathbb{R}^n, \quad \alpha \mapsto F(\alpha) = \omega,
\]
of the hyperurface \( \Sigma^{R,t}_x, \) where \( W \) denotes an open subset. Differentiating \( \Phi_{x,y} \) with respect to \( \alpha \) and setting the derivatives to zero gives the conditions \( \langle \kappa(x) - \kappa(g \cdot y), \partial F/\partial \alpha_i \rangle = 0 \) for \( i = 1, \ldots, n - 1, \) implying that \( \kappa(x) - \kappa(g \cdot y) \) must be normal to \( \Sigma^{R,t}_x \) at \( \omega. \) On the other side, the derivatives of \( \Phi_{x,y} \) with respect to \( g \) read \( \sum_{k=1}^n \omega_k (d\tilde{\beta}_k)_g, \) where \( \{X_1, \ldots, X_n\} \) denotes a basis of \( \mathfrak{g} \) and \( \{\tilde{X}_1, \ldots, \tilde{X}_n\} \) are the corresponding fundamental vector fields on \( M. \) Setting them to zero yields \( (g \cdot y, \omega) \in T^*_yY \cap \Omega \simeq N_{\omega,\mathcal{O}_y}, \) and we conclude that
\[
\text{Crit} \Phi_{x,y} = \left\{ (\omega, g) : (g \cdot y, \omega) \in \Omega, x = g \cdot y \in N_{\omega,\Sigma^{R,t}_x}\right\}.
\]
The second condition means that \( \kappa(x) - \kappa(g \cdot y) \) is co-linear to \( \text{grad}_y \zeta(t, \kappa(x), \omega) \). But in view of (2.7) we have the equality
\[
\langle \text{grad}_y \zeta(t, \omega), \omega \rangle = R > 0, \quad \omega \in \Sigma^{R,t}_x,
\]
so that if \( x \neq g \cdot y \) and \( \kappa(x) - \kappa(g \cdot y) \in N_x \sum^R \), we deduce the lower bound

\[
(3.10) \quad \left| \left\langle \kappa(x) - \kappa(g \cdot y), \omega \right\rangle \right| \geq C > 0
\]

for a uniform constant \( C > 0 \). Since the \( G \)-action on \( M \) is smooth, there is an invariant tubular neighbourhood around each \( G \)-orbit in \( M \), and we may assume that the chart \((\kappa, Y)\) is given in terms of such a neighbourhood around \( O_x \). Thus, let \( NO_x \) be the normal bundle to \( O_x \), and

\[
\tau := \exp \circ \gamma : NO_x \longrightarrow M
\]

an equivariant diffeomorphism onto some open neighborhood of \( O_x \), where \( \exp \) denotes the exponential map and \( \gamma : NO_x \to NO_x \) is certain contraction \([2] \) Theorem VI.2.2). In particular, note that

\[
d(\exp)_z : T_z(NO_x) \equiv N_zO_x \oplus T_zO_x = T_zM \longrightarrow T_zM, \quad z \in O_x,
\]

is the identity, where \( O_x \) is embedded as the zero section in \( NO_x \). If we now let \( Y \subset \tau(NO_x) \) be small enough, we can identify \( \tau^{-1}(Y) \) with an open neighbourhood of the origin in \( T \omega NO_x \) via the exponential map, and we put \( \kappa := (\tau^{-1})_\ast \gamma \).

To show \((a)\), let us assume that \( y \in O_x \). Then the vector \( \kappa(x) - \kappa(g \cdot y) \) must be approximately normal to \( (dx)_x(N_xO_x) \equiv N_xO_x \) for sufficiently small \( Y \), so that if \( (\omega, g) \in \text{Crit} \Phi_{x,y} \), which in particular means that \( \omega \in N_{x,y}O_x \), the vector \( \kappa(x) - \kappa(g \cdot y) \) must be approximately normal to \( \omega \), which would be a contradiction to the lower bound \((3.10)\), unless \( x = g \cdot y \). Thus, we conclude that \( \text{Crit} \Phi_{x,y} = C_{x,y} \).

In order to see that \( \text{Crit} \Phi_{x,y} \) is clean, note that with respect to the parametrization \((3.7)\) of \( \Sigma^R \) and canonical coordinates on \( G \) the Hessian \( \text{Hess} \Phi_{x,y}(\omega, g) \) of \( \Phi_{x,y} \) at a critical point \( (\omega, g) \in C_{x,y} \), as a symmetric bilinear form on \( T_{x,\Sigma^R} \times T_{y,G} \), is given by the matrix

\[
\begin{pmatrix}
0 & \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i}(d\tilde{x}_k)_x(\tilde{X}_j) \\
\sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j}(d\tilde{x}_k)_x(\tilde{X}_i) & -\frac{1}{2} \left\langle \tilde{X}_{i,x}(\tilde{X}_j(\kappa)) + \tilde{X}_{j,x}(\tilde{X}_i(\kappa)), \omega \right\rangle
\end{pmatrix}
\]

The kernel of the corresponding linear transformation is given by those \((\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d\) satisfying the conditions

\[
\sum_k \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i}(d\tilde{x}_k)_x(\tilde{X}(\tilde{s})) = 0 \quad \text{for all} \quad i = 1, \ldots, n - 1,
\]

\[
\sum_{j, k} \tilde{\alpha}_j \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j}(d\tilde{x}_k)_x(\tilde{X}_i) = 0 \quad \text{for all} \quad i = 1, \ldots, d,
\]

where we put \( X(\tilde{s}) := \sum_{j=1}^d \tilde{s}_j X_j \). Indeed, \((3.12)\) implies that

\[
(d\kappa)_x(\tilde{X}(\tilde{s})) = \left( (d\tilde{x}_1)_x(\tilde{X}(\tilde{s})), \ldots, (d\tilde{x}_n)_x(\tilde{X}(\tilde{s})) \right) \in (d\kappa)_x(T_xO_x)
\]

collinear to \( \text{grad}_t \zeta(t, \kappa(t, x), \omega) \). Furthermore, since \((d\kappa)_x(T_xO_x) \equiv T_xO_x \) is clearly normal to \((d\kappa)_x(N_xO_x) \equiv N_xO_x \), the vector \((d\kappa)_x(\tilde{X}(\tilde{s})) \) is normal to \( N_xO_x \). In view of \((3.9)\) and the fact that \((x, \omega) \in N_xO_x \) we would obtain a contradiction, unless \( \tilde{X}(\tilde{s}) \) vanishes at \( x \); in particular, this implies that \( \tilde{X}(\tilde{s})(\kappa), \omega \) has a zero of second order at \( x \) in orbit direction, so that

\[
\tilde{X}_{i,x} \left\langle \tilde{X}(\tilde{s})(\kappa), \omega \right\rangle + \tilde{X}(\tilde{s})_x \left\langle \tilde{X}(\kappa), \omega \right\rangle = 0.
\]

Thus, the coefficients in the fourth quadrant of the matrix \((3.11)\) do not contribute to Equations \((3.13)\), and the kernel in question is given by

\[
\{ (\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d \mid \sum_{j=1}^d \tilde{s}_j \tilde{X}_j = 0, \sum_{j, k} \tilde{\alpha}_j \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j}(d\tilde{x}_k)_x \in \text{Ann} \left( T_xO_x \right) \}
\]

which means that \( \text{Hess} \Phi_{x,y} \) is transversally non-degenerate on \( C_{x,y} \), yielding \((a)\).
In order to see (b), assume that \( y \notin O_x \), and let the chart \((\kappa, Y)\) be defined as above in terms of the tubular neighbourhood \( \tau : NO_x \to M \). Note that without loss of generality we can assume that \( y \in S_x \cap Y \), where \( S_x := \tau (N_x O_x) \). The first part of (b) is clear from (3.8). Now, assume that the co-spheres \( S^x_\ast M \) are strictly convex. For small \(|\epsilon|\), the hypersurfaces \( \Sigma_{R,t}^x \) will be strictly convex, too. In particular, \( \Sigma_{R,t}^x \) is orientable, and the Gauss map

\[
\mathcal{N} : \Sigma_{R,t}^x \ni \omega \mapsto \mathcal{N} (\omega) \in N_{\omega} \Sigma_{R,t}^x,
\]

which assigns to each point of \( \Sigma_{R,t}^x \) the outer normal unit vector to \( \Sigma_{R,t}^x \) at that point, is a global diffeomorphism. Therefore, for each \( x \neq y \in Y \) there is a unique \( \omega_y \in \Sigma_{R,t}^x \) such that

\[
\kappa (y) - \kappa (x) \| \kappa (y) - \kappa (x) \| = \mathcal{N} (\omega_y).
\]

Consequently, if \((\omega, g) \in \text{Crit } \Phi_{x,y}\), the vector \( \omega \) is locally uniquely determined by the condition

\[
\mathcal{N} (\omega) = \pm \mathcal{N} (\omega_{x,y}).
\]

Now, introduce the sets

\[
W_n := \tau (V_{1/n}), \quad V_{1/n} := \{ v \in NO_x \mid \| v \| < 1/n \}, \quad n \in \mathbb{N},
\]

and assume that for each \( n \in \mathbb{N} \) there is a \( y_n \in W_n \cap Y \cap S_x \) such that \( \Phi_{x,y_n} \neq G_{y_n} \), locally. In other words, assume that for each \( n \in \mathbb{N} \) there is a smooth curve

\[
\gamma_n : (\varepsilon_n, \varepsilon_n) \ni t \mapsto (\omega_n (t), g_n (t)) \in \text{Crit } \Phi_{x,y_n}, \quad \varepsilon_n > 0,
\]

parametrized such that \( \| \omega_n (t) \| = 1 \). In this way, we obtain for each \( n \in \mathbb{N} \) a curve \( \omega_n (t) \in \Sigma_{R,t}^x \) along which the unit normal vector field to \( \Sigma_{R,t}^x \) is determined by the direction of \( \kappa (x) - \kappa (g_n (t) \cdot y_n) \), so that \( \mathcal{N} (\omega_n (t)) = \pm \mathcal{N} (\omega_{x,y_n}(t)) \). In view of (3.10), the curves

\[
\{ g_n (t) \mid t \in (\varepsilon_n, \varepsilon_n) \} \subset Y
\]

converge to \( x \) as \( n \to \infty \), which in particular implies that \( \varepsilon_n \to 0 \). Similarly, due to the compactness of \( \Sigma_{R,t}^x \) the curves

\[
\{ \omega_n (t) \mid t \in (\varepsilon_n, \varepsilon_n) \} \subset \Sigma_{R,t}^x
\]

converge to at least one \( \omega_\infty \in \Sigma_{R,t}^x \cap N_\ast O_x \) after passing to a suitable convergent subsequence \( \omega_{n_k} (t) \). Now, assume that \( G \) acts on \( M \) with orbits of the same dimension \( \kappa \). If \( O_{\text{prin}} \) is a principal orbit and \( O \) a principal or exceptional orbit, there is an equivariant covering map \( O_{\text{prin}} \to O \), so that \( O_{\text{prin}} \) and \( O \) are locally diffeomorphic, compare [2, p. 181]. Therefore, we can assume that all orbits in \( Y \) are diffeomorphic, which implies that the more \( y_n \) approaches \( x \), the faster the direction of \( \kappa (x) - \kappa (g_n (t) \cdot y_n) \) changes as \( t \in (\varepsilon_n, \varepsilon_n) \) varies, and the faster \( \mathcal{N} (\omega_n (t)) \) changes as \( t \in (\varepsilon_n, \varepsilon_n) \) varies. Consequently, the Gaussian curvature of \( \Sigma_{R,t}^x \) at \( \omega_\infty \), which is given by the product of the principal curvatures, cannot stay bounded, compare Figure [3.1].

Thus, we have shown that for sufficiently small \( Y \) we locally have \( \text{Crit } \Phi_{x,y} \simeq G_y \), which implies that \( \text{Crit } \Phi_{x,y} \) is a smooth submanifold of codimension \( n - 1 + \dim O_y \). We are left with the task of showing that \( \text{Hess } \Phi_{x,y} \) is transversally non-degenerate. For this, we are going to show that for each fixed \((\omega, g) \in \text{Crit } \Phi_{x,y}\) one has \( \text{Ker Hess } \Phi_{x,y} (\omega, g) \simeq T_{(\omega,g)} \text{Crit } \Phi_{x,y} \). To do so, note that with respect to the coordinates introduced at the beginning, the Hessian \( \text{Hess } \Phi_{x,y} (\omega, g) \) of \( \Phi_{x,y} \) at a critical point \((\omega, g)\) is given by the matrix

\[
\begin{pmatrix}
\left( \kappa (x) - \kappa (g \cdot y), \frac{\partial^2 \mathcal{F}}{\partial \alpha \partial \alpha} (\alpha^{-1} (\omega)) \right) \\
\sum_{k=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_k} (\alpha^{-1} (\omega))(d\tilde{x}_k)_{g \cdot y} (\tilde{X}_j) \\
\sum_{k=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_k} (\alpha^{-1} (\omega))(d\tilde{x}_k)_{g \cdot y} (\tilde{X}_i) \\
-\frac{1}{2} \left( \tilde{X}_{i,g \cdot y} (\tilde{X}_j (\kappa)) + \tilde{X}_{j,g \cdot y} (\tilde{X}_i (\kappa)), \omega \right)
\end{pmatrix}.
\]

Since \( \kappa (g \cdot y) - \kappa (x) \in N_\ast \Sigma_{R,t}^x \), the submatrix in the first quadrant corresponds to a multiple of the second fundamental of \( \Sigma_{R,t}^x \)

\[
\Pi : T \Sigma_{R,t}^x \times T \Sigma_{R,t}^x \to C^\infty (\Sigma_{R,t}^x), \quad \Pi(\mathcal{X}, \mathcal{Y}) := \langle \nabla \mathcal{X}, \mathcal{N} \rangle = \langle \mathcal{X}, A \mathcal{Y} \rangle,
\]
To compute the kernel of the matrix (3.14), assume that the identifying (3.16)–(3.17) are equivalent to \( \tilde{\alpha} \) with \( \tilde{\alpha} \equiv X \),...,d, where we wrote again \( \tilde{\alpha} \) at \( \omega \) through different points \( y \); the black dotted lines represent normal spaces to the orbits. The three coloured ellipse segments depict different hypersurfaces \( \Sigma_{g,t}^R \in \mathbb{R}^n \) whose unit normal at \( \omega \in N_{g,t}O_y \cap \Sigma_{g,t}^R \), depicted by a colored dotted line, is determined by the corresponding colored line segments \( \kappa(x) - \kappa(g \cdot y) \).

where \( \nabla_X Y \equiv \mathcal{X}(Y) \) denotes the covariant derivative in Euclidean space \( \mathbb{R}^n \) and \( A : T\Sigma_{g,t}^R \to T\Sigma_{g,t}^R \) the symmetric endomorphism induced by \( II \) [17, Chapter VII, Section 3]. Indeed, assume that \( \kappa(x) - \kappa(g \cdot y) \) points in the direction of \( -N(\omega) \), and let \( \partial / \partial \alpha_{j|\omega} := \partial F(\alpha^{-1}(\omega)) / \partial \alpha_i \), \( 1 \leq i \leq n-1 \), be the coordinate frame given by the parametrization (3.7). Then, the entries of the submatrix in the first quadrant of (3.14) read

\[
- \| \kappa(x) - \kappa(g \cdot y) \| \Pi \left( \frac{\partial}{\partial \alpha_i|\omega}, \frac{\partial}{\partial \alpha_j|\omega} \right) = - \| \kappa(x) - \kappa(g \cdot y) \| \left( \frac{\partial}{\partial \alpha_i|\omega}, A \frac{\partial}{\partial \alpha_j|\omega} \right).
\]

To compute the kernel of the matrix (3.14), assume that the \( X_1, \ldots, X_d \in \mathfrak{g} \) are such that the vector fields \( \tilde{X}_1, \ldots, \tilde{X}_\kappa \) constitute an orthonormal basis of \( T_{g \cdot y}O_y \) at \( g \cdot y \), while the vector fields \( \tilde{X}_{\kappa+1}, \ldots, \tilde{X}_d \) vanish at \( g \cdot y \), and consider for \( (\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d \) the system of equations

\[
\sum_{j=1}^{n-1} \left( \kappa(x) - \kappa(g \cdot y) \right) \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j}(\alpha^{-1}(\omega)) \tilde{\alpha}_j + \sum_{k=1}^{n} \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i} (d\tilde{x}_k)_{g \cdot y}(\tilde{X}(\tilde{s})) = 0
\]

with \( i = 1, \ldots, n-1 \), as well as

\[
\sum_{k=1}^{n} \sum_{j=1}^{n-1} \tilde{\alpha}_j \left( \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i} (d\tilde{x}_k)_{g \cdot y}(X_{\tilde{i}}) - \frac{1}{2} \left( \tilde{X}_{i,g \cdot y}(\tilde{X}(\tilde{s})(\kappa)) + \tilde{X}(\tilde{s})_{g \cdot y}(\tilde{X}_i(\kappa)), \omega \right) \right) = 0
\]

with \( i = 1, \ldots, d \), where we wrote again \( \tilde{X}(\tilde{s}) := \sum_{j=1}^{d} \tilde{s}_j X_j \) for short. We have to show that Equations (3.16)–(3.17) are equivalent to \( \tilde{\alpha} = 0, \tilde{s}_1 = \cdots = \tilde{s}_\kappa = 0 \). Writing \( W_\omega(\tilde{\alpha}) := \sum_{j=1}^{n-1} \tilde{\alpha}_j \partial / \partial \alpha_j|\omega \) and identifying \( Y \) with \( \kappa(Y) \), the system of equations (3.16) reads

\[
-\|x - g \cdot y\| \left( \frac{\partial}{\partial \alpha_i|\omega}, A W_\omega(\tilde{\alpha}) \right) + \left( \frac{\partial}{\partial \alpha_i|\omega}, \tilde{X}(\tilde{s})_{g \cdot y} \right) = 0, \quad i = 1, \ldots, n-1,
\]

which is equivalent to

\[
W_\omega(\tilde{\alpha}) = \|x - g \cdot y\|^{-1} A^{-1}(\text{proj}_{|T_{g \cdot y}O_y}(\tilde{X}(\tilde{s})_{g \cdot y})),
\]

compare Figure 3.2.
Note that $A$ is invertible, since the Gaussian curvature of $\Sigma_{x,t}$ does not vanish. Furthermore, since $\Sigma_{x,t}$ is strictly convex, the eigenvalues of $A$, which are given by the principal curvatures of $\Sigma_{x,t}$ with respect to the outer unit normal vector field, are strictly negative.\footnote{Note that the sign convention used here is such that if $\Sigma_{x,t}$ equals the standard $(n-1)$-sphere $S^{n-1}(R)$ of radius $R$, then $A = -1/R$, where $I$ represents the identity transformation on $T_o S^{n-1}(R)$, see \cite[Chapter VII, Example 4.2]{17}.} Hence $A$ defines a non-positive operator on $T_o \Sigma_{x,t}$. On the other hand, \eqref{3.17} amounts to the equations

\begin{equation}
\left\langle \mathcal{W}_\omega(\tilde{a}), \tilde{X}_{i,g,y} \right\rangle = \frac{1}{2} \left( \tilde{X}_{i,g,y}(\tilde{X}(\tilde{s}),\omega) + \tilde{X}(\tilde{s})_{g,y}(\tilde{X}_i,\omega) \right), \quad i = 1, \ldots, d.
\end{equation}

Inserting \eqref{3.18} into \eqref{3.19} one obtains for all $i = 1, \ldots, d$

\begin{equation}
F^i_s(g \cdot y, \omega) = \|x - g \cdot y\| G^i_s(g \cdot y, \omega),
\end{equation}

where we set

\begin{align*}
F^i_s(z, \omega) &:= \left\langle A^{-1}(\text{proj}_{T_o \Sigma_{x,t}}) (\tilde{X}(\tilde{s}))_{z}, \tilde{X}_{i,z} \right\rangle, \quad G^i_s(z, \omega) := \frac{1}{2} \left( \tilde{X}_{i,z}(\tilde{X}(\tilde{s}),\omega) + \tilde{X}(\tilde{s})_{z}(\tilde{X}_i,\omega) \right).
\end{align*}

As functions on $\{ (z, \omega) \in Y \times \Sigma_{x,t} \mid \omega \in N_z O_z \}$, $F^i_s$ and $G^i_s$ are smooth and bounded from above. Furthermore, the projection from $T_o O_z$ to $T_o \Sigma_{x,t}$ has a trivial kernel if $\omega$ is normal to $O_z$ at $z$, since $\omega$ cannot be tangential to $\Sigma_{x,t}$ in view of \eqref{3.9}. Therefore,

\begin{equation}
F^i_s(z, \omega) = 0 \quad \text{for all } i = 1, \ldots, d \iff \tilde{X}(\tilde{s})_z = 0,
\end{equation}

$A$ being a non-positive operator. Let us now assume that $\tilde{X}(\tilde{s})_{g,y} \neq 0$. Choosing $Y$ sufficiently small, we deduce from \eqref{3.21} that there is at least one $i$ such that $|F^i_s(z, \omega)| \geq C^i_s$ on $Y$ for some uniform constant $C^i_s > 0$. But then, letting $Y$ become even smaller so that $1/\|x - g \cdot y\|$ becomes large compared to $|G^i_s(g \cdot y, \omega)/C^i_s$ we arrive at a contradiction in view of \eqref{3.20}. Thus, we must have $\tilde{X}(\tilde{s})_{g,y} = 0$, which in turn implies $\mathcal{W}_\omega(\tilde{a}) = 0$ by \eqref{3.18}. We have therefore shown that Equations \eqref{3.16}–\eqref{3.17} are fulfilled iff $\tilde{s}_1 = \cdots = \tilde{s}_k = 0$ and $\tilde{\alpha} = 0$, so that

$$\text{Ker Hess } \Phi_{x,y}(\omega, g) \simeq \{0\} \times \mathbb{R}^{d-k} \simeq T_{(\omega,g)} \text{Crit } \Phi_{x,y},$$

Figure 3.2. Concerning the cleanness of the critical set of $\Phi_{x,y}$ in case that $y \notin O_x$. Black circles represent $G$-orbits in $Y \equiv \kappa(Y) \subset \mathbb{R}^n$ through $x$ and $y$, respectively; the black dotted lines represent normal spaces to the orbits and tangent spaces to the hypersurface $\Sigma_{x,t}$, respectively, the latter being depicted by an ellipse. The red arrows represent points $\omega \in \Sigma_{x,t}$, the green arrows segments $\kappa(g \cdot y) - \kappa(x)$. The magenta arrows depict vectors $\tilde{X}(\tilde{s})_{g,y}$ and the blue arrows the corresponding vectors $\mathcal{W}_\omega(\tilde{a})$, compare \eqref{3.18}.\footnote{Note that the sign convention used here is such that if $\Sigma_{x,t}$ equals the standard $(n-1)$-sphere $S^{n-1}(R)$ of radius $R$, then $A = -1/R$, where $I$ represents the identity transformation on $T_o S^{n-1}(R)$, see \cite[Chapter VII, Example 4.2]{17}.}
and we obtain (b). An alternative proof of the fact that the Hessian of \( \Phi_{x,y} \) is transversally non-degenerate in the cases (a) and (b) will be given in Theorem 3.3 by explicitly computing the transversal Hessian.

In order to show (c), let \( x \in Y \cap M_{\text{prin}} \) and \( (\omega, g) \in C_{x,y} \). If \( x \) is of principal isotropy type, \( G_x \) acts trivially on \( N_x(G \cdot x) \) [2] pp. 308 and 181] and, via the identification \( T^*M \simeq TM \), also on \( \text{Ann}(T_x(G \cdot x)) \). But in view of (3.5) and (a) we have \( \omega \in \text{Ann}(T_x(G \cdot x)) \), so that \( g \cdot \omega = \omega \) in this case, and with (3.6) we obtain the desired inclusion and therefore (c). In particular, since \( \text{Crit} \Phi \) has codimension \( 2\kappa \), \( C_{x,y} \) has codimension \( 2\kappa \) as well.

\[ \square \]

Remark 3.2.

