Mixed three-point functions of conserved currents in three-dimensional superconformal field theory

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Abstract

We consider mixed three-point correlation functions of the supercurrent and flavour current in three-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal field theories. Our method is based on the decomposition of the relevant tensors into irreducible components to guarantee that all possible tensor structures are systematically taken into account. We show that only parity even structures appear in the correlation functions. In addition to the previous results obtained in [arXiv:1503.04961], it follows that supersymmetry forbids parity odd structures in three-point functions involving the supercurrent and flavour current multiplets.
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1 Introduction

It is a well-known property of conformal field theories that the functional form of two- and three-point functions of conserved currents such as the energy-momentum tensor and vector current are fixed up to finitely many parameters. In [1, 2] a systematic formalism was developed to construct two- and three-point functions of primary operators in diverse dimensions. The method was based on properly imposing the relevant symmetries arising from scale transformations and permutations of points as well as the conservation laws for the conserved currents, (see also refs. [3–10] for earlier work). More recently it was shown in [11] that a peculiar feature of three-dimensional (and perhaps in general, odd-dimensional) conformal field theories is the appearance of parity violating contributions in three-point functions of conserved currents. These structures were overlooked in the original study by Osborn and Petkou [1] (also [2]), and have since been shown to arise in Chern–Simons theories interacting with parity violating matter. Parity violating (or parity odd) structures were also studied in [12–20]. Recently they were also studied in momentum space [21].

In contrast with the non-supersymmetric case studied in [1, 2], supersymmetry imposes additional restrictions on the structure of three-point functions of conserved currents. In supersymmetric field theories the energy-momentum tensor is replaced with the supercurrent multiplet [31], which contains the energy-momentum tensor, the supersymmetry current and additional components such as the $R$-symmetry current. Similarly, a conserved vector current becomes a component of the flavour current supermultiplet. The general formalism to construct the two- and three-point functions of primary operators in three-dimensional superconformal field theories was developed in [32–35]. Within this formalism it was shown in [33] that the three-point function of the supercurrent (and, hence, of the energy-momentum tensor) in three-dimensional $\mathcal{N} = 1$ superconformal theory is comprised of only one tensor structure. It was also shown that the three-point function of the non-abelian flavour current (and, hence, the three-point function of conserved vector currents) also contains only one tensor structure. In both cases the tensor structures are parity even.

The aim of this paper is to apply the approach of [33] to the case of mixed correlators involving the supercurrent and flavour current multiplets. Our method is based on a systematic decomposition of the relevant tensors into irreducible components, which

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$^1$Parity even correlation functions in momentum space were discussed in [22–30].

$^2$A similar formalism in four dimensions was developed in [36–38] and in six dimensions in [39].
guarantees that all possible linearly independent structures are consistently taken into account. We demonstrate that these correlation functions contain only parity even structures, hence in combination with the results of [33] we conclude that supersymmetry forbids parity odd structures in the three-point functions of conserved low spin currents such as the energy momentum tensor, supersymmetry current and conserved vector current. In [40] Maldacena and Zhiboedov showed under quite general assumptions that if a three-dimensional conformal field theory possesses a conserved higher spin current then it is free. Since a free theory results in only parity even contributions to correlation functions, we arrive at the conclusion that if the assumptions of [40] are fulfilled, one cannot obtain parity odd structures in three-point functions of all conserved currents in supersymmetric conformal field theories.

The paper is organised as follows. In section 2 we review the construction of the two-point and three-point building blocks which appear in correlation functions of primary superfields. We also review the general form of two- and three-point correlation functions of primary operators. In section 3 we introduce a systematic approach to solve for correlation functions of conserved currents. We illustrate our method by reconsidering the flavour current three-point function which was previously computed in [33]. In section 4 we study three-point functions of mixed correlators involving both the supercurrent and the flavour current multiplet. We show that the three-point function involving one supercurrent and two flavour current multiplets is fixed by the $\mathcal{N} = 1$ superconformal symmetry up to an overall coefficient. We also show that the three-point function involving two supercurrents and one flavour current vanishes. In section 5 we present a systematic discussion regarding the absence of parity violating structures in our results. In section 6 we generalise our method to superconformal theories with $\mathcal{N} = 2$ supersymmetry. We show that both mixed correlators are fixed up to an overall coefficient. In appendix A we summarise our three-dimensional notation and conventions.

The non-vanishing of the three-point function of two supercurrents and one flavour current is quite a surprise given that a similar three-point function vanishes in the $\mathcal{N} = 1$ case. Naively it appears to be a contradiction, as any theory with $\mathcal{N} = 2$ supersymmetry is also a theory with $\mathcal{N} = 1$ supersymmetry. Hence the number of independent tensor structures cannot grow as one increases the number of supersymmetries. Nevertheless, we explain that our results in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases are fully consistent.

In this paper we concentrate on mixed correlators in theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal symmetry. Mixed correlators in conformal field theories with higher extended supersymmetry will be studied elsewhere.
2 Superconformal building blocks

The formalism to construct correlation functions of primary operators for conformal field theories in general dimensions was first elucidated in [1] using an efficient group theoretic formalism. In four dimensions the method was then extended to the case of $\mathcal{N} = 1$ supersymmetry in [36, 37, 41], and was later generalised to higher $\mathcal{N}$ in [38]. Here we review the pertinent details of the three-dimensional formalism [32, 33] necessary to construct correlation functions of the 3D supercurrent and flavour current multiplets.

2.1 Superconformal transformations and primary superfields

Let us begin by reviewing infinitesimal superconformal transformations and the transformation laws of primary superfields. This section closely follows the notation of [42–44]. Consider 3D $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2N}$, parameterised by coordinates $z^A = (x^a, \theta^\alpha_I)$, where $a = 0, 1, 2$, $\alpha = 1, 2$ are Lorentz and spinor indices, while $I = 1, \ldots, \mathcal{N}$ is the $R$-symmetry index. The 3D $\mathcal{N}$-extended superconformal group cannot act by smooth transformations on $\mathbb{M}^{3|2N}$, in general only infinitesimal superconformal transformations are well defined. Such a transformation

$$\delta z^A = \xi z^A \iff \delta x^a = \xi^a(z) + i(\gamma^a)_{\alpha\beta} \xi^\alpha_I(z) \theta^\beta_I, \quad \delta \theta^\alpha_I = \xi^\alpha_I(z),$$

is associated with the real first-order differential operator

$$\xi = \xi^A(z) \partial_A = \xi^a(z) \partial_a + \xi^\alpha_I(z) D^I_\alpha,$$

which satisfies the master equation $[\xi, D^I_\alpha] \propto D^J_\beta$. From the master equation we find

$$\xi^\alpha_I = \frac{i}{6} D^I_\beta \xi^{\alpha\beta} ,$$

which implies the conformal Killing equation

$$\partial_a \xi^a + \partial_b \xi^b = \frac{2}{3} \eta_{ab} \partial_c \xi^c .$$

The solutions to the master equation are called the conformal Killing supervector fields of Minkowski superspace [43, 45]. They span a Lie algebra isomorphic to the superconformal algebra $\text{osp}(\mathcal{N}|2; \mathbb{R})$. The components of the operator $\xi$ were calculated explicitly in [32], and are found to be

$$\xi^{\alpha\beta} = a^{\alpha\beta} - \lambda^\alpha_\gamma x^{\gamma\beta} - x^{\alpha\gamma} \lambda^\beta_\gamma + \sigma x^{\alpha\beta} + 4i \epsilon_{I}^{(a} \theta^{b)}_I + 2i \Lambda_{IJ} \theta^3_J \theta^3_I$$

$$+ x^{\alpha\gamma} x^{\delta\beta} \theta^\gamma_\delta + i b^3_\delta \theta^\delta_\beta \theta^2 - \frac{1}{4} b^{\alpha\beta} \theta^2 \theta^2 - 4i \eta_{IJ} x^{\gamma(\alpha} \theta^{\beta)}_I + 2 \eta_{I}^{(a} \theta^{b)}_I \theta^2 ,$$

\(2.5a\)
\( \xi_\alpha^I = \epsilon_\alpha^I - \lambda_\alpha^\beta \theta_\beta^I + \frac{1}{2} \sigma \theta_\beta^I + \Lambda_{IJ} \theta_\gamma^I + b_{\beta\gamma} x^{\beta\alpha}_I \theta^\gamma_I + \eta_{\beta\gamma} (2i \theta^\gamma_I \theta^\alpha_I - \delta_{IJ} x^{\beta\alpha}) \), \quad (2.5b)

\[ a_{\alpha\beta} = a_{\beta\alpha}, \quad \lambda_{\alpha\beta} = \lambda_{\beta\alpha}, \quad \lambda_\alpha^I = 0, \quad b_{\alpha\beta} = b_{\beta\alpha}, \quad \Lambda_{IJ} = -\Lambda_{JI}. \] \quad (2.6)

The bosonic parameters \( a_{\alpha\beta}, \lambda_{\alpha\beta}, \sigma, b_{\alpha\beta}, \Lambda_{IJ} \) correspond to infinitesimal translations, Lorentz transformations, scale transformations, special conformal transformations and \( R \)-symmetry transformations respectively, while the fermionic parameters \( \epsilon_\alpha^I \) and \( \eta_\alpha^I \) correspond to \( Q \)-supersymmetry and \( S \)-supersymmetry transformations. Furthermore, the identities

\[ D^I_{[\alpha} \xi^J_{\beta]} \propto \varepsilon_{\alpha\beta}, \quad D^I_{(\alpha} \xi^J_{\beta)} \propto \delta^I \xi^J, \quad D^I_{[\alpha} \xi^J_{\beta] \propto \delta^I \varepsilon_{\alpha\beta}, \quad (2.7) \]

imply that

\[ [\xi, D^I_{\alpha}] = -(D^I_{(\alpha} \xi^J_{\beta)} D^J_{\beta} = \lambda_{\alpha\beta} (z) D^I_{\beta} + \Lambda^I_{(\alpha} (z) D^J_{\beta} - \frac{1}{2} \sigma (z) D^J_{\alpha}, \quad (2.8) \]

\[ \lambda_{\alpha\beta} (z) = -\frac{1}{N} D^I_{(\alpha} \xi^J_{\beta)} , \quad \Lambda^I_{(\alpha} (z) = -2 D^I_{\alpha} \xi^J_{\alpha}, \quad \sigma (z) = \frac{1}{N} D^I_{\alpha} \xi^J_{\alpha}. \quad (2.9) \]

The local parameters \( \lambda_{\alpha\beta} (z), \Lambda_{IJ} (z), \sigma (z) \) are interpreted as being associated with combined special-conformal/Lorentz, \( R \)-symmetry and scale transformations respectively, and appear in the transformation laws for primary tensor superfields. For later use let’s also introduce the \( z \)-dependent \( S \)-supersymmetry parameter

\[ \eta_{(\alpha \beta} (z) = -\frac{1}{2} D^I_{\alpha} \sigma (z). \quad (2.10) \]

Explicit calculations of the local parameters give \[32\]

\[ \lambda_{\alpha\beta} (z) = \lambda_{\alpha\beta} - b^{\alpha\beta} x^2 - \frac{1}{2} b^{\alpha\beta} \theta^2 + 2i \eta^{(\alpha \beta}, \quad (2.11a) \]

\[ \Lambda_{IJ} (z) = \Lambda_{IJ} + 4i \eta_{IJ} (\alpha \beta) + 2b_{\alpha\beta} \theta^{\alpha} \theta^2, \quad (2.11b) \]

\[ \sigma (z) = \sigma + b_{\alpha\beta} x^{\alpha\beta} + 2i \theta^\alpha \eta_{(\alpha \beta}, \quad (2.11c) \]

\[ \eta_{IJ} (z) = \eta_{IJ} - b_{\alpha\beta} \theta^\alpha. \quad (2.11d) \]

Now consider a generic tensor superfield \( \Phi^T_A (z) \) transforming in a representation \( T \) of the Lorentz group with respect to the index \( A \), and in the representation \( D \) of the \( R \)-symmetry group \( O(N) \) with respect to the index \( J \). Such a superfield is called primary with dimension \( q \) if its superconformal transformation law is

\[ \delta \Phi^T_A = -\xi \Phi^T_A - q \sigma (z) \Phi^T_A + \lambda_{\alpha\beta} (z) (M_{\alpha\beta})_A^B \Phi^T_B + \Lambda^I (z) (R_{IJ})^T_J \Phi^T_A, \quad (2.12) \]

where \( \xi \) is the superconformal Killing vector, \( \sigma (z), \lambda_{\alpha\beta} (z), \Lambda_{IJ} (z) \) are the \( z \)-dependent parameters associated with \( \xi \), and the matrices \( M_{\alpha\beta} \) and \( R_{IJ} \) are the Lorentz and \( O(N) \) generators respectively.

\[ ^3\text{We assume the representations } T \text{ and } D \text{ are irreducible.} \]
2.2 Two-point functions

