2D Principal Chiral Field at Large N as a Possible Solvable 2D String Theory

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(based on the papers [1] written in collaboration with V. A. Fateev and P. B. Wiegmann)

Abstract

We present the exact and explicit solution of the principal chiral field in two dimensions for an infinitely large rank group manifold. The energy of the ground state is explicitly found for the external Noether’s fields of an arbitrary magnitude. At small field we found an inverse logarithmic singularity in the ground state energy at the mass gap which indicates that at \( N = \infty \) the spectrum of the theory contains extended objects rather than pointlike particles.
1 Introduction

The progress in the modern quantum field theory, especially in the solution of realistic problems like QCD, quantum gravity and strings in physical dimensions, seems to be considerably slowed down because of a very limited list of exactly solvable models of many interacting degrees of freedom. Currently, this list includes the following relatively well explored systems:

1. Quantum field theories and statistical mechanical spin models for the dimensions $D \leq 2$;
2. Strings or 2d gravity coupled to matter with central charge $c_m \leq 1$;
3. Quantum gravity for $d \leq 2$.

All the attempts to make some substantial progress on the way to realistic dimensions are usually confronted to the obstacles which every time seem to have a very similar mathematical origin. It is clear that some principally new mathematical tools will be needed to attack in the future all these problems. In the absence of them we may try to use the old tools to get an insight to the physics of realistic dimensions for the string theory.

One conceivable loophole would be to take some integrable two dimensional field theory of $N \times N$ matrix valued field. Its planar diagrams might describe fluctuating world sheets of a string propagating in the 1+1 dimensional space.

Essentially, there exists only one candidate - the principal chiral field (PCF) on, say, $SU(N)$ group manifold. It can be defined by the following action:

$$S = \frac{N}{2\lambda_0} \int d^2x \ tr [\partial_\mu g^\dagger \partial_\mu g]$$ (1)

where $g$ is an $N \times N$ unitary matrix. Its large $N$ solution has been anticipated for a long time to follow from its finite $N$ solution [2, 3, 4]. It was finally explicitly established in our paper [5]. We will follow the guidelines of this solution.

The partition function of the model reads as:

$$Z = \int [Dg(x)]_H \exp S$$ (2)

where the integration over $g$ goes with the Haar measure for $SU(N)$ at every point of the 2D space. To test physics in this asymptotically free theory we have to introduce somehow an energy scale. Technically the simplest way to do it is to take the theory on a cylinder with the space compactified on the space interval $[0, L]$ and to introduce the following twist of the field $g$:

$$g(x, t) \rightarrow e^{iH_L x} g(x, t) e^{iH_R x}$$ (3)
where \( H_{R(L)} = diag(h_1, h_2 - h_1, \ldots, h_{N-1} - h_{N-2}, -h_{N-1}) \) is an element of the Cartan subalgebra.

It amounts to introducing in eq. (1) the covariant derivative

\[
D_\mu g = \partial_\mu g - i\delta_\mu 0 (H_L g + g H_R)/2
\]

instead of the usual derivative \( \partial_\mu g \). In what follows, we shall consider only the case \( H_L = H_R = H \).

In the hamiltonian language one adds to the hamiltonian of the PCF a term \( tr(H_L Q_L + H_R Q_R)/2 \) to the hamiltonian of the theory corresponding to the lagrangian (1). Here \( Q_L = \int d^2x g_{0}^{-1} \) and \( Q_R = \int d^2x g^{-1} \partial_{0} g \) are the Noether’s left and right charges.

The model is completely integrable for any \( N \) by means of the Bethe ansatz approach \cite{3} or the bootstrap procedure for the physical S-matrix. The results show the presence of \( N-1 \) physical particles in the spectrum of the theory obeying the masses:

\[
m_l = m_{1} \frac{\sin(\frac{\pi l}{N})}{\sin(\frac{\pi}{N})}
\]

where \( l = 1, \ldots, N-1 \) is the rank of a fundamental representation and \( m = m_{1} \) is the mass of the vector particle. In the two-loop approximation it is

\[
m = \Lambda \frac{1}{\sqrt{\lambda_0}} \exp\left(\frac{-4\pi}{\lambda_0}\right)
\]

where \( \Lambda \) is a cutoff. These results are in complete agreement with the renormalization group predictions for the asymptotically free theory.

