Radial balanced metrics on the unit disk

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Abstract

Let $\Phi$ be a strictly plurisubharmonic and radial function on the unit disk $D \subset \mathbb{C}$ and let $g$ be the Kähler metric associated to the Kähler form $\omega = \frac{1}{2} \partial \bar{\partial} \Phi$. We prove that if $g$ is $g_{\text{eucl}}$-balanced of height 3 (where $g_{\text{eucl}}$ is the standard Euclidean metric on $\mathbb{C} = \mathbb{R}^2$), and the function $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to an entire analytic function on $\mathbb{R}$, then $g$ equals the hyperbolic metric. The proof of our result is based on an interesting characterization of the function $f(x) = 1 - x$.

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1 Introduction and statement of the main results

Let $\Phi : M \to \mathbb{R}$ be a strictly plurisubharmonic function on a $n$-dimensional complex manifold $M$ and let $g_0$ be a Kähler metric on $M$. Denote by $\mathcal{H} = L^2_{\text{hol}}(M, e^{-\Phi} \omega_0^n)$ the separable complex Hilbert space consisting of holomorphic functions $\varphi$ on $M$ such that

$$
\langle \varphi, \varphi \rangle = \int_M e^{-\Phi} |\varphi|^2 \frac{\omega_0^n}{n!} < \infty,
$$

where $\omega_0$ is the Kähler form associated to the Kähler metric $g_0$ (this means that $\omega_0(X,Y) = g_0(JX,Y)$, for all vector fields $X,Y$ on $M$, where $J$ is the complex structure of $M$). Assume for each point $x \in M$ there exists $\varphi \in \mathcal{H}$ non-vanishing at $x$. Then, one can consider the following holomorphic map into the $N$-dimensional ($N \leq \infty$) complex projective space:

$$
\varphi_{\Phi} : M \to \mathbb{C}P^N : x \mapsto [\varphi_0(x), \ldots, \varphi_N(x)],
$$

where $\varphi_j$, $j = 0, \ldots, N$, is an orthonormal basis for $\mathcal{H}$. In the case $N = \infty$, $\mathbb{C}P^\infty$ denote the quotient space of $l^2(\mathbb{C})$ (the space of sequences $z_j$ such
that \( \sum_{j=1}^{\infty} |z_j|^2 < \infty \), where two sequences \( z_j \) and \( w_j \) are equivalent iff there exists \( \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) such that \( w_j = \lambda z_j, \forall j \).

Let \( g \) be the Kähler metric associated to the Kähler form \( \omega = i \frac{1}{2} \partial \bar{\partial} \Phi \) (and so \( \Phi \) is a Kähler potential for \( g \)). We say that the metric \( g \) is \( g_0 \)-balanced of height \( \alpha, \alpha > 0 \), if \( \varphi_0^* g_{FS} = \alpha g \), or equivalently

\[
\varphi_0^* \omega_{FS} = \alpha \omega, \quad (3)
\]

where \( g_{FS} \) is the Fubini–Study metric on \( \mathbb{C}P^N \) and \( \omega_{FS} \) its associated Kähler form, namely

\[
\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \sum_{j=0}^{N} |Z_j|^2,
\]

for a homogeneous coordinate system \([Z_0, \ldots, Z_N]\) of \( \mathbb{C}P^N \) (note that this definition is independent from the choice of the orthonormal basis). Therefore, if \( g \) is a \( g_0 \)-balanced metric of height \( \alpha \), then \( \alpha g \) is projectively induced via the map (2) (we refer the reader to the seminal paper [5] for more details on projectively induced metrics). In the case a metric \( g \) is \( g \)-balanced, i.e. \( g = g_0 \), one simply call \( g \) a balanced metric.

The study of balanced and \( g_0 \)-balanced metrics is a very fruitful area of research both from mathematical and physical point of view (see [2], [6], [7], [11], [12], [13], [15], [16], [17] and [18]). The map \( \varphi_0 \) was introduced by J. Rawnsley [21] in the context of quantization of Kähler manifolds and it is often referred to as the coherent states map.

