A large-$N$ analysis of the local quantum critical point and the spin-liquid phase

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We study analytically the Kondo lattice model with an additional nearest-neighbor antiferromagnetic interaction in the framework of large-$N$ theory. We find that there is a local quantum critical point between two phases, a normal Fermi-liquid and a spin-liquid in which the spins are decoupled from the conduction electrons. The local spin susceptibility displays a power-law divergence throughout the spin liquid phase. We check the reliability of the large-$N$ results by solving by quantum Monte Carlo simulation the $N = 2$ spin-liquid problem with no conduction electrons and find qualitative agreement. We show that the spin-liquid phase is unstable at low temperatures, suggestive of a first-order transition to an ordered phase.

The effects of the vicinity to a quantum critical point (QCP) on the low-temperature physical properties of various systems are presently attracting great attention [1]. There is in particular growing consensus that the anomalous metallic properties observed in some heavy-fermion materials as well as the presence of superconductivity near magnetic phases could be interpreted in terms of the proximity to a QCP [2]. The issue of QCPs in superconducting cuprates is also a hotly debated subject [3].

In the traditional picture of the QCP in metallic systems [4,5] collective spin-fluctuations of specific wavevectors become critical at the QCP. A substantially different scenario has recently been suggested [6] in which the quantum phase transition involves all wavevectors simultaneously. At this local QCP the local degrees of freedom develop long-range correlations in time [8] leading to a divergent local susceptibility. At the onset of magnetism the energy scale that characterizes the coherent metallic phase vanishes.

It was proposed [6] that this peculiar type of behavior arises in a particular Kondo-lattice model with nearest-neighbor magnetic RKKY interactions. The model was studied within Extended Dynamical Mean-Field Theory (EDMFT) which reduces the problem to an effective impurity model in which a localized spin interacts with fermionic and bosonic baths [10,11]. However, some important aspects of the self-consistent procedure used in Ref. [6] to close the set of EDMFT equations require further scrutiny [8].

In this paper we use a large-$N$ approach to solve analytically the Kondo-lattice-RKKY model of Ref. [6] fully implementing the self-consistent EDMFT scheme. We supplement this large-$N$ analysis with a Quantum Monte Carlo (QMC) investigation of the Kondo-decoupled phase of interacting spins represented by the pure RKKY model.

We start from the following Hamiltonian,

$$H = \sum_{\langle i,j \rangle, \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + J_K \sum_i s_i \cdot S_i + \sum_{\langle i,j \rangle} J_{ij} S_i \cdot S_j \ . \ (1)$$

The operators $c_{i\sigma}^\dagger$ and $S_i$ represent respectively the creation of a conduction electron of spin $\sigma = \uparrow, \downarrow$ and a localized spin at the $i$th site of a regular lattice of size $N$. We denote by $s_i$ the local spin density of the conducting electrons. The hopping terms $t_{ij}$ correspond to a non interacting electronic density of states $\rho_0(\epsilon) = 1/N \sum_k \delta(\epsilon - \epsilon_k)$ and the antiferromagnetic RKKY couplings $J_{ij}$ are described by a spectral density $\rho_F(\epsilon) = 1/N \sum_k \delta(\epsilon - J_k)$ where $\epsilon_k$ and $J_k$ are the Fourier transforms of the nearest neighbors couplings $t_{ij}$ and $J_{ij}$, respectively. We assume for simplicity that $\rho_0^\dagger$ and $\rho_F$ are even functions that vanish outside the intervals $[-D, D]$ and $[-J, J]$, respectively.

In the EDMFT approach the above model is implemented on a lattice with a large coordination number $z$, so that both the hopping and the magnetic RKKY coupling are scaled as $t_{ij} \rightarrow t_{ij}/\sqrt{z}$, $J_{ij} \rightarrow J_{ij}/\sqrt{z}$.

To leading order in a large-$z$ expansion we obtain the following effective action for the local degrees of freedom on the 0-th site (say) [13]:

$$A = -\int_0^\beta \int_0^\beta \sum_{\sigma = \uparrow, \downarrow} d\tau d\tau' \ c_{\sigma}^\dagger(\tau) G^{-1}_0(\tau - \tau') c_\sigma(\tau')$$
\[ + J_K \int_0^\beta d\tau s(\tau) \cdot S(\tau) \]
\[ - \int_0^\beta \int_0^\beta d\tau d\tau' \tilde{\chi}_0^{-1}(\tau - \tau') S(\tau) \cdot S(\tau') \ , \ (2)\]

where the correlators $G^{-1}_0(\tau)$ and $\tilde{\chi}_0^{-1}(\tau)$ are self-consistently determined cavity fields [9-11].

