KOLMOGOROV COMPLEXITY AND
SYMMETRIC RELATIONAL STRUCTURES

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Abstract. We study partitions of Fraisse limits of classes of finite relational structures where the partitions are encoded by infinite binary strings which are random in the sense of Kolmogorov-Chaitin.

§1. Introduction. This paper follows on [6] where a study was made of the properties of combinatorial configurations which are encoded or generated by infinite binary strings which are random in the sense of Kolmogorov-Chaitin [14, 2] (to be referred to as KC-strings). We shall study countable homogeneous structures from this point of view. A relational structure $X$ is homogeneous if any isomorphism $f: A \rightarrow B$ between finite substructures of $X$ can be extended to an automorphism of $X$. This is perhaps the strongest symmetry condition one can impose on a relational structure. Our aim is to depict various situations where this kind of symmetry will be seen to be preserved by an arbitrary KC-string. Our work is based on Fraisse's well-known characterisation of countable homogeneous structures [7].

A well-known example of a countable homogeneous structure is the random graph $R$ of Rado [17]. We now illustrate some of the results of this paper with respect to the graph $R$. For a finite graph $\beta$, write $[R, \beta]$ for the set of copies of $\beta$ in $R$. We call a subset $Y$ of $[R, \beta]$ a $\beta$-organisation when $Y$ is exactly the set of all copies of $\beta$ in some subgraph $R'$ of $R$, where $R'$ is isomorphic to $R$. Now, $R$ has a simple recursive representation of the form $(\omega, E)$ where $E$ is a recursive subset of the set of 2-subsets of $\omega$. This implies that one can find a recursive enumeration $(\beta_j | j < \omega)$, without repetition, of the set $[R, \beta]$. Let $\varepsilon = \prod_{j=0}^{\infty} \varepsilon_j$ be a KC-string. If we define a 2-colouring $\chi_\varepsilon: [R, \beta] \rightarrow \{0, 1\}$ by giving each $\beta_j$ the colour $\varepsilon_j$, it will be shown that there always exists a monochromatic $\beta$-organisation $Y_\varepsilon$. Moreover, one can compute the $\beta$-organisation $Y_\varepsilon$ from $\varepsilon$ by means of a simple greedy algorithm. In this way, a KC-string has two aspects: (i) as a random partition of the copies of $\beta$ in $R$, and (ii) as a generator of a $\beta$-organisation in $R$ which is monochromatic under this partition. The symmetric structure $R$ is reflected (or preserved) by each KC-string in two distinct ways.

Similar results will be established for many other homogeneous structures. The main result is formulated in Section 2.7 and proved in Section 3.2. In Section 4 we

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apply this theory to the Fraïssé limits of what we shall call \textit{ranked diagrams}. It is also shown how a KC-string can be used to generate the Fraïssé limit in this case.

\section{Preliminaries.}

2.1. The composition of two functions $f$ and $g$, denoted by $fg$, is defined by $fg(x) = f(g(x))$. The set of non-negative integers is denoted by $\omega$. We view the elements of $\omega$ as finite ordinals, so that $n < \omega$ denotes the set $\{0, 1, \ldots, n - 1\}$. The cardinality of a finite set $A$ is denoted by $|A|$. We write $\cN$ for the product space $\{0, 1\}^\omega$. The set of words over the alphabet $\{0, 1\}$ is denoted by $\{0, 1\}^\ast$. If $\alpha = \alpha_0\alpha_1\alpha_2\ldots$ is in $\cN$ and $n < \omega$, we write $\bar{\alpha}(n)$ for the binary word $\prod_{i<n} \alpha_i$.

We use the usual recursion-theoretic terminology $\Sigma^0_\beta$, $\Pi^0_\beta$ and $\Delta^0_\beta$ for the description of the arithmetical subsets of $\omega^k \times \cN^k$ — see [10], for example. We write $\lambda$ for the Lebesgue measure on $\cN$. This is the unique probability measure that assigns the value $\frac{1}{2^n}$ to each of the events $A_i = \{\alpha \in \cN | \alpha_i = 1\}$ and under which the events $A_i$ are statistically independent.

2.2. A prefix algorithm is a partial recursive function $f$ from $\{0, 1\}^\ast$ to $\{0, 1\}^\ast$ whose domain is prefix-free, i.e. if $u, v \in \text{dom} f$ then neither is an initial segment of the other. It is well-known (and easy to prove) that there is an effective enumeration of prefix algorithms and, therefore, that there is some universal prefix algorithm $U$. For $s \in \{0, 1\}^\ast$ let $H(s)$, the \textit{Kolmogorov-complexity} of $s$, be the length of a shortest “program” $p \in \{0, 1\}^\ast$, such that $U(p) = s$. (For the history and underlying intuition of these notions, the reader is referred to [20]. See also [15], [2], [9] or [8].) An infinite binary string $e$ is said to be \textit{Kolmogorov-Chaitin complex} (KC-complex) if and only if

$$\exists m \forall n H(\bar{e}(n)) \geq n - m.$$ 

The set of KC-complex strings does not depend on the choice of the universal prefix algorithm $U$ and has $\lambda$-measure one. We denote this set by $KC$ and refer to the elements as KC-strings. We shall make frequent use of the following result.

