K-theory of quasi-toric manifolds

P.Sankaran and V.Uma

AMS Subject Classification: Primary: 55N15, Secondary: 14M25

keywords: Quasi-toric manifolds, K-theory, Bott towers, Bott-Samelson varieties

Abstract

In this note we shall give a description of the $K$-ring of a quasi-toric manifold in terms of generators and relations. We apply our results to describe the $K$-ring of Bott-Samelson varieties.

1 Introduction

The notion of quasi-toric manifolds is due to M.Davis and T.Januszkiewicz [7] who called them ‘toric manifolds’. The quasi-toric manifolds are a natural topological generalization of the algebraic geometric notion of non-singular projective toric varieties. However there are compact complex non-projective non-singular toric varieties which are quasi-toric manifolds. See [3]. Recently Civan [5] has constructed an example of a compact complex non-singular toric variety which is not a quasi-toric manifold.

In [15], we obtained, among other things, a description of the $K$-ring of projective non-singular toric varieties in terms of generators and relations. (In fact our result is applicable to slightly more general class of varieties.) The purpose of this note is to extend the $K$-theoretic results of [15] to the context of quasi-toric manifolds. As an application we obtain a description of the $K$-ring of Bott-Samelson varieties. The more difficult problem of computing the $KO$-theory has been solved by A.Bahri and M.Bendersky.

Let $G = (\mathbb{S}^1)^n$ be an $n$-dimensional compact torus and let $P \subset \mathbb{R}^n$ be a simple convex polytope of dimension $n$. That is, $P$ is a convex polytope in which exactly $n$ facets — codimension 1 faces of $P$ — meet at each vertex of $P$. A $G$-quasi-toric manifold over $P$ is a (smooth) $G$-manifold $M$ where the $G$-action is locally standard with projection $\pi : M \rightarrow M/G \cong P$. Here ‘local standardness’ means that every point of $M$ has an equivariant neighbourhood $U$ such that there exists an automorphism $\theta : G \rightarrow G$, an equivariant open subset $U' \subset \mathbb{C}^n$ where $G$ action on $\mathbb{C}^n$ is given by the standard inclusion of $G \subset U(n)$, and a diffeomorphism $f : U \rightarrow U'$ where $f(t \cdot x) = \theta(t)f(x)$ for all $x \in U, t \in G$. Any two points of
$$\pi^{-1}(p)$$ have the same isotropy group its dimension being codimension of the face of $$P$$ which contains $$p$$ in its relative interior. It is known that $$M$$ admits a CW-structure with only even dimensional cells. In particular $$M$$ is simply connected and hence orientable.

Let $$\mathcal{F}_P$$ (or simply $$\mathcal{F}$$) denote the set of facets of $$P$$ and let $$|\mathcal{F}| = d$$. For each $$F_j \in \mathcal{F}$$, let $$M_j = \pi^{-1}(F_j)$$ and $$G_j$$ be the (1-dimensional) isotropy subgroup at any ‘generic’ point of $$M_j$$. Then $$M_j$$ is orientable for each $$j$$. The subgroup $$G_j$$ determines a primitive vector $$v_j$$ in $$\mathbb{Z}^n = \text{Hom}(S^1, G)$$ which is unique up to sign. The sign is determined by choosing an omni-orientation on $$M$$, i.e. orientations on $$M$$ as well as one on each $$M_j$$, $$1 \leq j \leq d$$. Choosing such a $$v_j$$ for $$1 \leq j \leq d$$ defines the ‘characteristic map’ $$\lambda : \mathcal{F} \to \mathbb{Z}^n \cong \text{Hom}(S^1, G)$$ where $$F_j \mapsto v_j$$. Suppose that $$F_1, \ldots, F_d$$ are the facets of $$P$$, then writing $$v_j = \lambda(F_j)$$, the primitive vectors $$v_1, \ldots, v_d$$ are such that

$$\text{if } \bigcap_{1 \leq r \leq k} F_{j_r} \subset P \text{ is of codimension } k \text{ then}$$

$$v_{j_1}, \ldots, v_{j_k} \text{ extends to a } \mathbb{Z}\text{-basis } v_{j_1}, \ldots, v_{j_k}, w_1, \ldots, w_{n-k} \text{ of } \mathbb{Z}^n. \quad (1.1)$$

Fix an orientation for $$M$$. The omni-orientation on $$M$$ determined by $$\lambda$$ is obtained by orienting $$M_j$$ so that the oriented normal bundle corresponds to the 1-parameter subgroup given by $$v_j$$.

We shall call any map $$\lambda : \mathcal{F}_P \to \mathbb{Z}^n$$ that satisfies (1.1) a characteristic map.

Conversely, starting with a pair $$(P, \lambda)$$ where $$P$$ is any simple convex polytope and a characteristic map $$\lambda : \mathcal{F} \to \mathbb{Z}^n$$, there exists a quasi-toric manifold $$\tilde{M}$$ over $$P$$ whose characteristic map is $$\lambda$$. The data $$(P, \lambda)$$ determines the $$G$$-manifold $$\tilde{M}$$ and an omni-orientation on it. We refer the reader to [1] and [2] for basic facts concerning quasi-toric manifolds.

Suppose that $$P$$ is a simple convex polytope of dimension $$n$$ and that $$F_j \mapsto v_j, 1 \leq j \leq d$$ is a characteristic map $$\lambda : \mathcal{F} \to \mathbb{Z}^n$$. Assume that $$F_1 \cap \cdots \cap F_n$$ is a vertex of $$P$$ so that $$v_1, \ldots, v_n$$ is a $$\mathbb{Z}\text{-basis of } \mathbb{Z}^n$$. Let $$S$$ be any commutative ring with identity and let $$r_1, \ldots, r_n$$ be invertible in $$S$$.

