Topology of rotating stratified fluids with and without background shear flow

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Poincaré inertio-gravity modes described by the shallow water equations in a rotating frame have non-trivial topology, providing a new perspective on the origin of equatorially trapped Kelvin and Yanai waves. We investigate the topology of rotating shallow water equations and continuously stratified primitive equations with and without background shear flow. Continuously stratified fluids support waves are analogous to the edge modes of weak three-dimensional topological insulators. Background shear flow not only breaks the Hermiticity and homogeneity of the system but also leads to instabilities. By introducing a gauge-invariant winding number, we show that singularities in the phase of the Poincaré waves of the unforced shallow-water equations and primitive equations persist in the presence of both horizontal and vertical shear flows. Thus the bulk Poincaré bands have non-trivial topology and we expect and confirm the persistence of the equatorial waves in the presence of shear along the equator where the Coriolis parameter $f$ changes sign.

I. INTRODUCTION

Oceanic and atmospheric waves share fundamental physics with topological insulators and the quantum Hall effect, and topology plays an unexpected role in the movement of the atmosphere and oceans [1]. Topology guarantees the existence of unidirectional propagating equatorial waves on planets with atmospheres or oceans. In particular, there is a topological origin for two well-known equatorially trapped waves, the Kelvin and Yanai modes, caused by the breaking of time-reversal symmetry by planetary rotation. Coastal Kelvin waves have also been demonstrated to have a topological origin [2]; thus Kelvin’s 1879 discovery of such waves [3] likely marked the first time that edge modes of topological origin were uncovered in any context (though the topological nature remained hidden). Recently reanalysis observations of Poincaré-gravity waves in the stratosphere have been used to demonstrate the non-trivial topology of these waves[4]. In light of these discoveries, it is important to consider the generalization of the shallow water equations to the more general problem of continuously stratified fluids. At the same time, it is also crucial to consider fluids driven by shear flows and damped by friction. Such extensions bring greater realism to models of actual fluids both on Earth [5] and on other planets [6]. The extension to background shear may also pave the way to the treatment of nonlinearities through the use of the mean-field quasilinear approximation [7–9] that self-consistently treats the interaction of waves with mean flows.

The existence of topological edge modes can be understood, via the principle of bulk-interface correspondence, to be predicted by the non-trivial topology of bulk modes. Bulk-interface correspondence has been invoked for the quantum Hall effect and topological insulators [10, 11] as well as for a variety of classical wave systems, including nanophotonics [12–15], acoustics [16–18], mechanical systems [19, 20], continuum fluids [1, 2, 21–23] and plasmas [24–26]. The principle is clearest for Hermitian systems. Driving and dissipation however lead to non-Hermitian dynamics [27–31]. By continuity, weak damping and driving may be expected to only change the waves slightly, but what happens as the forcing increases? Efforts have been put into the topological classification of non-Hermitian systems [32–35]. Whether or not bulk-interface correspondence continues to hold remains a central problem. It has been argued that traditional bulk-interface correspondence breaks down in non-Hermitian systems [36, 37]. Alternatives to the topological Chern number have been proposed [33, 38–41]. Non-Hermitian bulk-interface correspondence has also been explored experimentally [42, 43]. Here, we show that the phase singularity in the bulk wavefunctions persists in the presence of shear flow. The phase of the bulk Poincaré modes exhibits a vortex or anti-vortex at the origin in the wavevector space, with a change in the phase winding number across the equator. We show that equatorial Yanai and Kelvin waves persist in the background shear, consistent with the continued applicability of the principle of bulk-interface correspondence in the non-Hermitian realm.

The paper is organized as follows. A brief introduction to topology in the context of fluid systems is presented in Section II. It includes references to some pedagogical reviews. We derive the shallow water equations in the
presence of shear and compare numerical and perturbative methods to find the wave spectrum in Section III. The
continuously stratified primitive equations with and without shear are analyzed in the f-plane approximation
in Section IV and the Chern number for each band is found following the procedure introduced in Ref. [1], demonstrating
a correspondence with weak 3D topological insulators. In Section V, we show that the system is unstable with
both horizontal and vertical shear. In Section VI we numerically calculate the winding number to demonstrate the
topology of the bulk. We first show that bulk-interface correspondence holds in the case of spatially varying Coriolis
parameters. (The reader may wish to look at Ref. [4] which attempts to make the topological concepts discussed
here accessible to climate scientists and geophysical fluid dynamicists.) We then show our main result, which is that
bulk-interface correspondence also appears to hold as background shear is turned on and the dynamics become non-
Hermitian. Discussion and concluding remarks are made in Section VII. Some details of the calculations are relegated
to Appendices.

II. TOPOLOGICAL INVARIANTS AND BULK-INTERFACE CORRESPONDENCE

Topology is the branch of mathematics concerned with the qualitative shapes of objects that remain unchanged
under continuous deformations. The topological equivalence of a donut and a coffee mug (both have a single hole) is
a commonly mentioned example, as is the fact that an Möbius strip cannot be made orientable without tearing the
paper and that it is impossible to comb the spines of a hedgehog (the Hairy Ball Theorem).

Topology finds noteworthy applications in fluids. Vortex rings for instance show persistence that is rooted in
topology. The persistence of vortex rings was striking enough for William Thomson (Lord Kelvin) to attempt to
develop a theory of atoms based upon vortex rings in the hypothetical aether. Kelvin’s circulation theorem states
that the circulation (the line integral of the fluid velocity) around a closed loop that is advected with the fluid and
thus deformed by the internal motion remains constant in the absence of viscosity and forcing. Tornadoes, hurricanes,
Jupiter’s red spot, and even cutoff low-pressure regions in Earth’s atmosphere and vortex loops in the ocean are all
eXamples of persistent vortices.

Topology may also be applied to more abstract mathematical spaces. In work recognized by the 2016 Nobel Prize in
Physics, David Thouless and his collaborators demonstrated that the quantized conductance of the integer quantum
Hall effect can be understood mathematically in terms of the topology of complex-valued wavefunctions that live
on a compact Brillouin zone [44]. The electrical Hall conductance is proportional to an integer Chern number that
characterizes the topology of the wave functions. This quantization has a physical interpretation as electrical currents
that propagate around the boundary of the semiconducting material in discrete modes, modes that owe their existence
to the principle of bulk-interface correspondence. The principle states that non-trivial topology away from a boundary
implies the existence of unidirectional waves trapped along the boundary. The quantum of resistance, \( h/e^2 \), can be
measured so precisely that it has now been adopted as the international standard of resistance.

The topology of linearized wave equations is frequently quantified in terms of the Chern number [45]. See Refs.
4, 46–48 for some pedagogical reviews. However, the Chern number has a number of drawbacks. In contrast to
systems on spatial lattices (where the Chern number was first applied), for continuous systems such as fluids the
Chern number need not be integer-valued as it depends on how an integral over the Berry curvature is regularized
at high wavevectors. This ambiguity can sometimes be avoided by compactification [1, 2]. Our viewpoint here is
that this is more of a mathematical problem than a physical one because at small scales dissipation becomes strong
providing a natural (albeit non-Hermitian) regularization at high wavenumbers. Ultimately at the smallest scales, the
fluid description passes over to Hamiltonian molecular dynamics. It is unclear how to extend the Chern number to
systems with dissipation, driving, or nonlinearities – all properties of real fluids.

