Computing $L$-Functions of Quadratic Characters at Negative Integers

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Abstract

We survey a number of different methods for computing $L(\chi, 1 - k)$ for a Dirichlet character $\chi$, with particular emphasis on quadratic characters. The main conclusion is that when $k$ is not too large (for instance $k \leq 100$) the best method comes from the use of Eisenstein series of half-integral weight, while when $k$ is large the best method is the use of the complete functional equation, unless the conductor of $\chi$ is really large, in which case the previous method again prevails.

1 Introduction

This paper can be considered as a complement of two of my old papers [2] and [3], updated to include new formulas, and surveying existing methods.

The general goal of this paper is to give efficient methods for computing the values at negative integers of $L$-functions of Dirichlet characters $\chi$. Since these values are algebraic numbers, more precisely belong to the cyclotomic field $\mathbb{Q}(\chi)$, we want to know their exact value. When $\chi(-1) = (-1)^{-r-1}$ we have $L(\chi, 1 - k) = 0$, so we always assume implicitly that $\chi(-1) = (-1)^r$. In addition, if $\chi$ is a non-primitive character modulo $F$ and $\chi_f$ is the primitive character associated to $\chi$, we have

$$L(\chi, 1 - k) = L(\chi_f, 1 - k) \prod_{p | F, p \nmid f} (1 - \chi_f(p)p^{-r-1}),$$

so we may assume that $\chi$ is primitive.

Note that we will not consider the slightly different problem of computing tables of $L(\chi, 1 - k)$, either for fixed $k$ and varying $\chi$ (such as $\chi = \chi_D$ the
quadratic character of discriminant $D$), or for fixed $\chi$ and varying $k$, although several of the methods considered here can be used for this purpose.

In addition to their intrinsic interest, these computations have several applications, for instance:

(1) Computing $\lambda$-invariants of quadratic fields (I am indebted to J. Ellenberg and S. Jain for this, see [7]).

(2) Computing Sato–Tate distributions for modular forms of half-integral weight, see [8] and [9].

(3) Computing Hardy–Littlewood constants of polynomials, see [1].

There exist at least five different methods for computing these quantities, some having several variants. We denote by $F$ the conductor of $\chi$.

(1) Bernoulli methods: one can express $L(\chi, 1 - k)$ as a finite sum involving $O(F)$ terms and Bernoulli numbers, so that the time required is $\tilde{O}(F)$ (we use the “soft-$O$” notation $\tilde{O}(X)$ to mean $O(X^{1+\varepsilon})$ for any $\varepsilon > 0$). This method has two variants: one which uses directly the definition of $\chi$-Bernoulli numbers, the second which uses recursions.

(2) Use of the complete functional equation. Using it, it is sufficient first to compute numerically $L(\chi, k)$ to sufficient accuracy (given by the functional equation), which is done using the Euler product, and second to know an upper bound on the denominator of $L(\chi, 1 - k)$, which is easy (and usually equal to 1). The required time is also $\tilde{O}(F)$, but with a much smaller implicit $O()$ constant.

(3) Use of the approximate functional equation, which involves in particular computing the incomplete gamma function or similar higher transcendental functions. The required time is $\tilde{O}(F^{1/2})$, but with a large implicit $O()$ constant.

(4) Use of Hecke-Eisenstein series (Hilbert modular forms) on the full modular group, which expresses $L(\chi, 1 - k)$ as a finite sum involving $O(F^{1/2})$ terms and (twisted) sum of divisors function. The required time is $\tilde{O}(F^{1/2})$ with a very small implicit $O()$ constant. A variant which is useful only for very small $k$ such as $k \leq 10$ uses Hecke-Eisenstein series on congruence subgroups of small level.

(5) Use of Eisenstein series of half-integral weight over $\Gamma_0(4)$, which again expresses $L(\chi, 1 - k)$ as a finite sum involving $O(F^{1/2})$ terms and (twisted) sum of divisors function, but different from the previous ones. The required time is again $\tilde{O}(F^{1/2})$, but with an even smaller implicit $O()$ constant. An important variant, valid for all $k$, is to use modular forms of half-integral weight on subgroups of $\Gamma_0(4)$. 

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The first three methods are completely general, but the last two are really efficient only if \( \chi \) is equal to a quadratic character or possibly a quadratic character times a character of small conductor. We will therefore present all five methods and their variants, but consider the last two methods only in the case of quadratic characters, and therefore compare them only in this case.

After implementing these methods and comparing their running times for various values of \( F \), we have arrived at the following conclusions: first, the two fastest methods are always either the fifth (Eisenstein series of half-integral weight) or the second (complete functional equation). Second, one should choose the second method only if \( k \) is large, for instance \( k \geq 100 \), except if \( F \) is large. Note also that the case \( F = 1 \) corresponds to the computation of Bernoulli numbers, and that indeed the fastest method for this is the use of the complete functional equation of the Riemann zeta function.

Because of these conclusions, we will give explicitly the formulas for the first, third, and fourth method, but only describe the precise implementations and timings for the second and fifth, which are the really useful ones.

## 2 Bernoulli Methods

### 2.1 Direct Formulas

**Proposition 2.1** Define the \( \chi \)-Bernoulli numbers \( B_k(\chi) \) by the generating function

\[
\frac{T}{e^{FT} - 1} \sum_{0 \leq r < F} \chi(r)e^{rT} = \sum_{k \geq 0} \frac{B_k(\chi)}{k!} T^k .
\]

Then

\[
L(\chi, 1 - k) = -\frac{B_k(\chi)}{k} - \chi(0)\delta_{k,1} .
\]

Note that since we assume \( \chi \) primitive, the term \( \chi(0)\delta_{k,1} \) vanishes unless \( F = 1 \) and \( k = 1 \), in which case \( L(\chi, 1 - k) = \zeta(0) = -1/2 \). Also, recall that for \( k \geq 2 \) we have \( B_k(\chi) = 0 \) if \( \chi(-1) \neq (-1)^k \).

**Proposition 2.2** Set \( S_n(\chi) = \sum_{0 \leq r < F} \chi(r)r^n \). We have

\[
B_k(\chi) = \frac{1}{F} \left( S_k(\chi) - \frac{kF}{2} S_{k-1}(\chi) + \sum_{1 \leq j \leq k/2} \binom{k}{2j} B_{2j} F^{2j} S_{k-2j}(\chi) \right)
\]

\[
= \frac{1}{F} \sum_{0 \leq r < F} \chi(r) \left( r^k - \frac{kF}{2} r^{k-1} + \sum_{1 \leq j \leq k/2} \binom{k}{2j} B_{2j} r^{k-2j} F^{2j} \right)
\]

\[
= \frac{1}{F} \sum_{1 \leq j \leq k+1} \frac{(-1)^{j-1}}{j} \binom{k+1}{j} \sum_{0 \leq r < F_j} \chi(r)r^k .
\]
2.2 Recursions

There are a large number of recursions for $B_k(\chi)$. The following three propositions give some of the most important ones:

**Proposition 2.3** We have the recursion

$$\sum_{0 \leq j < k} F^{k-j} \binom{k}{j} B_j(\chi) = kS_{k-1}(\chi),$$

where $S_n(\chi)$ is as above.

