NON STANDARD COHOMOLOGY FOR EQUIVARIANT SHEAVES: THE ROLE OF GENERIC MODELS

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Abstract. We introduce a new class of generic cohomologies and show how, in some cases, they simplify non standard cohomologies [5, 18]. For doing so, we use a previous generalization of the Generic Model Theorem for equivariant exact presheaves of structures; extending the results of Macintyre and Caicedo [6, 17].

Foreword
We work with an equivariant version of the sheaves of structures introduced by Comer [7] and Macintyre [17] and later further expanded by Caicedo [6], suitable for working with a transformation group $G$ acting on the sheaf. A $G$-structure is a structure with an action such that elements of $G$ commute with language symbols. Given a presheaf of $G$-structures $\mathcal{M}$ on $X$; through a previous extension of (topo)logical “truth” due to Caicedo [6] to the equivariant context, we show that

- If $\mathcal{M}$ is exact, then it has a generic $G$-model.
- If $(\mathcal{M}, d)$ is a differential presheaf of structures with generic model $\mathcal{M}^{\text{gen}}$, then there is a canonical generic cohomology structure $H^{\text{gen}}(X, \mathcal{M})$ with nice category-theoretic properties.

This is done through a simplification in the presentation of the pointwise forcing relation. The article follows this sequence: Sections 1 and 2 provide the basics on $G$-structures and equivariant presheaves with fibers of that kind. In section 3 we review the local semantics and their behavior. Section 4 is devoted to the construction of equivariant generic models and what we mean by their “genericity” (Theorem 4.4.3), so we extend both the Generic Model Theorem 5.2 of [6] and Theorem 3 of [17] to the equivariant context. We also introduce a new class of cohomologies related to differential generic models. Our aim here is therefore to establish the first results towards a Model Theoretic analysis of geometric structures beyond sheaves.

From now on, we fix a group $G$.

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1. G-STRUCTURES

We follow usual model theory conventions (see for example [14, 19]) in our definition (and notation) of a language $\mathcal{L} = (\mathcal{F}, \mathcal{R}, \mathcal{C})$ and a structure $\mathcal{M} = (M, \mathcal{F}^M, \mathcal{R}^M, \mathcal{C}^M)$. Relations, functions, constants, arities, formulas and semantics (such as $\mathcal{M} \models \varphi(a)$) are understood on this context.

1.1. A morphism $\alpha$ of structures $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$ is a map between universe sets $M \xrightarrow{\alpha} N$ that commutes with the language symbols. In particular, we only ask $\alpha (\mathcal{R}^M) \subset \mathcal{R}^N$ for each $R \in \mathcal{R}$. Geometric reasons for this will be appreciated soon. An isomorphism is a bijective morphism whose inverse is also a morphism.

1.2. A morphism $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$ is transfilled (short for “transversally filled”) iff $\alpha^{-1} (\mathcal{R}^N) \subset \mathcal{R}^M$ for each relation symbol $R \in \mathcal{R}$. An embedding (resp. a submersion) is an injective (resp. surjective) transfilled morphism. Given two structures $\mathcal{M}, \mathcal{N}$ such that $M \subset N$; we say that $\mathcal{M}$ is a substructure of $\mathcal{N}$ iff the inclusion map is an embedding, in that case we write $\mathcal{M} \leq \mathcal{N}$.

Given a transfilled morphism $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$; there is a unique substructure $\mathcal{I}(\alpha)$ of $\mathcal{N}$ whose universe $\text{Im}(\alpha)$ is the image set of $\alpha$. From the obvious equivalence relation on $M$ induced by $\alpha$ we also obtain a quotient model $\mathcal{M}/\sim$ whose universe is the quotient set $M/\sim$; the quotient projection $\mathcal{M} \xrightarrow{\pi} \mathcal{M}/\sim$ is a submersion and the induced arrow $\mathcal{M}/\sim \xrightarrow{\pi} \mathcal{I}(\alpha)$ is an isomorphism.

1.3. Fix some group $G$. A $G$-structure is a structure $\mathcal{M}$ such that

(1) The universe $M$ is endowed with an action $G \times M \xrightarrow{\Phi} M$.

(2) The action commutes with the language symbols. More precisely

(a) The set of constants is invariant: $GC^A = C^A$.

(b) Relations are invariant subsets: $G^nR^A = R^A$ for each $R \in \mathcal{R}$. In other words; given $(x_1, \ldots, x_n^R) \in R^A$ and $g_1, \ldots, g_n^R \in G$, we also have $(g_1x_1, \ldots, g_n^Rx_n^R) \in R^A$.

