**L_P SPECTRAL MULTIPLIERS ON THE FREE GROUP N_{3,2}**

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**Abstract.** Let $L$ be the homogeneous sublaplacian on the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators. We prove a theorem of Mihlin-Hörmander type for the functional calculus of $L$, where the order of differentiability $s > 6/2$ is required on the multiplier.

1. Introduction

The free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators is the simply connected, connected nilpotent Lie group defined by the relations

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2,$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ is a basis of its Lie algebra (that is, the Lie algebra of the left-invariant vector fields on $N_{3,2}$). In exponential coordinates, $N_{3,2}$ can be identified with $\mathbb{R}^3 \times \mathbb{R}^3$, where the group law is given by

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \wedge x'/2)$$

and $x \wedge x'$ denotes the usual vector product of $x, x' \in \mathbb{R}^3$. The family $(\delta_t)_{t>0}$ of automorphic dilations of $N_{3,2}$, defined by

$$\delta_t(x, y) = (tx, t^2y),$$

turns $N_{3,2}$ into a stratified group of homogeneous dimension $Q = 9$.

Let $L = -(X_1^2 + X_2^2 + X_3^2)$ be the homogeneous sublaplacian on $N_{3,2}$. $L$ is a self-adjoint operator on $L^2(N_{3,2})$, hence a functional calculus for $L$ is defined via spectral integration and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(N_{3,2})$ whenever the “spectral multiplier” $F$ is a bounded function. Here we are interested in giving a sufficient condition for the $L^p$-boundedness (for $p \neq 2$) of the operator $F(L)$, in terms of smoothness properties of the multiplier $F$.

Let $W^s_2(\mathbb{R})$ denote the $L^2$ Sobolev space of (fractional) order $s$. Then our main result reads as follows.

**Theorem 1.** Suppose that a function $F : \mathbb{R} \to \mathbb{C}$ satisfies

$$\sup_{t>0} \|\eta F(t \cdot)\|_{W^s_2} < \infty$$

for some $s > 6/2$ and some nonzero $\eta \in C^\infty_c([0, \infty[)$. Then the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(N_{3,2})$ for all $p \in [1, \infty[.$

**Remark.** Observe that the general multiplier theorem for homogeneous sublaplacians on stratified Lie groups by Christ [3] and Mauceri and Meda [16] requires the stronger regularity condition $s > Q/2 = 9/2$. To the best of our knowledge, in the case of $N_{3,2}$ none of the results and techniques known so far allowed one to go below the condition $s > Q/2$. Our result pushes the regularity assumption down to

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s > d/2 = 6/2, where d = 6 is the topological dimension of N_{3,2}. We conjecture that this condition is sharp.

The problem of $L^p$-boundedness for spectral multipliers on nilpotent Lie groups has a long history, and the theorem by Christ and Mauceri and Meda is itself an improvement of a series of previous results (see, e.g., [4, 8, 5]). Nevertheless it is still an open question, whether the homogeneous dimension in the smoothness condition may always be replaced by the topological dimension.

It has been known for a long time [10, 17] that such an improvement of the multiplier theorem holds true in the case of the Heisenberg and related groups (more precisely, for direct products of Métivier and abelian groups; see also [11, 14]). This class of groups, however, does not include $N_{3,2}$, nor any free 2-step nilpotent group $N_{n,2}$ on $n$ generators (see [20, §3] for a definition), except for the smallest one, $N_{2,2}$, which is the 3-dimensional Heisenberg group. The free groups $N_{n,2}$ have in a sense the maximal structural complexity among 2-step groups, since every 2-step nilpotent Lie group is a quotient of a free one. Our result should then hopefully shed some new light and contribute to the understanding of the problem for general 2-step nilpotent Lie groups.

2. Strategy of the proof

The sublaplacian $L$ is a left-invariant operator on $N_{3,2}$, hence any operator of the form $F(L)$ is left-invariant too. Let $K_{F(L)}$ then denote the convolution kernel of $F(L)$. As shown, e.g., in [14, Theorem 4.6], the previous Theorem 1 is a consequence of the following $L^1$-estimate.

Proposition 2. For all $s > 6/2$, for all compact sets $K \subseteq [0, \infty]$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that $\text{supp} \, F \subseteq K$,

$$\|K_{F(L)}\|_1 \leq C_{K,s}\|F\|_{W^{s,2}}.$$  

Let $|\cdot|_\delta$ be any $\delta$-homogeneous norm on $N_{3,2}$; take, e.g., $|(x, y)|_\delta = |x| + |y|^{1/2}$. The crucial estimate in the proof of [16] of the general theorem for stratified groups, that is,

$$\|(1 + |\cdot|_\delta)^\alpha K_{F(L)}\|_2 \leq C_{K,\alpha,\beta}\|F\|_{W^{\beta,2}}$$

for all $\alpha \geq 0$ and $\beta > \alpha$, implies (1) when $s > 9/2$, by Hölder’s inequality. In order to push the condition down to $s > 6/2$, here we prove an enhanced version of (2), that is,

$$\|(1 + |\cdot|_\delta)^\alpha w^r K_{F(L)}\|_2 \leq C_{K,\alpha,\beta,r}\|F\|_{W^{\beta,2}},$$

for some “extra weight” function $w$ on $N_{3,2}$, and suitable constraints on the exponents $\alpha, \beta, r$.

