Spherical and Planar Ball Bearings — Nonholonomic Systems with Invariant Measures

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Abstract—We first construct nonholonomic systems of \( n \) homogeneous balls \( B_1, \ldots, B_n \) with centers \( O_1, \ldots, O_n \) and with the same radius \( r \) that are rolling without slipping around a fixed sphere \( S_0 \) with center \( O \) and radius \( R \). In addition, it is assumed that a dynamically nonsymmetric sphere \( S \) of radius \( R + 2r \) and the center that coincides with the center \( O \) of the fixed sphere \( S_0 \) rolls without slipping over the moving balls \( B_1, \ldots, B_n \). We prove that these systems possess an invariant measure. As the second task, we consider the limit, when the radius \( R \) tends to infinity. We obtain a corresponding planar problem consisting of \( n \) homogeneous balls \( B_1, \ldots, B_n \) with centers \( O_1, \ldots, O_n \) and the same radius \( r \) that are rolling without slipping over a fixed plane \( \Sigma_0 \), and a moving plane \( \Sigma \) that moves without slipping over the homogeneous balls. We prove that this system possesses an invariant measure and that it is integrable in quadratures according to the Euler–Jacobi theorem.

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Dedicated to the memory of Professor Alexey Vladimirovich Borisov

1. INTRODUCTION

In this paper, we first construct nonholonomic systems of \( n \) homogeneous balls \( B_1, \ldots, B_n \) with centers \( O_1, \ldots, O_n \) and with the same radius \( r \) that are rolling without slipping around a fixed sphere \( S_0 \) with center \( O \) and radius \( R \). We assume that a dynamically nonsymmetric sphere \( S \) of radius \( R + 2r \) and the center that coincides with the center \( O \) of the fixed sphere \( S_0 \) rolls without slipping over the moving balls \( B_1, \ldots, B_n \). The rolling of the balls \( B_i \) and the sphere \( S \) are considered under the inertia and in the absence of external forces. We refer to this system as a spherical ball bearing (see Fig. 1).

As the second task, we consider the limit, when the radius \( R \) tends to infinity. In that way, we obtain a corresponding planar problem consisting of \( n \) homogeneous balls \( B_1, \ldots, B_n \) with centers \( O_1, \ldots, O_n \) and the same radius \( r \) that are rolling without slipping over a fixed plane \( \Sigma_0 \), and a moving plane \( \Sigma \) that moves without slipping over the homogeneous balls. We refer to this system as a planar ball bearing (see Fig. 2).

Although the rolling ball problems have been very well studied (see [4, 6–8]), the spherical and planar bearing problems do not seem to have been considered before. There are two nonholonomic systems which are close to the spherical ball bearings. One is the so-called spherical support system, introduced by Fedorov in [11]. It describes the rolling without slipping of a dynamically nonsymmetric sphere \( S \) over \( n \) homogeneous balls \( B_1, \ldots, B_n \) of possibly different radii, but with fixed centers. The second one is the rolling of a homogeneous ball \( B \) over a dynamically asymmetric
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Sphere $S$, introduced by Borisov, Kilin, and Mamaev in [5]. They considered in [5] both situations: when the center of $S$ is fixed, and when it is not.

In Section 2 we define the spherical ball bearing system: the configuration space $Q$, the nonholonomic distribution $\mathcal{D} \subset TQ$ and the Lagrangian that coincides with the kinetic energy of the system. The kinetic energy and the distribution are invariant with respect to an appropriate action of the Lie group $SO(3)^{n+1}$, and the system can be reduced to $\mathcal{M} = \mathcal{D}/SO(3)^{n+1}$. In Section 3 we derive the equations of motion of the reduced spherical ball bearing system in terms of the reaction forces and list some first integrals in Propositions 1 and 2. Proposition 1 implies that the centers $O_1, \ldots, O_n$ are at rest with respect to each other. Thus, there are no collisions of the balls $B_1, \ldots, B_n$.

In Section 4 we perform the second reduction by fixing the values of the $n$ first integrals from Proposition 2. We obtain the closed system of equations of motion of the reduced spherical ball bearing system on the space $\mathcal{N} = \mathbb{R}^3 \times (S^2)^n$ in Theorem 1. The complete set of non-reduced equations of motion is given in Corollary 1. Finally, we prove that the spherical ball bearing problem has an invariant measure in Theorem 2. The question of integrability in the spherical ball bearing problem will be studied in a separate paper.

In Section 6 we consider the planar ball bearing problem. To simplify notation, we consider the case $n = 3$ and refer to the system as the three balls planar bearing problem. However, all the statements and considerations from Section 6 hold for arbitrary $n$ in a straightforward manner.

For general $n$, the configuration space and the nonholonomic distribution $\mathcal{D}$ are of the same dimensions as in the spherical ball bearing problem, but the description of the system is slightly different. For $n = 3$, we derive the equations of motion on $\mathcal{D}/SO(3) \times SO(3) \times SO(3)$, see Theorem 3. We perform a second reduction to a space $\mathcal{Q} \subset \mathbb{R}^6$ defined by an algebraic inequality. We prove that the planar three balls bearing problem on $\mathcal{Q}$ has an invariant measure and four independent first integrals. Therefore, it is integrable according to the Euler–Jacobi theorem, see Theorem 4.

2. ROLLING OF A DYNAMICALLY NONSYMMETRIC SPHERE OVER $N$ MOVING HOMOGENEOUS BALLS AND A FIXED SPHERE

We consider the following spherical ball bearing problem: $n$ homogeneous balls $B_1, \ldots, B_n$ with centers $O_1, \ldots, O_n$ and the same radius $r$ roll without slipping around a fixed sphere $S_0$ with center $O$ and radius $R$. A dynamically nonsymmetric sphere $S$ of radius $R + 2r$ with the center that coincides with the center $O$ of the fixed sphere $S_0$ rolls without slipping over the moving balls $B_1, \ldots, B_n$.

For $n \geq 4$ there are initial positions of the balls $B_1, \ldots, B_n$ that imply the condition that the center of the moving sphere $S$ coincides with the center $O$ of the fixed sphere $S_0$. Let us reiterate that the configuration of the balls is congruent during the time evolution. In order to include all...
possible initial positions for arbitrary \( n \), the condition that \( O \) coincides with the center of the sphere \( S \) is assumed to be a holonomic constraint.

Let
\[
O e_1^0, e_2^0, e_3^0, \quad O e_1, e_2, e_3, \quad O_i e_1^i, e_2^i, e_3^i, \quad i = 1, \ldots, n
\]
be positively oriented reference frames rigidly attached to the spheres \( S_0, S \), and the balls \( B_i \), \( i = 1, \ldots, n \), respectively. By \( g, g_i \in SO(3) \) we denote the matrices that map the moving frames \( O e_1, e_2, e_3 \) and \( O_i e_1^i, e_2^i, e_3^i \) to the fixed frame \( O e_1^0, e_2^0, e_3^0 \):
\[
g_{jk} = (e_j^0, e_k^0), \quad g_{i,jk} = (e_j^i, e_k^i), \quad j, k = 1, 2, 3, \quad i = 1, \ldots, n.
\]

We apply the standard isomorphism between the Lie algebras \( (so(3), [\cdot, \cdot]) \) and \( (\mathbb{R}^3, \times) \) given by
\[
a_{ij} = -\varepsilon_{ijk}g_{jk}, \quad i, j, k = 1, 2, 3. \tag{2.1}
\]
The skew-symmetric matrices
\[
\omega = \dot{g}g^{-1}, \quad \omega_i = \dot{g}_i g_i^{-1}
\]
correspond to the angular velocities \( \tilde{\omega}, \tilde{\omega}_i \) of the sphere \( S \) and the \( i \)-th ball \( B_i \) in the fixed reference frame \( O e_1^0, e_2^0, e_3^0 \) attached to the sphere \( S_0 \). The matrices
\[
\Omega = g^{-1}g = g^{-1}\omega g, \quad W_i = g_i^{-1}g_i = g_i^{-1}\omega_i g_i
\]
correspond to the angular velocities \( \tilde{\Omega}, \tilde{W}_i \) of \( S \) and the \( B_i \) in the frames \( O e_1, e_2, e_3 \) and \( O_i e_1^i, e_2^i, e_3^i \) attached to the sphere \( S \) and the balls \( B_i \), respectively.

We have
\[
\tilde{\omega} = g\tilde{\Omega}, \quad \tilde{\omega}_i = g_i\tilde{W}_i.
\]

Let \( I \) be the inertia operator of the outer sphere \( S \). We choose the moving frame \( O e_1, e_2, e_3 \), such that \( O e_1, O e_2, O e_3 \) are the principal axes of inertia: \( I = \text{diag}(A, B, C) \). Let \( \text{diag}(I_1, I_i, I_i) \) and \( m_i \) be the inertia operator and the mass of the \( i \)-th ball \( B_i \). Then the configuration space and the kinetic energy of the problem are given by

\[
Q = SO(3)^{n+1} \times (S^2)^n \{ g, g_1, \ldots, g_n, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \},
\]
\[
T = \frac{1}{2}\langle I\tilde{\Omega}, \tilde{\Omega} \rangle + \frac{1}{2} \sum_{i=1}^n I_i \langle \tilde{W}_i, \tilde{W}_i \rangle + \frac{1}{2} \sum_{i=1}^n m_i \langle \tilde{v}_{O_i}, \tilde{v}_{O_i} \rangle
\]
\[
= \frac{1}{2}\langle I\tilde{\Omega}, \tilde{\Omega} \rangle + \frac{1}{2} \sum_{i=1}^n I_i \langle \tilde{\omega}, \tilde{\omega}_i \rangle + \frac{1}{2} \sum_{i=1}^n m_i \langle \tilde{v}_{O_i}, \tilde{v}_{O_i} \rangle.
\]

Here \( \tilde{\gamma}_i \) is the unit vector
\[
\tilde{\gamma}_i = \frac{\overrightarrow{O_i\tilde{\gamma}_i}}{|\overrightarrow{O_i\tilde{\gamma}_i}|}
\]
determining the position \( O_i \) of the center of the \( i \)-th ball \( B_i \) and \( \tilde{v}_{O_i} = (R+r)\tilde{\gamma}_i \) is its velocity, \( i = 1, \ldots, n \). The kinetic energy plays the role of the Lagrangian.

