Minimally Intersecting Set Partitions of Type $B$

William Y.C. Chen and David G.L. Wang
Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
chen@nankai.edu.cn, wgl@cfc.nankai.edu.cn

Abstract

Motivated by Pittel’s study of minimally intersecting set partitions, we investigate minimally intersecting set partitions of type $B$. We find a formula for the number of minimally intersecting $r$-tuples of $B_n$-partitions, as well as a formula for the number of minimally intersecting $r$-tuples of $B_n$-partitions without zero-block. As a consequence, it follows the formula of Benoumhan for the Dowling number in analogy to Dobiński’s formula.

Keywords: minimally intersecting $B_n$-partitions, Dobiński’s formula, the Dowling number

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1 Introduction

This paper is primarily concerned with the meet structure of the lattice of type $B_n$-partitions of the set $[\pm n] = \{\pm1, \pm2, \ldots, \pm n\}$, as well as of the meet-semilattice of type $B_n$-partitions without zero-block. The lattice structure of type $B_n$ set partitions has been studied by Reiner [8]. It can be regarded as a representation of the intersection lattice of the type $B$ Coxeter arrangements, see Björner and Wachs [3], Björner and Brenti [2] and Humphreys [6].

We establish a formula for the number of $B_n$-partitions $\pi'$ which minimally intersect a given $B_n$-partition $\pi$. Using the same technique, we derive a formula for the number of $B_n$-partitions $\pi'$ without zero-block which minimally intersect a given $B_n$-partition $\pi$ without zero-block. The ordinary case has been studied by Pittel [7]. In particular, if we take $\pi$ to be the minimal $B_n$-partition, our formula reduces to a formula of Benoumhan [1] for the number of $B_n$-partitions (called the Dowling number, see Dowling [5]), which is analogous to Dobiński’s formula for the number of partitions of a finite set.

In a more general setting, we derive two formulas for the number of minimally intersecting $r$-tuples of $B_n$-partitions and the number of minimally intersecting $r$-tuples of $B_n$-partitions without zero-block. Recall that Canfield [4] has found a relation
between the exponential generating function of the number of minimally intersecting \( r \)-tuples of partitions and the powers of the Bell numbers. We give a type \( B \) analogue of this relation.

Let us give an overview of relevant notation and terminology. A partition of a set \( S \) is a collection \( \{B_1, B_2, \ldots, B_k\} \) of subsets of \( S \) such that \( B_1 \cup B_2 \cup \cdots \cup B_k = S \) and \( B_i \cap B_j = \emptyset \) for any \( i \neq j \). A \emph{set partition of type} \( B_n \) is a partition \( \pi \) of the set \([\pm n]\) into blocks satisfying the following conditions:

(i) For any block \( B \) of \( \pi \), its opposite \( -B \) obtained by negating all elements of \( B \) is also a block of \( \pi \);

(ii) There is at most one \emph{zero-block}, which is defined to be a block \( B \) such that \( B = -B \).

We call \( \pm B \) a \emph{block pair} of \( \pi \) if \( B \) is a non-zero-block of \( \pi \). For example,

\[
\pi_1 = \{\{\pm 1, \pm 2, \pm 5, \pm 8, \pm 12\}, \{\pm 3, 11\}, \{\pm 4, -7, 9, 10\}, \{\pm 6\}\}
\]

is a \( B_{12} \)-partition consisting of 3 block pairs and the zero-block \( \{\pm 1, \pm 2, \pm 5, \pm 8, \pm 12\} \).

The total number of partitions of the set \([n] = \{1, 2, \ldots, n\}\) is called the Bell number and is denoted by \( B_n \), see Rota [11]. The type \( B \) analogue of the Bell numbers are the Dowling numbers \( |\Pi_n^B| \), where \( \Pi_n^B \) denotes the set of \( B_n \)-partitions. The sequence \( \{|\Pi_n^B|\}_{n \geq 0} \) is listed as A007405 in [12]:

\[
1, 2, 6, 24, 116, 648, 4088, 28640, 219920, 1832224, \ldots
\]

Let \( \pi \) and \( \pi' \) be two partitions of the set \([n]\). We say that \( \pi \) \emph{refines} \( \pi' \) if every block of \( \pi \) is contained in some block of \( \pi' \). The refinement relation is a partial ordering of the set \( \Pi_n \) of all partitions of \([n]\). Define the \emph{meet}, denoted \( \pi \wedge \pi' \), to be the largest partition which refines both \( \pi \) and \( \pi' \). Define their \emph{join}, denoted \( \pi \vee \pi' \), to be the smallest partition which is refined by both \( \pi \) and \( \pi' \). The poset \( \Pi_n \) is a lattice with the minimum element \( \hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\} \). We say that the partitions \( \pi_1, \pi_2, \ldots, \pi_r \) intersect minimally if \( \pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r = \hat{0} \).

Pittel [7] has found a formula for the number of partitions minimally intersecting a given partition. He also computed the number of minimally intersecting \( r \)-tuples of partitions.

**Theorem 1.1.** Let \( \pi \) be a partition of \([n]\), and let \( i_1, \ldots, i_k \) be the sizes of the blocks of \( \pi \) listed in any order. Then the number of partitions intersecting \( \pi \) minimally equals

\[
N(\pi) = i! \left[ x^i \right] \exp \left( \prod_{\alpha \in [k]} (1 + x_{\alpha}) - 1 \right),
\]
where $i! = \prod_{\alpha \in [k]} i_{\alpha}!$ and $[x^i]$ stands for the coefficient of $x^i = \prod_{\alpha \in [k]} x_{\alpha}^{i_{\alpha}}$ of a power series in $x_1, x_2, \ldots, x_k$. Let $r \geq 2$. The number $N_{n,r}$ of minimally intersecting $r$-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of partitions is given by

$$N_{n,r} = \frac{1}{e^r} \sum_{k_1, \ldots, k_r \geq 0} \frac{(k_1k_2 \cdots k_r)_n}{k_1!k_2! \cdots k_r!},$$

where the notation $(x)_n = x(x-1) \cdots (x-n+1)$ denotes the falling factorial.

