THE TOPOLOGICAL COMPLEXITY AND THE HOMOTOPY COFIBER OF THE DIAGONAL MAP FOR NON-ORIENTABLE SURFACES

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Abstract. We show that the Lusternik-Schnirelmann category of the homotopy cofiber of the diagonal map of non-orientable surfaces equals three. Also, we prove that the topological complexity of non-orientable surfaces of genus \( g > 4 \) is four.

1. Introduction

The topological complexity \( TC(X) \) of a space \( X \) was defined by Farber [F] as an invariant that measures the navigation complexity of \( X \) regarded as the configuration space for a robot motion planning. By a slightly modified definition \( TC(X) \) is the minimal number \( k \) such that \( X \times X \) admits a cover by \( k+1 \) open sets \( U_0, \ldots, U_k \) such that over each \( U_i \) there is a continuous motion planning algorithm \( s_i : U_i \to PX \), i.e., a continuous map of \( U_i \) to the path space \( PX = X^{[0,1]} \) with \( s_i(x,y)(0) = x \) and \( s_i(x,y)(1) = y \) for all \( (x,y) \in U_i \). Here we have defined the reduced topological complexity. The original (non-reduced) topological complexity is by one larger.

The topological complexity of an orientable surface of genus \( g \) was computed in [F]:

\[
TC(\Sigma g) = \begin{cases} 
2, & \text{if } g = 0,1, \\
4, & \text{if } g > 1.
\end{cases}
\]

For the non-orientable surfaces of genus \( g > 1 \) the complete answer is still unknown. What was known are the bounds: \( 3 \leq TC(N_g) \leq 4 \) and the equality \( TC(\mathbb{R}P^2) = 3 \). In this paper we show that \( TC(N_g) = 4 \) for \( g > 4 \).

The topological complexity is a numeric invariant of topological spaces similar to the Lusternik-Schnirelmann category. It is unclear if \( TC \) can be completely reduced to the LS-category. One attempt of such reduction ([GV2], [Dr2]) deals with the problem of whether the topological complexity \( TC(X) \) coincides with the Lusternik-Schnirelmann category \( \text{cat}(C_{\Delta X}) \) of the homotopy cofiber of the diagonal map \( \Delta : X \to X \times X, C_{\Delta X} = (X \times X)/\Delta X \). The coincidence of these two concepts was proven in [GV2] for a large class of spaces. Also in [GV1] the equality was proven for the weak in the sense of the Berstein and Hilton versions of \( TC \) and cat.
In this paper we prove that $\text{cat}(C_{\Delta N}) = 3$ for any non-orientable surface $N$. Thus, in view of the computation $TC(N_g) = 4$ for $g > 4$ we obtain counterexamples to the conjecture $TC(X) = \text{cat}(C_{\Delta X})$.

Since both computations are rather technical, at the end of the paper we present a short counterexample: Higman’s group. We show that $TC(BH) \neq \text{cat}(C_{\Delta BH})$ where $BH = K(H, 1)$ is the classifying space for Higman’s group $H$. The proof of that is short because the main difficulty there, the proof of the equality $TC(BH) = 4$, is hidden behind the reference [GLO].

2. Preliminaries

2.1. Category of spaces. By the definition the Lusternik-Schnirelmann category $\text{cat} X \leq k$ for a topological space $X$ if there is a cover $X = U_0 \cup \cdots \cup U_k$ by $k + 1$ open subsets each of which is contractible in $X$.

Let $\pi = \pi_1(X)$. We recall that the cup product $\alpha \smile \beta$ of twisted cohomology classes $\alpha \in H^i(X; L)$ and $\beta \in H^j(X; M)$ takes value in $H^{i+j}(X; L \otimes M)$ where $L$ and $M$ are $\pi$-modules and $L \otimes M$ is the tensor product over $\mathbb{Z}$ [Bro]. Then the cup-length of $X$, denoted as $c.l.(X)$, is defined as the maximal integer $k$ such that $\alpha_1 \smile \cdots \smile \alpha_k \neq 0$ for some $\alpha_i \in H^{n_i}(X; L_i)$ with $n_i > 0$. The following inequalities give estimates on the LS-category [CLOT]:

2.1.1. Theorem. $c.l.(X) \leq \text{cat} X \leq \dim X$.

If $X$ is $k$-connected, then $\text{cat} X \leq \dim X/(k + 1)$.

2.2. Category of maps. We recall that the LS-category of a map $f : Y \to X$ is the least integer $k$ such that $Y$ can be covered by $k + 1$ open sets $U_0, \ldots, U_k$ such that the restrictions $f|_{U_i}$ are null-homotopic for all $i$.

The following two facts are proven in [Dr3] (Proposition 4.3 and Theorem 4.4):

2.2.1. Theorem. Let $u : X \to B\pi$ be a map classifying the universal covering of a CW complex $X$. Then the following are equivalent:

1. $\text{cat}(u) \leq k$;
2. $u$ is homotopic to a map $f : X \to B\pi$ with $f(X) \subset B\pi^{(k)}$.

2.2.2. Theorem. Let $X$ be an $n$-dimensional CW complex whose universal covering $\tilde{X}$ is $(n - k)$-connected. Suppose that $X$ admits a classifying map $u : X \to B\pi$ with $\text{cat} u \leq k$. Then $\text{cat} X \leq k$.

2.3. Inessential complexes. One can extend Gromov’s theory of inessential manifolds [Gr] to simplicial complexes and, in particular, to pseudo-manifolds. We call an $n$-dimensional complex $X$ inessential if a map $u : X \to B\pi$ that classifies the universal covering of $X$ can be deformed to the $(n - 1)$-dimensional skeleton. Otherwise it is called essential.

2.3.1. Proposition. An $n$-dimensional complex $X$ is inessential if and only if $\text{cat} X \leq n - 1$.

Proof. Suppose that $\text{cat} X \leq n - 1$. Then $\text{cat}(u) \leq n - 1$ where $u : X \to B\pi$ is a classifying map. By Theorem 2.2.1 $X$ is inessential.

If $X$ is inessential, by Theorem 2.2.1 $\text{cat}(u) \leq n - 1$. We apply Theorem 2.2.2 to $X$ with $k = n - 1$ to obtain that $\text{cat} X \leq n - 1$. □

Remark. Proposition 2.3.1 in the case when $X$ is a closed manifold was proven in [KR].
2.4. Pseudo-manifolds. We recall that an \( n \)-dimensional pseudo-manifold is a simplicial complex \( X \) which is pure, non-branching and strongly connected. Pure means that \( X \) is the union of \( n \)-simplices. Non-branching means that there is a subpolyhedron \( S \subset X \) of dimension \( \leq n-2 \) such that \( X \setminus S \) is an \( n \)-manifold. Strongly connected means that every pair of \( n \)-simplices \( \sigma, \sigma' \) in \( X \) can be joined by a chain of simplices \( \sigma_0, \ldots, \sigma_m \) with \( \sigma_0 = \sigma, \sigma_m = \sigma' \), and \( \dim(\sigma_i \cap \sigma_{i-1}) = n-1 \) for \( i = 1, \ldots, m \). Note that every \( n \)-dimensional pseudo-manifold \( X \) admits a CW complex structure with one vertex and one \( n \)-dimensional cell.

A sheaf \( O_X \) on an \( n \)-dimensional pseudo-manifold \( X \) generated by the presheaf \( U \to H_n(X, X \setminus U) \) is called the orientation sheaf. We recall that in case of manifolds the orientation sheaf \( O_N \) on \( N \) is defined as the pull-back of the canonical \( \mathbb{Z} \)-bundle \( \mathcal{O} \) on \( \mathbb{R}P^\infty \) by the map \( w_1 : N \to \mathbb{R}P^\infty \) that represents the first Stiefel-Whitney class.

