Uniqueness of $U_q(N)$ as a quantum gauge group
and
representations of its differential algebra

I.Ya.Aref’eva
and
G.E.Arutyunov *
Steklov Mathematical Institute, Russian Academy of Sciences,
Vavilov st.42, GSP-1,117966, Moscow, Russia

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Abstract

To construct a quantum group gauge theory one needs an algebra which is
invariant under gauge transformations. The existence of this invariant algebra is
closely related with the existence of a differential algebra $\delta_H G_q$ compatible with
the Hopf algebra structure. It is shown that $\delta_H G_q$ exists only for the quantum
group $U_q(N)$ and that the quantum group $SU_q(N)$ as a quantum gauge group is
not allowed.

The representations of the algebra $\delta_H G_q$ are constructed. The operators corre-
responding to the differentials are realized via derivations on the space of all irreducible
$*$-representations of $U_q(2)$. With the help of this construction infinitesimal gauge
transformations in two-dimensional classical space-time are described.

*E-mail: arefeva@qft.mian.su, arut@qft.mian.su
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1 Introduction

Recently a construction of quantum group gauge theory (QGGT), i.e. the gauge theory with a quantum group playing the role of the gauge group, has been initiated [1]-[10].

In spite of the impressive successes of applying gauge theory to the description of all known physical interactions the natural question about the possibilities of extending the strict frames of gauge theories arises. One can think that an enlargement of the rigid framework of gauge theory would help to solve fundamental theoretical problems of spontaneous symmetry breaking and quark confinement. The theory of quantum groups looks rather attractive as the mathematical foundation of a new theory since the general requirements of symmetries of a physical system can be formulated on the language of quantum groups [1,2].

At the last two years attempts where made to understand the algebraic structure of QGGT [1]-[10]. The main efforts have been to keep the classical form of gauge transformations for the gauge potential $A$:

$$A \rightarrow A' = TAT^{-1} + dT^{-1}. \quad (1)$$

Roughly speaking, the problem is in the following. Assume that $T$ is an element of a quantum group. What differential calculus should we consider and from what algebra $A$ should be taken a gauge potential $A$ to guarantee that $A'$ also belongs to the algebra $A$?

One of suitable resolutions of this problem has been recently found by Isaev and Popovicz [10]. In their scheme $T$ and $dT$ are realized as generators of the differential extension $\delta H G_q$ of a quantum group $G_q$ compatible with the Hopf algebra structure. It is natural to try to find $\delta H$-extension for $SU_q(N)$ that would lead to the algebraic formulation of the $SU_q(N)$ gauge theory. It is known that the $\delta H$ extension of $GL_q(N)$ does exist [12,13,14]. In this paper we deal with the analogous construction for the quantum group $U_q(N)$. The quantum group $U_q(N)$ is one of the real forms of $GL_q(N)$ and may be obtained from $GL_q(N)$ by introducing $\ast$-involution operation. The $\delta H$ extension of $GL_q(N)$ also admits $\ast$-involution and we get $\delta H(U_q(N))$. To obtain the $\delta H$-extension of $SU_q(N)$ one needs to fix the quantum determinant equal to unity. However, at this point one obstacle arises. Namely, the quantum determinant is not a central element of $\delta H(U_q(N))$ and therefore cannot be fixed equal to unity.

Summing up, it is possible to present an algebraic construction of the quantum group gauge potential for $U_q(N)$ but the quantum group $SU_q(N)$ as a gauge group is not allowed. Thus, if one believes that the ordinary gauge theory is obtained as the classical limit, $q \rightarrow 1$, of some QGGT then one can speculate that QGGT predicts the group $U(2)$. From this point of view the fact that electroweak group is $U(2) = SU(2) \otimes U(1)$ looks rather promising.

Fields defined on the classical space-time and taking value on a quantum group or quantum algebra should be the natural object of the QGGT [1]. However, just at this point there is the problem for a straightforward application of the standard approach to quantum groups. The ordinary local gauge theory is based on the existence of a sufficiently wide class of differentiable maps from space-time into a group. This class can be easily constructed since a Lie group is a smooth manifold and so it is possible to regard its coordinates as functions on space-time. In the standard quantum group approach the space of c-number parameters numerating points of a quantum group is not available. Usually the theory of quantum groups is formulated in terms of the function
algebra $Fun(G_q)$ on a quantum group $G_q$. Adopting this view and trying to describe QGGT one can expect that ordinary gauge theory may be formulated in terms of the function algebra $Fun(G_q)$, i.e. on the dual language. However, it is not suitable for field theory applications. So it is clear that one cannot build QGGT in the framework of the standard quantum group approach. We need to extend the usual content of quantum groups by introducing in the theory new objects. In other words, to consider a map of the classical space-time $R^4$ into a quantum group we need a more liberated treatment of a quantum group or quantum plane than the ordinary theory offers. Such an approach was suggested in [1]. For consideration of this problem see also [13, 14, 17, 18, 19]. An example showing the necessity of introduction new objects is given by the analogue of the exponential map for quantum groups. It turns out that in addition to a quantum group one should introduce a set of generators taking value in so-called ”quantum superplane” [1].