All particles are bound states of the vector particles. They form the multiplets of the fundamental representations of \( SU(N) \) algebra (antisymmetric tensors corresponding to columns of the Young tableau). It follows from the last statement that \( h_k \) is precisely the chemical potential of the k-th type of the physical particles.

To demonstrate this last statement let us turn for a moment to the one-dimensional PCF and calculate its partition function in the periodic time \( (0, T) \) with the same twisted boundary conditions.

The functional integral representation of the partition function is

\[
Z(T, H_L, H_R) = \int [Dg(t)]_H e^{N_{tr} \int_0^T dt \partial_0 g^{-1} \partial_0 g},
\]

where \([Dg(t)]_H\) is the Haar measure on the unitary group \( U(N) \), and the holonomies at the end points of the time interval are \( g(0) = e^{iH_L} \) and \( g(T) = e^{iH_R} \) (with \( e^{iH_L} \) and \( e^{iH_R} \) the
eigenvalues of $g(0)$ and $g(T)$). Due to the invariance of the Haar measure at each moment of time, we have, after the introduction of a new variable (connection) $A(t) = ig^{-1}\partial_t g(t)$ the following representation of $Z(T, H_L, H_R)$:

$$Z(T, H_0, H_T) = \int DA(t)e^{-N\text{tr} \int_0^T dt A(t)^2} \delta \left( [Te^{i \int_0^T dt A(t)}] e^{iH_L}, e^{iH_R} \right).$$ (8)

Using the character expansion of the group $\delta$-function:

$$\delta(U, U') = \sum_R \chi_R(U)\chi_R(U')^*$$ (9)

and the fact that the character is the trace of the matrix element in a given representation $R$: $\chi_R(T \exp(i \int Adt)) = \text{tr}_R[T \exp(i \int A^a \tau^a_R dt)]$ ($\tau^a_R$ is the $a$th generator of $U(N)$ in the $R$th irreducible representation), we arrive after a simple gaussian integration in $A(t)$ (independent at each moment) at the conclusion that

$$Z(T, H_L, H_R) = \sum_R e^{-TC^2(R)} \chi_R(e^{iH_L})\chi_R(e^{iH_R})$$ (10)

where $C^2(R)$ is the second Casimir of the group $U(N)$ in the representation $R$. This is also equivalent to the partition function of two dimensional QCD on the cylinder with the boundary holonomies specified at either end of the cylinder by the two distributions $e^{iH_L}$ and $e^{iH_R}$.

Let us now recall the Weil formula for the character:

$$\chi_R(e^{iH}) = \frac{\det_{k,j}(e^{i(n_k-k+N)H_j})}{\Delta(e^{iH})},$$ (11)

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Van-der-Monde determinant.

Let us consider one of two characters entering the expression (10). Due to the antisymmetry of the characters in $n_k$’s we can retain in one of them only a diagonal term from the Weil determinant under the sum over irreps (in other words, over integers $n_k$):

$$\exp(i \sum_k H_k n_k) = \exp(i \sum_k (h_k - h_{k-1}) n_k) = \exp(-i \sum_k h_k (n_{k+1} - n_k))$$ (12)

Since $n_{k+1} - n_k$ is the number of columns of length $k$ in the Young tableau of the irrep $R$ (corresponding to the antisymmetric group of $k$ indices from the tensor describing the whole irrep $R$), we conclude that $h_k$ is the chemical potential of $k$-vectors - the $k$-th fundamental representations which constitute the whole irrep $R$. In the 2D PCF these $k$-vectors will correspond to the physical particles of the type $k$.
2  Ground state energy and beta-function of the PCF at large N

Let us turn back to two the two-dimensional PCF, formulate our main result and then discuss it in details.

