ON ALMOST REPRESENTATIONS OF PROPERTY (T) GROUPS

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ABSTRACT. Property (T) for groups means a dichotomy: a representation either has an invariant vector or all vectors are far from being invariant. We show that, under a stronger condition of Žuk, a similar dichotomy holds for almost representations as well.

1. INTRODUCTION

Let $\Gamma$ be a group generated by a symmetric finite set $S$ and let $\pi : \Gamma \to U(H_\pi)$ be a unitary representation of $\Gamma$. Suppose that $\Gamma$ has the property (T) of Kazhdan (i.e. the trivial representation is isolated in the dual space of $\Gamma$). We refer to [3] for basic information about property (T). It is well known [4] that the spectrum of $\pi(x) = \frac{1}{|S|} \sum_{s \in \Gamma} \pi(s)$ has a gap near 1:

$$\text{Sp}(\pi(x)) \subset [-1, 1 - c] \cup \{1\},$$

where $c$ is the Kazhdan constant for $\Gamma$ (with respect to $S$). In terms of the group $C^*$-algebra, this means that we can apply a continuous function $f$ such that $f(1) = 1$ and $f(t) = 0$ for any $t \in [-1, 1 - c]$ to $x = \frac{1}{|S|} \sum_{g \in S} g \in C^*(\Gamma)$ to obtain the canonical projection $p = f(x) \in C^*(\Gamma)$ corresponding to the trivial representation [10].

Our aim is to generalize the above property for the case of almost representations of $\Gamma$. Recall that, for $\varepsilon \geq 0$, an $\varepsilon$-almost representation $\pi$ of $\Gamma$ (with respect to the given set $S$ of generators) is the map $\pi : S \to U(H_\pi)$ satisfying

$$\|\pi(s_1s_2) - \pi(s_1)\pi(s_2)\| \leq \varepsilon$$

for any $s_1, s_2, s_1s_2 \in S$ and $\pi(s^{-1}) = \pi(s)^*$ for any $s \in S$. This definition appeared in [1] and then (in a slightly different form) in [7]. If $\varepsilon = 0$ (in the case, when $\Gamma$ is finitely presented and $S$ is sufficiently big) then a 0-almost representation obviously generates a genuine representation of $\Gamma$. It is known that for some applications it suffices for $\pi$ to be defined on $S$ only instead of the whole $\Gamma$. Any small perturbation of a genuine representation is an almost representation, but there exist almost representations that are far from any genuine representation [11]. One should distinguish almost representations from other group ‘almost’ notions, e.g. quasi-representations, almost actions etc. [9] [2], which are completely different.

If we have an asymptotic representation (i.e. a continuous family of $\varepsilon_t$-almost representations $(\pi_t)_{t \in [0, \infty)}$ with $\lim_{t \to \infty} \varepsilon_t = 0$) then it follows from the theory of $C^*$-algebra asymptotic homomorphisms that the spectrum of $\pi_t(x)$ has a gap for $t$ sufficiently great: there is a continuous function $\alpha = \alpha(t) > 0$ such that $\lim_{t \to \infty} \alpha(t) = 0$ and $\text{Sp}(\pi_t(x)) \subset [-1, 1 - c + \alpha(t)] \cup [1 - \alpha(t), 1]$. Unfortunately, if we are interested in a single almost representation, it may be impossible to include an almost representation into an asymptotic one [6], and we don’t know how to check existence of a spectral gap because there is no nice formula for the projection $p$.

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Nevertheless, there is a condition, which is only slightly stronger than the property (T) and which provides a gap in $\text{Sp}(\pi(x))$ for an almost representation $\pi$. The importance of this condition was discovered by A. Žuk [12]. Let us recall his construction.

It is supposed that the neutral element doesn’t belong to $S$. A finite graph $L(S)$ is assigned to the set $S$ of generators as follows: the set of vertices of $L(S)$ is $S$ and the set $T$ of edges of $L(S)$ is the set of all pairs $\{(s, s') : s, s', s^{-1}s' \in S\}$. By including some additional elements in $S$, one can assume that the graph $L(S)$ is connected [12]. For a vertex $s \in L(S)$, let $\deg(s)$ denote its degree, i.e. the number of edges adjacent to $s$. Let $\Delta$ be a discrete Laplace operator acting on functions $f$ defined on vertices of $L(S)$ by

$$\Delta f(s) = f(s) - \frac{1}{\deg(s)} \sum_{s' \sim s} f(s'),$$  \hspace{1cm} (1)

