Fractional Klein–Gordon equation for linear dispersive phenomena: analytical methods and applications

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Abstract—In this paper we discuss some explicit results related to the fractional Klein–Gordon equation involving fractional powers of the D’Alembert operator. By means of a space-time transformation, we reduce the fractional Klein–Gordon equation to a case of fractional hyper-Bessel equation. We find an explicit analytical solution by using the McBride theory of fractional powers of hyper-Bessel operators. These solutions are expressed in terms of multi-index Mittag-Leffler functions studied by Kiryakova and Luchko [8]. A discussion of these results within the framework of linear dispersive wave equations is provided. We also present exact solutions of the fractional Klein–Gordon equation in the higher dimensional cases. Finally, we suggest a method of finding travelling wave solutions of the nonlinear fractional Klein–Gordon equation with power law nonlinearities.

I. INTRODUCTION

In this paper we study the following fractional Klein–Gordon equation

\[ \left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right)^\alpha u_\alpha(x, t) = -\lambda^2 u_\alpha(x, t), \]  

(1)

where \( \alpha \in (0, 1] \), and \( x \in \mathbb{R}^N \). Hence we consider a space-time fractional order operator, that is a fractional power of the D’Alembert operator. In order to analyze the travelling wave-type solutions of (1), the main trick is based on the space-time transformation

\[ w = \sqrt{c^2 t^2 - \sum_{k=1}^{N} x_k^2}, \]

which converts (1) into a case of the fractional hyper-Bessel equation

\[ \left( \frac{d^2}{dw^2} + \frac{N}{w} \frac{d}{dw} \right)^\alpha u_\alpha(w) = -\frac{\lambda^2}{c^{2\alpha}} u_\alpha(w). \]  

(2)

In order to treat (2) we will use the theory developed by A.C. McBride in a series of papers on fractional power of hyper-Bessel-type operators. Here we recall one of his results by showing that the explicit representation of a general hyper-Bessel-type operator is given as a product of Erdélyi–Kober fractional integrals. This fact is also at the basis of the generalized fractional calculus developed by Kiryakova [6]. By means of this theory we find in an explicit form travelling wave solutions of the fractional Klein–Gordon equation. We consider both the one-dimensional and higher-dimensional cases. Similar results on fractional Klein–Gordon-type equations have been recently discussed in [3], where an application to the fractional telegraph-type processes has been investigated. A similar approach was adopted by Garra et al. [4] for the study of the fractional relaxation equation with time-varying coefficients. In view of these results, we also study the nonlinear fractional Klein–Gordon equation with power law nonlinearities. By recurring to the general theory, we are able to find in explicit form some particular solutions also in the nonlinear case. The main aim of this paper is to give new mathematical tools to solve linear and nonlinear space-time fractional equations that are strictly related to the propagation of linear dispersive waves. Moreover we show the way to treat
fractional-Bessel equations that have wide applications in different fields of physics.

II. FRACTIONAL HYPER-BESSEL OPERATORS

In this section we briefly recall some useful results on the fractional power of hyper-Bessel-type operators. We refer to the theory developed by McBride in a series of papers [13], [12].

The hyper-Bessel operator considered in [12] is defined as

\[ L = x^{a_1} D x^{a_2} \ldots x^{a_n} D x^{a_{n+1}}, \]

where \( n \) is a positive integer number, \( a_1, \ldots, a_{n+1} \) are complex numbers and \( D = d/dx \). Hereafter we assume that the coefficients \( a_j \), \( j = 1, \ldots, n+1 \), are real numbers. The operator \( L \) was first introduced and studied, also with its fractional powers, by Dimovski [2] and served as a base for the generalized fractional calculus in Kiryakova [6]. In this book the whole chapter 3 is devoted to the hyper-Bessel operators, the solutions of differential equations involving it and to the development of its theory in terms of products of Erdélyi-Kober operators. Fractional power of second order version of

\[ L_{B_n} = x^{-n} \prod_{i=k+1}^{n} d/dx, \]

are dealt with in section III and IV below.

