Virial expansion and TBA in $O(N)$ sigma models

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Abstract

We study the free energy of the 1 + 1 dimensional $O(N)$ nonlinear $\sigma$-models for even $N$ using the TBA equations proposed recently. We give explicit formulae for the constant solution of the TBA equations (Y-system) and calculate the first two virial coefficients. The free energy is also compared to the leading large $N$ result.
1 Introduction

A large class of integrable two-dimensional models can be described as perturbed conformal field theories (PCFT). If the perturbing relevant operator is chosen so that perturbation preserves integrability then the spectrum of the resulting (in general massive) integrable model and its exact S-matrix can be conjectured [1]. Further, the set of TBA equations [2] describing the ground state energy in finite volume (or equivalently the free energy of the system at finite temperature $T$) can be written down.

Non-linear $\sigma$-models (NLS) are another important class of integrable models. Their spectrum and S-matrix have been obtained by bootstrap methods [3] but the corresponding TBA system was not known until recently. In [4, 5] (based on earlier constructions [6, 7, 8]) NLS models are represented as limits of PCFT models. More concretely, the NLS model whose fields take value in the coset space $G/H$ can be described as the $k \rightarrow \infty$ limit of the $(G/H)_k$ conformal coset model [9] perturbed by a certain relevant operator. This way of representing NLS models gives additional insight into the structure of the model and is reminiscent of the description of the model as the infinite flavour limit of a certain fermionic model [10]. Moreover, it allows one to calculate the ground state energy of the NLS model by considering the limiting case of the TBA system, available for finite values of the Kac-Moody level $k$.

Because of the obvious importance of the NLS models in this paper we perform a number of tests for the above TBA system. For simplicity, we restrict our attention here to the $O(N)$ NLS models for even $N$. We show that in the $T \rightarrow \infty$ limit the correct UV central charge is reproduced. In the opposite (low temperature or equivalently low density) limit there exists a systematic expansion, the virial expansion. It is known [11] that the virial coefficients are completely determined by the scattering data alone. This is useful since the virial coefficients can also be calculated from the TBA equations by solving the linearized system in Fourier space. We work out the first two virial coefficients directly from the S-matrix and also from the TBA system and find that they agree completely. Finally, NLS models can also be solved in the $1/N$ expansion. We calculate the free energy in the leading large $N$ limit and compare it to the $N \rightarrow \infty$ limit of the virial coefficients calculated previously.

2 TBA equations

Here we study the TBA system proposed in [4] for the $O(N)$ nonlinear sigma model for the case of even $N$. The TBA equations are formulated in terms of the densities $\varepsilon_0(\theta)$ and $\varepsilon_m^{(a)}(\theta)$, where the index $a$ labels the nodes of the $D_r$ Dynkin diagram.
\((N/2 = r \geq 2)\) and the range of the lower index is \(m = 1, 2, \ldots, k - 1\) \((k \geq 2)\). Let us introduce

\[
A_m^{(a)}(\theta) = \frac{g}{4\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{\cosh \frac{g}{2}(\theta - \theta')} \ln \left[ 1 + e^{-\varepsilon_m^{(a)}(\theta')} \right],
\]

\[
A_0(\theta) = \frac{g}{4\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{\cosh \frac{g}{2}(\theta - \theta')} \ln \left[ 1 + e^{-\varepsilon_0(\theta')} \right],
\]

\[
B_m^{(a)}(\theta) = \frac{g}{4\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{\cosh \frac{g}{2}(\theta - \theta')} \ln \left[ 1 + e^{\varepsilon_m^{(a)}(\theta')} \right],
\]

where (1) and (3) are defined for all \(a = 1, 2, \ldots, r\) and \(m = 1, 2, \ldots, k - 1\). By convention we take \(B_0^{(a)}(\theta) = B_k^{(a)}(\theta) = 0\). \(g = 2r - 2\) is the Coxeter number of \(D_r\).