**Proposition 2.4** Let $\chi$ be a nontrivial primitive character of conductor $F$, set $\varepsilon = \chi(2)$ and

$$Q_k(\chi) = \sum_{1 \leq r < F/2} \chi(r)r^k.$$

We have the recursion

$$(2^k - \varepsilon)B_k(\chi) = -\left(k2^{k-1}Q_{k-1}(\chi) + \sum_{1 \leq j < k/2} \binom{k}{2j}(2^{k-1-2j} - \varepsilon)F^{2j}B_{k-2j}(\chi)\right).$$

**Proposition 2.5** Let $\chi$ be a nontrivial primitive character of conductor $F$.

1. If $\chi$ is even we have

$$\sum_{0 \leq j \leq (k-1)/2} \binom{k}{2j+1} F^{2j} B_{2k-2j}(\chi) = \frac{(-1)^k}{F} \sum_{0 \leq r < F/2} \chi(r)r^k(F-r)^k.$$

2. If $\chi$ is odd we have

$$\sum_{0 \leq j \leq (k-1)/2} \binom{k}{2j+1} F^{2j} B_{2k-1-2j}(\chi) = \frac{(-1)^{k-k}}{F} \sum_{0 \leq r < F/2} \chi(r)r^{k-1}(F-r)^{k-1}(F-2r).$$

In practice, it seems that the fastest way to compute $L(\chi, 1-k)$ using $\chi$-Bernoulli numbers is to use Proposition 2.4, but it is not competitive with the other methods that we are going to give.

3 Using the Complete Functional Equation

In this section and the next, we use approximate methods to compute $L(\chi, 1-k)$, which for simplicity we call transcendental methods, since they use transcendental functions.
Since our goal is to compute these values as exact algebraic numbers, and since we know that $L(\chi, 1 - k) \in \mathbb{Q}(\zeta_u)$, where $u$ is the order of $\chi$, we simply need to know an upper bound for the denominator of $L(\chi, 1 - k)$ as an algebraic number, and we need to compute simultaneously $L(\chi^j, 1 - k)$ for $k$ modulo $u$ and coprime to $u$, so that the individual values can then be obtained by simple linear algebra. A priori this involves $\phi(u)$ computations, but since $L(\chi^{-1}, 1 - k)$ is simply the complex conjugate of $L(\chi, 1 - k)$, only $\lceil \phi(u)/2 \rceil$ computations are needed. In particular, if $u = 1, 2, 3, 4, \text{ or } 6$, a single computation suffices.

Thus, we need two types of results: one giving the approximate size of $L(\chi, 1 - k)$, so as to determine the relative accuracy with which to do the computations, and second an upper bound for its denominator. The first result is standard, and the second can be found in Section 11.4 of [5]:

**Proposition 3.1** We have

$$L(\chi, 1 - k) = 2 \cdot (k - 1)! F^k \frac{\overline{L}(\chi, k)}{(-2i\pi)^k \mathfrak{g}(\chi)},$$

where $\mathfrak{g}(\chi)$ is the standard Gauss sum of modulus $|F|^{1/2}$ associated to $\overline{\chi}$.

**Corollary 3.2** As $k \to \infty$ we have

$$|L(\chi, 1 - k)| \sim 2 \cdot e^{-1/2} \left( \frac{kF}{2\pi e} \right)^{k-1/2}.$$

**Proof.** Clear from Stirling’s formula and the fact that $L(\overline{\chi}, k)$ tends to $1$ when $k \to \infty$. \hfill \Box

**Theorem 3.3** Denote by $u$ the order of $\chi$, so that $u | \phi(F)$ and $L(\chi, 1 - k) \in K = \mathbb{Q}(\zeta_u)$. We have $D(\chi, k)L(\chi, 1 - k) \in \mathbb{Z}[\zeta_u]$, where the “denominator” $D(\chi, k)$ can be chosen as follows:

1. If $F$ is not a prime power then $D(\chi, k) = 1$.
2. Assume that $F = p^v$ for some odd prime $p$ and $v \geq 1$.
   a. If $u \neq p^{v-1}(p - 1)/\gcd(p - 1, k)$ then $D(\chi, k) = 1$.
   b. If $u = p^{v-1}(p - 1)/\gcd(p - 1, k)$ then $D(\chi, k) = \frac{pk}{(p - 1)/u}$ if $v = 1$ or $D(\chi, k) = \chi(1 + p) - 1$ if $v \geq 2$.
3. If $F = 2^v$ for some $v \geq 2$ then $D(\chi, k) = 1$ if $v \geq 3$, while $D(\chi, k) = 2$ if $v = 2$.
4. If $F = 1$ then $D(\chi, k) = k \prod_{(p - 1)/k} p$.

Stronger statements are easy to obtain, see [5], but these bounds are sufficient.
To compute \( L(\chi, 1 - k) \) using these results, we proceed as follows. Let \( B \) be chosen so that

\[
B > (k - 1/2) \log(kF/(2\pi e)) + \log(|D(\chi, k)|) + 10,
\]

where 10 is simply a safety margin. Thanks to the above two results, computing \( L(\chi, 1 - k) \) to relative accuracy \( e^{-B} \) will guarantee that the coefficients of the algebraic integer \( D(\chi, k)L(\chi, 1 - k) \) on the integral basis \( (\zeta_j^B)_{0 \leq j < \phi(u)} \) will be correct to accuracy \( e^{-5} \), say, and since they are in \( \mathbb{Z} \), they can thus be recovered exactly.

Thanks to the functional equation, it is thus sufficient to compute \( L(\chi, k) \) to relative accuracy \( e^{-B} \), but since \( L(\chi, k) \) is close to 1, \( k \) being large, this is the same as absolute accuracy. Note from the above formula that \( B \) will be (considerably) larger than \( k \).

To compute \( L(\chi, k) \), we first compute \( \prod_{p \leq L(B, k)}(1 - \chi(p)/p^k) \), using an internal accuracy of \( e^{-kB/(k-1)} \) and limit \( L(B, k) = (e^{B}/(k-1))^{1/(k-1)} \). More precisely, we initially set \( P = 1 \), and for primes \( p \) going from 2 to \( L(B, k) \), we compute \( 1/p^k \) to \( p^k e^{-kB/(k-1)} \) of relative accuracy (this is crucial), and then set \( P \leftarrow P - P(1/p^k) \). It is clear that this will compute \( 1/L(\chi, k) \) to the desired precision, from which we immediately obtain \( L(\chi, k) \). Important implementation remark: to compute the accuracy needed in the intermediate computations, one does not compute \( \log(p^k) = k \log(p) \), but only some rough approximation, for instance by counting the number of bytes that the multiprecision integer \( p^k \) occupies in memory, or any other fast method.

Even though this method is designed to be fast for relatively large \( k \), we find that it is considerably faster than any of the Bernoulli methods, even for very small \( k \), the ratio increasing with increasing \( k \) and decreasing \( F \).