where $s' \sim s$ means that the vertex $s'$ is adjacent to the vertex $s$. Operator $\Delta$ is a non-negative, self-adjoint operator on the (finitesimal Hilbert space $l^2(L(S), \deg)$ and zero is a simple eigenvalue of $\Delta$. Let $l_1(L(S))$ be the smallest non-zero eigenvalue of $\Delta$. We say that a group $\Gamma$ with the generating set $S$ satisfies the Žuk’s condition if $l_1(L(S)) > \frac{1}{2}$. One of the main results of [12] claims that the Žuk’s condition implies property (T) with the Kazhdan constant $c = \frac{2}{3} \left( 2 - \frac{1}{l_1(L(S))} \right)$. We appreciate Žuk’s approach because it allows to work with almost representations as well. The main result of this paper is the following statement:

**Theorem 1.** Let $\Gamma$, $S$ satisfy the Žuk’s condition and let $c$ be as above. There is a continuous function $\alpha = \alpha(\varepsilon) \geq 0$ such that $\alpha(0) = 0$ and

$$\text{Sp} \left( \frac{1}{|S|} \sum_{s \in S} \pi(s) \right) \subset [-1, 1 - c/2 + \alpha(\varepsilon)] \cup [1 - \alpha(\varepsilon), 1]$$

for any $\varepsilon$-almost representation $\pi$.

**Corollary 2.** For any $\varepsilon$-almost representation $\pi$ there exists an $(\varepsilon + 6|S|\alpha(\varepsilon))$-almost representation $\pi'$ such that $\|\pi'(s) - \pi(s)\| \leq 3|S|\alpha(\varepsilon)$ for any $s \in S$ and $\pi' = \pi + \sigma$, where $\pi$ is a trivial representation and $\sigma$ is an $(\varepsilon + 6|S|\alpha(\varepsilon))$-almost representation satisfying

$$\text{Sp} \left( \frac{1}{|S|} \sum_{s \in S} \sigma(s) \right) \subset [-1, 1 - c/2 + (1 + 3|S|)\alpha(\varepsilon)].$$  \hspace{1cm} (2)

**Proof.** Let $H \subset \mathcal{H}_\pi$ be the range of the spectral projection of $\frac{1}{|S|} \sum_{s \in S} \pi(s)$ corresponding to the set $[1 - c + \alpha(\varepsilon), 1]$. Then $\|\pi(s)\xi - \xi\| \leq |S|\alpha(\varepsilon)\|\xi\|$ for any $s \in S$ and for any $\xi \in H$ and if we write $\pi(s)$ as a matrix $(A \ B)$ with respect to the decomposition $H \oplus H^\perp$ then $\|B\| \leq |S|\alpha(\varepsilon)$ and $\|C\| \leq |S|\alpha(\varepsilon)$, hence there exists a unitary $D'$ such that $\|D' - D\| \leq 2|S|\alpha(\varepsilon)$. Put $\pi'(s) = (A \ B^T)$. Then $\|\pi'(s) - \pi(s)\| \leq 3|S|\alpha(\varepsilon)$ and $\pi'$ is obviously an $(\varepsilon + 6|S|\alpha(\varepsilon))$-almost representation, which is trivial on $H$. Hence $H^\perp$ is $\pi(s)$-invariant for all $s \in S$. Since the restriction of $\frac{1}{|S|} \sum_{s \in S} \pi(s)$ onto $H^\perp$ satisfies $\left( \frac{1}{|S|} \sum_{s \in S} \pi(s) \right)_{H^\perp} \leq 1 - c/2 + \alpha(\varepsilon)$, we get (2).

The remaining part of the paper is devoted to the proof of Theorem 1. The proof follows the proof of Žuk for genuine representations, but has additional argument because relations for almost representations do not hold exactly, but only approximately. It will be seen from the proof that one can take $\alpha(\varepsilon) = O(\varepsilon^{2/5})$ in Theorem 1.
2. Proof of the theorem

The following Hilbert spaces and operators are defined exactly as in [12]. It doesn’t matter that \( \pi \) is not a representation here.

