On the quadratic Morse-Smale endomorphisms

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Abstract. For the quadratic map $F_\mu = (xy, (x - \mu)^2)$ we indicate an interval of parameter values, such that map $F_\mu$ with a parameter value in the indicated interval is a (nonsingular) Morse-Smale endomorphism.

1. Introduction
The bibliography of investigation of polynomial maps, and in particular, of quadratic maps is very extensive (see, for example, [1] – [10]). In this paper we consider the quadratic maps on the plane from the one-parameter family of the type

$$F_\mu(x, y) = (xy, (x - \mu)^2), \quad (1)$$

where $(x, y)$ is an arbitrary point of the plane $xOy$ and $\mu \in (0, 1]$. Different aspects of dynamics of the maps from one-parameter family (1) are investigated in [7] – [10]. This paper is a continuation of [7] – [10].

We prove here, that the map $F_\mu$ is a (nonsingular) Morse-Smale endomorphism for any $\mu \in (0, 1)$ and the map $F_1$ is a singular Morse-Smale endomorphism.

We will extend the definition of a (nonsingular) Morse-Smale endomorphism acting on a compact manifold [11] to the case of maps of a plane.

Nonsingular Morse-Smale flows were considered in [12]. Let us define a nonsingular Morse-Smale endomorphism by analogy with [12].

In the case of endomorphisms, the global stable manifold of a point is not necessarily connected. Therefore, in Definition 1 we use the local stable manifold $W^s_{loc}(p)$ of a $p$; we understand it as a connected component of the global stable manifold $W^s_p$ containing the point $p$ [13].

Definition 1. An endomorphism $F : R^2 \to R^2$ is called a (nonsingular) Morse-Smale endomorphism, if

(i) the nonwandering set $\Omega(F)$ is finite and consists of hyperbolic periodic points;
(ii) the local stable manifold $W^s_{loc}(p)$ and the global unstable manifold $W^u(q)$ of different periodic points $p$, $q$ intersect transversally, i.e., if $(x, y) \in W^s_{loc}(p) \cap W^u(q)$, then $T_{(x,y)} = T_{(x,y)} W^s_{loc}(p) \oplus T_{(x,y)} W^u(q)$ holds.

Definition 2. An endomorphism $F : R^2 \to R^2$ with a finite nonwandering set $\Omega(F)$ is called a singular Morse-Smale endomorphism, if at least one of the following conditions is fulfilled:
(i) the nonwandering set $\Omega(F)$ contains a nonhyperbolic periodic point;
(ii) there exist periodic points $p', q' \in \Omega(F)$ such that $W^u_{loc}(p')$ and $W^s(q')$ intersect nontransversally.

The main result of the paper work is the following.

**Theorem 1.** Let $F_\mu$ be a map of type (1). Then for each $\mu \in (0, 1)$ the map $F_\mu$ is a (nonsingular) Morse-Smale endomorphism, such that its nonwandering set $\Omega(F_\mu)$ consists of three fixed points (the sink $A_1(0; \mu^2)$, the source $A_2(\mu+1; 1)$ and the saddle point $A_3(\mu-1; 1)$) and periodic orbit $B$ of period two, which is formed by two sources $B_1(2 - \sqrt{2}; \frac{1}{1 + 2\sqrt{2}})$, $B_2(2 + \sqrt{2}; \frac{1}{1 + 2\sqrt{2}})$.

The map $F_1$ is a singular Morse-Smale endomorphism, such that its nonwandering set $\Omega(F_1)$ consists of two fixed points (the nonhyperbolic point $A_1(0; 1)$, the source $A_2(2; 1)$) and the periodic orbit $B$ of period two, which is formed by two sources $B_1(2 - \sqrt{2}; \frac{1}{1 + 2\sqrt{2}})$, $B_2(2 + \sqrt{2}; \frac{1}{1 + 2\sqrt{2}})$.

2. The main technical results

In this part of the paper we give the preliminary information that will be used.

We begin this section with a formulation of the main properties of map $F_\mu$.

Let us $F_\mu^n$ is the $n$-th iteration of map (1) and $f_{\mu,n}$ and $g_{\mu,n}$ are the first and the second coordinate functions of $F_\mu^n$ respectively, i.e. $F_\mu^n(x, y) = (f_{\mu,n}(x, y), g_{\mu,n}(x, y))$ for any $n \geq 1$.

Denote by $J(F_\mu(x, y)) = \frac{\partial F_\mu^n(x, y)}{\partial (x, y)}$ and $J(F_\mu^n(x, y))$ Jacobian matrix and Jacobian of $F_\mu^n$ respectively.

**Proposition 1.** Map (1) for every $\mu \in (0, 1]$ possesses the properties:

(i) The equality $J(F_\mu^n(x, y)) = 0$ ($n \geq 1$) holds off coordinates of a point $(x; y)$ satisfy one of the equations $f_{\mu,i}(x, y) = 0$ or $f_{\mu,i}(x, y) = \mu$, for some $0 \leq i \leq n - 1$; in particular, the critical set of the map $F_\mu$ consists of two straight lines $C_0$: $x = 0$ and $C_\mu$: $x = \mu$.

(ii) The following inclusions hold for every open quadrant $K_i$ ($i = 1, 2, 3, 4$) of the plane $xOy$: $F_\mu^n(K_1) \subset K_1$, $F_\mu^n(K_4) \subset K_2$, $F_\mu^n(K_2) \subset K_3$, $F_\mu^n(K_1) \subset K_1$, where $(\cdot)$ means the closure of a set.

(iii) Triangle $\Delta_\mu = \{(x, y) \in (x, x, y) : x, y \geq 0, \frac{x}{2\mu} + \frac{y}{\mu^2} \leq 1\}$ is $F_\mu$ - invariant set such that its legs $k_{\mu,x} = \{(x, 0) : 0 \leq x \leq 2\mu\}$ and $k_{\mu,y} = \{(0; y) : 0 \leq y \leq \mu^2\}$ satisfy the equalities $F_\mu(k_{\mu,x}) = k_{\mu,y}$, $F_\mu(k_{\mu,y}) = (0; \mu^2)$; in addition the restriction $F_\mu|_{h_\mu}$ on its hypotenuse $h_\mu$ is defined by the equality $F_\mu|_{h_\mu}(x, y) = (x\mu^2(x, x, y))^2; (\mu - \frac{2y}{\mu^2})^2$ such that the image $F_\mu(h_\mu)$ of hypotenuse $h_\mu$ is the closed interval of the line $2x + \mu y = \mu^3$, which lies in the triangle $\Delta_\mu$.

(iv) There are three fixed points $A_1(0; \mu^2)$, $A_2(\mu + 1; 1)$ and $A_3(\mu - 1; 1)$, such that $A_1(0; \mu^2)$ is a sink for any $\mu \in (0, 1)$ and is a non-hyperbolic point when $\mu = 1$, $A_2(\mu + 1; 1)$ is a source for any $\mu \in (0, 1]$ and point $A_3(\mu - 1; 1)$ is a saddle point for any $\mu \in (0, 1]$.

1 We say that set $A$, $A \subset R^2$ is $F_\mu$ - invariant set if the inclusion $F_\mu(A) \subset A$ holds.
(v) There exists a unique periodic orbit $B$ with the least period two, which formed by two sources $B_1(\frac{\mu^2 + 1 - \sqrt{\mu^2 + 1}}{\mu}, \frac{-\mu^2}{\sqrt{\mu^2 + 1}})$ and $B_2(\frac{\mu^2 + 1 + \sqrt{\mu^2 + 1}}{\mu}, \frac{-\mu^2}{\sqrt{\mu^2 + 1}})$.

