The weak converse of Zeckendorf’s theorem

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Abstract

By Zeckendorf’s Theorem, every positive integer is uniquely written as a sum of non-adjacent terms of the Fibonacci sequence, and its converse states that if a sequence in the positive integers has this property, it must be the Fibonacci sequence. If we instead consider the problem of finding a monotone sequence with such a property, we call it the weak converse of Zeckendorf’s theorem. In this paper, we first introduce a generalization of Zeckendorf conditions, and subsequently, Zeckendorf’s theorems and their weak converses for the general Zeckendorf conditions. We also extend the generalization and results to the real numbers in the interval (0, 1), and to \( p \)-adic integers.

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1 Introduction

Zeckendorf’s Theorem [18] states that each positive integer is expressed uniquely as a sum of distinct nonadjacent terms of the Fibonacci sequence \((1, 2, 3, 5, \ldots)\) where we reset \((F_1, F_2) = (1, 2)\). Similar to the binary expansion, each positive integer can be expressed as a sequence of 0 and 1 indicating whether the Fibonacci term is involved or not. For example, the natural number 100 corresponds to the Zeckendorf digits \((1000010100)_{\mathbb{Z}}\), meaning that \(F_{10} + F_{5} + F_{3} = 89 + 8 + 3 = 100\). Zeckendorf digits share the simplicity of representation with the binary expansion, but also they are quite curious in terms of the arithmetic operations, determining the 0th digits, the partitions in Fibonacci terms, and the minimal summand property of Zeckendorf expansions; see [4,8,11,13], and [17]. One of the most striking features of Zeckendorf’s Theorem is its converse and the questions it opens up.

**Theorem 1** (Daykin[5]) *If a sequence \(\{Q_k\}_{k=1}^{\infty}\) of positive integers uniquely expresses each positive integer as a sum of its distinct non-adjacent terms, then it is the Fibonacci sequence.*

This is called the converse of Zeckendorf’s Theorem, and we shall call the problem of finding monotone sequences rather than arbitrary sequences the weak converse of Zeckendorf’s Theorem. In this paper, we introduce:
1. a general approach to Zeckendorf conditions, generalizing the conditions introduced in [14];
2. Zeckendorf’s theorem for a general Zeckendorf condition, which includes cases of linear recurrences with negative coefficients;
3. results on their weak converses, not only for sequences in the positive integers, but also sequences in the real numbers and $p$-adic integers.

A general Zeckendorf condition shall be properly introduced in Section 2, and in this section let us introduce another example to help the reader be familiar with Zeckendorf conditions. The Nth order Fibonacci sequence $\{H_k\}_{k=1}^{\infty}$, whose name is coined in [9], is defined by $H_n = H_{n-1} + \cdots + H_{n-N}$ for all $n > N$ and $H_n = 2^{n-1}$ for all $1 \leq n \leq N$, and Zeckendorf’s Theorem for the Nth order Fibonacci sequence states that each positive integer is expressed uniquely as a sum of distinct terms of the Nth order Fibonacci sequence where no $N$ consecutive terms are used [2,14]. We may call the restriction of not allowing $N$ consecutive terms the Nth order Zeckendorf condition. The weak converse for this Zeckendorf condition can be stated as follows: The Nth order Fibonacci sequence is the only increasing sequence that represents $\mathbb{N}$ uniquely under the Nth order Zeckendorf condition, and a proof is found in [2].

Another interesting direction that the converse theorem opens up is investigating the unique existence of a sequence when a Zeckendorf condition and a set of numbers are given. We say that a set $X$ of numbers is represented by a sequence $\{Q_k\}_{k=1}^{\infty}$ uniquely under a Zeckendorf condition if each member of $X$ is uniquely expressed as a sum of terms of the sequence that satisfies the Zeckendorf condition, and each sum of terms of the sequence that satisfies the Zeckendorf condition is a member of $X$. For example, we may ask whether the set of positive odd integers can be represented by an increasing sequence under the second order Zeckendorf condition, and if so, whether such a sequence uniquely exists, which is the weak converse for the positive odd integers under the second order Zeckendorf condition. Let us introduce another representative example of this direction of research. Let $X$ be the open interval $(0, 1)$ of real numbers, and ask ourselves whether the interval can be represented by a decreasing sequence of positive real numbers uniquely under the second order Zeckendorf condition, and if so, does the weak converse for the interval under the second order Zeckendorf condition hold? We shall provide answers to these two questions in Sect. 2 along with our main results which are presented in a more general setting.

The remainder of the paper is organized as follows. In Sect. 2.1, general definitions of Zeckendorf conditions are introduced along with results on its formulation in terms of blocks. Introduced in Sect. 2.2 are main results on Zeckendorf’s Theorem and their weak converses for sets of numbers in $\mathbb{N}$, the interval $(0, 1)$ of real numbers, and $p$-adic integers in $\mathbb{Z}_p$. Examples are instrumental for properly understanding the general concepts of Zeckendorf conditions, and they are briefly introduced in Sect. 2. However, it is necessary to discuss more examples that are interesting, in order to present the full extent of the definition, and they are introduced in Sects. 3 and 4. The main results introduced in Sects. 2.1 and 2.2 are proved in Sect. 5.
2 Definitions and results

2.1 Definitions

In this paper, $R$ will denote one of the following sets of numbers: the natural numbers $\mathbb{N}$, the open interval $I := (0, 1)$ of real numbers, and the $p$-adic integers $\mathbb{Z}_p$. A sequence is usually denoted by $\{a_n\}_{n \geq 1}$. In this paper, a sequence of numbers in $R$ is identified with a list of numbers in the infinite product $\prod_{k=1}^{\infty} R$. We usually denote them by capital letters such as $Q$, and their terms are denoted by $Q_k$ for $k = 1, 2, 3, \ldots$. For example, if $Q_k = k$ for $k \geq 1$, then $Q = (1, 2, 3, \ldots)$. Given a function $\epsilon : \mathbb{N} \to \mathbb{R} \cup \{0\}$ and a sequence $Q$, we denote $\epsilon(k)$ by $\epsilon_k$, and define $\sum Q \epsilon$ to be the formal sum $\sum_{k=1}^{\infty} \epsilon_k Q_k$. In this context, $\epsilon$ is called a coefficient function. We also use the list notation to present the values of $\epsilon$, i.e., $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$, and the bar notation $\bar{\epsilon}$ denotes the repeating entries, e.g., $\epsilon = (1, 2, 3, \bar{0})$ meaning that $\epsilon_k = 0$ for all $k > 3$. If there is an index $M$ such that $\epsilon_k = 0$ for all $k > M$, $\epsilon$ is said to have a finite support, and we say, a coefficient function $\epsilon$ is supported on a subset of indices $A$ if $\epsilon_k = 0$ for all $k \notin A$. Note here that given a coefficient function $\epsilon$, such an index subset $A$ is not uniquely determined, and it is a subset we assign to a coefficient function. Let $\beta^i$ be the coefficient function such that $\beta^i_k = 0$ for all $k \neq i$ and $\beta^i_i = 1$, and call it the $i$th basis coefficient function.

If a subset of indices $A$ consists of consecutive indices $\{a, a + 1, \ldots, a + n\}$, we call it an interval of indices. Given an interval $I$ of indices and a coefficient function $\delta$, let both $\delta \mid J$ and $\delta \mid J$ denote $\sum_{k \in J} \delta_k \beta_k^i$, i.e., the restriction of $\delta$ on the indices in $J$. For example, $\delta \mid [a, b) = k = a \delta_k \beta_k^i$. For convenience, let us denote $\delta \mid [1, M]$ by $\delta \equiv M \mu \mid [a, b)$ and the relationship $\delta \equiv M \mu \mid [a, b)$ for all $M$. Note here that given a coefficient function $\epsilon$, such an index subset $A$ is not uniquely determined, and it is a subset we assign to a coefficient function. Let $\beta^i$ be the coefficient function such that $\beta^i_k = 0$ for all $k \neq i$ and $\beta^i_i = 1$, and call it the $i$th basis coefficient function.

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Let us consider lexicographical orders on the set of coefficient functions. Given two coefficient functions $\epsilon$ and $\epsilon'$, we define the descending lexicographical order as follows. If there is a smallest positive integer $k$ such that $\epsilon_j = \epsilon'_j$ for all $j < k$ and $\epsilon_k < \epsilon'_k$, then we denote the property by $\epsilon <_d \epsilon'$. For example, if $\epsilon = (1, 2, 10, 5, \ldots)$ and $\epsilon' = (1, 3, 1, 10, \ldots)$, then $\epsilon <_d \epsilon'$ since $\epsilon_2 < \epsilon'_2$ and $\epsilon_4 = \epsilon'_4$. Let us point out that the lexicographical order is defined on the set of coefficient functions, and it does not mean that the values of a coefficient function in the set form a decreasing sequence. For the representation of the real numbers in the open interval $I$, we shall use the descending lexicographical order on the set of coefficient functions. Given two coefficient functions $\mu$ and $\mu'$ with finite support, we define the ascending lexicographical order as follows. If there is a largest positive integer $k$ such that $\mu_j = \mu'_j$ for all $j > k$ and $\mu_k < \mu'_k$, then we denote the property by $\mu <_a \mu'$. For example, if $\mu = (1, 2, 10, 3, 7)$ and $\mu' = (1, 3, 1, 4, 7)$, then $\mu <_a \mu'$ since $\mu_4 < \mu'_4$ and $\mu_5 = \mu'_5$. As in the earlier case, it does not mean that the values of a coefficient function in the set form an increasing sequence. For the representation of the positive integers we shall use the ascending lexicographical order on the set of coefficient functions with finite support.

Given a set of numbers $R$ listed above, we define a collection $\mathcal{E}$ of coefficient functions under a lexicographical order to be a set of coefficient functions ordered by the same lexicographical order that contains the zero coefficient function and all basis coefficient functions $\beta^i$. We call the set an ascendingly-ordered collection of coefficient functions if it is under the ascending lexicographical order, and a descendingly-ordered collection of coefficient functions if it is under the descending lexicographical order. A member of $\mathcal{E}$ is called an $\mathcal{E}$-coefficient function, and a coefficient function is said to satisfy the $\mathcal{E}$-condition.
if it is a member of $\mathcal{E}$. For example, if $\mathcal{E}$ is the collection of coefficient functions $\mu$ with finite support such that $\mu_k$ is either 0 or 1 for all $k \geq 1$ and the list $\mu$ does not have two consecutive entries of 1, then $\mathcal{E}$ is the classical Zeckendorf condition used for writing positive integers as a sum of Fibonacci terms.

Let $\mathcal{E}$ be a collection of coefficient functions under the ascending or descending lexicographical order. Let $\delta$ be a coefficient function in $\mathcal{E}$, and we introduce the following terminology with respect to its lexicographical order. The smallest coefficient function in $\mathcal{E}$ that is greater than $\delta$, if (uniquely) exists, is called the immediate successor of $\delta$ in $\mathcal{E}$, and we denote it by $\hat{\delta}$. The largest coefficient function in $\mathcal{E}$ that is less than $\delta$, if (uniquely) exists, is called the immediate predecessor of $\delta$ in $\mathcal{E}$, and we denote it by $\tilde{\delta}$.

Let us introduce an order notation that will be instrumental throughout the paper, and it is intended to reflect the magnitude of a number expressed in terms of coefficient functions. Let $\epsilon$ be a non-zero coefficient function of a collection under the descending lexicographical order, which will be used for the real numbers in $\mathbb{R}$. The smallest index $n$ such that $\epsilon_n \neq 0$ is called the order of $\epsilon$, denoted by $\text{ord}(\epsilon)$. If $\epsilon = 0$, then we define $\text{ord}(\epsilon) = \infty$. For a non-zero function $\mu$ of a collection under the ascending lexicographical order that has finite support, the largest index $n$ such that $\mu_n \neq 0$ is called the order of $\mu$, denoted by $\text{ord}(\mu)$. If $\mu = 0$, we define $\text{ord}(\mu) = 0$.

Let us further introduce the notion of Zeckendorf collections of coefficient functions under the ascending lexicographical order, which will be used for positive integers. By definition, a collection of coefficient functions contains all the basis coefficient functions, i.e., $\beta^{n-1} \in \mathcal{E}$ for $n \geq 2$, and hence, the immediate predecessor $\beta^n$, if exists, has a non-zero value at index $n - 1$, i.e., $\text{ord}(\beta^n) = n - 1$.

**Definition 2** Let $\mathcal{E}$ be an ascendingly-ordered collection of coefficient functions with finite support. The collection is called Zeckendorf for positive integers if it satisfies the following:

1. For each $\mu \in \mathcal{E}$ there are at most finitely many coefficient functions that are less than $\mu$.
2. Given $\mu \in \mathcal{E}$, if its immediate successor $\hat{\mu}$ is not $\beta^1 + \mu$, then there is an index $n \geq 2$ such that $\mu \equiv \hat{\beta}^n \text{ res}[1, n]$ and $\hat{\mu} = \beta^n + \text{res}_{[n, \infty]}(\mu)$.

Definition 2, Part 2 says that each coefficient function of a Zeckendorf collection $\mathcal{E}$ for positive integers has a (unique) immediate successor in $\mathcal{E}$, and Definition 2, Part 1 implies that it has a (unique) immediate predecessor as well. Let us use the following lemma to explain this property.

**Lemma 3** Let $\mathcal{E}$ be a Zeckendorf collection for positive integers. Let $\tau^0$ be the zero coefficient function, and let $\tau^{n+1}$ be the immediate successor of $\tau^n$ in $\mathcal{E}$ for each $n \geq 0$. Then, $\tau^n$ is the immediate predecessor of $\tau^{n+1}$ in $\mathcal{E}$ for each $n \geq 0$, and $\{\tau^n : n \geq 0\} = \mathcal{E}$.

**Proof** Let $\tau^0$ be the zero coefficient function, and let $\tau^{n+1}$ be the immediate successor of $\tau^n$ for each $n \geq 0$. Suppose that there are $\delta \in \mathcal{E}$ and an integer $n \geq 0$ such that $\tau^n <_a \delta <_a \tau^{n+1}$. Then, it contradicts that $\tau^{n+1}$ is the immediate successor of $\tau^n$. Hence, this proves that $\tau^n$ is the immediate predecessor of $\tau^{n+1}$.

Notice that $S := \{\tau^n : n \geq 0\}$ is a subset of $\mathcal{E}$. Let $\mu$ be a coefficient function in $\mathcal{E}$ that is greater than $\tau^0$, and let us show $\mu \in S$. The subset $T := \{\alpha \in \mathcal{E} : \alpha \preceq_a \mu\}$ is a finite set by
Definition 2, Part 1. By the definition of the ascending lexicographical order, \( \tau^0 \in T \), and there is a largest element \( \tau^m \) of the nonempty finite subset \( S \cap T \). It suffices to show that \( \tau^m \) is the largest element of \( T \), which is true. Suppose that \( \tau^m \) is the largest element of \( T \). The collection \( \mathcal{E} \) is totally ordered under the ascending lexicographical order, i.e., either \( \tau^m \leq \gamma \) if \( \tau^m \leq \gamma \) is true. If \( \tau^m \leq \gamma \), then \( \tau^m \leq \gamma \leq \mu \), which contradicts the choice of \( \tau^m \). If \( \tau^m \leq \gamma \), then it contradicts that \( \tau^m \) is the immediate successor of \( \tau^m \). The implications of the above two cases contradict the existence of \( \gamma \in \mathcal{E} \) under the lexicographical order, and hence, we prove that \( \tau^m = \mu \in S \).

Before we introduce examples, let us extend the definition to \( p \)-adic integers. A coefficient function \( \mu \) is called the limit of a sequence of coefficient functions \( \mu^k \) with finite support for \( k \geq 1 \) if there is an increasing sequence of indices \( M_k \geq \text{ord}(\mu^k) \) for \( k \geq 1 \) such that \( \mu \equiv \mu^k \pmod{M_k} \). A collection of coefficient functions \( \mathcal{E} \) is called a Zeckendorf collection for \( p \)-adic integers if it has a Zeckendorf sub-collection \( \mathcal{E}_0 \) for positive integers such that \( \mathcal{E} \) is the set of coefficient functions that are the limits of sequences of coefficient functions in \( \mathcal{E}_0 \), and \( \mathcal{E} \) is also called the completion of \( \mathcal{E}_0 \).

Example 4 Let \( \mathcal{E}_0 \) be the ascendingly-ordered collection of coefficient functions \( \mu \) with finite support such that \( \mu_k \leq k \) and \( \mu_k = k + 1 \) implies \( \mu_k = 0 \) for all \( k \geq 1 \). Then, \( \mathcal{E}_0 \) is Zeckendorf for positive integers. For example, \( \hat{\beta}^7 = (0, 2, 0, 4, 0, 6, 0) \), and the immediate successor of \( \hat{\beta}^7 + 3\hat{\beta}^7 + 2\hat{\beta}^8 \) is \( 4\hat{\beta}^7 + 2\hat{\beta}^8 \). If \( \mathcal{E} \) is the completion of \( \mathcal{E}_0 \), then \( \sum_{k=1}^{\infty}(1 + 2k)\beta^{1+2k} \) and \( \sum_{k=1}^{\infty}(3k)\beta^{3k} \) are examples of coefficient functions in \( \mathcal{E} \).

Example 5 Let \( B = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : 0 \leq a_k \leq 1 \text{ for all } k \} \) \( - \{(1, 1, 0)\} \), and let \( \mathcal{E} \) be the ascendingly-ordered collection of coefficient functions \( \epsilon \) with finite support generated by concatenating some blocks in \( B \). Then, \( \mathcal{E} \) is a Zeckendorf collection for positive integers, whose immediate predecessors are given by \( \hat{\beta}^n = \sum_{k=1}^{n-1}\beta^k = (1, 1, \ldots, 1, 0) \) for \( n \neq 0 \) \( \pmod{3} \), and \( \hat{\beta}^3n = \sum_{k=1}^{3(n-1)}\beta^k + \hat{\beta}^{3n-1} = (1, \ldots, 1, 0, 1, 0) \) for \( n \geq 1 \).

Definition 2 accomplishes a concise description of Zeckendorf conditions in terms of properties the collection must satisfy, and when it is easy to determine the immediate successor of each member as in Example 4, it is useful for determining whether a collection is Zeckendorf or not. However, it turns out that a Zeckendorf collection is completely determined by the subset \( \{\hat{\beta}^n : n \geq 2\} \), and it is not so simple to see this fact, i.e., to find the immediate successor of \( \delta \) when a coefficient function \( \delta \) and \( \hat{\beta}^n \) for \( n \geq 2 \) are given. Theorem 7 and Corollary 9 will make this clear, and the proof will be given in Sect. 5. Theorem 7 and Corollary 9 will also show that Definition 2 generalizes the definition introduced in [14, Definition 1.1].

Definition 6 Let \( \delta^n : n \geq 2 \) be a set of coefficient functions under the ascending lexicographical order such that \( \text{ord}(\delta^n) = n - 1 \) for all \( n \geq 2 \). A coefficient function \( \zeta \) is called a proper \( \delta \)-block at index \( n \) if there is an index \( 1 \leq i \leq n \) such that \( 0 \leq \zeta_i < \delta_i^{n+1} \), \( \zeta \equiv \delta_i^{n+1} \pmod{\delta_i^n} \), and \( \zeta_j = 0 \) for all \( s \neq i \). We call the interval of indices \( [i, n] =: [i, \ldots, n] \) the support of a proper \( \delta \)-block at index \( n \). The coefficient function \( \delta^n \) is called the maximal \( \delta \)-block at index \( n - 1 \), and we call the interval of indices \( [1, n] \) the support of \( \delta^n \).
Note that the zero coefficient function is declared to be a proper $\delta$-block at index $n$ for any integer $n \geq 1$, for which the support is $[n, n]$, while the support of a nonzero proper $\delta$-block is uniquely determined. The basis coefficient function $\beta^n$ is a simple example of nonzero $\delta$-blocks at index $n \geq 1$, which may or may not be proper. Consider the collection $\mathcal{E}_0$ defined in Example 4. Then, $\zeta = (0, 0, 0, 0, 5)$ is an example of proper $\beta$-blocks, and its support is $[3, 5]$ since $i = 3$ is the index such that $\zeta_i < \beta^n_4 = 3$ and $\zeta_6 = \beta^n_5$ for $k = 4, 5$.

Our main interest for positive integers is an ascendingly-ordered collection $\mathcal{E}$ of coefficient functions with finite support such that the immediate predecessors $\hat{\beta}^n$ exist for each $n \geq 1$, and in general, a $\beta$-block $\zeta$ at index $n$ is not required to be a member of $\mathcal{E}$. If $\mathcal{E}$ is Zeckendorf, then by Theorem 7, Part 2, a proper $\beta$-block is a member of $\mathcal{E}$.

**Theorem 7** Let $\mathcal{E}$ be an ascendingly-ordered collection of coefficient functions with finite support such that the immediate predecessors $\hat{\beta}^n$ exist for each $n \geq 2$.

1. The collection $\mathcal{E}$ is Zeckendorf if and only if all of the following are satisfied:

   (a) For each $\mu \in \mathcal{E}$, there are a positive integer $M$, a unique $\hat{\beta}$-block $\hat{\gamma}^1$ with support $[i_1, n_1]$, and unique proper $\beta$-blocks $\zeta^m$ with support $[i_m, n_m]$ for $2 \leq m \leq M$ (if $M \geq 2$) such that $i_1 = 1$, $n_1 + 1 = i_{m+1}$ for all $1 \leq m \leq M - 1$, and $\mu = \sum_{m=1}^{M} \zeta^m$. We call the expression the $\hat{\beta}$-block decomposition.

   (b) Given $\mu \in \mathcal{E}$, if $\mu = \sum_{m=1}^{M} \zeta^m$ is the $\hat{\beta}$-block decomposition, then $\mu = \beta^1 + \mu$ if $\hat{\gamma}$ is not maximal, and $\mu = \beta^n + \sum_{m=2}^{M} \zeta^m$ if $\hat{\gamma}$ is $\beta^n$ for some $n \geq 2$.

2. Let $\mathcal{E}$ be Zeckendorf, and let $\zeta^m$ for $m = 1, \ldots, M$ be a sequence of $\hat{\beta}$-blocks with disjoint supports $[i_m, n_m]$ such that $\zeta^m$ are proper for $m \geq 2$ (if $M \geq 2$). Then, the coefficient function $\sum_{m=1}^{M} \zeta^m$ is a member of $\mathcal{E}$.

If $\mathcal{E}$ is Zeckendorf, then for convenience we may write the $\hat{\beta}$-block decomposition as $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ where all sufficiently large $\hat{\beta}$-blocks are zero coefficient functions. Let us introduce some examples. The collection $\mathcal{E}$ for the classical Zeckendorf condition is a Zeckendorf collection, e.g., $\mu = (0, 1, 0, 1, 0, 0, 1, 0)$ is decomposed into non-zero $\hat{\beta}$-blocks $(0, 1, 0, 1, 0) + \hat{\beta}^3$ where $\zeta^1 = (0, 1, 0, 1, 0)$ is the maximal $\hat{\beta}$-block at index 4, and $\mu = \beta^3 + \beta^7$. The coefficient functions $\epsilon := \sum_{k=1}^{\infty} \beta^{n+2k}$ for $n \geq 1$ are members of the completion $\hat{\mathcal{E}}$. For the completion of $\mathcal{E}$, an infinite sum of proper $\hat{\beta}$-blocks with disjoint supports is a member of $\mathcal{E}$, but some members of $\mathcal{E}$ such as $\epsilon$ defined above may not be written as an infinite sum of proper $\hat{\beta}$-blocks with disjoint supports.

