LOCAL MIRROR SYMMETRY FOR THE
TOPOLOGICAL VERTEX

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Abstract. For three-partition triple Hodge integrals related to
the topological vertex, we derive Eynard-Orantin type recursion
relations from the cut-and-join equation. This establishes a version
of local mirror symmetry for the local $\mathbb{C}^3$ geometry with three $D$-
branes, as proposed by Mariño [14] and Bouchard-Klemm-Mariño-
Pasquetti [4].

1. Introduction

This is a sequel to an earlier paper [15] in which we study a version
of local mirror symmetry for one-legged framed topological vertex. In
this paper we generalize the results to the general topological vertex
[1]. The motivations of this work and also that of [15] can be found
in the Introduction of that paper, so we will only briefly explain them
here. Physically the topological vertex is a formalism that computes
partition functions in the local A-theory of certain noncompact Calabi-
Yau 3-folds with D-branes, based on duality with Chern-Simons theory
developed in a series of work. A mathematical theory of the topologi-
cal vertex [13] has been developed, based on some earlier work on Hodge
integrals and localizations on relative moduli spaces. Recently a new
formalism for the local B-theory on the mirror of toric Calabi-Yau
threefolds has been proposed in [14] [4], inspired by the recursion pro-
cedure of [8] discovered first in the context of matrix models. It is then
very interesting to verify local mirror symmetry in arbitrary genera
using this new formalism of the local B-theory, and this has been done in
many cases in [14] and [4]. For the simplest case which is the one-legged
framed topological vertex, Bouchard and Mariño [5] made a conjecture
based on the proposed new formalism of the B-theory in [14] [4]. They
also made a similar conjecture for Hurwitz numbers, which has recently
been proved by Borot-Eynard-Safnuk-Mulase [3] and Eynard-Safnuk-
Mulase [7] by two different methods. More recently, two slightly dif-
ferent proofs for the Bouchard-Mariño Conjecture for the one-legged
framed topological vertex have appeared [6] [15]. They both use ideas
from [7]. In this paper we will make a generalization of [15] to study the case of general topological vertex. The following is our main result:

**Theorem 1.** The three-partition triple Hodge integrals related to the topological vertex satisfy some Eynard-Orantin type recursion relations, as proposed by Mariño [14] and Bouchard-Klemm-Mariño-Pasquetti [4].

Indeed, such recursion relations can be derived from the cut-and-join equation satisfied by these Hodge integrals by the same method as in [15].

The rest of this paper is arranged as follows. In §2 we elaborate on some constructions in the study of the local mirror symmetries of $\mathbb{C}^3$. For example, we show that some Lagrangian submanifolds correspond to complex submanifolds in the mirror manifold, but some do not. In §3 we derive some recursion relations for the three-partition triple Hodge integrals related to the topological vertex using a cut-and-join equation they satisfy. Finally in §4, we reformulate the recursion relations in terms of Eynard-Orantin type recursions following the proposal of Mariño [14] and Bouchard-Klemm-Mariño-Pasquetti [4].

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2. The Mirror Geometry of $\mathbb{C}^3$

For general cases of local mirror constructions, see [14, 10, 2, 4]. In this section we will go through some details of the constructions for $\mathbb{C}^3$.

2.1. T-duality on flat 3-tori. Let $V$ be a Euclidean space of dimension 3, and let $\Gamma \subset V$ be a lattice. Then $V/\Gamma$ is a flat 3-torus. Let $V^*$ be the dual space of $V$, endowed with the dual Euclidean metric. Let $\Gamma^* \subset V^*$ be the normalized dual lattice, i.e.,

$$\Gamma^* = \{ \varphi \in V^* | \frac{1}{4\pi^2} \langle \varphi, v \rangle \in \mathbb{Z}, \forall v \in \Gamma \}. \quad (1)$$

The torus $V^*/\Gamma^*$ will be referred to as the T-dual of $V/\Gamma$. Now because $(V^*)^* \cong V$ and $(\Gamma^*)^* \cong \Gamma$, the T-dual of $V^*/\Gamma^*$ is $V/\Gamma$.

**Example 2.1.** Let $V$ be $\mathbb{R}^3$ with the standard metric, $\Gamma$ be the lattice generated by $(2\pi r_1, 0, 0)$, $(0, 2\pi r_2, 0)$, and $(0, 0, 2\pi r_3)$. Then $V/\Gamma$ is a Riemannian product of three copies of circles, with radii $r_1$, $r_2$ and $2\pi r_3$, respectively. The dual space $V^*$ is still $\mathbb{R}^3$ with standard metric, $\Gamma^*$ is now the lattice generated by $(2\pi/r_1, 0, 0)$, $(0, 2\pi/r_2, 0)$, and $(0, 0, 2\pi/r_3)$, and $V^*/\Gamma^*$ is a Riemannian product of three copies of circles, with radii $1/r_1$, $1/r_2$ and $1/r_3$, respectively.
Let $\Gamma_1 \subset \Gamma$ be a sublattice, which spans a linear subspace $V_1 \subset V$, then $V_1/\Gamma_1$ is a subtorus of $V/\Gamma$. One can get a family of 2-tori by translating the subspace $V_1$ by vectors in $V$: Given any $v \in V_1$, each affine subspace $v + V_1$ determines a subtorus in $V/\Gamma$. Let $\Gamma_1^+ \subset \Gamma^*$ be the sublattice of $\Gamma^*$, orthogonal to $\Gamma_1$, i.e.,
\begin{equation}
\Gamma_1^+ = \{ \varphi \in \Gamma^* | \varphi(v) = 0, \forall v \in \Gamma_1 \}.
\end{equation}

It is easy to see that the linear subspace spanned by $\Gamma_1^+$ in $\Gamma^*$ is the subspace $V_1^+$, which is the orthogonal complement of $V_1$ in $V^*$. The subtorus $V_1^+/\Gamma_1^+$ will be referred as the $T$-dual of $V_1/\Gamma_1 \subset V/\Gamma$. We can also obtain from it a family of subtori by translating $V_1^+$ by vectors in $V^*$.

Note $(\Gamma_1^+)^\perp = \Gamma_1$ and $(V_1^+)^\perp = V_1$, the T-dual of $V_1^+ / \Gamma_1^+ \subset V^*/\Gamma^*$ is $V_1/\Gamma_1$.

**Example 2.2.** Let $V$ and $\Gamma$ be as in Example 2.1. Let $\Gamma_1$ be the sublattice generated by $(2\pi r_1, 4\pi r_2, 0)$. Then $\Gamma_1^+$ is the sublattice generated by $(-4\pi/r_2, 2\pi/r_1, 0)$ and $(0, 0, 2\pi/r_3)$. In this case $V_1/\Gamma_1$ is a circle, and $V_1^+/\Gamma_1^+$ is a 2-torus.

### 2.2. A degenerate torus fibration of $\mathbb{C}^3$

Consider the following natural torus action on $U(1)^3 \times \mathbb{C}^3 \to \mathbb{C}^3$:
\begin{equation}
(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \cdot (z_1, z_2, z_3) = (e^{i\alpha_1}z_1, e^{i\alpha_2}z_2, e^{i\alpha_3}z_3).
\end{equation}

This is a Hamiltonian action with moment map given by:
\begin{equation}
\mu : \mathbb{C}^3 \to \mathbb{R}^3_{\geq 0}, \quad \mu(z_1, z_2, z_3) = \frac{1}{2}(|z_1|^2, |z_2|^2, |z_3|^2),
\end{equation}

where $\mathbb{R}^3_{\geq 0}$ is the half line $\{x \in \mathbb{R} | x \geq 0\}$. The inverse image of $\mu$ is generically a flat 3-torus, but along the boundary of $\mathbb{R}^3_{\geq 0}$, it may degenerate to a 2-torus, a circle or a point. Let $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$. We will focus on $\mu : (\mathbb{C}^*)^3 \to \mathbb{R}^3_+$, where $\mathbb{C}^*$ is the set of nonzero complex numbers. This is a fibration with flat 3-tori as fibers. By adding the degenerate fibers corresponding to the points on $\partial \mathbb{R}^3_{\geq 0}$, one gets a partial compactification: The union of the degenerate fibers form a divisor with normal crossing in $\mathbb{C}^3$:

$$\{(z_1, z_2, 0) | z_1, z_2 \in \mathbb{C}\} \cup \{(z_1, 0, z_3) | z_1, z_3 \in \mathbb{C}\} \cup \{(0, z_2, z_3) | z_2, z_3 \in \mathbb{C}\}.$$

