Transformation design of in-plane elastic cylindrical cloaks, concentrators and lenses

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Abstract

We analyse the elastic properties of a class of cylindrical cloaks deduced from linear geometric transforms $x \rightarrow x'$ in the framework of the Milton-Briane-Willis cloaking theory [New Journal of Physics 8, 248, 2006]. More precisely, we assume that the mapping between displacement fields $u(x) \rightarrow u'(x')$ is such that $u'(x') = A^{-1}u(x)$, where $A$ is either the transformation gradient $F_{ij} = \partial x'_i / \partial x_j$ or the second order identity tensor $I$. The nature of the cloaks under review can be three-fold: some of them are neutral for a source located a couple of wavelengths away; other lead to either a mirage effect or a field confinement when the source is located inside the concealment region or within their coated region (some act as elastic concentrators squeezing the wavelength of a pressure or shear polarized incident plane wave in their core); a last category of cloaks is classified as an elastic counterpart of electromagnetic perfect cylindrical lenses. The former two categories require either rank-4 elastic tensor and rank-2 density tensor and additional rank-3 and 2 positive definite tensors ($A = F$) or a rank 4 elasticity tensor and a scalar density ($A = I$) with spatially varying positive values. However, the latter example further requires that all rank-4, 3 and 2 tensors be negative definite ($A = F$) or that the elasticity tensor be negative definite (and non fully symmetric) as well as a negative scalar density ($A = I$). We provide some illustrative numerical examples with the Finite Element package Comsol Multiphysics when $A$ is the identity.

Keywords: Cloaking; Anisotropic Heterogeneous elastic media; Geometric Transform; Numerical simulations

1. Introduction

There has been a growing interest over the past years in the analysis of elastic waves in thin plates in the metamaterial community with the theoretical proposal \cite{1,2}, and its subsequent experimental validation \cite{3,4} of a broadband cloak for flexural waves. Square \cite{5} and diamond \cite{6} cloaks are based on an

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improved transformed plate model, while form-invariance of the transformed equations in the framework of pre-stressed anisotropic plates is analyzed in [7, 8, 9].

There is currently a keen activity in transformation optics, whereby transformation based solutions to the Maxwell equations expressed in curvilinear coordinate systems travel along geodesics rather than in straight lines [10]. The fact that light follows shortest trajectories, the physical principle behind transformation optics, was formulated by de Fermat back in 1662. This minimization principle is applicable to ray optics, when the wavelength is much smaller than the size of the diffraction object. Leonhardt has shown in 2006 [11] that this allows for instance the design of invisibility cloaks using conformal mappings. Pendry, Schurig and Smith simultaneously reported that the same principle applies to electromagnetic waves, i.e. when the wavelength is in resonance with the scattering object, by creating a hole in the curved space [12]. Interestingly, the mathematicians Greenleaf, Lassas and Uhlmann proposed an earlier route to invisibility using an inverse problem approach in 2003 [13], and together with Kurylyev have been able since then to bridge the cloaking theory with Einstein theory of relativity, thereby suggesting possible avenues towards electromagnetic wormholes [14, 15]. Leonhardt and Philbin have further proposed an optical fibre experiment [16] for an analogue of Hawking’s famous event horizon in his theory of black holes [17]. It seems therefore fair to say that transformation optics offers a unique laboratory for thought experiments, leading to a plethora of electromagnetic paradigms. However, this would remain some academic curiosity without the practical side effect: so-called metamaterials, first introduced by Pendry in 1999 to obtain artificial magnetism in locally resonant periodic structures [18].

The first realization of an electromagnetic invisibility cloak [19] is a metamaterial consisting of concentric arrays of split-ring resonators. This structured material effectively maps a concealment region into a surrounding shell thanks to its strongly anisotropic effective permittivity and permeability which further fulfil some impedance matching with the surrounding vacuum. The cloak thus neither scatter waves nor induces a shadow in the transmitted field. Split ring resonators enable to meet among others the prerequisite artificial magnetism property, otherwise unobtainable with materials at hand [18]. This locally resonant micro-structured cloak was shown to conceal a copper cylinder around 8.5 GHz, as predicted by numerical simulations [19].

