Cyclic evolution of radio pulsars on the time scale of hundreds of years

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ABSTRACT

The recent massive measurements of pulsar frequency second derivatives have shown that they are 100-1000 times larger than expected for standard pulsar slowdown law. Moreover, the second derivatives as well as braking indices are even negative for about half of pulsars. We explain these paradoxical results on the basis of the statistical analysis of the rotational parameters $\nu$, $\dot{\nu}$ and $\ddot{\nu}$ of the subset of 295 pulsars taken mostly from the ATNF database. We have found strong correlation of $\ddot{\nu}$ and $\dot{\nu}$ either for $\ddot{\nu} > 0$ (correlation coefficient $r \approx 0.9$) and $\ddot{\nu} < 0$ ($r \approx 0.85$), and of $\nu$ and $\dot{\nu}$ ($r \approx 0.7$). We interpret these dependencies as evolutionary ones due to $\dot{\nu}$ being nearly proportional to characteristic age $\tau_{ch}$. The derived statistical relations as well as “anomalous” values of $\ddot{\nu}$ are well explained in the framework of the simple model of cyclic evolution of the rotational frequency of the pulsars. It combines the secular change of $\nu_{tr}(t)$, $\dot{\nu}_{tr}(t)$ and $\ddot{\nu}_{tr}(t)$ according to the power law with $n \approx 5$ and harmonic oscillations of 100–1000 years period with an amplitude from $10^{-3}$ Hz for young pulsars to $10^{-10}$ Hz for elder ones. The physical nature of these cyclic variations of the rotational frequency may be similar to the well-known red timing noise, however, with much larger characteristic time scale.

Subject headings: methods: data analysis — methods: statistical — pulsars: general

1. Introduction

The slowdown of radio pulsars is due to the electromagnetic emission energy losses. According to this ”classical” approach, their rotational frequencies $\nu$ evolve following the

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slowdown law $\dot{\nu} = -K\nu^n$, where $K$ is the positive constant that depends on the magnetic dipole moment and the moment of inertia of the neutron star, $n$ is the braking index. The latter can be determined observationally from measurements of $\nu$, $\dot{\nu}$ and $\ddot{\nu}$ as $n = \nu\ddot{\nu}/\dot{\nu}^2$. For pure dipole structure of pulsar magnetosphere $n = 3$, the pulsar wind decreases this value down to $n = 1$, while for multipole magnetic field it is up to $n = 5$ (Manchester & Taylor 1977; Manchester, Durdin & Newton 1985; Blandford & Romani 1988). For such law of pulsar evolution the second derivatives of their rotational frequencies (and therefore their braking indices) can’t be measured for objects with the characteristic age $\tau_{ch} = -\nu/2\dot{\nu} > 10^6$ years as their $\ddot{\nu} < 10^{-30}$ s$^{-3}$, that is much lower than the current detection limit (Livingstone et al. 2005). At the same time the massive measurements of pulsar frequency second derivatives $\ddot{\nu}$ have shown that their values are much larger than expected for standard spin-down law and are even negative for about half of all pulsars. The corresponding braking indices are from $-10^6$ to $10^6$ (D’Alessandro et al. 1993; Chukwude 2003; Hobbs et al. 2004).

These strange results are usually interpreted as an influence of a red timing noise, which manifests itself as irregular variations of pulse arrival times relative to the secular frequency evolution (Groth 1975; Cordes & Helfand 1980; Cordes & Downs 1985), described through Taylor series expansion as

$$\nu(t) = \nu_0 + \dot{\nu}_0(t - t_0) + \frac{1}{2}\ddot{\nu}_0(t - t_0)^2 + \ldots \quad (1)$$

