Classification of integrable vector equations of geometric type

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ABSTRACT. A complete classification of isotropic vector equations of the geometric type that possess higher symmetries is proposed. New examples of integrable multi-component systems of the geometric type are found.

1 Introduction

Consider evolution systems of the form

\begin{equation}
\frac{u_{i}}{t} = u_{xxx}^{i} + 3A_{jk}^{i}(u)u_{x}^{j}u_{x}^{k}u_{x}^{l} + 3A_{i}^{j}(u)u_{x}^{j}u_{x}^{k}u_{x}^{l}, \quad i, j, k, s = 1, \ldots, N,
\end{equation}

where $u = (u^{1}, \ldots, u^{N})$. Here and below, we assume that the summation is carried out over repeated indexes.

Integrable systems of this type are connected with various geometric and algebraic structures and are of interest by themselves. In addition, the most interesting of them play the role of infinitesimal symmetries for physically important hyperbolic systems of the form

\begin{equation}
\frac{u_{xy}^{i}}{t} = C_{jk}^{i}(u)u_{x}^{j}u_{y}^{k}.
\end{equation}

Having an efficient description of integrable systems (1.1), we can construct a class of integrable systems of the form (1.2) following the approach from the papers [1, 2].

An example of such type integrable system provides the following equation [3]

\begin{equation}
\frac{U_{t}}{t} = U_{xxx} - \frac{3}{2}U_{x}U^{-1}U_{xx} - \frac{3}{2}U_{xx}U^{-1}U_{x} + \frac{3}{2}U_{x}U^{-1}U_{x}U^{-1}U_{x},
\end{equation}

where $U(x, t)$ is an $m \times m$ matrix. In this case $N = m^{2}$. For any $m$ this system has infinitely many local symmetries and conservation laws.

It is convenient to rewrite (1.1) in the following way

\begin{equation}
\frac{u_{i}}{t} = u_{3}^{i} + 3A_{jk}^{i}(u)u_{x}^{j}u_{x}^{k}u_{x}^{l} + \left( \frac{\partial A_{lk}^{i}}{\partial u_{i}} + 2A_{i}^{s}A_{jk}^{s} - A_{i}^{s}A_{jk}^{s} + \beta_{jkl}^{i} \right)u_{x}^{j}u_{x}^{k}u_{x}^{l}.
\end{equation}
The class of systems (1.3) is invariant under the arbitrary point transformations \( u \to \Phi(u) \). It is easy to see that under such a change of coordinates, the functions \( A^i_{jk} \) and \( \beta^i_{jkm} \) are transformed just as components of an affine connection \( \Gamma \) and of a tensor \( \beta \), respectively.

**Example 1.** In the case \( N = 1 \) equation (1.3) has the form

\[
\frac{du}{dt} = u_{xxx} + 3A(u)u_xu_{xx} + \left[A'(u) + A(u)^2 + \beta(u)\right]u_x^3.
\]

Using the symmetry approach (see [4]), one can verify that this equation possesses higher symmetries iff \( \beta' = 2A\beta \). By a proper point transformation of the form \( u \to \Phi(u) \) the function \( A \) can be reduced to zero (for \( N = 1 \) any affine connection is flat) and the function \( \beta \) becomes a constant. The equation \( \frac{du}{dt} = u_{xxx} + \text{const} u_x^3 \) is known to be integrable and it is related to the mKdV equation by a potentiation.

Without loss of generality we assume that the tensor \( \beta \) is symmetric:

\[
\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)
\]

for any vectors \( X, Y, Z \). The functions \( \beta^i_{jkm} \) are defined by the values of \( \beta(X, X, X) \).

Suppose a system of the form (1.3) has higher symmetries and/or non-degenerate conservation laws and \( A^i_{jk} = A^i_{kj} \), i.e. the torsion tensor \( T \) is equal to zero. Then\(^1\) the corresponding affine connected space is symmetric \([6]\) which means that

\[
\nabla_X \left( R(Y, Z, V) \right) = 0,
\]

where \( R \) is the curvature tensor\(^2\). Let

\[
\sigma(X, Y, Z) \overset{\text{def}}{=} \beta(X, Y, Z) - \frac{1}{3} \left( R(X, Y, Z) + R(Z, Y, X) \right).
\]

Then

\[
\sigma(X, Y, Z) = \sigma(Z, Y, X),
\]

\[
\nabla_X \left( \sigma(Y, Z, V) \right) = 0,
\]

\[
R(X, Y, Z) = \sigma(X, Z, Y) - \sigma(X, Y, Z),
\]

and

\[
\sigma(X, \sigma(Y, Z, V), W) - \sigma(W, V, \sigma(X, Y, Z)) + \sigma(Z, Y, \sigma(X, V, W)) - \sigma(X, V, \sigma(Z, Y, W)) = 0.
\]

\(^1\)It was discovered by S. Svinilupov and V. Sokolov and was published without proof in the survey [5] dedicated to Sergey Svinolupov.

\(^2\)We use here the following formula for the curvature tensor:

\[
R^m_{ijk} = \frac{\partial}{\partial u_j} A^m_{ki} - \frac{\partial}{\partial u_k} A^m_{ij} + A^m_{js} A^s_{kj} - A^m_{ks} A^s_{kj}.
\]
The identities (1.5) and (1.8) mean that at any point \( u \) the tensor \( \sigma(u) \) defines a triple Jordan system \([7, 8]\).

**Conjecture 1.** If a symmetric \((T = 0)\) affine connection and a tensor \( \sigma \) satisfy identities (1.4) – (1.7) and (1.8) then the corresponding\(^3\) system (1.3) possesses infinitely many local symmetries and conservation laws.

Several integrable systems of the form (1.3) that correspond to symmetric connections can be found in [9] but no integrable models corresponding to the case \( T \neq 0 \) are known. In this paper we construct examples of integrable models (1.3) such that \( T \neq 0 \) and \( R = 0 \).

Our goal is to find all non-triangular integrable systems of the form (1.1), which belong to a special class of vector isotropic equations of the form

\[
\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x + f_0 \mathbf{u}, \tag{1.9}
\]

where \( \mathbf{u}(x, t) \) is an \( N \)-dimensional vector and the coefficients \( f_i \) are supposed to be functions of the following six independent scalar products:

\[
(u, u), \quad (u, u_x), \quad (u_x, u_x), \quad (u, u_{xx}), \quad (u_x, u_{xx}), \quad (u_{xx}, u_{xx}). \tag{1.10}
\]

Equations (1.9) are invariant with respect to the orthogonal group \( O_N \).