A pseudo-manifold \( X \) is locally orientable if \( O_X \) is locally constant with the stalks isomorphic to \( \mathbb{Z} \). For a locally orientable \( n \)-dimensional pseudo-manifold \( X \), \( H_n(X; O_X) = \mathbb{Z} \), and the \( n \)-dimensional cell (we may assume that it is unique) defines a generator of \( \mathbb{Z} \) called the fundamental class \([X]\) of \( X \).

2.4.1. Theorem. Let \( X \) be a locally orientable \( n \)-dimensional pseudo-manifold and let \( A \) be a \( \pi_1(X) \)-module. Then the cap product with \([X]\) defines an isomorphism 
\[
[X] \cap : H^n(X; A) \to H_0(X; A \otimes \mathbb{Z})
\]
where \( \mathbb{Z} \) stands for the \( \pi_1(X) \)-module \( \mathbb{Z} \) defined by the orientation sheaf \( O_X \).

Proof. We note that in these dimensions the proof of the classical Poincaré Duality for locally oriented manifolds (\cite{Bred}) works for pseudo-manifolds as well. \qed

2.4.2. Proposition. Suppose that a map \( f : M \to B\pi \) of an \( n \)-dimensional locally orientable pseudo-manifold induces an epimorphism of the fundamental groups. Suppose that the orientation sheaf on \( M \) is the image under \( f^* \) of a sheaf on \( B\pi \). Then \( f \) can be deformed to the \((n-1)\)-skeleton \( B\pi^{(n-1)} \) if and only if \( f_*([M]) = 0 \) where \([M]\) is the fundamental class.

Proof. The ‘only if’ direction follows from the dimensional reason and the fact that \( f_* \) does not change under a homotopy.

Let \( f_*([M]) = 0 \). We show that the primary obstruction \( o_f \) for deformation of \( f \) to the \((n-1)\)-skeleton is trivial. Since \( o_f \) is the image of the primary obstruction \( o' \) to deformation of \( B\pi \) to \( B\pi^{(n-1)} \), it suffices to prove the equality \( f^*(o') = 0 \). Note that 
\[
f_*([M] \cap f^*(o')) = f_*([M]) \cap o' = 0.
\]
Since \( f \) induces an epimorphism of the fundamental groups, it induces an isomorphism of 0-dimensional homology groups with any local coefficients. Hence, \([M] \cap f^*(o') = 0 \). Since in dimension 0 the Poincare Duality holds for locally orientable pseudo-manifolds, we obtain that \( f^*(o') = 0 \). \qed

2.5. Homology of projective space. We denote by \( \mathcal{O} \) the canonical local coefficient system on the projective space \( \mathbb{R}P^\infty \) with the fiber \( \mathbb{Z} \).

2.5.1. Proposition.
\[
H_i(\mathbb{R}P^\infty; \mathcal{O}) = \begin{cases} 
\mathbb{Z}_2, & \text{if } i \text{ is even}, \\
0, & \text{if } i \text{ is odd}.
\end{cases}
\]
Proof. Let \( Z \) denote a \( \mathbb{Z}_2 \)-module \( Z \) with the non-trivial \( \mathbb{Z}_2 \)-action. We note that \( H_\ast(\mathbb{R}P^\infty; \mathcal{O}) = H_\ast(\mathbb{Z}_2, Z) \). If \( \mathbb{Z}_2 = \{1, t\} \), then the homology groups \( H_\ast(\mathbb{Z}_2, Z) \) are the homology of the chain complex \((\text{Bro})\):

\[
\cdots \xrightarrow{1-t} \mathbb{Z} \xrightarrow{1+t} \mathbb{Z} \xrightarrow{1-t} \mathbb{Z} \xrightarrow{1+t} \mathbb{Z} \xrightarrow{1-t} \mathbb{Z}
\]

which is the complex

\[
\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}.
\]

2.6. Schwarz genus. We recall that the Schwarz genus \( g(f) \) of a fibration \( f : E \to B \) is the minimal number \( k \) such that \( B \) can be covered by \( k \) open sets on which \( f \) admits a section \([\text{Sch}]\). Then cat \( X + 1 = g(c : * \to X) \) and \( \text{TC}(X) + 1 = g(\Delta : X \to X \times X) \) where the constant map \( c \) and the diagonal map \( \Delta \) are assumed to be represented by fibrations. Schwarz related the genus \( g(f) \) with the existence of a section of a special fibration constructed out of \( f \) by means of an operation that generalizes the Whitney sum.

Here is the construction: Given two maps \( f_1 : X_1 \to Y \) and \( f_2 : X_2 \to Y \), we define the fiberwise join of spaces \( X_1 \) and \( X_2 \) as

\[ X_1 \ast_Y X_2 = \{tx_1 + (1-t)x_2 \mid f_1(x_1) = f_2(x_2)\} \]

and define the fiberwise join of \( f_1, f_2 \) as the map

\[ f_1 \ast_Y f_2 : X_1 \ast_Y X_2 \to Y, \quad \text{with } (f_1 \ast_Y f_2)(tx_1 + (1-t)x_2) = f_1(x_1) = f_2(x_2). \]

The iterated \( n \) times fiberwise join product of a map \( f : E \to B \) is denoted as \( \ast^n_B : \ast^n_B E \to B \).

2.6.1. Theorem \([\text{Sch}, \text{Theorem 3}]\). For a fibration \( f : E \to B \) the Schwarz genus \( g(f) \leq n \) if and only if \( \ast^n_B f : \ast^n_B \to E \) admits a section.

Also Schwarz proved the following \([\text{Sch}]\):

2.6.2. Theorem. Let \( \beta \) be the primary obstruction to a section of a fibration \( f : E \to B \). Then \( \beta^n \) is the primary obstruction to a section of \( \ast^n_B f \).

Let \( p^X : PX \to X \times X \) be the end points map: \( p^X(\phi) = (\phi(0), \phi(1)) \in X \times X \). Here \( PX \) is the space of all paths \( \phi : [0, 1] \to X \) in \( X \). Clearly, \( p^X \) is a Serre path fibration that represents the diagonal map \( \Delta : X \to X \times X \). Let \( P_0 X \subset PX \) be the space of paths issued from a base point \( x_0 \), and let \( \tilde{p}^X : P_0 X \to X \) be defined as \( \tilde{p}^X(\phi) = \phi(1) \). Then \( \tilde{p}^X \) is a fiber representation for the map \( x_0 \to X \).

For \( n > 0 \) we denote by \( p_n^X = \ast^{n+1}_{X \times X} p^X \) and \( \Delta_n(X) = \ast^{n+1}_{X \times X} PX \). Thus, elements of \( \Delta_n(X) \) can be viewed as formal linear combinations \( \sum_{i=0}^n t_i \phi_i \) where \( \phi_i : [0, 1] \to X \) with \( \phi_1(0) = \cdots = \phi_n(0), \phi_1(1) = \cdots = \phi_n(1), \) \( t_i \geq 0 \), and \( \sum t_i = 1 \).

Similarly we use the notation \( \tilde{p}_n^X = \ast^{n+1}_{X} \tilde{p}^X \) and \( G_n(X) = *^n_X P_0 X \). The fibration \( \tilde{p}_n^X \) is called the \( n \)-th Ganea fibration. For these fibrations Theorem 2.6.1 produces the following

2.6.3. Theorem. For a CW-space \( X \),

(1) \( \text{TC}(X) \leq n \) if and only if there exists a section of \( p_n^X : \Delta_n(X) \to X \times X \);

(2) \( \text{cat} X \leq n \) if and only if there exists a section of \( \tilde{p}_n^X : G_n(X) \to X \).
We note that the fiber of both fibrations $p^X$ and $\tilde{p}^X$ is the loop space $\Omega X$. The fiber $F_n = (p^X_n)^{-1}(x_0)$ of the fibration $p^X_n$ is the join product $\Omega X \ast \cdots \ast \Omega X$ of $n + 1$ copies of the loop space $\Omega X$ on $X$. So, $F_n$ is $(n - 1)$-connected.

