Energy scale independence of Koide’s relation for quark and lepton masses

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Abstract
Koide’s mass relation of charged leptons has been extended to quarks and neutrinos, and we prove here that this relation is independent of energy scale in a huge energy range from 1 GeV to $2 \times 10^{16}$ GeV. By using the parameters $k_u$, $k_d$ and $k_\nu$ to describe the deviations of quarks and neutrinos from the exact Koide’s relation, we also check the quark-lepton complementarity of masses such as $k_l + k_d \approx k_\nu + k_u \approx 2$, and show that it is also independent (or insensitive) of energy scale.

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I. INTRODUCTION

Despite splendid successes of the standard model of particle physics, some fundamental questions remained unanswered, among which the origin of fermion masses is one of the most important problems. In the standard model, these masses are taken as free parameters, and can only be extracted from experimental data. Phenomenological analysis aiming at discovering the numerical relations between fermion masses is useful and practical for the exploration of the mystery of lepton and quark masses. There have been some conjectures on this problem (for example, [1, 2]), in which Koide’s relation [3, 4] is the most accurate one, which links the masses of charged leptons together,

\[ m_e + m_\mu + m_\tau = \frac{2}{3} (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2, \]

where \( m_e, m_\mu, m_\tau \) are the masses of electron, muon, and tau, respectively. In order to see the accuracy of Koide’s relation, we can introduce a parameter \( k_l \),

\[ k_l = \frac{m_e + m_\mu + m_\tau}{\frac{2}{3} (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}. \]

With the data of PDG [7],

\[ m_e = 0.510998902 \pm 0.000000021 \text{ MeV}, \]
\[ m_\mu = 105.658357 \pm 0.000005 \text{ MeV} \]
\[ m_\tau = 1776.99^{+0.29}_{-0.26} \text{ MeV}, \]

we can get the range of \( k_l \), \( k_l = 1^{+0.000002635}_{-0.000002021} \), which is perfectly close to 1.

This relation was speculated on the basis of a composite model of quarks and leptons [3] and the extended technicolor-like model [4]. The fermion mass matrix in these models is taken as

\[ M_f = m_0^f G O_f G, \]

where \( G = \text{diag}(g_1, g_2, g_3) \). With the assumptions \( g_i = g^{(1)}_i + g^{(8)}_i \), \( \sum_i g^{(8)}_i = 0 \) and \( \sum_i (g^{(8)}_i)^2 = 3(g^{(1)}_i)^2 \), and the charged lepton mass matrix is 3×3 unit matrix, Koide obtained this relation. This relation is deduced in several other different models by Koide also [5]. (For a review, see [6].) Several explanations for this excellent relation were presented in the past ten years. Foot [8] gave a geometrical interpretation to it,

\[
\cos \theta_l = \frac{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}) \cdot (1, 1, 1)}{|(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})|| (1, 1, 1)|} = \frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{3} \sqrt{m_e + m_\mu + m_\tau}},
\]

where \( \theta_l \) is the angle between the points \((\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})\) and \((1, 1, 1)\). Commonly, we can extend \((1, 1, 1)\) to a more general mass square root vector \((\sqrt{m_0}, \sqrt{m_0}, \sqrt{m_0})\), however, the
value of \( m_0 \) does not affect explicitly the value of \( \theta_l \). Furthermore, we can find the relation between \( k_l \) and \( \theta_l \),

\[
k_l = \frac{1}{2 \cos^2 \theta_l},
\]

and \( \theta_l = \frac{\pi}{4} \).

Besides the miraculous success of Koide’s relation for charged leptons, two further questions emerge naturally,

1. whether this relation can be applied to quarks and neutrinos,
2. whether this relation holds well at different energy scales.

The first question has been discussed by Esposito and Santorelli \[9\], Rivero and Gsponer \[10\], and us \[11\]. In Ref. \[11\], we applied Koide’s relation to quarks and neutrinos, and estimated the masses of neutrinos. For the second question, due to the renormalization effect, the masses of quarks and leptons vary with the energy scale of interaction, so we must examine Koide’s relation at different energy scales explicitly, before one can take the extension of the relation to quarks and neutrinos seriously.