On $(\mathbb{C}^*)^3$ one can use the polar coordinates on each copies of $\mathbb{C}^*$, then the Euclidean metric becomes
\begin{equation}
g = \sum_{j=1}^{3} (dr_j^2 + r_j^2 d\theta_j^2).
\end{equation}
2.3. T-duality of the torus fibration. Note we have a fibration of flat 3-tori, so we can apply the T-duality to each fiber. More precisely, one can define a new metric on \((\mathbb{C}^*)^3\):

\[
g^\vee = \sum_{j=1}^{3} (dr_j^2 + \frac{1}{r_j^2}d\theta_j^2).\]

In this metric, the 3-tori in the fibers becomes their dual tori. This metric is no longer Hermitian with respect to the original complex structure. For the metric \(\hat{g}\) to be a Hermitian metric, the dual almost complex structure is taken to be:

\[
J^\vee \left( \frac{\partial}{\partial r_j} \right) = r_j \frac{\partial}{\partial \theta_j}, \quad J^\vee \left( \frac{\partial}{\partial \theta_j} \right) = -\frac{1}{r_j} \frac{\partial}{\partial r_j}.
\]

On differential 1-forms, the almost complex structure acts by:

\[
\begin{align*}
&dr_j \mapsto -\frac{1}{r_j}d\theta_j, & d\theta_j \mapsto r_j dr_j.
\end{align*}
\]

Note the action of the original almost complex structure acts as follows on tangent vectors and 1-forms:

\[
\begin{align*}
&\frac{\partial}{\partial r_j} \mapsto \frac{1}{r_j} \frac{\partial}{\partial \theta_j}, & \frac{\partial}{\partial \theta_j} \mapsto -r_i \frac{\partial}{\partial r_j},
&dr_j \mapsto -r_j d\theta_j, & d\theta_j \mapsto \frac{1}{r_j} dr_j.
\end{align*}
\]

In the dual almost complex structure, \(\frac{1}{2}r_j^2 + \sqrt{-1}\theta_j i\) are new complex local coordinates, because \(r_j dr_j + \sqrt{-1}d\theta_j\) are now of type \((1, 0)\). Because \(\theta_j\) is multiple valued, we take

\[
y_j = \exp\left( -\left( \frac{1}{2}r_j^2 + \sqrt{-1}\theta_j \right) \right).
\]

Denote by \(X^\vee\) the space \((\mathbb{C}^*)^3\) with the dual complex coordinates \(\{y_1, y_2, y_3\}\). Note because \(|y_j| < 1\), \(X^\vee\) is no longer a complex 3-torus \((\mathbb{C}^*)^3\), but instead a product of three copies of punctured open unit disc

\[
\{y \in \mathbb{C} \mid 0 < |y| < 1\}.
\]

We will denote this dual complex space by \(X^\vee\). In the dual complex coordinates the metric \(\hat{g}\) can be written as:

\[
\hat{g} = -\sum_{j=1}^{3} \frac{|dy_j|^2}{|y_j|^2 \cdot \ln |y_j|^2}.
\]
Note this is a complete metric, but no longer Ricci-flat. The dual symplectic form is given by
\begin{equation}
\omega^\vee = - \sum_{j=1}^{3} \frac{1}{r_j} dr_j \wedge d\theta_j = -\sqrt{-1} \sum_{j=1}^{3} \frac{dy_j \wedge d\bar{y}_j}{|y_j|^2 \cdot \ln |y_j|^2}.
\end{equation}

Now the 3-torus action (3) is given in polar coordinates by
\begin{equation}
r_j \mapsto r_j, \quad \theta_j \mapsto \theta_j + \alpha_j.
\end{equation}
Therefore, it induces the following action on the mirror manifold:
\begin{equation}
y_j \mapsto e^{i\alpha_j} y_j.
\end{equation}
This is again a Hamiltonian action with moment map \( \mu^\vee : X^\vee \to \mathbb{R}^3 \) given by:
\begin{equation}
\mu^\vee(y_1, y_2, y_3) = (\ln \ln |y_1|^2, \ln \ln |y_2|^2, \ln \ln |y_3|^2).
\end{equation}

2.4. Lagrangian submanifolds of Aganagic-Vafa type. We recall some Lagrangian submanifolds of \( \mathbb{C}^3 \) constructed by Aganagic and Vafa [2]. Let
\begin{align}
m_1\mu_1 + m_2\mu_2 + m_3\mu_3 &= \alpha_1, \\
n_1\mu_1 + n_2\mu_2 + n_3\mu_3 &= \alpha_2
\end{align}
be two planes in \( \mathbb{R}^3_+ \) that intersects along a ray \( l \) in \( \mathbb{R}^3 \). We assume that \( m_i, n_i \) are integers. This ray can also be described by:
\begin{equation}
(\mu_1, \mu_2, \mu_3) = (a_1, a_2, a_3) + t_1(k_1, k_2, k_3),
\end{equation}
for some integers \( k_1, k_2, k_3 \) and nonnegative integers \( a_1, a_2, a_3, t_2 \). For \( b_1, b_2, b_3 \in \mathbb{R} \),
\begin{equation}
\theta_i = m_i s + n_i t + b_i, \quad i = 1, 2, 3, \quad s, t \in \mathbb{R}
\end{equation}
determines a subtorus \( K \) of the 3-torus. It is clear that \( l \times K \) is a Lagrangian submanifold in \( \mathbb{C}^3 \).

**Example 2.3.** Consider the following complex conjugate involution on \( (\mathbb{C}^*)^3 \):
\begin{equation}
(z_1, z_2, z_3) \mapsto \left(\frac{a^2}{\bar{z}_1}, \bar{z}_3, \bar{z}_2\right),
\end{equation}
for some \( a > 0 \). The fixed point set is a Lagrangian submanifold given by
\begin{equation}
\{(ae^{i\theta_1}, r_2e^{i\theta_2}, r_2e^{-i\theta_2}) \mid \theta_1, \theta_2 \in \mathbb{R}, \  r_2 \in \mathbb{R}_+\}.
\end{equation}
This is a Lagrangian submanifold of Aganagic-Vafa type which corresponds to the following ray in $\mathbb{R}^3_+$:
\[
\left( \frac{1}{2}a^2, t, t \right), \quad t > 0.
\]
A Lagrangian submanifold will be said to be in phase I, phase II or phase III if it corresponds to a ray of the form $(a, t, t)$, $(t, a, t)$ or $(t, t, a)$ for fixed $a > 0$, respectively.

**Example 2.4.** One can also consider the following complex conjugate involution [12]:
\[
(z_1, z_2, z_3) \mapsto \left( \frac{1}{z_1}, \bar{z}_1 \bar{z}_3, \bar{z}_1 \bar{z}_2 \right).
\]
The fixed point set is a Lagrangian submanifold given by
\[
\{(e^{i\theta_1}, r_2 e^{i\theta_2}, r_2 e^{-i(\theta_1 + \theta_2)}) \mid \theta_1, \theta_2 \in \mathbb{R}, \quad r_2 \in \mathbb{R}_+ \}\.
\]
This is not of Aganagic-Vafa type.

### 2.5. Mirror $B$-branes of Lagrangian submanifold of Aganagic-Vafa type

One can apply the T-duality to each of the fibers in a Lagrangian submanifold of Aganagic-Vafa type, because they are subtori of the 3-tori. It was claimed in [2] that this will yield a complex submanifold. We now verify this claim by the discussion of the dual almost complex structure presented in §2.3. Take a dual subtorus $K$ of dimension 2 in $T^3$ as in (20). Its dual can be taken a circle given in the theta coordinates by
\[
\begin{align*}
  m_1 \theta_1 + m_2 \theta_2 + m_3 \theta_3 &= \beta_1, \\
  n_1 \theta_1 + n_2 \theta_2 + n_3 \theta_3 &= \beta_2,
\end{align*}
\]
or equivalently
\[
(\theta_1, \theta_2, \theta_3) = (c_1, c_2, c_3) + t_2(k_1, k_2, k_3).
\]
Combined with (20), one sees that the tangent space is given by the linear space of the following two vectors:
\[
\sum_{j=1}^{3} k_j \frac{\partial}{\partial r_j}, \quad \sum_{j=1}^{3} k_j \frac{\partial}{\partial \theta_j}
\]
This is $J^\vee$-invariant, therefore, we have proved the following

**Theorem 2.1.** The mirror dual $l \times K^\vee$ of a Lagrangian submanifold $l \times K$ of Aganagic type is a complex submanifold of $X^\vee$.