(vi) The unbounded set $D_{\mu,+\infty} = \{(x,y) \in K_1 : x \geq \mu + 1, y \geq 1\}$ and the closed set $D_{\mu,A_1} = \{(x,y) \in K_1 : 0 < x \leq \mu + 1, 0 < y \leq 1\}$ are $F_{\mu}$-invariant sets; for every point $(x,y) \in D_{\mu,+\infty} \setminus \{A_2(\mu + 1; 1)\}$ the following equalities hold:

$$
\lim_{n \to +\infty} f_{\mu,n}(x,y) = \lim_{n \to +\infty} g_{\mu,n}(x,y) = +\infty,
$$

and for any point $(x,y) \in D_{\mu,A_1} \setminus \{A_2(\mu + 1; 1)\}$ the following equalities hold:

$$
\lim_{n \to +\infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = \mu^2;
$$

in addition, the restriction $F_{\mu}|_{D_{\mu,+\infty}}$ is diffeomorphism of set $D_{\mu,+\infty}$ on its image $F_{\mu}(D_{\mu,+\infty})$ such that $F_{\mu}(D_{\mu,+\infty}) \subset D_{\mu,+\infty}$.

To prove Theorem 1 we need also the following property of the map $F_{\mu}$.

**Proposition 2 [8].** For any $\mu \in (0, 1]$ there exists a $C^1$-smooth strictly decreasing function $y = \Gamma_\mu(x)$ defined on the interval $(\mu, +\infty)$, such that $\Gamma_\mu((\mu, +\infty)) = (0, +\infty)$; moreover, the graph $\Gamma_\mu$ of this function is a $F_{\mu}$-invariant curve belonging to the unbounded noninvariant set $\{A_2(\mu + 1; 1)\} \cup (K_1 \setminus (D_{\mu,+\infty} \cup D_{\mu,A_1}))$.

The invariant curve $\Gamma_\mu$ makes it possible to construct the partition of the first quadrant and to prove that $F_{\mu}$ is a (nonsingular) Morse-Smale endomorphism for every $\mu \in (0, 1)$ and the map $F_1$ is a singular Morse-Smale endomorphism.

Note, that the source $A_2(\mu + 1; 1)$ is the limit point of the preimages of unbounded rank $\{F_{\mu}^{-1}(C_\mu \cap K_1)\}_{i \geq 1}$ of the part of critical line $C_\mu$ in the first open quadrant $K_1$ and $\{F_{\mu}^{-1}(C_\mu \cap K_1)\}_{i \geq 1}$ are preimages of unbounded rank of the part of critical line $C_0$ in $K_1$ $(\{F_{\mu}^{-1}(C_\mu \cap K_1)\}_{i \geq 1} \subset \{F_{\mu}^{-1}(C_0 \cap K_1)\}_{i \geq 1})$ since $F_{\mu}^2(C_\mu) = C_0$ for $\mu \in (0, 1]$. It is means, that there not exist universal neighborhood $U(A_2)$ of the source $A_2$, such that the restriction $F_{\mu}|_{U(A_2)}$ is diffeomorphism for all $n \geq 1$.

Calculating eigenvectors of the differential $\text{DF}_{\mu}(A_2)$ and using claim 6 of Proposition 1, we obtain, that one of unit eigenvectors $p_{A_2} = \left(\frac{-1 + \sqrt{9 + \sqrt{376}}}{\sqrt{20 + 8\mu + 2\sqrt{4 + 8\mu}}, \frac{4}{\sqrt{20 + 8\mu + 2\sqrt{4 + 8\mu}}}}\right)$ of $DF_{\mu}(A_2)$ of the map $F_{\mu}$ lies in the unbounded invariant set $D_{\mu,+\infty}$ and the other unit eigenvector $q_{A_2} = \left(\frac{1 + \sqrt{9 + \sqrt{376}}}{\sqrt{20 + 8\mu - 2\sqrt{4 + 8\mu}}, \frac{4}{\sqrt{20 + 8\mu - 2\sqrt{4 + 8\mu}}}}\right)$ lies in the unbounded noninvariant set $K_1 \setminus D_{\mu,+\infty}$.

Therefore, the standard technique of cones and normal forms can not be applied for the proof of existence of the invariant curve $\Gamma_{\mu}$.

For the proof of the Proposition 2 we use the nonlocal theorem of existence of unlocal strictly decreasing implicit functions $y = \eta_{\mu,n}(x)$, $n \geq 1$ [8]. The function $y = \eta_{\mu,n}(x)$ is the solution of equation $f_{\mu,n}(x,y) = \mu + 1$ on the interval $(\mu, +\infty)$ for any $n \geq 2$ and the function $y = \eta_{\mu,1}(x)$ is the solution of equation $f_{\mu,1}(x,y) = \mu + 1$ on the interval $(0, +\infty)$. The graph $\eta_{\mu,n}$ of this function for all $n \geq 1$ contains the unique fixed point – source $A_2(\mu + 1; 1)$ and doest not contain the common points with the periodic orbit $B$ of the period two. The properties of the function $y = \eta_{\mu,n}(x)$, $n \geq 1$ are formulated below, in Proposition 3.

**Proposition 3.** Let $F_{\mu}$ be the map (1), $\mu \in (0; 1]$. Then:

(i) for any $n \geq 1$ the following relations hold:

(a) $\eta_{\mu,2n-1} \prec \eta_{\mu,2n+1}, \eta_{\mu,2n+2} \prec \eta_{\mu,2n}$ for any $x \in (\mu, \mu + 1)^2$.

2 We say that $\eta_{\mu,k}$ is preceded $\eta_{\mu,l}$ for $k \neq l$, $k, l \geq 1$ and denote by $\eta_{\mu,k} \prec \eta_{\mu,l}$ if the inequality $\eta_{\mu,k}(x) < \eta_{\mu,l}(x)$ holds for any $x \in (\mu, \mu + 1)$ (for any $x \in (\mu + 1, +\infty)$).
\( \eta_{1,2n} < \eta_{1,2n+2}, \eta_{1,2n+1} < \eta_{1,2n-1} \) for any \( x \in (\mu + 1, +\infty) \).

(ii) the sequence of the restrictions \( \{ \eta_{1,2n}[\mu,\mu+1](x) \} \rightarrow \{ \eta_{1,2n-1}[\mu+1,\mu+3](x) \} \) of smooth strictly decreasing functions \( y = \eta_{1,2n}(x) \) for \( n \geq 1 \) on the interval \([\mu, \mu+1] \subset (\mu, \mu+1) \) converges in \( C^1 \) norm on the interval \([\mu, \mu+1] \) to the strictly decreasing function \( y = \Gamma_\mu(x) \) restricted on the interval \([\mu, \mu+1] \).

Note also, that the preimages of the different orders of the straight line \( x = \mu + 1 \) and the critical line \( C_\mu \) intersect each other under curve \( \Gamma_\mu \). Therefore, for the proof Theorem 1 we need the sets:

\[
D^\mu_1 = \{ (x, y) : x \in (\mu, +\infty), y > \Gamma_\mu \};
\]

\[
D^\mu_2 = \{ (x, y) : x \in (0, +\infty), 0 < y < \Gamma_\mu \} \quad \text{(see Fig. 1)}.
\]

**Figure 1.** Intersections of preimages of the straight line \( x = \mu + 1 \) with the preimages of the critical line \( C_\mu \) in double and triple points, which lie in the \( K_1 \).