A similar approach is adopted in the mentioned works on the Heisenberg and related groups. However, in [17] the extra weight $w$ is the full weight $1 + |\cdot|_\delta$, while [10] employs the weight $w(x, y) = 1 + |x|$. Here instead the weight $w(x, y) = 1 + |y|$ is used, and (3) is proved under the conditions $\alpha \geq 0, 0 \leq r < 3/2, \beta > \alpha + r$ (see Proposition 9 below).

The proof of (5) when $\alpha = 0$ is based on a careful analysis exploiting identities for Laguerre polynomials, somehow in the spirit of [4, 17, 19], but with additional complexity due, inter alia, to the simultaneous use of generalized Laguerre polynomials of different types. The estimate for arbitrary $\alpha$ is then recovered by interpolation with (2). An analogous strategy is followed in [15], where identities for Hermite polynomials are used in order to prove a sharp spectral multiplier theorem for Grushin operators.
3. A joint functional calculus

It is convenient for us to embed the functional calculus for the sublaplacian \( L \) in a larger functional calculus for a system of commuting left-invariant differential operators on \( N_{3,2} \). Specifically, the operators

\[
L_i = -iY_i, \quad i = 1, 2, 3
\]

are essentially self-adjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [13]).

If \( Y \) denotes the “vector of operators” \((-iY_1, -iY_2, -iY_3)\), then we can express the convolution kernel \( K_{G(L,Y)} \) of the operator \( G(L,Y) \) in terms of Laguerre functions (cf. [7]). Namely, for all \( n, k \in \mathbb{N} \), let

\[
L_n^{(k)}(u) = \frac{u^{-k}e^u}{n!} \left( \frac{d}{du} \right)^n (u^{k+n} e^{-u})
\]

be the \( n \)-th Laguerre polynomial of type \( k \), and define

\[
L_n^{(k)}(t) = 2^{(-1)^n} e^{-t} L_n^{(k)}(2t).
\]

Further, for all \( \eta \in \mathbb{R}^3 \setminus \{0\} \) and \( \xi \in \mathbb{R}^3 \), define \( \xi_0^\eta \) and \( \xi_1^\eta \) by

\[
\xi_0^\eta = \langle \xi, \eta/|\eta| \rangle, \quad \xi_1^\eta = \xi - \xi_0^\eta \eta/|\eta|.
\]

**Proposition 3.** Let \( G : \mathbb{R}^4 \to \mathbb{C} \) be in the Schwartz class, and set

\[
m(n, \mu, \eta) = G((2n+1)|\eta| + \mu^2, \eta),
\]

for all \( n \in \mathbb{N}, \mu \in \mathbb{R}, \xi, \eta \in \mathbb{R}^3 \) with \( \eta \neq 0 \). Then

\[
K_{G(L,Y)}(x,y) = \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n, \xi_0^\eta, \eta) L_n^{(0)}(|\xi_0^\eta / |\eta||) e^{i\langle \xi, x \rangle} e^{i\langle \eta, \mu \rangle} d\xi d\eta.
\]

**Proof.** For all \( \eta \in \mathbb{R}^3 \setminus \{0\} \), choose a unit vector \( E_\eta \in \eta^\perp \), and set \( \bar{E}_\eta = (\eta/|\eta|) \wedge E_\eta \); moreover, for all \( x \in \mathbb{R}^3 \), denote by \( x_0^\eta, x_1^\eta, x_\parallel^\eta \) the components of \( x \) with respect to the positive orthonormal basis \( E_\eta, \bar{E}_\eta, \eta/|\eta| \) of \( \mathbb{R}^3 \).

For all \( \eta \in \mathbb{R}^3 \setminus \{0\} \) and all \( \mu \in \mathbb{R} \), an irreducible unitary representation \( \pi_{\eta, \mu} \) of \( N_{3,2} \) on \( L^2(\mathbb{R}) \) is defined by

\[
\pi_{\eta, \mu}(x, y) \phi(u) = e^{i\langle y, \mu \rangle} e^{i\langle \eta(u+x_\parallel^\eta)/2, x_0^\eta \rangle} e^{i\eta/2} \phi(x_0^\eta + u)
\]

for all \( (x, y) \in N_{3,2}, u \in \mathbb{R}, \phi \in L^2(\mathbb{R}) \). Following, e.g., [11] §2, one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of \( N_{3,2} \), and the corresponding Fourier inversion formula:

\[
f(x,y) = (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \text{tr}(\pi_{\eta, \mu}(x,y) \pi_{\eta, \mu}(f)) \langle \eta \rangle d\mu d\eta
\]

for all \( f : N_{3,2} \to \mathbb{C} \) in the Schwartz class and all \( (x, y) \in N_{3,2} \), where

\[
\pi_{\eta, \mu}(f) = \int_{N_{3,2}} f(z) \pi_{\eta, \mu}(z^{-1}) dz.
\]

Fix \( \eta \in \mathbb{R}^3 \setminus \{0\} \) and \( \mu \in \mathbb{R} \). The operators \([11]\) are represented in \( \pi_{\eta, \mu} \) as

\[
d\pi_{\eta, \mu}(L) = -\partial_\mu^2 + |\eta|^2 u^2 + \mu^2, \quad d\pi_{\eta, \mu}(-iY_j) = \eta_j.
\]

