STONE-WEIERSTRASS APPROXIMATION REVISITED

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ABSTRACT. The aim of the present article is to extend the Stone-Weierstrass theorem to functions ranging in a lattice normed space and admitting order rather than topological approximation. We proceed with the machinery of Boolean-valued transfer from lattice normed space to normed space.

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1. Introduction. There are many generalizations of the classical Stone-Weierstrass approximation theorem which broaden the class of the continuous scalar or vector valued functions to be approximated. One of the most striking results belongs to Bishop; see [4]. The available proof of Bishop’s result utilizes some nontrivial tools of functional analysis. Machado formulated a version of the theorem for vector-valued functions and gave a completely elementary proof in [21]. Ransford [24] succeeded in finding the proof that is both elementary and concise. Prolla revised Machado’s proof in [23] and found some new forms of the Stone-Weierstrass theorem that are given below. A survey of some other generalizations can be found in Prolla’s book [22]. The two recent publication should be also mentioned: Timofte [25] presented a unified approach to vector-valued versions of the Stone-Weierstrass theorem based an appropriate factorization of a topological space rather then the traditional localizability; Asgarova [2] handled the case of real-valued continuous functions, but the approximation is carried out by two algebras instead of one.

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Throughout this paper, $S$ denotes a compact Hausdorff topological space, compact space for short. Let $X$ be a real or complex normed space and let $C(S, X)$ be the vector space of all continuous functions from $S$ to $X$ with the supremum norm

$$\|f\| := \sup\{\|f(s)\| : s \in S\} \quad (f \in C(S, X)).$$

A multiplier of a subset $W \subset C(S, X)$ is a continuous function $\varphi : S \to [0, 1]$ such that $\varphi f + (1_S - \varphi)g \in W$ for all $f, g \in W$, where $1_S$ is the identically 1 function on $S$. The next three results correspond respectively to Theorems 1, 2, and 3 in [23].

**Theorem 1.1.** Let $W$ be a nonempty subset of $C(S, X)$ such that the set of all multipliers of $W$ separates the points of $S$. Then for all $f \in C(S, X)$ and $0 < \varepsilon \in \mathbb{R}$ the following are equivalent:

1. There exists $g \in W$ with $\|f - g\| < \varepsilon$.
2. For every $s \in S$ there exists $g_s \in W$ such that $\|f(s) - g_s(s)\|_X < \varepsilon$.

**Theorem 1.2.** Let $W$ be a nonempty subset of $C(S, X)$ such that the set of all multipliers of $W$ separates the points of $S$. Then for every $f \in C(S, X)$ there exists $s \in S$ such that

$$\inf\{\|f - g\| : g \in W\} = \inf\{\|f(s) - x\|_X : x \in W(s)\}.$$

**Theorem 1.3.** Let $W$ be a vector subspace of $C(S, X)$ and $A := \{\varphi \in C(S, \mathbb{R}) : (\forall g \in W) \varphi g \in W\}$. Assume that $A$ separates the points of $S$ and $W(s)$ approximates every member of $X$ for all $s \in S$. Then $W$ is uniformly dense in $C(S, X)$. The aim of the present article is to extend the above results to the order rather than topological approximation of functions ranging in a lattice normed space.

We use the standard notation and terminology of Aliprantis and Burkinshaw [1] for the theory of vector lattices. Everywhere below $E$ and $F$ denote some Archimedean real vector lattices, while $\mathbb{B}(E)$ and $\mathbb{P}(E)$ stand for the Boolean algebras of all bands and band projections in $E$. Recall also that $x, y \in F$ are disjoint (in symbols $x \perp y$) if $|x| \land |y| = 0$. Also, $M^\perp := \{x \in F : (\forall y \in M) x \perp y\}$. Given a vector lattice $E$, we denote, by $E^\delta$ and $E^u$ the Dedekind completion and the universal completion of a vector lattice $E$; moreover, we assume that $E \subset E^\delta \subset E^u$.

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with join $\lor$, meet $\land$, complement $(\cdot)^\perp$, unity (top) $\mathbbm{1}$ and zero (bottom) $\mathbb{0}$. A partition of an element $b \in \mathbb{B}$ in a Boolean algebra $\mathbb{B}$ is a family $(b_\xi)$ in $\mathbb{B}$ such that $b_\xi \land b_\eta = 0$ for all $\xi \neq \eta$ and $b = \sup \xi b_\xi$ if $b$ is the top of $\mathbb{B}$ then $(b_\xi)$ is a partition of unity. We let $:=\text{ denote the assignment by definition, while } \mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

### 2. Preliminaries

In what follows, we will need some information about lattice normed spaces, approximating sets, extreme operators, and lattice homomorphisms.

**Lattice normed spaces.** We use the abbreviation LNS for “lattice normed space.” Consider a (real or complex) vector space $X$ and a real vector lattice $E$. ...
DEFINITION 2.1. An $E$-valued norm on $E$ is a mapping $|\cdot| : E \to E_+$ such that $|x| = 0$ implies $x = 0$, $|\lambda x| = |\lambda| |x|$, and $|x + y| \leq |x| + |y|$ for all $x, y \in E$, and $\lambda \in \mathbb{R}(\mathbb{C})$. The pair $(E, |\cdot|)$ is called an LNS over $E$. An $E$-valued norm $|\cdot|$ on $X$ as well as $X$ itself is $d$-decomposable if, for each decomposition $|x| = e_1 + e_2$ with disjoint $e_1, e_2 \in E_+$ and $x \in X$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k$ for all $k \in \{1, 2\}$.

If $X$ is a $d$-decomposable LNS over $E$, then there is a natural mapping that associates to each $\pi \in \mathcal{P}(E_\rho)$, with $E_\rho = \{ |x| : x \in X \}^\perp$, the linear projection $\hat{\pi}$ in $X$. Namely, $\hat{\pi}(x) = x_1$, where $|x| = \pi(|x|) + \pi^\perp(|x|) = e_1 + e_2$ and $x = x_1 + x_2$ with $|x_k| = e_k$ for all $k \in \{1, 2\}$.

The set $\mathcal{P}(X) := \{ \hat{\pi} : \pi \in \mathcal{P}(E) \}$ with the order relation defined by letting $\pi \leq \rho$ if and only if $\pi \circ \rho = \pi$ is a Boolean algebra and the mapping $\pi \mapsto \hat{\pi}$ is a Boolean isomorphism from $\mathcal{P}(E)$ onto $\mathcal{P}(X)$. Moreover,

$$|\hat{\pi} x + \hat{\pi}^\perp y| = \pi|x| + \pi^\perp|y| \quad (\pi \in \mathcal{P}(E); x, y \in X).$$  \hspace{1cm} (1)

In the sequel, we will assume that $\{ |x| : x \in X \}^{\perp\perp} = E$ for every LNS $X$ over $E$ and the two Boolean algebras $\mathcal{P}(E)$ and $\mathcal{P}(X)$ are identified; see (see [14, Section 2.1]).

DEFINITION 2.2. We say that a net $(x_\alpha)_{\alpha \in A}$ in $X$ norm $o$-converges or norm $r$-converges with regulator $e \in E_+$ to $x \in X$ and write $x = \no-lim x_\alpha$ or, respectively, $x = \nr-lim x_\alpha$ if $\lim_{\alpha} \|x - x_\alpha\| = 0$ or, respectively, $\lim_{\alpha} \|x - x_\alpha\| = 0$ with regulator $e \in E_+$. A net $(x_\alpha)$ is norm $o$-Cauchy or norm $r$-Cauchy with regulator $e \in E_+$ if the net $(x_\alpha - x_\beta)_{(\alpha, \beta) \in A \times A}$ norm $o$-vanishes or norm $r$-vanishes with regulator $e \in E_+$. The set of the $o$-limits or $r$-limits of all $o$-convergent or $r$-convergent nets in $X$ comprised of the elements of some subset $X_0 \subset X$ is the $o$-closure or $r$-closure of $X_0$.

DEFINITION 2.3. A lattice normed space $X$ is norm $o$-complete or norm $r$-complete if every norm $o$-Cauchy or, respectively, norm $r$-Cauchy net in $X$ norm $o$-converges or norm $r$-converges to an element of $X$. A Banach–Kantorovich space over a vector lattice $E$ is a vector space $X$ with a $d$-decomposable norm $|\cdot| : X \to E$ which is norm $o$-complete. A Banach–Kantorovich space over $E$ is universally complete if the vector lattice $E$ is universally complete. Given two LNSs $X$ and $Y$ over the same vector lattice, an operator $T : X \to Y$ is isometric if $|T x| = |x|$ for all $x \in X$.

