Differentially Private Gaussian Processes

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Abstract

A major challenge for machine learning is increasing the availability of data while respecting the privacy of individuals. Differential privacy is a framework which allows algorithms to have provable privacy guarantees. Gaussian processes are a widely used approach for dealing with uncertainty in functions. This paper explores differentially private mechanisms for Gaussian processes. We compare binning and adding noise before regression with adding noise post-regression. For the former we develop a new kernel for use with binned data. For the latter we show that using inducing inputs allows us to reduce the scale of the added perturbation. We find that, for the datasets used, adding noise to a binned dataset has superior accuracy. Together these methods provide a starter toolkit for combining differential privacy and Gaussian processes.

1 Introduction

As machine learning algorithms are applied to an increasing range of personal data types, interest is increasing in mechanisms that allow individuals to retain their privacy while the wider population can benefit from inferences drawn through assimilation of data. A differentially private algorithm (Dwork & Roth, 2014) allows queries to be performed while minimising the release of information about individual records. For example, one might add Laplacian noise to the sum of a database column to mask the influence of individual items.

This perturbation can be added at any of three stages in the learning process; to the (i) input data, prior to its use in the algorithm, (ii) to components of the calculation (such as to the gradients) or (iii) to the output of the algorithm (e.g. the slope and intersect of a linear regression).

Our longer term interest is driven by the concept of user-centric data models. In a user-centric data model each user retains control over their data, but can still assist a machine learner by revealing particular features, this can be thought of as distributing each entry in the database (or from a the perspective of a data design matrix, distributing the rows of the design matrix) across the users. Considering the three stages above, this concept of keeping the user’s data private, even from the algorithm, could be achieved by adding noise at stage (i) the input and (ii) the calculation components. However, option (iii) is excluded because it requires the algorithm has access to non-differentially private aspects of the user’s data.

We describe methods for integrating differential privacy with Gaussian processes. We exploit the ability of the Gaussian process to naturally assimilate summary measures in Section 2 by deriving the covariance function for fitting the model to binned data. We explore the extent to which user
data needs to be corrupted for use in Gaussian processes through bounding the scale of perturbations of the mean function to provide differential privacy for the training outputs (Section 3). We address challenges of high sensitivity to individual users by reformulating the model with inducing inputs that reduce the bound on the sensitivity of the mean function, allowing differential privacy to be more generally applicable (Section 4). Finally we compare the two strategies for inducing privacy (input or output perturbation) to investigate which method provides the most accuracy for a given privacy guarantee.

1.1 Differential Privacy

To query a database in a differentially private manner, a randomised algorithm $M$ is $(\varepsilon, \delta)$-differentially private if, for all possible query outputs $m$ and for all neighbouring databases $D$ and $D'$ (i.e. databases which only differ by one row),

$$P\left(M(D) = m\right) \leq e^{\varepsilon} P\left(M(D') = m\right) + \delta.$$  

I.e. we want each output value to be almost equally likely regardless of the value of one row. This makes intuitive sense for privacy: we don’t want one query to give an attacker strong evidence for a particular row’s value. $\varepsilon$ puts a bound on how much privacy is lost by the query, with a smaller $\varepsilon$ meaning more privacy. $\delta$ says this inequality only holds with probability $1 - \delta$.

2 Privacy for Model Inputs: Data Binning

Consider a set of children’s heights: to preserve anonymity, a dataset is aggregated into the means over age ranges: e.g. aged 24 to 36 months average 90cm, aged 36 to 48 months, 92cm, etc. Each average is then corrupted by addition of some noise. This process is designed to protect individual privacy. From a machine learning point of view we might wish to know what the most likely height is (with confidence intervals) for a child aged (for example) 38 months.

To proceed via Gaussian process regression (see e.g. Lawrence et al., 2006) we assume that there is some latent function, $f(t)$, that represents the values of weights as a function of age. The summary measures (average over age ranges) can then be derived through integrating across the latent function to give us the necessary average. Importantly, if the latent function is drawn from a Gaussian process then its integral is also jointly drawn from the same Gaussian process. This allows us to analytically map between the aggregated measure and the observation of interest.