Let us make a special choice of \( h_k \)'s which will be technically the simples one and quite suitable for the limit \( N = \infty \):

\[
h_k = h \frac{\sin \left( \frac{\pi}{N} k \right)}{\sin \left( \frac{\pi}{N} \right)}
\]  

(13)

Since every \( k \)-th type of particles is excited only if \( h_k \) exceeds \( m_k \), we will have for this choice of \( h_k \)'s no particles excited when \( h \leq m \), and all particles excited on equal footing when \( h > m \).

We show that the energy of the ground state is expressed in terms of modified Bessel functions:

\[
f(h) \equiv \frac{1}{N^2} \left( \mathcal{E}(h) - \mathcal{E}(0) \right) = -\frac{h^2}{8\pi} B^2 I_1(B) K_1(B)
\]  

(14)

where the parameter \( B \) is defined through

\[
\frac{m}{h} = BK_1(B)
\]  

(15)

The distribution of rapidities of physical particles will obey the simple semi-circle law with the support \( B \). The parameter \( B \) defines the value of rapidity corresponding to the Fermi momentum of the fused particles. We shall see, that \( B \) gives the most natural definition of the renormalized (running) coupling constant:

\[
\bar{\lambda}(h) = \frac{4\pi}{B}
\]  

(16)

With the definition (16) of the running charge one can find from eq. (13) the exact beta-function:

\[
\beta(\bar{\lambda}) = h \frac{\partial}{\partial h} \bar{\lambda} = -4\pi \frac{\partial B}{B^2 \partial \ln h/m} = -4\pi \frac{K_1(B)}{B^2 K_0(B)}
\]  

(17)

or

\[
\beta(\bar{\lambda}) = -\frac{1}{4\pi} \bar{\lambda}^2 \frac{K_1 \left( \frac{4\pi}{\bar{\lambda}} \right)}{K_0 \left( \frac{4\pi}{\bar{\lambda}} \right)} = -\frac{1}{4\pi} \bar{\lambda}^2 \sum_{n=2}^{\infty} b_n \left( \frac{\bar{\lambda}}{32\pi} \right)^n
\]  

(18)

where

\[
b_0 = 1, \quad b_1 = 4, \quad b_2 = -8, b_3 = 64, \quad b_4 = -5^2 \cdot 2^5, \quad b_5 = 13 \cdot 2^{10}, \quad b_6 = -1073 \cdot 2^8, \quad b_7 = 103 \cdot 2^{16}, \quad ...
\]  

(19)

\[
b_n \sim -(-1)^n \sqrt{8/(\pi n)} (4n/e)^n, \quad n \to \infty
\]  

(20)
3 Physical consequences of the exact solution: weak and strong coupling limits

Let us first look for the weak coupling regime and find the link with the results of the standard renormalized perturbation theory. In the asymptotically free theory weak coupling means the presence of a big energy scale, namely, big \( h/m \). From eq. (15) we conclude that it corresponds to \( B \to \infty \). It follows from the large \( B \) asymptotics of McDonald function

\[
\frac{h}{m} = \sqrt{\frac{2}{\pi}} \sqrt[2]{\sqrt[8]{B}} \left( 1 - \frac{3}{8B} + O(1/B^2) \right)
\]  (21)

Solving for \( B \) we obtain:

\[
B = \ln \frac{h}{m} + \frac{1}{2} \ln \ln \frac{h}{m} + \frac{1}{2} \ln \frac{\pi}{2} + O\left( \frac{1}{\ln \frac{h}{m}} \right)
\]  (22)

Using the large \( B \) asymptotics \( I_1(B) \) we finally find from (14) and (22):

\[
16\pi f(h) = -h^2 B + O\left( \frac{h^2}{B} \right) = -h^2 \left( \ln \frac{h}{m} + \frac{1}{2} \ln \ln \frac{h}{m} + \frac{1}{2} \ln \frac{\pi}{2} + O\left( \frac{1}{\ln \frac{h}{m}} \right) \right)
\]  (23)

This result reproduces correctly one- and two-loop terms of the perturbation theory as well as the universal non-perturbative constant \( \frac{1}{2} \ln \frac{\pi}{2} \). We also see from the comparison of eq.(22) with the standard 2-loop perturbation theory that our definition (16) of the running coupling was justified.

