Common Fixed Point Theorems in Metric spaces with Applications

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Abstract. In this paper, we investigate the existence and uniqueness of common fixed point theorems for certain contractive type of mappings. As an application the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming are discuss by using the our results.

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1. Introduction

Bellman and Lee [3] first introduced the basic form of the functional equations in dynamic programming is as follows:

\[ f(x) = \text{opt}_{y \in D} H(x, y, f(T(x, y))) \forall x \in S \]

where opt represent sup. or inf., \( x, y \) denote the state and decision vectors respectively, \( T \) stands for the transformation of the process and \( f(x) \) represents the optimal return function with the initial state \( x \). Afterwards, the existence and uniqueness of fixed point solutions for several classes of contractive mappings and functional equations studied by many investigators such as Bhakta and Mitra [5], Liu [15], Liu and Ume [20], Pathak and Fisher [21], Baskaran and Subhramanyam [1] and others.

Ray [22] proved two common fixed point theorems for three self mappings \( f, g \) and \( h \) in the complete metric space using the following contractive condition:

\[ d(fx, gy) \leq d(hx, hy) - w(d(hx, hy)), \forall x, y \in X \]

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Further Liu[15] established common fixed point theorem and introduced a class of mappings in a complete metric space as follows:

\[ d(fx, gy) \leq \max\{d(hx, hy), d(hx, fx), d(hy, gy)\} - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\} ). \]  

(3)

Recall that the notion of orbitally complete metric space and orbitally continuous mapping were introduced by Ciric [9] . These definitions were extended to the case of two or three mappings by Sastry et al.[12]. Some common fixed point results in this situation were obtained in [12] . We give now respective definitions for pairs of mappings given in literature.

2. Priliminaries

**Definition 1** (6). A self map \( f \) on a metric space \( (X, d) \) is said to be asymptotically regular at a point \( x \) in \( X \) if \( \lim_{n \to \infty} d(f^n(x), f^{n+1}(x)) = 0 \). Where \( f^n(x) \) denotes the \( n \)th iterate of \( f \) at \( x \).

**Definition 2** (6). Let \( f \) and \( g \) be two self mappings of \( X \) and \( \{x_n\} \) a sequence in \( X \), then \( \{x_n\} \) is said to be asymptotically \( g \)-regular with respect to \( f \) if \( \lim_{n \to \infty} d(fx_n, gx_n) = 0 \).

**Definition 3** (9). Let \( \{x_n\} \) is a sequence which is asymptotically \( g \)-regular with respect to \( f \), then \( O(f, x_n) = \{fx_1, fx_2, fx_3, ...fx_n, ...\} \) is called asymptotic orbit of \( f \).

**Definition 4** (12). \( X \) is said to be \( f \)-asymptotically complete if every Cauchy sequence of the form \( \{fx_n\} \) converges in \( X \).

**Definition 5** (12). A self map \( f \) is said to be asymptotically continuous if it is continuous on closure of \( O(f, x_n) \).

**Definition 6**. Two self maps \( g \) and \( h \) of \( X \) are said to be weakly commuting if \( d(ghx, hgx) \leq d(gx, hx) \forall x \in X \).

**Definition 7** (12). Let \( f, g \) and \( h \) be three self maps on a metric space \( X \).

(i) If for a point \( x_0 \in X \), there exists a sequence \( \{x_n\} \) in \( X \) such that \( fx_{2n} = hx_{2n+1} \) and \( gx_{2n+1} = hx_{2n+2}, n = 0, 1, 2,... \). Then the set \( O(x_0, f, g, h) = \{Tx_n|n = 0, 1, 2,...\} \) is called the orbit of \( (f, g, h) \) at \( x_0 \).

(ii) The space \( (X, d) \) is said to be \( (f, g, h) \)– orbitally complete if every Cauchy sequence in \( O(x_0, f, g, h) \) converges in \( X \).
(iii) The map \( h \) is said to be \( (f, g, h) \)-orbitally continuous at \( x_0 \) if it is continuous on \( O(x_0, f, g, h) \).

