Perfect Numbers and Groups

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Abstract

A number is perfect if it is the sum of its proper divisors; here we call a finite group ‘perfect’ if its order is the sum of the orders of its proper normal subgroups. (This conflicts with standard terminology but confusion should not arise.) The notion of perfect group generalizes that of perfect number, since a cyclic group is perfect just when its order is perfect. We show that, in fact, the only abelian perfect groups are the cyclic ones, and exhibit some non-abelian perfect groups of even order.

This article was originally composed in 1996 for Eureka, the journal of the Cambridge student mathematical society (but has yet to appear, as no issue has been published since). It is therefore written to be comprehensible to an undergraduate readership, and contains many reminders of basic facts.

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Introduction

Perfect numbers are an ancient object of study. A number is called perfect if it is the sum of its proper divisors—for instance, 6 is perfect, since $6 = 1 + 2 + 3$. It is straightforward to classify the even perfect numbers, but it is a long-standing question as to whether there are any odd perfect numbers at all.

This article generalizes the notion of ‘perfection’ from numbers to groups. We define what it means for a group to be perfect, explain in what sense this is a generalization of the notion for numbers, and go on to give some theory and examples of perfect groups. Signposts are provided for the reader not well versed in group theory, so that at least the rough shape of the ideas should be discernible.

The first properties of perfect numbers are summarized in Section 1. In Section 2 we give the definition of a perfect group and look at some examples. Section 3 is devoted to ‘multiplicativity’. This shows some close parallels with the world of numbers, including the results of Section 1, and the new theory also enables us to give some more interesting examples of perfect groups than was possible previously. The climax of the article, such as it is, is a theorem concerning the abelian quotients of perfect groups, a corollary of which classifies the perfect abelian groups. Two rather different proofs of this result are offered, one in each of Sections 4 and 5.

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I must also make two apologies. First of all, I crave the reader’s indulgence for the use of the term ‘perfect group’ when it is firmly established to mean something else. Faced with a group-theoretic concept generalizing that of perfect number, any other name seemed unnatural. My second apology is for the lack of pointers to the literature: some of the results included here are surely widely known, but I am not well enough educated to provide references.

1 Perfect Numbers

Here we go over the basic properties of perfect numbers.

For any number $n$, define $D(n) = \sum_{d|n} d$, the sum of the divisors of $n$, and call $n$ perfect if $D(n) = 2n$. By a ‘number’ I mean a positive integer.

1.1 Multiplicativity The function $D$ is multiplicative: that is to say, if $m_1$ and $m_2$ are coprime (have no common divisors other than 1) then $D(m_1m_2) = D(m_1)D(m_2)$. To see this, first observe that any divisor $d$ of $m_1m_2$ can be written uniquely as $d_1d_2$ where $d_i$ is a divisor of $m_i$ ($i = 1, 2$); conversely, if $d_i$ is a
divisor of \( m_i \) \((i = 1, 2)\) then \( d_1d_2 \) is a divisor of \( m_1m_2 \). Hence

\[
D(m_1m_2) = \sum_{d|m_1m_2} d
= \sum_{d_1|m_1, d_2|m_2} d_1d_2
= \left( \sum_{d_1|m_1} d_1 \right) \left( \sum_{d_2|m_2} d_2 \right)
= D(m_1)D(m_2),
\]
as required.

1.2 Even Perfect Numbers

It is easy to classify the even perfect numbers: they are precisely those numbers \( 2^r(2^r - 1) \) where \( r \geq 2 \) and \( 2^r - 1 \) is prime. (Of course, computing which values of \( r \) make \( 2^r - 1 \) prime is itself a hard problem.) The first three perfect numbers are \( 2 \times 3 = 6 \), \( 4 \times 7 = 28 \), and \( 16 \times 31 = 496 \).

In one direction, suppose that \( r \geq 2 \) and \( 2^r - 1 \) is prime: then

\[
D(2^r-1(2^r - 1)) = D(2^r-1)D(2^r - 1)
= (1 + 2 + 2^2 + \cdots + 2^{r-1})(1 + 2^r - 1)
= (2^r - 1)2^r
= 2[2^{r-1}(2^r - 1)],
\]
so \( 2^{r-1}(2^r - 1) \) is an even perfect number.

