Embeddings of Automorphism Groups of Free Groups into Automorphism Groups of Affine Algebraic Varieties

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To the memory of A. N. Parshin

Abstract—For every integer \( n > 0 \), we construct a new infinite series of rational affine algebraic varieties such that their automorphism groups contain the automorphism group \( \text{Aut}(F_n) \) of the free group \( F_n \) of rank \( n \) and the braid group \( B_n \) on \( n \) strands. The automorphism groups of such varieties are nonlinear for \( n \geq 3 \) and are nonamenable for \( n \geq 2 \). As an application, we prove that every Cremona group of rank \( \geq 3n - 1 \) contains the groups \( \text{Aut}(F_n) \) and \( B_n \). This bound is 1 better than the bound published earlier by the author; with respect to \( B_n \), the order of its growth rate is one less than that of the bound following from a paper by D. Krammer. The construction is based on triples \((G, R, n)\), where \( G \) is a connected semisimple algebraic group and \( R \) is a closed subgroup of its maximal torus.

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1. INTRODUCTION

The trend of the last decade has been to study the abstract-algebraic, topological, algebro-geometric, and dynamical properties of automorphism groups of algebraic varieties. The present paper is related to this topic and continues the research started in the author’s paper [11].

In [11], an infinite series of irreducible algebraic varieties is constructed such that the automorphism groups of the free groups \( F_n \) of rank \( n \) are embedded into the automorphism groups of these varieties. This construction has applications to the problems of linearity and amenability of automorphism groups of algebraic varieties and to the problems of embeddability of various groups into Cremona groups. To formulate the results obtained in this paper, we recall the construction introduced in [11].

Consider a connected algebraic group \( G \). Let \( X \) be the group variety of the algebraic group \( G^n := G \times \ldots \times G \) (\( n \) terms). We fix in \( F_n \) a free system of generators \( f_1, \ldots, f_n \). For any \( w \in F_n \) and \( x = (g_1, \ldots, g_n) \in X \), \( g_j \in G \) for all \( j \), denote by \( w(x) \) the element of \( G \) obtained from the word \( w \) in \( f_1, \ldots, f_n \) by replacing \( f_j \) with \( g_j \) for each \( j \). For each \( \sigma \in \text{Aut}(F_n) \), the mapping

\[
\sigma_X: X \to X, \quad x \mapsto (\sigma(f_1)(x), \ldots, \sigma(f_n)(x))
\]

is an automorphism of the algebraic variety \( X \) (but not of the group \( G^n \) in general). The mapping \( \sigma \mapsto (\sigma^{-1})_X \) is a group homomorphism \( \text{Aut}(F_n) \to \text{Aut}(X) \). It defines an action of \( \text{Aut}(F_n) \) by automorphisms of \( X \) that commutes with the diagonal action of \( G \) on \( X \) by conjugation. Let us
assume that for the restriction of this action to a closed subgroup $R$ of $G$, there is a categorical quotient

$$\pi_{X//R}: X \to X//R$$

(1.3)

(for example, this property holds if $R$ is finite, see [15, Ch. III, § 3, Proposition 19, Example 2]); it also holds if $G$ is affine and $R$ is reductive, see [12, Sect. 4.4]). Then it follows from the definition of a categorical quotient (see [12, Definition 4.5]) that $\sigma_X$ descends to a uniquely defined automorphism $\sigma_{X//R}$ of $X//R$ with the property

$$\pi_{X//R} \circ \sigma_X = \sigma_{X//R} \circ \pi_{X//R}.$$ 

(1.4)

In this case, there arises a group homomorphism

$$\text{Aut}(F_n) \to \text{Aut}(X//R), \quad \sigma \mapsto (\sigma^{-1})_{X//R},$$

(1.5)

which defines the action of the group $\text{Aut}(F_n)$ by automorphisms of the variety $X//R$. For some (but not all) $G$ and $R$, the homomorphism (1.5) is an embedding. Namely, in [11] it is proved that

(a) in the following cases, the homomorphism (1.5) is an embedding:

- $G$ is nonsolvable and $R$ is finite;
- $G$ is reductive, $R = G$, $n = 1$, and $G$ contains a connected simple normal subgroup of one of the types
  $$A_\ell \text{ with } \ell \geq 2, \quad D_\ell \text{ with odd } \ell, \quad E_6;$$

(b) in the following cases, the homomorphism (1.5) is not an embedding:

- $G$ is solvable, $R$ is finite, and $n \geq 3$,
- $G$ is reductive, $R = G$, and either $n \geq 2$, or $n = 1$ and for each of the types (1.6), $G$ contains no connected simple normal subgroups of this type.