(iv) The pair \( (f, g) \) is said to be asymptotically regular w.r.to \( h \) at \( x_0 \) if there exists a sequence \( \{x_n\} \) in \( X \) such that
\[
fx_{2n} = hx_{2n+1}, \quad gx_{2n+1} = hx_{2n+2} ; \quad n = 0, 1, 2, \ldots \quad \text{and} \quad d(hx_n, hx_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.
\]

Throughout in this paper, we assume that \( R^+ = [0, +\infty), \quad R = (-\infty, +\infty), \quad w \) and \( N \) denote the set of all non-negative and positive integers respectively.

\[
W = \{ w : w : R^+ \to R^+ \text{is continuous mappings with} \quad 0 < w(t) < t \quad \forall \ t > 0 \}
\]

Let \( \Phi = \{ \phi : \phi : [0, \infty) \to [0, \infty) \} \) satisfying the following conditions:
(i) \( \phi \) is continuous and non-decreasing
(ii) \( \phi(t) < \psi(t) \quad \forall t > 0 \)
(iii) \( \lim_{n \to \infty} \phi(t_n) = 0 \iff \lim_{n \to \infty} t_n = 0. \)

Let \( \Psi = \{ \psi : \psi : [0, \infty) \to [0, \infty) \} \) satisfying the following conditions:
(i) \( \psi \) is non-decreasing
(ii) \( \phi(t) < \psi(t) \quad \forall t > 0 \)
(iii) \( \psi(a + b) \leq \psi(a) + \psi(b) \quad \forall a, b \in [0, \infty) \)
(iv) \( \psi(t) < t \quad \forall t \in [0, \infty) \)

The aim of this paper is to provide the sufficient conditions for the existence and uniqueness of common fixed point for the following type of contractive mappings metric space \((X, d)\).

\[
\psi(d(fx, gy)) \leq \max\{\phi(d(hx, hy)), \phi(d(hx, fx)), \phi\left(\frac{1}{2}(d(hx, hy) + d(fx, gy))\right), \phi(d(hy, gy))\} - w\left(\max\{\phi(d(hx, hy)), \phi(d(hx, fx)), \phi(d(hy, gy))\}, \phi\left(\frac{1}{2}(d(hx, hy) + d(fx, gy))\right)\right).
\]

for all \( x, y \in X \). Where \( \psi \) and \( \phi \) are defined above.

As an applications, we discuss the existence and uniqueness of common solutions of the following functional equations arising in dynamic programming.

\[
f(x) = \text{opt}_{y \in D}\{u(x, y) + H(x, y, f(T(x, y)))\} \forall x \in S
\]

and

\[
f_i(x) = \text{opt}_{y \in D}\{u(x, y) + H_i(x, y, f_i(T(x, y)))\} \forall x \in S \& i \in \{1, 2, 3\}
\]
3. Main Results

**Theorem 1.** Let \( f, g \) and \( h \) be three self maps on a metric space \( X \) satisfying:

(i) either \( f \) commute with \( h \) or \( g \) commute with \( h \).

(ii) there exists \( w \in W \) such that (5) hold for all \( x, y \in X \).

(iii) The pair \((f, g)\) is asymptotically regular with respect to \( h \) at \( x_0 \).

(iv) The space \( X \) is \((f, g, h)\)-orbitally complete at \( x_0 \) and \( h \) is orbitally continuous at \( x_0 \).

Then \( f, g \) and \( h \) have a unique common fixed point in \( X \).

**Proof** Since \((f, g)\) is asymptotically respect to \( h \) at \( x_0 \), there exists a sequence \( \{x_n\} \) in \( X \) such that \( fx_{2n} = hx_{2n+1} \) and \( gx_{2n+1} = hx_{2n+2}, n = 0, 1, 2, \ldots \) and \( d(hx_n, hx_{n+1}) \to 0 \) as \( n \to \infty \).

