Shorter Signatures from Proofs of Knowledge for the SD, MQ, PKP and RSD Problems

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**Abstract.** The MPC in the head introduced in [IKOS07] has established itself as an important paradigm in order to design efficient digital signatures. In particular, it has been leveraged in the Picnic scheme [CDG+20] that is currently considered in the third round of NIST Post-Quantum Standardization process. In addition, it has been used in [Bee20] to introduce the Proof of Knowledge (PoK) with Helper paradigm. This construction permits to design shorter signatures but induces a non negligible performance overhead. In this paper, our contributions are twofold. Firstly, we introduce a new PoK with Helper for the Syndrome Decoding (SD) problem. This construction relies on ideas from [BGMK22] and [FJR21] and improve the latter using a new technique that can be seen as performing the cut and choose with a meet in the middle approach. Secondly and most importantly, we introduce a new paradigm to design PoK that brings improvements over the PoK with Helper one. Indeed, we show how one can substitute the Helper in these constructions by leveraging the underlying structure of the considered problem. This new approach does not suffer from the performance overhead inherent to the PoK with Helper paradigm hence offers different trade-offs between signature sizes and performances. Interestingly, our new approach is quite generic and can be applied to many problems and their associated PoK.

In order to demonstrate this versatility, we provide new PoK related to the SD, MQ, PKP and RSD problems. In practice, these PoK lead to shorter signatures for the aforementioned problems. Indeed, considering (public key + signature), we get sizes below 12 kB for our signature related to the SD problem, below 8 kB for our signature related to the MQ problem, below 9 kB for our signature related to the PKP problem and below 7 kB for our signature related to the RSD problem.

**Document status.** This draft have been made publicly available to provide a written record of this work following its preliminary presentation during a workshop. A revamped version of this draft will be released in due time.

1 Preliminaries

1.1 Hard problems

**Definition 1 (SD problem).** Given positive integers \((q, n, k, w)\), a random parity-check matrix \(H \leftarrow F^{(n-k) \times n}_q\) and a syndrome \(y \in F_q^{(n-k)}\), the syndrome
Theorem 2. If there exists a MQ problem with success probability \( F \) asks to find a solution \( x \in \mathbb{F}_q^n \) such that \( Hx^\top = y^\top \) and \( w_H(x) = w \).

**Definition 2 (QCSD problem).** Given positive integers \((n = \ell k, k, w, M)\), a random parity-check matrix of a quasi-cyclic code \( H \leftarrow \mathcal{QC}(\mathbb{F}_2^{(n-k) \times n}) \) and a syndrome \( y \in \mathbb{F}_2^{(n-k)} \), the quasi-cyclic syndrome decoding problem QCSD \((n, k, w)\) asks to find \( x \in \mathbb{F}_2^n \) such that \( Hx^\top = y^\top \) and \( w_H(x) = w \).

**Definition 3 (DiffSD problem).** Let \((n = \ell k, k, w, M)\) be positive integers, \( H \leftarrow \mathcal{QC}(\mathbb{F}_2^{(n-k) \times n}) \) be a random parity-check matrix of a quasi-cyclic code and foreach \( \mu \in [1, M], y_\mu \in \mathbb{F}_2^{n-k} \) be a syndrome such that \( Hx_\mu^\top = y_\mu^\top \) with \( w_H(x_\mu) = w \). The differential syndrome decoding problem DiffSD \((n, k, w, M)\) asks to find \((c, d_1, d_2) \in \mathbb{F}_2^{(n-k) \times \mathbb{F}_2^2 \times \mathbb{F}_2^2} \) such that foreach \( i \in [1, 2], H(d_i + c = \text{rot}_{n_i}(y_\mu^\top)) \) with \( w_H(d_i) = w \) for \( \kappa_i \in [1, k] \) and \( \mu_i \in [1, M] \).

**Theorem 1.** If there exists a PPT algorithm solving the DiffSD \((n, k, w, M)\) problem with probability \( p \), then there exists a PPT algorithm solving the QCSD \((n, k, w)\) problem with probability \( (1 - \frac{c}{2^{n-k+1}}) \cdot \frac{1}{M} \cdot p \).

Proof. Sketch of proof for the Theorem 1 can be found in Appendix A.

**Definition 4 (MQ problem).** Given positive integers \((q, m, n)\), a multivariate quadratic map \( F : \mathbb{F}_q^n \to \mathbb{F}_q^m \) of \( m \) quadratic polynomials in \( n \) variables defined over \( \mathbb{F}_q \) and \( y \in \mathbb{F}_q^m \), the multivariate quadratic problem MQ \((q, m, n)\) asks to find a solution \( x \in \mathbb{F}_q^n \) such that \( F(x) = y \).

**Definition 5 (MQH problem).** Given positive integers \((q, m, n)\), a multivariate quadratic map \( F : \mathbb{F}_q^n \to \mathbb{F}_q^m \) of \( m \) homogeneous quadratic polynomials in \( n \) variables defined over \( \mathbb{F}_q \) and \( y \in \mathbb{F}_q^m \), the homogeneous multivariate quadratic problem MQH \((q, m, n)\) asks to find a solution \( x \in \mathbb{F}_q^n \) such that \( F(x) = y \).

**Definition 6 (DiffMQH problem).** Let \((q, m, n, M)\) be positive integers, \( F : \mathbb{F}_q^n \to \mathbb{F}_q^m \) be a multivariate quadratic map of \( m \) homogeneous quadratic polynomials of degree 2 in \( n \) variables defined over \( \mathbb{F}_q \) and foreach \( \mu \in [1, M], y_\mu \in \mathbb{F}_q^m \) such that \( F(x_\mu) = y_\mu \) where \( x_\mu \in \mathbb{F}_q^n \). The differential homogeneous multivariate quadratic problem DiffMQH \((q, m, n)\) asks to find \((c, d_1, d_2) \in \mathbb{F}_q^m \times \mathbb{F}_q^2 \times \mathbb{F}_q^2 \) such that for each \( i \in [1, 2], F(d_i + c = \kappa_i^2 \cdot y_\mu) \).

**Theorem 2.** If there exists a PPT algorithm solving the DiffMQH \((q, m, n, M)\) problem with success probability \( p \), then there exists a PPT algorithm solving the MQH \((q, m, n)\) problem with success probability \( (1 - \frac{1}{q(m-n)}) \cdot \frac{1}{M} \cdot p \).
Definition 7 (PKP problem). Given positive integers \((q, m, n)\), a random matrix \(H \leftarrow \mathbb{F}_q^{m \times n}\) and a vector \(x \in \mathbb{F}_q^n\), the permuted kernel problem \(\text{PKP}(q, m, n)\) asks to find a permutation \(\pi \in S_n\) such that \(H(\pi[x]) = 0\).

Definition 8 (IPKP problem). Given positive integers \((q, m, n)\), a random matrix \(H \leftarrow \mathbb{F}_q^{m \times n}\), a vector \(x \in \mathbb{F}_q^n\) and a vector \(y \in \mathbb{F}_q^n\), the inhomogeneous permuted kernel problem \(\text{IPKP}(q, m, n)\) asks to find a permutation \(\pi \in S_n\) such that \(H(\pi[x]) = y\).

Definition 9 (RSD problem). Given positive integers \((q, m, n, k, w)\), a random parity-check matrix \(H \leftarrow \mathbb{F}_q^{(n-k) \times n}\) and a syndrome \(y \in \mathbb{F}_q^{(n-k)}\), the syndrome decoding problem \(\text{RSD}(q, m, n, k, w)\) asks to find \(x \in \mathbb{F}_q^n\) such that \(Hx^\top = y^\top\) and \(w_R(x) = w\).

Definition 10 (IRSD problem). Let \((q, m, n = \ell k, k, w)\) be positive integers, \(P \in \mathbb{F}_q[X]\) be an irreducible polynomial of degree \(k\), \(H \in \mathbb{F}_q^{(n-k) \times n}\) be a random parity check matrix under systematic form of an \([n, k]\) \(\ell\)-ideal code and let \(y \in \mathbb{F}_q^{(n-k)}\) be a syndrome. The ideal rank syndrome decoding problem \(\text{IRSD}(q, m, n, k, w)\) problem asks to find \(x \in \mathbb{F}_q^n\), such that \(Hx^\top = y^\top\) and \(w_R(x) = w\).

Proof. Sketch of proof for Theorem 2 can be found in Appendix B.

Definition 11 (RSL problem). Let \((q, m, n, k, w, M)\) be positive integers, \(H \leftarrow \mathbb{F}_q^{(n-k) \times n}\) be a random parity-check matrix, \(E\) a random subspace of \(\mathbb{F}_q^n\) of dimension \(\omega\) and \((y_1, \ldots, y_M) \in (\mathbb{F}_q^{n-k})^M\) be syndromes such that \(Hx_i^\top = y_i^\top\) where \(x_i\) is randomly chosen such that \(\text{Supp}(x_i) = E\). The rank support learning problem \(\text{RSL}(q, m, n, k, w, M)\) asks to find \((x_1, \ldots, x_M)\) given \((H, y_1, \ldots, y_M)\).

Definition 12 (IRSL problem). Let \((q, m, n, k, w, M)\) be positive integers, \(P \in \mathbb{F}_q[X]\) be an irreducible polynomial of degree \(k\), \(H \leftarrow \mathbb{F}_q^{(n-k) \times n}\) be a random parity-check matrix under systematic form of an \([n, k]\) \(\ell\)-ideal code, \(E\) be a random subspace of \(\mathbb{F}_q^n\) of dimension \(\omega\) and \((y_1, \ldots, y_M) \in (\mathbb{F}_q^{n-k})^M\) be syndromes such that \(Hx_i^\top = y_i^\top\) where \(x_i\) is randomly chosen such that \(\text{Supp}(x_i) = E\). The ideal rank support learning problem \(\text{IRSL}(q, m, n, k, w)\) asks to find \((x_1, \ldots, x_M)\) given \((H, y_1, \ldots, y_M)\).
Definition 13 (DiffIRSL problem). Let \((q, m, n, k, w, M) ≥ 2, Δ ≥ M\) be positive integers, \(P ∈ \mathbb{F}_q[X]\) be an irreducible polynomial of degree \(k\), \(H ← \mathbb{F}_q^{m×n}\) be a random parity-check matrix under systematic form of an \([n, k]\) \(ℓ\)-ideal code, \(E\) be a random subspace of \(\mathbb{F}_q^m\) of dimension \(ω\) and \((y_1, …, y_M) ∈ (\mathbb{F}_q^{m−k})^M\) be syndromes such that \(Hx_i^\top = y_i^\top\) where \(x_i\) is randomly chosen such that \(\text{Supp}(x_i) = E\). The differential ideal rank support learning problem \(\text{DiffIRSL}(q, m, n, k, w, M, Δ)\) asks to find \((c, d_1, …, d_Δ) ∈ \mathbb{F}_q^2 × (\mathbb{F}_q^2)^Δ\) such that for each \(l ∈ [1, Δ]\), \(Hd^\top_l + c = \sum_{(i, j) ∈ [1, M] × [1, k]} γ_{i,j} · \text{rot}_j(y^\top_i)\) with \(w_R(d_l) = w\) and \(γ_{i,j} ∈ \mathbb{F}_q\).

Theorem 3. If there exists a PPT algorithm solving the \(\text{DiffIRSL}(q, m, n, k, w, M)\) problem with probability \(p\), then there exists a PPT algorithm solving the \(\text{IRSL}(q, m, n, k, w)\) problem with probability \((1 − q^{m−ω} + mω−m(n−k)) · p\).

Proof. Sketch of proof for the Theorem 3 can be found in Appendix C.

2 New PoK with Helper for the SD problem

We describe in Figure 1 a new PoK with Helper for the SD problem that encompasses several ideas from [BGKM22], [FJR21] and [BGKS21]. Our protocol follows the same paradigm as PoK 2 from [BGKM22] as it introduces several permutations \((\pi_i)_{i ∈ [1, N]}\) and masks the secret \(x\) using the permutation \(\pi_α\) for \(α ∈ [1, N]\) while revealing the other permutations \((\pi_i)_{i ∈ [1, N]\setminusα}\). A notable difference comes from the fact that it also leverages the shared permutations paradigm from [FJR21] where the permutations \((\pi_i)_{i ∈ [1, N]}\) are nested around a global permutation \(π\) such that \(π = π_N ◦ … ◦ π_1\). As a consequence, our protocol masks the secret \(x\) using \(π_α ◦ … ◦ π_1[x]\) for \(α ∈ [1, N]\). This differs from [FJR21] where the secret is masked by \(π[x]\) and where \(π[u + x] + v\) is computed without revealing \(π\). This difference allows us to perform a cut and choose with meet in the middle where the recurrence relation related to \(π\) is used in both directions rather than just one. Thanks to this new property, our protocol can benefit from an optimization introduced in [BGKS21]. As a consequence, our protocol outperforms the protocols from [BGKM22] and [FJR21]. We defer the reader to Tables 2 and 3 for a comparison with existing protocols including the recent [FJR22] proposal.

Theorem 4. If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 1 is an honest-verifier zero-knowledge proof of knowledge with helper for the SD problem over \(\mathbb{F}_2\) with soundness error \(1/N\).

