Parasupersymmetry and $\mathcal{N}$-fold Supersymmetry in Quantum Many-Body Systems II. Third Order

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Abstract

Based on the general formalism of parafermionic algebra and parasupersymmetry proposed previously by us, we explicitly construct third-order parafermionic algebra and multiplication law, and then realize third-order parasupersymmetric quantum systems. We find some novel features in the third-order, namely, the emergence of a fermionic degree of freedom and of a generalized parastatistics. We show that for one-body cases the generalized Rubakov–Spiridonov model can be constructed also in our framework and find that it admits a generalized 3-fold superalgebra. We also find that a three-body system can have third-order parasupersymmetry where three independent supersymmetries are folded. In both cases, we also investigate the new concept of quasi-parasupersymmetry introduced by us and find that those of order $(3, 3)$ are indeed realized under less restrictive conditions than (ordinary) parasupersymmetric cases.

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I. INTRODUCTION

In our previous article [1], we proposed the general formalism of parafermionic algebra and parasupersymmetric quantum systems (for an extensive bibliography, see the references cited therein) without recourse to any specific matrix representation and any kind of deformed oscillator algebra so that we can investigate and discuss general aspects of them. Within the formalism, we showed generically that every parasupersymmetric system of order $p$ consists of $N$-fold supersymmetric pairs of component Hamiltonians with $N \leq q$ and thus they have isospectral property and weak quasi-solvability. The implication of the result in view of higher-dimensional quantum theories is that parasupersymmetric quantum field theory (cf. Refs. [2, 3]) as well as weak supersymmetric one [4], if exists as a consistent theory, would satisfy some kinds of perturbative non-renormalization theorems, as in the case of ordinary supersymmetric quantum field theory, since quasi-solvability is a one-dimensional analog and, in a sense, a generalization of them [5]. To illustrate how the formalism works, in the article [1] we constructed explicitly second-order parafermionic algebra based on solely the postulates of the formalism and some second-order parasupersymmetric quantum systems.

In this article, we proceed with the study of parasupersymmetry based on the previous formalism in Ref. [1] and focus on, in particular, third-order case. First of all, we would like to check whether the postulates proposed in Ref. [1] work consistently not only in the second-order case but also in higher-order cases. Second, we would like to observe what kind of novel features can arise when the parafermionic order increases. In particular, the new concept of quasi-parasupersymmetry did not produce any new results in the second-order case, and thus it is interesting to see the situation in higher-order cases. Third, the direct construction of parafermionic algebra based on the postulates must become more cumbersome as the parafermionic order increases. Hence, we would like to acquire more general features in lower-order cases so that we would be able to establish an inductive construction of parafermionic algebra and multiplication law for arbitrary order. In fact, it would be almost impossible to guess the form of them only by the knowledge of the second-order case.

Motivated by the above purposes, in this article we first construct third-order parafermionic algebra and multiplication law by following a similar way of the derivation of the second-order ones in Ref. [1]. We confirm that the postulates of the formalism work consistently in the third-order case too, and find some novel features in the third-order parafermionic algebra. We then construct some quantum systems with third-order parasupersymmetry and quasi-parasupersymmetry. For one-body systems, we find that the generalized Rubakov–Spiridonov model [6, 7, 8] of order 3 is realized also in our framework and further show that it admits a generalized 3-fold superalgebra. Furthermore, we find that a three-body system also admits third-order parasupersymmetry where three independent supersymmetries are folded. In both cases, we also investigate quasi-parasupersymmetry and find that order $(3, 2)$ cases are identical with (ordinary) parasupersymmetric ones while order $(3, 3)$ cases are indeed less restrictive than them due to the lack of a constraint on one of the component Hamiltonians.

We organize the article as follows. In the next section, we review the general formalism of parafermionic algebra and (quasi-)parasupersymmetry in quantum mechanical systems developed in Ref. [1]. In Section III we construct third-order parafermionic algebra solely based on the postulates in the formalism and derive the multiplication law. There some novel features of the third-order case are pointed out. In Section IV using the parafermionic
algebra and multiplication law of order 3, we investigate and construct two different para-supersymmetric quantum systems of order 3. The one consists of one-body Hamiltonians and is identical to the model in Refs. [7, 8]. We also show that this system admits a generalized 3-fold superalgebra in an analogous way the second-order Rubakov–Spiridonov model admits a generalized 2-fold superalgebra shown in Ref. [1]. The other third-order model consists of three-body Hamiltonians and has three independent supersymmetric structures. Furthermore, we investigate quasi-parasupersymmetry in each of the model. Finally, we summarize and discuss the results in Section V.

II. REVIEW OF (QUASI-)PARASUPERSYMMETRY

First of all, let us define parafermionic algebra of order \( p \) \((\in \mathbb{N})\). It is an associative algebra composed of the identity operator \( I \) and two parafermionic operators \( \psi^- \) and \( \psi^+ \) of order \( p \) which satisfy the nilpotency:

\[
(\psi^-)^p \neq 0, \quad (\psi^+)^p \neq 0, \quad (\psi^-)^{p+1} = (\psi^+)^{p+1} = 0.
\] (2.1)

Hence, we immediately have \( 2p + 1 \) non-zero elements \( \{I, \psi^-, \ldots, (\psi^-)^p, \psi^+, \ldots, (\psi^+)^p\} \). We call them the fundamental elements of parafermionic algebra of order \( p \). Parafermionic algebra is characterized by anti-commutation relation \( \{A, B\} = AB + BA \) and commutation relation \([A, B] = AB - BA\) among the fundamental elements. As a postulate we impose the following relation for arbitrary order \( p \):

\[
\{\psi^-, \psi^+\} + \{(\psi^-)^2, (\psi^+)^2\} + \cdots + \{(\psi^-)^p, (\psi^+)^p\} = pI.
\] (2.2)

We shall next define parafermionic Fock spaces \( V_p \) of order \( p \) on which the parafermionic operators act. The latter space is \((p + 1)\) dimensional and its \( p + 1 \) bases \( |k\rangle \) \((k = 0, \ldots, p)\) are defined by

\[
\psi^-|0\rangle = 0, \quad |k\rangle = (\psi^+)^k|0\rangle, \quad \psi^-|k\rangle = |k - 1\rangle \quad (k = 1, \ldots, p).
\] (2.3)

That is, \( \psi^- \) and \( \psi^+ \) act as annihilation and creation operators of parafermions, respectively. The state \( |0\rangle \) is called the parafermionic vacuum. The subspace spanned by each state \( |k\rangle \) \((k = 0, \ldots, p)\) is called the \( k \)-parafermionic subspace and is denoted by \( V_p^{(k)} \). The adjoint vector \( \langle k| \) of each \( |k\rangle \) is introduced as a linear operator which maps every vector in \( V_p \) into a complex number as follows:

\[
\langle k|l\rangle = \langle 0|(\psi^-)^k|l\rangle, \quad \langle 0|l\rangle = \langle 0|(\psi^+)^l|0\rangle = \delta_{0,l} \quad (k, l = 0, \ldots, p).
\] (2.4)

By the definitions (2.3) and (2.4), we immediately have a bi-orthogonal relation:

\[
\langle k|l\rangle = \delta_{k,l} \quad (k, l = 0, \ldots, p).
\] (2.5)

We can now define a set of projection operators \( \Pi_k : V_p \rightarrow V_p^{(k)} \) \((k = 0, \ldots, p)\) which satisfy

\[
\Pi_k|l\rangle = \delta_{k,l}|k\rangle, \quad \Pi_k\Pi_l = \delta_{k,l}\Pi_k, \quad \sum_{k=0}^{p} \Pi_k = I.
\] (2.6)
From the definitions (2.3) and (2.6), we obtain

\[ \Pi_{k+1} \psi^+ = \psi^+ \Pi_k, \quad \psi^- \Pi_{k+1} = \Pi_k \psi^-, \]  

(2.7)

where and hereafter we put \( \Pi_k \equiv 0 \) for all \( k < 0 \) and \( k > p \).

We now come back to the parafermionic algebra. Apparently, the relations (2.1) and (2.2) are not sufficient for the determination of the full algebra. To determine other multiplication relations we impose the following postulates:

1. First, the algebra must be consistent with Eq. (2.3). This requirement is indispensable for defining consistently the parafermionic Fock space \( V_p \).

2. Every projection operator \( \Pi_k \) \((k = 0, \ldots, p)\) can be expressed as a polynomial of the fundamental elements of the corresponding order \( p \) so that the algebra is consistent with the definition (2.6).

3. Every product of three fundamental elements can be expressed as a polynomial of at most second-degree in the fundamental elements. These formulas are called the multiplication law. In particular, we assume that the following relations

\[ \psi^- \psi^+ \psi^- = \psi^-, \quad \psi^+ \psi^- \psi^+ = \psi^+. \]  

(2.8)

hold for parafermionic operators of any order \( p \). As a consequence of this assumption, we immediately obtain for all \( m, n \in \mathbb{N} \)

\[ (\psi^-)^m \psi^+ (\psi^-)^n = (\psi^-)^{m+n-1}, \quad (\psi^+)^m \psi^- (\psi^+)^n = (\psi^+)^{m+n-1}, \]  

(2.9)

which also hold for arbitrary order.

4. We also assume that the following relations hold for arbitrary order:

\[ (\psi^-)^p \psi^+ \Pi_{p-1} = (\psi^-)^{p-1} \Pi_{p-1}, \quad \psi^+ (\psi^-)^p \Pi_p = (\psi^-)^{p-1} \Pi_p. \]  

(2.10)

We note that every polynomial composed of the fundamental elements can be reduced to a polynomial of at most second-degree in them as a consequence of the third postulate and the associativity. Hence, together with the second postulate it means in particular every projection operators must be expressed as a polynomial of second-degree in the fundamental elements.

