Conformal symmetry based relation between Bjorken and Ellis-Jaffe sum rules

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Abstract

The identity between perturbative expressions for the coefficient functions of the Bjorken and Ellis-Jaffe sum rules is derived in the conformal invariant limit of massless $U(1)$ theory, i.e. in the perturbative quenched QED model. It is also satisfied in the conformal invariant limit of the massless $SU(N_c)$ gauge theory with fermions. The latter limit is realized in the imaginable case, when all perturbative coefficients of the corresponding renormalization group $\beta$-function are equal to zero. The derivation is based on the comparison of results of application of the operator product expansion approach to the dressed triangle Green functions of singlet Axial vector-Vector-Vector and non-singlet Axial vector-Vector-Vector fermion currents in the limit, when the conformal symmetry remains unbroken. The expressions for the $O(\alpha_s^3)$ approximation of the non-singlet coefficient function, derived in the conformal invariant limit of $SU(N_c)$ group, is reminded. Its possible application in the phenomenological analysis of the experimental data for the Bjorken polarized sum rule is outlined.
I. INTRODUCTION

The definitions of the massless perturbative expressions for the Bjorken and Ellis-Jaffe sum rules of the polarised lepton-nucleon DIS are well-known and have the following form

$$Bjp(Q^2) = \int_0^1 (g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2))dx = \frac{1}{6}g_AC_{NS}(A_s(Q^2))$$  \hspace{1cm} (1)

$$EJ^{lp(n)}(Q^2) = C_{NS}(A_s(Q^2))(\pm \frac{1}{12}a_3 + \frac{1}{36}a_8) + C_{SI}(A_s(Q^2))\frac{1}{9}\Delta \Sigma(Q^2)$$  \hspace{1cm} (2)

where $a_3 = \Delta u - \Delta d = g_A$, $a_8 = \Delta u + \Delta d - 2\Delta s$, $\Delta u, \Delta d$ and $\Delta s$ are the polarised parton distributions and the subscript $lp(n)$ indicate the polarised DIS of charged leptons ($l$) on protons ($p$) and neutrons ($n$). In the $SU(N_c)$ colour gauge theory $A_s = \alpha_s/(4\pi)$. The order $O(A_s^3)$ and $O(A_s^4)$ perturbative expressions for the non-singlet (NS) coefficient function $C_{NS}(A_s)$ were analytically evaluated in [1] and [2] correspondingly, while the analytical expressions for the leading in the number of quarks flavours terms (renormalon contributions) were obtained in [3] (see [4] as well). The singlet (SI) contribution $C_{SI}$ to Eq.(2) contains the coefficient function, calculated in [3] at the $O(A_s^3)$- level, while the SI anomalous dimension term is known analytically from the $O(A_s^3)$ and $O(A_s^4)$ results of [6] and [5] respectively. In all these calculations the $\overline{MS}$-scheme was used. In this scheme the polarised gluon distribution $\Delta G$ does not enter into Eq.(2). Our main aim is to prove, that the analytical expressions for $C_{NS}$ and $C_{SI}$, defined in Eq.(2), are identical in all orders of perturbation theory in the conformal invariant limit of the massless $U(1)$ model with fermions, i.e. in the perturbative quenched QED (pqQED) approximation, and in the conformal invariant limit of the massless SU($N_c$) gauge model with fermions. The latter limit is realized in the imaginable case, when all perturbative coefficients $\beta$-function of the renormalization-group $\beta$-function of SU($N_c$) gauge theory with fermions are identically equal to zero.

While proving this identity we follow the pqQED studies, given in [7], where the classical Crewther relation [8], derived in the quark-parton era from the three-point Green function of the NS Axial vector-Vector-Vector(AVV) currents, is compared with the similar Crewther-type relation, which follows from the three-point Green function of singlet Axial vector-Vector-Vector currents. In the era of continuing understanding of the special features of the relations between NS characteristics of strong interactions, evaluated within perturbative approach in the the SU($N_c$) gauge group (see [3], [2], [10], [11]), the detailed considerations of the relations, which follow from the three-point Green functions of the NS AVV currents, were studied theoretically in [12], [13], [14]. The comment on possible phenomenological applications of the conformal-symmetry motivated expression for the Bjorken polarised sum-rule, which in QCD depends from the scale, fixed within principle of maximal conformality [23], [24], is given.

