Entanglement dynamics of generic quantum states

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Abstract

We theoretically analyze the dynamics of average entanglement measures of a typical quantum state of a complex system, with its reduced density matrix described by a generalized, multiparametric Wishart ensemble with unit trace. Our results indicate quantitatively different rate of increase of the average Renyi and Von-Neumann entropies with evolving complexity of the state as system parameters change. While the state itself is multiparametric, we find that the growth of the average measures can be described in terms of an information-theoretic function, referred as the complexity parameter that could also act as a possible tool for hierarchical arrangement of entangled states.
I. INTRODUCTION

A maximally entangled state of a many body quantum system is expected, by definition, to be ergodic; the latter corresponds to an equal a-priori probability in all basis states such that the information the knowledge of the state can provide about its subunits is minimum. This aspect renders the state a huge potential applicability in quantum information processes and has led to intense research seeking its existence/ engineering during past two decades.

A typical many body state is however expected to evolve in time and need not retain its initial entanglement characteristics. As the number of ways for a generic state to be entangled are more than unentangled, the entanglement measures in general tend to grow in time with a growth rate that can reveal many system specific characteristics e.g. many body localization, the existence of a local conserved charge etc. The studies on black hole entropy and thermalization in many body systems have also indicated bounds on "scrambling" of quantum information i.e the time beyond which quantum information can not be retrieved completely due to our inability to perform measurements [1–3, 10]. This has motivated in past many studies on time-dynamics of entanglement [5–12]. The system-conditions can however change over time, thus rendering it relevant to study the dynamics with changing system conditions and is primary focus of the present study.

The analysis of entanglement growth of a many body state requires a comparative study of a related measure at various stages of system conditions. Previous research studies have introduced many measures to characterize the entanglement in a multipartite state. Unfortunately however the criteria/ hierarchical arrangement of states with increasing entanglement predicted by different measures lacks unanimity i.e the relative degree of entanglement between two state indicated by one measure differs from other. Previously it was believed that the issue does not arise in bipartite systems and the degree of entanglement can be consistently described by any of the standard measure e.g. Von-Neumann entropy, Renyi entropy etc. Indeed, based on systems without conservation laws or disorder, previous studies predicted a linear growth of both the entropies [13–30]. The recent studies however indicate different growth rates of the two entropies with time if the quantum system is subjected to global physical constraints e.g. symmetry, conservation laws [7–9] or local ones e.g. dis-
order. For example, in systems exhibiting diffusive transport of some conserved quantity e.g. spin, charge, energy, the Renyi entropies grow much more slowly in time than the von Neumann entropy. This leads to a natural query regarding the role of physical constraints on the entanglement growth. The present work analyzes the issue in context of one global constraint i.e. the states with and without time-reversal symmetry.

The density operator $\rho$ for a pure state $|\Psi\rangle$ is expressed as $\rho = |\Psi\rangle \langle \Psi|$. For the bipartite representation of a many body system, the entanglement measures between its two parts, say $A$ and $B$, can be obtained from the reduced density matrices $\rho_A$ and $\rho_B$ obtained by projecting $\rho$ onto the orthogonal bases of $B$ and $A$ respectively: $\rho_A = \text{Tr}_B \rho$, $\rho_B = \text{Tr}_A \rho$.

For example, the Renyi entanglement entropy $R_\alpha$ for $\rho_A$ is defined as

$$R_\alpha = \frac{1}{1-\alpha} \log_b \text{Tr} (\rho_A)^\alpha = \frac{1}{1-\alpha} \log_b \sum_n \lambda_n^\alpha.$$  \hspace{1cm} (1)

where $\lambda_n$ refers to the $n^{th}$ eigenvalue of the reduced density matrix $\rho_A$. The limit $\alpha \to 1$ corresponds to the Von-Neumann entropy

$$R_1 = -\text{Tr} \rho_A \log_b \rho_A = -\sum_n \lambda_n \log_b \lambda_n$$  \hspace{1cm} (2)

with base $b$ referring to the dimension of the local Hilbert space of the subunits of the subsystems $A$ and $B$; (hereafter the base will be suppressed unless needed for clarity). Further the min-entropy $R_\infty(\rho_A)$, defined as $\lim_{\alpha \to \infty} R_\alpha(\rho_A)$, can be expressed in terms of the maximum eigenvalue: $R_\infty(\rho_A) = -\log_b \lambda_{max}$. As the above definitions indicate, the determination of entanglement entropies requires a prior knowledge of the reduced density matrix and therefore the components of the pure bipartite state in the basis of interest. It is however technically non-trivial, almost impossible, usually, to determine the components exactly if the state is that of a many body system. A lack of detailed information about the complicated interactions usually manifests through a randomization of the components with large fluctuations at a local scale. Based on Berry’s Gaussian random wave hypothesis in context of quantum chaotic systems [31], a typical ergodic many body eigenstate constitutes a random fock-space vector with independently distributed Gaussian entries [32]. This in turn leads to reduced density matrix for the subsystems described by the standard Wishart random matrix ensemble.

A typical many body state however need not be ergodic and can exhibit different degree of
localization e.g. localized, partially localized or extended as well based on system parameters. As a consequence, the reduced density matrix belong to a general class of Wishart random matrix ensembles where system-dependence appears through distribution parameters of the matrix elements. This in turn affects the distribution of the Schmidt eigenvalues and thereby the average behavior of the entanglement measures. This motivates us to pursue the present study with a primary objective to analyze the entanglement measures and their growth with changing system conditions.

The paper is organized as follows. The reduced density matrix of a bipartite quantum state can in general be written as a Wishart matrix. The latter belongs to a random matrix ensemble if the system is complex e.g one with complicated interactions. As the state in general depends on many system parameters, this manifests in the ensemble through distribution parameters and their variation can lead to a mutiparametric variation of the ensemble. The latter’s evolution in matrix space however is governed by the complexity parameter; this is discussed in section II.B. This is followed by a derivation of the complexity parameter governed evolution of the ensemble averaged measures in section IV. The details of a numerical verification of our theoretical predictions is presented in section V followed by conclusion in section VI. To avoid defocussing, the steps related to some of the derivations are moved from main text to the appendices.

II. COEFFICIENT MATRIX ENSEMBLE FOR A PURE STATE

Consider a composite system consisting of two subsystems $A$ and $B$, with $|a_i\rangle$ and $|b_j\rangle$, $i = 1 \rightarrow N_a, j = 1 \rightarrow N_b$, as the orthogonal basis states in the subspaces of $A$ and $B$ respectively. An arbitrary pure state $|\Psi\rangle$ of the composite system in the $N_a \times N_b$ product basis $|a_i\rangle|b_j\rangle$ can be written as

$$|\Psi\rangle = \sum_{i,j}^{N_a, N_b} C_{ij} |a_i\rangle |b_j\rangle$$

(3)

Representation of the density operator $\rho = |\Psi\rangle\langle\Psi|$ in the product basis gives pairwise products of the coefficients $C_{ij}$ as the elements of the density matrix $\rho$. A partial trace operation on $B$-subspace then leads to the reduced density matrix $\rho_A$ which turns out to be a Wishart matrix.
\[ \rho_A \equiv CC^\dagger \]

subjected to a fixed trace constraint \( \text{Tr}\rho_A = 1 \). As \( C \) consists of the coefficients (components) of the state \( \Psi \), we refer it as the coefficient matrix or just \( C \)-matrix.

### A. Structure of coefficient matrix

A physically motivated basis can often be chosen as the one that preserves the same global constraints (e.g. symmetry, conservation laws etc.) as the system. The components of any state of the system, represented in such a basis, then satisfy the transformation rules for the constraint class. As a consequence, the structure of the \( C \) matrix e.g. whether it is real or complex, the relation among the elements and their transformation rules etc depends on the global constraints e.g. symmetry, conservation laws etc.

For example, for systems with both time-reversal symmetry and integer angular momentum, a physically motivated basis can always be chosen as real. Thus \( C \) is a real matrix for systems with both time-reversal symmetry and integer angular momentum, a quaternion matrix for the systems with time-reversal symmetry and half integer angular momentum, and, a complex matrix in absence of the time-reversal symmetry. For clarity purposes, the present study is confined only to \( C \) real or complex with no other matrix constraints. The elements of \( C \) can then be written as \( C_{kl} = \sum_{s=1}^{\beta} (i)^s C_{kl;s} \) with \( \beta = 1 \) or 2 for \( C \) real or complex; taking \( N_a = N, N_b = N + \nu_0 \), we have \( k = 1 \to N, l = 1 \to (N + \nu_0) \). The generalization to quaternion \( C \) can be done following similar steps but is technically tedious and is therefore not included here).

The main influence on the structure of \( C \) matrix however comes from the nature of the interactions between the subsystems \( A \) and \( B \). For example, in ergodic limit or when almost all units of \( A \) are interacting with those of \( B \) with almost same average strength, the correlations between any orthogonal base pair, with one of the pair belonging to basis states of \( A \) and other to \( B \), can be almost same for all such pairs. This in turn leads to almost all matrix elements of \( C \) of the same order. On the contrary, the presence of interactions, local to subsytems only, are expected to cause correlations varying from one orthogonal base pair to the other. The strength of such correlations can also vary from the system to system e.g. decaying as power law or exponentially. We note that contrary
to a Hamiltonain matrix where a fully localized quantum dynamics leads to a diagonal matrix, the localization in this case affects the structure along the columns. For example, the $C$-matrix corresponding to a separable state has non-zero elements only along the first column. The decaying strength of matrix elements along columns therefore indicate a state with partial entanglement. Similarly a growth of entanglement reflects on the $C$-structure as increasing strength of the entries along the columns. The parts (b-d) of figure 1 display the schematic pictures corresponding to three examples of partial entanglement alongwith a fully entangled state and a separable one in part (a) and (e) respectively; we use these $C$-matrices later on for our numerical analysis too.

