ON SESHADRI CONSTANTS OF VARIETIES WITH LARGE FUNDAMENTAL GROUP

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ABSTRACT. Let $X$ be a smooth variety and let $L$ be an ample line bundle over $X$. If $\pi_1^{alg}(X)$ is large, we show that the Seshadri constant $\epsilon(p^*L)$ can be made arbitrarily large by passing to a finite étale cover $p : X' \to X$. This result answers affirmatively a conjecture of J.-M. Hwang. Moreover, we prove an analogous result when $\pi_1(X)$ is large and residually finite. Finally, when $X$ is a smooth surface we generalize these results to the case of big and nef line bundles.

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1. Introduction

Let $X$ be a $n$-dimensional projective variety and let $L$ be an ample line bundle over $X$. An interesting way of studying the local positivity of the line bundle is by estimating the so-called Seshadri constants of $L$. Recall that, given a nef line bundle $L$ and a point $x \in X$, J.-P. Demailly defines the numbers

\[
\epsilon(L; x) = \inf_{C \supseteq x} \frac{L \cdot C}{\text{mult}_x(C)}, \quad \epsilon(L) = \inf_{x \in X} \epsilon(L; x)
\]

to be respectively the Seshadri constant of $L$ at $x$ and the global Seshadri constant of $L$, where $C \subset X$ is an irreducible curve and $\text{mult}_x(C)$ is the multiplicity of such a curve at $x$. The following proposition explains the connection between separation of jets and Seshadri constants.

**Proposition 1.1** (Demailly, Proposition 6.8. in [Dem90]). Let $X$ be a $n$-dimensional projective variety and let $L$ be a nef line bundle. If $\epsilon(L; x) > n + s$, then the linear system $|K_X + L|$ separates $s$-jets at $x$. Moreover, if $\epsilon(L; x) > 2n$, then $K_X + L$ is very ample.

Recall that a line bundle $L$ separates $s$-jets at $x \in X$ if the evaluation map

\[
H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_X/m_x^{s+1})
\]
is surjective, where $m_x$ is the maximal ideal of the point $x \in X$. In the rest of the paper we will denote by $s(L;x)$ the largest natural number $s$ such that $L$ separates $s$-jets at $x$. One of Demailly’s fundamental observations is that $\epsilon(L;x)$ controls the asymptotic of $s(kL;x)$ as a function of $k$, see Proposition 2.1 for a precise statement or the original reference [Dem90] for more details.

Unfortunately, the Seshadri constants are in practice very hard to compute or even estimate. The main purpose of this paper is to show that, in the presence of a “large” fundamental group, they can be nicely estimated up to a finite cover at least when $L$ is an ample line bundle. The following notion for the fundamental group of a variety was introduced by J. Kollár in [Kol93].

**Definition 1.2.** Let $X$ be a smooth variety. Let $Z \subset X$ be an irreducible subvariety and let us denote by $n_Z : \bar{Z} \to Z$ its normalization. We say that $X$ has large algebraic fundamental group (resp. large fundamental group) if for any $Z \subset X$ the image $\pi_1^{alg}(\bar{Z}) \to \pi_1^{alg}(X)$ (resp. $\pi_1(\bar{Z}) \to \pi_1(X)$) is infinite.

Given Definition 1.2 we can state our first result.

**Theorem 1.3.** Let $X$ be a smooth variety and let $L$ be an ample line bundle over $X$. If $\pi_1^{alg}(X)$ is large, given a positive number $N > 0$ there exists a finite étale cover $p : X' \to X$ such that $\epsilon(p^* L) \geq N$.

Let us observe that Theorem 1.3 answers affirmatively a conjecture of J.-M. Hwang, see Problem 2.6.2 in [Bauer et al.12].

In complex differential geometry it is usually more convenient to work with the topological fundamental group, see for example Theorem 1.8 in Section 1.1. The strategy of the proof of Theorem 1.3 can be adapted to prove the following.

**Theorem 1.4.** Let $X$ be a smooth variety and let $L$ be an ample line bundle over $X$. If $\pi_1(X)$ is residually finite and large, given a positive number $N > 0$ there exists a finite étale cover $p : X' \to X$ such that $\epsilon(p^* L) \geq N$.

**Remark 1.5.** It is possible to show that if $\pi_1(X)$ is residually finite and large then $\pi_1^{alg}(X)$ has to be large as well. Thus, Theorem 1.4 is strictly speaking a particular case of Theorem 1.3. Nevertheless, we have decided to state Theorem 1.4 independently of Theorem 1.3 because in the geometric analysis literature $\pi_1^{alg}$ is rarely used. We refer to Theorem 1.8 below for an example of a classical result in this field.