(c) functions are $G$-equivariant: For each $f \in \mathcal{F}$ with arity $n$, $g \in G$ and $x_1, \ldots, x_n \in A$:

$$f^A(gx_1, \ldots, gx_n) = gf^A(x_1, \ldots, x_n)$$

1.4. Examples. Here there are some examples:

(1) A module $\mathcal{M} = (M, +, 0_\mathcal{M})$ over a principal ideal domain $(D, +, 0, 1)$ is a $(D, +, 0)$-structure. The equivariance of the map $+$ implies linearity of scalar stretching since $am + am' = +(am, am') = a \cdot [(m, m')] = a(m + m')$ for $a \in D$, $m, m' \in M$. Also $D \cdot 0_\mathcal{M} = 0_\mathcal{M}$ so the set of constants is invariant.
(2) For a compact group $G$ and a topologic $G$-space $X$; each $g \in G$ provides, by left multiplication, a homeomorphism $X \xrightarrow{g} X$. These induce a family of chain isomorphism on singular chains

$$SC_*(X) \xrightarrow{g} SC_*(X).$$

Notice that $G$ acts linearly on the singular chain groups $SC_*(X)$ and also on the homology groups $H_*(X)$ so these cases are similar to example (1). The same can be done for Lie groups, smooth manifolds, smooth forms and De Rham cohomology. It actually can be extended to more complicated (co)homology theories, see for instance [22].

(3) This notation will be useful in the sequel: For any set $Y$ and a cardinal $\kappa \leq |Y|$, write $Y^{[\kappa]} = \{A \subset Y : |A| = \kappa\}$. The sets $Y^{[<\kappa]}$ and $Y^{[\leq \kappa]}$ are defined in a similar way. A countable polyhedron is a pair $(S, \subset)$ in the language of posets $L = (\prec)$, where $S \subset \mathbb{N}^{[<\kappa]}$ is a family of finite subsets of the set of natural numbers $\mathbb{N}$ in which is hereditary in the following sense: if $u \subset v \in S$ then $u \in S$; see [20] for details. The standard $n$-simplex corresponds to $\Delta^n = (n, P(n))$ where, as usual, $0 = \emptyset \ n = \{0, \ldots, n-1\}$ for $n > 0$, and $P(n)$ is the set parts of $n$. For this finite polyhedron, the unique subgroup $G \leq S_n$ such that $\Delta^n$ is a $G$-structure is the trivial group $G = \{e\}$. On the other hand; the geometric boundary $\partial(\Delta^n) = P(n)\{n\}$, is a $G$-structure for any subgroup of $G \leq S_n$.

(4) We due the following example/remark to X. Caicedo. The orbit set $M/G$ of a $G$-structure $M$ is not necessarily a structure in the same language. For this to happen, we should consider a stronger notion $G$-structure, replacing 2(c) by more restrictive condition as, for instance, that each function $f^M$ to be coordinatewise equivariant; then the induced quotient functions would make sense in $(M/G)^n$. It is possible to construct generic orbit models coming from sheaves of "strong" $G$-structures; however the first two examples of this list would be excluded.

1.5. A morphism of $G$-structures is a $G$-equivariant morphism. A G-substructure of $M$ is a $G$-invariant substructure. The composition of $G$-equivariant morphisms (resp. embeddings, submersions, etc.) is an arrow of the same kind.

1.6. The family of $G$-structures and $G$-equivariant morphisms (resp. embeddings, submersions, elementary embeddings) is a category, we will denote it by $\mathcal{M}_G$ (resp. $\mathcal{M}_G^\leq$, $\mathcal{M}_G^\prec$, $\mathcal{M}_G^\succ$). For an inverse system of $G$-structures $\{M_i : i \in D\}$ and $a \in M_i$ for some $i \in D$ we write $[a]$ for the germ of $a$ in the colimit $M = \text{colim} M_i$. The limit action of $G$ on $M$ is well defined.

Proposition 1.6.1. Let $M = \text{colim} M_i$, be a colimit in $\mathcal{M}_G^\leq$, and $a \in M_i^n$ for some $i \in D$. If $\varphi(v)$ has no $\neg\forall; \text{ then } \models \varphi([a])$ if and only if there is some $j \leq i$ such that $M_j \models \varphi(\rho_j(a))$.

[Proof] ($\Rightarrow$) By induction on formulas. For instance:
If \( \varphi \) is \( t(v) = s(v) \): If \( M \models \varphi([a]) \) then \( t^M([a]) = s^M([a]) \) so \( t^M_i(a) = s^M_i(a) \). There is some \( j \leq i \) such that \( \rho_{j,i}(t^M_i(a)) = \rho_{j,i}(s^M_i(a)) \).

Since \( \rho_{j,i} \) is a morphism it commutes with terms, so \( t^M_j(\rho_{j,i}(a)) = s^M_j(\rho_{j,i}(a)) \); therefore \( M_j \models \varphi(\rho_{j,i}(a)) \).