Assume that a second function, $F(s, t)$, describes the integral between the ages $s$ and $t$ of $f(\cdot)$. Finally, we are given observations, $y(s, t)$, which are noisy samples from $F(s, t)$.

A Gaussian process assumption for a function specifies that any finite realisation of points from that function should be jointly distributed as a Gaussian density with particular mean and covariance matrix. A Gaussian density has the property that any linear combination of its samples will, in turn, be jointly distributed as Gaussian with the original density. Similarly for a Gaussian process, any linear operator (such as integration) applied to the original function will lead to a joint Gaussian process over the result of that linear operator and the original function. In other words there will be a joint Gaussian process between the two functions $f(t')$ and $F(s, t)$. Such a Gaussian process is specified, a priori, by its mean function and its covariance function. The mean function is often taken to be zero (although non-zero mean functions are easily incorporated into the framework) but it is the covariance function where the main interest lies.

To construct the joint Gaussian process posterior we need expressions for the covariance between values of $f(t)$ and $f(t')$ at different times ($t$ and $t'$), values of $F(s, t)$ and $F(s', t')$, i.e. the covariance between two integrals, and the ‘cross covariance’ between the the latent function $f(t')$ and the output of the integral $F(s, t)$.

For the underlying latent function we assume that the covariance between the values of the latent function $f(\cdot)$ are described by the exponentiated quadratic (EQ) form,

$$k_{ff}(u, u') = \alpha e^{-\frac{|u - u'|^2}{\ell^2}},$$

\footnote{Aggregation is not enough to protect individual privacy as two noise free aggregated values could be combined to reveal data that was supposed to be private.
}
where $\alpha$ is the scale of the output and $\ell$ is the (currently) one-dimensional length-scale. To compute the covariance for the integrated function, $F(\cdot)$ we integrate the original EQ form over both its time variables,

$$k_{FF}((s, t), (s', t')) = \alpha \int_s^t \int_{s'}^{t'} k_{ff}(u, u') \, du' \, du$$

substituting in our EQ kernel, and integrating;

$$k_{FF}((s, t), (s', t')) = \alpha^2 \frac{\ell^2}{2} \left[ g\left(\frac{t - s'}{\ell}\right) + g\left(\frac{t' - s}{\ell}\right) - g\left(\frac{t - t'}{\ell}\right) - g\left(\frac{s - s'}{\ell}\right) \right]$$

(1)

where we defined $g(z) = z \sqrt{\pi} \text{erf}(z) + e^{-z^2}$ and $\text{erf}(\cdot)$ is the Gauss error function.

Similarly we can calculate the cross-covariance between $F$ and $f$,

$$k_{Ff}((s, t), (t')) = \sqrt{\pi \ell^2} \left( \text{erf}\left(\frac{t - t'}{\ell}\right) + \text{erf}\left(\frac{t' - s}{\ell}\right) \right).$$

The same idea can be used to extend the input to multiple dimensions. Each kernel function contains a unique lengthscale parameter, with the bracketed kernel subscript index indicating these differences. In conclusion we can express the new kernel as the product of our one dimensional kernels:

$$k_{FF} = \prod_i k_{FF(i)}((s_i, t_i), (s'_i, t'_i)),$$

with the cross covariance given by

$$k_{Ff} = \prod_i k_{Ff(i)}((s_i, t_i), (s'_i, t'_i)).$$

2.1 Experiment on Age Data

We apply the approach to the age distribution of 255 people from a single output area (E00172420) from the 2011 UK census. We group the people into a histogram with equal ten year wide bins. Our implementation is based on the GPy software (GPy, since 2012), here we fixed the lengthscale we use GPy’s optimization algorithm to select the noise variance and kernel scale parameter (for both kernels) which maximize the log likelihood of the data.

Figure 1c illustrates the improvement over the EQ kernel, while figure 1a shows a GP fit to the histogram data. To aid intuition, figure 1d illustrates the same data binned into 20 year wide domains, and illustrates EQ and integral kernel GP means in which the sample noise is fixed at zero, and lengthscale fixed at 10 years. Note how, without noise, the EQ kernel would pass through the top-centre of each histogram bin, while the integral kernel will ensure the area under its curve is equal to the area of the bin.

