Article

Topological Indices of $m$th Chain Silicate Graphs

Jia-Bao Liu 1, Muhammad Kashif Shafiq 2*, Haidar Ali 2, Asim Naseem 3, Nayab Maryam 2 and Syed Sheraz Asghar 2

1 School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China; liujiabaoad@163.com
2 Department of Mathematics, Government College University Faisalabad (GCUF), Faisalabad 38023, Pakistan; haidarali@gcuf.edu.pk (H.A.); nayabamer81@gmail.com (N.M.); s.asghar9@gmail.com (S.S.A.)
3 Department of Mathematics, GC University Lahore, Lahore 54000, Pakistan; dr.asimnaseem@gcu.edu.pk
* Correspondence: kashif4v@gmail.com

Received: 18 November 2018; Accepted: 15 December 2018; Published: 4 January 2019

Abstract: A topological index is a numerical representation of a chemical structure, while a topological descriptor correlates certain physico-chemical characteristics of underlying chemical compounds besides its numerical representation. A large number of properties like physico-chemical properties, thermodynamic properties, chemical activity, and biological activity are determined by the chemical applications of graph theory. The biological activity of chemical compounds can be constructed by the help of topological indices such as atom-bond connectivity (ABC), Randić, and geometric arithmetic (GA). In this paper, Randić, atom bond connectivity (ABC), Zagreb, geometric arithmetic (GA), ABC4, and GA5 indices of the $m$th chain silicate $SL(m, n)$ network are determined.

Keywords: general Randić index; Zagreb index; atom-bond connectivity (ABC) index; geometric-arithmetic (GA) index; $SL(m, n)$

1. Introduction and Preliminary Results

Graph theory is a branch of numerical science in which we apply the apparatuses of the diagram hypothesis to demonstrate the phenomena of compounds scientifically. In chemical graph theory, the edges represent the covalent bonding between atoms, and the vertices of a molecular graph represent atoms. The significance of the molecular graph is that the hydrogen atom is omitted from it.

“A topological index is a quantity that is somehow calculated from the molecular graph and for which we believe that it reflects relevant structural features of the underlying molecule” [1]. Until the late 1970s, topological indices only had information on atom connectivity, but in this age, they have many properties of saturated hydrocarbons [2]. The whole theory of topological indices was started by Wiener [3] when he was doing the experiment on the boiling point of paraffin.

As a consequence of various experiments, we can claim that topological indices are useful in QSPR/QSAR studies. The correlation between physico-chemical properties (QSPR) and the biological activity relationships (QSAR) of the molecules was tested with the help of topological indices. It is pertinent to note that the topological indices have some major classes such as counted related topological indices, degree-based topological indices, and distance-based topological indices of graphs. The degree-based topological indices are very important in chemical graph theory to test the attributes of compounds and drugs, which have been mostly used in chemical and pharmacy engineering [4,5].

In 1975, Milan Randić put forward the topological index, which is based on degree, in his seminar paper [6]. His index was denoted by $R_{-\frac{1}{2}}(G)$ and defined as:

$$R_{-\frac{1}{2}}(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\deg(u)\deg(v)}}$$

(1)
Randić named it the “branching index”, but then it was re-named [7,8] to the “connectivity index”. Currently, it is called the “Randić index” [9]. Let $R_\alpha(G)$ be the general Randić index defined as:

$$R_\alpha(G) = \sum_{uv \in E(G)} (\deg(u)\deg(v))^\alpha$$ for $\alpha = 1, \frac{1}{2}, -1, -\frac{1}{2}$ (2)

“Ruder Boskovic” is the group for theoretical chemistry at the Institute of Zagreb, and Balaban et al. was a member of this group. Therefore, he named the $M_1$ and $M_2$ topological indices as “Zagreb group indices”. Later on, the “Zagreb group index” was abbreviated to “Zagreb index”, and now, $M_1$ is renamed as the “first Zagreb index”; on the other hand, $M_2$ is called the “second Zagreb index”. These indices are defined as:

$$M_1(G) = \sum_{uv \in E(G)} (\deg(u) + \deg(v))$$ (3)

$$M_2(G) = \sum_{uv \in E(G)} (\deg(u)\deg(v))$$ (4)

A new topological index that is obtained by the extension of Equation (1), introduced by Emesto Estrada, is called the “atom-bond connectivity index” and defined as:

$$\text{ABC}(G) = \sum_{uv \in E(G)} \sqrt{\frac{\deg(u) + \deg(v) - 2}{\deg(u)\deg(v)}}$$ (5)

The ABC$_4$ index is the fourth version of the atom-bond connectivity index. The ABC$_4$ index was formulated by Ghorbani et al. [10] and is defined as:

$$\text{ABC}_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_uS_v}}$$ (6)

Another topological index invented by Vukičević and Furtula [11] was named the “geometric-arithmetic index”. It is defined as:

$$\text{GA}(G) = \sum_{uv \in E(G)} \frac{2\sqrt{\deg(u)\deg(v)}}{(\deg(u) + \deg(v))}$$ (7)

$\text{GA}_5$ is the fifth version of the geometric arithmetic index [12]. The $\text{GA}_5$ index was proposed by Graovac et al. [13] and is defined as:

$$\text{GA}_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_uS_v}}{S_u + S_v}$$ (8)

In this article, graph $G$ is to be taken as a graph with edge set $E(G)$ and vertex set $V(G)$, and the degree of vertex $u \in V(G)$ is denoted as $d_u$ and $S_u = \sum_{v \in N_G(u)} S_v$ where $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$.

Hayat et al. worked on the degree-based topological index, such as for silicate, hexagonal, honeycomb, and oxide [14]. For more conclusive results related to topological indices of chemical graphs and their graph invariants, see [15–23].

**Construction of the Silicate Chain Graph:**

SiO$_4$ tetrahedra are found nearly in all the silicates. Silicates are immensely essential and complicated minerals. We get silicates from metal carbonates with sand or from fusing metal oxides. Silicates behave as the building blocks of the usual rock-forming minerals.
Consider a single tetrahedron (i.e., a pyramid having a triangular base). Place oxygen atoms at the four corners of a tetrahedron, and the silicon atom is bonded with equally-spaced atoms of oxygen. The resulting tetrahedron is a silicate tetrahedron, which is shown in Figure 1a, and when this tetrahedron joins with other tetrahedra linearly, then a single-row silicate chain is formed, as shown in Figure 1b. When two tetrahedra join together corner-to-corner, then each tetrahedron shares its oxygen atom with the other tetrahedron, as shown in Figure 1c. After this sharing, these two tetrahedra can be joined with two other tetrahedra, as in Figure 1d. Now, extend this structure in one direction, then double the silicate chain formed, as in Figure 1e, were \( m \) is the number of row lines and \( n \) is the number of edges in a row line.

![Figure 1. Chain silicate network.](image)

2. Results for \( m \)th Chain Silicate

In this section, we calculate the closed results for topological indices, which are based on vertex degrees of the \( m \)th silicate chain. We compute the general Randić for \( \alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}, \text{ABC}, \text{GA}, \text{ABC}_4, \) and \( GA_5 \) indices for the \( m \)th chain silicate in this section. The number of edges of the \( m \)th chain silicate are \( 6mn \). In the following theorem, the general Randić index for the \( m \)th chain silicate is computed.