If \( h_n \) is the \( n \)-th Hermite function, that is,

\[
h_n(t) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{t^2/2} \left( \frac{d}{dt} \right)^n e^{-t^2},
\]

and \( \bar{h}_{\eta,n} \) is defined by

\[
\bar{h}_{\eta,n}(u) = |\eta|^{1/4} h_n(|\eta|^{1/2} u),
\]
then \( \{ \hat{h}_{n,n} \}_{n \in \mathbb{N}} \) is a complete orthonormal system for \( L^2(\mathbb{R}) \), made of joint eigenfunctions of the operators \( \Pi \); in fact,
\[
(8) \quad d\pi_{\eta,\mu}(L)\hat{h}_{n,n} = (|\eta|(2n + 1) + \mu^2)\hat{h}_{n,n},
\]
\[
d\pi_{\eta,\mu}(-iY)\hat{h}_{n,n} = \eta \hat{h}_{n,n}.
\]
Moreover the corresponding diagonal matrix coefficients \( \varphi_{\eta,\mu,n} \) of \( \pi_{\eta,\mu} \) are given by
\[
(9) \quad \varphi_{\eta,\mu,n}(x,y) = \langle \pi_{\eta,\mu}(x,y)\hat{h}_{n,n}, \hat{h}_{n,n} \rangle
\]
\[
= e^{i\langle \eta, y \rangle}e^{i\mu x_1^n}|\eta|^{1/2} \int_{\mathbb{R}} e^{i|\eta|u x_2^n} h_n(|\eta|^{1/2}(u + x_1^n/2)) h_n(|\eta|^{1/2}(u - x_1^n/2))\,du.
\]
The last integral is essentially the Fourier-Wigner transform of the pair \( (h_n, h_n) \),
whose Fourier transform has a particularly simple expression (cf. [9, formula (1.90)]);
the parity of the Hermite functions then yields
\[
\varphi_{\eta,\mu,n}(x,y) = e^{i\langle \eta, y \rangle}e^{i\mu x_1^n} |\eta|^{1/2} \int_{\mathbb{R}} e^{iv_2 x_2^n} e^{iv_1 x_1^n} \times \int_{\mathbb{R}} e^{-it(2v_1/|\eta|^{1/2})} h_n(t + v_2/|\eta|^{1/2}) h_n(t - v_2/|\eta|^{1/2}) \, dt \, dv,
\]
that is,
\[
(10) \quad \varphi_{\eta,\mu,n}(x,y) = \frac{1}{2\pi|\eta|} e^{i\langle \eta, y \rangle}e^{i\mu x_1^n} \int_{\mathbb{R}} e^{iv_2 x_2^n} e^{iv_1 x_1^n} \mathcal{L}^0_n(|v|^2/|\eta|) \, dv
\]
(see [21, Theorem 1.3.4] or [9, Theorem 1.104]).

Note that \( K_{G(L,Y)} \in \mathcal{S}(N_{3,2}) \) since \( G \in \mathcal{S}(\mathbb{R}^4) \) (see [2, Theorem 5.2] or [12, §4.2]). Moreover
\[
\pi_{\eta,\mu}(K_{G(L,Y)})\hat{h}_{n,n} = G(|\eta|(2n + 1) + \mu^2, \eta)\hat{h}_{n,n}
\]
by \((8)\) and \([9, \text{Proposition 1.1}]\), hence
\[
\langle \pi_{\eta,\mu}(x,y)\pi_{\eta,\mu}(K_{G(L,Y)})\hat{h}_{n,n}, \hat{h}_{n,n} \rangle = m(n, \mu, \eta) \varphi_{\eta,\mu,n}(x,y).
\]

Therefore, by \((6)\) and \((9)\),
\[
K_{G(L,Y)}(x,y) = (2\pi)^{-5} \int_{\mathbb{R}^2 \setminus \{0\}} \int_{\mathbb{R}^2} m(n, \mu, \eta) \varphi_{\eta,\mu,n}(x,y) |\eta| \, d\mu \, d\eta
\]
\[
= (2\pi)^{-6} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} m(n, \xi_3, \eta) e^{i\langle \eta, y \rangle}e^{i\langle \xi, (x_1^n, x_2^n, x_3^n) \rangle} \mathcal{L}^0_n((\xi_1^2 + \xi_2^2)/|\eta|) \, d\xi \, d\eta.
\]
The conclusion follows by a change of variable in the inner integral. \(\square\)

4. Weighted estimates

For convenience, set \( \mathcal{L}_n^{(k)} = 0 \) for all \( n < 0 \). The following identities are easily obtained from the properties of Laguerre polynomials (see, e.g., [6, §10.12]).