More than that: we can use the known formulae for the 1/B expansion of the product \( I_1(B)K_1(B) \) to get the explicit coefficients of the renormalized weak coupling expansion:

\[
f(h)/h^2 = -\frac{1}{4\lambda} \left( 1 - \sum_{n=1}^{\infty} C_{2n} \left( \frac{\lambda}{4\pi} \right)^{2n} \right)
\]  (24)

where

\[
C_{2n} = \frac{2n + 1}{2n - 1} \left( \frac{(2n)!}{n!} \right)^3 \rightarrow_{n \to \infty} 2/\sqrt{\pi n} \left( \frac{n}{e} \right)^{2n}
\]  (25)

Note that the expansion goes only in even powers.

As we see, in spite of the fact that every coefficient represents a sum over renormalized planar graphs, it grows factorially with the order. Most probably this happens because of the renormalons (some subsequence of logarithmically divergent graphs) giving the main factorial contribution in each order noticed long time ago by 'tHooft. This means that we have an exponential number of graphs in each order but some of them give \((2n)!\) contribution after the momenta integration. More than that: the series is a non-signchanging one and
thus non-Borel summable. Nevertheless, the free energy perfectly exists for any finite $\lambda$. These phenomena seem to be imminent for any asymptotically free field theory. It sheds some doubts on the possibility to interpret the standard planar Feynman diagrams in this theory as the fluctuating world sheets of some string. The point is that in all examples of the solvable theories of non-critical strings with the central charge of the matter $c \leq 1$ the expansions in the cosmological constant (conjugated to the invariant area of the world sheet, or, in the discretized version of the string theory, to the number of vertices of the corresponding random (Feynman) graph) have usually a finite radius of convergency, which means that near the critical value of the cosmological coupling the graphs are big. In the present case we have a zero radius of convergency with respect to the renormalized coupling (because of renormalons), which prevents the usual procedure of the thermodynamical limit of a big world sheet.

The hope to interpret the model in terms of a new string theory gets revived if one considers an opposite, nonperturbative limit of small energies (of the order of the mass gap): $\Delta = h/m - 1 \rightarrow 0).$ This corresponds to $B \rightarrow 0$.

At small $B$ asymptotics of the Bessel functions give a singular behaviour on the threshold:

$$f(h) \simeq - (m/2\pi)^2 \frac{\Delta}{|\ln \Delta|}, \quad \Delta \rightarrow 0$$  (26)

It differs drastically from the threshold behaviour for a finite $N$ theory of massive particles, where we would have $3/2$ law (see e.g. [3]):

$$f_N(h) \sim - m^2(\Delta)^{3/2}$$  (27)

The reason for that is that we fixed the mass scale at $m_{N/2}$ at large $N$, and therefore the mass spectrum became continuous (we have $N-1$ different masses on the finite interval). The simpler limit would correspond to fixing $m_1$. In this case $m_k = km_1$ and we would not have any bound states.

It is interesting to compare the result (26) with the $c = 1$ matrix model [8] where the ground state energy behaves in a similar way with respect to the cosmological constant $\lambda$:

$$E(\lambda) \simeq \frac{(\lambda_c - \lambda)^2}{|\ln(\lambda_c - \lambda)|}, \quad \Delta \rightarrow 0$$  (28)

The mechanism by which the inverse logarithmic behaviour with respect to the cosmological constant occurs also requires a parametrization through the fermi level of the corresponding fermions (whose coordinates are the eigenvalues of a hermitean matrix field). The Fermi level plays the role of a ”hidden” parameter of the problem and the eigenvalues give rise to an extra (Liouville) degree of freedom of the theory.
4 Sketch of the solution

Let us now sketch out the major steps of the solution of the PCF.

