GLOBAL EXISTENCE AND BLOW-UP FOR
SEMILINEAR DAMPED WAVE EQUATIONS IN
THREE SPACE DIMENSIONS

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Abstract
We consider initial value problem for semilinear damped wave equations in three
space dimensions. We show the small data global existence for the problem without
the spherically symmetric assumption and obtain the sharp lifespan of the solutions.
This paper is devoted to a proof of the Takamura’s conjecture in [2] on the lifespan
of solutions.

Keywords: Semilinear damped wave equation, Blow-up, Lifespan, three space dimen-
sions.

1 Introduction
In this paper, we consider the Cauchy problem for semilinear damped wave equations

\begin{align}
(1.1) & \quad v_{tt}(x,t) - \Delta v(x,t) + \frac{2}{1+t}v_t(x,t) = |v(x,t)|^p \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times [0, \infty),
(1.2) & \quad v(x,0) = \epsilon f(x), \quad v_t(x,0) = \epsilon g(x) \quad \text{for} \quad x \in \mathbb{R}^n,
\end{align}

where $p > 1$, $n \in \mathbb{N}$ and $\epsilon > 0$.

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Let $\rho \geq 1$ and we assume that
\begin{equation}
\text{supp}\{f(x), g(x)\} \subset \{x \in \mathbb{R}^n \mid |x| \leq \rho\}.
\end{equation}

We study small data global existence and blowup for (1.1) and (1.2) with $n = 3$. Our aim is to obtain the lifespan of the solutions.

Before we proceed to our problem, we recall some known results. We begin with a semilinear wave equation
\begin{equation}
\frac{w_{tt} - \Delta w}{w} = |w|^p, \quad (x, t) \in \mathbb{R}^n \times [0, \infty),
\end{equation}
where $n \geq 1$ and $p > 1$. Let $p_S(n)$ be the positive root of the quadratic equation
\begin{equation}
\gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0.
\end{equation}
Then, for any $n \geq 2$, it is known that small data global existence holds for (1.4) if $p > p_S(n)$, while small data blowup occurs if $1 < p \leq p_S(n)$.

For (1.1) and (1.2), D’Abbicco, Lucente and Ressig [2] have determined a critical power
\begin{equation}
p_c(n) := \max\{p_F(n), p_S(n + 2)\} \quad \text{for } n \leq 3,
\end{equation}
where $p_F(n) = 1 + 2/n$ is the critical power for semilinear heat equation, $u_t - \Delta u = |u|^p$. In the proof of [2], they reduced the problem (1.1) and (1.2) to the following equations (1.6) and (1.7). By setting $u(x, t) = (1 + t)v(x, t)$, we derive the following semilinear wave equations
\begin{align*}
\Box u(x, t) &= (1 + t)^{-p-1}|u(x, t)|^p \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty), \\
u(x, 0) &= f(x), \quad u_t(x, 0) = \epsilon\{f(x) + g(x)\} \quad \text{for } x \in \mathbb{R}^n.
\end{align*}
D’Abbicco and Lucente [1] have also showed the global existence for $p_S(n + 2) < p < 1 + 2/\max\{2, (n - 3)/2\}$ to odd and higher dimensions ($n \geq 5$) under the spherically symmetric assumption.

We put
\begin{equation}
m(p) = \begin{cases} 
1 & (1 < p < 2), \\
2 & (p \geq 2).
\end{cases}
\end{equation}
We think of $C^{m(p)}$-solutions of the following integral equation associated with (1.6) and (1.7):
\begin{equation}
u(x, t) = u^0(x, t) + L[|u|^p](x, t), \quad (x, t) \in \mathbb{R}^3 \times [0, \infty),
\end{equation}
where
\[
L[w](x,t) = \frac{1}{4\pi} \int_0^t (t-s)ds \int_{|\eta|=1} (1+s)^{-(p-1)}w(x+(t-s)\eta,s)d\omega_\eta
\]
for \( w \in C(\mathbb{R}^3 \times [0,\infty)) \) and \( u^0 \) is a solution to the linear wave equations
\[
\begin{align*}
&u_{tt}(x,t) - \Delta u(x,t) = 0, \quad (x,t) \in \mathbb{R}^3 \times [0,\infty), \\
u(x,0) = \epsilon f(x), \quad u_t(x,0) = \epsilon \{f(x) + g(x)\}, \quad x \in \mathbb{R}^3.
\end{align*}
\]
We remark that if \( u \in C^2(\mathbb{R}^3 \times [0,\infty)) \) is the solution of (1.9) with \( p \geq 2 \), then \( u \) is the classical solution to the initial value problem (1.6) and (1.7) (See Lemma I in [6]).

To state our results, we define the lifespan \( T(\epsilon) \) of the solution of (1.9) by
\[
T(\epsilon) := \sup \{ T \in [0,\infty) \mid \text{There exists a unique solution } u \in C^{m(p)}(\mathbb{R}^3 \times [0,T)) \text{ of (1.9)} \}.
\]
for arbitrarily fixed \((f,g)\).

In the blowup case \( 1 < p \leq p_F(n) \), Wakasugi [10, 11] has showed that the upper bound of life span of the solutions for (1.6) and (1.7) is
\[
T(\epsilon) \leq C \epsilon^{-\frac{1}{2} - \delta} \left( \frac{p}{p_S(5)} \right)^{\frac{1}{2} - \frac{\delta}{p_S(5)}} \exp \left( C \epsilon^{-p(p-1)} \right) \quad (p = p_S(5)).
\]
with arbitrary small \( \delta > 0 \). In [2], for \( n = 3 \) and \((f,g) \neq (0,0)\), they remark the following Takamura’s conjecture.
\[
T(\epsilon) \sim \begin{cases} 
C \epsilon^{-\frac{2p(p-1)}{p_S(5)m(p)}} & (1 < p \leq p_S(5)) \\
\exp \left( C \epsilon^{-p(p-1)} \right) & (p = p_S(5))
\end{cases}
\]

In the following theorem, we establish the global existence without the spherically symmetric assumption.

**Theorem 1.1.** Let \( n = 3 \), \( p > p_S(5) = \frac{3+\sqrt{17}}{4} \), \( f \in C^2(\mathbb{R}^3) \) and \( g \in C^1(\mathbb{R}^3) \), where \( m(p) \) is given by (1.8). If \( \epsilon \) is sufficiently small, then (1.9) has a unique global solution \( u \in C^{m(p)}(\mathbb{R}^3 \times [0,\infty)) \).

**Remark 1.2.** In [2], for \( p > p_S(5) \), they have showed the global existence in \( C(\mathbb{R}^3 \times [0,\infty)) \cap C^2(\mathbb{R}^3 \setminus \{0\} \times [0,\infty)) \) with the radial symmetric condition.
We obtain the lower bound of the lifespan as follows.

**Theorem 1.3.** Let \( n = 3, \ 1 < p \leq p_S(5), \ f \in C^3_0(\mathbb{R}^3) \) and \( g \in C^2_0(\mathbb{R}^3) \). There exist positive constants \( A \) and \( \epsilon_0 \) such that for \( 0 < \epsilon \leq \epsilon_0 \), it holds

\[
T(\epsilon) \geq \begin{cases} 
A\epsilon^{-\frac{2p(p-1)}{3(p-5)}} & (1 < p < p_S(5)) \\
\exp(A\epsilon^{-p(p-1)}) & (p = p_S(5)) 
\end{cases}
\]

The following theorem shows that the optimality of the lower bound of Theorem 1.3.