In the other direction, suppose that \( n \) is an even perfect number. Write \( n = 2^s m \) where \( s \geq 1 \) and \( m \) is odd: then \( n \) being perfect says that

\[
D(2^s m) = 2 \times 2^s m,
\]
i.e.

\[
(2^{s+1} - 1)D(m) = 2^{s+1} m,
\]
i.e.

\[
(2^{s+1} - 1)(D(m) - m) = m. \quad (*)
\]
Hence \( D(m) - m \) is a divisor of \( m \), and since

\[
2^{s+1} - 1 > 2^{0+1} - 1 = 1,
\]
it is a proper divisor of \( m \). But \( D(m) - m \) is by definition the sum of the proper divisors of \( m \), so \( D(m) - m \) is the unique proper divisor of \( m \). Thus \( m \) is prime and \( D(m) - m = 1 \), and by \((*)\), the latter means that \( m = 2^{s+1} - 1 \). So \( n = 2^s(2^{s+1} - 1) \) with \( s \geq 1 \) and \( 2^{s+1} - 1 \) prime, as required.
2 Definition and First Examples of Perfect Groups

In this section we define the notion of a perfect group, and search for examples among some of the well-known families of groups (symmetric, alternating, . . .). In fact, the only examples of perfect groups we will find are cyclic, although by Section 3 we will have developed enough theory to be able to exhibit some more interesting examples.

Of the examples below, only the cyclic groups 2.1 and the symmetric and alternating groups 2.2 will be needed later on.

The reader is reminded that a normal subgroup of a group $G$ is a subset of $G$ which is the kernel of some homomorphism from $G$ to some other group; equivalently, it is a subgroup $N$ of $G$ such that $gng^{-1} \in N$ for all $n \in N$ and $g \in G$. We write $N \trianglelefteq G$ to mean that $N$ is a normal subgroup of $G$.

From here on, ‘group’ will mean ‘finite group’.

If $G$ is a group, define $D(G) = \sum_{N \trianglelefteq G} |N|$, the sum of the orders of the normal subgroups of $G$, and say that $G$ is perfect if $D(G) = 2|G|$.

2.1 Example: Cyclic Groups Let $C_n$ be the cyclic group of order $n$. Then $C_n$ has one normal subgroup of order $d$ for each divisor $d$ of $n$, and no others, so $D(C_n) = D(n)$ and $C_n$ is perfect just when $n$ is perfect. Thus perfect groups provide a generalization of the concept of perfect numbers, and $C_6, C_{28}$ and $C_{496}$ are all perfect groups.

2.2 Example: Symmetric and Alternating Groups None of the symmetric groups $S_n$ or alternating groups $A_n$ is perfect. If $n \geq 5$ then $A_n$ is simple and the only normal subgroups of $S_n$ are 1, $A_n$ and $S_n$, so $D(A_n)$ and $D(S_n)$ are too small. For $n \leq 4$, we have

$$D(A_1) = 1, \quad D(S_1) = 1,$$
$$D(A_2) = 1, \quad D(S_2) = 1 + 2 = 3,$$
$$D(A_3) = 1 + 3 = 4, \quad D(S_3) = 1 + 3 + 6 = 10,$$
$$D(A_4) = 1 + 4 + 12 = 17, \quad D(S_4) = 1 + 4 + 12 + 24 = 41.$$

2.3 Example: $p$-Groups A (finite) $p$-group is a group of order $p^r$, where $p$ is prime and $r \geq 0$. Lagrange’s Theorem says that the order of any subgroup of a group divides the order of the group, so if $G$ is a $p$-group then $D(G) \equiv 1 \pmod{p}$. Hence no $p$-group is perfect.

2.4 Example: Dihedral Groups Let $E_{2n}$ be the dihedral group of order $2n$: that is, the group of all isometries of a regular $n$-sided polygon. Of the $2n$ isometries, $n$ are rotations (forming a cyclic subgroup of order $n$) and $n$ are reflections. We examine the cases of $n$ odd and $n$ even separately.

$n$ odd: All reflections are in an axis passing through a vertex and the midpoint of the opposite side, and any reflection is conjugate to any other by a suitable rotation. Thus if $N \leq E_{2n}$ and $N$ contains a reflection, then $N$ contains
all reflections; but $1 \in N$ too, so $|N| \geq n+1$, so $N = E_{2n}$. So any proper normal subgroup is inside the rotation group $C_n$; conversely, any (normal) subgroup of $C_n$ is normal in $E_{2n}$. Thus

$$D(E_{2n}) = D(C_n) + 2n,$$

and $E_{2n}$ is perfect if and only if $n$ is a perfect number.