There are two steps in the implementation of the EDMFT procedure [9-11]. The first (and usually most difficult) one consists in solving the local impurity problem for fixed $G^{-1}_0$ and $\tilde{\chi}_0^{-1}$ in order to find the spin-symmetric local electronic Green function $G_e(\tau) =$
\(-\langle T c_\sigma(\tau) c^\dagger_\sigma(0) \rangle\) and the local magnetic susceptibility \(\chi_{\text{loc}}(\tau) = (\mathcal{T} S(\tau) \cdot S(0))\). Next, the condition that the impurity site is equivalent to any other lattice site is imposed through a set of self-consistency relations that allow us to express the cavity fields \(G_0^{-1}\) and \(\tilde{\chi}_0^{-1}\) in terms of \(G_c\) and \(\chi_{\text{loc}}\).

Here we implement this procedure in the framework of a large-\(N\) expansion [12]. The SU(2) spins are replaced by SU(\(N\)) operators in the fermionic representation \(S_{\sigma, \sigma'} = f_\sigma f_{\sigma'} - \delta_{\sigma, \sigma'}/2\) with \(\sigma, \sigma' = 1, \ldots, N\). The \(f\)-fermions are subject to the constraint \(\sum_{\sigma=1}^N f_\sigma^\dagger f_\sigma = N/2\), enforced through the introduction of a Lagrange multiplier \(\lambda f(\tau)\) [12]. The coupling constants must now be rescaled so that physical quantities remain finite in the \(\mathcal{N} \to \infty\) limit. A consistent rescaling of all the couplings in the Hamiltonian (1) is not possible without losing the dynamical character of the spin fluctuations. We adopt instead a different approach in which we rescale the couplings in the action (2) as \(J_K \to J_K/N\) and \(\tilde{\chi}_0 \to \lambda_0^{-1}/N\), and define the local susceptibility per spin component \(\chi_{\text{loc}}(\tau) = N^{-2} \sum_{\sigma=1}^N \langle f_\sigma^\dagger(\tau) f_\sigma(\tau) \rangle f_\sigma(0) f_\sigma(0)\). With this rescaling all the terms in Eq. (2) are \(\mathcal{O}(N)\).

In the \(\mathcal{N} \to \infty\) limit, the hybridization parameter \(\tau = -J_K(c_\sigma f_\sigma)\), the Lagrange multiplier \(\lambda\) and the chemical potential \(\mu\) are determined from the saddle point conditions that may be written in the compact form

\[
\left\{ -\frac{r}{J_K}, \frac{1}{2}, \frac{n_c}{2} \right\} = -\{ G_{fc}, G_f, G_c \} (\tau = 0) .
\]

\(G_f(\tau) = -\langle T f_\sigma(\tau) f_\sigma^\dagger(0) \rangle\), \(G_c(\tau) = -\langle T c_\sigma(\tau) c_\sigma^\dagger(0) \rangle\) and \(G_{fc}(\tau) = -\langle T f_\sigma(\tau) c_\sigma^\dagger(0) \rangle\) are the dressed \(f\)-fermion, conduction electron and mixed Green functions. The effect of spin fluctuations is embodied in a self-energy correction \(\Sigma_f(\tau) = -2\tilde{\chi}_0^{-1}(\tau) G_f(-\tau)\) to the bare \(f\) propagator, \(G_f^{-1}(\omega_n) \equiv i\omega_n + \lambda - \Sigma_f(\omega_n)\). The dressed Green functions are \(G_c(\omega_n) = G_0^0(i\omega_n + \mu + r^2 G_f(\omega_n))\), \(G_f(\omega_n) = G_f^0(\omega_n) + r^2 G_f^0(\omega_n) G_c(\omega_n)\) and \(G_{fc}(\omega_n) = r G_f^0(\omega_n) G_c(\omega_n)\), respectively. The usually difficult step of obtaining \(\chi_{\text{loc}}\) from the bath \(\tilde{\chi}_0^{-1}\) can easily be performed in the large-\(N\) limit in which

\[
\chi_{\text{loc}}(\tau) = -G_f(\tau) G_f(-\tau) .
\]

