NOTES ON PARAMETERS OF QUIVER HECKE ALGEBRAS

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Abstract. Varagnolo-Vasserot and Rouquier proved that, in a symmetric generalized Cartan matrix case, the simple modules over the quiver Hecke algebra with a special parameter correspond to the upper global basis. In this note we show that the simple modules over the quiver Hecke algebras with a generic parameter also correspond to the upper global basis in a symmetric generalized Cartan matrix case.

1. Introduction

Lascoux-Leclerc-Thibon ([8]) conjectured that the irreducible representations of Hecke algebras of type $A$ are controlled by the upper global basis ([4, 5]) (or dual canonical basis ([10])) of the basic representation of the affine quantum group $U_q(A^{(1)}_1)$. Then Ariki ([1]) proved this conjecture by generalizing it to cyclotomic Hecke algebras. The crucial ingredient in his proof was the fact that the cyclotomic Hecke algebras categorify the irreducible highest weight representations of $U(A^{(1)}_1)$. Because of the lack of grading on the cyclotomic Hecke algebras, these algebras do not categorify the representation of the quantum group.

Then Khovanov-Lauda and Rouquier introduced independently a new family of graded algebras, a generalization of affine Hecke algebras of type $A$, in order to categorify arbitrary quantum groups ([6, 7, 11]). These algebras are called Khovanov-Lauda-Rouquier algebras or quiver Hecke algebras.

Let $U_q(g)$ be the quantum group associated with a symmetrizable Cartan datum and let $\{R(\beta)\}_{\beta \in \mathbb{Q}^+}$ be the corresponding quiver Hecke algebras. Then it was shown...
in [6, 7] that there exists an algebra isomorphism

\[ U^-_A(g) \simeq \bigoplus_{\beta \in Q^+} K(R(\beta)\text{-proj}), \]

where \( U^-_A(g) \) is the integral form of the half \( U^-_q(g) \) of the quantum group \( U_q(g) \) with \( A = \mathbb{Z}[q, q^{-1}] \), and \( K(R(\beta)\text{-proj}) \) is the Grothendieck group of the category \( R(\beta)\text{-proj} \) of finitely generated projective graded \( R(\beta) \)-modules. The positive root lattice is denoted by \( Q^+ \). By the duality, we have

\[ U^-_A(g)^* \simeq \bigoplus_{\beta \in Q^+} K(R(\beta)\text{-gmod}), \tag{1.1} \]

where \( U^-_A(g)^* \) is the direct sum of the dual of the weight space \( U^-_A(g)_{-\beta} \) of \( U^-_A(g) \), and \( R(\beta)\text{-gmod} \) is the abelian category of graded \( R(\beta) \)-modules which are finite-dimensional over the base field \( k \).

When the generalized Cartan matrix is a symmetric matrix, Varagnolo and Vasserot ([13]) and Rouquier ([12]) proved that the upper global basis introduced by the author or Lusztig’s dual canonical basis corresponds to the isomorphism classes of simple \( R(\beta) \)-modules via the isomorphism \((1.1)\).

However, for a given generalized Cartan matrix, associated quiver Hecke algebras are not unique and depend on the parameters \( c \). Varagnolo-Vasserot and Rouquier have proved the above results for a very special choice \( c_0 \) of parameters (see \((2.8)\)). Let \( R(\beta)_{c_0} \) denote the quiver Hecke algebra with the choice \( c_0 \), and \( R(\beta)_{c_{\text{gen}}} \) the quiver Hecke algebra with a generic choice \( c_{\text{gen}} \) of parameters. When a simple \( R(\beta)_{c_0} \)-module is specialized at the special parameter \( c_0 \), it may be a reducible \( R(\beta)_{c_0} \)-module. The purpose of this note is to prove that the specialization of any simple \( R(\beta)_{c_{\text{gen}}} \)-module at \( c_0 \) remains a simple \( R(\beta)_{c_0} \)-module. In other words, the set of isomorphism classes of simple \( R(\beta)_{c_{\text{gen}}} \)-modules also corresponds to the upper global basis.

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2. Review on global bases and quiver Hecke algebras

2.1. Global bases. Let \( I \) be a finite index set. An integral square matrix \( A = (a_{i,j})_{i,j \in I} \) is called a symmetrizable generalized Cartan matrix if it satisfies (i) \( a_{i,i} = 2 \) (\( i \in I \)), (ii) \( a_{i,j} \leq 0 \) (\( i \neq j \)), (iii) \( a_{i,j} = 0 \) if \( a_{j,i} = 0 \) (\( i, j \in I \)), (iv) there is a diagonal matrix \( D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I) \) such that \( DA \) is symmetric.

A Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\) consists of
(1) a symmetrizable generalized Cartan matrix \( A \),
(2) a free abelian group \( P \) of finite rank, called the weight lattice,
(3) \( P^\vee := \text{Hom}(P, \mathbb{Z}) \), called the co-weight lattice,
(4) \( \Pi = \{ \alpha_i \mid i \in I \} \subset P \), called the set of simple roots,
(5) \( \Pi^\vee = \{ h_i \mid i \in I \} \subset P^\vee \), called the set of simple coroots,

satisfying the condition: \( \langle h_i, \alpha_j \rangle = a_{i,j} \) for all \( i, j \in I \).