In this paper we present an explicit realization of a differentiable local map from the classical space-time $R^2$ into the quantum group $U_q(2)$ supplied with $*$-involution and compatible with the bicovariant differential calculus on $U_q(2)$. We will see that for this purpose it is suitable to consider a quantum group as the set of all its irreducible unitary representations and think of parameters numerating these representations as ”coordinates” on a quantum group. Note that this consideration is in the line of the approach [1] (see also [10]) and has an implicit support in a definition of integral on a quantum group proposed in [20]. By the compatibility of a map $R^2 → U_q(2)$ with the bicovariant differential calculus on $U_q(2)$ we mean that exterior derivative $d$ acting on elements of the quantum group can be decomposed over a basis $\{\varepsilon^i\}$ of ordinary differential forms on $R^2$: $d = \varepsilon^i \otimes \partial_i$.

To get representations for the derivatives $\partial_i$ we start from a construction of representations for $(T, L)$-pair, where $L$ is a quantum gauge field with zero curvature $L = dTT^{-1} = \left( \begin{array}{cc} \omega_0 & \omega_+ \\ \omega_- & \omega_1 \end{array} \right)$. The simple consideration shows that it is impossible to realize the operators $\omega_0, \omega_+, \omega_-, \omega_1$ in the space of an irreducible representation of the algebra $Fun(U_q(2))$. This means that we have to extend the space of representation. We deal with a direct integral of Hilbert spaces over parameters labelled irreducible representations of $U_q(2)$. We find a simple formulae for the operators $\omega_0, \omega_+, \omega_-, \omega_1$ and then get the representations for $dT$. It turns out that there are two types of representations of $dT$ corresponding to differentiations over two parameters specified irreducible unitary representations of $U_q(2)$. For these two different representations we use notations $\partial_1 T$ and $\partial_2 T$. Thus we realize the derivatives $\partial_i$ as differentiations over parameters of representations of a quantum group itself, i.e. as differentiations over ”coordinates” on a quantum group.

Now having at hand an explicit form of derivatives of $U_q(2)$ elements one can locally construct a differentiable map $R^2 → U_q(2)$ as following:

$$T(x) = \left( \begin{array}{cc} a(x) & b(x) \\ c(x) & d(x) \end{array} \right) = \left( \begin{array}{cc} a + x^i \partial_i a & b + x^i \partial_i b \\ c + x^i \partial_i c & d + x^i \partial_i d \end{array} \right), \quad i = 1, 2.$$  (2)

Differentiation of operators $a(x), \ldots, d(x)$ with respect to $x^1, x^2$ gives the derivatives compatible with the bicovariant differential calculus. In this construction we have to limit ourselves by considering the two-dimensional classical space-time since the space of parameters of infinite dimensional unitary representations of $U_q(2)$ is two-dimensional.

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1This approach to integral on quantum group is used to define a lattice QGGT [21]
Therefore we have for the bicovariant differential calculus on $U_q(2)$ a usual ”classical” picture: if the quantum group is a set of its irreducible unitary representations then the quantum group derivatives are indeed derivatives with respect to the coordinates on $U_q(2)$.

The paper is organized as follows. In section 2 we describe an algebraic approach to constructing QGGT. In section 3 we introduce $*$-involution for the $\delta_H$-extension of the algebra $Fun(GL_q(N))$ and prove a no-go theorem that only $\delta_H(U_q(N))$ exists and the $\delta_H$-extension of $SU_q(N)$ is not allowed. In sections 4 and 5 two inequivalent $*$-representations of the $\delta_H$-extension of $U_q(2)$ in the Hilbert space are constructed. We use them in section 6 to write out an explicit form of a two-dimensional map into the quantum group $U_q(2)$.

## 2 Algebraic Scheme of QGGT

Let $T$ belongs to a quantum matrix group $G$, i.e. $T$ is the subject of the relations:

$$R_{12}T_1T_2 = T_2T_1R_{12}. \quad (3)$$

Here $R_{12}$ is a quantum R-matrix and $T_1 = T \otimes I, T_2 = I \otimes T$ (see [22, 23] for details). Recall that due to the existence of the Hopf algebra structure for quantum groups the product $gT$ of two elements being the subjects of equation (3) satisfies

$$R_{12}(gT)_1(gT)_2 = (gT)_2(gT)_1R_{12}, \quad (4)$$

if the entries of the matrices $g$ and $T$ mutually commute.

