Gapped solitons and periodic excitations in strongly coupled BECs

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Abstract
It is found that localized solitons in the strongly coupled cigar-shaped Bose–Einstein condensates (BECs), in the repulsive domain, form two distinct classes. The one without a background is an asymptotically vanishing, localized soliton, having a wave number, which has a lower bound in magnitude. Periodic soliton trains exist only in the presence of a background, where the localized soliton has a $W$-type density profile. This soliton is well suited for trapping of neutral atoms and is found to be stable under Vakhitov–Kolokolov criterion, as well as numerical evolution. We identify an insulating phase of this system in the presence of an optical lattice. It is demonstrated that the $W$-type density profile can be precisely controlled through trap dynamics.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Cigar-shaped Bose–Einstein condensates (BECs) provide an ideal venue for realizing different types of solitonic excitations in one dimension. Dark [1, 2], bright [3–6], grey [7] solitons and periodic soliton trains [6] have been experimentally observed. The full many-body quantum mechanical treatment of BECs [8] is known to result in coherent structures [9, 10] in one dimension, which can be classically interpreted as solitons. In the mean field approach, where quantum fluctuations are neglected, it is well established that the Gross–Pitaevskii (GP) equation [11] aptly captures the BEC dynamics [12]. Experimental occurrence of different types of solitons shows the relevance of the GP equation for the trapped cigar-shaped BECs. It needs to be mentioned that the same is not true in the case of optical lattice with deep wells, wherein quantum fluctuations play a significant role. In the weak coupling regime, the GP equation reduces to the well-studied nonlinear Schrödinger equation (NLSE), possessing soliton solutions. The fact that NLSE is an integrable system explains the observed soliton dynamics [13–16]. In one dimension, the possibility of using these localized excitations for producing atom laser and other applications [17–19], requiring macroscopic coherence, has made cigar-shaped BECs an area of significant current interest [12, 20–23]. The fact that scattering length can be controlled through Feshbach resonance [24] has given access to both weak and strong coupling sectors, as well as attractive and repulsive domains. The regulation of the transverse trap frequency $\omega_{\perp}$ can also be used to control the coupling in the cigar-shaped BECs, as has been demonstrated experimentally in the generation of the Faraday modes [25].

The strongly coupled cigar-shaped BECs, in the repulsive domain, is characterized by a quadratic nonlinearity, following the Thomas–Fermi approximation [26, 27]. In contrast to the weak coupling domain, the solitons in the strong coupling sector are not well studied. The grey soliton dynamics has been numerically investigated, which yielded a structure similar to that of the weak coupling regime [28]. Here, we study dark solitons, as well as soliton trains in this nonlinear system and find significant differences, with the weak coupling sector. We restrict ourselves to the repulsive domain, keeping in mind the
fact that strong coupled BECs will be prone to instability in the attractive sector.

In the strong repulsive domain, it is observed that unlike the case of the NLSE, the solitons exist in two distinct classes. Starting from a general ansatz solution with non-vanishing background, it is found that there can be two domains of localized solutions. The one with background is a $W$-type soliton, which can exist for all the values of the momentum of the envelope profile. The other solution is an asymptotically vanishing soliton, without a background, which requires a finite momentum to get excited. Both the solutions are shown to be stable under the numerical Crank–Nicolson finite difference method. Interestingly, periodic cnoidal waves can only exist with a background. Using a general Padé type ansatz, we identify more general solutions.

It is found that a unique localized solution can exist only with a real background. We also explore the structure of the solutions in the presence of an optical lattice and identify an insulating phase. Subsequently in section 4, their coherent control is analytically demonstrated, where both the amplitude and width of the $W$-type soliton can be exactly regulated. We conclude in the final section with a number of directions for future work.

