Unique determination of fractional order and source term in a fractional diffusion equation from sparse boundary data

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Abstract
In this article, for a two dimensional fractional diffusion equation, we study an inverse problem for simultaneous restoration of the fractional order and the source term from the sparse boundary measurements. By using a sequence of harmonic functions, we construct useful quantitative relation between the unknowns and measurements. From Laplace transform and the knowledge in complex analysis, the uniqueness theorem is proved.

Keywords: fractional diffusion equation, inverse problem, nonlinearity, sparse measurements, uniqueness, multiple unknowns

1. Introduction

1.1. Mathematical statement
In this article, an inverse problem in the fractional diffusion equation $\partial^\alpha_t u - \Delta u = F(x, t)$ in $\mathbb{R}^2 \times (0, \infty)$ is considered, where order $\alpha$ and source $F(x, t)$ are the unknowns. To recover $F(x, t)$, we will need the observation of $u$ on the whole domain $\mathbb{R}^2 \times (0, \infty)$, which seems impossible in practice. So, most existing work focus on the time dependent or space dependent cases, namely, $F(x, t) := p(x)$ or $F(x, t) := q(t)$ or as a product $F(x, t) := p(x)q(t)$ where either $p$ or $q$ is known.

We attempt to recover the source $F$ with more general formulation,

$$F(x, t) := \sum_{k=1}^{K} p_k(x) \chi_{(c_{k-1}, c_k]}.$$
with unknown \( \{p_k, c_k\} \). This can be regarded as semi-discrete form of general \( F(x, t) \), with piecewise constant discretization on time trace \( t \). Note that we do not require \( K \) to be finite, i.e. \( K \in \mathbb{N}^+ \cup \{\infty\} \), and the order \( \alpha \) in \( \partial^\alpha \) is also unknown. See below for a precise mathematical statement of this inverse problem.

The mathematical model is:

\[
\begin{align*}
\partial^\alpha_t u - \Delta u &= \sum_{k=1}^{K} p_k(x) \chi_{[c_k-1, c_k)}, \quad (x, t) \in \Omega \times (0, \infty), \\
{u(x, 0) = 0}, & \quad x \in \Omega, \\
{u(x, t) = 0}, & \quad (x, t) \in \partial \Omega \times (0, \infty).
\end{align*}
\]

Here \( \Omega \) is the unit disc in \( \mathbb{R}^2 \) and we choose \( \partial^\alpha \) as the Djrbashyan–Caputo derivative of order \( \alpha \in (0, 1) \), defined by

\[
\partial^\alpha_t \psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} \psi(\tau) d\tau, \quad t > 0.
\]

There are two common definitions of fractional derivatives, Riemann–Liouville version \( D^\alpha_t := \frac{d^\alpha}{dt^\alpha} \) and Djrbashyan–Caputo version \( \partial^\alpha_t := \int_0^t \frac{1}{\Gamma(\alpha-\beta)} \frac{d}{d\tau} \psi(\tau) d\tau \), where the Riemann–Liouville fractional integral \( I^\beta \psi \) of order \( \beta > 0 \) is defined as

\[
I^\beta \psi(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \psi(\tau) d\tau, \quad t > 0,
\]

\( \Gamma(\cdot) \) is the Gamma function, and the positive integer \( n \) satisfies \( n-1 \leq \beta < n \). To better utilize the initial condition, we choose the Djrbashyan–Caputo definition \( \partial^\alpha \) in this work.

In equation (1.1), \( \{p_k(x)\}_{k=1}^{K}, \{c_k\}_{k=0}^{K} \subset \mathbb{N}^+ \cup \{\infty\} \) are the unknowns. The order \( \alpha \) can reflect some of the inhomogeneity of the medium, which with the source term usually cannot be measured straightforwardly. Furthermore, in some cases, the observed points cannot be set in the interior of the domain \( \Omega \). Namely, we can only obtain the information of the solution \( u \) on a subset \( Z_{\text{ob}} \) of the boundary \( \partial \Omega \). Here, the boundary flux data \( \frac{\partial u}{\partial \nu} \) is used, and \( \nu \) is the unit outward normal vector of \( \partial \Omega \). The interested inverse problem is stated below.

**Problem 1.1.** Given the boundary flux data

\[
\frac{\partial u}{\partial \nu}(z, t), \quad t \in (0, \infty), \quad z \in Z_{\text{ob}} \subset \partial \Omega,
\]

can we uniquely determine the order \( \alpha \) of the fractional derivative, the spatial components \( \{p_k(x)\}_{k=1}^{K} \) and the time mesh \( \{c_k\}_{k=0}^{K} \) of the source term simultaneously?

This inverse problem contains several challenges. For instance, multiple unknowns, and the nonlinear relation between \( p_k \) and \( \chi_{[c_k-1, c_k]} \). In addition, people always want the size of observed area \( Z_{\text{ob}} \) as small as possible to save cost. This demand also increases the difficulty.

1.2. Background and literature

In the field of statistical mechanics, from Brownian motion, people can deduce the classical diffusion equation \( u_t = Cu_{xx} \), where \( u \) means the density of particles. Actually, we may generalize Brownian motion to continuous time random walk, in which the jump length and waiting time between two successive jumps will follow the given probability density functions, denoted by \( \lambda(x) \) and \( \psi(t) \), respectively. By the central limit theorem, with the assumption that both the
moments \( \int_0^\infty t \psi(t)dt \) and \( \int_0^\infty x^2 \lambda(x)dx \) are finite, the long-time behavior of the continuous time random walk will correspond to Brownian motion again.

However, within the last few decades, the collapse of the conditions \( \int_0^\infty t \psi(t)dt < \infty \) or \( \int_0^\infty x^2 \lambda(x)dx < \infty \) were found in more and more anomalous diffusion processes. For example, for a diffusive process in a heterogeneous medium, the particles may be absorbed to a low permeability zone which has a longer waiting time \( \psi(t) \sim t^{-1-\alpha}, t \to \infty, \alpha \in (0, 1) \). In this case, it follows a subdiffusion process whose mean squared displacement \( \langle x^2 \rangle \) will be proportional to time \( t^\alpha \) as \( t \) large, instead of \( \langle x^2 \rangle \sim t \) in the situation of classical diffusion process.

To capture such anomalous diffusion processes, people introduced fractional derivatives into differential equations. By assuming the waiting time distribution \( \psi(t) \) is independent of the jump length distribution \( \lambda(x) \), and that \( \psi(t) \) is of power-law distribution and \( \lambda(x) \) obeys Gaussian distribution as time large, from an abstract point of view, [26] derived the time-fractional diffusion equation in the framework of the continuous time random walk. In addition, the temporal fractional diffusion–advection equation is obtained by the time-changed Langevin equation with an inverse \( \alpha \)-stable subordinator in [25]. For other applications of fractional differential equations in anomalous diffusion phenomena and anomalous diffusion-like processes involving memory effects, see [1, 2, 5, 8, 14, 15, 34, 35] and the references therein.

The inverse problems of determining the fractional order or the unknown source term in fractional diffusion equations are well studied and considerable results are generated. For the determination of the fractional order, one can consult [3, 9, 16, 17, 19]. We refer to [11, 13, 29, 32, 36] for recovering the spatial unknown in the source term, and [23, 24] for the source with temporal unknown. For the extensive review, [12, 18, 20, 22] are suggested. In addition, the inverse source problem in classical diffusion equation, in which the source term \( p(x)q(t) \) contains two unknowns \( p, q \) and \( q \) is set to be a step function, is considered in [30].

1.3. Main result and outline

For problem 1.1, we prove the uniqueness theorem and give a positive answer. Meanwhile, the size of measured area \( \Omega_{ob} \) is limited to two appropriately chosen points, i.e. \( \Omega_{ob} = \{z_1, z_2 \} \subset \partial \Omega \). This reflects the sparsity in the title.

Before stating the main theorem, we list several restrictions on the unknowns \( \{ p_k, c_k, \alpha \} \).

Assumption 1.1.

(a) \( K \in \mathbb{N}^+ \cup \{ \infty \}, \quad 0 \leq c_0 < c_1 < \cdots < c_k < \cdots \) and \( \exists \eta > 0 \) such that \( \inf \{ |c_k - c_{k+1}| : k = 0, \ldots, K - 1 \} \geq \eta \);

(b) \( \exists \gamma > 0 \) such that \( p_k \in \mathcal{D}((-\Delta)^\gamma) \subset L^2(\Omega), \quad k = 1, \ldots, K, \) and \( \sum_{k=1}^K p_k(x) \chi_{[c_k+1, c_k)} \in L^1(0, \infty; \mathcal{D}((-\Delta)^\gamma)) \);

(c) \( \| p_k \|_{L^2(\Omega)} \neq 0 \) for \( k = 1, \ldots, K, \) and \( \| p_k - p_{k+1} \|_{L^2(\Omega)} \neq 0 \) for \( k = 1, \ldots, K - 1 \).