Let us denote the contact points of the balls \( B_1, \ldots, B_n \) with the spheres \( S_0 \) and \( S \) by \( A_1, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \), respectively. The condition that the balls \( B_1, \ldots, B_n \) and the sphere \( S \) roll without slipping leads to the nonholonomic constraints
\[
\tilde{v}_{O_i} + \tilde{\omega}_i \times \overrightarrow{O_i A_i} = 0, \quad \tilde{v}_{O_i} + \tilde{\omega}_i \times \overrightarrow{O_i B_i} = \tilde{\omega} \times \overrightarrow{O B_i}, \quad i = 1, \ldots, n,
\]
that is,
\[
\tilde{v}_{O_i} = r\tilde{\omega}_i \times \tilde{\gamma}_i, \quad \tilde{v}_{O_i} = (R+2r)\tilde{\omega} \times \tilde{\gamma}_i - r\tilde{\omega}_i \times \tilde{\gamma}_i, \quad i = 1, \ldots, n. \tag{2.2}
\]

The dimension of the configuration space \( Q \) is \( 5n + 3 \). There are \( 4n \) independent constraints in (2.2), defining a nonintegrable distribution \( \mathcal{D} \subset TQ \). Therefore, the dimension of the vector
subspaces of admissible velocities \( D_q \subset T_q Q \) is \( n + 3, q \in Q \). The phase space of the system has the dimension \( 6n + 6 \), which is the dimension of the bundle \( D \) as a submanifold of \( TQ \).

The equations of motion of the spherical ball bearing problem are given by the Lagrange–d’Alembert equations [1, 2]

\[
\delta T = \left( \frac{\partial T}{\partial q} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}} \right) \delta q = 0, \quad \text{for all virtual displacements } \delta q \in D_q. \tag{2.3}
\]

Instead of using the Lagrange–d’Alembert equations, below we will derive the equations of motion directly from the fundamental laws of classical mechanics and using the vector notation.

**Remark 1.** We will show in Proposition 1 that, if the initial conditions are chosen such that the distances between \( O_i \) and \( O_j \) are all greater than \( 2r \), \( 1 \leq i < j \leq n \), then the balls will not have collisions along the course of motion. This is the reason why we do not assume additional one-side constraints

\[
|\gamma_i - \gamma_j| \geq \frac{2r}{r + R}, \quad 1 \leq i < j \leq n. \tag{2.4}
\]

The kinetic energy and the constraints are invariant with respect to the \( SO(3)^{n+1} \)-action defined by

\[
(g, g_1, \ldots, g_n, \gamma_1, \ldots, \gamma_n) \mapsto (ag, ag_1a_1^{-1}, \ldots, ag_na_n^{-1}, a\gamma_1, \ldots, a\gamma_n), \tag{2.5}
\]

\( a, a_1, \ldots, a_n \in SO(3) \), representing a freedom in the choice of the reference frames

\( Oe_1^0, e_2^0, e_3^0, \quad O_i e_i^1, e_i^2, e_i^3, \quad i = 1, \ldots, n. \)

Indeed, for the extension of the transformation (2.5) to the tangent bundle \( TQ \) we have

\[
\Omega = g^{-1}g \mapsto (ag)^{-1}ag = \Omega,
\]

\[
\omega = gg^{-1} \mapsto ag(ag)^{-1} = a\omega a^{-1},
\]

\[
\omega_i = g_i g_i^{-1} \mapsto ag_ia_i^{-1}(ag_i)^{-1} = a\omega_i a^{-1},
\]

and, therefore,

\[
\vec{\Omega} \mapsto \vec{\Omega}, \quad \vec{\omega} \mapsto a\vec{\omega}, \quad \vec{\omega}_i \mapsto a\vec{\omega}_i, \quad \vec{v}_{O_i} \mapsto a\vec{v}_{O_i}. \tag{2.6}
\]

It is clear that the kinetic energy and the constraints are invariant with respect to the transformation (2.6). Also, note that (2.5) does not change the vectors \( \vec{\omega}_i, \vec{\gamma}_i, \vec{v}_{O_i} \) written in the moving frame \( Oe_1, e_2, e_3 \):

\[
\vec{\Omega}_i = g^{-1}_i \vec{\omega}_i \mapsto (ag)^{-1}a\vec{\omega}_i = \vec{\Omega}_i,
\]

\[
\vec{\Gamma}_i = g^{-1}_i \vec{\gamma}_i \mapsto (ag)^{-1}a\vec{\gamma}_i = \vec{\Gamma}_i \quad \text{(implying } \vec{\Gamma}_i \mapsto \vec{\Gamma}_i),
\]

\[
\vec{V}_{O_i} = g^{-1}_i \vec{v}_{O_i} = (R + r)g^{-1}_i \vec{\gamma}_i \mapsto (R + r)(ag)^{-1}a\vec{\gamma}_i = g^{-1}_i \vec{V}_{O_i} = \vec{V}_{O_i}.
\]

Thus, for the coordinates in the space \((TQ)/SO(3)^{n+1}\) we can take the angular velocities and the unit position vectors in the reference frame attached to the sphere \( S \):

\[
(TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \{ \vec{\Omega}, \vec{\Omega}_1, \ldots, \vec{\Omega}_n; \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \}.
\]

In the moving reference frame \( Oe_1, e_2, e_3 \), the constraints become

\[
\vec{V}_{O_i} = (R + 2r)\vec{O}_i \times \vec{\Gamma}_i - r\vec{\Omega}_i \times \vec{\Gamma}_i, \tag{2.7}
\]

\[
\vec{V}_{O_i} = r\vec{\Omega}_i \times \vec{\Gamma}_i, \quad i = 1, \ldots, n, \tag{2.8}
\]

defining the reduced phase space \( \mathcal{M} = D/\text{SO}(3)^{n+1} \subset (TQ)/\text{SO}(3)^{n+1} \) of dimension \( 3n + 3 \).

Since both the kinetic energy and the constraints are invariant with respect to the \( SO(3)^{n+1} \)-action (2.5), the equations of motion (2.3) are also \( SO(3)^{n+1} \)-invariant. Thus, they induce a well-defined system on the reduced phase space \( \mathcal{M} \).
3. THE KINEMATIC AND MOMENTUM EQUATIONS IN THE MOVING FRAME

The time derivative of $\dot{\Gamma}_i$ can be directly extracted from the constraints as follows.

**Lemma 1.** The kinematic part of the equations of motion of the spherical ball bearing system is

$$\dot{\Gamma}_i = \frac{R}{2R + 2r} \Gamma_i \times \bar{\Omega}, \quad i = 1, \ldots, n.$$  \hspace{1cm} (3.1)

**Proof.** Let us for the moment consider the fixed reference frame $O\bar{e}_1, \bar{e}_2, \bar{e}_3$. One has

$$\dot{\Omega} \bar{e}_i + \bar{\omega}_i \times \bar{\Omega} \bar{e}_i = 0.$$  

Therefore, $(R + r)\dot{\gamma}_i - r\dot{\omega}_i \times \gamma_i = 0$, or equivalently,

$$\dot{\gamma}_i = \frac{r}{R + r} \dot{\omega}_i \times \gamma_i.$$ \hspace{1cm} (3.2)

Equation (3.2) in the moving reference frame $O\bar{e}_1, \bar{e}_2, \bar{e}_3$ has the form

$$\dot{\Gamma}_i + \bar{\Omega} \times \Gamma_i = \frac{r}{R + r} \bar{\Omega} \times \Gamma_i.$$  

Thus, we get

$$\dot{\Gamma}_i = \left( \frac{r}{R + r} \bar{\Omega} - \bar{\Omega}_i \right) \times \Gamma_i.$$ \hspace{1cm} (3.3)

From the constraints (2.7) and (2.8), we obtain

$$\bar{\Omega}_i \times \Gamma_i = \frac{R + 2r}{2r} \bar{\Omega} \times \Gamma_i, \quad i = 1, \ldots, n.$$ \hspace{1cm} (3.4)

Finally, using (3.4), Eqs. (3.3) can be written in a more convenient form (3.1). \qed

As a consequence, we have:

**Proposition 1.** The following functions are the first integrals of motion:

$$\langle \Gamma_i, \Gamma_j \rangle = \gamma_{ij} = \text{const}, \quad i, j = 1, \ldots, n.$$  

**Proof.** By a direct differentiation, we get

$$\frac{d}{dt} \langle \Gamma_i, \Gamma_j \rangle = \langle \dot{\Gamma}_i, \Gamma_j \rangle + \langle \Gamma_i, \dot{\Gamma}_j \rangle = \frac{R}{2R + 2r} \left( \langle \Gamma_i \times \bar{\Omega}, \Gamma_j \rangle + \langle \Gamma_j \times \bar{\Omega}, \Gamma_i \rangle \right) = 0.$$ \hspace{1cm} \qed

In other words, the centers $O_i$ of the homogeneous balls $B_i$ are at rest with respect to each other. In particular, since $\langle \gamma_i, \gamma_j \rangle = \langle \bar{\Omega}_i, \bar{\Omega}_j \rangle$, the interior of the region (2.4) is invariant under the flow of the system.