By contrast, these ambiguities do not arise for a winding number invariant. To demonstrate the concept of the
winding number, it is necessary to define the gauge-invariant, complex-valued quantity \( \Xi \) in the frequency-wavevector
space:

\[
\Xi(k_x, k_y) = h^*(k_x, k_y) \, v(k_x, k_y),
\]

where \( h \) is the height and \( v \) is the meridional velocity. Normal wave modes, which are defined only up to an overall
phase and magnitude, have their overall phases cancel out in Eq. (1), leaving only the relative phase difference between
\( h \) and \( v \) and making \( \Xi \) gauge-invariant.

The idealized rotating shallow-water model on the f-plane is an illustration of the winding number. Figure 1 shows
the positive and negative frequency Poincaré modes (inertio-gravity waves) and the zero-frequency geostrophically
balanced mode. The geostrophically balanced mode becomes Rossby waves if the Coriolis parameter varies with lati-
tude. The topology of Poincaré-gravity modes is characterized by a vortex or antivortex in the frequency-wavevector
space, with winding number \( \pm 1 \). A winding number of \( +1 \) means \( \Xi \) increases (decreases) by \( 2\pi \) as one moves around
the center of the vortex in a clockwise (counterclockwise) sense. On the other hand, the winding number of the
geostrophic balanced mode is 0 (topologically trivial). The winding number, as an alternative to the Chern number, serves the same function by quantifying the topology of the bands.

A band inversion is a phenomenon where the winding number flips sign. This can occur for the Poincaré-gravity waves when either the Coriolis parameter or the wave frequency changes sign. According to the bulk-interface correspondence, the number of waves that traverse the otherwise forbidden region in the frequency space is the change in the winding number, which is 2 in this case.

Spectral flow in frequency-wavevector space as the zonal wavenumber increases shows that the negative frequency Poincaré band loses two modes, the geostrophic band gains and loses one mode and the positive frequency Poincaré band gains the two modes. These are the equatorial Kelvin and Yanai waves (The Yanai waves are also called mixed Rossby-gravity waves). The two equatorial modes move with an eastward group velocity at all zonal wavenumbers, and this unidirectional propagation is a consequence of the breaking of time-reversal invariance by the planetary rotation.

FIG. 1: Dispersion relation in frequency-wavevector space of the rotating shallow water equations in the $f$-plane approximation as a function of latitude. The upper and lower bands are positive and negative frequency modes of the Poincaré waves, and the color indicates the sign of the winding number of the upper band (blue = -1, red = +1) as shown by the plots of the argument of $\Xi(k_x, k_y)$ in the lower half of the figure (see text). At the equator $f = 0$, the frequency gap vanishes in a Weyl point, and a topological transition occurs (purple) as the two bands invert. The subinertial range has only a zero frequency band (black) containing modes in exact geostrophic balance. The inset shows the dispersion relation on the equatorial $\beta$-plane with the quasi-geostrophic Rossby waves, the Poincaré waves, and the unidirectional Kelvin and Yanai waves. The vectors plot corresponds to the south pole (left) and the north pole (right) respectively. (Figure and caption adapted from Ref. [4].)

III. ROTATING SHALLOW WATER EQUATIONS WITH HORIZONTAL SHEAR

We begin this section by presenting the linearized rotating shallow water equations in the presence of horizontal shear and later consider vertical shear in the continuously stratified primitive equations. (See Chapter 5 on “Zonally symmetry wave – mean interaction theory” of Ref. [49] for relevant background.) Note that $x$ and $y$ are zonal and meridional coordinates respectively. For simplicity we only consider shearing flow moving in the $x$-direction.
\( \mathbf{U}(y) = (U(y), 0) \). We first introduce the following dimensionless quantities:

\[
\hat{t} = 2\Omega t, \quad \hat{\eta} = \frac{\eta}{H}, \quad \hat{H}(y) = 1 + \frac{h(y)}{H}, \quad \hat{u} = \frac{u}{c}, \quad \mathbf{U} = \frac{U}{c}, \quad \hat{f}(y) = \frac{f(y)}{2\Omega}, \quad \hat{x} = \frac{x}{L_d}.
\] (2)

where \( c = \sqrt{gH} \) is the gravity waves speed in nonrotating shallow water equations, \( \Omega \) is the planet rotation rate, \( H \) is zonally averaged depth in the absence of shear, and \( L_d = c/2\Omega \) is the global Rossby radius of deformation. Note that we assume \( L_d \) is much smaller than the domain width, which allows us to treat the two equators independently. In terms of these nondimensionalized quantities and dropping the tildes for clarity, the shallow water equations after linearization and non-dimensionalization are given as follows (see Appendix A for the derivation):

\[
\begin{align*}
&\partial_t u + U(y)\partial_x u + v\partial_y U(y) + \partial_x \eta - f(y)v = 0, \\
&\partial_t v + U(y)\partial_x v + \partial_y \eta + f(y)u = 0, \\
&\partial_t \eta + H(y)(\partial_x u + \partial_y v) + v\partial_y H(y) + U(y)\partial_x \eta = 0,
\end{align*}
\] (3)

where \( u, v \) are respectively the \( x \) and \( y \) components of fluid velocity in the horizontal directions, \( f(y) \) is the Coriolis parameter, \( H(y) \) is the mean layer depth and \( \eta \) is the fluctuation in the depth about this mean; thus the total layer depth is given by \( h = H(y) + \eta \). Note that \( H \) here is a function of \( y \) due to the balance with the horizontal shear flow (see Eq. (5) below).

We now further specialize to the case of a background basic shear flow that oscillates sinusoidally in the \( y \)-direction:

\[
U(y) = U_0 \sin \left( \frac{2\pi y}{\Lambda} \right), \tag{4}
\]

where \( U_0 \) is the magnitude of the shear flow measured in units of \( c \equiv \sqrt{gh} \) and \( \Lambda \) is the wavelength of the shear. Note that linear shear \( U(y) \propto y \) is incompatible with the periodic boundary conditions that we adopt in the following to eliminate any boundaries from the bulk problem that would confuse the application of the bulk-interface correspondence principle, as the only boundaries that we consider here are those located where the Coriolis parameter vanishes. Geostrophically balancing the basic flow then determines the mean depth \( H(y) \), which satisfies:

\[
\frac{\partial H(y)}{\partial y} = -f(y)U(y). \tag{5}
\]

In the \( f \)-plane approximation \( f(y) = f_0 \) for a constant \( f_0 \) and the mean depth is:

\[
H(y) = 1 + \frac{U_0 f_0 \Lambda}{2\pi} \cos \left( \frac{2\pi y}{\Lambda} \right). \tag{6}
\]

### A. Waves on a planet with two equators

To investigate whether or not bulk-interface correspondence continues to hold in the presence of horizontal shear, we first examine the dispersion relation of shallow water waves in the presence of both rotation and shear. The wave frequencies are found numerically with the open-source pseudo-spectral \texttt{Dedalus} package [50]. We employ \( N_y = 61 \) spectral modes in the \( y \)-direction, sufficient to resolve the waves and odd in number so that symmetry about \( y = 0 \) can be preserved. We check that increasing the resolution \( N_y \) does not change the frequencies significantly, including the Rossby wave frequency and the dispersion of the geostrophic modes. We choose

\[
f(y) = \sin \left( \frac{2\pi y}{L_y} \right), \tag{7}
\]

and set \( L_y = 4\pi \) where \( L_y \) is the width of the periodic domain (Fig. 2). This choice respects the periodic boundary conditions and is sometimes called “a planet with two equators” as the Coriolis parameter changes sign twice across the domain [1].

