On Soft \textit{w}-Structures Defined by Soft Sets

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Abstract
In this work, we introduce the notion of soft \textit{w}-structure and investigate some basic properties of this new structure by using the concept of soft set. Moreover, we study the notions of soft \textit{w}-\textit{T}_0 (soft \textit{w}-\textit{T}_1, soft \textit{w}-\textit{T}_2).

Keywords: Soft set, Soft topology, Soft \textit{w}-structure, Soft \textit{w}-\textit{T}_0 (\textit{w}-\textit{T}_1, \textit{w}-\textit{T}_2).

1. Introduction and Preliminaries

In 1999, Molodtsov [1] initiated the notion of soft set theory as a new mathematical tool which is free from the complex problems. Later on Maji et al. [2] proposed several operations on soft sets and some basic properties and then Pei and Miao [3] investigated the relationships between soft sets and information systems.

In 2011, Shabir and Naz [4] introduced the notion of soft topological spaces and the author [5] corrected some their results. Zorlutuna et al. [6] continued to study the properties of soft topological spaces by defining the concepts of interior and soft neighborhoods in soft topological spaces. In 2011, Cagman et al. [7] defined soft topological spaces by modifying the soft set. Also, Roy and Samanta [8] strengthen the definition of the soft topological spaces presented in [7].

In 2017, with the aim of generalizing the notion of soft topology, Zakari et al. [9] introduced a soft weak structure. Recently, Al-Saadi and Min [10] investigated the notion of soft generalized closed sets in a soft weak structure.

Meanwhile, Min and Kim [11] introduced a new notion called weak structures as the following: Let \( X \) be a non-empty set and \( P(X) \) denote the power set of \( X \).

Definition 1.1 ([1]). For \( A \subseteq E \), a pair \((F,A)\) is called a soft set over \( X \), where \( F \) is a mapping given by \( F : A \rightarrow P(X) \). For \( e \in A \), \( F(e) \) may be considered as the set of \( e \)-approximate elements of the soft set \((F,A)\).

Definition 1.2 ([2]). A soft set \((F,A)\) over \( X \) is said to be:
1) A null soft set denoted by $\emptyset$ if $F(e) = \emptyset$ for all $e \in A$.

2) An absolute soft set denoted by $\bar{X}$ if $F(e) = X$ for all $e \in A$.

**Definition 1.3** ([2]). For any two soft sets $(F, A)$ and $(G, B)$ defined over a common universe $X$, we have:

1) $(F, A)\tilde{\subseteq}(G, B)$ iff $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$.

2) $(F, A)=(G, B)$ iff $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$.

3) $(F, A)\cup(G, B)=(H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B, \end{cases}$$

for all $e \in C$.

4) $(F, A)\cap(h(G, B) = (K, D)$ where $D = A \cap B$ and $K(e) = F(e) \cap G(e)$ for all $e \in C$.

5) $x \in (F, A)$ where $x \in X$ iff $x \in F(e)$ for all $e \in A$ and $x \not\in (F, A)$ whenever $x \not\in F(e)$ for some $e \in A$.

**Definition 1.4** ([12]). For a soft set $(F, A)$ over $X$, the relative complement of $(F, A)$ (denoted by $(F, A)'$) is defined by $(F, A)' = (F', A)$, where $F' : A \to P(X)$ is given by $F'(e) = X - F(e)$ for all $e \in A$.

**Definition 1.5** ([4]). Let $\tau$ be the collection of soft sets over $X$. Then $\tau$ is called a soft topology on $X$ if $\tau$ satisfies the following axioms:

1) $\emptyset, \bar{X}$ belong to $\tau$.

2) The union of any number of soft sets in $\tau$ belong to $\tau$.

3) The intersection of any two soft sets in $\tau$ belong to $\tau$.

The triple $(X, \tau, E)$ is called a soft topological space over $X$. The member of $\tau$ are said to be soft open in $X$. A soft set $(F, E)$ over $X$ is said to be soft closed in $X$ if its relative complement $(F, E)'$ belong to $\tau$.