We further define

\[
\psi_{ab}(\theta) = \int_{-\infty}^{\infty} d\omega e^{i\omega \theta} N_{ab}(\omega),
\]

where for \(r \geq 3\) \(N_{ab}(\omega)\) is the matrix inverse of

\[
\mathcal{M}_{ab}(\omega) = 2 \cosh \left( \frac{\pi \omega}{g} \right) \delta_{ab} - I_{ab}.
\]

Here \(I_{ab}\) is the incidence matrix of the \(D_r\) Dynkin diagram.

For later use we note that \(N_{a1}(\omega)\) is given by

\[
N_{a1}(\omega) = \frac{q^a + q^{2r-2-a}}{1 + q^{2r-2}} \quad a = 1, 2, \ldots, r - 2
\]

and

\[
N_{r-1,1}(\omega) = N_{r1}(\omega) = \frac{q^{r-1}}{1 + q^{2r-2}},
\]

where \(q = e^{-\frac{\pi |\omega|}{g}}\). We will assume that (7) holds also for \(r = 2\), by definition in this case.

The TBA equations proposed in [4] are

\[
\varepsilon_0(\theta) = \frac{1}{\mathcal{T}} (M \cosh \theta - H) - \frac{1}{2\pi} \sum_{a=1}^{r} \int_{-\infty}^{\infty} d\theta' \psi_{a1}(\theta - \theta') \ln \left[ 1 + e^{\varepsilon_1^{(a)}(\theta')} \right]
\]

and for \(a = 1, 2, \ldots, r\) and \(m = 1, 2, \ldots, k - 1\)

\[
\varepsilon_m^{(a)}(\theta) = A_0(\theta) (\delta_1 \delta_{m1} + \delta_2 \delta_{m1} \delta_{r2}) + B_{m+1}(\theta) + B_{m-1}(\theta) - \sum_{b=1}^{r} I_{ab} A_m^{(b)}(\theta).
\]
In (8) $M$ is the mass of the particles and $H$ is a chemical potential coupled to particle number.

Finally the pressure of this one-dimensional gas is given by

$$p = \frac{MT}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \ln \left[ 1 + e^{-\varepsilon_0(\theta)} \right].$$

(10)

The above TBA system describes the perturbed $(O(N)/O(N-1))_k$ coset model for finite $k$. The pressure of the $O(N)$ NLS model can be obtained from it by taking the $k \to \infty$ limit [4].

3 Virial expansion

We have introduced the chemical potential $H$ to facilitate the virial expansion of the pressure. In the limit $H \to -\infty$ the densities can be expanded in powers of the fugacity $y = e^{H/T}$ as

$$\varepsilon_0(\theta) = \frac{1}{T}(M \cosh \theta - H) + \sum_{\mu=0}^{\infty} \varepsilon^{(\mu)}_0(\theta)y^\mu,$$

(11)

$$\varepsilon^{(a)}_m(\theta) = \sum_{\mu=0}^{\infty} \varepsilon^{(a)(\mu)}_m(\theta)y^\mu,$$

(12)

whereas the pressure is expanded as

$$p = T \sum_{\mu=1}^{\infty} p^{(\mu)}y^\mu.$$  

(13)

Alternatively, one can get the same expansion coefficients (without introducing the chemical potential) by considering the low temperature expansion of the pressure. In this case the coefficient $p^{(\mu)}$ is of order $\exp(-\mu M/T)$.

It is known [11] that for the case of a system of $N$ particles (independently of integrability) the virial expansion coefficients are determined by the S-matrix. If all particles have the same mass $M$ then the first two virial coefficients are

$$p^{(1)} = \frac{n}{2\pi} \int_{-\infty}^{\infty} d\theta M \cosh \theta e^{-\frac{M}{T} \cosh \theta}$$

(14)

and $p^{(2)} = p^{(2)}_1 + p^{(2)}_2$, where

$$p^{(2)}_1 = \frac{-ns}{4\pi} \int_{-\infty}^{\infty} d\theta M \cosh \theta e^{-\frac{2M}{T} \cosh \theta}.$$