This leads to the following general question.

**Question.** Is it possible to classify the triples $(G, R, n)$, where $G$ is a connected reductive algebraic group, $R$ is a closed subgroup of $G$, and $n$ is a positive integer, for which the homomorphism (1.5) is an embedding?

The main result of the present paper is Theorem 1.1, in which we take the next step after [11] towards answering the question posed: we add one more class to the triples of the specified type found in [11].

**Theorem 1.1.** Let $G$ be a connected semisimple algebraic group, and let $R$ be a closed subgroup of its maximal torus. Then the homomorphism (1.5) is an embedding.

As an application we obtain the following Theorem 1.2, in which $B_n$ denotes the braid group on $n$ strands (by definition, for $n = 1$ this group is trivial).

**Theorem 1.2.** We keep the assumptions of Theorem 1.1. Then $X//R$ is an affine algebraic variety such that

(a) $\text{Aut}(X//R)$ contains $\text{Aut}(F_n)$;
(b) $\text{Aut}(X//R)$ is nonlinear for $n \geq 3$;
(c) $\text{Aut}(X//R)$ contains the braid group $B_n$;
(d) $\text{Aut}(X//R)$ is nonamenable for $n \geq 2$.

We also explore the rationality problem.

**Theorem 1.3.** The variety $X//R$ from Theorems 1.1 and 1.2 is rational.
As an application we strengthen by 1 the bounds obtained in [11, Corollary 9, Remark 10].

**Theorem 1.4.** For any integer \( n \geq 1 \), the Cremona group of rank \( \geq 3n - 1 \) contains the group \( \text{Aut}(F_n) \) as well as the braid group \( B_n \).

**Remark.** As proved in [4] by D. Krammer, the braid group \( B_n \) embeds into \( \text{GL}_{n(n-1)/2} \). Hence \( B_n \) embeds into the Cremona group of rank \( n(n-1)/2 \). The order of the growth rate of this bound on the minimal rank of the Cremona group containing \( B_n \) is 1 greater than that of the bound from Theorem 1.4.

The proofs of Theorems 1.1–1.4 are given in Section 6.

2. CONVENTIONS AND NOTATION

In what follows, algebraic varieties are considered over an algebraically closed field \( k \). We use the results of [6] and [12, Proposition 3.4] obtained under the condition \( \text{char}(k) = 0 \). Therefore, we also assume that this condition holds. With respect to algebraic geometry and algebraic groups, we follow [1].

The identity element of a group considered in multiplicative notation is denoted by \( e \) (it will be clear from the context which group is meant).

The statement that a group \( A \) contains a group \( B \) means the existence of a group monomorphism \( B \to A \), by which \( B \) is identified with its image.

\( \mathcal{C}(A) \) is the center of a group \( A \).

\( A \cdot m \) and \( A_m \) are, respectively, the orbit and the stabilizer of a point \( m \) with respect to a considered action of a group \( A \) (it will be clear from the context which action is meant).

The **kernel of an action** \( \alpha \): \( A \times M \to M \) of a group \( A \) on a set \( M \) is the following normal subgroup of \( A \):

\[ \ker(\alpha) := \{ a \in A \mid a \cdot m = m \text{ for all } m \in M \} \]

By homomorphisms of algebraic groups we mean algebraic homomorphisms, and by their actions on algebraic varieties we mean algebraic actions. In particular, for an algebraic group \( A \), we denote by \( \text{Aut}(A) \) the group of its algebraic automorphisms.

The multiplicative group \( k^\times \) of the field \( k \) is considered as the algebraic group \( \mathbb{G}_m \), and its additive group is considered as \( \mathbb{G}_a \).

3. TERMINOLOGY AND SOME GENERAL RESULTS

Recall (see [12]) the necessary terminology and results concerning an action \( \alpha \) of an algebraic group \( H \) on an irreducible algebraic variety \( Y \).

(a) An action \( \alpha \) is said to be **stable** if there is a nonempty open set in \( Y \) such that the \( H \)-orbit of its every point is closed in \( Y \).

