Decomposition of generalized O’Hara’s energies

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Abstract
O’Hara introduced several functionals as knot energies. One of them is the Möbius energy. We know its Möbius invariance from Doyle-Schramm’s cosine formula. It is also known that the Möbius energy was decomposed into three components keeping the Möbius invariance. The first component of decomposition represents the extent of bending of the curves or knots, while the second one indicates the extent of twisting. The third one is an absolute constant. In this paper, we show a similar decomposition for generalized O’Hara energies. We also extend the cosine formula for the Möbius energy to generalized O’Hara energies. It gives us a condition for which the right circle minimizes the energy under the length-constraint. Furthermore, it shows us how far the energy is from the Möbius invariant property. Using decomposition, the first and second variation formulae are derived.

Keywords
O’Hara energy · Möbius energy · Knot energy · Decomposition of energy · Variation formula · Cosine formula

Mathematics Subject Classification
53A04 · 58J70 · 49Q10

1 Introduction

Let \( f \) be a knot with length \( \mathcal{L} \), which is parameterized by the arc length. In Ref. [23], the following functional was introduced by O’Hara as a knot energy:

\[
\mathcal{E}(f) = \int \int_{(\mathbb{R}/\mathcal{L})^2} \left( \frac{1}{\|f(s_1) - f(s_2)\|_\mathbb{R}^3} - \frac{1}{\mathcal{D}(f(s_1), f(s_2))^\alpha} \right)^p \, ds_1 ds_2,
\]

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where $D(f(s_1), f(s_2))$ is the intrinsic distance between two points $f(s_1)$ and $f(s_2)$ on the curve $f$, \textit{i.e.}, the distance along the curve (see Fig. 1), and $\alpha$ and $p$ are positive constants. We refer to it as O’Hara’s $(\alpha, p)$ energy.

Since O’Hara’s paper [23], many related papers [10,11,22,24,25,31] etc. were published in the first decade. Among others, Freedman-He-Wang [11] found the invariance of $E$ for $(\alpha, p) = (2, 1)$ under Möbius transformations. The energy was called the Möbius energy after this property. When $\alpha p = 2$, including the Möbius energy, the energy is invariant under dilations. This causes a difficulty for showing minimizers in a given knot class. Nevertheless, using the Möbius invariance, they showed the existence of minimizers in knot classes of prime knots and trivial knots. Their method is not applicable to classes of composite knots. Kusner-Sullivan [22] conjectured that there are no minimizers in any composite knot class based on numerical calculations. As long as the authors know, this has not been resolved.

In the past two decades, a lot of related works of knot energies were published. For examples, the gradient flow was considered in Refs. [2,4,14]. The regularity of critical points was discussed in Refs. [7–9,28,29,32]. In particular, Ref. [9] brought the regularity problem of the Möbius energy to the end. Several discrete energies, \textit{i.e.}, energies defined for polygons, were proposed and analyzed in [5,6,19,20,26,27,30,31]. See also works [1,3,10,12], some of which play important roles in this article.

The Möbius energy has expressions other than the definition. One of them is the cosine formula (1) below, shown in Ref. [22]. Another is the Möbius invariant decomposition (2) shown by the present authors [15–18]. See also a related article [13]. The aim of this paper is to see that some O’Hara’s energy have a similar formula and a decomposition.

Although a knot is defined as a closed curve in $\mathbb{R}^3$ without self-intersections, the above energy can be defined for curves in $\mathbb{R}^n$. Therefore, we here consider $f$ as a closed curve in $\mathbb{R}^n$ without self-intersections. We denote the total length by $L$, while $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$.

Let $\Phi$ be a function from $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ to itself. We consider the energy

$$E_\Phi(f) = \int\int_{(\mathbb{R}/LZ)^2} \left( \frac{1}{\Phi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})} - \frac{1}{\Phi(D(f(s_1), f(s_2)))} \right) ds_1 ds_2$$

for closed curves in $\mathbb{R}^n$. The above energy is O’Hara’s $(\alpha, 1)$ energy if $\Phi(x) = x^\alpha$; therefore, it is natural to refer to it as generalized O’Hara’s energy. As non-negativity of this energy density is necessary, we suppose the following:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The intrinsic distance $D(f(s_1), f(s_2))$ (blue), and the extrinsic distance $\|f(s_1) - f(s_2)\|_{\mathbb{R}^3}$ (magenta)
\end{figure}
Decomposition of generalized O’Hara’s energies

(A.1) $\Phi$ is monotonically increasing.

In what follows, we use the notation $\Delta u$ to mean $u(s_1) - u(s_2)$ for a function on $\mathbb{R}/\mathcal{L}\mathbb{Z}$. In the case of $\Phi(x) = x^2$, it holds that

$$E_{x^2}(f) = \int\int_{(\mathbb{R}/\mathcal{L}\mathbb{R})^2} \frac{1 - \cos \varphi(s_1, s_2)}{\|\Delta f\|_{\mathbb{R}^n}} ds_1 ds_2 + 4,$$

(1)

according to Doyle-Schramm, where $\varphi$ is the conformal angle (see [22]). This is the cosine formula. The conformal angle is defined as follows. Let $C_{12}$ be the circle contacting a knot $\text{Im} f$ at $f(s_1)$ and passing through $f(s_2)$. We define the circle $C_{21}$ similarly. The angle $\varphi(s_1, s_2)$ is that between these two circles at $f(s_1)$ (and also at $f(s_2)$). Note that it is determined from $\Delta f, f'(s_1)$, and $f'(s_2)$. The existence of $f'$ almost everywhere follows from the finiteness of energy, as shown by Blatt [3]. The conformal angle is clearly invariant under Möbius transformations. The invariance of $\frac{ds_1 ds_2}{\|\Delta f\|^2_{\mathbb{R}^n}}$ under Möbius transformations follows from that of the cross ratio. Hence, we can easily read the Möbius invariance property of $E_{x^2}$ from (1).

Remark 1 The invariance of the $E_{x^2}$ energy was first proven by Freedman-He-Wang [11] using a different approach.

We showed that $E_{x^2}$ may be decomposed as follows:

$$E_{x^2}(f) = E_{x^2,1}(f) + E_{x^2,2}(f) + 4,$$

(2)

$$E_{x^2,1}(f) = \int\int_{(\mathbb{R}/\mathcal{L}\mathbb{R})^2} \frac{\|\Delta \tau\|^2_{\mathbb{R}^n}}{2\|\Delta f\|^2_{\mathbb{R}^n}} ds_1 ds_2,$$

$$E_{x^2,2}(f) = \int\int_{(\mathbb{R}/\mathcal{L}\mathbb{R})^2} \frac{2}{\|\Delta f\|^2_{\mathbb{R}^n}} \left( \tau(s_1) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}, \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \wedge^2_{\mathbb{R}^n} ds_1 ds_2,$$

with $E_{x^2,1}$ and $E_{x^2,2}$ being Möbius invariant ([15,17]). Here, $\tau = f'$ is the unit tangent vector, $\wedge$ is the wedge product between two vectors on $\mathbb{R}^n$, and $\langle \cdot, \cdot \rangle_{\wedge^2_{\mathbb{R}^n}}$ is the inner product on the space $\wedge^2_{\mathbb{R}^n}$ of 2-vectors. The first decomposed component $E_{x^2,1}$ represents the extent of bending of the curves or knots, while the second one $E_{x^2,2}$ indicates the extent of twisting. Therefore, it is natural to consider whether the generalized O’Hara’s energy may be decomposed. It appears that decomposition of the Möbius energy can be achieved by decomposing the cosine formula; however, our previous proof of that decomposition ([15]) does not depend on that formula. This suggests that the generalized O’Hara’s energy may be decomposed in a similar manner to the Möbius energy. Note that we do not consider Möbius invariance in this case.

We suppose further conditions for $\Phi$, as follows.

(A.2) $\int_x^\infty \frac{dt}{\Phi(t)} < \infty$ for $x > 0$.

(A.3) The function space $W_\Phi$ is defined by

$$W_\Phi = \left\{ u \in W^{1,2}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \left| \int\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\Delta u\|^2_{\mathbb{R}^n}}{\Phi(\text{dist}_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(s_1, s_2))} ds_1 ds_2 < \infty \right\}. $$

If $E_\Phi(f) < \infty$, then $f \in W_\Phi \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, $f$ is bi-Lipschitz.
Theorem 1

We suppose more, it follows that

\[ E \text{ without self-intersections.} \]

Taking these points into consideration, the scaling invariance of energy makes the constant independent of \( L \).

On the other hand, in [25], it is reported that an inequality

\[ a \text{ generalized cosine formula} \]

depends on \( \lambda \), except for \( \alpha \) sufficient conditions, we show that (A.1)–(A.5) hold if

\[ \Phi(\lambda, x) = C(\lambda, \mathcal{L}) \Phi(x), \]

(a) for any \( \lambda \in (0, 1) \) and any \( x \in \left(0, \frac{2}{\mathcal{L}}\right]\), there exists a constant \( C(\lambda, \mathcal{L}) > 0 \) such that \( \Phi(\lambda, x) \geq C(\lambda, \mathcal{L}) \Phi(x) \),

(b) \( \inf_{x \in \left(0, \frac{2}{\mathcal{L}}\right]} \left( \frac{1}{\Phi(x)} + \Lambda(x) \right) \geq 0 \).

In Sect. 5 we refer to the sufficient conditions of \( \Phi \) for these conditions to hold. Using the sufficient conditions, we show that (A.1)–(A.5) hold if \( \alpha \in [2, 3] \) in the case of \( \Phi(x) = x^\alpha \). In [25], it is reported that an inequality \( \alpha \geq 2 \) is a condition of self-repulsiveness of \( \mathcal{E}_{x^\alpha} \).

On the other hand, \( \alpha < 3 \) is a condition of \( \mathcal{E}_{x^\alpha} < \infty \) for any smooth closed curves without self-intersections. Taking these points into consideration, \( \mathcal{E}_{x^\alpha} \) is well-defined as the knot energy if \( \alpha \in [2, 3] \).

Our main theorem is as follows.

Theorem 1 We suppose (A.1)–(A.5) hold and set

\[
\mathcal{E}_{\Phi, 1}(f) = \int \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\Delta \tau\|^2_{\mathbb{R}^n}}{2\Phi(\|\Delta \tau\|_{\mathbb{R}^n})} \, ds_1 ds_2,
\]

\[
\mathcal{E}_{\Phi, 2}(f) = \int \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left( \frac{1}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} - \Lambda(\|\Delta f\|_{\mathbb{R}^n}) \right) \times \left( \|\Delta f\|_{\mathbb{R}^n} \right)^2 \, ds_1 ds_2.
\]

If \( \mathcal{E}_{\Phi}(f) < \infty \), then the integrals of \( \mathcal{E}_{\Phi, 1}(f) \) and \( \mathcal{E}_{\Phi, 2}(f) \) are absolutely convergent. Furthermore, it follows that

\[
\mathcal{E}_{\Phi}(f) = \mathcal{E}_{\Phi, 1}(f) + \mathcal{E}_{\Phi, 2}(f) + 2\mathcal{L} \int_{\frac{2}{\mathcal{L}}}^{\infty} \frac{dx}{\Phi(x)}.
\]

As noted above, this decomposition theorem corresponds to that obtained in Ref. [15] for \( \Phi(x) = x^2 \), for which the third part of decomposition is an absolute constant. The third part depends on \( \mathcal{L} \) except for \( \Phi(x) = cx^2 \), but this is quite natural. Indeed, when \( \Phi(x) = cx^2 \), the scaling invariance of energy makes the constant independent of \( \mathcal{L} \).

We prove our main theorem in Sect. 2.

