Quantum correlations in time

Tian Zhang,1,2 Oscar Dahlsten,2,1,3 and Vlatko Vedral1,4,5,6

1Department of Physics, Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, UK
2Institute for Quantum Science and Engineering, Department of Physics, Southern University of Science and Technology (SUSTech), Shenzhen 518055, China
3London Institute for Mathematical Sciences, 35a South Street, Mayfair, London, W1K 2XF, United Kingdom
4Centre for Quantum Technologies, National University of Singapore, Block S15, 3 Science Drive 2, Singapore 117543
5Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542
6I.S.I. Istituto Interscambio Scientifico, Via Chisola, 5, 10126 Torino TO, Italy

(Dated: February 25, 2020)

Abstract

We investigate quantum correlations in time in different approaches, under the assumption that temporal correlations should be treated in an even-handed manner with spatial correlations. We compare the pseudo-density matrix formalism with several other approaches: indefinite causal structures, consistent histories, generalised quantum games, out-of-time-order correlations (OTOCs), and path integrals. We establish relationships among these space-time approaches in non-relativistic quantum theory, and show that they are closely related and map into each other. This results in a unified picture in which temporal correlations in different space-time approaches are the same or operationally equivalent. However, the path integral formalism defines correlations in terms of amplitude measure rather than probability measure in other approaches, thus do not fit into the unified picture.
I. INTRODUCTION

Time remains a mysterious concept in physics. Whether time is absolute or relative, whether time is more or less or equally important as space, whether time is fundamental or just a parameter, all these remain important open questions. The problem of time \cite{1} is especially notorious in quantum theory as time cannot be treated as an operator in contrast with space.

Several attempts have been proposed to incorporate time into the quantum world in a more even-handed way to space, including: indefinite causal structures \cite{2,7}, consistent histories \cite{8,12}, generalised quantum games \cite{13,14}, spatio-temporal correlation approaches \cite{15,16}, path integrals \cite{17,18}, and pseudo-density matrices \cite{19,23}. Different approaches have their own advantages. Of particular interest here is the pseudo-density matrix approach for which one advantage is that quantum correlations in space and time are treated on an equal footing. The present work is motivated by the need to understand how the different approaches connect via temporal correlations, so that ideas and results can be transferred more readily.

We accordingly aim to identify mappings between these approaches and pseudo-density matrices. We ask what kind of relationship these space-time approaches hold in terms of...
temporal correlations. Are the allowed temporal correlations the same or different from each other? If the same, are they equal, or do they map with each other and what kind of mapping? If different, how different are they? More specifically, we take temporal correlations represented in different approaches and find that they are consistent with each other. Quantum correlations in time in these approaches are either exactly equal or operationally equivalent except those used in the path integral formalism. By operational equivalence of two formalisms, we mean the correlations or the probabilities of possible measurement outcomes with given inputs in these two formalisms are equal. We find several mappings and relations between these approaches, including (i) we map process matrices with indefinite causal order directly to pseudo-density matrices in four different ways; (ii) we show the diagonal terms of decoherence functionals in consistent histories are exactly the probabilities in temporal correlations of corresponding pseudo-density matrices; (iii) we show quantum-classical signalling games give the same probabilities as temporal correlations measured in pseudo-density matrices; (iv) the calculation of OTOCs reduces half numbers of steps by pseudo-density matrices; and (v) correlations in path integrals are defined as expectation values in terms of the amplitude measure rather than the probability measure as in pseudo-density matrices and are different from correlations in all the other approaches. A particular example via a tripartite pseudo-density matrix is presented to illustrate the unified picture of different approaches except path integrals. This applies to more complicated cases and provides a unified picture of these approaches. It also supports the further development of space-time formalisms in non-relativistic quantum theory. Difference in correlations between path integrals and other approaches also suggests the importance of measure in quantum theory.

The paper proceeds as follows. We introduce the pseudo-density matrix formalism in Section II. Then we compare it with indefinite causal order in terms of forms, causality violation, quantum switch and postselection in Section III. In Section IV, we establish the relation between pseudo-density matrix and decoherence functional in consistent histories. We further explore generalised non-local games and build pseudo-density matrices from generalised signalling games in Section V. In Section VI, we simplify the calculation of out-of-time-order correlations via pseudo-density matrices. We further provide a unified picture under a tripartite pseudo-density matrix in Section VII. In Section VIII, we argue that the path integral formalism defines correlations in a different way and does not fit into the
unified picture. Finally we summarise our work and provide an outlook.

II. PSEUDO-DENSITY MATRIX FORMALISM

We firstly introduce the pseudo-density matrix formalism [19–23] as a unified approach for quantum correlations in space and time. We review the definition of pseudo-density matrices for finite dimensions, continuous variables, and general measurement processes, and present their properties.

A. Finite dimensions: definition and properties

The pseudo-density matrix formalism is originally proposed as a finite-dimensional quantum-mechanical formalism which aims to treat space and time on an equal footing [19]. In general, this formulation defines an event via making a measurement in space-time and is built upon correlations from measurement results; thus, it treats temporal correlations just as spatial correlations from observation of measurements and unifies spatio-temporal correlations in a single framework. As a price to pay, the spacetime states represented by pseudo-density matrices may not be positive semi-definite.

An \( n \)-qubit density matrix can be expanded by Pauli operators \( \sigma_i \) in terms of Pauli correlations which are the expectation values of these Pauli operators. In spacetime, instead of considering \( n \) qubits, let us pick up \( n \) events; for each event a single-qubit Pauli operator is measured. The pseudo-density matrix is then defined as

\[
\hat{R} \equiv \frac{1}{2^n} \sum_{i_1=0}^{3} \ldots \sum_{i_n=0}^{3} \langle \{ \sigma_{i_j} \}_{j=1}^{n} \rangle \bigotimes_{j=1}^{n} \sigma_{i_j},
\]  

(1)

where \( \langle \{ \sigma_{i_j} \}_{j=1}^{n} \rangle \) is the expectation value of the product of these measurement results for a particular choice of events with measurement operators \( \{ \sigma_{i_j} \}_{j=1}^{n} \). Similar to a density matrix, a pseudo-density matrix is Hermitian and unit-trace; but it is not positive semi-definite as we mentioned before. If the measurements are space-like separated or local systems evolve independently, the pseudo-density matrix will reduce to a standard density matrix. Otherwise, for example if measurements are made in time, the pseudo-density matrix may have negative eigenvalues.
B. Generalisation of pseudo-density matrix formalism

The pseudo-density matrix formalism in continuous variables is given in various forms [22], including the Gaussian case, spacetime Wigner functions and corresponding spacetime density matrices, and for position measurements and weak measurements.

Gaussian states are fully characterised by the first two statistical moments of the quantum states, the mean value and the covariance matrix. The mean value \( \mathbf{d} \), is defined as the expectation value of the \( N \)-mode quadrature field operators \( \{ \hat{q}_k, \hat{p}_k \}_{k=1}^{N} \) arranged in \( \mathbf{x} = (\hat{q}_1, \hat{p}_1, \cdots, \hat{q}_N, \hat{p}_N)^T \), that is,

\[
    d_j = \langle \hat{x}_j \rangle_{\rho} = \text{Tr}(\hat{x}_j \rho),
\]

for the Gaussian state \( \hat{\rho} \). The elements in the covariance matrix \( \sigma \) are defined as

\[
    \sigma_{ij} = \langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle_{\rho} - 2 \langle \hat{x}_i \rangle_{\rho} \langle \hat{x}_j \rangle_{\rho}. \tag{3}
\]

Now we generalise this definition to the spacetime domain. A Gaussian spacetime state is defined in Ref. [22] via measurement statistics as being (i) a vector \( \mathbf{d} \) of \( 2N \) expectation values of the \( N \)-mode quadrature field operators \( \{ \hat{q}_k, \hat{p}_k \}_{k=1}^{N} \) arranged in \( \mathbf{x} = (\hat{q}_1, \hat{p}_1, \cdots, \hat{q}_N, \hat{p}_N)^T \), with \( j \)-th entry

\[
    d_j = \langle \hat{x}_j \rangle_{\rho} = \text{Tr}(\hat{x}_j \rho), \tag{4}
\]

and (ii) a covariance matrix \( \sigma \) with entries as

\[
    \sigma_{ij} = 2\langle \{ \hat{x}_i, \hat{x}_j \} \rangle_{\rho} - 2 \langle \hat{x}_i \rangle_{\rho} \langle \hat{x}_j \rangle_{\rho}. \tag{5}
\]

where \( \langle \{ \hat{x}_i, \hat{x}_j \} \rangle_{\rho} \) is the expectation value for the product of measurement results; specifically \( \{ \hat{x}_i, \hat{x}_j \} = \frac{1}{2}(\hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i) \) for measurements at the same time.

For general continuous variables and general measurement processes, see Ref. [22].

III. INDEFINITE CAUSAL STRUCTURES

The concept of indefinite causal structures was proposed as probabilistic theories with non-fixed causal structures as a possible approach to quantum gravity [24, 25]. There are different indefinite causal order approaches: quantum combs [2, 3], operator tensors [4, 26], process matrices [5, 27], process tensors [6, 28], and super-density operators [7, 29]. Also, other formalisms are equivalent in certain sense [30], for example, quantum channels.
with memories \[31\], general quantum strategies \[32\], multiple-time states \[33, 35\], general boundary formalism \[36\], and quantum causal models \[37, 38\]. Since there are clear maps among quantum combs, operator tensors, process tensors, and process matrices, we just take the process matrix formalism in order to learn from causality inequalities, quantum switch and post-selection. We will investigate its relation with the pseudo-density matrix and show what lessons we shall learn for pseudo-density matrices.

A. Comparison in form

In this subsection, we first review process matrices in finite dimensions. Then we compare them with pseudo-density matrices in finite dimensions and discuss the bipartite case in particular. Similar relations hold for the continuous-variable version and we omit the discussion. We further establish the relation between process matrix formalism and pseudo-density matrix formalism with general measurement processes.

1. Definition of process matrix

The process matrix formulation is one of the formalisms with indefinite causal structures assuming local quantum mechanics and well-defined probabilities. Since the causal order is not fixed, it may violate a causality inequality and allow quantum switch as we will discuss in Subsection B. It also allows post-selection naturally as Subsection C suggests.

Here we review the definition and characterisation of process matrices in finite dimensions \[5, 27\]. Assume that the laws of quantum mechanics hold in local laboratories. Associated to the local laboratory operated by Alice, there is an input Hilbert space \(\mathcal{H}^A_I\) and an output Hilbert space \(\mathcal{H}^A_O\). The map \(\mathcal{M}^A : \mathcal{L}^A_I \rightarrow \mathcal{L}^A_O\) is completely positive(CP) and trace non-increasing where \(\mathcal{L}^A_I/A_O\) is the space of Hermitian linear operators over the Hilbert space \(\mathcal{H}^A_I/A_O\). The measurement outcome \(a\) is represented in the map as \(\mathcal{M}^A_a\). For all possible measurement outcomes, \(\sum_a \mathcal{M}^A_a\) is completely positive and trace preserving(CPTP).

Completely positive maps have another representation according to the Choi-Jamiołkowski isomorphism \[39, 40\]: for a completely positive map \(\mathcal{M}^A_a : \mathcal{L}^A_I \rightarrow \mathcal{L}^A_O\), its corresponding Choi-Jamiołkowski matrix is given as \(M^A_a \equiv [\mathcal{I} \otimes \mathcal{M}^A_a(|1\rangle\langle 1|)]^T \in A_I \otimes A_O\), with \(\mathcal{I}\) as the identity map and \(|1\rangle = |1\rangle^{A_I/A_I} \equiv \sum_j |j\rangle^{A_I} \otimes |j\rangle^{A_I} \in \mathcal{H}^{A_I} \otimes \mathcal{H}^{A_I}\) is the non-normalised
maximally entangled state. \( T \) is the matrix transposition with the chosen orthonormal basis \( |j\rangle^{A_i} \). The inverse is given as \( \mathcal{M}(\rho^{A_i}) = \text{Tr}[(\rho^{A_i} \otimes 1^{A_O})M^{A_iA_O}]^T \) where \( 1^{A_O} \) is the identity matrix on \( H^{A_O} \).

For simplicity, consider two parties Alice and Bob first. The probability to observe the outcomes \( a \) and \( b \) respectively under operations \( \mathcal{M}^A_a \) and \( \mathcal{M}^B_b \) is

\[
P(\mathcal{M}^A_a, \mathcal{M}^B_b) = \text{Tr}[(M^{A_iA_O}_a \otimes M^{B_iB_O}_b)W],
\]

for some Hermitian operator \( W \in A_I \otimes A_O \otimes B_I \otimes B_O \). The linearity of probabilities is preserved by the assumption that quantum mechanics is valid in local laboratories. Now we require the non-negativity and normalisation of probabilities.

We require that probabilities are non-negative for any pair of completely positive maps \( \mathcal{M}^A \) and \( \mathcal{M}^B \); that is transformed to their CJ operators as

\[
\text{Tr}[W^{A_iA_OB_iB_O}(M^{A_iA_O} \otimes M^{B_iB_O})] \geq 0 \quad \forall M^{A_iA_O} \geq 0, M^{B_iB_O} \geq 0.
\]

(7)

If we further assume that any ancillary states can be shared independent of processes in local operations, we may gain an extended process matrix \( W^{A_iA_OB_iB_O} = \rho^{A_i'B_i'} \otimes W^{A_iA_OB_iB_O} \).