Definition 3 ([12]). For \( r = 0, 1 \) and 2 let \( C^r \) be the Hilbert spaces defined as follows:

\[
C^0 = \{ u : u \in \mathcal{H}_\pi \}; \quad \langle u, w \rangle_{C^0} = \langle u, w \rangle_{\mathcal{H}_\pi} |T| \quad \text{for } u, w \in C^0;
\]

\[
C^1 = \{ f : S \rightarrow \mathcal{H}_\pi : f(s^{-1}) = -\pi(s^{-1}) f(s) \text{ for all } s \in S \}; \quad \langle f, g \rangle_{C^1} = \sum_{s \in S} \langle f(s), g(s) \rangle_{\mathcal{H}_\pi} n(s);
\]

\[
C^2 = \{ g : T \rightarrow \mathcal{H}_\pi \}; \quad \langle f, g \rangle_{C^2} = \sum_{t \in T} \langle f(t), g(t) \rangle_{\mathcal{H}_\pi},
\]

where \( n(s) = \#\{ s' \in S : (s, s') \in T \} \).

Since the graph \( L(S) \) is connected, \( n(s) \geq 0 \) for every \( s \in S \) and \( n(s) = \deg(s) \). Moreover, it is easy to see that \( n(s) = n(s^{-1}) \) and \( \sum_{s \in S} n(s) = |T| \).

Definition 4 ([12]). Let us define linear operators \( d_1 : C^0 \rightarrow C^1 \) and \( d_2 : C^1 \rightarrow C^2 \) as follows:

\[
d_1 u(s) = \pi(s) u - u, \text{ for all } u \in C^0; \quad d_2 f((s, s')) = f(s) - f(s') + \pi(s) f(s^{-1} s'), \text{ for all } f \in C^1.
\]

Lemma 5 ([12]). One has \( d_1^* f = -2 \sum_{s \in S} f(s) \frac{n(s)}{|T|} \) for any \( f \in C^1 \) and \( \|d_1^*\|_{C^1 \rightarrow C^0} \leq 2 \).

In [12] it is shown that \( d_2 d_1 = 0 \) for any unitary representation. However, if \( \pi \) is only an almost representation then this doesn’t hold any more. One can only show that this composition is small.

Lemma 6. For any \( u \in C^0 \) and \( (s, s') \in T \) one has

\[
\|d_2 d_1 u((s, s'))\|_{\mathcal{H}_\pi} \leq \varepsilon \|u\|_{\mathcal{H}_\pi} \tag{3}
\]

Proof. By the definitions of \( d_1 \) and \( d_2 \), we have

\[
\|d_2 d_1 u((s, s'))\|_{\mathcal{H}_\pi} = \|d_1 u(s) - d_1 u(s') + \pi(s) d_1 u(s^{-1} s')\|_{\mathcal{H}_\pi} = \|\pi(s) u - \pi(s') u + \pi(s) \pi(s^{-1} s') u - u\|_{\mathcal{H}_\pi} \leq \varepsilon \|u\|_{\mathcal{H}_\pi},
\]

hence we have \( \|d_2 d_1 u((s, s'))\|_{\mathcal{H}_\pi} \leq \varepsilon \|u\|_{\mathcal{H}_\pi} \). \( \square \)

Corollary 7. \( \|d_2 d_1\|_{C^2 \rightarrow C^0} \leq \varepsilon \).

That’s why we have to introduce two more (sub)spaces.

Definition 8. For any \( \beta \geq 0 \) set

\[
B^0(\beta) = \{ P_\Omega(d_1^* d_1)(u) : u \in C^0 \} \subset C^0, \quad B^1(\beta) = \{ d_1 u : u \in B^0(\beta) \} \subset C^1,
\]

where \( P_\Omega \) is the spectral projection corresponding to \( \Omega = [\beta, +\infty) \).

It is clear that \( B^0(\beta) \) and \( B^1(\beta) \) are invariant subspaces for \( d_1^* d_1 \) and \( d_1 d_1^* \) respectively.
Proposition 9. If there exists $c > 0$ and $0 < \delta < c/2$ such that for every $f \in B^1(\frac{\delta^2}{|T|})$

$$\langle d_1d_1^* f, f \rangle_{C^1} > c \langle f, f \rangle_{C^1}$$  \tag{4}

then, for any $\varepsilon$-almost representation $\pi$, either there exists $u \in C^0$ such that

$$\|\pi(s)u - u\|_{H_s} < \delta \|u\|_{H_s} \quad \text{for any } s \in S$$  \tag{5}

or

$$\max_{s \in S} \|\pi(s)u - u\|_{H_s} \geq c/2 \|u\|_{H_s}$$  \tag{6}

for every $u \in C^0$.