To construct QGGT one can start with the consideration of gauge fields having zero curvature [14]:

$$L = dTT^{-1}. \quad (5)$$

In order to give a meaning to (5) we need to specify the differential $dT$ on a quantum group, i.e. to determine the differential calculus. Differential calculi on quantum groups were developed in [22, 20, 12, 24, 13]. The operator of exterior derivative is supposed to have the usual properties:

$$d^2 = 0, \quad d(AB) = (dA)B + A(dB). \quad (6)$$

If $T$ and $dT$ are understood as matrices with non-commutative entries then $L$ is also a matrix with non-commutative entries. It is interesting to know if there are permutation relations between the entries of $L$ that can be written in terms of R-matrix. For the case of $GL_q(N)$ the answer is yes if one considers a special differential calculus. It can be formulated by introducing the set of generators $dT_{ij}$ that satisfy the relations [10, 12, 14, 24]:

$$R_{12}(dT)_1T_2 = T_2(dT)_1R_{21}^{-1}, \quad (7)$$

$$R_{12}(dT)_1(dT)_2 = -(dT)_2(dT)_1R_{21}^{-1}. \quad (8)$$

The relations (3), (7) and the definition (5) yield the quadratic algebra for $L$:

$$R_{12}L_1R_{21}L_2 + L_2R_{12}L_1R_{12}^{-1} = 0. \quad (9)$$

The differential calculus [12, 10] is compatible with the Hopf algebra structure in the sense that the following equation is satisfied

$$R_{12}d(gT)_1(gT)_2 = d(gT)_2(gT)_1R_{21}^{-1} \quad (10)$$
(for more precise definition see section 3). For ordinary groups the field $L$ is transformed under gauge transformations

$$T \rightarrow gT$$

as follows:

$$L \rightarrow L' = gLg^{-1} + dgg^{-1}.$$  \tag{12}

It is remarkable that equations (11) and (12) are enough to guarantee the invariance of the algebra (12) under transformations (12). Thus to construct QGGT we have to extend our quantum group to the $\delta_\mathcal{H}$ Hopf algebra.

The next nontrivial step is to postulate for the gauge potential $A$ of the general form

$$R_{12}A_1R_{21}A_2 + A_2R_{12}A_1R_{12}^{-1} = 0.$$ \tag{13}

The relations (11), (14) are again enough for invariance of (13) under gauge transformations (12). Note that as in the case of zero curvature potential we assume that the entries of the matrices $A$ and $g$ are mutually commutative. In what follows we will regard the quadratic algebra (13) as the algebra $A$ of quantum group gauge potentials. The corresponding curvature has the form

$$F = dA - A^2$$

and it is transformed under gauge transformations as

$$F \rightarrow gFg^{-1}.$$ \tag{14}

To conclude the brief presentation of the formal algebraic QGGT construction one has to mention that the action should be taken in the form $tr_q F^2$, since the $q$-trace [22, 3] is invariant under (14). Note that $F$ also belongs to a quadratic algebra [10] defined by the reflection equations [25].

### 3 *-involution for the $\delta_\mathcal{H}$ extension of $Fun(U_q(2))$

The $\delta_\mathcal{H}$ extension $\delta_\mathcal{H}(GL_q(N))$ of the Hopf algebra $Fun(GL_q(N))$ is the Hopf algebra [3] itself with the comultiplication $\Delta$, the counity $\epsilon$ and the antipod $S$ which are defined by

\begin{align*}
\Delta(T) &= T \otimes T, \quad \epsilon(T) = 1, \quad S(T) = T^{-1}, \\
\Delta(dT) &= dT \otimes T + T \otimes dT, \quad \epsilon(dT) = 0, \quad S(dT) = -T^{-1}dT^{-1}.
\end{align*}

\tag{15}

Now we are going to show that $\delta_\mathcal{H}(GL_q(N))$ admits $*$-involution.

Recall the definition of $*$-involution of a Hopf algebra. An involution $*$ of a Hopf algebra $\mathcal{A}$ is a map $\mathcal{A} \rightarrow \mathcal{A}$ which is the algebra antiautomorphism and the coalgebra automorphism obeying two conditions:

1. $(a^*)^* = a$
2. $S(S(a^*)) = a$ for any $a \in \mathcal{A}$. 

5
Let $q$ be real. Supposing the existence of the $*$-involution for the Hopf algebra $A$ and applying $*$ to equation (4) one finds:

$$R_{12}(T^*)_2(dT^*)_1 = (dT^*)_1(T^*)_2R_{21}^{-1}$$

(16)

Using the Hopf algebra structure of $\delta_H(GL_q(N))$ (15) and defining relations (3)-(7) one can deduce that $T^*$ obeys equation:

$$R_{12}(S(T)^*)_2(S(dT)^*)_1 = (S(dT)^*)_1(S(T)^*)_2R_{21}^{-1}$$

(17)

where $t$ means the matrix transposition. Comparing (16) and (17) we see that it is possible to make the identification:

$$T^* = S(T)^t$$

and $$(dT)^* = S(dT)^t = -(T^{-1})^t(dT)^t(T^{-1})^t$$

(18)

One can check that the operation $*$ introduced in the last equation is the involution of the Hopf algebra $\delta_H(GL_q(N))$. The $*$-Hopf algebra arising in such a way is nothing but the $\delta_H$ extension $\delta_H(U_q(N))$.