We further require that the probabilities of this extended scheme are non-negative, then

\[
\text{Tr}[\rho^{A_i'B_i'} \otimes W^{A_iA_OB_iB_O}(M^{A_iA_iA_O} \otimes M^{B_i'B_i'B_O})] \geq 0
\]

\[
\forall M^{A_iA_iA_O}, M^{B_i'B_i'B_O}, \rho^{A_i'B_i'} \geq 0.
\]

(8)

This condition is equivalent to \( W^{A_iA_OB_iB_O} \) being positive semi-definite [41]:

\[
W^{A_iA_OB_iB_O} \geq 0.
\]

(9)

In addition, the probabilities should be normalised:

\[
1 = \sum_{ab} P(\mathcal{M}^A_a, \mathcal{M}^B_b) = P(\sum_a \mathcal{M}^A_a, \sum_b \mathcal{M}^B_b);
\]

(10)

that is,

\[
P(\mathcal{M}^A, \mathcal{M}^B) = 1, \quad \forall \text{ CPTP } \mathcal{M}^A, \mathcal{M}^B.
\]

A necessary and sufficient condition is that

\[
\text{Tr}[W^{A_iA_OB_iB_O}(M^{A_iA_O} \otimes M^{B_iB_O})] = 1
\]

\[
\forall M^{A_iA_O}, M^{B_iB_O} \geq 0 \quad \text{Tr}_{A_O} M^{A_iA_O} = 1^{A_i}, \text{Tr}_{B_O} M^{B_iB_O} = 1^{B_i}
\]

(12)
Extension to \( N \) parties is straightforward. We will refer to a matrix \( W \) that satisfies the linearity, non-negativity and normalisation of probabilities (that is, equivalent to the above conditions under assumptions) as the process matrix. Terms that can exist in a process matrix include states, channels, channels with memory; nevertheless, postselection, local loops, channels with local loops and global loops are not allowed [5].

A bipartite process matrix can be fully characterised in the Hilbert-Schmidt basis [5]. Define the signalling directions \( \preceq \) and \( \succeq \) as follows: \( A \preceq B \) means \( A \) is in the causal past of \( B \), \( A \succeq B \) means it is not; similar for \( \preceq \) and \( \succeq \). Any valid bipartite process matrix \( W^{AIAOBIBO} \) can be given in the Hilbert-Schmidt basis as

\[
W^{AIAOBIBO} = \frac{1}{d_A d_B} (1 + \sigma_{A \preceq B} + \sigma_{A \succeq B} + \sigma_{A \preceq \succeq B})
\]

where the matrices \( \sigma_{A \preceq B} \), \( \sigma_{A \succeq B} \), and \( \sigma_{A \preceq \succeq B} \) are defined by

\[
\sigma_{A \preceq B} \equiv \sum_{ij>0} c_{ij} \sigma_i^{AO} \sigma_j^{BI} + \sum_{ijk>0} d_{ijk} \sigma_i^{AI} \sigma_j^{AO} \sigma_k^{BI} \tag{14}
\]

\[
\sigma_{A \succeq B} \equiv \sum_{ij>0} e_{ij} \sigma_i^{AI} \sigma_j^{BO} + \sum_{ijk>0} f_{ijk} \sigma_i^{AI} \sigma_j^{BI} \sigma_k^{BO} \tag{15}
\]

\[
\sigma_{A \preceq \succeq B} \equiv \sum_{i>0} g_i \sigma_i^{AI} + \sum_{i>0} h_i \sigma_i^{BI} + \sum_{ij>0} l_{ij} \sigma_i^{AI} \sigma_j^{BI} \tag{16}
\]

Here \( c_{ij}, d_{ijk}, e_{ij}, f_{ijk}, g_i, h_i, l_{ij} \in \mathbb{R} \). That is, a bipartite process matrix of the system \( AB \) is a combination of an identity matrix, the matrices where \( A \) signals to \( B \), where \( B \) signals to \( A \), and where \( A \) and \( B \) are causally separated. It is thus a linear combination of three possible causal structures.

2. Process matrix to pseudo-density matrix: one lab to one event direct mapping

Now we analyse the relation between a process matrix and a pseudo-density matrix in finite dimensions.

A process matrix with a single-qubit Pauli measurement taken at each laboratory is mapped to a finite-dimensional pseudo-density matrix. Compare them in the bipartite case as an illustration. In the simplest temporal case, a maximally mixed qubit evolves under
the identity evolution between two times. The process matrix for this scenario is given as

\[ W = \frac{1}{2}^{A_I} \otimes [[1]]^{A_O B_I}, \]  

(18)

where 

\[ [[1]]^{XY} = \sum_{ij} |i\rangle \langle j|^{X} \otimes |i\rangle \langle j|^Y = \frac{1}{2}(I \otimes I + X \otimes X - Y \otimes Y + Z \otimes Z). \]  

At the same time, the corresponding pseudo-density matrix is

\[ R = \frac{1}{4}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z) = \frac{1}{2} = [[1]]^{PT}, \]  

(19)

where the swap operator 

\[ S = \frac{1}{2}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z) = [[1]]^{PT}, \]  

here PT is the partial transpose. For an arbitrary state \( \rho \) evolving under the unitary evolution \( U \), the process matrix is given as

\[ W = \rho^{A_I} \otimes [[U]]^{A_O B_I}, \]  

(20)

where \( [[U]] = (1 \otimes U)[[1]](1 \otimes U^\dagger) \) and the pseudo-density matrix as

\[ R = \frac{1}{4}(1 \otimes U)(\rho^A \otimes 1^B S + S \rho^A \otimes 1^B)(1 \otimes U^\dagger) = \rho^A \otimes 1^B [[U]]^{PT} + [[U]]^{PT} \rho^A \otimes 1^B, \]  

(21)

where the partial transpose is taken on the subsystem \( A \). Now we compare the correlations in the two formalisms and check whether they hold the same information.

The single-qubit Pauli measurement \( \sigma_i \) for each event in the pseudo-density matrix has the Choi-Jamiołkowski representation as

\[ \Sigma_i^{A_I A_O} = P_i^{A_I} \otimes P_i^{A_O} - P_i^{-A_I} \otimes P_i^{-A_O} \]  

(22)

where \( P_i^\pm = \frac{1}{2}(1 \pm \sigma_i) \); that is, to make a measurement \( P_i^\alpha (\alpha = \pm 1) \) to the input state and project the corresponding eigenstate to the output system. It is equivalent to

\[ \Sigma_i^{A_I A_O} = \frac{1}{2}(1^{A_I} \otimes \sigma_i^{A_O} + \sigma_i^{A_I} \otimes 1^{A_O}). \]  

(23)

In the example of a single qubit \( \rho \) evolving under \( U \), the correlations from the process matrix are given by

\[ p(\Sigma_i^{A_I A_O}, \Sigma_j^{B_I B_O}) = \text{Tr}[(\Sigma_i^{A_I A_O} \otimes \Sigma_j^{B_I B_O})W] = \frac{1}{2} \text{Tr}[\sigma_j U \sigma_i U^\dagger]; \]  

(24)

while the correlations from the pseudo-density matrix are given as

\[ \langle \{ \sigma_i, \sigma_j \} \rangle = \frac{1}{2} \left( \text{Tr}[\sigma_j U \sigma_i \rho U^\dagger] + \text{Tr}[\sigma_j U \rho \sigma_i U^\dagger] \right) = \frac{1}{2} \text{Tr}[\sigma_j U \sigma_i U^\dagger]. \]  

(25)
The last equality holds as a single-qubit $\rho$ is decomposed into $\rho = \frac{1}{2} + \sum_{k=1,2,3} c_k \sigma_k$. The allowed spatio-temporal correlations given by the two formalisms are the same; thus, pseudo-density matrices and process matrices are equivalent in terms of encoded correlations. In a general case of bipartite systems on $AB$, this equivalence holds for $A \preceq B$, $A \succeq B$, $A \preceq B$ and thus their superpositions for arbitrary process matrices. The only condition is that $A$ and $B$ make Pauli measurements in their local laboratories. Therefore, a process matrix where a single-qubit Pauli measurement is made at each laboratory corresponds to a finite-dimensional pseudo-density matrix since the correlations are equal.

For generalised measurements, for example, arbitrary POVMs, a process matrix is fully mapped to the corresponding generalised pseudo-density matrix; thus, a process matrix can be always mapped to a generalised pseudo-density matrix in principle. The process matrix and the corresponding generalised pseudo-density matrix just take the same measurement process in each laboratory or at each event. The analysis for correlations is similar.

For a given set of measurements, a process matrix where the measurement is made in each laboratory hold the same correlations as a generalised pseudo-density matrix with the measurement made at each event. Thus, a universal mapping from a process matrix to a pseudo-density matrix for general measurements is established.

However, a pseudo-density matrix in finite dimensions is not necessarily mapped back to a valid process matrix. As mentioned before, a valid process matrix excludes the possibilities for post-selection, local loops, channels with local loops and global loops. Pseudo-density matrices are defined operationally in terms of measurement correlations and may allow these possibilities. We will come back to this point in the discussion for post-selection and out-of-time-order correlation functions.

B. Causality violation

In this subsection, we introduce a causal inequality where the usual causality can be violated in the process matrix formalism as well as in the pseudo-density matrix formalism, and use an example of quantum switch to illustrate the existence of indefinite causal order.
1. Causal inequality: one lab to one event with double spaces mapping

Bell non-locality is defined by the violation of Bell inequalities where quantum correlations do not satisfy classical constraints anymore. Here we consider possible causal inequalities under the framework with indefinite causal order. First we introduce a causal game for this causal inequality based on Ref. [5]. Then we use process matrices and pseudo-density matrices, separately, to argue that causal inequalities may be violated without definite causal structure. Then we gain the mapping from a process for one laboratory to a pseudo-density matrix for one event with ancillary spaces.

Consider a communication task between Alice and Bob. When the task starts, Alice receives a random bit \( a \in \{0, 1\} \) and Bob receives two random bits, \( b, b' \in \{0, 1\} \). If \( b' = 0 \), Bob will have to communicate the bit \( b \) to Alice; if \( b' = 1 \), Bob will try his best to guess \( a \). We denote Alice’s guess for \( b \) by \( x \) and Bob’s guess for \( a \) by \( y \). The goal is to maximise the probability

\[
p_s = \frac{1}{2} [p(x = b|b' = 0) + p(y = a|b' = 1)]
\]

Consider that all the events happen in causal orders, i.e., at least one of the following holds: Alice cannot signal to Bob, or Bob cannot signal to Alice. Then we have

\[
p_{s}\text{causal} \leq \frac{3}{4}.
\]

Eqn. (27) is referred to as a causal inequality.

Now we consider a strategy to violate the causal inequality. Let us first formulate Alice’s and Bob’s labs in the process matrix formalism. Alice has no information about \( b' \). She gains a bit \( a \) from the input system \( A_I \) and sends a bit \( x \) to the output system \( A_O \). For a simple implementation, Alice makes a \( \sigma_z \) measurement on the incoming qubit with the result \( x = 0 \) for \( |z_+\rangle \) and \( x = 1 \) for \( |z_-\rangle \); then she prepares the qubit in the \( z \)-basis for \( a = 0 \) to \( |z_+\rangle \) and \( a = 1 \) to \( |z_-\rangle \) and sends the qubit away. This operation in the Choi-Jamiołkowski representation is given by \( M^{A_IA_O}(x,a) = \frac{1}{2}[\mathbb{1} + (-1)^x \sigma_z]^{A_I} \otimes [1 + (-1)^a \sigma_z]^{A_O} \). For Bob, consider \( b' = 0 \) and \( b' = 1 \) separately. If \( b' = 0 \), he measures the incoming qubit in the \( x \)-basis and assigns \( y = 0 \) for \( |x_+\rangle \) and \( y = 1 \) for \( |x_-\rangle \). For the result \( |x_+\rangle \) he prepares the qubit in the \( z \)-basis for \( b = 0 \) to \( |z_+\rangle \) and \( b = 1 \) to \( |z_-\rangle \); for the result \( |x_-\rangle \) he prepares the qubit in the \( z \)-basis with the output \( b = 0 \) to \( |z_-\rangle \) and \( b = 1 \) to \( |z_+\rangle \). Then he sends the qubit away. If \( b' = 1 \), he measures the incoming qubit in the \( z \)-basis and assigns \( y = 0, 1 \).
to $|z_+\rangle$, $|z_1\rangle$, respectively. Note that the reparation is not important here as the task is for Bob to read Alice’s bit. The operation in the Choi-Jamiołkowski representation is given by $M_{B_1B_0}(y,b) = (b' \oplus 1)M_{B_1B_0}(y,b|b' = 0) + b'M_{B_1B_0}(y,b|b' = 1)$, where $M_{B_1B_0}(y,b|b' = 0) = \frac{1}{2} [1 + (-1)^y \sigma_z^{B_1} \otimes [1 + (-1)^{b+y} \sigma_z^{B_0}], M_{B_1B_0}(y,b|b' = 1) = \frac{1}{2} [1 + (-1)^y \sigma_z^{B_1} \otimes \rho^{B_0}].$
For a process matrix in the form of

$$W^{A_1A_0B_1B_0} = \frac{1}{4} [1 + \frac{1}{\sqrt{2}} (\sigma_z^{A_0} \sigma_z^{B_1} + \sigma_z^{A_1} \sigma_z^{B_1} \sigma_z^{B_0})],$$

(28)

the task has the success probability as

$$p_s = \text{Tr} \left( [M^{A_1A_0}(x,a) \otimes M_{B_1B_0}(y,b)]_s W^{A_1A_0B_1B_0} \right) = \frac{2 + \sqrt{2}}{4} > \frac{3}{4}$$

(29)

where $s$ denotes the sum of half probability for $x = b$ when $b' = 0$ and half probability for $y = a$ when $b' = 1$. The causal inequality is violated.

