Non-Hermitian Hamiltonians in decoherence and equilibrium theory

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Abstract
There are many formalisms to describe quantum decoherence. However, many of them give a non-general and \textit{ad hoc} definition of ‘pointer basis’ or ‘moving preferred basis’, and this fact is a problem for the decoherence program. In this paper, we will consider quantum systems under a general theoretical framework for decoherence and we will present a tentative definition of the moving preferred basis. These ideas are implemented in a well-known open system model. The obtained decoherence and the relaxation times are defined and compared with those of the literature for the Lee–Friedrichs model.

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1. Introduction

From the appearance of quantum mechanics, many attempts have been made to recover the laws of classical mechanics through some classical limit. The most common scheme of this type includes \textit{quantum decoherence}. This process eliminates the terms of interference of the density matrix, that are classically inadmissible, since they prevent the use of a classical (Boolean) logic. In addition, decoherence gives a rule to select candidates for classical states.

Historically, decoherence was conceived to explain how the state of a quantum system goes from the frontier of the convex set of states to an interior point of this convex set (see [1, 2]). In this work, the decoherence is considered an interaction process between an open quantum system and its environment. This process called \textit{environment-induced decoherence (EID)} determines case by case which is the privileged basis, usually called \textit{moving preferred basis}, where decoherence takes place in a decoherence time \(t_D\) that is much smaller than the relaxation time \(t_R\) and it defines certain observables that acquire classical characteristics. This is the orthodox position on the subject [3]. The moving preferred basis was introduced, case by case,
in several papers (see [4]) in a non-systematic way. On the other hand, in [5] and [6] Roland Omnès introduces a rigorous and almost general definition of the moving preferred basis based on a reasonable choice of relevant observables and other physical considerations. Recently, it has become evident that dissipation from system to environment is not a necessary condition for decoherence [4] and the arrival at equilibrium of closed systems was also considered [7–14]. A closed system will be discussed at large elsewhere. In this work, we focus our attention on EID, which is a well-known theory, with well-established experimental verifications, which makes any further explanation unnecessary.

Non-unitary evolutions are essential to explain and study decoherence phenomena, quantum to classical limit and final equilibrium. These phenomena appear in the evolution of the quantum system, where decoherence time and relaxation time can be defined using non-unitary evolutions, pole theory and non-Hermitian Hamiltonians. We will consider a closed system \( U \) and define two subsystems: \( S \), the ‘proper or open system’, and \( E \), the environment. It is well known that in this case, the state of the proper system is obtained from the total density operator by tracing over the environmental degrees of freedom. If we consider the Hermitian Hamiltonian of a composed closed system \( U \) and the inner product of the evolved state with any observable, we can make its analytical continuation, in the energy variable into the lower complex half-plane, and in general we will find poles. These poles are complex eigenvalues of the non-Hermitian Hamiltonian \( H_{\text{eff}} \) that determines the system evolution. These complex eigenvalues define all the possible non-unitary decaying modes with characteristic decaying times proportional to the inverse of the imaginary part of the poles (see [15–22]). From these characteristic times, we can deduce that the relaxation time is the largest characteristic time and it is related to the pole closest to the real axis. We can also deduce the decoherence time that turns out to be a function of the imaginary part of the poles and the initial conditions of the system. Moreover, we will introduce a tentative definition of the moving preferred basis. All these definitions are considered in the Lee–Friedrichs model.

2. Toward a definition of the moving preferred basis

In this section, we will try to introduce a very general theory for the moving preferred basis in the case of a general distribution of poles and for any relevant observable space \( O_{R} \). For this purpose, it is necessary to consider the coordinates of observables and states in the Hamiltonian basis \( |\omega\rangle \) (i.e. the functions \( O(\omega, \omega') \) and \( \rho(\omega, \omega') \)) endowed with extra analytical properties in order to find the definition of a moving preferred basis.

It is well known that evolution toward equilibrium has two phases (there is also an initial non-exponential Zeno-period which is irrelevant in this paper).

(i) An exponential damping phase that can be described studying the analytical continuation of the Hamiltonian into the complex plane of the energy (see [15–22]).

(ii) A final decaying inverse polynomial in \( t^{-1} \) known as the long time evolution or Khalfin effect (see [23, 24]), which is difficult to detect experimentally (see [25]). The power-law decay for long times described by the Khalfin effect has no intrinsic parameter. It has no characteristic timescale. The Khalfin period is the one where the decaying exponential modes are not more dominant and only inverse powers of time modes remain. We can consider that the time characteristic of this period is infinite (or very long). Instead of using the word ‘infinite’, we will use Khalfin timescale.

These two phases will play an important role in the definition of the moving preferred basis. They can be identified by the theory of analytical continuation of vectors,
observables and states. To introduce the main equations, we will make a short abstract of papers [15] and [20].

2.1. Analytic continuations in the bra–ket language

We begin reviewing the analytical continuation for pure states. Let the Hamiltonian be \( H = H_0 + V \), where the free Hamiltonian \( H_0 \) satisfies (see [15] or [20])

\[
H_0|\omega\rangle = \omega|\omega\rangle, \quad \langle \omega|H_0 = \omega\langle \omega|, \quad 0 \leq \omega < \infty
\]  

(1)

and

\[
I = \int_0^\infty d\omega|\omega\rangle\langle \omega|, \quad \langle \omega|\omega'\rangle = \delta(\omega - \omega').
\]  

(2)

Then,

\[
H_0 = \int_0^\infty \omega|\omega\rangle\langle \omega| d\omega
\]  

(3)

and

\[
H = H_0 + V = \int_0^\infty \omega|\omega\rangle\langle \omega| d\omega + \int_0^\infty d\omega \int_0^\infty d\omega' V_{\omega\omega'}|\omega\rangle\langle \omega'| = \int_0^\infty \omega|\omega^+\rangle\langle \omega^+| d\omega,
\]  

(4)

where the \( |\omega^+\rangle \) are the eigenvectors of \( H \) that also satisfy equation (2). The eigenvectors of \( H \) are given by the Lippmann–Schwinger equations (see [15] equations (12) and (13))

\[
\langle \psi|\omega^+\rangle = \langle \psi|\omega \rangle + \langle \psi| \frac{1}{\omega + i0 - H} V|\omega\rangle, \quad \langle \omega^+|\varphi \rangle = \langle \omega|\varphi \rangle + \langle \omega| V\varphi \rangle \frac{1}{\omega - i0 - H}|\varphi \rangle.
\]  

(5)

Let us now endow the function of \( \omega \) with adequate analytical properties (see [16]). For example, let us consider that the state \( |\varphi \rangle \) (resp. \( \langle \psi| \)) is such that it does not create poles in the complex extension of \( \langle \omega|\varphi \rangle \) (resp. \( \langle \psi|\omega \rangle \)), and therefore, this function is analytic in the whole complex plane. The physical meaning of this hypothesis is that if the system were a non-interacting one it would never reach equilibrium. Moreover, we will consider that the complex extension of function \( \langle \omega^+|\varphi \rangle \) (resp. \( \langle \psi|\omega^+ \rangle \)) is analytic but with just one simple pole at \( z_0 = \omega_0 - \frac{i}{\gamma_0}, \gamma_0 > 0 \) in the lower half-plane (resp. another pole \( z_0^* = \omega_0 + \frac{i}{\gamma_0}, \gamma_0 > 0 \) on the upper half-plane) (see [10] for details). Then, in this paper, for the sake of simplicity, we will always use a model with just one pole and an integral that corresponds to the Khalfin effect. Then, we make an analytic continuation from the positive \( \omega \) axis to some curve \( \Gamma \) of the lower half-plane.