By taking $\pi = \hat{0}$, the above formula reduces to Dobiński’s formula

$$B_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}, \quad (1.1)$$

see Rota [11]. Wilf has obtained the following alternative formula

$$N_{n,r} = \sum_{j=1}^{n} B^r_j s(n, j), \quad (1.2)$$

where $s(n, j)$ is the Stirling number of the first kind. Denote the generating function of $N_{n,r}$ by

$$M_r(x) = \sum_{n \geq 0} N_{n,r} \frac{x^n}{n!}.$$

Canfield [4] has established the following connection between $M_r(x)$ and the Bell numbers:

$$M_r \left( e^x - 1 \right) = \sum_{n \geq 0} B^r_n \frac{x^n}{n!}. \quad (1.3)$$

We shall give type $B$ analogues of (1.2) and (1.3) based on type $B$ partitions without zero block.

This paper is organized as follows. In Section 2, we give an expression for the number of $B_n$-partitions that minimally intersect a $B_n$-partition $\pi$ of a given type, which contains Benoumihanı’s formula for the Dowling number as a special case. Moreover, we obtain a formula for the number of minimally intersecting $r$-tuples of $B_n$-partitions. In Section 3, we consider the enumeration of minimally intersecting $r$-tuples of $B_n$-partitions without zero-block, and give two formulas in analogy to (1.2), and (1.3).

## 2 Minimally intersecting $B_n$-partitions

The main objective of this section is to derive a formula for the number of minimally intersecting $r$-tuples of $B_n$-partitions. If $\pi \in \Pi^B_n$ has a zero-block $Z = \{\pm i_1, \pm i_2, \ldots, \pm i_k\}$, we say that $Z$ is of half-size $k$. The partition $\hat{0}^B = \{\{1\}, \{-1\}, \{2\}, \{-2\}, \ldots, \{n\}, \{-n\}$
is called the minimal partition, and \( \hat{1}^B = \{ \{ \pm 1, \pm 2, \ldots, \pm n \} \} \) is called the maximal partition. We say that \( \pi_1, \pi_2, \ldots, \pi_r \) are minimally intersecting if \( \pi_1 \land \pi_2 \land \cdots \land \pi_r = \emptyset^B \).

Let \( j = (j_1, j_2, \ldots, j_k) \) be a composition of \( n \). Let \( \pi \) be a \( B_n \)-partition consisting of \( k \) block pairs and a zero-block of half-size \( i_0 \). For the purpose of enumeration, we often assume that the block pairs of \( \pi \) are ordered subject to certain convention. We say that \( \pi \) is of type \( (i_0; j) \) if the block pairs of \( \pi \) are ordered such that the \( i \)-th block pair is of length \( j_i \).

We first consider the problem of counting the number of \( B_n \)-partitions with \( l \) block pairs which minimally intersects a given \( B_n \)-partition. As a special case, we are led to Benoumhani’s formula for the Dowling number

\[
|\Pi_n^B| = \frac{1}{\sqrt{e}} \sum_{k \geq 0} \frac{(2k + 1)^n}{(2k)!!},
\]

in analogy to Dobiński’s formula (1.1). Next, we find a formula for the number of ordered pairs of minimally intersecting \( B_n \)-partitions. In general, we give a formula for the number of minimally intersecting \( r \)-tuples of \( B_n \)-partitions.

**Theorem 2.1.** Let \( \pi \) be a \( B_n \)-partition consisting of a zero-block of half-size \( i_0 \) (allowing \( i_0 = 0 \)) and \( k \) block pairs of sizes \( i_1, i_2, \ldots, i_k \) (\( k \geq 1 \)) listed in any order. For any \( l \geq 1 \), the number of \( B_n \)-partitions \( \pi' \) containing exactly \( l \) block pairs that intersect \( \pi \) minimally equals

\[
N^B(\pi; l) = \frac{i!}{(2l - 2i_0)!!} \sum_{i'} \left[ x^{i'} \right] \left( \prod_{a \in [k]} (1 + x_a)^{i_a} - 1 \right)^{l-i_0} \prod_{a \in [k]} (1 + x_a)^{2i_a},
\]

where \( i' \) runs over all vectors \( (i'_1, i'_2, \ldots, i'_k) \) such that \( i'_a \in \{ i_a, i_a - 1 \} \) for any \( a \in [k] \), and \( x^{i'} = \prod_{a=1}^{k} x_a^{i'_a} \).

For example, \( \Pi_2^B \) contains 6 partitions:

\( \emptyset^B, \hat{1}^B, \{ \{ \pm 1 \}, \{ \pm 2 \} \}, \{ \{ \pm 2 \}, \{ \pm 1 \} \}, \{ \pm \{ 1, 2 \} \}, \{ \pm \{ 1, -2 \} \} \).

Let \( \pi = \{ \{ \pm 1 \}, \{ \pm 2 \} \} \). We have \( i_0 = 1, k = 1 \), and \( i_1 = 1 \). For \( l = 1 \), by (2.2), \( N^B(\pi; 1) = \sum_{i=0}^{1} [x^i] (1 + x)^2 = 3 \). The three \( B_2 \)-partitions which contain exactly 1 block pair and intersect \( \pi \) minimally are \( \{ \pm \{ 2 \}, \{ \pm 1 \} \}, \{ \pm \{ 1, 2 \} \}, \{ \pm \{ 1, -2 \} \} \).

