GRAPHS OF GROUPS WITH TOTALLY DISCONNECTED MORSE BOUNDARY

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Abstract. Let a finitely generated group \( G \) split as a graph of groups. If edge groups are undistorted and do not contribute to the Morse boundary \( \partial_M G \), we show that every connected component of \( \partial_M G \) with at least two points originates from the Morse boundary of a vertex group. When vertex groups have totally disconnected Morse boundary, this implies that \( \partial_M G \) is totally disconnected.

Under stronger assumptions on the edge groups (such as wideness in the sense of Druțu–Sapir), we show that Morse boundaries of vertex groups are topologically embedded in \( \partial_M G \).

1. Introduction.

Morse boundaries \( \partial_M G \) of finitely generated groups \( G \) were introduced by Charney, Sultan and Cordes [CS15, Cor17] in an attempt to extend to all groups some of the most fundamental properties of Gromov boundaries of hyperbolic groups [Gro87]. Importantly, every quasi-isometry between finitely generated groups extends to a homeomorphism of their Morse boundaries, which can provide a useful tool to distinguish quasi-isometry classes of groups.

When \( G \) is not Gromov-hyperbolic, the topology of \( \partial_M G \) is rather unwieldy (for instance, it is not 1st countable or compact), which often makes an explicit computation difficult. Despite this, Charney, Cordes and Sisto recently showed that essentially all known examples of infinite, totally disconnected Morse boundaries fall into just two homeomorphism classes: the Cantor space and the \( \omega \)-Cantor space [CCS20].

While groups whose Morse boundary is a Cantor space are fully classified (they are hyperbolic, hence virtually free), it appears that the class of groups with \( \omega \)-Cantor boundary is rather large. For instance, it includes all irreducible, non-free right-angled Artin groups, as well as all non-geometric graph manifold groups [CCS20].

In this work, we seek to expand the class of finitely generated groups known to have totally disconnected Morse boundary. A natural source of examples is provided by graphs of groups.

As a set, \( \partial_M G \) is defined as the collection of all Morse geodesic rays in a fixed Cayley graph of \( G \), identifying rays at finite Hausdorff distance. Our topology of choice will always be the direct limit topology from [CS15, Cor17].

We mention that an alternative topology on \( \partial_M G \) was introduced by Cashen and Mackay in [CM19]. The latter has the advantage of being metrisable, while retaining quasi-isometric invariance. However, it appears to be more complicated to describe explicitly, and we are not aware of a single non-empty Morse boundary of a non-hyperbolic group for which the Cashen–Mackay topology can be given “intrinsic” characterisations in the spirit of [CCS20].

The following relative version of Morse boundaries plays an important role in our results.

Definition (Relative Morse boundary). Let \( H \leq G \) be finitely generated groups, with \( H \) undistorted in \( G \). The relative Morse boundary of \( H \) in \( G \), denoted \( (\partial_M H, G) \), is the subset of \( \partial_M G \) consisting of points represented by rays that remain Morse in the Cayley graphs of \( G \).

We always endow \( (\partial_M H, G) \) with the subspace topology coming from \( \partial_M H \). This is the opposite of the convention adopted in [Kar21].
Note that we always have a natural injection $(\partial_M H, G) \hookrightarrow \partial_M G$. Our first result greatly extends the main theorem of [Kar21], while providing a significantly simpler proof.

**Theorem A.** Let a finitely generated group $G$ split as a graph of groups. Suppose that:
- all edge groups are finitely generated and undistorted in $G$;
- $(\partial_M E, G) = \emptyset$ for every edge group $E$.
If a connected component $C \subseteq \partial_M G$ is not a singleton, then $C$ is contained in the image of the natural injection $(\partial_M V, G) \hookrightarrow \partial_M G$ for a vertex group $V \leq G$.

The following consequence of Theorem A has some overlap with [CCS20, Theorem 1.2]. The advantage is that here we do not require acylindricity of the splitting, nor that the vertex groups have trivial Morse boundary.

**Corollary B.** Under the assumptions of Theorem A, suppose additionally that $(\partial_M V, G)$ is either empty or totally disconnected for every vertex group $V \leq G$. Then $\partial_M G$ is empty or totally disconnected.

Corollary B follows from Theorem A because, for every undistorted subgroup $H \leq G$, the natural inclusion $(\partial_M H, G) \hookrightarrow \partial_M G$ is an open map (see Lemma 2.3).

However, even in the setting Theorem A the inclusions $(\partial_M V, G) \hookrightarrow \partial_M G$ need not be continuous, as demonstrated by the following example.

**Example.** Consider the group $G = \mathbb{Z}^2 * \mathbb{Z} = \langle x, y \rangle * \langle z \rangle$. It admits the splitting $G = \langle x, y \rangle * (y) \langle y, z \rangle$.
Since $\langle y \rangle$ lies in the flat $\langle x, y \rangle$, it is undistorted and has trivial relative Morse boundary in $G$.

Consider the vertex group $V := \langle y, z \rangle \simeq F_2$, which is also undistorted. The inclusion:

$$(\partial_M V, G) \hookrightarrow \partial_M G$$

is not continuous. In order to see this, consider the rays labelled by $z^ny^nz^\infty$ and $z^\infty$, which all lie in $(\partial_M V, G)$. Since $V$ is hyperbolic, we have $z^ny^nz^\infty \to z^\infty$ in the topology of $\partial_M V$.

However, since the rays $z^ny^nz^\infty$ spend longer and longer in the flat $\langle x, y \rangle$, they are not uniformly Morse in $G$. It follows that they form a closed subset of $\partial_M G$ (every stratum of $\partial_M G$ contains only finitely many of them). Hence $z^ny^nz^\infty \neq z^\infty$ in the topology of $\partial_M G$. This last argument is taken from [Mur19, Section 5].

Of course, it would be desirable to have conditions ensuring that the injections $(\partial_M V, G) \hookrightarrow \partial_M G$ in Theorem A are topological embeddings, as this would allow us to fully characterise the Morse boundary of $G$ in terms of the boundaries of the vertex groups.

To this regard, note that a key feature of the above example is that, although $(\partial_M \langle y \rangle, V) = \emptyset$, we have $(\partial_M \langle y \rangle, V) \neq \emptyset$. This leads us to suspect that this kind of issue should not present itself if all edge groups have trivial relative Morse boundary in the incident vertex groups.

We prove this guess under the following, potentially stronger assumption.

**Definition** (Relatively wide). Let $H \leq G$ be finitely generated groups, with $H$ undistorted in $G$.

- The group $G$ is wide if none of its asymptotic cones $G_\omega$ have cut points [DS05].
- We say that $H$ is relatively wide in $G$ if, for every asymptotic cone $G_\omega$, no two points of the limit $H_\omega \subseteq G_\omega$ are separated by a cut point of $G_\omega$.

Note that $H$ is relatively wide in $G$ as soon as either $H$ or $G$ is wide.

Examples of wide groups include one-ended groups satisfying a law (e.g. solvable, uniformly amenable, Burnside, etc) [DS05], one-ended groups with infinite centre, and various higher-rank lattices [DMS10].

Wide groups have empty Morse boundary. It is a well-known open question whether the converse holds. We record here the relative version of this question, as it might be easier to find counterexamples in the relative case.
Question. Let $H \leq G$ be finitely generated groups, with $H$ undistorted in $G$ and $(\partial MH, G) = \emptyset$. Is $H$ relatively wide in $G$?

Conversely, it is easy to see that $(\partial MH, G) = \emptyset$ holds as soon as $H$ is relatively wide in $G$. We can now state our second main result.

**Theorem C.** Let a finitely generated group $G$ split as a graph of groups. Consider a vertex group $V \leq G$. Suppose that all incident edge groups $E \leq V$ are finitely generated, undistorted in $G$, and relatively wide in $V$. Then:

1. $V$ is undistorted in $G$ and $(\partial MV, G) = \partial MV$;
2. the inclusion $\partial MV \hookrightarrow \partial MG$ is a topological embedding.

**Corollary D.** If every vertex group satisfies the assumptions of Theorem C then every connected component of $\partial MG$ is either a singleton or homeomorphic to a connected component of the Morse boundary of a vertex group.

Corollary D gives many examples of groups with totally disconnected Morse boundary, but it does not further describe the topological spaces that may arise as boundaries. The remarkable Corollary 1.4] shows instead that, for any finitely generated group $G$, the boundary $\partial MG$ is an $\omega$–Cantor space as soon as it is totally disconnected, non-compact, $\sigma$–compact, and contains a Cantor subspace. The last property is generally not hard to obtain: for instance, using acylindrical hyperbolicity of $G$ (if given) to construct a stable free subgroup of $G$ [Sis16, DGO17], or by applying Theorem C to any Cantor subspace that boundaries of vertex groups may have.

This suggests studying the following problem.

**Question.** In the setting of Corollary D suppose that all vertex groups have $\sigma$–compact Morse boundary. Is $\partial MG$ then $\sigma$–compact?

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2. Preliminaries.

To economise on constants, we will speak of $C$–quasi-geodesics when referring to $(C, C)$–quasi-geodesics. In the whole section, $X$ and $Y$ are proper geodesic metric spaces.