The easiest way to start with is the construction of the physical S-matrix for the (vector × vector) particles \[4, 5\]. Following the general guidelines of the bootstrap method one arrives to the result:

\[ S = X(\theta)S(\theta) \otimes S(\theta). \]

Here \(\theta\) is the rapidity of a massive relativistic particle \((p^0 = m \cosh \theta, p^1 = m \sinh \theta)\) and \(X(\theta)\) is the CDD-ambiguity factor which cannot be determined by the factorization, unitary and crossing symmetry conditions. The \(SU(N)\) unitary, crossing invariant, factorized S-matrix of vector particles is well known. It is

\[ S(\theta) = u(\theta)(P^+ + \theta + i2\pi/N \theta - i2\pi/N P^-) \]

where \(P^\pm\) is the projection operator onto symmetric (antisymmetric) states.

\[ u(\theta) = \frac{\Gamma(1 - \frac{\theta}{2\pi i})\Gamma(\frac{1}{N} + \frac{\theta}{2\pi i})}{\Gamma(1 + \frac{\theta}{2\pi i})\Gamma(\frac{1}{N} - \frac{\theta}{2\pi i})} \]

Finally, the CDD factor \(X\) is chosen to cancel all double zeros and double poles on the physical sheet \(0 < \text{Im}\theta < \pi\):

\[ X(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{N})}{\sinh(\frac{\theta}{2} - \frac{i\pi}{N})} \]

This is the S-matrix of the vector particles. It has a pole on the physical sheet at \(\theta_b = 2\pi i/N\) in the antisymmetric channel. It corresponds to the first bound state (the second rank antisymmetric tensor) with a mass \(m_2 = m \sin(2\pi/N)/\sin(\pi/N)\). The S-matrix of these particles can be also found by tensoring the vector S-matrix (the fusion procedure). It also has a pole in the antisymmetric channel, and so on. In this way the whole mass spectrum \([5]\) can be generated.

To find the thermodynamical properties of the ground state we have to use the Bethe ansatz procedure. The idea is to put many vector particles into a periodic box of the size \(L\). In the thermodynamical limit they will self-organize into higher bound states, and to find the ground state energy we have to calculate the densities of physical particles \(\rho_k(\theta)\) for every type \(k = 1, 2, ..., N - 1\).

The next step (which can be justified within the thorough consideration of the Bethe procedure) consists from throwing away the isotopic index structure of the S-matrix and
using its scalar part $S(\theta) = u^2(\theta)X(\theta)$ to write down the phase balance eq. for a particle moving around the box:

$$\exp(i m L \sinh \theta_\alpha) = \prod_{\beta=1, \alpha \neq \beta}^{N} \exp(i \phi(\theta_\alpha - \theta_\beta)) \quad (32)$$

where $\exp(i \phi(\theta)) = u^2(\theta)X(\theta)$.

To restore the whole $SU(N) \times SU(N)$ structure of the physical states of the model we have to consider the solutions of this eq. for complex values of \(\theta\)'s. It is known that in the large $L$ limit the vector particles form the “strings” of complex rapidities: $\theta^{r,(l)} \rightarrow \theta^{(l)} + 2r \pi i/N$, where $\theta^l$ is a rapidity of the $l$-th particle and $r$ is an integer running between $-l/2$ and $l/2$. Substituting this into the Eq.(32) and multiplying equations over $r$ we shall obtain the equations for the rapidities of the state which contains $N_l$ particles of the kind $l$. Taking the logarithm of both sides the Eq.(32) we obtain

$$Lm_l \sinh \theta^{(l)}_\alpha = 2\pi J^{(l)}_\alpha + \sum_{n=1}^{N_l} \sum_{\alpha=1, \neq \beta} \phi_{ln}(\theta^{(l)}_\alpha - \theta^{(n)}_\beta) \quad (33)$$

where $\phi_{ln}(\theta) = \sum_{|r| < l/2, |r'| < n/2} \phi(\theta + 2r \pi i/N + 2r' \pi i/N)$ is the scattering phase of the $l$-th and the $n$-th particles, and integers $J$ are the quantum numbers of the states. The energy of this state is obviously

$$E = \frac{1}{L} \sum_{l=1}^{N-1} m_l \sum_{\alpha=1}^{N_l} \cosh \theta^{(l)}_\alpha \quad (34)$$