(1) Let \( y \notin O_x \). As an example where \( \text{Crit} \Phi_{x,y} \) is not isomorphic to \( G_y \), and does not have codimension \( n - 1 + \kappa \), consider the singular action of \( G = \text{SO}(2) \) on the standard 2-sphere \( M = S^2 \subset \mathbb{R}^3 \) by rotations around the poles \( x_N, x_S \), and assume that \( \Sigma_{x}^{R,t} = S^1 \). Let \( (Y, \kappa) \) be an invariant tubular neighborhood around the fixed point \( x_N \). Then, for any \( y \in Y - \{ x_N \} \) one has

\[ \text{Crit} \Phi_{x,y} = \{ (\omega, g) \mid (g \cdot y, \omega) \in N_{\phi y}(G \cdot y), \kappa(x_N) - \kappa(g \cdot y) \parallel \omega \} \simeq \text{SO}(2) \times \mathbb{Z}_2 \not\simeq \text{G}_y = \{ e \}, \]

which has codimension \( \kappa = 1 \) instead of 2, showing the necessity of the assumption in Lemma 3.1(b) that all \( G \)-orbits must have the same dimension.

(2) Note that Lemma 3.1(c) cannot hold in general for arbitrary \( x \in Y \cap (M_{\text{except}} \cup M_{\text{sing}}) \). In particular, if \( x \) were a fixed point we would have \( \Phi_{x,x} \equiv 0 \), so that \( \text{Crit} \Phi_{x,x} = \Sigma_{x}^{R,t} \times G \) in this case. Furthermore, Assertion (c) means that \( \Phi_{x,x} \) does not have secondary critical points for \( x \in Y \cap M_{\text{prin}} \), that is, critical points which do not arise from critical points of \( \Phi \).

From the previous lemma one now deduces

Theorem 3.3. Assume that \( G \) is a continuous compact Lie group acting on \( M \) with orbits of dimension less or equal \( n - 1 \), and consider the oscillatory integrals \( I_{x,y}(\mu) \) defined in (3.1).

(a) Let \( y \in O_x \). Then, for every \( \tilde{N} \) one has the asymptotic formula

\[ I_{x,y}(\mu) = (2\pi/\mu)^{\dim O_x} \sum_{k=0}^{\tilde{N}-1} Q_k(x,y) \mu^{-k} + R_{\tilde{N}}(x,y,\mu), \quad \mu \to +\infty, \]

with explicitly known coefficients and remainder. In particular,

\[ Q_0(x,y) = \int_{C_{x,y}} \frac{a(x,y,\omega,g)}{\text{det} \Phi_x^{a}(\omega,g)(N_{\phi y}C_{x,y})^{1/2}} \, dc_{x,y}(\omega,g), \]

where \( dc_{x,y} \) denotes the induced volume density. Furthermore, \( Q_k(x,y) \) and \( R_{\tilde{N}}(x,y,\mu) \) depend smoothly on \( R \) and \( t \), and satisfy the bounds

\[ |Q_k(x,y)| \leq C_{k,\Phi_{x,y}} \text{vol} \text{supp} a(x,y,\cdot,\cdot) \cap C_{x,y} \sup_{l \leq 2k} \| D^l a(x,y,\cdot,\cdot) \|_{\infty, C_{x,y}}, \]

\[ |R_{\tilde{N}}(x,y,\mu)| \leq \tilde{C}_{\tilde{N},\Phi_{x,y}} \text{vol} \text{supp} a(x,y,\cdot,\cdot) \sup_{l \leq 2\tilde{N}+\dim O_x+1} \| D^l a(x,y,\cdot,\cdot) \|_{\infty, \Sigma_x^{R,t} \times G} \mu^{-\tilde{N}}, \]

uniformly in \( R, t \) for suitable constants \( C_{k,\Phi_{x,y}} > 0 \) and \( \tilde{C}_{\tilde{N},\Phi_{x,y}} > 0 \), where \( D^l \) denote differential operators of order \( l \) on \( \Sigma_x^{R,t} \times G \). Moreover, as functions in \( x \) and \( y \), \( Q_k(x,y) \) and \( R_{\tilde{N}}(x,y,\mu) \) are smooth on \( Y \cap M_{\text{prin}} \), and the constants \( C_{k,\Phi_{x,y}} \) and \( \tilde{C}_{\tilde{N},\Phi_{x,y}} \) are uniformly bounded in \( x \) and \( y \) if \( M = M_{\text{prin}} \cup M_{\text{except}} \). If the amplitude factorizes according to \( a(x,y,\omega,g) = a_1(x,y,\omega) a_2(x,y,g) \), the remainder can also be estimated by

\[ |R_{\tilde{N}}(x,y,\mu)| \leq \tilde{C}_{\tilde{N},\Phi_{x,y}} \prod_{i=1,2} \text{vol} \text{supp} a_i(x,y,\cdot,\cdot) \sup_{l \leq 2\tilde{N}+\dim O_x+1} \| D^l a_i(x,y,\cdot) \|_{\infty, M_1}, \mu^{-\tilde{N}}, \]

where \( D^l_1 \) and \( D^l_2 \) denote differential operators of order \( l \) on \( M_1 = \Sigma_x^{R,t} \) and \( M_2 = G \), respectively.
(b) Let \( y \notin O_x \). Assume that \( M = M_{\text{prin}} \cup M_{\text{except}} \) and that the co-spheres \( S^*_y M \) are strictly convex. Then, for sufficiently small \( Y \) and every \( \bar{N} \in \mathbb{N} \) one has the asymptotic formula

\[
I_{x,y}(\mu) = \sum_{J \in \pi_c(\text{Crit } \Phi_{x,y})} \left( 2\pi / \mu \right)^{n+1-\kappa} e^{i \Phi_{x,y}^J} \sum_{k=0}^{\bar{N}-1} \langle Q_{J,k}(x,y) \mu^{-k} + \mathcal{R}_{J,\bar{N}}(x,y,\mu) \rangle
\]

as \( \mu \to +\infty \) with explicitly known coefficients and remainder, where \( \kappa := \dim M / G \). The coefficients \( Q_{J,k}(x,y) \) and the remainder terms \( \mathcal{R}_{J,\bar{N}}(x,y,\mu) \) are given by distributions depending smoothly on \( R, t \), and \( x, y \in Y \cap M_{\text{prin}} \) with support in \( \text{Crit } \Phi_{x,y} \) and \( \Sigma^{R,t}_{y} \times G \), respectively. Furthermore, they satisfy bounds analogous to the ones in (1), where now the constants \( C_{k,\Phi_{x,y}} \) and \( C_{\bar{N},\Phi_{x,y}} \) are no longer uniformly bounded, but satisfy

\[
C_{k,\Phi_{x,y}} \ll \text{dist}(y, O_x)^{-(n-1-\kappa)/2-k}, \quad C_{\bar{N},\Phi_{x,y}} \ll \text{dist}(y, O_x)^{-(n-1-\kappa)/2-\bar{N}}.
\]

Finally, \( 0 \Phi_{x,y}^J \) stands for the constant values of \( \Phi_{x,y} \) on the connected components \( J \) of its critical set, and is given by

\[
0 \Phi_{x,y}^J(R,t) = R e_{x,y}(t), \quad e_{x,y}(t) := \pm \frac{\|\kappa(x) - \kappa(g_{J,y})\|}{\|\nabla \zeta(t,\kappa(x),\omega_J)\|}, \quad (\omega_J, g_J) \in J.
\]

**Proof.** The asymptotic expansions for the integrals \( I_{x,y}(\mu) \), the smoothness of the coefficients \( Q_{J,k}(x,y) \), \( Q_{J,k}(x,y) \) and the remainder terms in the parameters \( R, t \), and \( x, y \in Y \cap M_{\text{prin}} \), as well as the bounds satisfied by them are a direct consequence of Lemma 3.1 and Theorem 7.6. To see that the constants \( C_{k,\Phi_{x,y}} \), \( C_{\bar{N},\Phi_{x,y}} \) satisfy the specified bounds, we have to compute the transversal Hessian of \( \Phi_{x,y} \) and its derivative in the two cases \( y \in O_x \) and \( y \notin O_x \). Recall the notation and the proof of Lemma 3.1 and let \( (\omega, g) \in \text{Crit } \Phi_{x,y} \) be a critical point. As a bilinear form on \( T_{\omega}\Sigma^{R,t}_{y} \times G \), the Hessian of \( \Phi_{x,y} \) can be written as the \( (n-1+d) \times (n-1+d) \)-matrix

\[
\begin{pmatrix}
-\|x - g \cdot y\| & \langle \frac{\partial}{\partial \alpha_{i,j}}, A \frac{\partial}{\partial \alpha_{i,j}} \rangle \\
\langle \frac{\partial}{\partial \alpha_{i,j}}, \tilde{X}_{i,j} \rangle & -\beta \left( \tilde{X}_{i,j} \right)
\end{pmatrix},
\]

compare (3.11) and (3.14), where we took into account (3.15), and made the identification \( k(Y) \simeq Y \). Note that for \( y \in O_x \) one has \( g \cdot y = x \). Now, recall that the projection from \( T_{\omega}g \cdot O_y \) to \( T_{\omega}T_{\omega}\Sigma^{R,t}_{y} \) has a trivial kernel, and choose vectors \( \tilde{\alpha}, \ldots, \tilde{\alpha}^* \in \mathbb{R}^{n-1} \) such that one has the decomposition

\[
T_{\omega}T_{\omega}\Sigma^{R,t}_{y} = \text{proj}_{T_{\omega}T_{\omega}\Sigma^{R,t}_{y}}(T_{\omega}g \cdot O_y) \oplus \text{Span} \{ W_{\omega}(\tilde{\alpha}) \in N_\omega g \cdot O_y \}
\]

where \( W_{\omega}(\tilde{\alpha}) := \alpha_{i,j} \partial / \partial \alpha_{i,j} \). Further, suppose that the \( X_j \in g \) have been chosen such that the vector fields \( \tilde{X}_{1}, \ldots, \tilde{X}_{\dim E} \) constitute an orthonormal basis of \( T_{\omega}g \cdot O_y \) at \( g \cdot y \), while the vector fields \( \tilde{X}_{\dim E+1}, \ldots, \tilde{X}_{d} \) vanish at \( g \cdot y \). Then, with respect to the basis \( \{ W_{\omega}(\tilde{\alpha}) \} \) the Hessian of \( \Phi_{x,y}(\omega, g) \) is essentially given by the \( (n-1 + \dim E) \times (n-1 + \dim E) \)-matrix

\[
M_{x,y}(\omega, g) :=
\begin{pmatrix}
-\|x - g \cdot y\| & \langle W_{\omega}(\tilde{\alpha}), A W_{\omega}(\tilde{\alpha}) \rangle \\
\langle W_{\omega}(\tilde{\alpha}), \tilde{X}_{i,j} \rangle & 0
\end{pmatrix}.
\]

since \( \langle \tilde{X}_{j}, \omega \rangle \) has a zero of second order at \( g \cdot y \) in orbit direction for \( j = \dim E + 1, \ldots, d \). If \( y \notin O_x \), the transversal Hessian of \( \Phi_{x,y}(\omega, g) \) is given by \( M_{x,y}(\omega, g) \); if \( y \in O_x \), it is given by the
which is obtained from $M_{x,y}(\omega, g)$ by removing the $(\dim O_x + 1)$-th, ..., $(n - 1)$-th columns and rows. Clearly, $\det M_{x,y}(\omega, g) = \det M'_{x,y}(\omega, g)$, where $M'_{x,y}(\omega, g)$ is the matrix obtained from $M_{x,y}(\omega, g)$ by setting the coefficients in the fourth quadrant equal to zero. A computation as in (3.12)-(3.13) then shows that the kernel of the linear transformation corresponding to $M'_{x,y}(\omega, g)$ is trivial, so that $\det M'_{x,y}(\omega, g) \neq 0$. Thus, in the case (a) we have shown again that the transversal Hessian of $\Phi_{x,y}$ is non-degenerate, as in Lemma 3.1 (a), and that

$$
\frac{1}{\det \operatorname{TransHess} \Phi_{x,y}(\omega, g)} = \frac{1}{\det M_{x,y}(\omega)} \ll 1
$$

uniformly in $x, y \in M_{\text{prin}} \cup M_{\text{except}}$, since principal and exceptional orbits are locally diffeomorphic [2, p. 181], and principal and exceptional isotropy groups infinitesimally isomorphic. On the other hand, in the case (b), suppose that the matrix $A$ has diagonal form with respect to the basis $W_\omega(\tilde{a}_i)$. Denote its entries, which correspond to the principal curvatures of $\Sigma_x$, by $(\varrho_1, \ldots, \varrho_{n-1})$. Then

$$
\det M_{x,y}(\omega, g) = \|x - g \cdot y\|^{n-1-\kappa}
$$

(3.22)

$$
\cdot \left( c_0 + \|x - g \cdot y\| c_1 + \cdots + \|x - g \cdot y\|^\kappa c_n \right), \quad c_i \in \mathbb{R},
$$

where

$$
c_0 = \pm \varrho_{k+1} \cdots \varrho_{n-1} \det M_{x,y}(\omega, g).
$$

Since $\det M_{x,y}(\omega)$ is uniformly bounded away from zero, we have $|c_0| \geq C > 0$ for a uniform constant $C > 0$. Taking $Y$ sufficiently small, it is mainly the term $c_0$ that contributes to $\det M_{x,y}(\omega, g)$ so that we conclude again that the Hessian of $\Phi_{x,y}$ is transversally non-degenerate in the case (b), compare Lemma 3.1 (b), and that

$$
\frac{1}{\det \operatorname{TransHess} \Phi_{x,y}(\omega, g)} = \frac{1}{\det M_{x,y}(\omega)} \ll \|x - g \cdot y\|^{-(n-1-\kappa)}
$$

uniformly in $x, y \in M = M_{\text{prin}} \cup M_{\text{except}}$. Summing up, we have shown on $M_{\text{prin}} \cup M_{\text{except}}$ the uniform bound

$$
\frac{1}{\det \operatorname{TransHess} \Phi_{x,y}(\omega, g)} \ll \begin{cases} 1, & y \in O_x, \\ \operatorname{dist}(y, O_x)^{-(n-1-\kappa)}, & y \notin O_x. \end{cases}
$$

By (4.3) it then follows that the constants $C_{k,\Phi_{x,y}}, \tilde{C}_{R,\Phi_{x,y}}$ satisfy the specified bounds. Regarding the values of $\Phi_{x,y}$ on its critical set, note that for $(\omega, g) \in \operatorname{Crit} \Phi_{x,y}$ one computes with (2.7)

$$
\Phi_{x,y}^0(R, t) = (\kappa(x) - \kappa(g \cdot y), \omega) = \pm c_{x,g,y}(t) \langle \operatorname{grad}_\eta \zeta(t, \kappa(x), \omega), \omega \rangle = \pm R c_{x,g,y}(t),
$$

since $\kappa(x) - \kappa(g \cdot y)$ must be co-linear to $\operatorname{grad}_\eta \zeta(t, \kappa(x), \omega)$. In particular notice that $c_{x,g,y}(t)$ is independent of $R$ due to the fact that $\zeta(t, \kappa(x), \eta)$ is homogeneous of degree 1 in $\eta$, so that $\operatorname{grad}_\eta \zeta(t, \kappa(x), \omega)$ only depends on the direction of $\omega$. □

As the previous theorem shows, the integrals $I_{x,y}(\mu)$ exhibit a caustic behaviour in their dependence on the variables $x$ and $y$, obeying different asymptotics in the cases $y \in O_x$ and $y \notin O_x$, respectively. In particular, in the latter case, the coefficients in the asymptotic expansion become singular as $y \to O_x$. In what follows, we shall derive a uniform asymptotic expansion for the integrals $I_{x,y}(\mu)$ that

\[ \text{See Appendix A for a discussion of the terminology.} \]
interpolates between these two different asymptotic behaviours. This result will be necessary for
deriving the equivariant $L^p$-bounds in Section 5.

**Theorem 3.4.** Consider the integrals $I_{x,y}(\mu)$ defined in (3.1). Assume that the continuous compact
Lie group $G$ acts on $M$ with orbits of the same dimension $\kappa \leq n-1$, and that the co-spheres $S^*_x M$
are strictly convex. Then, for sufficiently small $Y$ and arbitrary $\tilde{N}_1, \tilde{N}_2 \in \mathbb{N}$ one has the asymptotic
formula

$$I_{x,y}(\mu) = \sum_{J \in \pi_0(\text{Crit } \Phi_{x,y})} e^{\mu \Phi^J_{x,y}} \mathcal{P}(\mu) \left[ \sum_{k_1, k_2 = 0} \mathbb{Q}^J_{k_1, k_2}(x, y) \right]$$

as $\mu \to +\infty$. The coefficients and the remainder term

$$\mathcal{R}^J_{\tilde{N}_1, \tilde{N}_2}(x, y, \mu) = O \left( \mu^{-\tilde{N}_1(\mu\kappa(x) - \kappa(g_J \cdot y)) + 1} \right)$$

are given by distributions depending smoothly on $R, t$ with support in each of the components $J$ of
$\text{Crit } \Phi_{x,y}$ and $\Sigma^R_x \times G$, respectively. Furthermore, they are uniformly bounded in $x$ and $y$ by derivatives
of a with respect to $g$ up to order $2k_1$ and $2\tilde{N}_1 + \kappa + 1$, respectively, while $\Phi^J_{x,y} := R_{x,y}(t)$
denotes the constant value of $\Phi_{x,y}$ on $J$. If the amplitude factorizes according to $a(x, y, \omega, g) = a_1(x, y, \omega) a_2(x, y, g)$, the remainder can also be estimated by derivatives of $a$ with respect to $g$ up to order $2\tilde{N}_1 + [\kappa/2 + 1]$.

**Proof.** Let the notation be as before, and recall from Lemma 3.1 the description of the critical set of
$\Phi_{x,y}$ in the two cases $y \in O_x$ and $y \notin O_x$. For simplicity, let us assume that $G_y$ is connected. In the
first case, $\text{Crit } \Phi_{x,y}$ is given by the set

$$J = (\Sigma^R_x \cap N_y O_x) \times \{ g_J \cdot G_y \},$$

where $g_J \in G$ is determined by the condition $g_J \cdot y = x$, and is connected if $\kappa < n - 1$. In the second
case, each of the connected components of $\text{Crit } \Phi_{x,y}$ has the form

$$J = \{ \omega \} \times \{ g_J \cdot G_y \},$$

where $g_J \in G$, $\omega \in \Sigma^R_x \cap N_{g_J \cdot y} O_y$ are determined by the condition $\kappa(x) - \kappa(g_J \cdot y) \in N_{\kappa \Sigma^R_x \Sigma^R_x}$. In
both cases, each of the $J \in \pi_0(\text{Crit } \Phi_{x,y})$ is contained in a set of the form

$$\{ (\omega, g) \mid \omega \in \Sigma^R_x \cap N_{g_J \cdot y} O_y, g \in g_J \cdot G_y \}.$$
If \( \kappa = n - 1 \), the intersection \( N_{g \cdot y} \mathcal{O}_y \cap \mathcal{U}_g \) consists of isolated points, and \( d\omega' \) corresponds to the counting measure. Our intention is to apply the stationary phase principle first to the inner integral and then to the outer integral. For this, let \( \mathcal{J} \) and \( \omega' \) be fixed, and introduce the phase function
\[
\Phi^\mathcal{J}_{x,y,\omega}(\alpha', g) := \Phi_{x,y}(F^\mathcal{J}_\omega(\alpha', \omega'), g).
\]
We clearly have
\[
\text{Crit } \Phi^\mathcal{J}_{x,y,\omega} = \{(\alpha', g) \in (-\varepsilon, \varepsilon)^n \times G \mid \Phi^\mathcal{J}_\omega(\alpha', \omega') \in N_{g \cdot y} \mathcal{O}_y, \kappa(x) - \kappa(g \cdot y) \perp V_{F^\mathcal{J}_\omega(\alpha', \omega')} \},
\]
where we put \( V_{\omega} := \text{Span}\{\frac{\partial}{\partial \alpha_i} | \omega\} \). The two conditions imply that the vector \( \kappa(x) - \kappa(g \cdot y) \) cannot be roughly tangential to \( T_{g \cdot y} \mathcal{O}_y \), unless it is zero. Now, if \( y \in \mathcal{O}_x \), the mentioned vector becomes almost tangential to \( T_{g \cdot y} \mathcal{O}_y \) for small \( Y \), so that we must have \( x = g \cdot y \) and consequently
\[
(3.24) \quad \kappa(x) - \kappa(g \cdot y) \perp V_{\omega}, \quad \text{for all } \omega' \in N_{g \cdot y} \mathcal{O}_y \cap \mathcal{U}_g,
\]
showing the inclusion "\( \subset \)" in (3.24). To see the converse, assume as in the proof of Lemma 3.1(b) that for each \( n \in \mathbb{N} \) there is a point \( y_n \in W_n \cap Y \cap S_x \) approaching \( x \) and a smooth curve
\[
(-\varepsilon_n, \varepsilon_n) \ni t \mapsto (\alpha'_n(t), g_n(t)) \in \text{Crit } \Phi^\mathcal{J}_{x,y,\omega} \]
parametrized such that \( \|\alpha'_n(t)\| = 1 \). Since the vectors \( \kappa(x) - \kappa(g_n(t) \cdot y_n) \) are approximately normal to \( T_{g_n(t) \cdot y_n} \mathcal{O}_{y_n} \), the curves \( \{g_n(t) \cdot y_n \mid t \in (-\varepsilon_n, \varepsilon_n)\} \subset Y \) must converge to \( x \) as \( n \to \infty \), while \( \varepsilon_n \to 0 \). Similarly, due to the compactness of \( \Sigma^{R,t}_x \) the curves
\[
\{F^\mathcal{J}_\omega(\alpha'_n(t), \omega') \mid t \in (-\varepsilon_n, \varepsilon_n)\} \subset \Sigma^{R,t}_x
\]
converge to at least one \( F^\mathcal{J}_\omega(\alpha'_\infty, \omega') \in \Sigma^{R,t}_x \cap N_x \mathcal{O}_x \), eventually after passing to a suitable convergent subsequence. By assumption, \( G \) acts on \( M \) with orbits of the same dimension \( \kappa \), so that we can assume that all orbits in \( Y \) are diffeomorphic. Consequently, the more \( y_n \) approaches \( x \), the faster the direction of \( \kappa(x) - \kappa(g_n(t) \cdot y_n) \) changes in orbital direction as \( t \in (-\varepsilon_n, \varepsilon_n) \) varies. Thus, the Gaussian curvature of \( \Sigma^{R,t}_x \) at \( F^\mathcal{J}_\omega(\alpha'_\infty, \omega') \) cannot stay bounded in \( \alpha'_\infty \)-direction, and we have shown that for sufficiently small \( Y \) we locally have \( \text{Crit } \Phi^\mathcal{J}_{x,y,\omega} \simeq G_y \), obtaining the inclusion "\( \subset \)" in (3.24). In any of the cases \( y \in \mathcal{O}_x \) or \( y \in \mathcal{O}_x \) we see that \( \text{Crit } \Phi^\mathcal{J}_{x,y,\omega} \) is given by (3.24) and, in particular, a smooth submanifold of codimension \( 2\kappa \). To see that it is clean, note that with the notation and the arguments in the proof of the previous theorem the transversal Hessian of \( \Phi^\mathcal{J}_{x,y,\omega} \) is constant on its critical set, and can be represented by the \((2\kappa \times 2\kappa)\)-matrix
\[
\begin{pmatrix}
\|x - g \cdot y\| \text{ diag}(e_1, \ldots, e_\kappa) & \frac{\partial}{\partial \alpha_i | \omega', \tilde{X}_{J,g \cdot y}} \\
\langle \frac{\partial}{\partial \alpha_i | \omega', \tilde{X}_{J,g \cdot y}}, \tilde{X}_{i,g \cdot y} \rangle & -\frac{1}{2} \left( \tilde{X}_{i,g \cdot y}((\tilde{X}_j, \omega')) + \tilde{X}_{j,g \cdot y}((\tilde{X}_i, \omega')) \right)
\end{pmatrix}
\]
with \( \omega' = F^\mathcal{J}(0, \omega') \), where the vector fields \( \{\tilde{X}_1, \ldots, \tilde{X}_\kappa\} \) constitute an orthonormal basis of \( T_{g \cdot y} \mathcal{O}_y \) at \( g \cdot y \), while the vector fields \( \{\tilde{X}_{n+1}, \ldots, \tilde{X}_d\} \) vanish at \( g \cdot y \). As in (3.22) one then computes
\[
(3.26) \quad \det \text{Trans Hess } \Phi^\mathcal{J}_{x,y,\omega} = c_0 + \|x - g \cdot y\| e_1 + \cdots + \|x - g \cdot y\| \kappa c_\kappa
\]
with $c_i \in \mathbb{R}$ and $c_0 \neq 0$ uniformly in $x$ and $y$, showing the cleanness of $\text{Crit } \Phi^J_{x,y,\omega'}$ for sufficiently small $Y$. Now, applying Theorem A.1 to the inner integral in (3.23) we obtain

$$I^J_{x,y} (\mu) = \mu^{-\kappa} \int_{N_{x,y} \cap U_J} e^{\mu^0 \Phi^J_{x,y}(\omega')} \left[ N_{k=0}^{N_1-1} Q_{J,k} (x,y,\omega') \mu^{-k} + R_{J,N_1} (x,y,\omega',\mu) \right] d\omega',$$

where $0 \Phi^J_{x,y}(\omega') := (\kappa(x) - \kappa(g_{J'} \cdot y), \omega')$ stands for the constant value of $\Phi^J_{x,y,\omega'}$ on its critical set, and the coefficients and the remainder satisfy the usual estimates. Note that they are bounded from above by negative powers of the determinant of the transversal Hessian of $\Phi^J_{x,y,\omega'}$. But since by (3.20) this determinant of Trans Hess $\Phi^J_{x,y,\omega'}$ is uniformly bounded away from zero in $x$ and $y$, the coefficients $Q_{J,k} (x,y,\omega')$ and the remainder $R_{J,N_1} (x,y,\omega',\mu)$ must be uniformly bounded in $x$ and $y$. Furthermore, they are bounded by derivatives with respect to $(\alpha', g)$ of the amplitude $a_J$ up to order $2k_1$ and $2N_1 + \kappa + 1$, or, if the amplitude factors according to $a(x,y,\omega) = a_1(x,y,\omega) a_2(x,y,g)$, by derivatives up to order $2N_1 + \lceil \kappa/2 \rceil$, respectively, compare Remark A.2. If $y \in O_x$, we have $0 \Phi^J_{x,y}(\omega') = 0$ for all $\omega'$, and we recover the asymptotic expansion for $I^J_{x,y}(\mu)$ derived in Theorem 3.3 (a). Let us therefore assume that $y \notin O_x$. If $\kappa = n - 1$, the set $N_{g_J \cdot y} O_y \cap U_J$ consists only of $\omega_J$, and we are done. If $\kappa < n - 1$, in order to apply the stationary phase principle to the integral over $\omega'$, observe that

$$\text{Crit } 0 \Phi^J_{x,y} = \{ \omega' \in N_{g_J \cdot y} O_y \cap U_J \mid \kappa(x) - \kappa(g_{J'} \cdot y) \in N_{x,y} \Sigma_x^{R,t} \} = \{ \omega_J \},$$

$$\text{since } \kappa(x) - \kappa(g_{J'} \cdot y) \downarrow V_{\omega'} \text{ by (3.25), so that } \text{grad}_{\omega'} 0 \Phi^J_{x,y} = 0 \iff \kappa(x) - \kappa(g_{J'} \cdot y) \in N_{\omega} \Sigma_x^{R,t}.$$