Proof. Proof of Theorem 4 can be found in Appendix D.
**Helper(H, y)**

\[ \theta \in \{0, 1\}^*, \; \xi \in \{0, 1\}^* \]

for \( i \in [1, N] \) do

\[ \theta_i \in \{0, 1\}^*, \; \phi_i \in \{0, 1\}^* \]

\[ \pi_i \leftrightarrow s_i, \; \psi_i \leftrightarrow \frac{r_i}{r_0} \]

\[ r_{1,i} \in \{0, 1\}^*, \; \com_{1,i} = \text{Com}(r_{1,i}, \phi_i) \]

end

\[ \pi = \pi_N \odot \cdots \odot \pi_1 \]

\[ v = v_N + \sum_{i \in \mathbb{Z}\backslash\{N\}} \pi_i \odot \cdots \odot \pi_{i+1}[v_i] \]

\[ \mathbf{r} \leftarrow \left\{ \begin{array}{l} \frac{r_i}{r_0} \quad \text{if} \quad v_i \neq v \\ \frac{\pi_i}{\pi_0} \quad \text{if} \quad v_i = v \end{array} \right. \]

\[ q = H \left( \pi[a] + v \right) \]

\[ \com_1 = \text{Hash}(q) \| \text{com}_{1,i} \| |1:N| \}

Send \((\theta, \xi)\) to the Prover and \com_1\ to the Verifier

**Prover(x, H, y, \theta, \xi)**

Compute \((\theta, \pi, \psi_i)\|_{i \in [1,N]}\) and \u\ from \((\theta, \xi)\)

\( s_0 = u + x \)

for \( i \in [1, N] \) do

\( s_i = \pi_i[s_{i-1}] + \psi_i \)

end

\[ \com_2 = \text{Hash}(u + x) \| \text{com}_{2,i} \| |1:N| \}

**Verifier(H, y, \com_1)**

\[ \alpha \leftarrow \left\{ \begin{array}{l} k \quad \text{if} \quad |N| \backslash \alpha \end{array} \right. \]

Compute \((\delta_i, r_{1,i}, \phi_i)\|_{i \in [1,N]}\) from \com_2\;

Compute \(\mathbf{f}\) from \com_2\;

\( t_x = \mathbf{f} \)

for \( i \in \{N, \ldots, \alpha + 1\} \) do

\( t_{i-1} = \pi_i^{-1}[t_i - \psi_i] \)

end

\( s_0 = a_1, \; b_i = t_i + a_i \)

for \( i \in [1, N] \backslash \alpha \) do

\( s_i = \pi_i[s_{i-1}] + \psi_i \)

end

for \( i \in [1, N] \backslash \alpha \) do

\[ \com_{1,i} = \text{Com}(r_{1,i}, \delta_i) \]

end

\[ \com_{1,\alpha} = \text{com}_{1,i}, \; q = H s_1 = y \| \mathbf{f} \]

\[ \delta_1 = \left( \text{com}_1 = \text{Hash}(q) \| \text{com}_{1,i} \| |1:N| \} \right) \]

\[ \delta_2 = \left( \text{com}_2 = \text{Hash}(s_1 \| |1:N|) \right) \]

\[ \delta_3 = \left( w_0(x) - \omega \right) \]

return \( b_1 \wedge b_2 \wedge b_3 \)

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**Fig. 1:** ZK PoK with Helper for the SD problem over \(\mathbb{F}_2\)
3 New paradigm to design PoK

The PoK with Helper paradigm introduced in [Beu20] eases the design of PoK and has historically led to shorter signatures. It relies on introducing a trusted third party (the so called Helper) that is later removed using cut and choose. In this section, we introduce a new paradigm that can be seen as an alternate way to remove the Helper in such PoK. Formally, we no longer introduce the Helper and instead substitute it by using some structure related to the underlying problem. For instance, starting from the SD problem over $\mathbb{F}_2$, one may introduce the required extra structure by considering the QCSD problem over $\mathbb{F}_2$ or the SD problem over $\mathbb{F}_q$. We describe in Table 1 how our new paradigm can be applied to the PoK with Helper from Section 2, [Wan22] and [Beu20] in order to design new PoK related to the SD, MQ, PKP and RSD problem. Removing the Helper provides an improvement on performances that can led to smaller signature sizes when some additional conditions are verified. Hereafter, we discuss the advantages and limits of our new approach by describing its impact on soundness, performance, security and size of the resulting constructions. We denote by $C_1$ and $C_2$ the sizes of the two considered challenge spaces and by $\tau$ the number of iterations that one need to execute to achieve a negligible soundness error.

| Scheme       | First Challenge | Second Challenge | Problem            |
|--------------|-----------------|------------------|--------------------|
| MUDFISH      | Helper          | Linearity over $\mathbb{F}_q$ | MQ                 |
| [Beu20]      | Helper          | Shared Permutation | MQ                 |
| Section 4.3  | Homogeneity over $\mathbb{F}_q$ | Secret Sharing | MQ, PKP            |
| SUSHYFISH    | Helper          | Linearity over $\mathbb{F}_q$ | IPKP               |
| [Beu20]      | Helper          | Shared Permutation | IPKP               |
| Section 4.3  | Homogeneity over $\mathbb{F}_q$ | Secret Sharing | MQ, PKP            |
| Section 2    | Helper          | Shared Permutation | RSD                |
| Section 4.3  | Cyclicity       | Shared Permutation | IRSD               |
|              | Cyclicity & Linearity | Shared Permutation | IRSL              |

Table 1: Substitution of the Helper within PoK with Helper

**Impact on soundness error.** Removing the trusted setup induced by the Helper allows new cheating strategies for malicious provers. In the general case, this means that the soundness error is increased from $\max\left(\frac{1}{C_1}, \frac{1}{C_2}\right)$ for protocols using a helper to $\frac{1}{C_1} \cdot \left(1 - \frac{1}{C_2}\right) \cdot \frac{1}{C_2} = \frac{1}{C_2} + \frac{C_1 - 1}{C_1 C_2}$ for protocols without helper. One can see from this expression that for challenge spaces such that $C_2 \gg 1$, the resulting soundness error is close to $\frac{1}{C_2} + \frac{1}{C_2}$. In addition, if $C_1 \gg C_2$ then the resulting soundness error is close to $\frac{1}{C_2}$.
**Impact on performances.** Using a helper generally implies to repeat some operations $\tau \cdot C_1 \cdot C_2$ times during the trusted setup phase. While in practice the beating parallel repetition optimization from \cite{KKW18} allows to reduce the impact of the trusted setup phase, the latter still induces some performance overhead. By removing the helper, one generally reduces the cost of this phase to $\tau \cdot C_2$ operations. Nonetheless, one has to keep the 5-round structure of the underlying PoK as collapsing the proof from 5-round to 3-round would re-introduce the aforementioned performance overhead in most of the cases.

**Impact on security.** One should note that PoK following our new paradigm are slightly less conservative than PoK with Helper collapsed to 3-round. Indeed, the underlying 5-round structure can be exploited as demonstrated in \cite{KZ20}. In practice, parameters can be increased to avoid this attack. In addition, security proof for the PoK following our new paradigm might be a bit more involved. Indeed, one might need to introduce an intermediary problem and rely on a reduction from the targeted problem to the intermediary problem. This strategy was used in \cite{AGS11} with the introduction of the DiffSD problem and its reduction to the QCSD problem. More recently, such a strategy has also been used in \cite{FJR22} where the d-split SD problem is used as an intermediary problem along with a reduction to the SD problem. In practice, one may need to increase the underlying parameters in order for these reductions to hold securely.

**Impact on signature size.** Several of the aforementioned elements impact the signature size in conflicting ways. For instance, increasing the soundness may impact negatively the signature size while improving performances allows to look for better sizes at the same cost. In addition, using a 5-round structure reduces the number of commitments to be sent but requires to increase the number of iterations $\tau$ to consider. Moreover, removing the Helper reduces the number of seeds to be sent. One should note that with our new approach, the sizes of the considered challenge spaces are of paramount importance as the attack from \cite{KZ20} impacts significantly the security of the constructions (see next point for additional details). In practice, our new paradigm brings an improvement over comparable PoK with Helper when the parameters are chosen carefully.

**On the challenge spaces sizes.** While the size of some challenge spaces can be arbitrarily chosen, other challenge spaces may not offer such a flexibility as they are related to some structure such as the size of a finite field for instance. Nonetheless, one may need this flexibility in order to optimize performances or signature size. This was observed in \cite{Ben20} where it was suggested to consider subspaces of the actual challenge spaces in some cases. In our context, being able to increase the size of some challenge spaces is beneficial. In order to do so, one can introduce several instances of the considered hard problem in a given public key as suggested in \cite{BBBC21}. Using the SD problem for illustrative purposes, one replace the secret key $sk = (x)$ and public key $pk = (H, y)$ by $M$ instances such that $sk = (x_i)_{i \in [1, M]}$ and $pk = (H, (y_i)_{i \in [1, M]})$. Doing so, one increases the
size of the public key by a factor $M$ but may also increase the size of a challenge space related to the underlying problem by the same factor $M$.

4 New PoK for the SD, MQ, PKP and RSD problems

4.1 PoK for the QCSD problem over $\mathbb{F}_2$ and SD problem over $\mathbb{F}_q$

We apply our new paradigm described in Section on top of the PoK with Helper introduced in Section. We provide two variants that respectively leverage quasi-cyclicity over $\mathbb{F}_2$ and linearity over $\mathbb{F}_q$. This results in two PoK based on the QCSD over $\mathbb{F}_2$ as well as the SD over $\mathbb{F}_q$ that are respectively described in Figures and . Our first variant shares some similarity with the Quasi-Cyclic Stern protocol from [BGKS21] that improves the initial Stern protocol using quasi-cyclicity. Indeed, it is based on the DiffSD problem and its security relies on the reduction from the QCSD problem to the DiffSD problem.

Theorem 5. If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 2 is an honest-verifier zero-knowledge proof of knowledge for the QCSD problem over $\mathbb{F}_2$ with soundness error equal to $\frac{1}{N} + \frac{N-1}{MN}$.

Proof. Proof of Theorem 5 can be found in Appendix E.

Theorem 6. If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 3 is an honest-verifier zero-knowledge proof of knowledge for the SD problem over $\mathbb{F}_q$ with soundness error equal to $\frac{1}{N} + \frac{N-1}{N(q-1)}$.

Proof. Proof of Theorem 6 can be found in Appendix E.

Additional optimization. One should note that the protocol described in Sections and induce an overhead on the signing part. In order to mitigate it, one could consider a variant that use the cut and choose with meet in the middle approach from bottom to top or top to bottom adaptively depending if $\alpha$ is closer to 1 or closer to $N$.

Additional metric. Our protocols can be adapted to other metrics such as the rank metric, the Lee metric as well as the Hamming metric along with secrets of large weight. In order to do so, one need to use isometries tailored to the considered metric as explained in [BGKS21].

Additional variant. Our new paradigm apply straightforwardly to [LJR21] too which lead to another trade-off between signature size and performances.
4.2 PoK for the $\mathcal{MQ}_H$ problem

We apply our new paradigm to the $\mathcal{MQ}$ problem by restricting it to the $\mathcal{MQ}_H$ problem. Doing so, one can exploit the structure of homogeneous polynomials of degree 2 using the relation $F(\kappa \cdot x) = \kappa^2 \cdot y$. Our protocol is inspired by the recent PoK proposed by [Wan22] and its security relies on the reduction from the $\mathcal{MQ}_H$ problem to the Diff.$\mathcal{MQ}_H$ problem.

**Theorem 7.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 4 is an honest-verifier zero-knowledge proof of knowledge for the $\mathcal{MQ}_H$ problem with soundness error equal to $\frac{1}{N} + \frac{N-1}{M^N(q-1)}$.

**Proof.** Proof of Theorem 7 can be found in Appendix G.

**Variant with additional structure.** One could also consider adding cyclicity to the $\mathcal{MQ}$ problem in order to use additional structure. Such an alternative can improve the resulting signature size as discussed in Section 5.2.

4.3 PoK for the IPKP problem

We also apply our new technique to the IPKP problem leveraging the shared permutation from [FJR21].

**Theorem 8.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 5 is an honest-verifier zero-knowledge proof of knowledge for the IPKP problem with soundness error equal to $\frac{1}{N} + \frac{N-1}{M^N(q-1)}$.

**Proof.** Proof of Theorem 8 can be found in Appendix H.

4.4 PoK for the IRSL problem

As mentioned in Section 2, one can straightforwardly adapt the protocol described in Figure 2 to the rank metric setting by replacing permutations by isometries for the rank metric. In addition, by considering the IRSD or IRSL problem, one can fully exploit our new paradigm thanks to the inherent properties of the rank metric. Hereafter, we describe the case based on the IRSL problem which can be straightforwardly adapted to the less structured IRSD setting. Given several vector $x_i$ sharing the same support $E$, the support of any vector $x'$ constructed as a linear combination of the vectors $x_i$ is included in $E$. This property is very useful when paired with the optimization regarding challenges spaces sizes described Section 4.1. Indeed, using the IRSL problem instantiated with $M$ syndromes, one can increase the size of the underlying challenge space by a factor $q^{Mk}$ which is exponential in $M$. In practice, one has to ensure that
the considered linear combinations have the same weight than the secret which is not described hereafter for simplicity. This implies that the trade-off between public key size and signature size is way more efficient in the rank metric setting than in the Hamming metric one. In practice, this means that one only need a small increase on the number of repetitions $\tau$ of the underlying PoK in order to take into account the attack from [KZ20].