**Definition 1.1.** Consider the ideal $$\mathcal{I}$$ of the polynomial algebra $$S[x_1, \ldots, x_d]$$ generated by the following two types of elements:

$$x_{j_1} \cdots x_{j_k} \quad (1.2)$$

whenever $$F_{j_1} \cap \cdots \cap F_{j_k} = \emptyset$$, and the elements

$$z_u := \prod_{j, u(v_j) > 0} (1 - x_j)^{u(v_j)} - r_u \prod_{j, u(v_j) < 0} (1 - x_j)^{-u(v_j)} \quad (1.3)$$

where $$u \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \text{Hom}(G, S^1)$$ and $$r_u := \prod_{1 \leq i \leq n} r_i^{u(v_i)}$$. We denote the quotient $$S[x_1, \ldots, x_d]/\mathcal{I}$$ by $$\mathcal{R}(S; \lambda)$$ or simply by $$\mathcal{R}$. 

2
Let $E \rightarrow B$ be a principal $G$-bundle with base space $B$ a compact Hausdorff space. Denote by $E(M)$ the associated $M$-bundle with projection map $p : E \times_G M \rightarrow B$. The choice of the basis $v_1 \cdots, v_n$ for $\mathbb{Z}^n = \text{Hom}(S^1, G)$ yields a product decomposition $G = \prod_{1 \leq i \leq n} G_i \cong (S^1)^n$. Also one obtains principal $S^1$-bundles $\xi_i$, $1 \leq i \leq n$, over $B$ associated to the $i$-th projection $G = (S^1)^n \rightarrow S^1$. The projection $E \rightarrow B$ is then the projection of the bundle $\xi_1 \times \cdots \times \xi_n$ over $B$. For any $G$-equivariant vector bundle $V$ over $M$ denote by $\mathcal{V}$ the bundle over $E(M)$ with projection $E(V) \rightarrow E(M)$. We shall often denote the complex line bundle associated to a principal $S^1$ bundle $\xi$ also by the same symbol $\xi$.

Suppose that $V$ is the product bundle $M \times \mathbb{C}_\chi$, where $\mathbb{C}_\chi$ is the 1-dimensional $G$-representation given by the character $\chi : G \rightarrow S^1$. Then $\mathcal{V}$ is isomorphic to the pull-back of the bundle $p^*(E_\chi)$, where $E_\chi$ is obtained from $E \rightarrow B$ by ‘extending’ the structure group to $S^1$ via the character $\chi$. Writing $\chi = \sum_{1 \leq i \leq n} a_i \rho_i$, where $\rho_i : G \rightarrow S^1$ is the $i$-th projection, one has $\mathcal{V} \cong p^*(\xi_1^{a_1} \cdots, \xi_n^{a_n})$, where $\xi^a = (\xi^*)^{-a}$ when $a < 0$.

**Theorem 1.2.** Let $M$ be a quasi-toric manifold over a simple convex polytope $P \subset \mathbb{R}^n$ and characteristic map $\lambda : \mathcal{F} \rightarrow \mathbb{Z}^n$. Let $E \rightarrow B$ be a principal $G = (S^1)^n$ bundle over a compact Hausdorff space. With the above notations, there exist equivariant line bundles $L_j$ over $M$ such that, setting $r_i = [\xi_i] \in K(B)$, $1 \leq i \leq n$, one has an isomorphism of $K(B)$-algebras $\varphi : R(K(B); \lambda) \rightarrow K(E(M))$ defined by $x_j \mapsto (1 - [L_j])$.

The proof is given in §3. A technical lemma needed in the proof is established in §2. In §4, we apply our result to obtain the $K$-ring of Bott-Samelson varieties.

## 2 Generators of $R$

We keep the notations of the previous section. In this section we give a convenient generating set for $S$-module $R(S; \lambda)$ where $S$ is any commutative ring with identity and $\lambda : \mathcal{F}_p \rightarrow \mathbb{Z}^n$ a characteristic map, $P \subset \mathbb{R}^n$ being a simple convex $n$ dimensional polytope.

Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a linear map which is injective when restricted to the set $P_0$ of vertices of $P$. Then $h$ is a generic “height function” with respect to the polytope $P$. That is, $h$ is injective when restricted to any facet of $P$. The map $h$ induces an ordering on the set $P_0$ of vertices of $P$, where $w < w'$ in $P_0$ if $h(w) < h(w')$. The ordering on $P_0$ induces an orientation on the edges $P_1$ of $P$ in the obvious fashion. Since $P$ is simple, there are exactly $n$ edges which meet at each vertex of $P$. Given any $w \in P_0$, denote by $T_w$ the face of $P$ spanned by all those edges incident at $w$ which point away from $w$. Then the following property holds:

$$\text{if } w' \in P_0 \text{ belongs to } T_w, \text{ then } w \leq w'. \quad (2.1)$$

This is a consequence of the assumption that $P$ is simple and can be proved easily using for example Lemma 3.6 of [18]. Property (2.1) is ‘dual’ to property (*) of §5.1, [18].
It is shown in [7] that $M$ has a perfect cell decomposition with respect to which the
submanifolds $M_w := \pi^{-1}(T_w)$ are closed. (Cf. [2], and [13].)

If $Q$ is a proper face of $P$, we denote by $x(Q)$ the product $x_{j_1}\cdots x_{j_r}$
where $F_{j_1}, \ldots, F_{j_r}$ are the distinct facets of $P$ which contain $Q$.

**Lemma 2.1.** With the above notation, the elements $x(T_w), \ w \in P_0$, generate $\mathcal{R}$ as an
$S$-module. \hfill \Box

The proof is identical to that of lemma 2.2(iv), [15] using (2.1) in the place of property (*)
of [15]. Proof of lemma 2.2(i), [15], which was omitted, had an error. (See Errata [15].)
However it is redundant here since $z_u = 0$ in $\mathcal{R}$ in view of equation (1.3) in Definition [11]
above.

### 3 Proof of Theorem 1.2

We keep the notations of the previous section. Let $M$ be the quasi-toric manifold over a
simple convex polytope $P \subset \mathbb{R}^n$ with characteristic map $\lambda : \mathcal{F} \to \mathbb{Z}^n$. Let $|\mathcal{F}| = d$. As in the
previous section we shall assume that $\bigcap_{1 \leq i \leq n} F_i$ is a vertex of $P$ and $G = \prod_{1 \leq i \leq n} G_i \cong (S^1)^n$
the corresponding product decomposition.