(15)
and

\[
p_2^{(2)} = \frac{1}{4\pi^2} \sum_l \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\theta' \ M \cosh \theta \ \phi_l(\theta - \theta') e^{-\frac{M}{2}(\cosh \theta + \cosh \theta')}. \tag{16}
\]

Here \( n = N \), \( \phi_l(\theta) = -i \frac{d}{d\theta} \ln S_l(\theta) \), where \( \{S_l(\theta); \ l = 1, 2, \ldots, N^2\} \) are the eigenvalues of the 2-particle S-matrix and \( s \) is a factor related to particle statistics. For the case of diagonal scattering, \( s = \mp 1 \) for bosonic/fermionic type particles [12], whereas for all known non-diagonal cases (where all particles are of fermionic type) \( s = N \). Finally, for later use, we introduce the Fourier transform

\[
\sum_l \phi_l(\theta) = \int_{-\infty}^{\infty} d\omega e^{-i\omega\theta} \tilde{\phi}(\omega). \tag{17}
\]

For the \( O(N) \) nonlinear sigma model, using the eigenvalues of the bootstrap S-matrix [3] we get

\[
\tilde{\phi}(\omega) = N^2 \frac{e^{-\pi|\omega|} + e^{\frac{2\pi|\omega|}{N-2}}}{1 + e^{-\pi|\omega|}} - e^{-\pi|\omega|} - \frac{N^2 - N + 2}{2} e^{\frac{2\pi|\omega|}{N-2}}. \tag{18}
\]

## 4 Solution of the Y-system

Let us consider the generic \( r \geq 3 \) case first. The leading terms in the expansion (12) turn out to be \( \theta \)-independent and the constants

\[
\hat{x}_m^{(a)} = e^{\xi_m^{(a)(0)}(\theta)} \tag{19}
\]

can be determined by solving the equations

\[
(\hat{x}_m^{(a)})^2 \prod_{b=1}^{r} \left[ 1 + \frac{1}{\hat{x}_m^{(b)}} \right] I_{ab} = \left( 1 + \hat{x}_{m+1}^{(a)} \right) \left( 1 + \hat{x}_{m-1}^{(a)} \right) \tag{20}
\]

for \( a = 1, 2, \ldots, r \), \( m = 1, 2, \ldots, k - 1 \), where by convention \( \hat{x}_0^{(a)} = \hat{x}_k^{(a)} = 0 \). This ubiquitous equation is called the constant Y-system and its solution is well known for all classical Dynkin diagrams [13]. It is usually solved by relating it to the Q-system equations

\[
(Q_m^{(a)})^2 = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b=1}^{r} [Q_m^{(b)}] I_{ab} \tag{21}
\]

for \( a = 1, 2, \ldots, r \), \( m = 1, 2, \ldots, k - 1 \), with “boundary condition”

\[
Q_0^{(a)} = Q_k^{(a)} = 1. \tag{22}
\]
We are interested in positive solutions of (21), related to (20) by
\[ \hat{x}^{(a)}_m = \frac{Q^{(a)}_{m+1} Q^{(a)}_{m-1}}{\prod_{b=1}^{r} [Q^{(b)}_m]^I_{ab}}. \]  
(23)

We will need the explicit solution of (21) only for \(a = 1\). This can be written as
\[ Q^{(1)}_m = \frac{p(m + r - 1) p(r - 1)}{p(r - 1) \prod_{j=1}^{2r-3} p(m + j) p(j)}, \]  
(24)

with \(p(n) = \sin \frac{n\pi}{k + 2r - 2}\). In particular,
\[ Q^{(1)}_1 = \frac{p(1)p(2r - 2)}{p(r - 1)p(2r - 1)p(2r - 2)}, \quad \text{and} \quad Q^{(1)}_2 = \frac{p(r + 1)p(2r - 1)p(2r - 2)}{p(1)p(2)p(r - 1)}. \]  
(25)

We can now calculate the first virial coefficient. It is of the form (14), where the coefficient \(n\) can be expressed in terms of the constants (19) as
\[ n = \prod_{a=1}^{r} \left[ 1 + \hat{x}^{(a)}_1 \right]^{N_{a1}(0)}. \]  
(26)

If we use (13), (7) together with (23), (24) can be written as
\[ n = \prod_{a=1}^{r-1} \left[ 1 + \hat{x}^{(a)}_1 \right] = Q^{(1)}_1. \]  
(27)

From (25) we see that \( \lim_{k \to \infty} n = 2r = N \).