Hence, we solve the Takamura’s conjecture. In [4], they have obtained the following sharp upper bound of the life span (1.14) for \( p = p_S(5) \) by the test function method. In this paper, we prove the result by using the iteration argument with the slicing method.

**Theorem 1.4.** Let \( n = 3, \ 1 < p \leq p_S(5), \ f \in C^3_0(\mathbb{R}^3), \ g \in C^2_0(\mathbb{R}^3) \). We assume that \( f \equiv 0 \) and \( g \geq 0 \) (\( g \not\equiv 0 \)). There exists a positive constant \( \epsilon_0 \) such that for \( 0 < \epsilon \leq \epsilon_0 \), the solution of (1.9) blows up in a finite time \( T(\epsilon) \). Moreover, there exists a positive constant \( B \) independent of \( \epsilon \) such that

\[
T(\epsilon) \leq \begin{cases} 
B\epsilon^{-\frac{2p(p-1)}{3(p-5)}} & (1 < p < p_S(5)) \\
\exp(B\epsilon^{-p(p-1)}) & (p = p_S(5)) 
\end{cases}
\]

**Remark 1.5.** Wakasa [9] had obtained the optimal life span estimate for \( n = 1 \). The study of the lifespan of solutions to the semilinear damped wave equation with the general variable coefficient has been studied by many mathematicians (see [7, 8] and refer to the references cited therein).

The article is organized as follows. In Section 2, we show the decay estimate for the solution of linear wave equations and give the estimate of the integral operator (1.10). In Section 3, we prove Theorem 1.1 and Theorem 1.3. The global existence and lower bounds of the life span will be obtained by the weighted \( L^\infty-L^\infty \) estimates introduced in Lemma 3.1 and Lemma 3.2. In Section 4, we prove Theorem 1.4. We will show the upper bounds of lifespan by using the iteration argument and the slicing method.

## 2 Preliminaries

In this section, we study the decay estimate of solution for the homogeneous wave equation (1.11) and (1.12), and estimate the integral operator (1.10). The solution of (1.11) and (1.12) can be expressed by

\[
u^0(x, t) = \epsilon t \int_{|\xi|=1} \{f(x + t\xi) + g(x + t\xi)\}d\omega_\xi + \epsilon \frac{\partial}{\partial t} \left\{ \frac{t}{4\pi} \int_{|\xi|=1} f(x + t\xi)d\omega_\xi \right\}.
\]

We prepare the following decay estimate of solution of free wave equations.
Lemma 2.1. Assume a support property (1.3). Then, there exists a constant \( \gamma \) such that for \( \nu = 0, 1, 2, 3 \) we have

\[
\rho^{\vert \alpha \vert} |D_x^\alpha u^0(x, t)| \leq \epsilon \gamma \left( \frac{t + \rho}{\rho} \right)^{-1} N_\nu \quad \text{for} \quad x \in \mathbb{R}^3, \ t \geq 0, \ |\alpha| \leq \nu,
\]

where

\[
N_\nu = \sum_{|\alpha| \leq \nu + 2} \sup_{x \in \mathbb{R}^3} \rho^{\vert \alpha \vert} |D_x^\alpha f(x)| + \sum_{|\beta| \leq \nu + 1} \sup_{x \in \mathbb{R}^3} \rho^{\vert \beta \vert + 1} \{ |D_x^\beta f(x)| + |D_x^\beta g(x)| \}.
\]

Moreover, it holds

\[
u^0(x, t) = 0 \quad \text{for} \quad |t - |x|| \geq \rho.\]

Proof. This well-known fact can be found in Lemma 2.3 and Lemma 2.4 of [3]. We shall omit the proof.

We define

\[
\overline{w}(r, t) = \sup_{|x|=r} |w(x, t)|.
\]

We can estimate \( L[w] \) in terms of \( \overline{w} \).

Lemma 2.2. For \( t \geq 0 \), it holds

\[
|L[w](x, t)| \leq \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} (1 + s)^{-(p-1)} \overline{w}(\lambda, s) d\lambda.
\]

Hence, we have

\[
\overline{L}[w](r, t) \leq \int \int_{R(r, t)} \frac{\lambda}{2r} (1 + s)^{-(p-1)} \overline{w}(\lambda, s) d\lambda ds,
\]

where

\[
R(r, t) = \{ (\lambda, s) \mid t - r \leq s + \lambda \leq t + r, \ s - \lambda \leq t - r, \ s \geq 0 \}.
\]

Proof. The proof can be found in Lemma II of [3]. We shall omit the proof.

We assume hereafter that

\[
w(x, t) = 0 \quad \text{for} \quad |x| \geq t + \rho.
\]

It follows that

\[
\overline{w}(r, t) = 0 \quad \text{for} \quad r \geq t + \rho.
\]
Since $\rho \geq 1$, we have from Lemma 2.2 and (2.3)

$$
|L[w](x, t)| \leq \int_{0}^{t} ds \int_{I} \frac{\lambda}{2r} \left( \frac{s + \rho}{\rho} \right)^{-(p-1)} \bar{w}(\lambda, s) d\lambda d\lambda,
$$

(2.4)

where $I = [\|r - t + s\|, r + t - s] \cap [\|r - t + s\|, s + \rho]$ and

$$
\tilde{R}(r, t) = \{(\lambda, s) \mid 0 \leq s \leq t, \lambda \in I\}.
$$

We consider the case $0 \leq r \leq t - \rho$. Introducing new variables of integration

$$
\alpha = s + \lambda, \quad \beta = s - \lambda.
$$

(2.5)

For $\alpha + \beta \geq 0$ and $\beta \geq -\rho$, we have

$$
\frac{s + \rho}{\rho} \geq \frac{\alpha + \beta + 4\rho}{4\rho} \geq \frac{\alpha + 2\rho}{4\rho}.
$$

(2.6)

From (2.4) and (2.6), it follows that

$$
|L[w](x, t)| \leq \int_{-\rho}^{t-r} d\beta \int_{t-r}^{t+r} \frac{\alpha - \beta}{8r} \left( \frac{\alpha + 2\rho}{4\rho} \right)^{-(p-1)} \bar{w} \left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) d\alpha
$$

(2.7)

for $0 \leq r \leq t - \rho$.

In order to prove the basic estimates given by Lemma 3.1 and Lemma 3.2 below, we prepare the following lemma (for the proof, see [6]).

**Lemma 2.3.** Let $0 \leq s \leq t$ and $t - \rho \leq r \leq t + \rho$. It holds

$$
\frac{1}{2r} \int_{I} \lambda d\lambda \leq \frac{12\rho(s + \rho)}{t + \rho},
$$

where $I = [\|r - t + s\|, r + t - s] \cap [\|r - t + s\|, s + \rho]$.