By using operational methods, the integer power of the operator \( L \) can be explicitly given in terms of a product of Erdélyi–Kober fractional derivatives (for further details see [12], pag. 527 and [6] pag. 59–60). In what follows, we use the notations adopted in McBride works. Let us define the coefficients

\[ a = \sum_{k=1}^{n+1} a_k, \quad m = |a - n|, \]

\[ b_k = \frac{1}{m} \left( \sum_{i=k+1}^{n+1} a_i + k - n \right), \quad k = 1, \ldots, n. \]

It is possible to prove the following result which was first formulated in [12]

**Lemma 2.1**: Let \( r \) be a positive integer, \( a < n \),

\[ b_k \in A_{p,\mu,m} := \{ \eta \in \mathbb{C} : \Re(m\eta + \mu) + m \neq 1/p - ml, \quad l = 0, 1, 2, \ldots, \}, \quad k = 1, \ldots, n, \]

where \( (p, \mu) \in [1, +\infty) \times \mathbb{C} \). Then

\[ L^r f = m^{nr} x^{-m} \prod_{k=1}^{n} I_m^{b_k - r} f, \]

where, for \( \alpha > 0 \) and \( \Re(m\eta + \mu) + m > 1/p \)

\[ I_m^{\eta,\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha - 1} u^m f(u) \, du, \]

which is pratically an Erdélyi–Kober derivative in the sense of Kiryakova [6].

The couple of parameters \((p, \mu)\) is related to the functional space to which \( f \) belongs [12]. The fractional generalization \( L^\alpha \) of the operator \( L \) is consequently defined as follows (see [12], pag. 527).

**Definition 2.2**: Let \( m = n - a > 0, \alpha \in \mathbb{R}, b_k \in A_{p,\mu,m}, \) for \( k = 1, \ldots, n. \) Then,

\[ L^\alpha f = m^{\alpha} x^{-m} \prod_{k=1}^{n} I_m^{b_k - \alpha} f. \]

Note that, for \( n = 1, a_1 = a_2 = 0 \) and \( \alpha > 0 \), equation (5) coincides with the Riemann–Liouville fractional derivative of order \( \alpha \) (see [17] Section 2.3). Moreover, we observe that the topic of fractional Bessel equations has been considered in recent papers with a different approach (see e.g. [16] and the references therein). An application to the description of corneal topography has been also suggested in [15]. A complete study of different approaches to fractional Bessel equations and their applications should be object of a further research.

The following lemma plays a relevant role for the next calculations.

**Lemma 2.3**: Let be \( \eta + \frac{\beta}{m} + 1 > 0, m \in \mathbb{N}, \alpha \in \mathbb{R}, \) we have that

\[ I_m^{\eta,\alpha} x^\beta = \frac{\Gamma \left( \eta + \frac{\beta}{m} + 1 \right)}{\Gamma \left( \alpha + \eta + 1 + \frac{\beta}{m} \right)} x^\beta. \]

III. FRACTIONAL KLEIN–GORDON EQUATION

**A. The one-dimensional case**

Let us consider the following fractional Klein–Gordon equation

\[ \left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right)^\alpha u_\alpha(x, t) = -\lambda^2 u_\alpha(x, t), \]

\( x \in \mathbb{R}, \ t \geq 0, \ \alpha \in (0, 1]. \)

The classical Klein–Gordon equation (\( \alpha = 1 \)) emerges from the quantum relativistic energy equation. It is also used in the
analysis of wave propagation in linear dispersive media (see, for example, [10]). The fractional Klein–Gordon equation was recently studied in the context of nonlocal quantum field theory, within the stochastic quantization approach (see [9] and the references therein). The fractional power of D’Alembert operator has been considered by [11] and [19], with different approaches.

