Statistical moments in superposition models and strongly intensive measures

Wojciech Broniowski\textsuperscript{1,2} and Adam Olszewski\textsuperscript{2} \textsuperscript{†}

\textsuperscript{1}The H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, 31-342 Cracow, Poland
\textsuperscript{2}Institute of Physics, Jan Kochanowski University, 25-406 Kielce, Poland

(Dated: 4 April 2017)

PACS numbers: 25.75.-q, 25.75Gz, 25.75.Ld
Keywords: ultra-relativistic nuclear collisions, cumulants, factorial moments, superposition models, event-by-event fluctuations, QCD critical point, strongly intensive measures

I. INTRODUCTION

In recent years, intense activity has been focused on possible detection of critical phenomena in QCD. A basic tool of these investigations are the statistical moments of the multiplicities (or, in general, one-body observables such as momenta or charges) of the produced and experimentally detected particles, with the premise that the large fluctuations linked to critical phenomena will remain manifest in the experimentally detected particle distributions. The primary objects in these studies are the cumulant \cite{1} and factorial moments \cite{2}, vastly used in statistical studies in various domains of science. The very long history of applications of cumulants to particle physics includes, to mention a few, the studies of intermittency via factorial moments \cite{3}, analysis of fluctuations in gluodynamics \cite{4} or investigations of ratios of factorial to cumulant moments \cite{5}. In the field of ultra-relativistic heavy-ion collisions, the cumulants in the azimuthal angle have become a standard tool in studies of the harmonic flow \cite{6,7}. General mathematical features and the combinatorial interpretation of cumulants, including the multivariate case, have been recently reviewed in \cite{8}, with a stress on applications to harmonic flow.

From the practical point of view, important applications concern the unfolding of uninteresting fluctuations, as studied, e.g., in \cite{9,12}. Similar goals were addressed in \cite{13}, or more recently in \cite{14,15} by means of the so-called strongly intensive measures.

The focus of our study are the superpositions of distributions, also known in statistics as compound distributions (see, e.g., \cite{16}). The initial source emits independently particles with a certain distribution, which results in a final (measured) distribution of particles. More superposition steps \cite{17,18} may be needed in a realistic description of the production process, involving a hydrodynamic stage in the intermediate step.

The purpose of this paper is twofold: First, we bring standard techniques and results to the attention of practitioners in the field of particle/heavy-ion physics, where the material presented in a form of a concise glossary can be useful. In particular, we present exact relations between various kinds of moments (standard, cumulant, factorial, and factorial cumulant) to all orders, which involve Bell polynomials or the Stirling numbers (Sec. III).

Next, we present explicit formulas for the composed moments in superposition models to all orders. The structure of the composition laws for various kinds of moments follows directly from the composition properties of the corresponding generating functions, which are basic objects in our derivations. These composition properties are particularly simple for certain combinations of types of moments (Sec. III). We also consider the inverse problem, where one inverts the distribution of sources from the known distribution of particles and the overlaid distribution. This unfolding procedure eliminates the unwanted/trivial fluctuations. Two important cases are the unfolding of the detector efficiency \cite{19,20}, typically

\textsuperscript{1} The nature of the source depends on a particular model. The operational definition is that it emits particles independent from other sources, whereas the distribution of sources themselves may in general be correlated.
modeled with a superposed Bernoulli trial, and the removal of thermal fluctuations, which lead to an overlaid Poisson distribution.

Second (Sec. VII and VIII), we present a novel systematic way to derive the strongly intensive measures [13], which follows from the consideration of cumulants for the difference of multiplicities (or momenta or charges) of two types of particles. That way we are able to generalize the results of [14] [15] and obtain new relations involving higher-rank moments. The relations involve identical in form combinations of moments for the particles produced from a source and for the final particles, similarly to the case of the rank-2 formulas from [14].

Our results can be useful in the actively pursued investigations of mechanisms of particle production and event-by-event fluctuations, in particular in the search of the QCD phase transition at finite baryon density [11, 21–31]. An active search program is on the way by the NA61 Collaboration [32]. Other aspects of correlations and fluctuations in relativistic heavy-ion collisions are reviewed in [33]. Experimentally obtained cumulants of the net proton distributions [34] and the net charge distributions [35, 36] have been recently analyzed in [37–40]. A review of up-to-date lattice results can be found in [41], whereas a study in the UrQMD model of the STAR experimental data has been carried out in [42, 43].

Sections II–VI have mostly an introductory character, preparing grounds for Sec. VII and VIII where new results for the strongly intensive fluctuation measures are presented in the UrQMD model of the STAR experimental data. As the generating functions of various quantities indicate the given random variable.

The relation between the standard and cumulant moments is as follows:

\[
\mu^X_m = \sum_{k=1}^{m} B_{m,k}(\kappa^X_1, \ldots, \kappa^X_{m-k+1}),
\]

\[
\kappa^X_m = \sum_{k=1}^{m} (-1)^k (k-1)! B_{m,k}(\mu^X_1, \ldots, \mu^X_{m-k+1}),
\]

where \(B_{m,k}(x_1, \ldots, x_{m-k+1})\) denote the partial (a.k.a. incomplete) exponential Bell polynomials [44]. The combinatorial meaning of these polynomials lies in the encoding of the information on set partitions: the coefficients of the subsequent monomials in \(B_{m,k}\) are equal to the number of partitions of an \(m\)-element set into non-empty subsets. More precisely, the coefficient of the monomial \(x_1^{p_1} \cdots x_s^{p_s}\) is equal to the number of partitions into subsets with \(t_1, \ldots, t_s\) elements, where the subsets occur \(p_1, \ldots, p_s\) times. For instance, 

\[B_{4,2}(x_1, x_2, x_3) = 4x_1x_2+3x_1^2,\]

showing that we can partition a 4-element set into subsets of one- and three elements in 4 ways, and into two subsets of 2 elements in 3 ways. That way we can interpret the upper Eq. (3) as a decomposition of \(\mu^X_m\) into “connected” components \(\kappa^X_m\).

The generating function for the central moments \(\mu^X_m\) is

\[G^X(t) = e^{-\mu^X t} M^X(t) = 1 + \sum_{i=1}^{\infty} \mu^X_i t^i / i!,\]

with \(\mu^X = E_X X\) denoting the average. The relation with the cumulant moments is

\[\mu^X_m = \sum_{k=1}^{m} B_{m,k}(0, \kappa^X_2, \ldots, \kappa^X_{m-k+1}),\]

\[\kappa^X_m = \sum_{k=1}^{m} (-1)^{k+1} (k-1)! B_{m,k}(0, \mu^X_2, \ldots, \mu^X_{m-k+1}),\]

with \(m \geq 2\).