\textbf{Theorem 1.} [5] If $X$ is a $\Pi^0_2$-subset of $\cN$ and $\lambda(X) = 1$, then $X$ contains every KC-string $e$.

The proof of this result is based on Martin-Löf's description [16] of the set $KC$.

2.3. In the sequel, $\cL$ will stand for the signature of a relational structure. Moreover, $\cL$ will always be finite and the arities of the relational symbols will all be $\geq 1$. This has the implication that the empty set carries a unique $\cL$-structure. The definitions that follow were introduced by Fraïssé [7] in 1954. (For a general discussion of the results to be summarised, the reader is also referred to Hodges [11], Chapter 7).

The \textit{age} of an $\cL$-structure $X$, written $\text{Age}(X)$, is the class of all finite $\cL$-structures (defined on finite ordinals) which can be embedded as $\cL$-structures into $X$. The structure $X$ is \textit{homogeneous} (some authors say \textit{ultrahomogeneous}) if, given any isomorphism $f : A \rightarrow B$ between finite substructures of $X$, there is an automorphism $g$ of $X$ whose restriction to $A$ is $f$. The following result is due to Fraïssé. (See [11, Chapter 7] for a proof.)
PROPOSITION 1. The countable \( \mathcal{L} \)-structure \( X \) is homogeneous if and only if, for \( A, B \in \text{Age}(X) \) and embeddings \( f : A \to B, h : A \to X \), there is an embedding \( g : B \to X \) such that \( h = gf \). It suffices to require this when \( |B| = |A| + 1 \).

2.4. A class \( K \) of finite \( \mathcal{L} \)-structures has the amalgamation property if, for structures \( A, B_1, B_2 \) in \( K \) and embeddings \( f_1 : A \to B_1 \), \( f_2 : A \to B_2 \) there is a structure \( C \) in \( K \) and there are embeddings \( g_i : B_i \to C \) (i = 1, 2), such that \( g_1 f_1 = g_2 f_2 \).

Suppose \( K \) is a countable class of finite \( \mathcal{L} \)-structures, the domains of which are finite ordinals such that

1. if \( A \) is a finite \( \mathcal{L} \)-structure defined on some finite ordinal, if \( B \in K \) and if there is an embedding of \( A \) into \( B \), then \( A \in K \);
2. the class \( K \) has the amalgamation property.

Then, Fraïssé showed that there is a countable homogeneous structure \( X \) such that \( \text{Age}(X) = K \). Moreover, \( X \) is unique up to isomorphism. The unique \( X \) is called the Fraïssé limit of \( K \). We also recall that, conversely, the age \( K \) of a countable homogeneous structure has properties (i) and (ii).

2.5. In our study of partitions of a homogeneous structure \( X \) we shall require its age to be dense in \( X \) in the following sense: If \( A, B \in \text{Age}(X) \) and \( i : A \to B \) is an embedding, then there exist \( C \in \text{Age}(X) \) and embeddings \( f_1, f_2 : B \to C \) such that \( f_1 i = f_2 i \) and \( \text{Im} f_1 \cap \text{Im} f_2 = \text{Im} f_1 i = \text{Im} f_2 i \). The Fraïssé limit of finite graphs (the random graph of Rado [17]) and the Fraïssé limit of ranked diagrams (see Section 4) are examples of homogeneous structures with dense ages. For any \( n \), a disjoint union of countably many copies of the finite complete graph \( K_n \) is an example of a homogeneous structure whose age is not dense. (The complement of this structure does have a dense age.) The following combinatorial lemma plays a central role in the proof of Theorem 2.

**Lemma 1.** Suppose \( X \) is a countable homogeneous structure with a dense age. If \( U, V \) are disjoint subsets of \( X \), then there is a sequence \( (V_i | i < \omega) \) of pairwise disjoint subsets of \( X \) such that \( U \cap V_i = \emptyset \) and \( U \cup V_i \) and \( U \cup V \) inherit isomorphic \( \mathcal{L} \)-structures from \( X \), for all \( i < \omega \).

**Proof.** Set \( V_0 = V \) and suppose pairwise disjoint \( V_0, \ldots, V_{k-1} \) have been constructed with \( U \cap V_i = \emptyset \) and \( U \cup V_i \) isomorphic to \( U \cup V \) for all \( i < k \). Set \( W = \bigcup_{i < k} V_i \). Choose \( A, B \in \text{Age}(X) \) with \( A \subset B \) so that \( A \) is isomorphic to \( U \subset X \) via an isomorphism which extends to an isomorphism of \( B \) to \( U \cup W \subset X \). Since \( \text{Age}(X) \) is dense in \( X \), there exist \( C \in \text{Age}(X) \) and embeddings \( f_1, f_2 : B \to C \) such that \( A \subset C \) and \( f_1, f_2 \) are both the identity on \( A \), while \( \text{Im} f_1 \) and \( \text{Im} f_2 \) will have exactly the elements of \( A \) in common.