$n$ even: The reflections split into two conjugacy classes, $R_1$ and $R_2$, each of size $n/2$: those in an axis through two opposite vertices, and those in an axis through the midpoints of two opposite sides. Write $C_{n/2}$ for the group of rotations by 2 or 4 or ... or $n$ vertices, a subgroup of $E_{2n}$ which is cyclic of order $n/2$. Then we can show that the smallest subgroup of $E_{2n}$ containing $R_i$ is $R_i \cup C_{n/2}$, for $i = 1$ and 2. Moreover, $R_i \cup C_{n/2}$ is of order $n$, i.e. index 2, therefore normal in $E_{2n}$. So we have two different normal subgroups, $R_1 \cup C_{n/2}$ and $R_2 \cup C_{n/2}$, of order $n$. We also have the normal subgroups $\{1\}$ and $E_{2n}$, hence

$$D(E_{2n}) \geq 1 + n + n + 2n > 4n$$

and $E_{2n}$ is not perfect.

In summary, the perfect dihedral groups are in one-to-one correspondence with the odd perfect numbers—so it is an open question as to whether there are any.

### 3 Multiplicativity

We proved in [1.4] that the function $D(n)$, on numbers $n$, was multiplicative. The aim of this section is to prove an analogous result for groups, and then to give some examples of nonabelian perfect groups by using this result.

Some difficulties are present for the reader not acquainted with composition series and the Jordan-Hölder Theorem. However, it is still possible for him or her to understand an example [3.3] of a nonabelian perfect group, provided that the following fact is taken on trust: if $G_1$ and $G_2$ are groups whose orders are coprime, and $G_1 \times G_2$ their direct product, then $D(G_1 \times G_2) = D(G_1)D(G_2)$. This done, the reader may proceed to 3.3 straight away.

The Jordan-Hölder Theorem states that any two composition series for a group $G$ have the same set-with-multiplicities of factors, up to isomorphism of the factors. I shall write this set-with-multiplicities as $c(G)$, and use $+$ to denote the disjoint union (or ‘union counting multiplicities’) of two sets-with-multiplicities. Thus if

$$c(G) = \{C_2, C_2, C_5\} \text{ and } c(H) = \{C_2, A_6\}$$

then

$$c(G) + c(H) = \{C_2, C_2, C_5, A_6\}.$$

We will use the fundamental fact that if $K \subseteq X$ then

$$c(X) = c(X/K) + c(K).$$
A pair of groups will be called **coprime** if they have no composition factor in common; alternatively, we will say that one group is **prime to** the other. (In particular, if two groups have coprime orders then they are coprime.) We will prove that $D$ is **multiplicative**: that is, if $G_1$ and $G_2$ are coprime then $D(G_1 \times G_2) = D(G_1)D(G_2)$. First of all we establish the group-theoretic analogue of a number-theoretic result from Section 1—namely, the second sentence of 1.1.

### 3.1 Proposition

Let $G_1$ and $G_2$ be coprime groups. Then the normal subgroups of $G_1 \times G_2$ are exactly the subgroups of the form $N_1 \times N_2$, with $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$.

**Proof** If $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$ then $N_1 \times N_2 \triangleleft G_1 \times G_2$; conversely, suppose $N \triangleleft G_1 \times G_2$. Write $\pi_i : G_1 \times G_2 \to G_i$ ($i = 1, 2$) for the projections, and regard $G_1$ as a normal subgroup of $G_1 \times G_2$ by identifying it with $G_1 \times \{1\}$, and similarly $G_2$. We have

$$\pi_1 N \cong \frac{N}{\ker(\pi_1|_N)} = \frac{N}{G_2 \cap N},$$

so by the ‘fundamental fact’ above,

$$c(N) = c(\pi_1 N) + c(G_2 \cap N);$$

and therefore by symmetry

$$c(\pi_1 N) + c(G_2 \cap N) = c(\pi_2 N) + c(G_1 \cap N).$$

But $c(\pi_i N) \subseteq c(G_i)$ and $G_1$ and $G_2$ are coprime, so $c(\pi_1 N)$ and $c(\pi_2 N)$ have no element in common; similarly $c(G_i \cap N) \subseteq c(G_i)$, so $c(G_1 \cap N)$ and $c(G_2 \cap N)$ have no element in common. Hence $c(\pi_i N) = c(G_i \cap N)$. We also know that $c(X)$ determines the order of a group $X$ and that $G_i \cap N \subseteq \pi_i N$, so in fact $G_i \cap N = \pi_i N$. Thus