Based on the physical as well as the local Hilbert space dimensions (i.e the number of states of a typical subunit in a subsystem), the number and strength of the components of a typical state in the product basis can vary. This in turn can lead to different sparse structures, based on the location of the "cut" or "partition" i.e how the many body system is divided into two subsystems. Previous studies have often focussed on the evolution of the entanglement growth with changing location of the cut in time. Although the location also enters in our formulation, the present work is primary focussed on the evolution with changing system conditions while the cut is kept at a fixed location.

We note that the present analysis is not based on any real physical Hamiltonian and therefore role of the physical dimensions of the system or the subunits of each subsystem and the associated local Hilbert spaces is not directly obvious from the formulation. The theoretical results derived here and numerical examples considered are however applicable in general to any dimensionality and can be obtained by relevant changes. For example, if a subsystem X (=A or B) consists of $g_x$ subunits or particles, each associated with a local Hilbert spaces of dimensionality $b$, this implies $N_x = b^{g_x}$ and the logarithmic terms appearing in various entanglement measures are then defined on the base $b$.

B. Multi-parametric distribution of Coefficients: role of constraints

Based on Berry’s conjecture, an ergodic quantum state has independent and identical Gaussian distributed coefficients (in a basis in which the classical dynamics is ergodic); in terms of the $C'$ matrix elements, , this corresponds to the constraints on their first two moments only: $\langle |C_{kl}|^2 \rangle = \frac{1}{2\gamma}, \langle C_{kl} \rangle = 0$. Based on maximum entropy hypothesis, $C$-matrix
then belongs to a maximum information entropy ensemble. But, as predicted by Page’s law, an ergodic state is also maximally entangled. Thus an ensemble of $C$ with maximum information entropy (Shannon) here corresponds to maximum entanglement entropy of the state too.

A typical many body quantum state need not be ergodic and not all its coefficients (i.e $C_{ij}$) are, in general, of the same order. Indeed the statistical behavior of the $C$-elements depends on the local constraints too. Based on the physical aspects of the system, these can be of various types. While the global constraints (matrix constraints) affect nature of each matrix, the local constraints (ensemble constraints) influence distribution parameters of the matrix elements: disorder, dimensionality, boundary conditions, nature of dynamics (chaotic or non-chaotic), interactions among subunits of system etc. For example, assuming all independent elements, a most generic form of $\rho_c$ can be written as

$$\rho_c(C) = N \rho_c(C_{11}, C_{12}, \ldots, C_{NN}) F_c$$

with $F_c$ dependent on the matrix constraints including $\text{Tr}(C^\dagger C) = 1$. The corresponding ensemble of $N \times N$ reduced density matrices $\rho_A = C.C^\dagger$ then forms a generalized Wishart ensemble. With known constraints on the 1st and 2nd order moments of $C_{kl}$, a typical quantum state can be described by a Gaussian ensemble of $C$ matrices. The ensemble constraints on higher order moments of $C_{kl}$ in general lead to a non-Gaussian ensembles.

A reduced density matrix for a typical pure quantum state of a complex system in general belongs to a non-stationary Wishart ensemble with arbitrarily randomly distributed elements of $C$, with ensemble parameters governed by correlations among the basis states. The change in complexity e.g. many body interactions within the system can cause a variation of correlations between subsystems and thereby leading to evolution of the $C$-matrix elements. This is discussed in next section.

C. Diffusion of matrix elements: ensemble complexity parameter

A typical element of $C$-matrix describes the correlations between the two orthogonal bases with its distribution a measure of the inaccuracy (deterministic or due to disorder) arising due to complexity of the system. The changing system conditions over time can however lead to a variation of these correlations as well as the inaccuracy and thereby distribution
parameters. As a consequence, the ensemble density $\rho_c$ can evolve due to change of the matrix elements as well as the parameters. As discussed below, the evolution of the ensemble density in matrix space can be mimicked by that in the parameter-space; this in turn helps better insights in the behavior of entanglement measures.

As our analysis is based on many types of parameters, it is important to distinguish them at the onset: hereafter the term ”system parameters” refer to those which appear in a single matrix due to their presence in the corresponding operator or the chosen basis, the term ”distribution parameters” or ”ensemble parameters” refers to those characterizing an ensemble of the matrices. Another set of parameters, referred as the complexity parameters and an important aspect of our analysis, is introduced later.

For clear exposition of our ideas, we consider the elements of $C$ as independent Gaussian distributed, with arbitrary mean and variances:

$$\rho_c(C; h, b) = N \exp \left[ -\sum_{k,l,s} \frac{1}{2h_{kl,s}} (C_{kl,s} - b_{kl,s})^2 \right]$$  (6)

with $\sum_{k,l,s} \equiv \sum_{k=1}^{N} \sum_{l=1}^{N+\nu_0} \sum_{s=1}^{\beta}$ and $N$ as a normalization constant: $N = \prod_{k,l,s} (2h_{kl,s})^{-1/2}$. Here $h \equiv [h_{kl,s}]$ and $b \equiv [b_{kl,s}]$ refer to the matrices of variances and mean values of $C_{kl,s}$. Clearly, with different choices of $h$ and $b$-matrices, eq.(6) can give rise to many coefficient ensembles; some of them are used later in section VI for numerical verification of our results.

A small perturbation of the quantum state by change of the system parameters over time causes the matrix elements $C_{kl}$ undergo a dynamics, thereby leading to an evolution of $\rho_c$ in the matrix space. But as $h, b$ depend on system parameters too (i.e perturbation also affects the accuracy of each $C_{kl}$), $\rho_c$ also evolves in the ensemble space, due to change in the parameters $h_{kl,s} \rightarrow h_{kl,s} + \delta h_{kl,s}$ and $b_{kl} \rightarrow b_{kl} + \delta b_{kl}$ over time. As discussed in appendix A (also in [34] in context of chiral Hamiltonians), the multiparametric evolution of $\rho_c(C)$, generated by a combination of first order parametric derivatives $T \equiv \sum_{k\leq l,s} \left[ (1 - \gamma h_{kl,s}) \frac{\partial}{\partial h_{kl,s}} - \gamma b_{kl,s} \frac{\partial}{\partial b_{kl,s}} \right]$, leads to a diffusion with finite drift in $C$-matrix space. It is however possible to define a transformation of variables from $h, b$-set to another set of $M$ variables, namely, $Y, y_2, \ldots y_M$ that reduces $T\rho \equiv \frac{\partial \rho}{\partial Y}$ and thereby leading to a single parameter governed diffusion of $\rho = N_0 \rho_c$

$$\frac{\partial \rho}{\partial Y} = \sum_{k,l,s} \frac{\partial}{\partial C_{kl,s}} \left[ \frac{\partial \rho}{\partial C_{kl,s}} + \gamma C_{kl,s} \rho \right]$$  (7)
where $N_0 = e^{\int f_0(Y) \, dY}$, with $f_0(Y) = T \log N$ and $Y$ is known as the ensemble complexity parameter

$$
Y = \frac{1}{2M\gamma} \ln \left[ \prod_{k,l} |(1 - 2\gamma h_{kl,s})| |b_{kl,s}|^2 \right] + \text{const}
$$

(8)

with $\prod_{k,l}'$ implies a product over non-zero $b_{kl,s}$ as well as $x_{kl,s} = (1 - \gamma h_{kl,s})$, with $M$ as their total number; (for example for the case with all $x_{kl,s} \neq 0$ but $b_{kl,s} = 0$, we have $M = \beta N(N + \nu_0)$ and for case with all $x_{kl,s} \neq 0$ and $b_{kl,s} \neq 0$, we have $M = 2\beta N(N + \nu_0)$).

Further $\gamma$ is an arbitrary parameter, related to final state of the ensemble (giving the variance of matrix elements at the end of the evolution) and the constant in eq.(8) is determined by the initial state of the ensemble.

Eq.(7) describes the evolution of $\rho$, equivalently, $\rho(C;Y)$ at $Y = y_1(h,b)$ starting from an arbitrary initial ensemble density, say $\rho_0(C,Y_0)$ at $Y_0 = y_1(h_0,b_0)$ with $h_0,b_0$ as the initial ensemble parameters. As discussed in appendix A, (following from eq.(A4)), the evolution is subjected to conditions $y_k = \text{constant}$ for $k > 1$ and reaches a steady state when $\partial \rho / \partial Y \rightarrow 0$ with $\rho$ approaching the probability density of the stationary Wishart ensemble, $\rho \propto e^{-\gamma/2} \text{Tr} C^2$. As the system information in eq.(7) enters only through $Y$, its solution $\rho(C;Y)$ remains same for different ensembles, irrespective of the details of their $h$ and $b$ matrices, if they share same values for $y_1, y_2, \ldots, y_M$.

D. Diffusion of Schmidt eigenvalues

With $\rho_A$ as a $N \times N$ Hermitian matrix, its eigenvalue equation can be written as $\rho_A \, U = U \, \Lambda$ where $U$ is the $N \times N$ eigenvector matrix, unitary in nature i.e. $U^\dagger U = 1$ ($\rho_A$ being Hermitian) and $\Lambda$ is the the $N \times N$ diagonal matrix of its eigenvalues, $\Lambda_{mn} = \lambda_n \delta_{mn}$ also referred as the Schmidt eigenvalues. A small change $\delta Y$ in $Y$ (given by eq.(8)) results in diffusive dynamics of of $C$ that in turn manifests in the $\rho_A$-matrix space too and the moments for the matrix elements $\rho_{A,mn} = \sum_{k=1}^{N} C_{mk} C^*_{nk}$ can be calculated from those of $C$.