Remark 1.1. The subspace \( \mathcal{D}((-\Delta)^\gamma) \) is defined in (2.2). The condition \( \sum_{k=1}^K p_k(x) \chi_{[c_k+1, c_k)} \in L^1(0, \infty; \mathcal{D}((-\Delta)^\gamma)) \) will lead to \( \sum_{k=1}^K p_k(x) \chi_{[c_k+1, c_k)} \in L^2(0, \infty; \mathcal{D}((-\Delta)^\gamma)) \) by direct calculation and the fact \( \sum_{n=1}^\infty |b_n|^2 \leq (\sum_{n=1}^\infty |b_n|^2)^2 \).

Assumption 1.1 (a) is given to support the proof of lemma 3.1. We set assumption 1.1 (c) to make sure the source series \( \sum_{k=1}^K p_k(x) \chi_{[c_k+1, c_k)} \) cannot be simplified further. For example, assume that \( \| p_0 \|_{L^2(\Omega)} = \| p_{k-1} - p_k \|_{L^2(\Omega)} = 0 \), then the source series can be rewritten as

\[
\sum_{k \notin \{p_0, p_1, \ldots, p_k \}} p_k(x) \chi_{[c_k+1, c_k)} + p_k(x) \chi_{[c_k+1, c_k)}.
\]
Under assumption 1.1, we state the main theorem.

**Theorem 1.1.** Set \( z_\ell = (\cos \theta_\ell, \sin \theta_\ell) \in \partial \Omega, \ell = 1, 2 \) to be the boundary observation points and suppose the following condition is fulfilled.

\[
\theta_1 - \theta_2 \notin \pi \mathbb{Q}, \quad \mathbb{Q} \text{ is the set of rational numbers.}
\]

Let the two sets of unknowns \( \{ \alpha, c_0, p_k(x), c_k \}_{k=1}^K \) and \( \{ \tilde{\alpha}, \tilde{c}_0, \tilde{p}_k(x), \tilde{c}_k \}_{k=1}^{\tilde{K}} \) satisfy assumption 1.1, and assume that

\[
1/2 < \alpha, \tilde{\alpha} < 1.
\]

Denote the solutions of equation (1.1) corresponding to the two sets of unknowns by \( u \) and \( \tilde{u} \), respectively. If

\[
\frac{\partial u}{\partial \nu}(z_\ell, t) = \frac{\partial \tilde{u}}{\partial \nu}(z_\ell, t), \quad t \in (0, \infty), \quad \ell = 1, 2,
\]

then \( \alpha = \tilde{\alpha}, c_0 = \tilde{c}_0, K = \tilde{K} \) and

\[
\| p_k - \tilde{p}_k \|_{L^2(\Omega)} = 0, \quad c_k = \tilde{c}_k, \quad k = 1, \ldots, K.
\]

The remaining part of this manuscript is structured as follows. Section 2 collects the preliminary knowledge, such as fractional calculus, the Dirichlet eigensystem of \( -\Delta \) on \( \Omega \). In section 3, we build the measurement representation from Green’s identities. After taking Laplace transform on this representation, we give several auxiliary results and prove the main theorem, theorem 1.1. Finally, the concluding remark is given in section 4.

### 2. Preliminaries

#### 2.1. Fractional calculus

Letting \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), the Mittag–Leffler function is defined as follows

\[
E_{\alpha, \beta}(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)}, \quad y \in \mathbb{C}.
\]

When \( \alpha, \beta \in (0, \infty) \), \( E_{\alpha, \beta}(\cdot) \) is an entire function.

Since the Laplace transform of \( E_{\alpha, \beta}(\cdot) \) will generate the term \( s^{\alpha} \), \( s \in \mathbb{C} \), which is a multivalued function, we define the branch \( \Lambda \) to make sure the analyticity,

\[
\Lambda := \{ \rho e^{i\zeta} \in \mathbb{C} : \rho \in (0, \infty), \quad \zeta \in [0, 2\pi) \}, \quad \Lambda^+ := \{ s \in \Lambda : \text{Re} s > 0 \}.
\]

Now we give the lemma about Laplace transform of Mittag–Leffler function.

**Lemma 2.1** ([10, proposition 4]). For \( \lambda \geq 0, \alpha \in (0, 1) \),

\[
\mathcal{L} \left\{ t^{\alpha-1} E_{\alpha, \beta}(-\lambda t^\alpha); s \right\} = \frac{1}{s^\alpha + \lambda}, \quad s \in \Lambda^+.
\]

The next two lemmas about Mittag–Leffler function will be used in the future proof.

**Lemma 2.2** ([27, theorem 1.6]). Let \( 0 < \alpha < 2 \) and \( \beta \in \mathbb{R} \). We suppose \( \pi \alpha / 2 < \mu < \min\{ \pi, \pi \alpha \} \), then there exists a constant \( C = C(\alpha, \beta, \mu) > 0 \) such that

\[
|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi.
\]
Lemma 2.3 ([28, 31]). For $\lambda, \alpha > 0$,
\[
\frac{d}{dt} E_{\alpha,1}(-\lambda^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,0}(-\lambda^\alpha), \quad t > 0,
\]
and $E_{\alpha,0}(-\lambda^\alpha) \geq 0$. Then $\|\lambda t^{\alpha-1} E_{\alpha,0}(-\lambda^\alpha)\|_{L^2(\partial)} = 1$.

The lemmas below concern the fractional derivatives $\partial_t^\alpha$ and $D_\alpha^t$.

Lemma 2.4. Let $v(x) \in L^2(\Omega)$ and $\psi(t) \in C(0, \infty)$ with $D_\alpha^t \psi(t) \in C(0, \infty), I^{1-\alpha} \psi(0) = 0, \alpha \in (0, 1)$. Then for $v$ in equation (1.1), we have
\[
\int_0^t \int_\Omega \partial_t^\alpha u(x, \tau) v(x) \psi(t - \tau) d\tau dx = \int_0^t \int_\Omega u(x, \tau) D_\alpha^t \psi(t - \tau) d\tau dx, \quad t > 0.
\]

Proof. [6, 10] Provided the well-definedness of $\partial_t^\alpha u$ in equation (1.1) in the sense of $L^2(\Omega)$ and defined it as
\[
\langle \partial_t^\alpha u(\cdot, t), v(\cdot) \rangle_{L^2(\Omega)} = \partial_t^\alpha \langle u(\cdot, t), v(\cdot) \rangle_{L^2(\Omega)}.
\]
Here $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$. Then it follows that
\[
\int_0^t \int_\Omega \partial_t^\alpha u(x, \tau) v(x) \psi(t - \tau) d\tau dx = \int_0^t \int_\Omega u(x, \tau) D_\alpha^t \psi(t - \tau) d\tau dx,
\]
where Fubini’s theorem is used in the last equality. Using integration by parts implies
\[
\int_0^t \int_\Omega \partial_t^\alpha u(x, \tau) v(x) \psi(t - \tau) d\tau dx = \int_0^t \int_\Omega \langle u(x, \tau), v(x) \rangle_{L^2(\Omega)} D_\alpha^t \psi(t - \tau) d\tau dx, \quad t > 0,
\]
in view of the assumption $u(x, 0) = I^{1-\alpha} \psi(0) = 0$. We finish the proof. \hfill \square

Lemma 2.5 ([27, example 4.3]). For $0 < \alpha < 1$ and $\lambda > 0$, we have $\psi(t) := \lambda^\alpha t^{-\alpha} E_{\alpha,0}(-\lambda^\alpha)$ is the unique solution to the time-fractional ordinary differential equation
\[
D_\alpha^t \psi(t) + \lambda \psi(t) = 0, \quad t > 0,
\]
with the initial condition
\[
\lim_{t \to 0} I^{1-\alpha} \psi(t) = 1.
\]

2.2. Dirichlet eigensystem of $-\Delta$ and regularity of solution $u$

The eigensystem $\{\lambda_n, \varphi_n\}_{n=1}^\infty$ (multiplicity counted) of the operator $-\Delta$ on $\Omega$ with Dirichlet boundary condition is defined as follows:
\[
0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots \to \infty, \quad \text{as } n \to \infty,
\]
and $\varphi_n$ denotes the corresponding eigenfunction
\[
\varphi_n(\tau, \theta) = \omega_n J_{\nu(\alpha)}(\lambda_n^{1/2} \tau) e^{i \nu(\alpha)n \theta}, \quad n \in \mathbb{N}^+.
\]
which form an orthonormal basis of $L^2(\Omega)$. Here $(r, \theta)$ are the polar coordinates on $\Omega$, and $J_{m(n)}(\cdot)$ is the Bessel function of order $|m(n)|$ with $\lambda_n^{1/2}$ as its zero point. The Bessel orders $m$ depend on the choice of $n$ and we use the notation $m(n)$ to show the dependence (sometimes we may use $J_m$ for short).