Then (see [15] equation (29)), we can define

\[
\langle \tilde{f}_0|\psi \rangle \equiv \text{cont}_{\omega \rightarrow z_0} \langle \psi|\omega^+ \rangle, \quad \langle \psi|\tilde{f}_0 \rangle \equiv \langle \psi|\omega_0 \rangle - \langle \psi|\omega^+ \rangle
\]

(6)

and (see [15] equation (31))

\[
\langle \psi|\tilde{f}_z \rangle \equiv \text{cont}_{\omega \rightarrow z} \langle \psi|\omega^+ \rangle, \quad \langle \psi|\tilde{f}_z \rangle \equiv \langle \psi|\omega_0 \rangle + \langle \psi|\omega^+ \rangle, \quad z \in \Gamma, \forall \varphi \rangle \langle \psi |
\]  

(7)

where cont means analytic continuation. The tilde in \( \tilde{f}_0 \) originates from the fact that in the complex extension there is no one-to-one correspondence between bra and kets [15].

Finally, it can be proved that (see [15] equation (1.33) and [20] equation (82))

\[
H = z_0|\tilde{f}_0\rangle \langle \tilde{f}_0| + \int_\Gamma z|\tilde{f}_z\rangle \langle \tilde{f}_z| dz.
\]  

(8)

That is a simple extension of the eigendecomposition of \( H \) to the complex plane with the one-pole term and the integral term that produces the Khalfin effect.
When it is possible to neglect the Khalfin term (i.e. for not extremely long times) the Hamiltonian reads where we have only a complex energy $\epsilon_0$:

$$H_{\text{eff}} = \epsilon_0 |f_0\rangle \langle f_0|.$$  

(9)

This is the non-Hermitian Hamiltonian that determines the evolution of the system far from the Khalfin timescale.

2.2. Analytical continuation in the observables and states language

What we have said about the pure states and the Hamiltonian can be rephrased in the case of the states, observables and the Liouvillian operator $L$ (see a review in [27]). But we prefer to follow the line of [15] and keep the Hamiltonian framework and discuss the analytical continuation of $\langle O \rangle_{\rho(t)}$, that we will also symbolize as $\langle \rho(t) | O \rangle$. In fact, we know that this scalar is the main character of the play, so we will completely study its analytical properties.

So let us call

$$|\omega\rangle = |\omega\rangle \langle \omega| \quad \text{and} \quad |\omega, \omega'\rangle = |\omega\rangle \langle \omega'|.$$  

(10)

Then, a generic relevant observable is $O_R \in \mathcal{O}_R$:

$$O_R = \langle O_R \rangle = \int d\omega \langle \omega | \omega \rangle + \int d\omega \int d\omega' \langle \omega | O(\omega, \omega') | \omega, \omega' \rangle$$  

(11)

and the generic state is

$$\rho_R = \langle \rho_R \rangle = \int d\omega \rho(\omega) \langle \omega | \omega \rangle + \int d\omega \int d\omega' \rho(\omega, \omega') \langle \omega | \omega' \rangle,$$  

(12)

where (see also [20] equation (44) or [15] equation (45))

$$\langle \omega | \omega' \rangle = \delta(\omega - \omega') \quad \text{and} \quad \langle \omega, \omega' | \omega'', \omega''' \rangle = \delta(\omega - \omega') \delta(\omega' - \omega'').$$  

(13)

Then,

$$\langle \omega | O_R \rangle = O(\omega), \quad \langle \omega, \omega' | O_R \rangle = O(\omega, \omega').$$  

(14)

We will consider the subject as generally as possible, i.e. $O_R$ would be any observable such that $O_R \in \mathcal{O}_R$ and $\rho_R \in \mathcal{O}_R$. In fact, in the next subsection we will only consider the generic mean value $\langle \rho_R(t) | O_R \rangle$ for two paradigmatic models below. Model 1 with just one pole and the Khalfin effect and model 2 with two poles.

2.3. Model 1. One pole and the Khalfin term

We will use a formalism for states and observables which has been proposed by the Brussels school (led by Ilya Prigogine) in [26]. It can be proved (cf ([15]) equation (67)) that the evolution equation of the mean value $\langle \rho(t) | O \rangle$ is

$$\langle O \rangle_{\rho(t)} = \langle \rho(t) | O_R \rangle = \int_{0}^{\infty} \rho^*(\omega) O(\omega) \, d\omega + \int_{0}^{\infty} \int_{0}^{\infty} \rho^*(\omega, \omega') O(\omega, \omega') \, e^{i(\omega-\omega')t} \, d\omega \, d\omega'.$$  

(15)

i.e. this mean value in the case $V \neq 0$ reads

$$\langle \rho(t) | O_R \rangle = \int d\omega \rho(0) |\Phi_\omega\rangle \langle \Phi_\omega | O_R \rangle + \int d\omega \int d\omega' e^{i(\omega-\omega')t} \langle \rho(0) | \Phi_{\omega'} \rangle \langle \Phi_{\omega'} | O_R \rangle,$$  

(16)

where $O(\omega) = \langle \Phi_\omega | O_R \rangle$, $O(\omega, \omega') = \langle \Phi_{\omega'} | O_R \rangle$, $\rho^*(\omega) = \langle \rho_R(0) | \Phi_\omega \rangle$, $\rho^*(\omega, \omega') = \langle \rho_R(0) | \Phi_{\omega'} \rangle$. These $\Phi$ vectors are defined as

$$|\Phi_\omega\rangle = |\omega^+\rangle |\omega^\dagger\rangle, \quad |\Phi_{\omega'}\rangle = |\omega'^+\rangle |\omega'^\dagger\rangle$$  

