A Solovay-Kitaev Decomposition Strategy for Single-Qubit Channels

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Inspired by the Solovay-Kitaev decomposition for approximating unitary operations as a sequence of operations selected from a universal quantum computing gate set such as the Hadamard \(H\), \(\pi/8\) gate \(T\) and the controlled-not (CNOT), we introduce a method for approximating any single-qubit channel as a sequence of just \(H\), \(T\) and CNOT operations. Our approach decomposes the single-qubit channel into a convex combination of “quasi-extreme” channels belonging to the closure of the set of extreme points of the set of single-qubit channels, and our quantum simulator uses just one resettable ancillary qubit plus one classical bit as well as a universal quantum computing gate set.

Quantum computing requires the capability to efficiently approximate arbitrary quantum operations as a sequence of a finite set of operations. The celebrated Solovay-Kitaev theorem \([1, 2]\) addresses this problem by providing a strategy for approximating any unitary operation \(U\) within error tolerance \(\epsilon\) as a sequence of \(O(\text{polylog}(1/\epsilon))\) gates chosen from the finite set, such as the standard \(\{H, T = Z^{1/4}, \text{CNOT}\}\) \((H\) is the Hadamard gate and \(T\) is the single-qubit \(\pi/8\) gate). Dawson and Nielsen \([3]\) introduced an algorithm for the Solovay-Kitaev decomposition, and many improvements have appeared recently \([4–7]\). These algorithms are central to quantum simulation efforts, which is especially important because quantum simulation is regarded as the most promising direction for a non-trivial quantum computation \([8]\).

Closed-system (i.e., Hamiltonian-generated) quantum simulation is well established \([9–13]\), but open-system quantum simulation is embryonic with attention focused on simulating memoryless (Markovian) dynamics based on a Lindblad master equation \([14–19]\). Open-system quantum simulation is important to cool to the ground state \([20]\), prepare thermal states \([21, 22]\) and entangled states \([23, 24]\), study nonequilibrium quantum phase transitions \([25]\), and connect dissipative quantum computing to the quantum Turing machine \([18, 20]\).

Given the importance of open-system quantum simulation, efficiently approximating channels rather than just approximating unitary evolution is critical. Here we fully solve single-qubit channel simulation, thereby creating a foundation for ultimately simulating multi-qubit channels. Our channel simulator can be regarded as a primitive for simulating open-system dynamics, in the same way as single-qubit unitary gates are a primitive for closed-system dynamics.

An obvious direction for implementing a channel is applying Stinespring dilation to implement a channel as a unitary operator on an expanded Hilbert space. This resultant unitary transformation can then be implemented by standard techniques such as the Solovay-Kitaev Dawson-Nielsen (SKDN) algorithm \([3]\). The problem with this approach is that it requires implementing a general unitary operator on a space with dimension given by the cube of the Hilbert space dimension for the original system. In the case of a single-qubit channel, a unitary operation on three qubits would be required. Implementing a general unitary on three qubits requires a complicated circuit needing between 14 and 20 CNOT gates \([27, 28]\). Here we aim to present a more efficient scheme, in that it requires fewer CNOTs and only one ancilla qubit.

Our goal for channel simulation extends further than just determining a quantum algorithm for simulating efficiently. We aim to establish an efficient autonomous classical algorithm for designing this quantum algorithm, with the quantum algorithm expressed as a sequence of one- and two-qubit gates. By autonomous, we mean that the classical algorithm designs the quantum algorithm without any sentient intervention other than providing the input: initial state \(\rho\), channel parameters represented by \(E\), and error tolerance \(\epsilon\). The true output state is \(\rho'\) and the simulator’s approximation to this state is designated \(\tilde{\rho}'\), where we henceforth use ‘\(\tilde{}\)’ to designate computer approximations.

We show that the tolerance \(\epsilon\) is the Schatten one-norm distance between the simulated channel and the correct channel in any number of Hilbert-space dimensions \([18, 29]\). The classical and quantum algorithms we derive for single-qubit channel simulation are efficient in that their time and space costs are no worse than polylog \((1/\epsilon)\). Our algorithms and complexity results for channel simulation rely on decomposing the channel into a convex combination of simpler channels, dilating each of these channels to unitary mappings on two qubits \([13]\), and making use of the SKDN algorithm \([3]\).

A succinct statement of the problem we solve follows.