Notice that one can easily give an alternative definition of balanced metrics (not involving projectively induced Kähler metrics) in terms of the reproducing kernel of the Hilbert space \( \mathcal{H} \). Nevertheless the definition given here is motivated by the recent results on compact manifolds. In fact, it can be easily extended to the case when \((M, \omega)\) is a polarized compact Kähler manifold, with polarization \( L \), i.e., \( L \) is a holomorphic line bundle \( L \) over \( M \), such that \( c_1(L) = [\omega] \) (see e.g. [1] and [3] for details). In the quantum mechanics terminology the bundle \( L \) is called the quantum line bundle and the pair \((L, h)\) a geometric quantization of \((M, \omega)\). The problem of the existence and uniqueness of balanced metrics on a given Kähler class of a compact manifold \( M \) was solved by S. Donaldson [9] when the group of biholomorphisms of \( M \) which lifts to the quantum line bundle \( L \) modulo the \( \mathbb{C}^* \) action is finite and by C. Arezzo and the second author in the general case (see also [19]).

Nevertheless, many basic and important questions on the existence and uniqueness of balanced metrics on noncompact manifolds are still open. For
example, it is unknown if there exists a complete balanced metric on \( \mathbb{C}^n \) different from the euclidean metric. The case of \( g_0 \)-balanced metric on \( \mathbb{C}^n \), where \( g_0 = g_{\text{eucl}} \) is the Euclidean metric has been studied by the second author and F. Cuccu in [8]. There they proved the following.

**Theorem A** Let \( g \) be a \( g_{\text{eucl}} \)-balanced metric (of height one) on \( \mathbb{C}^n \). If \( \Phi \) is rotation invariant then (up to holomorphic isometries) \( g = g_{\text{eucl}} \).

In this paper we are concerned with the \( g_{\text{eucl}} \)-balanced metrics \( g \) on the unit disk \( D = \{ z \in \mathbb{C} \mid |z|^2 < 1 \} \), where \( g_{\text{eucl}} = dz \otimes d\bar{z} \) is the standard Euclidean metric on \( \mathbb{C} \). In this case, the Hilbert space \( \mathcal{H} \) consists of all holomorphic functions \( \varphi : D \to \mathbb{C} \) such that

\[
\int_D e^{-\Phi} |\varphi|^2 \frac{i}{2} dz \wedge d\bar{z} < \infty,
\]

where \( \Phi \) is a Kähler potential for \( g \). Therefore \( \mathcal{H} \) is the weighted Bergman space \( L^2_{\text{hol}}(D, e^{-\Phi} \omega_{\text{eucl}}) \) on \( D \) with weight \( e^{-\Phi} \). Notice that when \( g = g_{\text{hyp}} = \frac{dz \otimes d\bar{z}}{(1-|z|^2)^2} \) is the hyperbolic metric on \( D \), then \( \Phi(z) = -\log(1-|z|^2) \) is a Kähler potential for \( g_{\text{hyp}} \) and the Hilbert space \( \mathcal{H} = L^2_{\text{hol}}(D, e^{-\Phi} \omega_{\text{eucl}}) \) consists of holomorphic functions \( f \) on \( D \) such that

\[
\int_D (1-|z|^2)^2 |f|^2 \frac{i}{2} dz \wedge d\bar{z} < \infty.
\]

It is easily seen that \( \sqrt{(j+1)(j+2)} \pi \frac{z^j}{\pi}, j = 0, \ldots \) is an orthonormal basis of \( \mathcal{H} \). The map (2), in this case, is given by:

\[
\varphi : D \to \mathbb{CP}^\infty : z \mapsto [\ldots, \sqrt{(j+1)(j+2)} \pi z^j, \ldots].
\]

Thus,

\[
\varphi^* g_{FS} = \frac{i}{2} \partial \bar{\partial} \log \left[ \frac{1}{\pi} \sum_{j=0}^{+\infty} (j+1)(j+2)|z|^{2j} \right] = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{(1-|z|^2)^3} = 3 \omega_{\text{hyp}}
\]

and so \( g_{\text{hyp}} \) is a \( g_{\text{eucl}} \)-balanced (even balanced) metric of height \( \alpha = 3 \). Notice that the function \( \Phi = -\log(1-|z|^2) \) is a radial function and \( h(x) = e^{-\Phi(z)} = 1-|z|^2, x = |z|^2 \), is an entire analytic function defined on all \( \mathbb{R} \).

