RECOVERING $l$-ADIC REPRESENTATIONS

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Abstract. We consider the problem of recovering $l$-adic representations from a knowledge of the character values at the Frobenius elements associated to $l$-adic representations constructed algebraically out of the original representations. These results generalize earlier results in [Ra] concerning refinements of strong multiplicity one for $l$-adic representations, and a result of Ramakrishnan [DR] recovering modular forms from a knowledge of the squares of the Hecke eigenvalues. For example, we show that if the characters of some tensor or symmetric powers of two absolutely irreducible $l$-adic representation with the algebraic envelope of the image being connected, agree at the Frobenius elements corresponding to a set of places of positive upper density, then the representations are twists of each other by a finite order character.

1. Introduction

Let $K$ be a global field and let $G_K$ denote the Galois group over $K$ of an algebraic closure $\overline{K}$ of $K$. Let $F$ be a non-archimedean local field of characteristic zero, and $M$ an affine, algebraic group over $F$. Suppose

$$\rho_i : G_K \to M(F), \ i = 1, 2$$

are continuous, semisimple representations of the Galois group $G_K$ into $GL_n(F)$, unramified outside a finite set $S$ of places containing the archimedean places of $K$. Let

$$R : M \to GL_m$$

be a rational representation of $M$ into $GL_m$ defined over $F$. We assume that the kernel $Z_R$ of $R$, is contained inside the centre of $M$. In this paper, we are interested in the problem of recovering $l$-adic representations from a knowledge of $l$-adic representations constructed algebraically out of the original representations, as follows:

Question 1.1. Let $T$ be a subset of the set of places $\Sigma_K$ of $K$ satisfying,

$$(1.1) \quad T = \{v \not\in S \mid \text{Tr}(R \circ \rho_1(\sigma_v)) = \text{Tr}(R \circ \rho_2(\sigma_v))\},$$

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where $\rho_i(\sigma_v)$ for $v \not\in S$ denotes the Frobenius conjugacy classes lying in the image. Suppose that $T$ is a ‘sufficiently large’ set of places of $K$. How are $\rho_1$ and $\rho_2$ related?

More specifically, under what conditions on $T$, $R$ or the nature of the representations $\rho_i$, can we conclude that there exists a central abelian representation $\chi : G_K \to \mathbb{Z}_R(F)$, such that the representations $\rho_2$ and $\rho_1 \otimes \chi$ are conjugate by an element of $M(F)$?

Further we would also like to know the answer when we take $R$ to be a ‘standard representation’, for example if $R$ is taken to be $k^{th}$ tensor, or symmetric or exterior powers of a linear representation of $M$. The representation $R \circ \rho$ can be thought of as an $l$-adic representation constructed algebraically from the original representation $\rho$.

When $M$ is isomorphic to $GL_m$, and $R$ is taken to be the identity morphism, then the question is a refinement of strong multiplicity one and was considered in earlier papers [DR1], [Ra]. In [Ra], we proved Ramakrishnan’s conjecture [DR1], that if the upper density of $T$ is strictly greater than $1 - 1/2m^2$, then $\rho_1 \simeq \rho_2$. Further, if the upper density of $T$ is sufficiently large in relation with the number of connected components of the algebraic envelope of the image of $\rho_1$, then the representations are isomorphic upon restriction to the absolute Galois group of a finite extension of $K$. Moreover it was shown that if $\rho_1$ is absolutely irreducible, the algebraic envelope of the image of $\rho_1$ is connected and $T$ is of positive density, then there exists a character $\chi$ of $G_K$, such that $\rho_2 \simeq \rho_1 \otimes \chi$ (Theorem 3.1 below).

It was shown by Ramakrishnan [DR], that holomorphic newforms are determined up to a quadratic twist from knowing that the squares of the Hecke eigenvalues coincide at a set of places of density at least $17/18$. In this paper we use the methods and techniques of our earlier paper [Ra], to obtain generalisations in the context of $l$-adic representations, of the theorem of Ramakrishnan. In the above setting, Ramakrishnan’s theorem can be obtained by taking $M$ to be $GL_2$, and $R$ to be either the symmetric square or the two fold tensor product of the two dimensional regular representation of $GL(2)$. In the process we obtain a different proof (and also a generalisation) of Ramakrishnan’s theorem.

We break the general problem outlined above into two steps. First we use the results of [Ra], to conclude that $R \circ \rho_1$ and $R \circ \rho_2$ are isomorphic under suitable density hypothesis on $T$. We consider then the algebraic envelopes of the $l$-adic representations, and we try to answer the question of recovering algebraic representations with the role of the Galois group replaced by a reductive group. We first state a general theorem that under suitable density hypothesis on $T$, we can
conclude up to twisting by a central abelian representation, that the representation \( \rho_2 \) is essentially an ‘algebraic twist’ of \( \rho_1 \). Combining this with a theorem of Richardson \([R3]\), we obtain a finiteness result, that if we fix a \( l \)-adic representation \( \rho_1 \), then the collection of \( l \)-adic representations \( \rho_2 \) satisfying the hypothesis of the above question for some \( R \) fall into finitely many ‘algebraic classes’ up to twisting by an abelian representation commuting with \( \rho_1 \).

We then specialize \( R \) to be either symmetric, tensor power, adjoint and twisted product (Asai) representations of the ambient group \( GL_n \). In this situation, the ambiguity regarding algebraic twists can be resolved, and under suitable density hypothesis, we can essentially recover representations from knowing that the character values of the symmetric or tensor powers of them coincide. We discuss the situation when the representation is absolutely irreducible and the algebraic envelope of the image is not connected, with the expectation that it may be of use in understanding some aspects of endoscopy, as for instance arising in the work of Blasius \([B]\).

Specialising to modular forms we generalise the results of Ramakrishnan \([JR]\), where we consider arbitrary \( k \)-th powers of the eigenvalues for a natural number \( k \), and also \( k \)-th symmetric powers in the eigenvalues of the modular forms, and the Asai representations. We also present an application in the context of abelian varieties.

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2. Preliminaries

**Notation.** Denote by \( \Sigma_K \) the set of places of \( K \). For a nonarchimedean place \( v \) of \( K \), let \( p_v \) denote the corresponding prime ideal of \( \mathcal{O}_K \), and \( Nv \) the norm of \( v \) be the number of elements of the finite field \( \mathcal{O}_K/p_v \). Suppose \( L \) is a finite Galois extension of \( K \), with Galois group \( G(L/K) \). Let \( S \) denote a subset of \( \Sigma_K \), containing the archimedean places together with the set of places of \( K \) which ramify in \( L \). For each place
$v$ of $K$ not in $S$, and a place $w$ of $L$ lying over $v$, we have a canonical Frobenius element $\sigma_w$ in $G(L/K)$, defined by the following property:

$$\sigma_w(x) \equiv x^{Nv} \pmod{p_w}.$$ 

The set $\{\sigma_w \mid w \mid v\}$ form the Frobenius conjugacy class in $G(L/K)$, which we continue to denote by $\sigma_v$.

Let $M$ denote a connected, reductive algebraic group defined over $F$, and let $\rho$ be a continuous representation of $G_K$ into $M(F)$, where $F$ is a non-archimedean local field of residue characteristic $l$. Let $L$ denote the fixed field of $K$ by the kernel of $\rho$. Write $L = \cup \alpha L_\alpha$, where $L_\alpha$ are finite extensions of $K$. $\rho$ is said to be unramified outside a set of primes $S$ of $K$, if each of the extensions $L_\alpha$ is an unramified extension of $K$ outside $S$.

We will assume henceforth that all our linear $l$-adic representations of $G_K$ are continuous and semisimple, since we need to determine a linear representation from it’s character. By the results of [KR], it follows that the set of ramified primes is of density zero, and hence arguments involving density as in [Ra], go through essentially unchanged. $S$ will indicate a set of primes of density zero, containing the ramified primes of the (finite) number of $l$-adic representations under consideration, and the archimedean places of $K$.

Let $w$ be a valuation on $L$ extending a valuation $v \notin S$. The Frobenius elements at the various finite layers for the valuation $w \mid_{L_\alpha}$ patch together to give raise to the Frobenius element $\sigma_w \in G(L/K)$, and a Frobenius conjugacy class $\sigma_v \in G(L/K)$. Thus $\rho(\sigma_w)$ (resp. $\rho(\sigma_v)$) is a well defined element (resp. conjugacy class) in $M(F)$. If $\rho : G_K \to GL_m(F)$ is a linear representation, let $\chi_\rho$ denote the associated character. $\chi_\rho(\sigma_v)$ is well defined for $v$ a place of $K$ not in $S$.