(b) A subgroup \( H_\ast \) of \( H \) is called the **stabilizer in general position** of an action \( \alpha \) if there is a nonempty open set in \( Y \) such that for its every point \( y \) the subgroups \( H_y \) and \( H_\ast \) are conjugate in \( H \).

(c) If \( H \) is reductive and \( Y \) is smooth and affine, then

- the stabilizer in general position exists;
- the varieties \( Y \) and \( Y/H \) are endowed with the Luna stratifications defined as follows. Points \( a, b \in Y/H \) belong to the same Luna stratum if and only if the normal vector bundles to the unique \( H \)-orbits closed in the fibers \( \pi^{-1}_{Y/H}(a) \) and \( \pi^{-1}_{Y/H}(b) \) are \( H \)-equivariantly isomorphic.

The Luna strata in \( Y \) are the sets of the form \( \pi^{-1}_{Y/H}(L) \) with \( L \) a Luna stratum in \( Y/H \).
The Luna stratifications have the following properties:

(i) the set of all Luna strata is finite;

(ii) all Luna strata in the varieties $Y//H$ and $Y$ are smooth locally closed subvarieties of these varieties;

(iii) for any Luna stratum $L$ in $Y//H$ there exists an affine variety $F$ endowed with an action of $H$ such that the restriction of the morphism $\pi_{Y//H}$ to the stratum $\pi_{Y//H}^{-1}(L)$ (called the canonical morphism of the Luna stratum $\pi_{Y//H}^{-1}(L)$) is an étale trivial bundle $\pi_{Y//H}^{-1}(L) \to L$ with fiber $F$.

In view of properties (i) and (ii), there are (unique) open Luna strata in $Y//H$ and $Y$. They are called the principal strata and denoted by $(Y//H)_{pr}$ and $Y_{pr}$, respectively.

**Lemma 3.1.** We keep the previous notation $H$, $Y$, and $\alpha$. Let $y \in Y$ be a point such that

(y1) the orbit $H \cdot y$ is closed in $Y$;
(y2) $H_y = \ker(\alpha)$.

Let $\beta$ be an action of the group $H$ on an algebraic variety $Z$ such that

(k) $\ker(\alpha) = \ker(\beta)$,

and let $\varphi : Z \to Y$ be an $H$-equivariant morphism such that $\varphi^{-1}(H \cdot y) \neq \emptyset$. Then for every point $z \in \varphi^{-1}(H \cdot y)$ the following conditions hold:

(z1) the orbit $H \cdot z$ is closed in $Z$;
(z2) $H_z = \ker(\beta)$.

**Proof.** In view of condition (y1), the nonempty $H$-invariant subset $\varphi^{-1}(H \cdot y)$ is closed in $Z$ and hence contains the closure $\overline{H \cdot z}$ of the orbit $H \cdot z$. Assume that (z1) fails, i.e., $\overline{H \cdot z} \setminus H \cdot z \neq \emptyset$. Let $v \in \overline{H \cdot z} \setminus H \cdot z$. Then

$$\dim(H \cdot v) < \dim(H \cdot z).$$

(3.1)

The restriction of the morphism $\varphi$ to the orbit $H \cdot v$ is $H$-equivariant, and therefore it is a surjective morphism $H \cdot v \to H \cdot y$. So $\dim(H \cdot v) \geq \dim(H \cdot y)$, which together with (3.1) gives

$$\dim(H \cdot y) < \dim(H \cdot z).$$

(3.2)

On the other hand, since $H_z \supset \ker(\beta)$, conditions (y2) and (k) imply that $\dim(H \cdot y) \geq \dim(H \cdot z)$. This contradicts (3.2) and proves condition (z1).

The restriction of the morphism $\varphi$ to the orbit $H \cdot z$ is $H$-equivariant, and therefore it is a surjective morphism $H \cdot z \to H \cdot y$. Hence, there exists $h \in H$ for which $\varphi(h \cdot z) = y$. Therefore,

$$\ker(\alpha) \overset{(k)}{=} \ker(\beta) \subseteq H_{h \cdot z} \subseteq H_y \overset{(y2)}{=} \ker(\alpha);$$

hence, we have $H_{h \cdot z} = \ker(\beta)$. Since $H_{h \cdot z} = hh \cdot h^{-1}$ and $\ker(\beta)$ is normal in $H$, this proves condition (z2). $\square$

**Lemma 3.2.** Let a commutative group $H$ act transitively on a set $M$. Then

(e1) for every $H$-equivariant mapping $\varphi : M \to M$ there is an element $h \in H$ such that

$$\varphi(m) = h \cdot m \quad \text{for every point} \quad m \in M;$$

(3.3)