It is clear that any equation (1.9) whose component form belong to the class of equations (1.1) has the following structure:

\[
\mathbf{u}_t = \mathbf{u}_{xxx} + a_1 u_{[0,1]} \mathbf{u}_{xx} + (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[0,1]}^2) \mathbf{u}_x \\
+ (a_5 u_{[1,2]} + a_6 u_{[0,2]} u_{[0,1]} + a_7 u_{[1,1]} u_{[0,1]} + a_8 u_{[0,1]}^3) \mathbf{u}, \tag{1.11}
\]

where

\[
\mathbf{u}_{[i,j]} = (\partial_i^j \mathbf{u}, \partial_i^j \mathbf{u}), \quad 0 \leq i \leq j
\]

and the coefficients \( a_i \) are functions in one variable: \( a_i = a_i(u_{[0,0]}) \). In this case the components, the torsion and the curvature tensors for the corresponding affine connection are given by

\[
A(X, Y) = \frac{1}{3} (a_1 (u, X) Y + a_2 (u, Y) X + (a_5 (X, Y) + a_6 (u, X) (u, Y)) \mathbf{u}),
\]

\[
T(X, Y) = \frac{1}{3} (a_1 - a_2) ((u, X) Y - (u, Y) X)
\]

and

\[
R(X, Y, Z) = \frac{1}{9} (g (u, X) (u, Z) + p (X, Z) ) Y - \frac{1}{9} (g (u, X) (u, Y) + p (X, Y) ) Z + \\
\frac{r}{9} ((u, Y) (X, Z) - (u, Z) (X, Y)) \mathbf{u},
\]

\(^3\)It is clear that \( \beta(X, X, X) = \sigma(X, X, X) \).
where
\[ p = a_2 a_5 u^2 - 3a_2 + 3a_5, \quad q = a_2 a_6 u^2 + a_2^2 + 3a_6 - 6a'_2, \quad r = a_5 a_6 u^2 + a_5^2 - 3a_6 + 6a'_5. \]

To find all integrable equations (1.11), we use a version of the symmetry approach developed in [10] for vector equations.

In Section 2 we discuss necessary conditions [10] of the existence of higher symmetries for vector equations of the form (1.9). In Section 3 we present lists of integrable equations (1.11), formulate and prove classification statements. For some of these equations written in components of the vector \( u \) the torsion \( T \) is not zero. To justify the real integrability of equations found in Section 3, we detect (see Section 4) auto-Bäcklund transformations for these equations. Each of them is a new integrable semi-discrete model.

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2 Integrability conditions for vector equations

It was shown in [10] that if an equation of the form (1.9) has infinitely many vector higher symmetries

\[ u_\tau = f_n u_n + f_{n-1} u_{n-1} + \cdots + f_0 u, \quad \text{where} \quad u_k = \frac{\partial^k u}{\partial x^k}, \tag{2.1} \]

then an infinite series of special local\(^4\) conservation laws exists for equation (1.9). Their densities \( \rho_n, n = 0, 1, \ldots \) are called canonical.

The first two canonical densities are given by

\[ \rho_0 = -\frac{1}{3} f_2, \tag{2.2} \]
\[ \rho_1 = \frac{1}{9} f_2^2 - \frac{1}{3} f_1 + \frac{1}{3} D_x f_2. \tag{2.3} \]

Using a technique developed in the papers [11, 12], one can obtain the following recursion formula for other canonical densities for equations of the form (1.9):

\[ \rho_{n+2} = \frac{1}{3} \left[ \theta_n - f_0 \delta_{n,0} - 2 f_2 \rho_{n+1} - f_2 D_x \rho_n - f_1 \rho_n \right] \]
\[ - \frac{1}{3} \left[ f_2 \sum_{s=0}^{n} \rho_s \rho_{n-s} + \sum_{0 \leq s + k \leq n} \rho_s \rho_k \rho_{n-s-k} + \frac{1}{3} \sum_{s=0}^{n+1} \rho_s \rho_{n-s+1} \right] \]
\[ - D_x \left[ \rho_{n+1} + \frac{1}{2} \sum_{s=0}^{n} \rho_s \rho_{n-s} + \frac{1}{3} D_x \rho_n \right] \quad n \geq 0. \tag{2.4} \]

\(^4\)A conservation law \( D_t(\rho) = D_x(\theta) \) is called local if \( \rho \) and \( \theta \) are functions of variables (1.12).
Here the symbol $\delta_{i,j}$ denotes the Kronecker delta and the functions $\theta_i$ are fluxes of the canonical conservation laws

$$D_t \rho_n = D_x \theta_n, \quad n = 0, 1, 2, \ldots$$

(2.5)

In this formula $D_x$ and $D_t$ are the total derivatives of $x$ and $t$, respectively. For brevity, we call relation (2.5) $\rho_n$-integrability condition.

Using formulas (2.2)–(2.4), one can obtain the next density

$$\rho_2 = -\frac{1}{3} f_0 + \frac{1}{3} \theta_0 - \frac{2}{81} f_2^3 + \frac{1}{9} f_1 f_2 - D_x \left( \frac{1}{9} f_2^3 + \frac{2}{9} D_x f_2 - \frac{1}{3} f_1 \right)$$

(2.6)

and so on. Notice that the density $\rho_n$, $n \geq 2$ depends on the coefficients of (1.9) and on the fluxes $\{\theta_0, \theta_1, \ldots, \theta_{n-2}\}$. These fluxes are to be calculated from the previous conditions (2.5).

To eliminate the function $\theta_n$ from (2.5) one can apply the variational derivative

$$\frac{\delta}{\delta u} = \sum_{0 \leq i \leq j} \left[ (D_x)^i \left( u_j \frac{\partial}{\partial u_{[i,j]}} \right) + (D_x)^j \left( u_i \frac{\partial}{\partial u_{[i,j]}} \right) \right].$$

(2.7)

to both sides of (2.5) and use the fact that

$$\frac{\delta (D_x g)}{\delta u} = 0$$

for any function $g$ (see, for example [13], chapter 4) to obtain

$$\frac{\delta}{\delta u} (D_t \rho_n) = 0, \quad n = 1, 2, \ldots.$$  

(2.8)

Conditions (2.8) are most efficient for the cases $n = 1, 2$ since (2.2) and (2.3) do not depend on $\theta_i$.

### 3 Classification of integrable equations (1.11)

We are searching for non-triangular integrable equations of the form (1.11). In this section, integrability means the existence of an infinite sequence of higher symmetries [13, 4] of the form (2.1).