Set $\tilde{G} = (S^1)^d$ and let $\theta : \mathcal{F} \to \mathbb{Z}^d$ be defined by $F_j \mapsto e_j$, the standard basis vector, for
each $j \leq d$. For any face $F = F_{j_1} \cap \cdots \cap F_{j_k}$, set $\tilde{G}_F$ denote the subgroup \{(\(t_1, \ldots, t_d\)) \in \tilde{G} : t_i = 1, \quad i \neq j_1, \ldots, j_k\}. One has a $\tilde{G}$-manifold $Z_P := \tilde{G} \times P/ \sim$ where $(g, p) \sim (g', p')$
if and only if $p = p'$ and $g^{-1}g' \in \tilde{G}_F$ where $p$ is in the relative interior of the face $F \subset P$.
The action of $G$ on $Z_P$ is given by $g.[g', p] = [gg', p]$ for $g \in \tilde{G}$ and $[g', p] \in Z_P$. One has the projection map $p : Z_P \to P$. However $Z_P$ is not a quasi-toric manifold over $P$ since $d > n$.
When $P$ is clear from the context we shall denote $Z_P$ simply by $Z$.

Let $\tilde{\lambda} : \mathbb{Z}^d \to \mathbb{Z}^n$ be defined as $e_j \mapsto v_j := \lambda(F_j), \ 1 \leq j \leq d$. This corresponds to
a surjective homomorphism of groups $\Lambda : \tilde{G} \to G$ with kernel $\Lambda \cong \tilde{G}$ for the subgroup
corresponding to $\ker(\lambda)$. One has a splitting $\mathbb{Z}^d = \ker(\lambda) \oplus \mathbb{Z}^n$ induced by the injection
$\mathbb{Z}^n \to \mathbb{Z}^d$ defined as $v_i \mapsto e_i, 1 \leq i \leq n$. This injection corresponds to an imbedding
$\Gamma : G \to \tilde{G}$. Identifying $\mathbb{Z}^d$ with $Hom(S^1, \tilde{G})$, the splitting yields an identification $\tilde{G} \cong G \times H$,
$\tilde{g} = g.h$ where $g = \Gamma \circ \lambda(\tilde{g}) \in G$ and $h = g^{-1}\tilde{g}$. The group $H \cong (S^1)^{d-n}$ is the subgroup of
$\tilde{G}$ with $\ker(\tilde{\Lambda}) = Hom(S^1, H) \subset Hom(S^1, \tilde{G})$. We let group $H$ act freely on the right of $Z$
where $x.h = h^{-1}x \in Z$ for $x \in Z, \ h \in H \subset \tilde{G}$. The quotient of $Z$ by $H$ is the quasi-toric
manifold $M$. (cf. §4. [2].)

Let $\chi : H \to S^1$ be the restriction to $H$ of any character again denoted $\chi : \tilde{G} \to S^1$.
One obtains a $G$-equivariant complex line bundle $L_\chi$ over $M$ with projection $Z \times_H \mathbb{C}_\chi \to M$
where $\mathbb{C}_\chi$ denotes the 1-dimensional complex representation space corresponding to $\chi$. Here
the Borel construction $\mathcal{Z} \times_H \mathbb{C}_\chi$ is obtained by the identification

$$ (xh, z) \sim (x, \chi(h)z), \quad h \in H, \quad x \in \mathcal{Z}, \quad z \in \mathbb{C}. $$

Equivalently $\mathcal{Z} \times_H \mathbb{C}_\chi$ is the quotient of the diagonal action by $H$ on the left on $\mathcal{Z} \times \mathbb{C}_\chi$. The equivalence class of $(x, z)$ is denoted by $[x, z]$. The $G$-action on $L_\chi$ is given by $g.[x, z] := [gx, \chi(g)z]$ for $x \in \mathcal{Z}$, $z \in \mathbb{C}_\chi$.

When $\chi = \rho_j$ is the $j$-th projection $\tilde{G} \to S^1$, the corresponding $G$-line bundle on $M$ will be denoted $L_j$. Denote by $\pi_j : L_j \to M$ the projection of the bundle $L_j$.

Henceforth we shall identify the character group of $\tilde{G}$ with $\text{Hom}(\mathbb{Z}^d, \mathbb{Z})$ etc. If $u \in \text{Hom}(\mathbb{Z}^d, \mathbb{Z})$ vanishes on $\ker(\tilde{\chi})$, then the line bundle $L_u$ is isomorphic to the product bundle. However the $G$ action on it is given by the character $u|G$.

Given $u \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) = \text{Hom}(G, S^1)$, composing with surjection $\tilde{G} \to G$, we obtain a character of $\tilde{G}$ which is trivial on $H$. As an element of $\text{Hom}(\mathbb{Z}^d, \mathbb{Z})$, this is just the composition $\tilde{u} = u \circ \tilde{\chi}$. Let $e_1^*, \ldots, e_d^*$ be the dual of the standard basis for $\mathbb{Z}^d$. Note that the character $\tilde{G} \to S^1$ corresponding to $e_j^*$ is just the $j$-th projection $\rho_j$. Clearly,

$$ \tilde{u} = u \circ \tilde{\chi} = \sum_{1 \leq j \leq d} u(\tilde{\chi}(e_j))e_j^* = \sum_{1 \leq j \leq d} u(v_j)e_j^*. $$

Hence we obtain the following isomorphism of $G$-bundles:

$$ L_{\tilde{u}} = \prod_{1 \leq j \leq d} L_j^{u(v_j)}. \quad (3.2) $$

Note that since $\tilde{u}|H$ is trivial, $\mathcal{Z} \times_H \mathbb{C}_{\tilde{u}} = M \times \mathbb{C}$ and so $L_{\tilde{u}}$ is isomorphic to the product bundle.

