Penrose transform and monogenic functions.

Tomáš Salač

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675 Prague, Czech Republic

Abstract

Penrose transform gives an isomorphism between the kernel of a 2-Dirac operator and sheaf cohomology on the twistor space. The point of this paper is to write down this isomorphism explicitly. This gives a new insight into the structure of the kernel of the operator.

Keywords:
Monogenic spinors, Parabolic geometries, Penrose transform

1. Introduction.

Let us denote by $V_0^R(2, n + 4)$ the Grassmanian of null 2-planes in $\mathbb{R}^{n+2,2}$ with a quadratic form of signature $(n+2, 2)$. This space is the homogeneous space of a parabolic geometry. There is a sequence

$$
\Gamma(V_1) \xrightarrow{D_1} \Gamma(V_2) \xrightarrow{D_2} \Gamma(V_3) \xrightarrow{D_3} \Gamma(V_4) \rightarrow 0
$$

(1)

of operators living on $V_0^R(2, n + 4)$ which are invariant with respect to the action of the principal group of the geometry, see [TS]. This sequence belongs to the singular character and to the stable range. The first operator $D_1$ is a first order operator and is called the 2-Dirac operator (in the parabolic setting). Sections which belong to the kernel of $D_1$ are called monogenic sections. The sequence (1) is interesting from the point of the 2-Dirac operator (in the Euclidean setting) studied in Clifford analysis, see [CSSS].

The sequence (1) is coming from Penrose transform. The Penrose transform is explained in [BE] and [WW]. The Penrose transform for similar sequences in the unstable range was studied in [K]. Similar sequences were obtained by the Penrose transform also in [B]. The Penrose transform lives in the holomorphic category. We loose no information in holomorphic category about the sequence (1) since $V_0^R(2, n + 4)$ is algebraic variety and we consider real analytic sections which extend uniquely to holomorphic sections over small holomorphic thickenings of affine open subsets of $V_0^R(2, n + 4)$.

The Penrose transform tells us that the sequence (1) is exact (after adding the kernel of $D_1$ at the first spot in (1)) on every convex sufficiently small subset of $V_0^R(2, n + 4)$ and is a resolution of the sheaf of monogenic sections. In particular, the Penrose transform gives an isomorphism between the kernel of the operator $D_1$ on affine subset $U$ and $H^2(W, O_\lambda)$. However, general machinery of the Penrose transform does not yield this isomorphism explicitly. In this paper we write this isomorphism explicitly for the case $n = 6$. The explicit formula for the Penrose transform is in (14). This will give us some interesting insight into the space of monogenic sections. Moreover we decompose the space of monogenic sections with respect to a maximal reductive subgroup of parabolic subgroup, see Theorem 6.1.

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\[ \text{Email address: salac@karlin.mff.cuni.cz (Tomáš Salač)} \]
1.1. Notation.

Let $V_0^C(k, n)$ denotes the space of $k$-dimensional subspaces in $\mathbb{C}^n$ that are totally null with respect to some non-degenerate, symmetric, complex bilinear form. The space $V_0^C(k, n)$ is smooth algebraic variety and thus also complex manifold. We will denote by $M(k, n, \mathbb{C})$, resp. $M(k, \mathbb{C})$, resp. $A(k, \mathbb{C})$ the space of complex matrices of rank $k \times n$, resp. $k \times k$, resp. the space of skew-symmetric matrices of rank $k \times k$. We denote the span of vectors $v_1, \ldots, v_k$ by $\langle v_1, \ldots, v_k \rangle$. Affine charts on Grassmannian of $k$-planes in $\mathbb{C}^n$ will be denoted by usual notation. We will always work with some preferred basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$. These affine charts will correspond to affine subspaces in the space of matrices $M(n, k, \mathbb{C})$. If $A = (a_{ij})_{j=1, \ldots, k} \in M(n, k, \mathbb{C})$ is such a matrix then we associate to it the $k$-plane spanned by the vectors $u_1, \ldots, u_k$ where $u_i = \sum_j a_{ji}v_j$.

2. The parabolic geometry.

Let $h$ be a non-degenerate, symmetric, complex bilinear form on $\mathbb{C}^{10}$. We will work with basis $\{e_1, e_2, e_3, e_4, e_5, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_2\}$ such that for all $1 \leq i, j \leq 5 : h(e_i, \bar{e}_j) = \delta_{ij}, h(e_i, e_j) = h(\bar{e}_i, \bar{e}_j) = 0$.