We denote closed metric balls by $B(x, r)$ and closed metric neighbourhoods of subsets by $N(A, r)$. Where necessary, we may add a subscript $B_X(x, r)$ or $N_Y(A, r)$ to specify the relevant space.

2.1. Morse boundaries. We refer the reader to [Cor17] for further details.

A quasi-geodesic $\gamma \subseteq X$ is $N$–Morse for a function $N : [1, +\infty) \to [0, +\infty)$ if every $C$–quasi-geodesic with endpoints on $\gamma$ is contained in the $N(C)$–neighbourhood of $\gamma$. The function $N$ is usually referred to as a Morse gauge for $\gamma$.

Fix a basepoint $p \in X$. We define $\partial^N_M X_p$ as the set of $N$–Morse geodesic rays based at $p$, identifying rays at finite Hausdorff distance. Endowed with the compact-open topology, this space is compact and metrisable. Let $\partial_M X_p$ be the union of all spaces $\partial^N_M X_p$, as $N$ varies among all possible Morse gauges. We define a topology on $\partial_M X_p$ as follows: a subset $U \subseteq \partial_M X_p$ is open (resp. closed) if and only if all intersections $U \cap \partial^N_M X_p$ are open (resp. closed).

If $q \in X$ is a different basepoint, we have a natural homeomorphism $\partial_M X_p \rightarrow \partial_M X_q$ given by pairing rays at finite Hausdorff distance. Thus, the space $\partial_M X_p$ is independent of the choice of $p$ and we simply denote it by $\partial_M X$. We refer to $\partial_M X$ as the Morse boundary of $X$.

We record here the following standard properties of Morse quasi-geodesics for later use.
Lemma 2.1.

(1) Let $\alpha \subseteq X$ be an $N$–Morse geodesic. Let $\beta \subseteq X$ be a $C$–quasi-geodesic with endpoints at distance $\leq C$ from those of $\alpha$. Then $d_{\text{Haus}}(\alpha, \beta) \leq D$, where $D$ only depends on $C$ and $N$.

(2) Let $\alpha \subseteq X$ be an $N$–Morse geodesic ray. Let $\beta \subseteq X$ be $C$–quasi-geodesic ray with the same starting point and $d_{\text{Haus}}(\alpha, \beta) < +\infty$. Then $d_{\text{Haus}}(\alpha, \beta) \leq D$ and $\beta$ is $N'$–Morse, where $D$ and $N'$ only depend on $C$ and $N$.

(3) Let $\alpha \subseteq X$ be an $N$–Morse $C$–quasi-geodesic. Then there exists an $N'$–Morse geodesic $\beta \subseteq X$ with the same starting point and $d_{\text{Haus}}(\alpha, \beta) \leq D$, where $D$ and $N'$ only depend on $C$ and $N$.

(4) The restriction of an $N$–Morse $C$–quasi-geodesic to a sub-interval of its domain is always $N'$–Morse, where $N'$ only depends on $N$ and $C$.

Proof. Part (1) is easily deduced from [Cor17, Lemma 2.1]. Part (2) is [Cor17, Corollary 2.5]. Part (3) can be proved using [CS15, Lemma 2.5] as in the proof of [Cor17, Lemma 2.9]. Part (4) is [Liu19, Lemma 3.1]. □

Let $f : Y \to X$ be a quasi-isometric embedding. If $\gamma$ is a Morse geodesic in $Y$, the quasi-geodesic $f \circ \gamma$ might still not be Morse in $X$. This motivates the following notion of relative Morse boundary, which is equivalent to the one from the introduction.

Definition 2.2. Set $(\partial M Y, f) := \{[\gamma] \in \partial M Y \mid f \circ \gamma \text{ is Morse in } X\}$. We endow $(\partial M Y, f)$ with the subspace topology coming from $\partial M Y$.

If $[\gamma] \in (\partial M Y, f)$, then the quasi-geodesic ray $f \circ \gamma$ is Morse in $X$, hence at finite Hausdorff distance from a Morse geodesic ray by Lemma 2.1(3). This defines an injection:

$$f_* : (\partial M Y, f) \hookrightarrow \partial M X.$$

The example in the introduction shows that $f_*$ is not continuous in general. However, it is always a closed map (equivalently, an open map, since $f_*$ is injective).

Lemma 2.3. If $f : Y \to X$ is a quasi-isometric embedding, $f_* : (\partial M Y, f) \hookrightarrow \partial M X$ is a closed map.

Proof. Fix a basepoint $q \in Y$ and set $p := f(q)$. Let $A \subseteq (\partial M Y, f)$ be a closed subset. We need to show that the intersection $f_*(A) \cap \partial N^M X_p$ is closed for every Morse gauge $N$. In fact, since $\partial N^M X_p$ is metrisable, it suffices to show that $f_*(A) \cap \partial N^M X_p$ is sequentially closed.

Let $\alpha_n \subseteq Y$ be geodesic rays based at $q$, so that $[\alpha_n] \in A$ and $f_*[\alpha_n] \in \partial N^M X_p$ for some Morse gauge $N$. Suppose that $f_*[\alpha_n] \to \xi$ in $\partial M X$. We need to show that $\xi \in f_*(A)$.

Let $\beta_n \subseteq X$ be $N$–Morse geodesic rays based at $p$ representing $f_*[\alpha_n]$. By Lemma 2.1(2), the quasi-geodesic rays $f \circ \alpha_n$ are at uniformly finite Hausdorff distance from $\beta_n$, say $\leq D$, and they are uniformly Morse in $X$. Since $f$ is a quasi-isometric embedding, it follows that the $\alpha_n$ are uniformly Morse in $Y$, say $N'$–Morse.

Now, by the Arzelà–Ascoli theorem, we can pass to a subsequence and assume that the $\beta_n$ converge uniformly on compact sets to an $N$–Morse geodesic ray $\beta \subseteq X$ based at $p$. Similarly, the $\alpha_n$ converge to an $N'$–Morse ray $\alpha \subseteq Y$ based at $q$. In particular, $[\alpha_n] \to [\alpha]$ in the topology of $\partial M Y$, hence $[\alpha] \in A$. Since $d_{\text{Haus}}(f \circ \alpha_n, \beta_n) \leq D$ for every $n$ and $f$ is a quasi-isometric embedding, we have $d_{\text{Haus}}(f \circ \alpha, \beta) < +\infty$. Hence $\xi = [\beta] = f_*[\alpha] \in f_*(A)$, as required. □

2.2. Divergence. It was shown in [ACGH17] that Morse quasi-geodesics can equivalently be characterised as quasi-geodesics with completely super-linear divergence. The proof of Theorem C will require this equivalence to be effective, in the sense that the Morse gauge and the divergence function of the quasi-geodesic should determine each other. This can be deduced from the proofs of various results in [ACGH17], but it is not explicitly discussed there, so we briefly describe here how to obtain the required statements. We begin with the definition of divergence.
Lemma 2.5. Let $\gamma \subseteq X$ be an $L$–quasi-geodesic ray. Suppose that $\delta_r(r, \epsilon) \leq Cr$ for some $r, C$ and $\epsilon < \frac{1}{L}$. Then there exists $s \geq r$ such that $\gamma(s \pm r)$ are joined by an $L'$–quasi-geodesic $\mathcal{B}(\gamma(s), \epsilon'r)$, where the constants $L'$ and $\epsilon'$ depend only on $L, C, \epsilon$ (and not on $r$).

Corollary 2.6. Given constants $L, \epsilon < \frac{1}{L}$, and a Morse gauge $N$, there exists a weakly increasing, diverging function $f$ such that the following holds. For every $N$–Morse $L$–quasi-geodesic ray $\gamma \subseteq X$, we have $\delta_r(r, \epsilon) \geq r f(r)$ for all $r \geq 0$.

Proof. Fix $\tau \geq 0$. Define $g(\tau)$ as the infimum of the ratio $\delta_r(\tau, \epsilon)/\tau$ as $\gamma$ varies among all $N$–Morse $L$–quasi-geodesic rays in $X$. Let $\gamma$ be one such quasi-geodesic satisfying $\delta_r(\tau, \epsilon)/\tau \leq 2g(\tau)$.

Lemma 2.5 gives $s \geq \tau$ and an $L'$–quasi-geodesic joining $\gamma(s \pm \tau)$ avoiding the ball $\mathcal{B}(\gamma(s), \epsilon' \tau)$, where $\epsilon'$ and $L'$ depend only on $L, \epsilon$ and the value $g(\tau)$. Since $\gamma$ is $N$–Morse, we must have $\epsilon' \tau \leq N(L')$. This implies that $g(r)$ diverges as $r \to +\infty$.

Now, let $f$ be the largest weakly increasing function with $f \leq g$, namely:

$$f(x) = \inf_{t \geq x} g(t).$$

Since $g$ diverges, so does $f$. Finally, if $\gamma \subseteq X$ is an $N$–Morse $L$–quasi-geodesic ray, it is clear that we have $\delta_r(r, \epsilon) \geq r g(r) \geq r f(r)$ for all $r \geq 0$.