Figure 1c shows that for all values of the lengthscale parameter, the integral kernel produces a more accurate estimate of the original data. We believe that, in general, this kernel will be superior for regressing binned or histogram datasets. Both GP regression results were, with the right lengthscale, superior to the use of the binned means.

3 Privacy for Model Outputs

So far we have considered DP on the inputs to the GP: in particular when the data is first aggregated into a histogram of binned means or counts. We now turn our attention to DP guarantees on the output of a Gaussian process, which has been trained on the original, non-private, data.

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2Note that there is a $\sqrt{2}$ difference between our length-scale and that normally defined, this is for convenience in later integrals.
(a) GP fit to a perturbed histogram
(b) perturbed GP fitted to raw data
(c) Accuracy of different kernel functions
(d) Demonstration of integral kernel vs EQ kernel

Figure 1: (a) A Gaussian process fit to differentially private histogram, using the integral-kernel. Mean function of Gaussian process (solid curve) with 95% confidence intervals (dotted). Histogram’s differentially private noise indicated with 95% confidence error bars. Annual age counts indicated by blue crosses. (b) GP fit to the original data, then perturbed. The dashed blue line indicates the prediction from the original Gaussian process, while the red indicates the effect of adding noise to make the output differentially private. 95% confidence intervals take combine both GP and DP noise. ε = 1 in both these cases. (c) Root mean squared error of the predictions for individual years using the Gaussian process mean, for the EQ kernel (blue) and integral kernel (red). In black is the result if we just take the histogram frequency as our estimate. Curves generated from the average of repeated runs with different DP noise samples. Confidence intervals too narrow to show. (d) Comparison of regression provided by GPs with the two kernels, EQ (blue) and the integral kernel (red).

3.1 Differential Privacy for Functions

Hall et al. (2013) extend differential privacy to functions and functionals. Consider a function, $f$, that we want to release (with privacy guarantees on its inputs). If the family of functions from which this function is sampled lies in the reproducing kernel Hilbert space (RKHS) then one can consider the function as a point in the RKHS. We can consider another function, $f'$, which represents another function that’s been generated using identical data except for the perturbation of one row. The distance, $||f - f'||$, between these points is bounded by the sensitivity, $\Delta$. The norm is defined to be $||g|| = \sqrt{\langle g, g \rangle_H}$. Specifically the sensitivity is written

$$\Delta \geq \sup_{D \leadsto D'} ||f_D - f_{D'}||_H,$$

i.e. the sensitivity must be greater or equal to the largest distance between the functions (in RKHS).

Hall et al. (2013) showed that one can ensure that a version of $f$, function $\tilde{f}$ is $(\varepsilon, \delta)$-differentially private by adding a scaled sample from the Gaussian distribution $G$ (which uses the same kernel as $f$). We scale the sample by $\frac{\Delta c(\delta)}{\varepsilon}$, where $c(\delta) \geq \sqrt{2\log \frac{1.25}{\delta}}$.

In the Gaussian process case we have some data (at inputs $X$ and at outputs $Y$). Assume that the inputs are non-private columns, while the outputs are private.

The mean function of a Gaussian process posterior lies in the RKHS. We need to add the correctly scaled sample to ensure its differentially private release. It will become clear that the covariance function does not need perturbation as it does not contain direct reference to the output values. We first need to know the possible values that the output $y$ can take. For the age histogram example, the $y$ values are the result of a histogram query, and thus have a sensitivity, $\Delta_y$, of 1 (Dwork & Roth, 2014).

Using the notation of Williams & Rasmussen (2006), the conditional distribution from a Gaussian process at a test point $x_*$ has mean,

$$\tilde{f}_* = k^\top_* (K' + \sigma_n^2 I)^{-1} y,$$

and covariance,

$$V[f_*] = k(x_*, x_*) - k^\top_* (K' + \sigma_n^2 I)^{-1} k_*,$$
Returning to the calculation of the sensitivity, we can expand equation 4 substituted into 3:

\[\Delta y = \sum_{i=1}^{n} \alpha_i k(x_*, x_i) - \sum_{i=1}^{n} \alpha'_i k(x_*, x_i) = \sum_{i=1}^{n} k(x_*, x_i)(\alpha_i - \alpha'_i)\]  (4)

In the kernel density estimation example, in [Hall et al. (2013)], all but the last term in the two summations cancel as the \(\alpha\) terms were absent. In our case however they remain and, generally, \(\alpha_i \neq \alpha'_i\). We therefore need to provide a bound on difference between the values of \(\alpha\) and \(\alpha'\).