(17)
and
\[
\langle \Phi_{\omega} | = \int \mathrm{d} \varepsilon \left[ \langle \omega^+ | \varepsilon | \omega^+ \rangle - \delta (\omega - \varepsilon) \delta (\omega' - \varepsilon) \right] + \int \mathrm{d} \varepsilon' \int \mathrm{d} \varepsilon \left[ \langle \omega^+ | \varepsilon' | \omega^+ \rangle (\varepsilon, \varepsilon') \right].
\]
\[
\langle \Phi_{\omega} | = \langle \omega |.
\]

(18)

It should be emphasized that, according to definitions (10)–(14), \( \langle \omega | \neq (| \omega^+ \rangle | \omega^+ \rangle) \) and \( (\omega \omega') \neq (| \omega^+ \rangle | \omega^+ \rangle) \) in contrast to the case of discrete spectra (see [15] for details). Then, if we endow the functions with analytical properties and there is just one pole \( z_0 \) in the lower half-plane, we can prove that (see [15] equation (70))
\[
(\rho (t)|O_R) = \int \mathrm{d} \varepsilon (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R) + e^{i\varepsilon z_0 |^2} (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R)
\]
\[
+ \int_{\Gamma_{\omega}} \mathrm{d} z \int \mathrm{d} \varepsilon e^{i\varepsilon z_0 |^2} (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R)
\]
\[
+ \int_{\Gamma_{\omega}} \mathrm{d} z e^{i\varepsilon z_0 |^2} (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R),
\]
where \( z_0 \) is the simple pole in the lower half-plane, \( |\Phi_{\omega} \rangle, \langle \Phi_{\omega}| \), \( |\Phi_{\omega'} \rangle, \langle \Phi_{\omega'}| \), and \( \langle \Phi_{\omega'} | \) can be defined as in the case of equations (6) and (7). \( |\Phi_{\omega} \rangle, \langle \Phi_{\omega}| \) and \( |\Phi_{\omega'} \rangle, \langle \Phi_{\omega'}| \) can also be defined as a simple generalization of the vectors \( |f_0 \rangle, \langle f_0| \), \( |f_2 \rangle, \langle f_2| \) ([15], equation (42)).

Therefore, we can conclude that the last four terms of equation (19) vanish respectively with characteristic times
\[
\frac{1}{\gamma_0}; \frac{2}{\gamma_0}; \frac{2}{\gamma_0}; \infty.
\]
(20)

Let us observe that

(i) The vanishing of the second, third and fourth terms of equation (19) is an exponential decay corresponding to the first three terms of equation (20). This will also be the case in more complicated models with many poles.

(ii) The \( \infty \) in equation (20) means that the evolution of the last term of this equation corresponds to a polynomial in \( t^{-1} \), i.e. to the Khalfin evolution. This is a very weak effect detected in 2006 [25]. Therefore, if there is a finite number of poles and the curve \( \Gamma' \) is below them, the contribution of the integral along \( \Gamma' \) corresponds to the Khalfin effect. A closed system model for the Khalfin effect can be found in [28], section 6 and an EID-like model in [29], section 5.

Then, as we must have \( t_D \ll t_R \) and since from equation (20) we have just two characteristic times \( \gamma_0^{-1} \) and \( \infty \), the only possible choice is \( t_D = \gamma_0^{-1} \) and \( t_R = \infty \). In fact, for times \( t \gg t_D = \gamma_0^{-1} \), equation (19) reads
\[
(\rho (t)|O_R) = \int \mathrm{d} \varepsilon (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R) + \int_{\Gamma_{\omega}} \mathrm{d} z e^{i\varepsilon z_0 |^2} (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R);
\]
(21)
since here \( t \gg t_D = \gamma_0^{-1} \), the pole terms have vanished and we just have the Khalfin term. Let us now diagonalize \( \rho (t) \) of the last equation as
\[
\rho (t) = \int \mathrm{d} \varepsilon \rho (t)|i(t)\rangle \langle i(t)|,
\]
(22)
where \( \langle i(t)| \) is the moving eigenbasis of \( \rho (t) \). Now let us define a state \( (\rho_P (t)| \), the preferred state, such that, for all times, it is
\[
(\rho_P (t)|O_R) = \int \mathrm{d} \varepsilon (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R) + \int_{\Gamma_{\omega}} \mathrm{d} z e^{i\varepsilon z_0 |^2} (\rho (0)|\Phi_{\omega}) (\Phi_{\omega}|O_R).
\]
(23)
So $\rho_P(t)$ is a state that evolves in a model with no poles and with just the Khalfin term. The functional $(\rho_P(t))$ is defined by the inner product $(\rho_P(t)|O_R)$ as follows from the Riesz theorem.

It is quite clear that

(i) when $t < t_D$, $\rho(t) \neq \rho_P(t)$,
(ii) when $t \to t_D$, $\rho(t) \to \rho_P(t)$,
(iii) when $t \gg t_D$, $\rho(t) = \rho_P(t)$.

The eigenstates of the $\rho_P(t)$ are those that we will choose for the moving decoherence basis. In fact, diagonalizing $\rho_P(t)$ we have

$$\rho_P(t) = \sum_j \rho_j(t) |j(t)\rangle \langle j(t)|,$$

and when $t \to t_D = \gamma_0^{-1}$, we have that $\rho(t) \to \rho_P(t)$ so from equations (22) and (24), we see that the eigenbases of $\rho(t)$ and $\rho_P(t)$ also converge

$$\{|i(t)\rangle\} \to \{|j(t)\rangle\}.$$

Namely the basis $\{|i(t)\rangle\}$ converges to $\{|j(t)\rangle\}$ and therefore $\rho(t)$ becomes diagonal in $\{|j(t)\rangle\}$. Thus, $\{|j(t)\rangle\}$ is our definition for the moving preferred basis for this case, since $\rho(t)$ becomes diagonal in the just defined preferred basis $\{|j(t)\rangle\}$ when $t \to t_D$ and $t_D = \gamma_0^{-1}$ is really the definition of the decoherence time. In this model the relaxation time $t_R$ corresponds to the Khalfin term, i.e. an extremely long time, so that

$$t_D \ll t_R.$$

2.4. Model 2. Two poles without the Khalfin term

The Khalfin term is so small (see [25]) that it can be neglected in most experimental cases. Then, we can eliminate the Khalfin term since it corresponds to extremely long time. In this case, the Hamiltonian becomes non-Hermitian as in equations (9) and (64). So let us consider the case of two poles $z_0$ and $z_1$ (and no relevant Khalfin term) where equation (19) reads

$$(\rho(t)|O_R) = \int d\omega (\rho(0)|\Phi_\omega)(\tilde{\Phi}_\omega|O_R) + e^{i t \gamma_0^{-1} H}(\rho(0)|\Phi_0)|O_R\rangle$$