DEFINITION 2.4. By a universal completion or a norm completion of an LNS $(X, E)$ we mean a universally complete Banach–Kantorovich space $(X^u, E^u)$ or, respectively, a Banach–Kantorovich space $(\tilde{X}, E^\delta)$ together with a linear isometry $\iota : X \to X^u (\tilde{X})$ such that each universally complete subspace of $(X^u, E^u)$ or, respectively, each decomposable norm complete subspace of $(\tilde{X}, E^\delta)$ including $\iota(X)$ coincides with $X^u (\tilde{X})$. The $d$-decomposable hull of $X$ is the least $d$-decomposable subspace $\tilde{X} \subset X^u$ including $\iota(X)$.

LEMMA 2.5. For each LNS $X$ over a vector lattice $E$ the following hold:

(1) There is a universal completion of $X$ unique to within linear isometry.
(2) There is a norm $o$-completion of $X$ unique to within linear isometry.
(3) There is a $d$-decomposable hull of $X$ unique to within an isometry.
Approximating sets. We will briefly recall the concept of approximating subset in an LNS which was introduced by Gutman in [11] and turns out very useful in the general theory of LNSs as well as in the study of disjointness preserving operators; see also [12]. Let $X$ be an LNS over a Dedekind complete vector lattice $E$.

DEFINITION 2.6. If $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathcal{P}(X)$ and $(x_\xi)_{\xi \in \Xi}$ is a family in $X$, then $x \in X$ with $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$ is a mixing of $(x_\xi)$ by $(b_\xi)$ denoted by $\text{mix}_{\xi \in \Xi} b_\xi x_\xi$. The set of all mixings of arbitrary or finite families in $U$ is the cyclic hull or, respectively, finitely cyclic hull) of $U$ which is denoted by $\text{mix}(U)$ or, respectively, by $\text{mix}_{\text{fin}}(U)$. Say that $X$ is $d$-complete if, for each partition of unity $(\pi_\xi)$ in $\mathcal{P}(E)$ and norm bounded family $(x_\xi)$ ($\|x_\xi\| \leq e$ for all $\xi$ and some $e \in E$) in $X$, there is $x \in X$ with $x = \text{mix} \pi_\xi x_\xi$. It easy to verify that the (finitely) cyclic hull of a set $U$ is the smallest (finitely) cyclic set that includes $U$. It follows from (1) that for a set $U$ to be finitely cyclic, it suffices that $U$ contains the sums $\pi x + \pi y$ for all $x, y \in V$ and $\pi \in \mathcal{P}r(X)$.

Lemma 2.5 can be supplemented as follows: Each LNS has a $d$-completion $\bar{X}$ unique up to linear isometry.

DEFINITION 2.7. Let $U$ be a subset of an LNS $X$. We say that $U$ (orderly) approximates $x \in X$ if $\inf_{u \in U} |x - u| = 0$. We say that $U$ (orderly) approximates $W \subset X$ if $U$ approximates every element of $W$. A subset of $X$ is (order) approximating if it approximates $X$. The set $X_0$ is no-dense or nr-dense in $X$ if every member of $X$ is the no-limit (nr-limit) of some net in $X_0$.

LEMMA 2.8. Let $X$ be a $d$-decomposable LNS over a Dedekind complete vector lattice $E$. Given a subset $U$ and an element $u$ of $X$, the following hold:

1. $U$ approximates $u$ if and only if $u$ is the no-limit in $\bar{X}$ of some net of elements from $\text{mix}_{\text{fin}}(U)$.

2. $U$ approximates $u$ if and only if $u$ is the nr-limit in of elements from $\bar{X}$ of some net of elements of $\text{mix}(U)$.

3. If, moreover, $E$ has an order unity $\mathbb{1}$ then $U$ approximates $u$ if and only if $u$ is the nr-limit in $\bar{X}$ with regulator $\mathbb{1}$ of some net of elements of $\text{mix}(U)$.

Proof. See [11, Propositions 1.3, 1.6, and 1.8].

LEMMA 2.9. The following properties of a subset $U$ of an LNS $X$ are equivalent:

1. $U$ is an approximating subset of $X$.

2. $\text{mix}_{\text{fin}}(U)$ is an approximating subset of $\bar{X}$.

3. $\text{mix}(U)$ is an approximating subset of $\bar{X}$.

4. $\text{mix}(U)$ is norm $r$-dense in $\bar{X}$.

5. $\text{mix}_{\text{fin}}(U)$ is norm $o$-dense in $\bar{X}$. 

Proof. See [11, Propositions 1.3, 1.6, and 1.8].
Proof. See [11, Propositions 1.4, 1.7, and 1.9].

**Extreme operators and lattice homomorphisms.** The question of recovering convex sets of operators from extreme points was raised in the well-known paper of Bonsall, Lindenstrauss, and Phelps [5]. The general results on the geometric structure of the support sets of sublinear operators, as well as the facts of use below, can be found in [16, Chapter 2].

A sublinear operator $p : X \to E$ from a vector space $X$ to a vector lattice $E$ is a subadditive and positive homogeneous mapping; i.e., $p(x + y) \leq p(x) + p(y)$ and $p(rx) = rp(x)$ for all $x, y \in X$ and positive $r \in \mathbb{R}$. The collection of all linear operators from $X$ into $E$ dominated by $p$ is the support set or the subdifferential at zero of $p$ and denoted by $\partial p$. In symbols,

$$
\partial p := \{ T \in L(X, E) : (\forall x \in X) \, T x \leq p(x) \},
$$

where $L(X, E)$ is the space of all linear operators from $X$ into $E$. If $A$ is a subring and sublattice of the orthomorphism ring $\Lambda := \text{Orth}(E)$ and $p$ is a $A$-sublinear operator $(p(\pi x + \rho y) \leq \pi p(x) + \rho p(y)$ for all $x, y \in X$ and $\pi, \rho \in A_+$), then the members of $\partial(p)$ are automatically module homomorphisms; see [16, Theorem 2.3.15].

Denote by $l_\infty(S, E)$ the set of all (order) bounded mappings from $S$ into $E$; i.e., $f \in l_\infty(S, E)$ if and only if $f : S \to E$ and $\{ f(s) : s \in S \}$ is order bounded in $E$. It is easy to verify that $l_\infty(S, E)$ with the coordinate-wise algebraic operations and order is a Dedekind complete vector lattice if so is $E$. Moreover, $l_\infty(S, E)$ is a faithful module over $\Lambda$ with multiplication $(\lambda, f) \mapsto \lambda f := \lambda \circ f$ ($\lambda \in \Lambda, f \in l_\infty(S, E)$).

The operator $\varepsilon_{S,E}$ acting from $l_\infty(S, E)$ into a Dedekind complete $E$ by the rule

$$
\varepsilon := \varepsilon_{S,E} : f \mapsto \sup \{ f(s) : s \in S \} \quad (f \in l_\infty(S, E))
$$

is the canonical sublinear operator given by $S$ and $E$.

**Lemma 2.10.** For $e \in E$ denote by $\bar{e}$ the constant function $s \mapsto e$ ($s \in S$). Then

$$
\partial(\varepsilon_{S,E}) = \{ T \in L^+(l_\infty(S, E), E) : (\forall e \in E) \, T \bar{e} = e \},
$$

where $L^+(G, E)$ is the cone of all positive operators from $G$ to $E$.

**Proof.** See [16, 2.1.4].

**Definition 2.11.** Given $s \in S$, the $\delta$-function $\delta_s : l_\infty(S, E) \to E$ is defined as $\delta_s : f \mapsto f(s)$. The mixing $\sum_{s \in S} s \delta_s$ of a family $(\delta_s)_{s \in S}$ by a partition of unity $(\pi_s)_{s \in S}$ in $\mathcal{P}(E)$ is an $E$-valued pure state on $S$. (Here $\sum$ means the pointwise order sum.)

Clearly, all $\delta$-functions and all pure states are extreme points of $\partial(\varepsilon_S)$. Moreover, every extreme point of $\partial(\varepsilon_S)$ can be approximated by pure states.
Lemma 2.12. The set of extreme points of the support set \( \partial(\varepsilon S) \) coincides with the set of lattice homomorphisms from \( l_\infty(S, E) \) into \( E \) which belong to \( \partial\varepsilon S \). Moreover, all members of \( \partial(\varepsilon S) \) are \( A \)-linear.

Proof. See [19, Theorem 3] or [16, Theorem 2.2.9].

Lemma 2.13. Each extreme point of the convex set \( \partial(\varepsilon S) \) is a pointwise \( r \)-limit of a net of pure states.