For a pseudo-density matrix, we may adopt the same strategy. Alice has two systems $X$ and $A$, $A$ is prepared in the $z$-basis according to $a$. $X$ is measured in the $z$-basis with the result $x$; then $A$ is measured in the $z$-basis with the result $a$. The initial state of $A$ is $|a\rangle \langle a|^A$. The measurement for the systems $XA$ is given as $P^{XA}(x,a) = |x\rangle \langle x|^X \otimes |a\rangle \langle a|^A$. Bob has two systems $Y$ and $B$, $B$ is prepared as $|b\rangle \langle b|$. For $b' = 0$, $Y$ is measured in the $x$-basis given the result $y$; for $y = 0$ we apply the identity operator to $B$ and for $y = 1$ we flip $B$ to $|b \oplus 1\rangle \langle b \oplus 1|$. For $b' = 1$, $Y$ is measured in the $z$-basis and . Then the operation for $YB$ is given as $P^{YB}(y,b) = (b' \oplus 1)P^{YB}(y,b|b' = 0) + b'P^{YB}(y,b|b' = 1)$, where $P^{YB}(y,b|b' = 0) = \frac{1}{2} [1 + (-1)^y \sigma_z^Y \otimes |b \oplus y\rangle \langle b \oplus y|^B$ and for $y' = 1$ we flip $B$ to $y = b$ and $b' = 1$. Given a pseudo-density matrix

$$R^{XAYB} = \frac{1}{4} [(\frac{1}{\sqrt{2}} (\sigma_z^X \sigma_z^Y + \sigma_y^X \sigma_y^Y + \sigma_x^X \sigma_x^Y) \otimes |a\rangle \langle a|^A \otimes |b\rangle \langle b|^B$$

$$+ \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} \sigma_z^X \sigma_z^Y + \sigma_z^X \sigma_z^Y \sigma_z^B)],$$

(30)

the success probability is bounded similarly as Eqn. (29):

$$p_s = \text{Tr} \left\{ [P^{XA}(x,a) \otimes P^{YB}(y,b)] R^{XAYB} \right\}_s = \frac{2 + \sqrt{2}}{4} > \frac{3}{4}$$

(31)

Again the causal inequality is violated. This example also highlights another relationship for the mapping between a process matrix and a pseudo-density matrix. Instead of an input system and an output system in a process matrix, the corresponding pseudo-density matrix has an additional ancillary system for each event.
A process matrix which makes a measurement and reprepares the state in one laboratory
describes the same probabilities as a pseudo-density matrix with ancillary systems which
makes a measurement and reprepares the state at each event. Thus, another mapping from
a process matrix to a pseudo-density matrix is established by introducing ancillary systems.

2. Quantum switch: one lab to two events mapping

Quantum switch is originally proposed in the quantum comb formalism for quantum
networks [42]. It is a high-order transformation which takes two black boxes as the input
and the superposition of two orders to apply the two black boxes sequentially as the output
gate. A quantum switch of boxes takes quantum control over causal orders of operations.
For example, with unitary channels $U_f$ and $U_g$, the quantum switch of them is given as\[
\mathcal{W}_{f,g}(\rho) = W_{f,g} \rho W_{f,g}^\dagger \]
where\[
W_{f,g} = |0\rangle \langle 0| \otimes U_f U_g + |1\rangle \langle 1| \otimes U_g U_f. \tag{32}\]

If the channels take Kraus forms as $f(\rho) = \sum_i f_i \rho f_i^\dagger$ and $g(\rho) = \sum_j g_j \rho g_j^\dagger$, then the channel under quantum control is represented by $\mathcal{W}_{f,g}(\sigma) = \sum_{i,j} W_{f_i,g_j} \sigma W_{f_i,g_j}^\dagger$, where $W_{f_i,g_j} = |0\rangle \langle 0| \otimes f_i g_j + |1\rangle \langle 1| \otimes g_j f_i$.

It is argued that the quantum switch, or more general superpositions of orders for gates,
leads to a decrease in computational complexity; in a particular example in Ref. [43] the best-
known complexity of $O(n^2)$ queries is reduced to $O(n)$. Different experimental verifications
for quantum switch or indefinite causal order are given in Ref. [44–50].

The quantum switch with a fixed input state is represented by a tripartite process matrix.
The two parties $A$ and $B$ perform an arbitrary completely positive map each on the target
state as the black boxes. The ancilla $C$ has a trivial output but the input space is divided
into a target space and a control space. That is $\mathcal{H} = \mathcal{H}^{A_i} \otimes \mathcal{H}^{A_0} \otimes \mathcal{H}^{B_i} \otimes \mathcal{H}^{B_0} \otimes \mathcal{H}^{C_i}$, where $\mathcal{H}^{C_i} = \mathcal{H}^{C_T} \otimes \mathcal{H}^{C_C}$.

Now we formulate the quantum switch in terms of process matrix. Recall that an identity
channel from $A$’s output space to $B$’s input space is described by $|1\rangle\langle 1|^{A_0 B_i}$ where $|1\rangle^{A_0 B_i} = \sum_j |j\rangle^{A_0} |j\rangle^{B_i}$. Consider that $A$ receives a state $|\psi\rangle$, performs an arbitrary
operation, sends the output to $B$ under an identity channel; then $B$ performs an arbitrary
operation, sends the output to $C_T$. This is represented by the process matrix $|v\rangle \langle v|$ where
\[ |v\rangle = |\psi\rangle^{A_1} |1\rangle^{A_0B_1} |1\rangle^{B_0C_1}. \] Then the quantum switch with the control qubit in the state \( |\frac{0}{\sqrt{2}} + |\frac{1}{\sqrt{2}} \rangle \) is represented by the process matrix \( |w\rangle \langle w| \) where
\[
|w\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle^{A_1} |1\rangle^{A_0B_1} |1\rangle^{B_0C_1} |0\rangle^C + |\psi\rangle^{B_1} |1\rangle^{B_0A_1} |1\rangle^{A_0C_1} |1\rangle^C). \tag{33}
\]

Note that
\[
\langle U_A^* |A_1A_0 \langle U_A^* |B_1B_0 \cdot |w\rangle = \frac{1}{\sqrt{2}}(|0\rangle^C \otimes (U_B U_A \langle \psi\rangle)^C_T + |1\rangle^C \otimes (U_A U_B \langle \psi\rangle)^C_T],
\tag{34}
\]

where \( |U_A^* \rangle^{A_1A_0} = (1 \otimes U_A^*) |1\rangle^{A_1A_0}. \) Specifically,
\[
\text{Tr}_{C_1} |w\rangle \langle w| = \text{Tr}_{C_1} \langle \psi|^{A_1} \otimes |1\rangle \langle 1|^{A_0B_1} \otimes |1\rangle^{B_0} \text{ and } W^{B \leq A} = |\psi\rangle \langle \psi|^{B_1} \otimes |1\rangle \langle 1|^{B_0A_1} \otimes |1\rangle^{A_0}. \tag{35}
\]

For a pseudo-density matrix, we consider the same scenario of quantum switch. We may not be able to gain a quantum switch operator as clear as the process matrix formalism in which the evolution is made at each laboratory. For a process matrix, the channel evolution can be encoded into one laboratory; however, at each event of a pseudo-density matrix we can only make a measurement and evolution is for two events at different times. Thus it is not possible to use a single operator to represent the quantum switch in pseudo-density matrices. This shows the fundamental differences between process matrices and pseudo-density matrices. However, we can still characterise temporal correlations of quantum switch in the pseudo-density matrix formalism. In general, a pseudo-density matrix version of quantum switch is presented by a tripartite pseudo-density matrix \( R_{123}^{qw} \) with half probability of each causal order:
\[
R_{123}^{qw} = \frac{1}{2} R_{123}(f \rightarrow g) + \frac{1}{2} R_{123}(g \rightarrow f) \tag{36}
\]

where \( R_{123}(f \rightarrow g) \) is the pseudo-density matrix which describes an initial state \( \rho \) evolves under unitary channels \( U_f \) and \( U_g \) and \( R_{123}(g \rightarrow f) \) through \( U_g \) and \( U_f \). In particular, the correlation \( \langle \sigma_i, \sigma_j, \sigma_k \rangle_{f \rightarrow g} \) in \( R_{123}(f \rightarrow g) \) is given as
\[
\langle \sigma_i, \sigma_j, \sigma_k \rangle_{f \rightarrow g} = \frac{1}{4} \{ \text{Tr}[\sigma_k(f \rightarrow g)\sigma_j(f)\sigma_i\rho] + \text{Tr}[\sigma_i\sigma_k(f \rightarrow g)\sigma_j(f)\rho]
\]
\[
+ \text{Tr}[\sigma_j(f)\sigma_k(f \rightarrow g)\sigma_i\rho] + \text{Tr}[\sigma_i\sigma_j(f)\sigma_k(f \rightarrow g)\rho]\} \tag{37}
\]

where \( \sigma_k(f \rightarrow g) = U_f U_g \sigma_k U_g^\dagger U_f^\dagger \) and \( \sigma_j(f) = U_f \sigma_j U_f^\dagger. \) Then the correlation \( \langle \sigma_i, \sigma_j, \sigma_k \rangle_{qw} \) is given as
\[
\langle \sigma_i, \sigma_j, \sigma_k \rangle_{qw} = \frac{1}{2} \langle \sigma_i, \sigma_j, \sigma_k \rangle_{f \rightarrow g} + \frac{1}{2} \langle \sigma_i, \sigma_j, \sigma_k \rangle_{g \rightarrow f}. \tag{38}
\]
Thus, we find that for each laboratory with a unitary evolution, the process matrix corresponds to a pseudo-density matrix with two events. A process with each laboratory undergoing a quantum channel evolution is operationally equivalent to a pseudo-density matrix with two events.

Quantum switch is used as an example to illustrate the validity of indefinite causal order. However, it remains doubt whether it is truly physical. The existence of indefinite causal order remains to be tested or better justified. However, pseudo-density matrices have clear operational meanings and are built from measurement correlations. It is easier to justify the existence of pseudo-density matrices as they are generalisation of density matrices in time built upon spatial-temporal correlations. Thus, it is always testable from the operational and experimental perspectives. That is one advantage of pseudo-density matrices over other indefinite causal structures.

C. View from post-selection

Post-selection is conditioning on the occurrence of certain event in probability theory, or conditioning upon certain measurement outcome in quantum mechanics. It allows a quantum computer to choose the outcomes of certain measurements and increases its computational power significantly. In this subsection, we take the view from post-selection and show that a particular subset of post-selected two-time states correspond to process matrices in indefinite causal order. Post-selected closed timelike curves are presented as a special case.

1. Two-time quantum states

In this subsubsection, we review the two-time quantum states approach which fixes independent initial states and final states at two times. The two-time quantum state takes its operational meaning from post-selection. Consider that Alice prepares a state $|\psi\rangle$ at the initial time $t_1$. Between the initial time $t_1$ and the final time $t_2$, she performs arbitrary operations in her lab. Then she measures an observable $O$ at the final time $t_2$. The observable $O$ has a non-degenerate eigenstate $|\phi\rangle$. Taking $|\phi\rangle$ as the final state, Alice discards the experiment if the measurement of $O$ does not give the eigenvalue corresponding to the
Consider that Alice makes a measurement by the set of Kraus operators \( \{ \hat{E}_a = \sum_{k,l} \beta_{a,kl} |k\rangle \langle l| \} \) between \( t_1 \) and \( t_2 \). Note that \( \{ \hat{E}_a \} \) are normalised as \( \sum_a \hat{E}_a^\dagger \hat{E}_a = 1 \). The probability for Alice to gain the outcome \( a \) under the pre- and post-selection is given as

\[
p(a) = \frac{| \langle \phi | \hat{E}_a | \psi \rangle |^2}{\sum_{a'} | \langle \phi | \hat{E}_{a'} | \psi \rangle |^2}.
\]  

(39)

Now define the two-time state and the two-time version of Kraus operator as

\[
\Phi = A_2 \langle \phi | \otimes | \psi \rangle^A_1 \in \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1},
\]

\[
E_a = \sum_{kl} \beta_{a,kl} |k\rangle^A_2 \otimes |l\rangle^A_1 \in \mathcal{H}^{A_2} \otimes \mathcal{H}_{A_1},
\]  

(40)

where the two-time version of Kraus operator is denoted by \( E_a \) without the hat. An arbitrary pure two-time state takes the form

\[
\Phi = \sum \alpha_{ij} A_2 \langle i | \otimes | j \rangle^A_1 \in \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}.
\]  

(41)

Then the probability to obtain \( a \) as the outcome is given as

\[
p(a) = \frac{| \Phi \cdot E_a |^2}{\sum_{a'} | \Phi \cdot E_{a'} |^2}.
\]  

(42)

A two-time density operator \( \eta \) is given as

\[
\eta = \sum_r p_r \Phi_r \otimes \Phi_r^\dagger \in \mathcal{H}_{A_2} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}_{A_1}^\dagger \otimes \mathcal{H}_{A_2}^\dagger.
\]  

(43)

Consider a coarse-grained measurement

\[
J_a = \sum_{\mu} E_a^\mu \otimes E_a^{\mu \dagger} \in \mathcal{H}^{A_2} \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}_{A_2}^\dagger
\]  

(44)

where the outcome \( a \) corresponds to a set of Kraus operators \( \{ \hat{E}_a^\mu \} \). Then the probability to obtain \( a \) as the outcome is given as

\[
p(a) = \frac{\eta \cdot J_a}{\sum_{a'} \eta \cdot J_{a'}}.
\]  

(45)

2. **Connection between process matrix and pseudo-density matrix under post-selection**

Now consider post-selection applied to ordinary quantum theory. It is known that a particular subset of post-selected two-time states in quantum mechanics give the form of process
matrices within indefinite causal structures [35]. Here we first give a simple explanation for this fact and further analyse the relation between a process matrix and a pseudo-density matrix from the view of post-selection.

