Field-theoretical three-body relativistic equations for the multichannel $\pi N \leftrightarrow \gamma N \leftrightarrow \pi\pi N \leftrightarrow \gamma\pi N$ reactions\(^1\)

A. I. Machavariani\(^{\ddagger,*}\) and Amand Faessler\(^\dagger\)

\(^\dagger\) Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany,

\(^\ddagger\) Joint Institute for Nuclear Research, Moscow Region 141980 Dubna, Russia

\(^*\) High Energy Physics Institute of Tbilisi State University, University Str. 9 380086 Tbilisi, Georgia

Abstract

A new kind of the relativistic three-body equations for the coupled $\pi N$ and $\gamma N$ scattering reactions with the $\pi\pi N$ and $\gamma\pi N$ three particle final states are suggested. These equations are derived in the framework of the standard field-theoretical $S$-matrix approach in the time-ordered three dimensional form. Therefore corresponding relativistic covariant equations are three-dimensional from the beginning and the considered formulation is free of the ambiguities which appear due to a three dimensional reduction of the four dimensional Bethe-Salpeter equations. The solutions of the considered equations satisfy the unitarity condition and are exactly gauge invariant even after the truncation of the of the multiparticle ($n > 3$) intermediate states. Moreover the form of these three-body equations does not depend on the choice of the model Lagrangian and it is the same for the formulations with and without quark degrees of freedom. The effective potential of the suggested equations is defined by the vertex functions with two on-mass shell particles. It is emphasized that these INPUT vertex functions can be constructed from experimental data.

Special attention is given to the construction of the intermediate on shell and off shell $\Delta$ resonance states. These intermediate $\Delta$ states are obtained after separation of the $\Delta$ resonance pole contributions in the intermediate $\pi N$ Green function. The resulting amplitudes for the $\Delta \leftrightarrow N\pi; \Delta \leftrightarrow N\gamma; \Delta' \leftrightarrow \Delta\gamma$ transition have the same structure as the vertex functions for transitions between the on mass shell particle states with spin $1/2$ and $3/2$. Therefore it is possible to introduce the real value for the magnetic momenta for the $\Delta' \leftrightarrow \Delta\gamma$ transition amplitudes in the same way as it is done for the $N' \leftrightarrow N\gamma$ vertex function.

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1. INTRODUCTION

The problem of the relativistic description of an particle interactions in the framework of a potential picture is usually solved by relativistic generalization of the Lippmann-Schwinger type equation of the nonrelativistic collision theory [1, 2]. In quantum field theory potentials of the relativistic Lippmann-Schwinger type equation are constructed from more simple vertex functions which can be determined by a Lagrangian. Depending on the initial general relations in quantum field theory, one can distinguish three essentially different derivations of the field-theoretical generalizations of the Lippmann-Schwinger type equations which have different off mass shell behavior of the amplitudes, the potentials, the vertex functions and the propagators. The most popular source of the derivation of the such type field-theoretical equations is the Bethe-Salpeter equation [3, 4, 5, 6, 7]. The effective potential of these four-dimensional equations consists of the sum of the Feynman diagrams. The well known quasipotential method was applied [8, 9, 4, 5] in order to obtain the equivalent three-dimensional equations. The different three-dimensional representations of the Bethe-Salpeter equations, derived in the framework of the different quasipotential methods, have different free Green functions and diverse quasipotentials \( W \) which are constructed from the Bethe-Salpeter potential \( V \). For practical calculations the quasipotential is usually taken in the Born approximation \( W \approx V \). Therefore the results of these approximations depend on the form of the three-dimensional reduction. In addition for construction of \( V \) or \( W \) three-variable vertex functions are required as the “input” functions\(^2\). Therefore, in the practical calculations based on the Bethe-Salpeter equations or their quasipotential reductions the off-mass shell variables in the vertex functions are usually neglected or a separable form for all three variables is introduced [11, 12].

Another derivation of the relativistic Lippmann-Schwinger equation is based on the field-theoretical generalization of the Schrödinger equation in the form of the functional Tomonaga-Schwinger equations [13, 14]. In the framework of this method a covariant Hamiltonian theory for construction of the relativistic three-dimensional equations was given in Refs. [15, 16]. In this approach the covariant equations are three-dimensional from the beginning, and therefore they are free from the ambiguities of the three-dimensional reduction. The potential in these equations is constructed from the vertex functions with all particles on mass shell, i.e., the related “input” vertex functions are used dependent also on three variables. Apart from this problem in these equations one has the non-physical “spurious” degrees of freedom. Light-front reformulation of these equations was done in Refs. [17, 18]. The generalization of the Faddeev equations in this formulation was done in Ref.[19].

The third way of the derivation of the three-dimensional Lippmann-Schwinger type integral equations in the quantum field theory proceeds from the field-theoretical generalizations of the off-shell unitarity conditions. These conditions can be treated as a field-theoretical spectral decomposition of the scattering amplitude over the asymptotic “in(out)” states. This formulation, suggested first by F. Low [20] in the framework of the

\(^2\)The only exception is the quasipotential equation derived by F. Gross[10], where the input vertex functions are the two-variable vertices, because one particle in these vertices is on mass shell.
old perturbation theory, was developed afterwards in Refs. [14, 21, 22, 23, 24, 25, 26, 27]. In particular, in Refs. [14, 21, 22, 23] these three-dimensional, time-ordered and quadratically nonlinear equations were used for the evaluation of the $\pi N$ scattering amplitudes. In the Refs. [24, 26, 25] the explicit linearization procedures of these equations were suggested. The linearized equations have the form of relativistic Lippmann-Schwinger type equations and they were employed for the calculation of the low energy $\pi N$ and $NN$ scattering reactions. The corresponding equations are also three-dimensional and time ordered from the beginning and their potentials are constructed from completely dressed matrix elements with two on mass shell particles. Using the on-mass shell methods, such as the dispersion relations, sum rules, current algebra etc. one can determine the required “input” vertices with two on mass shell particles. The effective potential of the considered equations contains also the equal-time commutators of the two external interacting field operators. Therefore in this approach it is necessary to use some model Lagrangian in order to determine these equal-time commutators. The resulting operators calculated by the equal-time commutators are sandwiched between the real asymptotic on-mass shell states, i.e. these equal-time commutators are also determined by the one-variable vertex functions which can be considered as “input” vertices.

The equal-time term of a nonrenormalizable Lagrangian produce a number of contact terms [25]. But first of all our aim is to determine the minimal number of the terms from equal-time commutators (in the Chew-Low model these terms were omitted!) which are necessary for the description of the multichannel $\pi N$ and $\gamma N$ scattering reactions. These simplest Lagrangians can be improved to become closer to the more complete Lagrangian by inclusion of more complicated symmetries and by adding more terms, capable to describe different mechanisms of the interactions. In our previous works [26, 25, 27] about the $NN$ scattering we have estimated the contributions of the additional contact terms which arise after using the $\pi N$ Lagrangian with the nonrenormalizable pseudo-vector coupling. In papers [25, 30] we have considered the structure of the present field-theoretical approach with quark degrees of freedom. We have shown, that the structure of the present relativistic three-dimensional equations and their potentials with and without quark degrees of freedom is the same. Moreover for each Lagrangian (even in the formulation with quark degrees of freedom) the terms produced by the equal-time commutators are Hermitian and do not contribute in the unitarity condition.

Presently, the interest to investigate reactions with three-body final $\gamma \pi N$ states is stimulated by the proposal to determine the magnetic moment of the $\Delta^+$ resonance in the $\gamma p \to \gamma' \pi^0 p'$ reaction. The basic idea of this investigation [31] is to separate the contribution of the $\Delta \to \gamma' \Delta'$ vertex function which in analogy to the $N - \gamma' N'$ vertex, contains the magnitude of the $\Delta$ magnetic moment at threshold. The contribution of the $\Delta^+ \to \gamma' \Delta^+$ vertex function in the $\gamma p \to \gamma' \pi^0 p'$ reaction was numerically estimated in Refs. [32, 33, 34, 35] in order to study the dependence of the observables on the value of the $\Delta^+$ magnetic moment. First data about the $\gamma p \to \gamma' \pi^0 p'$ reaction were obtained in a recent experiment by the A2/TAPS collaboration at MAMI [28] and future experimental investigations of this reaction are planned by using the Crystal Ball detector at MAMI [29].
This paper is devoted to the three-body generalization of the two-body field-theoretical equations [14, 20, 21, 22, 23, 25, 30] for the case of the multichannel \( \pi N - \gamma N - \pi \pi N - \gamma \pi N \) reactions. The potential of these equations has minimal off shellness (i.e. only two of external particles are being off mass shell) and they are analytically connected with any other field-theoretical equations, based on the Bethe-Salpeter or the Tomonaga-Schwinger equations. The form of these equations do not depend on the choice of the Lagrangian, and they do not change their form even for the formulations with quark-gluon degrees of freedom. Therefore this formulation can help to clarify the difference between a number of theoretical models which describe with quite a good accuracy the experimental data of the two-body \( \pi N \) and \( \gamma N \) reactions in the framework of the different approximations without of the reproduction of the three-body \( \gamma \pi N \) and \( \pi \pi N \) data. From this point of view the unified description of the multichannel \( \pi N - \gamma N - \pi \pi N - \gamma \pi N \) reactions up to threshold of the creation of the third pi-meson can clarify the dynamic of the two-body \( \gamma N, \pi N, \gamma \pi \) and \( \pi \pi \) interactions in the low and intermediate energy region. In the recent investigations [12, 32] the importance of the choice of the form of the intermediate \( \Delta \) propagator was demonstrated by the description of the \( \pi N \) and multichannel \( \gamma N \) scattering reactions. Therefore the unified description of the \( \pi N - \gamma N - \pi \pi N - \gamma \pi N \) reactions can be employed also for the determination of the properties of the intermediate \( \Delta \) resonance propagation.

The organization of this papers is as follows. In Sec.2 we state the three-body spectral decomposition equations (which have the form of the off shell unitarity conditions [1]) for the amplitudes of the coupled \( \pi N - \gamma N - \pi \pi N - \gamma \pi N \) channels. These quadratically nonlinear three-dimensional equations are derived after extraction of the two external particles from the asymptotic in or out states, performed in the framework the standard S-matrix reduction formulas [3, 13, 14, 36]. In this section we consider also the connected and disconnected parts of the three-body amplitudes and the construction of the equal-time commutators. In Sec. 3 and in Appendix A the equivalence of the above quadratically nonlinear equations and the Lippmann-Schwinger type equations is outlined. The analytical expressions of the three-body potential of the presented Lippmann-Schwinger type equations is given Appendix B. The procedure of extraction of the intermediate on shell and off shell \( \Delta \)-isobar degrees of freedom from the intermediate \( \pi N \) states in the suggested three-body equation is developed in Sec. 4. In Sec. 5 we apply this procedure to the derivation of the three-body equations for the coupled \( \Delta - \pi \Delta - \gamma N - \gamma \Delta \) transition amplitudes. The unitarity and gauge invariance for the derived three-body equations are demonstrated in Sec. 6. Finally in Sec.7 we give some concluding remarks.

2. Spectral decomposition method of the multichannel \( \pi N \) and \( \gamma N \) scattering amplitudes over the complete set of the asymptotic "in" or "out" states.

Consider the S-matrix element \( S_{\alpha,\beta} \) and the scattering amplitude \( f_{\alpha,\beta} \) for the \( \alpha; \beta = 1, 2, 3, 4 \equiv \pi N, \gamma N, \pi \pi N, \gamma \pi N \) states

\[
S_{\alpha,\beta} = \langle \text{out}; \alpha | \beta; \text{in} \rangle = \langle \text{in}; \alpha | \beta; \text{in} \rangle = -(2\pi)^4 i \delta^{(4)}(P_\alpha - P_\beta) f_{\alpha,\beta} \tag{2.1}
\]
where \( P_\alpha \equiv (P^0_\alpha, \mathbf{P}_\alpha) \) stands for the complete four-momentum of the asymptotic state \( \alpha \) and

\[
f_{\alpha\beta} = - < \text{out}; \bar{\alpha}|j_\alpha(0)|\beta; \text{in} >
\]

\[
= < \text{out}; \bar{\alpha}|j_\alpha(0), a^+_b(0)|\beta; \text{in} > + i \int d^4x e^{-ip_\alpha x} < \text{out}; \bar{\alpha}|T(\dot{j}_\alpha(0)j_\alpha(x))|\beta; \text{in} >,
\]

(2.2)

\( a = \pi', \gamma' \); \( b = \pi, \gamma \) denotes the one particle pion or photon states extracted from the \( \alpha \) "out" and \( \beta \) "in" states correspondingly

\[
\alpha = \bar{\alpha} + a; \quad \beta = \bar{\beta} + b.
\]

(3.3)

The four-momentum of the asymptotic one-particle states \( a \) or \( b \) is \( p_\pi = (\sqrt{m^2_\pi + p^2_\pi}, p_\pi) \equiv (\omega_\pi(p_\pi), p_\pi) \) for pion with mass \( m_\pi \) or \( p_\gamma = (|k_\gamma|, k_\gamma) \equiv (\omega_\gamma(p_\gamma), p_\gamma) \) for photon with the observed (physical) mass \( m_\gamma = 0 \).

The expression (2.2) is defined through the meson and photon current operators

\[
j_{\pi}(x) = (\Box + m^2_\pi) \Phi_\pi(x); \quad J_{\eta_k}(x) = \epsilon^\eta_m(k) \Box A_\mu(x),
\]

(2.4a)

where \( \epsilon^\eta_m(k) = (\epsilon^0_\mu(k), \epsilon^\mu_\eta(k)) \) denotes the polarization four-vector of photon with the four momentum \( k = (|k|, k) \). The current operators in Eq.(2.4a) is defined through the \( \pi \) meson \( \Phi_\pi(x) \) and photon \( A_\mu(x) \) Heisenberg field operators as

\[
a^+_\pi(x_0) = -i \int d^3x e^{-ipx} \frac{\partial}{\partial x_0} \Phi_\pi(x),
\]

(2.4b)

\[
a^+_\gamma(x_0) = -i \epsilon^\eta_m(k) \int d^3x e^{-ikx} \frac{\partial}{\partial x_0} A_\mu(x).
\]

(2.4c)

These operators transforms into meson and photon creation or annihilation operators in the asymptotic regions \( x_0 \rightarrow \pm \infty \). Here and afterwards we use the definitions and normalization conditions from the Itzykson and Zuber's book [3].

The \( S \)-matrix element \( S_{\alpha,\beta} \) (2.1) and scattering amplitude \( f_{\alpha,\beta} \) (2.2) consists of the connected \( S^c_{\alpha,\beta} \), \( f^c_{\alpha,\beta} \) and disconnected \( S^d_{\alpha,\beta} \), \( f^d_{\alpha,\beta} \) parts

\[
S_{\alpha,\beta} = S^d_{\alpha,\beta} + S^c_{\alpha,\beta}; \quad f_{\alpha,\beta} = f^d_{\alpha,\beta} + f^c_{\alpha,\beta};
\]

(5.5)

where for the two-body and for the three-body channels we have

\[
S^d_{\alpha+N',b+N} = < \text{in}; a, N'|b, N; \text{in} >
\]

(2.6a)

\[
S^d_{\alpha+\pi',N',b+N} = -(2\pi)^4 i\delta^{(4)}(P_{\alpha+\pi'+N'} - P_{b+N}) f^d_{\alpha+\pi'+N',b+N}
\]

\[
= -(2\pi)^4 i\delta^{(4)}(P_{\alpha+\pi'+N'} - P_{b+N}) \left[ < \text{in}; N'|N; \text{in} > f^1_{\alpha+\pi',b} + < \text{in}; \pi'|b; \text{in} > f^2_{\alpha+N',N} \right]
\]

(2.6b)

\[
S^d_{\alpha+\pi'+N',b+\pi+N} = -(2\pi)^4 i\delta^{(4)}(P_{\alpha+\pi'+N'} - P_{b+\pi+N}) f^d_{\alpha+\pi'+N',b+\pi+N} =
\]
\[
< \text{in}; a, \pi', N'|b, \pi, N; \text{in} > -(2\pi)^4 i \delta^{(4)}(P_{a+\pi'+N'} - P_{b+\pi+N}) \\
< \text{in}; N'|N; \text{in} > f^1_{a+\pi', b+\pi} + < \text{in}; \pi'|b; \text{in} > f^2_{a+N', \pi+N} + < \text{in}; \pi'; \text{in} > f^3_{a+N', b+N}
\]

(2.6c)

Figure 1: The disconnected parts of the two-body S-matrix elements (2.6a) and three-body S-matrix (2.6b), (2.6c) correspondingly in Fig.1A and Fig.1B, Fig.1C. The curled line relates to the \( a = \gamma' \) or \( \pi' \) and \( b = \gamma \) or \( \pi \) asymptotic states, the dashed line describes pion and the solid line stands for the nucleon. The shaded circle corresponds to the vertex function.

where \( f^i \) stands for the following connected amplitudes

\[
f^1_{a+\pi', b} = - < \text{out}; \pi'|j_a(0)|b; \text{in} >; \quad f^2_{a+N', N} = - < \text{out}; N'|j_a(0)|N; \text{in} > \quad (2.7a)
\]

\[
f^1_{a+\pi', b+\pi} = - < \text{out}; \pi'|j_a(0)|b, \pi; \text{in} >; \quad f^2_{a+N', b+N} = - < \text{out}; N'|j_a(0)|b, N; \text{in} >
\]

\(^2\)This condition is one of the axioms in the axiomatic quantum field theory. It can be argued by the following chain of transformations \(|p_A; \text{in} >= \sum_n |n'; \text{out} > < \text{out}; n'|p_A; \text{in} >= \sum_{p'_A} |p'_A; \text{out} > < \text{out}; p'_A|p_A; \text{in} >= |p_A; \text{out} >\), where we have taken into account that the S matrix of the \( 1 \rightarrow n' \) transition does not disappear only for the transition \( 1 \rightarrow 1' \). Therefore \(< \text{out}; p'_A|p_A; \text{in} >= < \text{in}; p'_A|p_A; \text{in} >\).
\[ f_{a+N',\pi+N}^3 = - \langle \text{out}; N' \mid j_a(0) \rangle \pi, N; \text{in} \rangle . \]  

(2.7b)

The graphical representation of Eq. (2.6a,b,c) is given in Fig. 1. In the derivation of Eq. (2.6a,b,c) and Eq. (2.7a,b,c) we have used the one particle stability condition \[ |\alpha; \text{in} \rangle \equiv |p_{A}; \text{in} \rangle = |p_{A}; \text{out} \rangle \] and the condition \[ \langle 0 \mid j_{a}(0) \rangle |\alpha; \text{in} \rangle = 0 . \] It must be noted, that \[ S_{a+\pi',N'+b+N}^d = 0 \] due to the energy-momentum conservation rule and the asymptotic particle stability condition.