**Lemma 4.** For all \( k, n, n' \in \mathbb{N} \) and \( t \in \mathbb{R} \),
\[
(10) \quad \mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t),
\]
\[
(11) \quad \frac{d}{dt} \mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t),
\]
\[
(12) \quad \int_0^\infty \mathcal{L}_n^{(k)}(t) \mathcal{L}_{n'}^{(k)}(t) \, t^k \, dt = \begin{cases}
\left(\frac{n+k}{2}\right)^k & \text{if } n = n', \\
0 & \text{otherwise}.
\end{cases}
\]
We introduce some operators on functions $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$:

\[
\tau f(n, \mu, \eta) = f(n + 1, \mu, \eta),
\delta f(n, \mu, \eta) = f(n + 1, \mu, \eta) - f(n, \mu, \eta),
\partial_\mu f(n, \mu, \eta) = \frac{\partial}{\partial \mu} f(n, \mu, \eta),
\partial_\eta^\alpha f(n, \mu, \eta) = \left(\frac{\partial}{\partial \eta}\right)^\alpha f(n, \mu, \eta),
\]

for all $\alpha \in \mathbb{N}^3$. For all multiindices $\alpha \in \mathbb{N}^3$, we denote by $|\alpha|$ its length $\alpha_1 + \alpha_2 + \alpha_3$.

We set moreover $\langle t \rangle = 2|t| + 1$ for all $t \in \mathbb{R}$.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$; hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

\[
(1 + \tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f,
\]

and therefore the operator $(1 + \tau)^q$ is well-defined for all $q \in \mathbb{Z}$.

**Proposition 5.** Let $G : \mathbb{R}^4 \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and let $m(n, \mu, \eta)$ be defined by [3]. For all $\alpha \in \mathbb{N}^3$,

\[
\int_{\mathbb{R}^3} |y^\alpha \mathcal{K}_G(x,y)|^2 \, dx \, dy \\
\leq C_\alpha \sum_{\ell \in I_\alpha} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^\gamma \partial_\mu^\delta (1 + \tau)^{|\beta| - k} m(n, \mu, \eta)|^2 \\
\times \mu^{2h_1} |\eta|^{2|\gamma| - 2|\alpha| - 2k_1 + |\beta| + 1} |n|^{3|\beta|} \, d\mu \, d\eta,
\]

where $I_\alpha$ is a finite set and, for all $\ell \in I_\alpha$,

- $\gamma_1 \in \mathbb{N}_0$, $l, k_1 \in \mathbb{N}$, $\gamma_1 \leq \alpha$, $\min\{1, |\alpha|\} \leq |\gamma_1| + l + k_2 \leq |\alpha|$, \\
- $b_1 \in \mathbb{N}$, $\beta_1 \in \mathbb{N}_0$, $b_1 + |\beta_1| = l_1 + 2k_1$, $|\gamma_1| + l + b_1 \leq |\alpha|$.

**Proof.** Proposition [3] and integration by parts allow us to write

\[
y^{\alpha} \mathcal{K}_G(x,y) = \frac{i^{|\alpha|}}{(2\pi)^6} \int_{\mathbb{R}^3} \left( \frac{\partial}{\partial \eta} \right)^\alpha \sum_{n \in \mathbb{N}} m(n, \xi_1^\eta, \eta) \mathcal{L}_n^{(0)} \left( |\xi_1^\eta|^2 / |\eta| \right) \right] e^{i\langle x, \eta \rangle} \, d\xi d\eta.
\]

From the definition of $\xi_\parallel$ and $\xi_\perp$, the following identities are not difficult to obtain:

\[
\frac{\partial}{\partial n_1} \xi_\parallel = (\xi_1^\eta) \left( \frac{1}{|\eta|} \right), \quad \frac{\partial}{\partial n_2} (\xi_2^\eta) = -\xi_\parallel \frac{\partial}{\partial n_2} \frac{\eta_2}{|\eta|} - (\xi_2^\eta) \frac{\eta_2}{|\eta|^2},
\]

\[
\frac{\partial}{\partial n_2} \frac{\xi_2^\eta}{|\eta|} = -\xi_\parallel (\xi_2^\eta) \left( \frac{2}{|\eta|^2} - \frac{\eta_2^2}{|\eta|^5} \right).
\]

The multiindex notation will also be used as follows:

\[
(\xi_1^\eta)^{\beta_1} = (\xi_1^\eta_1)^{\beta_1} \cdot (\xi_1^\eta_2)^{\beta_2} \cdot (\xi_1^\eta_3)^{\beta_3}
\]

for all $\xi, \eta \in \mathbb{R}$, with $\eta \neq 0$, and all $\beta \in \mathbb{N}_0^3$; consequently

\[
|\xi_1^\eta|^2 = (\xi_1^\eta)^{(2,0,0)} + (\xi_1^\eta)^{(0,2,0)} + (\xi_1^\eta)^{(0,0,2)}.
\]
Via these identities, one can prove inductively that, for all $\alpha \in \mathbb{N}^3$,

$$\left( \frac{\partial}{\partial \eta} \right)^{\alpha} \sum_{n \in \mathbb{N}} m(n, \xi_\perp^n, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^n|^2/|\eta|)$$

$$= \sum_{\varsigma \in I_\alpha} \sum_{\eta \in \mathbb{N}} \partial_{\eta}^{\varsigma} \partial_{\mu}^{\varsigma} \delta^k \cdot m(n, \xi_\parallel^n, \eta) (\xi_\parallel^n)^{\varsigma} (\xi_\perp^n)^{\varsigma} \Theta_i(\eta) \mathcal{L}_n^{(k)}(|\xi_\perp^n|^2/|\eta|),$$