Let $G$ be a connected, reductive group. We have a decomposition,

$$G = C_G G_d,$$

where $C_G$ is the connected component of the center of $G$, $G_d$ the semisimple component of $G$, is the derived group of $G$, and $C_G \cap G_d$ is a finite central subgroup of $G_d$.

For a diagonalisable group $D$, let $X^*(D)$ denote the finitely generated dual abelian group of characters of $D$. $D$ is a torus if and only if $X^*(D)$ is a free abelian group. Given a morphism $\phi : D \to D'$ between diagonalisable groups, let $\phi^*$ denote the dual morphism $X^*(D') \to X^*(D)$. We recall that there is an (anti)-equivalence between the category of diagonalisable groups over $\bar{F}$ and the category of finitely generated abelian groups, where the morphism one way at the level of objects is given by sending $D$ to it’s character group $X^*(D)$. 


For a continuous morphism $\rho : G_K \to M(F)$, denote by $H_\rho$ denote the algebraic envelope in $M$ of the image group $\rho(G_K)$ inside $M(F)$, i.e., the smallest algebraic subgroup $H_\rho$ of $M$, defined over $F$, such that $\rho(G_K) \subset H_\rho(F)$. $H_\rho$ is also the Zariski closure over $F$ of $\rho(G_K)$ inside $M$. For $i = 1, 2$, let $H_i = H_{\rho_i}$, and let $H_i^0$ be the identity component of $H_i$.

We let $R : M \to GL_m$ denote a rational representation of $M$, and $Z$ a central subgroup of $M$ containing the kernel $Z_R$ of $R$.

### 3. Strong multiplicity one

We recall here the results of [Ra]. Let $R : M \to GL_m$ be a rational representation of $M$. The upper density $ud(P)$ of a set $P$ of primes of $K$, is defined to be the ratio,

$$ud(P) = \lim_{x \to \infty} \frac{\#\{v \in \Sigma_K \mid Nv \leq x, \ v \in P\}}{\#\{v \in \Sigma_K \mid Nv \leq x\}}.$$

Consider the following two hypothesis on the upper density of $T$, depending on the representations $\rho_1$ and $\rho_2$:

\begin{align*}
(3.1) & \quad \text{DH1} : \quad ud(T) > 1 - 1/2m^2 \\
(3.2) & \quad \text{DH2} : \quad ud(T) > \min(1 - 1/c_1, 1 - 1/c_2),
\end{align*}

where $c_i = |R(H_i)/R(H_i)^0|$ is the number of connected components of $R(H_i)$. As a consequence of the refinements of strong multiplicity one proved in [Ra, Theorems 1 and 2], we obtain

**Theorem 3.1.** i) If $T$ satisfies DH1, then $R \circ \rho_1 \simeq R \circ \rho_2$.

ii) If $T$ satisfies DH2, then there is a finite Galois extension $L$ of $K$, such that $R \circ \rho_1|_{GL} \simeq R \circ \rho_2|_{GL}$. The connected component $R(H_2^0)$ is conjugate to $R(H_1^0)$. In particular if either $H_1$ or $H_2$ is connected and $ud(T)$ is positive, then there is a finite Galois extension $L$ of $K$, such that $R \circ \rho_1|_{GL} \simeq R \circ \rho_2|_{GL}$.

**Proof.** For the sake of completeness of exposition, we present a brief outline of the proof and refer to [Ra] for more details. The proof reduces to taking $M = GL_m$ and $R$ to be the identity morphism. Let $\rho = \rho_1 \times \rho_2$, and let $G$ denote the image $\rho(G_K)$. Consider the algebraic subscheme

$$X = \{(g_1, g_2) \mid \text{Tr}(g_1) = \text{Tr}(g_2)\}.$$

It is known that if $C$ is a closed, analytic subset of $G$, stable under conjugation by $G$ and of dimension strictly smaller than that of $G$, then the set of Frobenius conjugacy classes lying in $C$ is of density 0.
Using this it follows that the collection of Frobenius conjugacy classes lying in $X$ has a density equal to

$$\lambda = \frac{|\{ \phi \in H/H^0 \mid H^\phi \subset X\}|}{|H/H^0|}. \quad (3.3)$$

Since this last condition is algebraically defined, the above expression can be calculated after base changing to $\mathbb{C}$. Let $J$ denote a maximal compact subgroup of $H(\mathbb{C})$, and let $p_1$, $p_2$ denote the two natural projections of the product $GL_m \times GL_m$. Assume that $p_1$ and $p_2$ give raise to inequivalent representations of $J$. (i) follows from the inequalities

$$2 \leq \int |\text{Tr}(p_1(j)) - \text{Tr}(p_2(j))|^2 d\mu(j) \leq (1 - \lambda)4m^2, \quad (3.4)$$

where $d\mu(j)$ denotes a normalized Haar measure on $J$. The first inequality follows from the orthogonality relations for characters. For the second inequality, we observe that the eigenvalues of $p_1(j)$ and $p_2(j)$ are roots of unity, and hence

$$|\text{Tr}(p_1(j)) - \text{Tr}(p_2(j))|^2 \leq 4m^2.$$

Combining this with the expression for the density $\lambda$ given by equation (3.3) gives us the second inequality.

To prove (ii), it is enough to show that $H^0 \subset X$. Let $c_1 < c_2$. The density hypothesis implies together with the expression (3.3) for the density, that there is some element of the form $(1,j) \in J \cap X$. The proof concludes by observing that the only element in the unitary group $U(m) \subset GL(m, \mathbb{C})$ with trace equal to $m$ is the identity matrix, and hence the connected component of the identity in $J$ (or $H$) is contained inside $X$.

Remark 3.1. In the automorphic context, assuming the Ramanujan-Petersson conjectures, it is possible to obtain the inequalities in (3.4), by analogous arguments, and thus a proof of Ramakrishnan’s conjecture in the automorphic context. The first inequality follows from replacing the orthogonality relations for characters of compact groups, by the Rankin-Selberg convolution of $L$-functions, and amounts to studying the behavior at $s = 1$ of the logarithm of the function,

$$L(s, '|\pi_1 - \pi_2|^2') := \frac{L(s, \pi_1 \times \tilde{\pi}_1)L(s, \pi_2 \times \tilde{\pi}_2)}{L(s, \pi_1 \times \tilde{\pi}_2)L(s, \tilde{\pi}_1 \times \pi_2)},$$

where $\pi_1$ and $\pi_2$ are unitary, automorphic representations of $GL_n(\mathbb{A}_K)$ of a number field $K$, and $\tilde{\pi}_1$, $\tilde{\pi}_2$ are the contragredient representations of $\pi_1$ and $\pi_2$. The second inequality follows from the Ramanujan hypothesis. For more details we refer to [Ra1].
4. Recovering representations

Theorem 3.1 allowed us to conclude under certain density hypothesis on $T$, that $R \circ \rho_1 \simeq R \circ \rho_2$. Our aim now is to find sufficient algebraic and group theoretic conditions in order to deduce that $\rho_1$ and $\rho_2$ determine each other up to twisting by a central abelian representation. The problem essentially boils down to considering extensions of morphisms between tori.

Let $\phi : H_1 \to H_2$ be an algebraic homomorphism between reductive groups defined over $F$. Suppose $H_2'$ is a reductive group over $F$, with a surjective homomorphism, 

$$\pi : H_2' \to H_2,$$

with $Z'_\pi = \text{Ker}(\pi)$ a central subgroup in $H_2'$. We say that a homomorphism $\psi : H_1 \to H_2'$ defined over $\bar{F}$, lifts $\phi$, if

$$\pi \circ \psi = \phi.$$ 

We have the following lemma giving sufficient conditions for a lift to exist:

**Lemma 4.1.** Suppose $\phi : H_1 \to H_2$ is a homomorphism between connected, reductive groups over $F$. With notation as above, we have the following:

1. There exists a connected, reductive group $H_1'$ together with an isogeny $f : H_1' \to H_1$, and a lift $\psi : H_1' \to H_2'$ of $\phi \circ f : H_1' \to H_2$.
2. Suppose the derived group $H_1$ is simply connected, and that the kernel $Z_\pi$ of $\pi$ is connected. Then a lift $\psi$ of $\phi$ exists.