The case \( \varphi(v) := [t(v) \in R] \) is similar. Next suppose that

- \( \varphi(v) = \psi(v) \land \theta(v) \): By induction, assume the statement for both \( \psi(v) \) and \( \theta(v) \). \( M \models \varphi([a]) \) iff \( M \models \psi([a]) \) and \( M \models \theta([a]) \). Then \( \exists k, k' \leq i \) such that \( M_k \models \psi(\rho_{k,i}(a)) \) and \( M_{k'} \models \theta(\rho_{k,i}(a)) \). Take any \( j \leq k, k' \) and \( \rho_{j,i}(a) = \rho_{k,i}(\rho_{k,i}(a)) = \rho_{k,i}(\rho_{k,i}(a)) \). Now, notice that all restrictions maps are embeddings and, by our assumptions, we can suppose that \( \psi, \theta \) are quantifier-free formulas. Since \( \rho_{k,i} \) is an embedding and \( M_k \models \psi(\rho_{k,i}(a)) \), we deduce that \( M_j \models \psi(\rho_{j,i}(a)) \). Similarly, we get \( M_j \models \theta(\rho_{j,i}(a)) \). Therefore, \( M_j \models \varphi(\rho_{j,i}(a)) \).

The other cases are similar. The converse (\( \Rightarrow \)) holds because morphisms preserve the validity of formulas without \( \neg, \forall \).

\[ \square \]

2. Sheaves of \( G \)-structures

2.1. The definitions (and notation) of presheaves, restrictions, and induced sheaves are directly taken from [3, 11]. A presheaf of \( G \)-structures on \( X \) is a presheaf \( (\mathcal{T}, \subset) \xrightarrow{M} \mathcal{M}_G \). Each open subset \( U \) of \( X \) is sent to some \( G \)-structure \( M_U \), and each inclusion of open subsets \( U \subset V \) is mapped to the corresponding equivariant restriction morphism \( M_V \xrightarrow{\rho_{UV}} M_U \). When \( G \) is trivial it is usual to talk about a “presheaf of structures”.

2.2. Examples. The following examples were inspired by Gendron [10]; we further develop them later in this paper as examples [13] and [14]. Let’s consider some examples on \( X = \mathbb{N} \) the set of natural numbers with the discrete topology.

(1) The sheaf of real sequences \( \mathcal{RS} \) is given as follows: For each \( U \subset \mathbb{N} \) define \( \mathcal{RS}_U = \mathbb{R}^U \) as the set of all maps from \( U \) to \( \mathbb{R} \). For each inclusion \( U \subset V \) there is a restriction map \( \mathcal{RS}_V \xrightarrow{\rho_{UV}} \mathcal{RS}_U \) given by \( \alpha \mapsto \alpha|_U \). Coherence and exactness are straightforward. The group \( \mathbb{Z} \) of integer numbers acts on \( \mathcal{RS} \) with the translations induced by its structure as an additive subgroup of \( \mathbb{R} \), more precisely \( \mathbb{Z} \times \mathcal{RS}_U \xrightarrow{\cdot} \mathcal{RS}_U \) is given by \( (n, \alpha)(i) = n + \alpha(i), \forall i \in U \).

(2) The presheaf \( \mathcal{G} \) of graphs on \( \mathbb{N} \) is given as follows: For each \( U \subset \mathbb{N} \) define \( \mathcal{G}_U = 2^{U^{[2]} \alpha} \). An element of \( \mathcal{G}_U \) is a function \( U^{[2]} \xrightarrow{\alpha} 2 \) that decides, for each possible edge \( e = \{u, v\} \subset U \), whether \( u, v \) are connected (\( \alpha(\{u, v\}) = 1 \)) or not (\( \alpha(\{u, v\}) = 0 \)). If \( U \subset V \) then \( U^{[2]} \subset V^{[2]} \), so the restriction \( \rho_{UV}(\alpha) = \alpha|_{i^{[2]}_U} \) makes sense: it corresponds to the graph obtained by dropping the vertices in \( U \setminus V \) [8]. This presheaf is exact but not coherent. It is possible to define in a
similar way a presheaf $\mathcal{P}^k$ of $k$-polyhedra, where $k > 0$ is the geometric dimension allowed; as $k$ grows, there are more possibilities to extend local $k$-polyhedra, so $\mathcal{P}^k$ becomes less coherent.