Here are the times obtained using this method. The reader will notice that the times for very small \( k \) are larger than for moderate \( k \) due to the very large number of Euler factors to be computed, the smallest being impossibly long. We use * to indicate very long times (usually more than 100 seconds), and on the contrary – to indicate a negligible time, less than 50 milliseconds.

\[
\begin{array}{|c|cccccccccc|}
\hline
D \backslash k & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
\hline
10^5 + 1 & * & 1.03 & 0.20 & 0.11 & 0.09 & 0.08 & 0.08 & 0.07 & 0.08 \\
10^7 + 1 & * & 14.4 & 2.36 & 1.19 & 0.93 & 0.81 & 0.79 & 0.73 & 0.77 \\
10^8 + 1 & * & * & 27.9 & 13.1 & 9.75 & 8.32 & 7.97 & 7.25 & 7.54 \\
10^9 + 1 & * & * & * & 105. & 87.2 & 81.7 & 73.5 & 75.5 & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|cccccccccc|}
\hline
D \backslash k & 20 & 40 & 60 & 80 & 100 & 150 & 200 & 250 & 300 \\
\hline
10^5 + 1 & - & - & - & - & - & 0.06 & 0.08 & 0.11 & 0.15 \\
10^6 + 1 & 0.08 & 0.12 & 0.17 & 0.22 & 0.29 & 0.48 & 0.68 & 1.01 & 1.32 \\
10^7 + 1 & 0.77 & 1.09 & 1.62 & 2.01 & 2.66 & 4.48 & 6.29 & 9.23 & 12.2 \\
10^8 + 1 & 7.52 & 10.3 & 15.1 & 18.8 & 24.7 & 41.3 & 58.5 & 85.5 & 114. \\
10^9 + 1 & 75.2 & 99.8 & * & * & * & * & * & * & * \\
\hline
\end{array}
\]
4 Using the Approximate Functional Equation

All the methods that we have seen up to now take time proportional to the conductor $F$, the main difference between them being the dependence in $k$ and the size of the implicit constant in the time estimates.

We are now going to study a number of methods which take time proportional to $F^{1/2+\varepsilon}$ for any $\varepsilon > 0$. The simplest version of the approximate functional equation that we will use is as follows:

**Theorem 4.1** Let $e = 0$ or 1 be such that $\chi(-1) = (-1)^k = (-1)^e$. For any complex $s$ we have the following formula, valid for any $A > 0$:

$$\Gamma\left(\frac{s + e}{2}\right) L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \Gamma\left(\frac{s + e}{2}, \frac{A\pi n^2}{F}\right) + \varepsilon(\chi) \sum_{n \geq 1} \frac{n \chi(n)}{n^{1-s}} \Gamma\left(1 - \frac{s + e}{2}, \frac{\pi n^2}{AF}\right),$$

where the root number $\varepsilon(\chi)$ is given by the formula $\varepsilon(\chi) = \frac{g(\chi)}{(i^e \sqrt{F})}$, where $g(\chi)$ is the Gauss sum attached to $\chi$, and $\Gamma(s, x)$ is the incomplete gamma function $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$.

Since $\Gamma(s, x) \sim x^{s-1} e^{-x}$ hence tends to 0 exponentially fast as $x \to \infty$, the above formula does lead to a $O(F^{1/2})$ algorithm for computing $L(\chi, s)$, not necessarily for a negative integer $s$. Note that this type of formula is available for any type of $L$-function with functional equation, not only those attached to a Dirichlet character.

The constant $A$ is included as a check on the implementation, since the left-hand side is independent of $A$, but once checked the optimal choice is $A = 1$. This constant can also be used to compute $\varepsilon(\chi)$ if it is not known (note that $\varepsilon(\chi) = 1$ if $\chi$ is quadratic), but there are better methods to do this.

Even though this method is in $O(F^{1/2})$, so asymptotically much faster than the first two methods that we have seen, its main drawback is the computation time of $\Gamma(s, x)$. Even though quite efficient methods are known for computing it, our timings have shown that in all ranges of the conductor $F$ and value of $k$, either the use of the full functional equation or the use of Eisenstein series of half-integral weight (methods (2) and (5)) are considerably faster, so we will not discuss this method further.
5 Using Hecke–Eisenstein Series

5.1 The Main Theorem

The main theorem comes from the computation of the Fourier expansion of Hecke–Eisenstein series in the theory of Hilbert modular forms, and is easily proved using the methods of [3]:

Theorem 5.1 Let \( K \) be a real quadratic field of discriminant \( D > 0 \), let \( \psi \) be a primitive character modulo \( F \) such that \( \psi(-1) = (-1)^k \), let \( N \) be a squarefree integer, and assume that \( \gcd(F, ND) = 1 \). If we set

\[
a_{k,\psi,N}(0) = \prod_{p \mid N} (1 - \psi\chi_D(p)p^{k-1}) \frac{L(\psi, 1-k)L(\psi\chi_D, 1-k)}{4},
\]

and for \( n \geq 1 \):

\[
a_{k,\psi,N}(n) = \sum_{d|n, \gcd(d,N)=1} \psi\chi_D(d)d^{k-1} \sum_{s \in \mathbb{Z}} \sigma_{k-1,\psi} \left( \frac{(n/d)^2 D - s^2}{4N} \right),
\]

where \( \sigma_{k-1,\psi}(m) = \sum_{d|m} \psi(d)d^{k-1} \), then

\[
\sum_{n \geq 0} a_{k,\psi,N}(n)q^n \in M_{2k}(\Gamma_0(FN), \psi^2).
\]

Note that in the above we set implicitly \( \sigma_{k-1,\psi}(m) = 0 \) if \( m \notin \mathbb{Z}_{\geq 1} \).

The restriction \( \gcd(F, N) = 1 \) is not important, since letting \( N \) have factors in common with \( F \) would not give more general results. Similarly for the restriction on \( N \) being squarefree. On the other hand, the restriction \( \gcd(F, D) = 1 \) is more important: similar results exist when \( \gcd(F, D) > 1 \), but they are considerably more complicated. Since we need them, we will give one such result below in the case \( \gcd(F, D) = 4 \).

We use this theorem in the following way. First, we must assume for practical reasons that \( k, F, \) and \( N \) are not too large. In that case it is very easy to compute explicitly a basis for \( M_{2k}(\Gamma_0(FN), \psi^2) \). Given this basis, it is then easy to express any constant term of an element of the space as a linear combination of \( u \) coefficients (not necessarily the first ones), where \( u \) is the dimension of the space. In particular, this gives \( a_{k,\psi,N}(0) \), and hence \( L(\psi\chi_D, 1-k) \), as a finite linear combination of some \( a_{k,\psi,N}(n) \) for \( n \geq 1 \).

Second, since the conductor of \( \psi \) must be small, the method is thus applicable only to compute \( L(\chi, 1-k) \) for Dirichlet characters \( \chi \) which are “close” to quadratic characters, in other words of the form \( \psi\chi_D \) with conductor of \( \psi \) small. Of course the quantities \( L(\psi, 1-k) \) are computed once and for all (using any method, since \( F \) and \( k \) are small). Note that the auxiliary integer \( N \) is used only to improve the speed of the formulas, as we will see below, but of course one can always choose \( N = 1 \) if desired.
5.2 The Case $k$ Even

For future use, define

$$S_k(m, N) = \sum_{s \in \mathbb{Z}} \sigma_k \left( \frac{m - s^2}{N} \right),$$

where for any arithmetic function $f$ such as $\sigma_k$ we set $f(x) = 0$ if $x \notin \mathbb{Z}_{\geq 1}$, i.e., if $x$ is either not integral or non-positive. Using the theorem with $F = N = 1$ we immediately obtain formulas such as

$$L(\chi_D, -1) = -\frac{1}{5} S_1(D, 4)$$
$$L(\chi_D, -3) = S_3(D, 4)$$
$$L(\chi_D, -5) = -\frac{1}{195} \left(24 + 2^5 \left(\frac{D}{2}\right)\right) S_5(D, 4) + S_5(D, 1)$$

To obtain a general formula we recall the following theorem of Siegel:

**Theorem 5.2** Let $r = \dim(M_k(\Gamma))$ and define coefficients $c_i^k$ by

$$\Delta^{-r} E_{12r - k + 2} = \sum_{i \geq -r} c_i^k q^i,$$

where by convention $E_0 = 1$. Then for any $f = \sum_{n \geq 0} a(n) q^n \in M_k(\Gamma)$ we have

$$\sum_{0 \leq n \leq r} a(n) c_n^k = 0,$$

and $c_0^k \neq 0$.

