Hilbert spaces of entire functions with trivial zeros

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Abstract

Let $H$ be a Hilbert space of entire functions. Let $H'$ be the space of the functions $f(z)/\prod_i(z - z_i)$ where $f$ belongs to $H$ and vanishes at $n$ given complex points $z_i$. We compute a suitable $E$ function for $H'$ when one is given for $H$.

1 Hilbert space of entire functions with vanishing conditions

Let $H$ be an Hilbert space, whose vectors are entire functions, and such that the evaluations at complex numbers are continuous linear forms, hence correspond to specific vectors $Z_z$:

$$Z_z \in H, \quad \forall f \in H \quad (Z_z, f) = f(z)$$

Scalar products $(g, f)$ in this paper will be complex linear in $f$ and conjugate linear in $g$. Using the Banach-Steinhaus theorem we know that evaluations of derivatives $f \mapsto f^{(k)}(z)$ are also bounded linear forms on the Hilbert space $H$. We will write $Z_{z,k}$ for the corresponding vectors.

If $H$ satisfies the axiomatic framework of $[\Pi]$ (and is not the zero space), there is an entire function $E$ (not unique) with the property:

$$\text{Im}(z) > 0 \implies |E(z)| > |E(\overline{z})|$$

and in terms of which the evaluators are given as:

$$Z_z(w) = \frac{E(z)E(w) - E^*(z)E^*(w)}{i(\overline{z} - w)}$$
We have written $E^*$ for the function $w \mapsto E(\overline{w})$.

Conversely to each entire function satisfying (2) there is associated a Hilbert space $H(E)$ with evaluators given by (3): the elements of $H(E)$ are the entire functions $f$ such that both $f/E$ and $f^*/E$ belong to the Hardy space of the upper half-plane ($f^*(z) = \overline{f(z)}$). A basic instance of this theory is the Paley-Wiener space $H = PW_x$ ($x > 0$) of entire functions, square integrable on the real line, and of exponential type at most $x$. For these classical Paley-Wiener spaces we use the scalar product $(f, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(z)|^2 \, dz$, and the presence here of a $\frac{1}{2\pi}$ is related to its absence in (3); we find this more convenient. An appropriate $E$ function for $PW_x$ is $E(z) = e^{-ixz}$.

One of the axioms of (1) is: (1) if $f \in H$ verifies $f(z_0) = 0$ then $\frac{1}{z-z_0} f(z)$ belongs to $H$ and has the same norm as $f$. The others are: (2) $H$ is a Hilbert space and evaluating at a complex point is a bounded linear form, and (3) for any $f \in H$, the function $f^*$ belongs to $H$ and has the same norm as $f$. From axiom (1) one sees that for $z_0$ non real, we can always find in $H$ (if it is not the zero space) a function not vanishing at $z_0$. So the evaluator $Z_{z_0}$ can not be zero if $z_0$ is non real (and if $H$ is not the zero space). Of course when $H$ is known as an $H(E)$, then $\text{Im}(z) \neq 0 \implies Z_z(z) = \|Z_z\|^2 > 0$ by (3) and (2).

Let $\sigma = (z_1, \ldots, z_n)$ be a finite sequence of non necessarily distinct complex numbers with associated evaluators $Z_1, \ldots, Z_n$ in $H$. More precisely, in the case where the $z_i$’s are not all distinct, we assemble in succession the indices corresponding to the same complex number, and if for example $z_1 = z_2 = z_3 \neq z_4$, we let $Z_1 = Z_{z_1}, Z_2 = Z_{z_1,1}, Z_3 = Z_{z_1,2},$ and $Z_4 = Z_{z_4}$ etc. . . Also, when $f$ is an arbitrary analytic function, we introduce a notation $f[z_j]$ such that in the example above $f[z_1] = f(z_1), f[z_2] = f'(z_2)(= f'(z_1)), f[z_3] = f''(z_3), f[z_4] = f(z_4)$ etc. . .

Let $H^\sigma$ be the closed subspace of $H$ of functions vanishing at the $z_i$’s, in other words this is the orthogonal complement to the $Z_z$’s, $1 \leq i \leq n$. For the classical Paley-Wiener spaces $H = PW_x$, such subspaces $PW_x^\sigma$ have been considered in the work of Lyubarskii and Seip (3), where however the sequences $\sigma$ arising are infinite. We restrict ourselves here to the finite case which is already interesting: as will be shown in the companion paper (2) this has given a way from the classical Paley-Wiener spaces to explicit Painlevé VI transcendent.