Since \( A \) is symmetrizable, there is a symmetric bilinear form \( \langle \quad \rangle \) on \( P \) satisfying
\[
(\alpha_i|\alpha_j) = d_i a_{i,j} \quad \text{and} \quad (\alpha_i|\lambda) = d_i \langle h_i, \lambda \rangle \quad \text{for all } i, j \in I, \; \lambda \in P.
\]

The free abelian group \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) is called the root lattice. Set \( Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset Q \) and \( Q^- = \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i \subset \mathbb{Q} \). For \( \beta = \sum_{i \in I} m_i \alpha_i \in \mathbb{Q} \), we set \( \text{ht}(\beta) = \sum_{i \in I} |m_i| \).

Let \( q \) be an indeterminate. Set \( q_i = q^{d_i} \) for \( i \in I \) and we define \([n]_i = (q_i^n - q_i^{-n})(q_i - q_i^{-1})^{-1}\) and \([n]_i! = \prod_{k=1}^{n} [k]_i \) for \( n \in \mathbb{Z}_{\geq 0} \).

**Definition 2.1.** The quantum algebra \( U_q(\mathfrak{g}) \) associated with a Cartan datum \( (A, P, \Pi, \Pi^\vee) \) is the algebra over \( \mathbb{Q}(q) \) generated by \( e_i, f_i \) \( (i \in I) \) and \( q^h \) \( (h \in P^\vee) \) satisfying following relations:

(i) \( q^0 = 1, \; q^h q^{h'} = q^{h+h'} \) for \( h, h' \in P^\vee \),
(ii) \( q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \; q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i \) for \( h \in P^\vee, i \in I \),
(iii) \( e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \) where \( K_i = q_i^{h_i} \),
(iv) \( \sum_{r=0}^{1-a_{i,j}} (-1)^r e_i \bigl( 1-a_{i,j}-r \bigr) e_j e_i^{(r)} = 0 \) if \( i \neq j \), where \( e_i^{(n)} = e_i^n / [n]_i! \),
(v) \( \sum_{r=0}^{1-a_{i,j}} (-1)^r f_i \bigl( 1-a_{i,j}-r \bigr) f_j f_i^{(r)} = 0 \) if \( i \neq j \), where \( f_i^{(n)} = f_i^n / [n]_i! \).

Let \( U^-_q(\mathfrak{g}) \) be the \( \mathbb{Q}(q) \)-subalgebra of \( U_q(\mathfrak{g}) \) generated by the elements \( f_i \). We define the endomorphisms \( e_i^\prime \) and \( f_i^\prime \) of \( U^-_q(\mathfrak{g}) \) by
\[
[e_i, a] = (q_i - q_i^{-1})^{-1}(K_i e_i a - K_i^{-1} e_i a) \quad \text{for } a \in U^-_q(\mathfrak{g}).
\]

Then \( e_i^\prime \) and the left multiplication of \( f_j \) satisfy the \( q \)-boson commutation relations
\[
e_i^\prime f_j - q_i^{-a_{i,j}} f_j e_i^\prime = \delta_{i,j}.
\]

Set \( A = \mathbb{Z}[q, q^{-1}] \) and let \( U^-_A(\mathfrak{g}) \) be the \( A \)-subalgebra of \( U^-_q(\mathfrak{g}) \) generated by the elements \( f_i^{(n)} \). Then \( U^-_A(\mathfrak{g}) \) has a weight decomposition \( U^-_A(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}^-} U^-_A(\mathfrak{g})_\beta \) where
Proposition 2.2 ([4, 5]). There exists a unique basis \( \{G^{up}(b)\}_{b \in B} \) of the \( \mathbf{A} \)-module \( U_{A}(g)^* \), called the upper global basis, which satisfies the following conditions:

(i) \( \phi \in \{G^{up}(b) \mid b \in B\} \),

(ii) for any \( b \in B \), \( G(b) \) belongs to \( \left(U_{A}(g)^*\right)^* \) for some \( \beta \in \mathbb{Q}^- \), which is denoted by \( wt(b) \),

(iii) Set \( \varepsilon_i(b) = \max \{n \in \mathbb{Z}_{\geq 0} \mid e_i^nG^{up}(b) \neq 0\} \). Then for any \( b \in B \) and \( i \in I \), there exists \( \tilde{f}_i b \in B \) such that, when writing

\[
\tilde{f}_i G^{up}(b) = \sum_{b' \in B} F^i_{b,b'} G^{up}(b') \quad \text{with } F^i_{b,b'} \in A,
\]

we have

(a) \( F^i_{b,\tilde{f}_i b} = q_i^{-\varepsilon_i(b)} \),

(b) \( \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1 \),

(c) \( F^i_{b,b'} = 0 \) if \( b' \neq \tilde{f}_i b \) and \( \varepsilon_i(b') \geq \varepsilon_i(b) + 1 \),

(d) \( F^i_{b,b'} \in q q_i^{-\varepsilon_i(b)} \mathbb{Z}[q] \) for \( b' \neq \tilde{f}_i b \).