Finally, we introduce the quantity of parafermionic degree of operators as follows:

\[ \text{deg } I = 0, \quad \text{deg } \psi^+ = 1, \quad \text{deg } \psi^- = p, \]  

(2.11)

\[ \text{deg } AB \equiv \text{deg } A + \text{deg } B \pmod{p+1}. \]  

(2.12)

For example, \( \text{deg}(\psi^+)^k = k \) and \( \text{deg}(\psi^-)^k = p + 1 - k \) \((k = 1, \ldots, p)\).

Parasupersymmetry of order 2 in quantum mechanics was first introduced by Rubakov and Spiridonov [6] and was later generalized to arbitrary order independently by Tomiya [9] and by Khare [7]. A different formulation for order 2 was proposed by Beckers and Debergh [10] and a generalization of the latter to arbitrary order was attempted by Chenaghlo and Fakhri [11]. Thus, we call them RSTK and BDCF formalism, respectively. To define a
pth-order parasupersymmetric system, we first introduce a pair of parasupercharges \( Q^\pm \) of order \( p \) which satisfy

\[
(Q^-)^p \neq 0, \quad (Q^+)^p \neq 0, \quad (Q^-)^{p+1} = (Q^+)^{p+1} = 0. \tag{2.13}
\]

A system \( H \) is said to have *parasupersymmetry of order* \( p \) if it commutes with the parasupercharges of order \( p \)

\[
[Q^-, H] = [Q^+, H] = 0, \tag{2.14}
\]

and satisfies the non-linear relations in the RSTK formalism

\[
\sum_{k=0}^{p} (Q^-)^{p-k}Q^+(Q^-)^k = C_p(Q^-)^{p-1}H, \quad \sum_{k=0}^{p} (Q^+)^{p-k}Q^-(Q^+)^k = C_p(H(Q^+)^{p-1}, \tag{2.15}
\]

or in the BDCF formalism

\[
\underbrace{[Q^-, \cdots, [Q^-, [Q^+, Q^-]] \cdots]}_{(p-1) \text{ times}} = (-1)^p C_p(Q^-)^{p-1}H, \tag{2.16a}
\]

\[
\underbrace{[Q^-, \cdots, [Q^-, [Q^-, Q^+]] \cdots]}_{(p-1) \text{ times}} = C_p(H(Q^+)^{p-1}, \tag{2.16b}
\]

where \( C_p \) is a constant. An apparent drawback of the BDCF formalism is that the relations (2.16) do not reduce to the ordinary supersymmetric anti-commutation relation \( \{Q^-, Q^+\} = C_1H \) when \( p = 1 \), in contrast to the RSTK relation (2.15). For this reason, we discard the BDCF formalism in this article though its defect may be amended by, e.g., replacing all the commutators in (2.16) by anti-commutators, graded commutators \( [A, B] = AB - (-1)^{\deg A \cdot \deg B} BA \), and so on (see also the third paragraph in Section V).

An immediate consequence of the commutativity (2.14) is that each \( n \)-th-power of the parasupercharges \( (2 \leq n \leq p) \) also commutes with the system \( H \)

\[
[(Q^-)^n, H] = [(Q^+)^n, H] = 0 \quad (2 \leq n \leq p). \tag{2.17}
\]

Hence, every parasupersymmetric system \( H \) satisfying (2.14) always has \( 2p \) conserved charges.

To realize parasupersymmetry in quantum mechanical systems, we usually consider a vector space \( \mathfrak{F} \times V_p \) where \( \mathfrak{F} \) is a linear space of complex functions such as the Hilbert space \( L_2^H \) in Hermitian quantum theory and the Krein space \( L_2^P \) in \( PT \)-symmetric quantum theory [12, 13]. A parafermionic quantum system \( H \) is introduced by

\[
H = \sum_{k=0}^{p} H_k \Pi_k, \tag{2.18}
\]

where \( H_k (k = 0, \ldots, p) \) are scalar Hamiltonians of \( p \) variables acting on \( \mathfrak{F} \):

\[
H_k = -\frac{1}{2} \sum_{i=1}^{p} \frac{\partial^2}{\partial q_i^2} + V_k(q_1, \cdots, q_p) \quad (k = 0, \ldots, p). \tag{2.19}
\]
Two parasupercharges $Q^\pm$ are defined by

$$Q^- = \sum_{k=0}^{p} Q_k^- \psi^- \Pi_k, \quad Q^+ = \sum_{k=0}^{p} Q_k^+ \Pi_k \psi^+, \quad (2.20)$$

where $Q_k^- (k = 0, \ldots, p)$ are first-order linear operators acting on $\mathfrak{F}$

$$Q_k^+ = \sum_{i=1}^{p} w_{k,i}(q_1, \ldots, q_p) \frac{\partial}{\partial q_i} + W_k(q_1, \ldots, q_p) \quad (k = 0, \ldots, p), \quad (2.21)$$

and for each $k$ $Q_k^-$ is given by a certain ‘adjoint’ of $Q_k^+$, e.g., the (ordinary) adjoint $Q_k^- = (Q_k^+)^\dagger$ in the Hilbert space $L^2$, the $\mathcal{P}$-adjoint $Q_k^- = \mathcal{P}(Q_k^+)^\dagger \mathcal{P}$ in the Krein space $L^2_\mathcal{P}$, and so on. For all $k \leq 0$ and $k > p$ we put $Q_k^\pm \equiv 0$. When $p = 1$, the triple $(H, Q^-, Q^+)$ defined in Eqs. (2.18) and (2.20) becomes

$$H = H_0 \psi^- \psi^+ + H_1 \psi^+ \psi^-, \quad Q^- = Q_1^- \psi^-, \quad Q^+ = Q_1^+ \psi^+, \quad (2.22)$$

and thus reduces to an ordinary supersymmetric quantum mechanical system [14]. The non-linear relation (2.15) together with the nilpotency (2.13) for $p = 1$ are just the anti-commutation relations between supercharges

$$\{Q^\pm, Q^\pm\} = 0, \quad \{Q^-, Q^+\} = C_1 H. \quad (2.23)$$

Hence, the parasupersymmetric quantum systems defined by Eqs. (2.13)–(2.21) provide a natural generalization of ordinary supersymmetric quantum mechanics.

It is easy to check that the parasupercharges $Q^\pm$ defined by Eq. (2.20) already satisfy the nilpotency (2.13) and that the commutativity (2.14) is satisfied if and only if

$$H_{k-1} Q_k^- = Q_k^- H_k, \quad Q_k^+ H_{k-1} = H_k Q_k^+, \quad \forall k = 1, \ldots, p. \quad (2.24)$$

That is, each pair of $H_{k-1}$ and $H_k$ must satisfy the intertwining relations with respect to $Q_k^-$ and $Q_k^+$. Similarly, the commutativity (2.17) implies that any pair of $H_{k-n}$ and $H_k$ $(1 \leq n \leq k \leq p)$ satisfies

$$H_{k-n} Q_{k-n+1} \cdots Q_{k-1} Q_k^- = Q_{k-n+1} \cdots Q_{k-1} Q_k^- H_k, \quad (2.25a)$$

$$Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+ H_{k-n} = H_k Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+, \quad (2.25b)$$

which means that $H_{k-n}$ and $H_k$ constitute a pair of $\mathcal{N}$-fold supersymmetry with $\mathcal{N} = n$. The relations (2.25) can be also derived by repeated applications of Eq. (2.24). Since $\mathcal{N}$-fold supersymmetry is essentially equivalent to weak quasi-solvability [13], parasupersymmetric quantum systems also possess weak quasi-solvability. To see the structure of weak quasi-solvability in the parasupersymmetric system $H$ more precisely, let us first define

$$\mathcal{V}_{n,k}^- = \ker(Q_{k-n+1}^- \cdots Q_k^-), \quad \mathcal{V}_{n,k}^+ = \ker(Q_k^- \cdots Q_{k-n+1}^+) \quad (1 \leq n \leq k \leq p). \quad (2.26)$$

By the definition (2.22), the vector spaces $\mathcal{V}_{n,k}^\pm$ for each fixed $k$ are related as

$$\mathcal{V}_{1,k}^- \subset \mathcal{V}_{2,k}^- \subset \cdots \subset \mathcal{V}_{k,k}^-, \quad \mathcal{V}_{1,k}^+ \subset \mathcal{V}_{2,k}^+ \subset \cdots \subset \mathcal{V}_{k,k}^+. \quad (2.27)$$
On the other hand, it is evident from the intertwining relations (2.25) that each Hamiltonian \( H_k \) \((0 \leq k \leq p)\) preserves vector spaces as follows:

\[
H_k \mathcal{V}_{n,k}^- \subset \mathcal{V}_{n,k}^- \quad (1 \leq n \leq k),
\]

\[
H_k \mathcal{V}_{n,k+n}^+ \subset \mathcal{V}_{n,k+n}^+ \quad (1 \leq n \leq p - k).
\]

From Eqs. (2.27) and (2.28), the largest space preserved by each \( H_k \) \((0 \leq k \leq p)\) is given by

\[
\mathcal{V}_{k,k}^- + \mathcal{V}_{p-k,p}^+ \quad (0 \leq k \leq p).
\]

Needless to say, each Hamiltonian \( H_k \) preserves the two spaces in Eq. (2.29) separately. The intertwining relations (2.24) and (2.25) ensure that all the component Hamiltonians \( H_k \) \((k = 0, \ldots, p)\) of the system \( H \) are isospectral outside the sectors \( \mathcal{V}_{n,k}^\pm \) \((1 \leq n \leq k \leq p)\).

The spectral degeneracy of \( H_k \) in these sectors depends on the form of each component of the parasupercharges, \( Q_k^\pm \) \((k = 1, \ldots, p)\), and its structure can be very complicated even in the case of second-order, see e.g. Refs. [16, 17].