Let
\[
\gamma(z) = \frac{1}{(z-z_1) \cdots (z-z_n)}
\] (4)

and define $H(\sigma) = \gamma(z)H^\sigma$:
\[
H(\sigma) = \{F(z) = \gamma(z)f(z) \mid f \in H, f[z_1] = \cdots = f[z_n] = 0\}
\] (5)

We call $F(z) = \gamma(z)f(z)$ the “complete” form of $f$ (for any $f$, in $H$ or not, vanishing
at the $z_i$'s), and call $z_1, \ldots, z_n$ the “trivial zeros”. We say that we switch from the space $H$ to the space $H(\sigma)$ by adding trivial zeros, but this is of course slightly misleading as the $z_i$’s are trivial zeros only for the incomplete functions $f(z)$, not for the complete functions $F(z)$ which are vectors in the space $H(\sigma)$.

We give $H(\sigma)$ the Hilbert space structure which makes $f \mapsto F$ an isometry with $H^\sigma$. Let us note that evaluations $F \mapsto F(z)$ are again continuous linear forms on this new Hilbert space of entire functions: this is immediate if $z \notin \sigma$ and follows from the Banach-Steinhaus theorem if $z \in \sigma$. Let $F \in H(\sigma)$ with incomplete form $f$. If $F(z_0) = 0$ then $f(z_0) = 0$, with multiplicity suitably increased if $z_0$ belongs to $\sigma$. The function $g(z) = \frac{\gamma(z)}{z - z_0} F(z)$ belongs to $H$ and still vanishes on $\sigma$ (multiplicities included), hence its complete form $\frac{\gamma(z)}{z - z_0} F(z)$ belongs to $H(\sigma)$. Finally, let us consider for $F \in H(\sigma)$ its conjugate in the real axis $F^*(z) = F(\overline{z})$. With $F(z) = \gamma(z)f(z)$ we thus have $F^*(z) = \gamma^*(z)f^*(z) = \gamma(z) \prod_{1 \leq i \leq n} \frac{\gamma(z_i)}{z - z_i} f^*(z)$. But the function $f^*$ belongs to $H$ (with the same norm as $f$) and has zeros at the $z_i$’s. Hence $\prod_{1 \leq i \leq n} \frac{\gamma(z_i)}{z - z_i} f^*(z)$ belongs to $H$ with the same norm. And it has zeros at the $z_i$’s, it is thus an element of $H^\sigma$ and its complete form is an element of $H(\sigma)$ (with the same norm as $F$). This completes the verification that $H(\sigma)$ verifies the axioms of $\mathbb{I}$ if $H$ does.

**Remark 1.** Let us suppose that $z_0$ is not real. From what precedes if we can find a non-zero element $F$ in $H(\sigma)$ we can find one with $F(z_0) \neq 0$. This proves in particular that if $Z_1, \ldots, Z_n$ do not already span $H$, then any evaluator $Z_{z_0}$ with $z_0$ non-real and distinct from the $z_i$’s is not a linear combination of $Z_1, \ldots, Z_n$.

Let us give a first formula (which does not use (3) but only (1)) for the evaluators $K_z$ in $H(\sigma)$ and their scalar products $(K_w, K_z) = K_z(w)$. Let $k_z \in H^\sigma$ be the incomplete form of $K_z$, so that $K_z(w) = \gamma(w)k_z(w)$. One has to be careful that for $f \in H^\sigma$, with complete form $F$, we have by definition $(k_z, f) = (K_z, F) = F(z) = \gamma(z)f(z) = \gamma(z)(Z_z, f)$. Hence $k_z$ is $\gamma(z)$ times the orthogonal projection $\pi(Z_z)$ of $Z_z$ in $H$ onto $H^\sigma \subset H$. As is well-known, orthogonal projections can be written in Gram determinantal form:

$$k_z = \frac{\gamma(z)}{\gamma(z)} \pi(Z_z) = \frac{\gamma(z)}{G_n} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & (Z_2, Z_1) & \ldots & (Z_2, Z_n) \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & (Z_2, Z_1) & \ldots & (Z_3, Z_n) \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ (Z_{n-1}, Z_1) & \ldots & (Z_{n-1}, Z_{n-1}) & (Z_n, Z_1) & \ldots & (Z_n, Z_{n-1}) \\ Z_1 & \ldots & Z_n & Z_1 & \ldots & Z_{n-1} \\ Z_2 & \ldots & Z_n & Z_2 & \ldots & Z_{n-1} \\ \end{vmatrix}$$

(6)

We wrote $G_n$ for the principal $n \times n$ minor. Then $(K_w, K_z) = K_z(w) = \gamma(w)k_z(w)$, and we have thus obtained, writing now $K_w^\sigma$ for $K_z$.