A continuous map $f : X \to Y$ for any $n$ defines the commutative diagram

$$
\begin{array}{c}
\Delta_n(X) \\
p_n^* \\
\downarrow \\
X \times X \\
\downarrow f \times f \\
Y \times Y.
\end{array}
$$

2.6.4. Corollary. If $TC(X) \leq n$, then for any $f : X \to Y$ the map $f \times f$ admits a lift with respect to $p_n^X$.

2.7. Berstein-Schwarz cohomology class. The Berstein-Schwarz class of a discrete group $\pi$ is the first obstruction $\beta_\pi$ to a lift of $B\pi = K(\pi,1)$ to the universal covering $E\pi$. Note that $\beta_\pi \in H^1(\pi, I(\pi))$ where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$ [Bre, Sch].

2.7.1. Theorem (Universality [DR, Sch]). For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of $\pi$-modules $I(\pi)^k \to L$ such that the induced homomorphism for cohomology takes $(\beta_\pi)^k \in H^k(\pi; I(\pi)^k)$ to $\alpha$ where $I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi)$ and $(\beta_\pi)^k = \beta_\pi \sim \cdots \sim \beta_\pi$.

3. Computation of the LS-category of the cofiber

For $X = \mathbb{R}P^n$, $n > 1$, the equality $TC(X) = \text{cat}(C_{\Delta X})$ was established in [GV2]. Together with the computation $TC(\mathbb{R}P^2) = 3$ from [FTV] it gives the following

3.0.2. Theorem.

$$
\text{cat}(\mathbb{R}P^2 \times \mathbb{R}P^2 / \Delta \mathbb{R}P^2) = 3.
$$

The goal of this section is to extend this result to all non-orientable surfaces.

3.1. Kunneth formula for twisted coefficients. For sheaves $\mathcal{A}$ and $\mathcal{B}$ on $X$ and $Y$ we use the notation $\mathcal{A} \hat{\otimes} \mathcal{B}$ for $pr_X^* \mathcal{A} \otimes pr_Y^* \mathcal{B}$ where $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ are the projections. Similarly we use notation $\mathcal{A} \hat{\otimes} \mathcal{B}$ for the periodic product $pr_X^* \mathcal{A} \ast pr_Y^* \mathcal{B}$. We recall that in the case when $\mathcal{A} \hat{\otimes} \mathcal{B} = 0$ for locally nice spaces $X$ and $Y$ there is the Kunneth formula [Bre]

$$(*) \quad H_n(X \times Y; \mathcal{A} \hat{\otimes} \mathcal{B}) = \bigoplus_i H_i(X; \mathcal{A}) \otimes H_{n-i}(Y; \mathcal{B}) \oplus \bigoplus_i H_{i-1}(X; \mathcal{A}) \ast H_{n-i}(Y; \mathcal{B}).$$

Note that for the stalk at $(x, y) \in X \times Y$ we have $(\mathcal{A} \hat{\otimes} \mathcal{B})_{(x, y)} = \mathcal{A}_x \ast \mathcal{B}_y$. Thus, the Kunneth formula holds for local coefficients if one of the coefficient modules is torsion free as an abelian group.

3.2. Free abelian topological groups. Let $\mathbb{A}(N)$ denote the free abelian topological group generated by $N$ (see [M, G] or [DFI]). Let $j : N \to \mathbb{A}(N)$ be the natural inclusion. By the Dold-Thom theorem [DT] (see also [DFI]), $\pi_i(\mathbb{A}(N)) = H_i(N)$ and $j_* : \pi_i(N) \to \pi_i(\mathbb{A}(N))$ is the Hurewicz homomorphism. Therefore, $\mathbb{A}(\mathbb{R}P^2)$ is homotopy equivalent to $\mathbb{R}P^\infty$. Moreover, for a non-orientable surface $N$ of genus $g$ the space $\mathbb{A}(N)$ is homotopy equivalent to $\mathbb{R}P^\infty \times T^{g-1}$ where $T^m = S^1 \times \cdots \times S^1$ denotes the $m$-dimensional torus.

Let $\mathcal{O}$ be the twisted coefficient system on $\mathbb{A}(N)$ that comes from the canonical system $\mathcal{O}$ on $\mathbb{R}P^\infty$ as the pull-back under the projection $\mathbb{R}P^\infty \times T^{g-1} \to \mathbb{R}P^\infty$. 
3.2.1. **Proposition.** For any non-orientable surface $N$ 

$$H_2(A(N); \hat{O}) = \oplus\mathbb{Z}_2.$$  

**Proof.** We replace $A(N)$ by a homotopy equivalent space $T^{g-1} \times \mathbb{R}P^\infty$. Note that $\hat{O} = \mathbb{Z} \otimes O$ where $\mathbb{Z}$ is the trivial sheaf. Since $\mathbb{Z} \ast \mathbb{Z} = 0$ we obtain that $\mathbb{Z} \ast O = 0$ and we can apply the Künneth formula. Then by the Künneth formula for local coefficients,

$$H_2(A(N); \hat{O}) = H_0(T^{g-1}) \otimes H_2(\mathbb{R}P^\infty; O)$$  

$$\oplus H_1(T^{g-1}) \otimes H_1(\mathbb{R}P^\infty; O)$$  

$$\oplus H_2(T^{g-1}) \otimes H_0(\mathbb{R}P^\infty; O).$$  

The periodic product part of the Künneth formula is trivial since the first factor has torsion free homology groups. Thus, taking into account Proposition 2.3.1 we obtain

$$H_2(A(N); \hat{O}) = H_2(\mathbb{R}P^\infty; O) \oplus (\mathbb{Z}_2 \otimes H_2(T^{g-1})) = \mathbb{Z}_2 \oplus H_2(T^{g-1}; \mathbb{Z}_2) = \oplus\mathbb{Z}_2.$$

□

For a topological abelian group $A$ we denote by $\mu_A = \mu : A \times A \to A$ the continuous group homomorphism defined by the formula $\mu(a, b) = a - b$.

3.2.2. **Proposition.** Let $N = \mathbb{R}P^2$. Then the pull-back $(j \times j)^*\mu^*(O)$ is the $\mathbb{Z}$-orientation sheaf for the manifold $\mathbb{R}P^2 \times \mathbb{R}P^2$ where $\mu = \mu_{A(\mathbb{R}P^2)}$ and $O$ is the canonical $\mathbb{Z}$-bundle over $A(\mathbb{R}P^2) = \mathbb{R}P^\infty$.

We make a forward reference to Corollary 3.3.3 for the proof.

3.2.3. **Proposition.** Let $a \in H_2(\mathbb{R}P^\infty; O)$ be a generator. Then $\mu_*(a \otimes a) = 0$ where $\mu = \mu_{A(\mathbb{R}P^2)}$.

**Proof.** Note that $\pi_1((\mathbb{R}P^2 \times \mathbb{R}P^2)/\Delta\mathbb{R}P^2) = \mathbb{Z}_2$ (see Proposition 3.4.1). Let

$$f : (\mathbb{R}P^2 \times \mathbb{R}P^2)/\Delta\mathbb{R}P^2 \to A(\mathbb{R}P^2)$$

be a map that induces an isomorphism of the fundamental groups. We claim that the map $\mu \circ (j \times j)$ is homotopic to $f \circ q$ where $q : \mathbb{R}P^2 \times \mathbb{R}P^2 \to (\mathbb{R}P^2 \times \mathbb{R}P^2)/\Delta\mathbb{R}P^2$ is the quotient map. This holds true since both maps induce the same homomorphism of the fundamental groups. In view of Theorem 3.0.2 and Proposition 2.3.1 the map $f$ is homotopic to a map with the image in the 3-dimensional skeleton. Therefore, $f_*q_*(b \otimes b) = 0$ where $b$ is a generator of $H_2(\mathbb{R}P^2; O_{\mathbb{R}P^2}) = \mathbb{Z}$. Note that $j_*(b) = a$. Then $\mu_*(a \otimes a) = \mu_*(j \times j)_*(b \otimes b) = f_*q_*(b \otimes b) = 0$. □

Let $N = N_g = \#_g \mathbb{R}P^2$ be a non-orientable surface of genus $g$. Let $\pi = \pi_1(N)$ and $G = Ab(\pi) = H_1(N) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$. We recall that by the Dold-Thom theorem [DT], [Dr1] the space $A(N)$ is homotopy equivalent to $K(G, 1) \sim \mathbb{R}P^\infty \times T^{g-1}$.