where $t_0$, $\nu_0$, $\dot{\nu}_0$ and $\ddot{\nu}_0$ are initial time moment, frequency and its derivatives at this moment. Historically, the secular evolution of pulsar rotation rate has been considered as the linear trend of the frequency behaviour, while quadratic term of the series (1) (with cubic behaviour in pulse arrival times), measured as $\ddot{\nu}$, has been treated as the signature of non-monotonic (on time scale of the observations) processes – red noise, post-glitch recovery, pulsar orbital motion and free precession (Groth 1975; Cordes & Helfand 1980; Shemar & Lyne 1996; Stairs, Lyne & Shemar 2000). However, recently the increase of the sensitivity of the receivers and time spans of the continuous timing (from several to thirty years) made it possible to estimate the higher-order coefficients of the series (1) (Camilo et al. 2000; Scott, Finger & Wilson 2003; Hobbs et al. 2004; Livingstone et al. 2005). The results of these studies show that $\ddot{\nu}$, while changing on the time scales shorter than observation time spans, on average characterizes the pulsar secular evolution rather than noise processes. The latter, causing the variations mentioned earlier, worsens the accuracy of the second derivative determination, while can’t change its mean value hundreds times (Camilo et al. 2000; Hobbs et al. 2004; Livingstone et al. 2005).

In this Letter we report the detection of correlation of $\ddot{\nu}$ and $\dot{\nu}$ either for $\ddot{\nu} > 0$ and $\ddot{\nu} < 0$ and correlation of $\nu$ and $\dot{\nu}$. We show that these statistical relations as well as ”anomalous” values of $\ddot{\nu}$ can be explained by the combination of secular decrease of pulsar frequencies
Fig. 1.— The $\dot{\nu} - \ddot{\nu}$ diagram. The figure shows the pulsars from the work (Hobbs et al. 2004) as circles, and the objects measured by other groups as squares. Empty symbols represent the pulsars associated with the supernova remnants, and therefore – relatively young ones. Analytical fits for both positive and negative branches are shown as solid lines.

with $n \approx 5$ and their cyclic variations on the time scale of several hundreds of years.

2. Statistical analysis of the ensemble of pulsars

Our statistical analysis is based on the assumption that numerous measurements of the pulsar frequency second derivatives reflect their secular evolution on the time scale larger than the duration of observations, and uses the parameters of nearly 300 pulsars.

From 389 objects of ATNF catalogue (Manchester et al. 2005) with known $\dot{\nu}$ we have compiled a list of "ordinary" radio pulsars with $P > 20$ ms and $\dot{P} > 10^{-17}$ s s$^{-1}$, excluding recycled, anomalous and binary pulsars, and with relative accuracy of second derivative
measurements better than 75%. It has been appended with 26 pulsars from other sources (D’Alessandro et al. 1993; Chukwude 2003). The parameters of all pulsars have been plotted on the $\dot{\nu} - \dot{\nu}$ diagram (Fig. 1). Here the circles are the objects with second derivatives measured at the Jodrell Bank Observatory (Hobbs et al. 2004), squares – the results of other groups (D’Alessandro et al. 1993; Chukwude 2003), and empty symbols represent young pulsars associated with supernova remnants.

The main result of the statistical analysis of these data is a significant correlation of $\ddot{\nu}$ and $\dot{\nu}$, either for 168 objects with $\ddot{\nu} > 0$ (correlation coefficient $r \approx 0.90$) and for 127 objects with $\ddot{\nu} < 0$ ($r \approx 0.85$). Either groups follow the nearly linear laws, however they are not exactly symmetric with respect to $\ddot{\nu} = 0$. Also, the young pulsars with $\ddot{\nu} < -10^{-11} \text{ s}^{-2}$ and $\dot{\nu} < 0$ are absent.

Figure 2 shows the dependence of $\dot{\nu}$ on the characteristic age $\tau_{ch}$. Of course, due to practical proportionality of these parameters there is a strong dependence of $\dot{\nu}$ on $\tau_{ch}$ either for pulsars with $\ddot{\nu} > 0$ (positive branch, $r \approx 0.85$) and for $\ddot{\nu} < 0$ (negative branch, $r \approx 0.75$) (Fig. 3a). Also, we have found the significant correlation of $n$ and $\tau_{ch}$ (Fig. 3b). Note some important points of these results.