In particular if the derived group $H_{1d}$ is simply connected, $H_2 = PGL(n)$, $H_2' = GL(n)$, then a lift exists.

**Proof.** 1) Let $H_{1s}$ be the simply connected cover of the derived group $H_{1d}$ of $H_1$. Let $H_1'' = C_{H_1} \times H_{1s}$. We have a natural finite morphism $f'' : H_1'' \to H_1$. Since $H_{1s}$ is simply connected, a lifting $\psi_s : H_{1s} \to H_2'$ of $\phi \circ f'' | H_{1s}$ exists. We have reduced the proof to the following statement:

- $H_1 = C_1 \times H_{1s}$, $H_{1s}$ simply connected and $C_1$ a torus.
- There are morphisms $\psi_s : H_{1s} \to H_2'$ and $\phi_c : C_1 \to H_2$, such that the image of $\phi_c$ commutes with the image of $\psi_s = \pi \circ \psi_s$.
- We need to produce a torus $C_1'$ together with an isogeny $f_c : C_1' \to C_1$ and a lift $\psi_c : C_1' \to H_2'$ of $\phi_c \circ f_c$, such that its image is contained in the commutant of the image group $\psi_s(H_{1s})$.

Let $T_2$ be the image torus $\phi_c(C_1)$. Since the kernel of $\pi$ is central, we can find a torus $T_2'$ contained inside $H_2'$ which surjects via $\pi$ onto $T_2$,
and commutes with the image $\psi_s(H_{1s})$. We have an injection at the
dual level $X^*(T_2) \xrightarrow{\pi} X^*(T'_2)$, and let

$$M(\pi) = \{ \alpha \in X^*(T'_2) \mid n\alpha \in \pi^*(X^*(T_2)) \text{ for some } n \in \mathbb{Z} \},$$

be the torsion closure of $\pi^*(X^*(T_2))$ inside $X^*(T'_2)$. $M(\pi)$ is a direct
summand of $X^*(T'_2)$. Form the diagram,

$$\begin{array}{ccc}
X^*(T_2) & \xrightarrow{(\phi \circ f)^*} & X^*(C_{H_1}) \\
\pi^* & \downarrow & \downarrow \\
M(\pi) & \longrightarrow & M_1
\end{array}$$

where $M_1$ is the maximal torsion-free subgroup of the group

$$\frac{M(\pi) \oplus X^*(C_{H_1})}{\text{Image}(\pi^* \oplus (\phi \circ f)^*)(X^*(T_2))}.$$
where the surjection follows from the fact that characters of closed subgroups of diagonalisable groups extend to the ambient group. Since $Z_\pi$ is connected, we have a splitting $s : X^*(Z_\pi) \to X^*(T')$ and
\[ X^*(T') = X^*(T) \oplus s(X^*(Z)). \]
We are reduced then to producing a lift from $s(X^*(Z_\pi))$ to $X^*(C_{H_1})$ compatible with the projection to $X^*(C_{H_1} \cap H_{1d})$. But since $Z_\pi$ is connected, $s(X^*(Z_\pi))$ is free, and by duality we obtain a lift as desired.

\[ \square \]

**Remark 4.1. (Rationality)** The question arises of imposing sufficient conditions in order to ensure that $\theta$ is defined over $F$. From the proof given above, the obstruction to the lift being defined over $F$ lies essentially at the level of the morphism restricted to the center. One sufficient condition is to assume that $C_{H_1}$ and $Z$ are split tori over $F$, and $T' \simeq T \times Z$ over $F$. For example, if $M = GL_n$, the representation $\rho_1$ is absolutely irreducible with $H_1$ connected and $H_{1d}$ simply connected, and $Z$ is the group of scalar matrices, then $\theta$ can be taken to be defined over $F$.

**Definition 4.1.** Let $\rho_1 : G_K \to M(F)$ be a $l$-adic representation as above, and let $H_1$ be the algebraic envelope of the image of $\rho_1$. A $l$-adic representation $\rho_2$ is an (algebraic) conjugate of $\rho_1$, if there exists an algebraic homomorphism $\theta : H_1 \to M$ defined over $\bar{F}$, such that
\[ \rho_2 = \theta \circ \rho_1. \]
Let $Z$ be a subgroup of the center of $M$. $\rho_2$ is said to be a conjugate of $\rho_1$ up to twisting by a central abelian representation with values in $Z$, if there exists a homomorphism $\chi : G_K \to \bar{Z}(\bar{F})$ such that for any $\sigma \in G_K$, we have
\[ \rho_2(\sigma) = (\theta \circ \rho_1)(\sigma)\chi(\sigma). \]
Let $R : M \to GL_m$ be a rational representation of $M$ such that the kernel $Z_R$ of $R$ is a central subgroup of $M$. Combining Lemma 4.1 and Theorem 3.1, we obtain the following general theorem giving sufficient conditions to recover $l$-adic representations under suitable density hypothesis for the corresponding characters:

**Theorem 4.1.** Let $\rho_i : G_K \to M(F)$, $i = 1, 2$ be $l$-adic representations as above. With the above notation, we have the following:

1) Suppose that $R \circ \rho_1$ and $R \circ \rho_2$ satisfy DH2. Then the following hold:
there exists a connected, reductive group $H'_1$ over $\bar{F}$ with a finite morphism $f : H'_1 \to H^0_1$ and an algebraic homomorphism $\theta : H'_1 \to M$.

- a finite extension $L$ of $K$, and a splitting $\rho'_1 : G_L \to H'_1(\bar{F})$ satisfying $f \circ \rho'_1 = \rho_1|G_L$.

- there exists a representation $\chi : G_L \to Z_R(\bar{F})$ such that,

  $$\rho_2|G_L = (\theta \circ \rho'_1) \otimes \chi.$$  

ii) Suppose that $H_1$ is connected, and the semisimple component $H^0_{1d}$ is simply connected. Let $Z$ be a connected, central subgroup of $M$ containing the kernel $Z_R$ of $R$.

  a) Suppose further that $R \circ \rho_1$ and $R \circ \rho_2$ satisfy DH2. Then there exists a finite extension $L$ of $K$, such that $\rho_2|G_L$ is an algebraic twist of $\rho_1|G_L$ up to twisting by a representation with values in $Z$.

  b) Suppose that $R \circ \rho_1$ and $R \circ \rho_2$ satisfy DH1. Then $\rho_2$ is an algebraic twist of $\rho_1$ up to twisting by a representation with values in $Z$.

Proof. By Theorem 3.1, we can assume by the density hypothesis DH2, that there exists a finite extension $L$ of $K$, such that

$$R \circ \rho_1|G_L \simeq R \circ \rho_2|G_L.$$  

If DH1 is satisfied, we can take $L = K$.

For $i = 1, 2$, let $\bar{H}_i$ denote the image of $H_i$ in the quotient group $M/Z_R$. The morphism $R$ factors via

$$R : M \xrightarrow{\pi} M/Z_R \xrightarrow{\bar{R}} GL_m,$$

where $\pi : M \to M/Z_R$ denotes the projection map, and $\bar{R}$ is the embedding of $M/Z_R$ into $GL_m$ given by $R$. Denote by

$$\bar{\rho}_1, \bar{\rho}_2 : G_K \to (M/Z_R)(F)$$

obtained from the inclusion $M(F)/Z_R(F) \subset (M/Z_R)(F)$. Since $R \circ \rho_1 \simeq R \circ \rho_2$, there is $A \in GL_m(F)$ such that

$$R \circ \rho_2(g) = A^{-1}(R \circ \rho_1)(g)A, \forall g \in G_K.$$  

Denote by $\theta_A$ the automorphism of $GL_m$, given by conjugation by $A$. The above equation translates to,

$$\theta_A \circ \bar{R} \circ \bar{\rho}_1 = \bar{R} \circ \bar{\rho}_2$$

as morphisms from $G_K$ to $GL_m(F)$. The image subgroups $\bar{\rho}_1(G_K)$ and $\bar{\rho}_2(G_K)$ are Zariski dense in $\bar{H}_1$ and $\bar{H}_2$ respectively. Thus $\theta_A \circ \bar{R}$ defines a homomorphism from $\bar{H}_1$ to $\bar{R}(H_2)$. Since $\bar{R}$ is an isomorphism of $M/Z_R$ onto its image, there exists an algebraic isomorphism

$$\bar{\theta} : \bar{H}_1 \to \bar{H}_2,$$
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satisfying

$\tilde{\theta} \circ \tilde{\rho}_1 = \tilde{\rho}_2$.