The above corollary shows that the divergence function of a Morse quasi-geodesic ray can be bounded uniformly (from below) in terms of the Morse gauge. In order to reverse this kind of result, we need to speak of contracting geodesics.

Definition 2.7. Let $\gamma \subseteq X$ be a quasi-geodesic with closed image.

1. The nearest-point projection $\pi_\gamma : X \to 2^\gamma$ is defined by $\pi_\gamma(x) = \{p \in \gamma \mid d(x, p) = d(x, \gamma)\}$.

   Since $X$ is proper and $\gamma$ is closed, the subset $\pi_\gamma(x) \subseteq \gamma$ is always non-empty.

2. If $\rho$ is a sublinear, weakly increasing, non-negative function, we say that $\gamma$ is $\rho$–contracting if, for all $x, y \in X$ with $d(x, y) \leq d(x, \gamma)$, we have $\text{diam}(\pi_\gamma(x) \cup \pi_\gamma(y)) \leq \rho(d(x, \gamma))$.

From the proof of [ACGH17] Proposition 5.5], we obtain:

Lemma 2.8. Let $\gamma \subseteq X$ be an $L$–quasi-geodesic ray with closed image. Consider points $x, y \in X$ with $d(x, y) \leq d(x, \gamma)$ and projections $x' \in \pi_\gamma(x)$, $y' \in \pi_\gamma(y)$. Then, if $d(x', y') \geq 5L^3$, we have:

$$4d(x, \gamma) \geq \delta_\gamma(r, \frac{1}{10L}),$$

for some $r \in \left[\frac{1}{10L} \cdot d(x', y'), L \cdot d(x', y')\right]$.

Finally, the following is [ACGH17] Proposition 4.1].

Proposition 2.9. Let $\gamma \subseteq X$ be a quasi-geodesic with closed image. If $\gamma$ is $\rho$–contracting, then it is $N$–Morse with $N$ depending only on $\rho$.

We will use the combination of the previous results in the following form.

Lemma 2.10. Let $\alpha \subseteq X$ and $\beta \subseteq Y$ be $L$–quasi-geodesic rays. Suppose that there exist constants $K \geq 0$ and $0 < \epsilon_1, \epsilon_2 \leq \frac{1}{10L}$ such that, for all $r > K$, we have:

$$\delta_\alpha(r, \epsilon_1) \leq K \cdot \delta_\beta(r, \epsilon_2).$$

If $\alpha$ is $N$–Morse in $X$, then $\beta$ is $N'$–Morse in $Y$, with $N'$ depending only on $N, L, K, \epsilon_1$. 5
Proof. We will show that \( \beta \) is \( \rho \)-contracting, with \( \rho \) depending only on \( N, L, K, \epsilon_1 \). The fact that \( \beta \) is \( N' \)-Morse then follows from Proposition 2.9.

By Corollary 2.10 there exists a weakly increasing, diverging function \( f \) such that \( \delta_\alpha(r, \epsilon_1) \geq rf(r) \) for all \( r \geq 0 \). Recall that \( f \) depends only on \( L, \epsilon_1, N \). Setting \( \eta(r) := rf(r)/K \), we have:

\[
\delta_\beta(r, \frac{1}{10r}) \geq \delta_\beta(r, \epsilon_2) \geq \frac{1}{K} \delta_\alpha(r, \epsilon_1) \geq \eta(r),
\]

for all \( r > K \). Note that \( \eta \) is strictly increasing and \( \eta(r)/r \to +\infty \). Define:

\[
\rho'(x) := \sup\{r \mid \eta(r) \leq x\}, \quad \rho(x) := \sup_{t \leq x} \rho'(t).
\]

Observe that \( \rho'(x)/x \to 0 \) for \( x \to +\infty \). In addition, \( \rho \) is weakly increasing and \( \rho \geq \rho' \). Observing that \( \rho' \) is bounded on bounded sets, the fact that \( \rho' \) is sublinear implies that \( \rho \) is sublinear as well.

Now, consider points \( x, y \in Y \) with \( d(x, y) \leq d(x, \beta) \) and projections \( x' \in \pi_\beta(x), \ y' \in \pi_\beta(y) \). Suppose that \( d(x', y') > \max\{5L^3, 4LK\} \). Recalling that \( \eta \) is monotone, Lemma 2.8 shows that:

\[
4d(x, \beta) \geq \eta(\frac{1}{4L} \cdot d(x', y')).
\]

It follows that:

\[
d(x', y') \leq 4L \cdot \rho'(4d(x, \beta)) \leq 4L \cdot \rho(4d(x, \beta)).
\]

This shows that \( \beta \) is \( \rho \)-contracting, where \( \rho \) depends only on \( \eta \), hence only on \( L, \epsilon_1, N, K \). \( \square \)

2.3. Relatively wide subgroups. Relatively wide subgroups were defined in the introduction. This property will be required in the proof of Theorem C in the following form.

Lemma 2.11. Let \( H \leq G \) be finitely generated groups, with \( H \) undistorted and relatively wide in \( G \). Fix a Cayley graph \( \Gamma \) for \( G \). For every \( C \geq 1 \), there exists \( K = K(C) \geq 1 \) with the following property. Let \( \alpha : [−r, r] \to \Gamma \) be a \( C \)-quasi-geodesic with \( \alpha(±r) \in H \subseteq \Gamma(0) \). If \( r > K \), then \( \alpha(±r) \) are joined by an edge path \( \gamma \subseteq \Gamma \) that is disjoint from the ball \( B_\Gamma(\alpha(0), \frac{r}{C}) \) and has length \( |\gamma| \leq K \cdot r \).

Proof. Suppose that the lemma fails for some constant \( C \). Then there exists a sequence of \( C \)-quasi-geodesics \( \alpha : [−r_n, r_n] \to \Gamma \) with \( r_n \to +\infty \), such that every path joining \( \alpha_n(±r_n) \in H \) in \( \Gamma \setminus B_\Gamma(\alpha_n(0), \frac{r_n}{C}) \) has length \( > nr_n \).

Fix a non-principal ultrafilter \( \omega \). Let \( \Gamma_\omega \) be the asymptotic cone of \( \Gamma \) given by basepoints \( \alpha_n(0) \) and scaling factors \( \frac{1}{r_n} \). Let \( x_-, x_0, x_+ \in \Gamma_\omega \) be the points determined by the sequences \( \alpha_n(−r_n), \alpha_n(0), \alpha_n(r_n) \), respectively. The \( \alpha_n \) converge to a \( C \)-bi-Lipschitz curve \( \alpha : [−1, 1] \to \Gamma_\omega \) with \( \alpha(−1) = x_-, \alpha(0) = x_0 \) and \( \alpha(1) = x_+ \).

Since \( H \) is relatively wide in \( G \) and \( x_\pm \) are limits of sequences in \( H \subseteq \Gamma \), the points \( x_\pm \) lie in the same connected component of \( \Gamma_\omega \setminus \{x_0\} \). Since \( \Gamma_\omega \) is a geodesic space, it is locally path connected, hence there exists a continuous path in \( \Gamma_\omega \setminus \{x_0\} \) joining \( x_\pm \). Being compact, this path misses the \( 4\epsilon \)-ball around \( x_0 \) for some \( \epsilon > 0 \). Thus, we can discretise it to a sequence \( x_1, \ldots, x_k \in \Gamma_\omega \setminus B_\Gamma(x_0, 4\epsilon) \) with \( d(x_i, x_{i+1}) \leq \epsilon \) and \( x_1 = x_-, x_k = x_+ \).

Choose approximations \( x_i(n) \in \Gamma \) with \( d(x_i(n), x_{i+1}(n)) \leq 2\epsilon r_n \) and \( d(x_i(n), \alpha(n)(0)) \geq 3\epsilon r_n \), so that \( x_1(n) = \alpha_n(−r_n) \) and \( x_k(n) = \alpha_n(r_n) \). Joining consecutive points by geodesics, we obtain a path of length \( \leq 2K \epsilon r_n \) avoiding \( B_\Gamma(\alpha_n(0), \epsilon r_n) \) and connecting \( \alpha_n(±r_n) \). For large \( n \), we have \( \frac{1}{n} < \epsilon \) and \( n > 2K \epsilon \), which contradicts our initial assumptions. \( \square \)

2.4. Graphs of groups. Throughout the paper, we are interested in finitely generated groups \( G \) that split as a graph of groups. This is equivalent to the fact that \( G \) admits a non-elliptic action on a simplicial tree \( G \curvearrowright T \) without inversions [Ser93]. In this case, there is a unique smallest \( G \)-invariant subtree [CM87]. Restricting to it, we can further assume that the action \( G \curvearrowright T \) is minimal, i.e. that no proper subtree is \( G \)-invariant.

In this subsection, we consider the following setting.
Assumption 2.12. Let $G$ be a group generated by a finite subset $S_G \subseteq G$ with $1 \in S_G$ and $S_G = S_G^{-1}$. Let $\Gamma$ be the corresponding Cayley graph of $G$, endowed with its graph metric $d_{\Gamma}$. Suppose that we have a non-elliptic, minimal action without inversions on a simplicial tree $G \acts T$.