Let $G := \{g \in \text{End}_\mathbb{C}(\mathbb{C}^{10})|v, v \in \mathbb{C}^{10} : h(gu, gv) = h(u, v), \det(g) = 1\}$. Let $x_0 := \langle e_1, e_2 \rangle$ and let $P := \{g \in G|g(x_0) = x_0\}$. The space $G/P$ is naturally isomorphic to the Grassmannian $V^C_0(2, 10)$. Let $G_0$ be the stabilizer of $x_0 := \langle e_3, e_4, e_5, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ in $P$. Then $G_0$ is maximal reductive subgroup of $P$ isomorphic to $\text{SO}(6, \mathbb{C}) \times \text{SO}(4, \mathbb{C})$. The Lie algebra $\mathfrak{g}$ of $G$ consists of all matrices of the form

$$
\begin{pmatrix}
A & Y_1 & Y_2 & Y_{12} \\
X_1 & B & D & -Y_2^T \\
X_2 & C & -B^T & -Y_1^T \\
X_{12} & -X_2^T & -X_1^T & -A^T
\end{pmatrix}
$$

where $A \in M(2, \mathbb{C}), B \in M(3, \mathbb{C}), C, D \in A(3, \mathbb{C}), Y_i, X_i^T \in M(2, 3, \mathbb{C}), X_{12}, Y_{12} \in A(2, \mathbb{C})$. There is $G_0$-invariant gradation

$$
\mathfrak{g} \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2
$$

such that $\mathfrak{g}_0$ is the Lie algebra of $G_0$ and the Lie algebra $\mathfrak{p}$ of $P$ is $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The subalgebra $\mathfrak{g}_0$ corresponds to the blocks $A, B, C, D, \mathfrak{g}_1$ to the blocks $Y_1, Y_2, \mathfrak{g}_2$ to the block $Y_{12}$, $\mathfrak{g}_{-2}$ to the block $X_{12}$ and $\mathfrak{g}_{-1}$ to the blocks $X_1, X_2$. Let us denote by $\mathfrak{g}_{-} := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Let $\pi : G \to G/P$ be the canonical projection. Let $G_{-} := \exp(\mathfrak{g}_{-})$ and let $U := \pi(G_{-})$. The map $\exp$ is biholomorphism between $\mathfrak{g}_{-}$ and $G_{-}$ and the map $\pi$ is biholomorphism between $G_{-}$ and $U$.

3. The double fibration.

The Penrose transform starts with two fibrations $\tau, \eta$ which fit into diagram

$$
\begin{array}{ccc}
G/Q & \to & G/P \\
\downarrow \eta & & \downarrow \tau \\
G/R & \to & G/P.
\end{array}
$$

The twistor space $G/R$ is the connected component of $z_0 := \langle e_1, \ldots, e_5 \rangle$ in the Grassmannian $V^C_0(5, 10)$. We set $R := \{g \in G|g(z_0) = z_0\}$. The Lie algebra of $R$ is the subspace of $\mathfrak{g}$ where the matrices $C, X_2, X_{12}$ from (2) are zero. Let $z_0 = \langle \bar{e}_1, \ldots, e_5 \rangle$ and let $R_0 := \{g \in R|g(z_0) = z_0\}$. Then $R_0$ is maximal reductive subgroup of $R$ isomorphic to $\text{GL}(5, \mathbb{C})$ and its Lie algebra consists of the matrices where $A, B, X_1, Y_1$ are arbitrary and the other matrices are zero.

The correspondence space $G/Q$ consists of the pairs $(z, x)$ with $x \in G/P, z \in G/R$ such that $x \subseteq z$. We choose $Q := P \cap R$ so that the Lie algebra $\mathfrak{q}$ of $Q$ sits in the blocks $A, B, D, Y_1, Y_2, Y_{12}$. Let $Q_0 := G_0 \cap R_0$. Then $Q_0$ is maximal reductive subgroup of $Q$ isomorphic to $\text{GL}(2, \mathbb{C}) \times \text{GL}(3, \mathbb{C})$. The Lie algebra $\mathfrak{q}_0$ of $Q_0$ corresponds to the blocks $A, B$.

Let us denote $\mathcal{V} := \tau^{-1}(U)$ and let $\mathcal{W} := \eta(\mathcal{V})$. 

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3.1. Projection $\tau$

Let us recall that $x_0 = \langle e_1, e_2 \rangle, x_0' = \langle e_3, \ldots, e_5, e_3, \ldots, e_5 \rangle, z_0 = \langle e_1, \ldots, e_5 \rangle$. Let us notice that $x_0' = x_0 \oplus x_0''$. The projection $\tau|_V : V \to U$ sends $(x, z) \mapsto x$. The fibre $\tau^{-1}(x_0)$ is isomorphic to the set of all null 5-planes in $G/R$ which contains the null 2-plane $x_0$. Thus $(z_0, x_0) \in \tau^{-1}(x_0)$. It is easy to see that any such null 5-plane is determined by an unique null 3-plane in $x_0'$. For example $z_0 = x_0 \oplus y_0$ where $y_0 := \langle e_3, e_4, e_5 \rangle$. The fibre $\tau^{-1}(x)$ is isomorphic to $\mathbb{P}/Q$ and in particular is connected. We deduce that the fibre is biholomorphic to the connected component of $y_0$ in the space of all null 3-planes in $x_0' \cong \mathbb{C}^6$. This connected component is biholomorphic to the family of $\alpha$-planes in $\mathbb{C}^6$.