To prove the first part, by Lemma 4.1, there exists a connected reductive group $H'_1$ over $\bar{F}$, a finite morphism $f : H'_1 \to H^0_1$, and a morphism $\theta : H'_1 \to H_2$ lifting $\tilde{\theta} \circ \pi_1$, where $\pi_1$ is the projection from $H_1 \to \bar{H}_1$. Since $f$ is a finite central isogeny, pulling back this central extension of $H^0_1(F)$ to $G_K$ by the morphism $\rho_1$, we obtain a finite, central extension of $G_K$ by a finite abelian group $A$. This gives an element $\alpha$ in $H^2(G_K, A) := \lim_{\rightarrow} H^2(G_K/G_L, A)$, where $L$ runs over the collection of finite Galois extensions of $K$ contained inside $\bar{K}$. Thus there is a finite extension $L$ of $K$, at which $\alpha$ can be considered as a class in $H^2(G_K/G_L, A)$, and then the restriction of $\alpha$ to $G_L$ becomes trivial. Hence the central extension splits after going to a finite extension, and so we obtain a splitting $\rho'_1 : G_L \to H'_1(\bar{F})$ satisfying $f \circ \rho'_1 = \rho_1 \mid G_L$.

Since $\tilde{\theta} \circ \tilde{\rho}_1 = \tilde{\rho}_2$, we obtain

$\pi \circ \theta \circ \rho'_1 = (\pi \circ \rho_2) \mid G_L$,

as morphisms from $G_L$ to $M(F)$. Since the kernel of $\pi$ at the level of $F$-points is $Z_R(F)$, we obtain the first part of the theorem.

To prove the second and the third parts of the theorem, since $Z_R \subset Z$, it follows from reasoning as above that there is an isomorphism $\theta$ between the images of $H_1$ and $H_2$ in $M/Z$. Using the fact that $H_2/H_2 \cap Z \simeq H_2 Z/Z$, we now apply the second part of Lemma 4.1 to obtain the proof in these cases.

Part ii b) of the foregoing theorem allows us to avoid base changing our original representation to an extension of the base field, but instead allow the values of the twisting character to lie in a connected, central group containing $Z_R$. We now give a criterion for extending an isomorphism between two representations.

**Lemma 4.2.** Let $\rho_1, \rho_2 : G_K \to GL_n(F)$ be $l$-adic representations of $G_K$. Suppose there is a finite extension $L$ of $K$ such that $\rho_1 | G_L \simeq \rho_2 | G_L$. Assume further that $\rho_1 | G_L$ is absolutely irreducible. Then there is a character $\chi : G_K \to F^*$ such that,

$\rho_2 \simeq \rho_1 \otimes \chi$.

**Proof.** By Schur’s lemma, the commutant of $\rho_1(G_L)$ inside the algebraic group $GL_n/F$, is a form of $GL_1$. By Hilbert Theorem 90, the commutant of $\rho_1(G_L)$ inside $GL_n(F)$, consists of precisely the scalar matrices.
For $\sigma \in G_K$, let $T(\sigma) = \rho_1(\sigma)^{-1}\rho_2(\sigma)$. Since $\rho_1|_{G_L} = \rho_2|_{G_L}$, $T(\sigma) = Id$ for $\sigma \in G_L$. Now for $\tau \in G_L$ and $\sigma \in G_K$,

$$T(\sigma)^{-1}\rho_1(\tau)T(\sigma) = \rho_2(\sigma)^{-1}\rho_1(\sigma)\rho_1(\tau)\rho_1(\sigma)^{-1}\rho_2(\sigma)$$

$$= \rho_2(\sigma)^{-1}\rho_1(\sigma\tau\sigma^{-1})\rho_2(\sigma)$$

$$= \rho_2(\sigma)^{-1}\rho_2(\sigma\tau\sigma^{-1})\rho_2(\sigma)$$

$$= \rho_2(\tau)$$

$$= \rho_1(\tau)$$

Thus $T(\sigma)$ is equivariant with respect to the representation $\rho_1|_{G_L}$, and hence is given by a scalar matrix $\chi(\sigma)$. Since $\chi(\sigma)$ is a scalar matrix, it follows that for $\sigma, \tau \in G_K$, $\chi(\sigma\tau) = \chi(\sigma)\chi(\tau)$, i.e., $\chi$ is character of $\text{Gal}(L/K)$ into the group of invertible elements $F^*$ of $F$, and $\rho_2(\sigma) = \chi(\sigma)\rho_1(\sigma)$ for all $\sigma \in G_K$.

\[\Box\]

Remark 4.2. In the context of Question 1.1, the above lemma is applicable, whenever $H_1d$ has a unique representation into $GL_n$, or as we shall see in the next section when $R$ is a tensor or symmetric power of the original representation and $L$ is such that the algebraic envelope of the image of $(\rho_1 \times \rho_2)|G_L$ is connected. In conjunction with Theorem 4.1, it allows us to impose a mild hypothesis- that $T$ has positive upper density for an answer to Question 1.1.

We now single out the case when $M = GL_n$. Other interesting examples can be obtained by specialising $M$ to be $GSp_n$ or $GO_n$. For the sake of simplicity we assume that $H_1d$ is simply connected. Observe that in the special case when $\rho_1$ is further irreducible, by Remark 4.1 following Lemma 4.1, we obtain that $\theta$ is defined over $F$.

Corollary 4.1. Let $M = GL_n$, and assume that the semisimple component $H_1d$ of the connected component of the algebraic envelope of the image of $\rho_1$ is simply connected. Let $R : GL_n \to GL_m$ be a rational representation, $c_i = |R(H_i)/R(H_i)^0|$, and let

$$T = \{v \in \Sigma_K - S \mid \text{Tr}(R \circ \rho_1(\sigma_v)) = \text{Tr}(R \circ \rho_2(\sigma_v))\}.$$

a) Suppose that $R \circ \rho_1$ and $R \circ \rho_2$ satisfy the following DH2:

$$ud(T) > \min(1 - 1/c_1, 1 - 1/c_2).$$

Then there exists an algebraic homomorphism $\theta : H_1 \to M$ defined over $F$, a finite extension $L$ of $K$, and a character $\chi : GL \to GL_1(F)$ such that,

$$\rho_2|_{G_L} = (\theta \circ \rho_1|_{G_L}) \otimes \chi.$$
b) Assume further that $H_1$ is connected and that $R \circ \rho_1$ and $R \circ \rho_2$ satisfy the following DH1:

$$ud(T) > 1 - 1/2m^2.$$ 

Then there is an algebraic homomorphism $\theta : H_1 \to M$ defined over $F$, and a character $\chi : G_K \to GL_1(F)$ such that,

$$\rho_2 = (\theta \circ \rho_1) \otimes \chi.$$ 

c) If further $H_{0d}^0$ has a unique absolutely irreducible representation to $GL_n$ up to equivalence, and if $ud(T)$ is positive, we have

$$\rho_2 = \rho_1 \otimes \chi.$$ 

Remark 4.3. Let $G$ be a semisimple algebraic group over $F$, and let $\theta_1, \theta_2, \cdots, \theta_m$ be the irreducible representations of $G$ to $GL_n$. Let $R$ denote the tensor product $(F^n)^{\otimes m}$, on which $G$ acts via the tensor product of the representations $\rho_1 \otimes \cdots \otimes \rho_m$. Suppose $\rho : G_K \to GL_n(F)$ is a Galois representation with the Zariski closure of the image being isomorphic to $G$. We do have for any $i$ that $R \circ \rho \simeq R \circ (\theta_i \circ \rho)$. Thus if we are dealing with $R$ general, it is not possible to limit $\theta$ further. In the next section, we will see that if we restrict $R$ to be the tensor or symmetric power of the original representation, then for ‘general’ representations (meaning non-induced), we can conclude under the hypothesis of the Theorem that $\theta$ can be taken to be conjugation by an element of $GL_n(F)$.