Theorem 1. The general Randić index for \( G \approx SL(m, n) \) is equal to, when \( m = n \):

\[
R_\alpha(G) = \begin{cases} 
36(3mn - 6n + 2) + 18(3mn + 3n - 4) + 9(3m + 2), & \alpha = 1; \\
6(3mn - 6n + 2) + 3\sqrt{2}(3mn + 3n - 4) + 3(3m + 2), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 6n + 2) + \frac{3mn + 3n - 4}{3\sqrt{2}} + \frac{1}{3}(3m + 2), & \alpha = -1; \\
\frac{1}{6}(3mn - 6n + 2) + \frac{3mn + 3n - 4}{3\sqrt{2}} + \frac{1}{3}(3m + 2), & \alpha = -\frac{1}{2}; 
\end{cases}
\]

when \( m < n \):

Case 1: When \( m \) is odd and \( n \) is even and vice versa. For \( m > 1 \),

(i) when \( m \) is odd and \( n \) is even:

\[
R_\alpha(G) = \begin{cases} 
36(3mn - 2n - 4m + 2) + 18(3mn + 2n + m - 5) + 9(3m + 3), & \alpha = 1; \\
6(3mn - 2n - 4m + 2) + 3\sqrt{2}(3mn + 2n + m - 5) + 3(3m + 3), & \alpha = \frac{1}{2}; \\
\frac{1}{10}(3mn - 2n - 4m + 2) + \frac{1}{15}(3mn + 2n + m - 5) + \frac{1}{3}(3m + 3), & \alpha = -1; \\
\frac{1}{6}(3mn - 2n - 4m + 2) + \frac{3mn + m - 5}{3\sqrt{2}} + \frac{1}{3}(3m + 3), & \alpha = -\frac{1}{2}; 
\end{cases}
\]
(ii) when $m$ is even and $n$ is odd:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 2n - 4m + 2) + 18(3mn + 2n + m - 8) + 9(3m + 6), & \alpha = 1; \\
6(3mn - 2n - 4m + 2) + 3\sqrt{2}(3mn + 2n + m - 8) + 3(3m + 6), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 2n - 4m + 2) + \frac{1}{18}(3mn + 2n + m - 8) + \frac{1}{3}(3m + 6), & \alpha = -1; \\
\frac{1}{6}(3mn - 2n - 4m + 2) + \frac{3mn + 2n - 8}{3\sqrt{2}} + \frac{1}{3}(3m + 6), & \alpha = -\frac{1}{2}.
\end{cases}$$

Case 2: When $m$ and $n$ both are even:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 2n - 4m + 2) + 18(3mn + 2n + m - 6) + 9(3m + 4), & \alpha = 1; \\
6(3mn - 2n - 4m + 2) + 3\sqrt{2}(3mn + 2n + m - 6) + 3(3m + 4), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 2n - 4m + 2) + \frac{1}{18}(3mn + 2n + m - 6) + \frac{1}{3}(3m + 4), & \alpha = -1; \\
\frac{1}{6}(3mn - 2n - 4m + 2) + \frac{3mn + 2n - 6}{3\sqrt{2}} + \frac{1}{3}(3m + 4), & \alpha = -\frac{1}{2}.
\end{cases}$$

Case 3: When $m$ and $n$ both are odd:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 6m - 2) + 18(3mn + 3m - 2) + 9(3m + 4), & \alpha = 1; \\
6(3mn - 6m - 2) + 3\sqrt{2}(3mn + 3m - 2) + 3(3m + 4), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 6m - 2) + \frac{1}{18}(3mn + 3m - 2) + \frac{1}{3}(3m + 4), & \alpha = -1; \\
\frac{1}{6}(3mn - 6m - 2) + \frac{3mn + 3m - 2}{3\sqrt{2}} + \frac{1}{3}(3m + 4), & \alpha = -\frac{1}{2}.
\end{cases}$$

when $m > n$:

Case 1: When $m$ is odd and $n$ is even and vice versa. For $m > 2$,

(i) when $m$ is odd and $n$ is even:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 4n - 2m) + 18(3mn + 4n - m - 1) + 9(3m + 1), & \alpha = 1; \\
6(3mn - 4n - 2m) + 3\sqrt{2}(3mn + 4n - m - 1) + 3(3m + 1), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 4n - 2m) + \frac{1}{18}(3mn + 4n - m - 1) + \frac{1}{3}(3m + 1), & \alpha = -1; \\
\frac{1}{6}(3mn - 4n - 2m) + \frac{3mn + 4n - m - 1}{3\sqrt{2}} + \frac{1}{3}(3m + 1), & \alpha = -\frac{1}{2}.
\end{cases}$$

(ii) when $m$ is even and $n$ is odd:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 4n - 2m) + 18(3mn + 4n - m - 4) + 9(3m + 4), & \alpha = 1; \\
6(3mn - 4n - 2m) + 3\sqrt{2}(3mn + 4n - m - 4) + 3(3m + 4), & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 4n - 2m) + \frac{1}{18}(3mn + 4n - m - 4) + \frac{1}{3}(3m + 4), & \alpha = -1; \\
\frac{1}{6}(3mn - 4n - 2m) + \frac{3mn + 4n - m - 4}{3\sqrt{2}} + \frac{1}{3}(3m + 4), & \alpha = -\frac{1}{2}.
\end{cases}$$

Case 2: When $m$ and $n$ both are even:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 4n - 2m - 2) + 18(3mn + 4n - m + 2) + 27m, & \alpha = 1; \\
6(3mn - 4n - 2m - 2) + 3\sqrt{2}(3mn + 4n - m + 2) + 9m, & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 4n - 2m - 2) + \frac{1}{18}(3mn + 4n - m + 2) + \frac{m}{3}, & \alpha = -1; \\
\frac{1}{6}(3mn - 4n - 2m - 2) + \frac{3mn + 4n - m + 2}{3\sqrt{2}} + m, & \alpha = -\frac{1}{2}.
\end{cases}$$

Case 3: When $m$ and $n$ both are odd. For $n > 1$:

$$R_{\alpha}(\mathcal{G}) = \begin{cases} 
36(3mn - 4n - 2m - 2) + 18(3mn + 4n - m + 2) + 27m, & \alpha = 1; \\
6(3mn - 4n - 2m - 2) + 3\sqrt{2}(3mn + 4n - m + 2) + 9m, & \alpha = \frac{1}{2}; \\
\frac{1}{5}(3mn - 4n - 2m - 2) + \frac{1}{18}(3mn + 4n - m + 2) + \frac{m}{3}, & \alpha = -1; \\
\frac{1}{6}(3mn - 4n - 2m - 2) + \frac{3mn + 4n - m + 2}{3\sqrt{2}} + m, & \alpha = -\frac{1}{2}.
\end{cases}$$
Proof. When \( m = n, m > 1 \):

Let \( G \) be the chain silicate Figure 2, based on degree \( n \). The \( E(G) \) can be divided into three partitions. The \( E_1(G) \) have \( 3m + 2 \) edges \( uv \); where \( \deg(u) = \deg(v) = 3 \). The \( E_2(G) \) have \( 3mn + 3n - 4 \) edges \( uv \), where \( \deg(u) = 3 \) and \( \deg(v) = 6 \). The \( E_3(G) \) have \( 3mn - 6n + 2 \) edges \( uv \); where \( \deg(u) = \deg(v) = 6 \). See Table 1 for the edge partitions of \( G \). Thus, from Equation (2), it follows that:

\[
R_\alpha(G) = \sum_{uv \in E(G)} (\deg(u) \deg(v))^\alpha
\]

For \( \alpha = 1 \):