**2. Soft $w$-Structures**

**Definition 2.1.** Let $sw$ be the collection of soft sets over $X$. Then $sw$ is called a soft $w$-structure on $X$ if $sw$ satisfies the following axioms:

- The intersection of any two soft sets in $sw$ belongs to $sw$.

The triple $(X, sw, E)$ is called a soft $w$-space over $X$. The member of $sw$ is said to be soft $w$-open in $X$. A soft set $(F, E)$ over $X$ is said to be soft $w$-closed in $X$ if its relative complement $(F, E)'$ belongs to $sw$.

**Remark 2.2.** Let $sw$ be a soft $w$-structure over $X$. The soft $w$-structure $sw$ is a kind of generalized soft topology and a stronger structure than a soft weak structure defined by Zakari et al. [9] as the following: Let $X$ be a non-empty set and $E$ a set of parameters. A collection $\omega$ of soft sets defined over $X$ with respect to $E$ is called a soft weak structure [9] iff $\emptyset \in \omega$.

**Example 2.3.** Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2\}$ and $sw = \{(\emptyset, \bar{X}, (F_1, E), (F_2, E), (F_3, E))\}$, where

- $F_1(e_1) = \{h_2, h_3\}$, $F_2(e_2) = \{h_1, h_2\}$;
- $F_2(e_1) = \{h_1, h_2\}$, $F_3(e_2) = \{h_1, h_3\}$;
- $F_3(e_1) = \{h_2\}$, $F_4(e_2) = \{h_1\}$.

Then $sw$ is a soft $w$-structure over $X$ with respect to $E$ but not a soft topology.

**Definition 2.4.** Let $sw$ be a soft $w$-structure over $X$ with respect to $E$. For a soft set $(F, E)$ over $X$, the soft $w$-closure of $(F, E)$ (simply, $c_{sw}(F, E)$) and the soft $w$-interior of $(F, E)$ (simply, $i_{sw}(F, E)$) are defined as the following:

- $i_{sw}(F, E) = \bigcup\{(G, E) : (G, E)\tilde{\subseteq}(F, E), (G, E) \in sw\}$.
- $c_{sw}(F, E) = \bigcap\{(H, E) : (F, E)\tilde{\subseteq}(H, E), (H, E)' \in sw\}$.

**Theorem 2.5.** Let $sw$ be a soft $w$-structure over $X$ with respect to the parameters set $E$ and $(F, E)$ a soft set. If there exists a soft $w$-open set $(G, E)$ such that $x \in (G, E)\tilde{\subseteq}(F, E)$, then $x \in i_{sw}(F, E)$

**Proof.** It is obvious.

**Example 2.6.** As in Example 2.3, consider the soft $w$-structure $sw$ over $X$ with respect to $E$ and a soft set $(F_4, E)$ as follows:

- $F_4(e_1) = \{h_1, h_2, h_3\}$,
- $F_4(e_2) = \{h_1, h_2, h_3\}$.

Then $(F_4, E) = i_{sw}(F_4, E)$. For $h_3 \in i_{sw}(F_4, E)$, there is no a soft $w$-open set containing $h_3$ in $sw$. So the converse of Theorem 2.5 is not always true.

**Theorem 2.7.** Let $sw$ be a soft $w$-structure over $X$ with respect to the parameters set $E$ and $(F, E)$ a soft set. If $x \in c_{sw}(F, E)$,
then $(G, E) \cap (F, E) \neq \emptyset$ for all $(G, E) \in sw$ such that $x \in (G, E)$.