The generating function for the factorial moments

\[f^X_i = E_X (X - 1) \ldots (X - i)\]

is

\[F^X(t) = E_X (1+t)^X = 1 + \sum_{i=1}^{\infty} f^X_i t^i / i!,\]

having the interpretation of the average number, average number of pairs, average number of triples, etc. From definition, it is related to the generating function for the cumulant moments via the change of variables:

\[F^X(t) = M^X[\log(1+t)] = e^{K^X[\log(1+t)]}.\]

Finally, the generating function for the factorial cumulant moments, \(\kappa^X_m\), is defined as

\[G^X(t) = K^X[\log(1+t)] = \log[F^X(t)] = \sum_{i=1}^{\infty} \kappa^X_i t^i / i!.\]
cumulant
factorial
argument may indicate advantages of a certain type of
related via linear transformations. Nevertheless, physical
to choose in a given analysis, as they are all directly
of convenience and convention which type of moments
polynomials (arrows indicate the transformations via the partial Bell
ments are summarized in Fig. 1, where the labels on the
The scheme of relations between various kinds of mo-
Bkinds of moments.

FIG. 1. Summary of the transformations between various
kinds of moments. B denotes the transformation with partial
Bell polynomials (as in Eq. (5) [12]), and S denotes the trans-
formation with the Stirling numbers (as in Eq. (11)).

The relation of factorial cumulant moments to factorial
interpretation as discussed above.

As the functions \( F^X \) and \( M^X \) in Eq. (9) and the func-
tions \( G^X \) and \( K^X \) in Eq. (10) are related through a simple
change of variables \( t \leftrightarrow \log(1+t) \), the corresponding mo-
ments are linked with a linear transformation involving
the Stirling numbers as the coefficients. Explicitly,

\[
f_m^X = \sum_{k=1}^{m} s(m,k) \mu_m^X, \quad \mu_m^X = \sum_{k=1}^{m} S(m,k) f_m^X, \tag{11}
\]

\[
\kappa_m^X = \sum_{k=1}^{m} s(m,k) \kappa_m^X, \quad \kappa_m^X = \sum_{k=1}^{m} S(m,k) \kappa_m^X, \tag{12}
\]

where \( s(m,k) \) are the signed Stirling numbers of the first
kind (cycle numbers), and \( S(m,k) \) are the Stirling num-
bers of the second kind (partition numbers). Finally, we
note that the relation between \( G^X \) and \( F^X \) in Eq. (9) has the same form as the relation between \( K^X \) and \( M^X \)
in Eq. (10), therefore

\[
f_m^X = \sum_{k=1}^{m} B_{m,k}(\kappa_1^X, \ldots, \kappa_{m-k+1}^X), \tag{12}
\]

\[
\kappa_m^X = \sum_{k=1}^{m} (-1)^{k+1}(k-1)! B_{m,k}(f_1^X, \ldots, f_{m-k+1}^X), \tag{12}
\]

The scheme of relations between various kinds of mo-
ments is fully analogous to the relation of cumulant
standard
f
factorial
cumulant
κ'
κ
SB
S
B
central
μ
μ'
B
III. GENERATING FUNCTIONS IN THE SUPERPOSITION MODEL

In superposition models\(^3\) of hadron production, the
number of particles \( N \), as registered in the experiment, is
composed from independent production from \( S \) sources;
the \( j \)-th source produces \( n_j \) particles, i.e.,

\[
N = \sum_{j=1}^{S} n_j. \tag{13}
\]

The variables \( n_j \) are random, and so is the number of
sources \( S \). All the variables \( n_j \) and the multiplicity of
sources \( S \) are, by assumption on the production mechani-
m, independent from one another, which is crucial in
the following derivation of the composition formulas. The
number of sources \( S \) fluctuates event-by-event, hence the
multiplicity distribution of the finally produced particles
reflects these fluctuations, as well as the fluctuations in
the random variables \( n_j \).

For simplicity, we assume that the production from
each source is the same, hence all \( n_j \) have the same dis-
tribution and a common cumulant generating function
\( K^n(t) \). We start with the case of one type of sources,
with the straightforward generalization to more types of
sources presented in Sec. VI.

One should bear in mind that what we call in this
document “multiplicity” may in fact refer to any additive
one-body characteristics, such as charge of momentum.
For instance, \( N \) could stand for the total charge in the
event, and \( n_i \) for the charge of particles produced by the
source \( i \).

We remark here that our notion of the source, whereby
emission from different sources is by definition indepen-
dent from each other, is quite restrictive from the point
of view of the global conservation laws of the energy-
momentum or charges (for reviews of the conservation
laws see, e.g., [45, 46] and for the influence on the
strongly intensive measures see [27, 48]). Naive imposi-
tion of such global conservation constraints on the
momenta or charges of the produced particles would neces-
sarily correlate production from different sources, which
would be at odds with the basic assumption. The issue
may be resolved by introducing various types of sources
(labeled, for instance, by the value of the charge) and
keeping track of the local conservation laws at the level
of the production from a given source. We return to this
issue at the end of the Conclusion section, where we in-
dicate how to introduce more types of sources and the
local conservation laws, generalizing the approach.

\(^3\) In other domains of application of statistics, these models are
frequently referred to as compound models [16].
Substitution of Eq. (13) into Eq. (2) yields immediately
\[ e^{K^N(t)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}^{S} \mathbb{E}^{S} \sum_{j=1}^{n} n_j = \mathbb{E}^{S} \mathbb{E}^{S} e^{S K^n(t)} = e^{K^S[K^n(t)]}. \] (14)

In the third equality we have used the fact that \( n_i \) and \( n_j \) are not correlated for \( i \neq j \), and in the last equality we have used the definition of the cumulant generating function for the distribution of the number of sources \( S \), which in the formula takes the argument \( K^n(t) \). Thus we have arrived at a known fact for the compound models, namely, that the independent superposition of distributions leads to the composition of the corresponding cumulant generating functions [16].

\[ K^N(t) = K^S[K^n(t)]. \] (15)

In addition to the composition law of Eq. (15), one may straightforwardly derive additional relations. For the standard moment generating functions the following composition law follows:
\[ M^N(t) = M^S(K^n(t)) = M^S[\log (M^n(t))], \] (16)

whereas for the central moment generating function we find
\[ C^N(t) = [C^n(t)]^{\mu S} C^S(K^n(t)) = [C^n(t)]^{\mu S} C^S[\log(C^n(t)) + \mu^t]. \] (17)