**Remark 2.1.** It is straightforward to extend this result to the case of $\mathbb{C}^n$ for arbitrary $n$ and substorus of any codimension.
In our case, \( l \times K^\vee \) can be explicitly given in the \( y_j \)-coordinates. By (20) and (27):

\[
y_j = e^{-(a_j + i\beta_j)} e^{-k_j (t_1 + it_2)},
\]

where \( t_1, t_2 > 0 \), or by (17), (18), (25), (26), we have

\[
y_{1m} y_{2m} y_{3m} = e^{-(\alpha_1 + i\beta_1)},
\]

\[
y_{1n} y_{2n} y_{3n} = e^{-(\alpha_2 + i\beta_2)}.
\]

**Example 2.5.** The mirror dual of the Lagrangian submanifold in Example 2.3 is given by:

\[
y_1 = e^{-(a_2/2 + c_1 i)}, \quad y_2 = e^{-c_2 i} e^{-(t_1 + it_2)}, \quad y_3 = e^{-c_3 i} e^{-(t_1 + it_2)}.
\]

**Example 2.6.** The mirror dual of the Lagrangian submanifold in Example 2.4 is given by

\[
\begin{align*}
r_1 &= 1, \\
r_2 &= r_3 = r, \\
\theta_1 &= \theta_2 = \theta.
\end{align*}
\]

Therefore, its tangent space is spanned by the following vectors;

\[
\frac{\partial}{\partial r_2} + \frac{\partial}{\partial r_3}, \quad \sum_{j=1}^{3} \frac{\partial}{\partial \theta_j}
\]

This is not \( J^\vee \)-invariant, and so \( l \times K^\vee \) is not a complex submanifold of \( X^\vee \).

**2.6. The mirror curve of \( \mathbb{C}^3 \).** The mirror geometry of \( \mathbb{C}^3 \) according to the construction of Hori-Iqbal-Vafa [10] is not the mirror manifold \( X^\vee \) above, but instead the subspace of \( \mathbb{C}^2 \times (\mathbb{C}^*)^3 \) defined by the following equation

\[
uv = y_1 + y_2 + y_3
\]

where \( u, v \in \mathbb{C}, y_1, y_2 \in \mathbb{C}^* \), modulo the following action by \( \mathbb{C}^* \):

\[
t \cdot (u, v, y_1, y_2, y_3) \mapsto (tu, v, ty_1, ty_2, ty_3).
\]

Denote this space by \( \tilde{X} \), and by \( [u, v, y_1, y_2, y_3] \) the equivalence class of a point \( (u, v, y_1, y_2, y_3) \) in this space. There is a natural projection map \( \pi : \tilde{X} \to \mathbb{P}^2 \) defined by

\[
\pi([u, v, y_1, y_2, y_3]) = [y_1 : y_2 : y_3] \in \mathbb{P}^2.
\]

The base space of this projection is the projectivized \( X^\vee \):

\[
\{[y_1 : y_2 : y_3] \in \mathbb{P}^2 \mid y_1, y_2, y_3 \in (\mathbb{C}^*)^2 \},
\]

and the fiber \( \pi^{-1}[y_1, y_2, y_3] \) is the space

\[
\{(\tilde{u}, v) \in \mathbb{C}^2 \mid \tilde{u}v = \frac{y_1}{y_3} + \frac{y_2}{y_3} + 1\}.
\]
This is a copy of $\mathbb{C}^*$ when the RHS does not vanish, but it becomes a copy of normal crossing singular set $\{(\tilde{u}, v) \in \mathbb{C}^2 | \tilde{u}v = 0\}$ when

$$y_1 + y_2 + y_3 = 0.$$  

One can embed $(\mathbb{C}^*)^3$ in this space as follows:

$$ (y_1, y_2, y_3) \mapsto (w_1, w_2, w_3, u, v) = (y_1, y_2, 1, y_3, \frac{y_1 + y_2 + 1}{y_3}),$$

The curve

$$\{[y_1 : y_2 : y_3] \in \mathbb{P}^2 | y_1, y_2, y_3 \in \mathbb{C}^*, y_1 + y_2 + y_3 = 0\}$$

is called the mirror curve of $\mathbb{C}^3$. This is a copy of $\mathbb{C}^*$, or equivalently, $\mathbb{P}^1$ with three points $0, -1, \infty$ removed. If we take $z = \frac{y_1}{y_3}$ as coordinate on the mirror curve, $z = 0, 1, \infty$ correspond to $[y_1 : y_2 : y_3] = [0 : -1 : 1]$, $[-1 : 1 : 0]$ and $[1 : -1 : 0]$ respectively. There are different ways to realize the mirror curve as a plane curve. If one takes

$$\tilde{X} = \frac{y_1}{y_3}, \quad \tilde{Y} = \frac{y_2}{y_3},$$

then one gets

$$\tilde{X} + \tilde{Y} + 1 = 0;$$

if one takes

$$X' = \frac{y_3}{y_1}, \quad Y' = \frac{y_2}{y_1},$$

then one gets

$$X' + Y' + 1 = 0$$

and

$$X' = \tilde{X}^{-1}, \quad Y' = \tilde{X}^{-1}\tilde{Y}.$$  

One can also take

$$X = \tilde{x}\tilde{y}^a, \quad Y = \tilde{y}$$

for any integer $a$, then one gets:

$$X + Y^a + Y^{a+1} = 0.$$  

Let $X = -(-1)^ax$ and $Y = -Y$, then one gets:

$$x = y^a - y^{a+1}.$$  

This is the equation for the framed mirror curve.
3. Recursion Relations from Cut-and-Join Equation for the Topological Vertex

In this section we will first recall the definition of the three-partition triple Hodge integrals related to the topological vertex. Then we will derive some recursions relations using the cut-and-join equation.

3.1. Partitions. Recall that a partition $\mu$ of a nonnegative integer $d$ is a sequence of positive integers $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0)$ such that

$$d = \mu_1 + \ldots + \mu_n.$$  

We write $|\mu| = d$, and $l(\mu) = n$. The following numbers associated with a partition $\mu$ are often used:

(48) $|\text{Aut}(\mu)| = \prod_j m_j(\mu)!$, where $m_j(\mu) = \#\{i : \mu_i = j\}$,

(49) $z_\mu = |\text{Aut}(\mu)| \cdot \prod_{i=1}^n \mu_i$,

(50) $\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1)$,

For convenience we also consider the partition of 0 and write it as $\emptyset$. We use the following conventions:

(51) $|\text{Aut}(\emptyset)| = 1$, $z_\emptyset = 1$, $\kappa_\emptyset = 0$.

Let $\mathcal{P}$ denote the set of partitions, and let

$$\mathcal{P}_+ = \mathcal{P} - \{\emptyset\}, \quad \mathcal{P}^2_+ = \mathcal{P}^2 - \{0, \emptyset\}, \quad \mathcal{P}^3_+ = \mathcal{P}^3 - \{(0, 0, 0)\}.$$

Given a triple of partitions $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}^3$, we define

$$l(\vec{\mu}) = \sum_{i=1}^3 l(\mu^i), \quad \text{Aut}(\vec{\mu}) = \prod_{i=1}^3 \text{Aut}(\mu^i).$$

3.2. Three-Partition Hodge Integrals related to the topological vertex. Let $T^3$ act on $\mathbb{C}^3$ by

(52) $$(e^{iw_1}, e^{iw_2}, e^{iw_3}) \cdot (z_1, z_2, z_3) = (e^{iw_1}z_1, e^{iw_2}z_2, e^{iw_3}z_3).$$

Elements of the subgroup of $T^3$ that preserves the holomorphic volume form $dz_1 \wedge dz_2 \wedge dz_3$ satisfy the following condition:

(53) $w_1 + w_2 + w_3 = 0$. 
We will focus on this subgroup and assume this condition throughout the rest of this section. For simplicity of notations, we will write:

\[(54) \quad w = (w_1, w_2, w_3), \quad w_4 = w_1.\]