**Proof.** Let $x \in c_{sw}(F, E)$. Suppose that there exists an element $(G, E) \in sw$ such that $x \in (G, E)$ and $(F, E) \cap (G, E) = \emptyset$. Then $(F, E) \subset (G, E)'$, so $c_{sw}(F, E) \subset \tilde{G}(G, E)'$ and $x \notin c_{sw}(F, E)$. So it is a contradiction. \[ \square \]

**Example 2.8.** Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E)\}$ where

\[
F_1(e_1) = \{h_2, h_3\}, \quad F_2(e_2) = \{h_1, h_2\}; \\
F_2(e_1) = \{h_1, h_2\}, \quad F_3(e_2) = \{h_1, h_3\}; \\
F_3(e_1) = \{h_2\}, \quad F_4(e_2) = \{h_1\}.
\]

Then $sw$ is a soft $w$-structure over $X$ with respect to $E$. Consider a soft set $(F_4, E)$ defined as:

\[
F_4(e_1) = \{h_1\}, \quad F_4(e_2) = \{h_3\}.
\]

Since $(F_4, E)$ is soft $w$-closed, $(F_4, E) = c_{sw}(F_4, E)$. For $h_1 \in X$, $(F_2, E)$ is the only soft $w$-open set and $(F_4, E) \cap (F_2, E) = \emptyset$, however, $h_3 \notin c_{sw}(F_4, E)$. So the converse of Theorem 2.7 is not always true.

**Theorem 2.9.** Let $sw$ be a soft $w$-structure defined over $X$ with respect to the parameters set $E$ and $(F, E)$ be a soft set.

- If $(F, E)$ is a soft $w$-open set, then $(F, E) = i_{sw}(F, E)$.
- If $(F, E)$ is a soft $w$-closed set, then $(F, E) = c_{sw}(F, E)$.

**Proof.** From the definitions of soft $w$-interior and soft $w$-closure, it is obvious. \[ \square \]

But the converses in Theorem 2.9 are not always true as shown the next example.

**Example 2.10.** Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$, where

\[
F_1(e_1) = \{h_3\}, \quad F_1(e_2) = \{h_2\}; \\
F_2(e_1) = \{h_2, h_3\}, \quad F_2(e_2) = \{h_1, h_2\}; \\
F_3(e_1) = \{h_1, h_2\}, \quad F_3(e_2) = \{h_1, h_3\}; \\
F_4(e_1) = \{h_1\}, \quad F_4(e_2) = \{h_2, h_3\}; \\
F_5(e_1) = \{h_2\}, \quad F_4(e_2) = \{h_1\}; \\
F_6(e_1) = \{h_1, h_3\}, \quad F_4(e_2) = \{h_2, h_3\}.
\]

Then $sw$ is a soft $w$-structure over $X$ with respect to $E$. For a soft set $(F_5, E)$, $c_{sw}(F_5, E) = (F_5, E)$ but $(F_5, E)$ is not soft $w$-closed. And, for a soft set $(F_6, E)$, $i_{sw}(F_6, E) = (F_6, E)$ but $(F_6, E)$ is not soft $w$-open.

**Theorem 2.11.** Let $sw$ be a soft $w$-structure over $X$ with respect to $E$. Let $(F, E)$ and $(G, E)$ be two soft sets over $X$. Then:

- $i_{sw}(F, E) \subset \tilde{G}(F, E)$.
- If $(F, E) \subset \tilde{G}(F, E)$, then $i_{sw}(F, E) \subset i_{sw}(G, E)$.
- $i_{sw}(F, E) = i_{sw}(F, E) \subset i_{sw}(G, E)$.
- $i_{sw}(F, E) \subset i_{sw}(F, E) = i_{sw}(F, E)$.

**Proof.** (1) and (2) are obvious.

(3) It is obvious that $i_{sw}(F, E) \subset \tilde{G}(F, E) \subset i_{sw}(G, E)$ from (2). For soft $w$-open sets $(U, E) \subset \tilde{G}(F, E)$ and $(V, E) \subset \tilde{G}(G, E)$, $(U, E) \cap (V, E)$ is a soft $w$-open set contained in $(F, E) \cap (G, E)$. This implies that $i_{sw}(F, E) \cap i_{sw}(G, E) = i_{sw}((F, E) \cap (G, E))$.