Another important application of Theorem 7 is constructing a Zeckendorf collection with the immediate predecessors $\hat{\delta}^n$.

**Definition 8** Let $\{\delta^n : n \geq 2\}$ be a set of coefficient functions under the ascending lexicographical order such that $\text{ord}(\delta^n) = n - 1$ for all $n \geq 2$. The ascendingly-ordered collection $\mathcal{E}$ of coefficient functions determined by $\{\delta^n : n \geq 2\}$ is defined to be the ascendingly-ordered collection $\mathcal{E}$ of coefficient functions consisting of $\delta$-block decompositions $\sum_{m=1}^{M} \zeta^m$ where $M$ is a positive integer.

**Corollary 9** If $\mathcal{E}$ is the ascendingly-ordered collection of coefficient functions determined by coefficient functions $\delta^n$ of order $n - 1$ for $n \geq 2$, then $\mathcal{E}$ is Zeckendorf, and $\hat{\beta}^n = \delta^n$ for all $n \geq 2$. Moreover, $\mathcal{E}$ is the only Zeckendorf collection for positive integers such that $\hat{\beta}^n = \delta^n$ for each $n \geq 2$. 
By Corollary 9, given a set of coefficient functions $\delta^n$ of order $n - 1$ for $n \geq 2$, we may define the Zeckendorf collection for positive integers with immediate predecessors $\hat{\delta}^n = \delta^n$ for $n \geq 2$ to be the collection defined in Definition 8.

Let us introduce Zeckendorf collections for the unit interval $I$.

**Definition 10** Let $\mathcal{E}$ be a descendingly-ordered collection of coefficient functions, and given an index $M \geq 1$, let $\mathcal{E}^M$ denote the collection consisting of $\epsilon$ res $M$ for $\epsilon \in \mathcal{E}$. The collection $\mathcal{E}$ is called Zeckendorf for the open interval $I$ if the following are satisfied:

1. For each $\mu \in \mathcal{E}^M$ there are at most finitely many coefficient functions in $\mathcal{E}^M$ that are less than $\mu$.
2. Given an index $n \geq 1$, there is a unique coefficient function $\hat{\beta}^n$ of order $n$ with infinite support, not necessarily a member of $\mathcal{E}$, such that $\hat{\beta}^n$ res $M$ is the immediate predecessor of $\beta^{n-1}$ res $M$ in $\mathcal{E}^M$ for all $M \geq n$ if $n \geq 2$, and it is maximal in $\mathcal{E}^M$ for all $M \geq n$ if $n = 1$. The coefficient functions $\hat{\beta}^n$ are called the maximal coefficient function of order $n$ for $\mathcal{E}$.
3. Given $\mu \in \mathcal{E}^M$ that is less than $\hat{\beta}^1$ res $M$, if its immediate successor $\hat{\mu}$ in $\mathcal{E}^M$ is not $\mu + \beta^M$, then there is an index $1 \leq n < M$ such that $\hat{\beta}^{n+1} = \mu$ res $(n,M)$ and $\hat{\mu} = \text{res}_{[1,M]}(\mu) + \beta^n$.
4. Let $\epsilon$ be a coefficient function in $\mathcal{E}$. Then, $\epsilon \in \mathcal{E}$ if and only if there are infinitely many indices $M \geq 1$ such that $\epsilon^M := \epsilon$ res $M$ is a member of $\mathcal{E}^M$ and the immediate successor of $\epsilon^M$ in $\mathcal{E}^M$ is given by $\epsilon + \beta^M$ res $M$.

**Example 11** Let $\mathcal{E}$ be the descendingly-ordered collection of coefficient functions $\mu$ such that $\mu_k \leq k$ and $\mu_k = k$ implies $\mu_{k+1} = 0$ for all $k \geq 1$. Then, $\mathcal{E}$ is Zeckendorf for $I$. The coefficient function $\tilde{\beta}^3 = (0, 0, 3, 0, 5, 0, 7, 0, \cdots)$ is an example of maximal coefficient functions for $\mathcal{E}$, and it is not a member of $\mathcal{E}$. The immediate successor of $\text{res}_{[1,6]}(\beta^2 + \tilde{\beta}^3) = (0, 1, 3, 0, 5, 0)$ in $\mathcal{E}^6$ is $2\beta^2$.

Notice that since $\hat{\beta}^n \in \mathcal{E}^M$ for all $M \geq n$, the property $\text{res}_{[1,M]}(\hat{\beta}^n) \prec_{d} \text{res}_{[1,M]}(\hat{\beta}^n)$ implies that $\hat{\beta}_n^1 \geq 1$, i.e., ord($\hat{\beta}^n$) = $n$. Also notice that $\hat{\beta}^n$ does not satisfy the existence of infinitely many indices $M$ described in Definition 10, Part 4, and hence, it is not a member of the Zeckendorf collection $\mathcal{E}$. This condition is motivated from the situation where we have two representations in the binary expansions, $1/2 = 1/2^2 + 1/2^3 + 1/2^4 + \cdots$. However, the fact that a coefficient function $\mu$ terminates with a maximal coefficient function of order $n$ does not imply that $\mu \notin \mathcal{E}$. For example, $\mu := \beta^1 + \tilde{\beta}^3$ may or may not be members of $\mathcal{E}$, and we shall explain this properly after Corollary 15. As in the case of Definition 2, a Zeckendorf collection for $I$ is completely determined by the maximal coefficient functions of order $n$ for $n \geq 1$, and it is proved by Theorem 13 and Corollary 15.

**Definition 12** Let $\{\delta^n : n \geq 1\}$ be a set of coefficient functions with infinite support under the descending lexicographical order such that ord$(\delta^n) = n$ for each $n \geq 1$. A coefficient function $\xi$ is called a proper $\delta$-block at index $n$ if there is an index $i \geq n$ such that $0 \leq \xi_i < \delta_i^n$, $\xi = \delta^n \text{ res } [n,i)$, and $\xi_s = 0$ for all $s \notin [n,i]$, and the interval of indices $[n,i]$ is called the support of a proper $\delta$-block at index $n$.
The zero coefficient function is declared to be a proper $\delta$-block at any index $n$ with the support $[n, n]$, and the basis coefficient function $\beta^n$ is a proper $\delta$-block at index $n$ for any integer $n \geq 1$ since $\delta^n$ has infinite support. Let $\mathcal{E}$ be the descendingly-ordered collection defined in Example 11, and let $\mu = (0, 0, 3, 0, 5, 0, 0, 0)$. Then, $\mu \in \mathcal{E}$, and it is a proper $\beta$-block at index 3 with support $[3, 7]$.

Our main interest is a descendingly-ordered collection $\mathcal{E}$ of coefficient functions such that the coefficient function $\beta^n$ defined in Definition 10, Part 2 exists for each $n \geq 1$. In general, a proper $\beta$-block is not required to be a member of $\mathcal{E}$. By Theorem 13, Part 2, if $\mathcal{E}$ is Zeckendorf, then the proper $\beta$-blocks are members of $\mathcal{E}$.

**Theorem 13** Let $\mathcal{E}$ be a descendingly-ordered collection of coefficient functions for the open interval $I$.

1. The collection $\mathcal{E}$ is Zeckendorf if and only if all of the following are satisfied:
   
   (a) Given an index $n \geq 1$, there is a maximal coefficient function $\beta^n$ of order $n$ as defined in Definition 10.
   
   (b) For each $\epsilon \in \mathcal{E}$, there are unique proper $\beta$-blocks $\zeta^m$ at index $n_m$ with support $[n_m, n_m]$ for all $m \geq 1$ such that $n_1 = 1$ and $n_m + 1 = n_{m+1}$, and $\epsilon = \sum_{m=1}^{\infty} \zeta^m$. We call the expression the $\beta$-block decomposition of $\epsilon$.
   
   (c) Given a $\beta$-block decomposition $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ of $\epsilon \in \mathcal{E}$ and an index $M \geq 1$, the immediate successor $\tilde{\epsilon}$ of $\epsilon$ res $M$ in $\mathcal{E}^M$ is given as follows, if $\epsilon \neq \beta^1$ res $[1, M]$. Let $K$ be the index such that the support $[n_K, i_K]$ of $\zeta^K$ contains $M$. If $n_K \leq M < i_K$, then $\tilde{\epsilon} = \sum_{m=1}^{K-2} \zeta^m + \zeta^{K-1} + \beta^{\alpha_K} - \beta$. If $M = i_K$, then $\tilde{\epsilon} = \sum_{m=1}^{K-1} \zeta^m + \zeta^K + \beta^{i_K}$.

2. Suppose that $\mathcal{E}$ is Zeckendorf, and let $\zeta^m$ be proper $\beta$-blocks at index $n_m$ with support $[n_m, i_m]$ such that $n_1 = 1$ and $i_m + 1 = n_{m+1}$ for all $m \geq 1$. Then, $\sum_{m=1}^{\infty} \zeta^m$ is a member of $\mathcal{E}$.

As in the case of Zeckendorf collections for positive integers, a Zeckendorf collection $\mathcal{E}$ for $I$ is completely determined by the set of maximal coefficient functions $\beta^n$.

**Definition 14** Let $\{\delta^n : n \geq 1\}$ be a set of coefficient functions with infinite support under the descending lexicographical order such that $\text{ord} (\delta^n) = n$ for each $n \geq 1$. The descendingly-ordered collection $\mathcal{E}$ of coefficient functions determined by $\{\delta^n : n \geq 1\}$ is defined to be the descendingly-ordered collection $\mathcal{E}$ of coefficient functions consisting of $\delta$-block decompositions $\sum_{m=1}^{\infty} \zeta^m$.

**Corollary 15** If $\mathcal{E}$ is the descendingly-ordered collection $\mathcal{E}$ of coefficient functions determined by coefficient functions $\delta^n$ of order $n$ with infinite support for $n \geq 1$, then $\mathcal{E}$ is Zeckendorf, and $\beta^n = \delta^n$ for all $n \geq 1$. Moreover, $\mathcal{E}$ is the only Zeckendorf collection for $I$ such that $\beta^n = \delta^n$ for each $n \geq 1$.

By Corollary 15, given a set of coefficient functions $\delta^n$ of order $n$ with infinite support for $n \geq 1$, we may define the Zeckendorf collection for $I$ determined by maximal coefficient functions $\beta^n = \delta^n$ for $n \geq 1$ to be the collection defined in Definition 14. For example, let $\mathcal{E}$ be the Zeckendorf collection for $I$ determined by maximal coefficient functions $\beta^n = \sum_{k=0}^{\infty} \delta^{n+2k} = (0, 1, 0, 1, 0, 1, 0, \ldots)$ for $n \geq 1$. Then, the collection is similar to the one for the classical Zeckendorf condition, but it allows infinitely many entries of 1.
Let us revisit the earlier example $\mu := \beta^1 + \beta^3$. Suppose that $\hat{\beta}^3 = \sum_{k=0}^{\infty} \beta^{3+2k}$ and $\hat{\beta}^n = \sum_{k=n}^{\infty} \beta^k$ for all $n \geq 4$ are the maximal coefficient functions of a Zeckendorf collection for $I$. If $\beta^1 = \beta^1 + \sum_{k=3}^{\infty} \beta^k$, then $\mu = (\beta^1 + \beta^3 + 0) + (\beta^3 + 0) + (\beta^7 + 0) + \cdots$ is an $\beta$-block decomposition, and hence, $\mu \in \mathcal{E}$ by Theorem 13. If $\hat{\beta}^n = \sum_{k=n}^{\infty} \beta^k$ for all $n \geq 1$, then $\mu = \beta^1 + \sum_{k=3}^{\infty} \beta^k$ does not have an $\beta$-block decomposition since the first proper $\beta$-block in $\mu$ is $(\beta^1 + 0)$, but there is no proper $\beta$-block at index 3 in $\mu$. Hence, $\mu$ is not a member of $\mathcal{E}$.

Let

$$Q_n = e_1Q_{n-1} + \cdots + e_NQ_{n-N}$$

(1)

be a linear recurrence for a sequence in $R$ where $N$ is a fixed positive integer and $e_k$ are integers independent of $n$. Let us review the standard Zeckendorf conditions on the coefficient functions associated with this recursion for sequences $Q$ in $\mathbb{N}$ in terms of immediate predecessors. The conditions in full generality are first introduced in [14], and introduced below would be a slight generalization toward adapting infinite expansions of numbers in $I$ and $\mathbb{Z}_p$. Let $L = (e_1, \ldots, e_N)$ be a finite list of non-negative integers where $e_1e_N > 0$, and we shall call it a Zeckendorf multiplicity list. Given an index $n \geq 2$, let $\delta^n = \sum_{k=1}^{n-1} \hat{\delta}_{rem(k)}\beta^{n-k} = (\ldots, e_2, e_1, e_N, e_{N-1}, \ldots, e_2, e_1, 0)$ be coefficient functions where $\text{rem}(k)$ denotes the least positive residue of $k$ mod $N$ and $\hat{e}_k = e_k$ for all $1 \leq k < N$ and $e_N := e_N - 1$. We may consider the Zeckendorf collection for positive integers with $\hat{\beta}^n = \delta^n$ for $n \geq 2$, and we denote it by $\mathcal{L}$. The completion of $\mathcal{L}$ for the $p$-adic integers is denoted by $\overline{\mathcal{L}}$. For the Zeckendorf collection for the open interval $I$, we consider $\hat{\beta}^n := \sum_{k=n}^{\infty} \hat{\delta}_{rem(k-n+1)}\beta^k = (0, e_1, e_2, \ldots, \hat{e}_N, e_1, e_2, \ldots)$, and we also denote the collection by $\overline{\mathcal{L}}$.

Given a set of numbers $R$ and a lexicographically ordered collection $\mathcal{E}$ of coefficient functions, a sequence $Q$ in $R$ is said to have the unique $\mathcal{E}$-representation property if $\sum \delta Q$ have distinct values for the coefficient functions $\delta \in \mathcal{E}$. Given a sequence $Q$ with unique $\mathcal{E}$-representation property, let us denote by $X_Q^\mathcal{E}$ the subset consisting of values of $\sum \delta Q$ for non-zero coefficient functions $\delta \in \mathcal{E}$, and we call it an $\mathcal{E}$-subset of $R$. When the collection $\mathcal{E}$ is understood in the context, we simply denote the subset by $X_Q$. Recall that if $L = (1, 1)$ is a Zeckendorf multiplicity list, the ascending $\mathcal{L}$-Zeckendorf condition on coefficient functions for positive integers coincides with the classical Zeckendorf condition on the Fibonacci sequence. For example, consider $Q_n = 2^{n-1}$ for $n \geq 1$ under the ascending $\mathcal{L}$-Zeckendorf condition. Then, $165 = Q_8 + Q_6 + Q_3 + Q_1$ is a member of $X_Q$ while 166 and 167 are not. However, if $\overline{L} = (1, 2)$, then the binary expansion of a positive integer is $\overline{\mathcal{L}}$-Zeckendorf, and $X_Q^{\overline{\mathcal{L}}} = \mathbb{N}$ while $X_Q^{\overline{\mathcal{L}}} = \mathbb{N}$ is a proper subset of $\mathbb{N}$. Let us introduce another interesting example. Let $L = (1, 1)$, and let $Y$ be the subset consisting of $\sum \mu F$ for $\mathcal{L}$-Zeckendorf coefficient functions $\mu$ with $\mu_1 = 0$ where $F = (1, 2, 3, \ldots)$ is the Fibonacci sequence, i.e., the positive integers whose classical Zeckendorf decompositions do not involve $F_1$. Then, obviously, $Y$ is represented uniquely under the $\mathcal{L}$-Zeckendorf condition by the sequence $Q$ given by $Q_k := F_{k+1}$ for $k \geq 1$. More interestingly, by Theorem 16, it turns out that it is the only increasing sequence with that property.

Given a lexicographically ordered collection $\mathcal{E}$ and a subset $Y$ of $R$, if there is a sequence $Q$ in $R$ with unique $\mathcal{E}$-representation property such that $X_Q = Y$, then $Q$ is called a fundamental sequence for the $\mathcal{E}$-subset $Y$. In addition, if the fundamental sequence $Q$ is an increasing sequence, then $Y$ is also called an increasing $\mathcal{E}$-subset of $\mathbb{N}$. If the fundamental
sequence $Q$ is a decreasing sequence in $R$ that is either $I$ or $\mathbb{Z}_p$, the subset $Y$ is called a decreasing $\mathcal{E}$-subset of $R$.

2.2 Results

Let us begin with the results on positive integers. In this paper, for simplicity an increasing sequence means a strictly increasing sequence in the usual sense.

**Theorem 16**

1. *(Zeckendorf's Theorem for positive integers)* Let $Q$ be an increasing sequence in $\mathbb{N}$ with $Q_1 = 1$.

   (a) Then, there are coefficient functions $\mu^n$ of order $n - 1$ for $n \geq 2$ such that
   
   \[ Q_n = Q_1 + \sum \mu^n Q. \]  

   (b) Suppose that a sequence of coefficient functions $\mu^n$ of order $n - 1$ for $n \geq 2$ satisfies the recursion (2) for all $n \geq 2$, and let $\mathcal{E}$ be the Zeckendorf collection with $\hat{\beta}^n := \mu^n$ as the immediate predecessor of $\beta^n$ for $n \geq 2$. Then, $Q$ is a fundamental sequence for the $\mathcal{E}$-set of numbers $\mathbb{N}$.

2. *(The weak converse)* Let $\mathcal{E}$ be an arbitrary ascendingly-ordered collection of coefficient functions with finite support.

   (a) If $\mathcal{E}$ is Zeckendorf, then $\mathbb{N}$ is an $\mathcal{E}$-set of numbers.

   (b) Each increasing $\mathcal{E}$-subset $Y$ of $\mathbb{N}$ has a unique increasing fundamental sequence $Q$. In addition, if $\mathcal{E}$ is Zeckendorf, and $Y = \mathbb{N}$, then $Q$ is given by $Q_n = \sum \hat{\beta}^n Q + 1$ for all $n \geq 2$ and $Q_1 = 1$ where $\hat{\beta}^n$ for $n \geq 2$ are the immediate predecessors of $\beta^n$.

**Example 17** If $\mathcal{L}$ is a Zeckendorf collection for $\mathbb{N}$ determined by Zeckendorf multiplicity list $L = (e_1, e_2, \ldots, e_N)$, the recursion (2) is reduced to $Q_n = e_1 Q_{n-1} + \cdots + e_N Q_{n-N}$ for all $n > N$, and $\mathcal{L}$-Zeckendorf's Theorem is proved in [14]. In Theorem 16, Part 2 (b), it is asserted that it is the only increasing fundamental sequence.

Recall the Zeckendorf collection $\mathcal{E}$ in Example 4. Then, the increasing fundamental sequence $Q$ is given by $Q_{n+2} = (n + 1)Q_{n+1} + Q_n$ for all $n \geq 1$ and $Q = (1, 2, 5, 17, \ldots)$. Below we list the first few small coefficient functions in $\mathcal{E}_0$:

\[
(1) <_a (0, 1) <_a (1, 1) <_a (0, 2) <_a (0, 0, 1) <_a (1, 0, 1) <_a (0, 1, 1) <_a (1, 1, 1)
<_a (0, 2, 1) <_a (0, 0, 2) <_a \cdots <_a (0, 2, 2) <_a (0, 0, 3) <_a (1, 0, 3) <_a (0, 0, 1)
\]

where the immediate predecessors $\hat{\beta}^n$ for $n = 2, \ldots, 5$ are given by $(1), (0, 2), (1, 0, 3)$, and $(0, 2, 0, 4)$. In terms of the values of $\sum \epsilon Q$, each value is obtained by adding $Q_1$ to $\sum \epsilon Q$ where $\hat{\epsilon}$ is the immediate predecessor of $\epsilon$. Presenting values of $\sum \epsilon Q$ with respect to the lexicographical order of $\epsilon$ as above trivially proves that $Q$ is a fundamental sequence.

**Example 18** Let $\mathcal{E}$ be the Zeckendorf collection for positive integers with $\hat{\beta}^n = \sum_{k=1}^{n-2} \beta^k + 2\beta^{n-1} = (1, 1, \ldots, 1, 1, 2)$ for $n \geq 2$. Then, the increasing fundamental sequence is given by $Q_n = 3Q_{n-1} - Q_{n-2}$ for $n \geq 3$ and $Q = (1, 3, 8, 21, 55, \ldots)$. Below we list the first few small coefficient functions in $\mathcal{E}$ determined by the immediate predeces-
sors:

\[
(1) <_a (2) <_a (0, 1) <_a (1, 1) <_a (2, 1) <_a (0, 2) <_a (1, 2) <_a (0, 0, 1) <_a (1, 0, 1)
\]

\[
<_a (2, 0, 1) <_a (0, 1, 1) <_a \cdots <_a (2, 0, 2) <_a (0, 2, 2) <_a (1, 1, 2) <_a (0, 0, 0, 1)
\]

The fundamental sequence in fact satisfies the equation (2), and the common “tail part” \( \sum_{k=1}^{n-1} Q_k \) of the equation allows us to derive the short recursion above.

By reversing the process of finding a linear recurrence from immediate predecessors with simple periodic tails as in Example 18, we obtain the following general result, and the proof follows immediately from Proposition 28 in Sect. 3.2.

**Theorem 19** Let \( N \) be a non-negative integer, and let \( Q_n = \sum_{k=1}^{N+1} c_k Q_{n-k} \) for all \( n \geq N + 2 \) be a linear recurrence where \( c_k \) are constants in \( \mathbb{Z} \) such that \( (c_k + \cdots + c_1) \geq 1 \) for all \( 1 \leq k \leq N + 1 \) and \( c_1 \geq 2 \). Then, there is a Zeckendorf collection \( \& \) and fixed non-negative coefficients \( e_1, \ldots, e_N \) and \( b \) for which \( e_1 \geq 2 \), \( \beta^n = \sum_{k=1}^{N-1} b \beta^k + \sum_{k=1}^{N} e_k \beta^{n-k} \) for all \( n \geq N + 2 \), and \( \beta^n = \sum_{k=1}^{N} e_k \beta^{n-k} \) for all \( 2 \leq n \leq N + 1 \), and there are initial values \((Q_1, \ldots, Q_{N+1})\) for which the recurrence defines a fundamental sequence for \( \mathbb{N} \) under the \( \&\)-Zeckendorf condition.

For example, let \( Q_n \) be a sequence given by \( Q_n = 8Q_{n-1} - 2Q_{n-2} - 3Q_{n-3} \) for \( n \geq 4 \) and \((Q_1, Q_2, Q_3) = (1, 8, 62)\). Then, the Zeckendorf collection described in Proposition 28 and Theorem 19 has \( \beta^n = \sum_{k=1}^{n-1} 2 \beta^k + 5 \beta^{n-2} + 7 \beta^{n-1} \) for \( n \geq 3 \), \( \beta^3 = 5 \beta^1 + 7 \beta^2 \), and \( \beta^2 = 7 \beta^1 \). If coefficients \( c_k \) do not satisfy the conditions in Theorem 19, but there are increasing initial values \((Q_1, \ldots, Q_{N+1})\) for which the recurrence defines an increasing sequence, there is still a Zeckendorf condition under which \( Q \) is an increasing fundamental sequence for \( \mathbb{N} \), as asserted in Theorem 16. It will be further explained in Algorithm 20, but we do not expect that its immediate predecessors have periodic tails. Theorem 16, Part 1 (a) can be proved by a trivial sequence of coefficient functions \( \mu^n \) for \( n \geq 2 \) given by

\[
\mu^n := (Q_n - Q_{n-1} - 1)\beta^1 + \beta^{n-1}, \tag{3}
\]

but we may use a greedy algorithm as described in Algorithm 20 to find \( \mu^n \) where the values of \( \mu^n_k \) are relatively smaller.