We will let

\[(55) \quad a_i := \frac{w_{i+1}}{w_i},\]

and write \(a_1 = a\). Then we have:

\[(56) \quad \frac{w_3}{w_2} = \frac{a + 1}{a}, \quad \frac{w_1}{w_3} = -\frac{1}{a + 1}.\]

For \(\bar{\mu} = (\mu^1, \mu^2, \mu^3) \in P^3_+\), we let

\[d^1_{\bar{\mu}} = 0, \quad d^2_{\bar{\mu}} = l(\mu^1), \quad d^3_{\bar{\mu}} = l(\mu^1) + l(\mu^2).\]

We define the three-partition triple Hodge integral to be

\[G_{g,\bar{\mu}}(w) = \left(\frac{\sqrt{-1}}{|\text{Aut}(\bar{\mu})|}\right)^3 \prod_{i=1}^{3} \prod_{j=1}^{l(\mu^i)} (\mu^i_{j+1}^{-1} (\mu^i_{j+1} w_i + k w_i) (\mu^i_{j+1} - 1)!)^{l(\mu^i)-1} \prod_{i=1}^{3} \prod_{j=1}^{l(\mu^i)} \frac{\Lambda_g^i(w_i) w_i^{l(\bar{\mu})-1}}{\Lambda_g^i(w_i) w_i^{l(\bar{\mu})-1}} \right),\]

where \(\Lambda_g^i(u) = u^\mu - \lambda_1 u^{\mu-1} + \cdots + (-1)^{\mu} \lambda_\mu\). See [13] for its relationship with the topological vertex. From the definition, we have the following cyclic symmetries:

\[(58) \quad G_{g,\mu^1,\mu^2,\mu^3}(w_1, w_2, w_3) = G_{g,\mu^2,\mu^3,\mu^1}(w_2, w_3, w_1).\]

Note \(\sqrt{-1}^{l(\bar{\mu})} G_{g,\bar{\mu}}(w)\) is a rational function in \(w_1, w_2, w_3\) with rational coefficients, and by a simple dimension counting is homogeneous of degree 0. We will write

\[G_{g,\bar{\mu}}(a) := G_{g,\bar{\mu}}(w) = G_{g,\mu}(1, a, -1 - a).\]

Several exceptional cases that play important roles need special care. Recall for \(n \geq 3\), the following identity holds:

\[(59) \quad \int_{\mathcal{M}_{0,n}} \frac{1}{(1 - a_1 \psi_1) \cdots (1 - a_n \psi_n)} = (a_1 + \cdots + a_n)^{n-3}.\]

This identity inspires the following useful conventions:

\[(60) \quad \int_{\mathcal{M}_{0,2}} \frac{1}{1 - a_1 \psi_1 (1 - a_2 \psi_2)} = \frac{1}{a_1 + a_2},\]

\[(61) \quad \int_{\mathcal{M}_{0,2}} \frac{1}{1 - a_1 \psi_1} = \frac{1}{a_1^2}.\]
By these conventions we have in the case of \( l(\bar{\mu}) = 1 \),

\[
G_{0,(m),\emptyset,\emptyset}(a) = -\sqrt{-1} \prod_{j=1}^{m-1} \frac{(ma + j)}{(m - 1)!} \frac{1}{m^2},
\]

and similar expressions for \( G_{0,\emptyset,(m),\emptyset}(a) \) and \( G_{0,\emptyset,\emptyset,(m)}(a) \) by changing \( a \) to \( -\frac{a+1}{a} \) and \( a_3 = \frac{1}{a+1} \) respectively. In the case of \( l(\bar{\mu}) = 2 \), we have

\[
G_{0,(m_1,m_2),\emptyset,\emptyset}(a) = \frac{a(a+1)}{1 + \delta_{m_1,m_2}} \prod_{i=1}^{2} \frac{\prod_{j=1}^{m_i-1} (m_i a + j)}{(m_i - 1)!} \cdot \frac{1}{m_1 + m_2},
\]

and similar expressions for \( G_{\emptyset,\emptyset,(m_1,m_2),\emptyset}(a) \) and \( G_{\emptyset,\emptyset,\emptyset,(m_1,m_2)}(a) \) by changing \( a \) to \( -\frac{a+1}{a} \) and \( -\frac{1}{a+1} \) respectively; furthermore,

\[
G_{0,(m_1),\emptyset,(m_2)}(a) = -\frac{1}{a_3} \prod_{i=1,2} \frac{\prod_{j=1}^{m_i-1} (m_i a_i + j)}{(m_i - 1)!} \cdot \frac{1}{m_i a_i + m_i},
\]

\[
G_{0,\emptyset,\emptyset,(m_2)}(a) = -\frac{1}{a_1} \prod_{i=2,3} \frac{\prod_{j=1}^{m_i-1} (m_i a_i + j)}{(m_i - 1)!} \cdot \frac{1}{m_i a_i + m_i},
\]

\[
G_{0,\emptyset,\emptyset,\emptyset}(a) = -\frac{1}{a_2} \prod_{i=3,1} \frac{\prod_{j=1}^{m_i-1} (m_i a_i + j)}{(m_i - 1)!} \cdot \frac{1}{m_i a_i + m_i}.
\]

### 3.3. The cut-and-join equation for three-partition Hodge integrals

Let \( p^i = (p^1_i, p^2_i, \ldots) \) be formal variables. Given a partition \( \mu \) we define \( p^i_\mu = p^1_i \cdot \cdots \cdot p^i_\ell(\mu) \) (note \( p^i_\emptyset = 1 \)). We write

\[
P = (p^1, p^2, p^3), \quad P_{\bar{\mu}} = p^1_{\mu_1} p^2_{\mu_2} p^3_{\mu_3}.
\]

Define the generating functions of the three-partition Hodge integrals by

\[
G_{\bar{\mu}}(\lambda; a) = \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\bar{\mu})} G_{g,\bar{\mu}}(a),
\]

\[
G(\lambda; P; a) = \sum_{\bar{\mu} \in P^3} G_{\bar{\mu}}(\lambda; a) P_{\bar{\mu}},
\]

where \( P^3 \) is the set of three-partitions of \( \mathbb{Z} \).
where \( \lambda \) is a formal variable. By the results in [13], the following cut-and-join equation is satisfied by \( G \):

\[
(66) \quad \frac{\partial G}{\partial a} = \frac{\sqrt{-1}}{2} \sum_{k=1}^{3} \frac{\partial}{\partial a} \left( \frac{w_{k+1}}{w_k} \right) 
\]

\[
\cdot \sum_{i,j \geq 1} \left( i j p^k_{i+j} \frac{\partial^2 G}{\partial p^k_i \partial p^k_j} + i j p^k_{i+j} \frac{\partial G}{\partial p^k_i} \frac{\partial G}{\partial p^k_j} + (i + j) p^k_{i+j} \frac{\partial G}{\partial p^k_{i+j}} \right).
\]

By (55) and (56),

\[
(67) \quad \frac{\partial}{\partial a} \left( \frac{w_2}{w_1} \right) = 1, \quad \frac{\partial}{\partial a} \left( \frac{w_3}{w_1} \right) = \frac{1}{a^2}, \quad \frac{\partial}{\partial a} \left( \frac{w_1}{w_3} \right) = \frac{1}{(a + 1)^2}.
\]

### 3.4. The symmetrized generating function for three-partition Hodge integrals.

One can also define

\[
G_g(p; a) = \sum_{\bar{\mu}} G_{g; \bar{\mu}}(a)p_{\bar{\mu}}.
\]

