Analytical solutions of a CQNLSE with a source

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Abstract. Motivated by the recent observation of quantum droplet and its beyond mean-field description, we investigate the existence of cnoidal waves in a driven cubic-quartic nonlinear Schrödinger equation (CQNLSE), as the most simplistic modeling of beyond mean-field formalism of Bose-Einstein condensate (BEC) can be traced in the framework of CQNLSE. To the best of our knowledge, a comprehensive analytical description of CQNLSE is nonexistent. We check for the existence of the analytical solutions of forced CQNLSE. The stabilization of the system necessitates the inclusion of the source term. We derive analytical cnoidal wave solutions which are both localized as well as nonlocalized type. Further we reveal the existence of bright soliton like structure, a flat valley like structure and $w$-type solitons. We are also able to see the non existence of sinusoidal modes.

1. Introduction

Nonlinear Schrödinger equation (NLSE) has drawn interest from diverse community for last several decades ranging from water waves to plasmas. Formally NLSE are the homogeneous second order nonlinear differential equation and they admit different classes of analytical solution. It is not difficult to map the NLSE to Jacobi Elliptic equation [1] which in turn allows us to use the 12 Jacobi elliptic functions as solutions. It must be noted that, the Jacobi elliptic functions can be derived from the amplitude function of Jacobi elliptic integrals [2, 3]. These solutions can be constant, periodic or localized based on the parameter $m$ as $0 \leq m \leq 1$. A broad class of the localized solutions are categorized as the solitons which are highly sought after in fiber optics communication system, as robust localized pulses with ability to retain the shape over a large distance, is highly amenable for long distance communication [4]. Apart from these, one can also find solitons in bosonic and fermionic superfluids [5, 6, 7, 8, 10, 11, 12, 13, 14] where these localized matter waves can play useful role in different fields such as atom lithography and atom optics.

The dynamics of Bose-Einstein condensate (BEC) is quite successfully captured via mean-field formalism developed by Gross and Pitaevskii and therefore NLSE associated with BEC dynamics is known as Gross-Pitaevskii (GP) equation [15, 16]. Since the first successful realization of the atomic BEC, numerous works on this unique quantum state have taken place [17, 18, 19] which has widened our understanding in manifold. Very recently, a group of experimentalists have observed the formation of liquid droplet like state in a BEC mixture [20]. This is quite surprising as the prevailing conception of the liquid state is highly influenced by the theory of van
der Waals. It asseverate that the liquid state arises when density is quite high due to an equilibrium between attractive inter-atomic forces and short-range repulsion. However, these newly emerged droplets in ultra-cold and extremely dilute atomic gases do not explicitly follow the usual van der Waals predicted perception [21]. This liquid like state is purely manifestation of the quantum fluctuations [22, 23]. These droplets are small clusters of atoms self-bound by the interplay of attractive and repulsive forces. The origin of the attractive force can be modeled in the purview of standard mean-field theory whereas the repulsive force originates from the beyond mean-field correction [24]. The underlying theory relies on the Lee-Huang-Yang’s (LHY) correction [25] to the mean-field Gross-Pitaevskii (GP) equation [15, 16] which is discussed in great detail in Ref. [26].

From the mathematical perspective, the problem boils down to a nonlinear equation whose nonlinearity is not limited to the cubic term but it also carries a quartic contribution [20]. From our survey of literature we realize that there exists a significant void in understanding the competition between cubic and quartic nonlinearity. We were only able to find out some discussion in Ref. [27], where the authors have studied the existence of soliton in cubic-quartic NLSE (CQNLSE) by using phase portrait analysis. However, no explicit solution was provided. Nevertheless, the recent observation of quantum droplets demands a more thorough investigation of CQNLSE and the interplay between the attractive and repulsive interactions emerging from beyond mean-field analysis as described above.

In this article, we are primarily interested to obtain analytical solution for CQNLSE. Since regular NLSE posses cnoidal solutions therefore we first attempt to obtain cnoidal solution in CQNLSE. However, to capture the essence of the recent experiments without being too specific, we employ a phenomenological source term which counter balances the quartic nonlinearity. This pathological attempt must be considered as a first step in understanding the competition between cubic and quartic nonlinearities. In Sec. 2 we derive the cnoidal solutions associated with a driven CQNLSE. Here, we like to point out that, to the best of our knowledge these solutions are provided for the very first time. Next, we concentrate in the localized modes in Sec. 3 and analyze the competition between cubic and quartic nonlinearity. We draw our conclusion in Sec. 4.