Combined with the main theorem (with $F = N = 1$), we obtain the following corollary:

**Corollary 5.3** Keep the above notation, let $k \geq 2$ be an even integer, and set $r = \dim(M_{2k}(\Gamma)) = \lfloor k/6 \rfloor + 1$. If $D > 0$ is a fundamental discriminant we have

$$L(\chi_D, 1 - k) = \frac{4k}{c_0^{2k} B_k} \sum_{1 \leq m \leq r} S_{k-1}(m^2 D, 4) \sum_{1 \leq d \leq r/m} d^{k-1} \left(\frac{D}{d}\right) c_{-d m}^{2k}.$$

For very small values of $k$ it is possible to improve on the speed of the above general formula by choosing $F = 1$ but larger values of $N$ in the theorem. Without entering into details, on average we can gain a factor of 3.95 for $k = 2$, of 1.6 for $k = 6$, and of 1.1 for $k = 8$, and I have found essentially no improvement for other values of $k$ including for $k = 4$.

The advantages of this method are threefold. First, it is by far the fastest method seen up to now for computing $L(\chi_D, 1 - k)$. Second, the universal
coefficients $c_k^j$ that we need are easily computed thanks to Siegel’s theorem. And third, the flexibility of choosing the auxiliary Dirichlet character $\psi$ in the theorem allows us to compute $L(\chi, 1 - k)$ for more general characters $\chi$ than quadratic ones.

The two disadvantages are first that the quantities $S_{k-1}(m^2D, 4)$ need to be computed for each $m$ (although some duplication can be avoided), and second that $m^2D$ becomes large when $m$ increases. These two disadvantages will disappear in the method using Eisenstein series of half-integral weight (at the expense of losing some of the advantages mentioned above), so we will not give the timings for this method.

5.3 The Case $k$ Odd

Thanks to the main theorem, although Hilbert modular forms in two variables are only for real quadratic fields, thus with discriminant $D > 0$, if we choose an odd character $\psi$ such as $\psi = \chi_{-3}$ or $\chi_{-4}$, it can also be used to compute $L(\chi_{D}, 1 − k)$ for $D < 0$, hence $k$ odd. I have not been able to find useful formulas with $\psi = \chi_{-3}$, so from now on we assume that $\psi = \chi_{-4}$, so $F = 4$. We first introduce some notation.

**Definition 5.4** We set 

$$
\sigma_k^{(1)}(m) = \sum_{d|m} \left( \frac{-4}{d} \right) d^k, \quad \sigma_k^{(2)}(m) = \sum_{d|m} \left( \frac{-4}{m/d} \right) d^k, \quad \text{and}
$$

$$
S_k^{(j)}(m, N) = \sum_{s \in \mathbb{Z}} \sigma_k^{(j)} \left( \frac{m - s^2}{N} \right),
$$

with the usual understanding that $\sigma_k^{(j)}(m) = 0$ if $m \notin \mathbb{Z}_{\geq 1}$.

Note that, as for $\sigma_k$ itself when $k$ is odd, for $k$ even these arithmetic functions occur naturally as Fourier coefficients of Eisenstein series of weight $k + 1$ and character $(\frac{-4}{.})$. More precisely, for $k \geq 3$ odd the series $E_k(\chi_{-4}, 1)$ and $E_k(1, \chi_{-4})$ form a basis of the Eisenstein subspace of $M_k(\Gamma_0(4), \chi_{-4})$, where

$$
E_k(\chi_{-4}, 1)(\tau) = \frac{L(\chi_{-4}, 1 - k)}{2} + \sum_{n \geq 1} \sigma_{k-1}^{(1)}(n) q^n \quad \text{and}
$$

$$
E_k(1, \chi_{-4})(\tau) = \sum_{n \geq 1} \sigma_{k-1}^{(2)}(n) q^n.
$$

To be able to use the theorem in general, it is necessary to assume the following:

**Conjecture 5.5** If $D > 1$ is squarefree (not necessarily a discriminant), $F = 4$, and $N = 1$, the statement of Theorem 5.1 is still valid verbatim.
This is probably easy to prove, and I have checked it on thousands of examples. Assuming this conjecture, applying the theorem to $\psi = \chi - 4$ and the Hecke operator $T(2)$ it is immediate to prove the following:

**Corollary 5.6** Let $D < -4$ be any fundamental discriminant. Set

$$a_{k,D}(0) = \left(1 - 2^{k-1}\left(\frac{D}{2}\right)\right) \frac{L(\chi_{-4},1-k)L(\chi_D,1-k)}{4},$$

and

$$a_{k,D}(n) = \sum_{d|n} \left(\frac{4D/\delta}{d}\right)^{k-1} S_{k-1}^{(1)}((n/d)^2|D/\delta|,1),$$

where $\delta = 1$ if $D \equiv 1 \pmod{4}$ and $\delta = 4$ if $D \equiv 0 \pmod{4}$. Then $\sum_{n \geq 0} a_{k,D}(n)q^n \in M_{2k}(\Gamma_0(2))$.

To use this result, we need an analogue of Siegel’s Theorem 5.2 for $\Gamma_0(2)$, and for this we need to introduce a number of modular forms.

**Definition 5.7** We set $F_2(\tau) = 2E_2(2\tau) - E_2(\tau)$, $F_4(\tau) = (16E_4(2\tau) - E_4(\tau))/15$, and $\Delta_4(\tau) = (E_4(\tau) - E_4(2\tau))/240$, where $E_2$ and $E_4$ are the standard Eisenstein series of weight 2 and 4 on the full modular group.

Note that $F_2 \in M_2(\Gamma_0(2))$ and $F_4$ and $\Delta_4$ are in $M_4(\Gamma_0(2))$.

**Theorem 5.8** Let $k \in 2\mathbb{Z}$ be a positive even integer, set $r = \lfloor k/4 \rfloor + 2$, $E = F_2F_4$ if $k \equiv 0 \pmod{4}$, $E = F_4$ if $k \equiv 2 \pmod{4}$, and write $E/\Delta_4' = \sum_{i \geq -r} c_i^k q^i$. Then for any $F = \sum_{n \geq 0} a(n)q^n \in M_k(\Gamma_0(2))$ we have

$$\sum_{0 \leq n \leq r} a(n)c_n^k = 0,$$

and in addition $c_0^k \neq 0$.