The set of intersection points of the above preimages is described in the following lemma.

**Lemma 1.** Let \( F_\mu \) be the map (1), \( \mu \in (0, 1] \). Then

(i) for any odd number \( n \geq 1 \) and for any \( x \in (0, \mu + 1) \) the curve \( f_{\mu,n}(x, y) = \mu + 1 \) has in \( D^\mu_1 \) at least one triple point \( M^{(3)}_n \) (i.e. a point of the intersection of three curves \( f_{\mu,n}(x, y) = \mu + 1, f_{\mu,n+1}(x, y) = \mu \) and \( f_{\mu,n+2}(x, y) = \mu \) (\( n \geq 1 \)), at least one double point \( M^{(2,1)}_n \) of the first type (i.e. a point of the intersection of two curves \( f_{\mu,n}(x, y) = \mu + 1 \) and \( f_{\mu,n-1}(x, y) = \mu \) (\( n \geq 1 \)) and a countable set of double points \( M^{(2,2)}_n \) of the second type (i.e. the points of intersection of the curve \( f_{\mu,n}(x, y) = \mu + 1 \) with every curve \( f_{\mu,j}(x, y) = \mu \) for any \( j \geq n + 3, n \geq 1 \)).
(ii) for any even number \( n \geq 1 \) and for any \( x \in (\mu + 1, +\infty) \) the curve \( f_{\mu,n}(x,y) = \mu + 1 \) has in \( D_{\Gamma}^n \) at least one triple point \( M_n^{(3)} \), at least one double point \( M_n^{(2,1)} \) of the first type and countable set of double points \( M_n^{(2,2)} \) of the second type;

(iii) the set \( D_{\Gamma}^n \) doesn’t contain points of intersection of the curves \( f_{\mu,n}(x,y) = \mu + 1 \) and \( f_{\mu,m}(x,y) = \mu \ (m,n \geq 1) \) other than above.

3. The special partition of the first quadrant \( K_1 \) on the plane

In this section we construct the partition in \( K_1 \) using the properties of the invariant curve \( \Gamma_\mu \), the properties of strictly decreasing functions \( y = \eta_{\mu,n}(x) \) and the preimages of the different orders of the critical line \( C_\mu \). This partition help us to investigate the asymptotic behaviour of \( F_\mu \)–trajectories in the sets \( D_{\Gamma}^n \) and \( D_{\Gamma}^n \).

**Proposition 4.** Let \( F_\mu \) be a map of type \( (1) \), \( \mu \in (0,1] \). Then

(i) the set \( D_{\Gamma}^0 \) is invariant set and for every point \( (x,y) \in D_{\Gamma}^0 \) the equalities hold

\[ \lim_{n \to +\infty} f_{\mu,n}(x,y) = +\infty, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = +\infty. \]

(ii) the set \( D_{\Gamma}^0 \) is invariant set and for every point \( (x,y) \in D_{\Gamma}^0 \) with exception of the periodic orbit \( B \) of period two and all of its preimages the equalities hold

\[ \lim_{n \to +\infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = \mu^2. \]

We devide the proof of Proposition 4 in some steps.

**I.** Let us consider the set \( D_{\Gamma}^0 \). For the proof of the claim 1 of Proposition 4 we define the sets:

\[ T_{D_{\Gamma}^0, 2n} = \{(x,y) \in D_{\Gamma}^0 : \eta_{\mu,2n+2}[a,\mu+1](x) \leq y \leq \eta_{\mu,2n+1}[a,\mu+1](x)\} \quad \text{where} \quad a \in (\mu, \mu + 1); \]

\[ T_{D_{\Gamma}^0, 2n-1} = \{(x,y) \in D_{\Gamma}^0 : \eta_{\mu,2n+1}[\mu+1,b](x) \leq y \leq \eta_{\mu,2n-1}[\mu+1,b](x)\} \quad \text{where} \quad b \in (\mu + 1, +\infty); \]

\[ T_{D_{\Gamma}, +\infty} = \{(x,y) \in D_{\Gamma}^0 : y \geq \eta_{\mu,1}\}; \quad T_{D_{\Gamma}, +\infty} = \{(x,y) \in D_{\Gamma}^0 : y \geq \eta_{\mu,2}\}. \]

Then from Proposition 3 we obtain the following equality

\[ D_{\Gamma}^0 = D_{\Gamma, +\infty} \cup T_{D_{\Gamma}, +\infty} \cup \bigcup_{n \geq 1} T_{D_{\Gamma}, 2n-1} \cup \bigcup_{n \geq 1} T_{D_{\Gamma}, 2n}, \tag{2} \]

such that the inclusions are valid

\[ F_\mu(T_{D_{\Gamma}, 2n}) \subset T_{D_{\Gamma, +\infty}}, \quad F_\mu(T_{D_{\Gamma}, 2n}) \subset T_{D_{\Gamma, +\infty}} \quad \text{and} \quad F_\mu(T_{D_{\Gamma}, 2n}) \subset D_{\Gamma, +\infty}. \tag{3} \]

**Lemma 2.** For all \( \mu \in (0,1] \) and any \( n \geq 1 \) the map \( F_\mu \) is the \( C^1 \) – diffeomorphism of the set \( T_{D_{\Gamma}, 2n} \) on the set \( T_{D_{\Gamma}, 2n-1} \).

**Proof.** Indeed, by Proposition 3 and Lemma 1 the restriction \( F_\mu|_{T_{D_{\Gamma}, 2n-1}} \) is surjective local diffeomorphism of the set \( T_{D_{\Gamma}, 2n} \) on the set \( T_{D_{\Gamma}, 2n-1} \). We show, that \( F_\mu : T_{D_{\Gamma}, 2n} \to T_{D_{\Gamma}, 2n-1} \) is injective map. Let us \((x_1,y_1)\) and \((x_2,y_2)\) are an arbitrary points in \( T_{D_{\Gamma}, 2n} \), such that \((x_1,y_1) \neq (x_2,y_2)\). The coordinate function \( f_{\mu,1}(x,y) = xy \) is the strictly monotone function for each variable and the coordinate function \( g_{\mu,1}(x,y) = (x-\mu)^2 \) is the strictly monotone function for variable \( x \ (x > \mu) \). We have

\[ (f_{\mu,1}(x_1,y_1),g_{\mu,1}(x_1,y_1)) \neq (f_{\mu,1}(x_2,y_2),g_{\mu,1}(x_2,y_2)). \]
Then \( F_\mu : T_{D_{\Gamma_\mu}}^\alpha,2n \to T_{D_{\Gamma_\mu}}^\alpha,2n-1 \) is bijection and there exists the inverse map \( (F_\mu|_{T_{D_{\Gamma_\mu}}^\alpha,2n-1})^{-1} \) to the map \( F_\mu|_{T_{D_{\Gamma_\mu}}^\alpha,2n} \) with the Jacobian matrix

\[
J((F_\mu|_{T_{D_{\Gamma_\mu}}^\alpha,2n-1})^{-1}(x,y)) = \left( \begin{array}{c}
0 \\
\frac{1}{\mu+\sqrt{\beta}} \\
\frac{\sqrt{\beta}}{\mu+\sqrt{\beta}}
\end{array} \right) |_{T_{D_{\Gamma_\mu}}^\alpha,2n-1}.
\]

The partial derivatives of coordinate functions of map \( (F_\mu|_{T_{D_{\Gamma_\mu}}^\alpha,2n-1})^{-1} \) are continuous on the set \( T_{D_{\Gamma_\mu}}^\alpha,2n-1 \). Therefore, \( F_\mu : T_{D_{\Gamma_\mu}}^\alpha,2n \to T_{D_{\Gamma_\mu}}^\alpha,2n-1 \) is \( C^1 \) – diffeomorphism. Lemma 2 is proved.