Remark 4.4. In the automorphic context, if we replace $G_K$ by the conjectural Langlands group $L_K$ whose admissible representations to the dual $L$-group of a reductive group $G$ over $K$, parametrize automorphic representations of $G(\mathbb{A}_K)$, then corresponding to $R$ as above, there is a conjectural functoriality lift. In this context the theorems above and in the next section, amount to finding a description of the fibres of the functoriality lifting with respect to $R$.

4.1. A finiteness result. Given a representation $\rho : G_K \to M(F)$ and an element $m \in M(F)$, define the twisted representation $\rho^{(m)} : G_K \to M(\bar{F})$ by

$$\rho^{(m)}(\sigma) = m^{-1} \rho(\sigma)m.$$ 

Combining Theorem 4.1 with a theorem of Richardson [Ri], we deduce a finiteness result up to twisting by representations with values in the center. For the sake of simplicity, we impose some additional hypothesis on $\rho_1$. 

Corollary 4.2. Fix a continuous representation $\rho_1 : G_K \to M(F)$ as above. Assume that $H_1$ is connected, $H_{1d}$ is simply connected, and that the center $Z$ of $M$ is connected. Then there exists finitely many representations $\rho_1, \rho_2, \ldots, \rho_d$ with the following property: suppose $\rho : G_K \to M(F)$ is a continuous representation as above such that $R \circ \rho_1$ and $R \circ \rho$ satisfy DH1, where $R : M \to GL_m$ is some rational representation of $M$ with kernel contained in the center $Z$ of $M$. Then there exists $m \in M(\overline{F})$ and a representation $\chi : G_K \to Z(\overline{F})$ with values in $Z$, such that

$$\rho \simeq \rho_i^{(m)} \otimes \chi.$$  

Proof. By Theorem 4.1, we obtain that there exists a morphism $\theta : H_{1d} \to M$ defined over $\overline{F}$, and a representation $\chi : G_K \to Z(\overline{F})$, such that under the hypothesis of the Theorem,

$$\rho \simeq \rho_i^{(m)} \otimes \chi.$$  

It follows from a theorem of Richardson [Ri], that upto conjugacy by elements in $M(\overline{F})$, there exists only finitely many representations of $H_{1d}$ into $M$. Choosing a representative in each equivalence class, we obtain the Theorem.

5. Tensor and Symmetric powers, Adjoint, Asai and Induced representations

We now specialize $R$ to some familiar representations. The algebraic formulation of our problem, allows us the advantage to base change and work over the complex numbers. For a linear representation $\rho$ of a group $G$ into $GL_n$, let $T^k(\rho)$, $S^k(\rho)$, $E^k(\rho)$ ($k \leq n$), $\text{Ad}(\rho)$ be respectively the $k^{th}$ tensor, symmetric, exterior product and adjoint representations of $G$.

5.1. Tensor powers.

Proposition 5.1. Let $G$ be a connected algebraic group over a characteristic zero base field $F$, and let $\rho_1$, $\rho_2$ be finite dimensional semisimple representations of $G$ into $GL_n$. Suppose that

$$T^k(\rho_1) \simeq T^k(\rho_2)$$  

for some $k \geq 1$. Then $\rho_1 \simeq \rho_2$.

Proof. We can work over $\mathbb{C}$. Let $\chi_{\rho_1}$ and $\chi_{\rho_2}$ denote respectively the characters of $\rho_1$ and $\rho_2$. Since $\chi_{\rho_1}^k = \chi_{\rho_2}^k$, $\chi_{\rho_1}$ and $\chi_{\rho_2}$ differ by a $k^{th}$ root of unity. Choose a connected neighbourhood $U$ of the identity in $G(\mathbb{C})$, where the characters are non-vanishing. Since $\chi_{\rho_1}(1) = \chi_{\rho_2}(1)$, and the characters differ by a root of unity, we have $\chi_{\rho_1} = \chi_{\rho_2}$ on $U$. 

Since they are rational functions on $G(\mathbb{C})$ and $U$ is Zariski dense as $G$ is connected, we see that $\chi_{\rho_1} = \chi_{\rho_2}$ on $G(\mathbb{C})$. Since the representations are semisimple, we obtain that $\rho_1$ and $\rho_2$ are equivalent.

**Example 5.1.** The connectedness assumption cannot be dropped. Fong and Greiss [FG] (see also Blasius[Bl]), have constructed for infinitely many triples $(n,q,m)$ homomorphisms of $PSL_n(\mathbb{F}_q)$ into $PGL_m(\mathbb{C})$, which are elementwise conjugate but not conjugate as representations. Here $\mathbb{F}_q$ is the finite field with $q$-elements. Lift two such homomorphisms to representations $\rho_1, \rho_2 : SL(n, \mathbb{F}_q) \to GL(m, \mathbb{C})$.

We obtain that for each $g \in SL(n, \mathbb{F}_q)$, $\rho_1(g)$ is conjugate to $\lambda \rho_2(g)$, with $\lambda$ a scalar. Let $l$ be an exponent of the group $SL(n, \mathbb{F}_q)$. Then $g^l = 1$ implies that $\lambda^l = 1$. It follows that the characters $\chi_1$ and $\chi_2$ of $\rho_1$ and $\rho_2$ respectively satisfy

$$\chi_1^l = \chi_2^l.$$ 

Thus the $l^{th}$ tensor powers of $\rho_1$ and $\rho_2$ are equivalent, but by construction there does not exist a character $\chi$ of $SL(n, \mathbb{F}_q)$ such that $\rho_2 \simeq \rho_1 \otimes \chi$.

**5.2. Symmetric powers.**

**Proposition 5.2.** Let $G$ be a connected reductive algebraic group over a characteristic zero base field $F$. Let $\rho_1, \rho_2$ be finite dimensional representations of $G$ into $GL_n$. Suppose that

$$S^k(\rho_1) \simeq S^k(\rho_2)$$

for some $k \geq 1$. Then $\rho_1 \simeq \rho_2$.

**Proof.** We can work over $\mathbb{C}$. Let $T$ be a maximal torus of $G$. Since two representations of a reductive group are equivalent if and only if their collection of weights with respect to a maximal torus $T$ are the same, it is enough to show that the collection of weights of $\rho_1$ and $\rho_2$ with respect to $T$ are the same. By Zariski density or Weyl’s unitary trick, we can work with a compact form of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}_r$. Let $t$ be a maximal torus inside $\mathfrak{g}_r$. The weights of the corresponding Lie algebra representations associated to $\rho_1$ and $\rho_2$ are real valued. Consequently we can order them with respect to a lexicographic ordering on the dual of $t$.

Let $\{\lambda_1, \cdots, \lambda_n\}$ (resp. $\{\mu_1, \cdots, \mu_n\}$) be the weights of $\rho_1$ (resp. $\rho_2$) with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (resp. $\mu_1 \geq \cdots \geq \mu_n$). The weights of $S^k(\rho_1)$
are composed of elements of the form
\[ \left\{ \sum_{1 \leq i \leq n} k_i \lambda_i \mid \sum_{1 \leq i \leq n} k_i = k \right\}, \]
and similarly for \( S^k(\rho_2) \).

By assumption the weights of \( S^k(\rho_1) \) and \( S^k(\rho_2) \) are same. Since \( k\lambda_1 \) (resp. \( k\mu_1 \)) is the highest weight of \( S^k(\rho_1) \) (resp. \( S^k(\rho_2) \)) with respect to the lexicographic ordering, we have \( k\lambda_1 = k\mu_1 \). Hence \( \lambda_1 = \mu_1 \).

By induction, assume that for \( j < l, \lambda_j = \mu_j \). Then the set of weights \( \left\{ \sum_{i<j} k_i \lambda_i \mid \sum_{i<j} k_i = k \right\} \) and \( \left\{ \sum_{i<j} k_i \mu_i \mid \sum_{i<j} k_i = k \right\} \) are same.

Hence the complementary sets \( T_1(i) \) (resp. \( T_2(i) \)) composed of weights in \( S^k(\rho_1) \) (resp. \( S^k(\rho_2) \)), where at least one \( \lambda_j \) (resp. \( \mu_j \)) occurs with positive coefficient for some \( j \geq i \) are the same.

The highest weight in \( T_1(i) \) is \((k - 1)\lambda_1 + \lambda_i\), and in \( T_2(i) \) is \((k - 1)\mu_1 + \mu_i\). Since \( \lambda_1 = \mu_1 \), we obtain \( \lambda_i = \mu_i \). Hence we have shown that the collection of weights of \( \rho_1 \) and \( \rho_2 \) are the same, and so the representations are equivalent.