To proceed further, it is important to note that eq.(7) is equivalent to the evolution of an arbitrary $N_a \times N$ (with $N_a \geq N$) rectangular matrix $C_0$ subjected to a random perturbation, of strength $t$, by another $N_a \times N$ rectangular matrix $V$ (real or complex for $L$ real-symmetric or complex Hermitian, respectively. with Gaussian density $\rho_v(V) =$
\[ \left( \frac{1}{2\pi v^2} \right)^{\beta N_a/2} e^{-\frac{1}{2v^2} \text{Tr}(VV^\dagger)} \]. The perturbed matrix \( C(t) \) is described as \( C(t) = \sqrt{f}(C_0 + t \ V) \) with \( f = (1 + \gamma_0 t^2)^{-1} \), \( C(0) = C_0 \) as a fixed random matrix and \( \gamma_0 \) as an arbitrary positive constant \([35]\). The evolution due to variation of strength \( t \) of the random perturbation is markovian if considered in terms of a rescaled parameter \( Y = -\frac{1}{2\gamma} \ln f = \frac{1}{2\gamma} \ln(1 + \gamma t^2) \) \([33, 35]\):

\[
C(Y) \equiv C(0) e^{-\gamma Y} + V(Y) \left( \frac{1 - e^{-\gamma Y}}{\gamma} \right)^{1/2}
\]  

As discussed in appendix B, relevant information for the moments of \( \lambda_n \) can subsequently be derived by using second order pertubation theory for Hermitian matrices which on substitution in standard Fokker Planck equation leads to the equation describing \( Y \)-governed evolution of the joint probability distribution function (\( jpdf \)) \( P_c(\lambda_1, \lambda_2, \ldots, \lambda_N; Y) \),

\[
\frac{\partial P_\lambda}{\partial Y} = \sum_{n=1}^{N} \left[ \frac{\partial^2 (\lambda_n P_\lambda)}{\partial \lambda_n^2} - \frac{\partial}{\partial \lambda_n} \left( \sum_{m=1}^{N} \frac{\beta \lambda_n}{\lambda_n - \lambda_m} - \frac{\beta \nu}{2} \gamma \lambda_n \right) P_\lambda \right]
\]

where \( \nu = (N_b - N_a - 1)/2 = (\nu_0 - 1)/2 \). (Here we set \( v^2 = 1/4 \) for simplification and without loss of generality). Hereafter unless required for clarity, we will use the notation \( f(\lambda) \) for \( f(\lambda_1, \ldots, \lambda_N) \) for any arbitrary function \( f \).

The above equation describes the diffusion of \( P_\lambda(\lambda, Y) \), with a finite drift, from an arbitrary initial state \( P_\lambda(\lambda_0, Y_0) \) at \( Y = Y_0 \). In limit \( \frac{\partial P_\lambda}{\partial Y} \to 0 \) or \( Y \to \infty \), the diffusion approaches a unique steady state:

\[
P_\lambda(\lambda; \infty) = C_\beta \prod_{m<n=1}^{N} |\lambda_m - \lambda_n|^\beta \prod_{k=1}^{N} |\lambda_k|^{2\nu \beta - 1} e^{-\gamma \sum_{k=1}^{N} \lambda_k}
\]

with \( C_\beta \) as the normalization constant. The above \( jpdf \) of the eigenvalues corresponds to that of a stationary Wishart ensemble and is consistent with the expectaion: the limit \( \frac{\partial \rho}{\partial Y} \to 0 \) of eq.(7) corresponds to \( \frac{\partial P_\lambda}{\partial Y} \to 0 \) for eq.(10), and, with \( \rho \) approaching stationary Wishart ensemble density in this limit, \( P_\lambda \) is expected to approach corresponding \( jpdf \) of the eigenvalues. We note that eq.(10) can also be derived by an exact diagonalization of eq.(7) (following the similar route as discussed in \([34]\) for the chiral ensembles).

As mentioned in section II.C, the reduced density matrix \( \rho_A \) is a Wishart matrix subjected to a fixed trace constraint \( \text{Tr} \rho_A = 1 \). The \( jpdf \) \( P_c(\lambda_1, \lambda_2, \ldots, \lambda_N; Y) \) (denoted as \( P_c(\lambda; Y) \) hereafter) as of the eigenvalues of \( \rho_A \) can then be given as.
\[ P_c(\lambda; Y) = C_{hs} \delta (\sum_n \lambda_n - 1) P_\lambda(\lambda, Y). \]  

(12)

with \( C_{hs} \equiv C_{hs}(Y) \) is the normalization constant for \( P_c(\lambda; Y) \); we note here that the constraint \( \sum_n \lambda_n = 1 \) confines \( P_c(\lambda; Y) \) to an area much less than that of \( P_\lambda(\lambda, Y) \). Further, the stationary Wishart ensemble subjected to the above constraint is also referred as the Hilbert-Schmidt ensemble.

### III. GROWTH OF AVERAGE ENTANGLEMENT ENTROPIES

Following from the definition (1), the average entanglement measure, say \( R_f \equiv f(\lambda) \) for a typical state can be expressed in terms of the jpdf \( P_c \) of the Schmidt eigenvalues

\[ \langle f \rangle = C_{hs} \int f(\lambda) P_\lambda(\lambda) \delta (\sum \lambda_n - 1) D\lambda. \]  

(13)

For simplification of the technical analysis, the above can be rewritten as

\[ \langle f(S_1 = 1) \rangle = C_{hs} \int f(\lambda) P_\lambda(\lambda) \delta(S_1 - \sum \lambda_n) D\lambda \]  

(14)

The above relation can now be used to derive the \( Y \)-governed evolution equation of \( \langle f \rangle \):

\[ \frac{\partial \langle f(S_1) \rangle}{\partial Y} = C_{hs} \int f(\lambda) \frac{\partial P_\lambda}{\partial Y} \delta(S_1 - \sum \lambda_n) D\lambda \]  

(15)

Substitution of eq.(10) then leads to

\[ \frac{\partial \langle f(S_1) \rangle}{\partial Y} = I_0 + I_1 + I_2 \]  

(16)

with \( \delta_1 \equiv \delta(S_1 - \sum \lambda_n) \), \( I_0 = \frac{\partial \log C_{hs}}{\partial Y} \langle f(S_1) \rangle \) and

\[ I_1 = -C_{hs} \sum_{n=1}^{N} \int D\lambda \delta_1 f(\lambda) \frac{\partial}{\partial \lambda_n} \left( \sum_{m=1}^{N} \frac{\beta \lambda_m}{\lambda_n - \lambda_m} - \beta \nu - 2\gamma \lambda_n \right) P_\lambda \]  

(17)

and

\[ I_2 = C_{hs} \sum_{n=1}^{N} \int D\lambda \delta_1 f(\lambda) \frac{\partial^2 (\lambda_n P_\lambda)}{\partial \lambda_n^2} \]  

(18)

As discussed in appendices C and D, using integration by parts along with \( \frac{\partial \delta_1}{\partial \lambda_n} = -\frac{\partial \delta_1}{\partial S_1} \), \( I_1 \) and \( I_2 \) can further be simplified leading to a close form equation for \( \langle f \rangle \) which can be solved,
in principle, to obtain the average entanglement measures for any arbitrary $Y$ value. Below we consider the cases for $f(\lambda) = R_1(\lambda)$ and $R_2(\lambda)$.

A. Average Von-Neumann Entropy

The ensemble average of Von-Neumann entropy $R_1 = -\sum_n \lambda_n \log \lambda_n$, defined in eq.(2), is given by $\langle R_1(1) \rangle$ where, from eq.13, we have

$$\langle R_1(S_1) \rangle = C_{hs} \int \left[ -\sum_n \lambda_n \log \lambda_n \right] P_\lambda(\lambda) \delta(\sum \lambda_n - 1) \delta(\lambda). \tag{19}$$

Using $f = -\sum_n \lambda_n \log \lambda_n$ in eq.(14) and substituting $I_1$ and $I_2$ from appendices C and D, the $Y$-governed growth of average Von Neumann entropy can be given as

$$\frac{\partial \langle R_1(S_1) \rangle}{\partial Y} = \alpha \langle R_1 \rangle + \frac{1}{2} \beta N_\nu \langle R_0 \rangle - \frac{1}{2} (\beta NN_\nu - 4 \gamma S_1 + 2(N - 2)) J + G_s \tag{20}$$

with $\alpha = \frac{\partial \log C_{hs}}{\partial Y}$, $\langle R_0 \rangle = -\langle \sum_n \log \lambda_n \rangle$, $N_\nu = N - 2\nu - 1$ and

$$G_s(S_1) = -\frac{1}{2} (\beta NN_\nu - 4 \gamma S_1) \frac{\partial \langle R_1 \rangle}{\partial S_1} + S_1 \frac{\partial^2 \langle R_1 \rangle}{\partial S_1^2} + 2 \frac{\partial J}{\partial S_1} \tag{21}$$

Noting that $R_1$ and $S_1$ vary at different rate with $Y$ as well as $N$, we assume $\langle R_1(S_1) \rangle = g(S_1) R_1(1)$ (based on multiplication of independent probabilities) with $g(S_1)$ as a function of $S_1$ only (with its exact form not needed for our qualitative analysis). Similarly, we can write $J(S_1) = g_0(S_1) J(1)$. This in turn gives $G_s(S_1) = \phi(S_1) \langle R_1(1) \rangle$ with

$$\phi(S_1) = -\frac{1}{2} (\beta NN_\nu - 4 \gamma S_1) g'(S_1) + S_1 g''(S_1) + 2g_0'(S_1) \tag{22}$$

with $g'(S_1) \equiv \frac{\partial g(S_1)}{\partial S_1}$, $g''(S_1) \equiv \frac{\partial^2 g(S_1)}{\partial S_1^2}$ and $g_0'(S_1) = \frac{\partial g_0}{\partial S_1}$. (Alternatively, we can also first expand $\langle R_1(S_1) \rangle$ in a Taylor’s series near $S_1 = 1$: $\langle R_1(S_1) \rangle = \langle R_1(1) \rangle + \sum_k \frac{\partial^k}{\partial S_1^k} \langle R_1 \rangle \big|_{S_1=1} \frac{(S_1-1)^k}{k!}$ (with notation $\partial_s \equiv \frac{\partial}{\partial S_1}$) and use the assumption that $\partial_s \langle R_1(S_1) \rangle = g'(S_1) \langle R_1(1) \rangle$. This leaves the final result unaffected). For later reference, we note that $g'(1) > 0$ (also depicted in Figure 5); (this can be seen directly by writing $R_1 = \sum_{n=1}^{N-1} \lambda_n \log \lambda_n + (S_1 - \sum_{n=1}^{N-1} \lambda_n) \log(S_1 - \sum_{n=1}^{N-1} \lambda_n)$ and then considering $\frac{\partial R_1}{\partial S_1}$).