\( R_\alpha(G) \) is the general Randić index, defined by:

\[
R_1(G) = \sum_{j=1}^{3} \sum_{uv \in E_j(G)} \deg(u) \cdot \deg(v)
\]

By using Table 1, we get the following:

\[
R_1(G) = 9|E_1(G)| + 18|E_2(G)| + 36|E_3(G)|
\]

\[\implies R_1(G) = 36(3mn - 6n + 2) + 18(3mn + 3n - 4) + 9(3m + 2)\]

For \( \alpha = \frac{1}{2} \):

We apply the formula of \( R_\alpha(G) \).

\[
R_\frac{1}{2}(G) = \sum_{j=1}^{3} \sum_{uv \in E_j(G)} \sqrt{\deg(u) \cdot \deg(v)}
\]

By using Table 1, we get:

\[
R_\frac{1}{2}(G) = 3|E_1(G)| + 2\sqrt{3}|E_2(G)| + 6|E_3(G)|
\]

\[\implies R_\frac{1}{2}(G) = 6(3mn - 6n + 2) + 3\sqrt{2}(3mn + 3n - 4) + 3(3m + 2)\]

Figure 2. \( m \)th chain silicate (SL(\( m, n \))).
For $\alpha = -1$:
We apply the formula of $R_\alpha(G)$.

$$R_{-1}(G) = \frac{1}{3} \sum_{j=1}^{3} \sum_{uv \in E_j(G)} \frac{1}{\deg(u) \cdot \deg(v)}$$

$$R_{-1}(G) = \frac{1}{9} |E_1(G)| + \frac{1}{18} |E_2(G)| + \frac{1}{36} |E_3(G)|$$

$$\implies R_{-1}(G) = \frac{1}{36} (3mn - 6n + 2) + \frac{1}{18} (3mn + 3n - 4) + \frac{1}{9} (3m + 2)$$

For $\alpha = -\frac{1}{2}$
We apply the formula of $R_\alpha(G)$.

$$R_{-\frac{1}{2}}(G) = \frac{1}{3} \sum_{j=1}^{3} \sum_{uv \in E_j(G)} \frac{1}{\sqrt{\deg(u) \cdot \deg(v)}}$$

$$R_{-\frac{1}{2}}(G) = \frac{1}{3} |E_1(G)| + \frac{1}{3\sqrt{2}} |E_2(G)| + \frac{1}{6} |E_3(G)|$$

$$\implies R_{-\frac{1}{2}}(G) = \frac{1}{6} (3mn - 6n + 2) + \frac{3mn + 3n - 4}{3\sqrt{2}} + \frac{1}{3} (3m + 2)$$

When $m < n$:
The proof of the following cases, (i) when $m$ is odd and $n$ is even, (ii) when $m$ is even and $n$ is odd, (iii) when $m$ and $n$ both are even, and (iv) when $m$ and $n$ both are odd, is the same as $m = n$ by using Tables 2–5, respectively.

When $m > n$:
The proof of the following cases, (i) when $m$ is odd and $n$ is even, (ii) when $m$ is even and $n$ is odd, (iii) when $m$ and $n$ both are even, and (iv) when $m$ and $n$ both are odd, is the same as $m = n$ by using Tables 6–9, respectively. \(\square\)

### Table 1. Edge partition of silicate chain (SL(m,n)), when $m = n$, based on the degrees of the end vertices of each edge.

| $(d_u, d_v)$ Where $uv \in E(G)$ | Number of Edges |
|-----------------------------|-----------------|
| (3,3)                       | $3m + 2$        |
| (3,6)                       | $3mn + 3n - 4$  |
| (6,6)                       | $3mn - 6n + 2$  |

### Table 2. Edge partition of silicate chain (SL(m,n)), when $m < n$ and $m$ is odd, based on the degrees of the end vertices of each edge.

| $(d_u, d_v)$ Where $uv \in E(G)$ | Number of Edges |
|-----------------------------|-----------------|
| (3,3)                       | $3m + 3$        |
| (3,6)                       | $3mn + m + 2n - 5$ |
| (6,6)                       | $3mn - 2n - 4m + 2$ |

### Table 3. Edge partition of silicate chain (SL(m,n)), when $m < n$ and $m$ is even, based on the degrees of the end vertices of each edge.

| $(d_u, d_v)$ Where $uv \in E(G)$ | Number of Edges |
|-----------------------------|-----------------|
| (3,3)                       | $3m + 6$        |
| (3,6)                       | $3mn + m + 2n - 8$ |
| (6,6)                       | $3mn - 2n - 4m + 2$ |
Table 4. Edge partition of silicate chain \((SL(m,n))\), when \(m < n\), \(m\) and \(n\) both are even, based on the degrees of the end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m + 4\)      |
| \((3,6)\)                           | \(3mn + m + 2n - 6\) |
| \((6,6)\)                           | \(3mn - 4m - 2n + 2\) |

Table 5. Edge partition of silicate chain \((SL(m,n))\), when \(m < n\), \(m\) and \(n\) both are odd, based on the degrees of the end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m + 4\)      |
| \((3,6)\)                           | \(3mn + 3m - 2\) |
| \((6,6)\)                           | \(3mn - 6m - 2\) |

Table 6. Edge partition of silicate chain \((SL(m,n))\), when \(m > n\) and \(m\) is odd, based on the degrees of end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m + 1\)      |
| \((3,6)\)                           | \(3mn - m + 4n - 1\) |
| \((6,6)\)                           | \(3mn - 2m - 4n\) |

Table 7. Edge partition of silicate chain \((SL(m,n))\), when \(m > n\) and \(m\) is even, based on the degrees of the end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m\)         |
| \((3,6)\)                           | \(3mn - m + 4n + 2\) |
| \((6,6)\)                           | \(3mn - 2m - 4n - 2\) |

Table 8. Edge partition of silicate chain \((SL(m,n))\), when \(m > n\), \(m\) and \(n\) both are even, based on the degrees of the end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m\)         |
| \((3,6)\)                           | \(3mn - m + 4n + 2\) |
| \((6,6)\)                           | \(3mn - 2m - 4n - 2\) |

Table 9. Edge partition of silicate chain \((SL(m,n))\), when \(m > n\), \(m\) and \(n\) both are odd, based on the degrees of the end vertices of each edge.

| \((d_u, d_v)\) Where \(uv \in E(G)\) | Number of Edges |
|-------------------------------------|-----------------|
| \((3,3)\)                           | \(3m\)         |
| \((3,6)\)                           | \(3mn - m + 4n + 2\) |
| \((6,6)\)                           | \(3mn - 2m - 4n - 2\) |

In this theorem, we find the result of the first Zagreb index for chain silicate.