**Theorem 9.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 2 is an honest-verifier zero-knowledge proof of knowledge for the IRSL problem with soundness error equal to $\frac{1}{N} + \frac{(N-1)(\Delta-1)}{q^m + 1}$. 

*Proof.* Proof of Theorem 9 can be found in Appendix II.

**Adaptation to other protocols.** Our idea leveraging challenge spaces size that can be made arbitrarily small in order to reduce signature size can be straightforwardly adapted to the protocols described in Sections 4.1 and 4.3. Additional details and corresponding numbers will be given in the final version of this work. Preliminary results shows that one could obtain an improvement up to 40% with respect to some of the numbers given in Section 5.
Fig. 2: ZK PoK for the QCSD problem over $\mathbb{F}_2$
**Prover(x, H, y)**

\[ \theta \leftarrow \{0, 1\}^{\lambda}, \ \xi \leftarrow \{0, 1\}^{\lambda} \]
for \( i \in [1, N] \) do
\[ \theta_i, \xi_i \leftarrow \{0, 1\}^{\lambda}, \ \phi_i \leftarrow \{0, 1\}^{\lambda} \]
\[ \pi_i, \psi_i \leftarrow \mathbb{P}_q \]
\[ r_{1,i}, \pi_{1,i} \leftarrow \{0, 1\}^{\lambda}, \ \text{com}_{1,i} = \text{Com}(r_{1,i}, \phi_i) \]
end
\[ \tau = \sum_{i=1}^{N} \pi_i \]
\[ v = v_N + \sum_{i=N}^{N+1} \tau_i \]
\[ \rho \leftarrow \mathbb{P}_q, \ \upsilon = \pi^{-1}[x - v] \]
\[ q = Hu \leftarrow \|\pi[u] + v\| \cdot |x| \]
\[ \text{com}_1 = \text{Hash}(q \| \text{com}_{1,i} \| \{1, N\}) \]

**Veriﬁer(H, y)**

\[ \text{com}_1 \]
\[ \kappa \leftarrow \mathbb{P}_q \]
\[ s_0 = u + \kappa \cdot x \]
for \( i \in [1, N] \) do
\[ s_i = \pi_i[s_{i-1}] + \upsilon_i \]
end
\[ \text{com}_2 = \text{Hash}(u + \kappa \cdot x \| \{s_i \}_{i \in [1, N]}) \]

\[ \alpha \leftarrow \mathbb{P}_q \]
\[ \text{com}_2 \]

Compute \((\bar{\phi}, \bar{r}_{1,i}, \bar{\pi}, \bar{\psi})_{i \in [1, N]}\) from \( z_3 \)
Compute \( \tau \) from \( z_3 \)
\[ t_N = \tau, \ \bar{b}_N = z_N \]
for \( i \in \{N, \ldots, \alpha + 1\} \) do
\[ t_{i-1} = \pi^{-1}[t - \bar{\psi}_i], \ \bar{b}_{i-1} = \pi^{-1}[b_i] \]
end
\[ s_0 = z_1, \ \bar{s}_0 = t_0 + \kappa \cdot \bar{b}_0 \]
for \( i \in [1, N] \setminus \alpha \) do
\[ s_i = \pi_i[s_{i-1}] + \upsilon_i \]
end
for \( i \in [1, N] \setminus \alpha \) do
\[ \text{com}_{1,i} = \text{Com}(r_{1,i}, \phi_i) \]
end
\[ \text{com}_{1,\bar{z}} = \text{com}_{1,z}, \ \bar{q} = H\bar{z}_1 - \kappa \cdot y \| \tau \| z_4 \]
\[ b_1 \leftarrow \langle \text{com}_{1,\bar{z}} \rangle = \text{Hash}(\bar{q} \| \langle \text{com}_{1,i} \rangle_{i \in [1, N]} \rangle) \]
\[ b_2 \leftarrow \langle \text{com}_{2,\bar{z}} \rangle = \text{Hash}(\bar{z}_1 \| \langle \bar{s}_i \rangle_{i \in [1, N]} \rangle) \]
\[ b_3 \leftarrow |u\cdot(z_1) = \omega| \]
return \( b_1 \wedge b_2 \wedge b_3 \)

**Fig. 3:** ZK PoK for the SD problem over \( \mathbb{F}_q \)
\[
\text{Prover}(\mathbf{s}_1(1, N), \mathbf{F}, \mathbf{F}(\mathbf{y}_1(1, N)))
\]

\begin{align*}
\text{if } & \alpha \in \{0, 1\} \text{ } \text{ then } \\
& \text{for } i \in [1, N - 1] \text{ do } \\
& \quad \theta_i \leftarrow \{0, 1\}^q, \phi_i \leftarrow \{0, 1\}^q \\
& \quad u_i \leftarrow \mathbb{F}_q^*, v_i \leftarrow \mathbb{F}_q^* \\
& \quad r_{1,i} \leftarrow \{0, 1\}^q, \text{ com}_{1,i} = \text{Com}(r_{1,i}, \phi_i) \\
& \text{end } \\
& \theta_N \leftarrow \{0, 1\}^q, \phi_N \leftarrow \{0, 1\}^q, u_N \leftarrow \mathbb{F}_q^*, v_N \leftarrow \mathbb{F}_q^* \\
& u = \sum_{i \in [1, N]} u_i, v = \mathcal{F}(u) - \sum_{i \in [1, N-1]} v_i, \\
& r_{1,N} \leftarrow \{0, 1\}^q, \text{ com}_{1,N} = \text{Com}(r_{1,N}, v_N || \phi_N) \\
& \text{com}_1 = \text{Hash}((\text{com}_{1,i})_{i \in [1,N]})
\end{align*}

\[
\text{Verifier}(\mathbf{y}_1(1, N))
\]

\begin{align*}
\text{com}_1 & \quad (\mu, \kappa) \leftarrow \{1, M\} \times \mathbb{F}_q^* \\
\text{com}_2 & \quad \alpha \leftarrow \{1, N\} \\
\text{rsp} & \quad \text{Compute } (\tilde{s}_i, r_{1,i}, u_i, \psi_i)_{i \in [1,N], \alpha} \text{ from } z_2 \quad \text{com}_1 \leftarrow \text{com}_{1,i} \\
& \text{for } i \in [1, N] \setminus \alpha \text{ do } \\
& \quad \tilde{s}_i = \mathcal{G}(u_i, s_i) + \psi_i \\
& \text{end } \\
& \text{for } i \in [1, N] \setminus \alpha \text{ do } \\
& \quad \text{if } i \neq N \text{ do } \\
& \quad \quad \text{com}_{1,i} = \text{Com}(s_i, \tilde{s}_i) \\
& \quad \text{else } \\
& \quad \quad \text{com}_{1,N} = \text{Com}(r_{1,N}, \psi_N || \tilde{s}_N) \\
& \quad \text{end } \\
& \text{com}_1 = \text{com}_{1,i} \\
& b_1 = \text{com}_1 = \text{com}_{1,i} \quad \text{com}_2 = \text{com}_2 = \text{Hash}((\text{com}_{2,i})_{i \in [1,N]}) \\
& \text{return } b_1 \wedge b_2
\end{align*}

Fig. 4: ZK PoK for the MQ\text{#} problem
\[ \text{Prover}(\pi, H, x, y) \]

\[
\begin{align*}
\theta & \leftarrow \{0, 1\}^k \\
\text{for } i \in \{N, \ldots, 1\} \\
& \text{do} \\
\theta_i & \leftarrow \{0, 1\}^k, \phi_i \leftarrow \{0, 1\}^k \\
\text{if } i \neq 1 \text{ do} \\
\pi_i & \leftarrow \Phi_0, \nu_i \leftarrow \Phi_0 \mathbb{G}_1^* \\
\varepsilon_i & \leftarrow \{0, 1\}^k, \text{com}_{i, 1} = \text{Com}(\varepsilon_i, \phi_i) \\
\text{else} \\
\pi_i & = \pi_1^{-1} \circ \cdots \circ \pi_i^{-1} \circ \pi_1, \nu_i \leftarrow \Phi_0 \mathbb{G}_1^* \\
\varepsilon_i & \leftarrow \{0, 1\}^k, \text{com}_{i, 1} = \text{Com}(\varepsilon_i, \pi_i \| \phi_i) \\
\end{align*}
\]

\[
\begin{align*}
\nu & = \nu_N + \sum_{i < [N - 1]} \pi_i \circ \cdots \circ \pi_N \alpha_i \nu_i \\
\text{com}_{i, 1} & = \text{Hash}(\pi \| (\text{com}_{i, 1})_{\alpha \in [1, N]}) \\
\end{align*}
\]

\[
\begin{align*}
\text{for } i \in [1, N] \\
& \text{do} \\
\alpha_i & = \pi_i \cdot \nu_i \\
\text{com}_{i, 2} & = \text{Hash}(\langle \alpha_i \rangle_{i \in [1, N]}) \\
\end{align*}
\]

\[
\begin{align*}
\alpha & \leftarrow \{0, 1\}^k \\
\text{if } i \neq 1 \text{ do} \\
\phi_i & = (\theta_i \| (\theta_i \| \pi_i \| \phi_i)_{\alpha \in [1, N]}) \\
\text{else} \\
\phi_i & = (\theta_i \| (\theta_i \| \pi_i \| \phi_i)_{\alpha \in [1, N]}) \\
\end{align*}
\]

\[
\begin{align*}
\text{rst} & = \langle \phi_i \rangle_{i \in [1, N], \alpha} \\
\end{align*}
\]

\[ \text{Verifier}(H, x, y) \]

\[
\begin{align*}
\kappa & \leftarrow \Phi_0 \mathbb{G}_1^* \\
\end{align*}
\]

\[
\begin{align*}
\text{Compute } (\tilde{\phi}_1, \tilde{r}_1, \tilde{\pi}_1, \tilde{\nu}_1, \tilde{\psi}_1)_{\alpha \in [1, N], \alpha} \text{ from } z_2 \\
\tilde{b}_0 & = \kappa \cdot x \\
\text{for } i \in [1, N] \setminus \alpha \text{ do} \\
\tilde{b}_i & = \tilde{b}_{i-1} \cdot \nu_i \\
\tilde{b} & = \langle \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4, \tilde{b}_5 \rangle \\
\text{for } i \in [1, N] \setminus \alpha \text{ do} \\
& \text{if } i \neq 1 \text{ do} \\
\text{com}_{i, 1} & = \text{Com}(\tilde{r}_i, \tilde{\phi}_i) \\
& \text{else} \\
\text{com}_{i, 1} & = \text{Com}(\tilde{r}_i, \tilde{\pi}_i \| \tilde{\phi}_i) \\
\end{align*}
\]

\[
\begin{align*}
\text{com}_{i, 0} & = \text{com}_{i, 2} - \text{Hash}(\tilde{b}) \\
& \text{if } i \neq 1 \text{ do} \\
\text{com}_{i, 1} & = \text{Com}(\tilde{r}_i, \tilde{\phi}_i) \\
& \text{else} \\
\text{com}_{i, 1} & = \text{Com}(\tilde{r}_i, \tilde{\pi}_i \| \tilde{\phi}_i) \\
\end{align*}
\]

\[
\begin{align*}
\text{com}_{i, 0} & = \text{com}_{i, 2} - \text{Hash}(\tilde{b}) \\
b_1 & = \text{Hash}(\tilde{b}_{\text{com}}) \\
b_2 & = \text{Hash}(\tilde{b}_{\text{com}}) \\
\text{return } b_1 \land b_2 \\
\end{align*}
\]

Fig. 5: ZK PoK for the IPKP problem
**Prover:** \( (x_z), \lambda_{(a,b)}, H, (r_z), \gamma_{(a,b)} \)  
\[ # \leftarrow \{0,1\}^k, \ x \leftarrow \{0,1\}^k \]
\[ \text{for } i \in [1, \lambda] \text{ do} \]
\[ a_i, \text{GL}_a(F), s_i, \text{GL}_s(F), v, \text{GL}_v(u), \gamma_i = \text{com}_i = \text{Com}(r_i, s_i) \]
\[ \text{end} \]
\[ P = P_{\lambda} \times \cdots \times P_1, \ Q = Q_{\lambda} \times \cdots \times Q_1, \ v = v_x + \sum_{i=1}^{\lambda} P_i \times \cdots \times P_{i-1} \ v_i, \ Q_0 = \cdots \ Q_{\lambda-1} \]
\[ r, = P^{\lambda-1}(r-v) Q^{-1} \]
\[ q = H_a[P \ u \ Q + v] \]
\[ \text{com}_1 = \text{Hash}(q|| (\text{com}_i), i \in [0, \lambda]) \]

\[ x_r = \sum_{i=0}^{\lambda-1}(\text{com}_i \cdot \text{rot}(s_i)) \]
\[ x_s = u + x_r \]
\[ \text{for } i \in [1, \lambda] \text{ do} \]
\[ x_i = P_i, x_{i-1}, Q_i, + v_i \]
\[ \text{end} \]
\[ \text{com}_2 = \text{Hash}(x_s x_r \text{|| } (\text{com}_i), i \in [0, \lambda]) \]

\[ x_0 = u + x_r, \ x_0 = (\text{com}_i), x_0 \]
\[ x_1 = x_2 = Q_0 \times \cdots \times Q_{\lambda-1}, x_1 \times \cdots \times Q_1 \]
\[ r_p = (x_1, x_2, x_3, \text{com}_{\gamma}) \]

**Verifier:** \( H, (r_z), \gamma_{(a,b)} \)
\[ \# = H_a x_{(a,b)} + \text{com}_i = \text{com}(r_i, s_i) \]
\[ \text{end} \]
\[ \text{com}_{\gamma} = \text{com}_{\gamma} \]
\[ \# = H_s - \sum_{i=0}^{\lambda-1}(\text{com}_i \cdot \text{rot}(s_i)) \]
\[ b_1 \leftarrow (\text{com}_1 = \text{Hash}(q|| (\text{com}_i), i \in [0, \lambda])) \]
\[ b_2 \leftarrow (\text{com}_2 = \text{Hash}(x_s x_r \text{|| } (\text{com}_i), i \in [0, \lambda])) \]
\[ b_0 \leftarrow (w_x = w) \]
\[ \text{return } b_0 \wedge b_1 \wedge b_2 \]

---

**Fig. 6:** ZK PoK for the $\text{IRSL}$ problem
5 Resulting signatures and comparison

PoK can be transformed into signatures using the Fiat-Shamir heuristic [FS86]. Several optimizations can be employed in the process, we defer the interested reader to [BGKM22] for additional details. Hereafter, we keep the inherent 5-round structure of our PoK (except the one from Section 2 that is collapsed to 3-round) hence parameters are chosen taking into account the attack from [KZ20]. Hereafter, we have only consider parameters that are believed to be secure for $\lambda = 128$ bits of security. For greater security level, one may need to evaluate whether collapsing to 3-round is more suitable or not. The commitments are instantiated using hash functions along with some randomness. For the signatures, random salts are added to the hash functions.