For an arbitrary bipartite process matrix $W \in \mathcal{H}^{A_I} \otimes \mathcal{H}^{A_O} \otimes \mathcal{H}^{B_I} \otimes \mathcal{H}^{B_O}$, we can expand it in some basis:

$$W_{ijkl, pqrs} = \sum_{ijkl, pqrs} w_{ijkl, pqrs} |ijkl\rangle \langle pqrs|.$$  \hspace{1cm} (46)

For the elements in each Hilbert space, we map them to the corresponding parts in a bipartite two-time state. For example, we map the input Hilbert space of Alice to the bra and ket space of Alice at time $t_1$, and similarly for the output Hilbert space for $t_2$. That is,

$$|i\rangle \langle p| \in \mathcal{L}(\mathcal{H}^{A_I}) \rightarrow \langle p| \otimes |i\rangle \in \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_1}$$  \hspace{1cm} (47)

$$|j\rangle \langle q| \in \mathcal{L}(\mathcal{H}^{A_O}) \rightarrow \langle q| \otimes |j\rangle \in \mathcal{H}^{A_2} \otimes \mathcal{H}^{A_2}$$  \hspace{1cm} (48)

Thus, a two-time state $\eta_{W^{A_1 A_2}} \in \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_1}$ is equivalent to a process matrix for a single laboratory $W^{A_1 A_2}$.

The connection with pre- and post-selection suggests one more interesting relationship between a process matrix and a pseudo-density matrix. For a process matrix, if we consider the input and output Hilbert spaces at two times, we can map it to a two-time state. That is, we connect a process matrix with single laboratory to a two-time state. A pseudo-density matrix needs two Hilbert spaces to represent two times. For a two-time state $\eta_{12}$, the corresponding pseudo-density matrix $R_{12}$ has the same marginal single-time states, i.e., $\text{Tr}_1 \eta_{12} = \text{Tr}_1 R_{12}$ and $\text{Tr}_2 \eta_{12} = \text{Tr}_2 R_{12}$. Then we find a map between a process matrix for a single event and a pseudo-density matrix for two events. Note that in the previous subsections, we have mapped a process matrix for two events to one pseudo-density matrix with half Hilbert space for two events, and mapped a process matrix for two events to a pseudo-density matrix with two Hilbert spaces at each of two events. This suggests that the relationship between a process matrix and a pseudo-density matrix is non-trivial with a few possible mappings.

One question arising naturally here concerns the pseudo-density matrices with post-selection. The definitions for finite-dimensional and Gaussian pseudo-density matrices guarantee that under the partial trace, the marginal states at any single time will give the state at that time. In particular, tracing out all other times in a pseudo-density matrix, we get
the final state at the final time. On the one hand, we may think that pseudo-density matrix formulation is kind of time-symmetric. On the other hand, the final state is fixed by evolution; that implies that we cannot assign an arbitrary final state, making it difficult for the pseudo-density matrix to be fully time-symmetric. For other generalisation of pseudo-density matrices like position measurements and weak measurements, the property for fixed final states does not hold. Nevertheless, we may define a new type of pseudo-density matrices with post-selection. We assign the final measurement to be the projection to the final state and renormalise the probability. For example, a qubit in the initial state $\rho$ evolves under a CPTP map $\mathcal{E} : \rho \rightarrow \mathcal{E}(\rho)$ and then is projected on the state $\eta$. We may construct the correlations $\langle \{\sigma_i, \sigma_j, \eta\} \rangle$ as

$$\langle \{\sigma_i, \sigma_j, \eta\} \rangle = \sum_{\alpha,\beta = \pm 1} \alpha \beta \text{Tr}[\eta P_j^\beta \mathcal{E}(P_i^\alpha \rho P_i^\alpha) P_j^\beta]/p_{ij}(\eta),$$

where $P_i^\alpha = \frac{1}{2}(1 + \alpha \sigma_i)$ and $p_{ij}(\eta) = \sum_{\alpha,\beta = \pm 1} \text{Tr}[\eta P_j^\beta \mathcal{E}(P_i^\alpha \rho P_i^\alpha) P_j^\beta]$. Then the pseudo-density matrix with post-selection is given as

$$R = \frac{1}{4} \sum_{i,j=0}^3 \langle \{\sigma_i, \sigma_j, \eta\} \rangle \sigma_i \otimes \sigma_j \otimes \eta.$$  

We further conclude the relation between a process matrix and a pseudo-density matrices with post-selection. A process matrix with postselection in a laboratory is operationally equivalent to a tripartite postselected pseudo-density matrix.

We briefly discuss post-selected closed timelike curves before we move on to a summary. Closed timelike curves (CTCs), after being pointed out by Gödel to be allowed in general relativity [51], have always been arising great interests. Deutsch [52] proposed a circuit method to study them and started an information theoretic point of view. Deustch’s CTCs are shown to have many abnormal properties violated by ordinary quantum mechanics. For example, they are nonunitary, nonlinear, and allow quantum cloning [53, 54]. Several authors [55–58] later proposed a model for closed timelike curves based on post-selected teleportation. It is studied that process matrices correspond to a particular linear version of post-selected closed timelike curves [59]. In pseudo-density matrices we can consider a system evolves in time and back; that is the case for calculating out-of-time-order correlation functions we will introduce later. For post-selected closed timelike curves, it is better to be illustrated by the pseudo-density matrices with post-selection.
D. Summary of the relation between pseudo-density matrix and indefinite causal structures

In this section, we have introduced the relation between pseudo-density matrices and indefinite causal order. We argue that the pseudo-density matrix formalism belongs to indefinite causal structures. So far, all other indefinite causal structures to our knowledge use a tensor product of input and output Hilbert spaces, while a pseudo-density matrix only assumes a single Hilbert space. For a simple example of a qudit at two times, the dimension used in other indefinite causal structures is $d^4$ but for pseudo-density matrix is $2d^2$. Though other indefinite causal structures assume a much larger Hilbert space, pseudo-density matrix should not be taken as a subclass of any indefinite causal structures which already exist.

There are certain non-trivial relation between pseudo-density matrices and other indefinite causal structures. As we can see from the previous subsections, it is possible to map a process matrix to a corresponding pseudo-density matrix in four different ways: one-lab to one-event direct map, one-lab to one-event with double Hilbert spaces map, one-lab to two-event map, and a post-selected version.

**Claim 1.** A process matrix and the corresponding pseudo-density matrix allow the same correlations or probabilities in four different mappings.

One obvious difference between a process matrix and a pseudo-density matrix is that, for each laboratory, a process matrix measures and reprepares a state while a pseudo-density matrix usually only makes a measurement and the state evolves into its eigenstate for each eigenvalue with the corresponding probability. The correlations given by process matrices and pseudo-density matrices are also the same. Examples in post-selection and closed time curves suggest further similarities. In general, we can understand that the pseudo-density matrix is defined in an operational way which does not specify the causal order, thus belongs to indefinite causal structures. We borrow the lessons from process matrices here to investigate pseudo-density matrices further. Maybe it will be interesting to derive a unified indefinite causal structure which takes the advantage of all existing ones.

Nevertheless, the ultimate goal of indefinite causal order towards quantum gravity is still far reaching. So far, all indefinite causal structures are linear superpositions of causal structures; will that be enough for quantising gravity? It is generally believed among indefinite
causal structure community that what is lacking in quantum gravity is the quantum uncertainty for dynamical causal structures suggested by general relativity. The usual causal order may be changed under this quantum uncertainty and there is certain possibility for a superposition of causal orders. It is an attractive idea; however, being criticised due to lack of evidence to justify the existence. One may argue that process matrices and quantum switch can describe part of the universe; however, such an approach to quantum gravity remains doubt. For example, indefinite causal structures restrict to linear superpositions of causal orders and can only describe linear post-selected closed timelike curves. Why is the ultimate theory of nature necessary to be linear?

IV. CONSISTENT HISTORIES

In this section we review on consistent histories and explore the relation between pseudo-density matrices and consistent histories.

A. Preliminaries for consistent histories

Consistent histories, or decoherent histories, is an interpretation for quantum theory, proposed by Griffiths [8, 9], Gell-Mann and Hartle [10, 11], and Omnes [12]. The main idea is that a history, understood as a sequence of events at successive times, has a consistent probability with other histories in a closed system. The probabilities assigned to histories satisfy the consistency condition to avoid the interference between different histories and that set of histories are called consistent histories [60–66].

Consider a set of projection operators \( \{P_\alpha\} \) which are exhaustive and mutually exclusive:

\[
\sum_\alpha P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\beta, \quad (51)
\]

where the range of \( \alpha \) may be finite, infinite or even continuous. For each \( P_\alpha \) and a system in the state \( \rho \), the event \( \alpha \) is said to occur if \( P_\alpha \rho P_\alpha = \rho \) and not to occur if \( P_\alpha \rho P_\alpha = 0 \). The probability of the occurrence of the event \( \alpha \) is given by

\[
p(\alpha) = \text{Tr}[P_\alpha \rho P_\alpha]. \quad (52)
\]

A projection of the form \( P_\alpha = |\alpha\rangle \langle \alpha| \) (\( \{|\alpha\rangle\} \) is complete) is called completely fine-grained, which corresponds to the precise measurement of a complete set of commuting observables.
Otherwise, for imprecise measurements or incomplete sets, the projection operator is called coarse-grained. Generally it takes the form \( P_\alpha = \sum_{\alpha \in \tilde{\alpha}} P_\alpha \).

In the Heisenberg picture, the operators for the same observables \( P \) at different times are related by

\[
P(t) = \exp(iHt/\hbar)P(0)\exp(-iHt/\hbar),
\]

with \( H \) as the Hamiltonian of the system. Then the probability of the occurrence of the event \( \alpha \) at time \( t \) is

\[
p(\alpha) = \text{Tr}[P_\alpha(t)\rho P_\alpha(t)].
\]

Now we consider how to assign probabilities to histories, that is, to a sequence of events at successive times. Suppose that the system is in the state \( \rho \) at the initial time \( t_0 \). Consider a set of histories \([\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_n]\) consisting of \( n \) projections \( \{P^k_{\alpha_k}(t_k)\}_{k=1}^n \) at times \( t_1 < t_2 < \cdots < t_n \). Here the subscript \( \alpha_k \) allows for different types of projections, for example, a position projection at \( t_1 \) and a momentum projection at \( t_2 \). Then the decoherence functional is defined as

\[
D([\alpha], [\alpha']) = \text{Tr}[P^n_{\alpha_n}(t_n) \cdots P^1_{\alpha_1}(t_1) \rho P^1_{\alpha'_1}(t_1) \cdots P^n_{\alpha'_n}(t_n)],
\]

where

\[
P^k_{\alpha_k}(t_k) = e^{i(t_k - t_0)H^k} P^k_{\alpha_k} e^{-i(t_k - t_0)H}. \tag{56}
\]

It is important in consistent histories because probabilities can be assigned to histories when the decoherence functional is diagonal. It is easy to check that

\[
D([\alpha], [\alpha']) = D([\alpha'], [\alpha])^*, \tag{57}
\]

\[
\sum_{[\alpha]} \sum_{[\alpha']} D([\alpha], [\alpha']) = \text{Tr} \rho = 1. \tag{58}
\]

The diagonal elements are the probabilities for the histories \( (\rho, t_0) \to (\alpha_1, t_1) \to \cdots \to (\alpha_n, t_n) \):

\[
p(\alpha_1, \alpha_2, \ldots, \alpha_n) = D(\alpha_1, \alpha_2, \ldots, \alpha_n | \alpha_1, \alpha_2, \ldots, \alpha_n) = D([\alpha], [\alpha]) \tag{59}
\]

Until now, we considered fine-grained projections \( P^k_{\alpha_k} \) for fine-grained histories. The coarse-grained histories are characterised by the coarse-grained projections \( \bar{P}^k_{\bar{\alpha}_k} \). To satisfy the probability sum rules, the probability for a coarse-grained history is the sum of the probabilities for its fine-grained histories. That is,

\[
p(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) = \sum_{[\bar{\alpha}] \in [\bar{\alpha}]} p(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{60}
\]
where

\[ \sum_{[\alpha] \in \bar{\alpha}} \sum_{\alpha_1 \in \bar{\alpha}} \sum_{\alpha_2 \in \bar{\alpha}} \cdots \sum_{\alpha_n \in \bar{\alpha}}. \]  

(61)

On the other hand, we gain the decoherence functional for coarse-grained histories by directly summing over the fine-grained projections as

\[ D([\bar{\alpha}], [\bar{\alpha}']) = \sum_{[\alpha] \in [\bar{\alpha}]} \sum_{[\alpha'] \in [\bar{\alpha}']} D([\alpha], [\alpha']). \]  

(62)

For the diagonal terms,

\[ D([\bar{\alpha}], [\bar{\alpha}]) = \sum_{[\alpha] \in [\bar{\alpha}]} D([\alpha]) \sum_{[\alpha'] \in [\bar{\alpha}]} D([\alpha], [\alpha']) + \sum_{[\alpha] \neq [\alpha'] \in [\bar{\alpha}]} \sum_{[\alpha'] \in [\bar{\alpha}]} D([\alpha], [\alpha']). \]  

(63)

where \([\alpha] \neq [\alpha']\) means \(\alpha_k \neq \alpha_k'\) for at least one \(k\).