**5.3. Adjoint and Generalized Asai representations.** For a \( G \)-module \( V \), let \( \text{Ad}(V) \) denote the adjoint \( G \)-module given by the natural action of \( G \) on \( \text{End}(V) \cong V^* \otimes V \), where \( V^* \) denotes the dual of \( V \).

**Example 5.2.** Let \( G \) be a semisimple group, and let \( V, W \) be non self-dual irreducible representations of \( G \). Let \( V_1 = V \otimes W \), \( V_2 = V \otimes W^* \), considered as \( G \times G \)-modules. Then as \( G \times G \) irreducible modules (or as reducible \( G \) modules), \( \text{Ad}(V_1) \cong \text{Ad}(V_2) \), but \( V_1 \) is neither isomorphic to \( V_2 \) or to the dual \( V_2^* \).

However when \( G \) is simple, the following proposition is proved in [Ra3].

**Theorem 5.3.** Let \( g \) be a simple Lie algebra over \( \mathbb{C} \). Let \( V_1, \ldots, V_n \) and \( W_1, \ldots, W_m \) be non-trivial, irreducible, finite dimensional \( g \)-modules. Assume that there is an isomorphism of the tensor products,
\[ V_1 \otimes \cdots \otimes V_n \cong W_1 \otimes \cdots \otimes W_m, \]
as \( g \)-modules. Then \( m = n \), and there is a permutation \( \tau \) of the set \( \{1, \ldots, n\} \), such that
\[ V_i \cong W_{\tau(i)}, \]
as \( g \)-modules.

In particular, if \( V, W \) are irreducible \( g \)-modules, and assume that
\[ \text{End}(V) \cong \text{End}(W), \]
as $g$-modules. Then $V$ is either isomorphic to $W$ or the dual $g$-module $W^*$. 

The above theorem arose out of the application to recovering representations knowing that the adjoint representations are isomorphic. The arithmetical application of the above theorem to Asai representations was suggested by D. Ramakrishnan’s work, who has proved a similar result as below for the usual degree two Asai representations, and I thank him for conveying to me his results. We consider now a generalisation of Asai representations. Let $K/k$ be a Galois extension with Galois group $G(K/k)$. Given $\rho$, we can associate the pre-Asai representation $A_s(\rho) = \otimes_{g \in G(K/K)} \rho^g$, where $\rho^g(\sigma) = \rho(\tilde{g}\sigma \tilde{g}^{-1})$, $\sigma \in G_K$, and where $\tilde{g} \in G_k$ is a lift of $g \in G(K/k)$. At an unramified place $v$ of $K$, which is split completely over a place $u$ of $k$, the Asai character is given by,

$$\chi_{A_s(\rho)}(\sigma_v) = \prod_{v | u} \chi_{\rho}(\sigma_v).$$

Hence we get that upto isomorphism, $A_s(\rho)$ does not depend on the choice of the lifts $\tilde{g}$. If further $A_s(\rho)$ is irreducible, and $K/k$ is cyclic, then $A_s(\rho)$ extends to a representation of $G_k$ (called the Asai representation associated to $\rho$ when $n = 2$ and $K/k$ is quadratic).

5.4. Exterior powers. Let $V$ and $W$ be $G$-modules. It does not seem possible to conclude in general from an equivalence of the form $E^k(V) \simeq E^k(W)$ as $G$-modules, that $V \simeq W$. Here $E^k(V)$ denotes the exterior $k^{th}$ power of $V$. For example, let $V$ be a non self-dual $G$-module of even dimension $2n$. If $G$ is semisimple, then $E^n(V)$ is self-dual, and also that $E^n(V)$ is dual to $E^n(V^*)$, where $V^*$ denotes the dual of $V$. Hence we have $E^n(V) \simeq E^n(V^*)$, but $V \not\simeq V^*$.

It would be interesting to know, for the possible applications in geometry, the relationship between two linear representations of a connected reductive group $G$ over $\mathbb{C}$ into $GL_n$, given that their exterior $k^{th}$ power representations for some positive integer $k < n$ are isomorphic.

5.5. We summarize the results obtained so far when $R$ is taken to be a special representation.

**Theorem 5.4.** Let $\rho_1, \rho_2 : G_K \to GL_n(F)$ be $l$-adic representations of $G_K$. Let $R$ be the representation $T^k$ or $S^k$ of $GL_n$, for some $k \geq 1$. Suppose that the representations $R \circ \rho_1$ and $R \circ \rho_2$ satisfy DH2. Then
a) There is a finite extension $L$ of $K$ such that $\rho_1 | G_L \simeq \rho_2 | G_L$.

b) Assume further that $H^0_1$ acts irreducibly on $F^n$. Then there is a character $\chi : G_K \to F^*$ such that,

$$\rho_2 \simeq \rho_1 \otimes \chi.$$ 

c) Suppose now that $H^0_1$ acts irreducibly on $F^n$ and that the Lie algebra of the semisimple component of $H^0_1$ is simple. Assume now that the representations $\text{Ad}(\rho_1)$ and $\text{Ad}(\rho_2)$ satisfy $\text{DH}_2$. Then there is a character $\chi : G_K \to F^*$ such that

$$\rho_2 \simeq \rho_1 \otimes \chi \quad \text{or} \quad \rho_2 \simeq \rho_1^\ast \otimes \chi.$$ 

d) Suppose now that $H^0_1$ acts irreducibly on $F^n$ and that the Lie algebra of the semisimple component of $H^0_1$ is simple. Let $G(K/k)$ be a finite group of automorphisms acting on $K$, with quotient field $k$. Assume now that the generalized Asai representations $\text{As}(\rho_1)$ and $\text{As}(\rho_2)$ satisfy $\text{DH}_2$. Then there is a character $\chi : G_K \to F^*$, and an element $g \in G(K/k)$ such that

$$\rho_2 \simeq \rho_1^g \otimes \chi.$$ 

e) Suppose that the representations $T^k(\rho_1)$ and $T^k(\rho_2)$ satisfy $\text{DH}_1$. Then if $\rho_1$ is irreducible, so is $\rho_2$.

Proof. Let $\rho := \rho_1 \times \rho_2 : G_K \to (GL_n \times GL_n)(F)$ be the product representation, and let $H = H_\rho$. Applying the above Propositions 5.1, 5.2, and 5.3 to $H^0$ and Lemma 4.2, we obtain parts a), b), c) of the theorem. Part d) follows, since by assumptions the twisted tensor product can be considered as different representations of the same algebraic group (since the algebraic envelopes of the twists are isomorphic), and applying Theorem 5.3. Part e) follows from Proposition 5.5 proved below.

5.6. Induced representations. Theorem 5.4 shows that it is easier to understand the generic situation when the algebraic envelope is connected and the representation is absolutely irreducible. The troublesome case then is to consider representations which are induced from a subgroup of finite index. We now discuss Question 1.1 in this context, and hope that this will help in clarifying some of the issues pertaining to endoscopy in the context of the work of Labesse and Langlands and Blasius [B]. We first present another application of the algebraic machinery.
Proposition 5.5. Let $G$ be an algebraic group over a characteristic zero base field $F$, and let $\rho_1, \rho_2$ be finite dimensional semisimple representations of $G$ into $GL_n$. Suppose that

$$T^k(\rho_1) \simeq T^k(\rho_2)$$

for some $k \geq 1$. Then if $\rho_1$ is irreducible, so is $\rho_2$.