Some equations (1.9) become triangular in the spherical coordinates, which is defined by the formulas

$$u = R v, \quad |v| = 1, \quad \text{where} \quad R = |u|.$$

Let

$$v_{[i,j]} = (\partial_x^i v, \partial_x^j v), \quad i \leq j.$$

Since $v_{[0,0]} = 1$, we have $D_x(v_{[0,0]}) = 2v_{[0,1]} = 0$. Moreover, $D_x v_{[0,1]} = v_{[0,2]} + v_{[1,1]} = 0$, i.e. $v_{[0,2]} = -v_{[1,1]}$ and so on. It is clear that all variables $v_{[0,k]}$ can be expressed in terms of the variables $v_{[i,k]}$, $1 \leq i \leq k < \infty$. 


We call equation (1.9) **triangular** if it can be rewritten in the spherical coordinates as
\[
v_t = v_{xxx} + g_2 v_{xx} + g_1 v_x + g_0 v,
\]
\[
R_t = R_{xxx} + S(v_{[1,1]}, v_{[1,2]}, v_{[2,2]}, R, R_x, R_{xx}),
\]
where the coefficients \( g_i \) depend on \( v_{[1,1]}, v_{[1,2]}, v_{[2,2]} \) only.

### 3.1 Classification statements

The class of equations of the form (1.11) is invariant with respect to the point transformations of the form
\[
u = v \varphi(v_{[0,0]}).
\]

Under such a transformation the coefficient \( a_1 \) changes as follows:
\[
\tilde{a}_1(v_{[0,0]}) = 2 \varphi^{-1} \varphi'(v_{[0,0]} a_1 \varphi^2 + 3) + a_1 \varphi^2,
\]
where \( a_1(u_{[0,0]}) = a_1(v_{[0,0]} \varphi^2) \).

It easy to see that if \( a_1 = -3 u_{[0,0]}^{-1} \), then we obtain \( \tilde{a}_1 = -3 v_{[0,0]}^{-1} \). For any function \( a_1 \) different from \(-3 u_{[0,0]}^{-1}\) we can choose the function \( \varphi \) such that \( \tilde{a}_1 \) vanishes. Thus, up to the point transformations we have two non-equivalent cases:

1. \( a_1 = 0 \) and 2. \( a_1 = -\frac{3}{u_{[0,0]}} \).

**Theorem 1.** Any non-triangular integrable equation of the form (1.11) with \( a_1 = 0 \) can be reduced to one of equations from the following List 1 by a scaling of the form \( u \to \lambda u \).

**List 1.**

\[
u_t = u_{xxx} - \frac{3}{z} u_x \left( \frac{u_{[0,1]}^2}{z + u_{[0,0]}} - u_{[1,1]} \right) + \frac{3}{z} u F,
\]
where \( \lambda = 1 \) or \( \lambda = \frac{1}{2} \),

\[
u_t = u_{xxx} - \frac{3}{z} u_x \left( \frac{z u_{[0,2]} - u_{[0,0]} u_{[1,1]}}{z + u_{[0,0]}} + \frac{u_{[0,0]} u_{[1,1]}^2}{(z + u_{[0,0]})^2} \right) + \frac{3}{z} u F,
\]

\[
u_t = u_{xxx} - \frac{3}{z} u_x \left( \frac{z u_{[0,2]} - (z - u_{[0,0]} u_{[1,1]})}{z + u_{[0,0]}} + \frac{(2z - u_{[0,0]}) u_{[1,1]}^2}{2 (z + u_{[0,0]})^2} \right) + \frac{3}{z} u F,
\]

where
\[
F = u_{[0,1]} \frac{u_{[0,2]} + u_{[1,1]}}{z + u_{[0,0]}} - u_{[1,2]} - \frac{u_{[0,1]}^3}{(z + u_{[0,0]})^2}.
\]

Here \( z \neq 0 \) is an arbitrary parameter.

**Theorem 2.** Any non-triangular integrable equation of the form (1.11) with \( a_1 = -\frac{3}{u_{[0,0]}} \) can be reduced to one of the equations from the following List 2 by a point transformation of the form (3.1).
3.2 Proof of Theorem 1

Hence, allows us to write the coefficient \( q \) where

\[ F = \left( \rho \rho \right) \]

Equation (3.7) is equivalent to the equation

\[ u_t = u_{xxx} - 3u_{xx} \frac{u_{[0,1]}}{u_{[0,0]}} - 3u_x \left( \frac{u_{[1,1]}}{u_{[0,0]}} - \frac{u_{[0,1]}^2}{u_{[0,0]}^2} \right) \]

(3.5)

\[ u_t = u_{xxx} - 3u_{xx} \frac{u_{[0,1]}}{u_{[0,0]}} - \frac{3}{2} u_x \left( \frac{2u_{[0,2]}}{u_{[0,0]}} + \frac{u_{[1,1]}}{u_{[0,0]}} \right) + 3u \left( \frac{u_{[1,2]}}{u_{[0,0]}} - \frac{u_{[0,1]} u_{[1,1]}}{u_{[0,0]}^2} + \frac{4u_{[0,1]}^3}{3 u_{[0,0]}^3} \right) \]

(3.6)

\[ u_t = u_{xxx} - 3u_{xx} \frac{u_{[0,1]}}{u_{[0,0]}} - \frac{3}{2} u_x \left( \frac{2u_{[0,2]}}{u_{[0,0]}} + \frac{u_{[1,1]}}{u_{[0,0]}} - \frac{4u_{[0,1]}^2}{u_{[0,0]}^2} \right) + 3u \left( \frac{u_{[1,2]}}{u_{[0,0]}} - \frac{u_{[0,1]} u_{[1,1]}}{u_{[0,0]}^2} + \frac{4u_{[0,1]}^3}{3 u_{[0,0]}^3} \right) \]

(3.7)

Remark 1. Equation (3.7) is equivalent to the equation

\[ u_t = u_{xxx} - 3u_{xx} \frac{u_{[0,1]}}{u_{[0,0]}} - \frac{3}{2} u_x \left( \frac{2u_{[0,2]}}{u_{[0,0]}} + \frac{u_{[1,1]}}{u_{[0,0]}} - \frac{4u_{[0,1]}^2}{u_{[0,0]}^2} \right) + 3u \left( \frac{u_{[1,2]}}{u_{[0,0]}} - \frac{u_{[0,1]} u_{[1,1]}}{u_{[0,0]}^2} + \frac{4u_{[0,1]}^3}{3 u_{[0,0]}^3} \right) \]

(3.8)

found in [5, formula (59)].

Remark 2. Using the formulas from Introduction, one can verify that for equations (3.2) and (3.7) the torsion \( T \) is equal to zero while for equations (3.3)–(3.6) we have \( T \neq 0 \), \( R = 0 \).