To obey the probability sum rules that all probabilities are non-negative and summed to 1, the sufficient and necessary condition is

\[ \Re[D(\alpha_1, \alpha_2, \ldots, \alpha_n | \alpha_1', \alpha_2', \ldots, \alpha_n')] = p(\alpha_1, \alpha_2, \ldots, \alpha_n) \delta_{\alpha_1 \alpha_1'} \cdots \delta_{\alpha_n \alpha_n'}. \]  

(64)

Eqn. (64) is called the consistency condition or decoherence condition. Sets of histories obeying the condition are referred to consistent histories or decoherent histories. A stronger version of consistency condition is

\[ D(\alpha_1, \alpha_2, \ldots, \alpha_n | \alpha_1', \alpha_2', \ldots, \alpha_n') = p(\alpha_1, \alpha_2, \ldots, \alpha_n) \delta_{\alpha_1 \alpha_1'} \cdots \delta_{\alpha_n \alpha_n'}. \]  

(65)

The decoherence functional has a path integral representation. With configuration space variables \(q^i(t)\) and the action \(S[q^i]\),

\[ D([\alpha], [\alpha']) = \int_{[\alpha]} \mathcal{D}q^i \int_{[\alpha']} \mathcal{D}q'^i \exp(iS[q^i] - iS[q'^i]) \delta(q'^i_f - q^i_f) \rho(q^i_0, q'^i_0), \]  

(66)

where the two paths \(q^i(t), q'^i(t)\) begin at \(q^i_0, q'^i_0\) respectively at \(t_0\) and end at \(q^i_f = q'^i_f\) at \(t_f\), and correspond to the projections \(P^{k}_{\alpha_k}, P^{k}_{\alpha_k'}\) made at time \(t_k\) \((k = 1, 2, \ldots n)\).

**B. Temporal correlations in terms of decoherence functional**

The relation with the \(n\)-qubit pseudo-density matrix is arguably obvious. For example, consider an \(n\)-qubit pseudo-density matrix as a single qubit evolving at \(n\) times. For each
event, we make a single-qubit Pauli measurement $\sigma_{i_k}$ at the time $t_k$. We can separate the measurement $\sigma_{i_k}$ into two projection operators $P_{i_k}^{-1} = \frac{1}{2}(I + \sigma_{i_k})$ and $P_{i_k}^+ = \frac{1}{2}(I - \sigma_{i_k})$ with its outcomes $\pm 1$. Corresponding to the history picture, each pseudo-density event with the measurement $\sigma_{i_k}$ corresponds to two history events with projections $P_{i_k}^{\alpha_k}(\alpha_k = \pm 1)$. A pseudo-density matrix is built upon measurement correlations $\langle \{\sigma_{i_k}\}_{k=1}^n \rangle$. These correlations can be given in terms of decoherence functionals as

$$\langle \{\sigma_{i_k}\}_{k=1}^n \rangle = \sum_{\alpha_1, \ldots, \alpha_n} \alpha_1 \cdots \alpha_n \text{Tr}[P_{i_n}^{\alpha_n} U_{n-1} \cdots U_1 P_{i_1}^{\alpha_1} \rho P_{i_1}^{\alpha_1} U_1^\dagger \cdots U_{n-1}^\dagger P_{i_n}^{\alpha_n}]$$

$$= \sum_{\alpha_1, \ldots, \alpha_n} \alpha_1 \cdots \alpha_n p(\alpha_1, \ldots, \alpha_n)$$

$$= \sum_{\alpha_1, \ldots, \alpha_n} \alpha_1 \cdots \alpha_n D([\alpha], [\alpha]), \quad (67)$$

where $D([\alpha], [\alpha])$ is the diagonal terms of decoherence functional with $[\alpha] = [\alpha_1, \ldots, \alpha_n]$. Note that here only diagonal decoherence functionals are taken into account, which coincides with the consistency condition.

Similar relations hold for the Gaussian spacetime states. For each event, we make a single-mode quadrature measurement $\hat{q}_k$ or $\hat{p}_k$ at time $t_k$. We can separate the measurement $\hat{x}_k = \int x_k |x_k\rangle \langle x_k| dx_k$ into projection operators $|x_k\rangle \langle x_k|$ with outcomes $x_k$. Then each Gaussian event with the measurement $\hat{x}_k$ corresponds to infinite and continuous history events with projections $|x_k\rangle \langle x_k|$.

$$\langle \{x_k\}_{k=1}^n \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n x_1 \cdots x_n$$

$$\text{Tr}[|x_n\rangle \langle x_n| U_{n-1} \cdots U_1 |x_1\rangle \langle x_1| \rho |x_1\rangle \langle x_1| U_1^\dagger \cdots U_{n-1}^\dagger |x_n\rangle \langle x_n|]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n x_1 \cdots x_n p(x_1, \ldots, x_n)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n x_1 \cdots x_n D([x], [x]), \quad (68)$$

where $D([x], [x])$ is the diagonal terms of decoherence functional with $[x] = [x_1, \ldots, x_n]$.

For general spacetime states for continuous variables, we make a single-mode measurement $T(\alpha_k)$ at time $t_k$ for each event. It separates into two projection operators $P_{i_k}^{\alpha_k}(\alpha_k)$ and $P_{i_k}^{-\alpha_k}(\alpha_k)$, then it follows as the $n$-qubit case.

The interesting part is to apply the lessons from consistent histories to the generalised pseudo-density matrix formulation with general measurements. We have argued that the
spacetime density matrix can be expanded diagonally in terms of position measurements as
\[
\rho = \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} dx_1 \cdots dx_n p(x_1, \ldots, x_n) |x_1\rangle \langle x_1| \otimes \cdots \otimes |x_n\rangle \langle x_n|.
\] (69)

It reminds us of the diagonal terms of the decoherence functional. It is possible to build a spacetime density matrix from all possible decoherence functionals as
\[
\rho = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 dx'_1 \cdots dx_n dx'_n D(x_1, \ldots, x_n | x'_1, \ldots, x'_n) |x_1\rangle \langle x'_1| \otimes \cdots \otimes |x_n\rangle \langle x'_n|.
\] (70)

Applying the strong consistency condition to the above equation, we gain Eqn. (69) again. This argues why it is effective to only consider diagonal terms in position measurements, which is originally taken for convenience.

Similarly, the spacetime Wigner function from weak measurements is easily taken as a generalisation for the diagonal terms of the decoherence functional allowing for general measurements. Recall that a generalised effect-valued measure is represented by
\[
\hat{f}(q, p) = C \exp \left[ -\alpha [(\hat{q} - q)^2 + \lambda(\hat{p} - p)^2] \right].
\] (71)

The generalised decoherence functional for weak measurements is then given by
\[
D(q, p, q', p'; \tau | \hat{\rho}) = \text{Tr} \left[ F(q, p, q', p'; \tau) \hat{\rho} \right],
\] (72)

where
\[
F(q, p, q', p'; \tau) \hat{\rho} = \int \text{d}\mu_G[q(t), p(t)] \int \text{d}\mu_G[q'(t), p'(t)] \delta \left( q - \frac{1}{\tau} \int_0^\tau dt q(t) \right) \delta \left( p - \frac{1}{\tau} \int_0^\tau dt p(t) \right) \delta \left( q' - \frac{1}{\tau} \int_0^\tau dt q'(t) \right) \delta \left( p' - \frac{1}{\tau} \int_0^\tau dt p'(t) \right) \exp \left[ -\frac{i}{\hbar} \hat{H} \tau \right] \mathcal{T} \exp \left[ -\frac{\gamma}{2} \int_0^\tau dt [(\hat{q}_H(t) - q(t))^2 + \lambda(\hat{p}_H(t) - p(t))^2] \right] \hat{\rho}
\]
\[
\mathcal{T}^* \exp \left[ -\frac{\gamma}{2} \int_0^\tau dt [(\hat{q}_H(t) - q'(t))^2 + \lambda(\hat{p}_H(t) - p'(t))^2] \right] \exp \left[ \frac{i}{\hbar} \hat{H} \tau \right],
\] (73)

here
\[
d\mu_G[q(t), p(t)] = \lim_{N \to \infty} \left( \frac{\gamma \sqrt{\lambda}}{\pi N} \prod_{s=1}^{N} dq(t_s) dp(t_s) \right),
\] (74)
\[
d\mu_G[q'(t), p'(t)] = \lim_{N \to \infty} \left( \frac{\gamma \sqrt{\lambda}}{\pi N} \prod_{s=1}^{N} dq'(t_s) dp'(t_s) \right),
\] (75)
and
\[ \hat{q}_H(t) = \exp \left[ \frac{i}{\hbar} \hat{H} t \right] \hat{q} \exp \left[ -\frac{i}{\hbar} \hat{H} t \right], \quad \hat{q}'_H(t) = \exp \left[ \frac{i}{\hbar} \hat{H} t \right] \hat{q}' \exp \left[ -\frac{i}{\hbar} \hat{H} t \right], \]
\[ \hat{p}_H(t) = \exp \left[ \frac{i}{\hbar} \hat{H} t \right] \hat{p} \exp \left[ -\frac{i}{\hbar} \hat{H} t \right], \quad \hat{p}'_H(t) = \exp \left[ \frac{i}{\hbar} \hat{H} t \right] \hat{p}' \exp \left[ -\frac{i}{\hbar} \hat{H} t \right]. \quad (76) \]

The diagonal terms under the strong consistency condition reduce to the form in Ref \[22\]:
\[ p(q, p, \tau | \hat{\rho}) = \text{Tr} \mathcal{F}(q, p; \tau) \hat{\rho}, \quad (77) \]
where
\[ \mathcal{F}(q, p; \tau) \hat{\rho} = \int dq(t) dp(t) \delta \left( q - \frac{1}{\tau} \int_0^\tau dtq(t) \right) \delta \left( p - \frac{1}{\tau} \int_0^\tau dtp(t) \right) \exp \left[ -\frac{i}{\hbar} \hat{H} \tau \right] \]
\[ \mathcal{T} \exp \left[ -\gamma \int_0^\tau dt \left[ (\hat{q}_H(t) - q(t))^2 + \lambda (\hat{p}_H(t) - p(t))^2 \right] \right] \hat{\rho} \]
\[ \mathcal{T}^* \exp \left[ -\gamma \int_0^\tau dt \left[ (\hat{q}'_H(t) - q(t))^2 + \lambda (\hat{p}'_H(t) - p(t))^2 \right] \right] \exp \left[ -i \hbar \hat{H} \tau \right]. \quad (78) \]

Now we conclude the relation between decoherence functionals in consistent histories and temporal correlations in pseudo-density matrices.

**Claim 2.** The decoherence functional in consistent histories is the probabilities in temporal correlations of pseudo-density matrices.

Thus, we establish the relationship between consistent histories and all possible forms of pseudo-density matrix. From the consistency condition, we also have a better argument for why spacetime states for general measurements are defined in the diagonal form. It is not a coincide.

V. GENERALISED NON-LOCAL GAMES

Game theory studies mathematical models of competition and cooperation under strategies among rational decision-makers \[67\]. Here we give an introduction to nonlocal games, quantum-classical nonlocal games, and quantum-classical signalling games. Then we show the relation between quantum-classical signalling games and pseudo-density matrices, and comment on the relation between general quantum games and indefinite causal order.
A. Introduction to non-local games

The interests for investigating non-local games start from interactive proof systems with two parties, the provers and the verifiers. They exchange information to verify a mathematical statement. A nonlocal game is a special kind of interactive proof system with only one round and at least two provers who play in cooperation against the verifier. In nonlocal games, we refer to the provers as Alice, Bob, \ldots, and the verifier as the referee. In Ref. [68], nonlocal games were formally introduced with shared entanglement and used to formulate the CHSH inequality [69]. Here we introduce the CHSH game as an example and then give the general form of a non-local game.

The CHSH game has two cooperating players, Alice and Bob, and a referee who asks questions and collects answers from the players. The basic rules of the CHSH game are as the following:

1) There are two possible questions \( x \in \{0, 1\} \) for Alice and two possible questions \( y \in \{0, 1\} \) for Bob. Each question has an equal probability as \( p(x, y) = \frac{1}{4}, \forall x, \forall y \).

2) Alice answers \( a \in \{0, 1\} \) and Bob \( b \in \{0, 1\} \).

3) Alice and Bob cannot communicate with each other after the game begins.

4) If \( a \oplus b = x \cdot y \), then they win the game, otherwise they lose.

For a classical strategy, that is, Alice and Bob use classical resources, they win with the probability at most \( \frac{3}{4} \). Alice and Bob can also adopt a quantum strategy. If they prepare and share a joint quantum state \( |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) and make local measurements based on the questions they receive separately, then they can achieve a higher winning probability \( \cos^2(\pi/8) \approx 0.854 \).