Let $1 \leq j \leq d$. Choose an affine map $h_j : \mathbb{R}^n \to \mathbb{R}$ such that $h_j$ vanishes on $F_j$ and $h_j(p) > 0$ for $p \in P \setminus F_j$. Since $\tilde{G}_{F_j} := \tilde{G}_j$ acts freely on $\mathcal{Z} \setminus p^{-1}(F_j)$, one has a well-defined trivialization $\sigma_j : \pi_j^{-1}(M - M_j) \to (M - M_j) \times \mathbb{C}_\chi$ given by $\sigma_j([x, z]) = ([x], \rho_j(g^{-1})z)$ where $x = [g, p] \in \mathcal{Z}$, $z \in \mathbb{C}_\chi$.

Using $\sigma_j$ and $h_j$ one obtains a well-defined section $s_j : M \to L_j$ by setting $s_j([x]) = [x, h_j(p)\rho_j(g)]$ where $x = [g, p] \in \mathcal{Z}$. Note that the section $s_j$ vanishes precisely on $M_j$. It is straightforward to verify that $s_j$ is $G$ equivariant.

Now let $1 \leq j_1, \ldots, j_k \leq d$ be such that $F_{j_1} \cap \cdots \cap F_{j_k} = \emptyset$. Thus $M_{j_1} \cap \cdots \cap M_{j_k} = \emptyset$. Consider the section $s : M \to V$ defined as $s(m) = (s_{j_1}(m), \ldots, s_{j_k}(m))$ where $V$ is (the total space of) the vector bundle $L_{j_1} \oplus \cdots \oplus L_{j_k}$. The section $s$ is nowhere vanishing: indeed $s(m) = 0 \iff s_{j_r}(m) = 0 \forall r \iff m \in M_{j_r} \forall r$. Since $\cap_{1 \leq r \leq k}M_{j_r} = \emptyset$, we see that $s$ is nowhere vanishing. Since $V$ has geometric dimension at most $k - 1$, applying the $\gamma^k$-operation to $[V] - k$ we obtain $\gamma^k([V] - k) = \gamma^k(\mathbb{R}^{j_1 \cup j_r \leq k}([L_{j_r}] - 1)) = \prod_{1 \leq r \leq k}([L_{j_r}] - 1)$. Therefore

$$ \prod_{1 \leq r \leq k} (1 - [L_{j_r}]) = 0 \quad (3.3) $$
whenever $\cap_{k} F_{j_{r}} = \emptyset$.

**Remark 3.1.** (i) Let $\tilde{L}_{j}$ denote the pull-back of $L_{j}$ by the quotient map $Z \to M$. Since $H$ acts freely on $Z$, $\tilde{L}_{j}$ is isomorphic to the product bundle. This is the same as dual of the bundle $L_{j}$ considered in §6.1 of [7]. A description of the stable tangent bundle of $M$ was obtained in Theorem 6.6 of [7]. It follows from their proof that the $L_{j}|_{M_{j}}$ is isomorphic to the normal bundle to the imbedding $M_{j} \subset M$. Therefore we have $c_{1}(L_{j}) = e(L_{j}) = \pm [M_{j}] \in H^{2}(M; \mathbb{Z})$ where $e(L_{j})$ denotes the Euler class of $L_{j}$ (see [14]). The omni-orientation corresponding to $\lambda$ is so chosen as to have $c_{1}(L_{j}) = +[M_{j}]$.

(ii) The complex projective $n$-space $\mathbb{P}^{n}$ is a quasi-toric manifold over the standard $n$-simplex $\Delta^{n} = \{x = \sum_{1 \leq i \leq n} x_{i} e_{i} \in \mathbb{R}^{n} \mid \sum_{1 \leq i \leq n} x_{i} \leq 1, 0 \leq x_{i} \leq 1 \ \forall i \geq 1\}$. The characteristic map $\lambda$ sends the facet $\Delta^{n}_{0} = \{x \in \Delta^{n} \mid x_{i} = 0\}$ opposite the vertex $e_{i}, 1 \leq i \leq n$, to the standard basis element $v_{i} = e_{i} \in \mathbb{Z}^{n}$ for $i > 0$ and sends the facet opposite the origin $\Delta^{n}_{0} = \{x \in \Delta^{n} \mid \sum_{1 \leq j \leq n} x_{j} = 1\}$ to the vector $v_{0} := -(e_{1} + \cdots + e_{n}) \in \mathbb{Z}^{n}$. The line bundle $L_{j}$ is then verified to be isomorphic to the dual of tautological bundle over $\mathbb{P}^{n}$ for $0 \leq j \leq n$. $M$ is canonically oriented as a complex manifold. The omni-orientation corresponding to this choice of $\lambda$ on $M$ determined by $\lambda$ is the orientation on $M_{j} \cong \mathbb{P}^{n-1}$ determined by its complex structure. (See also Example 5.19, [3].)

**Proposition 3.2.** With notations as above, let $\mathcal{R} = \mathcal{R}(\mathbb{Z}; \lambda)$ where $r_{i} = 1 \ \forall i \leq n$. One has a well-defined homomorphism $\psi : \mathcal{R} \to K(M)$ of rings which is in fact an isomorphism.

**Proof:** Relations (3.3) and (3.2) above clearly imply that $\psi$ is a well-defined algebra homomorphism.

From Theorem 4.14 [7], the integral cohomology of $M$ is generated by degree 2 elements. Indeed these can be taken to be dual cohomology classes $[M_{j}], 1 \leq j \leq d$, where $M_{j} = \pi^{-1}(F_{j})$, $F_{j}$ being the facets of $P$. As noted in Remark 3.1 $c_{1}(L_{j}) = [M_{j}], 1 \leq j \leq d$.

Now Lemma 4.1, [13], implies that $K(M)$ is generated by the line elements $[L_{j}], 1 \leq j \leq d$. This shows that $\psi$ is surjective. To show that it is injective, we observe that since $M$ is a CW complex with cells only in even dimensions, $K(X)$ is a free abelian group of rank $\chi(M)$ the Euler characteristic of $M$. But $\chi(M) = m$, the number of vertices of $P$. (In fact a $\mathbb{Z}$-basis for the integral cohomology of $M$ is the set of dual cohomology classes $[M_{w}], w \in P_{0}$.) Since by Lemma 2.1 the rank of $\mathcal{R}$, as an abelian group, is at most $m$, it follows that $\psi$ is in fact an isomorphism of rings.