In addition to those ‘power-type’ symmetries, every parasupersymmetric quantum system \( H \) defined in Eq. (2.18) can have ‘discrete-type’ ones. The conserved charges of this type are given by

\[
Q^{\pm}_{(n)} = \left\{ (\psi^-)^n, (\psi^+)^n \right\}, \quad Q^\pm_{[n]} = \left\{ [(\psi^-)^n, (\psi^+)^n], Q^\pm \right\} \quad (n = 1, \ldots, p).
\]

It follows from Jacobi identity that they indeed commute with \( H \):

\[
\{ Q^\pm_{(n)}, H \} = \{ Q^\pm_{[n]}, H \} = 0 \quad (n = 1, \ldots, p).
\]

We note, however, that they are in general not linearly independent and we cannot determine the number of linearly independent conserved charges without the knowledge of parafermionic algebra of each order.

The non-linear relations (2.15) can be also calculated in a similar way. The first non-linear relation in Eq. (2.15) is satisfied if and only if the following two identities hold:

\[
Q_1^- \cdots Q_p^- Q_p^+ + \sum_{k=1}^{p-1} Q_1^- \cdots Q_{p-k}^- Q_{p-k}^+ Q_{p-k}^- \cdots Q_{p-1}^- = C_p Q_{1}^- \cdots Q_{p-1}^- H_{p-1},
\]

\[
\sum_{k=1}^{p-1} Q_2^- \cdots Q_{p-k+1}^- Q_{p-k+1}^+ Q_{p-k+1}^- \cdots Q_p^+ + Q_1^+ Q_1^- \cdots Q_p^- = C_p Q_2^- \cdots Q_p^- H_p.
\]

The conditions for the second non-linear relation in Eq. (2.15) are apparently given by the ‘adjoint’ of Eqs. (2.32).

With a given pair of parasupercharges \( Q^\pm \) of order \( p \) which satisfy the nilpotency (2.13), a system \( H \) is said to have quasi-parasupersymmetry of order \((p, q)\) if there exists a natural number \( q \) \((1 \leq q \leq p)\) such that \((Q^\pm)^q\) commutes with \( H \) and the non-linear constraint (2.13) is satisfied. That is, it is characterized by the following algebraic relations:

\[
(Q^-)^p \neq 0, \quad (Q^+)^p \neq 0, \quad (Q^-)^{p+1} = (Q^+)^{p+1} = 0, \quad (Q^-)^q, H = (Q^+)^q, H = 0 \quad (1 \leq q \leq p),
\]

\[
\sum_{k=0}^{p} (Q^-)^{p-k} Q^+ (Q^-)^k = C_p (Q^-)^{p-1} H, \quad \sum_{k=0}^{p} (Q^+)^{p-k} Q^- (Q^+)^k = C_p H (Q^+)^{p-1}.
\]
By definition, quasi-parasupersymmetry of order \((p, q)\) reduces to (ordinary) parasupersymmetry when \(q = 1\). Thus, it can be regarded as a generalization of parasupersymmetry. A key ingredient of this new symmetry is that the commutativity \([([Q^\pm]^n, H] = 0\) for \(n < q\) is not necessarily fulfilled in contrast to (ordinary) parasupersymmetry. As a consequence, only the less restrictive \(q\)th-order intertwining relations \((2.25)\) with \(n = q\) should be satisfied between every pair of \(H_{k-q}\) and \(H_k\) in the case of quasi-parasupersymmetry of order \((p, q)\). The ‘power-type’ conserved charges in this case are apparently given by

\[
[[Q^-, H] = [[Q^+, H] = 0 \quad (2 \leq n \leq \binom{[2]}{[q]}),
\]

where \([x]\) is the maximum integer which does not exceed \(x\), and thus the number of conserved charges is reduced to \(2^\binom{n}{q}\). It is evident that parasupersymmetry of order \(p\) always implies quasi-parasupersymmetry of order \((p, q)\) for all \(q = 1, \ldots, p\). The ‘discrete-type’ conserved charges (cf. Eq. \((2.30)\)) are similarly defined.

### III. PARAFERMIONIC ALGEBRA OF ORDER 3

In this section, we shall construct parafermionic algebra of order 3 based on the postulates in Section II. The starting point is the relations \((2.1)\) and \((2.2)\) for \(p = 3\):

\[
(\psi^-)^4 = (\psi^+)^4 = 0,
\]

\[
\{\psi^-, \psi^+\} + \{(\psi^-)^2, (\psi^+)^2\} + \{(\psi^-)^3, (\psi^+)^3\} = 3I.
\]

We note that any formula derived from Eqs. \((3.1)\) and \((3.2)\) also hold when all the indices of \(+\) and \(−\) in the formula are simultaneously interchanged since the original algebra \((3.1)\) and \((3.2)\) is invariant under the interchange of \(+\) and \(−\). First, multiplying \((3.2)\) by three \(\psi^−\)s as \((\psi^-)^3 \times \psi^−\), \((\psi^-)^2 \times (\psi^-)^2 \times \psi^-\), \(\psi^- \times (\psi^-)^2 \times (\psi^-)^2\), and \((\psi^-)^3 \times (\psi^-)^3\), and applying the assumption \((2.3)\) and the nilpotency \((3.1)\), we immediately obtain

\[
(\psi^-)^3 (\psi^+)^3 (\psi^-)^3 = (\psi^-)^3 (\psi^+)^2 (\psi^-)^2 = (\psi^-)^2 (\psi^+)^2 (\psi^-)^3 = (\psi^-)^3,
\]

\[
(\psi^-)^3 (\psi^+)^2 (\psi^-)^2 = 0,
\]

and thus their sign-interchanged relations

\[
(\psi^+)^3 (\psi^-)^3 (\psi^+)^3 = (\psi^+)^3 (\psi^-)^2 (\psi^+)^2 = (\psi^+)^2 (\psi^-)^2 (\psi^+)^3 = (\psi^+)^3,
\]

\[
(\psi^+)^3 (\psi^-)^2 (\psi^+)^2 = 0.
\]

Next, multiplying \((3.2)\) by \((\psi^+)\) and \((\psi^-)\) as \((\psi^+)^2 (\psi^-)^3 \times \psi^+\), \((\psi^+)^2 \times (\psi^-)^2 \times (\psi^-)^2\), and \((\psi^+)^2 \times (\psi^-)^3\), and applying \((2.9)\), \((3.1)\), and the formulas \((3.3)\)–\((3.4)\), we have

\[
(\psi^+)^2 (\psi^-)^3 \psi^+ + (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 + (\psi^+)^3 (\psi^-)^3 = 2(\psi^+)^2 (\psi^-)^2,
\]

\[
(\psi^+)^3 (\psi^-)^3 + (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 + (\psi^+)^2 (\psi^-)^3 (\psi^+)^3 (\psi^-)^2 = 2(\psi^+)^2 (\psi^-)^2,
\]

\[
\psi^- (\psi^+)^3 (\psi^-)^2 + (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 (\psi^-)^2 + (\psi^+)^3 (\psi^-)^3 (\psi^-)^2 = 2(\psi^+)^2 (\psi^-)^2.
\]

From the above set of equations, we obtain

\[
(\psi^+)^2 (\psi^-)^3 \psi^+ = (\psi^+)^2 (\psi^-)^3 (\psi^+)^3 (\psi^-)^2 = \psi^- (\psi^+)^3 (\psi^-)^2.
\]

\[\text{(3.6)}\]
Multiplying (3.6) by \( \psi^+ \) from left or by \( \psi^- \) from right, and applying the formulas (3.3)–(3.4), we get
\[
(\psi^+)^3(\psi^-)^3\psi^+ = (\psi^+)^3(\psi^-)^2, \\
\psi^-(\psi^+)^3(\psi^-)^3 = (\psi^+)^2(\psi^-)^3,
\]
and thus their sign-interchanged ones
\[
(\psi^-)^3(\psi^+)^3\psi^- = (\psi^-)^3(\psi^+)^2, \\
\psi^+(\psi^-)^3(\psi^+)^3 = (\psi^-)^2(\psi^+)^3.
\]

As a consequence of the formulas (3.7)–(3.8), we immediately have
\[
(\psi^+)^3(\psi^-)^3(\psi^+)^2 = (\psi^+)^3(\psi^-)^2\psi^+, \\
(\psi^-)^2(\psi^+)^3(\psi^-)^3 = \psi^- (\psi^+)^2(\psi^-)^3, \\
(\psi^-)^3(\psi^+)^3(\psi^-)^2 = (\psi^-)^3(\psi^+)^2\psi^-, \\
(\psi^+)^2(\psi^-)^3(\psi^+)^3 = \psi^+ (\psi^-)^2(\psi^+)^3.
\]

Next, multiplying Eq. (3.2) by \( (\psi^+)^3\psi^- \) from left or by \( \psi^+(\psi^-)^3 \) from right, and applying (2.9) and (3.1), we have
\[
(\psi^+)^3(\psi^-)^2\psi^+ + (\psi^+)^3(\psi^-)^3(\psi^+)^2 = 2(\psi^+)^3\psi^-,
\]
\[
\psi^- (\psi^+)^2(\psi^-)^3 + (\psi^-)^2(\psi^+)^3(\psi^-)^3 = 2\psi^+(\psi^-)^3.
\]

From Eqs. (3.9) and (3.11), we obtain
\[
(\psi^+)^3(\psi^-)^2\psi^+ = (\psi^+)^3(\psi^-)^3(\psi^+)^2 = (\psi^+)^3\psi^-,
\]
\[
\psi^- (\psi^+)^2(\psi^-)^3 = (\psi^-)^2(\psi^+)^3(\psi^-)^3 = \psi^+(\psi^-)^3,
\]
and thus their sign-interchanged relations
\[
(\psi^-)^3(\psi^+)^2\psi^- = (\psi^-)^3(\psi^+)^3(\psi^-)^2 = (\psi^-)^3\psi^+,
\]
\[
\psi^+(\psi^-)^2(\psi^+)^3 = (\psi^+)^2(\psi^-)^3(\psi^+)^3 = \psi^- (\psi^+)^3.
\]