Next, Theorems 1.3 and 1.4 combined with Proposition 1.1 have an interesting corollary.

**Corollary 1.6.** Let $X$ be a smooth variety with ample canonical line bundle $K_X$. If $\pi_1(X)$ is residually finite and large or if $\pi_1^{alg}(X)$ is large, there exists a finite étale cover $p : X' \to X$ such that $2K_{X'}$ is very ample.

Finally, we can state a generalization of Theorems 1.3 and 1.4 when $X$ is a smooth surface and $L$ is a big and nef line bundle. This result was originally
proven in [DiC14] by using techniques which are peculiar to complex dimension two. We present here a different proof of this result which is along the lines of the proofs of Theorems 1.3 and 1.4. It is our hope that this proof will generalize to higher dimensions.

Recall that given a big line bundle $L$ we denote by $B_+(L)$ its augmented base locus, see [ELMNP06] for more details. In Section 3 we prove the following.

**Theorem 1.7.** Let $X$ be a smooth surface and let $L$ be a big and nef line bundle over $X$. If $\pi_1(X)$ is residually finite and large or if $\pi_1^{alg}(X)$ is large, given a positive number $N > 0$ there exists a finite étale cover $p : X' \to X$ such that $\epsilon(p^*L; x) \geq N$ for any $x \notin p^*B_+(L) = B_+(p^*L)$.

Let us observe that when $L$ is ample, i.e. $B_+(L)$ is empty, Theorem 1.7 recovers the two dimensional statements of Theorems 1.3 and 1.4. For a different proof of this result see Theorem 1.1 in [DiC14].

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### 1.1. Motivations and preliminaries.

The problem of estimating the Seshadri constants of a “positive” line bundle is extensively studied both in the algebraic geometry and geometric analysis literatures. As shown by Demailly in [Dem90], these numerical invariants are deeply connected with the theory of singular Hermitian metrics and Hörmander $L^2$-estimates. Therefore they can be studied with techniques coming from geometric analysis. This approach has been successfully explored mainly when the underlying variety $X$ admits Kähler metrics of non-positive sectional curvature. For example, if $(X, \omega_B)$ of $n$-dimensional complex hyperbolic manifolds equipped with the standard locally symmetric metric Bergman metric, J.-M. Hwang and W.-K. To in [HT99] were able to prove that $\epsilon(K_X; x) \geq (n + 1) \sinh(i_x)$ where $i_x$ is the injectivity radius of $\omega_B$ at $x$. Similarly, if $(X, \omega)$ is a $n$-dimensional compact Kähler manifold with non-positive sectional curvature, say $-a^2 \leq R_\omega \leq 0$, and $L$ is an ample line bundle over $X$, then S.-K. Yeung in [Yeu00] shows that $\epsilon(L; x) \geq \delta(i_x, a, L; n)$ where $\delta$ is an explicitly computable function of the injectivity radius $i_x$, the curvature bounds, the curvature of $L$ and the dimension. Moreover, if we fix a normalization for the curvature then $\delta \to \infty$ as $i_x \to \infty$. This interesting fact then implies the following.

**Theorem 1.8** (Yeung, Theorem 2 in [Yeu00]). Let $(X, \omega)$ be a $n$-dimensional Kähler manifold with non-positive sectional curvature and let $L$ be an ample line bundle over $X$. If $\pi_1(X)$ is residually finite, given any positive number $N > 0$ there exists a finite étale cover $p : X' \to X$ such that $\epsilon(p^*L) \geq N$. 
Observe that, by a classical result of H. Wu [GW79], the universal cover of a compact Kähler manifold with non-positive sectional curvature is a Stein manifold. Since Stein manifolds do not contain proper subvarieties, it is easy to show that the varieties considered in Theorem 1.8 have large fundamental groups, compare with Theorem 1.4. This paper started as an attempt of finding a proof of Theorem 1.8 which relies on the properties of the fundamental group rather than on curvature assumptions.

Concerning the organization of the paper, in Section 2 we recall the main results in [ELN96] and provide the proofs of Theorems 1.4 and 1.3. In Section 3, we give the details of a proof of Theorem 1.7.

2. PROOFS OF THE MAIN RESULTS

We start by recalling the connection between Seshadri constants and separation of jets. Let $L$ be a nef line bundle on $X$. For any $k \geq 0$ and given any $x \in X$, let us denote by $s(kL; x)$ the maximal integer such that the linear series $|kL|$ separates $s$-jets at $x$. The following proposition gives a nice lower bound for Seshadri constants in terms of the separation of jets of the linear series $|kL|$.