$$\pi_1 N \times \pi_2 N = (G_1 \cap N) \times (G_2 \cap N) \subseteq N,$$

and as always

$$N \subseteq \pi_1 N \times \pi_2 N,$$

so $N = \pi_1 N \times \pi_2 N$, with $\pi_1 N \triangleleft G_i$. \hfill $\Box$

### 3.2 Corollary

$D$ is multiplicative.

**Proof** This is a direct analogue of 1.1. For by 3.1, if $G_1$ and $G_2$ are coprime then

$$D(G_1 \times G_2) = \sum_{N_1 \triangleleft G_1, N_2 \triangleleft G_2} |N_1 \times N_2| = \sum_{N_1 \triangleleft G_1} \sum_{N_2 \triangleleft G_2} |N_1| |N_2| = D(G_1)D(G_2).$$

\hfill $\Box$
We can now exhibit three nonabelian perfect groups.

3.3 Example: \(S_3 \times C_5\). The group \(S_3 \times C_5\), of order 30, is perfect. For \(S_3\) and \(C_5\) have coprime orders (6 and 5), so are coprime, so

\[
D(S_3 \times C_5) = D(S_3)D(C_5) = (1 + 3 + 6)(1 + 5) = 60 = 2|S_3 \times C_5|.
\]

3.4 Example: \(A_5 \times C_{15128}\). We present this example (of order 907,680) along with the method by which it was found. Firstly, \(A_5\) is a simple group of order \(5!/2 = 60\). Now, let us try to find a perfect group \(G\) of the form \(G = A_5 \times G_1\) where \(G_1\) is some group prime to \(A_5\). Since

\[
D(A_5)/|A_5| = 61/60,
\]

we need to find a \(G_1\) such that

\[
D(G_1)/|G_1| = 120/61.
\]

Let us look for such a group \(G_1\) amongst those of the form \(G_1 = C_{61} \times G_2\), where \(G_2\) is prime to \(C_{61}\) and \(A_5\). Since

\[
D(C_{61})/|C_{61}| = 62/61,
\]

we need to find a \(G_2\) such that

\[
D(G_2)/|G_2| = 120/62 = 60/31.
\]

In turn, let us look for such a group \(G_2\) amongst those of the form \(G_2 = C_{31} \times G_3\), where \(G_3\) is prime to \(C_{31}\), \(C_{61}\) and \(A_5\). Since

\[
D(C_{31})/|C_{31}| = 32/31,
\]

we need to find a \(G_3\) such that

\[
D(G_3)/|G_3| = 60/32 = 15/8.
\]

This is satisfied by \(G_3 = C_8\), and the groups \(A_5\), \(C_{61}\), \(C_{31}\) and \(C_8\) are pairwise coprime. Thus if

\[
G = A_5 \times C_{61} \times C_{31} \times C_8
= A_5 \times C_{61 \times 31 \times 8}
= A_5 \times C_{15128}
\]

then \(G\) is perfect.
3.5 Example: \( A_6 \times C_{366776} \) By the same technique we get this next example, of order 132,039,360. This time, we start with the simple group \( A_6 \) of order \( 6!/2 = 360 \), and the sequence of groups \( A_6, C_{361}, C_{127}, C_8 \) ‘works’ in the sense of the previous example. The details are left to the reader; note that \( 361 = 19^2 \) and that 127 is prime.

4 The Abelian Quotient Theorem: Proof by Counting

In each of the next two sections we present a separate proof of our main classification result, the abelian quotient theorem. The two proofs have rather different flavours, and each produces its own insights, which is why both are included. We start with the more elementary of the two.

An abelian quotient of a group \( G \) is just a quotient of \( G \) which is abelian. That is, it’s an abelian group \( A \) for which there exists a surjective homomorphism \( G \rightarrow A \); alternatively, it’s an abelian group isomorphic to \( G/K \) for some normal subgroup \( K \) of \( G \). We will prove:

4.1 Abelian Quotient Theorem

If \( G \) is a group with \( D(G) \leq 2|G| \) then any abelian quotient of \( G \) is cyclic.