Note that since we will use this theorem for $M_{2k}(\Gamma_0(2))$ with $k$ odd, we have $2k \equiv 2 \pmod{4}$, so we will always use $E = F_4$. The analogue of Corollary 5.3 is then as follows:

**Corollary 5.9** Keep the above notation, let $k \geq 3$ be an odd integer, and set $r = (k+3)/2$. If $D < -4$ is a fundamental discriminant we have

$$L(\chi_D,1-k) = \frac{8}{A} \sum_{1 \leq m \leq r} S_{k-1}^{(1)}(m^2|D/\delta|,1) \sum_{1 \leq d \leq r/m} d^{k-1} \left(\frac{4D/\delta}{d}\right)^{2k},$$

with $A = c_0^k (2^{k-1}(\frac{D}{2}) - 1) E_{k-1},$ and where the $E_k$ are the Euler numbers ($E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$).

The advantages/disadvantages mentioned in the case $k$ even are the same here.
6 Using Eisenstein Series of Half-Integral Weight

We now come to the most powerful method known to the author for computing \( L(\chi_D, 1 - k) \): the use of Eisenstein series of half-integral weight. Once again, we will see a sharp distinction between \( k \) even and \( k \) odd. We first begin by recalling some basic results on \( M_w(\Gamma_0(4)) \) (we use the index \( w \) for the weight since it will be used with \( w = k + 1/2 \)). Later, we will see that it is more efficient to use modular forms on subgroups of \( \Gamma_0(4) \).

6.1 Basic Results on \( M_w(\Gamma_0(4)) \)

Recall that the basic theta function \( \theta(\tau) = \sum_{s \in \mathbb{Z}} q^{s^2} = 1 + 2 \sum_{s \geq 1} q^{s^2} \) satisfies for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \) the modularity condition \( \theta(\gamma(\tau)) = v_\theta(\gamma)(c\tau + d)^{1/2} \theta(\tau) \), where the theta-multiplier system \( v_\theta(\gamma) \) is given by

\[
v_\theta(\gamma) = \left( -\frac{4}{d} \right)^{-1/2} \left( \frac{c}{d} \right),
\]

and all square roots are taken with the principal determination. The space \( M_w(\Gamma_0(4), v_\theta^2) \) of holomorphic functions behaving modularly like \( \theta \) under \( \Gamma_0(4) \) and holomorphic at the cusps will be simply denoted \( M_w(\Gamma_0(4)) \) since there is no risk of confusion. Note, however, that if \( w \) is an odd integer and in the context of modular forms of integral weight, \( M_w(\Gamma_0(4)) \) is denoted \( M_w(\Gamma_0(4), \chi_{-4}) \).

We recall the following easy and well-known results (note that \( F_2 \) and \( \Delta_4 \) are not the same functions as those used above):

**Proposition 6.1** Define

\[
F_2(\tau) = \frac{\eta(4\tau)^8}{\eta(2\tau)^4} = -\frac{1}{24}(E_2(\tau) - 3E_2(2\tau) + 2E_2(4\tau)),
\]

\[
\Delta_4(\tau) = \frac{\eta(\tau)^8\eta(4\tau)^8}{\eta(2\tau)^8} = \frac{1}{240}(E_4(\tau) - 17E_4(2\tau) + 16E_4(4\tau)).
\]

(1) We have

\[
\bigoplus_w M_w(\Gamma_0(4)) = \mathbb{C}[\theta, F_2] \quad \text{and} \quad \bigoplus_w S_w(\Gamma_0(4)) = \theta \Delta_4 \mathbb{C}[\theta, F_2].
\]

(2) In particular we have the dimension formulas

\[
\dim(M_w(\Gamma_0(4))) = \begin{cases} 0 & \text{for } w < 0 \\ 1 + \lfloor w/2 \rfloor & \text{for } w \geq 0. \end{cases}
\]

\[
\dim(S_w(\Gamma_0(4))) = \begin{cases} 0 & \text{for } w \leq 4 \\ \lfloor w/2 \rfloor - 1 & \text{for } w > 4, w \notin 2\mathbb{Z} \\ \lfloor w/2 \rfloor - 2 & \text{for } w > 4, w \in 2\mathbb{Z}. \end{cases}
\]
We also recall that when $w \in 1/2 + \mathbb{Z}$, the Kohnen +–space of $M_w(\Gamma_0(4))$, denoted simply by $M_w^+$, is defined to be the space of forms $F$ having a Fourier expansion $F(\tau) = \sum_{n \geq 0} a(n)q^n$ with $a(n) = 0$ if $(-1)^{w-1/2}n \not\equiv 0, 1 \pmod{4}$. Note that we include Eisenstein series. It is clear that $M_{1/2}^+ = M_{1/2}(\Gamma_0(4))$ and $M_{3/2}^+ = \{0\}$, so we will always assume that $w \geq 5/2$. In that case there is a single Eisenstein series in $M_w^+$, due to the author, that we will denote by $H_k$: its importance is due to the fact that if we write $H_k(\tau) = \sum_{n \geq 0} a_k(n)q^n$, then if $D = (-1)^{w-1/2}n$ is a fundamental discriminant we have $a_k(n) = L(\chi_D, 1 - (w-1/2))$, so being able to compute efficiently the Fourier coefficients of $H_k$ automatically gives us a fast method for computing our desired quantities $L(\chi_D, 1 - k)$ with $k = w - 1/2$.

The remaining part of $M_w^+$, which we of course denote by $S_w^+$, is formed by the cusp forms belonging to $M_w^+$. One of Kohnen’s main theorems is that $S_w^+$ is Hecke-isomorphic to the space of modular forms of even weight $S_{2w-1}(\Gamma)$. In particular, note the following:

**Corollary 6.2** For $w \geq 5/2$ half-integral we have

$$\dim(M_w^+) = \begin{cases} 1 + \lfloor w/6 \rfloor & \text{if } 6 \nmid (w - 3/2), \\ \lfloor w/6 \rfloor & \text{if } 6 \mid (w - 3/2). \end{cases}$$

Notation:

1. Recall that if $a(n)$ is any arithmetic function (typically $a = \sigma_{k-1}$ or twisted variants), we define $a(x) = 0$ if $x \not\in \mathbb{Z}_{\geq 1}$.
2. If $F$ is a modular form and $d \in \mathbb{Z}_{\geq 1}$, we denote by $F[d]$ the function $F(d\tau)$.
3. We will denote by $D(F)$ the differential operator $qd/dq = (1/(2\pi i))d/d\tau$.

### 6.2 The Case $k$ Even using $\Gamma_0(4)$

The main idea is to use Rankin–Cohen brackets of known series with $\theta$ to generate $M_w^+$: indeed, $\theta$ and its derivatives are lacunary, so multiplication by them is much faster than ordinary multiplication, at least in reasonable ranges (otherwise Karatsuba or FFT type methods are faster to construct tables).

First note the following immediate result:

**Proposition 6.3** The form $\theta E_2[4] - 6D(\theta)$ is a basis of $M_{5/2}^+$ and the form $\theta E_4[4]$ is a basis of $M_{9/2}^+$.

In particular, we recover the formulas $L(\chi_D, -1) = (-1/5)S_1(D, 4)$ and $L(\chi_D, -3) = S_3(D, 4)$ already obtained using Hecke–Eisenstein series.

As in the case of Hecke–Eisenstein series, we will need to distinguish two completely different cases: the case $w - 1/2$ even, which is considerably simpler, and the case $w - 1/2$ odd, which is more complicated and less efficient. The reason for this sharp distinction is the following theorem:
Theorem 6.4 Assume that $w \geq 9/2$ is such that $k = w - 1/2 \in 2\mathbb{Z}$. The modular forms

$$([\theta, E_{k-2j}[4]]_j)_{0 \leq j \leq \lfloor k/6 \rfloor}$$

form a basis of $M^+_w$, where we recall that $[f, g]_n$ denotes the $n$th Rankin–Cohen bracket.