3.2.4. **Proposition.** There is a homomorphism of topological abelian groups

$$h : A(N_g) \to A(\mathbb{R}P^2) \times T^{g-1}$$

which is a homotopy equivalence.
We consider two cases.

(1) If \( g \) is odd, then \( N_g = M_{(g-1)/2}\#\mathbb{R}P^2 \). We define \( f \) as the composition

\[
M_{(g-1)/2}\#\mathbb{R}P^2 \xrightarrow{q} M_{(g-1)/2} \vee \mathbb{R}P^2 \xrightarrow{\phi\vee j} T^{g-1} \vee \mathbb{A}(\mathbb{R}P^2) \xrightarrow{i} T^{g-1} \times \mathbb{A}(\mathbb{R}P^2)
\]

where \( q \) is collapsing of the separating circle in the connected sum, \( \phi \) is a map that induces isomorphism of 1-dimensional homology, and \( i \) is the inclusion. It is easy to check that \( f \) induces an isomorphism \( f_* : H_1(N_g) \to H_1(\mathbb{A}(\mathbb{R}P^2) \times T^{g-1}) \).

(2) If \( g \) is even, then \( N_g = M_{(g-2)/2}\#K \) where \( K \) is the Klein bottle. There is a homotopy equivalence \( s : \mathbb{A}(\mathbb{K}) \to S^1 \times \mathbb{A}(\mathbb{R}P^2) \). We define \( f \) as the composition

\[
M_{(g-2)/2}\#K \xrightarrow{q} M_{(g-1)/2} \vee K \xrightarrow{\phi\vee j} T^{g-2} \vee (S^1 \times \mathbb{A}(\mathbb{R}P^2)) \xrightarrow{i} T^{g-2} \times S^1 \times \mathbb{A}(\mathbb{R}P^2)
\]

where \( q \) is collapsing of the connecting circle, \( \phi \) is a map that induces isomorphism of 1-dimensional homology, and \( i \) is the inclusion. One can check that \( f \) induces an isomorphism \( f_* : H_1(N_g) \to H_1(\mathbb{A}(\mathbb{R}P^2) \times T^{g-1}) \). \( \square \)

3.2.5. **Proposition.** For abelian topological groups \( A \) and \( B \), \( \mu_{A\times B} = \mu_A \times \mu_B \).

**Proof.** For all \( a, a' \in A \) and \( b, b' \in B \) we have

\[
\mu_{A\times B}(a \times b, a' \times b') = (a - a') \times (b - b') = \mu_A(a) \times \mu_B(b) = (\mu_A \times \mu_B)(a \times b).
\]

\( \square \)

3.3. **Twisted fundamental class.** The pull-back of the canonical \( \mathbb{Z} \)-bundle \( \mathcal{O} \) over \( \mathbb{R}P^\infty \) under the projection \( \mathbb{R}P^\infty \times T^{g-1} \to \mathbb{R}P^\infty \) defines a local coefficient system \( \tilde{\mathcal{O}} \) on \( \mathbb{A}(N) \) with the fiber \( \mathbb{Z} \). On the \( G \)-module level the action of the fundamental group on \( \mathbb{Z} \) is generated by the projection homomorphism \( p : \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1} \to \mathbb{Z}_2 \). We note that \( \mathcal{O}_N = j^*(\tilde{\mathcal{O}}) \).

For sheafs \( \mathcal{A} \) and \( \mathcal{B} \) on \( X \) and \( Y \) we use the notation \( \mathcal{A} \hat{\otimes} \mathcal{B} \) for \( pr_X^*\mathcal{A} \otimes pr_Y^*\mathcal{B} \) where \( pr_X : X \times Y \to X \) and \( pr_Y : X \times Y \to Y \) are the projections. We note that if \( \mathcal{O}_X \) and \( \mathcal{O}_Y \) are the orientation sheaf on manifolds \( X \) and \( Y \), then \( \mathcal{O}_X \hat{\otimes} \mathcal{O}_Y \) is the orientation sheaf on \( X \times Y \).

3.3.1. **Proposition.** The cross product \([N]\mathcal{O}_N \times [N]\mathcal{O}_N \) is a fundamental class for \( N \times N \).

**Proof.** Note that for the orientation sheaf \( \mathcal{O}_{N \times N} \) on the manifold \( N \times N \) we have \( H_4(N \times N; \mathcal{O}_{N \times N}) = \mathbb{Z} \). We note that \( \mathcal{O}_N \hat{\otimes} \mathcal{O}_N = 0 \). Then the Kunneth formula implies

\[
\mathbb{Z} = H_4(N \times N; \mathcal{O}_{N \times N}) = H_4(N \times N; \mathcal{O}_N \hat{\otimes} \mathcal{O}_N) = H_2(N; \mathcal{O}_N) \otimes H_2(N; \mathcal{O}_N) = \mathbb{Z} \otimes \mathbb{Z}.
\]

Thus \([N\mathcal{O}_N] \otimes [N\mathcal{O}_N] \) is a generator in \( H_4(N \times N; \mathcal{O}_{N \times N}) \). \( \square \)

3.3.2. **Proposition.** For \( \mu = \mu_{\mathbb{A}(N)} \),

\[
\mu^* \tilde{\mathcal{O}} = \tilde{\mathcal{O}} \hat{\otimes} \tilde{\mathcal{O}}.
\]
Proposition. 3.3.5. where $\alpha$ is well defined.

By the representation $j$ is well defined.

Corollary. I

For a non-orientable surface $N$ the homomorphism $\mu(N)_*\colon H_4(N \times N; O_{N \times N}) \to H_4(\mathbb{A}(N); \hat{O})$ is well defined and $\mu(N)_*([N \times N]_{O_{N \times N}}) = 0$ where $[N \times N]_{O_{N \times N}} \in H_4(N \times N; O_{N \times N})$ is the fundamental class.

Proof. By Corollary 3.3.3 $O_{N \times N} = (j \times j)^*\mu^*(\hat{O})$ and, hence, the homomorphism $\mu_*(j \times j)_* : H_4(N \times N; O_{N \times N}) \to H_4(\mathbb{A}(N); \hat{O})$ is well defined.

As before, we replace $\mathbb{A}(N)$ by $\mathbb{A}(\mathbb{R}P^2) \times T^{g-1}$.

By Proposition 3.3.1 the cross product $[N]_{O_N} \times [N]_{O_{N}} \in H_4(N \times N; O_N \hat{\otimes} O_N)$ is a fundamental class: $[N \times N]_{O_{N \times N}} = \pm [N]_{O_N} \times [N]_{O_{N}}$.

Since $O_N = j^*\hat{O}$, the homomorphism $j_* : H_2(N; O_N) \to H_2(\mathbb{A}(N); \hat{O})$ is well defined. In view of the Kunneth formula (see formula (*) in subsection 3.1) we obtain $j_*([N]_{O_N}) = a \otimes 1_B + A \otimes b \in (H_2(A(\mathbb{R}P^2); O_N) \otimes \mathbb{Z}) \otimes (\mathbb{Z}_2 \otimes H_2(T^{g-1})) = H_2(A(N); \hat{O})$. 