Assuming the sinusoidal horizontal shear Eq. (4), which is antisymmetric about the equator located at \( y = 0 \), has the same periodicity as the domain size (\( \Lambda = L_y \)) the mean depth is:

\[
H(y) = 1 + U_0 \left[ \frac{L_y}{8\pi} \sin \left( \frac{4\pi y}{L_y} \right) - \frac{y}{2} \right]. \tag{8}
\]
Similarly, if the shear is symmetric about the equator at $y = 0$, namely
\[ U(y) = U_0 \cos \left( \frac{2\pi y}{L} \right), \] (9)
from geostrophic balance, the mean depth is:
\[ H(y) = 1 + \frac{U_0 L_y}{8\pi} \cos \left( \frac{4\pi y}{L_y} \right). \] (10)
We consider both profiles in the following.

In the absence of shear, Fig. 2(a), equatorial Kelvin waves and Yanai waves appear in the gap between the high-frequency Poincaré and low-frequency planetary waves. These waves have a topological origin [1]. They propagate unidirectionally (their group velocity does not change sign for all $k_z$), as guaranteed by topology. Note that while the wave crest of the Rossby wave indeed always has a westward component, its group velocity can be both directions, as can be seen from the wave dispersion in Fig. 2(a). As there are two oppositely-oriented equators, there are both eastward and westward propagating modes localized respectively at each equator. When shear $U_0 \neq 0$ is turned on, the planetary Rossby waves are Doppler shifted and we observe continuous spectra near the zero-frequency [51]. The continuous spectrum spans $\omega = \pm U_0 k_x$. The dispersion of the Poincaré modes also changes with increasing $k_z$; see Figs. 2(b) and (c). The Kelvin and Yanai waves remain localized near the equators. We have also investigated spectra with larger values of $U_0$ and find that the Kelvin and Yanai waves persist so long as $U_0$ is not too large. If $U_0$ is too large, the bulk bands and the boundary modes become difficult to distinguish due to significant changes in the frequency of the bulk modes and the large Doppler shift of the planetary waves, especially in the case of the sine shear flow. We show below that the continued presence of the waves is consistent with the persistence of bulk-interface correspondence in the presence of shear.

**B. Bulk waves on the f-plane**

We now develop a purely spectral approach to including shear that is amenable to either direct diagonalization or a perturbative expansion. First, we briefly review shallow water waves on the f-plane in the absence of shear flow [1]. We expand the eigenmodes in the plane wave basis, $(u,v,\eta) = \Psi(k_x,k_y,f_0) = \Psi(ik_x x + ik_y y - i\omega t)$. In this basis, the linear wave operator is a $3 \times 3$ matrix:
\[ L_0(k_x,k_y,f_0) = \begin{pmatrix} 0 & ik_0 & k_x \\ -ik_0 & 0 & k_y \\ k_x & k_y & 0 \end{pmatrix}. \] (11)
The amplitudes of the normal modes $\Psi_{\pm,0}(k_x,k_y,f_0)$ with frequencies $\omega_{\pm,0}$ can be obtained by diagonalizing $L_0$. The positive Poincaré mode frequency is $\omega_+ = \sqrt{k_x^2 + k_y^2 + f_0^2}$ with the eigenmode:
\[ \Psi_+ = \begin{pmatrix} \frac{k_x}{k} + i\frac{f_0 k_y}{k \sqrt{k_x^2 + f_0^2}} \\ \frac{k_x}{k} - i\frac{f_0 k_y}{k \sqrt{k_x^2 + f_0^2}} \\ \frac{1}{\sqrt{k_x^2 + f_0^2}} \end{pmatrix}, \] (12)
where $k \equiv \sqrt{k_x^2 + k_y^2}$. A highly degenerate geostrophically-balanced mode appears at zero frequency, $\omega_0 = 0$ (the degeneracy is lifted when the Coriolis parameter varies with latitude or in the presence of shear):
\[ \Psi_0(k_x,k_y,f_0) = \frac{1}{\sqrt{k_x^2 + f_0^2}} \begin{pmatrix} -ik_y \\ ik_x \\ f_0 \end{pmatrix}. \] (13)
Finally the negative Poincaré mode has angular frequency $\omega_- = -\omega_+$ with corresponding wavefunction $\Psi_-(k_x,k_y,f_0) = \Psi_+(-k_x,-k_y,-f_0)$ reflecting the fact that the wave amplitudes in real space are real-valued.

For Poincaré-gravity waves, the gauge-invariant quantity displays a vortex or antivortex (depending on the signs of
FIG. 2: Numerical evaluation of the frequency-wavenumber dispersion of the linearized shallow water equations obtained with Dedalus with $N_y = 61$ spectral modes showing the spectral flow of the Kelvin and Yanai waves between bands. Colors show the projected real space position and $y^* = \langle \Psi | y | \Psi \rangle / L_y$. (a) No shear; (b) Imposed sine shear (Eq. (4)) with $U_0 = 0.2$, and (c) Cosine shear (Eq. (9)) with $U_0 = 0.2$. The Coriolis parameter varies sinusoidally (Eq. (7)) and changes sign at $y = 0$ ($y^* = -1$) and $y = \pm L_y / 2$ ($y^* = 1$). We set $L_y = 4\pi$. Black solid lines represent the frequency of the $k_y = 0$ Poincaré modes in the absence of shear and in the $f$-plane approximation $f = 1$: $\omega = \pm \sqrt{k_x^2 + f^2}$. Colors represent the proximity of the band wavefunctions to the two equators.

the frequency and the Coriolis frequency) centered at the origin in wavevector space:

$$\Xi_\pm(k_x, k_y) = \frac{k_y - i \text{sgn}(f_0) k_x}{f_0},$$

where we use the long-wavelength approximation $k^2 \ll f_0^2$. The vortex / antivortex has winding number $\pm 1$ which constitutes its topological charge. Representing the phase of $\Xi$ with an arrow makes these patterns evident as shown
in Figure 1. The zero-frequency geostrophic mode, by contrast, has in the same limit

$$\Xi_0(k_x, k_y) = \frac{ik_x}{f_0},$$

and thus has a domain wall at $k_x = 0$ and zero winding number. Its topological charge therefore vanishes.

### C. Horizontal Shear Flow on the f-plane

In the presence of shear flow the system is no longer translationally invariant along the $y$-direction. While the linear wave operator can still be expressed as a matrix in wavevector space, it is no longer composed of $3 \times 3$ block matrices along the diagonal. We first rewrite Eq. (3) in position space in the form of a matrix of differential operators,

$$\hat{L}(x, y, f_0, U_0) = i \begin{pmatrix} U(y)\partial_x & \frac{\partial U(y)}{\partial y} - f_0 & \partial_x \\ f_0 & U(y)\partial_y & \partial_y \\ H(y)\partial_x & H(y)\partial_y & \partial_x \end{pmatrix}.$$ (16)