Let us show that the quantum determinant $D$ for the $GL_q(N)$ is not a central element of the extended algebra $\delta_H(GL_q(N))$. It is well known (22) that $D$ can be written in the form:

$$D = \text{tr} \left( P^-(T \otimes T) \right) = \text{tr} \left( P^- T_1 T_2 P^- \right)$$

(19)

where $P^-$ is a projector:

$$P^- = -PR + qI$$

(20)

that can be treated as the quantum analog of symmetrizer in $C^n \otimes C^n$. Then we have:

$$(dT)_1 P_{23} T_2 T_3 P_{23} = P_{23}^{-1} R_{12}^{-1} R_{13}^{-1} T_2 T_3 (dT)_1 R_{31}^{-1} R_{21}^{-1} P_{23}$$

(21)

For the projector $P^-$ we have the relation that follows from definition (20):

$$P_{23}^{-1} R_{12} R_{13} = q P_{23}$$

(22)

Therefore (21) reduces to:

$$\frac{1}{q} P_{23}^{-1} T_2 T_3 (dT)_1 R_{31}^{-1} R_{21}^{-1} P_{23}$$

The transposition of (22) gives

$$R_{31}^{-1} R_{21}^{-1} P_{23} = \frac{1}{q} P_{23}^-$$

where the fact was used that $(P_{23}^-)^t = P_{23}$. Now equation (21) takes the form:

$$(dT)_1 P_{23}^{-1} T_2 T_3 P_{23} = \frac{1}{q^2} P_{23}^- T_2 T_3 P_{23}^- (dT)_1$$

Finally taking the trace we obtain:

$$dTD = \frac{1}{q^2} DdT$$

(23)

Thus $D$ is not a central element of $\delta_H(GL_q(N))$. Applying the involution to $D$ we find that $D$ is the unitary element of $Fun(U_q(N))$:

$$D^* D = DD^* = I.$$
4 Type I representation of the Hopf algebra
\( \delta_H(\text{Fun}(U_q(2))) \)

Let us consider the question about \(*\)-representations of the Hopf algebra \( \delta_H(\text{Fun}(U_q(2))) \) in the separable Hilbert space \( \mathcal{H} \). Since \( \text{Fun}(U_q(2)) \) has the completion that is a \( C^* \)-algebra the elements \( T_{ij} \) can be realized as bounded operators in \( \mathcal{H} \). Then \( \partial T_{ij} \) are unbounded operators. It can be proved by using (23). Let us suppose that \( \partial T_{ij} \) is a bounded operator and \( q < 1 \). Then the norm \( ||\partial T_{ij}|| \) is defined and

\[
D\partial T_{ij} D = \frac{1}{q^2} \partial T_{ij} .
\]

This allows one to write

\[
||D\partial T_{ij} D|| = ||\partial T_{ij} D|| \leq ||\partial T_{ij}|| ||D|| = ||\partial T_{ij}|| .
\]

So we obtain

\[
\frac{1}{q^2} ||\partial T_{ij}|| \leq ||\partial T_{ij}||
\]

and therefore \( 1/q^2 \leq 1 \) that contradicts to \( q < 1 \).

It is difficult to begin with constructing representations of the algebra (3), (7), (8) since the involution condition for \( \partial T_{ij} \) is rather complicated (we will come back to this question in the next section). But this difficulty can be solved by introducing the new set of generators \( L = \partial TT^{-1} \) on which the action of the involution is simple. From (3), (4) one can deduce the defining relation for \( T \) and \( L \):

\[
R_{12}L_1R_{21}T_2 = T_2L_1 .
\]

(24)

The pair \((T, L)\) is called the \((T, L)\)-pair [1]. The involution property for \( L \) is

\[
L^\dagger = -L .
\]

Now we are going to construct the special representation of the \((T, L)\)-pair in a Hilbert space for the case \( T \in \text{Fun}(U_q(2)) \). Let \( T = ||t_{ij}|| \) be the matrix of the form:

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

the inverse of which is

\[
T^{-1} = D^{-1} \begin{pmatrix} d & -\frac{1}{q}b \\ -qc & a \end{pmatrix} .
\]