For the factorial moment generating function one arrives, after a short calculation, at the composition law
\[ F^N(t) - 1 = F^S[F^n(t) - 1] - 1, \] (18)

which takes the same form as Eq. (15). For that reason all the general statements concerning the standard and cumulant moments also hold for the factorial and factorial cumulant moments. Finally, for the factorial cumulant moment generating function we find
\[ G^N(t) = G^S[F^n(t) - 1] = G^S[e^{G^n(t)} - 1]. \] (19)

Additional formulas are obtained via the replacement \( t \to \log(1 + t) \) in Eq. (15) [16], which yields
\[ G^N(t) = K^S[G^n(t)], \quad F^N(t) = M^S[G^n(t)]. \] (20)

The obtained composition laws are collected in Table I, which is central for this part of our paper. We list formulas both in the autonomous form, i.e., involving the generating functions for moments of one type only, as well as in the form of compositions of the generating functions of different types, which may also be useful in some applications.

### Table I. Composition laws for generating functions in the superposition model

| type of moments composition formula | central | standard | cumulant | factorial | factorial |
|------------------------------------|--------|----------|----------|-----------|-----------|
| \( C^N(t) = [(C^n(t)]^{\mu S} C^S[\log C^n(t)] + \mu^t] \) | \( M^N(t) = M^S[\log M^n(t)] = M^S[K^n(t)] \) | \( K^N(t) = K^S[K^n(t)] \) | \( F^N(t) - 1 = F^S[F^n(t) - 1] - 1 \) | \( F^N(t) = M^S[G^n(t)] \) | \( G^N(t) = G^S[e^{G^n(t)} - 1] \) |

### IV. MOMENTS IN THE SUPERPOSITION MODEL

We note from the formulas in Table I that certain compositions of generating functions for various types of moments have a simple form of a composite function: \( M^N(t) = M^S[K^n(t)], K^N(t) = K^S[K^n(t)], \) etc. Note that the formulas may involve various types of moments. When the composition has a generic form
\[ P^N(t) = Q^S[R^n(t)], \] (21)

where \( P, Q, R \) stands for \( M, K, F, 1, \) or \( G \), we may obtain the corresponding moments in a particularly simple manner [16]. Namely, one can use the Faà di Bruno’s formula for the \( n \)-th derivative of a composite function to arrive at the formula for the corresponding moments (denoted with the same generic symbol as the generating functions):
\[ P_m = \sum_{k=1}^{m} Q_k B_{m,k}(R_1, \ldots, R_{m-k+1}), \] (22)

where again we encounter the exponential partial Bell polynomials discussed in Sec. III. We have denoted the moments with the same generic symbol as for the generation function in Eq. (21), for instance, the moments corresponding to \( P^N(t) \) are \( P_m \). Explicit formulas for the first few values of \( m \) are listed in App. A.

The probabilistic interpretation of Eq. (22), directly related to the superposition model, is visualized in Fig. 2. The blobs connected with shaded regions indicate the

---

4 Of course, we may always obtain the composition law for the moments via the Maclaurin expansion of the generation function of any form.
where we wish to infer the corresponding moments of the source distributions. Hence, in analogy to Eq. (22),

\[ \lambda_j = \frac{1}{(R_1) j} \sum_{k=1}^{j-1} (-1)^k \frac{(j+k-1)!}{(j-1)!} B_{j-1,k}(\hat{R}_1, \ldots, \hat{R}_{j-k}), \]

for \( j \geq 2 \), where the scaled moments of \( n \) are

\[ \hat{R}_t = \frac{R_{t+1}}{(i+1)R_i}. \]

The combination of Eqs. (25) and (26) yields the inverse composition formulas for the moments of the sources.

 Explicit expressions for the first few values of \( m \) in Eq. (25) are presented in Eq. (A2) in App. A.

V. CASE OF SPECIAL OVERLAID DISTRIBUTIONS

There are two physically relevant cases where the composition laws assume a very simple form, because one of the types of the generating function is linear in \( t \). The first case occurs when the detector efficiency is modeled with a Bernoulli trial, with \( p \) denoting success of the observation of a particle, and \( q = 1 - p \) failure. For the Bernoulli trial the simplest generating function is for the factorial moments, \( F^n(t) - 1 = pt \). Then we find immediately from Eq. (8,18) and Eq. (10,19) that

\[ f^N_m = p^m f^S_m, \quad \kappa^N_m = p^m \kappa^S_m, \quad (\text{Bernoulli}) \]

which means a uniform scaling of the factorial and the factorial cumulant moments with powers of \( p \).

The other important case is the Poisson distribution, for which the simplest is the factorial cumulant generating function \( G(t) = \beta t \), with \( \beta \) denoting the mean. This case is encountered in modeling statistical hadronization from thermal sources. Then we find the simple relations

\[ f^N_m = \beta^m f^S_m, \quad \kappa^N_m = \beta^m \kappa^S_m, \quad (\text{Poisson}) \]

linking the factorial moments of particles with the standard moments of sources, and the factorial cumulant moments of particles with the cumulant moments of sources. The scale coefficients \( p \) or \( \beta \) in Eq. (28,29) disappear when appropriate scale-less ratios of the moments are

\[ \begin{align*}
\lambda_1 &= \frac{1}{R_1}, \\
\lambda_j &= \frac{1}{(R_1)^j} \sum_{k=1}^{j-1} (-1)^k \frac{(j+k-1)!}{(j-1)!} B_{j-1,k}(\hat{R}_1, \ldots, \hat{R}_{j-k}), \\
\text{for } j \geq 2, \text{ where the scaled moments of } n \text{ are } \\
\hat{R}_t &= \frac{R_{t+1}}{(i+1)R_i}.
\end{align*} \]
considered, for instance $\kappa_4^N/(\kappa_2^N)^2 = \kappa_2^S/(\kappa_2^S)^2$ for the Poisson case.

Other popular distributions, such as the binomial distribution, the Gamma distribution, of the negative binomial distribution, do not lead to composition laws as simple as Eq. (28) or (29), and in such cases one needs to use the general composition laws spelled out in App. A.

Multiplicity-dependent and non-Bernoulli trial efficiency corrections were considered in [39]. The case of the non-Bernoulli trial corrections may be analyzed according to the general formulas (A1). Note that a strong sensitivity to the detector features advocated in [39] may be attributed to large numerical coefficients appearing in Eqs. (A1), in particular for the higher-rank moments.