3.2 Proof of Theorem 1

The equation under consideration is the following:

\[ u_t = u_{xxx} + (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[0,1]}^2) u_x + (a_5 u_{[1,2]} + a_6 u_{[0,2]} u_{[0,1]} + a_7 u_{[1,1]} u_{[0,1]} + a_8 u_{[0,1]}^3) u. \]

(3.9)

For such equations the canonical densities (2.2), (2.3), and (2.6) are given by

\[ \rho_0 = 0, \quad \rho_1 = -\frac{1}{3} (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[0,1]}^2), \]

\[ \rho_2 = -\frac{1}{3} (a_5 u_{[1,2]} + a_6 u_{[0,2]} u_{[0,1]} + a_7 u_{[1,1]} u_{[0,1]} + a_8 u_{[0,1]}^3) + \frac{1}{3} D_x (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[0,1]}^2). \]

Consider the \( \rho_1 \)-condition. The equality (2.8) with \( n = 1 \) has the form \( \sum_{i=0}^4 q_i u_i = 0 \), where

\[ q_4 = 4 u_{[0,1]} (a_2' - a_3'). \]

Hence, \( a_3 = a_2 + c_1 \), where \( c_1 \) is a constant. Eliminating \( a_3 \), we find that \( q_3 \) vanishes, which allows us to write the coefficient \( q_2 \) as

\[ q_2 = (u_{[0,3]} + 3u_{[1,2]})( (a_5 u_{[0,0]} - 3)(2a_2' - a_4) - c_1 (a_2 + a_5) ) + u_{[0,1]} (u_{[0,2]} F_1 + u_{[1,1]} F_2 + u_{[0,1]}^2 F_3), \]

where \( F_i = F_i(u_{[0,0]}) \). The functions \( F_i \) are too cumbersome to be shown explicitly here while the difference \( F_1 - F_2 \) is very short:

\[ F_1 - F_2 = c_1 (a_2' + 2a_6 - 2a_7). \]
Thus from (2.8) with \( n = 1 \) we have obtained three simple relations

\[
a_3 = a_2 + c_1, \quad (a_5u_{[0,0]} - 3)(2a'_2 - a_4) - c_1(a_2 + a_5) = 0, \quad c_1(a'_5 + 2a_6 - 2a_7) = 0. \quad (3.10)
\]

Consider now the \( \rho_2 \)-condition. We obtain

\[
\frac{\delta}{\delta u}(D_t \rho_2) = \sum_{i=0}^{4} p_i u_i = 0,
\]

where

\[
p_4 = \left( (u_{[0,2]} + u_{[1,1]}) (a'_5 + a_6 - a_7) + 2u^2_{[0,1]}(a''_5 + a''_6 - a'_7) \right).
\]

Equating \( p_4 \) to zero, we find \( a_7 = a'_5 + a_6 \) and conclude that this implies \( p_3 = 0 \). Substituting \( a_7 \) into third of equations (3.10), we obtain that \( c_1a'_5 = 0 \). Equating now \( p_0 \) to zero, we find one more simple relation \( a'_6 - a_8 = 0 \). So the \( \rho_2 \)-condition implies

\[
a_7 = a'_5 + a_6, \quad c_1(a_6 - a_7) = 0, \quad a'_6 - a_8 = 0. \quad (3.11)
\]

Several more useful relations can be derived from the \( \rho_4 \)-condition. The density \( \rho_4 \) has the following structure:

\[
\rho_4 = \frac{1}{3} \theta_2 - \frac{1}{3} D_x(\theta_1) + R,
\]

where \( R \) does not depend on \( \theta_1 \) and \( \theta_2 \). The term with \( \theta_1 \) disappears when we apply the variational derivative in the formula (2.8) with \( n = 4 \). So to use the \( \rho_4 \)-condition, we have to specify the form of the function \( \theta_2 \) only.

Using (3.11), we obtain that \( \rho_2 \) is trivial: \( \rho_2 = D_x(S) \), where

\[
S = \frac{1}{6} \left( 2a_4u^2_{[0,1]} - a_6u^2_{[0,1]} + 2a_2u_{[0,2]} + 2a_2u_{[1,1]} - a_5u_{[1,1]} + 2c_1u_{[1,1]} \right).
\]

Therefore, \( \theta_2 = D_x(S) \). Taking into account this expression for \( \theta_2 \), we find that

\[
\frac{\delta}{\delta u}(D_t \rho_4) = \sum_{i=0}^{6} r_i u_i,
\]

where

\[
r_6 = 2 \left( 2a''_5u^2_{[0,1]} + a'_5(u_{[0,2]} + u_{[1,1]}) \right).
\]

This means that \( a_5 = c_2 \), where \( c_2 \) is a constant. Substituting \( a_5 = c_2 \) into the second equation of (3.10), we obtain

\[
(c_2u_{[0,0]} - 3)(2a'_2 - a_4) + (a_2 + c_2)c_1 = 0, \quad \text{or} \quad a_4 = 2a'_2 - \frac{c_1(a_2 + c_2)}{c_2u_{[0,0]} - 3}. \quad (3.12)
\]
Using (3.10), (3.11), (3.12), we express all coefficients in (3.9) in terms of \(a_2, a_6, c_1, c_2\). Then the coefficient \(r_5\) vanishes and \(r_4\) turns into

\[
r_4 = (u_{[0,4]} + 4u_{[1,3]})(a_6c_2u_{[0,0]} + c_2^2 - 3a_6) + 4u_{[0,1]}u_{[0,3]}c_1(a_2'(c_2u_{[0,0]} - 3) + a_2c_2).
\]

The equation \(r_4 = 0\) is then equivalent to relations

\[
a_6 = -\frac{c_2^2}{c_2u_{[0,0]} - 3} \quad \text{and} \quad c_1\left(a_2'(c_2u_{[0,0]} - 3) + a_2c_2\right) = 0 \quad (3.13)
\]

and we have proved the following:

**Lemma 1.** Any integrable equation (3.9) has the form

\[
\frac{\partial u}{\partial t} = u_{xxx} + u_x \left( a_2u_{[0,2]} + (a_2 + c_1)u_{[1,1]} + \left( 2a_2' - c_1\frac{a_2 + c_2}{c_2u_{[0,0]} - 3}\right)u_{[0,1]}^2 \right) + \\
u \left( c_2u_{[1,2]} - c_2^2u_{[0,1]} \frac{u_{[2,0]} + u_{[1,1]}}{c_2u_{[0,0]} - 3} + \frac{c_2^3u_{[0,1]}^3}{(c_2u_{[0,0]} - 3)^2} \right),
\]

where \(a_2 = a_2(u_{[0,0]})\) and \(c_i\) are constants.