- Measurements of different observational groups show the same laws, so the correlations discovered are not related to the peculiarities of the receiving devices of a single radio telescope.

- Young pulsars confidently associated with supernova remnants are systematically shifted to the left in Figure 1 and Figure 2 (empty symbols). The order of their physical ages roughly corresponds to that of characteristic ages. It means that any dependence on
\( \dot{\nu} \) or \( \tau_{\text{ch}} \) may be interpreted as the dependence on time.

- Separate analysis of the whole set of parameters of pulsars with \( \ddot{\nu} > 0 \) and \( \ddot{\nu} < 0 \) (galactic positions, fluxes, luminosities, magnetic fields, dispersion and rotation measures) shows the statistical identity of these sub-groups.

So, the \( \ddot{\nu} - \dot{\nu} \) diagram (Fig. 1) may be interpreted as an evolutionary one. In other words, each pulsar during its evolution moves along the branches of this diagram while increasing the value of its \( \dot{\nu} \) (which corresponds to the increase of its characteristic age). However, there is an obvious contradiction, as for \( \ddot{\nu} < 0 \) (negative branch), \( \dot{\nu} \), being negative, may only decrease with time, so the motion along the negative branch may be only backward! Certainly, this contradiction is easily solved by assuming non-monotonic behaviour of \( \nu(t) \), which may have some cyclic (quasi-periodic) component along with secular one. Figure 4 shows the example of such cyclic behaviour of a pulsar. The characteristic time scale \( T \) of such cycles must be much shorter than the pulsar life time and at the same time much larger than the time scale of the observations. Each pulsar during its evolution repeatedly changes the sign of \( \ddot{\nu} \) and spends roughly half of its life time on each branch. The asymmetry of the branches reflects the positive secular behaviour of \( \dot{\nu}_{\text{tr}}(t) \), and so, secular increase of \( \dot{\nu}(t) \). Systematic decrease of branches separation reflects the decrease of the oscillations amplitude and/or the increase of its characteristic time scale. Any non-monotonic variations of \( \nu(t) \), like post-glitch recovery, standard red noise or orbital motion, will manifest itself in a similar way on the \( \ddot{\nu} - \dot{\nu} \) diagram and lead to extremely high values of \( \ddot{\nu} \) (Groth 1975; Cordes & Helfand 1980; Shemar & Lyne 1996; Stairs, Lyne & Shemar 2000). However, their characteristic time scales vary from weeks to years, and they are detected immediately. In our case we have processes on time scales of hundreds of years, and their study is possible only by means of statistical methods, assuming that the ensemble of pulsars of various age models the behaviour of a single pulsar during its evolution.

Note that the correlations between \( \dot{P} \) and \( P \), similar to these discovered here (but without distinguishing between objects with \( \ddot{\nu} > 0 \) and \( \ddot{\nu} < 0 \)), were discussed earlier (Cordes & Downs 1985; Arzoumanian et al. 1994; Lyne 1999) as an evidence of higher timing noise of young pulsars.

3. The simplest model of pulsar cyclic evolution

Cyclic variations of the pulsar rotational frequency may be complicated – quasi-periodic, multi-harmonic, stochastic. As a rough initial approximation, we use a simple model of
harmonic oscillations superimposed on secular evolution. In this case,

\[ \nu(t) = \nu_{tr}(t) + A(t) \sin [\Omega(t)t + \phi_0], \]  

where \( A(t) \) is the amplitude of periodic frequency variations, \( \Omega(t) = 2\pi/T(t) \) – its frequency, and \( \nu_{tr}(t) \) is the secular evolution term. This leads to