Further, the Hessian of $0 \Phi^J_{x,y}$ at the critical point $\omega_J$ is given by

$$\text{Hess } 0 \Phi^J_{x,y}(\omega_J) = -\| \kappa(x) - \kappa(g_{J'} \cdot y) \| \text{ diag}(\varrho_n, \ldots, \varrho_{n-1}) |_{\omega_J}.$$

Consequently, $\omega_J$ is a non-degenerate critical point due to the strict convexity of $\Sigma_x^{R,t}$. In order to find an interpolation formula we proceed as in Appendix A and write

$$e^{i\mu 0 \Phi^J_{x,y}(\omega')} = e^{i(\| \kappa(x) - \kappa(g_{J'} \cdot y) \| + 1)^0 \Phi^J_{x,y}(\omega')} e^{-i 0 \Phi^J_{x,y}(\omega')} \Psi_{x,y}(\omega,g) := \frac{\Phi_{x,y}(\omega,g)}{\| \kappa(x) - \kappa(g_{J'} \cdot y) \|}.$$

Theorem A.1 then implies for $\kappa < n - 1$ with $0 \Phi^J_{x,y}$ as phase function and $\mu \| \kappa(x) - \kappa(g_{J'} \cdot y) \| + 1$ as asymptotic parameter the asymptotic expansion

$$\int_{N_{x,y} \cap U_J} e^{i\mu 0 \Phi^J_{x,y}(\omega')} Q_{J,k} (x,y,\omega') d\omega' = \left( \mu \| \kappa(x) - \kappa(g_{J'} \cdot y) \| + 1 \right)^{(n-\kappa-1)/2} e^{i\mu 0 \Phi^J_{x,y}}$$

$$\cdot \left[ N_{k=0}^{N_2-1} (\mu \| \kappa(x) - \kappa(g_{J'} \cdot y) \| + 1)^{-k} Q_{J,k} (x,y,\omega_J) + R_{J,N_2} (x,y,\omega_J,\mu) \right],$$

where $0 \Phi^J_{x,y}$ denotes the constant value of $0 \Phi^J_{x,y}(\omega')$ at $\omega_J$, which equals the constant value of $\Phi_{x,y}$ at $(\omega_J, g_{J'}) \in J$. Note that the coefficients $Q_{J,k} (x,y,\omega_J)$ and the remainder $R_{J,N_2} (x,y,\omega_J,\mu)$ are uniformly bounded in $x, y$ in view of (3.27), and are not bounded by additional derivatives with respect to $g$. Treating the remainders $R_{J,N_1}$ alike, the theorem follows.

To close this section, let us still consider the case of a finite group $G$. For this, one has to examine the asymptotic behavior of oscillatory integrals of the form

$$(3.28) I_z (\nu) := \int_{\Sigma} e^{i\nu \Psi_z (\omega)} a(\omega) d\Sigma(\omega), \quad z \in S^{n-1}, \quad \nu \to +\infty,$$

where $\Sigma \subset \mathbb{R}^n$ denotes a strictly convex $C^\infty_0$-hypersurface, $d\Sigma$ the induced volume density, and $\Phi_z$ the phase function $\Psi_z (\omega) := \langle z, \omega \rangle$, while $a \in C^\infty_0 (\Sigma)$ is an amplitude that might depend on $\nu$ and other parameters.
Lemma 3.5. For every $\hat{N} \in \mathbb{N}$ one has the asymptotic formula

$$I_z(\nu) = \sum_{\omega_0 \in \text{Crit } \Psi_z} \frac{e^{i\mu \Psi_z(\omega_0)}}{\left(\det (\nu \Pi_{\omega_0}/2\pi i)\right)^{1/2}} \left[\sum_{j=0}^{\hat{N}-1} Q_j(a, \Psi_z, \omega_0)\nu^{-j} + R_{\hat{N}}(a, \Psi_z, \omega_0, \nu)\right]$$

as $\mu \to +\infty$, where the critical set of $\Psi_z$ is given by

$$\text{Crit } \Psi_z = \left\{ \omega \in \Sigma \mid z \in N_\omega \Sigma \right\},$$

and only consists of non-degenerate, isolated points, while $\Pi$ denotes the second fundamental form of $\Sigma$. The coefficients and the remainder satisfy the bounds

$$|Q_j| \leq C_j \sup_{l \leq 2j} |D^l a(\omega_0)|, \quad |R_{\hat{N}}| \leq \bar{C}_{\hat{N}} \sup_{l \leq \left\lfloor (n-1)/2 + 1\right\rfloor + 2\hat{N}} \|D^l a\|_{\infty, \Sigma} \mu^{-\hat{N}}$$

for suitable constants $C_j, \bar{C}_{\hat{N}} > 0$ independent of $z$ and $\nu$, where $D^l$ denotes a differential operator on $\Sigma$ of order $l$. In particular,

$$Q_0(a, \Psi_z, \omega_0) = a(\omega_0).$$

Proof. The statement of the proposition is essentially known \cite[Theorem 7.7.14]{14}, but for completeness, we include a proof here. Consider a local parametrization

$$F : \mathbb{R}^{n-1} \supset U \longrightarrow \Sigma \subset \mathbb{R}^n, \quad \xi \longmapsto F(\xi) = \omega,$$

of the hypersurface $\Sigma$. If we compute the derivatives of $\Psi_z$ with respect to this parametrization and set them equal to zero, we arrive at the conditions $(z, \partial F/\partial \xi_i) = 0$ for $i = 1, \ldots, n-1$, implying that $z$ must be normal to $\Sigma$ at $\omega$. Thus, $\text{Crit } \Psi_z = \{ \omega \in \Sigma \mid z \in N_\omega \Sigma \}$. Since $\Sigma$ is strictly convex, the Gauss map $\mathcal{N} : \Sigma \ni \omega \longmapsto \mathcal{N}(\omega) \in \partial \Sigma$ is a global diffeomorphism, so that for each $\tilde{z} \in S^{n-1}$ there is a unique $\omega_z \in \Sigma$ such that $\tilde{z} = \mathcal{N}(\omega_z)$. Consequently, $\omega \in \text{Crit } \Psi_z$ is locally uniquely determined by the condition $\mathcal{N}(\omega) = \pm \mathcal{N}(\omega_z)$, so that $\omega$ is an isolated point. In order to see that $\text{Crit } \Psi_z$ consists of non-degenerate points, note that with respect to the parametrization \eqref{3.29} of $\Sigma$ the Hessian of $\Psi_z$ at a critical point $\omega$ is given by the matrix

$$\text{Hess } \Psi_z(\omega) \equiv \left( \langle z, \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} (F^{-1}(\omega)) \rangle \right)_{1 \leq i,j \leq n-1}.$$  

Since $z \in N_\omega \Sigma$, $\text{Hess } \Psi_z(\omega)$ corresponds to the second fundamental form II of $\Sigma$, compare \cite[Chapter VII, Section 3]{17}. Because $\Sigma$ is strictly convex, the eigenvalues of II at $\omega$, which are given by the principal curvatures of $\Sigma$ at that point, are all non-zero. Therefore, the determinant of $\text{Hess } \Psi_z(\omega)$ is non-zero, and $\omega$ must be a non-degenerate critical point. In conclusion, $\Psi_z$ has a clean critical set, so that the asymptotic formula for $I_z(\nu)$, together with the estimates for $Q_j$ and the remainder, follow directly by applying Theorem \ref{A.1} to $I_z(\nu)$. \hfill \square

We now have the following

Proposition 3.6. Consider the integrals $I_{x,y}(\mu)$ defined in \eqref{3.1}. Assume that $G$ is finite, and that the co-spheres $S^*_x M$ are strictly convex. Then, for arbitrary $\hat{N} \in \mathbb{N}$ one has the asymptotic formula

$$I_{x,y}(\mu) = \sum_{x \in G, \omega_0 \in \text{Crit } \Phi_{x,y,g}(\omega)} \frac{e^{i\mu \Phi_{x,y,g}(\omega_0)}}{\left(\mu \|\kappa(x) - \kappa(g \cdot y)\| + 1\right)^{n/2}} \left[\sum_{k=0}^{\hat{N}-1} \frac{Q_{\omega_0,k}(x, y, g)}{\mu \|\kappa(x) - \kappa(g \cdot y)\| + 1}^k + R_{\omega_0,\hat{N}}(x, y, g, \mu)\right]$$

as $\mu \to +\infty$, where $\Phi_{x,y,g}(\omega) := \Phi_{x,y}(\omega, g)$. The coefficients and the remainder term

$$R_{\omega_0,\hat{N}}(x, y, g, \mu) = O\left((\mu \|\kappa(x) - \kappa(g \cdot y)\| + 1)^{-\hat{N}}\right)$$

are explicitly given and depend smoothly on $R, t$. Furthermore, they are uniformly bounded in $x$ and $y$. 

Proof. For finite $G$, the integral (3.1) reads

$$I_{x,y}(\mu) := \sum_{g \in G} I_{x,y,g}(\mu), \quad I_{x,y,g}(\mu) := \int_{\Sigma_{x,t}} e^{i\mu \Phi_{x,y,g}(\omega)} a(x,y,\omega,g) d\Sigma_{x,t}(\omega).$$

Assume that $x \neq g \cdot y$. Writing

$$e^{i\mu \Phi_{x,y,g}(\omega)} = e^{i(\|\kappa(x) - \kappa(g \cdot y)\|/\mu + 1)} \Phi_{x,y,g}(\omega) e^{-i \Psi_{x,y,g}(\omega)}, \quad \Psi_{x,y,g}(\omega) := \frac{\Phi_{x,y,g}(\omega)}{\|\kappa(x) - \kappa(g \cdot y)\|},$$

the previous lemma implies with $\nu = \|\kappa(x) - \kappa(g \cdot y)\| \mu + 1$ as asymptotic parameter and $\Psi_{x,y,g}$ as phase function the expansion

$$I_{x,y,g}(\mu) = \sum_{\omega \in \text{Crit} \Psi_{x,y,g}} \frac{e^{i\nu \Psi_{x,y,g}(\omega)}}{(\det (\nu \Pi_{\omega_0}/2\pi i))^{1/2}} \left[ \sum_{j=0}^{N-1} Q_j(a(x,y,\cdot,g)) e^{-i \Psi_{x,y,g}(\omega)} \nu^{-j} \right.$$\left. + R_N(a(x,y,\cdot,g)) e^{-i \Psi_{x,y,g}(\omega)} \nu^0, \right]$$
as $\mu \to +\infty$, where

$$\text{Crit} \Psi_{x,y,g} = \left\{ \omega \in \Sigma_{x,t} \mid \kappa(x) - \kappa(g \cdot y) \in N_{\omega \Sigma_{x,t}} \right\}$$

and all expressions are uniformly bounded in $x$ and $y$. If $x = g \cdot y$,

$$I_{x,y,g}(\mu) = \int_{\Sigma_{x,t}} a(x,y,\omega,g) d\Sigma_{x,t}(\omega),$$

and the assertion of the proposition follows by setting $\text{Crit} \Phi_{x,y,g} := \Sigma_{x,t}$ and replacing the sum over $\text{Crit} \Phi_{x,y,g}$ by an integral over $\Sigma_{x,t}$ in this case. \hfill $\square$

4. The equivariant local Weyl law

Let us now come back to our initial question of finding an asymptotic description of the equivariant spectral function. With the notation of the previous sections we have

**Proposition 4.1 (Point-wise asymptotics for the kernel of the equivariant approximate projection).** For any fixed $x \in M$, $\gamma \in \hat{G}$, and $\hat{N} \in \mathbb{N}$ one has as $\mu \to +\infty$

$$K_{\tilde{\chi},\Pi,\gamma}(x,x) = \sum_{j \geq 0, e_j \in L_{\gamma}(M)} \varrho(\mu - \mu_j) |e_j(x)|^2$$

$$= \left( \frac{\mu}{2\pi} \right)^{-n - \dim O_{\gamma} - 1} d_{\gamma} \left[ \frac{N-1}{2\pi} \sum_{k=0}^{\hat{N}-1} \mathcal{L}_k(x,\gamma) \mu^{-k} + \mathcal{R}_{\gamma}(x,\gamma) \right]$$

with known coefficients and remainder that depend smoothly on $x \in M_{\text{prin}}$. If $G$ is continuous, they satisfy the bounds

$$|\mathcal{L}_k(x,\gamma)| \leq C_{k,x} \sup_{l \leq 2k} \|D^l \gamma\|_{\infty}, \quad |\mathcal{R}_{\gamma}(x,\gamma)| \leq \tilde{C}_{\gamma,x} \sup_{l \leq 2\hat{N} + [\dim O_{\gamma}/2+1]} \|D^l \gamma\|_{\infty} \mu^{-\hat{N}},$$

where $D^l$ denotes a differential operator on $G$ of order $l$, and the constants $C_{k,x}$, $\tilde{C}_{\gamma,x}$ are uniformly bounded in $x$ if $M = M_{\text{prin}} \cup M_{\text{except}}$; if $G$ is finite, similar bounds hold with $l = 0$. In particular, the leading coefficient is given by

$$\mathcal{L}_0(x,\gamma) = \hat{g}(0)[\pi_{\gamma}\chi_{G_x} : 1] \text{vol}([\Omega \cap S^*x M]/G),$$

where $S^*M := \{(x,\xi) \in T^*M \mid p(x,\xi) = 1\}$, which for finite $G$ simply reads

$$\hat{g}(0) \sum_{g \in G_x} \bar{\chi}(g) \text{vol}([S^*x M]/G).$$

If $\mu \to -\infty$, the function $K_{\tilde{\chi},\Pi,\gamma}(x,x)$ is rapidly decreasing in $\mu$. 

Proof. Let the notation be as in Corollary 2.2 and \( R, t \in \mathbb{R}, x \in Y \) be fixed. If \( G \) is continuous, one deduces as a direct consequence of Theorem 3.3 (a) for any \( \bar{N} \in \mathbb{N} \)

\[
\partial_{R,t}^{\beta} \Gamma_{\gamma}^{\alpha}(\mu, R, t, x, x) = (2\pi/\mu)^{\dim O_x} \sum_{k=0}^{N-1} Q_{\alpha,\beta}^{k}(R, t, x, \gamma) \mu^{-k} + O_{R,t,x,\gamma}(\mu^{-\dim O_x - \bar{N}}),
\]

where the coefficients and the remainder term are explicitly given by distributions depending smoothly on \( R, t \), and \( x \in Y \setminus M_{\text{prin}} \). Furthermore, both the coefficients \( Q_{\alpha,\beta}^{k}(R, t, x, \gamma) \) and the remainder are bounded by expressions involving derivatives of \( \gamma \) up to order \( 2k \) and \( 2\bar{N} + [\dim O_x/2+1] \), respectively, which are uniformly bounded in \( x \) if \( M = M_{\text{prin}} \cup M_{\text{except}} \). Note that \( \Phi_{x,x} \) vanishes on its critical set \( \text{Crit}_{R,t}(\Phi_{x,x}) := \Omega \cap \Sigma^{R,t}_{x} \times G_x \) no matter what values \( R \) and \( t \) take. Otherwise differentiation with respect to \( R \) and \( t \) of the factor \( e^{i\mu \psi_0} \) in (4.2) with \( \psi_0 \equiv \Phi_{x,x}|_{\text{Crit}_{R,t}(\Phi_{x,x})} \) would yield additional positive powers of \( \mu \). Furthermore, \( a_{\varepsilon} \equiv S^0_{\varepsilon} \) is a classical symbol of order 0, so that

\[
|\partial_{\alpha} a_{\varepsilon}(t, \kappa_i(x), \mu, \omega)| = |\mu|^{|a|} |(\partial_{\alpha} a_{\varepsilon})(t, \kappa_i(x), \mu, \omega)| \leq C|\omega|^{-|a|}.
\]

Consequently, the dependence of the amplitude on \( \mu \) in (2.10) does not interfere with the asymptotics, compare [8, Proposition 1.2.4]. Corollary 2.2 then implies the asymptotic expansion (4.1) with

\[
\mathcal{L}_{0}(x, \gamma) = \sum_{i} f_i(x) \hat{\varrho}(0) \int_{\text{Crit}_{1,0} \Phi_{x,x}} \gamma(g) \frac{\operatorname{det} \Phi''_{x,x}(\omega,g)_{|N(x,\omega) \cap \text{Crit}_{1,0} \Phi_{x,x}|^{1/2}}}{\operatorname{vol} O_x(x,\omega)} d(Crit_{1,0} \Phi_{x,x}(\omega, g)),
\]

since \( a(q(x, \omega)) = 1 \) on \( \Sigma^{1,0}_{x} \) and \( J(g, x) = 1 \) for \( g \in G_x \). In order to compute \( \mathcal{L}_{0}(x, \gamma) \), let us note that for any \( x \in Y \) and smooth, compactly supported function \( f \) on \( \Omega \cap \Sigma^{R,t}_{x} \) one has the formula

\[
\int_{\text{Crit}_{R,t} \Phi_{x,x}} \frac{\gamma(g) f(x, \omega)}{\operatorname{det} \Phi''_{x,x}(\omega,g)_{|N(x,\omega) \cap \text{Crit}_{R,t} \Phi_{x,x}|^{1/2}} d(Crit_{R,t} \Phi_{x,x}(\omega, g)) = [\pi_{\gamma}|_{G_x} : 1] \int_{\Omega \cap \Sigma^{R,t}_{x}} f(x, \omega) \frac{d(\Omega \cap \Sigma^{R,t}_{x})}{\operatorname{vol} O_x(x,\omega)},
\]

where we took into account that \( \int_{G_x} \gamma(g) dG_x(g) = [\pi_{\gamma}|_{G_x} : 1] \), compare [5, Lemma 7], [22, Proof of Theorem 9.5], and [8, Section 3.3.2], the map \( \text{Crit}_{R,t} \Phi_{x,x} \to \Omega \cap \Sigma^{R,t}_{x} \) being a submersion. As a consequence of this, we obtain for \( \mathcal{L}_{0}(x) \) the expression

\[
\mathcal{L}_{0}(x, \gamma) = \hat{\varrho}(0)[\pi_{\gamma}|_{G_x} : 1] \sum_{i} f_i(x) \int_{\Omega \cap \Sigma^{1,0}_{x}} d(\Omega \cap \Sigma^{1,0}_{x}) \frac{\operatorname{vol} O_x(x,\omega)}{\operatorname{vol} O_x(x,\omega)} = \hat{\varrho}(0)[\pi_{\gamma}|_{G_x} : 1] \operatorname{vol} [\Omega \cap S^0_{x} M] / G].
\]

The case when \( G \) is finite can be deduced from Proposition 3.6 in an analogous way, since then \( \Omega = T^* M \).

\[\square\]

Remark 4.2. Note that, if \( M = M_{\text{prin}} \cup M_{\text{except}} \), the previous proposition and the Cauchy-Schwarz inequality imply for \( \bar{N} = 0 \) with \( \kappa := \dim G/K \) the estimate

\[
K_{\chi_0 \circ \Pi, \Pi}(x, y) \leq \sqrt{K_{\chi_0 \circ \Pi, \Pi}(x, x)} \sqrt{K_{\chi_0 \circ \Pi, \Pi}(y, y)} \ll \mu^{n-\kappa-1} d_{\kappa/2+1} \sup_{t \leq [\kappa/2+1]} \|D^{t}\|_{\infty},
\]

uniformly in \( x \) and \( y \), \( \varrho \in S(\mathbb{R}, \mathbb{R}_+) \) being a positive function.

Using a standard Tauberian argument, we can now deduce from Proposition 4.1 our first main result.

Theorem 4.3 (Equivariant local Weyl law). Let \( M \) be a closed connected Riemannian manifold \( M \) of dimension \( n \) carrying an isometric and effective action of a compact Lie group \( G \), and \( P_0 \) a \( G \)-invariant elliptic classical pseudodifferential operator on \( M \) of degree \( m \). Let \( p(x, \xi) \) be its principal symbol, and assume that \( P_0 \) is positive and symmetric. Denote its unique self-adjoint extension by \( P \),
and for a given $\gamma \in \hat{G}$ let $e_\gamma(x, y, \lambda)$ be its induced spectral function. Further, let $\mathcal{J} : T^*M \to \mathfrak{g}^*$ be the momentum map of the $G$-action on $M$, and put $\Omega := \mathcal{J}^{-1}({\{0\}})$. Then, for fixed $x \in M$ one has

$$
(4.2) \quad \left| e_\gamma(x, x, \lambda) - \frac{d_\gamma}{2\pi} \int_{\{\xi \mid (x, \xi) \in \Omega, p(x, \xi) < 1\}} \frac{d\xi}{\operatorname{vol}(O_{(x, \xi)})} \right| \leq C_{x, \gamma} \lambda \frac{n-m}{m}
$$

as $\lambda \to +\infty$, where $n := \dim O_x$, $d_\gamma$ denotes the dimension of an irreducible $G$-representation $\pi_\gamma$, belonging to $\gamma$ and $[\pi_\gamma|_{G_x} : 1]$ the multiplicity of the trivial representation in the restriction of $\pi_\gamma$ to the isotropy group $G_x$ of $x$. If $G$ is continuous,

$$
(4.3) \quad C_{x, \gamma} = O_x \left( d_\gamma \sup_{\lambda \leq \dim C_x/2+3} \|D^l \gamma\|_\infty \right)
$$

is a constant that depends smoothly on $x \in M_{\text{prin}}$ and is uniformly bounded in $x$ if $M = M_{\text{prin}} \cup M_{\text{except}}$; if $G$ is finite, $C_{x, \gamma} = O(d_\gamma \|\|_{\infty})$.