In Section II, we check the application of Koide’s relation to quarks and charged leptons. By using the previously introduced parameters \( k_u, k_d \) to characterize the deviations of \( u \)-type and \( d \)-type quarks from Koide’s relation \[11\], we find that \( k_u \) and \( k_d \) keep almost invariant in a very wide energy range. This means that Koide’s relation is a universal result which is independent (or insensitive) of the running masses of quarks and leptons. In Section III, we apply Koide’s relation to neutrinos, with both the normal and inverted mass schemes considered. Furthermore, with some analogies and conjectures between quarks and leptons, the neutrino masses are predicted. Finally, in Section IV, we give some discussions to Koide’s relation.

II. KOIDE’S RELATION FOR QUARKS AND CHARGED LEPTONS

In order to see whether Koide’s relation holds well at different energy scales, we must get the values of quark and lepton masses first. Because of the renormalization effect, the scale dependence of the running quark masses is determined by \[12, 13\]

\[
\mu \frac{d}{d\mu} m(\mu) = -\gamma(\alpha_s) m(\mu), \quad (2)
\]
and
\[
\gamma(\alpha_s) = \gamma_0 \frac{\alpha_s}{\pi} + \gamma_1 \left( \frac{\alpha_s}{\pi} \right)^2 + \gamma_2 \left( \frac{\alpha_s}{\pi} \right)^3 + O(\alpha_s^4),
\]
\[
\gamma_0 = 2,
\]
\[
\gamma_1 = \frac{101}{12} - \frac{5}{18} n,
\]
\[
\gamma_2 = \frac{1}{32} \left[ 1249 - \left( \frac{2216}{27} + \frac{160}{3} \zeta(3) \right) n - \frac{140}{81} n^2 \right].
\]

where \( \mu \) is an energy scale, \( n \) is the effective number of quark flavors \[14\], and \( \alpha_s \) is the effective QCD coupling constant, which is also a \( \mu \)-dependent function,
\[
\alpha_s(\mu) = \frac{4\pi}{\beta_0} \frac{1}{L} \left\{ 1 - \frac{2\beta_1}{\beta_0^2} \ln L + \frac{4\beta_2^2}{\beta_0^4 L^2} \left[ \left( \ln L - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right] \right\} + O \left( \frac{\ln^2 L}{L^3} \right),
\]
where \( L = \ln(\mu^2/\Lambda^2) \), and \( \beta_0, \beta_1, \beta_2 \) are the coefficients of the renormalization group equation
\[
\mu \frac{d}{d\mu} \alpha_s = \beta(\alpha_s),
\]
and
\[
\beta_0 = 11 - \frac{2}{3} n,
\]
\[
\beta_1 = 51 - \frac{19}{3} n,
\]
\[
\beta_2 = 2857 - \frac{5033}{9} n + \frac{325}{27} n^2.
\]

Using Eq. \[2\], the numerical results of the running masses of quarks were obtained by Fusaoka and Koide \[13\], as summarized in Table 1, in which both low and high energy scales are calculated.

First, for the energy scales lower than the spontaneous symmetry breaking energy scale \( \Lambda_W \) of the electroweak gauge symmetry \( SU(2)_L \otimes U(1)_Y \), seven different energy scales are taken into account, i.e., \( \mu = 1 \) GeV, \( \mu = m_c, \mu = m_b, \mu = m_W, \mu = m_Z, \mu = m_t, \) and \( \mu = \Lambda_W \), where \( m_W \) and \( m_Z \) are the mass of W and Z bosons.

Second, for the energy scales extremely higher than \( \Lambda_W \), the evolution of the Yukawa coupling constants must be considered. In \[13\], two different models are presented. One is the standard model, and the other is the minimal SUSY model. In both of these two models, the Hamiltonian of the fermion fields can be written as
\[
\mathcal{H} = \sum_{ij} Y_{ij}^a \bar{\psi}_{Lai} \psi_{Raj} \phi_a^0 + H.c.,
\]
TABLE I: The masses of quarks at different energy scales. The upper seven rows in Table I are the cases of lower energy scales, the 8th and 9th rows are the cases of higher energy scales in the standard model, and the last two rows are the cases of higher energy scales in the minimal SUSY model.