Proof. First, we show that if there is no $u \in C^0$ satisfying \([5]\) then $B^0(\frac{\delta^2}{|T|}) = C^0$. If this is not true then there exists a non-zero vector $u^\perp$ orthogonal to $B^0(\frac{\delta^2}{|T|})$. Since $\|d_1^*d_1u^\perp\|_{C^0} < \frac{\delta^2}{|T|} \|u^\perp\|_{C^0}$, we have $\langle d_1u^\perp, d_1^*d_1u^\perp \rangle_{C^1} = \langle u^\perp, d_1^*d_1u^\perp \rangle_{C^0} \leq \|u^\perp\|_{C^0} \|d_1^*d_1u^\perp\|_{C^0} < \frac{\delta^2}{|T|} \|u^\perp\|_{C^0}$, which implies that $\|d_1u^\perp\|_{C^1} < \frac{\delta}{\sqrt{|T|}} \|u^\perp\|_{C^0}$. By definition of $\|\cdot\|_{C^1}$, it is easy to see that $\|\pi(s)u^\perp - u^\perp\|_{H_s} \leq \|d_1u^\perp\|_{C^1} < \frac{\delta}{\sqrt{|T|}} \sqrt{|T|} \|u^\perp\|_{H_s} = \delta \|u^\perp\|_{H_s}$ for any $s \in S$, which is in contradiction with the assumption.

Next we prove that \([4]\) implies that the operator $d_1d_1^* : B^1(\frac{\delta^2}{|T|}) \to B^1(\frac{\delta^2}{|T|})$ has a bounded inverse. By \([4]\), $d_1d_1^*(B^1(\frac{\delta^2}{|T|}))$ is closed in $B^1(\frac{\delta^2}{|T|})$. If $d_1d_1^*(B^1(\frac{\delta^2}{|T|}))$ were different from $B^1(\frac{\delta^2}{|T|})$, there would exist a non-zero vector $f \in B^1(\frac{\delta^2}{|T|})$ orthogonal to $d_1d_1^*(B^1(\frac{\delta^2}{|T|}))$. Then we would have, by \([4]\),

$$0 = \langle f, d_1d_1^*(f) \rangle_{C^1} > c \langle f, f \rangle_{C^1}$$

which is a contradiction.

Thus $d_1d_1^* : B^1(\frac{\delta^2}{|T|}) \to B^1(\frac{\delta^2}{|T|})$ has a bounded inverse $(d_1d_1^*)^{-1} : B^1(\frac{\delta^2}{|T|}) \to B^1(\frac{\delta^2}{|T|})$ and $\|(d_1d_1^*)^{-1}\|_{B^1(\frac{\delta^2}{|T|}) \to B^1(\frac{\delta^2}{|T|})} \leq c^{-1}$.

Now suppose that neither \([5]\) nor \([6]\) holds. Then there is some $\gamma < c/2$ and some $u \in H_s$ such that $\|u\|_{H_s} = 1$ and $\|\pi(s)u - u\|_{H_s} \leq \gamma$ for every $s \in S$. Therefore

$$\|d_1u\|_{B^1(\frac{\delta^2}{|T|})}^2 = \sum_{s \in S} \|d_1u(s)\|_{H_s}^2 n(s) = \sum_{s \in S} \|\pi(s)u - u\|_{H_s}^2 n(s) \leq \sum_{s \in S} \gamma^2 n(s) = \gamma^2|T|$$

which gives $\|d_1u\|_{B^1(\frac{\delta^2}{|T|})} \leq \gamma \sqrt{|T|}$. Then

$$\|d_1^*(d_1d_1^*)^{-1}d_1u\|_{C^0} \leq \|d_1^*\|_{B^1(\frac{\delta^2}{|T|}) \to C^0} \cdot \|(d_1d_1^*)^{-1}\|_{B^1(\frac{\delta^2}{|T|}) \to B^1(\frac{\delta^2}{|T|})} \cdot \|d_1u\|_{B^1(\frac{\delta^2}{|T|})} \leq 2 \cdot c^{-1} \cdot \gamma \sqrt{|T|} < \sqrt{|T|}.$$  

By definition of the norm in $C^0$ one has then $d_1^*(d_1d_1^*)^{-1}d_1u = u'$, whence $\|u'\|_{H_s} < 1$, so the vector $u - u'$ is non-zero. Finally,

$$d_1(u - u') = d_1u - d_1(d_1^*(d_1d_1^*)^{-1})d_1u = d_1u - d_1u,$$

which means that, for every $s \in S$,

$$\pi(s)(u - u') - (u - u') = 0.$$  

Thus $u - u'$ is a non-zero invariant vector and \([5]\) holds, which gives a contradiction. \qed
Following [12], define the operator $D : C^1 \rightarrow C^2$ by

$$Df((s_1, s_2)) = f(s_1) - f(s_2),$$

where $f \in C^1$ and $(s_1, s_2) \in T$.