For the case where only two first moments of the overlaid distribution, $R_1$ and $R_2$, are nonzero, such as for instance in the case of the normal distribution with $K^\mu(t) = \mu t + \sigma^2 t^2/2$, we find the following composition formulas:

$$P_n = \sum_{k=0}^{n-1} B(n,n-k) R_1^{n-2k} R_2^K Q_{n-k},$$

$$Q_n = \sum_{k=0}^{n-1} b(n,n-k)(n-k)! R_1^{-k+n-1} R_2^K P_{n-k},$$

where $B(n,k)$ and $b(n,k)$ are the Bessel numbers of the second and first kind [49], respectively. They are equal to

$$B(n,k) = \frac{2^{k-n}n!}{(2k-n)!(n-k)!},$$

$$b(n,k) = \frac{\left(-\frac{1}{2}\right)^{n-k} (2n-k-1)!}{(k-1)!(n-k)!}.$$  (31)

If we wish to unfold other types of distributions, then the relations are more complicated than in Eqs. (28,29), keeping the generic triangular form of Eq. (A2). This is for instance the case of the $\Gamma$ distribution or the negative binomial distribution, frequently used to model the early production from the initial sources. In that case we can explicitly use the cumulant moment generating functions of the form

$$K(t) = -a \log(1 - mt/a)$$

or

$$K_{\text{neg. bin.}}(t) = -n \log[(1 - q e^t)/(1 - q)]$$

(or the corresponding generating functions for other types of moments) in the composition formulas, and derive the appropriate relations via the Maclaurin expansion. Of course, one can alternatively use the general composition formulas of App. A.

VI. MORE KINDS OF SOURCES

In various models of particle production one may distinguish more kinds of sources. Examples are the wounded nucleon [50] or wounded quark [51] models, where we have emitting sources (wounded objects) associated to the two colliding nuclei, $A$ or $B$. Another case occurs in considering the rapidity bins in studies of the longitudinal correlations, or, in general, separated bins in the kinematic space. In that situation, if production of particles in different bins is independent from each other, one may formally treat the bins as (possibly correlated) sources. We present in detail the extension of the formalism to the case of two sources, as a generalization to more sources is obvious. We note that we now enter the domain of multivariate moments and cumulants [1], which have a similar combinatorial interpretation as the univariate case discussed in the previous sections.

Let the two kinds of sources be denoted as $A$ and $B$. These sources are, in general, not independent of each other (they may be correlated), but as before the particles produced from different sources are independent from one another, and also independent of the multiplicity of the sources, denoted as $S_A$ and $S_B$. The cumulant generating function for two variates (the total number of produced particles of type $A$ and $B$),

$$N_A = \sum_{j=1}^{S_A} n_j, \quad N_B = \sum_{j=1}^{S_B} n_j,$$

is defined as

$$K^{N_A,N_B}(t_A,t_B) = \log \left[ E_{N_A,N_B} e^{t_A N_A + t_B N_B} \right]$$

$$= \sum_{i,j=0}^{\infty} \kappa_{i,j}^{N_A,N_B} \frac{t_A^i t_B^j}{i! j!},$$  (35)

and similarly for the case of the sources $S_A$ and $S_B$. Note that the sum in Eq. (35) involves also the terms with $\kappa_{i,j}^{0,0} = \kappa_i^A$ and $\kappa_{i,j}^{0,0} = \kappa_j^B$. Generalizing the derivation of Sec. III] yields the relation

$$e^{K^{N_A,N_B}(t_A,t_B)} = \frac{E_{N_A,N_B}}{N_A,N_B} e^{t_A N_A + t_B N_B}$$

$$= \frac{E_{S_A,S_B}}{S_A,S_B} e^{S_A K^{A}(t_A) + S_B K^{B}(t_B)}$$

$$= e^{K_{S_A,S_B} [K^{A}(t_A), K^{B}(t_B)]},$$  (36)

hence

$$K^{N_A,N_B}(t_A,t_B) = K_{S_A,S_B} [K^{N_A}(t_A), K^{N_B}(t_B)].$$  (37)

We may now repeat the steps of Sec. [IV] to arrive at more generic composition laws between the generating functions of various kinds, in analogy to Table IV

$$P^{N_A,N_B}(t_A,t_B) = Q^{S_A,S_B} [R^{N_A}(t_A), R^{N_B}(t_B)].$$  (38)

For the inverse problem

$$Q^{S_A,S_B}(u_A,u_B) = P^{N_A,N_B} [(R^{N_A})^{-1}(u_A), (R^{N_B})^{-1}(u_B)].$$  (39)
For the corresponding moments we find
\[ P_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} Q_{k,l} B_{m,k}(R_{A,1}, \ldots, R_{A,m-k+1}) \times B_{n,l}(R_{B,1}, \ldots, R_{B,n-l+1}), \]
\[ Q_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} P_{k,l} B_{m,k}(\lambda_{A,1}, \ldots, \lambda_{A,m-k+1}) \times B_{n,l}(\lambda_{B,1}, \ldots, \lambda_{B,n-l+1}), \]

where an obvious generalization of the notation of Sec. [IV] has been used, and the prime in the summation symbol indicates that the term \( k = l = 0 \) is avoided. Generalization of the above formulas to the case with more particle types is immediate, with
\[ P^{N_{A},N_{B},N_{C}, \ldots}(t_{A}, t_{B}, t_{C}, \ldots) = Q^{S_{A},S_{B},S_{C}, \ldots}[R^{n_{A}}(t_{A}), R^{n_{B}}(t_{B}), R^{n_{C}}(t_{C}), \ldots], \]
and similarly for the inverse relation and the moments.

VII. MORE KINDS OF PARTICLES

In this section we consider the case where we have one type of sources, but the source can produce particles of two distinguishable types. In this context, the number of fluctuating sources has frequently been referred to as volume fluctuations. In measuring a fluctuating quantity, we obviously want to be insensitive to these spurious fluctuations, and separate them from a true correlation mechanism.