(iv) for \( b \in B \) such that \( \varepsilon_i(b) > 0 \), there exists \( \tilde{e}_i b \in B \) such that, when writing

\[
e_i G^{up}(b) = \sum_{b' \in B} E^i_{b,b'} G^{up}(b') \quad \text{with } E^i_{b,b'} \in A,
e_i G^{up}(b) = \sum_{b' \in B} E^i_{b,b'} G^{up}(b') \quad \text{with } E^i_{b,b'} \in A,
\]

we have

(a) \( E^i_{b,\tilde{e}_i b} = [\varepsilon_i(b)]_i \),

(b) \( \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \),

(c) \( E^i_{b,b'} = 0 \) if \( b' \neq \tilde{e}_i b \) and \( \varepsilon_i(b') \geq \varepsilon_i(b) - 1 \),

(d) any \( E^i_{b,b'} \) is invariant under the automorphism \( q \rightarrow q^{-1} \),

(e) \( E^i_{b,b'} \in q q_i^{1-\varepsilon_i(b)} \mathbb{Z}[q] \) for \( b' \neq \tilde{e}_i b \).

(v) \( \tilde{f}_i \tilde{e}_i b = b \) if \( \varepsilon_i(b) > 0 \), and \( \tilde{e}_i \tilde{f}_i b = b \).

Note that \( B \) has the weight decomposition

\[
B = \bigcup_{\beta \in \mathbb{Q}^-} B_{\beta} \quad \text{with } B_{\beta} := \{b \in B \mid wt(b) = \beta\}.
\]
There exists a unique involution (called the bar involution) \( - : U_A^-(\mathfrak{g})^* \to U_A^-(\mathfrak{g})^* \) such that
\[
\begin{align*}
(a) \quad (qu)^- &= q^{-1}u \quad \text{for any } u \in U_A^-(\mathfrak{g})^*, \\
(b) \quad - \circ e_i &= e_i \circ - \quad \text{for any } i, \\
(c) \quad \overline{\phi} &= \phi.
\end{align*}
\]

We have \( G^{up}(b) = G^{up}(b) \) for any \( b \in B \).

2.2. Quiver Hecke algebras. Let \((A, P, \Pi, P^\vee, \Pi^\vee)\) be a Cartan datum. In this subsection, we recall the construction of the quiver Hecke algebra associated with \((A, P, \Pi, P^\vee, \Pi^\vee)\). For \( i, j \in I \) such that \( i \neq j \), set
\[
S_{i,j} = \{(p, q) \in \mathbb{Z}_{\geq 0}^2 \mid (\alpha_i, \alpha_i)p + (\alpha_j, \alpha_j)q = -2(\alpha_i, \alpha_j)\}.
\]

Let \( k(A) \) be the commutative \( \mathbb{Z} \)-algebra generated by indeterminates \( \{t_{i,j;p,q}\} \) and the inverse of \( t_{i,j;p,q} \) where \( i, j \in I \) such that \( i \neq j \) and \( (p, q) \in S_{i,j} \). They are subject to the defining relations:
\[
t_{i,j;p,q} = t_{j,i;q,p}.
\]

Let us define the polynomials \((Q_{ij})_{i,j \in I}\) in \( k(A)[u, v]\) by
\[
(2.2) \quad Q_{ij}(u, v) = \begin{cases} 
0 & \text{if } i = j, \\
\sum_{(p, q) \in S_{i,j}} t_{i,j;p,q}u^pv^q & \text{if } i \neq j.
\end{cases}
\]

They satisfy \( Q_{i,j}(u, v) = Q_{j,i}(v, u) \).

We denote by \( S_n = \langle s_1, \ldots, s_{n-1} \rangle \) the symmetric group on \( n \) letters, where \( s_i := (i, i+1) \) is the transposition of \( i \) and \( i+1 \). Then \( S_n \) acts on \( P^n \).

**Definition 2.3** ([6, 11]). The quiver Hecke algebra \( R(n) \) of degree \( n \) associated with a Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\) is the \( \mathbb{Z} \)-graded algebra over \( k(A) \) generated by \( e(\nu) \) \((\nu \in P^n)\), \( x_k \) \((1 \leq k \leq n)\), \( \tau_l \) \((1 \leq l \leq n-1)\) satisfying the following defining relations:
\[
\begin{align*}
& e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), & & \sum_{\nu \in P^n} e(\nu) = 1, \\
& x_kx_l = x_lx_k, & & x_ke(\nu) = e(\nu)x_k, \\
& \tau_l e(\nu) = e(s_l(\nu))\tau_l, & & \tau_k\tau_l = \tau_l\tau_k \quad \text{if } |k - l| > 1, \\
& \tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),
\end{align*}
\]
\begin{align*}
(\tau_k x_l - x_{s_k(l)} \tau_k) e(\nu) &= \begin{cases} 
-e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\
e(\nu) & \text{if } l = k + 1, \nu_k = \nu_{k+1}, \\
0 & \text{otherwise},
\end{cases} \\
-\nu_k \tau_l e(\nu) &= \begin{cases} 
Q_{\nu_k \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k \nu_{k+1}}(x_{k+2}, x_{k+1}) e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}