3. Local Semantics

3.1. **Pointwise semantics.** Fix some presheaf of $G$-structures $\mathcal{M}$ on $X$ and a point $x \in X$. Let $\varphi(v)$ be a formula in free variables $v = (v_1, \ldots, v_n)$. We also fix (temporarily) an open set $U \ni x$ and some element $a \in M_U$; we say that $\mathcal{M}$ **forces** $\varphi(a)$ at $x$, and write $\mathcal{M} \vDash_x \varphi(a)$, according to the following induction:

1. $\varphi(v)$ has no $\neg, \forall$: Then $\mathcal{M} \vDash_x \varphi(a)$ if there is some open set $V \ni x$ with $x \in V$, such that $M_V \models \varphi(\rho_U(v))$. Notice that, by Proposition 1.6.1 this is equivalent to require that $M_x \models \varphi([a]_x)$.

2. $\varphi(v)$ is $\neg \psi(v)$: $\mathcal{M} \vDash_x \varphi(a)$ if there is some open set $V \ni x$ with $x \in V$, such that $\mathcal{M} \vDash_x \psi(a)$ for all $y \in V$.

3. $\varphi(v)$ is $\psi(v) \rightarrow \nu(v)$: $\mathcal{M} \vDash_x \varphi(a)$ if there is some open set $V \ni x$ with $x \in V$, such that, for all $y \in V$; if $\mathcal{M} \vDash_x \psi(y)$ then $\mathcal{M} \vDash_x \nu(a)$.

4. $\varphi(v)$ is $\forall u \psi(v, w)$: $\mathcal{M} \vDash_x \varphi(a)$ if there is some open set $V \ni x$ with $x \in V$, such that, for each $y \in V$ and each $b \in M_V$, we have $\mathcal{M} \vDash_y \psi(a, b)$.

**Proposition 3.1.1.** Pointwise semantics is equivalent to Caicedo’s local semantics at $[6]$.

[Proof] See Proposition 4.1.1; [21] p.6.

3.1.1. Given a point $x \in X$, an open set $U \ni x$, a formula $\varphi(v)$ in free variables $v = (v_1, \ldots, v_n)$ and some $a \in M_U$; the following properties hold:

(a) **Local Semantics:** $\mathcal{M} \vDash_x \varphi(a)$ iff there is some open set $U \ni V \ni x$ such that $\mathcal{M} \vDash_y \varphi(a)$ for all $y \in V$.

(b) **Classical Semantics:** For an isolated point $x \in X$ we have that $\mathcal{M} \vDash_x \varphi(a) \iff M_x \models \varphi([a]_x)$.

(c) **Excluded Middle Principle:** $\mathcal{M} \vDash_x \forall u \forall v(u = v \lor v \neq v)$ iff there is an open set $U \ni x$ such that at all elements $y \in U$, all pairs of sections $a, b$ defined at $y$ are forced at $y$ to be equal or different. When $\mathcal{M}$ is a sheaf and the base space $X$ is Hausdorff, this means exactly that the induced sheafspace is Hausdorff in some neighborhood of $x$.

3.2. **Open semantics.** Given a presheaf of $G$-structures $\mathcal{M}$, an open set $U \subset X$ and some $a \in M_U$; we say that $\mathcal{M}$ **forces** $\varphi(a)$ in $U$, and we write $\mathcal{M} \vDash_U \varphi(a)$, iff $\mathcal{M} \vDash_x \varphi(a)$ for all $x \in U$. By Proposition 3.1.1 (1), $\mathcal{M} \vDash_v \varphi(a) \iff$ there is some neighborhood $U \ni V \ni x$ such that $\mathcal{M} \vDash_v \varphi(a)$.

The validity of $\varphi(a)$ is related to the topology of $X$:

(a) **Restrictions:** If $U \ni V$ then $\mathcal{M} \vDash_v \varphi(a) \Rightarrow \mathcal{M} \vDash_U \varphi(a)$.

(b) **Coverings:** $\mathcal{M} \vDash_U \varphi(a) \Rightarrow \mathcal{M} \vDash_{\cap_i U_i} \varphi(a)$.
(c) Existential quantifier: $\mathcal{M} \models_{U} \exists \nu \varphi(a, \nu)$ iff there is an open covering $\bigcup_{i} U_{i} \supset U$ and some $b_{i} \in \mathcal{M}_{U_{i}}$ for each $i$; such that $\mathcal{M} \models_{U_{i}} \varphi(a, b_{i})$ for each $i$.

**Proposition 3.2.1. [Maximum principle]** Let $\mathcal{M}$ be an exact presheaf of $G$-structures on $X$, $x \in X$ a point, $U \ni x$ an open set and $a$ in $\mathcal{M}_{V}$. If $\mathcal{M} \models_{U} \exists \nu \varphi(a, \nu)$ then there is some open subset $V \subset U$ and $b \in \mathcal{M}_{V}$ such that $\mathcal{M} \models_{V} \varphi(a, b)$.

**Proof** This is a translation of Theorem 3.3 in [6, p.18]. See Proposition 4.2.2; [21, p.13]. □

### 4. Equivariant generic models

In this section we show how the models constructed at §4 are generic. Fix a presheaf of $G$-structures $\mathcal{M}$ on $X$.