Now we show that \( hx_n \) is Cauchy. On contrary suppose that \( hx_n \) is not Cauchy, then there exists an \( \epsilon > 0 \) and positive integers \( m_k \) and \( n_k \) with \( m_k < n_k \) such that \( d(hx_{m_k}, hx_{n_k}) \geq \epsilon \) and \( d(hx_{m_k}, hx_{n_k-1}) \leq \epsilon \) for all \( k = 0, 1, 2, \ldots \). Since \( d(hx_{m_k}, hx_{n_k}) \leq d(hx_{m_k}, hx_{n_k-1}) + d(hx_{n_k-1}, hx_{n_k}) \).

Then we obtain \( d(hx_{m_k}, hx_{n_k}) \to \epsilon \) as \( k \to \infty \).

Now there are four cases: (i) \( m_k \) is even and \( n_k \) is odd (ii) \( m_k \) is even and \( n_k \) is even (iii) \( m_k \) is odd and \( n_k \) is even (iv) \( m_k \) is odd and \( n_k \) is odd.

Suppose \( m_k \) is even and \( n_k \) is odd, we have

\[
\psi(d(hx_{m_k}, hx_{n_k})) \leq \psi(d(hx_{m_k}, hx_{m_k+1})) + \psi(d(hx_{m_k+1}, hx_{m_k+1})) + \psi(d(hx_{n_k+1}, hx_{n_k})) - \psi(d(hx_{m_k}, hx_{m_k})),
\]

\[
\phi(d(gx_{n_k}, hx_{m_k})), \phi\left(\frac{1}{2}d(hx_{m_k}, hx_{n_k}) + d(fx_{m_k}, gx_{n_k})\right),
\]

\[
\phi(\frac{1}{2}d(hx_{m_k}, hx_{n_k}) + d(fx_{m_k}, gx_{n_k})))\right) + \psi(d(hx_{n_k+1}, hx_{n_k}))
\]

Letting \( k \to \infty \), we obtain

\[
\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)
\]

a contradiction. In the remaining cases we have a similar situation. Hence \( \{hx_n\} \) is Cauchy. Since \( X \) is \((f, g, h)\)-orbitally complete at \( x_0 \), it follows that there exist \( z \in X \) s.t. \( hx_n \to z \) as \( n \to \infty \). Now, again

\[
\psi(d(fz, gx_{n+1})) \leq \max\{\phi(d(hz, hx_{n+1})), \phi(d(fz, hx_{n+1})), \phi(d(gz, hz)), \frac{1}{2}d(hx_{n+1}, \phi(d(fz, gx_{n+1})))\right) - w(\max\{\phi(d(hz, hx_{n+1})), \phi(d(fz, hx_{n+1})))
\]

\[
\phi(gz, hz), \phi(\frac{1}{2}d(hx_{n+1}, hz) + d(fz, gx_{n+1})))\}
\]

and

\[
\psi(d(fz, gx_{n+1})) \leq \max\{\phi(d(hz, hx_{n+1})), \phi(d(fz, hz)), \phi(d(gz, hz)), \phi(d(gz, hz)), \phi(\frac{1}{2}d(hx_{n+1}, hz) + d(fz, gx_{n+1})))\}
\]
\[
\phi \left( \frac{1}{2} \left[ d(hz, hx_{2n+1}) + d(fz, gx_{2n+1}) \right] \right) - w(\max \{ \phi(d(hz, hx_{2n+1})), \phi(d(fz, hz)) \}) - \phi(d(gx_{2n+1}, hx_{2n+1})) + \phi(\frac{1}{2} \left[ d(hz, hx_{2n+1}) + d(fz, gx_{2n+1}) \right] ) \]