3.2. The family of $\alpha$-planes in the Grassmanian $V_0^G(3, 6)$

The restriction of the $G$-invariant quadratic form on $\mathbb{C}^{10}$ induced by $h$ descends to non-degenerate quadratic form on $x_0' \cong \mathbb{C}^6$. It will be convenient to make the following identifications. We identify $x_0' \cong \Lambda^2 \mathbb{C}^4$ such that the basis $\{e_3, e_4, e_5, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$ goes to the basis

$$\{f_0 \wedge f_1, f_0 \wedge f_2, f_0 \wedge f_3, f_2 \wedge f_3, -f_1 \wedge f_3, f_1 \wedge f_2\},$$

of $\Lambda^2 \mathbb{C}^4$ where $\{f_0, f_1, f_2, f_3\}$ is the standard basis of $\mathbb{C}^4$. Under this identification, the quadratic form on $\mathbb{C}^6$ goes to the quadratic form $Q$ on $\Lambda^2 \mathbb{C}^4$ which is determined by $\alpha \wedge \alpha = Q(\alpha)f_0 \wedge f_1 \wedge f_2 \wedge f_3$ for all $\alpha \in \Lambda^2 \mathbb{C}^4$. This identifies $\text{SL}(4, \mathbb{C}) \cong \text{SPin}(6, \mathbb{C})$. The corresponding isomorphism $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C})$ is determined by

$$\begin{pmatrix} A_1 & E_{12} & 0 & 0 \\ E_{21} & A_2 & E_{23} & 0 \\ 0 & E_{32} & A_3 & E_{34} \\ 0 & 0 & E_{43} & A_4 \end{pmatrix} \rightarrow \begin{pmatrix} A_1 + A_2 & E_{23} & 0 & 0 & 0 & 0 \\ E_{32} & A_1 + A_3 & E_{34} & 0 & 0 & E_{12} \\ 0 & E_{43} & A_1 + A_4 & 0 & -E_{12} & 0 \\ 0 & 0 & 0 & -A_1 - A_2 & -E_{32} & 0 \\ 0 & 0 & -E_{21} & -E_{23} & -A_1 - A_3 & -E_{43} \\ 0 & E_{21} & 0 & 0 & -E_{34} & -A_1 - A_4 \end{pmatrix}.$$

The Grassmanian $V_0^G(3, 6)$ is the disjoint sum of two families. The first family, called the family of $\alpha$-planes, can be identified with $\mathbb{C}P^3$ by the following mapping. Let $\pi \in \mathbb{C}P^3$. Let $v \in \mathbb{C}^4$ be a representative of $\pi$. Let $\{v, v_1, v_2, v_3\}$ be a basis of $\mathbb{C}^4$. We assign to $\pi$ the 3-plane $\langle v \wedge v_1, v \wedge v_2, v \wedge v_3 \rangle, i = 1, 2, 3$. It is easy to see that the 3-plane is null and that the map is well defined. The latter family, called the family of $\beta$-planes, can be identified with $\mathbb{P}(\mathbb{C}^4)^*$ by the assignment $[w] \mapsto \langle v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3 \rangle$ where $\{v_1, v_2, v_3\}$ is a basis of $\text{Ker}(\omega)$. One can easily check that this map is well defined.

3.3. Affine coordinates on the family of $\alpha$-plane.

Since the family of $\alpha$-planes is biholomorphic to $\mathbb{C}P^3$ we know that there is an affine covering $\{U_0, \ldots, U_3\}$ of the family of $\alpha$-planes. Let us write down the affine charts on $U_0$ and $U_1$. Let $v = (v_0, v_1, v_2, v_3) \in \mathbb{C}^4$ be a non-zero vector and let us assume that $v_0 \neq 0$, resp. $v_1 \neq 0$. Let $w_0 := v_0^{-1}v = (1, \zeta_1, \zeta_2, \zeta_3)$, resp. $w_1 := v_1^{-1}v = (\rho_1, 1, \rho_2, \rho_3)$. Then the null 3-plane $\langle w_0 \wedge f_1, w_0 \wedge f_2, w_0 \wedge f_3 \rangle$, resp. $\langle w_1 \wedge f_0, w_1 \wedge f_2, w_1 \wedge f_3 \rangle$ has a unique basis of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\zeta_3 & \zeta_2 & 0 \\ -\zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix}, \text{ resp.}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ -\rho_2 & \rho_1 & 0 \\ 0 & -\rho_3 & \rho_1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

where we use the notation from [1]. Similarly for $v_3 \neq 0$ and $v_4 \neq 0$. The change of coordinates between $U_0$ and $U_1$ is

$$\zeta_1^{-1} = \rho_1, \zeta_2 \zeta_1^{-1} = \rho_2, \zeta_3 \zeta_1^{-1} = \rho_3.$$
3.4. The set $W$.