Because \( G_g(p; a) \) is a formal power series in \( p^k_1, p^k_2, \ldots, p^k_n, \ldots, k = 1, 2, 3 \), for each \( n = (n_1, n_2, n_3) \), one can obtain from it a formal power series \( \Phi_{g,n}(x^1_{[n_1]}; x^2_{[n_2]}; x^3_{[n_3]}; a) \) by applying the following symmetrization operator [9, 6]:

\[
p^k_{\mu_k} \rightarrow (\sqrt{-1})^{n_k} \delta_{i(\mu^k), n_k} \sum_{\sigma \in S_{n_k}} (x^k_{\sigma(1)})^{\mu^k_1} \cdots (x^k_{\sigma(n_k)})^{\mu^k_{n_k}}.
\]

\[
(68) \quad G_{g; \bar{\mu}}(w) = \frac{(-\sqrt{-1})^{\ell(\bar{\mu})}}{|\text{Aut}(\bar{\mu})|} \prod_{i=1}^{3} \prod_{j=1}^{\ell(\mu^i)} \prod_{a=1}^{\mu^i_j - 1} \left( \mu^i_j w_{i+1} + aw_i \right) \left( \mu^i_j - 1 \right)! w_i^{j-1}
\]

\[
\int_{\mathcal{M}_{g,\bar{\mu}(\bar{\pi})}} \prod_{i=1}^{3} \prod_{j=1}^{\ell(\mu^i)} \Lambda^v(w_i)w_i^{\ell(\bar{\mu}) - 1} \prod_{j=1}^{\ell(\mu^i)} (w_i(w_i - \mu^i_j \psi_{d_{\mu^i_j}}))
\]

From the definition, we have for \( 2g - 2 + n_1 + n_2 + n_3 > 0 \),

\[
\Phi_{g,n}(x^1_{[n_1]}; x^2_{[n_2]}; x^3_{[n_3]}; a)
\]

\[
= (-a(a + 1))^{n_1 + n_2 + n_3 - 1} \sum_{b_j \geq 0} \left( \prod_{i=1}^{3} \prod_{j=1}^{n_i} \tau_{b_j} \cdot T_g(a) \right) \prod_{i=1}^{3} \prod_{j=1}^{n_i} \phi_{b_j}(x^i_j; a_i)
\]

\[
(69) \quad \prod_{j=1}^{n_2} \frac{1}{a^{2+b_j}} \prod_{j=1}^{n_3} \frac{1}{(-a - 1)^{2+b_j}}
\]
\[ \langle \tau_1 \cdots \tau_n T_g(a) \rangle_g = \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{b_i} \cdot \Lambda_g^\vee(1) \Lambda_g^\vee(a) \Lambda_g^\vee(-1 - a), \]
\[ \phi_b(x; a) = \sum_{m \geq 1} \frac{\prod_{j=1}^{m-1} (ma + j)}{(m - 1)!} m^b x^m. \]

For simplicity of notations we will set
\[ \phi_1^b(x; a) = \phi_b(x; a), \]
\[ \phi_2^b(x; a_2) = \frac{1}{a^2 + x^2} \phi_b(x; a_2), \]
\[ \phi_3^b(x; a_3) = \frac{1}{(-a - 1)^2 + x^2} \phi_b(x; a_3). \]

### 3.5. Exceptional cases.

We now study several exceptional cases that will play a key role later. First we have
\[ \Phi_{0,1,0,0}(x_1^1; a) = \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m-1} (ma + j)}{(m - 1)!} m^{-2}(x_1^1)^m = \phi_{-2}(x_1^1; a); \]

similarly,
\[ \Phi_{0,0,1,0}(x_1^2; a) = \phi_{-2}(x_1^2; a_2), \]
\[ \Phi_{0,0,0,1}(x_1^3; a) = \phi_{-2}(x_1^3; a_3). \]

So we have
\[ x_1^1 \frac{\partial}{\partial x_1^1} \Phi_{0,1,0,0}(x_1^1; a) = \phi_{-1}(x_1^1; a) = -\ln y(x_1^1; a), \]
\[ x_1^2 \frac{\partial}{\partial x_1^2} \Phi_{0,0,1,0}(x_1^2; a) = \phi_{-1}(x_1^2; a_2) = -\ln y(x_1^2; a_2), \]
\[ x_1^3 \frac{\partial}{\partial x_1^3} \Phi_{0,0,0,1}(x_1^3; a) = \phi_{-1}(x_1^3; a_3) = -\ln y(x_1^3; a_3), \]

where
\[ y(x; a) = 1 - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-2} (na + j)}{n!} x^n. \]

Secondly,
\[ \Phi_{0,2,0,0}(x_1^1, x_2^1; a) = -a(a + 1) \sum_{m_1, m_2 \geq 1} \frac{\prod_{j=1}^{m_1-1} (m_1a + j)}{(m_1 - 1)!} \cdot \frac{(x_1^1)^{m_1}(x_2^1)^{m_2}}{m_1 + m_2}. \]
This has been treated in [6, 15]. It is easy to see that:

\[(80) \quad (x_1^1 \frac{\partial}{\partial x_1^1} + x_2^1 \frac{\partial}{\partial x_2^1}) \Phi_{0,2,0,0}(x_1^1, x_2^1; a) \]

\[= -a(a + 1) \frac{t(x_1^1; a) - 1}{a + 1} \cdot \frac{t(x_2^1; a) - 1}{a + 1} \]

\[= -a(a + 1) \frac{y_1 - 1}{(a + 1)y_1 - a} \cdot \frac{y_2 - 1}{(a + 1)y_2 - a}. \]

One can verify that:

\[(81) \quad \Phi_{0,2,0,0}(x_1^1, x_2^1; a) \]

\[= -\ln\left(\frac{y(x_2^1, a) - y(x_1^1; a)}{x_1^1 - x_2^1}\right) + \ln \frac{1 - y(x_1^1; a)}{x_1^1} + \ln \frac{1 - y(x_2^1; a)}{x_2^1}, \]

\[(82) \quad x_1^1 \frac{\partial}{\partial x_1^1} \Phi_{0,2,0,0}(x_1^1, x_2^1; a) \]

\[= \left(1 \frac{1}{y(x_1^1; a) - y(x_2^1; a)} - \frac{1}{y(x_1^1; a) - 1}\right) x_1^1 \frac{\partial y(x_1^1; a)}{\partial x_1^1} + \frac{x_2^1}{x_1^1 - x_2^1}. \]

\[(83) \quad \frac{\partial}{\partial x_1^1} \frac{\partial}{\partial x_2^1} \Phi_{0,2,0,0}(x_1^1, x_2^1; a) \]

\[= -\frac{1}{(y(x_1^1; a) - y(x_2^1; a))^2} \frac{\partial y(x_1^1; a)}{\partial x_1^1} \frac{\partial y(x_1^1; a)}{\partial x_2^1} + \frac{1}{(x_1^1 - x_2^1)^2}. \]

One can get similar results for \(\Phi_{0,0,2,0}(x_2^1, x_2^1; a)\) and \(\Phi_{0,0,0,2}(x_3^1, x_3^1; a)\) by changing the indices and changing a to \(a_2\) and \(a_3\) respectively. The case of \(\Phi_{0,1,1,0}(x_1^1; x_1^1; a)\), \(\Phi_{0,0,1,1}(x_2^1; x_2^1; a)\), and \(\Phi_{0,1,0,1}(x_3^1; x_3^1; a)\) can be treated in the same fashion. First,

\[\Phi_{0,1,1,0}(x_1^1; x_1^2; a) = -(a + 1) \sum_{m^1, m^2 \geq 1} 2 \prod_{j=1}^{m^1-1} \prod_{i=1}^{m^2} \frac{m^i a_i + j}{(m^i - 1)!} \cdot \frac{(x_1^1)^{m^1} (x_2^1)^{m^2}}{m^1 a + m^2}. \]

It is easy to see that:

\[(84) \quad (a x_1^1 \frac{\partial}{\partial x_1^1} + x_1^2 \frac{\partial}{\partial x_1^1}) \Phi_{0,1,1,0}(x_1^1; x_1^2; a) \]

\[= -(a + 1) \frac{t(x_1^1; a) - 1}{a + 1} \cdot \frac{t(x_1^2; a_2) - 1}{a_2 + 1} \]

\[= -(a + 1) \frac{y(x_1^1; a) - 1}{(a + 1)y(x_1^1; a) - a} \cdot \frac{y(x_1^2; a_2) - 1}{(a_2 + 1)y(x_1^2; a_2) - a}.\]
Write $y_1 = y(x_1^1; a)$ and $y_2 = y(x_1^2; a_2)$, one then gets:

$$
(85) \quad (ax_1^1 \frac{\partial}{\partial x_1^1} + x_1^2 \frac{\partial}{\partial x_1^2}) \Phi_{0, 1, 1, 0}(x_1^1; x_1^2; a) = \frac{a(y_1 - 1)(y_2 - 1)}{(y_1 - \frac{a}{a + 1})(y_2 - (a + 1))}.
$$