This result has the following corollaries, the second of which says that abelian perfect groups ‘are’ just perfect numbers:

4.2 Corollaries

a. If \( G \) is a perfect group then any abelian quotient of \( G \) is cyclic.

b. The perfect abelian groups are precisely the cyclic groups \( C_n \) of order \( n \) with \( n \) perfect.

Proof Part (a) is immediate. For (b), if \( A \) is perfect abelian then \( A \) is an abelian quotient of the perfect group \( A \), hence \( A \) is cyclic. But we have already seen (2.1) that the perfect cyclic groups correspond exactly to the perfect numbers.

(Those who know about such things will recognize that the theorem could be stated more compactly in this way: if \( G \) is a group with \( D(G) \leq 2|G| \) then \( G^{ab} \) is cyclic. Here \( G^{ab} \) is the abelianization of \( G \): it is an abelian quotient of \( G \) with the property that any abelian quotient of \( G \) is also a quotient of \( G^{ab} \). In particular, if \( A \) is abelian then \( A^{ab} \cong A \), which is how we would deduce Corollary (2.2) from this formulation.)

The proof of the abelian quotient theorem given in this section uses two ingredients. The first is a new way of evaluating \( D(G) \):
4.3 Lemma  For any group \(G\),
\[
D(G) = \sum_{g \in G} |\{\text{normal subgroups of } G \text{ containing } g\}|.
\]

Proof  We have
\[
D(G) = \sum_{N \leq G} |N| = |\{(N, g) : N \leq G, g \in N\}| = \sum_{g \in G} |\{N : N \leq G, g \in N\}|.
\]

\[\square\]

The second ingredient is the ‘standard’ fact that the inverse image (under a homomorphism) of a normal subgroup is a normal subgroup. For let \(\pi : G_1 \to G_2\) be a homomorphism of groups, and let \(N \leq G_2\). Then \(N\) is the kernel of the natural homomorphism \(\phi : G_2 \to G_2/N\), in other words, \(N = \phi^{-1}\{0\}\). So
\[
\pi^{-1}N = \pi^{-1}\phi^{-1}\{0\} = (\phi \circ \pi)^{-1}\{\phi(0)\},
\]
i.e. \(\pi^{-1}N\) is the kernel of the homomorphism \(\phi \circ \pi : G_1 \to G_2/N\). Thus \(\pi^{-1}N\) is a normal subgroup of \(G_1\).

We are now ready to assemble these ingredients into the following proposition, from which the abelian quotient theorem follows immediately. Two pieces of terminology will be used. An element \(h\) of \(G\) is called a normal generator of \(G\) if the only normal subgroup of \(G\) containing \(h\) is \(G\) itself. A group is called simple if it has precisely two normal subgroups—inevitably, the whole group and the one-element subgroup.

4.4 Proposition  Let \(G\) be a group.

a. If \(D(G) \leq 2|G|\) then \(G\) has a normal generator.

b. If \(G\) has a normal generator then any abelian quotient of \(G\) is cyclic.

Proof  

a. By Lemma [4.3], \(D(G) \leq 2|G|\) if and only if the mean over all \(g \in G\) of
\[
\nu(g) := |\{\text{normal subgroups of } G \text{ containing } g\}|
\]
is \(\leq 2\). If \(G\) is not simple or trivial then \(\nu(1_G) \geq 3\) (where \(1_G\) is the identity element of \(G\)); so for the mean to be \(\leq 2\), there must be some \(h \in G\) for which \(\nu(h) = 1\)—and this says exactly that \(h\) is a normal generator of \(G\). On the other hand, if \(G\) is simple then any nonidentity element of \(G\) is a normal generator, and if \(G\) is trivial then \(1_G\) is a normal generator. So (a) is proved in all cases.
b. Let $A$ be an abelian quotient of $G$, with $\pi : G \to A$ a surjective homomorphism, and let $h$ be a normal generator of $G$. Then $\pi(h)$ is a normal generator of $A$: for if $K \trianglelefteq A$ and $\pi(h) \in K$ then $\pi^{-1}K$ is a normal subgroup of $G$ containing $h$, so $\pi^{-1}K = G$; and since $\pi$ is surjective, this means that $K = A$. But $A$ is abelian, so all subgroups are normal, so the fact that $\pi(h)$ is a normal generator of $A$ says that the only subgroup of $A$ containing $\pi(h)$ is $A$ itself. And this in turn says exactly that the cyclic subgroup generated by $\pi(h)$ is $A$ itself. \hfill \Box

5 The Abelian Quotient Theorem: Proof by Prime-Index Subgroups

This last section is devoted to a second proof of the abelian quotient theorem, [4.1]. This time, the proof reveals something about the normal subgroup structure of a perfect group $G$: namely, that $G$ has at most one normal subgroup of each prime index (5.2(a)). It is a corollary of this that any abelian quotient of $G$ is cyclic.