**Problem.** Construct an efficient autonomous algorithm for designing an efficient quantum circuit, implemented from a small single-qubit universal gate set, that accurately simulates any completely-positive trace-preserving single-qubit mapping for any input state within pre-specified error tolerance \(\epsilon\) quantifying the distance between true and approximated states.

Our solution provides the following results: (i) a geometric look-up database for implementing the SKDN algorithm \([3]\) to decompose unitary operators; (ii) an exact decomposition

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of arbitrary single-qubit channels as a convex combination of quasi-extreme single-qubit channels; (iii) a cost reduction of single-qubit channel simulation from requiring a two-qubit ancillary state to one ancillary qubit plus one classical bit; and (iv) a proof of efficient simulation by showing that the time and space costs for both the classical algorithm for designing the quantum algorithm and the quantum circuit itself are at most polylog(1/ε).

Now we proceed to the technical aspects. The system is a single qubit whose state is a positive semidefinite operator \( \rho \in \mathcal{T}(\mathcal{H}^S) \) with \( \mathcal{H}^S \) the two-dimensional Hilbert space for the system and \( \mathcal{T}(\mathcal{H}) \) denoting the set of operators on Hilbert space \( \mathcal{H} \). The channel is

\[
\mathcal{E} : \mathcal{T}(\mathcal{H}^S) \to \mathcal{T}(\mathcal{H}^S) : \rho \mapsto \sum_i K_i \rho K_i^\dagger,
\]

with the summation at the end showing the operator-sum representation [34]. The operators \( \{K_i\} \) are called Kraus operators and satisfy \( \sum_i K_i^\dagger K_i = \mathbb{1} \).

The channel can be dilated to a unitary operator on the joint Hilbert space \( \mathcal{H}^{SE} = \mathcal{H}^S \otimes \mathcal{H}^E \) with \( \mathcal{H}^E \) denoting the environment (or ancillary space) being introduced to purify the dynamics. Conversion of channel \( \mathcal{E} \) to a Hamiltonian-generated unitary evolution can be achieved by performing a Stinespring dilation with unitary operator \( U : \mathcal{H}^{SE} \to \mathcal{H}^{SE} \), and

\[
U : \mathcal{T}(\mathcal{H}^{SE}) \to \mathcal{T}(\mathcal{H}^{SE}) : \rho \mapsto \rho^{SE} = U \rho^{SE} U^\dagger
\]

such that \( \text{tr}_E \rho^{SE} = \rho^S \), \( \text{tr}_E \rho^{SE} = \rho^S \) and \( \mathcal{E} : \rho^S \mapsto \rho^S \).

Specifically the Kraus operators [1] have representation \( K_i = \mathbb{E}(i|U|0)^E \) for \( |i\rangle^E \) an orthonormal basis state of the environment [31].

The unitary operator \( U \) is a minimal dilation of \( \mathcal{E} \) if \( U \) is a dilation such that \( \dim \mathcal{H}^E = (\dim \mathcal{H}^S)^2 \). For the case of a single qubit, \( \dim \mathcal{H}^E = 4 \) for minimal dilation. Although \( \mathcal{H}^E \) should have dimension that is the square of the dimension of \( \mathcal{H}^S \), hence four-dimensional, we will show that we only require a single resettable ancilla qubit so \( \dim \mathcal{H}^E = 2 \) [15].

We develop the algorithm for a general single-qubit completely-positive trace-preserving (CPTP) map using the four-vector state representation \( \rho = \frac{1}{2} \left[ I + b \cdot \sigma \right] \), with \( \sigma := (X,Y,Z) \), and the 4 × 4 matrix CPTP representation [32,33]

\[
\mathcal{E} \to \mathbb{T} = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}, \quad T_{ij} = \frac{1}{2} \text{tr} [\sigma_i \mathcal{E}(\sigma_j)], \quad \sigma_0 := \mathbb{1},
\]

with \( \mathbb{T} \) having twelve independent parameters. In this representation the channel is an affine map [34]

\[
\mathcal{E} : \rho \mapsto \frac{1}{2} \left( \mathbb{1} + b' \cdot \sigma \right), \quad b' = Tb + t,
\]

with \( \mathcal{E} \) being CPTP iff the Choi matrix satisfies \( C \geq 0 \), where \( C \) is defined by the involution [35,36] \( \langle i,k|C|j,l \rangle := \langle i,k|\mathbb{T}|j,l \rangle \). Geometrically, \( \mathcal{E} \) maps the state ball into an ellipsoid with \( t \) the shift from the ball’s origin and \( T \) a distortion matrix for the ball. The distortion matrix can be transferred into a diagonal form via a singular-value decomposition, so \( \mathcal{E} = U(\varphi) \mathcal{E}(\varphi) U(\varphi) \) for some \( \mathcal{E}^\prime \) with a diagonal \( T^\prime [34] \).