The transformation

\[
\begin{align*}
z_1 &= ct + x, \\
z_2 &= ct - x,
\end{align*}
\]

reduces (7) to the form

\[
(4c^2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2})^\alpha u_\alpha(z_1, z_2) = -\lambda^2 u_\alpha(z_1, z_2).
\]

The partial differential equation (8) involves in fact Riemann–Liouville fractional derivatives with respect to the variables \(z_1\) and \(z_2\). The further transformation \(w = \sqrt{z_1 z_2}\) gives the fractional Bessel equation

\[
\left(\frac{d^2}{dw^2} + \frac{1}{w} \frac{d}{dw}\right)^\alpha u_\alpha(w) = -\frac{\lambda^2}{c^{2\alpha}} u_\alpha(w).
\]

The Bessel operator

\[
L_B = \frac{d^2}{dw^2} + \frac{1}{w} \frac{d}{dw}
\]

appearing in (9) is a special case of \(L\), when \(n = 2, a_1 = -1, a_2 = 1, a_3 = 0\). By definition (5) and Lemma (2.1) we have that \(m = 2, b_1 = b_2 = 0\) and thus

\[
(L_B)^\alpha f(w) = 4^\alpha w^{-2\alpha} I_{2,0-\alpha}^\alpha f(w).
\]

We are now ready to state the following

Theorem 3.1: Let \(\alpha \in (0, 1]\), the fractional equation

\[
(L_B)^\alpha u_\alpha(w) = -\frac{\lambda^2}{c^{2\alpha}} u_\alpha(w),
\]

is satisfied by

\[
u_\alpha(w) = w^{2\alpha-2} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\lambda}{2\alpha c^\alpha} \right)^{2k} w^{2\alpha k} \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + \alpha)}
\]

By applying now the operator \((L_B)^\alpha\) to the function (12) we obtain (since \(\beta = 2\alpha k + 2\alpha - 2\))

\[
(L_B)^\alpha \left( w^{2\alpha-2} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\lambda}{2\alpha c^\alpha} \right)^{2k} \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + \alpha)} \right)^2 = 4^\alpha \sum_{k=0}^{\infty} (-1)^k \left( \frac{\lambda}{2\alpha c^\alpha} \right)^{2k} w^{2\alpha k - 2} \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + \alpha)}^{2k} = -\frac{\lambda^2}{c^{2\alpha}} u_\alpha(w),
\]

as claimed.

Remark 3.2: Let us note that our solution to equation (11) expressed in terms of the power series (12) can also be written by recurring to the multi-index Mittag-Leffler functions of Kiryakova and Luchko [8], that is defined as follows

\[
E_{(\alpha_i),n}^{(n)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \mu_1) \ldots \Gamma(\alpha_n k + \mu_n)}.
\]

Namely we have that (12) can be written as

\[
u_\alpha(w) = w^{2\alpha-2} E_{2,0-\alpha}^{(2)} \left[ -\left( \frac{\lambda w^\alpha}{2\alpha c^\alpha} \right)^2 \right]
\]

Returning to the original problem, the equation (7) admits the solution

\[
u_\alpha(x, t) = (c^2 t^2 - x^2)^{\alpha-1} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\lambda}{2\alpha c^\alpha} \right)^{2k} \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + \alpha)}
\]

which for \(\alpha = 1\), reduces to the Bessel function of the first kind

\[
u_1(x, t) = J_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right), \quad |x| < ct.
\]

Remark 3.3: We observe that within a similar approach, some particular solutions of the fractional wave equation with a source term of the type

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right)^\alpha u_\alpha(x, t) = f(x, t),
\]

can be simply achieved. This kind of fractional generalization of the wave equation is new and can be of interest for the applications in the fractional approach to the electromagnetic theory (see e.g. [18] and [20]).
B. Relation with the linear damped wave equation

We recall that the linear damped wave equation for waves propagating on an elastically supported string, when the string motion is damped by air friction, has the form

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 2\sigma \frac{\partial}{\partial t} \right) u = -u,
\]

(18)

where \( \sigma \) is the damping coefficient. It can be proved that by using the transformation

\[
u(x, t) = e^{-\sigma t} v(x, t),
\]