Taking \( k \to \infty \), we obtain

\[
\psi(d(z, gz)) \leq \max \{ \phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi(\frac{1}{2} [d(hz, z) + d(z, gz)]) \}
- w(\max \{ \phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi(\frac{1}{2} [d(hz, z) + d(z, gz)]) \}) \] (8)

and

\[
\psi(d(fz, z)) \leq \max \{ \phi(d(z, hz)), \phi(d(z, z)), \phi(d(gz, hz)), \phi(\frac{1}{2} [d(hz, z) + d(fz, z)]) \}
- w(\max \{ \phi(d(z, hz)), \phi(d(z, z)), \phi(d(gz, hz)), \phi(\frac{1}{2} [d(hz, z) + d(fz, z)]) \}) \] (9)

Since \( h \) is orbitally continuous at \( x_0 \) and \( fh = hf \), we infer that \( fhx_{2n} = hf_{2n} \to Tz \) as \( n \to \infty \). Similarly \( ghx_{2n+1} = hg_{2n+1} \to hz \) as \( n \to \infty \).

Again,

\[
\psi(d(fhx_{2n}, gx_{2n+1})) \leq \max \{ \phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fhx_{2n}, hhx_{2n})), \phi(d(gx_{2n+1}, hx_{2n+1})), \phi(\frac{1}{2} [d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})]) \}
- w(\max \{ \phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fhx_{2n}, hhx_{2n})), \phi(d(gx_{2n+1}, hx_{2n+1})), \phi(\frac{1}{2} [d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})]) \})
\]

Taking \( k \to \infty \), we obtain

\[
\psi(d(hz, z)) \leq \max \{ \phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi(\frac{1}{2} [d(hz, z) + d(hz, z)]) \}
- w(\max \{ \phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi(\frac{1}{2} [d(hz, z) + d(hz, z)]) \}) \]

implies that

\[
\psi(d(hz, z)) \leq \phi(d(z, hz)) - w(\phi(d(z, hz)) < \phi(d(z, hz)) \]

a contradiction. Hence \( hz = z \). Using (8) and (9) together with \( Tz = z \), we infer that \( fz = gz = hz = z \). Further uniqueness of common fixed point can easily prove.

Taking \( \psi(t) = t \) and \( \phi(t) = ht \) where \( h < 1 \), we state the following
Corollary 1. Let $A$, $B$ and $T$ be self maps on a metric space $(X,d)$ such that $T$ commutes with both $A$ and $B$ and the pair $(A,B)$ is asymptotically regular w.r.to $T$ at $x_0 \in X$. $X$ is orbitally complete and $T$ is orbitally continuous at $x_0$ and

$$d(Ax, By) \leq \phi\left(\max\{d(Tx, Ty), d(Ax, Tx)\}, d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)]\right)$$

$$- w(\max\{d(Tx, Ty), d(Ax, Tx)\}, d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)])$$

for all $x, y \in X$. Then $A, B$ and $T$ have unique common fixed point in $X$.

Theorem 2. Let $(X,d)$ be a metric space and $f$, $g$ and $h$ be self mappings on $X$ such that $f(X) \cup g(X) \subseteq h(X)$. If there exists a $w \in W$ satisfying (5). Then the pair $(f, h)$ and $(g, h)$ have a coincidence point in $X$, provided that (i) $X$ is $h$-asymptotically complete, (ii) $h$ is asymptotically continuous and (iii) $h$ is weakly commute with $f$ and $g$. Further $f$, $g$ and $h$ have a unique common fixed point in $X$.

Proof. Let $x_0 \in X$ be any point in $X$. Since $f(X) \cup g(X) \subseteq h(X)$. We choose sequence $\{x_n\} \in X$ such that $f x_{2n} = h x_{2n+1}$ and $g x_{2n+1} = h x_{2n+2}$ for all $n \in w$.

