Preprint

This is the submitted version of a paper published in *European journal of mechanics. B, Fluids*.

Citation for the original published paper (version of record):

Tammissola, O. (2016)
Optimal wavy surface to suppress vortex shedding using second-order sensitivity to shape changes.
*European journal of mechanics. B, Fluids*, : 10.1016/j.euromechflu.2016.12.006
https://doi.org/10.1016/j.euromechflu.2016.12.006

Access to the published version may require subscription.

N.B. When citing this work, cite the original published paper.

Permanent link to this version:
http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-199204
Optimal wavy surface to suppress vortex shedding using second-order sensitivity to shape changes

O. Tammisola

Faculty of Engineering, The University of Nottingham, University Park, Nottingham NG7 2RD UK

KTH Mechanics, KTH Royal Institute of Technology, SE-10044 Stockholm, Sweden

Abstract

A method to find optimal 2\textsuperscript{nd}-order perturbations is presented, and applied to find the optimal spanwise-wavy surface for the suppression of cylinder wake instability. As shown in recent studies [1, 2, 3], 2\textsuperscript{nd}-order perturbations are required to capture the stabilizing effect of spanwise waviness, which is ignored by standard adjoint-based sensitivity analyses. Here, previous methods are extended so that (i) 2\textsuperscript{nd}-order sensitivity is formulated for base flow changes satisfying the linearised Navier-Stokes, and (ii) the resulting method is applicable to a 2D global instability problem. This makes it possible to formulate the 2\textsuperscript{nd}-order sensitivity to shape modifications. This formulation is used to find the optimal shape to suppress the cylinder wake instability. The optimal shape is then perturbed by random distributions in full 3D stability analysis to confirm that it is a local optimal at the given amplitude and wavelength. At $Re = 100$, surface waviness of maximum height 1\% of the cylinder diameter is sufficient to stabilize the flow. The optimal surface creates streaks passively by extracting energy from the base flow derivatives and altering the tangential velocity component at the wall. This paper extends previous techniques to a fully two-dimensional method to find boundary perturbations which optimize the 2\textsuperscript{nd}-order drift. The method should be applicable to generic flow instability problems, and to different types of control, such as boundary forcing, shape modulation or suction.

Keywords:

1. Introduction

The present study deals with optimal second-order eigenvalue drifts, which may arise as a result of asymmetric control. One example is the flow around a cylinder with spanwise-sinusoidal boundary modulations — the demonstration case in the present study. Since a decade, spanwise waviness is known to efficiently suppress vortex shedding and reduce drag behind bluff bodies. [4] showed experimentally that a spanwise wavy trailing edge completely suppressed the vortex shedding around a rectangular cylinder at $Re = 40000$, resulting in a 30
% reduction of the mean drag. A similar effect was observed by \cite{5} numerically at \( Re = 100 - 500 \).

As pointed out by \cite{5}, the stabilizing effect of spanwise waviness may also be created by changing the wall boundary condition by bleed or transpiration. Through steady spanwise-alternating suction and blowing, \cite{6} shifted the Hopf bifurcation of the wake behind a circular cylinder from \( Re \approx 45 \) to \( Re > 140 \) in DNS. The instability could only be suppressed when the actuation had a spanwise wavelength of \( 5 - 6 \) cylinder diameters. The reason for the efficiency of medium wavelengths has been analysed in several subsequent works. \cite{1} examined the instability of a fixed wake profile superposed with spanwise waviness, and observed that in this model medium wavelengths were not absolutely unstable. \cite{3, 7} considered base flow modifications generated by spanwise-alternating suction. They concluded that the streaks generated by suction were optimally amplified by transient growth at medium wavelengths, and hence the base flow modification was also largest at medium wavelengths. \cite{2} considered formally modifications of global mode eigenvalues with spanwise-wavy base flow modifications, which required 2\textsuperscript{nd}-order perturbations. Wavelength selection was based on an eigenmode resonance at long wavelengths, and the strongest interaction with the 2\textsuperscript{nd}-order sensitivity core at medium wavelengths.

The optimal distribution of spanwise waviness has been studied much less than the optimal wavelength. However, for the flow around the circular cylinder, azimuthal location of the waviness is an important parameter. \cite{6} applied the spanwise-alternating suction from two slots placed on the top and the bottom of the cylinder; locations at the rear and front of the cylinder were mentioned to be inefficient. Moreover, the configuration in which the suction through the upper slot was in-phase with that through the lower slot was found to be much more effective than the anti-phase configuration, which was later explained using the mode resonance effect in \cite{2}. \cite{7} performed a 3D optimization of the azimuthal distribution of waviness in order to create strongest possible base flow streaks. Their optimal distribution also peaked at the top and bottom of the cylinder but was continuous, and stabilized the flow at a much lower peak suction amplitude (<1%) than the slots of \cite{6} (8%) at \( Re = 100 \). However, the optimization was performed on the streakiness of the base flow, and eigenvalue drift was not a part of the optimization. Hence, the method was dependent on the physics of this particular flow and the chosen control.

The aim of the present study is to extend the previous works into a method of mathematically optimizing the eigenvalue drift, so that the method would be applicable for other flow cases and other choices of (asymmetric) boundary control. \cite{8} computed optimal spanwise-wavy base flow modifications for a parallel flow in a mixing layer, accounting for the eigenvalue drift. The 2\textsuperscript{nd}-order perturbation system was written in matrix form and elegantly manipulated to form a Hessian matrix, and the most stabilizing perturbation found from its extremal eigenpairs. The manipulations involved forming an explicit inverse of a system matrix, which was possible since the flow was parallel with 1D eigenfunctions. The global wake instability problem considered here, however, has 2D eigenfunctions.
The present study introduces a new approach to compute optimal boundary perturbations at the 2nd order, accounting for both base flow change and eigenvalue drift. The perturbation system is projected on a smaller basis of boundary functions, and the optimal recovered using only 2D computations no larger than the original system. Using this method, we find the optimal spanwise-wavy cylinder surface to suppress vortex shedding around it. Spanwise-wavy shapes are already used to suppress vortex shedding around e.g. chimneys. The optimal spanwise-wavy shape, however, has not been examined yet.

The new attributes of this approach can be summarised as follows. In [2], base flow modifications induced by wall suction were computed and analysed a posteriori. In [8], 2nd order optimal base flow modifications were computed a priori. Their base flow sensitivity was the 2nd-order counterpart of generic base flow sensitivity[9], in particular, the base flow modifications did not satisfy Navier–Stokes equations. The present theory introduces base flow modifications which satisfy the (boundary-perturbed) Navier-Stokes equations, coupling this with the maximal eigenvalue drift, similarly to an adjoint base flow approach[9]. In addition, the projection to boundary basis functions makes it possible to apply the optimisation to two-dimensional problems. An extension to optimal shapes containing both 1st-order (spanwise constant) and 2nd-order (spanwise wavy) shape modifications is relatively straightforward.