**Theorem 2.** For the chain silicate \(G\), the first Zagreb index is equal to, when \(m = n\):

\[
M_1(G) = 12(3mn - 6n + 2) + 9(3mn + 3n - 4) + 6(3m + 2)
\]
when \( m < n \):

**Case 1:** When \( m \) is odd and \( n \) is even or vice versa. For \( m > 1 \),

(i) when \( m \) is odd and \( n \) is even:

\[
M_1(G) = 12(3mn - 2n - 4m + 2) + 9(3mn + 2n + m - 5) + 6(3m + 3)
\]

(ii) when \( m \) is even and \( n \) is odd:

\[
M_1(G) = 12(3mn - 2n - 4m + 2) + 9(3mn + 2n + m - 8) + 6(3m + 6)
\]

**Case 2:** When \( m \) and \( n \) both are even.

\[
M_1(G) = 12(3mn - 2n - 4m + 2) + 9(3mn + 2n + m - 6) + 6(3m + 4)
\]

**Case 3:** When \( m \) and \( n \) both are odd.

\[
M_1(G) = 12(3mn - 6m - 2) + 9(3mn + 3m - 2) + 6(3m + 4)
\]

when \( m > n \):

**Case 1:** When \( m \) is odd and \( n \) is even or vice versa. For \( m > 2 \),

(i) when \( m \) is odd and \( n \) is even:

\[
M_1(G) = 12(3mn - 4n - 2m) + 9(3mn + 4n - m - 1) + 6(3m + 1)
\]

(ii) when \( m \) is even and \( n \) is odd:

\[
M_1(G) = 12(3mn - 4n - 2m) + 9(3mn + 4n - m - 4) + 6(3m + 4)
\]

**Case 2:** When \( m \) and \( n \) both are even.

\[
M_1(G) = 12(3mn - 4n - 2m - 2) + 9(3mn + 4n - m + 2) + 18m
\]

**Case 3:** When \( m \) and \( n \) both are odd.

\[
M_1(G) = 12(3mn - 4n - 2m - 2) + 9(3mn + 4n - m + 2) + 18m
\]

**Proof.** When \( m = n \):

Let \( G \) denotes the chain silicate. By using the edge partition from Table 1, the result follows. From Equation (3), we have:

\[
M_1(G) = \sum_{uv \in E(G)} (\deg(u) + \deg(v)) = \sum_{j=1}^{3} \sum_{uv \in E_j(G)} (\deg(u) + \deg(v))
\]

\[
M_1(G) = 6|E_1(G)| + 9|E_2(G)| + 12|E_3(G)|
\]

By doing some calculation, we get the following:

\[
\implies M_1(G) = 12(3mn - 6n + 2) + 9(3mn + 3n - 4) + 6(3m + 2)
\]

When \( m < n \):
Corollary 1. $M_2(G)$ is always equal to $R_1(G)$.

Now, we calculate the results of the following topological indices of chain silicate.

**Theorem 3.** Let $G$ be the chain silicate, based on degree $n$, then we have, when $m = n$:

- $ABC(G) = \frac{1}{3} \sqrt{2} (3mn - 2m + 3n - 2) + \frac{1}{2} \sqrt{2} (3mn + 2m - 5) + \frac{2}{3} (3m + 3)$. For $m > 1$
- $GA(G) = 6mn + 3m - 3n$. For $m > 1$
- $ABC_4(G) = \frac{1}{15} \sqrt{2} (3mn - 12n - 4m + 26) + \frac{2}{15} \sqrt{7} (3m - 3) + \frac{1}{3} \sqrt{\frac{25}{30}} (3mn - 7n - 6m + 16) + \frac{2}{5} \sqrt{3} (m + n - 7) + \frac{7(m+n+1)}{16 \sqrt{2}} + \frac{1}{6} \sqrt{\frac{2}{7}} (2m + 2n + 2) + \frac{1}{9} \sqrt{\frac{2}{7}} (m + 3n - 11) + \frac{1}{3} \sqrt{\frac{12}{7}} (m + 5n - 17) + \frac{2}{5} \sqrt{3} (m + 3n - 11) + \frac{1}{3} \sqrt{10} (m + 5n - 17) + \frac{2}{3} \sqrt{5} (m + 7n - 23) - 11n + 4 \sqrt{2} + \frac{8 \sqrt{5}}{3} + \frac{8 \sqrt{5}}{3} + 24$. For $m > 2$
- $GA_3(G) = 3mn + \frac{1}{3} \sqrt{15} (3mn - 7n - 6m + 16) + \frac{12}{5} \sqrt{2} (m + n - 1) + \frac{1}{3} \sqrt{10} (2m + 2n + 2) + \frac{2}{5} \sqrt{6} (m + 3n - 11) + \frac{1}{5} \sqrt{10} (m + 5n - 17) + \frac{2}{5} \sqrt{5} (m + 7n - 23) - 11n + 4 \sqrt{2} + \frac{8 \sqrt{5}}{3} + \frac{8 \sqrt{5}}{3} + 24$. For $m > 2$

When $m < n$:

**Case 1:** When $m$ is odd and $n$ is even or vice versa.

(i) when $m$ is odd and $n$ is even:

- $ABC(G) = \frac{1}{3} \sqrt{2} (3mn - 2m - 4n + 2) + \frac{1}{2} \sqrt{2} (3mn + 2m + m - 5) + \frac{2}{3} (3m + 3)$. For $m > 1$
- $GA(G) = 6mn. For m > 1$
- $ABC_4(G) = \frac{1}{15} \sqrt{2} (3mn - 12n - 4m + 26) + \frac{2}{15} \sqrt{7} (3m - 3) + \frac{1}{3} \sqrt{\frac{25}{30}} (3mn - 7n - 6m + 16) + \frac{2}{5} \sqrt{3} (m + n - 7) + \frac{7(m+n+1)}{16 \sqrt{2}} + \frac{1}{6} \sqrt{\frac{2}{7}} (2m + 2n + 2) + \frac{1}{9} \sqrt{\frac{2}{7}} (m + 3n - 11) + \frac{1}{3} \sqrt{\frac{12}{7}} (m + 5n - 17) + \frac{2}{5} \sqrt{3} (m + 3n - 11) + \frac{1}{3} \sqrt{10} (m + 5n - 17) + \frac{2}{3} \sqrt{5} (m + 7n - 23) - 11n + 4 \sqrt{2} + \frac{8 \sqrt{5}}{3} + \frac{8 \sqrt{5}}{3} + 24$. For $m > 2$

(ii) when $m$ is even and $n$ is odd:

- $ABC(G) = \frac{1}{3} \sqrt{2} (2 - 4m + 3mn - 2n) + \frac{1}{2} \sqrt{2} (3mn + 2n + m - 8) + \frac{2}{3} (3m + 6)$. For $m > 1$
- $GA(G) = 6mn. For m > 1$
- $ABC_4(G) = \frac{1}{15} \sqrt{2} (3mn - 12n - 4m + 26) + \frac{2}{15} \sqrt{7} (3m - 3) + \frac{1}{3} \sqrt{\frac{25}{30}} (3mn - 8n - 5m + 18) + \frac{2}{5} \sqrt{3} (m + n - 7) + \frac{7(m+n+1)}{16 \sqrt{2}} + \frac{1}{6} \sqrt{\frac{2}{7}} (2m + 2n + 2) + \frac{1}{9} \sqrt{\frac{2}{7}} (m + 3n - 11) + \frac{1}{3} \sqrt{\frac{12}{7}} (m + 5n - 17) + \frac{2}{5} \sqrt{3} (m + 3n - 11) + \frac{1}{3} \sqrt{10} (m + 5n - 17) + \frac{2}{3} \sqrt{5} (m + 7n - 23) - 11n + 4 \sqrt{2} + \frac{8 \sqrt{5}}{3} + \frac{8 \sqrt{5}}{3} + 25$. For $m > 2$
• ABC\(^{(G)}\) = \(\frac{1}{\sqrt{2}}(3mn - 2n - 4m + 2) + \frac{1}{3}\sqrt{2}(3mn + 2n + m - 6) + \frac{2}{3}(4 + 3m)\)
• GA\(^{(G)}\) = 6mn
• ABC\(^{4(G)}\) = \(\frac{1}{18}\sqrt{2}(3mn - 16m + 8) + \frac{2}{17}\sqrt{7}(3m - 2) + \frac{1}{2}\sqrt{4}(3mn - 16n + 3m + 36) + \frac{2}{7}\sqrt{13}(m + n - 6) + \frac{2}{7}\sqrt{10}(m + 3n - 2) + \frac{1}{9}\sqrt{4}(m - 6) + \frac{1}{9}\sqrt{12}(m + 5n - 20) + \frac{2}{3}\sqrt{2}(m + 6n - 24) + \frac{2}{3}\sqrt{2} + \frac{1}{12}\sqrt{7} + \frac{1}{12}\sqrt{2} + \frac{2}{3}\sqrt{2} + \frac{2}{3}\sqrt{5}. \text{ For } m > 2\)
• GA\(^{2(G)}\) = 3mn + \(\frac{1}{4}\sqrt{15}(3mn - 16n + 3m + 36) + \frac{2}{7}\sqrt{10}(m + n - 2) + \frac{1}{3}\sqrt{6}(4m - 6) + \frac{2}{3}\sqrt{10}(m + 5n - 20) + \frac{2}{9}\sqrt{2}(m + 6n - 24) - 12m + n + 4\sqrt{2} + 8\sqrt{3} + 8\sqrt{5} + 8. \text{ For } m > 2\)