| $\mu$          | $m_u$ (MeV) | $m_c$ (MeV) | $m_t$ (GeV) | $m_d$ (MeV) | $m_s$ (MeV) | $m_b$ (GeV) |
|---------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1 GeV         | 4.88        | 1506        | 475         | 9.81        | 195.4       | 7.18        |
| $m_c = 1.302$ GeV | 4.18        | 1302        | 399         | 8.40        | 167.2       | 6.12        |
| $m_b = 4.339$ GeV | 3.17        | 949         | 272         | 6.37        | 126.8       | 4.34        |
| $m_W = 80.33$ GeV | 2.35        | 684         | 183         | 4.73        | 94.2        | 3.03        |
| $m_Z = 91.19$ GeV | 2.33        | 677         | 181         | 4.69        | 93.4        | 3.00        |
| $m_t = 170.8$ GeV | 2.23        | 646         | 171         | 4.49        | 89.4        | 2.85        |
| $\Lambda_W = 174.1$ GeV | 2.23        | 645         | 171         | 4.48        | 89.3        | 2.85        |
| $10^9$ GeV    | 1.28        | 371         | 109         | 2.60        | 51.9        | 1.51        |
| $2 \times 10^{16}$ GeV | 0.94        | 272         | 84          | 1.94        | 38.7        | 1.07        |
| $10^9$ GeV    | 1.47        | 427         | 149         | 2.28        | 45.3        | 1.60        |
| $2 \times 10^{16}$ GeV | 1.04        | 302         | 129         | 1.33        | 26.5        | 1.00        |

where $a = u, d$, $i, j = 1, 2, 3$, $Y^a_{ij}$ are the Yukawa coupling constants, and $\phi^0_a$ are the vacuum expectation values of the neutral components of the Higgs bosons $\phi_a$. In the standard model, $\phi^0 = \phi^0_u = \phi^0_d$, and in the minimal SUSY model which has two Higgs bosons, $\phi^0_u = \phi^0 \sin \beta(\mu)$ and $\phi^0_d = \phi^0 \cos \beta(\mu)$. The mass matrices at the energy scale $\mu = \Lambda_W$ of quarks are given by

$$m(\mu)_a = \frac{1}{\sqrt{2}} Y(\mu)_a v_a,$$

where $v_a$ are the vacuum expectation values of $\phi^0_a$, and $v_a = \sqrt{2} \langle \phi^0_a \rangle$. In the standard model, $v_u = v_d = \sqrt{2} \Lambda_W$, and in the minimal SUSY model, $\sqrt{v_u^2 + v_d^2} = \sqrt{2} \Lambda_W$.

We can see from Table I that all the quark masses decrease with the increase of the energy scales, and the masses in the minimal SUSY model are a little bit larger than those in the standard model at the same energy scales.

Now, we can examine whether Koide’s relation holds well for quarks with the help of
TABLE II: $x_u$, $y_u$ and $k_u$ at different energy scales.

| $\mu$       | $m_u$ (MeV) | $m_c$ (MeV) | $m_t$ (GeV) | $x_u$  | $y_u$  | $k_u$ |
|-------------|-------------|-------------|-------------|--------|--------|-------|
| 1 GeV       | 4.88        | 1506        | 475         | 313.75 | 98958  | 1.341 |
| 1.302 GeV   | 4.18        | 1302        | 399         | 311.48 | 95455  | 1.338 |
| 4.339 GeV   | 3.17        | 949         | 272         | 299.37 | 85804  | 1.333 |
| 80.33 GeV   | 2.35        | 684         | 183         | 291.06 | 77872  | 1.328 |
| 91.19 GeV   | 2.33        | 677         | 181         | 290.56 | 77682  | 1.328 |
| 170.8 GeV   | 2.23        | 646         | 171         | 288.69 | 76682  | 1.327 |
| 174.1 GeV   | 2.23        | 645         | 171         | 288.24 | 76682  | 1.327 |
| $10^9$ GeV  | 1.28        | 371         | 109         | 289.84 | 85156  | 1.335 |
| $2 \times 10^{16}$ GeV | 0.94  | 272         | 84          | 289.36 | 89362  | 1.339 |
| $10^9$ GeV  | 1.47        | 427         | 149         | 290.48 | 101361 | 1.347 |
| $2 \times 10^{16}$ GeV | 1.04  | 302         | 129         | 299.01 | 124038 | 1.359 |

The data in Table I. To characterize the deviation of quark masses from the exact Koide’s relation, here we introduce, as was done in Ref. [11], two parameters $k_u$ and $k_d$ similarly as in Eq. (1),

$$
k_u \equiv \frac{m_u + m_c + m_t}{\frac{2}{3}(\sqrt{m_u} + \sqrt{m_c} + \sqrt{m_t})^2} = \frac{1 + x_u + y_u}{\frac{2}{3}(1 + \sqrt{x_u} + \sqrt{y_u})^2},$$

and

$$
k_d \equiv \frac{m_d + m_s + m_b}{\frac{2}{3}(\sqrt{m_d} + \sqrt{m_s} + \sqrt{m_b})^2} = \frac{1 + x_d + y_d}{\frac{2}{3}(1 + \sqrt{x_d} + \sqrt{y_d})^2},$$

where $x_u = m_c/m_u$, $y_u = m_t/m_u$, $x_d = m_s/m_d$, and $y_d = m_b/m_d$.