In [12], the relation between $d_2$ and $D$ was investigated and it was shown that $\frac{1}{3}(d_2f, d_2f)_{C^2} = \langle Df, Df \rangle_{C^2} - \langle f, f \rangle_{C^1}$. Since here an almost representation is engaged, we have to estimate the difference between the left and the right hand side.

**Lemma 10.** For every $f \in C^1$ one has

$$\langle f, f \rangle_{C^1} = \sum_{(s, s') \in T} \langle f(s^{-1}s'), f(s^{-1}s') \rangle_{\mathcal{H}_\pi}; \quad (7)$$

$$\|d_2f((s, s')) + d_2f((s', s))\|_{\mathcal{H}_\pi} \leq \varepsilon \|f((s')^{-1}s)\|_{\mathcal{H}_\pi}; \quad (8)$$

$$d_2f((s, s')) = -\pi(s)d_2f((s^{-1}, s^{-1}s')); \quad (9)$$

$$\sum_{(s, s') \in T} \langle d_2f((s^{-1}, s^{-1}s')), -f(s^{-1}s') \rangle_{\mathcal{H}_\pi} = \sum_{(s, s') \in T} \langle d_2f((s, s')), -f(s') \rangle_{\mathcal{H}_\pi}; \quad (10)$$

$$\|d_2f((s, s')) - \pi(s')d_2f(((s')^{-1}, (s')^{-1}s))\|_{\mathcal{H}_\pi} \leq \varepsilon \|f((s')^{-1}s)\|_{\mathcal{H}_\pi}; \quad (11)$$

$$\left| \sum_{(s, s') \in T} \langle d_2f((s, s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} - \frac{1}{3}(d_2f, d_2f)_{C^2} \right| < \frac{5}{3}\varepsilon \|f\|_{C^1}^2; \quad (12)$$

**Proof.** The proof of (7) and (9) in [12] doesn’t depend on the property of $\pi$ being a representation.

$$d_2f((s, s')) = f(s) - f(s') + \pi(s)f(s^{-1}s'),$$

$$= -(f(s') - f(s) + \pi(s)\pi(s^{-1}s')f((s')^{-1}s)),$$

$$= -(f(s') - f(s) + \pi(s')f((s')^{-1}s) + \pi(s')f((s')^{-1}s) - \pi(s)\pi(s^{-1}s')f((s')^{-1}s)),$$

$$= -d_2f((s', s)) + (\pi(s')f((s')^{-1}s) - \pi(s)\pi(s^{-1}s')f((s')^{-1}s)),$$

hence

$$\|d_2f((s, s')) + d_2f((s', s))\|_{\mathcal{H}_\pi} = \|\pi(s')f((s')^{-1}s) - \pi(s)\pi(s^{-1}s')f((s')^{-1}s)\|_{\mathcal{H}_\pi}$$

$$\leq \varepsilon \|f((s')^{-1}s)\|_{\mathcal{H}_\pi},$$

which proves (8).

By (8) and (9),

$$\|d_2f((s, s')) - \pi(s')d_2f(((s')^{-1}, (s')^{-1}s))\|_{\mathcal{H}_\pi}$$

$$= \|d_2f((s, s')) + d_2f((s', s)) - d_2f((s', s)) - \pi(s')d_2f(((s')^{-1}, (s')^{-1}s))\|_{\mathcal{H}_\pi}$$

$$\leq \|d_2f((s, s')) + d_2f((s', s))\|_{\mathcal{H}_\pi} \leq \varepsilon \|f((s')^{-1}s)\|_{\mathcal{H}_\pi},$$

which proves (11).

Consider the mapping $M : T \rightarrow T$, $M((s, s')) = (s^{-1}, s^{-1}s') = (t, t')$. Then it is easy to see $M$ is a bijection. Hence,

$$\sum_{(s, s') \in T} \langle df((s^{-1}, s^{-1}s')), -f(s^{-1}s') \rangle_{\mathcal{H}_\pi} = \sum_{(t, t') \in T} \langle df((t, t')), -f(t') \rangle_{\mathcal{H}_\pi},$$