For \( i \in \{1, 2\} \), let \( A_i \) be the complement of \( A \) in \( \text{Im} f_1 \). Then \( A_1 \cap A_2 = \emptyset \) but \( A \cup A_i \) is isomorphic to \( B \) and hence also to \( U \cup W \subset X \). Moreover, \( A \cap A_i = \emptyset \). Let \( \alpha \) be an isomorphism (e.g. the one from the construction of \( A \) and \( B \) above) from \( B \) onto \( U \cup W \subset X \) that maps \( A \) onto \( U \). By Proposition 1 there is an embedding \( \beta \) such that the diagram in Figure 1 commutes. We can write \( \text{Im} \beta = U \cup W \cup W' \) with \( U \cup W' \) isomorphic to \( U \cup W \) and \( (U \cup W) \cap W' = \emptyset \). Let \( V_k \) be any subset of \( W' \) such that \( U \cup V_k \) is isomorphic to \( U \cup V \). (Such as exists by the isomorphism of \( U \cup W \) with \( U \cup W' \).) The sequence \( (V_i | i < \omega) \) constructed in this way has the required properties.
2.6. A recursive representation of a countable $\mathcal{L}$-structure $X$ is a bijection $\phi : X \to \omega$ such that, for each $R \in \mathcal{L}$, if the arity of $R$ is $n$, then the relation $R^\phi$ defined on $\omega^n$ by

$$R^\phi(x_1, x_2, \ldots, x_n) \leftrightarrow R(\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n))$$

is recursive. If we identify the underlying set of $X$ with $\omega$ via $\phi$ and each $R$ with $R^\phi$ we call the resulting structure a recursive $\mathcal{L}$-structure.

If $X$ is countable and homogeneous and if $\text{Age}(X)$ has an enumeration $A_0, A_1, A_2, \ldots$, possibly with repetition, with the property that there is a recursive procedure that yields, for each $i < \omega$, and $R \in \mathcal{L}$, the underlying set $n(i)$ of $A_i$ together with the interpretation of $R$ in $n(i)$, then we call $(A_i| i < \omega)$ a recursive enumeration of $\text{Age}(X)$. It follows from the construction of Fraïssé limits from their ages, as discussed in [11] (p329) that one can construct a recursive representation of $X$ from a recursive enumeration of its ages. (Conversely, it is trivial to derive a recursive enumeration of $\text{Age}(X)$ from a recursive representation of $X$.) It is therefore not difficult to find recursive representations for Fraïssé limits of classes $\mathbf{K}$ from recursive enumerations of their members.

2.7. Let $X$ be a countable, homogeneous structure with a recursive representation $\phi$. For $\beta \in \text{Age}(X)$, let $[X, \beta]$ be the set of copies (images under embeddings) of $\beta$ in $X$. Suppose, in addition, that $X$ has a dense age. We can use $\phi$ to find a recursive enumeration $\beta_0, \beta_1, \ldots$, without repetition, of the set $[X, \beta]$. The density of $\text{Age}(X)$ in $X$ ensures that $[X, \beta]$ is infinite (see Lemma 1) and the representation $\phi$ can be used to decide whether a given finite subset of $X$ inherits a structure isomorphic to $\beta$.

If $\alpha$ is an infinite binary string then $\alpha$ induces a 2-colouring $\chi_\alpha$ of $[X, \beta]$ where $\chi_\alpha$ assigns to the $i$-th copy $\beta_i$ of $\beta$ in $X$ the colour $\alpha_i$, the $i$-th bit of $\alpha$. The main theorem of the paper can now be formulated.

**Theorem 2.** Let $X$ be a recursive homogeneous structure with a dense age. For each $\beta \in \text{Age}(X)$ and each KC-string $\epsilon$, there exists an embedding $v : X \to X$ such that $\chi_\epsilon(\beta') = 1$ for each $\beta' \in [v(X), \beta]$. In addition, $v$ can be so constructed that it is recursive relative to $\epsilon$.