**Algorithm 20** Let \( Q \) be an increasing sequence in \( \mathbb{N} \) such that \( Q_1 = 1 \). Given \( Q_n \) for \( n \geq 2 \), let \( e_1 \) be the largest integer such that \((Q_{n-1}) - (Q_{n-1}) \geq 0 \) and for \( 2 \leq k \leq n - 1 \), recursively define \( e_k \) to be the largest integer such that \((Q_{n-1}) - (Q_{n-1}) + \cdots + e_k Q_{n-k} \geq 0 \). Then, since \( Q_1 = 1 \), the algorithm terminates with an equality \( Q_n = 1 + \sum_{k=1}^{n-1} e_k Q_{n-k} \).

By Corollary 9, we have the Zeckendorf \( \& \) for positive integers with \( \beta_n := (e_{n-1}, e_{n-2}, \ldots, e_1, 0) \) for \( n \geq 2 \) where \( e_1, \ldots, e_{n-1} \) depend on \( n \), and by Theorem 2.2, the sequence \( Q \) is the only increasing fundamental sequence for the \( \& \)-set of numbers \( \mathbb{N} \).

For example, let \( Q \) be a sequence given by \( Q_k = k! \) for all \( k \geq 1 \). If we apply Algorithm 20 using the identity \( \cdot 1 + 2! \cdot 2 + \cdots + n! \cdot n = (n + 1)! - 1 \), we obtain the Zeckendorf \( \& \) for positive integers with \( \beta^n = \sum_{k=1}^{n-1} k \beta^k = (1, 2, \ldots, n - 1, 0) \) for all \( n \geq 2 \). The trivial example of \( \mu^n \) mentioned in (3) yields immediate predecessors \( \beta^n := (Q_n - Q_{n-1} - 1)\beta^1 + \beta^{n-1} = ((n - 1)! \cdot (n - 1) - 1)\beta^1 + \beta^{n-1} \) for \( n \geq 2 \).
Let us also introduce the algorithm of finding the $\mathcal{E}$-expansion of a positive integer when $\mathcal{E}$ is Zeckendorf.

**Algorithm 21** Let $\mathcal{E}$ be a Zeckendorf collection for positive integers, and let $Q$ be the increasing fundamental sequence $\mathcal{E}$-set of numbers $\mathbb{N}$. If $x \in \mathbb{N}$, the unique non-zero $\hat{\beta}$-block decomposition $x = \sum_{m=1}^{M} \sum \zeta^{m}Q$ is given by a greedy algorithm in terms of $\hat{\beta}$-blocks. That is, for each $k = 1, 2, \ldots, M - 1$, the proper $\hat{\beta}$-block $\zeta^{M-k+1}$ is recursively defined to be the largest one for which $x - \sum_{m=1}^{k-1} \sum \zeta^{M-m+1}Q \geq \sum \zeta^{M-k+1}Q$, and $\zeta^{1}$ is the largest $\hat{\beta}$-block, proper or maximal, for which $x - \sum_{m=1}^{M-1} \sum \zeta^{M-m+1}Q = \sum \zeta^{1}Q$. It is a straightforward induction exercise on $x$ to show that the algorithm terminates with zero remainder.

To prove Theorem 16, Part 1 (b), we may use Algorithm 21. In fact, we use the algorithm to establish the existence and uniqueness of the $\mathcal{E}$-expansions of real numbers in the interval $I$, and the proof directly translates to the case of $\mathbb{N}$. In Sect. 5 we also introduce a different approach to proving the case of $\mathbb{N}$, for which we do not use a greedy algorithm at all.

At first, we were motivated to prove the weak converse for the $\mathcal{L}$-subsets of $\mathbb{N}$, and realized that the weak converse for subsets of $\mathbb{N}$ holds even for non-Zeckendorf collections of coefficient functions, as stated in Theorem 16, Part 2 (b). However, we learned later that Theorem 16, Part 2 (b) for non-Zeckendorf collections was noticed and proved for $\mathbb{N}$ in [12], and apparently it had been unnoticed in the later literature such as [2] where the weak converse for the $N$th order Zeckendorf set of numbers $\mathbb{N}$ is proved using a different method. Our proof is nearly identical to the one in [12], but we include our version in Sect. 5 as it is more generally for subsets of $\mathbb{N}$.

Theorem 16, Part 2 concerns proper $\mathcal{E}$-subsets of $\mathbb{N}$ as well. Let $L = (1, 1, 1)$ be a Zeckendorf multiplicity list. Then, the Zeckendorf collection $\mathcal{E}$ defines the 3rd order Zeckendorf condition. Let $Y := 7\mathbb{N}$, i.e., the subset of positive multiple of 7. Then, $Y = X_{Q}$ where $Q$ is given by $Q_{k} = 7H_{k}$ where $H$ is the third order Fibonacci sequence $H = (1, 2, 4, 7, 13, \cdots)$. Thus, $7\mathbb{N}$ is an $\mathcal{L}$-subset of $\mathbb{N}$, and by Theorem 16, the sequence $Q$ is the only increasing fundamental sequence for $7\mathbb{N}$. It turns out that other congruence classes mod 7 are not $\mathcal{E}$-subsets of $\mathbb{N}$ for any nontrivial Zeckendorf collections $\mathcal{E}$, and neither is the subset of positive odd integers, which was mentioned in Sect. 1. We shall further discuss this example and more in Sect. 4.

Recall from Example 17 the fundamental sequence $Q$ for the $\mathcal{L}$-set of numbers $\mathbb{N}$. If we instead choose initial values such that $Q_{n} > \sum_{k=1}^{n-1} e_{k}Q_{n-k}$ for all $2 \leq n \leq N$, then it is a straightforward exercise to show that $\mu < \epsilon$ implies $\sum \mu Q < \sum \epsilon Q$. This implies that $Q$ is an increasing sequence, and has the unique $\mathcal{L}$-representation property. We may also use the following general criteria to generate more $\mathcal{L}$-subsets of $\mathbb{N}$. If the initial values for the linear recurrence in Example 17 are increasing, then the fundamental sequence is increasing, and by Theorem 16, Part 2, these increasing $\mathcal{L}$-subsets satisfy the weak converse of $\mathcal{L}$-Zeckendorf’s theorem.

**Theorem 22** Let $N$ be a positive integer, and let $Q$ be a sequence in $\mathbb{N}$ given by the linear recurrence (1) where $e_{k} \geq 1$ for all $k = 1, \ldots, N$. Then, the sequence $Q$ has the unique $\mathcal{L}$-representation property if and only if the values of $\sum \sigma Q$ are distinct for all $\sigma \in \mathcal{L}$ of order $\leq 4N$. 
Let us demonstrate the theorem with \( L = (2,3) \) and \( Q = (5,3,21,\ldots) \). As mentioned earlier, the property \( Q_2 > 2Q_1 \) guarantees that the sequence has the unique \( L \)-representation property, but the inequality fails for \( Q \). However, computer calculations show that there are 6,560 \( L \)-Zeckendorf coefficient functions \( \mu \) of order \( \leq 8 \), and they generate distinct values of \( \sum \mu Q \). Hence, by Theorem 22, it has the unique \( L \)-representation property. See Sect. 5.5 for example of a sequence that does not have the unique \( L \)-representation property. If \( L \) is the \( N \)th order Zeckendorf condition, the search can be shortened to the coefficient functions of order \( \leq 2N \); see [3].

**Example 23** Let us demonstrate the theorem with \( L \) for some Zeckendorf conditions, i.e., there are two fundamental sequences for \( \mathbb{N} \) under the Zeckendorf condition. Let \( \mathcal{E} \) be the Zeckendorf collection of coefficient functions defined in Example 5, and let \( Q \) and \( Z \) be sequences in \( \mathbb{N} \) defined by the recurrence \( Q_n = 7Q_{n-3} \) and \( Z_n = 7Z_{n-3} \) such that \( Q = (1,2,3,\ldots) \) and \( Z = (2,1,3,\ldots) \). Then, both sequences are fundamental sequences for the \( \mathcal{E} \)-set of numbers \( \mathbb{N} \), and \( Z_{1+3n} > Z_{2+3n} \) for all \( n \geq 0 \).

Let us introduce our results for the unit interval \( I \) of real numbers.

**Theorem 24** 1. *(Zeckendorf’s Theorem for the unit interval)* Let \( Q \) be a decreasing sequence in \( I \), and let \( Q_0 = 1 \).

(a) If \( Q_n \to 0 \) as \( n \to \infty \), then there are coefficient functions \( \mu^n \) of order \( n \) with infinite support for \( n \geq 1 \) such that

\[
Q_{n-1} = \sum \mu^n Q
\]

for all \( n \geq 1 \).

(b) Let \( \mathcal{E} \) be a Zeckendorf collection for \( I \), and let \( \hat{\beta}^n \) for \( n \geq 1 \) be its maximal coefficient functions of order \( n \). Then, \( Q \) is a decreasing fundamental sequence for the \( \mathcal{E} \)-interval \( I \) if and only if \( Q \) satisfies (4) for all \( n \geq 1 \) where \( \mu^n = \hat{\beta}^n \) for each integer \( n \geq 1 \).

2. *(The weak converse)* Let \( \mathcal{L} \) be the Zeckendorf collection of coefficient functions for \( I \) determined by a Zeckendorf multiplicity list \( L = (e_1,\ldots,e_N) \), and let \( \omega \) be the (only) positive real zero of the polynomial \( e_N x^N + \cdots + e_1 x - 1 \). Then, \( I \) is an \( \mathcal{L} \)-set of numbers, and \( I \) has one and only one decreasing fundamental sequence \( Q \) given by \( Q_k = \omega^k \) for all \( k \geq 1 \).

Recall from the introduction the question on the weak converse for the \( \mathcal{L} \)-set of numbers \( I \) where \( L = (1,1) \). Theorem 24, Part 1 (b) provides an affirmative answer to the question, and it is given by \( Q_k = \omega^k \) for all \( k \geq 1 \) where \( \omega \) is the reciprocal of the golden ratio. In fact, the weak converse for the \( \mathcal{L} \)-set of numbers \( I \) is proved in [6] for the case of \( L = (1,0,0,\ldots,0,1) \) where \( e_2 = \cdots = e_{N-1} = 0 \). Our idea is similar to [6], and our proof relies on [6, Theorem 1], but also we improve [6, Theorem 1] in Proposition 27. We prove the proposition in Sect. 5 in a more general setting.

As demonstrated in Theorem 19, if \( \hat{\beta}^n \) for \( n \geq 1 \) have common “tails”, it admits a short recursion. Let \( \hat{\beta}^n := 2\hat{\beta}^n + \sum_{k=n+1}^{\infty} \beta^k \), which is an analogy of Example 18 for \( I \). Then, if exists, a decreasing fundamental sequence must satisfy \( Q_n = 3Q_{n+1} - Q_{n+2} \). Its characteristic polynomial is \( x^2 - 3x + 1 \), and if we impose the decreasing property to
Binet’s Formula for $Q_n$, we find that $Q_k = \omega^k$ for $k \geq 1$ where $\omega = \frac{1}{2}(3 - \sqrt{5})$ must be the case. Since $Q$ satisfies (4), by Theorem 24, Part 1 (b), it is a decreasing fundamental sequence, and in particular, it is the only decreasing fundamental sequence.

For Theorem 24, Part 2 originally we assumed the condition $e_1 \leq \cdots \leq e_N$ to obtain certain inequalities between the values of $\sum \epsilon Q$, which seemed necessary to derive the equality (4), but later we could derive the inequality without the assumption. The increasing condition on $Q_k$, prior to the work of [14], the Zeckendorf’s theorem was only known in the case where the Zeckendorf multiplicity list satisfies the Parry condition.

For Theorem 24, Part 1 (a), we use the greedy algorithm that is similar to Algorithm 20 to find coefficient functions $\mu^a$ for $n \geq 1$. Given an index $n \geq 0$, recursively define $e_k$ for $k = 1, 2, 3, \ldots$ to be the largest non-negative integer such that $Q_{n+1} = (e_1Q_{n+1} + \cdots + e_kQ_{n+k-1}) > e_kQ_{n+k}$; the strict inequality allows us to find infinitely many indices $k$ with $e_k \geq 1$. Then, $\mu^{n+1} = \sum_{k=1}^{\infty} e_k \beta^{n+k}$, and $Q$ satisfies the equality (4).

**Example 25** Let $Q$ be a sequence given by $Q_k = 1/(k+1)$ for all $k \geq 0$. Then,

$$Q_{n-1} = Q_n + Q_{n^2+n-1} \quad \text{for } n \geq 1,$$

and if we repeatedly apply the recurrence to the last term of the RHS of (5), we obtain $Q_{n-1} = Q_{n_1} + Q_{n_2} + Q_{n_3} + \cdots$ where $n_1 = n$ and $m_{k+1} = m_k(m_k + 1)$ for $k \geq 1$. Let $\mathcal{E}$ be the Zeckendorf collection for $I$ determined by $\hat{\beta}^n = \sum_{k=1}^{\infty} \beta^{m_k}$ for $n \geq 1$, e.g., $\hat{\beta}^3 = \beta^3 + \beta^2 + \beta^1 + \beta^0$. Then, $Q_{n-1} = \sum \hat{\beta}^n Q$ for $n \geq 1$, and hence, by Theorem 24, Part 1 (b), $Q$ is a decreasing fundamental sequence for the $\mathcal{E}$-set of numbers $I$. Moreover, if we use the identity (5) to apply the greedy algorithm described above that requires a strict inequality, we find that the expression $Q_{n-1} = \sum \hat{\beta}^n Q$ for $n \geq 1$ coincides with the one obtained by the greedy algorithm. Thus, the expansion of each real number obtained by applying the greedy algorithm with $Q$ makes a unique $\mathcal{E}$-expansion. For example, recall the well-known expansion $\frac{\pi}{8} = \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} = \frac{1}{3} + \frac{1}{35} + \frac{1}{63} + \cdots + = Q_2 + Q_{34} + Q_{62} + \cdots$, but since $34 \cdot 35 = 1190$, it is not $\mathcal{E}$-Zeckendorf. The greedy algorithm yields

$$\frac{\pi}{8} \approx Q_2 + Q_{16} + Q_{1844} + Q_{4683104} + \cdots$$

$$= \frac{1}{3} + \frac{1}{17} + \frac{1}{3^2 \cdot 5 \cdot 41} + \frac{1}{3^2 \cdot 5 \cdot 7 \cdot 14867} + \cdots$$

where $\beta^2 + 0 \cdot \beta^6$ is the first non-zero proper $\beta$-block, and this is the only way of expressing $\pi/8$ under the $\mathcal{E}$-Zeckendorf condition. The process of finding non-zero terms of $Q$ is similar to that of continued fraction expansions, i.e., $x \mapsto \frac{1}{x} - (\frac{1}{x} + \frac{1}{y})$ and $1/y \approx 1844.27$. In fact, the Zeckendorf collection $\mathcal{E}$ enjoys the finite expansions of all rational numbers in $I$ as in the continued fraction expansion.

Let us introduce our results for $p$-adic integers. A sequence $Q$ in $\mathbb{Z}_p$ is decreasing if $|Q_k|_p > |Q_{k+1}|_p$.

**Theorem 26** Let $p$ be a prime number.

1. (Zeckendorf’s Theorem for $p$-adic integers) Let $\mathcal{E}$ be an arbitrary collection of coefficient functions $\epsilon$ such that $\epsilon_n < p$ for all $n \geq 1$. If $Q$ is a decreasing sequence in $\mathbb{Z}_p$, then $Q$ has the unique $\mathcal{E}$-representation property.
2. (The weak converse) Let $\mathcal{E}_0$ be a Zeckendorf collection for positive integers, and let $\mathcal{E}$ be the completion of $\mathcal{E}_0$. If $\epsilon_n \leq \min\{\sqrt{p}, (p - 1)/2\}$ for all $n \geq 1$ and all $\epsilon \in \mathcal{E}$, then each decreasing $\mathcal{E}$-subset of $\mathbb{Z}_p$ has a unique decreasing fundamental sequence.

Let $p = 41$. Then, the golden ratio $\phi$ is defined in the $p$-adic integers $\mathbb{Z}_p$, and let us consider the sequence $Q$ in $\mathbb{Z}_p$ given by $Q_k = (\phi^k + 3\phi^k)p^{k-1}$. If we consider the Zeckendorf multiplicity list $L = (1, 1)$ as for the Fibonacci sequence, by Theorem 26, $Q$ is the only decreasing fundamental sequence for the $\mathcal{L}$-subset $X_Q$, and this can be considered a $p$-adic analogue of the Fibonacci sequence in terms of the weak converse. It satisfies the recurrence $Q_{n+2} = pQ_{n+1} + p^2Q_n$ for all $n \geq 1$, but $X_Q \neq \mathbb{Z}_p$. The sequences $Z$ defined by the classical Fibonacci recurrence $Z_{n+2} = Z_{n+1} + Z_n$ make some $\sum \epsilon Z$ divergent in $\mathbb{Z}_p$.

The weak converse fails for the $\mathcal{L}$-set of numbers $\mathbb{Z}_p$ if the values of coefficient functions are too large, so it does not satisfy the condition of Theorem 26, Part 2. Let $Q$ be the sequence given by $Q_k = p^{k-1}$ for $k \geq 1$, and let $L = (p - 1, p)$. The standard $p$-adic expansion of $\mathbb{Z}_p$ makes an $\mathcal{L}$-expansion $\sum \epsilon Q$, and it has the unique $\mathcal{L}$-representation property by Theorem 26, Part 1. However, if $Z_k = (p(p - 1))^k$, then $\text{ord}_p(Z_k) = k - 1$ implies that each $p$-adic integer is equal to $\sum \epsilon Z$ for a unique $\mathcal{L}$-Zeckendorf coefficient function $\epsilon$. Thus, there are two distinct decreasing fundamental sequences for $\mathbb{Z}_p$, and hence, the weak converse fails for $\mathbb{Z}_p$ under that $\mathcal{L}$-Zeckendorf-condition.

In proving Theorem 24, Part 2, we establish the following result under a more general condition on the coefficients. The result is also found in [15, Theorem 12.2], but our proof is more detailed, and we introduce the proof in Section 5.3.

If a polynomial $f(z)$ over the complex numbers has a unique complex root with largest modulus, it is called a dominant polynomial.

**Proposition 27** Let $f(z) = a_nz^n - a_{n-1}z^{n-1} - \cdots - a_1z - a_0$ be a polynomial in $\mathbb{R}[z]$ such that $a_k \geq 0$ for all $0 \leq k \leq n$ and $a_n a_0 \neq 0$. If there are indices $m$ and $\ell$ such that $1 \leq m < \ell \leq n - 1$, $\gcd(m, \ell) = 1$, and $a_m a_\ell \neq 0$, then $f(z)$ is a dominant polynomial.

By a classical theorem of [1], if $a_{n-1} \geq \cdots \geq a_0$, then the unique positive zero of the polynomial $f(z)$ in Proposition 27 is a Pisot number, and hence, it is a dominant polynomial. As proved in [7], most polynomials are dominant, but proving that a certain class of polynomials are dominant in general may require some work. Various constructions and tests for dominant polynomials are introduced in [7].

### 3 Examples of non-standard Zeckendorf conditions

Recall that by Theorem 16, given an increasing sequence in $\mathbb{N}$, there is a Zeckendorf condition under which the sequence is a fundamental sequence for $\mathbb{N}$, and it can be constructed using a greedy algorithm. In this section, we introduce some non-standard examples for which the Zeckendorf conditions are more concrete than the one abstractly given by a greedy algorithm.

#### 3.1 Fixed blocks

Recall the Zeckendorf collection defined in Example 5. The immediate predecessors are given by $\hat{\beta}^n = \sum_{k=1}^{n-1} \beta^k = (1, 1, \ldots, 1, 1, 0)$ for $n \neq 0$ mod 3, and $\hat{\beta}^{3n} = \sum_{k=1}^{3(n-1)} \beta^k + \beta^{3n-1} = (1, 0, 1, 0)$. The unique increasing fundamental sequence $Q$ is given by $Q_n :=$
$Q_{n-1} + \cdots + Q_2 + 2Q_1$ for \( n \not\equiv 0 \mod 3 \) where \((Q_1, Q_2) = (1, 2), Q_{3n} = Q_{3n-1} + Q_{3n-3} + \cdots + Q_2 + 2Q_1 \) for all \( n \geq 2 \), and \( Q_3 = 3 \). Let us show that these recursions reduce to \( Q_n = 7Q_{n-3} \) for \( n \geq 4 \) with \((Q_1, Q_2, Q_3) = (1, 2, 3) \).

Notice that the values of \( \beta \)-expansions supported on the indices \( \{3n-2, 3n-1, 3n\} \) are obtained by adding \( Q_{3n-2} \) each time as follows:

\[
0 < Q_{3n-2} < Q_{3n-1} < Q_{3n} < Q_{3n-2} + Q_{3n-1} < Q_{3n} < Q_{3n-2} + Q_{3n-1} \leq Q_{3n+1}.
\]

(6)

Hence, \( 7Q_{3n-2} = Q_{3n+1} \). Also notice that as we keep adding \( Q_{3n-2} \), we arrive at \( Q_{3n-1} \) and \( Q_{3n} \), and we have \( Q_{3n-1} = 2Q_{3n-2} \) and \( Q_{3n} = 3Q_{3n-2} \). Thus,

\[
Q_{3n-1} = 2Q_{3n-2}, \quad n \geq 1 \Rightarrow Q_{3n+2} = 2Q_{3n+1} = 7(2Q_{3n-2}) = 7Q_{3n-1},
\]

\[
Q_{3n} = 3Q_{3n-2}, \quad n \geq 1 \Rightarrow Q_{3n+3} = 3Q_{3n+1} = 7(3Q_{3n-2}) = 7Q_{3n},
\]

Let us introduce a different perspective, from which it is far easier to see the simple formula of \( Q_n \). First notice that \((1, 2, 3)\) is the only increasing sequence of three positive integers that uniquely represents the first consecutive positive integers \( \{1, 2, \ldots, 7\} \) under the \( \beta \)-Zeckendorf condition where \( M \) turns out to be 6. Given any positive integers \( n \), let \( n = \sum_{r=0}^{\infty} a_r 7^r \) be the base-7 expansion where \( 0 \leq a_r \leq 6 \), and write \( n = \sum_{r=0}^{\infty} (\mu_{r_1} + 2\mu_{r_2} + 3\mu_{r_3})7^r \) where \( (\mu_{r_1}, \mu_{r_2}, \mu_{r_3}) \) is the unique one of the seven blocks such that \( a_r = \mu_{r_1} + 2\mu_{r_2} + 3\mu_{r_3} \). Thus, they make an \( \beta \)-expansion.