2. Soliton solutions of the strongly coupled BECs

The three-dimensional GP equation describing the dynamics of BECs in a cylindrical harmonic trap, $V = \frac{1}{2}M\omega_0^2(x^2 + y^2)$, is given by

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + U_0|\Psi(r,t)|^2 + V \right] \Psi(r,t), \quad (1)$$

where $U_0 = 4\pi\hbar^2a/M$ is the effective two-body interaction, $a$ being the scattering length. The condensate is repulsive if $U_0$ (or $a$) is positive and attractive if negative. It is by now well established that the GP equation captures the mean field dynamics of BECs quite well. Methods to produce both one-dimensional and two-dimensional BECs have also been well established experimentally. For finding the relevant equation in lower dimensions, one assumes factorization of the wavefunction into two parts. For example, in the quasi-one-dimensional limit, the condensate wavefunction can be factorized [14, 26, 27]:

$$\Psi(r,t) = f(z,t)G(x,y,\sigma), \quad (2)$$

where $\sigma(z)$ is the local particle density. $G(x,y,\sigma)$ is the normalized equilibrium wavefunction for the transverse motion:

$$G(x,y,\sigma) = \frac{e^{-(x^2+y^2)/2\sigma^2}}{\pi^{1/2}\sigma^2}. \quad (3)$$

with

$$\sigma(z) = \int dx \, dy \, |\Psi(x,y,z)|^2 = |f(z,t)|^2. \quad (4)$$

It is assumed that $G(x,y,\sigma)$ varies negligibly in the $z$ direction [26], which is specifically justified in the strong coupling limit, with corresponding error in numerical energy estimation being $\sim 10^{-7}$ [27]. After eliminating $G(x,y,\sigma)$ from equation (1), we end up with [27]

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} + \frac{U_0}{2\pi a_r^2 \sqrt{1 + 2aN|f|^2}} |f|^2 + \frac{\hbar\omega_{\perp}}{2} \left( \frac{1}{\sqrt{1 + 2aN|f|^2}} + \sqrt{1 + 2aN|f|^2} \right) \right] f. \quad (5)$$

In the case of strong coupling, $2aN|f|^2 \gg 1$ (but $N|f|^2 \ll 1$ to satisfy the diluteness of BECs), one obtains the condensate equation in a suitably normalized form [28]:

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} + 2\hbar\omega_{\perp}a^{1/2} \left( |f| - \sigma_0^{1/2} \right) \right] f. \quad (6)$$

Here, $\sigma_0$ is the equilibrium density of the atoms far away from the axis. For comparison, in the weak coupling limit $2aN|f|^2 \ll 1$, after suitable normalization, we arrive at [28]

$$i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} + 2\hbar\omega_{\perp}a \left( |f|^2 - \sigma_0 \right) \right] f, \quad (7)$$

with a cubic nonlinear coupling. Respective solutions, therefore, differ appreciably. As already mentioned in the introduction, in the weak coupling regime, the GP equation reduces to the NLSE, possessing soliton solutions being an integrable system. On controlling the scattering length through Feshbach resonance [24], both weak and strong coupling sectors can be obtained, as well as attractive and repulsive domains. In cigar-shaped BECs, the transverse trap frequency $\omega_{\perp}$ can also be regulated to control the coupling. In [26–28], localized solutions of the non-polynomial equation (equation (5)) are obtained, which smoothly interpolates between localized solutions of the strong and weak coupling regimes. For the latter case, the obtained solution is a special case of the present general solution. These authors also check the validity of their reduction to one dimension numerically.