We shall find now the equations for the connected part of the scattering amplitude \[ f_{\alpha,\beta}^c \] (2.5). For this aim we insert the complete set of the asymptotic "in" states \[ \sum_n |n; \text{in} \rangle < \text{in}; n \rangle = \hat{1} \] between the current operators in expression (2.2) and after integration over \( x \) we get

\[
f_{\alpha\beta} = W_{\alpha\beta} + (2\pi)^3 \sum_{\gamma=1}^4 f_{\alpha\gamma} \frac{\delta^{(3)}(p_b + P_{\beta} - P_{\gamma})}{\omega_b(p_b) + P_{\beta} - P_{\gamma} + i\epsilon} F_{\gamma}\beta\alpha \]  

(2.8a)

where \( p_b = \omega_b(p_b) \) is the energy of the incoming \( \pi \) meson or photon, \( W_{\alpha\beta} \) contains all contributions of the intermediate states of the \( \beta \rightarrow \alpha \) reaction except the \( s \)-channel \( \gamma = \pi N, \gamma N, \pi \pi N, \gamma \pi N \) exchange diagrams which are included in the second term of the equation (2.8a).

\[
W_{\alpha\beta} = - \langle \text{out}; a | j_{a}(0), a_{b}^{+}(0) | \bar{\beta}; \text{in} \rangle \\
+ (2\pi)^3 \sum_{n=N,3\pi N,...} < \text{out}; a | j_{a}(0) | n; \text{in} \rangle \frac{\delta^{(3)}(p_b + P_{\beta} - P_{n})}{\omega_b(p_b) + P_{\beta} - P_{n} + i\epsilon} < \text{in}; n | j_{b}(0) | \bar{\beta}; \text{in} \rangle \\
+ (2\pi)^3 \sum_{n=N,3\pi N,2\pi N,3\pi N,...} < \text{out}; a | j_{b}(0) | n; \text{in} \rangle \frac{\delta^{(3)}(-p_b + P_{\alpha} - P_{n})}{-\omega_b(p_b) + P_{\alpha} - P_{n}} < \text{in}; n | j_{a}(0) | \bar{\beta}; \text{in} \rangle .
\]  

(2.9a)

This term consist also from the disconnected and connected parts

\[
W_{\alpha\beta} = W_{\alpha\beta}^c + W_{\alpha\beta}^d \]  

(2.9b)

The consistent procedure of extraction of the complete set of a connected terms for the three-dimensional expressions like (2.9a) is well known as cluster decomposition [37, 22]. In Appendix B this procedure is applied to the three-body potential (2.9a), where \( < \text{out}; a \rangle \) are replaced by \( < \text{in}; a \rangle \). The explicit formula for the two-body \( b + N \rightarrow a + N' \) potential are given in Appendix B by Eq.(B.4a)-(B.4h) which are depicted in Fig.2A-Fig.2H respectively. These diagrams have different chronological sequences of the absorption of the initial emission of the final particles. In particular, the \( s \)-channel diagram 2A corresponds to the chain of reactions, where firstly the initial nucleon and \( b = \pi \) or \( \gamma \) transforms into intermediate \( N'' \), \( 3\pi'' N'' \), ... states which afterwards produces the final nucleon and \( a = \pi' \) or \( \gamma' \)-particle states. On the diagram 2B at first the final \( N' \) and the intermediate states \( 2\pi'', 3\pi'', ... \) are generated from the initial \( bN \) states and next we obtain final \( a \) particle from the intermediate \( 2\pi'', 3\pi'', ... \) states. Unlike the diagram
2B, on the diagram 2C the intermediate $2\pi^{''}, 3\pi^{''}, ..$ states arise from the initial pion and afterwards these intermediate states generates the final $aN^{''}$ state together with the initial nucleon. In Fig. 2D we have first the creation of the final nucleon $N'$ with the following absorption of the initial nucleon $N$. Therefore, in this, so called $Z$ diagram [37], the antinucleon intermediate states are appearing. The combination of this $Z$ diagrams with the corresponding nucleon exchange diagrams produces the one nucleon exchange Feynman diagram. Thus the numbers of the particle and antiparticle exchange diagrams in the time-ordered formulations coincides.

Figure 2: The simplest on-mass shell $N, \bar{N}, \pi N, 2\pi, 3\pi, ...$ exchange diagrams which are taken into account in the second and the third term of effective potential (2.9a) for the binary reactions $bN \rightarrow aN'$. The intermediate $2\pi$ and $3\pi$ states can be replaced with the effective $\sigma, \rho, \omega, ...$ heavy meson states. All considered diagrams have the three-dimensional time-ordered form with the "dressed" renormalizable vertices. Therefore they differ from the Feynman diagrams.

The diagrams in Fig. 3 describe the effective potentials $W_{\alpha \beta}^{c}$ (B.8a)-(B.h) for the reactions with the three-body final states $a + \pi' + N'$. The remaining 8 diagram describing $b + N \rightarrow a + \pi' + N'$ reaction can be obtained from the diagrams in Fig. 3A - Fig.3H by transposition of the final $\pi'$ to the other vertex function.

The pure three-body potential $w_{\alpha \beta}^{c}$ of reaction $b + \pi + N \rightarrow a + \pi' + N'$ is determined by Eq.(B.12a)-(B.12h) and is depicted in Fig.4A - Fig.4H. The other 40 time-ordered
Figure 3: Same as in Fig.2, but for the $b + N \rightarrow a + \pi' + N'$ reaction. The next 8 diagrams have the same form, but with $\pi'$ emission from the first vertex function. In the s-channel diagram A the $\pi N, \pi\pi N, \gamma N, \gamma\pi N$ intermediate states are excluded, because they are taken into account in the second, driving term of Eq.(2.8c) or Eq. (2.13).

diagrams for this reaction can be reproduced after crossing of the initial $\pi, \gamma$ and $\pi', \gamma'$-mesons.

The disconnected parts of the potential (2.9a) $W^d_{\alpha\beta}$ together with the disconnected parts of the second term on the right hand of Eq.(2.8a) comprise the disconnected part of amplitudes (2.7a,b). For instance, if we compare the expressions with the noninteracting pion states in Eq.(2.7b) and in Eq.(2.8a), then we obtain

$$< in; \pi'|\pi; in > f^3_{a+N',b+N} \equiv - < in; \pi'|\pi; in > < out; p'_N|j_a(0)|p_N p_b; in >$$

$$= < in; \pi'|\pi; in > \left\{ W^c_{a+N',b+N}+(2\pi)^3 \sum_{n=N''_N,N''_N,...} f_{a+N',n} \frac{\delta^{(3)}(p_N + p_b - P_n)}{\omega_b(p_b)} + \frac{E_{p_N} - P_n + i\epsilon F_{b+N,n}}{E_{p_N} - P_n + i\epsilon} \right\}$$

This means that the disconnected parts on the both side of relation (2.8a) constitute independent equations. Therefore one can separate the connected parts of amplitudes and effective potentials in Eq.(2.8a) as
Figure 4: The first 8 diagrams for the three-body $b\pi N \leftrightarrow a\pi' N'$ reaction. The next 40 diagrams corresponds to the other chronological sequence of absorption of $\pi, \gamma$ and emission $\pi', \gamma'$.

\[
f^c_{\alpha\beta} = W^c_{\alpha\beta} + (2\pi)^3 \sum_{\gamma=1}^{4} f^c_{\alpha\beta\gamma} \frac{\delta^{(3)}(p_b + P_\beta - P_\gamma)}{\omega_b(p_b)} + \frac{P_{\alpha} - P_{\beta} + i\epsilon}{\beta\gamma} F^{\ast c}_{\beta\gamma}.
\] (2.8c)

Thus the disconnected and connected parts of the amplitudes (2.8a) and the effective potentials (2.9a) form an independent set of equations. On the other hand, it is well known, that the potential of a three-body Faddeev equation contains the sum of the disconnected parts and iteration of these disconnected parts contributes in the sought connected three-body amplitude. The same properties have also the effective potentials of the three-body Bethe-Salpeter equations [39, 40, 41] which are constructed in the framework of the graphical method [38]. In the considered $S$-matrix approach the effective potential consists the product of the two complete physical amplitudes and the last cut lemma of the graphical method does not work for this case. Therefore here the cluster decomposition method [37, 22] that separates analytically the connected and disconnected parts of amplitudes was used. As a result of this procedure the disconnected and connected parts of the amplitude (2.5) can be calculated independently from each other according to the Eq.(2.8b) and Eq.(2.8c). In other words, the contributions of the products of the disconnected amplitudes $f^d_{\alpha\beta}$ (Fig.1) are already taken into account in $w^c_{\alpha\beta}$ as it can be observed in Fig.3 and Fig.4. In particular, the combinations of the disconnected parts of amplitudes depicted in Fig.1B and Fig.1C constitute the connected term depicted in
Fig. 3B and in Fig. 4E with the one particle $N''$ or $\pi''$ intermediate states.

Eq. (2.8a) contains two type of transition amplitudes $f_{\alpha\beta}$ (2.2) and

$$F_{\alpha\beta} = -<\text{in}; \tilde{\alpha}|j_{a}(0)|\beta; \text{in}>.$$ (2.10)

The later includes only "in" asymptotic states in contrary to $f_{\alpha\beta}$. In particular, for the two-body states $\alpha = 1', 2' \equiv \pi'N', \gamma'N'$ and an arbitrary initial $\beta$ state, we have

$$F_{\pi'N',\beta} = f_{\pi'N',\beta} = -<\text{out}; p'_{N}|j_{\pi'}(0)|\beta; \text{in}>;$$
$$F_{\gamma'N',\beta} = f_{\gamma'N',\beta} = -<\text{out}; p'_{N}|J_{\kappa'(0)}|\beta; \text{in}>,$$ (2.11a)

because $<\text{out}; p'_{N}| = <\text{in}; p'_{N}|$, but

$$f_{\alpha\beta} \neq F_{\alpha\beta}$$ (2.11b)

for the three-body states $\alpha, \beta = 3, 4 \equiv \pi\pi N, \gamma\pi N$.

For $F_{\alpha\beta}$ (2.10) we can derive similar to Eq.(2.2) relations

$$F_{\alpha\beta} = <\text{in}; \tilde{\alpha}|j_{a}(0), a^{+}(0)|\beta; \text{in}> + i \int d^{4}e^{-ip_{\beta}x} <\text{in}; \tilde{\alpha}|T(j_{a}(0), j_{b}(x))|\beta; \text{in}>,$$ (2.12)

where after insertion of the completeness relation $\sum |n; \text{in}> <\text{in}; n| = \hat{1}$ between the current operators in (2.12) and subsequent integration over $x$ and separation of the connected parts, we get

$$F_{\alpha\beta}^c = w_{\alpha\beta}^c + (2\pi)^{3} \sum_{\gamma=1}^{4} F_{\alpha\gamma}^c \frac{\delta^{(3)}(P_{b} + P_{\beta} - P_{\gamma})}{\omega_{b}(P_{b}) + P_{\beta}^{\prime} - P_{\gamma}^{\prime} + i\epsilon} F_{\beta\gamma}^{c \ast},$$ (2.13)

where $w_{\alpha\beta}^c$ can be obtained from the $W_{\alpha\beta}$ (2.6) after replacement $<\text{out}; \tilde{\alpha}| \leftrightarrow <\text{in}; \tilde{\alpha}|$.

From Eq.(2.12) one can derive also another type of relations between the $F_{\beta\gamma}$ and $f_{\beta\gamma}$ amplitudes. Substituting the complete set of the "out" states $\sum_{n; \text{out}}^{\prime} \langle\text{out}; n| = \hat{1}$ between the current operators in (2.12), we obtain

$$F_{\alpha\beta}^c = \tilde{W}_{\alpha\beta}^c + (2\pi)^{3} \sum_{\gamma=1}^{4} f_{\beta\gamma}^{c \ast} \frac{\delta^{(3)}(P_{b} + P_{\beta} - P_{\gamma})}{\omega_{b}(P_{b}) + P_{\beta}^{\prime} - P_{\gamma}^{\prime} + i\epsilon} f_{\gamma\beta}^{c},$$ (2.14)

where $\tilde{W}_{\alpha\beta}^c$ differs from $w_{\alpha\beta}^c$ (2.6) by the intermediate "out" states.

Equations (2.8c) and (2.13), (2.14) represent the spectral decomposition formulas (or off shell unitarity conditions) for the three-body amplitudes in the standard quantum field theory. Such three-dimensional time-ordered relations are often considered in the textbooks in the quantum field theory [36, 3, 37] and in the nonrelativistic collision theory [1, 2] for the two-body reactions. Therefore, we will treat Eq. (2.8c) and Eq. (2.13), Eq. (2.14) as the three-body generalization of the field-theoretical spectral decomposition formulas (or off shell unitarity conditions) for the two-body amplitudes.
All of the above considered expressions are three-dimensional and time-ordered from the beginning, and the corresponding relativistic equations are often called as the equations of the "old perturbation theory". Equations (2.8c) and (2.13), (2.14) are formulated for the matrix representation of the physical (i.e. renormalized) current operators (2.4a) and their equal-time commutators with the Heisenberg field operators (2.4b,c). These expressions and the left hand sides of Eqs. (2.2), (2.8a) and (2.13) < out(in); \tilde{\alpha}[j_\alpha(0)]\beta; in > are Lorentz-invariant, but due to presence of the step functions \( \theta(\pm x^0) \) in the time-ordered expression (2.2) the propagators of the considered equations (2.8a,b,c) and (2.9a) violated the manifestly Lorentz-covariance form of the considered equations. A time-ordered product is a Lorentz-covariant object, since a Lorentz-transformation cannot change the order of the operators in the time-ordered product and the resulting expressions for the S-matrix [13]. In order to restore the manifestly Lorentz-covariance one can introduce the "co-variant time", i.e. instead of \( x_\alpha \) and \( y_\gamma \) in (2.2) one can use \( X_\alpha = \lambda_\mu x^\mu \) and \( Y_\gamma = \lambda_\mu y^\mu \), where in the c.m. frame \( p_b + p_\bar{\beta} = 0 \) four-vector \( \lambda_\mu \) is time-like unit vector \( \lambda_\mu = (1, 0, 0) \) and in an arbitrary system \( \lambda_\mu = (p_b + p_\bar{\beta})_\mu / |(p_b + p_\bar{\beta})| \). Then in the c.m. system we obtain the expression(2.2) again and in any arbitrary system we will have the covariant propagator with \( \omega_\gamma(k_\gamma) \Rightarrow \lambda_\mu k^\mu_\gamma, \omega_\xi(p_\xi) \Rightarrow \lambda_\mu p_\xi^\mu \) and \( E_{p_\xi} \Rightarrow \lambda_\mu p_\xi^\mu \) etc. [16, 25]. This procedure restores the explicit form of Lorentz-invariance of the considered equations.

Unlike the two-body case, in the three-body formulation it is necessary to operate with the two kind amplitudes \( f_{\gamma\beta} \) (2.2) and \( F_{\gamma\beta} \) (2.10). The advantage of Eq. (2.13) is that it contains only \( F_{\gamma\beta} \). Using the \( T \)-invariance and relations (2.11a), we see, that for the reactions with the two-body initial states \( \beta \equiv \pi N, \gamma N \) \( f_{\gamma\beta} \) (2.2) and \( F_{\gamma\beta} \) (2.10) coincides.

The important part of the effective potential \( w_{\alpha\beta}^c \) is the equal-time commutator

\[
Y_{\alpha\beta} = < out; \tilde{\alpha}[j_\alpha(0), a_\beta^+(0)]|\beta; in >. \tag{2.15}
\]

The explicit form of this expression can be determined from the \textit{a priori} given Lagrangian and equal-time commutations relation between the Heisenberg field operators. In the case of renormalized Lagrangian models or for nonrenormalizable simple phenomenological Lagrangians the equal-time commutators are easy to calculate [25]. In that case potential (2.15) consists of the off shell \textit{internal} one particle exchange potentials (see Fig.5A, Fig.5C and Fig.5E) and of the contact (overlapping) terms (Fig.5B, Fig.5D and Fig.5F). This is the only part of Eq.(2.13) which contains \textit{explicitely} the \textit{internal} particle exchange diagrams, since other terms in the effective potential (2.9a,b) and in Eq.(2.13) consists of the matrix elements of the source operators of the \textit{external} particle operators \( j_a(x), \) and \( j_b(x) \). In order to clarify the structure of the equal-time terms, we will consider Lagrangian of the linear \( \sigma \) model with the electromagnetic fields [3, 5, 37]

\[
\mathcal{L}_I = -\bar{\psi} \gamma^\mu \frac{1 + \tau_3}{2} \psi A_\mu - g_\pi \bar{\psi}[\Phi_\sigma + ig_5 \tau \Phi_\pi] \psi - ie A_\mu \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha \Phi_\beta \partial_\gamma \Phi_\delta + e^2 A_\mu A^\mu \Phi_\pi^2
\]

\[
- \frac{g_\pi^2 (m_\pi^2 - m_\sigma^2)}{4m_N^2} (\Phi_\sigma^2 + \Phi_\pi^2)^2 - \frac{g_\pi (m_\pi^2 - m_\sigma^2)}{2m_N} \Phi_\sigma (\Phi_\sigma^2 + \Phi_\pi^2), \tag{2.16}
\]
where $\Phi_\sigma$ is the auxiliary scalar field operator with mass $m_\sigma$ and $g_\pi$ is defined thought the pion decay constant $f_\pi$ as $g_\pi f_\pi = m_N$.

The corresponding current operator and the equation of motion are

$$\partial_\nu D^\nu A_\mu = J_\mu = -e\psi\gamma^\mu\frac{1+\gamma_5}{2}\psi - ie\varepsilon^{3ij}\Phi_\pi^i\partial_\mu\Phi_\pi^j + e^2 A_\mu\Phi_\pi^2. \quad (2.17a)$$

$$(\partial_\nu + \gamma^\nu\mathcal{M})\Phi_\pi^i = j^i = -g_\pi\bar{\psi}_5 \gamma^i \psi - 2ieA_\mu\varepsilon^{3ij}\Phi_\pi^i + e\partial_\nu A_\mu\varepsilon^{3ij}\Phi_\pi^j + e^2 A_\mu A^\mu\Phi_\pi^j - \frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{m_N^2}(\Phi_\sigma^2 + \Phi_\pi^2)\Phi_\pi^j = \frac{g_\pi(m_\sigma^2 - m_\pi^2)}{m_N}\Phi_\sigma\Phi_\pi^i. \quad (2.17b)$$

Using Eq. (2.4b,c) and the equal-time commutation relation between the pion fields $[\partial_\alpha\Phi_\pi^i(x), \Phi_\pi^j(y)]\delta(x_o - y_o) = -i\delta^{ij}\delta(4)(x - y)$, for $a = \pi'$ and $b = \pi$ we get from Eq.(2.15)

$$Y_{\alpha\beta} = 2e\bar{p}_{\pi}^\mu\varepsilon^{3ij} < \text{out}; \bar{\alpha}|A_\mu(0)|\bar{\beta}; \text{in} >_c + < \text{out}; \bar{\alpha}|e^2A_\nu(0)A^\nu(0) + e\partial_\nu A_\mu(0)|\bar{\beta}; \text{in} >_c$$

$$-\frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{2m_N^2} < \text{out}; \bar{\alpha}|(\Phi_\sigma^2(0) + \Phi_\pi^2(0))\delta^{ij} + 2\Phi_\pi^j(0)\Phi_\pi^j(0)|\bar{\beta}; \text{in} >_c$$

$$-\frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{m_N} < \text{out}; \bar{\alpha}|\Phi_\sigma(0)|\delta^{ij}\bar{\beta}; \text{in} >_c$$

$$\approx -\frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{2m_N^2} < \text{out}; \bar{\alpha}|(\Phi_\sigma^2(0) + \Phi_\pi^2(0))\delta^{ij} + 2\Phi_\pi^j(0)\Phi_\pi^j(0)|\bar{\beta}; \text{in} >_c$$

$$-\frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{m_N} \delta^{ij} < \text{out}; \bar{\alpha}|j_\mu(0)|\bar{\beta}; \text{in} >_c$$

$$-\frac{g_\pi^2(m_\sigma^2 - m_\pi^2)}{(P_\alpha - P_\beta)^2 - m_\pi^2} \delta^{ij} < \text{out}; \bar{\alpha}|A_\mu(0)|\Phi_\pi^i(0)\bar{\beta}; \text{in} >_c$$

$$\epsilon^\gamma_\eta(k_\gamma) \quad (2.18a)$$

where $\bar{\alpha} = \pi'N$ or $\pi'N'$ or $\gamma'N$; $\bar{\beta} = N$ or $\pi N$ or $\gamma N$ and we have omitted terms with the intermediate photons.