**Proposition 1.** Let $H$ be a Hilbert space of entire functions with continuous evaluators $Z_z$: $\forall f \in H \ f(z) = (Z_z, f)$. Let $\sigma = (z_1, \ldots, z_n)$ be a finite sequence of (non necessarily distinct) complex numbers with associated evaluators $Z_1, \ldots, Z_n$, assumed to be linearly
independent. Let $H(\sigma)$ be the Hilbert space of entire functions which are complete forms of the elements of $H$ vanishing on $\sigma$. The evaluators of $H(\sigma)$ are given by:

$$K_\sigma^\alpha(w) = \frac{\gamma(w)\gamma(z)}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & (Z_1, Z_z) \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & (Z_2, Z_z) \\ \vdots & \ldots & \vdots & \vdots \\ (Z_w, Z_1) & \ldots & (Z_w, Z_n) & (Z_w, Z_z) \end{vmatrix}$$

(7)

where $G_n^\sigma$ is the principal $n \times n$ minor of the matrix at the numerator. Of course this formula must be interpreted as a limit when $z$ or $w$ belongs to $\sigma$.

Assume that an $E$ function is known such that the evaluators in $H$ are given by formula [35]. We find a function $E_\sigma$ playing the analogous role for $H(\sigma)$:

**Theorem 2.** Let $E_\sigma(w)$ be the unique entire function such that its “incomplete form” (its product with $\prod_{1 \leq i \leq n} (w - z_i)$) differs from $E(w)$ by a finite linear combination of the evaluators $Z_1(w)$, $\ldots$, $Z_n(w)$. In other words, let

$$E_\sigma(w) = \frac{\gamma(w)}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & E[\gamma_1] \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & E[\gamma_2] \\ \vdots & \ldots & \vdots & \vdots \\ Z_1(w) & \ldots & Z_n(w) & E(w) \end{vmatrix}$$

(8)

where $G_n^\sigma$ is the principal $n \times n$ minor. The evaluator $K_\sigma^\alpha$ at $z$ for the space $H(\sigma)$ verifies:

$$K_\sigma^\alpha(w) = (K_\sigma^\alpha, K_\sigma^\alpha) = \frac{E_\sigma(z)E_\sigma(w) - E_\sigma^2(z)E_\sigma^*(w)}{(z - w)}$$

(9)

**Remark 2.** We mentioned earlier that, if $H$ is not already spanned by the $Z_i$, $1 \leq i \leq n$, any evaluator $Z_z$ with $\text{Im}(z) \neq 0$ is linearly independent from the $Z_i$’s. This implies $K_\sigma^\alpha \neq 0$, hence for $\text{Im}(z) > 0$ and by (8): $|E_\sigma(z)|^2 - |E_\sigma^*(z)|^2 > 0$, thus $E_\sigma$ given by (8) indeed verifies (2) if $H(\sigma)$ is not the zero space.

**Remark 3.** Let us write $F(w) = E^*(w) = \overline{E(w)}$ and similarly $F_\sigma(w) = E_\sigma^*(w) = \overline{E_\sigma(w)}$. It will be shown in the proof that $F_\sigma$ follows the same recipe as $E_\sigma:

$$F_\sigma(w) = \frac{\gamma(w)}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & F[\gamma_1] \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & F[\gamma_2] \\ \vdots & \ldots & \vdots & \vdots \\ Z_1(w) & \ldots & Z_n(w) & F(w) \end{vmatrix}$$

(10)

**Remark 4.** We did not see an immediate easy manipulation of determinants leading to (10) from (7) and (8). Even the compatibility of the two equations (8) and (10) with the relation
\( F_\sigma = E_\sigma^* \) does not seem to be immediately visible from easy manipulations of determinants. However, under the additional hypotheses that the space \( H \) has the additional symmetry \( f(z) \mapsto f(-z) \) and that the “trivial zeros” are purely imaginary and distinct, a relatively simple determinantal approach is proposed in [2]. It leads in fact to other determinantal expressions for \( E_\sigma \) and \( F_\sigma \) than (8) and (10), thus giving further determinantal identities.