5.1 Signatures based on the SD problem

Table 2 compares signature sizes of our new protocol with respect to existing ones. One can see that our signatures scale with a factor $1.5n \cdot \tau$ which bring an improvement with respect to previous (comparable) schemes that scale with a factor $2n \cdot \tau$. Table 3 provides a comparison to other code-based signatures constructed from PoK for the SD or QCSD problem over $\mathbb{F}_2$. Numbers are chosen taking into account results from [BJMM12] and [Sen11]. For our protocol from Section 2 we have used $(n = 1190, k = 595, \omega = 132)$ as well as $(N = 8, \tau = 49, M = 187)$ and $(N = 32, \tau = 28, M = 389)$. For our protocol from Section 4.1 we have used $(n = 1238, k = 619, \omega = 136)$ as well as $(N = 32, \tau = 43)$ and $(N = 256, \tau = 32)$. In addition, we have considered a variant that leverages the trade-off between public key and signature size described in Section 3. For this variant, parameters used are $(M = 16, N = 32, \tau = 37)$ and $(M = 11, N = 256, \tau = 27)$. Numbers for [FJR21] have been recomputed using the aforementioned parameters in order to provide a fair comparison. Table 4 provides a comparison with signatures constructed from PoK for the SD problem over $\mathbb{F}_q$. Parameters considered are $(n = 214, q = 256, \omega = 81)$ as well as $(N = 32, \tau = 47)$ and $(N = 256, \tau = 35)$. Numbers for [GPS21] and [FJR22] are from the original papers.

| Signature size | (BGKM22) (Sig. 2) | $3\lambda + \tau \cdot (2n + 3\lambda \cdot \log_2(N) + 3\lambda \cdot \log_2(M/\tau))$ |
|----------------|-------------------|----------------------------------------------------------------------------------|
| Signature size | (FJR21) (Sig. 3)  | $3\lambda + \tau \cdot (2n + \lambda \cdot \log_2(N) + 2\lambda + 3\lambda \cdot \log_2(M/\tau))$ |
| Signature size | (Our Work)        | $4\lambda + \tau \cdot (1.5n + \lambda \cdot \log_2(N) + 2\lambda + 3\lambda \cdot \log_2(M/\tau))$ |
| Signature size | (Our Work)        | $6\lambda + \tau \cdot (1.5n + \lambda \cdot \log_2(N) + 2\lambda)$ |

Table 2: Signature sizes (sorted by decreasing size)
| Type          | pk  | σ    | Structure | Security Assumption |
|--------------|-----|------|-----------|---------------------|
| BGKM22 (Sig. 2) | Fast | 0.1 kB | 26.4 kB  | 3-round SD over $F_2$ |
|              | Short | 0.1 kB | 20.5 kB  |                     |
| FJR21 / BGKM22 (Sig. 3) | Fast | 0.1 kB | 23.3 kB  | 3-round SD over $F_2$ |
|              | Short | 0.1 kB | 16.9 kB  |                     |
| Our Work (Section 2) | Fast | 0.1 kB | 19.7 kB  | 3-round SD over $F_2$ |
|              | Short | 0.1 kB | 14.9 kB  |                     |
| FJR22        | Fast | 0.1 kB | 17.0 kB  | 5-round SD over $F_2$ |
|              | Short | 0.1 kB | 11.8 kB  |                     |
| Our Work (Section 4.1) | Fast | 0.1 kB | 14.9 kB  | 5-round QCSD over $F_2$ |
|              | Short | 0.1 kB | 12.7 kB  |                     |
|              | Fast | 1.3 kB | 12.9 kB  | 5-round QCSD over $F_2$ |
|              | Short | 0.9 kB | 10.7 kB  |                     |

Table 3: Signatures based on SD over $F_2$ for $\lambda = 128$ (sorted by decreasing size)

| Type          | pk  | σ    | Structure | Security Assumption |
|--------------|-----|------|-----------|---------------------|
| GPS21        | Fast | 0.2 kB | 27.1 kB  | 3-round SD over $F_q$ |
|              | Short | 0.2 kB | 19.8 kB  |                     |
| Our Work (Section 4.1) | Fast | 0.2 kB | 20.4 kB  | 5-round SD over $F_q$ |
|              | Short | 0.2 kB | 16.9 kB  |                     |
| FJR22        | Fast | 0.2 kB | 11.5 kB  | 5-round SD over $F_q$ |
|              | Short | 0.2 kB | 8.3 kB   |                     |

Table 4: Signatures based on SD over $F_q$ for $\lambda = 128$ (sorted by decreasing size)

5.2 Signatures based on the MQ problem

The signature size of the protocol described in Section 4.2 is given in Table 5. Table 6 provides a comparison with respect to other PoK for the MQ problem. Numbers for [Ben20] and [Wan22] are the ones suggested by their respective authors. For our work, we consider $(m = 41, n = 40, q = 256)$ for parameters along with $(N = 32, \tau = 47)$ and $(N = 256, \tau = 35)$. In addition, we give numbers for a variant that leverages the trade-off between public key and signature size described in Section 3. The latter uses $(M = 12, N = 32, \tau = 39)$ and $(M = 14, N = 256, \tau = 28)$. Preliminary work suggests that one can further reduce these numbers by approximately 1.5 kB relying on the cyclic variant of the MQ$_N$ problem as mentioned in Section 4.2.
Table 5: Signature sizes for our MQ based constructions

| Type  | pk  | σ    | Structure | Security Assumption |
|-------|-----|------|-----------|---------------------|
| MUDFISH [Beu20] | -   | 14.4 kB | 3-round MQ | MQ                   |
| [Wan22] | Fast | 0.1 kB | 3-round MQ | MQ                   |
|        | Short | 0.1 kB | 8.8 kB | MQ                   |
| Our Work (Section 4.3) | Fast | 0.1 kB | 9.2 kB | MQH                  |
|        | Short | 0.1 kB | 8.6 kB | MQH                  |
|        | Fast | 0.6 kB | 7.7 kB | MQH                  |
|        | Short | 0.6 kB | 6.9 kB | MQH                  |

Table 6: Signatures based on MQ for $\lambda = 128$ (sorted by decreasing size)

5.3 Signatures based on the PKP problem

The signature size of the protocol described in Section 4.3 is given in Table 7. Table 8 provides a comparison with respect to other PoK for the PKP problem. For this comparison, we have considered parameters from [Beu20] namely $q = 997$, $n = 61$ and $m = 28$. We use $(N = 32, \tau = 42)$ for our fast instance and $(N = 256, \tau = 31)$ for our short instance.

Table 7: Signature sizes for our PKP based constructions

| Type  | pk  | σ    | Structure | Security Assumption |
|-------|-----|------|-----------|---------------------|
| MUDFISH [Beu20] | Fast | 18.1 kB | 3-round IPKP | IPKP               |
|        | Short | 12.1 kB | IPKP | IPKP               |
| Our Work (Section 4.3) | Fast | 10.0 kB | 5-round IPKP | IPKP               |
|        | Short | 8.9 kB | IPKP | IPKP               |

Table 8: Signatures based on PKP for $\lambda = 128$ (sorted by decreasing size)
5.4 Signatures based on the IRSL problem

The signature size of the protocols described in Section 4.1 (adapted to the rank metric setting) and Section 4.2 is given in Table 9. Table 10 provides a comparison with respect to other PoK for the RSD problem. Numbers for ABC+19 are the ones suggested by the authors. For our protocol from Section 2 we have used \((q = 2, m = 31, n = 30, k = 15, \omega = 9)\) as well as \((N = 8, \tau = 49, M = 187)\) and \((N = 32, \tau = 28, M = 389)\). For our protocol from Section 4.4 we have used \((q = 2, m = 37, n = 34, k = 17, \omega = 10)\) as well as \((N = 32, \tau = 35, M = 5, N = 64, \tau = 23)\) and \((M = 5, N = 1024, \tau = 14)\).

| Type               | pk     | σ       | Structure | Security Assumption |
|--------------------|--------|---------|-----------|--------------------|
| Durandal           | -      | 15.3 kB | 4.1 kB    | -                  |
| Fast               | 0.1 kB | 16.8 kB | 3-round RSD |
| Short              | 0.1 kB | 13.2 kB | RSD       |
| Our Work (Section 4.1) | 0.1 kB | 11.4 kB | 5-round IRSD |
| Fast               | 0.1 kB | 9.1 kB  | IRSD      |
| Short              | 0.1 kB | 8.0 kB  | IRSL      |
| Fast               | 0.5 kB | 5.8 kB  | IRSL      |

Table 9: Signature sizes for our RSD based constructions

| Signature size       |
|----------------------|
| Our Work (Section 2) |
| \(4\lambda + \tau \cdot (mn + \omega(m + n - 2\omega) + \lambda \cdot \log_2(N) + 2\lambda + 3\lambda \cdot \log_2(M/\tau))\) |
| Our Work (Section 4.4) |
| \(6\lambda + \tau \cdot (mn + \omega(m + n - 2\omega) + \lambda \cdot \log_2(N) + 2\lambda)\) |

Table 10: Signatures based on RSD for \(\lambda = 128\)
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A Proof of Theorem 1

Theorem 1. If there exists a PPT algorithm solving the DiffSD($n, k, w, M$) problem with probability $p$, then there exists a PPT algorithm solving the QCSD($n, k, w$) problem with probability $(1 - \frac{\binom{n}{w}}{2^{n-1}}) \cdot \frac{1}{\sqrt{M}} \cdot p$.

Sketch of proof. For the considered $[n, k]$ quasi-cyclic code $C$, we restrict our analysis (as well as our parameters choice) to the case where the weight $\omega$ is lower than the Gilbert-Varshamov bound associated to $C$ i.e. the value for which the number of words of weight less or equal $w$ corresponds to the number of syndromes. As a consequence, given a syndrome $y$, the probability that there exists a pre-image $x$ of $y$ such that $\mathbf{H} x = y$ and $w_H(x) = \omega$ is $\binom{n}{\omega}/2^{(n-k)}$ namely the number of possible words of weight $\omega$ divided by the number of syndromes. Given a solution of the DiffSD problem, we consider two different cases depending on the value of the constant $c$.

If $c = (0, \ldots, 0)$, then there exists $d_i$ such that $w_H(d_i) = \omega$ and $\mathbf{H} d_i = \text{rot}_{\omega}(\mathbf{y}^{\top}_{\mu_i})$ for some $\kappa_i \in [1, k]$ and $\mu_i \in [1, M]$. This implies that $x = \text{rot}_{n-\kappa_i}(d_i)$ is solution of the QCSD instance $\mathbf{H} x = y^{\top}_{\mu_i}$. As $M$ syndromes are available to the adversary, he can find such a solution with an advantage $\sqrt{M}$ with respect to the case where only one syndrome is given following the result from [Sen11].

If $c \neq (0, \ldots, 0)$, then there exists two values $d_i$ such that $w_H(d_i) = \omega$ and $\mathbf{H} d_i = \text{rot}_{\kappa_i}(\mathbf{y}^{\top}_{\mu_i}) - c$ for some $\kappa_i \in [1, k]$ and $\mu_i \in [1, M]$. An adversary can easily solve one of these equations by fixing $c$ to the appropriate value. For the remaining equation, the adversary has to solve a syndrome equation given a random syndrome. Such an equation has a second solution with average probability $\binom{n}{\omega}/2^{(n-k)}$ as discussed previously. In these cases, one does not recover any solution of the considered QCSD instance with $M$ syndromes.