Let us consider the following two branches

**A.** \(c_1 = 0\) and **B.** \(c_1 \neq 0\).

In Case **A** the \(\rho_5\)-condition leads to \(c_2 = 0\) and \(a_2 = 0\) and the linear equation \(u_t = u_3\) appears.

Case **B** we separate into two following subcases:

**B.1.** \(a_2 = 0\) and **B.2.** \(a_2 \neq 0\).

Consider Case **B.1.** If \(c_2 = 0\), then we arrive at the equation \(u_t = u_3 + u_1c_1u_{[1,1]}\). This equation is not integrable since the \(\rho_5\)-condition leads to a contradiction.

If \(c_2 \neq 0\), then equation (3.14) coincides with equation (3.2), where

\[
c_2 = -\frac{3}{\lambda}, \quad c_1 = -\frac{3\lambda}{\lambda}.
\]

The \(\rho_5\)-condition gives rise to the following equation

\[
(2\lambda - 1)(\lambda - 1) = 0.
\]

Consider Case **B.2.** The coefficient \(r_0\) in the \(\rho_4\)-condition is given by

\[
r_0 = -\frac{4}{9}u_{[0,1]}u_{[1,6]}c_1(3a_2'^2 - a_2^2) - \frac{4u_{[0,2]}u_{[0,6]}}{3(c_2u_{[0,0]} - 3)^2}a_2c_1(a_2(c_2u_{[0,0]} - 3) + 3c_2).
\]
Therefore,

\[ c_1 (3 a'_2 - a_2^2) = 0, \quad a_2 c_1 \left( a_2 (c_2 u_{[0,0]} - 3) + 3 c_2 \right) = 0 \]

and we have

\[ a_2 = - \frac{3 c_2}{c_2 u_{[0,0]} - 3}. \]

Then \( \rho_5 \)-condition provides the following equation

\[ (c_1 + c_2)(2 c_1 + c_2) = 0. \]

The two possibilities \( c_1 = -c_2 \) and \( c_1 = -\frac{c_2}{2} \) correspond to equations (3.3) and (3.4), where \( c_2 = -\frac{3}{z} \). \( \Box \)

**Remark 3.** We have verified that all equations of List 1 satisfy the \( \rho_n \)-conditions with \( n \leq 7 \). It turns out that \( \rho_n \), where \( n = 0, 2, 4, 6 \), are total \( x \)-derivatives. In accordance with a general statement from [10] this is an indication of the existence of infinite series of local conservation laws. The canonical conservation laws, corresponding to \( n = 1, 3, 5, 7 \), have the orders 1, 2, 3, 4, respectively. Moreover, each equation from List 1 possesses a fifth order symmetry.

### 3.3 Proof of Theorem 2

Consider equations of the form

\[ u_t = u_{xxx} - 3 \frac{u_{[0,1]}}{u_{[0,0]}} u_{xx} + (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[2,1]}^2) u_x + \]

\[ (a_5 u_{[1,2]} + a_6 u_{[2,0]} u_{[0,1]} + a_7 u_{[1,1]} u_{[0,0]} + a_8 u_{[0,1]}^3) u. \]  \( \text{(3.15)} \)

The simplest canonical densities (2.2), (2.3), and (2.6) are given by

\[ \rho_0 = \frac{1}{2} D_x(u_{[0,0]}), \quad \rho_1 = \frac{u_{[0,1]}^2}{u_{[0,0]}} - \frac{1}{3} (a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[2,1]}^2) - D_x \left( \frac{u_{[0,1]}}{u_{[0,0]}} \right), \]

\[ \rho_2 = \frac{1}{3} D_x \left( \frac{u_{[0,2]}}{u_{[0,0]}} - \frac{7}{2} \frac{u_{[1,1]}^2}{u_{[0,0]}} - \frac{1}{2} \frac{u_{[1,1]}}{u_{[0,0]}} + 2 D_x \left( \frac{u_{[0,1]}}{u_{[0,0]}} \right) + a_2 u_{[0,2]} + a_3 u_{[1,1]} + a_4 u_{[2,1]}^2 \right). \]

Using the same line of reasoning as in Section 3.2, we derive short relations from the \( \rho_1 - \rho_5 \) conditions. Namely, it follows from the \( \rho_1 \)-condition that

\[ a_3 = a_2 + c_1 u_{[0,0]}^{-1}, \]

\( \text{(3.16)} \)

\[ (a_5 u_{[0,0]} - 3)(2 u_{[0,0]} a'_2 - a_4 u_{[0,0]}^2 + 3) = c_1 u_{[0,0]} (a_2 + a_5), \]

\( \text{(3.17)} \)

where \( c_1 \) is a constant. The \( \rho_3 \) and \( \rho_4 \)-conditions implies

\[ c_1 \left( 6 a'_2 u_{[0,0]} - (a_2 u_{[0,0]} + 3) (a_6 u_{[0,0]} + a_5) - a_2 (a_2 u_{[0,0]} - 3) \right) = 0, \]

\( \text{(3.18)} \)
and the $\rho_5$-condition leads to the following relations:

\[
(a_5 a_2 u_{[0,0]} + 3) - 3 a_2 = (c_1 + 3)(2c_1 + 3),
\]

\[
2c_1 u_{[0,0]}(a_5 a_2 u_{[0,0]} + 3) - 3 a_2 = (c_1 + 3)(2c_1 + 3),
\]

It follows from (3.21) that $c_1 \neq 0$ and we may reduce (3.18) and (3.19) by the factor $c_1$.

Let us simplify the equation (3.15) by an appropriate point transformation of the form (3.1).

It is more convenient for computations to rewrite it as

\[
\mathbf{u} = \left( \frac{f}{v_{[0,0]}} \right)^{1/2} \mathbf{v}, \quad u_{[0,0]} = f(v_{[0,0]}).
\]

One can verify that under this transformation the coefficient $a_2$ transforms as

\[
\tilde{a}_2 = \frac{\partial f}{\partial v_{[0,0]}} \frac{a_2 f + 3}{f} - \frac{3}{v_{[0,0]}}.
\]

It follows from this formula that we can reduce $a_2$ to zero with the exception of the case $a_2(u_{[0,0]}) = -\frac{3}{u_{[0,0]}}$.

**Case A:** $a_2 = 0$. It follows from (3.20), (3.18) and (3.19) that $a_5 = a_6 = a_7 = 0$. From (3.17) we obtain $a_4 = \frac{3}{u_{[0,0]}^2}$. Moreover, relation (3.21) leads to $(c_1 + 3)(2c_1 + 3) = 0$. Substituting $a_i, \ 2 \leq i \leq 7$ into the $\rho_1$-condition, we obtain that $a'_8 = -3a_8 u_{[0,0]}$ or $a_8 = k/u_{[0,0]}^3$. Finally, $\rho_5$-condition gives rise to $k = 0$.

