THE UNIVERSAL UNRAMIFIED MODULE FOR \( GL(n) \) AND
THE IHARA CONJECTURE

GILBERT MOSS

Abstract. Let \( F \) be a finite extension of \( \mathbb{Q}_p \). Let \( W(k) \) denote the Witt vectors of an algebraically closed field \( k \) of characteristic \( \ell \) different from \( p \), and let \( Z \) be the spherical Hecke algebra for \( GL_n(F) \) over \( W(k) \). Given a Hecke character \( \lambda : Z \to R \), where \( R \) is an arbitrary \( W(k) \)-algebra, we introduce the universal unramified module \( M_{\lambda,k} \). We show \( M_{\lambda,k} \) embeds in its Whittaker space and is flat over \( R \), resolving a conjecture of Lazarus. It follows that \( M_{\lambda,k} \) has the same semisimplification as any unramified principal series with Hecke character \( \lambda \).

In the setting of mod-\( \ell \) automorphic forms of \([\text{CHT}08]\), Clozel, Harris, and Taylor formulate a conjectural analogue of Ihara’s lemma. It predicts that every irreducible submodule of a certain cyclic module \( V \) of mod-\( \ell \) automorphic forms is generic. Our result on the Whittaker model of \( M_{\lambda,k} \) reduces the Ihara conjecture to the statement that \( V \) is generic.

1. The universal unramified module

1.1. Main results. Let \( F \) be a finite extension of \( \mathbb{Q}_p \) with residue field of order \( q \), and ring of integers \( \mathcal{O}_F \). Let \( G := GL_n(F) \) and let \( K := GL_n(\mathcal{O}_F) \). Let \( k \) be an algebraically closed field of positive characteristic \( \ell \neq p \), and \( \ell \neq 2 \).

Given a \( k[G] \)-module \( \pi \) (always presumed to be smooth), the spherical Hecke algebra \( k[K \backslash G/K] \) acts on the submodule \( \pi^K \) of \( K \)-fixed vectors via double-coset operators. Denote this action by \( \ast \).

Let \( k[G/K] \) denote the space of finitely supported functions on the set \( G/K \), and let \( G \) act by left-translation. Then \( k[G/K] \) admits a natural action of \( k[K \backslash G/K] \) and, in fact, the map \( k[K \backslash G/K] \to \text{End}_{k[G]}(k[G/K]) \) is an isomorphism ([Laz98, Prop 1.16]).

Let \( \lambda : k[K \backslash G/K] \to k \) be a homomorphism. We define the universal unramified module for \( \lambda \):

\[
\mathcal{M}_{\lambda,k} := k[G/K] \otimes_{k[K \backslash G/K]} k.
\]

It is universal in the following sense: for any \( k[G] \)-module \( V \), denote

\[
\pi^{K,\lambda} := \{ v \in \pi^K : z \ast v = \lambda(z)v \text{ for all } z \in k[K \backslash G/K] \},
\]

then there is an isomorphism \( \text{Hom}_{k[G]}(\mathcal{M}_{\lambda,k}, \pi) \cong \pi^{K,\lambda} \), where the map \( 1_K \otimes 1 \mapsto v \) corresponds to \( v \in \pi^{K,\lambda} \).

The module \( M_{\lambda,k} \) has been studied in [Kat81, SL96, Laz98, Laz00, Bel02, BO03, CHT08, Gro14]. In [SL96, CHT08], its properties were applied in the global setting of mod-\( \ell \) automorphic representations. The goal of this article is to examine the

Date: December 18, 2020.

2010 Mathematics Subject Classification. 11F33, 22E50, 22E55.
structure of $\mathcal{M}_{\lambda,k}$ and its Whittaker model, and apply our findings toward several outstanding questions in both the local and global settings.

Let $B = TU$ be the standard Borel subgroup, where $T$ is the diagonal torus. There is an isomorphism of rings, due to Satake,

$$k[K\backslash G/K] \cong k[T/T(\mathcal{O}_F)]^{W_C},$$

where $W_G$ is the Weyl group of $G$, and the ring structure on $k[K\backslash G/K]$ is given by convolution. Any map $\lambda : k[K\backslash G/K] \rightarrow k$ corresponds to a Weyl orbit of unramified characters $\chi : T \rightarrow k^\times$, and we can ask about the connection between $\mathcal{M}_{\lambda,k}$ and the associated unramified principal series representations $i^G_{\lambda,\chi} := \{\phi : G \rightarrow k \text{ locally constant} : \phi(tug) = \delta_B^{-1}(t)\chi(t)\phi(g), t \in T, u \in U, g \in G\}$. The modulus character $\delta_B : B \rightarrow k^\times$ is defined as

$$\delta_B(b) := [bK b^{-1} : bK b^{-1} \cap K]/[K : bK b^{-1} \cap K],$$

for any choice of compact open subgroup $K$ with pro-order prime to $\ell$ (Vig96 I.2.6). The following conjecture was made by Lazarus.

**Conjecture 1.1** ([Laz98], §2 Remarque). There is an equality of Jordan–Holder multisets $JH(\mathcal{M}_{\lambda,k}) = JH(i^G_{\lambda,\chi})$.

Conjecture 1.1 was already known in many cases: when $n = 2$ it follows from results of Serre in [SL96], which are proved with Bruhat–Tits theory. It is proved in [Laz98] when the characteristic $\ell$ is banal for $G$ (i.e. $\ell = 0$ or $\ell \nmid \#\text{GL}_n(F_q)$) and in [CHT08] Lemma 5.1.4 for $\ell > n$ and $q \equiv 1 \bmod \ell$. It appears in [BO03] for arbitrary $\ell \neq p$ for $G\ell_3$, again using Bruhat–Tits theory.

**Theorem 1.2.** Conjecture 1.1 is true.

We prove the conjecture by showing $W(k)[G/K]$ is flat over $W(k)[K\backslash G/K]$, where $W(k)$ is the Witt vectors. This flatness was conjectured by Lazarus [Laz00 Conjecture 1.0.5], and was previously known for $GL_3$ ([BO03 §1.3]). We emphasize that Theorem 1.2 does not contain an assumption that $\ell$ is banal.

It is natural to ask whether the arrangement of the Jordan–Hölder constituents of $\mathcal{M}_{\lambda,k}$ exhibits a consistent structure as we deform $\lambda$. For example, when $n = 2$ it was shown by Serre in [SL96] that $\mathcal{M}_{\lambda,k}$ always has a unique irreducible submodule that is infinite-dimensional.

Fix an additive character $\psi : F \rightarrow k^\times$ with conductor zero, and let $\psi$ also denote the usual extension of $\psi$ to a nondegenerate character $U \rightarrow k^\times$. We say a $k[G]$-module $\pi$ is generic if $\text{Hom}_{k[G]}(\pi, \text{Ind}_G^U \psi) \neq 0$, where $\text{Ind}_G^U \psi$ is the space

$$\{W : G \rightarrow k \text{ locally constant} : W(ug) = \psi(u)W(g), \ u \in U, \ g \in G\}.$$

The Shintani formula for spherical Whittaker functions (Shi76) implies that the map

$$\text{ev}_1 : \text{Ind}_G^U \psi \rightarrow k$$

$$W \mapsto W(1)$$

induces an isomorphism $(\text{Ind}_G^U \psi)^{K,\lambda} \rightarrow k$ (see Section 2). Let $W_0^0 \in (\text{Ind}_G^U \psi)^{K,\lambda}$ denote the preimage of 1. The universal property of $\mathcal{M}_{\lambda,k}$ gives a canonical morphism $\Lambda : \mathcal{M}_{\lambda,k} \rightarrow \text{Ind}_G^U \psi$ sending $1_K \otimes 1$ to $W_0^0$.

**Theorem 1.3.** For every $\lambda$, the map $\Lambda : \mathcal{M}_{\lambda,k} \rightarrow \text{Ind}_G^U \psi$ is injective.
In fact, we prove a stronger result over $W(k)$: see Theorem 7.1 below.

When $n = 1$, $k[G/K]$ can be identified with the universal unramified character $F^\times \to k[K\backslash G/K]^\times \cong k[X^\pm 1]^\times$, and the result is immediate. When $n = 2$, Theorem 1.3 is easily deduced from Serre’s description [SL96] and the fact that irreducible representations of $GL_2(F)$ are generic if and only if they are infinite-dimensional. Theorem 1.3 was tentatively conjectured by Clozel, Harris, and Taylor ([CHT08, end of §5.1]).

Only one irreducible Jordan–Holder constituent of $\mathcal{M}_{\lambda,k}$ is generic (this follows from Theorem 1.2 or Proposition 3.1 below). But Theorem 1.3 tells us the unique generic constituent must occur as a submodule, and that it is the only irreducible submodule. Representations with this property are essentially absolutely irreducible generic in the terminology of Emerton and Helm in [EH14, §3.2], where they were studied in the context of formulating the local Langlands correspondence in families.

1.2. Further questions. These questions have been posed in more general settings. The assumption that $\text{char}(F) = 0$ is a relic of the same assumption appearing in the reference [Hel16], which we cite to prove admissibility of the universal module over its endomorphism ring; it is almost certainly not needed there, but this would need to be checked. The assumption $\ell \neq 2$ is mostly for convenience in the introduction, as it is made in [Laz98] to ensure $k$ contains a square root of $q$ for the Satake isomorphism. In the rest of this paper, all the arguments are over $W(k)$, and remain valid when $\ell = 2$ by replacing $W(k)$ with $W(k)[\sqrt{q}]$.

Beyond $GL_n$, Lazarus has conjectured a criterion for the flatness of $R[G/K]$ over $R[K\backslash G/K]$ for $G$ an unramified reductive $p$-adic group and $K$ hyperspecial, in terms of the Weyl action on unramified characters of a minimal Levi ([Laz00 Conjecture 1.0.5]). The criterion is established for banal $\ell$ ([Laz00], and flatness has been shown for arbitrary reductive groups of rank one ([Bel02, IV]). One can also consider the compact induction $c\text{-Ind}_K^G \rho$ for nontrivial $\rho$. In [Gro14], a flatness criterion was investigated for arbitrary reductive $p$-adic groups, nontrivial $\rho$, and $\ell = p$.

In future work we will try to extend the methods of the present article to these more general settings. We remark that new insights are required to go beyond $GL_n$, or at least to groups $G(F)$ having non-generic cuspidal irreducible representations. This is because Lemma 4.1, which links Jacquet functors to Whittaker models in the induction arguments for Theorems 4.2 and 7.1, would fail. For example, there are cuspidal irreducible representations of $Sp_4(F)$ that do not admit a Whittaker model [HPS79].

1.3. Consequences in the mod-$\ell$ representation theory of $GL_n(F)$. We can now characterize those unramified principal series with embeddings into $\text{Ind}_B^K \psi$. For $v \in (i_B^K \chi)^K$, let $p_v : \mathcal{M}_{\lambda,k} \to i_B^K \chi$ be the map $1_K \otimes 1 \mapsto v$, then,

$v$ is a generator for $i_B^K \chi$ $\iff$ $p_v$ is surjective
$\iff$ $p_v$ is injective (since $\text{JH}(\mathcal{M}_{\lambda,k}) = \text{JH}(i_B^K \chi)$)
$\iff$ the socle of $i_B^K \chi$ is irreducible generic.

In other words, we have the following corollary.

**Corollary 1.4.** For any unramified character $\chi : T \to k^\times$, the following are equivalent:
GILBERT MOSS

(1) \(i_B \chi\) is cyclic, generated by a spherical vector,
(2) the unique generic constituent of \(i_B \chi\) occurs as a submodule, and \(i_B \chi\) has no other irreducible submodules,
(3) \(i_B \chi\) is isomorphic to \(M_{\lambda,k}\) where \(\lambda : k[K\backslash G] \to k\) is the spherical Hecke character associated to \(\chi\).

When the characteristic of \(k\) is zero, every Weyl orbit of unramified characters contains a \(\chi\) such that \(i_B \chi\) is isomorphic to \(M_{\lambda,k}\) (it is the \(\chi\) satisfying the “does-not-proceed” condition on segments [CHT08, 4.3.2]). However, when \(k\) has positive characteristic, and \(i_B \chi\) is reducible, there may not exist a character \(\chi\) in the Weyl orbit such that \(i_B \chi\) is isomorphic to \(M_{\lambda,k}\). For example, in the “limit” case \(\ell > n\) and \(q \equiv 1 \mod \ell\), Vignéras has shown that \(i_B \chi\) is semisimple ([CHT08, Appendix B, Thm 1 (7)]), hence will not exhibit the structure of \(M_{\lambda,k}\) for any \(\chi\), if reducible.

Since no proper quotient of \(M_{\lambda,k}\) is generic, we deduce another striking corollary.

**Corollary 1.5.** Suppose \(\pi\) is a smooth \(k[G]\)-module such that
(1) \(\pi\) has a generator \(v\) in \(\pi^K,\lambda\) for some homomorphism \(\lambda : k[K\backslash G] \to k\), and
(2) \(\pi\) is generic.

Then the canonical surjection \(1_K \otimes 1 \rightarrow v : M_{\lambda,k} \rightarrow \pi\) is an isomorphism (in particular, \(\pi\) has finite length).

**1.4. Application to Ihara’s lemma.** In the global setting of mod-\(\ell\) automorphic forms of [CHT08], Clozel, Harris, and Taylor formulate a conjecture known as “Ihara’s lemma” ([CHT08 Conjecture I]). When \(n = 2\) it is deduced easily from strong approximation, but is open for \(n > 2\). Assuming the truth of Ihara’s lemma, the authors give a proof of a non-minimal \(R = T\) theorem. The weaker statement \(R^{red} = T\), where \(R^{red}\) is the reduced quotient of \(R\), was later obtained unconditionally using Taylor’s “Ihara avoidance” method ([Tay08]), and was enough for applications to the Sato–Tate conjecture. However, the full \(R = T\) theorem would have applications to special values of the adjoint L-function, would imply that \(R\) is a complete intersection, and it would be useful for generalizing the local-global compatibility results of [Eme11]. Ihara’s lemma remains a conspicuous missing piece in our understanding of congruences among algebraic automorphic forms of different levels.

In Section 9 we apply Corollary 1.5 to reduce Ihara’s lemma to an easier statement. For the sake of this introduction, we give an informal summary of the punchline, postponing the detailed discussion until Section 9.

In this subsection, let \(F_{w_0}\) be the completion at a place \(w_0\) of the CM field \(F\) appearing in the setting of [CHT08] (or Section 8 of this paper). Given a mod-\(\ell\) automorphic form \(f\) (as in [CHT08, 3.4]), having level \(K = GL_n(O_{F,w_0})\) at the place \(w_0\), one can form the cyclic \(k[GL_n(F_{w_0})]\)-submodule
\[
(GL_n(F_{w_0}) \cdot f)
\]
inside the space of mod-\(\ell\) automorphic forms having arbitrary level at \(w_0\). If \(f\) is an eigenform for a “non-Eisenstein” maximal ideal \(m\) of a certain global Hecke algebra away from \(w_0\), the Ihara conjecture predicts that all irreducible submodules of \((GL_n(F_{w_0}) \cdot f)\) are generic (see Conjecture 1.1 below for the precise statement).