The first term of Eq.(2.18a) corresponds to the contact (overlapping) potential which is depicted in Fig.5B, Fig.5D and Fig.5F. Second term describes the isoscalar $\sigma$-meson exchange interaction and it is given in Fig.5A, Fig.5C and Fig.5E. For the other kind equal-time commutators with $b = \pi$ ($\bar{\beta} = N$ or $\pi N$ or $\gamma N$) and $a = \gamma'$ ($\bar{\alpha} = N'$ or $\pi'N'$) we have

$$Y_{\alpha\beta} = 2\left[-ie\varepsilon^{3ij} < \text{out}; \bar{\alpha}|(i\partial_\mu - (p_\pi)_\mu)\Phi_\pi^j(0)|\bar{\beta}; \text{in} >_c + e^2 < \text{out}; \bar{\alpha}|A_\mu(0)\Phi_\pi^j(0)|\bar{\beta}; \text{in} >_c\right] \epsilon^\gamma_\eta(k_\gamma)$$

$$=-2e\varepsilon^{3ij}(P_\alpha - P_\beta)_\mu < \text{out}; \bar{\alpha}|j_\mu(0)|\bar{\beta}; \text{in} >_c \epsilon^\gamma_\eta(k_\gamma) + 2e^2 < \text{out}; \bar{\alpha}|A_\mu(0)\Phi_\pi^j(0)|\bar{\beta}; \text{in} >_c \epsilon^\gamma_\eta(k_\gamma). \quad (2.18b)$$

For $b = \gamma$ and $a = \pi'$
\[ Y_{\alpha\beta} = -2ie^{3ij} < \text{out}; \bar{\alpha}|\partial_\mu \Phi^j(0)|\bar{\beta}; \text{in} > e^\mu(k_\gamma) + 2e^2 < \text{out}; \bar{\alpha}|A_\mu(0)\Phi^j(0)|\bar{\beta}; \text{in} > e^\mu(k_\gamma) \]

\[ = -2ie^{3ij}(P_\alpha - P_\beta)^\mu < \text{out}; \bar{\alpha}|j^j_\pi(0)|\bar{\beta}; \text{in} > e^\mu(k_\gamma) + 2e^2 < \text{out}; \bar{\alpha}|A_\mu(0)\Phi^j(0)|\bar{\beta}; \text{in} > e^\mu(k_\gamma), \]

(2.18c)

Figure 5: The graphical representation of the equal-time commutators (2.15) in the effective potential (2.9a). This term is depicted separately for the binary reactions A, B, for the reactions 2 \(\Rightarrow\) 3' C, D and for the three-body reactions 3 \(\Leftarrow\Rightarrow\) 3' E, F. Diagrams A, C, E correspond to one off-mass shell particle exchange interactions. The vertex functions between the \((\gamma, \pi)\) and \((\gamma', \pi')\) states are given in the lowest, tree approximation. Diagrams B, D, F describe the contact (overlapping) interaction which does not contain the intermediate hadron propagation between hadron states.
and finally for $b = \gamma$ and $a = \gamma'$

$$Y_{\alpha\beta} = e^2 < q | \Phi_\pi(0) | \tilde{\beta}; in > \epsilon_{\mu\nu}(k'_{\gamma}) \epsilon_{\gamma}(k_{\gamma})$$

(2.18d)

consists of a contact (overlapping) term only. If we replace $\Phi_\pi^2$ by a new type auxiliary scalar field $\sigma'$, then expression (2.18d) transforms into one $\sigma'$ scalar meson exchange diagram. Using more complete Lagrangian, one can obtain also heavy $\rho, \omega$ meson exchange diagrams [25, 24]. Moreover, in the Ref.[26, 25] the One Boson Exchange (OBE) Bonn model of $NN$ interaction was exactly reproduced from the equal-time commutator. For the three-body asymptotic states equal-time commutators (2.15) produce more complicated contact terms (Fig. 5C, Fig.5D and Fig. 5E, Fig.5F) which can be treated as pure three-body forces. For these terms with the $\sigma, \rho, \omega, ...$ field operators it is necessary to use corresponding derivation of the spectral decomposition formulas like Eq.(2.13). Note, that for the two-body reactions equal-time commutators are determined by the one-variable vertex functions with two on mass shell particles. For the three-body reactions the equal-time commutators are defined through the scattering amplitudes with three or four on-mass shell particles.

Thus starting from the Lagrangian of the well-known linear $\sigma$-model (2.16) we have obtained the one-particle exchange potentials (Fig. 5A, Fig. 5C and Fig. 5E) and contact (overlapping) terms. For the binary reactions these terms contribute in the potential of the considered equations. Unfortunately, the explicit expression of the one-variable vertices on Fig. 5A and Fig. 5B are not well determined. Using the model Lagrangians we can extract the coupling constants i.e.e we can obtain the threshold values of these vertex functions. For the $\gamma NN$ and $\gamma \pi \pi$ systems these vertices are determined from the experimental data. The $\pi NN$ vertex functions can be constructed from the dispersion relations. Apart from this, we can determine the asymptotic behavior of these vertex functions based on the quark counting rules[42, 43] and dispersion relations [44, 45, 46, 47, 48] or on the Regge trajectories theory [50]. Another possibility to determine the equal-time commutators in the two-body reactions, using the inverse scattering method, is considered in Ref.[49].

Presently, there exist numerous phenomenological models of the $NN$, $\pi N$ and $\gamma N$ reactions, where the pieces of Lagrangian (2.16) are used by construction of the effective potentials by the description of the corresponding reactions. Therefore in the framework of the considered formulation it seems possible to achieve a unified description of the coupled $\pi N - \gamma N - \pi \pi N - \gamma \pi N$ reactions proceeding from the generalized Lagrangian (2.16) with the $\rho, \omega, ...$ degrees of freedom. On the other hand, the Lagrangian of the linear $\sigma$-model can be reproduced from a set of more complicated and complete QCD motivated Lagrangians. Thus the unified description of the coupled $\pi N - \gamma N - \pi \pi N - \gamma \pi N$ reactions allows us to determine the form of the equivalent Lagrangians for the $\gamma \pi N$ interactions which are sufficient and necessary for a description of the experimental observables in the low and intermediate energy region.

In the quantum field theory with the quark-gluon degrees of freedom one can construct the hadron creation and annihilation operators as well as the Heisenberg field operators of hadrons from the quark-gluon fields in the framework of the Haag-Nishijima-Zimmermann [51, 52, 53, 54] treatment. In this case the form of the hadron quantum field operators
is changed, but equations (2.8c), (2.13), (2.14) and (2.15) remain the same [25, 32, 30] and one can separate again the one off-mass shell meson exchange diagrams in Fig.5A, Fig.5C and Fig.5E [30]. The contribution of the overlapping (contact) terms in Fig.5B, Fig.5D and Fig.5F can be estimated using the pure quark-gluon exchange or overlapping (contact) terms.

3. The relativistic Lippmann-Schwinger type equations for the multichannel amplitude $F_{\alpha\beta}$

The Lippmann-Schwinger type equations for the multichannel scattering $t$-matrix $T_{\alpha\beta}(E)$ with the Hermitian potential $V = V^+$ have the following form [1, 2]

$$T_{\alpha\beta}(E) \equiv \langle in; \alpha | T(E) | \beta \rangle; \langle in \rangle = V_{\alpha\beta} + \sum_\gamma \frac{1}{E_{\beta} - E_{\gamma} + i\epsilon} T_{\gamma\beta}(E)$$  \hspace{1cm} (3.1a)

$$= V_{\alpha\beta} + \sum_{\gamma, \gamma'} V_{\alpha\gamma} \langle in; \gamma \rangle \frac{1}{E_{\beta} - H + i\epsilon} | \gamma'; \langle in \rangle > V_{\gamma'\beta},$$  \hspace{1cm} (3.1b)

where $\langle in; \alpha |$ denotes the asymptotic $\alpha$-channel wave function with the energy $E_\alpha$ and quantum numbers $\alpha$, $\sum_\gamma$ stands for the integration over the momenta and the summation over the quantum numbers of the complete set intermediate $\gamma$-channel states, $H$ is the full Hamiltonian which has the complete set of the eigenfunctions $H | \Psi_\gamma > = E_\gamma | \Psi_\gamma >$; $\sum_\gamma | \Psi_\gamma > < \Psi_\gamma | = 1$. Using the decomposition formula of the full Green function $G(E) = 1/(E - H + i\epsilon)$ over the complete set of the functions $| \Psi_\gamma >$

$$G(E) = \sum_\gamma \frac{| \Psi_\gamma > < \Psi_\gamma |}{E - E_\gamma + i\epsilon},$$  \hspace{1cm} (3.2)

equations (3.1a) and (3.1b) can be written as the quadratically nonlinear integral equations [1, 2]

$$T_{\alpha\beta}(E) = V_{\alpha\beta} + \sum_\gamma T_{\alpha\gamma} \frac{1}{E_{\beta} - E_{\gamma} + i\epsilon} T_{\gamma\beta}^*,$$  \hspace{1cm} (3.3)

where we have taken into account the relation $T_{\alpha\beta}(E) = \langle in; \alpha | V | \Psi_\beta >$.

The three-dimensional equations (3.3) have the same form as equations (2.13). In spite of the great complexity of the nonlinear equations (3.3), they have very attractive properties, since they can be considered as the off shell generalization of the unitarity conditions [1] and, unlike the Lippmann-Schwinger type equations (see the book of Goldberger and Watson, ch.5, Eq. (88)-Eq.(100)), they are free from the difficulty of construction of the orthonormal set of the intermediate channel states. Therefore they may be considered as a basis of derivation of the special Lippmann-Schwinger type equations which have no inherent pathological properties. In the present formulation the nonlinear field-theoretical equations (2.13) are considered as the origin for a derivation of the linearized relativistic Lippmann-Schwinger type equations.
Unlike the nonrelativistic case, potential $w^c_{\alpha\beta}$, is not Hermitian due to the propagators of intermediate states. Nevertheless, in appendix A we show that quadratically nonlinear equations (2.10) are equivalent to the following Lippmann-Schwinger type equations

$$T_{\alpha\beta}(E_\beta) = U_{\alpha\beta}(E_\beta) + \sum_{\gamma=1}^{4} U_{\alpha\gamma}(E_\beta) \frac{1}{\omega_{\beta}(p_\beta) - P_\beta^0 + i\epsilon} T_{\gamma\beta}(E_\gamma),$$

(3.4)

where $P_\beta^0 \equiv E_\beta = \omega_\beta(p_\beta) + P_\beta^0$ and for the sake of simplicity we have omitted the total three-momentum conservation $\delta$ function $(2\pi)^3 \delta(P_\beta - P_\gamma)$.

The explicit form of the linear energy depending potential

$$U_{\alpha\beta}(E) = A_{\alpha\beta} + E B_{\alpha\beta}$$

(3.5)

with Hermitian $A$ and $B$

$$A_{\alpha\beta} = A_{\beta\alpha}^*; \quad B_{\alpha\beta} = B_{\beta\alpha}^*,$$

(3.6)

is defined in the Appendix B. $U_{\alpha\beta}(E)$ is simply connected with the $w^c_{\alpha\beta}$-potential (B.4a)-(B.4h), (B.8a)-(B.8h) and (B.12a)-(B.12h)

$$U_{\alpha\beta}(E_\alpha) = w^c_{\alpha\beta}.$$  

(3.7)

Therefore, for any field-theoretical potential $w^c_{\alpha\beta}$ on can unambiguously construct $U_{\alpha\beta}(E)$. The graphical form of the potential is depicted in Fig.2, Fig.3 and Fig.4 for the two-body $\alpha, \beta = 1, 2 = \pi N, \gamma N$ and the three-body $\alpha, \beta = 3, 4 = \pi \pi N, \gamma \pi N$ transition channels.

On energy shell solutions of the equations (2.10) and (3.4) coincide

$$T_{\alpha\beta}(E_\beta = E_\alpha) = F^c_{\alpha\beta}|_{E_\beta=E_\alpha}$$

(3.8)
and in the half on energy shell region these amplitudes are simply connected

$$F^c_{\alpha\beta} \equiv -<in; \bar{\alpha}|j_a(0)|\beta; in>_{\text{connected}}=w^c_{\alpha\beta} + \sum_{\gamma=1}^4 w^c_{\alpha\gamma} \frac{1}{E_{\beta} - E_{\gamma} + i\epsilon} T_{\gamma\beta}(E_{\beta}).$$ (3.9)

Equations (3.4) are our final equations for the multichannel $\pi N \leftrightarrow \gamma N \leftrightarrow \gamma \pi N$ scattering amplitudes, because their solution allows us to determine $F^c_{\alpha\beta}$ (2.8c) that on energy shell coincides with the solution of Eq. (2.13), and in the off energy shell region $F^c_{\alpha\beta}$ is determined through $T_{\gamma\beta}(E_{\beta})$ according to Eq. (3.9). For construction of the complete $3 \rightarrow 3'$ transition amplitude it is necessary to take into account the disconnected parts (Fig. 1C) and to use Eq. (2.14) or (2.8). Equation (3.4) have the form of the multichannel Lippmann-Schwinger equations with the connected potentials $U_{\alpha\beta}(E)$ that are single-valued determined by the potentials of the initial field-theoretical equation (2.13) $w^c_{\alpha\beta}$. The structure of the system of relativistic equations (3.4) is more simple as the field-theoretical generalizations of the Faddeev equations, based on the Bethe-Salpeter equations [40, 41] or in the framework of the other relativistic three-dimensional formulations (see the list of the corresponding quotations in book [39]), because $U_{\alpha\beta}(E)$ does not contain any disconnected parts and the complications, coming from double counting problems (i.e. from the iterations of the disconnected parts of the three-body potentials), does not take a part in this approach.\(^4\)

The essential difference between the nonrelativistic Lippmann-Schwinger equation [1, 2] their relativistic field-theoretical generalization (3.4) is the nonlinearity of the field-theoretical equations. The most famous nonlinearity was investigated in the nonlinear

\(^4\)Instead of Eq. (2.13) we can consider the quadratically-nonlinear three-body equations for the sum of the connected and the disconnected parts of amplitudes

$$F_{\alpha\beta} = w_{\alpha\beta} + (2\pi)^3 \sum_{\gamma=1}^4 F_{\alpha\gamma} \frac{\delta^{(3)}(p_\alpha + p_\beta - p_\gamma)}{\omega_{\beta}(p_\beta) + P_{\beta}^o - P_{\gamma}^o + i\epsilon} F_{\gamma\beta}^*, \hspace{1cm} (2.13')$$

where $F_{\alpha\beta} = F^c_{\alpha\beta} + F^d_{\alpha\beta}$ denotes the complete $\beta \rightarrow \alpha$-transition amplitude and Eq. (2.13’) have the form of the off-shell unitarity conditions [1]. Equation (2.13’) can be considered also as the basis for the derivation of the field-theoretical generalizations of the Faddeev-type equations. For this aim we must derive Eq. (2.13’) from the Faddeev-type equations after rearrangement of the three body amplitude $T_{\alpha=3'; \beta=3}(E)$ over the auxiliary amplitudes $T_{\alpha=3'; \beta=3}(E) = \sum_{i=1,3} T_{\gamma \beta,3}^{123}(E) = T_{\gamma \beta,3}^{123}(E) + T_{\gamma \beta,3}^{123}(E)$, where $T_{\alpha\beta}(E)$ denotes the disconnected part of amplitude and $T_{\gamma \beta,3}^{123}(E)$ is introduced for the pure three-body interactions [1]. This means that we must construct the complete operator $U_{\alpha\beta}(E)$ with the connected and the disconnected parts from $w_{\gamma \beta,3}$. $U_{\alpha\beta}(E)$ can contain only the three-body irreducible diagrams, i.e. diagrams which does not include the 1,2,3-particle exchange contributions in the $s$-channel. As result we will reproduce the Lippmann-Schwinger-type equations

$$T_{\alpha\beta}(E_{\beta}) = U_{\alpha\beta}(E_{\beta}) + \sum_{\gamma} U_{\alpha\gamma}(E_{\beta}) \frac{1}{\omega_{\beta}(p_{\beta}) + P_{\beta}^o - P_{\gamma}^o + i\epsilon} T_{\gamma\beta}(E_{\beta}), \hspace{1cm} (3.4')$$

The detailed investigation of this equation and their comparison with Eq. (3.4) we plane in the forthcoming papers.
Chew-Low equations [14, 21, 22, 23, 24, 25] for the $\pi N$ scattering problem. It was demonstrated, that even the $s$-channel nonlinearity in Eq.(3.3) generates the infinite sum of the Castilleho-Dalitz-Dyson (CDD) poles [55] in the $\pi N$ amplitude. Subsequently, the position of these poles was determined in the dispersion theory with the quark-gluon degrees of freedom [56] and in Ref.[23] one of the CDD poles was used to reproduce the $\Delta$ resonance. In the book of Goldberger and Watson [2] it was argued the uniqueness of the solution of nonlinear (3.3) and linear (3.1a) equations. The argumentation of such type assertion were given also in Ref.[57], where it was shown, that if the spectrum of the free Hamiltonian does not contain any compound or exited states, then the conditions of completeness and orthonormality of the wave functions, that are the solution of the Lippmann-Schwinger equations (3.1a), are compatible with the solution of the nonlinear Chew-Low equations without CDD poles i.e. solutions of linear Lippmann-Schwinger equations (3.4) and nonlinear equations (3.3) are identical. Also in Ref.[58] for the Chew-Low equations it was shown, that under special conditions imposed on the coupling constants and vertex functions the Chew-Low type equations have a unique solution.

From the explicit form of the $u^c_{\alpha\beta}$-potentials (B4a)-(B.4h), (B.8a)-(B.8h) and (B.12a)-(B.12h) and their graphical representation in Fig. 2, Fig.3, Fig.4 and Fig.5 we see, that besides the $u$-channel crossing terms in Fig.2E, Fig.3E and Fig.4E, the unknown amplitudes placed also in all other potential terms except the diagrams with the antinucleon exchange. Even the terms with $\pi$-meson intermediate states from the equal-time commutators in Fig.5C, Fig.5D, Fig.5E and Fig.4G generate the nonlinearity. In Ref.[24] it was shown, that also the potential in the Bethe-Salpeter equation contains the nonlinearity in the crossed $u$-channel diagram for the $\pi N$ scattering reactions. One can show, that the other kind of nonlinearities, depicted in Fig.2, Fig.3, Fig.4 and Fig.5 arise in the Bethe-Salpeter equation too. The source of such type nonlinearities is the field-theoretical nature of the considered approaches, that originate the essential non-linear functional equations [13, 3], which are the base of the above integral equations. Therefore the problem of the role of these nonlinearities in the considered reactions is the important problem for the future investigations.

4. $\Delta$ degrees of freedom.