In the case $c_1 = -3$ we arrive at equation (3.5) while $c_1 = -\frac{3}{2}$ leads to equation (3.6). It follows from (3.24) that in Case A the only admissible point transformations are $\mathbf{u} \to \text{const} \mathbf{u}$ and therefore equations (3.5) and (3.6) are non-equivalent.

**Case B:** $a_2 = -\frac{3}{u_{[0,0]}}$. Taking into account (3.16), we find that the equation has the following form

\[
\mathbf{u}_t = \mathbf{u}_{xxx} - 3 \frac{u_{[0,1]}}{u_{[0,0]}} \mathbf{u}_x + \mathbf{u}_x \left( a_4 u_{[0,1]}^2 - 3 \frac{u_{[0,2]}}{u_{[0,0]}} \frac{u_{[1,1]}}{u_{[0,0]}} + (c_1 - 3) \frac{u_{[1,1]}}{u_{[0,0]}} \right) + \mathbf{u} \left( a_5 u_{[1,2]} + a_6 u_{[0,2]} u_{[0,1]} + a_7 u_{[1,1]} u_{[0,1]} + a_8 u_{[0,1]}^3 \right). \tag{3.25}
\]

Relations (3.17) and (3.21) can be rewritten as

\[
(c_1 - 3)(2c_1 - 3) = 0, \quad (a_5 u_{[0,0]} - 3) (a_4 u_{[0,0]}^2 + c_1 - 9) = 0. \tag{3.26}
\]

For equations of the form (3.25) the $\rho_1$-condition provides the following additional relations:

\[
(a_4 u_{[0,0]}^2 + c_1 - 9)(a_6 u_{[0,0]}^2 - a_7 u_{[0,0]}^2 - 3) = 0, \tag{3.27}
\]

\[
a'_4 = \frac{a_6}{3u_{[0,0]}} (a_4 u_{[0,0]}^2 + c_1 - 9) - 2 \frac{a_4}{u_{[0,0]}}. \tag{3.28}
\]
Moreover it follows from the $\rho_5$-condition that
\[
(a_4u_{[0,0]}^2 + c_1 - 9)(2c_1 - 3) = 0, \quad (3.29)
\]
\[
(a_4u_{[0,0]}^2 + c_1 - 9)(u_{[0,0]}^4a_4^2 + 6a_4^2u_{[0,0]}^3 - 9a_8u_{[0,0]}^3 - 6a_4u_{[0,0]}^2 - 6a_6u_{[0,0]}^2 + 36) = 0. \quad (3.30)
\]

Under transformations (3.23) the coefficient $a_4$ changes as follows:
\[
\tilde{a}_4 = \frac{f'^2}{f^2} (a_4f^2 + c_1 - 9) + \frac{9 - c_1}{u_{[0,0]}^2}.
\]
The condition $\tilde{a}_4 = 0$ is a differential equation for $f$, which has a non-constant solution except for the case $a_4u_{[0,0]}^2 + c_1 - 9 = 0$. So we arrive at the following two cases:

**B.1.** $a_4 = 0$ and **B.2.** $a_4 = \frac{9 - c_1}{u_{[0,0]}^2}$.

In the case **Case B.1.** it follows from (3.26) – (3.30) that
\[
a_4 = 0, \quad a_5 = \frac{3}{u_{[0,0]}}, \quad a_6 = 0, \quad a_7 = \frac{3}{u_{[0,0]}^2}, \quad a_8 = \frac{4}{u_{[0,0]}^3}, \quad c_1 = \frac{3}{2}.
\]

Substituting all these coefficients into equation (3.25), we obtain equation (3.7).

Consider the **Case B.2.** According to equations (3.26) and (3.28) we have the following equation:
\[
\mathbf{u}_t = \mathbf{u}_3 - 3\mathbf{u}_2\frac{u_{[0,1]}}{u_{[0,0]}} + \mathbf{u}_1 \left( (9 - c_1)\frac{u_{[0,1]}^2}{u_{[0,0]}^2} - 3\frac{u_{[0,2]}}{u_{[0,0]}} + (c_1 - 3)\frac{u_{[1,1]}}{u_{[0,0]}} \right)
\]
\[+ \mathbf{u} \left( a_5u_{[1,2]} + a_6u_{[0,1]}u_{[0,2]} + a_7u_{[0,1]}u_{[1,1]} + a_8u_{[0,1]}^3 \right), \quad (3.31)
\]
where $(c_1 - 3)(2c_1 - 3) = 0$. It can be verified that in the spherical coordinates equation (3.31) has the form
\[
\mathbf{v}_t = \mathbf{v}_3 + c_1\mathbf{v}_1v_{[1,1]} + 3\mathbf{v}v_{[1,2]}, \quad (c_1 - 3)(2c_1 - 3) = 0, \quad (3.32)
\]
\[
R_t = R_3 + \frac{R_1R_2}{R}(R^4b_6 + R^2b_5 - 6) - R_1v_{[1,1]}(R^4b_6 - R^4b_7 - R^2b_5 - c_1)
\]
\[+ \frac{R_3}{R^2}(R^6b_8 + R^4b_7 + 6) + Rv_{[1,2]}(R^2b_5 - 3), \quad (3.33)
\]
where $b_i(R) = a_i(u_{[0,0]}) \equiv a_i(R^2)$. So, the system (3.31) is triangular. \(\square\)