2. Perturbation analysis

Let us consider a general eigensystem of the form:
\[
L^{(0)}\{q^{(0)}\} = \sigma^{(0)}q^{(0)},
\]
where \(q^{(0)}\) is an eigenvector, and \(\sigma^{(0)}\) an eigenvalue, and the curly bracket indicates that the operator \(L^{(0)}\) acts on \(q^{(0)}\) — this notation is adopted throughout the paper. After introducing a small boundary modification denoted by \(\epsilon h\), where \(\epsilon\) is an amplitude parameter and \(h\) is normalized to unity in a given boundary norm, we write:
\[
L(\epsilon h) \{q(h)\} = \sigma(h)q(h)
\]
The solution may be expanded in a perturbation series where \(\epsilon\) denotes the amplitude of \(h\) (e.g. [10]):
\[
\left( L^{(0)} + \epsilon L^{(1)} + \epsilon^2 L^{(2)} + O(\epsilon^3) \right) \left( \sum_{n=0}^{2} \epsilon^n q^{(n)} + O(\epsilon^3) \right) = \left( \sum_{n=0}^{2} \sigma^{(n)} + O(\epsilon^3) \right) \left( \sum_{n=0}^{2} \epsilon^n q^{(n)} + O(\epsilon^3) \right).
\]
where the superscripts in parenthesis, (0), (1), (2), denote indices in the perturbation series (while powers are written without parenthesis). By grouping together terms of any given power of \(\epsilon\), we can generate approximations of the eigenvalue drift accurate up to that order.
1st order. : When collecting terms of the order $\epsilon^1$, we obtain:

$$
\left( \mathcal{L}^{(0)} - \sigma^{(0)} I \right) \{q^{(1)}\} = -\mathcal{L}^{(1)} \{q^{(0)}\} + \sigma^{(1)} q^{(0)},
$$  \hspace{1cm} (4)

where $I$ is the identity operator. By projecting this equation under inner product $\langle \cdot , \cdot \rangle$ with the adjoint eigenmode $q^{(0)+}$, it can be shown \[10\] that the left hand side is zero: $\langle q^{(0)+} , (\mathcal{L}^{(0)} - \sigma^{(0)} I) \{v\} \rangle = 0$ for any vector $v$. By equating the right-hand side with zero, the 1st-order eigenvalue drift $\sigma^{(1)}$ is found to be:

$$
\sigma^{(1)} = \langle q^{(0)+} , \mathcal{L}^{(1)} \{q^{(0)}\} \rangle,
$$  \hspace{1cm} (5)

which is equivalent to the integrated sensitivity used to estimate an eigenvalue drift with respect to control in numerous previous studies (see e.g. \[11\] for a review).

2nd order. : When collecting terms of the order $\epsilon^2$, we obtain:

$$
\left( \mathcal{L}^{(0)} - \sigma^{(0)} I \right) \{q^{(2)}\} = -\mathcal{L}^{(1)} \{q^{(1)}\} - \mathcal{L}^{(2)} \{q^{(0)}\} + \sigma^{(1)} q^{(1)} + \sigma^{(2)} q^{(0)},
$$  \hspace{1cm} (6)

Again, the left hand side is zero when projected by $q^{(0)+}$. By equating the right hand side to zero, $\sigma^{(2)}$ takes the form:

$$
\sigma^{(2)} = \langle q^{(0)+} , \mathcal{L}^{(1)} \{q^{(1)}\} \rangle + \langle q^{(0)+} , \mathcal{L}^{(2)} \{q^{(0)}\} \rangle,
$$  \hspace{1cm} (7)

where $q^{(1)}$ is the 1st-order eigenvector correction obtained from Eq. (4), normalized so that $\langle q^{(0)+} , q^{(1)} \rangle = 1$. The second term represents a change in the 1st-order eigenvalue drift, in the case that $\mathcal{L}$ depends quadratically on the boundary modification. To find $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$, the governing linear operator $\mathcal{L}(h)$ needs to be Taylor-expanded so that:

$$
\mathcal{L} = \mathcal{L}^{(0)} + \epsilon \mathcal{L}^{(1)} + \epsilon^2 \mathcal{L}^{(2)} + O(\epsilon^3)
$$  \hspace{1cm} (8)

where $\mathcal{L}^{(1)}$ is linear in $h$, and $\mathcal{L}^{(2)}$ is a symmetric bilinear form in $h$, in the boundary inner product to be defined soon.

Shape changes, boundary suction, or mass injection at the cylinder can all be addressed by the method presented next with minimal adjustments to the boundary conditions. It is common in shape optimization to parameterize the boundary, to reduce the degrees of freedom, but also obtain robust optimal shapes which are easy to manufacture \[12\]. Let us parameterize the displacement of the cylinder wall using $N$ basis functions:

$$
\epsilon h = \sum_{n=1}^{N} a_n h_n.
$$  \hspace{1cm} (9)

\[^1]\text{which implies that } \sigma_1(q_0^+, q_1) = 0
Then, by substituting the above sum into Eq. (7):

\[ \epsilon^2 \sigma_2 \]  

\[ = \langle q^{(0) +}, L^{(1)} (\epsilon h) \{ q^{(1)} (\epsilon h) \} \rangle + \langle q^{(0) +}, L^{(2)} (\epsilon^2 h^2) \{ q^{(0)} \} \rangle \]

\[ = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \left[ \langle q^{(0) +}, L^{(1)} (h_n) \{ q^{(1)} (h_m) \} \rangle + \langle q^{(0) +}, L^{(2)} (h_n, h_m) \{ q^{(0)} \} \rangle \right] \]

\[ = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \bar{S}_{nm} \]  

(10)

The reason that the sums could be moved outside the brackets is that both terms in \( \langle \cdot \rangle \) are quadratic forms. This can be seen from that both terms are of the 2nd order in \( h \), but also follows more generally in this Hilbert space, because \( \sigma^2 \) is a Hessian of \( \sigma \) with respect to \( h \). Every quadratic form can be written using a symmetric matrix. To rewrite \( \sigma^2 \) using a symmetric 2nd-order sensitivity operator matrix \( \bar{S} \), set \( \bar{S} = 0 \) to obtain

\[ \sigma^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \bar{S}_{nm} \]  

(11)

Eq. (11) gives the eigenvalue drift for any given choice of \( \epsilon h \), but the question of the optimal \( \epsilon h \) for a given norm (or equivalently, the optimal \( h \) of unit norm) needs to be addressed. To this end, let us denote the chosen inner product by \( \langle \cdot \rangle_B \) so that for any boundary function \( \phi \):

\[ ||\phi||_B = \langle \phi, \phi \rangle_B \]

If the basis functions \( h_n \) are non-orthonormal in \( \langle \cdot \rangle_B \), then a "mass matrix" \( M \) needs to be applied so that:

\[ M_{mn} = \langle h_m, h_n \rangle, \]  

and hence

\[ \epsilon^2 = || \sum_{n=1}^{N} a_n h_n ||^2 = a_m M_{mn} a_n = a^T Ma \]

Doing so, we obtain the result below (where the mathematical manipulation of the mass matrix is identical to the manipulation of the energy matrix in a transient growth analysis\[13\]). In the following, \( \Re \) denotes the real part of a complex number:

\[ \min_{||h||_B=1} \Re \sigma_2 = \min_{||\epsilon h||_B=\epsilon} \Re \epsilon^2 \sigma_2 = \min_a a^T \Re \bar{S} a = \min_a a^T FF^T a \]  

5
\[
\begin{align*}
\mathbf{M} = \mathbf{F}^T \mathbf{F}^T & \quad \text{the Cholesky decomposition of the mass matrix, and} \\
\mathbf{b} = \mathbf{F}^T \mathbf{a} & \quad \text{The largest decrease in growth rate is achieved when the weighted coefficient vector } \mathbf{b} \text{ is the eigenvector of the matrix } (\mathbf{F}^T)^{-1} \hat{\mathbf{S}}(\mathbf{F})^{-1} \text{ corresponding to the eigenvalue with the most negative real part. For orthonormal basis sets (which are preferred for simplicity), the mass matrix } \mathbf{F} \text{ is the identity matrix,} \\
\text{and the optimal simplifies to } & \quad \min (\Re(\text{eig} \left[ \left( \mathbf{F}^T \right)^{-1} \hat{\mathbf{S}}(\mathbf{F})^{-1} \right])) \quad \text{where } \mathbf{a} \text{ is the eigenvector corresponding to the eigenvalue with smallest real part.}
\end{align*}
\]

Summarizing, the elements of the sensitivity operator (matrix) \( \hat{\mathbf{S}} \) are formed by computing eigenvalue drifts for \( 0.5N^2 \) pairs of basis functions \( (h_n, h_m) \). The most stabilizing shape change at the 2\textsuperscript{nd} order for orthonormal shape function bases is the eigenvector of the small matrix \( \Re(\hat{\mathbf{S}}) \) corresponding to the most negative eigenvalue. Correspondingly, the most destabilizing shape change at the 2\textsuperscript{nd} order is the eigenvector of the small matrix \( \Re(\hat{\mathbf{S}}) \) corresponding to the most positive eigenvalue. It is worth noting that this method is fundamentally equivalent to the one in \[8\], where the sensitivity operator \( \sigma_2 = \langle \delta \mathbf{U}, \mathbf{S} \{ \delta \mathbf{U} \} \rangle \) was formed, with the difference that here the base flow modifications result from boundary modifications and satisfy the Navier-Stokes equations.