The following theorem, which is the main result of this paper, shows that the hyperbolic metric on the unit disk can be characterized by the previous data.
Theorem 1.1 Let $g$ be a Kähler metric on the unit disk $D$. Assume that $g$ admits a (globally) defined Kähler potential $\Phi$ which is radial and such that the function $h(x) = e^{-\Phi(z)}$, $x = |z|^2$, extends to a (real valued) entire analytic function on $\mathbb{R}$. If the metric $g$ is $g_{euc}$-balanced of height 3, then $g = g_{hyp}$.

The proof of Theorem 1.1 is based on the following characterization of the function $f(x) = 1 - x$ very interesting on its own sake.

Lemma 1.2 Let $\lambda$ be a positive real number, and let $f: \mathbb{R} \to \mathbb{R}$ be an entire analytic function such that $f(x) > 0$ for all $x \in (0,1)$. Define

$$I_j = \int_0^1 f(t) t^j \, dt \quad \text{for } j \in \mathbb{N}. \quad (4)$$

If the series $\sum_{j=0}^{+\infty} \frac{x^j}{I_j}$ converges for every $x \in (-1,1)$, and if

$$\frac{2\lambda^2}{f^3(x)} = \sum_{j=0}^{+\infty} \frac{x^j}{I_j} \quad \text{for all } x \in (0,1), \quad (5)$$

then $f(x) = \lambda(1 - x)$ for all $x \in \mathbb{R}$.

Despite the very natural statement the proof of Lemma 1.2 is far to be trivial, being based on a careful analysis of the behaviour of $f(x)$ and its derivatives as $x \to 1^-$.

In view of this lemma the authors believe the validity of the following conjecture which could be an important step towards the classification of $g_{euc}$-balanced metrics of height $\alpha$ on the complex hyperbolic space $\mathbb{C}H^n$, namely the unit ball $B^n \subset \mathbb{C}^n$ equipped with the hyperbolic form $\omega_{hyp} = -\frac{i}{2} \partial \overline{\partial} \log(1 - \|z\|^2)$, $z \in B^n$.

Conjecture: Fix a positive integer $n$ and let

$$D_n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | 0 < x_1 + \cdots + x_n < 1, \ x_j > 0\}.$$ 

Suppose that there exists an integer $\alpha > n + 1$, a positive real number $\lambda$ and an entire analytic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in D_n$. 

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such that
\[
\frac{(\alpha - 1) \cdots (\alpha - n) \lambda^2}{f^\alpha(x)} = \sum_J \frac{x^J}{I_J(\alpha)}, \quad \forall J = (j_1, \ldots, j_n) \in \mathbb{N}^n,
\]
where
\[
I_J(\alpha) = \int_{D_n} f^{\alpha-(n+1)}(x)x^J dx_1 \cdots dx_n.
\]
Then \(f(x) = \lambda(1 - x_1 - \cdots - x_n)\).

Notice that Lemma 1.2 shows the validity of the previous conjecture for \(n = 1\) and \(\alpha = 3\).

**Remark 1.3** The studies of balanced metrics on the unit ball \(B^n \subset \mathbb{C}^n\) is far more complicated that one of studying the geud-balanced metrics (we refer the reader to a recent paper of Miroslav Engliš [14] for the study of radial balanced metrics on \(B^n\)). The situation is similar in the compact case where there are no obstructions for the existence of \(g_0\)-balanced metrics (where \(g_0\) is a fixed metric) on a given integral Kähler class of a compact complex manifold \(M\) while the existence of balanced metric on \(M\) is subordinated to the existence of a constant scalar curvature metric in that class (cf. [3] and [4]).