Considering the relation between (1) and (2), we may think that Theorem 1 goes back to a generalized cosine formula

\[
\mathcal{E}_{\Phi}(f) = \int \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{1 - \cos \Phi(s_1, s_2)}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} \, ds_1 ds_2 + 2\mathcal{L} \int_{\frac{2}{\mathcal{L}}}^{\infty} \frac{dt}{\Phi(t)} \tag{3}
\]
for $E$ with some angle $\phi$ determined from $\Delta f$, $\tau(s_1)$, and $\tau(s_2)$. The answer is affirmative; we state (3) precisely, and prove it in Sect. 3. Abrams-Cantarella-Fu-Ghomi-Howard [1] established a condition under which the right circle minimizes the knot energies. The cosine formula (3) establishes another type of condition. Furthermore, the formula indicates how far the energy is from the Möbius invariant.

Previously, in Ref. [16], we gave first and second variational formulae for the case of $\Phi(x) = x^2$ by use of the decomposition and presented estimates for some function spaces. In Sect. 4 of this study, we derive those formulae for the generalized O’Hara’s energy. For the variation formulae of the $(\alpha, p)$ energies with $p > 1$, see the recent work [21].

## 2 Proof of the main theorem

In this section, we prove our main theorem. We denote the energy densities of $E_\Phi(f), E_{\Phi,1}(f)$, and $E_{\Phi,2}(f)$ as $\mathcal{M}_\Phi(f), \mathcal{M}_{\Phi,1}(f)$, and $\mathcal{M}_{\Phi,2}(f)$, respectively. As $E_\Phi$ is non-negative from condition $(A.1)$, it holds that

$$E_\Phi(f) = \lim_{\epsilon \to +0} \int_{\mathbb{R}/LZ} \left( \int_{s_1-\epsilon}^{s_1+\frac{\epsilon}{2}} + \int_{s_1+\frac{\epsilon}{2}}^{s_1+\epsilon} \right) \mathcal{M}_\Phi(f) \, ds_2 \, ds_1.$$ 

We deform $\mathcal{M}_\Phi(f)$, which is a function of $(s_1, s_2)$. Assuming that condition $(A.2)$ holds, we set

$$\Psi(x) = -\int \left( \int_x^\infty \frac{dt}{\Phi(t)} \right) dx.$$ 

An integral with regard to $dx$ is an indefinite integral.

**Lemma 1** If $s_1$ and $s_2$ satisfy $0 < |s_1 - s_2| < \frac{\epsilon}{2}$, then it follows that

$$\mathcal{M}_\Phi(f) = \mathcal{M}_{\Phi,1}(f) + \mathcal{M}_{\Phi,2}(f)$$

$$+ \frac{\partial^2}{\partial s_1 \partial s_2} \left( \Psi(\mathcal{D}(f(s_1), f(s_2)) - \Psi(\|f(s_1) - f(s_2)\|_F)) \right).$$

**Proof** When $|s_1 - s_2| \leq \frac{\epsilon}{2}$, $\mathcal{D}(f(s_1), f(s_2))^2 = (s_1 - s_2)^2$ holds. Therefore, if $0 < |s_1 - s_2| < \frac{\epsilon}{2}$, it follows that

$$\frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_1} = \frac{1}{2\mathcal{D}(f(s_1), f(s_2))} \frac{\partial \mathcal{D}(f(s_1), f(s_2))^2}{\partial s_1} = \frac{s_1 - s_2}{\mathcal{D}(f(s_1), f(s_2))},$$

$$\frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_2} = \frac{1}{2\mathcal{D}(f(s_1), f(s_2))} \frac{\partial \mathcal{D}(f(s_1), f(s_2))^2}{\partial s_2} = \frac{s_2 - s_1}{\mathcal{D}(f(s_1), f(s_2))},$$

$$\frac{\partial^2 \mathcal{D}(f(s_1), f(s_2))}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_1} \mathcal{D}(f(s_1), f(s_2))$$

$$= \frac{1}{\mathcal{D}(f(s_1), f(s_2))} - \frac{s_2 - s_1}{\mathcal{D}(f(s_1), f(s_2))^2} \frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_1}$$

$$= \frac{1}{\mathcal{D}(f(s_1), f(s_2))} + \frac{(s_1 - s_2)^2}{\mathcal{D}(f(s_1), f(s_2))^3} \frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_1}$$

$$= 0.$$
Therefore, we obtain

\[
\frac{\partial^2}{\partial s_1 \partial s_2} \Psi(\mathcal{D}(f(s_1), f(s_2)))
= \frac{\partial}{\partial s_1} \left( \Psi'(\mathcal{D}(f(s_1), f(s_2)) \frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_2} \right)
= \Psi''(\mathcal{D}(f(s_1), f(s_2)) \frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_1} \frac{\partial \mathcal{D}(f(s_1), f(s_2))}{\partial s_1}
+ \Psi'(\mathcal{D}(f(s_1), f(s_2)) \frac{\partial^2 \mathcal{D}(f(s_1), f(s_2))}{\partial s_1 \partial s_2}
= -\frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))}.
\]

Hence, we show that

\[
\mathcal{M}_\Phi(f) = \frac{1}{\Phi(||f(s_1) - f(s_2)||_{\mathbb{R}^n})} - \frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))}
= \frac{1}{\Phi(||f(s_1) - f(s_2)||_{\mathbb{R}^n})} \frac{\partial^2}{\partial s_1 \partial s_2} \Psi(||f(s_1) - f(s_2)||_{\mathbb{R}^n})
+ \frac{\partial^2}{\partial s_1 \partial s_2} \left( \Psi(\mathcal{D}(f(s_1), f(s_2)) - \Psi(||f(s_1) - f(s_2)||_{\mathbb{R}^n}) \right).
\]

Using the above calculation, we obtain

\[
\frac{\partial}{\partial s_1} ||f(s_1) - f(s_2)||_{\mathbb{R}^n} = \frac{\tau(s_1) \cdot (f(s_1) - f(s_2))}{||f(s_1) - f(s_2)||_{\mathbb{R}^n}} = \tau(s_1) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}},
\]
\[
\frac{\partial}{\partial s_2} ||f(s_1) - f(s_2)||_{\mathbb{R}^n} = -\frac{\tau(s_2) \cdot (f(s_1) - f(s_2))}{||f(s_1) - f(s_2)||_{\mathbb{R}^n}} = -\tau(s_2) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}},
\]
\[
\frac{\partial^2}{\partial s_1 \partial s_2} ||f(s_1) - f(s_2)||_{\mathbb{R}^n}
= -\frac{\tau(s_1) \cdot \tau(s_2)}{||f(s_1) - f(s_2)||_{\mathbb{R}^n}} + \frac{\{\tau(s_1) \cdot (f(s_1) - f(s_2))\} \cdot \{\tau(s_2) \cdot (f(s_1) - f(s_2))\}}{||f(s_1) - f(s_2)||_{\mathbb{R}^n}^2}
= -\frac{1}{||\Delta f||_{\mathbb{R}^n}} \left\{ \tau(s_1) \cdot \tau(s_2) - \left( \tau(s_1) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right) \left( \tau(s_2) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right) \right\}
= -\frac{1}{||\Delta f||_{\mathbb{R}^n}} \left\{ \tau(s_1) \wedge \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}}, \tau(s_2) \wedge \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right\} \wedge 2_{\mathbb{R}^n}.
\]

Therefore,

\[
\frac{\partial}{\partial s_1} ||f(s_1) - f(s_2)||_{\mathbb{R}^n} \frac{\partial}{\partial s_2} ||f(s_1) - f(s_2)||_{\mathbb{R}^n}
= -\tau(s_1) \cdot \tau(s_2) + \tau(s_1) \cdot \tau(s_2) - \left( \tau(s_1) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right) \left( \tau(s_2) \cdot \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right)
= -\tau(s_1) \cdot \tau(s_2) \left\{ \tau(s_1) \wedge \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}}, \tau(s_2) \wedge \frac{\Delta f}{||\Delta f||_{\mathbb{R}^n}} \right\} \wedge 2_{\mathbb{R}^n}.
\]
Finally, we have
\[
\frac{\partial^2}{\partial s_1 \partial s_2} \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})
\]
\[
= -\Psi''(\|\Delta f\|_{\mathbb{R}^n})(\tau(s_1) \cdot \tau(s_2))
\]
\[
+ \left( \Psi''(\|\Delta f\|_{\mathbb{R}^n}) - \frac{\Psi''(\|\Delta f\|_{\mathbb{R}^n})}{\|\Delta f\|_{\mathbb{R}^n}} \right) \left( \tau(s_1) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} , \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \wedge^2 \mathbb{R}^n
\]
\[
= - \frac{\tau(s_1) \cdot \tau(s_2)}{\Phi(\|\Delta f\|_{\mathbb{R}^n})}
\]
\[
+ \left( \frac{1}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} - \Lambda(\|\Delta f\|_{\mathbb{R}^n}) \right) \left( \tau(s_1) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} , \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \wedge^2 \mathbb{R}^n
\]
\[
= - \frac{\tau(s_1) \cdot \tau(s_2)}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} + \mathcal{M}_{\Phi, 2}(f),
\]
and we conclude that
\[
\frac{1}{\Phi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})} + \frac{\partial^2}{\partial s_1 \partial s_2} \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})
\]
\[
= \frac{1 - \tau(s_1) \cdot \tau(s_2)}{\Phi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})} + \mathcal{M}_{\Phi, 2}(f)
\]
\[
= \mathcal{M}_{\Phi, 1}(f) + \mathcal{M}_{\Phi, 2}(f).
\]

\[\square\]

**Proposition 1** Under the assumptions (A.1)–(A.4), it holds that
\[
\lim_{\epsilon \to +0} \int_{\mathbb{R}/\mathbb{Z}} \left( \int_{s_1 - \frac{\xi}{2} + \epsilon}^{s_1} + \int_{s_1 + \epsilon}^{s_1 + \frac{\xi}{2} - \epsilon} \right) \frac{\partial^2}{\partial s_1 \partial s_2} \left( \Psi(\mathcal{D}(f(s_1), f(s_2)) - \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})) \right)
\]
\[
- \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})) \right) ds_1 ds_2
\]
\[
= 2C \int_0^{\infty} \frac{dx}{\Phi(x)}.
\]

**Proof** If \(0 < |s_1 - s_2| < \frac{\xi}{2}\), the simple calculation
\[
\frac{\partial}{\partial s_1} \left( \Psi(\mathcal{D}(f(s_1), f(s_2)) - \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})) \right)
\]
\[
= \Psi'(\|\Delta s\|) \frac{\partial \|\Delta s\|}{\partial s_1} - \Psi'(\|\Delta f\|_{\mathbb{R}^n}) \frac{\partial \|\Delta f\|_{\mathbb{R}^n}}{\partial s_1}
\]
\[
= \Psi'(\|\Delta s\|) \frac{\Delta s}{\|\Delta s\|} - \Psi'(\|\Delta f\|_{\mathbb{R}^n}) \left( \tau(s_1) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right)
\]
\[
= \Delta s \left\{ -\Lambda(\|\Delta s\|) + \Lambda(\|\Delta f\|_{\mathbb{R}^n}) \left( \tau(s_1) \cdot \frac{\Delta f}{\Delta s} \right) \right\}
\]

yields
\[
\left( \int_{s_1 - \frac{\xi}{2} + \epsilon}^{s_1 - \epsilon} + \int_{s_1 + \epsilon}^{s_1 + \frac{\xi}{2} - \epsilon} \right) \frac{\partial^2}{\partial s_1 \partial s_2} \left( \Psi(\mathcal{D}(f(s_1), f(s_2)) - \Psi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})) \right) ds_2
\]