**Remark 4.** Both equations (3.32) are integrable equations on the sphere [10]. They have infinitely many conservation laws depending on the variables $v_{[i,j]}$. This is a reason why all conditions from Section 2 are satisfied for any functions $a_5 - a_8$. However, we can use the geometric integrability conditions (1.4) - (1.8) for the classification of triangular systems (3.31) (see Appendix 6).
Remark 5. It turns out that the equations of List 1 can be simplified by the point transformation (3.23) with
\[ f = \frac{z v_{[0,0]}}{a - v_{[0,0]}}, \]
where \( a \neq 0 \) is an arbitrary constant. As a result, the coefficients of \( u \) vanish and the equations (3.2), (3.3) and (3.4) transform to equations
\[
\dot{u}_t = u_3 - 3u_2 \frac{w_{[0,1]}}{a + w_{[0,0]}} - 3u_1 \left( \frac{w_{[0,2]}}{a + w_{[0,0]}} - \frac{w_{[1,1]}(\lambda - 1)}{a + w_{[0,0]}} + \frac{w_{[2,1]}(\lambda - 3)}{(a + w_{[0,0]})^2} \right), \tag{3.34}
\]
where \( \lambda = 1 \) or \( \lambda = \frac{1}{2} \),
\[
\dot{u}_t = u_3 - 3u_2 \frac{w_{[0,1]}}{a + w_{[0,0]}} - 3u_1 \left( \frac{w_{[1,1]}}{a + w_{[0,0]}} - \frac{w_{[2,1]}}{(a + w_{[0,0]})^2} \right), \tag{3.35}
\]
and
\[
\dot{u}_t = u_3 - 3u_2 \frac{w_{[0,1]}}{a + w_{[0,0]}} - \frac{3}{2} u_1 \left( \frac{w_{[1,1]}}{a + w_{[0,0]}} - \frac{2w_{[2,1]}}{(a + w_{[0,0]})^2} \right), \tag{3.36}
\]
respectively. Equations written in this form appeared in [14] (see formulas (3.12), (3.13), (3.16) and (3.17)).

4 Auto-Bäcklund transformations

An auto-Bäcklund transformation of the first order for a vector equation of the form (1.9) is defined by the formula
\[
\dot{u}_x = h \, v_x + f \, u + g \, v,
\]
where \( u \) and \( v \) are solutions of (1.9). The functions \( f, g \) and \( h \) are (scalar) functions of variables
\[
w_{[0,0]} \overset{\text{def}}{=} (u, u), \quad v_{[i,j]} \overset{\text{def}}{=} (v_i, v_j), \quad w_{i,j} \overset{\text{def}}{=} (u_i, v_j), \quad i, j \geq 0.
\]

Remark 6. If the auto-Bäcklund transformation depends on an arbitrary parameter \( \mu \), one can construct exact multi-parameter solutions of equation (1.9) by applying the transformation several times to a trivial solution.

Remark 7. The existence of a vector auto-Backlund transformation with an arbitrary parameter is the most easily verifiable evidence for the integrability of a vector equation.

For equations (3.2)–(3.4) we use the canonical forms (3.34)–(3.36) since the auto-Bäcklund transformations for them look more elegant.
The auto-Bäcklund transformations for equation (3.34) with \( \lambda = 1 \) and with \( \lambda = 1/2 \) are given by the formulas

\[
\mathbf{u}_x = \frac{p}{q} \mathbf{v}_x + \frac{p(q w_{[0,1]} - p v_{[0,1]})}{q^2(a - pq + w_{[0,0]})}(\mathbf{u} - \mathbf{v}) - a \frac{p}{q}(\mathbf{u} - \mathbf{v}),
\]

(4.1)

and

\[
\mathbf{u}_x = \frac{p}{q} \mathbf{v}_x + \frac{p(q w_{[0,1]} - p v_{[0,1]})}{q^2(a - pq + w_{[0,0]})}(\mathbf{u} - \mathbf{v}) + \frac{\mu p^{3/2}(\mathbf{u} - \mathbf{v})}{q^{1/2}(a - pq + w_{[0,0]})},
\]

(4.2)

respectively. Here \( p = \sqrt{u_{[0,0]} + a} \), \( q = \sqrt{v_{[0,0]} + a} \) and \( \mu \) is an arbitrary parameter.

The auto-Bäcklund transformations for equations (3.35) and (3.36) have the following form:

\[
\mathbf{u}_x = \frac{p}{q} \mathbf{v}_x + \frac{\mu p (p - q)(\mathbf{u} - \mathbf{v})}{a - pq + w_{[0,0]}},
\]

(4.3)

and

\[
\mathbf{u}_x = \frac{p}{q} \mathbf{v}_x + \frac{\mu p(\mathbf{u} - \mathbf{v})}{\sqrt{a - pq + w_{[0,0]}}},
\]

(4.4)

The auto-Bäcklund transformations for equation (3.5), (3.6) and (3.8) have the following form:

\[
\mathbf{u}_x = \frac{\sqrt{u_{[0,0]}}}{\sqrt{v_{[0,0]}}} \mathbf{v}_x + \frac{\sqrt{v_{[0,0]}}}{\sqrt{v_{[0,0]}}}(w_{[0,1]} - \sqrt{u_{[0,0]} v_{[0,1]}})(\mathbf{u} \sqrt{v_{[0,0]}} - \mathbf{v} \sqrt{u_{[0,0]}}) + \mu \sqrt{u_{[0,0]}} \sqrt{v_{[0,0]}} \mathbf{u},
\]

(4.5)

\[
\mathbf{u}_x = \frac{\sqrt{u_{[0,0]}}}{\sqrt{v_{[0,0]}}} \mathbf{v}_x + \frac{\sqrt{v_{[0,0]}}}{\sqrt{v_{[0,0]}}}(w_{[0,1]} - \sqrt{u_{[0,0]} v_{[0,1]}})(\mathbf{u} \sqrt{v_{[0,0]}} - \mathbf{v} \sqrt{u_{[0,0]}}) + \mu |u_{[0,0]}|^{1/4} |v_{[0,0]}|^{1/4} \mathbf{u},
\]

(4.6)

and

\[
\mathbf{u}_x = \frac{\sqrt{u_{[0,0]}}}{\sqrt{v_{[0,0]}}} \mathbf{v}_x + \mu \mathbf{u} \left( \sqrt{u_{[0,0]} \sqrt{v_{[0,0]}} - w_{[0,0]}} \right)^{1/2} + \mu \frac{u_{[0,0]} \mathbf{v} - w_{[0,0]} \mathbf{u}}{\sqrt{u_{[0,0]} \sqrt{v_{[0,0]}} - w_{[0,0]}}}^{1/2} + \frac{(\mathbf{u} \sqrt{v_{[0,0]}} - \mathbf{v} \sqrt{u_{[0,0]}})(\sqrt{u_{[0,0]} v_{[0,1]}} - \sqrt{v_{[0,0]} w_{[0,1]}})}{\sqrt{u_{[0,0]} \sqrt{v_{[0,0]}} - w_{[0,0]}}},
\]

(4.7)

respectively.