Next, multiplying (3.2) by two \( \psi^- \)s as \( (\psi^-)^2 \times (3.2) \), \( \psi^- \times (3.2) \times \psi^- \), and \( (3.2) \times (\psi^-)^2 \), and applying (2.9), (3.1), and the formulas (3.12)–(3.13), we get
\[
\{\psi^+, (\psi^-)^3\} = (\psi^-)^2(\psi^+)^2(\psi^-)^2 = (\psi^-)^2,
\]
\[
\{\psi^-, (\psi^+)^3\} = (\psi^+)^2(\psi^-)^2(\psi^+)^2 = (\psi^+)^2.
\]

As a consequence of the formulas (3.14)–(3.15), we immediately have
\[
\psi^+(\psi^-)^3\psi^+ = \psi^+(\psi^-)^2 - (\psi^-)^2(\psi^+)^3 = (\psi^-)^2(\psi^+)^2 - (\psi^-)^3(\psi^+)^2,
\]
\[
\psi^- (\psi^+)^3\psi^- = \psi^- (\psi^+)^2 - (\psi^-)^2(\psi^+)^3 = (\psi^+)^2(\psi^-)^2 - (\psi^-)^3(\psi^+)^2,
\]
and as by products
\[
[\psi^+, (\psi^-)^3] = [(\psi^+)^2, (\psi^-)^3], \\
[\psi^-, (\psi^+)^3] = [(\psi^-)^2, (\psi^+)^3].
\]

From Eqs. (3.5) and (3.14), we have
\[
(\psi^+)^2(\psi^-)^3\psi^+ = (\psi^+)^2(\psi^-)^3(\psi^-)^2 = (\psi^+)^2(\psi^-)^2 - (\psi^-)^3(\psi^+)^3,
\]
\[
(\psi^-)^2(\psi^+)^3\psi^- = \psi^+(\psi^-)^3(\psi^+)^2 = (\psi^-)^2(\psi^+)^2 - (\psi^-)^3(\psi^+)^3.
\]
Next, we examine the combination \( (\psi^-)^2(\psi^+)^3(\psi^-)^2 \). From the associativity and the formulas \((3.13), (3.16),\) and \((3.20)\) we have on one hand

\[
(\psi^-)^2(\psi^+)^3(\psi^-)^2 = ((\psi^-)^2(\psi^+)^3) \psi^- = (\psi^+(\psi^-)^3(\psi^+)^2) \psi^- = \psi^+((\psi^-)^3(\psi^+)^2) \psi^- = (\psi^-)^2(\psi^+ - (\psi^-)^3(\psi^+)^2).
\]  

(3.21)

On the other hand, from the formulas \((3.8)\) and \((3.20)\) we have

\[
(\psi^-)^2(\psi^+)^3(\psi^-)^2 = ((\psi^-)^2(\psi^+)^3) \psi^- = ((\psi^-)^2(\psi^+)^2 - (\psi^-)^3(\psi^+)^3) \psi^- = (\psi^-)^2(\psi^+)^2 \psi^- - (\psi^-)^3(\psi^+)^2.
\]  

(3.22)

In a similar way, we further obtain for the same combination

\[
(\psi^-)^2(\psi^+)^3(\psi^-)^2 = \psi^+(\psi^-)^2 - (\psi^+)^2(\psi^-)^3 = \psi^-(\psi^+)^2(\psi^-)^2 - (\psi^+)^2(\psi^-)^3.
\]  

(3.23)

Comparing Eqs. \((3.21)-(3.23)\), we finally obtain

\[
(\psi^-)^2(\psi^+)^3(\psi^-)^2 = \psi^+(\psi^-)^2 - (\psi^+)^2(\psi^-)^3 = (\psi^-)^2(\psi^+ - (\psi^-)^3(\psi^+)^2),
\]  

(3.24)

\[
(\psi^-)^2(\psi^+)^2 \psi^- = (\psi^-)^2(\psi^+)^2 \psi^- = (\psi^-)^2(\psi^+)^2 \psi^- = \psi^+(\psi^-)^2.
\]  

(3.25)

and their sign-interchanged relations

\[
(\psi^+)^2(\psi^-)^3(\psi^+)^2 = \psi^-(\psi^+)^2 - (\psi^-)^2(\psi^+)^3 = (\psi^+)^2(\psi^-)^2 - (\psi^+)^3(\psi^-)^2,
\]  

(3.26)

\[
(\psi^+)^2(\psi^-)^2 \psi^+ = (\psi^+)^2(\psi^-)^2 \psi^+ = (\psi^+)^2(\psi^-)^2 \psi^+ = \psi^+(\psi^-)^2.
\]  

(3.27)

For the combination \((\psi^-)^2(\psi^+)^3 \psi^+\), we have with the aid of Eqs. \((3.17)\) and \((3.19)\)

\[
(\psi^-)^2(\psi^+)^3 \psi^- = \psi^- (\psi^+)^2(\psi^-)^3 \psi^- = \psi^- ((\psi^+)^2(\psi^-)^2 - (\psi^+)^3(\psi^-))^2 = \psi^-(\psi^+)^2(\psi^-)^2 + (\psi^+)^3(\psi^-)^3.
\]  

(3.28)

Comparing Eqs. \((3.20)\) and \((3.28)\), we obtain

\[
\psi^-(\psi^+)^2 \psi^- = \psi^+(\psi^-)^2 \psi^+ = \{(\psi^-)^2, (\psi^+)^2\} - \{(\psi^-)^3, (\psi^+)^3\}.
\]  

(3.29)

Finally, multiplying \((3.2)\) by \(\psi^-\) from left or right, and applying \((2.9), (3.1), (3.7), (3.8),\) and \((3.25)\), we get

\[
\{\psi^+, (\psi^-)^2\} + \{(\psi^+)^2, (\psi^-)^3\} = 2\psi^-,
\]  

(3.30)

\[
\{\psi^-, (\psi^+)^2\} + \{(\psi^-)^2, (\psi^+)^3\} = 2\psi^+.
\]  

(3.31)

Multiplying the formula \((3.30)\) by \(\psi^+\) from left, and applying Eqs. \((3.20)\) and \((3.29)\), we further obtain

\[
2\{(\psi^-)^2, (\psi^+)^2\} = 2\left(\psi^+ \psi^- + (\psi^-)^3 (\psi^+)^3\right) = 2\left(\psi^- \psi^+ + (\psi^+)^3 (\psi^-)^3\right) = \{\psi^-, \psi^+\} + \{(\psi^-)^3, (\psi^+)^3\}.
\]  

(3.32)
Hence, the relation (3.2) is decomposed as
\[ \{\psi^-,\psi^+\} + \{(\psi^-)^3,(\psi^+)^3\} = 2I, \quad (\psi^-)^2, (\psi^+)^2\} = I. \] (3.33)

The second equality in Eq. (3.32) can be expressed as
\[ [\psi^-,\psi^+] = [(\psi^-)^3, (\psi^+)^3]. \] (3.34)

Summarizing the results, we have derived third-order parafermionic algebra as follows:
\[ (\psi^-)^4 = (\psi^+)^4 = 0, \] (3.35)
\[ \{\psi^-,(\psi^+)^3\} = (\psi^+)^2, \quad \{\psi^+,(\psi^-)^3\} = (\psi^-)^2, \] (3.36)
\[ \{\psi^-,(\psi^+)^2\} + \{(\psi^-)^2,(\psi^+)^3\} = 2\psi^+, \quad \{\psi^+,(\psi^-)^2\} + \{(\psi^+)^2,(\psi^-)^3\} = 2\psi^-, \] (3.37)
\[ \{\psi^-,\psi^+\} + \{(\psi^-)^3,(\psi^+)^3\} = 2I, \quad \{(\psi^-)^2,(\psi^+)^2\} = I, \] (3.38)
\[ [\psi^-,(\psi^+)^2] = [(\psi^-)^2, (\psi^+)^3], \quad [\psi^+, (\psi^-)^2] = [(\psi^+)^2, (\psi^-)^3], \] (3.39)
\[ [\psi^-,\psi^+] = [(\psi^-)^3, (\psi^+)^3]. \] (3.40)

The relations (3.37) and (3.39) together imply important formulas
\[ (\psi^-)^2\psi^+ + (\psi^+)^2(\psi^-)^3 = \psi^+(\psi^-)^2 + (\psi^-)^3(\psi^+)^2 = \psi^-, \] (3.41)
\[ \psi^- (\psi^+)^2 + (\psi^+)^3(\psi^-)^2 = (\psi^+)^2\psi^- + (\psi^-)^2(\psi^+)^3 = \psi^+. \] (3.42)

Similarly, the relations (3.38) and (3.40) together imply
\[ \psi^-\psi^+ + (\psi^+)^3(\psi^-)^3 = \psi^+\psi^- + (\psi^-)^3(\psi^+)^3 = I. \] (3.43)

Classifying the formulas (3.3)–(3.4), (3.7)–(3.10), (3.12)–(3.17), (3.19)–(3.20), (3.24)–(3.27), and (3.29) with respect to parafermionic degrees, we have the following multiplication law for the parafermionic operators of order 3:

- **Degree 0:**
  \[ (\psi^-)^3\psi^+(\psi^-)^2 = 0, \quad \psi^+(\psi^-)^3(\psi^+)^2 = \Pi_1, \]
  \[ (\psi^-)^3(\psi^+)^2(\psi^-)^3 = 0, \quad \psi^+(\psi^-)^2\psi^+ = \Pi_1 + \Pi_2, \]
  \[ (\psi^-)^2\psi^+(\psi^-)^3 = 0, \quad (\psi^+)^2(\psi^-)^3\psi^+ = \Pi_2, \]
  \[ (\psi^-)^2(\psi^+)^3\psi^- = \Pi_1, \quad (\psi^+)^2\psi^-(\psi^-)^3 = 0, \]
  \[ \psi^- (\psi^+)^2\psi^- = \Pi_1 + \Pi_2, \quad (\psi^+)^3(\psi^-)^2(\psi^+)^3 = 0, \]
  \[ \psi^- (\psi^+)^3(\psi^-)^2 = \Pi_2, \quad (\psi^+)^3\psi^- (\psi^-)^2 = 0. \]