where $I_\alpha$, $\gamma$, $l$, $b$, $\beta$ are as in the statement above, while $\Theta_i : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $|\gamma| - |\alpha| - k$. For the inductive step, one employs Leibniz’ rule, and when a derivative hits a Laguerre function, the identity \((11)\) together with summation by parts is used.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$,

$$\sum_{n \in \mathbb{N}} f(n, \mu, \eta) \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau) f(n, \mu, \eta) \mathcal{L}_n^{(k+1)}(t),$$

by \((10)\). Since $1 + \tau$ is invertible, simple manipulations and iteration yield the more general identity

$$\sum_{n \in \mathbb{N}} f(n, \mu, \eta) \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau)^{k-k'} f(n, \mu, \eta) \mathcal{L}_n^{(k')}(t),$$

for all $k, k' \in \mathbb{N}$. This formula allows us to adjust in \((16)\) the type of the Laguerre functions to the exponent of $\xi_\perp$, and to obtain that

$$\left( \frac{\partial}{\partial \eta} \right)^{\alpha} \sum_{n \in \mathbb{N}} m(n, \xi_\parallel^n, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^n|^2/|\eta|)$$

$$= \sum_{\varsigma \in I_\alpha} \sum_{\eta \in \mathbb{N}} \partial_{\eta}^{\varsigma} \partial_{\mu}^{\varsigma} \delta^k \cdot (1 + \tau)^{|\beta^\perp| - k} \cdot m(n, \xi_\parallel^n, \eta) (\xi_\parallel^n)^{\varsigma} (\xi_\perp^n)^{\varsigma} \Theta_i(\eta) \mathcal{L}_n^{(|\beta^\parallel|)}(|\xi_\perp^n|^2/|\eta|),$$

By plugging this identity into \((14)\) and exploiting Plancherel’s formula for the Fourier transform, the finiteness of $I_\alpha$ and the triangular inequality, we get that

$$\int_{N_{3,2}} |y^a K_{G(L, X)}(x, y)|^2 \, dx \, dy$$

$$\leq C_o \sum_{\varsigma \in I_\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{N}} \left| \partial_{\eta}^{\varsigma} \partial_{\mu}^{\varsigma} \delta^k \cdot (1 + \tau)^{|\beta^\perp| - k} \cdot m(n, \mu, \eta) \mathcal{L}_n^{(|\beta^\parallel|)}(\xi_\perp^n/|\eta|) \right|^2 \, m(\mu, \eta) \, d\mu \, d\eta \, \, d\zeta$$

A passage to polar coordinates in the $\zeta$-integral and a rescaling then give that

$$\int_{N_{3,2}} |y^a K_{G(L, X)}(x, y)|^2 \, dx \, dy$$

$$\leq C_o \sum_{\varsigma \in I_\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| \partial_{\eta}^{\varsigma} \partial_{\mu}^{\varsigma} \delta^k \cdot (1 + \tau)^{|\beta^\perp| - k} \cdot m(n, \mu, \eta) \mathcal{L}_n^{(|\beta^\parallel|)}(s) \right|^2 \, s^{2|\beta^\perp|} \, ds \, d\mu \, d\eta,$$

and the conclusion follows by applying the orthogonality relations \((12)\) for the Laguerre functions to the inner integral. \hfill $\Box$

Note that $\tau f(\cdot, \mu, \eta)$, $\delta f(\cdot, \mu, \eta)$ depend only on $f(\cdot, \mu, \eta)$; in other words, $\tau$ and $\delta$ can be considered as operators on functions $\mathbb{N} \to \mathbb{C}$. The next lemma will be useful in converting finite differences into continuous derivatives.
Lemma 6. Let \( f : \mathbb{N} \to \mathbb{C} \) have a smooth extension \( \tilde{f} : [0, \infty] \to \mathbb{C} \), and let \( k \in \mathbb{N} \). Then

\[
\delta^k f(n) = \int_{J_k} \tilde{f}^{(k)}(n + s) \, d\nu_k(s)
\]

for all \( n \in \mathbb{N} \), where \( J_k = [0, k] \) and \( \nu_k \) is a Borel probability measure on \( J_k \). In particular

\[
|\delta^k f(n)|^2 \leq \int_{J_k} |\tilde{f}^{(k)}(n + s)|^2 \, d\nu_k(s)
\]

for all \( n \in \mathbb{N} \).

Proof. Iterated application of the fundamental theorem of integral calculus gives

\[
\delta^k f(n) = \int_{[0,1]^k} \tilde{f}^{(k)}(n + s_1 + \cdots + s_k) \, ds.
\]

The conclusion follows by taking as \( \nu_k \) the push-forward of the uniform distribution on \([0,1]^k\) via the map \((s_1, \ldots, s_k) \mapsto s_1 + \cdots + s_k\), and by Hölder’s inequality. \( \square \)

We give now a simplified version of the right-hand side of (13), in the case where we restrict to the functional calculus for the sublaplacian \( L \). In order to avoid divergent series, however, it is convenient at first to truncate the multiplier along the spectrum of \( Y \).

