Breather, coupled and periodic coupled soliton solutions for modified KdV equation

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We present the discovery of a class of exact spatially localized as well as periodic solutions within the framework of the modified Korteweg-de Vries equation. This class comprises breather and coupled soliton solutions as well as periodic wave solutions. The functional forms of these solutions include a joint parameter which can take both positive and negative values of unity. It is found that the existence of those closed form solutions depend strongly on whether the cubic nonlinearity parameter should be considered positive or negative. The derived wave structures show interesting properties that are important for practical applications. The propagation behavior of these nonlinear waves is strongly dependent on the values of parameters \( p \) and \( q \) of obtained solutions.

PACS numbers: 05.45.Yv, 42.65.Tg, 42.81.Qb

The modified Korteweg-de Vries (mKdV) equation is an important model which applies to the description of wave dynamics in different physical systems, such as soliton propagation in lattices [1], meandering ocean currents [2], the dynamics of traffic flow [3–5], nonlinear Alfvén waves propagating in plasma [6], and ion acoustic soliton experiments in plasmas [7]. This model is also relevant for nonlinear waves in distributed Schottky barrier diode transmission lines [8] and internal waves in stratified fluids [9]. Recent results have also demonstrated that the propagation of optical pulses consisting of a few cycles in Kerr-type media can be described beyond the slowly varying envelope approximation by using the mKdV equation [10–13]. Additionally, this equation has gained further importance recently, mainly because of its effectiveness in modeling supercontinuum generation in nonlinear optical fibers [14].

The mKdV equation has been successfully used to model the evolution of long waves in the critical case of vanishing quadratic nonlinearity [15]. Due to the wide-ranging potential applications of this nonlinear wave evolution equation, various powerful methods have been employed to search for its explicit solutions. This because wave solutions are helpful for a better understanding of physical phenomena modeled by this equation such as the stability of nonlinear wave propagation. Having solutions in analytic form is relevant not only for determining certain important physical quantities and serving as diagnostics for simulations but also even for comparing experimental results with theory. Particularly, Wadati derived the exact N-soliton solutions of the mKdV equation by using the inverse-scattering transform scheme [16]. In addition, Kevrekidis et al. [17] obtained some classes of periodic solutions of this model. Another class of nonlinear wave solutions that conserve their energy during evolution -breathers (oscillatory wave packets)- has been also found for this model (see, e.g., [18–20]).

A challenging problem is the search for other types of nonlinear wave solutions for the real mKdV equation. In contrast to the complex mKdV equation, it is not straightforward to find the exact analytical spatially localized solutions (breather and solitons) for its real variant. Obtaining more explicit solutions for this equation should significantly widen its applicability in various branches of contemporary physics.

In this Letter, we predict the existence of three types of nonlinear wave structures through discovery of physically important exact solutions of the mKdV model that includes the cubic nonlinearity term with either positive or negative sign. Significant classes of breather and periodic waves as well as coupled soliton solutions are presented for the first time. Remarkably, breather and periodic waves formation is observed in the case of a negative coefficient of the cubic nonlinear term while the soliton solution is formed when this coefficient is positive.

We start by considering the mKdV equation in standard dimensionless form as

\[
  u_t + u_{xxx} + 6\mu u^2 u_x = 0,
\]

where \( u(x, t) \) is the real function. The parameter \( \mu \) (\( \pm 1 \)) denotes the type of nonlinearity, i.e., +1 for focusing type of nonlinearity and −1 for defocusing nonlinearity. The mKdV is a fully integrable equation which means that it has an infinite number of conserved invariants [21].