3. Optimal boundary perturbations of global modes of Navier-Stokes

This study considers the stabilization of the primary wake instability eigenmode by means of spanwise-wavy boundary perturbations with wavelength \( \beta_B \), parameterized as:

\[
\epsilon h = \sum_{n=1}^{N} a_n h_n = \sum_{n=1}^{N} a_n \tilde{h}_n \cos \beta_B z = \sum_{n=1}^{N} a_n \tilde{h}_n \left( \exp \{ i\beta_B z \} + \exp \{ -i\beta_B z \} \right).
\]

The \( L_2 \)-inner product applied to (real-valued) boundary functions \( \phi_a, \phi_b \) is:

\[
\langle \phi_a, \phi_b \rangle_B = \lim_{L_z \to \infty} L_z^{-1} \int_{-L_z/2}^{L_z/2} \int_{B(x,y)} \phi_a \phi_b \, dx \, dy \, dz
\]

so that the amplitude of the boundary perturbation becomes:

\[
A = ||\epsilon h||_2 = \sqrt{\langle \epsilon h, \epsilon h \rangle_B}
\]

\[\text{(13)}\]
Similarly, the $L_2$-inner product is applied to the velocity fields:

\[
\langle u_a, u_b \rangle = \lim_{L \to \infty} L^{-1} \int_{-L/2}^{L/2} \int_{D(x,y)} u^*_a u_b \, dx dy dz
\]

\[
= \lim_{L \to \infty} L^{-1} \int_{-L/2}^{L/2} \int_{D(x,y)} (u^*_a u_b + v^*_a v_b + w^*_a w_b) \, dx dy dz,
\]

where $D(x,y)$ is the 2D flow domain, and $^*T$ denotes the transpose conjugate. The steady base flow $\left( U^{(0)}, P^{(0)} \right)$, for a straight cylinder satisfies the two-dimensional, steady, nonlinear, incompressible Navier-Stokes equations:

\[
U^{(0)} \cdot \nabla U^{(0)} + \nabla P^{(0)} - Re^{-1} \nabla^2 U^{(0)} = 0.
\]

The global linear eigenmodes $q^{(0)} = (u^{(0)}, p^{(0)})$ which develop around the base flow satisfy the linearized Navier–Stokes:

\[
\mathcal{L}^{(0)} q^{(0)} = -U^{(0)} \cdot \nabla u^{(0)} - u^{(0)} \cdot \nabla U^{(0)} - \nabla p^{(0)} + Re^{-1} \nabla^2 u^{(0)} = \sigma^{(0)} u^{(0)},
\]

The adjoint eigenmodes $q^{(0)+} = (u^{(0)+}, p^{(0)+})$ satisfy

\[
\mathcal{L}^{(0)+} q^{(0)+} = - (\nabla U)^T \cdot \hat{u}^{(0)+} + U^{(0)} \cdot \nabla \hat{u}^{(0)+} - \nabla \hat{p}^{(0)+} + Re^{-1} \nabla^2 \hat{u}^{(0)+} = \sigma^{(0)} \ast u^{(0)+},
\]

where $\ast$ is a complex conjugate. Next, we will perturb equations (17) and (18) at different orders in $\epsilon$.

1\textsuperscript{st} and 2\textsuperscript{nd} order base flow modification. A detailed derivation of the base flow modifications equations is included in Appendix A, but the final equations are stated below.

When perturbing the base flow equation and the boundary condition at the cylinder wall, grouping terms of the order $\epsilon^1$, and introducing the decomposition (Eq. 13) we obtain:

\[
U^{(1)}_n \cdot \nabla U^{(0)} + U^{(1)}_n \cdot \nabla U^{(0)} + \nabla P^{(1)}_n - Re^{-1} \nabla^2 U^{(1)}_n = 0.
\]

with a nonzero Dirichlet boundary condition cylinder wall:

\[
U^{(1)}_n = -h_n \frac{\partial U^{(0)}}{\partial r}
\]

and $U^{(1)}_n = 0$ at all other boundaries.

When grouping the terms of order $\epsilon^2$, we obtain:

\[
U^{(0)} \cdot \nabla U^{(2)}_{nm} + U^{(1)}_n \cdot \nabla U^{(1)}_m + U^{(2)}_{nm} \cdot \nabla U^{(0)} + \nabla P^{(2)}_{nm} - Re^{-1} \nabla^2 U^{(2)}_{nm} = 0.
\]

with the boundary condition at the cylinder wall:

\[
U^{(2)}_{nm} = -h_n \frac{\partial U^{(1)}_m}{\partial r} - h_m h_n \frac{\partial^2 U^{(0)}_m}{\partial r^2} = 0
\]

and $U^{(2)}_{nm} = 0$ at all other boundaries.
Figure 1: The three first surface modification basis functions: \( \hat{h}_1 \) (blue solid), \( \hat{h}_2 \) (red dash) and \( \hat{h}_3 \) (green dash-dot): (a) Fourier basis (orthonormal), (b) Gaussian RBF basis (non-orthogonal). The cylinder radius is modified as \( \delta r(\theta) = \hat{h}(\theta) \cos(\beta z) \).

1\textsuperscript{st} order eigenvector correction. In what follows, we adopt the notation

\[
h_n = h_{n+} + h_{n-}, \quad h_{n\pm} = 0.5h_n \exp \{ \pm i\beta_B z \}.
\]

As \( U_n^{(1)} \) and the first order eigenvector correction \( u_n^{(1)} \) are linear in \( h \), they divide similarly:

\[
U_n^{(1)} = U_{n+}^{(1)} + U_{n-}^{(1)}, \quad U_{n\pm}^{(1)} = \hat{U}_n^{(1)} \exp \{ \pm i\beta_B z \}
\]
\[
u_n^{(1)} = u_{n+}^{(1)} + u_{n-}^{(1)}, \quad u_{n\pm}^{(1)} = \hat{u}_n^{(1)} \exp \{ \pm i\beta_B z \},
\]

where the eigenvector corrections are obtained from:

\[
\left( \mathcal{L} - \sigma(0) \mathcal{I} \right) \{ q^{(1)} \} = \{-\hat{u}_n^{(1)} \cdot \nabla U^{(0)} - U^{(0)} \cdot \nabla \hat{u}_n^{(1)} - \nabla p^{(1)} + Re^{-1}\nabla^2 \hat{u}_n^{(1)} - \sigma(0) \hat{u}_n^{(1)} \}
\]

\[
= -L_1 \{ q^{(0)} \} = \{ \hat{u}_n^{(0)} \cdot \nabla U^{(1)} + U^{(1)} \cdot \nabla \hat{u}^{(0)} \}
\]