This linear operator preserves the parity-time (PT) symmetry despite the broken Hermiticity, and spontaneous PT-symmetry breaking has been known to lead to instabilities [52, 53]. However, note that if the shear flow has dependence on both $x$ and $y$ (i.e., $U(x, y)$), PT symmetry would be broken. Substituting in the sine shear flow $U(y)$ from Eq. (4) with $H(y)$ satisfying the geostrophic balance in Eq. (5), we obtain

$$\hat{L}(x, y, f_0, U_0) = i \begin{pmatrix} \frac{U_0 \sin \left(\frac{2\pi y}{\Lambda}\right) \partial_x}{f_0} & \frac{2\pi U_0 \cos \left(\frac{2\pi y}{\Lambda}\right) - f_0 \partial_x}{2\pi} & \partial_y \\ \left[1 + \frac{U_0 f_0 \Lambda}{2\pi} \cos \left(\frac{2\pi y}{\Lambda}\right)\right] \partial_x & \left[1 + \frac{U_0 f_0 \Lambda}{2\pi} \cos \left(\frac{2\pi y}{\Lambda}\right)\right] \partial_y - U_0 f_0 \sin \left(\frac{2\pi y}{\Lambda}\right) & \partial_x \end{pmatrix}.$$ (17)

Note that the linear wave operator has a $y$-dependence, which means that when expanding $H$ in wavevector space, different modes with different $k_y$’s can mix. Without the loss of generality, we assume $\Lambda = 1$. We can consider the simplest case where there are only three modes, $k_y, k_y = \pm 2\pi$, in the basis. In this case, the full linear wave operator is a $9 \times 9$ matrix that can be decomposed into $3 \times 3$ blocks, which can be formally represented as follows,

$$\mathcal{L}_{9 \times 9}(k_x, k_y, f_0, U_0) = \begin{pmatrix} L_0(k_x, k_y + 2\pi, f_0) & T_1(k_x, k_y, f_0, U_0) & 0 \\ T_2(k_x, k_y + 2\pi, f_0, U_0) & L_0(k_x, k_y, f_0) & T_1(k_x, k_y - 2\pi, f_0, U_0) \\ 0 & T_2(k_x, k_y, f_0, U_0) & L_0(k_x, k_y - 2\pi, f_0) \end{pmatrix},$$ (18)

where $L_0$ is given in Eq. (11) and $T_1$ and $T_2$ are the transition matrices between modes:

$$T_1(k_x, k_y, f_0, U_0) = \langle k_x, k_y + 2\pi | \hat{L} | k_x, k_y \rangle$$

$$= \frac{U_0}{2} \begin{pmatrix} ik_x & 0 & 2\pi i \\ 0 & 0 & 0 \\ f_0 \frac{k_y}{2\pi} + k_0 f_0 & f_0 & ik_x \end{pmatrix},$$

$$T_2(k_x, k_y, f_0, U_0) = \langle k_x, k_y - 2\pi | \hat{L} | k_x, k_y \rangle$$

$$= \frac{U_0}{2} \begin{pmatrix} -ik_x & 0 & 2\pi i \\ 0 & 0 & 0 \\ f_0 \frac{k_y}{2\pi} - k_0 f_0 & f_0 & -ik_x \end{pmatrix}.$$ (19)

The derivation of $T_1$ and $T_2$ can be found in Appendix B. The matrix $T_1(k_x, k_y, f_0, U_0)$ connects wavenumber $k_y$ to $k_y + 2\pi$ and $T_2(k_x, k_y, f_0, U_0)$ connects $k_y$ to $k_y - 2\pi$. Note that $T_1 \neq T_2^\dagger$ and the linear wave operator is non-Hermitian. The frequency spectrum and the eigenvectors can then be obtained by diagonalizing the full matrix $\mathcal{L}(k_x, k_y, f_0, U_0)$. We validate our results by comparing our spectra with the ones obtained with Dedalus [50] in Appendix C.
D. Perturbative treatment of shear

We also consider a perturbative expansion of the eigenfunctions/values in powers of the shear [54–56]. We may treat the shear flow perturbatively by considering the quantity \( \delta L = L - L_0 \), namely the off-diagonal blocks in Eq. (18). The correction to the frequency of the Poincaré mode first appears at second order in the shear:

\[
\omega_n = \omega_n^{(0)} + \sum_{m \neq n} \frac{\delta L_{mn} \delta L_{nm}}{\omega_n^{(0)} - \omega_m^{(0)}},
\]

where \( \delta L_{mn} = \langle m | \delta L | n \rangle \), and \( m \) and \( n \) are indices that label a wavevector state with some \( k_y \). The wavefunctions including the first-order correction are given as follows:

\[
|n\rangle = |n^{(0)}\rangle + \sum_{m \neq n} \frac{\delta L_{mn}}{\omega_n^{(0)} - \omega_m^{(0)}} |m^{(0)}\rangle,
\]

where \( |n^{(0)}\rangle \) and \( |m^{(0)}\rangle \) are unperturbed wavefunctions corresponding to some \( k_y \). The perturbed eigenmodes are still labelled by wavevector \((k_x, k_y)\) despite the fact that they contain contributions from modes at other \( k_y \). To second order in the shear \( U_0 \), the frequencies only involve intermediate modes at wavevectors \((k_x, k_y \pm 2\pi)\); higher orders of perturbations involve increasing departures of the wavenumber away from \( k_x = 0 \). Figure 3 compares the frequency obtained from full diagonalization of the \( 9 \times 9 \) linear wave operator to the spectrum from second-order perturbation theory. The two spectra agree well with each other. Through comparing the perturbative spectrum and the full diagonalization, we show that firstly, the bulk can be classified by \((k_x, k_y)\) and secondly, the change of the bulk Poincaré wave is smooth as a function of \( U_0 \). Therefore, we argue that using the bulk-interface correspondence is valid despite the broken Hermiticity. As discussed later in Section VI, the first and second order perturbative corrections to the wavefunctions do not alter their topological properties.

E. Wave dynamics

Figures 4 and 5 show snapshots of the propagation of wavenumber 2 \( (k_x = 4\pi / L_x) \) Kelvin and Yanai waves subjected to sine and cosine shear. The waves remain localized near the \( y = 0 \) equator as they propagate. The wave amplitude grows in time with the sine shear (Figs. 4 (b) and (d)) and decays in time with the cosine shear (Figs. 5 (b) and (d)), consistent with the imaginary part of the frequency eigenvalues that correspond to growth and decay respectively for the two types of the shear. Note that since the sine shear is odd in \( y \), the Kelvin wave also becomes asymmetric in \( y \) as time evolves (Fig. 4(b)).
FIG. 4: Time evolution of the $\eta$-component of (a, b) the Kelvin wave and (c, d) the Yanai wave for sine shear with $U_0 = 0.1$, $N_y = 121$, $N_x = 71$, $L_y = 20\pi$, $L_x = 10$.

FIG. 5: Time evolution of the $\eta$-component of (a, b) the Kelvin wave and (c, d) the Yanai wave for cosine shear with $U_0 = 0.1$, $N_y = 121$, $N_x = 71$, $L_y = 20\pi$, and $L_x = 10$.

IV. PRIMITIVE EQUATIONS WITH AND WITHOUT SHEAR

We turn next to the continuously stratified primitive equations. It has been shown that non-rotating stratified fluids with profiles of stratification that transition with increasing depth from marginally unstable to stable have a wave of topological origin along the interface [57]. We make the standard Boussinesq approximation, and the vertical
velocity or variation in the buoyancy replaces the depth as one of the dynamical fields.

We first analyze the topological character of the linear stratified equations in the absence of shear by calculating the topological invariant within the bulk $f$-plane approximation. The linearized and non-dimensionalized equations can be derived from the underlying hydrostatic equations (see Appendix D for the detailed derivation):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -U(y) \frac{\partial u}{\partial x} - v \frac{\partial U(y)}{\partial y} + f(y) v - \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} &= -f(y) u - U(y) \frac{\partial v}{\partial x} - \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} &= -w - U(y) \frac{\partial^2 \eta}{\partial x \partial z},
\end{align*}
\]

where $w$ is the vertical velocity and the vertical depth variation $\eta$ and the buoyancy $b$ are related by the diagnostic relationship $\partial_z \eta = b$.