**Proof of Theorem 2.1.** Let \( Z_1 \) be the zero-block of \( \pi \), and \( \pm B_1, \pm B_2, \ldots, \pm B_k \) be the block pairs of \( \pi \). Let \( Z_2 \) be the zero-block of \( \pi' \), and \( \pm B'_1, \pm B'_2, \ldots, \pm B'_l \) be the block pairs of \( \pi' \). To ensure that \( \pi \) and \( \pi' \) are minimally intersected, it is necessary to characterize the intersecting relations for all pairs \( (B, B') \) where \( B \) is a block of \( \pi \) and \( B' \) is a block of \( \pi' \).

First, we observe that the intersection \( B \cap B' \) contains at most one element subject to the minimally intersecting property. In particular, \( Z_1 \cap Z_2 = \emptyset \). If \( B = Z_1 \) and
$B' \neq Z_2$, then the two intersections $Z_1 \cap B'$ and $Z_1 \cap (-B')$ are a pair of opposite subsets. This observation allows us to disregard $Z_1 \cap (-B')$ in our consideration. Since the cardinality of $B \cap B'$ is either zero or one, we can represent $B \cap B'$ by

$$F(k; l) \prod_{\beta \in [l]} (1 + z_1 w_\beta) \prod_{\alpha \in [k]} (1 + x_\alpha z_2),$$

where

$$F(k; l) = \prod_{\alpha \in [k], \beta \in [l]} (1 + x_\alpha y_\beta)(1 + x_\alpha \bar{y}_\beta). \quad (2.3)$$

Here we use $x_i$ ($w_i$, resp.) to represent one of the two blocks in the $i$-th block pair of $\pi$ ($\pi'$, resp.), and we use $y_i$ and $\bar{y}_i$ to represent the two blocks in the $i$-th block pair of $\pi'$.

The above argument allows us to generate all $B_n$-partitions that minimally meet with $\pi$. Let us consider the generating function of such $B_n$-partitions. Set

$$x = (x_1, x_2, \ldots, x_k), \quad i = (i_1, i_2, \ldots, i_l), \quad x^i = \prod_{\alpha \in [k]} x^{i_\alpha};$$

$$w = (w_1, w_2, \ldots, w_l), \quad a = (a_1, a_2, \ldots, a_l), \quad w^a = \prod_{\beta \in [l]} w^{a_\beta};$$

$$y = (y_1, y_2, \ldots, y_l), \quad b = (b_1, b_2, \ldots, b_l), \quad y^b = \prod_{\beta \in [l]} y^{b_\beta};$$

$$\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_l), \quad c = (c_1, c_2, \ldots, c_l), \quad \bar{y}^c = \prod_{\beta \in [l]} \bar{y}^{c_\beta}.$$

Let $j_0$ be a nonnegative integer and $j = (j_1, j_2, \ldots, j_l)$ a composition of $n - j_0$. Denote by $N^B(\pi; j_0, j)$ the number of $B_n$-partitions $\pi'$ of type $(j_0; j)$ such that $\pi'$ meets $\pi$ minimally. In the above notation, we have

$$N^B(\pi; j_0, j) = c \cdot \sum_{a+b+c=j} \left[ x^i z_1^j z_2^j w^a y^b \bar{y}^c \right] F(k; l) \prod_{\beta \in [l]} (1 + z_1 w_\beta) \prod_{\alpha \in [k]} (1 + x_\alpha z_2), \quad (2.4)$$

where $c = i! \cdot (2i_0)!!/(2l)!!$. Denote by $\binom{S}{m}$ the collection of all $m$-subsets of $S$. Since

$$[z_1^j] \prod_{\beta \in [l]} (1 + z_1 w_\beta) = \sum_{Y \in \binom{[l]}{j_0}} \prod_{\beta \in Y} w_\beta, \quad (2.5)$$

$$[z_2^j] \prod_{\alpha \in [k]} (1 + x_\alpha z_2) = \sum_{X \in \binom{[k]}{j_0}} \prod_{\alpha \in X} x_\alpha, \quad (2.6)$$

substituting (2.5) and (2.6) into (2.4), we obtain that

$$N^B(\pi; j_0, j) = c \cdot \sum_{a+b+c=j} \left[ x^i w^a y^b \bar{y}^c \right] \left( \sum_{Y \in \binom{[l]}{j_0}} \prod_{\beta \in Y} w^{a_\beta} \right) \left( \sum_{X \in \binom{[k]}{j_0}} \prod_{\alpha \in X} x^{i_\alpha} \right) F(k; l),$$

where $b = (b_1, b_2, \ldots, b_l), \quad y^{b_\beta} = \prod_{\beta \in [l]} y^{b_\beta}, \quad x_\alpha = (x_1, x_2, \ldots, x_k), \quad x^{i_\alpha} = \prod_{\alpha \in [k]} x^{i_\alpha}, \quad \bar{y}_{\beta} = \prod_{\beta \in [l]} \bar{y}_{\beta}.$
where $\chi$ is the characteristic function defined by $\chi(P) = 1$ if $P$ is true, and $\chi(P) = 0$ otherwise. Therefore

$$
N^B(\pi; l) = \sum_{j_0, j_1, \ldots, j_l \geq 1} N^B(\pi; j_0, j_1, \ldots, j_l) = c \cdot \sum_{j_0, X} \left[ \prod_{\alpha} x^{i_\alpha}_\alpha - \chi(\alpha \in X) \right] \sum_{j_0, j_1, \ldots, j_l \geq 1} f(j), \quad (2.7)
$$

where

$$f(j) = \sum_{Y, b} \left[ y^b \prod_{\beta} y_{\beta}^{j_\beta - b_\beta - \chi(\beta \in Y)} \right] F(k; l).
$$