(e2) for every element $h \in H$, the map $\varphi : M \to M$ defined by (3.3) is $H$-equivariant.
Proof. (e1) Fix a point $m_0 \in M$. Since the action is transitive, for any point $m \in M$ there is an element $z \in H$ such that $m = z \cdot m_0$. In particular, there is $h \in H$ for which $\varphi(m_0) = h \cdot m_0$. Since $\varphi$ is $H$-equivariant and $H$ is commutative, we then have
\[
\varphi(m) = \varphi(z \cdot m_0) = z \cdot \varphi(m_0) = z \cdot (h \cdot m_0) = zh \cdot m_0 = hz \cdot m_0 = h \cdot (z \cdot m_0) = h \cdot m.
\]
(e2) This follows directly from (3.3) in view of the commutativity of $H$. \hfill \Box

4. REDUCTION

The proof of Theorem 1.1 is based on the following geometric description of the kernel of the homomorphism (1.5).

Lemma 4.1. Let $G$ be a connected affine algebraic group, and let $R$ be a closed reductive subgroup of $G$. The following properties of an element $\sigma \in \text{Aut}(F_n)$ are equivalent:

(a) $\sigma$ lies in the kernel of the homomorphism (1.5);

(b) $\sigma_X(O) = O$ for every closed $R$-orbit $O$ in $X$.

Proof. In this case, the variety $X$ is affine, which implies (see [9, §1.2 and Appendix B to Ch. 1]) that the morphism $\pi$ is surjective, its fibers are $R$-invariant, and for every point $b \in X//R$ the fiber $\pi^{-1}(b)$ contains a unique closed $R$-orbit $O_b$. It follows from (1.4) that the restriction of the morphism $\sigma_X$ to the fiber $\pi^{-1}(b)$ is its $R$-equivariant isomorphism with the fiber $\pi^{-1}(\sigma_{X//R}(b))$. In view of the uniqueness of closed orbits in the fibers, this means that $\sigma_X(O_b) = O_{\sigma_X//R(b)}$. Therefore, $\sigma_{X//R}(b) = b$ if and only if $\sigma_X(O_b) = O_b$. \hfill \Box

Under the conditions of Lemma 4.1, the algebra $k[X]^R$ of all $R$-invariant elements of the algebra $k[X]$ of regular functions on $X$ is finitely generated, $X//R$ is an affine algebraic variety with the algebra of regular functions $k[X//R] = k[X]^R$, and the comorphism corresponding to the morphism (1.3) is the identity embedding $k[X]^R \hookrightarrow k[X]$. This implies that for any reductive closed subgroup $S$ of $G$ containing $R$, the identity embedding $k[X]^S \hookrightarrow k[X]^R$ determines a dominant morphism $X//R \to X//S$. This morphism is $\text{Aut}(F_n)$-equivariant. Therefore, the kernel of the action of the group $\text{Aut}(F_n)$ on $X//R$ lies in the kernel of its action on $X//S$.

In the situation considered in Theorem 1.1, this gives the following. By the assumption, in it, $R$ is a subgroup of some maximal torus $T$ of the group $G$. Therefore, it follows from what has been said that it suffices to prove Theorem 1.1 for
\[
R = T.
\]

In what follows, we assume that the group $G$ satisfies the conditions of Theorem 1.1, i.e., it is connected and semisimple. Note that the kernel of the action of the group $T$ on $X$ is $\mathcal{C}(G)$, since $\mathcal{C}(G) \subset T$ (see [1, Sect. 13.17, Corollary 2(d)]).

5. PRINCIPAL LUNA STRATUM FOR THE ACTION OF $T$ ON $X$

The variety $X$ is smooth, and the group $T$ is reductive. Therefore, the diagonal action of the torus $T$ on $X$ by conjugation determines the Luna stratifications of the varieties $X$ and $X//T$. In what follows, by $X_{pr}$ we denote the principal stratum of this stratification of the variety $X$.