**Remark 2.1.** \{ωₙ\} are the normalized coefficients to make sure $\|\varphi_n\|_{L^2(\Omega)} = 1$. From Bourget’s hypothesis, proved in [33], there exist no common positive zeros between two Bessel functions with different nonnegative integer orders. Also recall that $J_{-m}(r) = (-1)^m J_m(r)$, given an eigenvalue $\lambda_0\$, $\lambda_n^{1/2}$ can only be the zero of $J_{\pm m(n)}(\cdot)$. Hence the multiplicity for $\lambda_0$ is two if $m(n_0)$ is nonzero, otherwise, it will be one.

In the case of $m(n_0) \neq 0$, by setting $\lambda_0 = \lambda_{n_0} + 1$, the corresponding eigenpairs are given as

$$
\left( \lambda_{n_0}, \omega_{n_0} J_{m(n_0)} \left( \lambda_{n_0}^{1/2} r \right) e^{i m(n_0) \theta} \right), \quad \left( \lambda_{n_0} + 1, \omega_{n_0} J_{m(n_0)} \left( \lambda_{n_0}^{1/2} r \right) e^{-i m(n_0) \theta} \right).
$$

Now setting $m(n_0) = m(n_0) = -m(n_0 + 1)$, the representation (2.1) is consistency and $m$ is uniquely determined by the value of $n$. See [7] for details about the structure of \{ωₙ\}.

With \{(ωₙ, ϕₙ)\}ₙ=₁, we can define the subspace $D((-\Delta)^2) \subset L^2(\Omega)$, $\gamma > 0$ as:

$$
D((-\Delta)^2) := \left\{ \psi \in L^2(\Omega): \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |\langle \psi, \varphi_n(\cdot) \rangle_{L^2(\Omega)}|^2 < \infty \right\}. \tag{2.2}
$$

Here $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$. Since $\Omega$ is the unit disc in $\mathbb{R}^2$, $D((-\Delta)^2) \subset H^{2\gamma}(\Omega)$.

The next lemma concerns the values of the normalized parameters \{ωₙ\}.

**Lemma 2.6.** ωₙ has the form $\omega_n = \pi^{-1/2} \left[ J_{|m(n)|+1}(\lambda_n^{1/2}) \right]^{-1}$, \(n \in \mathbb{N}^+\).

**Proof.** Firstly we list some properties of Bessel functions,

$$
\begin{align*}
2mJ_m(x)/x &= J_{m-1}(x) + J_{m+1}(x), \\
2[J_m(x)]' &= J_{m-1}(x) - J_{m+1}(x), \\
[x^{m+1}J_{m+1}(x)]' &= x^{m+1}J_{m}(x).
\end{align*}
$$

Since $\|\varphi_n\|_{L^2(\Omega)} = 1$, then

$$
\omega_n^2 \left[ \int_0^{2\pi} |e^{im(n)\theta}|^2 d\theta \right] \left[ \int_0^1 J_m^2(\lambda_n^{1/2} r) dr \right] = 1.
$$

Not hard to see that $\int_0^{2\pi} |e^{im(n)\theta}|^2 d\theta = 2\pi$. Also, with (2.3) and the fact that $\lambda_n^{1/2}$ is the zero of $J_{|m|}(r)$,

$$
\int_0^1 J_m^2(\lambda_n^{1/2} r) dr = \lambda_n^{-1} \frac{\lambda_n^{1/2}}{0} J_m^2(\lambda_n^{1/2} r) dr
\begin{align*}
&= \lambda_n^{-1} \left[ r^2 J_{|m|+1}(r)/2 + r^2 J_{|m|}^2(r)/2 - |m|r J_{|m|}(r)J_{|m|+1}(r) \right]^{1/2} \\
&= J_{|m|+1}^2(\lambda_n^{1/2})/2.
\end{align*}
$$
Remark 2.2. Not hard to check for
\[ \text{Lemma 2.7 (\cite{[21], theorem 9.4]), gives } \partial_{\nu}(\partial u) \mid_{\Omega} \text{ for a.e. } t \in (0,\infty). \]

\[ \text{Lemma 2.8. } \text{Let } u \text{ satisfy equation (1.1), for } z \in \partial \Omega, \text{ we have} \]
\[ \lim_{N \to \infty} \int_{\partial \Omega} \delta^N_z(x) \frac{\partial u}{\partial \nu}(x,t) \, dx = \frac{\partial u}{\partial \nu}(z,t), \quad a.e. t \in (0,\infty). \]

Proof. From the definition (2.4) of the function \( \delta^N_z \), it follows that
\[ \sum_{i=-N}^{N} \int_{\partial \Omega} \xi_i(z) \xi_i(x) \frac{\partial u}{\partial \nu}(x,t) \, dx = \sum_{i=-N}^{N} \left\langle \frac{\partial u}{\partial \nu}, \xi_i(t) \right\rangle_{L^2(\partial \Omega)} \xi_i(z), \quad t > 0. \]

The result \( \frac{\partial u}{\partial \nu}, \xi(t) \in C^{0,2}(\partial \Omega) \) in remark 2.2 implies that the Fourier series of \( \frac{\partial u}{\partial \nu}, \xi(t) \) converges pointwisely on \( \partial \Omega \) for a.e. \( t \in (0,\infty) \). Hence,
\[ \lim_{N \to \infty} \sum_{i=-N}^{N} \left\langle \frac{\partial u}{\partial \nu}, \xi_i(t) \right\rangle_{L^2(\partial \Omega)} \xi_i(z) = \frac{\partial u}{\partial \nu}(z,t), \quad a.e. t \in (0,\infty), \]
which completes the proof. \( \square \)

Since \( \delta^N_z \in L^2(\Omega) \), then we can express \( \delta^N_z \) in \( L^2(\Omega) \) topology as
\[ \delta^N_z(x) = \sum_{n=1}^{\infty} (\delta^N_z, \varphi_n)_{L^2(\Omega)} \varphi_n(x). \]
By Lemma 2.6 and (2.3), the Fourier coefficients \( (\delta^N_z, \varphi_n)_{L^2(\Omega)} \) are calculated as

\[
(\delta^N_z, \varphi_n)_{L^2(\Omega)} = \begin{cases} \pi^{-1/2} \lambda_n^{-1/2} e^{-i\text{Im}(\alpha)t}, & \text{if } |N| \geq |m(n)|, \\ 0, & \text{otherwise.} \end{cases}
\]  

(2.5)

We assume \( u^N_z \) is the solution of the following initial-boundary value problem

\[
\begin{align*}
D^\alpha_t u^N_z(x, t) &= \Delta u^N_z(x, t), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial}{\partial t} u^N_z(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, \infty), \\
\Gamma^{1-\alpha} u^N_z(x, t) &= -\delta^N_z(x), \quad (x, t) \in \Omega \times \{0\}.
\end{align*}
\]  

(2.6)

In view of the fact that \( \delta^N_z \) is the linear combination of harmonic functions on \( \Omega \), we see that \( \Delta \delta^N_z = 0 \), then \( u^N_z(x, t) := u^N_z(x, t) + \frac{\rho^{-1}}{\Gamma(\alpha)} \delta^N_z(x) \) satisfies the following initial-boundary value problem

\[
\begin{align*}
D^\alpha_t u^N_z(x, t) &= \Delta u^N_z(x, t), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial}{\partial t} u^N_z(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, \infty), \\
\Gamma^{1-\alpha} u^N_z(x, t) &= -\delta^N_z(x), \quad (x, t) \in \Omega \times \{0\}.
\end{align*}
\]  

(2.7)

By Lemma 2.5, (2.5) and (2.6), the representation of \( u^N_z \) is given as

\[
u^N_z(x, t) = \sum_{|m(n)| \leq |N|} \pi^{-1/2} \lambda_n^{-1/2} e^{-i\text{Im}(\alpha)t} \left[ \frac{1}{\Gamma(\alpha)} - E_{\alpha, \alpha}(-\lambda_n t) \right] \varphi_n, \quad t > 0.
\]  

(2.8)

3. Uniqueness theorem

Now we will establish the proof of Theorem 1.1. Throughout this section, assumption 1.1 is supposed to be valid. For \( k = 1, \ldots, K, n \in \mathbb{N}^+, z = (\cos \theta_z, \sin \theta_z) \in \partial \Omega \), we denote

\[
p_{k,n} := (p_k(\cdot), \varphi_n(\cdot))_{L^2(\Omega)}, \quad a_n(z) := \pi^{-1/2} \lambda_n^{-1/2} e^{i\text{Im}(\alpha)t}.
\]

Then the following absolute convergence result can be proved, which will be used in the uniqueness proof.