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\[
\begin{align*}
&= \left[ \Delta s \left\{ -A(|\Delta s|) + A(\|\Delta f\|_{\mathbb{R}^n}) \left( \tau(s_1) \cdot \frac{\Delta f}{\Delta s} \right) \right\} \right]_{s_2=s_1-\epsilon}^{s_2=s_1+\epsilon} \\
&+ \left[ \Delta s \left\{ -A(|\Delta s|) + A(\|\Delta f\|_{\mathbb{R}^n}) \left( \tau(s_1) \cdot \frac{\Delta f}{\Delta s} \right) \right\} \right]_{s_2=s_1+\epsilon}^{s_2=s_1+\epsilon-\epsilon} \\
&= \epsilon \left\{ -A(\epsilon) + A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left( \tau(s_1) \cdot \frac{f(s_1) - f(s_1 - \epsilon)}{\epsilon} \right) \right\} \\
&- \left( \frac{\mathcal{L}}{2} - \epsilon \right) \left\{ -A \left( \frac{\mathcal{L}}{2} - \epsilon \right) + A \left( \left\| f(s_1) - f \left( s_1 - \frac{\mathcal{L}}{2} + \epsilon \right) \right\|_{\mathbb{R}^n} \right) \right\} \\
&- \left( \frac{\mathcal{L}}{2} - \epsilon \right) \left\{ -A \left( \frac{\mathcal{L}}{2} - \epsilon \right) + A \left( \left\| f(s_1) - f \left( s_1 + \frac{\mathcal{L}}{2} - \epsilon \right) \right\|_{\mathbb{R}^n} \right) \right\} \\
&= -2\epsilon A(\epsilon) + A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left\{ (\tau(s_1) - \tau(s_1 - \epsilon)) \cdot (f(s_1) - f(s_1 - \epsilon)) \right\} \\
&+ A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left\{ \tau(s_1 - \epsilon) \cdot (f(s_1) - f(s_1 - \epsilon)) \right\} \\
&- A(\|f(s_1) - f(s_1 + \epsilon)\|_{\mathbb{R}^n}) \left\{ \tau(s_1) \cdot (f(s_1) - f(s_1 + \epsilon)) \right\} \\
&- 2 \left( \frac{\mathcal{L}}{2} - \epsilon \right) A \left( \frac{\mathcal{L}}{2} - \epsilon \right) \\
&- A \left( \left\| f(s_1) - f \left( s_1 - \frac{\mathcal{L}}{2} + \epsilon \right) \right\|_{\mathbb{R}^n} \right) \tau(s_1) \cdot \left( f(s_1) - f \left( s_1 - \frac{\mathcal{L}}{2} + \epsilon \right) \right) \\
&+ A \left( \left\| f(s_1) - f \left( s_1 + \frac{\mathcal{L}}{2} - \epsilon \right) \right\|_{\mathbb{R}^n} \right) \tau(s_1) \cdot \left( f(s_1) - f \left( s_1 + \frac{\mathcal{L}}{2} - \epsilon \right) \right). \\
\end{align*}
\]

We define $\tilde{A}$ by

\[ A(x) = \tilde{A}(\sqrt{x}). \]

From

\[ A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left\{ (\tau(s_1) - \tau(s_1 - \epsilon)) \cdot (f(s_1) - f(s_1 - \epsilon)) \right\} \]

\[ = \frac{1}{2} \frac{d}{ds_1} \int_{\mathbb{R}^n} A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \tilde{A}(x) \, dx, \]

we know that

\[ \int_{\mathbb{R}/\mathbb{LZ}} A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left\{ (\tau(s_1) - \tau(s_1 - \epsilon)) \cdot (f(s_1) - f(s_1 - \epsilon)) \right\} \, ds_1 = 0. \]

As $f$ is periodic, we obtain

\[ \int_{\mathbb{R}/\mathbb{LZ}} \left[ A(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \left\{ \tau(s_1 - \epsilon) \cdot (f(s_1) - f(s_1 - \epsilon)) \right\} - A(\|f(s_1) - f(s_1 + \epsilon)\|_{\mathbb{R}^n}) \left\{ \tau(s_1) \cdot (f(s_1) - f(s_1 + \epsilon)) \right\} \right] \, ds_1 \]

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Therefore,

$\int_{\mathbb{R}/LZ} \left[ -2\epsilon A(\epsilon) + \Lambda(\|f(s_1) - f(s_1 - \epsilon)\|_{\mathbb{R}^n}) \{\tau(s_1) - \epsilon \cdot (f(s_1) - f(s_1 - \epsilon))\} \right. \\
- \Lambda(\|f(s_1) - f(s_1 + \epsilon)\|_{\mathbb{R}^n}) \{\tau(s_1) - (f(s_1) - f(s_1 + \epsilon))\} \right] ds_1 \\
= 2\epsilon \int_{\mathbb{R}/LZ} (\Lambda(\|f(s_1) - f(s_1 + \epsilon)\|_{\mathbb{R}^n}) - \Lambda(\epsilon)) ds_1 \\
- \int_{\mathbb{R}/LZ} \Lambda(\|f(s_1 + \epsilon) - f(s_1)\|_{\mathbb{R}^n}) \int_{s_1}^{s_1+\epsilon} \|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}^2 ds_2 ds_1$

holds. If $\epsilon \to +0$, the above amount converges to 0 from (A.4) and, obviously,

$-\int_{\mathbb{R}/LZ} 2 \left( \frac{L}{2} - \epsilon \right) A \left( \frac{L}{2} - \epsilon \right) ds_1 \to -L^2 A \left( \frac{L}{2} \right) = 2L \int_{\frac{L}{2}}^{\infty} \frac{dx}{\Phi(x)}$

follows. From Lebesgue’s convergence theorem, it holds that

$\int_{\mathbb{R}/LZ} \left\{ -A \left( \|f(s_1) - f(s_1 - \frac{L}{2} + \epsilon)\|_{\mathbb{R}^n} \right) \tau(s_1) \cdot \left( f(s_1) - f(s_1 - \frac{L}{2} + \epsilon) \right) \\
+ A \left( \|f(s_1) - f(s_1 + \frac{L}{2} - \epsilon)\|_{\mathbb{R}^n} \right) \tau(s_1) \cdot \left( f(s_1) - f(s_1 + \frac{L}{2} - \epsilon) \right) \right\} ds_1 \to 0$

as $\epsilon \to +0$. Here we have used $f(s_1 + \frac{L}{2}) = f(s_1 - \frac{L}{2})$. □

From this lemma, a principal value integral

$\lim_{\epsilon \to +0} \int_{\mathbb{R}/LZ} \left( \int_{s_1 - \frac{L}{2} + \epsilon}^{s_1 - \epsilon} + \int_{s_1 + \epsilon}^{s_1 + \frac{L}{2} - \epsilon} \right) (M_{\Phi,1}(f) + M_{\Phi,2}(f)) ds_2 ds_1$

converges under the conditions (A.1)–(A.4) and

$\mathcal{E}_\Phi(f) = \lim_{\epsilon \to +0} \int_{\mathbb{R}/LZ} \left( \int_{s_1 - \frac{L}{2} + \epsilon}^{s_1 - \epsilon} + \int_{s_1 + \epsilon}^{s_1 + \frac{L}{2} - \epsilon} \right) (M_{\Phi,1}(f) + M_{\Phi,2}(f)) ds_2 ds_1 \\
+ 2L \int_{\frac{L}{2}}^{\infty} \frac{dx}{\Phi(x)}$ (4)

holds. We can regard this as a preliminary step for a decomposed theorem; that is, this is equivalent to the cosine formula in the Möbius energy case. Next, we show that $M_{\Phi,1}(f)$ and $M_{\Phi,2}(f)$ belongs to $L^1((\mathbb{R}/LZ)^2)$ when we assume the condition (A.5).
First, we show the absolute integrability of $\mathcal{M}_{\Phi,1}(f)$. There exists a constant $\lambda \in (0,1)$ with $\lambda \text{dist}_{\mathbb{R}/\mathbb{Z}}(s_1, s_2) = \mathcal{D}(f(s_1), f(s_2)) \leq \Delta f\|_{\mathbb{R}^n}$, as $f$ is bi-Lipschitz. We obtain

$$0 \leq \mathcal{M}_{\Phi,1}(f) = \frac{\|\Delta \tau\|_{\mathbb{R}^n}^2}{2\Phi(\|\Delta f\|_{\mathbb{R}^n})} \leq \frac{\|\Delta \tau\|_{\mathbb{R}^n}^2}{2\Phi(\lambda \text{dist}_{\mathbb{R}/\mathbb{Z}}(s_1, s_2))} \leq \frac{\|\Delta \tau\|_{\mathbb{R}^n}^2}{2C(\lambda, \mathcal{L})\Phi(\text{dist}_{\mathbb{R}/\mathbb{Z}}(s_1, s_2))}.$$  

Condition (A.3) yields $f \in W_\Phi$; therefore, we know that $\mathcal{M}_{\Phi,1}(f) \in L^1((\mathbb{R}/\mathbb{Z})^2)$. The following elementary identity shows the absolute integrability of $\mathcal{M}_{\Phi,2}(f)$.

**Lemma 2** For $a, b, c \in \mathbb{R}^n$, it holds that

$$|(a - b) \cdot c|^2 + \| (a + b) \wedge c \|^2_{\mathbb{R}^n} = \| (a - b) \|^2_{\mathbb{R}^n} \cdot \| c \|^2_{\mathbb{R}^n} + 4(a \wedge c, b \wedge c) \vee_{\mathbb{R}^n}.$$  

**Proof** From a straightforward calculation, we conclude that

$$|(a - b) \cdot c|^2 + \| (a + b) \wedge c \|^2_{\mathbb{R}^n} = \| a - b \|^2_{\mathbb{R}^n} \cdot \| c \|^2_{\mathbb{R}^n} + 4(a \wedge c, b \wedge c) \vee_{\mathbb{R}^n}.$$  

We set

$$a = \tau(s_1), \quad b = \tau(s_2), \quad c = \Delta f = \|\Delta f\|_{\mathbb{R}^n}.$$  

Then, a simple calculation

$$\left|\Delta \tau \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right|^2 + \left\| (\tau(s_1) + \tau(s_2)) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right|^2$$

yields

$$\mathcal{M}_{\Phi,1}(f) + \mathcal{M}_{\Phi,2}(f)$$

$$= \frac{1}{4} \left( \frac{1}{\Phi(x)} - A(x) \right) \left\{ \left|\Delta \tau \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right|^2 + \left\| (\tau(s_1) + \tau(s_2)) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right|^2 \right\}$$

$$+ \frac{1}{4} \left( \frac{1}{\Phi(x)} + A(x) \right) \|\Delta \tau\|_{\mathbb{R}^n}^2.$$  

From the fact that $\Phi(x) > 0$ and $A(x) < 0$, the right hand side is a non-negative function if (A.5)(b) holds. Accordingly, $\mathcal{M}_{\Phi,1}(f) + \mathcal{M}_{\Phi,2}(f)$ is absolutely integrable from the integrability in the sense of the principal value. Therefore, so too is $\mathcal{M}_{\Phi,2}(f) = (\mathcal{M}_{\Phi,1}(f) + \mathcal{M}_{\Phi,2}(f)) - \mathcal{M}_{\Phi,1}(f)$. Thus, we have proven Theorem 1.

\(\square\)
Example 1 When $\Phi(x) = x^\alpha$ ($\alpha \in [2, 3]$), it follows that

$$E_{x^\alpha}(f) = E_{x^\alpha, 1}(f) + E_{x^\alpha, 2}(f) + \frac{2^\alpha}{(\alpha - 1) \mathcal{L}^{\alpha-2}},$$

$$E_{x^\alpha, 1}(f) = \iint_{(\mathbb{R}/\mathbb{L})^2} \frac{\|\Delta \tau\|^2_{\mathbb{R}^n}}{2\|\Delta f\|^2_{\mathbb{R}^n}} \, ds_1 ds_2,$$

$$E_{x^\alpha, 2}(f) = \iint_{(\mathbb{R}/\mathbb{L})^2} \frac{\alpha}{(\alpha - 1) \|\Delta f\|^2_{\mathbb{R}^n}} \left(\tau(s_1) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}, \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right) \wedge_{\mathbb{R}^n} ds_1 ds_2.$$

In particular, the case of $\alpha = 2$ corresponds to the decomposition described in Sect. 1.