**Proposition 2.1.** Let $X$ be a smooth variety of dimension $n$ and let $L$ be a nef line bundle. For any $k > 0$, we have

$$\epsilon(L; x) \geq \frac{s(kL; x)}{k}.$$  

**Proof.** Let $C \subset X$ be a reduced irreducible curve containing $x \in X$. By definition for any $k \geq 0$, the linear series $|kL|$ generates $s_k$-jets at $x$, where $s_k = s(kL; x)$. We can then find a divisor $D_x \in |kL|$ which does not contain $C$ such that $\text{mult}_x(D_x) \geq s_k$. We then conclude that

$$\epsilon(L; x) \geq \frac{L \cdot C}{\text{mult}_x(C)} \geq \frac{s_k}{k}.$$  

The proof is then complete. \qed

Let us observe that when $L$ is ample then the asymptotic growth of jet separation actually computes $\epsilon(L; x)$ for any given $x \in X$, for more details see Proposition 6.3 in [Dem90] or Chapter V in [Laz04]. Nevertheless, for our purposes Proposition 2.1 will be sufficient.

Next, we need to state a result of Ein, Lazarsfeld and Nakamaye proved in [ELN96]. This theorem is a variant of the main theorem in the celebrated work of Demailly [Dem93].

**Theorem 2.2** (Ein-Lazarsfeld-Nakamaye [ELN96]). Let $X$ be a smooth variety of dimension $n$ and let $L$ be an ample divisor on $X$ satisfying $L^n > (n+s)^n$. Let $b$ a non-negative number such that $bL - K_X$ is nef. Suppose that $m_0$ is a positive integer such that $|m_0L|$ is base point free. Then for any point $x \in X$ either

- $|K_X + L|$ separates $s$-jets at $x$, or
there exists a codimension c subvariety V containing x such that 
\[ L^{\dim(V)} \cdot V \leq \left( b + m_0(n - c) + \frac{nl}{c!} \right)^c (n + s)^n. \]

Now that we recalled these basic results, we can proceed with the proofs of Theorems 1.3 and 1.4. The first step is the construction of a suitably “large” étale cover of X.

**Lemma 2.3.** Let X be a smooth variety. If \( \pi_1(X) \) is residually finite and large or if \( \pi_1^{\text{alg}}(X) \) is large, given a positive number \( N > 0 \) there exists a finite étale cover \( q : X' \to X \) such that for any irreducible subvariety \( Z \subseteq X' \) the map \( Z \to q(Z) \) has degree \( \geq N \).

**Proof.** Assume \( \pi_1^{\text{alg}}(X) \) to be large. Let \( \hat{\Gamma} \) be the kernel of the map \( \pi_1(X) \to \pi_1^{\text{alg}}(X) \) and let us denote by \( \hat{\rho} : \hat{X} \to X \) the algebraic universal cover, where \( \hat{X} = \tilde{X}/\hat{\Gamma} \) and where \( \tilde{X} \) is the topological universal cover. Fix a Kähler metric \( \omega \) on \( X \) and observe that \( \hat{\rho} : \hat{X} \to X \) is injective and \( \hat{\rho} : \hat{X} \to X \) is surjective. Thus for all \( k \), there exists \( \gamma_k \in \hat{\Gamma} \) such that \( \hat{\rho}(\gamma_k z_k) = q \) for some positive constant \( M \). Let \( D \) be a fundamental domain for \( X \in \hat{X} \), in other words \( \hat{\rho} : D \to X \) is injective and \( \hat{\rho} : D \to X \) is surjective. Thus for all \( k \), there exists \( g_k \in \hat{\Gamma}_0 \) such that \( g_k z_k \in D \). Let us define \( \hat{\gamma}_k' = g_k \hat{\gamma}_k z_k \) and \( \hat{\gamma}_k = g_k \hat{\gamma}_k g_k^{-1} \), where \( \hat{\gamma}_k' \in \hat{\Gamma}_k \) since \( \hat{\Gamma}_k \) is by construction a normal subgroup of \( \hat{\Gamma}_0 \). By compactness of \( D \), there exists a subsequence \( \{ \hat{\gamma}_k' \} \) converging to a point \( \tilde{z} \in \hat{D} \). Since \( d(\hat{\gamma}_k' z_k, \hat{\gamma}_k z_k) \leq d(z_k, \hat{\gamma}_k z_k) \), we have that 
\[ d(\hat{\gamma}_k' z_k, \hat{\gamma}_k z_k) \leq d(z_k, \hat{\gamma}_k z_k) + 2M. \]