Proof. Let $p : \mathbb{Z}_g \oplus \mathbb{Z}^{g-1} \to \mathbb{Z}_2$ be the projection. The sheaf on the left is defined by the representation $p\mu_* : \pi_1(\mathbb{A}(N) \times \mathbb{A}(N)) \to Aut(\mathbb{Z}) = \mathbb{Z}_2$. The sheaf on the right is defined by the representation $\alpha(p \times p) : \pi_1(\mathbb{A}(N)) \times \pi_1(\mathbb{A}(N)) \to \mathbb{Z}_2$

where $\alpha : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $\alpha(x, y) = x + y$, is the addition homomorphism. It is easy to check that these two coincide on the generating set $(\pi_1(\mathbb{A}(N)) \times 1) \cup (1 \times \pi_1(\mathbb{A}(N))) \subset \pi_1(\mathbb{A}(N) \times \mathbb{A}(N))$.

Hence, they coincide on $\pi_1(\mathbb{A}(N) \times \mathbb{A}(N))$. Therefore, these sheafs are equal. 

3.3.4. Proposition. Let $I : \mathbb{A}(N) \to \mathbb{A}(N)$, $I(x) = -x$, be the taking inverse map. Then $I$ fixes every local system $M$ on $\mathbb{A}(N)$ and defines the identity homomorphism in homology $I_* : H_*(\mathbb{A}(N); M) \to H_*(\mathbb{A}(N); M)$.

Proof. In view of Proposition 3.2.3 it suffices to prove it for $\mathbb{A}(\mathbb{R}P^2) \times T^k$. Note that the inverse homomorphism $I : \mathbb{A}(\mathbb{R}P^2) \times T^k \to \mathbb{A}(\mathbb{R}P^2) \times T^k$ is the product of the inverse homomorphisms $I^1$ and $I^2$ for $\mathbb{A}(\mathbb{R}P^2)$ and $T^k$ respectively. Also note that both $I^1 : \mathbb{A}(\mathbb{R}P^2) \to \mathbb{A}(\mathbb{R}P^2)$ and $I^2 : T^k \to T^k$ are homotopic to the identity. Thus, $I$ defines the identity automorphism of the fundamental group and, hence, fixes $M$. Then the homomorphism $I_* : H_*(\mathbb{A}(N); M) \to H_*(\mathbb{A}(N); M)$ is defined and $I_* = 1$. 

3.3.5. Proposition. For a non-orientable surface $N$ the homomorphism $\mu(N)_*\colon H_4(N \times N; O_{N \times N}) \to H_4(\mathbb{A}(N); \hat{O})$ is well defined and 

$\mu(N)_*([N \times N]_{O_{N \times N}}) = 0$ where $[N \times N]_{O_{N \times N}} \in H_4(N \times N; O_{N \times N})$ is the fundamental class.
with $a \in H_2(\mathbb{A}(\mathbb{R}P^2); \mathcal{O}) = H_2(\mathbb{R}P^\infty; \mathcal{O})$ being the generator, $b \in H_2(T^{9-1}; \mathbb{Z})$, and generators $1_A \in H_0(\mathbb{A}(\mathbb{R}P^2); \mathcal{O}) = \mathbb{Z}_2$ and $1_B \in H_0(T^{9-1}; \mathbb{Z}) = \mathbb{Z}$. Let $\bar{a} = a \otimes 1_B$ and $\bar{b} = 1_A \otimes b$.

We apply Proposition 3.2.5 with $\mu = \mu_\mathcal{H}(\mathbb{R}P^2) \times T^{9-1}$ and $[N] = [N]_{\mathcal{O}N}$ to obtain:

$$
\mu_* (j \times j)_* ([N] \times [N]) = \mu_* (j_* ([N]) \times j_* ([N]) = \mu_* (\bar{a} \times \bar{b} \times (\bar{a} + \bar{b}))
$$

$$
= \mu_* (\bar{a} \times \bar{a} + \bar{a} \times \bar{b} + \bar{b} \times \bar{a} + \bar{b} \times \bar{b}) = \mu_* (\bar{a} \times \bar{a} + \mu_* (\bar{a} \times \bar{b} + \bar{b} \times \bar{a}) + \mu_* (\bar{b} \times \bar{b})
$$

$$
= \mu_* (a \times a) \times 1_B + \mu_* (a \times 1_A) \times \mu_2 (1_B \times b) + \mu_1 (a \times a) \times \mu_2 (b \times 1_B) + 1_A \times \mu_2 (b)\times b)
$$

where $\mu^1 = \mu_\mathcal{H}(\mathbb{R}P^2)$ and $\mu^2 = \mu_{T^{9-1}}$. By Proposition 3.2.5 $\mu_*^1 (a \times a) \times 1_B = 0$.

We recall that a $\mathbb{Z}$-twisted homology class in a space $X$ with a local system $\rho : E \to X$ is defined by a cycle in $X$ with coefficients in the sections of $\rho$ on (singular) simplices in $X$. One can assume that the sections are taken in the $\pm 1$-subbundle of the $\mathbb{Z}$-bundle $\rho$. This implies that every homology class is represented by a continuous map $f : M \to X$ of a pseudo-manifold that admits a lift $f' : M \to E$ with value in the $\pm 1$-subbundle of $\rho$.

One can show that the homology class $a \in H_2(\mathbb{A}(\mathbb{R}P^2); \mathcal{O})$ is realized by a map $f : S^2 \to \mathbb{A}(\mathbb{R}P^2)$ that admits a lift to the $\pm 1$-subbundle of $\mathcal{O}$. The homology class $1_A \in H_0(\mathbb{A}(\mathbb{R}P^2); \mathcal{O})$ can be realized by the point representing the unit $0 \in \mathbb{A}(\mathbb{R}P^2)$. Then the class $\mu_*^1 (a \times 1_A)$ is realized by the same map

$$
f = \mu^1 (f, 0) : S^2 \to \mathbb{A}(\mathbb{R}P^2),$$

i.e., $\mu_*^1 (a \times 1_A) = a$, whereas the class $\mu_*^1 (1_A \times a)$ is realized by the composition $I \circ f$. By Proposition 3.3.3, $\mu_*^1 (1_A \times a) = I_* (a) = a$. Similarly, $\mu_*^2 (b \times 1_B) = b$, and $\mu_*^2 (1_B \times b) = I_* (b) = b$. Therefore, $\mu_* (\bar{a} \times \bar{b} + \bar{b} \times \bar{a}) = 2(a \times b)$ is divisible by 2.

Then by Proposition 3.2.4, $\mu_* (\bar{a} \times \bar{b} + \bar{b} \times \bar{a}) = 0$.

Next we show that $\mu_*^2 (b \times b) = 0$. In view of Proposition 3.2.4 it suffices to show that $\mu_*^2 (b \times b) = 0 \mod 2$. We recall that the Pontryagin product defines the structure of an exterior algebra on $H_*(T^r) = \Lambda [x_1, \ldots, x_r]$. Thus, $b = \sum_{i<j} \lambda_{ij} x_i \wedge x_j$. Then

$$
b \times b = \sum_{i<j, k<l} \lambda_{ij} \lambda_{kl} (x_i \wedge x_j) \times (x_k \wedge x_l).
$$

Let $\{i, j\} \cap \{k, l\} = \emptyset$. By the argument similar to the above using Propositions 3.2.5 and 3.3.4 we obtain that

$$
\mu_*^2 ((x_i \wedge x_j) \times (x_k \wedge x_l) + (x_k \wedge x_l) \times (x_i \wedge x_j))
$$

is divisible by 2.

If $|\{i, j, k, l\}| \leq 3$, then the problem can be reduced to a 3-torus. Then

$$
\mu_*^2 ((x_i \wedge x_j) \times (x_k \wedge x_l)) = 0
$$

by the dimensional reason.