In general given a positive integer \( N \geq 1 \), we may consider a random list of immediate predecessors \( \hat{\beta}^k \) for \( 2 \leq k \leq N+1 \). They determine an increasing finite sequence \( (Q_1, \ldots, Q_N) \) where \( Q_1 = 1 \) and \( Q_n = Q_1 + \sum \hat{\beta}^n Q \) for \( 2 \leq n \leq N \), and let \( B \) be the list of blocks of length \( N \) that were used in representing each integer from 1 to \( \sum \hat{\beta}^{N+1} Q \). Then, the collection \( \beta \) generated by concatenating blocks in \( B \) is Zeckendorf. Consider the sequence \( Q \) given by \( Q_n = bQ_{n-N} \) where \( b = Q_{N+1} \) and the initial values \( (Q_1, \ldots, Q_N) \) that were the ones determined earlier. Then, as argued in the earlier example where \( b = 7 \) and \( N = 3 \), it is easy to see via the base \( b \)-expansions that \( Q \) is an increasing fundamental sequence for \( \mathbb{N} \). By Theorem 16, under the \( \beta \)-Zeckendorf condition, \( \mathbb{N} \) has a unique increasing fundamental sequence, and hence, the system of equalities \( Q_{n+1} = Q_1 + \sum \hat{\beta}^n Q \) must imply the short recursion \( Q_n = bQ_{n-N} \) for all \( n \geq N+1 \). However, we can also use the method of listing expressions using all blocks as in (6), i.e.,

\[
Q_{nN+1} = \cdots < \sum_{k=nN+1}^{(n+1)N} \hat{\beta}_k^{N+1} Q_k < Q_{(n+1)N+1},
\]

(7)

and as we list each block, we obtain

\[
Q_{nN+k} = c_k Q_{nN+1} \quad \text{for} \ 1 \leq k \leq N
\]

(8)

where \( c_k \) is independent of \( n \). This proves that \( Q_{nN+N} = bQ_n \) for all \( n \geq 1 \) where \( b = Q_{N+1} \).

Let us consider Zeckendorf collections constructed with these fixed blocks for the unit interval \( I \). Let \( B \) be the descendingly-ordered list of blocks \( \{(a_1, a_2, a_3) \in \mathbb{Z}^3 : 0 \leq a_k \leq 1, \ k = 1, 2, 3\} - \{(0, 1, 1)\} \), and let us declare the maximal coefficient functions of order \( n \)
by concatenating these seven blocks. For \( n \geq 0 \), define

\[
\tilde{b}^{3n+1} := b^{3n+1} + b^{3n+2} + \sum_{k=3n+4}^{\infty} b^k
\]

\[
\tilde{b}^{3n+2} := b^{3n+2} + \sum_{k=3n+4}^{\infty} b^k
\]

\[
\tilde{b}^{3n+3} := b^{3n+3} + \sum_{k=3n+4}^{\infty} b^k,
\]

and let \( \mathcal{E} \) be the Zeckendorf collection determined by \( \tilde{b}^n \) for \( n \geq 1 \). If exists, by Theorem 24, a decreasing fundamental sequence \( Q \) must satisfy the following for \( n \geq 0 \) since \( Q_{3n+3} = \sum_{k=4}^{\infty} Q_{3n+k} \):

\[
\begin{align*}
Q_{3n} &= Q_{3n+1} + Q_{3n+2} + 2Q_{3n+3} \\
Q_{3n+1} &= Q_{3n+2} + Q_{3n+3} \\
Q_{3n+2} &= 2Q_{3n+3}.
\end{align*}
\]

(10)

The situation is similar to the case of positive integers. Recall the subcollection \( \mathcal{E}^{3n+3} \), which is finite, from Definition 10. The following seven values are obtained from using coefficient functions in \( \mathcal{E}^{3n+3} \), and they are equal to consecutive integer multiples of \( Q_{3n+3} \):

\[
0 < Q_{3n+3} < Q_{3n+2} < Q_{3n+1} < \cdots < Q_{3n+3} + Q_{3n+2} + Q_{3n+1} < Q_{3n}
\]

and \( Q_{3n+2} = 2Q_{3n+3} \) and \( Q_{3n+1} = 3Q_{3n+3} \) for all \( n \geq 1 \). Hence, \( Q_n = 7Q_{n+3} \) for all \( n \geq 1 \).

If we use the same idea of decomposing coefficients of the base \( 1/7 \)-expansion, it is easy to see that \( (Q_1, Q_2, Q_3) = (3/7, 2/7, 1/7) \) and \( Q_n = 7Q_{n+3} \) for all \( n \geq 1 \) define a decreasing fundamental sequence for \( I \) under the \( \mathcal{E} \)-Zeckendorf condition.

Let us consider the problem of the weak converse under the \( \mathcal{E} \)-Zeckendorf condition, i.e., the problem of determining whether this is the only decreasing fundamental sequence. Since the characteristic polynomial for the recursion is \( 7x^3 - 1 \), and its zeros have the moduli equal to each other, [6, Theorem 1] does not allow us to conclude that a decreasing fundamental sequence is given by \( a/\sqrt[3]{7}^n \) for \( n \geq 1 \). In fact, there are complex numbers \( \alpha_k \) for \( k = 1, 2, 3 \) such that the terms of the above fundamental sequence \( Q \) are given by

\[
Q_n = \alpha_1 \frac{1}{\sqrt[3]{7}^n} + \alpha_2 \frac{2^{3n/3}}{\sqrt[3]{7}^n} + \alpha_3 \frac{2^{3n/3}}{\sqrt[3]{7}^n} \quad \text{for all } n \geq 0 \text{ where } \alpha_1 \approx 0.96, \alpha_2 \approx 0.02 + 0.07i, \text{ and } \alpha_3 \approx 0.02 - 0.07i.
\]

It is possible to use Binet’s formula and consider other possibilities for \( \alpha_k \) to prove that \( Q \) is the only decreasing fundamental sequence, but also we can specialize the system (10) with \( n = 0 \) to obtain \( (Q_1, Q_2, Q_3) = (3/7, 2/7, 1/7) \). This proves that \( Q \) is the only decreasing fundamental sequence under the \( \mathcal{E} \)-Zeckendorf condition, i.e., the weak converse holds for the \( \mathcal{E} \)-interval \( I \), but as in the integer case Example 23, the full converse fails since the initial values \( (Q_1, Q_2, Q_3) = (3/7, 1/7, 2/7) \) makes a fundamental sequence as well.

The setup (9) can be generalized as follows. Let \( N \) be a positive integer, and given integer \( k = 1, 2, \ldots, N \), let \( a_k \) be a positive integers and \( b_{kj} \) be non-negative integers for \( j = k+1, \ldots, N \). Let \( \mathcal{E} \) be the Zeckendorf collection for \( I \) determined by maximal coefficient functions \( \tilde{b}^m \) of order \( m \) such that

\[
\tilde{b}^{N_{n+k}} = a_k b^{N_{n+k}} + \sum_{j=k+1}^{N} b_{kj} b^{N_{n+j}} + \tilde{b}^{N_{(n+1)+1}}
\]
for \( k = 1, \ldots, N \) and \( n \geq 0 \). Then, we have a corresponding system of linear equations for \( Q_1, \ldots, Q_N \) we obtain as in (10) with \( n = 0 \). Its row reduction becomes a nonhomogeneous system since \( Q_0 = 1 \), and the row-reduced system has no shifts in leading positions, i.e., it is nonsingular. Thus, it is clear that, if exists, the first \( N \) values of a decreasing fundamental sequence \( Q \) are uniquely determined, and it must be given by \( Q_n = bQ_{n+N} \) according to the principle introduced in (7) and (8). By Theorem 24, it is indeed a decreasing fundamental sequence for the \( \mathcal{E} \)-interval \( I \), and hence, the weak converse holds.

3.2 Linear recurrence with negative coefficients

Recall Example 18. The recurrence was obtained by considering \( \hat{\beta}^n = \sum_{k=1}^{n-2} \beta^k + 2\beta^{n-1} \) where \( \sum_{k=1}^{n-2} \beta^k \) allows \( Q_n \) and \( Q_{n-1} \) to have a common tail, i.e., \( Q_n = 2Q_{n-1} + \sum_{k=1}^{n-2} Q_k \) and \( Q_{n-1} = 2Q_{n-2} + \sum_{k=1}^{n-3} Q_k \) have a common tail \( \sum_{k=1}^{n-3} Q_k \), and this allows us to derive a short recursion \( Q_n = 3Q_{n-1} - Q_{n-2} \). We have the following general result.

**Proposition 28** Let \( N \) and \( b \) be non-negative integers such that \( b > 0 \) if \( N = 0 \), and let \( e_k \) for \( 1 \leq k \leq N \) be non-negative integers such that \( e_1 \geq 1 \) if \( N > 0 \). Let \( \mathcal{E} \) be a Zeckendorf collection for positive integers determined by immediate predecessors \( \hat{\beta}^n = \sum_{k=1}^{n-1} e_k \beta^{n-k} \) for \( 2 \leq n \leq N + 1 \), and for \( n \geq N + 2 \), \( \hat{\beta}^n = \sum_{k=1}^{n-N} b\beta^k + \sum_{k=1}^{N} e_k \beta^{n-k} \) (where coefficients \( e_k \) are independent of \( n \)). Let \( Q \) be the increasing fundamental sequence for the \( \mathcal{E} \)-set of numbers \( \mathbb{N} \). Then, for \( n \geq N + 2 \),

\[
Q_n = (e_1 + 1)Q_{n-1} + \sum_{k=2}^{N} (e_k - e_{k-1})Q_{n-k} + (b - e_N)Q_{n-N-1}.
\]

**Proof** If \( n \geq N + 2 \),

\[
Q_n = Q_1 + \sum_{k=1}^{n-N-1} b\beta^k Q_k + \sum_{k=1}^{N} e_k Q_{n-k} \\
= Q_1 + \sum_{k=1}^{n-N-2} b\beta^k Q_k + bQ_{n-N-1} + \sum_{k=1}^{N} e_k Q_{n-k},
\]

\[
Q_{n-1} = Q_1 + \sum_{k=1}^{n-N-2} b\beta^k Q_k + \sum_{k=1}^{N} e_k Q_{n-k-1}
\]

\[
\Rightarrow Q_n - Q_{n-1} = e_1 Q_{n-1} + \sum_{k=2}^{N} (e_k - e_{k-1})Q_{n-k} + (b - e_N)Q_{n-N-1}
\]

\[
\Rightarrow Q_n = (e_1 + 1)Q_{n-1} + \sum_{k=2}^{N} (e_k - e_{k-1})Q_{n-k} + (b - e_N)Q_{n-N-1}.
\]

The structure of the coefficients in (11) immediately implies Theorem 19, and we leave the proof to the reader.

3.3 Linear recurrence with non-constant coefficients

Let us introduce immediate predecessors \( \hat{\beta}^n \) that are written in terms of polynomials with positive coefficients. It is still based on the same idea of using a common tail in
Get an immediate predecessor as in Section 3.2. Let \( \tau = \sum_{k=1}^{\infty} k \beta^k \) be a coefficient function, and let \( \tau^n \) denote \( \text{res}_{[1,n]}(\tau) \) for \( n \geq 1 \). Let \( \mathcal{E} \) be the Zeckendorf collection for positive integers with \( \hat{\beta}^n = \tau^{n-2} + n\beta^{n-1} = (1, 2, 3, \ldots, n-2, n, 0) \) for all \( n \geq 2 \). Let \( Q \) be the increasing fundamental sequence for the \( \mathcal{E} \)-set of numbers \( \mathbb{N} \), so that it is given by \( Q_n = Q_1 + \sum \hat{\beta}^n Q \). Then, using the same idea introduced in Section 3.2, we find \( Q_n = (n+1)Q_{n-1} - Q_{n-2} \) for all \( n \geq 3 \). For the same coefficient function \( \tau \), if we instead consider immediate predecessors \( \hat{\beta}^n = \tau^{n-3} + 2n\beta^{n-2} + 3n\beta^{n-1} \), then the recursion turns out to be \( Q_n = (3n+1)Q_{n-1} - (n-3)Q_{n-2} - (n+1)Q_{n-3} \) for all \( n \geq 4 \).

Recall from Sect. 2.2 the example of a fundamental sequence \( Q \) given by \( Q_{n+2} = (n+1)Q_{n+1} + Q_n \) for all \( n \geq 1 \) and \( Q = (1, 2, 5, 17, \ldots) \). This recursion is obtained from \( \hat{\beta}^n = \sum_{k=0}^{[(n-1)/2]} (n-1-2k)\beta^{n-1-2k} = (\ldots, 0, n-5, 0, n-3, 0, n-1, 0) \) for all \( n \geq 2 \), and if we use the same idea demonstrated in this section and the expression \( \hat{\beta}^n = (n-1)\beta^{n-1} + \beta^{n-2} \) for all \( n \geq 4 \), i.e.,

\[
Q_n = (n-1)Q_{n-1} + (n-3)Q_{n-3} + \cdots, \\
Q_{n-2} = (n-3)Q_{n-3} + (n-5)Q_{n-5} + \cdots.
\]

then we obtain \( Q_{n+2} = (n+1)Q_{n+1} + Q_n \) for all \( n \geq 1 \).

Conversely, if we have

\[
Q_{n+3} = f_1(n+2)Q_{n+2} + f_2(n+1)Q_{n+1} + f_3(n)Q_n
\]

for all \( n \geq 1 \), for example, where \( f_k(n) \) for \( k = 1, 2, 3 \) are non-zero polynomials with non-negative integer coefficients, we may recursively define immediate predecessors as follows: \( \hat{\beta}^n = f_1(n-1)\beta^{n-1} + f_2(n-2)\beta^{n-2} + (f_3(n-3) - 1)\beta^{n-3} + \hat{\beta}^{n-3} \) for all \( n \geq 5 \), \( \hat{\beta}^4 = f_1(3)\beta^3 + f_2(2)\beta^2 + (f_3(1) - 1)\beta^1 \), \( \hat{\beta}^3 = f_1(2)\beta^2 + f_2(1)\beta^1 \), and \( \hat{\beta}^2 = f_1(1)\beta^1 \). For example,

\[
\hat{\beta}^6 = f_1(5)\beta^5 + f_2(4)\beta^4 + (f_3(3) - 1)\beta^3 + f_1(2)\beta^2 + f_2(1)\beta^1.
\]

4 *Examples of \( \mathcal{E} \)-Zeckendorf subsets*

Recall from Sect. 2 that \( a\mathbb{N} \) is an \( \mathcal{L} \)-subset of \( \mathbb{N} \) if \( L = (1, 1, 1) \) and \( a \) is a positive integer. In fact, \( a\mathbb{N} \) is an \( \mathcal{E} \)-subset of \( \mathbb{N} \) for all Zeckendorf collections \( \mathcal{E} \) since \( \mathbb{N} \) is an \( \mathcal{E} \)-set of numbers for all Zeckendorf collections \( \mathcal{E} \) by Theorem 16. It turns out that other congruence classes are essentially not an \( \mathcal{E} \)-subset for any Zeckendorf collection \( \mathcal{E} \) that contains more than the basis coefficient functions \( \beta^n \) and the zero coefficient function. We shall call such a collection a nontrivial Zeckendorf collection.

**Proposition 29** Given an integer \( a > 1 \) and an integer \( 0 < j < a \) such that \( \gcd(j, a) = 1 \), the subset \( j + a\mathbb{N} \) is not an \( \mathcal{E} \)-subset if \( \mathcal{E} \) is not trivial.

**Proof** Let \( \mathcal{E} \) be a nontrivial Zeckendorf collection for \( \mathbb{N} \), and suppose that \( j + a\mathbb{N} \) is an \( \mathcal{E} \)-subset of \( \mathbb{N} \), so that it has a fundamental sequence \( Q \). Since \( \mathcal{E} \) is not trivial, there must be a coefficient function \( \mu \) whose immediate successor is given by \( \hat{\mu} = \beta^1 + \mu \). Then, \( \{Q_1, \sum \mu Q \} < j + a\mathbb{N} \) implies \( \sum \hat{\mu} Q = Q_1 + \sum \mu Q \equiv j + j \mod a \). Since \( \sum \hat{\mu} Q \in j + a\mathbb{N} \), it follows that \( 2j \equiv j \mod a \), and since \( \gcd(j, a) = 1 \), it implies that \( 2 \equiv 1 \mod a \), which is a contradiction. This proves the proposition.

In fact, if \( \mathcal{E} \) is trivial, then we may define \( Q_k := j + ak \) and \( j + a\mathbb{N} = X_Q \). The only \( \mathcal{E} \)-expansions are \( Q_k \) since \( \mathcal{E} = \{\beta^k : k \geq 1\} \cup \{0\} \). The case of \( \gcd(j, a) > 1 \) is reduced
\[ j + aN = d(j_0 + a_0N) \] where \( d = \gcd(j, a) \). If \( a_0 > 1 \), then the proposition implies that \( \mathcal{E} \) must be trivial, but if \( a_0 = 1 \), then the answer is different.

**Proposition 30** Let \( j \) be a positive integer. Then, \( j + \mathbb{N} \) is an \( \mathcal{E} \)-subset of \( \mathbb{N} \) for a nontrivial Zeckendorf collection \( \mathcal{E} \).

**Proof** Let \( \hat{\beta}^n = \beta^{n-1} \) for \( 2 \leq n \leq j + 1 \), and \( \hat{\beta}^n = \beta^i + \beta^{n-1} \) for \( n \geq j + 2 \). Let \( \mathcal{E} \) be the Zeckendorf collection determined by immediate predecessors \( \hat{\beta}^n \) for \( n \geq 2 \). Let \( Q \) be an increasing sequence given by \( Q_k = j + k \) for \( 1 \leq k \leq j \), and \( Q_n = j(n - j) \) for \( n \geq j + 1 \). Then, \( (Q_1, \ldots, Q_j) \) makes a complete residue system mod \( j \), and the terms \( Q_k \) for \( k \geq j + 1 \) represent multiples of \( j \). Hence, given a number \( j + x \in j + \mathbb{N} \), by the long division algorithm, there are unique integers \( k \) and \( q \) such that \( x = k + jq \) and \( 1 \leq k \leq j \). If \( q = 0 \), i.e., \( 1 \leq x \leq j \), then \( j + x = Q_k \), and if \( q > 0 \), i.e., \( x > j \), then \( j + x = Q_k + Q_n \)

First notice that the following are typical sequences of immediate successors:

\[ \beta^1 <_a \beta^2 <_a \cdots <_a \beta^i, \]

Given \( n \geq j + 1 \),

\[ \beta^n <_a \beta^1 + \beta^n <_a \beta^2 + \beta^n <_a \cdots <_a \beta^i + \beta^n <_a \beta^{n+1}. \]

Then, given a number \( j + x \in j + \mathbb{N} \), by the earlier decomposition using the long division algorithm, there is a unique coefficient function \( \mu \in \mathcal{E} \) such that \( j + x = \sum \mu Q \). This proves that \( j + x \in \mathcal{E} \). Since all non-zero coefficient functions \( \mu \in \mathcal{E} \) are \( \beta^k \) or \( \beta^k + \beta^n \) for \( 1 \leq k \leq j \) and \( n \geq j + 1 \), clearly \( \sum \mu Q \in j + \mathbb{N} \), i.e., \( X_Q \subset j + \mathbb{N} \).

Let us introduce another method of generating \( \mathcal{E} \)-subsets of \( \mathbb{N} \), and it is inspired by the shifting function introduced in [13]. Let \( Z \) be the sequence given by \( Z_k := Q_k + 1 \) where \( Q \) is a fundamental sequence for the \( \mathcal{E} \)-set of numbers \( \mathbb{N} \), and let \( \psi_Q : \mathbb{N} \to \mathbb{N} \) be the function defined by \( \psi_Q(\sum \epsilon Q) = \sum \epsilon Z \). The unique representation property of \( Q \) under the \( \mathcal{E} \)-Zeckendorf condition implies that the function is well-defined. For example, if \( L = (1, 1) \) is a Zeckendorf multiplicity list and \( Q \) is the Fibonacci sequence under the \( \mathscr{L} \)-Zeckendorf condition, then \( \psi_Q(100) = \psi(F_{10} + F_5 + F_3) = F_{11} + F_6 + F_4 = 162 \), and in general, \( \psi_Q(n) = \phi n + O(1) \) where \( \phi \) is the golden ratio. Clearly the subset \( \psi_Q(\mathbb{N}) \) is an \( \mathscr{L} \)-subset, and \( \psi_Q(\mathbb{N}) = (\{ \epsilon Q : \epsilon \in \mathscr{L} \& \epsilon_1 = 0 \} \) has a positive proportion of \( \mathbb{N} \) in the following sense: \( \frac{1}{\epsilon} \# \{ m \leq x : m \in \psi_Q(\mathbb{N}) \} \to \phi - 1 \) as \( x \to \infty \). If \( \mathcal{E} \) is an arbitrary Zeckendorf collection, there is no difficulty in defining the shifting function \( \psi_Q : \mathbb{N} \to \mathbb{N} \) where \( Q \) is a fundamental sequence for the \( \mathcal{E} \)-set of numbers \( \mathbb{N} \), and \( \psi_Q(\mathbb{N}) \) and \( \psi_Q(\cdots (\psi_Q(\mathbb{N}))) \) generate a good deal of interesting \( \mathcal{E} \)-subsets of \( \mathbb{N} \).

Another interesting way of constructing \( \mathcal{E} \)-subsets are as follows. Let \( \widetilde{\mathcal{E}} \) be a Zeckendorf collection, and suppose that it has a proper Zeckendorf subcollection \( \mathcal{E} \). If \( Q \) is a decreasing fundamental sequence for the \( \widetilde{\mathcal{E}} \)-set of numbers \( R \), then we may investigate the subset \( X_Q^{\widetilde{\mathcal{E}}} \). Recall from Section 2 the sequence \( Q \) given by \( Q_k := 2^k - 1 \) for \( k \geq 1 \), which is an increasing fundamental sequence under the \( \mathscr{L} \)-Zeckendorf condition determined by a Zeckendorf multiplicity list \( \widetilde{L} = (1, 2) \), and the subset \( X_Q^{\mathcal{E}} \) where \( \mathcal{E} \) is the Zeckendorf collection determined by \( L = (1, 1) \). The \( \mathscr{L} \)-subset \( X_Q^{\mathcal{E}} \) contains nearly zero proportion of \( \mathbb{N} \) since \( \# \{ m \leq x : m \leq 2^k \} = F_{k+1} \) which implies \( \# \{ m \in X_Q^{\mathcal{E}} : m < x \} = O(x^{0.82}(\phi)) \) where \( \phi \approx 1.6 \) is the golden ratio.
Let us consider Zeckendorf collections \( \tilde{J} \) and \( J \) of coefficient functions for \( I \) determined by Zeckendorf multiplicity lists \( \tilde{L} = (1, 2) \) and \( L = (1, 1) \), respectively. Then, the sequence \( Q \) given by \( Q_k = 1/2^k \) for \( k \geq 1 \) is a decreasing fundamental sequence for the \( \tilde{J} \)-interval \( I \). As in the case of \( \mathbb{N} \), the subset generated by \( Q \) under the \( L \)-Zeckendorf condition is small in the following sense:

**Proposition 31** Let \( Q \) be the sequence given by \( Q_k = 1/2^k \) for all \( k \geq 1 \), and let \( L = (1, 1) \). Then, the \( L \)-subset \( X_Q \) is Lebesgue measurable, and it has Lebesgue measure zero.

**Proof** Let \( \tilde{L} = (1, 2) \) be a Zeckendorf multiplicity list, and let \( \tilde{J} \) be the Zeckendorf collection determined by \( \tilde{L} \). Let \( S \) be the subset of \( X_Q \) consisting of \( \sum \epsilon Q \) such that \( \epsilon \in L \) is non-zero, and has finite support. Given a positive integer \( b \), let \( S_b \) be the subset of \( S \) consisting of \( \epsilon \) such that \( \epsilon_b = 1 \) and \( \epsilon_k = 0 \) for all \( k > b \). Then, \( S \) is the disjoint union of \( S_b \) where \( b \) varies over the positive integers.