2. CQNLSE With a Source

Based on our discussion in the previous section, we realize that, understanding the interplay of cubic and quartic nonlinearity in a CQNLSE is need of the hour. Though the recent experiments were performed for binary BEC [20] and dipolar BEC [23], we consider a general CQNLSE with an external source term. The objective of our investigation is, first obtain localized solution for a general CQNLSE and later concentrate on the exact experimental situation. Hence, we start with a CQNLSE, however to model the additional contributions we assume the external source term. We expect that, this formalism can be experimentally realized as well by applying external optical force via pico second lasers. Hece, we starts from the following CQNLSE with a source,

\[ i \frac{\partial \psi(x,t)}{\partial t} + A \frac{\partial^2 \psi(x,t)}{\partial x^2} + B|\psi(x,t)|^2 \psi(x,t) + C|\psi(x,t)|^3 \psi(x,t) = D(x,t). \] (1)
Here, $A$, $B$ and $C$ are coefficients. In GP equation $A$ is equivalent to $\hbar^2/2m$ where as in nonlinear fiber optics it defines the dispersion. $B$ and $C$ are the coefficient of the nonlinearities which defines the strength of the nonlinearities. $D$ defines the external source. Here, we are interested in analyzing the static solutions and for that purpose we define, $\psi(x,t) = \phi(x)e^{-iut}$ and the source is phase locked with the solution such that $\mathcal{D} = F(x)e^{-iut}$. It must be noted that the traveling solutions are beyond the purview of the current investigation however they are expected to play a very important role in our future analysis. So, taking into account the above considerations, Eq.(1) leads to,

\[
\frac{d^2\phi(x)}{dx^2} + \alpha\phi(x) + \beta|\phi(x)|^2\phi(x) + \gamma|\phi(x)|^3\phi(x) - \eta(x) = 0, \tag{2}
\]

where, $\alpha = \mu/A$, $\beta = B/A$, $\gamma = C/A$ and $\eta = F(x)/A$.

From our prior knowledge about NLSE we know that, it is possible to map NLSE to Jacobi elliptic equation [1,3] which poses a variety of solutions in the form of 12 Jacobian elliptic functions [2]. Taking cue from the NLSE results, we assume an ansatz as a function of cnoidal solution for our case as Eq.(2) can be mapped to NLSE when the quartic nonlinearity is absent and no external force is applied. Since, cnoidal functions can lead to localized as well as sinusoidal modes based on the parameter value $m$ therefore it is always beneficial if we can obtain a set of cnoidal solutions.

We are primarily interested in finding the solutions in terms of the copolar trio, i.e., “cn”, “sn” and “dn”. Further, we know that the cnoidal functions at $m = 0$ and 1 can be written as, $\text{cn}(x,0) = \cos x$, $\text{cn}(x,1) = \text{sech } x$, $\text{sn}(x,0) = \sin x$, $\text{sn}(x,1) = \tanh x$, $\text{dn}(x,0) = 1$ and $\text{dn}(x,1) = \text{sech } x$. Thus, $\text{dn}(x,1) = \text{cn}(x,1)$ and $\text{dn}(x,0)$ is constant. This allows us to concentrate only on “cn” and “sn” solutions without any loss of generality.

We assume an ansatz of the form $\phi(z) = A + B\text{cn}(z, m)$ where $z = \zeta x$ and $\zeta$ is the inverse of coherence length. We consider the source as $\eta = F\text{cn}^2(z, m)$. The choice of the function is from the fact that it will help in stabilizing the additional nonlinearity (quartic). Due to the change of variable Eq.(2) modifies to

\[
\zeta^2 \frac{d^2\phi(z)}{dz^2} + \alpha\phi(z) + \beta|\phi(z)|^2\phi(z) + \gamma|\phi(z)|^3\phi(z) - \eta(z) = 0. \tag{3}
\]

Applying the ansatz in Eq.(3) we obtain a set consistency conditions,

\[
A^3\gamma + A^2\beta + \alpha = 0
\]

\[
4A^2\gamma + 3A^2\beta + \alpha + (2m - 1)\zeta^2 = 0
\]

\[
A = -\frac{\beta}{2\gamma}, \quad 4AB^2\gamma + B^2\beta - 2m\zeta^2 = 0, \quad F = B^4\gamma. \tag{4}
\]