By (5), we conclude that

$$\psi(d(h x_{2n+1}, h x_{2n+2})) = \psi(d(f x_{2n}, g x_{2n+1}))$$

$$\leq \max\{\phi(d(h x_{2n}, h x_{2n+1}), \phi(d(h x_{2n}, f x_{2n})), \phi(d(h x_{2n+1}, g x_{2n+1})), \phi(\frac{1}{2}(d(h x_{2n}, h x_{2n+1}) + d(f x_{2n}, g x_{2n+1})))\} - w(\max\{\phi(d(h x_{2n}, h x_{2n+1})), \phi(d(h x_{2n}, f x_{2n})), \phi(d(h x_{2n+1}, g x_{2n+1}), \phi(\frac{1}{2}(d(h x_{2n}, h x_{2n+1}) + d(f x_{2n}, g x_{2n+1}))\})$$

This yields

$$\psi(d_{2n+1}) \leq \max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1}))\} - w(\max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1}))\}).$$

Suppose $d_{2n+1} > d_{2n}$, then $\phi(d_{2n+1}) > \phi(d_{2n})$. Using (5), we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n+1}) - w(\phi(d_{2n+1})) < \phi(d_{2n+1})$$

a contradiction. Consequently, we have $d_{2n+1} \leq d_{2n}$, from (5) we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n}) - w(\phi(d_{2n})) < \phi(d_{2n})$$

for any $n \in w$. Similarly, we have $\psi(d_{2n}) \leq \phi(d_{2n-1}) - w(\phi(d_{2n-1})) < \phi(d_{2n-1})$ for all $n \in N$. It follows that

$$\psi(d_n) \leq \phi(d_{n-1}) - w(\phi(d_{n-1})) \forall n \in N \quad (10)$$
From (10), we have
\[ \sum_{i=0}^{n} w(\phi(d_i)) \leq \phi(d_0) - \psi(d) < \phi(d_0) \quad \forall n \in N \]
Thus the sequence \( \{d_n\} \) is decreasing sequence whereas the series \( \sum_{n=0}^{\infty} w(\phi(d_n)) \) and \( \{\phi(d_n)\} \) are convergent. It is clear that \( \lim_{n \to \infty} w(\phi(d_n)) = 0 \). Since sequence \( \{d_n\} \) is decreasing so there exists \( p \in \mathbb{R}^+ \) such that \( \lim_{n \to \infty} d_n = p \). By continuity of \( \phi \) and \( w \) we have \( \lim_{n \to \infty} w(\phi(d_n)) = w(\phi(p)) = 0 \). Thus \( p = 0 \). Therefore \( \lim_{n \to \infty} d(hx_{2n}, hx_{2n+1}) = 0 \) implies that \( \lim_{n \to \infty} d(hx_{2n}, hx_{2n+1}) = 0 \) means that \( \lim_{n \to \infty} d(hx_{n}, f x_{2n}) = 0 \) and \( \lim_{n \to \infty} d(hx_{2n+1}, g x_{2n+1}) = 0 \). i.e. the sequence \( \{x_n\} \) is asymptotically \( h \)-regular with respect to \( f \) and \( g \).

Next we show that \( \{hx_n\} \) is Cauchy sequence in \( X \). We need only to show that \( \{hx_{2n}\} \) is Cauchy sequence. On contrary suppose \( \{hx_n\} \) is not Cauchy. Then there exists some \( \epsilon > 0 \) such that for any even integers \( 2m(k) \) and \( 2n(k) \) with \( 2m(k) > 2n(k) > 2k \) and \( d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon \). Further, let \( 2m(k) \) denote the least even positive integer exceeding \( 2n(k) \) which satisfies that \( 2m(k) > 2n(k) > 2k \)

\[ d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \epsilon \quad \text{and} \quad d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon. \]  

(11)

Note that for any \( k \in N \)

\[ d(hx_{2m(k)}, hx_{2n(k)}) \leq d_{2m(k)-1} + d_{2m(k)-2} + d(hx_{2m(k)-2}, hx_{2n(k)}). \]

\[ |d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})| \leq d_{2n(k)}. \]

\[ |d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})| \leq d_{2m(k)}. \]

\[ |d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})| \leq d_{2n(k)+1}. \]