Next, let $\bar{F}_{B_i}$ and $\bar{F}_{A_i}$ be the reaction forces that act on the ball $B_i$ at the points $B_i$ and $A_i$, respectively. The reaction force at the point $B_i$ on the sphere $S$ is then $-\bar{F}_{B_i}$.

By using the laws of change of angular momentum and momentum of a rigid body in the moving reference frame for the balls $B_i$ and the sphere $S$ (e.g., see [1]), we get:

**Lemma 2.** The dynamical part of the equations of motion of the spherical ball bearing system is

$$I_i \ddot{\Omega}_i = I_i \bar{\Omega}_i \times \bar{\Omega} + r \Gamma_i \times (\bar{F}_{B_i} - \bar{F}_{A_i}),$$  \hspace{1cm} (3.5)

$$m_i \ddot{V}_{O_i} = m_i \bar{V}_{O_i} \times \bar{\Omega} + \bar{F}_{B_i} + \bar{F}_{A_i}, \quad i = 1, \ldots, n$$ \hspace{1cm} (3.6)

$$I \ddot{\Omega} = I \bar{\Omega} \times \bar{\Omega} - \sum_{i=1}^{n} (R + 2r) \Gamma_i \times \bar{F}_{B_i}.$$ \hspace{1cm} (3.7)
For \( n = 1 \) and the absence of the interior fixed sphere \( S_0 \), i.e., \( \mathbf{F}_{A_0} = 0 \), see [5].

We still need to calculate the torques of reaction forces. Prior to that, we formulate and prove the following important statement.

**Proposition 2.** The projections of the angular velocities \( \vec{\Omega}_i \) to the directions \( \vec{\Gamma}_i \) are the first integrals of motion:

\[
\langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = \text{const}, \quad i = 1, \ldots, n.
\]

**Proof.** From (3.5) and (3.3) we get

\[
\frac{d}{dt} \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = \langle \dot{\vec{\Omega}}_i, \vec{\Gamma}_i \rangle + \left( \frac{1}{I_i} \frac{r}{\vec{\Gamma}_i} \right) \times \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle + \left( \frac{r}{I_i} \right) \times (\mathbf{F}_{B_i} - \mathbf{F}_{A_i}) \cdot \vec{\Gamma}_i
\]

\[
+ \left( \vec{\Omega}_i, \frac{r}{R + r} \frac{r}{\vec{\Gamma}_i} \vec{\Omega}_i \times \vec{\Gamma}_i \right) - \langle \vec{\Omega}_i, \frac{r}{R + r} \vec{\Omega}_i \times \vec{\Gamma}_i \rangle = 0.
\]

\[
\square
\]

4. THE REDUCED SYSTEM

From the constraints written in the moving frame (2.7), (2.8) and (3.4), we get

\[
\langle \vec{\Omega} \times \vec{\Gamma}_i, \vec{\Omega}_i \rangle = 0.
\]

That means that the vectors \( \vec{\Gamma}_i, \vec{\Omega}, \vec{\Omega}_i \) are coplanar. Moreover, we obtain

\[
\vec{\Omega}_i = \langle \vec{\Gamma}_i, \vec{\Omega} \rangle \vec{\Gamma}_i + \frac{R + 2r}{2r} \frac{r}{\vec{\Gamma}_i} \vec{\Omega} - \frac{R + 2r}{2r} (\vec{\Gamma}_i, \vec{\Omega}) \vec{\Gamma}_i.
\]  

(4.1)

Further, from Proposition 2, we find that the reduced phase space \( M = D / SO(3)^n + 1 \) is foliated by \( 2n + 3 \)-dimensional invariant varieties

\[
\mathcal{M}_c : \quad \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = \text{const}, \quad i = 1, \ldots, n.
\]

On the invariant variety \( \mathcal{M}_c \), the vector-functions \( \vec{\Omega}_i \) can be uniquely expressed as functions of \( \vec{\Omega}, \vec{\Gamma}_i \) using Eq. (4.1):

\[
\vec{\Omega}_i = c_i \vec{\Gamma}_i + \frac{R + 2r}{2r} \vec{\Omega} - \frac{R + 2r}{2r} (\vec{\Gamma}_i, \vec{\Omega}) \vec{\Gamma}_i.
\]  

(4.2)

Whence, \( \vec{\Omega} \) determines all velocities of the system on \( \mathcal{M}_c \) and \( \mathcal{M}_c \) is diffeomorphic to the second reduced phase space

\[
\mathcal{N} = \mathbb{R}^3 \times (S^2)^n \{ \Omega, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \}.
\]

This can be seen as follows. Consider the natural projection

\[
\pi : (TQ)/SO(3)^n + 1 \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \to \mathcal{N},
\]

\[
\pi(\vec{\Omega}, \vec{\Omega}_1, \ldots, \vec{\Omega}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n) = (\vec{\Omega}, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n),
\]

and let \( \pi_c \) be the restriction to \( \mathcal{M}_c \subset \mathcal{M} \subset (TQ)/SO(3)^n + 1 \) of \( \pi \). Then the projection

\[
\pi_c : \mathcal{M}_c \mapsto \mathcal{N}
\]

is a bijection.

Thus, instead of the derivation of the torques of all reaction forces in (3.5) and (3.7), it is sufficient to find the torque in Eq. (3.7) on a given invariant variety \( \mathcal{M}_c \).
To simplify Eqs. (3.1) and (4.2), we introduce the parameters
\[ \varepsilon = \frac{R}{2R + 2r} \quad \text{and} \quad \delta = \frac{R + 2r}{2r}. \] (4.3)

We define the **modified operator of inertia** \( I \) as
\[ I = I + \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2) pr_i, \] (4.4)
where \( pr_i : \mathbb{R}^3 \to \tilde{\Gamma}_i^\perp \) is the orthogonal projection to the plane orthogonal to \( \tilde{\Gamma}_i \). We set
\[ \bar{M} = I \bar{\Omega} = I \bar{\Omega} + \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2) \bar{\Omega} - \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2) (\bar{\Gamma}_i, \bar{\Omega}) \tilde{\Gamma}_i, \] (4.5)
\[ \bar{N} = \delta \sum_{i=1}^{n} I_i c_i \tilde{\Gamma}_i. \] (4.6)

**Theorem 1.** The reduction of the spherical ball bearing problem to \( \mathcal{M}_c \cong \mathcal{N} \) is described by the equations
\[ \frac{d}{dt} \bar{M} = \bar{M} \times \bar{\Omega} + (1 - \varepsilon) \bar{N} \times \bar{\Omega}, \] (4.7)
\[ \frac{d}{dt} \bar{\Gamma}_i = \varepsilon \bar{\Gamma}_i \times \bar{\Omega}, \quad i = 1, \ldots, n. \] (4.8)

Note that the kinetic energy of the system takes the form
\[ T = \frac{1}{2} \langle \bar{M}, \bar{\Omega} \rangle + \frac{1}{2} \sum_{i=1}^{n} I_i c_i^2. \]

Also, since
\[ \frac{d}{dt} \bar{N} = \varepsilon \bar{N} \times \bar{\Omega}, \]
Eq. (4.7) is equivalent to
\[ \frac{d}{dt} (\bar{M} + \bar{N}) = (\bar{M} + \bar{N}) \times \bar{\Omega}. \] (4.9)

**Proof (of Theorem 1).** From Eqs. (3.5) and (3.6) one has
\[ \bar{\Gamma}_i \times \bar{\mathcal{F}}_{Bi} = \frac{1}{2r} (I_i \bar{\Omega}_i + \bar{\Omega} \times (I_i \bar{\Omega}_i)) + \frac{m_i}{2} \bar{\Gamma}_i \times \bar{V}_{Oi} + \frac{m_i}{2} (\bar{\Gamma}_i \times (\bar{\Omega} \times \bar{V}_{Oi})). \]

By plugging the last expression into the third equation of motion (3.7), it becomes
\[ I \dot{\bar{\Omega}} + \bar{\Omega} \times I \ddot{\bar{\Omega}} = - \sum_{i=1}^{n} \left[ \frac{R + 2r}{2r} (I_i \dot{\bar{\Omega}}_i + \bar{\Omega} \times (I_i \dot{\bar{\Omega}}_i)) \right. \]
\[ \left. + \frac{m_i (R + 2r)}{2} \bar{\Gamma}_i \times \dot{\bar{V}}_{Oi} + \frac{m_i (R + 2r)}{2} (\bar{\Gamma}_i \times (\bar{\Omega} \times \bar{V}_{Oi})) \right]. \] (4.10)

From (2.8), (3.1), and (3.3), we get \( \dot{\bar{\Gamma}}_i \times \bar{V}_{Oi} = 0 \) and, therefore,
\[ \frac{d}{dt} (\bar{\Gamma}_i \times \bar{V}_{Oi}) = \bar{\Gamma}_i \times \dot{\bar{V}}_{Oi}. \]
Also, we have
\[ \bar{\Gamma}_i \times (\bar{\Omega} \times \bar{V}_{Oi}) = \bar{\Omega} \times (\bar{\Gamma}_i \times \bar{V}_{Oi}). \]
With the last two expressions in mind, Eq. (4.10) becomes
\[ \frac{d}{dt} \left( I \dot{\Omega} + \sum_{i=1}^{n} \left( \frac{R + 2r}{2r} I_i \dot{\Omega}_i + \frac{m_i (R + 2r)}{2} \Gamma_i \times \dot{V}_i \right) \right) = \]
\[ - \ddot{\Omega} \times \left( I \ddot{\Omega} + \sum_{i=1}^{n} \left( \frac{R + 2r}{2r} I_i \ddot{\Omega}_i + \frac{m_i (R + 2r)}{2} \Gamma_i \times \ddot{V}_i \right) \right). \]  

(4.11)

Finally, using (4.2), constraints (3.4), the definitions (4.3), (4.5), and (4.6) of parameters \( \varepsilon \) and \( \delta \) and the vectors \( \vec{M} \) and \( \vec{N} \), Eq. (4.11) takes the form (4.9).