Two soliton branches exist for the mKdV equation (1) in the case of a positive coefficient of the cubic nonlinear term (i.e., \( \mu = 1 \)), they are defined as

\[
  u(x, t) = p + \frac{q^2}{\Lambda \cosh(q(x - x_0 - vt)) + 2p},
\]
where $\Lambda = \pm \sqrt{4p^2 + q^2}$, $v = 6p^2 + q^2$, and $p, q, x_0$ are the arbitrary constants. In the case with $\mu = 1$ also exists the algebraic solitary wave solution as

$$u(x, t) = p - \frac{p}{p^2(x - x_0 - vt)^2 + 1/4},$$

(3)

where $v = 6p^2$, and $p, x_0$ are the arbitrary constants. In the limit when $q \gg p$, the solution in Eq. (2) tends to the familiar sech-shaped soliton family, which is known to be stable with respect to small perturbations (see, e.g., [22]).

The breather’s expression for $\mu = 1$ was obtained from inverse scattering transform by Pelinovsky and Grimshaw in [15] and later in Ref. [23]. Note that the breather has much more complicated dynamics than soliton. Recently Slunyaev and Pelinovsky [24] also presented an explicit breather solution for the mKdV equation (1) with $\mu = 1$ as

$$u(x, t) = 2pq \left( \frac{p\sinh(\theta) \sin(\phi) - q\cosh(\theta) \cos(\phi)}{p^2 \sin^2(\phi) + sq^2 \cosh^2(\theta)} \right),$$

(4)

when $s = -1$. Here $\theta$ and $\phi$ which control the wave envelope and the inner wave respectively, are given by

$$\theta = p(x - x_0) + p(3q^2 - p^2)t + \theta_0,$$

(5)

$$\phi = q(x - x_0) + q(q^2 - 3p^2)t + \phi_0,$$

(6)

where $p, q$ and $\theta_0, \phi_0, x_0$ are the arbitrary real parameters. We have remarked that this breather solution still also existing for the case when $s = +1$ with the same relations of $\theta$ and $\phi$ as those given in Eqs. (5) and (6). Thus the mKdV equation (1) with the focusing type of nonlinearity possesses two exact breather solutions (4) corresponding to the values $s = \pm 1$.

**FIG. 1:** Coupled soliton solution of mKdV equation defined by Eq. (7) ($\mu = 1$) for the values $p = 1$, $q = 0.2$ and $s = 1$. 
In what follows, we report what is to our knowledge the first analytical demonstration of existence of a class of breather, periodic and coupled soliton solutions for the mKdV equation with either focusing or defocusing types of nonlinearity. More precisely, using the inverse scattering method (the details can be published somewhere) we have found three types of exact solutions for the modified KdV equation (1) with \( \mu = \pm 1 \). These solutions are given in Eq. (7) (for \( \mu = 1 \)), and Eqs. (10) and (13) (for \( \mu = -1 \)). A first class of localized waves in the form of coupled soliton solution is obtained for Eq. (1) in the case of focusing cubic nonlinearity (i.e., \( \mu = 1 \)) as

\[
\mathcal{u}(x, t) = 2pq \left( \frac{p \sin(\theta) \sinh(\phi) - q \cosh(\theta) \cosh(\phi)}{p^2 \sinh^2(\phi) + sq^2 \cosh^2(\theta)} \right),
\]

where \( s = \pm 1 \) and \( \theta, \phi \) are given by

\[
\theta = p(x - x_0) - p(3q^2 + p^2)t + \theta_0,
\]

\[
\phi = q(x - x_0) - q(3p^2 + q^2)t + \phi_0.
\]

Here and below \( p, q, \theta_0, \phi_0, x_0 \) are the arbitrary real parameters. A typical example of coupled soliton profile is shown in Fig. 1 (with \( \mu = 1 \)) for the case \( p = 1, q = 0.2 \) and \( s = 1 \). Here we choose the initial soliton position \( x_0 \) and phases \( \theta_0 \) and \( \phi_0 \) equal to zeros. From this figure, we can see that the wave solution (7) takes the shape of a coupled pair of bright and dark solitons. Compared with the well-known sech-shaped soliton which is located on a zero background, this mKdV soliton solution containing a dark localized mode in the middle of the structure sits on a nonzero background. We notice that the solution (11) with \( s = -1 \) also takes the shape of coupled soliton structure.