In view of the expression (2.3), the total degree of $x_\alpha$’s agrees with the sum of the total degrees of $y_\beta$’s and $\bar{y}_\beta$’s in $F(k; l)$. In other words,

$$
\sum_{\alpha \in [k]} i_\alpha - \chi(\alpha \in X) = \sum_{\beta \in [l]} b_\beta + (j_\beta - b_\beta - \chi(\beta \in Y)),
$$

namely, $j_0 + j_1 + \cdots + j_l = n$. So we may drop this condition in the inner summation of (2.7). For any $A \subseteq [l]$, let

$$S(A) = \sum_{j_0, j_1, \ldots, j_l \geq 1} f(j) = \sum_{Y} \sum_{b, j_\gamma \geq 0} \prod_{\gamma \in A} y_{\gamma}^{b_\gamma - j_\gamma} \bar{y}_{\gamma}^{j_\gamma - \chi(\gamma \in Y)} F(k; A),$$

where

$$F(k; A) = \prod_{\alpha \in [k], \gamma \in A} (1 + x_\alpha y_\gamma)(1 + x_\alpha \bar{y}_\gamma).$$

Since $j_\gamma$ and $b_\gamma$ run over all nonnegative integers, the exponent $j_\gamma - b_\gamma - \chi(\gamma \in Y)$ can considered as a summation index. It follows that

$$S(A) = \sum_{Y \in \binom{\alpha}{i_0}} \sum_{b, c_\gamma \geq 0, \gamma \in A} \left[ \prod_{\gamma \in A} y_{\gamma}^{b_\gamma} \bar{y}_{\gamma}^{c_\gamma} \right] F(k; A) = \binom{|A|}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2|A|}.
$$

By the principle of inclusion-exclusion, we have

$$\sum_{j_1, \ldots, j_l \geq 1} f(j) = \sum_{A \subseteq [l]} (-1)^{|A|-1} S(A) = \sum_{m} \binom{l}{m} (-1)^{l-m} \binom{m}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2m}$$

$$= \binom{l}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_\alpha} \left( \prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^{l-i_0}.
$$

Now, employing (2.7) we find that $N^B(\pi; l)$ equals

$$\frac{i!}{(2l - 2i_0)!} \sum_{X \subseteq [k]} \left[ \prod_{\alpha \in [k]} x^{i_\alpha}_\alpha - \chi(\alpha \in X) \right] \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_\alpha} \left( \prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^{l-i_0}, \quad (2.8)$$

which can be rewritten in the form of (2.2). This completes the proof.
The formula (2.8) will also be used in the proof of Corollary 3.1. Summing (2.2) over \( l \geq i_0 \), we obtain the following formula.

**Corollary 2.2.** The number \( N^B(\pi) \) of \( B_n \)-partitions that minimally intersect \( \pi \) is

\[
N^B(\pi) = \frac{i!}{\sqrt{e}} \sum_{i'} [x^{i'}] F(x),
\]

where

\[
F(x) = \left( \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0} \right) \exp \left( \frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right).
\]

Setting \( \pi = \hat{0}^B \), (2.9) reduces to (2.1), since

\[
N^B(\hat{0}^B) = \frac{1}{\sqrt{e}} \sum_{i'_\alpha \in \{0,1\}} [x_1^{i'_1} \cdots x_n^{i'_n}] \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha=1}^n (1 + x_\alpha)^{2j}.
\]

In fact, the number \( N^B(\pi) \) can be expressed in terms of an infinite sum.

**Corollary 2.3.**

\[
N^B(\pi) = \frac{i!}{\sqrt{e}} \sum_{j \geq 0} \frac{(2i_0 + 2j + 1)!}{(2j)!!} \prod_{\alpha \in [k]} \frac{1}{(2i_0 + 2j + 1 - i_\alpha)!}.
\]

**Proof.** From (2.10) it follows that

\[
F(x) = \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2(i_0 + j)}.
\]

Hence

\[
N^B(\pi) = \frac{i!}{\sqrt{e}} \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \left( \binom{2(i_0 + j)}{i_\alpha} + \binom{2(i_0 + j)}{i_\alpha - 1} \right)
\]

\[
= \frac{i!}{\sqrt{e}} \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \binom{2(i_0 + j) + 1}{i_\alpha},
\]

which gives (2.11). This completes the proof.

**Corollary 2.4.** Let \( N^B_{n,2}(i_0; k) \) denote the number of ordered pairs \( (\pi, \pi') \) of minimally intersecting \( B_n \)-partitions such that \( \pi \) consists of exactly \( k \) block pairs and a zero-block of half-size \( i_0 \) (allowing \( i_0 = 0 \)). Then

\[
N^B_{n,2}(i_0; k) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}} [x^{n-i_0}] \sum_{j \geq 0} \frac{1}{(2j)!!} (1 + x)^{2i_0 + 2j + 1 - 1}.
\]
Proof. By a simple combinatorial argument we see that the number of \( B_n\)-partitions of type \((i_0; i_1, \ldots, i_k)\) equals
\[
c = \binom{n}{i_0, i_1, \ldots, i_k} \frac{2^{n-i_0-k}}{k!} = \frac{(2n)!!}{(2i_0)!!(2k)!!} \cdot \frac{1}{i!}.
\]
Thus by (2.9), we have
\[
N_{n,2}^{B}(k) = \sum_{i_0 + i_1 + \cdots + i_k = n, \ i_1, \ldots, i_k \geq 1} c \cdot N^B(\pi) = \frac{(2n)!!}{(2i_0)!!(2k)!!} \sqrt{e} \sum_{i_0 + i_1 + \cdots + i_k = n, \ i_1, \ldots, i_k \geq 1} \sum_{i'} \left[ x^i \right] F(x). \quad (2.13)
\]
For any \( A \subseteq [k] \), consider
\[
S(A) = \sum_{i_0 + i_1 + \cdots + i_k = n, \ i_1, \ldots, i_k \geq 0} \sum_{i' \in A, i'' \in A^c} \left[ x^i \right] F(x) = \sum_{i_0 + \sum_{\alpha \in A} i_0 = n, \ i_0 \geq 0, \ i'' \in A} \sum_{i' \in A} \left[ x^{i'} \right] F(x|_A),
\]
where \( x|_A \) (resp. \( i'|_A \)) denotes the vector obtained by removing all \( x_\alpha \) (resp. \( i'_\alpha \)) such that \( \alpha \notin A \) from the vector \( x \) (resp. \( i' \)). Let \( t \) be the number of \( \alpha \)'s such that \( i'_\alpha = i_\alpha - 1 \) in the inner summation. Noting that
\[
F(x|_A) = \left( \prod_{\alpha \in A} (1 + x_\alpha)^{2i_0} \right) \exp \left( \frac{1}{2} \prod_{\alpha \in A} (1 + x_\alpha)^2 \right),
\]
we can transform \( S(A) \) to
\[
S(A) = \left( \sum_t \binom{|A|}{t} \left[ x^{n-i_0-t} \right] \right) (1 + x)^{2i_0|A|} \exp \left( \frac{1}{2} (1 + x)^{|A|} \right)
\]
\[
= \left[ x^{n-i_0} \right] (1 + x)^{(2i_0+1)|A|} \exp \left( \frac{1}{2} (1 + x)^{|A|} \right).
\]
In view of the principle of inclusion-exclusion, we deduce from (2.13) that
\[
N_{n,2}^{B}(k) = \frac{(2n)!!}{(2i_0)!!(2k)!!} \sqrt{e} \sum_{A \subseteq [k]} (-1)^{|A|} S(A),
\]
which gives (2.12). This completes the proof. \( \blacksquare \)