Given two superspace points \( z_1 \) and \( z_2 \), we can define the two-point functions
\[
\mathbf{x}_{12}^{\alpha \beta} = (x_1 - x_2)^{\alpha \beta} + 2i \theta_{11}^\alpha \theta_{21}^\beta - i \theta_{12}^\alpha \theta_{12}^\beta, \quad \theta_{12}^{\alpha I} = \theta_1^{\alpha I} - \theta_2^{\alpha I}, \tag{2.13}
\]
which transform under the superconformal group as follows
\[
\tilde{\delta} \mathbf{x}_{12}^{\alpha \beta} = \left( \frac{1}{2} \delta_{\gamma}^{\alpha \gamma} \sigma(z_1) - \lambda_{\gamma}^{\alpha \gamma}(z_1) \right) \mathbf{x}_{12}^{\gamma \beta} + \left( \frac{1}{2} \delta_{\gamma}^{\beta \gamma} \sigma(z_2) - \lambda_{\gamma}^{\beta \gamma}(z_2) \right), \tag{2.14a}
\]
\[
\tilde{\delta} \theta_{12}^\alpha = \left( \frac{1}{2} \delta_{\beta}^{\alpha \beta} \sigma(z_1) - \lambda_{\beta}^{\alpha \beta}(z_1) \right) \theta_{12}^\beta - \theta_{12}^\gamma \theta_{12}^\beta, \tag{2.14b}
\]
Here the total variation \( \tilde{\delta} \) is defined by its action on an \( n \)-point function \( \Phi(z_1, ..., z_n) \) as
\[
\tilde{\delta} \Phi(z_1, ..., z_n) = \sum_{i=1}^n \xi_i \Phi(z_1, ..., z_n). \tag{2.15}
\]
It should be noted that (2.14b) contains an inhomogeneous piece in its transformation law, hence it will not appear as a building block in two- or three-point functions. Due to the useful property,
\[
\mathbf{x}_{21}^{\alpha \beta} = -\mathbf{x}_{12}^{\beta \alpha}, \tag{2.16}
\]
is recognised as the bosonic part of the standard two-point superspace interval. Next let us introduce the two-point objects
\[
\hat{x}_{12}^{\alpha \beta} = \mathbf{x}_{12}^{\alpha \beta} + \frac{i}{2} \varepsilon^{\alpha \beta} \theta_{12}^2, \quad \theta_{12}^2 = \theta_{12}^{\gamma I} \theta_{12}^\gamma I. \tag{2.17}
\]
The symmetric component
\[
x_{12}^{\alpha \beta} = (x_1 - x_2)^{\alpha \beta} + 2i \theta_{11}^\alpha \theta_{21}^\beta, \tag{2.18a}
\]
is recognised as the bosonic part of the standard two-point superspace interval. Next let us introduce the two-point objects
\[
\hat{x}_{12}^{\alpha \beta} = \mathbf{x}_{12}^{\alpha \beta} \sqrt{x_{12}^2} , \quad \hat{x}_{12}^{\alpha \gamma} \hat{x}_{12}^{\gamma \beta} = \delta_{\alpha \beta}. \tag{2.18b}
\]
Hence, we find
\[
(x_{12}^{-1})^{\alpha \beta} = -\frac{x_{12}^{\beta \alpha}}{x_{12}^2}. \tag{2.19}
\]
Under superconformal transformations, (2.18a) transforms with local scale parameters, while (2.18b) transforms with local Lorentz parameters
\[
\tilde{\delta} x_{12}^2 = (\sigma(z_1) + \sigma(z_2)) x_{12}^2, \tag{2.20a}
\]
\[
\tilde{\delta} \hat{x}_{12}^{\alpha \beta} = -\lambda_{\gamma}^{\alpha \gamma}(z_1) \hat{x}_{12}^{\gamma \beta} - \hat{x}_{12}^{\alpha \gamma} \lambda_{\gamma}^{\beta}(z_2). \tag{2.20b}
\]
Thus, both objects are essential in the construction of correlation functions of primary superfields. We also have the useful differential identities

\[ D_{(1)\gamma}^{I} x_{12}^{\alpha\beta} = -2i\theta_{12}^{I\beta}\delta_{\gamma}^{\alpha}, \quad D_{(1)\alpha}^{I} x_{12}^{\alpha\beta} = -4i\theta_{12}^{I\beta}, \]  

(2.21)

where \( D_{(i)\alpha}^{I} \) is the standard covariant spinor derivative \([A,16]\) acting on the superspace point \( z_{i} \). Finally, for completeness, the \( SO(N) \) structure of primary superfields in correlation functions is addressed by the \( N \times N \) matrix

\[ u_{IJ}^{12} = \delta_{IJ} + 2i\theta_{12}^{I\alpha}(x_{12}^{-1})_{\alpha\beta}\theta_{12}^{I\beta}, \]  

(2.22)

which is orthogonal and unimodular,

\[ u_{IK}^{12} u_{KJ}^{12} = \delta_{IJ}, \quad \det u_{12} = 1. \]  

(2.23)

The infinitesimal variation of this matrix is

\[ \delta u_{IJ}^{12} = \Lambda_{IK}^{12}(z_{1}) u_{KJ}^{12} - u_{IK}^{12} \Lambda_{JL}^{12}(z_{2}). \]  

(2.24)

Hence, \([2.22]\) is expected to appear in the construction of correlation functions of primary superfields with \( SO(N) \) indices.

The two-point correlation function of a primary superfield \( \Phi_{A}^{T} \) and its conjugate \( \Phi_{J}^{R} \) is fixed by the superconformal symmetry as follows

\[ \langle \Phi_{A}^{T}(z_{1})\Phi_{J}^{R}(z_{2}) \rangle = c T_{A}^{B}(x_{12}^{-1})D_{J}^{T}(u_{12}) (x_{12}^{2})^{q}, \]  

(2.25)

where \( c \) is a constant coefficient. The denominator of the two-point function is determined by the conformal dimension of \( \Phi_{A}^{T} \), which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

### 2.3 Three-point functions

Given three superspace points \( z_{i}, \ i = 1, 2, 3 \), one can define the three-point building blocks \( Z_{i} = (X_{i}, \Theta_{i}) \) as follows:

\[ X_{1\alpha\beta} = -(x_{21}^{-1})_{\alpha\gamma}x_{23}^{\gamma\delta}(x_{13}^{-1})_{\delta\beta}, \quad \Theta_{1\alpha} = (x_{21}^{-1})_{\alpha\beta}\theta_{12}^{I\beta} - (x_{31}^{-1})_{\alpha\beta}\theta_{13}^{I\beta}, \]  

(2.26a)

\[ X_{2\alpha\beta} = -(x_{32}^{-1})_{\alpha\gamma}x_{31}^{\gamma\delta}(x_{21}^{-1})_{\delta\beta}, \quad \Theta_{2\alpha} = (x_{32}^{-1})_{\alpha\beta}\theta_{23}^{I\beta} - (x_{12}^{-1})_{\alpha\beta}\theta_{21}^{I\beta}, \]  

(2.26b)

\[ X_{3\alpha\beta} = -(x_{13}^{-1})_{\alpha\gamma}x_{12}^{\gamma\delta}(x_{32}^{-1})_{\delta\beta}, \quad \Theta_{3\alpha} = (x_{13}^{-1})_{\alpha\beta}\theta_{31}^{I\beta} - (x_{23}^{-1})_{\alpha\beta}\theta_{32}^{I\beta}. \]  

(2.26c)
These objects, along with their corresponding transformation laws, may be obtained from one-another by cyclic permutation of superspace points. The building blocks transform covariantly under the action of the superconformal group:

\[ \tilde{\delta} X_{1\alpha\beta} = \lambda_\alpha^\gamma(z_1) X_{1\gamma\beta} + X_{1\alpha\gamma} \lambda_\gamma^\beta(z_1) - \sigma(z_1) X_{1\alpha\beta}, \]  

\[ \tilde{\delta} \Theta^I_{1\alpha} = \left( \lambda_\alpha^\beta(z_1) - \frac{1}{2} \delta_\alpha^\beta \sigma(z_1) \right) \Theta^I_{1\beta} + \Lambda^I_\gamma(j_1) \Theta^J_{1\alpha}. \]

Therefore (2.26a), (2.26b) and (2.26c) will appear as building blocks in three-point correlations functions. It should be noted that under scale transformations of superspace, \( z^A = (x^a, \theta^\alpha) \mapsto z'^A = (\lambda^{-2} x^a, \lambda^{-1} \theta^\alpha) \), the three-point building blocks transform as \( Z = (X, \Theta) \mapsto Z' = (\lambda^2 X, \lambda \Theta) \). Next we define

\[ X^2_1 = -\frac{1}{2} X_{1\alpha\beta} X_{1\alpha\beta}^{\beta\alpha} = \frac{x_{23}^2}{x_{13}^2 x_{12}^2}, \quad \Theta^2_1 = \Theta^{I\alpha} \Theta^I_{1\alpha}, \]

which, due to (2.27a) and (2.27b), have the transformation laws

\[ \tilde{\delta} X^2_1 = -2 \sigma(z_1) X^2_1, \quad \tilde{\delta} \Theta^2_1 = -\sigma(z_1) \Theta^2_1. \]

We also define the inverse of \( X_1 \),

\[ (X^{-1}_1)^{\alpha\beta} = -\frac{X_{1\alpha\beta}}{X^2_1}, \]

and introduce useful identities involving \( X_i \) and \( \Theta_i \) at different superspace points, e.g.,

\[ x_{13}^{\alpha\alpha'} X_{3\alpha'\beta} x_{31}^{\beta\gamma} = -(X^{-1}_1)^{\beta\alpha}, \]

\[ \Theta^I_{1\gamma} x_{13}^{\gamma\delta} X_{3\delta\beta} = u_{I3}^J \Theta^J_{3\beta}. \]

As a consequence of (2.29), we can identify the three-point superconformal invariant

\[ \frac{\Theta^2_1}{\sqrt{X^2_1}} \Rightarrow \tilde{\delta} \left( \frac{\Theta^2_1}{\sqrt{X^2_1}} \right) = 0. \]

Hence, the superconformal symmetry fixes the functional form of three-point correlation functions up to this combination. Indeed, using (2.31a) and (2.31b) one can show that the superconformal invariant is also invariant under permutation of superspace points, i.e.

\[ \frac{\Theta^2_1}{\sqrt{X^2_1}} = \frac{\Theta^2_2}{\sqrt{X^2_2}} = \frac{\Theta^2_3}{\sqrt{X^2_3}}. \]
The three-point objects (2.26a), (2.26b) and (2.26c) have many properties similar to those of the two-point building blocks. After decomposing $X_1$ into symmetric and antisymmetric parts similar to (2.16) we have

$$X_{1\alpha\beta} = X_{1\alpha\beta} - \frac{i}{2} \varepsilon_{\alpha\beta} \Theta_1^2, \quad X_{1\alpha\beta} = X_{1\beta\alpha},$$

(2.34)

where the symmetric spinor $X_{1\alpha\beta}$ can be equivalently represented by the three-vector $X_{1m} = \frac{1}{2}(\gamma_m)^{\alpha\beta}X_{1\alpha\beta}$. It is now convenient to introduce analogues of the covariant spinor derivative and supercharge operators involving the three-point objects,

$$D_{(1)\alpha} = \partial_{\Theta_1^I} + i(\gamma^m)_{\alpha\beta} \Theta_1^I \partial_{X_1^m}, \quad Q_{(1)\alpha} = i \partial_{\Theta_1^I} + (\gamma^m)_{\alpha\beta} \Theta_1^I \partial_{X_1^m},$$

(2.35)

which obey the standard commutation relations

$$\{D_{(i)\alpha}, D_{(j)\beta}\} = \{Q_{(i)\alpha}, Q_{(j)\beta}\} = 2i \delta^{IJ}(\gamma^m)_{\alpha\beta} \partial_{X_1^m}. \quad (2.36)$$

Some useful identities involving (2.35) are

$$D_{(1)\gamma}X_{1\alpha\beta} = -2i \varepsilon_{\gamma\beta} \Theta_{1\alpha}^I, \quad Q_{(1)\gamma}X_{1\alpha\beta} = -2i \varepsilon_{\gamma\alpha} \Theta_{1\beta}^I. \quad (2.37)$$

We must also account for the fact that various primary superfields obey certain differential equations. Using (2.21) we arrive at the following

$$D_{(1)\gamma}X_{3\alpha\beta} = 2i(x_{13}^{-1})_{\alpha\gamma} u_{13}^J \Theta_{3\beta}^J, \quad D_{(1)\alpha} \Theta_{3\beta}^J = -(x_{13}^{-1})_{\alpha\beta} u_{13}^J, \quad (2.38a)$$

$$D_{(2)\gamma}X_{3\alpha\beta} = 2i(x_{23}^{-1})_{\beta\gamma} u_{23}^J \Theta_{3\beta}^J, \quad D_{(2)\alpha} \Theta_{3\beta}^J = (x_{23}^{-1})_{\beta\alpha} u_{23}^J. \quad (2.38b)$$

Now given a function $f(X_3, \Theta_3)$, there are the following differential identities which arise as a consequence of (2.37), (2.38a) and (2.38b):

$$D_{(1)\gamma}f(X_3, \Theta_3) = (x_{13}^{-1})_{\alpha\gamma} u_{13}^J D_{(3)\alpha}^J f(X_3, \Theta_3), \quad (2.39a)$$

$$D_{(2)\gamma}f(X_3, \Theta_3) = i(x_{23}^{-1})_{\alpha\gamma} u_{23}^J Q_{(3)\alpha}^J f(X_3, \Theta_3). \quad (2.39b)$$

These will prove to be essential for imposing differential constraints on correlation functions, e.g. those arising from conservation equations in the case of correlators involving the supercurrent and flavour current multiplets.

Finally, for completeness, let us introduce the three-point objects which take care of the $R$-symmetry structure of correlation functions. We define

$$U_{1}^{IJ} = u_{12}^{IK} u_{23}^{KL} u_{31}^{LJ} = \delta^{IJ} + 2i \Theta_{1\alpha}^I (X_{1}^{-1})_{\alpha\beta} \Theta_{1\beta}^J, \quad (2.40)$$
which transforms as an $O(N)$ tensor at $z_1$,

$$\tilde{\delta}U^{IJ}_1 = \Lambda^{IK}(z_1) U^{KJ}_1 - U^{IK}_1 \Lambda^{KJ}(z_1). \quad (2.41)$$

and is orthogonal and unimodular by construction. The others are obtained by cyclic permutation of superspace points, and are related by the useful identities

$$U^{IJ}_2 = u^{IK}_2 U^{KL}_1 u^{LJ}_2, \quad U^{IJ}_3 = u^{IK}_3 U^{KL}_1 u^{LJ}_3. \quad (2.42)$$

As concerns three-point correlation functions; let $\Phi$, $\Psi$, $\Pi$ be primary superfields with conformal dimensions $q_1$, $q_2$ and $q_3$ respectively. The three-point function may be constructed using the general expression

$$\langle \Phi^{a_1}_{\mathcal{A}_1}(z_1) \Psi^{a_2}_{\mathcal{A}_2}(z_2) \Pi^{a_3}_{\mathcal{A}_3}(z_3) \rangle = T^{(1)}_{\mathcal{A}_1 B_1}(x_{13}) T^{(2)}_{\mathcal{A}_2 B_2}(x_{23}) D^{(1)}_{\mathcal{A}_1 J_1}(u_{13}) D^{(2)}_{\mathcal{A}_2 J_2}(u_{23}) H^{J_1 J_2 J_3}_{B_1 B_2 A_3}(X_3, \Theta_3, U_3), \quad (2.43)$$

where the tensor $H^{J_1 J_2 J_3}_{A_1 A_2 A_3}$ is highly constrained by the superconformal symmetry as follows:

(i) Under scale transformations of superspace the correlation function transforms as

$$\langle \Phi^{a_1}_{\mathcal{A}_1}(z_1') \Psi^{a_2}_{\mathcal{A}_2}(z_2') \Pi^{a_3}_{\mathcal{A}_3}(z_3') \rangle = (\lambda^2)^{q_1+q_2+q_3} \langle \Phi^{a_1}_{\mathcal{A}_1}(z_1) \Psi^{a_2}_{\mathcal{A}_2}(z_2) \Pi^{a_3}_{\mathcal{A}_3}(z_3) \rangle, \quad (2.44)$$

which implies that $H$ obeys the scaling property

$$H^{J_1 J_2 J_3}_{A_1 A_2 A_3}(\lambda^2 X, \lambda \Theta, U) = (\lambda^2)^{q_3-q_2-q_1} H^{J_1 J_2 J_3}_{A_1 A_2 A_3}(X, \Theta, U), \quad \forall \lambda \in \mathbb{R} \setminus \{0\}. \quad (2.45)$$

This guarantees that the correlation function transforms correctly under conformal transformations.