We can now easily achieve our goal. First, we compute the Fourier expansions of the basis given by the theorem up to the Sturm bound. Then to compute $L(\chi_D, 1 - k)$ with $k = w - 1/2$, we do as follows: we compute the Fourier expansion of $H_k$ up to the Sturm bound, and using the basis coefficients we deduce a linear combination of the form

$$H_k = \sum_{0 \leq j \leq \lfloor k/6 \rfloor} c^k_j [\theta, E_{k-2j}[4]]_j.$$ 

We can easily compute the coefficients of these brackets:

Proposition 6.5 Let $F_r = -B_r E_r/(2r)$ be the Eisenstein series of level 1 and weight $r$ normalized so that the Fourier coefficient $q^1$ is equal to 1. We have $[\theta, F_r[4]]_n = \sum_{m \geq 0} b_{n,r}(m)q^m$, with

$$b_{n,r}(m) = m^n \sum_{s \in \mathbb{Z}} \mathcal{P}_n(r)(s^2/m)\sigma_{r-1}\left(\frac{m - s^2}{4}\right),$$

where

$$\mathcal{P}_n(r)(X) = \sum_{\ell = 0}^n (-1)^\ell \begin{pmatrix} n - 1/2 \\ \ell \end{pmatrix} \begin{pmatrix} 2n + r - \ell - 3/2 \\ n - \ell \end{pmatrix} X^{n - \ell},$$

are Gegenbauer polynomials.

In particular, if we generalize a previous notation and set for any polynomial $P_n$ of degree $n$

$$S_k(m, N, P_n) = m^n \sum_{s \in \mathbb{Z}} \mathcal{P}_n(r)(s^2/m)\sigma_k\left(\frac{m - s^2}{N}\right),$$

we have

$$L(\chi_D, 1 - k) = \sum_{0 \leq j \leq \lfloor k/6 \rfloor} c^k_j S_{k-2j-1}(D, 4, P_{j,k-2j}).$$

The biggest advantage of this formula compared to the one coming from Hecke–Eisenstein series is that the different $S_{k-2j-1}$ can be computed simultaneously since they involve factoring the same integers $(D - s^2)/4$, and in addition these integers stay small, contrary to the former method where the integers were of the form $(n^2D - s^2)/4$.

The two disadvantages are that first, it is not easy (although possible) to generalize to general characters $\chi$, but mainly because for large $k$ the computation of $c^k_j$ involves solving a linear system of size proportional to $k$, so when $k$ is...
in the thousands, this becomes prohibitive. It is possible that there is a much faster method to compute them analogous to Siegel’s theorem which expresses the constant term of a modular form as a universal (for a given weight) linear combination of higher degree terms, but I do not know of such a method.

As already mentioned, this gives the fastest method known to the author for computing $L(\chi_D, 1 - k)$, at least when $k$ is not unreasonably large.

6.3 The Case $k$ Even using $\Gamma_0(4N)$

We can, however, do better by using subgroups of $\Gamma_0(4)$, i.e., brackets with $E_{k-2j}[4N]$ for $N > 1$. Recall that in the case of Hecke–Eisenstein series this allowed us to give faster formulas only for very small values of $k$ ($k = 2, 6$ and $8$). On the contrary, we are going to see that here we can obtain faster formulas for all $k$, only depending on congruence and divisibility properties of the discriminant $D$.

After considerable experimenting, I have arrived at the following conjecture, which I have tested on tens of thousands of cases and proved in small weights. All of these identities can in principle be proved.

**Conjecture 6.6** For $N = 4, 8, 12$ and $16$ and any even integer $k \geq 2$ there exist universal coefficients $c_{j,N}^k$ such that for all positive fundamental discriminants $D$ (which in addition must be congruent to $1$ modulo $8$ when $N = 16$) we have

$$
\left(1 + \left(\frac{D}{N/4}\right)\right) L(\chi_D, 1 - k) = \sum_{0 \leq j \leq \lfloor k/m_N \rfloor} c_{j,N}^k S_{k-2j-1}(D, N, P_{j,k-2j}),
$$

with $m_N = 6, 4, 3, 4$ respectively and the same polynomials $P$ as above.

By what we said above this conjecture is proved for $N = 4$ (with $c_{j,4}^k = 2 c_{j}^k$), and should be easy to prove using the finite-dimensionality of the corresponding modular form spaces together with the Sturm bounds, but I have not done these proofs. It is also easy to prove for small values of $k$.

It is clear that if we can choose a larger value of $N$ than $N = 4$ (i.e., when $1 + \left(\frac{D}{N/4}\right) \neq 0$) the number of terms involved in $S_{k-2j-1}$ will be smaller. Computing that number leads to the following algorithm:

If $3 \mid D$ use $N = 12$, otherwise if $D \equiv 1 \pmod{8}$ use $N = 16$, otherwise if $4 \mid D$ use $N = 8$, otherwise if $D \equiv 1 \pmod{3}$ use $N = 12$, otherwise use $N = 4$.

Note, however, that the size of the linear system which needs to be solved to find the coefficients $c_{j,N}^k$ is larger when $N > 4$, so one must balance the time to compute these coefficients with the size of $D$: for very large $D$ it may be worthwhile, but for moderately large $D$ it may be better to always choose $N = 4$ (see the second table below).

As before, we give tables of timings using these improvements. Note that they are only an indication, since congruences modulo 16 and 3 may improve the times.
In the next table, we use the improvements for larger \( N \) only when \( D \) is sufficiently large, and the corresponding timings have a *; all the others are obtained only with \( N = 4 \):

| \( D \setminus k \) | 20   | 40   | 60   | 80   | 100  | 150  | 200  | 250  | 300  | 350  |
|-------------------|------|------|------|------|------|------|------|------|------|------|
| \( 10^6 + 1 \)    | -    | -    | -    | -    | -    | 0.07 | 0.14 | 0.29 | 0.51 |
| \( 10^7 + 1 \)    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    |
| \( 10^8 + 1 \)    | -    | -    | -    | 0.06 | 0.10 | 0.25 | 0.50 | 0.89 | 1.50 | 2.28 |
| \( 10^9 + 1 \)    | 0.06 | 0.12 | 0.19 | 0.30 | 0.74 | 1.59 | 2.85 | 4.96 | 7.56 |
| \( 10^{10} + 1 \) | 0.12 | 0.23 | 0.39 | 0.64 | 1.00 | 2.48 | 5.20 | 9.08 | 15.0 | 22.4 |
| \( 10^{11} + 1 \) | 0.49 | 0.86 | 1.47 | 2.37 | 3.67 | 9.09 | 18.9 | 32.8 | 52.8 | 77.2 |
| \( 10^{12} + 1 \) | 2.84 | 4.04 | 6.01 | 9.04 | 13.4 | 31.8 | 64.6 | *    | *    | *    |
| \( 10^{13} + 1 \) | 12.3 | 16.5 | 23.4 | 34.2 | 49.9 | *    | *    | *    | *    | *    |
| \( 10^{14} + 1 \) | 60.8 | 74.8 | 98.8 | *    | *    | *    | *    | *    | *    | *    |