### 3 Global existence and Lower bound of the lifespan

In this section, we prove Theorem 1.1 and Theorem 1.3. We denote a weighted $L^\infty$ norm by

$$
\|u\| = \sup_{x \in \mathbb{R}^3} \sup_{0 \leq t < T} w(|x|, t)|u(x, t)|
$$

(3.1)

$$
= \sup_{r \geq 0} \sup_{0 \leq t < T} w(r, t)\bar{u}(r, t).
$$
Here,

\[
  w(r, t) = \begin{cases} 
    \rho^{-2(p-1)}(t + r + 2\rho)^{\frac{1}{2}} & (p \neq \frac{3}{2}) \\
    \rho^{-1}(t + r + 2\rho) \left\{ \log \frac{2(t + r + 2\rho)}{t - r + 2\rho} \right\}^{-1} & (p = \frac{3}{2}) 
  \end{cases},
\]

where

\[
  q = \max\{2(p-1), 1\} \quad \text{and} \quad \overline{q} = \max\{0, 2p - 3\}. \tag{3.3}
\]

Our first step to prove Theorem 1.1 and Theorem 1.3 is the following a priori estimate.

**Lemma 3.1.** Let \( p > 1 \). Assume that \( u \in C(\mathbb{R}^3 \times [0, T]) \) with \( \supp u \subset \{(x, t) \in \mathbb{R}^3 \times [0, T] \mid |x| \leq t + \rho \} \). Then, it holds

\[
  \|L[|u|^p]\| \leq C_1 \rho^2 \|u\|^p D(T), \tag{3.4}
\]

where \( D(T) \) is defined by

\[
  D(T) = \begin{cases} 
    \left( \frac{2T + 3\rho}{\rho} \right)^{\gamma(p, 5)/2} & (1 < p < p_S(5)) \\
    \log \left( \frac{T + 2\rho}{\rho} \right) & (p = p_S(5)) \\
    1 & (p > p_S(5))
  \end{cases}, \tag{3.5}
\]

**Proof.** From (3.1) and (3.2), it follows that

\[
  \overline{u}(\lambda, s) \leq \|u\| \times \begin{cases} 
    \left( \frac{s + \lambda + 2\rho}{\rho} \right)^{-q} & (p \neq \frac{3}{2}) \\
    \left( \frac{s - \lambda + 2\rho}{s - \lambda + 2\rho} \right)^{-\overline{q} - 1} & (p = \frac{3}{2})
  \end{cases}, \tag{3.6}
\]

and

\[
  \overline{u}(\lambda, s) = 0 \quad \text{for} \quad \lambda \geq s + \rho. \tag{3.7}
\]

We put

\[
  S_1 = \{(r, t) \mid t - \rho \leq r \leq t + \rho\}, \quad S_2 = \{(r, t) \mid 0 \leq r \leq t - \rho\}.
\]

First, we consider the case \((r, t) \in S_1\). We see

\[
  1 \leq \frac{t - r + 2\rho}{\rho} \leq 3 \quad \text{and} \quad \frac{t + r + 2\rho}{3} \leq t + \rho \quad \text{for} \quad (r, t) \in S_1. \tag{3.8}
\]

From (3.6) and (3.8), we get

\[
  \overline{u}(\lambda, s) \leq \|u\| \left( \frac{s + \rho}{\rho} \right)^{-q} \eta(t) \quad \text{for} \quad (\lambda, s) \in S_1, \tag{3.9}
\]

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\[ \eta(t) = \begin{cases} \frac{1}{\log \frac{6(t+r)}{\rho}} & (p \neq \frac{3}{2}) \\ \frac{1}{p} & (p = \frac{3}{2}) \end{cases} \]

Noticing that \( \hat{R}(r, t) \subset S_1 \) and substituting (3.9) into (2.4), we obtain from Lemma 2.3, (3.3) and (3.8)

\[ L[|u|^p](x, t) \leq \|u\|^p \eta(t)^p \int_0^t ds \left( \frac{s + \rho}{\rho} \right)^{-(q+1)p+1} \int_t^\infty \frac{\lambda}{2r} d\lambda \]
\[ \leq \frac{12\rho^2 \|u\|^p \eta(t)^p}{t + \rho} \int_0^t \left( \frac{s + \rho}{\rho} \right)^{-(q+1)p+2} ds \]
\[ \leq C \rho^2 \|u\|^p \eta(t)^p + \left( \frac{t + \rho}{\rho} \right)^{1+\max\{-2p^2+p+3,0\}} \]
\[ \leq C \rho^2 \|u\|^p \left( \frac{t + r + 2\rho}{\rho} \right)^{-q} \times \begin{cases} \left( \frac{T+r}{\rho} \right)^{-2p^2+3p} & (1 < p < \frac{3}{2}) \\ \left\{ \log \frac{6(T+r)}{\rho} \right\}^\frac{5}{2} & (p = \frac{3}{2}) \\ 1 & (p > \frac{3}{2}) \end{cases} \]

Hence, we get from (3.8) and (3.5)

(3.10) \[ L[|u|^p](x, t) \leq C \rho^2 \|u\|^p \left( \frac{t + r + 2\rho}{\rho} \right)^{-q} \left( \frac{t - r + 2\rho}{\rho} \right)^{-q} D(T) \text{ in } S_1. \]

Next, we consider the case \((r, t) \in S_2\). We divide the proof into two cases, \(p \neq \frac{3}{2}\) and \(p = \frac{3}{2}\).

(i) Estimation in the case of \(p \neq \frac{3}{2}\).

Substituting (3.6) into (2.7), we get

(3.11) \[ L[|u|^p](x, t) \leq 2^{(p-1)} \rho^{2p^2-p-1} \|u\|^p \times \frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-p(q+1)+2} d\alpha \times \int_{t-r}^{t-r} (\beta + 2\rho)^{-p\beta} d\beta. \]

We consider the case \(t - r + 2\rho \geq \frac{1}{2}(t + r + 2\rho)\). We have

(3.12) \[ \frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-p(q+1)+2} d\alpha \leq \frac{C}{r} \int_{t-r}^{t+r} d\alpha \times (t + r + 2\rho)^{-p(q+1)+2} \leq \begin{cases} C(t + r + 2\rho)^{-2p^2+p+2} & (1 < p < \frac{3}{2}) \\ C(t + r + 2\rho)^{-1} (t - r + 2\rho)^{-2p^3} & (p > \frac{3}{2}) \end{cases} \]
Next, we consider \( t - r + 2\rho \leq \frac{1}{2}(t + r + 2\rho) \). In other words, for \( t + 2\rho \leq 3r \), we obtain

\[
\frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-p(\rho+1)+2} d\alpha \leq \frac{4}{t + r + 2\rho} \times \begin{cases} 
\frac{1}{2p-3} (t + r + 2\rho)^{-2p + p + 3} & (1 < p < \frac{3}{2}) \\
\frac{1}{2p-3} (t - r + 2\rho)^{-2p + p + 3} & (p > \frac{3}{2})
\end{cases}
\]

We have from (3.12) and (3.13)

\[
\frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-p(\rho+1)+2} d\alpha \leq \begin{cases} 
C(t + r + 2\rho)^{-2p + p + 2} & (1 < p < \frac{3}{2}) \\
C(t + r + 2\rho)^{-1} (t - r + 2\rho)^{-2p + p + 3} & (p > \frac{3}{2})
\end{cases}
\]

We get from (3.1)

\[
\int_{t-r}^{t+r} (\beta + 2\rho)^{-r(p+1)\rho} d\beta \leq \begin{cases} 
t - r + \rho & (1 < p < \frac{3}{2}) \\
\frac{2}{\gamma(p,5)} (t - r + 2\rho)^{\gamma(p,5)/2} \log \frac{t - r + 2\rho}{\rho} & (\frac{3}{2} < p < p_S(5)) \\
C\rho^{\gamma(p,5)/2} & (p = p_S(5)) \\
C\rho^{\gamma(p,5)/2} & (p > p_S(5))
\end{cases}
\]

Hence, from (3.11), (3.14) and (3.15), it follows that

\[
L[|u|^p](x, t) \leq C\rho^p \|u\|^p \left( \frac{t + r + 2\rho}{\rho} \right)^{-q} \left( \frac{t - r + 2\rho}{\rho} \right)^{-q} D(T) \quad \text{in} \quad S_2.
\]

(ii) Estimation in the case of \( p = \frac{3}{2} \).