As our interest here is in case $S_1 = 1$ only, eq.(23) can be rewritten as (using notation $\langle R_1 \rangle \equiv \langle R_1(1) \rangle$ and $J(1) = 1$,
\[
\frac{\partial \langle R_1 \rangle}{\partial Y} = v_2 - v_1 \langle R_1 \rangle
\]

(23)

with \(v_1(Y) = -(\alpha - 4\gamma)g(S_1) - \phi(1)\) and \(v_2(Y) = \frac{1}{2}\beta N_\nu \langle R_0 \rangle + \frac{1}{2}(\beta NN_\nu + 2(N - 1) - 4 \gamma)\)

The solution of the above equation for arbitrary initial condition and for finite \(Y\) is

\[
\langle R_1(Y) \rangle = e^{-\int_{Y_0}^Y dv_1(y)} \left[ \int_{Y_0}^Y dt \, v_2(t) e^{\int_{Y_0}^Y v_1(t) dt} + R_1(Y_0) \right]
\]

(24)

A better insight in the above form can be gained by taking a large \(N\)-limit: as \(v_2(Y) \approx \frac{1}{2} \beta N_\nu \langle R_0 \rangle\), \(v_1(Y) \approx -\phi(1) \approx \frac{1}{2} \beta NN_\nu g'(1)\) in the limit, the above solution can now be rewritten as

\[
\langle R_1(Y) \rangle \approx \frac{\beta N_\nu}{2} e^{\phi(1)(Y - Y_0)} \int \langle R_0 \rangle \, e^{-\phi(1)(t - Y_0)} \, dt + \langle R_1(0) \rangle \, e^{\phi(1)(Y - Y_0)}
\]

(25)

As \(\langle R_0 \rangle\) rapidly becomes almost constant with respect to \(Y\) (as displayed in figure 4), the above can be approximated as

\[
\langle R_1(Y) \rangle \approx \frac{\langle R_0 \rangle}{Ng'(1)} \left( 1 - e^{-\frac{1}{2} \beta NN_\nu g'(1)(Y - Y_0)} \right) + \langle R_1(Y_0) \rangle \, e^{-\frac{1}{2} \beta NN_\nu g'(1)(Y - Y_0)}
\]

(26)

Here again for separable state chosen as initial one, we have \(\langle R_1(0) \rangle = 0\). For small \(Y\), the above equation then given a linear increase of the entropy with \(Y\):

\[
\langle R_1(Y) \rangle \approx \frac{1}{2} \beta N_\nu \langle R_0 \rangle (Y - Y_0).
\]

(27)

For large \(Y\), the growth however is again rapid with \(\langle R_1(Y) \rangle\) approaching a constant value.

The \(Y \to \infty\) limit of the above solution gives

\[
\langle R_1(\infty) \rangle = \frac{v_2}{v_1} \approx \frac{\langle R_0 \rangle}{Ng'(1)}
\]

(28)

Our numerical analysis based on three different ensembles indicates \(\langle R_0 \rangle \sim N \log N\) for large \(Y\) and large \(N\) (displayed in figure 4, details given in section V). This in turn gives expected limit \(\langle R_1(\infty) \rangle \sim \log N\) for a maximally entangled state; (as \(\lambda_n \approx \frac{1}{N}\) in large \(Y\) limit, the logarithmic \(R_1\) behavior is consistent with its expectation).

As eq.(26) indicates, the growth of \(R_1(Y)\) with changing complexity is indeed sensitive to the underlying symmetry conditions (through parameter \(\beta\)) as well as to the "size cut" of subsystems through parameter \(\nu = (\nu_0 - 1)/2 = (N_b - N_a - 1)/2\). We note that, for
system conditions changing with time, \( Y \) indeed is a function of time; the above analysis then describes the time-dependent growth too.

It is worth noting here the important role played by the terms originating from the derivative \( \frac{\partial R_1(S_1)}{\partial S_1} \big|_{S_1=1} \) i.e the terms which reflects the role of the constraint \( S_1 = \sum_n \lambda_n \). These terms are necessary to reach the correct stationary limit of \( \langle R_1 \rangle \) i.e the one for the Hilbert-Schmidt ensemble and thereby indicate the role played by the collective dynamics of eigenvalues. In the unconstrained stationary limit i.e stationary Wishart ensemble, \( \lambda_n \) are free to move on positive real axis although the Gaussian term in eq.(11) for \( P_\lambda(\lambda) \) confines them from escaping to infinity; the competition between repulsion term and confinement due to Gaussian term results in an average \( \lambda_n \sim \frac{1}{\sqrt{N}} \) and thereby \( R_1(\lambda) \sim \frac{1}{2} \sqrt{N} \log N \). With \( R_1 \sim \log N \) for the constrained case, clearly the Von-Neumann entropy is reduced due to the constraint.

B. Average 2nd order Renyi Entropy

As in previous case, the ensemble average of 2nd Renyi entropy \( R_2 \), defined in eq.(2), is given by \( \langle R_2(S_1 = 1) \rangle \) where, from eq.(13), we have

\[
\langle R_2(S_2) \rangle = C_{hs} \int \delta(S_1 - \sum \lambda_n) \left[ -\log \sum_n \lambda_n^2 \right] P_c(\lambda) \, D\lambda
\]  

(29)

Using \( f = R_2 \) in eq.(14) and substituting \( I_1 \) and \( I_2 \) from appendices C and D, the \( Y \)-governed growth of average 2nd order Renyi entropy can be given as

\[
\frac{\partial \langle R_2(S_1) \rangle}{\partial Y} = (\alpha + 2\gamma \langle R_2 \rangle - (2\beta(N - \nu - 1) + 2) S_1 \langle \frac{1}{S_2} \rangle + 4 \langle \frac{S_3}{S_2^2} \rangle + 4 \gamma J + G_s
\]  

(30)

with

\[
G_s(S_1) = 4 \frac{\partial J}{\partial S_1} + (2\gamma S_1 + 2 - (1/2)\beta NN_\nu) \frac{\partial \langle R_2 \rangle}{\partial S_1} + S_1 \frac{\partial^2 \langle R_2 \rangle}{\partial S_1^2}
\]  

(31)

Proceeding again as in the previous case and rewriting \( G_s(S_1) = \phi(S_1) \langle R_2(1) \rangle \) with

\[
\phi(S_1) \equiv S_1 g'_0(S_1) + 4 (2\gamma S_1 + 2 - (1/2)\beta NN_\nu) g'(S_1) + S_1 g''(S_1),
\]  

(32)
We note that, contrary to previous case, now \( g'(1) < 0 \) (also displayed in figure 5). (The latter can also be seen directly by writing \( R_2 \) as \( R_2 = -\log S_2 \) and \( S_2 = \sum_{n=1}^{N-1} \lambda_n^2 + (S_1 - \sum_{n=1}^{N-1} \lambda_n)^2 \) which gives \( \frac{\Delta R_2}{\Delta S_2} = -1 \)over\( S_2 \\frac{\Delta S_2}{\Delta S_1} = \frac{2}{S_2} < 0 \).

we have

\[
\frac{\partial (R_2)}{\partial Y} = u_2 - u_1 (R_1)
\]

with \( u_1(Y) = - (\alpha + 2 \gamma) g(S_1) - \phi(1) \) and \( u_2(Y) = - (2 \beta (N - \nu - 1) + 2) \langle \frac{1}{S_2} \rangle + 4 \langle \frac{S_1}{S_2} \rangle + 4 \gamma + 1 \), the general solution for \( \langle R_2 \rangle \) for arbitrary initial condition and for finite \( Y \) can be written as

\[
\langle R_2(Y) \rangle = e^{-\int_{Y_0}^{Y} u_1(Y) dY} \left[ \int_{Y_0}^{Y} u_2(t) e^{\int_{t}^{Y} u_1(t) dt} dt + R_2(0) \right].
\]

As in the case of \( \langle R_2 \rangle \), here again we consider a large \( N \)-limit of the above solution: as \( \frac{1}{N^2} \leq \langle S_3 \rangle \leq 1 \), we have \( u_2(Y) \approx -2 \beta N \langle \frac{1}{S_2} \rangle, u_1(Y) \approx -\phi(1) \approx 2 \beta N \nu g'(1) \). The above solution can now be approximated as

\[
\langle R_2(Y) \rangle \approx -2 \beta N e^{\phi(1)} \langle Y - Y_0 \rangle \int \langle \frac{1}{S_2} \rangle e^{-\phi(1)} dt \langle R_2(0) \rangle e^{\phi(1)} (Y - Y_0).
\]

It is important to note here that the term \( \langle \frac{1}{S_2} \rangle = \langle e^{R_2} \rangle \) varies rapidly with \( Y \), for small \( Y \), and cannot be treated as constant while integrating over \( Y \). Based on our numerical analysis of three different ensembles (discussed in section V), we find \( \langle \frac{1}{S_2} \rangle \sim (Y - Y_0)^{\alpha} \) with \( \alpha \) a system-dependent positive constant: \( \alpha \approx 1/2 \). For initial state chosen as separable, we have \( \langle R_2(Y_0) \rangle = 0 \). For small \( Y - Y_0 \), the above equation then gives

\[
\langle R_2(Y) \rangle \approx (Y - Y_0)^{\alpha} \left| g'(1) \right| N \nu
\]

For large \( Y - Y_0 \), however, the growth rapidly approaches a constant value. The \( Y \to \infty \) limit of eq.\((34)\) gives

\[
\langle R_2(\infty) \rangle = \frac{u_2}{u_1} = \frac{(2 \beta (N - \nu - 1) + 2) \langle \frac{1}{S_2} \rangle - 4 \langle \frac{S_1}{S_2} \rangle - 4 \gamma - 1}{\alpha + 2 \gamma + \phi(1)}.
\]

With \( \langle S_3 \rangle \approx \frac{1}{N^2} \), we have \( \langle R_2(\infty) \rangle = \frac{1}{2 \left| g'(1) \right| N \nu} \langle \frac{1}{S_2} \rangle \). Our numerical analysis based on three different ensembles indicates \( \langle \frac{1}{S_2} \rangle \sim N \log N \) for large \( Y \) and large \( N \) (displayed in figure...
4, details discussed in section V). This implies $\langle R_2(\infty) \rangle \sim \log N$, the expected limit for a maximally entangled state (as $\lambda_n \approx \frac{1}{N}$ in large $Y$ limit, the logarithmic $R_2$ behavior is consistent with its expectation).