Case 3: When \(m\) and \(n\) both are odd.
• ABC\(^{(G)}\) = \(\frac{1}{\sqrt{2}}(3mn - 6m - 2) + \frac{1}{3}\sqrt{2}(3mn + 3m - 2) + \frac{2}{3}(4m + 4)\)
• GA\(^{(G)}\) = 6mn
• ABC\(^{4(G)}\) = \(\frac{1}{18}\sqrt{2}(3mn - 14m - 2m + 12) + \frac{2}{17}\sqrt{7}(m + 2n - 3) + \frac{1}{2}\sqrt{4}(3mn - 5m - 8n + 11) + \frac{2}{7}\sqrt{13}(m + n - 9) + \frac{2}{7}\sqrt{10}(m + 3n + 7) + \frac{1}{9}\sqrt{4}(m + 3n - 9) + \frac{1}{9}\sqrt{12}(m + 5n - 11) + \frac{2}{3}\sqrt{2}(m + 7n - 17) + \frac{1}{3}\sqrt{2} + \frac{1}{12}\sqrt{7} + \frac{2}{3}\sqrt{2} + \frac{2}{3}\sqrt{5}. \text{ For } m > 4\)
• GA\(^{2(G)}\) = 3mn + \(\frac{1}{4}\sqrt{15}(3mn - 8m - 5n + 11) + \frac{2}{7}\sqrt{2}(m + n + 1) + \frac{2}{3}\sqrt{10}(m + 3n + 7) + \frac{2}{3}\sqrt{6}(m + 3n - 9) + \frac{2}{3}\sqrt{10}(m + 5n - 11) + \frac{2}{9}\sqrt{2}(m + 7n - 17) - 11n + 4\sqrt{2} + 8\sqrt{3} + 8\sqrt{5} + 8. \text{ For } m > 4\)

when \(m > n\):
Case 1: When \(m\) is odd and \(n\) is even or vice versa.
(i) when \(m\) is odd and \(n\) is even:
• ABC\(^{(G)}\) = \(\frac{1}{\sqrt{2}}(3mn - 4n - 2m) + \frac{1}{3}\sqrt{2}(3mn + 4m - n - 1) + \frac{2}{3}(3m + 1). \text{ For } m > 2\)
• GA\(^{(G)}\) = 6mn. For \(m > 2\)
• ABC\(^{4(G)}\) = \(\frac{1}{18}\sqrt{2}(3mn - 14n - 2m + 12) + \frac{2}{17}\sqrt{7}(m + 2n - 3) + \frac{1}{2}\sqrt{4}(3mn - 5m - 8n + 11) + \frac{2}{7}\sqrt{13}(m + n - 9) + \frac{7}{18}\sqrt{10}(m + 3n + 7) + \frac{1}{9}\sqrt{4}(m + 3n - 9) + \frac{1}{9}\sqrt{12}(m + 5n - 11) + \frac{2}{3}\sqrt{2}(m + 7n - 17) + \frac{1}{3}\sqrt{2} + \frac{1}{12}\sqrt{7} + \frac{2}{3}\sqrt{2} + \frac{2}{3}\sqrt{5}. \text{ For } m > 4\)
• GA\(^{2(G)}\) = 3mn + \(\frac{1}{4}\sqrt{15}(3mn - 8m - 5n + 11) + \frac{2}{7}\sqrt{2}(m + n + 1) + \frac{2}{3}\sqrt{10}(m + 3n + 7) + \frac{2}{3}\sqrt{6}(m + 3n - 9) + \frac{2}{3}\sqrt{10}(m + 5n - 11) + \frac{2}{9}\sqrt{2}(m + 7n - 17) - 11n + 4\sqrt{2} + 8\sqrt{3} + 8\sqrt{5} + 8. \text{ For } m > 4\)

(ii) when \(m\) is even and \(n\) is odd:
• ABC\(^{(G)}\) = \(\frac{1}{\sqrt{2}}(3mn - 4n - 2m) + \frac{1}{3}\sqrt{2}(3mn + 4m - n - 4) + \frac{2}{3}(3m + 4). \text{ For } m > 2\)
• GA\(^{(G)}\) = 6mn. For \(m > 2\)
• ABC\(^{4(G)}\) = \(\frac{1}{18}\sqrt{2}(3mn - 14n - 2m + 12) + \frac{2}{17}\sqrt{7}(m + 2n - 3) + \frac{1}{2}\sqrt{4}(3mn - 5m - 8n + 11) + \frac{2}{7}\sqrt{13}(m + n - 9) + \frac{7}{18}\sqrt{10}(m + 3n + 7) + \frac{1}{9}\sqrt{4}(m + 3n - 9) + \frac{1}{9}\sqrt{12}(m + 5n - 11) + \frac{2}{3}\sqrt{2}(m + 7n - 17) + \frac{1}{3}\sqrt{2} + \frac{1}{12}\sqrt{7} + \frac{2}{3}\sqrt{2} + \frac{2}{3}\sqrt{5}. \text{ For } m > 4\)
• GA\(^{2(G)}\) = 3mn + \(\frac{1}{4}\sqrt{15}(3mn - 8m - 5n + 12) + \frac{2}{7}\sqrt{2}(m + n + 1) + \frac{2}{3}\sqrt{10}(m + 3n - 5) + \frac{2}{3}\sqrt{6}(m + 3n - 9) + \frac{2}{3}\sqrt{10}(m + 5n - 11) + \frac{2}{9}\sqrt{2}(m + 7n - 17) - 11n + 4\sqrt{2} + 16\sqrt{3} + 16\sqrt{5} + 9. \text{ For } m > 4\)

Case 2: When \(m\) and \(n\) both are even.
• ABC\(^{(G)}\) = \(\frac{1}{\sqrt{2}}(3mn - 4n - 2m - 2) + \frac{1}{3}\sqrt{2}(3mn + 4n - m + 2) + 2m. \text{ For } m > 2\)
• GA\(^{(G)}\) = 6mn + 4. For \(m > 2\)
Case 3: When $m$ and $n$ both are odd.