**Remark 1.4** Lemma 1.2 should be compared with the following characterization of the exponential function due to Miles and Williamson [20] which is the main tool in [8] in order to prove Theorem A: let \(f(x) = \sum_j b_j x^j\) be an entire function on \(\mathbb{R}\) such that \(b_0 = 1, \ b_j > 0, \ \forall j \in \mathbb{N}\), and
\[
\int_{\mathbb{R}} \frac{b_j t^j}{f(t)} dt = 1, \quad \forall j \in \mathbb{N},
\]
then \(f(x) = e^x\).

The paper contains another section where we prove Lemma 1.2 and Theorem 1.1.

**2 Proof of the main results**

In the proof of Lemma 1.2 we need the the following elementary result.
Lemma 2.1 Let \( r_0 \in \mathbb{N} \). If a sequence \( \lbrace c_j \rbrace \) satisfies \( c_j = O(j^{r_0}) \) as \( j \to +\infty \), then the power series \( \sum_{j=0}^{+\infty} c_j x^j \) converges in the interval \((-1,1)\) to a function \( S(x) = O((1-x)^{-r_0-1}) \) as \( x \to 1^- \).

**Proof:** If \( r_0 = 0 \) then the conclusion follows from the definition of the symbol \( O \) and the fact that \( \sum_{j=0}^{+\infty} x^j = (1-x)^{-1} \). If, instead, \( r_0 > 0 \) then the conclusion follows similarly after the observation that \( a_j = O((j+1) \cdot \ldots \cdot (j+r_0)) \) and \( \sum_{j=0}^{+\infty} (j+1) \cdot \ldots \cdot (j+r_0) x^j = r_0! (1-x)^{-r_0-1} \). \( \square \)

2.1 Proof of Lemma 1.2

By replacing \( f(x) \) with \( \lambda f(x) \) we may assume \( \lambda = 1 \). Unless otherwise stated, the variable \( x \) ranges in the interval \((0,1)\). The starting idea of the proof of Lemma 1.2 is the following. From the Taylor series of \( f(x) \) at \( x_0 = 1 \),

\[
 f(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{f^{(k)}(1)}{k!} (1-x)^k,
\]

we obtain an asymptotic estimate of the left-hand side of (5) (with \( \lambda = 1 \)) as \( x \to 1^- \). Moreover, by repeatedly integrating by parts we obtain, for every \( j, k_0 \in \mathbb{N} \)

\[
 I_j = \sum_{k=0}^{k_0} \frac{(-1)^k f^{(k)}(1)}{(k+1) \ldots (j+k+1)} + \frac{(-1)^{k_0+1}}{(j+1) \ldots (j+k_0+1)} \int_0^1 f^{(k_0+1)}(t) t^j/k_0 \, dt.
\]

Passing to the reciprocal \( 1/I_j \) and using Lemma 2.1 we obtain an asymptotic estimate of the right-hand side of (5) (with \( \lambda = 1 \)). Since equality holds, we subsequently determine \( f(1), f'(1), f''(1) \). Then, the proof is concluded by means of a more sophisticated argument.

Step 1: \( f(1) = 0 \) and \( f'(1) = -1 \). Denote by \( k_0 \in \mathbb{N} \) the smallest natural number such that \( f^{(k_0)}(1) \neq 0 \). By (6) we get \( f(x) = \frac{1}{k_0!} (-1)^k f^{(k)}(1) (1-x)^{k_0} (1 + O(1-x)) \). In the sequel we will make often use of the following elementary expansion:

\[
 (1 + t)^p = 1 + pt + O(t^2) \text{ as } t \to 0, \ p \in \mathbb{R},
\]

which implies, in particular, \( (1 + O(1-x))^{-3} = 1 + O(1-x) \). Taking this into account, we deduce

\[
 \frac{2}{f^3(x)} = \frac{2(k_0!)^3 (1 + O(1-x))}{(-1)^{k_0} (f^{(k_0)}(1))^3 (1-x)^{3k_0}}
\]
Since we are assuming \( f^{(k)}(1) = 0 \) for \( k < k_0 \), and since the integral in (7) tends to zero at least as fast as \( 1/j \) as \( j \to +\infty \), we may write

\[
I_j = \frac{(-1)^{k_0} f^{(k_0)}(1)}{(j + 1) \cdot \ldots \cdot (j + k_0 + 1)} (1 + O(1/j)),
\]

which in turn, by (8), implies

\[
\frac{1}{I_j} = \frac{(j + 1) \cdot \ldots \cdot (j + k_0 + 1)}{(-1)^{k_0} f^{(k_0)}(1)} + O(j^{k_0}).
\]