Proof. This follows directly by taking $\bar{N} = 1$ in (4.1) and integrating with respect to $\mu$ from $-\infty$ to $\sqrt{\lambda}$ with the arguments given in [9, Proof of Eq. (2.25)]. \]

Remark 4.4.

(1) Note that in view of (3.5) the integral in the leading term can also be written as

$$
\lambda \frac{n-m}{m} \int_{\{\xi \mid (x, \xi) \in \Omega, p(x, \xi) < 1\}} \frac{d\xi}{\operatorname{vol}(O_{(x, \xi)})} = \int_{\{\xi \mid (x, \xi) \in \Omega, p(x, \xi) < \lambda^{1/m}\}} \frac{d\xi}{\operatorname{vol}(O_{(x, \xi)})}.
$$

(2) The formula (4.2) is shown by proving first the estimate

$$
|e_\gamma(x, x, \lambda + 1) - e_\gamma(x, x, \lambda)| \leq C_{x, \gamma} \lambda \frac{n-m}{m}, \quad x \in M,
$$

compare [9, Lemma 2.3]. Since $e_\gamma(x, y, \lambda + 1) - e_\gamma(x, y, \lambda)$ is the kernel of a positive operator, one immediately infers from this with the Cauchy-Schwarz inequality the bound

$$
|e_\gamma(x, y, \lambda + 1) - e_\gamma(x, y, \lambda)| \leq \sqrt{C_{x, \gamma} \lambda \frac{n-m}{m}} \sqrt{C_{y, \gamma} \lambda \frac{n-m}{m}}, \quad x, y \in M.
$$

From this, it is not difficult to deduce a corresponding equivariant local Weyl law for $e_\gamma(x, y, \lambda)$ in a neighborhood of the diagonal, see [13, pp. 210] or [27, Section 21].

Remark 4.5. In case that $G$ is a connected compact semisimple Lie group, the bound (4.3) of the previous theorem can be rephrased using the Cartan-Weyl classification of unitary irreducible representations of $G$. In fact, let $\mathfrak{g}$ be its Lie algebra, and $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$. Denote by $\mathfrak{g}_C$ and $\mathfrak{t}_C$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Then $\mathfrak{t}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$, and we write $\Sigma(\mathfrak{g}_C, \mathfrak{t}_C)$ for the corresponding system of roots and $\Sigma^+$ for a set of positive roots. Since any element in $G$ is conjugated to an element of $T$, a character $\gamma \in \hat{G}$ is fully determined by its restriction to $T$. Now, as a consequence of the Cartan-Weyl classification of irreducible finite-dimensional representations of reductive Lie algebras over $\mathbb{C}$ one has the identification

$$
\hat{G} \simeq \{ \Lambda \in \mathfrak{t}_C^* \mid \Lambda \text{ is dominant integral and } T\text{-integral} \},
$$

compare [35], and we write $\Lambda_\gamma \in \mathfrak{t}_C^*$ for the highest weight corresponding to $\gamma \in \hat{G}$ given by this isomorphism. Weyl’s dimension formula then implies that $d_\gamma = O(|\Lambda_\gamma|^{\dim \mathcal{O}_x/2+3})$, while from Weyl’s character formula one infers that if $D^l$ is a differential operator on $G$ of order $l$,

$$
\|D^l \gamma\|_{\infty} = O(|\Lambda_\gamma|^{l+|\Sigma^+|}), \quad |\Lambda_\gamma| \to \infty,
$$

compare [23, Eq. (3.5)]. Consequently, the bound (4.3) can be rewritten as

$$
C_{x, \gamma} = O_x \left( |\Lambda_\gamma|^{\dim \mathcal{O}_x/2+3+2|\Sigma^+|} \right)
$$

As a first consequence of Theorem 4.3 let us note that the estimate (4.4) is equivalent to the following bound for spectral clusters.
Corollary 4.6 (Point-wise bounds for isotypic spectral clusters). In the situation of Theorem 4.3 we have
\[ \sum_{\lambda_j \leq \lambda} |e_j(x)|^2 \leq C_{x,\gamma} \lambda^{\frac{n-\kappa_x-1}{m}}, \quad x \in M, \]
where \( \{e_j\} \) denotes an orthonormal basis of \( L^2(M) \) consisting of eigenfunctions of \( P \) with eigenvalues \( \{\lambda_j\} \).

\[ \square \]

A further implication of Theorem 4.3 is the following Kuznecov sum formula for periods of \( G \)-orbits, which generalizes the classical Kuznecov formula for periods of closed geodesics [36].

Corollary 4.7 (Generalized Kuznecov sum formula for periods of \( G \)-orbits). In the setting of Theorem 4.3 we have
\[ \left| \sum_{\lambda_j \leq \lambda} \left| \int_G e_j(g^{-1} \cdot x) \, dg \right|^2 - \frac{\text{vol} G_x}{(2\pi)^n} \lambda^{\frac{n-\kappa_x}{m}} \int_{\{\xi | (x,\xi) \in \Omega, p(x,\xi) < 1\}} \frac{d\xi}{\text{vol} \Omega(x,\xi)} \right| \leq C_x \lambda^{\frac{n-\kappa_x-1}{m}} \]
for some constant \( C_x > 0 \) depending on \( x \).

\[ \text{Proof.} \] Let \( \gamma = \gamma_{\text{triv}} \) correspond to the trivial representation. Then
\[ e_{\gamma_{\text{triv}}}(x, x, \lambda) = \sum_{\lambda_j \leq \lambda, e_j \in L^2_{\text{triv}}(M)} |e_j(x)|^2 = \sum_{\lambda_j \leq \lambda} \left| \int_G e_j(g^{-1} \cdot x) \, dg \right|^2, \]
and the assertion follows from the previous theorem with \( d_{\gamma_{\text{triv}}} = 1 \) and
\[ [\pi_{\gamma_{\text{triv}}|G_x} : 1] = \int_{G_x} \gamma_{\text{triv}}(g) \, dG_x(g) = \text{vol} G_x. \]

\[ \square \]

In case that \( \widetilde{M} := M/G \) is an orbifold we essentially recover the description of the spectral function of a Riemannian orbifold given by Stanhope and Uribe in [32]. More precisely, we infer

Corollary 4.8 (Local Weyl law for Riemannian orbifolds). In the situation of Theorem 4.3, assume that \( G \) acts on \( M \) with finite isotropy groups. Then, for fixed \( x \in M \) and \( \gamma \in \hat{G} \) the asymptotic formula [12] holds with \( n-\kappa_x \equiv n-\kappa \) being equal to the dimension of \( \widetilde{M} \). Moreover, let \( \gamma_{\text{triv}} \) be the trivial representation. Then \( e_{\gamma_{\text{triv}}}(x, x, \lambda) \) is \( G \)-invariant, and descends to a function on \( \widetilde{M} \times \widetilde{M} \) satisfying
\[ |e_{\gamma_{\text{triv}}} (\tilde{x}, \tilde{x}, \lambda) - \frac{|G_{\tilde{x}}|}{(2\pi)^{\dim M}} \lambda^{\frac{\dim \widetilde{M} - 1}{m}} \text{vol} (S^*_{p,\tilde{x}}(\widetilde{M}))| \leq C_{\tilde{x}} \lambda^{\frac{\dim \widetilde{M} - 1}{m}}, \quad \tilde{x} \in \widetilde{M}, \]
where \( (G_{\tilde{x}}) \) denotes the isotropy type of \( \tilde{x} := G \cdot x, |G_{\tilde{x}}| \) its cardinality, while \( S^*_{p,\tilde{x}}(\widetilde{M}) \) equals the fiber over \( \tilde{x} \) of the orbifold bundle \( S^*_p(\widetilde{M}) := \{ (\tilde{x}, \xi) \in T^* \widetilde{M} | \overline{p}(\tilde{x}, \xi) = 1 \} \), \( \overline{p} \) being the function on \( \widetilde{M} \) induced by \( p \).

\[ \text{Proof.} \] The first assertion is clear, since all \( G \)-orbits on \( M \) have the same dimension \( \kappa \), so that no singular orbits are present. To see the second note that since \( G_x \) is finite, one computes
\[ [\pi_{\gamma_{\text{triv}}|G_x} : 1] = \int_{G_x} \gamma_{\text{triv}}(g) \, dG_x(g) = \sum_{l=1}^{|G_x|} 1 = |G_x|, \]
\( dG_x \) being the counting measure. For the volume factor, see [18, Remark 6.2].

\[ \square \]
Example 4.9. Let us consider the case where $M = T^2 \subset \mathbb{R}^3$ is the standard 2-torus on which $G = \text{SO}(2)$ acts by rotations around the symmetry axis. Then all orbits are 1-dimensional and of principal type, and Theorem 4.3 yields with the identification $\mathbb{Z} \simeq \widehat{\text{SO}}(2)$ for the reduced spectral function of the Laplace-Beltrami operator

$$e_m(x, x, \lambda) - \frac{1}{2\pi} \sqrt{\lambda} \int_{\xi \in \{ (x, \xi) \in \Omega, \|x, \xi\| < 1 \}} \frac{d\xi}{\text{vol} \mathcal{O}(x, \xi)} = O(1 + |m|^3), \quad m \in \mathbb{Z},$$

uniformly in $x \in T^2$, the irreducible characters of $\text{SO}(2)$ being given by the exponentials $\theta \mapsto e^{im\theta}$, $\theta \in [0, 2\pi) \simeq \text{SO}(2)$, $m \in \mathbb{Z}$.

Example 4.10. Consider a connected semisimple Lie group $G$ with finite center and Lie algebra $\mathfrak{g}$, together with a discrete co-compact subgroup $\Gamma$. In particular, $\Gamma$ might have torsion, meaning that there are non-trivial elements of $\Gamma$ conjugate in $G$ to an element of $K$. Let $K$ be a maximal compact subgroup of $G$, and choose a left-invariant metric on $G$ given by an $\text{Ad}(K)$-invariant bilinear form on $\mathfrak{g}$. The quotient $M := \Gamma \setminus G$ is a compact manifold without boundary, and has a Riemannian structure induced by the one of $G$. Furthermore, $K$ acts on $\Gamma \setminus G$ from the right in an isometric and effective way, and the isotropy group of a point $\Gamma g$ is conjugate to the finite group $gKg^{-1} \cap \Gamma$. Hence, all $K$-orbits in $\Gamma \setminus G$ are either principal or exceptional, $\Gamma \setminus G \setminus K$ is an orbifold, and Corollary 4.8 applies.

Example 4.11. Let us now consider a case where singular orbits are present, and $M = S^2 \subset \mathbb{R}^3$ be the standard 2-sphere on which $G = \text{SO}(2) \subset \text{SO}(3)$ acts by rotations around the $x_3$-axis with fixed points $x_N = (0, 0, 1)$ and $x_S = (0, 0, -1)$. In this case the phase function of $I_x(\mu)$ reads $\Phi_x(\omega, g) = \langle x - g \cdot x, \omega \rangle$ with respect to standard coordinates in $\mathbb{R}^3$. For $x = x_N, x_S$ it simply vanishes, so that $I_x(\mu)$ is independent of $\mu$ in this case, which is consistent with the asymptotics

$$I_x(\mu) = \begin{cases} O(\mu^0), & \mu = x_N, x_S, \\ O(\mu^{-1}), & \text{otherwise,} \end{cases}$$

implied by Theorem 3.3. Let us now apply Theorem 4.3 to the Laplace-Beltrami operator $-\Delta$ on $S^2$, and notice for this that the orbit volume $\text{vol} \mathcal{O}(x, \xi)$ is of order $\sqrt{\xi_1^2 + \xi_2^2 + \sqrt{x_1^2 + x_2^2}}$ for arbitrary $x$ and $\xi$. By Theorem 4.3 and with the identification $\text{SO}(2) \simeq \mathbb{Z}$ the reduced spectral function satisfies on $S^2_{\text{prin}} = S^2 - \{x_N, x_S\}$ the estimate

$$|e_m(x, x, \lambda) - \frac{\sqrt{\lambda}}{2\pi} \int_{\{ (x, \xi) \in \Omega, \|x, \xi\| < 1 \}} \frac{d\xi}{\text{vol} \mathcal{O}(x, \xi)}| \leq C_x (1 + |m|^3), \quad x \in S^2_{\text{prin}}, m \in \mathbb{Z}. \tag{4.5}$$

In this case, $\Omega \cap T^*_x(S^2)$ is 1-dimensional; the integral in (4.5) is finite, but as $S^2_{\text{prin}} \ni x \to x_N$ or $x_S$ the orbit volume becomes of order $\sqrt{\xi_1^2 + \xi_2^2}$, so that the mentioned integral goes to infinity. On the other hand, for the fixed points $x = x_N, x_S$ the space $\Omega \cap T^*_x(S^2) = T^*_{x_N}S^2$ is 2-dimensional and Theorem 4.3 yields

$$|e_m(x, x, \lambda) - \frac{\pi_{m|G} : 1}{(2\pi)^2} \lambda \int_{\{ (x, \xi) \in \Omega, \|x, \xi\| < 1 \}} \frac{d\xi}{\text{vol} \mathcal{O}(x, \xi)}| \leq C_x (1 + |m|^3) \sqrt{\lambda}, \quad x = x_N, x_S, m \in \mathbb{Z}, \tag{4.6}$$

where

$$[\pi_{m|G} : 1] = \begin{cases} 1, & m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, at the fixed points only the trivial representation contributes to the main term in the asymptotic formula for the spectral function given by the local Weyl law (1.1). Further note that, though for $x = x_N, x_S$ the orbit volume is proportional to $\sqrt{\xi_1^2 + \xi_2^2}$, its inverse is still locally integrable on $T^*_xS^2$, and the integral in (4.6) certainly exists. Ultimately, the leading coefficient in (4.5) must blow up as $x$ approaches the fixed points in order to compensate for the fact that the leading power changes abruptly from $\sqrt{\lambda}$ to $\lambda$ at the fixed points. Note that the remainder estimates in (4.5) and (4.6) are consistent with the asymptotics (1.16) for the spherical function $Y_{k,0}$. 
5. Equivariant $L^p$-bounds of eigenfunctions for non-singular group actions

Let the notation be as in the previous sections. From the asymptotic formula for the equivariant spectral function proved in Theorem 4.3 we already deduced in Corollary 4.6 point-wise bounds for isotypic spectral clusters. Similarly, one immediately obtains in the non-singular case the following equivariant $L^\infty$-bounds for eigenfunctions.

**Proposition 5.1** ($L^\infty$-bounds for isotypic spectral clusters). Assume that $G$ acts on $M$ with orbits of the same dimension $\kappa$, and denote by $\chi_{\lambda}$ the spectral projection onto the sum of eigenspaces of $P$ with eigenvalues in the interval $(\lambda, \lambda + 1]$. Then, for any $\gamma \in G$,

\[
\| (\chi_{\lambda} \circ \Pi_{\gamma}) u \|_{L^\infty(M)} \leq C_{\gamma}(1 + \lambda)^{\frac{n-\kappa}{2m}} \| u \|_{L^2(M)}, \quad u \in L^2(M),
\]

where, if $G$ is continuous,

\[
C_{\gamma} = O \left( \sqrt{\frac{d_\gamma}{\sup_{\ell \leq \lceil \kappa/2+1 \rceil} \| D^\gamma \|_\infty}} \right).
\]

If $G$ is finite, one simply has $C_{\gamma} = O(\sqrt{d_\gamma \| \gamma \|_\infty})$. In particular, we obtain

\[
\| u \|_{L^\infty(M)} \ll C_{\gamma} \lambda^{\frac{n-\kappa-1}{2m}}
\]

for any eigenfunction $u \in L^2(M)$ of $P$ with eigenvalue $\lambda$ satisfying $\| u \|_{L^2} = 1$.

**Proof.** The assertion is a direct consequence of Theorem 4.3. In fact, standard arguments \cite[Eq. (3.2.6)]{30} imply that

\[
\| \chi_{\lambda} \circ \Pi_{\gamma} \|_{L^2 \rightarrow L^\infty}^2 = \left[ \sup_x \left( \int_M |K_{\chi_{\lambda} \circ \Pi_{\gamma}}(x, y)|^2 dM(y) \right)^{1/2} \right]^2
\]

\[
= \sup_x K_{\chi_{\lambda} \circ \Pi_{\gamma}}(x, x) = \sup_x \left[ e_{\gamma}(x, x, \lambda) - e_{\gamma}(x, x, \lambda) \right].
\]

Since $M = M_{\text{prin}} \cup M_{\text{except}}$, the assertion follows from (4.2). \(\square\)

It is instructive to see how Proposition 5.1 can be deduced directly from Proposition 4.1 by transferring the arguments given in \cite[pp. 50]{30} to the equivariant setting. By duality, the estimate 5.1 is equivalent to

\[
\| (\chi_{\lambda} \circ \Pi_{\gamma}) u \|_{L^2(M)} \leq C_{\gamma}(1 + \lambda)^{\frac{n-\kappa}{2m}} \| u \|_{L^1(M)}.
\]

In order to show the latter estimate, one considers again a Schwartz function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ satisfying $\varrho(0) = 1$ and $\text{supp } \varrho \subset (-\delta/2, \delta/2)$ for a given $\delta > 0$. If $\tilde{\chi}_{\lambda}$ denotes the corresponding approximate spectral projection, one then shows that (5.2) is implied by

\[
\| (\tilde{\chi}_{\lambda} \circ \Pi_{\gamma}) u \|_{L^2(M)} \leq C_{\gamma}(1 + \lambda)^{\frac{n-\kappa}{2m}} \| u \|_{L^1(M)}.
\]

Thus, one is left with the task of proving (5.3). Now, the $L^1 \rightarrow L^2$ operator norm can be estimated according to

\[
\| \tilde{\chi}_{\lambda} \circ \Pi_{\gamma} \|_{L^1 \rightarrow L^2}^2 = \sup_{y \in M} \int_M |K_{\tilde{\chi}_{\lambda} \circ \Pi_y}(x, y)|^2 dM(x)
\]

\[
= \sup_{y \in M} \sum_{j \geq 0, e_j \in L^2(M)} (\varrho(\lambda - \lambda_j)^2 |e_j(y)|^2 \leq \| \varrho \|_{L^\infty(\mathbb{R})} \sup_{y \in M} K_{\tilde{\chi}_{\lambda} \circ \Pi_y}(y, y).
\]

Hence, everything is shown, since by Proposition 4.1 we have the uniform bound

\[
|K_{\tilde{\chi}_{\lambda} \circ \Pi_y}(y, y)| \ll d_\gamma \sup_{\ell \leq \lceil \kappa/2+1 \rceil} \| D^\gamma \|_\infty (1 + \lambda)^{\frac{n-\kappa}{m}}, \quad y \in M = M_{\text{prin}} \cup M_{\text{except}},
\]
with \( l = 0 \) for \( G \) finite, and we obtain again \((5.1)\) with the slightly better estimate
\[
C_{\gamma} = O \left( \sqrt{d_{\gamma}} \frac{\sup_{t \leq \kappa \cdot 2^{\gamma+1}} \|D^{\gamma}u\|_{\infty}}{} \right). 
\]

**Remark 5.2.** If \( G \) is a connected compact semisimple Lie group, the bound \((5.4)\) can be rewritten in terms of the highest weight \( \Lambda_{\gamma} \in \mathfrak{t}^*_C \) of \( \gamma \in \hat{G} \), and we obtain
\[
C_{\gamma} = O \left( \sqrt{\|\Lambda_{\gamma}^{2\lambda^2+\left\lfloor \kappa/2+1 \right\rfloor}} \right),
\]
compare Remark 4.3.

**Example 5.3.** In the situation of Example 4.9, where \( M = T^2 \subset \mathbb{R}^3 \) is the standard 2-torus on which \( G = \text{SO}(2) \) acts by rotations, Proposition 5.1 and \((5.4)\) imply the bound
\[
\|u\|_{L^\infty(T^2)} = O \left( \sqrt{1 + |m|} \right), \quad u \in L^2_m(T^2), \quad \|u\|_{L^2} = 1,
\]
for any eigenfunction of \( P \) in a specific isotypic component, which in case of the Laplace-Beltrami operator \( \Delta \) are well-known. Indeed, via the identification
\[
\mathbb{R}^2 / \mathbb{Z}^2 \cong T^2 \simeq S^1 \times S^1, (x_1, x_2) \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2}),
\]
the standard orthonormal basis of \( \Delta \) is given by \( \{ e^{2\pi i k_1 x_1} e^{2\pi i k_2 x_2} \mid (k_1, k_2) \in \mathbb{Z}^2 \} \), showing that the above bound is sharp in the eigenvalue but not in the isotypic aspect.

In what follows, we shall derive refined \( L^p \)-bounds for isotypic spectral clusters using complex interpolation techniques. For this, we shall need the additional assumption that the co-spheres \( S^*_2 M \) are strictly convex. In essence, the proof is an elaboration of arguments from [26] applied to the equivariant setting. While for the proof of the \( L^\infty \)-bounds in the previous proposition it was sufficient to consider the asymptotic behaviour of the integrals \( I_{x,y}(\mu) \) in case that \( x = y \), the proof of \( L^p \)-estimates actually requires estimates for the integrals \( I_{x,y}(\mu) \) in a neighborhood of the diagonal, making things significantly more involved. This leads us to our second main result.