Theorem 5.1. Let $G$ be a connected semisimple algebraic group with a maximal torus $T$ acting diagonally by conjugation on the group variety $X$ of the group $G^n$. Then

(a) the kernel of the specified action is $\mathcal{C}(G)$;

(b) the following properties of a point $x \in X$ are equivalent:

(b1) $x \in X_{pr}$;

(b2) the orbit $T \cdot x$ is closed in $X$, and $T_x = \mathcal{C}(G)$;
(c) every fiber of the canonical morphism of the Luna stratum $X_{pr}$ is a $T$-orbit equivariantly isomorphic to $T/\mathcal{C}(G)$;
(d) $\text{codim}_X(X \setminus X_{pr}) \geq n$.

**Proof.** (a) This statement is obvious.

(b) Denote by $\mathcal{V}$ the trivial $\text{codim}_X(T)$-dimensional vector bundle over $T/\mathcal{C}(G)$. Note that $\dim(T/\mathcal{C}(G)) = \dim(T)$ since the group $\mathcal{C}(G)$ is finite (see [1, Sect. 14.2, Corollary (a)]).

- It suffices to prove that
  - (i) the action of the torus $T$ on $X$ under consideration is stable;
  - (ii) $\mathcal{C}(G)$ is its stabilizer in general position,

or, in other words, that there is a nonempty open subset of $X$ such that property (b2) holds for all points $x$ of $X$. Indeed, suppose such a subset exists. Since it is open, its intersection with the open set $X_{pr}$ is nonempty. Let $x$ be a point of this intersection. Since $\mathcal{C}(G)$ is the kernel of the action of the group $T$ on $X$, it follows from property (b2) that the normal bundle of the orbit $T \cdot x$ is equivariantly isomorphic to $\mathcal{V}$. This and the definition of the Luna strata imply that a closed $T$-orbit from $X$ lies in $X_{pr}$ if and only if its normal bundle is equivariantly isomorphic to $\mathcal{V}$. In particular, the dimension of this orbit is $\dim(T)$. It remains to note that the $T$-orbit of any point $y \in X_{pr}$ is closed. Indeed, if this were not the case, then the unique closed $T$-orbit in the fiber $\pi^{-1}_{X/T}(\pi_{X/T}(y))$ that lies in its closure would have dimension strictly less than $\dim(T \cdot y) \leq \dim(T)$, which contradicts the $\dim(T)$-dimensionality of this closed orbit.

- Let us now prove that properties (i) and (ii) indeed hold. It suffices to prove them for $n = 1$.

  - Indeed, suppose that for $n = 1$ they are proved, i.e., there is a nonempty open subset of $G$ such that the $T$-orbit of its every point $x$ is closed in $G$ and $T_x = \mathcal{C}(G)$. Then, as explained above, $G_{pr}$ is the set of all such points $x$. Let $\pi_i: X = G^n \rightarrow G$ be the natural projection onto the $i$th factor. Applying Lemma 3.1 to it, we infer that property (b2) holds for every point $x$ of a nonempty set $\pi^{-1}_{i}(G_{pr})$, which is open in $X$; this means that properties (i) and (ii) hold.

- It remains to prove that properties (i) and (ii) hold for $n = 1$. In [16, 6.11], it is proved that the action of $G$ on itself by conjugation is stable and its stabilizer in general position is $T$. From [5, Theorem and Sect. 3] and the reductivity of $T$, it follows that the natural action of $T$ on $G/T$ is stable. These two facts imply, according to [10, Proposition 6], that property (i) holds for $n = 1$.

- Let $\Phi$ be the root system of the group $G$ with respect to the torus $T$ in which subsystems of positive and negative roots with respect to some base in $\Phi$ are fixed. For any $\alpha \in \Phi$, there is an embedding of algebraic groups $\varepsilon_\alpha: G_a \hookrightarrow G$ such that
  \[ t \varepsilon_\alpha(x) t^{-1} = \varepsilon_\alpha(\alpha(t)x) \quad \text{for all} \quad t \in T, \quad x \in G_a \]  
  (5.1)

(see [3, 26.3, Theorem] and [16, 2.1]). Consider in $G$ the “big cell” $\Theta$ (see [3, 28.5, Proposition]), i.e., the set of all elements of the form
  \[ \prod_{\alpha < 0} \varepsilon_\alpha(x_\alpha) t \prod_{\alpha > 0} \varepsilon_\alpha(x_\alpha), \quad x_\alpha \in G_a, \quad t \in T, \]  
  (5.2)