We shall now prove the main theorem.

**Proof of Theorem 1.2** Note that the complex line bundles $L_{j}$ are $G$-equivariant. Denote by $\mathcal{L}_{j}$ the bundle $E(L_{j}) := E \times_{G} L_{j}$ over $E(M) = E \times_{G} M$. Since the sections $s_{j}$ are equivariant, so is the section $s = (s_{j_{1}}, \cdots, s_{j_{k}})$ of $V = L_{j_{1}} \oplus \cdots \oplus L_{j_{k}}$. Hence we obtain
a section $E(s): E(M) \rightarrow E(V)$. If $\bigcap_{1 \leq r \leq k} F_{j_r} = \emptyset$, then $s$ and hence $E(s)$ is nowhere vanishing. Again by applying the $\gamma^k$-operation to $[E(V)] - k \in K(E(M))$, we conclude that

$$\prod_{1 \leq r \leq k} ([L_{j_r}] - 1) = 0$$

whenever $\bigcap_{1 \leq r \leq k} F_{j_r} = \emptyset$.

Since the isomorphism in equation (3.2) is $G$-equivariant, one obtains an isomorphism $L_u = \prod_{1 \leq j \leq d} L_j^{u(v_j)}$. Since $L_u$ is the product bundle $M \times \mathbb{C}_u \rightarrow M$, the bundle $L_u \cong p^*(\xi_1^{a_1} \cdots \xi_n^{a_n})$ where $a_i = u(v_i)$. (See §1.) It follows that, in the $K(B)$-algebra $K(E(M))$ one has

$$\prod_{1 \leq j \leq d} [L_j]^{u(v_j)} = [\xi_1^{a_1} \cdots \xi_n^{a_n}].$$

(3.5)

Setting $r_i = [\xi_i]$ for $1 \leq i \leq n$, in view of (3.4) and (3.5) we see that there is a well-defined homomorphism of $K(B)$-algebras $\psi: \mathcal{R}(K(B); \lambda) \rightarrow K(E(M))$ which maps $x_j$ to $(1 - [L_j])$ for $1 \leq j \leq d$.

It follows from Prop. 3.2 that, the monomials in the $L_j$ generate $K(M)$. Hence the fibre-inclusion $M \rightarrow E(M)$ is totally non-cohomologous to zero in $K$-theory as the bundles $L_j$ restrict to $L_j$. As $B$ is compact Hausdorff and $K(M)$ is free abelian, we observe that the hypotheses of Theorem 1.3, Ch. IV, [12], are satisfied. It follows that $K(E(M)) \cong K(B) \otimes K(M)$ as a $K(B)$-module. In particular, we conclude that $\psi$ is surjective. To see that $\psi$ is a monomorphism, note that $K(E(M))$ is a free module over $K(B)$ of rank $\chi(M) = m$, the number of vertices in $P$. Since, by Lemma 2.1, $\mathcal{R}(K(B), \lambda)$ is generated by $m$ elements, it follows that $\psi$ is an isomorphism.

To conclude this section, we illustrate the above theorem in the case when $M$ is the $n$-dimensional complex projective space. We remark that this special case follows immediately from Theorem 2.16, Ch. VI of [12] as well.

The complex projective $n$-space $\mathbb{P}^n$ is a quasi-toric manifold over the standard $n$-simplex $\Delta^n = \{ x = \sum_{1 \leq i \leq n} x_i e_i \in \mathbb{R}^n \mid \sum_i x_i \leq 1, \ 0 \leq x_i \leq 1 \ \forall i \geq 1 \}$. The characteristic map $\lambda$ sends the $i$-th facet — the face opposite the vertex $e_i$ — to the standard basis element $v_i := e_i \in \mathbb{Z}^n$ for $i \geq 1$ and sends the $0$-th facet which is opposite the vertex $v_0 := -(e_1 + \cdots + e_n) \in \mathbb{Z}^n$. The space $E(\mathbb{P}^n)$ is just the projective space bundle $\mathbb{P}(1 \oplus \xi_1 \oplus \cdots \oplus \xi_n)$ over $B$. Here $1$ denotes the trivial complex line bundle $\mathbb{C}e_0 \rightarrow B$ over $B$. Indeed the map which sends $[e, x] = ([w_1, \cdots, w_n], [x_0, \cdots, x_n])$ to the complex line spanned by the vector $x_0 e_0 + x_1 w_1 + \cdots + x_n w_n$ in the fibre $\mathbb{P}(\mathbb{C}e_0 + \mathbb{C}w_1 + \cdots + \mathbb{C}w_n)$ over $\pi(e) \in B$, where $e = (w_1, \cdots, w_n)$, is a well defined bundle isomorphism.

Example 3.3. (Cf. Chapter IV, Theorem 2.16, [12].) $K(E(\mathbb{P}^n)) \cong K(B)[y]/\langle \prod_{0 \leq i \leq n} (y - [\xi_i]) \rangle$ where $\xi_0 := 1$ and $y$ is the class of the canonical bundle over $E(\mathbb{P}^n)$ which restricts to the tautological bundle on each fibre of $E(\mathbb{P}^n) \rightarrow B$.  

7
**Proof:** By choosing \( u \in Hom(\mathbb{Z}^n, \mathbb{Z}) \) to be the dual basis element \( e_i^* \), \( i \geq 1 \), relation (1.3) gives \( [L] = [\xi][L_0] \) in \( R(K(B); \lambda) \) since \( \xi_0 = 1 \). It can be seen easily that other choices of \( u \) in relation (1.3) do not lead to any new relation. Substituting for \([L]\) in relation (1.2), we obtain \( \prod_{0 \leq i \leq n}(1 - [\xi_i][L_0]) = 0 \). Setting \( y = [L_0]^{-1} \) we obtain \( \prod((\xi_i) - y) = 0 \). By theorem \ref{1.2} it follows that \( K(E(\mathbb{P}^n)) \cong K(B)[y]/(\prod((\xi_i) - y)) \). Since \( y = [L_0] \), the proof is completed by observing that \( L_0 \) restricts to the tautological bundle on the fibres \( \mathbb{P}^n \). \( \square \)