In our approach, the channel is constructed from two simpler channels, each of which can be simulated using only one ancillary qubit. Any single-qubit channel can be decomposed into a convex combination of two channels belonging to the closure of the set of extreme points of the set of single-qubit channels [32]. It turns out that these quasi-extreme channels, denoted as \( \mathcal{E}^\text{qe} \), can be simulated using only one ancillary qubit. In addition, the convex combination is easy to implement, simply by probabilistically implementing one or the other of the quantum channels.

The channel \( \mathcal{E}^\text{qe} \) can have its distortion matrix diagonalized by initial and final unitary operators. Then one finds that, for some \( \alpha \) and \( \beta \), the shift vector is \( t_{qe} = (0,0,\sin(\alpha + \beta) \sin(\alpha - \beta))^T \), and the distortion matrix is \( T_{qe} = \text{diag}(\cos(\alpha - \beta), \cos(\alpha + \beta), \cos(\alpha - \beta) \cos(\alpha + \beta)) \). The map can then be obtained via two Kraus operators

\[
K_0 = \begin{pmatrix} \cos \beta & 0 \\ 0 & \cos \alpha \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sin \alpha \\ \sin \beta & 0 \end{pmatrix}.
\]

The channel \( \mathcal{E}^\text{qe} \) is a generalization of the amplitude damping channel, namely, channel \( \mathcal{E}^\text{qe} \) reduces to the amplitude “attenuation” channel for \( \beta = 0 \), and reduces to the amplitude “amplification” channel for \( \alpha = 0 \).

The circuit to implement the channel \( \mathcal{E}^\text{qe} \) is depicted in Fig. 1. The rotation takes the form \( R_{\gamma}(2\gamma) = \exp(iY\gamma) = \mathbb{1} \cos \gamma + iY \sin \gamma \). The measurement in the computational basis with outcome \( |0\rangle \) (|1\rangle) corresponds to realization of Kraus operator \( K_0 \) (\( K_1 \)). There are only two CNOTs required because the final operation is just a classically controlled X operation.

In contrast, the direct approach to simulate a single-qubit channel is to use Stinespring dilation to construct a unitary acting on the system qubit and two ancillary qubits. This approach is somewhat inefficient, as a large number of gates are needed to implement a three-qubit unitary. In contrast to the best known technique using 20 CNOTs [28] and requiring at least 14 [27], our technique succeeds with only two such gates. Our result is now summarized in Prop. 1.

**Proposition 1.** Any single-qubit CPTP channel \( \mathcal{E} \) can be decomposed into the convex combination

\[
\mathcal{E} = p\mathcal{E}_1^{\text{qe}} + (1-p)\mathcal{E}_2^{\text{qe}},
\]

with \( 0 \leq p \leq 1 \), and be simulated with one ancillary qubit, two CNOTs and four single-qubit operations.
within one cube of a period–requires at most a sequence of gates from the finite set \( \{1, 2\} \).

For this purpose \([3–7]\). However, for concreteness we consider a variation of the Dawson-Nielsen approach \([3]\) with the gate techniques used for decomposing single-qubit gates are useful in the look-up database represented as rotations of the sphere. (c) The radius–π/2 ball of single-qubit unitary operations (note that this is a different to the Bloch ball). Each look-up database gate element \( G_n \) is located within one cube of a period–\( \frac{1}{2\sqrt{3}} \) cubic lattice. At the boundary, a cube’s center can lie outside the ball but still must be a legitimate domain for the search algorithm.