Proof. This fact was established in [15, The Main Theorem]; see [16, Proposition 2.4.8 and Theorem 2.4.11] for details.

Lemma 2.14. Let \( E \) and \( F \) be two vector lattices with \( F \) Dedekind complete. If \( G \) is a majorizing vector sublattice of \( E \) and \( T : G \to F \) is a lattice homomorphism, then \( T \) extends to all of \( E \) as a lattice homomorphism.

Proof. This is the well-known Lipecki–Luxemburg–Schep Theorem see [1, Theorem 2.29].

3. Main results. From now on, we will assume that \( E \) is a Dedekind complete vector lattice and \( X \) is an LNS over \( E \), while \( \mathcal{B} = \mathcal{P}(E) \) and \( \Lambda := \text{Orth}(E) \). Then \( \Lambda \) is a Dedekind complete \( f \)-algebra (under composition) with unity \( I := I_E \); see [1, Theorems 2.45 and 2.59]. Before stating the results, we will introduce the main object of study, i.e., the space of uniformly norm continuous functions with values in \( X \). Let \(( S, \mathcal{U}) \) be a uniform space with uniformity \( \mathcal{U} \).

Definition 3.1. Let \( X \) be an LNS over a Dedekind complete vector lattice \( E \). A vector-function \( f : S \to X \) is uniformly order continuous if

\[
\inf_{U \in \mathcal{U}} \sup \{|f(u_1) - f(u_2)| : (u_1, u_2) \in U\} = 0.
\]

This amounts to saying that \( f \) is norm bounded on \( S \) (i.e., there exists \( e \in E \) with \( |f(s)| \leq e \) for all \( s \in S \)) and if \( |f| := \sup\{|f(s)| : s \in S\} \), then for all \( e \) there exists a partition of unity \( (\pi_\alpha)_{\alpha \in \mathcal{U}} \) in \( \mathcal{P}(X) \) such that \( \pi_\alpha|f(s_1) - f(s_2)| \leq e e \) for all \( \alpha \) and \( (s_1, s_2) \in \alpha \). We denote by \( C_{ro}(S, X) \) the vector space of all uniformly order continuous mappings from \( S \) into \( X \) endowed with the \( E \)-valued norm \( f \mapsto |f| \) (see [7]). Obviously, \( C_{ro}(S, \Lambda) \subset l_\infty(S, \Lambda) \).

For a compact space \( S \) there exists exactly one uniformity \( \mathcal{U} \) on the \( S \) that induces the original topology of \( S \); moreover this uniformity is totally bounded and complete [8, Theorems 8.3.13 and 8.3.16]. Clearly, \( C_{ro}(S, X) = C(S, X) \) is the space of continuous functions with values in a normed space \( X \) whenever \( E = \mathbb{R} \).

The vector space \( C_{ro}(S, \Lambda) \) will be considered with pointwise multiplication: given \( \varphi, \psi \in C_{ro}(S, \Lambda) \), we put \( \varphi \psi : s \mapsto \varphi(s)\psi(s) \) \( (s \in S) \). We also equip \( C_{ro}(S, X) \) with multiplication by elements of \( C_{ro}(S, \Lambda) \) by putting \( \varphi f : s \mapsto \varphi(s)f(s) \) \( (s \in S) \).
If $X$ is a decomposable LNS or a Banach–Kantorovich space over $E$ then so is $C_{ro}(S, X)$. Moreover, $C_{ro}(S, \Lambda)$ is an $f$-algebra and $C_{ro}(S, X)$ is a module over an $f$-algebra $C_{ro}(S, \Lambda)$ with $|\varphi f| \leq |\varphi||f|$ for all $\varphi \in C_{ro}(S, \Lambda)$ and $f \in C_{ro}(S, X)$.

Proof. This is immediate from Theorem 4.9 and Lemma 5.3 and can be derived without Boolean valued analysis on using [14, Theorem 2.2.3].

Lemma 3.3. Let $\varepsilon^c_{S, \Lambda}$ be the restriction to $C_{ro}(S, \Lambda)$ of the sublinear operator $\varepsilon_{S, \Lambda} : l_\infty(S, \Lambda) \to \Lambda$. Every lattice homomorphism in the convex set $\partial(\varepsilon^c_{S, \Lambda})$ is a pointwise $r$-limit of a net of pure states.

Proof. Note that $C_{ro}(S, \Lambda)$ is a majorizing sublattice of $l_\infty(S, \Lambda)$ as for every $e \in E$ the function $\overline{e} : s \mapsto e$ belongs to $C_{ro}(S, \Lambda)$. It follows that a lattice homomorphism $\sigma : C_{ro}(S, \Lambda) \to \Lambda$ extends to a lattice homomorphism $\hat{\sigma} : l_\infty(S, \Lambda) \to \Lambda$ by Lemma 2.14. If $\sigma \in \partial(\varepsilon^c_{S, \Lambda})$ then $\sigma(\overline{e}) = e$ for all $e \in E$ and hence $\sigma \in \partial(\varepsilon^c_{S, \Lambda})$ by Lemma 2.10. It remains to observe that $\hat{\sigma}$ is an extreme point of $\partial(\varepsilon^c_{S, \Lambda})$ in view of Lemma 2.12 and appeal to Lemma 2.13.

Lemma 3.4. Let $W$ be a nonempty subset of $C_{ro}(S, X)$. A function $\varphi \in C_{ro}(S, \Lambda)$ with $0 \leq \varphi(s) \leq I$ for all $s \in S$ is a multiplier of $W$ if $\varphi f + (I - \varphi) g \in W$ for every pair of elements $f, g \in W$. The set of all multipliers of $W$ is denoted by $\mu(W)$.

It is clear that if $\varphi, \psi \in \mu(W)$, then $I - \varphi \in W$ and $\varphi \psi \in \mu(W)$.

Denote by $\Sigma := \Sigma(S, \Lambda)$ the set of all extreme points of the convex set $\partial(\varepsilon^c_{S, \Lambda})$. Each extreme point of $\partial(\varepsilon^c_{S, \Lambda})$ extends to an extreme point of $\partial(\varepsilon^c_{S, \Lambda})$ by Milman’s theorem for support sets (see [16, Theorem 2.210]). Thus, taking Lemmas 2.10 into account, we see that $\Sigma$ comprises all lattice homomorphisms $\sigma : C_{ro}(S, \Lambda) \to \Lambda$ with $\sigma(\overline{\lambda}) = \lambda$, where $\lambda : s \mapsto \lambda (s \in S, \lambda \in \Lambda)$. Given $s \in S$, the restriction of the corresponding $\delta$-function to $C_{ro}(S, E)$ will be denoted by the same symbol $\delta_s$. Clearly, the mapping $s \mapsto \delta_s$ is an injection of $S$ into $\Sigma$. This is how we will identify $S$ with the corresponding subset of $\Sigma$. Given $u, v \in E$ and $\sigma, \tau \in \Sigma$, define

$$
[u = v] := \bigvee \{b \in B : bu = bv\}, \quad [u \neq v] := I_E - [u = v];
$$

(2)

$$
[\sigma = \tau] := \bigvee \{b \in B : b\sigma = b\tau\}, \quad [\sigma \neq \tau] := I_E - [\sigma = \tau].
$$

(3)

Let $\hat{X}$ be the norm completion of an LNS $X$. The Boolean isomorphism $\pi \mapsto \hat{\pi}$ from $\mathcal{P}(E)$ onto $\mathcal{P}(X)$ can be extended to a monomorphism of the ring $\Lambda$ into the ring of endomorphisms of the additive group of $\hat{X}$. Hence, $\hat{X}$ admits a faithful module structure over $\Lambda$. Show that every uniformly order continuous function $f \in C_{ro}(S, X)$ extends canonically to some function $\hat{f} : \Sigma \to \hat{X}$.

Definition 3.5. Given $f \in C_{ro}(S, X)$ and a pure state $\sigma = \sum_{s \in S} \pi_s \delta_s$ with a partition of unity $(\pi_s)_{s \in S}$ in $\mathbb{B}$, we put $\hat{f}(\sigma) := \sum_{s \in S} \pi_s f(s)$ and note that $|\hat{f}(\sigma)| = \sum_{s \in S} |\pi_s| |f(s)| = \sigma(\varphi) \leq |f|$. By Lemma 2.13 an arbitrary member $\sigma$ of $\Sigma$ is the pointwise $r$-limit of some net $(\sigma_\alpha)$ of pure states, and so we put $\hat{f}(\sigma) := \lim_\alpha \hat{f}(\sigma_\alpha)$. 