where the factors in the products are taken with respect to some fixed orders on the sets of positive and negative roots. The set $\Theta$ is open in $G$ and each of its elements can be uniquely written as (5.2) (see [1, Sect. 14.5, Proposition (2), Sect. 14.14, Corollary] and [16, 2.2, 2.3]). In view of (5.1), it is $T$-invariant. The set $\Theta^0$ of all elements of the form (5.2) with $x_\alpha \neq 0$ for each $\alpha \in \Phi$ has the same properties. Let $a \in \Theta^0$ and $c \in T$. It follows from (5.1) and the indicated uniqueness that the condition $c \in T_a$ is equivalent to the condition
  \[ c \in \ker(\alpha) \quad \text{for all} \quad \alpha \in \Phi. \]  
  (5.3)
In turn, it follows from (5.1), (5.2), and the openness of $\Theta^0$ that (5.3) is equivalent to the property that $c$ belongs to the kernel of the action of $T$ on $G$; i.e., (5.3) is equivalent to the inclusion $c \in \mathcal{C}(G)$. This proves that $T_n = \mathcal{C}(G)$. Hence, property (ii) holds for $n = 1$. This completes the proof of statement (b).

(c) This follows from statement (b), since every fiber of the canonical morphism of any Luna stratum in $X$ contains a unique orbit closed in $X$.

(d) As explained in the proof of statement (b), the set $X_{pr}$ contains the set $\bigcup_{i=1}^{n} \pi_{-1}^{-1}(G_{pr})$, which implies

$$X \setminus X_{pr} \subseteq (G \setminus G_{pr}) \times \ldots \times (G \setminus G_{pr}) \quad (n \text{ factors}).$$

(5.4)

From (5.4) it follows that $\dim(X \setminus X_{pr}) \leq n(\dim(G) - 1) = \dim(X) - n$. This proves statement (d). □

6. PROOFS OF THEOREMS 1.1–1.4

Proof of Theorem 1.1. As explained in Section 4, we can (and will) assume that equality (4.1) holds. Arguing by contradiction, suppose that the kernel of the homomorphism $\sigma$ contains an element $\sigma \in \text{Aut}(F_n)$, $\sigma \neq e$. The cases $n = 1$ and $n \geq 2$ will be considered separately; in each of them the proof is based on properties that do not hold in the other case.

Case $n = 1$. The order of $\text{Aut}(F_1)$ is 2 and $\sigma(f_1) = f_1^{-1}$, so

$$\sigma_X(g) = g^{-1} \quad \text{for every} \quad g \in G = X. \quad (6.1)$$

For any element $t \in T$, we have $T \cdot t = t$. By Lemma 4.1, this implies that $\sigma_X(t) = t$. Together with (6.1) this shows that $t^2 = e$ for any $t \in T$. This conclusion contradicts the fact that the set of orders of elements of the torsion subgroup of any torus of positive dimension is not upper bounded (see [1, Sect. 8.9, Proposition]).

Case $n \geq 2$. The proof is divided into several steps.

- Since the kernel of the considered action of the torus $T$ on $X$ is $\mathcal{C}(G)$ (see Theorem 5.1(a)), this action defines a faithful (that is, with trivial kernel) action on $X$ of the torus

$$S := T/\mathcal{C}(G). \quad (6.2)$$

The orbits of this action of the torus $S$, and hence the categorical quotient and the Luna stratifications, are the same as those of the action of the torus $T$. Below, instead of the original action of the torus $T$, we consider the indicated action of the torus $S$.

- Theorem 5.1(b) and Lemmas 4.1 and 3.2(e1) imply the existence of a set-theoretic mapping $\psi: (X//T)_{pr} \to S$ such that

$$\sigma_X(x) = \psi(\pi_X//T(x)) \cdot x \quad \text{for every} \quad x \in X_{pr}. \quad (6.3)$$

Let us prove that the set-theoretic mapping

$$X_{pr} \to S, \quad x \mapsto \psi(\pi_X//T(x)) \quad (6.4)$$

is a morphism of algebraic varieties. According to Theorem 5.1(b) and property (iii) in Section 3, the canonical morphism $X_{pr} \to (X//T)_{pr}$ is an étale trivial bundle with fiber $S$. Since algebraic tori are special groups in the sense of Serre (see [14, Proposition 14]), this bundle is locally trivial in the Zariski topology. Hence, $X_{pr}$ is covered by $S$-invariant open sets for which there are $S$-equivariant isomorphisms of them with varieties of the form $U \times S$, where $U$ is an open subset of $(X//T)_{pr}$ and
the torus $S$ acts through translations of the second factor. If we identify them by these isomorphisms, then the restriction of the mapping (6.4) to any of these open sets has the form