**Theorem 5.4 (\( L^p \)-bounds for isotypic spectral clusters).** Let \( M \) be a closed connected Riemannian manifold \( M \) of dimension \( n \) on which a compact Lie group \( G \) acts effectively and isometrically with orbits of the same dimension \( \kappa \). Further, let \( P \) be the unique self-adjoint extension of a \( G \)-invariant elliptic positive symmetric classical pseudodifferential operator on \( M \) of degree \( m \), and assume that its principal symbol \( p(x, \xi) \) is such that the co-spheres \( S^*_2 M := \{(x, \xi) \in T^* M \mid p(x, \xi) = 0\} \) are strictly convex. Denote by \( \chi_{\gamma} \) the spectral projection onto the sum of eigenspaces of \( P \) with eigenvalues in the interval \((\lambda, \lambda + 1] \), and by \( \Pi_{\gamma} \) the projection onto the isotypic component \( L^2_{\gamma}(M) \), where \( \gamma \in \hat{G} \). Then, for \( u \in L^2(M) \)
\[
\|(\chi_{\lambda} \ast \Pi_{\gamma}) u\|_{L^q(M)} \leq \begin{cases} C_{\gamma} \frac{(n-\kappa)(n-2\lambda)}{m} \|u\|_{L^2(M)}, & 2(n-\kappa+1) \leq q \leq \infty, \\ C_{\gamma} \frac{1+1}{4m^q} \|u\|_{L^2(M)}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}
\]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \),
\[
\delta_{n-\kappa}(q) := \max \left( n - \kappa, \left\lfloor \frac{1}{2} - \frac{1}{1}, - \frac{1}{2} \right\rfloor \right),
\]
and the constant \( C_{\gamma} > 0 \) satisfies the bound \((5.4)\) if \( G \) is continuous, or \( C_{\gamma} = O(\sqrt{d_{\gamma}} \|\gamma\|_{\infty}) \) in case that \( G \) is finite. In particular,
\[
\|u\|_{L^q(M)} \leq \begin{cases} C_{\gamma} \frac{(n-\kappa)(n-2\lambda)}{m}, & 2(n-\kappa+1) \leq q \leq \infty, \\ C_{\gamma} \frac{1+1}{4m^q}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}
\]
for any eigenfunction \( u \in L^2_{\gamma}(M) \) of \( P \) with eigenvalue \( \lambda \) satisfying \( \|u\|_{L^2} = 1 \).
Proof. By duality, (5.5) is equivalent to

\[
\| (\chi_\mu \circ \Pi_\gamma) u \|_{L^2(M)} \leq \begin{cases} 
C_\gamma \mu^{\delta_{n-\kappa}(p)} \| u \|_{L^p(M)} , & 1 \leq p \leq \frac{2(n-\kappa+1)}{n-\kappa+3} , \\
C_\gamma \mu^{(n-\kappa-1)2-p/n} \| u \|_{L^p(M)} , & \frac{2(n-\kappa+1)}{n-\kappa+3} \leq p \leq 2 ,
\end{cases}
\]

where \( \chi_\mu \) denotes the spectral projection onto the sum of eigenspaces of \( Q := \sqrt{\mathcal{P}} \) with eigenvalues in the interval \( (\mu, \mu + 1] \), \( \mu = \sqrt{\lambda} \). The case \( p = 1 \) follows from the equivariant local Weyl law, and has already been dealt with in (5.2). On the other hand, orthogonality arguments immediately imply

\[
\| (\chi_\mu \circ \Pi_\gamma) u \|_{L^2(M)} \leq \| u \|_{L^2(M)} .
\]

By the Riesz interpolation theorem \[33\] Chapter V, Theorem 1.3] it therefore suffices to prove (5.6) in case that \( p = \frac{2(n-\kappa+1)}{n-\kappa+3} \), which can be inferred from the corresponding bound

\[
\| (\chi_\mu \circ \Pi_\gamma) u \|_{L^2(M)} \leq C_\gamma \mu^{\delta_{n-\kappa}(p)} \| u \|_{L^p(M)} , 
\]

for the approximate spectral projection \( \tilde{\chi}_\mu \) defined in (2.1). Now, by Hölder’s inequality one computes

\[
\| (\tilde{\chi}_\mu \circ \Pi_\gamma) u \|_{L^2(M)}^2 = \int_M \left| \sum_{j=0, e_j \in L^2(M)} \vartheta(\mu - \mu_j) E_j u(x) \right|^2 dM(x) 
= \int_M \sum_{j=0, e_j \in L^2(M)} \vartheta^2(\mu - \mu_j) E_j u(x)u(x) dM(x) 
\leq \| (\tilde{\chi}_\mu \circ \Pi_\gamma) u \|_{L^{p'}(M)} \| u \|_{L^p(M)} ,
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \), and we put \( \tilde{\chi}_\mu u := \sum_{j=0}^{\infty} \vartheta^2(\mu - \mu_j) E_j u \) for \( u \in L^2(M) \). In order to see (5.7) it is therefore sufficient to prove

\[
\| (\tilde{\chi}_\mu \circ \Pi_\gamma) u \|_{L^{p'}(M)} \leq C_\gamma \mu^{2\delta_{n-\kappa}(p)} \| u \|_{L^p(M)} , 
\]

In order to show the latter, we shall use analytic interpolation \[33\] Chapter V, Theorem 4.1], and consider the analytic family of operators

\[
\tilde{\chi}_\mu^z := \frac{e^{iz\partial}}{2\pi} \int_{\mathbb{R}} \vartheta^2(t) e^{it\mu} (t-i0)^z U(t) \, dt , \quad z \in \mathbb{C},
\]

where \( (t-i0)^z \) denotes the distribution limit as \( z \to 0^+, (t-iz)^z \). Clearly, \( \tilde{\chi}_\mu^z = \tilde{\chi}_\mu \) if \( z = 0 \), and since \( 2\delta_{n-\kappa}(2(n-\kappa + 1)/(n-\kappa + 3)) = (n-\kappa-1)/(n-\kappa + 1) \), analytic interpolation theory implies that (5.8) would follow if we were able to show that

\[
\| (\tilde{\chi}_\mu^z \circ \Pi_\gamma) u \|_{L^2(M)} \leq C_\gamma \| u \|_{L^2(M)} , \quad \text{Re } z = -1 ,
\]

(5.9)

\[
\| (\tilde{\chi}_\mu^z \circ \Pi_\gamma) u \|_{L^\infty(M)} \leq C_\gamma \mu^{\frac{n-\kappa-1}{2}} \| u \|_{L^1(M)} , \quad \text{Re } z = \frac{n-\kappa-1}{2} .
\]

(5.10)

The crucial observation for the following estimates is that the Fourier transform of the distribution \( \tau_+^z/\Gamma(z+1) \) is given by the formula

\[
\int_{\mathbb{R}} e^{-it\tau} \frac{\tau_+^z}{\Gamma(z+1)} \, d\tau = e^{-i\pi(z+1)/2}(t-i0)^{-z-1} , \quad z \in \mathbb{C},
\]

where \( \Gamma \) denotes the Gamma function, see \[14\] Example 7.1.17; in particular, the singularity of \( \tau_+^z/\Gamma(z+1) \) at \( \tau = 0 \) determines the asymptotic behaviour of \( (t-i0)^{-z-1} \) as \( t \to \infty \), and viceversa. From this (5.9) immediately follows. The non-trivial bound to be proven is (5.10), which would follow if we were able to show that the Schwartz kernel of \( \tilde{\chi}_\mu^z \circ \Pi_\gamma \) satisfies

\[
|K_{\tilde{\chi}_\mu^z \circ \Pi_\gamma}(x,y)| \leq C_\gamma \mu^{\frac{n-\kappa-1}{2}} , \quad \text{Re } z = \frac{n-\kappa-1}{2} ,
\]

(5.12)
uniformly in $x, y \in M$. Note that, in contrast, by Remark 4.2 we have the uniform bound $|K_{\xi, \omega}(x, y)| \leq C_\gamma \mu^{n-\kappa-1}$. Furthermore, it is not possible to reduce the proof of (5.12) to the case $x = y$, since $\hat{\chi}_\mu^2$ is not a positive operator, compare Remark 4.4 (2). Now, it is clear from (9.8) that

$$K_{\xi, \omega}(x, y) = \frac{\mu^n d_x e^{z^2}}{(2\pi)^{n+1}} \int \int e^{i\mu(t-Rt)(t-i0)^2} I_\gamma(\mu, R, t, x, y) dt dR$$

where $I_\gamma(\mu, R, t, x, y)$ is as in (2.10) with $\varrho$ replaced by $\varrho^2$. Due to the presence of the distribution $(t-i0)^2$ we cannot apply the stationary phase theorem to the $(R, t)$-integral. Instead, we shall apply the stationary phase principle to the integrals $I_\gamma(\mu, R, t, x, y)$ first, and then use (5.11) to deal with the $(R, t)$-integral. Let us first consider the case of a continuous group $G$. If $x \notin Y_i$ or $O_{y_i} \cap Y_i = \emptyset$, $I_\gamma(\mu, R, t, x, y) = 0$. Otherwise, one deduces from Theorem 3.4 for fixed $R, t \in \mathbb{R}$, and any $\hat{N}_i \in \mathbb{N}$ the asymptotic expansion

$$I_\gamma(\mu, R, t, x, y) = \sum_{J \in \pi_0(\text{Crit } \Phi_{i,x,y})} \frac{e^{i\mu \Phi_{i,x,y}(R,t)}}{\mu^n(\mu \| \kappa_i(x) - \kappa_i(g_J \cdot y)\| + 1)^2} \left[ \sum_{k_1, k_2 = 0}^{\hat{N}_1 - 1, \hat{N}_2 - 1} \sum_{k_1, k_2 = 0}^{\hat{N}_1 - 1, \hat{N}_2 - 1} \Phi_{i,x,y}(R, t, x, y) + R_{i,J,\hat{N}_1,\hat{N}_2}(R, t, x, y, \mu) \right] + O_{R,t}(\mu^{-\hat{N}_1 + |\hat{N}_2/2 + 1|}, \| \kappa_i(x) - \kappa_i(g_J \cdot y)\| + 1)^{-\hat{N}_2})$$

are given by distributions depending smoothly on $R, t$ with support in the component $J$ of $\text{Crit } \Phi_{i,x,y}$ and $\Sigma_{i,x} \times G$, respectively. Furthermore, they and their derivatives with respect to $R, t$ are uniformly bounded in $x$ and $y$ by derivatives of $\gamma$ up to order $2k_1$ and $2\hat{N}_1 + |\kappa/2 + 1|$, respectively, while

$$\Phi_{i,x,y}(R, t) := g_{x,g_J \cdot y}(t)$$

denotes the constant value of $\Phi_{i,x,y}$ on $J$. If $y \notin O_x$ one has $x = g_J \cdot y$, so that up to remainder terms the kernel $K_{\xi, \omega}(x, y)$ is given by a linear combination of terms of the form

$$\mu^{n-\kappa-1} d_x e^{z^2} \int \int e^{i\mu(t-Rt)(t-i0)^2} Q_{i,J,k_1,k_2}(R, t, x, y) dt dR,$$

and if $y \notin O_x$, up to remainder terms by a linear combination of terms of the form

$$\mu^{n-\kappa-1} \Phi_{i,x,y}(R, t) Q_{i,J,k_1,k_2}(R, t, x, y) dt dR.$$

Now, as a consequence of (5.11), one has for any $f \in C^\infty_c(\mathbb{R} \times \mathbb{R})$, that might depend on $\mu$ as a parameter, and $z \in \mathbb{C}$

$$\langle (t-i0)^2, e^{i\mu(1-R)t} f(R, t) \rangle = \frac{e^{-i\pi z/2}}{\Gamma(-z)} \left\langle \tau_+^{z-1}, f(R, \cdot)(\tau - \mu(1-R)) \right\rangle.$$

Let us consider first the case when $z = 0, 1, 2, 3, \ldots$, and write $-l := -z - 1$. Since $\tau_+^{-l}/\Gamma(-l + 1) = \delta_0^{(l-1)}$, compare [13 (3.2.17)], partial integration yields

$$\int \int e^{i\mu(1-R)t}(t-i0)^2 f(R, t) dt dR = e^{-i\pi z/2}(1)^{-l-1} \int \int (-it)^l e^{i\mu(1-R)} f(R, t) dt dR = e^{-i\pi z/2} \mu^{-l+1} \int \int e^{i\mu(1-R)}(\partial_R^{-l-1} f)(R, t) dt dR.$$
The relevant integrals in (5.13) and (5.14) therefore read
\[
e^{-\pi z/2} \mu^{l+1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\mu(1-R)} \partial_R^{l+1} \left[ e^{i\mu \Phi^{\gamma_{i,j}}(R,t)} Q^{\gamma_{i,j,k},k_2}_{\gamma_{j,k}}(R,t,x,y) \right] dt dR.
\]
Since similar considerations also hold for the remainder terms, an application of the classical stationary phase theorem \[10\, Proposition 2.3\] to the \((R,t)\)-integral allows us to deduce for \(z = 0, 1, 2, 3, \ldots\) the uniform bound \[5.12\]. Indeed, if \(y \in O_x\), the phase function in \[5.13\] simply reads \(t(1-R)\), and the only critical point is \((R_0, t_0) = (1,0)\), which is non-degenerate, the determinant of the Hessian being \(-1\). If \(y \not\in O_x\), the phase function is given by \(t(1-R) + \Phi^{\gamma_{i,j}}(R,t)\), and a computation shows that the determinant of the matrix of its second derivatives is given by
\[
- \left(1 + c'_{x,g,\gamma}(t)\right)^2 \approx -(1 + O(\|\mu(x) - \mu(y)\|)) \]
since \(c_{x,y}(t) = \pm \|\mu(x) - \mu(y)\|/\|\nabla_\gamma \zeta(t, \mu(x), \omega)\|\). By choosing the charts \(Y_i\) sufficiently small so that \(|\mu(x) - \mu(y)|\) is small, we can therefore achieve that in a sufficiently small neighborhood of \((R,t) = (1,0)\), which is where \(Q^{\gamma_{i,j,k},k_2}_{\gamma_{j,k}}(R,t,x,y)\) is supported, the phase function \(t(1-R) + \Phi^{\gamma_{i,j}}(R,t)\) has, if at all, only non-degenerate, hence isolated, critical points. If we now apply the stationary phase theorem to the integral \[5.15\] with respect to the phase function \(t(1-R)\) and \(t(1-R) + \Phi^{\gamma_{i,j}}(R,t)\), respectively, treating the remainder terms alike, we obtain
\[
|K_{\Phi^{\gamma_{i,j}}}(x,y)| \leq C_\gamma \mu^{n-k-1}, \quad y \in O_x,
\]
as well as
\[
|K_{\Phi^{\gamma_{i,j}}}(x,y)| \leq C_\gamma \mu^{n-k-1} \left( \mu \|\mu(x) - \mu(y)\| + 1 \right) \sum_{\nu=0}^{l-1} \left( \mu \|\mu(x) - \mu(y)\| \right)^{l-\nu} \]
\[
\leq C_\gamma \mu^{n-k-1}, \quad y \not\in O_x,
\]
yielding \[5.12\] for \(z = 0, 1, 2, 3, \ldots\). Next, let us turn to the case where \(z \neq 0, 1, 2, 3, \ldots\), and note that by homogeneity of \(\tau^z\) one has
\[
\left( t - i0 \right)^z e^{i\mu(1-R)t} f(R,t) = \frac{e^{-\pi z/2}}{\Gamma(-z)} \left( t^{z-1}, \mu^{-1} f(R,t) \right) \left( \tau^z \right) \left( \tau - 1 + R \right)
\]
\[
e^{-\pi z/2} \frac{1}{\Gamma(-z)} \left( t^{z-1}, \mu^{-1} f(R,t) \right) \left( \tau^z \right) \left( \tau - 1 + R \right) \]
\[
\text{compare} [14 (3.2.7)]. By definition of \(\tau^z\) and partial integration one computes
\[
- z(z + 1) \ldots (-z - 1 + l)(-1)^l \int_{\mathbb{R}} \int_{\mathbb{R}} \tau^{z-1} f(R,t) (\tau - 1 + R) d\tau dR
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \tau^{z-1+l} \partial_{\tau} \left[ f(R,t) (\tau - 1 + R) \right] d\tau dR
\]
\[
= (-1)^l \mu \int_{\mathbb{R}} \tau^{z-1+l} \left[ \int_{\mathbb{R}} e^{-i\mu(\tau - 1 + R)} (\partial_{R} f)(R,t) dR \right] d\tau,
\]
where \(l > \Re z\) is a sufficiently large positive integer, so that \(\tau^{z-1+l}\) becomes locally integrable. Note that we have, as we may, interchanged the integrals over \(\tau\) and \(R\), while the integrals over \(\tau\) and \(t\) cannot be interchanged. As a consequence, the relevant integrals in \[5.13\] and \[5.14\] are given by linear combinations of terms of the form
\[
\mu^{-z} \int_{\mathbb{R}} \tau^{z-1+l} \left[ \int_{\mathbb{R}} e^{-i\mu(\tau - 1 + R)} (\partial_{R} f)(R,t) \right] d\tau.
\]
Again, let us examine the \((R,t)\)-integral by means of the stationary phase. If \(y \in O_x\), the phase function is given by \(t(\tau - 1 + R)\), the only critical point is \((R_0, t_0) = (1, -\tau, 0)\), and we obtain for \[5.17\]
the estimate
\[
2\pi\mu^{-z-1} \int_{\mathbb{R}_{+}}^{\tau_{+}^{-1}+t} \left( \partial_{R}^{2} Q_{i,j,k_{1},k_{2}}^{\gamma}(1 - \tau, 0, x, y) + O_{\gamma, \tau}(\mu^{-1}) \right) d\tau = O_{\gamma}(\mu^{-\Re z-1})
\]
uniformly in \( x, y \), the remainder \( O_{\gamma, \tau}(\mu^{-1}) \) being rapidly falling in \( \tau \), since \( Q_{i,j,k_{1},k_{2}}^{\gamma} \) has compact \((R,t)\)-support. Now, if \( y \notin \mathcal{O}_x \), the phase function reads \( t(1-R) + \Phi_{i,x,y}^{\gamma}(R, t) - t\tau \), and the determinant of the matrix of its second derivatives is again given by \((5.16)\). By the previous arguments, we can therefore assume that in a sufficiently small neighborhood of \((R, t) = (1, 0)\) the phase function \( t(1-R) + \Phi_{i,x,y}^{\gamma}(R, t) - t\tau \) has only one non-degenerate critical point \((R_0, t_0)\). It satisfies the relations
\[
t_0 = c_{x,g,J,y}(t_0) \approx 0, \quad R_0 = \frac{1 - \tau}{1 - \epsilon_{x,g,J,y}(t_0)} \approx 1 - \tau,
\]
and at this point, the phase function takes the value \( t_0(1 - R_0) + \Phi_{i,x,y}^{\gamma}(R_0, t_0) - t_0\tau = t_0(1 - \tau) \).
Taking into account that for any \( w \in \mathbb{C} \) with \( \Re w > -1 \) and \( g \in \mathcal{S}(\mathbb{R}) \) one has
\[
\int_{\mathbb{R}} e^{-i\mu\tau} t_0^{w}(\gamma, t) d\tau = O\left((1 + \mu)^{-\Re w-1}\right), \quad \mu \geq 0,
\]
compare \((5.11)\), we obtain for \((5.17)\) the bound
\[
2\pi\mu^{-z-1} \int_{\mathbb{R}}^{\tau_{+}^{-1}+t} \sum_{l'=l}^{l'=l} c_{l', l''}^{\gamma, \mu} \left( (i\mu c_{x,g,J,y}(t_0))^{l'} (\partial_{R}^{2} Q_{i,j,k_{1},k_{2}}^{\gamma}(R_0, t_0, x, y) + O_{\gamma, \tau}(\mu^{-1} \|\kappa_{i}(x) - \kappa_{i}(g_{\mathcal{F}} \cdot y)\|^{l'}) \right) d\tau
\]
\[
= O_{\gamma}\left(\mu^{-\Re z-1} (1 + \mu \|\kappa_{i}(x) - \kappa_{i}(g_{\mathcal{F}} \cdot y)\|) \Re z^{-1} \sum_{l'=0}^{l} (\mu \|\kappa_{i}(x) - \kappa_{i}(g_{\mathcal{F}} \cdot y)\|^{l'}) \right)
\]
\[
= O_{\gamma}\left(\mu^{-\Re z-1} (1 + \mu \|\kappa_{i}(x) - \kappa_{i}(g_{\mathcal{F}} \cdot y)\|) \Re z\right)
\]
uniformly in \( x, y \), where the \( c_{l', l''}^{\gamma, \mu} \) are certain coefficients, and the remainder is rapidly falling in \( \tau \). Treating the remainder terms alike, we have shown \((5.12)\) for \( z \neq 0, 1, 2, 3, \ldots \) as well. This completes the proof of Theorem 5.4 in case that \( G \) is continuous. The finite group case follows in an analogous way using Proposition 3.6 instead of Theorem 5.4.

\[\Box\]

**Example 5.5.** Let us resume Example 4.10 of a connected semisimple Lie group \( G \) with finite center, discrete co-compact subgroup \( \Gamma \), and maximal compact subgroup \( K \). The group \( K \) acts on \( \Gamma \setminus G \) with orbits of principal and exceptional type, all orbits having the dimension \( \dim K \), and we deduce from Proposition 5.1 for each \( \gamma \in \hat{K} \) the estimate
\[
\|u\|_{L^{\infty}(\Gamma \setminus G)} \leq C_{\gamma} \lambda^{\dim G/K - 1}, \quad u \in L_{c}^{2}(\Gamma \setminus G), \quad \|u\|_{L^{2}} = 1,
\]
for any eigenfunction \( u \) of a \( K \)-invariant elliptic positive symmetric classical pseudodifferential operator \( P \) on \( \Gamma \setminus G \) of degree \( m \) with eigenvalue \( \lambda \). More generally, with \( \frac{1}{q} + \frac{1}{q'} = 1 \) and
\[
\delta(q) := \max\left( \dim G/K \left| \frac{1}{2} - \frac{1}{q} - \frac{1}{2}, 0 \right| \right)
\]
we have by Theorem 5.4 the bound
\[
\|u\|_{L^{q}(\Gamma \setminus G)} \leq \left\{ \begin{array}{ll}
C_{\gamma} \lambda^{\delta(q)/q}, & 2(\dim G/K + 1) \leq q \leq \infty, \\
C_{\gamma} \lambda^{(\dim G/K - 1)(2 - q')/4m}, & 2 \leq q \leq 2(\dim G/K + 1),
\end{array} \right.
\]
provided that \( P \) satisfies the strict convexity assumption in Theorem 5.4. In case that \( \Gamma \) has no torsion, \( \Gamma \setminus G/K \) is a locally symmetric space, and eigenfunctions of the Beltrami-Laplace operator on \( \Gamma \setminus G/K \) correspond exactly to \( K \)-invariant eigenfunctions of the Beltrami-Laplace operator on \( \Gamma \setminus G \),
the space $L^2(\Gamma \backslash G/\mathcal{K}) \simeq L^2(\Gamma \backslash G)^K$ being isomorphic to the trivial isotypic component in the Peter-Weyl decomposition of $L^2(\Gamma \backslash G)$. Thus, our results generalize the classical $L^p$-bounds on $\Gamma \backslash G/\mathcal{K}$ to arbitrary $\mathcal{K}$-types.

6. The desingularization process

As already noted, the asymptotic formula for the reduced spectral function $e_\gamma(x, x, \lambda)$ given in Theorem 4.3 depends in a highly non-smooth way on $x \in M$ if non-principal orbits are present. Moreover, if $G$ is continuous, the mentioned formula does not give a precise description of the caustic behaviour of $e_\gamma(x, x, \lambda)$ near singular orbits, leaving it unclear if the coefficients in the expansion of $e_\gamma(x, x, \lambda)$ are integrable in $x$, and how one could deduce from Theorem 4.3 asymptotics for the equivariant spectral counting function $N_\gamma(\lambda) := \int_M e_\gamma(x, x, \lambda) \, dM(x)$. In what follows, we shall therefore examine the case of a continuous group $G$ more closely. Our goal is to derive a description of $e_\gamma(x, x, \lambda)$ that interpolates between the asymptotics for different values of $x$, and in particular to characterize the behaviour of the leading coefficient and the remainder term in Theorem 4.3 as $x \in M_{\text{prin}}$ approaches singular orbits. For this, we shall make use of resolution of singularities. As we shall see, the major difficulty resides in the fact that, unless the Hamiltonian $G$-action on $T^*M$ is free, so that the corresponding momentum map becomes a submersion, $\Omega$ and the critical set (6.6) of the phase function $\Phi$ are not smooth manifolds. To overcome this difficulty, it was shown in [22] that by constructing a strong resolution of the set

\begin{equation}
N := \{ (x, g) \in M \times G \mid g \cdot x = x \}
\end{equation}

a partial desingularization

\begin{equation}
Z : \tilde{X} \to X := T^*M \times G
\end{equation}

of the critical set $\text{Crit } \Phi$ can be achieved, and after applying the stationary phase theorem in the resolution space $\tilde{X}$, an asymptotic description of the integrals $I(\mu)$ defined in (3.3) can be obtained, leading to an asymptotic formula for $N(\lambda)$. In the ensuing sections, we shall use the partial desingularization (6.2) to obtain an asymptotic formula for the integrals $I(\mu)$ defined in (3.2) that allows us to describe the caustic behaviour of the coefficients $Q_k(x, x)$ in Theorem 3.3 (a) as one approaches singular orbits. One can deduce from this the asymptotic description of the integrals $I(\mu)$ given in [22], but the converse implication is more subtle and not straight-forward. For this reason, a careful re-examination of the results of [22] is needed in order to obtain a precise description of the coefficients in the asymptotic formula for the integrals $I(\mu)$ and, ultimately, of the leading coefficient in the asymptotic formula for the equivariant spectral function.