The existence of limits in $\tilde{X}$ and the soundness of the above definition follow from Lemma 5.3. Moreover, $|f| = |\tilde{f}| := \sup\{|\tilde{f}(\sigma)|_X : \sigma \in \Sigma\}$ and $[\sigma = \tau] \leq [\tilde{f}(\sigma) = \tilde{f}(\tau)]$ for all $\sigma, \tau \in \Sigma$, as can be easily seen from Lemmas 5.3 and 5.4. In the sequel, we will write $f(\sigma)$ instead of $\tilde{f}(\sigma)$.

**Lemma 3.6.** Given $\sigma, \tau \in \Sigma$ and $\mathcal{F} \subset C_{ro}(S, \Lambda)$, the following are equivalent:

1. $[\sigma \neq \tau] \leq \vee\{[\sigma(\varphi) \neq \tau(\varphi)] : \varphi \in \mathcal{F}\}$.
2. $[\sigma = \tau] \geq \bigwedge\{[\sigma(\varphi) = \tau(\varphi)] : \varphi \in \mathcal{F}\}$.
3. If $b\sigma \neq b\tau$ for some $b \in \mathbb{B}$ then there exists $\varphi \in \mathcal{F}$ such that $b\sigma(f) \neq b\tau(f)$.
4. For every $b \in \mathbb{B}$ we have $b\sigma = b\tau$ whenever $b\sigma(\varphi) = b\tau(\varphi)$ for all $\varphi \in \mathcal{F}$.

**Proof.** The equivalence $(1) \iff (2)$ is an immediate consequence of the infinite De Morgan laws in $\mathbb{B}$, while $(3) \iff (4)$ follows easily from the logical equivalence of $A \rightarrow B$ and $\neg B \rightarrow \neg A$. If there is $b \in \mathbb{B}$, $b\sigma \neq b\tau$, and $(1)$ holds; then $0 < b \leq [\sigma \neq \tau]$ and hence $b_0 := b \land [\sigma(\varphi) \neq \tau(\varphi)] > 0$ for some $\varphi \in \mathcal{F}$. It follows that $b_0\sigma(f) \neq b_0\tau(f)$ is true, but then so is $b\sigma(f) \neq b\tau(f)$ which implies $(1) \rightarrow (3)$. If $(4)$ holds and $b\sigma(\varphi) = b\tau(\varphi)$ or, equivalently, $b \leq [\sigma(\varphi) = \tau(\varphi)]$ for all $\varphi \in \mathcal{F}$, then $[\sigma = \tau]$, whence $(4) \rightarrow (2)$. 

**Definition 3.7.** Say that a subset $\mathcal{F} \subset C_{ro}(S, \Lambda)$ separates the points of $\Sigma$ if, given any points $\sigma, \tau \in \Sigma$, one and, hence, all conditions $(1)–(4)$ of Lemma 3.6 are satisfied.

Now we have all prerequisites to formulating the main results.

**Theorem 3.8.** Let $X$ be an LNS over a Dedekind complete vector lattice, let $\tilde{X}$ be the norm completion if $X$, and let $W$ be a nonempty subset of $C_{ro}(S, X)$. If the set $\mu(W)$ of all multipliers of $W$ separates the points of $\Sigma$; then, for every $f \in C_{ro}(S, X)$, the following are equivalent:

1. $W$ approximates $f$ in $C_{ro}(S, X)$.
2. $W(\sigma) := \{g(\sigma) \in \tilde{X} : g \in W\}$ approximates $f(\sigma)$ in $\tilde{X}$ for all $\sigma \in \Sigma$.

**Corollary 3.9.** Let $X$ be a Banach–Kantorovich space and let $W$ be a nonempty subset of $C_{ro}(S, X)$ and $f \in C_{ro}(S, X)$. If $\mu(W)$ separates the points of $\Sigma$, then the following are equivalent:

1. $f$ is the norm o-limit in $C_{ro}(S, X)$ of some net in $\text{mix}_{\text{fin}}(W)$.
2. $f$ is the norm r-limit in $C_{ro}(S, X)$ of some net from $\text{mix}(W)$.
3. For every $\sigma \in \Sigma$ the value $f(\sigma)$ is the norm o-limit in $X$ of some net in $\text{mix}_{\text{fin}}(W(\sigma))$.
4. For every $\sigma \in \Sigma$ the value $f(\sigma)$ is the norm r-limit in $X$ of some net from $t\text{mix}(W(\sigma))$.

**Proof.** This is immediate from Theorem 3.8 and Lemma 2.8. 

$\square$
Remark 3.10. It is worth highlighting the two extreme cases of Theorem 3.8 and Corollary 3.9:

1. If \( E = \mathbb{R} \), then \( \mathbb{P}(X) = \{0, I_X\} \), \( X = \bar{X} = X \), \( \text{mix}(U) = U \) for every \( U \subset X \), and \( S = \Sigma \). In this case, there is no need to involve \( \bar{X} \), since the extension operator \( f \mapsto \hat{f} \) in Definition 3.5 is the only place where the completeness of \( \bar{X} \) is needed. Thus, we may assume that \( X \) is a normed space, and hence we arrive at Prolla's result; see [23, Theorem 1 and Corollary 1].

2. Another extreme case is \( X = E = \mathbb{R} \) and \( |x| = |x| \) for all \( x \in E \). In this event \( \hat{f}(\sigma) = \sigma(f) \) and we obtain a new version of the Stone-Weierstrass theorem for vector-functions with values in a Dedekind complete vector lattice: If \( W \) is a nonempty subset of \( C_{ro}(S, E) \) such that \( \mu(W) \) separates the points of \( \Sigma \), then for every \( f \in C_{ro}(S, E) \) the following are equivalent:

   (i) \( \inf\{|f - g| : g \in W\} = 0 \) in \( C_{ro}(S, E) \).
   (ii) \( \inf\{|(\sigma(f) - \sigma(g)) : g \in W\} = 0 \) in \( E \) for all \( \sigma \in \Sigma \).

Let us formulate this result with approximation in terms of order convergence.

Corollary 3.11. Let \( W \) be a nonempty subset of \( C_{ro}(S, E) \) such that the set \( \mu(W) \) of all multipliers of \( W \) separates the points of \( \Sigma \). Then for each \( f \in C_{ro}(S, E) \) the following are equivalent:

1. \( f \) is the norm \( o \)-limit in \( C_{ro}(S, E) \) of some net in \( \text{mix}_{\text{fin}}(W) \).
2. For every \( \sigma \in \Sigma \) the value \( \sigma(f) \) is the \( o \)-limit in \( E \) of some net in \( \text{mix}_{\text{fin}}(\sigma(W)) \), where \( \sigma(W) = \{\sigma(g) \in E : g \in W\} \).

Theorem 3.12. Let \( W \) be a nonempty subset of \( C_{ro}(S, X) \) such that the set \( \mu(W) \) of all multipliers of \( W \) separates the points of \( \Sigma \). Then for each \( f \in C_{ro}(S, X) \) there exists \( \sigma \in \Sigma \) such that

\[ \inf\{|f - g| : g \in W\} = \inf\{|(f(\sigma) - x)_{\bar{X}} : x \in W(\sigma)\} \].

Theorem 3.13. Let \( X \) be a Banach–Kantorovich space, let \( W \) be a vector subspace of \( C_{ro}(S, X) \) and \( A := \{\varphi \in C_{ro}(S, \Lambda) : \forall g \in W \varphi g \in W\} \). Assume that \( A \) separates the points of \( \Sigma \) and \( W(\sigma) \) approximates every member of \( X \) for all \( \sigma \in \Sigma \). Then the following hold:

1. \( \text{mix}_{\text{fin}}(W) \) is norm \( o \)-dense in \( C_{ro}(S, X) \);
2. if, moreover, \( E \) has an order unity \( 1 \);, then \( \text{mix}(W) \) is norm \( r \)-dense in \( C_{ro}(S, X) \) with the same regulator 1.