To reiterate, \(\alpha = K^{-1}y\). So the difference between the two vectors is,

\[\alpha - \alpha' = K^{-1}y - K^{-1}y' = K^{-1}(y - y')\]  (5)

We note that all the values of \(y\) and \(y'\) are equal except for the last element which differs by at most \(\Delta_y\). I.e. equation 5 is bounded by \(\Delta_y\) times the maximum possible sum of the last column in \(K^{-1}\) (the infinity-norm, \(\|K^{-1}\|_\infty\)).

As \(K\) doesn’t contain private information itself (it is dependent purely on the input and the features of the kernel) we can find the exact value of \(\|K^{-1}\|_\infty\). We shall call this value \(b(K)\). See the supplementary material for a general upper bound.

### 3.1.1 Upper Bound Calculation

Returning to the calculation of the sensitivity, we can expand equation 4 substituted into 3:

\[\|f_D(x_*) - f'_D(x_*)\|^2 = \left(\sum_{i=1}^{n} \alpha_i k(x_*, x_i), \sum_{i=1}^{n} \alpha'_i k(x_*, x_i)\right)\]  (6)

We now use our constraint that the chosen kernel has a maximum value of one\(^3\), so the weighted sum of \(\alpha_i - \alpha'_i\) will be less than or equal to the sum of \(\alpha_i - \alpha'_i\), which we already know have an upper bound of \(\Delta_y b(K)\). This means that an upper bound on the sensitivity is,

\[\|f_D(x_*) - f'_D(x_*)\|^2 \leq \Delta_y^2 b(K)^2.\]  (7)

We again use the age-histogram from the UK’s census, but in this case leave the input data disaggregated. At this stage we are fixing the kernel parameters, as estimating them from the data will

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\(^3\)This might in practice restrict us to stationary covariance functions.
require additional privacy protections. We’ve described in the supplementary material how the exponential mechanism can be applied to select the parameters using the differential framework. We assume an EQ kernel with lengthscale of 8 years, and a homoscedastic noise variance of 10 people and a one dimensional data set with 50 points placed uniformly 2 years apart. We can find the empirical value of the infinity norm of the inverse covariance matrix, \( b(K) \), without compromising privacy. For this example data set and kernel parameters it is equal to 0.130. This results in the sensitivity (as \( \Delta_w = 1 \)). The scaling factor that we need to multiply by our GP sample is \( \frac{c(\delta)}{\varepsilon} \) where \( c(\delta) \geq \sqrt{2 \log \frac{1}{\delta}} \). We continue our example with values of \( \varepsilon = 1 \) and \( \delta = 0.00625 \). Substituting in our given values: \( c(\delta) = 3.26 \), so the scaling factor we should multiply our sampled Gaussian with is 0.42. Figure 1B illustrates an example data set with a non-private Gaussian process mean and the differentially private mean. The private mean function appears to trace a path that describes the data set reasonably, suggesting that for such data sets with small sensitivities, this method may be useful. Note that negative counts appear in the private function, but that this is expected under a DP mechanism.

4 Inducing Variables and Sensitivity

The previous data sets were histogram-queries, where individual output sensitivities were equal to one (because of the counts). For other data sets however, the output sensitivities may be higher, which means the amount of noise required to ensure privacy may be too large for the method to be practical. In this section we show how this excessive scaling can be partly solved by the introduction of inducing variables. Inducing variables were originally developed to make large scale problems for Gaussian processes more tractable. The idea is to compress the information in the data to a small number of carefully placed inputs.