$$\alpha: U \times S \to S, \quad (u, s) \mapsto \psi(u).$$

The issue therefore boils down to proving that $\alpha$ is a morphism of algebraic varieties. To this end, note that since $\sigma$ is a morphism,

$$U \times S \to U \times S, \quad (u, s) \mapsto (u, \psi(u)s)$$

is also a morphism in view of (6.3). Hence

$$\beta: U \times S \to S \times S, \quad (u, s) \mapsto (s, \psi(u)s)$$

is also a morphism. Moreover,

$$\gamma: S \times S \to S, \quad (s_1, s_2) \mapsto s_1^{-1}s_2$$

is also a morphism. It remains to note that $\alpha = \gamma \circ \beta$.

- Thus, there exists a rational mapping

$$\theta: X \to S,$$

which is defined everywhere on the open set $X_{pr}$ and coincides on it with the morphism (6.4). Since $n \geq 2$, it follows from Theorem 5.1(d) that

$$\text{codim}_X(X \setminus X_{pr}) \geq 2. \quad (6.6)$$

The torus $S$ can be identified with the product of several copies of the group $k^\times$. Then $\theta$ is given by a set of rational functions $\theta_i: X \to k$ that are compositions of the mapping $\theta$ with the projections of this product onto the factors. Each $\theta_i$ is regular and does not vanish on $X_{pr}$. Since $X$ is smooth, it follows from this and (6.6) that the divisor of $\theta_i$ on $X$ is zero, that is, $\theta_i$ is regular and does not vanish on the whole of $X$. Thus, we have a morphism $\theta_i: X \to k^\times$. Since $X$ is the group variety of the connected algebraic group $G^n$, it follows from this and from [13, Theorem 3] that $\theta_i$ is the product of a character of this group and a constant. But being semisimple, $G^n$ has no nontrivial characters. Hence $\theta_i$ is a constant. This means that there is an element $s \in S$ for which $\theta(X) = s$.

- Fix an element $t \in T$ mapped to $s$ under the natural surjection $T \to S$ (see (6.2)). We have proved that $\sigma_X(x) = t \cdot x$ for every point $x \in X_{pr}$. Since $X_{pr}$ is open in $X$, this means that

$$\sigma_X(x) = t \cdot x \quad \text{for every point} \quad x \in X. \quad (6.7)$$

Since $\sigma \neq e$, it follows from [11, Theorem 2(b1)] that $\sigma_X \neq \text{id}_X$. In view of (6.7) and Theorem 5.1(a), this gives

$$t \notin \mathcal{C}(G). \quad (6.8)$$

It follows from (6.7), (1.1), and (1.2) that for each $i \in \{1, \ldots, n\}$ the following group identity holds:

$$\sigma(f_i)(g_1, \ldots, g_n) = tg_i t^{-1} \quad \text{for any} \quad g_1, \ldots, g_n \in G. \quad (6.9)$$

In particular, for every $g \in G$ the equality obtained by substituting $g_1 = \ldots = g_n = g$ into (6.9) holds. Since $\sigma(f_i)$ is a noncommutative Laurent monomial in $f_1, \ldots, f_n$, this means that there exists an integer $d$ such that the following group identity holds:

$$g^d = tgt^{-1} \quad \text{for every} \quad g \in G. \quad (6.10)$$
Notice that
\[ d \neq 1 \quad \text{and} \quad d \neq -1. \] (6.11)

Indeed, in view of (6.10), if \( d = 1 \), then \( t \in \mathscr{C}(G) \) contrary to (6.8). If \( d = -1 \), then the equality
\[ h^{-1}g^{-1} = (gh)^{-1} \quad \text{(6.10)} \]
holds for any \( g, h \in G \), which means that the group \( G \) is commutative and contradicts its semisimplicity.

Further, if \( r \) is a positive integer, then the following group identity holds:
\[ t^r g t^{-r} = g^{dr} \quad \text{for every} \quad g \in G. \] (6.12)

Indeed, (6.12) becomes (6.10) for \( r = 1 \). Arguing by induction, from
\[ t^r g t^{-r} = (t^r g t^{-r+1})t^{-1} = tg^{dr-1}t^{-1} \quad \text{(6.10)} \]
as stated.