- $ABC_4(G) = \frac{1}{15} \sqrt{\frac{29}{2} (3mn - 14n - 2m + 4) + \frac{7}{15} \sqrt{7} (3m - 6) + \frac{1}{3} \sqrt{\frac{29}{10} (3mn - 6n - 7m + 12) + \frac{2}{5} \sqrt{13} (m + n - 10) + \frac{7(m+n+2)}{18\sqrt{2}}} + \frac{1}{5} \sqrt{\frac{29}{6} (m + 3n + 10) + \frac{1}{3} \sqrt{\frac{29}{10} (m + 3n - 8) + \frac{2}{5} \sqrt{13} (m + 7n - 14) - 13n + 2m + 4\sqrt{2} + \frac{8\sqrt{3}}{5} + \frac{8\sqrt{5}}{5} - 4}$. For $m > 6$

- $G_{A_{2k}}(G) = 3mn + \frac{1}{4} \sqrt{15} (3mn - 6n - 7m + 12) + \frac{1}{12} \sqrt{2} (m + n + 2) + \frac{4}{13} \sqrt{10} (m + 3n + 10) + \frac{2}{5} \sqrt{6} (m + 3n - 8) + \frac{6}{19} \sqrt{10} (m + 5n - 8) + \frac{3}{7} \sqrt{5} (m + 7n - 14) - 13n + 2m + 4\sqrt{2} + \frac{8\sqrt{3}}{5} + \frac{8\sqrt{5}}{5} + 6$. For $m > 5$

**Proof.** When $m = n$:

By using Table 1, we get,

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{\deg(u) + \deg(v) - 2}{\deg(u) \cdot \deg(v)}} = \sum_{f=1}^{3} \sum_{uv \in E_f(G)} \sqrt{\frac{\deg(u) + \deg(v) - 2}{\deg(u) \cdot \deg(v)}}.$$

$$ABC(G) = \frac{2}{3} |E_1(G)| + \frac{1}{3} \sqrt{2} |E_2(G)| + \sqrt{10} |E_3(G)| \tag{9}$$

By doing some calculation, we get $\implies ABC(G) = \frac{1}{3} \sqrt{\frac{5}{2} (3mn - 6n + 2) + \frac{1}{3} \sqrt{\frac{7}{2} (3mn + 3n - 4) + \frac{2}{5} (3m + 2)}$.

From Equation (7), we get:

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{\deg(u) \cdot \deg(v)}}{(\deg(u) + \deg(v))} = \sum_{f=1}^{3} \sum_{uv \in E_f(G)} \frac{2\sqrt{\deg(u) \cdot \deg(v)}}{(\deg(u) + \deg(v))}.$$

After some simplification, we get

$$GA(G) = \frac{2}{3} |E_1(G)| + \frac{2}{3} \sqrt{2} |E_2(G)| + \sqrt{5} |E_3(G)| \tag{10}$$

By doing some calculation, we get:

$$GA = 6mn + 3m - 3n \tag{11}$$

For $S_u$ and $S_v$, the edge set $E(G)$ can be divided into fourteen edge partitions. The $E_1(G)$ contains six edges $uv$; where $S_u = S_v = 12$. The $E_2(G)$ have $3m - 4$ edges $uv$, where $S_u = S_v = 15$. The $E_3(G)$ have $4m + 4$ edges $uv$, where $S_u = 15$ and $S_v = 24$. The $E_4(G)$ have six edges $uv$; where $S_u = 12$ and $S_v = 24$. The $E_5(G)$ have $4m + 4n - 20$ edges $uv$, where we have $S_u = 15$ and $S_v = 27$. The $E_6(G)$ have $2m$ edges $uv$, where $S_u = 24$ and $S_v = 27$. The $E_7(G)$ have two edges $uv$, where $S_u = S_v = 24$. The $E_8(G)$ have $2m - 8$ edges $uv$, where $S_u = S_v = 27$. The $E_9(G)$ have two edges $uv$, where $S_u = 18$.
and \( S_v = 24 \). The \( E_{10}(G) \) have \( 2m + 2n - 10 \) edges \( uv \), where \( S_u = 18 \) and \( S_v = 27 \). The \( E_{11}(G) \) have \( 3mn - 4m - 9n + 14 \) edges \( uv \), where \( S_u = 18 \) and \( S_v = 30 \). The \( E_{12}(G) \) have two edges \( uv \), where \( S_u = 24 \) and \( S_v = 30 \). The \( E_{13}(G) \) have \( 3m + 3n - 14 \) edges \( uv \), where \( S_u = 27 \) and \( S_v = 30 \). The \( E_{14}(G) \) have \( 3mn - 16m + 20 \) edges \( uv \), where \( S_u = S_v = 30 \). Table 2 shows such an edge partition of \( G \).

From Equation (6), we get:

\[
ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}} = \sum_{j=1}^{14} \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}.
\]

By using the edge partition given in Table 10, we get the following:

\[
ABC_4(G) = \sqrt{\frac{22}{15}}|E_1(G)| + \frac{2\sqrt{2}}{5} |E_2(G)| + \frac{1}{5} \sqrt{\frac{37}{10}} |E_3(G)| + \frac{1}{17} \sqrt{\frac{33}{4}} |E_4(G)| + \frac{2\sqrt{10}}{905} |E_5(G)| + \frac{7}{18\sqrt{2}} |E_6(G)| + \sqrt{\frac{3}{2}} |E_7(G)| + \frac{2\sqrt{7}}{9} |E_8(G)| + \frac{1}{5} \sqrt{\frac{27}{10}} |E_9(G)| + \frac{1}{5} \sqrt{\frac{27}{10}} |E_{10}(G)| + \frac{1}{3} \sqrt{\frac{45}{2}} |E_{11}(G)| + \frac{1}{5} \sqrt{\frac{3}{5}} |E_{12}(G)| + \frac{1}{5} \sqrt{\frac{33}{5}} |E_{13}(G)| + \frac{7\sqrt{2}}{34} |E_{14}(G)|.
\]

Table 10. Edge partition of silicane chain \((SL(m,n))\) based on the sum of degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \( uv \in E(G) \) | Number of Edges | \((S_u, S_v)\) Where \( uv \in E(G) \) | Number of Edges |
|---|---|---|---|
| (12, 12) | 6 | (27, 27) | 2m - 8 |
| (15, 15) | 3m - 4 | (18, 24) | 2 |
| (15, 24) | 4m + 4 | (18, 27) | 2m + 2n - 10 |
| (12, 24) | 6 | (18, 30) | 3mn - 4m - 9n + 14 |
| (15, 27) | 4m + 4n - 20 | (24, 30) | 2 |
| (24, 27) | 2m | (27, 30) | 3m + 3n - 14 |
| (24, 24) | 2 | (30, 30) | 3mn - 16m + 20 |

After some calculation, we get
\[
\Rightarrow ABC_4(G) = \frac{1}{15} \sqrt{\frac{22}{2}} (3mn - 16m + 20) + \frac{2}{17} \sqrt{\frac{7}{2}} (3m - 4) + \frac{1}{3} \sqrt{\frac{37}{10}} (3mn - 9n - 4m + 14) + \frac{2}{5} \sqrt{\frac{3}{2}} (2m - 8) + \frac{2m}{9\sqrt{2}} + \frac{1}{5} \sqrt{\frac{27}{10}} (4m + 4) + \frac{1}{5} \sqrt{\frac{27}{10}} (2m + 2n - 10) + \frac{1}{3} \sqrt{\frac{45}{2}} (3m + 3n - 14) + \frac{7}{9} \sqrt{2} (4m + 4n - 20) + \frac{2\sqrt{7}}{9} + \frac{2\sqrt{7}}{9} + \frac{1}{5} \sqrt{\frac{33}{5}} + \frac{7\sqrt{2}}{34}
\]

