A construction of Courant algebroids on foliated manifolds

by

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ABSTRACT. For any transversal-Courant algebroid \( E \) on a foliated manifold \((M, \mathcal{F})\), and for any choice of a decomposition \( TM = TF \oplus Q \), we construct a Courant algebroid structure on \( TF \oplus T^*F \oplus E \).

1 Preliminaries

General Courant algebroids were studied first in a paper by Liu, Weinstein and Xu [2], which appeared in 1997 and became the object of an intensive research since then. Courant algebroids provide the framework for Dirac structures and generalized Hamiltonian formalisms. In [3] we have introduced the notions of transversal-Courant and foliated Courant algebroid, thereby extending the framework to bases that are a space of leaves of a foliation rather than a manifold. In the present note we show that a transversal-Courant algebroid over a foliated manifold can be extended to a foliated Courant algebroid. A similar construction for Lie algebroids (which is a simpler case) was given in [4]. We assume that the reader has access to the paper [4], from which we also take the notation, and he will consult [4] for the various definitions and results that we use here. In this paper we assume that all the manifolds, foliations, mappings, bundles, etc., are \( C^\infty \)-differentiable.

A Courant algebroid over the manifold \( M \) is a vector bundle \( E \to M \) endowed with a symmetric, non degenerate, inner product \( g_E \in \Gamma \otimes^2 E^* \), with a bundle morphism \( \sharp_E : E \to TM \) called the anchor and a skew-symmetric bracket \( \{ , \}_E \) on \( E \), such that the following conditions (axioms) are satisfied:

1) \( \sharp_E [e_1, e_2]_E = [\sharp_E e_1, \sharp_E e_2] \),

2) \( \text{im}(\sharp_E \circ t^*_E) \subseteq \ker \sharp_E \),

3) \( \sum C_{ycl}[[e_1, e_2]_E, e_3]_E = (1/3)\partial_E \sum C_{ycl} g_E([e_1, e_2]_E, e_3), \) \( \partial_E = (1/2)\sharp_E \circ t^*_E : T^*M \to E \), \( \partial_E f = \partial_E (df) \),

4) \( [e_1, fe_2]_E = f[e_1, e_2]_E + (\sharp_E e_1(f)) e_2 - g(e_1, e_2) \partial_E f \).
5) \((\xi_E)(g_E(e_1, e_2)) = g_E((e, e_1)_E + \partial_E g(e, e_1), e_2) + g_E(e_1, [e, e_2]_E + \partial_E g(e, e_2))\).

In these conditions, \(e, e_1, e_2, e_3 \in \Gamma E, f \in C^\infty(M)\) and \(t\) denotes transposition. Notice also that the definition of \(\partial_E\) is equivalent with the formula

\[ g_E(e, \partial_E f) = \frac{1}{2} \xi_E e(f). \] (1.1)

The index \(E\) will be omitted if no confusion is possible.

The basic example of a Courant algebroid was studied in [1] and it consists of the big tangent bundle \(T^{\text{big}} M = \mathcal{T} M \oplus T^* M\), with the anchor \(\xi(X \oplus \alpha) = X\) and with

\[ g(X_1 \oplus \alpha_1, X_2 \oplus \alpha_2) = \frac{1}{2} (\alpha_1(X_2) + \alpha_2(X_1)), \] (1.2)

\[ [X_1 \oplus \alpha_1, X_2 \oplus \alpha_2] = [X_1, X_2] \oplus (L_{X_1} \alpha_2 - L_{X_2} \alpha_1 + \frac{1}{2} d(\alpha_1(X_2) - \alpha_2(X_1))]. \] (1.3)

(The notation \(X \oplus \alpha\) instead of the accurate \(X + \alpha\) or \((X, \alpha)\) has the advantage of showing the place of the terms while avoiding some of the parentheses. The unindexed bracket of vector fields is the usual Lie bracket.)

