Crossing the phantom divide line in the holographic dark energy model in a closed universe

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Abstract Conditions needed to cross the phantom divide line in an interacting holographic dark energy model in a closed Friedmann–Robertson–Walker universe are discussed. The probable relationship between this crossing and the coincidence problem is studied.

1 Introduction

One of the candidates proposed to explain the present acceleration of the universe [1–4] is the dark energy model which assumes that nearly 70% of the universe is filled by an exotic energy component with negative pressure. Based on observations, the density of (dark) matter and the dark energy component must be of the same order today (known as the coincidence problem) [5–35]. Also, based on recent data, the dark energy component seems to have an equation of state parameter $w < -1$ at the present epoch, while $w > -1$ in the past [36–52]. One way to explain these data is to consider dynamical dark energy with proper interaction with matter [53–77].

In [78], it was found that the formation of black holes requires a relationship between ultraviolet and infrared cutoffs. In this context the total energy, $E$, in a region of size $L$, must be less than (or equal to) the energy of a black hole of the same size, i.e., $E \leq L M_p^2$, where $M_p$ is the Planck mass. In terms of the energy density, $\rho$, this inequality can be rewritten as $\rho \leq M_p^2 L^{-2}$. Based on this result, in [79] an expression for the dynamical dark energy (dubbed the holographic dark energy) was proposed: $\rho_d = 3c^2 M_p^2 L^{-2}$, where $c$ is a numerical constant. Different choices may be adopted for the infrared cutoff of the universe, e.g., the particle horizon, the Hubble horizon, the future event horizon and so on [80–100]. In a noninteracting model, if we take the particle horizon as the infrared cutoff, we are unable to explain the accelerated expansion of the universe [79]. Besides an appropriate equation of state parameter for dark energy or dark matter cannot be derived if one chooses the Hubble horizon as the cutoff [101]. Instead, if we choose the future event horizon, although the present accelerated expansion of the universe may be explained [79], the coincidence problem is still unsolved. This problem can be alleviated by considering a suitable interaction between dark matter and holographic dark energy.

In this paper we consider a closed Friedmann–Robertson–Walker (FRW) universe (we do not restrict ourselves to the small spatial curvature limit) and assume that the universe is composed of two interacting perfect fluids: holographic dark energy and cold (dark) matter. A general (as far as possible) interaction between these components is considered. We allow the infrared cutoff to lie between future and particle event horizons. After some general remarks on the properties of the model, we discuss the conditions needed to cross the phantom divide line (transition from quintessence to phantom phase). We show that this crossing poses some conditions on the parameters of the model and, using an example, we show that this can alleviate the coincidence problem (at least) at the transition epoch.

We use units $\hbar = G = k_B = c = 1$ throughout the paper.

2 General properties of the model

The FRW metric,

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right), \tag{1}$$

describes a homogeneous and isotropic closed space time with scale factor $a(t)$. We assume that this universe is filled with perfect fluids and that its energy momentum tensor is...
given by
\[ T_{\mu\nu} = (P + \rho) U_{\mu} U_{\nu} + P g_{\mu\nu}, \]
where \( U^\mu = (1, 0, 0, 0) \), is the normalized four velocity of the fluids in the comoving coordinates, and \( \rho \) and \( P \) are energy density and pressure of the total fluid respectively. Using Einstein’s equation, one can obtain the Friedmann equations:
\[ H^2 = \frac{8\pi}{3} \sum_i \rho_i - \frac{1}{a^2(t)}, \]
\[ \dot{H} = -4\pi \sum_i (P_i + \rho_i) + \frac{1}{a^2(t)}. \]