4 Bott-Samelson varieties

In this section we illustrate our theorem in the case of Bott-Samelson manifolds which were first constructed in \[4\] to study cohomology of generalized flag varieties. M. Demazure and D. Hansen used it to obtain desingularizations of Schubert varieties in generalized flag varieties. M. Grossberg and Y. Karshon \[10\] constructed Bott towers, which are iterated fibre bundles with fibre at each stage being \( \mathbb{P}^1_C \). They also showed that Bott-Samelson variety can be deformed to a toric variety. The ‘special fibre,’ of this deformation is a Bott tower. The underlying differentiable structure is preserved under the deformation. It follows that Bott-Samelson varieties have the structure of a quasi-toric manifold with quotient polytope being the \( n \)-dimensional cube \( I^n \) where \( n \) is the complex dimension of the Bott-Samelson variety. This quasi-toric structure has been explicitly worked out by Grossberg-Karshon \[10\] and by M. Willems \[16\]. In this section we use Example 3.3 to describe the \( K \)-ring of the Bott towers in terms of generators and relations. Perhaps our theorem \ref{4.2} is well-known to experts but we could not find it explicitly stated in the literature.

Let \( C = (c_{i,j}) \) denote an \( n \)-by-\( n \) unipotent upper triangular matrix with integer entries. The matrix \( C \) determines a Bott tower \( M(C) \) of (real) dimension \( 2n \). Using the notation of \$3\$, it turns out that \( Z = (S^1)^n \times I^n / \sim \) is the space \( (S^3)^n \subset C^{2n} \). The quasi-toric manifold \( M(C) \) is the quotient of \( (S^3)^n \) by the action of \( H = (S^1)^n \) on the right of \( Z \) where

\[
(z_1, w_1, \cdots, z_n, w_n).t_i = (z_1, w_1, \cdots, z_i t_i, w_i t_i, \cdots, z_n, w_n t_i^{c_{i,n}})
\]

for \( t_i \) in the \( i \)-th factor of \( H \). The group \( G = (S^1)^n \) acts on the left of \( Z = (S^3)^n \) by \( t.(z_1, w_1, \cdots, z_n, w_n) = (z_1, t_1^{-1} w_1, \cdots, z_n, t_n^{-1} w_n) \). This descends to an action on \( M(C) \).

For \( 1 \leq i \leq n \) denote by \( L_i \) the complex line bundle over \( M(C) \) associated to the character \( \rho_i : H \rightarrow S^1 \) defined as the projection to the \( i \)-th coordinate. Thus the total space of \( L_i \) is obtained as the fibre product \( S^3 \times_H C \) where \( ((z_1, w_1, \cdots, z_n, w_n), t, \lambda) \sim ((z_1, w_1, \cdots, z_n, w_n), t_i \lambda) \) for \( t = (t_1, \cdots, t_n) \in H \) and \( \lambda \in C \). We shall refer to \( L_i \) as the \( i \)-th canonical bundle over \( M(C) \).

Let \( C_i \) be the matrix obtained as the top \( i \)-by-\( i \) diagonal block of \( C \), \( 1 \leq i \leq n \). Let \( M(C_0) \) be the space consisting of a single point and let \( M(C_1) = S^2 = \mathbb{P}^1 \). We shall denote by \( H_i = (S^1)^i \) the subgroup of \( H \) where the last \( n - i \) coordinates are the identity element. For each \( i \geq 1 \), one has the corresponding Bott tower \( M(C_i) \). Let \( L_j \) denote the \( j \)-th canonical
bundle over $M(C_i)$, $1 \leq j \leq i$. Consider the projection $\pi_i : M(C) \to M(C_i)$ induced by the projection $\left(S^3\right)^n \to \left(S^3\right)^i$ onto the first $i$-coordinates. Then $\pi_i^*(L_j) \cong L_j$ for $1 \leq j \leq i$. Indeed one has a commuting diagram

$$\begin{array}{ccc}
(S^3)^n \times_H \mathbb{C} & \to & (S^3)^i \times_H \mathbb{C} \\
\downarrow & & \downarrow \\
M(C) & \to & M(C_i)
\end{array}$$

where top horizontal map is the ‘bundle map’ defined as $[(z_1, w_1, \ldots, z_n, w_n), \lambda] \mapsto [(z_1, w_1, \ldots, z_i, w_i), \lambda]$, the vertical maps are projections of the bundle $L_j$ and $L_j$. It follows from Lemma 3.1, [14] that $\pi_i^*(L_j) \cong L_j$. In view of this, by abuse of notation, we shall denote by the same symbol $L_j$ the $j$-th canonical bundle $L_j$ on $M(C_i)$.

Consider the map $\pi_{i+1} : M(C_{i+1}) \to M(C_i)$ induced from the projection onto the first $i$-factors $(S^3)^{i+1} \to (S^3)^i$. The map $\pi_{i+1}$ is the projection of the $\mathbb{S}^2 = \mathbb{P}_C^1$-bundle associated with the complex vector bundle $1 \oplus L_i$ where $L_i$ is given by the character $(c_1, \ldots, c_{i+1}) \in \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \text{Hom}(H^4, \mathbb{S}^1)$ (see relation (3.1) in §3). Thus

$$L_i \cong L_{i+1}^{c_{i+1}} \cdots L_i^{c_i}.$$  \hspace{1cm} (4.1)

Denote by $\eta$ the complex line bundle over $M(C_{i+1})$ which restricts to the dual of the tautological bundle on the fibres of $\pi_{i+1} : M(C_{i+1}) \to M(C_i)$. Observe that $\eta$ is just the bundle associated to the character $\rho_{i+1} : H^{i+1} \to \mathbb{S}^1$. Hence $\eta = L_{i+1}$ on $M(C_{i+1})$.