**Remark 2.** If we formally set \( \varepsilon = 1 \) in the system (4.7)–(4.8), we obtain the equation of the spherical support system introduced by Fedorov in [11]. The system describes the rolling without slipping of a dynamically nonsymmetric sphere \( S \) over \( n \) homogeneous balls \( B_1, \ldots, B_n \) of possibly different radii, but with fixed centers. It is an example of a class of non-Hamiltonian L+R systems on Lie groups with an invariant measure (see [13, 14, 17]). On the other hand, if we set \( \vec{N} = 0 \), we obtain an example of the \( \varepsilon \)-modified L+R system studied in [18].

Since \( \langle \vec{\omega}_i, \vec{\gamma}_i \rangle = \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle \), we have that \( D \) is foliated by invariant varieties
\[ D_c: \langle \vec{\omega}_i, \vec{\gamma}_i \rangle = c_i, \quad i = 1, \ldots, n, \quad \dim D_c = 5n + 6 \]
and \( M_c = D_c/ SO(3)^{n+1} \). As a result, we obtain the following diagram:

\[ \begin{array}{ccc}
D_c & \xleftarrow{\pi} & D \\
/ SO(3)^{n+1} & \downarrow & / SO(3)^{n+1} \\
M_c & \xleftarrow{\pi} & M \\
/ (TQ)/ SO(3)^{n+1} & \downarrow & / (TQ)/ SO(3)^{n+1} \\
N = \mathbb{R}^3 \times (S^2)^n & \cong & \mathbb{R}^{3(n+1)} \times (T^* S^n)
\end{array} \]

which implies that \( D_c \) and \( \mathbb{R}^3 \times SO(3)^{n+1} \times (S^2)^n \{ \Omega, \vec{g}, \vec{g}_1, \ldots, \vec{g}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \} \) are diffeomorphic:
\[ D_c \cong \mathbb{R}^3 \times SO(3)^{n+1} \times (S^2)^n \{ \Omega, \vec{g}, \vec{g}_1, \ldots, \vec{g}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \}. \]

**Corollary 1.** The complete equations of motion of the sphere \( S \) and the balls \( B_1, \ldots, B_n \) of the spherical bearing problem on the invariant manifold \( D_c \) are given by
\[ \dot{\vec{M}} = \vec{M} \times \vec{\Omega} + (1 - \varepsilon) \vec{N} \times \vec{\Omega}, \]
\[ \dot{\vec{g}} = \vec{g} \vec{\Omega}, \]
\[ \dot{\vec{g}}_i = \vec{g} \vec{\Omega}_i (\vec{\Omega}, \vec{\Gamma}_i, c_i) \vec{g}_i, \]
\[ \dot{\vec{\Gamma}}_i = \varepsilon \vec{\Gamma}_i \times \vec{\Omega}, \quad i = 1, \ldots, n, \]
where \( \vec{M} \) and \( \vec{N} \) are given by (4.5) and (4.6). Here \( \vec{\Omega} \) and \( \vec{\Omega}_i (\vec{\Omega}, \vec{\Gamma}_i, c_i) \) are skew-symmetric matrices related to \( \vec{\Omega} \) and \( \vec{\Omega}_i \) after the identification (2.1); \( \vec{\Omega}_i (\vec{\Omega}, \vec{\Gamma}_i, c_i) \) as in the Eq. (4.2).

5. THE ASSOCIATED SYSTEM ON \( \mathbb{R}^3 \times SYM(3) \) AND AN INVARIANT MEASURE

Let
\[ \Gamma = - \delta^2 \sum_{i=1}^{n} (I_i + m_i r_i^2) pr_i \]
be the symmetric operator using which the definition of the modified inertia operator $I$, Eq. (4.4) can be rewritten as

$$I = I - \Gamma, \quad \Gamma = \delta^2 \sum_{i=1}^{n} (I_i + m_i r_i^2)(\tilde{\Gamma}_i \otimes \tilde{\Gamma}_i - E), \quad E = \text{diag}(1, 1, 1).$$

Along the flow of the system, $\Gamma$ satisfies the equation

$$\frac{d}{dt} \Gamma = \varepsilon [\Gamma, \Omega], \quad (5.1)$$

where $\Omega$ is the skew-symmetric matrix that corresponds to the angular velocity $\tilde{\Omega}$ via isomorphism (2.1).

Let us consider a special case when $c_1 = 0, \ldots, c_n = 0$, i.e., the invariant manifold $M_0$. This means that there is no twisting of the balls, i.e., the vectors $\tilde{\Omega}_i$ and $\tilde{\Gamma}_i$ are orthogonal to each other. However, note that these conditions are not nonholonomic constraints, but first integrals of motion.

As a result, we obtain the associated system

$$\dot{\tilde{M}} = \tilde{M} \times \tilde{\Omega}, \quad \dot{\tilde{\Omega}} = \Gamma \tilde{\Omega}, \quad \dot{\tilde{\Gamma}} = \varepsilon [\tilde{\Gamma}, \tilde{\Omega}] \quad (5.2)$$

on the space $\mathbb{R}^3 \times \text{Sym}(3)$, where $\text{Sym}(3)$ are $3 \times 3$ symmetric matrices. The system belongs to the class of $\varepsilon$-modified L+R systems studied in [18].

Let $d\Omega$ and $d\Gamma$ be the standard measures on $\mathbb{R}^3\{\tilde{\Omega}\}$ and $\text{Sym}(3)\{\tilde{\Gamma}\}$. The system (5.2) possesses the invariant measure $\mu(\tilde{\Omega})d\Omega \wedge d\Gamma$ with the density $\mu(\tilde{\Gamma}) = \sqrt{\text{det}(I)}$ (see Theorem 4, [18]). Therefore, $\mu = \sqrt{\text{det}(I)}$ is a natural candidate for the density of an invariant measure of the system (4.7)–(4.8) when the constants $c_i$ are different from zero. Indeed, we have

**Theorem 2.** For arbitrary values of parameters $c_i$, the reduced system (4.7)–(4.8) has the invariant measure

$$\mu(\Gamma_1, \ldots, \Gamma_n)d\Omega \wedge \sigma_1 \wedge \cdots \wedge \sigma_n, \quad \mu = \sqrt{\text{det}(I)} = \sqrt{\text{det}(I - \Gamma)}, \quad (5.3)$$

where $d\Omega$ and $\sigma_i$ are the standard measures on $\mathbb{R}^3\{\tilde{\Omega}\}$ and $\mathbb{S}^2\{\tilde{\Gamma}_i\}, i = 1, \ldots, n.$

The proof of the theorem we are going to present is a variant of a corresponding proof for $\varepsilon$-modified L+R systems. It is given below for completeness of exposition. In what follows we use

**Lemma 3.** Let $A$ be a symmetric matrix and let $\tilde{\Omega} \in \mathbb{R}^3$ and $\Omega \in so(3)$ be related by (2.1). Then:

(i) the symmetric part of the matrix $\partial (A\tilde{\Omega} \times \tilde{\Omega})/\partial \tilde{\Omega}$ is equal to $\frac{1}{2}[A, \Omega]$;

(ii) $A\tilde{\Omega} \times \tilde{\Omega} = [A, \Omega]\tilde{\Omega}$.

**Proof (of Theorem 2).** We can consider the system

$$\frac{d}{dt}(I\tilde{\Omega}) = I\tilde{\Omega} \times \tilde{\Omega} + (1 - \varepsilon)\tilde{N} \times \tilde{\Omega}, \quad I = I - \Gamma, \quad (5.4)$$

$$\frac{d}{dt}\tilde{\Gamma}_i = \varepsilon \tilde{\Gamma}_i \times \tilde{\Omega}, \quad i = 1, \ldots, n, \quad (5.5)$$

as extended in the Euclidean space $\mathbb{R}^{3n+3}\{\tilde{\Omega}, \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_n\}$ as well. The extended system also has first integrals $(\tilde{\Gamma}_i, \tilde{\Gamma}_j) = \gamma_{ij}$. In particular, by taking $\gamma_{ii} = 1, i = 1, \ldots, n$, we find that the reduced system is the restriction of the extended system (5.4)–(5.5) from $\mathbb{R}^{3n+3}$ to the invariant variety $\mathcal{N}$. Therefore, it is sufficient to prove that the extended system preserves the measure

$$\nu = \mu(\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_n)d\Omega \wedge d\Gamma_1 \wedge \cdots \wedge d\Gamma_n, \quad \mu = \sqrt{\text{det}(I)},$$

where $d\Gamma_i$ is the standard measure in $\mathbb{R}^3\{\tilde{\Gamma}_i\}$. 
This is a standard construction: if a system has an invariant measure, then the restriction of the system to an invariant manifold also has an invariant measure induced by the measure from the ambient space. Let

\[ X = (\mathring{\Omega}, \mathring{\Gamma}_1, \ldots, \mathring{\Gamma}_n) \]

be the vector field on \( \mathbb{R}^{3n+3} \) defined by Eqs. (5.4) and (5.5) and assume that the Lie derivative \( \mathcal{L}_X \) of \( \nu \) vanishes. It is well known that for \( \mathring{\Gamma}_i \neq 0 \) the volume form in \( \mathbb{R}^3 \{ \mathring{\Gamma}_i \} \) can be written as 

\[ d \mathring{\Gamma}_i = \alpha_i \wedge \sigma_i, \]

where \( \sigma_i \) is the standard measure on the unit sphere and 

\[ \alpha_i = |\mathring{\Gamma}_i|^2 d(|\mathring{\Gamma}_i|) = \frac{1}{3} d(\mathring{\Gamma}_i, \mathring{\Gamma}_i)^2, \quad i = 1, \ldots, n. \]

Since \( \mathcal{L}_X (\alpha_i) = 0 \), we have 

\[ \mathcal{L}_X (\nu) = \text{const} \cdot \alpha_1 \wedge \cdots \wedge \alpha_n \wedge \mathcal{L}_X (\mu d \Omega \wedge \sigma_1 \wedge \cdots \wedge \sigma_n) = 0, \]

\( \mathring{\Gamma}_i \neq 0, \, i = 1, \ldots, n \). Therefore, the reduced system preserves the measure (5.3).

