Energy Dissipation in the Smagorinsky Model of Turbulence

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Abstract

The Smagorinsky model often severely over-dissipates flows and, consistently, previous estimates of its energy dissipation rate blow up as $Re \to \infty$. This report estimates time averaged model dissipation, $\langle \varepsilon_S \rangle$, under periodic boundary conditions as

$$\langle \varepsilon_S \rangle \leq 2\frac{U^3}{L} + Re^{-1} \frac{U^3}{L} + \frac{32}{27} C_S^2 \left( \frac{\delta}{L} \right)^2 \frac{U^3}{L},$$

where $U, L$ are global velocity and length scales and $C_S \approx 0.1, \delta < 1$ are model parameters. Thus, in the absence of boundary layers, the Smagorinsky model does not over dissipate.

1. Introduction

The Smagorinsky model, used for turbulent flow simulations, [S63],[S01], is given by

$$u_t + u \cdot \nabla u - \nu \Delta u - \nabla \cdot \left( (C_S \delta)^2 |\nabla u| \nabla u \right) + \nabla p = f(x) \& \nabla \cdot u = 0 . \quad (1)$$

It is related to the Ladyzhenskaya model [La67], [DG91], [P92] and the von Neumann-Richtmyer method for shocks. In (1), $\nu$ is the kinematic viscosity, $\delta \ll 1$ is a model length scale, the Reynolds number is $Re = LU/\nu$ where $U, L$ are given in (3), and $C_S \approx 0.1$. Experience with the model, e.g., [S01], indicates it over dissipates, often severely, consistent with estimates of model energy dissipation rates for shear flows in [L02]. Model refinements aim at reducing model dissipation occur as early as 1975, [S75], and currently,
Section 4.3 p. 117 onwards, include approaches such as dynamic parameter selection, structural sensors, the accentuation technique and damping functions. Perhaps surprisingly, Theorem 1.1 below establishes that the Smagorinsky model does not over dissipate in the absence of boundary layers.

Let \( \Omega = (0, L_\Omega)^3 \). For \( \phi = u, u_0, f, p \) impose periodicity

\[
\phi(x + L_\Omega e_j, t) = \phi(x, t) \quad j = 1, 2, 3 \quad \text{and} \quad \int_\Omega \phi dx = 0. \tag{2}
\]

The data \( u_0(x), f(x) \) are smooth, periodic, have zero mean and satisfy \( \nabla \cdot u_0 = 0 \) and \( \nabla \cdot f = 0 \). The model energy dissipation rate from (5) below is

\[
\epsilon_S(u) := |\Omega|^{-1} \int_\Omega \nu|\nabla u(x, t)|^2 + (C_S \delta)^2 |\nabla u(x, t)|^3 dx
\]

and the long time average of a function \( \phi(t) \) is defined by

\[
\langle \phi \rangle := \lim \sup_{T \to \infty} \frac{1}{T} \int_0^T \phi(t) dt.
\]

The estimate below, \( \langle \epsilon_S \rangle \simeq U^3/L \), is consistent as \( Re \to \infty, \delta \to 0 \) with both phenomenology, [Po00], and the rate proven for the Navier-Stokes equations in [CD92], [W97], [DF02].

**Theorem 1.** \( \langle \epsilon_S \rangle \) satisfies: for any \( 0 < \alpha < \frac{2}{3} \),

\[
\langle \epsilon_S(u) \rangle \leq \frac{1}{1 - \alpha} \frac{U^3}{L} + \frac{1}{4\alpha(1 - \alpha)} Re^{-1} \frac{U^3}{L} + \frac{4}{27(1 - \alpha)\alpha^2} C_S^2 \left( \frac{\delta}{L} \right)^2 \frac{U^3}{L}.
\]

1.1. Related work

The energy dissipation rate is a fundamental statistic of turbulence, e.g., [Po00], [V15]. In 1992 Constantin and Doering [CD92] established a direct link between phenomenology and NSE predicted energy dissipation. This work builds on [B78], [H72] (and others) and has developed in many important directions subsequently.
2. Proof of Theorem 1.1