2\textsuperscript{nd}-order sensitivity. The 2\textsuperscript{nd}-order sensitivity contains an integral in the span-wise \( z \)-direction, and hence, any products where the superposed wavenumber is nonzero will cancel out. This leads to:

\[
\tilde{S}_{nm} = S_{nm+} + S_{nm-} + S_{mn+} + S_{mn-},
\]

where

\[
S_{ij\pm} = \int_V \{-\hat{u}_i^{(0)} + \hat{u}_j^{(2)} \cdot \nabla U^{(1)} + U^{(1)} \cdot \nabla \hat{u}_j^{(1)} + \hat{u}_j^{(0)} \cdot \nabla U^{(2)} + \hat{u}_j^{(0)} \cdot \nabla U^{(2)} \cdot \nabla \hat{u}_j^{(0)} \} dV
\]

where the first integral is the same as in [2]. The second integral accounts for the base flow changes at the 2\textsuperscript{nd} order, which were neglected in [2] by numerical evidence, as will be done for the results in the cylinder wake demonstration case presented in this paper. We can observe that only 2\textsuperscript{nd}-order base flow changes.
which are spanwise constant \((i.e.\) a zero spanwise wavenumber) can contribute to \(\sigma_2\). It will be shown later that the induced spanwise constant base flow changes are small for this wavy cylinder flow, compared to the base flow changes at the chosen \(\beta_B\). More importantly, in \cite{2}, it was shown that the eigenvalue is 30-60 times more sensitive to spanwise base flow changes at \(\beta_B = 0.5 - 2\) than at \(\beta_B = 0\). This implies that even if the spanwise-constant modification would have been of the same order, it would still have contributed 30-60 times less to \(\sigma_2\).

Assumption of negligible \(U_{i \pm, j \mp}^{(2)}\) is not expected to hold in general flows or general asymmetric control, and hence in general \(U_{i \pm, j \mp}^{(2)}\) should be computed and the second integral included in \(\sigma_2\). This will somewhat increase the computational time, but all computations will still remain two-dimensional.

4. Numerical method

To formulate the problem numerically, we need to start by choosing \(N\) basis functions onto which the azimuthal surface modification distribution can be projected. The method in this paper can be performed with an arbitrary set of basis functions. In this work, two different basis sets are examined. The first set which called FR\(_1\)-FR\(_N\) (Fig. 1a) is an orthonormal Fourier expansion in the azimuthal angle \(\theta\), similarly to\cite{7}:

\[
Tn = \begin{cases} 
1, & n = 1 \\
\sqrt{2} \cos \left(\frac{(n-1)}{2} \theta\right), & \forall n > 1 \text{ odd} \\
\sqrt{2} \sin \left(\frac{n}{2} \theta\right), & \forall n > 1 \text{ even} 
\end{cases} 
\] (27)

The second basis called RBF\(_1\)-RBF\(_N\) (Fig. 1b) is a radial basis function basis, which can be shown to be spectrally accurate if the function to be approximated is sufficiently smooth (see e.g. \cite{14} regarding the choice of a RBF basis). The basis consists of Gaussians which describe the normal (radial) displacement of the surface point \(r = [\cos\theta, \sin\theta]\) (where \(\theta\) is an azimuthal angle) as:

\[
T_hn = \exp\{-50||r - r_{0,n}||\} 
\] (28)

and the midpoints of the Gaussians are evenly spaced over the cylinder circumference:

\[
r_{0,n} = [\cos(2\pi n/N), \sin(2\pi n/N)] 
\] (29)

Convergence of the optimal with respect to the number of basis functions for both sets will be demonstrated in Sec. 5.

The numerical solution of the optimal wavy shape is fully two-dimensional and consists of five steps:

1. base flow
2. direct and adjoint eigenmodes (Eq. 18-19)
3. base flow modifications
   \((N\) solutions of Eq. 20-21)
   and in general also \(0.5N^2\) solutions of Eq. 22-23
4. eigenvector corrections ($N$ solutions of Eq. 24)
5. inner products (evaluations of the $0.5N^2 + 0.5N$ integrals in Eq. 10).

For simplicity, the whole method is implemented into the matrix-based open-source finite-element software FreeFem++ (a more detailed description of the method can be found in [2]). It is worth mentioning that the method is software-independent and could be implemented in connection to any linear stability solver (whether matrix-based or a time-stepper such as Nek5000).

After derivation of the variational formulation of the governing equations, the associated sparse matrices are built by FreeFem++. The grid consists of 97010 Taylor-Hood elements ($P_2 - P_1$). Steps (a) and (b) are identical to [2]. In (a), the base flow for the non-wavy cylinder is generated by solving the time-independent (steady) Navier-Stokes equations using a Newton-Raphson method. In (b), the direct and adjoint eigenproblems for the non-wavy cylinder are solved by the Arnoldi solver in FreeFem++ combined and UMFPACK. In step (c), the steady linear equation systems (24) for the base flow modifications are solved directly using the sparse LU solver UMFPACK. In step (d), the steady eigenvector correction equations (24) are solved again with UMFPACK. These computations are needed to form the small $N \times N$-matrix $\tilde{S}$. The eigenpairs of this small matrix can be found using any standard QZ-algorithm, e.g. `eig` of MATLAB used in the present work. The reason a QZ-algorithm is preferred over other methods is that $\tilde{S}$ may be nearly singular in the general case. The eigenvalues of $\tilde{S}$ represent global mode eigenvalue drifts for different basis function combinations, and many combinations may have negligible effect on the eigenvalue (resulting in near-zero eigenvalues of $\tilde{S}$).

The verification of the above method is performed by a direct computation of 3D base flows and tri-global eigenmodes with a wavy boundary using a spectral element method (SEM) implemented in Nek5000 [15], which has an efficient parallelization scaling linearly up to millions of processors. The grids for the wavy cylinders are generated by starting from a mesh for a flow around a straight cylinder. The outer domain boundaries form a rectangular block with a diagonal from $[x, y, z] = [-20, -20, -\pi/\beta_B]$ to $[x, y, z] = [50, 20, \pi/\beta_B]$. The element distribution is uniform in $z$, and finest close to the cylinder in the $x$-$y$-plane. Periodic boundary conditions are imposed at $z = \pm \pi/\beta_B$. In this work, three different meshes have been constructed: one for $\beta_B = 0.8$ (52224 spectral elements, 12 elements in $z$), $\beta_B = 1.26$ (34817 elements, 8 elements in $z$), and $\beta_B = 2$ (26112 elements, 6 elements in $z$). Another option would have been to stretch the same mesh in the $z$-direction. In a second step, the mesh is mapped onto the wavy cylinder boundary by solving a Laplace equation for the mesh displacements in the $x$-$y$-plane, where the boundary conditions are a given displacement at the cylinder boundary and zero displacement at the outer boundaries. The polynomial order of the elements is $p = 6$. Firstly, the base flows around wavy cylinders are obtained by integrating the nonlinear Navier-Stokes equation forward in time, and converging towards the steady solution by applying selective frequency damping [10]. Secondly, the full 3D eigenpairs of the Linearized Navier–Stokes operator are computed using the linearized DNS.
time-stepper available in Nek5000 coupled with an Arnoldi method as in [17, 2].

5. Results

The basic method for the base flow and eigenmode computations has been cross-validated between the two solvers (FreeFem++ and Nek5000), and against results from the literature. For example at $Re = 50$, $\sigma = 0.0143 + 0.7492i$ ($St = 0.119$) for FreeFem++ and $\sigma = 0.0128 + 0.746i$ ($St = 0.119$) for Nek5000. Both compare well to the eigenvalue in Fig. 7 at $Re = 50$ in [18] with $\sigma_r = 0.013$ and $St = 0.119$. The method of 2nd-order perturbations was introduced in [2], and the eigenvalue drift reproduced well against 3D computations. Here, we focus on validating the new features of this work, which are: (1) Computation of base flow modifications inside the algorithm, and (2) Optimal boundary modulations inducing a maximal 2nd-order eigenvalue drift.