- **Degree 1:**
  \[ (\psi^-)^3\psi^+\psi^- = (\psi^-)^3, \quad \psi^+(\psi^-)^3(\psi^+)^3 = (\psi^-)^2(\psi^+)^3, \]
  \[ (\psi^-)^3(\psi^+)^3(\psi^-)^3 = (\psi^-)^3, \quad \psi^+(\psi^-)^3\psi^+ = \psi^+, \]
  \[ \psi^- \psi^+(\psi^-)^3 = (\psi^-)^3, \quad (\psi^+)^3(\psi^-)^3\psi^+ = (\psi^+)^3(\psi^-)^2, \]
  \[ (\psi^-)^3(\psi^+)^2(\psi^-)^2 = (\psi^-)^3, \quad \psi^+(\psi^-)^2(\psi^+)^2 = \psi^-(\psi^+)^2, \]
  \[ (\psi^-)^2\psi^+(\psi^-)^2 = (\psi^-)^3, \quad (\psi^+)^2(\psi^-)^2(\psi^+)^2 = \varphi^+, \]
  \[ (\psi^-)^2(\psi^+)^2(\psi^-)^3 = (\psi^-)^3, \quad (\psi^+)^2(\psi^-)^2\psi^+ = (\psi^+)^2\psi^-, \]
  \[ \psi^- (\psi^+)^3\psi^- = \varphi^+, \quad (\psi^+)^3\psi^- (\psi^+)^3 = 0. \]
• Degree 2:

\[
\begin{align*}
(\psi^-)^3(\psi^+)^2\psi^- &= (\psi^-)^3\psi^+, \\
(\psi^-)^3(\psi^+)^3(\psi^-)^2 &= (\psi^-)^3\psi^+, \\
(\psi^-)^2(\psi^+)^2\psi^- &= (\psi^-)^2, \\
(\psi^-)^2(\psi^+)^3(\psi^-)^3 &= \psi^+(\psi^-)^3, \\
\psi^-\psi^+(\psi^-)^2 &= (\psi^-)^2, \\
(\psi^-)^2(\psi^+)^2(\psi^-)^3 &= \psi^+(\psi^-)^3, \\
\psi^-\psi^+(\psi^-)^2 &= (\psi^-)^2, \\
\psi^+(\psi^-)^2(\psi^+)^3 &= \psi^+(\psi^-)^3.
\end{align*}
\]

\[\psi^+(\psi^-)^3 = \psi^+(\psi^-)^3, \quad \psi^+(\psi^-)^3 = \psi^+(\psi^-)^3.\]

In the above, \(\varphi^-, \varphi^+, \Pi_1, \) and \(\Pi_2\) are defined by

\[
\begin{align*}
\varphi^- &= \psi^+(\psi^-)^2 - (\psi^-)^2(\psi^-)^3 = (\psi^-)^2\psi^+ - (\psi^-)^3(\psi^+)^2, \\
\varphi^+ &= \psi^-(\psi^+)^2 - (\psi^+)^2(\psi^+)^3 = (\psi^+)^2\psi^- - (\psi^+)^3(\psi^-)^2, \\
\Pi_1 &= \psi^+\psi^- - (\psi^+)^2(\psi^-)^2 = (\psi^-)^2(\psi^+)^2 - (\psi^-)^3(\psi^+)^3, \\
\Pi_2 &= (\psi^+)^2(\psi^-)^2 - (\psi^+)^3(\psi^-)^3 = \psi^-\psi^+ - (\psi^-)^2(\psi^+)^2.
\end{align*}
\]

The above \(\Pi_1\) and \(\Pi_2\) are indeed the projection operators into the 1- and 2-parafermionic subspaces, respectively. The other projection operators \(\Pi_0\) and \(\Pi_3\) are given by

\[\Pi_0 = (\psi^-)^3(\psi^+)^3, \quad \Pi_3 = (\psi^+)^3(\psi^-)^3.\]

We can easily check with the aid of the multiplication law and the formula (3.43) that the operators \(\Pi_i\) \((i = 0, 1, 2, 3)\) in Eqs. (3.46)–(3.48) satisfy the definition (2.6) for \(p = 3\). The intertwining relations (2.7) can be easily checked as

\[
\begin{align*}
\Pi_1\psi^+ &= \psi^+\Pi_0 = (\psi^-)^2(\psi^+)^3, \\
\Pi_2\psi^+ &= \psi^+\Pi_1 = \varphi^+, \\
\Pi_3\psi^+ &= \psi^+\Pi_2 = (\psi^+)^3(\psi^-)^2, \\
\psi^-\Pi_1 &= \Pi_0\psi^- = (\psi^-)^3(\psi^+)^2, \\
\psi^-\Pi_2 &= \Pi_1\psi^- = \varphi^-, \\
\psi^-\Pi_3 &= \Pi_2\psi^- = (\psi^+)^2(\psi^-)^3.
\end{align*}
\]

We also note that the third-order parasuperalgebra (3.35)–(3.43) is consistent with Eq. (2.3). From the relations (3.36), (3.42), and (3.43) we have

\[
\begin{align*}
\psi^-|1\rangle &= \psi^-\psi^+|0\rangle = (I - (\psi^+)^3)(\psi^-)|0\rangle = |0\rangle, \\
\psi^-|2\rangle &= \psi^-\psi^+(\psi^-)^3|0\rangle = (\psi^+ - (\psi^+)^3(\psi^-)^2)|0\rangle = |1\rangle, \\
\psi^-|3\rangle &= \psi^-\psi^+(\psi^-)^3|0\rangle = ((\psi^+)^2 - (\psi^+)^3(\psi^-)|0\rangle = |2\rangle.
\end{align*}
\]
which are exactly Eq. (2.3). The assumption (2.10) for \( p = 3 \) is also satisfied as
\[
(\psi^-)^3 \psi^+ \Pi_2 = (\psi^-)^2 \Pi_2 = (\psi^-)^3 \psi^+, \quad \psi^+ (\psi^-)^3 \Pi_3 = (\psi^-)^2 \Pi_3 = \psi^+ (\psi^-)^3.
\]
Therefore, we have confirmed that all the postulates in Section II are fulfilled. In contrast to the second-order, the trilinear relations of parafermionic statistics [18, 19] are not satisfied since
\[
[\psi^+, [\psi^+, \psi^+]] = 2 \psi^+ \psi^+ \psi^+ - \{\psi^+, (\psi^+)^2\} = 2 \psi^+ - \{\psi^+, (\psi^+)^2\},
\]
and the anti-commutators \( \{\psi^+, (\psi^+)^2\} \) themselves do not any more reduce to simpler forms in the third-order. However, we find that the following quadrilinear relations are instead satisfied:
\[
\{\psi^+, [\psi^+, [\psi^+, \psi^+]]\} = -[\psi^+, [\psi^+, \{\psi^+, \psi^+\}]] = [\psi^+, \{\psi^+, [\psi^+, \psi^+]\}] = (\psi^+)^2. \quad (3.49)
\]
It is evident that the first quadrilinear term in Eq. (3.49) is proportional to \( (\psi^+)^2 \) if the trilinear relations of parafermionic statistics \( [\psi^+, [\psi^+, \psi^+]] \propto \psi^+ \) hold. Hence, the quadrilinear relations (3.49) can be regarded as generalized parafermionic statistics.

It is worth noticing that if we restrict the fundamental elements to the set \( \{I, (\psi^-)^2, (\psi^+)^2\} \), the third-order parafermionic algebra and multiplication law only consist of
\[
\{(\psi^-)^2, (\psi^+)^2\} = \{(\psi^+)^2, (\psi^+)^2\} = 0, \quad \{(\psi^-)^2, (\psi^+)^2\} = I, \quad (3.50)
\]
\[
(\psi^-)^2 (\psi^+)^2 (\psi^-)^2 = (\psi^-)^2, \quad (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 = (\psi^+)^2, \quad (3.51)
\]
which are exactly equivalent to the ordinary fermionic relations. Therefore, in the third-order parafermionic system, \( (\psi^-)^2 \) and \( (\psi^+)^2 \) behave as ordinary fermions.

IV. THIRD-ORDER PARASUPERSYMMETRIC QUANTUM SYSTEMS

We are now in a position to construct third-order parasupersymmetric quantum systems by using the third-order parafermionic algebra just derived in the previous section. From Eqs. (3.40)–(3.48), and the multiplication law, the triple \( (H, Q^-, Q^+) \) in Eqs. (2.18) and (2.20) for \( p = 3 \) is expressed as
\[
H = H_0 (\psi^-)^3 (\psi^+)^3 + H_1 (\psi^+ \psi^- - (\psi^+)^2 (\psi^-)^2) + H_2 ((\psi^+)^2 (\psi^-)^2 - (\psi^+)^3 (\psi^-)^3) + H_3 (\psi^+)^3 (\psi^-)^3, \quad (4.1)
\]
\[
Q^- = Q_1^- (\psi^-)^3 (\psi^+)^2 + Q_2^- \varphi^- + Q_3^- (\psi^+)^2 (\psi^-)^3, \quad (4.2)
\]
\[
Q^+ = Q_1^+ (\psi^-)^2 (\psi^+)^3 + Q_2^+ \varphi^+ + Q_3^+ (\psi^+)^3 (\psi^-)^2. \quad (4.3)
\]
We recall the fact that the above third-order parasupercharges (4.2) and (4.3) already satisfy the nilpotent condition (2.13) for \( p = 3 \), \( (Q^-)^4 = (Q^+)^4 = 0 \), as we have previously mentioned in Section II.