From above inequalities, we infer that

\[ \epsilon = \lim_{n \to \infty} d(hx_{2m(k)}, hx_{2n(k)}) \]

\[ = \lim_{n \to \infty} d(hx_{2m(k)}, hx_{2n(k)+1}) \]

\[ = \lim_{n \to \infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \]

\[ = \lim_{n \to \infty} d(hx_{2m(k)+1}, hx_{2n(k)+2}). \]
Again from (5), we have

\[
\psi(d(fx_{2m(k)}, gx_{2n(k)+1})) \leq \max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \phi(d_{2n(k)+1}), \phi(\frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1})]) - w(\max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \\
\phi(d_{2n(k)+1}), \phi(\frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1})])\}\}
\]

Taking \( k \to \infty \), we deduce that

\[
\psi(\epsilon) \leq \max\{\phi(\epsilon), \phi(0), \phi(0), \phi(\frac{1}{2}[\epsilon + \epsilon])\} - w(\max\{\phi(\epsilon), \phi(0), \phi(0), \phi(\frac{1}{2}[\epsilon + \epsilon])\})
\]

\[
\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)
\]
a contradiction and hence \( \{hx_n\} \) is Cauchy sequence. Since \( X \) is asymptotically \( h \)-complete implies that the sequence \( \{hx_n\} \) converges to a point \( z \in X \). We infer that \( \{hx_{2n}\} \) and \( \{hx_{2n+1}\} \) also converges to \( z \). \( h \) is asymptotically continuous implies that

\[
vhx_{2n} \to hz, vhx_{2n+1} \to hz, hfx_{2n} \to hz, hgx_{2n+1} \to hz \text{ as } n \to \infty.
\]

Now using weakly commutativity and sub additivity of \( \psi \), we have

\[
\psi(d(fhx_{2n}, hz)) \leq \psi(d(fhx_{2n}, hfx_{2n})) + \psi(d(hfx_{2n}, hz)) \leq \psi(d(fhx_{2n}, hx_{2n})) + \psi(d(hfx_{2n}, hz))
\]

Taking \( n \to \infty \) implies that \( fhx_{2n} \to hz \). Similarly \( ghx_{2n+1} \to hz \). Suppose \( d(hz, gz) > 0 \). Then

\[
\psi(d(hz, gz)) \leq \psi(d(hz, fhx_{2n})) + \psi(d(fhx_{2n}, gz)) \leq \psi(d(hz, fhx_{2n})) + \max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fhx_{2n})), \phi(d(hz, gz)), \\
\phi(\frac{1}{2}[d(hhx_{2n}, hz) + d(fh_{2n}, gz)])\} - w(\max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fhx_{2n})), \\
\phi(d(hz, gz)), \phi(\frac{1}{2}[d(hhx_{2n}, hz) + d(fh_{2n}, gz)])\})
\]

Taking \( n \to \infty \), we infer that

\[
\psi(d(hz, gz)) \leq \phi(d(hz, gz))
\]
a contradiction. Hence \( hz = gz \). Similarly we can easily prove that \( hz = fz \). Next we have to show that \( z \) is fixed point of \( h \). Suppose that \( d(hz, z) > 0 \). From (5), we have

\[
\psi(d(fx_{2n}, ghx_{2n})) \leq \max\{\phi(d(hx_{2n}, hhx_{2n})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hhx_{2n}, ghx_{2n}))\},
\]
\[
\phi\left(\frac{1}{2}d(hx_{2n},hhx_{2n}) + d(fx_{2n},ghx_{2n})\right) - w(\max\{\phi(d(hx_{2n},hhx_{2n})), \phi(d(hx_{2n},fx_{2n}))\})
\]

Taking \(n \to \infty\), we infer that

\[
\psi(d(z,hz)) \leq \max\{\phi(d(z,hz)), \phi(0), \phi(0), \phi\left(\frac{1}{2}[d(z,hz) + d(z,hz)]\right)\}
\]

implies that

\[
\psi(d(z,hz)) \leq \phi(d(z,hz)) - w(\phi(d(z,hz))) < \phi(d(z,hz))
\]
a contradiction. Hence \(hz = z\). Therefore \(fz = gz = hz = z\), i.e. \(z\) is common fixed point of \(f, g\) and \(h\).