5 The second virial coefficient

Encouraged by the agreement of the first virial coefficient with the expected form we now go one step further and calculate the second virial coefficient. Using the expansions (11) and (12) in (45), (46) and going to Fourier space we find that \(p^{(2)}_1\) is of the form (15) with \(s = n\) and \(p^{(2)}_2\) is of the form (16) with
\[ \tilde{\phi}(\omega) = n^2 \sum_{a=1}^{r} N_{a1}(\omega) S^{(a)}_1(q). \]  
(28)
Here the functions \( S_1^{(a)}(q) \) can be calculated by solving the linear equations

\[
(q + q^{-1}) \left(1 + \frac{1}{\hat{x}_m} \right) S_m^{(a)} = \delta_a^1 \delta_{m1} + S_m^{(a)} + S_{m+1}^{(a)} + \sum_{b=1}^{r} I_{ab} \frac{1}{\hat{x}_m} S_m^{(b)}
\]  

(29)

for \( a = 1, 2, \ldots, r \), \( m = 1, 2, \ldots, k - 1 \), where \( S_0^{(a)} = S_k^{(a)} = 0 \).

The solution of the linear equations (29) can be given in terms of the numbers \( Q_m^{(a)} \) satisfying (21). Although their explicit expression is known [13], it is actually easier to verify the validity of the solution given below by using the \( Q \)-system equations (21) directly.

The solution for \( S_m^{(a)}(q) \) is given as follows. Using the definitions

\[
b_m^{(a)} = \frac{Q_m^{(a)}}{Q_{m-1}^{(a)}} \quad m = 1, 2, \ldots, k
\]

(30)

and

\[
I_m^{(1)} = b_m^{(1)}
\]

(31)

\[
I_m^{(a)} = \frac{b_m^{(a+1)}}{b_m^{(a-1)}} \quad a = 2, \ldots, r - 2
\]

(32)

\[
I_m^{(r-1)} = \frac{b_m^{(r-1)} b_m^{(r-2)}}{b_m^{(r-2)}} + \frac{b_m^{(r-1)}}{b_m^{(r-2)}}
\]

(33)

\[
I_m^{(r)} = 0
\]

(34)

for \( m = 1, 2, \ldots, k - 1 \) (for \( m = 1 \) the second term in (33) is absent) together with \( I_0^{(a)} = I_k^{(a)} = 0 \) we first define

\[
H_m^{(a)}(q) = I_m^{(a)} q^{m+a-1} - I_m^{(a+1)} q^{m+a+1}, \quad a = 1, 2, \ldots, r - 1
\]

(35)

\[
H_m^{(r)}(q) = H_m^{(r-1)}(q)
\]

(36)

for \( m = 0, 1, \ldots, k \). Next we define

\[
\hat{S}_m^{(a)}(q) = \frac{1}{n} \left[ H_m^{(a)}(q) - q^{k+g} H_{k-m}^{(a)} \left( \frac{1}{q} \right) \right].
\]

(37)

Note that both \( H_m^{(a)}(q) \) and \( \hat{S}_m^{(a)}(q) \) are finite polynomials in \( q \).

The solution of (24) is finally given by

\[
S_m^{(a)}(q) = \frac{1}{1 - q^{2k+2g}} \left[ \hat{S}_m^{(a)}(q) - q^{2k+2g} \hat{S}_m^{(a)} \left( \frac{1}{q} \right) \right].
\]