Substituting (3.6) into (2.7), we get

\[
L[|u|^p](x, t) \leq \|u\|^\frac{3}{2} \int_{t-r}^{t+r} d\beta \int_{t-r}^{t+r} \frac{\alpha - \beta}{8r} \left( \frac{\alpha + 2\rho}{4\rho} \right)^{-\frac{1}{2}} \left( \frac{\alpha + 2\rho}{\rho} \right)^{-\frac{3}{2}} \left\{ \log \frac{2(\alpha + 2\rho)}{\beta + 2\rho} \right\}^\frac{3}{2} d\alpha
\leq C\rho^2 \|u\|^\frac{3}{2} \times r \int_{t-r}^{t+r} d\beta \int_{t-r}^{t+r} (\alpha + 2\rho)^{-1} \left\{ \log \frac{2(\alpha + 2\rho)}{\beta + 2\rho} \right\}^\frac{3}{2} d\alpha
\]

\[
\leq C\rho^2 \|u\|^\frac{3}{2} \left\{ J_1 + J_2 \right\}, \tag{3.17}
\]

where

\[
J_1 := \left\{ \log \frac{2(t + r + 2\rho)}{\rho} \right\}^\frac{3}{2} \times \frac{1}{r} \int_{t-r}^{t+r} d\beta \int_{t-r}^{t+r} (\alpha + 2\rho)^{-1} d\alpha,
\]

\[
J_2 := \frac{1}{r} \int_{t-r}^{t+r} d\beta \int_{t-r}^{t+r} (\alpha + 2\rho)^{-1} \left\{ \log \frac{2(\alpha + 2\rho)}{\beta + 2\rho} \right\}^\frac{3}{2} d\alpha.
\]

By the same calculation as (3.12) and (3.13), we have

\[
J_1 \leq C \left\{ \log \frac{2(t + r + 2\rho)}{\rho} \right\}^\frac{3}{2} \left( \frac{t + r + 2\rho}{\rho} \right)^{-1} \log \frac{2(t + r + 2\rho)}{t - r + 2\rho}
\]

\[
\leq C \log^\frac{3}{2} \left\{ \log \frac{2(2T + 3\rho)}{\rho} \right\}^\frac{3}{2} \tag{3.18}
\]

\[9\]
Next we evaluate $J_2$. Introducing new variables of integration $\sigma$, $\theta$ in $J_2$ by

$$2(\alpha + 2\rho) = (t - r + 2\rho)\sigma, \quad \beta + 2\rho = (t - r + 2\rho)\theta\sigma$$

and setting

$$A = t + r + 2\rho, \quad B = t - r + 2\rho,$$

we have

$$J_2 = \frac{B}{r} \int_2^\frac{2\pi}{r} d\sigma \int_0^{\frac{1}{2r}} (-\log \theta) \frac{3}{2}d\theta$$

$$\leq \frac{2(A - B)}{r} \int_0^{\frac{1}{2r}} (-\log \theta) \frac{3}{2}d\theta$$

$$(3.19)$$

We obtain from (3.17), (3.18) and (3.19)

$$L\left[|u|^p(x, t)\right] \leq C\rho^2\|u\|_3 w(r, t)^{-1} D(T) \quad \text{in} \quad S_2.$$ 

Hence, from (3.10), (3.16) and (3.20), we get (3.4). This completes the proof.

**Proof of theorem 1.1.** We define

$$X = \left\{ u(x, t) \mid D^\alpha_x u(x, t) \in C(\mathbb{R}^3 \times [0, \infty)), \quad \|D^\alpha_x u\| < \infty \quad (|\alpha| \leq m(p)) \right\},$$

where $D^\alpha_x = D^\alpha_1 D^\alpha_2 D^\alpha_3$ ($\alpha = (\alpha_1, \alpha_2, \alpha_3)$) and $D_k = \frac{\partial}{\partial x_k}$ ($k = 1, 2, 3$).

We can verify easily that $X$ is complete with respect to the norm

$$\|u\|_X = \sum_{|\alpha| \leq m(p)} \|D^\alpha_x u\|.$$ 

We define the sequence of functions $\{u_n\}$ by

$$u_0 = u^0, \quad u_{n+1} = u^0 + L[|u_n|^p] \quad \text{for} \quad n \geq 0.$$

We have from (2.1) and (2.2)

$$(3.21) \quad \|D^\alpha u^0\| \leq \epsilon \gamma 3^{2(p-1)} \rho^{-\nu} N_\nu \quad (|\alpha| = \nu \leq 3).$$

As in [6], from Lemma 3.1 and (3.21), we see that if $u^0$ satisfies

$$C_1 \rho^2 \|u^0\|^{p-1} \leq \frac{1}{p^2\rho},$$

then $\{u_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists a function $u \in X$ such that $D^\alpha_x u_n$ converges uniformly to $D^\alpha_x u$, $n \to \infty$. Clearly $u$ satisfies (1.9). In view of (1.9) and (1.10), we note that $\partial u/\partial t$ can be expressed in $D^\alpha_x u$ ($|\alpha| \leq 1$). Thus Theorem 1.1 is proved by taking $\epsilon$ is small.
To prove Theorem 1.3, we prepare the following lemma.

Lemma 3.2. Let $1 < p \leq p_S(5)$ and $0 \leq \nu \leq p - 1$. Assume that $u^0, u \in C(\mathbb{R}^3 \times [0, T))$ with (2.2) and supp $u \subset \{(x, t) \in \mathbb{R}^3 \times [0, T) \mid |x| \leq t + \rho\}$. It holds

$$
\|L[u^0]^\nu |u|^{\nu}\| \leq C\rho^2 \|u^0\|_0 \|u\|_\nu \times \begin{cases}
D(T) \left(\frac{2\nu(2p+3)}{\nu(p+5)}\right)^{\nu} & (1 < p < \frac{3}{2}) \\
\log \left(\frac{6(T+\rho)}{\rho}\right)^{\nu} & (p = \frac{3}{2}) \\
1 & (\frac{3}{2} < p \leq p_S(5))
\end{cases},
$$

where

$$
\|u^0\|_0 := \sup_{x \in \mathbb{R}^3} \rho^{-1}(t + |x| + 2\rho)|u^0(x, t)|.
$$

Proof. We obtain

$$
0 \leq \overline{u}^0(\lambda, s) \leq \|u^0\|_0 \left(\frac{s + \lambda + 2\rho}{\rho}\right)^{-1}
$$

and

$$
\overline{u}^0(\lambda, s) = 0 \quad \text{for} \quad |\lambda - s| \geq \rho.
$$

First, we consider the case $(r, t) \in S_1$. We get

$$
\overline{u}^0(\lambda, s)^{\nu} u(t, s) \leq \|u^0\|_0^{\nu} \|u\|_\nu \eta(t)^{\nu} \left(\frac{s + \rho}{\rho}\right)^{-p+\nu(1-q)}
$$

in $S_1$.