With the numerical results of quark masses in Table I, we can calculate all the $x_u$, $y_u$, $x_d$ and $y_d$ of both the $u$-type and $d$-type quarks at different energy scales, and then get $k_u$ and $k_d$ at different energy scales straightforwardly. These results are listed in Table II and Table III.

We can see from Table II that both $x_u$ and $y_u$ decrease with the increase of $\mu$ at low energy scales and increase slightly at high energy scales. However, we find that in spite of the change of $x_u$ and $y_u$, $k_u$ is almost invariant (approximately 1.33) at all energy scales, which means that $k_u$ is a constant independent of the running of quark masses. This is an
TABLE III: $x_d$, $y_d$ and $k_d$ at different energy scales.

| $\mu$          | $m_d$ (MeV) | $m_s$ (MeV) | $m_b$ (GeV) | $x_d$ | $y_d$ | $k_d$ |
|----------------|-------------|-------------|-------------|-------|-------|-------|
| 1 GeV          | 9.81        | 195.4       | 7.18        | 19.92 | 731.91| 1.068 |
| 1.302 GeV      | 8.40        | 167.2       | 6.12        | 19.90 | 728.57| 1.067 |
| 4.339 GeV      | 6.37        | 126.8       | 4.34        | 19.91 | 681.32| 1.057 |
| 80.33 GeV      | 4.73        | 94.2        | 3.03        | 19.92 | 640.75| 1.048 |
| 91.19 GeV      | 4.69        | 93.4        | 3.00        | 19.91 | 639.66| 1.048 |
| 170.8 GeV      | 4.49        | 89.4        | 2.85        | 19.91 | 634.74| 1.046 |
| 174.1 GeV      | 4.48        | 89.3        | 2.85        | 19.93 | 636.16| 1.047 |
| $10^9$ GeV     | 2.60        | 51.9        | 1.51        | 19.96 | 580.77| 1.032 |
| $2 \times 10^{16}$ GeV | 1.94       | 38.7        | 1.07        | 19.95 | 551.55| 1.025 |
| $10^9$ GeV     | 2.28        | 45.3        | 1.60        | 19.87 | 701.75| 1.062 |
| $2 \times 10^{16}$ GeV | 1.33       | 26.5        | 1.00        | 19.92 | 751.88| 1.072 |

interesting phenomenon, and indicates that Koide’s relation is a universal result.

Furthermore, we can find from Table II that $k_u$ is not 1 as $k_l$ of charged leptons. This means that Koide’s relation should be improved before being applied to the case of quarks. In [11] we calculated the range of $k_u$, and got $1.1 < k_u < 1.4$ (with the mean value of 1.25). However, in [11] we did not consider the renormalization effect of quark masses. With this effect taken into account, we find that $k_u$ changes a little, from 1.25 to 1.33.

From Table III, we can find that $k_d$ also keeps invariant with the change of energy scales, just as $k_u$, and its approximate value is 1.05.

Similarly, in Table IV, $k_l$ with the increase of $\mu$ at high energy scales are listed, both in the standard model and in the minimal SUSY model, as we know that the values in [7] should only be taken at low energy scale.

At the same time, we can see in Table IV that $k_l$ at high energy scales is still quite close to 1 as at the low energy scale, which means that Koide’s relation is suitable for charged leptons at all the energy scales, just as quarks. Also, we can find in Table IV that $k_l$ in the minimal SUSY model is a little larger than that in the standard model.
TABLE IV: $x_l$, $y_l$ and $k_l$ at different energy scales. The upper three rows are the cases in the standard model, and the lower three rows are the cases in the minimal SUSY model.