(38)
Using the solution (38) we can calculate
\[ \tilde{\phi}(\omega) = \frac{1}{1 + q^{g} \frac{1}{1 - q^{2k+2g}}} \left[ \hat{K}(q) - q^{2k+3g} \hat{K} \left( \frac{1}{q} \right) \right], \]
where
\[ \hat{K}(q) = n^{2} \sum_{a=1}^{r-1} (q^{a} + q^{g-a}) \hat{S}_{1}^{(a)}(q). \]
Putting everything together we have
\[ \hat{K}(q) = Q_{2}^{(1)} q^{2} + (n^{2} - 1)q^{g} - \left( n^{2} - Q_{2}^{(1)} \right) q^{g+2} - q^{2g} \]
and using also
\[ \lim_{k \to \infty} Q_{2}^{(1)} = (r + 1)(2r - 1) = \frac{N^{2} + N - 2}{2} \]
we see that (39) indeed reproduces (18) in the limit \( k \to \infty \).

### 6 Calculation of the UV central charge

For this purpose it turns out to be useful to rewrite the TBA equations a little. Let us introduce the new variables
\[ \epsilon_{0}(\theta) = \epsilon_{0}(\theta), \quad L_{0}(\theta) = \ln \left[ 1 + e^{-\epsilon_{0}(\theta)} \right], \]
\[ \epsilon_{m}^{(a)}(\theta) = -\epsilon_{m}^{(a)}(\theta), \quad L_{m}^{(a)}(\theta) = \ln \left[ 1 + e^{-\epsilon_{m}^{(a)}(\theta)} \right]. \]
In terms of these new variables the TBA equations take the standard form
\[ \epsilon_{0} = \frac{1}{T} (M \cosh \theta - H) - \sum_{a=1}^{r} \psi_{a1} * L_{1}^{(a)}, \]
\[ \epsilon_{m}^{(a)} = \sum_{b,c=1}^{r} \psi_{ab} I_{bc} * L_{m}^{(c)} - \psi_{a1} \delta_{m1} * L_{0} - \sum_{b=1}^{r} \psi_{ab} * \left( L_{m+1}^{(b)} + L_{m-1}^{(b)} \right). \]

Our convention for convolution is
\[ (f * g)(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta') \]
and \( L_{0}^{(a)}(\theta) = L_{k}^{(a)}(\theta) = 0 \) by convention.
We note that the physical particles correspond to $\epsilon_0$. They are of mass $M$ and are coupled to the chemical potential $H$. All other excitations corresponding to the densities $\epsilon_m^{(a)}$ are “magnons”. They are massless and are not coupled to the chemical potential.

The standard form of the TBA equations allows us to calculate the UV central charge the standard way [12]. It can be written as $c_{\text{UV}} = c^{(1)} - c^{(2)}$ with

$$c^{(1)} = \frac{6}{\pi^2} \left\{ \mathcal{L} \left( \frac{x_0}{1 + x_0} \right) + \sum_{a=1}^{r-1} \sum_{m=1}^{k-1} \mathcal{L} \left( \frac{x_m^{(a)}}{1 + x_m^{(a)}} \right) \right\}$$

and

$$c^{(2)} = \frac{6}{\pi^2} \sum_{a=1}^{r} \sum_{m=1}^{k-1} \mathcal{L} \left( \frac{x_m^{(a)}}{1 + x_m^{(a)}} \right),$$

where $\mathcal{L}(z)$ is Roger’s dilogarithm. In [13] the set of numbers $\{\hat{x}_m^{(a)}\}$ satisfy (20) and are the same as occurring in the virial expansion. They can be found in [13] where not only their values, but also the corresponding sum of dilogarithms is evaluated (conjectured):

$$c^{(2)} = \frac{k(k-1)r}{k+2r-2}. \quad (50)$$

The numbers $\{x_m^{(a)}\}$ are defined as the solution of a similar problem. They satisfy the equations

$$(x_m^{(a)})^2 \prod_{b=1}^{r} \left[ 1 + \frac{1}{x_{m}^{(b)}} \right]^{f_{ab}} = \left( 1 + x_{m+1}^{(a)} \right) \left( 1 + x_{m-1}^{(a)} \right)$$

for $a = 1, 2, \ldots, r$, $m = 1, 2, \ldots, k - 1$ with “boundary conditions”