5.1. Base flow modifications

For the results shown in this paper, for simplicity, only the linear component of the base flow modification $L_1$ has been computed. Hence, the optimals shown here are accurate assuming that the 2nd-order base flow modification $L_2$ is small. This assumption was based on the physics of the chosen flow case, and will not hold for 2nd-order optimals in arbitrary flows. The method itself does not require this (the 2nd-order base flow changes can also be computed two-dimensionally as described in Sec. 2–4), but this made the implementation somewhat easier, and the computations cheaper.

The predicted and exact base flow modifications are compared in Fig. 2. The exact modification is created as follows. Firstly, a 3D base flow around the wavy cylinder is computed in Nek5000. The difference is then formed between the base flow velocity around the wavy cylinder, and the base flow velocity around the non-wavy cylinder. This gives the exact base flow modification (Fig. 2 a). This can be compared against $U_1$ predicted by FreeFem++ (Fig. 2 b).

The assumption of linear base flow changes is supported next by numerical evidence from the exact (Nek5000) base flow modification, even though the most important argument relies on the sensitivity of the eigenvalue drift. The overall norm and the norm of the three velocity components with amplitude are shown in Fig. 3 (a,b,c) for three different different shapes and two different spanwise wavelengths. The shapes chosen are the optimal shape at $\beta_B = 0.8$ (Fig. 3 c), the first Fourier basis function $FR_1$ at $\beta_B = 2$ (Fig. 3 a), and the first RBF basis function $RBF_1$ at $\beta_B = 0.8$. All velocity components for all three cases remain essentially linear in the small-amplitude range ($A < 0.04$). As expected, the optimal induces much larger base flow changes than the other shapes: > 5 times larger than RBF$_1$ (at $\beta_B = 0.8$), and > 12 times larger.

\footnote{This value contained a typo in the previous version}
Figure 2: Spanwise velocity induced by surface waviness, $A \approx 0.02$, $\beta_B = 0.8$, $Re = 50$: (a) Optimal, exact (Nek5000 in 3D), (b) Optimal, predicted (FreeFem++ in 2D), (c) RBF$_1$, exact, (d) RBF$_1$, predicted.
Figure 3: Relative norms of the different components of base flow change due to waviness, Nek5000: (a) $L_2$-norm of the base flow change for RBF1, $\beta_B = 0.8$, (b) $L_2$-norm of the base flow change for FR1 at $\beta_B = 2$, (c) $L_2$-norm of the base flow change for the optimal at $\beta_B = 0.8$, (d) Relative amplitudes of the linear ($\beta = \beta_B$) and quadratic spanwise constant ($\beta = 0$) base flow changes.
than FR₁ (at \( \beta_B = 2 \)). For RBF₁ which has a high peak, the 3D mesh quality started to become too poor to continue to higher amplitudes. For FR₁, which is constant in the azimuthal direction, it was possible to increase the amplitude until \( A = 0.2 \) (top-to-top amplitude 22% of the cylinder diameter) without the mesh quality getting too poor, and the high-amplitude range is also shown.

In Sec. 2–3 it was shown mathematically that only zero-wavenumber non-linear base flow change can give a contribution to the 2\(^{\text{nd}}\)-order drift \( \sigma_2 \). The zero-wavenumber component of the base flow change has also been extracted from Nek5000 for the optimal shape, and is shown against the linear component \( (\beta = \beta_B) \) and the total norm in Fig. 3 (d). In the amplitude range analysed in this paper \( (A \leq 0.02) \), the zero-wavenumber component is maximally 4% of the linear component. However, more importantly, [2] showed that for a wake flow, the global mode eigenvalue is 30 – 60 times more sensitive to base flow changes at \( \beta_B = 0.5 – 1 \) than it is to base flow changes at \( \beta_B = 0 \). This is the real reason why the nonlinear zero-wavenumber component can be neglected in this specific flow case.

5.2. Optimal shape

Now, we will address the optimal wavy surface shape near bifurcation at \( Re = 50 \) for validation purposes. To make the stabilization more challenging, the wavelength of the waviness is chosen to be \( \beta_B = 0.8 \), longer than the wavelength
Figure 5: Optimal shape at \( Re = 50 \) obtained with Fourier vs. Gaussian RBF basis, both with \( N = 24 \).

| Case                          | \( \sigma \) in Nek5000          |
|-------------------------------|----------------------------------|
| Original straight cylinder    | 0.0128 + 0.7467i                 |
| Optimal unperturbed           | -0.02205 + 0.7380i               |
| Optimal perturbed by RAND1-FR | -0.0208 + 0.7385i                |
| Optimal perturbed by RAND2-FR | -0.0203 + 0.7378i                |
| Optimal perturbed by RAND3-FR | -0.0205 + 0.7381i                |
| Optimal perturbed by RAND4-FR | -0.0203 + 0.7377i                |
| Optimal perturbed by RAND5-FR | -0.0205 + 0.7384i                |
| Optimal perturbed by RAND6-FR | -0.0202 + 0.7381i                |
| Optimal perturbed by RAND7-FR | -0.0202 + 0.7383i                |
| Optimal perturbed by RAND8-FR | -0.0196 + 0.7386i                |
| Optimal perturbed by RAND9-FR | -0.0207 + 0.7379i                |
| Optimal perturbed by RAND10-FR| -0.0204 + 0.7379i                |

Table 1: Eigenvalues from 3D computations in Nek5000 of: (a) the optimal wavy shape normalized to \( A = 0.03 \) (row 1), (b) the optimal wavy shape superposed with a random perturbation at \( A = 0.01 \) and re-normalized to \( A = 0.03 \) (rows 2-11). The 20 coefficients of the random vectors are obtained in succession from MATLAB’s \texttt{rand}-function with default settings.
Table 2: Eigenvalue growth rates (columns 1 and 2) and drifts column (3 and 4) for the straight cylinder, optimal cylinder and 10 random shapes, in Nek5000 and FreeFem++. The eigenvalue drift gives the absolute value, and percent of the optimal drift in parenthesis. All values are rounded to four decimal places.

| Case            | Nek5000: $\sigma_r$ | FreeFem++: $\sigma_r$ | Nek5000: $\delta\sigma_r$ | FreeFem++ : $\delta\sigma_r$ |
|-----------------|----------------------|------------------------|-----------------------------|-------------------------------|
| Straight cylinder | 0.0128               | 0.0140                 | 0.0                         | 0.0                           |
| Optimal         | -0.02205             | -0.0171                | -0.0348                     | -0.0311                       |
| RAND1-RBF       | 0.0130               | 0.0137                 | 0.0001                      | -0.0003                       |
| RAND2-RBF       | 0.0129               | 0.0140                 | 0.0000                      | 0.0000                        |
| RAND3-RBF       | 0.0098               | 0.0100                 | -0.0003                     | -0.0004                       |
| RAND4-RBF       | 0.0058               | 0.0060                 | -0.0070                     | -0.0080                       |
| RAND5-RBF       | 0.0101               | 0.0109                 | -0.0003                     | -0.0003                       |
| RAND6-RBF       | 0.0125               | 0.0136                 | -0.0003                     | -0.0004                       |
| RAND7-RBF       | 0.0131               | 0.0144                 | -0.0002                     | 0.0004                        |
| RAND8-RBF       | 0.0129               | 0.0140                 | 0.0000                      | 0.0000                        |
| RAND9-RBF       | 0.0007               | 0.0005                 | -0.0121                     | -0.0135                       |
| RAND10-RBF      | 0.0109               | 0.0117                 | -0.0019                     | -0.0023                       |

Figure 6: Wavy cylinder shapes. Left: Random waviness RBFR9, Middle: Random waviness RBFR10, Right: Optimal waviness at $Re = 50$. Upper row: 3D illustration, with waviness amplitude exaggerated by a factor 3. Lower row: 2D wavy cross-section in scale (solid line, red online). Circular cylinder shape shown for comparison (dashed line, blue online).
Figure 7: The optimal wavy cylinder at $Re = 100$ (in real scale), and streamwise velocity at selected cross-sections shown by both colours and contours with spacing $\Delta U = 0.16$.