Vertex stabilisers for the action $G \acts T$ will be referred to as vertex groups, and usually denoted by $V$. Similarly, edge groups are $G$–stabilisers of edges of $T$, and will usually be denoted by $E$ when the corresponding edge is not specified. To avoid confusion between edges of $\Gamma$ and $T$, we will denote the latter by fraktur letters $\mathfrak{e}$.

Remark 2.13. The action $G \acts T$ is cocompact. Indeed, fixing a basepoint $p \in T$, let $K$ be the convex hull of $S_G \cdot p$. This is a compact subtree of $T$. Since $S_G$ generates $G$, the subset $G \cdot K$ is connected, hence a $G$–invariant subtree. By minimality, we obtain $T = G \cdot K$.

The following results are certainly well-known to experts, but we were not able to locate proofs in the literature at the required level of generality. Our arguments are inspired by the proof of [Bow98 Proposition 1.2] in the case of hyperbolic groups with quasi-convex edge groups.

Lemma 2.14. Consider a vertex $p \in T$ and let $V$ be its stabiliser. There exist a constant $D_p$ and stabilisers $E_1, \ldots, E_k$ of edges incident to $p$ such that the following holds. Every path $\gamma \subseteq \Gamma$ joining points of $V$ can be decomposed as a concatenation $\gamma_1 \gamma_2 \ldots \gamma_m$ with the following property. The endpoints of each $\gamma_i$ lie in the $D_p$–neighbourhood in $\Gamma$ of a coset $v_i E_{j_i}$ with $v_i \in V$ and $1 \leq j_i \leq k$.

Proof. By Remark 2.13 there are finitely many $V$–orbits of edges of $T$ incident to $p$. Let $e_1, \ldots, e_k$ be a finite list of representatives and let $E_1, \ldots, E_k$ be their stabilisers. Let $K \subseteq T$ be the convex hull of $S_G \cdot p$. If $e \subseteq T$ is an edge, define $\Omega(e) := \{ g \in G \mid e \subseteq gK \}$.

Claim: there exists a constant $D_p$ such that, for every edge $e \subseteq T$ incident to $p$, the set $\Omega(e)$ is contained in the $D_p$–neighbourhood in $\Gamma$ of some coset $vE_i$ with $v \in V$ and $1 \leq i \leq k$.

Proof of claim. Observe that $\Omega(e)$ is a union of at most $N$ right cosets of $E$, where $E$ is the stabiliser of $e$ and $N$ is the number of edges in the compact tree $K$. Thus, there exists a constant $D_p$ such that, for each $1 \leq i \leq k$, the set $\Omega(e_i)$ is contained in the $D_p$–neighbourhood of $E_i$ in $\Gamma$.

For an arbitrary edge $e \ni p$, we can write $e = v e_i$ for some $v \in V$ and some $i$. In this case, $\Omega(e) = \Omega(v e_i) = v \Omega(e_i)$ is contained in the $D_p$–neighbourhood of the left coset $vE_i$.

Now, we define a continuous $G$–equivariant map $f_p : \Gamma \to T$ as follows. For every $s \in S_G$, let $\pi_s \subseteq K$ be the geodesic from $p$ to $sp$. If $g \in G$, we set $f_p(g) = gp$. Then, on each edge $[g, gs] \subseteq \Gamma$ with $g \in G$ and $s \in S_G$, we define $f_p$ as the linear parametrisation of the path $g \pi_s$.

Up to decomposing $\gamma$, we can assume that $\gamma$ meets $V$ only at its endpoints. We can also suppose that $\gamma$ is not a single edge, or the statement is clear. Thus, $f_p \circ \gamma$ is a nontrivial path in the tree $T$ beginning and ending at the basepoint $p$.

Let $e_1, \ldots, e_m \subseteq \Gamma$ be all edges of $\gamma$ for which $f_p(e_i)$ contains $p$, in the order in which they appear along $\gamma$. Thus, $e_1$ and $e_m$ are necessarily the initial and terminal edge of $\gamma$, respectively. Note that the geodesic $f_p(e_i) \subseteq T$ meets $p$ exactly once and not at its endpoints, except for $e_1$ and $e_m$.

Define $\gamma_i \subseteq \gamma$ as the sub-segment between the initial vertex of $e_i$ and the initial vertex of $e_{i+1}$. The portion of the path $f_p(\gamma_i \cup e_{i+1})$ between the two occurrences of $p$ is contained in a connected component of $T \setminus \{p\}$. Thus, it must begin and end by crossing the same edge of $T$ incident to $p$.

We name this edge $f_i$. Say the endpoints of $e_i \subseteq \Gamma$ are $g_i \in G$ and $g_is_i$ with $s_i \in S_G$. Since $f_i \subseteq f_p(e_i) \cap f_p(e_{i+1})$, we have $f_i \subseteq g_i \pi_{s_i} \cap g_{i+1} \pi_{s_{i+1}}$, hence $\{ g_i, g_{i+1} \} \subseteq \Omega(f_i)$. Since $g_i$ and $g_{i+1}$ are the endpoints of $\gamma_i$, the Claim concludes the proof.

If $\alpha \subseteq \Gamma$ is a path, we denote by $|\alpha|$ its length, i.e. the number of edges that it contains. Recall that a finitely generated subgroup of $G$ is undistorted if the inclusion in $G$ is a quasi-isometric embedding.
Lemma 2.15. Let \( p \in T \) be a vertex with stabiliser \( V \). Suppose that the stabiliser of every edge \( p \in e \subseteq T \) is finitely generated and undistorted in \( T \). Then \( V \) is finitely generated and undistorted.

Proof. We will prove the following.

**Claim:** there exists a constant \( L \) such that, for every \( v \in V \), there exists a path \( \alpha \subseteq \Gamma \) from the identity to \( v \) such that \( \alpha \) is contained in the \( L \)-neighbourhood of \( V \) and \( |\alpha| \leq L \cdot d_{\Gamma}(1,v) \).

Assuming the claim, we define \( \Sigma \) as the intersection of \( V \) and the \((2L+1)\)-ball in \( \Gamma \) centred at the identity. Since \( \Gamma \) is locally finite, the set \( \Sigma \) is finite. Denoting by \( d_{\Sigma} \) the word metric on \( V \) induced by \( \Sigma \), the claim shows that \( d_{\Sigma}(1,v) \leq L \cdot d_{\Gamma}(1,v) \) for every \( v \in V \). Thus, \( \Sigma \) generates \( V \) and \( V \) is undistorted in \( G \).

Now, let us prove the claim. Let the constant \( D_{p} \) and the edge groups \( E_{1}, \ldots, E_{k} \leq V \) be those provided by Lemma 2.14. Let \( \gamma \subseteq \Gamma \) be a geodesic from the identity to some element \( v \in V \). Decompose \( \gamma = \gamma_{1}\gamma_{2} \ldots \gamma_{m} \) as in Lemma 2.14, with the endpoints of \( \gamma_{i} \) in the \( D_{p} \)-neighbourhood of a coset \( v_{i}E_{j_{i}} \).

Since \( E_{1}, \ldots, E_{k} \) are finitely generated and undistorted, there exists a constant \( L \) such that, for all \( i \) and all \( x, y \in E_{i} \), the points \( x \) and \( y \) are joined by a path \( \beta \subseteq \Gamma \) contained in the \( L \)-neighbourhood of \( E_{i} \) and satisfying \( |\beta| \leq L \cdot d_{\Gamma}(x,y) \). The same holds if \( x \) and \( y \) lie in a left coset of some \( E_{i} \).

It follows that we can replace each \( \gamma_{i} \) with a path \( \gamma'_{i} \) with the same endpoints so that \( \gamma'_{i} \) is contained in the \( L \)-neighbourhood of \( v_{i}E_{j_{i}} \subseteq V \) and \( |\gamma'_{i}| \leq L \cdot |\gamma_{i}| \). Define \( \alpha \) as the concatenation of the \( \gamma'_{i} \). Then, it is clear that \( \alpha \) is contained in the \( L \)-neighbourhood of \( V \) and \( |\alpha| \leq L \cdot |\gamma| = L \cdot d_{\Gamma}(1,v) \). This proves the claim, hence the lemma.

**Corollary 2.16.** If all edge groups are finitely generated and undistorted in \( G \), then all vertex groups are finitely generated and undistorted in \( G \).

3. PROOF OF THEOREM [A]

**Assumption 3.1.** Let \( G \) be a finitely generated group. Let \( G \rhd T \) be a minimal action on a simplicial tree without inversions. Suppose that all edge-stabilisers \( E \leq G \) are finitely generated, undistorted, and satisfy (\( \partial_{M}E, G \)) = \( \emptyset \).

Fix a finite generating set for \( G \) and let \( \Gamma \) be the corresponding Cayley graph. Choose a basepoint \( p \in T(0) \) and let \( f_{p} : G \to T \) denote the orbit map \( f_{p}(g) = gp \).

The complement in \( T \) of an open edge has exactly two connected components, which we refer to as **halffaces**. Let \( \mathcal{E}(T) \) denote the set of (unoriented) edges of \( T \), and let \( \mathcal{H}(T) \) be the set of halffaces. Note that \( \mathcal{H}(T) \) is naturally in bijection with the set of orientations of \( T \). In particular, every edge \( e \in \mathcal{E}(T) \) gives rise to two halffaces \( h, h^{*} \). The complement of \( h \) is always denoted \( h^{*} \).