Let $x \in \mathcal{U}$ and let $g \in G_-$ be the unique element such that $\pi(g) = x$. Let $x^c := g(x_0^c)$. Then $x^c$ is complement of $x$ in $x^+ = x^+ \setminus x^c$ with basis $\{g(e_3), g(e_4), g(e_5), g(e_6), g(e_7), g(e_8), g(e_9), g(e_{10})\}$. Thus for any $x \in \mathcal{U}$ we have the preferred isomorphisms $x^c \cong x_0^c \cong \mathbb{C}^6$. Any null 5-plane $z' \in W$ has a null orthogonal basis $\{v_1, \ldots, v_5\}$ such that $x' := \langle v_1, v_2 \rangle \in \mathcal{U}$ and that the null 3-plane $y' := \langle v_3, v_4, v_5 \rangle$ belongs to $(x')^c$. The null 3-plane $y'$ belongs to the family of $\alpha$-planes in $(x_0^c)$. If $y' \in \mathcal{U}_0$, resp. $y' \in \mathcal{U}_1$ then $z'$ admits the unique basis of the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad \text{resp.}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

(8)

where $\ast$ are determined by the other entries. Similarly for $y' \in \mathcal{U}_0$ or $y' \in \mathcal{U}_3$. Let us denote the corresponding affine subsets of $W$ by $W_i$ where $i = 0, 1, 2, 3$. Thus $\mathcal{W} := \{W_i | i = 0, 1, 2, 3\}$ is an affine covering of $W$. We write the left hand side of (8) in the block form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\quad \text{and let}
\begin{pmatrix}
0 & z_0 \\
0 & 0 \\
0 & z_1 \\
0 & z_2 \\
0 & z_3 \\
\end{pmatrix}
$$

(9)

The change of coordinates on $W_0 \cap W_1$ is

$$
w_0 = z_0 + z_2z_1 \xi_1^{-1} - z_2z_1 \xi_1^{-1}, \quad w_{11} = z_{11} + z_{31} \xi_1 \xi_3^{-1} + z_{21} \xi_2 \xi_4^{-1}
$$

(10)

and those in (7).

4. Sections of the bundle $\mathcal{O}_\lambda$ over the set $W$.

Let $\mathcal{C}_\lambda$ be an one-dimensional R-module with highest weight $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2})$. Let $\mathcal{O}_\lambda$ be the sheaf of holomorphic sections of the bundle $\mathcal{C}_\lambda := G \times \mathbb{C} \mathcal{C}_\lambda$. Let $\mathcal{M} \subset G/R$ be an open subset. Let $\mathcal{O}_\lambda(\mathcal{M})$ be the space of holomorphic sections of the bundle $\mathcal{C}_\lambda$ over $\mathcal{M}$ and let $\mathcal{O}_\lambda(\mathcal{M})$ be the space of holomorphic functions on $\mathcal{M}$.

Let $z \in W_0$ and let $\{v_1, \ldots, v_5\}$ be the preferred basis of $z$ from (8). Then $\{v_1, \ldots, v_5, e_3, e_4, e_5, e_6, e_7\}$ is null orthonormal basis of $\mathbb{C}^6$. Let $g \in \text{End}_{\mathbb{C}}(\mathbb{C}^6)$ be the linear map such that for all $1 \leq i \leq 5; g(e_i) = v_i, g(e_i) = \hat{e}_i$. Then $g \in G$ and the map $z \mapsto \rho_0(z) := g$ is section of the principal $R$-bundle over $W_0$. We define similarly for $i = 1, 2, 3$ sections $\rho_i$ over $W_i$. Let us write transition function on $\mathcal{C}_\lambda$ between $W_0$ and $W_1$ in the preferred trivializations $\rho_0$ and $\rho_1$. Let $f_0 \in \mathcal{O}(W_0), f_1 \in \mathcal{O}(W_1)$. Then $f_0, f_1$ defines an element of $\mathcal{O}_\lambda(W_0 \cup W_1)$ iff

$$
f_0(z_0, z_{ij}, \xi_i) = \xi_1^{-5} f_1(w_0, w_{ij}, \rho_i)
$$

(11)

on $W_0 \cap W_1$. The transition functions on $\mathcal{C}_\lambda$ between all sets $W_i$ are rational and thus we can also consider rational sections of $\mathcal{C}_\lambda$ over $W$. 

4
4.1. Cohomology groups.

The Penrose transform gives an isomorphism

\[ P : H^3(W, O_\lambda) \cong \text{Ker}(D_1, U) \]  

(12)

between the sheaf cohomology of \( O_\lambda \) over the set \( W \) and the kernel of \( D_1 \) on \( U \) consisting of holomorphic sections where the bundles from (11) are \( V_1 = G \times \mathbb{P}(C_\mu \otimes S_+), V_2 = G \times \mathbb{P}(C_\mu \otimes S_-) \) where \( GL(2, \mathbb{C}) \)-module \( C_\mu \), resp. \( C_\mu^* \) has highest weight \( (\frac{\lambda}{2}, \frac{\lambda}{2}) \), resp. \( (\frac{\lambda}{2}, \frac{\lambda}{2}) \) and \( S_+ \cong \mathbb{C}^4, S_- \cong (\mathbb{C}^4)^* \) as \( SL(4, \mathbb{C}) \)-modules.