One can verify that:

$$
(86) \quad \Phi_{0, 1, 1, 0}(x_1^1; x_1^2; a) = -\ln \frac{\tilde{y}(x_1^1; a) - y(x_1^2; a_2)}{\tilde{y}(x_1^1; a) - 1}.
$$

Indeed,

$$
(87) \quad -(ax_1^1 \frac{\partial}{\partial x_1^1} + x_1^2 \frac{\partial}{\partial x_1^2}) \ln \frac{\tilde{y}(x_1^1; a) - y(x_1^2; a_2)}{\tilde{y}(x_1^1; a) - 1} = -\frac{ax_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1} - x_1^2 \frac{\partial y(x_1^1; a)}{\partial x_1^1} + ax_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1}}{\tilde{y}(x_1^1; a) - y(x_1^2; a_2)} + \frac{ax_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1}}{\tilde{y}(x_1^1; a) - 1}.
$$

Recall

$$
x_1^2 \frac{\partial y(x_1^2; a_2)}{\partial x_1^2} = \frac{y(x_1^2; a_2)(1 - y(x_1^2; a_2))}{a_2 - (a_2 + 1)y(x_1^2; a_2)} = -a_2(1 - y_2) = \frac{a_2(1 - y_2)}{(a + 1) - y_2}.
$$

Now we use $x_1^1 = y(x_1^1; a)^a(1 - y(x_1^1; a))$ to get:

$$
(88) \quad \tilde{y}(x_1^1; a) = \frac{1}{x_1^1} y(x_1^1; a)^a = \frac{1}{1 - y(x_1^1; a)} = \frac{1}{1 - y_1}.
$$

It follows that:

$$
x_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1} = x_1^1 \frac{\partial}{\partial x_1^1} \left( \frac{1}{1 - y(x_1^1; a)} \right) = \frac{1}{(1 - y(x_1^1; a))^2} x_1^1 \frac{\partial}{\partial x_1^1} \tilde{y}(x_1^1; a) = \frac{y(x_1^1; a)}{(1 - y(x_1^1; a)) \cdot (a - (a + 1)y(x_1^1; a))} = \frac{y_1}{(1 - y_1)(a - (a + 1)y_1)}.
$$

Therefore, the RHS of (87) is

$$
-\frac{a_2(1 - y_2)}{(1 - y_1)(a + 1) - y_2} + a_2 \frac{y_2}{1 - y_1 - y_2} = a_2 \frac{y_2}{1 - y_1 - (y_1 - \frac{a}{a + 1})(a + 1)}.
$$

This matches with the RHS of (85). From (86) one easily gets:

$$
(89) \quad x_1^1 \frac{\partial}{\partial x_1^1} \Phi_{0, 1, 1, 0}(x_1^1; x_1^2; a) = -\frac{x_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1}}{\tilde{y}(x_1^1; a) - y(x_1^2; a_2)} + \frac{x_1^1 \frac{\partial \tilde{y}(x_1^1; a)}{\partial x_1^1}}{\tilde{y}(x_1^1; a) - 1},
$$
(90) \[
\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1^2} \Phi_{0,1,1,0}(x_1^1; x_1^2; a) = -\frac{\partial \tilde{y}(x_1^1; a) \partial y(x_1^2; a_2)}{\partial x_1 \partial x_1^2} \frac{\partial y(x_1^1; a)}{\tilde{y}(x_1^1; a) - y(x_1^2, a_2))^2}.
\]

By the cyclic symmetries one gets from (86):

(91) \[
\Phi_{0,0,1,1}(x_1^2; x_1^3; a) = -\ln \frac{\tilde{y}(x_1^2; a_2) - y(x_1^3; a_3)}{\tilde{y}(x_1^2; a_2) - 1},
\]

and

(92) \[
\Phi_{0,1,0,1}(x_1^1; x_1^3; a) = -\ln \frac{\tilde{y}(x_1^3; a_3) - y(x_1^1; a_1)}{\tilde{y}(x_1^3; a_3) - 1}.
\]

And so we have

(93) \[
\frac{\partial}{\partial x_1^2} \frac{\partial}{\partial x_1^3} \Phi_{0,0,1,1}(x_1^2; x_1^3; a) = -\frac{\partial \tilde{y}(x_1^2; a_2) \partial y(x_1^3; a_3)}{\partial x_1^2 \partial x_1^3} \frac{\partial y(x_1^1; a)}{\tilde{y}(x_1^2; a_2) - y(x_1^3; a_3))^2},
\]

and

(94) \[
x_1^1 \frac{\partial}{\partial x_1^1} \Phi_{0,1,0,1}(x_1^1; x_1^3; a) = \frac{x_1^1 \partial y(x_1^1; a)}{\tilde{y}(x_1^3; a_3) - y(x_1^1; a_1)}.
\]

(95) \[
\frac{\partial}{\partial x_1^1} \frac{\partial}{\partial x_1^3} \Phi_{0,1,0,1}(x_1^1; x_1^3; a) = -\frac{\partial \tilde{y}(x_1^2; a_3) \partial y(x_1^1; a)}{\partial x_1^1 \partial x_1^3} \frac{\partial y(x_1^1; a)}{\tilde{y}(x_1^3; a_3) - y(x_1^1; a))^2}.
\]

Remark 3.1. Without the proposal in [4], it will be very difficult for the author to find the explicit expressions (86), (91), and (92).
3.6. The symmetrized cut-and-join equation for three-partition Hodge integrals. Using the analysis in [9], one can obtain the symmetrized version of the cut-and-join equation (66) as follows:

\[
\frac{\partial}{\partial a} \Phi_{g,n}(x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a) = \frac{1}{2} \sum_{k=1}^{n_1} \frac{\partial}{\partial z_1} \Phi_{g-1;n_1+1,n_2,n_3}(z_1, z_2, x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a) \big|_{z_1, z_2 = x_k^1}
\]

\[
+ \frac{1}{2} \sum_{k=1}^{n_1} \sum_{g_1 + g_2 = g} \frac{\partial}{\partial x_k^1} \Phi_{g_1;|A_1|+1,|A_2|,|A_3|}(x_k^1, x_{A_1}^1; x_{A_2}^2; x_{A_3}^3; a) \cdot \frac{\partial}{\partial x_k^1} \Phi_{g_2;|B_1|+1,|B_2|,|B_3|}(x_k^1, x_{B_1}^2; x_{B_2}^3; a)
\]

\[
- \sum_{k=1}^{n_1} \sum_{j \in [n_1]_k} \frac{x_j^1}{x_k^1 - x_j^1} \cdot \frac{x_j^1}{x_k^1} \frac{\partial}{\partial x_k^1} \Phi_{g;n_1-1,n_2,n_3}(x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a)
\]

+ \ldots .

Here we have omitted the terms that correspond to cut-and-join operation in \(x_i^2\) and \(x_i^3\).

3.7. Symmetrized cut-and-join equation in the \(v\)-coordinates. As in [7 8 13], introduce the \(v\)-coordinates by

\[
x_j^i = e^{-i_3^2}/2.
\]

Now for \(2g - 2 + n_1 + n_2 + n_3 > 0\), (69) becomes

\[
\Phi_{g,n}(x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a) = (-a(a + 1))^{n_1+n_2+n_3-1} \sum_{b_j > 0} \prod_{i=1}^{3} \prod_{j=1}^{n_i} \xi_{b_j}^i(v_j; a_i)
\]

\[
(98) = (-a(a + 1))^{n_1+n_2+n_3-1} \sum_{b_j > 0} \prod_{i=1}^{3} \prod_{j=1}^{n_i} \xi_{b_j}^i(v_j; a_i)
\]

where

\[
(99) \xi_{b}^1(v; a) = \xi_{b}(v; a),
\]

\[
(100) \xi_{b}^2(v; a) = \frac{1}{a^2+b} \xi_{b}(v; a_2),
\]

\[
(101) \xi_{b}^3(v; a) = \frac{1}{(-a - 1)^{2+b}} \xi_{b}(v; a_3).
\]
The first term on the right-hand side is now:

\[
\frac{1}{2} \sum_{k=1}^{n_1} z_1 \frac{\partial}{\partial z_1} z_2 \frac{\partial}{\partial z_2} \Phi_{g-1;n_1+1,n_2,n_3}(z_1, z_2, x_{[n_1]}^1, x_{[n_2]}^2, x_{[n_3]}^3; a)|_{z_1,z_2=x_k}^1
\]

\[
= \frac{1}{2} (-a(a + 1))^{n_1+n_2+n_3} \sum_{k=1}^{n_1} \sum_{b_i,c_i \geq 0} \langle \tau_{b_i} \prod_{i=1}^{3} \tau_{c_i} \cdot T_g(a) \rangle_g \cdot \psi_{b+1}(v_k^1; a) \psi_{c+1}(v_k^1; a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i \atop (i,j) \neq (1,k)} \xi_{b_j^i}(v_j^i; a_i).
\]