The \(\mathbb{Z}\)-grading on \(R(n)\) is given by
\begin{equation}
\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k} | \alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_k} | \alpha_{\nu_{k+1}}).
\end{equation}

Note that \(R(n)\) has an anti-involution \(\psi\) that fixes the generators \(x_k, \tau_l\) and \(e(\nu)\).

For \(n \in \mathbb{Z}_{\geq 0}\) and \(\beta \in \mathbb{Q}^+\) such that \(\text{ht}(\beta) = n\), we set
\[I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n | \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.\]

We define
\begin{equation}
\begin{aligned}
e(\beta) &= \sum_{\nu \in I^\beta} e(\nu), \\
R(\beta) &= R(n) e(\beta) = \bigoplus_{\nu \in I^\beta} R(n) e(\nu).
\end{aligned}
\end{equation}

The algebra \(R(\beta)\) is called the quiver Hecke algebra at \(\beta\).

Similarly, for \(\beta, \gamma \in \mathbb{Q}^+\) with \(m = \text{ht}(\beta)\) and \(n = \text{ht}(\gamma)\)
\begin{equation}
\begin{aligned}
e(\beta, \gamma) &= \sum_{\nu} e(\nu) \in R(m + n) \\
\text{where } \nu \text{ ranges over the set of } \nu \in I^{m+n} \text{ such that} \\
\sum_{k=1}^m &\alpha_{\nu_k} = \beta \text{ and } \sum_{k=m+1}^{m+n} \alpha_{\nu_k} = \gamma.
\end{aligned}
\end{equation}

Then \(R(m + n) e(\beta, \gamma)\) is a graded \((R(\beta + \gamma), R(\beta) \otimes R(\gamma))\)-bimodule. For a graded \(R(\beta)\)-module \(M\) and a graded \(R(\gamma)\)-module \(N\), we define their convolution \(M \circ N\) by
\[M \circ N = R(\beta + \gamma) e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N).\]

For \(\ell \in \mathbb{Z}_{\geq 0}\), We define the graded \(R(\ell \alpha_i)\)-module \(L(i^\ell)\) by
\[L(i^\ell) = q_i^{\ell(\ell - 1)/2} \left( R(\ell \alpha_i) / \left( \sum_{k=1}^{\ell} R(\ell \alpha_i) x_k \right) \right).\]
Here \( q: \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta)) \) is the grade-shift functor:

\[
(qM)_k = M_{k-1},
\]

and \( q_i = q^{(\alpha_i/\alpha_i)} \).

For a commutative ring \( k \) and a ring homomorphism \( c: k(A) \rightarrow k \), we denote by \( R(\beta)_k \) the algebra \( k \otimes_{k(A)} R(\beta) \).

Let us denote by \( X(A) \) the scheme \( \text{Spec}(k(A)) \). For \( x \in X(A) \), let us denote by \( k(x) \) the residue field of the local ring \( (\mathcal{O}_{X(A)})_x \) and denote by \( R(\beta)_x \) the \( k(x) \)-algebra \( k(x) \otimes_{k(A)} R(\beta) \).

Let us take a commutative field \( k \) and a homomorphism \( k(x) \rightarrow k \). For \( \beta \in \mathbb{Q}^+ \), let us denote by \( R(\beta)_k \)-gmod the abelian category of graded \( R(\beta)_k \)-modules finite-dimensional over \( k \). Then the set of isomorphism classes of simple objects in \( R(\beta)_k \)-gmod is isomorphic to the one for \( R(\beta)_x \)-gmod by \( S \mapsto k \otimes_{k(x)} S \) (see \([6, \text{Corollary 3.19}]\)).

For \( i \in I \) and \( x \in X(A) \) we have functors

\[
R(\beta)_x \text{-gmod} \xrightarrow{F_i} R(\beta + \alpha_i)_x \text{-gmod}.
\]

Here these functors are defined by

\[
F_i M = M \circ L(i) \simeq (R(\beta + \alpha_i)_x e(\beta, \alpha_i)/R(\beta + \alpha_i)_x e(\beta, \alpha_i)x_{n+1}) \otimes_{R(\beta)_x} M,
\]

\[
E_i N = e(\beta, \alpha_i) N \simeq \text{Hom}_{R(\beta + \alpha_i)_x}(R(\beta + \alpha_i)_x e(\beta, \alpha_i), N)
\]

\[
\simeq e(\beta, \alpha_i) R(\beta + \alpha_i)_x \otimes_{R(\beta + \alpha_i)_x} N
\]

for \( M \in R(\beta)_x \)-gmod and \( N \in R(\beta + \alpha_i)_x \)-gmod. Then we have

\[
E_i F_i \simeq q^{-(\alpha_i/\alpha_i)} F_i E_i \bigoplus \text{id},
\]

\[
E_i F_j \simeq q^{-(\alpha_i/\alpha_j)} F_j E_i \quad \text{for } i \neq j;
\]

which immediately follows from \([3, \text{Theorem 3.6}]\).