On the $f$-plane it is again natural to switch to a basis of plane waves which decouples the modes with different wavenumber in the $z$-direction, $k_z$. The incompressibility constraint in this basis takes the form $\nabla \cdot u = i(k_x u + k_y v + k_z w) = 0$ permitting the replacement of $w$ and $\eta$ with $b$, $u$ and $v$. In the absence of shear, Eq. (22) now corresponds in this basis to the linear wave operator

\[
L_0 = \begin{pmatrix}
0 & -i \frac{k_z}{k_x} & \\
-i \frac{k_z}{k_x} & 0 & -i \frac{k_z}{k_x} \\
\frac{k_z}{k_x} & \frac{k_z}{k_x} & 0
\end{pmatrix}.
\]  

(23)

The eigenfrequencies of Eq. (23) are $\omega_\pm = \pm \sqrt{f_0^2 + k^2/k_x^2}$ and $\omega_0 = 0$ with corresponding eigenvectors:

\[
\Psi_\pm = \frac{1}{N_1} \begin{pmatrix}
+i k_x k_y \sqrt{f_0^2 k_x^2 + k^2} + f_0 k_z^2 k_y \\
+i k_x k_y \sqrt{f_0^2 k_x^2 + k^2} - f_0 k_z^2 k_x \\
0
\end{pmatrix},
\]

\[
\Psi_0 = \frac{1}{N_2} \begin{pmatrix}
-k_x k_y \\
k_x^2 \\
f_0 k_x k_z
\end{pmatrix},
\]

(24)

where $k^2 = k_x^2 + k_y^2$, and $N_{1,2}$ are normalization constants.

We consider the positive frequency eigenvector at fixed non-zero $k_z$. Letting $f_z = f_0 k_z$ and dividing $\Psi_+$ by $k_z$, we have

\[
\Psi_\pm = \frac{1}{N_3} \begin{pmatrix}
+i k_x \sqrt{f_0^2 + k^2} + f_z k_y \\
+i k_y \sqrt{f_0^2 + k^2} - f_z k_x \\
0
\end{pmatrix},
\]

(25)

where $N_3$ is the new normalization constant. The Berry connection is

\[
\text{Im}(\Psi_+ | \nabla_K | \Psi_+) = (-2 f_z k_y \sqrt{f_0^2 + k^2}, 2 f_z k_x \sqrt{f_0^2 + k^2}, 0),
\]

(26)

where $K = (k_x, k_y, f_z)$. The result is the same as the Berry connection of the positive Poincaré mode of the shallow water equations [1]. The difference of the Chern number between the two hemispheres, $\Delta C_\pm$, can be calculated analytically by integrating the Berry curvature over the unit sphere in $(k_x, k_y, f_z)$ space. For the Poincaré modes, the difference $\Delta C_\pm = \pm 2$. By bulk-interface correspondence, for each $k_z$, there are two pairs of boundary Kelvin and Yanai modes (one pair each for the two oppositely oriented equators). These stacks of boundary modes are analogous to the edge modes found in weak three-dimensional topological insulators [45].

With sinusoidal horizontal shear flow, the eigenmodes of Eq. (22) can be obtained by the methods outlined in Section III.C. The method again shows excellent agreement with the result obtained from Dedalus (not shown in the paper).
With a vertical shear flow, the primitive equations are modified to be the following:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -U(z)\frac{\partial u}{\partial x} - f(y)v - \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} &= -f(y)u - U(z)\frac{\partial v}{\partial x} - \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial x} &= -w - U(z)\frac{\partial^2 \eta}{\partial x \partial z}.
\end{align*}
\]

(27)

In this work, we consider a linear vertical shear flow, namely, \( U(z) = U_0z \). We numerically simulated the spectra in the \((y, z)\) space using Dedalus and is shown in Fig. 6 with \( U_0 = 0.05 \). Similar to Fig. 2, we use a sinusoidal Coriolis parameter \( f(y) = \sin(2\pi y/L_y) \) with \( L_y = 10\pi \). Both Yanai and Kelvin waves are present at different values of vertical wavenumber \( k_z \).

![Image of spectral flow](image)

**FIG. 6:** The spectral flow of Kelvin and Yanai waves between the band gaps exhibited by the linearized primitive equations with a linear vertical shear \( U_0 = 0.01 \) obtained from Dedalus with \( N_y = 24, N_z = 24, L_y = 10\pi, L_z = 2\pi \) using Fourier basis. The vertical wavenumber is (a) \( k_z = 1 \); (b) \( k_z = 2 \); and (c) \( k_z = 3 \). The solid black lines are the dispersion relation for the f-plane approximation with \( f = 1 \), Eq. (24). As in Fig. 2, the color indicates proximity to the two equators. The missing scattered points are due to the difficulty in separating out the modes that correspond to different vertical wavenumbers \( k_z \).

### V. SHEAR INDUCED INSTABILITY

To investigate the stability of the waves in the presence of horizontal shear we follow Ref. [58]. Introducing the background potential vorticity \( Q(y) = \frac{f(y) - \partial y U(y)}{H(y)} \), perturbations are bounded if there exists some constant \( \alpha \in \mathbb{R} \) such that the following two conditions hold for all \( y \in [-L_y, L_y] \):

1. \( |\alpha - U(y)| \partial_y Q(y) \geq 0 \) and \( |\alpha - U(y)|^2 \leq H(y) \).

For the sine horizontal shear flow condition (ii) can be satisfied, but condition (i) requires that the function \( g(U_0, y) = U_0 \sin(2\pi y) \) to be greater or equal to zero over the entire domain, but this condition is violated for any \( U_0 \neq 0 \). The analysis is similar with a cosine shear. Thus the bulk modes are always unstable in the presence of horizontal shear.

We numerically confirm the instability of the bulk modes by presenting the imaginary part of the frequency spectrum in Fig. 7. When \( U_0 \neq 0 \), the spectrum has a non-zero imaginary part that grows linearly in \( U_0 \) for small shear. The instability is most prominent in the planetary Rossby modes.

In the presence of the linear vertical shear with rigid-lid boundaries, since the derivative of \( U(z) \) has the same sign at the upper and lower boundaries, Eady instabilities are present at low wavenumbers [5]. We numerically verified the presence of Eady instabilities by simulating the primitive equations and observed that the spectra are unstable at low wavenumbers.

Despite the presence of instabilities with both horizontal and shear flows, the gauge-invariant phase is a robust method of quantifying the topological nature of the system.
VI. NUMERICAL CALCULATION OF BULK WINDING NUMBERS

For Hermitian systems, bulk-interface correspondence \cite{10, 59} establishes a relationship between the topological invariant, the Chern number of the bulk, and the number of edge modes. It states that the difference in the number of counterpropagating edge modes equals the difference in the Chern number in two bulk regions that are connected at a boundary: \[ \Delta C = n_L - n_R, \]
where \( n_L \) and \( n_R \) are the number of left-moving and right-moving modes.

The Chern number can be calculated analytically for the rotating shallow water equations. Each of the 3 bands may be parametrized on the unit sphere \( (k_x, k_y, f) \). The Chern number may then be found by integrating the Berry curvature over the surface of the sphere with a fixed radius \( \sqrt{k_x^2 + f^2} \) \cite{1}.