Remark 2 When $\Phi(x) = x^\alpha$ ($\alpha \in [2, 3]$), the constant $\frac{2^\alpha}{(\alpha - 1) \mathcal{L}^{\alpha-2}}$ is not the energy value of the right circles, except in the case of $\alpha = 2$. However, we can calculate the energy value of the right circles using the decomposition theorem, as follows. Let $f$ be a right circle. It follows that

$$\left(\Delta \tau \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right)^2 + \left(\tau(s_1) + \tau(s_2) \wedge \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}}\right)^2 = 0.$$

If $\Phi(x) = x^\alpha$, the relation $\frac{1}{\Phi(x)} + \Lambda(x) = \frac{\alpha - 2}{(\alpha - 1)x^\alpha}$ yields

$$E_{x^\alpha}(f) = \iint_{(\mathbb{R}/\mathbb{L})^2} \left(\mathcal{M}_{x^\alpha, 1}(f) + \mathcal{M}_{x^\alpha, 2}(f)\right) ds_1 ds_2 + \frac{2}{(\alpha - 1) \mathcal{L}^{\alpha-2}}$$

$$= \frac{\alpha - 2}{2(\alpha - 1)} E_{x^\alpha, 1}(f) + \frac{2}{(\alpha - 1) \mathcal{L}^{\alpha-2}}$$

from (5). It holds that

$$\|\Delta f\|^2_{\mathbb{R}^n} = \frac{\mathcal{L}^2}{\pi^2} \sin^2 \frac{\pi}{\mathcal{L}} \Delta s, \quad \|\Delta \tau\|^2_{\mathbb{R}^n} = 4 \sin^2 \frac{\pi}{\mathcal{L}} \Delta s$$

for a right circle with total length $\mathcal{L}$. We obtain

$$E_{x^\alpha, 1}(f) = \frac{2\pi^\alpha}{\mathcal{L}^{\alpha}} \int_0^\mathcal{L} \left(\int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \sin^2 \frac{\pi}{\mathcal{L}} (s + h) \right)^{1-\frac{\alpha}{2}} dh ds$$

$$= 4 \pi^\alpha \int_0^{\frac{\mathcal{L}}{2}} \sin^{2-\alpha} \frac{\pi}{\mathcal{L}} s ds$$

$$= 4 \pi^{\alpha-1} \int_0^{\frac{\mathcal{L}}{2}} \sin^{2-\alpha} \theta d\theta$$

$$= \frac{2\pi^{\alpha-1}}{\mathcal{L}^{\alpha-2}} \frac{\Gamma \left(\frac{3-\alpha}{2}\right)}{\Gamma \left(\frac{4-\alpha}{2}\right)},$$

using

$$\int_0^{\frac{\pi}{2}} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta = \frac{\Gamma(x) \Gamma(y)}{2 \Gamma(x + y)}.$$

Consequently, we have

$$E_{x^\alpha}(f) = \frac{1}{(\alpha - 1) \mathcal{L}^{\alpha-2}} \left\{ (\alpha - 2) \pi^{\alpha-1} \frac{\Gamma \left(\frac{3-\alpha}{2}\right)}{\Gamma \left(\frac{4-\alpha}{2}\right)} + 2^\alpha \right\}.$$
This value also follows from the result reported by Brylinski [10]. Note that the global minimizers of \( \Phi_{x^a} \) with fixed total length are right circles, as shown by Abrams et al. [1].

### 3 Cosine formula and its consequences

Let \( \varphi(s_1, s_2) \) be the conformal angle, and let \( \psi(s_1, s_2) \) be the angle between \( \tau(s_1) \) and \( \tau(s_2) \). We set

\[
\Theta_\varphi(t) = \frac{1}{2} (1 + \Phi(t) \Lambda(t))
\]

The following is an extension of the cosine formula for generalized O’Hara energies.

**Theorem 2** 1. Under (A.1)–(A.4), a generalized cosine formula

\[
\mathcal{E}_\varphi(f) = \text{p.v.} \int_{\mathbb{R}/\mathbb{Z}^2} \frac{1 - \cos \varphi(s_1, s_2)}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} ds_1 ds_2 + 2 \mathcal{L} \int_{\frac{\pi}{2}}^{\infty} \frac{dx}{\Phi(x)}
\]

holds. Here \( \text{p.v.} \int \int = \lim_{\epsilon \to +0} \int_{|s_1 - s_2| \geq \epsilon} \int \) is the integration in the principal value sense.

2. In addition to (A.1)–(A.4), if we assume (A.5) (b), then

\[
\varphi_\varphi(s_1, s_2) = \arccos [(1 - \Theta_\varphi(\|\Delta f\|_{\mathbb{R}^n})) \cos \psi(s_1, s_2) + \Theta_\varphi(\|\Delta f\|_{\mathbb{R}^n}) \cos \psi(s_1, s_2)]
\]

can be defined (see Fig. 2), and it holds that

\[
\mathcal{E}_\varphi(f) = \int_{\mathbb{R}/\mathbb{Z}^2} \frac{1 - \cos \varphi(s_1, s_2)}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} ds_1 ds_2 + 2 \mathcal{L} \int_{\frac{\pi}{2}}^{\infty} \frac{dx}{\Phi(x)}
\]

**Proof** We can see that the conformal angle is given as

\[
\cos \varphi(s_1, s_2) = -\tau(s_1) \cdot \tau(s_2) + 2 \left( \tau(s_1) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \left( \tau(s_2) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right).
\]

Hence we have

\[
\tau(s_1) \cdot \tau(s_2) = \left( \tau(s_1) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \left( \tau(s_2) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) = -\frac{1}{2} (1 - \tau(s_1) \cdot \tau(s_2))
\]

\[
+ \frac{1}{2} \left\{ 1 + \tau(s_1) \cdot \tau(s_2) - 2 \left( \tau(s_1) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \left( \tau(s_2) \cdot \frac{\Delta f}{\|\Delta f\|_{\mathbb{R}^n}} \right) \right\}
\]

\[
= -\frac{1}{4} \|\Delta \tau\|_{\mathbb{R}^n}^2 + \frac{1}{2} (1 - \cos \varphi(s_1, s_2)).
\]

Using

\[
\|\Delta \tau\|_{\mathbb{R}^n}^2 = 2 (1 - \tau(s_1) \cdot \tau(s_2)) = 2(1 - \cos \psi(s_1, s_2)),
\]
we obtain the first assertion of Theorem 2. Indeed, the integrand of (4) is

\[
\frac{1}{2} \left( \frac{1}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} - A(\|\Delta f\|_{\mathbb{R}^n}) \right) (1 - \cos \varphi(s_1, s_2)) \\
+ \frac{1}{4} \left( \frac{1}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} + A(\|\Delta f\|_{\mathbb{R}^n}) \right) \|\Delta \tau\|_{\mathbb{R}^n}^2
\]

\[
= \frac{1}{2\Phi(\|\Delta f\|_{\mathbb{R}^n})} \left\{ (1 - \Phi(\|\Delta f\|_{\mathbb{R}^n})A(\|\Delta f\|_{\mathbb{R}^n})) (1 - \cos \varphi(s_1, s_2)) \\
+ (1 + \Phi(\|\Delta f\|_{\mathbb{R}^n})A(\|\Delta f\|_{\mathbb{R}^n})) (1 - \cos \psi(s_1, s_2)) \right\}
\]

\[
= \frac{1}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} \left\{ (1 - \Theta_\Phi(\|\Delta f\|_{\mathbb{R}^n})) (1 - \cos \varphi(s_1, s_2)) \\
+ \Theta_\Phi(\|\Delta f\|_{\mathbb{R}^n})(1 - \cos \psi(s_1, s_2)) \right\}.
\]

If we also assume (A.5) (b), then

\[0 \leq \Theta(x) \leq \frac{1}{2}.
\]

Therefore, the integrand

\[(1 - \Theta_\Phi(\|\Delta f\|_{\mathbb{R}^n})) (1 - \cos \varphi(s_1, s_2)) + \Theta_\Phi(\|\Delta f\|_{\mathbb{R}^n})(1 - \cos \psi(s_1, s_2))
\]
is non-negative, and
\[
| (1 - \Theta_{\phi}(\|\mathbf{A}f\|_{\mathbb{R}^n})) \cos \varphi(s_1, s_2) + \Theta_{\phi}(\|\mathbf{A}f\|_{\mathbb{R}^n}) \cos \psi(s_1, s_2)) | \leq 1.
\]
Hence, the integration in the principal value sense becomes one in the usual $L^1$ sense. Furthermore, the angle $\varphi_{\phi}$ is defined. This shows the second assertion of Theorem 2. □

We do not need (A.5) (a).

Theorem 2 establishes a condition under which the right circle minimizes the knot energies different from that of Abrams-Cantarella-Fu-Ghomi-Howard [1]. Assume (A.1)–(A.4) and (A.5) (b). Set
\[
\mathcal{E}_{\phi, 3}(f) = \int \int_{(\mathbb{R}/\mathbb{Z})^2} (1 - \Theta_{\phi}(\|\mathbf{A}f\|_{\mathbb{R}^n})) (1 - \cos \varphi(s_1, s_2)) \frac{ds_1ds_2}{\Phi(\|\mathbf{A}f\|_{\mathbb{R}^n})},
\]
\[
\mathcal{E}_{\phi, 4}(f) = \int \int_{(\mathbb{R}/\mathbb{Z})^2} \Theta_{\phi}(\|\mathbf{A}f\|_{\mathbb{R}^n})(1 - \cos \psi(s_1, s_2)) \frac{ds_1ds_2}{\Phi(\|\mathbf{A}f\|_{\mathbb{R}^n})}.
\]
Let $C$ be a circle with a circumstance that is the same as the total length $L$ of Im$f$. As the conformal angle for $C$ vanishes identically, we have $\mathcal{E}_{\phi, 3}(C) = 0$. Since the energy density of $\mathcal{E}_{\phi, 3}$ is non-negative, we have
\[
\mathcal{E}_{\phi}(f) = \mathcal{E}_{\phi, 3}(f) + (\mathcal{E}_{\phi, 4}(f) - \mathcal{E}_{\phi, 4}(C)) + \mathcal{E}_{\phi}(C) \equiv (\mathcal{E}_{\phi, 4}(f) - \mathcal{E}_{\phi, 4}(C)) + \mathcal{E}_{\phi}(C).
\]
Consequently we obtain

**Corollary 1** Assume (A.1)–(A.4), and (A.5) (b). If $C$ is a minimizer of $\mathcal{E}_{\phi, 4}$ under the length-constraint, then it minimizes $\mathcal{E}_{\phi}$ under the constraint.