The effectiveness of the transformation based invisibility cloak was demonstrated theoretically by Leonhardt [11] solving the Schrödinger equation. Note that this equation is not only valid to compute ray trajectories (geodesics) in the geometrical optic limit, but also for matter waves in the quantum theory framework. Zhang et al. used this analogy to propose a quantum cloak based upon ultracold atoms within an optical lattice [20]. Greenleaf et al. subsequently discussed resonances (so-called trapped modes) occurring at a countable set of discrete frequencies inside the quantum cloak, using a spectral theory approach [21].

Using analogies between the Helmholtz and the Schrödinger equations, Cum-
mer and Schurig demonstrated that pressure acoustic waves propagating in a fluid also undergo the same geometric transform in 2D [22]. Chen and Chan further extended this model to 3D acoustic cloaks [23], followed by an independent derivation of the acoustic cloak parameters in [24]. Such meta-fluids require an effective anisotropic mass density as in the model of Torrent and Sanchez-Dehesa [25]. However, an acoustic cloak for linear surface water waves studied experimentally and theoretically in [26], only involves an effective anisotropic shear viscosity.

Nevertheless, transformation based invisibility cloaks cannot be applied in general to elastodynamic waves in structural mechanics as there is a lack of one-to-one correspondence between the equations of elasticity and the Schrödinger equation [27]. Bigoni et al. actually studied such neutral inclusions in the elastostatic context using asymptotic and computational methods in the case of anti-plane shear and in-plane coupled pressure and shear polarizations [28], but when one moves to the area of elastodynamics, geometrical transforms become less tractable and neutrality breaks down: there are no conformal maps available in that case, and one has to solve inherently coupled tensor equations.

More precisely, Milton, Briane and Willis have actually shown that there is no symmetric rank 4 elasticity tensor describing the heterogeneous anisotropic medium required for an elastodynamic cloak in the context of Cauchy elasticity [27]. However, so-called Willis’s equations, discovered by the British applied mathematician John Willis in the early 80’s [29, 30], offer a new paradigm for elastodynamic cloaking, as they allow for introduction of additional rank-3 and rank-2 tensors in the equations of motion that make cloaking possible.

Nevertheless, Brun, Guenneau and Movchan have shown [31] that it is possible to design an elastic cloak without invoking Willis’s equations for in-plane coupled shear and pressure waves with a metamaterial described by a rank 4 elasticity tensor preserving the main symmetries, as well as a scalar density. Importantly, both elasticity tensor and density are spatially varying, and the former one becomes singular at the inner boundary of the cloak [31]. Some design based on a homogenization approach for polar lattices has been proposed by Nassar, Chen and Huang [32] and Garau et al. [33]. Achaoui et al. have proposed an alternative design making use of elastic swiss-rolls [34]. Diatta and Guenneau [35] have shown that a spherical elastodynamic cloak can be designed using the same route as in [31], but the corresponding metamaterial design remains an open problem. There is an alternative, pre-stress, route to elastic cloaking proposed by Norris and Parnell that greatly relaxes constraints on material properties compared to the previous routes [36, 37, 38].

In the present article, we further investigate cylindrical cloaks for in-plane elastic waves using a radially symmetric linear geometric transform which depends upon a parameter. Depending upon the value of the parameter, the transform is applied to the design of neutral (invisibility) cloaks, elastic concentrators or cylindrical lenses. We discuss their underlying mechanism using a finite element approach which is adequate to solve the Navier equations in anisotropic heterogeneous media.
2. Governing equations and elastic properties of cloaks

2.1. The equations of motion

The propagation of in-plane elastic waves is governed by the Navier equations. Assuming time harmonic \( \exp(-i \omega t) \) dependence, with \( \omega \) as the wave frequency, allows us to work directly in the spectral domain. Such dependence is assumed henceforth and suppressed, leading to

\[
\nabla \cdot C : \nabla u + \rho \omega^2 u + b = 0, \tag{1}
\]

where, considering cylindrical coordinates \((r, \theta)\), \(u = (u_r, u_\theta)\) is the in-plane displacement, \(\rho\) the density and \(C_{ijkl}(i, j, k, l = r, \theta)\) the fourth-order elasticity tensor of the (possibly heterogeneous anisotropic) elastic medium. In eqn. (1) \(b\) is the body force.