Using Lemma 2, formula (2), inclusions (6) and claim 6 of Proposition 1 we obtain the following result.

**Lemma 3.** For any \( \mu \in (0,1] \) set \( D_{\Gamma_\mu}^\alpha \) is \( F_\mu \) – invariant set and for every point \( (x,y) \in D_{\Gamma_\mu}^\alpha \), the following equalities hold:

\[
\lim_{n \to +\infty} f_{\mu,n}(x,y) = +\infty, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = +\infty.
\]

Thus, the claim 1 of Proposition 4 is proved.

**II.** Our next step is to prove, that the set \( D_{\Gamma_\mu}^\alpha \) is invariant set, such that the trajectories of the points under \( \Gamma_\mu \) satisfy equalities

\[
\lim_{n \to +\infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x,y) = \mu^2.
\]

**II (a).** We begin from the construction the partition of the set \( D_{\Gamma_\mu}^\alpha \). For this goal we need the sets:

\[
T_{D_{\Gamma_\mu}}^\alpha,2n-1 = \{(x,y) \in D_{\Gamma_\mu}^\alpha : \eta_{\mu,2n-1}|_{[a,b]}(x) \leq y \leq \eta_{\mu,2n+1}|_{[a,b]}(x)\}, \text{ where } [a,b] \subset (\mu,\mu+1), n \geq 1;
\]

\[
T_{D_{\Gamma_\mu}}^\alpha,2n = \{(x,y) \in D_{\Gamma_\mu}^\alpha : \eta_{\mu,2n}|_{[a,b]}(x) \leq y \leq \eta_{\mu,2n+2}|_{[a,b]}(x)\}, \text{ where } [a,b] \subset (\mu+1,\infty), n \geq 1;
\]

\[
T_{D_{\mu,A_1}}^l = \{(x,y) \in K_1 : 0 < x < \mu + 1, 1 < y \leq \frac{\mu + 1}{x}\};
\]

\[
T_{D_{\mu,A_1}}^r = \{(x,y) \in K_1 : x > \mu + 1, 0 < y \leq \frac{\mu + 1}{x(x-\mu)^2}\}.
\]

**Lemma 4.** For any \( \mu \in (0,1] \) the inclusions

\[
F_\mu(T_{D_{\mu,A_1}}^l) \subset D_{\mu,A_1}, F_\mu(T_{D_{\mu,A_1}}^r) \subset T_{D_{\mu,A_1}}^l
\]

are valid.

**Proof.** We prove the first of the inclusions. Note that, the critical line \( C_\mu \) intersect the set \( T_{D_{\mu,A_1}}^l \). Therefore, we represent set \( T_{D_{\mu,A_1}}^l \) as the union of two sets:

\[
T_{D_{\mu,A_1}}^{l,1} = \{(x,y) \in T_{D_{\mu,A_1}}^l : x \in (0,\mu)\};
\]

\[
T_{D_{\mu,A_1}}^{l,2} = \{(x,y) \in T_{D_{\mu,A_1}}^l : x \in (\mu,\mu + 1)\};
\]

i. e. \( T_{D_{\mu,A_1}}^l = T_{D_{\mu,A_1}}^{l,1} \cup T_{D_{\mu,A_1}}^{l,2} \).
We show, that the following inclusions \( F_\mu(T_{D_{\mu,A_1}}^{d,1}) \subset D_{\mu,A_1} \), \( F_\mu(T_{D_{\mu,A_1}}^{d,2}) \subset D_{\mu,A_1} \) are valid. Let \((x; y)\) is an arbitrary point of the set \( T_{D_{\mu,A_1}}^{d,1} \). By definition of the set \( T_{D_{\mu,A_1}}^{d,1} \) the following inequalities hold: \( 0 < xy < \mu + 1, (x - \mu)^2 < 1 \). Hence, \( F_\mu(T_{D_{\mu,A_1}}^{d,1}) \subset D_{\mu,A_1} \).

If \((x; y) \in T_{D_{\mu,A_1}}^{d,2} \) then \( \mu < xy < \mu + 1, (x - \mu)^2 < 1 \) and the inclusion \( F_\mu(T_{D_{\mu,A_1}}^{d,2}) \subset D_{\mu,A_1} \) is valid. Therefore, \( F_\mu(T_{D_{\mu,A_1}}^{d,1}) \subset D_{\mu,A_1} \). Analogously above we prove, that \( F_\mu(T_{D_{\mu,A_1}}^{d,1}) \subset D_{\mu,A_1} \).

Lemma 4 is proved.

Using Lemma 4 and claim 6 of the Proposition 1 we obtain the following result.

**Lemma 5.** The set \( D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{d,1} \cup T_{D_{\mu,A_1}}^{d,2} \) is \( F_\mu \) - invariant set and for every point \((x; y) \in D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{d,1} \cup T_{D_{\mu,A_1}}^{d,2} \setminus \{A_2(\mu + 1; 1)\}\) the following equalities hold:

\[
\lim_{n \to +\infty} f_{\mu,n}(x, y) = 0, \quad \lim_{n \to +\infty} g_{\mu,n}(x, y) = \mu^2.
\]

**II (b).** In this part of the work we investigate the properties of the set

\[ G = \{(x; y) \in K_1 : x \in (0, \mu + 1), \eta_{\mu,1}(x) < y < \Gamma_\mu \} \cup \{(x; y) \in K_2 : x > \mu + 1, \eta_{\mu,2}(x) < y < \Gamma_\mu \}. \]

Using Lemma 5 we obtain

**Corollary 1.** Let \((x; y)\) is an arbitrary point of the set \( G \). Then the preimages of the point \((x; y)\) with respect to the restriction \( F_\mu|_{K_1} \) does not belong to the set \( D_{\mu,A_1} \cup T_{D_{\mu,A_1}}^{d,1} \cup T_{D_{\mu,A_1}}^{d,2} \).

By definitions of the sets \( G \) and \( T_{D_{\mu,A_1}}^{2n-1} \), \( T_{D_{\mu,A_1}}^{2n} \) the following equality hold

\[
\bigcup_{n=1}^{+\infty} (T_{D_{\mu,A_1}}^{2n-1} \cup T_{D_{\mu,A_1}}^{2n}) = G.
\]

Our next step is the investigate of \( F_\mu \) - trajectories of the points of the sets \( T_{D_{\mu,A_1}}^{2n-1} \) and \( T_{D_{\mu,A_1}}^{2n} \). For this goal, we consider the existence problem for implicit functions defined by the equations

\[ f_{\mu,n}(x, y) = \mu \] (4)

with initial condition

\[ f_{\mu,n}(x^*, y^*) = \mu \] (5)

on the interval \((\mu, +\infty)\), where \((x^*; y^*)\) is the point of one of the types described in Lemma 1. Using the existence theorem for a nonlocal \( C^1 \) - smooth implicit function (see [7]) we obtain the following claim.