The nonlinear excitations of the strongly interacting system can be probed through an ansatz solution

$$f(z,t) = e^{i(kz-\omega t)} \rho(\xi), \quad (8)$$

with a fast-moving component and slowly-varying envelope profile $\rho(\xi)$. Here, $\xi = \alpha(z-vt)$ and $v = \frac{\hbar}{M\kappa}$, with $\rho(\xi)$ satisfying

$$\alpha^2 \rho'' + g \rho^2 + \epsilon \rho = 0, \quad (9)$$
where
\[ g = -4M\omega_\perp a^{1/2}/\hbar, \]
\[ \epsilon = 2M\omega/\hbar + 4M\omega_\perp (\sigma_\alpha)^{1/2}/\hbar - k^2. \] (10)

It is straightforward to check that the following ansatz solution,
\[ \rho(\xi) = A + B cn(\xi, m), \] (11)
solves equation (9), where \( cn(\xi, m) \) is the cnoidal function, with \( m \) being the modulus parameter \((0 \leq m \leq 1)\) [30], and \( A, B \) are the constant parameters to be determined. We note that \( cn(\xi, 0) = \cos(\xi) \) and \( cn(\xi, 1) = \sech(\xi) \). \( A \) serves here as the background. On substitution, we arrive at the consistency conditions
\[ -2B\alpha^2(m - 1)A + gA^2 + \epsilon A = 0, \]
\[ -4\alpha^2(1 - 2m) + 2gA + \epsilon = 0, \]
\[ -6m\alpha^2 + gB = 0, \] (12)
leading to
\[ A = \frac{1}{2g}[4\alpha^2(1 - 2m) - \epsilon], \]
\[ B = \frac{6}{g}\alpha^2m, \] (13)
along with a relation between the width \( \alpha \) and \( \epsilon \):
\[ \epsilon^2 = 16\alpha^4(m^2 - m + 1). \] (14)

As is known, the BEC is characterized by a negative chemical potential, which appears as the linear term in the mean field equation (GP equation). For the case of BECs with profile as equation (8), the effective linear term is given by equation (10) (the effective chemical potential \( \epsilon \)), which is a sum of three terms, only the second one being positive definite. Hence, both the roots of the above equation are physically allowed.

The general periodic solution can then be written as
\[ \rho(\xi) = \frac{2\alpha^2}{g}[(1 - 2m) \pm \sqrt{m^2 - m + 1 + 3mcn^2(\xi, m)}], \] (15)
representing a soliton train. Consideration of the special case \( m = 1 \) yields localized solutions. It corresponds to two specific values, \( \epsilon = \pm 4\alpha^2 \). The positive root requires the presence of the background \( A \), and the envelope takes the form
\[ \rho(\xi) = -\frac{\epsilon}{g} \left[ 1 - \frac{3}{2} \sech^2(\xi) \right], \] (16)
representing a localized \( W \)-type soliton.

The VK criterion points out that the integral \( N(\epsilon) = \int |\rho(\xi)|^2 d\xi \), when varied with respect to the effective chemical potential \( \epsilon \), indicates the stability of the solution \( \rho(\xi) \). In the present case,
\[ \frac{dN(\epsilon)}{d\epsilon} = -6\epsilon/g^2, \] (17)
requiring that \( \epsilon > 0 \) for the stability of the solution, which is consistent with \( \epsilon = 4\alpha^2 \).

For this case we find
\[ k^2 \geq \frac{2M\omega}{\hbar} - |\epsilon|, \] (18)
setting a lower limit \( \frac{\hbar|\epsilon|}{2M} \) for \( \omega \), if it is positive, in order to generate the \( W \)-type soliton. Such a bound is not there if the frequency is negative. The physical motivation for negative frequency is drawn from the fact that the Schrödinger equation, having first derivative in time, admits both positive and negative values of energy \( E = \hbar\omega \), which form two distinct sectors.

The density profiles of the localized \( W \)-type soliton are depicted in figure 1. The two points where the order parameter vanishes may be useful for trapping of neutral atoms. In section 4, the control of barrier height and locations of the minima of this soliton will be discussed.