$$+ e^{i t \gamma_1^{-1} H}(\rho(0)|\Phi_1)(\tilde{\Phi}_1|O_R),$$

where $z_0 = \omega_0 - \gamma_0 > 0$, $z_1 = \omega_1 - \gamma_1 > 0$, and we will also consider that $\gamma_0 \ll \gamma_1$ (see [30] section 3, for details). Then, the characteristic times (20) now read

$$\frac{1}{\gamma_1}; \frac{1}{\gamma_1 + \gamma_0} = \frac{1}{\gamma_1} \approx \frac{1}{\gamma_1}.\footnote{All these formulas are confirmed by the coincidence of results with other methods, e.g., those used to study a ^{208}\text{Pb}(2d_{5/2})}$

$$\text{proton state in a Woods–Saxon potential (see[15] figure 3).}$$

Then, we must choose $t_D = \gamma_1^{-1}$ and $t_R = \gamma_0^{-1}$. Now for times $t \gg t_D = \gamma_1^{-1}$, equation (21) reads

$$(\rho(t)|O_R) = \int d\omega (\rho(0)|\Phi_\omega)(\tilde{\Phi}_\omega|O_R) + e^{i t \gamma_0^{-1} H}(\rho(0)|\Phi_0)|O_R\rangle$$

and we can define a state $(\rho_P(t))$ such that it would be

$$(\rho_P(t)|O_R) = \int d\omega (\rho(0)|\Phi_\omega)(\tilde{\Phi}_\omega|O_R) + e^{i t \gamma_0^{-1} H}(\rho(0)|\Phi_0)|O_R\rangle.$$
for all times. Repeating the reasoning from equations (21) to (25), we can see that diagonalizing this last equation, as in equation (24), we obtain the moving preferred basis. Then, in this case we see that the relaxation is obtained by an exponential damping (not a Khalfin term) and

\[ t_R = \frac{1}{\gamma_0} \gg t_D = \frac{1}{\gamma_1}. \]  

(31)

Again, in this case when \( t \to t_D = \gamma_0^{-1} \), we have that \( \rho_R(t) \to \rho_P(t) \), and we can conclude that the eigenbases of \( \rho(t) \) and \( \rho_P(t) \) also converge as in equation (25). Namely \( \rho(t) \) becomes diagonal in the moving preferred basis in a time \( t_D \).

2.5. The general case

Let us now consider the general case of a system with \( N + 1 \) poles at \( z_i = \omega_i^j - i\gamma_i \). In this case, it is easy to see that equation (27) (with no Khalfin term) becomes

\[ (\rho(t)|O_R) = (\rho_0|O_R) + \sum_{i=0}^{N} \alpha_i(t) \exp(-\gamma_i t) = (\rho_{R,e}|O_R) + f(t), \]

(32)

where \( (\rho_0|O_R) \) is the final equilibrium value of \( (\rho(t)|O_R) \) and the \( \alpha_i(t) \) are oscillating functions. In the most general case, the \( z_i \) will be placed either at random or following some laws. Anyhow in both cases they can be ordered as

\[ \gamma_0 \ll \gamma_1 \ll \gamma_2 \ll \cdots. \]

(33)

Then, if \( \gamma_0 \ll \gamma_1 \), it is quite clear that the relaxation time is

\[ t_R = \frac{1}{\gamma_0}. \]

(34)

So the relaxation time is defined with no ambiguity. Let us now consider the decoherence time. Really, each pole \( z_i \) defines a decaying mode with characteristic time \( t_i = \gamma_i^{-1} \). Moreover, the poles contain the essence of the decaying phenomenon and the definition of the decoherence time depends on their distribution and other data like the initial conditions. In fact, the initial conditions seem essential for the definition of \( t_D \). To introduce these conditions, let us define

\[ f(t) = \sum_{i=0}^{N} \alpha_i(t) e^{-\gamma_i t}, \quad f'(t) = \sum_{i=0}^{N} \alpha_i'(t) e^{-\gamma_i t} - \alpha_i(t) \gamma_i e^{-\gamma_i t}, \]

(35)

so at \( t = 0 \), we can write the initial conditions as

\[ f(0) = \sum_{i=0}^{N} \alpha_i(0), \quad f'(0) = \sum_{i=0}^{N} \alpha_i'(0) - \sum_{i=0}^{N} \alpha_i(0) \gamma_i. \]

(36)

Let us call \( f(t) = \text{const.} \exp \frac{g(t)}{t} \), \( g(t) \sim \exp g(t) \), and let us make a Taylor expansion of \( g(t) \) as

\[ g(t) = g(0) + g'(0)t + \frac{1}{2} g''(0)t^2 + \cdots. \]

(37)

So let us postulate the reasonable hypothesis that the decoherence time is \( t_D \ll t_R \). Then, in the period before decoherence that we are considering, precisely \( t < t_D \ll t_R \), we have \( \frac{t}{t_R} \ll 1 \). With this condition, we have the approximation

\[ g(t) = g(0) + g'(0)t. \]

(38)

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4 For simplicity, we will only consider the case \( \gamma_0 \ll \gamma_1 \ll \gamma_2 \ll \cdots \). Other special cases will be considered elsewhere.
where
\[ g(0) = \log f(0), \quad g'(0) = \frac{f'(0)}{f(0)} \]  
(39)

These equations contain the initial conditions. Then, in this approximation
\[ f(t) = e^{g(0)} e^{g'(0) t} = f(0) \exp \left( \sum_{i=0}^{N} \frac{a_i}{\gamma_i} t \right) \exp \left( - \sum_{i=0}^{N} \frac{a_i \gamma_i}{\gamma_i} t \right). \]  
(40)

So we define
\[ \bar{a}_i(t) = f(0) \exp \left( \sum_{i=0}^{N} \frac{a_i}{\gamma_i} t \right) \text{ and } \gamma_{\text{eff}} = \sum_{i=0}^{N} \frac{a_i \gamma_i}{\gamma_i} \]  
(41)

and (40) becomes
\[ f(t) = \bar{a}_i(t) \exp \left( - \gamma_{\text{eff}} t \right). \]  
(42)

The decoherence time is
\[ t_D = \frac{1}{\gamma_{\text{eff}}}. \]  
(43)

Then, \( \gamma_{\text{eff}} \) and \( t_D \) are both functions of the initial conditions. We will see that this \( t_D \) coincides with the one of the Omnès example in the following subsection.