Proof. The proof follows by base changing to complex numbers and considering the characters $\chi_1, \chi_2$ of the representations restricted to a maximal compact subgroup. The characters differ by a root of unity in each connected component, and hence $<\chi_1, \chi_1> = <\chi_2, \chi_2>$. Hence it follows by Schur orthogonality relations that if one of the representations is irreducible, so is the other. □

We continue with the hypothesis of the above proposition. By Proposition 5.1, we can assume that $\rho_1|G^0$ and $\rho_2|G^0$ are isomorphic, which we will denote by $\rho^0$. Write

$$\rho^0 = \oplus_{i \in I} r_i,$$

where $r_i$ are the representations on the isotypical components, and $I$ is the indexing set of the isotypical components of $\rho^{(0)}$. The group $\Phi := G/G^0$ acts via the representations $\rho_1$ and $\rho_2$ to give raise to two permutation representations $\sigma_1$ and $\sigma_2$ on $I$. For each $\phi \in \Phi$, $l = 1, 2$ let

$$S_l(\phi) = \{ i \in I \mid \sigma_l(\phi)(i) = i \}.$$

Denote by $V_l^\phi$ the representation space of $H^0$ obtained by taking the direct sum of the representations indexed by the elements occurring in $S_l(\phi)$. For $\phi \in \Phi$, let $G^\phi$ denote the corresponding connected component (identity component is $G^0$). Let $p_l^\phi$, $p_l^0$ denote respectively the projections of $G^\phi$, $G^0$ to $GL(V_l^\phi)$. With these assumptions we have,

Proposition 5.6. Let $G$ be an algebraic group over a characteristic zero base field $F$, and let $\rho_1, \rho_2$ be finite dimensional semisimple representations of $G$ into $GL_n$. Suppose that

$$T^k(\rho_1) \simeq T^k(\rho_2)$$

for some $k \geq 1$. Assume further that either $\rho_1$ or $\rho_2$ is absolutely irreducible. With notation as above, we have for all $\phi \in \Phi$, $S_1(\phi) = S_2(\phi)$. In particular we have $\text{Ker}(\sigma_1) = \text{Ker}(\sigma_2)$.

Assume further that one of the representations is irreducible. Then $\rho_1$ and $\rho_2$ are induced respectively from representations $r'_1$, $r'_2$ of the same subgroup $G'$ of $G$, such that the restriction of $r'_1$ and $r'_2$ to $G^0$ is an isotypical component of $\rho^0$. 

Proof. Let $\eta_i$ denote the character of $r_i$. Let $G(\phi)$ denote the coset of $G^0$ in $G$ corresponding to $\phi \in \Phi$. Denote by $\chi_1$ and $\chi_2$ the characters of $\rho_1$ and $\rho_2$ respectively. For an element $g\phi \in G^0$, $g \in G^0$, $l = 1, 2$ we have

$$\chi_l(g\phi) = \sum_{i \in S_l(\phi)} \eta_i(g).$$

Since $\chi_1^k = \chi_2^k$, we obtain

$$\sum_{i \in S_1(\phi)} \eta_i = \zeta \sum_{i \in S_2(\phi)} \eta_i,$$

for some $\zeta$ a $k^{th}$ root of unity. The first part of the lemma follows from linear independence of irreducible characters of a group.

To prove the second assertion, fix an isotypical component say $r_{i_0}$ of $\rho^0$. The subgroup $G'$ of $G$ which stabilizes $r_{i_0}$ is the same for both the representations. Let $r'_1$, $r'_2$ be the extension of $r_{i_0}$ as representations of $G'$ associated respectively to $\rho_1$ and $\rho_2$. It follows by theorems of Clifford that the representations $\rho_1$ and $\rho_2$ are induced from $r'_1$ and $r'_2$ respectively.

Assume now that the constituents $r_i$ of $\rho^0$ are absolutely irreducible, i.e, the irreducible representations occur with multiplicity one. In the notation of Proposition 5.6, the assumption of multiplicity one on $\rho^0$, implies that $r'_2 = r'_1 \otimes \chi$ for some character $\chi \in \text{Hom}(G', F'^*)$ trivial upon restriction to $G^0$, where $F'$ is a finite extension of $F$. Assuming the hypothesis of Proposition 5.5, we would like to know whether $\rho_2$ and $\rho_1$ differ by a character. This amounts to knowing that the character $\chi$ extends to a character of $G$, since the representations are induced. Assume from now onwards that $G'$ is normal in $G$. Then the question of extending $\chi$ amounts first to showing that $\chi$ is invariant and then to show that invariant characters extend.

Remark 5.1. We rephrase the problem in a different language, with the hope that it may shed further light on the question. For $\sigma \in G$, let $T(\sigma) = \rho_1(\sigma)^{-1}\rho_2(\sigma)$ be as in Lemma 4.2. The calculations of Lemma 4.2 with now $\sigma \in G, \tau \in G^0$, show that $T(\sigma)$ takes values in the commutant of $\rho^0$. Let $S$ denote the $F'$-valued points of commutant torus of $\rho^0$ which is defined over a finite extension $F'$ of $F$. Since we have assumed that $\rho_1$ is irreducible, $G$ acts on $S$ via $\sigma_1$ transitively as a permutation representation on indexing set $I$. Hence $I$ can be taken to be $G/G'$ and $S$ is isomorphic to the induced module $\text{Ind}_{G'}^G(F'^*)$, where the action of $G'$ on $F'^*$ is trivial. We have for $\sigma, \tau \in G$,

$$T(\sigma\tau) = \rho_1(\tau)^{-1}T(\sigma)\rho_1(\tau)T(\tau),$$
i.e., $T$ is a one cocycle on $G$ with values in $S$. Since $S$ is induced, the ‘restriction’ map $H^1(G, S) \to H^1(G', F'^*)$ is an isomorphism. The invariants of the $G$-action on $S$ is given by the diagonal $F'$ sitting inside $S$, and the composite map, $\operatorname{Hom}(G, F'^*) \to H^1(G, S) \to \operatorname{Hom}(G', F'^*)$ is the restriction map. To say that $\chi$ extends to a character of $G$, amounts to knowing that $\chi$ lies in the image of this composite map.

Example 5.3. We consider the example given by Blasius [Bl] in this context. Let $n$ be an odd prime, and let $H_n$ be the finite Heisenberg group, with generators $A, B, C$ subject to the relations: $A^n = B^n = C^n = 1$, $AC = CA$, $BC = CB$, $AB = CBA$. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{C}^n$, and let $\xi_n$ be a primitive $n^{th}$ root of unity. For each integer $a$ coprime to $n$, define the representation $\rho_a : H_n \to \operatorname{GL}_n(\mathbb{C})$ by,

$$
\begin{align*}
\rho_a(A)e_i &= \xi_n^{(i-1)a}e_i \\
\rho_a(B)e_i &= e_{i+1} \\
\rho_a(C)e_i &= \xi_ne_i,
\end{align*}
$$

where the notation is that $e_{n+1} = e_1$. It can be seen that $\rho_a$ are irreducible representations, and that the corresponding projective representations for any pair of integers $a, b$ not congruent modulo $n$, are inequivalent. Further for any element $h \in H_n$, the images of $\rho_a(h)$ and $\rho_b(h)$ in $\operatorname{PGL}(n, \mathbb{C})$ are conjugate. Hence it follows that for some positive integer $k$ (which we can take to be $n$) the representations $T^k(\rho_a)$ and $T^k(\rho_b)$ are isomorphic.

Let $T$ be the abelian normal subgroup of index $n$ generated by $A$ and $C$. There exists a character $\chi_{ab}$ of $T$ such that $\rho_a|T \simeq \rho_b|T \otimes \chi_{ab}$. From the theory of induced representations, it can be further checked that $\rho_a|T$ has multiplicity one, and that $\rho_a$ is induced from a character $\psi_a$ of $T$. However we have that there does not exist any character $\eta$ of $H_n$ such that $\rho_a \simeq \rho_b \otimes \eta$.

With this example in mind, we now present proposition in the positive direction (which applies in particular to CM forms of weight $\geq 2$):

Proposition 5.7. Let $G$ be an algebraic group over a characteristic zero base field $F$, and let $\rho_1, \rho_2$ be finite dimensional semisimple representations of $G$ into $\operatorname{GL}_n$. Suppose that

$$
T^k(\rho_1) \simeq T^k(\rho_2)
$$

for some $k \geq 1$. Assume further that either $\rho_1$ or $\rho_2$ is absolutely irreducible, and the following assumptions:
the representation \( \rho^0 := \rho|_{G^0} \) can be written as a direct sum of irreducible representations \( \oplus_{i \in I} r_i \) with multiplicity one, i.e., each of the isotypical components \( r_i \) are irreducible.

- the subgroup \( G' \) is normal in \( G \).

Let \( r_1', r_2' \) be representations of \( G' \) as in the proof of Proposition 5.6. Let \( \chi \) be a character of \( G' \) such that \( r_2' \simeq r_1' \otimes \chi \) (this exists because of the assumption of multiplicity one).