**Proof.** From Theorem 14 in Ref. \([32]\), any single-qubit channel \( \mathcal{E} \) can be decomposed into the convex combination \( \mathcal{E} = p\mathcal{E}_1^p + (1 - p)\mathcal{E}_2^e \), with \( 0 \leq p \leq 1 \). Note that channels \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) can be diagonalized, but the unitary operators to do so may be different in the two cases. The quasi-extreme channels \( \mathcal{E}_3^e \) can be realized by using the appropriate initial unitary operator, then applying the circuit above with corresponding angles \( \alpha_i, \beta_i \), and then applying the final unitary operator. Then the channel \( \mathcal{E} \) can be simulated by randomly implementing the two quasi-extreme channels according to a classical random number generator with probabilities \( p \) and \( 1 - p \). The above circuit uses two CNOTs, two rotations, a classically controlled \( \chi \) gate, and two additional unitary operators to diagonalize \( \mathcal{E}_3^e \). The final diagonalizing unitary \( U(\phi) \) may be combined with the \( X \) gate, so only four single-qubit unitary operators are needed.

In order to complete the decomposition of the channel into a universal gate set, it is now necessary to decompose the single-qubit unitary operators in Prop. \([\mathbf{1}] \) into the gate set. Any of the techniques used for decomposing single-qubit gates are useful for this purpose \([\mathbf{3}, \mathbf{7}] \). However, for concreteness we consider a variation of the Dawson-Nielsen approach \([\mathbf{3}] \) with the gate set \( \{H, T, \text{CNOT}\} \). Figure \( \mathbf{2}(a) \) depicts the Solovay-Kitaev strategy by which any unitary operator

\[
U(\theta) = e^{-i\tilde{H}/\hbar} = e^{i\theta_0} \exp \{ -i\theta \cdot \sigma \}
\]

(7)
can be approximately (within \( \epsilon \)) decomposed into a unitary operator \( \tilde{U} = \cdots U_2 U_1 U_0 \) comprising a polylog(\( 1/\epsilon \)) sequence of gates from the finite set \([\mathbf{1, 2}] \).

The SKDN algorithm provides an explicit construction that requires at most \( O(\log^{2.71}(1/\epsilon)) \) time and \( O(\log^{3.97}(1/\epsilon)) \) gates \([\mathbf{3}] \) but requires a database of single-qubit gates \( \{G_n\} \) depicted schematically in Fig. \( \mathbf{2}(b) \). This database gives each \( G_n \) as a sequence of \( T \) and \( H \) gates. However, Dawson and Nielsen do not discuss the size of the database (number of gates \( G_n \)), how it can be searched, and the length of the sequence of \( T \) and \( H \) gates \([\mathbf{3}] \). We explicitly provide an efficient geometric search technique depicted in Fig. \( \mathbf{2}(c) \) and described below.

Ignoring global phase, each \( U \) can be identified with coordinate \( \theta \in \mathbb{R}^3 \). As \( U(\theta) = U(\theta(1 - \pi/|\theta|)) \), the space \( \mathbb{R}^3 \) can be reduced to a radius–π/2 ball as depicted in Fig. \( \mathbf{2}(c) \). We therefore embed a cubic lattice into \( \mathbb{R}^3 \) to use as a lookup table. That is, we construct a database such that, for each cube, there is a sequence of gates that produces a unitary operation within that cube. Then, if we require a sequence of operators to approximate a given unitary operator, we identify which cube in the lattice this unitary operator occupies and then select the corresponding sequence of operators from the database. Each cube has side length \( \frac{1}{2\sqrt{3}} \), thereby ensuring a maximum separation of \( 1/32 \) between the unitary and the approximating sequence, which is sufficient for the SKDN algorithm. The portion of the cubic lattice filling the sphere has approximately 175 cubes along the diameter and a total of 2878147 \( \approx \frac{\pi}{4} (\frac{175}{2})^3 \) cubes, hence 2878147 entries in the database. We use \( T \) and \( H \) gates, and find that no more than 36 are required. (An alternative database lookup procedure is given in Ref. \([\mathbf{3}] \).)

Using this database construction with the SKDN algorithm and Prop. \([\mathbf{1}] \) we have an explicit algorithm to decompose a single-qubit channel into the gate set \( \{H, T, \text{CNOT}\} \).

As output the algorithm delivers the description \([\mathbf{C}] \) of the quantum algorithm implemented as a sequence of gates from the instruction set.