Given \( \gamma \in S \), find the index \( b \) such that \( \gamma \in S_b \). Let \( \gamma_0 := \gamma + 1/2^{b+1} \), and

\[
I_\gamma := (\gamma_0, \gamma_0 + 1/2^{b+1}) = \{ \gamma_0 + \sum \delta Q : \delta \in \tilde{J}, b + 1 < \text{ord}(\delta) < \infty \}.
\]

Then, the binary expansions of a real number \( x \) in the intervals \( I_\gamma \) are in the form of \( \sum_{k=1}^{b-1} \epsilon_k 2^{-k} + 2^{−b} + 2^{−b−1} + \sum_{k=b+2}^{\infty} \delta_k 2^{−k} \) where \( \epsilon \in L \) and \( \delta \in \tilde{J} \), which are non-zero. Since \( \epsilon \in L \), the sum \( 2^{−b} + 2^{−b−1} \) is the first adjacent terms in the expansion of \( x \). Conversely, if the binary expansion of a number in \( I_\gamma \) is given, we can identify \( \gamma \) by finding the first two adjacent terms. If \( \gamma \) and \( \gamma' \) are two numbers in \( S \), and \( I_\gamma \cap I_{\gamma'} \neq \emptyset \), then the first two adjacent terms of the (unique) binary expansion of a number in \( I_\gamma \cap I_{\gamma'} \) determine \( \gamma \) and hence, \( \gamma = \gamma' \). Thus, \( \bigcup_{\gamma \in S} I_\gamma \) forms a disjoint (countable) union of open intervals. For all \( \gamma \in S \), the left endpoint \( \gamma_0 \) of the interval \( I_\gamma \) is not a member of \( X_Q \), but the right endpoint may or may not be a member of \( X_Q \). Let \( J_\gamma^0 \) denote \( [\gamma_0, \gamma_0 + 1/2^{b+1}] \) if the endpoint is not in \( X_Q \), and \( [\gamma_0, \gamma_0 + 1/2^{b+1}] \) otherwise. Then since \( S \) is countable, the union \( \bigcup_{\gamma \in S} J_\gamma^0 \) is Lebesgue measurable, and \( I - X_Q = \bigcup_{\gamma \in S} J_\gamma^0 \). It is a straightforward induction exercise to show that given an integer \( b \geq 1 \), the Fibonacci term \( F_{b−1} \) where \( F_0 = 1 \) is the number of coefficient functions in \( S_b \). Let \( \text{msr} \) denote the Lebesgue measure. Then, it follows

\[
\text{msr}(I - X_Q) = \text{msr} \left( \bigcup_{\gamma \in S} J_\gamma^0 \right) = \text{msr} \left( \bigcup_{\gamma \in S} I_\gamma \right) = \sum_{b=1}^{\infty} \sum_{\gamma \in S_b} \text{msr}(I_\gamma) = \sum_{b=1}^{\infty} \sum_{b=1}^{\infty} F_{b−1} 2^{b+1} = 1.
\]

Therefore, \( X_Q \) is Lebesgue measurable, and \( \text{msr}(X_Q) = 0 \).

\[ \square \]

### 5 Proofs

We prove the results that are introduced in Sect. 2. The following lemma is fundamental to the connection between the two descriptions of Zeckendorf collections introduced in Theorem 7.

**Lemma 32** Let \( \delta \) be a Zeckendorf collection for positive integers, and let \( \delta = \alpha + \rho \in \delta \) where \( \alpha \) and \( \rho \) are coefficient functions such that \( \alpha \neq 0 \) and \( \rho \equiv 0 \) \( \text{res } [1, r] \) where \( r := \text{ord}(\alpha) \). Then, there is an index \( t \) such that \( t > r, \text{res}_{[t, t]}(\rho) \) is a proper \( \beta \)-block at index \( t − 1 \), and one of the successors of \( \delta \) in the lexicographical order is \( \beta^1 + \text{res}_{[1, \infty]}(\rho) \).

**Proof** Let \( \delta^1 := \delta \), and let \( \delta^k \) be the immediate successor of \( \delta^{k−1} \) for \( k = 2, \ldots \), which are obtained by adding \( \beta^1 \) or replacing a copy of \( \beta^u \) for some index \( u \). Let \( S \) be the
set of indices $k$ such that $\delta^k \equiv \delta \text{ res } (r, \infty)$. Since $1 \in S$, the set $S$ is non-empty, and by Definition 2, it is a finite set. We denote by $K$ the largest index in $S$. By the choice of $K$, we must have $\delta^{K+1} \neq \delta^K + \beta^1$, because otherwise we would have $K + 1 \in S$. Then, by Part 2 of Definition 2, there is an index $t \geq 2$ such that $\delta^K \equiv \beta^1 \text{ res } [1, t)$ and $\delta^{K+1} = \beta^1 + \text{ res}_{[t, \infty)}(\delta^K)$. Let us claim that $t > r$. If $t \leq r$, then $\delta^{K+1} \equiv \delta^K \equiv \delta \text{ res } (r, \infty)$, which implies $K + 1 \in S$.

Write $\delta^K = \alpha^K + \rho$. Since $\delta^K \equiv \delta \equiv \rho \text{ res } (r, \infty)$, we have $\alpha^K \equiv 0 \text{ res } (r, \infty)$. Notice that $\beta^1 \equiv \alpha^K + \rho \text{ res } [1, t) \Rightarrow \beta^1 \equiv \alpha^K + \rho \text{ res } [r, t)$. Since $\alpha^K \equiv 0 \text{ res } (r, \infty)$, it follows that $\beta^1 \equiv \alpha^K$ and $\beta^1 = \alpha^K + \rho \equiv \rho \text{ res } (r, t)$. By the definition of the lexicographical order, $\alpha^K \geq r$, and hence, and $\beta^1 = \alpha^K + \rho > \rho$. Thus, $\rho \text{ res } [r, t)$ is a proper $\beta$-block at index $t - 1$. In particular, if $t = r + 1$ and $\rho_r = 0$, then the proper $\beta$-block at index $t - 1$ is the zero coefficient function.

Corollary 33 Let $\rho$ be the coefficient function defined in Lemma 32. Then $\rho$ has a decomposition into proper $\beta$-blocks.

Proof Let $t_0 := r$. By Lemma 32, there is an index $t_1 > t_0$ such that $\zeta^1 := \text{ res}_{[t_0, t_1)}(\rho)$ is a proper $\beta$-block at index $t_1 - 1$. Also since $\beta^{t_1} + \text{ res}_{[t_1, \infty)}(\rho)$ is a member of $\mathcal{E}$, the lemma with $\alpha = \beta^{t_1}$ again implies that there is an index $t_2 > t_1$ such that $\zeta^2 := \text{ res}_{[t_1, t_2)}(\rho)$ is a proper $\beta$-block at index $t_2 - 1$ where $\beta^{t_2} + \text{ res}_{[t_2, \infty)}(\rho) \in \mathcal{E}$. This process continues and generates an infinite sequence of coefficient functions $\beta^{t_k} + \text{ res}_{[t_k, \infty)}(\rho)$ in $\mathcal{E}$ where $\text{ res}_{[t_k-1, t_k]}(\rho)$ is a proper $\beta$-block at index $t_k - 1$ for $k \geq 1$. Then, the expression $\rho = \sum_{k=1}^{\infty} \text{ res}_{[t_k-1, t_k]}(\rho)$ serves as a decomposition into proper $\beta$-blocks where all $\beta$-blocks with sufficiently large $k$ are the zero coefficient function (since $\rho$ has finite support).

The following lemma is also fundamental to the connection described in Theorem 7.

Lemma 34 Let $E$ be the ascendingly-ordered collection of coefficient functions determined by $\{\delta^n : n \geq 2\}$ defined in Definition 8. Then, the following are true:

1. Each coefficient function $\mu$ in $E$ has a unique $\delta$-block decomposition, and the largest $\delta$-block in the decomposition is given at index ord($\mu$).
2. Each coefficient function in $E$ has a unique immediate successor in $E$, and it is given by the rule described in Theorem 7, Part 1 (b) where $\beta^n = \delta^n$ for each $n \geq 2$. In particular, for each $n \geq 2$, $\delta^n$ is the immediate predecessor of $\beta^n$, and the rule satisfies Definition 2, Part 2.
3. Given $\mu \in E$, there are only finitely many coefficient functions in $E$ that are less than $\mu$.

In particular, $E$ is a Zeckendorf collection for positive integers with $\beta^n = \delta^n$ for each $n \geq 2$.

Proof Let us prove Part 3. If $\mu' <_a \mu$ are two coefficient functions in $E$, then ord($\mu'$) $\leq$ ord($\mu$), and $\mu'$ is written in terms of $\delta$-blocks of order $\leq$ ord($\mu$). Since there are only finitely many $\delta$-blocks of order $\leq$ ord($\mu$), we prove Part 3.

Let us use the mathematical induction on the order of coefficient functions to prove Part 1. The case of ord($\epsilon$) = 0 is trivial. Assume that the coefficient functions $\epsilon \in E$ with ord($\epsilon$) $= s \leq M$ for some $M \geq 0$ have a unique $\delta$-block decomposition, and let $\mu$ be a coefficient function in $E$ with ord($\mu$) = $M + 1$. Let $\mu = \sum_{m=1}^{A} \xi^m = \sum_{m=1}^{B} \xi^m$ be
two \( \delta \)-block decompositions with supports \([i_m, n_m]\) and \([j_m, \ell_m]\), respectively, such that \( i_1 = 1, n_m + 1 = i_{m+1} \) for \( 1 \leq m \leq A, \), \( \zeta^A \neq 0, j_1 = 1, \ell_m + 1 = j_{m+1} \) for \( 1 \leq m \leq B, \). 

Since \( \text{ord}(\mu) = \text{ord}(\xi^A) = \text{ord}(\xi^B) \), both \( \xi^A \) and \( \xi^B \) are non-zero \( \delta \)-blocks at index \( c := n_A = t_B \), and for convenience, let \([a, c] \) and \([b, c] \) denote the supports of \( \xi^A \) and \( \xi^B \), respectively. If \( b < a \), then \( \delta_{a+1}^\xi = \delta_{a+1}^\xi = \delta_{a+1}^\xi < \delta_{a+1}^\xi \), which is a contradiction, and \( a < b \) implies a similar contradiction. Thus, we conclude \( a = b \), and after cancelling \( \xi^A \) and \( \xi^B \), by induction hypothesis, we prove the uniqueness property. Given a unique \( \delta \)-block decomposition \( \mu = \sum_{m=1}^A \zeta^m \) where \( \text{ord}(\mu) = \text{ord}(\xi^A) \), it is clear that the largest \( \delta \)-block is clearly given by \( \xi^A \).

Let us prove Part 2. First let us prove that \( \delta^n \) is the immediate predecessor of \( \beta^n \) for each \( n \geq 2 \). By Part 3, an immediate predecessor of coefficient function greater than \( \beta^1 \) uniquely exists. Let \( \mu := \sum_{m=1}^M \zeta^m \) be the \( \delta \)-block decomposition of the immediate predecessor of \( \beta^n \). Then, \( \beta^{n-1} \leq_a \zeta^M \), and hence, \( \zeta^M \) has index \( n-1 \). If \( \zeta^M \) is the maximal \( \delta \)-block at index \( n-1 \), and hence, \( M = 1 \), then \( \zeta^M = \delta^n \), which proves the assertion. If \( \zeta^M \) is a proper \( \delta \)-block at index \( n-1 \), then there is an index \( i \leq n-1 \) such that \( \zeta_i^M \leq \delta_i^n \) and \( \zeta_i^M = \delta_i^n \) for \( i < k \leq n-1 \). However, this implies that \( \mu \leq_a \delta^n \), which contradicts that \( \mu \) is not the immediate predecessor of \( \beta^n \). This concludes the proof of \( \delta^n = \hat{\beta}^n \).

Let \( \mu := \sum_{m=1}^M \zeta^m \) be the \( \delta \)-block decomposition of \( \mu \in E \) with supports \([i_m, n_m]\) with \( n_m + 1 = i_{m+1} \) for \( 1 \leq m \leq M - 1 \). If \( \zeta^1 \) is a proper \( \delta \)-block, then clearly \( \beta^1 + \mu \) is the unique immediate successor of \( \mu \) in \( E \). Suppose that \( \zeta^1 = \delta^{n+1} \), and there is \( \mu^1 \in E \) such that

\[
\mu := \zeta^1 + \sum_{m=2}^M \zeta^m <_a \mu^1 <_a \mu^2 := \beta^{n+1} + \sum_{m=2}^M \zeta^m = \beta_i^2 + \sum_{m=2}^M \zeta^m.
\]

Then, \( \mu_k^1 = \mu_k^2 \) for all \( k > i_2 \Rightarrow \mu_k^1 = \mu_k^2 = \zeta_{i_2}^2 \) for all \( k > i_2 \),

and \( \mu_{i_2} = \zeta_{i_2}^2 <_a \mu_{i_2}^1 = \mu_{i_2}^2 <_a 1 + \zeta_{i_2}^2 \).

If \( \mu_{i_2}^1 = \mu_{i_2}^2 \), then \( \mu_k^1 < \mu_k^2 = 0 \) for some \( k < i_2 \), which is not possible. So, it follows that \( \mu_{i_2} = \mu_{i_2}^1 = \zeta_{i_2}^2 \), and hence, there is a coefficient function \( \tau \) of order \( n_1 \) that is larger than \( \zeta^1 = \delta^{n+1} \) such that \( \mu^1 = \tau + \sum_{m=1}^M \xi^m \). If \( M \geq 2 \), by Part 1, \( \zeta^M \) is an \( \delta \)-block appearing in the \( \delta \)-block decomposition of \( \mu^1 \), and \( \tau + \sum_{m=1}^{M-1} \xi^m \) remains to be a member of \( E \). By repeating this process, we find that \( \tau \) is a member of \( E \), which is true for \( M = 1 \) as well. However, \( \delta^{n+1} <_a \tau <_a \beta^{n+1} \) contradicts that \( \delta^{n+1} \) is the immediate predecessor of \( \beta^{n+1} \). Therefore, we conclude that \( \mu^2 \) is the immediate successor of \( \mu \) in \( E \).

Let us prove that \( E \) is Zeckendorf. Part 3 of this lemma proves the property described in Definition 2, Part 1. Part 2 of this lemma proves the property described in Definition 2, Part 2.

\[ \square \]

Proposition 35 Let \( \mathcal{E} \) be the collection defined in Lemma 32 with immediate predecessors \( \hat{\beta}^n \) for \( n \geq 2 \), and let \( E \) be the collection defined in Lemma 34 where \( \delta^n = \hat{\beta}^n \) for each \( n \geq 2 \). Then, the following are true:

1. If \( \mu \in E \cap \mathcal{E} \), then its immediate successor in \( E \) is equal to its immediate successor in \( \mathcal{E} \).
2. The collection \( E \) is a subset of \( \mathcal{E} \).
**Proof** Since $0 \in E \cap E'$, the intersection is non-empty. Let $\mu^0$ be a coefficient function in $E \cap E'$, and let $\mu^1$ be the immediate successor of $\mu^0$ in $E$. Let us show that the immediate successor of $\mu^0$ in $E'$ is $\mu$. Let $\mu^0 = \sum_{m=1}^{M} \xi^m$ be the $\hat{\beta}$-block decomposition. Suppose that $\xi^1$ is a proper $\hat{\beta}$-block. Then, by Part 2 of Lemma 34, we have $\mu^1 = \beta^1 + \mu^0$. By contradiction, assume that the immediate successor $\mu^0$ in $E'$ is not $\mu^1$. Then, by Part 2 of Definition 2, there is $n \geq 2$ such that $\mu^0 \equiv \hat{\beta}^n \text{ res } [1, n)$ and the immediate successor of $\mu^0$ in $E'$ is given by $\beta^t + \text{ res } [t, \infty)(\mu^0)$, where $\beta^t$ is an $\hat{\beta}$-block decomposition into proper $\hat{\beta}$-blocks for $t = 1, \ldots, T$, and hence, $\mu^0 = \beta^t + \sum_{k=1}^{T} \xi^m$, which is an $\hat{\beta}$-block decomposition. However, this violates the uniqueness of $\hat{\beta}$-block decompositions since the decomposition $\mu^0 = \sum_{m=1}^{M} \xi^m$ begins with a proper $\hat{\beta}$-block, and hence, the immediate successor of $\mu^0$ in $E'$ is given by $\beta^1 + \mu^0$, which coincides with $\mu^1$ by Lemma 34, Part 2.

Suppose that $\xi^1 = \hat{\beta}^{m+1}$. Then, by Part 2 of Lemma 34, we have $\mu^1 = \beta^{m+1} + \sum_{m=3}^{M} \xi^m$. By contradiction, assume that the immediate successor of $\mu^0$ in $E'$ is given by $\beta^1 + \mu^0 \in E'$. Then, by Corollary 33, $\text{ res } [1, \infty)(\mu^0)$ has the $\hat{\beta}$-block decomposition into proper $\hat{\beta}$-blocks, i.e., $\mu^0 = \sum_{k=1}^{T} \xi^m$ where $\xi^m$ are proper. However, since the smallest $\hat{\beta}$-block of $\mu^0 = \sum_{m=1}^{M} \xi^m$ is maximal, it violates the uniqueness of $\hat{\beta}$-block decompositions of members of $E$, and hence, the immediate successor of $\mu^0$ in $E'$ is given by $\beta^t + \text{ res } [t, \infty)(\mu^0)$ for some $t$ where $\beta^t \equiv \mu^0 \text{ res } [1, t)$. By Corollary 33, $\text{ res } [t, \infty)(\mu^0) = \sum_{m=1}^{\infty} \theta^m$ is a decomposition into proper $\hat{\beta}$-blocks, and hence, $\mu^0 = \beta^t + \sum_{m=1}^{\infty} \theta^m$ is an $\hat{\beta}$-block decomposition. By the uniqueness of the $\hat{\beta}$-block decomposition, we have $t = n + 1$, and the immediate successor of $\mu^0$ in $E'$ coincides with $\mu^1$ by Lemma 34, Part 2. For both cases, $\mu^1$ is the immediate successor of $\mu^0$ in $E'$ as well as in $E$.

Let us show that $E \subseteq E'$. By Lemma 34, $E$ is Zeckendorf, and by Lemma 3, $E = \{ \tau^n : n \geq 0 \}$ where $\tau^n$ are the coefficient functions defined in Lemma 3. By Part 2 above, if $\tau^n \in E \cap E'$, then $\tau^{n+1} \in E \cap E'$. Since $\tau^0 \in E \cap E'$, by induction, we prove $E \subseteq E'$.

**Lemma 36** Let $\mathcal{E}$ and $E$ be the collections defined in Proposition 35. Then, $\mathcal{E}$ is a subset of $E$. In particular, $\mathcal{E} = E$.

**Proof** Let $\mu$ be a coefficient function in $\mathcal{E}$. Let us show that $\mu$ has an $\hat{\beta}$-block decomposition. If $\tilde{\mu} = \beta^1 + \mu$, then by Corollary 33, $\mu$ has a decomposition into proper $\hat{\beta}$-blocks, and hence, $\mu \in E$. Suppose that $\beta^T \equiv \mu \text{ res } [1, T)$ and $\tilde{\mu} = \beta^T + \text{ res } [T, \infty)(\mu)$. Then, by Corollary 33, we conclude that $\text{ res } [T, \infty)(\mu) \in \mathcal{E}$ has a block decomposition into proper $\hat{\beta}$-blocks $\xi^m$ for $m \geq 1$. Thus, $\mu = \beta^T + \sum_{m=1}^{\infty} \xi^m$ makes an $\hat{\beta}$-block decomposition, and hence, $\mu \in E$. By Proposition 35, we prove that $\mathcal{E} = E$.

**5.1 Proofs of Theorem 7 and 13**

Let’s prove Theorem 7, Part 2. Let $\mathcal{E}$ be a Zeckendorf collection for positive integers, and let $E$ be the ascendingly-ordered collection determined by the immediate predecessors $\hat{\beta}^n$ of $\mathcal{E}$ for $n \geq 2$. If $\tau^m$ for $m = 1, \ldots, M$ are $\hat{\beta}$-blocks defined in Theorem 7, Part 2, then $\mu := \sum_{m=1}^{M} \tau^m$ is a member of $E$, and by Lemma 36, $\mu$ is a member of $\mathcal{E}$.

Let’s prove Theorem 7, Part 1. Let $\mathcal{E}$ be a Zeckendorf collection for positive integers, and let $E$ be the ascendingly-ordered collection determined by $\hat{\beta}^n$ of $\mathcal{E}$ for $n \geq 2$. Then by Lemma 36, $\mathcal{E} = E$, and this immediately proves the property of Theorem 7, Part 1 (a). The property of Part 1(b) follows from Lemma 34, Part 2 since $\mathcal{E} = E$. Let us prove the if-part
of Theorem 7, Part 1. Let $\mathcal{E}$ be an ascendingly-ordered collection of coefficient functions satisfying properties described in Theorem 7, Part 1. Then, Theorem 7, Part 1(a) implies that $\mathcal{E}$ is the subset of the ascendingly-ordered collection $E$ determined by $\hat{\beta}^n$ for $n \geq 2$. On the other hand, by Lemma 34, $E$ is Zeckendorf, and by Lemma 3, $E = \{\tau^n : n \geq 0\}$ where $\tau^n$ are the coefficient functions defined in Lemma 3. By Lemma 34, Part 2, each $\tau^n$ is obtained by applying the rule given in Theorem 7, Part 1(b), beginning with $\tau^0$, and hence, $E$ is a subset of $\mathcal{E}$. Thus, $E = \mathcal{E}$ is Zeckendorf.

Let us prove Corollary 9. Let $\mathcal{E}$ be the collection defined in the corollary. Then by Lemma 34, $\mathcal{E}$ is Zeckendorf, and $\hat{\beta}^n = \delta^n$ for each $n \geq 2$. If $\mathcal{F}$ is another Zeckendorf collection for positive integers with the immediate predecessors $\hat{\beta}^n = \delta^n$ for $n \geq 2$, then by Lemma 36, $\mathcal{F} = \mathcal{E}$, and hence, the ascendingly-ordered collection $\mathcal{E}$ is the only Zeckendorf collection for positive integers such that $\hat{\beta}^n = \delta^n$ for each $n \geq 2$.

Let us prove Theorem 13. Let $\mathcal{E}$ be a Zeckendorf collection for the interval $I$. The proofs of Corollary 33, Lemma 34, Proposition 35, and Lemma 36 remain valid for coefficient functions in $\mathcal{E}^M$ for $M \geq 1$ simply by reversing the order of the values of the coefficient functions. Let us make references to these versions by the descending version of Statement $X$, e.g., the descending version of Corollary 33.

Suppose that $\mathcal{E}$ is Zeckendorf for the interval $I$, and let us prove Theorem 13, Part 1 (b). Let $\epsilon \in \mathcal{E}$, and let $M_k$ for $k \geq 1$ be the finest sequence guaranteed by Definition 10, Part 4, i.e., $M_1$ is the smallest choice, and recursively, $M_{k+1}$ is the smallest choice that is greater than $M_k$. By the descending version of Corollary 33, for each $\epsilon^k := \epsilon \mod M_k$, we find a decomposition of $\epsilon^k$ into proper $\hat{\beta}$-blocks. Notice that if $\epsilon \neq 0$ and $\epsilon^k = \sum_{m=1}^{T} \zeta^m$ is the $\hat{\beta}$-block decomposition, then by the descending version of Lemma 34, Part 1, the largest non-zero $\hat{\beta}$-block of the $\hat{\beta}$-block decomposition of $\epsilon^{k+1}$ for sufficiently large $k$ is equal to the largest non-zero $\hat{\beta}$-block of $\epsilon^k$. This implies that $\epsilon^{k+1} = \sum_{m=1}^{T+1} \zeta^m$ is the $\hat{\beta}$-block decomposition for some proper $\hat{\beta}$-block $\zeta^{T+1}$ while the smaller $\hat{\beta}$-blocks are the same ones of $\epsilon^k$. This proves Theorem 13, Part 1 (b). Let $E$ be the descendingly-ordered collection determined by $\{\hat{\beta}^n : n \geq 1\}$. The descending version of Lemma 36 applied to $\mathcal{E}^M$ and $E^M$ for each $M \geq 1$ implies Part 1 (c), and this concludes the proof of the only-if part of Theorem 13, Part 1.