Furthermore, let \(\mathcal{F}\) be a foliation of the manifold \(M\). We denote the tangent bundle \(T \mathcal{F}\) by \(F\) and define the transversal bundle \(\nu \mathcal{F}\) by the exact sequence

\[ 0 \rightarrow F \xrightarrow{\iota} \mathcal{T} M \xrightarrow{\psi} \nu \mathcal{F} \rightarrow 0, \]

where \(\iota\) is the inclusion and \(\psi\) is the natural projection. We also fix a decomposition

\[ \mathcal{T} M = F \oplus Q, Q = \text{im}(\varphi : \nu \mathcal{F} \rightarrow \mathcal{T} M), \psi \circ \varphi = \text{id}, \] (1.4)

which implies

\[ T^* M = Q^* \oplus F^*, \quad Q^* = \text{ann} \ F, \quad F^* = \text{ann} Q \approx T^* M/\text{ann} \ F, \] (1.5)

where the last isomorphism is induced by the transposed mapping \(\iota\). The decompositions (1.4), (1.5) produce a bigrading \((p, q)\) of the Grassmann algebra bundles of multivector fields and exterior forms where \(p\) is the \(Q\)-degree and \(q\) is the \(F\)-degree [3].

The vector bundle \(T^{\text{big}} \mathcal{F} = F \oplus (T^* M/\text{ann} \ F)\) is the big tangent bundle of the manifold \(M^F\), which is the set \(M\) endowed with the differentiable structure of the sum of the leaves of \(\mathcal{F}\). Hence, \(T^{\text{big}} \mathcal{F}\) has the corresponding Courant structure (1.2), (1.3). A cross section of \(T^{\text{big}} \mathcal{F}\) may be represented as \(Y \oplus \bar{\alpha}\) \((Y \in \chi(M^F), \alpha \in \Omega^1(M^F))\), where the bar denotes the equivalence class of \(\alpha\) modulo \(\text{ann} \ F\) (this bar-notation is always used hereafter); generally, these cross sections are differentiable on the sum of leaves. If we consider \(Y_l \oplus \bar{\alpha}_l\) \((l = 1, 2)\) such that \(Y_l \in \chi(M)\) and \(\alpha_l \in \Omega^1(M)\) are differentiable with respect to the initial differentiable structure of \(M\) we get the inner product and Courant bracket

\[ g_F(Y_1 \oplus \bar{\alpha}_1, Y_2 \oplus \bar{\alpha}_2) = \frac{1}{2} (\alpha_1(Y_2) + \alpha_2(Y_1)), \] (1.6)

\[ [Y_1 \oplus \bar{\alpha}_1, Y_2 \oplus \bar{\alpha}_2] = ([Y_1, Y_2] \oplus (L_{Y_1} \alpha_2 - L_{Y_2} \alpha_1 + \frac{1}{2} d(\alpha_1(Y_2) - \alpha_2(Y_1)))). \] (1.7)
where the results remain unchanged if \( \alpha_l \to \alpha_l + \gamma_l \) with \( \gamma_l \in \text{ann} F \). Formulas (1.6), (1.7) show that \( T^\text{big} F \to M \), where \( M \) has its initial differentiable structure, is a Courant algebroid with the anchor given by projection on the first term. Alternatively, we can prove the same result by starting with (1.6), (1.7) as definition formulas and by checking the axioms of a Courant algebroid by computation.

We will transfer the Courant structure of \( T^\text{big} F \) by the isomorphism
\[
\Phi = \text{id} \oplus \iota : F \oplus (T^* M/\text{ann} F) \to F \oplus \text{ann} Q,
\]
i.e.,
\[
\Phi(Y \oplus \alpha) = Y \oplus \alpha_{0,1}, \quad (Y \in F, \alpha = \alpha_{1,0} + \alpha_{0,1} \in T^* M).
\]
This makes \( F \oplus \text{ann} Q \) into a Courant algebroid, which we shall denote by \( Q = T^\text{big} QF \), with the anchor equal to the projection on \( F \), the metric given by (1.6) and the bracket
\[
[Y_1 \oplus \bar{\alpha}_1, Y_2 \oplus \bar{\alpha}_2]_Q = [Y_1, Y_2] \oplus \text{pr}_{\text{ann} Q}(L_{Y_1} \alpha_2 - L_{Y_2} \alpha_1 + \frac{1}{2} d(\alpha_1(Y_2) - \alpha_2(Y_1))
\]
\( \alpha_1, \alpha_2 \in \text{ann} Q \). Using the formula \( L_Y = i_Y d + di_Y \) and the well known decomposition \( d = d'_1 + d''_0 + d''_1 + \partial_{-1} \), the expression of the previous bracket becomes
\[
[Y_1 \oplus \bar{\alpha}_1, Y_2 \oplus \bar{\alpha}_2]_Q = ([Y_1, Y_2] \oplus (i_Y d'' \alpha_2 - i_Y d'' \alpha_1) + \frac{1}{2} d''(i_Y \alpha_2 - i_Y \alpha_1)) \quad (\alpha_1, \alpha_2 \in T^*_{0,1} M).
\]

\[\text{2 The extension theorem}\]