Corollary 1.5 gives two reformulations of the Ihara conjecture.

**Corollary 1.6.** The following are equivalent:
(1) \( \langle GL_n(F_{w_0}) \cdot f \rangle \) has a unique irreducible submodule, which is generic, and has no other generic constituents (i.e. it is “essentially absolutely irreducible generic” in the sense of \([EH14]\)),

(2) all irreducible submodules of \( \langle GL_n(F_{w_0}) \cdot f \rangle \) are generic (i.e. the Ihara conjecture is true),

(3) \( \langle GL_n(F_{w_0}) \cdot f \rangle \) is generic.

The implications (1) \( \implies \) (2) \( \implies \) (3) are immediate; the main point is (3) \( \implies \) (1). If \( f \) is an eigenform for a non-Eisenstein maximal ideal \( m \) of the Hecke algebra at split places away from \( w_0 \), it turns out (by looking at the lift of the associated Galois representation) that \( f \) must also be an eigenvector for the action of the spherical Hecke algebra at \( w_0 \) (this is shown in \([CHT08]\)—see Theorem 9.2 below). In particular, there is a homomorphism

\[ Z_{w_0} := k[GL_n(O_{F_{w_0}}) \setminus GL_n(F_{w_0})/GL_n(O_{F_{w_0}})] \rightarrow k, \]

depending on \( m \), such that \( z \ast f = \lambda(z) f \) for \( z \in Z_{w_0} \). Therefore, the representation \( \langle GL_n(F_{w_0}) \cdot f \rangle \) satisfies conditions (1) and (2) of Corollary 1.5, and (1) follows.

For the application of Ihara’s lemma to the \( R = T \) theorem in \([CHT08]\) it suffices to know the truth of Ihara’s lemma in the quasi-banal setting: \( q \equiv 1 \mod \ell \) and \( \ell > n \), or \( \ell \) banal (c.f. \([CHT08, Prop\ 5.3.5]\)). In the quasi-banal setting we give a sufficient condition for the genericity of \( \langle GL_n(F_{w_0}) \cdot f \rangle \) in terms of the dimension of the span of the images of \( f \) under certain Iwahori–Hecke operators at \( w_0 \) (c.f. Corollary 9.5).

In the literature, there have recently been some results on modified versions of Ihara’s lemma beyond \( GL_2 \). For a very specific set of Satake parameters, Thorne proved it in the banal case using torsion vanishing results on the cohomology of Shimura varieties (\([Tho14]\)). A reformulation was given in the banal case by Sorensen (\([Sor16]\)). Boyer proved it under stronger hypotheses than irreducibility of the associated modular Galois representation (\([Boy20]\)), and it was recently generalized to Shimura curves, under stronger hypotheses, by Shotton and Manning (\([MS]\)).

Acknowledgements

The author is grateful for many helpful conversations with Ramla Abdellatif, Jean-François Dat, David Helm, Alberto Minguez, Stefan Patrikis, Gordan Savin, Vincent Sécherre, and Claus Sorensen. The author is grateful to David Helm for suggesting the induction strategy that ultimately led to the proof of Theorem 7.1.

Many thanks to Guy Henniart for pointing out a mistake in an earlier version.

2. Whittaker functions of spherical Hecke eigenvectors

Let \( Z := W(k)[K \setminus G/K] \) be the spherical Hecke algebra over \( W(k) \), let \( R \) be a ring, and let \( \lambda : Z \to R \) be a homomorphism. Define \( \text{Ind}_G^Z \psi_R \) to be the set of locally constant functions \( W : G \to R \) satisfying \( W(ug) = \psi(u)W(g), u \in U, g \in G \).

Let \( \varpi \) be a uniformizer of \( F \) and let \( T^{(j)} \) denote the element of \( Z \) given by the \( K \)-double coset operator

\[ \text{diag}(\varpi, \ldots, \varpi, 1, \ldots, 1). \]
Given any $n$-tuple $\mu \in \mathbb{Z}^n$, we let $\varpi^\mu$ denote the matrix diag$(\varpi^{\mu_1}, \ldots, \varpi^{\mu_n})$. If $W : G \to R$ is an element of $(\text{Ind}_U^G \psi_R)^K$, it follows from the Iwasawa decomposition that $W$ is entirely determined by its values on the set $\{\varpi^{\mu} : \mu \in \mathbb{Z}^n\}$.

Given a partition $\mu$ of length $n$, we define the Schur polynomial

$$S_\mu(X_1, \ldots, X_n) := \frac{\prod_{i<j} (X_i - X_j)}{\prod_{i} (X_i^{\mu_i + n - i})}.$$  

It is a symmetric function in the variables $X_1, \ldots, X_n$. If we let $T_1, \ldots, T_n$ denote the elementary symmetric functions in the variables $X_1, \ldots, X_n$, then $T_1, \ldots, T_n$ generate the ring of symmetric functions, and thus we may write $S_\mu$ as a polynomial in $T_1, \ldots, T_n$ (this dictionary is given explicitly by the Jacobi–Trudi identities in combinatorics). We will let $S_\mu(T_1, \ldots, T_n)$ denote the Schur polynomial $S_\mu$ expressed as a polynomial in the $T_i$’s.

The Satake isomorphism ([Laz98, Prop 1.6]) gives an isomorphism of rings:

$$Z := W(k)[K \setminus G/K] \cong W(k)[T/T(\mathcal{O}_F)]^{W_0} \cong W(k)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n},$$

where $q^{(i-1)/2} T(i)$ is sent to the $i$’th elementary symmetric function in the $X_i$’s.

The following proposition is a generalization of the main result of [Shi76], and the proof is nearly identical.

**Proposition 2.1.** Let $\lambda : Z \to R$ be a homomorphism, and let $W$ be an element of $(\text{Ind}_U^G \psi_R)^K$. Suppose that for each $T^{(j)} \in Z$,

$$T^{(j)}*W = \lambda(T^{(j)})W.$$

Then

$$W(\varpi^\mu) = q^{\sum_{i=1}^n(i-n)\mu_i} S_\mu(\lambda(T^{(1)}), \ldots, q^i(\lambda(T^{(2)}), \ldots, q^{n(n-1)/2} \lambda(T^{(n)})) \cdot W(1).$$

**Proof.** We will abbreviate $W(\mu) := W(\varpi^\mu)$. Set $\overline{W}(\mu) = q^{\sum_{i=1}^n(i-n)\mu_i} W(\mu)$. Let

$$I(j) := \{\epsilon \in \mathbb{Z}^n : \epsilon_i \in \{0, 1\} \text{ and } \sum_i \epsilon_i = j\}.$$

As a function $\mathbb{Z}^n \to R$, we claim that $\overline{W}$ satisfies the following conditions:

1. $\overline{W}((0, \ldots, 0)) = W((0, \ldots, 0))$
2. $\overline{W}(\mu) = 0$ if $\mu$ is non-dominant,
3. $q^{(j-1)/2} \lambda(T^{(j)})\overline{W}(\mu) = \sum_{\epsilon \in I(j)} \overline{W}(\mu + \epsilon)$ if $\mu$ is dominant, for $1 \leq j \leq n$.

The first condition is obvious, and the second follows from the conductor of $\psi$ being 0.

For the third condition, set $N_0 := N \cap K$ and $N_{0,\epsilon} := N_0 \cap \varpi^\epsilon K \varpi^{-\epsilon}$. Then by [Shi76 Sublemma], we have the following decomposition into single cosets:

$$K \varpi^\epsilon K = \bigcup_{\epsilon \in I(j)} \bigcup_{x \in N_0/N_{0,\epsilon}} x \varpi^\epsilon K.$$  

Since $T^{(j)}*W = \lambda(T^{(j)})W$, the third condition follows after computing the order of $N_{0}/N_{0,\epsilon}$ (cf. [Shi76 p.181]).

As in [Shi76 p.182] (or by an easy induction argument), a function $\overline{W} : \mathbb{Z}^n \to R$ satisfying conditions (1), (2), and (3) is uniquely determined. Since the function

$$\mu \mapsto S_\mu(\lambda(T^{(1)}), \ldots, q^{i(n-1)/2} \lambda(T^{(n)})) \cdot W((0, \ldots, 0))$$
also satisfies (1), (2), and (3), by the elementary properties of Schur polynomials, we have proved the result. □

**Corollary 2.2.** Let \( \lambda : Z \to R \) be a homomorphism. The map \( ev_1 : W \mapsto W(1) \) defines an isomorphism \((\text{Ind}^G_U \psi_R)^{K,\lambda} \cong R)\.

**Proof.** The injectivity is Proposition 2.1. The surjectivity is simply observing that, for any \( r \in R \), the Whittaker function defined by the equation

\[
W^0(\varpi \cdot) := q^{\sum_{i=1}^n (i-n) \mu_i} s_{\mu}(\lambda(T(1)), q\lambda(T(2)), \ldots, q^{n(n-1)/2} \lambda(T(n))) \cdot r
\]

is a preimage of \( r \) in the map \( W \mapsto W(1) \). □

Given a \( Z \)-module structure \( \lambda : Z \to R \), let \( W^0 \) denote the preimage of 1 in \( R \) in the map

\( ev_1 : (\text{Ind}^G_U \psi_R)^{K,\lambda} \to R \).

More explicitly,

\[
W^0(\varpi \cdot) := q^{\sum_{i=1}^n (i-n) \mu_i} s_{\mu}(\lambda(T(1)), q\lambda(T(2)), \ldots, q^{n(n-1)/2} \lambda(T(n)))
\]

As part of an induction argument below, we will require a version of Corollary 2.2 for Levi subgroups. Let \( P = MN \) be a proper standard parabolic subgroup of \( G \) with Levi \( M \) and unipotent radical \( N \). Let \( K_M := K \cap M \), let \( U_M := U \cap M \) and denote by \( Z_M \) the ring \( W(k)[K_M \backslash M/K_M] \). If \( V \) is a smooth \( W(k)[M] \)-module, \( Z_M \) acts via double-coset operators on the \( K_M \)-invariants \( V^{K_M} \). Given \( z \in Z_M \), denoted this action by \( z * v \), for \( v \in V^{K_M} \). There is a natural inclusion \( \iota : Z \to Z_M \), which can be realized via Satake as the inclusion \( W(k)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] S_n \hookrightarrow W(k)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] W_M \), where \( W_M \) is the subgroup of the Weyl group \( W_G \cong S_n \) corresponding to the Levi \( M \).

For a homomorphism \( \dot{\lambda} : Z_M \to R' \), we will consider the space

\[
(\text{Ind}^M_{U_M} \psi_{R'})^{K_M,\dot{\lambda}} := \{ W \in (\text{Ind}^M_{U_M} \psi_{R'})^{K_M} : z * W = \dot{\lambda}(z)W \text{ for all } z \in Z_M \}.
\]

**Lemma 2.3.** The map \( W \mapsto W(1) \) defines an isomorphism

\[
(\text{Ind}^M_{U_M} \psi_{R'})^{K_M,\dot{\lambda}} \to R'.
\]

**Proof.** By the Iwasawa decomposition applied to \( M \), any element \( W \in (\text{Ind}^M_{U_M} \psi_{R'})^{K_M} \) is determined by its values on weights which are dominant within each Levi component. If \( M = GL_{n_1} \times \cdots \times GL_{n_r} \), we can identify \( Z_M \cong Z_{n_1} \times \cdots \times Z_{n_r} \), where \( Z_{n_i} \) is the spherical Hecke algebra for \( GL_{n_i}(F) \). The result then follows from the same argument as in Proposition 2.1 and Corollary 2.2 applied to each Levi factor. □

3. Properties of the universal module

We will consider the universal unramified module in a more general setting than the introduction. Let \( W(k) \) denote the Witt vectors of \( k \). Given a commutative ring \( R \) and a homomorphism \( \lambda : Z \to R \), we define the universal unramified module to be

\[
\mathcal{M}_{\lambda,R} := W(k)[G/K] \otimes_{Z,\lambda} R.
\]

In this section we establish some basic properties of \( \mathcal{M}_{\lambda,R} \) that will be essential in what follows.
3.1. **Top derivative of the universal module.** Given a $W(k)$-algebra $R$, we define the functor $(-)^{(n)} : R[G]-\text{Mod} \to R-\text{Mod}$ to be the $U, \psi$-coinvariants, 

$$V^{(n)} := V_{U, \psi} := V/V(U, \psi),$$

where $V(U, \psi)$ is the sub-$R$-module generated by $\{uv - \psi(u)v : u \in U, v \in V\}$. This is the $n$th “derivative” of the Bernstein-Zelevinsky formalism introduced in [BZ77]. The derivative is exact. By the definition of $V(U, \psi)$ and right-exactness of tensor product, $(-)^{(n)}$ is compatible with extension of scalars in the sense that $(V \otimes_{W(k)} E)^{(n)} \cong V^{(n)} \otimes_{W(k)} E$ for any $W(k)$-module $E$.

Considering $W(k)[G/K]$ as a $Z[G]$-module, we can compute its $n$th derivative.

**Proposition 3.1.** The $Z$-module $(W(k)[G/K])^{(n)}$ is free of rank one. In particular, $(M_{A, R})^{(n)} \cong R$, and if $E$ is a $Z$-module, then $(W(k)[G/K] \otimes_Z E)^{(n)} \cong E$.

**Proof.** Let $\lambda = \text{id} : Z \to Z$ be the identity map. By Corollary 2.2, the space $Z \cdot W_0^\lambda = (\text{Ind}_U^G \psi_Z)^{K, \text{id}}$ is a free $Z$-module of rank 1. On the other hand, by the universal property of the universal unramified module $M := W(k)[G/K]$, we have

\begin{align*}
(\text{Ind}_U^G \psi_Z)^{K, \text{id}} & \cong \text{Hom}_Z(M, \text{Ind}_U^G \psi_Z) \\
& \cong \text{Hom}_Z(M^{(n)}, Z) 
\end{align*}

We would be done if we could show that $M^{(n)}$ is a reflexive module, i.e. $M \cong \text{Hom}_Z(\text{Hom}_Z(M^{(n)}, Z))$.