(ii) If any of the fields $\Phi$, $\Psi$, $\Pi$ obey differential equations, such as conservation laws in the case of conserved current multiplets, then the tensor $H$ is also constrained by differential equations. Such constraints may be derived with the aid of identities (2.39a), (2.39b).

(iii) If any (or all) of the superfields $\Phi$, $\Psi$, $\Pi$ coincide, the correlation function possesses symmetries under permutations of superspace points, e.g.

$$\langle \Phi^{a_1}_{\mathcal{A}_1}(z_1) \Phi^{a_2}_{\mathcal{A}_2}(z_2) \Pi^{a_3}_{\mathcal{A}_3}(z_3) \rangle = (-1)^{c(\Phi)} \langle \Phi^{a_2}_{\mathcal{A}_2}(z_2) \Phi^{a_1}_{\mathcal{A}_1}(z_1) \Pi^{a_3}_{\mathcal{A}_3}(z_3) \rangle, \quad (2.46)$$
where $\epsilon(\Phi)$ is the Grassmann parity of $\Phi$. As a consequence, the tensor $H$ obeys constraints which will be referred to as “point-switch identities”. To analyse these constraints, we note that under permutations of any two superspace points, the three-point building blocks transform as

\begin{align}
X_{3\alpha\beta} \xrightarrow{1+2} & -X_{3,\beta\alpha}, \\
\Theta^I_{3\alpha} \xrightarrow{1+2} & -\Theta^I_{3,\alpha}, \quad (2.47a) \\
X_{3\alpha\beta} \xrightarrow{2+3} & -X_{2,\beta\alpha}, \\
\Theta^I_{3\alpha} \xrightarrow{2+3} & -\Theta^I_{2,\alpha}, \quad (2.47b) \\
X_{3\alpha\beta} \xrightarrow{1+3} & -X_{1,\beta\alpha}, \\
\Theta^I_{3\alpha} \xrightarrow{1+3} & -\Theta^I_{1,\alpha}. \quad (2.47c)
\end{align}

The constraints above fix the functional form of $H$ (and therefore the correlation function) up to finitely many parameters. Hence the procedure described above reduces the problem of computing three-point correlation functions to deriving the tensor $H$ subject to the above constraints. In the next sections, we will apply this formalism to compute three-point correlation functions involving the supercurrent and flavour current multiplets.

3 Correlation functions of conserved currents in $\mathcal{N} = 1$ superconformal field theory

3.1 Supercurrent and flavour current multiplets

The 3D, $\mathcal{N} = 1$ conformal supercurrent is a primary, dimension 5/2 totally symmetric spin-tensor $J_{\alpha\beta\gamma}$, which contains the three-dimensional energy-momentum tensor along with the supersymmetry current \[45–47\]. It obeys the conservation equation

$$D^\alpha J_{\alpha\beta\gamma} = 0,$$

and has the following superconformal transformation law:

$$\delta J_{\alpha\beta\gamma} = -\xi J_{\alpha\beta\gamma} - \frac{5}{2} \sigma(z) J_{\alpha\beta\gamma} + 3\lambda(z) \delta_{(\alpha} J_{\beta\gamma)\delta}.$$

The $\mathcal{N} = 1$ supercurrent may be derived from, for example, supergravity prepotential approaches \[45\] or the superfield Noether procedure \[48, 49\].

The general formalism in section 2 allows the two-point function to be determined up to a single real coefficient:

$$\langle J_{\alpha\beta\gamma}(z_1) J^{\alpha'\beta'\gamma'}(z_2) \rangle = i b_{\mathcal{N}=1} \frac{x_{12}^{\alpha} x_{12}^{\alpha'} x_{12}^{\beta} x_{12}^{\beta'} x_{12}^{\gamma} x_{12}^{\gamma'}}{(x_{12}^2)^4}. \quad (3.3)$$
It is then a simple exercise to show that the two-point function has the right symmetry properties under permutation of superspace points

\[ \langle J_{\alpha'\beta'\gamma'}(z_1) J_{\alpha'\beta'\gamma'}(z_2) \rangle = -\langle J_{\alpha'\beta'\gamma'}(z_2) J_{\alpha'\beta'\gamma'}(z_1) \rangle, \] (3.4)

and also satisfies

\[ D_{(1)}^\alpha \langle J_{\alpha'\beta'\gamma'}(z_1) J_{\alpha'\beta'\gamma'}(z_2) \rangle = 0. \] (3.5)

Next let’s consider the 3D $\mathcal{N} = 1$ flavour current, which is represented by a primary, dimension $3/2$ spinor superfield $L_{\alpha}$ obeying the conservation equation

\[ D^\alpha L_{\alpha} = 0. \] (3.6)

It transforms covariantly under the superconformal group as

\[ \delta L_{\alpha} = -\xi L_{\alpha} - \frac{3}{2} \sigma(z) L_{\alpha} + \lambda(z)_\alpha^\beta L_{\beta}. \] (3.7)

We can also consider the case when there are several flavour current multiplets (represented by the flavour index, $\bar{a}$) corresponding to a simple flavour group. According to general formalism in section 2, the two-point function for $\mathcal{N} = 1$ flavour current multiplets is fixed up to a single real coefficient $a_{\mathcal{N}=1}$

\[ \langle L_{\alpha}(z_1) L_{\beta}(z_2) \rangle = ia_{\mathcal{N}=1} \frac{\delta^{\bar{a}\bar{b}} \mathbf{x}_{12\alpha\beta}}{(x_{12}^2)^2}. \] (3.8)

It is easy to see that the two-point function obeys the correct symmetry properties under permutation of superspace points, $\langle L_{\alpha}(z_1) L_{\beta}(z_2) \rangle = -\langle L_{\beta}(z_2) L_{\alpha}(z_1) \rangle$. One can also check that it satisfies the conservation equation (3.6)

\[ D_{(1)}^\alpha \langle L_{\alpha}(z_1) L_{\beta}(z_2) \rangle = 0. \] (3.9)

Three-point correlation functions of the flavour current and particularly the supercurrent are considerably more complicated, and were derived in [33, 34]. However, correlators of combinations of these fields (mixed correlators) were not studied previously and will be analysed in section 4.

---

4The tensor structure and the conservation law of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ flavour currents follow from the structure of unconstrained prepotentials for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector multiplets [50, 53].
3.2 Correlation functions of conserved current multiplets

The possible three-point correlation functions that may be constructed from the conserved $N=1$ supercurrent and flavour current multiplets are:

\[ \langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle, \quad \langle J_A(z_1)J_B(z_2)J_C(z_3) \rangle, \]

(3.10)

\[ \langle L_{\bar{a}a}(z_1)J_A(z_2)L_{\bar{b}b}(z_3) \rangle, \quad \langle J_A(z_1)J_B(z_2)L_{\bar{a}a}(z_3) \rangle, \]

(3.11)

where $A, B, C$ each denote a totally symmetric combination of three spinor indices. The correlators $\langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle$ and $\langle J_A(z_1)J_B(z_2)J_C(z_3) \rangle$ were studied in [33]. Before we compute the mixed correlators, let us demonstrate our method on the three-point function $\langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle$, which is comparatively straightforward.

The general form of the flavour current three-point function is:

\[ \langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle = f_{\bar{a}b\bar{c}} \mathcal{H}_{\alpha'\beta'\gamma}(X, \Theta), \]

(3.12)

The correlation function is required to satisfy the following properties:

(i) Scaling constraint:

Under scale transformations the correlation function must transform as

\[ \langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle = (\lambda^2)^{9/2} \langle L_{\bar{a}a}(z'_1)L_{\bar{b}b}(z'_2)L_{\bar{c}c}(z'_3) \rangle, \]

(3.13)

which gives rise to the homogeneity constraint on $\mathcal{H}$:

\[ \mathcal{H}_{\alpha'\beta'\gamma}(\lambda^2 X, \lambda \Theta) = (\lambda^2)^{-3/2} \mathcal{H}_{\alpha\beta\gamma}(X, \Theta). \]

(3.14)

(ii) Differential constraints:

The conservation equation for the flavour current results in

\[ D^\alpha_{(1)}\langle L_{\bar{a}a}(z_1)L_{\bar{b}b}(z_2)L_{\bar{c}c}(z_3) \rangle = 0. \]

(3.15)

Using identities (2.39a), (2.39b), we obtain a differential constraint on $\mathcal{H}$:

\[ D^a \mathcal{H}_{\alpha\beta\gamma}(X, \Theta) = 0. \]

(3.16)

We need not consider the conservation law at $z_2$ as we can use an algebraic constraint instead.

\[ ^5 \] Here we consider only the contribution proportional to the totally antisymmetric structure constants $f_{\bar{a}b\bar{c}}$. Similarly, one can consider the contribution totally symmetric in flavour indices. However, this contribution vanishes so it is omitted here.
(iii) Point permutation symmetry:

The symmetry under permutation of points \((z_1 \text{ and } z_2)\) results in the following constraint on the correlation function:

\[
\langle L^a_\alpha(z_1)L^b_\beta(z_2)L^c_\gamma(z_3) \rangle = -\langle L^b_\beta(z_2)L^a_\alpha(z_1)L^c_\gamma(z_3) \rangle,
\]  

which constrains the tensor \(H\) so that

\[
H_{\alpha\beta\gamma}(X, \Theta) = H_{\beta\alpha\gamma}(-X^T, -\Theta). \tag{3.18}
\]

On the other hand the symmetry under permutation of points \(z_1 \text{ and } z_3\) results in

\[
\langle L^a_\alpha(z_1)L^b_\beta(z_2)L^c_\gamma(z_3) \rangle = -\langle L^c_\gamma(z_3)L^b_\beta(z_2)L^a_\alpha(z_1) \rangle,
\]  

which gives rise to the point-switch identity

\[
H_{\alpha\beta\gamma}(X_3, \Theta_3) = \frac{x_{13}^{\gamma'}(x_{13}^{-1})^\alpha x_{13}^\beta\gamma X_3^{\gamma}X_3^{\alpha\beta}}{X_3^{\gamma}X_3^{\alpha\beta}} H_{\gamma'\beta'\alpha'}(-X^T_1, -\Theta_1). \tag{3.20}
\]

To solve this problem systematically let’s decompose the tensor \(H\) into irreducible components:

\[
H_{\alpha\beta\gamma}(X, \Theta) = \sum_i c_i H_i \alpha\beta\gamma(X, \Theta). \tag{3.21}
\]

It is also more convenient to work with \(X_m\) instead of \(X_{\alpha\beta}\). We have

\[
\begin{align*}
H_1 \alpha\beta\gamma &= \varepsilon_{\alpha\beta}\Theta_\gamma A(X), \tag{3.22a} \\
H_2 \alpha\beta\gamma &= \varepsilon_{\alpha\beta}(\gamma^a)^\gamma \Theta_\delta B_{\alpha}(X), \tag{3.22b} \\
H_3 \alpha\beta\gamma &= (\gamma^a)_{\alpha\beta}\Theta_\delta C_{\alpha}(X), \tag{3.22c} \\
H_4 \alpha\beta\gamma &= (\gamma^a)_{\alpha\beta}(\gamma^b)^\delta \Theta_\delta D_{ab}(X). \tag{3.22d}
\end{align*}
\]

Here we have used the fact that every matrix anti-symmetric in \(\alpha, \beta\) is proportional to \(\varepsilon_{\alpha\beta}\), every matrix symmetric in \(\alpha, \beta\) is proportional to a gamma-matrix, and that since \(H\) is Grassmann odd it follows that \(H\) is linear in \(\Theta\) due to \(\Theta_\alpha \Theta_\beta \Theta_\gamma = 0\). Due to the scaling property (3.14) it follows that the functions \(A, B, C, D\) have dimension \(-2\). From eq. (3.18) it also follows that

\[
\begin{align*}
A(X) &= A(-X), \quad B_{\alpha}(X) = B_{\alpha}(-X), \tag{3.23a} \\
C_{\alpha}(X) &= -C_{\alpha}(-X), \quad D_{ab}(X) = -D_{ab}(-X). \tag{3.23b}
\end{align*}
\]
It is easy to see that the conservation equation (3.16) splits into the two independent equations

\[ \partial^\alpha \mathcal{H}_{\alpha \beta \gamma} = 0, \quad \text{(3.24a)} \]
\[ (\gamma^t)^{\alpha \tau} \partial_\tau \mathcal{H}_{\alpha \beta \gamma} = 0. \quad \text{(3.24b)} \]

Imposing (3.24a) results in the algebraic equations

\[ A(X) = -D^a_a(X), \quad C_a(X) = B_a(X) + \epsilon_a^{\ mn} D_{mn}(X). \quad \text{(3.25)} \]

While on the other hand from (3.24b) we obtain

\[ \partial^a \{ B_a(X) + C_a(X) - \epsilon_a^{\ mn} D_{mn}(X) \} = 0, \quad \text{(3.26a)} \]
\[ \partial_\tau A(X) + \epsilon_t^{\ ma} \partial_m B_a(X) - \epsilon_t^{\ ma} \partial_m C_a(X) \]
\[ - \partial^m D_{mt}(X) + \partial_t D^a_a(X) - \partial^m D_{tm}(X) = 0. \quad \text{(3.26b)} \]

Using eqs. (3.25), (3.26a), (3.26b) we obtain that \( B_a \) and \( D_{ab} \) satisfy

\[ \partial^a B_a(X) = 0, \quad \partial^a D_{ab}(X) = 0. \quad \text{(3.27)} \]

Thus, the problem is reduced to finding transverse tensors \( B_a \) and \( D_{ab} \) of dimension \(-2\) satisfying (3.23b). The tensors \( A \) and \( C \) are then found using eq. (3.25). It is not difficult to show that the solution to this problem is given by

\[ A(X) = 0, \quad B_a(X) = 0, \quad \text{(3.28a)} \]
\[ C_a(X) = \frac{X_a}{X^3}, \quad D_{ab}(X) = \epsilon_{abe} \frac{X_c}{X^3}. \quad \text{(3.28b)} \]

with \( c_3 = -2c_4 \). Hence this correlation function is fixed up to a single real coefficient which we denote \( d_{N=1} \). Converting back to spinor notation we find

\[ \mathcal{H}_{\alpha \beta \gamma}(X, \Theta) = \frac{id_{N=1}}{X^3} \left\{ X_{\alpha \beta} \Theta_{\gamma} - \epsilon_{\alpha \gamma} X_{\beta}^{\ \delta} \Theta_\delta - \epsilon_{\beta \gamma} X_{\alpha}^{\ \delta} \Theta_\delta \right\}. \quad \text{(3.29)} \]