The parameter \(r\) describes the binding of the \(c\) electrons to the \(f\) ("spin") degrees of freedom. If \(r\) is finite there is Kondo compensation of the localized spin. For sufficiently small values of \(J\) we find a Fermi-liquid (FL) phase with \(r \neq 0\) for temperatures \(T \leq T_K(J)\), the Kondo temperature of the system. In this low-\(T\) and low-\(J\) region, the FL is characterized by a finite coherence energy scale \(\epsilon_{\text{FL}} \propto T_K\) and a large Fermi surface containing \(1 + n_c\) states (Luttinger theorem). The physical properties of this heavy Fermi liquid are similar to those discussed in Ref. [13]. In particular, the local magnetic susceptibility \(\chi_{\text{loc}}(T = 0, \omega = 0)\) remains finite.

Upon increasing \(J\), the physical quantities characterizing the Kondo-screened FL \((\tau, T_K, \epsilon_{\text{FL}}\) gradually decrease and vanish at a critical value of the magnetic coupling \(J_c \sim T_K^0\), the Kondo temperature for \(J = 0\). At the same time \(\chi_{\text{loc}}(T = 0, \omega = 0)\) increases continuously and diverges at \(J = J_c\). The paramagnetic solution with \(r = 0\) can be described in terms of a gas of free \(c\) electrons with a small Fermi surface enclosing \(n_c\) states decoupled from a spin-liquid (SL) of strongly correlated \(f\) fermions. Notice that, since our large-\(N\) approach neglects the fluctuations of \(r\), we cannot distinguish between the critical behavior at the QCP and that inside the SL phase.

Of course, a global magnetic instability may occur at a finite temperature \(T = T_c\) before this local instability of the Fermi surface has a chance to develop. This will happen if \(\chi(q = Q, \omega = 0, T)\) diverges at \(T = T_c\), where \(J_Q = -J\) corresponds to the lower edge of the RKKY spectral density \(\rho_I\). In this case, the local QCP behavior will be hidden by the magnetic instability but its effects may still be observable provided that spin ordering is established at sufficiently low temperatures.

In order to proceed we must fix the actual prescription used to close the self-consistency loop relating \(\tilde{\chi}_0^{-1}\) to \(\chi_{\text{loc}}\). Following the EDEFT scheme of Refs. [6,10,11], we introduce the local conduction electron and magnetic self-energies, \(\Sigma_c(\omega_n) = G_0^{-1}(\omega_n) - G_c^{-1}(\omega_n)\), and \(M(\omega_n) = \chi_0^{-1}(\omega_n) + \chi_{\text{loc}}^{-1}(\omega_n)\). These quantities may be expressed in terms of \(\mathcal{O}(N^0)\) physical quantities as

\[
G_c(\omega_n) = \frac{1}{N} \sum_k G_c(k, \omega_n) = \int_{-\infty}^{+\infty} \frac{d\epsilon}{i\omega_n - \Sigma_c(\omega_n) - \epsilon} ,
\]

\[
\chi_{\text{loc}}(\omega_n) = \frac{1}{N} \sum_q \chi(q, \omega_n) = \int_{-\infty}^{+\infty} \frac{d\epsilon}{M(\omega_n) + \epsilon} .
\]

with \(\chi^{-1}(q, \omega_n) = M(\omega_n) + J_q\) and \(G_c^{-1}(k, \omega_n) = \omega_n - \Sigma_c(\omega_n) - \epsilon_k\).

The results depend crucially on the behavior of \(\rho_I(\epsilon)\) near the band edge. We discuss first the case in which the RKKY spectral density \(\rho_I(\epsilon)\) is finite at the lower edge of the band, \(\rho(-J) \equiv \rho_I\), a situation that arises when the spin correlations have a two-dimensional character [6,10]. Then we may explicitly integrate Eq. (6) to find

\[
\chi_{\text{loc}}(\omega_n) = \rho_I \ln \left[ \frac{M(\omega_n) + J}{M(\omega_n) - J} \right] - R(\omega_n) ,
\]

where \(R(\omega_n)\) is a regular function of frequency. Notice that in this case a second-order magnetic transition [signaled by \(M(\omega_n = 0) = J\)] is necessarily accompanied by a divergence of \(\chi_{\text{loc}}\). Such a divergence at \(T = 0\) can only take place when \(r\) vanishes, that is at the quantum critical point, \(J = J_c\), and inside the SL phase \(J > J_c\).