To get some intuition as to why the inducing-inputs reduce the sensitivity, it is useful to consider what the inverse covariance (precision) matrix represents, both between training inputs and between training and inducing-inputs. Between training inputs, one can consider the diagonal elements, \( (i, i) \), of the precision matrix to be the inverse variance (precision) of each variable, controlling for the remaining variables. If another input \( k \) explains most of the variance of \( i \) then the remaining variance will be very small, causing this inverse-variance to be very large. The off-diagonal elements \( (i, j) \) and \( (j, j) \) are the negative partial correlations scaled by the root product of the two corresponding diagonal elements, \( (i, i) \) and \( (j, j) \), which we’ve seen will be large if \( k \) explains much of \( i \). In the inducing-input case, we are interested in how much inducing-inputs explain each other. Placed far apart, there is relatively little correlation, so the remaining unexplained variance for a given input is large, causing the diagonals of the precision matrix to be small. In this paper we show that using a small number of \( m \) inducing-inputs can reduce the bound on the sensitivity of the function, potentially substantially, allowing differential privacy to be used for a larger range of data sets.

Quickly we introduce inducing variables (see Snelson & Ghahramani (2005); Quiñonero Candela & Rasmussen (2005); Titsias (2009); Damianou et al. (2016)). Given a test input \( x_* \), data set \( D \) and inducing-inputs \( X \) we make an prediction of the output \( y_* \),

\[
p(y_*|x_*, D, X) = N(y_*|\mu_*, \sigma_*^2)
\]

We define \( K_{ff} \) as the covariance matrix between training inputs, \( [K_{ff}]_{i,i'} = K(x_i, x_{i'}) \), \( K_{fu} \) as the covariance matrix between training and inducing variables and \( K_{uu} \) as the covariance between inducing variables. Defining the covariance between the test and inducing variables as \( k_* \) and between the \( i \)th training input and the inducing variables as \( k_i \). The mean and variance of the posterior are,

\[
\mu_* = k_*^\top Q_{uu}^{-1} K_{uf} (\Lambda + \sigma^2 I)^{-1} y \quad \text{and} \quad \sigma_*^2 = K_{**} - k_*^\top K_{uu}^{-1} k_* + \sigma^2,
\]

where \( Q_{uu} = K_{uu} + K_{uf} (\Lambda + \sigma^2 I)^{-1} K_{fu} \) and the matrix \( \Lambda = \text{diag}(\lambda) \) where the diagonal elements are \( \lambda_i = K_{ii} - k_i^\top K_{uu}^{-1} k_i \).

In section 3 we rewrote the expression for the posterior mean, \( \mu_* = k_*^\top K^{-1} y \), as \( \sum \alpha_i k(x_i, x_*) \) where \( \alpha = K^{-1} y \). To find the sensitivity we considered the maximum sum of the absolute terms in a row of \( K^{-1} \) (its infinity-norm).

We can rewrite our expression for the mean of the IV Gaussian process the same way, and find the infinity-norm again,

\[
\alpha = Q_{uu}^{-1} K_{uf} (\Lambda + \sigma^2 I)^{-1} y.
\]
Figure 2: Lengths and ages of a species of fish. On the left is the result using a GP with inducing variables, and on the right a standard Gaussian process. The posterior means of the two GPs are indicated by the dashed black line. A differentially private sample is indicated with the solid black line and is surrounded by dotted black lines, indicating the 95% confidence interval, when both the DP noise and GP variance is combined. Four other DP mean samples are included, in light grey.

Figure 3: Heights, weights and ages of a group of children in Malawi (genders mixed). The blue circles are the original data samples, their diameter indicate the children’s weights (from 9.8kg to 21.5kg). The width of the grey squares indicate the averages centred on each bin.

So we want the infinity norm of,

\[ Q_{uu}^{-1} K_{uf} (\Lambda + \sigma^2 I)^{-1} \]  

(8)

To demonstrate, we consider 401 one-dimensional training inputs equally spaced between 0 and 4, with an EQ kernel with lengthscale 1, and sample variance 0.01. The inverse covariance matrix in this case has an infinity norm of 293.7. For comparison we place 5 inducing variables equally spaced between 0 and 4. We find the infinity norm of the matrix in expression 8 is only 3.33, an 88 fold decrease in the bound on the sensitivity of the function, allowing us to add less noise for the same privacy guarantee.