Here \( D = ad - qbc \) is the quantum determinant. Taking into account the involution one can write: \( a^* = D^{-1}d \) and \( c^* = -\frac{1}{q}D^{-1}b \). \( L \) is the matrix:

\[
L = \begin{pmatrix} \omega_0 & \omega_+ \\ \omega_- & \omega_1 \end{pmatrix} ,
\]

where the entries obey the following involution relations \( \omega_0^* = -\omega_0, \omega_+^* = -\omega_-, \omega_1^* = -\omega_1 \), i.e. \( \omega_0 \) and \( \omega_1 \) are antihermitian. The explicit form of the permutation relations for the \((T, L)\)-pair is

\[
a\omega_0 = q^2 \omega_0 a, \quad a\omega_1 = \omega_0 c ,
\]
\[ b \omega_0 = q^2 \omega_0 b, \quad d \omega_0 = \omega_0 d, \] (25)

\[ a \omega_+ = q \omega_+ a, \quad c \omega_+ = q \omega_+ c + \mu \omega_0 a, \]
\[ b \omega_+ = q \omega_+ b, \quad d \omega_+ = q \omega_+ d + \mu \omega_0 b, \] (26)

\[ c \omega_- = q \omega_- c, \quad a \omega_- = q \omega_- a + \mu \omega_0 c, \]
\[ d \omega_- = q \omega_- d, \quad b \omega_- = q \omega_- b + \mu \omega_0 d, \] (27)

\[ a \omega_1 = \omega_1 a + \mu c \omega_+, \quad b \omega_1 = \omega_1 b + \mu d \omega_+, \]
\[ c \omega_1 = q^2 \omega_1 c + q \mu \omega_0 a, \quad d \omega_1 = q^2 \omega_1 d + q \mu \omega_0 b, \] (28)

where \( \mu = q - \frac{1}{q} \).

Let \( \pi \) be a \( \ast \)-representation of the algebra \( \text{Fun}(U_q(N)) \) in the separable Hilbert space \( l_2 \). The operators \( \pi(t_{ij}) \) are supposed to be continuous ones. In [26, 27] it was proved that every irreducible \( \ast \)-representation \( \pi \) of \( \text{Fun}(U_q(N)) \) is unitary equivalent to the one of the following two series:

1. One dimensional representations \( \xi_\psi \) given by the formulae:

\[ \xi_\psi(a) = e^{i \psi}, \quad \xi_\psi(c) = 0, \quad \psi \in \mathbb{R}/2\pi \mathbb{Z}. \]

2. Infinite dimensional representations \( \rho_{\theta, \phi} \) in a Hilbert space with orthonormal basis \( \{e_n\}_{n=0}^\infty \):

\[ \rho_{\theta, \phi}(a)e_0 = 0, \quad \rho_{\theta, \phi}(a)e_n = e^{i(\theta + \phi)} \sqrt{1 - e^{-2nh}e_{n-1}}, \quad \rho_{\theta, \phi}(c)e_n = e^{i\theta} e^{-nh} e_n \]
\[ \rho_{\theta, \phi}(d)e_n = \sqrt{1 - e^{-2(n+1)h}e_{n+1}}, \quad \rho_{\theta, \phi}(b)e_n = -e^{i\phi} e^{-(n+1)h} e_n \] (29)

Here \( \theta, \phi \in [0, 2\pi) \), \( q = e^{-h} \).

Thus for \( \text{Fun}(U_q(N)) \) the set \( \hat{\mathcal{F}} \) of equivalence classes of irreducible unitary representations consists of two separate components each of these is numerated by continuous parameters \( \theta, \phi \in T^2 = S^1 \times S^1 \) playing the role of coordinates. We shall concentrate our attention on the infinite-dimensional component since only a trivial representation of differentials corresponds to one-dimensional representations of \( U_q(2) \).

The straightforward algebraic consideration shows that it is impossible to realise the operators \( \omega_0, \omega_+, \omega_-, \omega_1 \) in the space of an irreducible representation of the algebra \( \text{Fun}(U_q(2)) \). This means that one have to extend the space of representation or in other words to work with reducible representations of \( \text{Fun}(U_q(2)) \). It turns out that a suitable construction deals with a direct integral of Hilbert spaces. Let us consider the Hilbert space \( \mathcal{H} \) of functions on a circle taking value in \( l_2 \). It is known [28] that there exists the canonical isomorphism \( \mathcal{H} = l_2 \otimes \mathcal{L}_2(S^1) \) and

\[ \mathcal{H} = \int_{S^1} \mathcal{H}(\phi) d\phi, \]
where \( L_2(S^1) \) is the space of square integrable functions on a circle obeying the condition:

\[
\int_{-\pi}^{\pi} |f(\phi)|^2 d\phi < \infty ,
\]

for any \( f \in L_2(S^1) \) and \( \mathcal{H}(\phi) = l_2 \). Now the reducible representation of \( \text{Fun}(U_q(2)) \) in \( \mathcal{H} \) can be defined in the following manner:

\[
\hat{a}(e_0 \otimes f) = 0 ,
\]

\[
\hat{a}(e_n \otimes f) = \sqrt{1 - e^{-2nh}e_{n-1} \otimes e^{i(\theta + \phi)} f} ,
\]

\[
\hat{d}(e_n \otimes f) = \sqrt{1 - e^{-2(n+1)h}e_{n+1} \otimes f} ,
\]

\[
\hat{b}(e_n \otimes f) = -e^{-(n+1)h}e_n \otimes e^{i\phi} f ,
\]

\[
\hat{c}(e_n \otimes f) = e^{-nh}e_n \otimes e^{i\theta} f ,
\]

where the operators \( \hat{a}, \hat{b}, \hat{c}, \hat{d} \) correspond to \( a, b, c, d \). On putting \( \phi \) equal to some value \( \phi_0 \) the irreducible representation \( \pi_{\theta,\phi_0} \) of \( \text{Fun}(U_q(N)) \) stands out.