One can think of the mappings $\chi_\alpha : [X, \beta] \to 2$ as random partitions. It follows from Theorem 2 that when $[X, \beta]$ is subjected to a random partitioning then, with probability 1, one can find copies $X_0, X_1$ of $X$ in $X$ such that $\chi_\alpha$ is of colour $i$ on $[X_i, \beta]$ ($i = 0, 1$). This is because $\alpha$ is in $KC$ with probability 1. Moreover, when $\alpha$ is a KC-string, we can effectively generate, relative to $\alpha$, the automorphic copies $X_0$ and $X_1$ of $X$. The proof of the theorem appears in Section 3.2.
Recall that Ramsey's Theorem [18] says that for $X = K_\omega$, the complete graph on the natural numbers, for $\beta = K_n$, the complete graph on $n$ points, and $\varepsilon$ an arbitrary binary sequence, there exists an embedding $v : X \rightarrow X$ such that $[v(X), \beta]$ is monochromatic under the 2-colouring of $[X, \beta]$ induced by $\varepsilon$. E. Specker [19] has observed that there exists a recursive sequence $\varepsilon$ such that, for the colouring of $[X, K_2]$ induced by $\varepsilon$, there exists no recursive copy $X'$ of $X$ such that $[X', K_2]$ is monochromatic. This has been further refined by C.G. Jockusch [12] who showed that there exists a recursive sequence $\varepsilon$ such that, for the colouring of $[X, K_n]$ induced by $\varepsilon$, there is no $\Sigma^0_n$ copy $X'$ of $X$ for which $[X', \beta]$ is monochromatic. However, for any recursive $\varepsilon$, there always exists a $\Pi^0_n$ copy $X'$ of $X$ for which $[X', \beta]$ is monochromatic. It follows, however, from Theorem 2 that when $\varepsilon$ is a KC-string, one can find a monochromatic $X'$ which is recursive in $\varepsilon$. This emphasises that Jockusch's results exploit the non-random nature of recursive partitions.

§3. Complex partitions of Fraïssé limits.

3.1. In the following we will denote the class of all finite subsets of a set $Y$ by $\text{Fin}_Y$. If $w \in \text{Fin}_\omega$ we denote the largest element of $w$ by $\max w$. If $w$ is empty, then $\max w = -1$. If $n \in \omega$, then by $wn$ we mean $w \cup \{n\}$. We write $v < w$ if there is a $t \neq \emptyset$ with $w = v \cup t$ and $\max v < \min t$.

**Definition 1.** Let $Y$ be a countably infinite set. An encoding of $Y$ is a function $\pi : \text{Fin}_\omega \rightarrow \text{Fin}_Y$ such that

(i) $\pi(\emptyset) = \emptyset$ and for some $w_0 \in \text{Fin}_\omega$,

$$\pi(w_0) \neq \emptyset,$$

(ii) whenever $n > m > \max w$

$$\pi(wn) \cap \pi(wm) = \pi(w);$$

(iii) for each $w$ with $\pi(w) \neq \emptyset$,

$$\sum 2^{-|\pi(wk)\setminus\pi(w)|} = \infty$$

where the summation is over all $k > \max w$ such that $\pi(wk) \neq \pi(w)$.

**Definition 2.** An encoding $\pi$ is called effective relative to a bijection $\sigma : \omega \rightarrow Y$ when there exist a recursive binary relation $R_\sigma$ and a recursive function $f$, such that, for $i \in \omega$ and $w \in \text{Fin}_\omega$,

(i) $R_\sigma(i, w) \leftrightarrow \sigma(i) \in \pi(w)$, and also

(ii) $f(w) = |\pi(w)|$.

These definitions have been adapted from [6]. The next theorem is a generalization of Theorem A of [6].

**Theorem 3.** If the encoding $\pi : \text{Fin}_\omega \rightarrow \text{Fin}_Y$ is effective relative to $\sigma$ and if $\varepsilon \in KC$, then there exists a strictly increasing sequence

$$w_1 < w_2 < w_3 < \cdots$$

in $\text{Fin}_\omega$ such that

$$\varepsilon(j) = 1 \text{ whenever } \sigma(j) \in \bigcup_{n \geq 1} \pi(w_n).$$

There exists an oracle computation of this sequence from $\varepsilon$. 
PROOF. Let \( \pi \) be an encoding which is effective relative to \( \sigma \), as defined above. Apply (1) to fix \( w_0 = v_0k \in \text{Fin}\omega \), where \( k = \max w_0 \), such that \( \pi(v_0) = 0 \) but \( \pi(v_0k) \neq 0 \).

Let \( \varepsilon \) be in \( \text{KC} \). We construct a strictly increasing sequence in \( \text{Fin}\omega \) by induction so that for each \( n \)

\[
w_0 < w_1 < \cdots < w_n \text{ and } \varepsilon(j) = 1 \text{ for all } \sigma(j) \in \bigcup_{k=1}^{n} \pi(w_k).
\]

The construction will be recursive in \( \varepsilon \). This will suffice to prove the theorem.

Suppose \( n \geq 0 \) and \( w_0, \ldots, w_n \) have been constructed. For every \( k > \max w_n \), we define \( B_k \subseteq \mathcal{N} \) by:

\[
\alpha \in B_k \iff (\forall j)[\sigma(j) \in \pi(w_nk) \setminus \pi(w_n) \rightarrow \alpha_j = 1].
\]

By Definition 2, \( R_{\sigma}(i,w) \) and the function \( w \mapsto |\pi(w)| \) are both recursive, so there exists a total recursive function \( \psi : \omega \to \omega \) such that \( j \leq \psi(k) \) whenever \( \sigma(j) \in \pi(w_nk) \). The function \( \psi \) could, for example, compute the largest \( j \) so that \( \sigma(j) \in \pi(w_nk) \) when \( w_n \) and \( k \) have been given. Now,

\[
\alpha \in B_k \iff (\forall j \leq \psi(k))[R_{\sigma}(j, w_nk) \land \neg R_{\sigma}(j, w_n) \rightarrow \alpha_j = 1].
\]

It now follows that the relation \( \alpha \in B_k \) is recursive in \( k \) and \( \alpha \).