$$x_k^{(a)} = 0, \quad x_0^{(a)} = \delta_{a1} x_0, \quad (52)$$

where

$$x_0 = \prod_{a=1}^{r} \left( 1 + x_1^{(a)} \right)^{N_{a1}(0)}. \quad (53)$$

Similarly to the previous case, it is also useful to introduce the “Q-system” here. We denote the corresponding numbers by $\{R_m^{(a)}\}$ and define for $m = 0, 1, \ldots, k$

$$x_m^{(a)} = \frac{R_m^{(a)} R_{m+1}^{(a)}}{\prod_{b=1}^{r} \left[ R_{m}^{(b)} \right]^{f_{ab}}}. \quad (54)$$
The numbers \( R^{(a)}_m \) are assumed to be nonnegative and satisfy the Q-system equations

\[
(R^{(a)}_m)^2 = R^{(a)}_{m+1} R^{(a)}_{m-1} + \prod_{b=1}^r [R^{(b)}_m] I_{ab} \tag{55}
\]

for \( a = 1, 2, \ldots, r, \ m = 0, 1, \ldots, k \) with boundary condition

\[
R^{(a)}_{k+1} = 0, \quad R^{(a)}_{-1} = \delta_{a1} \prod_{b=1}^r (R^{(b)}_0) I_{1b} . \tag{56}
\]

We have found the solution of the modified Q-system (55), (56). The components corresponding to the fork part of the Dynkin diagram can be written as

\[
R^{(r-1)}_m = R^{(r)}_m = \prod_{\alpha \in \Delta_+} \frac{\sin \langle \alpha \mid (k-m) \omega_r + \rho \rangle \pi}{\sin \langle \alpha \mid \rho \rangle \pi} \prod_{l=0}^{r-2} \frac{\sin \langle (k-m+2l+1) \omega_r \rangle \pi}{\sin \langle (k-m+2l+1) \rangle \pi} \sin \frac{(2l+1)\pi}{k+g-1} \sin \frac{(k+1)\pi}{k+g-1} . \tag{57}
\]

Here \( \Delta_+ \) is the set of positive roots, \( \rho \) is half the sum of the positive roots, \( \omega_r \) is the \( r \)-th fundamental weight and scalar product is normalized in weight space so that \( \langle \alpha \mid \alpha \rangle = 2 \) for the roots \( \alpha \). For \( m = 0 \) the ratio

\[
\frac{\sin \langle (k-m+g-1) \rangle \pi}{\sin \langle (g-1) \rangle \pi} \tag{58}
\]

has to be omitted from the first factor in (57) together with the simultaneous omission of its inverse from the last factor.

The remaining components, \( \{ R^{(a)}_m \} \) for \( a = r-2, \ldots, 2, 1 \), can be calculated from the equations (55) with \( a = r-1, \ldots, 3, 2 \). We have verified that the last Q-system equation (for \( a = 1 \)) is then also satisfied.

Based on extensive numerical study, we conjecture that if we use the solution of the “modified Y-system” (51), (52) obtained this way in the dilogarithm sum (48) we get

\[
c^{(1)} = \frac{k(k-1)r + k(r-1)}{k+2r-3} . \tag{59}
\]

Putting (59) and (50) together, we see that \( c_{UV} \) can also be written in the suggestive form

\[
c_{UV} = c(k, N) - c(k, N-1) , \tag{60}
\]

where

\[
c(k, N) = \frac{kN(N-1)}{2} \frac{2}{k+N-2} \tag{61}
\]

is the central charge of the \( O(N) \) WZNW conformal model at level \( k \).
7 The $O(4)$ model

The $r = 2$ case is somewhat special and we discuss it separately. From (4) and (7) it follows that in this case

$$\psi_{\alpha 1}(\theta) = \frac{1}{\cosh \theta} = \psi(\theta).$$

(62)

We now define the new variables

$$\epsilon_0(\theta) = \epsilon_0(\theta), \quad \epsilon_m(\theta) = -\epsilon_m^{(1)}(\theta), \quad \epsilon_{-m}(\theta) = -\epsilon_m^{(2)}(\theta).$$