The equation \( \mathcal{L}_X (\nu) = 0 \) in \( \mathbb{R}^{3n+3} \) can be written in the equivalent form

\[ \dot{\mu} + \mu \text{div}(X) = \dot{\mu} + \mu \text{tr} \frac{\partial \mathring{\Omega}}{\partial \Omega} + \mu \sum_{i=1}^{n} \text{tr} \frac{\partial \mathring{\Gamma}_i}{\partial \mathring{\Omega}_i} = 0. \tag{5.6} \]

We have the following equalities:

\[
\begin{align*}
\mathbf{I} \dot{\Omega} + \mathbf{I} \dot{\Omega} &= \mathbf{I} \dot{\Omega} \times \dot{\Omega} - \mathbf{I} \mathring{\Omega} \times \mathring{\Omega} + (1 - \varepsilon) \mathbf{N} \times \mathring{\Omega}, \\
\dot{\Omega} &= \mathbf{I} \dot{\Omega} \times \dot{\Omega} - \mathbf{I} \mathring{\Omega} \times \mathring{\Omega} + \varepsilon [\mathbf{I}, \Omega] \mathring{\Omega} + (1 - \varepsilon) \mathbf{N} \times \mathring{\Omega} \\
&= \mathbf{I} \dot{\Omega} \times \dot{\Omega} - (1 - \varepsilon) \mathbf{I} \mathring{\Omega} \times \mathring{\Omega} + (1 - \varepsilon) \mathbf{N} \times \mathring{\Omega}, \quad \text{item (ii) of Lemma 3).}
\end{align*}
\]

Thus,

\[
\dot{\Omega} = \mathbf{I}^{-1}(\mathbf{I} \dot{\Omega} \times \dot{\Omega} - (1 - \varepsilon) \mathbf{I} \mathring{\Omega} \times \mathring{\Omega} + (1 - \varepsilon) \mathbf{N} \times \mathring{\Omega}),
\]

\[
\text{tr} \frac{\partial \mathring{\Omega}}{\partial \Omega} = \text{tr} \left( \mathbf{I}^{-1} \frac{\partial}{\partial \Omega} (\mathbf{I} \dot{\Omega} \times \dot{\Omega} - (1 - \varepsilon) \mathbf{I} \mathring{\Omega} \times \mathring{\Omega} + (1 - \varepsilon) \mathbf{N} \times \mathring{\Omega}) \right). \tag{5.7}
\]

The matrix \( \partial(\mathbf{N} \times \mathring{\Omega})/\partial \mathring{\Omega} \) is skew-symmetric. Since \( \mathbf{I}^{-1} \) is symmetric, only the symmetric part of the expression in parentheses in the last equation matters. Whence, using item (i) of Lemma 3, we get

\[
\text{tr} \frac{\partial \mathring{\Omega}}{\partial \Omega} = \text{tr} \left( \mathbf{I}^{-1} \left( \frac{1}{2} [\mathbf{I}, \Omega] - \frac{1 - \varepsilon}{2} [\mathbf{I}, \mathring{\Omega}] \right) \right) \tag{5.7}
\]

\[
= \text{tr} \left( \mathbf{I}^{-1} \frac{1}{2} [\mathbf{I}, \Omega] + \frac{\varepsilon}{2} \mathbf{I}^{-1} [\mathbf{I}, \mathring{\Omega}] \right) = \frac{\varepsilon}{2} \text{tr} (\mathbf{I}^{-1} [\mathbf{I}, \mathring{\Omega}]). \]

On the other hand,

\[
\dot{\mu} = \frac{1}{2 \sqrt{\det(\mathbf{I})}} \frac{d}{dt} (\det(\mathbf{I})) = \frac{1}{2 \sqrt{\det(\mathbf{I})}} \det(\mathbf{I}) \text{tr} (\mathbf{I}^{-1} \frac{d}{dt} (\mathbf{I} - \mathring{\Omega})) \tag{5.8}
\]

\[
= -\frac{1}{2} \mu \text{tr} (\mathbf{I}^{-1} \varepsilon [\mathbf{I}, \mathring{\Omega}]).
\]

Here we have used a well-known formula \( \frac{d}{dt} \det(\mathbf{I}) = \det(\mathbf{I}) \text{tr} (\mathbf{I}^{-1} \dot{\mathbf{I}}) \).

Since the matrices \( \partial \mathring{\Gamma}_i/\partial \mathring{\Omega}_i \) are skew-symmetric and have zero traces, Eqs. (5.7) with (5.8) imply the required condition (5.6). \( \square \)
Note that the existence of an invariant measure for nonholonomic problems is well studied in many classical problems [4, 6]. After Kozlov’s theorem on obstruction to the existence of an invariant measure for the version of the classical Suslov problem (see, e.g., [9, 14]) on Lie algebras [20], general existence statements for nonholonomic systems with symmetries are obtained in [21] and [15].

A closely related problem is the integrability of the nonholonomic systems [1]. Here we have the following statement.

**Proposition 3.** The system (4.7)–(4.8) always has the following first integrals:

\[ F_1 = \frac{1}{2} \langle \dot{\vec{M}}, \vec{\Omega} \rangle, \quad F_2 = \langle \dot{\vec{M}} + \vec{\dot{N}}, \vec{\dot{M}} + \vec{\dot{N}} \rangle, \quad F_{ij} = \langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle, \quad 1 \leq i < j \leq n. \]

Thus, in the special case \( n = 1 \), we have the 5-dimensional phase space \( \mathcal{N} = \mathbb{R}^3 \times S^2 \{ \vec{\Omega}, \vec{\Gamma}_1 \} \), and the system has two first integrals and an invariant measure. For the integrability, one needs to find a third independent first integral. We will study integrability in the spherical ball bearing systems in arbitrary dimension \( \mathbb{R}^m, m > 3 \) (e.g., see [12, 14, 16, 17, 19]), or the systems where the homogeneous balls \( \vec{B}_i \) are replaced by the systems of the form (ball + gyroscope), which satisfy the Zhukovskii conditions (see [10, 22]).

### 6. PLANAR SYSTEM — THE THREE BALLS BEARINGS PROBLEM

#### 6.1. Definition of the Planar Three Balls Bearing Problem

Consider the limit, when the radii of the spheres \( \vec{S}_0 \) and \( \vec{S} \) both tend to infinity. For simplicity, we consider the case \( n = 3 \). As a result, we obtain rolling without slipping of three homogeneous balls \( \vec{B}_1, \vec{B}_2, \vec{B}_3 \) of radius \( r \) and masses \( m_1, m_2, m_3 \) over the fixed plane \( \vec{S}_0 \), together with the moving plane \( \vec{\Sigma} \) of mass \( \vec{M} \) that is placed over the balls, such that there is no slipping between the balls and the moving plane. We will refer to the system as the three balls bearing problem. Note that all considerations of the section can be easily adopted for the case of the planar ball bearing with rolling of \( n \) homogeneous balls.

Let \( \vec{O}_0 \) be the fixed point of the plane \( \vec{S}_0 \), \( \vec{O}, \vec{O}_1, \vec{O}_2, \vec{O}_3 \) be the centers of mass of the plane \( \vec{\Sigma} \) and the balls \( \vec{B}_1, \vec{B}_2, \vec{B}_3 \), respectively. Let also

\[ \vec{O}_0 \vec{e}_1^0, \vec{e}_2^0, \vec{e}_3^0, \quad \vec{O}\vec{e}_1^i, \vec{e}_2^i, \vec{e}_3^i, \]

be positively oriented reference frames rigidly attached to the fixed plane \( \vec{S}_0 \), the moving plane \( \vec{\Sigma} \), and the ball \( \vec{B}_i \) \((i = 1, 2, 3)\), respectively. Here \( \vec{e}_3^i = \vec{e}_3^0 \) is the unit vector orthogonal to \( \vec{\Sigma} \) and \( \vec{S}_0 \).