The relativistic field-theoretical equations (2.13) and (3.4) are the multichannel generalization of the equations for the elastic $\pi N$ scattering from ref.[24, 25], where all of $\pi N$ partial waves up to 300 MeV, including the resonance $\pi N P33$ partial wave, were described. The effective $\pi N$ potentials there was constructed from the one-variable phenomenological $\pi N - N$, $\sigma N - N$, $\rho N - N$ and $\rho - \pi\pi$ vertex functions and corresponding simplest phenomenological Lagrangians. In the considered generalization of the Chew-Low model [20, 21, 22, 23] the contributions of the intermediate $\Delta$ resonance in the low-energy $\pi N$ scattering reaction was reproduced without any additional assumptions. Another way to taken into account the intermediate $\Delta$-resonance effects is to introduce $\Delta$-degrees of freedom, i.e. to construct the intermediate $\Delta$-resonance in the same manner as the usual one-particle hadron states. Unfortunately, it is not possible to construct the
representation of the Poincare group for the unstable states, because the asymptotic "out" or "in" states (i.e., the Fock space) is determined for the stable particles in the asymptotic region \( x_o \to \pm \infty \). Therefore, we will treat the \( \Delta \) degrees of freedom as intermediate \( \pi N \) cluster states using the procedure of the separation of the \( \Delta \) resonance poles from the intermediate \( \pi N \) Green functions in the \( P33 \) partial states. In particular, following our previous papers [32, 35], we firstly separate the intermediate full \( \pi N \) Green functions in the matrix elements (2.2) for the transition between the \( \hat{\alpha} + a \) and \( \hat{\beta} + b \) states. For instance the matrix element \( \langle \text{out}; \hat{\alpha}|j_a(0)\theta(-x_o)j_b(x)|\hat{\beta}; \text{in} \rangle > \) with the \( \pi N \) intermediate state can be transformed as

\[
\sum_{\pi N} \langle \text{out}; \hat{\alpha}|j_a(0)|p_{\pi}p_{\pi N}; \text{in} \rangle > \frac{\delta(p_{\beta} + p_{\pi_N} - p_{\pi} - p_{\pi N})}{p_{\beta}^o - \omega_b(p_b) - \omega(p_\pi) - E_{p_N} + i\epsilon} < \text{in}; p_{\pi}p_{\pi N}|j_b(0)|\hat{\beta}; \text{in} \rangle >
\]

\[
= \sum_{\pi N} \langle \text{out}; \hat{\alpha}|j_a(0)|p_{\pi}p_{\pi N}; \text{in} >_{\pi N \text{ irreducible}} \rangle > \frac{\mathcal{G}_{\pi N}(E = p_{\beta}^o + \omega_b(p_b))}{p_{\beta}^o - \omega_b(p_b) - \omega(p_\pi) + i\epsilon} < \text{in}; p_{\pi}p_{\pi N}|j_b(0)|\hat{\beta}; \text{in} >_{\pi N \text{ irreducible}}, \tag{4.1}
\]

where we have to distinguish the full \( \pi N \) wave function \( |\Psi_{\pi N} > \) and full \( \pi N \) Green function

\[
\mathcal{G}^{\pi N}(E, \mathbf{P}) = \int d^3p_{\pi}d^3p_{\pi N}(2\pi)^3 \delta(\mathbf{P} - \mathbf{p}_N - \mathbf{p}_{\pi}) \frac{|\Psi_{\pi N}^{p_{\pi p_{\pi N}}} > < \tilde{\Psi}_{\pi N}^{p_{\pi p_{\pi N}}}|}{E - E_{p_N} - \omega(\mathbf{p}_\pi) + i\epsilon} \tag{4.2a}
\]

using the "\( \pi N \) irreducible" matrix elements which does not contain the intermediate \( \pi N \) states

\[
\langle \text{out}; \hat{\alpha}|j_a(0)|p_{\pi}p_{\pi N}; \text{in} \rangle = \left\{ \langle \text{out}; \hat{\alpha}|j_a(0) \rangle \right\}_{\pi N \text{ irreducible}} |\Psi_{\pi N}^{p_{\pi N}p_{\pi}} >; \tag{4.3a}
\]

\[
\langle \pi N \rangle_{\pi N} \text{ irreducible} \]

\[
\langle \text{in}; p_{\pi}p_{\pi N}|j_b(0)|\hat{\beta}; \text{in} \rangle = \langle \tilde{\Psi}_{\pi N}^{p_{\pi N}p_{\pi}}|j_b(0)|\hat{\beta}; \text{in} \rangle \left\{ \langle j_b(0)|\hat{\beta}; \text{in} \rangle \right\}_{\pi N \text{ irreducible}}. \tag{4.3b}
\]

The wave functions \( |\Psi_{\pi N} > \) are simply connected with the \( t \)-matrices \( T_{\alpha\beta}(E) \) (3.4)

\[
T_{\pi N;\pi N}(E_{\pi N}) \equiv T_{11}(E_1) = \sum_{\gamma = 1}^4 \langle \pi' N \pi' p_{\pi}|U_{1\gamma}(E_{\pi N})|\gamma; \text{in} \rangle < \text{in}; \gamma|\Psi_{\pi N}^{p_{\pi N}p_{\pi}} >. \tag{4.4}
\]

where \( E_{\pi N} = E_{p_N} + \omega_\pi(\mathbf{p}_\pi) \).

According to Eq. (A.1) and normalization condition (A.5b) the wave function \( \langle \text{in}; \beta|\Psi_{\alpha=1=\pi N} > \) satisfies the following equation of motion

\[
\left( E_{p'_{\pi}} + \omega_\pi(\mathbf{p}'_{\pi}) - E_{p_N} - \omega_\pi(\mathbf{p}_\pi) \right) \langle \text{in}; p'_{\pi}p'_{\pi}|\Psi_{\pi N}^{p_{\pi N}p_{\pi}} > = \sum_{\gamma = 1}^4 \langle \text{in}; p'_{\pi}p'_{\pi}|U_{1\gamma}(E_{p_N} + \omega_\pi(\mathbf{p}_\pi))|\gamma; \text{in} \rangle < \text{in}; \gamma|\Psi_{\pi N}^{p_{\pi N}p_{\pi}} >. \tag{4.5a}
\]
and the normalization condition
\[ < \Psi_{\pi'N'} | (1 - B) | \Psi_{\pi N} > \equiv < \tilde{\Psi}_{\pi'N'} | \Psi_{\pi N} > = \delta_{\pi'N',\pi N}. \] (4.6)

Next we consider two kinds of extensions (or projections) the equation (2.5a) for the initial \( \pi N \) state energy in the complex region as
\[ E_{\pi N} = E_{p_N} + \omega_{\pi}(p_{\pi}) \implies E_{\Delta} \] (4.7a)
and
\[ E_{\pi N} = E_{p_N} + \omega_{\pi}(p_{\pi}) \implies E_{o \Delta} + \Sigma_{\Delta}(E, P_{\Delta}) \] (4.7b)
where \( P_{\Delta} = p_N + p_{\pi}, E_{o \Delta} = \sqrt{M_{\Delta}^2 + P_{\Delta}^2} \) and \( E_{\Delta} = \sqrt{M^2 + \tilde{P}_{\Delta}^2} \) are the energies of the bare and observed energies of \( \Delta \) with the bare \( M_{\Delta} \) and the Breit-Wigner (physical) mass \( M_{\Delta} = m_{\Delta} + i \Gamma_{\Delta}/2 \) mass of \( \Delta \) respectively, where \( m_{\Delta} = 1232\text{MeV} \) and \( \Gamma_{\Delta} = 120\text{MeV} \) are the observed mass and full width of \( \Delta \) resonances, \( \Sigma_{\Delta}(E, P_{\Delta}) \) is the complex function of \( E \) which satisfies the conditions
\[
Re \left[ E - E_{o \Delta} - \Sigma_{\Delta}(E, P_{\Delta}) \right]_{E = E_{o \Delta}} = 0 \] (4.8a)
\[
Im \left[ E - E_{o \Delta} - \Sigma_{\Delta}(E, P_{\Delta}) \right]_{E = E_{o \Delta}, P_{\Delta} = 0} = \Gamma_{\Delta}/2 \] (4.8b)

Using the extension (4.7a) we obtain the wave function of the on shell \( \Delta \)
\[ |\Psi_{\pi N_{PN}}\rangle \xrightarrow{E_{\pi N} \rightarrow E_{\Delta}} |\Psi_{\Delta}\rangle \] (4.9a)
which obeys the equation
\[
\left( E_{\pi N}(p'_{N}) + \omega_{\pi}(p'_{\pi}) - E_{\Delta} \right) |\Psi_{\Delta}\rangle = \sum_{\gamma=1}^{4} \langle in; p'_{N}p'_{\pi} | U_{1'\gamma}(E_{\Delta}) | \gamma \rangle |\Psi_{\Delta}\rangle \] (4.5b)

Similarly, the off shell projection of Eq.(4.5a) according to Eq.(4.7b) allows us to determine the explicit form of the wave function \( |\Psi_{\Delta}\rangle \)
\[ |\Psi_{PN_{PN}}\rangle \xrightarrow{E_{\pi N} \rightarrow E_{o \Delta} + \Sigma_{\Delta}(E, P_{\Delta})} |\Psi_{\Delta}(E)\rangle . \] (4.9b)
as the solution of the extended equation
\[
\left( E - E_{o \Delta} - \Sigma_{\Delta}(E, P_{\Delta}) \right) |\Psi_{\Delta}(E)\rangle = \sum_{\gamma=1}^{4} \langle in; p'_{N}p'_{\pi} | U_{1'\gamma}(E_{o \Delta} + \Sigma_{\Delta}(E, P_{\Delta})) | \gamma \rangle |\Psi_{\Delta}(E)\rangle . \] (4.5c)
Now one can pick out the $\Delta$ resonance part from the fully $\pi N$ Green function (4.2a) using the determination of the resonances as poles on the complex $\pi N$ energy sheet. For this aim it is enough to separate the contribution of the $\Delta$ resonance singularity at $E_{P\Delta}$ (4.7a) or at $E_{o\pi N}^{\Delta} + \Sigma_{\Delta}(E, P_{\Delta})$ (4.7b) in the integral (4.2a). The residue of this pole consists of the product of the two $\pi N - \Delta$ wave functions which are analytically continued with the $\pi N$ wave function (4.5a) according to projections (4.7a) or (4.9b). Thus from the Eq.(4.2a) we obtain

$$G^{\pi N}(E, P) = \int d^3P_{\Delta}(2\pi)^3 \delta(P - P_{\Delta}) \sum_{\Delta} \left| \frac{\Psi_{P\Delta}^\Delta}{E - E_{P\Delta}} \right| + \text{nonresonant part.} \quad (4.2b)$$

and

$$G^{\pi N}(E, P) = \int d^3P_{\Delta}(2\pi)^3 \delta(P - P_{\Delta}) \sum_{\Delta} \left| \frac{\Psi_{P\Delta}^\Delta(E)}{E - E_{o\pi N}^{\Delta} - \Sigma_{\Delta}(E, P_{\Delta})} \right| + \text{nonresonant part.} \quad (4.2c)$$

The essential feature of the different representation of the $\pi N$ Green function (4.2b) and (4.2c) is that they differently take into account $\Delta$ degrees of freedom in the intermediate states. For instance, the general formula (4.1) for the $s$-channel $b + \bar{\beta} \to a + \bar{\alpha}$ transitions, after replacement of the fully $\pi N$ Green function (4.2a) with their resonance part (4.2b), takes the form

$$\sum_{\Delta} <\text{out}; \bar{\alpha}|j_a(0)|\Psi_{P\Delta}^\Delta >_{\pi N \text{ irreducible}} \frac{\delta(P_{b} + P_{\bar{\beta}} - P_{\Delta})}{P_{\bar{\beta}}^\rho + \omega_b(P_b) - E_{P\Delta}} <\bar{\Psi}_{P\Delta}^\Delta|j_b(0)|\bar{\beta}; \text{in} >_{\pi N \text{ irreducible}}$$

and from Eq.(4.2c) we obtain

$$\sum_{\Delta} <\text{out}; \bar{\alpha}|j_a(0)|\Psi_{P\Delta}^\Delta (P_{\bar{\beta}}^\rho + \omega_b(P_b)) >_{\pi N \text{ irreducible}}$$

$$\frac{\delta(P_{b} + P_{\bar{\beta}} - P_{\Delta})}{P_{\bar{\beta}}^\rho + \omega_b(P_b) - E_{P\Delta}^{\bar{\beta}} - \Sigma_{\Delta}(P_{\bar{\beta}}^\rho + \omega_b(P_b), P_{\Delta})} <\bar{\Psi}_{P\Delta}^\Delta (P_{\bar{\beta}}^\rho + \omega_b(P_b))|j_b(0)|\bar{\beta}; \text{in} >_{\pi N \text{ irreducible}}.$$  

(4.10a)

(4.10b)

In the both expressions (4.10a) and (4.10b) we have neglected the nonresonant part of the $P_{33}$ $\pi N$ partial wave contributions.

In Eq.(4.10a) intermediate $\Delta$ propagators can be considered as on shell propagators, because only the Breit-Wigner mass and width are taken into account in the energy of $\Delta$. The more general expression (4.10b) includes the renormalization effects in the mass operator $\Sigma_{\Delta}(E, P_{\Delta})$ which satisfies the conditions (4.8a,b). The explicit form of $\Sigma_{\Delta}(E, P_{\Delta})$ depends on the model of interaction of the ingredient quark-gluon and pion-nucleon fields. An overview of these models is out of the scope of the present paper.
In order to demonstrate the suggested recipe of the construction of the intermediate \( \Delta \) propagators we consider the separable model for the \( \pi N \) potential \( V \) and \( t \)-matrix for the \( P33 \) partial wave in the c.m. frame \( p_N = -p_\pi = p \)

\[
V(p', p) = \lambda g(p')g(p); \quad t(p', p, E) = \frac{\lambda g(p')g(p)}{D(E)}; \quad (4.11a)
\]

\[
D(E) = \lambda \int \frac{|g(p)|^2 p^2 dp}{E + i\epsilon - E_{p_N} - \omega_\pi(p)}, \quad (4.11b)
\]

where the \( t \)-matrix satisfies the usual Lippman-Schwinger equation which is valid also for the \( \pi N \) wave function

\[
\langle p' | \Psi_p \rangle = \frac{\delta(p' - p)}{p'p} + \int E_{p_N} + \omega_\pi(p) + i\epsilon - E_{q_N} - \omega_\pi(q) < q | \Psi_p > \quad (4.12a)
\]

and has the following solution

\[
\langle p' | \Psi_p \rangle = \frac{\delta(p' - p)}{p'p} + \lambda \frac{g(p')g(p)}{D(E_{p_N} + \omega_\pi(p))}. \quad (4.12b)
\]

Using the procedure (4.7a,b) we get

\[
|\Psi_{\pi N}^{p_N p_\pi} \rangle \xrightarrow{E_{\pi N} \rightarrow M_\Delta} |\Psi_\Delta \rangle = \lambda \frac{g(p')g(M_\Delta)}{D(M_\Delta)} \quad (4.13a)
\]

\[
|\Psi_{\pi N}^{p_N p_\pi} \rangle \xrightarrow{E_{\pi N} \rightarrow M_\Delta + \Sigma_\Delta(E)} |\Psi_\Delta \left(M_\Delta + \Sigma_\Delta(E)\right) \rangle = \lambda \frac{g(p')g(M_\Delta + \Sigma_\Delta(E))}{D(M_\Delta + \Sigma_\Delta(E))} \quad (4.13b)
\]

and instead of (4.11a) we obtain

\[
t(p', p; E) \approx \left[ \begin{array}{c}
V + \sum_{\Delta} V_\Delta > \frac{1}{E - E_\Delta} < \Psi_\Delta | V \end{array} \right] |p > \\
= \lambda g(p')g(p) + \lambda^2 g(p')g(p)|g(M_\Delta)|^2 \left| \frac{1}{D(M_\Delta)^2} \right| \frac{1}{E - M_\Delta}, \quad (4.14a)
\]

or

\[
t(p', p, E) \approx \lambda g(p')g(p) + \lambda^2 g(p')g(p)|g(M_\Delta + \Sigma_\Delta(E))|^2 \left| \frac{1}{D(M_\Delta + \Sigma_\Delta(E))^2} \right| \frac{1}{E - M_\Delta - \Sigma_\Delta(E)}, \quad (4.14b)
\]

Comparing (4.14a) with (4.14b) we see that they have sufficiently different form. The advantage of the representations (4.14a) and (4.14b) is that in these expressions the propagation of the intermediate \( \Delta \) is taken into account exactly. In addition it is important
to note, that the \( g(p) \) and \( D(E) \) functions can be constructed directly from the \( \pi N \) phase shifts [39]. In the Section 6 we will consider the restrictions which generates the unitarity condition for the \( \Sigma_\Delta(E) \).

5. Three-body equation with the \( \Delta \) degrees of freedom

The main result of the previous section is the recipe of construction the amplitudes sandwiched by the \( \Delta \) resonance states. In particular, for the amplitude of the reaction \( \pi + N \rightarrow \pi' + N' + \gamma' \), according to Eq.(3.9) and Eq.(4.5a), we get

\[
F_{\gamma'\pi'N'}^{c} = F_{\gamma'\pi'N'}^{c}^{\pi N;\pi N} \equiv - \langle \text{in}; p_N' p_\pi'|J_\mu(0)|p_N p_\pi; \text{in} \rangle_{\text{connected}} = \sum_{\sigma=1}^{4} w_{\gamma'\pi'N',\sigma}^{c} c_{\pi N,\sigma}^{<} \langle \text{in}; \sigma|\Psi_{\pi N}^{\pi N} > .
\]

(5.1)

In virtue of the extensions (4.9a,b) and Eq.(4.5b,c) we can obtain following expressions for the \( \Delta \leftrightarrow \gamma'\pi'N' \) transition amplitudes with on shell \( \Delta \)

\[
F_{\gamma'\pi'N';\Delta}^{c} = - \langle \text{in}; p_N' p_\pi'|J_\mu(0)|\Psi_{\Delta}^{\Delta} > = \sum_{\sigma=1}^{4} w_{\gamma'\pi'N',\sigma}^{c} c_{\pi N,\sigma}^{<} \langle \text{in}; \sigma|\Psi_{\Delta}^{\Delta} > ,
\]

(5.2a)

and for off shell \( \Delta \)

\[
F_{\gamma'\pi'N';\Delta}(E) = - \langle \text{in}; p_N' p_\pi'|J_\mu(0)|\Psi_{\Delta}^{\Delta}(E) > = \sum_{\sigma=1}^{4} w_{\gamma'\pi'N',\sigma}^{c} c_{\pi N,\sigma}^{<} \langle \text{in}; \sigma|\Psi_{\Delta}^{\Delta}(E) > ,
\]

(5.2b)

where instead of the two three-momenta \( p_N \) and \( p_\pi \) in Eq.(5.1) we have only one three momentum \( P_\Delta = p_N + p_\pi \), since the extensions (4.9a,b) imply the following transformations:

1. Projection of the \( P33 \) partial wave states which remove the dependence on the relative \( \pi N \) angles in Eq.(5.1).