Now that we have had the third-order parafermionic algebra and multiplication law, we can explicitly construct the ‘discrete-type’ conserved charges. In the case of third-order, we can have twelve additional charges \( Q_{[n]}^\pm \) and \( Q_{[n]}^\pm (n = 1, 2, 3) \) defined by Eq. (2.30). Among
them, we find that there are essentially four linearly independent new conserved charges, e.g.,

\[ Q^-_2 = 2Q^-_2 \psi^-, \quad Q^-_3 = Q^-_1 (\psi^-)^3 (\psi^+)^2 - Q^-_3 (\psi^+)^2 (\psi^-)^3, \quad (4.4a) \]
\[ Q^+_2 = -2Q^+_2 \psi^+, \quad Q^+_3 = -Q^+_1 (\psi^-)^2 (\psi^+)^3 + Q^+_3 (\psi^+)^3 (\psi^-)^2, \quad (4.4b) \]

and the others are expressed as

\[ Q^-_{\{1\}} = -Q^-_{\{3\}}, \quad Q^-_{\{2\}} = 0, \quad 2Q^-_{\{1\}} = 2Q^-_{\{3\}} = 2Q^- - Q^-_2, \quad (4.5a) \]
\[ 2Q^+_\{1\} = -2Q^+_\{3\} + Q^+_2, \quad Q^+_\{2\} = 0, \quad (4.5b) \]
\[ 4Q^+_\{1\} = -2Q^+_2 - 2Q^+_\{3\}, \quad 2Q^+_\{3\} = -2Q^+_2 - Q^+_\{2\}. \quad (4.5c) \]

Needless to say, any linear combination of the original and new parasupercrages, \( Q^\pm, Q^\pm_{\{n\}} \), and \( Q^\pm_{\{n\}} \), is also conserved. In particular, the following combinations

\[ Q^-_2 \equiv Q^+ - Q^-_2 = Q^-_1 (\psi^-)^3 (\psi^+)^2 - Q^-_2 \psi^- + Q^-_3 (\psi^+)^2 (\psi^-)^3, \quad (4.6a) \]
\[ Q^-_3 \equiv \frac{1}{2}Q^-_2 + Q^-_3 = Q^-_1 (\psi^-)^3 (\psi^+)^2 + Q^-_2 \psi^- - Q^-_3 (\psi^+)^2 (\psi^-)^3, \quad (4.6b) \]

and their ‘adjoint’ ones are exactly the conserved charges (of order 3) reported in Refs. [7, 8], which are generalizations of the second-order ones in Ref. [20].

From Eqs. (2.24) and (2.32), the commutativity (2.14) and the non-linear constraints (2.15) for \( p = 3 \)

\[ (Q^-)^3 Q^+ + (Q^-)^2 Q^+ Q^- + Q^- Q^+(Q^-)^2 + Q^+(Q^-)^3 = C_3 (Q^-)^2 H, \quad (4.7a) \]
\[ (Q^+)^3 Q^- + (Q^+)^2 Q^- Q^- + Q^+ Q^-(Q^-)^2 + Q^- (Q^-)^3 = C_3 H (Q^+)^2, \quad (4.7b) \]

hold if and only if the following conditions

\[ H_0 Q^-_1 = Q^-_1 H_1, \quad H_1 Q^-_2 = Q^-_2 H_2, \quad H_2 Q^-_3 = Q^-_3 H_3, \quad (4.8) \]
\[ Q^-_1 Q^-_2 Q^-_3 + Q^-_1 Q^-_2 Q^-_2 + Q^-_1 Q^-_1 Q^-_2 = C_3 Q^-_1 Q^-_2 H_2, \quad (4.9) \]
\[ Q^-_2 Q^-_3 Q^-_3 + Q^-_2 Q^-_2 Q^-_3 + Q^-_2 Q^-_1 Q^-_3 = C_3 Q^-_1 Q^-_2 H_3, \quad (4.10) \]

and their ‘adjoint’ relations

\[ Q^+_1 H_0 = H_1 Q^+_1, \quad Q^+_2 H_1 = H_2 Q^+_2, \quad Q^+_3 H_2 = H_3 Q^+_3, \quad (4.11) \]
\[ Q^+_1 Q^+_2 Q^+_3 + Q^+_1 Q^+_2 Q^+_2 + Q^+_1 Q^+_1 Q^+_2 = C_3 H_2 Q^+_2 Q^+_1, \quad (4.12) \]
\[ Q^+_2 Q^+_3 Q^+_3 + Q^+_2 Q^+_2 Q^+_3 + Q^+_2 Q^+_1 Q^+_3 = C_3 H_3 Q^+_3 Q^+_2, \quad (4.13) \]

are satisfied. We note that when a solution to Eq. (4.9) and (4.12) are given by

\[ C_3 Q^-_H_2 = Q^+_1 Q^-_1 Q^-_2 + Q^-_2 Q^-_2 Q^-_2 + Q^-_2 Q^-_3 Q^-_3, \quad (4.14) \]

and its ‘adjoint’ relation, the conditions (4.10) and (4.13) are automatically satisfied so long as the third intertwining relations in Eqs. (4.8) and (4.11) hold. Thus, in this case it is sufficient to solve the intertwining relations (4.8) and (4.11). In general, we do not need to solve the ‘adjoint’ conditions.
For the third-order case, we have two different quasi-parasupersymmetries, namely, those of order \((3,2)\) and \((3,3)\). The conditions are given by Eqs. (4.8)–(4.13) but the first-order intertwining relations (4.8) and (4.11) are replaced in the case of order \((3,2)\) by the second-order intertwining relations

\[
H_0 Q^{-1}_1 Q_2^- = Q_1^- Q_2^- H_2, \quad H_1 Q_2^- Q_3^- = Q_2^- Q_3^- H_3, \quad (4.15a)
\]

\[
Q_2^+ Q_1^+ H_0 = H_2 Q_2^+ Q_1^+, \quad Q_3^+ Q_2^+ H_1 = H_3 Q_3^+ Q_2^+, \quad (4.15b)
\]

and in the case of order \((3,3)\) by the third-order ones

\[
H_0 Q_1^- Q_2^- Q_3^- = Q_1^- Q_2^- Q_3^- H_3, \quad Q_3^+ Q_2^+ Q_1^+ H_0 = H_3 Q_3^+ Q_2^+ Q_1^+. \quad (4.16)
\]

In the followings, we will show two different representations for the system \((H_k, Q_k^\pm)\) which satisfies the parasupersymmetric conditions (4.8)–(4.13) or quasi-parasupersymmetric ones, (4.15) or (4.16).

**A. One-Variable Representation**

First, we shall realize a third-order parasupersymmetric quantum system of one degree of freedom. Let us put

\[
H_k = -\frac{1}{2} \frac{d^2}{dq^2} + V_k(q), \quad Q_k^\pm = \pm \frac{d}{dq} + W_k(q). \quad (4.17)
\]

The general solutions to Eqs. (4.8) and (4.11) are given by

\[
2H_0 = Q_1^- Q_1^+ - 2R_1, \quad 2H_1 = Q_1^+ Q_1^- - 2R_1 = Q_2^+ Q_2^- - 2R_2, \quad (4.18a)
\]

\[
2H_2 = Q_2^+ Q_2^- - 2R_2 = Q_3^+ Q_3^- - 2R_3, \quad 2H_3 = Q_3^+ Q_3^- - 2R_3, \quad (4.18b)
\]

where \(R_i (i = 1,2,3)\) are constants and the functions \(W_k (k = 1,2,3)\) must satisfy

\[
W_1' + W_{k+1}' + W_k^2 - W_{k+1}^2 = 2R_k - 2R_{k+1} \quad (k = 1,2). \quad (4.19)
\]

Substituting them into Eqs. (4.9) and (4.12), we find that the conditions (4.9) and (4.12) are satisfied if and only if

\[
C_3 = 6, \quad R_1 + R_2 + R_3 = 0. \quad (4.20)
\]

In this case, the relation (4.14) and its ‘adjoint’ hold. Hence, the remaining conditions (4.10) and (4.13) are automatically satisfied. The third-order parasupersymmetric quantum system (4.17)–(4.20) is equivalent to the generalized Rubakov–Spiridonov model of order 3 constructed in Ref. [7, 8].

For the second-order parasupersymmetric quantum system of Rubakov–Spiridonov type, we found previously [1] that the system admits a generalized 2-fold superalgebra. In what follows, we show that the present third-order system analogously admits a generalized 3-fold superalgebra. To begin with, we construct zeroth-degree operators composed of only \(Q_k^\pm\)
and $Q^\pm$. Owing to the nilpotency $(Q^\pm)^4 = 0$, we can easily show that any such operator should be expressed as a function of the following six composite operators:

\begin{align*}
Q^- Q^+ &= Q_1^- Q_1^+ \Pi_0 + Q_2^- Q_2^+ \Pi_1 + Q_3^- Q_3^+ \Pi_2, \\
Q^+ Q^- &= Q_1^+ Q_1^- \Pi_1 + Q_2^+ Q_2^- \Pi_2 + Q_3^+ Q_3^- \Pi_3,
\end{align*}

\begin{align*}
(Q^-)^2(Q^+)^2 &= Q_1^- Q_1^+ Q_1^- Q_1^+ \Pi_0 + Q_2^- Q_2^+ Q_2^- Q_2^+ \Pi_1, \\
(Q^+)^2(Q^-)^2 &= Q_2^+ Q_2^- Q_2^+ Q_2^- \Pi_2 + Q_3^+ Q_3^- Q_3^+ Q_3^- \Pi_3,
\end{align*}

\begin{align*}
(Q^-)^3(Q^+)^3 &= Q_1^- Q_2^- Q_3^- Q_1^+ Q_2^+ Q_3^+ \Pi_0, \\
(Q^+)^3(Q^-)^3 &= Q_3^- Q_2^- Q_1^- Q_2^+ Q_1^+ \Pi_3.
\end{align*}