In the case of O’Hara $\Phi(x) = x^\alpha$, the function $\Theta_{\phi}$ is a constant, and the assertion of Theorem 2 becomes
\[
\mathcal{E}_{x^\alpha}(f) = \left\{ 1 - \frac{\alpha - 2}{2(\alpha - 1)} \right\} \int \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{1 - \cos \varphi(s_1, s_2)}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^{\alpha}} + \frac{\alpha - 2}{2(\alpha - 1)} \int \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{1 - \cos \psi(s_1, s_2)}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^{\alpha}} + \frac{2^\alpha}{(\alpha - 1)L^{\alpha - 2}}.
\]
This coincides with (1) when $\alpha = 2$. Now consider the normalized O’Hara energy:
\[
L^{\alpha - 2} \mathcal{E}_{x^\alpha}(f)
\]
\[
= \left\{ 1 - \frac{\alpha - 2}{2(\alpha - 1)} \right\} \int \int_{(\mathbb{R}/\mathbb{Z})^2} \left( \frac{L}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \right)^{\alpha - 2} \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^2} (1 - \cos \varphi(s_1, s_2)) \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^{\alpha}} + \frac{\alpha - 2}{2(\alpha - 1)} \int \int_{(\mathbb{R}/\mathbb{Z})^2} \left( \frac{L}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \right)^{\alpha - 2} \frac{1 - \cos \psi(s_1, s_2)}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^2} + \frac{2^\alpha}{\alpha - 1}.
\]
It is scale-invariant. The quantities
\[
1 - \cos \varphi(s_1, s_2), \quad \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^2}, \quad \int \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{1 - \cos \psi(s_1, s_2)}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \frac{ds_1ds_2}{\|\mathbf{A}f\|_{\mathbb{R}^n}^2}
\]
are Möbius invariant (for the last one, see [15,17]), but not
\[
\left( \frac{L}{\|\mathbf{A}f\|_{\mathbb{R}^n}} \right)^{\alpha - 2}
\]
Consequently, we can see from our formula how far the energy is from the Möbius invariant property.
4 Variational formulae

In this section, we derive the variational formulae of the decomposed energies. For a function \( v \) on \( \mathbb{R}/\mathbb{LZ} \), we use notation \( v_i \) to mean \( v(s_i) \). Setting

\[
Q_{1,i} v = \Delta v' = v'_1 - v'_2, \\
Q_{2,i} v = (-1)^{i-1}2\{v'_i - (R_1 f \cdot \tau_i)R_1 v\}, \\
R_1 v = \frac{\|\Delta s\|\Delta v}{\|\Delta f\|_{\mathbb{R}^n}\Delta s}, \\
R_2 v = \frac{1}{2}(v'_1 + v'_2), \\
\Phi_1(x) = 2\Phi(x), \\
\Phi_2(x) = -4\left(\frac{1}{\Phi(x)} - \Lambda(x)\right)^{-1},
\]

we rewrite \( \mathcal{M}_{\Phi,j}(f) \) as follows:

\[
\mathcal{M}_{\Phi,j}(f) = \frac{Q_{j,1}(f) \cdot Q_{j,2}(f)}{\Phi_j(\|\Delta f\|_{\mathbb{R}^n})}.
\]

Furthermore, we set

\[
S_{1,i}(v, w) = R_2 v \cdot Q_{1,i} w + Q_{1,i} v \cdot R_2 w, \\
S_{2,i}(v, w) = R_1 v \cdot Q_{2,i} w + Q_{2,i} v \cdot R_1 w.
\]

We assume that \( \Phi_j \) is differentiable and define

\[
\Xi_j(x) = \frac{x\Phi'_j(x)}{\Phi_j(x)}.
\]

We use \( \mathcal{B}_{\Phi,j} \) and \( \mathcal{H}_{\Phi,j} \) to denote the integrands of the first and second variational formulae of \( \mathcal{M}_{\Phi,j} \) respectively; that is,

\[
\mathcal{B}_{\Phi,j}(f)[\phi] \, ds_1 ds_2 = \delta(\mathcal{M}_{\Phi,j}(f) \, ds_1 ds_2)[\phi], \\
\mathcal{H}_{\Phi,j}(f)[\phi] \, ds_1 ds_2 = \delta^2(\mathcal{M}_{\Phi,j}(f) \, ds_1 ds_2)[\phi, \psi].
\]

**Theorem 3** For any \( \mathcal{L} > 0 \), we suppose the conditions (A.1)–(A.5) hold. If \( \Phi_j \in C^1(0, \infty) \), it holds that

\[
\mathcal{B}_{\Phi,j}(f)[\phi] = \frac{Q_{j,1}f \cdot Q_{j,2}\phi + Q_{j,2}f \cdot Q_{j,1}\phi}{\Phi_j(\|\Delta f\|_{\mathbb{R}^n})} - \frac{\mathcal{M}_{\Phi,j}(f)\Phi'_j(\|\Delta f\|_{\mathbb{R}^n})\|\Delta f\|_{\mathbb{R}^n}^2 \Phi_f(\|\Delta f\|_{\mathbb{R}^n})}{\|\Delta f\|_{\mathbb{R}^n}^2 \Phi_f(\|\Delta f\|_{\mathbb{R}^n})}.
\]

If \( \Phi_j \in C^2(0, +\infty) \), then it holds that

\[
\mathcal{H}_{\Phi,j}(f)[\phi, \psi] = \frac{Q_{j,1}\phi \cdot Q_{j,2}\psi + Q_{j,2}\phi \cdot Q_{j,1}\psi}{\Phi_j(\|\Delta f\|_{\mathbb{R}^n})} - \frac{\mathcal{B}_{\Phi,j}(f)[\phi] \, \Delta f \cdot \Delta \psi}{\|\Delta f\|_{\mathbb{R}^n}} - \frac{\mathcal{B}_{\Phi,j}(f)[\psi] \, \Delta f \cdot \Delta \phi}{\|\Delta f\|_{\mathbb{R}^n}^2}.
\]
\[ - \mathcal{M}_{\Phi,j}(f) \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) \frac{\Delta \psi \cdot \Delta \psi}{\| \Delta f \|^2_{\mathbb{R}^n}} \]

\[ + \mathcal{M}_{\Phi,j}(f) \left( 2 \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) - \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n})^2 - \| \Delta f \|_{\mathbb{R}^n} \mathcal{E}_j'(\| \Delta f \|_{\mathbb{R}^n}) \right) \times \frac{(\Delta f \cdot \Delta \phi)(\Delta f \cdot \Delta \psi)}{\| \Delta f \|^2_{\mathbb{R}^n}}. \]

**Proof** Here, we use the same method to prove this theorem as that used for the Möbius energy in [16]. From

\[ \mathcal{E}_{\Phi,j}(f)[\phi] ds_1 ds_2 \]

\[ = \delta(\mathcal{M}_{\Phi,j}(f) ds_1 ds_2)[\phi] \]

\[ = \delta \left( \frac{Q_{j,1}(f) \cdot Q_{j,2}(f)}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} ds_1 ds_2 \right)[\phi] \]

\[ = \left\{ \delta \left( \frac{Q_{j,1}(f) \cdot Q_{j,2}(f)}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \right)[\phi] \right\} ds_1 ds_2 \]

\[ + \frac{Q_{j,1}(f) \cdot Q_{j,2}(f)}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \delta(ds_1 ds_2)[\phi] \]

\[ = \left\{ \frac{Q_{j,1}(f) \cdot Q_{j,2}(f) + Q_{j,2}(f) \cdot Q_{j,1}(f)}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} - \frac{(Q_{j,1}(f) \cdot Q_{j,2}(f)) \Phi_j'(\| \Delta f \|_{\mathbb{R}^n})}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})^2} \right\} ds_1 ds_2, \]

it holds that

\[ \mathcal{E}_{\Phi,j}(f)[\phi] = \frac{Q_{j,1}(f) \cdot Q_{j,2}(f) + Q_{j,2}(f) \cdot Q_{j,1}(f)}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} - \frac{\mathcal{M}_{\Phi,j}(f) \Phi_j'(\| \Delta f \|_{\mathbb{R}^n})}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})^2} \Delta f \cdot \Delta \phi. \]

We set

\[ \mathcal{P}_j(f)[\phi] = Q_{j,1}(f) \cdot Q_{j,2}(f) + Q_{j,2}(f) \cdot Q_{j,1}(f). \]

Then, we obtain

\[ \mathcal{E}_{\Phi,j}(f)[\phi] = \mathcal{P}_j(f)[\phi] - \frac{\mathcal{M}_{\Phi,j}(f) \Phi_j'(\| \Delta f \|_{\mathbb{R}^n})}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \Delta f \cdot \Delta \phi, \]

using \( \mathcal{P}_j \) and \( \mathcal{E}_j \). The calculation

\[ \delta(\mathcal{E}_{\Phi,j}(f)[\phi])[\psi] \]

\[ = \delta \left( \frac{\mathcal{P}_j(f)[\phi]}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \right)[\psi] + \mathcal{P}_j(f)[\phi] \delta \left( \frac{1}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \right)[\psi] \]

\[ - \left\{ \left[ \delta(\mathcal{M}_{\Phi,j}(f)[\phi])[\psi] \right] \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) + \mathcal{M}_{\Phi,j}(f) \left\{ \delta(\mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n})[\psi]) \right\} \Delta f \cdot \Delta \phi \right\} \frac{\| \Delta f \|^2_{\mathbb{R}^n}}{\| \Delta f \|^2_{\mathbb{R}^n}} \]

\[ - \mathcal{M}_{\Phi,j}(f) \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) \delta \left( \frac{\Delta f \cdot \Delta \phi}{\| \Delta f \|^2_{\mathbb{R}^n}} \right)[\psi] \]

\[ = \frac{1}{\Phi_j(\| \Delta f \|_{\mathbb{R}^n})} \left\{ Q_{j,1}(f) \cdot Q_{j,2}(f) + Q_{j,2}(f) \cdot Q_{j,1}(f) \right\} \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) \]

\[ - \mathcal{P}_j(f)[\phi] \left( \tau_1 \cdot \tau_1 \cdot \psi_2' - \tau_2 \cdot \psi_2' \right) - (S_{j,1}(f, \Phi) S_{j,2}(f, \psi) + S_{j,2}(f, \Phi) S_{j,1}(f, \psi)) \]

\[ - \frac{\mathcal{P}_j(f)[\phi] \mathcal{E}_j(\| \Delta f \|_{\mathbb{R}^n}) \Delta f \cdot \Delta \psi}{\| \Delta f \|^2_{\mathbb{R}^n} \Phi_j(\| \Delta f \|_{\mathbb{R}^n})}. \]
\[- \left\{ \mathcal{g}_{\Phi,j}(f)[\psi] - \mathcal{M}_{\Phi,j}(f) (\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2) \right\} \Xi_j(||\Delta f||_{\mathbb{R}^n}) \frac{\Delta f \cdot \Delta \phi}{||\Delta f||^2_{\mathbb{R}^n}} \]

\[- \mathcal{M}_{\Phi,j}(f) \frac{\Xi_j'(||\Delta f||_{\mathbb{R}^n}) \Delta f \cdot \Delta \psi}{||\Delta f||^2_{\mathbb{R}^n}} \]

\[- \mathcal{M}_{\Phi,j}(f) \Xi_j(||\Delta f||_{\mathbb{R}^n}) \left\{ \frac{\Delta \phi \cdot \Delta \psi}{||\Delta f||_{\mathbb{R}^n}^2} - \frac{2(\Delta \phi \cdot \Delta \phi)(\Delta f \cdot \Delta \psi)}{||\Delta f||_{\mathbb{R}^n}^4} \right\} \]

\[- \mathcal{g}_{\Phi,j}(f)[\psi] (\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2) \]

\[+ \frac{Q_{j,1} \phi \cdot Q_{j,2} \psi + Q_{j,2} \phi \cdot Q_{j,1} \psi - (S_{j,1}(f, \phi)S_{j,2}(f, \psi) + S_{j,2}(f, \phi)S_{j,1}(f, \psi))}{\Phi_j(||\Delta f||_{\mathbb{R}^n})} \]

\[- \mathcal{g}_{\Phi,j}(f)[\psi] \Xi_j(||\Delta f||_{\mathbb{R}^n}) \frac{\Delta f \cdot \Delta \phi}{||\Delta f||^2_{\mathbb{R}^n}} \]

\[- \mathcal{g}_{\Phi,j}(f)[\psi] \Xi_j(||\Delta f||_{\mathbb{R}^n}) \frac{(\Delta f \cdot \Phi)(\Delta f \cdot \Delta \psi)}{||\Delta f||^4_{\mathbb{R}^n}} \]

\[- \mathcal{M}_{\Phi,j}(f) \Xi_j(||\Delta f||_{\mathbb{R}^n}) \left\{ \frac{\Delta \phi \cdot \Delta \psi}{||\Delta f||_{\mathbb{R}^n}^2} - \frac{2(\Delta \phi \cdot \Delta \phi)(\Delta f \cdot \Delta \psi)}{||\Delta f||_{\mathbb{R}^n}^4} \right\} \]