The scalar product in \( \mathcal{H} \) is given by

\[
< (e_n \otimes f), (e_m \otimes g) > = (e_n, e_m) \int_{-\pi}^{\pi} f \bar{g} d\phi .
\]

Introduce the following hermitian operator \( K \) defined on a dense region in \( L_2(S^1) \):

\[
(Kf) (\theta, \phi) = \sum_n a_{nm}q^{2n} \gamma^n = \left(e^{-2ih\frac{\partial}{\partial \phi}} f\right)(\theta, \phi) ,
\]

where \( f = \sum_n a_n \gamma^n \) is an arbitrary element of \( \mathcal{H} \), \( \gamma = e^{i\phi} \).

Now let us take \( \omega_0 \) to be the operator in \( \mathcal{H} \):

\[
\omega_0(e_n \otimes f) = ie_n \otimes e^{-2ih\frac{\partial}{\partial \phi}} f .
\]

Then \( \omega_0 \) is antihermitian as it is required. The permutation relations of \( \omega_0 \) with the operators \( \hat{a}, \ldots, \hat{d} \) are precisely (25). For the operator \( \omega_+ \) one can choose the realization:

\[
\omega_+(e_n \otimes f) = -ie^{(n+2)h} \sqrt{1 - e^{-2nh}e_{n-1} \otimes e^{i\phi} e^{-2ih\frac{\partial}{\partial \phi}} f} .
\]

Introducing formally the inverse operators \( \hat{b}^{-1} \) and \( \hat{c}^{-1} \):

\[
\hat{b}^{-1}(e_n \otimes f) = -e^{(n+1)h}e_n \otimes e^{-i\phi} f ,
\]

\[
\hat{c}^{-1}(e_n \otimes f) = e^{nh}e_n \otimes e^{-i\theta} f ,
\]

it is possible to rewrite \( \omega_+ \) in terms of \( \hat{a}, \ldots, \hat{c}^{-1} \):

\[
\omega_+ = -q^{-3} \hat{c}^{-1} \hat{a} e^{-2ih\frac{\partial}{\partial \phi}} = -q^{-3} \hat{c}^{-1} \hat{a}\omega_0 .
\]

By straightforward calculations one can check the fulfilment of the permutation relations (24) on a dense region where all operators coming in (35) are defined. Taking \( \omega_- \) to be the hermitian conjugation of \( \omega_+ \) with respect to the scalar product (31):

\[
\omega_- = \hat{d} \hat{b}^{-1} \omega_0 ,
\]
we find that the relations (27) are satisfied. Now the question arises how to find the operator \( \omega_1 \) that must be antihermitian and obey (28). We choose for \( \omega_1 \) the following ansatz:

\[
\omega_1 = \mathcal{P}(\hat{a}, \ldots, \hat{c}^{-1}) \omega_0, \tag{37}
\]

where \( \mathcal{P}(\hat{a}, \ldots, \hat{c}^{-1}) \) is a polynomial in \( \hat{a}, \ldots, \hat{c}^{-1} \). Then the first line in (28) reads:

\[
\hat{a}\mathcal{P} = \frac{1}{q^2} \mathcal{P}\hat{a} - \frac{\mu}{q^3} \hat{a}, \quad \hat{b}\mathcal{P} = \frac{1}{q^2} \mathcal{P}\hat{b} - \frac{\mu}{q^3} \hat{c}^{-1}\hat{a}, \tag{38}
\]

and the second one is achieved from the first by hermitian conjugation. Equations (38) have the simple solution \( \mathcal{P} = -\frac{1}{q^2} \hat{b}^{-1} \hat{d} \hat{c}^{-1} \hat{a} \). Therefore \( \omega_1 \) takes the form:

\[
\omega_1 = -\frac{1}{q^2} \hat{b}^{-1} \hat{c}^{-1} \hat{a} \tag{39}
\]

The found operators \( \omega_0, \omega_+, \omega_-, \omega_1 \) combined with (30) give \(*\)-representation of the \((T,L)\)-pair. Coming back to the derivatives \( \partial T = LT \) we see that

\[
\begin{align*}
\partial a &= 0, \quad \partial b = -\frac{1}{q^2} c^{-1} D \omega_0, \\
\partial c &= 0, \quad \partial d = -\frac{1}{q^2} d(b \hat{c}^{-1})^{-1} D \omega_0. \tag{40}
\end{align*}
\]

This means that our differentials can be treated as elements of the algebra \( \text{Fun}(U_q(2)) \) which is extended by adding the new element \( \omega_0 \) provided that the elements \( b \) and \( c \) are invertible. The equalities \( \partial a = 0 \) and \( \partial c = 0 \) seem rather restrictive and give a hint that other \(*\)-representations for which \( \partial a \neq 0, \partial c \neq 0 \) should also exist. In the next section we will construct one of such examples.