|   | $\mu$ (GeV) | $m_e$ (MeV) | $m_\mu$ (MeV) | $m_\tau$ (GeV) | $x_l$ | $y_l$ | $k_l$ |
|---|-------------|-------------|--------------|--------------|------|------|------|
|   | 91 GeV      | 0.48684727  | 102.75138   | 1.7467       | 211.05465 | 3587.78 | 1.001881 |
|   | $10^9$ GeV  | 0.51541746  | 108.78126   | 1.8492       | 211.05467 | 3587.77 | 1.001881 |
|   | $2 \times 10^{16}$ GeV | 0.49348567 | 104.15246   | 1.7706       | 211.05468 | 3587.95 | 1.001888 |
|   | 91 GeV      | 0.48684727  | 102.75138   | 1.7467       | 211.05465 | 3587.78 | 1.001881 |
|   | $10^9$ GeV  | 0.40850306  | 86.21727    | 1.4695       | 211.05661 | 3597.28 | 1.002277 |
|   | $2 \times 10^{16}$ GeV | 0.32502032 | 68.59813    | 1.1714       | 211.05797 | 3604.08 | 1.002560 |

In summary, we can conclude that Koide’s relation is a result independent (or insensitive) of energy scales of interaction, from low to extremely high energies.

III. KOIDE’S RELATION FOR NEUTRINOS

After testing Koide’s relation for quarks and charged leptons, and finding that $k_u$, $k_d$ and $k_l$ are independent of energy scales, a natural question is what about this relation for neutrinos. Here we again introduce the parameter $k_\nu$ as in Ref. [11], and discuss this problem. Moreover, we try to find the relations between the four parameters $k_u$, $k_d$, $k_\nu$ and $k_l$, and finally we can get the neutrino masses with some conjectures.

It is quite difficult to verify whether Koide’s relation is suitable for neutrinos because of the long-existing inaccuracy of the experimental data of neutrinos. Due to the untiring efforts by the numerous neutrino experiments, the oscillations and mixings of neutrinos have been strongly established now. The solar neutrino deficit is caused by the oscillation from $\nu_e$ to a mixture of $\nu_\mu$ and $\nu_\tau$ with a mixing angle approximately of $\theta_{\text{sol}} \approx 34^\circ$ in the KamLAND [15] and SNO [16] experiments. Also, the atmospheric neutrino anomaly is due to the $\nu_\mu$ to $\nu_\tau$ oscillation with almost the largest mixing angle of $\theta_{\text{atm}} \approx 45^\circ$ in the K2K [17] and SuperKamiokande [18] experiments. However, the non-observation of the disappearance of $\bar{\nu}_e$ in the CHOOZ [19] experiment showed that the mixing angle $\theta_{\text{chz}}$ is smaller than $5^\circ$ at the best fit point [20, 21].
These experiments also measured the mass-squared differences of the neutrino mass eigenstates. According to the global analysis of the experimental results, we have (the allowed ranges at 3σ) \[21\]

\[1.4 \times 10^{-3} \text{ eV}^2 < \Delta m_{\text{atm}}^2 = |m_3^2 - m_2^2| < 3.7 \times 10^{-3} \text{ eV}^2, \tag{5}\]

and

\[5.4 \times 10^{-5} \text{ eV}^2 < \Delta m_{\text{sol}}^2 = |m_2^2 - m_1^2| < 9.5 \times 10^{-5} \text{ eV}^2, \tag{6}\]

where \(m_1, m_2, m_3\) are the masses of the three mass eigenstates of neutrinos, and the best fit points are \(|m_3^2 - m_2^2| = 2.6 \times 10^{-3} \text{ eV}^2\), and \(|m_2^2 - m_1^2| = 6.9 \times 10^{-5} \text{ eV}^2 \[21\].

Due to Mikheyev-Smirnov-Wolfenstein \[22\] matter effect of solar neutrinos, we already know that \(m_2 > m_1\). Hence we have

\[m_1^2 = m_2^2 - \Delta m_{\text{sol}}^2, \tag{7}\]

and

\[m_3^2 = m_2^2 \pm \Delta m_{\text{atm}}^2. \tag{8}\]

So there are two mass schemes, (1) the normal mass scheme \(m_3 > m_2 > m_1\), and (2) the inverted mass scheme \(m_2 > m_1 > m_3\). We will discuss both of them in the following.

Now we apply Koide’s relation to neutrinos. First we take the normal mass scheme for example. If Koide’s relation holds well for neutrinos, we have

\[m_1 + m_2 + m_3 = \frac{2}{3} (\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2. \tag{9}\]

Substituting Eqs. (7) and (8) into Eq. (9), we get

\[\sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + m_2 + \sqrt{m_2^2 + \Delta m_{\text{atm}}^2} = \frac{2}{3} \left( \sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + \sqrt{m_2^2 + \Delta m_{\text{atm}}^2} \right)^2. \]

Solving this equation, we find that there is no real root for \(m_2\) with the restrictions in Eqs. (5) and (6). This means that no matter what value \(m_2\) is, Koide’s relation does not hold for neutrinos. So is the inverted mass scheme.