2 Adding one zero

We establish the case \( n = 1 \) of Theorem 2 by direct computation. Appropriate notations are needed in order to complete this deceptively simple looking task. We define:

\[
E(w) = \frac{1}{w - z_1} \left( E(w) - E(z_1) \frac{Z_1(w)}{(Z_1, Z_1)} \right)
\]

(11)

\[
F(w) = \frac{1}{w - z_1} \left( F(w) - F(z_1) \frac{Z_1(w)}{(Z_1, Z_1)} \right)
\]

(12)

\[
e_1 = E(z_1) \quad f_1 = F(z_1)
\]

(13)

hence:

\[
E(w) = (w - z_1)E(w) + e_1 \frac{\overline{E(w)} - \overline{F(w)}}{i(Z_1, Z_1)(\overline{w} - w)}
\]

(14)

\[
F(w) = (w - \overline{z_1})E^*(w) - \overline{e_1} \frac{F(w) - f_1 E(w)}{i(Z_1, Z_1)(z_1 - w)}
\]

(15)

on the other hand:

\[
F(w) = (w - z_1)F(w) + f_1 \frac{\overline{E(w)} - \overline{F(w)}}{i(Z_1, Z_1)(\overline{w} - w)}
\]

(16)

We multiply the last identity by \( \overline{w} - w \), the one before by \( z_1 - w \) and substract:

\[
(z_1 - \overline{z_1})F(w) = (w - z_1)(\overline{z_1} - w)(E^*(w) - F(w)) - \frac{|e_1|^2 - |f_1|^2}{i(Z_1, Z_1)} F(w)
\]

(17)

Thus \((w - z_1)(\overline{z_1} - w)(E^*(w) - F(w)) = 0\) and we have established:

\[
F(w) = E^*(w)
\]

(18)

We will also need the following identity:

\[
\overline{E(w)} - \overline{F(w)} = -iZ_1(w)
\]

(19)

Indeed, from (14) and (16):

\[
\overline{E(w)} - \overline{F(w)} = (w - z_1)(\overline{E(w)} - \overline{F(w)}) + (|e_1|^2 - |f_1|^2) \frac{Z_1(w)}{(Z_1, Z_1)}
\]

(20)
\[ i(\overline{\sigma} - w)Z_1(w) = (w - z_1)(\overline{\sigma} E(w) - \overline{f_1} F(w)) + i(\overline{\sigma} - z_1)Z_1(w) \] (21)

This proves (19).

Let us now compute the determinant
\[
\begin{vmatrix}
(Z_1, Z_1) & (Z_1, Z_2) \\
(Z_w, Z_1) & (Z_w, Z_2)
\end{vmatrix}
= (Z_1, Z_1) \frac{E(z)E(w) - F(z)F(w)}{i(\overline{\sigma} - w)} - \overline{Z_1(z)}Z_1(w) \] (22)

We first consider:
\[
(Z_1, Z_1) \frac{E(z)E(w) - (Z_1, Z_1) F(z)F(w) - i(\overline{\sigma} - w)Z_1(z)Z_1(w)}{i(\overline{\sigma} - w)} - \overline{Z_1(z)}Z_1(w) \] (23)
\[
= (Z_1, Z_1) \frac{E(z)E(w) - (Z_1, Z_1) F(z)F(w) - \overline{Z_1(z)}(\overline{\sigma} - 1)}{i(\overline{\sigma} - w)} - \overline{Z_1(z)}Z_1(w) \] (24)
\[
= (Z_1, Z_1) \overline{E(z)}E(w) - \overline{F(z)}F(w) \] (25)

Using (14) and (10) this is equal to
\[
(Z_1, Z_1)(\overline{\sigma} - z_1) \left( \overline{E(z)}(w - z_1)E(w) + \overline{E(z)}Z_1(w) \right) - \overline{F(z)}(w - z_1)F(w) - \overline{F(z)}Z_1(w) \] (26)
\[
= (Z_1, Z_1)(\overline{\sigma} - z_1) \overline{E(z)}E(w) - \overline{F(z)}F(w) + (\overline{E(z)} Z_1(w)) \] (27)
\[
= (Z_1, Z_1)(\overline{\sigma} - z_1) \overline{E(z)}E(w) - \overline{F(z)}F(w) + i(\overline{\sigma} - w) / \overline{\sigma} \] (28)

where (19) was used. Identity of (23) and (28) gives:
\[
(Z_1, Z_1) E(z) E(w) - (Z_1, Z_1) F(z) F(w) - i(\overline{\sigma} - w) / \overline{\sigma} \] (29)