Let $\| \cdot \|, (\cdot, \cdot), \| \cdot \|_p$ denote the usual $L^2(\Omega)$ norm, inner product and $L^p(\Omega)$ norm. The force, large scale velocity and length scales, $F, U, L$, are

\begin{align*}
F &= \left(\frac{1}{|\Omega|} |f|^2\right)^{\frac{1}{2}} ,
U &= \left\langle \frac{1}{|\Omega|} |u|^2 \right\rangle^{\frac{1}{2}} \quad \text{and} \\
L &= \min \left\{ \left|\frac{1}{\Omega}\right|^{\frac{1}{2}}, \frac{F}{|\Omega|^{\frac{1}{2}}}, \frac{F}{F}, \frac{F}{\left(\frac{1}{|\Omega|} |\nabla f|^2\right)^{\frac{1}{2}}}, \frac{F}{\left(\frac{1}{|\Omega|} |\nabla f|^3\right)^{\frac{1}{3}}} \right\}. \quad (3)
\end{align*}

It is easy to check that $L$ has units of length and satisfies

\begin{align*}
||\nabla f||_\infty &\leq \frac{F}{L}, \quad \frac{1}{|\Omega|} |\nabla f|^2 \leq \frac{F^2}{L^2} \quad \text{and} \quad \frac{1}{|\Omega|} |\nabla f|^3 \leq \frac{F^3}{L^3}. \quad (4)
\end{align*}

Solutions to (1) (2) are known, e.g., [DG91], [La67], [P92], to be unique strong solutions and satisfy the energy equality

\begin{align*}
\frac{1}{2|\Omega|} |u(T)|^2 + \int_0^T \varepsilon_S(u) dt = \frac{1}{2|\Omega|} |u_0|^2 + \int_0^T \frac{1}{|\Omega|} (f, u(t)) dt. \quad (5)
\end{align*}

Here $\varepsilon_S(u) = \varepsilon_0(u) + \varepsilon_\delta(u)$, $\varepsilon_0 = |\Omega|^{-1} \nu |\nabla u(t)|^2$ and $\varepsilon_\delta = |\Omega|^{-1} (C_S \delta)^2 |\nabla u(t)|_3^3$. From (5) and standard arguments it follows that

\begin{align*}
\sup_{t \in (0, \infty)} |u(t)|^2 \leq C(data) < \infty \quad \text{and} \quad \frac{1}{T} \int_0^T \varepsilon_S(u) dt \leq C(data) < \infty. \quad (6)
\end{align*}

Averaging (5) over $[0, T]$, applying Cauchy-Schwarz in time and (6) yield

\begin{align*}
\frac{1}{T} \int_0^T \varepsilon_S(u) dt \leq O\left(\frac{1}{T}\right) + F \left(\frac{1}{T} \int_0^T \frac{1}{|\Omega|} |u|^2 dt\right)^{\frac{1}{2}}. \quad (7)
\end{align*}

To bound $F$, take the inner product of (1) with $f(x)$, integrate by parts and average over $[0, T]$. This gives

\begin{align*}
F^2 &= \frac{(u(T) - u_0, f)}{T|\Omega|} - \frac{1}{T|\Omega|} \int_0^T (uu, \nabla f) dt + \\
&\quad + \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} (\nabla u, \nabla f) dt + \frac{1}{T|\Omega|} \int_0^T (C_S \delta)^2 |\nabla u| |\nabla u, \nabla f| dt. \quad (8)
\end{align*}
The first term on the RHS is $O(1/T)$ by (6). The second and third are bounded by the Cauchy-Schwarz inequality and (4) for $0 < \beta < 1$ by

$$
\left| \frac{1}{T|\Omega|} \int_0^T (uu, \nabla f) dt \right| \leq ||\nabla f||_\infty \frac{1}{T|\Omega|} \int_0^T ||u||^2 dt \leq \frac{F}{L} \frac{1}{T} \int_0^T \frac{1}{|\Omega|} ||u||^2 dt,
$$