Let \((M, \mathcal{F})\) be a foliated manifold. If the definition of a Courant algebroid is modified by asking the anchor to be a morphism \( E \to \nu \mathcal{F} \), by asking \( E, g, \sharp_E \) to be foliated, by asking only for a bracket \( [\ , \ ]_E : \Gamma_{\text{fol}} E \times \Gamma_{\text{fol}} E \to \Gamma_{\text{fol}} E \) and by asking the axioms to hold for foliated cross sections and functions, then, we get the notion of a \textit{transversal-Courant algebroid} \((E, g_E, \sharp_E, [\ , \ ]_E)\) over \((M, \mathcal{F})\) \[4\]. (The index \textit{fol} denotes foliated objects, i.e., objects that either project to or are a lift of a corresponding object of the space of leaves.)

On the other hand, a subbundle \( B \) of a Courant algebroid \( A \) over \((M, \mathcal{F})\) is a \textit{foliation} of \( A \) if: i) \( B \) is \( g_A \)-isotropic and \( \Gamma B \) is closed by \( A \)-brackets, ii) \( \sharp_A(B) = T \mathcal{F} \), iii) if \( C = B^\perp_{g_A} \), then the \( A \)-Courant structure induces the structure of a transversal-Courant algebroid on the vector bundle \( C/B \); then, the pair \((A, B)\) is called a \textit{foliated Courant algebroid} (see \[4\] for details).

In this section we prove the announced result:

**Theorem.** Let \( E \) be a transversal-Courant algebroid over the foliated manifold \((M, \mathcal{F})\) and let \( Q \) be a complementary bundle of \( F \) in \( TM \). Then \( E \) has a natural extension to a foliated Courant algebroid \( A \) with a foliation \( B \) isomorphic to \( F \).
Proof. The proof of this theorem requires a lot of technical calculations. We will only sketch the path to be followed, leaving the actual calculations to the interested reader. We shall denote the natural extension that we wish to construct, and its operations, by the index 0. Take $A_0 = T_{\mathbb{Q}}^0F \oplus E = Q \oplus E$ with the metric $g_0 = g_F \oplus g_E$ and the anchor $\sharp_0 = pr_F \oplus \rho$, where $\rho = \varphi \circ \sharp_E$ with $\varphi$ defined by (1.1), therefore, $\psi \circ \rho = \sharp_E$. Notice that this implies
\[
\partial_0 \lambda = (0, \lambda|_F) + \frac{1}{2} \sharp_{g_E}(\lambda \circ \rho) = (0, \lambda|_F) + \partial_E(i\varphi \lambda) \quad (\lambda \in T^*M) \quad (2.1)
\]
and, in particular,
\[
\partial_0 f = \partial_Q(d'f) \oplus \partial_E(d'f) = (0, d'f) \oplus \partial_E(d'f) \quad (f \in C^\infty(M)).
\]
Then, inspired by the case $T_{\mathbb{Q}}^0M = Q \oplus \nu F$ where the formulas below hold, we define the bracket of generating cross sections $Y \oplus \alpha \in \Gamma Q, e \in \Gamma_{fol}E$ by
\[
[Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2]_0 = [Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2]_Q
\]
\[
\oplus \frac{1}{2} \sharp_{g_E}((L_{Y_12,2} \alpha_2 - L_{Y_2,2} \alpha_1 + \frac{1}{2} d(\alpha_1(Y_2) - \alpha_2(Y_1)) \circ \rho)
\]
\[
= ([Y_1, Y_2] \oplus 0) + \partial_0([L_{Y_1,2} \alpha_2 - L_{Y_2,2} \alpha_1 + \frac{1}{2} d(\alpha_1(Y_2) - \alpha_2(Y_1))]),
\]
\[
[c, Y \oplus \alpha]_0 = ([\rho e, Y] \oplus (L_{\rho e, \alpha})|_F) \oplus \frac{1}{2} \sharp_{g_E}((L_{\rho e, \alpha}) \circ \rho)
\]
\[
= ([\rho e, Y] \oplus 0) + \partial_0 L_{\rho e, \alpha},
\]
\[
[e_1, e_2]_0 = (((\rho e_1, \rho e_2) - \rho [e_1, e_2]|_E) \oplus 0) \oplus [e_1, e_2]|_E.
\]
The first term of the right hand side of the second formula belongs to $\Gamma Q$ since $e \in \Gamma_{fol}E$ implies $[\rho e, Y] \in \Gamma F$. The first term of the right hand side of the third formula belongs to $\Gamma Q$ since we have
\[
\psi([\rho e_1, \rho e_2] - \rho [e_1, e_2]|_E) = \psi([\rho e_1, \rho e_2]) - \sharp_E[e_1, e_2]|_E
\]
\[
= \psi([\rho e_1, \rho e_2]) - \sharp_E[e_1, e_2]|_{\nu F} = 0.
\]
Furthermore, we extend the bracket (2.2) to arbitrary cross sections in agreement with the axiom 4) of Courant algebroids, i.e., for any functions $f, f_1, f_2 \in C^\infty(M)$, we define
\[
[Y \oplus \alpha, f e]_0 = f[Y \oplus \alpha, e]_0 \oplus (Y f)e,
\]
\[
[f_1 e_1, f_2 e_2]_0 = f_1 f_2 [e_1, e_2]_0 + f_1 (\rho e_1(f_2)) e_2 - f_2 (\rho e_2(f_1)) e_1 - g_E(e_1, e_2)(f_1 \partial_0 f_2 - f_2 \partial_0 f_1)
\]
\[
(Y \in \Gamma F, \alpha \in \text{ann } Q, e, e_1, e_2 \in \Gamma_{fol}E). \quad \text{It follows easily that formulas (2.2) and}
\]
\[
(2.3)
\]
\[
\text{give the same result if } f \in C^\infty_{\text{fol}}(M, F).
\]
We have to check that the bracket defined by (2.2), (2.3) satisfies the axioms of a Courant algebroid and it is enough to do that for every possible combination of arguments of the form $Y \oplus \alpha \in Q$ and $f e, e \in \Gamma_{fol}E, f \in C^\infty(M)$.
To check axiom 1), apply the anchor \( \sharp_0 = pr_E + \rho \) to each of the five formulas (2.2), (2.3) and use the transversal-Courant algebroid axioms satisfied by \( E \). To check axiom 2), use formula (2.1). The required results follow straightforwardly. It is also easy to check axiom 4) from (2.3) and from axiom 4) for \( Q \) and \( E \).

Furthermore, technical (lengthy) calculations show that if we have a bracket such that axioms 1), 2), 4) hold, then, if 5) holds for a triple of arguments, 5) also holds if the same arguments are multiplied by arbitrary functions. Therefore, in our case it suffices to check axiom 5) for the following six triples: (i) \( (e,e) \), (ii) \( (Y \oplus \alpha, e_1, e_2) \), (iii) \( (e, Y \oplus \alpha, e') \), (iv) \( (Y' \oplus \alpha', e) \), (v) \( (e, Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2) \), (vi) \( (Y \oplus \alpha, Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2) \), where all \( Y \oplus \alpha \in \Gamma Q \) and all \( e \in \Gamma_{\text{fol}} E \). In cases (i), (vi) the result follows from axiom 5) satisfied by \( E, Q \), respectively. In the other cases computations involving evaluations of Lie derivatives will do the job.

Finally, we have to check axiom 3). If we consider any vector bundle \( E \) with an anchor and a bracket that satisfy axioms 1), 2), 4), 5), then, by applying axiom 5) to the triple \( (e, e_1, \partial f, e_2) \) \( (f \in C^\infty(M)) \) we get

\[
[e, \partial f]_E = \frac{1}{2} \partial(\sharp_E e(f)),
\]

whence (using local coordinates, for instance) the following general formula follows

\[
[e, \partial_E \alpha]_E = \partial_E (L_{\sharp_E \alpha} e - \frac{1}{2} d(\alpha e)).
\]