Taking Lemma 2.1 into account, multiplication by \( x^j \) followed by summation over \( j \) yields

\[
\sum_{j=0}^{+\infty} \frac{x^j}{I_j} = \frac{(k_0 + 1)!}{(-1)^{k_0} f^{(k_0)}(1)} \frac{1}{1-x^{k_0+2}} + O((1-x)^{-k_0-1}) \text{ as } x \to 1^-.
\]

By comparing the last equality with (9) it follows that \( k_0 \) must satisfy \( 3k_0 = k_0 + 2 \), and therefore \( k_0 = 1 \). This implies \( f(1) = 0 \) and \( (f'(1))^3 = f'(1) \). Since \( f(x) > 0 \) for \( x \in (0,1) \), \( f'(1) \) must be negative and we conclude \( f'(1) = -1 \).

Step 2: \( f''(1) = 0 \). By Taylor expansion we have \( f(x) = (1-x)[1 + \frac{x}{2} f''(1)(1-x) + O((1-x)^2)] \). Using (8) we get

\[
\frac{2}{f^3(x)} = \frac{2}{(1-x)^3} - \frac{3f''(1)}{(1-x)^2} + O((1-x)^{-1}).
\]

Choosing \( k_0 = 2 \) in (7) and arguing as before, we also find \( 1/I_j = (j+1)(j+2) - (j+1)f''(1) + O(1) \) and therefore by Lemma 2.1

\[
\sum_{j=0}^{+\infty} \frac{x^j}{I_j} = \frac{2}{(1-x)^3} - \frac{f''(1)}{(1-x)^2} + O((1-x)^{-1}).
\]

By comparing the last estimate with (10) for \( x \to 1^- \) we deduce \( f''(1) = 0 \).

At this point one could try to obtain the higher order derivatives \( f^{(k)}(1) \), \( k \geq 3 \), as in Steps 1 and 2. Unfortunately this does not work. Indeed one can easily verify that by iterating the previous procedure one gets \( f^{(k)}(1) \), \( k \geq 4 \) in terms of \( f^{(3)}(1) \) but the latter remains undetermined. In order to overcome this problem notice that the previous steps imply that the function

\[
z(x) := \frac{2}{f^3(x)} - \frac{2}{(1-x)^3} - \frac{f''(1)}{1-x},
\]

\[
z(x) := \frac{2}{f^3(x)} - \frac{2}{(1-x)^3} - \frac{f''(1)}{1-x}.
\]
is real analytic in a neighbourhood of $x = 1$. Indeed, we have $f(x) = (1-x)[1 - \frac{1}{4} f'''(1)(1-x)^2 + (1-x)^3 \varphi(x)]$ for an entire analytic function $\varphi(x)$. Furthermore, $(1+t)^{-3} = 1 - 3t + t^2 \psi(t)$, where $\psi(t)$ is analytic for $t \in (-1, +\infty)$ and the claim follows.

Further, by (5) (with $\lambda = 1$), $z(x)$ admits the following expansion around the origin $z(x) = \sum_{j=0}^{+\infty} a_j x^j$, where

$$a_j = \frac{1}{I_j} - (j+1)(j+2) - f'''(1), \quad \text{for } j \in \mathbb{N}. \quad (12)$$

The proof of the lemma will be completed by showing that $z(x)$ vanishes identically. Indeed this is equivalent to

$$\frac{1}{I_j} = (j+1)(j+2) + f'''(1), \quad \text{for } j \in \mathbb{N}, \quad (13)$$

which plugged into (5) (with $\lambda = 1$) gives

$$\frac{2}{f^3(x)} = \frac{2}{(1-x)^3} + \frac{f'''(1)}{1-x}. \quad (14)$$

This shows that $f(1-t)$ is an odd function of $t$ and therefore $f^{(4)}(1) = 0$. Taking this into account, and using (7) with $k_0 = 4$ we obtain

$$I_j = \frac{1}{(j+1)(j+2)} - \frac{f'''(1)}{(j+1) \cdots (j+4)} + O(j^{-6}),$$

which in turn implies $1/I_j = (j+1)(j+2) + f'''(1) - 4 f'''(1)/j + O(j^{-2})$. By comparing the last expansion with (13) we deduce $f'''(1) = 0$. This and (14) imply $f(x) = 1 - x$ and this concludes the proof of the lemma.