Comparison with (22) gives the final result:
\[
\frac{\gamma_1(w)\gamma_1(z)}{(Z_1, Z_1)} \left| \begin{vmatrix}
(Z_1, Z_1) & (Z_1, Z_2) \\
(Z_w, Z_1) & (Z_w, Z_2)
\end{vmatrix}\right| = \frac{E(z)E(w) - F(z)F(w)}{i(\overline{\sigma} - w)} \] (30)

As we know from (19) that \( F = \mathcal{E}^* \) this completes the proof of Theorem 2 in the case \( n = 1 \).

3 General case with distinct trivial zeros

An induction establishes Theorem 2 when the \( z_i \)'s are distinct. Let us suppose it true for the \( n - 1 \) added "trivial zeros" \( z_1, \ldots, z_{n-1} \). From (7) we know that for any \( z \in \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\}, \end{align*}
\( \prod_{1 \leq i \leq n-1} (w - z_i)K_{z_i}^{z_{i+1}, \ldots, z_{n-1}}(w) \) is a linear combination of the original evaluators \( Z_1(w), \ldots, Z_{n-1}(w) \) and \( Z_z(w) \). This applies in particular to \( z = z_n \). The induction hypothesis
tells us that \( \prod_{1 \leq i < n} (w - z_i)E^{z_1, \ldots, z_{n-1}}(w) \) differs from \( E(w) \) by a linear combination of the original evaluators \( Z_1(w), \ldots, Z_{n-1}(w) \). The case \( n = 1 \) tells us that \( (w - z_n)E^{z_1, \ldots, z_n}(w) \) differs from \( E^{z_1, \ldots, z_{n-1}}(w) \) by a multiple of \( K_{z_n}^{\ast} \). Hence \( \prod_{1 \leq i \leq n} (w - z_i)E^{z_1, \ldots, z_n}(w) \) differs from \( E(w) \) by a linear combination of the original evaluators \( Z_1(w), \ldots, Z_{n-1}(w) \) and \( Z_n(w) \), as was to be established. This linear combination is fixed in a unique manner by evaluating at the trivial zeros of \( \prod_{1 \leq i \leq n} (w - z_i)E^{z_1, \ldots, z_n}(w) \). This completes the proof of Theorem 2. Furthermore we proved that the same iterative recipe as for \( E_\sigma \) works for the construction of \( F_\sigma = E_\sigma^* \). Hence the formula (10) holds.

4 General case with multiplicities

We introduce some notations for the case of repetitions among \( z_1, \ldots, z_n \). We define \( k_1, \ldots, k_n \) such that \( i' = i - k_i \) is the first index with \( z_{i'} = z_i \). For example if \( z_1 = z_2 = z_3 \neq z_4 = z_5 \), \( k_1 = 0, k_2 = 1, k_3 = 2, k_4 = 0, k_5 = 1 \), etc. Then, for \( f \) analytic function, we recall the notation \( f[z_i] := f^{(k_i)}(z_i) \). For \( f \in H \) we also have the scalar product \( (Z_i, f) = f[z_i] \).

Let \( k^\sigma(z, w) \) be the incomplete form of the reproducing kernel \( K^\sigma(z, w) \) in \( H(\sigma) \):

\[
k^\sigma(z, w) = \frac{1}{G_n} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & (Z_1, Z_z) \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & (Z_2, Z_z) \\ \vdots & \ldots & \vdots & \vdots \\ (Z_w, Z_1) & \ldots & (Z_w, Z_n) & (Z_w, Z_z) \end{vmatrix} = Z_z(w) - \sum_{1 \leq j \leq n} \beta_j^\sigma Z_j(w)
\] (31)

where the coefficients \( \beta_1^\sigma, \ldots, \beta_n^\sigma \) (which are also functions of \( z \)) are determined by the constraints:

\[
\forall i \quad \sum_{1 \leq j \leq n} \beta_j^\sigma Z_j[z_i] = Z_z[z_i] \quad (= (Z_i, Z_z))
\] (32)