Overall, if one considers random instances of the DiffSD, it shows that solving an instance of the DiffSD problem with $M$ syndromes implies solving one of the associated QCSD instances up to a factor $(1 - \binom{n}{w}/2^{(n-k)}) \cdot 1/\sqrt{M}$ which completes the proof of Theorem 1.

B Proof of Theorem 2

Theorem 2. If there exists a PPT algorithm solving the DiffMQSD($q, m, n, M$) problem with success probability $p$, then there exists a PPT algorithm solving the MQSD($q, m, n$) problem with success probability $(1 - \frac{1}{q^{m-n}}) \cdot \frac{1}{M} \cdot p$.

Sketch of proof. We consider $\mathcal{F} : \mathbb{F}_q^m \rightarrow \mathbb{F}_q^n$ a multivariate quadratic map of $m$ homogeneous quadratic polynomials of degree 2 in $n$ variables defined over $\mathbb{F}_q$. For a given $y \in \mathbb{F}_q^n$, the probability that there exist a solution $x \in \mathbb{F}_q^n$ such that $\mathcal{F}(x) = y$ is roughly $1/q^{(m-n)}$. Given a solution of the DiffMQSD problem, we consider two different cases depending on the value of the constant $c$. 

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If \( c = (0, \ldots, 0) \), then there exists \( d_i \) such that \( F(d_i) = \kappa_i^2 \cdot y_\mu \) for some \( \kappa_i \in F_q^* \) and \( \mu \in [1, M] \). This implies that \( x = \kappa_i^{-1} \cdot d_i \) is solution of the \( MQ_H \) instance \( F(x) = y_\mu \). As \( M \) values \( y_\mu \) are given, one find a solution of an \( MQ_H \) instance of its choice by doing \( M \) queries to the Diff\( MQ_H \) solver in average.

If \( c \neq (0, \ldots, 0) \), then there exists two values \( d_i \) such that \( F(d_i) + c = \kappa_i^2 \cdot y_\mu \), for some \( \kappa_i \in F_q^* \) and \( \mu \in [1, M] \). An adversary can easily solve one of these equations by fixing \( c \) to the appropriate value. For the remaining equation, the adversary has to solve a random multivariate quadratic instance. Such an equation has a second solution with average probability \( 1/q^{(m-n)} \) as discussed previously. In these cases, one does not recover any solution of the considered \( MQ_H \) instance with \( M \) values \( (y_\mu)_{\mu \in [1, M]} \).

Overall, if one considers random instances of the Diff\( MQ_H \), it shows that solving an instance of the Diff\( MQ_H \) problem with \( M \) values \( y_\mu \) implies solving one of the associated \( MQ_H \) instances up to a factor \( (1 − 1/q^{(m-n)}) \cdot 1/M \) which completes the proof of Theorem 3.

C Proof of Theorem 3

**Theorem 3.** If there exists a PPT algorithm solving the DiffIRSL\((q, m, n, k, w, M)\) problem with probability \( p \), then there exists a PPT algorithm solving the IRSL\((q, m, n, k, w)\) problem with probability \( (1 − q^{(m−\omega)+\omega−m(n−k)}) \cdot p \).

**Sketch of proof.** For the considered \([n, k]\) ideal code \( C \), we restrict our analysis (as well as our parameters choice) to the case where the weight \( \omega \) is lower than the Rank Gilber-Varshanov bound associated to \( C \) i.e. the value for which the number of words of weight less or equal \( w \) corresponds to the number of syndromes. As a consequence, given a syndrome \( y \), the probability that there exists a pre-image \( x \) of \( y \) such that \( Hx^\top = y^\top \) and \( w_R(x) = \omega \) is \( q^{m(\omega−\omega)} \) where \( q^{m(\omega−\omega)} \) is an approximation of the Gaussian binomial which counts the number of vector spaces of dimension \( \omega \) in \( F_q^m \). \( q^{\omega} \) is the number of words in a basis of dimension \( \omega \) and \( q^{m(n−k)} \) is the number of syndromes. As such, this probability describes the number of codewords of rank weight \( \omega \) divided by the number of possible syndromes. Given a solution of the DiffIRSL problem, we consider two different cases depending on the value of the constant \( c \).

If \( c = (0, \ldots, 0) \), then there exists \( M \) values \( d_i \) such that for each \( i \in [1, M] \), \( H d_i^\top = \sum_{(i,j) \in [1, M] \times [1, k]} \gamma_{i,j} \cdot \text{rot}_j(y_i^\top) \) with \( w_R(d_i) = w \) and \( \gamma_{i,j} \in F_q \). Knowing \( M \) solutions permits by using the ideal structure to construct \( Mk \) preimages of weight \( \omega \) of \( Mk \) vectors belonging to the \( F_q \)-vector space generated by the \( Mk \) syndromes (\( M \) given syndromes times \( k \) due to cyclicity induced by the ideal structure). With a good probability the image by \( H \) of the \( Mk \) reconstructed vectors of weight \( \omega \) form a basis of the given \( Mk \) shifted syndromes hence it is possible to reconstruct a preimage of any \( F_q \)-linear combination of syndromes thus solving the considered IRSL instance.

If \( c \neq (0, \ldots, 0) \), then there exists \( M \) values \( d_i \) such that for each \( i \in [1, M] \), \( H d_i^\top + c = \sum_{(i,j) \in [1, M] \times [1, k]} \gamma_{i,j} \cdot \text{rot}_j(y_i^\top) \) with \( w_R(d_i) = w \) and \( \gamma_{i,j} \in F_q \).
Whenever $M \geq 2$, the attacker knows a preimage $d_i$ of weight $\omega$ for at least two syndrome equations. An adversary can easily solve one of these equations by fixing $c$ to the appropriate value. For the remaining equations, the adversary has to solve a syndrome equation given a random syndrome. Such an equation admits a second solution with average probability $1 - q^{m(m-\omega)+n\omega-m(n-k)}$ as discussed previously. In these cases, one does not recover any solution of the considered RSL instance.

Overall, if one considers random instances of the DiffRSL, it shows that solving an instance of the DiffRSL problem implies solving one of the associated RSL instances up to a factor $1 - q^{m(m-\omega)+n\omega-m(n-k)}$ which completes the proof of Theorem 3.

D Proof of Theorem 4

**Theorem 4.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 7 is an honest-verifier zero-knowledge proof of knowledge with helper for the SD problem over $\mathbb{F}_2$ with soundness error $1/N$.

**Proof.** We prove the correctness, special soundness and special honest-verifier zero-knowledge properties below.

**Correctness.** The correctness follows from the protocol description once the cut and choose with meet in the middle property of commitment is an honest-verifier zero-knowledge proof of knowledge with helper for the RSL scheme used is binding and hiding, then the protocol depicted in Figure 1 is correct.

**Special soundness.** We now show that the output is a solution to the given SD problem. One can compute $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]}$ from $z_2$ and $z_1'$. From the binding property of the commitments $(\text{com}_1, i)_{i \in [1,N]}$, one has $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]} = (\bar{\pi}_i, \bar{v}_i)_{i \in [1,N]}$. From the binding property of commitment $\text{com}_1$, one has $H(z_i - u) = y$ and $t_N = \pi[u] + v$. Using $t_N$ and $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]}$, one can compute $t_i = \pi_\alpha \circ \cdots \circ \pi_{i+1}[z_i]$. As $z_4 = \pi_\alpha \circ \cdots \circ \pi_1[x]$, one can conclude that $s_i = t_i + z_4$. 

We now show that the output is a solution to the given SD problem. One can compute $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]}$ from $z_2$ and $z_1'$. From the binding property of the commitments $(\text{com}_1, i)_{i \in [1,N]}$, one has $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]} = (\bar{\pi}_i, \bar{v}_i)_{i \in [1,N]}$. From the binding property of commitment $\text{com}_1$, one has $H(z_i - u) = y$ and $t_N = \pi[u] + v$. Using $t_N$ and $(\bar{t}_i, \bar{v}_i)_{i \in [1,N]}$, one get $t_i = \pi_\alpha \circ \cdots \circ \pi_{i+1}[v_i]$. From the binding property of commitment $\text{com}_2$, one has $s_0 = s_0' = z_1$. In addition, one has $s_i = \bar{\pi}_i[s_{i-1}] + \bar{v}_i$ for all $i \in [1,N] \setminus \alpha$ as well as $s_i' = \bar{\pi}_i[s_{i-1}] + \bar{v}_i$ for all $i \in [1,N] \setminus \alpha$. Therefore, the protocol is a zero-knowledge proof of knowledge with helper for the SD problem.
\[ \pi_i[\bar{s}_i^{-1}] + \bar{v}_i \] for all \( i \in [1, N] \setminus \alpha' \). Using the binding property of commitment \( \text{com}_2 \) once again, one can deduce that \( \bar{s}_i = \pi_i[\bar{s}_i^{-1}] + \bar{v}_i \) for all \( i \in [1, N] \) hence \( \bar{s}_\alpha = \pi_\alpha \circ \cdots \circ \pi_1[z_4] + v_\alpha + \sum_{i \in [1, \alpha-1]} \pi_\alpha \circ \cdots \circ \pi_{i+1}[v_i] \). From the binding property of commitment \( \text{com}_2 \), one has \( \bar{s}_\alpha = \bar{t}_\alpha + z_4 \) hence \( z_1 - u = \pi_1^{-1} \circ \cdots \circ \pi_\alpha^{-1}[z_4] \). As a consequence, one has \( H(\pi_1^{-1} \circ \cdots \circ \pi_\alpha^{-1}[z_4]) = y \) along with \( w_H(z_4) = \omega \) thus \( \pi_1^{-1} \circ \cdots \circ \pi_\alpha^{-1}[z_4] \) is a solution of the considered SD problem instance.

**Special Honest-Verifier Zero-Knowledge.** We start by explaining why valid transcripts do not leak anything on the secret \( x \). A valid transcript contains \((u + x, (\pi_i, v_i)_{i \in [1, N]} \setminus \alpha}, \pi[u] + v, \pi_\alpha \circ \cdots \circ \pi_1[x], \text{com}_1, \alpha)\) namely the secret \( x \) is masked either by a random value \( u \) or by a random permutation \( \pi_\alpha \). The main difficulty concerns the permutation \( \pi_\alpha \) as the protocol requires \( \pi_\alpha \circ \cdots \circ \pi_1[u + x] \) to be computed while both \((u + x)\) and \((\pi_i)_{i \in [1, \alpha-1]} \) are known. To overcome this issue, the protocol actually computes \( \pi_\alpha \circ \cdots \circ \pi_1[u + x] + v_\alpha + \sum_{i \in [1, \alpha-1]} \pi_\alpha \circ \cdots \circ \pi_{i+1}[v_i] \) for some random value \( v_\alpha \) hence does not leak anything on \( \pi_\alpha \). In addition, if the commitment used is hiding, \( \text{com}_1, \alpha \) does not leak anything on \( \pi_\alpha \) or \( v_\alpha \). Formally, one can build a PPT simulator \( \text{Sim} \) that given the public values \((H, y)\), random seeds \((\theta, \xi)\) and a random challenge \( \alpha \) outputs a transcript \((H, y, \text{com}_1, \text{com}_2, \alpha, \text{rsp})\) such that \( \text{com}_1 = \text{Setup}(\theta, \xi) \) that is indistinguishable from the transcript of honest executions of the protocol:

1. Compute \((\pi_i, v_i)_{i \in [1, N]}\) and \( u \) from \((\theta, \xi)\)
2. Compute \( \tilde{x}_1 \) such that \( H\tilde{x}_1 = y \) and \( \tilde{x}_2 \leftarrow \mathcal{S}_{\omega}(\mathbb{F}_2^n) \)
3. Compute \( \bar{s}_0 = u + \tilde{x}_1 \) and \( \bar{s}_i = \pi_i[\bar{s}_i^{-1}] + \bar{v}_i \) for all \( i \in [1, \alpha-1] \)
4. Compute \( \bar{s}_\alpha = \pi_\alpha \circ \cdots \circ \pi_1[u + \tilde{x}_2] + v_\alpha + \sum_{i \in [1, \alpha-1]} \pi_\alpha \circ \cdots \circ \pi_{i+1}[v_i] \)
5. Compute \( \bar{s}_i = \pi_i[\bar{s}_i^{-1}] + \bar{v}_i \) for all \( i \in [1, N] \)
6. Compute \( \text{com}_2 = H(\tilde{x}_1 \parallel (\bar{s}_i)_{i \in [1, N]}) \)
7. Compute \( \tilde{z}_1 = u + \tilde{x}_1, z_2 = (\theta_i)_{i \in [1, N]} \setminus \alpha, z_3 = \xi, z_4 = \pi_\alpha \circ \cdots \circ \pi_1[\tilde{x}_2] \)
8. Compute \( \text{rsp} = (\tilde{z}_1, z_2, z_3, \tilde{z}_4, \text{com}_1, \alpha) \) and output \((H, y, \text{com}_1, \text{com}_2, \alpha, \text{rsp})\)

The transcript generated by the simulator \( \text{Sim} \) is \((H, y, \text{com}_1, \text{com}_2, \alpha, \text{rsp})\) where \( \text{com}_1 \leftarrow \text{Setup}(\theta, \xi) \). Since \( \tilde{x}_1 \) and \( x \) are masked by a random mask \( u \) unknown to the verifier, \( \tilde{z}_1 \) and \( z_1 \) are indistinguishable. Similarly, since \( \tilde{x}_2 \) and \( x \) have the same Hamming weight and are masked by a random permutation \( \pi_\alpha \) unknown to the verifier, \( \tilde{z}_4 \) and \( z_4 \) are indistinguishable. As \( \tilde{z}_2 \) and \( z_2 \) are indistinguishable, \( \bar{s}_\alpha \) and \( s_\alpha \) are also indistinguishable for all \( i \in [1, \alpha-1] \). Since \( \bar{s}_\alpha \) and \( s_\alpha \) both contains a random mask \( v_\alpha \) unknown to the verifier, they are indistinguishable. As \( \bar{s}_\alpha \) and \( s_\alpha \) are indistinguishable, do \( \bar{s}_i \) and \( s_i \) for all \( i \in [\alpha+1, N] \). Finally, \( z_2 \) and \( z_3 \) are identical in both cases and \( \text{com}_1, \alpha \) does not leak anything if the commitment is hiding. As a consequence, \((\text{rsp}, \text{com}_2)\) in the simulation and \((\text{rsp}, \text{com}_2)\) in the real execution are indistinguishable. Finally, \( \text{Sim} \) runs in polynomial time which completes the proof.