Let the multiplicities of the two types of produced particles be
\[ N_{a} = \sum_{j=1}^{S} n_{a,j}, \quad N_{b} = \sum_{j=1}^{S} n_{b,j}, \]

where \( a \) and \( b \) label the particle types. The particles emitted from the same source are, in general, correlated, but as before, particles emitted from different sources are uncorrelated. Then, repeating the derivation of the previous sections, we readily find
\[ K^{N_{a},N_{b}}(t_{a}, t_{b}) = K^{S}[K^{n_{a},n_{b}}(t_{a}, t_{b})], \]
and, for the more general composition of the generating functions, in analogy to Table [I]
\[ P^{N_{a},N_{b}}(t_{a}, t_{b}) = Q^{S}[R^{n_{a},n_{b}}(t_{a}, t_{b})]. \]

In the present case the interesting inverse problem concerns the production mechanism from the sources,
\[ R^{n_{a},n_{b}}(t_{a}, t_{b}) = (Q^{S})^{-1}[P^{N_{a},N_{b}}(t_{a}, t_{b})]. \]

Interestingly, the problems stated in this Section, and in particular Eq. [44], are related to the construction of the strongly intensive measures [13] [15] [52] of the event-by-event fluctuations, i.e., combinations of moments of \( n_{a} \) and \( n_{b} \) expressed via moments of \( N_{a} \) and \( N_{b} \) in a form-invariant way where the moments of the sources \( S \) do not appear. In App. [B] we show how to derive the following combinations of moments: the obvious one
\[ \frac{P_{01}}{P_{10}} = \frac{R_{01}}{R_{10}}, \]
two rank-2 relations
\[ Q_{1} \left( \frac{P_{20}}{P_{10}^{2}} - \frac{P_{02}}{P_{01}^{2}} \right) = \frac{R_{20}}{R_{10}^{2}} - \frac{R_{02}}{R_{01}^{2}}, \]
\[ Q_{1} \left( \frac{P_{30}}{P_{10}^{2}} - 2 \frac{P_{11}}{P_{01}} + \frac{P_{03}}{P_{01}^{2}} \right) = \frac{R_{30}}{R_{10}^{2}} - 2 \frac{R_{11}}{R_{10} R_{01}} + \frac{R_{02}}{R_{01}^{2}}, \]
and one rank-3 relation
\[ \frac{P_{30}}{P_{10}^{3}} - \frac{3 P_{21}}{P_{10}^{2} P_{01}} + \frac{3 P_{12}}{P_{10} P_{01}^{2}} - \frac{P_{03}}{P_{01}^{3}} = \frac{R_{30}}{R_{10}^{3}} - \frac{3 R_{21}}{R_{10}^{2} R_{01}} + \frac{3 R_{12}}{R_{10} R_{01}^{2}} - \frac{R_{02}}{R_{01}^{3}}. \]

A non-trivial feature of the above formulas, crucial for the application, is that the structural form of the \( R \) and \( P \) moments appearing on both sides of the equalities is exactly the same.

One can use Eq. [B1] to rewrite Eq. [47,48] in an alternative form
\[ \frac{1}{Q_{1}} \left( \frac{P_{01} P_{20} - P_{10} P_{02}}{P_{10}^{2}} \right) = \frac{R_{01} R_{20} - R_{10} R_{02}}{R_{10}^{2}}, \]
\[ \frac{1}{Q_{1}} \left( \frac{P_{01} P_{20} - 2 P_{11} + P_{10} P_{02}}{P_{10}^{2}} \right) = \frac{R_{01} R_{20} - 2 R_{11} + R_{02}}{R_{10}^{2}}, \]
and the rank-3 equation as
\[ \frac{P_{01} P_{30}}{P_{10}^{3}} - \frac{3 P_{21}}{P_{10}^{2} P_{01}} + \frac{3 P_{12}}{P_{10} P_{01}^{2}} - \frac{P_{03}}{P_{01}^{3}} = \frac{R_{01} R_{30}}{R_{10}^{3}} - \frac{3 R_{21}}{R_{10}^{2} R_{01}} + \frac{3 R_{12}}{R_{10} R_{01}^{2}} - \frac{R_{02}}{R_{01}^{3}}. \]

In Eq. [49] we readily recognize the \( \Sigma \) and \( \Delta \) measures introduced in [14]. The appearance of \( Q_{1} \) may be canceled by a multiplication or Eqs. [49] side-by-side with the equation
\[ Q_{1}/(P_{10} \pm P_{01}) = 1/(R_{10} \pm R_{01}), \]
or
\[ Q_{1}/P_{10} P_{01} = 1/\sqrt{R_{10} R_{01}}, \]
or, in general, an equation of this form involving \( Q_{1} \) and any intensive quantities for the \( P \) and \( R \) moments [15].
We note that a form analogous to Eq. [50] was proposed a long time ago in [53] as a rank-3 generalization of the $\Phi_{PE}$ measure used for the rank-2 moments [13].

Finally, we note that Eqs. (47,48) can be stated in a more compact form for the case of cumulant moments of the scaled numbers of particles, $N_i = N_i/\langle N_i \rangle$, and the scaled moments of particles produced from a source, $\hat{n}_i = n_i/\langle n_i \rangle$:

$$Q_1 \left[ \kappa_2(\hat{N}_a) - \kappa_2(\hat{N}_b) \right] = \kappa_2(\hat{n}_a) - \kappa_2(\hat{n}_b),$$

$$Q_1^2 \kappa_2(\hat{N}_a - \hat{N}_b) = \kappa_2(\hat{n}_a - \hat{n}_b),$$

$$Q_1^3 \kappa_3(\hat{N}_a - \hat{N}_b) = \kappa_3(\hat{n}_a - \hat{n}_b),$$

which can be verified explicitly. Higher-rank relations of this type are obtained in Sec. [VIII].

### VIII. CUMULANTS FOR DIFFERENCES OF PARTICLE SPECIES AND NEW STRONGLY INTENSIVE MEASURES

As suggested by the simplicity of Eq. [53], we now consider in a greater detail the scaled moments of the difference of particles of type $a$ and $b$ produced from a single type of sources,

$$\hat{N}_- = \hat{N}_a - \hat{N}_b, \quad \hat{n}_- = \hat{n}_a - \hat{n}_b.$$  

The cumulant generating function satisfies the composition law

$$e^{K_{S,-}(t)} = \mathbb{E} e^{\langle \hat{n}_a \rangle} \sum_{i=1} S \hat{n}_a - t \langle \hat{n}_a \rangle \sum_{i=1} S \hat{n}_b$$

$$= \mathbb{E} e^{\langle \hat{n}_a \rangle} \sum_{i=1} S \hat{n}_a - \hat{n}_b = e^{K_{S,-}(t/\langle S \rangle)},$$

where we have used the fact that $\langle N_i \rangle = \langle S \rangle \langle n_i \rangle$. In our generic notation

$$P^\hat{N}_-(t) = Q^S \left[ R^{\hat{N}_-}(t/Q_1) \right],$$

thus we recover the structure of the composition law for the univariate case of Sec. [III].

Since the variable $t$ is rescaled, in the formulas given this section we have $R$ corresponding to the cumulant moments, whereas $P$ and $Q$ relate to the cumulant or standard moments (cf. rows 2 and 3 of Table I). One may then always pass to the desired type of moments according to the scheme of Fig. I.