Let \( \epsilon > 0 \) and \( z_i^\epsilon = z_i - k_i \epsilon \) for \( 1 \leq i \leq n \). We let \( k^\epsilon(z, w) \) be the (completely) incomplete form of the reproducing kernel in \( H(z_1^\epsilon, \ldots, z_n^\epsilon) \). It thus has the shape:

\[
k^\epsilon(z, w) = Z_z(w) - \sum_{1 \leq j \leq n} \alpha_j Z_j^\epsilon(w)
\] (33)

with the constraints

\[
\forall i \quad k^\epsilon(z, z_i - k_i \epsilon) = 0
\] (34)

Let us use the definitions

\[
Z_j^\epsilon := e^{-k_j} \sum_{0 \leq m \leq k_j} (-1)^m \binom{k_j}{m} z_{j-m \epsilon}
\] (35)
to obtain vectors $Z_j^* \in H$ which span the same subspace of $H$ as the $Z_{z_j} = Z_{z_j - t_j \epsilon}$. So there are coefficients $\beta_1^\epsilon, \ldots, \beta_n^\epsilon$ such that

$$k^\epsilon(z, w) = Z_z(w) - \sum_{1 \leq j \leq n} \beta_j^\epsilon Z_j^*(w)$$

(36)

with the constraints

$$\forall i \sum_{1 \leq j \leq n} \beta_j^\epsilon Z_j^*(z_i - t_k i) = Z_z(z_i - t_k i)$$

(37)

We define the symbol, for any arbitrary analytic function on $C$:

$$f[z_i] := \epsilon^{-k_i} \sum_{0 \leq l \leq k_i} (-1)^l \binom{k_i}{l} f(z_i - t_l \epsilon)$$

(38)

With the help of these symbols, the constraints on the $\beta_j^\epsilon$ can be equivalently rewritten:

$$\forall i \sum_{1 \leq j \leq n} \beta_j^\epsilon Z_j^*[z_i] = Z_z[z_i]$$

(39)

We examine the behavior for $\epsilon \to 0$ of the quantities $Z_j^*[z_i]$. In terms of the reproducing kernel $Z(z, w) = Z_z(w) = (Z_w, Z_z)$, which from equation (3) is analytic in $w$ and anti-analytic in $z$ we have:

$$Z_j^*[z_i] = \epsilon^{-k_i+k_j} \sum_{0 \leq l \leq k_i} (-1)^l \binom{k_i}{l} \binom{k_j}{m} Z(z_j - t_m \epsilon, z_i - t_l \epsilon)$$

(40)

Thus, with $(\partial F)(w_1, w_2) = \frac{\partial}{\partial w_2} F(w_1, w_2)$, $(\delta F)(w_1, w_2) = \frac{\partial}{\partial w_1} F(w_1, w_2)$:

$$\lim_{\epsilon \to 0} Z_j^*[z_i] = (\partial^{k_i} \delta^{k_j} Z)(z_j, z_i)$$

(41)

On the other hand we have:

$$(Z_i, Z_j) = \frac{\partial^{k_i}}{\partial w^{k_i}} \bigg|_{w=z_i} Z_j(w) = \frac{\partial^{k_i}}{\partial w^{k_i}} \bigg|_{w=z_i} (Z_j, Z_w) = \frac{\partial^{k_i}}{\partial w^{k_i}} \bigg|_{w=z_i} \frac{\partial^{k_j}}{\partial \omega^{k_j}} \bigg|_{\omega=z_j} Z(w, \omega)$$

$$= \frac{\partial^{k_i}}{\partial w^{k_i}} \bigg|_{w=z_i} \frac{\partial^{k_j}}{\partial \omega^{k_j}} \bigg|_{\omega=z_j} Z(\omega, w) = (\partial^{k_i} \delta^{k_j} Z)(z_j, z_i)$$

(42)

which gives

$$(Z_i, Z_j) = (\partial^{k_i} \delta^{k_j} Z)(z_j, z_i) = \lim_{\epsilon \to 0} Z_j^*[z_i]$$

(43)