The Leray theorem states that \( H^i(W, O_\lambda) \cong \hat{H}^i(W, O_\lambda) \) where \( \hat{H}^* \) are the cohomology groups computed with respect to the affine covering \( \mathfrak{W} \), see for example [WW]. The co-chains groups are \( C^4(W, O_\lambda) = 0, C^i(W, O_\lambda) = \{ \{f_{i=0} W_i, f| f \in O_\lambda(\bigcap_{i=0} W_i)\}, C^2(W, O_\lambda) = \ldots \). By definition we have that \( \hat{H}^3(W, O_\lambda) := C^3(W, O_\lambda)/\text{Im}(\delta^2) \) where \( \delta^2 \) is the Čech co-differential. We will denote the cohomology classes by \([ \text{ ]}\). Let us make some simple observations about \( \hat{H}^3(W, O_\lambda) \). We will work with the affine chart on \( W_0 \).

Let us first notice that \( \bigcap_{i=0,1,2,3} W_i = \{ z \in W_0| \zeta_1 \neq 0, \zeta_2 \neq 0, \zeta_3 \neq 0 \} \). Thus if \( f \in O_\lambda(\bigcap_{i=0,1,2,3} W_i) \) is a holomorphic section then \( f \) is the converging sum of rational sections \( f(s_0, s_{ij}, r_k) = \sum_{i} \prod_{j} \zeta_{ij} \zeta_r \zeta_{ij}^2 \zeta_{ij}^2 \zeta_{ij}^3 \) where \( s_0, s_{ij} \geq 0 \) and \( r_1, r_2, r_3 \in \mathbb{Z} \). It is easy to see that \( [f(s_0, s_{ij}, r_k)] = 0 \) if \( r_1 < 0 \) or \( r_2 < 0 \) or \( r_3 < 0 \).

From the formula (11) follows that if \( 5 + s_0 + \sum s_{ij} > r_1 + r_2 + r_3 \) then \( f(s_0, s_{ij}, r_k) \) extend to a rational section on \( W_1 \cap W_2 \cap W_3 \) and thus \( [f(s_0, s_{ij}, r_k)] = 0 \). However notice that this does not characterize \( \hat{H}^3(W, O_\lambda) \). For example the cohomology class of the section \( s_{ij}^4 \zeta_{ij}^{-1} \zeta_2^{-1} \zeta_3^{-3} \) is trivial for any \( i \geq 0 \) although the relation does not hold. The full characterization of the 3-rd sheaf cohomology group of rational sections will be given in Theorem [61] where we give it as a direct sum of \( G_0 \)-modules.

5. The correspondence \( x \in U \mapsto \eta \circ \tau^{-1}(x) \subset \mathfrak{W} \).

Let us write the correspondence on \( W_0 \). Then

\[
\begin{pmatrix}
1_2 \\
X_1 \\
X_2 \\
X_{12} = \frac{1}{2}(X_1^2 X_2 + X_2^2 X_1)
\end{pmatrix} \in U \mapsto 
\begin{pmatrix}
1_2 \\
X_2 - \zeta X_1 \\
X_{12} + \frac{1}{2}(X_1^2 X_2 - X_2^2 X_1) + X_1^2 \zeta X_{12} - X_1 X_{12}^2 \zeta
\end{pmatrix} \in W_0
\]

(13)

where the blocks of the matrix on the right hand side are those from (11). In particular \( X_1, X_2 \in M(3, 2, \mathbb{C}), X_{12} \in A(2, \mathbb{C}), \zeta \in A(3, \mathbb{C}) \). Notice that the fibres of the correspondence are biholomorphic to \( \mathbb{C}P^3 \).

5.1. Integration.

The isomorphism (12) is given by integrating over the fibres of the correspondence (13). Let \( f \in C^3(W, O_\lambda) \). We write \( f = f(B_0, B_1, B_2) \) where \( B_0, B_1, B_2 \) are the matrices from (9). Then the integral formula in the affine chart on \( W_0 \) is

\[
P(f)_A(X_{12}, X_1, X_2) = \frac{1}{(2\pi i)^3} \int_{(S^1)^3} (1, \zeta_1, \zeta_2, \zeta_3) f(X_{12} + \frac{1}{2}(X_1 X_2 - X_2 X_1) + X_1^2 \zeta X_2, X_2 - \zeta X_1, \zeta) d\zeta_1 d\zeta_2 d\zeta_3
\]

(14)

where \( d\zeta_1 d\zeta_2 d\zeta_3 \) is the holomorphic top form on the fibres which is homogeneous of degree 4 in the homogeneous coordinates. The section \( f \) is homogeneous of degree 5 in the homogeneous coordinates. Thus the integrand is homogeneous of degree zero and the integration does not depend on the choice of trivialization of the fibres of the correspondence. For example

\[
P(\zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-1}) = \frac{1}{(2\pi i)^3} \int_{(S^1)^3} (1, \zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3 = (1, 0, 0, 0)
\]

is constant spinor on \( U \).
6. Decomposition of monogenic sections into irreducible $G_0$-modules.