We denote by \( G_{h} \leq G \) the stabiliser of the halfface \( h \). Since \( G \) acts without inversions, \( G_{h} \) coincides with the stabiliser \( G_{e} \) of the edge \( e \) associated to \( h \).

Note that every \( h \in \mathcal{H}(T) \) gives rise to a \( G_{h} \)-invariant partition \( G = f_{p}^{-1}(h) \sqcup f_{p}^{-1}(h^{*}) \).

**Lemma 3.2.** We can choose a subgraph \( \Gamma(e) \subseteq \Gamma \) for every \( e \in \mathcal{E}(T) \) so that the following hold:

1. \( \Gamma(e) \) is connected, \( G_{e} \)-invariant and \( G_{e} \)-cocompact;
2. \( \Gamma (ge) = g\Gamma(e) \) for all \( g \in G \) and \( e \in \mathcal{E}(T) \);
3. if \( h, h^{*} \) are the two halffaces determined by \( e \), then \( \Gamma(e) \) contains every edge of \( \Gamma \) with an endpoint in \( f_{p}^{-1}(h) \) and the other in \( f_{p}^{-1}(h^{*}) \);
4. for every \( r \geq 0 \), we have \( \mathcal{N}_{\Gamma}(f_{p}^{-1}(h), r) \cap \mathcal{N}_{\Gamma}(f_{p}^{-1}(h^{*}), r) \subseteq \mathcal{N}_{\Gamma}(\Gamma(e), r) \);
5. each edge \( e \subseteq \Gamma \) lies in the subgraph \( \Gamma(e) \) for only finitely many edges \( e \in \mathcal{E}(T) \).

**Proof.** Recall that edge-stabilisers are finitely generated. If \( S_{e} \) is a finite generating set of \( G_{e} \), we can construct a subgraph \( \Gamma(e) \subseteq \Gamma \) satisfying (1) by choosing paths joining the identity \( 1 \in G \) to the elements of \( S_{e} \), and taking the union of all their \( G_{e} \)-translates. Defining \( \Gamma(e) \) in this fashion
for one edge \( e \) in every \( G \)-orbit of edges of \( T \), and then setting \( \Gamma(g\varepsilon) := g\Gamma(\varepsilon) \), guarantees that condition (2) is also satisfied.

Now, let us ensure that (3) holds. For this, it suffices to show that there are only finitely many \( G_\varepsilon \)-orbits of edges of \( \Gamma \) with an endpoint in \( f_p^{-1}(h) \) and the other in \( f_p^{-1}(h^*) \), as these can then be added to \( \Gamma(\varepsilon) \). If \( S_G \) is the finite generating set of \( G \) giving rise to \( \Gamma \), then every such edge is of the form \([g, gs] \) with \( gp \in h \), \( gsp \in h^* \) and \( s \in S_G \). Note that \( g^{-1}\varepsilon \) is then one of the finitely many edges separating \( p \) and \( sp \) for some \( s \in S_G \). It follows that \( g \) lies in a finite union of right cosets of \( G_\varepsilon \), as required.

Note that (4) follows from (3): if a point \( x \in \Gamma \) lies in the \( r \)-neighbourhood of both \( f_p^{-1}(h) \) and \( f_p^{-1}(h^*) \), then it lies on a path of length \( \leq r \) joining \( f_p^{-1}(h) \) to \( f_p^{-1}(h^*) \). Since \( G = f_p^{-1}(h) \cup f_p^{-1}(h^*) \), condition (3) implies that such a path must contain an edge of \( \Gamma(\varepsilon) \). Hence \( x \) lies in the \( r \)-neighbourhood of \( \Gamma(\varepsilon) \).

Finally, (5) follows from (1). Fix some \( \varepsilon \in \mathcal{E}(T) \). Suppose that \( \varepsilon \subseteq g\Gamma(\varepsilon) \) for some \( g \in G \). Since \( g^{-1}\varepsilon \subseteq \Gamma(\varepsilon) \) and \( G_\varepsilon \) acts cocompactly on \( \Gamma(\varepsilon) \), we deduce that \( g \) must lie in a finite union of left cosets of \( G_\varepsilon \). In other words, \( \varepsilon \) is contained in \( \Gamma(\varepsilon') \) for only finitely many edges \( \varepsilon' \) in the \( G \)-orbit of \( \varepsilon \). Since there are only finitely many \( G \)-orbits in \( \mathcal{E}(T) \), by Remark 2.13 this proves (5). □

Let us choose \( \Gamma(\varepsilon) \subseteq \Gamma \) for every \( \varepsilon \in \mathcal{E}(T) \) as in Lemma 3.2. If \( h \) is one of the two halfspaces determined by \( \varepsilon \), it is convenient to define \( \Gamma(h) \subseteq \Gamma \) as the set \( f_p^{-1}(h) \setminus \Gamma(\varepsilon) \subseteq G \) along with all (half-open) edges of \( \Gamma \setminus \Gamma(\varepsilon) \) that it intersects. Thus, we have a \( G_\varepsilon \)-invariant partition:

\[
\Gamma = \Gamma(h) \cup \Gamma(\varepsilon) \cup \Gamma(h^*).
\]

**Lemma 3.3.** For every Morse gauge \( N \), there exists a constant \( D(N) \) with the following property. If \( h \) is an \( N \)-Morse geodesic with endpoints in \( \Gamma(\varepsilon) \) for some \( \varepsilon \in \mathcal{E}(T) \) (resp. in \( \Gamma(h) \) for some \( h \in \mathcal{H}(T) \)), then \( \varepsilon \) is contained in the \( D(N) \)-neighbourhood of \( \Gamma(\varepsilon) \) (resp. of \( \Gamma(h) \)).

**Proof.** By Lemma 2.14(4) and Lemma 3.2(3), the statement about \( \Gamma(h) \) follows from that on \( \Gamma(\varepsilon) \).

Let \( \varepsilon_1, \ldots, \varepsilon_k \) be representatives for the orbits of \( G \sim \mathcal{E}(T) \) (recall Remark 2.13). Since edge-stabilisers are undistorted, there exists a constant \( L \) such that any two points of \( \Gamma(\varepsilon_i) \) can be connected by a path entirely contained in \( \Gamma(\varepsilon_i) \) that is an \( L \)-quasi-geodesic in \( \Gamma \).

Now, suppose then that \( \beta \) has endpoints \( x, y \in \Gamma(\varepsilon) \). There exist \( g \in G \) and \( 1 \leq i \leq k \) such that \( \Gamma(\varepsilon_i) = g\Gamma(\varepsilon_i) \). Thus, there exists an \( L \)-quasi-geodesic \( \alpha \subseteq \Gamma \) connecting \( x \) and \( y \) within \( \Gamma(\varepsilon) \). By Lemma 2.11(1), the Hausdorff distance between \( \beta \) and \( \alpha \) is at most \( D(N) \), where \( D(N) \) depends only on \( N \) and \( L \). Since \( \alpha \subseteq \Gamma(\varepsilon) \), this proves the lemma. □

**Corollary 3.4.** Let \( \gamma \subseteq \Gamma \) be a Morse geodesic ray. Then:

1. (for every edge \( \varepsilon \in \mathcal{E}(T) \), the intersection \( \gamma \cap \Gamma(\varepsilon) \) is compact;
2. (for every halfspace \( h \in \mathcal{H}(T) \), a sub-ray of \( \gamma \) is entirely contained in either \( \Gamma(h) \) or \( \Gamma(h^*) \);
3. if a sub-ray of \( \gamma \) is contained in \( \Gamma(h) \), then \( \gamma \) must get arbitrarily far from \( \Gamma(h^*) \).

**Proof.** If \( \gamma \cap \Gamma(\varepsilon) \) were non-compact, it would be unbounded and Lemma 3.3 would show that \( \gamma \) stays at bounded distance from \( \Gamma(\varepsilon) \). This would contradict the assumption that \( \partial_M G_\varepsilon(G) = \emptyset \). This proves part (1). Parts (2) and (3) then follow, respectively, from properties (3) and (4) in Lemma 3.2 □

By Corollary 3.4(3), which of the sets \( \Gamma(h) \) and \( \Gamma(h^*) \) contains a sub-ray of \( \gamma \) does not change if we replace \( \gamma \) with a ray at finite Hausdorff distance. This leads us to consider the following (well-defined) subset of \( \partial_M G \) for each \( h \in \mathcal{H}(T) \):

\[
M(h) = \{ \gamma \in \partial_M G \mid \gamma \text{ is eventually contained in } \Gamma(h) \}.
\]

**Lemma 3.5.** For every \( h \in \mathcal{H}(T) \), we have a partition \( \partial_M G = M(h) \cup M(h^*) \) into closed subsets.
Lemma 3.7. Let $h$ be part (1). If $e \in E$ then contain $\sigma$. Conversely, every ultrafilter $\sigma$ is an ultrafilter [Sag14, Definition 2.1].