We shall define a sequence \( X_0, X_1, X_2, \ldots \) of statistically independent random variables on the probability space \( (\mathcal{N}, \Sigma, \lambda) \) where \( \Sigma \) is the collection of Borel subsets of \( \mathcal{N} \) and \( \lambda \) the Lebesgue measure, as before. Let \( X_i(\alpha) = \alpha_i \) for \( \alpha \in \mathcal{N} \) and \( i \in \omega \). If \( k > \ell > \max w_n \) and both \( \pi(w_nk) \neq \pi(w_n) \) and \( \pi(w_n\ell) \neq \pi(w_n) \), then the events \( B_k \) and \( B_\ell \) are statistically independent. To see this, note that \( B_k \) belongs to the \( \sigma \)-algebra generated by

\[
\{X_j|\sigma(j) \in \pi(w_nk) \setminus \pi(w_n)\},
\]

and \( B_\ell \) belongs to the \( \sigma \)-algebra generated by

\[
\{X_j|\sigma(j) \in \pi(w_n\ell) \setminus \pi(w_n)\}.
\]

Independence follows from the fact that \( \pi(w_nk) \cap \pi(w_n\ell) = \pi(w_n) \).

Since the probability

\[
P(\alpha \in B_k) = 2^{-|\pi(w_nk) \setminus \pi(w_n)|}
\]

and we know, by (3), that the sum of the probabilities of these independent events diverges, it follows from the second Borel-Cantelli lemma that the event \( B_k \), with \( \pi(w_nk) \neq \pi(w_n) \), occurs infinitely often with probability 1. In particular, if we define \( B \) by

\[
\alpha \in B \iff \exists k (k > \max w_n \land \pi(w_nk) \neq \pi(w_n) \land \alpha \in B_k)
\]

then \( \lambda(B) = 1 \). But \( B \) is a \( \Sigma^0_1 \)-set and \( \Sigma^0_1 \subseteq \Pi^0_1 \), so it follows directly from Theorem 1 that \( \varepsilon \in B \). Choose the smallest \( k > \max w_n \) for which \( \pi(w_nk) \neq \pi(w_n) \) and such that \( \varepsilon \in B_k \). Set \( w_{n+1} = w_nk \). Then \( \varepsilon(j) = 1 \) for all \( j \) with \( \sigma(j) \in \bigcup_{\ell \leq n+1} \pi(w_{\ell}). \)

Every step—including this last one—is effective relative to \( \varepsilon \). \( \square \)
3.2. We now proceed to prove the main theorem of the paper (Theorem 2).

PROOF. Let $X$ be a recursive homogeneous structure with a dense age. There is a universal procedure that yields, for finite subsets $U, V$ of $X$ with $U \cap V = \emptyset$ and each $k < \omega$ a set $V_k$ such that the sequence $(V_k)_{k < \omega}$ is as in the conclusion of Lemma 1. This is evident from the proof of Lemma 1 since the inductive constructions of the $V_k$ can be done recursively for a given recursive structure $X$.

Since $X$ is recursive we can identify its domain with $\omega$. Our aim is to construct a function $\mu : \text{Fin}\omega \to \text{Fin}\omega$ such that, for $w \in \text{Fin}\omega$, there is an embedding $v(\omega)$ from the $\mathcal{L}$-structure on $|w| \subset X$ to an $\mathcal{L}$-structure $\mu(w) \subset X$ such that, for $k > \max w$, the embedding $v(v(w))$ will be an extension of $v(w)$.

The construction will be such that if $n > m > \max w$ then

$$\mu(w) \cap \mu(wm) = \mu(w)$$

and $\mu(wm)$ will always contain a copy of $\beta$ which is not in $\mu(w)$. Finally, we shall ensure that that the embeddings $v(w)$ will depend recursively on $w$. The construction is as follows:

1. Set $\mu(0) = 0$ and $v(0) = 0$.

2. Assume $\mu(w), v(w)$ and $k > \max w$ are given. Construct $V$ (which will be a finite set) such that $V \cap \mu(w) = \emptyset$ and if we set $Z = \mu(w) \cup V$ then $Z$ contains a copy of $|w| + 1$, extending the copy of $|w|$ in $\mu(w)$, and $Z$ contains a copy of $\beta$ not in $\mu(w)$. (Proposition 1 shows that we can extend $|w|$ and Lemma 1 implies that there are infinitely many copies of $\beta$.) Next, construct a pairwise disjoint sequence $V_0, V_1, V_2, \ldots$ (again using Lemma 1) which are all also disjoint from $\mu(w)$, such that if we set $Z_j = \mu(w) \cup V_j$ then $Z_j$ is isomorphic to $Z$. Finally, set $\mu(wk) = Z_k$ and let $v(v(wk))$ be an embedding of $|w| + 1$ into $Z_k$ which extends $v(w)$.