The use of scaled variables leads to simplification, as from construction, for the difference of the scaled moments we have

$$R_1 \equiv \left\langle \frac{N_a}{\langle N_a \rangle} - \frac{N_b}{\langle N_b \rangle} \right\rangle = 0.$$  

In consequence, from Eqs. (A1) we obtain the following hierarchy of equations

$$P_1 = 0,$$

$$Q_1 P_2 = R_2,$$

$$Q_1^2 P_3 = R_3,$$

$$Q_1^3 P_4 = 3Q_1 R_2^2 + R_4,$$

$$Q_1^4 P_5 = 10Q_1 R_2 R_3 + R_5,$$

$$Q_1^5 P_6 = 15Q_1 R_2^2 + Q_2 (10R_2^2 + 15R_2 R_4) + R_6,$$

$$Q_1^6 P_7 = 105Q_1 R_2^3 R_3 + Q_2 (35R_3 R_4 + 21R_2 R_5) + R_7,$$

$$\ldots,$$

where for brevity we also have introduced the scaled moments of the sources

$$\hat{Q}_n = Q_n/Q_1, \quad n = 2, 3, \ldots$$

We notice that the second and third equality in (58) are the same as the corresponding equalities in Eq. [53]: the second one is the $\Delta$ measure [14], and the third one is its generalization to rank 3, analogous to the relation derived in [53] for $\Phi_{PE}$.

We now pass to deriving new strongly intensive fluctuation measures. Eliminating $\hat{Q}_2$ form the fourth and fifth equations, which is possible when $R_3 \neq 0$, we arrive at the relation

$$Q_1 \left[ \frac{P_4}{3P_2^2} - \frac{P_3}{10P_2 P_3} \right] = \frac{R_4}{3R_2^2} - \frac{R_5}{10R_2 R_3},$$

involving moments up to rank 5. The next order relation comes via elimination of $Q_3$ from the sixth and seventh equalities in Eqs. (58), and then eliminating $Q_2$ using the fourth equality. This requires $Q_5 \neq 0$ and $Q_3 \neq 0$. The result is

$$Q_1 \left[ \frac{7P_3 P_6 - P_2 P_7}{70P_3^2 + 70P_2 P_4 P_3 - 21P_2^2 P_5} - (1-a) \frac{P_4}{3P_2^2} - a \frac{P_5}{10P_2 P_3} \right] = \frac{7R_3 R_6 - R_4 R_7}{70R_3^2 + 70R_2 R_4 R_3 - 21R_2^2 R_5} - (1-a) \frac{R_4}{3R_2^2} - a \frac{R_5}{10R_2 R_3},$$

where $a$ is any real parameter; the form is not unique, as we may use Eq. (60) to alter the coefficients in front of the terms involving $P_4$ and $P_5$, or $R_4$ and $R_5$, respectively.

The procedure may possibly be continued to yet higher orders, but it becomes tedious (see the discussion in App. B). We should also bear in mind that potential practical significance of the formulas decreases with the degree of complication and the increasing rank, as higher order moments are subject to larger experimental uncertainties. Present analyses use moments up to rank 4, which is sufficient to apply the second and third equality in Eq. (58). A usage of Eqs. (60,61) would require going up to rank 5 and 7, respectively, for which a very large data statistics would be necessary.

Division of Eq. (60,61) with Eqs. (51,52), or any equation dependent on $Q_1$ in a similar way, removes the dependence on $Q_1$, in full analogy to the construction of the $\Sigma$ and $\Delta$ [14] measures.
As mentioned in the Introduction, important physical applications of moments of differences of particles produced in ultra-relativistic heavy-ion collisions, or the corresponding strongly intensive measures, are linked to the quest of the QCD phase transition at a finite baryon density.

IX. CONCLUSION

We have reviewed the formalism of generating functions in the application to superposition models used in production of particles in hadronic or nuclear high-energy collisions. We have indicated that simple composition laws hold for appropriate types of functions (for the standard, cumulant, factorial, and factorial cumulant moments), which allows us for a simple derivation of the composition laws for the moments themselves to any order. We have recalled the exact transformations between various types of moments, as well as provided the inverse transformation, i.e., obtaining the moments of sources from the moments of particles. We have drawn attention to the fact that the composition laws hold for numerous combinations of types of moments, as summarized in Table I.

We have then considered the following simple cases: 1) two kinds of sources and a single particle type (e.g., wounded nucleon/quark model, or correlations between multiplicities in different kinematic bins), and 2) one type of sources and two kinds of particles (or two kinds of characteristics, such as multiplicity, charge, or transverse momentum). This case arises, e.g., when one considers the net baryon number or charge (the alleged probes of the QCD phase transition).

A generalization of the master composition formula for the generating functions for more types of sources and particles is straightforward

\[ P_{n_a}^{N_a, n_b} \cdots (t_a, t_b, \ldots) = \] (62)

\[ Q_{S_A}^{S_B, \cdots} R_{n_a}^{n_b} \cdots (t_a, t_b, \ldots), \] where \( n_j \) is the distribution of particles of type \( j \) produced from the source of type \( I \). The corresponding formulas for the composition of moments can be obtained via the Maclaurin expansion of Eq. (62).

Moreover, the quantities used in our statistical study need not be multiplicities themselves, as used throughout the paper for the simplicity of notation, but any additive one-body observable, for instance charge or the transverse momentum. We note that the correlations of the transverse momenta and multiplicities are actively pursued experimentally (see, e.g., [23]) with the use of the strongly-intensive measures.

We have used the framework to consider the scaled moments of the difference of multiplicities of two kinds of particles and found a straightforward derivation and a simple algebraic interpretation of the strongly intensive fluctuation measures. With this method we have derived new relations of that type, relating moments of higher rank. Hopefully, these relations can be applied to high statistics data samples, thus will become useful in analyses of the event-by-event fluctuations.

The generalization of the superposition framework to more types of sources, as in Eq. (62), allows for incorporation of global conservation laws. The conserved quantity (e.g., charge) may be distributed over the sources of different type (labeled with the carried charge). As the total charge of all sources is constrained, the distribution of sources will reflect this, which will show up in the \( Q \) moments. The production from a source should also be conserving, which will affect in the \( R \) moments. Nevertheless, the generic structure of Eq. (62) remains valid. In the case where there are more particle types than source types, one would get an over-determined system of equations in analogy to Eqs. (51), and relations between the particle moments \( P \) and source moments \( R \), without a reference to \( Q \) moments, may be obtained along the lines of App. B.