Let us prove the if-part of Theorem 13. Let $\mathcal{E}$ be an descendingly-ordered collection of coefficient functions satisfying properties described in Theorem 13, Part 1, and let $E$ be the descendingly-ordered collection determined by $\{\hat{\beta}^n : n \geq 1\}$. As argued for the collections for positive integers, we find $\mathcal{E}^M = E^M$ for each $M \geq 1$, and hence, $\mathcal{E} = E$. The descending version of Lemma 34 implies Definition 10, Part 1, and Definition 10, Part 2 and 3 follow from the statements of Theorem 13, Part 1 (a) and (c) since $\mathcal{E} = E$. Let us prove Definition 10, Part 4. Let $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ be the proper $\hat{\beta}$-block decomposition as in Theorem 13, Part 1 (b) with support $[n_m, i_m]$. Then, the increasing sequence $M_k := i_k$ satisfies the condition described in Definition 10, Part 4. Suppose that there are a coefficient function $\epsilon$ and an increasing sequence $M_k := i_k$ such that the condition described in Definition 10, Part 4 is satisfied, and let us prove that $\epsilon \in \mathcal{E}$. For each $M_k$, the descending versions of Corollary 33 and Lemma 34 imply the existence and uniqueness of proper $\hat{\beta}$-blocks $\zeta^j$ for $j = 1, 2, \ldots$ such that $\epsilon \equiv \sum_{j=1}^{\infty} \zeta^j \mod M_k$ for $k = 1, 2, \ldots$. Hence, $\epsilon = \sum_{j=1}^{\infty} \zeta^j \in E \equiv \mathcal{E}$. 


Let us prove Theorem 13, Part 2, and let $\mathcal{E}$ be Zeckendorf. If $\epsilon = \sum_{j=1}^{\infty} \zeta^j$ is the sum of disjoint proper $\hat{\beta}$-blocks with support $[n_{m_1}, n_{m_2}]$, then the increasing sequence $M_k := i_k$ for $k \geq 1$ satisfies the condition of Definition 10, Part 4, and hence, $\epsilon \in \mathcal{E}$.

Let us prove Corollary 15. Let $\mathcal{E}$ be the collection defined in the corollary. By the definition of the descendingly-ordered collection determined by $\{\delta^n : n \geq 2\}$, the property described in Theorem 13, Part 1(a,b) is trivially satisfied. By the descending version of Lemma 34, Part 2, the properties described in Theorem 13, Part 1(c) are satisfied for two $Z_n$.

By the descending version of Lemma 34, Part 2, the immediate successor of $\delta$ is trivially satisfied. By the descending version of Lemma 34, Part 2, the immediate successor of $\delta$ is Zeckendorf. Let $M$ be an arbitrary positive integer. By the descending version of Lemma 34, Part 2, the immediate successor of $\delta$ is Zeckendorf. Let $M$ be an arbitrary positive integer. By the descending version of Lemma 34, Part 2, the immediate successor of $\delta$ is Zeckendorf. Let $M$ be an arbitrary positive integer. Hence, $\hat{\beta}^n = \delta^n$ for each $n \geq 1$. Let $\mathcal{F}$ be another Zeckendorf collection for $I$ with the same $\hat{\beta}^n$ for each $n \geq 1$. Then, by the descending version of Lemma 36, $\mathcal{F}^M = \mathcal{E}^M$ for each $M \geq 1$. Hence, $\mathcal{F} = \mathcal{E}$.

5.2. The integer case

Let us prove Theorem 16. Part 1(a) follows immediately from Algorithm 20, and we prove Part 1(b) here. Recall that given $\delta \in \mathcal{E}$, we denote by $\hat{\delta}$ the immediate successor of $\delta$ in $\mathcal{E}$.

**Lemma 37** Let $\mathcal{E}$ and $Q$ be the collection and the sequence defined in Theorem 16, Part 1(b), respectively. If $\delta \in \mathcal{E}$, then $Q_1 + \sum \delta Q = \sum \hat{\delta} Q$.

**Proof** If $\delta = \beta^1 + \delta$, and the statement $Q_1 + \sum \delta Q = \sum \hat{\delta} Q$ is clearly true. If $\delta \neq \beta^1 + \delta$, then $\delta = \hat{\beta}^n + \text{res}_{[n,\infty)}(\delta)$, and $\hat{\delta} = \beta^n + \text{res}_{[n,\infty)}(\delta)$. Thus,

$$Q_1 + \sum \delta Q = Q_1 + \sum \hat{\beta}^n Q + \sum_{k=n}^{\infty} \delta_k Q_k = Q_n + \sum_{k=n}^{\infty} \delta_k Q_k = \sum \hat{\delta} Q$$

where the second equality is obtained by using the recursive definition of $Q$. 

Let us prove that the function $f : \mathcal{E} \setminus \{0\} \to \mathbb{N}$ given by $f(\delta) = \sum \delta Q$ is bijective. Let $\delta^1 := \beta^1$, and recursively define $\delta^{k+1}$ to be the immediate successor of $\delta^k$ for $k \geq 1$, so that $\mathcal{E} = \{0\} \cup \{\delta^k : k \geq 1\}$ by Lemma 3. Let us use the induction to prove $f(\delta^n) = n$ for all $n \geq 1$. First, $f(\delta^1) = \sum \delta^1 Q = \sum \beta^1 Q = 1$, and suppose that there is an index $n \geq 1$ such that $f(\delta^k) = k$ for all $k \leq n$. Then, $f(\delta^{n+1}) = \sum \delta^{n+1} Q = Q_1 + \sum \delta^n Q$ by Lemma 37, and hence, by the induction hypothesis, $f(\delta^{n+1}) = Q_1 + n = n + 1$. Thus, each integer $n \in \mathbb{N}$ is equal to $\sum \delta Q$ for a unique non-zero coefficient function $\delta \in \mathcal{E}$. This concludes the proof of Part 1(b).

Part 2(a) follows immediately from Part 1. Let us prove Part 2(b). Note that for the first sentence of Part 2(b), the collection is not necessarily Zeckendorf. Let $Q$ and $Z$ be increasing fundamental sequences in $\mathbb{N}$ that generate a subset $Y$ under the $\mathcal{E}$-condition, i.e., $X_Q^\mathcal{E} = X_Z^\mathcal{E}$. Let us use the induction to show that $Q_n = Z_n$ for each $n \geq 1$. Since $Q_1$ and $Z_1$ are the smallest integers in $Y$, we have $Q_1 = Z_1$. Suppose that $Q_k = Z_k$ for all $1 \leq k < n$ where $n \geq 2$. Suppose that $Z_n < Q_n$. Then, there is a coefficient function $\mu \in \mathcal{E}$ such that $Z_n = \sum \mu Q$. If $\text{ord}(\mu) \geq n$, then $Z_n = \sum \mu Q \geq Q_n$, which contradicts $Z_n < Q_n$. Thus, $\text{ord}(\mu) < n$. By the induction hypothesis, $Z_n = \sum \mu Q = \sum_{k=1}^{n-1} \mu_k Q_k = \sum \mu Z$. Since $Z_n$ and $\sum \mu Z$ are distinct $\mathcal{E}$-representations, it contradicts that $Z$ has the unique $\mathcal{E}$-representation property. We conclude that $Q_n \leq Z_n$. If $Q_n < Z_n$, a similar contradiction
is derived, and we conclude that $Q_n = Z_m$. The second sentence of Part 2 (b) follows immediately from the choice of $Q$ we made for Part 2 (a), and the uniqueness of increasing sequences we just proved above.

### 5.3 The real number case

We prove Theorem 24 in this section. Let $E$ be a Zeckendorf collection for the unit interval $I$, and recall from Sect. 2 the maximal coefficient function $\hat{\beta}^n$ of order $n$. For convenience of stating results in this section, we define $Q_0 = 1$.

**Lemma 38** Let $Q$ be a decreasing fundamental sequence for the $E$-interval $I$. If $n \geq 1$, then $\sum \hat{\beta}^n Q \leq Q_{n-1}$.

**Proof** For $n = 1$, by definition of being a fundamental sequence for the $E$-interval $I$, we have $\sum_{k=1}^{M} \hat{\beta}_k Q_k < 1$ for large index $M$, and hence, $\sum \hat{\beta}^n Q = \lim_{M \to \infty} \sum_{k=1}^{M} \hat{\beta}_k Q_k \leq 1$.

For some $n \geq 2$, and suppose that $\sum \hat{\beta}^n Q > Q_{n-1}$, then, there is a proper $\hat{\beta}$-block $\xi$ at index $n$ such that $Q_{n-1} < \sum \xi Q$. Since there are only finitely many proper $\hat{\beta}$-blocks that are less than $\xi$, find the largest proper $\hat{\beta}$-block $\xi$ with index $[n, M]$ such that

$$\sum \xi Q \leq Q_{n-1} < \sum (\xi + \beta^M)Q.$$  

(12)

Then, by the uniqueness of $E$-expansions with the fundamental sequence, we have a strict inequality $\alpha := \sum \xi Q < Q_{n-1} < Q_n + Q_M$. Thus, $0 < Q_{n-1} - \alpha < Q_M$, and by the existence of $E$-expansions of numbers in $I$, we have $0 < Q_{n-1} - \alpha = \sum \delta Q < Q_M$ for $\delta \in E$, and $\ell := \text{ord}(\delta) > M$; if $\ell \leq M$, then $\sum \delta Q > \delta \ell Q \ell > Q_M$. However, $Q_{n-1} = \alpha + \sum \delta Q = \sum \xi Q + \sum \delta Q$, and by Theorem 13, the last expansion is an $E$-expansion. Since $Q_{n-1}$ is an $E$-expansion, we conclude that $\sum \hat{\beta}^n Q \leq Q_{n-1}$. \hfill $\Box$

Let us introduce a lemma as a preparation for the proof of Proposition 40.

**Lemma 39** Let $Q$ be a decreasing fundamental sequence for the $E$-interval $I$. Let $\sum_{k=1}^{\infty} \xi^k$ be an $\hat{\beta}$-block decomposition with support $[m_k, a_k]$ such that $m_k - 1 = a_{k-1}$ for all $k \geq 2$. Then, $\sum_{k=1}^{\infty} \xi^k Q < Q_{m_k-1}$.

**Proof** By the definition of a proper $\hat{\beta}$-block, $0 < \xi^k < \hat{\beta}^{m_k}_{a_k}$, which implies that $\hat{\beta}^{m_k}_{a_k} > 0$. Then, for each $k = 1, 2, \ldots$

$$\sum \xi^k Q = \sum_{j=m_k}^{a_k} \hat{\beta}^{m_k}_{j} Q_j - b_k Q_a, \quad \text{for some integer } b_k \geq 1.$$

By Lemma 38, $\sum \hat{\beta}^{m_k}_{j} Q \leq Q_{m_k-1}$, and it follows that

$$\sum \xi^k Q \leq \sum \hat{\beta}^{m_k}_{j} Q - Q_{a_k} \leq \sum_{j=a_k+1}^{\infty} \hat{\beta}^{m_k}_{j} Q_j < Q_{m_k-1} - Q_{a_k}$$

where $\sum_{j=a_k+1}^{\infty} \hat{\beta}^{m_k}_{j} Q_j > 0$ since $\hat{\beta}^{m_k}_{j}$ has infinite support. Notice that $Q_{m_k-1} - Q_{a_k} = Q_{a_{k-1}} - Q_{a_k}$ for $k \geq 2$, and that the above inequalities hold for $\xi^k = 0$ as well.

$$\Rightarrow \sum_{k=1}^{\infty} \sum \xi^k Q = \sum \xi^1 Q + \sum_{k=2}^{\infty} \sum \xi^k Q$$

$$< Q_{m_1-1} - Q_{a_1} + \sum_{k=2}^{\infty} (Q_{a_{k-1}} - Q_{a_k})$$
Simplifying the telescoping sum, we have

\[ Q_{m-1} - Q_{a_1} + Q_{a_1} = Q_{m-1}. \]  

(13)

**Proposition 40** Let \( Q \) be a decreasing fundamental sequence for the \( \delta \)-interval \( I \). Then \( \sum \hat{\beta}^m Q = Q_{m-1} \) for each index \( m \geq 1 \), and \( \sum \delta Q < \sum \hat{\beta}^m Q \) for all \( \delta \in \delta \) of order \( m \).

**Proof** By Lemma 38, \( Q_{m-1} \geq \sum \hat{\beta}^m Q \). Suppose that \( Q_{m-1} > \sum \hat{\beta}^m Q \) for some \( m \geq 1 \), and let \( \sum \hat{\beta}^m Q = \sum_{k=1}^{\infty} \hat{\beta}^Q k Q \) be the \( \hat{\beta} \)-block decomposition into proper \( \hat{\beta} \)-blocks \( \hat{\beta}^Q k Q \) with support \([m_k, a_k]\) such that \( \zeta^1 \) is a non-zero \( \hat{\beta} \)-block and \( m_k - 1 = a_k - 1 \) for \( k \geq 2 \). Then, \( Q_{m-1} > \sum \hat{\beta}^m Q \) implies \( m_1 > m - 1 \), i.e., \( m_1 \geq m \).

Let \( Q \) be a decreasing fundamental sequence for the \( I \)-interval \( I \), and let \( \delta \)-interval \( I \) be the \( \hat{\beta} \)-block decomposition where \( \text{ord}(\delta) = \text{ord}(\zeta^1) =: m \). Then, by Lemma 39, we find \( \sum_{k=2}^{\infty} \delta Q \) for all \( k \), and hence,

\[ \Rightarrow Q_{a_1} < Q_{a_1} + \sum_{k=a_1+1}^{\infty} \hat{\beta}^m Q \leq \sum_{k=2}^{\infty} \delta Q \leq Q_{m-1} = Q_{a_1}. \]

Since this is a contradiction, we conclude \( \sum \hat{\beta}^m Q \geq Q_{m-1} \), and by Lemma 38, we prove that it is an equality.

Let us prove the second statement. Let \( \sum \delta Q = \sum_{k=1}^{\infty} \delta k Q \) be the \( \hat{\beta} \)-block decomposition where \( \text{ord}(\delta) = \text{ord}(\zeta^1) =: m \). Then, by Lemma 39,

\[ \sum \delta Q = \sum_{k=1}^{\infty} \delta k Q \leq Q_{m-1} = \sum \hat{\beta}^m Q. \]

\[ \square \]

### 5.3.1 Proof of Theorem 24, Part 1

Let us prove the only-if part of Theorem 24, Part 1 (b). Assume the notation and context of Theorem 24, Part 1 (b), and let \( Q \) be a decreasing fundamental sequence for \( I \). Then, by Proposition 40, the sequence satisfies the equality (4).

Let us prove the if-part of Part 1 (b). For the existence of an \( \delta \)-expansion, we shall use the greedy algorithm in terms of proper \( \hat{\beta} \)-blocks. Let \( x \in I \). Then, there is an index \( n \geq 1 \) such that \( Q_n < x < Q_{n-1} \). By the equality (4), \( \sum \hat{\beta}^n Q = Q_{n-1} \), and hence, there is a proper \( \hat{\beta} \)-block \( \zeta \) of order \( n \) such that \( x < \sum \zeta Q < Q_{n-1} \). Thus, by the finiteness of the number of proper \( \hat{\beta} \)-blocks at index \( n \) that are less than \( \zeta \), there must be a largest proper \( \hat{\beta} \)-block \( \zeta \) at index \( n \) such that \( \sum \zeta Q \leq x \). Note here that \( \zeta \) is in fact the largest proper \( \beta \)-block in \( \delta \) such that \( \sum \zeta Q \leq x \). Let \( \zeta^1 := \zeta \), \( x_1 := x - \sum \zeta^1 Q \), and \( x_0 := x \), and recursively define \( x_m := x_{m-1} - \sum \zeta^m Q \) for \( m \geq 1 \) where \( \zeta^m \) is the largest proper \( \hat{\beta} \)-block such that \( \sum \zeta^m Q \leq x_{m-1} \). Then, \( x = \sum_{m=1}^{M} \zeta^m Q + x_M \) for some \( x_M \in [0, 1) \) for all \( M \geq 1 \). By the choice of \( \zeta^m \), we have \( \zeta^m \neq 0 \) if \( x_{m-1} > 0 \), and if \( x_{M+1} = 0 \) for some \( M \geq 1 \), then \( \zeta^m = 0 \) for all \( m \geq M \).
Given $m \geq 1$, suppose that $\zeta^m$ and $\zeta^{m+1}$ are non-zero $\beta$-blocks, and have supports $[n, j]$ and $[n', j']$, respectively. Let us prove that $j < n'$. Suppose that $n' \leq j$, and note that $\zeta^{m+1}_n \geq 1$. Then, $x_{m+1} = x_m - \sum_{k=n}^{j} \zeta^{m+1} Q = x_m - Q_n - \sum_{k=n}^{j} \zeta^{m} Q - Q_j$. Notice that $\sum_{k=n}^{j} \zeta^m Q + Q_j = \sum_{k=n}^{j} \zeta^m Q_k + Q_j$. If $\zeta^m + \beta' < \hat{\beta}^n$, then $\zeta^m + \beta'$ forms a proper $\beta$-block with the same support $[n, j]$. If $\zeta^m + \beta' = \hat{\beta}^n$, since $\hat{\beta}^n$ has infinite support, and there is a smallest index $\ell > j$ such that $\hat{\beta}^\ell \geq 1$. Then, $\zeta^\ell := \zeta^m + \beta'$ also forms a proper $\beta$-block with the support $[n, \ell]$ where $\hat{\beta}^\ell = 0 < \hat{\beta}^n$. Hence, $0 \leq x_{m+1} \leq x_{m-1} - \sum \zeta^m Q$ where $\zeta^m \leq \hat{\beta}$ $\zeta$. This contradicts the choice of $\zeta^m$. Thus, $\zeta^k$ for $k \geq 1$ are decreasing proper $\beta$-blocks with disjoint supports.

Recall $x = \sum_{m=1}^{M} \sum \zeta^m Q + x_M$ for all $M \geq 1$, and let us prove that $x_M \to 0$ as $M \to \infty$. If $s$ is the largest index of the support of $\zeta^s$, then $x_M \leq Q_s$; otherwise, we would have chosen a proper $\beta$-block larger than or equal to $\zeta^M + \beta'$. Since $Q_s \to 0$ as $M \to \infty$, we prove that $x_M \to 0$ as well, and hence, $x = \sum_{m=1}^{\infty} \sum \zeta^m Q$. Thus, we proved that $x$ has an $\mathcal{E}$-expansion. The uniqueness of such an expansion follows immediately from the property that if $y$ is a real number in $I$ and $y = \sum_{m=1}^{\infty} \sum \zeta^m Q$ is an $\beta$-block decomposition such that $\hat{\xi}^1 \neq 0$, then $\hat{\xi}^1 \beta$-block decomposition such that $\sum \zeta^1 Q \leq y$. Let us prove this property. Let $\zeta^m$ be the proper $\beta$-blocks described above. Write $y = \sum_{m=1}^{\infty} \sum \zeta^m Q = \sum \zeta^1 Q + \sum_{m=2}^{\infty} \sum \zeta^m Q$, and let $[N, K]$ be the support of $\zeta^1$. If $\sum_{m=2}^{\infty} \sum \zeta^m Q$ is zero, then we are done, and if not, then by Proposition 40, this infinite sum is $< Q_{k_0}$ for some $k_0 \geq K$. It follows that $y < \sum \zeta^1 Q + Q_K$. Since there is no proper $\beta$-block $\zeta$ such that $\hat{\xi}^1 < a \zeta < \hat{\beta}^1 \beta_k$, the inequalities $\sum \zeta^1 Q \leq y < \sum \zeta^1 Q + Q_K$ implies that $\zeta^1 \beta$-block decomposition such that $\sum \zeta^1 Q \leq y$. Therefore, if $x = \sum_{m=1}^{\infty} \sum \zeta^m Q$ is the non-zero $\beta$-block decomposition, then each $\zeta^m$ for $m \geq 1$ is the largest proper $\beta$-block that goes into $x_{m-1}$, and hence, $\zeta^m = \zeta^m$ for all $m \geq 1$. This concludes the proof of Theorem 24, Part 1 (b).

For Part 1 (a), we use the greedy algorithm as explained around Example 25, and we leave it to the reader.

### 5.3.2 Proof of Theorem 24, Part 2

**Lemma 41** Let $Q$ be a decreasing sequence in $I$ converging to $0$. Let $\mathcal{L}$ be the Zeckendorf collection determined by a Zeckendorf multiplicity list $L = (e_1, \ldots, e_N)$. The sequence $Q$ is a decreasing fundamental sequence for the $\mathcal{L}$-interval $I$ if and only if $Q$ is a sequence defined by the linear recurrence $Q_n = \sum_{k=1}^{N} e_k Q_{n+k} - e_1 Q_{n+1} + \cdots + e_N Q_{n+N}$ for all $n \geq 0$.

**Proof** Suppose that $Q$ is a decreasing fundamental sequence for $I$, and recall $Q_0 = 1$. Given $n \geq 1$, let $\hat{\beta}^{n+1}$ be the maximal coefficient function of order $n + 1$ for the $\mathcal{L}$-Zeckendorf condition. Then, by Theorem 24, Part 1 (b), $Q_n = \sum_{k=1}^{N} \hat{\beta}^{n+1} Q = \sum_{k=1}^{N} e_k Q_{n+k} + e_N Q_{n+N} + \sum_{k=n+N+1}^{\infty} \hat{\beta}^{n+1} Q_k$. Since the last term of the RHS is the maximal coefficient function of order $n + N + 1$, by Proposition 40, it follows that

$$Q_n = \sum_{k=1}^{N} e_k Q_{n+k} + e_N Q_{n+N} + Q_{n+N},$$

which is the short linear recurrence.

Suppose that $Q$ satisfies the recurrence (14). By repeatedly applying the recurrence to the terms $Q_{n+kN}$ for $k \geq 1$, we find $Q_n = \sum \hat{\beta}^{n+1} Q$ for $n \geq 0$, and $\sum \hat{\beta}^{n+1} > Q_{n+1}$.
implies that it is a decreasing sequence. By Theorem 24, Part 1 (b), it is a fundamental sequence for \( I \).

\( \square \)

The following theorem is available in [6, Theorem 1].

**Theorem 42 (Daykin)** Let \( Q \) is a decreasing sequence of positive real numbers given by the \( L \)-linear recurrence given in Lemma 41. If the polynomial \( f(z) = e_N z^N + \cdots + e_1 z - 1 \) has a complex root \( \omega \) whose modulus is smaller than other complex roots, then \( \omega \) is a real number in \( I \), and there is a positive real number \( a \) such that \( Q_k = a \omega^k \) for all \( k \geq 1 \).

We shall show below that the hypothesis of Theorem 42 is always satisfied for Zeckendorf multiplicity lists, and we begin with the following lemma.