\[ \dot{\nu}(t) = \dot{\nu}_{tr}(t) + A(t)\Omega(t) \cos [\Omega(t)t + \phi_0], \]  

\[ \ddot{\nu}(t) = \ddot{\nu}_{tr}(t) - A(t)\Omega^2(t) \sin [\Omega(t)t + \phi_0], \]

where we neglected the terms with \( A(t) \) and \( \Omega(t) \) derivatives due to its relative smallness. From (3) and (4) it is easy to get the relation between \( \ddot{\nu} \) and \( \dot{\nu} \) in the cycloidal form

\[ (\ddot{\nu} - \ddot{\nu}_{tr})^2 = \Omega^4(t)A^4(t) - \Omega^2(t)[\dot{\nu}(t) - \dot{\nu}_{tr}(t)]^2, \]

which is plotted in Figure 4.

By marking with “+” and “-” subscripts the values related to the positive and negative branches on the \( \ddot{\nu} - \dot{\nu} \) diagram (Fig. 1), correspondingly, and pointing to the amplitude values of \( \nu \) and \( \dot{\nu} \) (i.e. assuming \( \sin [\Omega(t)t + \phi_0] = \mp 1 \)), we get

\[ \nu_\pm(t) = \nu_{tr}(t) \mp A(t) \]  

\[ \dot{\nu}_\pm(t) = \dot{\nu}_{tr}(t) \]  

\[ \ddot{\nu}_\pm(t) = \ddot{\nu}_{tr}(t) \pm A(t)\Omega^2(t). \]

So, due to (6), the branches of \( \ddot{\nu} - \dot{\nu} \) diagram describe the dependence of pulsar frequency second derivative \( \ddot{\nu} \) on the secular behaviour of its first derivative \( \dot{\nu}_{tr} \), which corresponds to the dependence on time.

Relations (2) – (7) describe the apparent evolution of some ”average” pulsar. Of course, in the ensemble of all pulsars the scatter of points on the \( \ddot{\nu} - \dot{\nu} \) diagram reaches 4 orders of magnitude due to distribution of the neutron star parameters and initial conditions. Due to this fact it is impossible to determine \( \ddot{\nu}_{tr}(t) \) directly as the half-sum of \( \ddot{\nu}_+(t) \) and \( \ddot{\nu}_-(t) \) from (7) and Fig. 1, as it requires the subtraction of two close values with significant errors.

First of all we compute the behaviour of \( \nu_{tr}(t) \) (or, \( \nu_{tr}(\dot{\nu}_{tr}) \)) by plotting the studied pulsar group on the \( \dot{\nu} - \nu \) diagram (Fig. 5). The objects with \( \dot{\nu} > 0 \) and \( \ddot{\nu} < 0 \) are marked as filled and empty circles, correspondingly. It is easily seen that the behaviour of these two sub-groups is the same, i.e. \( A(t) \) according to (5) is significantly smaller than the intrinsic scatter of \( \nu(\dot{\nu}) \). However, a strong correlation between \( \nu_{tr} \) and \( \dot{\nu}_{tr} \) (\( r \approx 0.7 \)) is seen, and

\[ \dot{\nu}_{tr} = -C\nu_{tr}^n, \]

where \( C = 10^{-15.26\pm1.38} \) and \( n = 5.13 \pm 0.34 \). So, the secular evolution of the ”average” pulsar is according to the ”standard” spin-down law with \( n \approx 5! \) This value may suggest the
importance of multipole components of pulsar magnetic field, or the deviation of the angle between pulsar rotational and dipole axes from $\pi/2$ (Manchester & Taylor 1977; Contopoulos & Spitkovsky 2005). From (8) we may easily get the relation between $\ddot{\nu}_{tr}$ and $\dot{\nu}_{tr}$

$$\ddot{\nu}_{tr}(\dot{\nu}_{tr}) = nC_{1}^{\frac{1}{4}}(-\dot{\nu}_{tr})^{2-\frac{4}{n}},$$

which is shown in Figure 6 as a thick dashed line. The amplitude of the $\ddot{\nu}(t)$ oscillations, $A_{\ddot{\nu}}(t)$, may be easily computed from (7) using fits for $\ddot{\nu}_{+}$ and $\ddot{\nu}_{-}$ in Figure 1,

$$A_{\ddot{\nu}}(t) = A(t)\Omega^{2}(t) = \frac{1}{2}[\ddot{\nu}_{+}(t) - \ddot{\nu}_{-}(t)].$$