Set $\pi(w) = [\mu(w), \beta]$. We now show that $\pi$ is an encoding of $Y = [X, \beta]$ in the sense of Definition 1. By the construction we see immediately that $\pi$ satisfies conditions (1) and (2) of Definition 1. In order to verify condition (3), we note that if $n > \max w$ then $\pi(wn) \setminus \pi(w)$ is non-empty and its size is independent of $n$ (again by Step 2 of the construction). The divergence of the series follows.

Let $\beta_0, \beta_1, \ldots$ be an effective enumeration without repetition of $Y$. For $i < \omega$, set $\sigma(i) = \beta_i$. Note that, since we have an effective representation of $X$, the straightforward (greedy!) algorithm for giving $\mu$ and $\pi$, respectively, shows that both are recursive. Since $\pi$ is recursive, so is the mapping $w \mapsto [\pi(w)]$. Also, whether $[\sigma(i) \in \pi(w)]$ holds, can be determined by listing and comparing the elements of $\sigma(i)$ and $\mu(w)$, where $\mu$ is as above. Therefore, $\pi$, as defined, is effective relative to $\sigma$ (in the sense of Definition 2).

Theorem 3 now gives an oracle computation of a strictly increasing sequence $w_1 < w_2 < w_3 < \cdots$ from $e$ such that $e(j) = 1$ whenever $\sigma(j) \in \bigcup \pi(w_n)$. In other words, since $\mu(w_n)$ is increasing in $n$, if $\sigma(j) \subset \bigcup \mu(w_n)$ then $\sigma(j) = 1$.

The embeddings $v(w_n) : |w_n| \to \mu(w_n)$ are mutually compatible and thus define an embedding $v : X \to X$ such that $\text{Im} v \subset \bigcup_n \mu(w_n)$. This embedding $v$ is the required embedding, which is indeed recursive relative to $e$ since $w \mapsto v(w)$ is recursive and the sequence $w_1 < w_2 < w_3 < \cdots$ is recursive relative to $e$. \qed
§4. Ranked diagrams.

4.1. In [6] it was shown that partitioning the edges of the complete countable graph $K_m$ into two classes $E_0, E_1$ by means of a KC-string $e$ yields two graphs $(\omega, E_0)$ and $(\omega, E_1)$ both of which are isomorphic to the Fraïssé limit of finite graphs. In this section we want to do the same for so-called ranked diagrams. These structures can be viewed as the Hasse diagrams of posets.

4.2. An $\aleph_0$-categorical first-order theory of ranked diagrams. In the sequel, $\ell \geq 2$ is fixed. Let $\mathcal{F}$ be the signature having $\ell$ unary relations, $L_0 \ldots L_{\ell-1}$ (denoting the levels of the ranked diagram), and one binary relation, $S$ (succession). The theory, $RD_\ell$, of ranked diagrams on $\ell$ levels ($\ell$-diagrams), has the following three axioms:

(i) For all $x$: $L_0(x) \vee \cdots \vee L_{\ell-1}(x)$.

(ii) For all $x$:
$$\bigwedge_{0 \leq i < j < \ell} \neg[L_i(x) \land L_j(x)].$$

(iii) For all $x$ and $y$:
$$S(x, y) \rightarrow \bigwedge_{i=0}^{\ell-2} [L_i(x) \rightarrow L_{i+1}(y)].$$

The preceding axioms imply that there exists, for each $x$, a unique $L_i$ such that $L_i(x)$ holds (or—in different notation—$x \in L_i$) and also that $S(x, y)$ can hold only if $x$ and $y$ are on adjacent levels, $y$ being “above” $x$. A model of the theory $RD_\ell$ is an $\ell$-diagram. (A special class of these diagrams, namely the $k$-layered posets, has been investigated in [1].)

We shall identify a class of countable $\ell$-diagrams, having the property that each one of them also contains a copy of every other countable $\ell$-diagram. This class is defined by an $\aleph_0$-categorical first-order theory consisting of the axioms of $RD_\ell$ as well as a collection of extension axioms—similar to the extension axioms used by Compton [4] in his proof of the fact that the class of partial orders has a (labelled) first order 0-1 law. In view of the result of Kleitman and Rothschild [13], showing that a finite partial order will be ranked and of height 3 with labelled asymptotic probability 1, it makes sense to investigate random partial orders via $\ell$-diagrams.