Lemma 7. Let \( \chi \in C^\infty_c(\mathbb{R}) \) be supported in \([1/2, 2]\), \( K \subseteq [0, \infty[ \) be compact and \( M \in [0, \infty[ \). If \( F : \mathbb{R} \to \mathbb{C} \) is smooth and supported in \( K \), and \( F_M : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \) is given by

\[
F_M(\lambda, \eta) = F(\lambda) \chi(|\eta|/M),
\]

then, for all \( r \in [0, \infty[ \),

\[
\int_{N_{3,2}} |y|^r K_{F_M(L,Y)}(x, y)|^2 \, dx \, dy \leq C_{K,\chi,r} M^{3-2r} \| F \|_{W^2_3}^2.
\]

Proof. We may restrict to the case \( r \in \mathbb{N} \), the other cases being recovered a posteriori by interpolation. Hence we need to prove that

\[
\int_{N_{3,2}} |y|^\alpha K_{F_M(L,Y)}(x, y)|^2 \, dx \, dy \leq C_{K,\chi,\alpha} M^{3-2\alpha} \| F \|_{W^2_3}^{2\alpha}
\]

for all \( \alpha \in \mathbb{N}^3 \). On the other hand, if

\[
m(n, \mu, \eta) = F(|\eta|/\mu) \chi(|\eta|/M),
\]

then the left-hand side of (17) can be majorized by (13), and we are reduced to proving

\[
\sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial^\nu_\eta \partial^\mu_x \delta^{k_i} (1 + \tau)^{[\beta^i]-k_i} m(n+\mu, \eta) |^2 \mu^{2k_i} |\eta|^2 [\gamma^j-2\alpha_k-2\beta^i+1 + \beta^j] \times \langle \eta \rangle^{[\beta^j]} d\mu \, d\eta \leq C_{K,\chi,\alpha} M^{3-2\alpha} \| F \|_{W^2_3}^{2\alpha}
\]

for all \( \tau \in I_\alpha \).

Consider first the case \( [\beta^i] \geq k_i \). A smooth extension \( \tilde{m} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \) of \( m \) is defined by

\[
\tilde{m}(t, \mu, \eta) = F(|\eta|(2t+1) + \mu^2) \chi(|\eta|/M).
\]

Then, by Lemma 6

\[
\partial^\nu_\eta \partial^\mu_x \delta^{k_i} (1 + \tau)^{[\beta^i]-k_i} m(n+\mu, \eta)
\]

\[
= \sum_{j=0}^{[\beta^i]-k_i} \binom{[\beta^i]-k_i}{j} \int_{J_k} \partial^\nu_\eta \partial^\mu_x \delta^{k_i} \tilde{m}(n+j+s, \mu, \eta) \, dv_\alpha(s),
\]
where $J_t = [0, k_t]$ and $r_t$ is a suitable probability measure on $J_t$; consequently (15) will be proved if we show that

$$
\sum_{n \in \mathbb{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial^\alpha_\eta \partial^\beta_\mu \partial^\chi_\nu \tilde{m}(n + s, \mu, \eta)|^2 \mu^{2b_\mu} |\eta|^{2|\alpha| - 2k_t + |\beta| + 1} \times (n)^{\beta_\nu} d\mu d\eta \leq C_{K, \chi, \alpha} M^{3-2|\alpha|} \|F\|^2_{W_2^{\alpha}}
$$

for all $s \in [0, |\beta_\nu|]$. On the other hand, it is easily proved inductively that

$$
|\partial^\alpha_\eta \partial^\beta_\mu \partial^\chi_\nu \tilde{m}(t, \mu, \eta)|^2 \leq C_{\chi, \alpha} \sum_{n=1}^{t} \sum_{l=0}^{\lfloor \alpha \rfloor} \sum_{v=0}^{\lfloor \beta \rfloor} \sum_{s=0}^{\lfloor \chi \rfloor} |\langle t \rangle^\nu \mu^{2r-l} M^{-q} F(k_t, v + r) |\langle \eta \rangle (t) + \mu^2 \rangle \chi^{|\nu|} (\langle \eta \rangle / M),
$$

where $\tilde{\chi}$ is the characteristic function of $[1/2, 2]$, and we are using the fact that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta| / M) \neq 0$. Consequently the left-hand side of (19) is majorized by

$$
C_{\chi, \alpha} \sum_{r=1}^{t} \sum_{l=0}^{\lfloor \alpha \rfloor} \sum_{v=0}^{\lfloor \beta \rfloor} \sum_{s=0}^{\lfloor \chi \rfloor} |\langle n \rangle^{\beta_\nu} |(n + s)^{2v} \times \int_{\mathbb{R}} \int_{\mathbb{R}} |F(k_t, v + r) (|\eta| + s + \mu^2)^{2} \mu^{2b_\mu + 4r - 2l} \tilde{\chi}(\langle \eta \rangle / M) d\mu d\eta
$$

$$
\leq C_{\chi, \alpha} \sum_{r=1}^{t} \sum_{l=0}^{\lfloor \alpha \rfloor} \sum_{v=0}^{\lfloor \beta \rfloor} \sum_{s=0}^{\lfloor \chi \rfloor} |\langle n \rangle^{\beta_\nu} |(n + s)^{2v} \times \int_{0}^{\infty} \int_{0}^{\infty} |F(k_t, v + r) (\rho(n + s + \mu^2)^{2} \mu^{2b_\mu + 4r - 2l} \tilde{\chi}(\rho / M) d\mu d\rho
$$