The result is shown in Figure 6 as a solid line. The amplitude $A_{\ddot{\nu}}$ is decreasing almost linearly with the increase of $\dot{\nu}_{tr}$, i.e. life time. This may be related to the decrease of either $A(t)$ or $\Omega(t)$.

From Figure 6 it is seen that our simple model formulated in (2) – (7) describes the observational data rather well, at least for the pulsars with $\dot{\nu} > -10^{-11}$ s$^{-2}$, i.e. with $\tau_{ch} > 10^{4}$ years. For $\dot{\nu} < -7\cdot10^{-12}$ s$^{-2}$, the $A_{\ddot{\nu}}$ is smaller than $\dot{\nu}_{tr}$, and during the oscillations $\dot{\nu} > 0$ always, that explains the absence of pulsars of the negative branch of $\ddot{\nu} - \dot{\nu}$ diagram in this region. The situation changes dramatically for $\dot{\nu} > 10^{-11}$ s$^{-2}$, i.e. $\tau_{ch} > 10^{4}$ years. Here $A_{\ddot{\nu}}$ becomes larger than $\dot{\nu}_{tr}$ and pulsars with $\dot{\nu} < 0$ and $n < 0$ appear. In other words, the cyclic behaviour according to the model described by (2) – (7) begins to manifest itself in the observing quantities, $-\ddot{\nu}$ and $n$ begins to deviate significantly from the predictions of the ”standard” spin-down law evolution.

On the basis of these assumptions it is possible to get a rough estimate of the maximal period of the cyclic behaviour as the several times of the time scale of the vanishing of $\dot{\nu}$ for the young pulsars with the measured $\dot{\nu}$ as $T_{\max} \approx -4\ddot{\nu}/\dot{\nu}$. For PSR B1509-58 (with $\ddot{\nu} \approx 2 \cdot 10^{-21}$ s$^{-3}$ and $\dot{\nu} \approx -1.3 \cdot 10^{-31}$ s$^{-4}$) (Livingstone et al. 2005) $T_{\max} \approx 2 \cdot 10^{3}$ years ($\Omega_{\min} \approx 10^{-10}$ s$^{-1}$). For Crab (with $\ddot{\nu} \approx 1.2 \cdot 10^{-20}$ s$^{-3}$ and $\dot{\nu} \approx -1.4 \cdot 10^{-30}$ s$^{-4}$) (Scott, Finger & Wilson 2003) $T_{\max} \approx 10^{3}$ years ($\Omega_{\min} \approx 2 \cdot 10^{-10}$ s$^{-1}$). On the other hand, the estimations of $\Omega$ may be done using the relations (2) – (4), for the pulsars with several measurements of rotational parameters at different epochs with enough separation. So, the lower limit is $\Omega_{\min} \approx \sqrt{|\Delta \ddot{\nu}/\Delta \dot{\nu}|}$. From the results of observations of the old ($\tau \propto 10^{6}$ years) pulsars, B0823+26, B1706-16, B1749-28 and B2021+51 with 7 to 10 years separation, we have obtained the same estimate of $T_{\max} \approx 10^{3} \div 2 \cdot 10^{3}$ years. Finally, the period of cyclic oscillations is likely to be in the range of 100 – 1000 years ($2 \cdot 10^{-10} < \Omega < 2 \cdot 10^{-9}$ s$^{-1}$). Here we have suggested that the minimal cycle period has to be several times the pulsar observation period – 30 years.
Taking into account that $A_\nu = \Omega^2 A$ is changing several orders of magnitude, our initial assumption of relative constancy of $\Omega$ and significant decrease of $A$ during the evolution is somehow justified. Assuming $\Omega \approx 10^{-9}$ s$^{-1}$, we get for the amplitude of frequency oscillations $A = A_\nu/\Omega^2$ values from $4 \cdot 10^{-4}$ s$^{-1}$ for the youngest pulsars till $10^{-10}$ s$^{-1}$ for the eldest.