Finally, we show that \(z\) is unique fixed point of \(f, g\&h\). Suppose \(z'\) be another fixed point. Then from (5), we obtain

\[
\psi(d(fz,gz')) \leq \max\{\phi(d(hz,hz')), \phi(d(hz,fz)), \phi(d(hz',gz')), \phi(\frac{1}{2}[d(hz,hz') + d(fz,gz')])\}
\]

we infer that

\[
\psi(d(hz,hz')) \leq \phi(d(hz,hz')) - w(\phi(d(hz,hz'))) < \phi(d(hz,hz'))
\]
a contradiction. Hence \(z = z'\).

**Corollary 2.** Let \((X,d)\) be a complete metric space and \(f, g\) and \(h\) be self mappings on \(X\) such that \(f(X) \cup g(X) \subseteq h(X)\). If there exists a \(w \in W\) satisfying (5). Then the pair \((f,h)\) and \((g,h)\) have a coincidence point in \(X\), provided that (i) \(h\) is continuous and (iii) \(h\) commutes with both \(f\) and \(g\). Further, \(f, g\) and \(h\) have a unique common fixed point in \(X\).

Taking \(\psi(t) = t = \phi(t)\), in cor.2 we obtain the following
Corollary 3. Let \((X, d)\) be a complete metric space. Let \(f, g\) and \(h\) be self maps on \(X\) such that \(f(X) \cup g(X) \subseteq h(X)\) and \(h\) is continuous and commute with both \(f\) and \(g\). If there exists a \(w \in W\) satisfying the following condition:

\[
d(fx, gy) \leq \max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)]\}
- w(\max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)]\})
\]

corresponding to all \(x, y \in X\). Then the pair \((f, h)\) and \((g, h)\) have coincidence point, further \(f\), \(g\) and \(h\) have unique common fixed point in \(X\).

Taking \(h = I\), in cor.3 we gain the following

Corollary 4. Let \((X, d)\) be a complete metric space. Let \(f\) and \(g\) be self maps on \(X\). If there exists a \(w \in W\) satisfying the following condition:

\[
d(fx, gy) \leq \max\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)]\}
- w(\max\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)]\})
\]

corresponding to all \(x, y \in X\). Then \(f\) and \(g\) have unique common fixed point in \(X\).

4. An Application

Throughout in this section, let \(X\) and \(Y\) be Banach spaces \(S \subseteq X\) be the state space and \(D \subseteq Y\) be decision space. \(B(S)\) denotes the set of all real-valued bounded functions on \(S\) and \(d(a, b) = \sup_{x \in S}|a(x) - b(x)|, \forall a, b \in B(S)\). It is obvious that \((B(S), d)\) is a complete metric space. Define \(u : S \times D \rightarrow R, T : S \times D \rightarrow S\) and \(H_i : S \times D \times R \rightarrow R\) for \(i = \{1, 2, 3\}\).

Now we study those conditions which guarantee the existence and uniqueness of common solutions of functional equations (7).

Theorem 3. If the following conditions are satisfied
\((C_1) u\) and \(H_i\) are bounded for \(i = \{1, 2, 3\}\)
\((C_2)\) there exist \(\phi \in \Phi, \psi \in \Psi\) and \(w \in W\) satisfying

\[
\psi[H_1(x, y, a(t))] - H_2(x, y, b(t))| \leq \max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\}
- w(\max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\})
\]
for all \((x, y) \in S \times D\); \(a, b \in B(S)\) and \(t \in S\). Where \(f, g\) and \(h\) are defined as follows:

\[
\begin{align*}
\forall x \in S, a_i \in B(S) \text{ and } i = \{1, 2, 3\} \\
\quad f(a_1(x)) &= \omega_{y \in D}(u(x, y) + H_1(x, y, a(T(x, y)))) \\
\quad g(a_2(x)) &= \omega_{y \in D}(u(x, y) + H_2(x, y, a(T(x, y)))) \\
\quad h(a_3(x)) &= \omega_{y \in D}(u(x, y) + H_3(x, y, a(T(x, y))))
\end{align*}
\]

\((C_3)\) \(f(B(S)) \cup g(B(S)) \subseteq h(B(S))\) and \(h\) is asymptotically continuous and weakly commute with both \(f\) and \(g\).