In general, a non-local game \( G \) is formulated by \((\pi, l)\) on

\[
\overrightarrow{nl} = \langle \mathcal{X}, \mathcal{Y}; \mathcal{A}, \mathcal{B}; l \rangle,
\]

where \( \mathcal{X}, \mathcal{Y} \) are question spaces of Alice and Bob and \( \mathcal{A}, \mathcal{B} \) are answer spaces of Alice and Bob. Here \( \pi(x, y) \) is a probability distribution of the question spaces for Alice and Bob in the form \( \pi : \mathcal{X} \times \mathcal{Y} \to [0, 1] \), and \( l(a, b|x, y) \) is a function of question and answer spaces for Alice and Bob to decide whether they win or lose in the form \( l : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to [0, 1] \); for example, if they win, \( l = 1 \); otherwise lose with \( l = 0 \). For any strategy, the probability distribution for answers \( a, b \) of Alice and Bob given questions \( x, y \), respectively, is referred
to as the correlation function $p(a, b|x, y)$ of the form

$$p : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to [0, 1].$$  \hfill (80)

with the condition $\sum_{a,b} p(a, b|x, y) = 1$. With a classical source,

$$p_c(a, b|x, y) = \sum_{\lambda} \pi(\lambda) d_A(a|x, \lambda) d_B(b|y, \lambda),$$  \hfill (81)

where $d_A(a|x, \lambda)$ is the probability of answering $a$ given the parameter $\lambda$ and the question $b$ and similar for $d_B(b|y, \lambda)$; with a quantum source,

$$p_q(a, b|x, y) = \text{Tr}[\rho_{AB}(P_{a|x}^A \otimes Q_{b|y}^B)].$$  \hfill (82)

where $\rho_{AB}$ is the quantum state shared by Alice and Bob, $P_{a|x}^A$ is the measurement made by Alice with the outcome $a$ given $x$, $Q_{b|y}^B$ is the measurement made by Bob with the outcome $b$ given $y$. Then the optimal winning probability is given by

$$E_{\mathcal{M}}[s] \equiv \max \sum_{x,y} \pi(x, y) \sum_{a,b} l(a, b|x, y) p_{c/q}(a, b|x, y).$$  \hfill (83)

B. Quantum-classical non-local & signalling games

First we introduce a generalised version of non-local games where the referee asks quantum questions instead of classical questions (therefore this type of non-local games are refereed to quantum-classical) \cite{13}. Then we give the temporal version of these quantum-classical non-local games as quantum-classical signalling games \cite{14}.

1. Quantum-classical non-local games

We now recap the model of quantum-classical non-local games \cite{13}, in which the questions are quantum rather than classical. More specifically, the referee sends quantum registers to Alice and Bob instead of classical information.

For a non-local game, with the question spaces $\mathcal{X} = \{x\}$ and $\mathcal{Y} = \{y\}$, the referee associates two quantum ancillary systems $X$ and $Y$ such that $\dim \mathcal{H}_X \geq |\mathcal{X}|$, $\dim \mathcal{H}_Y \geq |\mathcal{Y}|$, the systems are in the states $\tau_X^x = |x\rangle \langle x|$ and $\tau_Y^y = |y\rangle \langle y|$ with the questions $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Assume that Alice and Bob share a quantum state $\rho_{AB}$. Given the answer sets
\[ A = \{a\} \text{ and } B = \{b\} \text{ and quantum systems } XA \text{ and } YB, \text{ Alice and Bob can make the corresponding POVMs } P_{XA}^a \text{ and } Q_{YB}^b \text{ in the linear operators on the Hilbert space } \mathcal{H}_{XA} \text{ and } \mathcal{H}_{YB}, \text{ such that } \sum_a P_{XA}^a = 1_{XA} \text{ and } \sum_b Q_{YB}^b = 1_{YB}. \text{ Then the probability distribution for the questions and answers of Alice and Bob, that is, the correlation function } P(a, b|x, y), \text{ is given by}

\[ P(a, b|x, y) = \text{Tr}[(P_{XA}^a \otimes Q_{YB}^b)(\tau_X^x \otimes \rho_{AB} \otimes \tau_Y^y)]. \tag{84} \]

Quantum-classical non-local games replace classical inputs with quantum ones, formulated by \((\pi(x, y), l(a, b|x, y))\) on

\[ \overrightarrow{qcnl} = \langle \{\tau_x^x\}, \{\omega_y^y\}; A, B; l \rangle. \tag{85} \]

The referee picks \(x \in X\) and \(y \in Y\) with the probability distribution \(\pi(x, y)\) as the classical-classical non-local game. With a classical source,

\[ p_c(a, b|x, y) = \sum_{\lambda} \pi(\lambda) \text{Tr}[\tau_X^x \otimes \omega_Y^y (P_{XA}^{a|\lambda} \otimes Q_{YB}^{b|\lambda})]; \tag{86} \]

with a quantum source,

\[ p_q(a, b|x, y) = \text{Tr}[\tau_X^x \otimes \rho_{AB} \otimes \omega_Y^y (P_{XA}^{a} \otimes Q_{BY}^{b})]. \tag{87} \]

The optimal winning probability is, again, given by

\[ \mathbb{E}_{\overrightarrow{qcnl}[\ast]} \equiv \max \sum_{x,y} \pi(x, y) \sum_{a,b} l(a, b|x, y) p_{c/q}(a, b|x, y). \tag{88} \]

### 2. Quantum-classical signalling games

In quantum-classical signalling games \([14]\), instead of two players Alice and Bob, we consider only one player Abby at two successive instants in time. Then quantum-classical signalling games change the Alice-Bob duo to a timelike structures of single player Abby with

\[ \overrightarrow{qcs\hat{g}} = \langle \{\tau_x^x\}, \{\omega_y^y\}; A, B; l \rangle. \tag{89} \]

With unlimited classical memory,

\[ p_c(a, b|x, y) = \sum_{\lambda} \pi(\lambda) \text{Tr}[\tau_X^x P_{XA}^{a|\lambda}] \text{Tr}[\omega_Y^y Q_{YB}^{b|\lambda}]. \tag{90} \]
For admissible quantum strategies, suppose Abby at $t_1$ receives $\tau^x_X$ and makes a measurement of instruments $\{\Phi^{a,\lambda}_{X \to A}\}$, and gains the outcome $a$. Then the quantum output goes through the quantum memory $N: A \to B$. The output of the memory and $\omega^y_Y$ received by Abby at $t_2$ are fed into a measurement $\{\Psi^{b,a,\lambda}_{BY}\}$, with outcome $b$. Then

$$p_q(a, b|x, y) = \sum_\lambda \pi(\lambda) \text{Tr}[\{(N_{A \to B} \circ \Phi^{a,\lambda}_{X \to A})(\tau^x_X)\} \otimes \omega^y_Y] \Psi^{b,a,\lambda}_{BY}].$$

(91)

The optimal payoff function is, again, given by

$$E_{\gamma_{QCB}[*]} \equiv \max \sum_{x,y} \pi(x, y) \sum_{a,b} l(a, b|x, y)p_{c/q}(a, b|x, y).$$

(92)

C. Payoff functions as temporal correlations

To compare quantum-classical signalling games with pseudo-density matrices, first we generalise the finite-dimensional pseudo-density matrices from Pauli measurements to general positive-operator valued measures (POVMs). Recall that a POVM is a set of Hermitian positive semi-definite operator $\{E_i\}$ on a Hilbert space $\mathcal{H}$ which sum up to the identity $\sum_i E_i = 1_{\mathcal{H}}$. Instead of making a single-qubit Pauli measurement at each event, we make a measurement $E_i = M_i^a M_i^a$ with the outcome $a$. For each event, there is a measurement $\mathcal{M}_i: \mathcal{L}(\mathcal{H}^X) \to \mathcal{L}(\mathcal{H}^A), \tau^x_X \mapsto \sum_i M_i^a \tau^x_X M_i^{a\dagger}$ with $\sum M_i^{a\dagger} M_i^a = 1_{\mathcal{H}^X}$.

Now we map the generalise pseudo-density matrices to quantum-classical signalling games. Assume $\omega^y_Y$ to be trivial. For Abby at the initial time and the later time, we consider $\Phi^{a}_{X \to A}: \tau^x_X \to \sum_i M_i^a \tau^x_X M_i^{a\dagger}, \sum M_i^{a\dagger} M_i^a = 1_{\mathcal{H}^A}$. Between two times, the transformation from $A$ to $B$ is given by $N: \rho_A \to \sum_j N_j \rho_A N_j^\dagger$ with $\sum_j N_j^\dagger N_j = 1_{\mathcal{H}^A}$. Then

$$p_q(a, b|x, y) = \text{Tr}[\{(N_{A \to B} \circ \Phi^{a}_{X \to A})(\tau^x_X)\} \Psi^{b,a}_{B}]$$

$$= \sum_{ik} \text{Tr}[N\{M_i^a \tau^x_X M_i^{a\dagger}\} \Psi^{b,a}_{B}]$$

$$= \sum_{ijk} \text{Tr}[N_j M_i^a \tau^x_X M_i^{a\dagger} N_j^\dagger \Psi^{b,a}_{B}]$$

(93)

$$\langle\{\Phi, \Psi\}\rangle = \sum_{a,b} abp_q(a, b|x, y)$$

(94)

It is the temporal correlation given by pseudo-density matrices. That is, a quantum-classical signalling game with a trivial input at later time corresponds to a pseudo-density matrix with quantum channels as measurements.
Claim 3. The probability in a quantum-classical signalling game with a trivial input at later time corresponds to the probability in a pseudo-density matrix with quantum channels as measurements.

It is also convenient to establish the relation between generalised games in time and indefinite causal structures with double Hilbert spaces for each event. For completeness, we also mention that Gutoski and Watrous [32] proposed a general theory of quantum games in terms of the Choi-Jamiołkowski representation, which is an equivalent formulation of indefinite causal order.

VI. OUT-OF-TIME-ORDER CORRELATIONS (OTOCs)

In this section we introduce out-of-time-order correlation functions, find a simple method to calculation these temporal correlations via the pseudo-density matrix formalism.

A. Brief introduction to OTOCs

Consider local operators $W$ and $V$. With a Hamiltonian $H$ of the system, the Heisenberg representation of the operator $W$ is given as $W(t) = e^{iHt}W e^{-iHt}$. Out-of-time-order correlation functions (OTOCs) [15, 16] are usually defined as

$$\langle VW(t)V^\dagger W^\dagger(t)\rangle = \langle VU(t)^\dagger WU(t)V^\dagger(t)U^\dagger(t)W^\dagger U(t)\rangle,$$

where $U(t) = e^{-iHt}$ is the unitary evolution operator and the correlation is evaluated on the thermal state $\langle\cdot\rangle = \text{Tr}[e^{-\beta H}\cdot]/\text{Tr}[e^{-\beta H}]$. Note that OTOC is usually defined for the maximally mixed state $\rho = \frac{1}{d}$. Consider a correlated qubit chain. Measure $V$ at the first qubit and $W$ at the last qubit. Since the chain is correlated in the beginning, we have OTOC as 1 at the early time. As time evolves and the operator growth happens, OTOC will approximate to 0 at the late time.

B. Calculating OTOCs via pseudo-density matrices

In this subsection we make a connection between OTOCs and pseudo-density matrix formalism. If we consider a qubit evolving in time and backward, we can get a tripartite
pseudo-density matrix. In particular, we consider measuring $A$ at $t_1$, $B$ at $t_2$ and $A$ again at $t_3$ and assume the evolution forwards is described by $U$ and backward $U^\dagger$. Then the probability is given by

$$\text{Tr}[AU^\dagger BU A^\dagger U^\dagger B^\dagger UA] = \text{Tr}[AB(t) A^\dagger B^\dagger(t) A^\dagger]$$

(96)

If we assume that $AA^\dagger = A$, $\rho = \frac{1}{d}$, Eqn. (96) will reduce to the OTOC.

**Claim 4.** OTOCs can be represented as temporal correlations in pseudo-density matrices with half numbers of steps for calculation; for example, a four-point OTOC, usually calculated by evolving forwards and backwards twice, is represented by a tripartite pseudo-density matrix with only once evolving forwards and backwards.

**VII. A UNIFIED PICTURE**

Now we consider a unified picture in which temporal correlations serve as a connection for indefinite causal order, consistent histories, generalised quantum games and OTOCs. Given a tripartite pseudo-density matrix, a qubit in the state $\rho$ evolves in time under the unitary evolution $U$ and then back in time under $U^\dagger$. The correlations in the pseudo-density matrix are given as

$$\langle \sigma_i, \sigma_j, \sigma_k \rangle = \sum_{\alpha, \beta, \gamma = \pm 1} \alpha \beta \gamma \text{Tr}[P^\gamma_k U^\dagger P^\beta_j U P^\alpha_i \rho P^\alpha_i U^\dagger P^\beta_j U]$$

(97)

where $P^\alpha_i = \frac{1}{2}(\mathbb{I} + \alpha \sigma_i)$, $P^\beta_j = \frac{1}{2}(\mathbb{I} + \beta \sigma_j)$ and $P^\gamma_k = \frac{1}{2}(\mathbb{I} + \gamma \sigma_k)$. As the pseudo-density matrix belongs to indefinite causal order, we won’t discuss the transform for indefinite causal order.