#### 4.1. For the definitions and existence of filters and ultrafilters see [16, p.83,135].

The following definition is from [6]. A (non trivial) filter generated by a family of open subsets $F$ in $X$ is generic with respect to the presheaf $\mathcal{M}$ iff:

1. For each formula $\varphi(v)$ in free variables $v = (v_{1}, \ldots, v_{n})$, $U \in F$ and $a \in M_{U}$; there is some $U \supset V \in F$ such that $\mathcal{M} \models_{V} \varphi(\sigma)$ or $\mathcal{M} \models_{V} \neg \varphi(\sigma)$.
2. For each formula $\varphi(v, w)$ in free variables $v = (v_{1}, \ldots, v_{n})$ and $w = (w_{1}, \ldots, w_{m})$; each $U \in F$ and $a \in M_{U}$; if $\mathcal{M} \models_{U} \exists w \varphi(a, w)$ then there is some $U \supset V \in F$ and $b \in M_{V}$ such that $\mathcal{M} \models_{V} \varphi(a, b)$.

**Proposition 4.1.1. [Existence of generic filters]** If $\mathcal{M}$ is an exact presheaf of $G$-structures; then every ultrafilter in $X$ (generated by a family of open subsets) is generic with respect to $\mathcal{M}$.

**Proof** For condition 4.1.1-(1) apply the same proof of Theorem 5.1 at [6, p.27]. For condition 4.1.1-(2) the main argument is the maximum principle which, in our context, only requires the hypothesis of exactness. □

#### 4.2. A generic model for a presheaf of $G$-structures $\mathcal{M}$ is the colimit structure $\mathcal{M}^{\text{gen}} = \text{coLim}_{U \in F} \mathcal{M}_{U}$ on a generic filter $F$.

**Corollary 4.2.1. [Existence of equivariant generic $G$-models]** Every exact presheaf of $G$-structures $\mathcal{M}$ has a generic $G$-model $\mathcal{M}^{\text{gen}}$.

#### 4.3. Example.

For the sheaf $\mathcal{R}S$ of real sequences on $\mathbb{N}$; fix an ultrafilter $\mathcal{F}$ in $\mathbb{N}$. Given $U \subset \mathbb{N}$, the quotient projection $\mathcal{R}S_{U} \xrightarrow{q_{U}} \mathcal{R}S^{\text{gen}}$ sends each $\alpha$ to its $\mathcal{F}$-germ $[\alpha]$. In particular, the map $\mathcal{R}S_{\mathbb{N}} \xrightarrow{q_{\mathbb{N}}} \mathcal{R}S^{\text{gen}}$ is surjective: For each $[\alpha] \in \mathcal{R}S^{\text{gen}}$, $\alpha \in \mathcal{R}S_{U}$; take $\beta \in \mathbb{R}^{\mathbb{N}}$ defined as $\beta(i) = \alpha(i)$ if $i \in U$, and $\beta(i) = 0$ if $i \notin U$. Then $[\alpha] = [\beta] = q_{\mathbb{N}}(\beta)$. We deduce that $\mathcal{R}S^{\text{gen}}$ is the quotient of $\mathbb{R}^{\mathbb{N}}$ by the equivalence modulo the ultrafilter $\mathcal{F}$, i.e. the ultraproduct of $\mathbb{R}$ which leads to
the structure of non standard real numbers $\mathcal{R}\mathcal{S}^{\text{gen}} = \mathcal{R}$. We should also notice that this is a sheaf of strong $\mathbb{Z}$-structures, in the sense of §1.4-(4). Then

$$(\mathcal{R}\mathcal{S}/\mathbb{Z})^{\text{gen}} \cong \frac{(\mathbb{R}/\mathbb{Z})^\mathbb{N}}{\sim_p} = \mathcal{S}^1$$

is the non-standard unit circle group. Genericity, in this case, is just the universal semantic property of ultraproducts.

4.4. Let us show the behavior of the forcing relation under double negations. We start with two easy statements, the proofs are left to the reader who can go to [6] for more details.

**Lemma 4.4.1.** Let $\varphi(v)$ be a positive formula. Then $\mathcal{M} \models_{v} \neg(\neg \varphi(a))$ iff there is some open set $V \subset U$ such that $V$ is dense in $U$ and $\mathcal{M} \models_{v} \varphi(a)$.

**Lemma 4.4.2.** Let $F$ be a maximal filter of open sets in $X$, and $U \in F$. If $V \subset U$ is open and dense in $U$, then $V \in F$.