In order to prove that the the sequence $\{a_j\}$ vanishes identically we need the following steps.

Step 3. For every integer $k_1$ there exists a rational function $Q_{k_1}(j)$ such that

$$a_j = Q_{k_1}(j) + O(j^{-k_1}) \text{ as } j \to +\infty. \quad (15)$$

Observe, firstly, that if (15) holds for a particular $k_1 = \overline{k}_1$, then it also holds for every $k_1 < \overline{k}_1$ with $Q_{k_1} = Q_{\overline{k}_1}$. Hence, it suffices to prove (15) for $k_1 \geq 1$. Letting $k_0 = k_1 + 2$ in (7) we obtain:

$$I_j = [(j+1)(j+2)]^{-1}[1 + \tilde{Q}_{k_1}(j) + O(j^{-k_1-2})],$$

\[\text{where } \tilde{Q}_{k_1}(j) = \frac{Q_{k_1}(j)}{(j+1)(j+2)}\]
where
\[ \hat{Q}_{k_1}(j) = \sum_{k=3}^{k_1+2} \frac{(-1)^k f^{(k)}(1)}{(j+3)\cdots(j+k+1)} = \frac{-f'''(1)}{(j+3)(j+4)} + O(j^{-3}). \]

Therefore
\[ 1/I_j = (j+1)(j+2)[1 + \hat{Q}_{k_1}(j) + O(j^{-k_1-2})], \]
where \( \hat{Q}_{k_1}(j) = -\frac{\hat{Q}_{k_1}(j)}{1+Q_{k_1}(j)}. \)

Letting \( Q_{k_1}(j) = (j+1)(j+2)\hat{Q}_{k_1}(j) - f'''(1), \) the claim follows by the definition (12) of \( a_j. \) In the next step we will need the observation that
\[ Q_{k_1}(j) = O(j^{-1}). \] (16)

Step 4. The sequence \( \{a_j\} \) defined before tends to zero faster than every rational function of \( j, \) namely \( a_j = O(j^{-k_1}) \) as \( j \to +\infty \) for every integer \( k_1. \) This is proved by showing that for every \( k_1 \geq 2 \) and every rational function \( Q_{k_1} \) satisfying (15) we have \( Q_{k_1}(j) = O(j^{-k_1}). \) Suppose that this is not the case. Then, by (16), there exist positive integers \( d < k_1 \) and a rational function \( Q_{k_1} \) satisfying (15) such that the limit \( \lim_{j\to +\infty} j^d Q_{k_1}(j) \) is a finite \( c \neq 0. \) This and (15) imply \( a_j = c j^{-d} + O(j^{-d-1}). \) Now recall that the sum of the series \( \sum_{j=1}^{+\infty} x^j/j \) is the unbounded function \( -\log(1-x), \) while the series \( \sum_{j=1}^{+\infty} x^j/j^2 \) converges to a bounded function in the interval \([-1,1]. \) By comparison with these elementary series it follows that the \((d-1)\)-th derivative of \( \sum_{j=0}^{+\infty} a_j x^j \) is unbounded for \( x \) close to \( 1 \). But this is impossible because the last series converges to \( z(x), \) which is analytic in a neighbourhood of \( x = 1. \) This contradiction shows that \( Q_{k_1}(j) = O(j^{-k_1}) \) and the claim follows.