We have obtained another exact solution for the mKdV equation (11) with \( \mu = -1 \) in the form of a periodic wave solution as,

\[
\mathcal{u}(x, t) = 2pq \left( \frac{p \sin(\theta) \sin(\phi) + q \cos(\theta) \cos(\phi)}{p^2 \sin^2(\phi) - sq^2 \cos^2(\theta)} \right),
\]

FIG. 2: Periodic wave solution of mKdV equation defined by Eq. (10) ( \( \mu = -1 \)) for the values \( p = 0.5, q = 0.45 \) and \( s = -1 \).
where \( s = \pm 1 \) and \( \theta, \phi \) are

\[
\theta = p(x - x_0) + p(3q^2 + p^2)t + \theta_0, \tag{11}
\]

\[
\phi = q(x - x_0) + q(3p^2 + q^2)t + \phi_0. \tag{12}
\]

Figure 2 depicts an example of this periodic solution (with \( \mu = -1 \)) for the case \( p = 0.5, q = 0.45 \) and \( s = -1 \). Here the initial parameters \( x_0, \theta_0 \) and \( \phi_0 \) are chosen equal to zeros. We observe that the nonlinear waveform presents an oscillating behavior during the process of wave evolution.

We have also found that the mKdV equation \( \text{(1)} \) with the defocusing type of nonlinearity (i.e., \( \mu = -1 \)) satisfies an exact breather solution of the form:

\[
u(x, t) = 2pq \left( \frac{p\sin(\theta) \sinh(\phi) + q\cos(\theta) \cosh(\phi)}{p^2 \sinh^2(\phi) - sq^2 \cos^2(\theta)} \right), \tag{13}\]

where \( s = \pm 1 \) and \( \theta, \phi \) are

\[
\theta = p(x - x_0) + p(p^2 - 3q^2)t + \theta_0, \tag{14}
\]

\[
\phi = q(x - x_0) + q(3p^2 - q^2)t + \phi_0. \tag{15}
\]

![Breather solution of mKdV equation defined by Eq. (13) (\( \mu = -1 \)) for the values \( p = 1, q = 0.98 \) and \( s = -1 \).](image)

It is clear that our breather solution \( \text{(13)} \) which includes two nonlinear waveforms corresponding to the values \( s = \pm 1 \) markedly differ from the solution \( \text{(4)} \). Compared with the breather solution \( \text{(4)} \) which is formed in the case of
focusing nonlinearity ($\mu = 1$), the present solution exits in a defocusing situation ($\mu = -1$). To our knowledge, exact breather solutions to the mKdV equation with the defocusing type of nonlinearity are firstly reported in this paper. From these important results, one may conclude that a physical system described by the mKdV equation could allow a breather evolution in either the focusing or the defocusing nonlinearity. A typical example of the breather profile is shown Fig. 3 (with $\mu = -1$) for the values $p = 1$, $q = 0.98$ and $s = -1$. For this case, the initial parameters $x_0$, $\theta_0$ and $\phi_0$ are chosen equal to zeros. It can be seen that the breather structure is periodic in $t$ and the positions of the peaks are opposed to each other.

In conclusion, we have presented three types of spatially localized and periodic wave solutions for the mKdV equation describing propagation of nonlinear long waves in many physics areas when there is polarity symmetry. A class of exact soliton solutions is firstly obtained for the model. Unlike the reported usual soliton solution of the mKdV equation, the novel one can describe the propagation of coupled bright and dark solitons in the same time and is located on a constant background. We have also found a number of exact periodic and breather type solutions for the equation. The results showed that the formation of those closed form solutions is determined by the sign of cubic nonlinearity parameter solely. Undoubtedly, these new solutions will be useful for recognizing physical phenomena and dynamical processes in various physical systems where the mKdV equation can provide a realistically accurate description of the waves.

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