We now single out those $\ell$-diagrams that are not only models of $RD_\ell$ but also satisfy the following countable collection of axioms (indexed by the cardinalities of $X, Y, X', Y', Z$, for example):

(iv) (Extension Axioms) For each $i < \ell$ and configuration of non-negative integers, $(n_1, n_2, n_3, n_4, n_5)$, an axiom stating that when $X, Y$ are disjoint subsets of $L_{i+1}$, $Z$ is a subset of $L_i$ and $X', Y'$ disjoint subsets of $L_{i-1}$ such that
$$([X], [Y], [Z], [X'], [Y']) = (n_1, n_2, n_3, n_4, n_5)$$
then, for some $z \in L_i$ such that $z \not\in Z$, we have
$$S(z, x), \quad S(x', z), \quad \neg S(z, y) \quad \text{and} \quad \neg S(y', z)$$
for all $x \in X, x' \in X', y \in Y$ and $y' \in Y'$ respectively. (See Figure 2.) We think of $L_{i-1}$, respectively $L_{i+1}$, as a name for the empty set when $i = 0$, respectively $i = \ell - 1$. 
These extension axioms guarantee that we can extend a given arbitrary finite configuration on levels \( i - 1, i, i + 1 \) in the required way (to a new \( \ell \)-diagram) by just finding an appropriate \( z \) on level \( i \). Axioms (i)-(iv) all together give a countable collection of first-order sentences in our language \( S_\ell \). These make up a theory \( T_\ell \). We shall call its countable models the generic \( \ell \)-diagrams. Instances of the form \( X = X' = Y = Y' = \emptyset \) of (iv) guarantee that in any model of \( T_\ell \), the unary relations \( L_i \) are modelled by infinite sets, so that any countably infinite model necessarily has infinitely many elements on each level.

4.3. **Explicit construction of an generic \( \ell \)-diagram.** We now give an example of how to construct a recursive object that represents a generic \( \ell \)-diagram. A similar construction can be given for Rado's random graph [17]. Let \( A = \ell \times \omega \) be our underlying set and fix a collection

\[
P(i, n) \in \ell, \ n < \omega
\]

of distinct odd primes. Now define the binary relation \( P_\ell \) on \( \ell \times \omega \) by

\[
(i, n)P_\ell(i + 1, m) \leftrightarrow \begin{cases} 
m \neq 0 \text{ and } p(i, n) | m \\
n \neq 0 \text{ and } p(i + 1, m) | n \end{cases}
\]

In order to verify that \( (A, P_\ell) \) is generic, we need to check the extension property (iv). We first assume \( 0 < i < \ell - 1 \). Take any finite subsets

\[
X = \{(i + 1, x_0), \ldots, (i + 1, x_p)\}
\]
\[
Y = \{(i + 1, y_0), \ldots, (i + 1, y_q)\}
\]
\[
Z = \{(i, z_0), \ldots, (i, z_r)\}
\]
\[
X' = \{(i - 1, x'_0), \ldots, (i - 1, x'_t)\}
\]
\[
Y' = \{(i - 1, y'_0), \ldots, (i - 1, y'_t)\}
\]

of \( \ell \times \omega \) such that \( X \cap Y = \emptyset = X' \cap Y' \). It is sufficient to show that there exists \( z \notin \{0, z_0, \ldots, z_r\} \) so that

\[
p(i + 1, x_k) | z, k \leq p, \quad p(i - 1, x'_k) | z, k \leq s
\]
\[
p(i + 1, y_k) \nmid z, k \leq q, \quad p(i - 1, y'_k) \nmid z, k \leq t
\]
and also
\[
p(i, z) \uparrow y_k \text{ or } y_k = 0, \quad k \leq q
\]
\[
p(i, z) \uparrow y'_k \text{ or } y'_k = 0, \quad k \leq t.
\]
This can be achieved by setting
\[
z = \left( \prod_{k \leq p} p(i + 1, x_k) \right) \cdot \left( \prod_{k \leq s} p(i - 1, x'_k) \right) \cdot 2^w
\]
where \( w \) has been chosen sufficiently large to make \( z \neq z_0, \ldots, z_r \) and for \( p(i, z) \) not to divide any of the non-zero second components of elements of \( Y \cup Y' \). This determines a \( z \) with the required properties. The cases \( i = 0 \) and \( i = \ell - 1 \) are similarly dealt with.

4.4. Application of Theorem 2 to ranked diagrams. Let \( X \) be a generic \( \ell \)-diagram.

If \( A \) is a finite \( \ell \)-diagram and \( f : A \to X \) any embedding, and if \( B \) is a \( \ell \)-diagram with \( |B| = |A| + 1 \) and \( B \supset A \), then it follows directly from the extension axioms (iv) that \( f \) can be extended to an embedding of \( B \) into \( X \). Since each singleton \( \ell \)-diagram can be embedded into \( X \), it thus follows upon induction that any finite \( \ell \)-diagram can be embedded into \( X \). Finally, it follows from Proposition 1 that \( X \) is homogeneous. We conclude that \( X \) is the Fraisse limit of finite \( \ell \)-diagrams. We note that \( \text{Age}(X) \) is dense in \( X \) so that Theorem 2 also applies to generic \( \ell \)-diagrams.