In the presence of shear, however, the linear wave operator is no longer Hermitian, and a rigorous bulk-interface correspondence principle is not in hand. We may still investigate the topological properties of the bulk wavefunctions and compare with the boundary mode spectrum to test whether or not bulk-interface correspondence continues to operate. However, the presence of shear breaks translational invariance in the \( y \)-direction and the integral of the Berry curvature becomes difficult to evaluate. As an alternative, we instead look for singularities in the phase of the wavefunctions which appear as vortices in wavevector space \cite{60}. In the context of polarization physics, it has been shown that the winding of the polarization azimuth, or the wavefunction phase, equals the enclosed Chern number \cite{61, 62}. We set the Coriolis parameter such that it is in the bulk (namely, it does not change sign), and examine the phase of the wavefunctions in \( (k_x, k_y) \) to check whether there is a vortex or antivortex in the phase.

A. Spatially varying Coriolis parameter

Delplace et al. \cite{1} uses an f-plane approximation to analytically calculate the Chern number to show the nontrivial topology of the equatorial waves. However, realistically, Coriolis parameter is a function of the latitude and translational invariance is always broken in the bulk. Here, we first verify that translational invariance in the bulk is not required. To do this we preserve Hermiticity by considering a spatially varying Coriolis parameter in absence of the shear flow and find the winding number of the Poincaré modes. We choose

\[ f(y) = f_0 + \Delta f \sin\left(\frac{2\pi y}{L_y}\right), \]

so that we may adapt the formalism introduced in Eq. (18) to write the linear wave operator in wavevector space.
with transition blocks:

\[ T_1(\Delta f) = \frac{\Delta f}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2(\Delta f) = T_1(k_x, k_y, \Delta f)^T. \]  

(29)

By diagonalizing the linear wave operator, we can obtain the spectrum of shallow water equations with the \( y \)-dependent \( f(y) \). We choose \( \Delta f \) and \( f_0 \) such that \( f(y) \) does not change sign anywhere; thus we remain in the bulk and no edge modes should arise. Figure 8 shows the bulk spectrum with \( \Delta f = 0.5 \) and \( f_0 = 1 \). Frequencies obtained by diagonalization (Fig. 8(a)) in wavevector space agree with those obtained with Dedalus (Fig. 8(b)) and confirm that there are no Kelvin or Yanai waves.

![Figure 8](image_url)

**FIG. 8**: Numerical calculation of the bulk eigenfrequencies for the spatially varying Coriolis parameter. (a) Diagonalization of the \( 69 \times 69 \) wavevector space linear wave operator. (b) Dedalus with \( N_y = 23 \) for \( \Delta f = 0.5, f_0 = 1 \) and \( L = 4\pi \). Black dotted lines in (a) and solid lines in (b) represent the frequency of Poincaré modes in the \( \xi \)-plane approximation with \( f_0 = 1 \):

\[ \omega = \pm \sqrt{k_x^2 + f_0^2}. \]

Colors in (b) indicate proximity to the two oppositely oriented equators.

### B. Gauge invariant phase

We proceed to calculate the topological index of the bands by searching for singularities in the phase of the frequency eigenfunctions in wavevector space. The eigenfunctions have gauge freedom, as the phase of the three components can be rotated together by an arbitrary amount \( \phi(k) \) at each point in wavevector space:

\[ \Psi_{\pm,0}(k) \rightarrow e^{i\phi(k)}\Psi_{\pm,0}(k). \]  

(30)

As mentioned previously in Section II we remove the gauge redundancy by multiplying the \( \nu \)-component of the Poincaré modes by the complex conjugate of the \( \eta \)-component, \( \eta^*(k) = \eta(-k) \):

\[ \Xi_{\pm}(k) \equiv \nu_\pm(k) \eta_\pm(-k) \]  

(31)

leaving only the internal phase difference between the two amplitudes. Figure 9 depicts the argument of \( \Xi_{\pm}(k) \), \( \tan^{-1}(\text{Re}(\Xi)/\text{Im}(\Xi)) \), of the positive Poincaré modes as a function of \( k_x \) and \( k_y \) for the spatially varying Coriolis parameter of Eq. (28) where the eigenmodes are obtained by diagonalizing the \( 69 \times 69 \) linear operator. The positive Poincaré bands exhibit respectively a vortex and an anti-vortex centered at the origin in wavevector space where the phase cannot be uniquely defined for positive and negative \( f_0 \) respectively. The difference in the winding number between the two bands equals 2. The difference in the winding number for either Poincaré band changes by 2 going between the two hemispheres. The planetary waves have no vortex as expected (Fig. 10).

By virtue of the single-valuedness of \( \Xi_{\pm}(k) \), the winding number must be integer-valued and thus topological in character. Unlike the calculation of the Chern number which is found by integrating the Berry curvature over wavevector space, no integrals are required for the calculation of the winding number, and the non-compact nature of wavevector space for continuous fluids does not cloud its interpretation.

The Chern number equals the negative of the total winding within a closed domain so \( \Delta C = -\nu_+ - \nu_- \), where \( \nu_{\pm} \) is the winding number of the positive/negative frequency Poincaré mode and a vortex/anti-vortex corresponds to a winding number \( \pm 1 \) [61]. Thus \( \Delta C_+ = -2 \) for \( f_0 > 0 \) and \( \Delta C_- = 2 \) for \( f_0 < 0 \), in agreement with the Chern
numbers found for the f-plane [1]. By bulk-interface correspondence [10, 59], the difference in the number of prograde and retrograde moving edge modes at the equatorial interface where \( f \) changes sign equals the change in the Chern number \( \{ \Delta C_+, \Delta C_0, \Delta C_- \} \), consistent with 2 modes of topological origin localized near each equator. The localized Yanai and Kelvin waves in Fig. 2(a) thus have their origin in topology, just as they do for the shallow water equations using an f-plane approximation. [1].

C. Sinusoidal horizontal shear

Next we find the winding number of the Poincaré modes in the shallow water equations subjected to the sinusoidal horizontal shear. Figure 11 shows the phase of \( \Xi \pm (k) = v \pm (k) \eta \pm (-k) \) for \( U_0 = 0.3 \) and constant Coriolis parameter \( f_0 = \pm 1 \) showing qualitatively similar vortices as those in Fig. 9. Again the positive frequency Poincaré modes exhibit a vortex for \( f_0 > 0 \) and an anti-vortex for \( f_0 < 1 \) at the origin in wavevector space (the phase singularity is absent for the planetary modes). The change in the winding number of 2 is consistent with the number of edge modes seen in the spectrum (Fig. 2(b) and (c)). This result suggests that the localized Kelvin and Yanai modes that traverse the gap between Rossby modes and the bulk Poincaré modes have a topological origin like the equatorial modes in the absence of shear.
$U_0 = 0.3$

FIG. 11: Arrows representing argument of $\Xi_\pm(k) = v_\pm(k)\eta_\pm(-k)$ of the lowest positive frequency Poincaré modes as indicated by the direction of the arrows for the case of sinusoidal horizontal shear $U_0 = 0.3$ within the $f$-plane approximation for $f_0 = -1$ (left) and $f_0 = 1$ (right) with $L_y \to \infty$. The length of the arrows is rescaled to be equal. Colors represent normalized magnitude $|\Xi|$ in arbitrary units.

This is the main result of the paper, and we note that the result also holds in perturbation theory with the $9 \times 9$ linear wave operator, as the perturbative corrections to the wavefunction do not alter the winding number. The appearance of Kelvin and Yanai waves along the equators shown in Section IIIA is thus consistent with the persistence of the bulk-interface correspondence in the presence of shear.