yields the following conclusion:

\[\mathcal{H}_{\Phi,j}(f)[\phi, \psi] \]

\[= \delta \left( \mathcal{g}_{\Phi,j}(f)[\psi] \right) + \mathcal{g}_{\Phi,j}(f)[\psi] (\tau_1 \cdot \psi'_1 + \tau_2 \cdot \psi'_2) \]

\[+ \frac{Q_{j,1} \phi \cdot Q_{j,2} \psi + Q_{j,2} \phi \cdot Q_{j,1} \psi - (S_{j,1}(f, \phi)S_{j,2}(f, \psi) + S_{j,2}(f, \phi)S_{j,1}(f, \psi))}{\Phi_j(||\Delta f||_{\mathbb{R}^n})} \]

\[- \mathcal{g}_{\Phi,j}(f)[\psi] \Xi_j(||\Delta f||_{\mathbb{R}^n}) \frac{\Delta f \cdot \Delta \phi}{||\Delta f||^2_{\mathbb{R}^n}} \]

\[- \mathcal{g}_{\Phi,j}(f)[\psi] \Xi_j(||\Delta f||_{\mathbb{R}^n}) \frac{(\Delta f \cdot \Phi)(\Delta f \cdot \Delta \psi)}{||\Delta f||^4_{\mathbb{R}^n}} \]

\[- \mathcal{M}_{\Phi,j}(f) \Xi_j(||\Delta f||_{\mathbb{R}^n}) \left\{ \frac{\Delta \phi \cdot \Delta \psi}{||\Delta f||_{\mathbb{R}^n}^2} - \frac{2(\Delta \phi \cdot \Delta \phi)(\Delta f \cdot \Delta \psi)}{||\Delta f||_{\mathbb{R}^n}^4} \right\} \]
\[ -\mathcal{M}_{\Phi,j}(f)\|\Delta f\|_2^2\mathcal{E}_j\left(\|\Delta f\|_2^2\right) \frac{(\Delta f \cdot \Phi)(\Delta f \cdot \Delta \psi)}{\|\Delta f\|_2^4} \]

\[ -\mathcal{M}_{\Phi,j}(f)\mathcal{E}_j\left(\|\Delta f\|_2^2\right) \left\{ \frac{\Delta \Phi \cdot \Delta \psi}{\|\Delta f\|_2^2} - \frac{2(\Delta f \cdot \Delta \Phi)(\Delta f \cdot \Delta \psi)}{\|\Delta f\|_2^4} \right\} \]

\[ = O_{j,1}\Phi \cdot Q_{j,2} + Q_{j,2}\Phi \cdot Q_{j,1} - (S_{j,1}(f, \Phi)S_{j,2}(f, \psi) + S_{j,2}(f, \Phi)S_{j,1}(f, \psi)) \]

\[ - \mathcal{B}_{\Phi,j}(f)[\Phi]\mathcal{E}_j\left(\|\Delta f\|_2^2\right) \frac{\Delta \Phi \cdot \Delta \psi}{\|\Delta f\|_2^2} \]

\[ - \mathcal{M}_{\Phi,j}(f)\mathcal{E}_j\left(\|\Delta f\|_2^2\right) \frac{\Delta \Phi \cdot \Delta \psi}{\|\Delta f\|_2^2} \]

\[ + \mathcal{M}_{\Phi,j}(f) \left( 2\mathcal{E}_j\left(\|\Delta f\|_2^2\right) - \mathcal{E}_j\left(\|\Delta f\|_2^2\right)^2 - \|\Delta f\|_2^2\mathcal{E}_j'\left(\|\Delta f\|_2^2\right) \right) \]

\[ \times \frac{(\Delta f \cdot \Delta \Phi)(\Delta f \cdot \Delta \psi)}{\|\Delta f\|_2^4}. \]

\[ \square \]

**Remark 3** \( \mathcal{E}_j \) is a constant when \( \Phi(x) = x^\alpha \); hence, the above equation is simplified.

We have

\[ \delta \mathcal{E}_\Phi(f)[\Phi] = \sum_{i=1}^2 \int \int_{(\mathbb{R}/\mathbb{L})^2} \mathcal{H}_{\Phi,i}(f)[\Phi] \, ds_1 \, ds_2 + \delta \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} \right) [\Phi], \]

\[ \delta^2 \mathcal{E}_\Phi(f)[\Phi, \psi] = \sum_{i=1}^2 \int \int_{(\mathbb{R}/\mathbb{L})^2} \mathcal{H}_{\Phi,i}(f)[\Phi, \psi] \, ds_1 \, ds_2 + \delta^2 \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} \right) [\Phi, \psi]. \]

It is easy to see

\[ \delta \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} \right) [\Phi] = \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} - \frac{\mathcal{L}}{\Phi \left( \frac{\mathcal{L}}{2} \right)} \right) \delta \mathcal{L}(f)[\Phi], \]

\[ \delta^2 \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} \right) [\Phi, \psi] = \left( 2\mathcal{L} \int_\frac{\mathcal{L}}{2}^\infty \frac{dx}{\Phi(x)} - \frac{\mathcal{L}}{\Phi \left( \frac{\mathcal{L}}{2} \right)} \right) \delta^2 \mathcal{L}(\Phi, \psi) \]

\[ + \left( \frac{\mathcal{L} \Phi' \left( \frac{\mathcal{L}}{2} \right)}{2\Phi \left( \frac{\mathcal{L}}{2} \right)^2} - \frac{2}{\Phi \left( \frac{\mathcal{L}}{2} \right)} \right) \delta \mathcal{L}(f)[\Phi] \delta \mathcal{L}(f)[\psi], \]

and the variational formulae for \( \mathcal{L} \) are well-known. Thus we obtain those for \( \mathcal{L}_\Phi. \)

### 5 Sufficient conditions for (A.3) and (A.4)

In this section, we consider the self-repulsiveness of \( W_\Phi \) and the sufficient conditions of \( \Phi \) for assumptions (A.3) and (A.4). We set

(A.6) \( \Phi(x) = O(x^2) \) (\( x \to +0 \)).

**Proposition 2** If conditions (A.1) and (A.6) hold, then \( W_\Phi \) is self-repulsive.
Proof We prove this assertion in the same manner as reported in Ref. [25]. Here, denoting the existence of \( s_\ast \neq s_\dagger \) satisfying \( f(s_\ast) = f(s_\dagger) \), we show that the energy is infinite. As the energy density of \( E_\Phi \) is non-negative,

\[
E_\Phi(f) \geq \int \int_{|s_1 - s_\ast|^2 + |s_2 - s_\ast|^2 \leq \epsilon^2} \left( \frac{1}{\Phi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})} - \frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))} \right) ds_1 ds_2
\]

holds for sufficiently small \( \epsilon > 0 \). There exists a \( \delta > 0 \) that satisfies

\[
\mathcal{D}(f(s_1), f(s_2)) \geq \delta
\]

independent of \( s_1 \) and \( s_2 \), which belong to the interval of integration as \( s_\ast \neq s_\dagger \). Hence,

\[
- \frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))} \geq - \frac{1}{\Phi(\delta)}
\]

follows. As there exists \( C > 0 \) with \( \Phi(x) \leq Cx^2 \) when \( x \in (0, \frac{\epsilon}{2}] \), it holds that

\[
\frac{1}{\Phi(\|f(s_1) - f(s_2)\|_{\mathbb{R}^n})} - \frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))} \geq \frac{1}{C\|f(s_1) - f(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\Phi(\delta)}.
\]

It is sufficient to show that

\[
\int \int_{|s_1 - s_\ast|^2 + |s_2 - s_\ast|^2 \leq \epsilon^2} \frac{ds_1 ds_2}{\|f(s_1) - f(s_2)\|_{\mathbb{R}^n}^2} = \infty
\]

and then \( E_\Phi(f) = \infty \) contradicts the fact. We denote \( s_1 - s_\ast \) and \( s_2 - s_\dagger \) simply by \( s_1 \) and \( s_2 \), respectively. The simple calculation

\[
\|f(s_1 + s_\ast) - f(s_2 + s_\dagger)\|_{\mathbb{R}^n} = \|f(s_1 + s_\ast) - f(s_\ast) + f(s_\ast) - f(s_2 + s_\dagger)\|_{\mathbb{R}^n}
\]

\[
\leq \|f(s_1 + s_\ast) - f(s_\ast)\|_{\mathbb{R}^n} + \|f(s_\ast) - f(s_2 + s_\dagger)\|_{\mathbb{R}^n}
\]

\[
\leq |s_1| + |s_2|
\]

yields

\[
\int \int_{s_1^2 + s_2^2 \leq \epsilon^2} \frac{ds_1 ds_2}{\|f(s_1 + s_\ast) - f(s_2 + \delta)\|_{\mathbb{R}^n}^2} \geq C \int \int_{s_1^2 + s_2^2 \leq \epsilon^2} \frac{ds_1 ds_2}{s_1^2 + s_2^2} = \infty.
\]

\( \square \)

The following three propositions are based on Blatt’s method, as reported in Ref. [3].

Proposition 3 We assume (A.1) and (A.6). If \( f \in W_\Phi \) and \( E_\Phi(f) < \infty \), \( f \) is bi-Lipschitz with regard to the arc-length parameter.

Proof As \( \|\Delta f\|_{\mathbb{R}^n} \leq \mathcal{D}(f(s_1), f(s_2)) \) is obvious for the arc-length parameter, we show the opposite inequality. From \( f \in W_\Phi \), we have

\[
\int_{\mathbb{R}/L\mathbb{Z}} \int \frac{\frac{\ell}{\Phi(|s_2|)} \|\tau(s_1 + s_2) - \tau(s_1)\|_{\mathbb{R}^n}^2}{\Phi(|s_2|)} ds_2 ds_1 < \infty.
\]

For \( r \in (0, \frac{\ell}{2}] \),

\[
\|\tau(s_1 + s_2) - \tau(s_1)\|_{\mathbb{R}^n} \chi_{[-r, r]}(s_2) \leq \frac{\|\tau(s_1 + s_2) - \tau(s_1)\|_{\mathbb{R}^n}^2}{\Phi(|s_2|)}
\]

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holds and, from Lebesgue’s convergence theorem, we obtain
\[
\lim_{r \to 0} \int_{\mathbb{R}/LZ} \int_{-r}^{r} \frac{\|\tau(s_1 + s_2) - \tau(s_1)\|_{\mathbb{R}^n}^2}{\Phi(|s_2|)} ds_2 ds_1 = 0.
\]

Hence, there exists \( \delta \in (0, \min\{1, L\}) \), which satisfies
\[
\sup_{s \in \mathbb{R}/LZ} \int_{s-r}^{s+r} \int_{-r}^{r} \frac{\|\tau(s_1 + s_2) - \tau(s_1)\|_{\mathbb{R}^n}^2}{\Phi(|s_2|)} ds_2 ds_1 \leq \left( \frac{1}{2} \right)^2 \inf_{x \in (0, \frac{L}{2})} \frac{x^2}{\Phi(x)}
\]
for \( r \leq \delta \). Then, we obtain
\[
\frac{1}{2r} \int_{s-r}^{s+r} \frac{\|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}}{\Phi(|s_2|)} ds_1 \leq \frac{1}{4r^2} \int_{s-r}^{s+r} \int_{s-r}^{s+r} \|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n} ds_1 ds_2
\]
\[
\leq \left( \frac{1}{4r^2} \int_{s-r}^{s+r} \int_{s-r}^{s+r} \|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}^2 ds_1 ds_2 \right)^{\frac{1}{2}}.
\]