Thus we must introduce a parameter \(k_\nu \[11\] to character the deviation of neutrinos from Koide’s relation,

\[k_\nu \equiv \frac{m_1 + m_2 + m_3}{\frac{2}{3} (\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}. \tag{10}\]
Since $k_{\nu} \neq 1$, we must determine its range, and this can help us to find the relations between $k_u$, $k_d$, $k_{\nu}$ and $k_l$, and fix the neutrino masses. We now check the situations for the two mass schemes, respectively.

1. For the normal mass scheme, $m_3 > m_2 > m_1$, we have

$$k_{\nu} = \frac{\sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + m_2 + \sqrt{m_2^2 + \Delta m_{\text{atm}}^2}}{2 \left( \sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + \sqrt{m_2^2 + \Delta m_{\text{atm}}^2} \right)^2}.$$  \hspace{1cm} (11)

We can see that $k_{\nu}$ is the function of $m_2$ if $\Delta m_{\text{sol}}^2$ and $\Delta m_{\text{atm}}^2$ are fixed. Due to the inaccuracy of the experimental data, we take $\Delta m_{\text{sol}}^2$ and $\Delta m_{\text{atm}}^2$ as their best fit values here. The range of $k_{\nu}$ is shown in Fig. 1.

We can see from Fig. 1 that $0.50 < k_{\nu} < 0.85$, and $k_{\nu}$ decreases with the increase of $m_2$. So $k_{\nu} < 1$ for neutrinos.

2. For the inverted mass scheme, $m_2 > m_1 > m_3$, we have

$$k_{\nu} = \frac{\sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + m_2 + \sqrt{m_2^2 - \Delta m_{\text{atm}}^2}}{2 \left( \sqrt{m_2^2 - \Delta m_{\text{sol}}^2} + \sqrt{m_2^2 - \Delta m_{\text{atm}}^2} \right)^2}.$$  \hspace{1cm} (12)

The range of $k_{\nu}$ is shown in Fig. 2.

We can see from Fig. 2 that $0.50 < k_{\nu} < 0.65$. Again, $k_{\nu} < 1$ for neutrinos.

Altogether, $0.50 < k_{\nu} < 0.85$ for both of these two mass schemes. And $k_{\nu}$ of the normal mass scheme is larger than that of the inverted mass scheme.
Conclusively, the values of $k_u$, $k_d$, $k_\nu$ and $k_l$ can be summarized as follows,

$$
\begin{pmatrix}
  \nu_e \\
  e
\end{pmatrix}
\begin{pmatrix}
  \nu_\mu \\
  \mu
\end{pmatrix}
\begin{pmatrix}
  \nu_\tau \\
  \tau
\end{pmatrix}

\begin{pmatrix}
  u \\
  d
\end{pmatrix}
\begin{pmatrix}
  c \\
  s
\end{pmatrix}
\begin{pmatrix}
  t \\
  b
\end{pmatrix}

\begin{align*}
  k_\nu &< 1, \\
  k_l & = 1, \\
  k_u & > 1, \\
  k_d & \approx 1.
\end{align*}

(13)

We believe that the problem of the origin of the masses of leptons must be solved together with that of quarks. Since $k_l = 1$ and $k_d \approx 1$, we may conjecture that $k_l + k_d \approx 2$. At the same time, since $0.50 < k_\nu < 0.85$ and $k_u \approx 1.33$, we may analogize the conjecture of $k_l$ and $k_d$, and propose the hypothesis that

$$
  k_\nu + k_u \approx 2.
$$

(14)

This is from the speculation that there must be some relation between $k_u$, $k_d$, $k_\nu$ and $k_l$. Of course, this Ansatz is not the unique one of the relations between $k_u$, $k_d$, $k_\nu$ and $k_l$. For example, we may also assume that

$$
  k_l k_d \approx k_\nu k_u \approx 1,
$$

(15)

or

$$
  \frac{1}{k_l} + \frac{1}{k_d} \approx \frac{1}{k_\nu} + \frac{1}{k_u} \approx 2.
$$

(16)

Eq. (16) is based on the assumption that $\theta_l + \theta_d \approx \theta_\nu + \theta_u \approx \frac{\pi}{2}$ in Foot’s geometrical interpretation.

All these hypotheses can show the balance between $k_\nu$ and $k_u$ intuitively and transparently. This situation seems to be similar to the quark-lepton complementarity between
mixing angles of quarks and leptons, and we may call it a quark-lepton complementarity of masses.