Let $M$ be a closed connected Riemannian manifold and $G$ a continuous compact Lie group acting on $M$ by isometries. In what follows, we shall recall the construction of the partial desingularization (6.2) of the critical set $\mathcal{C} := \{ (x, \eta, g) \in (\Omega \cap T^*M) \times G \mid g \in G(x, \eta) \}$ performed in [22]. The desingularization process presented here is exactly the same, only that we apply it now to the study of the integrals (3.1) instead of the integrals (3.3). For details, the reader is referred to [22]. Consider the decomposition of $M$ into orbit types

\begin{equation}
M = M(H_1) \cup \cdots \cup M(H_L),
\end{equation}

where we suppose that the isotropy types are numbered in such a way that $(H_i) \geq (H_j)$ implies $i \leq j$, $(H_L)$ being the principal isotropy type, see Figure 6.1.

To construct (6.2), an iterative process along the strata of the $G$-action on $M$ is set up, where the centers of the blow-ups are successively chosen as isotropy bundles over unions of maximally singular orbits. For simplicity, one assumes that at each step the union of maximally singular orbits is connected.
Beginning of iteration. Let \( f_k : \nu_k \to M_k \) be an invariant tubular neighborhood of \( M_k(H_k) \) in

\[
M_k := M - \bigcup_{i=1}^{k-1} f_i(\bar{D}_{1/2}(\nu_i)), \quad k = 1, \ldots, L,
\]
a manifold with corners on which \( G \) acts with the isotropy types \( (H_k), (H_{k+1}), \ldots, (H_L) \). Here \( \nu_k \)
denotes the normal \( G \)-vector bundle of \( M \) is an equivariant diffeomorphism given in terms of the exponential map, and

\[
f_k(p^{(k)}, v^{(k)}) := (\exp_{p^{(k)}} \circ \gamma^{(k)})(v^{(k)}), \quad p^{(k)} \in M_k(H_k), \ v^{(k)} \in (\nu_k)p^{(k)},
\]
is an equivariant diffeomorphism given in terms of the exponential map, and

\[
\gamma^{(k)}(v^{(k)}) := \frac{F_k(p^{(k)})}{(1 + \|v^{(k)}\|^2)^{1/2}} v^{(k)},
\]
where \( F_k : M_k(H_k) \to \mathbb{R} \) is a smooth, \( G \)-invariant, positive function, see [2] p. 306. Let \( S_k \) be the
unit sphere bundle over \( M_k(H_k) \), and put \( W_k := f_k(\bar{D}_{1/2}(\nu_k)), W_L := M, \) so that we obtain the open covering

\[
M = W_1 \cup \cdots \cup W_L.
\]

Fix an inner product on \( \mathfrak{g} \), which induces a Riemannian structure on \( G \), and consider for each \( k \) and
\( p^{(k)} \in M_k(H_k) \) the decomposition

\[
T_e G \simeq \mathfrak{g} = \mathfrak{g}_{p^{(k)}} \oplus \mathfrak{g}_{p^{(k)}}^\perp,
\]
where \( \mathfrak{g}_{p^{(k)}} \simeq T_e G_{p^{(k)}} \) denotes the Lie algebra of the stabilizer \( G_{p^{(k)}} \) of \( p^{(k)} \), and \( \mathfrak{g}_{p^{(k)}}^\perp \), its orthogonal complement with respect to the above Riemannian structure. Now, introduce a partition of unity \( \{\chi_k\}_{k=1,\ldots,L} \) subordinated to the covering \((6.4)\), and define

\[
I_k(x, \mu) := \chi_k(x) I_x(\mu)
\]
with \( I_x(\mu) \) as in \((3.2)\). By Theorem \((3.3)\) \( (a) \) the asymptotic expansion for \( I_L(x, \mu) \) depends smoothly
on \( x \in W_L \cap Y \). Let us therefore turn to the case when \( 1 \leq k \leq L - 1 \) and \( W_k \cap Y \neq \emptyset \). For fixed \( k \)
and \( x = f_k(p^{(k)}, v^{(k)}) \in W_k \cap Y \) Lemma \((3.1)\) \( (a) \) implies that

\[
\text{Crit } \Phi_x = \{ (\omega, g) \in \Sigma_x^R \times G \mid (x, \omega) \in \Omega, g \cdot x = x \} \subset \Sigma_x^R \times G_{p^{(k)}}.
\]
Up to non-stationary contributions, it will therefore suffice to evaluate the integrals \( I_k(x, \mu) \) in a neighborhood of \( G_{p^{(k)}} \). To this end, consider the isotropy bundle \( \text{Iso} M_k(H_k) \to M_k(H_k) \) over \( M_k(H_k) \),
as well as the canonical projection

\[
\pi_k : W_k \to M_k(H_k), \quad f_k(p^{(k)}, v^{(k)}) \mapsto p^{(k)}.
\]
Further, let

\[
\pi_k^* \text{Iso} M_k(H_k) = \left\{ (f_k(p^{(k)}, v^{(k)}), h^{(k)}) \in W_k \times G \mid h^{(k)} \in G_{p^{(k)}} \right\}
\]
be the induced bundle. Let $U_k$ be a sufficiently small tubular neighborhood of $\pi_1^*\text{Iso}(H_k)$ in $W_k \times G$, and note that the fiber of the normal bundle $N \pi_1^*\text{Iso}(H_k)$ at a point $(f_k(p^{(k)}), u^{(k)})$, $h^{(k)})$ may be identified with the fiber of the normal bundle to $G_{p^{(k)}}$ at the point $h^{(k)}$. Consider further an orthonormal basis $\{A_1(p^{(k)}), \ldots, A_d(p^{(k)})\}$ of $g_{p^{(k)}}^+$, and introduce canonical coordinates of the second kind
\[(6.5) \quad \mathbb{R}^{d(k)} \times G_{p^{(k)}} \ni (\alpha_1^{(k)}, \ldots, \alpha_d^{(k)}, h^{(k)}) \mapsto e^\sum \alpha_i^{(k)} A_i(p^{(k)}) h^{(k)} \]
in a neighborhood of $G_{p^{(k)}}$, see [11, p. 146]. Denote by $b_\mu$ the amplitude $a$ multiplied by a smooth cut-off function with support in $U_k$, which is equal to 1 in a small neighborhood of $\pi_1^*\text{Iso}(H_k)$. Taking into account the non-stationary phase theorem [14, Theorem 7.7.1] one computes
\[(6.6) \quad I_k(x, \mu) = \chi_k(x) \int_{G_{p^{(k)}}} e^{i\mu \Phi} b_\mu d(\Sigma^{R_I}(\omega)) dA^{(k)} dh^{(k)} + O(\mu^{-\infty}), \]
where $dA^{(k)}$, $dA^{(k)}$ are suitable volume densities on the sets $G_{p^{(k)}}$ and $g_{p^{(k)}}^+$, respectively, such that $dg \equiv dA^{(k)} dh^{(k)}$, compare [22, (5.4)], and the remainder estimate is uniform in $x$.

We shall now successively resolve the singularities of (6.1) in order to obtain a factorization of $\Phi$. Note that by [22, Eq. (5.1)]
\[
\mathcal{N} = \mathcal{N}_L \cup \bigcup_{k=1}^{L-1} \mathcal{N}_k,
\]
where $\mathcal{N}_k := \mathcal{N} \cap U_k$, $\mathcal{N}_L := \text{Iso}(W_L)$, $\text{Iso}(W_L) \to W_L$ being the isotropy bundle over $W_L$. While $\mathcal{N}_L$ is a smooth submanifold, $\mathcal{N}_k$ is in general singular. In particular, if $\dim H_k \neq \dim H_L$, $\mathcal{N}_k$ has a maximal singular locus given by $\text{Iso}(M_k(H_k))$. One then performs for each $k \in \{1, \ldots, L-1\}$ a blow-up
\[
\zeta_k : B_{Z_k}(U_k) \to U_k
\]
with center $Z_k := \text{Iso}(M_k(H_k)) \subset \mathcal{N}_k$, and by piecing these transformations together one obtains the global blow-up
\[
\zeta^{(1)} : B_{Z^{(1)}} \mathcal{M} \to \mathcal{M}, \quad Z^{(1)} := \bigcup_{k=1}^{L-1} Z_k,
\]
where we put $\mathcal{M} := M \times G$, compare [22, p. 56]. To get a local description, fix $k$, let $\{v_1^{(k)}, \ldots, v_c^{(k)}\}$ be an orthonormal frame in $v_k$, and $(\theta_1^{(k)}, \ldots, \theta_c^{(k)})$ be coordinates in $\gamma^{(k)}(\{v_k\}^{(k)})$. Similarly, consider the coordinates $(\alpha_1^{(k)}, \ldots, \alpha_d^{(k)})$ introduced in (6.5). If one now covers $B_{Z_k}(U_k)$ with standard projective charts $\{(\phi_k^g, G_k)\}$ one obtains in the so-called $\theta^{(k)}$-charts $\{G_k^g\}_{1 \leq g \leq c^{(k)}}$, in which the $\theta_g^{(k)}$-coordinate is non-zero, for $\zeta_k$ the local expressions
\[(6.7) \quad \zeta_k^g = \zeta_k \circ (\phi_k^g)^{-1} : (p^{(k)}, \tau_k, \tilde{\nu}^{(k)}, A^{(k)}, h^{(k)}) \mapsto \left(\exp_{p^{(k)}} \tau_k \nu^{(k)}, e^{\tau_k A^{(k)}} h^{(k)}\right) = (x, g), \]
where
\[
p^{(k)} \in M_k(H_k), \quad A^{(k)} \in g_{p^{(k)}}^+, \quad h^{(k)} \in G_{p^{(k)}}, \quad \tilde{\nu}^{(k)} \in \gamma^{(k)}((S_k^+)^{p^{(k)})}),
\]
and $S_k^+ := \left\{v \in v_k \mid v : = \sum s_i v_i^{(k)}, s_i > 0, \|v\| = 1\right\}$, while $\tau_k \in (-1, 1)$, see [22, Eq. (5.6)]. A similar description of $\zeta_k$ is given in the so-called $a^{(k)}$-charts $\{G_k^g\}_{c^{(k)}+1 \leq g \leq c^{(k)}+d^{(k)}}$, in which the $\alpha_g^{(k)}$-coordinate does not vanish. By performing Taylor expansion at $\tau_k = 0$ one can then show that the phase function (3.4) factorizes according to
\[(6.8) \quad \Phi \circ (\text{id}_g \otimes \zeta_k^g) = (k) \tilde{\Phi}^{\text{tot}} = \tau_k \cdot (k) \tilde{\Phi}^{\text{wk}}, \]
where $(k) \tilde{\Phi}^{\text{tot}}$ and $(k) \tilde{\Phi}^{\text{wk}}$ being the total and weak transform of the phase function $\Phi$, respectively, see [22, Eqs. (5.8) and (5.9)]. Since $\zeta_k$ is a real-analytic surjective proper map, which is a diffeomorphism on the complement of $\zeta_k^{-1}(Z_k)$, we can lift the integral $I_k(x, \mu)$ along the restriction of $\zeta_k$ to the fiber over $\{x\} \times G$ to the resolution space $B_{Z_k}(U_k)$. To obtain local expressions, introduce a compactly supported
partition \( \{ u_k^g \} \) of unity subordinate to the covering \( \{ O_k^p \} \), set \( a_k^g := (u_k^g \circ (\phi_k^g)^{-1}) \cdot [(b, \chi_k) \circ (\text{id}_{\omega} \otimes \zeta_k^g)] \), and define for \( x = \exp_{p(\xi)} \tau_k \tilde{\varrho} \) \( I_k(x) \in W_k \cap Y \) and \( 1 \leq p \leq c(\xi) \) the integrals
\[
I^k_k(x, \mu) := |\tau_k|^{d(\xi)} \int_{G_{p(\xi)} \times G_{q(\xi)}} e^{i\mu \tau_k \tilde{\varrho}(\xi)} \tilde{\varrho}^{(\xi)} a_k^g d(\Sigma^{R_D}_x) dA^{(\xi)} dh^{(\xi)},
\]
and for \( c(\xi) + 1 \leq p \leq c(\xi) + d(\xi) \) corresponding integrals \( \bar{I}_k(x, \mu) \). Here \( \tilde{\varrho}^{(\xi)} \) denotes the weak transform regarded as a function of the variables \( \omega, A^{(\xi)}, h^{(\xi)} \), while \( \tau_k, p(\xi), \tilde{\varrho}(\xi) \) are considered as parameters. Let us emphasize that the amplitudes \( a_k^g \) are compactly supported. In view of (6.6) we arrive for \( x \in W_k \) at the decomposition
\[
I_k(x, \mu) = \sum_{\varrho = 1}^{c(\xi)} I^\varrho_k(x, \mu) + \sum_{\varrho = c(\xi) + 1}^{d(\xi)} \bar{I}^\varrho_k(x, \mu)
\]
up to terms of order \( O(\mu^{-\infty}) \), compare [22], p. 57]. As we shall see in Corollary (7.2) the weak transforms \( \tilde{\varrho}^{(\xi)}(\varrho, \nu, \theta(\xi)) \) have no critical points in the \( \alpha(\xi) \)-charts, which will imply that the integrals \( \bar{I}^\varrho_k(x, \mu) \) contribute to \( I(x, \mu) \) with terms of order \( O(\mu^{-\infty}) \). If \( G \) acts on \( S_k \) only with isotropy type \( (H_k) \), we shall see in the next section that in each of the \( \psi(\xi) \)-charts the weak transforms \( \tilde{\varrho}^{(\xi)}(\varrho, \nu, \theta(\xi)) \) have clean critical sets, so that one can apply the stationary phase theorem in order to obtain asymptotics for each of the \( I^\varrho_k(x, \mu) \). But in general, \( G \) will act on \( S_k \) with singular orbit types, so that neither \( N_k \) is resolved, nor do the weak transforms \( \tilde{\varrho}^{(\xi)}(\varrho, \nu, \theta(\xi)) \) have clean critical sets, and we are forced to continue with the iteration.

**Iteration step from \( N - 1 \) to \( N \).** Denote by \( \Lambda \leq L \) the maximal number of elements that a totally ordered subset of the set of isotropy types can have. Assume that \( 2 \leq N < \Lambda \), and let \( \{ (H_1), \ldots, (H_N) \} \) be a totally ordered subset of the set of isotropy types such that \( i_1 < \cdots < i_N < L \). Let \( f_{i_1}, S_{i_1} \), as well as \( p^{(i_1)} \) in \( M_{i_1}(H_{i_1}) \) be defined as at the beginning of the iteration, and assume that \( f_{i_1}, i_1, S_{i_1}, i_1, p^{(i_1)} \), \ldots have already been defined for \( j < N \). For every fixed \( p^{(i_N-1)} \), denote by \( \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \) the submanifold with corners of the closed \( G_{p^{(i_N-1)}} \)-manifold \( \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \) from which all orbit types less than \( G/H_{i_N} \) have been removed, and define \( \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \) analogously. Consider the invariant tubular neighborhood
\[
f_{i_1 \cdots i_N} := \exp \circ \gamma_{(i_N-1)}^{(i_N-1)} : \nu_{i_1 \cdots i_N} \to \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N}
\]
of the set \( \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \), where \( \nu_{i_1 \cdots i_N} \) denotes its normal \( G_{p^{(i_N-1)}} \)-vector bundle, and \( \exp \circ \gamma_{(i_N-1)}^{(i_N-1)} \) the corresponding equivariant diffeomorphism, and define \( S_{i_1 \cdots i_N}^+ \) as the sphere subbundle in \( \nu_{i_1 \cdots i_N} \), while
\[
S_{i_1 \cdots i_N}^+ := \{ v \in S_{i_1 \cdots i_N} : v = \sum s_{i_1 \cdots i_N} s_{i_1 \cdots i_N} s_{i_1 \cdots i_N} > 0 \}
\]
for some \( q_{i_1 \cdots i_N} \). Put
\[
W_{i_1 \cdots i_N} := f_{i_1 \cdots i_N}(D_1(\nu_{i_1 \cdots i_N})), \quad W_{i_1 \cdots i_N L} := \text{Int}(\gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N}),
\]
and denote the corresponding integrals in the decomposition of \( I_{1 \cdots i_{N-1}}^{(i_1 \cdots i_{N-1})}(x, \mu) \) by \( I_{1 \cdots i_{N-1}}^{(i_1 \cdots i_{N-1})}(x, \mu) \) and \( I_{1 \cdots i_{N-1}}^{(i_1 \cdots i_{N-1})} L(x, \mu) \), respectively. Here we can assume that, modulo terms of order \( O(\mu^{-\infty}) \), the \( W_{i_1 \cdots i_N} \times G_{p^{(i_N-1)}} \)-support of the integrand in \( I_{1 \cdots i_{N-1}}^{(i_1 \cdots i_{N-1})}(x, \mu) \) is contained in a compactum of a tubular neighborhood of the induced bundle \( \pi_{i_1 \cdots i_N} \text{Iso} \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \), where \( \pi_{i_1 \cdots i_N} : W_{i_1 \cdots i_N} \to \gamma_{(i_N-1)}^{(i_N-1)}((S_{i_1 \cdots i_{N-1}}_1)_{p^{(i_N-1)}})_{i_N} \) denotes the canonical projection. For a given point \( p^{(i_N)} \in \gamma_{(i_N-1)}^{(i_N-1)}((S^+_{i_1 \cdots i_{N-1}})_{p^{(i_N-1)}})_{i_N} \), consider further the decomposition
\[
g_{p^{(i_N-1)}} = g_{p^{(i_N-1)}} \oplus g_{p^{(i_N-1)}}^\perp,
\]
and set \(d^{(i_N)} := \dim g_{p^{(i_N)}}^{(i_N)}, e^{(i_N)} := \dim g_{p^{(i_N)}}^{(i_N)}\). This yields the decomposition

\[
\mathcal{g} = g_{p^{(i_1)}}^{(i_1)} \oplus g_{p^{(i_2)}}^{(i_2)} \oplus (g_{p^{(i_3)}}^{(i_3)} \oplus g_{p^{(i_4)}}^{(i_4)}) \oplus \cdots = g_{p^{(i_N)}}^{(i_N)} \oplus g_{p^{(i_{i_N})}}^{(i_{i_N})} \oplus \cdots \oplus g_{p^{(i_{i_N})}}^{(i_{i_N})}.
\]

Denote by \(\{A_{(i_N)}^{(i_N)}(p^{(i_N)}), \ldots, p^{(i_N)}\}\) an orthonormal basis of \(g_{p^{(i_N)}}^{(i_N)}\), and let \((\alpha_{d^{(i_N)}}, \ldots, \alpha_{d^{(i_N)}})\) be corresponding coordinates. Further, let \(\{v_{p^{(i_{i_N})}}^{(i_{i_N})}, \ldots, v_{c^{(i_{i_N})}}^{(i_{i_N})}\}\) be an orthonormal frame in \(\nu_{i_1 \ldots i_{i_N}}\), and \((\beta_{c^{(i_{i_N})}}, \ldots, \beta_{c^{(i_{i_N})}})\) corresponding coordinates. Now, let the blow-up \(\zeta^{(j)}\) be defined as in the beginning of the iteration, and assume that the blow-ups \(\zeta^{(j)}\) have already been defined for \(j < N\). Put \(\mathcal{M}^{(j)} := B_{\zeta^{(j)}}(\mathcal{M}^{(j-1)}), \mathcal{M}^{(0)} := \mathcal{M} = M \times G\), and consider the blow-up

\[
\zeta^{(N)} : B_{\zeta^{(N)}}(\mathcal{M}^{(N-1)}) \rightarrow \mathcal{M}^{(N-1)}, \quad Z^{(N)} := \bigcup_{i_1 < \cdots < i_{i_N} < L} Z_{i_1 \ldots i_{i_N}},
\]

where the union is over all totally ordered subsets \(\{(H_{i_1}), \ldots, (H_{i_{i_N}})\}\) of \(N\) elements with \(i_1 < \cdots < i_{i_N} < L\), and

\[
Z_{i_1 \ldots i_{i_N}} \cong \bigcup_{p^{(i_1)}, \ldots, p^{(i_{i_N})}} (-1,1)^{N-1} \times \text{Iso} \gamma^{(i_{i_N})}((S_{i_1 \ldots i_{i_N}-1})_{p^{(i_{i_N})}})_{i_{i_N}}(H_{i_{i_N}})
\]

are the possible maximal singular loci of \((\zeta^{(1)} \circ \cdots \circ \zeta^{(N)})^{-1}(\mathcal{N}), \text{compare } [22, \text{Eq. (5.14)}].\) Denote by \(\theta_{i_1}^{(1)} \circ \cdots \circ \theta_{i_{i_N}}^{(N)}\) a local realization of the sequence of blow-ups \((\zeta^{(1)} \circ \cdots \circ \zeta^{(N)})\) corresponding to the totally ordered subset \(\{(H_{i_1}), \ldots, (H_{i_{i_N}})\}\) in a set of charts labeled by the indices \(i_1, \ldots, i_{i_N}\). As a consequence, we obtain local factorizations of the phase function according to

\[
\Phi \circ ((\theta_{i_1}^{(1)} \circ \cdots \circ \theta_{i_{i_N}}^{(N)}_{i_1 \ldots i_{i_N}}) \otimes \text{id}_{\mathcal{N}}) = (i_{i_1 \ldots i_{i_N}}) \Phi_{\text{tot}} = \tau_{i_1} \cdots \tau_{i_{i_N}} (i_{i_1 \ldots i_{i_N}}) \Phi_{\text{w}},
\]

see [22, pp. 67]. Assume now that the indices \(i_1, \ldots, i_{i_N}\) correspond to a set of \((\theta^{(1)}), \ldots, (\theta^{(i_{i_N})})\)-charts. Then \(\theta_{i_1}^{(1)} \circ \cdots \circ \theta_{i_{i_N}}^{(N)}\) is explicitly given by

\[
(\tau_{i_1}, \ldots, \tau_{i_{i_N}}, p^{(i_1)}, \ldots, p^{(i_{i_N})}, g^{(i_1)}, A^{(i_1)}, \ldots, A^{(i_{i_N})}, h^{(i_{i_N})}) \mapsto (x^{(i_1 \ldots i_{i_N})}, g^{(i_1 \ldots i_{i_N})} = (x, g),
\]

where we set

\[
x_{p^{(i_1 \ldots i_{i_N})}}^{(i_1 \ldots i_{i_N})} := \exp_{p^{(i_1 \ldots i_{i_N})}}[\tau_{i_1} \exp_{p^{(i_1 \ldots i_{i_N})}}[\tau_{i_1} \exp_{p^{(i_1 \ldots i_{i_N})}}[\cdots \exp_{p^{(i_1 \ldots i_{i_N})}}(\tau_{i_{i_N}} h^{(i_{i_N})})] \cdots]],
\]

\[
g^{(i_1 \ldots i_{i_N})} := e^{\tau_{i_1} \cdots \tau_{i_{i_N}} A^{(i_1 \ldots i_{i_N})}} e^{\tau_{i_1} \cdots \tau_{i_{i_N}} A^{(i_1 \ldots i_{i_N})}} e^{\tau_{i_1} \cdots \tau_{i_{i_N}} h^{(i_{i_N})}}.
\]

In this situation we define

\[
I^{(i_1 \ldots i_{i_N})}_{\mu_{i_1 \ldots i_{i_N}}}(x, \mu) := \prod_{j=1}^{N} |\tau_{i_j}| \sum_{r=1}^N d^{(i_j)} \int_{\Theta_{i_1 \ldots i_{i_N}}^{(i_1 \ldots i_{i_N})} \times \Sigma_{x, r}^{(i_j)}} \int_{(1 \ldots i_1 \ldots i_{i_N})} \theta_{i_1 \ldots i_{i_N}}^{\theta^{(i_1 \ldots i_{i_N})}} \theta_{i_1 \ldots i_{i_N}}^{\theta^{(i_1 \ldots i_{i_N})}} \theta_{i_1 \ldots i_{i_N}}^{\theta^{(i_1 \ldots i_{i_N})}} d\mu dA^{(i_1)} \ldots dA^{(i_{i_N})} dh^{(i_{i_N})},
\]

compare [22, Eq. (5.15)], where

- \(\tilde{X}_{i_1 \ldots i_{i_N}}^{\theta^{(i_1 \ldots i_{i_N})}} := G_{p^{(i_1 \ldots i_{i_N})}} \times g_{p^{(i_1 \ldots i_{i_N})}}^{(i_1 \ldots i_{i_N})} \times \cdots \times g_{p^{(i_{i_N})}}^{(i_{i_N})}\),
- \((1 \ldots i_{i_N})_{\tilde{w}^{(i_1 \ldots i_{i_N})}}\tilde{v}^{(i_{i_N})}\) denotes the weak transform regarded as a function on \(\tilde{X}_{i_1 \ldots i_{i_N}}^{\theta^{(i_1 \ldots i_{i_N})}} \times \Sigma_{x, r}^{(i_j)},\) while the \(\tau_{i_1} p^{(i_1 \ldots i_{i_N})}\) are regarded as parameters,
- the \(a^{\theta^{(i_1 \ldots i_{i_N})}}\) are amplitudes with compact support in a system of \((\theta^{(i_1)}, \ldots, \theta^{(i_{i_N})})\)-charts labeled by the indices \(i_1, \ldots, i_{i_N},\)
- \(dA^{(i_1)}, \ldots, dA^{(i_{i_N})}, dh^{(i_{i_N})}\) are suitable measures on \(g_{p^{(i_1)}}^{(i_1)}, \ldots, g_{p^{(i_{i_N})}}^{(i_{i_N})}\), and \(G_{p^{(i_{i_N})}}\), respectively.