Let \( K(R(\beta)_x \text{-gmod}) \) denote the Grothendieck group of the abelian category \( R(\beta)_x \)-gmod. Then, it has a structure of a \( \mathbb{Z}[q, q^{-1}] \)-module induced by the grade-shift functor on \( R(\beta)_x \)-gmod.

Then the following theorem holds.
Theorem 2.4 ([6]). There exists a unique $\mathbb{Z}[q, q^{-1}]$-linear isomorphism

\begin{equation}
\bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)_x\text{-gmod}) \xrightarrow{\sim} U_{\mathbb{A}}(\mathfrak{g})^*
\end{equation}

such that

(i) the induced actions $[E_i]$ and $[F_i]$ by $E_i$ and $F_i$ correspond to $e_i$ and $f_i'$,
(ii) $\phi \in U_{\mathbb{A}}(\mathfrak{g})^*$ corresponds to the regular representation of $R(0)_x$.

Let $D: R(\beta)_x\text{-gmod} \to (R(\beta)_x\text{-gmod})^{\text{opp}}$ be the duality functor $M \mapsto M^*$ induced by the antiautomorphism $\psi$ of $R(\beta)_x$. We can easily see by the characterization (2.1) of the bar involution that the induced endomorphism $[D]$ of $\bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)_x\text{-gmod})$ corresponds to the bar involution $-\delta$ of $U_{\mathbb{A}}(\mathfrak{g})^*$.

The Grothendieck group $K(R(\beta)_x\text{-gmod})$ is a free $\mathbb{Z}$-module with the basis consisting of $[S]$ where $S$ ranges over the set of isomorphism classes of simple graded $R(\beta)_x$-modules. Khovanov-Lauda ([6]) proved that for any simple graded $R(\beta)_x$-module $S$, there exists $r \in \mathbb{Z}$ such that $D(q^r S) \simeq q^r S$. Let $\text{Irr}(R(\beta)_x)$ be the set of isomorphism classes of simple graded $R(\beta)_x$-modules $S$ such that $D(S) \simeq S$. Then $K(R(\beta)_x\text{-gmod})$ is a free $\mathbb{Z}[q, q^{-1}]$-module with $\{[S] \mid S \in \text{Irr}(R(\beta)_x)\}$ as a basis.

For a simple graded module $S$, let us denote by $\varepsilon_i(S)$ the largest integer $k$ such that $E_i^k S \neq 0$. Recall that $q$ denotes the shift-functor and $q_i = q^{[\alpha_i]}$.

Proposition 2.5 ([9, 6]). Let $x \in X(\mathfrak{A})$, $\beta \in \mathbb{Q}^+$ and $S$ a simple graded $R(\beta)_x$-module.

(i) The cosocle of $F_i S$ is a simple module. Its image under $q_i^{\varepsilon_i(S)}$ is denoted by $\widetilde{F}_i S$.
(ii) If $\varepsilon_i(S) > 0$ then the socle of $E_i S$ is simple. Its image under $q_i^{1-\varepsilon_i(S)}$ is denoted by $\widetilde{E}_i S$.
(iii) $\widetilde{F}_i \widetilde{E}_i S \simeq S$ if $\varepsilon_i(S) > 0$, and $\widetilde{E}_i \widetilde{F}_i S \simeq S$.
(iv) If $S$ is invariant by the duality $D$, then so are $\widetilde{F}_i S$ and $\widetilde{E}_i S$.
(v) The set $\bigsqcup_{\beta \in \mathbb{Q}^+} \text{Irr}(R(\beta)_x)$ is isomorphic to $B$, and $\widetilde{E}_i$ and $\widetilde{F}_i$ correspond to $\tilde{e}_i$ and $\tilde{f}_i$ by this isomorphism.

Hence, the cosocle of $F_i S$ is isomorphic to $q_i^{-\varepsilon_i(S)} \widetilde{F}_i S$, the socle of $E_i S$ is isomorphic to $q_i^{\varepsilon_i(S)-1} \widetilde{E}_i S$ and the cosocle of $E_i S$ is isomorphic to $q_i^{-\varepsilon_i(S)-1} \widetilde{E}_i S$.

For $b \in B_{-\beta}$, let us denote by $L_x(b)$ the corresponding simple graded $R(\beta)_x$-module in $\text{Irr}(R(\beta)_x)$. 

Now assume that $A$ is symmetric and consider a $k$-valued point $c_0$ of $X(A)$ given by
\begin{equation}
Q_{i,j}(u, v) = b_{i,j}(u - v)^{-a_{i,j}} \quad \text{for } i \neq j
\end{equation}
where $k$ is a field of characteristic 0 and $b_{i,j} \in k^\times$.