Let $p$ be a prime ideal of $Z$ and let $\lambda : Z \to \kappa$ denote the map to its residue field. Corollary 2.2 and the universal property again show

$$(\text{Ind}_U^G \psi_Z)^{K, \lambda} \cong \text{Hom}_\kappa((M \otimes_\kappa \kappa)^{(n)}, \kappa),$$

from which it follows that $(M \otimes \kappa)^{(n)} = M^{(n)} \otimes \kappa$ is one-dimensional over $\kappa$. By Nakayama, the localization $(M^{(n)})_p$ is cyclic. Applying this to the generic point $\eta = \{0\}$ of $Z$, we have that $(M^{(n)})_\eta \neq 0$, hence the annihilator of each localization $(M^{(n)})_p$ is zero in $Z_p$, hence $(M^{(n)})_p$ is free of rank 1, in particular it is reflexive. It follows that $M^{(n)}$ is reflexive. ☐

3.2. **Admissibility of the universal module.** As $W(k)[G/K]$ is the $K$-invariant subspace of $W(k)[G]$ under right-translation, $Z$ acts on $W(k)[G/K]$ on the right by convolution. This action commutes with left-$G$-translation, so there is a morphism

$$Z \to \text{End}_{W(k)[G]}(W(k)[G/K])^{\text{op}},$$

which is an isomorphism ([Laz98 Prop 1.16]). We will subsequently omit the “op,” as everything is commutative.

**Proposition 3.2.** $M_{A, R}$ is admissible as an $R[G]$-module for any $R$.

**Proof.** We only need to prove it when $R = Z$ since admissibility is preserved by extension of scalars.

This follows from the results of [Hel16]. Let $A$ be the center of the category of smooth $W(k)[G]$-modules. $A$ is a commutative ring, which by definition acts $G$-equivariantly on each object in the category in a way that commutes with all morphisms in the category. It is proven in [Hel16] that each finitely generated $W(k)[G]$-module is admissible as an $A[G]$-module (as explained on p. 4 of [Hel16]), this is an immediate consequence of the construction of faithfully projective objects.
in each block of the category \[\text{Hel10, Cor 11.18, 11.19}\], which are admissible over the centers of their respective blocks [\text{Hel16, Prop 12.7}]).

Since \(W(k)[G/K]\) is a cyclic \(W(k)[G]\)-module, it is admissible as an \(A[G]\)-module. Since the map \(Z \to \text{End}_{W(k)[G]}(W(k)[G/K])\) is an isomorphism, the action of \(A\) on \(W(k)[G/K]\) factors through a ring homomorphism \(A \to Z\). If \(H\) is any compact open subgroup and \(v_1, \ldots, v_r\) is a set of generators for \(W(k)[G/K]^H\) as an \(A\)-module, then \(v_1, \ldots, v_r\) is also a set of generators as a \(Z\)-module. \(\square\)

3.3. Jacquet module of the universal module. Let \(P = MN\) be a proper parabolic subgroup of \(G\), with Levi component \(M\) and unipotent radical \(N\). If \(R\) is a \(W(k)\)-algebra, let \(r^G_P : R[G]\text{-Mod} \to R[M]\text{-Mod}\) be the un-normalized parabolic restriction functor, and for any \(V \in R[G]\text{-Mod}\), we let \(p_N : V \to r^G_P V\) denote the canonical quotient map of \(R[M]\text{-modules}.\) Note that \(r^G_P\) commutes with arbitrary extension of scalars, for the same reasons as \((-)^{(n)}\) (c.f. Subsection 3.1).

**Lemma 3.3.** There is a map \(\Phi : W(k)[G/K] \to W(k)[M/K_M]\) which is surjective and induces an isomorphism of \(Z[M]\text{-modules}

\[r^G_P(W(k)[G/K]) \cong W(k)[M/K_M].\]

Moreover, \(\Phi(1_K) = 1_{K_M}\).

**Proof.** We turn to \[\text{BK98, §10}, \text{ or } \text{Ka98, 2.3}\] for the normalized version. Let \(dn\) denote the Haar measure on \(N\) normalized so that \(K \cap N\) has measure 1. The map given in \[\text{BK98, Lemma 10.3}\] by

\[(\Phi f)(m) = \delta_P(m) \int_N f(mn)dn, \quad \text{for } m \in M\]

makes sense over the base ring \(W(k)\), since \(W(k)\) contains a square root of \(q\). Its \(Z\)-equivariance is immediate. The proof that it induces an isomorphism

\[r^G_P(W(k)[G/K]) \cong W(k)[M/K_M]\]

exactly follows the proof of \[\text{BK98, Lemma 10.3}\] except it is simpler because we are in the special case where the representation of \(K\) under consideration is the trivial character on \(W(k)\). The fact that \(\Phi(1_K) = 1_{K_M}\) follows directly from the explicit description of the map \(\Phi\). \(\square\)

4. Flatness of the universal module

In this section we prove, for general linear groups, a conjecture of Lazarus that \(\mathcal{M}_{\lambda,R}\) is flat over \(R\) (c.f. \[\text{Laz98, Laz00, Bel02, BO03}\]). This section is a good warm-up for Section 7.

We require the following lemma generalizing the fact that cuspidal representations are generic. We will also use it in Section 7.

**Lemma 4.1.** Let \(R\) be any \(W(k)\)-algebra, let \(V\) be an admissible \(R[G]\)-module such that \(r^G_P V = 0\) for all proper parabolic subgroups \(P\). Then either \(V = 0\) or \(V^{(n)} \neq 0\).

**Proof.** If \(m\) is a maximal ideal then \(r^G_P(V_m) = (r^G_P V)_m\) and \((V^{(n)})_m = V^{(n)}_m\), so it suffices to prove the result after assuming \(R\) is a local ring.

If the result holds for all finitely generated submodules of \(V\), it also holds for \(V\) itself, thus without loss of generality we may replace \(V\) by a submodule that is finitely generated over \(R[G]\). In particular, \(V \otimes \kappa(m)\) is admissible and finitely generated, hence of finite length.
Since \( r_P^G V \) is zero, so is \( r_P^G(V \otimes_R \kappa(m)) \) for all proper parabolics \( P \). Hence the socle \( S \) of \( V \otimes \kappa(m) \) satisfies \( r_P^G S = 0 \) for all proper parabolics. Therefore \( S \) is either zero, or a finite direct sum of irreducible cuspidal \( \kappa \) are generic, we have \( S \) where

\[
\text{then}
\]

For any \( \lambda \), \( W \) by induction. For \( n = 1 \), \( W(k)[G/K] = Z \) and \( Z \cong W(k)[X_1^{\pm 1}] \).

The module \( W(k)[G/K] \) is free of rank one over \( Z \), hence flat. Since flatness is preserved by extension of scalars, so is \( M_{\wedge} R \).

For \( n > 1 \), it suffices to prove that for any injection \( \phi : E \hookrightarrow E' \) of finitely generated \( Z \)-modules, the map

\[
W(k)[G/K] \otimes_Z E \rightarrow W(k)[G/K] \otimes_Z E'
\]

\[
1_K \otimes e \mapsto 1_K \otimes \phi(e)
\]

is injective. Let \( V \) be the kernel of this map.

By the compatibility of \((-)^{(n)} \) with change of scalars, we can identify

\[
(W(k)[G/K] \otimes E)^{(n)} \cong E
\]

and similarly for \( E' \). The map on derivatives

\[
(W(k)[G/K] \otimes E)^{(n)} \rightarrow (W(k)[G/K] \otimes E')^{(n)}
\]

is given by \( E \mapsto E' \) which is injective, therefore \( V^{(n)} = 0 \).

Let \( P = MN \) be a proper parabolic subgroup. By the compatibility of the Jacquet functor \( r_P^G \) with change of scalars we can identify

\[
r_P^G(W(k)[G/K] \otimes_Z E) \cong r_P^G(W(k)[G/K]) \otimes_Z E,
\]

and similarly for \( E' \). By Lemma 3.3 we may further identify

\[
r_P^G(W(k)[G/K] \otimes_Z E) \cong W(k)[M/K_M] \otimes_Z E.
\]

As \( P \) is a proper parabolic subgroup, \( M \) is a product of groups \( G_1 \times \cdots \times G_r \) along the diagonal, with \( G_i := GL_{n_i}(F) \) for \( n_i < n \). We have a decomposition

\[
W(k)[M/K_M] = W(k)[G_1/K_1] \otimes \cdots \otimes W(k)[G_r/K_r]
\]

compatible with the decomposition \( Z_M = Z_{G_1} \otimes \cdots \otimes Z_{G_r} \), and by applying the induction hypothesis we conclude that \( W(k)[M/K_M] \) is flat over \( Z_M \). Since, in addition, \( Z_M \) is flat over \( Z \) we conclude that \( W(k)[M/K_M] \) is flat over \( Z \). Thus the map

\[
W(k)[M/K_M] \otimes_Z E \rightarrow W(k)[M/K_M] \otimes_Z E'
\]

is injective. Therefore \( r_P^G(V) = 0 \) for all \( P \), and the Theorem follows by Lemma 4.1.

\[ \square \]

**Corollary 4.3.** For any homomorphism \( \lambda : Z \rightarrow W(k) \), the \( W(k) \)-module \( M_{\lambda,W(k)} \) is free.
**Proof.** \( \mathcal{M}_{\lambda,W(k)} \) is the direct sum of the submodules \((\mathcal{M}_{\lambda,W(k)})_i \) constructed in [EH14 §2.1]. By admissibility, each \((\mathcal{M}_{\lambda,W(k)})_i \) is finitely generated over \( W(k) \) ([EH14 2.1.5]). Since direct summands of flat modules are flat, \((\mathcal{M}_{\lambda,W(k)})_i \) is flat over \( W(k) \) by Theorem 4.2, hence free.  

We now deduce Theorem 1.2 of the introduction from Corollary 4.3.

**Corollary 4.4.** For any \( \chi \) in the Weyl orbit corresponding to \( \lambda : \mathbb{Z} \to k \), \( \mathcal{M}_{\lambda,k} \) and \( \iota^G_B \chi \) have the same semisimplification.

**Proof.** Choose a lift \( \tilde{\chi} : T \to W(k)^\times \) and a corresponding lift \( \tilde{\lambda} : \mathbb{Z} \to W(k) \). Let \( \overline{K} \) be an algebraic closure of the fraction field of \( W(k) \) (it is isomorphic to \( \mathbb{C} \) by virtue of its cardinality and characteristic). Without loss of generality, suppose \( \chi \) and its lift \( \tilde{\chi} \) are chosen in the Weyl orbit in such a way that \( \iota^G_B \tilde{\chi} \otimes_{W(k)} \overline{K} \) has all of its subrepresentations generic (this is the does-not-proceed condition of [EH14 4.3.2]). Then \( W(k)[G/K] \otimes_{\mathbb{Z},\tilde{\lambda}} \overline{K} \) embeds in \( \iota^G_B \tilde{\chi} \otimes_{W(k)} \overline{K} \) by Theorem 1.3 in characteristic 0 (c.f. [EH14 Lemma 3.2.2(4)]). On the other hand, the map \( W(k)[G/K] \otimes_{\mathbb{Z},\tilde{\lambda}} \overline{K} \to \iota^G_B \tilde{\chi} \otimes_{W(k)} \overline{K} \) is also a surjection because for our choice of \( \chi \), \( \iota^G_B \tilde{\chi} \otimes_{W(k)} \overline{K} \) is also generated by its spherical vector by [Laz98] Prop 5.1. (Alternatively, one could apply the result we are currently proving, since it is already known in characteristic zero by [Laz98]). Hence

\[
W(k)[G/K] \otimes_{\mathbb{Z},\tilde{\lambda}} \overline{K} \cong \iota^G_B \tilde{\chi} \otimes_{W(k)} \overline{K}.
\]

By Corollary 1.3, \( W(k)[G/K] \otimes_{\mathbb{Z},\tilde{\lambda}} \mathbb{K} \) is an integrable lattices. On the other hand, \( \iota^G_B \tilde{\chi} \) is an integral structure since it is admissible and torsion-free over \( W(k) \) (hence free). Now apply the Brauer–Nesbitt theorem ([Vig04]) to the two \( W(k) \)-lattices \( W(k)[G/K] \otimes_{\mathbb{Z},\tilde{\lambda}} \mathbb{K} \) and \( \iota^G_B \tilde{\chi} \), and conclude that their mod-\( \ell \) reductions have the same semisimplifications. \( \square \)

### 5. Jacquet module of the Whittaker space

In this section we will investigate the Jacquet module of the space \( \text{Ind}^G_B \psi_R \). First we will recall a result of Bernstein and Zelevinsky from [BZ77] §5, which holds over a \( W(k) \)-algebra \( R \).

**5.1. Composition of restriction and induction.** In this subsection, we will extend the classical (twisted) restriction-induction result from [BZ77] §5 to arbitrary \( W(k) \)-algebras \( R \). We refer to [BZ77] for any unexplained notation.

Let \( G \) be a locally profinite group possessing a compact open subgroup with preorder invertible in \( W(k) \). Let \( \mathcal{C}(G) \) be the category whose objects consist of pairs \( (R,V) \), where \( R \) is a \( W(k) \)-algebra and \( V \) is a smooth \( R[G] \)-module, and whose morphisms consist of pairs \( (\lambda,\iota) \), where \( \lambda : R \to R' \) is a \( W(k) \)-algebra morphism and \( \iota : V \otimes_{R,A} R' \to V' \) is a morphism of \( R'[G] \)-modules.

Let \( B, T, U, \tilde{P}, M, \tilde{N} \) be closed subgroups, \( \psi \) a \( W(k) \)-valued character of \( U \), and \( \eta \) a \( W(k) \)-valued character of \( \tilde{N} \). This choice of letters aligns with the rest of the present paper, but not with [BZ77].

| Our Notation | Notation in [BZ77] |
|--------------|------------------|
| \( B = TU \) | \( P = MU \) |
| \( \tilde{P} = M\tilde{N} \) | \( Q = NV \) |
| \( \psi : U \to W(k)^\times \) | \( \theta : U \to \mathbb{C}^\times \) |
| \( \eta : \tilde{N} \to W(k)^\times \) | \( \psi : V \to \mathbb{C}^\times \) |
We make the following assumptions:

1. \( B = TU, \bar{P} = MN, T \cap U = M \cap \bar{N} = \{1\} \), \( T \) normalizes \( U \) and \( \psi \), and \( M \) normalizes \( \bar{N} \) and \( \eta \). Under this assumption, the compatibility of twisted induction and restriction with tensor products gives the normalized functors \( i_{U,\psi} : \mathcal{C}(T) \to \mathcal{C}(G) \) and \( r_{\bar{N},\eta} : \mathcal{C}(G) \to \mathcal{C}(M) \), respectively.

2. \( G \) is countable at infinity, \( U \) and \( \bar{N} \) are limits of compact subgroups.

3. the action of \( \bar{P} \) on the coset space \( X := B \backslash G \) via \( g \cdot Bh = Bh\bar{g}^{-1} \), has a finite number of orbits. In this case there is an ordering \( Z_1, \ldots, Z_k \) of the orbits such that \( Z_1, Z_1 \cup Z_2, \ldots, Z_1 \cup \cdots \cup Z_k \) are open.