It is convenient to introduce a grading on the space of all polynomial spinors on $U$. We trivialize the $P$-bundle over $U$ by the map $\phi_0 : \mathcal{U} \to G_\infty \to G$. We write coordinates on $\mathfrak{g}_-$ and thus also on $\mathcal{U}$ as $X_1 = (x_{ij})_{i,j=1,2,3}$, $X_2 = (x_{ij}^2)_{i,j=1,2,3}$, $X_{12} = \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix}$. We will denote polynomials on the affine space $\mathfrak{g}_-$ with the same letters as the coordinates.

Let us first define degree of linear polynomials by setting $\text{deg}(x_{12}) := 2$, $\text{deg}(x_{ij}^2) := 1$. Let us extend it to the set of monomials in $\mathbb{C}[x_{12}, x_{ij}^2]$ by requiring that $\text{deg}$ is morphism of $(\mathbb{C}[x_{12}, x_{ij}^2], \cdot)$ and $(\mathbb{Z}, +)$. Let $N_k$ be the vector space generated by the monomials of the degree $k$. Finally, let $M_k$ be the space of monogenic spinors whose components belong to $N_k$. If we extend this gradation naturally also to $\Gamma(\mathcal{U})$ over $U$, then the operator $D_1$ is homogeneous of degree $-1$.

The space of linear monogenic spinors is $G_0$-irreducible. The space of quadratic monogenic spinors $M_2$ decompose into $W(\mathbb{Z}, \frac{2}{3}) \otimes V(3,2,0,0) \oplus W(\mathbb{Z}, \frac{1}{3}) \otimes V(3,1,1,0) \oplus W(\mathbb{Z}, \frac{2}{3}) \otimes V(1,0,0,0)$, where $W_{(a,b)}$, resp. $V_{(a,b,c,d)}$ stands for irreducible $GL(2, \mathbb{C})$, resp. $SL(4, \mathbb{C})$-module with highest weight $(a,b)$, resp. $(a,b,c,d)$. Let us write the dimensions of the modules, highest weight vectors from $\mathcal{H}(\mathfrak{g}_l, \mathcal{O}_\lambda)$ and the corresponding monogenic spinors. We have

\[
\begin{align*}
W(\mathbb{Z}, \frac{2}{3}) \otimes V(3,2,0,0) & : 180, \\
W(\mathbb{Z}, \frac{1}{3}) \otimes V(3,1,1,0) & : 36, \\
W(\mathbb{Z}, \frac{2}{3}) \otimes V(1,0,0,0) & : 4, \\
W(\mathbb{Z}, \frac{1}{3}) \otimes V(0,0,0,0) & : (3x_{12} + \frac{1}{2} \sum_{i=1}^{3} (x_{1i}^2, x_{12} x_{ij}^2) - x_{12}^2 x_{1i}^2, x_{12}^2 x_{1i}^2, -x_{12}^2 x_{12} x_{ij}^2, x_{12}^2 x_{12} x_{ij}^2, -x_{12}^2 x_{1i}^2, x_{12}^2 x_{1i}^2, x_{12}^2 x_{1i}^2, -x_{12}^2 x_{12} x_{ij}^2).
\end{align*}
\]

In general we have the following theorem.

**Theorem 6.1.** Let us keep the notation as above. Then the space $M_k$ of monogenic spinors of degree $k$ on $U$ decomposes into irreducible $G_0$-modules.

\[
M_k \cong \bigoplus_{a,b,l \geq 0, 2a+b+2l=k} W(\frac{1}{3}+l+a+b, \frac{2}{3}+l+a) \otimes V(2a+b+1, a+b, 0).
\]

The decomposition of algebraic monogenic spinors into irreducible $G_0$-modules is multiplicity free.

**Proof:** Let us recall that the action of $G_0$ on $\mathcal{H}(\mathfrak{g}_l, \mathcal{O}_\lambda)$ is induced by the left action on the total space of the parabolic geometry. Let us now compute the weight of

\[
f = \sum_{i,j} x_{ij}^2 \varepsilon_{ij}^s \varepsilon_{ij}^l \in \mathcal{C}(\mathcal{W}, \mathcal{O}_\lambda).
\]

Let us denote for $j = 1, 2 : c_j := s_{1j} + s_{2j} + s_{3j}$, for $i = 1, 2, 3 : s_i := s_{i1} + s_{i2}$ and let $r := r_1 + r_2 + r_3, s := s_1 + s_2 + s_3$. We write weights as $\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$-weights. We find that the weight of $f$ is

\[
(c_1 + s_0 + \frac{5}{2}, c_2 + s_0 + \frac{5}{2}) \oplus (5 + s - r, r_1 + s_1, r_2 + s_2, r_3 + s_3).
\]

Let us recall (5). Let us write the $\mathbb{R}$-module structure on $\mathcal{C}_\lambda$ by $\sigma$. Let $A_{12}$ be a standard positive root in $\mathfrak{sl}(2, \mathbb{C})$. We find that all simple roots beside $E_{12}$ preserve the image of the section $\rho_0$ of the principal
Lemma 6.2. Let \( A_{12} \in f \) represent the cohomology class is non-zero.