2. Replacement of the relative \( \pi N \) momenta by the expression \( p^2 = ((s - m_N^2)(s - m_N^2)m_\pi^2)/4s \), where \( s = (m_\Delta + i\Gamma_\Delta/2)^2 \) or \( s = \left[ M_\Delta^2 + \Sigma_s(E, P_\Delta = 0) \right] \). in the c.m. frame of the \( \pi N \) system. It is convenient to suppose that the \( s \) variables in the above extensions (4.7a,b) are real

\[
s = (m_\Delta + i\Gamma_\Delta/2)(m_\Delta + i\Gamma_\Delta/2)^* = m_\Delta^2 + \frac{\Gamma_\Delta^2}{4} \quad (5.3a)
\]

\[
s = \left[ M_\Delta^2 + \Sigma_s(E, P_\Delta = 0) \right] \left[ M_\Delta^2 + \Sigma_s(E, P_\Delta = 0) \right]^* \quad (5.3b)
\]

The advantage of the projection procedure with the real variables (5.3a,b) is that the \( u, \pi, \bar{\pi} \) and other channel parts of the potentials \( U_{\alpha\beta} \) (3.4) or \( w_{\alpha\beta}^{c} \) (2.13) do not transform into
complex potentials after this type projection and there will not appear the nonphysical
contribute in the unitarity condition.

From the $\Delta \rightarrow \gamma'\pi'N'$ transition amplitudes (5.2a) and (5.2b) we can construct the
$\Delta \rightarrow \gamma'\Delta'$ transition amplitude

$$F_{\Delta'\gamma',\Delta} = -\lim_{E_{s',N'} \rightarrow E_{P',\Delta}} \langle in; p'_{N}p'_{\pi}|J_{\mu}(0)|\Psi_{P,\Delta}^{\Delta} > = \lim_{E_{s',N'} \rightarrow E_{P',\Delta}} \sum_{\sigma=1}^{4} w_{\gamma'\pi'N',\sigma}^{\Delta} < in; \sigma|\Psi_{P,\Delta}^{\Delta} >$$

(5.4a)

$$F_{\Delta'\gamma',\Delta}(E', E) = -\lim_{E_{s',N'} \rightarrow (E_{P',\Delta} + \Sigma_{\Delta}(E', p')_{\Delta})} \langle in; p'_{N}p'_{\pi}|J_{\mu}(0)|\Psi_{P,\Delta}^{\Delta}(E) >$$

$$= \lim_{E_{s',N'} \rightarrow (E_{P',\Delta} + \Sigma_{\Delta}(E', p')_{\Delta})} \sum_{\sigma=1}^{4} w_{\gamma'\pi'N',\sigma}^{\Delta} < in; \sigma|\Psi_{P,\Delta}^{\Delta}(E) >,$$

(5.4b)

where also the projection on the $P33$ partial wave states is assumed.

Expression (5.4a) depends only on the two four momenta $P_{\Delta} = (E_{P,\Delta}, p_{\Delta})$ and $P_{\Delta}' = (E_{P',\Delta}, p_{\Delta}')$ and in expressions (5.3b) there are also only two independent variables $P_{\Delta} = (E_{P,\Delta} + \Sigma_{\Delta}(E, p_{\Delta}), p_{\Delta})$ and $P_{\Delta}' = (E_{P',\Delta} + \Sigma_{\Delta}(E', p_{\Delta}'), p_{\Delta}')$. From Eq.(5.4a,b) we see, that there exists two ways of construction of the $\Delta' \rightarrow \gamma \Delta$ vertex functions: the projection of the $\pi'N' - \gamma \pi N$ transition matrix or the projection of the corresponding multichannel equation (3.4).

The functions $F_{\Delta'\gamma',\Delta}$ (5.4a) and $F_{\Delta'\gamma',\Delta}(E', E)$ (5.4b) can be treated as the covariant vertex functions with two on mass shell particles which have masses $s$ and spin $3/2$. Therefore we can use the representation of the $\Delta' \gamma' - \Delta$ vertices [59]

$$F_{\Delta'\gamma',\Delta}(P_{\Delta}', P_{\Delta}) = \rho^{\Delta}(s', P_{\Delta}')V_{\sigma\mu\rho}(P_{\Delta}', P_{\Delta})u^{\rho}(s, P_{\Delta})$$

(5.5a)

$$V_{\sigma\mu\rho}(P_{\Delta}', P_{\Delta}) = g_{\rho\sigma} \left[ F_{1}(Q^{2})\gamma_{\mu} + \frac{F_{2}(Q^{2})}{2M_{\Delta}} R_{\mu} \right] + Q_{\sigma} Q_{\rho} \left[ \frac{F_{3}(Q^{2})}{M_{\Delta}^{2}} \gamma_{\mu} + \frac{F_{4}(Q^{2})}{2M_{\Delta}^{2}} R_{\mu} \right],$$

(5.5b)

where $g_{\rho\sigma}$ is the metric tensor, $u^{\rho}(s, P_{\Delta})$ denotes the spinor for the spin $3/2$ particle with mass $s$, $Q = P_{\Delta} - P_{\Delta}$ and $R = P_{\Delta}' + P_{\Delta}$. The form factors $F_{i}(Q^{2})$ are simply connected with the charge monopole $G_{C0}(Q^{2})$, the magnetic dipole $G_{M1}(Q^{2})$, the electric quadrupole $G_{E2}(Q^{2})$ and the magnetic octupole $G_{M3}(Q^{2})$ form factors of the $\Delta$ resonance. The threshold values of these formfactors are determined by the physical constants. For instance $G_{C0}(0) = e$ stands for the electric charge and $G_{M1}(0) = \mu_{\Delta}$ can be used for the determination of the magnetic moment of $\Delta$.

The relativistic field-theoretical construction of the $\Delta - \gamma'\Delta'$ vertex function is significant by definition of the magnetic moment of $\Delta$. In particular, if we use the quantum-mechanical definition of $\mu_{\Delta}$ through the space components of the photon current operator sandwiched by the $\Delta$ wave functions $|\Psi_{\Delta} >$ [61], then we obtain the complex magnitude
for $\mu_\Delta$. Moreover, if we calculate the magnetic moment of the nucleon in the framework of this quantum-mechanical method, where we treat nucleon as the $\pi N$ cluster state for the $P11$ partial wave, then the resulting effective $\mu_N$ will be complex too.

We emphasize that the considered field-theoretical definition of $\mu_\Delta$ through the vertex function $F_{\Delta',\Delta}(P'_\Delta, P_\Delta)$ (5.5a) is analogue to the accepted definition of the magnetic moment of nucleon [14, 36, 3, 5]. In particular, if we include the $s$-channel one-nucleon exchange diagram in the second term of Eq.(2.8c) and Eq.(2.13) (i.e. we take five intermediate states $\sigma = N, 1, 2, 3, 4$ instead of four $s$-channel terms), then we obtain the analog to $\Delta'$ vertex function $F_{N',\Delta,N}(P'_N, P_N) = -<p' | J_\mu(0) | p>$ and $F_{N',\gamma',N}(P'_N, P_N) = \sum_{\sigma'} w_{N',\sigma'}^{c} \langle \sigma'; \Psi^N_P | \sigma \rangle$. Therefore, the other important difference between considered and quantum-mechanical [61] definitions of the $\Delta$ magnetic moment is that in the considered definition are included the contributions of the intermediate pion and nucleon magnetic momenta.

Certainly, in the considered formulation the ambiguity by determination of the explicit form of the intermediate $\Delta$ propagators in Eq.(4.2b) and in Eq.(4.2c) according to the off shell extensions (4.7a,b) arises. For instance, we can take $E_{P_\Delta} = \sqrt{m^2_\Delta + P^2_\Delta + i\Gamma_\Delta/2}$ for the on shell $\Delta$ and for the off shell $\Delta$ we can choose $E_{P_\Delta} = \sqrt{M^2_\Delta + i\Sigma_\Delta (E, P_\Delta)/2} + P^2_\Delta$.

In Ref.[12] it was demonstrated, that the sufficient different values for the coupling constants and cut-off parameters are necessary to use for the description of the $\pi N$ phase shifts up to 360 MeV pion Laboratory energy in the framework of the Bethe-Salpeter equations with the different form of the $\Delta$ propagators. The strong sensitivity of the description of the $\gamma p - \gamma p$, $\gamma p - \pi^0 p$ and $\gamma p - \gamma \pi^0 p$ reaction on the choice of the form of the intermediate $\Delta$ propagator was shown also in our previous paper [35], where the calculations were performed in the Born approximation of the analogous to that considered here a three-dimensional field-theoretical equations. One can hope, that the unified description of the multichannel $\gamma p - \pi N$ processes allows us to determine the form of $\Delta$ propagator.

Finally in this section we consider the representation of equation (3.4) in the framework of the isobar model, where instead of the intermediate $\pi N$ and multimeson states we will take $\Delta$ and heavy meson $h = \sigma, \rho, \omega, ..$ states. This means, that the $\pi N$ one $\Delta + \pi$ and $N + h$ nucleon and heavy meson states. As the initial states we have $\beta = 1,2 \equiv \pi N, \gamma p$ states.

\[
T_{\alpha\beta}(E_\beta) = U_{\alpha\beta}(E_\beta) + \sum_{\Lambda=\Delta',\pi',\Delta''} U_{\alpha\Lambda}(E_\beta) \frac{1}{E_\beta - \mathcal{E}_\Lambda(E_\beta)} T_{\Lambda\beta}(E_\beta), \quad (5.6)
\]

where we omitted intermediate states $\gamma N$ and $\gamma \pi N$ with photon, because they are of higher order in $\epsilon^2$. $\mathcal{E}_\Lambda(E_\beta)$ denotes the full energy of the intermediate $\Lambda$ cluster states

\[
\mathcal{E}_\Lambda(E_\beta) = \begin{cases} 
E_{P''_\Delta} + \Sigma_\Delta(E_\beta, P''_\Delta) & \text{if } \Lambda = \Delta'' \\
\omega_\pi(P''_\pi) + E_{P''_\pi} + \Sigma_\Delta(E_\beta, P''_\Delta) & \text{if } \Lambda = \pi'' + \Delta'' \\
E_{P''_N} + E_{P''_h} + \Sigma_h(E_\beta, P''_h) & \text{if } \Lambda = h'' + N'' \ (h = \sigma, \rho, \omega, ...)
\end{cases} \quad (5.7)
\]
Now in order to derive the two-body equation for the $\mathcal{T}_{\Lambda\beta}$ (4.6) transition amplitudes between the $\beta = \pi N, \gamma N$ and $\Lambda = \Delta, \pi\Delta, h + N$ states, we will use again the projection procedure (4.7a,b) of the resonances $\Delta'$ and $h'$ from the $\pi'N'$ and $\pi'\pi'$ final states. Then $\alpha = 1', 2', 3', 4' \equiv \pi'N', \gamma'N', \pi'\pi'N', \gamma'\pi'N'$ states will be replaced by

$$\alpha = 1' \equiv \pi'N' \implies \Delta'; \quad \alpha = 3' \equiv \pi'N' \implies \{\pi'\Delta'; h'N'\}; \quad \alpha = 4' \equiv \gamma'\pi'N' \implies \gamma'\Delta'. \quad (5.8)$$

and we get

$$\mathcal{T}_{\mathcal{K}\beta}(E_{\beta}) = U_{\mathcal{K}\beta}(E_{\beta}) + \sum_{\Lambda=\Delta', \pi'\Delta', h'N'} U_{\mathcal{K}\Lambda}(E_{\beta}) \frac{1}{E_{\beta} - E_{\Lambda}(E_{\beta})} \mathcal{T}_{\Lambda\beta}(E_{\beta}), \quad (5.9)$$

where $\mathcal{K} = \Delta', \gamma'N', \pi'\Delta', h'N', \gamma'\Delta' = \Lambda, \gamma'N', \gamma'\Delta'$.

![Figure 7](image-url)  
**Figure 7:** The graphical representation of Eq.(5.9) for the $b + N \implies a + \Delta'$ transition amplitude with the $\Delta, \pi\Delta, h\Delta$ intermediate states.

Substituting the solution of the two-body equations (5.9) in Eq.(5.6), we obtain the transition amplitudes into the three-body final states $\alpha = \pi'\pi'N', \gamma'\pi'N'$. Note, that one can obtain similar to Eq.(5.9) equation for $F_{c\gamma'\Delta;\Delta} = - < in; p'p'N|J_\mu(0)|\Psi_{\Delta} > \implies F_{c\gamma'\Delta;\Delta} = - < n; p'|J_\mu(0)|\Psi_{\Delta} >$ from the quadratically-nonlinear, spectral decomposition formula (2.13)

$$F_{K\beta} = w_{K\beta} + \sum_{\Lambda=\Delta', \pi'\Delta', h'N'} F_{K\Lambda}(E_{\beta}) \frac{1}{E_{\beta} - E_{\Lambda}(E_{\beta})} F_{\beta\Lambda}^* \quad (5.10)$$

which allows us to operate with the $\Delta \to \gamma'\Delta'$ vertex function.

The structure of the two-body equations (5.9) is illustrated in Fig.7. At first sight these equation have the same form as the relativistic equation derived from the Bethe-Salpeter equation in the framework of the Aaron, Amado and Young (AAY) model with
the two-body separable amplitudes [60, 39]. However in our approach only the Green
functions (4.2a,b,c) have the separable form due to the resonance pole. The effective
potential \( U_{\Lambda \Lambda}(E) \) consists of the sum of all connected diagrams (depicted in Fig.2, Fig.3,
Fig.4 and Fig.5), that are continued at the resonance poles. After these continuations,
\( U_{\Lambda \Lambda}(E) \) with three-body states \( \lambda = \pi N, \pi \pi N \) transforms into two-body form. Thus in
Eq.(5.9) the two-body potential \( U_{\Lambda \Lambda}(E) \) is constructed from the three particle potential
\( U_{\alpha \beta}(E) \) after the projection procedure (4.7a,b) to the resonance pole position.

6. Unitarity and gauge invariance

A. Unitarity

The three-dimensional quantum field-theoretical equations (2.13) have the form of
generalized unitarity conditions. Therefore for these multi-channel equations the unitarity
condition is fulfilled automatically. Moreover, the equivalent linearized equations (3.4)
with the potential \( U_{\alpha \beta}(E) \) satisfies the unitarity condition also in the off energy shell
region \( (E = E_{\beta}) \). However the unitarity condition for the two-body equations (5.6), (5.9)
and (5.10) with the complex propagators for the intermediate resonance states need a
special consideration. For this aim it is convenient to rewrite Eq.(3.4) in the form

\[
\mathcal{T}_{\alpha \beta}(E) = U_{\alpha \beta}(E) + \sum_{\gamma = \pi^\prime N, \pi^\prime \pi^\prime N^\prime} \frac{\mathcal{T}_{\alpha \gamma}(E)}{E - P_\gamma + i\epsilon} \mathcal{T}_{\beta \gamma}(E)^*,
\]

which using the same approximation as by derivation of Eq.(5.6), takes the form

\[
\mathcal{T}_{\alpha \beta}(E) \approx U_{\alpha \beta}(E) + \sum_{\Lambda = \Delta^\prime, \pi^\prime \Delta^\prime, h^\prime N^\prime} \frac{\mathcal{T}_{\alpha \Lambda}(E)}{E - \mathcal{E}_\Lambda(E)} \mathcal{T}_{\beta \Lambda}(E)^*.
\]

From Eq.(6.1) and (6.2) we can obtain the following condition for the \( \mathcal{E}_\Lambda(E) \)

\[
\mathcal{T}_{\alpha \beta}(E) - \mathcal{T}_{\beta \alpha}(E)^* = -2\pi i \sum_{\gamma = \pi^\prime N^\prime, \pi^\prime \pi^\prime N^\prime} \mathcal{T}_{\alpha \gamma}(E) \delta(E - P_\gamma) \mathcal{T}_{\beta \gamma}(E)^*
\]

\[
\approx \sum_{\Lambda = \Delta^\prime, \pi^\prime \Delta^\prime, h^\prime N^\prime} \mathcal{T}_{\alpha \Lambda}(E) \left[ \frac{1}{E - \mathcal{E}_\Lambda(E)} - \frac{1}{E - \mathcal{E}_\Lambda(E)^*} \right] \mathcal{T}_{\beta \Lambda}(E)^*.
\]

\[
\approx \sum_{\Lambda = \Delta, \pi \Delta, h \pi N} \sum_{\Lambda' = \Delta', \pi' \Delta', h' \pi N'} \mathcal{T}_{\alpha \Lambda}(E) \left[ \frac{1}{E - \mathcal{E}_\Lambda(E)} N_{\Lambda \Lambda'}(E) \frac{1}{E - \mathcal{E}_{\Lambda'}(E)^*} \right] \mathcal{T}_{\beta \Lambda'}(E)^*.
\]

where

\[
N_{\Lambda \Lambda'}(E) = -2\pi i \sum_{\gamma = \pi^\prime N^\prime, \pi^\prime \pi^\prime N^\prime} \mathcal{T}_{\gamma \Lambda}(E) \delta(E - P_\gamma) \mathcal{T}_{\gamma \Lambda'}(E)
\]
Equations (6.3c) and (6.3d) provide simple relations for the intermediate cluster propagators

$$\delta_{\Lambda \Lambda}^\prime \left[ \frac{1}{E - E_\Lambda(E)} - \frac{1}{E - E_\Lambda(E)^*} \right] \approx \frac{1}{E - E_\Lambda(E)} N_{\Lambda \Lambda}^\prime(E) \frac{1}{E - E_{\Lambda}\prime(E)^*}$$

which looks like the “unitarity condition” for the propagators of the intermediate resonances. Unfortunately, the validity of the condition (6.3e) is not enough for the validity of the unitarity condition (6.3a) for the observed amplitudes $T_{\alpha\beta}(E)$. In particular, after comparison of Eq.(6.3a) and Eq.(6.3b) we see that Eq.(6.3a) is given in the half on energy shell region due to the $\delta(E - P^\alpha_\gamma)$ function. Contrary to this in Eq.(6.3b) the intermediate states are depending on $E$. This means, that the unitarity condition (6.3a) can be approximately valid only for the resonance energies $E = E^R_{\Delta_n} = \sqrt{m_{\Delta}^2 + \mathbf{P}_{\Delta}^2}$, $E = E^R_{\pi \Delta_n} \equiv \omega_{\pi}(p^n) + \sqrt{m_{\Delta}^2 + \mathbf{P}_{\Delta}^2}$ and $E = E^R_{h'' N''} \equiv E_{p'' N''} + \sqrt{m_{h}^2 + \mathbf{P}_{h}^2}$ (see restrictions (4.8a,b)) and in the neighbor area of these resonance energies.

For instance, from the (6.3b) we can obtain the following formula

$$T_{\alpha\beta}(E) - T_{\beta\alpha}(E)^* \approx -2\pi i \sum_{\Lambda = \Delta_n, \pi \Delta_n, h'' N''} T_{\alpha\Lambda}(E) \delta(E - E^R_{\Lambda}) T_{\beta\Lambda}(E)^*$$

if $\Sigma_{\Delta}$ and $\Sigma_{h}$ in Eq.(5.7) tend to zero as

$$\lim_{E \to E^R} \Sigma_{\Delta}(E, \mathbf{P}_{h}^n) = i\epsilon$$ and

$$\lim_{E \to E^R} \Sigma_{h}(E, \mathbf{P}_{h}^n) = i\epsilon$$

Thus, the unitarity condition (6.4) with the resonance amplitudes $T_{\alpha\Lambda}(E)$ have the approximate form even in the resonance energy region $E \approx E_r$ in spite of the conditions (4.8a,b). According to Eq.(6.5), the approximation of the amplitudes or Green functions (4.2b) by the resonance part only has an acceptable accuracy for narrow resonances, or if $\Gamma_r/E_r \ll 1$.