2.2. The transformed equations of motion

Let us now consider the radial linear geometric transform \((r, \theta) \rightarrow (r', \theta')\), with \(\theta' = \theta\), shown in Fig. 1

\[
r' = \begin{cases} 
(1 + \alpha) r_1 - \alpha r_0 & \text{for } r' \leq r'_0 \text{ (domain A),} \\
(1 + \alpha) r_1 - \alpha r & \text{for } r'_0 \leq r' \leq r'_1 \text{ (domain B),} \\
r & \text{for } r' \geq r'_1 \text{ (domain C),}
\end{cases} \tag{2}
\]

where \(\alpha = -(r'_1 - r'_0)/(r_1 - r_0)\) is a real parameter and \(r'_0 = (1 + \alpha) r_1 - \alpha r_0\), \(r'_1 = r_1\). The transformation gradient is 
\[\nabla' = F \nabla\]
the gradient operator in the transformed coordinates. In particular, we stress that we assume an identity gauge transformation\([31, 4]\).

Design of in-plane transformation-based elastic cloaks has been first discussed in \[31\] when \(\alpha = -1 + r'_0/r'_1 \) \((r_0 = 0)\): in that case, \[2\] simplifies into the geometric transform for an invisibility cloak \(r' = r'_0 + \frac{r'_1 - r'_0}{r'_1} r\) in the domain (B) \([12, 13]\), where \(r'_0\) and \(r'_1\), respectively, denote the inner and outer radii of the circular cloak. However, other values of the parameter \(\alpha\) lead to equally interesting cloaks, such as neutral concentrators, first studied in the context of electromagnetism \[39\], and we would like to discuss these in the sequel.

We now need to consider two cases for the transformed equations of motion.

2.2.1. Gauge transform \(u'(r', \theta') = u(r, \theta)\)

By application of transformation \[2\] with the Gauge \(u'(r', \theta') = u(r, \theta)\) the Navier eqns.\([1]\) are mapped into the equations

\[
\nabla' \cdot C' : \nabla' u' + \rho' \omega^2 u' + b' = 0, \tag{3}
\]

where \(u'(r', \theta')\) and \(b'(r', \theta')\) are the transformed displacement and body force, respectively, and \(\nabla' = F \nabla\) the gradient operator in the transformed coordinates. In particular, we stress that we assume an identity gauge transformation\([31]\).
The stretched density is the scalar field

\( \rho' = \begin{cases} 
\frac{(1+\alpha)r'_1 - r'_0}{\alpha r'_0} \rho & \text{in A,} \\
\frac{r' - (1+\alpha)r'_1}{\alpha^2 r'} \rho & \text{in B,} \\
\rho & \text{in C,}
\end{cases} \) (4)

homogeneous in A and C. The transformed linear elasticity tensor has components

\[ C'_{ijkl} = J^{-1} C_{mnop} F_{im} F_{jn} \delta_{kp} \delta_{lp}, \] (5)

where \( (i,j,k,l = r',\theta') \), \( (m,n,o,p = r,\theta) \), \( \delta_{jn} \) is the Kronecker delta and the usual summation convention over repeated indices is used. In particular, if before transformation the material is isotropic, i.e. \( C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \) \( (i,j,k,l = r,\theta) \), with \( \lambda \) and \( \mu \) the Lamé moduli, the transformed elasticity tensor \( C' \) has non zero cylindrical components

\[
\begin{aligned}
C'_{rr'rr'} &= \frac{r' - (1+\alpha)r_1}{r'} (\lambda + 2\mu), \\
C'_{\theta\theta'\theta\theta'} &= \lambda, \\
C'_{r\theta'\theta'\theta'} &= C'_{\theta\theta'\theta'r'} = \mu,
\end{aligned}
\] (6)

in B and \( C' = C \) in A and C.

The transformation and the corresponding transformed density \( \rho' \) and elasticity tensor \( C' \) are broadband, they do not depend on the applied frequency \( \omega \).

2.2.2. Gauge transform \( u'(r',\theta') = F^{-t} u(r,\theta) \)

As noted in [27], by application of transformation [2] with the Gauge \( u'(r',\theta') = F^{-t} u(r,\theta) \), where \( F \) is the transformation gradient, the Navier eqns.[1] are mapped into the equations

\[
\nabla' : \left( C'' : \nabla' \mathbf{u}' + \mathbf{D}' : \mathbf{u}' \right) + \mathbf{S}' : \nabla' \mathbf{u}' + \omega^2 \rho' \mathbf{u}' + \mathbf{b}' = \mathbf{0} . \tag{7}
\]

The transformed rank-4 elasticity tensor \( C'' \) has components

\[ C''_{ijkl} = J^{-1} F_{im} F_{jn} C_{mnop} F_{ko} F_{lp}, \] (8)

where \( (i,j,k,l = r',\theta') \), \( (m,n,o,p = r,\theta) \).