Remark 3.6. If $\xi, \eta \in \partial_M G$ are in the same connected component, then $\sigma(\xi) = \sigma(\eta)$. Indeed, Lemma 3.5 implies that, for every $h \in \mathcal{H}(T)$, we have either $\{\xi, \eta\} \subseteq M(h)$ or $\{\xi, \eta\} \subseteq M(h^*)$.

Lemma 3.7. Let $\alpha \subseteq \Gamma$ be an $N$–Morse geodesic line whose endpoints at infinity $\alpha^\pm \in \partial_M G$ lie in the same connected component of $\partial_M G$. Set $\sigma := \sigma(\alpha^+)$, which coincides with $\sigma(\alpha^-)$ by Remark 3.6. Then, the following hold:

1. $\alpha$ is contained in the intersection of the $D(N)$–neighbourhoods of the subsets $\Gamma(h) \subseteq \Gamma$ with $h \in \sigma$ (here $D(N)$ is the constant introduced in Lemma 3.3);
2. there exists a vertex $x \in T$ such that $\sigma = \sigma_x$;
3. $\alpha$ stays at bounded distance from the stabiliser $G_x \leq G$.

Proof. Part (1) is immediate from Lemma 3.3. We will prove part (2) by showing that $\sigma$ does not contain any infinite descending chains of halfspaces.

Fix a vertex $g \in \alpha \subseteq \Gamma$. Consider the ultrafilter $\sigma_{gp} \subseteq \mathcal{H}(T)$ determined by the vertex $gp \in T$. If $h \in \sigma \setminus \sigma_{gp}$, then $g \in f_p^{-1}(h^*) \cap \alpha$, while $\alpha$ is contained in the $D(N)$–neighbourhood of $\Gamma(h)$ by part (1). If $e$ is the edge corresponding to $h$, Lemma 3.2(4) guarantees that $\Gamma(e)$ meets the $D(N)$–ball around $g$. By Lemma 3.2(5), the latter can occur only for finitely many edges of $T$.

This shows that the set $\sigma \setminus \sigma_{gp}$ is finite. Since $\sigma_{gp}$ does not contain infinite descending chains, neither does $\sigma$. This proves part (2).

Finally, let us prove part (3). Let $x \in T$ be the vertex provided by part (2). Let $\tau_x \subseteq \sigma_x$ be the subset of halfspaces whose corresponding edge of $T$ is incident to $x$. Let $\Omega \subseteq \Gamma$ be the intersection of the $D(N)$–neighbourhoods of the sets $\Gamma(h)$ with $h \in \tau_x$. Part (1) guarantees that $\alpha \subseteq \Omega$. Since $G_x$ fixes $x$, it leaves invariant $\tau_x$, and thus it also leaves invariant $\Omega \subseteq \Gamma$. In order to prove part (3), it suffices to show that the action $G_x \cap \Omega$ is cocompact. This implies that the Hausdorff distance between $\Omega$ and $G_x$ is finite, so $\alpha$ then stays at bounded distance from $G_x$.

Observe that $\{x\} = \bigcap_{h \in \tau_x} h$, hence $f_p^{-1}(x) = \bigcap_{h \in \tau_x} f_p^{-1}(h)$. The latter is either empty (if $x$ is not in the same $G$–orbit as our basepoint $p \in T$), or a single $G_x$–orbit. Thus, it suffices to show that $G_x$ acts with finitely many orbits on the difference $\Omega^{(0)} \setminus f_p^{-1}(x)$. 

Proof. From the above discussion, it is clear that $\mathcal{M}(h)$ and $\mathcal{M}(h^*)$ are disjoint and cover $\partial_M G$. We only need to prove that they are closed. Choosing some $x \in \Gamma(h)$ as basepoint, it suffices to show that $\mathcal{M}(h) \cap \partial_M^N \Gamma_x$ is sequentially closed for every Morse gauge $N$.

Let $\gamma_n \subseteq \Gamma$ be a sequence of $N$–Morse geodesic rays based at $x$ that converge uniformly on compact sets to an $N$–Morse geodesic ray $\gamma \subseteq \Gamma$. If $[\gamma_n] \in \mathcal{M}(h)$ for every $n$, then Lemma 3.3 guarantees that the $\gamma_n$ are all contained in the $D(N)$–neighbourhood of $\Gamma(h)$. The same neighbourhood must then contain $\gamma$ and, by Corollary 3.4(3), we conclude that $[\gamma] \in \mathcal{M}(h)$. 

If $x \in \partial_M G$, we introduce the following subset of $\mathcal{H}(T)$:

$$
\sigma(\xi) := \{h \in \mathcal{H}(T) \mid \xi \in \mathcal{M}(h)\} = \{h \in \mathcal{H}(T) \mid \text{if } \xi = [\gamma], \text{then } \gamma \text{ is eventually contained in } \Gamma(h)\}.
$$

Note that $\sigma(\xi)$ satisfies the following two properties:

1. if $e \in \mathcal{E}(T)$, then exactly one of the two halfspaces determined by $e$ lies in $\sigma(\xi)$;
2. if $h_1, h_2 \in \sigma(\xi)$, then $h_1 \cap h_2 \neq \emptyset$.

A subset of $\mathcal{H}(T)$ with these properties is known as an ultrafilter [Sag14, Definition 2.1].

For every vertex $x \in T^{(0)}$, an important example of ultrafilter is the set:

$$
\sigma_x := \{h \in \mathcal{H}(T) \mid x \in h\}.
$$

Conversely, every ultrafilter $\sigma \subseteq \mathcal{H}(T)$ that does not contain infinite descending chains of halfspaces is of the form $\sigma_x$ for a vertex $x \in T^{(0)}$ (see for instance [Sag14, Proposition 2.1]).
For every vertex \( y \in \Omega^{(0)} \setminus f_p^{-1}(x) \), there exists \( h \in \tau_x \) such that \( y \) lies in the intersection of \( f_p^{-1}(h^*) \) and the \( D(N) \)-neighbourhood of \( \Gamma(h) \). If \( e \) is the edge corresponding to \( h \), Lemma 3.2(4) shows that \( y \) lies in the \( D(N) \)-neighbourhood of \( \Gamma(e) \). Recall that there are only finitely many \( G_x \)-orbits of edges \( e \) corresponding to elements of \( \tau_x \), that these edges satisfy \( G_e \leq G_x \), and that \( G_e \) acts cocompactly on \( \Gamma(e) \). This shows that \( G_x \) acts with finitely many orbits on \( \Omega^{(0)} \setminus f_p^{-1}(x) \), as required.

**Corollary 3.8.** If a connected component \( \mathcal{C} \subseteq \partial_M G \) is not a singleton, then there exists a vertex \( x \in T \) such that \( \mathcal{C} \) is contained in the image of the natural inclusion \((\partial_M G_x, G) \hookrightarrow \partial_M G\).

**Proof.** By Remark 3.6 there exists an ultrafilter \( \sigma \subseteq \mathcal{H}(T) \) such that \( \sigma = \sigma(\xi) \) for all \( \xi \in \mathcal{C} \). Any two points of \( \mathcal{C} \) are endpoints at infinity of a Morse geodesic line in \( \Gamma \), by [Cor17, Proposition 3.11]. Thus, Lemma 3.7(2) shows that \( \sigma = \sigma_x \) for some \( x \in T^{(0)} \), and Lemma 3.7(3) guarantees that every point of \( \mathcal{C} \) is represented by a ray at bounded distance from \( G_x \).

Corollary 3.8, Lemma 2.3 and Corollary 2.10 have the following immediate consequence.

**Corollary 3.9.** If \((\partial_M G_x, G)\) is totally disconnected for every \( x \in T^{(0)} \), then \( \partial_M G \) is totally disconnected.

The following is not required in the proof of the main theorems, but it looks like a useful observation (for instance, when studying whether \( \partial_M G \) is \( \sigma \)-compact).

**Remark 3.10.** Recall that we have defined an ultrafilter \( \sigma(\xi) \subseteq \mathcal{H}(T) \) for every point \( \xi \in \partial_M G \). This yields a map \( f: \partial_M G \to T := T^{(0)} \sqcup \partial T \), where \( \partial T \) is the set of ends of \( T \).

Every halfspace \( h \in \mathcal{H}(T) \) can be extended to a subset \( \overline{h} \subseteq \overline{T} \) by adding the ends of \( T \) that it contains. This yields a partition \( \overline{T} = \overline{h} \sqcup \overline{h}^* \). It is customary to endow \( \overline{T} \) with the topology having the collection of sets \( \overline{h} \) with \( h \in \mathcal{H}(T) \) as a subbasis.

Lemma 3.5 implies that \( f: \partial_M G \to \overline{T} \) is continuous. Moreover, for every vertex \( x \in T \), the proof of Lemma 3.7(3) shows that \( f^{-1}(x) \) is exactly the set of points of \( \partial_M G \) that are represented by rays in \( \Gamma \) at bounded distance from the stabiliser \( G_x \). If \( \xi \in \partial T \), it is possible to show that \( f^{-1}(\xi) \) is either empty or a singleton, but this requires a bit more work.