**Lemma 43** Let \( f(z) = a_n z^n + \cdots + a_1 z + a_0 \) be a polynomial in \( \mathbb{R}[z] \), and suppose that \( a_k \geq 0 \) for all \( 1 \leq k \leq n \) such that there are indices \( m < \ell \) with \( \gcd(m, \ell) = 1 \) and \( a_m a_\ell \neq 0 \). If \( f(\omega) = 0 \) for some positive real number \( \omega \), then there is no other complex zero off on the circle centered at the origin with radius \( \omega \).

**Proof** Suppose that there is a complex root \( \alpha = \omega e^{i\theta} \) where \( 0 < \theta < 2\pi \), and it implies that its complex conjugate \( \bar{\alpha} = \omega e^{-i\theta} \) is also a root of the polynomial.

\[
0 = f(\omega e^{i\theta}) = a_0 + \sum_{k=1}^{n} a_k \omega^k e^{i(k\theta)} \quad \text{and} \quad 0 = f(\omega e^{-i\theta}) = a_0 + \sum_{k=1}^{n} a_k \omega^k e^{-i(k\theta)}
\]

By adding the RHS to each other, we have

\[
0 = 2a_0 + \sum_{k=1}^{n} a_k \omega^k (e^{i(k\theta)} + e^{-i(k\theta)}) = 2a_0 + \sum_{k=1}^{n} a_k \omega^k \cdot 2 \cos(k\theta) \Rightarrow 0 = a_0 + \sum_{k=1}^{n} a_k \omega^k \cos(k\theta)
\]

On the other hand, \( 0 = f(\omega) \Rightarrow 0 = a_0 + \sum_{k=1}^{n} a_k \omega^k \). By subtracting the above equation from this one, we have

\[
0 = a_0 + \sum_{k=1}^{n} a_k \omega^k - \left( a_0 + \sum_{k=1}^{n} a_k \omega^k \cos(k\theta) \right) = \sum_{k=1}^{n} a_k \omega^k (1 - \cos(k\theta)).
\]

Since both \( a_m \) and \( a_\ell \) are positive and \( 1 - \cos(k\theta) \geq 0 \) for all \( 1 \leq k \leq n \), it follows \( \cos(m\theta) = \cos(\ell\theta) = 1 \). This implies that \( \theta = 2m'\pi / m \) and \( \theta = 2\ell'\pi / \ell \) for two positive integers \( m' \) and \( \ell' \) with \( m' < m \) and \( \ell' < \ell \). However, it implies that \( m' \ell' = m\ell \), and this equality on the natural numbers cannot happen if \( \gcd(\ell, m) = 1 \) and \( m' < m \) and \( \ell' < \ell \).

\( \square \)

For example, \( 3z^n + z - 1 \) has a unique positive real zero \( \omega \) by Descartes’ Rule of Sign, and by Lemma 43, it has no other complex roots on the circle passing through \( \omega \) centered at the origin since \( \gcd(a_n, a_1) = 1 \).

Let us prove Theorem 24, Part 2. For the characteristic polynomial \( e_N z^N + \cdots + e_1 z - 1 \) for the linear recurrence, we have a unique positive real zero \( \omega \), and Rouché’s Theorem on the setting \( |e_N z^N + \cdots + e_1 z| < 1 \) on a circle with radius smaller than \( \omega \) implies that \( \omega \) has the smallest modulus. Moreover, by Lemma 43, it is the only one with the smallest
modulus. By Theorem 42, we find that if \( Q \) is a decreasing fundamental sequence for the \( \mathcal{L} \)-interval, then there is a positive real number \( a \) such that \( Q_k = a \omega^k \) for all \( k \geq 1 \). By Lemma 41, \( 1 = Q_0 = \sum_{k=1}^{N} e_k Q_k = a \sum_{k=1}^{N} e_k \omega^k = a \), and hence, \( a = 1 \). It remains to show that the sequence \( Q \) given by \( Q_k = \omega^k \) for all \( k \geq 1 \) is a fundamental sequence. Since the sequence clearly satisfies the recurrence (14) given in Lemma 41, it is a decreasing fundamental sequence.

5.3.3 **Proof of Proposition 27**

The application of Rouché’s Theorem demonstrated in Section 5.3.2 relies on the constant term \( e_0 \) being positive, not particularly on \( e_0 = 1 \), so we can say, \( e_N z^N + \cdots + e_1 z - e_0 \) has no zeros inside the circle of radius \( \omega \) if \( \omega \) is the only positive zero of this polynomial that has coefficients in \( \mathbb{R} \). If we use Lemma 43 with its conditions imposed on \( e_N, \ldots, e_1 \), we prove that \( \omega \) is a unique zero with smallest modulus. Since the polynomial in Proposition 27 is the reciprocal version of \( e_N z^N + \cdots + e_1 z - e_0 \), it proves that it is a dominant polynomial.

5.4 **The \( p \)-adic number case**

When non-zero coefficient functions \( \mu \) are used for \( p \)-adic integers, the smallest index \( n \) such that \( \mu_n \neq 0 \) is called the \( p \)-adic order of \( \mu \), denoted by \( \text{ord}_p(\mu) \). Let \( \text{ord}_p(0) = \infty \). This definition is well-suited for the order of the \( p \)-adic numbers, \( \text{ord}_p(x) \) for \( x \in \mathbb{Z}_p \).

5.4.1 **Proof of Theorem 26, Part 1**

Let \( Q \) be a decreasing sequence such that \( \sum \epsilon Q = \sum \delta Q \) for non-zero coefficient functions \( \epsilon \) and \( \delta \) in \( \mathcal{E} \), and let us claim \( n := \text{ord}_p(\epsilon) = \text{ord}_p(\delta) \). If \( n = \text{ord}_p(\epsilon) < n' := \text{ord}_p(\delta) \), then \( \text{ord}_p(\sum \epsilon Q) = \text{ord}_p(Q_n) < \text{ord}_p(Q_{n'}) = \text{ord}_p(\sum \delta Q) \) since \( \epsilon_n \) and \( \delta_{n'} \) are \( < p \) and \( Q \) is decreasing. This contradicts that \( \text{ord}_p(\sum \epsilon Q) = \text{ord}_p(\sum \delta Q) \). Without loss of generality, we conclude that \( n = \text{ord}_p(\epsilon) = \text{ord}_p(\delta) \). Suppose that there is a smallest positive index \( s \geq n \) such that \( \epsilon_s < \delta_s \). Then, \( \epsilon_s Q_s \equiv \delta_s Q_s \mod p^{m+1} \) where \( m = \text{ord}_p(Q_s) \geq 0 \), and hence, \( \epsilon_s(Q_s/p^m) \equiv b_s(Q_s/p^m) \mod p \). Since \( Q_s/p^m \neq 0 \mod p \), we have \( \epsilon_s \equiv \delta_s \mod p \). Since the values are \( < p \), we find that \( \epsilon_s = \delta_s \), which is a contradiction. Thus, without loss of generality, we prove that \( \epsilon = \delta \).

5.4.2 **Proof of Theorem 26, Part 2**

**Lemma 44** Let \( p \) be prime, and let \( \mathcal{E} \) be an arbitrary collection of coefficient functions \( \epsilon \) such that \( \epsilon_k < p \) for all \( k \geq 1 \). Let \( Q \) and \( Z \) be decreasing sequences in \( \mathbb{Z}_p \) such that \( X_Q = X_Z \). Then,

\[
\text{ord}_p(Q_k) = \text{ord}_p(Z_k) \quad \text{for all } k \geq 1. \tag{15}
\]

**Proof** Suppose that there is a smallest index \( k \geq 1 \) such that \( \text{ord}_p(Q_k) < \text{ord}_p(Z_k) \). Write \( Q_k = \sum \epsilon Z \) for a non-zero coefficient function \( \epsilon \in \mathcal{E} \), and \( n := \text{ord}_p(\epsilon) \). Then, \( \epsilon_j < p \) for all \( j \geq 1 \) implies that \( \text{ord}_p(Q_k) = \text{ord}_p(\sum \epsilon Z) = \text{ord}_p(Z_n) \). If \( n = k \), then \( n = k \) contradicts the choice of \( k \), and if \( n < k \), then \( \text{ord}_p(Q_k) = \text{ord}_p(Z_n) = \text{ord}_p(Q_n) \) by the choice of \( k \), which contradicts the decreasing property of \( Q \). Thus, \( n > k \), and it implies the following contradiction: \( \text{ord}_p(Q_k) > \text{ord}_p(Z_n) > \text{ord}_p(Z_k) > \text{ord}_p(Q_k) \). Using a similar argument, we also derive a contradiction from \( \text{ord}_p(Z_k) < \text{ord}_p(Q_k) \). Therefore, we prove (15). \( \square \)
Lemma 45 Let $\mathcal{E}$ be a Zeckendorf collection for $p$-adic integers. If $\epsilon \in \mathcal{E}$, then $\text{res}_{[n,\infty)}(\epsilon) \in \mathcal{E}$ for all indices $n \geq 1$.

Proof Let $\mathcal{E}_0$ be a Zeckendorf collection for positive integers whose completion is $\mathcal{E}$. Let $\epsilon \in \mathcal{E}$. Then, by definition, there is an increasing sequence of $M_k$ for $k \geq 1$ such that $\text{res}_{[1,M_k]}(\epsilon) \in \mathcal{E}_0$. Notice that if $\zeta$ is an $\hat{\beta}$-block at index $\ell$, then $\text{res}_{[n,\infty)}(\zeta)$ is an $\hat{\beta}$-block at index $\ell$ for all indices $n \geq 1$. Let $\text{res}_{[1,M_k]}(\epsilon) = \sum_{t=1}^{T} \epsilon^t$ be the unique $\hat{\beta}$-block decomposition as described in Theorem 7. For arbitrary positive integer $n \leq M_k$, there is an $\hat{\beta}$-block $\zeta^k$ with $1 \leq s \leq T$ such that the support of $\zeta^k$ is $[a, b)$ and $a \leq n \leq b$. Thus, $\text{res}_{[n,M_k]}(\epsilon) = \text{res}_{[n,b]}(\zeta^k) + \sum_{t=s+1}^{T} \epsilon^t$, and this implies that given $n \geq 1$, we have $\mu^k := \text{res}_{[n,M_k]}(\epsilon) \in \mathcal{E}_0$ for $k \geq 1$ and $M_k \geq \text{ord}(\mu^k)$. Since $\text{res}_{[n,\infty)}(\epsilon) \equiv \mu^k \text{ res } M_k$, the coefficient function is the limit of a sequence of coefficient functions in $\mathcal{E}_0$, and hence, $\text{res}_{[n,\infty)}(\epsilon) \in \mathcal{E}$.

\[ \square \]

Lemma 46 Let $\mathcal{E}$ be the Zeckendorf collection described in Theorem 26, Part 2. Let $Q$ and $Z$ be decreasing sequences such that $X_Q = X_Z$. Then, for each $n = 1, 2, \ldots$, there is an $\hat{\beta}$-block $\zeta$ such that $\zeta_n = \sum \zeta_Q \text{ ord}_p(\zeta) = n$, and $\zeta_n = 1$.

Proof Let $\mathcal{E}_0$ be a Zeckendorf collection for positive integers such that $\mathcal{E}$ is the completion of $\mathcal{E}_0$, and let $n$ be a positive integer. Since $\zeta_n \in X_Q$, by the definition of the completion, there is a non-zero $\hat{\beta}$-block $\zeta^1$ at index $\ell$ and a coefficient function $\epsilon$ such that $\zeta_n = \sum \epsilon^1 Q + \sum \epsilon Q$ and $\epsilon_s = 0$ for $1 \leq s \leq \ell$. Since $m := \text{ord}_p(\zeta_n) = \text{ord}_p(Q_n)$, $\text{ord}_p(\epsilon^1) = n$, and by Lemma 45, $\epsilon \in \mathcal{E}$. Recall that coefficient functions in $\mathcal{E}_0$ are ascendingly ordered, and $\text{ord}_p(\epsilon) \geq \ell + 1$. Since $\epsilon^1 \in \mathcal{E}$, it follows that $\sum \epsilon^1 Q = \zeta_n - \sum \epsilon Q \in X_Q = X_Z$. It is clear that $\zeta_n - \sum \epsilon Q = \sum \sigma Z$ for some $\sigma \in \mathcal{E}$ with $\text{ord}_p(\sigma) = n$ since $\text{ord}_p(\sum \epsilon Q) \geq \text{ord}_p(Q_{n+1})$ where $\text{ord}_p(\epsilon) = \text{ord}_p(\sum \epsilon Q) = \infty$ if $\epsilon = 0$. Then, $\zeta_n - \sum \epsilon Q = \sum \sigma Z$ implies $\zeta_n \equiv \sigma \zeta_n \text{ mod } p^{\mu+1}$, i.e., $1 \equiv \sigma_n \text{ mod } p^1$. Since $\sigma_n < p$, we find $\sigma_n = 1$. Write $\sigma^0 := \text{res}_{[n,\infty)}(\sigma)$, and write $\sum \epsilon Q = \sum \epsilon Z$ for some $\delta \in \mathcal{E}$, so that $\text{ord}_p(\epsilon) = \text{ord}_p(\delta) \geq \ell + 1$. Then, $\zeta_n + \sum \sigma^0 Z = \zeta_n - \sum \epsilon Q = \zeta_n - \sum \delta Z$, and hence, $0 = \sum \sigma^0 Z + \sum \delta Z = \sum_{k=1}^{\ell+1} \sigma^0_k Z_k + \sum_{k=1}^{\ell} \delta_k + \sigma^0_k Z_k$. Notice that $0 \leq \delta_k + \sigma^0_k \leq 2 \cdot (p-1)/2 = p-1$ for all $k \geq \ell + 1$ and $0 \leq \sigma^0_k \leq (p-1)/2$ for all $n+1 \leq k \leq \ell$. Since $Z$ is a decreasing sequence in $\mathcal{E}_p$, it follows that $\delta_k = \sigma^0_k = 0$ for all $k \geq \ell + 1$, and $\sigma^0_k = 0$ for all $n+1 \leq k \leq \ell$. Hence,

$$\sum \xi^1 Q = \zeta_n - \sum \delta Z = \zeta_n. \quad (16)$$

Let us show that $\xi^1 = 1$. Write $\zeta_n = \sum \alpha Z = \alpha_n Z_n + \sum \alpha^0 Z$ where $\alpha \in \mathcal{E}$ with $\text{ord}_p(\alpha) = n$ and $\alpha^0 := \text{res}_{[n,\infty)}(\alpha)$. Recall $m := \text{ord}_p(\zeta_n)$, then $\zeta_n = \sum \alpha Z \equiv \alpha_n Z_n \text{ mod } p^{\mu+1}$, and hence,

$$\zeta_n = \xi^1 \alpha_n Z_n \equiv \alpha_n Z_n \text{ mod } p^{\mu+1} \Rightarrow Z_n \equiv \xi^1 \alpha_n Z_n \text{ mod } p^{\mu+1} \Rightarrow 1 \equiv \xi^1 \alpha_n \text{ mod } p$$

$$\Rightarrow \xi^1 \alpha_n = ps + 1 \text{ for } s \geq 0. \quad (17)$$

On the other hand, $\xi^1 \alpha_n < \sqrt{p} = p$, and the equation (17) implies that $s = 0$ and $\xi^1 = 1$.

Let us prove Theorem 26, Part 2. Given a positive integer $n$, by Lemma 46, we have $\zeta_n = \sum \xi Q$ and $Q_n = \sum \xi Z$ where $\text{ord}_p(\zeta) = \text{ord}_p(\xi) = n$, $\xi_n = \xi_n = 1$, and $\xi$ and $\xi$ are at index $s$ and $t$, respectively. So,
\[ Z_n = \sum \zeta Q = Q_n + \sum_{k=n+1}^s \zeta_k Q_k \Rightarrow Z_n = \left( Z_n + \sum_{k=n+1}^t \xi_k Z_k \right) + \sum_{k=n+1}^s \zeta_k Q_k \]

\[ \Rightarrow 0 = \sum_{k=n+1}^t \xi_k Z_k + \sum_{k=n+1}^s \zeta_k Q_k. \]  

(18)

Let us show that \( \xi_k = \zeta_k = 0 \) for all \( k \geq n + 1 \). Suppose that there is a smallest index \( \ell \geq n + 1 \) such that \( \xi_\ell \neq 0 \) or \( \zeta_\ell \neq 0 \); without loss of generality, assume that \( \ell = n + 1 \) and \( \xi_{n+1} \neq 0 \). By Lemma 46, there is an \( \hat{\beta} \)-block \( \theta \) at index \( u \) with \( \operatorname{ord}_p(\theta) = n + 1 \) and \( \theta_{n+1} = 1 \) such that \( Z_{n+1} = \sum \theta Q \). Thus,

\[ 0 = \xi_{n+1} Z_{n+1} + \sum_{k=n+2}^t \xi_k Z_k + \theta_{n+1} Q_{n+1} + \sum_{k=n+1}^s \zeta_k Q_k \]

\[ = \xi_{n+1} \left( Q_{n+1} + \sum_{k=n+2}^u \theta_k Q_k \right) + \sum_{k=n+2}^t \xi_k Z_k + \theta_{n+1} Q_{n+1} + \sum_{k=n+2}^s \zeta_k Q_k \]

If \( r := \operatorname{ord}_p(Q_{n+1}) \), then it follows

\[ 0 \equiv (\xi_{n+1} + \zeta_{n+1}) Q_{n+1} \mod p^{r+1}. \]

Since \( \xi_{n+1} + \zeta_{n+1} \leq p - 1 \), we find \( \xi_{n+1} = \zeta_{n+1} = 0 \). Therefore, \( Q_n = Z_n \), and this concludes the proof of Theorem 26, Part 2.

5.5 The unique \( L \)-representation property

In this section we prove Theorem 22. Recall that a Zeckendorf multiplicity list \( L = (e_1, \ldots, e_N) \) for \( N \geq 2 \) considered in the theorem is an arbitrary list of positive integers \( e_k \) for \( k = 1, \ldots, N \), and let \( \mathcal{L} \) denote the Zeckendorf collection determined by \( L \).

Let \( \theta \) be a coefficient function \( \theta \in \mathcal{L} \). The coefficient function is called a proper \( \ell \)-block at index \( n \) if \( \theta \) is a proper \( \hat{\beta} \)-block at index \( n \) with support \([s, n]\) such that \( n - N + 1 \leq s \leq n \). If \( \theta \) has support \([n - N, n]\) where \( n - N \geq 1 \) and \( \theta_{n-N} = 0 \), we call \( \theta \) the maximal \( \ell \)-block at index \( n \), and if \( \theta \) is a maximal \( \hat{\beta} \)-block at index \( n \) where \( n \leq N \), it is also called a maximal \( \ell \)-block at index \( n \). The \( \ell \)-block interval of an \( \ell \)-block \( \theta \) is defined to be the interval of indices \([s, n]\) if \( \theta \) is a proper \( \hat{\beta} \)-block with support \([s, n]\). The \( \ell \)-block interval of the maximal \( \ell \)-block at index \( n \) is defined to be \([n - N + 1, n]\) if \( n > N \), and \([1, n]\) if \( n \leq N \).

For example, if \( L = (2, 3, 2) \), then \( \theta^0 = \beta^4 + 3\beta^5 + 2\beta^6 \) is the maximal \( \ell \)-block at index 6 with \( \ell \)-block interval \([2, 6]\) while \( \theta^0 \) is not a maximal \( \hat{\beta} \)-block at index 6, and the support of the proper \( \hat{\beta} \)-block \( \theta^0 = 0 \cdot \beta^3 + \beta^4 + 3\beta^5 + 2\beta^6 \) is \([3, 6]\). If \( \theta \) is a proper \( \ell \)-block, then the \( \ell \)-block interval of \( \theta \) coincides with the support of the proper \( \hat{\beta} \)-block \( \theta \).

For each coefficient function \( \mu \in \mathcal{L} \), there are unique \( \ell \)-blocks \( \theta^m \) for \( m \geq 1 \) with \( \ell \)-block interval \([s_m, n_m]\) such that \( \mu = \sum_{m=1}^M \theta^m, s_1 = 1 \), and \( n_m + 1 = s_{m+1} \). We call it the \( \ell \)-block decomposition of \( \mu \). In addition, if we use non-zero \( \ell \)-blocks only, it is called the non-zero \( \ell \)-block decomposition of \( \mu \). If \( \mu = \sum_{m=1}^M \theta^m \) is the non-zero \( \ell \)-block decomposition and \([i_m, n_m]\) is the \( \ell \)-block intervals of \( \theta^m \) such that \( n_m + 1 = i_{m+1} \) for all \( 1 \leq m \leq M - 1 \), then the decomposition is said to have no gaps between the \( \ell \)-block intervals. The sum of any \( \ell \)-blocks with disjoint \( \ell \)-block intervals is a member of \( \mathcal{L} \), and it is called a sum of disjoint \( \ell \)-blocks.
Lemma 47  Let $Q$ be a sequence in $\mathbb{N}$ given by the recurrence (1).

1. Let $\theta$ be an $L$-block with $L$-block interval $[a, \ell]$ where $\ell \geq N$, and let $\hat{\theta}$ be the maximal $L$-block at index $\ell$. Then, $\sum \theta Q \leq \sum \hat{\theta} Q$, and $Q_a + \sum \theta Q \leq Q_{\ell + 1}$. In particular, $\sum \theta Q < Q_{\ell + 1}$.

2. Let $\mu^1$ be a non-zero coefficient function in $\mathcal{L}$ with the non-zero $L$-block decomposition $\mu^1 = \sum_{m=1}^{M} \theta^m$ where $\text{ord}(\theta^1) \geq N$. Let $[a, \ell]$ be the $L$-block interval of $\theta^1$, and $n := \text{ord}(\theta^M)$.

   (a) If all the $L$-blocks are maximal and there are no gaps between them, then $Q_a + \sum \mu^1 Q = Q_{n+1}$.

   (b) If there are no gaps between the $L$-block intervals, and there is a smallest non-zero non-maximal $L$-block with $L$-block interval $[b, j]$, then $Q_a + \sum \mu^1 Q = Q_b + \sum \text{res}_{[b, j]}(\mu^1) Q < Q_{n+1}$.

   (c) If there is a smallest index $c$ such that $\ell < c < n$, and $c$ is not contained in any of the $L$-block intervals, then $Q_a + \sum \mu^1 Q \leq Q_c + \sum \text{res}_{[c, n]}(\mu^1) Q < Q_{n+1}$.

   (d) For all cases, $Q_a + \sum \mu^1 Q \leq Q_{n+1}$, and hence, $\sum \mu^1 Q < Q_{n+1}$.

3. Let $\mu^0$ be a coefficient function in $\mathcal{L}$ with the $L$-block decomposition $\mu^0 = \sum_{m=1}^{M} \theta^m$. If $n := \text{ord}(\theta^M) < N$, then $\sum \mu^0 Q < Q_{2N}$.