4.1 Fish lengths example

We consider a data set of 33 fish lengths and ages from [Freund & Minton (1979)], with a sub-population of stunted fish removed. In this example we are interested in protecting the privacy of the fish lengths, but we are willing to release their ages. We select the inducing variables based purely on the input data, and the intuition that the function will not be overly complicated. We space five inducing variables equally between 0 and 160 days of age. One could however place the inputs using the exponential mechanism, as illustrated in the supplementary material for kernel parameters. Figure 2 illustrates the original output and the improved output using the inducing variables.

A large value of \( \varepsilon = 20 \) was still required for the output curve to remain reasonably close to the true inducing variable mean, however one can see that the inducing variable differentially private output is significantly better than the original DP GP mean, allowing us to release a more accurate result with the same privacy guarantee. Note also that the improvement for this example is relatively modest due to the large sample-noise variance.
Table 1: RMSE for the Malawi child data set, using the four methods (integral and EQ kernels fitted to noisy binned means and GP and inducing variable GP (IV GP) fitted to the original data and perturbed). The confidence intervals are one S.E. based on multiple runs of the DP algorithm.

5 Numerical Comparison

We finally look at which of the above methods provide the most accurate reconstruction of the original data, given the same privacy guarantees. We use data from (McLellan et al., 2010) in which heights, weights and ages of 89 children in rural Malawi were recorded. We assume, in this example, that we are willing to release the ages and heights of the children, but we assume that their weights should remain private. We selected the inducing variables based purely on the input data, and the intuition that the mean function of height vs age would be quite smooth, one could again place the inducing variables using the exponential mechanism. Figure 3 illustrates the data set, with the inducing variables marked.

For the ‘input noise’ method, we bin the data into a grid of means over the data set, spaced 6 months and 5cm apart. For the ‘output noise’ method, we will use both a normal and an IV GP to the original data, then perturb the means with DP noise, as described in section 3. We chose lengthscales and noise parameters using the inputs and intuition around this type of data, without using the weight values, this avoids the additional costs associated with parameter selection.

Table 1 summarises the RMSE for the four algorithms for various levels of privacy. When differential privacy is applied the noisy input methods are both superior to the noisy output methods. However, due to the binning, when the output is not made private, these can fair less well. Due to the smoothness of the data, and relatively small bins, this isn’t clearly shown in this example, but it does suggest that, for datasets with low sensitivities, fitting a perturbed GP may be the most effective, depending on the size of the bins. Again, due to the smoothness of the data the integral kernel only performs better than the EQ kernel for the low input noise case, its degradation in the face of DP noise suggests it has poor noise immunity, possibly due to the effect noise has on a derivative. Of the two output-noise methods, the inducing variable method does, predictably, better than the full GP, due to its lower sensitivity, however, if the dataset has a low sensitivity in its outputs (which is equivalent to a large $\varepsilon$) the inaccuracies caused by the inducing input approximation will dominate the RMSE (which occurs in this example where $\varepsilon = \infty$).

6 Conclusions and Further Work

The most accurate method among those tested here, for introducing DP noise, is the adding of noise to the inputs. This matches the result of Berlioz et al. (2015), in which they found that input perturbation offers the best accuracy in the domain of matrix factorisation. Further refinements in the bounding of the sensitivity of the GP mean may change this in the future. The ‘output-noise’ DP GP may have other advantages that haven’t been tested here. With sufficiently large $\varepsilon$ they may preserve the structure of a data set more precisely, than the binned data would allow.

Other strategies exist for reducing the sensitivity of the DP GP. First, we could manipulate the sample noise for individual output values. By adjusting the noise for individual elements we can control the infinity-norm. For example those columns of the inverse covariance matrix with values greater than a threshold could be reduced by increasing the uncertainty associated with them, and thus their contribution will be less to the posterior mean.

The new kernel, which allows binned means to be used to predict a latent function could be extended to non-cuboid integrals by the provision of a mask function. This will allow patches (e.g. from a map) to be used as outputs, without the approximations of weighted centroids (or similar).
One important issue which we haven’t addressed is how to make a Gaussian process’ predictions differentially private if we want to protect the values of the input. Critically we need to find a bound on the inverse-covariance function, some suggestions are provided in the supplementary material.