We note that \( C'' \) in (8) has all the symmetries, unlike \( C' \) in (6), which has the major but not the minor simmetries.

The rank-3 tensors \( \mathbf{D}' \) and \( \mathbf{S}' \) in (8) have elements

\[ D'_{ijk} = J^{-1} F_{im} F_{jn} C_{mnop} \frac{\partial^2 x_k'}{\partial x_a \partial x_p} = D'_{ijk}, \] (9)
and
\[ S'_{ijk} = J^{-1} \frac{\partial^2 x'_i}{\partial x_m \partial x_n} C_{mnop} F_{jm} F_{kp} = S''_{ijk}. \tag{10} \]

Finally, the transformed density \( \rho' \) in \( \mathbb{L} \) is matrix valued
\[ \rho'_{ij} = J^{-1} \rho F_{im} F_{jm} + J^{-1} \frac{\partial^2 x'_i}{\partial x_m \partial x_n} C_{mnop} \frac{\partial^2 x'_j}{\partial x_o \partial x_p} = \rho''_{ij}. \tag{11} \]

These expressions were first derived in [27].

Now, if before transformation the material is isotropic, then the transformed elasticity tensor \( C'' \) has non zero cylindrical components
\[
C''_{rrrr} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2} \lambda \lambda + 2\mu, \\
C''_{\theta\theta\theta\theta} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2} (\lambda + 2\mu), \\
C''_{rr\theta\theta} = C''_{\theta\theta r\theta} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2} \lambda, \\
C''_{rrrr} = C''_{\theta\theta r\theta} = C''_{rr\theta\theta} = C''_{\theta\theta r\theta} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2} \mu.
\tag{12}
\]

in \( B \) and \( C'' = C \) in \( A \) and \( C \).

On the other hand, the rank-3 tensors \( D' \) and \( S' \) have non zero cylindrical components
\[
D'_{rrr} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2 (r'' - (1+\alpha)r_1)} \lambda = -S''_{rrr}, \\
D'_{\theta\theta\theta} = D'_{rr\theta} = 2 \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2 (r'' - (1+\alpha)r_1)} \mu = -S''_{\theta\theta\theta} = -S''_{rr\theta}, \\
D'_{\theta\theta\theta} = \frac{r'' - (1+\alpha)r_0}{(r_1 - r_0)^2 (r'' - (1+\alpha)r_1)} (2\mu + \lambda) = -S''_{\theta\theta\theta}.
\tag{13}
\]

Similar expressions can be derived for the transformed density. Expressions in [12] and [13] are more intricate than those in [6]; thus, in the sequel, we focus on the transformed equations of motion [3].

2.3. Interface conditions for Gauge \( u'(r', \theta') = u(r, \theta) \)

Perfect cloaking and perfect concentrator require additional conditions on displacements and tractions at the interfaces between the domains with different material properties introduced by the transformation [2]. In the transformed problem [3] there are two interfaces, between domains \( A \) and \( B \), at \( r' = r'_0 \) and \( r = r_0 \), and at the cloak’s outer boundary, between domains \( B \) and \( C \), at \( r' = r = r_1 \). Transformed equations [3] together with the assumption \( u'(r', \theta') = u(r, \theta) \) assure that displacements and tractions in the inhomogeneous transformed domain at a point \( (r', \theta') \) coincide with displacements and tractions at the corresponding point \( (r, \theta) \) in the original homogeneous problem [1] where no interfaces between different materials are present. In particular, if we introduce the Cauchy stress tensors \( \sigma' = C' \mathbf{\nabla} u' \) and \( \sigma = C \mathbf{\nabla} u \) for the transformed and original problem, respectively, it is verified the following equality between tractions
\[ \sigma' \cdot e'_r = \sigma \cdot e_r, \tag{14} \]
2.4. Perfect cloak. Singularity at the inner interface

We note that for the perfect cloak[31], i.e. \( r_0 = 0 \) and \( \alpha = -1 + r'_0 / r'_1 \), a point at \( r = 0 \) is mapped into a disk of radius \( r'_0 \). This is a singular transformation and, at the cloak inner boundary, \( r' = (1 + \alpha) r_1 \to 0 \). Therefore, at \( r' = r'_0 \), from (4) and (6) one can see that \( \rho' \to 0 \), \( C'_{r'^{+}r'^{+}} \), \( C'_{r'^{+}\theta'^{+}} \to 0 \), while \( C''_{\theta'^{+}\theta'^{+}} \), \( C''_{\theta'^{+}r'^{+}} \to \infty \).