Using
\[
1 \leq \frac{\Phi(2r)}{\Phi(|s_1 - s_2|)},
\]
we find
\[
\left( \frac{1}{4r^2} \int_{s-r}^{s+r} \int_{s-r}^{s+r} \|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}^2 ds_1 ds_2 \right)^{\frac{1}{2}} \leq \left\{ \frac{\Phi(2r)}{4r^2} \int_{s-r}^{s+r} \int_{s-r}^{s+r} \frac{\|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}^2}{\Phi(|s_1 - s_2|)} ds_1 ds_2 \right\}^{\frac{1}{2}}
\]
\[
\leq \left( \sup_{x \in (0, \frac{L}{2})} \frac{\Phi(x)}{x^2} \right)^{\frac{1}{2}} \left\{ \int_{s-r}^{s+r} \int_{s-r}^{s+r} \frac{\|\tau(s_1) - \tau(s_2)\|_{\mathbb{R}^n}^2}{\Phi(|s_1 - s_2|)} ds_1 ds_2 \right\}^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2},
\]
As \( \left\| \frac{1}{2r} \int_{s-r}^{s+r} \tau(s_2) ds_2 \right\|_{\mathbb{R}^n} \leq 1 \) holds, we have
\[
\inf_{\|v\|_{\mathbb{R}^n} \leq 1} \frac{1}{2r} \int_{s-r}^{s+r} \|\tau(s_1) - v\|_{\mathbb{R}^n} ds_1 \leq \frac{1}{2}.
\]
We consider the case in which \( s_j \in \mathbb{R}/\mathbb{L}\mathbb{Z} \) satisfies \( |s_1 - s_2| = 2r \leq 2\delta \), and suppose that \( s_3 \in \mathbb{R}/\mathbb{L}\mathbb{Z} \) satisfies \( \mathcal{D}(f(s_1), f(s_3)) = \mathcal{D}(f(s_3), f(s_2)) = r \). Then, it holds that
\[
\|f(s_1) - f(s_2)\|_{\mathbb{R}^n} = \left\| \int_{s_3-r}^{s_3+r} \tau(s) \, ds \right\|_{\mathbb{R}^n} = \sup_{\|\tau\|_{\mathbb{R}^n} \leq 1} \int_{s_3-r}^{s_3+r} \tau(s) \cdot f \, ds
\]
\[
= \sup_{\|\tau\|_{\mathbb{R}^n} \leq 1} \int_{s_3-r}^{s_3+r} \tau(s) \cdot (\tau(s) + (\tau(s) - \tau(s))) \, ds
\]
\[
\geq \left( 1 - \inf_{\|\tau\|_{\mathbb{R}^n} = 1} \frac{1}{2r} \int_{s_3-r}^{s_3+r} \|\tau(s) - \tau(s)\|_{\mathbb{R}^n} \, ds \right) |s_1 - s_2|
\]
\[
\geq \frac{1}{2} |s_1 - s_2|.
\]

We set \( I_\delta = \{(s_1, s_2) \in \mathbb{R}/\mathbb{L}\mathbb{Z} \times \mathbb{R}/\mathbb{L}\mathbb{Z} \mid \mathcal{D}(f(s_1), f(s_2)) \geq 2\delta \} \). As \( f \) is a continuous closed curve from the previous proposition, it follows that
\[
\inf_{(s_1, s_2) \in I_\delta} \|f(s_1) - f(s_2)\|_{\mathbb{R}^n} > 0.
\]

\[\square\]

We add the following conditions.

(A.7) \quad \bullet \quad \Phi \in C^1(0, \infty), \Phi(\sqrt{x}) \text{ are convex.}

\[\bullet \quad \text{There exists } C(\mathcal{L}) > 0 \text{ with } \int_{x \in \left(0, \frac{\epsilon}{2}\right]} \frac{x \Phi'(x)}{\Phi(x)} \geq C(\mathcal{L}).\]

\[\bullet \quad \text{There exists a positive constant } C(\mathcal{L}) \text{ and a function } \chi \text{ that satisfy} \]
\[\sup_{x \in \left(0, \frac{\epsilon}{2}\right]} \frac{\Phi(x)}{t \Phi(t^{-1} x)} \leq C(\mathcal{L}) \chi(t) \text{ and } \int_0^\epsilon \chi(t) \, dt = o(\epsilon) \text{ as } \epsilon \to +0.\]

\[\bullet \quad C^\infty(\mathbb{R}/\mathbb{L}\mathbb{Z}) \text{ is dense in } W_\Phi.\]

**Proposition 4** We assume (A.1) and (A.7). If \( \mathcal{E}_\Phi(f) < \infty \), then \( f \in W_\Phi \).

**Proof** It follows that
\[
\mathcal{E}_\Phi(f) = \int_{\mathbb{R}/\mathbb{L}\mathbb{Z}} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{1}{\Phi(|\Delta s|) - \Phi(\|\Delta f\|_{\mathbb{R}^n})} \Phi(\|\Delta f\|_{\mathbb{R}^n}) \, ds_1 \, ds_2
\]
\[
= \int_{\mathbb{R}/\mathbb{L}\mathbb{Z}} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{\Phi(|\Delta s|) - \Phi(\|\Delta f\|_{\mathbb{R}^n})}{\Phi(\|\Delta f\|_{\mathbb{R}^n})} \, ds_1 \, ds_2.
\]

As \( \Phi \) is in \( C^1 \) and as \( \Phi(\sqrt{x}) \) is convex, it follows that
\[
\Phi(|\Delta s|) - \Phi(\|\Delta f\|_{\mathbb{R}^n}) = \int_{\|\Delta s\|^2_{\mathbb{R}^n}}^{\|\Delta s\|^2_{\mathbb{R}^n}} \frac{d}{dx} \Phi(\sqrt{x}) \, dx \geq \int_{\|\Delta f\|^2_{\mathbb{R}^n}}^{\|\Delta s\|^2_{\mathbb{R}^n}} \frac{d}{dx} \Phi(\sqrt{x}) \bigg|_{x=\|\Delta f\|^2_{\mathbb{R}^n}} \, dx
\]
\[
= \frac{\Phi'(\|\Delta f\|_{\mathbb{R}^n})}{2\|\Delta f\|_{\mathbb{R}^n}} \left(\|\Delta s\|^2 - \|\Delta f\|^2_{\mathbb{R}^n}\right)
\]
\[
= \frac{\Phi'(\|\Delta f\|_{\mathbb{R}^n})}{4\|\Delta f\|_{\mathbb{R}^n}} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \|\tau(s_3) - \tau(s_4)\|^2_{\mathbb{R}^n} \, ds_3 \, ds_4.
\]

Consequently, we obtain
\[ E(f) \geq \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\phi(\|\Delta f\|_n)}{\phi(|\Delta s|)} \cdot \left( \int_{s_1}^{s_1} \|\tau(s_3) - \tau(s_4)\|^2_n ds_3 ds_4 ds_1 ds_2 \right) \]

Let \( \epsilon \in (0, \frac{C}{2}) \). Decomposing the interval of integration, we gain

\[
\int \int_{(\mathbb{R}^2) } \frac{\|\Delta \tau\|^2_n}{\phi(|\Delta s|)} ds_1 ds_2
\]

\[
= \int \int_{\mathbb{R}^2} \left( \int_{s_2-\frac{C}{2}}^{s_2} \|\Delta \tau\|^2_n ds_1 ds_2 + \int \int_{s_2}^{s_2 + \frac{C}{2}} \|\Delta \tau\|^2_n ds_1 ds_2 \right)
\]

\[
\leq \frac{2}{\phi(\epsilon)} \int \int_{(\mathbb{R}^2) } (\|\tau(s_1)\|^2_n + \|\tau(s_2)\|^2_n) ds_1 ds_2 = \frac{4C^2}{\phi(\epsilon)}.
\]

We perform the deformation as follows

\[
\int \int_{\mathbb{R}^2} \frac{\|\Delta \tau\|^2_n}{\phi(|\Delta s|)} ds_1 ds_2
\]

\[
\leq \int \int_{\mathbb{R}^2} \frac{\|\tau(s_1) - \tau(s_2)\|^2_n}{\phi(|\Delta s|)} ds_1 ds_2
\]

\[
\leq \frac{C}{\epsilon^2} \int \int_{\mathbb{R}^2} \frac{1}{\phi(|\Delta s|)} \cdot \left( \frac{2}{\epsilon} \int_0^{s_2} \frac{2}{\epsilon} \int_0^{s_2} (\|\tau(s_1 - (\Delta s)t_1) - \tau(s_2 + (\Delta s)t_2)\|^2_n + \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|^2_n + \|\tau(s_2 + (\Delta s)t_2) - \tau(s_2)\|^2_n) dt_1 dt_2 ds_1 ds_2 \right)
\]

Let \( \epsilon > 0 \) be sufficiently small. Changing variables as \( s_3 = s_1 - (\Delta s)t_1 \) and \( s_4 = s_2 + (\Delta s)t_2 \), we obtain

\[
\frac{C}{\epsilon^2} \int \int_{\mathbb{R}^2} \frac{1}{\phi(|\Delta s|)} \cdot \left( \frac{2}{\epsilon} \int_0^{s_2} \frac{2}{\epsilon} \int_0^{s_2} (\|\tau(s_1 - (\Delta s)t_1) - \tau(s_2 + (\Delta s)t_2)\|^2_n dt_1 dt_2 ds_1 ds_2 \right)
\]

\[
\leq \frac{C}{\epsilon^2} \int \int_{\mathbb{R}^2} \frac{1}{\phi(|\Delta s|)} \cdot \left( \frac{2}{\epsilon} \int_0^{s_2} \frac{2}{\epsilon} \int_0^{s_2} (\|\tau(s_3) - \tau(s_4)\|^2_n ds_3 ds_4 ds_1 ds_2 \right)
\]

\[
\leq \frac{C}{\epsilon^2} \int \int_{(\mathbb{R}^2) } \frac{1}{\phi(|\Delta s|)} \cdot \left( \frac{2}{\epsilon} \int_0^{s_2} \frac{2}{\epsilon} \int_0^{s_2} (\|\tau(s_3) - \tau(s_4)\|^2_n ds_3 ds_4 ds_1 ds_2 \right)
\]

A simple inequality

\[
\inf_{x \in (0, \frac{C}{2})} \frac{\phi'(x)}{\phi(x)} \geq C(L) > 0
\]
yields
\[
\frac{C}{\epsilon^2} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
= \frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
= \frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2.
\]

Then, it holds that
\[
\frac{C}{\epsilon^2} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2.
\]

As there exists \( \chi \) with
\[
\sup_{x \in (0, \frac{\epsilon}{2})} \frac{\Phi(x)}{\Phi(t - 1) x} \leq C(\mathcal{L}) \chi(t) < \infty, \quad \int_0^{\frac{\tau}{\epsilon}} \chi(t) dt = o(\epsilon) \quad (\epsilon \to +0),
\]
for sufficiently small \( t > 0 \), it holds that
\[
\frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_0^{\frac{\tau}{\epsilon}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1) - \tau(s_1 - (\Delta s)t_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2.
\]

In a similar manner, we have
\[
\frac{C}{\epsilon^2} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1 + (\Delta s)t_1) - \tau(s_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2 \\
\leq \frac{C(\mathcal{L})}{\epsilon} \int_{\mathbb{R}/\mathbb{L}} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{1}{\Phi(\|\Delta s\|)} \int_0^{\frac{\tau}{\epsilon}} \int_0^{\frac{\tau}{\epsilon}} \|\tau(s_1 + (\Delta s)t_1) - \tau(s_1)\|_{\mathbb{R}_n}^2 ds_1 dt_1 ds_2.
\]
and, hence,
\[
\left(1 - \frac{C(L)}{\epsilon} \int_0^{\frac{2}{\epsilon}} \chi(t) \, dt \right) \int_{\mathbb{R}/LZ^2} \frac{\| \Delta \tau \|^2_{\mathbb{R}^n}}{\Phi(|\Delta s|)} \, ds_1 \, ds_2 \leq C(L) \left( \frac{E(f)}{\epsilon^2} + \frac{1}{\Phi(\epsilon)} \right)
\]
is obtained if
\[
\int_{\mathbb{R}^2} \int_{s_2 - \epsilon}^{s_2 + \epsilon} \frac{\| \Delta \tau \|^2_{\mathbb{R}^n}}{\Phi(|\Delta s|)} \, ds_1 \, ds_2 < \infty.
\]
As a consequence, if \( f \) is smooth and \( E(f) < \infty \), then, by taking \( \epsilon \) sufficiently small, it holds that \( f \in W_\Phi \). As \( C^\infty(\mathbb{R}/L) \) is assumed to be dense in \( W_\Phi \), \( f \in W_\Phi \) follows from \( E_\Phi(f) < \infty \).