$$
\leq C_{\chi, \alpha} \sum_{r=1}^{t} \sum_{l=0}^{\lfloor \alpha \rfloor} \sum_{v=0}^{\lfloor \beta \rfloor} \sum_{s=0}^{\lfloor \chi \rfloor} |\langle n \rangle^{\beta_\nu} |(n + s)^{2v - 1} \chi(\rho (\langle n + s \rangle M)) d\mu d\rho.
$$

by passing to polar coordinates and rescaling. The last sum in $\nu$ is easily controlled by $(\rho / M)^{|\beta_\nu| + 2v}$, hence the left-hand side of (19) is majorized by

$$
C_{\chi, \alpha} M^{3-2|\alpha|} \sum_{r=1}^{t} \sum_{l=0}^{\lfloor \alpha \rfloor} \sum_{v=0}^{\lfloor \beta \rfloor} \sum_{s=0}^{\lfloor \chi \rfloor} |\langle n \rangle^{\beta_\nu} |(n + s)^{2v} \times \mu^{2b_\mu + 4r - 2l} \sup_{u \in [0, \max K]} |F(k_t, v + r) (\rho + u)^{2} d\mu d\rho
$$

because $2b_\mu + 4r - 2l \geq 0$ and $|\beta_\nu| + 2v \geq 0$ if $r$ and $v$ are in the range of summation, and $\text{sup} F \subseteq K$. Since moreover $k_t + v + r \leq k_t + \gamma_\nu + l_t \leq |\alpha|$, the last integral is dominated by $\|F\|^2_{W_2^{\alpha}}$ uniformly in $\nu, v, u$, and (19) follows.
Consider now the case $|\beta'| < k_i$. Via the identity
\[(1 + \tau)^{-1} = (1 - \tau)(1 - \tau^2)^{-1} = -\delta(1 - \tau^2)^{-1} = -\delta \sum_{j=0}^{\infty} \tau^{2j},
\]

then together with Lemma 3 we obtain that
\[(21) \quad \partial_{\eta}^{k_i} \partial_{\mu}^{k_i} (1 + \tau)^{|\beta'| - k_i} m(n, \mu, \eta) \]
\[= (-1)^{k_i - |\beta'|} \sum_{j=0}^{\infty} \left(\sum_{k_i-|\beta'|}^{j+k_i-|\beta'|+1}\right) \partial_{\eta}^{k_i} \partial_{\mu}^{2k_i-|\beta'|} \bar{m}(n + 2j + s, \mu, \eta) \, dv_{\nu}(s),
\]

where $J_i = [0, 2k_i - |\beta'|]$ and $\nu_i$ is a suitable probability measure on $J_i$. Note that, because of the assumptions on the supports of $F$ and $\chi$, the sum on $j$ in the right-hand side of (21) is a finite sum, that is, the $j$-th summand is nonzero only if $(n + 2j) \leq 2M^{-1} \max K$; consequently, by applying the Cauchy-Schwarz inequality to the sum in $j$, and by (20),
\[
|\partial_{\eta}^{k_i} \partial_{\mu}^{k_i} (1 + \tau)^{|\beta'| - k_i} m(n, \mu, \eta)|^2
\leq C_{K,\alpha} M^{1+2|\beta'|-2k_i} \sum_{j=0}^{\infty} \int_{J_i} |\partial_{\eta}^{k_i} \partial_{\mu}^{2k_i-|\beta'|} \bar{m}(n + 2j + s, \mu, \eta)|^2 \, dv_{\nu}(s)
\]
\[\leq C_{K,\alpha} \sum_{r \in [l_i/2]} \sum_{\nu \in 0} \sum_{n,j \in \mathbb{N}} (n + 2j + s)^{2v} (n)^{|\beta'|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} M^{2+2v-2|\alpha|+|\beta'|}
\times \mu^{2b_i + 4r - 2l_i} \left| F(2k_i - |\beta'| + v + r)(\eta(n + 2j + s) + \mu^2)^2 \bar{\chi}(\eta/M) \right| \, d\eta \, dv_{\nu}(s)
\]
\[\leq C_{K,\alpha} \sum_{r \in [l_i/2]} \sum_{\nu \in 0} \sum_{n,j \in \mathbb{N}} (n + 2j + s)^{2v+|\beta'|} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} M^{4+2v-2|\alpha|+|\beta'|}
\times \mu^{2b_i + 4r - 2l_i} \left| F(2k_i - |\beta'| + v + r)(\rho(n + 2j + s) + \mu^2)^2 \bar{\chi}(\rho/M) \right| \, d\rho \, dv_{\nu}(s)
\]
\[\leq C_{K,\alpha} \sum_{r \in [l_i/2]} \sum_{\nu \in 0} \sum_{n,j \in \mathbb{N}} (n + 2j + s)^{2v+|\beta'|} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |F(2k_i - |\beta'| + v + r)(\rho + \mu^2)|^2
\times \mu^{2b_i + 4r - 2l_i} \int_{J_i} \sum_{(n,j) \in \mathbb{N}^2} (n + 2j + s)|^{2v+|\beta'|+1} \bar{\chi}(\rho/(n + 2j + s)M) \, dv_{\nu}(s) \, d\mu \, d\rho.
\]