(4) From (1), it follows $i_{sw}(F, E) \subset i_{sw}(F, E) \subset i_{sw}(F, E)$. For any soft $w$-open set $(U, E)$ such that $(U, E) \subset i_{sw}(F, E)$, $(U, E) = i_{sw}(U, E) \subset i_{sw}(i_{sw}(F, E))$, and so $i_{sw}(F, E) \subset i_{sw}(i_{sw}(F, E))$. Consequently, we have $i_{sw}(i_{sw}(F, E)) = i_{sw}(F, E)$. \[ \square \]

**Theorem 2.12.** Let $sw$ be a soft $w$-structure defined over $X$ with respect to $E$. If $(F, E)$ and $(G, E)$ are two soft sets over $X$, then:

- $(F, E) \subset \tilde{c}_{sw}(F, E)$.
- If $(F, E) \subset \tilde{c}_{sw}(F, E)$, then $c_{sw}(F, E) \subset c_{sw}(G, E)$.
- $c_{sw}(F, E) \subset c_{sw}(G, E)$.
- $c_{sw}(c_{sw}(F, E)) = c_{sw}(F, E)$.

**Proof.** It is similar to the proof of Theorem 2.11. \[ \square \]

Now, we introduce the separation axioms in soft $w$-space with a soft $w$-structure $sw$.

**Definition 2.13.** Let $sw$ be a soft $w$-structure over $X$ with respect to $E$. A soft $w$-space $(X, sw, E)$ is called:
\[ w - T_0 \] if for each \( x, y \in X \) such that \( x \neq y \), there exists a soft \( w \)-open set \((F, E)\) such that \( x \in (F, E) \) and \( y \notin (F, E) \) or \( x \notin (F, E) \) and \( y \in (F, E) \).

\[ w - T_1 \] if for each \( x, y \in X \) such that \( x \neq y \), there exist soft \( w \)-open sets \((F, E)\) and \((G, E)\) such that \( x \in (F, E) \) and \( y \notin (F, E) \) and \( x \notin (G, E) \) and \( y \in (G, E) \).

\[ w - T_2 \] if for each \( x, y \in X \) such that \( x \neq y \), there exist soft \( w \)-open sets \((F, E)\) and \((G, E)\) such that \( x \in (F, E) \) and \( y \in (G, E) \) and \( (F, E) \cap (G, E) = \emptyset \).

We have the following diagram:

\[
\text{soft } w - T_2 \Rightarrow \text{soft } w - T_1 \Rightarrow \text{soft } w - T_0.
\]

**Example 2.14.** Let \( X = \{h_1, h_2, h_3\} \), \( E = \{e_1, e_2\} \) and \( \text{sw} = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\} \), where

\[
\begin{align*}
F_1(e_1) &= \{h_3\}, & F_1(e_2) &= \{h_3\}; \\
F_2(e_1) &= \{h_1, h_2\}, & F_2(e_2) &= \{h_1, h_3\}; \\
F_3(e_1) &= \{h_2, h_3\}, & F_3(e_2) &= \{h_1, h_2\}; \\
F_4(e_1) &= \{h_2\}, & F_4(e_2) &= \{h_1\}; \\
F_5(e_1) &= \{h_3\}, & F_5(e_2) &= \emptyset; \\
F_6(e_1) &= \emptyset, & F_6(e_2) &= \{h_3\}.
\end{align*}
\]

Then \( \text{sw} \) is a soft \( w \)-structure over \( X \) with respect to \( E \). It is obviously a soft \( w - T_1 \) space. For \( h_1, h_2 \in X \), \((F_2, E)\) and \((F_3, E)\) are unique soft \( w \)-open sets of \( h_1, h_2 \), respectively. But \((F_2, E) \cap (F_3, E) \neq \emptyset \). So \((X, \text{sw}, E)\) is not soft \( w - T_2 \).