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E Proof of Theorem 5

Theorem 5. If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 2 is an honest-verifier zero-knowledge proof of knowledge for the QCSD problem over $F_2$ with soundness error equal to $\frac{1}{N} + \frac{N-1}{N} \cdot \frac{1}{N}$. 

Proof. The proofs of the correctness and honest-verifier zero-knowledge properties follow the same arguments as the proofs given in Appendix A. Hereafter, we provide a proof for the $(Mk, N)$-special soundness property.

$(Mk, N)$-special soundness. To prove the $(Mk, N)$-special soundness, one need to build an efficient knowledge extractor $\text{Ext}$ which returns a solution of the QCSD instance defined by $(H, y)$ with high probability given a $(Mk, N)$-tree of accepting transcripts. In our case, we build $\text{Ext}$ as a knowledge extractor for the DiffSD problem and use it as extractor for the QCSD problem thanks to Theorem 1. We start by describing the structure of the $(Mk, N)$-tree of accepting transcripts. The root node of the tree represents the first commitment message. It has $Mk$ children nodes which represent the $Mk$ second commitment messages that can be generated from the protocol. Each of these children node has $N$ leaf nodes which represent the $N$ subsequent responses that can be generated from the protocol. One can see that each accepting transcript can be mapped to a path from the root of the tree to one of its leaf where the tree nodes represent the prover messages while the tree edges represent the verifier messages. One only need a subset of the tree to complete the proof namely $4$ leafs corresponding to challenges $(\mu_j, \kappa_j, \alpha_i)_{j \in [1, 2]}$. $\text{Ext}$ computes the solution as:

1. Compute $(\pi_i)_{i \in [1, n]}$ from $z_2^{(\mu_1, \kappa_1, \alpha_1)}$ and $z_2^{(\mu_1, \kappa_1, \alpha_2)}$
2. Compute $c_1 = H_z^{(\mu_1, \kappa_1)}(y_{\mu_1}) = H_z^{(\mu_2, \kappa_2)}(y_{\mu_2})$
3. Compute $c_2 = \pi_{\alpha_1} \circ \cdots \circ \pi_{\alpha_1} [z_1^{(\mu_1, \kappa_1)}] - z_1^{(\mu_1, \kappa_1)} = \pi_{\alpha_1} \circ \cdots \circ \pi_{\alpha_1} [z_1^{(\mu_2, \kappa_2)}] - z_1^{(\mu_2, \kappa_2)}$
4. Compute $c_3 = H(v_{\pi^1_{\alpha_1} \circ \cdots \circ \alpha_1} [c_2]) - c_1$
5. Compute $d_1 = \pi_{\alpha_1} \circ \cdots \circ \pi_{\alpha_1} [z_1^{(\mu_1, \kappa_1)}]$ for all $j \in [1, 2]$
6. Output $(c_3, d_1, d_2)$

One can compute $(\tilde{\pi}_i^{(\mu_j, \kappa_j)}, \tilde{v}_i^{(\mu_j, \kappa_j)})_{j \in [1, 2]}$ from $(z_2^{(\mu_j, \kappa_j, \alpha_i)})_{j \in [1, 2]}$. From the binding property of the commitments $(\text{com}_{1, i})_{i \in [1, N]}$, one can see that $(\pi_i, v_i)_{i \in [1, N]} = (\tilde{\pi}_i^{(\mu_1, \kappa_1)}, \tilde{v}_i^{(\mu_1, \kappa_1)})_{i \in [1, N]} = (\tilde{\pi}_i^{(\mu_2, \kappa_2)}, \tilde{v}_i^{(\mu_2, \kappa_2)})_{i \in [1, N]}$. From the binding property of commitment $\text{com}_2$, one has $s_0^{(\mu_j, \kappa_j, \alpha_2)} = s_0^{(\mu_j, \kappa_j, \alpha_2)} = \tilde{z}_1^{(\mu_j, \kappa_j)}$ for all $j \in [1, 2]$. In addition, one has $s_i^{(\mu_j, \kappa_j, \alpha_2)} = \tilde{\pi}_i^{(\mu_j, \kappa_j, \alpha_2)} + \tilde{v}_i$ for all $i \in [1, N] \setminus \alpha_2$ and all $j \in [1, 2]$ as well as $s_i^{(\mu_j, \kappa_j, \alpha_2)} = \tilde{\pi}_i^{(\mu_j, \kappa_j, \alpha_2)} + \tilde{v}_i$ for all $i \in [1, N] \setminus \alpha_2$ and all $j \in [1, 2]$. Using the binding property of commitment $\text{com}_2$ once again, one can deduce that $s_i^{(\mu_j, \kappa_j)} = \tilde{\pi}_i^{(\mu_j, \kappa_j)} + \tilde{v}_i$ for all $i \in [1, N]$ and $j \in [1, 2]$ hence $s_0^{(\mu_j, \kappa_j)} = \pi_{\alpha_1} \circ \cdots \circ \pi_{\alpha_1} [z_1^{(\mu_j, \kappa_j)}] + v_{\alpha_1} + \sum_{i \in [1, \alpha_1-1]} \pi_{\alpha_1} \circ \cdots \circ \pi_{i+1} [v_i]$. 

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for all $j \in [1, 2]$. From the binding property of commitment $\text{com}_1$, one has $c_1 = H_z^1(\mu_1, \kappa_1) - \text{rot}_{\kappa_1}(y_{\mu_1}) = H_z^1(\mu_2, \kappa_2) - \text{rot}_{\kappa_2}(y_{\mu_2})$. In addition, one has $\overline{r}(\mu_1, \kappa_1) = \overline{r}(\mu_2, \kappa_2)$ which implies that $t_{\alpha_1} = t_{\alpha_1}^{\mu_1, \kappa_1} = t_{\alpha_1}^{\mu_2, \kappa_2}$. From the binding property of commitment $\text{com}_2$, one has $\sigma_{a_1}(\mu_1, \kappa_1) = t_{\alpha_1}^{\mu_1, \kappa_1} + z_4^{\mu_1, \kappa_1}$ for all $j \in [1, 2]$. Using $t_{\alpha_1}^{\mu_1, \kappa_1} = t_{\alpha_2}^{\mu_2, \kappa_2}$, one can deduce that $c_2 = t_{\alpha_1} - v_{\alpha_1} - \sum_{i \in [1, \alpha_1 - 1]} \pi_{\alpha_1} \cdots \pi_{t_{i+1}} \cdot [v_i] = \pi_{\alpha_1} \cdots \pi_{t_{\alpha_1}} \cdot [z_1^{\mu_1, \kappa_1}] - z_4^{\mu_1, \kappa_1} = \pi_{\alpha_1} \cdots \pi_{t_{\alpha_2}} \cdot [z_1^{\mu_2, \kappa_2}] - z_4^{\mu_2, \kappa_2}$.

It follows that $z_1^{\mu_1, \kappa_1} = \pi_{\alpha_1} \cdots \pi_{t_{\alpha_1}} \cdot [c_2 - z_4^{\mu_2, \kappa_2}]$ for all $j \in [1, 2]$. As $c_1 = H_z^1(\mu_1, \kappa_1) - \text{rot}_{\kappa_1}(y_{\mu_1})$, one has $c_1 = H(\pi_1^{-1} \cdots \pi_{\alpha_1}^{-1} [c_2 - z_4^{\mu_2, \kappa_2}]) - \text{rot}_{\kappa_1}(y_{\mu_1})$ hence $H(\pi_1^{-1} \cdots \pi_{\alpha_1}^{-1} [z_4^{\mu_1, \kappa_1}]) + c_3 = \text{rot}_{\kappa_1}(y_{\mu_1})$ for all $j \in [1, 2]$. Given that $w_H(z_4^{\mu_1, \kappa_1}) = \omega$ for all $j \in [1, 2]$, one can conclude that $(c_3, d_1, d_2)$ is a solution of the considered DiffSD problem instance. One completes the proof by using Theorem 1.

## F Proof of Theorem 6

**Theorem 6.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 3 is an honest-verifier zero-knowledge proof of knowledge for the SD problem over $F_q$ with soundness error equal to $\frac{1}{N} + \frac{N-1}{N(q-1)}$.

**Proof.** The proofs of the correctness and honest-verifier zero-knowledge properties follow the same arguments as the proofs given in Appendix 1. Hereafter, we provide a proof for the $(q - 1, N)$-special soundness property.

**(q - 1, N)-special soundness.** To prove the $(q - 1, N)$-special soundness, one need to build an efficient knowledge extractor $\text{Ext}$ which returns a solution of the SD instance defined by $(H, y)$ given a $(q - 1, N)$-tree of accepting transcripts. We start by describing the structure of this tree. The root node of the tree represents the first commitment message. It has $q - 1$ children nodes which represent the $q - 1$ second commitment messages that can be generated from the protocol. Each of these children node has $N$ leaf nodes which represent the $N$ subsequent responses that can be generated from the protocol. One can see that each accepting transcript can be mapped to a path from the root of the tree to one of its leaf where the tree nodes represent the prover messages while the the tree edges represent the verifier messages. One only need a subset of the tree to complete the proof namely the four leaves corresponding to challenges $(\kappa, \alpha_1), (\kappa, \alpha_2), (\kappa', \alpha_1) \text{ and } (\kappa', \alpha_2)$ where $\kappa \neq \kappa'$ and $\alpha_1 \neq \alpha_2$. The knowledge extractor $\text{Ext}$ computes the solution as:

1. Compute $(\pi_i)_{i \in [1, n]}$ from $z_2^{(\kappa, \alpha_1)}$ and $z_2^{(\kappa, \alpha_2)}$
2. Compute $\pi = \pi_N \cdots \pi_1$
3. Output $\pi^{-1}[z_4]$
One can compute \((\overline{\pi}_i^{(\kappa)}, \overline{\nu}_i^{(\kappa)})_{i \in [1,N]}\) and \((\overline{\pi}_i^{(\kappa)}, \overline{\nu}_i^{(\kappa)})_{i \in [1,N]}\) from \((\overline{\pi}_2^{(\kappa,\alpha_1)}, \overline{\nu}_2^{(\kappa,\alpha_1)})_{i \in [1,2]}\) and \((z_2^{(\kappa,\alpha_1)})_{i \in [1,2]}\) respectively. From the binding property of the commitments \((\text{com}_1, i)_{i \in [1,N]}\), one has \((\overline{\pi}_i, \overline{\nu}_i)_{i \in [1,N]} = (\overline{\pi}_1^{(\kappa)}, \overline{\nu}_1^{(\kappa)})_{i \in [1,N]} = (\overline{\pi}_i^{(\kappa)}, \overline{\nu}_i^{(\kappa)})_{i \in [1,N]}\). From the binding property of commitment \text{com}_2, one has \(s_{0}^{(\kappa,\alpha_1)} = s_{0}^{(\kappa,\alpha_2)} = z_{1}^{(\kappa)}\). In addition, one has \(\overline{s}_i^{(\kappa,\alpha_1)} = \overline{s}_i s_{i-1}^{(\kappa,\alpha_1)}\) + \(\overline{v}_i\) for all \(i \in [1,N] \setminus \alpha_1\) as well as \(\overline{s}_i^{(\kappa,\alpha_2)} = \overline{s}_i s_{i-1}^{(\kappa,\alpha_2)}\) + \(\overline{v}_i\) for all \(i \in [1,N] \setminus \alpha_2\). Using the binding property of commitment \text{com}_2 once again, one can deduce that \(s_i^{(\kappa)} = \overline{s}_i s_{i-1}^{(\kappa,\alpha_1)} + \overline{v}_i\) for all \(i \in [1,N]\) hence \(s_0^{(\kappa)} = \pi_{\alpha_1} \circ \cdots \circ \pi_1 (\overline{z}_1^{(\kappa)}) + v_{\alpha_1} + \sum_{i \in [1,\alpha_1-1]} \pi_{\alpha_1} \circ \cdots \circ \pi_{i+1} (\overline{v}_i)\). From the binding property of commitment \text{com}_1, one has \(H_{z_1^{(\kappa)}} - \kappa \cdot y = H_{z_1^{(\kappa')} - \kappa' \cdot y}\) hence \(H(z_1^{(\kappa)} - z_1^{(\kappa')}) = (\kappa - \kappa') \cdot y\). In addition, one has \(\overline{r}_i^{(\kappa)} = \overline{r}_i^{(\kappa')}\) and \(\overline{z}_i^{(\kappa')} = \overline{z}_i^{(\kappa')}\) which imply respectively \(\overline{t}_i^{(\kappa)} = \overline{t}_i^{(\kappa')}\) and \(\overline{b}_i^{(\kappa')} = \overline{b}_i^{(\kappa')}\). From the binding property of commitment \text{com}_2, one has \(\overline{s}_i^{(\kappa)} = \overline{t}_i^{(\kappa)} + \kappa' \cdot \overline{b}_i^{(\kappa')}\) as well as \(\overline{s}_i^{(\kappa')} = \overline{t}_i^{(\kappa')} + \kappa' \cdot \overline{b}_i^{(\kappa')}\). Using \(\overline{t}_i^{(\kappa)} = \overline{t}_i^{(\kappa')}\), one can deduce that \(\pi_{\alpha_1} \circ \cdots \circ \pi_1 (\overline{z}_1^{(\kappa)}) = (\kappa - \kappa') \cdot \overline{b}_i^{(\kappa')}\). Using \(\overline{b}_i^{(\kappa')} = \overline{b}_i^{(\kappa')}\), one has \(\pi_{\alpha_1} \circ \cdots \circ \pi_1 (\overline{z}_1^{(\kappa)}) - z_1^{(\kappa')} = (\kappa - \kappa') \cdot \pi_{\alpha_1} \circ \cdots \circ \pi_{N} (\overline{z}_1^{(\kappa)})\). As a consequence, one has \(H(\pi_{\alpha_1} \circ \cdots \circ \pi_{N} (\overline{z}_1^{(\kappa)})) = y\) along with \(w_H(\overline{z}_1^{(\kappa)}) = \omega\) thus \(\pi_{\alpha_1} \circ \cdots \circ \pi_{N} (\overline{z}_1^{(\kappa)})\) is a solution of the considered SD problem instance.