We emphasise that the sets \( T^{(0)} \) and \( \partial T \) are neither open nor closed in \( \overline{T} \), since \( T \) will normally be locally infinite in our setting.

**Remark 3.11.** The attentive reader will surely have noticed that the majority of the proof of Theorem A works more generally when \( G \) acts cocompactly on a \( \text{CAT}(0) \) cube complex and all hyperplane-stabilisers are finitely generated, undistorted and with trivial relative Morse boundary in \( G \). We have chosen to restrict to actions on trees (the main case of interest) in order to avoid additional difficulties in the following two spots.

1. If \( G \) acts essentially and cocompactly on a \( \text{CAT}(0) \) cube complex with undistorted hyperplane-stabilisers, it is not clear if vertex-stabilisers will have to be undistorted as well (cf. Corollary 2.16). It is possible that this can be shown along the lines of [GM21, Theorem A].
2. The last paragraph of the proof of Lemma 3.7(3) would be a bit more delicate in a \( \text{CAT}(0) \) cube complex. One would probably need to require stabilisers of all intersections of pairwise-transverse hyperplanes to be finitely generated.

4. Proof of Theorem C

In order to prove that the Morse boundary of the vertex group \( V \) topologically embeds in the Morse boundary of the ambient group \( G \), we will show that uniformly Morse rays \( \alpha \subseteq V \) remain uniformly Morse in \( G \) (see Corollary 4.3 below).
One possible approach to this would consider a quasi-geodesic \( \gamma \subseteq G \) with endpoints on \( \alpha \) and use this to construct a quasi-geodesic \( \gamma' \subseteq V \) with the same endpoints and still getting roughly as far from \( \alpha \) as \( \gamma \). Lemma 2.11 allows us to decompose \( \gamma \) as a concatenation \( \gamma_1 \gamma_2 \ldots \gamma_m \), where the endpoints of each \( \gamma_i \) are near a coset of an edge group \( v_i E_j \subseteq V \). We might thus hope to replace each \( \gamma_i \) with a quasi-geodesic \( \gamma'_i \subseteq v_i E_j \) with nearby endpoints, in order to form a quasi-geodesic \( \gamma' \subseteq V \). Since \( \alpha \) is Morse in \( V \), the quasi-geodesic \( \gamma' \) would then stay close to \( \alpha \) and, if edge groups have trivial relative Morse boundary in \( V \), the \( \gamma'_i \) would all have to be quite short. Hence the \( \gamma_i \) would also be short and \( \gamma \) would stay close to \( \gamma' \) and \( \alpha \), showing that \( \alpha \) is Morse in \( G \).

The problem with this strategy is that, even if \( \gamma \) and the \( \gamma'_i \) are quasi-geodesics, there is no guarantee that the concatenation \( \gamma' = \gamma'_1 \gamma'_2 \ldots \gamma'_m \) will be a quasi-geodesic. Instead, we will follow a different approach based on divergence, which is the content of Proposition 4.2.

Assuming that \( \alpha \) has “slow” divergence function in \( G \), we will show that it also has slow divergence function in \( V \) (which contradicts the assumption that \( \alpha \) is Morse in \( V \)). The proof proceeds along similar lines to the above sketch. If \( \gamma \subseteq G \) were a short path with endpoints on \( \alpha \) avoiding a large ball, then we could decompose \( \gamma \) as \( \gamma_1 \gamma_2 \ldots \gamma_m \), and replace each \( \gamma_i \) with a short path in \( V \), using the assumption that edge groups are relatively wide in \( V \) (in the form of Lemma 2.11). The new path \( \gamma' \) still avoids a ball of roughly the same radius. Here, the advantage is that it does not matter whether or not \( \gamma' \) is a quasi-geodesic. Of course, asking that edge groups be relatively wide in \( V \) might be stronger than simply asking that edge groups have trivial relative Morse boundary in \( V \).

We now give precise proofs, considering the following setting.

**Assumption 4.1.** Let \( G \) be a finitely generated group with a non-elliptic, minimal action without inversions on a simplicial tree \( G \rtimes T \). Fix a base vertex \( p \in T \) with stabiliser \( V \). We assume that the stabiliser of every edge \( \epsilon \subseteq T \) incident to \( p \) is finitely generated and undistorted in \( G \).

Let \( D_p \) and \( E_1, \ldots, E_k \) be as in Lemma 2.11. Choose finite generating sets \( S_{E_i} \subseteq E_i \), \( S_V \subseteq V \) and \( S_G \subseteq G \), ensuring that:

\[
S_{E_1} \cup \cdots \cup S_{E_k} \subseteq S_V \subseteq S_G.
\]

Let \( \Theta_i \subseteq \Delta \subseteq \Gamma \) be the corresponding Cayley graphs of \( E_i, V \) and \( G \), respectively. We denote their intrinsic path metrics by \( d_{\Theta_i}, d_{\Delta} \) and \( d_\Gamma \).

The inclusions \( (\Theta_i, d_{\Theta_i}) \hookrightarrow (\Delta, d_{\Delta}) \hookrightarrow (\Gamma, d_\Gamma) \) are all \( 1 \)-Lipschitz. In view of Lemma 2.15, we can fix a constant \( C \geq 1 \) such that they are all \( C \)-bi-Lipschitz.

We assume that \( E_1, \ldots, E_k \) are relatively wide in \( V \). Let \( K \geq 1 \) be the maximum of the constants \( K(C) \) provided by Lemma 2.11 in relation to the pairs \( E_i \leq V \) with \( 1 \leq i \leq k \) and the Cayley graph \( \Delta \).

We write \( \delta_{\Delta}(\cdot, \cdot) \) and \( \delta_{\Gamma}(\cdot, \cdot) \) when we need to specify the graph used to compute divergence.

**Proposition 4.2.** There exist \( \eta \in (0,1) \) and \( K' \), both depending only on \( C, D_p, K \), such that, for every geodesic ray \( \alpha \subseteq (\Delta, d_{\Delta}) \) and every \( r > K' \), we have:

\[
\delta_{\Delta}(r, \eta \cdot \frac{1}{16C}) \leq K' \cdot \delta_{\Gamma}(r, \eta \cdot \frac{1}{16C}).
\]

**Proof.** Let \( \alpha \subseteq \Delta \) be a geodesic ray based at a vertex. Recall that \( \alpha \) is \( C \)-bi-Lipschitz as a path in \( (\Gamma, d_\Gamma) \). Suppose that there exist integers \( s, r \) such that the vertices \( \alpha(s \pm r) \) can be joined by a path \( \gamma \subseteq \Gamma \) that avoids the ball \( B_\Gamma(\alpha(s), \frac{r}{C}) \).

Our goal is to construct a path \( \gamma' \) joining \( \alpha(s + r) \), so that \( \gamma' \) is entirely contained in \( \Delta \), avoids the ball \( B_\Delta(\alpha(s), \frac{r}{16C}) \), and has length \( |\gamma'| \leq K' \cdot |\gamma| \). The constants \( K' \) and \( \eta \) will be determined at the end of the proof. Note that we are only interested in the situation where \( r > K' \).

By Lemma 2.11, we can decompose \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_m \) with the endpoints \( \gamma_1^{\pm} \) in the neighbourhood \( N_\Gamma(v_i E_j, D_p) \) for some \( v_i \in V \) and \( 1 \leq j \leq k \). Let \( \beta_i \subseteq v_i \Theta_j \) be a geodesic (for \( d_{\Theta_i} \)) whose endpoints \( \beta_i^{\pm} \) satisfy \( d_\Gamma(\beta_i^{\pm}, \gamma_i^{\pm}) \leq D_p \). Note that \( \beta_i \) is contained in \( \Delta \) and is \( C \)-bi-Lipschitz for \( d_\Delta \).
Suppose that \( r > 32C(D_p + K) \). Choose \( \epsilon > 0 \) small enough that:
\[
\frac{1 - \epsilon}{16C} - \epsilon K \geq \frac{1}{32C}.
\]

**Claim:** there exists a path \( \beta_i' \subseteq \Delta \) with the same endpoints as \( \beta_i \), such that \( \beta_i' \) avoids the ball \( B := B_\Delta(\alpha(s), \epsilon \cdot \frac{r}{16C}) \) and has length \( |\beta_i'| \leq K|\beta_i| \).

**Proof of claim.** If \( \beta_i \) is disjoint from \( B \), we can simply take \( \beta_i' = \beta_i \).

Otherwise, choose a vertex \( y \in \beta_i \cap B \) and set \( B' := B_\Delta(y, 2\epsilon \cdot \frac{r}{16C}) \). Observe that \( B \subseteq B' \). Thus, it suffices to construct the path \( \beta_i' \) so that it avoids the ball \( B' \).

Parametrise \( \beta_i \) by arc-length so that \( \beta_i(0) = y \). Suppose first that the domain of \( \beta_i \) contains an interval \([-\rho, \rho]\) satisfying the following three inequalities:
\[
\rho > K, \quad \rho > \frac{\epsilon r}{8C}, \quad \rho > \frac{\epsilon r}{8}.
\]

Since \( \rho > K \) and \( \beta_i \) is contained in a left coset of \( E_{\beta_i} \) in \( V \), we can apply Lemma 2.11 which provides a path \( \beta_i' \subseteq \Delta \) connecting the points \( \beta_i(\pm r) \), avoiding \( B_\Delta(y, \frac{r}{K}) \) and with length \( |\beta_i'| \leq K \rho \).