Similarly, at \( r' = r'_0 \), from (12) and (13) one can see that \( C''_{r'^{+}r'^{+}} \to 0 \), while \( C''_{\theta'^{+}\theta'^{+}} = C''_{\theta'^{+}r'^{+}} = C''_{\theta'^{+}\psi'^{+}} = C''_{r'^{+}\psi'^{+}} = C''_{r'^{+}r'^{+}} \to \infty \). Moreover, one notes that the rate of divergence is faster for \( C'' \) than \( C' \), and thus anisotropy is even more extreme in the neighborhood of the inner boundary for \( C'' \). All expressions in (13) diverge when \( r' - (1 + \alpha) r_1 \to 0 \).

The required extreme anisotropy physically means that pressure and shear waves propagate with an infinite velocity in the azimuthal \( \theta' \)-direction and zero velocity in the radial \( r' \)-direction along the inner boundary, which results in a vanishing phase shift between a wave propagating in a homogeneous elastic space and another one propagating around the coated region.

Clearly, the presence of unbounded physical properties poses limitations on possible realizations and numerical implementation of the model; regularization techniques have been proposed introducing the concept of near cloak [42, 43, 44], but the realization of such elastodynamic cloaks remains a challenge.

2.5. General transformation

We now wish to extend first the proposal of Rahm et al. [39] of an omnidirectional electromagnetic concentrator to the elastic setting and second to consider a more general transformation including folded transformed geometries, as proposed by for quasi static equations of electromagnetism by Milton et al. [45]. We recall that the transformation [2] compress/expand a disc with radius \( r_0 \) at the expense of an expansion/compression of the annulus between \( r_0 \) and \( r_1 \). The inner disk is expanded for \(-1 < \alpha < -1 + r'_0 / r'_1 \) with the limiting cases \( \alpha = -1 \) corresponding to an identity and \( \alpha = -1 + r'_0 / r'_1 \) to perfect cloaking. On the contrary the disk is compressed for \( \alpha < -1 \) and \( \alpha > 0 \), and, in the case of positive value of the topological parameter \( \alpha \), \( r_0 > r_1 \) and a folding of the original geometry is obtained.

The material remains homogeneous and isotropic in the inner disk A where only the density is changed. In the annulus region B the material is heterogeneous and elastically anisotropic. Consistently with Brun et al. [31] and differently from Milton et al. [27] the density remains a scalar field. We stress that the heterogeneity is smoothly distributed and the material is functionally graded with the absence of any jump in the material properties leading to possible scattering effects. As detailed above interface conditions are automatically satisfied and do not introduce any scattering.
It is important to note that, excluding the perfect cloaking case, for bounded values of $\alpha$ all the elastic rigidities and the density are bounded leading to possible physical and numerical implementation of the metamaterial structure.

2.6. The radial field concentrator. Unbounded $\alpha$

The limiting cases $\alpha \to \pm \infty$, where $r_0 = r_1$, correspond asymptotically to the radial transformation

$$r' = \begin{cases} 
\frac{r}{r_1} & \text{in A}, \\
\frac{r}{r_1} & \text{in B}, \\
r & \text{in C}.
\end{cases} \quad (15)$$

In such a case in the annulus region B $C_{\theta'\phi\psi'}$, $C_{\theta'\psi'\phi'} \to \infty$, $\rho' \to \infty$ and all the other elastic rigidities components are unchanged. In such material, independently on the external field impinging the metamaterial region $r' \leq r'_1$, the elastic fields in B are radially independent and depend only on the azimuthal coordinate $\theta'$. However, the harmonic behavior cannot be reached in a finite time after the transient since the density $\rho'$ is unbounded.
Figure 1: Geometric transform of eqn (2). (a) Representation of the transform $r \to r'$ for different values of the parameter $\alpha$. The domains A ($r' \leq r'_0$), B ($r'_0 \leq r' \leq r'_1$) and C ($r' \geq r'_1$) are indicated. (b) Transformation of the geometry for $\alpha > 0$, $\alpha < 1$ and $\alpha = -1 + r'_0/r'_1$ (perfect cloak).
3. Numerical results and discussion