which is just a matter of notation. This proves (10).
\[
\sum_{(s,s') \in T} \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi}
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + \langle -\pi(s) d_2 f((s^{-1}, s^{-1} s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + \langle \pi(s') d_2 f(((s')^{-1}, (s')^{-1} s)), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} \right) + D_1
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + \langle d_2 f((s^{-1}, s^{-1} s')), -f(s^{-1} s') \rangle_{\mathcal{H}_\pi} \right) + D_1 + D_2
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + \langle d_2 f((s, s')), -f(s') \rangle_{\mathcal{H}_\pi} \right) + D_1 + D_2
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + \langle d_2 f((s, s')), f(s) \rangle_{\mathcal{H}_\pi} \right) + D_1 + D_2 + D_3
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), f(s) - f(s') + \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} + D_1 + D_2 + D_3 \right)
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \left( \langle d_2 f((s, s')), d_2 f((s', s')) \rangle_{\mathcal{H}_\pi} + D_1 + D_2 + D_3 \right)
\]
\[= \frac{1}{3} \sum_{(s,s') \in T} \langle d_2 f, d_2 f \rangle_{C^2} + D_1 + D_2 + D_3,
\]
where
\[D_1 = \frac{1}{3} \sum_{(s,s') \in T} \langle d_2 f((s, s')) - \pi(s') d_2 f(((s')^{-1}, (s')^{-1} s)), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi},\]
\[D_2 = \frac{1}{3} \sum_{(s,s') \in T} \langle d_2 f(((s')^{-1}, (s')^{-1} s)), \pi((s')^{-1}) \pi(s) - \pi((s')^{-1} s) \rangle_{\mathcal{H}_\pi},\]
\[D_3 = \frac{1}{3} \sum_{(s,s') \in T} \langle d_2 f((s, s')) + d_2 f((s', s')), -f(s) \rangle_{\mathcal{H}_\pi},\]
hence
\[\sum_{(s,s') \in T} \langle d_2 f((s, s')), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_\pi} - \frac{1}{3} \sum_{(s,s') \in T} \langle d_2 f, d_2 f \rangle_{C^2} = D_1 + D_2 + D_3.
\]
By Cauchy inequality, definition of $\|\cdot\|_{C^1}$ and (7)—(11), we have

\[
|D_1| = \frac{1}{3} \left| \sum_{(s, s') \in T} \langle d_2 f((s, s')) - \pi(s') d_2 f(((s')^{-1}, (s')^{-1} s)), \pi(s) f(s^{-1} s') \rangle_{\mathcal{H}_{\pi}} \right|
\]

\[
\leq \frac{1}{3} \sum_{(s, s') \in T} \| d_2 f((s, s')) - \pi(s') d_2 f(((s')^{-1}, (s')^{-1} s)), \pi(s) f(s^{-1} s') \|_{\mathcal{H}_{\pi}} \| \pi(s) f(s^{-1} s') \|_{\mathcal{H}_{\pi}}
\]

\[
\leq \frac{1}{3} \sum_{(s, s') \in T} \varepsilon \| f((s')^{-1} s) \|_{\mathcal{H}_{\pi}} \| f(s^{-1} s') \|_{\mathcal{H}_{\pi}}
\]

\[
\leq \frac{1}{3} \varepsilon \left( \sum_{(s, s') \in T} \| f((s')^{-1} s) \|^2_{\mathcal{H}_{\pi}} \right)^{1/2} \left( \sum_{(s, s') \in T} \| f(s^{-1} s') \|^2_{\mathcal{H}_{\pi}} \right)^{1/2} = \frac{1}{3} \varepsilon \| f \|^2_{C^1},
\]

\[
|D_2| = \frac{1}{3} \left| \sum_{(s, s') \in T} \langle d_2 f(((s')^{-1}, (s')^{-1} s)), \pi((s')^{-1}) \pi(s) - \pi((s')^{-1} s)) f(s^{-1} s') \rangle_{\mathcal{H}_{\pi}} \right|
\]

\[
\leq \frac{1}{3} \sum_{(s, s') \in T} \| d_2 f(((s')^{-1}, (s')^{-1} s)) \|_{\mathcal{H}_{\pi}} \| \pi((s')^{-1}) \pi(s) - \pi((s')^{-1} s)) f(s^{-1} s') \|_{\mathcal{H}_{\pi}}
\]

\[
\leq \frac{1}{3} \sum_{(s, s') \in T} \| f((s')^{-1}) - f((s')^{-1} s) + \pi((s')^{-1}) f(s) \|_{\mathcal{H}_{\pi}} \varepsilon \| f(s^{-1} s') \|_{\mathcal{H}_{\pi}}
\]