G Proof of Theorem 7

**Theorem 7.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 4 is an honest-verifier zero-knowledge proof of knowledge for the MQ_H problem with soundness error equal to \(\frac{1}{N} + \frac{N^{-1}}{M^2 \cdot (q-1)}\).

**Proof.** We prove the correctness, special soundness and special honest-verifier zero-knowledge properties below.

**Correctness.** The correctness of the protocol relies on the polar form of \(F\) that ensures that \(G(a, b) = F(a+b) - F(a) - F(b)\). As \(\sum_{i \in [1,N]} s_i = G(\sum_{i \in [1,N]} u_i, s_0) + \sum_{i \in [1,N]} v_i\), one has \(G(u, s_0) + F(u) = \kappa \cdot y_\mu - F(z_1)\). As \(z_1 = s_0 = \kappa \cdot x_\mu - u\), one has \(G(U, \kappa \cdot x_\mu - u) + F(U) = \kappa \cdot y_\mu - F(\kappa \cdot x_\mu - u)\) which is equivalent to \(F(\kappa \cdot x_\mu) - F(U) - F(\kappa \cdot x_\mu - u) + F(u) = \kappa \cdot y_\mu - F(\kappa \cdot x_\mu - u)\). This gives \(F(\kappa \cdot x_\mu) = \kappa \cdot y_\mu\) which is equivalent to \(\kappa \cdot x_\mu) = \kappa \cdot y_\mu\) as homogeneous polynomials of degree 2 are used. Once this observation has been made, the correctness follows from the protocol description.

**\((M(q-1), N)\)-special soundness.** To prove the \((M(q-1), N)\)-special soundness, one need to build an efficient knowledge extractor \(\text{Ext}\) which returns a
solution of the \(\mathcal{MQ}_H\) instance defined by \((\mathcal{F}, \mathbf{y})\) with high probability given a \((M(q - 1), N)\)-tree of accepting transcripts. In our case, we build \(\text{Ext}\) as a knowledge extractor for the Diff\(\mathcal{MQ}_H\) problem and use it as extractor for the \(\mathcal{MQ}_H\) problem thanks to Theorem 2. We start by describing the structure of the \((M(q - 1), N)\)-tree of accepting transcripts. The root node of the tree represents the first commitment message. It has \(M(q - 1)\) children nodes which represent the \(M(q - 1)\) second commitment messages that can be generated from the protocol. Each of these children node has \(N\) leaf nodes which represent the \(N\) subsequent responses that can be generated from the protocol. One can see that each accepting transcript can be mapped to a path from the root of the tree to one of its leaf where the tree nodes represent the prover messages while the tree edges represent the verifier messages. One only need a subset of the tree to complete the proof namely the 4 leafs corresponding to challenges \((\mu_j, \kappa_j, \alpha_j)\) for all \(j \in [1, 2]\). The knowledge extractor \(\text{Ext}\) computes the solution as:

1. Compute \((\bar{u}_i, \bar{v}_i)_{i \in [1, N]}\) from \(z_2^{(\mu_1, \kappa_1, \alpha_1)}\) and \(z_2^{(\mu_1, \kappa_1, \alpha_2)}\)
2. Compute \(c_1 = \sum_{i \in [1, N]} \bar{u}_i\) and \(c_2 = \sum_{i \in [1, N]} \bar{v}_i\)
3. Compute \(c_3 = c_2 - \mathcal{F}(c_1)\)
4. Compute \(d_j = c_1 + z_j^{(\mu_j, \kappa_j)}\) for all \(j \in [1, 2]\)
5. Output \((c_3, d_1, d_2)\)

One can compute \((\bar{u}_i^{(\mu_i, \kappa_i)}, \bar{v}_i^{(\mu_i, \kappa_i)})_{j \in [1, 2]}\) from \((z_2^{(\mu_i, \kappa_i, \alpha)})_{i \in [1, 2]}\). From the binding property of the commitments \((\text{com}_1, i)_{i \in [1, N]}\), one can see that \((\bar{u}_i, \bar{v}_i)_{i \in [1, N]} = (\bar{u}_i^{(\mu_i, \kappa_i)}, \bar{v}_i^{(\mu_i, \kappa_i)})_{i \in [1, N]}\). From the binding property of commitment \(\text{com}_2\), one has \(s_1^{(\mu_j, \kappa_j, \alpha_1)} = s_0^{(\mu_j, \kappa_j, \alpha_2)} = z_1^{(\mu_j, \kappa_j)}\) for all \(j \in [1, 2]\). In addition, one has \(s_1^{(\mu_i, \kappa_i, \alpha_1)} = \mathcal{G}(\bar{u}_i, z_1^{(\mu_i, \kappa_i)}) + \bar{v}_i\) for all \(i \in [1, N] \setminus \alpha_1\) and all \(j \in [1, 2]\) as well as \(s_1^{(\mu_j, \kappa_j, \alpha_2)} = \mathcal{G}(\bar{u}_i, z_1^{(\mu_j, \kappa_j)}) + \bar{v}_i\) for all \(i \in [1, N] \setminus \alpha_2\) and all \(j \in [1, 2]\). Using the binding property of commitment \(\text{com}_2\) once again, one can deduce that \(s_1^{(\mu_j, \kappa_j)} = \mathcal{G}(\bar{u}_i, z_1^{(\mu_j, \kappa_j)}) + \bar{v}_i\) for all \(i \in [1, N]\) and \(j \in [1, 2]\) hence \(s_0^{(\mu_i, \kappa_i)} = \mathcal{G}(\bar{u}_i, z_1^{(\mu_i, \kappa_i)}) + \bar{v}_i\) for all \(i \in [1, N]\) and \(j \in [1, 2]\). This gives \(\sum_{i \in [1, N]} \mathcal{G}(\bar{u}_i, z_1^{(\mu_j, \kappa_j)}) = \mathcal{G}(\sum_{i \in [1, N]} \bar{u}_i, z_1^{(\mu_j, \kappa_j)}) = \kappa_j^2 \cdot y_{\mu_j}\) for all \(j \in [1, 2]\).

Special Honest-Verifier Zero-Knowledge. We start by explaining why valid transcripts do not leak anything on the secrets \((x_\mu)_{\mu \in [1, M]}\). Without loss of generi-
icery, we can omit the index μ in the following arguments. A valid transcript contains \((κ \cdot x – u, (π_i, v_i)_{i \in [1,N]} \| α, com_1, α)\) where the secret \(x\) is hidden by the unknown value \(u\). In the protocol, the verifier can recompute \(s_α\) however this does leak anything on the secret thanks to the use of the random value \(v_α\). The value \(v_α\) cannot be recomputed from the other \((v_i)_{i \in [1,N]}\) as it is masked by \(F(u)\) where \(u\) is not known. Finally, \(u\) cannot be recomputed from \((u_i)_{i \in [1,N]}\) as \(u_α\) is not known. In addition, if the commitment used is hiding, \(com_{1,α}\) does not leak anything on \(u_α\) nor \(v_α\). Formally, one can build a PPT simulator \(Sim\) that given the public values \((F, y)\), random challenges \((κ, α)\) outputs a transcript \((F, y, com_1, κ, com_2, α, rsp)\) that is indistinguishable from the transcript of honest executions of the protocol:

1. Compute \(\tilde{x} \leftarrow \mathbb{F}_q^n\)
2. Compute \((u_i)_{i \in [1,N]}\) and \((v_i)_{i \in [1,N-1]}\) as in the real protocol
3. Compute \(\tilde{v}_N = F(u) - \sum_{i \in [1,N-1]} v_i + κ^2 \cdot y - F(κ \cdot \tilde{x})\)
4. Compute \(\tilde{s}_0 = κ \cdot \tilde{x} – u\) and \(\tilde{s}_i = G(u_i, \tilde{s}_0) + \tilde{v}_i\) for all \(i \in [1, N]\)
5. Compute \(com_1, \tilde{z}_1\) and \(\tilde{z}_2\) as in the real protocol
6. Compute \(rsp = (\tilde{z}_1, \tilde{z}_2, com_1, α)\) and output \((F, y, com_1, κ, com_2, α, rsp)\)

The transcript generated by the simulator \(Sim\) is \((F, y, com_1, κ, com_2, α, rsp)\). Since \(\tilde{z}_1\) (in the simulation) and \(z_1\) (in the real world) are masked by a random value \(u\) unknown to the verifier, they are indistinguishable one from the other. In addition, since \(\tilde{v}_N\) is masked by a random value \(v_α\) unknown to the verifier, \(\tilde{z}_2\) and \(z_2\) are indistinguishable. Finally, \(com_{1,α}\) does not leak anything on \(u_α\) nor \(v_α\) if the commitment is hiding. As a consequence, \((com_1, com_2, rsp)\) (in the simulation) and \((com_1, com_2, rsp)\) (in the real execution) are indistinguishable. Finally, \(Sim\) runs in polynomial time which completes the proof.

H  Proof of Theorem 8

**Theorem 8.** If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 5 is an honest-verifier zero-knowledge proof of knowledge for the IPKP problem with soundness error equal to \(\frac{1}{N} + \frac{N – 1}{q – 1}\).

**Proof.** We prove the correctness, special soundness and special honest-verifier zero-knowledge properties below.

**Correctness.** The correctness follows from the protocol description once it is observed that
\(s_N = π[κ \cdot x] + v\) which implies that
\(Hs_N - κ \cdot y = Hπ[κ \cdot x] + Hv - κ \cdot y = Hv\).

\((q – 1, N)\)-special soundness. To prove the \((q – 1, N)\)-special soundness, one need to build an efficient knowledge extractor \(Ext\) which returns a solution of
the PKP instance defined by \((\mathbf{H}, \mathbf{x}, \mathbf{y})\) with high probability given a \((q - 1, N)\)-tree of accepting transcripts. We start by describing the structure of this tree. The root node of the tree represents the first commitment message. It has \(q - 1\) children nodes which represent the \(q - 1\) second commitment messages that can be generated from the protocol. Each of these children node has \(N\) leaf nodes which represent the \(N\) subsequent responses that can be generated from the protocol. One can see that each accepting transcript can be mapped to a path from the root of the tree to one of its leaf where the tree nodes represent the prover messages while the the tree edges represent the verifier messages. One only need a subset of the tree to complete the proof namely the four leafs corresponding to challenges \((\kappa, \alpha_1), (\kappa, \alpha_2), (\kappa', \alpha_1)\) and \((\kappa', \alpha_2)\) where \(\kappa \neq \kappa'\) and \(\alpha_1 \neq \alpha_2\). The knowledge extractor \(\mathbf{Ext}\) computes the solution as:

1. Compute \((\bar{\pi}_i)_{i \in [1, n]}\) from \(z_i^{(\kappa, \alpha_1)}\) and \(z_i^{(\kappa, \alpha_2)}\)
2. Compute \(\bar{\pi} = \bar{\pi}_N \circ \cdots \circ \bar{\pi}_1\)
3. Output \(\bar{\pi}\)

One can compute \((\bar{\pi}_i^{(\kappa)}, \bar{v}_i^{(\kappa)})_{i \in [1, \lceil 1, N \rceil]}\) and \((\bar{\pi}_i^{(\kappa)}, \bar{v}_i^{(\kappa')})_{i \in [1, \lceil 1, N \rceil]}\) and \((\bar{z}_i^{(\kappa, \alpha_1)})_{i \in [1, 2]}\) and \((\bar{z}_i^{(\kappa, \alpha_2)})_{i \in [1, 2]}\) respectively. From the binding property of the commitments \((\mathbf{com}_{1,i})_{i \in [1, N]}\), one has \((\bar{\pi}_i, \bar{v}_i)_{i \in [1, N]} = (\bar{\pi}_i^{(\kappa)}, \bar{v}_i^{(\kappa)})_{i \in [1, N]}\) and \((\bar{\pi}_i^{(\kappa)}, \bar{v}_i^{(\kappa')})_{i \in [1, N]}\).