ACKNOWLEDGMENTS

WB is grateful to Stanisław Mrówczyński for helpful discussions concerning the strongly intensive measures. This research was supported by the Polish National Science Centre grant 2015/19/B/ST2/00937.

Appendix A: Explicit formulas

In this Appendix we give a glossary of formulas following from the composition laws discussed in the body of the paper for the case of a single type of sources and a single type of the produced particles. These formulas can be useful in analyses in the superposition approach. For the first few values of \( m \) one finds directly from Eq. (22):

\[ P_1 = Q_1 R_1, \] (A1)
\[ P_2 = Q_2 R_1^2 + Q_1 R_2, \]
\[ P_3 = Q_3 R_1^3 + 3Q_2 R_1 R_2 + Q_1 R_3, \]
\[ P_4 = Q_4 R_1^4 + 6Q_3 R_1 R_2 R_1 + Q_2 (3R_2^2 + 4R_1 R_3) + Q_1 R_4, \]
\[ P_5 = Q_5 R_1^5 + 10Q_4 R_2 R_1^3 + Q_3 (10R_3 R_1^3 + 15R_2 R_1 R_4) + Q_2 (10R_2 R_3 + 5R_1 R_4) + Q_1 R_5, \]
\[ P_6 = Q_6 R_1^6 + 15Q_5 R_2 R_1^2 + Q_4 (20R_3 R_1^3 + 45R_2^2 R_1^2) + Q_3 (15R_2^3 + 60R_1 R_3 R_4 + 45R_4^2 R_1) + Q_2 (10R_3^2 + 15R_2 R_4 + 6R_1 R_5) + Q_1 R_6, \]
\[ \ldots \]

For the case of the inverse problem [23], the first few
terms are

\[
R_1 Q_1 = P_1, \\
R_1^2 Q_2 = 2P_2 R_1 - P_1 R_2, \\
R_1^3 Q_3 = 6P_3 R_1^2 - 6P_2 R_2 R_1 + P_1 (3R_2^2 - R_1 R_3), \\
R_1^4 Q_4 = 24P_4 R_1^3 - 36P_3 R_2 R_1^2 + P_2 (30R_1 R_2^2 - 8R_2^2 R_3) + P_1 (-15R_2^3 + 10R_1 R_3 R_2 - R_1^2 R_4), \\
\ldots
\]

**Appendix B: Derivation of intensive measures**

In this Appendix we list the explicit formulas for the case of a single type of sources and two kinds of the produced particles. From the form of Eqs. (44) one can, via the Maclaurin expansion, obtain the following hierarchy of equations:

\[
P_{01} = Q_1 R_{01}, \\
P_{10} = Q_1 R_{10}, \\
P_{20} = Q_2 R_{10}^2 + Q_1 R_{20}, \\
P_{11} = Q_2 R_{10} R_{01} + Q_1 R_{11}, \\
P_{02} = Q_2 R_{01}^2 + Q_1 R_{02},
\]

\[
P_{30} = Q_3 R_{10}^3 + 3Q_2 R_{10} R_{01} R_{20} + Q_1 R_{30}, \\
P_{21} = Q_3 R_{10} R_{01} + Q_2 (2R_{10} R_{11} + R_{20} R_{01}) + Q_1 R_{21}, \\
P_{12} = Q_3 R_{01} R_{10} + Q_2 (2R_{01} R_{11} + R_{02} R_{10}) + Q_1 R_{12}, \\
P_{03} = Q_3 R_{01}^3 + 3Q_2 R_{01} R_{02} + Q_1 R_{03},
\]

\[
P_{40} = Q_4 R_{10}^4 + 6Q_3 R_{20} R_{10}^2 + Q_2 (3R_{20}^2 + 4R_{10} R_{30}) + Q_1 R_{40}, \\
P_{31} = Q_4 R_{01} R_{10}^3 + Q_3 (3R_{11} R_{10}^2 + 3R_{01} R_{20} R_{10}) + Q_2 (3R_{11} R_{20} + 3R_{02} R_{21} + R_{01} R_{30}) + Q_1 R_{31}, \\
P_{22} = Q_4 R_{01} R_{01}^2 + Q_3 (R_{20} R_{11} + 4R_{10} R_{11} R_{01} + R_{02} R_{10}^2) + Q_2 (2R_{11}^2 + 2R_{10} R_{12} + R_{02} R_{20} + 2R_{01} R_{21}) + Q_1 R_{22}, \\
P_{13} = Q_4 R_{10} R_{01}^3 + Q_3 (3R_{11} R_{01}^2 + 3R_{02} R_{10} R_{01}) + Q_2 (R_{03} R_{10} + 3R_{02} R_{11} + 3R_{01} R_{12}) + Q_1 R_{13}, \\
P_{04} = Q_4 R_{01}^4 + 6Q_3 R_{02} R_{01}^2 + Q_2 (3R_{02}^2 + 4R_{01} R_{03}) + Q_1 R_{04}.
\]

etc. Algebraically, the above equations can be viewed as an over-determined set of equations for the variables \(Q_i\), thus one can find conditions for existence of a solution. From the Rouche–Capelli theorem it is clear, that Eq. \((B2)\) leads to 2 conditions between the \(P\) and \(R\) moments, as well as \(Q_1\), Eq. \((B3)\) leads to 3 additional conditions, and so on. However, we are seeking the conditions that can be written with the same structural forms for the \(P\) moments as for the \(R\) moments, for instance as in Eqs. \((17, 18)\). The problem is tedious for higher rank cases and we were not able to settle it down in general.

The lowest-rank relations are, however, straightforward to obtain. Eliminating \(Q_2\), \(Q_3\), \ldots, from Eq. \((47, 48)\) one finds the desired combinations. For instance, eliminating \(Q_2\) form the first and third Eq. \((B2)\) one arrives at the first formula in Eq. \((17)\).

We note that the derivation of the form of the strongly intensive measures is simpler along the lines of Sec. \(\text{VIII}\) where we use the generating function for the difference of scaled numbers of particles. However, in that case the problem of arriving at the same structural forms of the \(P\) and \(R\) moment combinations also becomes algebraically complicated at higher rank.