Furthermore, assuming again that axioms 1), 2), 4), 5) hold and using (1.1) and (2.3), a lengthy but technical calculation shows that, if axiom 3) holds for a triple \( (e_1, e_2, e_3) \), it also holds for \( (e_1, e_2, f e_3) \) \( (f \in C^\infty(M)) \) provided that

\[
\mathcal{E} := g([e_1, e_2], e_3) + \frac{1}{4} e_2 (g(e_1, e_3)) - \frac{1}{4} e_1 (g(e_2, e_3))
\]

\[
= \frac{1}{3} \sum_{C \in \text{gcd}} g([e_1, e_2], e_3)
\]

(:= denotes a definition). But, if the last two terms in \( \mathcal{E} \) are expressed by axiom 5) for \( E \) followed by (2.3), and after we repeat the same procedure one more time, we get

\[
\mathcal{E} = \frac{1}{4} \sum_{C \in \text{gcd}} g([e_1, e_2], e_3) + \frac{1}{4} \mathcal{E},
\]

whence we see that (2.4) holds for any triple \( (e_1, e_2, e_3) \).

Hence, it suffices to check axiom 3) for the following cases: (i) \( (e_1, e_2, e_3) \), (ii) \( (e_1, e_2, Y \oplus \alpha) \), (iii) \( (e, Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2) \), (iv) \( (Y_1 \oplus \alpha_1, Y_2 \oplus \alpha_2, Y_3 \oplus \alpha_3) \), where all \( Y \oplus \alpha \in \Gamma Q \) and all \( e \in \Gamma_{\text{fol}} E \). In case (i), using the second and third formula (2.2), we get

\[
[\rho e_1, \rho e_2]_E = [\rho e_1, e_3]_0 = (\rho [\rho e_1, e_2] - \rho [\rho e_1, e_3]_0 + [\rho e_1, e_2]_E, e_3]_E
\]

and the required result follows in view of the Jacobi identity for vector fields and of axiom 3) for \( E \) (in this case, the right hand side of axiom 3) for \( A_0 \) reduces to the one for \( E \)).

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To check the result in the other cases simpler, we decompose
\[ Y \oplus \alpha = (Y \oplus 0) + (0 \oplus \alpha) \]
and check axiom 3) for each case induced by this decomposition.

For a triple \((e_1, e_2, Y \oplus 0)\) the right hand side of axiom 3) is zero and the left-hand side is
\[
\left( ([\rho e_1, \rho e_2], Y) + ([\rho e_2, Y], \rho e_1) + ([Y, \rho e_1], \rho e_2) \right) \oplus 0 = 0
\]
by the Jacobi identity for vector fields.

For a triple \((e_1, e_2, 0 \oplus \alpha)\), after cancelations, the right hand side of axiom 3) becomes \((1/2)\alpha([\rho e_1, \rho e_2])\). The same result is obtained for the left hand side if we use the second form of the first two brackets defined by (2.2) and formula (2.5).

For a triple \((e, Y_1 \oplus 0, Y_2 \oplus 0)\) the two sides of axiom 3) vanish (the left hand side reduces to the Jacobi identity for the vector fields \((\rho e, Y_1, Y_2)\)), hence the axiom holds.

For a triple \((e, 0 \oplus \alpha_1, 0 \oplus \alpha_2)\), using the second form of the second bracket and formula (2.5), axiom 3) reduces to 0 = 0, i.e., the axiom holds.

For a triple \((e, Y_1 \oplus 0, 0 \oplus \alpha)\), if we notice that \(0 \oplus \alpha = \partial_0 \alpha\) (see (2.1)), we see that the two sides of the equality required by axiom 3) are equal to \((1/2)\partial_0(\alpha([\rho e, Y_1]) - (1/2)\rho e(\alpha(Y)))\), hence the axiom holds.

The case \((Y_1 \oplus 0, Y_2 \oplus 0, Y_3 \oplus 0)\) is trivial. In the case \((Y_1 \oplus 0, Y_2 \oplus 0, 0 \oplus \alpha = \partial_0 \alpha)\) similar computations give the value \((1/4)\partial_0(\alpha([Y_1, Y_2]) - d\alpha(Y_1, Y_2))\) for the two sides of the corresponding expression of axiom 3). Finally, in the cases \((Y \oplus 0, \partial_0 \alpha_1, \partial_0 \alpha_2)\) and \((\partial_0 \alpha_1, \partial_0 \alpha_2, \partial_0 \alpha_3)\) the two sides of the required equality are 0 since the image of \(\partial_0\) is isotropic and the restriction of the bracket to this image is zero (use axiom 2) and formula (2.5)).

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