Then the following theorem is proved by Varagnolo-Vasserot ([13]) and Rouquier ([12]).

**Theorem 2.6.** Assume that the generalized Cartan matrix $A$ is symmetric. Then the basis $\{[L_{c_0}(b)]\}_{b \in B}$ corresponds to the upper global basis $\{G^{up}(b)\}_{b \in B}$ by the isomorphism $\bigoplus_{\beta \in Q^+} K(R(\beta)_{c_{gen}}\text{gmod}) \cong U^-_A(\mathfrak{g})^*$.

For $M \in R(\beta)_{x'}\text{gmod}$, let us define its character $\text{ch}(M)$ by
$$
\text{ch}(M) = \sum_{\nu \in I^\beta, k \in \mathbb{Z}} \dim(e(\nu)M)_k q^k e(\nu) \in \bigoplus_{\nu \in I^\beta} \mathbb{Z}[q, q^{-1}]e(\nu).
$$
Then we have
\begin{equation}
\text{ch}(L_{c_0}(b)) = \sum_{\nu \in I^\beta} (e_{\nu_1} \cdots e_{\nu_n} G^{up}(b)) e(\nu) \quad \text{for } b \in B_{-\beta}.
\end{equation}

3. **Main results**

3.1. Let $c_{gen}$ be the generic point of $X(A)$. For $\beta \in Q^+$ and $b \in B_{-\beta}$, let us consider the simple graded $R(\beta)_{c_{gen}}\text{gmod}$ module $L_{c_{gen}}(b)$.

**Proposition 3.1.** The set $U_b := \{x \in X(A) \mid \text{ch}(L_x(b)) = \text{ch}(L_{c_{gen}}(b))\}$ is a Zariski open subset of $X(A)$ and there exists a graded $\mathcal{O}_{U_b} \otimes_{k(A)} R(\beta)\text{-gmod} \mathcal{L}(b)$ defined on $U_b$ such that it is locally free as an $\mathcal{O}_{U_b}$-module and the stalk of $\mathcal{L}(b)$ at any $x \in U_b$ is isomorphic to $L_x(b)$.

**Proof.** We shall prove it by induction on $ht(\beta)$. We may assume $\beta \neq 0$. Take an $i \in I$ such that $\ell := \varepsilon_i(b) \neq 0$. Set $\beta' = \beta - \ell \alpha_i$ and $b' = e_i^\ell b$. For any $x \in X(A)$, the graded $R(\beta')_{x'}\text{-gmod}$ module $L_x(b)$ is a simple cosocle of $L_x(b') \circ L(i^\ell)$. Moreover the kernel of $L_x(b') \circ L(i^\ell) \to L_x(b)$ is $\{s \in L_x(b') \circ L(i^\ell) \mid e(\beta', \ell \alpha_i) R(\beta)s = 0\}$.

By the induction hypothesis, there exists an $\mathcal{O}_{U_{b'}} \otimes_{k(A)} R(\beta')\text{-gmod} \mathcal{L}(b')$ as above. Set $\mathcal{R} = \mathcal{O}_{U_{b'}} \otimes_{k(A)} R(\beta)$ and we shall denote by $\mathcal{M}$ the $\mathcal{R}$-module $L(b') \circ L(i^\ell)$. Let $f$ be the composition
$$
\mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{O}_{X(A)}|U_{b'}}(\mathcal{R}, \mathcal{M}) \\
\longrightarrow \mathcal{H}om_{\mathcal{O}_{X(A)}|U_{b'}}(\mathcal{R}, \mathcal{M}/(1 - e(\beta', \ell \alpha_i)\mathcal{M})).
$$
Then the kernel of $f$ coincides with the sheaf
$$\{ u \in \mathcal{M} \mid e(\beta', \ell \alpha_i) \mathcal{R} u = 0 \}.$$The homomorphism $f$ factors through
$$\mathcal{M} \xrightarrow{\mathcal{T}} \mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{R}/\mathcal{R}_{\geq m}, \mathcal{M}/(1 - e(\beta', \ell \alpha_i))\mathcal{M}) \xrightarrow{\mathcal{T}} \mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{R}, \mathcal{M}/(1 - e(\beta', \ell \alpha_i))\mathcal{M})$$
for a sufficient large integer $m$. Here $\mathcal{R}_{\geq m} = \bigoplus_{k \geq m} \mathcal{R}_k$. Therefore $\mathcal{T}$ is a morphism of vector bundles on $U'$. On the other hand, $U_b$ is the set of $x \in U'$ such that the rank of $\mathcal{T}$ at $x$ is equal to its rank at the generic point. Hence $U_b$ is an open subset of $X(A)$ and the image of $\mathcal{T}|_{U_b}$ satisfies the condition for $L(b)$.

3.2. For $x \in X(A)$ and $b \in B$, let us consider the condition

(3.1) $L_x(b)$ corresponds to the upper global basis $G_{\text{up}}(b)$ by the isomorphism (2.7).