One may also check that this solution satisfies the point-switch identity (3.20). This agrees with the result in [33], which was computed in a different way. Our method has the advantage that it systematically takes care of all possible irreducible components of \( \mathcal{H} \) and, hence, is more useful when \( \mathcal{H} \) is a tensor of high rank.

\[ ^6 \text{Note that since } \Theta_\alpha \Theta_\beta \Theta_\gamma = 0 \text{ we can replace } X \text{ with } X \text{ in (3.29).} \]
4 Mixed correlators in $\mathcal{N} = 1$ superconformal field theory

4.1 The correlation function $\langle L J L \rangle$

Let us first consider the correlation function $\langle L_{\bar{a}}^{\alpha}(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_{\bar{b}}^{\beta}(z_3) \rangle$. Using the general expression (2.43), it has the form

$$\langle L_{\bar{a}}^{\alpha}(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_{\bar{b}}^{\beta}(z_3) \rangle = \delta_{\bar{a} \bar{b}} \hat{x}_{13}^{\alpha' \gamma_1} \hat{x}_{23}^{\gamma_2} \hat{x}_{23}^{\gamma_3} H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(X_3, \Theta_3), \quad (4.1)$$

where $H$ is totally symmetric in three of its indices, $H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3} = H_{\alpha\beta,(\gamma_1 \gamma_2 \gamma_3)}$. The correlation function is also required to satisfy:

(i) Scaling constraint:

Under scale transformations the correlation function transforms as

$$\langle L_{\bar{a}}^{\alpha}(z'_1) J_{\gamma_1 \gamma_2 \gamma_3}(z'_2) L_{\bar{b}}^{\beta}(z'_3) \rangle = (\lambda^2)^{11/2} \langle L_{\bar{a}}^{\alpha}(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_{\bar{b}}^{\beta}(z_3) \rangle, \quad (4.2)$$

which implies that we have the following homogeneity constraint on $H$:

$$H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(\lambda^2 X, \lambda \Theta) = (\lambda^2)^{-5/2} H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(X, \Theta). \quad (4.3)$$

(ii) Differential constraints:

The differential constraints on the flavour current and supercurrent result in the following constraints on the correlation function:

$$D_{(1)}^\alpha \langle L_{\bar{a}}^{\alpha}(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_{\bar{b}}^{\beta}(z_3) \rangle = 0, \quad (4.4a)$$

$$D_{(2)}^\gamma \langle L_{\bar{a}}^{\alpha}(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_{\bar{b}}^{\beta}(z_3) \rangle = 0. \quad (4.4b)$$

Using identities (2.39a), (2.39b), these result in the following differential constraints on $H$:

$$D^\alpha H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(X, \Theta) = 0, \quad (4.5a)$$

$$Q^{\gamma_1} H_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(X, \Theta) = 0. \quad (4.5b)$$

(iii) Point permutation symmetry:
The symmetry under permutation of points \((z_1 \text{ and } z_3)\) results in the following constraint on the correlation function:

\[
\langle L_\alpha^\dagger(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_\beta^\dagger(z_3) \rangle = -\langle L_\beta^\dagger(z_3) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L_\alpha^\dagger(z_1) \rangle,
\]

which results in the point-switch identity

\[
\mathcal{H}_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3}(X_3, \Theta_3) = \frac{-x_{13}^{\gamma_1} x_{13}^{-1} x_{13}^{\gamma_1} x_{13}^{\gamma_2} x_{13}^{\gamma_3}}{x_3^8 x_{13}^8} \times \mathcal{H}_{\alpha',\gamma_1' \gamma_2' \gamma_3'}(-X_1^T, -\Theta_1).
\]

Thus we need to solve for the tensor \(\mathcal{H}\) subject to the constraints (4.3), (4.5a), (4.5b) and (4.7). To start with we combine two of the three \(\gamma\) indices into a vector index, and impose a \(\gamma\)-trace constraint to remove the component anti-symmetric in \(\gamma_1, \gamma_2, \gamma_3\):

\[
\mathcal{H}_{\alpha\beta,\gamma_1 \gamma_2 \gamma_3} = (\gamma^m)_{\gamma_2 \gamma_3} \mathcal{H}_{\alpha\beta,\gamma_1 m}, \quad (\gamma^m)^{\gamma\gamma} \mathcal{H}_{\alpha\beta,\gamma_1 m} = 0.
\]

Since our correlator is Grassmann odd the function \(\mathcal{H}_{\alpha\beta,\gamma_1 m}\) must be linear in \(\Theta\). Just like the flavour current three-point function, linearity in \(\Theta\) implies that the differential constraints (4.9a) and (4.9b) are respectively equivalent to

\[
\partial^\alpha \mathcal{H}_{\alpha\beta,\gamma_1 m} = 0, \quad (\gamma^\gamma)^{\alpha\gamma} \Theta \partial_{\gamma} \mathcal{H}_{\alpha\beta,\gamma_1 m} = 0,
\]

\[
\partial^\gamma \mathcal{H}_{\alpha\beta,\gamma_1 m} = 0, \quad (\gamma^\gamma)^{\gamma\gamma} \Theta \partial_{\gamma} \mathcal{H}_{\alpha\beta,\gamma_1 m} = 0.
\]

Now let us decompose \(\mathcal{H}\) into irreducible components

\[
\mathcal{H}_{\alpha\beta,\gamma_1 m} = \sum_i c_i \mathcal{H}_{i\alpha\beta,\gamma_1 m},
\]

where

\[
\mathcal{H}_{1\alpha\beta,\gamma_1 m} = \epsilon_{\alpha\beta} \Theta \gamma A_m(X),
\]

\[
\mathcal{H}_{2\alpha\beta,\gamma_1 m} = \epsilon_{\alpha\beta} (\gamma^a)_{\gamma}^\delta \Theta \delta B_{ma}(X),
\]

\[
\mathcal{H}_{3\alpha\beta,\gamma_1 m} = (\gamma^a)_{\alpha\beta} \Theta \gamma C_{ma}(X),
\]

\[
\mathcal{H}_{4\alpha\beta,\gamma_1 m} = (\gamma^a)_{\alpha\beta} (\gamma^b)_{\gamma} \delta \Theta \delta D_{mab}(X).
\]

It follows from eq. (4.3) that the dimension of \(A, B, C, D\) is \(-3\). We now impose the differential constraints (4.9a) and (4.9b), along with the gamma-trace constraint (4.8). After imposing (4.9a), (4.9b) the terms \(O(\Theta^0)\) imply

\[
A_m(X) = 0, \quad C_{mn}(X) = 0,
\]

\[
B_{ma}(X) = -\epsilon_{nra} D_{mnr}(X), \quad \eta^{na} D_{mna}(X) = 0,
\]

\[
17
\]
while the terms $O(\Theta^2)$ give the differential constraints
\begin{align}
\partial^t B_{mt}(X) &= 0, \\
\partial^t D_{mnt}(X) &= 0, \\
\partial^t \{ B_{mt}(X) + \epsilon_t a m D_{mna}(X) \} &= 0, \\
\partial^t \{ D_{mnt}(X) + D_{mtn}(X) - \eta_lm m^a(X) + \epsilon_n t^a B_{ma}(X) \} &= 0.
\end{align}

(4.13a)\hspace{1cm}(4.13b)\hspace{1cm}(4.13c)\hspace{1cm}(4.13d)

Imposing the gamma-trace condition (4.8) results in
\begin{align}
\eta^{ma} B_{ma}(X) &= 0, \\
\epsilon^{ama} B_{ma}(X) &= 0, \\
\eta^{ma} D_{mna}(X) &= 0, \\
\epsilon^{ama} D_{mna}(X) &= 0.
\end{align}

(4.14a)\hspace{1cm}(4.14b)

One may show that the differential and algebraic constraints above are mutually consistent and reduce to:
\begin{align}
\partial^t B_{mt}(X) &= 0, \quad \partial^t D_{mnt}(X) = 0, \\
\eta^{na} D_{mna}(X) &= 0, \quad \eta^{ma} D_{mna}(X) = 0, \\
B_{ma}(X) &= -\epsilon_n r a D_{mnt}(X),
\end{align}

(4.15a)\hspace{1cm}(4.15b)\hspace{1cm}(4.15c)

where $B_{ma}$ is symmetric and traceless, $D_{mna}$ is symmetric in the first and last index. After some calculation one can show that general solutions consistent with the scaling property (4.3) and the above constraints is
\begin{align}
B_{ma}(X) &= \frac{\eta_{ma}}{X^3} - \frac{3X_m X_a}{X^5}, \\
D_{mna}(X) &= \epsilon_{ndm} \frac{X^d X_a}{X^5} + \epsilon_{nda} \frac{X^d X_m}{X^5},
\end{align}

(4.16)\hspace{1cm}(4.17)

with $c_2 = c_4$. Hence, the three-point correlation function is determined up to a single free parameter which we denote $c_N=1$. Our solution is then
\begin{align}
\langle L^a(z_1) J_{\gamma_1 \gamma_2 \gamma_3}(z_2) L^b_{\beta}(z_3) \rangle_{\alpha} = \frac{\delta^{ab} x_{13 \alpha} x_{23} \gamma_1 \gamma_2 x_{23} \gamma_3}{(x_{13}^2)^2 (x_{23}^2)^4} \mathcal{H}_{\alpha \beta, \gamma_1 \gamma_2 \gamma_3}(X_3, \Theta_3),
\end{align}

(4.18)

where
\begin{align}
\mathcal{H}_{\alpha \beta, \gamma_1 \gamma_2 \gamma_3}(X, \Theta) &= (\gamma^m)_{\gamma_2 \gamma_3} \mathcal{H}_{\alpha \beta, \gamma_1 m}(X, \Theta), \\
\mathcal{H}_{\alpha \beta, \gamma m}(X, \Theta) &= i c_{N=1} (\gamma^a)_{\gamma} \delta_{\delta} \left\{ \varepsilon_{\alpha \beta} B_{ma}(X) + (\gamma^a)_{\alpha \beta} D_{mna}(X) \right\},
\end{align}

(4.19)\hspace{1cm}(4.20)
with $B$ and $D$ given in eqs. (4.16), (4.17). In spinor notation, this is equivalent to

$$
\mathcal{H}_{\alpha\beta,\gamma\delta}(\mathbf{X}, \Theta) = i\epsilon_{\mathbf{X}} \left\{ \varepsilon_{\alpha\beta}^{x} \varepsilon_{\gamma\delta}^{x} \Theta + \varepsilon_{\gamma\delta}^{x} \Theta_{\gamma} + \frac{1}{\mathbf{X}} \varepsilon_{\gamma\delta}^{x} \varepsilon_{\gamma\delta}^{x} \Theta_{\delta} \right\} (4.21)
$$

$$
+ \varepsilon_{\gamma\delta}^{x} x_{\alpha\gamma} x_{\delta\gamma}^{x} \Theta + \varepsilon_{\gamma\delta}^{x} x_{\alpha\gamma} x_{\delta\gamma}^{x} \Theta + \varepsilon_{\gamma\delta}^{x} x_{\alpha\gamma} x_{\delta\gamma}^{x} \Theta_{\delta} - \varepsilon_{\gamma\delta}^{x} x_{\alpha\gamma} x_{\delta\gamma}^{x} \Theta_{\delta} - x_{\alpha\gamma} x_{\delta\gamma}^{x} \Theta_{\gamma} - 3\varepsilon_{\alpha\beta} x_{\alpha\gamma} x_{\beta\gamma} x_{\gamma\delta}^{x} \Theta_{\delta} \right\}.
$$

Finally, one must check that this solution also satisfies the point-switch identity. With the aid of identities (2.31a), (2.31b), it is a relatively straightforward exercise to show that the point-switch identity (4.7) is indeed satisfied.

### 4.2 The Correlation Function $\langle J_{JL} \rangle$

Let us now discuss the remaining mixed correlation function

$$
\langle J_{J_{1}, J_{2}, J_{3}}(z_{1}) J_{\gamma_{12}, \gamma_{3}}(z_{2}) L_{\alpha}(z_{3}) \rangle.
$$

(4.22)

Here the correlator can exist only if the flavour group contains $U(1)$-factors, so will assume that the flavour group is just $U(1)$. At the component level this correlation function contains $\langle T_{ab}(x_{1}) T_{mn}(x_{2}) L_{c}(x_{3}) \rangle$, which was shown to vanish in any conformal field theory after imposing all differential constraints and symmetries [11]. As we will show, the same occurs in the supersymmetric theory. However, we will see that (4.22) vanishes without needing to impose the conservation equation for $L_{\alpha}(z_{3})$. The general expression for this correlation function is

$$
\langle J_{J_{1}, J_{2}, J_{3}}(z_{1}) J_{\gamma_{12}, \gamma_{3}}(z_{2}) L_{\alpha}(z_{3}) \rangle = \frac{x_{13}^{\gamma_{1}\gamma_{2}\gamma_{3}} x_{13}^{\gamma_{1}\gamma_{2} \gamma_{3}} x_{23}^{\gamma_{1} \gamma_{2} \gamma_{3}} x_{23}^{\gamma_{1} \gamma_{2} \gamma_{3}}}{(x_{13}^{2})^{5/2} (x_{23}^{2})^{5/2}} \times \mathcal{H}_{\lambda_{1}, \lambda_{2}, \lambda_{3}}^{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}}(x_{3}, \Theta_{3}),
$$

(4.23)

where $\mathcal{H}$ has the symmetry property $\mathcal{H}_{\lambda_{1}, \lambda_{2}, \lambda_{3}}^{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}} = \mathcal{H}_{\lambda_{1}, \lambda_{2}, \lambda_{3}}^{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}}$. The correlation function is required to satisfy:

(i) **Scaling constraint:**

Under scale transformations it transforms as

$$
\langle J_{J_{1}, J_{2}, J_{3}}(z_{1}) J_{\gamma_{12}, \gamma_{3}}(z_{2}) L_{\alpha}(z_{3}) \rangle = (\lambda^{2})^{13/2} \langle J_{J_{1}, J_{2}, J_{3}}(z_{1}) J_{\gamma_{12}, \gamma_{3}}(z_{2}) L_{\alpha}(z_{3}) \rangle
$$

(4.24)

which results in the constraint

$$
\mathcal{H}_{\lambda_{1}, \lambda_{2}, \lambda_{3}}^{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}}(\lambda^{2} \mathbf{X}, \lambda \Theta) = (\lambda^{2})^{-7/2} \mathcal{H}_{\lambda_{1}, \lambda_{2}, \lambda_{3}}^{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}}(\mathbf{X}, \Theta).
$$

(4.25)
(ii) **Differential constraint:**

The conservation law on the supercurrent implies

\[
D_{(1)}^{\beta_j} \langle J_{\beta_1,\beta_2,\beta_3}(z_1) J_{\gamma_1,\gamma_2,\gamma_3}(z_2) L_{\alpha}(z_3) \rangle = 0 ,
\]

which results in a differential constraint on \( H \):

\[
D^{\beta_1} H_{\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) = 0 .
\]

(iii) **Point permutation symmetry:**

The symmetry under permutation of points \( z_1 \) and \( z_2 \) implies the following constraint on the correlation function:

\[
\langle J_{\beta_1,\beta_2,\beta_3}(z_1) J_{\gamma_1,\gamma_2,\gamma_3}(z_2) L_{\alpha}(z_3) \rangle = -\langle J_{\gamma_1,\gamma_2,\gamma_3}(z_2) J_{\beta_1,\beta_2,\beta_3}(z_1) L_{\alpha}(z_3) \rangle ,
\]

which results in the identity

\[
H_{\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) = -H_{\gamma_1,\gamma_2,\gamma_3,\beta_1,\beta_2,\beta_3,\alpha}(-X^T, -\Theta) .
\]

Thus, we need to solve for the tensor \( H \) subject to the constraints (4.25), (4.27) and (4.29). Note that we also must impose one more differential constraint

\[
D_\alpha^{(3)} \langle J_{\beta_1,\beta_2,\beta_3}(z_1) J_{\gamma_1,\gamma_2,\gamma_3}(z_2) L_{\alpha}(z_3) \rangle = 0 ,
\]

which is quite non-trivial in this formalism. Fortunately, constraints (4.25), (4.27) and (4.29) are sufficient to show that correlator (4.22) vanishes, hence we will not need to consider (4.30).