There also holds, for any $z$:

$$(Z_i, Z_z) = \lim_{\epsilon \to 0} Z_z[z_i]$$

(44)
So in the limit $\epsilon \to 0$ the linear constraints (39) on $(\beta_j^\epsilon)_{1 \leq j \leq n}$ become the constraints on the coefficients $\beta_1^\epsilon, \ldots, \beta_n^\epsilon$ which give in (31) the incomplete reproducing kernel $k^\sigma(z, w)$. This shows in passing that for $\epsilon \neq 0$ small the vectors $Z_j^\epsilon$ are also linearly independent, and proves $\forall j \lim_{\epsilon \to 0} \beta_j^\epsilon = \beta_j^\sigma$. We have further

$$Z_j(w) = \lim_{\epsilon \to 0} Z_j^\epsilon(w)$$ (45)

and this finally establishes:

$$\lim_{\epsilon \to 0} k^\epsilon(z, w) = k^\sigma(z, w)$$ (46)

In the exact same manner we can examine the quantities:

$$e_\sigma(w) = \frac{1}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & E[z_1] \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & E[z_2] \\ \vdots & \ldots & \vdots & \vdots \\ Z_1(w) & \ldots & Z_n(w) & E(w) \end{vmatrix}$$ (47)

and

$$e_\epsilon(w) = \frac{1}{G_n^\epsilon} \begin{vmatrix} (Z_{z_1}, Z_{z_1}) & \ldots & (Z_{z_1}, Z_{z_n}) & E(z_1) \\ (Z_{z_2}, Z_{z_1}) & \ldots & (Z_{z_2}, Z_{z_n}) & E(z_2) \\ \vdots & \ldots & \vdots & \vdots \\ Z_{z_1}(w) & \ldots & Z_{z_n}(w) & E(w) \end{vmatrix}$$ (48)

and prove

$$\lim_{\epsilon \to 0} e_\epsilon(w) = e_\sigma(w)$$ (49)

There is also an immediate limit to be taken in the gamma factor, and in the end we obtain the reproducing kernel formula (9) for the space $H(\sigma)$. The formula $E^*_\sigma = F_\sigma$ with $F_\sigma$ given by (10) is proven in the same manner.

### 5 An example

We take $H = PW_x (x > 0)$, the Paley-Wiener space of entire functions of exponential type at most $x$, square integrable on the real line ($(f, f) = \frac{1}{2\pi} \int_{\mathbb{R}} |f(z)|^2 \, dz$). The evaluators are given by the formula

$$Z_z(w) = (Z_w, Z_z) = \frac{\sin(\pi z - w)x}{\pi z - w} \left( = \int_{-x}^{x} e^{iwt} e^{-ixt} \, dt \right)$$ (50)

We can choose $E(z) = e^{-ixz}$, $F(z) = E^*(z) = e^{+ixz}$. Let $z_1, z_2, \ldots, z_n$ and $z$ be distinct complex numbers and define
\[ G_n = \begin{vmatrix} 2\sin((\pi - z_1)x) & 2\sin((\pi - z_1)x) & \cdots & 2\sin((\pi - z_1)x) \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sin((\pi - z_n)x)}{z_1 - z_n} & \frac{\sin((\pi - z_n)x)}{z_1 - z_n} & \cdots & \frac{\sin((\pi - z_n)x)}{z_1 - z_n} \end{vmatrix} \]  

\[
G_n(z, z) = \begin{vmatrix} 2\sin((\pi - z_1)x) & 2\sin((\pi - z_1)x) & \cdots & 2\sin((\pi - z_1)x) & 2\sin((\pi - z_1)x) \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} \end{vmatrix} \]  

\[
e_n(z) = \begin{vmatrix} 2\sin((\pi - z_1)x) & 2\sin((\pi - z_1)x) & \cdots & 2\sin((\pi - z_1)x) & e^{-iz_1} \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & e^{-iz_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & e^{-iz_n} \end{vmatrix} \]  

\[
f_n(z) = \begin{vmatrix} 2\sin((\pi - z_1)x) & 2\sin((\pi - z_1)x) & \cdots & 2\sin((\pi - z_1)x) & e^{iz_1} \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_2} & e^{iz_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & \cdots & \frac{\sin((\pi - z_1)x)}{z_1 - z_n} & e^{iz_n} \end{vmatrix} \]  

Then

\[ G_n(z, z)G_n = \frac{|e_n(z)|^2 - |f_n(z)|^2}{2 \text{Im}(z)} \quad \text{and} \quad f_n(z) = \prod_{1 \leq i \leq n} \frac{z - z_i}{z - \overline{z_i} e_n(z)} \]  

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