Solving Eq.(4) we obtain, $A = -\beta/2\gamma$, $\alpha = -\beta^3/8\gamma^2$, $\zeta = \pm iB\sqrt{\beta}/\sqrt{2m}$ and $B = \pm i\frac{\beta}{\sqrt{2m}\zeta\sqrt{\beta}}$. Thus the strength of the driving force can now be evaluated as, $F = \frac{16(2m-1)^2\gamma}{m^2\beta^4\gamma}$. Since coherence length is a real and positive quantity therefore $\zeta > 0$ which leads to $\beta < 0$ and $m > 0$. This implies that the cubic nonlinearity is attractive and the sinusoidal modes are strictly prohibited. Further we note that the solution does not exists for $m = 1/2$ as $B$ blows up. Hence, the final solution can be written as,

\[
\phi(z) = -\frac{|\beta|}{2\gamma} \left(1 \mp \frac{\sqrt{m}}{\sqrt{1-2m}} \text{cn}(z, m)\right). \tag{5}
\]
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Figure 1. (Color Online) The figures depict the behavior of the obtained “cn” and “sn” solution in (a) and (b) respectively for $0 < m < 1$ from Eq. 5. The solid line corresponds to $m = 2/3$ and the dashed line is for $m = 1/3$. Here, $\phi_0 = |\beta|/2\gamma$.

In Fig. 1(a) we plot the solution from Eq.(5) for two different values of $m$ such as $2/3$ and $1/3$ which yields oscillatory behavior. The solutions boils down to localized one when we use $m = 1$. The above calculation also points to the fact that the character of the cubic and quartic nonlinearity is opposite in nature. To be precise the cubic nonlinearity is attractive where as the quartic term is repulsive. So a competition between them along with the source results in yielding nontrivial cnoidal solution.

Next we use, $\phi(z) = A + B \text{sn}(z, m)$ as ansatz and $\eta(z) = F \text{sn}^4(z, m)$. Applying the ansatz and the source term in Eq.(2), we obtain,

\[
A = -\frac{\beta}{2\gamma}, \quad \alpha = -\frac{\beta^3}{8\gamma}, \quad \zeta = \pm \frac{B\sqrt{\gamma}}{2\sqrt{m}},
\]

\[
B = \pm \frac{\beta\sqrt{m}}{2\sqrt{m+1}} \quad \text{and} \quad F = \frac{\beta^4 m^2}{16(m+1)^2 \gamma^2}.
\]

Here also we observe that for $m = 0$ the coherence length reduces to zero as $\zeta \to \infty$ which implies nonexistence of sinusoidal modes. Thus the cnoidal wave solution reads,

\[
\phi(z) = -\frac{\beta}{2\gamma} \left( 1 \pm \frac{\sqrt{m}}{\sqrt{m+1}} \text{sn}(z, m) \right).
\]

In Fig. 1(b) the “sn” solution for $m = 1/3$ and $2/3$ is plotted. It must be noted from Eq. 6 that, for “sn” solution the cubic interaction is need not be attractive in nature. However, the repulsive nature of all the coefficient hinders the existence of localized modes as described in Fig. 2. In this figure we have plotted $|\phi(z)|^2/|\phi_0|^2$ against $z$ for $m = 1$. As mentioned earlier, $\text{cn}(z, 1) = \text{sech}(z)$ and $\text{sn}(z, 1) = \tanh(z)$ and applying these results in Eq.(5) and (6) we obtain localized as well as nonlocalized modes. The existence of localized mode of $\text{sech}(z)$ type motivates us to investigate further in this direction which we perform in the following section.

3. Localized Modes

Extending the discussion from previous section, here we elaborate about the localized solutions for CQNLSE. We consider an ansatz as $\phi(z) = A + B \text{sech}(z)$ and the driving force as $\eta(z) = F \text{sech}^4(z)$. Here, $z = \zeta x$ and $\zeta$ being the inverse of coherence length. Applying the ansatz in Eq.(2) we obtain a set of coefficient equation which can be equated to zero. The set of equation can be noted as,

\[
A^3 \gamma + A^2 \beta + \alpha = 0
\]
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Figure 2. (Color Online) The localized nature of \( \text{cn}(z, 1) \) solution (solid line) and the nonlocalised \( \text{sn}(z, 1) \) (dashed line) solution of driven CQNLSE is depicted. Here, \( \phi_0 = |\beta|/2\gamma \).

\[
4A^3\gamma + 3A^3\beta + \alpha + \zeta^2 = 0
\]
\[
A = -\frac{\beta}{2\gamma}, \quad \zeta = \pm i \sqrt{\frac{\beta}{2}} B, \quad F = B^4\gamma
\]

Now, solving for \( \alpha \) and \( B \), we obtain, \( \alpha = -\frac{\beta^3}{8\gamma^2} \) and \( B = \pm \frac{\beta}{2\gamma} \). Thus the strength of the driving force can now be evaluated as, \( F = \frac{\beta^4}{16\gamma^3} \). Thus, \( \phi(z) = -\frac{\beta}{2\gamma} (1 \mp \text{sech}(z)) \).