(19)
equation (18) is trasformed into the linear Klein–Gordon equation

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) v(x, t) = (\sigma^2 - 1)v(x, t),
\]

(20)
when \( \sigma^2 < 1 \). For \( \sigma^2 > 1 \), we obtain the Helmholtz equation which is strictly related to the telegraph equation (see e.g. [3]). In our case we consider a space-time-fractional operator in the linear Klein–Gordon equation. From the point of view of the applications to the propagation of damped waves, our idea is to take into account damping effects in the classical way, i.e. with an exponential damping term such as in (19) and fractional effects in the wave propagation by directly generalizing the linear Klein–Gordon equation (20).

C. Higher-dimensional case

Higher dimensional fractional Klein–Gordon equations can be analyzed in a similar way. Let us consider the \( N \)-dimensional fractional Klein–Gordon equation, i.e.

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right)^\alpha u_\alpha(x, t) = -\lambda^2 u_\alpha(x, t),
\]

(21)
\( \alpha \in (0, 1) \), \( x \in \mathbb{R}^N \).

By means of the transformation

\[
w = \left( c^2 t^2 - \sum_{k=1}^{N} x_k^2 \right)^{1/2},
\]

where \( x_k \) is the \( k \)-th coordinate of the \( N \)-dimensional vector \( x \), we transform (21) into

\[
\left( \frac{d^2}{dw^2} + \frac{N}{w} \frac{d}{dw} \right)^\alpha u_\alpha(w) = -\lambda^2 \frac{c^{2\alpha}}{w^{2\alpha}} u_\alpha(w).
\]

(22)
The operator appearing in (22) can be considered again as a specific case of the operator (3) with \( a_1 = -N \), \( a_2 = \frac{N-1}{2}, a_3 = 0, a = 0 \), \( n = m = 2, b_1 = \frac{N-1}{2} \) and \( b_2 = 0 \). Hence, from (5) we have that

\[
\left( \frac{d^2}{dw^2} + \frac{N}{w} \frac{d}{dw} \right)^\alpha u_\alpha(w) = 4^\alpha w^{-2\alpha} I_2^{-\alpha} I_2^{-\alpha} u_\alpha(w).
\]

By using arguments similar to those of the previous section, we can prove the following

**Theorem 3.4**: A solution to the \( N \)-dimensional fractional Klein–Gordon equation (21), is given by

\[
u_\alpha(x, t) = \sum_{k=0}^{\infty} \left( \frac{\lambda}{2^{2\alpha} c^{\alpha}} \right)^k (-1)^k \frac{c^2 t^2 - \sum_{k=1}^{N} x_k^2}{\Gamma(\alpha k + \alpha + \frac{2\alpha}{2\alpha})} \Gamma(\alpha k + \alpha) \times E^{(2)}_{(\alpha, \alpha), (\alpha, \alpha + \frac{2\alpha}{2\alpha})} \left[ -\left( \frac{\lambda}{2^{2\alpha} c^{\alpha}} \right)^2 \right].
\]

(23)

We observe that for \( \alpha = 1 \), the solution of Theorem 3.2 reduces to the Bessel function

\[
u_1(x, t) = \frac{J_{\frac{\lambda}{2\sqrt{c^2 t^2 - \sum_{k=1}^{N} x_k^2}}}}{\left( \sqrt{c^2 t^2 - \sum_{k=1}^{N} x_k^2} \right)^{N-1}}.
\]

(24)

For \( N = 1 \) we retrieve result (16).

IV. THE NONLINEAR CASE

Here we consider the one-dimensional nonlinear fractional Klein–Gordon equation with power law nonlinearity,

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right)^\alpha u_\alpha(x, t) = \lambda u_\alpha^s(x, t),
\]

(24)
\( x \in \mathbb{R}, t \geq 0, \alpha \in (0, 1], \lambda \in \mathbb{R}, s \neq 1 \).