Similarly, one defines analogous integrals \(\tilde{I}_{\mu_{i_1 \ldots i_{i_N}}}^{\theta^{(i_1 \ldots i_{i_N})}}(x, \mu)\) in the \((\theta^{(i_1)}, \ldots, \theta^{(i_{i_N})}, \alpha^{(i_{i_N})})\)-charts. As we shall see, \(I_{\mu}(\mu)\) will be given by a sum involving integrals of the type \(\tilde{I}_{\mu_{i_1 \ldots i_{i_N}}}^{\theta^{(i_1 \ldots i_{i_N})}}(x, \mu)\), compare (7.1).

Now, for each \(p^{(i_{i_N})} \), the isotropy group \(G_{p^{(i_{i_N})}}\) acts on \(\gamma^{(i_{i_N})}((S_{i_1 \ldots i_{i_N}-1})_{p^{(i_{i_N})}})_{i_{i_N}}\) by the isotropy types \((H_{i_1}), \ldots, (H_{i_{i_N}})\). The types occurring in \(W_{i_1 \ldots i_{i_N}}\) constitute a subset of these, and
$G_{p^{i(N-1)}}$ acts on the sphere bundle $S_{i_1...i_N}$ over the submanifold $\gamma^{i(N-1)}((S_{i_1...i_{N-1}},)_{p^{i(N-1)}}i_N(H_i) \subset W_{i_1...i_N}$ with one type less.

**End of iteration.** After $N = \Lambda - 1$ steps, the end of the iteration is reached, yielding a strong desingularization of $\mathcal{N}$, see [22 Theorem 5.1], and a factorization of the phase function $\Phi_x$ that will allow us to interpolate between the different asymptotics for the integrals $I_x(\mu)$ described in Theorem 3.3 (a).

7. **The singular equivariant local Weyl law. Caustics and concentration of eigenfunctions**

We are now ready to give an asymptotic formula for the integrals (6.11) that will result in a corresponding description of the integrals (3.1) in case that $x = y$. With the notation as before, consider for fixed $1 \leq N \leq \Lambda - 1$ a maximal, totally ordered subset $\{ (H_i),...,(H_{i_N}) \}$ of non-principal isotropy types in the sense that if there is an isotropy type $(H_{i_{N+1}})$ with $i_N < i_{N+1}$ such that $\{ (H_i),...,(H_{i_N}) \}$ is a totally ordered subset, then $(H_{i_{N+1}}) = (H_L)$. Assign to each such subset the sequence of consecutive local blow-ups

$$Z_{i_1...i_N}^{\delta_{i_1},...\delta_{i_N}} := (\xi_{i_1}^{\delta_{i_1}} \circ \cdots \circ \xi_{i_N}^{\delta_{i_N}} \circ (\delta_{i_1,...i_N} \otimes \text{id})) \otimes \text{id}_\eta$$

where $\delta_{i_1,...i_N}$ denotes the sequence of local quadratic transformations

$$\delta_{i_1,...i_N} : (\sigma_{i_1},...\sigma_{i_N}) \mapsto \sigma_{i_1}(1,\sigma_{i_2},...\sigma_{i_N}) = (\sigma'_{i_1},...\sigma'_{i_N}) \mapsto \sigma'_{i_2}(\sigma'_{i_1},1,...\sigma'_{i_N}) = (\sigma''_{i_1},...\sigma''_{i_N}) \mapsto \sigma''_{i_2}(\sigma''_{i_1},1,...\sigma''_{i_N}) = \cdots = (\tau_{i_1},...\tau_{i_N}).$$

The global morphism induced by the local transformations $Z_{i_1...i_N}^{\delta_{i_1},...\delta_{i_N}}$ is then denoted by

$$Z : \tilde{X} \to X := T^*M \times G,$$

and constitutes a partial desingularization of the critical set $\mathcal{C}$, see [22 Section 9]. Pulling the phase function (3.4) back along the maps $Z_{i_1...i_N}^{\delta_{i_1},...\delta_{i_N}}$ then yields the local factorization

$$\Phi \circ Z_{i_1...i_N}^{\delta_{i_1},...\delta_{i_N}} = (i_1...i_N)^{\text{tot}} \tilde{\Phi} = \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma)(i_1...i_N)^{\text{wk}},$$

where the $\tau_{i_j}$ are monomials in the desingularization parameters $\sigma_{i_1},...,\sigma_{i_N}$. The principal result in [22] is

**Theorem 7.1.** In any of the $(\theta^{(i_1)},...\theta^{(i_N)})$-charts, the critical sets of the weak transforms $(i_1...i_N)^{\text{wk}}$ are smooth submanifolds in the resolution space of codimension $2\kappa$, and the Hessians $\text{Hess}^{(i_1...i_N)}\tilde{\Phi}^{\text{wk}}$ are transversally non-degenerate. In other words, the weak transforms $(i_1...i_N)^{\text{wk}}$ have clean critical sets in the mentioned charts. On the other hand, the weak transforms $(i_1...i_N)^{\text{wk}}$ have no critical points in any of the $(\theta^{(i_1)},...\theta^{(i_{N-1})},\alpha^{(i_N)})$-charts.

**Proof.** See [22] Theorems 6.1 and 7.2, as well as p. 90. \hfill $\Box$

In order to prove Theorem 7.1 for the $(\theta^{(i_1)},...\theta^{(i_N)})$-charts one first shows that

$$\partial_{\sigma_{i_1},...\sigma_{i_N},p^{(i_1)},...p^{(i_N)},\tilde{\sigma}^{(i_N)}}(i_1...i_N)^{\text{wk}} = 0 \implies \partial_{\sigma_{i_1},...\sigma_{i_N},p^{(i_1)},...p^{(i_N)},\tilde{\sigma}^{(i_N)}}(i_1...i_N)^{\text{wk}} = 0,$$

see [22] p. 80. If therefore

$$(i_1...i_N)^{\text{wk}} = \sigma_{i_j},p^{(i_j)},\tilde{\sigma}^{(i_N)}(\alpha^{(i_j)},h^{(i_N)},\eta)$$

denotes the weak transform of $\Phi$ regarded as a function of the variables $(\alpha^{(i_1)},...\alpha^{(i_{N-1})},h^{(i_N)},\eta)$ alone, while the variables $(\sigma_{i_1},...\sigma_{i_N},p^{(i_1)},...p^{(i_{N-1})},\tilde{\sigma}^{(i_N)})$ are kept fixed at constant values, its critical set is given by the transversal intersection

$$\text{Crit}((i_1...i_N)^{\text{wk}}) = \text{Crit}((i_1...i_N)^{\text{wk}}) \cap \{ \sigma_{i_j},p^{(i_j)},\tilde{\sigma}^{(i_N)} = \text{constant} \}.$$
In fact, $\text{Crit}(\tilde{\theta})$ turns out to be a fibre bundle [22, p. 78], and the critical set of the phase function $(\tilde{\theta})$ is equal to the fiber over $(\tilde{\sigma}, p, z)$ of this bundle, in particular being a smooth submanifold. Furthermore, [22, Lemma 7.1] implies that the transversal Hessian of $(\tilde{\theta})$ is non-degenerate if the transversal Hessian of $(\tilde{\theta})$ is non-degenerate, and the latter fact being proved in [22, Proposition 7.4] for the critical case $\sigma_i \cdots \sigma_i = 0$. Thus, we arrive at

**Corollary 7.2.** In any of the $(\theta^{(i)}, \ldots, \theta^{(i)})$-charts, the weak transforms $(\tilde{\theta})$ have clean critical sets of codimension $2\kappa$ as functions on $X_i \times \Sigma^{R_i}$, They do not have critical points in the $(\theta^{(i)}, \ldots, \theta^{(i)})$-charts.

**Proof.** The assertion is a direct consequence of the foregoing explanations and transversality arguments like those given in [22, Section 7].

From this we immediately deduce

**Proposition 7.3.** For every $N \in \mathbb{N}$, $\varepsilon > 0$, any $(\theta^{(i)}, \ldots, \theta^{(i)})$-chart labeled by the indices $\vartheta_i, \ldots, \vartheta_i$, and $x = x_i \cdots \vartheta_i$ in $Y$ (or in $Y \cap M_{\text{prim}}$ and $\varepsilon > 0$) one has the asymptotic formula

$$f_{\theta^{(i)}, \ldots, \theta^{(i)}}(x, \mu) = \prod_{j=1}^N|\tau_j|^{\text{dim} G - \text{dim} H_j} \sum_{k=0}^{N-1} \sum_{\mu} \frac{k^{\varepsilon}}{|\mu|^{\tau_i \cdots \tau_i} + \varepsilon} + R_N(x, \mu),$$

where the $k^{\varepsilon}$ and $R_N(x, \mu)$ are explicitly known coefficients that are uniformly bounded in $x$ by $(A^{(i)}, h^{(i)})$-derivatives of the amplitude $a^{\theta^{(i)}, \ldots, \theta^{(i)}}$ up to order $2k$ and $2N + \kappa + 2$, respectively, and

$$R_N(x, \mu) = O((\mu|\tau_i \cdots \tau_i| + \varepsilon)^{-\kappa - 1}).$$

In particular, with $\Phi^{w} := (\tilde{\theta})^{w}$ we have

$$\tilde{\Phi}^{w} := (\tilde{\theta})^{w}$$

If the amplitude $\mathbf{a}$ factors according to $a(x, y, \omega, \mu) = a_1(x, \mu) a_2(x, y, \mu)$, the remainder can also be estimated by derivatives of $a^{\theta^{(i)}, \ldots, \theta^{(i)}}$ with respect to $(A^{(i)}, h^{(i)})$ up to order $2\kappa + [\kappa / 2 + 1]$.

**Proof.** By definition we have $d^{(i)} = \text{dim} H_{i-1} - \text{dim} H_i$, and $H_i := G$. Consequently, $\sum_{j=1}^N d^{(i)} = \text{dim} G - \text{dim} H_j$. By Corollary 7.2 we can apply Theorem A.1 and the final remarks in the appendix to the integral (6.11) with asymptotic parameter $\mu|\tau_i \cdots \tau_i| + \varepsilon$, yielding the assertion, since $e^{-\varepsilon \Phi^{w}} = 1$ on $\text{Crit} \Phi^{w}$. In particular, (A.3) implies that the coefficients and the remainder in the expansion are uniformly bounded in $x$, since $\text{det} \Phi^{w}$ is uniformly bounded away from zero with respect to the parameters $\sigma_i$, $p^{(i)}$, $z^{(i)}$. Furthermore, $\text{Crit} \Phi^{w}$ is given as a Cartesian product of $G^{(i)(N)}$ with a certain subspace in $T^*_x M$ intersected with $\Sigma^{R_i}$, compare [22, p. 78], so that Remark A.2 applies.

**Proposition 7.4.** In any $(\theta^{(i)}, \ldots, \theta^{(i-1)}, \alpha^{(i)})$-chart labeled by the indices $\vartheta_i, \ldots, \vartheta_i$, one has

$$\tilde{\Phi}^{w} := (\tilde{\theta})^{w}$$

uniformly in $x$.

**Proof.** This is an immediate consequence of the previous corollary and the non-stationary phase principle [14, Theorem 7.7.1].
Now, let us consider the oscillatory integral $I_x(\mu)$ introduced in \[(3.2)\]. Transforming it under the
global morphism $Z$ we obtain with our previous notation for $x = x_{\theta_1 \cdots \theta_N}$ the decomposition
\[(7.1)\] $I_x(\mu) = \sum_{N=1}^{\Lambda-1} \left( \sum_{\theta_1 \cdots \theta_{N-1} \subset L} I_{\theta_1 \cdots \theta_{N-1}}^{\theta_1 \cdots \theta_N}(x, \mu) + \sum_{\theta_1 \cdots \theta_{N-1} \subset L} I_{\theta_1 \cdots \theta_N}(x, \mu) \right) + R(x, \mu),$
where the first multiple sum is one over arbitrary totally ordered subsets of non-principal isotropy
types and corresponding charts, while the second multiple sum is one over arbitrary maximal totally ordered
sets of non-principal isotropy types and corresponding charts, and $R(\mu, x)$ denotes the
non-stationary contributions of order $O(\mu^{-\infty})$ that arise by localizing the relevant integrals to tubular
neighborhoods of the relevant critical sets, or correspond to integrals over charts of the resolution
spaces where the weak transforms of the phase functions do not have critical points, compare \[22, Eq.
(9.1)]. Here $I_{\theta_1 \cdots \theta_N}(x, \mu) = 0$ unless $x = x_{\theta_1 \cdots \theta_N}$ lies in the corresponding chart, and similarly for
$I_{\theta_1 \cdots \theta_{N-1}}^{\theta_1 \cdots \theta_N}(x, \mu)$. Since the latter integrals have an analogous asymptotic description than the one
given for the integrals $I_{\theta_1 \cdots \theta_N}(x, \mu)$ in Proposition 7.3 we arrive at

**Theorem 7.5.** For every $N \in \mathbb{N}$, $x \in Y$ and $\varepsilon > 0$ (or $x \in Y \cap M_{\text{prin}}$ and $\varepsilon \geq 0$) one has

$I_x(\mu) = \sum_{N=1}^{\Lambda-1} \left( \sum_{\theta_1 \cdots \theta_{N-1} \subset L} \prod_{i=1}^{N-1} |\tau_{i}|^{\text{dim } G - \text{dim } H_i} \sum_{k=0}^{N-1} \frac{k\phi_{\theta_1 \cdots \theta_{N-1}L}(x)}{(\mu|\tau_1 \cdots \tau_{N-1}| + \varepsilon)^{\kappa + k}} + O((\mu|\tau_1 \cdots \tau_{N-1}| + \varepsilon)^{-\kappa - \tilde{N}}) \right)$

$+ \sum_{N=1}^{\Lambda-1} \left( \sum_{\theta_1 \cdots \theta_N \subset Y} \prod_{i=1}^{N} |\tau_{i}|^{\text{dim } G - \text{dim } H_i} \sum_{k=0}^{N-1} \frac{k\phi_{\theta_1 \cdots \theta_N}(x)}{(\mu|\tau_1 \cdots \tau_N| + \varepsilon)^{\kappa + k}} + O((\mu|\tau_1 \cdots \tau_N| + \varepsilon)^{-\kappa - \tilde{N}}) \right)$

up to terms of order $O(\mu^{-\infty})$, where the multiple sums run over arbitrary totally ordered subsets and
arbitrary maximal totally ordered subsets of non-principal isotropy types, respectively. Furthermore,
all coefficients and remainders are given explicitly in terms of distributions on the resolution space,
and are uniformly bounded in $x$ by $G$-derivatives of the corresponding amplitudes up to order $2k$ and
$2\tilde{N} + \kappa + 1$, respectively. If the amplitude $a$ factorizes according to $a(x, y, \omega, g) = a_1(x, y, \omega) a_2(x, y, g),$
the remainder can also be estimated by $G$-derivatives up to order $2\tilde{N} + \lfloor \kappa/2 \rfloor + 1$.

\[\Box\]

Theorem 7.5 gives a simultaneous description of the competing asymptotics $\mu \to +\infty$ and $\tau_{i} \to 0$, and
for $\varepsilon > 0$ it yields a description of the singular behaviour of the coefficients in the expansion of
$I_x(\mu)$ in Theorem 3.3 (a) as $x \in M_{\text{prin}}$ approaches singular orbits. Note that the factors $|\tau_{i}|^{\text{dim } G - \text{dim } H_i}$ in the expansion of
Theorem 7.5 reflect the fact that the coefficients become more singular as the dimension of the stabilizer
groups $H_i$ become large, that is, as one approaches more and more singular orbits, answering for the
different asymptotics in Theorem 3.3 (a) given by the exponents $\kappa_\chi = \text{dim } O_\chi$. For an exceptional orbit of type $(H_i)$ one has $\dim G - \dim H_i = \kappa$, so that the corresponding factors $|\tau_{i}|^{\kappa}$ cancel each other,
in concordance with Theorem 3.3 (a), by which the summands in the expansion of $I_x(\mu)$ in

\[\text{indeed, assume that } M_{\text{prin}} \ni x_{\theta_1 \cdots \theta_N} \to y \in (M(H_i)) \text{ in such a way that the index } \tau_{i} \text{ goes to zero with rate}
\]

$\tau_{i} \approx \mu^{-1} \to 0$. Then, if $\kappa = \text{dim } G$, 

$\prod_{i=1}^{N} |\tau_{i}|^{\text{dim } G - \text{dim } H_i} \approx \prod_{i=1}^{N} \frac{|\tau_{i}|^{\text{dim } H_i}}{\mu^\kappa} \approx O(\mu^{\text{dim } G - \text{dim } H_i}) = O(\mu^{\text{dim } O_\chi}).$
Proposition 7.6 (Singular point-wise asymptotics for the kernel of the equivariant approximate projection). For arbitrary integers \( \bar{N}_1, \bar{N}_2 = 0, 1, 2, 3, \ldots \), fixed \( \gamma \in \mathcal{G}, x \in M \) and \( \varepsilon > 0 \) (or \( x \in M_{\text{prin}} \cup M_{\text{except}} \) and \( \varepsilon \geq 0 \)) one has for \( \mu \to +\infty \) the asymptotic expansion

\[
K_{\bar{x},\circ \Pi_x}(x, x) = \frac{\mu^{n-1}d_x}{(2\pi)^{n-\kappa}} \sum_{j=0}^{\bar{N}_1-1} \mu^{-j} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{\ell=1}^{N} |\tau_{i_\ell}|^{\text{dim } G - \text{dim } H_{i_\ell}} \cdot \left[ \sum_{k=0}^{\bar{N}_2-1} \frac{\mathcal{L}_{i_1 \cdots i_N}^j(x, \gamma)}{(\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{\kappa + k}} + O_x((\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{-\kappa - \bar{N}_2}) \right]
\]

up to terms of order \( O(\mu^{n-\bar{N}_1-1}) \), where the multiple sum runs over all possible totally ordered subsets \( \{(H_{i_1}), \ldots, (H_{i_N})\} \) of singular isotropy types, and all coefficients and remainders are explicitly given by distributions on the resolution space bounded uniformly in \( x \) by derivatives of \( \gamma \) up to order \( 2k \) and \( 2\bar{N}_2 + \lfloor \kappa/2 \rfloor + 1 \), respectively. For \( \mu \to -\infty \), the function \( K_{\bar{x},\circ \Pi_x}(x, x) \) is rapidly decreasing in \( \mu \).

Proof. The assertion follows from Corollary 2.2 by applying Theorem 7.3 to the integrals (2.10), combining the coefficients \( kP_{i_1 \cdots i_N-1}^m(x) \) and \( kQ_{i_1 \cdots i_N-1}^m(x) \) in the expansion of \( I_x(\mu) \), and collecting the terms from different charts corresponding to the same subset of isotropy types. Then, one merges the contributions from exceptional and principal isotropy types, taking into account that by Theorem 3.3 (a) the summands in Theorem 7.5 must stay bounded as one approaches exceptional orbits, all coefficients and remainders in the expansions being smooth in \( R, t \) and uniformly bounded in \( x \) by derivatives of \( \gamma \) up to order \( 2k \) and \( 2\bar{N}_2 + \lfloor \kappa/2 \rfloor + 1 \), respectively.

Using standard Tauberian arguments we obtain as our third main result

Theorem 7.7 (Singular equivariant local Weyl law). Let \( M \) be a closed connected Riemannian manifold \( M \) of dimension \( n \) with an isometric and effective action of a continuous compact Lie group \( G \) and \( \mathcal{P}_0 \) a \( G \)-invariant elliptic classical pseudodifferential operator on \( M \) of degree \( m \). Let \( p(x, \xi) \) be its principal symbol, and assume that \( \mathcal{P}_0 \) is positive and symmetric. Denote its unique self-adjoint extension by \( \mathcal{P} \), and for a given \( \gamma \in \mathcal{G} \) let \( e_\gamma(x, y, \lambda) \) be its reduced spectral counting function. Write \( \kappa \) for the dimension of the \( G \)-orbit in \( M \) of principal type and \( d_\gamma \) for the dimension of an irreducible \( G \)-representation \( \pi_\gamma \) of class \( \gamma \). Then, for \( x \in M_{\text{prin}} \cup M_{\text{except}} \) one has the asymptotic formula

\[
\left| e_\gamma(x, x, \lambda) - \frac{d_\gamma \lambda^{n-m}}{(2\pi)^{n-\kappa}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{\ell=1}^{N} |\tau_{i_\ell}|^{\text{dim } G - \text{dim } H_{i_\ell} - \kappa} \mathcal{L}_{i_1 \cdots i_N}^{0, 0}(x, \gamma) \right| \leq C_\gamma \lambda^{n-m-1} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{\ell=1}^{N} |\tau_{i_\ell}|^{\text{dim } G - \text{dim } H_{i_\ell} - \kappa - 1}
\]

as \( \lambda \to +\infty \), where the multiple sum runs over all possible totally ordered subsets \( \{(H_{i_1}), \ldots, (H_{i_N})\} \) of singular isotropy types, and the coefficients satisfy the bounds \( \mathcal{L}_{i_1 \cdots i_N}^{0, 0}(x, \gamma) \ll \|\gamma\|_\infty \) uniformly in \( x \), while

\[
C_\gamma \ll d_\gamma \sup_{\|D^l\gamma\|_\infty} \|D^l\|_\infty
\]

is a constant independent of \( x \) and \( \lambda \), the \( D^l \) are differential operators on \( G \) of order \( l \), and the \( \tau_{i_\ell} = \tau_{i_\ell}(x) \) parameters satisfying \( |\tau_{i_\ell}| \approx \text{dist}(x, M(H_{i_\ell})) \).
Proof. The assertion follows by integrating the expression for $K_{\Pi, \phi(x, x)}$ in Proposition 7.6 with respect to $\mu$ from $-\infty$ to $\sqrt[\mu]{\lambda}$ for the values $\mu = 0, N_1 = \kappa + 1, N_2 = 1$ with the arguments given in the proof of Theorem 4.3, noting that $\dim G - \dim H_{ii} = 0$ for all $i$. □

Remark 7.8. Again, if $G$ is a connected compact semisimple Lie group, in terms of the highest weight $\Lambda_\gamma \in t^*_G$ of $\gamma \in \hat{G}$ we have $C_\gamma \ll |\Lambda_\gamma|^{2|\Sigma^+| + |\Sigma^-|}$, compare Remark 4.5.