Step 5. The sequence \( \{a_j\} \) is identically zero. Define \( w_j = [(j+1)(j+2) + f'''(1)]I_j - 1. \) Since \( w_j = -I_j a_j \) and \( I_j \) is positive, it suffices to show that \( w_j = 0 \) for all \( j \in \mathbb{N}. \) This is achieved by representing \( w_j \) as the limit \( \lim_{k_0 \to +\infty} S_{k_0}(j) \) of the sum \( S_{k_0}(j) \) defined below, an then by showing that \( S_{k_0}(j) \) is infinitesimal as \( k_0 \to +\infty. \) Taking into account that \( \int_0^1 (1-t)^k t^j \, dt = k! j!/(j+k+1)! , \) multiplication of (6) by \( t^j \) followed by termwise integration over the interval \((0,1)\) yields
\[ I_j = \sum_{k=0}^{+\infty} \frac{(-1)^k f^{(k)}(1)}{(j+1)\cdots(j+k+1)}. \]

Notice that it makes sense to integrate over the interval \((0,1)\) since, by assumption, \( f \) is entire (cf. Remark 2.2). Since \( f(1) = f''(1) = 0 \) and
We get \( m \) have to investigate the coefficients \( R \) and the remainder \( S - R \) which shows that \( j \) in the variable \( j \), equality, and since the coefficients of \( By \) summation of the two rational functions in the right-hand side of the last equality, and since \( w_j = S_k_0(j) + R_k_0(j) \), where the partial sum \( S_k_0(j) \) and the remainder \( R_k_0(j) \) are given by

\[
S_k_0(j) = \frac{f''(1)}{(j+1)(j+2)} + \sum_{k=3}^{3} \frac{(-1)^k [(j+1)(j+2)+f''(1)] f^{(k)}(1)}{(j+1) \ldots (j+k+1)},
\]

(17)

and

\[
R_k_0(j) = \sum_{k=k_0+1}^{\infty} \frac{(-1)^k [(j+1)(j+2)+f''(1)] f^{(k)}(1)}{(j+1) \ldots (j+k+1)}.
\]

By (7), the remainder \( R_k_0(j) \) also admits the following representation:

\[
R_k_0(j) = \frac{(-1)^{k_0+1} [(j+1)(j+2)+f''(1)]}{(j+1) \ldots (j+k_0+1)} \int_0^1 f^{(k_0+1)}(t) t^{j+k_0+1} dt,
\]

which shows that \( R_k_0(j) = O(j^{-k_0}) \) as \( k_0 \to +\infty \). Furthermore, since \( w_j = -I_j a_j \) and \( I_j \) is bounded, by Step 4 we have, in particular, \( w_j = O(j^{-k_0}) \). It follows that \( S_k_0(j) = O(j^{-k_0}) \) and by (17) we may write

\[
S_k_0(j) = \frac{P_k_0(j)}{(j+1) \ldots (j+k_0+1)},
\]

(18)

where \( P_k_0(j) = m_k_0 j + q_k_0 \) is a convenient polynomial of degree \( \deg P_k_0(j) \leq 1 \) in the variable \( j \). In order to show that \( S_k_0(j) \) is infinitesimal as \( k_0 \to +\infty \) we have to investigate the coefficients \( m_k_0, q_k_0 \). Observe, firstly, that from (17) we get

\[
S_k_0+1(j) = S_k_0(j) + \frac{(-1)^{k_0+1} [(j+1)(j+2)+f''(1)] f^{(k_0+1)}(1)}{(j+1) \ldots (j+k_0+2)}.
\]

This and (18) yield

\[
\frac{P_k_0+1(j)}{(j+1) \ldots (j+k_0+2)} = \frac{P_k_0(j)}{(j+1) \ldots (j+k_0+1)} + \frac{(-1)^{k_0+1} [(j+1)(j+2)+f''(1)] f^{(k_0+1)}(1)}{(j+1) \ldots (j+k_0+2)}.
\]