4. We say a group \( H \) is decomposed with respect to \( (M, \bar{N}) \), or \( (T, U) \) respectively if \( H \cap M \bar{N} = (H \cap M)(H \cap \bar{N}), \) or \( H \cap TU = (H \cap T)(H \cap U) \).

For each orbit \( Z \subset X \) choose \( \bar{w} \in G \) such that \( B\bar{w}^{-1} \) is a point in \( Z \), and let \( w \) denote the inner automorphism \( w(g) := \bar{w}gw^{-1} \).

Define \( \Phi_Z \) by \( \Phi_Z := \psi(\bar{w}^{-1} \cdot \bar{w}) \) and \( \eta \) coincide on \( w(U) \cap \bar{N} \).

\[ T' = T \cap w^{-1}(M); \quad M' = w(T) \cap M = w(T') \]
\[ U' = M \cap w(U); \quad \bar{N}' = T \cap w^{-1}(\bar{N}); \quad \eta' = w^{-1}(\eta)|_{\bar{N}'} \]

Define \( \Phi_Z \) by \( \Phi_Z := i_{U',\psi'} \circ \varepsilon_2 \circ w \circ \varepsilon_1 \circ r_{\bar{N}',\eta'} : \mathcal{C}(T) \to \mathcal{C}(M) \), where \( \varepsilon_1 \) denotes twisting by \( \delta^{1/2}_{U'} \delta^{-1/2}_{U \cap w^{-1}(\bar{P})} \) and \( \varepsilon_2 \) denotes twisting by \( \delta^{1/2}_{\bar{N}'} \delta^{-1/2}_{\bar{N} \cap w(B)} \).

**Theorem 5.1** (§5.2, [BZ77]). Under the conditions (1)-(4), the functor

\[ F = r_{\bar{N},\eta} \circ i_{U,\psi} : \mathcal{C}(T) \to \mathcal{C}(M) \]

is glued from the functors \( \Phi_Z \) where \( Z \) runs through the \( \bar{P} \)-orbits on \( X \). More precisely, there is a filtration \( 0 = F_0 \subset \cdots \subset F_k = F \) such that \( F_i/F_{i-1} \cong \Phi_{Z_i} \).

**Proof.** With only one exception, the argument in [BZ77] §5 involves the geometry of the group \( G \), and goes through verbatim after replacing \( \mathbb{C} \) with an arbitrary \( W(k) \)-algebra \( R \). The exception is the following special case, which is case IV in [BZ77]: \( U = \{1\}, \ G = \bar{P}, \ M \subset T, \ \rho \in \text{Rep}_R(T) \). The map \( p_{\bar{N}'} : \rho \to r_{\bar{N}',\eta}(\rho) \) is the canonical projection, and the morphism

\[ \overline{A} : i_{\{1\},1}(\rho) \to r_{\bar{N}',\eta}(\rho) \]
\[ f \mapsto \int_{\bar{N}\setminus \bar{N}'} \eta^{-1}(v)p(f(v))dv, \]

induces a morphism of \( R \)-modules

\[ A : r_{\bar{N},\eta}(i_{\{1\},1}(\rho)) \to \varepsilon_2 r_{\bar{N}',\eta}(\rho) =: \Phi_Z(\rho). \]
From the functoriality of $r_{N',\eta}$ and its compatibility with extension of scalars, $\overline{A}$ and $A$ in fact define natural transformations of functors. It remains to check that $A$ is an isomorphism. Following loc. cit., we may reduce to the case $M = \{1\}$, $T = N'$, $\eta = 1$, and $\rho$ is the regular representation of $N'$ on the space of $R$-valued locally constant compactly support functions, $C^c_\infty(N',R)$. Then by transitivity of induction, $I_{\{1\},1}(\rho)$ is the regular representation of $\overline{N}$ on the space $C^\infty_c(\overline{N},R)$. A Haar integral on $\overline{N}$ is a morphism $C^\infty_c(\overline{N},R) \to R$ factoring through $r_{\overline{N},1}(C^\infty_c(\overline{N},R))$, and likewise for $N'$. Thus existence and uniqueness of Haar integrals ([BZ77] I.2.4) implies $r_{\overline{N},1}(C^\infty_c(\overline{N},R))$ and $r_{N,1}(C^\infty_c(N,R))$ are free of rank one over $R$. To check $A$ is an isomorphism we need to show it is surjective (not just nonzero, as in [BZ77]). Let $K$ be a compact open subgroup of $\overline{N}$ with pro-order invertible in $W(k)$, let $g$ in $C^\infty_c(\overline{N}',R)$ be the characteristic function of $K \cap \overline{N}'$. Then the Haar integral sending $g$ to 1 defines an isomorphism $r_{\overline{N},1}(C^\infty_c(\overline{N}',R)) \cong R$ that sends $p(g)$ to 1. If $f \in C^\infty_c(\overline{N},R)$ is $[K : K \cap N']^{-1}$ times the characteristic function of $K \cap \overline{N}'$ then $\overline{A}(f) = \int_{\overline{N}\backslash \overline{N}} p([K : K \cap N']^{-1} \cdot g)$, which is sent to 1 in $R$. Hence $A$ is surjective.

5.2. An argument of Bushnell and Henniart. We now return to the setting of the rest of the paper. Let $P = MN$ be a standard parabolic subgroup of $G = GL_n(F)$, with Levi component $M$ and unipotent radical $N$. If $R$ is a $W(k)$-algebra, let $r_{\overline{P}} : C(G) \to C(M)$ be the un-normalized parabolic restriction functor. If we let $U_M := U \cap M$, then $\psi|_{U_M}$ is again a nontrivial character and we can form the representation $\text{Ind}_U^M \psi_R$, and the compact induction $c\text{-Ind}_U^M \psi_R$, respectively. Given an object $(R,V)$ in $C(\{1\})$, it defines an object of $C(U)$ via the action of $\psi$, which we denote $\psi_V$. In this way we define functors $\text{Ind}_U^P \psi$, $r_{\overline{P}}(\text{Ind}_U^P \psi)$, and $c\text{-Ind}_U^P \psi$, respectively, with source $C(\{1\})$ and target $C(G)$, $C(M)$, and $C(M)$, respectively.

In the notation of Subsection 5.1, we let $T = \{1\}$ and $B = U$. We will take $\overline{P} = M \overline{N}$ to be the $M$-opposite parabolic to $P$, and $\eta$ to be the trivial character.

We start by recording a lemma appearing in [BH93], whose proof works verbatim over an arbitrary $W(k)$-module $R$.

Lemma 5.2 (Lemma 2.3, [BH93]). Let $\overline{P} = M \overline{N}$ be the opposite parabolic to $P$ and let $R$ be any $W(k)$-module. If the set $U \overline{P}$ supports a function $f : G \to R$ such that $f(uxn) = \psi(u)f(x)$ for $u \in U$, $x \in G$, $n \in \overline{N}$, then $g$ lies in $U \overline{P}$.

If an orbit $U \overline{P}$ satisfied condition (★) in Subsection 5.1 there would exist a well-defined function $f : U \overline{P} \to R$ satisfying the hypothesis of Lemma 5.2. Hence the only orbit satisfying (★) is the trivial orbit $Z = U \overline{P}$. Although we will not need it, we record the following corollary of Theorem 5.1.

Corollary 5.3. The functor $r_{N,1} \circ i_{U,\psi} : C(\{1\}) \to C(M)$ is equivalent to the functor $\Phi_{U,P} = i'_{U'},\psi'$, where $U' = U \cap M$ and $\psi' = \psi|_{U'}$. In particular, we also get the un-normalized version $r_{P}\text{-Ind}_U^P \psi \cong c\text{-Ind}_U^M \psi$.

Note that a $\overline{P}$-orbit $U \overline{P}$ in $U \backslash G$ satisfies condition (★) if and only if the $U$-orbit $P \overline{u} U$ in $P \backslash G$ satisfies the analogue of condition (★) after switching the roles of $(B,T,U,\psi)$ with $(P,M,\overline{N},\eta)$. Thus, after switching the roles of $(B,\{1\},U,\psi)$ with $(\overline{P},M,\overline{N},1)$, it follows from Lemma 5.2 that the only orbit satisfying (★) is $PU$. We get the following corollary by applying Theorem 5.1 with the roles of $(B,\{1\},U,\psi)$ and $(P,M,\overline{N},1)$ swapped:
Corollary 5.4. The functor \( r_{U',\psi} \circ i_{\mathcal{N},1} : \mathcal{C}(M) \to \mathcal{C}(\{1\}) \) is equivalent to the functor \( \Phi_{PU} = r_{U',\psi'} \). In particular, the un-normalized version is \( (i^G_P(-))_{U,\psi} = (-)_{U',\psi'} \).

Explicitly, given an object \((R, \rho)\) in \( \mathcal{C}(M) \), the isomorphism \( r_{U',\psi} \circ i_{\mathcal{N},1}(\rho) \cong r_{U',\psi'}(\rho) \) is induced by the homomorphism \( \overline{A} : i_{\mathcal{N},1}(\rho) \to r_{U',\psi'}(\rho) \) satisfying the following. If \( p_{U',\psi} : \rho \to r_{U',\psi'} \) denotes the canonical projection, and \( f \in i_{\mathcal{N},1}(\rho) \), then

\[
\overline{A}(f) = \int_{U \cap P \setminus U} \psi^{-1}(u)p_{U',\psi}(f(u))du.
\]

Let \((R', V)\) be any object in \( \mathcal{C}(\{1\}) \). Dualizing, we have

(4) \( \text{Hom}_{\mathcal{C}(\{1\})}((R, \rho)_{U',\psi'}, (R', V)) \cong \text{Hom}_{\mathcal{C}(\{1\})}((i^G_P(R, \rho))_{U,\psi}, (R', V)) \)

and applying Frobenius reciprocity, we have

(5) \( \text{Hom}_{\mathcal{C}(M)}((R, \rho), \text{Ind}^M_{U,M} \psi_{(R', V)}) \cong \text{Hom}_{\mathcal{C}(G)}(i^G_P(R, \rho), \text{Ind}^G_U \psi_{(R', V)}). \)

Explicitly, if \( \xi \mapsto W_\xi \) denotes a morphism \( \rho \otimes R R' \to \text{Ind}^M_{U,M} \psi_{(R', V)} \) in \( \mathcal{C}(M) \), and \( f \mapsto W_f \) denotes the corresponding morphism \( i^G_P(\rho \otimes R R') \to \text{Ind}^G_U \psi_{(R', V)} \), then for \( f \in i^G_P(\rho \otimes R R') \), \( W_f \) restricts on \( M \) to the function:

(6) \( W_f(m) = \int_{U \cap M \setminus U} \psi^{-1}(u)W_f(u)(m)du. \)

Finally, we apply Bernstein’s second adjointness theorem, which holds when \( p \) is invertible in \( R \), and when \( G \) is the \( F \)-points of a general linear group (or a classical group with \( p \neq 2 \), or a reductive group of relative rank 1 over \( F \)) by Theorem 1.5. The second adjointness property gives

(7) \( \text{Hom}_{\mathcal{C}(G)}(i^G_P(R, \rho), \text{Ind}^G_U \psi_{(R', V)}) \cong \text{Hom}_{\mathcal{C}(M)}((R, \rho), r_{U,M}^G \text{Ind}^G_U \psi_{(R', V)}). \)

Combining this with Equation (5), we have, for each \((R', V)\) in \( \mathcal{C}(\{1\}) \), an isomorphism of functors

\[
\text{Hom}_{\mathcal{C}(M)}(-, \text{Ind}^M_{U,M} \psi_{(R', V)}) \cong \text{Hom}_{\mathcal{C}(M)}(-, r_{U,M}^G \text{Ind}^G_U \psi_{(R', V)}).
\]

Yoneda’s lemma gives a natural isomorphism of functors

\[
\text{Ind}^M_{U,M} \psi \cong r_{U,M}^G \text{Ind}^G_U \psi.
\]

We summarize this result in the following proposition.

Proposition 5.5. Consider \( \text{Ind}^G_U \psi, \text{Ind}^M_{U,M} \psi, \) and \( r_{U,M}^G(\text{Ind}^G_U \psi) \) as functors \( \mathcal{C}(\{1\}) \to \mathcal{C}(M) \),

there exists a surjective natural transformation \( \Phi' : \text{Ind}^G_U \psi \to \text{Ind}^M_{U,M} \psi \), which induces an isomorphism

\[
r_{U,M}^G(\text{Ind}^G_U \psi) \cong \text{Ind}^M_{U,M} \psi.
\]

In particular, for \((R, V)\) and \((R', V')\) in \( \mathcal{C}(\{1\}) \) and a morphism \( \lambda : R \to R' \), \( \iota : V \otimes R R' \to V' \) between them, the following diagram commutes:

\[
\begin{array}{ccc}
\text{Ind}^G_U \psi_V & \xrightarrow{\Phi'(R, V)} & \text{Ind}^M_{U,M} \psi_V \\
\downarrow{\iota_*} & & \downarrow{\iota_*} \\
\text{Ind}^G_U \psi_{V'} & \xrightarrow{\Phi'(R', V')} & \text{Ind}^M_{U,M} \psi_{V'},
\end{array}
\]
where \( \iota \) denotes pushforward of Whittaker functions along \( V \to V \otimes_R R' \overset{\iota}{\to} V' \).

We will only ever use Proposition 5.5 in the special case where \( V = R \) and \( V' = R' \), at which point we will abbreviate the morphism \( \Phi'_{R,R'} \) by simply \( \Phi'_{R} \).

We note that the isomorphism \( r_{G}^{K}(\text{Ind}_U^{G} \psi) \cong \text{Ind}_{U_{\mathfrak{a}}}^{M} \psi \) depends on the choice of Haar measure in the map \( \overline{\mathcal{A}} \). Unfortunately, we are currently unable to write down a more explicit description of the map \( \Phi' \). However, its existence, naturality, and the useful properties of the next subsection fill in everything we will need.

6. Hecke algebras

6.1. Iwahori–Hecke algebras. We first summarize some well-known facts about Iwahori–Hecke algebras that we will need. The standard Iwahori subgroup \( I \) consists of the matrices in \( K \) whose reduction modulo \( (\mathfrak{a}) \) is upper triangular. Define

\[
\Lambda = T/T(\mathcal{O}_F) = \left\{ \left( \begin{array}{cccc}
\overline{\mu_1} & & & \\
& \ddots & & \\
& & \ddots & \\
& & & \overline{\mu_n}
\end{array} \right) : \mu_1, \ldots, \mu_n \in \mathbb{Z} \right\}.
\]

The Satake map induces an isomorphism

\[
S : \mathcal{Z} \xrightarrow{\sim} W(k)[\Lambda]^{W_G},
\]

\[
f \mapsto \left[ t \mapsto \delta_B(t)^{-1/2} \int_{U} f(tu) \, du \right]
\]

where \( W_G \) denotes the Weyl group of \( G \), and the algebra structure on the left is convolution of Hecke operators. The Haar measure \( du \) is normalized so that the measure of \( U \cap K \) is 1.