**Lemma 6.** Let \( F_\mu \) be a map \((1), \mu \in (0, 1] \). Then for every \( n \geq 1 \) there exists the \( C^1 \) - smooth function \( y = \psi_{\mu,n}(x) \) which is the solution of the problem \((4)\) with the initial condition \((5)\) on the interval \((\mu, +\infty)\). Moreover, the graph \( \psi_{\mu,n} \) of the function \( y = \psi_{\mu,n}(x) \) passes through the triple point \( M_{n-1}^3 \) and the double point of the first type \( M_{n+1}^{(2,1)} \) for \( n \geq 2 \); for \( n \geq 5 \) and \( k \geq 3 \) the graph \( \psi_{\mu,n} \) of the function \( y = \psi_{\mu,n}(x) \) passes through the double points of the second type \( M_{n-k}^{(2,2)} \).

Let us note, that the ray \( x = 2\mu \) for \( \mu \in (0, 1] \) and its preimages of the different orders plays an important role in investigation of \( F_\mu \) - trajectories from the set \( G \).

Thus, we consider also the existence problem for implicit functions defined by the equations

\[ f_{\mu,n}(x, y) = 2\mu \] (6)
with initial conditions
\[ f_{\mu,n}(x_{Q_n}, y_{Q_n}) = 2\mu \]  \hspace{1cm} (7)
onumber
on the interval \((\mu, +\infty)\).

The point \(Q_n(x_{Q_n}, y_{Q_n})\) is the unique point of intersection of the connected component of the full preimage of the ray \(x = 2\mu\) of the \((n-1)\) order with the unique smooth strictly monotone branch of the \(n\)th preimage of the ray \(x = \mu + 1\), which passes through the fixed point \(A_2(\mu + 1; 1)\) \((n \geq 1)\).

Let \(Q_1(2\mu, \frac{\mu+1}{2\mu})\) is the unique point of intersection of the ray \(x = 2\mu\) with unique smooth strictly decreasing branch of the first preimage of the ray \(x = \mu + 1\), which passes through the fixed point \(A_2(\mu + 1; 1)\). Note that, for \(x > \mu\) there exist the unique point \(Q_2(\mu + \frac{\mu+1}{2\mu}; \frac{2\mu}{\mu+1})\) of intersection of the first preimage of the ray \(x = 2\mu\) with the unique smooth strictly monotone branch of the second preimage of the ray \(x = \mu + 1\), which passes through the point \(A_2\) and the following equality hold \(F_{\mu}(Q_2) = Q_1\). Denote by \(Q_n(n \geq 1)\) the unique point of intersection of connected component of the full preimage of the ray \(x = 2\mu\) of the \((n-1)\) order with the unique continuous strictly monotone branch of the \(n\)th preimage of the ray \(x = \mu + 1\), which passes through the point \(A_2\) such that the following equality hold \(F_{\mu}(Q_{n+1}) = Q_n\) for any \(n \geq 1\). The point \(Q_n\) for any \(n \geq 1\) is the correctly defined.

Analogously above we apply the existence theorem of a nonlocal \(C^1\) – smooth implicit function [7] and obtain the following result.

**Lemma 7.** Let \(F_{\mu}\) be a map \((1), \mu \in (0, 1)\). Then there exists \(C^1\) – smooth monotone function \(y = \phi_{\mu,n}(x)\), which defined on the interval \((\mu, +\infty)\) by equality (6). The graph \(\phi_{\mu,n}\) of this function passes through the point \(Q_n\).

Lemmas 1, 6 and definition of the set \(\{Q_n, n \geq 1\}\) makes it possible to construct the partition of the set \(D_{r_\mu}^{\mu,2n-1} \subset G, n \geq 1\), such that boundaries of elements of this partition may contain the curves \(\eta_{\mu,n}, \phi_{\mu,m}, \psi_{\mu,k}\) (where \(m \geq 0, k \geq 0\) are defined by Lemma 1 and the set \(\{Q_n, n \geq 1\}\)), the critical line \(C_{\mu}\) and the ray \(x = 2\mu\).

For determination, we construct the partition of the set \(D_{r_\mu}^{\mu,2n-1}\). Let \([a, b] \subset (\mu, \mu + 1)\) is an arbitrary closed interval. For \(n = 1\) we have

\[ T_{D_{r_\mu}^{\mu,1}} = \{(x; y) \in D_{r_\mu}^{\mu} : \eta_{n,1}|_{[a,b]}(x) \leq y \leq \eta_{n,3}|_{[a,b]}(x)\}. \]

The critical line \(C_{\mu}\) and the line \(x = 2\mu\) intersect set \(D_{r_\mu}^{\mu,1}\) in the double point of the first type \(\{M_{1}^{(2,1)}(x_{M_{1}^{(2,1)}}, y_{M_{1}^{(2,1)}}), M_{1}^{(2,1)}(x_{Q_1}, y_{Q_1})\}\) and in the point \(Q_1\) respectively, such that the equality hold

\[ T_{D_{r_\mu}^{\mu,1}} = \bigcup_{i=1}^{3} T_{D_{r_\mu}^{\mu,1}}^{i}, \]

where
\[ T_{D_{r_\mu}^{\mu,1}}^{1} = \{(x; y) \in D_{r_\mu}^{\mu,1} : x \in (0, x_{M_{1}^{(2,1)}})\}; \]
\[ T_{D_{r_\mu}^{\mu,1}}^{2} = \{(x; y) \in D_{r_\mu}^{\mu,1} : x \in (x_{M_{1}^{(2,1)}}, x_{Q_1})\}; \]
\[ T_{D_{r_\mu}^{\mu,1}}^{3} = \{(x; y) \in D_{r_\mu}^{\mu,1} : x \in (x_{Q_1}, \mu + 1)\}. \]

From Lemmas 1, 6 follows, that the boundary of the set \(D_{r_\mu}^{\mu,1}\) contains the triple point \(\{M_{1}^{(3)}(x_{M_{1}^{(3)}}, y_{M_{1}^{(3)}}), M_{1}^{(3)}(x_{Q_1}, y_{Q_1})\}\), which is the point of intersection of three branches: the monotone branch \(\eta_{n,1}\) of the first preimage of the line \(x = \mu + 1\) and the monotone branches \(\psi_{\mu,2}\) and \(\psi_{\mu,3}\) of
the second and the third preimages of the critical line $C_\mu$ respectively; in addition, the abscissa $x_{M_1(3)}$ of the point $M_1(3)$ satisfies inequality

$$\mu < x_{M_1(3)} < 2\mu \text{ if } \mu \in (0, \frac{\sqrt{5} - 1}{2})$$

or satisfies inequality

$$2\mu < x_{M_1(3)} < \mu + 1 \text{ if } \mu \in \left(\frac{\sqrt{5} - 1}{2}, 1\right).$$

For the case $\mu \in (0, \frac{\sqrt{5} - 1}{2})$ we define the sets

$$T_{D_{1\mu}}^{2,1} = \{(x; y) \in T_{D_{1\mu}}^u : x \in (x_{M_1(2,1)}, x_{M_1(3)}), y < \psi_{1\mu,3}(x)\};$$

$$T_{D_{1\mu}}^{2,2} = \{(x; y) \in T_{D_{1\mu}}^u : x \in (x_{M_1(2,1)}, x_{M_1(3)}), \psi_{1\mu,3}(x) < y < \psi_{1\mu,2}(x)\};$$

$$T_{D_{1\mu}}^{2,3} = \{(x; y) \in T_{D_{1\mu}}^u : x \in (x_{M_1(3)}, x_{Q_1}), y > \psi_{1\mu,2}(x)\},$$

such that $T_{D\mu}^{2,1} = \bigcup_{i=1}^{3} T_{D_{1\mu}}^{2,i}$. From this follows the equality

$$T_{D\mu}^u = T_{D_{1\mu}}^u \cup T_{D_{1\mu}}^{3} \cup \bigcup_{i=1}^{3} T_{D_{1\mu}}^{2,i} \text{ (see Figure 2)}.$$

Figure 2. The partition of the set $T_{D\mu}^{2,1}$ with using critical line $C_\mu$, ray $x = 2\mu$ and the monotone branches $\psi_{1\mu,2}, \psi_{1\mu,3}$ for $\mu \in (0, \frac{\sqrt{5} - 1}{2})$.