5 Appendix. Geometric properties of equations with \( T = 0 \)

The affine connections that correspond to equation (3.8) and to two equations (3.34) have zero torsion: \( T = 0 \). We verified that they satisfy the integrability conditions (1.4)-(1.8). In the appendix we present explicit formulas for these equations.
Example 2. In the equations (3.34) we have

\[ A(X, Y) = -\frac{(u, X) Y + (u, Y) X}{a + u_{[0,0]}}, \]

where \( u_{[0,0]} = (u, u) \). One can check that this connection is the Levi-Civita affine connection of the metric

\[ g(X, Y) = \frac{(X, Y)}{a + u_{[0,0]}} - \frac{(u, X)(u, Y)}{(a + u_{[0,0]})^2}. \]

The tensors \( \beta, R \) and \( \sigma \) can be expressed in terms of \( g \) as follows:

\[ \beta(X, Y, Z) = \frac{3\lambda - 1}{3} \left( g(X, Y) Z + g(Y, Z) X + g(Z, X) Y \right), \]

\[ R(X, Y, Z) = g(X, Z) Y - g(X, Y) Z^5, \]

and

\[ \sigma(X, Y, Z) = \lambda g(X, Y) Z + \lambda g(Z, Y) X + (\lambda - 1) g(X, Z) Y. \quad (5.1) \]

It can be easily verified that for any bi-linear form \( g \) formula (5.1) defines a triple Jordan system iff \( \lambda = 1 \) or \( \lambda = \frac{1}{2} \) (cf. (3.34)). Both these triple systems are known to be simple [8].

Example 3. An elegant description [9] of all geometric objects for equation (3.8) can be done in terms of the simple triple Jordan system (cf. (5.1))

\[ S(X, Y, Z) \overset{\text{def}}{=} (X, Y) Z + (Z, Y) X - (X, Z) Y. \]

We have

\[ A(X, Y) = S(X, F, Y), \quad \text{where} \quad F \overset{\text{def}}{=} -\frac{u}{u_{[0,0]}}, \]

\[ \sigma(X, Y, Z) = -\frac{1}{2} S(X, S(F, Y, F), Z), \]

\[ R(X, Y, Z) = \sigma(X, Z, Y) - \sigma(X, Y, Z), \quad \beta(X, X, X) = \sigma(X, X, X). \]

The tensor \( \beta(X, Y, Z) \) can be obtained from \( \beta(X, X, X) \) by the symmetrization.

6 Appendix. Integrable triangular systems

Since \( T = 0 \) for triangular systems of the form (3.31), we may use the integrability conditions (1.4) - (1.8) for the classification of triangular systems.

Lemma 2. Using a transformation of the form (3.23), we can reduce the coefficient \( a_5 \) in (3.31) to

- Case a: \( a_5 = 0; \)

\footnote{This formula means that we are dealing with the space of the constant curvature \( k = 1 \).}
• Case b: \( a_5(x) = \frac{3}{x} \).

In the Case a the conditions (1.4) - (1.8) are equivalent to \( a_6 = a_7 = a_8 = a_9 = 0 \) and we arrive at the systems

\[
\mathbf{u}_t = \mathbf{u}_3 - 3\mathbf{u}_2 \frac{u_{[0,1]}}{u_{[0,0]}} + \mathbf{u}_1 \left( (9 - c_1) \frac{u_{[0,1]}^2}{u_{[0,0]}^2} - 3 \frac{u_{[0,2]}}{u_{[0,0]}} + (c_1 - 3) \frac{u_{[1,1]}}{u_{[0,0]}} \right), \quad c_1 = 3, \frac{3}{2}
\]

In the Case b we may use transformations (3.23) to vanish \( a_6 \). Transformations (3.23) with

\[
f(x) = k_1 x^{k_2},
\]

where \( k_i \) are arbitrary constants, preserve the normalization \( a_6 = 0 \). From conditions (1.4) - (1.8) it follows that

**Case b_1**: \( a_7(x) = -\frac{3}{x^2} \) or **Case b_2**: \( a_7(x) = -\frac{c_1 + 3}{x^2} \).

In the Case b_1 conditions (1.4) - (1.8) imply \( c_1 = \frac{3}{2} \) and \( a_8(x) = -\frac{1}{x^3} \) and we obtain the equation

\[
\mathbf{u}_t = \mathbf{u}_3 - 3\mathbf{u}_2 \frac{u_{[0,1]}}{u_{[0,0]}} + \mathbf{u}_1 \left( \frac{15}{2} \frac{u_{[0,1]}^2}{u_{[0,0]}^2} - 3 \frac{u_{[0,2]}}{u_{[0,0]}} + \frac{3}{2} \frac{u_{[1,1]}}{u_{[0,0]}} \right) + \mathbf{u} \left( 3 \frac{u_{[1,2]}}{u_{[0,0]}} - 3 \frac{u_{[0,1]} u_{[1,1]}}{u_{[0,0]}^2} - \frac{u_{[0,1]}^3}{u_{[0,0]}^2} \right).
\]

This equation is invariant with respect to the group of transformations (3.23), (6.1).

In the Case b_2 we get \( a_8 = \frac{k}{x^3} \), where \( k \) is a constant. By a transformation (3.23), (6.1) we can bring \( k \) to zero. As a result we obtain

\[
\mathbf{u}_t = \mathbf{u}_3 - 3\mathbf{u}_2 \frac{u_{[0,1]}}{u_{[0,0]}} + \mathbf{u}_1 \left( (9 - c_1) \frac{u_{[0,1]}^2}{u_{[0,0]}^2} - 3 \frac{u_{[0,2]}}{u_{[0,0]}} + (c_1 - 3) \frac{u_{[1,1]}}{u_{[0,0]}} \right) + \\
\mathbf{u} \left( 3 \frac{u_{[1,2]}}{u_{[0,0]}} - (c_1 + 3) \frac{u_{[0,1]} u_{[1,1]}}{u_{[0,0]}^2} \right), \quad c_1 = 3, \frac{3}{2}.
\]

Both of these equations admit a total separation of variables in the spherical coordinates: the equation with \( c_1 = 3 \) is converted to

\[
\mathbf{v}_t = \mathbf{v}_{xxx} + 3\mathbf{v}_x v_{[1,1]} + 3\mathbf{v} v_{[1,2]}, \quad R_t = R_{xxx} - 3 \frac{R_x R_{xx}}{R}
\]

while the equation with \( c_1 = \frac{3}{2} \) turns into

\[
\mathbf{v}_t = \mathbf{v}_{xxx} + \frac{3}{2} \mathbf{v}_x v_{[1,1]} + 3\mathbf{v} v_{[1,2]}, \quad R_t = R_{xxx} - 3 \frac{R_x R_{xx}}{R} + \frac{3}{2} \frac{R_x^3}{R^2}
\]

In both cases the scalar equation for \( R \) is point equivalent to the integrable equation \( \tilde{R}_t = \tilde{R}_{xxx} + \tilde{R}_x^3 \).

The equation from Case b_1 and the equations from Case a admit a partial separation of variables in the spherical coordinates.
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