In this section, we report the finite element computations performed in the COMSOL multiphysics package. Normalized material parameters are used. A cloak of density $\rho'$ and elasticity tensor $C'$ is embedded in an infinite isotropic elastic material with normalized Lamé moduli $\lambda = 2.3$ and $\mu = 1$, and mass density $\rho = 1$. The cloak has inner and outer radii $r'_0 = 0.2$ and $r'_1 = 0.4$, respectively. The disc inside the cloak consists of the same elastic material as the outer medium but different density. We further consider a harmonic unit concentrated force applied either in the direction $x_1$ or $x_2$ which vibrates with an angular frequency $\omega = 40$. This force is sometimes located outside the cloak (cf. Fig. 2-4), sometimes inside the coating (cf. Fig. 5) or within the central disc (cf. Fig. 6), depending upon whether we are looking for some neutrality feature, lensing/mirage effect or some localization.

Before we start looking at the cloaks’s features depending upon the ranges of values of the parameter $\alpha$, we briefly discuss the implementation of elastic perfectly matched layers (PMLs), in the framework of transformation elasticity.

3.1. Implementation of elastic cylindrical PMLs

A perfectly matched layer has been implemented in order to model the infinite elastic medium surrounding the cloak (cf. outer ring in Figs. 2-6); this has been obtained by application of the geometric transform [46],

$$x''_i = (1 + a)\hat{x}_i - ax_i, \quad i = 1, 2,$$

for $|x_i| > |\hat{x}_i|$, where $a$ is now a complex number whose imaginary part accounts for the decay of the elastic waves and $\hat{x}_i = \pm 1$ in Fig. 2-6. The corresponding (complex) density $\rho''$ and elasticity tensor $C''$ are still given by (4) and (6).

The accuracy of the PMLs has been numerically validated when $a = i - 1$, by comparison with the Green’s function in homogeneous elastic space (cf. Eq. 17 and Fig. 2b, c).

We can therefore confidently carry out computations with these PMLs. We start by the study of an invisibility cloak for in-plane elastic waves, whereby the point source considered in [31] now lies in the close vicinity of the cloak (intense near field limit when the acoustic ray picture breaks down).

3.2. Neutrality for a point source outside the cloak

We report in Fig. 2 and Fig. 3 the computations for a point force applied at a distance $r = 0.42$ away from the center of the cloak and close to the cloak itself of outer radius $r'_1 = 0.4$. The force is applied in the horizontal direction in Fig. 2 and in the vertical direction in Fig. 3. In both upper panels (a), we clearly see that both the wave patterns of the magnitude of the displacement field are smoothly bent around the central region within the cloak (where the magnitude is uniform).

The comparative analyses between panels (a) and (b) of Fig. 2 shows that the wave patterns in the external homogeneous domain is not perturbed by the
Figure 2: Elastic field generated by an horizontal unit force applied in the external homogenous region; \( \alpha = -1 + \frac{r_0}{r_1} = -0.5 \), \( \omega = 40 \), source \( x_0 = (-0.42, 0) \). Magnitude \( u = \sqrt{u_1^2 + u_2^2} \) of the displacement field in the system with inclusion and cloaking (a) and in an homogenous system (b). Comparison between the displacement magnitude \( u \) computed in Comsol for a cloaked inclusion (black line) and the analytical Green’s function in an infinite homogeneous linear elastic and isotropic material (gray line), see Eq. (17).
Figure 3: Elastic field generated by a vertical unit force applied in the external homogeneous region. \( \alpha = -1 + r'/r'_{1} = -0.5, \omega = 40, \) source \( x_{0} = (-0.42, 0). \) (a) Magnitude \( u \) of the displacement field; (b) Comparison between the displacement magnitude \( u \) computed in Comsol for a cloaked inclusion (black line) and the analytical Green’s function in an infinite homogeneous linear elastic and isotropic material (gray line).

presence of the inclusion and cloaking interface. This is verified quantitatively in Fig. 2 panel (c) and in Fig. 3 panel (b) where the numerically computed wave pattern is compared with the Green’s function in homogeneous elastic space

\[
G(x) = \frac{i}{4\mu} \left\{ \frac{H_{0}^{(1)}(k_{s}r)}{\omega^{2}} \nabla \nabla \left[ H_{0}^{(1)}(k_{s}r) - H_{0}^{(1)}(k_{p}r) \right] \right\}
\]