**Example 2.15.** Let \( X = \{h_1, h_2, h_3\} \), \( E = \{e_1, e_2\} \) and \( \text{sw} = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\} \), where

\[
\begin{align*}
F_1(e_1) &= \{h_1\}, & F_1(e_2) &= \{h_1\}; \\
F_2(e_1) &= \{h_2\}, & F_2(e_2) &= \{h_2\}; \\
F_3(e_1) &= \{h_1, h_3\}, & F_3(e_2) &= X; \\
F_4(e_1) &= \emptyset, & F_4(e_2) &= \{h_2\}.
\end{align*}
\]

Then \( \text{sw} \) is a soft \( w \)-structure over \( X \) with respect to \( E \). It is obviously a soft \( w - T_0 \) space but it is not soft \( w - T_1 \).

Let \( \text{sw} \) be a soft \( w \)-structure over \( X \) with respect to \( E \). A soft \( w \)-space \((X, \text{sw}, E)\) is called relative soft \( w - T_0 \) if for each \( x, y \in X \) such that \( x \neq y \), there exists a soft \( w \)-open set \((F, E)\) such that \( x \in (F, E) \) and \( y \notin (F, E) \) or \( x \notin (F, E) \) and \( y \in (F, E) \).

**Theorem 2.16.** Let \( \text{sw} \) be a soft \( w \)-structure on \( X \). If \( X \) is a relative soft \( w - T_0 \) space, then for each \( x, y \in X \) such that \( x \neq y \), we have \( c_{\text{sw}}(x, E) \neq c_{\text{sw}}(y, E) \).

**Proof.** Let \( X \) be a relative soft \( w - T_0 \) and \( x, y \in X \) such that \( x \neq y \). Then there exists a soft \( w \)-open set \((F, E)\) such that \( x \in (F, E) \) and \( y \notin (F, E) \). Therefore \((F, E)'\) is a soft \( w \)-closed set such that \( x \notin (F, E)' \) and \( y \in (F, E)' \). Since \( c_{\text{sw}}(y, E) \) is the intersection of all soft \( w \)-closed subsets containing \((y, E)\), \( c_{\text{sw}}(y, E) \cap (F, E)' \) and hence \( x \notin c_{\text{sw}}(y, E) \). Thus \( c_{\text{sw}}(x, E) \neq c_{\text{sw}}(y, E) \).

**Theorem 2.17.** Let \( \text{sw} \) be a soft \( w \)-structure on \( X \). If \( y \in c_{\text{sw}}(x, E) \), then for each soft \( w \)-open set \((G, E)\) containing \( y \), there exists a parameter \( e \in E \) such that \( x \in G(e) \).

**Proof.** Let \( y \in c_{\text{sw}}(x, E) \). Then by Theorem 2.7, \((G, E) \cap (x, E) \neq \emptyset \) for all \((G, E) \in \text{sw} \) such that \( y \in (G, E) \). Since \((G, E) \cap (x, E) \neq \emptyset \), there exists a parameter \( e \in E \) such that \( x \in G(e) \).

**Theorem 2.18.** Let \( \text{sw} \) be a soft weak structure on \( X \). A soft \( w \)-space \((X, \text{sw}, E)\) is soft \( w - T_1 \) if \((x, E)\) is soft \( w \)-closed set for all \( x \in X \).

**Proof.** Let \( x, y \in X \) such that \( x \neq y \). Then \((x, E)'\) and \((y, E)\)' are soft \( w \)-open sets such \( y \in (x, E)' \), \( x \in (x, E)' \) and \( y \notin (y, E)' \), \( x \in (y, E)' \). Hence \( X \) is soft \( w - T_1 \).

3. Conclusions

The author introduced the notion of soft \( w \)-structure and investigated some basic properties of this new structure. In the next research, the author will introduce the associated soft \( w \)-structures induced by soft topologies and study the relationship between soft \( w \)-structures and associated soft \( w \)-structure induced by soft topologies.

**Conflict of Interest**

No potential conflict of interest relevant to this article was reported.

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