Now, substituting the relations (4.18) into the above (4.21)–(4.26), we obtain

\begin{align*}
Q^- Q^+ &= 2(H_0 + R_1)\Pi_0 + 2(H_1 + R_2)\Pi_1 + 2(H_2 + R_3)\Pi_2, \\
Q^+ Q^- &= 2(H_1 + R_1)\Pi_1 + 2(H_2 + R_2)\Pi_2 + 2(H_3 + R_3)\Pi_3, \\
(Q^-)^2(Q^+)^2 &= 4(H_0 + R_1)(H_0 + R_2)\Pi_0 + 4(H_1 + R_2)(H_1 + R_3)\Pi_1, \\
(Q^+)^2(Q^-)^2 &= 4(H_2 + R_1)(H_2 + R_2)\Pi_2 + 4(H_3 + R_2)(H_3 + R_3)\Pi_3, \\
(Q^-)^3(Q^+)^3 &= 8(H_0 + R_1)(H_0 + R_2)(H_0 + R_3)\Pi_0, \\
(Q^+)^3(Q^-)^3 &= 8(H_3 + R_1)(H_3 + R_2)(H_3 + R_3)\Pi_3.
\end{align*}

From these formulas, we can easily find non-linear relations as follows:

\begin{align*}
(Q^-)^3(Q^+)^3 &= 8(H + R_1)(H + R_2)(H + R_3). 
\end{align*}

As in the case of second-order, they can be regarded as generalizations of 3-fold superalgebra. Indeed, if we restrict the linear space $\mathfrak{g} \times V_3$ on which the system $H$ acts to $\mathfrak{g} \times (V_3^{(0)} + V_3^{(3)})$ (cf. the definition between Eqs. (2.3) and (2.4)), we have

\begin{align*}
\{(Q^-)^3, (Q^+)^3\} = 8(H + R_1)(H + R_2)(H + R_3)|_{\mathfrak{g} \times (V_3^{(0)} + V_3^{(3)})}. 
\end{align*}

This, together with the trivial (anti-)commutation relations

\begin{align*}
\{(Q^-)^3, (Q^-)^3\} = \{(Q^+)^3, (Q^+)^3\} = \{(Q^\pm)^3, H\} = 0,
\end{align*}

collects a type of 3-fold superalgebra in the sector $\mathfrak{g} \times (V_3^{(0)} + V_3^{(3)})$.

Next, we shall examine quasi-parasupersymmetry in the system given by Eq. (4.17). For the purpose, we first solve the conditions (4.9) and (4.12). Substituting the expression (4.17) into the conditions (4.9) or (4.12), we can easily find that they are satisfied if and only if $C_3 = 6$ and

\begin{align*}
6V_2 &= W_1'' + 3W_2'' - W_3'' + W_1^2 + W_2^2 + W_3^2, \\
W_1'' + W_2'' + 2W_1W_1' - 2W_2W_2' &= 0.
\end{align*}
Similarly, if we substitute Eq. (4.17) into the conditions (4.10) or (4.13) with \( C_3 = 6 \), we
find
\[
6V_3 = W'_3 + 3W''_2 + 5W'_2 + W^2_1 + W^2_2 + W^2_3, \tag{4.38}
\]
\[
W''_1 + 2W''_2 + W''_3 + 2W'_1W'_1 - 2W'_3W'_3 = 0, \tag{4.39}
\]
and
\[
W'''' + W''_2 + 2W'_1W''_2 + 2W''W_2^2 - 2W''W_2 - 2W_2'W_3 = 0. \tag{4.40}
\]
It is easy to check that the condition (4.37) implies (4.40). From Eqs. (4.37) and (4.39), we
obtain exactly the same conditions as Eq. (4.19). Thus, from the potential forms (4.36) and
(4.38), and the condition (4.19) we have
\[
6H_2 = 3Q_2^+Q_2^- + 2(R_1 - 2R_2 + R_3) = 3Q_3^+Q_3^- + 2(R_1 + R_2 - 2R_3), \tag{4.41a}
\]
\[
6H_3 = 3Q_3^+Q_3^- + 2(R_1 + R_2 - 2R_3). \tag{4.41b}
\]
We now first consider quasi-parasupersymmetry of order (3, 2). The remaining conditions to
be satisfied are Eqs. (4.15). Substituting the expressions (4.17) and (4.41) into the conditions
(4.15), we find that they are satisfied if and only if
\[
2V_0 = W^2_2 - 2W'_2 - W_2 + \frac{2}{3}(R_1 - 2R_2 + R_3), \tag{4.42a}
\]
\[
2V_1 = W^2_2 - 2W'_3 - W_3 + \frac{2}{3}(R_1 + R_2 - 2R_3), \tag{4.42b}
\]
and Eq. (4.19) hold. Combining all the results (4.19), (4.41), and (4.42), we finally derive the
necessary and sufficient condition for the system (4.17) to have quasi-parasupersymmetry of
order (3, 2) as follows:
\[
2H_0 = Q_1^+Q_1^- - 2\bar{R}_1, \quad 2H_1 = Q_2^+Q_1^- - 2\bar{R}_1 = Q_3^+Q_2^- - 2\bar{R}_2, \tag{4.43a}
\]
\[
2H_2 = Q_2^+Q_2^- - 2\bar{R}_2 = Q_3^+Q_3^- - 2\bar{R}_3, \quad 2H_3 = Q_3^+Q_3^- - 2\bar{R}_3, \tag{4.43b}
\]
where the new set of constants \( \bar{R}_k \) (\( k = 1, 2, 3 \)) are introduced by
\[
3\bar{R}_1 = 2R_1 - R_2 - R_3, \quad 3\bar{R}_2 = -R_1 + 2R_2 - R_3, \quad 3\bar{R}_3 = -R_1 - R_2 + 2R_3, \tag{4.44}
\]
and thus satisfy
\[
\bar{R}_1 + \bar{R}_2 + \bar{R}_3 = 0. \tag{4.45}
\]
Hence, it is completely equivalent to the (ordinary) parasupersymmetric system (4.18). It is
evident that the same generalized 3-fold algebra (4.33) also holds (with \( R_k \to \bar{R}_k \)).

We next consider quasi-parasupersymmetry of order (3, 3). We recall the fact that we
have already solved the conditions (4.9)–(4.10) and (4.12)–(4.13) to obtain Eqs. (4.19) and
(4.41). Thus, the remaining conditions to be satisfied are Eq. (4.16). Substituting the
expressions (4.17) and (4.41) into the conditions (4.16), we find that they are satisfied if and
only if
\[
2V_0 = W_3^2 - 2W'_3 - 2W'_3 + \frac{2}{3}(R_1 + R_2 - 2R_3), \tag{4.46}
\]
\[
W'''' + 3W''_2 + 2W''_3 + 2W'_1W'_1 + 2W_2W'_2 - 4W_3W'_3 = 0, \tag{4.47}
\]
where $R$ and (4.52c) are compatible if and only if $f$ where $\bar{R}$, order (3 necessary and sufficient condition for the system (4.17) to have quasi-parasupersymmetry of order (3, 3) is as follows:

$$2H_0 = Q_1^+ Q_1^- - 2\bar{R}_1, \quad Q_2^+ Q_2^- - 2\bar{R}_1 = Q_3^+ Q_3^- - 2\bar{R}_2; \quad 2H_2 = Q_2^+ Q_2^- - 2\bar{R}_2 = Q_3^+ Q_3^- - 2\bar{R}_3, \quad 2H_3 = Q_3^+ Q_3^- - 2\bar{R}_3,$$

(4.49b)

where $\bar{R}_k (k = 1, 2, 3)$ are the same as Eq. (4.44). We note in particular that there are no constraints on the form of the Hamiltonian $H_1$ in contrast to the cases of order (3, 1) and (3, 2).

**B. Three-Variable Representation**

As another example, let us next consider a three-body system given by

$$H_k = -\frac{1}{2} \sum_{i=1}^{3} \partial^2 \partial q_k^i + V_k (q_1, q_2, q_3), \quad Q_k^\pm = \pm \partial \partial q_k + W_k (q_1, q_2, q_3).$$

(4.50)

The first-order intertwining relations (4.8) and (4.11) are satisfied if and only if

$$W_k (q_1, q_2, q_3) = W_k (q_k) \quad (k = 1, 2, 3),$$

(4.51)

and

$$2V_0 (q_1, q_2, q_3) = W_1 (q_1) - W_1' (q_1) + f_1 (q_2, q_3),$$

(4.52a)

$$2V_1 (q_1, q_2, q_3) = W_1 (q_1)^2 + W_1' (q_1) + f_1 (q_2, q_3)$$

$$= W_2 (q_2)^2 - W_2' (q_2) + f_2 (q_1, q_3),$$

(4.52b)

$$2V_2 (q_1, q_2, q_3) = W_2 (q_2)^2 + W_2' (q_2) + f_2 (q_1, q_3)$$

$$= W_3 (q_3)^2 - W_3' (q_3) + f_3 (q_1, q_2),$$

(4.52c)

$$2V_3 (q_1, q_2, q_3) = W_3 (q_3)^2 + W_3' (q_3) + f_3 (q_1, q_2),$$

(4.52d)

where $f_k (k = 1, 2, 3)$ are certain functions of two variables. The two equalities in Eqs. (4.52b) and (4.52c) are compatible if and only if

$$f_1 (q_2, q_3) = W_2 (q_2)^2 - W_2' (q_2) + W_3 (q_3)^2 - W_3' (q_3) - 2R,$$

(4.53a)

$$f_2 (q_1, q_3) = W_1 (q_1)^2 + W_1' (q_1) + W_3 (q_3)^2 - W_3' (q_3) - 2R,$$

(4.53b)

$$f_3 (q_1, q_2) = W_1 (q_1)^2 + W_1' (q_1) + W_2 (q_2)^2 + W_2' (q_2) - 2R,$$

(4.53c)

where $R$ is a constant. Substituting (4.53) into (4.52), we have

$$2H_0 = Q_1^+ Q_1^- + Q_2^+ Q_2^- + Q_3^+ Q_3^- - 2R,$$

(4.54a)

$$2H_1 = Q_1^+ Q_1^- + Q_2^+ Q_2^- + Q_3^+ Q_3^- - 2R,$$

(4.54b)
Finally, substituting (4.54) into the conditions (4.9) or (4.12), we find that they are satisfied if and only if

\[ C_3 = 2, \quad R = 0. \] (4.55)