From Equation (8), we get:

\[
GA_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{(S_u + S_v)} = \sum_{j=1}^{14} \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{(S_u + S_v)}
\]

By using the edge partition given in Table 10, we have
\[
GA_5(G) = |E_1(G)| + \frac{14}{17} |E_2(G)| + \frac{4\sqrt{15}}{15} |E_3(G)| + \frac{2\sqrt{2}}{5} |E_4(G)| + \frac{3\sqrt{5}}{17} |E_5(G)| + \frac{12\sqrt{2}}{905} |E_6(G)| + |E_7(G)| + \frac{4\sqrt{5}}{5} |E_8(G)| + \frac{2\sqrt{7}}{9} |E_9(G)| + \frac{1}{5} \sqrt{\frac{45}{2}} |E_{10}(G)| + \frac{3\sqrt{5}}{4} |E_{11}(G)| + \frac{4\sqrt{7}}{9} |E_{12}(G)| + \frac{6\sqrt{10}}{10} |E_{13}(G)| |G) + |E_{14}(G)|
\]

After calculation, we get
\[
\Rightarrow GA_5(G) = 3mn + \frac{1}{15} \sqrt{\frac{22}{2}} (3mn - 9n - 4m + 14) + \frac{24\sqrt{2}}{9} m + \frac{4}{13} \sqrt{\frac{37}{10}} (4m + 4) + \frac{2}{5} \sqrt{6} (2m + 2n - 10) + \frac{4\sqrt{10}}{6} (3m + 3n - 14) + \frac{2}{5} \sqrt{5} (4m + 4n - 20) - 11m + 4\sqrt{2} + \frac{8\sqrt{5}}{3} + \frac{8\sqrt{5}}{9} + 16
\]

When \( m < n \):

The proof of the following cases, (i) when \( m \) is odd and \( n \) is even, (ii) when \( m \) is even and \( n \) is odd, (iii) when \( m \) and \( n \) both are even, and (iv) when \( m \) and \( n \) both are odd, is the same as \( m = n \) by using Tables 2 and 11, Tables 3 and 12, Tables 4 and 13, and Tables 5 and 14, respectively.

When: \( m > n \):
The proof of the following cases, (i) when \( m \) is odd and \( n \) is even, (ii) when \( m \) is even and \( n \) is odd, (iii) when \( m \) and \( n \) both are even, and (iv) when \( m \) and \( n \) both are odd, is the same as \( m = n \) by using Tables 6 and 15, Tables 7 and 16, Tables 8 and 17, and Tables 9 and 18, respectively. □

Table 11. Edge partition of silicate chain (\( SL(m, n) \)), when \( m < n \) and \( m \) is odd, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges |
|---|---|---|---|
| (12, 12) | 6 | (27, 27) | \( m + n - 7 \) |
| (15, 15) | \( 3m - 3 \) | (18, 24) | 2 |
| (15, 24) | \( 2m + 2n + 2 \) | (18, 27) | \( m + 3n - 11 \) |
| (12, 24) | 6 | (18, 30) | \( 3mn - 6m - 7n + 16 \) |
| (15, 27) | \( m + 7n - 23 \) | (24, 30) | 2 |
| (24, 27) | \( m + n - 1 \) | (27, 30) | \( m + 5n - 17 \) |
| (24, 24) | 2 | (30, 30) | \( 3mn - 4m - 12n + 26 \) |

Table 12. Edge partition of silicate chain (\( SL(m, n) \)), when \( m < n \) and \( m \) is even, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges |
|---|---|---|---|
| (12, 12) | 12 | (27, 27) | \( m + n - 7 \) |
| (15, 15) | \( 3m - 6 \) | (18, 24) | 4 |
| (15, 24) | \( m + 3n - 11 \) | (18, 27) | \( m + 3n - 11 \) |
| (12, 24) | 12 | (18, 30) | \( 3mn - 5m - 8n + 18 \) |
| (15, 27) | \( m + 7n - 23 \) | (24, 30) | 4 |
| (24, 27) | \( m + n - 1 \) | (27, 30) | \( m + 5n - 17 \) |
| (24, 24) | 0 | (30, 30) | \( 3mn - 4m - 12n + 26 \) |

Table 13. Edge partition of silicate chain (\( SL(m, n) \)), when \( m < n \) and \( m \) and \( n \) are even, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges |
|---|---|---|---|
| (12, 12) | 6 | (27, 27) | \( m + n - 6 \) |
| (15, 15) | \( 3m - 2 \) | (18, 24) | 2 |
| (15, 24) | \( m + 3n - 2 \) | (18, 27) | \( 4m - 6 \) |
| (12, 24) | 6 | (18, 30) | \( 3mn + 3m - 16n + 36 \) |
| (15, 27) | \( 2m + 6n - 24 \) | (24, 30) | 2 |
| (24, 27) | \( m + n - 2 \) | (27, 30) | \( m + 5n - 20 \) |
| (24, 24) | 2 | (30, 30) | \( 3mn - 16m + 8 \) |

Table 14. Edge partition of silicate chain (\( SL(m, n) \)), when \( m < n \) and \( m \) and \( n \) are odd, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \( uv \in E(G)\) | Number of Edges |
|---|---|---|---|
| (12, 12) | 6 | (27, 27) | \( m + n - 6 \) |
| (15, 15) | \( 3m - 2 \) | (18, 24) | 2 |
| (15, 24) | \( m + 3n - 2 \) | (18, 27) | \( m + 3n - 12 \) |
| (12, 24) | 6 | (18, 30) | \( 3mn - 13m + 4 \) |
| (15, 27) | \( m + 7n - 26 \) | (24, 30) | 2 |
| (24, 27) | \( m + n - 2 \) | (27, 30) | \( m + 5n - 20 \) |
| (24, 24) | 2 | (30, 30) | \( 3mn + 4m - 20n + 48 \) |
Our research work has continued to derive new architectures from the chain silicate formulas. These results help from the chemical point of view, as well as that of pharmaceutical science.

Table 15. Edge partition of silicate chain (\(SL(m, n)\)), when \(m > n\) and \(m\) is odd, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges |
|---------------------------------|----------------|---------------------------------|----------------|
| (12, 12)                        | 6              | (27, 27)                        | \(m + n - 9\)  |
| (15, 15)                        | \(m + 2n - 3\) | (18, 24)                        | 2              |
| (15, 24)                        | \(m + 3n + 7\) | (18, 27)                        | \(m + 3n - 9\) |
| (12, 24)                        | 6              | (18, 30)                        | \(3mn - 5m - 8n + 11\) |
| (15, 27)                        | \(m + 7n - 17\) | (24, 30)                        | 2              |
| (24, 27)                        | \(m + n + 1\)  | (27, 30)                        | \(m + 5n - 11\) |
| (24, 24)                        | 2              | (30, 30)                        | \(3mn - 2m - 14n + 12\) |

Table 16. Edge partition of silicate chain (\(SL(m, n)\)), when \(m > n\) and \(m\) is even, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges |
|---------------------------------|----------------|---------------------------------|----------------|
| (12, 12)                        | 12             | (27, 27)                        | \(m + n - 9\)  |
| (15, 15)                        | \(m + 2n - 6\) | (18, 24)                        | 4              |
| (15, 24)                        | \(m + 3n - 5\) | (18, 27)                        | \(m + 3n - 9\) |
| (12, 24)                        | 12             | (18, 30)                        | \(3mn - 5m - 8n + 12\) |
| (15, 27)                        | \(m + 7n - 17\) | (24, 30)                        | 4              |
| (24, 27)                        | \(m + n + 1\)  | (27, 30)                        | \(m + 5n - 11\) |
| (24, 24)                        | 0              | (30, 30)                        | \(3mn - 2m - 14n + 12\) |