Finally, we study the phase of the gauge-invariant quantity $\Xi_\pm(k)$ for the linearized primitive equations with and without forcing from sinusoidal horizontal shear. The phase singularity of $\Xi_\pm(k)$ for primitive equations (not shown) is similar to that depicted in Figs. 9 and 11. Without shear, the positive and negative frequency Poincaré modes have opposing winding numbers, and the winding number also changes polarity when $f$ changes sign in agreement with the analytic calculation of the Chern number. The vortex of the bulk Poincaré modes continues to be robust in the presence of shear, despite the combined effects of broken translational invariance, non-Hermiticity, and instability. We have verified that the dispersion relation of the shear-forced primitive equations on the planet with two equators continues to exhibit spectral flow of the Kelvin and Yanai waves across the band gaps.

D. Linear vertical shear

We proceed to calculate the winding number of the Poincaré modes in the primitive equators subject to the linear vertical shear flow. Primitive equations with a vertical shear flow $U(z)$ are given as Eq. (27), which we simulate using Dedalus. Figure 12 shows that in the presence of linear vertical shear flow, the bulk Poincaré mode exhibits phase singularity, and the winding number depends on the sign of the Coriolis parameters for all vertical wavenumbers $k_z$'s. This suggests that the Yanai and Kelvin waves in Fig. 6 are topologically nontrivial. Note that while Fig. 12 is obtained with a Fourier basis and thus the effect of the rigid-lid boundary is removed, we verified that the winding numbers are similar to Fig. 12 with a no-slip boundary condition using a Chebyshev basis for each coefficient. Therefore, the topological nature of the boundary waves is robust against the presence of the Eady instability.

VII. DISCUSSION AND CONCLUSION

We investigated the topological properties of rotating shallow water equations and stratified primitive equations in the presence of shear flow that breaks translational invariance in the meridional direction and Hermiticity and introduces instabilities. The winding number of the phase of $\Xi_\pm(k)$ serves as a convenient probe of topological properties of the wavefunctions. This alternative to calculating the Chern number remains computationally tractable in the absence of translational invariance and Hermiticity. It may find application to experimental and observational data as
FIG. 12: Arrows representing argument of $\Xi_\pm (k) = v_\pm (k) \eta_\pm (-k)$ of a positive Poincaré mode as indicated by the direction of the arrows for the case of a linear vertical shear flow with $U_0 = 0.05$ within the f-plane approximation for $f_0 = -1$ (left) and $f_0 = 1$ (right). (a) $k_z = 1$ (b) $k_z = 2$ (c) $k_z = 3$. Colors represent normalized $|\Xi|$ in arbitrary units. Obtained using Dedalus with $N_z = 20$ and $L_z = 2\pi$ using a Fourier basis.

Our main result is that the winding number for both the shallow water equations and primitive equations remains unchanged in the presence of forcing by background shear flow. The difference in the winding number of the Poincaré well as to idealized theoretical models such as those studied here, as it can be obtained from the (usually neglected) phase information of the cross-periodogram between different fields such as the zonal velocity and geopotential height. To verify that the method yields sensible results, we studied the bulk modes in the presence of a spatially varying Coriolis parameter that does not change sign and demonstrated consistency with the standard calculation of the Chern number on the f-plane [1]. An alternative and equivalent way of quantifying the topological invariant is through the spectral index by counting the number of upward-going eigenvectors for increasing momentum, which is useful when we have access to the wavefunction [46, 63].
bands on opposite sides of the equator matches with the number of unidirectional waves localized at the equator, consistent with a topological origin for these forced Kelvin and Yanai waves. For the stratified primitive equations, there are topologically protected modes at each allowed vertical wavenumber in analogy to the physics of weak three-dimensional topological insulators.

We note that we do not rigorously prove the bulk-interface correspondence for the shear flows, nor topological protection. However, we show that the bulk spectrum in f-plane approximation evolves smoothly with increasing $U_0$ and the phase singularities persist in both the numerically found eigenmodes and in low-order perturbation theory, at least if $U_0$ is not too large. It may be possible to generalize the approach taken in Ref. [64] for frictionally damped shallow water waves to the problem of background shear. That system, and the problems investigated here, are invariant under the combined operation of parity and time-reversal (PT). We leave this, and an investigation of the maximum shear that will support equatorial waves of topological origin, for future work.

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Appendix A LINEARIZED SHALLOW WATER EQUATIONS IN THE PRESENCE OF HORIZONTAL SHEAR

We begin with the nonlinear shallow-water equations in the presence of rotation:

\[
\begin{align*}
\frac{\partial \mathbf{u}_{\text{tot}}}{\partial t} + (\mathbf{u}_{\text{tot}} \cdot \nabla) \mathbf{u}_{\text{tot}} &= -g \nabla h - \mathbf{f} \times \mathbf{u}_{\text{tot}}, \\
\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}_{\text{tot}}) &= 0,
\end{align*}
\]

where $\mathbf{u}_{\text{tot}} = \mathbf{u} + \mathbf{U}$, $\mathbf{u} = (u, v)$, $\mathbf{U} = (U(y), 0)$ is the shear flow along the zonal direction, $\mathbf{f} = f(y)\hat{z}$ is the Coriolis parameter, and $h = \eta + H(y)$. To the linear order, Eq. (32) can be written as follows,

\[
\begin{align*}
\frac{\partial u}{\partial t} + U(y) \frac{\partial u}{\partial x} + v \frac{\partial U(y)}{\partial y} + g \frac{\partial \eta}{\partial x} - f(y)v &= 0, \\
\frac{\partial v}{\partial t} + U(y) \frac{\partial v}{\partial x} + g \frac{\partial \eta}{\partial y} + f(y)u &= 0, \\
\frac{\partial \eta}{\partial t} + H(y) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + v \frac{\partial H(y)}{\partial y} + U(y) \frac{\partial \eta}{\partial x} &= 0.
\end{align*}
\]

A deformation length scale $L_d$ and gravity wave speed $c$ are defined to be:

\[
L_d \equiv \frac{c}{2\Omega}, \quad c \equiv \sqrt{gH},
\]

where $H$ is the zonally averaged depth without shear ($H(y) = H + h(y)$). Introducing the dimensionless quantities $\tilde{t} = 2\Omega t$, $\tilde{\eta} = \frac{\eta}{H}$, $\tilde{H}(y) = 1 + \frac{h(y)}{H}$, $\tilde{\mathbf{u}} = \frac{u}{v}$, $\tilde{\mathbf{U}} = \frac{U}{v}$, $\tilde{f}(y) = \frac{f(y)}{2\Omega}$, and $\tilde{x} = \frac{x}{L_d}$, the linearized equations of motion (Eq. (33)) around the basic state ($u = 0, h = H$) can then be written as follows:

\[
\begin{align*}
\partial_t \tilde{u} + \tilde{U}(y) \partial_{\tilde{z}} \tilde{u} + \tilde{v} \partial_{\tilde{y}} \tilde{U}(y) + \partial_{\tilde{z}} \tilde{\eta} - \tilde{f}(y) \tilde{v} &= 0, \\
\partial_t \tilde{v} + \tilde{U}(y) \partial_{\tilde{z}} \tilde{v} + \tilde{v} \partial_{\tilde{y}} \tilde{U}(y) + \tilde{f}(y) \tilde{u} &= 0, \\
\partial_t \tilde{\eta} + \tilde{H}(y) \left( \partial_{\tilde{z}} \tilde{u} + \partial_{\tilde{y}} \tilde{v} \right) + \tilde{v} \partial_{\tilde{y}} \tilde{H}(y) + \tilde{U}(y) \partial_{\tilde{z}} \tilde{\eta} &= 0.
\end{align*}
\]