3.4. Inessentiality of the cofiber.

3.4.1. Proposition. For a connected CW complex $X$ the fundamental group $G = \pi_1 (X \times X / \Delta X)$ is isomorphic to the abelianization of $\pi_1 (X)$ and the induced homomorphism $\pi_1 (X \times x_0) \to \pi_1 ((X \times X) / \Delta X)$ is surjective.
Proof. Let \( q : X \times X \to (X \times X)/\Delta X \) be the quotient map. Since \( q \) has connected point preimages, it induces an epimorphism of the fundamental groups. Suppose that \( g = q_*(a, b), g \in G, (a, b) \in \pi_1(X) \times \pi_1(X) = \pi_1(X \times X) \). Then
\[
g = q_*(a, b) = q_*(b, b)q_*(b^{-1}a, 1) = eq_*(b^{-1}a, 1)
\]
where \( e \in G \) is the unit. This proves the second part.

Let \( g, h \in G \). By the second part of the proposition, \( g = q_*(a, 1) \) and \( h = q_*(1, b) \). Since \((a, 1)(1, b) = (a, b) = (1, b)(a, 1), \) we obtain \( gh = hg \). Thus, \( G \) is abelian. We show that \( \pi_1(X \times x_0) \to \pi_1((X \times X)/\Delta X) \) is the abelianization homomorphism.

Note that the kernel of \( q_* \) is the normal closure of the diagonal subgroup \( \Delta \pi_1(X) \) in \( \pi_1(X) \times \pi_1(X) \). Thus every element
\[
(x, 1) \in K = ker\{(\pi_1(X \times x_0) \to \pi_1((X \times X)/\Delta X))\}
\]
can be presented as the product
\[
(x, 1) = (a_1^{y_1}, a_1^{z_1})(a_2^{y_2}, a_2^{z_2}) \cdots (a_n^{y_n}, a_n^{z_n})
\]
for \( a_i, y_i, z_i \in \pi_1(X) \) where \( a^g = gag^{-1} \). This equality implies that
\[
(a_1^{-1})^{z_1}(a_2^{-1})^{z_2} \cdots (a_n^{-1})^{z_n} = 1.
\]
Then \( x = (a_1^{-1})^{z_1} \cdots (a_n^{-1})^{z_n}a_1^{y_1} \cdots a_n^{y_n} \) lies in the kernel of the abelianization map. Therefore, \( K \subset [\pi_1(X), \pi_1(X)] \). \( \-box \)

3.4.2. Proposition. For any \( g \) the pseudo-manifold \((N_g \times N_g)/\Delta N_g\) is locally orientable and inessential.

Proof. We use the notation \( N = N_g \). To check the local orientability it suffices to show that \( H_4(W, \partial W) = \mathbb{Z} \) for a regular neighborhood of the diagonal \( \Delta N \) in \( N \times N \). Since \( H_4(W) = H_3(W) = 0 \), the exact sequence of pair implies \( H_4(W, \partial W) = H_3(\partial W) \). Note that the boundary \( \partial W \) is homeomorphic to the total space of the spherical bundle for the tangent bundle on \( N \). The spectral sequence for this spherical bundle implies that
\[
H_3(\partial W) = E^2_{2,1} = H_2(N; \underline{H_1(S^1)})
\]
where the local system \( \underline{H_1(S^1)} \) is the orientation sheaf on \( N \). Thus, we obtain \( H_3(\partial W) = \mathbb{Z} \).

Next, we show that the map \( \mu \circ (j \times j) \) is homotopic to \( f \circ q \) where \( q : N \times N \to (N \times N)/\Delta N \) is the quotient map and \( f \) is a map classifying the universal covering for \( (N \times N)/\Delta N \). Note that for the fundamental groups, \( ker(q_*) \) is the normal closure of the diagonal \( \Delta \pi \) in \( \pi \). Therefore, \((j \times j)_*(ker(q_*)) \subset \Delta(\text{Ab}(\pi)) = ker(\mu_*) \). Hence there is a homomorphism \( \phi : \text{Ab}(\pi) \to \text{Ab}(\pi) \) such that \( \mu_* \circ (j \times j)_* = \phi q_* \):
\[
\begin{array}{ccc}
\pi \times \pi & \xrightarrow{q} & \pi_1(N \times N/\Delta N) \\
\downarrow{(j \times j)_*} & & \downarrow{\phi} \\
\text{Ab}(\pi) \times \text{Ab}(\pi) & \xrightarrow{\mu_*} & \text{Ab}(\pi).
\end{array}
\]
By Proposition 3.4.1 \( \pi_1(N \times N/\Delta N) = \text{Ab}(\pi) = \mathbb{Z}g^{-1} \oplus \mathbb{Z}_2 \). Since \( \phi \) is surjective the homomorphism \( \phi \) is an isomorphism. The homomorphism \( \phi \) can be realized by a map \( f : (N \times N)/\Delta N \to \mathbb{A}(N) \). Since \( f \) induces an isomorphism of the fundamental groups \( f \) is a classifying map. Since the maps \( \mu \circ (j \times j) \) and \( f \circ q \) with
the target space $K(\text{Ab}(\pi), 1)$ induces isomorphisms of the fundamental groups, they are homotopic.

Finally, we note that the fundamental class of $(N \times N)/\Delta N$ is the image of that of $N \times N$. Then we apply Proposition 3.3.5 and Proposition 2.4.2. \hfill \Box

3.4.3. **Corollary.** $\text{cat}((N \times N)/\Delta N) \leq 3$.

**Proof.** We apply Proposition 2.3.1. \hfill \Box

3.4.4. **Theorem.** For a non-orientable surface of genus $g$,

$$\text{cat}((N_g \times N_g)/\Delta N_g) = 3.$$ 

**Proof.** We take $x \in H^1(N_g; \mathbb{Z}_2)$ such that $x^2 \neq 0$. Note that $(x \times 1 + 1 \times x)^2 = x^2 \times 1 + 1 \times x^2$ in $H^* (N_g \times N_g; \mathbb{Z}_2)$. Then

$$(x \times 1 + 1 \times x)^3 = (x^2 \times 1 + 1 \times x^2)(x \times 1 + 1 \times x) = x \times x^2 + x^2 \times x \neq 0.$$ 

The restriction of $x \times 1 + 1 \times x$ to the diagonal $\Delta N_g \subset N_g \times N_g$ equals

$$x \cup 1 + 1 \cup x = 2x$$

where $\cup$ is the cup product. Since $2x = 0$ in $H^*(N_g; \mathbb{Z}_2)$, we obtain $x \times 1 + 1 \times x = q^*(y)$ for some $y \in H^*(N_g \times N_g)/\Delta N_g; \mathbb{Z}_2)$. Therefore, $y^3 \neq 0$ in $H^*((N_g \times N_g)/\Delta N_g; \mathbb{Z}_2)$. By the cup-length estimate (Theorem 2.1.1),

$$\text{cat}((N_g \times N_g)/\Delta N_g) \geq 3.$$ 

This together with Corollary 3.4.3 implies the required equality. \hfill \Box

4. **Topological complexity of non-orientable surfaces**

Let $M$ be an orientable surface, $P = \mathbb{R}P^2$ be the projective plane, and let $q : M \vee P \to P$ denote the collapsing $M$ map. We denote by $O' = (q \times q)^* O_{P \times P}$ the pull-back of the orientation sheaf on $P \times P$.

4.0.5. **Proposition.** The 4-dimensional homology group of $(M \vee P)^2$ with coefficients in $O'$ equals

$$H_4((M \vee P)^2; O') = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$ 

Moreover, the inclusions of manifolds $\xi_i : W_i \to (M \vee P)^2$, $i = 1, \ldots, 4$ induce isomorphisms $H_4(W_i; \xi^* O') \to H_4((M \vee P)^2; O')$ onto the summands where $W_1 = M^2$, $W_2 = P^2$, $W_3 = M \times P$, and $W_4 = P \times M$.