Lemma 3.1. \( \sum_{k=1}^{K} \sum_{n=1}^{\infty} a_n(z)p_{k,n} \) is absolutely convergent for each \( z \in \partial \Omega \).

Proof.

\[
\sum_{k=1}^{K} \sum_{n=1}^{\infty} |a_n(z)p_{k,n}| \leq \sum_{k=1}^{K} \left[ \sum_{n=1}^{\infty} a_n^2(z) \lambda_n^{-2\gamma} \right]^{1/2} \left[ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_{k,n}^2 \right]^{1/2} = \left[ \sum_{n=1}^{\infty} a_n^2(z) \lambda_n^{-2\gamma} \right]^{1/2} \left[ \sum_{k=1}^{K} \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_{k,n}^2 \right]^{1/2}.
\]
Lemma 3.2. Weyl’s law gives that \( \lambda_n = O(n) \), as \( n \to \infty \), then \( |a_2^{(z)}(\lambda_n^{-2})| \leq C n^{-1-2\gamma} \), which implies \( \sum_{n=1}^{\infty} |a_2^{(z)}(\lambda_n^{-2})| \leq C \sum_{n=1}^{\infty} n^{-1-2\gamma} \). Also, from assumption 1.1 (b), we have

\[
\left\| \sum_{k=1}^{K} p_k(x) \chi_{\Omega_k}(x) \right\|_{L^{1}(0,1; L^{2}(-\Delta Z))} = \sum_{k=1}^{K} (c_k - c_{k-1}) \left\| \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_k(x) \right\|_{L^{2}}^{1/2} < \infty,
\]

which with assumption 1.1 (a) gives

\[
\eta \sum_{k=1}^{K} \left( \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_k(x) \right)^{1/2} \leq \sum_{k=1}^{K} (c_k - c_{k-1}) \left( \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_k(x) \right)^{1/2} < \infty,
\]

i.e. \( \sum_{k=1}^{K} \left( \sum_{n=1}^{\infty} \lambda_n^{2\gamma} p_k(x) \right)^{1/2} \) \( < \) \( \infty \). Hence, it holds that \( \sum_{k=1}^{K} \sum_{n=1}^{\infty} |a_n(z)p_k(x)| \) \( < \) \( \infty \) and the proof is complete.

### 3.1. Measurement representation

In this subsection we will build a connection between the flux measurements and the unknowns.

**Lemma 3.2.** Assume \( z \in \partial \Omega \), and let \( u \) and \( w^N \) be the solutions of (1.1) and (2.7) respectively, then

\[
- \left( I - \frac{\partial}{\partial t} \right) (u, t) = \int_{0}^{1} \lim_{N \to \infty} \left[ \int_{\Omega} \sum_{k=1}^{K} p_k(x) \chi_{\Omega_k}(x) w^N_k(x, t - \tau) dx \right] d\tau, \quad t > 0.
\]

**Proof.** Equation (2.7) and Green’s identities yield that for \( v \in H^1_0(\Omega) \),

\[
\int_{\Omega} D^\alpha u^N(x, t) v(x) + \nabla u^N(x, t) \cdot \nabla v(x) dx = 0, \quad t > 0.
\]

On the other hand, taking convolution of \( w^N_k(x, t) \) and equation (1.1), we see that

\[
I_N(t) := \int_{0}^{t} \int_{\Omega} \sum_{k=1}^{K} p_k(x) \chi_{\Omega_k}(x) w^N_k(x, t - \tau) dx d\tau
\]

\[
= \int_{0}^{t} \int_{\Omega} \left[ \partial_{\tau}^\alpha u(x, \tau) - \Delta u(x, \tau) \right] w^N_k(x, t - \tau) dx d\tau.
\]

With Green’s identities, we have

\[
I_N(t) = \int_{0}^{t} \int_{\Omega} \partial_{\tau}^\alpha u(x, \tau) w^N_k(x, t - \tau) dx d\tau + \int_{0}^{t} \int_{\Omega} \Delta u(x, \tau) \cdot \nabla w^N_k(x, t - \tau) dx d\tau
\]

\[
- \int_{0}^{t} \int_{\partial \Omega} \frac{\partial u}{\partial \nu}(x, \tau) w^N_k(x, t - \tau) dx d\tau.
\]

With (2.8) and \( I^{1-\alpha} w^N_k(x, 0) = 0 \), then from lemma 2.4, we arrive at the equality

\[
\int_{0}^{t} \int_{\Omega} \partial_{\tau}^\alpha u(x, \tau) w^N_k(x, t - \tau) dx d\tau = \int_{0}^{t} \int_{\Omega} u(x, \tau) D^\alpha w^N_k(x, t - \tau) dx d\tau.
\]
Finally, we get
\[
I_N = \int_0^t \int_\Omega \left[ D^2 u^N_t(x, t - \tau)u(x, \tau) + \nabla u^N_t(x, t - \tau) \cdot \nabla u(x, \tau) \right] \, dx \, d\tau \\
- \frac{1}{\Gamma(\alpha)} \int_0^t \int_\Omega (t - \tau)^{\alpha-1} \frac{\partial u}{\partial \nu}(x, \tau) \delta^N_t(x) \, dx \, d\tau \\
= - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{\partial u}{\partial \nu}(x, \tau) \delta^N_t(x) \, dx \, d\tau.
\]

Now from lemma 2.8 and realizing the ‘almost everywhere’ can be neglected in integration, it follows that
\[
\lim_{N \to \infty} \int_0^t \lim_{k \to \infty} \left[ \int_\Omega \sum_{k=1}^K p_k(x) \chi_{\tau \in [k-1,k]}(t - \tau) \right] \delta^N_t(x, t - \tau) \, dx \, d\tau = - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{\partial u}{\partial \nu}(z, \tau) \, d\tau,
\]
which completes the proof. \(\square\)

With the above lemma, we can show the result below straightforwardly.

**Lemma 3.3.** For \( z \in \partial \Omega, \) we have for \( t > 0, \)
\[
- \left( r_0 \frac{\partial u}{\partial \nu} \right)(z, t) = \int_0^t \sum_{k=1}^K \chi_{\tau \in [k-1,k]}(t - \tau) \alpha \int_0^\infty \sum_{n=1}^\infty p_{k,n}a_n(z) \left[ \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^\alpha) \right] \, d\tau.
\]

**Proof.** For a fixed \( t \in (0, \infty), \) there are only finite \( k \) satisfying \( c_k \leq t. \) Sequentially, the summation in the result of lemma 3.2 is finite and we denote it by \( \sum_{k=1}^{k_i} \ldots \) with \( k_i < \infty. \) Then we have
\[
\int_\Omega \sum_{k=1}^K \int_0^t p_k(x) \chi_{\tau \in [k-1,k]}(t - \tau) \delta^N_t(x, t - \tau) \, dx \, d\tau = \int_\Omega \sum_{k=1}^K p_k(x) \chi_{\tau \in [k-1,k]}(t - \tau)u^N_t(x, t - \tau) \, dx
\]
\[
= \sum_{k=1}^K \chi_{\tau \in [k-1,k]}(t - \tau)u^N_t(\cdot, t - \tau) \in L^2(\Omega).
\]