Proof. Demonstration may proceed along the lines of Theorems 3.8 and 3.12 by Boolean valued analysis. However, for diversity, we will deduce Theorem 3.13. from Corollary 3.9 by the standard means. Observe first that \( A \) is a subalgebra of \( C_{ro}(S, \Lambda) \) and \( \mu(W) = \{\varphi \in A : 0 \leq \varphi \leq 1\} \), where \( I := I_E \in \Lambda \). Prove that \( \mu(W) \) separates the points of \( \Sigma \). Given \( \sigma, \tau \in \Sigma \) and \( \varphi \in A \), put \( \varphi_0(\omega) := \varphi(\omega) - \varphi(\sigma) \) and \( \psi(\omega) := a\varphi_0^2(\omega) \) for all \( \omega \in \Sigma \), where \( a \) is the unique member of \( \Lambda^a \) such that \( a|\varphi_0^2| = \pi \) with \( \pi = |a| = |[\varphi_0^2]| \) and \( |a| \) being the order projection onto the band
Clearly, $\psi \in A$, $\psi(\sigma) = 0$ and $0 \leq \psi(\omega) \leq |a\varphi_0^2| = a\sup\{\omega(\varphi_0^2) : \omega \in \Sigma\} = a|\varphi_0^2| = \pi \leq I_E$. It follows that $\psi \in \mu(W)$. Moreover,

$$[\varphi(\tau) = \varphi(\sigma)] = [\varphi_0(\tau) = 0] = [\varphi_0^2(\tau) = 0] = [a|\varphi_0^2| = 0] = [\psi(\tau) = 0] = [\psi(\tau) = \psi(\sigma)].$$

Hence, for every $\varphi \in A$ there exists $\psi \in \mu(W)$ such that $[\varphi(\tau) = \varphi(\sigma)] = [\psi(\tau) = \psi(\sigma)]$. By hypothesis, $A$ separates the points of $\Sigma$, which means by Lemma 3.6 that $[\sigma = \tau] \geq \bigwedge\{[\sigma(\varphi) = \tau(\varphi)] : \varphi \in A\}$. Consequently, $[\sigma = \tau] \geq \bigwedge\{[\sigma(\psi) = \tau(\psi)] : \varphi \in \mu(W)\}$ and $\mu(W)$ separates the points of $\Sigma$. Denote by $W$ the norm $o$-closure of $\text{mix}_{\text{fin}}(W)$ in $C_{ro}(S, X)$. By Corollary 3.9, $W = C_{ro}(S, X)$. It remains to note that the norm $o$-closure of $\text{mix}_{\text{fin}}(W)$ coincide with the norm $r$-closure of $\text{mix}(W)$ and, if there is an order unity $\mathbb{1}$ in $E$, the norm $r$-closure can be taken with respect to the same regulator $\mathbb{1}$. □

4. Boolean valued requisites. In the sequel, $B$ is a complete Boolean algebra, and $V^{(B)}$ is a corresponding Boolean valued model of set theory. As the standard model of set theory, we consider the von Neumann universe $V$. We need some properties of $V$ and $V^{(B)}$ as well as some relationships between them; the detailed presentation can be found in [3, 17, 18].

There is a natural way of assigning to each statement $\phi$ about $x_1, \ldots, x_n \in V^{(B)}$ the Boolean truth-value $[\phi(x_1, \ldots, x_n)] \in B$. The sentence $\phi(x_1, \ldots, x_n)$ is called true within $V^{(B)}$ if $[\phi(x_1, \ldots, x_n)] = \mathbb{1}$. All axioms and rules of inference of the first-order predicate calculus with equality are true in $V^{(B)}$. In particular,

$$[u = v] \land [\phi(u)] \leq [\phi(v)] \tag{4}$$

for all $u, v \in V^{(B)}$ and every formula $\phi(x)$. It follows that all theorems of ZFC (Zermelo–Fraenkel set theory with the axiom of choice) are true in $V^{(B)}$. This statement is known as the Boolean valued transfer principle or transfer for short. There is also the maximum principle, which enables us to construct all particular objects in $V^{(B)}$.

Moreover, there is a smooth mathematical technique for interplay between the interpretations of any fact in the two models $V$ and $V^{(B)}$. The relevant ascending- and-descending machinery rests on the functors of canonical embedding (or standard name) $X \mapsto X^\downarrow$ and ascent $X \mapsto X^\uparrow$, both acting from $V$ to $V^{(B)}$, and the functor of descent $X \mapsto X_\downarrow$, acting from $V^{(B)}$ to $V$; see [3, 17, 18] for details.

Observe some simple properties of the standard name mapping we need in the sequel:

**Lemma 4.1.** (1) Given $x \in V$ and a formula $\varphi$ of ZF (Zermelo–Fraenkel set theory), we have

$$[\exists y \in x^\uparrow] \varphi(y) = \bigvee\{[\varphi(z^\uparrow)] : z \in x\},$$

$$[\forall y \in x^\uparrow] \varphi(y) = \bigwedge\{[\varphi(z^\uparrow)] : z \in x\}.$$
(2) The standard name mapping is injective. Moreover, for all \( x, y \in \mathbb{V} \) we have
\[
x \in y \iff \mathbb{V}(\mathbb{B}) \models x^\uparrow \in y^\uparrow,
\]
\[
x = y \iff \mathbb{V}(\mathbb{B}) \models x^\uparrow = y^\uparrow.
\]

**Lemma 4.2.** Denote by \( \mathcal{P}_{\text{fin}}(X) \) the collection of all finite sunsets of \( X \in \mathbb{V} \) and let \( [\mathcal{P}_{\text{fin}}(Y)] \) the collection of all finite subsets of \( Y \) \( \equiv 1 \) with \( Y \in \mathbb{V}(\mathbb{B}) \). Then
\[
\mathbb{V}(\mathbb{B}) \models \mathcal{P}_{\text{fin}}(X^\uparrow) = \mathcal{P}_{\text{fin}}(X)^\uparrow.
\]

**Lemma 4.3.** Let \( \emptyset \neq X \in \mathbb{V}, Y \in \mathbb{V}(\mathbb{B}) \), and \( [Y \neq \emptyset] = 1 \). Denote by \( \mathcal{F}(X,Y^\downarrow) \) the set of all functions from \( X \) to \( Y^\downarrow \) and let \( \mathcal{F}(X^\uparrow,Y)^\downarrow \) stand for the set of all functions from \( X^\uparrow \) to \( Y \) within \( \mathbb{V}(\mathbb{B}) \). Then the following hold:

1. If \( [g \text{ is a function from } X^\uparrow \text{ to } Y] = 1 \), then there exists a function \( g^\downarrow \) from \( X \) to \( Y^\downarrow \) uniquely determined by
\[
[g^\downarrow(x) = g(x^\uparrow)] = 1 \quad (x \in X).
\]
2. If \( f \) is a function from \( X \) to \( Y^\downarrow \), then there exists a function \( f^\uparrow \) from \( X^\uparrow \) to \( Y \) within \( \mathbb{V}(\mathbb{B}) \) determined uniquely by
\[
[f^\uparrow(x^\uparrow) = f(x)] = 1 \quad (x \in X).
\]
3. The mappings \( f \mapsto f^\uparrow \) and \( g \mapsto g^\downarrow \) are inverse to one another and establish bijections between \( \mathcal{F}(X,Y^\downarrow) \) and \( \mathcal{F}(X^\uparrow,Y)^\downarrow \).
4. \( [f^\uparrow(A^\uparrow) = f(A)^\uparrow] = 1 \) for every \( A \subset X \).

The above functors are applicable, in particular, to algebraic structures. Applying the transfer and maximum principles to the ZFC-theorem on the existence of the field of reals, we will find \( \mathcal{R} \in \mathbb{V}(\mathbb{B}) \), called the realis within \( \mathbb{V}(\mathbb{B}) \) satisfying
\[
[\mathcal{R} \text{ is the reals}] = 1 \quad \text{and} \quad [1^\uparrow \in \mathcal{R}^\uparrow \subset \mathcal{R}] = 1,
\]
where \( \mathbb{R} \in \mathbb{V} \) is the (standard) field of reals with unity 1. Gordon’s theorem [9] establishes the relationship between \( \mathbb{R}, \mathcal{R}, \) and \( \mathcal{R}^\downarrow \).

**Theorem 4.4.** (Gordon, 1977) The descent \( \mathcal{R}^\downarrow \) of \( \mathcal{R} \) (with the descended operations and order) is a universally complete vector lattice with weak order unity \( 1 := 1^\uparrow \). Moreover, the field \( \mathcal{R} \in \mathbb{V}(\mathbb{B}) \) can be chosen so that \( [\mathcal{R}^\uparrow \text{ is a dense subfield of } \mathcal{R}] = 1 \).

The detailed presentation of the proofs of Gordon’s theorem and the following two corollaries can be found in [18, Theorems 2.2.4 and 2.3.2].