---

[1] R. A. Fisher, [Proceedings of the London Mathematical Society] s2-30, 199 (1930), ISSN 1460-244X, http://dx.doi.org/10.1112/plms/s2-30.1.199
[2] J. Riordan, *An Introduction to Combinatorial Analysis* (Wiley, New York, 1958)
[3] A. Bialas and R. B. Peschanski, *Nucl. Phys.* B273, 703 (1986)
[4] I. M. Dremin, *Phys. Lett.* B313, 209 (1993)
[5] N. Suzuki, M. Białyjima, G. Wilk, and Z. Włodarczyk, *Phys. Rev. C* 58, 1720 (Sep 1998), http://link.aps.org/doi/10.1103/PhysRevC.58.1720M. Rybczyński, G. Wilk, Z. Włodarczyk, M. Białyjima, and N. Suzuki(1999), arXiv:hep-ph/9909380 [hep-ph]
[6] N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, *Phys. Rev. C* 63, 054906 (2001)
[7] A. Bilandzic, R. Snellings, and S. Voloshin, *Phys. Rev.* C83, 044913 (2011)
[8] P. Di Francesco, M. Guilbaud, M. Luzum, and J.-Y. Ollitrault(2016), arXiv:1612.05634 [nucl-th]
[9] J.-h. Fu and L.-s. Liu, *Phys. Rev.* C68, 064904 (2003)
[10] A. Bialas, Acta Phys. Polon. B35, 683 (2004)
[11] B. Ling and M. A. Stephanov, *Phys. Rev.* C93, 034915 (2016)
[12] T. Nonaka, M. Kitazawa, and S. Esumi(2017), arXiv:1702.07106 [physics.data-an]
[13] M. Gaździcki and S. Mrówczyński, *Z. Phys.* C54, 127 (1992)
[14] M. I. Gorenstein and M. Gaździcki, *Phys. Rev.* C84, 014904 (2011)
[15] M. Gaździcki, M. I. Gorenstein, and M. Mackowiak-Pawłowska, *Phys. Rev.* C88, 024907 (2013)
[16] M. A. Bean, *Probability: The Science of Uncertainty:*
with Applications to Investments, Insurance, and Engineering (American Mathematical Society, 2001) ISBN 0821847929

[17] A. Olszewski and W. Broniowski, Phys. Rev. C88, 044913 (2013)

[18] A. Olszewski and W. Broniowski, Phys. Rev. C92, 024913 (2015)

[19] J. Whitmore, Phys. Rept. 27, 187 (1976)

[20] L. Foa, Phys. Rept. 22, 1 (1975)

[21] M. A. Stephanov, K. Rajagopal, and E. V. Shuryak, Phys. Rev. Lett. 81, 4816 (1998)

[22] M. A. Stephanov, K. Rajagopal, and E. V. Shuryak, Phys. Rev. D60, 114028 (1999)

[23] M. A. Stephanov, Phys. Rev. Lett. 102, 032301 (2009)

[24] M. Cheng et al., Phys. Rev. D79, 074505 (2009)

[25] S. Gupta, Proceedings, 5th International Workshop on Critical point and onset of deconfinement (CPOD 2009): Upton, USA, June 8-12, 2009, PoS CPOD2009, 025 (2009)

[26] C. Athanasiou, K. Rajagopal, and M. Stephanov, Phys. Rev. D82, 074008 (2010)

[27] S. Gupta, X. Luo, B. Mohanty, H. G. Ritter, and N. Xu, Science 332, 1525 (2011)

[28] V. Begun, Phys. Rev. C94, 054904 (2016)

[29] V. Begun(2016), arXiv:1606.05358 [nucl-th]

[30] P. Braun-Munzinger, A. Kalweit, K. Redlich, and J. Stachel, Proceedings, 25th International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions (Quark Matter 2015): Kobe, Japan, September 27-October 3, 2015, Nucl. Phys. A956, 805 (2016)

[31] P. Braun-Munzinger, A. Rustamov, and J. Stachel, Nucl. Phys. A960, 114 (2017)

[32] M. Gaździcki and P. Seybold, Acta Phys. Polon. B47, 1201 (2016)

[33] V. Koch, in Chapter of the book

[34] M. M. Aggarwal et al. (STAR), Phys. Rev. Lett. 105, 022302 (2010)

[35] L. Adamczyk et al. (STAR), Phys. Rev. Lett. 113, 092301 (2014)

[36] A. Adare et al. (PHENIX), Phys. Rev. C93, 011901 (2016)

[37] A. Bzdak and V. Koch, Phys. Rev. C86, 044904 (2012)

[38] A. Bzdak, V. Koch, and V. Skokov, Phys. Rev. C87, 014901 (2013)

[39] A. Bzdak, R. Holzmann, and V. Koch, Phys. Rev. C94, 064907 (2016)

[40] G. Almasi, B. Friman, and K. Redlich(2017), arXiv:1703.05947 [hep-ph]

[41] F. Karsch, Proceedings, 16th International Conference on Strangeness in Quark Matter (SQM 2016): Berkeley, California, United States, J. Phys. Conf. Ser. 779, 012015 (2017)

[42] J. Xu, S. Yu, F. Liu, and X. Luo, Phys. Rev. C94, 024901 (2016)

[43] S. He and X. Luo(2017), arXiv:1704.00423 [nucl-ex]

[44] Wikipedia, “Bell polynomials — Wikipedia, the free encyclopedia,” (2017), [Online; accessed 15-March-2017], https://en.wikipedia.org/wiki/Bell_polynomials

[45] S. Jeon and V. Koch(2003), arXiv:hep-ph/0304012 [hep-ph]

[46] S. Mrówczynski, 4th Workshop on Particle Correlations and Femtoscopy (WPCF 2008) Crakow, Poland, September 11-14, 2008, Acta Phys. Polon. B40, 1053 (2009)

[47] J. Zaranek, Phys. Rev. C66, 024905 (2002)

[48] S. Mrówczynski, Phys. Rev. C66, 024904 (2002)

[49] Wikipedia, “Bell polynomials — Wikipedia, the free encyclopedia,” (2017), [Online; accessed 15-March-2017], https://en.wikipedia.org/wiki/Bell_polynomials

[50] A. Białoś, M. Bleszyński, and W. Czyż, Nucl. Phys. B111, 461 (1976)A. Białoś, J. Phys. G35, 044053 (2008)

[51] A. Białoś, W. Czyż, and W. Furmański, Acta Phys. Polon. B8, 585 (1977)V. V. Anisovich, Yu. M. Shabelski, and V. M. Shekhter, Nucl. Phys. B133, 477 (1978)

[52] E. Sangaline(2015), arXiv:1505.00261 [nucl-th]

[53] S. Mrówczyński, Phys. Lett. B465, 8 (1999)

[54] A. Aduszkiewicz et al. (NA61/SHINE), Eur. Phys. J. C76, 635 (2016)