Since \( u^N_t, \delta^N_t \in L^2(\Omega) \) for a.e. \( t > 0, \) so does \( u^N_t. \) Also assumption 1.1 ensures \( p_k \in L^2(\Omega), \) \( k = 1, \ldots, K. \) So with (2.8), we have for a.e. \( \tau \in (0, t), \)
\[
\langle p_k(\cdot, \delta^N_t(\cdot, t - \tau)) \rangle_{L^2(\Omega)} = (t - \tau)^{\alpha-1} \sum_{|m(n)| \leq N} p_{k,m}a_m(z) \left[ \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_m(t - \tau)^\alpha) \right],
\]
which leads to
\[
\lim_{N \to \infty} \int_\Omega \sum_{k=1}^K p_k(x) \chi_{\tau \in [k-1,k]}(t - \tau)u^N_t(x, t - \tau) \, dx = \lim_{N \to \infty} \sum_{k=1}^{k_i} \chi_{\tau \in [k-1,k]}(t - \tau)u^N_t(\cdot, t - \tau) \langle p_k(\cdot, \delta^N_t(\cdot, t - \tau)) \rangle_{L^2(\Omega)}
\]
\[
= \sum_{k=1}^{k_i} \chi_{\tau \in [k-1,k]}(t - \tau)u^N_t(\cdot, t - \tau) \langle p_k(\cdot, \delta^N_t(\cdot, t - \tau)) \rangle_{L^2(\Omega)}
\]
\[
\times \left[ \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_m(t - \tau)^\alpha) \right].
\]
Now given $\epsilon > 0$, lemma 3.1 yields that there exists $N_0 > 0$ such that $\sum_{n=N_0+1}^{\infty} |a_n(z)p_{n,x}| < \epsilon$, $k = 1, \ldots, K_i$. Let $N_1 = \max \{|m(n)|: n = 1, \ldots, N_0\}$, then for $N \geq N_1$, we have

$$\left| \left( \sum_{n=1}^{\infty} - \sum_{|m(n)| \leq N} \right) p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right) \right|$$

$$= \left| \left( \sum_{n=N_0+1}^{\infty} - \sum_{n>N_0} \right) p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right) \right|$$

$$\leq C \sum_{n=N_0+1}^{\infty} |a_n(z)p_{n,x}| < C\epsilon.$$

Estimate in lemma 2.2 is used above. Now we have proved for a.e. $\tau \in (0, t)$,

$$\lim_{N \to \infty} \int_{\Omega} \sum_{k=1}^{K} p_k(x) \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} w_N(x, t - \tau) dx$$

$$= \sum_{k=1}^{K} \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} (t - \tau)^{\alpha-1} \sum_{n=1}^{\infty} p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right)$$

$$= \sum_{k=1}^{K} \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} (t - \tau)^{\alpha-1} \sum_{n=1}^{\infty} p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right),$$

which together with lemma 3.2 completes the proof. \(\square\)

### 3.2. Laplace transform argument

The convolution structure in the result of lemma 3.3 encourages us to apply Laplace transform. From lemmas 2.2 and 3.1, it holds that

$$\left| \int_0^t \sum_{k=1}^{K} \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} (t - \tau)^{\alpha-1} \sum_{n=1}^{\infty} p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right) d\tau \right|$$

$$\leq \int_0^t \sum_{k=1}^{K} \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} (t - \tau)^{\alpha-1} \sum_{n=1}^{\infty} |p_{n,x} a_n(z)| \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right) d\tau$$

$$\leq C \int_0^t (t - \tau)^{\alpha-1} d\tau \leq Ct^\alpha.$$

Also we can see $|e^{-st}|$ is integrable on $(0, \infty)$ for $s \in \Lambda^+$. Then by dominated convergence theorem and lemma 2.1, taking Laplace transform on the result in lemma 3.3 yields that

$$\mathcal{L} \left\{ - \left( \Gamma \frac{\partial u}{\partial \nu} \right) (z, t); s \right\} = \int_0^\infty e^{-st} \int_0^t \sum_{k=1}^{K} \chi_{\tau \in [k^{1/\delta}, (k+1)^{1/\delta}]} (t - \tau)^{\alpha-1} \times \sum_{n=1}^{\infty} p_{n,x} a_n(z) \left( \frac{1}{\Gamma(\alpha)} - E_{\alpha,\alpha}(-\lambda_n(t - \tau)^{\alpha}) \right) d\tau d\tau$$

$$\times d\tau.$$
is holomorphic on

Inverse Problems

For $a$, fix $\Lambda$ which gives

of complex branch

which implies the uniform convergence.

∑  

Given $\omega$, we denote the set of distinct eigenvalues by $\{\lambda_j\}_{j=1}^\infty$ with increasing order. From the definition of complex branch $\Lambda$ and condition (1.3), every pole $(-\lambda_j)^{1/\alpha}$, $j \in \mathbb{N}^+$ in (3.1) is included by $\Lambda$. This is crucial in the proof of lemma 3.6, which can be seen later.

Next, after deducing (3.1), we need to show the well-definedness and analyticity of the complex series in it.

Lemma 3.4. Under assumption 1.1, the following properties hold,

(a) Define $\Lambda_R := \{s \in \Lambda : |s| < R, R > 0\}$, then for $k = 1, \ldots, K$, the series

$$
\sum_{n=1}^\infty a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1}
$$

is uniformly convergent for $s \in \Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$.

(b) $\sum_{n=1}^\infty a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1}$ is analytic on $\Lambda \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$ for $k = 1, \ldots, K$.

(c) $\sum_{k=1}^K (e^{-c k - \tau} - e^{-c k}) \left[ \sum_{n=1}^\infty a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1} \right]$ is analytic on $\Lambda^+ \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$.

Proof. For (a), fix $k$ and $R$, since $0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$, there exists a large $N_1 > 0$ such that $\lambda_n > 2R^\alpha$ for $n \geq N_1$. Then for $s \in \Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$ and $n \geq N_1$,

$$
|s^\alpha + \lambda_n| \geq |Re s^\alpha + \lambda_n| = \lambda_n + Re s^\alpha \geq \lambda_n - R^\alpha > 0,
$$

which gives

$$
|\lambda_n(s^\alpha + \lambda_n)^{-1}| = \lambda_n|s^\alpha + \lambda_n|^{-1} \leq \lambda_n(\lambda_n - R^\alpha)^{-1} < 2.
$$

Given $\epsilon > 0$, lemma 3.1 yields that there exists $N_2 > 0$ such that for $l \geq N_2$, $\sum_{n=1}^\infty |a_n(z)p_{k,n}| < \epsilon$. So, for $l \geq \max\{N_1, N_2\}$ and $s \in \Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$,

$$
\left| \sum_{n=1}^\infty a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1} \right| \leq \sum_{n=1}^\infty |a_n(z)p_{k,n}| |\lambda_n(s^\alpha + \lambda_n)^{-1}| \leq 2\sum_{n=1}^\infty |a_n(z)p_{k,n}| < 2\epsilon,
$$

which implies the uniform convergence.

For (b), with the definition of $\Lambda$ and $s^\alpha = e^{\alpha \ln s}$, it is clear that $a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1}$ is holomorphic on $\Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$. Then the uniform convergence gives that the series $\sum_{n=1}^\infty a_n(z)p_{k,n}\lambda_n(s^\alpha + \lambda_n)^{-1}$ is holomorphic, i.e. analytic on $\Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty$ for each
Given \( s \in \Lambda \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty \), we can find \( R > 0 \) such that \( s \in \Lambda_R \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty \), which means \( \sum_{n=1}^\infty a_n(z)p_{k,n}(s^\alpha + \lambda_n)^{-1} \) is analytic on \( \Lambda \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty \), and completes the proof.

For (c), on \( (\Lambda_R \cap \Lambda^+) \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty \), we see
\[
\sum_{k=1}^K \left| e^{-s^\alpha z} - e^{-s^\alpha t} \right| \left( \sum_{n=1}^\infty |a_n(z)p_{k,n}| |\lambda_n(s^\alpha + \lambda_n)^{-1}| \right) \leq 2 \sum_{k=1}^K \sum_{n=1}^\infty |a_n(z)p_{k,n}| |\lambda_n(s^\alpha + \lambda_n)^{-1}|.
\]

Then the proof for (a) and lemma 3.1 give the uniform convergence of the above series on \( (\Lambda_R \cap \Lambda^+) \setminus \{(-\lambda_j)^{1/\alpha}\}_{j=1}^\infty \) for \( R > 0 \). Let \( R \) be sufficiently large, the proof for (b) ensures the analyticity result and completes the proof. \( \square \)

3.3. Auxiliary results

**Lemma 3.5.** Define \( z_\ell := (\cos \theta_\ell, \sin \theta_\ell) \in \partial \Omega, \ell = 1, 2 \) satisfying condition (1.2). Then \( p_\ell = 0, n \in \mathbb{N}^+ \), provided that
\[
\sum_{\lambda_n = \lambda_j} a_n(z)p_\ell = 0, \quad j \in \mathbb{N}^+, \quad \ell = 1, 2.
\]

**Proof.** Given \( j \in \mathbb{N}^+ \), if \( m(n(j)) \neq 0 \), letting \( n(j), n(j) + 1 \) be the integers such that \( \lambda_n = \lambda_j \), then
\[
\sum_{\lambda_n = \lambda_j} a_n(z)p_\ell = \pi^{-1/2}\lambda_j^{-1/2}(e^{i|m|\theta_\ell}p_{\ell,j} + e^{-i|m|\theta_\ell}p_{\ell,j+1}) = 0, \quad \ell = 1, 2,
\]
which gives
\[
\begin{bmatrix}
e^{i|m|\theta_1} & e^{-i|m|\theta_1} \\
e^{i|m|\theta_2} & e^{-i|m|\theta_2}
\end{bmatrix}
\begin{bmatrix}
p_{\ell,j} \\
p_{\ell,j+1}
\end{bmatrix}
= \begin{bmatrix}0 \\
0\end{bmatrix}.
\]

The determinant of the matrix is
\[
e^{i|m|\theta_1} - e^{-i|m|\theta_2} = 2i \sin(m(\theta_1 - \theta_2)) \neq 0,
\]
by condition (1.2) and \( m \neq 0 \). Hence, we have \( p_{\ell,j} = p_{\ell,j+1} = 0 \).