**Proof.** We note that from the Mayer-Vietoris exact sequence for the decomposition $(M \vee P)^2 = A \cup B$ with $A = M^2 \cup P^2$ and $B = (M \times P) \cup (P \times M),

\cdots \to H_4(A; O'|_A) \oplus H_4(B; O'|_B) \xrightarrow{\psi} H_4((M \vee P)^2; O') \to H_3(M \vee M \vee P \vee P; O'|_\ldots)$

and dimensional reasons it follows that $\psi$ is an isomorphism. Note that the intersections $M^2 \cap P^2$ and $(M \times P) \cap (P \times M)$ in $(M \vee P)^2$ are singletons. Hence, $A = M^2 \vee P^2$ and $B = M \times P \vee P \times M$. Thus, $\psi$ defines the required isomorphism. \hfill \Box

4.0.6. **Corollary.**

$$H_4((M \vee P)^2; O') = H_4(M^2) \oplus H_4(P^2; O_{P}) \oplus H_4(M \times P; O_{M \times P}) \oplus H_4(P \times M; O_{P \times M}).$$ 

**Proof.** The proof is a verification that the restriction of $O'$ to each $W_i$, $i = 1, 2, 3, 4$, is the orientation sheaf. \hfill \Box
Let the map \( f : M \# P \to (M \# P)/S^1 = M \vee P \) be the collapsing of the connected sum circle. Note that the composition \( q \circ f \) with the above \( q \) takes the orientation sheaf \( \mathcal{O}_P \) to the orientation sheaf \( \mathcal{O}_{M \# P} \).

4.0.7. Proposition. \((f \times f)_*([([M \# P])^2]) = [M^2] + [P^2] + [M \times P] + [P \times M].\)

Proof. Let \( B \subset \xi(W_i) \) be a 4-ball. Then we claim that the homomorphism

\[
H_4((M \vee P)^2; \mathcal{O}') \to H^4((M \vee P)^2; (M \vee P)^2 \setminus \tilde{B}; \mathcal{O}') = H_4(B, \partial B) = \mathbb{Z}
\]

generated by the map of pairs and the excision is the projection of the direct sum \( H_4((M \vee P)^2; \mathcal{O}') = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) onto the \( i \)-th summand. This follows from the commutative diagram

\[
\begin{array}{ccc}
H_4((M \vee P)^2; \mathcal{O}') & \longrightarrow & H_4((M \vee P)^2; (M \vee P)^2 \setminus \tilde{B}; \mathcal{O}') \\
\uparrow \varepsilon_i & & \downarrow \\
H_4(W_i; \xi^* \mathcal{O}') & \longrightarrow & H_n(W_i, W_i \setminus \xi_i^{-1}(\tilde{B}); \xi^* \mathcal{O}') \\
\end{array}
\]

The commutative diagram

\[
\begin{array}{ccc}
H_4((M \# P)^2; \mathcal{O}_{(M \# P)^2}) & \longrightarrow & H_4((M \# P)^2; (M \# P)^2 \setminus \tilde{B}; \mathcal{O}_{(M \# P)^2}) \\
\downarrow \\
H_4((M \vee P)^2; \mathcal{O}') & \longrightarrow & H_4((M \vee P)^2; (M \vee P)^2 \setminus \tilde{B}; \mathcal{O}') \\
\end{array}
\]

shows that the projection of the image \( f_*([M \# P]) \) of the fundamental class onto the \( i \)-th summand, \( i = 1, 2, 3, 4 \), is a fundamental class.

The proof of the following proposition is straightforward.

4.0.8. Proposition. A retraction \( r : X \to A, A \subset X \), defines a fiberwise retraction \( \tilde{r} : (p^X)^{-1}(A) \to PA \). Moreover, for each \( k \) it defines a fiberwise retraction \( \tilde{r}_k : (p_k^X)^{-1}(A) \to \Delta_k(A) \) of the fiberwise joins:

\[
\begin{array}{ccc}
\Delta_k(X) & \longleftarrow & (p_k^X)^{-1}(A) \\
\downarrow p_k & & \downarrow p_k \\
X \times X & \longleftarrow & A \times A \\
\end{array}
\]

We denote

\[
g = (1 \vee j)^2 : (M \vee \mathbb{R}P^2)^2 \to (M \vee \mathbb{R}P^\infty)^2.
\]

It is easy to see that the sheaf \( \mathcal{O}' \) on \((M \vee \mathbb{R}P^\infty)^2\) is the pull-back under \( g \) of a sheaf \( \mathcal{O} \) on \((M \vee \mathbb{R}P^\infty)^2\) which comes from the pull-back of the tensor product \( \mathcal{O} \otimes \mathcal{O} \) of the canonical \( Z \)-bundles on \( \mathbb{R}P^\infty \).

4.0.9. Proposition. Let \( \kappa \in H^4((M \vee \mathbb{R}P^\infty)^2; \mathcal{F}) \) be the primary obstruction to a section of

\[
\tilde{p} = p_3^{M \vee \mathbb{R}P^\infty} : \Delta_3(M \vee \mathbb{R}P^\infty) \to (M \vee \mathbb{R}P^\infty)^2
\]

where \( M \) is an orientable surface of genus \( \geq 2 \). Then

1. \([M^2] \cap g^*(\kappa) \neq 0,
2. \([([\mathbb{R}P^2]^2] \cap g^*(\kappa) = 0, \) and
3. \(([M \times \mathbb{R}P^2] + [\mathbb{R}P^2 \times M]) \cap g^*(\kappa) = 0.\)
Proof. (1) Assume that $[M^2] \cap g^*(\kappa) = 0$. Then, $g_*([M^2]) \cap \kappa = 0$. This means that the map $\tilde{p}$ admits a section over $M^2 \subset (M \vee \mathbb{R}P^\infty)^2$. The collapsing $\mathbb{R}P^\infty$ to a point defines a retraction $r : M \vee \mathbb{R}P^\infty \to M$. By Proposition 4.0.8, the retraction $r$ defines a fiberwise retraction of $\tilde{p}^{-1}(M^2)$ onto $\Delta_3(M)$. This implies that $p_3^* : \Delta_3(M) \to M^2$ admits a section. Hence, by Theorem 2.6.3, $TC(M) \leq 3$. This contradicts to the fact that $TC(M) = 4$.

(2) Since $TC(\mathbb{R}P^2) = 3$, by Corollary 2.6.3, the map $g$ restricted to $(\mathbb{R}P^2)^2$ admits a lift with respect to $\bar{r}$. Hence the primary obstruction $\sigma'$ to such a lift is zero. Note that $\sigma' = (g^*\kappa)|_{(\mathbb{R}P^2)^2}$ is the restriction to $(\mathbb{R}P^2)^2$ of the image of $\kappa$ under $g^*$. Hence,

$$[\mathbb{R}P^2] \cap g^*(\kappa) = [\mathbb{R}P^2] \cap (g^*\kappa)|_{(\mathbb{R}P^2)^2} = 0.$$ 

(3) Let $\sigma : (M \vee \mathbb{R}P^\infty)^2 \to (M \vee \mathbb{R}P^\infty)^2$ be the natural involution: $\sigma(x, y) = (y, x)$. We may assume that the map $\sigma$ is cellular. It defines an involution $\bar{\sigma}$ on the path space $P(M \vee \mathbb{R}P^\infty)$ and involutions $\bar{\sigma}_k$ on the iterated fiberwise joins $\Delta_k(M \vee \mathbb{R}P^\infty)$.