**Corollary 4.5.** There is a Boolean isomorphism \( \chi \) from \( \mathbb{B} \) onto \( \mathbb{P}(\mathcal{R}^\downarrow) \) such that for all \( x, y \in \mathcal{R}^\downarrow \) and \( b \in \mathbb{B} \) we have
\[
\chi(b)x = \chi(b)y \iff b \leq [x = y],
\]
\[
\chi(b)x \leq \chi(b)y \iff b \leq [x \leq y].
\]
Corollary 4.6. The universally complete vector lattice \( R \downarrow \) with the descended multiplication is a semiprime \( f \)-algebra with the order and ring unity \( 1 := 1^\downarrow \). Moreover, for every \( b \in B \) the band projection \( \chi(b) \) acts as multiplication by the \( \chi(b) \).

Lemma 4.7. The following equivalences hold for a nonempty set \( A \subset R \downarrow \) and all \( a \in R \) and \( b \in B \):

\[
\begin{align*}
b \leq \left[ a = \sup(A) \right] & \iff \chi(b)a = \sup \chi(b)(A), \quad (6) \\
b \leq \left[ a = \inf(A) \right] & \iff \chi(b)a = \inf \chi(b)(A). \quad (7)
\end{align*}
\]

Definition 4.8. Take a normed space \( X := (X, \rho) \) within \( V(B) \), that is \( \rho: X \to R \) is a norm on a (real or complex) vector space \( X \). The descent \( X \downarrow \) of \( X \) is a pair \( (X \downarrow, \cdot) \), where \( \cdot := \rho \downarrow(\cdot) : X \downarrow \to R \downarrow \) is the descent of the internal norm \( \rho \).

If \( X \) is a Banach space within \( V(B) \), then the descent \( X := X \downarrow \) is a universally complete Banach–Kantorovich space over \( R \downarrow \); see [14, Theorem 5.4.1].

Theorem 4.9. For every LNS \( X \) over \( E \) with \( E = X \uparrow \perp \) and \( B := P(E) \) there exists a Banach space \( X' \) within \( V(B) \) unique to within a linear isometry and called the Boolean valued representation of \( X \) whose descent \( X' \downarrow \) is the universal completion of \( X \).

Proof. The proof can be found in [14, Theorem 8.3.2] and [17, Theorem 5.4.2]. \( \Box \)

5. Proofs. In this section \( B := P(E) \), and \( V(B) \) is the corresponding Boolean valued model of set theory.

Lemma 5.1. For every compact space \( S \) there exists a compact space \( \tilde{S} \) within \( V(B) \) such that \( \iota: S \to \tilde{S} \downarrow \) unique (up to homeomorphism) and such that \( \left[ \iota(S) \right] \) is dense in \( \tilde{S} \). The embedding \( \iota: S \to \tilde{S} \downarrow \) is defined as \( \iota: s \mapsto s^\wedge (s \in S) \).

Proof. For a compact space \( S \) there is exactly one uniformity \( \mathcal{U} \) on \( S \) that induces the original topology of \( S \); moreover, \( \mathcal{U} \) is totally bounded and complete [8, Theorems 8.3.13 and 8.3.16]. Working within \( V(B) \), we observe that \( \mathcal{U}^\wedge \) may fail to be a uniformity on \( S^\wedge \). However, \( S^\wedge \) becomes a uniform space when endowed with the uniformity base \( \mathcal{U}^\wedge \); this uniformity we will denote by the same symbol \( \mathcal{U}^\wedge \). Define the embedding \( \iota: S \to S^\wedge \) as \( s \mapsto s^\wedge (s \in S) \). By transfer every uniform space has exactly one (up to a uniform isomorphism) completion, [8, Theorems 8.3.12]. Denote by \( (\tilde{S}, \mathcal{W}^\wedge) \) the completion of this uniform space \( (S^\wedge, \mathcal{U}^\wedge) \) within \( V(B) \). This and the equation \( \left[ \iota(S) \right] = 1 \) (see Lemma 4.3(4)) imply that \( \iota(S) \) is dense in \( \tilde{S} \). Now, to ensure that \( \tilde{S} \) is compact it suffices to prove that \( (S^\wedge, \mathcal{U}^\wedge) \) is totally bounded; see [8, Corollary 8.3.17]. The latter amounts to
checking that for every \( U \in \mathcal{U}^\wedge \) there is a finite subset \( \theta \subset \mathcal{P}_{\text{fin}}(S^\wedge) \) that is \( U \)-dense in \((S^\wedge, \mathcal{U}^\wedge)\). In symbols,
\[
[(\forall U \in \mathcal{U}^\wedge) (\exists \theta \in \mathcal{P}_{\text{fin}}(S^\wedge)) S^\wedge \subset U(\theta)] = 1,
\]
where \( \mathcal{P}_{\text{fin}}(S^\wedge) \) is the collection of all finite subsets of \( S^\wedge, U(\theta) = \bigcup_{s \in \theta} U(s) \), and \( U(s) := \{ t \in S : (s, t) \in U \} \). Appreciating the equalities \( \mathcal{P}_{\text{fin}}(S^\wedge) = \mathcal{P}_{\text{fin}}(S^\wedge) \) (see Lemma 4.2) and \( U(\theta)^\wedge = U(\theta)^\wedge \) (see [17, Theorem 3.1.5(2)]), the simple rules for calculating Boolean truth values for quantifiers over standard names (see Lemma 4.1(1)), we arrive at the equivalent statement
\[
1 = \bigvee \{ [S^\wedge \subset U(\theta)^\wedge] : \theta \in \mathcal{P}_{\text{fin}}(S^\wedge) \}.
\]
Since the Boolean truth value \([S^\wedge \subset U(\theta)^\wedge]\) can take only the two values \( 0 \in \mathbb{B} \) and \( 1 \in \mathbb{B} \) (Lemma 4.1(2)), we obtain another equivalent form of the desired statement: For every \( U \in \mathcal{U} \) there is \( \theta \in \mathcal{P}_{\text{fin}}(S^\wedge) \) with \( S \subset U(\theta) \); i.e., \( S \) is totally bounded. The last claim is true by our hypothesis.

**Remark 5.2.** The internal compact space \( \tilde{S} \) is often called the *Boolean extension* of \( S \). The general theory of Boolean extensions of uniform spaces was built by Gordon and Lyubetskii [10]. The Boolean valued interpretation of compactness gives rise to the notion of a *cyclically compact space* in such a way that the descent \( \tilde{S} \downarrow \) turns out to be cyclically compact; see [14, Section 8.5]. The equivalent concept of *mix-compact* subset of an LNS space was treated in Gutman and Lisovskaya [13]; see also [18, 2.12.B, 2.12.C, and 2.12.13].

Define \( C(\tilde{S}, \mathcal{X}) \in \mathbb{V}^{(\mathbb{B})} \) to be the set of continuous functions \( f : \tilde{S} \to \mathcal{X} \) within \( \mathbb{V}^{(\mathbb{B})} \); i.e., \([C(\tilde{S}, \mathcal{X}) \text{ is the space of continuous functions from } \tilde{S} \text{ to } \mathcal{X} \text{ with the supremum norm } \| f \| := \sup_{s \in \tilde{S}} \| f(s) \|_{\mathcal{X}}] = 1\).

**Lemma 5.3.** Let \( X \) be a Banach–Kantorovich space and let \( \mathcal{X} \) be the Boolean valued representations of \( X \). For every \( f \in C_{\text{ro}}(S, X) \) there exists a unique \( \tilde{f} \in \mathbb{V}^{(\mathbb{B})} \) such that \([ \tilde{f} \in C(\tilde{S}, \mathcal{X}) \] = 1 and \([ f(u)^\wedge = f(u)^\wedge] = 1 \) for all \( u \in S \). The mapping \( f \mapsto \tilde{f} \) is an linear isometry (in the sense of lattice valued norms) from \( C_{\text{ro}}(S, X) \) into \( C(\tilde{S}, \mathcal{X}) \).