In the case of \( m(n(j)) = 0 \), it holds that
\[
\sum_{\lambda_n = \lambda_j} a_n(z)p_\ell = \pi^{-1/2}\lambda_j^{-1/2}p_{\ell,j} = 0, \quad \ell = 1, 2,
\]
sequentially \( p_{\ell,j} = 0 \). Since \( j \) is chosen arbitrarily, the proof is complete. \( \square \)

**Lemma 3.6.** Under conditions (1.2) and (1.3), assume \( \alpha \neq \tilde{\alpha} \) and the series \( \{\sum_{n=1}^\infty a_n(z)p_n, \sum_{n=1}^\infty a_n(z)p_n : \ell = 1, 2\} \) are absolutely convergent, if there exists \( \epsilon > 0 \) such that for \( t \in (0, \epsilon), \ell = 1, 2, \)
\[
\sum_{n=1}^\infty a_n(z)\lambda_n|p_n|^{\alpha}E_{\alpha,\alpha}(-\lambda_nt^\alpha) - |\tilde{p}_n|^{\alpha}E_{\tilde{\alpha},\tilde{\alpha}}(-\lambda_n\tilde{t}^\alpha)| = 0, \quad (3.2)
\]
then \( p_n = \tilde{p}_n = 0, n \in \mathbb{N}^+ \).

**Proof.** First we claim that the series in (3.2) are real analytic on \((\epsilon/2, \infty)\). In order to prove it, we utilize the knowledge in complex analysis and extend \( t \) to the branch
\( \Lambda_0 := \{ \rho e^k \in \mathbb{C} : \rho \in (0, \infty), \zeta \in (-\pi, \pi) \} \subset \mathbb{C} \). Obviously, \( r^\ell, r^{\ell-1}, \tilde{r}^\ell, \tilde{r}^{\ell-1} \) are holomorphic on \( \Lambda_0 \). Also, by lemma 2.2, there exists a small enough \( \zeta_0 > 0 \) depending on \( \alpha, \tilde{\alpha} \) such that \( \lambda_\alpha p^{\ell-1}_\alpha E_{\alpha, \alpha}(-\lambda_\alpha t^\alpha), \lambda_{\tilde{\alpha}} \tilde{p}^{\ell-1}_{\tilde{\alpha}} E_{\tilde{\alpha}, \tilde{\alpha}}(-\lambda_{\tilde{\alpha}} t^{\tilde{\alpha}}) \) are uniformly bounded on the subset \( \{ \rho e^k \in \Lambda_0 : \rho \in (\epsilon/2, \infty), |\zeta| \leq \zeta_0 \} \). These and the absolute convergence conditions yield that the series in (3.2) is analytic on \( \{ \rho e^k \in \Lambda_0 : \rho \in (\epsilon/2, \infty), |\zeta| \leq \zeta_0 \} \). Since \( (\epsilon/2, \infty) \) is included by this subset, the claim is valid.

From this claim and (3.2), we conclude that

\[
\sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ p_n r^{\ell-1}_{\alpha} E_{\alpha, \alpha}(-\lambda_\alpha t^\alpha) - \tilde{p}_n \tilde{r}^{\ell-1}_{\tilde{\alpha}} E_{\tilde{\alpha}, \tilde{\alpha}}(-\lambda_{\tilde{\alpha}} t^{\tilde{\alpha}}) \right] = 0, \quad t \in (0, \infty), \quad \ell = 1, 2.
\]

Taking Laplace transform on the above equality, lemma 2.3 and the absolute convergence conditions that dominated convergence theorem can be used. Then by termwise calculation, we have for \( s \in \Lambda \setminus \{ (-\lambda_j)_{1/\alpha}, (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty}, \ell = 1, 2 \),

\[
0 = \mathcal{L} \left\{ \sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ p_n r^{\ell-1}_{\alpha} E_{\alpha, \alpha}(-\lambda_\alpha t^\alpha) - \tilde{p}_n \tilde{r}^{\ell-1}_{\tilde{\alpha}} E_{\tilde{\alpha}, \tilde{\alpha}}(-\lambda_{\tilde{\alpha}} t^{\tilde{\alpha}}) \right] ; t \right\} \\
= \sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ p_n (s^\alpha + \lambda_\alpha)^{-1} - \tilde{p}_n (s^{\tilde{\alpha}} + \lambda_{\tilde{\alpha}})^{-1} \right].
\]

Following the proof of lemma 3.4, we have that the above series is analytic on \( \Lambda \setminus \{ (-\lambda_j)_{1/\alpha}, (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty}, \ell = 1, 2 \),

\[
\sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ p_n (s^\alpha + \lambda_\alpha)^{-1} - \tilde{p}_n (s^{\tilde{\alpha}} + \lambda_{\tilde{\alpha}})^{-1} \right] = 0, \\
s \in \Lambda \setminus \{ (-\lambda_j)_{1/\alpha}, (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty}, \quad \ell = 1, 2.
\]

Not hard to see that \( \{ (-\lambda_j)_{1/\alpha} \}_{j=1}^{\infty} \cap \{ (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty} = \emptyset \). Assume not, then there exist \( j_1, j_2 \) such that \( (-\lambda_{j_1})_{1/\alpha} = (-\lambda_{j_2})_{1/\tilde{\alpha}} \). This gives \( \pi(\alpha^{-1} - \tilde{\alpha}^{-1}) = 0 \), which leads to \( \alpha = \tilde{\alpha} \) and contradicts with our assumption. Also, due to \( \{ \lambda_j \}_{j=1}^{\infty} \) is strictly increasing and tends to infinity, \( \{ (-\lambda_j)_{1/\alpha} \}_{j=1}^{\infty} \) and \( \{ (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty} \) do not contain accumulation points. Furthermore, the definition of \( \Lambda \) and condition (1.3) ensure that \( \{ (-\lambda_j)_{1/\alpha}, (-\lambda_j)_{1/\tilde{\alpha}} \}_{j=1}^{\infty} \subset \Lambda \). These and the proof of lemma 3.4 (a) give that for each \( j \in \mathbb{N}^+ \),

\[
\lim_{\rho \to (\epsilon/2) \alpha^{-1}} \left| \sum_{\lambda_n = \lambda_j} a_n(z) \lambda_n p_n (s^\alpha + \lambda_\alpha)^{-1} \right| = \lim_{\rho \to (\epsilon/2) \tilde{\alpha}^{-1}} \left| \sum_{\lambda_n = \lambda_j} a_n(z) \lambda_n \tilde{p}_n (s^{\tilde{\alpha}} + \lambda_{\tilde{\alpha}})^{-1} \right| \\
- \sum_{\lambda_n \neq \lambda_j} a_n(z) \lambda_n p_n (s^\alpha + \lambda_\alpha)^{-1} < \infty,
\]

which leads to \( \sum_{\lambda_n = \lambda_j} a_n(z) p_n = 0, j \in \mathbb{N}^+, \ell = 1, 2 \). Similarly, we can obtain \( \sum_{\lambda_n = \lambda_j} a_n(z) \tilde{p}_n = 0, j \in \mathbb{N}^+, \ell = 1, 2 \). From lemma 3.5, these give the desired result and complete the proof. \( \Box \)
Lemma 3.7. Keep the same conditions in lemma 3.6 and assume
\[
\left\{ \sum_{n=1}^{\infty} a_n(z) p_n, \sum_{n=1}^{\infty} a_n(z) \tilde{p}_n : j \in \mathbb{N}^+, \quad \ell = 1, 2 \right\} \subset \mathbb{R}. \tag{3.3}
\]

Given \( \epsilon > 0 \), then
\[
\lim_{\text{Re} s \to \infty} \frac{e^{\epsilon s}}{\arg s} \lim_{n \to \infty} a_n(z) [p_n(s^\ell + \lambda_n)^{-1} - \tilde{p}_n(s^\ell + \lambda_n)^{-1}] = 0, \quad \ell = 1, 2,
\]
leads to \( p_n = \tilde{p}_n = 0, \ n \in \mathbb{N}^+ \).