Then the system of functional equations possess a unique common solution in \(B(S)\).

**Proof** From \((C_1)\) and \((C_2)\), \(f, g\) and \(h\) be self maps on \(B(S)\). Let \(a, b \in B(S)\) and \(x \in S\). For any \(\epsilon > 0\) there exist \(y, z \in D\) satisfying

\[
\begin{align*}
\quad f(a(x)) &< u(x, y) + H_1(x, y, a(T(x, y))) + \epsilon \\
\quad g(b(x)) &< u(x, z) + H_2(x, z, b(T(x, z))) + \epsilon \\
\quad f(a(x)) &\geq u(x, z) + H_1(x, z, a(T(x, z))) + \epsilon \\
\quad g(b(x)) &\geq u(x, y) + H_2(x, y, b(T(x, y))) + \epsilon
\end{align*}
\]

Combining above inequalities with \((C_2)\), we obtain the following:

\[
\begin{align*}
\psi(|f(a(x)) - g(b(x))|) &\leq \psi(\epsilon) + \psi(\max\{|H_1(x, y, a(T(x, y))) - H_2(x, y, b(T(x, y)))|, \\
\quad |H_1(x, z, a(T(x, z))) - H_2(x, z, b(T(x, z)))|\}) \\
&\leq \psi(\epsilon) + \max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\} \\
&\quad - \omega(\max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\})
\end{align*}
\]

Letting \(\epsilon \to \infty\) we get

\[
\begin{align*}
\psi(|f(a(x)) - g(b(x))|) &\leq \max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\} \\
&\quad - \omega(\max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\})
\end{align*}
\]

Therefore, Theorem 3.1 ensures that \(f, g\) and \(h\) have a unique common fixed point in \(B(S)\). That is, the system of functional equations \((7)\) possesses a unique common solution \(B(S)\). Similarly we can change the condition \((C_2)\) in Theorem 3.1 and get the common solution using corollaries. Taking \(h = I\) in Theorem, we conclude that
Theorem 4. Let the following condition hold: $(C_4)$ $u$ and $H_i$ are bounded for $i \in \{1, 2\}$.
$(C_5)$ there exist a $w \in \{W\}$ satisfying

$$
|H_1(x, y, a(t)) - H_2(x, y, b(t))| \leq \max \{d(a, b), d(a, fa), d(b, gb), \frac{1}{2}[d(a, b) + d(fa, gb)]\}
$$
$$
- w(\max \{d(a, b), d(a, fa), d(b, gb), \frac{1}{2}[d(a, b) + d(fa, gb)]\})
$$

for all $(x, y) \in S \times D$; $a, b \in B(S)$ and $t \in S$. Where $f$ and $g$ are defined as follows:

for all $x \in S$, $a_i \in B(S)$ and $i = \{1, 2, 3\}$

$$
f(a_1(x)) = \text{opt}_{y \in D} \{u(x, y) + H_1(x, y, a_1(T(x, y)))\}
$$
$$
g(a_2(x)) = \text{opt}_{y \in D} \{u(x, y) + H_2(x, y, a_2(T(x, y)))\}
$$

for all $x \in S$, $a_1, a_2 \in B(S)$. Then the system of functional equations

$$
f(x) = \text{opt}_{y \in D} \{u(x, y) + H_1(x, y, f(T(x, y)))\}
$$
$$
g(x) = \text{opt}_{y \in D} \{u(x, y) + H_2(x, y, g(T(x, y)))\}
$$

possesses a unique common solution in $B(S)$.

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