For the negative root of equation (14), with \( m = 1 \), the background vanishes, and we get
\[ \rho(\xi) = -\frac{3\epsilon}{2g} \sech^2(\xi), \] (19)
which, under the VK criterion, yields
\[ \frac{dN(\epsilon)}{d\epsilon} = 6\epsilon/g^2, \] (20)
and hence is stable as \( \epsilon < 0 \). In this case, \( k^2 \geq |\epsilon| + \frac{2M\omega}{\hbar} \), indicating that a finite wave number is needed to excite the solution for the positive frequency case. Such a condition does not arise if the frequency is negative unless \( |\omega| > \hbar|\epsilon|/2M \).

This type of velocity-restricted soliton has been identified in the higher order NLSE relevant for optical fibre pulses in the

**Figure 1.** Localized soliton with a double-well density profile for an atomic BEC of mass \( M = M_N \) with (a) \( \omega = 28 \) (red full) and (b) scattering length 189 au (blue dashed) and 1134 au (red full). \( \omega \) is in the unit of \( 10^{-3} \)au.
unconditionally stable. In this analysis, the initial profile has been checked that the evolution was unitary conserving the number of particles, up to second order in $d$. The temporal evolution indicates the stability of the W-soliton.

We have numerically evolved the W-type solution, using the Crank–Nicolson finite difference method, which is unconditionally stable. In this analysis, the initial profile has been taken as $\psi(z, t = 0) = \psi(z, t = 0) + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is a function, which assumes a random value at each point. The analysis was carried out with $d_z = 0.0003$, $dr = 0.0001$ for 1000 iterations. Figure 2 shows that the W-type soliton remains unchanged with its form showing minor perturbation, where $\tilde{\epsilon}$ is taken to be 10% of the peak value of $\psi$. The minima positions and the width remained unaltered. We have also checked that the evolution was unitary conserving the number of particles, up to second order in $dr$.

3. Complex envelope Bloch solitons

It is known that grey solitons [7, 26] are associated with complex envelopes. These are analogues of Bloch solitons in condensed matter systems. A general ansatz for identifying complex envelope Bloch-type solitons can be written as

$$\rho(\xi) = (A + iB) + (C + iD)cn^2(\xi, m),$$  \hspace{1cm} (21)

where $A, B, C, D$ are real. The corresponding consistency conditions lead to

$$g(A + iB)\sqrt{A^2 + B^2} - \epsilon(A + iB) + 2\alpha^2(C + iD)(m - 1) = 0,$$

$$[g^2(A^2 + B^2) - \epsilon^2](A + iB)(C + iD) + g^2(AC + BD)$$

$$\times(A + iB)^2 - 8\alpha^2(C + iD)^2(m - 1)(1 - 2m)$$

$$- 2\alpha^2C^2(C + iD)^2(m - 1) + 2(A + iB)$$

$$\times(1 - 2m)(C + iD) = 0,$$

$$g^2[(C^2 + D^2)(A + iB)^2 + (A^2 + B^2)(C + iD)^2$$

$$+ 4(AC + BD)(A + iB)(C + iD)] - \epsilon^2(C + iD)^2$$

$$- 4\alpha^2(C + iD)^2(3(A + iB)m^2(C + iD)(1 - 2m) +$$

$$8\alpha^2(C + iD)^2[2(1 - 2m)^2 + 3m(m - 1)] = 0,$$

$$g^2[(AC + BD) + (C - iD)(A + iB)] - 6\epsilon\alpha^2m$$

$$- 24\alpha^2m(1 - 2m) = 0,$$

$$g^2(C^2 + D^2) - 36\alpha^4m^2 = 0.$$  \hspace{1cm} (22)

Solutions of these consistency conditions can be found through a tedious, but straightforward calculation:

$$A = (p - B^2)^{1/2}, \quad C = [q^2 - (r/p)^2B^2]^{1/2},$$

$$D = (r/p)B,$$

$$B^2 = [p(g\sqrt{p} + \epsilon) - 4\alpha^2(m - 1)^2]$$

$$/ [(g\sqrt{p} + \epsilon) - 4\alpha^2(m - 1)^2(r/p)^2]$$  \hspace{1cm} (23)

where $p = 1/(3\epsilon^2), q = 3\epsilon\alpha^2m^2/g^2, r = 3\epsilon\alpha^2m^2/[4\alpha^2(1 - 2m) + \epsilon]$  \hspace{1cm} (24)

and $\epsilon^2 = 16\alpha^4$. In the case of localized solitons, i.e. for $m = 1$, we arrive at the interesting condition that $A = 0 = C$, leading to the previously found W-type soliton. However, periodic cnoidal wave solutions can exist with complex background parameters.