In this case, the Hamiltonian \( H_2 \) satisfies Eq. (4.14) and thus the remaining conditions (4.10) and (4.13) are automatically fulfilled. Hence, the necessary and sufficient conditions for the system (4.50) to have parasupersymmetry of order 3 are given by Eqs. (4.51), (4.54), and (4.55). We can easily see that there are three independent supersymmetries folded in the system (4.54). We note that the parasupersymmetric system \( H \) in this case is given by

\[ 2H = (Q_1^-Q_1^+ + Q_2^-Q_2^+ + Q_3^-Q_3^+)\Pi_0 + (Q_1^+Q_1^- + Q_2^+Q_2^- + Q_3^+Q_3^-)\Pi_1 \]
\[ + (Q_1^+Q_1^- + Q_2^+Q_2^- + Q_3^+Q_3^-)\Pi_2 + (Q_1^+Q_1^- + Q_2^+Q_2^- + Q_3^+Q_3^-)\Pi_3. \] (4.56)

Comparing it with Eqs. (4.21)–(4.26), we observe that any function of \( H \) cannot be expressed as a function of \( Q^- \) and \( Q^+ \), in contrast to the previous one-body case. The situation is analogous to that of the two-body second-order parasupersymmetric systems in Ref. 1.

Next, we shall examine quasi-parasupersymmetry in the system given by Eq. (4.50). For the purpose, we first solve the conditions (4.9) and (4.12). Substituting the expression (4.50) into the conditions (4.9) or (4.12), we can easily find that they are satisfied if and only if \( C_3 = 2 \) and

\[ 2V_2 = W_1^2 + (\partial_1 W_1) + W_2^2 + (\partial_2 W_2) + W_3^2 - (\partial_3 W_3), \] (4.57)
\[ (\partial_1 W_2) = 0, \quad (\partial_1 \partial_2 W_1) + 2W_1(\partial_2 W_1) = 0, \] (4.58)

where \((\partial_1 f) = \partial f/\partial q_i \). Similarly, if we substitute Eq. (4.50) into the conditions (4.10) or (4.13) with \( C_3 = 2 \), we find

\[ 2V_3 = W_1^2 + (\partial_1 W_1) + W_2^2 + (\partial_2 W_2) + W_3^2 + (\partial_3 W_3), \] (4.59)
\[ (\partial_1 W_2) = (\partial_1 W_3) = (\partial_2 W_3) = 0, \quad (\partial_1 \partial_2 W_1) + 2W_1(\partial_2 W_1) = 0, \] (4.60)
\[ (\partial_1 \partial_3 W_1) + 2W_1(\partial_3 W_1) + (\partial_2 \partial_3 W_2) + 2W_2(\partial_3 W_2) = 0. \] (4.61)

We now first consider quasi-parasupersymmetry of order \((3,2)\). The remaining conditions to be satisfied are Eqs. (4.15). Substituting the expressions (4.50) into the conditions (4.15) and using Eqs. (4.57)–(4.61), we find that they are satisfied if and only if

\[ 2V_0 = W_1^2 - (\partial_1 W_1) + W_2^2 - (\partial_2 W_2) + W_3^2 - (\partial_3 W_3), \] (4.62)
\[ 2V_1 = W_1^2 + (\partial_1 W_1) + W_2^2 - (\partial_2 W_2) + W_3^2 - (\partial_3 W_3), \] (4.63)
\[ (\partial_2 W_1) = (\partial_3 W_1) = (\partial_3 W_2) = 0. \] (4.64)

It is evident that the conditions (4.58), (4.60), (4.61), and (4.64) are altogether equivalent to Eq. (4.51). In this case, the formulas (4.57), (4.59), (4.62), and (4.63) are identical to Eqs. (4.54) with \( R = 0 \). Hence, for the system given by Eq. (4.50) quasi-parasupersymmetry of order \((3,2)\) is again completely equivalent to the (ordinary) parasupersymmetry.

We next consider quasi-parasupersymmetry of order \((3,3)\). We have already solved the conditions (4.9)–(4.10) and (4.12)–(4.13) to obtain Eqs. (4.51). Thus, the remaining conditions to be satisfied are Eq. (4.16). Substituting the expressions (4.50) into the conditions (4.16) and applying Eqs. (4.57)–(4.61), we find that they are satisfied if and only
\[ 2V_0 = W_1^2 - (\partial_1 W_1) + W_2^2 - (\partial_2 W_2) + W_3^2 - (\partial_3 W_3), \quad (4.65) \]
\[ (\partial_2 W_1) = (\partial_3 W_1) = (\partial_3 W_2) = 0. \quad (4.66) \]

Hence, for the three-body system \([4.50]\) parasupersymmetry of order \((3,3)\) is identical to those of order \((3,1)\) and \((3,2)\) except for the fact that there are no constraints on the form of the Hamiltonian \(H_1\), as in the case of the one-body system in Section \([V.A]\).

V. DISCUSSION AND SUMMARY

In this article, we have investigated third-order parafermionic algebra and parasupersymmetric quantum systems based on the general formalism in our previous work. We have found that the postulates of the formalism work well also in the case of third order and enable us to construct systematically the parafermionic algebra and the multiplication law. We have constructed the two different third-order parasupersymmetric quantum systems, the one consists of one-body Hamiltonians and the other consists of three-body ones. They are respectively natural generalizations of the one-body and two-body second-order systems in our previous article \([1]\). We have also investigated quasi-parasupersymmetry in those two systems and found that the order \((3,2)\) cases are equivalent to the order \((3,1)\), namely, the ordinary third-order parasupersymmetric cases, while the order \((3,3)\) cases are realized with the less restrictive conditions by dropping the constraint on the component Hamiltonian \(H_1\).

Although most of the features in the third-order case we have found in this article have strong resemblance to those in the second-order case, there are some novel features in the former which do not appear in the latter. One is the splitting of the anti-commutation relation \([3.2]\) into the odd- and even-degree parts, namely, Eq. \([3.38]\). As a result, the part of the fundamental elements, \(\{I, (\psi^-)^2, (\psi^+)^2\}\) behaves as an ordinary fermionic system. From this result, we conjecture that a similar decomposition would take place in arbitrary odd-order parafermionic algebra and in particular the part \(\{I, (\psi^-)^p, (\psi^+)^p\}\) in \((2p-1)\)th-order for all \(p = 1, 2, 3, \ldots\) would also behave as a fermionic system. It is also interesting to see whether some decomposition of the relation \([2.2]\) takes place in even \((2p)\)th-order cases for \(p \geq 2\).

Another new feature in the third-order is the emergence of the generalized parafermionic statistics characterized by the quadrilinear relations \([3.49]\). From this result, we conjecture that in our formalism parafermionic operators of order \(p\) are characterized by \((p + 1)\)-tuple linear relations. We have not appreciated whether such a generalized statistics is compatible with other physical requirements, especially in view of the canonical formulation of quantum theory (cf. Refs. \([18, 19, 21]\)). But we hope it could provide a new possibility in both physical and mathematical studies. We also note that the expected \((p + 1)\)-tuple linear relations in \(p\)th-order could be a clue to improving the BDCF formalism, that is, how to modify the left-hand sides of Eqs. \([2.16]\). We have found that the quasi-parasupersymmetric quantum systems of order \((3, 2)\) obtained in this article are identical to the corresponding parasupersymmetric ones. This result, together with the one that all the quasi-parasupersymmetric quantum systems of order \((2, 2)\) obtained in Ref. \([1]\) are also identical with the corresponding parasupersymmetric ones, indicates that the conditions of parasupersymmetry is in fact too strong, at least, for the second- and third-order cases. On the other hand, the peculiar feature of the order \((3, 3)\)
cases where only the component Hamiltonian $H_i$ has no restriction stems from the fact that the non-linear constraints (2.32) impose the restrictions only on $H_{p-1}$ and $H_p$ among the component Hamiltonians $H_k$ ($k = 0, \ldots, p$). In other words, the conditions (2.32) unnaturally violate equality of component Hamiltonians. These observations may suggest that there is a more suitable definition or formulation of parasupersymmetry.

The discovery of generalized $\mathcal{N}$-fold superalgebra in the one-body parasupersymmetric quantum systems of generalized Rubakov–Spiridonov type for second- and third-order clearly indicates that it also exists in this type of models for arbitrary order $p$. Indeed, the component Hamiltonians $H_0$ and $H_p$ in this type are always a one-body $\mathcal{N}$-fold supersymmetric pair ($\mathcal{N} = p$) with respect to the components of the ‘$\mathcal{N}$-fold’ supercharges $(Q^\pm)^p$ and thus in the subsector $\mathfrak{F} \times (V_p^0 + V_p^p)$ the triple $(H, (Q^-)^p, (Q^+)^p)$ must satisfy an $\mathcal{N}$-fold superalgebra. Hence, in the whole space $\mathfrak{F} \times V_p$ we can naturally expect some generalized form of it. We also note that the absence of any additional algebraic relation in the two- and three-body parasupersymmetric quantum systems would explain the reason why there have been no satisfactory formulation of $\mathcal{N}$-fold supersymmetry in quantum many-body systems solely in terms of commutators and anti-commutators. The parasuperalgebra (2.13)–(2.15) thus can be an alternative framework to realize higher-order intertwining relations among many-body Hamiltonians.

Gathering altogether the knowledge so far obtained in the second- and third-order cases, we hope we would be able to report some inductive study on higher-order cases in the near future.

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