Let $i \geq 0$. Note that one has $G$-equivariant sections $\sigma_i, \sigma'_i : M(C_i) \to M(C_{i+1})$ of the bundle $M(C_{i+1}) \to M(C_i)$ defined as $\sigma_i([z_1, \ldots, z_i, w_i]) = [z_i, w_1, \ldots, z_i, w_i, 1, 0]$ and $\sigma'_i([z_1, \ldots, z_i, w_i]) = [z_i, w_1, \ldots, z_i, w_i, 0, 1]$. The images of these sections are imbedded submanifolds of $M(C_{i+1})$ which correspond to the facets $I^i \times \{0\}$ and $I^i \times \{1\}$ of $I^{i+1}$ respectively. We regard $M(C_i)$ as a submanifold of $M(C_{i+1})$ via $\sigma_i$. The normal bundle to the imbedding $M(C_i) \subset M(C_{i+1})$ is just the bundle $L_{i+1}|M_i$. (Cf. Remark 3.1.) It follows that

$$c_1(L_{i+1}) = [M(C_i)] \in H^2(M(C_{i+1}; \mathbb{Z}).$$ \hspace{1cm} (4.2)

For $1 \leq i \leq n$ set $M_i := \pi_{i-1}^{-1}(M(C_{i-1}))$. Using the observations made above, the fact that $\pi_i$ is the projection of a fibre bundle, and equation (4.2), it follows that the line bundle $L_i|M_i$ is normal to the imbedding $M_i \subset M(C)$ and

$$c_1(L_i) = [M_i] \in H^2(M(C); \mathbb{Z}).$$ \hspace{1cm} (4.3)

The structure of $M(C)$ as an iterated 2-sphere bundle enables one to compute its $K$-ring using Example 3.3.

Theorem 4.1. We keep the above notations. Let $C$ be any $n \times n$ unipotent upper triangular matrix over $\mathbb{Z}$. The map $y_i \mapsto [L_i^+]$, $1 \leq i \leq n$ defines an isomorphism of rings from $K_n := \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]/J$ to $K(M(C))$ where $J$ is the ideal generated by the elements: $(y_i - 1)(y_i - y_0y_1^{-c_{i-1}} \cdots y_{i-1}^{-c_{i-1}})$, $1 \leq i \leq n$, where $y_0 := 1$. One has $c_1(y_i) = -[M_i] \in H^2(M(C); \mathbb{Z})$.  \hspace{1cm} 9
**Proof:** This follows from Example 3.3 by induction. Indeed, when \( n = 1 \), \( K(S^2) = K(\mathbb{P}^1) \cong \mathbb{Z}[y_1]/\langle (y_1 - 1)^2 \rangle \cong K_1 \) since \( y_1 \) is invertible in \( \mathbb{Z}[y_1]/\langle (y_1 - 1)^2 \rangle \) where \( y_1 \) is the class of the dual of the tautological bundle over \( \mathbb{P}^1 \).

Let \( i \geq 1 \). By induction assume that \( K_i \cong K(M(C_i)) \) where \( y_j \mapsto [L_j^i] \) for \( 1 \leq j \leq i \) and that \( y_j - 1 \in K_i \) is nilpotent for \( j \leq i \). Since the \((i + 1)\)-st canonical bundle over \( M(C_{i+1}) \) is the bundle that restricts to the dual of the tautological bundle along the fibres \( \mathbb{P}^1 \) of \( \pi_{i+1} : M(C_{i+1}) \to M(C_i) \), from Example 3.3 we obtain that \( K(M(C_{i+1})) \cong K_i[y_{i+1}]/\langle (y_{i+1}-1)(y_{i+1}-L_1) \rangle \) under the \( K \)-algebra map that sends \( y_{i+1} \) to \([L_{i+1}^i]\). Substituting for \([L_i]\) from equation (4.1) we obtain the relation \( (y_{i+1}-1)(y_{i+1}-y_{-c_{1,i+1}} \cdots y_{-c_{i+1}}) = K_{i+1} \). Since \( y_{i+1} - 1 \) has been shown to be nilpotent, the induction step is complete. The assertion about the first Chern class of \( y_{i+1} \) is immediate from equation (4.3).

Let \( \mathcal{G} \) be a complex semisimple linear algebraic group, \( B \) a Borel subgroup. Fix an algebraic maximal torus \( T \cong (\mathbb{C}^*)^l \), \( l \) being the rank of \( \mathcal{G} \), contained in \( B \) and let \( \Phi^+, \Delta \) denote the corresponding system of positive roots and simple roots respectively. Denote by \( W \) the Weyl group of \( \mathcal{G} \) with respect to \( T \) and \( S \subset W \) the set of simple reflections \( s_\alpha, \alpha \in \Delta \). For \( \gamma \in \Delta \), denote by \( P_\gamma \supset B \) the minimal parabolic subgroup corresponding to \( \gamma \) so that \( P_\gamma/B \cong \mathbb{P}^1 \). Let \( \alpha_1, \ldots, \alpha_n \) be any sequence of simple roots. Consider the Bott-Samelson variety \( M = P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n} \times_B \{pt\} \). Explicitly \( M \) is the quotient of \( \mathcal{P} := P_{\alpha_1} \times \cdots \times P_{\alpha_n} \) by the action of \( B^n \) given by \((p_1, \ldots, p_n).b = (p_1b_1^{-1}p_2b_2, \ldots, b_{n-1}p_nb_n) \) for \((p_1, \ldots, p_n) \in \mathcal{P}, (b_1, \ldots, b_n) \in B^n \). When \( w = s_{\alpha_1} \cdots s_{\alpha_n} \in W \) is a reduced expression for \( w \), one has a surjective birational morphism \( M \to X(w) \) which maps \([p_1, \ldots, p_n]\) to the coset \( p_1 \cdots p_nB \) in the Schubert variety \( X(w) \subset \mathcal{G}/B \). In this case, \( M \) is the Bott-Samelson-Demazure-Hansen \([4, 8, 11]\) desingularization of the Schubert variety associated to the reduced expression \( w = s_{\alpha_1} \cdots s_{\alpha_n} \). The Bott tower, which arises as the special fibre of a certain deformation of the complex structure of \( M_t \) is associated to the unipotent upper triangular matrix \( (c_{ij}) \) where \( c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle, i < j \). The polytope which arises as the quotient of the Bott tower by the \( n \)-dimensional (compact) torus action is the \( n \)-cube \( I^n \). See \([10] \) or \([16]\) for details. Feeding this data into Theorem 4.1 we obtain explicit description of the \( K \)-ring of a Bott-Samelson variety which is diffeomorphic to the Bott tower. Alternatively one could use Example 3.3 and induction to obtain the same result.