By construction, one has \(\bar{s}_{0}^{(\kappa, \alpha_1)} = s_{0}^{(\kappa, \alpha_2)} = \kappa \cdot \mathbf{x}\). In addition, one has \(\bar{s}_{i}^{(\kappa, \alpha_1)} = \bar{\pi}_i \bar{s}_{i - 1}^{(\kappa, \alpha_1)} + \bar{v}_i\) for all \(i \in \lceil 1, N \rceil \setminus \alpha_1\) as well as \(\bar{s}_{i}^{(\kappa, \alpha_2)} = \bar{\pi}_i \bar{s}_{i - 1}^{(\kappa, \alpha_2)} + \bar{v}_i\) for all \(i \in \lceil 1, N \rceil \setminus \alpha_2\). From the binding property of commitment \(\mathbf{com}_2\), one can deduce that \(\bar{s}_{i}^{(\kappa)} = \bar{\pi}_i \bar{s}_{i - 1}^{(\kappa)} + \bar{v}_i\) for all \(i \in \lceil 1, N \rceil\) hence \(\bar{s}_{N}^{(\kappa)} = \bar{\pi}_{\lceil \kappa \cdot \mathbf{x} \rceil} + \bar{v}_{\lceil \kappa \cdot \mathbf{x} \rceil} \). Following a similar argument, one also has \(\bar{s}_{N}^{(\kappa')} = \bar{\pi}_{\lceil \kappa' \cdot \mathbf{x} \rceil} + \bar{v}_{\lceil \kappa' \cdot \mathbf{x} \rceil} \). From the binding property of commitment \(\mathbf{com}_1\), one has \(\mathbf{H}s_{N}^{(\kappa)} - \kappa \cdot \mathbf{y} = \mathbf{H}s_{N}^{(\kappa')} - \kappa' \cdot \mathbf{y}\). It follows that \(\mathbf{H}(\bar{\pi}_{\lceil \kappa \cdot \mathbf{x} \rceil} + \bar{v}) - \kappa \cdot \mathbf{y} = \mathbf{H}(\bar{\pi}_{\lceil \kappa' \cdot \mathbf{x} \rceil} + \bar{v}) - \kappa' \cdot \mathbf{y}\) hence \((\kappa - \kappa') \cdot 2 \mathbf{H}[\bar{\pi}] = (\kappa - \kappa') \cdot \mathbf{H}[\mathbf{y}]\). This implies that \(\mathbf{H}[\bar{\pi}] = \mathbf{y}\) thus \(\bar{\pi}\) is a solution of the considered PKP problem.

**Special Honest-Verifier Zero-Knowledge.** We start by explaining why valid transcripts do not leak anything on the secret \(\pi\). A valid transcript contains \((\bar{s}_\alpha, (\bar{\pi}_i, \bar{v}_i)_{i \in [1, N] \setminus \alpha}, \mathbf{com}_{1, \alpha})\) where the secret \(\pi\) is hidden by the unknown permutation \(\pi_\alpha\). In our protocol, one need to compute \(\pi[\mathbf{x}]\) without leaking anything on the secret \(\pi\). To overcome this issue, the protocol actually computes \(\pi[\mathbf{x}] + \mathbf{v}\) for some value \(\mathbf{v}\) that is masked by the unknown random value \(v_\alpha\). In addition, if the commitment used is hiding, \(\mathbf{com}_{1, \alpha}\) does not leak anything on \(\pi_\alpha\) nor \(v_\alpha\). Formally, one can build a PPT simulator \(\mathbf{Sim}\) that given the public values \((\mathbf{H}, \mathbf{x}, \mathbf{y})\), random challenges \((\kappa, \alpha)\) outputs a transcript \((\mathbf{H}, \mathbf{x}, \mathbf{y}, \mathbf{com}_{1, \kappa}, \mathbf{com}_{2, \alpha}, \mathbf{rsp})\) that is indistinguishable from the transcript of honest executions of the protocol.
1. Compute \((\pi_1, v_1, c\not=_{1,i})\) as in the real protocol except for \(\tilde{\pi}_1 \leftarrow \$ S_n\)
2. Compute \(\tilde{\pi} = \pi_N \circ \cdots \circ \tilde{\pi}_1\)
3. Compute \(v\) and \(c\not=_{1,2}\) as in the real protocol
4. Compute \(\tilde{\mathbf{x}}\) such that \(H\tilde{\mathbf{x}} = \kappa \cdot y\)
5. Compute \(s_0 = \kappa \cdot x\) and \(\tilde{s}_i = \pi_1[\tilde{s}_{i-1}] + v_i\) for all \(i \in [1, \alpha - 1]\)
6. Compute \(\tilde{s}_\alpha = \pi_\alpha[\tilde{s}_{\alpha-1}] + v_\alpha + \pi_{\alpha+1}^{-1} \circ \cdots \circ \pi_N^{-1}[\tilde{\mathbf{x}} - \pi[\kappa \cdot x]]\)
7. Compute \(\tilde{s}_i = \pi_i[\tilde{s}_{i-1}] + v_i\) for all \(i \in [\alpha + 1, N]\)
8. Compute \(c\not=_{1,\alpha} = Hash(\kappa \cdot x || (\tilde{s}_i)_{i \in [1,N]} )\) and \(\bar{z}_1 = \tilde{s}_\alpha\)
9. Compute \(\bar{z}_2 = \tilde{\pi}_1 || (\theta_i)_{i \in [1,N]\setminus \alpha}\) if \(i \neq 1\) or \(\bar{z}_2 = (\theta_i)_{i \in [1,N]\setminus \alpha}\) otherwise
10. Compute \(r\not= = (\bar{z}_1, \bar{z}_2, c\not=_{1,\alpha})\) and output \((\mathbf{H}, \mathbf{x}, y, c\not=_{1,2}, \alpha, r\not=)\)

The transcript generated by the simulator \Sim\ is \((\mathbf{H}, \mathbf{x}, y, c\not=_{1,2}, \alpha, c\not=_{1,2}, \alpha, r\not=)\).

Since \(\tilde{s}_\alpha\) (in the simulation) and \(s_\alpha\) (in the real world) are masked by a random mask \(v_\alpha\) unknown to the verifier, \(\bar{z}_1\) and \(z_1\) are indistinguishable. In addition, since \(\tilde{\pi}_1\) is sampled uniformly at random in \(S_n\), \(\bar{z}_2\) and \(z_2\) are indistinguishable. Finally, \(c\not=_{1,\alpha}\) does not leak anything on \(\pi_\alpha\) nor \(v_\alpha\) if the commitment is hiding. As a consequence, \((c\not=_{1,2}, c\not=_{1,2}, r\not=)\) (in the simulation) and \((c\not=_{1,2}, c\not=_{1,2}, r\not=)\) (in the real execution) are indistinguishable. Finally, \Sim\ runs in polynomial time which completes the proof.

I Proof of Theorem 9

Theorem 9. If the hash function used is collision-resistant and if the commitment scheme used is binding and hiding, then the protocol depicted in Figure 7 is an honest-verifier zero-knowledge proof of knowledge for the IRSL problem with soundness error equal to \(\frac{1}{N} + \frac{(N-1)(\Delta-1)}{q^{M_k}-1}\).

Proof. The proofs of the correctness and honest-verifier zero-knowledge properties follow the same arguments as the proofs given in Appendix D. Hereafter, we provide a proof for the \((q^{M_k} - 1, N)\)-special soundness property.

\((q^{M_k} - 1, N)\)-special soundness. To prove the \((q^{M_k} - 1, N)\)-special soundness, one need to build an efficient knowledge extractor \Ext\ which returns a solution of the IRSL instance defined by \((\mathbf{H}, y_1, \ldots, y_M)\) with high probability given a \((q^{M_k} - 1, N)\)-tree of accepting transcripts. In our case, we build \Ext\ as a knowledge extractor for the DiffIRSL problem and use it as extractor for the IRSL problem thanks to Theorem 3. We start by describing the structure of the \((q^{M_k} - 1, N)\)-tree of accepting transcripts. The root node of the tree represents the first commitment message. It has \(q^{M_k} - 1\) children nodes which represent the \(q^{M_k} - 1\) second commitment messages that can be generated from the protocol. Each of these children node has \(N\) leaf nodes which represent the \(N\) subsequent responses that can be generated from the protocol. One can see that each accepting transcript can be mapped to a path from the root of the tree to one of its leaf where the tree nodes represent the prover messages while the tree edges
represent the verifier messages. One only need a subset of the tree to complete the proof namely $2\Delta$ leafs corresponding to challenges $(\mu_j, \kappa_j, \alpha_j)_{j \in [1, \Delta]}$. Ext computes the solution as:

1. Compute $(\pi_i)_{i \in [1, N]}$ from $(z^{(\mu_j, \kappa_j, \alpha_j)}_i)_{j \in [1, \Delta]}$. From the binding property of the commitments $(\text{com}_1, i)_{i \in [1, N]}$, one can see that $(\pi_i, v_i)_{i \in [1, N]} = (\pi^{(\mu_j, \kappa_j, \alpha_j)}_i, v^{(\mu_j, \kappa_j, \alpha_j)}_i)_{i \in [1, N]}$ as well as $\tilde{\pi}^{(\mu_j, \kappa_j, \alpha_j)}_i = \tilde{\pi}^{(\mu_j, \kappa_j, \alpha_j)}_i + \tilde{\nu}_i$ for all $i \in [1, N] \setminus \alpha_1$ and all $j \in [1, \Delta]$ as well as $\tilde{\pi}^{(\mu_j, \kappa_j, \alpha_2)}_i = \tilde{\pi}^{(\mu_j, \kappa_j, \alpha_2)}_i + \tilde{\nu}_i$ for all $i \in [1, N] \setminus \alpha_2$ and all $j \in [1, \Delta]$. Using the binding property of commitment com$_2$, one has $\tilde{s}^{(\mu_j, \kappa_j, \alpha_1)}_i = \tilde{\pi}^{(\mu_j, \kappa_j, \alpha_1)}_i$ for all $i \in [1, \Delta]$. In addition, one can deduce that $\tilde{s}^{(\mu_j, \kappa_j)}_i = \tilde{\pi}^{(\mu_j, \kappa_j)}_i + \tilde{\nu}_i$ for all $i \in [1, N]$ and $j \in [1, \Delta]$ hence $\tilde{s}^{(\mu_j, \kappa_j)}_i = \pi^{(\mu_j, \kappa_j)}_i$.

2. Compute $c_1 = \text{HZ}_1^{(\mu_1, \kappa_1)} - \sum_{(i, j) \in [1, \Delta] \times [1, k]} \gamma_{i, j} \cdot \text{rot}_{\kappa_i}(y_{\mu_1})$ for all $j \in [1, \Delta]$. Using $\tilde{t}_i^{(\mu_1, \kappa_1)} = \pi_i^{(\mu_1, \kappa_1)}$, one can deduce that $c_2 = \tilde{c}_1 - v_1 = \pi_1^{(\mu_1, \kappa_1)} - \sum_{i \in [1, \Delta]} \pi_1^{(\mu_1, \kappa_1)} + \pi_i^{(\mu_1, \kappa_1)} - z^{(\mu_1, \kappa_1)}_i = \pi_1^{(\mu_1, \kappa_1)} - z^{(\mu_1, \kappa_1)}_i$. It follows that $z^{(\mu_2, \kappa_2)}_i = \pi_1^{(\mu_2, \kappa_2)} + \pi_1^{(\mu_2, \kappa_2)} - c_2$ for all $j \in [1, \Delta]$. As $c_3 = \text{HZ}_2^{(\mu_2, \kappa_2)} - \sum_{(i, j) \in [1, \Delta] \times [1, k]} \gamma_{i, j} \cdot \text{rot}_{\kappa_i}(y_{\mu_2})$, one has $c_4 = \text{HZ}_2^{(\mu_2, \kappa_2)} - \sum_{(i, j) \in [1, \Delta] \times [1, k]} \gamma_{i, j} \cdot \text{rot}_{\kappa_i}(y_{\mu_2})$ for all $j \in [1, \Delta]$. Given that $w_H(z^{(\mu_j, \kappa_j)}) = \omega$ for all $j \in [1, \Delta]$, one can conclude that $(c_3, d_1, \ldots, d_\Delta)$ is a solution of the considered DiffISSL problem instance. One completes the proof by using Theorem 3.