In the fixed reference frame, the positions of the points \( \vec{O}, \vec{O}_1, \vec{O}_2, \) and \( \vec{O}_3 \) are, respectively, given by

\[ \vec{O}(x, y, 2r), \quad \vec{O}_1(x_1, y_1, r), \quad \vec{O}_2(x_2, y_2, r), \quad \vec{O}_3(x_3, y_3, r). \]

We denote by \( \vec{g} \in SO(2) \subset SO(3) \) the rotation matrix that maps \( \vec{O}\vec{e}_1^i, \vec{e}_2^i, \vec{e}_3^i \) to \( \vec{O}_0\vec{e}_1^0, \vec{e}_2^0, \vec{e}_3^0 \), and by \( \vec{g}_i \in SO(3) \) the matrix that maps the moving frame \( \vec{O}_i\vec{e}_1^i, \vec{e}_2^i, \vec{e}_3^i \) to the fixed frame \( \vec{O}_0\vec{e}_1^0, \vec{e}_2^0, \vec{e}_3^0 \), \( i = 1, 2, 3 \). As above, the skew-symmetric matrices

\[ \vec{\omega} = \vec{g} \vec{g}^{-1}, \quad \vec{\omega}_i = \vec{g}_i \vec{g}_i^{-1}, \]

after the identification (2.1), correspond to the angular velocities \( \vec{\omega} \) and \( \vec{\omega}_i \) of the plane \( \vec{\Sigma} \) and the ball \( \vec{B}_i \) relative to the fixed coordinate system. Note that

\[ \vec{g} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vec{\omega} = \begin{pmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \vec{\omega} = (0, 0, \dot{\varphi}). \]

Then the configuration space of the planar three balls bearing problem is

\[ \mathcal{Q} = SO(3) \times SO(3) \times SO(3) \times \mathbb{R}^2 \times SO(2) \times (\mathbb{R}^2)^3 \{ \vec{g}_1, \vec{g}_2, \vec{g}_3, x, y, \varphi, x_1, x_2, y_1, y_2, x_3, y_3 \}, \]

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while the kinetic energy is

\[ T = \frac{1}{2} I \dot{\varphi}^2 + \frac{1}{2} m (\vec{v}_O, \vec{v}_O) + \frac{1}{2} \sum_{i=1}^{3} I_i (\vec{\omega}_i, \vec{\omega}_i) + \frac{1}{2} \sum_{i=1}^{3} m_i (\vec{v}_{Oi}, \vec{v}_{Oi}), \]

where \( \text{diag}(I_1, I_2, I_3) \) is the inertia operator of the ball \( B_i \), \( i = 1, 2, 3 \), \( I \) is the moment of inertia of the plane \( \Sigma \) for the \( O\vec{e}_3 \)-axis through the mass center \( O \).

Let \( A_1 \) and \( A \) be the points of contact of \( B_1 \) with fixed plane \( \Sigma_0 \) and with plane \( \Sigma \), \( B_1 \) and \( B \) be the contact points of \( B_2 \), and \( C_1 \) and \( C \) be the contact points of \( B_3 \) with those planes. We have the following nonholonomic constraints written in the fixed reference frame \( O_0\vec{e}_0^1, \vec{e}_0^2, \vec{e}_0^3 \):

\[
\begin{align*}
\vec{v}_{O_1} - r \vec{\omega}_1 \times \vec{\gamma} &= 0, \\
\vec{v}_{O_2} - r \vec{\omega}_2 \times \vec{\gamma} &= 0, \\
\vec{v}_{O_3} + r \vec{\omega}_3 \times \vec{\gamma} &= \vec{v}_O + \vec{\omega} \times \vec{O}A, \\
\vec{v}_{O_2} + r \vec{\omega}_2 \times \vec{\gamma} &= \vec{v}_O + \vec{\omega} \times \vec{OB}, \\
\vec{v}_{O_3} + r \vec{\omega}_3 \times \vec{\gamma} &= \vec{v}_O + \vec{\omega} \times \vec{OC},
\end{align*}
\]

(6.1)

where \( \vec{\gamma} = (0, 0, 1) \) is the unit vector orthogonal to the planes \( \Sigma_0 \) and \( \Sigma \), i.e.,

\[ \vec{\gamma} = \vec{e}_3^0 = \vec{e}_3. \]

The first three vector constraints are obtained from the condition that the velocities of the contact points \( A_1, B_1, C_1 \) with the fixed plane \( \Sigma_0 \) are zero. The remaining ones follow from the condition that there is no sliding between the balls and the plane \( \Sigma \). This means that the velocities of points \( A, B \) and \( C \) are the same as the velocities of the corresponding points at the plane \( \Sigma \).

The configuration space \( Q \) is 18-dimensional and there are twelve independent nonholonomic constraints among (6.1). Hence, the vector subspaces of the admissible velocities \( D_q \subset T_q Q, q \in Q \), are six–dimensional.

Note that the dimensions of the configuration space \( Q \) and the constraints manifold \( \mathcal{D} \) in the problem of spherical ball bearing for \( n = 3 \) and the planar three balls bearing problem coincide. Now, the additional one-side constraints read

\[
\begin{align*}
|\vec{O}_1\vec{O}_2| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \geq 2r, \\
|\vec{O}_2\vec{O}_3| &= \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \geq 2r, \\
|\vec{O}_3\vec{O}_1| &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} \geq 2r.
\end{align*}
\]

(6.2)
6.2. The Equations of Motion

As in the case of the spherical ball bearing problem, we will derive the equations of motion in the planar case in the vector form. The corresponding reaction forces will be expressed in terms of Lagrange multipliers $\lambda_1, \ldots, \lambda_{12}$ that correspond to twelve independent nonholonomic constraints.

The equations of motion of the planar three balls bearing system relative to the fixed coordinate system are

$$
\begin{align*}
 m_1 \ddot{v}_{O_1} &= (\lambda_1, \lambda_2, 0) + (\lambda_7, \lambda_8, 0), \\
 m_2 \ddot{v}_{O_2} &= (\lambda_3, \lambda_4, 0) + (\lambda_9, \lambda_{10}, 0), \\
 m_3 \ddot{v}_{O_3} &= (\lambda_5, \lambda_6, 0) + (\lambda_{11}, \lambda_{12}, 0), \\
 m \ddot{v}_O &= -(\lambda_7, \lambda_8, 0) - (\lambda_9, \lambda_{10}, 0) - (\lambda_{11}, \lambda_{12}, 0),
\end{align*}
$$

(6.3)

and

$$
\begin{align*}
 I_1 \ddot{\omega}_1 &= -r \vec{\gamma} \times ((\lambda_1, \lambda_2, 0) - (\lambda_7, \lambda_8, 0)), \\
 I_2 \ddot{\omega}_2 &= -r \vec{\gamma} \times ((\lambda_3, \lambda_4, 0) - (\lambda_9, \lambda_{10}, 0)), \\
 I_3 \ddot{\omega}_3 &= -r \vec{\gamma} \times ((\lambda_5, \lambda_6, 0) - (\lambda_{11}, \lambda_{12}, 0)), \\
 I \ddot{\omega} &= -\overrightarrow{OA} \times (\lambda_7, \lambda_8, 0) - \overrightarrow{OB} \times (\lambda_9, \lambda_{10}, 0) - \overrightarrow{OC} \times (\lambda_{11}, \lambda_{12}, 0).
\end{align*}
$$

(6.4)

By differentiating the first three constraints from (6.1) and using the first three equations of motion in (6.3) and (6.4), we get

$$
\begin{align*}
(\lambda_1, \lambda_2, 0) &= \frac{m_1 r^2}{m_1 r^2 + I_1} (\lambda_7, \lambda_8, 0), \\
(\lambda_3, \lambda_4, 0) &= \frac{m_2 r^2}{m_2 r^2 + I_2} (\lambda_9, \lambda_{10}, 0), \\
(\lambda_5, \lambda_6, 0) &= \frac{m_3 r^2}{m_3 r^2 + I_3} (\lambda_{11}, \lambda_{12}, 0).
\end{align*}
$$

(6.5)

Therefore, Eqs. (6.3) and (6.4) can be written as

$$
\begin{align*}
 \ddot{v}_{O_1} &= \frac{2r^2}{m_1 r^2 + I_1} \overrightarrow{F}_1, \\
 \ddot{v}_{O_2} &= \frac{2r^2}{m_2 r^2 + I_2} \overrightarrow{F}_2, \\
 \ddot{v}_{O_3} &= \frac{2r^2}{m_3 r^2 + I_3} \overrightarrow{F}_3, \\
 m \ddot{v}_O &= -\overrightarrow{F}_1 - \overrightarrow{F}_2 - \overrightarrow{F}_3,
\end{align*}
$$

(6.6)

and

$$
\begin{align*}
 \ddot{\omega}_1 &= \frac{2r}{m_1 r^2 + I_1} \vec{\gamma} \times \overrightarrow{F}_1, \\
 \ddot{\omega}_2 &= \frac{2r}{m_2 r^2 + I_2} \vec{\gamma} \times \overrightarrow{F}_2, \\
 \ddot{\omega}_3 &= \frac{2r}{m_3 r^2 + I_3} \vec{\gamma} \times \overrightarrow{F}_3, \\
 I \ddot{\omega} &= -\overrightarrow{OA} \times \overrightarrow{F}_1 - \overrightarrow{OB} \times \overrightarrow{F}_2 - \overrightarrow{OC} \times \overrightarrow{F}_3,
\end{align*}
$$

(6.7)

where

$$
\begin{align*}
 \overrightarrow{F}_1 &= (\lambda_7, \lambda_8, 0), \\
 \overrightarrow{F}_2 &= (\lambda_9, \lambda_{10}, 0), \\
 \overrightarrow{F}_3 &= (\lambda_{11}, \lambda_{12}, 0).
\end{align*}
$$
By differentiating the remaining constraints, we get the following linear system of six equations in the Lagrange multipliers $\lambda_7, \ldots, \lambda_{12}$:

\[
\begin{align*}
\frac{4Ir^2}{m_1r^2 + I_1} \mathbf{F}_1 &= (\mathbf{O}\mathbf{A} \wedge \mathbf{F}_1 + \mathbf{O}\mathbf{B} \wedge \mathbf{F}_2 + \mathbf{O}\mathbf{C} \wedge \mathbf{F}_3)\mathbf{O}\mathbf{A} \\
&+ \mathbf{\bar{\omega}} \times (\mathbf{\bar{v}}_{O1} - \mathbf{\bar{v}}_O) - \frac{I}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3), \\
\frac{4Ir^2}{m_2r^2 + I_2} \mathbf{F}_2 &= (\mathbf{O}\mathbf{A} \wedge \mathbf{F}_1 + \mathbf{O}\mathbf{B} \wedge \mathbf{F}_2 + \mathbf{O}\mathbf{C} \wedge \mathbf{F}_3)\mathbf{O}\mathbf{B} \\
&+ \mathbf{\bar{\omega}} \times (\mathbf{\bar{v}}_{O2} - \mathbf{\bar{v}}_O) - \frac{I}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3), \\
\frac{4Ir^2}{m_3r^2 + I_3} \mathbf{F}_3 &= (\mathbf{O}\mathbf{A} \wedge \mathbf{F}_1 + \mathbf{O}\mathbf{B} \wedge \mathbf{F}_2 + \mathbf{O}\mathbf{C} \wedge \mathbf{F}_3)\mathbf{O}\mathbf{C} \\
&+ \mathbf{\bar{\omega}} \times (\mathbf{\bar{v}}_{O3} - \mathbf{\bar{v}}_O) - \frac{I}{m} (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3).
\end{align*}
\]

(6.8)

One can easily see that the system (6.8) determines the Lagrange multipliers $\lambda_7, \ldots, \lambda_{12}$ uniquely, and, at the same time, uniquely determines $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$.