Let us now consider the definition of the moving preferred basis. It is clear that, for the time \( t \gg t_D \), the modes with characteristic times \( \gamma_i < \gamma_{\text{eff}} \) (i.e. \( \gamma_i > \gamma_{\text{eff}} \)), that we will call the fast modes, have become negligible in equation (32). Then, we can define the functional \( (\rho_P(t) | O_B) \) as
\[ (\rho_P(t) | O_B) = (\rho_\gamma | O_B) + \sum_{i=0}^{M} a_i(t) \exp \left( - \gamma_i t \right), \]  
(44)

where the sum in this equation only contains \( M < N \) poles such that \( \gamma_i < \gamma_{\text{eff}} \), where the \( \gamma_i \) correspond to the slow modes. This is our adiabatic choice since we have selected the slow modes of decaying to define \( \rho_P(t) \) and rejected the fast modes. Our adiabatic choice corresponds to keeping the slow modes and disregarding the fast ones. Thus, for us the robust modes are the slow modes since they are the less affected by the interaction with the environment, that creates the poles, if compared with the fast modes, and it is usual to say that these robust modes are those that define the moving preferred basis. In fact:

(i) If the Hamiltonian were only \( H_0 \) (cf equation (3)), then there would be no poles (and this is the usual case in the literature). But the complex extension of the complete Hamiltonian \( H \) (cf equation (3)) certainly has poles. Therefore, the poles are created by the interaction Hamiltonian \( V \).

(ii) Thus, the slow modes and the fast ones are defined by these poles, and in the case we are considering, i.e. EID, the poles are defined by the interaction with the environment.

(iii) Then it is reasonable to call the slow modes robust, since the environment interaction has smaller influence on these poles, and we conclude that these are the modes that define the moving preferred basis.

This is our definition of robustness. If we compute the linear entropy, we know that if we only consider the slow modes, we will have a slower variation of this entropy, than if we consider all the modes (including the fast ones). This would be our minimization of the linear entropy: the moving preferred basis evolution only contains the slow modes.

Moreover, when \( t \gg t_D \), the motions produced by the fast modes, such that \( \gamma_i > \gamma_{\text{eff}} \), namely those with motions faster than that of the evolution of equation (41), are no longer
relevant for $\rho(t)$, and $\rho_P(t) \to \rho(t)$. Then, we diagonalize $\rho_P(t)$ and obtain the moving preferred basis $\{|j(t)\rangle\}$. The only influence in the evolution of $\rho_P(t)$ is given the poles such that $\gamma_i < \gamma_{\text{eff}}$. When $t \to t_D$, $\{|i(t)\rangle\} \to \{|j(t)\rangle\}$ the eigenbasis of $\rho(t)$ where $0 \leq t \leq \infty$. This $\{|j(t)\rangle\}$ is our candidate for a general definition of moving preferred basis.

3. The Omnès or Lee–Friedrichs model

Our more complete and simplest example of decoherence in open systems is the Omnès ‘pendulum’ (i.e. oscillator [31]) in a bath of oscillators, that we will compare with the pole theory in the following subsections. In fact, the Omnès model could be considered a pole model if we retain the poles and neglect the Khalfin term. Moreover, in Omnès philosophy the moving preferred basis must be related to some ‘collective variables’ in such a way that they would be experimentally accessible. In this case, this variable is the center of mass of the pendulum, i.e. the mean value of the position of a coherent state. In [31, p 285], a one-dimensional ‘pendulum’ (the system) in a bath of oscillators (the environment) is considered. Then, the Hamiltonian reads

$$H = \omega a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + \sum_k (\lambda_k a^\dagger b_k + \lambda_k^* b_k^\dagger a), \quad (45)$$

where $a^\dagger$ ($a$) is the creation (annihilation) operator for the system, $b_k^\dagger$ ($b_k$) are the creation (annihilation) operators for each mode of the environment, $\omega$ and $\omega_k$ are the energies of the system and of each mode of the environment and $\lambda_k$ are the interaction coefficients.

Then, let us consider a state

$$|\psi(t)\rangle = a|\alpha_1(t)\rangle \prod_k |\beta_{k1}(t)\rangle + b|\alpha_2(t)\rangle \prod_k |\beta_{k2}(t)\rangle, \quad (46)$$

where $|\alpha_1(0)\rangle$, $|\alpha_2(0)\rangle$ are coherent states for the ‘system’ corresponding to the operator $a^\dagger$, with center in $x_1(0)$ and $x_2(0)$ respectively, and $|\beta_{k1}(0)\rangle$ and $|\beta_{k2}(0)\rangle$ are coherent states for the environment corresponding to the operator $b_k^\dagger$. Let the initial conditions be

$$|\psi(0)\rangle = a|\alpha_1(0)\rangle \{\beta_{k1}(0) = 0\} + b|\alpha_2(t)\rangle \{\beta_{k2}(0) = 0\}. \quad (47)$$

Moreover, Omnès shows that, under reasonable hypotheses and approximations, the relaxation time of the system is

$$t_R = 1/\gamma, \quad (48)$$

where

$$\gamma = \pi \int n(\upsilon') d\upsilon' \lambda_0^2 \delta(\omega - \upsilon'), \quad (49)$$

where $n(\upsilon') d\upsilon' = dk$. On the other hand, the decoherence time of the system is (see [31, pp 289–91])

$$t_D \sim \frac{1}{m_0 \omega_0^2 L_0^2} t_R, \quad (50)$$

where $L_0 = |x_1(0) - x_2(0)|$. In the following subsection, we will attempt to recover these results using the polar technique.
3.1. The characteristic times from the polar technique

A particular important model can be studied, like the one in [21], with the Hamiltonian

\[ H = \omega_0 a^\dagger a + \int \omega_k b_k^\dagger b_k \, dk + \int \lambda_k (a^\dagger b_k + b_k^\dagger a) \, dk, \]  

(51)
i.e. a continuous version of (45). In this continuous version, we are forced to endow the scalar \( \rho(t)|O_R \) with some analyticity conditions. Precisely, function \( \lambda_k \) (where \( k = \omega_k = |k| \)) is chosen in such a way that

\[ \eta_\pm(\omega_k) = \omega_k - \omega_0 - \int \frac{dk\lambda_k^2}{\omega_k - \omega_k \pm i0} \]  

(52)
does not vanish when \( k \in \mathbb{R}^+ \), and its analytic extension \( \eta_+(z) \) in the lower half-plane only has a simple pole at \( z_0 \). This fact will have influence on the poles of \( \rho(t)|O_R \) as in the last section and we know that the study of \( \rho(t)|O_R \) is the essential way to understand the whole problem.