For larger values of \( k \) the time to compute the coefficients dominate, so we first give a table giving these timings:

| \( N \setminus k \) | 100  | 200  | 300  | 400  | 500  | 600  | 700  | 800  | 900  | 1000 |
|---------------------|------|------|------|------|------|------|------|------|------|------|
| 4                   | -    | 0.04 | 0.20 | 0.69 | 1.95 | 4.04 | 7.54 | 13.3 | 22.4 | 34.4 |
| 8                   | -    | 0.17 | 0.87 | 2.77 | 6.95 | 14.7 | 28.4 | 49.2 | 83.5 | *    |
| 12                  | -    | 0.32 | 1.90 | 5.77 | 14.5 | 32.0 | 61.6 | *    | *    | *    |
| 16                  | -    | 0.20 | 1.13 | 3.64 | 9.59 | 20.5 | 31.4 | 53.4 | 89.8 | *    |

As already mentioned, these timings would become much smaller if we had a method analogous to Siegel’s theorem to compute them.
6.4 The Case $k$ Odd using $\Gamma_0(4N)$

In this case, the Kohnen $+$-space, to which $H_k$ belongs, is the space of modular forms $\sum_{n \geq 0} a(n)q^n$ such that $a(n) = 0$ if $n \equiv 1$ or 2 modulo 4. Thus, we cannot hope to directly obtain elements in this space using brackets with $\theta$. What we can do is the following: as above, for $\ell \geq 1$ odd consider the two Eisenstein series

$$E^{(1)}_\ell := E_\ell(\chi_{-4}, 1)(\tau) = \frac{L(\chi_{-4}, 1 - \ell)}{2} + \sum_{n \geq 1} \sigma^{(1)}_{\ell-1}(n)q^n$$

and

$$E^{(2)}_\ell := E_\ell(1, \chi_{-4})(\tau) = \sum_{n \geq 1} \sigma^{(2)}_{\ell-1}(n)q^n,$$

which belong to $M_\ell(\Gamma_0(4))$ (using our notation, otherwise we should write $M_\ell(\Gamma_0(4), \chi_{-4})$). It is clear that for $u = 1$ and 2 the $j$-th brackets $[\theta, E^{(u)}_{\ell - 2j}]_j$ belong to $M_{k+1/2}(\Gamma_0(4))$, and it should be easy to prove that they generate this space (I have extensively tested this, and if it was not the case the implementation would detect it). We can therefore express any modular form, in particular $H_k$, as a linear combination of these brackets, and therefore again obtain explicit formulas for $L(\chi_D, 1 - k)$.

However, we can immediately do considerably better in two different ways. First, by Shimura theory we know that $T(4)H_k$ still belongs to $M_{k+1/2}(\Gamma_0(4))$, and by definition it is equal to $\sum_{n \geq 0} H_k(4n)q^n$. Expressing it as a linear combination of the above brackets again gives formulas for $L(\chi_D, 1 - k)$, but where the coefficients involve $|D|/4$ instead of $|D|$, so much faster (and of course applicable only for $D \equiv 0$ (mod 4)). Note that this trick is not applicable in the case of even $k$ because $T(4)H_k$ is not anymore in the Kohnen $+$-space, so we would lose all the advantages of having a space of small dimension.

The second way in which we can do better is to consider brackets of $\theta$ with $E^{(u)}_{\ell}[N]$ (where we replace $q^n$ by $q^{Nn}$) for suitable values of $N$: note that these modular forms belong to $M_{k+1/2}(\Gamma_0(4N))$. It is then necessary to apply a Hecke-type operator to reduce the dimension of the spaces that we consider. More precisely, if we only look at coefficients $a(|D|)$ with given $(\frac{D}{2})$, we see experimentally that there is a linear relation between $H_k$ and the above brackets. This leads to the following analogue for $k$ odd of Conjecture 6.6, where generalizing the notation $S^{(j)}_{k}(m, N)$ used above for $j = 1$ and 2 we also use

$$S^{(j)}_{k}(m, N, P_n) = m^n \sum_{s \in \mathbb{Z}} P_n(s^2/m)\sigma^{(j)}_{\ell-1}(m - s^2)N,$$

where $P_n$ is a polynomial of degree $n$.

**Conjecture 6.7** For $N = 1, 2, 3, 5, 6, 7, 8$, any odd integer $k \geq 3$, and $e \in \{-1, 0, 1\}$, there exist universal coefficients $c^{k}_{j,N,e}$ such that for all negative fundamental discriminants $D$ such that $(\frac{D}{2}) = e$ we have

$$\left(1 + \left(\frac{|D|}{N_2}\right)\right)L(\chi_D, 1 - k) = \sum_{0 \leq j \leq m(k,N,e)} c^{k}_{j,N,e} S^{(1+j_0)}_{k-j_1-1}(|D|/\delta, N, P_{j_1}, k-j_1),$$

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where $N_2 = N/2$ if $N$ is even and $N_2 = N$ if $N$ is odd, $\delta = 4$ if $4 \mid D$ and $\delta = 1$ otherwise, we write $j = 2j_1 + j_0$ with $j_0 \in \{0, 1\}$, upper bounds for $m(k, N, e)$ will be given below, and where we must assume $e \neq -1$ if $N = 6$ and on the contrary $e = -1$ if $N = 7$.

Upper bounds for $m(k, N, e)$ are given for $e = -1$, 0, and 1 as follows: $((k - 1)/4, (k - 1)/3, (k - 3)/4)$ for $N = 1$, $((k - 1)/4, (k - 1)/2, (k - 3)/4)$ for $N = 2$, $((k - 1)/2, (2k - 1)/3, (k - 1)/2)$ for $N = 3$, $((3k - 2)/4, k - 1, (3k - 5)/4)$ for $N = 5$, $(*, k - 1, k - 1)$ for $N = 6$, and $(k - 1, *, *)$ for $N = 7$, where $*$ denotes impossible cases.

For concreteness, we give the special case $k = 3$, $e = 1$: if $D \equiv 1 \pmod{8}$ is a negative fundamental discriminant, we have

$$L(\chi_D, -2) = \frac{1}{35} S_2^{(1)}(|D|, 1) = \frac{1}{7} S_2^{(1)}(|D|, 2) ,$$

$$(1 - \left(\frac{D}{2}\right))L(\chi_D, -2) = -\frac{2}{63} (S_2^{(1)}(|D|, 3) + 14 S_2^{(2)}(|D|, 3)) ,$$

$$(1 + \left(\frac{D}{2}\right))L(\chi_D, -2) = -\frac{2}{3} (S_2^{(1)}(|D|, 5) + 4 S_2^{(2)}(|D|, 5)) ,$$

$$(1 - \left(\frac{D}{4}\right))L(\chi_D, -2) = \frac{1}{14} (-52 S_2^{(1)}(|D|, 6) + 5 S_2^{(1)}(|D|, 6, 1 - 3x)) .$$

Similarly to the case of even $k$, computing the number of terms involved in the sums leads to the following algorithm:

1. When $D \equiv 0 \pmod{4}$: if $3 \mid D$ use $N = 6$, otherwise if $5 \mid D$ use $N = 5$, otherwise if $D \equiv 2 \pmod{3}$ use $N = 6$, otherwise if $D \equiv \pm 1 \pmod{5}$ use $N = 5$, otherwise use $N = 2$.