Substituting \( g = t \) into (6.10) and taking into account (6.11), we conclude that \( t \) is an element of finite order. Let \( r \) in (6.12) be equal to this order. Then (6.12) turns into the group identity
\[ e = g^{dr-1} \quad \text{for every} \quad g \in G. \] (6.13)

In view of (6.11), we have \( dr - 1 \neq 0 \). Hence the group identity (6.13) implies that \( G \), and hence \( T \) as well, is a torsion group whose elements have upper bounded orders. We have arrived at the same contradiction as in the case \( n = 1 \). This completes the proof of Theorem 1.1. □

**Proof of Theorem 1.2.** Theorem 1.1 implies statement (a). The group \( \text{Aut}(F_n) \) contains the group \( B_n \) (see [8, Sect. 3.7]), and for \( n \geq 3 \) it is nonlinear (see [2]). By statement (a), this implies statements (b) and (c). Since the group \( \mathscr{C}(F_n) \) is trivial for \( n \geq 2 \) (see [7, Ch. I, Proposition 2.19]), the group \( \text{Int}(F_n) \) is isomorphic to \( F_n \) and hence is not amenable. This implies statement (d). □

**Proof of Theorem 1.3.** Let us prove that the group variety of the group \( G \) is birationally \( T \)-equivariantly isomorphic to some \( T \)-module. To this end, consider the open set \( \Theta \) in \( G \) introduced in the proof of Theorem 5.1 and fix the following objects:

- a one-dimensional \( T \)-module \( L_\alpha \) for every \( \alpha \in \Phi \) on which \( T \) acts by the formula
  \[ t \cdot \ell = \alpha(t) \ell, \quad t \in T, \quad \ell \in L_\alpha; \] (6.14)

- a nonzero element \( \ell_\alpha \in L_\alpha \);

- a trivial \( T \)-module \( F \) of dimension \( \dim(T) \);

- an open embedding of algebraic varieties
  \[ \iota: T \hookrightarrow F \]
  (it exists because \( T \) is a torus).

Consider the \( T \)-module
\[ V := \bigoplus_{\alpha > 0} L_\alpha \oplus F \oplus \bigoplus_{\alpha < 0} L_\alpha, \]
where the summands in the direct sums are taken with respect to some fixed orders on the sets of positive and negative roots. In view of (5.1) and (6.14), the mapping \( \tau: \Theta \to V \) that sends each element (5.2) to the vector
\[ \bigoplus_{\alpha > 0} x_\alpha \ell_\alpha \oplus \iota(t) \oplus \bigoplus_{\alpha < 0} x_\alpha \ell_\alpha \]
is the required birational morphism (see [1, Sect. 14.4, Remark]).

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Consequently,

\[ \tau^n := \tau \times \cdots \times \tau : \Theta^n := \Theta \times \cdots \times \Theta \to V^n := V \oplus \cdots \oplus V \quad (n \text{ components}) \]

is also a \( T \)-equivariant, and therefore \( R \)-equivariant, birational morphism.

Since \( R \) and \( V^n \) are a diagonalizable group and an \( R \)-module, respectively, the field \( k(V)^R \) is rational over \( k \) (see [12, Sect. 2.9]). Since \( \Theta^n \) is open in \( X \), this implies that the field \( k(X)^R \) is also rational over \( k \). But the action of \( T \) on \( X \) is stable, and \( \mathcal{O}(G) \) is the stabilizer in general position for it by Theorem 5.1(b). Since \( R \) is reductive, it follows from [5, Theorem and Sect. 3] that the natural action of \( R \) on \( T/\mathcal{O}(G) \) is stable. Hence, according to [10, Proposition 6], the action of \( R \) on \( X \) is stable. By [12, Proposition 3.4], this implies that \( k(X)^R \) is the field of fractions of the algebra \( k[X]^R = k[X//R] \). This is what the rationality of the variety \( X//R \) means. \( \square \)

**Proof of Theorem 1.4.** Let \( G = \text{SL}_2 \), so that \( \dim(G) = 3 \) and \( \dim(T) = 1 \). It follows from here and from Theorem 5.1(c) that \( \dim(X//T) = 3n - 1 \). Hence, since the variety \( X//T \) is rational (Theorem 1.3), the group \( \text{Aut}(X//T) \) embeds into the Cremona group of rank \( 3n - 1 \). The claim of the theorem now follows from Theorem 1.2 and the fact that every Cremona group embeds into any Cremona group of higher rank. \( \square \)

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