By summation of the two rational functions in the right-hand side of the last equality, and since the coefficients of \( j^2, j^3 \) in the numerator must equal the corresponding ones in the left-hand side, we deduce

\[
0 = m_k_0 + (-1)^{k_0+1} f^{(k_0+1)}(1),
\]

\[
m_k_0+1 = (k_0 - 1) m_k_0 + q_k_0,
\]

\[
q_k_0+1 = (k_0 + 2) q_k_0 - [2 + f''(1)] m_k_0.
\]
Since the series (6) converges together with all its derivatives at \( x = 0 \), it follows that for every \( h \in \mathbb{N} \) we have \( m_{k_0} = o(k_0! k_0^{-h}) \) as \( k_0 \to +\infty \). The same holds for \( q_{k_0} \) because \( q_{k_0+1} = (k_0 + 2) [m_{k_0+1} - (k_0 - 1) m_{k_0}] - [2 + f'''(1)] m_{k_0} \). Hence, by (18), it follows that \( S_{k_0}(j) \to 0 \) as \( k_0 \to +\infty \) and therefore \( w_j = 0 \) for all \( j \), as claimed. \( \square \)

**Remark 2.2** The assumption in Lemma 1.2 that \( f \) is an entire analytic function can be relaxed. If equality (6) holds in the interval \((−\varepsilon, 2 + \varepsilon)\) for some \( \varepsilon > 0 \), then the same proof shows that \( f(x) = \lambda (1 - x) \) in that interval.

### 2.2 Proof of Theorem 1.1

Since the function \( h(x) = e^{-\Phi(z)} \), \( x = |z|^2 \), extends to all real numbers it follows that \( e^{-\Phi(z)} \) does not blow up at the boundary of \( D \). This implies that the monomials \( z^j \), \( j = 0, 1 \ldots \) are an orthogonal basis of \( \mathcal{H} = L^2_{hol}(D, e^{-\Phi} \omega_{eucl}). \) Hence the sequence \( \sqrt{b_j} z^j \), \( j = 0, \ldots \), with

\[
b_j = \left( \int_D e^{-\Phi} |z|^{2j} \frac{i}{2} dz \wedge d\bar{z} \right)^{-1},
\]

is an orthonormal basis of \( \mathcal{H} \) and the Kähler metric \( g \) is \( g_{eucl}\)-balanced of height 3 iff

\[
\frac{i}{2} \partial \bar{\partial} \log(\sum_{j=0}^\infty b_j |z|^{2j}) = 3\omega = 3\frac{i}{2} \partial \bar{\partial} \Phi.
\]

This implies that the function \( \Phi(z) - \log(\sum_{j=0}^\infty b_j |z|^{2j})^{\frac{1}{2}} \) is a radial harmonic function on \( D \) and hence equals a constant, say \( \Phi_0 \). By setting \( f(x) = (\sum_{j=0}^\infty b_j x^j)^{-\frac{1}{2}}, \ x \in [0, 1), \) and by the definition of the \( b_j \)’s one then gets

\[
e^{-\Phi_0 b_j} \int_D f(|z|^2) |z|^{2j} \frac{i}{2} dz \wedge d\bar{z} = 1, \ \forall j \in \mathbb{N}.
\] (19)

Observe that again the assumption that \( h(x) = e^{-\Phi(z)}, \ x = |z|^2 \), extends to an entire analytic function on \( \mathbb{R} \) implies the same property for \( f(x) = h(x)e^{\Phi_0}. \) By passing to polar coordinates \( z = \rho e^{i\theta}, \ \rho \in [0, +\infty), \ \theta \in [0, 2\pi) \) and by the change of variables \( t = \rho^2 \) one obtains:

\[
\pi e^{-\Phi_0 b_j} \int_0^1 f(t)t^j dt = 1, \ \forall j \in \mathbb{N}.
\]
By setting \( \lambda^2 = \frac{\pi e^{-\Phi_0}}{2}, I_j = \frac{1}{2\lambda^2 b_j} \) and by the definition of \( f(x) \) one gets (4) and (5). Therefore, by Lemma 1.2, \( f(x) = \lambda(1 - x) \), i.e. \( \Phi(z) = \Phi_0 - \log \lambda - \log(1 - |z|^2) \), which implies \( \omega = \frac{i}{2} \partial \bar{\partial} \Phi = -\frac{i}{2} \partial \bar{\partial} \log(1 - |z|^2) = \omega_{hyp} \) and this concludes the proof of the theorem.

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