If $\mu \in (\frac{\sqrt{5} - 1}{2}, 1)$ we need the following sets

$$T_{D_{1\mu}}^{2,1} = \{(x; y) \in T_{D_{1\mu}}^u : x \in (x_{M_1(2,1)}, x_{Q_1}), y < \psi_{1\mu,3}(x)\};$$
Then we obtain

$$T_{D_{1}^{\mu},1}^{2,2} = \{(x, y) \in T_{D_{1}^{\mu},1}^{2,2} : x \in (x_{M_{1}^{(2,1)}}, x_{Q_{1}}), \psi_{\mu,3}(x) < y < \psi_{\mu,2}(x)\};$$

$$T_{D_{1}^{\mu},1}^{2,3} = \{(x, y) \in T_{D_{1}^{\mu},1}^{2,3} : x \in (x_{M_{1}^{(2,1)}}, x_{Q_{1}}), y > \psi_{\mu,2}(x)\};$$

$$T_{D_{1}^{\mu},1}^{3,1} = \{(x, y) \in T_{D_{1}^{\mu},1}^{3,1} : x \in (x_{Q_{1}}, x_{M_{1}^{(3)}}), y < \psi_{\mu,3}(x)\};$$

$$T_{D_{1}^{\mu},1}^{2,2} = \{(x, y) \in T_{D_{1}^{\mu},1}^{2,2} : x \in (x_{Q_{1}}, x_{M_{1}^{(3)}}), \psi_{\mu,3}(x) < y < \psi_{\mu,2}(x)\};$$

$$T_{D_{1}^{\mu},1}^{3,3} = \{(x, y) \in T_{D_{1}^{\mu},1}^{3,3} : x \in (x_{Q_{1}, \mu + 1}), y > \psi_{\mu,2}(x)\}.$$

Then we obtain

$$T_{D_{1}^{\mu},1} = T_{D_{1}^{\mu},1}^{1} \cup \left( \bigcup_{i=1}^{3} T_{D_{1}^{\mu},1}^{2,i} \right) \cup \left( \bigcup_{j=1}^{3} T_{D_{1}^{\mu},1}^{3,j} \right) \quad \text{(see Figure 3)}.$$

Further, we consider the second case in more detail.

![Figure 3](image-url)

**Figure 3.** The partition of the set $T_{D_{1}^{\mu},1}^{2,2}$ with using critical line $C_{\mu}$, ray $x = 2\mu$ and the monotone branches $\psi_{\mu,2}, \psi_{\mu,3}$ for $\mu \in (\sqrt{\frac{5}{2}} - 1, 1)$.

Let $\mu \in (\sqrt{\frac{5}{2}} - 1, 1)$. On the first step, describe above, we obtain the partition of the set $T_{D_{1}^{\mu},1}^{2,2}$. In this partition there exist elements, which contains the preimages of critical line $C_{\mu}$ of the order $k \geq 2$ (for example, the set $T_{D_{1}^{\mu},1}^{2,2}$) and the set $T_{D_{1}^{\mu},1}^{3,3}$ contain the connected components of the preimages of critical line $C_{\mu}$ of the order 2 and 4 respectively, see Figure 3).

Suppose that after $n$ steps, we constructed the partition of the set $T_{D_{1}^{\mu},2n-1}^{2,2}$ on the following subsets:

$$T_{D_{1}^{\mu},2n-1}^{1} = \{(x, y) \in T_{D_{1}^{\mu},2n-1}^{1} : x \in (\mu, x_{M_{2n-1}^{(2,1)}}, y < \psi_{\mu,2n-2}(x))\};$$

$$T_{D_{1}^{\mu},2n-1}^{2,1} = \{(x, y) \in T_{D_{1}^{\mu},2n-1}^{2,1} : x \in (x_{M_{2n-1}^{(2,1)}}, x_{Q_{2n-1}}), \psi_{\mu,2n-2}(x) < y < \psi_{\mu,2n+1}(x)\};$$
\[ T^2_{D_F^1,2n-1} = \{(x;y) \in T_{D_F^1,2n-1}^u : x \in (x_{M(2)} , x_{Q_{2n-1}}), \psi_2,2n(x) < y < \psi_2,2n(x)\}; \]

\[ T^2_{D_F^2,2n-1} = \{(x;y) \in T_{D_F^2,2n-1}^u : x \in (x_{M(2)} , x_{Q_{2n-1}}), \psi_2,2n(x) < y < \phi_2,2n(x)\}; \]

\[ T^{3,1}_{D_F^2,2n-1} = \{(x;y) \in T_{D_F^2,2n-1}^u : x \in (x_{Q_{2n-1}}, x_{M(3)}), \phi_{2n-2}(x) < y < \psi_{2n+1}(x)\}; \]

\[ T^{3,2}_{D_F^2,2n-1} = \{(x;y) \in T_{D_F^2,2n-1}^u : x \in (x_{Q_{2n-1}}, x_{M(3)}), \psi_{2n+1}(x) < y < \psi_{2n}(x)\}; \]

\[ T^{3,3}_{D_F^2,2n-1} = \{(x;y) \in T_{D_F^2,2n-1}^u : x \in (x_{Q_{2n-1}}, \mu + 1), y > \psi_{2n}(x)\}; \]

Let us note that every element of this partition can contain the connected components of the preimages of critical line \( C_\mu \) of the order \( k \geq 2n \).

Let us describe \( n \)th step. Using definition of the points \( Q_n, n \geq 1 \), and definition of \( C_1 \) smooth decreasing monotone functions \( y = \eta_{\mu,n}(x), n \geq 1 \), we obtain that the monotone branches \( \psi_{2n} \) and \( \phi_{2n} \) intersects set \( T_{D_F^2,2n+1} \); in addition, the following equality

\[ T_{D_F^3,2n+1} = T_{D_F^3,2n+1}^1 \cup T_{D_F^3,2n+1}^2 \cup T_{D_F^3,2n+1}^3 \]

hold, where the sets \( T_{D_F^3,2n+1}^1, T_{D_F^3,2n+1}^2, T_{D_F^3,2n+1}^3 \) are defined by equalities

\[ T_{D_F^3,2n+1}^1 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (0,x_M^{2n+1})\}; \]

\[ T_{D_F^3,2n+1}^2 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (x_M^{2n+1},x_{Q_{2n}})\}; \]

\[ T_{D_F^3,2n+1}^3 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (x_{Q_{2n}},\mu+1)\}. \]

For all values of parameter \( \mu \) from the interval \( \left( \frac{\sqrt{5} - 1}{2},1 \right) \) the inequalities are fulfilled \( x_{Q_{2n}} < x_M^{2n+1} < \mu + 1 \). Hence, the equality

\[ T_{D_F^3,2n+1} = T_{D_F^3,2n+1}^1 \cup \bigcup_{i=1}^3 T_{D_F^3,2n+1}^i \cup \bigcup_{j=1}^3 T_{D_F^3,2n+1}^j \]

holds, where

\[ T_{D_F^3,2n+1}^1 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (x_M^{2n+1},x_{Q_{2n+1}}), \psi_2,2n(x) < y < \psi_2,2n(x)\}; \]

\[ T_{D_F^3,2n+1}^2 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (x_M^{2n+1},x_{Q_{2n+1}}), \psi_2,2n(x) < y < \psi_2,2n(x)\}; \]

\[ T_{D_F^3,2n+1}^3 = \{(x;y) \in T_{D_F^3,2n+1}^u : x \in (x_{Q_{2n+1}},\mu+1), y > \psi_{2n+2}(x)\}; \]

Thus, the partition of the set \( T_{D_F^3,2n} \) (for every \( n \geq 1 \)) is constructed.