Proof  Let us prove Part 1. Since $\ell \geq N$, by the linear recurrence, $Q_{\ell + 1} = \sum_{k=1}^{N} e_k Q_{\ell - k + 1} = \sum \hat{\theta} Q + Q_{\ell - N + 1}$

$$\Rightarrow Q_{\ell + 1} - \sum \theta Q - Q_a = \sum_{k=1}^{N} (e_k - \theta_{\ell - k + 1}) Q_{\ell - k + 1} - Q_a = (e_{\ell - a + 1} - \theta_a) Q_a + \sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} - Q_a.$$

Since $e_{\ell - a + 1} - \theta_a \geq 1$,

$$Q_{\ell + 1} - \sum \theta Q - Q_a \geq Q_a + \sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} - Q_a = \sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} \geq 0,$$

This proves $Q_{\ell + 1} \geq \sum \theta Q + Q_a > \sum \theta Q$. Also, $Q_{\ell + 1} = Q_{\ell - N + 1} + \sum \hat{\theta} Q$ implies

$$\sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} \leq Q_{\ell + 1} - \sum \theta Q - Q_a = (Q_{\ell - N + 1} + \sum \hat{\theta} Q) - \sum \theta Q - Q_a$$

$$\Rightarrow \sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} + Q_a - Q_{\ell - N + 1} \leq \sum \hat{\theta} Q - \sum \theta Q$$

If $a = \ell - N + 1$, then $\sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} = 0$, and $\sum \hat{\theta} Q - \sum \theta Q \geq 0$. If $a > \ell - N + 1$, then $\sum_{k=\ell - a + 2}^{N} e_k Q_{\ell - k + 1} \geq e_N Q_{\ell - N + 1}$, so we have $\sum \hat{\theta} Q - \sum \theta Q \geq Q_a > 0$ as well.

We shall use the mathematical induction on $M$ to prove Part 2. Since Part 2 (c) makes sense only for $M \geq 2$, the initial cases of the induction are $M = 1, 2$. Since the proofs of $M = 1, 2$ are very similar to the proof of the main induction step with arbitrary $M$, we leave it to the reader. Assume that Part 2 is true for some $M \geq 2$. Let $\mu^1 = \sum_{m=1}^{M+1} \theta^m$ be a non-zero $L$-block decomposition, and let $[i_m, n_m]$ be the $L$-block interval of $\theta^m$. Suppose there are no gaps between the $L$-block intervals. If the $L$-block intervals are all maximal,
then by the linear recurrence, $Q_i + \sum_{m=1}^{M+1} \theta^m Q = Q_{i+1} + \sum_{m=2}^{M+1} \theta^m Q$. By the induction hypothesis for Part 2 (a), $Q_i + \sum_{m=1}^{M+1} \theta^m Q = Q_{i M+1}$. So, Part 2 (a) for $M + 1$ is proved.

If there is a smallest index $t \geq 1$ such that $\theta^t$ is not maximal, then $Q_i + \sum_{m=1}^{M+1} \theta^m Q = Q_t + \sum_{m=1}^{M+1} \theta^m Q$ by the induction hypothesis for Part 2 (a). Since $n_t \geq N$, we have $Q_i + \sum_{m=t+1}^{M+1} \theta^m Q = Q_{i+1} + \sum_{m=t+1}^{M+1} \theta^m Q$ by Part 1. Thus, by the induction hypothesis for Part 2 (d), $Q_i + \sum_{m=1}^{M+1} \theta^m Q < Q_{i+1} + \sum_{m=t+1}^{M+1} \theta^m Q \leq Q_{M+1}$. This proves Part 2 (b) for $M + 1$. Suppose that there is a smallest gap $c$ between the $L$-block intervals. Then, $n_k + 1 = c < i_{k+1}$ for some $1 \leq k \leq M$, and by the induction hypothesis for Part 2 (d),

$Q_i + \sum_{m=1}^{M+1} \theta^m Q = Q_{i+1} + \sum_{m=1}^{k} \theta^m Q + \sum_{m=k+1}^{M+1} \theta^m Q \leq Q_c + \sum_{m=k+1}^{M+1} \theta^m Q.$

Since $c \geq N$, by the linear recurrence, $Q_c < Q_{k+1}$, so $Q_c + \sum_{m=k+1}^{M+1} \theta^m Q < Q_{k+1} + \sum_{m=k+1}^{M+1} \theta^m Q \leq Q_{M+1}$ where the induction hypothesis for Part 2 (d) is used for the last inequality. This proves Part 2 (b) for $M + 1$. Since any non-zero $L$-block decomposition satisfies one of the three possibilities for the gaps between $L$-block intervals described in Part 2 (a,c), we proved Part 2 (d) for $M + 1$.

Let us prove Part 3. Notice that the linear recurrence implies that $Q_k < Q_{k+N}$ for all $k \geq 1$, and hence, $\sum_{i=0}^{N-1} \mu_k^i Q_k < \sum_{i=0}^{N-1} \mu_k^i Q_{k+N}$. Let $\mu_k = \sum_{i=0}^{N-1} \mu_k^i$. Then, $\sum_{i=0}^{N-1} \mu_k^i Q_{k+N} = \sum_{i=0}^{N-1} \mu_k^i Q_k < Q_{k+N}$, and $\mu_k$ has a decomposition into $L$-blocks as described in Part 2. Hence, by Part 2 (d), we have $\sum_{i=0}^{N-1} \mu_k^i Q_k < Q_{k+N} < \sum_{i=0}^{N-1} \mu_k^i Q_{k+N} \leq Q_{N+1} \leq Q_{2N}$.

Let us prove Theorem 22. In fact, we prove a more specific version of the theorem, and it is stated below.

**Theorem 48** Let $L$ be a Zeckendorf multiplicity list $(e_1, \ldots, e_N)$ such that $e_k \geq 1$ for all $1 \leq k \leq N$, and let $Q$ be a sequence in $\mathbb{N}$. Let $\sigma$ and $\mu$ be two non-zero coefficient functions in $L$ such that $\sigma \in L \sum \mu Q$.

1. If $\sigma_k = \mu_k$ for all $k > 3N$, then there are $L$-Zeckendorf coefficient functions $\alpha$ and $\gamma$ of order $\leq 4N - 1$ and $\rho \in L$ such that $\sigma = \gamma + \rho$ and $\mu = \alpha + \rho$, and the $L$-block intervals of non-zero $L$-blocks in $\rho$ are contained in $(M, \infty)$ where $M = \max(\text{ord}(\gamma), \text{ord}(\sigma))$.

2. If there is a largest integer $K > 3N$ such that $\sigma_K > \mu_K$, then there are coefficient functions $\alpha$ and $\gamma$ in $L$, and an index $a$ such that $\text{ord}(\alpha) \leq 3N - 2$, $\text{ord}(\gamma) \leq 2N - 1$, $\text{max}(\text{ord}(\alpha) + 1, 2N) \leq a \leq 3N$, and

$$\sigma = \gamma + (c + 1)\beta^{1+\ell+\ell N} + \rho$$

$$\mu = \alpha + \hat{\theta} + c\beta^{1+\ell+\ell N} + \rho$$

where $\theta$ is a non-zero $L$-block with $L$-block interval $[a, \ell]$, $\hat{\theta} = \text{res}_{1+\ell+\ell N}(\beta^{1+\ell+\ell N})$, and $\rho$ is a coefficient function supported on $(1+\ell+\ell N, \infty)$ such that $(c+1)\beta^{1+\ell+\ell N} + \rho$ is a sum of disjoint $L$-blocks for some integer $c \geq 0$. Moreover, the equality $\sum \sigma Q = \sum \mu Q$ is reduced under the $L$-linear recurrence to the equation $\sum (\sigma + \theta)Q = \sum \gamma Q + Q_{\ell+1}$. 
For example, if \( L = (11, 3) \) and \( (Q_1, Q_2) = (19, 3) \), then \( 2Q_3 = 3Q_2 + 9Q_1 \), and

\[
(11Q_4 + Q_3) + 2Q_3 = (11Q_4 + Q_3) + 3Q_2 + 9Q_1 \\
\Rightarrow Q_5 = 11Q_4 + Q_3 + 3Q_2 + 9Q_1 \\
\Rightarrow Q_7 = 11Q_6 + 2Q_5 + 11Q_4 + Q_3 + 3Q_2 + 9Q_1.
\]

Then, the last expression fits into Part 2 of Theorem 48 with \( \gamma = \rho = 0, \theta = \beta^3 + 11\beta^4, \) \( \dot{\theta} = 2\beta^5 + 11\beta^6, \) and \( \alpha = 9\beta^1 + 3\beta^2. \)

**Proof.** Let \( Q \) be a sequence in \( \mathbb{N} \) such that \( \sum \mu Q = \sum \sigma Q \) where \( \mu \) and \( \sigma \) are distinct non-zero coefficient functions in \( \mathcal{L} \), and \( \mu <_a \sigma \). Let us prove Part 1. Suppose that \( \sigma_k = \mu_k \) for all \( k > 3N \). Then, the \( L \)-block intervals of (non-zero) \( L \)-blocks of order \( \geq 4N \) are contained in \([3N + 1, \infty)\), and hence, the set of \( L \)-blocks of order \( \geq 4N \) in \( \sigma \) is equal to the set of \( L \)-blocks of order \( \geq 4N \) in \( \mu \). Thus, after cancelling the \( L \)-blocks of order \( \geq 4N \), the equality reduces to \( \sum \sigma Q = \sum \mu Q \) where \( \sigma' \) and \( \mu' \) are distinct members of \( \mathcal{L} \), and their orders are \( \leq 4N - 1 \). So, the theorem is proved for this case.

Let us prove Part 2. Suppose that there is a largest index \( K > 3N \) such that \( \sigma_K > \mu_K \). Then, let us claim that there is an \( L \)-block in \( \mu \) with \( L \)-block interval \([K, s]\). Notice that \( \sigma_k = \mu_k \) for all \( k > K \), and let \( \theta^1 \) and \( \theta^2 \) be the smallest \( L \)-blocks of \( \sigma \) and \( \mu \) at index \( K \), respectively, including zero \( L \)-blocks. Lemma 34, Part 1 is valid for \( L \)-block decompositions as well, and since \( \sigma_k = \mu_k \) for \( k > K \), the two blocks \( \theta^1 \) and \( \theta^2 \) must have the same index \( s \). Notice that the choice of \( \theta^1 \) implies that the \( L \)-block interval of \( \theta^1 \) is either \([K + 1, s]\) or \([t, s]\) for \( t \leq K \), and we have the inequality \( \mu_K < \sigma_K \leq e_j \) for some \( 1 \leq j \leq N \). If it is \([K + 1, s]\), then it follows that \( \mu_K \beta^K \) where \( \mu_K < e_1 \) is an \( L \)-block of \( \mu \) with \( L \)-block interval \([K, K]\), and if it is \([t, s]\) for \( t \leq K \), then it follows that \( \theta^2 \) is an \( L \)-block of \( \mu \) with \( L \)-block interval \([K, s]\). It proves the claim, and let \( \mu^* := \text{res}_{K, \infty}(\mu) \). Then, \( \mu^* \) is a sum of disjoint \( L \)-blocks, \( \sigma \equiv \mu^* \text{ res}(K, \infty) \), and \( \sigma_K \geq \mu_K^* + 1 \).

Let \( \mu^0 \) be the sum of \( L \)-blocks in \( \mu \) whose \( L \)-block intervals are contained in \([1, N)\), and let \( \mu^1 := \mu - \mu^* - \mu^0 \). Then, \( \mu^1 \in \mathcal{L} \) since it is a sum of disjoint \( L \)-blocks, and \( \mu = \mu^* + \mu^1 + \mu^0 \). Let \( \sigma^0 \) be the sum of \( L \)-blocks in \( \sigma \) whose \( L \)-block intervals are contained in \([1, N)\), and let \( \sigma^1 := \sigma - \sigma^0 \). Then, \( \sigma^1 \in \mathcal{L} \), and \( \sigma = \sigma^1 + \sigma^0 \). Also, \( \sum \sigma^1 Q \geq \sum \mu^0 Q + Q_K \).

By Lemma 47, Part 3, we have \( \sum \mu^0 Q - \sum \sigma^0 Q \leq \sum \mu^0 Q < Q_{2N} \), and hence,

\[
\sum \sigma^1 Q = \sum \mu^* Q + \sum \mu^1 Q + \sum \mu^0 Q - \sum \sigma^0 Q < \sum \mu^* Q + \sum \mu^1 Q + Q_{2N}.
\] (19)

Let us further decompose \( \mu^1 \). Let \( \mu^{11} \) be the sum of all \( L \)-blocks in \( \mu^1 \) whose \( L \)-block intervals are contained in \([2N, \infty)\), and let \( \mu^{10} := \mu^1 - \mu^{11} \in \mathcal{L} \). Then, \( \mu^1 = \mu^{11} + \mu^{10} \). Since the number of indices in the \( L \)-block interval of an \( L \)-block is \( \leq N \), we have \( \text{ord}(\mu^{10}) \leq 3N - 2 \); otherwise, the \( L \)-block interval of a largest non-zero \( L \)-block in \( \mu^{10} \) is contained in \([2N, \infty)\).

Let us prove that \( \mu^{11} \) is non-zero. Assume that \( \mu^{11} = 0 \), and let \( m := \text{ord}(\mu^{10}) \leq 3N - 2 \). By Lemma 47, Part 2 (d), \( \sum \mu^{10} Q < Q_{m+1} \leq Q_{3N-1} \), and hence, the inequality (19) implies \( \sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{10} Q + Q_{2N} < \sum \mu^* Q + Q_{2N} + Q_{3N} + Q_{2N} \). Since \( Q_{2N} + Q_{2N} \) is the sum of distinct \( L \)-blocks, Lemma 47, Part 2 (d) implies that \( Q_{3N} + Q_{2N} < Q_{3N+1} \leq Q_K \).
and it yields \( \sum \sigma^1 Q < \sum \mu^* Q + Q_K \leq \sum \sigma^1 Q \). Since it implies \( \sum \sigma^1 Q < \sum \sigma^1 Q \), we prove that \( \mu^{11} \) is non-zero.

Let \( \theta \) be the smallest non-zero \( L \)-block in \( \mu^{11} \), let \([a, \ell]\) be its \( L \)-block interval, and let \( n := \text{ord}(\mu^{11}) \). Let us prove that \( \mu^{11} = \theta + \text{res}_{(\ell,K)}(\beta^K) \). Suppose that there is a smallest zero \( L \)-block in \( \mu^{11} \) at index \( c \) such that \( a \leq \ell < c < n \). Then, by Lemma 47, Part 2 (c),

\[
\sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{11} Q + \sum \mu^{10} Q + Q_{2N} \\
\leq \sum \mu^* Q + \sum \mu^{11} Q + Q_a + \sum \mu^{10} Q \\
\leq \sum \mu^* Q + \sum \text{res}_{(\ell,n)}(\mu^{11}) Q + Q_c + \sum \mu^{10} Q.
\]

(20)

Since \( e_2 \geq 1 \), the coefficient function \( \beta^c \) makes an \( L \)-block with \( L \)-block interval \([c, c]\) or \([c - 1, c]\). Notice that \( \theta + \mu^{10} \) is the sum of disjoint \( L \)-blocks, and \( c - 1 \geq a \). Thus, the coefficient function of the last three terms of (20) is the sum of disjoint \( L \)-blocks. By Lemma 47, Part 2 (d), we have \( \sum \sigma^1 Q < \sum \mu^* Q + Q_{n+1} \leq \sum \mu^* Q + Q_K \leq \sum \sigma^1 Q \).

Thus, the non-zero \( L \)-blocks in \( \mu^{11} \) have no gaps between their \( L \)-block intervals. In fact, even if some \( e_j \) are zeros, \( \theta \) being a non-zero \( L \)-block implies that \( \beta^c + \mu^{10} \) is the sum of disjoint \( L \)-blocks, but we do require \( e_k \geq 1 \) for the next part.

Suppose that \( \mu^{11} \) has a smallest non-zero \( L \)-block \( \theta^0 \) with \( L \)-block interval \([b, s]\) such that \( \theta^0 \) is not maximal, and \( a \leq \ell < b \). Then, by Lemma 47, Part 2 (b),

\[
\sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{11} Q + \sum \mu^{10} Q + Q_{2N} \\
\leq \sum \mu^* Q + \sum \mu^{11} Q + Q_a + \sum \mu^{10} Q \\
\leq \sum \mu^* Q + \sum \text{res}_{(b,n)}(\mu^{11}) Q + Q_b + \sum \mu^{10} Q \\
= \sum \mu^* Q + \sum \text{res}_{(b,n)}(\mu^{11}) Q + (\theta^0 + \beta^b) Q + \sum \mu^{10} Q.
\]

There is a non-negative integer \( v \) such that \( \theta^0 = \sum_{k=1}^{s-b} e_k \beta^{s-k+1} + v \beta^b \) and \( v < e_{s-b+1} \). If \( s - b + 1 = N \), then \( v < e_{N-1} \) since the \( L \)-block \( \theta^0 \) is non-maximal, and hence, \( \theta^0 + \beta^b \) is an \( L \)-block with \( L \)-block interval \([b, s]\). Otherwise, the \( L \)-block interval is either \([b, s]\) or \([b - 1, s]\) since \( e_{s-b+1} \geq 1 \). Since \( b - 1 \geq a \), the last three terms of the last expression is the sum of disjoint \( L \)-blocks. By Lemma 47, Part 2 (d), we have \( \sum \sigma^1 Q < \sum \mu^* Q + Q_{n+1} \leq \sum \mu^* Q + Q_K \leq \sum \sigma^1 Q \).

Thus, \( \mu^{11} = \theta + \text{res}_{(\ell,K)}(\beta^{n+1}) \). Let us show that \( n = K - 1 \). If \( n < K - 1 \), then by Lemma 47, Part 2 (d),

\[
\sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{11} Q + \sum \mu^{10} Q + Q_{2N} \\
\leq \sum \mu^* Q + \sum \mu^{11} Q + Q_a + \sum \mu^{10} Q \\
\leq \sum \mu^* Q + Q_{n+1} + \sum \mu^{10} Q < \sum \mu^* Q + Q_{n+2} \\
\leq \sum \mu^* Q + Q_K \leq \sum \sigma^1 Q.
\]

This concludes the proof of \( \mu^{11} = \theta + \text{res}_{(\ell,K)}(\beta^K) \). Moreover, if \( \mu_K + 1 < \sigma_K \), then \( \sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{11} Q + Q_a + \sum \mu^{10} Q \leq \sum \mu^* Q + Q_K + \sum \mu^{10} Q < \sum \mu^* Q + Q_K + Q_{m+1} < \sum \mu^* Q + Q_K + Q_K \) by Lemma 47, Part 2 (d) where \( m = \text{ord}(\mu^{10}) \leq 3N - 2 \). Then, \( \sum \sigma^1 Q < \sum \mu^* Q + 2Q_K \leq \sum \sigma^1 Q \). Thus, we have \( \mu_K + 1 = \sigma_K \) as well.
Let us also show that \( a \leq 3N \). If \( a \geq 3N + 1 \), then
\[
\sum \sigma^1 Q < \sum \mu^* Q + \sum \mu^{11} Q + Q_{2N} + \sum \mu^{10} Q
\]
\[
< \sum \mu^* Q + \sum \mu^{11} Q + Q_{3N} + \sum \mu^{10} Q.
\]
The coefficient function \( \beta^{3N} \) or \( \beta^{3N} + \beta^{3N-1} \) makes an \( L \)-block since \( e_2 \geq 1 \), and since \( \text{ord}(\mu^{10}) \leq 3N-2 \) and \( a \geq 3N + 1 \), the coefficient function \( \mu^* + \mu^{11} + \beta^{3N} + \mu^{10} \) is the sum of disjoint \( L \)-blocks. By Lemma 47, Part 2 (d), the previous inequality implies \( \sum \sigma^1 Q < \sum \mu^* + Q_K \leq \sum \sigma^1 Q \).

Let us claim that \( \sigma_k = 0 \) for all \( 2N \leq k < K \). Suppose that \( \sigma_r > 0 \) for some \( 2N \leq r < K \). Then, \( \sigma_K^{1} = \mu_K^{*} + 1 \) and \( Q_K \geq \sum \mu^{11} Q + Q_a \) imply
\[
\sum \sigma^1 Q - \sum \mu^* Q - \sum \mu^{11} Q > Q_r \geq Q_{2N},
\]
and this holds for \( \mu^{10} = 0 \) as well. On the other hand, \( \sum \sigma^1 Q - \sum \mu^* Q - \sum \mu^{11} Q = \sum \mu^0 Q - \sum \sigma^0 Q < Q_{2N} \), which contradicts the above inequality. Therefore, \( \sigma_k = 0 \) for all \( 2N \leq k < K \).

Recall that \( \hat{e}_k := e_k \) if \( 1 \leq k \leq N-1 \), and \( \hat{e}_{N-1} := e_N - 1 \). Then, the linear recurrence implies
\[
Q_K = \sum_{k=1}^{N} \hat{e}_k Q_{K-k} + \sum_{k=1}^{N} \hat{e}_k Q_{K-N-k} + \cdots + \sum_{k=1}^{N} e_k Q_{\ell-k+1}
\]
\[
\Rightarrow \sum \mu^0 Q - \sum \sigma^0 Q
\]
\[
= \sum \sigma^1 Q - \sum \mu^* Q - \sum \mu^{11} Q - \sum \mu^{10} Q
\]
\[
= \sum \text{res}_{\{1,2N\}}(\sigma^1) Q + Q_K - \sum \theta Q - \sum \text{res}_{\{\ell,K\}}(\hat{e}^K) Q - \sum \mu^{10} Q
\]
\[
= \sum \text{res}_{\{1,2N\}}(\sigma^1) Q + \sum_{k=1}^{N} e_k Q_{\ell-k+1} - \sum \theta Q - \sum \mu^{10} Q
\]
\[
\Rightarrow \sum \mu^0 Q + \sum \mu^{10} Q + \sum \theta Q
\]
\[
= \sum \sigma^0 Q + \sum \text{res}_{\{1,2N\}}(\sigma^1) Q + Q_{\ell+1}
\]
\[
= \sum \text{res}_{\{1,2N\}}(\sigma) Q + Q_{\ell+1}.
\]

Notice that \( Q_{\ell+1} \) or \( 0 \cdot Q_{\ell} + Q_{\ell+1} \) makes an \( L \)-block. Since \( 2N + 1 \leq a + 1 \leq \ell + 1 \leq a + N \leq 4N \), the coefficient function \( \text{res}_{\{1,2N\}}(\sigma) + Q_{\ell+1} \) is a sum of disjoint \( L \)-blocks, and it has order \( \leq 4N \). The coefficient function of the LHS is a sum of disjoint \( L \)-blocks, which has order \( \leq 4N-1 \). This concludes the proof of Part 2.

\[ \square \]

Theorem 48 proves the if-part of Theorem 22, and the only-if-part is trivial.
6 Future work

Recall Theorem 1, the full converse of Zeckendorf’s Theorem, and that the full converse fails for \( \mathbb{N} \) under the Zeckendorf condition \( E \) defined in Example 23. We may ask ourselves whether the full converse holds for all \( E \)-subsets of \( \mathbb{N} \) if it holds for \( \mathbb{N} \), or which additional conditions on Zeckendorf collections will guarantee the full converse. For example, if \( \mathcal{L} \) is the Zeckendorf collection defined by the Zeckendorf multiplicity list \( L = (1, 1) \) and \( Q \) is a sequence given by \( Q_k = 2^{k-1} \) for \( k \geq 1 \), does \( X^{\mathcal{L}}_Q \) have a unique fundamental sequence?

We proved the weak converse for \( I \) under the Zeckendorf conditions determined by certain short linear recurrences considered in Theorem 24, Part 2, but the weak converse for other Zeckendorf conditions remains to be investigated. Even for the Zeckendorf conditions considered in Theorem 24, Part 2, we may ask ourselves whether the weak converse holds for \( \mathcal{L} \)-subsets of \( I \). For example, if \( Q \) is a sequence given by \( Q_k = 1/2^k \) for \( k \geq 1 \) and \( L = (1, 1) \), does \( X^{\mathcal{L}}_Q \) have a unique decreasing fundamental sequence?

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