For $(r, t) \in S_1$, noticing $\tilde{R}(r, t) \subset S_1$ and substituting (3.26) into (2.4), we obtain from Lemma 2.3 and (3.8), we obtain from

$$
L[u^0]^\nu |u|^{\nu}(x, t) \leq \|u^0\|_0^{\nu} \|u\|_\nu \eta(t)^{\nu} \int_0^t ds \left(\frac{s + \rho}{\rho}\right)^{-2p+1+\nu(1-q)} \int_1^{\rho \over 2} d\lambda \\
\leq 12\rho^2 \|u^0\|_0^{\nu} \|u\|_\nu \eta(t)^{\nu} \int_0^t \left(\frac{s + \rho}{\rho}\right)^{-2(p-1)+\nu(1-q)} ds \\
\leq C\rho^2 \|u^0\|_0^{\nu} \|u\|_\nu \eta(t)^{\nu+1} \left(\frac{t + \rho}{\rho}\right)^{-1 + \max\{-2p+3+\nu(1-q), 0\}}
$$

$$
\leq C\|u^0\|_0^{\nu} \|u\|_\nu \rho^2 w(r, t)^{-1} \times \begin{cases}
\left(\frac{T+\rho}{\rho}\right)^{\nu(-2p+3)} & (1 < p < \frac{3}{2}) \\
\log \left(\frac{6(T+\rho)}{\rho}\right)^{\nu} & (p = \frac{3}{2}) \\
1 & (\frac{3}{2} < p \leq p_S(5))
\end{cases}.
$$
Next, we consider the case \((r, t) \in S_2\).
We get from (2.7), (3.6), (3.24) and (3.25)
\[
L[u^0]^{p-\nu}[u^\nu](x, t) \leq \|u^0\|_0^{p-\nu}\|u\|^{\nu}\eta(t)^{\nu} \int_{-\rho}^{\rho} \int_{t-r}^{t+r} \frac{\alpha - \beta}{8r} \left(\frac{\alpha + 2\rho}{4\rho}\right)^{-(p-1)} \frac{d\beta}{\nu} \int_{t-r}^{t+r} \frac{\alpha + 2\rho}{(\alpha + 2\rho)^{-(\nu-\nu)}} d\alpha \times \left(\frac{\alpha + 2\rho}{\rho}\right)^{-(p-\nu)-\nu} \left(\frac{\beta + 2\rho}{\rho}\right)^{-\nu} d\beta
\]
\[
\leq 2^{2p-5} \rho^{2p-1-\nu(1-q-\nu)}\|u^0\|_0^{p-\nu}\|u\|^{\nu}\eta(t)^{\nu} \times \frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-2(p-1)+\nu(1-q)} d\alpha
\]
\[
\times \int_{-\rho}^{\rho} (\beta + 2\rho)^{-\nu} d\beta
\]
\[
(3.28)
\]
By the same calculation as (3.12) and (3.13), we obtain
\[
(3.29) \quad \frac{1}{r} \int_{t-r}^{t+r} (\alpha + 2\rho)^{-2(p-1)+\nu(1-q)} d\alpha \leq \begin{cases} 
C(t + r + 2\rho)^{-2(p-1)+\nu(1-q)} & (p \neq \frac{3}{2}) \\
C(t + r + 2\rho)^{-1} \log \frac{2(t+r+2\rho)}{t-r+2\rho} & (p = \frac{3}{2})
\end{cases}
\]
We have from (3.28) and (3.29)
\[
(3.30)
\]
\[
L[u^0]^{p-\nu}[u^\nu](x, t) \leq C\rho^2\|u^0\|_0^{p-\nu}\|u\|^{\nu}w(r, t)^{-1} \times \begin{cases} 
\left(\frac{2T+3\rho}{\rho}\right)^{\nu(2p-3)} & (1 < p < \frac{3}{2}) \\
\log \left(\frac{6(T+\rho)}{\rho}\right)^{\nu} & (p = \frac{3}{2}) \\
1 & (\frac{3}{2} < p \leq ps(5))
\end{cases}
\]
From (3.27) and (3.30), we get (3.22). This completes the proof. □

From Lemma 3.2, we obtain the following estimate.

**Lemma 3.3.** Suppose that the assumptions in Lemma 3.2 are fulfilled. Then we have
\[
(3.31) \quad \|L[u^0]^{p-\nu}[u^\nu]\| \leq C2^p\|u^0\|_0^{p-\nu}\|u\|^{\nu} \times \begin{cases} 
1 & (\nu = 0) \\
D(T)^{1/p} & (\nu = 1) \\
D(T)^{(p-1)/(p+1)} & (\nu = p - 1)
\end{cases}
\]

**Proof.** When \(\frac{3}{2} \leq p \leq ps(5)\), from Lemma 3.2 and (3.3), the estimate (3.31) is trivial. When \(1 < p < \frac{3}{2}\), we see that \(\frac{2(-2p+3)}{\gamma(p,5)} \leq \frac{1}{p+1}\). Hence, from Lemma 3.2 and (3.5), we get (3.31). This completes the proof. □

**Proof of Theorem 1.3** We consider the following integral equation:
\[
(3.32) \quad U = L[u^0 + U^p] \quad \text{in} \quad \mathbb{R}^3 \times [0, T).
\]
Suppose we obtain the solution of (3.32). Then, putting \( u = U + u^0 \), we get the solution of (1.9), and its lifespan is the same as that of \( U \). Thus we have reduced the problem to the analysis of (3.32). Let \( Y \) be the norm space defined by

\[
Y = \left\{ U(x, t) \mid D^\alpha_x U(x, t) \in C(\mathbb{R}^3 \times [0, T]), \quad \|D^\alpha_x U\| < \infty \quad (|\alpha| \leq 1) \right\},
\]

which is equipped with the norm

\[
\|U\|_Y = \sum_{|\alpha| \leq 1} \|D^\alpha_x U\|.
\]

We shall construct a solution of the integral equation (3.32) in \( Y \). We define the sequence of functions \( \{U_n\} \) by

\[
U_0 = 0, \quad U_{n+1} = L[u^0 + U_n]^p \quad (n = 0, 1, 2, \ldots).
\]

From Lemma 2.1, (3.23) and (3.8), we see that there exists a positive constant \( C_0 \) such that

\[
\|D^\alpha_x u^0\|_0 \leq C_0 \epsilon \quad (|\alpha| \leq 1).
\]

We put

\[
C_3 := (2^{2p+2})^{\frac{1}{p-1}} \max \left\{ C_1 \rho^2 M_0^{p-1}, (C_2 \rho^2 C_0^{p-1})^p, (C_2 \rho^2 M_0^{p-2} C_0) \right\}
\]

and

\[
M_0 := 2^p \rho^2 C_0^p \max \{C_1, C_2\},
\]

where \( C_1 \) and \( C_2 \) are constants given in Lemma 3.1 and Lemma 3.3.