Analogously above we construct the partition of the set \( T_{D_F^3,2n} \) for every \( n \geq 1 \).

The properties of elements of the above partitions of the sets \( T_{D_F^1,2n-1} \) and \( T_{D_F^2,2n} \) for the case \( \mu \in \left( \frac{\sqrt{5} - 1}{2},1 \right) \) are formulated in following claim.

**Lemma 8.** Let \( \mu \in \left( \frac{\sqrt{5} - 1}{2},1 \right) \). Then map \( F_\mu \) is the diffeomorphism:
(i) of the set $T_{D_{ri}^\mu}^{2,i}2n$ on the set $T_{D_{ri}^\mu}^{2,i}2n-1$, $i = 1, 2, 3$;
(ii) of the set $T_{D_{ri}^\mu}^{3,i}2n$ on the set $T_{D_{ri}^\mu}^{3,i}2n-1$, $j = 1, 2, 3$;
(iii) of the set $T_{D_{ri}^\mu}^{1,i}2n$ on the set $T_{D_{ri}^\mu}^{1,i}2n-1$.

Proof. For determination, we give the proof of the claim 1. As it follows from the definition of sets $T_{D_{ri}^\mu}^{i}2n$ and $T_{D_{ri}^\mu}^{i}2n-1$ ($i = 1, 2, 3$) the restriction $F_{\mu}|_{T_{D_{ri}^\mu}^{i}2n}$ is local diffeomorphism.

Let us prove, that $F_{\mu} : T_{D_{ri}^\mu}^{2,i}2n \rightarrow T_{D_{ri}^\mu}^{2,i}2n-1$ is surjection for every $i = 1, 2, 3$. In fact, by the formula (1) the connected component of the full preimage of the closed interval $(x_0, y_0)$ by the formula (1) the connected component of the full preimage of the closed interval $(x_0, y_0)$ contains the closed interval $(x_0, y_0)$. Arbitrarily take and fix a leave $y = c$, where $c \in T_{D_{ri}^\mu}^{2,i}2n-1$, $c > 0$. From definition of the set $T_{D_{ri}^\mu}^{2,i}2n-1$ follows, that the absicces of the points $(x_1; c)$ and $(x_2; c) \in T_{D_{ri}^\mu}^{2,i}2n-1$ satisfy the inequalities:

$$x_{N_2(1)} < x_1 < x_2 < x_{N_2(3)}.$$ Using formula (1), we obtain, that there exist the points $(c^*; y_1)$ and $(c^*; y_2) \in T_{D_{ri}^\mu}^{2,i}2n$ such that the equalities $F_{\mu}(c^*, y_1) = (x_1; c)$, $F_{\mu}(c^*, y_2) = (x_2, c)$ are fulfilled. It means, that $F_{\mu}$ is surjection of the set $T_{D_{ri}^\mu}^{2,i}2n$ on the set $T_{D_{ri}^\mu}^{2,i}2n-1$ for $i = 1, 2, 3$.

We show, that $F_{\mu} : T_{D_{ri}^\mu}^{2,i}2n \rightarrow T_{D_{ri}^\mu}^{2,i}2n-1$ is injection for every $i = 1, 2, 3$. Using the monotonicity of the function $f_{\mu}(x, y)$ by the variable $x$ (resp. $y$), we obtain, that for the arbitrary two points $(x_1; y_1)$, $(x_2; y_2) \in T_{D_{ri}^\mu}^{2,i}2n$, such that $x_1 < x_2$ (resp. $y_1 < y_2$) the inequality is fulfilled $F_{\mu}(x_1, y_1) \neq F_{\mu}(x_2, y_2)$, where $F_{\mu}(x_1, y_1)$, $F_{\mu}(x_2, y_2) \in T_{D_{ri}^\mu}^{2,i}2n-1$ ($i = 1, 2, 3,$) Hence, $F_{\mu} : T_{D_{ri}^\mu}^{2,i}2n \rightarrow T_{D_{ri}^\mu}^{2,i}2n-1$ is injection. Therefore, we obtain, that $F_{\mu} : T_{D_{ri}^\mu}^{2,i}2n \rightarrow T_{D_{ri}^\mu}^{2,i}2n-1$ is bijection ($i = 1, 2, 3$).

Then, the inverse map $F_{\mu}^{-1} : T_{D_{ri}^\mu}^{2,i}2n-1 \rightarrow T_{D_{ri}^\mu}^{2,i}2n$ is defined, and the partial derivaties of its coordinate functions are continuous in $T_{D_{ri}^\mu}^{2,i}2n-1$. The claim 1 of Lemma 8 is proved.

Lemma 8 is proved.

By Lemmas 8 the following statement hold.

**Corollary 2.** The set $G$ is $F_{\mu}$ - invariant set for every $\mu \in (0, 1]$.

**II (c)** In this part of the paper we describe asymptotic behaviour of $F_{\mu}$ - trajectories of the points of the set $G$.

Note that set $G$ contains periodic orbit $B$ of period 2. Prove the following statement.

**Lemma 9.** The map $F_{\mu}$ have the unique periodic orbit $B$ with the least period two, which formed by two sources $B_1$ and $B_2$ such that $B_1 \in T_{D_{ri}^\mu}^{1,i}2n$, $B_2 \in T_{D_{ri}^\mu}^{2,i}2n$.

Proof.

(i) Existence of unique periodic orbit $B$ is proved in [8].
(ii) Let us prove, that $B_1 \in T_{D_{ri}^\mu}^{1,i}2n$. Note, that for any $\mu \in (0, 1]$ for the coordinate of point $B$ the equalities holds:

$$\frac{\mu^2 + 1 - \sqrt{\mu^2 + 1}}{\mu} = \frac{(\sqrt{\mu^2 + 1} - 1)\sqrt{\mu^2 + 1}}{\mu} = \frac{\mu^2 + 1}{\sqrt{\mu^2 + 1} + 1}$$
Proposition 4 is proved.

Using inclusion (9) and Lemmas 4, 5, 8–10 we obtain the following result.

Note, that the trajectory \( \lim_{n \to \infty} x_B = \mu_1 \) holds.

By Lemma 9 and Corollary 2 the preimages of periodic orbit \( B \) with respect to restriction \( \{ (x,y) : x = \mu, y > \frac{\mu + 1}{x} \} \) and by unbounded continuous arc \( \{ (x,y) : 0 < x < \mu, y = \frac{\mu + 1}{x} \} \). Consequently, \( B_1 \in T^1_{D_{\mu^*}} \) (see definition of set \( T^1_{D_{\mu^*}} \)). Using Lemma 6 for any \( \mu \in (0,1] \) we obtain, that \( B_2 \in T^1_{D_{\mu^*}} \).