By construction \( d(\hat{\gamma}_k' z_k, \hat{\gamma}_k z_k) \to 0 \), we then conclude that, up to a subsequence, \( \hat{\gamma}_k' \) converges to a point \( q \in B(\tilde{z}; 2M + \epsilon) \) for some \( \epsilon > 0 \). This implies that 
\[ \hat{\rho}(\hat{\gamma}_k' \tilde{z}) \to \hat{\rho}(q) \]
and then
\[ (\hat{\gamma}_k' \cdot \hat{\gamma}) q = \hat{\gamma}_k' \tilde{z} \to q. \]

for some \( \hat{\gamma} \in \hat{\Gamma}/\hat{\Gamma} \). Now the action of \( \hat{\Gamma}_k \) on \( \tilde{X} \) is properly discontinuous, we then conclude that \( \hat{\gamma}_k' \cdot \hat{\gamma} = \{ 1 \} \) for all \( j \) sufficiently large. Thus, we must have \( \hat{\gamma} = \{ 1 \} \) which then implies the contradiction \( \hat{\gamma}_k' = \{ 1 \} \). The proof is then complete. □
By using Lemma 2.4, we are now ready to conclude the proof. By contradiction assume that for any \( k \), we can always find a subvariety \( Z_k \) such that the degree of the map \( Z_k \to q_k(Z_k) \) is less than \( N \), where \( q_k : X_k \to X \) is the finite \( \acute{e} \) tale cover associated to \( \Gamma_k \). For any \( k \), let us define by \( \tilde{p}_k : \hat{X} \to X_k \) the algebraic universal covering map. By definition of the numerical invariant \( \hat{r}_k \), the map \( \tilde{p}_k : B(z; \frac{1}{k^2}) \to \hat{p}_k(B(z; \frac{1}{k})) \) is a biholomorphism for any \( z \in \hat{X} \). Thus for large \( k \) we must have \( Z_k \subset \tilde{p}_k(B(z; \frac{1}{k})) \) for some ball in \( \hat{X} \). We can then find a copy of \( Z_k \) inside the algebraic universal cover \( \hat{X} \). This fact contradicts the assumption that \( \pi_1^{\text{alg}}(X) \) is large. Recall in fact that \( \pi_1^{\text{alg}}(X) \) is large if and only if \( \hat{X} \) does not contain any proper subvariety, see Proposition 2.12 in [Kol93].

The proof under the assumptions that \( \pi_1(X) \) is residually finite and large is completely analogous. More precisely, let us consider a sequence \( \{ \Gamma_k \} \) of finite index normal subgroups in \( \Gamma = \pi_1(X) \) with the property that \( \bigcap_{k=0}^{\infty} \Gamma_k = \{1\} \). For any \( k \geq 0 \), we can define the numerical invariant
\[
r_k := \inf\{ d(z, \gamma z) \mid z \in \hat{X}, \gamma \in \Gamma_k, \gamma \neq 1 \},
\]
where \( \hat{X} \) is the topological universal cover, which satisfies \( r_k \to \infty \) as \( k \to \infty \). The proof then proceeds exactly as before once we recall that \( \pi_1(X) \) is large if and only if \( \hat{X} \) does not contain any proper subvariety, see again Proposition 2.12 in [Kol93].

Lemma 2.4 now implies the following.

**Corollary 2.5.** Let \( L \) be an ample line bundle on a smooth variety \( X \). If \( \pi_1(X) \) is residually finite and large or if \( \pi_1^{\text{alg}}(X) \) is large, given a positive number \( N > 0 \) there exists a finite \( \acute{e} \) tale cover \( q : X' \to X \) such that \((q^*L)^{\dim(V)}.V \geq N\) for any irreducible subvariety \( V \subseteq X' \).

**Proof.** Let \( q : X' \to X \) be as in Lemma 2.4. For any irreducible subvariety \( V \subseteq X' \), let us define \( V' := q(V) \). Since \( L \) is ample, we have that \((L^{\dim(V')}.V') \geq 1\) for any \( V' \subseteq X \). We therefore conclude that \((q^*L)^{\dim(V)}.V \geq N\).

We are now ready to prove Theorem 1.3. The proof of Theorem 1.4 is identical and we leave its details to the interested reader.