This section assumes some more sophisticated group theory than the last.

5.1 Lemma Let $G$ be a group and $p$ a prime: then the number of normal subgroups of $G$ with index $p$ is

$$\frac{p^r - 1}{p - 1} = 1 + p + \cdots + p^{r-1},$$

for some $r \geq 0$.

Remark ‘Usually’ $r = 0$, in which case both sides of the equation evaluate to 0.

Proof For this proof we write the cyclic group of order $p$ additively, as $\mathbb{Z}/p\mathbb{Z}$. We also write $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ for the set of all homomorphisms $G \to \mathbb{Z}/p\mathbb{Z}$, and $\text{Aut}(\mathbb{Z}/p\mathbb{Z})$ for the set of all automorphisms of the group $\mathbb{Z}/p\mathbb{Z}$ (that is, invertible homomorphisms $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$).

The key observation is that a normal subgroup of $G$ of index $p$ is just the kernel of a surjection from $G$ to $\mathbb{Z}/p\mathbb{Z}$.

All but one element of $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ is surjective, and the remaining one is trivial. Two surjections $\pi, \phi : G \to \mathbb{Z}/p\mathbb{Z}$ have the same kernel if and only if $\pi = \alpha \circ \phi$ for some $\alpha \in \text{Aut}(\mathbb{Z}/p\mathbb{Z})$; moreover, if such an $\alpha$ exists for $\pi$ and $\phi$ then it is unique. So the nontrivial elements of $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ have

$$\frac{|\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})| - 1}{|\text{Aut}(\mathbb{Z}/p\mathbb{Z})|}$$

different kernels between them. In other words, there are this many index-$p$ normal subgroups of $G$. We now just have to evaluate $|\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})|$ and $|\text{Aut}(\mathbb{Z}/p\mathbb{Z})|$. 9
Firstly, $\mathbb{Z}/p\mathbb{Z}$ is cyclic with $p - 1$ generators, so $|\text{Aut}(\mathbb{Z}/p\mathbb{Z})| = p - 1$.

Secondly, $\mathbb{Z}/p\mathbb{Z}$ is abelian, so $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ forms an abelian group under pointwise addition. Each element has order 1 or $p$, so $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ can be given scalar multiplication over the field $\mathbb{Z}/p\mathbb{Z}$, and thus becomes a finite vector space over $\mathbb{Z}/p\mathbb{Z}$. This vector space has a dimension $r \geq 0$, and then $|\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})| = p^r$. (Alternatively, Cauchy’s Theorem gives this result.)

The lemma is now proved. ✷

Let us temporarily call a group $G$ **tight** if for each prime $p$, $G$ has at most one normal subgroup of index $p$. Putting together the three parts of the following proposition gives us our second proof of the abelian quotient theorem.

5.2 Proposition

a. A group $G$ with $D(G) \leq 2|G|$ is tight.

b. A quotient of a tight group is tight.

c. A tight abelian group is cyclic.

Proof

a. For each prime $p$, we have

$$2|G| \geq D(G) \geq |G| + \frac{p^r - 1}{p - 1} \cdot \frac{|G|}{p}$$

where $r$ is as in Lemma 5.1. If $r \geq 2$ then

$$\frac{p^r - 1}{p - 1} \cdot \frac{|G|}{p} \geq (p + 1) \cdot \frac{|G|}{p} > |G|,$$

giving a contradiction. Thus $r$ is 0 or 1, and so $\frac{p^r - 1}{p - 1}$ is 0 or 1.

b. Let $\pi : G_1 \rightarrow G_2$ be a surjective homomorphism. If $N$ and $N'$ are distinct normal subgroups of $G_2$ with index $p$, then $\pi^{-1}N$ and $\pi^{-1}N'$ are distinct normal subgroups of $G_1$ with index $p$.