The module \( W(k)[I \setminus G] \) of finitely supported functions on \( I \setminus G \) has an action of \( G \) by right translation and its \( K \)-fixed vectors, \( W(k)[I \setminus G]^K = W(k)[I \setminus G/K] \) carry a Hecke action of \( \mathcal{Z} \). The quotient map \( I \setminus G \to K \setminus G \) induces an inclusion

\[
W(k)[K \setminus G] \hookrightarrow W(k)[I \setminus G],
\]

which is \((\mathcal{Z}, \ast)\)-equivariant on \( K \)-fixed vectors. By [Laz98 Prop 1.12], the Satake isomorphism extends to an isomorphism of \((\mathcal{Z}, \ast)\)-modules:

\[
W(k)[I \setminus G/K] \xrightarrow{\sim} W(k)[\Lambda].
\]

The Iwahori Hecke algebra \( \mathcal{H}(G, I) = W(k)[I \setminus G/I] \) admits a presentation due to Iwahori–Matsumoto. Let \( \{s_1, \ldots, s_{n-1}\} \) denote the transposition matrices, and let \( t \) denote the matrix

\[
t = \left( \begin{array}{ccc}
1 & & \\
& \ddots & \\
& & 1
\end{array} \right) \overline{\ast}.
\]

The Iwahori double cosets in \( I \setminus G/I \) can be uniquely represented as \( t^a w \) where \( w \) is product of \( s_i \)'s and their inverses. Let \( S_i := [I s_i I] \) and \( T = [I t I] \) in \( \mathcal{H}(G, I) \), where \( [-] \) denotes the characteristic function. Then \( \mathcal{H}(G, I) \) is generated as an algebra by \( \{S_1, \ldots, S_{n-1}\} \cup \{T, T^{-1}\} \), subject to the usual braid relations and quadratic relations, along with the relations \( T S_i = S_{i-1} T \) and \( T^2 S_i = S_{n-1} T^2 \) (c.f. [Vig96 1.3.14]). The subalgebra of \( \mathcal{H}(G, I) \) generated by \( \{S_1, \ldots, S_{n-1}\} \) is precisely
We denote this subalgebra $\mathcal{H}_W$. We let $1 : \mathcal{H}_W \to W(k)^\times$ denote the “trivial” character $w \mapsto [IwI : I]$.

$\mathcal{H}(G, I)$ admits another presentation due to Bernstein. Let

$$t_j := \begin{pmatrix} \varpi I_j \\ I_{n-j} \end{pmatrix}, \ 0 \leq j \leq n,$$

and let $T_j = [I t_j I] \in \mathcal{H}(G, I)$. Then $T_j$ is invertible in $\mathcal{H}(G, I)$, we can define $X_j := q^{j-(n+1)/2} T_j T_j^{-1}$, and $\mathcal{H}(G, I)$ is generated by $\{S_1, \ldots, S_{n-1}\} \cup \{X_1^{\pm 1}, \ldots, X_n^{\pm 1}\}$ subject to the relations

1. $W(k)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is commutative,
2. $X_j S_i = S_i X_j$ except if $j = i$ or $i + 1$,
3. $S_i X_{i+1} S_i = q X_i$
4. $X_j S_i = S_i X_{i+1} + (q - 1) X_i$
5. $X_{i+1} S_i = S_i X_i - (q - 1) X_i$
6. the quadratic relations and braid relations on the $S_i$’s.

If $\mu_1 \geq \cdots \geq \mu_n$ is a dominant weight, and $\varpi^\mu := \text{diag}(\varpi^{\mu_1}, \ldots, \varpi^{\mu_n}) \in \Lambda$, then $[I \varpi^\mu I]$ is invertible in $\mathcal{H}(G, I)$. If $\mu$ is non-dominant, it can be written $\mu' - \mu''$ for $\mu', \mu''$ dominant, and the product $[I \varpi^\mu I] [I \varpi^\mu' I]^{-1}$ depends only on $\mu$. This defines an injective map of algebras

$$W(k)[\Lambda] \hookrightarrow \mathcal{H}(G, I),$$

$$\varpi^\mu \mapsto q^{-l(\varpi^\mu)/2} [I \varpi^\mu I] [I \varpi^\mu' I]^{-1},$$

where $l(\varpi^\mu) = l(w)$, after writing $\varpi^\mu = t^a w$ for $w \in W_G$, and $l(w)$ denotes the length of $w$. Note that if $\varpi^{\mu_j} := \text{diag}(1, \ldots, 1, \varpi, 1, \ldots, 1)$, with $\varpi$ in the $j$’th position, this map sends $\varpi^{\mu_j} \mapsto X_j$ (despite this map, note that $X_j$ is not $[I \varpi^\mu I]$).

The image of the algebra $W(k)[\Lambda]$ is the subring $\mathcal{A} := W(k)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ of $\mathcal{H}(G, I)$, which is a Laurent polynomial ring. It identifies the center of $\mathcal{H}(G, I)$ with the image of the subring $W(k)[\Lambda]^{W_G}$, i.e. the Laurent polynomials in $\mathcal{A}$ invariant under permuting variables (Vig90, Lemme I.3.15).

We identify the ring $\mathcal{Z} := W(k)[K \backslash G / K]$ and its module $W(k)[K \backslash G / I]$ with the ring $W(k)[\Lambda]^{W_G}$ and its module $W(k)[\Lambda]$ via the Satake isomorphisms of Equations (8) and (9). By the Bernstein presentation, we can and do consider them as subalgebras, $\mathcal{Z}$ and $\mathcal{A}$, of $\mathcal{H}(G, I)$.

Let $P = MN$ be a standard parabolic subgroup. It is the set of block-upper triangular matrices associated to an ordered partition $n = n_1 + \cdots + n_r$, and $M \cong GL_{n_1}(F) \times \cdots \times GL_{n_r}(F)$. We have, for each factor $G_i := GL_{n_i}(F)$, the Bernstein presentation of its Iwahori–Hecke algebra $\mathcal{H}(G_i, I_i)$, with generators that we will denote $X_{i,1}^{(i)}, \ldots, X_{i,n_i}^{(i)}$, $S_{i,1}^{(i)}, \ldots, S_{i,n_i-1}^{(i)}$. Let $I_M := I \cap M$. There is a unique algebra homomorphism

$$j := j^G_M : \mathcal{H}(M, I_M) \to \mathcal{H}(G, I)$$

$$X_j^{(i)} \mapsto X_{n_1 + \cdots + n_i + j}$$

$$S_j^{(i)} \mapsto S_{n_1 + \cdots + n_i + j}$$
If $W_M$ denotes the Weyl group of $M$, considered as a subgroup of $W_G$, then it follows ([MWS86 I.3]) from the presentation that

$$\mathcal{H}(G, I) = \bigoplus_{w \in W_M \setminus W_G} j_P^M(\mathcal{H}(M, I_M)) \ast [IwI].$$

We identify $Z_M := W(k)[K_M \setminus M / K_M]$ with $W(k)[\Lambda]^{W_M}$ via Satake, and with the center of $\mathcal{H}(M, I_M)$ via the Bernstein presentation. Each $j := j_P^G$ gives an embedding $Z \hookrightarrow Z_M \hookrightarrow \mathcal{H}(G, I)$ which agrees with the Bernstein map $Z \hookrightarrow \mathcal{H}(G, I)$.

If $I_M$ denotes $I \cap M$, the natural map $V \to r_P^G V$ induces, in characteristic zero or banal characteristic, an isomorphism of $\mathcal{H}(M, I_M)$-modules $V^I \cong (r_P^G V)^{I_M}$, showing that the analogue of $r_P^G$ for Iwahori–Hecke modules is restriction of scalars from $\mathcal{H}(G, I)$ to $\mathcal{H}(M, I_M)$. Actually, this is also true if $R$ is a commutative ring with $q \in R\setminus \{0\}$.

**Lemma 6.1.** Let $G$ be a general linear group or a classical group (with $p \neq 2$), and suppose $q$ is invertible in $R$. If $I$ is the Iwahori subgroup of $K$, and $I_M := I \cap M$, then for any $R[G]$-module $V$, the quotient map $p_N : V \to r_P^G V$ induces an isomorphism $V^I \cong (r_P^G V)^{I_M}$.

**Proof.** The injectivity was proved for arbitrary reductive groups in [Vig96 II.3.1]. The surjectivity is [Dat09 Cor 3.9(i)], on general linear groups and (when $p \neq 2$) classical groups. □

**6.2. Additional structure in the quasi-banal setting.** In this entire section, we assume that $\ell$ is quasi-banal, which means:

- $\ell$ is banal for $GL_n(F)$
- or
- $\ell > n$ and $q \equiv 1 \mod \ell$.

We will use this subsection later, in a global context, to deduce Corollary [9.5]. Let $\mathcal{H}(G, I)_k = \mathcal{H}(G, I) \otimes_{W(k)} k = k[I \setminus G/I]$ denote the Iwahori–Hecke algebra over $k$, and similarly let $\mathcal{A}_k$, $\mathcal{H}_{W,k}$, and $Z_k$ denote the reductions mod-$\ell$ of the subrings $\mathcal{A}$, $\mathcal{H}_W$, and $Z$. The relations (1)-(6) in the Bernstein presentation simplify when $q = 1 \mod \ell$. In the quasi-banal setting, Vigneras has proved the following mod-$\ell$ analogue of the well-known result over $\mathbb{C}$:

**Proposition 6.2** (Vigneras, [CHT08 Appendix B]). Let $\ell$ be quasi-banal. The functor $V \mapsto V^I$ defines an equivalence of categories from the abelian subcategory of smooth $k[G]$-modules generated by their $I$-fixed vectors to the category of $\mathcal{H}(G, I)_k$-modules.

**Proposition 6.3.** Let $\ell$ be quasi-banal. There is an isomorphism of $\mathcal{H}(G, I)_k$-modules $k[G/K]^I \cong \mathcal{H}(G, I)_k \otimes_{\mathcal{H}_{W,k}} 1$.

**Proof.** To ease notation we will drop $k$'s from the subscripts, so for example $\mathcal{H}(G, I) = \mathcal{H}(G, I)_k$. The module $k[G/I]$ has $I$-fixed vectors $k[I \setminus G/I]$, which is cyclic as an
$\mathcal{H}(G, I)$-module, generated by $1_I$. There is a surjection $k[G/I]^I \to k[G/K]^I$ given by

$$
\phi(x) \mapsto \frac{1}{(K : I)} \sum_{g \in K/I} f(gx).
$$

The map is well-defined and surjective because $(K : I)$ is invertible in $k$ as a result of the quasi-banal hypothesis. It follows that $k[G/K]^I$ is a cyclic $\mathcal{H}(G, I)$-module, generated by the characteristic function $1_K$.

Since $1_K$ is fixed by $K$, the action of $\mathcal{H}_w$ on $1_K$ is via $1$. Thus $1 \mapsto 1_K$ defines a morphism of $\mathcal{H}_w$-modules $1 \to k[G/K]^I$. By the adjunction

$$
\text{Hom}_{\mathcal{H}(G, I)}(\mathcal{H}(G, I) \otimes \mathcal{H}_w, 1, k[G/K]^I) \cong \text{Hom}_{\mathcal{H}_w}(1, k[G/K]^I),
$$

we get a morphism of $\mathcal{H}(G, I)$-modules $\Theta : \mathcal{H}(G, I) \otimes \mathcal{H}_w 1 \to k[G/K]^I$ given by sending $h \otimes 1$ to $h * 1_K$. Since $1_K$ is a cyclic generator of $k[G/K]^I$, the map $\Theta$ is surjective. However, since $A \cong k[\Lambda]$ is free of rank $n!$ over $Z$ as $Z$-module, we conclude that $\mathcal{H}(G, I) \otimes \mathcal{H}_w 1$ is free of rank $n!$ over $Z$. On the other hand, Equation 9 identifies $k[G/K]^I \cong k[1 \setminus G/K] \cong k[\Lambda]$, showing that $k[G/K]^I$ is also free of rank $n!$ as a $Z$-module. It follows that $\Theta$ is an isomorphism.

**Corollary 6.4.** Let $M_k$ be an $\mathcal{H}(G, I)_k$ module via the projection

$$
\mathcal{H}(G, I)_k \to \mathcal{H}(G, I)_k \otimes_{\mathcal{H}_{w,k}} 1 \cong M_k.
$$

Let $\lambda : Z_k \to k$ be a homomorphism. Then $M^I_{\lambda, k}$ is isomorphic as an $\mathcal{H}(G, I)_k$-module to the $n!$-dimensional $k$-algebra $A_k \otimes_{Z_k, \lambda} k$.

**Proof.** Using the proof of [CHT08 Lemma 5.1.4], we find that

$$
M^I_{\lambda, k} := (k[G/K] \otimes_{Z_k, \lambda} k)^I = (k[G/K]^I) \otimes_{Z_k, \lambda} k.
$$

Since $k[G/K]^I \cong \mathcal{H}(G, I)_k \otimes_{\mathcal{H}_{w,k}} 1$ is isomorphic to $M_k$ as an $A_k$-module, the result follows.

### 6.3. Iwahori-fixed Whittaker functions

Let $W^0_\lambda$ be the canonical element of $\text{Ind}_{U}^G \psi_R$ guaranteed by Lemma 2.1, and let

$$
\Lambda_R : M_{\lambda, R} \to \text{Ind}_{U}^G \psi_R
$$

be the associated Whittaker model $1_K \otimes 1 \to W^0_\lambda$. The largest technical obstacle to proving our main result, Theorem 7.1 in Section 7, is showing that $\Lambda_R$ is compatible with parabolic restriction, in the following sense. Let $P = MN$ be a proper parabolic subgroup. On one hand, there is the map $\Psi$ such that the following diagram of $Z[M]$-modules commutes:

\begin{equation}
\begin{array}{c}
W(k)[G/K] \\
\Phi \downarrow \Phi' \\
W(k)[M/K_M]
\end{array} \xrightarrow{\Lambda_Z} \begin{array}{c}
\text{Ind}_{U}^G \psi_Z \\
\text{Ind}_{U,M}^M \psi_Z
\end{array}
\end{equation}

where the vertical maps are given by Lemmas 3.3 and Proposition 5.5. Namely $\Psi$ is the composition

$$
W(k)[M/K_M] \cong r_P^G(W(k)[G/K]) \xrightarrow{r_P^G \Lambda_Z} r_P^G(\text{Ind}_{U}^G \psi_Z) \cong \text{Ind}_{U,M}^M \psi_Z,
$$
and $\Psi(1_{K_M}) = \Phi_\kappa(W_{\text{id}}^0)$, where $\text{id} : Z \to Z$ is the identity map and $W_{\text{id}}^0$ is the canonical Whittaker function given by Corollary 2.2.