The obtained solution allows us to comment on the physical situation. We can clearly see that \( \zeta \) will be real only if \( \beta < 0 \) which means the two body low energy scattering should be attractive in nature. However, the beyond mean field interaction can be attractive as well as repulsive. This freedom is derived from the fact that the phenomenological source should have repulsive characteristics. Hence, the static solution can be noted as, \( \phi(z) = \phi_0(1 \pm \text{sech}(z)) \), where \( \phi_0 = \frac{|\beta|}{2\gamma} \), \( \zeta = \mp \frac{1}{\gamma} \left( \frac{|\beta|}{2\gamma} \right)^{3/2} \).

We must note that the nature of \( F \) depends on the nature of \( \gamma \), i.e., if \( F < 0 \) for \( \gamma < 0 \) and \( F > 0 \) for \( \gamma > 0 \).

Fig. 3 described two distinct situation. If \( \beta \) as well as \( \gamma \) is attractive then we obtain a bright soliton like density peak mode. However, as discussed in Ref. [21], for competing interactions, i.e., \( \beta < 0 \) and \( \gamma > 0 \), we observe a valley like structure. It must be noted that though the density peak is more like a bright soliton however, unlike bright soliton, the asymptotic density does not go to zero rather saturates at finite background governed by the interactions \( (|\phi_0|^2 = \beta^2/4\gamma^2) \). Interestingly similar nature has already been reported in the case of soliton generation with two and three-body interaction [28] where the competition between short range two body and phenomenological three body interaction plays the defining roles.

Next, we check for solution of the type \( \text{sech}^2(z) \) and consider an ansatz as \( \phi(z) = A + B \text{sech}^2(z) \). As the role of the source is to stabilize the CQNLSE or more precisely balance the quartic contribution we assume the source is proportional.
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Figure 3. (Color Online) Localized modes as a solution of CQNLSE with a source is depicted here. The dashed line describes a density notch with flatten valley whereas the solid line shows a bright soliton like density peak. Here, \( z = \zeta x \) and \( \phi_0 = |\beta|/2\gamma \).

Applying the ansatz in Eq. (2) we obtain the coefficient equations which leads to the suitable solution as,

\[
A = -\frac{\beta}{4\gamma}, \quad B = \frac{5\beta}{16\gamma}, \quad \zeta = \pm i \sqrt{\frac{B\beta}{4\sqrt{\gamma}}},
\]

\[
\alpha = -\frac{3\beta^3}{64\gamma^2}, \quad F = \frac{625\beta^4}{65536\gamma^3}.
\]  

(8)

The set of equations described in Eq. (8) clearly suggests that \( \beta < 0 \) for a localized solution otherwise the coherence length would be complex which will lead to an unstable trigonometric solution (secant). A significant discretion from the previous solution is that here \( |A| \neq |B| \). This actually leads to the possibility of generation of \( w \)-soliton as depicted in Fig. 4. It is also worth mentioning that the coherence length depends on the coefficients of both the nonlinearities for \( A + B \operatorname{sech}(z) \) solution however it depends only on \( \beta \) for \( A + B \operatorname{sech}^2(z) \) solution. The unequal \( A \) and \( B \) results in the generation of \( w \)-soliton which has been reported for strongly coupled BEC [29] (quadratic nonlinearity).

4. Conclusion

In this article, we have shown for the very first time, the existence of cnoidal solutions in a CQNLSE with a source. Though the cnoidal solutions of regular NLSE and cubic-quintic NLSE are quite well known albeit there exists a significant void in the context of CQNLSE. Recent observation of quantum droplets and its theoretical description via beyond-mean-field contribution motivated us to look for an analytical solution of a second order nonlinear (cubic and quartic) differential equation with a source. The source plays a role to stabilize the CQNLSE. We are able to derive analytical solutions
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Figure 4. (Color Online) Depiction of $w$ soliton as a solution of CQNLSE with a source. Here, $z = \zeta x$ and $\phi_0 = |\beta|/4\gamma$.

as a function of Jacobi “cn” and Jacobi “sn” which leads us to conclude the possibility of existence of localized as well as nonlocalized modes. It also rules out the possibility of existence of sinusoidal modes.

Next, we concentrate on localized solution and obtain localized density peak and density notch like solutions based on the character of the coefficient of the nonlinearities. Further, we are able to obtain $w$-soliton for competing interactions. Considering the nascent stage of quantum liquid research we expect our findings will be of high significance in analyzing this new phase of matter. The obvious extension of this work lies deriving the traveling soliton solution for CQNLSE and fitting this mathematical model in the actual physical picture. We expect that will lead to a more accurate characterization of the droplets.

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