We now explore more general solution space, through a Padé type ansatz [32]:

$$\rho(\xi) = A + B/1 + C/1 + D.$$  \hspace{1cm} (25)

$A, B$ and $C$ are real parameters to be determined from the consistency conditions, with $AC - B \neq 0$. It can be seen that these conditions are four in number, indicating the constrained nature of the general solution. The following consistency conditions, with $\alpha^2 \equiv \beta$, can be deduced:

$$A^2 + 2BC(m - 1)\beta + A(-2C^2(m - 1)\beta + \epsilon) = 0,$$

$$A^2Cg + B((2m - 1)\beta + \epsilon) + A(2Bg + C(\beta - 2m\beta + 2\epsilon)) = 0,$$

$$B^2g + AC(-\beta + 2m\beta + \epsilon) + BC(2Ag + \beta - 2m\beta + 2\epsilon) = 0,$$

$$B^2Cg - 2Bm\beta + 2ACm\beta + BC^2\epsilon = 0.$$  \hspace{1cm} (26)

We concentrate on the localized solutions, because of their physical interest:

$$\rho(\xi) = -\frac{\epsilon}{g} \left( \frac{1 - \text{sech}(2\xi)}{1 + \text{sech}(2\xi^2/8)} \right).$$ \hspace{1cm} (27)

Like the previous case, the width and the amplitude of the solution are coupled. It is easy to see that the profile can be rewritten as

$$\rho(\xi) = -\frac{\epsilon}{g} \left( 1 - \frac{3}{2} \text{sech}^2(\xi) \right).$$ \hspace{1cm} (28)

which is identical to the W-type soliton.

The above solution can be identified as the unique separatrix in the phase space of the solutions of equation (9), separating the periodic solutions with closed orbits from the unbounded ones, represented by open orbits [33].

The presence of periodic solutions motivates us to explore the nature of the solution in the presence of an optical lattice, $V(x) = V_0 \cos^2 x$, which necessitates the solutions to possess sinusoidal character. For the purpose of comparison [34], the 1D GP equation is normalized in the form

$$i\partial_t \psi(z, t) = \left(-\frac{1}{2}\partial_z^2 + g|\psi(z, t)|^2 + V(x) - \mu \right) \psi(z, t),$$  \hspace{1cm} (29)

where $g$ is the normalized two-body interaction. The solution is found to be of the form $\psi(z, t) = (a + b \cos^2 z)e^{-i\omega t}$. The parameter values for this insulating phase are given by...
case $[40–42]$, the GP equation is cast in the form
\[ \psi \left( z, t \right) = \frac{1}{2} a \left( t \right) \psi \left( z, t \right) \]  
\[ \frac{\partial \psi}{\partial z} + \frac{1}{2} M \left( t \right) \frac{\partial^2 \psi}{\partial z^2} + \frac{\kappa \left( t \right)}{2} \psi \]  
(30)