Summing over \( 0 \leq k \leq n - i_0 \) and \( 0 \leq i_0 \leq n \), we obtain the number of ordered pairs of minimally intersecting \( B_n\)-partitions.

Corollary 2.5. The number \( N_{n,2}^{B} \) of ordered pairs \((\pi, \pi')\) of minimally intersecting \( B_n\)-partitions is given by
\[
N_{n,2}^{B} = \frac{2^n}{e} \sum_{k,t \geq 0} \frac{(2kl + k + l)_n}{(2k)!!(2l)!!}.
\]
For example, $N^B_{1,2} = 3$, $N^B_{2,2} = 23$, $N^B_{3,2} = 329$, $N^B_{4,2} = 6737$. In general, we have the following theorem, which is the main result of this paper.

**Theorem 2.6.** Let $r \geq 2$. The number of minimally intersecting $r$-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of $B_n$-partitions equals

$$N^B_{n,r} = \frac{2^n}{e^{r/2}} \sum_{l_1,l_2,\ldots,l_r} \left( \frac{(f_r)_n}{(2l_1)!!(2l_2)!! \cdots (2l_r)!!} \right),$$

where

$$f_r = \frac{1}{2} \left( \prod_{t \in [r]} (2l_t + 1) - 1 \right).$$

**Proof.** For any $t \in [r]$, let $i_t$ be a nonnegative integer and $j_t = (j_{t,1}, j_{t,2}, \ldots, j_{t,k_t})$ be a composition of $n$. Let $\pi_t$ be a $B_n$-partition of type $(i_t; j_t)$. The condition that $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting leads us to consider the intersecting relations for all $r$-tuples $(B_1, B_2, \ldots, B_r)$ where $B_t$ is a block of $\pi_t$.

First, we observe that the intersection

$$B_1 \cap B_2 \cap \cdots \cap B_r$$

contains at most one element because of the minimally intersecting requirement. In particular, (2.15) is empty when $B_1, B_2, \ldots, B_r$ are all zero-blocks. We now consider the case that not all of $B_1, B_2, \ldots, B_r$ are zero-blocks. In this case, there exists a number $t \in [r]$ such that $B_1, \ldots, B_{t-1}$ are zero-blocks but $B_t$ is a non-zero-block. This number $t$ will play a key role in determining the intersection (2.15).

In fact, the partial intersection $B_1 \cap B_2 \cap \cdots \cap B_{t-1}$ is of the form $\{\pm i_1, \ldots, \pm i_j\}$. Thus for any non-zero-block $B$ of $\pi_t$, the two intersections

$$B_1 \cap \cdots \cap B_{t-1} \cap B \quad \text{and} \quad B_1 \cap \cdots \cap B_{t-1} \cap (-B)$$

form a pair of opposite subsets. This observation allows us to consider $B$ as a representative of the block pair $\pm B$. Since the cardinality of the intersection (2.15) is either zero or one, we can represent (2.15) by

$$f = 1 + z_1 \cdots z_{t-1} x_{t,\alpha_1} Y_{t+1} \cdots Y_r,$$

where

$$Y_p \in \{z_p, y_{p,1}, \bar{y}_{p,1}, \ldots, y_{p,k_p}, \bar{y}_{p,k_p}\}$$

for $p \geq t + 1$. Here we use $z_t$ to represent the zero-block of $\pi_t$, $x_{t,i}$ to represent one of the two blocks in the $i$-th block pair of $\pi_t$, $y_{p,i}$ and $\bar{y}_{p,i}$ to represent the two blocks in
Denote by $N^B(\pi_1; i_2, j_2; \ldots; i_r, j_r)$ the number of $(r - 1)$-tuples $(\pi_2, \ldots, \pi_r)$ of $B_n$-partitions such that $\pi_s$ $(2 \leq s \leq r)$ is of type $(i_s, j_s)$ and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. In the notation of $f$ in (2.16), we get

$$N^B(\pi_1; i_2, j_2; \ldots; i_r, j_r) = c \left[ x_t^{i_1} y_t^{j_1} \right] \sum_{2 \leq s \leq r} \left[ x_s^{a_s} y_s^{b_s} y_s^{c_s} z_s^{i_s} \right] F_r,$$

where

$$c = j_1! \cdot (2i_1)!! \prod_{2 \leq s \leq r} (2k_s)!!^{-1},$$

$$F_r = \prod_{t \in [r]} \prod_{a \in [k_t]} \prod_{y_p \in \{z_{p-1}, y_{p-1}, \ldots, y_p, k_p, z_p, g_p, k_p\}} \prod_{l+1 \leq p \leq r} f.$$ 