We have presented novel methods for combining differential privacy and Gaussian processes; both on the input and output of the algorithm. Gaussian processes are a highly flexible approach for a range of challenges in machine learning, in the longer term we believe a comprehensive set of methodologies could be developed to enhance their applicability in privacy preserving learning, this first paper has given a flavour of some of the challenges and their potential solutions. We have developed a new kernel for use with any cuboid binned data set (differentially private or not), applied DP for functions theory developed by [Hall et al., 2013] to Gaussian processes, and shown that using inducing variables can massively reduce the sensitivity of the function.

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Supplementary Material

Privacy on the inputs

To make the inputs private we need a general bound on the infinity norm of the covariance function. Varah (1975) show that if $J$ is strictly diagonally dominant\footnote{A matrix, $J$, is strictly diagonally dominant if $\Delta_i(J) > 0$ for all $1 \leq i \leq n$.} then:

$$||J^{-1}||_\infty \leq \max_{1 \leq i \leq n} \frac{1}{\Delta_i(J)} = b(J)$$

where we’ve defined this bound as $b(J)$. We also define $\Delta_i(J) = |J_{ii}| - \sum_{j \neq i} |J_{ij}|$, i.e. the sum of the off diagonal elements in row $i$ subtracted from the diagonal element.

So if $K$ is strictly diagonally dominant (which is achieved if the inputs are sufficiently far apart, and/or if sufficient uncertainty exists on the outputs), then we have a bound on the sums of its rows. The above bound means that,

$$\sum_{i=1}^{n} \alpha_i - \alpha'_i \leq \Delta_y b(J) \quad (9)$$

To ensure sufficient distance between inputs, we could use inducing variables, which can be arbitrarily placed, so that the above constraints on the covariance matrix are observed.

Kernel Parameter Estimation

In practice we often want to automatically estimate kernel parameters. This is usually done by calculating the gradient of the log-likelihood of the outputs, given the parameters and inputs, and performing a gradient ascent to maximise the likelihood (Williams & Rasmussen, 2006). Problematically, the log-likelihood is influenced by the output values, and thus contains private information. As mentioned in Hall et al. (2013) one can use the exponential mechanism to select the parameters. After calculating the log-likelihood at the vertices of a grid over the parameter values. One then selects a vertex using the exponential mechanism, to gain as large a likelihood as possible, while still keeping the outputs private.

The Exponential Mechanism

The exponential mechanism (McSherry & Talwar, 2007; Dwork & Roth, 2014) solves the problem of selecting an output as optimally as possible, in a differentially private manner, in which different outputs have different utilities. We want to select a parameter combination using the log-likelihood values for each combination but keep the individual values protected by differential privacy. The utility function in this case is provided by the log-likelihood itself. Given the above database, $D$, we randomly sample an output, $o$, with probability,

$$\frac{e^{\epsilon u(D,o)}}{\sum_{q \in \text{outputs}} e^{\epsilon u(D,q)}}$$

Before we can use the exponential mechanism for our application we need to calculate the sensitivity of the utility function. The expression for the log-likelihood is (Williams & Rasmussen, 2006): \[ \log p(y|X, \theta) = -\frac{1}{2} y^\top K_y^{-1} y - \frac{1}{2} \log |K_y| - \frac{n}{2} \log 2\pi \]

where $K_y$ is the covariance matrix for the noisy samples; $K_y = K_f + \sigma_n^2 I$. As before we can put a bound on $Y$ and we can calculate $K_y$ empirically.

Considering an upper bound on the values in $y$ of $y_{max}$, their maximum change between databases of $\Delta$ and the inverse covariance matrix, $G$, gives an upper bound on the sensitivity of the likelihood of:

$$2y_{max} \Delta \max_i \left[ \sum_j G_{ij} \right] + G_{nn} y_{max} \Delta$$
We can use the exponential mechanism now to select the parameters of the kernel.