For convenience, we drop the tilde in the main text.
Appendix B  THE SHALLOW WATER LINEAR WAVE OPERATOR IN WAVEVECTOR SPACE

The matrix elements of the linear wave operator Eq. (16) in wavevector space may be written using Dirac braket notation as $\langle k'_{x}, k'_{y} | \hat{L} | k_{x}, k_{y} \rangle$. Since the linear wave operator has no dependence on $x$ these matrix elements are non-zero only for $k'_{x} = k_{x}$. Along the $y$-direction, we make use of the following relations,

$$\frac{1}{L_{y}} \int_{-L_{y}/2}^{L_{y}/2} dy \sin \left( \frac{2\pi y}{L_{y}} \right) e^{i(k'_{y} - k_{y})y} = \frac{1}{2i} \left[ \delta_{k'_{y}, k_{y} - 2\pi/L_{y}} - \delta_{k'_{y}, k_{y} + 2\pi/L_{y}} \right],$$

and

$$\frac{1}{L_{y}} \int_{-L_{y}/2}^{L_{y}/2} dy \cos \left( \frac{2\pi y}{L_{y}} \right) e^{i(k'_{y} - k_{y})y} = \frac{1}{2} \left[ \delta_{k'_{y}, k_{y} - 2\pi/L_{y}} + \delta_{k'_{y}, k_{y} + 2\pi/L_{y}} \right].$$

(36)

(37)

In the absence of shear ($U_{0} = 0$), the linear wave operator in $k$-space is a block-diagonal matrix, with the diagonal blocks being $L_{0}(k_{x}, k_{y}, f)$ and with no off-diagonal blocks. The $3 \times 3$ linear wave operators $L_{0}$ at wavevectors $(k_{x}, k_{y})$ and $(k_{x}, k_{y} \pm 2\pi)$ are connected by the sinusoidal horizontal shear as a wave at wavevector $k_{y}$ mixes with modes $k'_{y} = k_{y} \pm 2\pi$. For a given $k_{x}$, we need to diagonalize the full matrix in the basis of $k_{y}, k_{y} \pm 2\pi, k_{y} \pm 4\pi, ...$ imposing a finite cutoff in $|k'_{y}|$ to keep the dimension of the matrix finite.

Appendix C  COMPARISON WITH DEDALUS

We validate our diagonalization scheme by comparing with Dedalus [50]. Figure 13 compares the spectra from diagonalizing a $69 \times 69$ linear wave operator corresponding to the 23 retained wavevectors in the $y$-direction with the spectrum obtained from Dedalus. To enable the comparison, the linear wave operator has been truncated to finite dimension in wavenumber space to match the total number of equations in Dedalus. The full diagonalization captures both the spread of the geostrophic modes and the bulk Poincaré modes. Note that the small difference in the geostrophic modes is due to the fact that the sample points along the $y$-direction in Dedalus is non-uniform whereas in the direct diagonalization, $k_{y}$’s are sampled uniformly. Figure 14 compares the positive frequency modes obtained from full diagonalization versus those found using Dedalus. The two methods show an excellent agreement. The frequency of the Poincaré modes increases with increasing shear and remain distinct beyond $U_{0} = 0.6$. We can apply the same procedure to obtain the transition matrices $T_{1}$ and $T_{2}$ for the cosine shear, and the spectra agrees with Figs. 13 and 14, as expected.

FIG. 13: Frequency spectra of the shallow water equations in the f-plane approximation with $f = 1$ and subjected to sine shear $U_{0} = 0.5$. The frequencies are obtained by (a) diagonalizing the $69 \times 69$ wavevector space linear wave operator and from (b) Dedalus with $N_{y} = 23$. 
FIG. 14: Comparison of the frequencies of the positive Poincaré and planetary modes. (a) Full diagonalization of the $69 \times 69$ linear wave operator. (b) Dedalus with $N_y = 23$. (c) The difference between frequencies of the lowest positive Poincaré mode obtained from full diagonalization, $\omega_F$, and Dedalus, $\omega_D$ in (a) and (b).

Appendix D  PRIMITIVE EQUATIONS

Using the same non-dimensionalization as Appendix A, the non-dimensional Boussinesq primitive equations are given as follows [5]:

$$ R_0 \frac{Du}{Dt} + f(y) \times u = -\nabla \phi, $$

$$ R_0 \frac{Db}{Dt} + \left( \frac{L_d}{L_y} \right)^2 N^2 w = 0 $$

$$ \partial_z \phi = b, $$

$$ \partial_x u + \partial_y v + \partial_z w = 0. $$

(38)

where $w$ is the vertical velocity, $\phi$ is the kinetic pressure, and $b$ is the buoyancy fluctuation about an average stratification, $N^2 = \partial b/\partial z$, and $L_d$ is the deformation radius, and $R_0$ is the Rossby number. We consider the linearized equations

$$ R_0 \frac{\partial u}{\partial t} + f(y) \times u = -\nabla \phi, $$

$$ R_0 \frac{\partial b}{\partial t} + \left( \frac{L_d}{L_y} \right)^2 N^2 w = 0, $$

$$ \partial_z \phi = b, $$

$$ \partial_x u + \partial_y v + \partial_z w = 0. $$

(39)

Here, $R_0$ and $NL_d/L_y$ can be set to unity by appropriate re-scaling of the variables. In the Fourier space, $-ik_x \phi = b$ and $ik_x u + ik_y v + ik_z w = 0$. Therefore, we can eliminate $\phi$ and $w$ by writing them in terms of $b$, $u$ and $v$. In the f-plane approximation, the dispersion relation for the Poincaré modes is $\omega^2 = f^2 + (k_x^2 + k_y^2)/k_z^2$.

Finally we consider the imposition of sinusoidal horizontal shear flow. We assume the system is periodic in the zonal and meridional direction and has rigid lids at $z = 0$ and $z = L_z$, where $L_z$ is a constant. Let $u_{tot} = u + U$, $u = (u, v)$, $U = (U(y), 0)$ and $\phi = \eta + H(y)$. From geostrophic balance, $U(y)$ and $H(y)$ must satisfy Eq. (5). Substituting $u_{tot}$
and $\phi$ into Eq. (38) and discarding non-linear terms, we obtain:
\[
\frac{\partial u}{\partial t} = -U(y) \frac{\partial u}{\partial x} - v \frac{\partial U(y)}{\partial y} + f(y) v - \frac{1}{R_0} \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} = -f(y) u - U(y) \frac{\partial v}{\partial x} - \frac{1}{R_0} \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} = -w - U(y) \frac{\partial^2 \eta}{\partial x \partial z}.
\]
(40)

Again $R_0$ and $N^2(L_d/L_y)^2$ may be set to unity. By doing so, Eq. (40) simplifies to
\[
\frac{\partial u}{\partial t} = -U(y) \frac{\partial u}{\partial x} - v \frac{\partial U(y)}{\partial y} + f(y) v - \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} = -f(y) u - U(y) \frac{\partial v}{\partial x} - \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} = -w - U(y) \frac{\partial^2 \eta}{\partial x \partial z}.
\]
(41)

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