Now, let $N^B(\pi_1, k_2, \ldots, k_r)$ be the number of $(r - 1)$-tuples $(\pi_2, \ldots, \pi_r)$ of $B_n$-partitions such that $\pi_s$ contains exactly $k_s$ block pairs and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. Then

$$N^B(\pi_1, k_2, \ldots, k_r) = \sum_{i_s \geq 0, j_{s,1}, \ldots, j_{s,k_s} \geq 1} N^B(\pi_1; i_2, j_2; \ldots; i_r, j_r) \quad (2.17)$$

We claim that the condition $j_{s,1} + \cdots + j_{s,k_s} + i_s = n$ can be dropped in the above summation. In fact, the factor $f$ in (2.16) contributes to $x_1$ or $z_1$ at most once with respect to the degree, and the contribution of $f$ to $x_1$ or $z_1$ equals the contribution of $f$ to $x_s$, $y_s$, $\bar{y}_s$, or $z_s$, for any $2 \leq s \leq r$. Therefore the sum of the degrees of $x_s$, $y_s$, $\bar{y}_s$, and $z_s$, equals the sum of the degrees of $x_1$ and $z_1$, that is, for any $2 \leq s \leq r$,

$$i_s + j_{s,1} + \cdots + j_{s,k_s} = i_1 + j_{1,1} + \cdots + j_{1,k_1} = n \quad (2.18)$$

Hence we can ignore the conditions (2.18) in (2.17). This implies that

$$N^B(\pi_1, k_2, \ldots, k_r) = c \left[ x_t^{i_1} z_t^{j_1} \right] \sum_{i_s \geq 0, a_s + b_s + c_s \geq 1} \left[ x_s^{a_s} y_s^{b_s} y_s^{c_s} z_s^{i_s} \right] F_r.$$
where \(a_s + b_s + c_s \geq 1\) indicates that \(a_s,h_s + b_s,h_s + c_s,h_s \geq 1\) for any \(1 \leq h_s \leq k_s\). We will compute \(\sum \left[ x_s^{a_s} y_s^{b_s} z_s^{c_s} \right] F_s\) for \(s = 2, 3, \ldots, r\) by the following procedure. First, for \(s = 2\), we have

\[
\sum_{i_2 \geq 0, a_2 + b_2 + c_2 \geq 1} x_2^{a_2} y_2^{b_2} z_2^{c_2} F_r = \sum_{l_2} \binom{k_2}{l_2} (-1)^{k_2 - l_2} F_{r,2},
\]

where

\[
F_{r,2} = \prod_{\alpha_1, Y_p} (1 + x_1^{\alpha_1} Y_3 \cdots Y_r)^{2l_2 + 1} \prod_{Y_p} (1 + z_1 Y_3 \cdots Y_r)^{l_2} \prod_{t \geq 3, \alpha_t, Y_p} (1 + z_1 z_3 \cdots z_t x_t^{\alpha_t} Y_{t+1} \cdots Y_r).
\]

So \(N^B(\pi_1, k_2, \ldots, k_r)\) equals

\[
c \left[ x_1^{k_1} z_1^{l_1} \right] \sum_{l_2} \binom{k_2}{l_2} (-1)^{k_2 - l_2} \sum_{i_2 \geq 0, a_2 + b_2 + c_2 \geq 1} x_s^{a_s} y_s^{b_s} z_s^{c_s} F_{r,2}.
\]

(2.19)

To compute the inner summation, let

\[
g_s = \frac{1}{2} \left( \prod_{2 \leq i \leq s} (2l_i + 1) - 1 \right).
\]

For any \(s \geq 2\), it is clear that

\[(2l_{s+1} + 1)g_s + l_{s+1} = g_{s+1}.
\]

Starting with (2.19), we can continue the above procedure to deduce that for \(2 \leq h \leq r - 1\),

\[
N^B(\pi_1, k_2, \ldots, k_r) = c \left[ x_1^{k_1} z_1^{l_1} \right] \sum_{l_2, \ldots, l_h} \prod_{2 \leq i \leq h} \binom{k_i}{l_i} (-1)^{k_i - l_i} \sum_{i_2 \geq 0, a_2 + b_2 + c_2 \geq 1} \sum_{h+1 \leq s \leq r} x_s^{a_s} y_s^{b_s} z_s^{c_s} F_{r,h},
\]

where

\[
F_{r,h} = \prod_{\alpha_1, Y_p} (1 + x_1^{\alpha_1} Y_{h+1} \cdots Y_r)^{\prod_{2 \leq i \leq h} (2l_i + 1)} \prod_{Y_p} (1 + z_1 Y_{h+1} \cdots Y_r)^{g_h} \prod_{t \geq h+1, \alpha_t, Y_p} (1 + z_1 z_3 \cdots z_t x_t^{\alpha_t} Y_{t+1} \cdots Y_r).
\]

In particular, for \(h = r - 1\), we have

\[
N^B(\pi_1, k_2, \ldots, k_r) = c \left[ x_1^{k_1} z_1^{l_1} \right] \sum_{l_2, \ldots, l_{r-1}} \left( \prod_{2 \leq i \leq r-1} \binom{k_i}{l_i} (-1)^{k_i - l_i} \right) G
\]

(2.20)
where

\[ G = \sum_{a_r + b_r + c_r \geq 1} \left[ x_r^a y_r^b z_r^c \right] \prod_{\alpha_1, y_r} (1 + x_{1}^{\alpha_1}) \prod_{y_r} (1 + z_1^{g_r-1}) \prod_{\alpha_r} (1 + z_1 x_{1}^{\alpha_r}) \]

\[ = \sum_{l_r} \binom{k_r}{l_r} (-1)^{k_r-l_r} (1 + z_1)^{g_r} \prod_{\alpha_1} (1 + x_1^{\alpha_1}) \sum_{l_r} (2l_r + 1) \prod_{l_r} (1 + x_1). \]