Unitarity condition (6.4) with the intermediate resonance states can be improved if one takes into account the nonresonant contributions to the resonant $\pi N$ and $\pi \pi$ interactions as it was done in Ref.[12], where the contributions of the nonresonant Feynman diagrams for the $\pi N P33$ partial waves was investigated. Also in Ref.[62] the resonant ($\Delta$) and nonresonant parts of the $\pi N$ interactions was separated proceeding from the effective three-dimensional Hamiltonian method and the corresponding Lippmann-Schwinger and Dyson equations were solved in the framework of the separable potential model. In the nonrelativistic approach the multichannel equations with the intermediate resonance and nonresonance parts and the explicit form of the corresponding unitarity conditions were considered in book of Bohr and Mottelson.[63].

B. Gauge invariance

The choice of the Coulomb gauge $\nabla_i A^\gamma_i(x) = 0$ for the considered three-dimensional time-ordered formulation allows us to exclude the non-physical degrees of freedom of
photon \Lambda(x) matrix function in the considered approximation, one can show that every transition in the amplitude of Eq.(2.13) or Eq.(3.4) satisfies the current conservation condition \( J(2.2) \) with the gauge transformation of the photon field operator. 

According to the book of Bjorken and Drell [36], in order to redefine the above equations in the Coulomb gauge one must first redefine the reduction formula for the S-matrix using transversal quantization rules, transversal photon fields, transversal source operators etc. In particular, the transversal photon current operator in Eq.(2.2) and (2.4a) can be constructed from the initial four-dimensional current operator as 

\[
J_{\mu=i=1,2,3}(x) \mapsto J^t_{i}(x) = J_i(x) - \frac{\nabla_i \partial_o}{\nabla^2} J_o(x).
\] (6.6a)

The conservation of the transverse current \( \nabla^i J^t_i(x) = 0 \) follows from the conservation of the four-dimensional current \( \partial^\mu J_\mu(x) = 0 \). In the same manner one can construct the photon field operator in the Coulomb gauge \( A_\mu^c(x) \) from the photon field operator \( A_\mu(x) \) in the Lorentz gauge \( \partial^\mu A_\mu = 0 \)

\[
A_\mu(x) \mapsto A_{i=1,2,3}(x) = A_i(x) - \frac{\nabla_i \partial_o}{\nabla^2} A_o(x),
\] (6.6b)

where the time component of \( A_\mu \) is determined through \( J_o \) from the equation of motion \( \partial_o \partial^\nu A_\nu(x) = J_o(x) \) and instead of the Eq.(2.4a) we have \( \partial^\nu \partial_o A_\nu^c(x) = J^t_\nu(x) \).

The quantization rules of the photon field operators in the Coulomb gauge are [36]

\[
[\partial_{\nu} A_i^c(x), A_j^c(y)]_{x_o=y_o} = -i \delta_{ij}^\nu (x-y)
\] (6.6c)

where \( \delta_{ij}^\nu(x) = (\delta_{ij} - \nabla_i \nabla_j/\nabla^2) \delta(x) \).

The redefined photon current operator (6.6a) and the quantization rules (6.6c) allow us to rewrite the S-matrix reduction formulas and corresponding equations (2.1) and (2.2) with the \( J_i^t(x) \) and equal-time commutators between the photon field operators in the Coulomb gauge [36]. The form of the three-dimensional time-ordered equations (2.13) or (3.4) with the fields in the Coulomb gauge remains the same. Now one can demonstrate the validity of the current conservation condition and invariance under the gauge transformation of the photon field operator

\[
A_i^c(x) = A_i^c(x) + \nabla_i \Lambda(x) \quad \text{or} \quad A_\mu^c(x) = A_\mu(x) + \partial_\mu \Lambda(x)
\] (6.7a)

or similarly, invariance under the gauge transformation of photon polarization vector

\[
\epsilon_i^\nu(k) = \epsilon_i^\nu(k) + \lambda k_i \quad \text{or} \quad \epsilon_\nu^\mu(k) = \epsilon_\nu^\mu(k) + \lambda k_\mu
\] (6.7b)

for an arbitrary \( \Lambda(x) \) and \( \lambda \). In particular, due to three-momentum conservation at every vertex function in the considered approximation, one can show, that every transition matrix \( <n|J_\mu^t(0)|m> \) with arbitrary \( n \) and \( m \) in(out) states in the effective potential or in the amplitude of Eq.(2.13) or Eq.(3.4) satisfies the current conservation condition

\[
0 =< n|\nabla^j J^t_j(0)|m> = (P_N - P_m)^j < n|J^t_j(0)|m> = k^j < n|J^t_j(0)|m>
\] (6.8)
where according to (6.6a)

\[ < n | J_{tr}^i(0) | m > = < n | J_i(0) | m > - \frac{\left( P_n - P_m \right)_i}{\left( P_n - P_m \right)^2} \left( P_n^o - P_m^o \right) < n | J_o(0) | m > . \]  

(6.9)

Thus the three-momentum conservation condition in Eq.(2.13) or Eq.(3.4) allows us to reduce the action of \( k^j_\gamma \) to the action of \( i\nabla^i \) on the photon current operator \( J_{tr}^r(0) \) for every term of Eq.(2.13) or (3.4). For example, for the second \( u \)-channel term in Eq.(2.9a) with \( a = \gamma' \) and \( b = \gamma \) in virtue of Eq.(6.8) we have

\[ \sum_{n=N,\pi N...} < \text{out}; \bar{\alpha} | J_{tr}^r(0) | n; \text{in} > \frac{\delta^{(3)}(-k_\gamma + P_\alpha - P_n)}{|k_\gamma| + P_\alpha^o - P_n^o} < n; \text{in} > | J_{tr}^r(0) | \bar{\beta}; \text{in} > = \sum_{n=N,\pi N...} < \text{out}; \bar{\alpha} | J_{tr}^r(0) | n; \text{in} > \frac{\delta^{(3)}(-k_\gamma + P_\alpha - P_n)}{|k_\gamma| + P_\alpha^o - P_n^o} < n; \text{in} > i\nabla_{x_j} J_{tr}^r(0) | \bar{\beta}; \text{in} > = 0 \]

In the same manner for the equal-time commutators we get

\[ k^j_\gamma < \text{out}; \bar{\alpha} | J_{tr}^r(0), a_j^+(0) | \bar{\beta}; \text{in} > = < \text{out}; \bar{\alpha} | J_{tr}^r(0), k^j_\gamma a_j^+(0) | \bar{\beta}; \text{in} > = 0. \]  

(6.10)

Thus we have demonstrated that

\[ k^j_\gamma \left[ f_{\gamma'\bar{\alpha};\gamma\bar{\beta}} \right]_{ij} = -k^j_\gamma < \text{out}; \bar{\alpha} | J_{tr}^r(0) | k_\gamma j; \bar{\beta}; \text{in} > = k^j_\gamma \left[ W_{\gamma'\bar{\alpha};\gamma\bar{\beta}} + (2\pi)^3 \sum_{\sigma=1}^4 f_{\gamma'\bar{\alpha};\sigma} \frac{\delta^{(3)}(k_\gamma + P_\beta - P_\gamma)}{|k_\gamma| + P_\beta^o - P_\gamma^o + i\epsilon F^{*\gamma\bar{\beta}}_{\gamma\bar{\sigma}}}_{ij} \right] = 0 \]  

(6.11a)

for an arbitrary Lagrangian by calculation of the equal-time commutator (2.15) and any number of the intermediate states in \( W_{\alpha\beta} \) (2.9a). Note that equations (2.8a), (2.9a), (2.13), (3.4) etc. are not depend on the three-momentum \( p'_a = k'_\gamma \) if \( a = \gamma' \). Therefore we can assume, that \( k'_\gamma = k_\gamma + P_\beta - P_\alpha \). Then in the same way as (6.11a) we obtain

\[ k'^j_\gamma \left[ f_{\gamma'\bar{\alpha};\gamma\bar{\beta}} \right]_{ij} = -k'^j_\gamma < \text{out}; \bar{\alpha} | J_{tr}^r(0) | k_\gamma j; \bar{\beta}; \text{in} > = k'^j_\gamma \left[ W_{\gamma'\bar{\alpha};\gamma\bar{\beta}} + (2\pi)^3 \sum_{\sigma=1}^4 f_{\gamma'\bar{\alpha};\sigma} \frac{\delta^{(3)}(k_\gamma + P_\beta - P_\gamma)}{|k_\gamma| + P_\beta^o - P_\gamma^o + i\epsilon F^{*\gamma\bar{\beta}}_{\gamma\bar{\sigma}}}_{ij} \right] = 0. \]  

(6.11b)

This completes the proof of the current conservation condition of the above equations in the Coulomb Gauge. It is easy to see that the current conservation condition (6.11a,b) for the both sides of Eq.(2.13) or Eq.(3.4) is sufficient for the validity of the invariance of these equation under the gauge transformation (6.7b) for \( e^{\mu'}_\mu(k) \). The gauge transformation (6.7a) \( A^i_\gamma(x) = A^i_\gamma(x) + \nabla_i \Lambda(x) \) leads to the gauge transformation (6.7b) if we
taken into account the conditions $\nabla^2 \Lambda(x) = \nabla^i A_i^c(x)$ [36, 5]. In the general sense the gauge invariance means the independence of results on the gauge used. In other words, if we represent the gauge condition as $n^\mu A^\mu(x) = 0$ with $n_\mu = k_\mu$ for the Lorentz gauge, $n_\mu = (0, k)$ for the Coulomb gauge, $n_\mu = (k_\mu, 0, 0, 0)$ for the axial gauge etc., then gauge invariance reduces to the independence of the observables on the choice of $n^\mu$. The proof of this general gauge invariance is out of the scope of our paper.

7. Conclusion

In this paper we have derived three-dimensional covariant scattering equations for the coupled system of the amplitudes of the $\pi N \leftrightarrow \gamma N \leftrightarrow \pi\pi N \leftrightarrow \gamma\pi N$ reactions. The basis of these three-body relativistic equations is the standard field-theoretical $S$-matrix reduction formulas. After decomposition over the complete set of the asymptotic "in" states the quadratically nonlinear three-dimensional equations (2.13) were obtained.

These equations were replaced by the equivalent Lippmann-Schwinger type equation (3.4). Unlike the three-body Faddeev-type equations, the potentials of the suggested three-body equations (3.4) consists of the connected parts only and thereby they have the form of the relativistic Lippmann-Schwinger type equations with the well defined connected three-body potential. This sufficient difference follows from the field-theoretical derivation of the considered equations, where the disconnected parts of the three-body amplitudes (2.2) $f_{\alpha\beta} = - \langle \text{out}; \tilde{\alpha}\mid j_\alpha(0)\mid \beta; \text{in} \rangle$ coincide with the disconnected part from the right side of these equations, i.e. Eq.(2.2) or Eq.(2.8a) or Eq.(2.13) consist of independent sets of equations for the connected and disconnected parts of amplitudes. In particular, if we note, that the potentials of Eq.(3.4) (or Eq.(2.13)) consist of the product of the two renormalized (physical) amplitudes or vertex functions with the corresponding propagator of the particles, then it is obvious, that the graphical method of Taylor, based on the last cut lemma, does not work in this approach. The analytical cluster decomposition leads to independent sets for the disconnected amplitudes in Eq.(2.8a) or in Eq.(2.13).

The potentials of the suggested equations require as their input the vertex functions with two on mass shell particles. For the two-body $\pi N \leftrightarrow \gamma N$ reactions these input functions are exactly the phenomenological one variable vertex functions that can be determined from dispersion relations or from two-body observables. The equal-time commutators (2.15) offer different opportunities to investigate the off-shell effects resulting from chosen model Lagrangian. Thus the number of the one off-mass shell particle exchange diagrams and contact (overlapping) terms in expression (2.15) (see Fig.5) depend on the model Lagrangian. Certainly, if one includes the higher order derivatives and some nonlocal effects in the effective Lagrangians, then in the equal-time commutators (2.15) arise numerous contact terms which will not be easy to take into account. Usually, in practical calculations simple Lagrangians which generate a minimal number of contact terms are used. On the other hand these simple Lagrangians with “effective” $\sigma, \rho, \omega, \ldots$-particle degrees of freedom could help us to estimate the more complicated effects contained in
the much more complicated Lagrangians like the Lagrangian of the nonlinear $\sigma$ model with vector mesons, some QCD motivated Lagrangians etc. Thus, if it is possible to describe the connected $\pi N \leftrightarrow \gamma N \leftrightarrow \pi\pi N \leftrightarrow \gamma\pi N$ reactions using some simple Lagrangian, then one can find a number of more complicated Lagrangians which lead to the description of the same data. On the other hand, one can, in principle, construct a two-body potential, coming from the equal-time commutators using the inverse scattering methods [49]. This link between the effective Lagrangians and the potentials of the solved field-theoretical equations which arise due to the equal-time commutators, can be considered as an additional tool for the investigation of the correlations between a class of Lagrangians and the calculated experimental data.

The present three-dimensional formulation is not more complicated than the Bethe-Salpeter equations. In the four-dimensional formulation intermediate particle and antiparticle degrees of freedom are combined in the same diagram. Therefore on the tree-level approximation Bethe-Salpeter equations are more simple. But if one wants to use renormalized physical amplitudes and vertices and if one wants to take into account the rescattering effects, then the suggested formulation is simpler, because the corresponding equations are three-dimensional from the beginning and in the amplitudes and in the vertex functions two of the external particles are on mass shell.

The important features of the suggested three-body field-theoretical equations are the following:

1. **Unitarity.** The final three-body equations (3.4) are equivalent to the nonlinear equations (2.13) which fulfill the off shell unitarity conditions in the nonrelativistic collision theory [1, 2]. Therefore these equations automatically satisfy the unitarity conditions for the complete set of equations with the infinity intermediate $|n; \text{in} >$ states or for the truncated set of equations with two and three particle intermediate states $n$. It is important to note, that unlike the four-dimensional Bethe-Salpeter equations, in the considered approach only the on-mass shell physical $"\text{in(out)}"$ states are truncated in the completeness condition $|n; \text{in(out)} > < \text{in(out)}; n| = 1$.

2. **Current conservation conditions.** In the considered formulation with the Coulomb gauge we have demonstrated current conservation and invariance of Eq. (2.13) and Eq.(3.4) and corresponding potentials under gauge transformations (6.7a) or (6.7b) for every number of the truncated intermediate states. In order to achieve gauge invariance in this formulation it is not necessary to use additional approximations like the tree approximation with a gauge invariant combination of terms or the construction of approximate auxiliary gauge-invariance-preserving currents or to use the special representation of the off-mass shell $\Delta$ propagator and the corresponding construction of the gauge invariant electromagnetic $\Delta$ vertex function. The only requirement which is necessary in the considered approach to attain the invariance under the gauge transformation (6.7a) or (6.7b) is the existence of the conserved currents $J_\mu(x)$ (2.4a) and the corresponding model Lagrangian.

Another aspect of the considered three-body equations is the intermediate $\Delta$ resonance degrees of freedom. We have construct the $\pi N \leftrightarrow \Delta$ wave functions using the extension of the corresponding Lippmann-Schwinger equations in the complex region for
the observed (physical) $\Delta$ pole position (i.e. for the Breit-Wigner mass $m_\Delta = 1232\, MeV$ and full width $\Gamma_\Delta = 120\, MeV$). We considered two cases, (i) with the “on shell $\Delta$”, when $E_{P_\Delta} = \sqrt{M_\Delta^2 + P_\Delta^2}$ and $M_\Delta = m_\Delta + i\Gamma_\Delta/2$ (4.7a) and (ii) “off shell $\Delta$”, where $E_{P_\Delta} = E_{P_\Delta}^0 + \Sigma_\Delta(E_{P_\Delta}, P_\Delta)$ (4.7b). These $\pi N \leftrightarrow \Delta$ wave functions were used for the construction of the intermediate full $\pi N$ Green function with the $\Delta$ resonance pole (4.2b,c) and for the equation of motion with intermediate $\Delta$ degrees of freedom (see Eq.(4.10a,b), (5.6) and (5.9)). Moreover, the same extension procedure generates the $\Delta \gamma - \Delta$ vertex functions (5.5a,b), that can be considered as the generalization of the $N\gamma - N'$ vertex function. Thus in contrast to other formulations we have not introduced the effective spin $3/2$ Lagrangian in order to introduce the intermediate $\Delta$'s, where additional conditions are necessary in order to determine the actual off-mass shell behavior of the amplitude.

Appendix A: Equivalence of quadratically nonlinear equations (2.10) and Lippmann-Schwinger type equations (3.4) with the linear energy depending potential (3.5).

By solving of the Lippmann-Schwinger type equations (3.4) we can define the corresponding wave function

$$<in; \alpha | \Psi_\beta > = < in; \alpha | \beta; in > + \frac{1}{E_\beta - E_\alpha + i\epsilon} \sum_{\gamma=1}^{4} U_{\alpha\gamma}(E_\beta) < in; \gamma | \Psi_\beta >.$$  \hspace{1cm} (A.1)

which satisfies the Schrödinger equation

$$\left( H_o + U(E_\beta) \right) | \Psi_\beta > = E_\beta | \Psi_\beta >.$$ \hspace{1cm} (A.2)

where in accordance with the definition of the free Hamiltonian, we have $H_o|\beta; in > = E_\beta|\beta; in >$.

If we taken into account the energy dependence of potential (3.5) $U_{\alpha\beta}(E_\beta) = A_{\alpha\beta} + E_\beta B_{\alpha\beta}$, then we can rewrite Eq. (A.2) as follows

$$\left( H_o + A \right) | \Psi_\beta > = E_\beta (1 - B) | \Psi_\beta >.$$ \hspace{1cm} (A.3)

Below we shall assume that exists $(1 - B)^{-1}$, i.e. the operator $1 - B$ is smooth enough. This allows us represent Eq. (A.3) in the form of the Schrödinger equation with Hermitian Hamiltonian $\hat{h}$

$$\hat{h} | \chi_\beta > = E_\beta | \chi_\beta > \hspace{1cm} (A.4a)$$

$$\hat{h} = (1 - B)^{-1/2}(H_o + A)(1 - B)^{-1/2} \hspace{1cm} (A.4b)$$

$$| \chi_\beta > = (1 - B)^{1/2} | \Psi_\beta > \hspace{1cm} (A.4c)$$
From the hermiticity of the Hamiltonian $h$ there follows the conditions of the completeness and ortho-normality of its eigenstates $|\chi_\beta \rangle$. Therefore the state vectors $|\Psi_\beta \rangle$ satisfy following completeness conditions

$$
\sum_{\gamma=1}^{4} |\Psi_\gamma \rangle < \Psi_\gamma | = (1 - B)^{-1}
$$  \hspace{1cm} (A.5a)

and of the ortho-normality

$$
|\Psi_\alpha \rangle |(1 - B)|\Psi_\beta \rangle = \delta_{\alpha\beta}
$$  \hspace{1cm} (A.5b)

Now if we insert the identity $U_{\alpha\beta}(E_\beta) = W_{\alpha\beta} + (E_\beta - E_\alpha)B_{\alpha\beta}$ in Eq.(A.2) and use Eq.(3.9) $F_{\alpha\beta} = \sum_{\gamma=1}^{4} W_{\alpha\gamma} < in; \gamma |\Psi_\beta \rangle$, then after some simple algebraic transformations we obtain

$$
< in; \alpha |(1 - B)|\Psi_\beta \rangle = < in; \alpha |\beta; in > + \frac{1}{E_\beta - E_\alpha + i\epsilon} F_{\alpha\beta},
$$  \hspace{1cm} (A.6a)

and

$$
< \Psi_\alpha |(1 - B)|\beta; in \rangle = < in; \alpha |\beta; in > - \frac{1}{E_\beta - E_\alpha + i\epsilon} F^*_\alpha\beta.
$$  \hspace{1cm} (A.6b)

Here we multiply relation (A.6b) by $F_{\sigma\alpha} = < in; \sigma |W|\Psi_\alpha \rangle$ and using the completeness condition (A.5a), after summation over $\alpha$ we get

$$
W_{\sigma\beta} = F_{\sigma\beta} - \sum_{\alpha=1}^{4} F_{\sigma\alpha} \frac{1}{E_\beta - E_\alpha + i\epsilon} F^*_\alpha\beta.
$$  \hspace{1cm} (A.7)

Equation (A.7) is identical with Eq.(2.10). The same derivation of the quadratically-nonlinear integral equations with bound states from the Lippmann-Schwinger equation for the one-channel case, was given in [25].