The Bott-Samelson variety \( M \) has an algebraic cell decomposition, i.e., a CW structure where the open cells are affine spaces contained in \( M \). The closed cells of real codimension 2 are the divisors \( M_j \), defined as the image of \( \{(p_1, \ldots, p_n) \in \mathcal{P} \mid p_j = 1\} \) under the canonical map \( \mathcal{P} \to M \). Note that the integral cohomology ring of \( M \) is generated by the dual cohomology classes \([M_j] \in H^2(M; \mathbb{Z}), 1 \leq j \leq n \).

Lemma 4.2 of \([12]\) implies that the forgetful map \( K(M) \to K(M) \) is an isomorphism of rings where \( K(M) \) is the Grothendieck \( K \)-ring of \( M \). The following theorem is established
Theorem 4.2. Let $M$ be the (generalized) Bott-Samelson variety $P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n} \times B \{pt\}$. Let $c_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle, 1 \leq i < j \leq n$. The Grothendieck ring $K(M)$ of algebraic vector bundles on $M$ is isomorphic to $\mathbb{Z}\left[y_1^{\pm 1}, \ldots, y_n^{\pm 1}\right]/\langle (y_i - 1)(y_i - y_0 y_1^{c_{1,i}} \cdots y_{i-1}^{c_{i-1,i}}) ; 1 \leq i \leq n \rangle$ where $y_0 := 1$. The class $y_i$ is represented by the algebraic line bundle $O(-M_i)$ for $1 \leq i \leq n$. The forgetful ring homomorphism $K(M) \to K(M)$ is an isomorphism. \hfill \Box

Remark 4.3. One has a well-defined involution $y_i \mapsto w_i = y_i^{-1}$ of the algebra $K(M(C))$. Indeed multiplying the two factors in generating relation $(y_i - 1)(y_i - y_1^{-c_{1,i}} \cdots y_{i-1}^{-c_{i-1,i}}) = 0$ by $y_i^{-1}$ and $y_i^{-1} y_1^{c_{1,i}} \cdots y_{i-1}^{c_{i-1,i}} = 0$ we get the same relation with the $y_j$’s replaced by $y_j^{-1} = w_j$: that is, $(w_i - 1)(w_i - w_1^{-c_{1,i}} \cdots w_{i-1}^{-c_{i-1,i}}) = 0$. Consequently, one could let $y_i$ to be the class of $O(M_i)$ in the above theorem.

Note: After this work was completed, we come across the papers of Civan and Ray [6] and M.Willems [17]. Civan and Ray determine the ring structures of the generalized cohomology theories arising from complex oriented ring spectra for Bott towers. They also determine the $KO$-rings of Bott towers using entirely different methods. Willems [17] has computed the equivariant K-ring of Bott towers and Bott-Samelson varieties. While we consider only $K$-ring, our results apply to the more general class of quasi-toric manifolds.

Acknowledgements: We thank Prof. V.Balaji for valuable discussions. We thank Prof. M.Masuda for his valuable comments and for careful reading of an earlier version of this paper.

References

[1] A.Bahri and M.Bendersky, The $KO$-theory of toric manifolds, Trans. Amer. Math. Soc. 352, 1191-1202.

[2] A.Bronsted, An introduction to convex polytope, (1983), Springer-Verlag, NY.

[3] V.M.Buchstaber and T.E.Panov, Torus actions and their applications in topology and combinatorics, Univ. Lect. Series-24, (2002), AMS, Providence, RI.

[4] R.Bott and H.Samelson, Applications of the theory of Morse to symmetric spaces. Amer. J. Math. 80, 1958, 964–1029.

[5] Y.Civan, Some examples in toric geometry, arXiv:math.AT/0306029 v2.

[6] Y.Civan and N.Ray, Homotopy decompositions and $K$-theory of Bott towers, arXiv:math.AT/0408261v1.
[7] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. Jour. 62,(1991), 417-451.

[8] M. Demazure, Désingularisation des variétés de Schubert généralisées, Ann. Sci. École Norm. Sup. 7 (1974), 53–88.

[9] W. Fulton, Introduction to toric varieties, Ann Math Studies 131, (1993), Princeton Univ. Press, Princeton, NJ.

[10] M. Grossberg and Y. Karshon, Bott towers, complete integrability, and the extended character of representations, Duke Math. Jour. 76, (1994), 23–58.

[11] H.C. Hansen, On cycles in flag manifolds, Math. Scand. 33 (1973), 269–274 (1974).

[12] M. Karoubi, K-Theory, Grundlehren der Mathematischen Wissenschaften 226, Springer-Verlag, Berlin, 1978.

[13] A.G. Khovanskii, Hyperplane sections of polyhedra, Funct. Analys. Appl. 20, (1986), 41-50.

[14] J.W. Milnor, J.D. Shasheff, Characteristic classes, Ann. Math. Studies 76, (1974), Princeton Univ. Press, Princeton, NJ.

[15] P. Sankaran and V. Uma, Cohomology of toric bundles, Comment. Math. Helv., 78, (2003), 540-554. Errata, 79, (2004), 840-841.

[16] M. Willems, Cohomologie et K-théorie équivariantes des tours de Bott et des variétés de drapeaux. Application au calcul de Schubert, arXiv:math.AG/0311079 v1.

[17] M. Willems, K-theorie équivariante des variétés de Bott-Samelson. Application à la structure multiplicative de la K-theorie équivariante des variétés de drapeaux, arXiv:math.AG/0412152.

[18] G. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995.

Institute of Mathematical Sciences
CIT Campus, Chennai 600 113, INDIA
E-mail: sankaran@imsc.res.in
uma@imsc.res.in

Current Address (V.U.):
Institut Fourier
100, rue des Maths, BP74
38402 St Martin d’Heres
Cedex, FRANCE
E-mail: uma@mozart.ujf-grenoble.fr