\[ \rho_0(z(t)) = \exp(-E_{12} \rho_0(z,t)) \] for a unique \( r(z,t) \in R \) for \( t \) sufficiently small. If we differentiate we find out that \( E_{12}f \) is the usual derivation of rational function which obeys the Leibniz rule plus the term \( \hat{\sigma}(E_{12})f(z) := \frac{d}{dt}[\sigma(r^{-1}(t,z))f(z)] \). We find that

\[
A_{12}\zeta_2 = \zeta_1, E_{34}\zeta_2 = -\zeta_3, E_{34}\zeta_3 = \zeta_2, E_{23}\zeta_4 = -\zeta_5, E_{23}\zeta_5 = \zeta_4, E_{43}\zeta_3 = -\zeta_4, E_{43}\zeta_4 = \zeta_3, E_{32}\zeta_4 = -\zeta_3, E_{32}\zeta_2 = \zeta_3, E_{21}\zeta_1 = 1, E_{12}\zeta_1 = -\zeta_2 z_{24} - \zeta_3 z_{23}, E_{12}\zeta_2 = \zeta_2 z_{23} - \zeta_2 z_{24}, \hat{\sigma}(E_{12})f = 5\zeta_1 f, \text{for} i = 2, 3 : E_{12}\zeta_i = \zeta_i \zeta_1, E_{12}z_{ij} = z_{ij}\zeta_1.
\]

while all other terms are zero. This and Leibniz rule allows us to compute easily the action of \( g_0 \) on \( C^3(\mathcal{W}, \mathcal{O}_\lambda) \).

Lemma 6.1. Let \( f \) be a \( q_0 \)-highest weight vector in the space of polynomials on the block \( B_1 \) in \( \mathcal{W} \), i.e. \( f \in \mathbb{C}[z_{ij}] \). Then \( f \) is \( A(z_{11}z_{23} - z_{21}z_{12})^a z_{11}^b \) for some \( A \in \mathbb{C}, a, b = 0, 1, 2, \ldots \)

Proof: See [GW]. \( \square \)

Lemma 6.2. Let \( f \) be the rational section from [10] such that the weight of \( f \) is dominant and \( s_0 = 0, r_i \geq 1 \). Then the class \( [E_{12}^{r_1}E_{23}^{r_2}E_{34}^{r_3}]f \) \( \in H^3(\mathcal{W}, \mathcal{O}_\lambda) \) is non-zero.

Proof: We have that

\[
E_{34}^{r_3-1} f = A \frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3}} + \zeta_3^{-2}(...)
\]
\[
E_{23}^{r_2-2} E_{34}^{r_3-1} f = B \frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3}} + \zeta_2^{-2}(...) + \zeta_3^{-2}(...)
\]
\[
E_{23}^{r_2} E_{34}^{r_3-3} E_{23}^{r_2} E_{34}^{r_3-1} f = C \frac{\prod_{ij} z_{ij}^{s_{ij}}}{\zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3}} + \zeta_1^{-2}(...) + \zeta_2^{-2}(...) + \zeta_3^{-2}(...),
\]

where \( \ldots \) denotes sections where \( \zeta_i \) appear only in denominators and

\[
A = (-1)^{r_3-1} r_2 (r_2 + 1) \ldots (r_2 + r_3 - 2)
\]
\[
B = (-1)^{r_2+r_3-2} r_1 (r_1 + 1) \ldots (r_1 + r_2 + r_3 - 3) A
\]
\[
C = AB(s_2 + s_3 + 5 - r)(s_2 + s_3 + 5 - (r - 1)) \ldots (s_2 + s_3 + 1).
\]

Since the weight of \( f \) is by assumption dominant, then \( 5 + s - r \geq r_1 + s_1 > 0 \) and thus \( 5 + s_2 + s_3 - r > 0 \). It follows that \( C \neq 0 \) and thus also

\[
\mathcal{P}(E_{12}^{r_1}E_{23}^{r_2}E_{34}^{r_3} f) = C(\prod_{ij} (x_{ij}^2)^{s_{ij}}, 0, 0, 0) + \ldots
\]

where \( \ldots \) denotes some spinors whose first components are different from \( \prod_{ij} (x_{ij}^2)^{s_{ij}} \). In particular we get that the cohomology class is non-zero. \( \square \)

Lemma 6.3. Let

\[
f = \sum_{k=1}^{K} g_k, \quad \text{where} \quad g_k = \frac{f_k}{\prod_i \zeta_i^{r_i}},
\]

be a highest weight vector in \( H^3(\mathcal{W}, \mathcal{O}_\lambda) \) such that all \( f_k \in \mathbb{C}[z_{ij}] \). Then \( K = r_1 + 1 = r_2 + 1 = r_3 + 1 \) and \( f_1 \) is a \( q_0 \)-maximal polynomial given in Lemma 6.2.
Proof: Each summand in (20) satisfy the assumptions of Lemma (6.2). Let us notice that from (17) follows that for all $1 \leq j, k \leq K : r_1^j + r_2^j + r_3^j = r_1^k + r_2^k + r_3^k$ and that $\deg(f_j) = \deg(f_k)$. We can choose in (20) indexation by $j$ such that for all $k > 1$ the following holds: $r_1^j > r_1^k$ or $r_1^j \neq r_1^k$ and $r_2^j > r_2^k$ or $r_1^j = r_1^k$ and $r_2^j = r_2^k$ and $r_3^j > r_3^k$.