5 Type II \(*\)-representation of \( \delta\mathcal{H}(\text{Fun}(U_q(2))) \)

The permutation relations between the elements \( a, b, c, d \) and their derivatives follow from (7). In particular we have:

\[
\begin{align*}
a(\partial a) &= q^2 (\partial a) a, \quad c(\partial a) = q (\partial a) c, \\
b(\partial a) &= q (\partial a) b, \quad d(\partial a) = (\partial a) d. \tag{41}
\end{align*}
\]

Let us consider now the Hilbert space of square integrable functions on a torus \( T^2 = S^1 \times S^1 \) taking value in \( l_2 \). The scalar product in \( \mathcal{H} \) has the form:

\[
(f, g) = \int_{T^2} <f(\theta, \phi), g(\theta, \phi)> d\theta d\phi, \tag{42}
\]

where \( <,> \) is a scalar product in \( l_2 \). As in the previous section the reducible representation of \( \text{Fun}(U_q(2)) \) in \( \mathcal{H} \) can be defined by the formulas (30) where \( \theta \) is no longer a parameter and \( f = f(\theta, \phi) \in \mathcal{H} \).

Note that equations (41) are compatible with the condition \( \partial a = (\partial a)^* \). This allows one to choose for \( \partial a \) the simple realization by the hermitian unbounded operator:

\[
(\partial a)(\theta, \phi) = \sum_{nm} a_{nm} q^{n+m} \gamma_1^n \gamma_2^m = e^{-ih\frac{\partial}{\partial \theta} - ih\frac{\partial}{\partial \phi}} f(\theta, \phi), \tag{43}
\]

\[
\]
where \( f = \sum_{nm} a_{nm} \gamma_1^n \gamma_2^m \) is an arbitrary element of \( \mathcal{H} \), \( \gamma_1 = e^{i\theta}, \gamma_2 = e^{i\phi} \) and \( \partial \hat{a} \) is the operator that corresponds to \( a \).

One can go further and require the condition \( \partial a = (\partial a)^* \) to be consistent with the involution \( \mathcal{H} \) whose explicit form is

\[
(\partial a)^* = -q^2(D^{-1}) \left(q^2 d^2(\partial a) - bd(\partial c) - qdc(\partial b) + \frac{1}{q} bc(\partial d)\right),
\]

\[
(\partial d)^* = -q^2(D^{-1}) \left((q^2 ad - D)(\partial a) - qab(\partial c) - qac(\partial b) + a^2(\partial d)\right),
\]

\[
(\partial b)^* = -q^2(D^{-1}) \left(-q^2 cd(\partial a) + qad(\partial c) + q^2 c^2(\partial b) - ac(\partial d)\right),
\]

\[
(\partial c)^* = -q^2(D^{-1}) \left(-q^2 db(\partial a) + b^2(\partial c) + \frac{1}{q} ad(\partial b) - \frac{1}{q} ba(\partial d)\right).
\]

In this way we can express \( \partial d \) in terms of \( \partial a, \partial b \) and \( \partial c \):

\[
\partial d = q(bc)^{-1} \left(-q^2 d^2 + \frac{1}{q^2} D^2(\partial a) + bd(\partial c) + qdc(\partial b)\right).
\]

Therefore to specify \( \partial d \) we need the explicit realization of \( \partial b \) and \( \partial c \) that are compatible with \( \mathcal{H} \), \( \mathcal{J} \). The permutation relations between \( \partial b, \partial c \) and the elements of \( \text{Fun}(U_q(2)) \) are

\[
a(\partial c) = q^2(\partial c)a + q\mu(\partial c), \quad c(\partial c) = q^2(\partial c)c,
\]

\[
b(\partial c) = (\partial c)b + \mu(\partial a)d, \quad d(\partial c) = (\partial c)d,
\]

\[
a(\partial b) = q(\partial b)a + \mu(\partial b)c, \quad c(\partial b) = (\partial b)c + \mu d(\partial a),
\]

\[
b(\partial b) = q^2(\partial b)b, \quad d(\partial b) = q(\partial b)d.
\]

To solve equations \( \mathcal{H} \) and \( \mathcal{J} \) we take as in the previous section:

\[
\partial b = \mathcal{I}\partial a \quad , \quad \partial c = \mathcal{J}\partial a
\]

where \( \mathcal{I} \) and \( \mathcal{J} \) are polynomials in \( \hat{a}, \ldots, \hat{a} \) that should be defined. By simple computations we find:

\[
\partial \hat{a} = q\hat{c}^{-1}\hat{a}\partial a \quad , \quad \partial \hat{c} = \hat{b}^{-1}\partial \hat{a}.
\]