We have the following analogue of Propositions 1 and 2.

**Proposition 4.** The moving triangles $\triangle O_1O_2O_3(t)$ and $\triangle ABC(t)$ are congruent to the triangle formed by the centers of the balls under the initial condition.

**Proof.** From the constraints (6.1) we get

\[
\begin{align*}
2\frac{d}{dt} \mathbf{AB} &= \mathbf{\bar{\omega}} \times \mathbf{AB}, \\
2\frac{d}{dt} \mathbf{BC} &= \mathbf{\bar{\omega}} \times \mathbf{BC}, \\
2\frac{d}{dt} \mathbf{CA} &= \mathbf{\bar{\omega}} \times \mathbf{CA}.
\end{align*}
\]

Therefore,

\[
\frac{d}{dt} (\mathbf{AB}, \mathbf{AB}) = (\mathbf{AB}, \mathbf{\bar{\omega}} \times \mathbf{AB}) = 0.
\]

Similarly, we have $(\mathbf{BC}, \mathbf{BC}) = \text{const}$, $(\mathbf{CA}, \mathbf{CA}) = \text{const}$. \qed

Thus, as in the spherical case, if the initial condition is within the interior of the region (6.2), the system remains within the interior of the region (6.2) along the motion.

Also, from (6.7) and $\gamma = 0$ we get

**Proposition 5.** The projections of the angular velocities $\mathbf{\bar{\omega}}_i$ to $\mathbf{\gamma}$ are conserved along the motion:

\[
\omega_{i3} = \langle \mathbf{\bar{\omega}}_i, \mathbf{\gamma} \rangle = c_i, \quad i = 1, 2, 3.
\]

**6.3. Reduction**

Set $v_\varphi = \dot{\varphi}$, $v_x = \dot{x}$, $v_y = \dot{y}$. By using the constraints (6.1) we can obtain a closed system of the equations of motion on the space

\[
P = T\mathbb{R}^2 \times TSO(2) \times (\mathbb{R}^3)^3 \{ v_x, v_y, v_\varphi, x, y, \varphi, x_1, y_1, x_2, y_2, x_3, y_3 \}, \quad \dim P = 12.
\]

Note that we have a diffeomorphism

\[
P / SO(3) \times SO(3) \cong P \times \mathbb{R}^3 \{ \omega_{13}, \omega_{23}, \omega_{33} \}.
\]
where the $SO(3) \times SO(3) \times SO(3)$-action on $\mathcal{D} \subset TQ$, as in the spherical case, is given by the right trivialization of the tangent bundle of the Lie group $SO(3) \times SO(3) \times SO(3)$.

It is interesting that, contrary to the spherical case, the equations on $\mathcal{P}$ do not depend on the integrals $c_i$. By using the constraints (6.1), the kinetic energy for $c_1 = c_2 = c_3 = 0$ on $\mathcal{P}$ takes the form

$$T = \frac{1}{2} \dot{v}_\varphi^2 + \frac{1}{2} m (v_x^2 + v_y^2) + \frac{1}{2} \left( \frac{I_1 + r^2 m_1}{4r^2} \right) \langle \vec{v}_O + \vec{\omega} \times \vec{O} \vec{A}, \vec{v}_O + \vec{\omega} \times \vec{O} \vec{A} \rangle + \frac{1}{2} \left( \frac{I_2 + r^2 m_2}{4r^2} \right) \langle \vec{v}_O + \vec{\omega} \times \vec{O} \vec{B}, \vec{v}_O + \vec{\omega} \times \vec{O} \vec{B} \rangle + \frac{1}{2} \left( \frac{I_3 + r^2 m_3}{4r^2} \right) \langle \vec{v}_O + \vec{\omega} \times \vec{O} \vec{C}, \vec{v}_O + \vec{\omega} \times \vec{O} \vec{C} \rangle.$$

Let us denote

$$\vec{N} = \delta_1 \vec{O} \vec{A} + \delta_2 \vec{O} \vec{B} + \delta_3 \vec{O} \vec{C},$$

$$M = \delta_1 (\vec{O} \vec{A}, \vec{O} \vec{A}) + \delta_2 (\vec{O} \vec{B}, \vec{O} \vec{B}) + \delta_3 (\vec{O} \vec{C}, \vec{O} \vec{C}),$$

$$\delta_1 = \frac{m_1 r^2 + I_1}{4r^2}, \quad \delta_2 = \frac{m_2 r^2 + I_2}{4r^2}, \quad \delta_3 = \frac{m_3 r^2 + I_3}{4r^2}, \quad \delta = \delta_1 + \delta_2 + \delta_3.$$

Note that $\vec{N}(t)$ determines the trajectory of the mass center $S(t)$ of the moving triangle $\triangle ABC(t)$ with masses $\delta_1, \delta_2, \delta_3$ placed at the vertices $A, B, C$: $\vec{O} \vec{S} = \frac{1}{2} \vec{N}$. Also, by definition, $\vec{N}$ and $M$ satisfy the inequality $\delta M \geq \langle \vec{N}, \vec{N} \rangle = N_1^2 + N_2^2$.

The equality would imply that the points $A, B, C$ coincide. Therefore, in the region of admissible motions (6.2) we have

$$\delta M > N_1^2 + N_2^2.$$

With the above notation, the formula for the kinetic energy simplifies to

$$T = \frac{1}{2} (I + M) \dot{v}_x^2 + \frac{1}{2} (m + \delta) (v_x^2 + v_y^2) + v_\varphi (N_1 v_y - N_2 v_x).$$

As in the problem of spherical ball bearing, we will derive the equations of motion in the planar case without calculating the explicit formulae for $\vec{F}_1, \vec{F}_2, \vec{F}_3$.

**Theorem 3.** The equations of motion of the planar three balls bearing problem on $\mathcal{P}$ are given by

\[
\begin{align*}
\dot{\varphi} &= v_\varphi, \quad \dot{x} = v_x, \quad \dot{y} = v_y, \\
2(\dot{x}_1, \dot{y}_1) &= (v_x, v_y) + (-v_\varphi (y_1 - y), v_\varphi (x_1 - x)), \\
2(\dot{x}_2, \dot{y}_2) &= (v_x, v_y) + (-v_\varphi (y_2 - y), v_\varphi (x_2 - x)), \\
2(\dot{x}_3, \dot{y}_3) &= (v_x, v_y) + (-v_\varphi (y_3 - y), v_\varphi (x_3 - x)),
\end{align*}
\]

and

\[
\begin{align*}
(m + \delta) \dot{v}_x &= \frac{1}{2} N_1 v_\varphi^2 - \frac{\delta}{2} v_\varphi v_y + N_2 \dot{v}_x, \\
(m + \delta) \dot{v}_y &= \frac{1}{2} N_2 v_\varphi^2 + \frac{\delta}{2} v_\varphi v_x + N_1 \dot{v}_x, \\
(I + M) \dot{v}_\varphi &= \frac{1}{2} v_\varphi (N_1 v_x + N_2 v_y) + N_2 \dot{v}_x - N_1 \dot{v}_y,
\end{align*}
\]

where $\vec{N}, M, \delta$ are given by (6.9).

An explicit form of Eqs. (6.11) is given below in (6.16).
Proof.  The kinematic equations (6.10) follow directly from the constraints (6.1).

From the last equations in (6.6) and (6.7), we have
\[(m\ddot{v}_x, m\ddot{v}_y, I\ddot{v}_z) = -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 - \overrightarrow{OA} \times \vec{F}_1 - \overrightarrow{OB} \times \vec{F}_2 - \overrightarrow{OC} \times \vec{F}_3, \tag{6.12}\]
where \(\vec{F}_1, \vec{F}_2, \vec{F}_3\), from (6.8) and (6.10), are written in terms of variables on \(\mathcal{P}\).