The Gödel translation $\varphi_c$ of some formula $\varphi$ is defined, by induction, as follows:

- $\varphi_c$ is $\neg(\neg \varphi)$ for an atomic formula $\varphi$.
- $(\varphi \land \psi)_c = \varphi_c \land \psi_c$.
- $(\varphi \lor \psi)_c = \neg(\neg \varphi \land \neg \psi_c)$.
- $(\neg \varphi)_c = \neg(\varphi_c)$.
- $(\forall v \varphi)_c = \forall (\varphi_c)$.
- $(\exists v \varphi)_c = \neg \forall (\neg \varphi_c)$.

**Theorem 4.4.3.** [Equivariant generic model theorem] Let $\mathcal{M}$ be a sheaf of $G$-structures on $X$ and $\mathcal{M}^{\text{gen}}$ the generic model induced by some generic filter $\mathcal{F}$ on $X$. For each formula $\varphi(v)$, $U \in \mathcal{F}$ and $a \in \mathcal{M}_U$; the following statements are equivalent:

1. $\mathcal{M}^{\text{gen}} \models [\varphi([a])]$.
2. $\mathcal{M} \models_{v} \varphi_c(a)$ for some $U \supset V \in \mathcal{F}$.
3. $\{x \in U : \mathcal{M} \models_{x} \varphi_c(a)\} \in \mathcal{F}$.

[Proof] See Theorem 4.3.3 at [21].

5. Generic cohomology

The model theory of cohomology has had some earlier results (see [18]). Our notion is adapted to sheaves and $G$-sheaves - we exploit the functoriality of generic models in our definition and provide various examples that show that this extends classical notions of cohomology, and furthermore allows us to define new extensions.

5.1. Let us fix two exact presheaves $\mathcal{M}, \mathcal{N}$ of $G$-structures. Then;

1. By colimit properties; $\mathcal{M}^{\text{gen}}$ inherits a natural action, so it is a $G$-structure.
(2) Each morphism of presheaves (i.e. natural transformation) \( \mathcal{M} \rightarrow \mathcal{N} \)
induces a morphism \( \mathcal{M}^{gen} \rightarrow \mathcal{N}^{gen} \) of structures. If the first arrow is
equivariant then the second is amorphism of \( G \)-structures.

5.2. When \( \mathcal{M} \) an abelian category there is a zero-structure. A

differential of \( \mathcal{M} \) is a transfilled endomorphism \( \mathcal{M} \xrightarrow{d} \mathcal{M} \) such that
\( d^2 = 0 \). We define the
generic cohomology of \( X \) with values in \( (\mathcal{M},d) \) as
\[
H^{gen}(X,\mathcal{M}) := H(\mathcal{M}^{gen},d) = \ker \left[ \mathcal{M} \xrightarrow{d} \mathcal{M} \right] / \text{Im} \left[ \mathcal{M} \xrightarrow{d} \mathcal{M} \right]
\]
These “cohomology structures” are well defined by \( \mathcal{M} \); they extend the usual notion
of cohomology. Since the space \( X \) belongs to any ultrafilter, there is a quotient
map \( \mathcal{M} \rightarrow \mathcal{M}^{gen} \) which commutes with the differential \( d \), and a well defined
morphism of cohomology structures
\[
H(X,\mathcal{M}) \rightarrow H^{gen}(X,\mathcal{M})
\]

5.3. Usual examples come from algebraic topology context. Among them,

(1) Singular homology groups for \( G \)-spaces \([4]\).

(2) De Rham cohomology, for smooth \( G \)-manifolds \([2,3,12]\).

(3) Intersection homology \([13]\) and cohomology \([22]\).

(4) \((N,q)\)-cohomologies of amplitude \( 1 \leq k < N \), and \( N > 0 \); \([9,15]\).

The relation between these presheaves is usually given given in terms of
cohomology spectral sequences.

5.4. Example: Non standard generic cohomology. For similar examples
coming from other contexts, see \([5]\). Consider the sheaf \( \mathcal{Z}_nS \) of sequences on
(subsets of) \( \mathbb{N} \) with values on the ring \( \mathbb{Z}_n \) of integers modulo some fixed \( n > 0 \); let
\( n = p_1^{m_1} \cdots p_s^{m_s} \) be its prime decomposition.

For each \( U \subset \mathbb{N} \) define \( \mathcal{Z}_nS_U = \mathcal{Z}_n^{U} \) and for each inclusion \( U \subset V \) let \( \rho_{UV}(\alpha) = \alpha|_U \) as usual. A linear map \( \mathcal{Z}_n^{h} \xrightarrow{d} \mathcal{Z}_n^{n} \) will induce a sheaf endomorphism
\( \mathcal{Z}_nS \xrightarrow{d} \mathcal{Z}_nS \) iff \( d \) is diagonal; i.e. iff for each \( i \in \mathbb{N} \), the element \( e_i \in \mathcal{Z}_n^{N} \) of the
canonical basis satisfies \( d(e_i) = \pi_i e_i \) for some \( \pi_i \in \mathcal{Z}_n \). Notice \( d^N(\pi) = \left( \pi_i^{N} \right)_{i \in \mathbb{N}} \)
for all \( x \in \mathcal{Z}_n^{N} \), \( N > 0 \); so \( d \) is nilpotent iff there is some \( N > 0 \) such that we can solve
\( a_i^{N} \equiv 0(n) \) for all \( i \in \mathbb{N} \).