c. For this we invoke the classification theorem for finite abelian groups, which tells us that for any abelian group $A$ there exist primes $p_1, \ldots, p_n$ and numbers $t_1, \ldots, t_n \geq 1$ such that

$$A \cong C_{p_1^{t_1}} \times \cdots \times C_{p_n^{t_n}}.$$  

Suppose that $p_i = p_j$ (= $p$, say) for some $i \neq j$. Then, since $t_i \geq 1$, $C_{p_i^{t_i}}$ has a (normal) subgroup $N_i$ of index $p$; and similarly $C_{p_j^{t_j}}$. Hence $N_i \times C_{p_j^{t_j}}$ and $C_{p_i^{t_i}} \times N_j$ are distinct index-p subgroups of $C_{p_i^{t_i}} \times C_{p_j^{t_j}}$, and $C_{p_i^{t_i}} \times C_{p_j^{t_j}}$ is not tight. Since $C_{p_i^{t_i}} \times C_{p_j^{t_j}}$ is a quotient of $A$, part (b) implies that $A$ is not tight either. Thus if $A$ is tight then all the $p_k$’s are distinct, so that

$$A \cong C_{p_1^{t_1}} \times \cdots \times C_{p_n^{t_n}}.$$  

✷
There are still other lines of proof for the abelian quotient theorem. In part (b) of the Proposition, the fact that \( p \) was prime was quite irrelevant, and in just the same manner we can prove that

\[
\frac{D(G_1)}{|G_1|} \geq \frac{D(G_2)}{|G_2|}
\]

whenever \( G_2 \) is a quotient of \( G_1 \). (If \( \pi : G_1 \to G_2 \) is the quotient map, with kernel of order \( k \), then a normal subgroup \( N \) of \( G_2 \) gives rise to a normal subgroup \( \pi^{-1}N \) of \( G_1 \) of order \( k|N| \).) Thus if \( G \) is a group with \( D(G) \leq 2|G| \) and \( A \) is an abelian quotient of \( G \) then \( D(A) \leq 2|A| \). So we have reduced the abelian quotient theorem to the abelian case: if \( A \) is abelian and \( D(A) \leq 2|A| \) then \( A \) is cyclic. Certainly this is provable by methods derived from one of the two proofs of the general case, but other approaches exist; I leave that for the reader.

**Further Thoughts**

We finish with some general speculative thoughts, roughly in order of the material above.

The chosen definition of the function \( D \), and therefore of perfect group, is one amongst many candidates. We defined \( D \) to be the sum of the orders of the normal subgroups, but we could change ‘normal subgroups’ to ‘subgroups’, ‘characteristic subgroups’, ‘subnormal subgroups’, . . . , or we could define \( D \) to be the sum of the *indices* of the normal subgroups, etc. In all cases we preserve the identity \( D(C_n) = D(n) \), but only in some of them does \( D \) remain multiplicative (a feature we probably like).

More abstractly, this article was about lifting the classical function \( D : \{ \text{numbers} \} \to \{ \text{numbers} \} \) to a function \( D : \{ \text{groups} \} \to \{ \text{numbers} \} \). We might consider it natural to go the whole hog and create a function assigning not just a number, but some kind of algebraic structure, to each group \( G \). I do not know of any very useful way to do this.

In number theory there is a whole body of work on multiplicative functions of integers, which include the number-of-divisors function, the sum-of-divisors function, the Euler function \( \phi \), and the Möbius function \( \mu \). In the world of groups we have at least the beginning of an analogue. For let \( F \) be a multiplicative function from groups to numbers: then just as in Corollary 3.2, the function \( F' : G \mapsto \sum_{N \triangleleft G} F(N) \) is multiplicative. For instance, if \( F \) is the function with constant value 1 then \( F' \) gives the number of normal subgroups of a group, and is multiplicative.

The abelian quotient theorem says that if \( D(G) \leq 2|G| \) then \( G \) has some special property expressible in standard group-theoretic terms. We can prove this in at least two ways, but it seems rather more challenging to prove something in the other direction: that if \( D(G) \) is ‘too big’ then \( G \) has a certain form.

Finally, we can make various conjectures on perfect groups, based on the skimpy evidence above: for instance, ‘there are no odd-order perfect groups’,
or ‘there are infinitely many nonabelian perfect groups’. Example 2.4 on the
dihedral groups, tells us that classifying the even-order perfect groups is at least
as hard as determining whether there are any odd perfect numbers. Clearly such
problems are unlikely to be easy to solve.