A complete solution of the model at the QCP and in the SL phase can be obtained in the large-\(N\) approach by solving Eqs. (3) and (4) for \(r = 0\) supplemented with the relation
\( \chi_0^{-1}(\omega_n) = J \coth \left[ \frac{\chi_{\text{loc}}(\omega_n) + R(\omega_n)}{2\rho_I} \right] - \frac{1}{\chi_{\text{loc}}(\omega_n)} , \tag{8} \)

which follows easily from Eq. (7).

Since the solution obtained from the RG treatment for the local impurity problem [6] yields a power-law behavior for \( M \) and a logarithmic divergence for \( \chi_{\text{loc}} \) at low \( \omega \) and \( T \), we first check the validity of these results within our large-\( N \) scheme. We thus assume that, at low frequency, \( M(\omega_n) - J \sim (|\omega_n|/J)^\gamma \) and \( \chi_{\text{loc}}(\omega_n) = \gamma \rho_I \ln (J/|\omega_n|) \). Analytic continuation to real frequencies leads to:

\[
\text{Im} \left[ \chi_0^{-1}(\omega) \right] = \frac{\pi \text{sgn}(\omega)}{2\gamma \rho_I \ln^2 |J/\omega|}.
\tag{9}
\]

The imaginary part of the self-energy \( \Sigma_f(\omega) \) for real frequencies is

\[
\Sigma_f''(\omega) = 2 \int_{-\infty}^{\infty} d\varepsilon \text{Im} \left[ \chi_0^{-1}(\varepsilon) \right] \rho_f(\omega - \varepsilon) \times \left[ n_B(\varepsilon) + n_F(\omega - \varepsilon) \right].
\tag{10}
\]

In the low-frequency regime, with the (self-consistently verified) assumption that \( \Sigma_f(\omega) \) dominates \( \omega \), the corresponding fermionic Green function is given by \( G_f(\omega) \approx -\Sigma_f^{-1}(\omega) \). Taking then \( \rho_f(\omega) = \sqrt{\rho_I/(2\pi)} \ln |J/\omega|/|\omega| \) corresponding to \( \Sigma_f''(\omega) = -\sqrt{|\omega|/\left( \sqrt{\gamma \rho_I} \ln |J/\omega| \right)} \), and with the correlator of Eq. (9), one can check that Eq. (10) is verified. Using these results in Eq. (4) we find \( \chi_{\text{loc}}(\omega_n) = (\gamma \rho_I/6\pi) \ln |J/\omega_n|^3 \). We now notice that this finding is inconsistent with the previously assumed simply logarithmic behavior of \( \chi_{\text{loc}} \). On the other hand, one can further check that an Ansatz \( \rho_f(\omega) \sim 1/\sqrt{|J/\omega|} \) (necessary to get a purely logarithmic \( \chi_{\text{loc}} \)) does not verify Eq. (10). All the above shows that, within our large-\( N \) scheme, a correlator of the form \( \chi_{\text{loc}}^{-1}(\omega_n) \propto \ln^{-1}(J/|\omega_n|) \) produces additional logarithmic corrections to the local susceptibility, which prevent self-consistency.

On the other hand, one can still find self-consistent solutions of the problem. Starting from the Ansatz

\[
\chi_{\text{loc}}(\omega_n) = (A/J) |\omega_n|/J)^{-\delta},
\tag{11}
\]

\[
G_f(\omega_n) = -i(B/J) |\omega_n|/J)^{-\alpha},
\tag{12}
\]

diverging at low frequencies, one finds \( \text{Im} \chi_0^{-1}(\omega) \approx (J/A)(|\omega|/J)^\delta \cos(\delta \pi/2) \). Inserting these expressions in Eq. (10) one can determine \( (T = 0, \omega > 0) \)

\[
\Sigma_f''(\omega) = -(2JB/\pi A)(\omega/J)^{-\delta -1} \\
\times \sin (\pi \delta/2) \cos (\pi \alpha/2) \int_0^1 dx x^{\delta}(1-x)^\alpha.
\tag{13}
\]

Assuming that \( \delta < \alpha \) (that will be self-consistently verified), one has \( G_f(\omega) \approx -\Sigma_f^{-1}(\omega) \). Compatibility between Eqs. (12) and (13) requires

\[
2\alpha = \delta + 1 , \tag{14}
\]

\[
\frac{\pi A}{B^2} \sin \left( \frac{\pi \delta}{2} \right) = \frac{\pi A}{B^2} \frac{\pi \alpha}{2} \int_0^1 \frac{dx}{(1-x)^\alpha}.
\tag{15}
\]