Let us fix \( \epsilon_0 \) as

\[
C_3 \epsilon_0^{p(p-1)} \leq \begin{cases} \frac{1}{2} \left( \frac{1}{2} \right)^{\frac{1}{p-1}} & (1 < p < p_S(5)) \\ \frac{1}{4} \left( \frac{\log 2}{\rho} \right)^{\frac{1}{p-1}} & (p = p_S(5)) \end{cases}.
\]

For \( 0 < \epsilon \leq \epsilon_0 \), we assume that

\[
C_3 \epsilon^{p(p-1)} \leq \begin{cases} \left( \frac{4T}{\rho} \right)^{\frac{1}{p-1}} & (1 < p < p_S(5)) \\ \left( \frac{\log 2T}{\rho} \right)^{\frac{1}{p-1}} & (p = p_S(5)) \end{cases}.
\]

Then, we have

\[
C_3 \epsilon^{p(p-1)} D(T) \leq 1.
\]

Therefore as in \( \mathbb{5} \), from Lemma 3.1 and Lemma 3.3 we see that \( \{U_n\} \) is a Cauchy sequence in \( Y \), provided (3.34) holds. We can verify easily that \( Y \) is complete. Hence, there exists a function \( U \) such that \( U_n \) converges to \( U \) in \( Y \). Therefore \( U \) satisfies the integral equation (3.32). The lower bound estimate (1.13) follows immediately from (3.34). This completes the proof. \( \square \)
4 Upper bound of the lifespan

In this section, we prove Theorem 1.4. Our proof is based on the iteration argument which was introduced by [6]. For the critical case, we apply the slicing method which was introduced by [3].

We define

\[ \tilde{w}(r, t) = \frac{1}{4\pi} \int_{|\xi|=1} w(r\xi, t)d\omega. \]  

(4.1)

We get the following representation formula (4.2) (for the proof, see [6]).

Lemma 4.1. Let \( L \) be the linear integral operator defined by (1.17). Assume that \( w \in C(\mathbb{R}^3 \times [0, T]) \). Then, For \( (r, t) \in [0, \infty) \times [0, T) \), it holds

\[ \tilde{L}[w](r, t) = \int \int_{R(r,t)} \frac{\lambda}{2r}(1+s)^{-(p-1)}\tilde{w}(\lambda, s)d\lambda. \]  

(4.2)

Since \( p > 1 \), \(|u|^p\) is a convex function with \( u \). By using the Jensen’s inequality and Lemma 4.1, it follows from (1.9) that

\[ \tilde{u}(r, t) \geq \tilde{u}^0(r, t) + \int \int_{R(r,t)} \frac{\lambda}{2r}(1+s)^{-(p-1)}|\tilde{u}(\lambda, s)|^pd\lambda. \]  

(4.3)

We define the following domains:

\[ \Sigma_j = \{(r, t) \mid l_j\rho \leq t - r \leq r\}, \quad \Sigma_\infty = \{(r, t) \mid 2\rho \leq t - r \leq r\}, \]  

where

\[ l_j = 1 + \frac{1}{2} + \cdots + \frac{1}{2j} \quad (j = 0, 1, 2, \cdots). \]  

(4.4)

We derive a lower bound of the solution to (1.9), which is a first step of our iteration argument.

Lemma 4.2. We assume that \( f \equiv 0 \) and \( g \geq 0 \) (\( g \neq 0 \)). Let \( u \) be a solution of (1.9). Then there exists a positive constant \( M \) independent of \( \epsilon \) such that

\[ \tilde{u}(r, t) \geq \frac{Me^p}{(t+r)(t-r)^{2p-3}} \]  

in \( \Sigma_0 \).

(4.5)

Proof. From (1.11) and (4.1), the spherical averages \( \tilde{u}^0(r, t) \) satisfy \((\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2})ru^0(r, t) = 0\). By using the d’Alembert’s formula and the assumption \( f \equiv 0 \), we obtain

\[ \tilde{u}^0(r, t) = \frac{H(t+r) - H(t-r)}{2r}, \]  

(4.6)
where

\[ H(s) = -\epsilon \int_s^\infty \sigma \tilde{g}(\sigma) d\sigma. \]

From \((1.3)\), we have \(H(s) = 0\) for \(s \geq \rho\). Since \(g(x) \geq 0\) \((\not\equiv 0)\), there exist constants \(M_0 > 0, a, b\ (0 < a < b < \rho)\) such that

\[ H(s) \leq -2M_0\epsilon \ (a \leq s \leq b). \]

From \((1.6)\), for \(t + r \geq \rho, a \leq t - r \leq b\), we get

\[ \tilde{u}^0(r, t) = -\frac{H(t - r)}{2r} \geq \frac{M_0\epsilon}{r}. \]

(4.7)

We put

\[ S(r, t) = \{(\lambda, s) \mid t - r \leq \lambda, s + \lambda \leq 3(t - r), a \leq s - \lambda \leq b\}. \]

For \((r, t) \in \Sigma_0\), we see \(t + r \geq 3(t - r)\). Then, it follows that

(4.8) \[ S(r, t) \subset \{(\lambda, s) \mid s + \lambda \geq \rho, a \leq s - \lambda \leq b\} \quad \text{in} \ \Sigma_0. \]

Then, for \((r, t) \in \Sigma_0\), we have from \((1.3), (1.7)\) and \((1.8)\)

(4.9) \[ \bar{u}(\lambda, s) \geq \frac{M_0\epsilon}{\lambda} \quad \text{in} \ S(r, t). \]

Noticing that \(S(r, t) \subset R(r, t)\) for \((r, t) \in \Sigma_0\) and substituting \((4.9)\) into \((4.3)\), we get

\[ \bar{u}(r, t) \geq \int \int_{S(r, t)} \frac{\lambda}{2r} (1 + s)^{-(p-1)} \left| \frac{M_0\epsilon}{\lambda} \right|^p d\lambda ds \quad \text{in} \ \Sigma_0. \]

Using the variables of integration \(\alpha, \beta\) from \((2.5)\) and since \(t - r \geq \rho \geq 1\), we get

\[ \bar{u}(r, t) \geq \frac{M_0^p \epsilon^p}{4r} \int_a^b d\beta \int_{2(t-r)+\beta}^{3(t-r)} \left( \frac{\alpha - \beta}{2} \right)^{1-p} \left( \frac{2 + \alpha + \beta}{2} \right)^{1-p} d\alpha \]

\[ \geq \frac{M_0^p \epsilon^p}{4r} \int_a^b d\beta \int_{2(t-r)+\beta}^{3(t-r)} (3(t-r))^{1-p}(6(t-r))^{1-p} d\alpha \]

\[ = \frac{18^{-p} M_0^p \epsilon^p}{4r(t-r)^{2(p-1)}} \int_a^b (t-r-\beta) d\beta \]

\[ \geq \frac{18^{-p}(b - a) M_0^p \epsilon^p (t-r-b)}{2(t+r)(t-r)^{2(p-1)}} \]

\[ \geq \frac{M \epsilon^p}{(t+r)(t-r)^{2(p-3)}} \quad \text{in} \ \Sigma_0, \]

where \(M = 2^{-1} 18^{-p}(b - a)(1 - b/\rho) M_0^p\). This completes the proof.
In order to prove Theorem 1.4, we replace the integral domain in (4.3). We set

\[ Q_j(r, t) = \{(\lambda, s) \mid t - r \leq \lambda, \ s + \lambda \leq 3(t - r), \ l_j \rho \leq s - \lambda \leq t - r\} \quad (j = 0, 1, 2, \cdots). \]

For \((r, t) \in \Sigma_0\), we see \(t + r \geq 3(t - r)\). Then, we have

\[ Q_j(r, t) \subset R(r, t) \quad \text{in } \Sigma_0, \]
\[ Q_j(r, t) \subset \Sigma_j \quad \text{in } \Sigma_0. \]

Since \(\Sigma_j \subset \Sigma_0\), we have from (4.3) and (4.10)

\[ \tilde{u}(r, t) \geq \int \int_{Q_j(r,t)} \frac{\lambda}{2r}(1 + s)^{-(p-1)}|\tilde{u}(\lambda, s)|^pd\lambda \quad \text{in } \Sigma_j. \]

Proof of Theorem 1.4. We divide the proof into two cases, \(1 < p < p_S(5)\) and \(p = p_S(5)\).