On the other hand, there is the canonical map

$$\Lambda^M_{Z_M} : W(k)[M/K_M] \to \text{Ind}^M_{U_M} \psi Z_M$$

$$1_{K_M} \mapsto W_{\text{id}}^0,$$

where $\text{id} : Z_M \to Z_M$ is the identity and $W_{\text{id}}^0$ is the canonical element of $(\text{Ind}^M_{U_M} \psi Z_M)^{K_M, \text{id}}$ satisfying $W_{\text{id}}^0(1) = 1$, given by Corollary 2.3. To make the induction argument work in Section 7, we need to show the two maps agree in the following sense.

**Proposition 6.5.** Let $j^* : \text{Ind}^M_{U_M} \psi Z \to \text{Ind}^M_{U_M} \psi Z_M$ be pushforward along the canonical inclusion $j : Z \hookrightarrow Z_M$. The composition

$$W(k)[M/K_M] \xrightarrow{\Psi} \text{Ind}^M_{U_M} \psi Z \xrightarrow{j^*} \text{Ind}^M_{U_M} \psi Z_M$$

differs from $\Lambda^M_{Z_M}$ by multiplication by a unit in $Z_M$.

The strategy involves extending scalars to $\kappa := \text{Frac}(Z)$, where it is easier to check the compatibility of the two maps because $\text{Frac}(Z)$ is a field of characteristic zero. We will make repeated use of the fact that $\text{Frac}(Z_M) = Z_M \otimes Z \kappa$. In particular, if a $Z_M$-module is torsion-free when restricted to $Z$, it is also torsion-free as a $Z_M$-module.

For any smooth $W(k)[G]$-module $V$ recall that $*$ denotes the action of $Z$ on $V^K$ via double-coset convolution operators. The Bernstein presentation gives an inclusion of algebras $Z \cong W(k)[A]W_G \to \mathcal{H}(G, I)$ whose image is the center of $\mathcal{H}(G, I)$. By virtue of this inclusion, $Z$ acts on any $\mathcal{H}(G, I)$-module. Similarly, the embedding $j : Z \to Z_M$ gives an action of $Z_M$ on any $Z_M$-module, for example, $Z$ acts on $X^{K_M}$ for any $X \in \text{Rep}_{W(k)}(M)$. Finally, $Z$ also acts on $\mathcal{H}(M, I_M)$-modules, via the embeddings $Z \hookrightarrow Z_M \hookrightarrow \mathcal{H}(M, I_M)$. Recall that the Bernstein presentation gives an embedding $Z_T \hookrightarrow \mathcal{H}(G, I)$ whose image we denote $A$. We will denote all these convolution operator actions by $*$.

On the other hand, the Whittaker spaces $\text{Ind}^G_U \psi Z$, $\text{Ind}^M_{U_M} \psi Z$, $\text{Ind}^M_{U_M} \psi Z_M$, $\text{Ind}^G_U \psi \kappa$, $\text{Ind}^M_{U_M} \psi \kappa$, $\text{Ind}^M_{U_M} \psi \kappa \otimes Z M$ carry actions of $Z$, $\kappa$, and $Z_M$, $\kappa$, and $\text{Frac}(Z_M)$, respectively, by multiplication of the values of Whittaker functions. We will denote all these actions by $\cdot$. If no action is specified it is implicitly assumed to be the $\cdot$ action.

We will compare the actions $*$ and $\cdot$ by computing the subspace of $(\text{Ind}^M_{U_M} \psi \kappa)^{K_M}$ where the two $Z_M$-actions coincide on the subring $Z$:

$$(\text{Ind}^M_{U_M} \psi \kappa)^{K_M, (Z, \cdot) = (Z, *)} := \{ v \in (\text{Ind}^M_{U_M} \psi \kappa)^{K_M} : z \cdot v = z \cdot \cdot \cdot, z \in Z \}.$$

To save notation, abbreviate to $W^0$ the Whittaker function $W_{\text{id}}^0 \in (\text{Ind}^G_U \psi Z)^{K, \text{id}}$ given by Corollary 2.2. Let $W^0|_M$ be its restriction to $M$, considered as an element of $(\text{Ind}^M_{U_M} \psi Z)^{K_M}$. Let $X := Z_M \ast W^0|_M$ be the $(Z_M, \ast)$-submodule of $(\text{Ind}^M_{U_M} \psi Z)^{K_M}$ generated by $W^0|_M$.

**Lemma 6.6.** The image of $X \otimes Z \kappa$ in the embedding

$$(\text{Ind}^M_{U_M} \psi Z)^{K_M} \otimes Z \kappa \hookrightarrow (\text{Ind}^M_{U_M} \psi \kappa)^{K_M}$$

$$W \otimes \xi \mapsto W \cdot \xi$$
equals \((\text{Ind}_{U_M}^M \psi_k)^{K_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\).

**Proof.** Let \(Y := Z_T \otimes_{\mathbb{Z}} \kappa\) be the \((\mathbb{Z}_T,*)\)-submodule of \((\text{Ind}_{U_M}^M \psi_Z)^{I_M}\) generated by \(W^0|_M\). The Lemma follows from two claims.

**Claim 1:** The image of \(Y \otimes_{\mathbb{Z}} \kappa\) in the embedding
\[
(\text{Ind}_{U_M}^M \psi_Z)^{I_M} \otimes_{\mathbb{Z}} \kappa \to (\text{Ind}_{U_M}^M \psi_k)^{I_M}
\]
is the space \((\text{Ind}_{U_M}^M \psi_k)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\).

Note that \(I_T = K_T = T \cap K\), so Claim 1 implies the Lemma in the case \(M = T\). The isomorphism of Lemma \([6,1]\) is not only an isomorphism of \(\kappa\)-vector spaces, it is equivariant with respect to the \(*\) action of \(Z_M\). Combined with Lemma \([6,1]\), we have
\[
(\text{Ind}_{U_M}^M \psi_k)^{I_M} \cong (\text{Ind}_{U_M}^T \psi_k)^{K_T}
\]
as \(\kappa\)-vector spaces and as \((\mathbb{Z}_M,*\))-modules. As \(Z\) acts via the natural embeddings \(\mathbb{Z} \hookrightarrow Z_M \hookrightarrow Z_T \hookrightarrow \mathbb{H}(T, I_T)\), we have
\[
(\text{Ind}_{U_M}^M \psi_k)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*) \cong (\text{Ind}_{U_M}^T \psi_k)^{K_T,(\mathbb{Z},*)} = (\mathbb{Z},*).
\]
An element of \((\text{Ind}_{U_M}^T \psi_k)^{K_T}\) is a smooth function on the lattice \(\omega^{\mathbb{Z}}\). It is proved in \([Laz98, \text{Thm 3.2}]\) that adding the additional condition that the actions \((\mathbb{Z},*)\) and \((\mathbb{Z},*)\) be equivalent cuts out a space of dimension \(n!\) over \(\kappa\). So the space \((\text{Ind}_{U_M}^M \psi_k)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\) has dimension \(n!\) over \(\kappa\).

On the other hand, by definition of \(W^0\), the restriction \(W^0|_M\) lies in the space
\[
(\text{Ind}_{U_M}^M \psi_Z)^{K_M,(\mathbb{Z},*)} = (\mathbb{Z},*)
\]
which is stable under the \(*\) action of \(Z_T\). Thus the cyclic module \(Y\) is contained in \((\text{Ind}_{U_M}^M \psi_Z)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\). Since the action of \(Z\) via \(\cdot\) on \((\text{Ind}_{U_M}^M \psi_Z)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\) is torsion-free, so is the action of \(Z\) via \(*\) and hence so is the action of \(Z_T\) via \(*\). Hence \(Y\) is free of rank one over \(Z_T \cong W(k)[\Lambda] \cong cA\), and \(Y \otimes_{\mathbb{Z}} \kappa\) has dimension \(n!\) over \(\kappa\). Since it is contained in \((\text{Ind}_{U_M}^M \psi_k)^{I_M,(\mathbb{Z},*)} = (\mathbb{Z},*)\), we have proven Claim 1.

**Claim 2:** \(Y \otimes_{\mathbb{Z}} \kappa)^{K_M} = X \otimes_{\mathbb{Z}} \kappa\).

Note that \(\ell\) is invertible in \(\kappa\), hence the action of \(\mathbb{H}(M,I_M)\) factors through \(\mathbb{H}(M,I_M)[\frac{1}{\ell}] = \mathbb{H}(M,I_M) \otimes_{W(k)} \mathbb{K}\), where \(\mathbb{K} = \text{Frac}(W(k))\). We will abbreviate
\[
(\mathbb{K},\cdot) := (\mathbb{K}) \otimes_{W(k)} \mathbb{K}.
\]
Let \(Y'\) be the cyclic \(\mathbb{H}(M,I_M)\)-submodule of \((\text{Ind}_{U_M}^M \psi_Z)^{I_M}\) generated by \(W^0|_M\). Then we can identify \(Y' \otimes_{\mathbb{Z}} \kappa\) with the \(\mathbb{H}(M,I_M)_{\mathbb{K}}\)-submodule of \((\text{Ind}_{U_M}^M \kappa)^{I_M}\) generated by \(W^0|_M\).

Since \(W^0|_M\) is fixed by \(K_M\), the subring \(\mathbb{H}_{W_M} = \mathbb{K}[I_M\backslash K_M/K_M]\) of \(\mathbb{H}(M,I_M)_{\mathbb{K}}\) acts by the “trivial” character \(1 : \mathbb{H}_{W_M} \to \mathbb{K}^X\), which sends \([IwI]\) to \([IwI : I]\), which is a power of \(q\), and invertible everywhere. Hence we have an equality of \(\mathbb{H}(M,I_M)_{\mathbb{K}}\)-modules:
\[
Y' \otimes_{\mathbb{Z}} \kappa := \mathbb{H}(M,I_M)_{\mathbb{K}} \ast W^0|_M = A_{\mathbb{K}} \ast W^0|_M = (Z_T)_{\mathbb{K}} \ast W^0|_M = Y \otimes_{\mathbb{Z}} \kappa.
\]
Since \(Z_T\) acts torsion-free on \(W^0|_M\), we have \(Y \otimes_{\mathbb{Z}} \kappa \cong (Z_T)_{\mathbb{K}}\), and Claim 2 follows since \((Z_T)_{\mathbb{K}} = (Z_T)^{\mathbb{H}_{W_M}} = \mathbb{K}[\Lambda]^{W_M} = (Z_M)_{\mathbb{K}}\).
\(\square\)
We conclude this section by proving Proposition 6.5.

**Proof of Proposition 6.5.** Since $\Psi$ is $M$-equivariant, its restriction to $K_M$-fixed vectors is $(Z_M, \ast)$-equivariant, and hence also $(Z, \ast)$-equivariant. In particular $\Psi(1_{K_M})$ defines an element of the space

$$(\text{Ind}_{U_M}^M \psi_Z)^{K_M, (Z, \ast) = (Z, \ast)}$$

defined before Lemma 6.6. By the conclusion of Lemma 6.6 there is an element $z$ of $Z_M$ and an element $z_0$ of $Z$ such that $z_0 \Psi(1_{K_M}) = z \ast W_0 |_{M}$. If we consider $W_0 |_{M}$ as an element of $\text{Ind}_{U_M}^M \psi_{Z_M}$ by pushing it forward along the inclusion $j: Z \to Z_M$, we have $z_0 j (\Psi(1_{K_M})) = z \ast W_0 |_{M}$. By Proposition 5.5, $j (\Psi(1_{K_M})) = \Phi_{Z_M}^* (W_0^0)$. It follows from Lemma 6.1 that $\Phi_{Z_M}^* (W_0^0)$, and hence $z$, are nonzero.

On the other hand, the canonical element

$W_{id}^0 \in (\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M, \text{id}} =: (\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M, (Z, \ast) = (Z, \ast)}$

also lies in the bigger space $(\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M, (Z, \ast) = (Z, \ast)}$. Extending scalars from $\kappa$ to Frac($M$) in the conclusion of Lemma 6.6, we find there is some $z' \in Z_M$ and $z_1 \in Z$ such that $z_1 W_{id}^0 = (z') \ast W_0 |_{M}$. Since $W_{id}^0 (1) = 1$, $z'$ is nonzero.

Since Frac($Z_M$) = $Z_M \otimes Z \text{Frac}(Z)$, we can write

$$\frac{z}{z'} = \frac{z''}{z_2} \in \text{Frac}(Z_M), \text{ for some } z'' \in Z_M, z_2 \in Z.$$ 

We have

$$z_0 j (\Psi(1_{K_M})) = z'' (\frac{z_1}{z_2} \cdot W_{id}^0) = \frac{z''}{z_2} z_1 \cdot W_{id}^0 \in (\text{Ind}_{U_M}^M \psi_{\text{Frac}(Z_M)})^{K_M}.$$ 

In other words,

$$z_0 z_2 j (\Psi(1_{K_M})) = z'' z_1 \cdot W_{id}^0 \in (\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M}.$$ 

Thus the action of $Z_M$ on $z_0 z_2 j (\Psi(1_{K_M}))$ via $\ast$ is equivalent to that via $\cdot$, since it is true for any multiple of $W_0 |_{M}$. It follows from $Z$-torsion freeness that the same is true for $j (\Psi(1_{K_M}))$ itself. Thus $j (\Psi(1_{K_M}))$ lies in the space

$$(\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M, (Z, \ast) = (Z, \ast)} = (\text{Ind}_{U_M}^M \psi_{Z_M})^{K_M, \text{id}} = Z_M \cdot W_{id}^0.$$ 

By Lemma 6.3, we already know that $j (\Psi(1_{K_M})) = \Phi_{Z_M}^* (W_0^0)$ is nonzero modulo every maximal ideal of $Z_M$, hence the same must be true of $j (\Psi(1_{K_M})) (1)$ by Lemma 2.3. In particular $j (\Psi(1_{K_M})) (1)$ is a unit in $Z_M$. By Lemma 2.3, $j (\Psi(1_{K_M}))$ differs from $W_{id}^0$ by multiplication by $j (\Psi(1_{K_M})) (1)$. In other words, $j (\Psi(1_{K_M}))$ is equivalent to $\Lambda_{Z_M}^0 (1_{K_M})$ after multiplying by a unit in $Z_M$. Since the maps $j \circ \Psi$ and $\Lambda_{Z_M}^0$ are $W(k)[M]$-equivariant, and $W(k)[M/K_M]$ is generated as a $W(k)[M]$-module by $1_{K_M}$, the statement of the Proposition follows. 