The Hamiltonian (51) is sometimes called the Lee–Friedrichs Hamiltonian and it is characterized by the fact that it contains different numbers of mode sectors (numbers of particle sectors in QFT). In fact, \( a^\dagger \) and \( b_k^\dagger \) are creation operators that allow us to define these numbers of mode sectors. For example, the one-mode sector will contain states such as \( a^\dagger |0\rangle \) and \( b_k^\dagger |0\rangle \) (where \( a|0\rangle = b_k|0\rangle = 0 \)). Then, the action of \( \exp(-Ht) \) (or simply that of \( H \)) will conserve the number of modes of this sector in just one mode, since in (51), all the annihilation operators are preceded by a creation operator. This is also the case for the \( n \)-mode sector.

3.1.1. The Friedrichs model and the relaxation time. In the case of the one-mode sector, this model is the so-called Friedrichs–Fano–Anderson or Friedrichs model. For a complete discussion on this model see [32]. The Hamiltonian of the Friedrichs model is

\[ H_F = \omega_0 |1\rangle\langle 1| + \int \omega_k |\omega\rangle\langle \omega| \, d\omega + \int \left( \lambda(\omega) |\omega\rangle\langle 1| + \lambda^*(\omega) |1\rangle\langle \omega| \right) \, d\omega \]  

(53)
(this Hamiltonian, which is similar to that of equation (4), is expressed just in the variable \( \omega \), the one that will be analytically continued). As a consequence of the analyticity condition mentioned above, this simple Friedrichs model only shows one resonance. In fact, this resonance is produced in \( z_0 \). In paper [32], we can see that the poles we compute here are the same as the poles of Green’s function. Let \( H_F \) be the Hamiltonian of the complex-extended Friedrichs model, i.e. the Hamiltonian of equation (8), then

\[ H_F|z_0\rangle = z_0|z_0\rangle, \quad H_F|z\rangle = z|z\rangle, \]  

(54)
where \( z_0 = \omega_0 + \delta \omega_0 - i\gamma_0 = \omega_0' - i\gamma_0 \) is the only pole and \( z \in \Gamma \) corresponds to the integral term and to the Khalfin effect.

The Lee–Friedrichs model, describing the interaction between a quantum oscillator and a scalar field, is extensively analyzed in the literature. Generally, this model is studied by analyzing the one excited mode sector, i.e. the Friedrichs model. Then, if we compute the pole, of this last model, up to the second order in \( \lambda_k \), we obtain that

\[ z_0 = \omega_0 + \int \frac{dk\lambda_k^2}{\omega_0 - \omega_k + i0}. \]  

(55)

Only symbolically, the poles really belong to the scalar \( \rho(t)|O \), as in the last section.
So the pole (that will correspond to the one closest to the real axis in the Lee–Friedrichs model) can be computed (see [33] equation (42)). These results coincide (mutatis mutandis) with one in the Omnès books [31, p 288], for the relaxation time. In fact,

\[ \frac{1}{\omega_0 - \omega'} = P \left( \frac{1}{\omega_0 - \omega'} \right) - i\pi \delta(\omega_0 - \omega'), \]

(56)

where \( P \) symbolizes the 'principal part', so

\[ z_0 = \omega_0 + P \int \frac{dk'\lambda_{k'}^2}{\omega_0 - \omega_k} - i\pi \int dk'\lambda_{k'}^2 \delta(\omega_0 - \omega_k). \]

(57)

Then, if \( dk = n(\omega') d\omega \), we have

\[ \delta\omega = P \int \frac{n(\omega') d\omega'}{\omega_0 - \omega}, \quad \gamma = \pi \int n(\omega') d\omega' \lambda_{k'}^2 \delta(\omega_0 - \omega'), \]

(58)

where \( \delta\omega \) is a shift and \( \gamma \) a damping coefficient, then the system would arrive at a state of equilibrium, namely the results of [31, p 288], and the one contained in equation (49) yields

\[ z_0 = (\omega_0 + \delta\omega) - i\gamma = \omega'_0 - i\gamma. \]

(59)

So the Omnès result for the relaxation time coincides, as we have already said, with that obtained by the pole theory, precisely (see (48))

\[ t_R = \frac{1}{\gamma}. \]

(60)

3.1.2. Other poles of the Lee–Friedrichs model. Let us now consider the Lee–Friedrichs Hamiltonian (51) for the many mode sectors. Then, as an example for the three mode sector (with just the unique pole \( z_0 \) and \( z_1, z_2 \) or \( z_3 \) 'real continuous eigenvalues' transported to the curve \( \Gamma \)), we have

\[ H|_{z_0, z_1, z_2} = (z_0 + z_1 + z_2)|_{z_0, z_1, z_2}, \]

(61)

where \((z_0 + z_1 + z_2)\) is the eigenvalue. Then, \( z_1, z_2, z_3 \in \Gamma \) are the Khalpin terms (i.e. they belong to the complex contour on the lower complex energy plane), and let \( z_0 \) be the pole of one-particle sector. So in the real complex plane, the spectrum of \( H \) contains

1. Eigenvalues \((z_1 + z_2 + z_3)\) with three points of the curve \( \Gamma \).
2. Eigenvalues \((z_1 + z_2 + z_0)\), \((z_1 + z_0 + z_3)\) and \((z_0 + z_2 + z_3)\), with two points of the curve \( \Gamma \) and the pole \( z_0 \).
3. Eigenvalues \((z_1 + z_0 + z_0)\), \((z_0 + z_2 + z_0)\) and \((z_0 + z_0 + z_3)\), with a pole at \( 2z_0 \) and one point of the curve \( \Gamma \).
4. Eigenvalue \((z_0 + z_0 + z_0)\), with a pole at \( 3z_0 \).

These values appear in the expression of the mean value as \( \sim e^{-i\frac{\pi}{2}t} \) (like in equation (19) second term on the lhs) or as \( \sim \int \int \int e^{-i\frac{\pi}{2}t} f(z) dz \) (like in equation (19) three last terms on the lhs). Then, we have that the four cases above become

1. \( \int_{\Gamma} \int_{\Gamma} \int_{\Gamma} e^{-i\frac{\pi}{2}(z_1+z_2+z_3)} f(z_1, z_2, z_3) dz_1 dz_2 dz_3, \)
2. \( \int_{\Gamma} \int_{\Gamma} e^{-i\frac{\pi}{2}(z_0+z_2+z_3)} f(z_1, z_2) dz_1 dz_2, \) and the same for the combinations \((z_1 + z_0 + z_3), \)

(11)
(3) \( \int_{\Gamma} e^{-\frac{i(tz_0)}{\hbar}} f(z_1) \, dz_1 \), and the same for the combinations \((z_2 - 2z_0), \ (z_3 + 2z_0)\).

(4) \( e^{-\frac{i\pi}{2\hbar}} \).

Then, if we neglect the Khalfin effect, we just have point (4).