Since the number of \( B_n \)-partitions of type \( j_1 \) equals

\[ c' = \binom{n}{t_1} \binom{n-i_1}{j_1} \frac{2^{n-i_1-k_1}}{k_1!} = \frac{(2n)!}{(2t_1)!!(2k_1)!!}, \]

by (2.20), we obtain

\[ N_{n,r}^B = \sum_{\substack{j_1, j_2, \ldots, j_k \geq 1 \\nu \tilde{j} \geq n \\nu \tilde{j} \geq 1}} c' \sum_{k_1, \ldots, k_r} N_{\pi, k_2, \ldots, k_r}^B \]

\[ = (2n)! \sum_{k_2 \leq n} \left( \prod_{l_2} \binom{k_2}{l_2} \frac{(-1)^{k_2-l_2}}{(2k_2)!!} \right) \sum_{l_2, k_1} \frac{1}{(2k_1)!!} \left[ z_1^{i_1} (1 + z_1)^{g_r} H \right] \]

where

\[ H = \sum_{\substack{i_1 + j_1 + \cdots + j_k = n \\nu \tilde{j} \geq 1}} \left[ x_1^i \right] \prod_{\alpha_1} (1 + x_1^{\alpha_1}) \prod_{\alpha_1} (1 + z_1^{g_r}) \prod_{\alpha_1} (1 + z_1 x_1^{\alpha_1}) \]

\[ = \sum_{l_1} \binom{k_1}{l_1} (-1)^{k_1-l_1} \left[ x^{n-i_1} \right] (1 + x)^{l_1} \prod_{\alpha_1} (1 + z_1^{g_r}) \]

Using the identity

\[ \sum_{k} \binom{k}{l} \frac{(-1)^{k-l}}{(2k)!!} = \frac{e^{-1/2}}{(2l)!!}, \]

we can simplify the summation over \( k_1, k_2, \ldots, k_r \geq 0 \) in (2.21) to deduce that

\[ N_{n,r}^B = (2n)! \sum_{k_1, k_2, \ldots, k_r} \left( \prod_{l_1} \binom{k_1}{l_1} \frac{(-1)^{k_1-l_1}}{(2k_1)!!} \right) \sum_{l_1} \left[ x^{n-i_1} z_1^{i_1} \right] (1 + z_1)^{g_r} (1 + x)^{l_1} \prod_{\alpha_1} (1 + z_1^{g_r}) \prod_{\alpha_1} (1 + z_1 x_1^{\alpha_1}) \]

\[ = (2n)! \sum_{l_1, l_2, \ldots, l_r} \frac{1}{(2l_1)!!(2l_2)!! \cdots (2l_r)!!} \left[ x^n \right] (1 + x)^{g_r+l_1} \prod_{\alpha_1} (1 + z_1^{g_r+l_1}) \prod_{\alpha_1} (1 + z_1 x_1^{\alpha_1}) \]

To further simplify the above summation, we observe that

\[ g_r + l_1 \prod_{2 \leq i \leq r} (2l_i + 1) = \frac{1}{2} \left( \prod_{l_i \in [r]} (2l_i + 1) - 1 \right). \]

Substituting (2.24) into (2.23), we arrive at (2.14). This completes the proof. □

For example, we have \( N_{1,r} = 2^r - 1 \) and \( N_{2,3}^B = 187. \)
3 Minimally intersecting $B_n$-partitions without zero-block

In this section, we investigate the meet-semilattice of $B_n$-partitions without zero-block. Note that the minimal $B_n$-partition without zero-block is $\hat{0}^B$. Inspecting the proof of Theorem 2.1, we can restrict our attention to the set of $B_n$-partitions without zero-block by setting $i_0 = 0$ and $X = \emptyset$ in (2.8).

Corollary 3.1. Let $\pi$ be a $B_n$-partition consisting of $k$ block pairs of sizes $i_1, i_2, \ldots, i_k$ listed in any order. For a given $l \geq 1$, the number $N^D(\pi; l)$ of $B_n$-partitions $\pi'$ consisting of $l$ block pairs, which intersects $\pi$ minimally, is equal to

$$N^D(\pi; l) = \frac{i!}{(2l)!!} \left[ x^l \right] \left( \prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^l. \quad (3.1)$$

The total number of $B_n$-partitions without zero-block that intersect $\pi$ minimally is given by

$$N^D(\pi) = \frac{i!}{\sqrt{e}} \left[ x^l \right] \exp \left( \frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right). \quad (3.2)$$

For example, let $n = 3$, $\pi = \{\pm\{2\}, \pm\{1, -3\}\}$ and $l = 2$. Then (3.1) yields $N^D(\pi; 2) = 5$. In fact, the $B_n$-partitions consisting of 2 block pairs which intersect $\pi$ minimally are exactly the 5 partitions consisting of two block pairs except for $\pi$ itself.

Let $N_n$ be the number of $B_n$-partitions without zero-block. Taking $\pi = \hat{0}^B$ in (3.2), we obtain the following formula.

Corollary 3.2. We have

$$N_n = \frac{1}{\sqrt{e}} \sum_{k \geq 0} \frac{(2k)^n}{(2k)!!}. \quad (3.3)$$

Let $N_n(k)$ denote the number of $B_n$-partitions containing $k$ block pairs but no zero-block. It should be noted that the formula (3.3) can be easily deduced from the relation

$$N_n(k) = 2^{n-k} S(n, k),$$

where $S(n, k)$ are the Stirling numbers of the second kind, and the following identity on the Bell polynomials [9, 10]:

$$\sum_{k=0}^{n} S(n, k) x^k = \frac{1}{e^x} \sum_{k \geq 0} \frac{k^n}{k!} x^k.$$
The sequence \( \{N_n\}_{n \geq 0} \) is A004211 in [12]:

\[
1, 1, 3, 11, 49, 257, 1539, 10299, 75905, 609441, \ldots
\]

The proof of Corollary 2.4 implies the following corollary.