Then, from (6.6) and (6.1), we get
\[
\begin{align*}
\vec{F}_1 &= 2\delta_1 \dot{v}_{O1} = \delta_1 \frac{d}{dt}(\vec{v}_O + \vec{\omega} \times \overrightarrow{OA}), \\
\vec{F}_2 &= 2\delta_2 \dot{v}_{O2} = \delta_2 \frac{d}{dt}(\vec{v}_O + \vec{\omega} \times \overrightarrow{OB}), \\
\vec{F}_3 &= 2\delta_3 \dot{v}_{O3} = \delta_3 \frac{d}{dt}(\vec{v}_O + \vec{\omega} \times \overrightarrow{OC}).
\end{align*}
\]

Thus, the last equation in (6.6) can be rewritten as
\[
\frac{d}{dt} \left( (m + \delta)\vec{v}_O + \vec{\omega} \times \overrightarrow{N} \right) = 0, \quad \delta = \delta_1 + \delta_2 + \delta_3. \tag{6.13}
\]

On the other hand, we have
\[
\overrightarrow{OA} \times \vec{F}_1 = \delta_1 \overrightarrow{OA} \times \frac{d}{dt}(\vec{v}_O + \vec{\omega} \times \overrightarrow{OA})
\]
\[
= \frac{d}{dt}(\delta_1 \overrightarrow{OA} \times (\vec{v}_O + \vec{\omega} \times \overrightarrow{OA})) - \delta_1 \left( \vec{v}_{O1} - \vec{v}_O \right) \times \left( \vec{v}_O + \vec{\omega} \times \overrightarrow{OA} \right)
\]
\[
= \frac{d}{dt}(\delta_1 \overrightarrow{OA} \times (\vec{v}_O + \vec{\omega} \times \overrightarrow{OA})) - \frac{\delta_1}{2} \left( \vec{\omega} \times \overrightarrow{OA} - \vec{v}_O \right) \times \left( \vec{v}_O + \vec{\omega} \times \overrightarrow{OA} \right)
\]
\[
= \frac{d}{dt}(\delta_1 \overrightarrow{OA} \times \vec{v}_O + \delta_1 \vec{\omega} \times \overrightarrow{OA}) + \vec{v}_O \times (\vec{\omega} \times \delta_1 \overrightarrow{OA})
\]

Similar equations hold for \(\overrightarrow{OB} \times \vec{F}_2\) and \(\overrightarrow{OC} \times \vec{F}_3\). Therefore, the last equation in (6.7) takes the form
\[
\frac{d}{dt} \left( I\vec{\omega} + \vec{N} \times \vec{v}_O + M\vec{\omega} \right) + \vec{\omega}(\vec{v}_O, \vec{N}) = 0. \tag{6.14}
\]

The time derivatives of \(M\) and \(\vec{N}\) along the motion are given by
\[
\dot{M} = 2\delta_1 \langle \overrightarrow{OA}, \vec{v}_{O1} - \vec{v}_O \rangle + 2\delta_2 \langle \overrightarrow{OB}, \vec{v}_{O2} - \vec{v}_O \rangle + 2\delta_3 \langle \overrightarrow{OC}, \vec{v}_{O3} - \vec{v}_O \rangle
\]
\[
= \delta_1 \langle \overrightarrow{OA}, \vec{\omega} \times \overrightarrow{OA} - \vec{v}_O \rangle + \delta_2 \langle \overrightarrow{OB}, \vec{\omega} \times \overrightarrow{OB} - \vec{v}_O \rangle + \delta_3 \langle \overrightarrow{OC}, \vec{\omega} \times \overrightarrow{OC} - \vec{v}_O \rangle
\]
\[
= -\langle \vec{v}_O, \vec{N} \rangle,
\]
\[
\dot{\vec{N}} = \frac{\delta_1}{2} (\vec{\omega} \times \overrightarrow{OA} - \vec{v}_O) + \frac{\delta_2}{2} (\vec{\omega} \times \overrightarrow{OB} - \vec{v}_O) + \frac{\delta_3}{2} (\vec{\omega} \times \overrightarrow{OC} - \vec{v}_O)
\]
\[
= \frac{1}{2} (\vec{\omega} \times \vec{N} - \delta \vec{v}_O).
\]

Finally, Eqs. (6.13) and (6.14) can be written as
\[
(m + \delta)\ddot{v}_O + (I + M)\dot{\vec{\omega}} = \frac{d}{dt}(\vec{N} \times (\vec{\omega} - \vec{v}_O))
\]
\[
= \frac{1}{2} (\vec{\omega} \times \vec{N}) + \frac{1}{2} (\vec{v}_O, \vec{N})\vec{\omega} + \frac{\delta}{2} \vec{\omega} \times \vec{v}_O + \vec{N} \times (\vec{\omega} - \vec{v}_O),
\]
which proves (6.11).  \(\square\)
6.4. Invariant Measure

It is clear that we can pass from $\mathcal{P}$ to the space

$$\mathcal{Q} = \{(v_x, v_y, v_\varphi, N_1, N_2, M) \in \mathbb{R}^6 \mid \delta M > N_1^2 + N_2^2\},$$

(6.15)

with the induced system described by Eqs. (6.11) and

$$\dot{\bar{N}} = \frac{1}{2}(\bar{\omega} \times \bar{N} - \delta \bar{v}_O), \quad \dot{M} = -\langle \bar{v}_O, \bar{N} \rangle.$$

If we introduce

$$\mathbf{v} = (v_x, v_y, v_\varphi), \quad \mathbf{n} = (N_1, N_2, M),$$

$$\mathbf{m} = \frac{1}{2}(N_1 v_\varphi^2 - \delta v_\varphi v_y, N_2 v_\varphi^2 + \delta v_\varphi v_x, v_\varphi(N_1 v_x + N_2 v_y)),$$

$$\mathbb{I} = \begin{pmatrix} m + \delta & 0 & -N_2 \\ 0 & m + \delta & N_1 \\ -N_2 & N_1 & I + M \end{pmatrix}, \quad \mathbb{J} = -\frac{1}{2} \begin{pmatrix} \delta & 0 & N_2 \\ 0 & \delta & -N_1 \\ 2N_1 & 2N_2 & 0 \end{pmatrix},$$

then the reduced equations of motion on $\mathcal{Q}$ (6.15) become

$$\dot{\mathbf{v}} = \mathbb{I}^{-1} \mathbf{m}, \quad \dot{\mathbf{n}} = \mathbb{J} \mathbf{v}.$$  

(6.16)

**Remark 3.** Since

$$\det(\mathbb{I}) = (m + \delta)((m + \delta)I + mM + (\delta M - (N_1^2 + N_2^2)) > 0|_\mathcal{Q},$$

the matrix $\mathbb{I}$ is invertible on $\mathcal{Q}$ and we have

$$\mathbb{I}^{-1} = \frac{1}{\det(\mathbb{I})} \begin{pmatrix} (m + \delta)(I + M) - N_1^2 & -N_1 N_2 & (m + \delta)N_2 \\ -N_1 N_2 & (m + \delta)(I + M) - N_2^2 & -(m + \delta)N_1 \\ (m + \delta)N_2 & -(m + \delta)N_1 & (m + \delta)^2 \end{pmatrix}.$$

**Theorem 4.** Equations (6.16) have the following first integrals:

$$f_1 = (m + \delta)v_x - v_\varphi N_2,$$

$$f_2 = (m + \delta)v_y + v_\varphi N_1,$$

$$f_3 = \delta M - (N_1^2 + N_2^2),$$

$$f_4 = T = \frac{1}{2}(I + M)v_\varphi^2 + \frac{1}{2}(m + \delta)(v_x^2 + v_y^2) + v_\varphi(N_1 v_y - N_2 v_x)$$

(6.17)

and they possess an invariant measure

$$\sqrt{\det(\mathbb{I})} dv_x \wedge dv_y \wedge dv_\varphi \wedge dN_1 \wedge dN_2 \wedge dM.$$

The system (6.16) can be solved by quadratures.

**Proof.** It is clear that the functions (6.17) are the first integrals of the system (6.16). Note that $f_3 > 0$ on $\mathcal{Q}$. Next, at the invariant level set

$$\mathcal{Q}_d: \quad f_1 = d_1, \quad f_2 = d_2, \quad f_3 = d_3,$$

we have

$$v_x = \frac{v_\varphi N_2 + d_1}{m + \delta}, \quad v_y = \frac{-v_\varphi N_1 + d_2}{m + \delta}, \quad M = \frac{1}{\delta}(N_1^2 + N_2^2) + \frac{d_3}{\delta},$$

$$\det(\mathbb{I}) = (m + \delta)((m + \delta)I + \frac{m}{\delta}(N_1^2 + N_2^2) + \frac{md_3}{\delta} + d_3).$$
We obtain a closed system in the space $\mathbb{R}^3\{v_\varphi, N_1, N_2\}$ given by

\[
\dot{v}_\varphi = \frac{mv_\varphi(N_1d_1 + N_2d_2)}{2\det(\mathbb{I})},
\]

\[
\dot{N}_1 = -\frac{m + 2\delta}{2(m + \delta)}N_2v_\varphi - \frac{\delta d_1}{2(m + \delta)},
\]

\[
\dot{N}_2 = \frac{m + 2\delta}{2(m + \delta)}N_1v_\varphi - \frac{\delta d_2}{2(m + \delta)}.
\]

(6.18)

Using similar arguments as in the proof of Theorem 2, it is sufficient to prove that $\mu|_{\mathcal{Q}_d}$ is the density of an invariant measure of the reduced system (6.18).

Let $X = (\dot{v}_\varphi, \dot{N}_1, \dot{N}_2)$. Then

\[
\text{div}(X) = \frac{m(N_1d_1 + N_2d_2)}{2\det(\mathbb{I})}.
\]

On the other hand,

\[
\frac{d}{dt}\det(\mathbb{I}) = -m(N_1d_1 + N_2d_2).
\]

Therefore, the function $\mu = \sqrt{\det(\mathbb{I})}$ satisfies the equation

\[
\dot{\mu} + \mu \text{div}(X) = 0,
\]

and the system (6.18) preserves the measure $\mu dv_\varphi \wedge dN_1 \wedge dN_2$. Integrability in quadratures follows according the Euler–Jacobi theorem [1].

\[\square\]

Remark 4. By setting $d_1 = d_2 = 0$, we find that

\[
v_\varphi = \text{const} \quad \text{and} \quad N_1^2 + N_2^2 = \text{const}.
\]

Thus, Eqs. (6.18) can be solved in terms of trigonometric functions.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.
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