4. Discussion and conclusions

The physical reasons of the discussed cyclic variations of the pulsar rotational frequency may be similar to the ones of the well-known red timing noise. In the first place, the free precession (Stairs, Lyne & Shemar 2000) is the periodic effect most suitable for the proposed harmonic model. Note that the interaction of the superfluid core and the crust of the neutron star does not depress its development (Alpar 2005). Several processes had been proposed for the explanations of the red noise (Cordes & Helfand 1980; Cordes & Greenstein 1981; Cheng 1987) – from the collective effects in the superfluid core till the electric current fluctuations in the pulsar magnetosphere. The possibility of its utilization for the explanation of the frequency variations on the time scale of hundreds of years is yet to be analyzed.

The argument towards the similarity between the discussed variations and the red timing noise is the coincidence of the red noise $\ddot{\nu}$ amplitude extrapolated according to its power spectrum slope (Baykal et al. 1999) to the time scale of hundreds of years, with $A_\nu$ in our model for the same $\dot{\nu}$, i.e. the same ages (see Fig. 6).

Finally, note that the real evolution of the pulsar rotational frequencies may be significantly more complicated than the proposed simple model. The variations may be non-harmonic, their amplitudes, phases and frequencies may change randomly. The principal point is that all the pulsars evolve this way, and the time scale of such variations must significantly exceed several tens of years. That explains the anomalous values of the observed $\dot{\nu}$ and braking indices.

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Fig. 3.— (a) The correlation of $\dot{\nu}$ and $\tau_{ch}$. Empty symbols represent the young pulsars associated with the supernova remnants. It is clear that the pulsar characteristic age roughly corresponds to the real one. (b) The correlation of $n$ and $\tau_{ch}$.
Fig. 4.— The sample of pulsar cyclic behaviour on the $\nu - \dot{\nu}$ diagram. The solid line shows the evolution of a pulsar according to the simple harmonic model presented in (2) – (7) with the oscillation period $T = 1000$ years. The dashed line shows the corresponding secular evolution term $\ddot{\nu}_{\text{tr}}(t)$, and the oscillations are around it, symmetric in linear scale. For $\dot{\nu} < -7 \cdot 10^{-12}$ s$^{-2}$, the amplitude of the oscillations is smaller than the secular term, thus $\ddot{\nu}$ is always positive, for larger values it changes its sign repeatedly, spending approximately half of the time in the negative region.
Fig. 5.— The $\dot{\nu} - \nu$ diagram for the pulsars with the measured second derivative. The filled symbols are objects with positive $\dot{\nu}$, the empty ones – with negative. The behaviour of both subsets is the same. The solid line represents the best fit, corresponding to the $n \approx 5$ braking index, the dotted ones – 1-$\sigma$ range.
Fig. 6.— The $\dot{\nu} - \ddot{\nu}$ diagram with the simple harmonic oscillations model. The solid line is the amplitude $A_{\nu}$ of the frequency second derivative variations according to (10), the dashed – the secular term $\ddot{\nu}_{tr}$ from (9), and the dot-dashed lines are the envelopes of the oscillations $\ddot{\nu}_{tr} \pm A_{\nu}$ with 1-$\sigma$ ranges (dotted lines). Each pulsar spends the majority of its life time at or very near the envelopes.