**Corollary 3.3.** Let \( N_{n,2}^D(k) \) denote the number of ordered pairs \((\pi, \pi')\) of minimally intersecting \(B_n\)-partitions without zero-block such that \(\pi\) consists of exactly \(k\) block pairs. Then

\[
N_{n,2}^D(k) = \frac{(2n)!}{(2k)!!} \sqrt{e} \sum_{j \geq 0} \frac{1}{(2j)!!} [(1 + x)^{2j} - 1]^k.
\]

The number \( N_{n,2}^D \) of ordered pairs \((\pi, \pi')\) of minimally intersecting \(B_n\)-partitions without zero-block is given by

\[
N_{n,2}^D = \frac{2^n}{e} \sum_{k,l \geq 0} \frac{(2kl)_n}{(2k)!!(2l)!!}.
\]

For example, \( N_{1,2}^D = 1 \), \( N_{2,2}^D = 7 \), \( N_{3,2}^D = 75 \). The following theorem is an analogue of Theorem 2.6 with respect to the meet-semilattice of \(B_n\)-partitions without zero-block.

**Theorem 3.4.** For \( r \geq 2 \), the number of minimally intersecting \( r \) -tuples \((\pi_1, \pi_2, \ldots, \pi_r)\) of \(B_n\)-partitions without zero-block equals

\[
N_{n,r}^D = \frac{2^n}{e^{r/2}} \sum_{k_1, k_2, \ldots, k_r} \frac{(2r-1)_{k_1} \cdots k_r n}{(2k_1)!!(2k_2)!! \cdots (2k_r)!!}
\]

(3.4)

**Proof.** Let \( 1 \leq t \leq r \). Let \( j_t = (j_{t,1}, j_{t,2}, \ldots, j_{t,k_t}) \) be a composition of \( n \). Assume that \(\pi_t\) is of type \((0; j_t)\). Let \( N^D(\pi_1, j_2, \ldots, j_r)\) be the number of \((r - 1)\)-tuples \((\pi_2, \ldots, \pi_r)\) of such \(B_n\)-partitions such that \((\pi_1, \pi_2, \ldots, \pi_r)\) is minimally intersecting. By the argument in the proof of Theorem 2.1, we find

\[
N^D(\pi_1, j_2, \ldots, j_r) = c \cdot [x^{j_1}] \sum_{b_1, c_1 = j_1} \left[ y_2^{b_2} y_2^{c_2} \cdots y_r^{b_r} y_r^{c_r} \right] f(j),
\]

(3.5)

where

\[
c = j_1! \prod_{2 \leq s \leq r} (2k_s)!!^{-1},
\]

\[
f(j) = \prod_{a \in [k_1]} (1 + x_a Y_2 Y_3 \cdots Y_r).
\]
Let $N^D(\pi_1, k_2, \ldots, k_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of $B_n$-partitions such that $\pi_s$ consists of $k_s$ block pairs, and $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting. It follows from (3.5) that

$$N^D(\pi_1, k_2, \ldots, k_r) = c \cdot \left[ x^1 \right] \sum_{b_1+c_s=j_s \geq 1} \left[ y_2^{b_2} \cdots y_r^{c_r} \right] f(j)$$

$$= j_1! \sum_{l_2, \ldots, l_r} \left[ x^1 \right] \prod_{\alpha \in [k_1]} (1 + x_\alpha)^{2^{r-1}l_2-\cdots-l_r} \prod_{2 \leq s \leq r} \left( \frac{k_s}{l_s} \right) \frac{(-1)^{k_s-l_s}}{(2k_s)!!}.$$

Consequently,

$$N_{n,r}^D = \sum_{k_1} \frac{1}{(2k_1)!!} \sum_{j_1, \ldots, j_r = n} 2^n n! \sum_{k_2, \ldots, k_r} N^D(\pi_1, k_2, \ldots, k_r)$$

$$= (2n)!! \sum_{k_1, k_2, \ldots, k_r} \prod_{1 \leq s \leq r} \left( \frac{k_s}{l_s} \right) \frac{(-1)^{k_s-l_s}}{(2k_s)!!} \left[ x^n \right] (1 + x)^{2^{r-1}l_2-\cdots-l_r}.$$

Applying (2.22), we can restate the above formula in the form of (3.4). This completes the proof.

For example, when $n = 2$ and $r = 3$, by (3.4) we find that $N_{2,3}^D = 25$. In fact, there are $3$ $B_2$-partitions without zero-block, that is,

$$0^B, \pi_1 = \{\pm\{1, 2\}\}, \pi_2 = \{\pm\{1, -2\}\}.$$

Among all $3^3 = 27$ 3-tuples of $B_2$-partitions without zero-block, only $(\pi_1, \pi_1, \pi_1)$ and $(\pi_2, \pi_2, \pi_2)$ are not minimally intersecting.

**Corollary 3.5.** We have

$$N_{n,r}^D = \sum_{j=1}^n N^r_{j} 2^{n-j} s(n, j),$$

where $s(n, j)$ are the Stirling numbers of the first kind. Moreover,

$$M_r^D \left( \frac{e^{2x} - 1}{2} \right) = \sum_{n \geq 0} N_{n,r}^{x^n} \frac{x^n}{n!},$$

where

$$M_r^D(x) = \sum_{n \geq 0} N_{n,r}^D x^n \frac{x^n}{n!}.$$

The formula (3.6) can be considered as a type $B$ analogue of Wilf’s formula (1.2), whereas (3.7) is analogous to Canfield’s formula (1.3).

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