Lemma 9 is proved.

By Lemma 9 and Corollary 2 the preimages of periodic orbit \( B \) of the arbitrary order \( n \geq 1 \) (with respect to restriction \( F_\mu|K_1 \)) belong to the set \( \bigcup_{n=1}^{\infty} (T^1_{D_{\mu^*}} \cup T^1_{D_{\mu^*}}) \).

Denote by \( \{(F_\mu|K_1)^{-n}(x_B; y_B)\} \) the subset of the full \( n \)th preimage of the point \( B_1 \) which does not contain the periodic orbit \( B \). From Lemmas 8, 9 follows, that \( \{(F_\mu|K_1)^{-n}(x_B; y_B)\} \) is single point set. The unique point of this set we denote by \( (x_B^1; y_B^1) \).

By the claim 3 of Proposition 3, Lemmas 8, 9 and Corollary 2 the following statement holds.

**Lemma 10.** Let \( F_\mu \) be a map of type (1), \( \mu \in (0,1] \). Then for any \( \mu \in (0,1] \) the limit equalities \( \lim_{n \to \infty} x_B = \mu + 1 \), \( \lim_{n \to \infty} y_B = 1 \) are valid.

Note, that the trajectory \( \{(F_\mu|K_1)^{-2n}(x_B; y_B)\}_{n \in \mathbb{Z}} \) \( \{(F_\mu|K_1)^{-2n}(x_B; y_B)\}_{n \in \mathbb{Z}} \) is heteroclinic, such that it is attracted to the point \( B_1( B_2) \) and it is repelled from the point \( A_2 \).

By definition of sets \( T^1_{D_{\mu^*}} \) and \( T^1_{D_{\mu^*}} \) the following inclusion

\[
F_\mu(T^1_{D_{\mu^*}}) \subset T^1_{D_{\mu^*}}
\]

holds.

From Lemmas 5 and 8 we obtain the following statement.

**Corollary 3.** The set \( D_{\mu^*} = G \cup D_{\mu,A_1} \cup T^1_{D_{\mu^*}} \cup T^1_{D_{\mu^*}} \) is invariant set.

Using inclusion (9) and Lemmas 4, 5, 8–10 we obtain the following result.

**Lemma 11.** Let \( F_\mu \) be a map of type (1), \( \mu \in (0;1] \). Then for every point \( (x,y) \in G \) with exception of periodic orbit \( B \) of period two and all of its preimages the equalities hold

\[
\lim_{n \to \infty} f_{\mu,n}(x,y) = 0, \quad \lim_{n \to \infty} g_{\mu,n}(x,y) = \mu^2.
\]

Claim 2 of Proposition 4 follows immediately from Lemmas 5, 11 and Corollary 3.

Proposition 4 is proved.
4. Completion of the proof of Theorem 1

In this part we describe nonwandering set $\Omega(F_\mu)$ of map $F_\mu$ and complete the proof of Theorem 1.

From Proposition 4 follows, that for any $\mu \in (0, 1)$ nonwandering set $\Omega(F_\mu|_{K_1})$ of the map $F_\mu|_{K_1}$ is finite and consists of the fixed point $A_2(\mu + 1; 1)$ and the periodic orbit $B$ of period two. Let us describe behaviour of $F_\mu$ - trajectories in $K_2$. As it follows from the claims 1 and 2 of Proposition 1, $F_\mu|_{K_2} : K_2 \to F_\mu(K_2)$ is a map of type (1).

By the claim 4 of Proposition 4 we obtain, that for any $\mu \in (0, 1)$ the set $K_2$ contains two fixed points: the saddle point $A_3(\mu - 1; 1)$ and the sink $A_1(0; \mu^2)$. Therefore, there exist continuous functions $y = \gamma_i^u$, $y = \gamma_i^s$, $i = 1, 2$, such that the graphs $\gamma_i^u$, $\gamma_i^s$ are the nonstable separatrix of saddle point $A_3$ and the graphs $\gamma_1^s$, $\gamma_2^s$ are the stable separatrix of saddle point $A_3$ [14]. Moreover, the equalities holds $\gamma_1^u \cup \gamma_2^u = W^u(A_3)$, $\gamma_1^s \cup \gamma_2^s = W^s(A_3)$. Let us define the sets

$$K_{21} = \{(x, y) : K_2 : y > \gamma_1^u(x), y > \gamma_2^u(x)\};$$

$$K_{22} = \{(x, y) : K_2 : \gamma_2^s(x) < y < \gamma_1^s(x)\};$$

$$K_{23} = \{(x, y) : K_2 : y < \gamma_2^u(x), y < \gamma_1^u(x)\};$$

$$K_{24} = \{(x, y) : K_2 : \gamma_2^s(x) < y < \gamma_1^s(x)\},$$

such that $\bigcup_{i=1}^4 K_{2i} = K_2$.

**Lemma 12.** Let $F_\mu$ be a map of type (1), $\mu \in (0, 1)$. Then

(i) for any point $(x, y) \in K_{22} \cup K_{23}$ the following equalities holds

$$\lim_{n \to +\infty} f_{\mu, n}(x, y) = 0, \quad \lim_{n \to +\infty} g_{\mu, n}(x, y) = \mu^2;$$

(ii) for any point $(x, y) \in K_{21} \cup K_{24}$ the following equalities holds

$$\lim_{n \to +\infty} f_{\mu, n}(x, y) = -\infty, \quad \lim_{n \to +\infty} g_{\mu, n}(x, y) = +\infty.$$

Let us describe the nonwandering set $\Omega(F_\mu|_{\overline{K}_2})$ of the map $F_\mu|_{\overline{K}_2}$, $\mu \in (0, 1)$.

From Lemma 12 follows, that for any $\mu \in (0, 1)$ set $\Omega(F_\mu|_{\overline{K}_2})$ is finite and consists of fixed points $A_3(\mu - 1; 1)$, $A_1(0; \mu^2)$; for $\mu = 1$ the equality is fulfilled $A_3(\mu - 1; 1) = A_1(0; \mu^2)$ and the set $\Omega(F_1|_{\overline{K}_2})$ consists of the single point $A_1(0; 1)$.

By formula (1) we obtain, that for any $\mu \in (0, 1]$ the following equalities holds

$$F_\mu(Ox) = \{0\} \times [0, +\infty), \quad F_\mu(Oy) = (0; \mu^2).$$

Therefore, for any $\mu \in (0, 1]$ the equality is fulfilled

$$(Ox \cup Oy) \cap \Omega(F_\mu) = A_1(0; \mu^2).$$

In addition, for any $\mu \in (0, 1)$ the global stable manifold $W^s(A_1)$ of sink $A_1$ and the global unstable manifold $W^u(A_3)$ of saddle point $A_3$ intersect transversally (see Figure 4). Therefore, we obtain, that for any $\mu \in (0, 1)$ the map $F_\mu$ is a (nonsingular) Morse-Smale endomorphism. For $\mu = 1$ there exist the nonhyperbolic point $A_1(0; 1)$ and map $F_1$ is the singular Morse-Smale endomorphism.

Thus, Theorem 1 is proved.

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Figure 4. Asymptotic behaviour of $F_{\mu}$ – trajectories in $K_2$. 

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