Figure 8: (a) Optimal spanwise-wavy shape distributions $h(\theta)$ compared with the optimal spanwise-wavy suction distribution $U_s$, all obtained by the method in Sec. 2. (b) Tangential ($u_{i,t}$) and normal ($u_{i,n}$) surface velocity distributions induced by $h(\theta)$, compared with $U_s$. 
$\beta_B \approx 1 - 1.5$) found to be optimal in previous studies of boundary suction. After validation, we will proceed to look at the optimal shape at $Re = 100$ at the near-optimal wavelength of $\beta_B = 1.26$. It needs to be emphasized that the optimal wavelength selection results might not carry over from boundary suction to boundary shape; however, if physical stabilization mechanisms are similar one might expect qualitatively similar wavelength selection. Further wavenumber variations will be omitted here for brevity; this paper presents a method for computing a 2nd-order optimal stabilizing distribution of waviness for a given spanwise wavenumber.

Firstly, we confirm that the optimal is independent of the chosen boundary basis, and the number of basis functions (as this is only one iteration of the optimization, based on the local Hessian, it cannot converge to different local optima). Both radial basis functions (Fig. A 4) and Fourier basis functions (Fig. B 4) are converged when $N = 20$, and result in identical optimal shape (Fig. C 4). It should be noted that Ref. [7] obtained convergence with only 6 Fourier basis functions for their optimal suction distribution obtained by another method. The reason that we need more functions here is that the shape optimum has four turning points instead of two for the suction optimum, and also that Ref. [7] used only the symmetric functions, assuming that the optimum would be symmetric.

The most stabilizing azimuthal distribution is predicted in FreeFem++ using the 2D 2nd-order sensitivity method outlined in Sec. 3–4. Now, this optimal distribution will be validated by direct computation of eigenvalues for 3D wavy cylinders in Nek5000. The validation is presented in Table 1. The straight cylinder eigenvalue at $Re = 50$ in Nek5000 is found to be unstable with growth rate 0.0128. The eigenvalue for the cylinder with predicted optimal waviness at $A = 0.021$ stabilizes to −0.0221 in Nek5000. This shows that the predicted optimal waviness stabilizes the flow in Nek5000. Next, we investigate whether the predicted shape is an optimal (most stabilizing) shape, at least for small waviness amplitudes. If the predicted shape is a local optimal, then any small deviation from it should produce less stable growth rates than the optimal shape does in a full 3D (tri-global) stability computation. To confirm this, we superpose the optimal shape with a random shape distribution of a small amplitude ($A = 0.004$), and re-normalise the result to $A = 0.021$. This test has been done for 10 different random azimuthal shape distributions called RAND-FR1,...,RAND-FR10, using the Fourier basis with $N = 24$, at the same spanwise wavelength $\beta_B = 0.8$. Only symmetric functions were perturbed. The coefficient vectors $a$ (Eq. 9) for these random shapes are zero for sines, and the first 12 obtained from the rand-function in MATLAB with the default settings for cosines. The superposition of the optimal shape with each of the random shapes was created in Nek5000, and the 3D eigenvalues computed, summarized in Table 1. In all cases, the eigenvalue is more stabilized by the optimal shape than the perturbed optimal. This confirms that the predicted optimal is at least a local optimal. If the theory is correct, then the predicted optimal is also a global optimal.

It is also interesting to see how much better the optimal performs compared to random waviness with $\beta_B = 0.8$. To see this, we have performed a second
test where the random shape distributions were applied alone (without the optimal), this time perturbing the RBF functions. The eigenvalues are again listed in table 1 first column. They show that none of the random shapes stabilize the flow at $A = 0.021$. The optimal decreases the growth rate three times as much as the best random wavy shape at $A = 0.021$. The predicted growth rates from FreeFem++ for each shape are shown in the second column. The predicted growth from FreeFem++ remains higher throughout, because the original unperturbed growth is higher (0.014 compared to 0.0128 in Nek5000). However, the differences in the eigenvalue drift (column 3 and 4) are smaller than the differences in the original eigenvalue. For shapes RAND-RBF3 and RAND-RBF7, the sign differs, but in those cases the drift is two orders of magnitude smaller than the optimal drift. The quantitative differences are probably attributed to the two different discretizations (SEM vs. FEM). In the literature, seldomly an exact correspondence has been reported between predicted and computed global mode eigenvalue drift (even within the same numerical method), due to the highly non-normal nature of linear global modes. The qualitative similarity of the eigenvalue drifts in the two numerical methods explains that the optimal found in FreeFem++ was confirmed to be optimal also in Nek5000 (table 1).

The optimal distribution of waviness is illustrated in Fig. 6, right column, alongside two of the random shapes (left and middle columns). The amplitude of the waviness is exaggerated in the 3D illustrations (top). The optimal distribution is symmetric with respect to $y = 0$ ("in-phase" as expected from previous spanwise-wavy suction studies) and attains its maximum displacement near $\theta = \pm 90^\circ$ ($y = \pm 0.5$). Random shape 9 (Fig. 6 left) also has a substantial amplitude near $\theta = \pm 90^\circ$, and is the second most efficient with growth rate decrease $\delta\sigma_r = -0.01$. Random shape 10 (Fig. 6 right) distributes its amplitude at the upstream and downstream ends of the cylinder, and is inefficient with $\delta\sigma_r = -0.002$. For square cylinders, both leading edge and trailing edge waviness has been considered previously [5]. The present results for a circular cylinder indicate that waviness near the wake separation has the largest effect; if this can be generalized to a square cylinder, then the best position for the waviness would be at the corners of the trailing edge. Summarizing, the results so far show clearly two features: (i) the method produces optimal shapes to a good approximation, and (ii) the distribution of the waviness is important — random shapes are less efficient than optimal shape at a fixed spanwise wavelength.

Going forward, we investigate changes in the optimal shape and its performance when increasing the Reynolds number to $Re = 100$. Both 2D predictions and 3D validations confirm that the waviness is even more efficient at $Re = 100$ than at $Re = 50$; an amplitude of $A = 0.01$ (maximum displacement $\approx 1\%$ of the cylinder diameter) is sufficient to stabilize the flow in both analyses at $Re = 100$. The optimal shape distributions at $Re = 50$ and $Re = 100$ are depicted together in Fig. 8(a), showing that the optimal distribution of waviness

\[ \text{A few random shapes will most probably stabilize the flow at higher amplitudes, as the eigenvalue drift is quadratic.} \]
remains qualitatively similar.

Finally, design robustness of the optimal solution deserves to be mentioned, because if the optimality range is very narrow, the stabilizing influence might not be observed in real-life applications. Indications of the design robustness can be obtained by comparing the magnitudes of different eigenvalues of the Hessian matrix \( \tilde{\sigma} \). The eigenvectors of the Hessian form an orthonormal basis. If the most positive (destabilizing) Hessian eigenvalue is of larger or similar magnitudes as the most negative (stabilizing) Hessian eigenvalue, then the optimal shape is not robust. The reason is that small components of destabilizing eigenvectors could counteract the stabilizing influence. At \( Re = 50 \), the ratio between the most stabilizing and most destabilizing eigenvalues of the real part of the Hessian is 20, which seems relatively robust. At \( Re = 100 \), this ratio is 197, which is considerably more robust.