(63)

We further define

$$l_m(\theta) = \ln \left[ 1 + e^{-\epsilon_m(\theta)} \right]$$

(64)

for $m = 1 - k, \ldots, 0, \ldots, k - 1$ and $l_{\pm k}(\theta) = 0$. This allows us to write the TBA equations (8) and (9) in the compact form

$$\epsilon_m + \psi * (l_{m+1} + l_{m-1}) = \frac{1}{T} (M \cosh \theta - H) \delta_{m0}$$

(65)

for $m = 1 - k, \ldots, 0, \ldots, k - 1$.

In this form it is easy to recognize that (63) is the standard TBA system [4] associated to the $A_{2k-1}$ Dynkin diagram with $m = 0$ corresponding to the single massive node. This observation allows us to determine the UV central charge immediately from

$$c_{UV} = c(A_{2k-1}) - 2 c(A_{k-1}).$$

(66)

Using the well-known formula [13]

$$c(A_l) = \frac{l(l + 1)}{l + 3}$$

(67)

we get

$$c_{UV} = \frac{3k^2}{(k + 1)(k + 2)},$$

(68)

which is the expected result since it can also be written as

$$c_{UV} = c(k, 4) - c(2k, 3).$$

(69)

Note that the case of the $O(4)/O(3)$ sigma model is exceptional in that the KM level corresponding to $O(3)$ is doubled. For $N = 3$ (61) is not valid either, we have $c(k, 3) = 3k/(k + 2)$ instead.
The Y-system equations in this case simplify to
\[ \dot{x}_m^2 = (1 + \dot{x}_{m+1})(1 + \dot{x}_{m-1}) \] (70)
for \( m = 1,2,\ldots,k-1 \) and \( \dot{x}_0 = \dot{x}_k = 0 \). They are solved in terms of the Q-system as
\[ \dot{x}_m = Q_{m+1}Q_{m-1}, \quad \text{where} \quad Q_m^2 = 1 + Q_{m+1}Q_{m-1}. \] (71)
The Q-system has a simple solution in this case. It is
\[ Q_m = \frac{p(m+1)}{p(1)}. \] (72)
It is easy to calculate the virial coefficients \( p^{(1)} \) and \( p_1^{(2)} \). They are of the form (14) and (15) respectively with
\[ s = n = (1 + \dot{x}_1) = Q_1^2, \] (73)
which indeed reproduces \( N = 4 \) in the \( k \to \infty \) limit.
The Fourier transform (17) entering the second virial coefficient in this case is
\[ \tilde{\phi}(\omega) = \frac{2n^2q^2}{1+q^2} S_1(q), \] (74)
where \( S_1(q) \) is determined by solving
\[ \left( q + \frac{1}{q} \right) \left( 1 + \frac{1}{\hat{x}_m} \right) S_m(q) = \delta_{m1} + S_{m+1}(q) + S_{m-1}(q) \] (75)
for \( m = 1,2,\ldots,k-1 \) with boundary condition \( S_0(q) = S_k(q) = 0 \). The solution of (75) is
\[ S_m(q) = \frac{1}{Q_1} \left[ \frac{1}{1 - q^{2k+4}} \left( \hat{S}_m(q) - q^{2k+4} \hat{S}_m \left( \frac{1}{q} \right) \right) \right], \] (76)
for \( m = 1,2,\ldots,k-1 \), where
\[ \hat{S}_m(q) = \frac{Q_{m+1}}{Q_m} q^m - \frac{Q_{m-1}}{Q_m} q^{m+2}. \] (77)
Finally we find
\[ \tilde{\phi}(\omega) = \frac{2nq^2}{1+q^2} \left[ \frac{1}{1 - q^{2k+4}} \left( Q_2 - q^2 + \frac{1}{q^2} \right) \right], \] (78)
which reduces to the expected result
\[ \tilde{\phi}(\omega) = \frac{8q^2(3-q^2)}{1+q^2} \] (79)
in the \( k \to \infty \) limit.
8 Large $N$ limit