The higher dimensional case can be handled in a similar way. In the recent literature, some attempts to find specific solutions to the nonlinear fractional Klein–Gordon equation were discussed (see e.g. [3]). However, these papers are based on the application of approximate methods such as the homotopy perturbation method and related to a different formulation of the fractional Klein–Gordon equation. In view of the previous discussion, we are going to find an explicit travelling wave solution of (24).
Theorem 4.1: A travelling wave solution of (24) is given by
\[
u_a(x,t) = \left(4^\alpha \left[ \frac{\Gamma \left(1 + \frac{s}{2} \right)}{\Gamma \left(1 - \alpha + \frac{s}{2} \right)} \right]^{2} \right)^{\frac{1}{2}} \times \left( c^2 t^2 - x^2 \right)^{\alpha/(1-s)}.
\]

Proof: We are going to study exact solutions in the travelling wave form \(\nu_a(\sqrt{c^2 t^2 - x^2})\). By means of the transformation
\[w = \left( c^2 t^2 - x^2 \right)^{1/2},\]
we transform equation (24) in
\[
\left( \frac{d^2}{dw^2} + \frac{1}{w} \frac{dw}{dw} \right) \nu_a(w) = \lambda \nu_a^\alpha(w).
\]
Assuming that the solution we are searching is in the form
\[
\nu_a(w) = kw^\beta,
\]
where \(\beta\) and \(k\) are real parameters that will be fixed in the next. Substituting (27) in (24) and using (13) we obtain
\[
4^\alpha \left[ \frac{\Gamma \left(1 + \frac{s}{2} \right)}{\Gamma \left(1 - \alpha + \frac{s}{2} \right)} \right]^{2} kw^{\beta-2\alpha} = \lambda k^s w^{\beta s},
\]
that is satisfied when
\[
\begin{align*}
\beta &= \frac{2\alpha}{s} \\
k &= \left( \frac{4^\alpha}{\lambda} \left[ \frac{\Gamma \left(1 + \frac{s}{2} \right)}{\Gamma \left(1 - \alpha + \frac{s}{2} \right)} \right]^{2} \right)^{\frac{1}{\alpha-s}},
\end{align*}
\]
as claimed.

We observe that, for \(s < 1\) we have bounded solutions for \(|x| \leq ct\), while for \(s > 1\) the solutions are bounded for \(|x| < ct\).

We recall that similar specific solutions to the nonlinear Klein–Gordon equation (in the non fractional case) were investigated by [11]. In particular, for \(\alpha = 1\), we recover the solution (4.7a) in the two-dimensional case (space and time), that is
\[
u_1(x,t) = \left( \frac{4}{\lambda (s-1)^2} \right)^{\frac{1}{s-1}} \times \left( c^2 t^2 - x^2 \right)^{1/(1-s)}.
\]
Note that \(c = 1\) in the original paper [11].

Similarly to Theorem 4.1, an exact solution of the nonhomogeneous equation
\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) \nu_a(x,t) = \lambda \nu_a^\alpha(x,t) + \gamma \left( c^2 t^2 - x^2 \right)^{\alpha/(1-s)},
\]
\(x \in \mathbb{R}, t \geq 0, \alpha \in (0,1], u \neq 1, \gamma, \lambda \in \mathbb{R},\)
can be found.

A case of particular physical interest is \(s = 3\), where (24), for \(\alpha = 1\), is strictly related to scalar \(\phi^4\) theory. In the fractional case, we have the specific solution
\[
\nu_a(x,t) = \left( \frac{4^\alpha}{\lambda} \left[ \frac{\Gamma \left(1 + \frac{3}{2} \right)}{\Gamma \left(1 - \frac{3}{2} \alpha \right)} \right]^{2} \right)^{\frac{1}{2}} \times \left( c^2 t^2 - x^2 \right)^{-\alpha/2},
\]
which, for \(\alpha = 1\), becomes
\[
u_1(x,t) = [\lambda (c^2 t^2 - x^2)]^{-1/2},
\]
that is the so-called meron solution in gauge theory.

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