We emphasize that the map

$$j \circ \Psi: W(k)[M/K_M] \to \text{Ind}_{U_M}^M \psi_{Z_M}$$

was a priori only $Z[M]$-equivariant, but we have now established that it is in fact $Z_M[M]$-equivariant, as it coincides with $\Lambda_{Z_M}^0$ up to a unit. We also emphasize that the scalar in $Z_M$, by which $j \circ \Psi$ differs from $\Lambda_{Z_M}^0$, depends on the choice of isomorphism $\tau_{j}^0 (\text{Ind}_{U_M}^M \psi_{Z_M}) \cong \text{Ind}_{U_M}^M \psi_{Z_M}$ made in the definition of $\Phi'$ (hence of $\Psi$) in Lemma 5.5.
7. The main theorem

Recall that $Z := W(k)[K \setminus G/K]$, $R$ is a ring, $\lambda$ is a homomorphism $Z \to R$, and $\mathcal{M}_{\lambda, R} := W(k)[G/K] \otimes_{Z, \lambda} R$.

Let $W^0_\lambda$ be the canonical element of $\text{Ind}_U^G \psi_R$ guaranteed by Lemma 2.4, and let $\Lambda_R : \mathcal{M}_{\lambda, R} \to \text{Ind}_U^G \psi_R$ be the canonical map given by $1_K \otimes 1 \mapsto W^0_\lambda$.

**Theorem 7.1.** Let $R$ be a Noetherian ring. The map $\Lambda_R : \mathcal{M}_{\lambda, R} \to \text{Ind}_U^G \psi_R$ is injective.

The proof of Theorem 7.1 will occupy the remainder of this section. The strategy is to use induction, combined with Lemma 3.3, Lemma 4.1, Proposition 5.5, and Proposition 6.5.

**Proof of Theorem 7.1.** Let $n = 1$. Then

$$Z = W(k)[F^\times / O_F^\times] \cong W(k)[\mathbb{A}^\times] \cong W(k)[X_1^{\pm 1}].$$

The module $W(k)[G/K] = Z$ is free of rank 1 over $Z$. Therefore $\mathcal{M}_{\lambda, R} \cong R$ with the action of $F^\times$ given by the unramified character $\lambda$. Since $U = \{1\}$, we have $\text{Ind}_U^G \psi_R = C^\infty(F^\times, R)$, and

$$W^0_\lambda(x) = \begin{cases} \lambda(X_1^{v_F(x)}) & \text{if } v_F(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the $F^\times$-equivariant map $\Lambda_R : R \to C^\infty(F^\times, R)$ given by $1 \mapsto W^0_\lambda$ is certainly injective since $W^0_\lambda(1) = 1$.

Now let $n$ be arbitrary. Choose a proper parabolic subgroup with Levi decomposition $P = MN$ and consider the map $\Phi$ defined by Eqn (10). By Proposition 6.5, the following diagram commutes, up to a scalar multiple in $Z^\times_M$:

$$\begin{array}{ccc}
W(k)[G/K] & \xrightarrow{\Lambda_Z} & \text{Ind}_U^G \psi_Z \\
\downarrow \phi & & \downarrow \phi' \\
W(k)[M/K] & \xrightarrow{\Lambda_M^Z} & \text{Ind}_U^M \psi_Z.
\end{array}$$

Let $R'$ be the $Z_M$-algebra $R \otimes Z Z_M$. We make the identification of $Z$-modules $W(k)[M/K] \otimes Z R \cong (W(k)[M/K] \otimes_{Z_M} Z_M) \otimes Z R \cong W(k)[M/K] \otimes_{Z_M} R'$. By tensoring with $R$ and composing with the natural maps

$$(\text{Ind}_U^G \psi_Z) \otimes_Z R \to \text{Ind}_U^G \psi_R$$

$W \otimes \xi \mapsto \lambda^* W \cdot \xi,$

we obtain the following commutative-up-to-unit-scalar diagram of $Z$-modules:

$$\begin{array}{ccc}
W(k)[G/K] \otimes_Z R & \xrightarrow{\Lambda_R} & \text{Ind}_U^G \psi_R \\
\downarrow \phi \otimes \text{id} & & \downarrow \phi' \otimes \text{id} \\
W(k)[M/K] \otimes_{Z_M} R' & \xrightarrow{\Lambda_M^R} & \text{Ind}_U^M \psi_R.
\end{array}$$

Because the diagram is commutative up to a unit scalar in $R'$, we have

$$(\Lambda_M^R \circ (\Phi \otimes \text{id}))(\ker \Lambda_R) = 0.$$
But $M$ is a proper Levi subgroup, so it is a product of groups $GL_{n_i}(F)$ along the diagonal, for $n_i < n$. The ring $R'$ is Noetherian because $R$ is. Thus we can apply the induction hypothesis to

$$W(k)[M/K_M] \otimes_{Z_M} R'$$

to conclude $\Lambda^M_{R'}$ is injective. Therefore $(\Phi \otimes \id)(\ker \Lambda_R) = 0$. Since $\Phi \otimes \id$ factors as the composition

$$\mathcal{M}_{\lambda,R} \overset{p_n}{\longrightarrow} \mathcal{M}^G_{\lambda,R} \cong W(k)[M/K_M] \otimes_{Z_M} R',$$

we conclude that $p_n(\ker \Lambda_R) = 0$, and hence $\mathcal{M}^G_{\lambda,R}$ is admissible over $R$. Thus since $R$ is Noetherian Lemma \[\text{[1,1]}\] tells us that either $\ker \Lambda_R = 0$ or $(\ker \Lambda_R)^{(n)} \neq 0$.

We conclude the proof of Theorem \[\text{[7.1]}\] by showing that $(\ker \Lambda_R)^{(n)} = 0$. Indeed, the sequence

$$0 \to \ker \Lambda_R \to \mathcal{M}_{\lambda,R} \to \mathrm{Ind}_{U}^{G} \psi_R$$

is exact, hence so is

$$0 \to (\ker \Lambda_R)^{(n)} \to (\mathcal{M}_{\lambda,R})^{(n)} \overset{\Lambda_R^{(n)}}{\longrightarrow} (\mathrm{Ind}_{U}^{G} \psi_R)^{(n)}.$$ 

If we let $\text{ev}_1 : \mathrm{Ind}_{U}^{G} \psi_R \to R$ denote the map $W \mapsto W(1)$, then $\text{ev}_1 \circ \Lambda_R$ factors through the composition

$$\mathcal{M}_{\lambda,R} \to (\mathcal{M}_{\lambda,R})^{(n)} \overset{\Lambda_R^{(n)}}{\longrightarrow} (\mathrm{Ind}_{U}^{G} \psi_R)^{(n)} \overset{\text{ev}_1^{(n)}}{\longrightarrow} R.$$ 

But $\text{ev}_1 \circ \Lambda_R$ is precisely the map inducing the isomorphism

$$(\text{ev}_1 \circ \Lambda_R)^{(n)} : (\mathcal{M}_{\lambda,R})^{(n)} \cong R$$

of Proposition \[\text{[8.1]}\] In particular, $\Lambda_R^{(n)}$ is injective, and $(\ker \Lambda_R)^{(n)}$ is zero. \[\square\]

8. The global setup of the Ihara conjecture

8.1. Unitary groups. Let $B$ be a division algebra, equipped with an involution $*: B \to B$. Let $F$ denote the center of $B$, and let $F^+$ denote the subfield of $F$ fixed by $*$. The unitary group $U(B,*)_{/F^+}$ is the algebraic group over $F^+$ representing the functor:

$$R \mapsto \{g \in (B \otimes_{F^+} R)^{\times} : g^*g = 1\}, \text{ for } F^+\text{-algebras } R.$$ 

The involution $*$ is said to be of the second kind if $F \neq F^+$. Then $[F : F^+] = 2$ and $B \otimes_{F^+} F^+ = (B \otimes_{F^+} F) \otimes_{F} F^+ \cong M_n(F^+) \times M_n(F^+)$, where $n^2 = \dim_F(B)$. After possibly modifying this isomorphism by an inner automorphism, $*$ is given on the target by $(X,Y) \mapsto (tY,tX)$ (\[\text{[PR94]}\] 2.3.3]). Thus, we have

\[U(B,*)(F^+) \cong \{(g,h) \in GL_n(F^+) \times GL_n(F^+) : (g,h)(t,h^t g) = (I,I)\}\]

$$= \{(g,t, g^{-1}) \in GL_n(F^+) \times GL_n(F^+)\}$$

$$\cong GL_n(F^+) \text{ (non-canonically)},$$

so $U(B,*)$ is an inner form of $GL_n$.

Now let $F^+$ be a fixed totally real number field, and $F = EF^+$ an imaginary quadratic extension that is unramified everywhere, and let $c$ denote conjugation in $\text{Gal}(F/F^+)$. We will construct a division algebra $B$ with center $F$, equipped with an involution $*$ extending $c$, of the second kind, such that $U(B,*)$ has certain
properties desirable for studying automorphic forms.

**Note:** The letter $F$ henceforth denotes a number field, whereas in all previous sections it denoted a $p$-adic field. We will eventually apply the results of the previous sections to the completions $F_w$ of $F$.

We follow the notation and choices of [CHT08 §3.3-3.5]. Let $\ell > n > 1$ be a prime that splits in $F/F^+$ and let $S_\ell$ be the set of places of $F^+$ above $\ell$.

Choose a nonempty finite set of places $S(B)$, each of which splits in $F/F^+$, none of which lies in $S_\ell$, and such that, when $n$ is even, $\#S(B)$ has the same parity as $\frac{n}{2}[F^+ : \mathbb{Q}]$. Then there exists a division algebra $B$ with center $F$, of dimension $n^2$ over $F$, which is non-split at the places lying over $S(B)$, and such that $B^{op} \cong B \otimes_{E, c} E$. According to [CHT08 §3.3], the assumption that $\#S(B)$ has the same parity as $\frac{n}{2}[F^+ : \mathbb{Q}]$ for even $n$ makes it possible to extend $c$ to an involution $\ast : B \to B$ of the second kind, in such a way that the group $G = U(B, \ast)$ satisfies:

- the unitary group $G$ is *compact at infinity*, i.e., $G(F_v^+) \cong U(n)$ for all $v \in S_\infty$,
- at all the finite places $v \notin S(B)$ that do not split in the quadratic extension $F/F^+$, the unitary group is *quasi-split*, i.e., $G(F_v^+)$ is a quasi-split unitary group over $F_v^+$.

For each place $v$ of $F^+$ that splits in $F$, the two places $w, \bar{w}$ lying over $F$ are exchanged by $c$, and the completions $F_w = F_{\bar{w}}$ are both equal to $F_v^+$. Choosing a place $w$ lying over $v \notin S(B)$ is equivalent to choosing an isomorphism

$$(11) \quad G(F_v^+) = \{(g, h) \in GL_n(F_w) \times GL_n(F_{\bar{w}}) : (g, h)(i^t h, i^t g) = (I, I)\}$$

$$= \{(g, i^tg^{-1}) \in GL_n(F_w) \times GL_n(F_{\bar{w}})\}$$

$$\cong GL_n(F_w),$$

and the two isomorphisms differ by $i(-)^{-1}$. Moreover, it is possible to choose an order $O_B$ in $B$ that is stable under $\ast$ and such that $O_{B, w}$ is maximal for all places $w$ of $F$ lying over split places of $F^+$; this gives a model of $G$ defined over $O_{F^+}$, such that $G(O_{F^+, w})$ is isomorphic under $(11)$ to $GL_n(O_{F, w})$ at split places $v \notin S(B)$.

### 8.2. Automorphic forms on definite unitary groups

Let $H$ be a connected reductive group over the totally real field $F^+$. Let $\mathbb{A}_{F^+}$ denote the ring of $F^+$-adeles and $\mathbb{A}_{F^+}^\infty$ the ring of finite adeles. Then $H(\mathbb{A}_{F^+})$ decomposes as a restricted direct product $\prod_v H(F_v^+)$, over all places $v$ of $F^+$, where the components lie in $H(O_{F^+, v})$ for all but finitely many $v$. Given a subset $S$ of places, a superscript $X^S$ will always denote $\prod_{v \notin S} X_v$ and and a subscript $X_S$ will denote $\prod_{v \in S} X_v$.

If $K_\infty$ is a maximal compact subgroup of the infinite adeles $H(F^+_\infty)$ and $U$ is a compact open subgroup of the finite adeles $H(\mathbb{A}_{F^+}^\infty)$, a classical (weight 0) automorphic form of level $K_\infty U$ on $H$ is a $\mathbb{C}$-valued function on the double coset space

$$H(F^+) \backslash H(\mathbb{A}_{F^+})/K_\infty U,$$

that satisfies various smoothness and growth conditions. $H$ is called *definite* if $H(F^+_\infty)$ is itself compact, in which case $K_\infty = H(F^+_\infty)$, and it follows that

$$H(F^+) \backslash H(\mathbb{A}_{F^+})/K_\infty U = H(F^+) \backslash H(\mathbb{A}_{F^+}^\infty)/U,$$
which by a theorem of Borel and Harish-Chandra is finite. It was observed by Gross in [Gro99] that automorphic forms on definite groups provide a good framework for investigating algebraic properties of automorphic forms, such as congruences.

Note that our particular group $G = U(B, \ast)$ is a definite unitary group because $G(F_v^+) \cong U(n)$ for all $v \in S_\infty$. Beyond its being definite, Clozel, Harris, and Taylor consider this particular species of unitary group in [CHT08] because its automorphic forms are well-studied, especially with respect to base-change to $GL_n$, and its algebraic automorphic forms give rise to Galois representations valued in a group $G$ closely connected to $GL_n$.

Algebraic automorphic forms on $G$ of level $U$ (and weight 0) are functions on

$$G(F^+) \backslash G(A_{F_v^+})/U$$

valued in rings more general than $\mathbb{C}$. In this section only, let $k$ be an arbitrary algebraic extension of $\mathbb{F}_p$, let $W(k)$ denote the Witt vectors and let $K = \text{Frac}(W(k))$ or a “sufficiently large” finite extension of Frac($W(k)$). Let $O$ denote the ring of integers in $K$. Fix an isomorphism $\iota : \mathbb{K} \cong \mathbb{C}$.

We will specify a level, i.e., a compact open subgroup $U = \prod_v U_v \subset G(A_{F_v^+})$ by fixing various sets of places and requiring that $U_v$ satisfy certain conditions for $v$ in those sets. We would like $U$ to be sufficiently small, which means some $U_v$ contains no non-trivial elements of finite order. Fix a finite nonempty set $S_\infty$ of finite places, each of which is split in $F/F^+$, such that $S_\infty$ is disjoint from $S_f \cup S$ and such that, if $v|p$, then $[F(\zeta_p) : F] > n$. We assume that $U_v \cong I + \omega_v M_n(O_{F_v^+})$ for $v \in S_\infty$, and this guarantees that $U$ is sufficiently small.