**Applied to parameter estimation**

For the age-distribution example, we have three parameters to set: The EQ’s scale, its lengthscale and the sample-noise variance. We assume no prior knowledge, and so sample a 3d grid of values spaced logarithmically for each of these parameters from 0.01 to 12000. For each parameter combination we found the log-likelihood and the sensitivity (which only depends on the inputs). We then simply applied the exponential mechanism to select a parameter combination which was likely to result in a high log-likelihood. With \( \varepsilon = 1 \) the expected log-likelihood is -474, compared to an optimal -174. By chance the expected likelihood would be -895. With a less private value of \( \varepsilon = 30 \) we achieve an expected log-likelihood of -228. The utility of this method depends on the prior range of parameters and the sensitivity of the likelihood on the parameter selection. An alternative, and potentially much quicker and more efficient (with regards privacy loss) is the differentially private Bayesian optimisation method developed by [Kusner et al.](2015), that could be used to set the parameters instead.

**The Multidimensional Binning Kernel**

Note that if the kernel is isotropic (i.e. lengthscales in each dimension are equal) it, can be written, 

\[
k_f = \alpha e^{r^2} \quad \text{where} \quad r^2 = (u_1 - u_1')^2 + (u_2 - u_2')^2 + \ldots
\]

Substituting in our expression for \( r \) and noting that \( e^a e^b = e^{a+b} \), and dropping our isotropic assumption, we can write the kernel as a product of kernels, 

\[
k_f = e^{-\frac{(u_1-u_1')^2}{t_1^2}} e^{-\frac{(u_2-u_2')^2}{t_2^2}} \times \ldots
\]

Integrating this new kernel over two dimensions, 

\[
k_{FF} = \alpha \int_{s_2}^{t_2} \int_{s_2'}^{t_2'} \int_{s_1}^{t_1} \int_{s_1'}^{t_1'} e^{-\frac{(u_1-u_1')^2}{t_1^2}} e^{-\frac{(u_2-u_2')^2}{t_2^2}} \, du_1' \, du_1 \, du_2' \, du_2
\]

One can see that the \( \exp \left[ -\frac{(u_2-u_2')^2}{t_2^2} \right] \) term is constant wrt \( u_1 \) and \( u_1' \), so we can move it outside the two inner integrals, 

\[
k_{FF} = \alpha \int_{s_2}^{t_2} \int_{s_2'}^{t_2'} e^{-\frac{(u_2-u_2')^2}{t_2^2}} \left[ \int_{s_1}^{t_1} \int_{s_1'}^{t_1'} e^{-\frac{(u_1-u_1')^2}{t_1^2}} \, du_1' \, du_1 \right] \, du_2' \, du_2
\]

leaving us with the bracketed expression with have previously calculated, and will replace with 

\[
k_{FF}((s_1, t_1), (s_1', t_1')).
\]

\[
k_{FF} = \alpha \int_{s_2}^{t_2} \int_{s_2'}^{t_2'} e^{-\frac{(u_2-u_2')^2}{t_2^2}} \, k_{FF}((s_1, t_1), (s_1', t_1')) \, du_2' \, du_2
\]

A similar logic applies here, \( k_{FF}((s_1, t_1), (s_1', t_1')) \) is constant wrt \( u_2 \) and \( u_2' \) and so can move outside the integrals, leaving us with another \( k_{FF} \), this time in terms of \( s_2 \) and \( t_2 \).

In conclusion we can express the new kernel as the product of our one dimensional kernels: 

\[
k_{FF} = k_{FF(1)}((s_1, t_1), (s_1', t_1')) \times k_{FF(2)}((s_2, t_2), (s_2', t_2')) \times \ldots
\]

The same reasoning applies to the cross-covariance:

\[
k_{Ff} = k_{Ff(1)}((s_1, t_1), (s_1', t_1')) \times k_{Ff(2)}((s_2, t_2), (s_2', t_2')) \times \ldots
\]

The derivative wrt length scale \( \ell_i \) is

\[
\frac{\delta k_{FF}}{\delta \ell_i} = \frac{\delta k_{FF(i,j)}}{\delta \ell_i} \prod_{i \neq j} k_{FF(j)}((s_j, t_j), (s_j', t_j'))
\]

Note the other terms are constant wrt \( \ell_i \) which is why the simple product remains. We have already derived an expression for the derivative in one dimension.