Proof. The absolute convergence conditions ensure that dominated convergence theorem can be used here. Then the order of integration and summation can be exchanged and we will not emphasize it in each step.

Lemma 2.1 gives that for \( \ell = 1, 2, s \in \Lambda^+ \),
\[
\sum_{n=1}^{\infty} a_n(z) \lambda_n [p_n(s^\ell + \lambda_n)^{-1} - \tilde{p}_n(s^\ell + \lambda_n)^{-1}] = \sum_{n=1}^{\infty} a_n(z) \lambda_n \int_0^{\infty} e^{-t} [p_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - \tilde{p}_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)] \, dt,
\]
which leads to
\[
e^{\epsilon s} \sum_{n=1}^{\infty} a_n(z) \lambda_n [p_n(s^\ell + \lambda_n)^{-1} - \tilde{p}_n(s^\ell + \lambda_n)^{-1}]
\]
\[
= \sum_{n=1}^{\infty} a_n(z) \lambda_n \int_0^{\epsilon} e^{(\epsilon-t)s} [p_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - \tilde{p}_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)] \, dt
\]
\[
= \sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ \int_0^{\epsilon} \ldots \right] + \sum_{n=1}^{\infty} a_n(z) \lambda_n \left[ \int_{\epsilon}^{\infty} \ldots \right]
\]
\[
=: I_1^\ell(s) + I_2^\ell(s).
\]

From the definition of \( I_1^\ell(s) \), lemma 2.3 and the proof of lemma 3.4, \( I_1^\ell(s), \ell = 1, 2 \) are well-defined and holomorphic on \( \mathbb{C} \), namely, entire functions. Lemma 2.2 and the absolute convergence conditions give that
\[
|I_2^\ell(s)| \leq C \int_{\epsilon}^{\infty} |e^{(\epsilon-t)s}| s^{-1} \, dt \leq C \epsilon^{-1} \int_{\epsilon}^{\infty} e^{(\epsilon-t)\text{Re}s} \, dt = C/\text{Re}s, \quad s \in \Lambda^+, \quad \ell = 1, 2.
\]

Hence \( I_2^\ell(s) \) tends to 0 as \( \text{Re} s \to \infty \). Recalling the limit assumption in this lemma, we have
\[
\lim_{\text{Re} s \to \infty, \arg s \in [0, \pi/2)} I_1^\ell(s) = \lim_{\text{Re} s \to \infty, \arg s \in [0, \pi/2)} [I_1^\ell(s) + I_2^\ell(s)] = \lim_{\text{Re} s \to \infty, \arg s \in [0, \pi/2)} I_2^\ell(s) = 0.
\]

This means that \( I_1^\ell(s) \) is bounded on the quarter plane \( \{ s \in \mathbb{C} : \arg s \in [0, \pi/2) \} \), and we denote the upper bound by \( C_0 \). For \( s \in \mathbb{C} \) with \( \arg s \in (-\pi/2, 0) \), obviously its conjugate satisfies \( \arg s \in [0, \pi/2) \). Then the straightforward calculation and condition (3.3) give that \( I_1^\ell(\bar{s}) \) is the
complex conjugate of $I'_1(s)$, which leads to $|I'_1(s)| = |I'_1(\overline{s})| \leq C_0$. In the case of $\text{Re } s \leq 0$, not hard to show that $I'_1(s)$ is bounded also, in view of lemma 2.3 and the absolute convergence conditions.

Thus, $I'_1(s)$, $\ell = 1, 2$ are bounded entire functions, and by Liouville’s theorem, $I'_1(s) \equiv C$. Considering the limit result, it holds that $I'_1(s) \equiv 0$, $s \in \mathbb{C}$, $\ell = 1, 2$. This gives for $\text{Re } s > 0$,  

$$0 = e^{-it}I'_1(s) = \sum_{n=1}^{\infty} a_n(z_\ell)\lambda_n \int_0^{\ell} e^{-st} \left[ p_n t^{\ell-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - \tilde{p}_n t^{\ell-1} E_{\alpha,\tilde{\alpha}}(-\lambda_n t^{\tilde{\alpha}}) \right] dt$$

$$= \mathcal{L} \left\{ \chi_{\mathbb{C} \setminus \{0,\ell\}} \sum_{n=1}^{\infty} a_n(z_\ell)\lambda_n \left[ p_n t^{\ell-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - \tilde{p}_n t^{\ell-1} E_{\alpha,\tilde{\alpha}}(-\lambda_n t^{\tilde{\alpha}}) \right] ; s \right\},$$

which leads to

$$\sum_{n=1}^{\infty} a_n(z_\ell)\lambda_n \left[ p_n t^{\ell-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - \tilde{p}_n t^{\ell-1} E_{\alpha,\tilde{\alpha}}(-\lambda_n t^{\tilde{\alpha}}) \right] \equiv 0, \quad t \in (0, \epsilon), \quad \ell = 1, 2.$$  

This and lemma 3.6 yield the desired result.  

Letting $\tilde{p}_n = 0$, $n \in \mathbb{N}^+$ in lemma 3.7, the next corollary can be deduced straightforwardly.

**Corollary 3.8.** **With the conditions in lemma 3.7,**  

$$\lim_{\substack{\text{Re } s \to \infty \\text{arg } s \in (0, \pi/2)}} e^{is} \sum_{n=1}^{\infty} a_n(z_\ell)\lambda_n(s^\alpha + \lambda_n)^{-1} = 0, \quad \ell = 1, 2, \quad \epsilon > 0,$$

implies that $p_n = 0$, $n \in \mathbb{N}^+$.

### 3.4. Proof of theorem 1.1

Now we are ready to prove our main theorem.

**Proof of theorem 1.1.** To shorten our proof, we define

$$P'_1(s) := \sum_{n=1}^{\infty} a_n(z_\ell)p_n, \lambda_n(s^\alpha + \lambda_n)^{-1}, \quad \tilde{P}'_1(s) := \sum_{n=1}^{\infty} a_n(z_\ell)\tilde{p}_n, \lambda_n(s^{\tilde{\alpha}} + \lambda_n)^{-1}.$$  

Then from (3.1) and lemma 3.4, we have for $s \in \Lambda^+ \setminus \{(-\lambda_j)^{1/\alpha}, (-\lambda_j)^{1/\tilde{\alpha}}\}_{j=1}^{K}, \ell = 1, 2,$

$$\sum_{k=1}^{K} (e^{-ck^{-\frac{1}{\alpha}}} - e^{-ck}) P'_1(s) = \sum_{k=1}^{\tilde{K}} (e^{-\tilde{c}k^{-\frac{1}{\tilde{\alpha}}}} - e^{-\tilde{c}k}) \tilde{P}'_1(s).$$

We first prove that $c_0 = \tilde{c}_0$. If not, assume $c_0 < \tilde{c}_0$ without loss of generality, then multiplying $e^{(c_0 - \epsilon)\alpha}$ with sufficiently small $\epsilon > 0$ such that $\epsilon < \min\{c_0 - c_1, 1 - c_0\}$ on (3.4) gives

$$e^{is} P'_1(s) = e^{i(c_0 - \epsilon)\alpha} P'_1(s) - \sum_{k=1}^{K} (e^{i(c_0 - \epsilon)\alpha} - e^{i(c_0 - \epsilon)\alpha}) P'_1(s)$$

$$+ \sum_{k=1}^{\tilde{K}} (e^{i(c_0 - \epsilon)\alpha} - e^{i(c_0 - \epsilon)\alpha}) \tilde{P}'_1(s).$$
For $s \in \Lambda^+$ with $\arg s \in [0, \pi/2)$, not hard to see $\arg s^\alpha = \alpha \arg s \in [0, \pi/2)$, i.e. $\Re s^\alpha > 0$. Then

$$|\lambda_\alpha (s^\alpha + \lambda_\alpha)^{-1}| \leq \lambda_\alpha (\Re s^\alpha + \lambda_\alpha)^{-1} \leq 1.$$  