To start, we combine two of the three \( \beta, \gamma \) indices into a vector index, and impose \( \gamma \)-trace constraints to remove antisymmetric components

\[
H_{\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) = (\gamma^a)_{\beta_2,\beta_3} (\gamma^b)_{\gamma_1,\gamma_2,\gamma_3} H_{\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) ,
\]

\[
(\gamma^a)^{\tau \beta} H_{\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) = 0 ,
\]

\[
(\gamma^b)^{\tau \gamma} H_{\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha}(X, \Theta) = 0 .
\]

Now let us split \( H \) into symmetric and antisymmetric parts in the first and second pair of indices

\[
H_{\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha} = H_{\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha} + H_{[\beta_1,\gamma_1,\gamma_2,\gamma_3,\alpha} .
\]
Due to the symmetry properties, (4.29) implies that $H_{(\sigma a, \gamma b), \alpha}$ is an even function of $X$, while $H_{[\beta a, \gamma b], \alpha}$ is odd. Therefore they do not mix in the conservation law (4.27) and may be considered independently. In irreducible components, $H_{(\beta a, \gamma b), \alpha}$ has the decomposition

$$H_{(\beta a, \gamma b), \alpha} = \sum_i H_i (\beta a, \gamma b), \alpha$$

(4.34)

where

$$H_1 (\beta a, \gamma b), \alpha = \varepsilon_{\beta \gamma} \Theta_{\alpha} A_{[ab]}(X),$$

(4.35a)

$$H_2 (\beta a, \gamma b), \alpha = \varepsilon_{\beta \gamma} (\gamma^m)_{\alpha} \delta B_{m[ab]}(X),$$

(4.35b)

$$H_3 (\beta a, \gamma b), \alpha = (\gamma^m)_{\beta \gamma} \Theta_{\alpha} C_{m(ab)}(X),$$

(4.35c)

$$H_4 (\beta a, \gamma b), \alpha = (\gamma^m)_{\beta \gamma} (\gamma^n)_{\alpha} \delta D_{mn(ab)}(X).$$

(4.35d)

Here we have made explicit the algebraic symmetry properties of $A, B, C$ and $D$, which by virtue of (4.29) are all even functions of $X$. Now due to linearity in $\Theta$, the differential constraint (4.27) is equivalent to the pair of equations

$$\partial^\beta H_{\beta a, \gamma b, \alpha} = 0, \quad (\gamma^\gamma)_{\beta \gamma} \Theta_{\alpha} \partial_\gamma H_{\beta a, \gamma b, \alpha} = 0.$$  

(4.36)

After imposing (4.36), the terms $O(\Theta^0)$ imply

$$A_{m[ab]}(X) = 0, \quad B_{m[ab]}(X) = 0,$$

(4.37a)

$$C_{m(ab)}(X) + \epsilon_m^{\; rs} D_{rs(ab)}(X) = 0,$$

(4.37b)

$$\eta^{mn} D_{mn(ab)}(X) = 0, \quad \eta^{ma} D_{mn(ab)}(X) = 0,$$

(4.37c)

so $H_1 (\beta a, \gamma b), \alpha = H_2 (\beta a, \gamma b), \alpha = 0$. The terms $O(\Theta^2)$ then result in the differential constraints

$$\partial^m \{- C_{m(ab)}(X) + \epsilon_m^{\; rs} D_{rs(ab)}(X)\} = 0,$$

(4.38a)

$$\epsilon_c^{\; tm} \partial_t C_{m(ab)}(X) - \partial^m D_{mc(ab)}(X) - \partial^m D_{cm(ab)}(X) = 0.$$  

(4.38b)

Imposing the gamma-trace condition (4.32) results in

$$\eta^{ma} C_{m(ab)}(X) = 0, \quad \epsilon_c^{\; ma} C_{m(ab)}(X) = 0,$$

(4.39a)

$$\eta^{ma} D_{mn(ab)}(X) = 0, \quad \epsilon_c^{\; ma} D_{mn(ab)}(X) = 0.$$  

(4.39b)

As in the previous case, our correlator is Grassmann odd which means we can replace $X$ with $X$. 

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Altogether (4.37b), (4.38a) and (4.39a) imply that $C$ is a totally symmetric, traceless, transverse and even function of $X$. Let’s try to construct such a tensor by analysing its irreducible components. To determine which irreducible components are permitted, let us trade each vector index for a pair of spinor indices. Since $C$ is completely symmetric and traceless, it is equivalent to $C(\alpha_1 \ldots \alpha_6)$. In addition since $C$ is even in $X_{\alpha\beta}$ only irreducible structures (that is, totally symmetric tensors) of rank 4 and 0 in $X_{\alpha\beta}$ can contribute to the solution. Going back to vector indices, let us denote these components of $C$ as $C_1(mn)(X)$ and $C_2(X)$.

Since it is not possible to construct a rank three tensor $C_{(mnk)}$ out of $C_1(mn)(X)$ and $C_2(X)$, the tensor $C_{mnk}$ vanishes. Hence, $\mathcal{H}_3(\beta_\alpha, \gamma^b, \alpha) = 0$.

Given this information, the remaining set of equations imply that $D$ is now a totally symmetric, traceless and transverse tensor that is even in $X$. Following a similar argument, the symmetries imply that it has irreducible components $D_1(mnab)(X)$, $D_2(mn)(X)$ and $D_3(X)$. We are now equipped with enough information to construct an explicit solution for $D$. Using the symmetries and the scaling property (4.25) we have the most general ansatz

$$D_{(mnab)}(X) = \frac{d_1}{X^4} \left[ \eta_{ma} \eta_{nb} + \eta_{mb} \eta_{na} + \eta_{mn} \eta_{ab} \right]$$
$$+ \frac{d_2}{X^6} \left[ \eta_{mn} X_a X_b + \eta_{ma} X_n X_b + \eta_{mb} X_a X_n \right. $$
$$\left. + \eta_{na} X_m X_b + \eta_{nb} X_m X_a + \eta_{ab} X_m X_n \right] + \frac{d_3}{X^8} X_m X_n X_a X_b .$$

(4.40)

Requiring that $D$ be traceless and transverse fixes all the $d_i$ to 0. Hence, $D = 0$, and $\mathcal{H}_{(\beta_\alpha, \gamma^b, \alpha)}$ vanishes.

In a similar way we consider $\mathcal{H}_{[\beta_\alpha, \gamma^b], \alpha}$ for which we have the following decomposition

$$\mathcal{H}_{[\beta_\alpha, \gamma^b], \alpha} = \sum_i \mathcal{H}_i [\beta_\alpha, \gamma^b], \alpha ,$$

(4.41)

where

$$\mathcal{H}_1 [\beta_\alpha, \gamma^b], \alpha = \varepsilon_{\beta\gamma} \Theta_\alpha A_{(ab)}(X) ,$$

(4.42a)

$$\mathcal{H}_2 [\beta_\alpha, \gamma^b], \alpha = \varepsilon_{\beta\gamma} (\gamma^m)_\alpha \delta_\beta B_{m(ab)}(X) ,$$

(4.42b)

$$\mathcal{H}_3 [\beta_\alpha, \gamma^b], \alpha = (\gamma^m)_\beta \Theta_\alpha C_{m(ab)}(X) ,$$

(4.42c)

$$\mathcal{H}_4 [\beta_\alpha, \gamma^b], \alpha = (\gamma^m)_\beta \Theta_\alpha \delta_\delta D_{mn(ab)}(X) .$$

(4.42d)
In this case, $A, B, C, D$ are now odd functions in $X$. Imposing the conservation equations and vanishing of the $\gamma$-trace we obtain the following set of constraints:

\begin{align}
A_{(ab)}(X) &= 0, \quad B_{m(ab)}(X) = 0 \quad (4.43a) \\
D_{m[ab]}^m(X) &= 0, \quad C_{m[ab]}(X) + \epsilon^{rs}_{m} D_{rs[ab]}(X) = 0, \quad (4.43b) \\
C_{m[mb]}^m(X) &= 0, \quad D_{mn[ab]}^m(X) = 0, \quad (4.43c) \\
\epsilon^{cma}_{m} C_{m[ab]}(X) &= 0, \quad (4.43d) \\
\epsilon^{cma}_{m} D_{mn[ab]}(X) &= 0. \quad (4.43e)
\end{align}

We see that the functions $A$ and $B$ vanish. To show that $C_{m[ab]}$ vanishes we consider eq. (4.43d) and use the fact that in three dimensions an antisymmetric tensor is equivalent to a vector

\[ C_{m[ab]}(X) = \epsilon_{ab}^{\ q} \tilde{C}_{mq}(X). \quad (4.44) \]

Hence from (4.43d) it follows that

\[ \tilde{C}_{ab}(X) - \eta_{ab} \tilde{C}_{d}^{d}(X) = 0. \quad (4.45) \]

Contracting with $\eta^{ab}$ we find that $\tilde{C}_{d}^{d} = 0$, hence $\tilde{C}_{ab} = 0$. It also implies that $C_{m[ab]}=0$. In a similar way using eq. (4.43e) one can show that $D_{mn[ab]} = 0$. This means that $H_{\beta a, \gamma b, \alpha} = 0$. Hence the three-point function of two supercurrents and one flavour current (4.22) vanishes.

## 5 Comments on the absence of parity violating structures

In [11] it was shown that correlation functions of conserved current in three-dimensional conformal field theories can have parity violating structures. Specifically, it was defined as follows. Given a conserved current

\[ J_{\alpha_1 \alpha_2 \ldots \alpha_{2s-1} \alpha_{2s}}(x) = (\gamma^{m_1})_{\alpha_1 \alpha_2} \cdots (\gamma^{m_s})_{\alpha_{2s-1} \alpha_{2s}} J_{m_1 \ldots m_s}(x) \quad (5.1) \]

we can construct

\[ J_s(x, \lambda) = J_{\alpha_1 \alpha_2 \ldots \alpha_{2s-1} \alpha_{2s}}(x) \lambda^{\alpha_1} \cdots \lambda^{\alpha_{2s}}, \quad (5.2) \]

where $\lambda^\alpha$ are auxiliary commuting spinors. The action of parity is then $x \to -x, \lambda \to i\lambda$. In theories with a parity symmetry, $J_{\mu_1 \ldots \mu_s}(x)$ acquires a sign $(-1)^s$ under parity
and $J_s(x, \lambda)$ is invariant. However, as was shown in \cite{11} correlation functions admit contributions which are odd under parity. In particular, it was shown that a parity odd contribution to the mixed correlator of the energy-momentum tensor $T_{mn}$ and two flavour currents $L^k_x$ can arise. Translating their result into our notation it can be written as follows

$$
\langle T_{mn}(x_1) L^a_k(x_2) L^b_p(x_3) \rangle_{odd} = \frac{\delta^{\bar{a} \bar{b}}}{x_{13}^{2/3} x_{12}^{2/3} x_{23}} I_{mn,m'n'}(x_{13}) I_{kk'}(x_{23}) t_{m'n'k'p}(X_3),
$$

where

$$
t_{mnkp}(X) = \epsilon_{npq} \frac{X^q X_m X_k}{X^3} + \epsilon_{nkp} \frac{X^q X_m X_p}{X^3}.
$$

Here $X_i$ are three-point building blocks introduced by Osborn and Petkou in \cite{11}, while the object $I_{mn}(x)$ is the inversion tensor, and $I_{mn,m'n'}(x)$ is an inversion tensor which extracts the symmetric traceless component. They are defined as follows

$$
I_{mn}(x) = \eta_{mn} - 2 \frac{x_m x_n}{x^2},
$$

$$
I_{mn,m'n'}(x) = \frac{1}{2} \{ I_{mn}(x) I_{mm'}(x) + I_{mn'}(x) I_{mm'}(x) \} - \frac{1}{3} \eta_{mm'} \eta_{nn'}.
$$

An important and specific feature of all parity violating terms is appearance of the $\epsilon$-tensor.