Given \( \pi \neq 0 \) in \( \mathcal{Z}_n \); let \( 1 \leq a = b(q) \leq n - 1 \); where \( (q,n) = 1 \), i.e. \( q \) is the coprime
part of \( a \), so \( b = p_1^{m_1} \cdots p_s^{m_s} \) for some \( m_1, \ldots, m_s \). Then \( \pi \) is nilpotent in \( \mathcal{Z}_n \) iff
p_j/a (or, equivalently, 1 \leq m_j \leq r_j) for each j = 1, \ldots, s. Since the multiplication on \(Z_n\) by a unit \(\overline{\alpha}\) is an isomorphism; the order of \(\overline{\alpha}\) and \(\overline{\beta}\) coincides, so we will assume without loss of generality that \(q = 1\).

Assume that \(d \neq 0\) and suppose that the eigenvalues \(\{\alpha_i^j : i \in \mathbb{N}\}\) of \(d\) satisfy those conditions. The representative elements \(a_i\) are taken from a finite subset of \(\{1, \ldots, n - 1\}\). For each \(i \in \mathbb{N}\) pick the first \(k_i > 0\) such that \(a_i^{k_i} \equiv 0(n)\). Since the set \(\{k_i : i \in \mathbb{N}\}\) is finite; the order of \(d\) is \(N = \max\{k_i : i \in \mathbb{N}\}\).

The generic model \(Z_n S^{\text{gen}} = Z_n\) is the set of non-standard \(Z_n\)-sequences, i.e. the ultrapower of \(Z_n\) modulo some ultrafilter \(F\) in \(\mathbb{N}\). We get a well defined differential morphism \(Z_n S^{\text{gen}} \xrightarrow{d} Z_n S^{\text{gen}}\) given by \(d([\overline{x}]) = [d(\overline{x})]\) where \([\overline{x}]\) is the germ of an element \(\overline{x} \in Z_n S_U, U \subset \mathbb{N}\). Following [9, 15] \((Z_n, d)\) is a \(N\)-complex. For \(1 \leq m \leq N - 1\), the \(m\)th-amplitude generic cohomology is

\[H_m^{\text{gen}}(X, Z_n S) := H_m\left(\overset{\sim}{Z_n}\right) = \frac{\ker\left(\overset{\sim}{Z_n} \xrightarrow{d^k} \overset{\sim}{Z_n}\right)}{\operatorname{Im}\left(\overset{\sim}{Z_n} \xrightarrow{d(N-k)} \overset{\sim}{Z_n}\right)}\]

Interesting examples arise as \(n\) has many divisors. For instance:

- \(n = 12\): Take \(d([\overline{x}]) = ([6x])\). Since \(6^2 = 36 = 0, d^2 = 0\) and \(d\) is a usual differential operator. \(\ker(d) = \{\overline{x} \in Z_{12} : x_i = \text{even } \forall i\} \cong Z_n^\ast\). Similarly, it can be seen that \(\operatorname{Im}(d) \cong Z_3\) so \(H^{\text{gen}}(Z_{12} S, d = \overline{6}) = \overset{\sim}{Z_n} / \overset{\sim}{Z_3} = \overset{\sim}{Z_3}\).

- \(n = 48\): The eigenvalues \(\overline{\alpha}_i\) of \(d\) correspond to the integers \(\{a_i = 6, 12, 24\}\). The cohomology \(H^{\text{gen}}(X, Z_n S)\) will depend, on the dimension of the subspaces of eigenvectors. Since at least one of them must have infinite dimension, there always be a non-standard factor. Up to some adjusts, we can rearrange the canonical base as the disjoint union of subspaces of eigenvectors. We obtain the following table of (usual) generic cohomologies \((N = 2)\):

| \(d\) | \(H^{\text{gen}}(X, Z_{48} S)\) |
|------|----------------------------------|
| \(\overline{12}\) | \(Z_3\) |
| \(\overline{24}\) | \(Z_{12}\) |
| \(\overline{12} \oplus \overline{24}\) | \(Z_3^\ast \oplus Z_{12}\), or \(Z_3 \oplus Z_{12}^\ast\), or \(Z_3 \oplus Z_{12}^\ast\) |

In the last row, the factor with a superscript \(\kappa \in \mathbb{N}\) corresponds to a finite-dimensional subspace of eigenvectors, whenever it is the case. The order of the (diagonal) differential operator \(d\) induced by the eigenvalue \(\overline{6}\) is \(N = 4\). The amplitude generic cohomology \(H^{\text{gen}}(X, Z_{48} S)\) for \(m = 1, 2, 3\) is always \(\overset{\sim}{Z_n}\). Finally; the differential operators \(\overline{6} \oplus \overline{12}, \overline{6} \oplus \overline{24}\) and \(\overline{6} \oplus \overline{12} \oplus \overline{24}\) can be treated in a similar way.