Let us assume that $r_1^j r_2^j r_3^j \geq 2$. Let $E := E_{12}^{r_1^j+r_2^j+r_3^j} E_{23}^{r_1^j-r_2^j-2} E_{34}^{r_3^j-1}$. The formula (19) reveals that $\mathcal{P}(\mathcal{E}(g_1)) \neq 0$. Similar manipulations give that $\mathcal{P}(\mathcal{E}(g_1)) \neq -\mathcal{P}(\mathcal{E}(f - g_1))$ and thus $\mathcal{P}(E, f) \neq 0$ and thus $f$ is not highest weight vector. Thus the only possibility is that $K = r_1^j = r_2^j = r_3^j = 1$ and $f_1$ is $q_0$-maximal. □

Lemma 6.4. Let

\[ f = \sum_{i=0}^{s_0} z_0^{s_0-i} f_i = z_0^{s_0} f_0 + z_0^{s_0-1} f_1 + \ldots \]  

be a maximal highest weight vector such that $f_i \in C^3(\mathcal{M}, \mathcal{O}_\lambda)$ are rational sections which do not depend on $z_0$ and $[f_0] \neq 0$. Then $f_0$ is also highest weight vector and $f$ is uniquely determined by $s_0$ and $f_0$. Conversely given a non-zero highest weight vector $f_0 \in C^3(\mathcal{M}, \mathcal{O}_\lambda)$ that does not depend on the variable $z_0$ and $s_0 \geq 0$, then there exists a unique highest weight vector $f$ of the form as in (22) for some $f_i, i = 1, \ldots, s_0$.

Proof: We easily check that if $f$ is highest weight vector then also $f_0$ is highest weight vector. Thus $f_0$ is a multiple non-zero of $z_0^{s_0} (z_{11} z_{22} - z_{12} z_{21})^{s_0} \zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-1}$ for some $a, b \geq 0$. Let us check uniqueness of $f$ given $f_0$ and $s_0$. Let $f, \tilde{f}$ be two highest weight vectors of the same weight such that $\tilde{f} = z_0^{s_0} f_0 + z_0^{s_0-1} (\ldots)$ and $f = z_0^{s_0} f_0 + z_0^{s_0-2} (\ldots)$. Then $\tilde{f} - f = z_0^{s_0} \tilde{f}_0 + z_0^{s_0-1} \tilde{f}_1 + \ldots$ with $t_0 < s_0$ has to be a highest weight vector with the same weight as $f$ and $f'$. Thus $f_0$ is a multiple of $z_0^{s_0} (z_{11} z_{22} - z_{12} z_{21})^{s_0} \zeta_1^{-1} \zeta_2^{-1} \zeta_3^{-1}$ for some $c, d \geq 0$. But the formula (17) shows that then $a = c, b = d$ and thus also $f_0 = s_0$. Contradiction.

Let us consider a filtration $\{F_i \mid i \geq 0\}$ of $C^3(\mathcal{M}, \mathcal{O}_\lambda)$ given by the degree of $z_0$, i.e. $F_i := \{g \in C^3(\mathcal{M}, \mathcal{O}_\lambda) \mid \partial_{z_0}^i g = 0\}$ where $\partial_{z_0}$ is the coordinate vector field corresponding to the variable $z_0$. From the table (13) follows that $G_0$ preserve this filtration. Let $f_{\text{top}} = z_0^{s_0} f_0$ be the highest part of $f$. Let $V = G_0 f_{\text{top}} = \{ \sum_{j=1}^{M} g_j f_{\text{top}} \mid g_j \in G_0, M < \infty \}$. Then clearly $V$ is the smallest $G_0$-module which contains the vector $f_{\text{top}}$. Moreover $V \subset F_{s_0}$ and from the table (13) follows that $V/F_{s_0-1}$ is spanned by $f_{\text{top}}$. We have that $V = \oplus V_i$ for some irreducible $G_0$-modules $V_i$. Let $h_i$ be a maximal vector of $V_i$. Since the filtration $F_i$ is $G_0$-equivariant, there exists $i$ such that $h_i = f_{\text{top}} + \text{l.o.t.}$ where $\text{l.o.t.}$ means lower order terms in $z_0$-variables. From the uniqueness we have that $h_i$ is up to a multiple the unique highest weight vector with the leading term $z_0^{s_0} f_0$. □

Thus we have that any highest weight vector is uniquely determined by its leading term $f_{\text{top}}$ with respect to the variable $z_0$. If $f_{\text{top}} = z_0^{s_0} (z_{11} z_{22} - z_{12} z_{21})^{s_0} \zeta_{11}$, then $f$ is highest weight vector of the module $W_{(\frac{s_0}{2} + a + b + s_0, \frac{s_0}{2} + s_0 + b)} \otimes V_{(2a + b + 1, a, b, 0)}$. □

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