\[
\leq \frac{1}{3} \varepsilon \left( \sum_{(s, s') \in T} \| f((s')^{-1}) \|^2_{\mathcal{H}_{\pi}} \right)^{1/2} \left( \sum_{(s, s') \in T} \| f(s^{-1} s') \|^2_{\mathcal{H}_{\pi}} \right)^{1/2}
\]

\[
+ \frac{1}{3} \varepsilon \left( \sum_{(s, s') \in T} \| f((s')^{-1} s) \|^2_{\mathcal{H}_{\pi}} \right)^{1/2} \left( \sum_{(s, s') \in T} \| f(s^{-1} s') \|^2_{\mathcal{H}_{\pi}} \right)^{1/2}
\]

\[
+ \frac{1}{3} \varepsilon \left( \sum_{(s, s') \in T} \| f(s) \|^2_{\mathcal{H}_{\pi}} \right)^{1/2} \left( \sum_{(s, s') \in T} \| f(s^{-1} s') \|^2_{\mathcal{H}_{\pi}} \right)^{1/2}
\]

\[
= \frac{1}{3} \varepsilon \| f \|^2_{C^1} + \frac{1}{3} \varepsilon \| f \|^2_{C^1} + \frac{1}{3} \varepsilon \| f \|^2_{C^1} = \varepsilon \| f \|^2_{C^1},
\]

\[
|D_3| = \frac{1}{3} \left| \sum_{(s, s') \in T} \langle d_2 f((s, s')) + d_2 f((s', s)), -f(s) \rangle_{\mathcal{H}_{\pi}} \right|
\]

\[
\leq \frac{1}{3} \sum_{(s, s') \in T} \| d_2 f((s, s')) + d_2 f((s', s)) \|_{\mathcal{H}_{\pi}} \| f(s) \|_{\mathcal{H}_{\pi}}
\]
\[ \leq \frac{1}{3} \sum_{(s,s') \in T} \varepsilon \left\| f((s')^{-1}s) \right\|_{\mathcal{H}_\pi} \| f(s) \|_{\mathcal{H}_\pi} \]
\[ \leq \frac{1}{3} \varepsilon \left( \sum_{(s,s') \in T} \left\| f((s')^{-1}s) \right\|_{\mathcal{H}_\pi}^2 \right)^{1/2} \left( \sum_{(s,s') \in T} \| f(s) \|_{\mathcal{H}_\pi}^2 \right)^{1/2} = \frac{1}{3} \varepsilon \| f \|_{C^1}^2. \]

So
\[ \left| \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} - \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} \right| = |D_1 + D_2 + D_3| \]
\[ \leq |D_1| + |D_2| + |D_3| \leq \frac{5}{3} \varepsilon \| f \|_{C^1}^2, \]
which proves \((12)\).

**Proposition 11.** For every \( f \in C^1 \) one has
\[ |\langle Df, Df \rangle_{C^2} - \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} - \langle f, f \rangle_{C^1} | \leq \frac{10}{3} \varepsilon \langle f, f \rangle_{C^1} \]

**Proof.** By definition of operator \( D \), we have
\[ \langle Df, Df \rangle_{C^2} = \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s'), d_2 f((s,s')) \rangle_{\mathcal{H}_\pi} \]
\[ = \sum_{(s,s') \in T} \langle d_2 f((s,s')), d_2 f((s,s')) \rangle_{\mathcal{H}_\pi} + \sum_{(s,s') \in T} \langle f(s^{-1}s'), f(s^{-1}s') \rangle_{\mathcal{H}_\pi} - 2 \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} \]
\[ = \langle d_2 f, d_2 f \rangle_{C^2} + \langle f, f \rangle_{C^1} - \frac{2}{3} \langle d_2 f, d_2 f \rangle_{C^2} + \frac{2}{3} \langle d_2 f, d_2 f \rangle_{C^2} - 2 \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} \]
\[ = \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} + \langle f, f \rangle_{C^1} + 2 \left( \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} - \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} \right), \]

hence
\[ |\langle Df, Df \rangle_{C^2} - \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} - \langle f, f \rangle_{C^1} | \]
\[ = 2 \left| \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} - \sum_{(s,s') \in T} \langle d_2 f((s,s')), \pi(s)f(s^{-1}s') \rangle_{\mathcal{H}_\pi} \right| \leq \frac{10}{3} \varepsilon \langle f, f \rangle_{C^1}, \]
which ends the proof of Proposition 11. \( \square \)