Table 17. Edge partition of silicate chain (\(SL(m, n)\)), when \(m > n\), \(m\) and \(n\) are even, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges |
|---------------------------------|----------------|---------------------------------|----------------|
| (12, 12)                        | 6              | (27, 27)                        | \(m + n - 10\) |
| (15, 15)                        | \(3m - 6\)    | (18, 24)                        | 2              |
| (15, 24)                        | \(m + 3n + 10\) | (18, 27)                        | \(m + 3n - 8\) |
| (12, 24)                        | 6              | (18, 30)                        | \(3mn - 7m - 6n + 12\) |
| (15, 27)                        | \(m + 7n - 14\) | (24, 30)                        | 2              |
| (24, 27)                        | \(m + n + 2\)  | (27, 30)                        | \(m + 5n - 8\) |
| (24, 24)                        | 2              | (30, 30)                        | \(3mn - 2m - 14n + 4\) |

Table 18. Edge partition of silicate chain (\(SL(m, n)\)), when \(m > n\), \(m\) and \(n\) are odd, based on the sum of the degrees of the end vertices of each edge.

| \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges | \((S_u, S_v)\) Where \(uv \in E(G)\) | Number of Edges |
|---------------------------------|----------------|---------------------------------|----------------|
| (12, 12)                        | 6              | (27, 27)                        | \(m + n - 10\) |
| (15, 15)                        | \(3n\)         | (18, 24)                        | 2              |
| (15, 24)                        | \(m + 3n + 10\) | (18, 27)                        | \(m + 3n - 8\) |
| (12, 24)                        | 6              | (18, 30)                        | \(3mn - 2m - 11n + 2\) |
| (15, 27)                        | \(m + 7n - 14\) | (24, 30)                        | 2              |
| (24, 27)                        | \(m + n + 2\)  | (27, 30)                        | \(m + 5n - 8\) |
| (24, 24)                        | 2              | (30, 30)                        | \(3mn - 4m - 12n + 8\) |

3. Conclusions

In this paper, we have studied and computed some degree-based topological indices for the \(m\)th chain silicate \(SL(m, n)\) network for the very first time. We have also calculated their structural closed formulas. These results help from the chemical point of view, as well as that of pharmaceutical science. Our research work has continued to derive new architectures from the chain silicate \(SL(m, n)\) network.
Author Contributions: The authors made equal contributions in the article. All authors read and approved the final manuscript.

Funding: This research was funded by the China Postdoctoral Science Foundation under Grant 2017M621579; the Postdoctoral Science Foundation of Jiangsu Province under Grant 1701081B; Project of Anhui Jianzhu University under Grant no. 2016QD116 and 2017dc03.

Acknowledgments: The authors would like to thank all the respected reviewers for their suggestions and useful comments, which resulted in an improved version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Diudea, M.V.; Gutman, I.; Lorentz, J. Molecular Topology; Nova: Huntington, NY, USA, 2001.
2. Devillers, J.; Balaban, A.T. Topological Indices and Related Descriptors in QSAR and QSPR; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1999.
3. Wiener, H. Structural determination of paraffin boiling points. *J. Am. Chem. Soc.* 1947, 69, 17–20. [CrossRef] [PubMed]
4. Liu, J.B.; Pan, X.F. Minimizing Kirchoff index among graphs with a given vertex bipartitineness. *Appl. Math. Comput.* 2016, 291, 84–88.
5. Liu, J.B.; Pan, X.F.; Hu, F.T.; Hu, F.F. Asymptotic Laplacian-energy-like invariant of lattices. *Appl. Math. Comput.* 2015, 253, 205–214.
6. Randić, M. On Characterization of molecular branching. *J. Am. Chem. Soc.* 1975, 97, 6609–6615. [CrossRef]
7. Kier, L.B.; Hall, L.H.; Murray, W.J.; Randic, M. Molecular connectivity. I: Relationship to nonspecific local anesthesia. *J. Pharm. Sci.* 1975, 64, 1971–1974. [CrossRef] [PubMed]
8. Kier, L.B.; Hall, L.H. Molecular Connectivity in Chemistry and Drug Research; Academic Press: New York, NY, USA, 1976.
9. Gutman, I. Degree based topological indices. *Croat. Chem. Acta* 2013, 86, 351–361. [CrossRef]
10. Ghorbani, M.; Hosseiniazadeh, M.A. Computing $ABC_4$ index of nanostar dendrimers. *Optoelectron. Adv. Mater. Rapid Commun.* 2010, 4, 1419–1422.
11. Vukičević, D.; Furtula, B. Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges. *J. Math. Chem.* 2009, 46, 1369–1376. [CrossRef]
12. Imran, M.; Baig, A.Q.; Ali, H. On topological properties of dominating devid derived networks. *Can. J. Chem.* 2016, 94, 137–148. [CrossRef]
13. Hayat, S.; Imran, M. Computation of topological indices of certain networks. *Appl. Math. Comput.* 2014, 240, 213–228. [CrossRef]
14. Bača, M.; Horváthová, J.; Mokrišová, M.; Suhányiová, A. On topological indices of fullerenes. *Appl. Math. Comput.* 2015, 251, 154–161. [CrossRef]
15. Ali, H.; Siddiqui, H.M.A.; Shafiq, M.K. On Degree-Based Topological Descriptors of Oxide and Silicate Molecular Structures. *MAGNT Res. Rep.* 2016, 4, 135–142.
16. Iranmanesh, A.; Zareaatkar, M. Computing GA index for some nanotubes, *Optoelectron. Adv. Mater. Rapid Commm.* 2010, 4, 1852–1855.
17. Javaid, M.; Jung, C.Y. M-Polynomials and Topological Indices of Silicate and Oxide Networks. *Int. J. Pure Appl. Math.* 2017, 115, 129–152. [CrossRef]
18. Lin, W.; Chen, J.; Chen, Q.; Gao, T.; Lin, X.; Cai, B. Fast computer search for trees with minimal ABC index based on tree degree sequences. *MATCH Commun. Math. Comput. Chem.* 2014, 72, 699–708.
19. Liu, J.-B.; Ali, H.; Shafiq, M.K.; Munir, U. On Degree-Based Topological Indices of Symmetric Chemical Structures. *Symmetry* 2018, 10, 619. [CrossRef]
20. Manuel, P.D.; Abd-El-Barr, M.I.; Rajasingh, I.; Rajan, B. An efficient representation of Benes networks and its applications. *J. Discret. Algorithms* 2008, 6, 11–19. [CrossRef]
21. Palacios, J.L. A resistive upper bound for the ABC index. *MATCH Commun. Math. Comput. Chem.* 2014, 72, 709–713.
22. Simonraj, F.; George, A. Topological Properties of few Poly Oxide, Poly Silicate, DOX and DSL Networks. *Int. J. Futur. Comput. Commun.* 2013, 2, 90–95. [CrossRef]

23. Rajan, B.; William, A.; Grigorious, C.; Stephen, S. On certain topological indices of silicate, honeycomb and hexagonal networks. *J. Comput. Math. Sci.* 2012, 3, 498–556.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).