**Proof.** Without loss of generality we may assume that \( X = \mathcal{X} \downarrow \) for some Banach space within \( \mathbb{V}^{(\mathbb{B})} \). Let \( f \) be a uniformly order continuous function from \( S \) to \( X \). Let the function \( f_\uparrow : S^\wedge \to X \) within \( \mathbb{V}^{(\mathbb{B})} \) be defined as in Lemma 4.3(2). If \( f_\uparrow \) is uniformly continuous then \( f_\uparrow \) extends uniquely to some continuous functions \( \tilde{f} : \tilde{S} \to \mathcal{X} \). By transfer the restriction operator \( g \mapsto g|_{S^\wedge} \) from \( C(\tilde{S}, \mathcal{X}) \) to \( C_{\text{ro}}(S^\wedge, \mathcal{X}) \) is an isometric lattice isomorphism of Banach lattices within \( \mathbb{V}^{(\mathbb{B})} \). Then the Banach–Kantorovich lattices \( C(\tilde{S}, \mathcal{X}) \downarrow \) and \( C_{\text{ro}}(S^\wedge, \mathcal{X}) \downarrow \) are also isometrically lattice isomorphic. So it remains to prove that \( C_{\text{ro}}(S, X) \) and \( C_{\text{ro}}(S^\wedge, \mathcal{X}) \downarrow \) are isometrically lattice isomorphic. In virtue of Lemma 4.3(3) it suffices to ensure that \( f_\uparrow \) is uniformly continuous if and only if \( f \) is uniformly order continuous. Define \( D_U, d_U \in \mathbb{V} \) and \( \delta_U, \hat{\delta}_U \in \mathbb{V}^{(\mathbb{B})} \) by
\[
D_U := \{|f(s_1) - f(s_2)| : (s_1, s_2) \in U\}, \quad d_U := \sup(D_U) \in E;
\]
$$\mathcal{D}_U := \{ \| f\uparrow(s_1) - f\uparrow(s_2) \| : (s_1, s_2) \in U^\uparrow \}, \quad \delta_U := \sup \mathcal{D} \in \mathcal{R}.$$ Observe that $\|D_U\uparrow = \mathcal{D}_U\| = \mathcal{1}$ by Lemma 4.3(4) and $\|d_U\uparrow = \delta_U\| = \mathcal{1}$ by Lemma 4.7 (formula (6)). Applying Lemma 4.7 again (formula (7)), we conclude that $\inf(D_U) = 0$ if and only if $\inf(\mathcal{D}_U) = 0 = \mathcal{1}$, as required.

Lemma 5.4. Let $\tilde{\Sigma} \in \mathcal{V}(B)$ be defined as $\tilde{\Sigma} = \{ \delta_s : s \in \tilde{S} \}$, where $\delta_s : C(\tilde{S}, \mathcal{R}) \to \mathcal{R}$ is the point evaluation $\varphi \mapsto \varphi(s)$ within $\mathcal{V}(B)$ determined by a given $s \in \tilde{S}$. Then the mapping $j : s \mapsto (\delta_s)\downarrow$ is a bijection from $\tilde{\Sigma}\downarrow$ onto $\Sigma$. Moreover, if $j(\tilde{\sigma}) = \sigma$ and $j(\tilde{\tau}) = \tau$ for some $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Sigma}$, then $[\tilde{\sigma} = \tilde{\tau}] = [\sigma = \tau]$. Proof. The mapping $\varkappa : \varphi \mapsto \tilde{\varphi}$ is a linear isometry (in the sense of lattice valued norms) from $C_{ro}(S, \mathcal{R}\downarrow)$ onto $C(\tilde{S}, \mathcal{R}\downarrow)$; see Lemma 5.3. Moreover, $\varkappa$ is a lattice isomorphism in view of the easily verifiable fact that $\varphi \in C_{ro}(S, \mathcal{R}\downarrow)$ is positive if and only if so is $\tilde{\varphi}$ within $\mathcal{V}(B)$. In this way we may identify $C_{ro}(S, \mathcal{R}\downarrow)$ and $C(\tilde{S}, \mathcal{R}\downarrow)$. If $\sigma$ is a $\mathcal{R}\downarrow$-linear positive operator from $C_{ro}(S, \mathcal{R}\downarrow)$ to $\mathcal{R}\downarrow$ and $\tilde{\sigma} := \sigma \circ \varkappa$, then $\tilde{\sigma}\uparrow$ is a positive linear functional on $C(\tilde{S}, \mathcal{R})$ within $\mathcal{V}(B)$. By transfer, $\tilde{\sigma}\uparrow \in \tilde{\Sigma}$ if and only if $\tilde{\sigma}\uparrow$ is a lattice homomorphism and $\tilde{\sigma}\uparrow(\mathcal{1}_\Sigma) = 1$. It can be easily checked that the latter means that $\sigma$ is a lattice homomorphism and $\sigma(\tilde{e}) = e$ for all $e \in \mathcal{R}\downarrow$. The equality $[\tilde{\sigma} = \tilde{\tau}] = [\sigma = \tau]$ is immediate from (5).  

Lemma 5.5. Let $E = \mathcal{R}\downarrow$ and $X = \mathcal{X}\downarrow$ with $\mathcal{X}$ being a Banach space within $\mathcal{V}(B)$, Consider $W \subset C_{ro}(S, X)$ and define $\tilde{W} \in \mathcal{V}(B)$ by the formula $\tilde{W} := \{ \tilde{f} : f \in W \} \uparrow$. Then the following hold:

1. $[\tilde{W} \subset C_{ro}(\tilde{S}, \mathcal{X}\downarrow)] = \mathcal{1}$ and $[\tilde{\mu}(W) \subset \mu(W)] = \mathcal{1}$.
2. If $\mu(W)$ separates the points of $\Sigma$, then $[\mu(W) \subset \mathcal{R}\uparrow(\mathcal{1}_\Sigma)] = \mathcal{1}$.

Proof. (1) : The first inclusion is obvious. To prove the second, start with observing that the mapping $\psi \mapsto \tilde{\psi}$ is a bijection from $C(\tilde{S}, [0, 1]\downarrow)$ onto $C_{ro}(S, [0, 1]\downarrow)$. Indeed, if $\varphi = \tilde{\psi}$, then $\varphi(\mathcal{S}\uparrow) = \psi(S)\uparrow$ by Lemma 4.3(4); and, as $\varphi(\tilde{S})$ is the closure of $\varphi(\mathcal{S}\uparrow)$, we have $\varphi(\tilde{S}) \subset [0, 1]$ if and only if $\psi(S)\uparrow = \operatorname{mix}(\psi(S)) \subset [0, 1]$ if and only if $\psi(S) \subset [0, 1]$. Now, using (4) and the formulas for computing Boolean truth values for quantifiers over ascents (see [18, 1.6.2]), for every $\varphi \in C(\tilde{S}, [0, 1]\downarrow)$ we have

$$[\varphi \in \tilde{\mu}(W)] = \bigvee_{\psi \in \mu(W)} [\varphi = \tilde{\psi}]$$

$$= \bigvee_{\psi \in \mu(W)} [\varphi = \tilde{\psi}] \land [\tilde{\psi}W + (1 - \tilde{\psi})\mathcal{W} \subset \mathcal{W}]$$

$$\leq \bigvee_{\psi \in \mu(W)} [\varphi\mathcal{W} + (1 - \varphi)\mathcal{W} \subset \mathcal{W}]$$

$$= [\varphi\mathcal{W} + (1 - \varphi)\mathcal{W} \subset \mathcal{W}]$$

$$= [\varphi \in \mu(W)].$$
Proof of Theorem 3.12.

This is equivalent to the equation \( \sigma \neq \tau \Rightarrow \forall \{ \sigma(\varphi) \neq \tau(\varphi) : \varphi \in \mu(W) \} = 1 \) for all \( \sigma \tau \in \Sigma \). Note also that if \( \varphi = \psi \), then \( \{ \sigma(\varphi) = \tau(\varphi) \} = \{ \sigma = \tau \} \) and \( \{ \sigma(\varphi) = \tau(\varphi) \} = \{ \sigma(\psi) = \tau(\psi) \} \) by (2), (3), and (5). Denote \( b := [\mu(W) \text{ separates the points of } \Sigma] \). Easy calculations of Boolean truth values yield

\[
b = [\forall \sigma, \tau \in \tilde{\Sigma}(\sigma \neq \tau) \Rightarrow (\exists \varphi \in \mu(W) \varphi(\sigma) \neq \varphi(\tau))]
\]

\[
= \bigwedge_{\sigma, \tau \in \Sigma} \left( [\sigma \neq \tau] \Rightarrow \bigvee_{\psi \in \mu(W)} [\sigma(\psi) \neq \tau(\psi)] \right)
\]

\[
= \bigwedge_{\sigma, \tau \in \Sigma} \left( [\sigma \neq \tau] \Rightarrow \bigvee_{\psi \in \mu(W)} [\sigma(\psi) \neq \tau(\psi)] \right)
\]

\[
= 1,
\]

where \( c \Rightarrow d := c^+ \lor d \) for all \( c, d \in \mathbb{B} \). By (1) \( \mu(\tilde{W}) \) is wider then \( \mu(W) \); so that \( [\mu(\tilde{W}) \text{ separates the points of } \tilde{\Sigma}] = 1 \), as claimed. \( \square \)

Proof of Theorem 3.8.