Here $\psi$, $t$ and $z$ have been scaled, respectively, by $a_B^{1/2}$, $a_B$ and 1/$a_B$, making them dimensionless. $\gamma = 2 \left( a(t)/a_B \right)^{1/2}$ is a time-dependent nonlinearity coefficient, controllable through Feshbach resonance; $M(t) = \omega_0^2(t)/\omega_0^2$ is related to the axial trap frequency, which can be made time dependent. $\kappa(t) = \eta(t)/h \omega_0$ is time-dependent loss/gain. The oscillator length in the transverse direction is defined as $a_B = (h/\omega_0)^{1/2}$ and $a_B$ is the Bohr radius. We consider the following ansatz solution:
\[ \psi \left( z, t \right) = B(t) F \left( \xi \right) e^{i \Phi \left( z, t \right) + i \frac{1}{2} G \left( \xi \right) / \epsilon}. \]  
(31)

where $G(t) = \int_0^t \kappa \left( t' \right) \, dt'$. The phase has been taken in the form $\Phi \left( z, t \right) = a(t) + b(t) z - \frac{i}{2} c(t) z^2$, where $a(t)$ is a $z$ independent phase term: $a(t) = a_0 + \frac{i}{2} \int_0^t A(t') \, dt'$. The solutions are necessarily chirped in time and space, with time varying amplitude and width. $b(t)$ is a time-dependent

\begin{align*}
a &= V_0/2g \quad \text{and} \quad b = -V_0/g, \quad \text{with} \quad \omega = 1 + \frac{i}{2} V_0 - \mu. \quad \text{This phase is insulating, since the current density vanishes. This type of insulating state also emerges in the weak coupling GP equation. The dynamical superfluid to insulator transition here is completely classical [35], unlike the Mott insulator, where the phase transition was quantum fluctuation driven [36]. In contrast to the weak coupling case [37], where analytic solutions have been obtained for both superfluid and insulating phases, for this case, we have been able to identify only an insulating phase, analytically. It is worth observing that in this case, the competition between the lattice potential and the nonlinearity yields sinusoidal solutions, whereas, for the localized soliton solutions, the nonlinearity and the dispersion are responsible for the existence of the solutions. In the repulsive domain, the above solutions exist both for positive and negative values of $V_0$. The corresponding energy (in dimensionless units) is found to be $E = 4 \pi V_0^2 + d \mu V_0^3 - 2 \pi \mu V_0^2$, where $d = 1/8 g^2$.
\end{align*}  

We note that the contribution from the interaction term does not explicitly contribute to the energy, although the coupling parameter appears in $E$ through the solutions. The average atom number density, $\nu = V_0^2/8 g^2$, is a constant and can be controlled by tuning the lattice potential and the scattering length.

4. Coherent control of the solitons

We now investigate the effect of time-dependent nonlinearity and gain or loss on the $W$-type soliton profile. The two points of this soliton, where the order parameter vanishes, may be useful for trapping of neutral atoms. The barrier height and the locations of the minima can be controlled by changing the frequency and the scattering length, which can be manipulated through Feshbach resonance [38, 39]. Keeping this in mind, in the following section, we study the control of this localized excitation. For the sake of comparison with the weak coupling case [40–42], the GP equation is cast in the form
\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial z^2} + \gamma(t) |\psi|^2 \psi + \frac{i \kappa(t)}{2} \psi. \]  
(30)

Here $\psi$, $t$ and $z$ have been scaled, respectively, by $a_B^{1/2}$, $a_B$ and 1/$a_B$, making them dimensionless. $\gamma = 2 \left( a(t)/a_B \right)^{1/2}$ is a time-dependent nonlinearity coefficient, controllable through Feshbach resonance; $M(t) = \omega_0^2(t)/\omega_0^2$ is related to the axial trap frequency, which can be made time dependent. $\kappa(t) = \eta(t)/h \omega_0$ is time-dependent loss/gain. The oscillator length in the transverse direction is defined as $a_B = (h/\omega_0)^{1/2}$ and $a_B$ is the Bohr radius. We consider the following ansatz solution:
\[ \psi \left( z, t \right) = B(t) F \left( \xi \right) e^{i \Phi \left( z, t \right) + i \frac{1}{2} G \left( \xi \right) / \epsilon}. \]  
(31)

where $G(t) = \int_0^t \kappa \left( t' \right) \, dt'$. The phase has been taken in the form $\Phi \left( z, t \right) = a(t) + b(t) z - \frac{i}{2} c(t) z^2$, where $a(t)$ is a $z$ independent phase term: $a(t) = a_0 + \frac{i}{2} \int_0^t A(t') \, dt'$. The solutions are necessarily chirped in time and space, with time varying amplitude and width. $b(t)$ is a time-dependent

\begin{align*}
momentum and $c(t)$ balances the oscillator, leading to the Riccati equation:
\[ \frac{dc(t)}{dt} - c^2(t) = M(t). \]  
(32)
\end{align*}  