Then \( \chi \) is invariant with respect to the action of \( G \) on the characters of \( G' \). In particular, if invariant characters of \( G' \) extend to invariant characters of \( G \) (which happens if \( G/G' \) is cyclic), then we have \( \rho_2 \simeq \rho_1 \otimes \chi \).

Proof. Restricting \( \rho_1 \) and \( \rho_2 \) to \( G' \), have by our assumptions

\[
\rho_1|G' = \oplus_{\phi \in G/G'} r_1^\phi, \quad \text{and} \quad \rho_2|G' = \oplus_{\phi \in G/G'} (r_1^\phi \chi)\phi,
\]

where \( \chi \) is a character of \( G' \) trivial on \( G^0 \). Let \( \mu_1 \) denote the character of the representation \( r_1 \) of \( G^0 \). Let \( \tau \) be an element of \( G - G' \). If \( \chi \neq \chi^\tau \), choose an element \( \theta \) of \( G' \) such that \( \chi(\theta) \neq \chi^\tau(\theta) \). It follows from our hypothesis of multiplicity one, that any element \( x \) in the coset of \( G^0 \) defined by \( \theta \) can be written as \( x = zy \), where \( z \) is a fixed element in the center of \( G' \) depending only on \( \theta \) and not on \( x \), and \( y \in G^0 \). We obtain from our assumption \( T^k(\rho_1) \simeq T^k(\rho_2) \), that for some \( \zeta \in \mu_k \), we have

\[
\zeta \sum_{\phi \in G/G'} (\mu_1(z)\mu_1(y))^\phi = \sum_{\phi \in G/G'} (\mu_1(z)\mu_1(y)\chi(\theta))^\phi,
\]

for all \( y \in G^0 \). But the assumption of multiplicity one implies that the irreducible characters \( \mu_1^\phi \) of \( G^0 \) are linearly independent, and hence the above equality forces the character \( \chi \) to be invariant.

6. Applications to Modular Forms and Abelian Varieties

We consider now applications of the general theory developed so far to the \( l \)-adic representations attached to holomorphic (Hilbert) modular forms and to abelian varieties. As a corollary to Theorem 5.1 and Proposition 5.4, we have

**Corollary 6.1.** Let \( K \) be a number field and let \( \rho_1, \rho_2 : G_K \to GL_2(F) \) be continuous \( l \)-adic representations as above. Suppose that the algebraic envelope \( H_1 \) of the image \( \rho_1(G_K) \) contains a maximal torus. Let \( R : GL_2 \to GL_m \) be a rational representation with kernel contained in the center of \( GL(2) \), and \( T \) be as in Question 1.1. Then the following holds:
a) Assume that the algebraic envelope of the image of $\rho_1$ contains $SL_2$. If $ud(T)$ is positive, then there exists a character $\chi : G_K \to GL_1(\overline{F})$ such that $\rho_2 \simeq \rho_1 \otimes \chi$.

b) Suppose that $ud(T) > 1/2$. Then there exists a finite extension $L$ of $K$ and a character $\chi : GL_L \to GL_1(\overline{F})$ such that $\rho_2|_{GL_L} \simeq \rho_1|_{GL_L} \otimes \chi$.

c) Suppose that the representations $T_k(\rho_1)$ and $T_k(\rho_2)$ satisfy $DH_{1,2}$ for some positive integer $k$ and $\rho_1$ is irreducible. Then there exists a character $\chi : G_K \to GL_1(\overline{F})$ such that $\rho_2 \simeq \rho_1 \otimes \chi$.

We now make explicit the application to classical holomorphic modular forms. A similar statement as below can be made for the class of Hilbert modular forms too. For any pair of positive integers $N, k \geq 2,$ and Nebentypus character $\omega : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$, denote by $S_k^0(N, \omega)$ the set of normalized newforms of weight $k$ on $\Gamma_1(N)$. The assumption on the weights imply that the Zariski closure of the image of the $l$-adic representation $\rho_f$ of $G_{\mathbb{Q}}$ attached to $f$ contains a maximal torus, thus satisfying the hypothesis of the above corollary. For $f \in S_k^0(N, \omega)$ and $p$ coprime to $N$, let $a_p(f)$ denote the corresponding Hecke eigenvalue.

**Corollary 6.2.** Let $f \in S_k^0(N, \omega)$ and $f' \in S_k^0(N', \omega')$ and let $\rho_f, \rho_{f'}$ be the associated $l$-adic representations of $G_{\mathbb{Q}}$. Let $R : GL_2 \to GL_m$ be a rational representation with kernel contained in the center of $GL(2)$, and $T$ be as in Question 1.1. Then the following holds:

a) If $f$ is not a CM-form and $ud(T)$ is positive, then there exists a Dirichlet character $\chi$ such that for all $p$ coprime to $NN'$, we have

$$a_p(f) = a_p(f')\chi(p).$$

In particular $k = k'$ and $\omega = \omega'\chi^2$.

b) Suppose that for some positive integer $l$, we have

$$a_p(f)^l = a_p(f')^l,$$

for a set of primes of density strictly greater than $1 - 2^{-((2l+1)l)}$. Assume also that if $f$ is dihedral, then it is not of weight one. Then there exists a Dirichlet character $\chi$ such that for all $p$ coprime to $NN'$, we have

$$a_p(f) = a_p(f')\chi(p).$$

If we further assume that the conductors $N$ and $N'$ are squarefree in a) and b) above, we can conclude that $f = f'$.

This gives us the generalization to classical modular forms of the theorems of Ramakrishnan [DR]. We now consider an application to Asai representations associated to holomorphic Hilbert modular forms. With notation as in Part d) of Theorem 5.4, assume further that $K/k$ is a quadratic extension of totally real number fields, and that $\rho_1, \rho_2$
are $l$-adic representations attached respectively to holomorphic Hilbert modular forms $f_1$, $f_2$ over $K$. Let $\sigma$ denote a generator of the Galois group of $K/k$, and further assume that $f_1$ (resp. $f_2$) is not isomorphic to any twist of $f_1^\sigma$ (resp. $f_2^\sigma$) by a character, where $f_i^\sigma$ denotes the form $f_i$ twisted by $\sigma$. Then the twisted tensor automorphic representations of $GL_2(\mathbb{A}_K)$, defined by $f_i \otimes f_2^\sigma$ (with an abuse of notation) is irreducible. Further it has been shown to be modular by D. Ramakrishnan [DR2], and descend to define automorphic representations denoted respectively by $As(f_1)$ and $As(f_2)$ on $GL_2(\mathbb{A}_k)$. As a simple consequence of Part d) of Theorem 5.4, we have the following corollary:

Corollary 6.3. Assume further that $f_1$ and $f_2$ are non-CM forms, and that the Fourier coefficients of $As(f_1)$ and $As(f_2)$ are equal at a positive density of places of $k$, which split in $K$. Then there is an idele-class character $\chi$ of $K$ satisfying $\chi\chi^\sigma = 1$ such that,

$$f_2 = f_1 \otimes \chi \quad \text{or} \quad f_2 = f_1^\sigma \otimes \chi.$$ 

We limit ourselves now to providing a simple application of the theorems in the context of abelian varieties. Let $K$ be a number field, and $A$ be an abelian variety of dimension $d$ over $K$, and let $\rho_A$ be the associated $l$-adic representation of $G_K$ on the Tate module $V_l(A) = \varprojlim A_{l^n} \otimes \mathbb{Q} \simeq GL_2(\mathbb{Q}_l)$, where $A_{l^n}$ denotes the group of $l^n$ torsion points of $A$. As applications of our general theorems, we have

Corollary 6.4. Let $A, B$ be abelian varieties as above.

a) Let $R$ be either $T^k$ or $S^k$ for some positive integer $k$. Suppose that $R \circ \rho_A$ and $R \circ \rho_B$ satisfy the density hypothesis DH2. Then $A$ and $B$ are isogenous over $\bar{K}$.

b) Suppose that we are in the ‘generic’ case, i.e., $H_A = GSp_d$. Let $R$ be any representation of $GSp_d$ with finite kernel, such that $Tr(R \circ \rho_A)(\sigma_v) = Tr(R \circ \rho_B)(\sigma_v)$ on a set of places $v$ of $K$ of positive upper density. Then $A$ and $B$ are isogenous over $\bar{K}$, and there exists a Dirichlet character $\chi$ of $K$ such that $\rho_B \simeq \rho_A \otimes \chi$.

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