For consistent histories, we assume the state in $\rho$ at the initial time and construct a set of histories $[\chi] = [\alpha \rightarrow \beta \rightarrow \gamma]$ with projections $\{P^\alpha_i, P^\beta_j, P^\gamma_k\}$. Then the decoherence functional is given as

$$D([\xi], [\xi]) = \text{Tr}[P^\gamma_k U^\dagger P^\beta_j U P^\alpha_i \rho P^\alpha_i U^\dagger P^\beta_j U P^\gamma_k]$$

(98)

When we apply the consistency conditions, it is part of Eqn. (97) as

$$D([\xi], [\xi]) = \text{Tr}[P^\gamma_k U^\dagger P^\beta_j U P^\alpha_i \rho P^\alpha_i U^\dagger P^\beta_j U P^\gamma_k],$$

(99)

$$\langle \sigma_i, \sigma_j, \sigma_k \rangle = \sum_{\alpha, \beta, \gamma = \pm 1} \alpha \beta \gamma D([\xi], [\xi]).$$

(100)
A quantum-classical signalling game is described in terms of one player Abby at two times in a loop, or one player Abby at three times with evolution $U$ and $U^\dagger$. The quantum-classical signalling game is formulated by $(\pi(x, y), l(a, b|x, y))$ on

$$ \overrightarrow{qcs} = \langle \{\tau^x\}, \{\omega^y\}, \{\eta^z\}; A, B, C; l \rangle. \quad (101) $$

The referee associates three quantum systems in the states $\tau^x$, $\omega^y$ and $\eta^z$ with the questions chosen from the question spaces $x \in X$, $y \in Y$, and $z \in Z$. Suppose Abby at $t_1$ receives $\tau^x_X$ and makes a measurement of instruments $\{M^a_i\}_i$ with the outcome $a$. From $t_1$ to $t_2$, the quantum output evolves under the unitary quantum memory $U: A \rightarrow B$. After that, Abby receives the output of the channel and $\omega^y$, and makes a measurement of instruments $\{N^b_j\}_j$ with the outcome $b$. Then, we can consider that either the quantum memory goes backwards to $t_1$ or evolves under $U^\dagger: B \rightarrow C$ to $t_3$. Abby receives the output of the channel again and $\eta^z$, and makes a measurement of instruments $\{O^c_k\}_k$ with the outcome $c$. Then we have

$$ p_q(a, b, c|x, y, z) = \sum_{\lambda, i, j, k} \pi(\lambda) \text{Tr}[O^c_k U^\dagger N^b_j U M^a_i \rho M^a_i U^\dagger N^b_j U O^c_k]. \quad (102) $$

If we properly choose the measurements, we will have the decoherence functionals and the probabilities in the correlations of pseudo-density matrix.

What is more, the tripartite pseudo-density matrix we describe is just the one we used to construct OTOC. Thus, through this tripartite pseudo-density matrix, we gain a unified picture for indefinite causal order, consistent histories, generalised quantum games and OTOCs in which temporal correlations are the same or operationally equivalent. Thus all these approaches are mapping into each other directly in this particular case via temporal correlations. Generalisation to more complicated scenarios are straightforward.

VIII. PATH INTEGRALS

The path integral approach [17] is a representation of quantum theory, not only useful in quantum mechanics but also quantum statistical mechanics and quantum field theory. It generalises the action principle of classical mechanics and one computes a quantum amplitude by replacing a single classical trajectory with a functional integral of infinite numbers of possible quantum trajectories. Here we argue that the path integral approach of quantum
mechanics use amplitude as the measure in correlation functions rather than probability measure in the above formalisms.

A. Introduction to path integrals

Now we briefly introduce path integrals and correlation functions. Consider a bound operator in a Hilbert space $U(t_2, t_1)(t_2 \geq t_1)$ as the evolution from time $t_1$ to $t_2$, which satisfies the Markov property in time as

$$ U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1), \forall t_3 \geq t_2 \geq t_1 \quad U(t, t) = 1. \quad (103) $$

We further assume that $U(t, t')$ is differentiable and the derivative is continuous:

$$ \left. \frac{\partial U(t, t')}{\partial t} \right|_{t=t'} = -H(t)/\hbar $$

where $\hbar$ is a real parameter, and later identified with Planck’s constant; $H = i\tilde{H}$ where $\tilde{H}$ is the quantum Hamiltonian. Then

$$ U(t'', t') = \prod_{m=1}^{n} U[t' + m\epsilon, t' + (m - 1)\epsilon], \quad n\epsilon = t'' - t'. \quad (105) $$

The position basis for $\hat{q} |q\rangle = q |q\rangle$ is orthogonal and complete: $\langle q' | q \rangle = \delta(q - q')$, $\int dq |q\rangle \langle q| = 1$. We have

$$ \langle q'' | U(t'', t') | q' \rangle = \int \prod_{k=1}^{n-1} dq_k \prod_{k=1}^{n} \langle q_k | U(t_k, t_{k-1}) | q_{k-1} \rangle \quad (106) $$

with $t_k = t' + k\epsilon, q_0 = q', q_n = q''$. Suppose that the operator $H$ is identified with a quantum Hamiltonian of the form

$$ H = \hat{p}^2/2m + V(\hat{q}, t) \quad (107) $$

where $p, q \in \mathbb{R}^d$. We have

$$ \langle q | U(t, t') | q' \rangle = \left( \frac{m}{2\pi\hbar(t - t')} \right)^{d/2} \exp[-S(q)/\hbar] \quad \text{exp} \quad (108) $$

where

$$ S(q) = \int_{t'}^{t} d\tau [\frac{1}{2} m q^2(\tau) + V(q(\tau), \tau)] + O((t - t')^2), \quad (109) $$

and

$$ q(\tau) = q' + \frac{\tau - t'}{t - t'} (q - q'). \quad (110) $$
We consider short time slices, then
\[
\langle q'\rangle \mid U(t'', t') \mid q' \rangle = \lim_{n \to \infty} \left( \frac{m}{2 \pi \hbar} \right)^{dn/2} \int \prod_{k=1}^{n-1} d^d q_k \exp[-S(q, \epsilon)/\hbar],
\] (111)

with
\[
S(q, \epsilon) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} dt \left[ \frac{1}{2} m \dot{q}^2(t) + V(q(t), t) \right] + O(\epsilon^2).
\] (112)

Introducing a linear and continuous trajectory
\[
q(t) = q_k + \frac{t - t_k}{t_{k+1} - t_k} (q_{k+1} - q_k) \quad \text{for} \quad t_k \leq t \leq t_{k+1},
\] (113)

we can rewrite Eqn. (112) as
\[
S(q, \epsilon) = \int_{t'}^{t''} dt \left[ \frac{1}{2} m \dot{q}^2(t) + V(q(t), t) \right] + O(n\epsilon^2).
\] (114)

Taking \( n \to \infty \) and \( \epsilon \to 0 \) with \( n\epsilon = t'' - t' \) fixed, we have
\[
S(q) = \int_{t''}^{t'} dt \left[ \frac{1}{2} m \dot{q}^2(t) + V(q(t), t) \right]
\] (115)
as the Euclidean action. The path integral is thus defined as
\[
\langle q'' \rangle \mid U(t'', t') \mid q' \rangle = \int_{q(t'') = q''}^{q(t') = q'} [dq(t)] \exp(-S(q)/\hbar),
\] (116)

where a normalisation of \( \mathcal{N} = \left( \frac{m}{2 \pi \hbar} \right)^{dn/2} \) is hidden in \( [dq(t)] \).

The quantum partition function \( \mathcal{Z}(\beta) = \text{Tr} \, e^{-\beta H} \) (\( \beta \) is the inverse temperature) can be written in terms of path integrals as
\[
\mathcal{Z}(\beta) = \text{Tr} \, e^{-\beta H} = \text{Tr} \, U(h\beta, 0) = \int dq''dq' \delta(q'' - q') \, \langle q'' \rangle \mid U(h\beta, 0) \mid q' \rangle
\]
\[
= \int_{q(0) = q(h\beta)} [dq(t)] \exp[-S(q)/\hbar],
\] (117)

The integrand \( e^{-S(q)/\hbar} \) is a positive measure and defines the corresponding expectation value as
\[
\langle \mathcal{F}(q) \rangle = \mathcal{N} \int [dq(t)] \mathcal{F}(q) \exp[-S(q)/\hbar],
\] (118)

where \( \mathcal{N} \) is chosen for \( \langle 1 \rangle = 1 \). Moments of the measure in the form as
\[
\langle q(t_1)q(t_2) \cdots q(t_n) \rangle = \mathcal{N} \int [dq(t)] q(t_1)q(t_2) \cdots q(t_n) \exp[-S(q)/\hbar]
\] (119)
are the \( n \)-point correlation function. Suppose for the finite time interval \( \beta \) periodic boundary conditions hold as \( q(\beta/2) = q(-\beta/2) \). The normalisation is given as \( \mathcal{N} = Z^{-1}(\beta) \). Then we define

\[
Z^{(n)}(t_1, \ldots, t_n) = \langle q(t_1) \cdots q(t_n) \rangle. \tag{120}
\]

The generating functional of correlation functions is

\[
\mathcal{Z}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \cdots dt_n Z^{(n)}(t_1, \ldots, t_n) f(t_1) \cdots f(t_n)
= \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \cdots dt_n \langle q(t_1) \cdots q(t_n) \rangle f(t_1) \cdots f(t_n)
= \left\langle \exp \left[ \int dt q(t) f(t) \right] \right\rangle. \tag{121}
\]

What is more, the \( n \)-point quantum correlation functions in time appear as continuum limits of the correlation functions of \( 1D \) lattice in classical statistical models. The path integral, thus, represent a mathematical relation between classical statistical physics on a line and quantum statistical physics of a point-like particle at thermal equilibrium. This is the first example of the quantum-classical correspondence which maps between quantum statistical physics in \( D \) dimensions and classical statistical physics in \( D + 1 \) dimensions [18].

**B. Temporal correlations in path integrals are different**

Here we take two-point correlations functions:

\[
\langle q(t_1) q(t_2) \rangle = \frac{\int [dq(t)] q(t_1) q(t_2) \exp[-S(q)/\hbar]}{\int [dq(t)] \exp[-S(q)/\hbar]} \tag{122}
\]

In the Gaussian representation of pseudo-density matrices, temporal correlation for \( q_1 \) at \( t_1 \) and \( q_2 \) at \( t_2 \) with the evolution \( U \) and the initial state \( |q_1\rangle \) is given as

\[
\langle \{q_1, q_2\} \rangle = \int dq_1 dq_2 \langle q_2 | U | q_1 \rangle |^2 = \int dq_1 dq_2 q_1 q_2 \left| \int_{q(t_1) = q_1}^{q(t_2) = q_2} [dq(t)] \exp[-S(q)/\hbar] \right|^2 \tag{123}
\]

Correlations are defined as the expectation values of measurement outcomes. However, path integrals and pseudo-density matrices use different positive measure to calculate the expectation values. The correlations in path integrals use the amplitude as the measure, while in pseudo-density matrices the measure is the absolute square of the amplitude, or we say the probability.
To see the difference, we consider a quantum harmonic oscillator. The Hamiltonian is given as \( H = \hat{p}^2 / 2m + m\omega^2 \hat{q}^2 / 2 \). Note that the quantum amplitude of a quantum harmonic oscillator is given as

\[
\langle q_2 | U(t_2, t_1) | q_1 \rangle = \left( \frac{m\omega}{2\pi \hbar \sinh \omega \tau} \right)^{1/2} \exp \left\{ - \frac{m\omega}{2\hbar \sinh \omega \tau} \left[ (q_1^2 + q_2^2) \cosh \omega \tau - 2q_1q_2 \right] \right\},
\]

where \( \tau = t_2 - t_1 \). In the Gaussian representation of pseudo-density matrices, temporal correlations are represented as

\[
\langle \{ q_1, q_2 \} \rangle = \int dq_1 dq_2 q_1 q_2 \langle q_2 | U | q_1 \rangle \langle q_2 | U | q_1 \rangle = \frac{\hbar}{8m\omega \sinh^2 \omega \tau}.
\]

However, in the path integral formalism, we consider

\[
\text{Tr} \ U_G(\tau/2, -\tau/2; b) = \int [dq(t)] \exp[-S_G(q, b)/\hbar]
\]

with

\[
S_G(q, b) = \int_{-\tau/2}^{\tau/2} dt \left[ \frac{1}{2} m\dot{q}(t)^2 + \frac{1}{2} m\omega^2 q(t)^2 - b(t)q(t) \right]
\]

and periodic boundary conditions \( q(\tau/2) = q(-\tau/2) \). We have

\[
Z_G(\beta, b) = \text{Tr} \ U_G(\hbar\beta/2, -\hbar\beta/2; b) = Z_0(\beta) \langle \exp \left[ \frac{1}{\hbar} \int_{-\hbar\beta/2}^{\hbar\beta/2} dt b(t)q(t) \right] \rangle_0 \]

where \( \langle \bullet \rangle_0 \) denotes the Gaussian expectation value in terms of the distribution \( e^{-S_t/\hbar} / Z_0(\beta) \) and periodic boundary conditions. Here \( Z_0(\beta) \) is the partition function of the harmonic oscillator as

\[
Z_0(\beta) = \frac{1}{2 \sinh(\beta\omega/2)} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}.
\]

Then two-point correlations functions are given as

\[
\langle q(t_1)q(t_2) \rangle = Z_0^{-1}(\beta) \hbar^2 \frac{\delta^2}{\delta b(t)\delta b(u)} Z_G(\beta, b) \bigg|_{b=0} = \frac{\hbar}{2\omega \tanh(\omega \tau/2)}.
\]

It is no surprise that the temporal correlations are distinct from each other in this example.

**Claim 5.** In general, temporal correlations in path integrals do not have the operational meaning as those in pseudo-density matrices since they use different measures, with exception of path-integral representation for spacetime states and decoherence functionals.

That indicates a fundamental difference of temporal correlations in path integrals and other spacetime approaches, and raises again the question whether probability or amplitude...
serves as the measure in quantum theory. It is natural that amplitudes interferes with each other in field theory and expectation values of operators are defined with amplitudes interference. Thus temporal correlations in path integrals cannot be operationally measured as pseudo-density matrices. However, spacetime states defined via position measurements and weak measurements in pseudo-density matrix formulation are motivated by the path integral formalism and have path-integral representations naturally. In addition, consistent histories also have a path-integral representation of decoherence functionals as we mentioned earlier.