This with lemma 3.1 yields that

$$\lim_{\Re s \to \infty, \arg s \in [0, \pi/2)} \left| \sum_{k=2}^{K} (e^{(e+c_0-c_k \iota)\alpha} - e^{(e+c_0-c_k \iota)\alpha})P_k^\alpha (s) \right| \leq \lim_{\Re s \to \infty, \arg s \in [0, \pi/2)} 2e^{(e+c_0-c_1 \iota)\alpha} \Re s \lim_{k=2}^{K} \sum_{n=1}^{\infty} |a_n(z^\alpha) p_{k,n}| = 0.$$  

Analogously, we can show other terms in the right side of (3.5) tend to zero as $\Re s \to \infty, s \in \Lambda, \arg s \in [0, \pi/2)$. Now we have

$$\lim_{\Re s \to \infty, \arg s \in [0, \pi/2)} e^{\alpha \iota} P_\ell^\alpha (s) = 0, \quad \ell = 1, 2, \quad \epsilon > 0.$$  

Not hard to check that $\{a_n(z) p_{1,n} : n \in \mathbb{N}^+, \ell = 1, 2\}$ satisfies condition (3.3). Then with corollary 3.8, it holds that $p_{1,n} = 0, n \in \mathbb{N}^+$, i.e. $\|p_1\|_{L^2(\Omega)} = 0$, which contradicts with assumption 1.1. Hence, we have $c_0 = \tilde{c}_0$.

Next we prove $\alpha = \tilde{\alpha}$. Assume $\alpha < \tilde{\alpha}$ without loss of generality and pick the small $\epsilon > 0$ satisfying $\epsilon < \min\{c_1 - c_0, \tilde{c}_1 - \tilde{c}_0\}$. Multiplying $e^{\epsilon (c_1 - \iota)\alpha}$ on (3.4) and considering the result $c_0 = \tilde{c}_0$ yield that

$$e^{\epsilon \alpha} [P_1^\alpha (s) - \tilde{P}_1^\alpha (s)] = e^{(e+c_0-c_1 \iota)\alpha} P_1^\alpha (s) - e^{(e+c_0-\tilde{c}_1 \iota)\alpha} \tilde{P}_1^\alpha (s)$$

$$- \sum_{k=2}^{K} (e^{(e+c_0-c_k \iota)\alpha} - e^{(e+c_0-\tilde{c}_k \iota)\alpha})P_k^\alpha (s)$$

$$+ \sum_{k=2}^{K} (e^{(e+c_0-c_k \iota)\alpha} - e^{(e+c_0-\tilde{c}_k \iota)\alpha})\tilde{P}_k^\alpha (s).$$

The similar proof gives that for $\ell = 1, 2$,

$$\lim_{\Re s \to \infty, \arg s \in [0, \pi/2)} e^{\epsilon \alpha} [P_\ell^\alpha (s) - \tilde{P}_\ell^\alpha (s)] = 0,$$

which together with lemma 3.7 gives $p_{1,n} = \tilde{p}_{1,n} = 0, n \in \mathbb{N}^+$. This leads to $\|p_1\|_{L^2(\Omega)} = \|\tilde{P}_1\|_{L^2(\Omega)} = 0$, which contradicts with assumption 1.1. So, $\alpha = \tilde{\alpha}$.

Inserting $\alpha = \tilde{\alpha}$ into (3.6) leads to

$$\lim_{\Re s \to \infty, \arg s \in [0, \pi/2)} e^{\epsilon \alpha} \sum_{n=1}^{\infty} a_n(z^\alpha) (p_{1,n} - \tilde{p}_{1,n}) \lambda_\alpha (s^\alpha + \lambda_\alpha)^{-1} = 0, \quad \ell = 1, 2.$$
From this and corollary 3.8, it follows that $p_{1,n} - \tilde{p}_{1,n} = 0$, $n \in \mathbb{N}^+$, namely, $\|p_1 - \tilde{p}_1\|_{L^2(\Omega)} = 0$. Now subtracting $e^{-i\omega}P^j_1(s)$ from (3.4), it holds that
\[
e^{-i\omega}[P^j_2(s) - P^j_1(s)] + e^{-i\omega}[\tilde{P}^j_1(s) - \tilde{P}^j_2(s)]
= e^{-i\omega}P^j_2(s) - e^{-i\omega}\tilde{P}^j_2(s) - \sum_{k=3}^{K} (e^{-i\omega_{k-1}^s} - e^{-i\omega_{k+1}^s})P^j_1(s)
+ \sum_{k=3}^{K} (e^{-i\omega_{k-1}^s} - e^{-i\omega_{k+1}^s})\tilde{P}^j_1(s).
\]

Assume $c_1 \neq \tilde{c}_1$ and let $c_1 < \tilde{c}_1$ without loss of generality. Picking $\epsilon > 0$ satisfying $\epsilon < \min\{\tilde{c}_1 - c_1, c_2 - c_1\}$, from the proof for showing $c_0 = \tilde{c}_0$ we can derive that $p_{2,n} - \tilde{p}_{2,n} = 0$, $n \in \mathbb{N}^+$, i.e. $\|p_2 - \tilde{p}_2\|_{L^2(\Omega)} = 0$, which contradicts with assumption 1.1. Hence, $c_1 = \tilde{c}_1$.

Now we have $c_0 = \tilde{c}_0$, $c_1 = \tilde{c}_1$, $\alpha = \tilde{\alpha}$, $\|p_1 - \tilde{p}_1\|_{L^2(\Omega)} = 0$. Inserting them into (3.4) gives that for $s \in \Lambda^+ \setminus \{(-\lambda)^{1/\alpha}\}_{j=1}^\infty$, $\ell = 1, 2$,
\[
\sum_{k=2}^{K} (e^{-i\omega_{k-1}^s} - e^{-i\omega_{k+1}^s})P^j_1(s) = \sum_{k=2}^{K} (e^{-i\omega_{k-1}^s} - e^{-i\omega_{k+1}^s})\tilde{P}^j_1(s).
\]

Following the proof above, we can deduce that $c_2 = \tilde{c}_2$, $\|p_2 - \tilde{p}_2\|_{L^2(\Omega)} = 0$. Continuing this procedure, we can get
\[
\alpha = \tilde{\alpha}, \quad c_0 = \tilde{c}_0, \quad c_k = \tilde{c}_k, \quad \|p_k - \tilde{p}_k\|_{L^2(\Omega)} = 0, \quad 1 \leq k \leq \min\{K, \tilde{K}\}. \tag{3.7}
\]

Finally, we need to show $K = \tilde{K}$ (the case of $K = \infty$, $\tilde{K} = \infty$ is considered as $K = \tilde{K}$). Assume not, then we set $K < \tilde{K}$ without loss of generality. (3.4) and (3.7) give that
\[
\sum_{k=K+1}^{K} (e^{-i\omega_{k-1}^s} - e^{-i\omega_{k+1}^s})\tilde{P}^j_1(s) = 0, s \in \Lambda^+ \setminus \{(-\lambda)^{1/\alpha}\}_{j=1}^\infty, \quad \ell = 1, 2.
\]

Now pick $\epsilon > 0$ fulfilling $\epsilon < \tilde{c}_{K+1} - \tilde{c}_K$, the previous proof will give $\|\tilde{p}_{K+1}\|_{L^2(\Omega)} = 0$, which is a contradiction. Therefore, $K = \tilde{K}$, which together with (3.7) completes the proof of theorem 1.1. \qed

4. Concluding remark and future work

In this work, we prove that sparse flux data on boundary can uniquely determine the order $\alpha$ and semi-discrete source term simultaneously. This is the theoretical basis for numerical reconstruction, which is one of our future work.

In numerical aspect, we do not need to worry about condition (1.2), which seems impossible in programming. This is because that in practice we can only utilize finitely many eigenvalues, so that the index $|n|$ has an upper bound. By this and the proof of lemma 3.5, condition (1.2) can be weakened to the one that $(\theta_1 - \theta_2)/\pi$ is not in a subset of rational numbers. For example, we use $\{\varphi_n\}_{n=1}^N$ to approximate the unknowns, and set $M := \max\{|m(n)| : n = 1, \ldots, N\}$. Then
we can choose $\theta_1, \theta_2 \in [0, 2\pi)$ such that

$$\frac{\theta_1 - \theta_2}{\pi} \notin \left\{ \frac{2k}{k} \mid k \in \mathbb{Z}, k \neq 0 \right\}.$$ 

Furthermore, for the theoretical analysis, we will investigate this inverse source problem with a more general source term, or consider equation (1.1) in the unit sphere of $\mathbb{R}^3$, or even in a manifold.

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