From Table II, we can see that the value of $k_u$ is approximately 1.33. Thus from the hypothesis $k_\nu + k_u \approx 2$, we get that $k_\nu \approx 0.67$. This is consistent with the normal mass scheme and in conflict with the inverted mass scheme. This indicates that the three masses of neutrino mass eigenstates are heavier in order, which is the same as quarks and charged leptons.

Now we can estimate the absolute masses of neutrinos. Substituting $k_\nu = 0.67$, $\Delta m^2_{\text{atm}} = 2.6 \times 10^{-3}$ eV$^2$, and $\Delta m^2_{\text{sol}} = 6.9 \times 10^{-5}$ eV$^2$ into Eq. (11), we obtain the value of $m_2$,

$$0.67 = \frac{\sqrt{m_2^2 - 6.9 \times 10^{-5} \text{eV}^2} + m_2 + \sqrt{m_2^2 + 2.6 \times 10^{-3} \text{eV}^2}}{\frac{2}{3} \left(\sqrt{m_2^2 - 6.9 \times 10^{-5} \text{eV}^2} + \sqrt{m_2^2 + \frac{4}{3} \sqrt{m_2^2 + 2.6 \times 10^{-3} \text{eV}^2}}\right)}^2,$$ (17)

and we get $m_2 = 0.0089$ eV.

Straightforwardly, we get

$$m_1 = \sqrt{m_2^2 - \Delta m^2_{\text{sol}}} = 0.0032 \text{ eV},$$ (18)

and

$$m_3 = \sqrt{m_2^2 + \Delta m^2_{\text{atm}}} = 0.052 \text{ eV}.\quad (19)$$

Now we discuss the uncertainty of $m_1$, $m_2$ and $m_3$. In Fig. 1, we can see that the slope of the curve in very large where $0.65 < k_\nu < 0.8$, so the value of $m_2$ is not very sensitive to the error of $k_\nu$. $m_2$ will approximately be $8.5 \sim 8.9 \times 10^{-3}$ eV even if the value of $k_\nu$ charges from 0.65 to 0.8, so the value of $m_2$ is precise to a fairly good degree of accuracy.

Similarly, the value of $m_3$ will be about 0.052 eV to a good degree of accuracy too, because $m_3 = \sqrt{m_2^2 + \Delta m^2_{\text{atm}}}$, and $\Delta m^2_{\text{atm}} \gg m_2^2$. The only point desired to be mentioned here is the range of $m_1$. Because if $m_2^2$ is rather closed to $\Delta m^2_{\text{sol}}$, and due to the big uncertainty of $\Delta m^2_{\text{sol}}$, the value of $m_1$ may change largely with $k_\nu$. We can see this in the other two hypotheses in Eqs. (15) and (16).

1. In Eq. (15), where $k_l k_d \approx k_\nu k_u \approx 1$, we have $k_\nu \approx 0.75$, and

$$m_1 = 0.0012 \text{ eV},$$

$$m_2 = 0.0084 \text{ eV},$$

$$m_3 = 0.050 \text{ eV}.\quad (20)$$
2. In Eq. (16), where $\frac{1}{k_1} + \frac{1}{k_d} \approx \frac{1}{k_\nu} + \frac{1}{k_u} \approx 2$, we have $k_\nu \approx 0.80$, and

$$m_1 = 1.0 \times 10^{-5} \text{ eV},$$

$$m_2 = 0.0084 \text{ eV},$$

$$m_3 = 0.050 \text{ eV}. \quad (21)$$

From Eqs. (18), (20) and (21), we can see that the value $m_1$ is only a rough estimate of the first step till now, and its effective number and order of magnitude may change with the more and more precise experimental data in the future. However, the values of $m_2$ and $m_3$ are consistent in these three hypotheses.

Finally, we should point out that Koide also gave an interpretation of his relation as a mixing between octet and singlet components in a nonet scheme of the flavor $U(3)$, and got the neutrino masses as $m_1 = 0.0026 \text{ eV}, m_2 = 0.0075 \text{ eV}$ and $m_3 = 0.050 \text{ eV}$ [25]. We can see that his results are strongly consistent with ours, especially with the results from Eq. (14). And recently, he got the three masses as $m_1 = 0.0039 \text{ eV}, m_2 = 0.0088 \text{ eV}$ and $m_3 = 0.053 \text{ eV}$ in a seesaw mass matrix model of quarks and leptons with flavor-triplet Higgs scalars [26], which is even closer to our results from Eq. (14).