The second term on the right-hand side has several cases. Case 1. The splitting is stable, i.e.,

\[
2g_1 - 1 + |A^1| + |A^2| + |A^3| > 0,
\]

\[
2g_2 - 1 + |B^1| + |B^2| + |B^3| > 0.
\]

Then we get a term of the form:

\[
\frac{1}{2} x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{g_1; |A^1|+1,|A^2|,|A^3|}(x_k^1, x_{A^1}^1, x_{A^2}^2, x_{A^3}^3; a)
\]

\[
\cdot x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{g_2; |B^1|+1,|B^2|,|B^3|}(x_k^1, x_{B^1}^1, x_{B^2}^2, x_{B^3}^3; a)
\]

\[
= \frac{1}{2} (-a(a + 1))^{n_1+n_2+n_3-1}
\]

\[
\cdot \sum_{b_i,c_i \geq 0} \langle \tau_{b_i} \prod_{i=1}^{3} \tau_{c_i} \cdot T_{g_1}(a) \rangle_{g_1} \cdot \langle \tau_{c_i} \prod_{i=1}^{3} \tau_{b_i} \cdot T_{g_2}(a) \rangle_{g_2}
\]

\[
\cdot \psi_{b+1}(v_k^1; a) \psi_{c+1}(v_k^1; a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i \atop (i,j) \neq (1,k)} \xi_{b_j^i}(v_j^i; a_i).
\]

Case 2. There are terms that involve exceptional terms of the form:

\[
\sum_{k=1}^{n_1} x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{0,1,0,0}(x_k^1; a) \cdot x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{g, a}(x_{[n_1]}^1, x_{[n_2]}^2, x_{[n_3]}^3; a)
\]

\[
= \sum_{k=1}^{n_1} \xi_{-1}(v_k^1; a) \cdot (-a(a + 1))^{n_1+n_2+n_3-1} \sum_{b_j \geq 0} \langle \prod_{i=1}^{3} \prod_{j=1}^{n_i} \tau_{b_j^i} \cdot T_g(a) \rangle_g
\]

\[
\cdot \prod_{i=1}^{3} \prod_{j=1}^{n_i} \xi_{b_j^i+\delta_k, \epsilon_k}(v_j^i; a_i).
\]

Here we have used (76).
Case 3. We have some unstable terms which combined with the third term on the right-hand side gives us:

\[
\sum_{k=1}^{n_1} \sum_{l \in [n_1]} (x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{0,2,0,0} (x_k^1, x_l^1; a) - \frac{x_j^1}{x_k^1 - x_j^1})
\]

\[
\cdot x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{g,n_1-1,n_2,n_3} (x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a)
\]

\[
= - \sum_{k=1}^{n_1} \sum_{l \in [n_1]} \left( \frac{1}{y(x_k^1; a) - y(x_l^1; a)} - \frac{1}{y(x_k^1; a) - 1} \right) x_k \frac{\partial y(x_k^1; a)}{\partial x_k^1}
\]

\[
\cdot (-a(a + 1))^{n_1+n_2+n_3-2} \sum_{b_j \geq 0} \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i} \tau_{b_j} \cdot T_g(a) \rangle_g
\]

\[
\prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (1,l)} \xi_{b_j, \delta_{i,j,k}} (v_i^j; a_i).
\]

Here we have used (82).

Case 4.

\[
\sum_{k=1}^{n_1} \sum_{l=1}^{n_2} x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{1,1,0}^0 (x_k^1, x_l^1; a) \cdot x_k^1 \frac{\partial}{\partial x_k^1} \Phi_{n_1,n_2-1,n_3}^g (x_{[n_1]}^1; x_{[n_2]}^2; x_{[n_3]}^3; a)
\]

\[
= \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} \left( -x_k^1 \frac{\partial \tilde{y}(x_k^1; a)}{\partial x_k^1} + \frac{x_k^1 \partial \tilde{y}(x_k^1; a)}{\tilde{y}(x_k^1; a) - 1} \right)
\]

\[
\cdot (-a(a + 1))^{n_1+n_2+n_3-2} \sum_{b_j \geq 0} \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i} \tau_{b_j} \cdot T_g(a) \rangle_g
\]

\[
\prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (2,l)} \xi_{b_j, \delta_{i,j,k}} (v_i^j; a_i).
\]

Here we have used (89).
Case 5.

\[
\sum_{k=1}^{n_1} \sum_{l=1}^{n_3} x_k^{1} \frac{\partial}{\partial x_k^{1}} \Phi_{1,0,1}^{9}(x_k^{1}; x_l^{3}; a) \cdot x_k^{1} \frac{\partial}{\partial x_k^{1}} \Phi_{n_1,n_2,n_3-1}^{9}(x_{[n_1]}^{1}; x_{[n_2]}^{2}; x_{[n_3]}^{3}; a)
\]

\[
= \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} x_k^{1} \frac{\partial y(x_k^{1}; a_3)}{\partial x_k^{1}} - y(x_k^{1}; a_1) \cdot (-a(a+1))^{n_1+n_2+n_3-2}
\]

\[
\cdot \sum_{b_i \geq 0} \prod_{1 \leq i \leq 3; 1 \leq j \leq n_i} \tau_{b_i} \cdot T_g(a) \cdot \prod_{1 \leq i \leq 3; 1 \leq j \leq n_i; (i,j) \neq (3,l)} \xi_{b_j + \delta_{i,k}} (v_j^{i}; a_k).
\]

Here we have used (94).

So far we have only considered the terms corresponding to cut-and-join in \(x_k^{1}\) variables. The terms for \(x_k^{2}\) and \(x_k^{3}\) can be obtained by cyclic symmetry.

As in [15], we regard both sides of the equation (96) as meromorphic functions in \(v_1^{i}\), take the principal parts and then take only the even
powers in $v_i^1$. The left-hand side and the \ldots terms have no contributions. So we get:

\[
\begin{align*}
\xi^i_{b_j^0} (v_i^1; a) \cdot \sum_{b_j^0 \geq 0} \frac{3}{4} \frac{n_i}{i=n_i} \sum_{j=1}^{3} \tau_{b_j^0} \cdot T_{g}(a) \cdot \frac{3}{4} \frac{n_i}{i=n_i} \sum_{j=1}^{3} \xi^i_{b_j^0 + \delta_i \delta_i} (v_i^1; a_i) \\
= \frac{1}{2} a(a+1) \sum_{b,c,b_j^0 \geq 0} \tau_{b} \cdot \tau_{b_j^0} \cdot T_{g}(a) \cdot T_{g_1}(a) \cdot T_{g_2}(a) \\
\cdot \psi_{c+1} (v_i^1; a) \psi_{c+1} (v_i^1; a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (1,1)} \xi^i_{b_j^0} (v_i^1; a_i) \\
- \frac{1}{2} \sum_{b,c,b_j^0 \geq 0} \tau_{b} \cdot \tau_{b_j^0} \cdot T_{g_1}(a) \cdot T_{g_2}(a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (1,1)} \xi^i_{b_j^0} (v_i^1; a_i) \\
+ \frac{1}{2} \sum_{b,c} \tau_{b} \cdot T_{g}(a) \cdot \tau_{c} \cdot T_{g}(a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (1,1)} \xi^i_{b_j^0 + \delta_i \delta_i} (v_i^1; a_i) \\
- \frac{1}{2} \sum_{b,c} \tau_{b} \cdot T_{g}(a) \cdot \tau_{c} \cdot T_{g}(a) \cdot \prod_{1 \leq i \leq 3, 1 \leq j \leq n_i, (i,j) \neq (1,1)} \xi^i_{b_j^0 + \delta_i \delta_i} (v_i^1; a_i) \\
\end{align*}
\]
modulo terms analytic in $v_1$. We now take $\prod_{i=2}^{n_1} x_j^i \frac{\partial}{\partial x_j} \prod_{j=1}^{n_1} x_j^i \frac{\partial}{\partial x_j}$
onumber on both sides then dividing both sides by $\xi_{o_1}(v; a)$. This gives us:

\[
\sum_{b'_j \geq 0} \sum_{i=1}^{3} \tau_{b'_j} \cdot T_g(a) \cdot \sum_{j=1}^{3} \Xi_{b'_j+1}(v^i_j; a_i) = \frac{1}{\xi_{o_1}(v^1_1; a)} \cdot \left( \frac{1}{2} \cdot \sum_{b,c,b'_j \geq 0} \langle \tau_b \tau_c \prod_{i=1}^{3} \tau_{b'_j} \cdot T_g(a) \rangle_g \cdot \sum_{j=1}^{3} \Xi_{b'_j+1}(v^i_j; a_i) \right)
\]

\[
= \frac{1}{a(a+1)} \sum_{l \in [n_1]} \langle \tau_{b'_j} \cdot T_g(a) \rangle_g \cdot \sum_{j=1}^{3} \Xi_{b'_j+1}(v^i_j; a_i)
\]

\[
= \frac{1}{a(a+1)} \sum_{l \in [n_1]} \langle \tau_{b'_j} \cdot T_g(a) \rangle_g \cdot \sum_{j=1}^{3} \Xi_{b'_j+1}(v^i_j; a_i)
\]

modulo a term with at most a simple pole at 0 in $v^1_1$. 

4. EYNARD-ORANTIN RECURRENCE RELATIONS FOR THREE-PARTITION TRIPLE HODGE INTEGRALS

In this section we reformulate the recursion relations for three-partition triple Hodge integrals derived in the end of last section as Eynard-Orantin type recursion relations. This verifies a version of local mirror symmetry proposed by Bouchard-Klemm-Marino-Pasquetti \[4\] for the topological vertex.

4.1. DIFFERENTIALS ASSOCIATED TO THREE-PARTITION TRIPLE HODGE INTEGRALS. Define

\[
W_g(x_1^{1}; x_2^{2}; x_3^{3}; a) = (-1)^{g-1} \prod_{i=1}^{3} n_i \sum_{j=1}^{n_i} \frac{\partial}{\partial x_j^g} \Phi_{n_1,n_2,n_3}(x_1^{1}; x_2^{2}; x_3^{3}; a) \cdot \prod_{i=1}^{3} n_i \, dx_j^i.
\]

Then we have

\[
W_g(x_1^{1}; x_2^{2}; x_3^{3}; a) = (-1)^{g+n_1+n_2+n_3(a(a+1))} \prod_{i=1}^{3} \tau_{ij} \cdot T_g(a) \cdot \prod_{i=1}^{3} n_i d\Phi_{ij}(x_i^a; a_i) \cdot \prod_{j=1}^{n_2} \frac{1}{a^{2+b_j}} \cdot \prod_{j=1}^{n_3} \frac{1}{(a-1)^{2+b_j}},
\]

for \(2g-2+n > 0\). By (76), (77) and (78), we have

\[
W_0(x_1^1; a) = \ln y(x_1^1; a) \frac{dx_1^1}{x_1^1},
\]

\[
W_0(x_2^2; a) = \ln y(x_2^2; a_2) \frac{dx_2^2}{x_2^2},
\]

\[
W_0(x_3^3; a) = \ln y(x_3^3; a_3) \frac{dx_3^3}{x_3^3}.
\]

By (81) we get:

\[
W_0(x_1^1, x_2^1; a) = \frac{dy(x_1^1; a)dy(x_2^1; a)}{(y(x_1^1; a) - y(x_2^1; a))^2} - \frac{dx_1^1 dx_2^1}{(x_1^1 - x_2^1)^2}.
\]

Similarly,

\[
W_0(x_1^2, x_2^2; a) = \frac{dy(x_1^2; a_2)dy(x_2^2; a_2)}{(y(x_1^2; a_2) - y(x_2^2; a_2))^2} - \frac{dx_1^2 dx_2^2}{(x_1^2 - x_2^2)^2},
\]

\[
W_0(x_1^3, x_2^3; a) = \frac{dy(x_1^3; a_3)dy(x_2^3; a_3)}{(y(x_1^3; a_3) - y(x_2^3; a_3))^2} - \frac{dx_1^3 dx_2^3}{(x_1^3 - x_2^3)^2}.
\]
By (90), (93) and (95), we have

\[ W_0(x_1; x_1^2; a) = \frac{dy(x_1^1; a)dy(x_1^2; a_2)}{(y(x_1^1; a) - y(x_1^2; a_2))^2}, \]

\[ W_0(x_1^2; x_1^3; a) = \frac{dy(x_1^2; a_2)dy(x_1^3; a_3)}{(y(x_1^2; a_2) - y(x_1^3; a_3))^2}, \]

\[ W_0(x_1^1; x_1^3; a) = \frac{dy(x_1^3; a_3)dy(x_1^1; a)}{(y(x_1^3; a_3) - y(x_1^1; a))^2}. \]

4.2. Eynard-Orantin formalism for the topological vertex. According to the proposal in [14], the differentials \( W_g(x_{[n_1]}; x_{[n_2]}; x_{[n_3]}; a) \) can be computed recursively by the Eynard-Orantin formalism for the framed mirror curve given by (17). By abuse of notations, we will write the differential \( W_g(x_{[n_1]}; x_{[n_2]}; x_{[n_3]}; a) \) as \( W_g(y_{[n_1]}; y_{[n_2]}; y_{[n_3]}; a) \), where \( y_i = y(x_i, a_i) \). The initial values are given by (104)-(112), and the recursion is given by:

\[ W_g(y_{[n_1]}; y_{[n_2]}; y_{[n_3]}; a) = \text{Res}_{z=0} \frac{dE_z(y^1_1)}{\omega(z)} \left( W_{g-1}(y(z), y(P(z)), y_{[n_1]}^1; y_{[n_2]}^2; y_{[n_3]}^3; a) \right. \]

\[ + \sum_{g_1+g_2=g} \sum_{A_1B_1=[n_1]} A_1 \cdot B_1 \cdot y_{[n_2]} \cdot y_{[n_3]} \cdot a \cdot B_2 \cdot y_{[n_2]}^2 \cdot y_{[n_3]}^3 \cdot a \left. \right) \cdot W_{g_2}(y(P(z)), y_{[n_1]}^1; y_{[n_2]}^2; y_{[n_3]}^3; a), \right) \]

where

\[ \omega(z) = (\ln y(z) - \ln y(P(z))) \cdot \frac{dx(z)}{x(z)}, \]

\[ dE_z(y^1_1) = \frac{1}{2} \left( \frac{1}{y(z^1_1) - y(z)} - \frac{1}{y(z^1_1) - y(P(z))} \right) dy_1. \]

Here for simplicity of notations, we write

\[ x(z) = x\left(\frac{a}{a+1} + z; a\right) = \left(\frac{1}{a+1} - z\right)\left(\frac{a}{a+1} + z\right)^a, \]

\[ y(z) = \frac{a}{a+1} + z, \quad y^1_1 = y(z^1_1). \]

On the right-hand side of (113), \( W_0\left(\frac{a}{a+1} + z; a\right) \) and \( W_0\left(\frac{a}{a+1} + P(z); a\right) \) are understood as 0. Therefore, (113) is a recursion relation that determines \( W_g(y_{[n_1]}^1; y_{[n_2]}^2; y_{[n_3]}^3; a) \) for \( n_1 > 0 \). When \( n_1 = 0 \), one can use the cyclic symmetry (58) to reduce to the \( n_1 > 0, n_3 = 0 \) case.
Theorem 1 in the Introduction can be rephrased more precisely as

**Theorem 4.1.** The differentials $W_g(y_{[n_1]}^1; y_{[n_2]}^2; y_{[n_3]}^3; a)$ satisfy the recursion relations (113).

When $n_2 = n_3 = 0$, this has been proved in [15] and [6] using ideas from [7]. For the proof of the general case, one needs to show that (113) is equivalent to the recursion relations derived from the cut-and-join equation in the end of §3. It is almost a verbatim straightforward generalization of the treatment in [15], so we will omit it.

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