$$
\left| \frac{1}{T} \int_0^T \frac{\nu}{|\Omega|} (\nabla u, \nabla f) dt \right| \leq \left( \frac{1}{T} \int_0^T \frac{\nu^2}{|\Omega|} ||\nabla u||^2 dt \right)^{1/2} \left( \frac{1}{T} \int_0^T \frac{1}{|\Omega|} ||\nabla f||^2 dt \right)^{1/2}
\leq \left( \frac{1}{T} \int_0^T \varepsilon_0(u) dt \right)^{1/2} \sqrt{\nu} F \leq \beta \frac{2}{3} U^{-1} \frac{F}{T} \int_0^T \varepsilon_0(u) dt + \frac{3}{8\beta} UF \nu L^2.
$$

The fourth term is estimated using Hölder’s inequality as

$$
\left| \frac{1}{T|\Omega|} \int_0^T ((C_S \delta)^2 |\nabla u| \nabla u, \nabla f) dt \right| \leq \frac{(C_S \delta)^2}{|\Omega|} \frac{1}{T} \int_0^T ||\nabla u||^3 dt ||\nabla f||_{L^3}
\leq \frac{F}{L} (C_S \delta)^{2/3} \frac{1}{T} \int_0^T \varepsilon_\delta(u)^{3/2} dt.
$$

Using $ab \leq (2/3)a^{3/2} + (1/3)b^3$ gives, for any $0 < \beta < 1$,

$$
\frac{1}{T} \int_0^T \left( \frac{U^2}{\beta^{3/2} L} (C_S \delta)^{2/3} \right) \left( \frac{\beta^{3/2}}{U^{3/2}} \varepsilon_\delta(u)^{3/2} \right) dt \leq \frac{2\beta}{3UT} \int_0^T \varepsilon_\delta(u) dt + \frac{U^2 (C_S \delta)^2}{3\beta^2 L^3}.
$$

Using these three estimates in (8) yields

$$
F \leq O\left( \frac{1}{T} \right) + \frac{1}{LT} \int_0^T \frac{1}{|\Omega|} ||u||^2 dt + \frac{3}{8\beta} U \nu \frac{L^2}{T} + \frac{1}{3} \frac{1}{U} \int_0^T \varepsilon_S(u) dt + \frac{U^2 (C_S \delta)^2}{3\beta^2 L^3}.
$$

This estimate for $F$ in the RHS of (7) gives

$$
\frac{1}{T} \int_0^T \varepsilon_S(u) dt \leq O\left( \frac{1}{T} \right) + \left( \frac{1}{T|\Omega|} \int_0^T ||u||^2 dt \right)^{1/2} \times
\left( \frac{1}{LT|\Omega|} \int_0^T ||u||^2 dt + \frac{3U\nu}{8\beta L^2} + \frac{2\beta}{3} U \frac{1}{T} \int_0^T \varepsilon_S(u) dt + \frac{U^2 (C_S \delta)^2}{3\beta^2 L^3} \right).
$$

The limit superior as $T \to \infty$, which exists by (6), yields the claimed bound after rescaling $\alpha = \frac{2}{3}\beta$

$$
(1 - \alpha) \langle \varepsilon_S(u) \rangle \leq \frac{U^3}{L} + \frac{1}{4\alpha} \frac{U^3 \nu}{L L U} + \frac{U^3 (C_S \delta)^2}{27\alpha^2 L^2}.
$$
3. Conclusions

Comparing \( \langle \varepsilon_S \rangle \simeq U^3/L \) for periodic boundary conditions with \( \langle \varepsilon_S \rangle \simeq [1 + C^2_S(\delta/L)^2(1 + Re^2)]U^3/L \) in [L02] for shear flows strongly suggests model over-dissipation is due to the action of the model viscosity in boundary layers rather than in interior small scales generated by the turbulent cascade. Analysis of \( \langle \varepsilon_S \rangle \) for shear flows including modifications such as, [S01] Section 4.3, dynamic parameter selection, structural sensors, the accentuation technique and damping functions is therefore an important open problem.

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