Proof. There is no loss of generality in assuming that \( \Lambda = R \downarrow \) and \( X = X \downarrow \), where \( X \) is a Banach space within \( V(B) \). By Lemma 5.5, \( [\tilde{W} \subset C_{ro}(\tilde{S}, \tilde{X})] = 1 \) and \( [\mu(\tilde{W}) \text{ separates the points of } \tilde{S}] = 1 \). Take \( f \in C_{ro}(\tilde{S}, X) \) and note that \( \tilde{f} \in C_{ro}(\tilde{S}, \tilde{X}) \) by Lemma 5.3. Using transfer enables us to apply Theorem 1.1 within \( V(B) \), so that for all \( 0 < \varepsilon \in \mathbb{R}^\infty \) the following are equivalent:

1. There exists \( \tilde{g} \in \tilde{W} \) with \( \| \tilde{f} - \tilde{g} \| < \varepsilon \).
2. For every \( s \in \tilde{S} \) there exists \( \tilde{g}_s \in \tilde{W} \) such that \( \| \tilde{f}(s) - \tilde{g}_s(s) \|_{\tilde{X}} < \varepsilon \).

According to Lemma 5.3 there exist \( g \in C_{ro}(S, X) \) such that \( [\tilde{g}(u^\uparrow) = g(u)] = 1 \) for all \( u \in S \). Clearly, \( [\tilde{f} - \tilde{g} = (f - g)] = 1 \) and \( [\| f - g \| = \| \tilde{f} - \tilde{g} \|] = 1 \), so (1) is equivalent to \( \inf \{ \| f - g \| : g \in W \} = 0 \) in view of Lemma 4.4. Indeed, if \( A := \{ \| f - g \| : g \in W \} \) and \( B = \{ \| \tilde{f} - \tilde{g} \| : \tilde{g} \in \tilde{W} \} \), then for arbitrary \( a \in \mathbb{R}^\downarrow \) we have

\[
[a \in A^\uparrow] = \bigvee \{ [a = u] : u \in A \}
\]

\[
= \bigvee \{ [a = \| f - g \|] : g \in W \} = \bigvee \{ [a = \| \tilde{f} - \tilde{g} \|] : \tilde{g} \in \tilde{W} \} = [a \in B],
\]

hence \( A^\uparrow = B \). Taking into account Lemma 5.5, we can show in a similar way that \( [\tilde{f}(s) - \tilde{g}(s) = (f - g)(s)] = 1 \) and \( [\| f(s) - g(s) \| = \| \tilde{f}(s) - \tilde{g}(s) \|] = 1 \), so the assertion (2) is equivalent to \( \inf \{ \| f(s) - g(s) \| : g \in W \} = 0 \) for all \( s \in \Sigma \). \( \square \)

Proof of Theorem 3.12.
Proof. Using the above notation, we see that $W$ and $f$ satisfy the conditions of Theorem 1.2 within $\mathcal{V}(B)$. Define $A, B(\sigma) \in \mathcal{V}$ and $\tilde{A}, \tilde{B}(\sigma) \in \mathcal{V}(B)$ by

$$A := \{|f - g| : g \in W\}, \quad B(\sigma) := \{|f(\sigma) - x|_X : x \in W(\sigma)\};$$

$$\tilde{A} := \{\|\tilde{f} - g\| : g \in \tilde{W}\}, \quad \tilde{B}(\sigma) := \{\|\tilde{f}(\sigma) - x\|_X : x \in \tilde{W}(\sigma)\}.$$ 

It can be shown as above that $[\tilde{A} = A^\uparrow] = 1$ and $[\tilde{B}(\sigma) = B(\sigma)^\uparrow] = 1$. By transfer there exists $\sigma \in \tilde{\Sigma}$ such that $[\tilde{A} = \tilde{B}(\sigma)] = 1$. It remains to apply Lemma 4.4 to obtain the required equality $A = B(\sigma)$.

Remark 5.6. Theorems 3.8 and 3.12 contain Theorems 1.1 and 1.2 as particular cases. At the same time, as seen from the above, Theorems 3.8 and 3.12 are the Boolean-valued interpretations of Theorems 1.1 and 1.2. Of course, the proofs avoiding Boolean valued analysis are also possible.

Using the same technique, we can formulated negative results on order approximation by the cyclic hulls of sublattices and Grothendieck subspaces; cp. [18, Chapter 3].

Theorem 5.7. Let $E$ be a Dedekind complete vector lattice, while $S$ is a compact space, $\Lambda = \text{Orth}(E)$, and $X = C_{ro}(S, E)$. Then

(1) a sublattice $L$ of $X$ is such that $\text{mix}(L)$ does not coincide with $X$ if and only if there are $\Lambda$-linear lattice homomorphisms $S, T : X \to E$ satisfying $L \subset \ker(S - T);$  
(2) A Grothendieck subspace $L$ of $X$ is such that $\text{mix}(L)$ does not coincide with $X$ if and only if there are $\Lambda$-linear lattice homomorphisms $S, T : X \to E$ satisfying $L \subset \ker(S + T).$

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References

1. C.D. Aliprantis and O. Burkinshaw, Positive Operators, Springer, Dordrecht, 2006.
2. A.Kh. Asgarova, On a generalization of the Stone–Weierstrass theorem, Ann. Math. Quèbec 42 (2018), 1–6.
3. J.L. Bell, Boolean-Valued Models and Independence Proofs in Set Theory, Clarendon Press, New York etc., 1985.
4. E. Bishop, A generalization of the Stone-Weierstrass theorem, Pacific J. Math. 11 (1961), 777–783.
5. F.F. Bonsall, J. Lindenstrauss, and R.R. Phelps, Extreme positive operators on algebras of functions, Math. Scand. 18(2) (1966), 161–182.
6. G. Buskes and A. Rooij, Hahn-Banach for Riesz homomorphisms, *Indag. Math.* **92** (1989), 25–34.

7. E.Y. Emelyanov, S.G. Gorokhova, and S.S. Kutateladze, Unbounded order convergence and the Gordon theorem, *Vladikavkaz Mat. Zh.* **21**(4) (2019), 56–62.

8. R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

9. E.I. Gordon, Real numbers in Boolean-valued models of set theory, and $K$-spaces, (Russian), *Dokl. Akad. Nauk SSSR* **237** (1977), 773–775.

10. E.I. Gordon and V.A. Lyubetskii, Some applications of nonstandard analysis in the theory of Boolean-valued measures, (Russian), *Dokl. Akad. Nauk SSSR* **256** (1981), 1037–1041.

11. A.E. Gutman, Banach bundles in the theory of lattice-normed spaces, III, Approximating sets and bounded operators, *Siberian Adv. Math.* **4** (1994), 54–75.

12. __________, Disjointness preserving operators, In: *Vector Lattices and Integral Operators*, (Ed. S.S. Kutateladze), pp. 359–454, Kluwer, Dordrecht, 1996.

13. A.E. Gutman and S.A. Lisovskaya, The boundedness principle for lattice-normed spaces, *Siberian Math. J.* **50** (2009), 830–837.

14. A.G. Kusraev, *Dominated Operators*, Kluwer, Dordrecht, 2000.

15. A.G. Kusraev and S.S. Kutateladze, Analysis of subdifferentials with the aid of Boolean-valued models, *Dokl. Akad. Nauk SSSR* **265** (1982), 1061–1064.

16. __________, *Subdifferentials: Theory and Applications*, Kluwer Academic Publishers, Dordrecht, 1995.

17. __________, *Boolean Valued Analysis*, Dordrecht, Kluwer, 1999.

18. __________, *Boolean Valued Analysis: Selected Topics*, Vladikavkaz, SMI VSC RAS (2014). (Trends in Science: The South of Russia. A Math. Monogr. 6.)

19. S.S. Kutateladze, The Krein-Mil’man theorem and its inverse, *Siberian Math. J.* **21** (1980), 130–138.

20. W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces*, Vol. 1, North Holland, Amsterdam/London, 1971.

21. S. Machado, On Bishop’s generalization of the Stone-Weierstrass theorem, *Indag. Math.* **39** (1977), 218–224.

22. J.B. Prolla, *Weierstrass-Stone: The Theorem*, Peter Lang, Frankfurt am Main, 1993.

23. __________, The Weierstrass-Stone theorem, *J. Approx. Theory* **78** (1994), 299–313.

24. T.J. Ransford, A short elementary proof of the Bishop-Stone-Weierstrass theorem, *Math. Proc. Camb. Phil. Soc.* **96** (1984), 309–309.

25. V. Timofte, Stone-Weierstrass theorems revisited, *J. Approx. Theory* **136** (2005), 45–59.

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