5.5. **A final comment.** A description of \(H^{\text{gen}}\) as a loose cohomology and its relation to Weil cohomologies, in the terms of Macintyre [15], is still a pending
assignment. The above examples are similar to those non standard cohomologies of Brünjes and Serpé, [5]. New examples of generic cohomologies should appear from structure sheaves such as those provided by Abramsky [1]. An additional reason of interest in this subject, from a model theoretic perspective, is the development of stability theoretical tools for the classification of sheaves. These question are not treated here. The authors hope to fulfill these tasks in a forthcoming article.

References

[1] ABRAMSKY, S.; MANSFIELD, S. & SOARES, R. The Cohomology of NonLocality and Contextuality. 8th. Int. Workshop on QPL. EPTCS 95 (2012) pp. 1-14.
[2] BOTT, R. & LU T. Differential Forms in Algebraic Topology. Graduate Texts in Math. Vol. 82. Springer-Verlag (1982).
[3] BREDON, G. Sheaf theory. Graduate Texts in Mathematics Vol. 170 2nd. Ed. Springer-Verlag. Berlin (1997).
[4] BREDON, G. Introduction to Compact Transformation Groups. Pure and Applied Mathematics Vol. 46. Academic Press. New York 1972.
[5] BRUNJES, L. & C. SERPÉ, Non standard étale cohomology. Jour. Pure App. Algebra Vol. 214 Nro. 6, (2010) pp. 960-981.
[6] CAICEDO, X. Lógica de los haces de estructuras. Rev. Acad. Colomb. Cienc. 19-74 (1995) pp.569-586.
[7] COMER, S. Elementary properties of structures of sections. Bol. Soc. Mat. Mexicana. 19 (1974).
[8] DIESTEL, R. Graph Theory. Graduate Texts in Mathematics Vol. 173 Springer-Verlag. New York-Heidelberg-Berlin 2000.
[9] DUBOIS-VIOLETTE, M. Generalized homologies for $d^N = 0$ and graded $q$-differential algebras. Secondary calculus and cohomological physics. Contemp. Math. 219 (1998) pp.69-79.
[10] GENDRON, T. Real algebraic number theory II. Arxiv.math NT.1208.4334 (2012).
[11] GODEMENT, R. Topologie Algébrique et Théorie des Faisceaux. Hermann, Paris 1958.
[12] GREUB, W.; HALPERIN, S. & VANSTONE, R. Connections, curvature and cohomology. Pure and Applied Mathematics Vol. 47. Academic Press, New York 1972.
[13] GORESKY, M. & MACPHERSON, R. Intersection homology theory. Topology. Vol. 9. 135-162 (1980).
[14] HODGES, w. Model theory. Encyclopedia of Mathematics and its Applications. Vol. 42 Cambridge Univ. Press, Cambridge 1993.
[15] KAPRANOV, M. On the $q$-Analog of Homological Algebra. Arxiv Math. AT. 9611005. (1996).
[16] KELLEY, J. General Topology. D. Van Nostrand, 1955.
[17] MACINTYRE, A. Model completeness for sheaves of structures. Fundamenta Mathematicae 81.1, (1973) pp.73-89.
[18] MACINTYRE, A. Nonstandard Analysis and Cohomology. In Nonstandard Methods and Applications in Mathematics Editors: Nigel J. Cutland, Mauro Di Nasso, David A. Ross. Lecture Notes in Logic, 25. Association for Symbolic Logic. Providence, RI, 2006.
[19] MARKER, D. Model theory, an introduction. Graduate Texts in Mathematics. Vol. 217 Springer-Verlag, New York 2002.
[20] MIJARES, J. & PADILLA, G. A Ramsey space of infinite polyhedra and the infinite random polyhedron. Arxiv Math.LO 1209.6421v4 (2016).
[21] PADILLA, G. & VILLAVECES, A.; Sheaves of G-structures and generic G-models. Arxiv Math.LO 1304.2477v2 (2013).
[22] SARALEGI, M. Homological Properties of Stratified Spaces. Illinois J. Math. 38, (1994) pp.47-70.

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