One can note here that the ground state of the system can be described by a big Young tableau where every column of a length $k$ corresponds to a physical particle of the type $k$ which transforms as $k$-th fundamental representation of $SU(N)$.

The next step is to find rapidities to minimize the energy (34) in the thermodynamic limit $N_l/L = n_l$, while $L \rightarrow \infty$. We assume that in the ground state $\theta$'s are distributed smoothly between $-B_l$ and $B_l$ with a distribution functions $\rho_l(\theta)$ Than eq.(33) implies the spectral equations

$$\frac{1}{2\pi} m_l \cosh \theta = \sum_n \int_{-B_l}^{B_l} R_{ln}(\theta - \theta') \rho_n(\theta') d\theta' \quad (35)$$

where $R_{ln}(\theta) = \delta_{ln} - \frac{1}{2\pi} d\phi_{ln}(\theta)/d\theta$. The Fermi rapidities $B_l$ are determined by the number of particles in the $l$ th representation: $\int_{-B_l}^{B_l} \rho_l(\theta) d\theta = n_l$. The energy of the state is then

$$E = \sum_l \int_{-B_l}^{B_l} m_l \cosh \theta \rho_l(\theta) d\theta \quad (36)$$
An explicit form of the scattering kernel $R_{ln}$ was found in [3]. Its Fourier transformation is

$$R_{ln}(\omega) = 2 \frac{\sinh \left( \pi \omega \left(1 - \frac{l}{N} \right) \right) \sinh \left( \frac{\pi \omega n}{N} \right)}{\sinh \pi \omega}$$  \hspace{1cm} (37)$$

at $l > n$ and $R_{ln} = R_{nl}$.

Now we pass to the large N solution (found in [1]) of this rather complicated system of integral equations.

At large $N$ we can consider a particular distribution of fields $h_l$ which creates all different particles on equal footing, namely one which follows the spectrum of masses (3): $h_l = (h/m)m_l$. This field creates $N_l = N(m_l/m)$ particles in the $l$-th representation (the most representative Young tableau). In this case all Fermi momenta are equal: $B_l = B$ and $\rho_l = \frac{1}{N}(m_l/m)\rho$. Then the spectral equations (35) can be easily diagonalized. They reflect the structure of the Cartan matrix and moreover have the same eigenvectors:

$$\sum_{l,n=1}^{N-1} \chi_l^{(p)} R_{ln}(\omega) \chi_n^{(p')} = R^{(p)}(\omega)\delta^{p,p'}$$

$$R^{(p)}(\omega) = \frac{2N}{\pi} \sum_{r=-\infty}^{\infty} \frac{|\omega|}{\omega^2 + (p + rN)^2}$$  \hspace{1cm} (38)$$

where $\chi_l^{(p)} = \sqrt{2/N} \sin \frac{\pi p l}{N}$, $p = 1, 2, ..., N - 1$ ($\chi^{(1)}$ is the Perron-Frobenius mode) is the orthogonal set of eigenfunctions of the Cartan matrix $C_{ln} = 2\delta_{ln} - \delta_{l,n+1} - \delta_{l+1,n}$.

Let us note that the expression (38) for the kernel of the integral equation reminds the propagator of the free motion of a particle on the space consisting from the physical space (the time plus the 1-dimensional physical space described by rapidities) plus the discrete periodized space represented by the Dynkin diagram. In this sense it looks like the effective space of our model becomes three dimensional, the phenomenon familiar to us from the experience with the c=1 bosonic string (where it effectively passes from 1D to 2D).