(17)

with \( H_{0}^{(1)} \) the Hankel function, \( \mathbf{I} \) the second order identity tensor, \( \nabla \) the gradient operator, \( k_{p} = \omega/c_{p}, k_{s} = \omega/c_{s}, Q = (1/c_{p}^{2} + 1/c_{s}^{2})^{-1}(\lambda + \mu)/(\lambda + 2\mu), c_{p} = \sqrt{(\lambda + 2\mu)/\rho}, c_{s} = \sqrt{\mu/\rho}. \) The plot are reported along the horizontal line \( x_{2} = 0 \) passing from the point of application of the force. The absence of forward or backward scattering is demonstrated by the excellent agreement between the two fields in the external homogeneous domain \( r > 0.4. \) Clearly, the profile is much different when the coating is removed and the inner disc is clamped or freely vibrating. We also see that the field in the cloaking region has the same amplitude as the one in homogeneous case but shifted following
the transformation $r'(r)$; in the inner disk the field is homogeneous. Finally the effectiveness of the PML domains can also be appreciated.

In Fig. 4 the deformation fields are also reported for both cases, where the force is applied in horizontal (first column) and vertical (second column) direction. In the upper (a,b) and central (c,d) panels the skew-symmetric nature of the components $\varepsilon_{11}$ and $\varepsilon_{22}$ of the deformation tensor reveals the tensor nature of the problem. The component $\varepsilon_{12}$ leads to a non-intuitive pattern whereby fully-coupled shear and pressure components create the optical illusion of interferences. Again, the effect of the cloaking is shown and also for deformation fields waves are bent around the cloaking region without backward and forward scattering.

3.3. Mirage effect for a point source in the coating

In this section, we look at the case of a point force located inside the coating. In a way similar to what was observed for an electromagnetic circular cylindrical cloak [47], we observe in Fig. 5 a mirage effect: the point force seems to radiate from a location shifted towards the inner boundary (further away from an observer) as given by

$$r = \frac{(1 + \alpha)r_1 - r'}{\alpha}, \theta = \theta', \quad (18)$$

as also shown in panel (b).

Importantly, the profile of the shifted point source in homogeneous elastic space is superimposed with that of the point source located inside the coating, but only outside the cloak. In the invisibility region i.e. the disc at the center of the cloak, the field is constant and this suggest that the central region behaves as a cavity. We study this cavity phenomenon in the next section.

The example in Fig. 5 reveals that any object located inside the coating would appear as a different elastic material with a different shape to an observer.

3.4. Confinement for a point source in the central region

We now consider a point force inside the invisibility region. Interestingly, a point source located in the invisibility zone always radiates outside the cloak as if it was located at the origin and this is quite natural as the central disc is simply the image of the origin via the geometric transform (18). The fact that the central disc behaves as a closed cavity is also intuitive, as the elasticity tensor $C'$ is singular on the boundary of the disc.

3.5. Squeezing the wavelength with an elastic concentrator

We report the effects associated to an increase in the magnitude of the parameter $\alpha$ describing the linear transformation (2). In Fig. 7 the effect of the cylindrical coating on the inclusion is given for a pressure plane wave $u = (A \exp(ik_x t), 0)$ propagating in the horizontal direction $x_1$. A decrease of $\alpha$ from $\alpha = -1 + r'_0/r'_1 = -0.5$ introduces wave propagation within the inclusion with progressive shorter wavelengths while the amplitude of the wave remains
Figure 4: Elastic deformation fields generated by a unit force applied in the external homogeneous region. $\alpha = -1 + r_0/r_1 = -0.5$, $\omega = 40$, source = $(-0.42, 0)$. (a), (c), (e) Force applied in the horizontal direction $x_1$. (b), (d), (f) Force applied in the vertical direction $x_2$. (a), (b) Component $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$; (c), (d) Component $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$; (e), (f) Component $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})$. 

14
unchanged. From Fig. 7 it is evident that, when $\alpha < -1$, the interface acts as an energy concentrator within the inclusion increasing the energy flux. The energy crossing the inclusion region $r \leq r'_0 = 0.2$ equals the energy crossing the larger region $r \leq r_0$, in a homogeneous material. In the interval $-1 > \alpha > -\infty$, $r'_0 < r_0 < r'_1$.