Finally, we consider the $N \to \infty$ limit, where the $O(N)$ $\sigma$-models are exactly solvable. This limit was studied in [14], where the exact formula

$$\ln \frac{M(R)}{M} = F(z) = 2 \sum_{m=1}^{\infty} K_0(mz) + \mathcal{O}\left(\frac{1}{N}\right)$$  \hspace{1cm} (80)

for the mass gap $M(R)$ in finite volume (corresponding to a circle of radius $R$) was found. Here $K_0$ is a modified Bessel function and $z = M(R)R$, which can be obtained, as a function of $\zeta = MR$, by solving this implicit equation.

Using similar techniques, we calculated the $N \to \infty$ limit of the pressure. This requires the calculation of the ground state energy in finite volume [12]. We find

$$p(R) = \frac{N}{8\pi} \left\{ M^2 - M^2(R) \right\} + \frac{N}{2\pi} M^2(R) \sum_{m=1}^{\infty} K_2(mz) + \mathcal{O}(1),$$  \hspace{1cm} (81)

where $K_2$ is a modified Bessel function and the temperature is given by $T = 1/R$.

We first study the $R \to 0$ limit of the effective central charge

$$\tilde{c}(R) = \frac{6R^2}{\pi} p(R).$$  \hspace{1cm} (82)

For short distances $F(z) \sim \pi/z$ implying [14]

$$z \sim \frac{\pi}{\ln\left(\frac{1}{\zeta}\right)}$$  \hspace{1cm} (83)

consistently with AF perturbation theory. This can be used to determine the short distance (or high temperature) limit of the effective central charge. We find

$$\tilde{c}(R) = N \left\{ 1 - \frac{3}{2 \ln\left(\frac{1}{\zeta}\right)} + \mathcal{O}\left(\frac{1}{\ln^2\left(\frac{1}{\zeta}\right)}\right) \right\} + \mathcal{O}(1),$$  \hspace{1cm} (84)

as expected.

In the opposite ($R \to \infty$) limit (80) can be expanded as

$$\frac{M(R)}{M} = 1 + \sum_{\nu=1}^{\infty} \Delta_\nu(\zeta),$$  \hspace{1cm} (85)

where $\Delta_\nu(\zeta) = \mathcal{O}(e^{-\nu\zeta})$. The first two terms of this expansion are

$$\Delta_1(\zeta) = 2K_0(\zeta) \quad \text{and} \quad \Delta_2(\zeta) = 2K_0^2(\zeta) + 2K_0(2\zeta) - 4\zeta K_0(\zeta)K_1(\zeta).$$  \hspace{1cm} (86)
Similarly, the pressure $p(R)$ can be expanded as

$$p(R) = \frac{1}{R} \sum_{\nu=1}^{\infty} p^{(\nu)}(\zeta), \quad (87)$$

where $p^{(\nu)}(\zeta) = \mathcal{O}(e^{-\nu\zeta})$. As explained above, these expansion coefficients must be the same as the virial coefficients of $I(3)$. Using (86) in the expansion of (81), for the first two expansion coefficients we get

$$p^{(1)}(\zeta) = \frac{NM}{\pi} K_1(\zeta) + \mathcal{O}(1), \quad (88)$$
$$p^{(2)}(\zeta) = \frac{NM}{2\pi} \left[ K_1(2\zeta) - 2\zeta K_2^0(\zeta) \right] + \mathcal{O}(1). \quad (89)$$

The agreement of (88) and (89) with the first two virial coefficients of $I(3)$ in the large $N$ limit is obvious for $p^{(1)}$, since (88) and (I3) are exactly the same for any model (independently of the interaction). It is not so obvious for the second coefficients. First one has to note that the (constant) $\mathcal{O}(N^2)$ term of (18) exactly cancels (15). There remains the $\mathcal{O}(N)$ term

$$N \left( \frac{1}{2} + \pi|\omega| - \frac{2\pi|\omega|}{1 + e^{-\pi|\omega|}} \right), \quad (90)$$

which, after Fourier transformation, indeed reproduces (89).

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