IX. CONCLUSION AND DISCUSSION

We conclude that there is not much difference in different spacetime approaches for non-relativistic quantum mechanics under this comparison of temporal correlations except path integrals. They are closely related compared with pseudo-density matrices and formulate temporal correlations in the same way or operationally equivalent. However, the path integral approach of quantum mechanics give temporal correlation in a different way. Via the pseudo-density matrix formalism, we establish the relations among different spacetime formulations like indefinite causal structures, consistent histories, generalised nonlocal games, out-of-time-order correlation functions, and path integrals. As we can see, all these relations are rather simple. The big surprise we learn from these relations is that almost everything we know about space-time in non-relativistic quantum mechanics so far is connected with each other but path integrals are not. Thus, it shows the possibility of a unified picture of non-relativistic quantum mechanics in spacetime and a gap to relativistic quantum field theory. We claim:

(1) A process matrix and the corresponding pseudo-density matrix allow the same correlations or probabilities in four different mappings.
(2) The decoherence functional in consistent histories is the probabilities in temporal correlations of pseudo-density matrices.
(3) The probability in a quantum-classical signalling game with a trivial input at later time corresponds to the probability in a pseudo-density matrix with quantum channels as measurements.
(4) OTOCs can be represented as temporal correlations in pseudo-density matrices with
half numbers of steps for calculation; for example, a four-point OTOC, usually calculated by evolving forwards and backwards twice, is represented by a tripartite pseudo-density matrix with only once evolving forwards and backwards.

(5) In general, temporal correlations in path integrals do not have the operational meaning as those in pseudo-density matrices since they use different measures, with exception of path-integral representation for spacetime states and decoherence functionals.

A unified theory for non-relativistic quantum mechanics is suggested; nevertheless, how to move on to relativistic quantum information, or further to quantum gravity, is still a big gap worth exploring.

ACKNOWLEDGMENTS

The authors thank Lucien Hardy, Giulio Chiribella, Kavan Modi, Fabio Costa, and David Felce for discussion on indefinite causal structures; thank Seth Lloyd, Robert Spekkens, and Wojciech Zurek for introducing consistent histories and decoherence functional; thank Francesco Buscemi and Denis Rosset for discussion on quantum-classical signalling games; thank Beni Yoshida and Nick Hunter-Jones for discussion on OTOCs; thank Zi-Wen Liu for the suggestion to write up the paper; thank Rafael Sorkin for discussion on path integrals and quantum measure; and thank Lee Smolin for general discussion on time and correlations. T.Z. thanks the visiting graduate fellow program at Perimeter Institute for Theoretical Physics.

[1] E. Anderson, The problem of time in quantum gravity (2010), arXiv:1009.2157 [gr-qc].
[2] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Quantum circuit architecture, Phys. Rev. Lett. 101, 060401 (2008).
[3] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Theoretical framework for quantum networks, Phys. Rev. A 80, 022339 (2009).
[4] L. Hardy, The operator tensor formulation of quantum theory, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 370, 3385 (2012).
[5] O. Oreshkov, F. Costa, and Č. Brukner, Quantum correlations with no causal order, Nature communications 3, 1092 (2012).
[6] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Non-markovian quantum processes: Complete framework and efficient characterization, Phys. Rev. A 97, 012127 (2018).

[7] J. Cotler, C.-M. Jian, X.-L. Qi, and F. Wilczek, Superdensity operators for spacetime quantum mechanics, Journal of High Energy Physics 2018, 93 (2018).

[8] R. B. Griffiths, Consistent histories and the interpretation of quantum mechanics, Journal of Statistical Physics 36, 219 (1984).

[9] R. B. Griffiths, Consistent quantum theory (Cambridge University Press, 2003).

[10] M. Gell-Mann and J. B. Hartle, Quantum Mechanics in the Light of Quantum Cosmology, (1989), arXiv:1803.04605 [gr-qc].

[11] M. Gell-Mann and J. B. Hartle, Classical equations for quantum systems, Phys. Rev. D 47, 3345 (1993).

[12] R. Omnés, From hilbert space to common sense: A synthesis of recent progress in the interpretation of quantum mechanics, Annals of Physics 201, 354 (1990).

[13] F. Buscemi, All entangled quantum states are nonlocal, Phys. Rev. Lett. 108, 200401 (2012).

[14] D. Rosset, F. Buscemi, and Y.-C. Liang, Resource theory of quantum memories and their faithful verification with minimal assumptions, Phys. Rev. X 8, 021033 (2018).

[15] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, JHEP 08, 106 arXiv:1503.01409 [hep-th].

[16] D. A. Roberts and B. Yoshida, Chaos and complexity by design, JHEP 04, 121 arXiv:1610.04903 [quant-ph].

[17] R. P. Feynman, A. R. Hibbs, and D. F. Styer, Quantum mechanics and path integrals (Courier Corporation, 2010).

[18] J. Zinn-Justin, Path integrals in quantum mechanics (Oxford University Press, 2010).

[19] J. F. Fitzsimons, J. A. Jones, and V. Vedral, Quantum correlations which imply causation, Scientific reports 5, 18281 (2015).

[20] Z. Zhao, R. Pisarczyk, J. Thompson, M. Gu, V. Vedral, and J. F. Fitzsimons, Geometry of quantum correlations in space-time, Phys. Rev. A 98, 052312 (2018).

[21] R. Pisarczyk, Z. Zhao, Y. Ouyang, V. Vedral, and J. F. Fitzsimons, Causal limit on quantum communication, Phys. Rev. Lett. 123, 150502 (2019).
[22] T. Zhang, O. Dahlsten, and V. Vedral, Different instances of time as different quantum modes: quantum states across space-time for continuous variables, *New Journal of Physics* (2020).

[23] T. Zhang, Z. Zhu, F. Tennie, X. Yang, X. Peng, and V. Vedral, Quantum correlation under time translation symmetry, In Preparation (2020).

[24] L. Hardy, Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure, *Journal of Physics A: Mathematical and Theoretical* 40, 3081 (2007).

[25] L. Hardy, Quantum gravity computers: On the theory of computation with indefinite causal structure, in *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: Essays in Honour of Abner Shimony* (Springer Netherlands, Dordrecht, 2009) pp. 379–401.

[26] L. Hardy, The construction interpretation: a conceptual road to quantum gravity, arXiv preprint arXiv:1807.10980 (2018).

[27] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi, and Č. Brukner, Witnessing causal nonseparability, *New Journal of Physics* 17, 102001 (2015).

[28] S. Milz, F. A. Pollock, and K. Modi, An introduction to operational quantum dynamics, *Open Systems & Information Dynamics* 24, 1740016 (2017).

[29] J. Cotler, X. Han, X.-L. Qi, and Z. Yang, Quantum causal influence, *Journal of High Energy Physics* 2019, 42 (2019).

[30] F. Costa, M. Ringbauer, M. E. Goggin, A. G. White, and A. Fedrizzi, Unifying framework for spatial and temporal quantum correlations, *Phys. Rev. A* 98, 012328 (2018).

[31] D. Kretschmann and R. F. Werner, Quantum channels with memory, *Phys. Rev. A* 72, 062323 (2005).

[32] G. Gutoski and J. Watrous, Toward a general theory of quantum games, in *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing* (ACM, 2007) pp. 565–574.

[33] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Time symmetry in the quantum process of measurement, *Phys. Rev. 134*, B1410 (1964).

[34] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, Multiple-time states and multiple-time measurements in quantum mechanics, *Phys. Rev. A* 79, 052110 (2009).

[35] R. Silva, Y. Guryanova, A. J. Short, P. Skrzypczyk, N. Brunner, and S. Popescu, Connecting processes with indefinite causal order and multi-time quantum states, *New Journal of Physics* 19, 103022 (2017).
[36] R. Oeckl, A “general boundary” formulation for quantum mechanics and quantum gravity, *Physics Letters B* **575**, 318 (2003).

[37] F. Costa and S. Shrapnel, Quantum causal modelling, *New Journal of Physics* **18**, 063032 (2016).

[38] J.-M. A. Allen, J. Barrett, D. C. Horsman, C. M. Lee, and R. W. Spekkens, Quantum common causes and quantum causal models, *Phys. Rev. X* **7**, 031021 (2017).

[39] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, *Reports on Mathematical Physics* **3**, 275 (1972).

[40] M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra and its Applications* **10**, 285 (1975).

[41] H. Barnum, C. A. Fuchs, J. M. Renes, and A. Wilce, Influence-free states on compound quantum systems (2005), arXiv:quant-ph/0507108 [quant-ph].

[42] G. Chiribella, G. M. D’Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, *Phys. Rev. A* **88**, 022318 (2013).

[43] M. Araújo, F. Costa, and Č. Brukner, Computational advantage from quantum-controlled ordering of gates, *Phys. Rev. Lett.* **113**, 250402 (2014).

[44] L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. A. Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Č. Brukner, and P. Walther, Experimental superposition of orders of quantum gates, *Nature Communications* **6**, 7913 (2015).

[45] K. Goswami, C. Giarmatzi, M. Kewming, F. Costa, C. Branciard, J. Romero, and A. G. White, Indefinite causal order in a quantum switch, *Phys. Rev. Lett.* **121**, 090503 (2018).

[46] G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, Č. Brukner, and P. Walther, Experimental verification of an indefinite causal order, *Science Advances* **3**, e1602589 (2017).

[47] G. Rubino, L. A. Rozema, F. Massa, M. Araújo, M. Zych, Časlav Brukner, and P. Walther, Experimental entanglement of temporal orders, in *Quantum Information and Measurement (QIM) V: Quantum Technologies* (Optical Society of America, 2019) p. S3B.3.

[48] K. Goswami, J. Romero, and A. White, Communicating via ignorance, arXiv preprint arXiv:1807.07383 (2018).
[49] Y. Guo, X.-M. Hu, Z.-B. Hou, H. Cao, J.-M. Cui, B.-H. Liu, Y.-F. Huang, C.-F. Li, and G.-C. Guo, Experimental investigating communication in a superposition of causal orders, arXiv preprint arXiv:1811.07526 (2018).

[50] K. Wei, N. Tischler, S.-R. Zhao, Y.-H. Li, J. M. Arrazola, Y. Liu, W. Zhang, H. Li, L. You, Z. Wang, Y.-A. Chen, B. C. Sanders, Q. Zhang, G. J. Pryde, F. Xu, and J.-W. Pan, Experimental quantum switching for exponentially superior quantum communication complexity, Phys. Rev. Lett. 122, 120504 (2019).

[51] K. Gödel, An example of a new type of cosmological solutions of einstein’s field equations of gravitation, Rev. Mod. Phys. 21, 447 (1949).

[52] D. Deutsch, Quantum mechanics near closed timelike lines, Phys. Rev. D 44, 3197 (1991).

[53] D. Ahn, C. R. Myers, T. C. Ralph, and R. B. Mann, Quantum-state cloning in the presence of a closed timelike curve, Phys. Rev. A 88, 022332 (2013).

[54] T. A. Brun, M. M. Wilde, and A. Winter, Quantum state cloning using deutschian closed timelike curves, Phys. Rev. Lett. 111, 190401 (2013).

[55] C. Bennett and B. Schumacher, Teleportation, simulated time travel, and how to flirt with someone who has fallen into a black hole, QUPON, Wien (2005).

[56] G. Svetlichny, Time travel: Deutsch vs. teleportation, International Journal of Theoretical Physics 50, 3903 (2011).

[57] T. A. Brun and M. M. Wilde, Perfect state distinguishability and computational speedups with postselected closed timelike curves, Foundations of Physics 42, 341 (2012).

[58] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, Closed timelike curves via postselection: Theory and experimental test of consistency, Phys. Rev. Lett. 106, 040403 (2011).

[59] M. Araújo, P. A. Guérin, and A. Baumeler, Quantum computation with indefinite causal structures, Phys. Rev. A 96, 052315 (2017).

[60] C. Isham and N. Linden, Continuous histories and the history group in generalized quantum theory, Journal of Mathematical Physics 36, 5392 (1995).

[61] C. J. Isham, N. Linden, K. Savvidou, and S. Schreckenberg, Continuous time and consistent histories, Journal of Mathematical Physics 39, 1818 (1998).
[62] C. Isham, Quantum logic and decohering histories, Topics in Quantum Field Theory: Modern Methods in Fundamental Physics, Editor D. Trakian, World Scientific, 30 (1995).

[63] C. J. Isham, Quantum logic and the histories approach to quantum theory, Journal of Mathematical Physics 35, 2157 (1994).

[64] J. J. Halliwell, A review of the decoherent histories approach to quantum mechanics, arXiv preprint gr-qc/9407040 (1994).

[65] H. F. Dowker and J. J. Halliwell, Quantum mechanics of history: The decoherence functional in quantum mechanics, Phys. Rev. D 46, 1580 (1992).

[66] F. Dowker and A. Kent, On the consistent histories approach to quantum mechanics, Journal of Statistical Physics 82, 1575 (1996).

[67] R. B. Myerson, Game theory (Harvard University Press, 2013).

[68] R. Cleve, P. Hoyer, B. Toner, and J. Watrous, Consequences and limits of nonlocal strategies, in Proceedings. 19th IEEE Annual Conference on Computational Complexity, 2004 pp. 236–249.

[69] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. 23, 880 (1969).