Finally, we consider the relation between the error for approximating the single-qubit unitary operators and the error for approximating the channel. For closed-system quantum simulation, the error is simply the worse-case two-norm distance between the true and approximate pure states in the system Hilbert space. For open-system quantum simulation of channel evolution \( \mathcal{E} \), the error of the approximate channel evolution \( \tilde{\mathcal{E}} \) is the Schatten one-norm

\[
\|\mathcal{E} - \tilde{\mathcal{E}}\|_{1 \to 1} := \max_{\rho} \|\mathcal{E}(\rho) - \tilde{\mathcal{E}}(\rho)\|_1, \quad \|\cdot\|_1 := \text{tr}\sqrt{\cdot^\dagger \cdot}. \tag{8}
\]

The following proposition establishes that the channel-simulation error condition is satisfied if the error bound for the dilated unitary operator \( \tilde{U} \) is \( \epsilon/2 \) with respect to the operator norm \( \| \cdot \| : \mathcal{H} \to \mathcal{H} \); i.e.,

\[
\|\tilde{U} - \tilde{U}\| := \max_{\psi} \|\tilde{U}(\tilde{U}^\dagger\psi)\| < \epsilon/2 \tag{9}
\]

for \( \tilde{U} \) the dilation of \( \tilde{\mathcal{E}} \).

**Proposition 2.** For all \( \epsilon \in \mathbb{R}^+ \), \( \rho \in \mathcal{T}(\mathcal{H}) \) and CPTP maps \( \mathcal{E}, \tilde{\mathcal{E}} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \) with respective minimal dilations \( U, \tilde{U} \), then \( \|\tilde{U} - \tilde{U}\| \leq \epsilon/2 \iff \|\mathcal{E} - \tilde{\mathcal{E}}\|_{1 \to 1} \leq \epsilon \).
This proposition is proved with the aid of the following proposition, which employs trace distance denoted by $D_t(\bullet, \bullet)$.

**Proposition 3.** For any unitary operators

$$U, \bar{U} : \mathcal{H} \otimes \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H}'$$

and any states $\rho \in T(\mathcal{H})$ and $\rho' \in T(\mathcal{H}')$, then

$$\|U - \bar{U}\| \geq \max_{\rho} D_t \left( U(\rho \otimes \rho') U^\dagger, \bar{U}(\rho \otimes \rho') \bar{U}^\dagger \right).$$

**Proof.** This result is based on the inequality

$$\|U - \bar{U}\| \geq \max_{\psi} \| \langle \psi | (U - \bar{U}) \psi \rangle \|$$

with $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}'$. For mixed state $\sigma = \rho \otimes \rho'$, with $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ and $\sum_i p_i = 1$, we obtain

$$\|U - \bar{U}\| = \max_{|\psi\rangle} \| (U - \bar{U}) |\psi\rangle \|$$

$$\geq \max_{\rho} \sum_i p_i \| U - \bar{U} |\psi_i\rangle \|$$

$$\geq \max_{\rho} \sum_i p_i D_t \left( U|\psi_i\rangle \langle \psi_i U^\dagger, \bar{U}|\psi_i\rangle \langle \psi_i \bar{U}^\dagger \right)$$

$$\geq \max_{\rho} D_t (U \sigma U^\dagger, \bar{U} \sigma \bar{U}^\dagger).$$

In the above inequalities, the last step employs the strong convexity of trace distance under a CPTP map. As maximizing over only $\rho$, rather than $\rho$ and $\rho'$, can reduce the maximum, we obtain inequality (11). \qed

Now we can use this result to prove Prop. 2. Note that both Props. 2 and 3 are not restricted to the single-qubit case and, therefore, will be useful for future studies of simulating general qudit quantum channels.

**Proof of Proposition 2** The trace-distance contraction property $[11]$

$$D_t(\mathcal{E}(\rho), \mathcal{E}(\rho')) \leq D_t( \rho \otimes \rho', U \rho \otimes \rho' U^\dagger)$$

and Prop. 3 combined with Eq. (8), yield

$$\|U - \bar{U}\| \geq \sup_{\rho} D_t(\mathcal{E}(\rho), \mathcal{E}(\rho)) = \frac{1}{2} \| \mathcal{E} - \mathcal{E} \|_{1 \rightarrow 1}.$$ 

For any $\epsilon \in \mathbb{R}^+$, if $\|U - \bar{U}\| \leq \epsilon/2$, then $\| \mathcal{E} - \mathcal{E} \|_{1 \rightarrow 1} \leq \epsilon$. \qed

We now articulate our complete result for the decomposition of the channel into the universal gate set.

**Proposition 4.** Any single-qubit channel $\mathcal{E}$ can be approximated within one-norm distance $\epsilon$ using $O(\log^{3.97}(1/\epsilon))$ computer time and gates from the gate set $\{H, T, CNOT\}$, and using one resettable ancillary qubit and one classical bit.
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