**Proof of Theorem 1.3.** Let \( n \) be the dimension of the smooth variety \( X \). Given the ample line bundle \( L \) on \( X \), let us fix an integer \( a = a(L) \) such that \( aL - K_X \) is ample and \( aL - 2K_X \) is nef. Moreover, let us define \( L' = aL - K_X \). Observe that by Anghel-Reih [AS95], we have that for \( m_0 \geq \binom{n+1}{2} \) then \( m_0L' \) is base point free. Next, let us define \( N' := \max\left\{ \left(1 + \binom{n+1}{2}(n-c) + \frac{n}{2}\right)^c (n+s)^n + 1 \mid c = 0, \ldots, n \right\} \) where \( s > 0 \) is a fixed parameter to be determined later. By Corollary 2.5 we can find a finite regular \( \acute{e} \) tale cover \( q_* : X_s \to X \) such that \((q_*^*L')^{\dim(V)} \cdot V \geq N' \) for any subvariety \( V \subseteq X_s \). Moreover, we have that by construction
\[
q_*^*(L' - K_X) = q_*^*L' - K_X
\]
is nef and
\[
m_0q_*^*L' = q_*^*(m_0L')
\]
is base point free. By Theorem 2.2 for any \( x \in X_s \) the linear system \( |K_{X_s} + p^*L'| \) separates \( s \)-jets at \( x \). Since

\[ K_{X_s} + q^*_s L' = a q^*_s L \]

by Proposition 2.1 we conclude that \( \epsilon(q^*_s L) \geq s a \). Thus, given any \( N > 0 \) we simply find \( s \) such that \( s a \geq N \) and the associated cover \( q_s : X_s \to X \), which we rename to be \( p' : X' \to X \), satisfies the requirements of the theorem. The proof is then complete. \( \square \)

3. Generalizations in dimension two

In this section, we prove Theorem 1.7 by using a generalization of Reider’s theorem which deals with separation of higher order jets. For a proof of the following theorem we refer to Lazarsfeld’s lecture notes [Laz97].

**Theorem 3.1** (Corollary 7.5 in [Laz97]). Let \( X \) be a smooth surface. Fix a positive integer \( s \) and a point \( x \in X \). Let \( L \) be a nef divisor such that \( L^2 > (s+2)^2 \). Suppose

\[ L \cdot C \geq s^2 + 3s + 3, \]

for any curve \( C \) passing through \( x \). Then \( K_X + L \) separates \( s \)-jets at \( x \).

Theorem 3.1 estimates the separation of jets of adjoint line bundles of the form \( K_X + L \). Since we are interested in estimating the separation of jets of the line bundle \( L \), we need a lemma which, at least up to a large multiple, relates the two linear series.

**Lemma 3.2.** Let \( L \) be a big and nef divisor on \( X \). Then there exists a positive integer \( m \) such that

\[ s(kmL; x) \geq s(k(K_X + L); x), \]

for any \( x \notin B_+(L) \) and any \( k \geq 0 \).

**Proof.** Let \( m_0 \) be a positive integer such that \( m_0L \) is base point free for any \( x \notin B_+(L) \). Recall that

\[ B_+(L) = \bigcap_{i=1}^k \text{Supp } E_i, \]

where \( L \sim \bigoplus A_i + E_i \) with \( A_i \) ample and \( E_i \) effective. For any \( i \), fix \( m_i \) a positive multiple of \( m_0 \) such that \( m_iA_i - K_X \) is very ample. Let \( m := \sum m_i \). We claim that \( mL - K_X \) is base point free outside \( B_+(L) \). Let \( x \notin B_+(L) \). Then there exists an index \( i \) such that \( m_iL - K_X \) has a section that does not vanish at \( x \), in other words let us choose \( i \) such that \( x \notin E_i \). Moreover, \( (m - m_i)L \) is a positive multiple of \( m_0L \) which has a non-vanishing section at \( x \). This proves the claim.

Let \( g \in |mL - K_X| \) be a section with \( g(x) \neq 0 \). If we multiply \( K_X + L \) by \( g \), we obtain that \( s(k(K_X + L); x) \leq s(k(m + 1)L; x) \) for any \( k \geq 0 \) and any \( x \notin B_+(L) \). \( \square \)

We can now prove Theorem 1.7.
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Proof of Theorem 1.7. Let \( m \) be the positive number given by Lemma 3.2. Let \( s = mN \). By Lemma 2.3, we can find a finite étale cover \( q : X' \to X \) such that \( L' = q^*L \) satisfies Theorem 3.1. In particular, we know that \( K_{X'} + L' \) separates \( s \)-jets outside \( B_+(L) \). Note that Lemma 3.2 is satisfied on \( X' \) with the same \( m \).

Thus, we conclude that
\[
\epsilon(L'; x) \geq \frac{1}{m} s(mL'; x) \geq \frac{1}{m} s(K_{X'} + L'; x) \geq N.
\]
The proof is then complete. \( \square \)

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