Of course in the general case \( 3 \to n \), we would have \( e^{-i\pi z_0^r} \) (for the point \( n \)), plus many integrals on the curve \( \Gamma \) (for the points \( 1, 2, \ldots, n - 1 \)) corresponding to Khalfin terms. Then, if we neglect the integrals that produce the Khalfin effect, since this effect corresponds to extremely long times, the \( \Gamma \) term disappears and we simply have a pole at \( z_0 = nz_0 \). This elimination (in the case of just one pole \( z_0 \)) introduces in the model the structure of a complex oscillator. Then, we can introduce a non-Hermitian effective Hamiltonian

\[
H_{\text{eff}} = z_0 (a_0^\dagger a_0 + \frac{1}{2}) = z_0 (N_0 + \frac{1}{2}),
\]

where \( a_0^\dagger \) and \( a_0 \) are the creation and annihilation operators and \( N_0 = a_0^\dagger a_0 \) is the number of mode operators and

\[
N_0|n\rangle = n|n\rangle. \tag{63}
\]

In the case of large \( n \), \( H_{\text{eff}} \) becomes extremely close to

\[
H_{\text{eff}} = z_0 a_0^\dagger a_0 = z_0 N_0. \tag{64}
\]

Moreover, we can call

\[
z_n = nz_0 = n(\omega_0 - iy_0), \tag{65}
\]

and we will find the evolutions

\[
\exp(-iH_{\text{eff}}t)|n\rangle = \exp(-inz_0t)|n\rangle = \exp(-iz_n t)|n\rangle. \tag{66}
\]

So, in this approximation, the effective Lee–Friedrichs Hamiltonian \( H_{\text{eff}} \) simply is a (non-Hermitian) version of \( H \) with just damping terms. Below we will use this structure.

### 3.1.3. The initial conditions

As initial conditions, \( |\alpha_1(0)\rangle, |\alpha_2(0)\rangle \), it is possible to choose any linear combination of the elements \(|n\rangle\) where \( n = 0, 1, \ldots, \infty \). So we can choose coherent states

\[
|\alpha_i(0)\rangle = e^{-\frac{|\alpha_i(0)|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_i(0))^n}{\sqrt{n!}} |n\rangle. \tag{67}
\]

Then, let us choose the initial conditions as the sum of two coherent states, namely

\[
|\Phi(0)\rangle = a |\alpha_1(0)\rangle + b |\alpha_2(0)\rangle. \tag{68}
\]

Thus, the initial state operator is

\[
\rho(0) = |a|^2 |\alpha_1(0)\rangle \langle \alpha_1(0)| + |b|^2 |\alpha_2(0)\rangle \langle \alpha_2(0)| + ab^* |\alpha_1(0)\rangle \langle \alpha_2(0)| + a^* b |\alpha_2(0)\rangle \langle \alpha_1(0)|. \tag{69}
\]

We choose the two Gaussians (67) with the center at \( p_{1,2}(0) = 0 \) (see [31] equation (7.15) p 284), and

\[
\alpha_1(0) = \frac{m\omega}{\sqrt{2m\omega}} x_1(0) \quad \text{and} \quad \alpha_2(0) = \frac{m\omega}{\sqrt{2m\omega}} x_2(0). \tag{70}
\]

So \( \alpha_1(0) \) and \( \alpha_2(0) \) are real numbers. With a change of coordinates, we can choose \( x_1(0) \) and \( x_2(0) \) without loss of generality. So we can consider that the \( \alpha_1(0) \) and \( \alpha_2(0) \) are both positive.
For this reason, we will interchange \( \alpha_i(0) \) and \( |\alpha_i(0)| \) below. Without loss of generality, we can choose
\[
\alpha_1(0) = 0 \quad \text{and} \quad \alpha_2(0) = \frac{m_0}{\sqrt{2m_0}} L_0. \tag{71}
\]

The macroscopic case

It is easy to prove that for macroscopic initial conditions, i.e. when the peaks of the two Gaussians are far from each other, that is, to say \( |\alpha_1(0) - \alpha_2(0)| \rightarrow \infty \), the states \( \{|\alpha_1(0)\rangle, \{|\alpha_2(0)\rangle\} \) are a quasi-orthogonal basis:
\[
\langle \alpha_1(0)|\alpha_2(0)\rangle \cong \langle \alpha_2(0)|\alpha_1(0)\rangle \cong e^{\frac{(\alpha_1(0) - \alpha_2(0))^2}{2}} \cong 0. \tag{72}
\]
Then, the macroscopic condition for the initial conditions is \( |\alpha_1(0) - \alpha_2(0)| \gg 1 \). So we have,
\[
|\alpha_1(0) - \alpha_2(0)| = \alpha_2(0) \gg 1 \quad \frac{m_0}{\sqrt{2m_0}} L_0 \gg 1. \tag{73}
\]

3.1.4. Components of the non-diagonal part of the state. The evolved state is
\[
\rho(t) = |a|^2 \langle \alpha_1(t)| \langle \alpha_1(t) | + |b|^2 \langle \alpha_2(t) | \langle \alpha_2(t) | + ab^* \langle \alpha_1(t) | \langle \alpha_2(t) | + a^*b \langle \alpha_2(t) | \langle \alpha_1(t) |angle. \tag{74}
\]
Let us not consider the non-diagonal part of \( \rho(t) \), \( \rho^{(\text{ND})}(t) \), in the basis of the initial conditions \( \{|\alpha_1(0)\rangle, \{|\alpha_2(0)\rangle\} \). Then, we have
\[
\rho^{(\text{ND})}(t) = \rho^{(\text{ND})}_{12}(t) |\alpha_1(0)\rangle \langle \alpha_2(0)| + \rho^{(\text{ND})}_{21}(t) |\alpha_2(0)\rangle \langle \alpha_1(0)|. \tag{75}
\]
Since the basis \( \{|\alpha_1(0)\rangle, \{|\alpha_2(0)\rangle\} \) is quasi-orthogonal, from equation (74), we have
\[
\rho^{(\text{ND})}_{ij}(t) = ab^* \langle \alpha_1(t)|\alpha_1(t)\rangle \langle \alpha_2(t)|\alpha_2(t)\rangle + a^*b \langle \alpha_2(t)|\alpha_2(t)\rangle \langle \alpha_1(t)|\alpha_1(t)\rangle. \tag{76}
\]
If we consider the evolution given by the non-Hermitian Hamiltonian
\[
|\alpha_i(t)\rangle = e^{-iH_{\text{eff}}t} |\alpha_i(0)\rangle = e^{-\frac{\sqrt{|\alpha_i(0)|^2}}{2n!} \sum_{n=0}^{\infty} \left( \frac{\alpha_i(0)}{\sqrt{n!}} \right)^n e^{-i\omega nt} |n\rangle, \tag{77}
\]
we can compute these products \( \langle \alpha_i(0)|\alpha_j(t)\rangle \) and can replace them in (76) to obtain
\[
\rho^{(\text{ND})}_{12}(t) \cong a b^* e^{-i|\alpha_2(0)|^2(1-e^{-i\omega t})},
\rho^{(\text{ND})}_{21}(t) \cong a^*b e^{-i|\alpha_2(0)|^2(1-e^{-i\omega t})}. \tag{78}
\]