Note that every \( f \in C^1 \) can be considered as a function on \( L(S) \). It was shown in [12] (and the proof doesn’t depend on the property of \( \pi \) to be a representation) that \( \langle f, f \rangle_{C^1} = \langle f, f \rangle_{L(S)} \)
\[ \langle Df, Df \rangle_{C^2} = \sum_{(s,s') \in T} \langle f(s) - f(s'), f(s) - f(s') \rangle_{\mathcal{H}_\pi} = 2 \langle \Delta f, f \rangle_{L(S)} \]
Theorem 1 should satisfy
\[ \alpha \text{ and } \parallel \]

\[ \text{Proof. By (13), we have} \]
\[ \text{Lemma 13. For any} \]
\[ \text{Corollary 14. For every} \]
\[ \text{proof (14).} \]
\[ \text{Proof. By Lemma 6, for any} \]
\[ \text{Corollary 14 provides the constant} \]
\[ \text{therefore, we denote } \lambda_1(L(S)) \text{ by } \lambda_1. \]

For every \( f \in C^1 \) one has
\[ \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} + \frac{1}{2} \lambda_1 \langle d_1^* f, d_1^* f \rangle_{C^0} \geq (2 \lambda_1 - 1 - \frac{10}{3} \varepsilon) \langle f, f \rangle_{C^1}. \] (14)

\[ \text{Proof. By (13), we have} \]
\[ \text{2} \lambda_1 \langle f, f \rangle_{C^1} - \langle f, f \rangle_{C^1} \leq \langle \Delta f, f \rangle_{L(S)} - \langle f, f \rangle_{C^1} + \frac{\lambda_1}{2} \langle d_1^* f, d_1^* f \rangle_{C^0}. \]
\[ \text{Since} \]
\[ \text{\langle Df, Df \rangle}_{C^2} = 2 \langle \Delta f, f \rangle_{L(S)}, \]
\[ \text{2} \lambda_1 \langle f, f \rangle_{C^1} - \langle f, f \rangle_{C^1} \leq \langle Df, Df \rangle_{C^2} - \langle f, f \rangle_{C^1} + \frac{\lambda_1}{2} \langle d_1^* f, d_1^* f \rangle_{C^0} \]
\[ \leq \frac{1}{3} \langle d_2 f, d_2 f \rangle_{C^2} + \frac{10}{3} \varepsilon \langle f, f \rangle_{C^1} + \frac{\lambda_1}{2} \langle d_1^* f, d_1^* f \rangle_{C^0}, \]
\[ \text{which proves (14).} \]

For any \( f \in B^1(\frac{\varepsilon^2}{|T|}) \) one has
\[ \| d_2 f \|_{C^2}^2 \leq \frac{4 |T| \varepsilon^2}{\delta^4} \| f \|_{C^1}. \]

\[ \text{Proof. By Lemma 6, for any} \]
\[ \| d_2 d_1 u \|_{C^2}^2 = \sum_{(s, s') \in T} \| d_2 d_1 u((s, s')) \|_{H_x}^2 \leq \varepsilon^2 \sum_{(s, s') \in T} \| u \|_{H_x}^2 = \varepsilon^2 |T| \| u \|_{H_x}^2 = \varepsilon^2 \| u \|_{C^0}^2. \]
\[ \text{Since} \]
\[ \text{\langle d_1(B^0(\frac{\varepsilon^2}{|T|})) \rangle \text{ is dense in } B^1(\frac{\varepsilon^2}{|T|}), \text{ for any} \}
\[ \text{it also holds that} \]
\[ \| d_2 f \|_{C^2}^2 \leq \frac{4 |T| \varepsilon^2}{\delta^4} \| f \|_{C^1}. \]

For every \( f \in C^1 \) one has
\[ \langle d_1^* f, d_1^* f \rangle_{C^0} \geq \left( 4 - \frac{2}{\lambda_1} - \frac{20 \varepsilon}{3 \lambda_1} - \frac{8 |T| \varepsilon^2}{3 \lambda_1 \delta^4} \right) \langle f, f \rangle_{C^1}. \]

Corollary 14 provides the constant \( c \) for Proposition 9. Now the function \( \alpha(\varepsilon) \) from Theorem 1 should satisfy \( \alpha(\varepsilon) = \max \left\{ \frac{10 \varepsilon}{3 \lambda_1} + \frac{8 |T| \varepsilon^2}{3 \lambda_1 \delta^4}, \delta \right\} \). In order to get a continuous function with \( \alpha(0) = 0 \) one may take \( \delta = \varepsilon^{2/5} \). Then \( \alpha(\varepsilon) = O(\varepsilon^{2/5}) \) and Theorem 1 directly follows from Corollary 14 and Proposition 9.

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