(i) Estimation in the case of \(1 < p < p_S(5)\).

We define the sequences \(\{a_j\}, \{b_j\}\) and \(\{D_j\}\) by

\[ a_0 = 1, \quad b_0 = 2(p - 1), \quad D_0 = M\epsilon^p \]

and

\[ a_{j+1} = pa_j + 2, \quad b_{j+1} = pb_j + 2(p - 1), \quad D_{j+1} = \frac{18^{-p}}{2a_{j+1}^2}D_j^p \quad (j = 0, 1, 2, \cdots). \]

By using the induction argument, we derive

\[ \tilde{u}(r, t) \geq \frac{D_j(t - r - \rho)^{a_j}}{(t + r)(t - r)^{b_j}} \quad \text{in } \Sigma_0 \quad (j = 0, 1, 2, \cdots). \]

From (4.3), it holds (4.15) with \(j = 0\). We assume that (4.15) holds for one natural number \(j\) and \((r, t) \in \Sigma_0\). Noticing that \(Q_0(r, t) \subset \Sigma_0\) for \((r, t) \in \Sigma_0\) and putting (4.15) into (4.12), we get

\[ \tilde{u}(r, t) \geq \int \int_{Q_0(r,t)} \frac{\lambda}{t + r}(1 + s)^{-(p-1)} \left| \frac{D_j(s - \lambda - \rho)^{a_j}}{(s + \lambda)(s - \lambda)^{b_j}} \right|^p d\lambda ds \quad \text{in } \Sigma_0. \]

Then using the variable of integration \(\alpha, \beta\) from (2.5), we obtain

\[ \tilde{u}(r, t) \geq \frac{1}{2(t + r)} \int_{\rho}^{t-r} d\alpha \int_{\rho}^{t-r+\beta} 2^{-(p-1)} \left| \frac{D_j(\beta - \rho)^{a_j}}{\alpha^{b_j}} \right|^p d\alpha \]
\[ \geq \frac{D_j^p}{4(t + r)}((6(t - r))^{-(p-1)} \int_{\rho}^{t-r} d\beta \beta^{-p_bj}(\beta - \rho)^{p_{a_j}} \int_{\rho}^{t-r+\beta} \frac{\alpha - \rho}{\alpha^p} d\alpha \]
\[ \geq \frac{D_j^p}{4(t + r)}(6(t - r))^{-(p-1)}(t - r)^{-p_b} \int_{\rho}^{t-r} d\beta (\beta - \rho)^{p_{a_j}} \int_{\rho}^{t-r+\beta} 2(t - r) \frac{2(t - r)}{3p(t - r)^p} d\alpha \]
\[ = \frac{18^{-p}D_j^p}{2(t + r)(t - r)^{p_bj + 2(p - 1)}} \int_{\rho}^{t-r} (\beta - \rho)^{p_{a_j}}(t - r - \beta) d\beta. \]
Using integration by parts, it follows that
\[
\int_{\rho}^{t-r} (\beta - \rho)^{pa_j} (t - r - \beta) d\beta = \frac{1}{pa_j + 1} \int_{\rho}^{t-r} (\beta - \rho)^{pa_j+1} d\beta \geq \frac{1}{a_{j+1}^2} (t - r - \rho)^{pa_j+2}.
\]
(4.18)

Therefore, from (4.17) and (4.18), (4.15) holds for all natural number.

Solving (4.13) and (4.14) yields
\[
a_j = \frac{p^j(p+1) - 2}{p-1}, \quad b_j = 2(p^{j+1} - 1). \quad (j = 0, 1, 2, \cdots).
\]
(4.19)

Hence, we get from (4.14) and (4.19)
\[
D_{j+1} \geq \frac{F D_j^p}{p^{2(j+1)}},
\]
where \( F = \frac{18 - p}{2} (\frac{p-1}{p+1})^2 \). Hence we have
\[
\log D_j \geq p^j \left[ \log D_0 + \sum_{k=1}^{j} \frac{p^{k-1} \log F - 2k \log p}{p^j} \right].
\]
(4.20)

By using the d’Alembert’s criterion, we see that the sum part in (4.20) converges as \( j \to \infty \). Hence, from (4.13), there exists a constant \( q \) such that it holds
\[
D_j \geq \exp \{ p^j \log (Me^{q}e^{q}) \}. \tag{4.21}
\]

Therefore, we have from (4.15), (4.19) and (4.21)
\[
\tilde{u}(r, t) \geq \exp \left[ p^j J(r, t) \right] \frac{(t - r)^2}{(t + r)(t - r - \rho)^{p+1}} \quad \text{in } \Sigma_0.
\]
(4.22)

Here,
\[
J(r, t) = \log \left\{ e^{p} Me^{q} \frac{(t - r - \rho)^{p+1}}{(t - r)^{2p}} \right\}.
\]
(4.23)

We take \( \epsilon_0 > 0 \) so small that
\[
B \epsilon_0 \frac{2p(p-1)}{\gamma(p, 5)} \geq 8\rho,
\]
where
\[
B = \left( 2^{2(1+2p-1)} Me^{q} \right)^{\frac{2p(p-1)}{\gamma(p, 5)}}. \tag{4.24}
\]
Next, for a fixed $\epsilon \in (0, \epsilon_0]$, we suppose that

\[
\tau > B \epsilon^{\frac{2p(p-1)}{3(p-3)}} \geq 8\rho.
\]

(4.24)

Let $(r, t) = (\tau/2, \tau)$. Then $(r, t) \in \Sigma_0$ and $t - r - 2\rho \geq (t - r)/2$. Hence we get from (4.23) and (4.24)

\[
J(\tau/2, \tau) \geq \log \left(e^p (B^{-1} \tau)^{\frac{2(p-5)}{2(p-3)}}\right) > 0.
\]

(4.25)

Therefore, from (4.22) and (4.25), we get $\tilde{u}(\tau/2, \tau) \to \infty (j \to \infty)$. Hence, $T(\epsilon) \leq B \epsilon^{\frac{2p(p-1)}{3(p-3)}}$ for $0 < \epsilon \leq \epsilon_0$.

(ii) Estimation in the case of $p = p_S(5)$.