3.1.5. Decoherence time. Since the contributions \( \gamma_0 \) of individual poles \( z_n \) do not appear explicitly in equation (78), we may think that such poles are not involved in the outcome.
However, if we express the exponential of (78) as its Taylor series, we have
\[
\rho^{(\text{ND})}_{12}(t) \cong a b^* e^{-i|\alpha_2(0)|^2} \sum_{n=0}^{\infty} \frac{(|\alpha_2(0)|^2)^n}{n!} e^{-i\omega nt} = ab^* \sum_{n=0}^{\infty} c_n(t) e^{-i\omega nt},
\rho^{(\text{ND})}_{21}(t) \cong a^*b e^{-i|\alpha_2(0)|^2} \sum_{n=0}^{\infty} \frac{(|\alpha_2(0)|^2)^n}{n!} e^{-i\omega nt} = a^*b \sum_{n=0}^{\infty} c_n(t) e^{-i\omega nt}, \tag{79}
\]
where
\[
c_n(t) = e^{-i|\alpha_2(0)|^2} \frac{(|\alpha_2(0)|^2)^n}{n!} e^{i\omega nt} \tag{80}
\]
and these equations show that all the diagonal terms vanish when $t \to \infty$ showing that there is decoherence. Now we would like to know the decoherence time, then we must find $\gamma_{\text{eff}}$. So we analyze the decay of

$$
|\rho_{12}^{(\text{ND})}(t)|^2 = \rho_{12}^{(\text{ND})}(t)(\rho_{12}^{(\text{ND})}(t))^* = |ab|^2 e^{g(t)},
$$

where

$$
g(t) = \ln \left( \sum_{n=0}^{\infty} c_n(t) e^{-\gamma_n t} \right) \left( \sum_{j=0}^{\infty} c_j'(t) e^{-\gamma_j t} \right).\tag{82}
$$

Let us now expand $e^{g(t)}$ as

$$
e^{g(t)} = e^{g(0)+g'(0)t+\frac{1}{2}g''(0)t^2+\cdots}.	ag{83}
$$

As the decoherence time is a very short, $t_D \ll t_R$, let us neglect it from the quadratic term. Now, from equation (82) we have that

$$
g(0) = 0,
$$

$$
g'(0) = -2e^{-|\alpha_2(0)|^2} \sum_{n=0}^{\infty} \left( |\alpha_2(0)|^2 \right)^n \frac{1}{n!} \gamma_n = \gamma_{\text{eff}},
$$

then from (81) and (84), we have

$$
|\rho_{12}^{(\text{ND})}(t)| \equiv |ab| \exp (-\gamma_{\text{eff}} t).	ag{85}
$$

This is precisely the interpolation that corresponds to equation (42). Now we have the decoherence time

$$
t_D = \frac{1}{\gamma_{\text{eff}}} = \frac{2}{m_{\omega0}} \frac{1}{t_R}.	ag{86}
$$

In fact, this $t_D^{-1}$ turns out to be a weighted average of the imaginary part of the poles $z_n$. The same time was found by Omnès in [31] (or in equation (50) of this paper) and corresponds to definition (43) of the last section. So in fact, we have found the same result. Also in [31], the result for $t_D$ is only valid for small $t$ as in the last section. So the coincidence of both formalisms is proved.

Let us now consider the mathematical definition of moving preferred basis. The basis $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\} = |\alpha(t)\rangle$ is orthonormal when $L \to \infty$ and the reasoning below is done under this condition. Then, let us diagonalize $\rho_\nu(0)$ (always when $L \to \infty$) as

$$
\rho_\nu(t) = \sum_{i=1,2} \rho_\nu(t) |i(t)\rangle \langle i(t)| = \rho_\nu(t) |\tilde{1}(t)\rangle \langle \tilde{1}(t)| + \rho_\nu(t) |\tilde{2}(t)\rangle \langle \tilde{2}(t)|,
$$

where $|\tilde{i(t)}\rangle$ is our orthogonal moving pointer basis. But for times $t \gg t_D$ $\rho_\nu(t) = \rho(t)^{(D)}$, we have

$$
\rho_\nu(t) = \rho^{(D)}(t) = |a|^2 |\alpha_1(t)\rangle \langle \alpha_1(t)| + |b|^2 |\alpha_2(t)\rangle \langle \alpha_2(t)|.\tag{88}
$$

Then, since a linear orthonormal decomposition is unique, we find the moving pointer basis $|\tilde{1}(t)\rangle = |\alpha_1(t)\rangle$, $|\tilde{2}(t)\rangle = |\alpha_2(t)\rangle$.

Then, Omnès basis coincides with $|\tilde{1}(t)\rangle$, $|\tilde{2}(t)\rangle$, but this also is our moving preferred basis since it evolves under the slow motion pole evolution. So Omnès basis and our basis coincide (always when $L \to \infty$).

So we have proved that all the characters of the Omnès model, $t_R$, $t_D$ and the moving preferred basis, coincide with our definitions of the last section.
4. Conclusions

In this work we have

(i) Discussed a general scheme for decoherence that in principle could be used in many examples.

(ii) We have given a quite general definition of moving preferred basis \( \{ |\tilde{j}(t)\rangle \} \), and of relaxation and decoherence times for a generic system.

(iii) We have proved that our definitions coincide with those of the Omnès model.

We hope that these general results will produce some light in the general problem of decoherence.

The Omnès formalism of [5], [6] and [31] contains the most general definition of moving preferred basis of the literature on the subject. Our basis has another conceptual frame: the catalogue of decaying modes in the non-unitary evolution of a quantum system. But since the Omnès formalism is the best available, it would be very important for us to show, in the future, the coincidence of both formalisms, as we have at least done for one model in this paper.

Of course, we are fully aware that, to prove our proposal, more examples must be added, as we will do elsewhere. But we also believe that we have a good point of departure. In fact, probably the coincidences that we have found in the Omnès model could be a general feature of the decoherence phenomenon. Essentially, because the pole catalog contains all the possible decaying modes of the non-unitary evolutions, since relaxation and decoherence are non-unitary evolutions, necessarily they must be contained in this catalog.

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