In $\mathcal{N} = 1$ supersymmetric theories the supercurrent $J_{\alpha \beta \gamma}$ and the flavour current multiplet $L^a_\alpha$ contain the following conserved currents

$$
T_{\alpha \beta \gamma \delta} = D(\delta J_{\alpha \beta \gamma})|, \quad T_{\alpha \beta \gamma \delta} = (\gamma^m)_{(\alpha \beta}(\gamma^n)_{)\gamma \delta) T_{mn}, \quad \partial^m T_{mn} = 0, \quad \eta^{mn} T_{mn} = 0,
$$

$$
Q_{\alpha \beta \gamma} = J_{\alpha \beta \gamma}|, \quad Q_{\alpha \beta \gamma} = (\gamma^m)_{\alpha \beta} Q_{\gamma \gamma}, \quad \partial^m Q_{\alpha \beta} = 0, \quad (\gamma^m)_{\alpha \beta} Q_{\gamma \gamma} = 0,
$$

$$
V^a_{\alpha \beta} = D(\alpha L^a_\beta)|, \quad V^a_{\alpha \beta} = (\gamma^m)_{\alpha \beta} V^a_m, \quad \partial^m V^a_m = 0,
$$

where $T_{mn}$ is the energy-momentum tensor, $Q_{\gamma \gamma}$ is the supersymmetry current and $V^a_m$ is a vector current. Hence, the mixed correlators studied in the previous section give rise to the following correlators in terms of components

$$
\langle T_{mn}(x_1) T_{pq}(x_2) V_k(x_3) \rangle, \quad \langle Q_{\alpha \beta}(x_1) Q_{\gamma \gamma}(x_2) V_k(x_3) \rangle, \quad \langle V^a_{\alpha \beta}(x_1) T_{mn}(x_2) V^b_p(x_3) \rangle.
$$

The first two correlators vanish because the entire superspace correlator (4.22) vanishes. The last one is, in general, non-zero and fixed up to one overall coefficient. It can be computed using eqs. (4.18), (4.21) using the superspace reduction procedure

$$
\langle V^a_{\alpha \beta}(x_1) T_{mn}(x_2) V^b_p(x_3) \rangle = \frac{1}{16} \langle \gamma_k^{a_1 a_2} (\gamma_{m})^{\beta_1 \beta_2} (\gamma_n)^\beta_3 \beta_4 (\gamma_p)^{\gamma_1 \gamma_2} D(1)_{a_1} D(2)_{\beta_1} D(3)_{\gamma_1} (L^a_{\alpha_2} (z_1)) J_{\beta_2 \beta_3 \beta_4} (z_2) L^b_{\gamma_2} (z_3) \rangle.
$$
Here the bar-projection denotes setting the fermionic coordinates \( \theta_\alpha \) to zero. We will not perform the reduction explicitly, instead we will indirectly determine whether \((5.9)\) is even or odd under parity. For this it is sufficient to study whether or not the \( \epsilon \)-tensor appears upon reduction. Since

\[
\epsilon_{mnp} = \frac{1}{2} \text{tr}(\gamma_m \gamma_n \gamma_p), \tag{5.10}
\]

it is enough to count the number of gamma-matrices: If the number of \( \gamma \)-matrices appearing in the superspace reduction is even the \( \epsilon \)-tensor cannot arise and the contribution is parity even, if the number of \( \gamma \)-matrices is odd the contribution is parity odd. Let us perform the counting. Since in \((5.9)\) we act with just three covariant derivatives before setting all \( \theta_i = 0 \) (where \( i = 1, 2, 3 \) is the index labelling the three point) only term linear and cubic in \( \theta_i \) will contribute. Let us concentrate on the terms linear on \( \theta_i \). Since the function \( \mathcal{H} \) in \((4.21)\) is already linear in \( \theta_i \) we can set \( \theta_i = 0 \) in \( x_{ij} \) and \( X \). This makes \( x_{ij} \) and \( X \) symmetric and proportional to a gamma-matrix. Now we have four gamma-matrices in \((5.9)\), four gamma-matrices coming from \( x_{ij} \) in eq. \((4.18)\), zero or two gamma-matrices coming from \( \mathcal{H} \) in \((4.21)\) and also one more gamma-matrix contained in \( \Theta_3 \), see eq. \((2.26c)\). Overall we have odd number of gamma-matrices at this point. However, superspace covariant derivatives also contain gamma-matrices, see eq. \((A.16)\). Since we are considering terms linear in \( \theta_i \) and setting \( \theta_i = 0 \) upon differentiating it is easy to realise that in the three derivatives \( D^{(1)} \alpha_1 D^{(2)} \beta_1 D^{(3)} \gamma_1 \) we must take one derivative with respect to \( x_i \) and two derivatives with respect to \( \theta_i \). This gives one more gamma-matrix making the total number even. Terms cubic in \( \theta_i \) can be considered in a similar way. They also yield an even number of gamma-matrices. Hence, the entire contribution \((5.9)\) is parity even.

In a similar way we can count the number of gamma-matrices in the superspace reduction of \((3.12), (3.29)\):

\[
\langle V^a_m(x_1) V^b_n(x_2) V^c_k(x_3) \rangle = \frac{1}{8} (\gamma_m)^{\alpha_1 \alpha_2} (\gamma_n)^{\beta_1 \beta_2} (\gamma_k)^{\gamma_1 \gamma_2} D^{(1) \alpha_1} D^{(2) \beta_1} D^{(3) \gamma_1} \langle L^a_{\alpha_2} (z_1) L^b_{\beta_2} (z_2) L^c_{\gamma_2} (z_3) \rangle \text{.} \tag{5.11}
\]

An analysis similar to the above shows that this contribution is also parity even. Finally, one can also consider the superspace reduction of the three-point function of the supercurrent

\[
\langle T_{mn}(x_1) T_{kl}(x_2) T_{pq}(x_3) \rangle = \frac{1}{64} (\gamma_m)^{\alpha_1 \alpha_2} (\gamma_n)^{\alpha_3 \alpha_4} (\gamma_k)^{\beta_1 \beta_2} (\gamma_l)^{\beta_3 \beta_4} (\gamma_p)^{\gamma_1 \gamma_2} (\gamma_q)^{\gamma_3 \gamma_4} D^{(1) \alpha_1} D^{(2) \beta_1} D^{(3) \gamma_1} \langle J^{\alpha_2 \alpha_3 \alpha_4} (z_1) J^{\beta_2 \beta_3 \beta_4} (z_2) J^{\gamma_2 \gamma_3 \gamma_4} (z_3) \rangle \text{.} \tag{5.12}
\]
\[ \langle T_{mn}(x_1)Q_{k\beta}(x_2)Q_{\gamma}(x_3) \rangle = \frac{1}{16} (\gamma_m)^{(\alpha_1 \alpha_2)(\gamma_h)^{(\alpha_3 \alpha_4)}(\gamma_k)^{\beta_1 \beta_2}(\gamma_p)^{\gamma_1 \gamma_2} D_{(1)\alpha_1} (J_{\alpha_2 \alpha_3 \alpha_4}(z_1)J_{\beta_1 \beta_2}(z_2)J_{\gamma_1 \gamma_2}(z_3)) \right) . \] (5.13)

The three-point function of the supercurrent was found in [33]. We will not repeat it here since the expression for it is quite long. However, a similar analysis shows that the contributions (5.12) and (5.13) are parity even.\(^8\)

This means that no parity violating structures can arise in three-point functions of \(T_{mn}, Q_m\) and \(V^a_m\) in superconformal field theories. Maldacena and Zhiboedov proved in [40] that if a three-dimensional conformal field theory possesses a higher spin conserved current then it is essentially a free theory. Since a free theory has only parity even contributions to the three-point functions of conserved currents, the correlators involving one or more higher spin conserved currents admit only parity even structures. This leads us to conclude that \(\mathcal{N} = 1\) supersymmetry forbids parity violating structures in all three-point functions of conserved currents unless the assumptions of the Maldacena–Zhiboedov theorem are violated. The strongest assumption of the theorem is that the theory under consideration contains unique conserved current of spin two which is the energy-momentum tensor. Some properties of theories possessing more than one conserved current with spin two were discussed in [40]. In supersymmetric theory the energy-momentum tensor is a component of the supercurrent. One can also consider a different supermultiplet containing a conserved spin two current, namely

\[ J_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)} , \quad D^{\alpha_1} J_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)} = 0 . \] (5.14)

The lowest component of \(J_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)}\) is a conserved spin two current which is not the energy-momentum tensor. Note that \(J_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)}\) also contains a conserved higher-spin current. It will be interesting to perform a systematic study of three-point functions of \(J_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)}\) to see if they allow any parity violating structures.

\section{Mixed correlators in \(\mathcal{N} = 2\) superconformal field theory}

Now we will generalise our method to mixed three-point functions in superconformal field theory with \(\mathcal{N} = 2\) supersymmetry. A specific feature of three-dimensional \(\mathcal{N} = 2\)

\(^8\)In general, if a superspace three-point function is fixed up to an overall coefficient it is expected to be parity even because this contribution is expected to exist in a free theory of a real scalar superfield.
superconformal field theories is contact terms in correlation functions of the conserved currents \[54, 55\]. In this paper, we study correlation functions at non-coincident points where the contact terms do not contribute.

### 6.1 Supercurrent and flavour current multiplets

The 3D, \( \mathcal{N} = 2 \) supercurrent was studied in \[53, 56–58\]. It is a primary, dimension 2 symmetric spin-tensor \( J_{\alpha\beta} \), which obeys the conservation equation

\[
D^I_{\alpha} J_{\alpha\beta} = 0 ,
\]

and has the following superconformal transformation law:

\[
\delta J_{\alpha\beta} = -\xi J_{\alpha\beta} - 2\sigma(z)J_{\alpha\beta} + 2\lambda(z)(\gamma_{\alpha} J_{\beta}\gamma) .
\]

The general formalism in section 2 allows the two-point function to be determined up to a single real coefficient

\[
\langle J_{\alpha\beta}(z_1) J^{\alpha',\beta'}(z_2) \rangle = b_{\mathcal{N}=2} \frac{\mathbf{x}_{12(\alpha} \mathbf{x}_{12\beta')}}{(\mathbf{x}_{12}^2)^3} .
\]

It’s then a simple exercise to show that the two-point function has the right symmetry properties under permutation of superspace points

\[
\langle J_{\alpha\beta}(z_1) J_{\alpha',\beta'}(z_2) \rangle = \langle J_{\alpha',\beta'}(z_2) J_{\alpha\beta}(z_1) \rangle ,
\]

and also satisfies the conservation equation

\[
D^I_{(1)}\langle J_{\alpha\beta}(z_1) J^{\alpha',\beta'}(z_2) \rangle = 0 , \quad z_1 \neq z_2 .
\]

Similarly, the 3D \( \mathcal{N} = 2 \) flavour current is a primary, dimension 1 scalar superfield \( L \), which obeys the conservation equation

\[
\left( D^\alpha(t) D_{\alpha}^l - \frac{1}{2} \delta^{lJ} D^{\alpha K} D^K_{\alpha} \right) L = 0 ,
\]

and transforms under the superconformal group as

\[
\delta L = -\xi L - \sigma(z) L .
\]

As in the \( \mathcal{N} = 1 \) case, we assume the \( \mathcal{N} = 2 \) superconformal field theory in question has a set of flavour currents \( L^a \) associated with a simple flavour group. Due to the absence of
spinor or $R$-symmetry indices, the $\mathcal{N} = 2$ flavour current two-point function is fixed up to a single real coefficient $a_{\mathcal{N}=2}$ as follows
\[
\langle L^\alpha(z_1) L^\beta(z_2) \rangle = a_{\mathcal{N}=2} \frac{\delta^{\alpha\beta}}{x_{12}^2}. \tag{6.8}
\]
The two-point function obeys the correct symmetry properties under permutation of superspace points, $\langle L^\alpha(z_1) \bar{L}^\beta(z_2) \rangle = \langle \bar{L}^\beta(z_2) L^\alpha(z_1) \rangle$, and also satisfies the conservation equation
\[
(D^{(I)}_\alpha D^I_\alpha - \frac{1}{2} \delta^{IJ} D^{(K)}_\alpha D^K_{(1)\alpha}) \langle L^\alpha(z_1) L^\beta(z_2) \rangle = 0, \quad z_1 \neq z_2. \tag{6.9}
\]
In the next section we will compute the mixed correlation functions associated with the $\mathcal{N} = 2$ supercurrent and flavour current multiplets. There are two possibilities to consider, they are
\[
\langle L^\alpha(z_1) J_{\alpha\beta}(z_2) \bar{L}^\beta(z_3) \rangle, \quad \langle J_{\alpha\beta}(z_1) J_{\gamma\delta}(z_2) L(z_3) \rangle. \tag{6.10}
\]
Note that in second case we are considering a $U(1)$ flavour current.

### 6.2 The correlation function $\langle L J L \rangle$

First let us consider the $\langle L J L \rangle$ case first. Using the general ansatz, we have
\[
\langle L^\alpha(z_1) J_{\alpha\beta}(z_2) \bar{L}^\beta(z_3) \rangle = \delta^{\alpha\beta} \frac{x_{23}^\alpha x_{23}^{\alpha'}}{x_1^2 (x_{23}^2)^3} \mathcal{H}_{\alpha'\beta'}(X_3, \Theta_3), \tag{6.11}
\]
where $\mathcal{H}_{\alpha\beta} = \mathcal{H}_{(\alpha\beta)}$. The correlation function is also required to satisfy the following:

(i) **Scaling constraint:**

Under scale transformations the correlation function transforms as
\[
\langle L^\alpha(z'_1) J_{\alpha\beta}(z'_2) \bar{L}^\beta(z'_3) \rangle = (\lambda^2)^4 \langle L^\alpha(z_1) J_{\alpha\beta}(z_2) \bar{L}^\beta(z_3) \rangle, \tag{6.12}
\]
from which we find the homogeneity constraint
\[
\mathcal{H}_{\alpha\beta}(\lambda^2 X, \lambda \Theta) = (\lambda^2)^{-2} \mathcal{H}_{\alpha\beta}(X, \Theta). \tag{6.13}
\]

(ii) **Differential constraints:**

The differential constraints on the flavour current and supercurrent result in the following constraints on the correlation function:
\[
(D^{(I)}_\alpha D^I_{(1)\alpha} - \frac{1}{2} \delta^{IJ} D^{(K)}_{(1)\alpha} D^K_{(1)\alpha}) \langle L^\alpha(z_1) J_{\alpha\beta}(z_2) \bar{L}^\beta(z_3) \rangle = 0, \tag{6.14a}
\]
\[
D^{(2)}_\alpha \langle L^\alpha(z_1) J_{\alpha\beta}(z_2) \bar{L}^\beta(z_3) \rangle = 0. \tag{6.14b}
\]
These result in the following differential constraints on $\mathcal{H}$:

$$
(D^{\sigma}I_D^{\sigma}) - \frac{1}{2}\delta^{I,J}D^{\sigma^K}D^{\sigma^J}K\mathcal{H}_{\alpha\beta}(X, \Theta) = 0 ,
$$

(6.15a)

$$
Q^I\mathcal{H}_{\alpha\beta}(X, \Theta) = 0 .
$$

(6.15b)

(iii) **Point permutation symmetry:**

The symmetry under permutation of points ($z_1$ and $z_3$) results in the following constraint on the correlation function:

$$
\langle L^{a}(z_1)J^{\alpha\beta}(z_2)L^{\bar{b}}(z_3) \rangle = \langle L^{\bar{b}}(z_3)J^{\alpha\beta}(z_2)L^{a}(z_1) \rangle ,
$$