Let $K = (M \vee \mathbb{R}P^\infty)^2$ be a $\sigma$-invariant CW complex structure with an invariant subcomplex $Q = (M \times \mathbb{R}P^\infty) \vee (\mathbb{R}P^\infty \times M)$. We claim that there is a section $s : K(3) \to \Delta_3(M \vee \mathbb{R}P^\infty)$ which is $\sigma$-equivariant on $Q(3)$. First we fix an invariant section at the wedge point

$$s(x_0, x_0) = c_{x_0} + 0 + 0 + 0 \in \Delta_3(M \vee \mathbb{R}P^\infty)$$

where $c_{x_0}$ is the constant path at $x_0$. Then we define our section $s$ on $Q(3)$ by induction on dimension of simplices. We note that $\sigma(e) \neq e$ for all cells in $Q$ except the wedge vertex. Assume that an equivariant section $s$ is defined on the $i$-skeleton $Q(i), i < 3$. Then for all distinct pairs of $i$-cells $e, \sigma(e)$ we do an extension of $s$ to $e$ and define it on $\sigma(e)$ to be $\bar{\sigma}_i s e$. Note that an extension of $s$ to $e$ exists since the fiber of $\tilde{p}$ is 2-connected. Also, in view of the 2-connectedness of the fiber the section $s$ on $Q(3)$ can be extended to $K(3)$.

Thus, we may assume that the restriction of the obstruction cocycle to $Q$ is symmetric. Hence, for the obstruction cohomology class we obtain $(\sigma^*\kappa)|_Q = \sigma_0^*(\kappa|_Q) = \kappa|_Q$ where $\sigma_0 = \sigma|_Q$.

Let

$$q_0 : (M \times \mathbb{R}P^\infty) \vee (\mathbb{R}P^\infty \times M) \to M \times \mathbb{R}P^\infty$$

be the projection to the orbit space of the $\sigma$-action, i.e., the folding map. Let $\kappa' = P^\infty \times M_{\sigma}$. Then $\kappa|_Q = q_0^*(\kappa')$. Note that $\bar{\sigma}$ restricted to $(M \times \mathbb{R}P^\infty) \vee (\mathbb{R}P^\infty \times M)$ equals $g_0^* \bar{\sigma} |_{M \times \mathbb{R}P^\infty}$. Hence the homomorphism in homology $(q_0)_*$ is well defined. Since $q_0$ induces an epimorphism of the fundamental groups and takes both classes $g_*[M \times \mathbb{R}P^2]$ and $g_*[\mathbb{R}P^2 \times M]$ to $g_*[M \times \mathbb{R}P^2]$, we obtain

$$(q_0)_*([g_*[M \times \mathbb{R}P^2] + g_*[\mathbb{R}P^2 \times M]] \cap \kappa) = 2g_*[M \times \mathbb{R}P^2] \cap \kappa' = 0.$$ 

The last equality follows from the fact that $[\mathbb{R}P^2]$ has order 2 in $\mathbb{R}P^\infty$ (see Proposition 3.2.1).

Since $q_0$ induces an epimorphism of the fundamental groups, we obtain

$$((g_*[M \times \mathbb{R}P^2] + g_*[\mathbb{R}P^2 \times M]) \cap \kappa = 0.$$ 

Therefore, $([M \times \mathbb{R}P^2] + [\mathbb{R}P^2 \times M]) \cap g^*(\kappa) = 0$. \qed
Proof. First we recall that

\begin{equation}
\text{Theorem.}
\end{equation}

be presented explicitly (in terms of \(\beta\) where \(\alpha\) if necessary). Indeed, by Schwarz’ Theorem 2.6.2, with respect to \(f\), \(\alpha\) is well defined. Note that \(g_\ast([M^2]+([RP^2]^2)+[M \times RP^2]+[RP^2 \times M]) \cap \kappa \neq 0\). Since \(g_\ast\) is an isomorphism in dimension 0, we derive the result from Corollary 4.0.10. \(\square\)

4.0.11. Corollary.

\[ g_\ast([M^2]+([RP^2]^2)+[M \times RP^2]+[RP^2 \times M]) \cap \kappa \neq 0. \]

Proof. First we consider the case when \(g\) is odd. Then \(N_g = M \# \# P^2\) for an orientable surface \(M\) of genus \(>1\). Let \(f : N_g = M \# \# P^2 \rightarrow M \vee P^2\) be a map that collapses the connected sum circle. Clearly, \(f\) induces an epimorphism of the fundamental groups. Note that the orientation sheaf \(\mathcal{O}_{N_g}\) is the pull-back \(\pi^\ast \mathcal{O}_{P^2}\) where \(\pi : M \vee P^2 \rightarrow P^2\) is the collapsing map.

We show that the map \(g \circ (f \times f) = (1 \vee j) f \times (1 \vee j) f\) does not admit a lift with respect to

\[ \bar{p} = p_3^{M \vee P\infty} : \Delta_3(M \vee P^\infty) \rightarrow (M \vee P^\infty)^2. \]

Then, by Corollary 2.6.2, we obtain the inequality \(TC(N_g) \geq 4\).

The primary obstruction \(\pi\) to such a lift is the image \((f \times f)^\ast g^\ast(\kappa)\) of the primary obstruction to a section. Note that by Proposition 4.0.7 and Corollary 4.0.11, \(g_\ast(f \times f)_\ast([N^2_g] \cap o) = g_\ast(f \times f)_\ast([N^2_g] \cap \kappa) = g_\ast([M^2]+[P^2]+[M \times P]+[P \times M]) \cap \kappa \neq 0\) where \(P = P^2\). Therefore, \([N^2_g] \cap o \neq 0\). By the Poincaré duality (with local coefficients) we obtain that \(o \neq 0\).

When \(g > 4\) is even, \(N_g = M \# \# P^2 \# \# P^2\) for an orientable surface \(M\) of genus \(>1\). We consider the map \(f : N_g \rightarrow M \vee P^2\) which is the composition of the quotient map \(N_g \rightarrow M \vee P^2 \vee P^2\) and union of the folding map \(P^2 \vee P^2 \rightarrow P^2\) and the identity map on \(M\). For such \(f\) the orientation sheaf on \(N_g\) can be pushed forward and the above argument works. \(\square\)

1. Remark. Implicitly our proof of the inequality \(TC(N_g) \geq 4\) is based on the zero-divisors cup-length estimate as in [F]. Indeed, by Schwarz’ Theorem 2.6.2 \(\kappa = \beta^4\) where \(\beta \in H^1((M \vee P^\infty)^2; \mathcal{F}_0)\) is the primary obstruction for the section of the fibration

\[ p_3^{M \vee P\infty} : P(M \vee P^\infty) \rightarrow (M \vee P^\infty)^2. \]

Therefore, the restriction of \(\beta\) to the diagonal equals zero. We have proved that \(\alpha^4 \neq 0\) for the element \(\alpha = (f \times f)^\ast g^\ast(\beta) \in H^1(N_g \times N_g; \mathcal{F}_0)\) which is trivial on the diagonal. The local coefficient system \(\mathcal{F}_0\) as well as a cocycle \(a\) representing \(\alpha\) can be presented explicitly (in terms of \(\pi_1(N_g)\)-modules and cross homomorphisms) as it was done in [C].

This is not surprising in view of Theorem 2.1, part (b) of Farber and Grant [FG].

2. Remark. The above technique can be pushed to get \(TC(N_4) = 4\) (see a version of this paper in ArXiv). The technique does not seem to be applicable to the Klein bottle nor to \(N_3 = P^2 \# P^2 \# P^2\).
5. Higman’s group

Higman introduced a group $H$ generated by 4 elements $a, b, c, d$ with the relations
\[a^{-1}ba = b^2, \quad b^{-1}cb = c^2, \quad c^{-1}dc = d^2, \quad d^{-1}ad = a^2.\]

Among others the group $H$ has the following properties: It is acyclic and it has finite 2-dimensional Eilenberg-Maclane complex $K(H, 1)$.

5.0.13. Theorem. Let $K = K(H, 1)$ where $H$ is Higman’s group. Then
\[2 = \text{cat}(C_{\Delta K}) < TC(K) = 4.\]

Proof. By Proposition 3.4.1 $\pi_1(C_{\Delta K}) = H_1(H) = 0$. Then by Theorem 2.1.1
\[\text{cat}(C_{\Delta K}) \leq (\dim C_{\Delta K})/2 = 2.\]

The equality $TC(K(H, 1)) = 4$ is a computation by Grant, Lupton and Oprea [GLO].

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