Eq. (35) describes the growth of $R_2(Y)$ with changing complexity: similar to $R_1$, $R_2$ also depends, for finite $Y$ cases, on the symmetry parameter $\beta$ as well as the "size cut" $\nu$.

IV. PHYSICAL INTERPRETATION OF COMPLEXITY PARAMETER

To decipher the relevance of $Y$-formulation, it is important to first deconstruct eq. (A2) and seek the physical interpretation of each of its parts. While a point in $h, b$-space corresponds to an ensemble of $C$-matrices, the one in the matrix space corresponds to a single $C$-matrix. With $\rho_c$ a function of ensemble parameters (i.e. $h, b$) as well as matrix elements ($C_{kl}$), the generators $T$ and $L$ in eq. (A2) describe the dynamics in different spaces: $T$ generates an evolution of $\rho_c$ in the $h, b$ space while $C_{kl}$ ($\forall k, l$) are kept fixed, $L$ describes it in the matrix space (i.e $C_{kl}$ now varying) with a fixed $h, b$-values. Thus eq. (A2) describes the dynamics when a small perturbation of $\rho_c$ in the matrix space can be exactly mimicked by the one in the ensemble parameter space. This is expected on grounds that the dynamics in both spaces originates by variation of the system parameters. (The latter are not identified here as we begin with an ensemble directly). We note that, for an ensemble to appropriately represent a complex system statistically, its parameters must be derived from the system parameters e.g. strength of local interactions, disorder, dimensionality, boundary conditions etc. A knowledge of relation between the two then gives $Y$ as a function of various system parameters.

The $h, b \rightarrow y$ transformation, defined by eq. (A4), implies that the evolution of $\rho_c$ in the matrix space can also be mimicked by the dynamics in the complexity parameter space (consisting of $y_1, \ldots, y_M$). As eq. (7) indicates, the variation of $\rho_c$, depends on $h_{kl}, b_{kl}$ (for all $k, l$) only through a function $Y \equiv y_1(h, b)$ while $y_2(h, b), \ldots, y_M(h, b)$ remain fixed at their initial values. This encourages following implication: the $y$-space can essentially be viewed as a "center of mass" frame of reference, with $y_1$ acting as a "center of mass" of the ensemble parameters (corresponding to average uncertainty associated in determination of the matrix $C$). The remaining parameters $y_2, \ldots, y_M$, acting as the relative distances, are basically the functions of uncertainties which remain constant.
An important implication of the above is following: each point in the path defined by $y_1(h,b), \ldots, y_M(h,b)$ in $y$-space can represent many different $C$-ensembles and thereby different quantum states. Two states are predicted to share the same statistics of the Schmidt eigenvalues and thereby the entanglement entropy if they belong to same $y_k \forall k$, equivalently if they have same $y_1$ and evolve from a statistically same initial state; the latter ensures same values of $y_2, \ldots, y_M$. The latter intuitively appears to be possible for the quantum states belonging to same class of global constraints e.g. symmetry, conservation laws; a rigorous analysis is however needed in this context. The set of complexity parameters can then be used as a classification criteria of the entanglement for quantum states.

The above insight gives rise to an important query: is it always possible to define a transformation of the set of $M$ non-zero parameters $h$ and $b$ to another set \{ $y_1, \ldots, y_M$ \} i.e $h_{kl} = h_{kl}(y_1, \ldots, y_M)$ and $b_{kl} = b_{kl}(y_1, \ldots, y_M)$ such that $y_2, \ldots, y_M$ remain constant? The answer lies in the type of the perturbation of the quantum state: if the perturbation, due to variation of system parameters, leaves at least $M - 1$ matrix constraints invariant, the latter, or their combinations, can then be identified with $y_2, \ldots, y_M$. For example, if the perturbation is analyzed in a fixed basis, $y_2, \ldots, y_M$ can be chosen as the functions of basis parameters (as the number $M$ of ensemble parameters depends on the number of basis states). A similar formulation in context of Hermitian operators is discussed in detail in [38, 39].

An important consequence of the complex parameteric formulation is that it identifies the hidden, relevant driver as well as constants of the evolution of the entanglement measures: variation of the system conditions can cause a change of entanglement measures of a typical state only if $Y$ changes. It is interesting to note that $Y$ has a form of an information theoretic function: it is a sum over logarithmic values of the uncertainties associated with pairwise correlations of the basis-states and is therefore a measure of total information content, (also called surprise or self-information) of basis state correlations. It is therefore not surprising that $Y$ governs the evolution of entanglement entropies.

Another important query arises regarding the role of $T$ i.e the particular combination of first order parameteric derivatives, in $Y$ governed evolution of $\rho_c$ and thereby the entanglement measures. The choice of $T$ permits a linear transformation from $h,b$-space to $y$-space resulting $Y$ in the form of a information theoretic function. This also ensures that the Gaussians densities spread linearly by the same rate and their mean values too shift in a same way, finally reaching to a stationary state corresponding to maximum entanglement.
It is then natural to wonder how far a Gaussian choice of $\rho_c$ is necessary for the complexity parameter formulation? For example, is it possible to reach to same deductions by defining another $T$-operator if $\rho_c$ is non-gaussian? A similar query in context of the Hermitian operators represented by non-Gaussian ensembles [32] indicates that the evolution in Hermitian matrix space can be retained in same form (generator $L$ same as in Gaussian cases) if one considers the higher order parametric derivatives i.e by changing the form of generator $T$. It is however not clear so far whether a single parameter evolution can still be defined. Alternatively, by keeping only the first order parametric derivatives i.e $T$ same, the evolution in matrix space is no longer constant diffusion with finite drift (i.e $L$ no longer of same form).

V. NUMERICAL VERIFICATION OF COMPLEXITY PARAMETER BASED FORMULATION OF THE ENTROPIES

Based on the complexity parametric formulation, different reduced matrix ensembles subjected to same global constraints e.g symmetry and conservation laws are expected to undergo similar statistical evolution of the Schmidt eigenvalues. This in turn implies an analogous evolution of their entanglement measures. Intuitively this suggests the following: the underlying complexity of the system wipes out details of the correlations between two sub-bases, leaving their entanglement to be sensitive only to an average measure of complexity i.e $Y - Y_0$. Besides fundamental significance, the complexity parameter based formulation is useful also for the following reason: for states evolving along the same path, $Y$ can be used as a hierarchical criteria even if they belong to different complex systems but same global constraints.

In this section, we verify the analogy of evolution by numerically comparing the entanglement measures of three different complex systems, represented by three multi-parametric Gaussian ensembles of real $C$ matrices with different variance types (with $h_{kl} = \langle C_{kl}^2 \rangle - (\langle C_{kl} \rangle)^2$, $b_{kl} = \langle C_{kl} \rangle$). (We recall here that $C_{kl}$ corresponds to a component of the state $\Psi$ in product basis $|kl\rangle$ consisting of eigenstate $|k\rangle$ of subsystem A and $|l\rangle$ of subsystem B, a choice of variance of $C_{kl}$ changing with $l$ implies change of correlation between two subunits). Further details are as follows (with schematic images given in Figure 1)

(i) **Components with same variance along higher columns (EB):** The ensemble parameters in this case are same for all elements except those in first column:
\[ h_{k1} = 1, \quad h_{kl} = \frac{1}{(1 + \mu)}, l \neq 1, b_{kl,s} = 0 (\forall k,l). \]  

(38)

The substitution of the above in eq. (8) leads to

\[ Y = -\frac{N(N - 1)}{2M\gamma} \left[ \ln \left( 1 - \frac{2\gamma}{1 + \mu} \right) \right] + c_0 \]

with constant \( c_0 \) determined by the initial state of the ensemble.

Choosing initial condition with \( \mu \to \infty \) corresponds to an ensemble of \( C \)-matrices with only first column elements as non-zero; this in turn gives \( Y_0 = c_0 \).

(ii) **Components-variance decaying as a Power law along columns (EP):** The variance of the \( C_{kl} \) now changes, as a power law, across the column as well as row but its mean is kept zero:

\[ h_{kl} = \frac{1}{1 + \frac{k(l - 1)}{ab}}, \quad b_{kl} = 0 \quad \forall \ k,l \]

(40)

where \( a \) and \( b \) are arbitrary parameters. Eq. (8) then gives

\[ Y = -\frac{1}{2M\gamma} \left[ \sum_{r_1=1}^{N-1} \sum_{r_2=1}^{N} \ln \left( 1 - \frac{2\gamma}{1 + \frac{r_1 r_2}{r_1 r_2}} \right) \right] + c_0 \]

(41)

Choosing initial condition with \( b, a \to \infty \) again corresponds to an ensemble of \( C \)-matrices with only first column elements as non-zero and thereby \( Y_0 = c_0 \).

(iii) **Component variance with exponential decay along columns (EE):** Here again the mean is kept zero for all elements but the variance changes exponentially across the column as well as row:

\[ h_{kl} = \exp \left( -\frac{k(l - 1)}{ab} \right), \quad b_{kl} = 0 \quad \forall \ k,l \]

(42)

with \( b \) as an arbitrary parameter. Eq. (8) now gives

\[ Y = -\frac{1}{2M\gamma} \left[ \sum_{r_1=1}^{N-1} \sum_{r_2=1}^{N} \ln \left( 1 - \frac{2\gamma}{\exp \left( \frac{r_1 r_2}{a b} \right)} \right) \right] + c_0 \]

(43)

with \( M = N^2 \). Here again the initial choice of parameters \( b, a \to \infty \) leads to a \( C \)-matrix ensemble same as in above two cases and same \( Y_0 \).