5.3. Relation between the optimal shape and the optimal suction distributions

The optimal suction distribution on a circular cylinder was considered in [3, 7]. They observed that the suction changes the base flow by creating streaks through a lift-up effect. The suction distribution which creates streaks of the maximal amplitude for a given suction amplitude was found. The peak amplitude of this distribution occurred at \( \theta = \pm 90 \), the same as the spanwise-wavy surface here, suggesting that the same mechanism may be active. Fig. 7 shows the optimal wavy cylinder at \( Re = 100 \) and \( A = 0.01 \) in scale, together with the streamwise velocity around this cylinder from DNS. The figure illustrates that the flow is steady, and that the wavy cylinder is a very efficient streak generator, which may explain its stabilizing influence. The surface waviness is so small that it seems unobservable. The waviness-induced streaks, i.e. the variation of the streamwise velocity, are however strong. The strength of the streaks is shown by the spanwise variation of colours in the horizontal cross-section, and length of the streaks is indicated by the displacement of the velocity contours in the horizontal plane. The reverse flow velocity has a spanwise variation from \( U = -0.4 \) to \( U = -0.01 \) at \( z \approx 3 \), nearly breaking the zone into separate recirculation cells. The vertical velocity variation is small in comparison to the streamwise velocity variation, as expected for streaks. This is shown by that contours in the vertical planes are displaced very little.

Optimal spanwise-suction distribution has been obtained for the cylinder flow by [7]. Their boundary suction was optimized to maximize the amplitude of streaks in the cylinder base flow. They showed that small-amplitude spanwise-wavy suction was able to generate high-amplitude streaks through lift-up effect, and hence assumed that maximal streaks would also generate maximal stabilizing effect. Later, [8] showed that in a parallel mixing layer, the structures which induce optimal transient growth also induce maximal 2\( \text{nd} \)-order eigenvalue drift. Figures 3 and 2 indeed reveal that the optimal indeed generates base flow changes which are several times stronger than for arbitrarily chosen functions.

It would therefore be interesting to compute an optimal suction distribution using the present method, and compare that both to the optimal suction distri-
bution by [7] to see whether the results obtained by the different methods agree, but also to the optimal shape distribution already obtained. To this end, we also computed the optimal suction distribution at $Re = 100$ using the method of 2nd-order optimals. The suction velocity is represented in the same $h_n$-basis, but leading to new $U_1$ with a different boundary condition: $\delta U = (0, -h_n)$ (wall-normal suction), and therefore also new eigenvector corrections. The optimal suction distribution at $Re = 100$ obtained this way is shown by a dashed line in Fig. 8 (a,b). The distribution seems quite identical to the one shown in [7]. This confirms that the most stabilizing spanwise-wavy suction is indeed the one which creates the strongest streaks, as suggested by [3, 7, 19] based on physical intuition. The mechanism by which blowing and suction creates streaks efficiently was already well discussed in these papers. Since the optimal shape changes also create strong streaks (Fig. 7), this indicates that the most stabilizing mechanism for shape changes is very similar.

As discussed in Sec. 2, the effect of the shape change is modelled with a velocity boundary condition at the original surface position. It may be interesting to compare this equivalent velocity distribution at the surface to the optimal steady suction distribution. The equivalent velocity distribution of the wavy cylinder surface is depicted in Fig. 8 (b). It is immediately clear that the tangential component (solid line, red online) dominates the normal component (dashed line, green online). Hence, the surface waviness acts on the tangential velocity component, by extracting energy from the base flow derivatives at the cylinder surface. Moreover, the velocity changes sign at $\theta \approx 60^\circ$, and a high tangential velocity is induced around this point. This implies that the fluid near the surface is pushed alternatively towards and away from the line $\theta \approx 60^\circ$. The optimal suction, in contrast, changes the normal (vertical) velocity component at the top of the cylinder.

6. Conclusions

A method to compute optimal second-order perturbations of global instability problems is formulated, and demonstrated by creating an optimal spanwise-wavy cylinder surface to passively stabilize vortex shedding. This combines the previous methods of second-order perturbation [2], second-order sensitivity [8], and projection to boundary basis functions [7], to a fully two-dimensional method which can be applied to optimizes both base flow and global mode eigenvalue drift at the 2nd order. The base flow and eigenvalue systems are both perturbed up to the second order, and it is shown that the fundamental and zero-wavenumber base flow modifications are the only ones may influence the eigenvalue drift at the 2nd order. In the cylinder case in the present parameter regime, the zero-wavenumber change (mean flow distortion) is shown to be small and hence neglected in the actual optimization.

The optimal distribution of waviness is found for a given spanwise wave-length. The wake around a circular cylinder is stabilized in global modes and DNS at $Re = 100$ and $Re = 50$, with the maximal surface displacement ca. 1% and 2%, respectively. The surface displacement required for stabilization is
of the same order as the smallest stabilizing spanwise-wavy suction amplitude
reported in [7]. To investigate this point, the optimal suction distribution is
also computed here, and the distribution is very similar to [7]. This is interesting, as those authors optimized the base flow streak amplitude created by
spanwise-wavy suction. The present study recovers the same optimal suction
distribution but by optimizing eigenvalue drift directly, without a hypothesis
about the physical mechanisms involved. Similar same identity between tran-
sient growth optimal and growth rate stabilisation control optimal was obtained
in [8] regarding optimal spanwise modifications of a parallel mixing layer.

We note that the optimal surface displacement (i.e. optimal shape) distribution, however, is significantly different from the optimal suction distribution.
Nevertheless, it is shown that the optimal shape distribution also creates streaks
of a large amplitude, by changing the tangential velocity near the wall. The
method is based on perturbation theory of linearized Navier-Stokes equations,
and hence should be applicable for a wide class of flows where 2\textsuperscript{nd}-order pertur-
bations are relevant. As a next step, the analysis should be applied to other flows
where the eigenvalue drift is confirmed to be of the 2\textsuperscript{nd} order, and where the
boundary modulation is not due to streak generation (for example the stenotic
flow in[20]). Also it is hoped to see an experimental confirmation of the optimal
shapes presented here.

References

[1] Y. Hwang, J. Kim, H. Choi, Stabilization of absolute instability in spanwise
wavy two-dimensional wakes, Journal of Fluid Mechanics 727 (2013) 346–
378.

[2] O. Tammisola, F. Giannetti, V. Citro, M. P. Juniper, Second-order per-
turbation of global modes and implications for spanwise wavy actuation,
Journal of Fluid Mechanics 755 (2014) 314–335.

[3] G. Del Guercio, C. Cossu, G. Pujals, Stabilizing effect of optimally am-
plified streaks in parallel wakes, Journal of Fluid Mechanics 739 (2014)
37–56.

[4] P. Bearman, J. Owen, Reduction of bluff-body drag and suppression of
vortex shedding by the introduction of wavy separation lines, Journal of
Fluids and Structures 12 (1998) 123–130.

[5] R. M. Darekar, S. J. Sherwin, Flow past a bluff body with a wavy stagnation
face, Journal of Fluids and Structures 15 (2001) 587–596.

[6] J. Kim, H. Choi, Distributed forcing of flow over a circular cylinder, Physics
of Fluids 17 (2005) 033103.

[7] G. Del Guercio, C. Cossu, G. Pujals, Optimal streaks in the circular cylin-
der wake and suppression of the global instability, Journal of Fluid Me-
chanics 752 (2014) 572–588.
[8] E. Boujo, A. Fani, F. Gallaire, Second-order sensitivity of parallel shear flows and optimal spanwise-periodic flow modifications, Journal of Fluid Mechanics 782 (2015) 491–514.