The above can be cast as the familiar Schrödinger eigenvalue equation:
\[ -\phi'''(t) - M(t) \phi(t) = 0, \]  
(33)
via a change of variable: $c(t) = -\frac{\partial \ln \phi(t)}{\partial t}$. The constant part of $M(t)$ acts as the eigenvalue. A number of variations in the oscillator frequency can be analytically incorporated from solvable quantum mechanical problems. The oscillator can also be made explosive. The location of the condensate profile satisfies $d\bar{c}(t)/dt - c(t) \bar{c}(t) = b(t)$. Other parameters are obtained using the following consistency conditions: $B(t) = B_0 \exp \left( \int_0^t c(t') \, dt' \right)$, $\gamma(t) = -\frac{1}{2} B_0 A(t) B(t) \exp (-G(t)/2)$ and $A(t) = B^2(t) = b(t)$. The real part of the GP equation yields
\[ \frac{d^2 \phi}{dt^2} + g \phi^2 \phi + e \phi = 0, \]  
(34)
which takes the form of equation (9), with $g = -2 \frac{\partial \ln \phi}{\partial t}$, $e = -\lambda$ and $T = A(t) \left[ \bar{c}(t) \right]$. One can find various singular and non-singular solutions for $(T, e)$, using the previous procedure, provided $g$ is constant. We now illustrate two specific cases of interest.

(1) First, we consider the condition $M(t) = M_0^2$, with a constant oscillator frequency. This yields a periodic solution, with period $\pi/2 M_0$:
\[ \psi \left( z, t \right) = -\frac{\epsilon}{g} \sqrt{\lambda \sec \left( M_0 \right)} \times \frac{3}{2} \text{sech}^2 \left( T/2 \right) \]  
(35)
\[ \times G \left( \frac{3}{2} \right) \phi \left( z, t \right) + i \frac{1}{2} G \left( \xi \right) / \epsilon}. \]  
(31)

The soliton can be compressed and accelerated through the time dependence of coupling. The presence of $\sec (M_0 t)$ in the amplitude and width of the soliton profile leads to its compression and amplification, as time increases. The temporal evolution of the density profile is shown in figure 3.
(2) We now turn our attention to the solution in an expulsive potential \( \langle M(t) = -M_0^2 \rangle \), where the W-type density profile is given by

\[
\psi(z, t) = -\frac{\epsilon}{g} \frac{A_0 \text{sech}(M_0 \ell)}{1 - \frac{3}{2} \text{sech}^2(T/2)} e^{i\Phi(z, t)}.
\]

The above profile has a transient character and the behaviour in time for the expulsive potential is very different from the regular case. One observes gradual reduction in amplitude as time progresses. It is worth mentioning that expulsive potentials are experimentally realizable [6].

5. Conclusion

In conclusion, the cigar-shaped BECs in the strong coupling sector lead to two different types of stable localized solitons, absent in the weak coupling regime. Localized solitons, with no background, require a finite momentum to exist for positive frequency. There is no such restriction for the soliton trains, which can propagate only on a background. The solutions are found to be stable under VK criterion. It is shown that the localized soliton can be effectively controlled through scattering length and trap frequencies, which makes it useful for trapping of atoms. It is interesting to note that the asymptotically vanishing localized solution is also velocity restricted. This type of solution exists in higher order nonlinear problems.

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