For numerical analysis of various entropies, we exactly diagonalize (using Lapack subroutine for real matrices based on Lanczos algorithm) each ensembles for many matrix sizes
and for many values of the ensemble parameters $a, b$ but with fixed $\gamma = 1/4$. We note that in each of the three ensembles (schematic illustration in figure 1 for visual guidance), the change of variance along the columns ensures the variation of entanglement from an initial separable state to maximally entangled state. Here the separable state corresponds to very small $a, b$ (equivalently $Y \to Y_0$) and the maximally entangled state corresponds to the ensemble with large $a, b$ ($Y > 1$). For simplification, the $C$ matrix chosen for all cases is a $N \times N$ square matrix, thus implying $\nu_0 = 0$. The obtained Schmidt eigenvalues are then used to numerically derive the Von-Neumann and Renyi entropies (using log base 2) for each matrix of the ensemble under consideration.

Figures 2 and 3 illustrates the $Y$-governed evolution of the ensemble averaged Von Neuman as well as 2nd Renyi entropy for three ensembles and for two symmetry conditions, namely, the quantum state with/ without time-reversal symmetry, respectively. As clear from these figures, although both entropies grow with increasing $Y$ (as expected due to $Y = Y_0$ corresponding to the separable limit and a large $Y$ to the maximum entanglement), their growth rate follows a quantitatively different although qualitatively similar routes with same functional dependence; this is consistent with our theoretical prediction too. (It must be emphasized here that same $Y_0$ for the above three ensembles does not by itself imply same values for other parameters i.e $y_2, \ldots, y_M$ in the complexity parameter set and therefore they need not fall on the same evolutionary path). More specifically, (i) both entropies show sensitivity to symmetry conditions (as can be seen from difference of $Y$ dependent growth for $\beta = 2$ from $\beta = 1$ depicted in figure 3, (ii) both entropies on average show almost linear increase, growing to a maximum limit dependent on the global constraints on the ensembles, (iii) the behavior in the intermediate region i.e $Y \sim 1$ however is dependent on the ensemble details that is local constraints: while the growth is rapid/ sudden in ensemble with constant coefficients for higher columns, it is slower and relatively smooth other two ensembles. We also find different growth rates, at least quantitatively, of the two entropies, with $R_2$ always slower than $R_1$ for all three cases.

Figure 4 describes the average behavior of two additional measures namely $\langle R_0 \rangle$ and $\langle \frac{1}{S_2} \rangle$ (with $R_0 = -\sum_n \log \lambda_n$ and $S_2 = \sum_n \lambda_n^2$) with respect to $Y$ as well as subsystem size $N$. The figure confirms the expected behavior in large $N$ as well as large $Y$ limit: $\langle R_0 \rangle \propto N \log_2 N$ and $\langle \frac{1}{S_2} \rangle \propto N \log_2 N$. Although here we display the results for the ensemble (iii) only, similar results are confirmed for the ensembles (i) and (ii) too. Figure 5 depicts the dependence of
\( R_1(S_1) \) and \( R_2(S_1) \) on \( S_1 \) for a Wishart ensemble subject to constraint \( S_1 = \sum_n \lambda_n \); as clear from the figure, \( \frac{\partial R_1}{\partial S_1} > 0 \) and \( \frac{\partial R_2}{\partial S_1} < 0 \).

VI. CONCLUSION

In the end, we summarize with a brief discussion of our main idea, results and open questions. We have theoretically analyzed the system-dependence of the evolution of the average entanglement entropies of a typical quantum state in a bipartite basis. While previous theoretical studies have mostly considered ergodic states leading to a standard Wishart ensemble representation of the reduced density matrix, our primary focus has been on the non-ergodic states, specifically the states with their component (in bipartite basis) as Gaussian distributed with arbitrary variances and mean values and subjected to symmetry constraints. This in turn gives the reduced density matrix as a multiparametric Wishart ensemble with fixed trace and permits an analysis of the entanglement with changing system conditions i.e complexity of the system. Supported by detailed numerical analysis presented here, our theoretical results for ensemble averaged Von-Neumann as well as second order Renyi entropies indicate a linear behavior only for small \( Y \)-ranges i.e near separability limit. In large \( Y \)-limit, both entropies approach a constant value, latter dependent on the global constraints, but the change from linear to constant behavior is exponentially rapid. The growth rates of the two entropies with respect to changing complexity are however different. We also find the results to be sensitive to global symmetry conditions with different growth rates of entanglement for the states with and without time-reversal symmetry as well sensitivity to subsystem "cut".

An important ingredient of our approach is the common mathematical formulation of the joint probability distribution of the Schmidt eigenvalues for different quantum states which in turn leads to the evolution equations for the distributions of the Renyi and Von-Neumann entropies in terms of a single function of all system parameters. The formulation, referred as complexity parameter formulation, is a generalization of a similar formulation reported in [33, 34] (for unconstrained Wishart and chiral ensembles) and not only leads to a technical simplification but also provides fundamental insights about the entanglement relations between different quantum states (including states belonging to different local system constraints). For states, where explicit time-dependence of the system parameters
is known, the complexity parameter can be expressed as a function of time and therefore can be used to derive information about the time-dependent evolution of the entanglement measures.

The present study still leaves many open questions e.g. whether a knowledge of the average entanglement measures is sufficient for a typical state of a complex system or their fluctuations should also be taken into account? The consideration would have significant impact on any hierarchical arrangement of states. Another important question is regarding the explicit role played by the system parameters in the entanglement evolution. The complexity parameter depends on the system parameters through distribution parameters. A knowledge of their exact relation however requires a prior knowledge of the quantum Hamiltonian, its matrix representation in the product basis as well as the nature and distribution parameters of the appropriate ensemble. The knowledge leads to the determination of the appropriate ensemble of its eigenstates. We intend to answer some of these queries in near future.

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Appendix A: Complexity parameter formulation

A perturbation of the state by a change of the parameters \( h_{kl,s} \rightarrow h_{kl,s} + \delta h_{kl,s} \) and \( b_{kl} \rightarrow b_{kl} + \delta b_{kl} \) over time causes the matrix elements \( C_{kl} \) to undergo a dynamics in the matrix space. We consider a combination of multiparametric variations defined as

\[
T \rho_c \equiv \sum_{k \leq l, s} \left[ \frac{2}{\tilde{g}_{kl}} x_{kl,s} \left( \frac{\partial \rho_c}{\partial h_{kl,s}} - \gamma b_{kl,s} \frac{\partial \rho_c}{\partial b_{kl,s}} \right) \right] \quad (A1)
\]

For such a combination along with its Gaussian form, the multi-parametric evolution of \( \rho_c(C) \) can then be described exactly in terms of a diffusion, with finite drift, in \( C \)-matrix space:

\[
T \rho_c = L \rho_c + (T \log N) \rho_c \quad (A2)
\]

with
\[ L\rho_c = \sum_{k,l,s} \frac{\partial}{\partial C_{kl,s}} \left[ \frac{\partial \rho_c}{\partial C_{kl,s}} + \gamma C_{kl,s} \rho_c \right]. \] (A3)

As the above equation is difficult to solve for generic parametric values, we seek a transformation from the set of \( M \) parameters \( \{h_{kl,s}, b_{kl,s}\} \) to another set \( \{y_1, \ldots, y_M\} \) such that only \( y_1 \) varies under the evolution governed by the operator \( T \) and rest of them i.e \( y_2, \ldots, y_M \) remain constant: \( T\rho_c \equiv \frac{\partial \rho_c}{\partial y_1} \). This in turn requires

\[ Ty_1 = 1, Ty_k = 0 \quad \forall \ k > 1. \] (A4)

(The steps for the transformation are similar as for the Hermitian operators, discussed in details with examples in a series of studies \[38, 39\] and not included here to avoid repetition).

By defining \( \rho = e^{\int f_{\text{dyn}} \rho_c} \) with \( f_0 = T \log \mathcal{N} = \sum_{kl,s} \frac{1 - \gamma h_{kl,s}}{2b_{kl,s}} \), eq.\((A3)\) can be written as the diffusion equation for \( \rho \)

\[ \frac{\partial \rho}{\partial y_1} = \sum_{k,l,s} \frac{\partial}{\partial C_{kl,s}} \left[ \frac{\partial \rho}{\partial C_{kl,s}} + \gamma C_{kl,s} \rho \right]. \] (A5)

**Appendix B: Diffusion of reduced density matrix \( \rho_A \)**

As expected, the diffusive dynamics of the matrix elements \( C_{kl} \) manifests itself in the \( \rho_A \)-matrix space and the moments for the matrix elements \( \rho_A; mn = \sum_{k=1}^{N_A} C_{mk} C_{nk}^* \) can be calculated from those of \( C \):

As discussed in \[33, 35\], eq.\((9)\) can be rewritten as

\[ C(Y + \delta Y) \equiv \frac{C(Y) + \sqrt{2 \delta Y} \sqrt{V(Y)}}{\sqrt{\gamma + 2 \gamma \delta Y}} \]

\[ \approx C(Y) (1 - \gamma \delta Y) + \sqrt{2 \delta Y} V(Y) + O((\delta Y)^{3/2}). \] (B2)

Here the symbol " \( \equiv \) " implies the equivalence of the ensembles of matrices on two sides. The ensemble approaches to equilibrium as \( Y \to \infty \). The equivalence of eq.\((9)\) and eq.\((B1)\) along with the derivation of the diffusion equation for \( C(Y) \) is discussed in \[35\].