On the other hand, self-consistency requires that \( \chi_{\text{loc}} \) calculated from Eq. (4) for the specific \( G_f \) considered here coincides with Eq. (11). This leads to the additional condition

\[
\frac{\pi A}{B^2} \sin \left( \frac{\pi \delta}{2} \right) = \frac{\pi A}{B^2} \frac{\pi \alpha}{2} \int_0^1 \frac{dx}{(1-x)^\alpha} \tag{16}
\]

\[
= 2\sin^2 \left( \frac{\pi \delta}{2} \right) \int_0^1 \frac{dx}{x^\delta (1-x)^\alpha}, \tag{17}
\]

where the last equality follows from Eq. (15). Surprisingly, this equation (in conjunction with Eq. (14)) can be solved analytically. We find \( \delta = 2\alpha - 1 = 1/3 \). We have tested these large-\( N \) findings by a numerical approach based on the QMC algorithm of Ref. [14]. Since no algorithm exists capable of treating the Kondo and RKKY interactions on the same footing, we confine ourselves to
the investigation of the paramagnetic Kondo-decoupled SL phase. Therefore we solve the self-consistency equations for the pure RKKY model represented by the last term of the action Eq. (2) for standard SU(2) spins with a constant $\rho_1(\epsilon) = 1/(2J)$. The local susceptibility in imaginary time thus obtained is shown in Fig. 1(a). The low-$T$, long-time behavior deduced from the numerical results is $\chi_{\text{loc}}(\tau) \sim [\sin(\pi\tau/\beta)]^{-1/3}$. This corresponds to the following scaling form

$$\chi''_{\text{loc}}(\omega) \sim J^{-1}(\omega/J)^{-\delta}F_\delta(\omega/T)$$  \hspace{1cm} (18)

with $F_\delta(x) = x^{\delta} \Gamma(1/2 + i(x^2/2\delta)^{1/2})^2 \sinh(x^2/2\delta)$ and $\delta = 2/3$. Eq. (18) gives a power-law divergence at low energy, $\chi''_{\text{loc}}(T = 0, \omega) \sim J^{-1} \omega^{-\delta}$ and $\chi'_{\text{loc}}(T, \omega = 0) \sim J^{-1} \omega^{-\delta}$. The $T$-dependence of $\chi_{\text{loc}}(\omega = 0)$, displayed in Fig. 1(b), shows indeed this behavior. Quite interestingly, apart from the difference in the value of the exponent $\delta$ (that was expected as critical exponents do depend on $N$) the “exact” QMC results for SU(2) spins are qualitatively similar to those of the simpler large-$N$ approach. We expect on general grounds that the inclusions of $1/N$ corrections should improve the agreement between the $1/N$ expansion and QMC results. It is important to notice that if the solution (18) for $\chi_{\text{loc}}$ is used to determine $M(\omega)$ from Eq. (8), the resulting $M''(\omega)$ exhibits relaxations below a temperature $T^*$, a behavior that violates the condition of thermodynamic stability $\omega M''(\omega) \leq 0$. We have determined $T^*$ in two different ways, from our analytic expressions in Eqs. (8) and (18), and by analytically continuing our numerical results on the imaginary frequency axis using the method of Padè approximants. The two methods yield $T^* \approx 0.03J$. This is proportional to, but substantially smaller than $J$, so that a wide range $T^* < T < J$ of temperatures still exists within which the power-law given by Eq.(18) holds. The instability found at $T^*$ could signal that the magnetic transition is first-order as suggested by other authors [15] (see also Ref. [16]); we will address this issue elsewhere [17].