It is simple to show that \( \Phi(x) = x^\alpha (\alpha \in [2, 3)) \) satisfies (A.1), (A.6), and (A.7). Then, we have the following corollary.

**Corollary 2** We suppose that (A.1), (A.6), and (A.7) hold. Then, we have (A.3). In particular, (A.3) holds when \( \Phi(x) = x^\alpha (\alpha \in [2, 3)) \).

We assume that

(A.8) \[ \Phi \in C^1(0, \infty), \quad \frac{1}{\Phi(\sqrt{x})} \text{ is convex.} \]

\[ \bullet \text{ There exists } C(L) > 0 \text{ with } \sup_{x \in (0, \frac{L}{2}]} \frac{\Phi'(x)}{x} < C(L). \]

**Proposition 5** Supposing that (A.1) and (A.8) hold, we have \( E_\Phi(f) < \infty \) if \( f \in W_\Phi \).

**Proof** It holds that
\[
E_\Phi(f) = \int_{\mathbb{R}/LZ^2} \frac{d}{d x} \frac{1}{\Phi(\sqrt{x})} \, dx \, ds_1 \, ds_2 = \int_{\mathbb{R}/LZ^2} \frac{\mathcal{D}(f(s_1), f(s_2))^2}{\| \Delta f \|^2_{\mathbb{R}^n}} \, d x \, ds_1 \, ds_2.
\]

From the assumption of \( \Phi \),
\[
\frac{d}{d x} \frac{1}{\Phi(\sqrt{x})} \bigg|_{x = \| \Delta f \|^2_{\mathbb{R}^n}} \leq \frac{d}{d x} \frac{1}{\Phi(\sqrt{x})} \leq 0
\]
holds when \( x \in \left[ \| \Delta f \|^2_{\mathbb{R}^n}, \| \mathcal{D}(f(s_1), f(s_2)) \|^2_{\mathbb{R}^n} \right] \) and
\[
E_\Phi(f) \leq \int_{\mathbb{R}/LZ^2} \frac{d}{d x} \frac{1}{\Phi(\sqrt{x})} \bigg|_{x = \| \Delta f \|^2_{\mathbb{R}^n}} \, dx \, ds_1 \, ds_2 = \frac{1}{2} \int_{\mathbb{R}/LZ^2} \frac{\Phi'(\| \Delta f \|^2_{\mathbb{R}^n})}{\Phi(\| \Delta f \|^2_{\mathbb{R}^n})} \left( \| \mathcal{D}(f(s_1), f(s_2)) \|^2_{\mathbb{R}^n} - \| \Delta f \|^2_{\mathbb{R}^n} \right) \, ds_1 \, ds_2
\]
\[
= \frac{1}{4} \int_{\mathbb{R}/LZ^2} \frac{\Phi'(\| \Delta f \|^2_{\mathbb{R}^n})}{\Phi(\| \Delta f \|^2_{\mathbb{R}^n})} \left( \int_{s_1}^{s_2} \|	au(s_3) - \tau(s_4)\|^2_{\mathbb{R}^n} \, ds_3 \, ds_4 \right) \, ds_1 \, ds_2
\]

\( \Box \) Springer
holds. From \(|s_3 - s_4| \leq |\Delta s|\), we have
\[
\mathcal{E}_\Phi(f) \leq \frac{1}{4} \int \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{\Phi'(\|\Delta f\|_{\mathbb{R}^n})}{\|\Delta f\|_{\mathbb{R}^n}} \left( \int_{s_1}^{s_2} \int_{s_1}^{s_2} \frac{\|\tau(s_3) - \tau(s_4)\|_{\mathbb{R}^n}^2}{\Phi(|s_3 - s_4|)} ds_3 ds_4 \right) ds_1 ds_2.
\]

If
\[
\sup_{x \in (0, \frac{\mathcal{L}}{x})} \frac{\Phi'(x)}{x} \leq C(\mathcal{L})
\]
holds, then it is shown that
\[
\mathcal{E}_\Phi(f) \leq C(\mathcal{L})^2 \int \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{\|\tau(s_3) - \tau(s_4)\|_{\mathbb{R}^n}^2}{\Phi(|s_3 - s_4|)} ds_3 ds_4.
\]

Lastly, we provide a sufficient condition of \(\Phi\) for (A.4).

(A.9) For any \(\lambda \in (0, 1)\), we assume that
\[
\limsup_{\epsilon \to +0} \epsilon^2 \sup_{x \in [\lambda^2 \epsilon^2, \epsilon^2]} \frac{d}{dx} \Lambda(\sqrt{x}) \sup_{y \in [0, \epsilon]} \Phi(y) < \infty.
\]

**Proposition 6** We suppose that (A.1), (A.2), and (A.9). If \(f \in W_\Phi\) and \(f\) is bi-Lipschitz, then the \((*)\) of (A.4) holds.

**Proof** As
\[
\Lambda(x) - \Lambda(y) = \int_{y^2}^{x^2} \frac{d}{dt} \Lambda(\sqrt{t}) dt
\]
follows for \(x \geq y > 0\), it holds that
\[
|\Lambda(x) - \Lambda(y)| = \sup_{t \in [y^2, x^2]} \left| \frac{d}{dt} \Lambda(\sqrt{t}) \right| |x^2 - y^2|.
\]

From the bi-Lipschitz estimate, there exists a positive constant \(\lambda \in (0, 1)\) independent of \(s_1\) and \(s_2\), and it holds that
\[
\lambda \mathcal{D}(f(s_1), f(s_2)) \leq \|\Delta f\|_{\mathbb{R}^n} \leq \mathcal{D}(f(s_1), f(s_2)).
\]

From
\[
\lambda^2 \epsilon^2 \leq \|f(s_1 + \epsilon) - f(s_1)\|_{\mathbb{R}^n}^2 \leq \epsilon^2,
\]

\[
|\Lambda(\|f(s_1 + \epsilon) - f(s_1)\|_{\mathbb{R}^n}) - \Lambda(\epsilon)|
\]
\[
\leq \sup_{x \in [\lambda^2 \epsilon^2, \epsilon^2]} \left| \frac{d}{dx} \Lambda(\sqrt{x}) \right| \left( \epsilon^2 - \|f(s_1 + \epsilon) - f(s_1)\|_{\mathbb{R}^n}^2 \right)
\]
\[
= \sup_{x \in [\lambda^2 \epsilon^2, \epsilon^2]} \left| \frac{d}{dx} \Lambda(\sqrt{x}) \right| \left( \int_{s_1}^{s_1+\epsilon} \int_{s_1}^{s_1+\epsilon} (1 - \|\tau(s_3) - \tau(s_4)\|_{\mathbb{R}^n}^2) ds_3 ds_4 \right)
\]
\[
= \frac{1}{2} \sup_{x \in [\lambda^2 \epsilon^2, \epsilon^2]} \left| \frac{d}{dx} \Lambda(\sqrt{x}) \right| \left( \int_{s_1}^{s_1+\epsilon} \int_{s_1}^{s_1+\epsilon} \|\tau(s_3) - \tau(s_4)\|_{\mathbb{R}^n}^2 ds_3 ds_4 \right)
\]
\[
\tag{\text{(*)}}
\]
holds and, hence, we have

\[\epsilon \int_{R/L^2} (\Lambda(\|\mathbf{f}(s_1) - \mathbf{f}(s_1 + \epsilon)\|_{R^n}) - \Lambda(\epsilon)) \, ds_1 \]

\[\leq \frac{\epsilon}{2} \sup_{x \in [\lambda^2e^2, e^2]} \left| \frac{d}{dx} \Lambda(\sqrt{x}) \right| \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \int_{s_1}^{s_1 + \epsilon} \left\| \mathbf{r}(s_3) - \mathbf{r}(s_4) \right\|^2_{R^n} \, ds_3 \, ds_4 \, ds_1.
\]

Changing the order of integration, we have

\[\epsilon \int_{R/L^2} (\Lambda(\|\mathbf{f}(s_1) - \mathbf{f}(s_1 + \epsilon)\|_{R^n}) - \Lambda(\epsilon)) \, ds_1 \]

\[= \frac{\epsilon}{2} \sup_{x \in [\lambda^2e^2, e^2]} \left| \frac{d}{dx} \Lambda(\sqrt{x}) \right| \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \int_{s_1}^{s_1 + \epsilon} \left\| \mathbf{r}(s_3) - \mathbf{r}(s_4) \right\|^2_{R^n} \, ds_3 \, ds_4 \, ds_1.
\]

From the absolute integrability of integration,

\[\lim_{\epsilon \to +0} \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \left\| \mathbf{r}(s_3) - \mathbf{r}(s_4) \right\|^2_{R^n} \, ds_3 \, ds_4 = 0.
\]

(A.10) For any \(\lambda \in (0, 1)\), we assume that \(\limsup_{\epsilon \to +0} \sup_{x \in (0, \epsilon]} \sup_{y \in (\lambda \epsilon, \epsilon]} |\Lambda(y)| < \infty\).

**Proposition 7** We suppose that (A.1), (A.2), and (A.10) hold. If \(f \in W_\phi\) and \(f\) is bi-Lipschitz, then the \((\dagger)\) of (A.4) holds.

**Proof** There exists \(\lambda \epsilon \leq \|f(s_1 + \epsilon, s_1) - f(s_1)\|_{R^n} \leq \epsilon\) from the bi-Lipschitz estimate. Therefore, we have

\[\int_{R/L^2} |\Lambda(\|f(s_1 + \epsilon) - f(s_1)\|_{R^n})| \int_{s_1}^{s_1 + \epsilon} \left\| \mathbf{r}(s_1) - \mathbf{r}(s_3) \right\|^2_{R^n} \, ds_3 \, ds_1 \]

\[\leq \sup_{x \in (0, \epsilon]} \Phi(x) \sup_{y \in (\lambda \epsilon, \epsilon]} |\Lambda(y)| \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \frac{\left\| \mathbf{r}(s_1) - \mathbf{r}(s_3) \right\|^2_{R^n}}{|s_3 - s_1|} \, ds_3 \, ds_1 \]

\[\leq C(\lambda) \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \frac{\left\| \mathbf{r}(s_1) - \mathbf{r}(s_3) \right\|^2_{R^n}}{|s_1 - s_3|} \, ds_3 \, ds_1.
\]

If \(f \in W_\phi\), then

\[\lim_{\epsilon \to +0} \int_{R/L^2} \int_{s_1}^{s_1 + \epsilon} \frac{\left\| \mathbf{r}(s_1) - \mathbf{r}(s_3) \right\|^2_{R^n}}{|s_1 - s_3|} \, ds_3 \, ds_1 = 0
\]

holds from the absolute continuity of integration. \qed
It is easily shown that $\Phi(x) = x^\alpha$ ($\alpha \in [2, 3]$) satisfies (A.1), (A.2), (A.10), and (A.11); therefore, we have the next corollary.

**Corollary 3** We suppose that (A.1), (A.2), (A.10), and (A.11) hold; then, (A.4) holds. In particular, (A.4) holds when $\Phi(x) = x^\alpha$ ($\alpha \in [2, 3]$).

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