As expected, the diffusive dynamics of the matrix elements \( C_{kl} \) manifests itself in the \( \rho_A \)-matrix space and the moments for the matrix elements \( \rho_A; mn = \sum_{k=1}^{N_A} C_{km}^* C_{kn} \) can be calculated from those of \( C \). The above equations along with relation between the elements
of $\rho_A$ and $C$ gives the moments of the matrix elements of $L$. As discussed in appendix B, the $1^{st}$ moment is same for both $\beta = 1$ or 2:

$$\langle \delta \rho_{A;mn} \rangle = 2 (\beta v^2 N_a \delta_{mn} - \gamma \rho_{A;mn}) \delta Y$$  \hspace{1cm} (B3)

but the $2^{nd}$ moment depends on $\beta$ as follows

**Case $\beta = 1$**

$$\langle \delta \rho_{A;mn} \delta \rho_{A;kl}^* \rangle = \langle \delta \rho_{A;mn} \delta \rho_{A;kl} \rangle = 2 v^2 [\rho_{A;mk} \delta_{nl} + \rho_{A;nl} \delta_{mk} + \rho_{A;nk} \delta_{ml} + \rho_{A;ml} \delta_{nk}] \delta Y$$ \hspace{1cm} (B4)

**Case $\beta = 2$**

$$\langle \delta \rho_{A;mn} \delta \rho_{A;kl}^* \rangle = 4 v^2 [\rho_{A;mk} \delta_{nl} + \rho_{A;nl} \delta_{mk}] \delta Y$$

$$\langle \delta \rho_{A;mn} \delta \rho_{A;kl} \rangle = 4 v^2 [\rho_{A;nl} \delta_{nk} + \rho_{A;nk} \delta_{ml}] \delta Y$$  \hspace{1cm} (B5)

with $\beta = 1, 2$ for $\rho_A$ real-symmetric or complex Hermitian, respectively.

**Appendix C: Diffusion of Schmidt eigenvalues**

The diffusion of the matrix elements of $\rho_A$ manifests itself in the dynamics of its eigenvalues and eigenfunctions. The evolution equation for the joint probability density function (JPDF) of all eigenvalues can now be derived as follows.

Let $U$ be the $N \times N$ eigenvector matrix of $\rho_A(Y)$, unitary in nature i.e $U^\dagger U = 1$ ($L$ being Hermitian) and $\lambda$ be the $N \times N$ diagonal matrix of its eigenvalues, $\lambda_{mn} = \lambda_n \delta_{mn}$. A small change $\delta Y$ in parameter $Y$ changes $\rho_A$ and its eigenvalues and eigenfunctions. Using standard perturbation theory for Hermitian operators and by considering matrix $\rho_A + \delta \rho_A$ in the eigenfunction representation of matrix $W$, a small change $\delta \lambda_n$ in the eigenvalue $\lambda_n$ can be given as

$$\delta \lambda_n = \delta \rho_{A;nn} + \sum_{m \neq n} \frac{|\delta \rho_{A;mn}|^2}{\lambda_n - \lambda_m} + o((\delta \rho_{A;nn})^3)$$  \hspace{1cm} (C1)

where $\rho_{A;mn} = \lambda_n \delta_{mn}$ at value $Y$ of complexity parameter (due to $\rho_A + \delta \rho_A$ being considered in the diagonal representation of $\rho_A$). Eq. (C1) gives, up to first order of $\delta Y$ (see appendix B),
\[
\langle \delta \lambda_n \rangle = 2 \beta v^2 \left[ N_a - \frac{\gamma}{\beta v^2} \lambda_n + \sum_{m=1, m \neq n}^N \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \right] \delta Y \tag{C2}
\]

\[
\langle \delta \lambda_n \delta \lambda_m \rangle = 8 v^2 \lambda_n \delta_{nm} \delta Y \tag{C3}
\]

In general, assuming Markovian process, the parametric diffusion of the joint probability distribution \(P_x(x_1, \ldots, x_N; Y)\) of \(N\) variables \(x_n, n = 1, \ldots, N\) from an arbitrary initial condition, with \(Y\) as the parameter, is given by the standard Fokker-Planck equation

\[
\frac{\partial P_x}{\partial Y} \delta Y = \frac{1}{2} \sum_{k,l=1}^N \frac{\partial^2}{\partial x_k \partial x_l} (\langle \delta x_k \delta x_l \rangle P_x) - \sum_{k=1}^N \frac{\partial}{\partial x_k} (\langle \delta x_k \rangle P_x) \tag{C4}
\]

Using the above, the diffusion equation for the joint probability density for \(P_{ev}(\{\lambda_n\}; Y)\) at perturbation strength \(Y\) where \(\{\lambda_n\}\) refer to the sets of all eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_N\) can now be given as

\[
\frac{\partial P_\lambda}{\partial Y} \delta Y = \sum_n \frac{\partial}{\partial \lambda_n} \left[ \frac{1}{2} \frac{\partial}{\partial \lambda_n} (\langle (\delta \lambda_n)^2 \rangle - \langle \delta \lambda_n \rangle) \right] P_\lambda \tag{C5}
\]

Note here \(P_\lambda\) is subjected to following boundary condition: \(P_\lambda \to 0\) for \(\lambda_n \to [0, \infty)\) for \(n = 1 \to N\); this follows because the higher order moments of the ensemble density are assumed to be negligible. Substitution of the moments from eq.(C2) and eq.(C3) lead to the diffusion equation for \(P(\lambda)\) given by eq.(10).

**Appendix D: Derivation of eq.(17)**

For \(f(\lambda)\) as an arbitrary function of eigenvalues \(\lambda_1 \ldots \lambda_N\), \(I_1\) in eq.(17) can be written as

\[
I_1 = -C_{hs} \sum_{n=1}^N \int_0^\infty \delta_1 f(\lambda) \frac{\partial}{\partial \lambda_n} \left( \sum_{m=1}^N \frac{\beta \lambda_n}{\lambda_n - \lambda_m} - \beta \nu - 2 \gamma \lambda_n \right) P_\lambda \text{ D} \lambda. \tag{D1}
\]

with \(P_\lambda(\lambda) \to 0\) at the two integration limits \(0 \leq \lambda_n \leq \infty \; \forall n = 1 \to N\). We note here that the constraint \(\delta_1\) effectively reduces the limits to \(\frac{1}{N} \leq \lambda_n \leq \infty\).

Integration by parts now gives \(I_2 = A + B\) where

\[
A = C_{hs} \sum_{n=1}^N \int_0^\infty \left[ \frac{\partial \delta_1}{\partial \lambda_n} f \right] \left( \sum_{m=1}^N \frac{\beta \lambda_n}{\lambda_n - \lambda_m} - \beta \nu - 2 \gamma \lambda_n \right) P_\lambda \text{ D} \lambda \tag{D2}
\]

\[
B = C_{hs} \sum_{n=1}^N \int_0^\infty \left[ \delta_1 \frac{\partial f}{\partial \lambda_n} \right] \left( \sum_{m=1}^N \frac{\beta \lambda_n}{\lambda_n - \lambda_m} - \beta \nu - 2 \gamma \lambda_n \right) P_\lambda \text{ D} \lambda \tag{D3}
\]
Again using $\frac{\partial \delta_1}{\partial \lambda_n} = -\frac{\partial \delta_1}{\partial S_1}$, $A$ can be rewritten as

$$A = -C_{hs} \frac{\partial}{\partial S_1} \left( \sum_{n=1}^{N} \lambda_n \right) P_{\lambda} \, D\lambda$$

with

$$\langle f \rangle \equiv \langle f(S_1) \rangle = C_{hs} \int \delta_1 \, f \, P_{\lambda} \, D\lambda$$

Similarly

$$B = \beta J_2 - C_{hs} \int \delta_1 \left( \sum_{n=1}^{N} \frac{\partial f}{\partial \lambda_n} \right) P_{\lambda} \, D\lambda$$

where

$$J_2 \equiv J_2(S_1) = C_{hs} \int \delta_1 \left( \sum_{m,n=1}^{N} \frac{\partial f}{\partial \lambda_n} \right) P_{\lambda} \, D\lambda$$

Based on the details of function $f(\lambda)$, the integrals $A$ and $B$ can further be reduced. Here we give the results for $R_1$ and $R_2$.

**Case** $f = R_1(\lambda)$: Following from above, we have for $f = -\sum_n \lambda_n \log \lambda_n$,

$$A = \frac{\partial}{\partial S_1} \left( 2 \gamma S_1 - \frac{1}{2} \beta N(N - 2\nu - 1) \right) \langle R_1 \rangle$$

and

$$B = -C_{hs} \int \delta_1 \left( \sum_{m,n=1}^{N} \frac{\beta \lambda_n (1 + \log \lambda_n)}{\lambda_n - \lambda_m} - \beta \nu \sum_n (1 + \log \lambda_n) - 2\gamma \sum_n \lambda_n (1 + \log \lambda_n) \right) P_{\lambda} \, D\lambda$$

Eq. (D8) can be rewritten as

$$B = -\beta J_2 - \frac{1}{2} \beta N(N - 2\nu - 1) J - \beta \nu \langle R_0 \rangle + 2\gamma \, \langle S_1 \, J + \langle R_1 \rangle \rangle$$

where $R_0(\lambda) \equiv -\sum_n \log \lambda_n$. To express $J_2$ in form of $\langle R_1 \rangle$, we note that $|\lambda_n - \lambda_m| \leq 1$ and therefore we can approximate $\log \lambda_n \approx \log \lambda_m$. This leads to following relation

28.
\[
\sum_{m,n=1}^{N} \frac{\lambda_n \log \lambda_m}{\lambda_n - \lambda_m} = \frac{1}{2} \sum_{m,n=1}^{N} \frac{\lambda_n \log \lambda_n - \lambda_m \log \lambda_m}{\lambda_n - \lambda_m}
\]  
(D13)

\[
\approx \frac{1}{2} \sum_{m,n=1}^{N} \frac{(\lambda_n - \lambda_m) \log \lambda_n}{\lambda_n - \lambda_m}
\]  
(D14)

\[
\approx \frac{1}{2} (N - 1) \sum_n \log \lambda_n \approx -\frac{1}{2} (N - 1) R_0
\]  
(D15)

Substitution of the above relation in eq.(D16) leads to

\[
B = -\frac{1}{2} \beta N (N - 2\nu - 1) J + \frac{1}{2} \beta (N - 2\nu - 1) \langle R_0 \rangle + 2\gamma (S_1 J - \langle R_1 \rangle)
\]  
(D16)