By passing to polar coordinates and rescaling, the sum in $(n, j)$ is dominated by $(\rho/M)^{2v+|\beta'|+1}$, uniformly in $s \in J_i$, and moreover supp $F \subseteq K$. Therefore the left-hand side of (15) is majorized by
\[
C_{K,\alpha} M^{3-2|\alpha|} \sum_{r \in [l_i/2]} \sum_{\nu \in 0} \sup_{n \in [0, \max K]} \int_{\mathbb{R}^3} |F(2k_i - |\beta'| + v + r)(\rho + u)|^2 \, d\rho.
\]

On the other hand, $b_i + |\beta'| = l_i + 2k_i$, hence $2k_i - |\beta'| + v + r \leq 2k_i - |\beta'| + |\gamma'| + l_i = b_i + |\gamma'| \leq |\alpha|$ if $r$ and $v$ are in the range of summation, therefore the last integral is dominated by $\|F\|_{W_2^{\alpha}}$ uniformly in $r, v, u$, and (13) follows. \qed
Proposition 8. Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that $\text{supp} F \subseteq K$ for some compact set $K \subseteq [0, \infty]$. For all $r \in [0, 3/2]$, 
\[
\int_{N_{3,2}} \left| (1 + |y|)^r \mathcal{K}_{F(L)}(x, y) \right|^2 \, dx \, dy \leq C_{K,r} \|F\|^{2}_{W^{3}_{2}}.
\]

Proof. Take $\chi \in C_{c}^{\infty}(\mathbb{R})$ such that $\text{supp} \chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t \in [0, \infty]$. Note that, if $(\lambda, \eta)$ belongs to the joint spectrum of $L, Y$, then $|\eta| \leq \lambda$. Therefore, if $k_{K} \in \mathbb{Z}$ is sufficiently large so that $2^{k_{K} - 1} > \max K$, and if $F_{M}$ is defined for all $M \in [0, \infty]$ as in Lemma 7, then 
\[
F(L) = \sum_{k \in \mathbb{Z}, k \leq k_{K}} F_{2^{k}}(L, Y)
\]
(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski’s inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^{k}}}(L, Y)$ given by Lemma 7. If $r < 3/2$, then the series $\sum_{k \leq k_{K}} (2^{k})^{3/2 - r}$ converges, thus 
\[
\int_{N_{3,2}} \left| (1 + |y|)^r \mathcal{K}_{F(L)}(x, y) \right|^2 \, dx \, dy \leq C_{K,r} \|F\|^{2}_{W^{3}_{2}}.
\]

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. \qed

Recall that $\cdot \cdot |_{\delta}$ denotes a $\delta$-homogeneous norm on $N_{3,2}$, thus $||(x, y)|_{\delta} \sim |x| + |y|^{1/2}$. Interpolation then allows us to improve the standard weighed estimate for a homogeneous sublaplacian on a stratified group.

Proposition 9. Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that $\text{supp} F \subseteq K$ for some compact set $K \subseteq [0, \infty]$. For all $r \in [0, 3/2]$, $\alpha \geq 0$ and $\beta > \alpha + r$, 
\[
(22) \quad \int_{N_{3,2}} \left| (1 + |(x, y)|)^{\alpha} (1 + |y|)^{\beta} \mathcal{K}_{F(L)}(x, y) \right|^2 \, dx \, dy \leq C_{K,\alpha,\beta,r} \|F\|^{2}_{W^{3}_{2}}.
\]

Proof. Note that $1 + |y| \leq C(1 + |(x, y)|)^{2}$. Hence, in the case $\alpha > 0$, $\beta > \alpha + 2r$, the inequality (22) follows by the standard estimate [10] Lemma 1.2. On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (22) is given by Proposition 8. The full range of $\alpha$ and $\beta$ is then obtained by interpolation (cf. the proof of [10] Lemma 1.2). \qed

We can finally prove the fundamental $L^{1}$-estimate, and consequently Theorem 1.

Proof of Proposition 9. Take $r \in [9/2 - s, 3/2]$. Then $s - r > 3/2 + 3 - 2r$, hence we can find $\alpha_{1} > 3/2$ and $\alpha_{2} > 3 - 2r$ such that $s - r > \alpha_{1} + \alpha_{2}$. Therefore, by Proposition 9 and Hölder’s inequality, 
\[
\|\mathcal{K}_{F(L)}\|_{1}^{2} \leq C_{K,s} \|F\|^{2}_{W^{3}_{2}} \int_{N_{3,2}} (1 + |(x, y)|)^{-2\alpha_{1} - 2\alpha_{2}} (1 + |y|)^{-2r} \, dx \, dy.
\]
The integral on the right-hand side is finite, because $2\alpha_{1} > 3$, $\alpha_{2} + 2r > 3$, and 
\[
(1 + |(x, y)|)^{-2\alpha_{1} - 2\alpha_{2}} (1 + |y|)^{-2r} \leq C_{s}(1 + |x|)^{-2\alpha_{1}} (1 + |y|)^{-\alpha_{2} - 2r},
\]
and we are done. \qed

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