In this subsection, we shall prove the following theorem.

**Theorem 3.2.** Let $c_0$ be a point of $X(A)$ satisfying (3.1) for any $b \in B$. Then $c_0$ belongs to $U_b$ for any $b \in B$. Hence (3.1) holds also for any $x \in U_b$.

**Proof.** It is enough to show that $c_{\text{gen}}$ satisfies (3.1). We shall take a triple $(K, \mathcal{O}, k)$ such that $K = k(c_{\text{gen}})$, $\mathcal{O}$ is a discrete valuation ring, $K$ coincides with the fraction field of $\mathcal{O}$, $k$ is the residue field of $\mathcal{O}$, $(\mathcal{O}_{X(A)})_{x_0} \subset \mathcal{O}$ and $(\mathcal{O}_{X(A)})_{x_0} \subset \mathcal{O} \rightarrow k$ factors through $k(x_0)$. Such a triple exists (see [2, (7.1.7)]).

We have the reduction map
$$\text{Red}_{K, k}: K(R(\beta)_K) \longrightarrow K(R(\beta)_k)$$
by assigning $[K \otimes_{\mathcal{O}} L] \in K(R(\beta)_K)$ to $[k \otimes_{\mathcal{O}} L] \in K(R(\beta)_k)$ for a graded $R(\beta)_\mathcal{O}$-module $L$ that is finitely generated and torsion-free as an $\mathcal{O}$-module. The homomorphism $\text{Red}_{K, k}$ commutes with the duality $D$. Also it is compatible with the correspondence (2.7), namely we have a commutative diagram:

$$\bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)_K\text{-gmod}) \xrightarrow{\text{Red}_{K, k}} \bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)_k\text{-gmod})$$

$$\xrightarrow{\sim} U_A(\emptyset)^* \xleftarrow{\sim}$$
For \( b \in B \), set \( L(b)_K := L_{c_0b}(b) \) and \( L(b)_k := k \otimes_{k(c_0)} L_{c_0}(b) \). Take \( b \in B_{-\beta} \), and let \( L(b)_\mathcal{O} \) be an \( R(\beta)_\mathcal{O} \)-lattice of \( L(b)_K \), i.e., a finitely generated graded \( R(\beta)_\mathcal{O} \)-submodule \( L(\beta)_\mathcal{O} \) of \( L(b)_K \) such that \( K \otimes_{\mathcal{O}} L(\beta)_\mathcal{O} = L(b)_K \). In order to see the theorem, it is enough to show that \( k \otimes_{\mathcal{O}} L(b)_\mathcal{O} \simeq L(b)_k \).

We shall prove it by induction on \( h_t(\beta) \). Take an \( i \in I \) such that \( \varepsilon_i(b) > 0 \) and set \( b' = \varepsilon_i b \). Then \( [L(b')_K] \) corresponds to \( G^{up}(b') \) by the induction hypothesis. We take an \( R(\beta')_\mathcal{O} \)-lattice \( L(b')_\mathcal{O} \) of \( L(b')_K \). Then by the induction hypothesis, we have \( L(b')_k \simeq k \otimes_{\mathcal{O}} L(b')_\mathcal{O} \). The image of \( q^{i,0}_i F_i L(b')_\mathcal{O} \) by \( q^{i,0}_i F_i L(b')_K \to L(b)_K \) is an \( R(\beta)_\mathcal{O} \)-lattice of \( L(b)_K \), and we can take it as \( L(b)_\mathcal{O} \). Since \( q^{i,0}_i F_i L(b')_k \simeq q^{i,0}_i k \otimes_{\mathcal{O}} F_i L(b')_\mathcal{O} \to k \otimes_{\mathcal{O}} L(b)_\mathcal{O} \), the simples in a Jordan-Holder series of \( k \otimes_{\mathcal{O}} L(b)_\mathcal{O} \) appears in the one of \( q^{i,0}_i F_i L(b')_k \).

Now assume that \( q^r L(b)_k \) appears in \( \text{Red}_{K,K} L(b)_K = [k \otimes_{\mathcal{O}} L(b)_\mathcal{O}] \) for \( r \in \mathbb{Z} \) and \( b_1 \in B_{-\beta} \). Then \( q^r G^{up}(b_1) \) appears in \( q^{i,0}_i f_i G^{up}(b') \) by the assumption that \( c_0 \) satisfies (3.1). In particular, \( L(b)_K \) appears in \( [k \otimes_{\mathcal{O}} L(b)_\mathcal{O}] \) exactly once by (iiiia) in Proposition 2.2. Now assume that \( (r, b_1) \neq 0, b \). Then (iiiia) and (iiid) in Proposition 2.2 imply that \( r > 0 \). Since \( L(b)_K \) is stable by the duality functor \( D \), \( q^{-r} L(b)_k \simeq D(q^r L(b)_k) \) also appears in \( \text{Red}_{K,K} L(b)_K \). Hence \( -r > 0 \). It is a contradiction. This shows the desired result: \( k \otimes_{\mathcal{O}} L(b)_\mathcal{O} \simeq L(b)_k \). This completes the proof of Theorem 3.2. \( \square \)