The subscript \( i \) stands for the \( i \)th perfect fluid and \( H = \frac{\dot{a}(t)}{a(t)} \) is the Hubble parameter. In this paper the universe is assumed to be composed of a dark energy component with pressure \( P_d \) and an energy density \( \rho_d \), and the cold (dark) matter whose energy density is \( \rho_m \). Although these components, due to their interaction, are not conserved,
\[ \dot{\rho}_d + 3H(\rho_d + P_d) = -Q, \]
\[ \dot{\rho}_m + 3H\rho_m = Q, \]
but we have vanishing of the covariant divergence of the energy momentum tensor (2), this yields the conservation equation
\[ \dot{\rho} + 3H(\rho + P) = 0, \]
for the whole system. In this two-component universe the Friedmann equations reduce to
\[ H^2 = \frac{8\pi}{3} (\rho_m + \rho_d) - \frac{1}{a^2(t)}, \]
\[ \dot{H} = -4\pi (P_d + \rho_d + \rho_m) + \frac{1}{a^2(t)}. \]
The first equation can be rewritten as
\[ \Omega_m + \Omega_d = 1 + \Omega_K, \]

where \( \Omega_m = \frac{\rho_m}{3H^2}, \Omega_d = \frac{\rho_d}{3H^2} \) and the geometrical parameter is defined through \( \Omega_K = \frac{1}{a^2(t)H^2} \). The critical energy density \( \rho_c \) is defined by \( \rho_c = \frac{3H^2}{8\pi} \). Note that for a flat universe, \( \rho = \rho_c \) and \( \Omega_m + \Omega_d = 1 \). Different models have been proposed for the dark energy component of the universe. Here we adopt holographic dark energy, which in terms of the infrared cutoff of the universe, \( L \), can be expressed as
\[ \rho_d = \frac{3c^2}{8\pi L^2}. \]

In [102] the infrared cutoff was chosen as \( L_f = a(t) \sin y_f \) where \( y_f \) is
\[ y_f = \int_t^\infty \frac{dt}{a(t)} \]
\[ = \int_0^{L_f} \frac{dr}{\sqrt{1 - r^2}}. \]

In this way \( L_f \) is the radius of the future event horizon measured on the sphere of the horizon [102]. In the presence of a big rip [103, 104] at \( t = t_s, \infty \) in (9) must be replaced with \( t = t_s \). In the flat case this cutoff reduces to \( L_f = R_h = a(t) \int_t^\infty \frac{dt}{a(t)} \). A similar choice is to take an infrared cutoff based on the particle horizon, i.e., \( L_p = a(t) \sin y_p \), where
\[ y_p = \int_t^{L_p} \frac{dt}{a(t)} \]
\[ = \int_0^{L_p} \frac{dr}{\sqrt{1 - r^2}}. \]

As proposed in [105, 106], in general, \( L \) can be taken as a combination of both \( L_p \) and \( L_f \). In this paper we take the cutoff as
\[ L = aL_p + \beta L_f; \quad 0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1. \]

For \( \alpha = 0, \beta = 1 \), we get \( L = L_f \) and \( L = L_p \) is obtained when \( \beta = 0, \alpha = 1 \). If we attribute an entropy, \( S \), to the surface \( A = 4\pi L^2 \):
\[ S = \frac{A}{4} = \pi L^2, \]
then the second law of thermodynamics implies \( \dot{L} > 0 \). Using
\[ \dot{L} = HL - \beta \cos y_f + \alpha \cos y_p, \]
we find that the second law of thermodynamics is satisfied when
\[ X < 1, \]
where
\[ X = \frac{\beta \cos y_f - \alpha \cos y_p}{c} \Omega_d^\frac{1}{2}. \]

Note that
\[ y_p + y_f = \int_0^\infty \frac{dt}{a(t)} := \gamma \]
is a functional of \( a(t) \) and is time independent: \( \dot{\gamma} = 0 \). In the limit \( \Omega_K \ll \Omega_d \),
\[ \alpha \sin y_p + \beta \sin y_f = \frac{L}{a(t)} = c \left( \frac{\Omega_K}{\Omega_d} \right)^\frac{1}{2} \]
implies: \( y_f, y_p \ll 1 \).