For any $O$-algebra $A$, let $S(U, A)$ be the set of locally constant functions

$$\{f : G(F^+) \backslash G(A_{F_v^+}) \rightarrow A, f(gu) = f(g), u \in U, g \in G(A_{F_v^+})\}$$

for all $u \in U$. Since $U$ is sufficiently small this is a finite free $A$-module ([CHT08 p.98]). It is a space of $\ell$-integral automorphic forms in the sense that

$$S(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{K} \cong \bigoplus_{\pi \text{ finite}} \pi_v^{U_v} \otimes_{m_\pi}$$

$$f \mapsto [\phi : g \mapsto \iota(f(\phi))]$$

where $\pi$ runs over all classical automorphic representations of $G(A_{F_v^+})$ over $\mathbb{C}$ such that $\pi_{\infty}$ is the trivial representation, and $m_\pi$ is the multiplicity of $\pi$.

From [Gro99 Prop 9.2], this construction is compatible with reduction mod-$\ell$:

$$S(U, \mathcal{O}) \otimes_{\mathcal{O}} k = S(U, k)$$

$$\text{End}_\mathcal{O}(S(U, \mathcal{O})) \otimes k \cong \text{End}_k(S(U, k)).$$

8.3. Global Hecke algebras. Let $T$ be a finite set of finite places containing $S_f \cup S_B \cup S_\infty$, all of which split in $F/F^+$. Assume $U_v \cong G(O_{F_v^+}) \cong GL_n(O_{F_v^+})$ for all split places $v \notin T$, and $U_v$ is hyperspecial for all non-split places $v \notin T$. Each of the two divisors $w|v$, for $v \notin T$, gives an isomorphism

$$G(F_v^+) \cong GL_n(F_w) = GL_n(F_v^+).$$

Define the Hecke operators $T^{(j)}_w$, $j = 1, \ldots, n$ to be the double coset operators

$$T^{(j)}_w = [GL_n(O_{F_v}) \left( \omega_w I_j I_{n-j} \right) GL_n(O_{F_w})].$$
Define

$$\mathcal{T}^T(U) := \mathcal{O} \left[ T^{(1)}_w, \ldots, T^{(n)}_w, (T^{(n)}_w)^{-1} : w \nmid T, \ v \text{ split} \right]$$

to be the $\mathcal{O}$-subalgebra of $\text{End}_\mathcal{O}(S(U, \mathcal{O}))$ generated by the operators

$$T^{(1)}_w, \ldots, T^{(n)}_w, (T^{(n)}_w)^{-1},$$

for any choice of $w|v$ where $v$ ranges over split places not in $T$. Then $\mathcal{T}^T(U)$ is a reduced commutative ring which is finite free as a $\mathcal{O}$-module ([CHT08 §3.4]), and does not depend on the choices of $w|v$.

Let $\Gamma_F$ be the absolute Galois group. To each maximal ideal $m \subset \mathcal{T}^T(U)$ there is associated a unique continuous semisimple Galois representation

$$\pi_m : \Gamma_F \to GL_n(\mathcal{T}^T(U)/m) = GL_n(k),$$

unramified away from $T$, such that the $j$’th coefficient of the characteristic polynomial of Frobenius away from $T$ is $(-1)^j (# k(v))^{(j-1)/2} T^{(j)}_w$ mod $m$, where $#k(v)$ denotes the order of the residue field of $F^+_w$ ([CHT08 Prop 3.4.2]).

If $\pi_m$ is absolutely irreducible, it has a natural continuous lifting $r_m$ to the localization $\mathcal{T}^T(U)_m$. We say $m$ is non-Eisenstein if $\pi_m$ is absolutely irreducible.

**Proposition 8.1** ([CHT08 Cor 3.4.5]). Suppose $m$ is non-Eisenstein and $v_0 \in T - (S_t \cup S(B))$ and $U_{v_0} = G(O_{F^+_v, v_0})$. If $w$ is a prime of $F$ above $v_0$ then there exist $t_1, \ldots t_n \in \mathcal{T}^T(U)_m$ such that

$$T^{(j)}_w * f = t_j f, \ \text{for} \ j=1, \ldots, n$$

for any $f$ in $S(U, \mathcal{O})_m$.

9. **Application of Theorem 7.1 to Ihara**

The $R = \mathbb{T}$ theorem of [CHT08] is proven conditionally on a conjecture, known as Ihara’s lemma. As explained in the introduction, we can apply Theorem 7.1 to reduce this conjecture to an easier statement. We now give more details.

From this section onward, we reinstate our assumption that $k$ is algebraically closed.

**Conjecture 9.1** ([CHT08 §5.3: weak Ihara’s lemma]). Let $U \subset G(A^\infty_{F^+})$ be a sufficiently small open subgroup. Suppose

- $v_0 \in T - (S_t \cup S(B) \cup S_\alpha)$ is a place where $U_{v_0} \cong G(O_{F^+_v})$,
- $m \subset \mathcal{T}^T(U)$ is a non-Eisenstein maximal ideal,
- $f$ is an element of $S(U, k)[m]$.

If $\pi$ is an irreducible $k[G(F^+_v)]$-submodule of

$$\langle G(F^+_v) \cdot f \rangle \subset S(U^{v_0}, k),$$

then $\pi$ is generic.

Toward the goal of Conjecture 9.1 we have the following.

**Theorem 9.2.** For $U, v_0$, and $m$ as in Conjecture 9.1, $S(U, k)[m]$ is a spherical Hecke eigenspace at $v_0$. More precisely if $w|v_0$, there is a homomorphism

$$\lambda : k[GL_n(O_{F^+_w}) \backslash GL_n(F^+_w)/GL_n(O_{F^+_w})] \to k$$

such that

$$T^{(j)}_w * f = \lambda(T^{(j)}_w) f \ \text{ for } j = 1, \ldots, n,$$

for all $f \in S(U, k)[m]$. 
Proof. This is just a formal consequence of Proposition \[8.1\]. Take \(U, v_0\), and \(m\) as in Conjecture \[9.1\].

Since \(T^T(U)\) is finite free over the complete DVR \(O\), it is semilocal, and we can write \(T^T(U) = \prod_m T^T(U)_m\), a product over the localizations at each maximal ideal. Then \(S(U, O)\) and \(S(U, k)\) decompose as the product of their localizations \(\prod_m S(U, O)_m\) and \(\prod_m S(U, k)_m\), respectively. In particular, \(S(U, k)[m] = S(U, k)[m]\).

We also have \(S(U, O)_m \otimes k = S(U, k)_m\), and there is a natural map
\[
\text{End}_O(S(U, O)_m) \to \text{End}_k(S(U, k)_m).
\]

The image of each Hecke operator \(T^{(j)}_w\) in this map is the Hecke operator \(T^{(j)}_w\), by definition.

Thus we conclude that the action of the Hecke operator \(T^{(j)}_w\) on \(S(U, k)_m\) is given by the reduction mod-\(\ell\) of the scalar \(t_j \in T^T(U)_m\) appearing in Proposition \[8.1\].

The action of \(T^T(U)_m \otimes_O k\) on \(S(U, k)[m]\) factors through the residue field \(T^T(U)_m/m = k\). We conclude that there are scalars \(t_j \in k\) such that the action of \(T^{(j)}_w\) on \(S(U, k)[m]\) is given by multiplication by \(t_j\). The result follows. \(\square\)

We now state a weaker conjecture:

**Conjecture 9.3.** Assume the setup of Conjecture \[9.1\]. The \(k[G(F^+_{v_0})]-\)module \(\langle G(F^+_{v_0}) \cdot f \rangle\) is generic.

As a corollary of Theorem \[7.1\] we obtain the following.

**Corollary 9.4.** Conjecture \[9.3\] and Conjecture \[7.1\] are equivalent, and both are equivalent to \(\langle G(F^+_{v_0}) \cdot f \rangle\) having exactly one irreducible Jordan–Holder constituent that is generic.

Proof. Theorem \[7.2\] shows that the \(k[G(F^+_{v_0})]-\)module \(\langle G(F^+_{v_0}) \cdot f \rangle\) satisfies hypotheses (1) and (2) of Corollary \[1.5\]. Thus the result is given by the conclusion of Corollary \[1.3\] combined with the conclusion of Theorem \[7.1\]. \(\square\)

For its applications to \(R = \mathbb{T}\) theorems, it suffices to know the truth of Conjecture \[9.1\] in the quasi-balanced setting (c.f. [CHT08 Prop 5.3.5]), where we can refine our result slightly thanks to our work in Subsection \[9.2\]. In the notation of Conjecture \[9.1\] let \(I_{v_0}\) be the Iwahori subgroup of \(G(F^+_{v_0}) \cong GL_n(F_{v_0})\), let \(H(G(F^+_{v_0}), I_{v_0})\) be the local Iwahori–Hecke algebra at \(v_0\), and let \(A_{v_0}\) be the subalgebra \(k[I_{v_0}] G(F^+_{v_0})/G(O_{F^+,v_0}]\).

**Corollary 9.5.** Let \(v_0, f, U\) be as in Conjecture \[9.1\], suppose \(\ell\) is quasi-balanced for \(\#k(v_0)\), and let \(\bar{U} := (\prod_{v \neq v_0} U_v) \times I_{v_0}\). Then Conjecture \[9.1\] is equivalent to the following statement: the cyclic \(A_{v_0}\)-submodule of \(S(\bar{U}, k)\) generated by \(f\) has dimension \(n\).

Proof. By Theorem \[9.2\] there is some \(\lambda : \mathbb{Z} \to k\) and a map \(M_{\lambda,k} \to \langle G(F^+_{v_0}) \cdot f \rangle\) whose image contains \(f\). Thus \(\langle G(F^+_{v_0}) \cdot f \rangle\) is a quotient of \(M_{\lambda,k}\). Hence \(\langle G(F^+_{v_0}) \cdot f \rangle\) is a quotient of \(\mathcal{M}_{\lambda,k}\). Thus \(\langle G(F^+_{v_0}) \cdot f \rangle\) is cyclic as an \(H(G,I)\)-module. Since \(H_{\mathcal{M}}\) acts trivially on \(f\), we have that \(\langle G(F^+_{v_0}) \cdot f \rangle\) isomorphic to \(M_{\lambda,k}\) and \(\langle G(F^+_{v_0}) \cdot f \rangle\) would be isomorphic to \(M_{\lambda,k}\). The other direction is immediate, since we have already
established that Conjecture[1.1] is equivalent to the map $\mathcal{M}_{\lambda,k} \rightarrow \langle G(F^+_{v_0}) \cdot f \rangle$ being an isomorphism, and $\mathcal{M}_{\lambda,k}^i$ has dimension $n!$. □

References

[Bel02] Joël Bellaïche. Congruences endoscopiques et représentations Galoisiennes. PhD thesis, L’Université Paris XI-Orsay, 2002.

[BH03] Colin J. Bushnell and Guy Henniart. Generalized Whittaker models and the Bernstein center. American Journal of Mathematics, 125:513–547, 2003.

[BK98] Colin J. Bushnell and Philip C. Kutzko. Smooth representations of reductive $p$-adic groups: structure theory via types. Proc. London Math. Soc., 77(3):582–634, 1998.

[BO03] Joël Bellaïche and Ania Otwinowska. Platitude du module universel pour $GL_3$ en caractéristique non banale. Bull. Soc. Math. France, 131:507–525, 2003.

[Boy20] Pascal Boyer. Ihara’s lemma and level raising in higher dimension. Journal de l’Inst. Math. de Jussieu, 2020. to appear.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $p$-adic groups. I. Annales scientifiques de l’É.N.S., 1977.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some $l$-adic lifts of automorphic mod-$l$ Galois representations. Publ. math. IHES, (108), 2008.

[Dat09] Jean-François Dat. Finitude pour les représentations lisses des groupes $p$-adiques. J. Inst. Math. Jussieu, 8:261–333, 2009.

[EH14] Matthew Emerton and David Helm. The local Langlands correspondence in families. Ann. Sci. E.N.S., 2014.

[Eme11] Matthew Emerton. Local-global compatibility in the $p$-adic langlands programme for $GL_2$. 2011.

[Gro99] Benedict Gross. Algebraic modular forms. Israel Journal of Mathematics, 113:61–93, 1999.

[Gro14] Elmar Große-Klönne. On the universal module of $p$-adic spherical hecke algebras. American Journal of Mathematics, 136(3):599–652, 2014.

[Hel16] David Helm. The Bernstein center of the category of smooth $W(k)[GL_n(F)]$-modules. Forum of math, sigma, 4, 2016.

[HP87] R. Howe and I.I. Piatetski-Shapiro. A counterexample to the “generalized Ramanujan conjecture” for (quasi)-split groups. In Automorphic forms, representations, and $L$-functions (Proc. Sympos. Pure Math., Vol 33, Part 1, Corvallis 1977), pages 315–322, Providence, R.I., 1979. Amer. Math. Soc.

[Kat81] Shin-ichi Kato. On eigenspaces of the Hecke algebra with respect to a good maximal compact subgroup of a $p$-adic reductive group. Math. Ann., (257):1–7, 1981.

[Laz98] Xavier Lazarus. Module universel en caractère $l > 0$ associé à un caractère de l’algèbre de Hecke de $GL(n)$ sur un corps $p$-adique, avec $l \neq p$. Journal of algebra, 213:662–686, 1998.

[Laz00] Xavier Lazarus. Module universel non ramifié pour un groupe réductif $p$-adique. PhD thesis, L’Université Paris-Sud, 2000.

[MS] Jeffrey Manning and Jack Shotton. Ihara’s lemma for shimura curves over totally real fields via patching. Math. Annalen, to appear.

[MW86] C. Moeglin and J. L. Waldspurger. Journál für die reine und angewandte Mathematik, 372:136–177, 1986.

[PR94] Vladimir Platonov and Andrei Rapinchuk. Algebraic groups and number theory. Academic press, inc., San Diego, 1994.

[Shi76] Takuro Shintani. On an explicit formula for class-1 “whittaker functions” on $GL_n$ over $\mathfrak{q}$-adic fields. Proc. Japan Acad., (52), 1976.

[SL96] J.-P. Serre and R. Livné. Two letters on quaternions and modular forms (mod $p$). Israel Journal of Mathematics, 95:281–299, 1996.

[Sor16] Claus M. Sorensen. The local langlands correspondence in families and Ihara’s lemma for $U(n)$. Journal of Number Theory, 164:127–165, 2016.

[Tay08] Richard Taylor. Automorphy for some $l$-adic lifts of automorphic mod $l$ representations. II. Pub. Math. IHES, (108):183–239, 2008.

[Tho14] Jack Thorne. Raising the level for $GL(n)$. Forum of mathematics sigma, 2, 2014.
[Vig96] Marie-France Vignéras. *Representations $\ell$-modulaires d’un groupe reductif $p$-adique avec $\ell$ different de $p$*. Boston: Birkhauser, 1996.

[Vig04] Marie-France Vignéras. On highest Whittaker models and integral structures. *Contributions to Automorphic Forms, Geometry, and Number theory*, 2004.