(6.16)

which results in the point-switch identity

$$
\mathcal{H}_{\alpha\beta}(X, \Theta_3) = \frac{\mathbf{x}^{\alpha\alpha'}\mathbf{x}_3^{\alpha\alpha'}\mathbf{x}^{\bar{b}\bar{b}'}\mathbf{x}_3^{\bar{b}\bar{b}'}}{\mathbf{X}^{6}x_3^{6}} \mathcal{H}_{\sigma\rho}(-X^T_1, -\Theta_1) .
$$

(6.17)

The symmetry properties of $\mathcal{H}$ allow us to trade the spinor indices for a vector index

$$
\mathcal{H}_{\alpha\beta}(X, \Theta) = (\gamma^m)_{\alpha\beta} \mathcal{H}_m(X, \Theta) .
$$

(6.18)

The most general expansion for $\mathcal{H}_m(X, \Theta)$ is then

$$
\mathcal{H}_m(X, \Theta) = A_m(X) - \frac{1}{2}\Theta^2B_m(X) + (\Theta\Theta)^nC_{mn}(X) + \frac{1}{8}\Theta^4D_m(X) ,
$$

(6.19)

where we have defined

$$
(\Theta\Theta)_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}(\Theta\Theta)_{\alpha\beta} ,
$$

(6.20)

and accounted for the $\mathcal{N} = 2$ identity

$$
\Theta^2\Theta^{I}_{\alpha}\Theta^{J}_{\beta}\epsilon_{IJ} = 0 .
$$

(6.21)

The prefactors in front of $B$ and $D$ have been chosen for convenience, and as in the $\mathcal{N} = 1$ case it is more convenient to work with $X^m$ instead of $X^{\alpha\beta}$. Imposing (6.15b) results in the differential constraints

$$
\partial^mA_m(X) = 0 ,
$$

(6.22a)

$$
\partial^mB_m(X) = 0 ,
$$

(6.22b)

$$
\epsilon^{mni}\partial_nC_{mt}(X) = 0 ,
$$

(6.22c)

$$
B_q(X) + \epsilon_{qmn}\partial^nA^m(X) = 0 ,
$$

(6.22d)

$$
D_q(X) - \epsilon_{qmn}\partial^nB^m(X) = 0 ,
$$

(6.22e)

$$
\partial^m\{ C_{mt}(X) + C_{tm}(X) - \eta_mC^n_{\alpha}(X) \} = 0 ,
$$

(6.22f)
and the algebraic constraints
\[ C^a_a(X) = 0, \quad (6.23a) \]
\[ \epsilon^{qm} C_{mt}(X) = 0, \quad (6.23b) \]
which imply that \( C \) is symmetric and traceless. Furthermore the scaling condition (6.13) allows us to construct the solutions
\[ A_m(X) = a \frac{X_m}{X^3}, \quad (6.24a) \]
\[ B_m(X) = b \frac{X_m}{X^4}, \quad (6.24b) \]
\[ C_{mn}(X) = c \left( \frac{\eta_{mn}}{X^3} - \frac{3X_mX_n}{X^5} \right), \quad (6.24c) \]
\[ D_m(X) = d \frac{X_m}{X^5}. \quad (6.24d) \]
Together (6.22) imply \( B_m(X) = D_m(X) = 0 \), while \( a \) and \( c \) remain as two free parameters. Hence the solution for \( H \) becomes
\[ H_{\alpha\beta}(X, \Theta) = \tilde{c}_{N=2} \frac{X_{\alpha\beta}}{X^3} \left\{ \Theta_{I\alpha} \Theta_{J\beta} \epsilon_{IJ} \frac{X^3}{X^3} + 3 \frac{X_{\alpha\beta} X^{\gamma\delta} \Theta_{I\gamma} \Theta_{J\delta} \epsilon_{IJ}}{2 X^5} \right\}. \quad (6.25) \]
After some lengthy calculation it turns out that only the second structure satisfies the conservation equation (6.15a). Hence there is only one linearly independent structure in the correlation function that is compatible with the differential constraints. Therefore we find that the final solution is
\[ \langle L^{\bar{a}}(z_1) J_{\alpha\beta}(z_2) L^{\bar{a}}(z_3) \rangle = \frac{\delta^{ab} x_{23\alpha}^\alpha x_{23\beta}^\beta}{x_{13}^2 (x_{23}^2)^3} \mathcal{H}_{\alpha'\beta'}(X_3, \Theta_3), \quad (6.26) \]
with
\[ \mathcal{H}_{\alpha\beta}(X, \Theta) = i c_{N=2} \left\{ \Theta_{I\alpha} \Theta_{J\beta} \epsilon_{IJ} \frac{X^3}{X^3} + 3 \frac{X_{\alpha\beta} X^{\gamma\delta} \Theta_{I\gamma} \Theta_{J\delta} \epsilon_{IJ}}{2 X^5} \right\}. \quad (6.27) \]
In deriving this result, we Taylor expanded the denominator in (6.25) using \( X^2 = X^2 - \frac{1}{4} \Theta^4 \), which follows from (2.28), (2.34), and then used the \( N = 2 \) identity (6.21). It may also be shown that this structure satisfies the point-switch identity (6.17).

The supercurrent \( J_{\alpha\beta} \) leads to the following \( N = 1 \) supermultiplets (here the bar-projections denotes setting \( \theta^{I=2} \) to zero and \( D^\alpha = D_{\alpha J=1} \))
\[ S_{\alpha\beta} = J_{\alpha\beta} \, |, \quad D^\alpha S_{\alpha\beta} = 0, \quad (6.28a) \]
\[ J_{\alpha\beta\gamma} = i D_{(\alpha}^2 J_{\beta\gamma)}, \quad D^\alpha J_{\alpha\beta\gamma} = 0. \quad (6.28b) \]

\(^{9}\)From here we will use bold \( R \)-symmetry indices to distinguish them from other types of indices.
In these equations $J_{\alpha\beta\gamma}$ is the $\mathcal{N} = 1$ supercurrent and $S_{\alpha\beta}$ is the additional $\mathcal{N} = 1$ supermultiplet containing the second supersymmetry current and the $R$-symmetry current. Similarly, the $\mathcal{N} = 2$ flavour current leads to

$$S = L^a, \quad (6.29a)$$

$$L^a_\alpha = iD_\alpha^2 L^a, \quad D^\alpha L^a_\alpha = 0, \quad (6.29b)$$

where $L^a_\alpha$ is the $\mathcal{N} = 1$ flavour current and $S$ is unconstrained. Hence, the $\mathcal{N} = 2$ three-point function $\langle L^a(z_1)J_{\alpha\beta}(z_2)L^b(z_3) \rangle$ contains three-point functions of the following conserved component currents: the energy-momentum tensor, conserved vector currents, the supersymmetry currents and the $R$-symmetry current. All these three-point functions can be found by superspace reduction and are fixed by the $\mathcal{N} = 2$ superconformal symmetry up to one overall coefficient (or vanish). A simple gamma-matrix counting procedure similar to the one discussed in the previous section shows that all these correlators are parity even.

### 6.3 The correlation function $\langle J_JL \rangle$

For this example, the general ansatz gives

$$\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle = \frac{x_{13}^{\alpha'i} x_{13}^{\beta'} x_{23}^{\gamma'} x_{23}^{\delta'}}{(x_{13}^2)^3(x_{23}^2)^3} \mathcal{H}_{\alpha'i\beta'\gamma'\delta'}(X_3, \Theta_3), \quad (6.30)$$

where $\mathcal{H}_{\alpha\beta\gamma\delta} = \mathcal{H}_{(\alpha\beta)(\gamma\delta)}$. The correlation function is required to satisfy the following:

(i) Scaling constraint:

Under scale transformations the correlation function transforms as

$$\langle J_{\alpha\beta}(z_1')J_{\gamma\delta}(z_2')L(z_3') \rangle = (\lambda^2)^5 \langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle, \quad (6.31)$$

from which we find the homogeneity constraint

$$\mathcal{H}_{\alpha\beta\gamma\delta}(\lambda X, \lambda \Theta) = (\lambda^2)^{-3} \mathcal{H}_{\alpha\beta\gamma\delta}(X, \Theta). \quad (6.32)$$

(ii) Differential constraints:

The differential constraints on the flavour current and supercurrent result in the following constraints on the correlation function:

$$D_{(1)}^{I\alpha} \langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle = 0. \quad (6.33a)$$

$$\left(D_{(3)}^{(I} d_{(3)}^{J)} - \frac{1}{2} \delta^{IJ} D_{(3)}^{\sigma K} D_{(3)}^{\kappa \sigma} \right) \langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle = 0. \quad (6.33b)$$

31
The first equation results in the following differential constraints on $\mathcal{H}$:

$$\mathcal{D}^{\alpha} \mathcal{H}_{\alpha\beta\gamma\delta}(X, \Theta) = 0.$$  \hfill (6.34)

The second constraint (6.33b) is more difficult to handle in this formalism, however we will demonstrate how to deal with it later.

(iii) **Point permutation symmetry:**

The symmetry under permutation of points $z_1$ and $z_2$ results in the following constraint on the correlation function:

$$\langle J_{\alpha\beta}(z_1) J_{\gamma\delta}(z_2) L(z_3) \rangle = \langle J_{\gamma\delta}(z_2) J_{\alpha\beta}(z_1) L(z_3) \rangle,$$ \hfill (6.35)

which results in the point-switch identity

$$\mathcal{H}_{\alpha\beta\gamma\delta}(X, \Theta) = \mathcal{H}_{\gamma\delta\alpha\beta}(-X^T, -\Theta).$$ \hfill (6.36)

Now due to the symmetry properties of $\mathcal{H}$, we may trade pairs of symmetric spinor indices for vector indices

$$\mathcal{H}_{(\alpha\beta)(\gamma\delta)}(X, \Theta) = (\gamma^m)_{\alpha\beta} (\gamma^n)_{\gamma\delta} \mathcal{H}_{mn}(X, \Theta).$$ \hfill (6.37)

Now if we split $\mathcal{H}_{mn}$ into symmetric and anti-symmetric parts

$$\mathcal{H}_{mn}(X, \Theta) = \mathcal{H}_{(mn)}(X, \Theta) + \mathcal{H}_{[mn]}(X, \Theta)$$

$$= \mathcal{H}_{(mn)}(X, \Theta) + \epsilon_{mnt} \mathcal{H}^t(X, \Theta),$$ \hfill (6.38)

then the point-switch identity implies

$$\mathcal{H}_{(mn)}(X, \Theta) = \mathcal{H}_{(mn)}(-X^T, -\Theta), \quad \mathcal{H}_t(X, \Theta) = -\mathcal{H}_t(-X^T, -\Theta).$$ \hfill (6.39)

General expansions consistent with the index structure and symmetries are

$$\mathcal{H}_{(mn)}(X, \Theta) = A_{(mn)}(X) + \Theta^2 B_{(mn)}(X) + (\Theta \Theta)^s C_{(mn)s}(X) + \Theta^4 D_{(mn)}(X),$$ \hfill (6.40a)

$$\mathcal{H}_t(X, \Theta) = A_t(X) + \Theta^2 B_t(X) + (\Theta \Theta)^s C_{t s}(X) + \Theta^4 D_t(X).$$ \hfill (6.40b)

All the tensors comprising $\mathcal{H}_{(mn)}$ are even functions of $X$, while those in the expansion for $\mathcal{H}_t$ are odd functions of $X$. Furthermore, due to symmetry arguments the tensors $\mathcal{H}_{(mn)}$ and $\mathcal{H}_t$ do not mix in the conservation law (6.34), hence they may be considered
independently. First let us analyse $\mathcal{H}_{(mn)}$; imposing (6.15a) results in the differential constraints
\[
\partial^m A_{(mn)}(X) = 0, \quad (6.41a)
\]
\[
\partial^m B_{(mn)}(X) = 0, \quad (6.41b)
\]
\[
\epsilon^{mrs} \partial_r C_{(mn)s}(X) = 0, \quad (6.41c)
\]
\[
2B_{(qn)}(X) + i \epsilon^q_{mt} \partial_t A_{(mn)}(X) = 0, \quad (6.41d)
\]
\[
4D_{(qn)}(X) + i \epsilon^q_{mt} \partial_t B_{(mn)}(X) = 0, \quad (6.41e)
\]
\[
\partial^m \{ C_{(mn)s}(X) + C_{(sn)m}(X) - \eta_{ms} C^a_{na}(X) \} = 0, \quad (6.41f)
\]
and the algebraic constraints
\[
C^m_{nm}(X) = 0, \quad (6.42a)
\]
\[
\epsilon^{mrs} C_{(mn)s}(X) = 0. \quad (6.42b)
\]
The scaling condition (6.32), along with (6.42) imply that $C$ is totally symmetric, traceless and even in $X$. Following the argument presented in section 4.2 we find that no such tensor exists, hence $C = 0$. Furthermore, evenness in $X$ allows us to identify solutions for the remaining tensors
\[
A_{(mn)}(X) = a_1 \frac{\eta_{mn}}{X^3} + a_2 \frac{X_m X_n}{X^5}, \quad (6.43a)
\]
\[
B_{(mn)}(X) = b_1 \frac{\eta_{mn}}{X^4} + b_2 \frac{X_m X_n}{X^6}, \quad (6.43b)
\]
\[
D_{(mn)}(X) = d_1 \frac{\eta_{mn}}{X^5} + d_2 \frac{X_m X_n}{X^7}. \quad (6.43c)
\]
Imposing (6.41a) and (6.41b) results in $a_2 = -3a_1$, $b_2 = -2b_1$, however for this choice of coefficients (6.41d) implies $B = 0$, while the tensor $A$ survives. It is then easy to see that (6.41e) implies $D = 0$. Therefore the only solution is
\[
A_{(mn)}(X) = a \left( \frac{\eta_{mn}}{X^3} - \frac{3X_m X_n}{X^5} \right). \quad (6.44)
\]
Now let us direct our attention to $\mathcal{H}_t$; imposing (6.34) results in the set of equations
\[
\epsilon^{mnt} \partial_m A_t(X) = 0, \quad (6.45a)
\]
\[
\epsilon^{mnt} \partial_m B_t(X) = 0, \quad (6.45b)
\]
\[
\partial^m \{ C_{mn}(X) - \eta_{mn} C^s s(X) \} = 0, \quad (6.45c)
\]
\[
2\epsilon_{qt}^s B_s(X) - i \partial_t A_q(X) + i \eta_{qt} \partial^s A_s(X) = 0, \quad (6.45d)
\]
\[
4\epsilon_{qt}^s D_s(X) - i \partial_t B_q(X) + i \eta_{qt} \partial^s B_s(X) = 0, \quad (6.45e)
\]
and the algebraic constraints

\[
\begin{align*}
\epsilon_n^maC_{ma}(X) &= 0, \\
C_{mn}(X) - \eta_{mn}C^s_s(X) &= 0.
\end{align*}
\] (6.46a) (6.46b)