2. When $D \equiv 1 \pmod{4}$: if $7 \mid D$ and $D \equiv 5 \pmod{8}$ use $N = 7$, otherwise if $3 \mid D$ and $D \equiv 1 \pmod{8}$ use $N = 6$, otherwise if $5 \mid D$ use $N = 5$, otherwise if $D \equiv 5 \pmod{8}$ and $D \equiv 3, 4, 6 \pmod{7}$ use $N = 7$, otherwise if $D \equiv 2 \pmod{3}$ and $D \equiv 1 \pmod{8}$ use $N = 6$, otherwise if $3 \mid D$ (hence $D \equiv 5 \pmod{8}$) use $N = 3$, otherwise if $D \equiv \pm 1 \pmod{5}$ use $N = 5$, otherwise use $N = 2$.

As in the case of $k$ even, care must be taken in choosing $N > 1$ since the size of the linear system to be solved in order to compute the universal coefficients $c^k_{j, N, e}$ is larger, so the above algorithm is valid only if this time is negligible.

We thus give a table of timings using this algorithm. Note that $-10^4 - 3$ is usually (but not always) slower than $-10^4 - 4$ since in the latter case the sums involve $|D|/4$ instead of $|D|$, and that a lot depends on divisibilities by 3, 5, and 7, so the tables are only an indication:
Note that we have included the case $k = 1$ which will be discussed below.

As we have done in the case $k$ even, for larger values of $k$ the time to compute the coefficients dominate, so we first give a table giving these timings:

| $(\frac{D}{N}, N) \backslash k$ | 81 | 101 | 151 | 201 | 251 | 301 | 351 | 401 | 451 | 501 |
|-----------------------------|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $(1,1)$                     | -  | -   | 0.07| 0.20| 0.53| 1.22| 2.28| 3.76| 6.15| 9.37|
| $(1,2)$                     | -  | -   | 0.16| 0.48| 1.23| 3.00| 5.50| 8.83| 14.3| 21.3|
| $(1,3)$                     | -  | 0.10| 0.58| 1.73| 4.39| 9.31| 17.5| 30.1| 50.4| 80.6|
| $(1,5)$                     | 0.21| 0.57| 2.57| 7.43| 18.4| 37.3| 70.8| *   | *   | *   |
| $(1,6)$                     | -  | 0.07| 0.38| 1.44| 3.90| 10.2| 18.7| 39.4| 71.7| *   |
| $(-1,1)$                    | -  | -   | 0.07| 0.22| 0.54| 1.27| 2.30| 3.90| 6.33| 9.70|
| $(-1,2)$                    | -  | -   | 0.16| 0.51| 1.23| 3.11| 5.51| 9.15| 14.4| 22.0|
| $(-1,3)$                    | -  | 0.11| 0.62| 1.82| 4.59| 9.69| 18.1| 31.4| 52.5| 64.1|
| $(-1,5)$                    | 0.13| 0.34| 1.53| 4.95| 10.1| 20.4| 38.5| 67.2| 110. | *   |
| $(-1,7)$                    | 0.28| 0.61| 2.73| 8.15| 20.4| 41.6| 77.4| *   | *   | *   |
| $(0,1)$                     | -  | -   | 0.12| 0.39| 1.05| 1.91| 3.34| 5.80| 9.33| 14.1|
| $(0,2)$                     | -  | 0.07| 0.40| 1.15| 2.67| 5.76| 10.6| 19.6| 29.7| 52.8|
| $(0,3)$                     | 0.08| 0.18| 0.95| 2.82| 6.80| 14.6| 27.2| 50.6| 77.1| *   |
| $(0,5)$                     | 0.25| 0.53| 2.28| 6.78| 16.0| 33.2| 60.7| 105. | *   | *   |
| $(0,6)$                     | 0.29| 0.58| 2.64| 7.66| 19.0| 41.3| 75.6| 110. | *   | *   |

For future reference, we observe that the times are very roughly

$$10^{-10} k^4(1.5, 3.6, 12, 46.8, 12.5) \text{ for } e = 1,$$
$$10^{-10} k^4(1.5, 3.6, 12, 25.4, 51) \text{ for } e = -1, \text{ and}$$
$$10^{-10} k^4(2.2, 7, 18, 40, 50) \text{ for } e = 0,$$

where as usual $e = (\frac{D}{N})$.

In the next table we use $N = 2$ only when $|D|$ is sufficiently large, and the corresponding timings have a *; all the other timings are obtained with $N = 1$.  

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7 The Case $k = 1$

In this brief section, we consider the case $k = 1$, i.e., the problem of computing $L(\chi, 0)$ for an odd character $\chi$. Of course the Bernoulli method as well as the approximate functional equation are still applicable in general. But in the case $\chi = \chi_D$ with $D < 0$ there are still methods coming from modular forms. Note that in that case for $D < -4$ we have $L(\chi_D, 0) = h(D)$ which can therefore be computed using subexponential algorithms, but it is still interesting to look at modular-type formulas. Note that $\mathcal{H}_1$ is not quite but almost a modular form of weight $3/2$, so it is not surprising that the method given above also works for $k = 1$.

For instance, we have the following result, where we refer to Definition 5.4 for the definition of $S_0^{(1)}$ (note that $S_0^{(2)} = S_0^{(1)}$):

**Proposition 7.1** Let $D$ be a negative fundamental discriminant $D$.

1. Set $e = \left(\frac{D}{\chi}\right)$. We have

\[
\frac{S_0^{(1)}([D], N)}{L(\chi_D, 0)} = \begin{cases} 
3(1 - e) & \text{when } N = 1 \text{ and } N = 2, \\
(1 - \left(\frac{D}{\chi}\right))(5 - e)/2 & \text{when } N = 3, \\
(1 + \left(\frac{D}{\chi}\right))(1 - e)/2 & \text{when } N = 5, \\
(1 - \left(\frac{D}{\chi}\right))(1 + e)/2 & \text{when } N = 6, \\
(1 - \left(\frac{D}{\chi}\right)) & \text{when } N = 7 \text{ and } e = -1.
\end{cases}
\]
(2) If $4 \mid D$, we also have

$$S^{(1)}_n(|D|/4, N) = \begin{cases} 3 & \text{when } N = 1, \\ 1 & \text{when } N = 2, \\ (1 - \left(\frac{D}{3}\right))/2 & \text{when } N = 3 \text{ and } N = 6, \\ (1 + \left(\frac{D}{5}\right))/2 & \text{when } N = 5. \end{cases}$$

In particular, Conjecture 6.7 is valid for $k = 1$ with $m(1, N, e) = 0$, $c^1_{0,N,e} = 2/(3(1 - e))$, $2/(3(1 - e))$, $2/(5 - e)$, $2/(1 - e)$, $2/(1 + e)$, and 1 when $\delta = 1$ for $N = 1, 2, 3, 5, 6,$ and 7 respectively, and $c^1_{0,N,0} = 2/3, 2, 2, 2,$ and 2 when $\delta = 4$ and $N = 1, 2, 3, 5,$ and 6 respectively.

Since we can efficiently compute $L(\chi_D, 0)$ by using class numbers this result has no computational advantage, but is simply given to show that the formulas that we obtained above for $k \geq 3$ odd have analogs for $k = 1$.

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