We discuss next the case in which $\rho_1(\epsilon)$ has a square root singularity at the band edge which corresponds to a three dimensional spin-wave spectral density. A particularly simple case is that of the semi-circular density of states, $\rho_1(\epsilon) = (J\pi/2)^{-1/2} \sqrt{1 - (\epsilon/J)^2}$. Then it is easy to show that $\chi_{\text{loc}}^{-1}(\omega_n) = (J^2/4)\chi_{\text{loc}}(\omega_n)$ [9]. This particular model was studied in Ref. [13] in the context of the SL state first discussed in Ref. [18]. At the local QCP, the fermionic Green function diverges like $G_f(\omega) \sim 1/\sqrt{|J\omega|}$ which yields $\chi_{\text{loc}}(\omega) \sim J^{-1} \ln(J/|\omega|)$ allowing to close the self-consistency procedure without the appearance of additional logarithmic corrections. These results hold as long as the magnetic self-energy $M'(\omega_n)$ remains outside the interval $[-J, J]$, so that no magnetic instability occurs. However, in this case $M'(\omega_n = 0) = J$, $J\chi_{\text{loc}}(\omega_n = 0) = 2$, a condition satisfied at a finite temperature $T_c \sim J$. Since at the transition $J < T_K^0$, it occurs while $T_K$ (and $r$) are still finite. Then, as pointed out in Ref. [6], in the three dimensional case the local QCP is masked by the finite temperature magnetic transition. The analysis of this QCP, expected to be of the Millis-Hertz type [4,5], lies beyond the power of our lowest-order large-$N$ theory.

In summary, we carried out a large-$N$ implementation of the EDMFT equations for the Kondo-lattice-RKKY model. This simple scheme allows us to find self-consistent solutions of the EDMFT equations without any ad hoc input. We find in agreement with Ref. [6] that there exists a local QCP separating a FL and a SL phase. However, this approximation, which neglects the fluctuations of $r$, does not allow us to distinguish between the properties of the QCP and those of the SL phase which correspond respectively to the unstable and stable fixed points of the RG equations of Ref. [6]. We showed that the large-$N$ results for the SL phase are reliable by comparing them with those of an exact numerical solution of the SL problem for $N = 2$. Both our methods (in agreement with the RG results of Ref. [7]) show that a correlator $\chi_0^{-1} \sim \omega^\delta$ corresponds to a local susceptibility $\chi_{\text{loc}} \sim \omega^{-\delta}$. On the other hand we find that a correlator $\chi_0^{-1} \sim \ln^{-1} \omega$ does not correspond to a simply logarithmic susceptibility, showing that the limit $\delta \to 0$ can not be taken trivially. Although this conclusion can only be safely drawn in connection with the SL problem, it suggests that caution must be exerted when taking the same limit at the QCP [19].

Finally, we showed that the power-law solution that we found in the SL phase is thermodynamically unstable below a temperature $T^* \ll J$. This suggests that a first-order phase transition could take place at low temperature as suggested by others [15].

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[1] S. Sachdev, Quantum Phase Transitions, Cambridge University Press, Cambridge (1999).
[2] See, e.g., P. Coleman, Physica B 250-261, 353 (1999).
[3] A. Chubukov, et al., Phys. Rev. B 49, 11919 (1994); C. Castellani, et al., Phys. Rev. Lett. 75, 4650 (1995); C. M. Varma, ibid. 83, 3538 (1999) and references therein; M. Vojta et al., ibid. 85, 4940 (2000).
[4] J. A. Hertz, Phys. Rev. B 14, 1165 (1976).
[5] A. J. Millis, Phys. Rev. B 48, 7183 (1993).
[6] Q. Si et al., Nature 413, 804 (2001); Q. Si et al. cond-mat/0202411.
[7] I. L. Zhu and Q. Si, Phys. Rev. B 66, 024426 (2002).
[8] For a general discussion of this issue see, e.g., P. Coleman, Nature 413, 788 (2001).
[9] A. Georges et al., Rev. Mod. Phys. 68, 13 (1996).
[10] J. L. Smith and Q. Si, Phys. Rev. B 61, 5184 (2000).
[11] R. Chitra and G. Kotliar, Phys. Rev. Lett. 84, 3678 (2000).
[12] See, e.g., P. Coleman, Phys. Rev. B 35, 5072 (1987); D. M. Newns and N. Read, Adv. Phys. 36, 799 (1987).
[13] S. Burdin, D. R. Grempel, and A. Georges, Phys. Rev. B 66, 045111 (2002).
[14] D. R. Grempel and M.J. Rozenberg, Phys. Rev. Lett. 80, 389, (1998).
[15] S. Pankov, G. Kotliar, and Y. Motome, cond-mat/0112083.
[16] K. Haule, et al. cond-mat/0205347.
[17] S. Burdin et al., unpublished.
[18] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993).
[19] Notice, however, that usually scaling properties holding at a critical point prevent the scaling quantities from acquiring (logarithmic) corrections except at the upper or lower critical dimensions (for an example see, e.g., Appendix D, in the second Ref. [6]).