**Example 3.3.** Let us give an example of a simple \( R(\beta) \)-module which does not correspond to any element in the upper global basis. Let \( g = A_1^{(1)} \) with \( I = \{0, 1\} \), \((\alpha_0|\alpha_0) = (\alpha_1|\alpha_1) = -(\alpha_0|\alpha_1) = 2\), and \( Q_{0,1}(u, v) = u^2 + auv + v^2 \). Here \( k \) is an arbitrary field and \( a \in k \). Set \( \delta = \alpha_0 + \alpha_1, b' = f_1 f_0 \phi \) and \( N = L(b') \). Then \( N = kv \) with \( x_1 v = x_2 v = \tau_1 v = 0 \) and \( v = e(01) v \). Set \( M = N \circ N \), and \( u = v \otimes v \in M \). Then \( \text{ch}(M) = 2e(0101) + [2]^2 e(0011) \). Here \( e(0101) M = ku \otimes kw \) with \( w := \tau_2 \tau_3 \tau_1 \tau_2 u \). By the weight consideration, \( \tau_k e(0101) M = 0 \) for \( k = 1, 3 \) and \( \tau_k e(0101) M = 0 \) for \( 1 \leq k \leq 4 \). Easy calculations show that \( \tau_2 w = -a \tau_2 u \). Hence \( y := w + au \) is annihilated by all \( x_k \)'s and \( \tau_k \)'s and \( ky \) is an \( R(2\delta) \)-submodule of \( M \). Set \( M_0 = M/k y \). Then \([M_0]\) corresponds to \( G^{up}(b) \) with \( b := f_1 f_0 \phi \). It is easy to see that \( M_0 \) is a simple \( R(2\delta) \)-module if \( a \neq 0 \). When \( a = 0 \), \( e(0011) M_0 \) is a simple \( R(2\delta) \)-submodule of \( M_0 \) and \( L(b) = e(0011) M_0 \). Note that the case (2.8) is when \( a = \pm 2 \).

**Example 3.4.** Let us give another example of a simple \( R(\beta) \)-module which does not correspond to any element in the upper global basis. Let \( g = A_2^{(1)} \) with \( I = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\} \) with \((\alpha_i|\alpha_i) = 2\) and \((\alpha_i|\alpha_j) = -1\) for \( i \neq j \) and \( Q_{i,i+1}(u, v) = a_i u + b_{i+1} v \).
(i \in I)$ with $a_i, b_i \in \mathbb{k}^\times$, where $\mathbb{k}$ is an arbitrary field. Set $\delta = \alpha_0 + \alpha_1 + \alpha_2$, $b' = \tilde{f}_2 \tilde{f}_1 \tilde{f}_0 \phi$ and $N = L(b')$. Then $N = \mathbb{k}v$ with $x_kv = \tau_k v = 0$ and $v = e(012)v$. Set $M = N \circ N$ and $u = v \otimes v \in M$. Then $\text{ch}(M) = 2e(01212) + [2]^3 e(01122) + [2]^2 e(010122) + [2]^2 e(010122) + 2e(01212)$. Here $e(01212)M = ku \oplus kw$ with $w' := \tau_3 \tau_4 \tau_5 \tau_2 \tau_3 \tau_4 \tau_1 \tau_2 \tau_3 u$.

By the weight consideration $\tau_k e(01212)M = 0$ for $k \neq 3$ and $x_k e(01212)M = 0$ for $1 \leq k \leq 6$. By calculations, we have $\tau_3 w = -\gamma \tau_3 u$ where $\gamma = a_0 a_1 a_2 - b_0 b_1 b_2$. Hence $y := w + \gamma u$ is annihilated by all $x_k$’s and $\tau_k$’s and $\mathbb{k}y$ is an $R(2\delta)$-submodule of $M$. Set $M_0 = M/\mathbb{k}y$. Then $[M_0]$ corresponds to $G_{\text{up}}(b)$ with $b := \tilde{f}_2^3 \tilde{f}_1^2 \tilde{f}_0 \phi$. It is easy to see that $M_0$ is a simple $R(2\delta)$-module if $\gamma \neq 0$. When $\gamma = 0$, $S := (1 - e(01212))M_0 = R(2\delta)\tau_3 u$ is a simple $R(2\delta)$-module and $L(b) = S$ and $\text{ch}(M_0/S) = e(01212)$. Note that the case (2.8) corresponds to $a_0 a_1 a_2 + b_0 b_1 b_2 = 0$.

**Remark 3.5.** If we assume

(3.2) the simple modules of $R(\beta)_{crit}$ correspond to the upper global basis,

then $G_{\text{up}}(b) \in \sum_{S \in \text{Tr}(\beta)} \mathbb{Z}_{\geq 0}[q, q^{-1}][S]$ for any $x \in X(A)$ and $b \in B$. We can ask if this positivity assertion still holds without the assumption (3.2).

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