Appendix B: The explicit form of the potentials $w^c_{\alpha\beta}$ and $U_{\alpha\beta}(E)$ in the multichannel equations (2.13) and (3.4)

In order to determine the structure of the potentials $w^c_{\alpha\beta}$ (2.13 or $U_{\alpha\beta}(E)$ (3.4) firstly we must separate the connected and disconnected parts from $W_{\alpha\beta}$ (2.9a). In the present formulation this transformation is equivalent to the so called "cluster decomposition" procedure [37, 22, 25] which allows us to distinguish the equations for the connected parts of scattering amplitudes $f^c_{\alpha\beta}$ and for the connected parts of the effective potential $w^c_{\alpha\beta}$. In the other words, we will represent $w^c_{\alpha\beta}$ and corresponding $U_{\alpha\beta}(E)$ in the terms of the connected parts of the transition matrix elements only. In particular, from Eq.(2.2) after insertion of the complete "in" states we get

$$
f^c_{\alpha\beta} = \{- < out; \bar{\alpha} | [j_a(0), a^+_b(0)] |\bar{\beta}; in >
$$

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Equations (2.8a) and (2.9a) come out from (B.1a) after separation of the states. For the amplitude we have

\[ \tilde{\alpha}|j_a(0)|n; \text{in} \]

\[ \tilde{\alpha}|j_a(0)|n; \text{in} \]

\[ \tilde{\alpha}|j_a(0)|n; \text{in} \]

\[ \tilde{\alpha}|j_a(0)|n; \text{in} \]

Equations (2.8a) and (2.9a) come out from (B.1a) after separation of the states. For the amplitude \( F_{\alpha\beta} \) (2.10) all "out" states are replaced by the "in" states and we have

\[ F_{\alpha\beta}^c = \left\{ - < \text{in}; \tilde{\alpha}|j_a(0), a_{\alpha}^+(0)|\tilde{\beta}; \text{in} > \right\} \]

The special case of the disconnected parts of the amplitude \( < \text{in}; \tilde{\alpha}|j_a(0)|\tilde{\beta}; \text{in} > \) are defined according to Eq.(2.6a,b,c), Eq.(2.7a,b) and they are depicted in the Fig.1. The separation of the disconnected parts for the amplitude \( < \text{in}; \tilde{\alpha}|j_a(0)|n''; \text{in} > \) with asymptotic \( \tilde{\alpha} \) and \( N \) states means

\[ < \text{in}; \tilde{\alpha}|j_a(0)|n''; \text{in} >= < \text{in}; \tilde{\alpha}|j_a(0)|n''; \text{in} >_c + \]

\[ < \text{in}; N'|\text{in} > < \text{in}; \tilde{\alpha}_{N'}|j_a(0)|n''; \text{in} ; \text{in} >_c + < \text{in}; \pi'|\pi''; \text{in} > < \text{in}; N'|j_a(0)|n''; \text{in} >_c \]

where the subscript "c" stands for the connected part of the amplitude, \( < \text{in}; N'|\text{in} > \) denotes the one particle nucleon \( N' \) state and the second term with disconnected in Eq.(B.2) arise only for the three-particle states, when \( < \text{in}; \alpha|\equiv < \text{in}; a, \pi' N'|\), since \( a = \pi' \) or \( \gamma' \) and \( |n''; \text{in} > = |n''; \pi''; \text{in} > \).

The cluster decomposition for the expression (B.1b) is the same as exclusion of the disconnected parts for the product of the amplitudes according to Eq.(B.2). Thus, after the separation of the connected parts in the second and third parts of Eq.(B.1b) according to the (B.2), we obtain the following representation for the first connected s channel term in (B.1b)

\[ \sum_{n=N,\pi N,\pi N,3\pi N,...} < \text{in}; \tilde{\alpha}|j_a(0)|n; \text{in} > \frac{\delta^{(3)}(P_{\beta} + P_{\alpha} - P_n)}{\omega_b(P_{\beta}) + P_{\alpha}^0 - P_n^0 + i\epsilon} < \text{in}; j_b(0)|\tilde{\beta}; \text{in} > \]

\[ = \sum_{n=N,\pi N,2\pi N,...} < \text{in}; \tilde{\alpha}|j_a(0)|n; \text{in} > c \frac{\delta^{(3)}(P_{\beta} + P_{\alpha} - P_n)}{\omega_b(P_{\beta}) + P_{\alpha}^0 - P_n^0 + i\epsilon} < \text{in}; j_b(0)|\tilde{\beta}; \text{in} > c + \]

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\[
\sum_{m=\text{mesons},...} <in; \tilde{\alpha}_{N'}|j_a(0)|m; in > c \frac{\delta^{(3)}(P_\beta + P_b - P_m - P'_N)}{\omega_b(P_b) + P^0_b - P_m - E_{P'_N} + i\epsilon} < in; m, N'|j_b(0)|\tilde{\beta}; in > c
\]

+ terms with the all other transposition of \(\tilde{\alpha}, N'\) and \(\tilde{\beta}, N\)  \hspace{1cm} (B.3)

where \(P'_N = \left( E_{P'_N} = \sqrt{m_N^2 + p'_N^2}, p'_N \right)\) stands for the four-momentum of the nucleon and \(P_\beta = \left( P^0_\beta, P_\beta \right)\) is the four-momentum of the asymptotic \(\tilde{\beta}\) state.

Next we will determine the explicit form of the \(u^e_{\alpha\beta}\) in Eq.(2.13) and the corresponding \(U_{\alpha\beta}(E)\) (2.13) separately for the \(2 \rightarrow 2', 2 \rightarrow 3'\) and \(3 \rightarrow 3'\) reactions

For the two-body \(\pi N \leftrightarrow \gamma N\) reactions cluster decomposition is the same as the transposition of the nucleon from the asymptotic \(< \tilde{\alpha} = p'_N > \) or \(|\tilde{\beta} = p_N >\) states using the separation of connected and disconnected parts in Eq.(B.2) [37, 22, 25, 24]. Thus instead of the two \(s\) and \(u\)-channel terms (2.6) there will appear eight terms with connected amplitudes

\[
W^e_{\alpha+N',b+N} = W^e_{\alpha+N',b+N} = -< in; p'_N|j_a(0), a_b^+(0)|p_N; in >
\]

\[
+(2\pi)^3 \sum_{n=N''} < in; p'_N|j_a(0)|p_{N''}; in > \frac{\delta^{(3)}(p_b + p_N - p_{N''})}{\omega_b(p_b) + E_{p_N} - E_{p_{N''}} + i\epsilon} < in; p_{N''}|j_b(0)|p_N; in > \hspace{1cm} (B.4a)
\]

\[
+(2\pi)^3 \sum_{m=\text{mesons},NN} < 0|j_a(0)|m; in > \frac{\delta^{(3)}(p_b + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} - p^0_m - E_{p'_N} + i\epsilon} < in; p'_N, m|j_b(0)|p_N; in > c \hspace{1cm} (B.4b)
\]

\[
+(2\pi)^3 \sum_{m=\text{mesons},NN} < in; p'_N|j_a(0)|p_N, m; in > c \frac{\delta^{(3)}(p_b - p_m)}{\omega_b(p_b) - p^0_m} < in; m|j_b(0)|0 > \hspace{1cm} (B.4c)
\]

\[-(2\pi)^3 \sum_{NN,..} < 0|j_a(0)|p_Np_{N'}; in > \frac{\delta^{(3)}(p_b - p'_{N''} - p_{N'})}{\omega_b(p_b) - E_{p'_N} - E_{p'_{N''}}} < in; p'_N, p_{N'}|j_b(0)|0 > \hspace{1cm} (B.4d)
\]

\[
+(2\pi)^3 \sum_{n=N'''} < in; p'_N|j_b(0)|n; in > c \frac{\delta^{(3)}(-p_b + p'_{N''} - p_{N})}{-\omega_b(p_b) + E_{p'_N} - p^0_n} < in; n|j_a(0)|p_N; in > c \hspace{1cm} (B.4e)
\]

\[
+(2\pi)^3 \sum_{m=\text{mesons},NN} < 0|j_b(0)|m; in > \frac{\delta^{(3)}(-p_b - p_m)}{-\omega_b(p_b) - p^0_m} < in; p'_N, m|j_b(0)|p_N; in > c \hspace{1cm} (B.4f)
\]

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\[+(2\pi)^3 \sum_{m=\text{mesons},N,N'} \langle in; p'_N|j_b(0)|p_N,m; in \rangle \frac{\delta^{(3)}(-p_b - p_m + p'_N)}{-\omega_b(p_b) - E_{p_N} - P_m + E_{p'_N}} < in; m|j_a(0)|0 > (B.4g)\]

\[-(2\pi)^3 \sum_{N,...} < 0|j_b(0)|p_Np_{N'}; in > \frac{\delta^{(3)}(-p_b - p_N - p_{N'})}{-\omega_b(p_b) - E_{p_N} - E_{p_{N'}}} < in; p'_Np_{N'}|j_a(0)|0 >, (B.4h)\]

where the high order over \(e^2\) intermediate terms are omitted. In addition, we have eliminated the four-particle intermediate states from our consideration. Therefore from the s-channel term (B.4a) are excluded intermediate \(n = 3\pi^m N, \ldots\) states.

The terms (B.4a)-(B.4h) are represented in the Fig.2A-Fig.2H correspondingly. Unlike to the analogical relations for the \(\pi N\) scattering [22, 24, 25], equations (B.4a)-(B.4h) includes in the asymptotic \(\gamma N\) states, because \(a = \pi' N', \gamma' N'\) and \(b = \pi N, \gamma N\), i.e. Eq.(B.4a)-(B.4h) are derived for the coupled \(\pi N \leftrightarrow \gamma N\) processes.

In order to transform \(w_{\alpha\beta}^{c}(B.4a)-(B.4h)\) in the form of the Hermitian potential \(U(E)_{\alpha\beta}\) (3.5) we will use following identities for the propagators

\[\frac{1}{\omega_b(p_b) + E_{p_N} - E_{p_{N'}}} \equiv \frac{1}{\omega_b(p_b) + E_{p_N} - E_{p_{N'}}} \left[\omega_a(p'_a) + E_{p'_N}\right] - E_{p_{N'}} \tag{B.5a}\]

\[\frac{1}{\omega_b(p_b) + E_{p_N} - P_m - E_{p'_N} + i\epsilon} \equiv \frac{1}{\omega_b(p_b) + E_{p_N} - P_m - E_{p'_N} + i\epsilon} \left[\omega_a(p'_a) + E_{p'_N}\right] - P_m - E_{p_{N'}} \tag{B.5b}\]

\[\frac{1}{\omega_b(p_b) - P_m} \equiv \frac{1}{\omega_b(p_b) - P_m} \left[\omega_a(p'_a) + E_{p'_N}\right] - P_m - E_{p_{N'}} \tag{B.5c}\]

\[\frac{1}{\omega_b(p_b) - E_{p'_N} - E_{p_{N'}}} \equiv \frac{1}{\omega_b(p_b) - E_{p'_N} - E_{p_{N'}}} \left[\omega_a(p'_a) + E_{p'_N}\right] - E_{p'_N} - E_{p_{N'}} - E_{p_{N'}} \tag{B.5d}\]

\[\frac{1}{-\omega_b(p_b) + E_{p'_N} - P_m} \equiv \frac{1}{-\omega_b(p_b) + E_{p'_N} - P_m} \left[-\omega_a(p'_a) + E_{p'_N}\right] + E_{p_{N'}} - P_m \tag{B.5e}\]

\[\frac{1}{-\omega_b(p_b) - P_m} \equiv \frac{1}{-\omega_b(p_b) - P_m} \left[-\omega_a(p'_a) - E_{p'_N} - P_m + E_{p_{N'}} \tag{B.5f}\right]
Here we wish to stress that only the common factor \( \left[ \omega_a(p'_a) + E_{p'_N} \right] \) destroys the hermiticity of the propagators (B.5e)-(B.5h).

All of identities (B.5a)-(B.5h) contain the same expression of the final state \( \alpha = a + N' \) energy \( E_\alpha = \omega_a(p'_a) + E_{p'_N} \) in the square parenthesis. After substitution of the identities (B.5a)-(B.5h) in the relations (B.4a)-(B.4h) we define the sum of all terms with \( E_\alpha = \omega_a(p'_a) + E_{p'_N} \) as the matrix \( B_{\alpha\beta} \).

\[
B_{a+N',b+N} \equiv B_{a+N',b+N}(p'_N p'_a; \pi N p_b) =
\]

\[
\frac{1}{-\omega_b(p_b) - E_{p_N} - P_m + E_{p'_N}} \equiv \frac{1}{-\omega_b(p_b) - E_{p_N} - P_m + E_{p'_N}} - \left[ \omega_a(p'_a) + E_{p'_N} \right] + E_{p'_N} - P_m
\]

\[
(B.5g)
\]

\[
\frac{1}{-\omega_b(p_b) - E_{p_N} - E_{p'_N}} \equiv \frac{1}{-\omega_b(p_b) - E_{p_N} - E_{p'_N}} - \left[ \omega_a(p'_a) + E_{p'_N} \right] - E_{p'_N}
\]

\[
(B.5h)
\]

where the propagators (B.5e)-(B.5h) are taken from the corresponding expressions (B.4a)-(B.4h).

The terms in the left side of the identities (B.4a)-(B.4h) that are not proportional of \( \omega_a(p'_a) + E_{p'_N} \) compose the \( A_{a+N',b+N} \) matrix

\[
A_{a+N',b+N} \equiv A_{a+N',b+N}(p'_N p'_a; \pi N p_b) = -\left< \text{out}; N'\left| j_a(0), a_0^+(0) \right| N; \text{in} \right>
\]

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- \sum_{n=N'} E_{p_{N'}} \langle in; p'_N|j_a(0)|p_{N'}; in > \frac{(2\pi)^3 \delta^{(3)}(p_b + p_N - p_{N'})}{\omega_b(p_b) + E_{p_N} - E_{p_{N'}}}

- \sum_{m=mesons} (E_{p'_N} + P^o_m) < 0|j_a(0)|m; in > \frac{(2\pi)^3 \delta^{(3)}(p_b + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N} + i\epsilon}

- \sum_{m=mesons} (E_{p_N} + P^o_m) \langle in; p'_N|j_a(0)|p_N, m; in > \frac{(2\pi)^3 \delta^{(3)}(p_b - P^o_m)}{\omega_b(p_b) - P^o_m - E_{p_N} - i\epsilon}

+ \sum_{N,...} (E_{p'_N} + E_{p_N} + E_{p_{N'}}) < 0|j_a(0)|p_Np_{N'}; in > \frac{(2\pi)^3 \delta^{(3)}(p_b - p'_N - p_{N'})}{\omega_b(p_b) - E_{p'_N} - E_{p_{N'}} - E_{p_{N'}}}

+ a and b crossing terms.

It easy to observe, that the first and the fourth terms of Eq.(B.7) are Hermitian, but the second term (B.7b) is Hermitian conjugate of the third term (B.7c). Note, that these Hermitian conjugate terms (B.7b) and (B.7c) have the Hermitian conjugate singular propagators. Also the u channel propagators (B.5f) and (B.5g) are Hermitian conjugate.

We have included in \( A_{a+N',b+N} \) the equal time commutator because equal-time commutators produces the Hermitian terms for the case of the simplest renormalized Lagrangians in \( \phi^3 \) field theory. The more complicated types of the equal-time commutators which produces the contribution in the both matrices \( A_{a+N',b+N} \) and \( B_{a+N',b+N} \) are considered in our previous papers [25, 32].

For the \( 3 \leftrightarrow 2 \) transition amplitude \( \langle in; p'_N, p'_\pi|j_a(0)|p_N; in > \) which corresponds to the \( a + \pi' + N' \leftrightarrow b + N \) reaction with \( a = \pi', \gamma' \) and \( b = \pi, \gamma \), the cluster decomposition of the expression (B.1b) generates the following 16 terms.

\[ w_{a+\pi'+N',b+N}^c = - \langle in; p'_\pi, p'_N|[j_a(0), a_b^+(0)]|p_N; in > \]

\[ + \sum_{n=N''} \langle in; p'_\pi, p'_N|j_a(0)|p_{N''}; in > \frac{(2\pi)^3 \delta^{(3)}(p_b + p_N - p_{N''})}{\omega_b(p_b) + E_{p_N} - E_{p_{N''}}} \langle in; p_{N''}|j_b(0)|p_N; in > \]

\[ + \sum_{m=mesons} < p'_\pi|j_a(0)|m; in > c \frac{(2\pi)^3 \delta^{(3)}(p_b + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N} + i\epsilon} \langle in; p'_N, m|j_b(0)|p_N; in > c \]
\[
+ \sum_{m=\text{mesons}} <in; p'_\pi, p'_N|j_a(0)|p_N, m; in> \frac{(2\pi)^3 \delta(3)(p_b - p_m)}{\omega_b(p_b) - P_m} <in; m|j_b(0)|0 >
\]

(B.8c)

\[
- \sum_{N} <p'_\pi|j_a(0)|p_N p_N'; in> \frac{(2\pi)^3 \delta(3)(p_b - p'_N - p_N')}{\omega_b(p_b) - E_{p'_N} - E_{p_N'}} <in; p'_N, p_N'|j_b(0)|0 >
\]

(B.8d)

\[
+ \sum_{n=N', \ldots} <in; p'_N|j_a(0)|n; in> \frac{(2\pi)^3 \delta(3)(p_b + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} - P_m - \omega_{p'_N} + i\epsilon} <in; p'_\pi, n|j_b(0)|p_N; in> c
\]

(B.8e)

\[
+ \sum_{m=\text{mesons}} <0|j_a(0)|m; in> \frac{(2\pi)^3 \delta(3)(p_b + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} - P_m - \omega_{p'_N} + i\epsilon} <in; p'_\pi, p'_N, m|j_b(0)|p_N; in>
\]

(B.8f)

\[
+ \sum_{m=\text{mesons}} <in; p'_N|j_a(0)|p_N, m; in> \frac{(2\pi)^3 \delta(3)(p_b - p_m - p'_N)}{\omega_b(p_b) - P_m - \omega_{p'_N}} <in; p'_\pi, m|j_b(0)|0 >
\]

(B.8g)

\[
- \sum_{N, \ldots} <0|j_a(0)|p_N p_N'; in> \frac{(2\pi)^3 \delta(3)(p_b - p'_N - p_N - p'_\pi)}{\omega_b(p_b) - E_{p'_N} - E_{p_N} - \omega_{p'_\pi}} <in; p'_\pi, p'_N, p_N'|j_b(0)|0 >
\]

(B.8h)

\[+
\text{a and b crossing 8 terms.}
\]

These terms are depicted in Fig. 3A - Fig.3H correspondingly.