As an immediate consequence this yields

Corollary 7.9 (Singular point-wise bounds for isotypic spectral clusters). In the setting of Theorem 7.7 we have

$$
\sum_{\lambda_j \in (\Lambda, \lambda + 1), \ e_j \in L_2^2(M)} |e_j(x)|^2 \leq \begin{cases} C \lambda^{\frac{n-1}{n}}, & x \in M_{\text{sing}}, \\ C \lambda^{\frac{n-k-1}{n}} \sum_{N=1}^{N-1} \sum_{i_1 < \cdots < i_N} \prod_{i=1}^N |\tau_{ii}|^{n-G-\dim H_{ii}-k-1}, & x \in M - M_{\text{sing}}, \end{cases}
$$

with $C > 0$ independent of $\gamma$. In particular, the bound holds for each individual $e_j \in L_2^2(M)$ with $\lambda_j \in (\lambda, \lambda + 1)$.

We would like to remark that the expansion in Theorem 7.7 is only meaningful if $\lambda$ is sufficiently large compared to the desingularization parameters $\tau_{ii}$, more precisely, if

$$
\lambda^{1/m} \prod_i |\tau_{ii}| > 1
$$

for all possible combinations of the $\tau_{ii}$. While (4.2) describes the asymptotics of the equivariant spectral function for arbitrary, but fixed $x \in M$, Theorem 7.7 gives a uniform description of the behaviour of the coefficients as $x \in M_{\text{prin}}$ approaches singular orbits.

An asymptotic formula for $e_{\gamma}(x, x, \lambda)$ that interpolates between the various asymptotic behaviours in Theorem 4.3, in the same way as Theorem 7.5 interpolates between the different asymptotics in Theorem 3.3 (a) can be obtained by integrating the expression for $K_{\Pi, \phi(x, x)}$ in Proposition 7.6 with respect to $\mu$ from $-\infty$ to $\sqrt[\mu]{\lambda}$ for the values $\mu = 1, N_1 = \kappa + 1, N_2 = 1$ with the arguments given in the proof of Theorem 4.3. This leads to expressions for $e_{\gamma}(x, x, \lambda)$ which involve the hypergeometric function, in the same way than the associated Legendre polynomials are given in terms of that function [12] p. 188.

Example 7.10. To illustrate the desingularization process and our results, let us resume Example 4.11 where we considered the action of $G = \text{SO}(2)$ on the standard 2-sphere $M = S^2 \subset \mathbb{R}^3$ by rotations around the $x_3$-axis. The isotropy types are $H_1 = \text{SO}(2)$ and $H_2 = \{e\}$, and the set of maximally singular orbits $M_1(H_1) = \{x_N, x_S\}$ is disconnected in this case. Instead of working with the covering (6.4), we can cover $S^2$ with the two charts $Y_1 := S^2 - \{x_N\}$ and $Y_2 := S^2 - \{x_S\}$ by introducing geodesic polar coordinates $x = \exp_{x_{\tilde{v}}} (x_1 \tilde{v})$ and $x = \exp_{x_{\tilde{v}}} (x_2 \tilde{v})$ around the poles, respectively, where $\tilde{v} \in S^1$, and $\tau_i > 0$ equals the induced Riemannian distance of $x$ to the corresponding pole. Note that $\mathfrak{g}^\perp = \mathfrak{g}^\perp_{x_{\tilde{v}}} = \{0\}$, so that it is not necessary to perform a blow-up in the group variables, and no additional $O(\mu^{-\infty})$-terms arise. After one iteration, the action is desingularized, and one obtains in agreement with Theorem 7.5 for arbitrary $N \in \mathbb{N}$ and $\varepsilon \geq 0$ the asymptotic formula

$$
I_{\varepsilon}(\mu) = \sum_{i=1,2} \left[ \sum_{k=0}^{N-1} k \mathcal{Q}_i(x) (\mu \tau_i + \varepsilon)^{-1-k} + O((\mu \tau_i + \varepsilon)^{-1}) \right],
$$

all coefficients being bounded in $x$. In particular, setting $\varepsilon = 0$ one sees that the leading coefficient in Theorem 3.3 (a) is given by

$$
2\pi \mathcal{Q}_0(x) = \frac{1}{\tau_1} \mathcal{Q}_1(x) + \frac{1}{\tau_2} \mathcal{Q}_2(x), \quad x \neq x_N, x_S,
$$
which describes its singular behaviour as one approaches the fixed points. This implies for the reduced spectral counting function of the Laplace-Beltrami operator $-\Delta$ on $S^2$ the asymptotics

$$e_m(x, x, \lambda) = \frac{\sqrt{\lambda}}{2\pi} \frac{\mathcal{L}(x)}{\text{dist}(x, \{x_N, x_S\})} \ll \frac{1 + |m|^3}{\text{dist}^2(x, \{x_N, x_S\})}, \quad m \in \mathbb{Z}, x \neq x_N, x_S,$$

$\mathcal{L}(x)$ being bounded in $x$, provided that $\sqrt{\lambda} \text{dist}(x, \{x_N, x_S\}) > 1$, in agreement with Theorem 7.7. From this, we immediately deduce the following pointwise bounds for spherical harmonics. Let $Y_{k,m}$ be the classical spherical functions with $k \in \mathbb{N}, m \in \mathbb{Z} \simeq \text{SO}(2)$, $|m| \leq l$ satisfying

$$-\Delta Y_{k,m} = \lambda_k Y_{k,m}, \quad \lambda_k = k(k + 1).$$

Then, from

$$e_m(x, x, \lambda + 1) - e_m(x, x, \lambda) = \sum_{\lambda_k \in \{\lambda, \lambda + 1\}} |Y_{k,m}(x)|^2$$

one directly infers for fixed $m$ the point-wise bounds

$$|Y_{k,m}(x)|^2 \ll \begin{cases} (1 + |m|^3) \sqrt{\lambda_k}, & x = x_N, x_S, \\ (1 + |m|^3) \text{dist}^2(x, \{x_N, x_S\}), & x \neq x_N, x_S, \end{cases}$$

as $k \to \infty$, where we took into account the bound (4.6). In particular, this is consistent with (1.16). Thus, spherical harmonics with fixed $m$ concentrate on the poles as $k$ becomes large. This fact is in accordance with the probability of finding a classical particle of zero angular momentum near singular orbits and the shape of the corresponding equivariant quantum limits, see [18, Section 9.2]. Furthermore, if $c$ denotes a closed geodesic on $S^2$ we obtain for the restriction of $Y_{k,m}$ to $c$ the $L^\infty$-bounds

$$\|Y_{k,m}|c\|_{\infty} = \begin{cases} 0, & x_N, x_S \in c, \\ O(n, e(1)), & \text{otherwise}, \end{cases}$$

as $k \to \infty$. The foregoing considerations can be immediately generalized to surfaces of revolution diffeomorphic to the 2-sphere.

8. Sharpness

To conclude, we show that the obtained bounds are sharp and that, as in the classical case [11] and [30, Section 3.4], they are already attained on the 2-dimensional sphere. Denote by $M = S^n$ the standard sphere in $\mathbb{R}^{n+1}$ endowed with the induced metric, and let $\Delta$ be the Laplace-Beltrami operator on $S^n$. The eigenvalues of $-\Delta$ are given by the numbers $\lambda_k = k(k + n - 1)$, where $k = 0, 1, 2, 3, \ldots$ and the corresponding $d_k$-dimensional eigenspaces $\mathcal{H}_k$ are spanned by the classical spherical functions $Y_{kl}$, $1 \leq l \leq d_k$, so that

$$-\Delta Y_{kl} = \lambda_k Y_{kl}.$$  

The $Y_{kl}$ are orthonormal to each other, and by the spectral theorem we have the decomposition $L^2(M) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. Now, let $G \subset \text{SO}(n)$ be a subgroup of the isotropy group of a point in $S^n \simeq \text{SO}(n+1)/\text{SO}(n)$, and

$$\mathcal{H}_k = \bigoplus_{\gamma \in G} \mathcal{H}_k^\gamma$$

be the decomposition of the eigenspace $\mathcal{H}_k$ into its isotypic components. It is clear that $d_k = \sum_{\gamma \in G} m_{\gamma}(k) d_{\gamma}$, where $m_{\gamma}(k)$ denotes the multiplicity of $\pi_\gamma \in \gamma$ in $\mathcal{H}_k$. Let $\{Z_{kl}^\gamma\} \subset \text{Span} \{Y_{kl}\}_{l=1}^{d_k}$ be an orthonormal basis of $\mathcal{H}_k^\gamma$ so that with $\mu = \mu_k - 1$, $\mu_k = \sqrt{\lambda_k}$,

$$K_{\chi_\mu \circ \Pi_{\gamma}}(x, y) = \sum_{j=1}^{m_{\gamma}(k)d_{\gamma}} Z_{kl}^\gamma(Z_{kl}^\gamma(y) Z_{kl}^\gamma(y),$$

$\chi_\mu \circ \Pi_{\gamma}$ being the projection onto $\mathcal{H}_k^\gamma$. By Theorem 4.3 we have the bound

$$|K_{\chi_\mu \circ \Pi_{\gamma}}(x, y)| = |e_\gamma(x, x, \mu_k) - e_\gamma(x, x, \mu_k - 1)| \leq C_{x, \gamma} \mu_k^{n-\kappa_{\gamma} - 1}, \quad C_{x, \gamma} > 0, \ x \in S^n,$$
while the behaviour near singular orbits is described in Theorem 7.7 We now define for fixed $x \in S^n$ the isotypic zonal eigenfunction
\[ e^\gamma_{\mu_k} : S^n \ni y \mapsto \sum_{j=1}^{m_k(x)k} Z^\gamma_{k_j}(x)Z^\gamma_{k_j}(y) \in \mathbb{C}, \]
which is an eigenfunction of $\sqrt{-\Delta}$ for the eigenvalue $\mu_k$ and satisfies
\[ \|e^\gamma_{\mu_k}\|_{L^2} = \left( \sum_{j=0}^{m_k(x)k} |Z^\gamma_{k_j}(x)|^2 \right)^{1/2} = (K_{x,\gamma}(x,x))^{1/2}. \]

In order to examine the sharpness of the bounds obtained, we specialize to the case where $n = 2$ and $G = \text{SO}(2)$ acts by rotations around the symmetry axis through the poles. In this case, $\mathcal{H}_k$, $\gamma \equiv m \in \mathbb{Z}$, $|m| \leq k$ is spanned by the spherical function
\[ Y_{k,m}(\phi, \theta) = \sqrt{\frac{2k+1}{4\pi (k+m)!}} P_{k,m}(\cos \theta)e^{im\phi}, \quad 0 \leq \phi < 2\pi, \ 0 \leq \theta < \pi, \]
where $P_{k,m}$ is the associated Legendre polynomial.

Furthermore, for the Legendre polynomials $P_k(\cos \theta)$ one has the asymptotics
\[ P_k(\cos \theta) = \sqrt{\frac{2}{\pi k \sin \theta}} \cos \left( \left( k + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right) + O \left( \frac{1}{(k \sin \theta)^{3/2}} \right), \]
where the remainder is uniform in $\theta$ on any interval $[\varepsilon, \pi - \varepsilon]$ with $0 < \varepsilon$ small.\[ \text{[12 p. 303]} \] Thus, in the special case where $m = 0$ we see that with $\mu = \mu_k - 1$ one has in the limit $k \to \infty$
\[ K_{x,\gamma}(x,x) = |Y_{k,0}(x)|^2 = \frac{2k+1}{4\pi} |P_{k,0}(\cos \theta)|^2 \approx \begin{cases} \frac{\sqrt{\lambda_k}}{\sin \theta} \approx \frac{1}{\text{dist}(x,\{x_N, x_S\})}, & x = x_N, x_S, \\ \frac{1}{\sin \theta} \approx \frac{1}{\sqrt{\lambda_k}}, & x \in S^2 - \{x_N, x_S\}, \end{cases} \]
where $x_N$ and $x_S$ denote the poles. Consequently, we conclude that the remainder estimates in Theorems 4.3 and 7.7 are sharp in the spectral parameter $\lambda$, but not optimal in the desingularization parameters $\tau_i$, since in the present case we have $\lambda \approx k^2$, $\sin \theta \approx \theta \approx \tau_i$, compare also Example 7.10. Nevertheless, the estimate given in Theorem 7.7 qualitatively reflects the singular behaviour of $Y_{k,0}(x)$ as $x$ approaches the poles, and suggests that the asymptotic formula (8.1) should have a structural explanation in terms of caustics of oscillatory integrals. On the other hand, the bound for $|Y_{k,0}(x)|$ implies similar bounds for $e^\gamma_{\mu_k}(y) = Y_{k,0}(x)Y_{k,0}(y)$, and that for an eigenfunction $f \in L^2(S^2)$ of $-\Delta$ belonging to a specific isotypic component with $\|f\|_{L^2} = 1$ and eigenvalue $\lambda$ the estimate
\[ |f(x)| \leq C_{x,\gamma} \lambda^{\frac{n-1}{4}} - |m|, \quad x \in S^2, \]
in Corollary 4.6 cannot be improved in the eigenvalue aspect.

To close, let us mention that in the considered case $M = S^2$ and $G = \text{SO}(2)$ the previous considerations imply for the equivariant counting function $N_\gamma(\lambda)$ of the Beltrami-Laplace operator with $\gamma \equiv m$ the estimate
\[ N_\gamma(\lambda) = d_\gamma \sum_{\lambda_k \leq \lambda} m_k(k) \approx \sum_{k(k+1) \leq \lambda, |m| \leq k} 1 \approx \sum_{|m| \leq k \leq \sqrt{\lambda}} 1 \approx \sqrt{\lambda} - |m|, \]
\[ \text{[There is even an asymptotic expansion of } P_k(\cos \theta), \text{provided that } k \sin \theta > 1.} \]
as $\lambda \to +\infty$. From this one recovers the classical Weyl law

$$N(\lambda) = \sum_{k(k+1) \leq \lambda} \dim \mathcal{H}_k = \sum_{\gamma \in \hat{G}} N_\gamma(\lambda) \approx \sum_{|m| \leq \sqrt{\lambda}} (\sqrt{\lambda} - |m|) \approx (2\sqrt{\lambda} + 1)\sqrt{\lambda} - 2\frac{\sqrt{\lambda}(\sqrt{\lambda} + 1)}{2} = \lambda.$$ 

The asymptotic formula \[8.2\] implies that the equivariant Weyl law proved in \[22\] Theorem 9.5 is sharp up to a logarithmic factor in the remainder estimate, but shows that the remainder estimates in Theorems 4.3 and 7.7 are not optimal in $\gamma \in \hat{G}$.

**Appendix A. Stationary phase principle and caustics**

Our analysis relies on the generalized stationary phase principle, which we state below. Sketches of proofs can be found in \[3\] Theorem 3.3 and \[34\] Theorem 2.12. For a detailed proof, which includes explicit expressions for the coefficients and the remainder term in the stationary phase expansion, see \[22\] Theorem 4.1 and Remark 4.2.

**Theorem A.1** (Generalized stationary phase principle). Consider an $n$-dimensional Riemannian manifold $M$ with volume density $dM$, a phase function $\psi \in C^\infty(M, \mathbb{R})$, and set

$$I(\mu) = \int_M e^{i\mu\psi(m)} a(m) \, dM(m), \quad \mu > 0,$$

where $a(m) \in C^\infty_c(M)$ is an amplitude. In addition, assume that the critical set

$$C := \text{Crit}(\psi) = \{ m \in M \mid \psi_* : T_m M \to T_{\psi(m)} \mathbb{R} \text{ is zero} \}$$

of the phase function $\psi$ is clear\[\footnote{That is, $C$ is a smooth submanifold and the Hessian of $\psi$ is non-degenerate on $M \setminus N$ for all $m \in C$. In this case, we shall also say that the Hessian is \textit{transversally non-degenerate} or that the \textit{transversal Hessian} is non-degenerate at $m \in C$.}^6$ meaning that $\psi$ is a Morse–Bott function. Then, for all $\hat{N} \in \mathbb{N}$ one has the asymptotic formula

$$I(\mu) := e^{i\mu\psi_0} (2\pi/\mu)^{\frac{n-p}{2}} \left[ \sum_{r=0}^{\hat{N}-1} \mu^{-r} Q_r(\psi, a) + R_{\hat{N}}(\psi, a; \mu) \right],$$

where $p$ denotes the dimension of $C$, $\psi_0$ is the constant value of $\psi$ on $C$, and the expressions $Q_r(\psi, a)$ and $R_{\hat{N}}(\psi, a; \mu)$ can be computed explicitly. Furthermore, there exist constants $C_{r, \psi} > 0$ and $\hat{C}_{\hat{N}, \psi, \epsilon} > 0$ such that

$$|Q_r(\psi; a)| \leq C_{r, \psi} \text{vol}(\text{supp } a \cap C) \sup_{l \leq 2r} \|D^l a\|_{\infty, C},$$

$$|R_{\hat{N}}(\psi, a; \mu)| \leq \hat{C}_{\hat{N}, \psi, \epsilon} \mu^{-\hat{N}} \int_C \sup_{l \leq 2\hat{N}} \|D^l a\|_{H^{(n-p)/2+\epsilon}(N_m \mathbb{C})} \, d\sigma_C(m),$$

for any $\epsilon > 0$, where $D^l$ are differential operators on $M$ transversal to $C$ of order $l$ independent of $\psi$, $H^s$ denotes the $s$-th Sobolev space, and

$$C_{r, \psi} \leq \sup_{m \in C \cap \text{supp } a} \left\| \left( \psi''(m) |_{N_m \mathbb{C}} \right)^{-1} \right\|^r \cdot |\text{det } \psi''(m) |_{N_m \mathbb{C}}|^{-1/2}$$

with a similar bound for $\hat{C}_{\hat{N}, \psi, \epsilon}$. In particular,

$$Q_0(\psi, a) = \int_C \frac{a(m)}{|\text{det } \psi''(m) |_{N_m \mathbb{C}}|^{1/2}} d\sigma_C(m) e^{i\frac{\pi}{2} \sigma_{\psi''}},$$

where $d\sigma_C$ stands for the induced volume density on $C$ and $\sigma_{\psi''}$ for the constant value of the signature of the transversal Hessian $\psi''(m) |_{N_m \mathbb{C}}$ on $C$. 

\[\square\]
Remark A.2. In the setting of the previous theorem, suppose that \( M = M_1 \times M_2 \) is a product manifold, as well as \( C = C_1 \times C_2 \), where \( C_i \subset M_i \) are submanifolds of codimension \( q_i \), and that the amplitude factorizes according to \( a(m) = a_1(m_1) a_2(m_2) \), \( m = (m_1, m_2) \in M \). Then, the remainder term can be estimated according to

\[
| R_N(\psi, a; \mu) | \leq \sum_{i=1,2} \int_{C_i} \left\| D_t^l a_i \right\|_{H^{\frac{1}{2} + \varepsilon}(N, c_i)} d\sigma_{c_i}(m_i)
\]

for any \( \varepsilon > 0 \), the \( D_t^l \) being differential operators on \( M_i \) transversal to \( C_i \) of order \( l \). This allows one to estimate the remainder term by derivatives of the amplitudes \( a_i \) of lower order.

Remark A.3. As stated, the expansion (A.2) is valid for arbitrary \( \mu > 0 \), though the case of interest is when \( \mu \to +\infty \), since then the error becomes smaller than the other terms. In essence, the point is that by Taylor’s formula one has

\[
\left| e^{it} - \sum_{k=0}^{N-1} \frac{(it)^k}{k!} \right| = O(|t|^N) \quad \text{for arbitrary } t \in \mathbb{R},
\]

no matter how large \( |t| \) is, though the estimate is only meaningful for \( |t| < 1 \).

One of the main concerns of this paper is extrapolating between stationary phase expansions of different orders. Thus, consider an integral of the form (A.1) with a clean critical set, let \( \tau \geq 0 \) be an additional parameter, and define the integral

\[
I(\mu, \tau) := \int_M e^{i\mu \tau \psi(m)} a(m) \, dM(m).
\]

Depending on the value of \( \tau \), it will exhibit different asymptotic behaviours in \( \mu \). Indeed, for \( \tau > 0 \) the integral \( I(\mu, \tau) \) decreases with order \( O(\mu^{-\frac{N}{2} - \frac{\varepsilon}{\mu^2}}) \), while for \( \tau = 0 \) it is actually independent of \( \mu \). This behaviour is reflected in the fact that if we apply the previous theorem to the integral \( I(\mu, \tau) \), either with \( \mu \tau \) as asymptotic parameter, or with \( \tau \psi \) as phase function, we would arrive at an expansion of the form (A.2) in which the coefficients in the expansion blow up as \( \tau \to 0 \) due to the abrupt change of the critical set of the phase function \( \tau \psi(m) \) when \( \tau \) becomes zero. In general, if \( \psi \in C^\infty(M, \mathbb{R}) \) denotes a family of phase functions depending on a parameter \( \varepsilon \) such that \( \text{Crit}(\psi) \) is clean for generic values of \( \varepsilon \), one understands by a caustic point for this family a parameter value \( \varepsilon \) such that \( \text{Crit}(\psi) \) is not clean or where \( \text{Crit}(\psi) \) changes drastically its dimension, compare [34]. With this terminology, in the situation above \( \tau = 0 \) constitutes a caustic point. Nevertheless, it is possible to derive an adequate asymptotic expansion for \( I(\mu, \tau) \) that smoothly interpolates between the different asymptotics, and takes into account the competing asymptotics \( \mu \to +\infty \) and \( \tau \to 0 \), based on the following simple idea. Let \( \varepsilon \geq 0 \) be a fixed positive real number, and consider the integral

\[
I_\varepsilon(\mu) := \int_M e^{i\mu \varepsilon \psi(m)} e^{-i\varepsilon \psi(m)} a(m) \, dM(m).
\]

Clearly, \( I(\mu) = I_\varepsilon(\mu + \varepsilon) \). Since \( e^{-i\varepsilon \psi} \) is independent of \( \mu \), we can apply the previous theorem with \( \mu + \varepsilon \) as parameter, obtaining for each \( \varepsilon \in \mathbb{N} \) and each \( \varepsilon \geq 0 \) the asymptotic formula

\[
I(\mu) = e^{i(\mu + \varepsilon)\psi_0} \left( \frac{2\pi}{\mu + \varepsilon} \right)^{\frac{N-1}{2}} \sum_{r=0}^{\varepsilon \mu - 1} \left( \frac{-\varepsilon}{\mu} \right)^k (\mu + \varepsilon)^{-r} Q_r(\psi; e^{-i\varepsilon \psi} a) + R_N(\psi, e^{-i\varepsilon \psi} a; \mu + \varepsilon).
\]

Because

\[
\frac{1}{\mu + \varepsilon} = \frac{1}{\mu} \left( 1 + \frac{\varepsilon}{\mu} \right) = \frac{1}{\mu} \sum_{k=0}^{\infty} \left( \frac{-\varepsilon}{\mu} \right)^k = \frac{1}{\mu} - \frac{\varepsilon}{\mu^2} + \frac{\varepsilon^2}{\mu^3} - \cdots, \quad \varepsilon/\mu < 1,
\]

the expansion (A.2) is consistent with the expansion (A.4), the respective corrections being of lower order. Now, if we apply the previous argument to \( I(\mu, \tau) = I(\mu \tau) \) we obtain

\[
I(\mu, \tau) = e^{i(\mu \tau + \varepsilon)\psi_0} \left( \frac{2\pi}{\mu \tau + \varepsilon} \right)^{\frac{N-1}{2}} \sum_{r=0}^{\varepsilon \mu \tau - 1} \left( \mu \tau + \varepsilon \right)^{-r} Q_r(\psi; e^{-i\varepsilon \psi} a) + R_N(\psi, e^{-i\varepsilon \psi} a; \mu \tau + \varepsilon)
\]
as $\mu \to +\infty$. The formula is only meaningful for $\tau \mu + \varepsilon > 1$, and simultaneously describes the asymptotic behaviour of $I(\mu, \tau)$ in the competing parameters $\tau$ and $\mu$. For $\varepsilon > 0$, it interpolates between the asymptotics $O(\mu^{-\frac{n-2}{2}})$ and $O(\mu^0)$ in a smooth way; in fact, for $\tau = 0$ it simply collapses to $\int_M a \ dM$.  

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E-mail address, Pablo Ramacher: ramacher@mathematik.uni-marburg.de

Philipps-Universität Marburg, FB 12 Mathematik und Informatik, Hans-Meerwein-Str., 35032 Marburg