We define the sequences $\{d_j\}$ and $\{E_j\}$ by

\[
d_0 = 0, \quad E_0 = Me^p
\]

and

\[
d_{j+1} = pd_j + 1, \quad E_{j+1} = \frac{18^{-p} E_j^p}{2^{j+3}d_{j+1}} \quad (j = 0, 1, 2, \cdots).
\]

(4.26)

(4.27)

First, by using the induction argument, we will show

\[
\tilde{u}(r, t) \geq \frac{E_j}{(t + r)(t - r)^{2p-3}} \left(\log \frac{t - r}{l_j \rho}\right)^{\frac{d_j}{p}} \quad \text{in } \Sigma_j \quad (j = 0, 1, 2, \cdots).
\]

(4.28)

From (4.5), it holds (4.28) with $j = 0$. We assume that (4.28) holds for one natural number $j$ and $(r, t) \in \Sigma_{j+1}$. From $p = p_S(5)$, we have $p(2p - 3) = 1$. Noticing (4.11) and substituting (4.28) into (4.12), we obtain

\[
\tilde{u}(r, t) \geq \frac{1}{2(t + r)} \int_{l_j \rho}^{t-r} d\beta \int_{2(t-r)+\beta}^{3(t-r)+\beta} \frac{\alpha - \beta (2 + \alpha + \beta)^{1-p}}{2} \left(\log \frac{\beta}{l_j \rho}\right)^{\frac{d_j}{p}} d\alpha
\]

\[
\geq \frac{E_j^p}{2(t + r)} \int_{l_j \rho}^{t-r} d\beta \int_{2(t-r)+\beta}^{3(t-r)+\beta} \frac{6^{1-p}(t - r)^{1-p}}{3p(t - r)^{p-1}3^{p(2p-3)}} \left(\log \frac{\beta}{l_j \rho}\right)^{pd_j} d\alpha
\]

\[
\geq \frac{18^{-p} E_j^p}{2(t + r)(t - r)^{2(p-1)}} \int_{l_j \rho}^{t-r} \frac{t - r - \beta}{\beta} \left(\log \frac{\beta}{l_j \rho}\right)^{pd_j} d\beta.
\]

(4.29)
Since $\rho \leq \frac{t-r}{l_j+1}$, we get from (4.29) and (4.27)

$$\tilde{u}(r,t) \geq \frac{18^{-p}E_j^p}{2(t+r)(t-r)^{2(p-1)}} \int_{l_j \rho}^{t-r} (t-r-\beta) \left[ \frac{1}{pd_j+1} \left( \log \frac{\beta}{l_j \rho} \right)^{pd_j+1} \right]' \, d\beta$$

$$= \frac{18^{-p}E_j^p}{2(t+r)(t-r)^{2(p-1)}} \times \frac{1}{pd_j+1} \int_{l_j \rho}^{t-r} \left( \log \frac{\beta}{l_j \rho} \right)^{pd_j+1} \, d\beta$$

$$\geq \frac{18^{-p}(1 - \frac{l_j}{l_j+1})E_j^p}{2d_{j+1}(t+r)(t-r)^{2p-3}} \int_{l_j(t-r)}^{t-r} \left( \log \frac{t-r}{l_{j+1} \rho} \right)^{d_{j+1}} \, d\beta$$

From $1 - \frac{l_j}{l_j+1} = \frac{1}{2^{j+1}l_{j+1}} \geq \frac{1}{2^{j+1}}$, we get

$$\tilde{u}(r,t) \geq \frac{18^{-p}E_j^p}{2^{j+3}d_{j+1}(t+r)(t-r)^{2p-3}} \left( \log \frac{t-r}{l_{j+1} \rho} \right)^{d_{j+1}}$$

$$= \frac{E_{j+1}}{(t+r)(t-r)^{2p-3}} \left( \log \frac{t-r}{l_{j+1} \rho} \right)^{d_{j+1}}$$

in $\Sigma_{j+1}$.

Therefore, (4.28) holds for all natural number.

Solving (4.26) and (4.27) yields

$$d_j = \frac{p^j - 1}{p - 1} \quad (j = 0, 1, 2, \ldots)$$

Hence, we get

$$E_{j+1} \geq G \frac{E_j^p}{(2p)^j},$$

where $G = \frac{18^{-p}(p-1)^p}{2^p p}$. Therefore, it follows that

$$\log E_j \geq p^j \left[ \log E_0 + \sum_{k=1}^{j} \frac{p^{k-1} \log G - (k-1) \log (2p)}{p^j} \right].$$

The sum part in (4.31) converges as $j \to \infty$ by the d’Alembert’s criterion. Hence, there exists a constant $q$ such that it holds from (4.26)

$$E_j \geq \exp\{p^j \log (Me^{q_0}e^p)\}.$$
where

\[(4.33) \quad J(r, t) = \log \left( e^p \left( B^{-1} \log \frac{t - r}{2\rho} \right)^{\frac{1}{p-1}} \right), \quad B = (Me^q)^{-(p-1)}. \]

We take \(\varepsilon_0 > 0\) so small that

\[(4.34) \quad B\varepsilon_0^{-p(p-1)} \geq \log(4\rho). \]

For a fixed \(\varepsilon \in (0, \varepsilon_0]\), we suppose that \(\tau\) satisfies

\[(4.35) \quad \tau > \exp(2B\varepsilon^{-p(p-1)}) (> 4\rho). \]

From (4.35) and (4.34), it follows that

\[(4.36) \quad \tau > 4\rho \exp(B\varepsilon^{-p(p-1)}). \]

We get (4.33) and (4.36)

\[(4.37) \quad J(\tau/2, \tau) = \log \left( e^p \left( B^{-1} \log \frac{\tau}{4\rho} \right)^{\frac{1}{p-1}} \right) > 0. \]

Since \((\tau/2, \tau) \in \Sigma_\infty\), from (4.32) and (4.37), we get \(u(\tau/2, \tau) \to \infty (j \to \infty)\). Hence, \(T(\varepsilon) \leq \exp(2B\varepsilon^{-p(p-1)})\) for \(0 < \varepsilon \leq \varepsilon_0\). This completes the proof. \(\square\)

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References

[1] M. D’Abbicco and S. Lucente, NLWE with a special scale invariant damping in odd space dimension, Dyn. Syst. Differential Equations Appl. AIMS Proc. (2015) 312–319.

[2] M. D’Abbicco, S. Lucente and M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015), no. 10, 5040–5073.

[3] R. Agemi, Y. Kurokawa and H. Takamura, Critical curve for p -q systems of nonlinear wave equations in three space dimensions, J. Differential Equations 167 (2000), no. 1, 87–133.

[4] M. Ikeda, M. Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, Mathematische Annalen (in press) [arXiv:1709.00440].
[5] T. Imai, M. Kato, H. Takamura, K. Wakasa, *The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions*, in: Proceeding of the international conference Asymptotic Analysis for Nonlinear Dispersive and Wave Equations of a volume in Advanced Study of Pure Mathematics, (in press) arXiv:1610.05913.

[6] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. 28 (1979), 235–268.

[7] N.-A. Lai, H. Takamura, *Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case*, Nonlinear Anal. 168 (2018), 222-237.

[8] N.-A. Lai, H. Takamura, K. Wakasa, *Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent*, J. Differential Equations 263 (2017), no. 9, 5377–5394.

[9] K. Wakasa, *The lifespan of solutions to semilinear damped wave equations in one space dimension*, Commun. Pure Appl. Anal. 15 (2016), no. 4, 1265–1283.

[10] Y. Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant damping*, Fourier Analysis, Trends Math., Birkhäuser/Springer, Cham (2014), 375–390.

[11] Y. Wakasugi, *On the Diffusive Structure for the Damped Wave Equation with Variable Coefficients* (Doctoral thesis), Osaka University, 2014.

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