The algebraic constraints (6.46) imply that \( C = 0. \) Now since \( A, B \) and \( D \) are odd in \( X \) we can construct the solutions

\[
\begin{align*}
A_t(X) &= a \frac{X_t}{X^4}, \\
B_t(X) &= b \frac{X_t}{X^5}, \\
D_t(X) &= d \frac{X_t}{X^6}.
\end{align*}
\] (6.47a) (6.47b) (6.47c)

However it is not too difficult to show that imposing (6.45d), (6.45e) requires that \( A, B \) and \( D \) must all vanish. Hence \( \mathcal{H}_t(X, \Theta) = 0. \)

So far we have found a single solution consistent with the supercurrent conservation equation and the point-switch identity,

\[
\mathcal{H}_{mn}(X, \Theta) = a \left( \frac{\eta_{mn}}{X^3} - \frac{3X_mX_n}{X^5} \right),
\] (6.48)

\[
\mathcal{H}_{\alpha\beta\gamma\delta}(X, \Theta) = (\gamma^m)_{\alpha\beta}(\gamma^n)_{\gamma\delta}\mathcal{H}_{mn}(X, \Theta) - d_{N=2} \left( \frac{\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}}{X^3} + \frac{3X_{\alpha\beta}X_{\gamma\delta}}{X^5} \right).
\] (6.49)

Therefore the correlation function is

\[
\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle = \frac{x_{13\alpha\alpha'}x_{13\beta\beta'}x_{23\gamma\gamma'}x_{23\delta\delta'}}{(x_{13})^3(x_{23})^3} \mathcal{H}_{\alpha'\beta'\gamma'\delta'}(X_3, \Theta_3),
\] (6.50)

where, after writing our solution in terms of the variable \( X, \)

\[
\mathcal{H}_{\alpha\beta\gamma\delta}(X, \Theta) = d_{N=2} \left\{ \frac{\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}}{X^3} + \frac{\epsilon_{\alpha\delta}\epsilon_{\beta\gamma}}{X^3} + \frac{3\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\Theta^4}{8X^5} + \frac{3\epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\Theta^4}{8X^5} + \frac{3X_{\alpha\beta}X_{\gamma\delta}}{X^5} + \frac{3i\epsilon_{\alpha\beta}X_{\gamma\delta}\Theta^2}{2X^5} + \frac{3i\epsilon_{\gamma\delta}X_{\alpha\beta}\Theta^2}{2X^5} - \frac{3\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\Theta^4}{4X^5} + \frac{15X_{\alpha\beta}X_{\gamma\delta}\Theta^4}{8X^5} \right\}.
\] (6.51)

However it remains to check whether this solution satisfies the flavour current conservation equation. As mentioned earlier it is difficult to check conservation laws on the
third superspace point in this formalism as there are no identities that allow differential operators acting on the $z_3$ dependence to pass through the prefactor of (2.43). To deal with this we will re-write our solution in terms of the three-point building block $X_1$ using identities (2.31a), (2.33). This ultimately has the effect

$$\langle J_{\alpha\beta}(z_1)J_{\gamma\delta}(z_2)L(z_3) \rangle \rightarrow \langle L(z_3)J_{\gamma\delta}(z_2)J_{\alpha\beta}(z_1) \rangle .$$

Written in terms of the variable $X_1$, the correlation function is found to be

$$\langle L(z_3)J_{\gamma\delta}(z_2)J_{\alpha\beta}(z_1) \rangle = x_{21}^{\gamma\gamma'}x_{21}^{\delta\delta'}x_{23}^{\gamma\delta}H_{\gamma\delta\alpha\beta}(X_1, \Theta_1),$$

where

$$H_{\gamma\delta\alpha\beta}(X, \Theta) = d_{N=2} \left\{ \frac{X_{\gamma\alpha}X_{\delta\beta}}{X^3} + \frac{X_{\gamma\beta}X_{\alpha\delta}}{X^3} + \frac{3}{8} \frac{X_{\gamma\alpha}X_{\delta\beta}\Theta^4}{X^5} + \frac{3}{8} \frac{X_{\gamma\beta}X_{\alpha\delta}\Theta^4}{X^5} - \frac{3}{2} \frac{X_{\gamma\delta}X_{\alpha\beta}\Theta^2}{X^3} - \frac{3}{2} \frac{X_{\gamma\delta}X_{\alpha\beta}\Theta^2}{X^3} + \frac{3}{4} \frac{X_{\gamma\delta}X_{\alpha\beta}\Theta^4}{X^3} - \frac{15}{8} \frac{X_{\alpha\beta}X_{\gamma\delta}\Theta^4}{X^5} \right\}.$$ (6.54)

We are now able to check the conservation equation (6.33b), which after using identities equivalent to (2.39a) becomes the constraint

$$\left( D^\sigma(I^J) - \frac{1}{2} \delta_{J}^{I} D^\sigma D^K \sigma \right) H_{\gamma\delta\alpha\beta}(X, \Theta) = 0 .$$

After a very lengthy calculation one can show that the solution above satisfies this conservation equation, hence this correlation function is non-trivial and is determined up to a single parameter.

This is a peculiar result, as it was shown in section 4.2 that the correlation function $\langle JJL \rangle$ vanishes for $N = 1$. At first glance this appears to be a contradiction since any theory with $N = 2$ supersymmetry is also $N = 1$ supersymmetric. However, as was discussed in the previous subsection, the $N = 2$ current supermultiplets $J_{\alpha\beta}$ and $L$ contain not only the $N = 1$ supercurrent and flavour currents, but also the unconstrained scalar superfield $S$ and the supermultiplet of currents $S_{\alpha\beta}$. Hence, non-vanishing of the $N = 2$ three-point function (6.49), (6.50) implies non-vanishing of some of three-point functions involving these additional $N = 1$ currents. For example, from eqs. (6.49), (6.50) it follows that the following $N = 1$ correlator is, in general, non-zero:

$$\langle S_{\alpha_1\alpha_2}(z_1)J_{\beta_1\beta_2\beta_3}(z_2)L_{\gamma}(z_3) \rangle = -D^2_{(2)(\beta_1} D^2_{(3)\gamma} \langle J_{\alpha_1\alpha_2}(z_1)J_{\beta_2\beta_3})(z_2)L(z_3) \rangle ,$$

(6.56)
where the bar-projection means setting $\theta_i^2$ to zero. In components this correlator contains (among others) $\langle R_m(x_1)T_{pq}(x_2)V_s(x_3) \rangle$, where $R_m$ is the $U(1)$ $R$-symmetry current which exists in theories with $\mathcal{N} = 2$ supersymmetry. In theories with $\mathcal{N} = 1$ supersymmetry such a correlator does not exist because there is no $R$-symmetry current. On the other hand, the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction (6.57)

$$\langle J_{\alpha_1 \alpha_2}(z_1)J_{\beta_1 \beta_2}(z_2)L(z_3) \rangle \rightarrow \langle J_{\alpha_1 \alpha_2 \alpha_3}(z_1)J_{\beta_1 \beta_2 \beta_3}(z_2)L_(\gamma_3) \rangle,$$

must give zero to be consistent with the result of the previous subsection. Let us check that this is indeed the case. To perform the reduction we compute

$$- iD_{(1)(\alpha_1}^2D_{(2)(\beta_1}^2D_{(3)\gamma}^2\langle J_{\alpha_2 \alpha_3}(z_1)J_{\beta_2 \beta_3}(z_2)L(z_3) \rangle |.$$

That is we must act with three covariant derivatives with respect to $\theta_i^2$ and then set all $\theta_i^2$ to zero. From the explicit form of the correlator $\langle J_{\alpha_1 \alpha_2}(z_1)J_{\beta_1 \beta_2}(z_2)L(z_3) \rangle$ in eqs. (6.49), (6.50) it follows that it depends on $\theta_i^2 \theta_j^2$. Since it is Grassmann even it contains only even powers of $\theta_i^2$. Therefore, acting on $\langle J_{\alpha_1 \alpha_2}(z_1)J_{\beta_1 \beta_2}(z_2)L(z_3) \rangle$ with three derivatives as in (6.58) will give a result either linear or higher order in $\theta_i^2$, so it vanishes when we set $\theta_i^2 = 0$. This shows that despite being non-zero our result (6.49), (6.50) is consistent with vanishing of the similar correlator in the $\mathcal{N} = 1$ case.

**Acknowledgements**

The authors would like to thank Jessica Hutomo, Sergei Kuzenko and Michael Ponds for valuable discussions. The work of E.I.B. is supported in part by the Australian Research Council, project No. DP200101944. The work of B.S. is supported by the Bruce and Betty Green Postgraduate Research Scholarship under the Australian Government Research Training Program.

\[\text{\footnotesize{Note that all component three-point functions contained in (6.49), (6.50) are parity even.}}\]
A 3D conventions and notation

For the Minkowski metric we use the “mostly plus” convention: $\eta_{mn} = \text{diag}(-1, 1, 1)$. Spinor indices are then raised and lowered with the SL$(2, \mathbb{R})$ invariant anti-symmetric $\varepsilon$-tensor

$$
\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_{\alpha}^{\beta},
$$
(A.1)

$$
\phi_\alpha = \varepsilon_{\alpha\beta} \phi^\beta, \quad \phi^{\alpha} = \varepsilon^{\alpha\beta} \phi_\beta.
$$
(A.2)

The $\gamma$-matrices are chosen to be real, and are expressed in terms of the Pauli matrices $\sigma$ as follows:

$$
(\gamma_0)_\alpha^\beta = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_1)_\alpha^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$
(A.3a)

$$
(\gamma_2)_\alpha^\beta = -\sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
$$
(A.3b)

$$
(\gamma_m)_\alpha^\beta = \varepsilon_{\beta\delta}(\gamma_m)_\alpha^\delta, \quad (\gamma_m)^\alpha_\beta = \varepsilon^{\alpha\delta}(\gamma_m)_\delta^\beta.
$$
(A.4)

The $\gamma$-matrices are traceless and symmetric

$$
(\gamma_m)_\alpha^\alpha = 0, \quad (\gamma_m)_\alpha^\beta = (\gamma_m)^\beta_\alpha,
$$
(A.5)

and also satisfy the Clifford algebra

$$
\gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn}.
$$
(A.6)

Products of $\gamma$-matrices are then

$$
(\gamma_m)_\alpha^\rho(\gamma_n)_\rho^\beta = \eta_{mn}\delta_\alpha^\beta + \epsilon_{mnp}(\gamma_p)_\alpha^\beta,
$$
(A.7a)

$$
(\gamma_m)_\alpha^\rho(\gamma_n)_\rho^\sigma(\gamma_p)_\sigma^\beta = \eta_{mn}(\gamma_p)_\alpha^\beta - \eta_{np}(\gamma_m)_\alpha^\beta + \eta_{mp}(\gamma_n)_\alpha^\beta + \epsilon_{mnp}\delta_\alpha^\beta,
$$
(A.7b)

where we have introduced the 3D Levi-Civita tensor $\varepsilon$, with $\varepsilon^{012} = -\varepsilon_{012} = 1$. It satisfies the following identities:

$$
\epsilon_{mnp}\epsilon_{m'np'} = -\eta_{mn'}(\eta_{n'm'p'} - \eta_{n'p'}\eta_{np'}) - (n' \leftrightarrow m') - (m' \leftrightarrow p'),
$$
(A.8a)

$$
\epsilon_{mnp}\epsilon^{m'n'p'} = -\eta_{mn'}\eta_{n'p'} + \eta_{np'}\eta_{np'},
$$
(A.8b)

$$
\epsilon_{mnp}\epsilon^{mnp} = -2\eta_{np'},
$$
(A.8c)

$$
\epsilon_{mnp}\epsilon^{mnp} = -6.
$$
(A.8d)
We also have the orthogonality and completeness relations for the $\gamma$-matrices

\begin{equation}
(\gamma^m)_{\alpha\beta}(\gamma^m)^{\rho\sigma} = -\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\alpha, \quad (\gamma^m)_{\alpha\beta}(\gamma^m)_{\alpha\beta} = -2\eta_{mn} .
\end{equation}

Finally, the $\gamma$-matrices are used to swap from vector to spinor indices. For example, given some three-vector $x_m$, it may equivalently be expressed in terms of a symmetric second-rank spinor $x_{\alpha\beta}$ as follows:

\begin{equation}
x^{\alpha\beta} = (\gamma^m)^{\alpha\beta} x_m, \quad x_m = -\frac{1}{2}(\gamma^m)^{\alpha\beta} x_{\alpha\beta} ,
\end{equation}

\begin{equation}
det(x_{\alpha\beta}) = \frac{1}{2} x^{\alpha\beta} x_{\alpha\beta} = -x^m x_m = -x^2 .
\end{equation}

The same conventions are also adopted for the spacetime partial derivatives $\partial_m$

\begin{equation}
\partial^{\alpha\beta} = \partial^m (\gamma^m)^{\alpha\beta} , \quad \partial_m = -\frac{1}{2}(\gamma^m)^{\alpha\beta} \partial_{\alpha\beta} ,
\end{equation}

\begin{equation}
\partial_m x^n = \delta^n_m, \quad \partial_{\alpha\beta} x^{\rho\sigma} = -\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\alpha ,
\end{equation}

\begin{equation}
\xi^m \partial_m = -\frac{1}{2} \xi^{\alpha\beta} \partial_{\alpha\beta} .
\end{equation}

We also define the supersymmetry generators $Q^I_\alpha$

\begin{equation}
Q^I_\alpha = i \frac{\partial}{\partial \theta^I_\alpha} + (\gamma^m)_{\alpha\beta} \theta^I_\beta \frac{\partial}{\partial x^m} ,
\end{equation}

and the covariant spinor derivatives

\begin{equation}
D^I_\alpha = \frac{\partial}{\partial \theta^I_\alpha} + i(\gamma^m)_{\alpha\beta} \theta^I_\beta \frac{\partial}{\partial x^m} ,
\end{equation}

which anti-commute with the supersymmetry generators, $\{ Q^I_\alpha, D^J_\beta \} = 0$, and obey the standard anti-commutation relations

\begin{equation}
\{ D^I_\alpha, D^J_\beta \} = 2i \delta^{IJ}(\gamma^m)_{\alpha\beta} \partial_m .
\end{equation}

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