4.5. Binary sequences that generate generic \( \ell \)-diagrams. Fix some canonical recursive bijection
\[
\psi : (\ell - 1) \times \omega \times \omega \to \omega.
\]
Given \( \alpha \in \mathcal{N} \) we generate a ranked diagram \( S_\alpha \) on the underlying set \( A = \ell \times \omega \) by putting
\[
(i, n)S_\alpha (i + 1, m) \text{ whenever } \alpha_{\psi(i,n,m)} = 1.
\]
(5)

We would now like to know for which \( \alpha \in \mathcal{N} \), the ranked diagram \((A, S_\alpha)\) generated by the binary sequence \( \alpha \) is \( \ell \)-generic, where \( A = \ell \times \omega \), as before. Let
\[
G = \{ \alpha \in \mathcal{N} | (A, S_\alpha, \{0\} \times \omega, \ldots, \{\ell - 1\} \times \omega) \text{ is a model for } T_\ell \}.
\]
The construction of \( S_\alpha \), as in equation (5), is already such that the axioms (i)-(iii) of \( T_\ell \) are automatically satisfied for all \( \alpha \).

Let \( P(\alpha, X, Y, Z, X', Y', z) \) stand for the predicate over \( \mathcal{N} \times (\text{Fin} A)^5 \times A \) which states that \( z \not\in Z \) and
\[
S_\alpha(z, x), \quad S_\alpha(x', z), \quad \neg S_\alpha(z, y) \quad \text{and} \quad \neg S_\alpha(y', z)
\]
hold, for all \( x \in X, x' \in X', y \in Y \) and \( y' \in Y' \) respectively. If we identify \( \text{Fin} A \) with \( \omega \) via a recursive bijection, then it is clear that \( P \) is a recursive predicate. Set \( K_i = \{i\} \times \omega \) for \( i < \ell \) and \( K_{\ell - 1} = K_{\ell - 1} = \emptyset \). Let \( Q(\alpha) \) be the predicate
\[
(\forall 0 \leq i < l)(\forall X \in \text{Fin}K_{i+1})(\forall Y \in \text{Fin}K_{i+1})(\forall Z \in \text{Fin}K_i)(\forall X' \in \text{Fin}K_{i-1})
(\forall Y' \in \text{Fin}K_{i-1})(\exists z \in K_i)(X \cap Y = X' \cap Y' = \emptyset \rightarrow P(\alpha, X, Y, Z, X', Y', z))
\]
which is to say that \( Q(\alpha) \) holds if and only if \( \alpha \) codes a generic \( \ell \)-diagram. It is clear that \( Q \) is a \( \Pi^0_3 \)-predicate. We have thus shown that

**Lemma 2.** \( G \) is a \( \Pi^0_3 \)-set.
Let us now consider the probability that a uniformly randomly generated $\alpha$ will give an $\ell$-generic RD on $A$, where our probability measure is the Lebesgue measure $\lambda$, as before.

**LEMMA 3.** With probability 1, a sequence $\alpha \in \mathcal{N}$ defines a generic $\ell$-diagram.

**PROOF.** We have to show that $\lambda(G) = 1$. Note that

$$G = \bigcap_{z \in K_i} \bigcup \{\alpha | P(\alpha, X, Y, Z, X', Y', z)\}$$

where the intersection runs over all $i, X, Y, Z, X', Y'$ such that $0 \leq i < \ell$; $X, Y \in \text{Fin}K_{i+1}; Z \in \text{Fin}K_i; X', Y' \in \text{Fin}K_{i-1}$ such that $X \cap Y = X' \cap Y' = \emptyset$.

Since this is a countable intersection, we can henceforth regard all parameters, save $z$, as fixed, and need only prove that

$$\lambda \left( \bigcup_{z \in K_i \setminus Z} \{\alpha | P(\alpha, X, Y, Z, X', Y', z)\} \right) = 1$$

when $X, Y, X', Y'$ are as above.

Now, if $z'$ and $z''$ are distinct elements of $K_i \setminus Z$, then $P(\alpha, X, Y, Z, X', Y', z')$ holding for $\alpha$ and $P(\alpha, X, Y, Z, X', Y', z'')$ holding for $\alpha$ are independent events. For, the evaluation of these two instances of the predicate reference disjoint (finite) sets of digits in the sequence $\alpha$ ($\psi$ above being one-to-one). In each case, the probability that $P$ holds is $2^{-n}$ where $n = |X| + |Y| + |X'| + |Y'|$. We may therefore apply the second Borel-Cantelli lemma to conclude that the union,

$$\bigcup_{z \in K_i \setminus Z} \{\alpha | P(\alpha, X, Y, Z, X', Y', z)\}$$

does indeed have measure 1, which proves the lemma.

Theorem 1 together with Lemmas 2 and 3 now immediately give the following theorem.

**THEOREM 4.** If $\alpha$ is a KC-string, then the ranked diagram $(A, S_\alpha)$ is $\ell$-generic.

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