We also note that, when $\alpha \neq -0.5$ the transformation is regular and material parameters remain bounded indicating additional advantages in technological and numerical implementations of the model. Last but not least, the field in the external domain remains unchanged.

3.6. Folding transformation. Superconcentration of an elastic wave with a cylindrical lens

We finally report in Fig. 8 an enhanced energy concentration effect obtained from a folding transformation ($\alpha > 0$). In such a case all the energy crossing a circular region larger than the region delimited by the cloaking interface is concentrated into the inner inclusion. In Fig. 8, $\alpha = 0.94$ and the radius of the circular region in the homogenous space is $3.06$ times the radius of the inner inclusion. Again, such an effect is obtained by an increase in the energy flux leaving unperturbed both the wave amplitude and the fields in the external region.
Figure 5: Elastic field generated by a vertical unit force applied in the cloaking region. \( \alpha = -1 + r'_0/r'_1 = -0.5, \omega = 40, \) source \( x_0 = (-0.3, 0). \) (a) Magnitude \( u \) of the displacement field; (b) Comparison between the displacement magnitude \( u \) computed in Comsol for a cloaked inclusion (black line) and the analytical Green’s function in an infinite homogeneous linear elastic and isotropic material (gray line), corresponding to a force applied in a shifted source point \( x_0 = (-2, 0). \)
Figure 6: Elastic field generated by a vertical unit force applied in the internal inclusion. $\alpha = -1 + r'_0/r'_1 = -0.5$, $\omega = 40$, source $x_0 = (-0.17, 0)$. (a) Magnitude $u$ of the displacement field; (b) Comparison between the displacement magnitude $u$ computed in Comsol for a cloaked inclusion (black line) and the analytical Green’s function in an infinite homogeneous linear elastic and isotropic material (gray line), corresponding to a force applied at the origin $x_0 = (0, 0)$. 
Figure 7: Elastic field generated by a pressure plane wave $\mathbf{u} = (A \exp(i k_p t), 0)$ with $\omega = 60$. Left column: magnitude $u$ of the displacement field. Right column: comparison between the displacement magnitude $u$ computed in Comsol for a cloaked inclusion (black line) and the pressure plane wave in an infinite homogeneous linear elastic and isotropic material (gray line), results are plotted along an horizontal line passing from the center of the inclusion. (a) $\alpha = -0.5$, (b) $\alpha = -0.6$, (c) $\alpha = -1$, (d) $\alpha = -5$
Figure 8: Elastic field generated by a pressure plane wave $\mathbf{u} = (A \exp(i k_p t), 0)$ with $\omega = 60$ and $\alpha = 0.94$. (a) Magnitude $u$ of the displacement field. (b) Displacement magnitude $u$ computed in Comsol for a cloaked inclusion plotted along an horizontal line passing from the center of the inclusion.
4. Conclusion

We have proposed to use stretched coordinates in order to design an elastic cloak bending the trajectory of in-plane coupled shear and pressure waves around an obstacle, concentrating them in its core, or focussing them. We investigated the transformed equations of motion for the Mitlon-Briane-Willis

\[ u'(r', \theta') = F^{-1} u(r, \theta) \]

and the Brun-Guenneau-Movchan

\[ u'(r', \theta') = u(r, \theta) \]

(see [40]). The former leads to Willis’s equations with more extreme anisotropic parameters in the cloak than for the latter. However, the latter requires a transformed elasticity tensor without the minor symmetries, which is another hurdle for a metamaterial design.

We have studied various limiting cases for the value of a parameter in the considered radially symmetric linear geometric transforms. These transforms are applied to the design of neutral (invisibility) cloaks, elastic concentrators or cylindrical lenses.

We have numerically explored all the above for the gauge \( u'(r', \theta') = u(r, \theta) \) leading to a non-fully symmetric transformed elasticity tensor, and have notably shown that a source located inside the anisotropic heterogeneous elastic coating seems to radiate from a shifted location, and can also lead to anamorphism.

We believe that our space folding based design of elastic cylindrical lenses can lead to an in-plane counterpart of the external cylindrical cloak for anti-plane shear waves introduced in [48] and applied periodically in [49].

We hope our results might open new vistas in cloaking devices for elastodynamic waves. Whereas their governing equations do not generally retain their form under geometric transforms, unlike for electromagnetic and acoustic waves, one can choose specific gauges that can make the transformed equations of motions easier to handle.

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