The conjugate 2 ⇐ 3 transition amplitude \(< in; p'_N|j_a(0)|p_b p_\pi p_N; in >\) relates to the potential

\[w'_c = \langle j_a(0), a'_b(0) | p_N p_\pi; in > \]

(B.9a)

\[
+ \sum_{n=N'} <in; p'_N|j_a(0)|p_N'; in> \frac{(2\pi)^3 \delta(3)(p_b + p_N + p_\pi - p_{N'})}{\omega_b(p_b) + E_{p_N} + \omega_{p_\pi} - E_{p_{N'}}} <in; p_{N'}|j_b(0)|p_N p_\pi; in> c
\]

(B.9b)

\[
+ \sum_{m=\text{mesons}} <0|j_a(0)|m; in> \frac{(2\pi)^3 \delta(3)(p_b + p_\pi + p_N - p_m - p'_N)}{\omega_b(p_b) + E_{p_N} + \omega_{p_\pi} - P_m - E_{p'_N} + i\epsilon} <in; p'_N, m|j_b(0)|p_\pi p_N; in> c
\]

(B.9b)
\[
+ \sum_{m=\text{mesons}} \frac{\langle in; \mathbf{p}'_N | j_a(0) | \mathbf{p}_N, m; in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_\pi - \mathbf{P}_m)}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - P_m^0 + i\epsilon} \langle in; m | j_b(0) | \mathbf{p}_\pi; in \rangle > \tag{B.9c}
\]

\[-\sum_N \frac{\langle 0| j_a(0) | \mathbf{p}_N \mathbf{p}_N^\pi, in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_\pi - \mathbf{p}'_N - \mathbf{p}_N^\pi)}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{p'_N} - E_{p_N^\pi} + i\epsilon} \langle in; \mathbf{p}'_N, \mathbf{p}_N^\pi | j_b(0) | \mathbf{p}_\pi; in \rangle > \tag{B.9d}
\]

\[+ \sum_{n=\text{mesons}} \frac{\langle in; \mathbf{p}'_N | j_a(0) | \mathbf{p}_\pi, n; in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_N - \mathbf{p}_m)}{\omega_b(\mathbf{p}_b) - E_{p_N} + P_m^0 + E_{p'_N} + i\epsilon} \langle in; n | j_b(0) | \mathbf{p}_N; in \rangle > \tag{B.9e}
\]

\[+ \sum_{m=\text{mesons}} \frac{\langle 0| j_a(0) | \mathbf{p}_\pi \mathbf{p}_N \mathbf{p}_N^\pi, m; in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b - \mathbf{p}_m)}{\omega_b(\mathbf{p}_b) + E_{p_N} - P_m^0 - E_{p'_N} + i\epsilon} \langle in; m | j_b(0) | 0 \rangle > \tag{B.9f}
\]

\[-\sum_N \frac{\langle 0| j_a(0) | \mathbf{p}_\pi \mathbf{p}_N \mathbf{p}_N^\pi, in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b - \mathbf{p}'_N - \mathbf{p}_N^\pi)}{\omega_b(\mathbf{p}_b) - E_{p'_N} - E_{p_N^\pi}} \langle in; \mathbf{p}'_N, \mathbf{p}_N^\pi | j_b(0) | 0 >
\]

\[+ a \text{ and } b \text{ crossing 8 terms.} \]

Using the generalization of the identities (B.5a)-(B.5h) and combining (B.8b), (B.8e) terms with (B.9c), (B.9f) correspondingly, we obtain

\[
B_{a+\pi'+N',b+N} \equiv B_{a+\pi'+N',b+N}(\mathbf{p}'_N \mathbf{p}'_\pi \mathbf{p}'_a; \mathbf{p}_N \mathbf{p}_b) =
\]

\[\sum_{n=N''} \frac{\langle in; \mathbf{p}'_N | j_a(0) | \mathbf{p}_N \mathbf{p}_N''; in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_N - \mathbf{p}_N'')} {\omega_b(\mathbf{p}_b) + E_{p_N} - E_{p_{N''}}} \langle in; \mathbf{p}_N | j_b(0) | \mathbf{p}_N'\rangle > \]

\[+ \sum_{m=\text{mesons}} \frac{\langle \mathbf{p}'_N | j_a(0) | m; in \rangle (2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_N - \mathbf{p}_m - \mathbf{p}'_N) }{\omega_b(\mathbf{p}_b) + E_{p_N} - P_m^0 + E_{p'_N} + i\epsilon} \langle in; \mathbf{p}_N | m | j_b(0) | \mathbf{p}_N; in \rangle > \]

\[\omega_a(\mathbf{p}_a') + \omega_\pi(\mathbf{p}_\pi') - P_m^0 - i\epsilon \tag{B.10b}
\]
\[ B_{\alpha+\pi'+N',b+N}(p'_N p'_\pi'; p_N p_b) = B_{b+N',a+\pi+N}(p_N p_b; p_N p_\alpha p_\pi), \]

i.e. the multichannel matrix \( B_{\alpha\beta} \) contains the Hermitian conjugate non-diagonal parts. For the other part of the \( U_{\alpha\beta}(E) \) potential we have

\[ A_{a+\pi'+N',b+N} \equiv A_{a+\pi'+N',b+N}(p'_N p'_\pi; p_N p_b) = -<in; p'_\pi, p'_N|j_a(0), a^+_b(0)|p_N; in> \]

where each term of expression (B.10a)-(B.10h) is nonhermitian. Nevertheless

\[ B_{a+\pi'+N',b+N}(p'_N p'_\pi'; p_N p_b) = B_{b+N',a+\pi+N}(p_N p_b; p_N p_\alpha p_\pi), \]

i.e. the multichannel matrix \( B_{\alpha\beta} \) contains the Hermitian conjugate non-diagonal parts. For the other part of the \( U_{\alpha\beta}(E) \) potential we have

\[ A_{a+\pi'+N',b+N} \equiv A_{a+\pi'+N',b+N}(p'_N p'_\pi; p_N p_b) = -<in; p'_\pi, p'_N|j_a(0), a^+_b(0)|p_N; in> \]
The pure three-body amplitude 

\[ < in; \mathbf{p}'_N, m|j_b(0)|\mathbf{p}_N; in > \]

\[ \frac{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N} + i\epsilon}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p'_N} - i\epsilon} \]

\[ (B.11b) \]

\[ (2\pi)^3\delta(3) (p_b - P_m) < in; m|j_b(0)|0 > \]

\[ \frac{\omega_b(p_b) - P^o_m}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p'_N} - i\epsilon} \]

\[ (B.11c) \]

\[ \frac{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N} + i\epsilon}{\omega_a(p'_a) + \omega_\pi(p'_\pi) + i\epsilon} \]

\[ (B.11d) \]

\[ \frac{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N}}{\omega_a(p'_a) + E_{p'_N} - P^o_m - i\epsilon} \]

\[ (B.11e) \]

\[ \frac{\omega_b(p_b) + E_{p_N} - P^o_m - E_{p'_N} - \omega_\pi(p'_\pi) + i\epsilon}{\omega_a(p'_a) + E_{p'_N} - P^o_m} \]

\[ (B.11f) \]

\[ \frac{\omega_b(p_b) + E_{p_N} - P^o_m}{\omega_a(p'_a) + E_{p'_N} - P^o_m - i\epsilon} \]

\[ (B.11g) \]

\[ \frac{\omega_b(p_b) - E_{p'_N} - E_{p_{N'}}}{\omega_a(p'_a) - E_{p_{N}} - E_{p_{N'}}} \]

\[ (B.11h) \]

\[ + a \text{ and } b \text{ crossing } 8 \text{ terms.} \]

The pure three-body amplitude 

\[ < in; \mathbf{p}'_N, m|j_b(0)|\mathbf{p}_N; in > \]

\[ c \] that describes the \( a + \pi' + N' \) \( \iff \) \( b + \pi + N \) reaction with \( a = \pi' \) or \( \gamma' \) and \( b = \pi \) or \( \gamma \) relates to.
the three-body potential \( w_{a + \pi' + N', b + \pi + N} \) (2.13), that contains the following 48 potential terms after the cluster decomposition of the expression (B.1b)

\[
w_{a + \pi' + N', b + \pi + N} = - \left< in; \mathbf{p}'_{\pi}, \mathbf{p}'_{N} \bigg| j_a(0), a_b^+(0) \bigg| \mathbf{p}_{\pi} \mathbf{p}_{N}; in \right>
\]

\[
+ \left\{ \sum_{n=N''} < in; \mathbf{p}'_{\pi}, \mathbf{p}'_{N} | j_a(0) | \mathbf{p}_{N''}; in > \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_b + \mathbf{p}_\pi + \mathbf{p}_N - \mathbf{P}_{N''})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) + E_{\mathbf{p}_N} - E_{\mathbf{p}_{N''}}} < in; \mathbf{p}_{N''} | j_b(0) | \mathbf{p}_{\pi}; in > \right\}
\]

\[
+ \sum_{m={\text{mesons}}} < in; \mathbf{p}'_{\pi}, \mathbf{p}'_{N} | j_a(0) | \mathbf{p}_{N}, m; in > \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_\pi + \mathbf{p}_b - \mathbf{P}_m - \mathbf{P}'_{\pi})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{\mathbf{p}_N} - E_{\mathbf{p}_{N'}} + i\epsilon} < in; \mathbf{p}'_{N}, \mathbf{p}_{N} | j_b(0) | \mathbf{p}_{\pi}; in >
\]

\[
+ \sum_{m={\text{mesons}}} < in; \mathbf{p}'_{\pi}, \mathbf{p}'_{N} | j_a(0) | \mathbf{p}_{N}, m; in > e \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_\pi + \mathbf{p}_b - \mathbf{P}_m - \mathbf{P}'_{\pi})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{\mathbf{p}_N} - E_{\mathbf{p}_{N'}} + i\epsilon} < in; \mathbf{p}'_{N}, \mathbf{p}_{N} | j_b(0) | \mathbf{p}_{\pi}; in >
\]

\[
- \sum_{N} < \mathbf{p}'_\pi | j_a(0) | \mathbf{p}_{N} \mathbf{p}_{N'}; in > \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_\pi + \mathbf{p}_b - \mathbf{P}'_{N'} - \mathbf{P}_{N'})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{\mathbf{p}_N} - E_{\mathbf{p}_{N'}} - \omega_\pi(\mathbf{p}'_{\pi}) + i\epsilon} < in; \mathbf{p}'_{\pi}, \mathbf{p}_N | j_b(0) | \mathbf{p}_{\pi}; in >
\]

\[
+ \sum_{m={\text{mesons}}} < in; \mathbf{p}'_{\pi} | j_a(0) | \mathbf{p}_{N}, m; in > \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_\pi + \mathbf{p}_b - \mathbf{P}_m - \mathbf{P}'_{\pi})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{\mathbf{p}_N} - E_{\mathbf{p}_{N'}} + i\epsilon} < in; \mathbf{p}'_{\pi}, \mathbf{p}_N | j_b(0) | \mathbf{p}_{\pi}; in >
\]

\[
- \sum_{N} < 0 | j_a(0) | \mathbf{p}_{N} \mathbf{p}_{N'}; in > \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}_\pi + \mathbf{p}_b - \mathbf{P}'_{N'} - \mathbf{P}_{N'})}{\omega_b(\mathbf{p}_b) + \omega_\pi(\mathbf{p}_\pi) - E_{\mathbf{p}_N} - E_{\mathbf{p}_{N'}} - \omega_\pi(\mathbf{p}'_{\pi}) + i\epsilon} < in; \mathbf{p}'_{\pi}, \mathbf{p}_N | j_b(0) | \mathbf{p}_{\pi}; in >
\]
three-body generalization of the identities (B.5a)-(B.5h) we obtain

\[ B_{a+\pi'+N',b+\pi'+N} \equiv B_{a+\pi'+N',b+\pi'+N}(p'_a p'_\pi p'_N; p_b p_\pi p_N) = \]

\[ \sum_{n=N''} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) + E_{p_N} - E_{p_{N''}}} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) + E_{p_N} - E_{p_{N'}} + i\epsilon} \]

\[ - \sum_{N} \frac{\langle in; p'_\pi| j_a(0)| p_N p_N\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) - E_{p_N'} - E_{p_{N''}}} \]

\[ + \sum_{n=N''} \frac{\langle in; p'_\pi| j_a(0)| n \rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi| j_a(0)| p_N p_N\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ \{ \}

\[ + 8 \text{ terms with } \pi \text{ transposition and } + 8 \text{ terms with the both pion transposition. } \]

\[ + a \text{ and } b \text{ crossing } 24 \text{ terms.} \]

Expressions (B.12a)-(B.12h) are depicted in Fig.4A-Fig.4H correspondingly. Using the three-body generalization of the identities (B.5a)-(B.5h) we obtain

\[ B_{a+\pi'+N',b+\pi'+N} \equiv B_{a+\pi'+N',b+\pi'+N}(p'_a p'_\pi p'_N; p_b p_\pi p_N) = \]

\[ \sum_{n=N''} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) + E_{p_N} - E_{p_{N''}}} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) + E_{p_N} - E_{p_{N'}} + i\epsilon} \]

\[ - \sum_{N} \frac{\langle in; p'_\pi| j_a(0)| p_N p_N\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi) - E_{p_N'} - E_{p_{N''}}} \]

\[ + \sum_{n=N''} \frac{\langle in; p'_\pi| j_a(0)| n \rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi| j_a(0)| p_N p_N\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N''}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ + \sum_{m=\text{mesons}} \frac{\langle in; p'_\pi, p'_N| j_a(0)| p_{N'}\rangle}{\omega_b(p_b) + \omega_\pi(p_\pi)} \]

\[ \{ \}

\[ + 8 \text{ terms with } \pi \text{ transposition and } + 8 \text{ terms with the both pion transposition. } \}

\[ + a \text{ and } b \text{ crossing } 24 \text{ terms.} \]
Here expressions (B.13a), (B.13d), (B.13e) and (B.13h) are Hermitian. And the expressions (B.13b), (B.13f) are Hermitian conjugate of (B.13c), (B.13g) correspondingly. Therefore the complete potential \( B_{a+a'N',b+b'\pi+N} \) is Hermitian.

\[
A_{a+a'N',b+b'\pi+N} \equiv A_{a+a'N',b+b'\pi+N}(p'_{N}p'_{\pi};p_{0}p_{N}) = -<in; p'_{\pi}, p'_{N',a}|j_{a}(0), a_{b}^{+}(0)|p_{N}; in>
\]

\[
\left\{ -\sum_{n=N''}E_{p_{N''}} <in; p'_{\pi}, p'_{N'}|j_{a}(0)|p_{N''}; in >_{c} (2\pi)^{3}\delta^{(3)}(p_{b} + p_{\pi} + p_{N} - p_{N''}) \right. \]

\[
\left. \frac{\omega_{b}(p_{b}) + \omega_{\pi}(p_{\pi}) + E_{p_{N}} - E_{p_{N''}}}{\omega_{a}(p'_{a}) + \omega_{\pi}(p'_{\pi}) - E_{p_{N'}}} \right.
\]

\[
\left. + \sum_{m=mesons} \left[ p_{m}^{0} + E_{p_{N}} \right] <in; p'_{\pi}|j_{a}(0)|m; in >_{c} (2\pi)^{3}\delta^{(3)}(p_{b} + p_{\pi} + p_{N} - p_{m} - p_{N'}) \right. \]

\[
\left. \frac{\omega_{b}(p_{b}) + \omega_{\pi}(p_{\pi}) + E_{p_{N}} - E_{p_{N'}} - i\epsilon}{\omega_{a}(p'_{a}) + \omega_{\pi}(p'_{\pi}) - E_{p_{N'}}} \right.
\]

\[
\left. + \sum_{N} \left[ E_{p_{N'}} + E_{p_{N}} + E_{p'_{N}} \right] <p'_{\pi}|j_{a}(0)|p_{N}p_{N'}; in > (2\pi)^{3}\delta^{(3)}(p_{b} + p_{\pi} - p'_{N'} - p_{N'}) \right. \]

\[
\frac{\omega_{b}(p_{b}) + \omega_{\pi}(p_{\pi}) + E_{p_{N}} - E_{p_{N'}} - i\epsilon}{\omega_{a}(p'_{a}) + \omega_{\pi}(p'_{\pi}) - E_{p_{N'}} - E_{p_{N'}}} \right.
\]
\[
\langle \text{in}; p'_N, p'_\pi | j_\alpha(0) | p_\pi \rangle >_c \\
\frac{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p_N} - E_{p'_\pi}}{
\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p_N} - E_{p'_\pi}}
\]

(B.14d)

\[
- \sum_{n=N'} \left[ p_n^0 + \omega_\pi(p_\pi) \right] \langle \text{in}; p'_N | j_\alpha(0) | p_\pi \rangle >_c (2\pi)^3 \delta^{(3)} \frac{(p_0 + p_\pi + p_N - p'_N - p'_\pi)}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p_N} - E_{p'_\pi} - \omega_\pi(p_\pi) + i\varepsilon}
\]

(B.14e)

\[
- \sum_{m=\text{mesons}} \left[ E_{p_N} + p_m^0 + \omega_\pi(p_\pi) \right] \langle \text{in}; p'_N | j_\alpha(0) | p_\pi \rangle >_c (2\pi)^3 \delta^{(3)} \frac{(p_0 + p_\pi + p_N - p'_N - p'_\pi - p'_N - p'_\pi)}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p_N} - E_{p'_\pi} - \omega_\pi(p_\pi) + i\varepsilon}
\]

(B.14f)

\[
+ \sum_{N} \left[ E_{p'_N} + E_{p_N} + E_{p'_\pi} + \omega_\pi(p_\pi) \right] \langle \text{in}; p'_N | j_\alpha(0) | p_\pi \rangle >_c (2\pi)^3 \delta^{(3)} \frac{(p_0 + p_\pi - p'_N - p'_\pi)}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p'_N} - E_{p'_\pi} - \omega_\pi(p'_\pi) + i\varepsilon}
\]

(B.13g)

\[
+ \sum_{N} \left[ E_{p'_N} + E_{p_N} + E_{p'_\pi} + \omega_\pi(p_\pi) \right] \langle \text{in}; p'_N | p_N | p'_\pi \rangle >_c (2\pi)^3 \delta^{(3)} \frac{(p_0 + p_\pi - p'_N - p'_\pi)}{\omega_a(p'_a) + \omega_\pi(p'_\pi) - E_{p'_N} - E_{p'_\pi} - \omega_\pi(p'_\pi) + i\varepsilon}
\]

(B.13h)

\[
+ 8 \text{ terms with } \pi \text{ transposition and } + 8 \text{ terms with the both pion transposition. }
\]

\[
+ a \text{ and } b \text{ crossing } 24 \text{ terms},
\]

where \( A_{a+\pi'+N',b+\pi+N} \) has the same structure as \( B_{a+\pi'+N',b+\pi+N} \). But there one must taken into account an additional combinations expressions (B.14e),(B.14f),(B.14g),(B.14h) wit other 4 terms, where the initial pion \( \pi \) is transposed instead of the final pion \( \pi' \).
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