Substitution of eq.(D10) and eq.(D16) in eq.(D1), we have for \( f = R_1 \),

\[
I_1 = -\frac{1}{2} \beta N (N - 2\nu - 1) J + \frac{1}{2} \beta (N - 2\nu - 1) \langle R_0 \rangle + 2\gamma S_1 J +
\]
\[
+ (2 \gamma S_1 - \frac{1}{2} \beta N (N - 2\nu - 1)) \frac{\partial \langle R_1 \rangle}{\partial S_1}
\]  
(D17)

**Case** \( f = R_2 \): Proceeding similarly for \( f = R_2 = -\log \sum_n \lambda_n^2 \), we have, from eq.(D5) and eq.(D8),

\[
A = \frac{\partial}{\partial S_1} (2 \gamma S_1 - \frac{1}{2} \beta N (N - 2\nu - 1)) \langle R_2 \rangle
\]  
(D18)

and

\[
B = -2 C_{hs} \int \delta_1 \frac{1}{S_2} \left( \sum_{m,n=1}^{N} \frac{\beta \lambda_n^2}{\lambda_n - \lambda_m} - \beta \nu \sum_{n=1}^{N} \lambda_n - 2\gamma \sum_{n=1}^{N} \lambda_n^2 \right) P_\lambda \, d\lambda
\]  
(D19)

Rewriting \( 2 \sum_{m,n=1}^{N} \frac{\lambda_n^2}{\lambda_n - \lambda_m} = \sum_{m,n=1}^{N} \frac{\lambda_n^2 - \lambda_m^2}{\lambda_n - \lambda_m} = 2(N - 1) \sum_n \lambda_n \), the above equation can be rewritten as

\[
B = -C_{hs} \int \delta_1 \frac{1}{S_2} (2(N - 1 - \nu) \beta S_1 - 4\gamma S_2) P_\lambda \, d\lambda
\]  
(D20)

substitution of the above in eq.(D1) gives

\[
I_1 = -2 \beta (N - \nu - 1) S_1 \left( \frac{1}{S_2} \right) + 2\gamma (\langle R_2 \rangle + 2J) + (2 \gamma S_1 - \frac{1}{2} \beta N (N - 2\nu - 1)) \frac{\partial \langle R_2 \rangle}{\partial S_1}
\]  
(D21)
Appendix E: Derivation of eq. (18)

For \( f(\lambda) \) as an arbitrary function of eigenvalues \( \lambda_1 \ldots \lambda_N \), \( I_2 \) defined in eq. (18) can be written as

\[
I_2 = C_{hs} \sum_{n=1}^{N} \int \delta_1 f(\lambda) \frac{\partial^2 (\lambda_n^2 P_{\lambda})}{\partial \lambda_n^2} \, d\lambda \tag{E1}
\]

\[
= C_{hs} \sum_{n=1}^{N} \int \left[ \delta_1 \frac{\partial^2 f}{\partial \lambda_n^2} \lambda_n + 2 \frac{\partial \delta_1}{\partial \lambda_n} \frac{\partial f}{\partial \lambda_n} \lambda_n + \frac{\partial^2 \delta_1}{\partial \lambda_n^2} f \lambda_n \right] P_{\lambda} \, d\lambda \tag{E2}
\]

Now using \( \frac{\partial \delta_1}{\partial \lambda_n} = -\frac{\partial \delta_1}{\partial S_1} \) the above can be rewritten as

\[
I_2 = C_{hs} \int D\lambda \delta_1 P_{\lambda} F_1 - 2 \frac{\partial}{\partial S_1} \int D\lambda \delta_1 P_{\lambda} F_2 + \frac{\partial^2}{\partial S_1^2} (S_1 \langle f \rangle) \tag{E3}
\]

where \( F_1(f) = \sum_{n=1}^{N} \frac{\partial^2 f}{\partial \lambda_n^2} \lambda_n \) and \( F_2(f) = \sum_{n=1}^{N} \frac{\partial f}{\partial \lambda_n} \lambda_n \) and last term in eq. (E3) is obtained by the relation

\[
C_{hs} \int D\lambda \delta_1 P_{\lambda} \left[ \sum_{n=1}^{N} \lambda_n \right] f = C_{hs} S_1 \int D\lambda \delta_1 P_{\lambda} f = S_1 \langle f \rangle. \tag{E4}
\]

**Case** \( f = R_1 \): substituting \( f = -\sum_n \lambda_n \log \lambda_n \) in eq. (E3) and using \( F_1 = -\sum_n \frac{\lambda_n}{\lambda_n} = -N \) and \( F_2 = -\sum_{n=1}^{N} (\log \lambda_n + 1) \lambda_n \), \( I_2 \) can be written as

\[
I_2 = -N J + 2 \frac{\partial}{\partial S_1} \left( S_1 J - \langle R_1 \rangle \right) + \frac{\partial^2}{\partial S_1^2} (S_1 \langle R_1 \rangle) \tag{E5}
\]

\[
= -(N - 2) J + S_1 \frac{\partial^2 \langle R_1 \rangle}{\partial S_1^2} \tag{E6}
\]

where \( \langle R_1 \rangle \equiv \langle R_1(S_1) \rangle \) and \( J \equiv J(S_1) = \int D\lambda \delta_1 P_{\lambda} \).

**Case** \( f = R_2 \): Proceeding similarly for \( f = R_2 = -\log \sum_n \lambda_n^2 \), we have (now \( \langle R_2 \rangle \equiv \langle R_2(S_1) \rangle \))

\[
I_2 = 4 \left( \frac{S_3}{S_2} \right) - 2 S_1 \left( \frac{1}{S_2} \right) + 4 \frac{\partial J}{\partial S_1} + \frac{\partial^2}{\partial S_1^2} (S_1 \langle R_2 \rangle) \tag{E7}
\]

\[
= 4 \left( \frac{S_3}{S_2} \right) - 2 S_1 \left( \frac{1}{S_2} \right) + 4 \frac{\partial J}{\partial S_1} + 2 \frac{\partial \langle R_2 \rangle}{\partial S_1} + S_1 \frac{\partial^2 \langle R_2 \rangle}{\partial S_1^2} \tag{E8}
\]
FIG. 1. Schematic Representation of C-matrix: For visual guidance, the different protocols for variance distribution in the C-matrix used in our numerics are illustrated here: (a) Identical distribution (matrix taken from stationary Wishart ensemble), (b) Constant variance except the first column (a matrix taken from ensemble eq. (38)), (c) power-law-decay (a matrix taken from eq. (40)) and (d) exponential-decay along rows and columns (a matrix taken from eq. (42)). The details for each case are given in section V.
FIG. 2. Growth of Average entanglement entropy for a typical quantum state with time-reversal symmetry ($\beta = 2$): We study the scaling of average entanglement, comparing the von Neumann (VN) and second-order Rényi entropy ($R_2$), in the three ensembles (each of size 1000 $C$-matrices), with the complexity parameter $Y$. The ensemble averaging is done over 1000 $C$-matrices for each case, for a subsystem matrix size $N = 1024$, and the measures are calculated using the *QuantumInformation.jl* package in Julia programming language [40]. The scaling behavior is fitted to the function $R(Y) = \alpha[1 - (1 + \beta Y + \gamma Y^2) \exp(-\delta Y)]$ with the parameters ($\alpha, \beta, \gamma, \delta$) as follows: (a) $\langle R_1 \rangle$: (9.3, −5.5, 60.6, −180), $\langle R_2 \rangle$: (9.0, −8.5, 5.6, −75); (b) $\langle R_1 \rangle$: (9.2, −11.5, 71.5, −18.5), $\langle R_2 \rangle$: (9.0, −9.5, 65.5, −16.5); (c) $\langle R_1 \rangle$: (9.2, −9.5, 90.0, −17.5), $\langle R_2 \rangle$: (9.0, −9.0, 67.0, −14). The scaling for smaller values of $Y$ is also shown for the three cases in the inset.
FIG. 3. Growth of Average entanglement entropy for a typical quantum state without time-reversal symmetry ($\beta = 2$): The dynamics of average entanglement, comparing the von Neumann ($R_1$) and second-order Rényi entropy ($R_2$), is shown, when $C$-matrix is complex valued, for (a) ensemble with constant variance, (b) ensemble with a power-law decay, (c) ensemble with exponential-decay. The scaling behavior is fitted to the function $R(Y) = \alpha[1 - (1 + \beta Y + \gamma Y^2) \exp(-\delta Y)]$ with the parameters ($\alpha, \beta, \gamma, \delta$) as follows: (a) $\langle R_1 \rangle$: (9.3, -17.0, 84.5, -65.5), $\langle R_2 \rangle$: (9.0, -15.0, 80.0, -25.0); (b) $\langle R_1 \rangle$: (9.2, -12.0, 62.0, -12.5), $\langle R_2 \rangle$: (9.0, -8.0, 34.0, -9.0); (c) $\langle R_1 \rangle$: (9.2, -6.0, 19.0, -7.0), $\langle R_2 \rangle$: (9.0, -5.5, 18.5, -6.5). The comparison for two symmetry conditions is also show in the inset: the growth in complex case is slightly slower for each case (but different).
FIG. 4. *N and Y-dependence of* \( \langle R_0 \rangle \) *and* \( \langle \frac{1}{S_2} \rangle \) : Average behavior of \( R_0 \) and \( \frac{1}{S_2} \) for the ensemble (iii), with \( R_0 = -\sum_n \log \lambda_n \) and \( S_2 = \sum_n \lambda_n^2 \) with respect to \( Y \) as well as subsystem size \( N \). The figure confirms the expected behavior in large \( N \) as well as large \( Y \) limit: \( \langle R_0 \rangle \propto N \log_2 N \) and \( \langle \frac{1}{S_2} \rangle \propto N \log_2 N \).
FIG. 5. Here the variation of $\langle R_1 \rangle$ and $\langle R_2 \rangle$ with $S_1$ is shown for the standard Wishart ensemble, for $N = 512$. 