Then the density $\rho$ obeys the equation:

$$\frac{1}{N} \int_{-B}^{B} R^{(1)}(\theta - \theta')\rho(\theta') d\theta' = \frac{m}{2\pi} \cosh \theta.$$  \hspace{1cm} (39)$$

Further simplifications occur in the large $N$ limit

$$R^{(1)}(\omega) \approx \frac{2N}{\pi} \frac{|\omega|}{\omega^2 + 1}$$  \hspace{1cm} (40)$$
Now the density $\rho$ may be found in a closed form. To see this, let us apply the operator $(-\frac{\partial^2}{\partial \theta^2} + 1)$ on both sides of the equation. As a result we obtain an integral equation with the Cauchy kernel $(\theta - \theta')^{-2}$. This equation is solvable:

$$\rho(\theta) = \frac{m}{4K_0(B)\sqrt{B^2 - \theta^2}}$$

(41)

where $K_0(B)$ is the Bessel function. Note that, in the large $N$ limit $R^{(p)}(\omega)$ vanishes at large $\omega$, whereas at finite $N$ it approaches 1 (see eq.(28)). This implies a singular behaviour of $\rho(\theta)$ at the Fermi point $\pm B$. As a result the physics on the threshold $h \sim m$ will be changed drastically.

The value of the Fermi rapidities as a function of number of particles can now be obtained from

$$n = \int_{-B}^{B} \rho(\theta) d\theta = \frac{\pi m}{4K_0(B)}$$

(42)

In turn the energy of the state with a given number of particles is

$$E/N^2 = \frac{m}{2\pi^2} \int_{-B}^{B} \cosh \theta \rho(\theta) d\theta = \frac{m^2}{8\pi} \frac{I_0(B)}{K_0(B)}$$

(43)

And finally, the field $h = -2\pi^2/N^2 dE/dn$ which corresponds to a given number of particles and the energy as a function of the field $E = E - \sum h_i n_i = E - N^2/(2\pi^2) \ h n$ are given by the formulae (13) and (14).

Let us note that in the large $N$ limit any virtual and real processes involve all particles, since minimal energies are greater then a minimal separation between masses. A reasonable external field $h_l \sim m_l$ excites all of them on equal footing and leads to collective effects.

5 Conclusions

Let us now list the main lessons which can be drawn out from the solution:

1. Renormalized planar graphs do not model the world sheets of any string (they have the zero radius of convergency, due to renormalons).

2. A possible “stringy” behaviour with the 2 dimensional physical target space may occur only in the non-perturbative strong coupling regime (near the threshold $(h \sim m)$). In this domain the physics looks similar to the $c = 1$ bosonic string described by the matrix quantum mechanics [3]. At least, the ground state energy contains the similar inverse logarithmic behaviour with the scale parameter, and the fermionic spectrum of excitations which becomes classical in the large $N$ limit, is also common for two models.
3. The Bethe Ansatz solution of the PCF can be effectively described as having (1+1+1) dimensions: \((\text{time} \times \text{space} \times \text{Dynkin diagram})\), according to the form of the integral equations for densities of particles. The picture is similar to \((\text{time} \otimes \text{Liouville (eigenvalue) mode})\) forming a (1+1) dimensional effective space in the \(c=1\) bosonic string.

Among the most obvious questions which are left we list the following ones:

1. Elaboration of the \(1/N\) expansion and of the double scaling limit around the threshold \(h \sim m\), where a possibility for the “stringy” behaviour exists.

2. One has to understand the role of the 3-rd dimension (Dynkin diagram) in the model. May be, the theory is “almost free” in the (1+1+1) dimensional effective space.

3. Main challenge: to solve the model by simpler (matrix model) methods directly in terms of the original chiral field \(g(x)\). The character expansion used for the 1D principal chiral field (equivalent to the heat kernel on the group \(SU(N)\) manifold) [7, 8] might be very useful for it.

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