By the second inequality, this path avoids the ball \( B' \), which is contained in \( B_\Delta(y, \frac{r}{K}) \).

We can then prolong \( \beta_i' \) to \( \beta_i \) until it reaches \( \beta_i^\pm \). Since \( \beta_i \) is \( C \)-bi-Lipschitz with respect to \( d_\Delta \), the third inequality ensures that \( \beta_i \) can only meet the ball \( B' \) at times in the interval \((-\rho, \rho)\). Thus, \( \beta_i' \) avoids \( B' \) even after prolonging. This is the path required by the claim.

We are left to consider the case when the domain of \( \beta_i \) does not contain a sufficiently long interval \([-\rho, \rho] \). Suppose without loss of generality that \( d_\Gamma(y, \beta_i^-) \leq d_\Gamma(y, \beta_i^+) \). Then, recalling that \( \beta_i \) is \( 1 \)-Lipschitz with respect to \( d_\Gamma \), we must have:
\[
\max\{K, \frac{\epsilon r}{8C}, \frac{\epsilon r}{8}\} \geq d_\Gamma(y, \beta_i^-) \geq d_\Gamma(y, \gamma_i^-) - D_p \geq d_\Gamma(\gamma_i^-, \alpha(s)) - d_\Gamma(y, \alpha(s)) - D_p \\
\geq (1 - \epsilon)\frac{r}{16C} - D_p.
\]

The last inequality is due to the fact that \( \gamma_i^- \) lies on \( \gamma \), hence outside \( B_\Gamma(\alpha(s), \frac{r}{16C}) \), whereas \( y \) lies within \( B \), hence also in the larger ball \( B_\Gamma(\alpha(s), \epsilon \cdot \frac{r}{16C}) \).

Given our choice of \( \epsilon \), and recalling that \( r > 32C(D_p + K) \), we obtain a contradiction:
\[
(1 - \epsilon)\frac{r}{16C} - D_p \geq \left(\frac{1}{32C} + 2\epsilon K\right) \cdot r - D_p > K + \epsilon Kr \geq \max\{K, \frac{\epsilon r}{8C}, \frac{\epsilon r}{8}\}.
\]

Now, we complete the proof of the proposition. Recall that \( \beta_i' \) has the same endpoints as \( \beta_i \), namely \( \beta_i^\pm \). Let \( \zeta_i \subseteq \Delta \) be a shortest path connecting the points \( \beta_i^+ \) and \( \beta_i^- \). Joining the paths \( \beta_i' \) provided by the claim by the paths \( \zeta_i \), we form a path \( \gamma_i' \) entirely contained in \( \Delta \) and with the same endpoints as \( \gamma \).

Recalling that \( \gamma_i^+ = \gamma_i^- + 1 \) and \( d_\Gamma(\beta_i^\pm, \gamma_i^\pm) \leq D_p \), we have:
\[
d_\Delta(\beta_i^\pm, \gamma_i^\pm) \leq C \cdot d_\Gamma(\beta_i^\pm, \gamma_i^\pm) \leq 2CD_p.
\]
It follows that \( |\zeta_i| \leq 2CD_p \) for each \( i \).

Now, suppose that \( r > 32C^2D_p \). Then, since each \( \beta_i' \) avoids the ball \( B = B_\Delta(\alpha(s), \epsilon \cdot \frac{r}{16C}) \), while each \( \zeta_i \) has length at most \( 2CD_p \), the path \( \gamma_i' \) avoids the ball:
\[
B_\Delta(\alpha(s), \epsilon \cdot \frac{r}{16C} - 2CD_p) \supseteq B_\Delta(\alpha(s), \frac{r}{2} \cdot \frac{r}{16C}).
\]

Thus, setting \( \eta \) such that \( \frac{r}{2} \cdot \frac{r}{16C} \), the path \( \gamma_i' \) avoids the ball \( B_\Delta(\alpha(s), \eta \cdot \frac{r}{16C}) \).

We are left to bound the length of \( \gamma_i' \):
\[
|\gamma_i'| \leq \sum |\zeta_i| + \sum |\beta_i'| \leq m \cdot 2CD_p + K \cdot \sum |\beta_i| = 2mCD_p + K \cdot \sum d_\alpha(v_i^{-1} \beta_i^-, v_i^{-1} \beta_i^+) \\
\leq 2CD_p \cdot |\gamma_i| + 2K \cdot \sum d_\Gamma(\beta_i^-, \beta_i^+) \\
\leq 2CD_p \cdot |\gamma_i| + 2K \cdot (|\zeta_i| + 2D_p) \leq (2CD_p + 2K^2 + 2K^2D_p) |\gamma_i|.
\]
In conclusion, taking

$$K' := \max\{32C(D_p + K), \frac{32C^2 D_p}{\epsilon}, 2CD_p + KC^2 + 2KC^2 D_p\}$$

and assuming that $r > K'$, we have constructed a path $\gamma'$ with the same endpoints as $\gamma$, entirely contained in $\Delta$, avoiding the ball $B_{\Delta}(\alpha(s), \eta \cdot \frac{1}{16C})$, and with length $|\gamma'| \leq K' \cdot |\gamma|$. This shows that

$$\delta_\alpha^\gamma(r, \eta \cdot \frac{1}{16C}) \leq K' \cdot \delta_\alpha^\Gamma(r, \frac{1}{16C}),$$

as required.

Along with Lemma 2.10, Proposition 4.2 has the following consequence.

**Corollary 4.3.** For every Morse gauge $N$, there exists a Morse gauge $N'$ such that every $N$–Morse geodesic ray $\alpha \subseteq \Delta$ is an $N'$--Morse quasi-geodesic in $\Gamma$.

We are now ready to prove Theorem C.

**Proof of Theorem C.** Corollaries 2.16 and 4.3 imply that $V$ is undistorted and $(\partial_M V, G) = \partial_M V$. In addition, Corollary 4.3 guarantees that the natural inclusion $\partial_M V \hookrightarrow \partial_M G$ is a Morse preserving map in the sense of [Cor17, Definition 4.1]. Thus, the fact that $\partial_M V \hookrightarrow \partial_M G$ is a topological embedding follows from [Cor17, Proposition 4.2].

**REFERENCES**

[ACGH17] Goulnara N. Arzhantseva, Christopher H. Cashen, Dominik Gruber, and David Hume. Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction. Doc. Math., 22:1193–1224, 2017.

[Bow98] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. Acta Math., 180(2):145–186, 1998.

[CCS20] Ruth Charney, Matthew Cordes, and Alessandro Sisto. Complete topological descriptions of certain Morse boundaries. arXiv:1908.03542v2, 2020.

[CM87] Marc Culler and John W. Morgan. Group actions on $\mathbb{R}$-trees. Proc. London Math. Soc. (3), 55(3):571–604, 1987.

[CM19] Christopher H. Cashen and John M. Mackay. A metrizable topology on the contracting boundary of a group. Trans. Amer. Math. Soc., 372(3):1555–1600, 2019.

[Cor17] Matthew Cordes. Morse boundaries of proper geodesic metric spaces. Groups Geom. Dyn., 11(4):1281–1306, 2017.

[CS15] Ruth Charney and Harold Sultan. Contracting boundaries of $\text{CAT}(0)$ spaces. J. Topol., 8(1):93–117, 2015.

[DGO17] François Dahmani, Vincent Guirardel, and Denis V. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. Mem. Amer. Math. Soc., 245(1156):v+152, 2017.

[DMS10] Cornelia Drutu, Shahar Mozes, and Mark Sapir. Divergence in lattices in semisimple Lie groups and graphs of groups. Trans. Amer. Math. Soc., 362(5):2451–2505, 2010.

[DS05] Cornelia Drutu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. Topology, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.

[GM21] Daniel Groves and Jason F. Manning. Hyperbolic groups acting improperly. arXiv:1808.02325v4, 2021.

[Gro87] Misha Gromov. Hyperbolic groups. In Essays in group theory, volume 8 of Math. Sci. Res. Inst. Publ., pages 75–263. Springer, New York, 1987.

[Kar21] Annette Karrer. Right-angled Coxeter groups with totally disconnected Morse boundaries. arXiv:2105.04093v2, 2021.

[Liu19] Qing Liu. Dynamics on the Morse boundary. arXiv:1905.01404v1, 2019.

[Mur19] Devin Murray. Topology and dynamics of the contracting boundary of cocompact $\text{CAT}(0)$ spaces. Pacific J. Math., 299(1):89–116, 2019.

[Sag14] Michah Sageev. $\text{CAT}(0)$ cube complexes and groups. In Geometric group theory, volume 21 of IAS/Park City Math. Ser., pages 7–54. Amer. Math. Soc., Providence, RI, 2014.

[Ser03] Jean-Pierre Serre. Trees. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[Sis16] Alessandro Sisto. Quasi-convexity of hyperbolically embedded subgroups. Math. Z., 283(3-4):649–658, 2016.
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