[9] O. Marquet, D. Sipp, L. Jacquin, Sensitivity analysis and passive control of cylinder flow, Journal of Fluid Mechanics 615 (2008) 221–252.

[10] E. J. Hinch, Perturbation Methods, Cambridge University Press, Cambridge, 1991.

[11] P. Luchini, A. Bottaro, Adjoint equations in stability analysis, Annual Review of Fluid Mechanics 46 (2014) 493–517.

[12] A. Jameson, Aerodynamic shape optimization using the adjoint method, in: VKI Lecture Series on Aerodynamic Drag Prediction and Reduction, von Karman Institute of Fluid Dynamics, Rhode St Genese, 2003, pp. 3–7.

[13] P. J. Schmid, D. S. Henningson, Stability and Transition in Shear Flows, Springer Verlag, New York, 2001.

[14] J. P. Boyd, The near-equivalence of five species of radial basis functions (RBFs): Asymptotic approximations to the RBF cardinal functions on a uniform, unbounded grid, Vol. 230, 2011.

[15] P. F. Fischer, An Overlapping Schwarz Method for Spectral Element Solution of the Incompressible Navier-Stokes Equations, Journal of Computational Physics 133 (1997) 84–101.

[16] E. Åkervik, L. Brandt, D. S. Henningson, J. Hoepffner, O. Marxen, P. Schlatter, Steady solutions of Navier–Stokes equations by selective frequency damping, Physics of Fluids 18 (2006) 068102.

[17] I. Lashgari, O. Tammisola, V. Citro, L. Brandt, M. P. Juniper, The planar X-junction flow: Stability and control., Journal of Fluid Mechanics 753 (2014) 1–28.

[18] F. Giannetti, P. Luchini, Structural sensitivity of the first instability of the cylinder wake, Journal of Fluid Mechanics 581 (2007) 167–197.

[19] G. Del Guercio, C. Cossu, G. Pujals, Optimal perturbations of non-parallel wakes and their stabilizing effect on the global instability, Physics of Fluids 26 (2014) 024110.

[20] J. Samuelsson, O. Tammisola, M. P. Juniper, Breaking axi-symmetry in stenotic flow decreases the critical transition reynolds number, Physics of Fluids 27 (2015) 104103.

[21] B. Mohammadi, O. Pironneau, Shape optimization in fluid mechanics, Annual Review of Fluid Mechanics 36 (2004) 11.1–11.25.
Appendix A. Derivation of base flow modulations due to surface modifications

Base flow solution at different orders of $\epsilon h$. The base flow solution is expanded to later find the operators $L_1(h_n)$ and $L_2(h_n, h_m)$:

$$U = U^{(0)} + \epsilon U^{(1)} + \epsilon^2 U^{(2)} + O(\epsilon^3)$$

$$P = P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + O(\epsilon^3)$$

The base flow equation is:

$$0 = U^{(0)} \cdot \nabla U^{(0)} + \nabla P^{(0)} - Re^{-1} \nabla^2 U^{(0)}$$

$$+ \epsilon \left( U^{(0)} \cdot \nabla U^{(1)} + U^{(1)} \cdot \nabla U^{(0)} + \nabla P^{(1)} - Re^{-1} \nabla^2 U^{(1)} \right)$$

$$+ \epsilon^2 \left( U^{(0)} \cdot \nabla U^{(2)} + U^{(1)} \cdot \nabla U^{(1)} + U^{(2)} \cdot \nabla U^{(0)} + \nabla P^{(2)} - Re^{-1} \nabla^2 U^{(2)} \right) + O(\epsilon^3)$$

$$= U^{(0)} \cdot \nabla U^{(0)} + \nabla P^{(0)} - Re^{-1} \nabla^2 U^{(0)}$$

$$+ \epsilon \sum_{n=1}^N a_n \left( U^{(0)} \cdot \nabla U^{(1)}_n + U^{(1)}_n \cdot \nabla U^{(0)} + \nabla P^{(1)}_n - Re^{-1} \nabla^2 U^{(1)}_n \right)$$

$$+ \epsilon^2 \sum_{n=1}^N \sum_{m=1}^N a_n a_m \left( U^{(0)} \cdot \nabla U^{(2)}_{nm} + U^{(1)}_n \cdot \nabla U^{(1)}_m + \nabla P^{(2)}_{nm} - Re^{-1} \nabla^2 U^{(2)}_{nm} \right) + O(\epsilon^3)$$
\[ + \varepsilon^2 \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \left[ U^{(0)} \cdot \nabla U^{(2)}_{nm} + U^{(1)}_n \cdot \nabla U^{(1)}_m + U^{(2)}_n \cdot \nabla U^{(0)} + \nabla P^{(2)}_{nm} - Re^{-1} \nabla^2 U^{(2)}_{nm} \right] \]

\[ \text{(A.3)} \]

The boundary condition also needs to be Taylor expanded \[21, 22\]:

\[ 0 = U(r = 1 + h) = U(r = 1) + \varepsilon h \frac{\partial U}{\partial r}(r = 1) + \frac{\varepsilon^2 h^2}{2} \frac{\partial^2 U}{\partial r^2}(r = 1) + O(\varepsilon^3) \]

\[ = U^{(0)} + \varepsilon \left( U^{(1)} + h \frac{\partial U^{(0)}}{\partial r} \right) + \varepsilon^2 \left( U^{(2)} + h \frac{\partial U^{(1)}}{\partial r} + \frac{h^2}{2} \frac{\partial^2 U^{(0)}}{\partial r^2} \right) + O(\varepsilon^3) \]

\[ = U_0 + \varepsilon \sum_{n=1}^{N} a_n \left( U^{(1)}_n + h_n \frac{\partial U^{(0)}}{\partial r} \right) \]

\[ + \varepsilon^2 \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \left[ U^{(2)}_{nm} + h_n \frac{\partial U^{(1)}_m}{\partial r} + h_m h_n \frac{\partial^2 U^{(0)}}{\partial r^2} \right] + O(\varepsilon^3) \quad \text{(A.4)} \]

Apart from gradient-based shape optimization, Taylor expansion of Dirichlet boundary conditions with respect to a boundary displacement is very common in receptivity studies (e.g. \[23, 24\]) and interfacial flows (e.g. \[25, 26, 27\]).

Collecting the terms of the order \( \varepsilon^1 \) in Eq. \[A.3\] we obtain the equation system for \( U^{(1)}_n \):

\[ U^{(1)}_n \cdot \nabla U^{(0)} + U^{(1)}_n \cdot \nabla U^{(0)} + \nabla P^{(1)}_n - Re^{-1} \nabla^2 U^{(1)}_n = 0. \quad \text{(A.5)} \]

with boundary condition at the cylinder wall:

\[ U^{(1)}_n = -h_n \frac{\partial U^{(0)}}{\partial r} \]

and \( U^{(1)}_n = 0 \) at all other boundaries.

Collecting the terms of of the order in \( \varepsilon^2 \) in Eq. \[A.3\] we obtain the equation system for \( U^{(2)}_{nm} \):

\[ U^{(0)} \cdot \nabla U^{(2)}_{nm} + U^{(1)}_n \cdot \nabla U^{(1)}_m + U^{(2)}_n \cdot \nabla U^{(0)} + \nabla P^{(2)}_{nm} - Re^{-1} \nabla^2 U^{(2)}_{nm} = 0. \]

\[ \text{(A.6)} \]

with the boundary condition at the cylinder wall:

\[ U^{(2)}_{nm} = -h_n \frac{\partial U^{(1)}_m}{\partial r} - h_m h_n \frac{\partial^2 U^{(0)}}{\partial r^2} = 0 \]

and \( U^{(2)}_{nm} = 0 \) at all other boundaries.