IV. SUMMARY

Finally, we give some discussions.

1. Since Koide’s relation is such a wonderful result for charged leptons at low energy scale, to explore its behavior at high energy is straightforward. We carefully test whether it is energy scale independent in Section II, and find that it is really independent or insensitive of energy scale in a huge energy range and is almost the same in both the standard model and the minimal SUSY model, which proves that Koide’s relation is a universal result in particle physics.

2. Seesaw mechanism may give the origin of neutrino masses, in which the right-handed very-heavy neutrinos is included into the Lagrangian of the standard model, i.e.,

$$
\begin{pmatrix}
\nu_L & \nu_L^C \\
0 & m_D \\
m_D^T & M_R
\end{pmatrix}
\begin{pmatrix}
\nu_R^C \\
\nu_R
\end{pmatrix},
\quad (22)
$$
where the scale of $m_D$ is characterized to be the energy scale $v$ of the electroweak spontaneous breaking, and $M_R$ is the right-handed very-heavy neutrino mass matrix.

From Eq. (22), we can get

$$m_\nu = -m_D M_R m_D^T,$$

where the neutrino mass eigenvalues are $m_1 = m_u^2/M_1$, $m_2 = m_c^2/M_2$ and $m_3 = m_u^2/M_3$. Thus, the smallness of neutrino masses is due to the large values of $M_1$, $M_2$ and $M_3$. (For example, $m_\nu \sim 0.1$ eV if $M_R$ is taken as $10^{14}$ GeV.)

However, seesaw mechanism can only present an illustrative interpretation of the origin of neutrino masses, without accurate prediction of the masses of neutrino mass eigenstates. To obtain those, we must extend our theory and make some speculation, i.e., we examine whether Koide’s relation of charged leptons also holds well for neutrinos, and we find that not all quarks and leptons obey Koide’s relation precisely. So we introduce the parameters $k_u$, $k_d$ and $k_\nu$ to describe the deviations of quarks and neutrinos from the exact Koide’s relation. With this improved relation and the conjecture of a quark-lepton complementarity of masses such as $k_l + k_\nu \approx k_\nu + k_u \approx 2$, $k_l k_d \approx k_\nu k_u \approx 1$ or $\frac{1}{k_l} + \frac{1}{k_d} \approx \frac{1}{k_\nu} + \frac{1}{k_u} \approx 2$, we can determine the absolute masses of the neutrino mass eigenstates. Due to the inaccuracy of the experimental data of neutrinos nowadays, these results (especially the value of $m_1$) should be only taken as primary estimates. It is also possible that seesaw mechanism is responsible for the deviation of the neutrino masses from the exact Koide’s relation.

3. There remain some open questions to be answered. Such as

(1). Which hypothesis between $k_u$, $k_d$, $k_\nu$ and $k_l$ is really the relation between them? For example, if $\frac{1}{k_l} + \frac{1}{k_d} \approx \frac{1}{k_\nu} + \frac{1}{k_u} \approx 2$, we have $\theta_l + \theta_d \approx \theta_\nu + \theta_u \approx \frac{\pi}{2}$ in Foot’s geometrical interpretation, is there really some deeper reason behind it, just like the quark-lepton complementarity of their mixing angles [23]?

(2). We can see from Table II that $k_u$ is approximately 1.33, so may it be $\frac{4}{3}$ exactly? If so, we have $k_\nu = \frac{2}{3}$ in Eq. (14), $k_\nu = \frac{2}{4}$ in Eq. (15), and $k_\nu = \frac{4}{5}$ in Eq. (16). All these $k_u$ and $k_\nu$ are of special values, and is there some unknown principle leading this? Moreover, is $k_d = 1$ exactly as $k_l$, or deviates from 1 slightly? If so, why?

In conclusion, we believe that there must be some deeper principle behind the elegant Koide’s relation for charged leptons, and thus it is meaningful to check whether this relation is also applicable to quarks and neutrinos, at least at some degree. We show in this paper that the Koide’s relation with its deviation characterized by the parameters $k_u$ and $k_d$ is
also applicable to quarks without sensitivity to energy scale. By using the improved Koide’s relation with an Ansatz of quark-lepton complementarity of masses, we can also determine neutrino masses. If the prediction of neutrino masses is tested to be consistent with the precise experiments in the future, it would be a big success of Koide’s relation, which may shed light on our way to a grand unification theory of quarks and leptons.

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