Then \( \mathcal{H} \) reduces to the form:

\[
\partial \hat{a} = (\partial \hat{c})^{-1} (q^3 d^2 - \frac{1}{q} D^2)\partial \hat{a}.
\]

Using the found representation of the derivatives \( \mathcal{H}, \mathcal{J} \) one can show that the permutation relations for \( \partial d \):

\[
a(\partial d) = (\partial d)a + \mu(\partial b)c + \mu b(\partial c), \quad c(\partial d) = q(\partial d)c + \mu d(\partial c),
\]

\[
b(\partial d) = q(\partial d)b + \mu(\partial b)d, \quad d(\partial d) = q^2(\partial d)d.
\]

are satisfied. The conjugation of the operators \( \partial b, \partial c, \partial d \) with respect to the scalar product \( \mathcal{H} \) gives

\[
(\partial b)^* = -qab^{-1}(\partial a),
\]

\[
(\partial c)^* = -\frac{1}{q^2} c^{-1}(a\partial a),
\]

\[
(\partial d)^* = q(a^2 - 1)(bc)^{-1}(\partial a).
\]

The consistency of equations \( \mathcal{H} \) with \( \mathcal{H}, \mathcal{J} \) can be easily checked. Hence the formulae \( \mathcal{H}, \mathcal{J} \) give the other *-representation of the \( \delta \)-extension of \( \text{Fun}(U_q(2)) \). We refer to this representation as the type II.
6 Two-dimensional Local Gauge Transformations

Now having at hand an explicit representations of the \( \delta_H \)-extension of \( U_q(N) \) we are able to construct two-dimensional infinitesimal local gauge transformations. As was mentioned in Introduction it seems reasonable to identify the representations of types I and II with derivations in two linear independent directions of space-time. To construct an explicit realization of two-dimensional differentials of \( U_q(2) \) elements it is convenient to introduce the operators \( K_1 \) and \( K_2 \) acting on \( H \) in the following manner:

\[
\begin{align*}
(K_1 f)(\theta, \varphi) &= \sum_{nm} a_{nm} q^{2n} \gamma_1^n \gamma_2^m f(\theta, \varphi), \\
(K_2 f)(\theta, \varphi) &= \sum_{nm} a_{nm} q^{n+m} \gamma_1^n \gamma_2^m f(\theta, \varphi)
\end{align*}
\]

(55)

(56)

where \( \gamma_1 = e^{i\theta}, \gamma_2 = e^{i\varphi} \) and \( f \in H \). Then according to our conjecture we may regard the formulae (40) as the derivatives of the elements of \( U_q(2) \) with respect to the coordinate \( x^1 \) on \( \mathbb{R}^2 \), and the formulae (43),(51),(52) as the derivatives in the \( x^2 \) direction. Thus the decomposition of differentials on \( U_q(2) \) over the basis \( \{dx^i\} \) of differential forms on \( \mathbb{R}^2 \) is

\[
\begin{align*}
\delta a &= K_2 \otimes dx^2, \\
\delta b &= -\frac{1}{q^3} c^{-1} DK_1 \otimes dx^1 + q c^{-1} dK_2 \otimes dx^2, \\
\delta c &= db^{-1} K_2 \otimes dx^2, \\
\delta d &= -\frac{1}{q^3} d(bc)^{-1} DK_1 \otimes dx^1 + (bc)^{-1} \left(q^3 d^2 - \frac{1}{q} D^2\right) K_2 \otimes dx^2.
\end{align*}
\]

(57)

(We use \( \delta \) for exterior derivative rather than \( d \) to avoid misunderstanding with the element \( d \)).

Let us make a comment about equation (8). Formally this equation follows from (6) and the identity \( d^2 = 0 \). However, we have not define in the above construction an action of the operator \( d \) on \( dT \). Nevertheless, \( dT = \begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix} \) does satisfy the relation (8). This is non-trivial since the right hand side of (8) contains two differentials \( dx^1 \) and \( dx^2 \) and is the result of direct calculations.

The meaning of the formulae (57) is that locally we have a two-parameter map \( \mathbb{R}^2 \to U_q(2) \) compatible with the bicovariant differential calculus on \( U_q(2) \). For example, one can write

\[
b(x^1, x^2) = b - \frac{1}{q^3} c^{-1} DK_1 \otimes x^1 + q c^{-1} dK_2 \otimes x^2,
\]

then the usual derivations of \( b \) with respect to \( x^1 \) or \( x^2 \) give the operators on \( H \) with desirable properties (7),(8). It would be interesting to construct a global map \( \mathbb{R}^2 \to U_q(2) \) and this is the subject of further investigations.

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