II. PROOF OF THE IDENTITY

Theoretical considerations of [8] are based on the property that in the conformal invariant limit the dressed expression for the three-point Green functions of NS AVV currents is proportional to the 1-loop expression of the related three-point diagram [13]. In the momentum space this means, that

$$T_{\mu\alpha\beta}^{abc}(p,q) = i\int <0|TA^a_{\mu}(y)V^b_{\alpha}(x)V^c_{\beta}(0)|0> e^{ipx+iqy}dxdy = d_{abc}^{\mu\alpha\beta}(1\text{-loop})(p,q)$$  \hspace{1cm} (3)
where $A^a_\mu(y) = \overline{\psi}(y)\gamma_\mu(\lambda^a/2)\gamma_5\psi(y)$ and $V^b_\alpha(x) = \overline{\psi}(x)\gamma_\mu(\lambda^b/2)\psi(x)$ are the NS Axial-vector and Vector currents. In the same limit it is possible to write-down the similar expression for the three-point Green function of SI Axial vector-NS Vector-Vector currents.

$$T_{\mu\alpha\beta}^{ab}(p, q) = i\int <0|T A_\mu(y)V^a_\alpha(x)V^b_\beta(y)|0> e^{ipx+iqy}\frac{1}{d\sigma} = \delta^{ab}\Delta_{\mu\alpha\beta}^{(1-loop)}(p, q)$$

where $A_\mu(y) = \overline{\psi}(y)\gamma_\mu\gamma_5\psi(y)$. Thus, the cancellation of one-loop corrections to the three-point AVV Green function, which was demonstrated by the explicit calculations, preformed in Ref.[17] within dimensional regularization [18], can be understood using the concept of the conformal symmetry and demonstrate the validity of the theoretical work of Ref.[15].

The SI coefficient function of the Ellis-Jaffe sum rule is defined as the coefficient function $A^a_\mu(y)$ which enters into operator-product of the three-point Green function of Eq.(4) as

$$i\int TV^a_\alpha(x)V^b_\beta(0)\delta^{ab}\epsilon_{\alpha\beta\rho\sigma}\frac{1}{p^2}C_{EJp}(A_\mu(x))\frac{1}{A_\rho(0)} + \ldots$$

The expression should be compared with the definition of the NS coefficient function which enters into operator-product of the three-point Green function of Eq.(4) as

$$i\int TV^a_\alpha(x)V^b_\beta(0)\delta^{abc}\epsilon_{\alpha\beta\rho\sigma}\frac{1}{p^2}C_{NS}(A_\mu(x))\frac{1}{A_\rho(0)} + \ldots$$

Taking now the limit $q^2 \to \infty$ in Eq.(4) we get the following Crewther-type identity in the SI channel

$$C_{SI}(A_\mu) \times C_{SI}^{NS}(A_\mu) \equiv 1$$

It should be compared with the classical NS Crewther identity, namely

$$C_{NS}(A_\mu) \times C_{NS}^{NS}(A_\mu) \equiv 1$$

It follows from the $x$-space studies of the NS AVV three-point function [8] (see [9] as well). In the momentum space it was re-derived in [12] by considering the same three-point function of Eq.(3). Note, that $C_{SI}^{SI}(A_\mu)$ and $C_{NS}^{NS}(A_\mu)$ are the coefficient functions of the massless axial-vector and vector Adler D-functions, defined by taking derivative $Q^2\frac{d}{dq^2}$ of the mass-independent terms in the correlator of SI axial-vector currents

$$i\int <0|TA_\mu(x)A_\nu(0)|0> e^{iqx}d^4x = \Pi^{'SI}_{\mu\nu}(Q^2) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi^{SI}(Q^2)$$

and of the correlator of NS axial-vector currents

$$i\int <0|TA_\mu^{(a)}(x)A_\nu^{(b)}(0)|0> e^{iqx}d^4x = \Pi^{NS}_{\mu\nu}(Q^2) = \delta^{ab}(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi^{NS}(Q^2)$$

where $Q^2 = -q^2$ is the Euclidean momentum transfer. The exact chiral invariance of the massless perturbative expressions for the coefficient functions implies, that $C_{SI}^{SI}(A_\mu) \equiv C_{NS}^{NS}(A_\mu)$. Keeping this in mind and comparing l.h.s. of Eq.(7) and Eq.(8), we get the following relation

$$C_{NS}(A_\mu) \equiv C_{SI}(A_\mu)|_{conformal\,\,\,invariant\,\,\,limit}$$

where $A_\mu$ is fixed. Eq.(11) is valid in the conformal-invariant limit of the $SU(N_c)$ gauge model and in the pqQED model in all orders of perturbative expansion in the fixed expansion parameter $A = \alpha/(4\pi)$. In the latter case Eq.(11) was proved in [1].
III. CONFORMAL INVARIANT LIMIT OF THE THIRD ORDER PERTURBATIVE SERIES

In the pqQED, using the detailed considerations of Ref.\[7\], it is possible to demonstrate explicitly the validity of the identity of Eq. (11) at level of third order corrections. In the process of these studies the following $O(A^3)$ pqQED expressions were used: the order $O(A^3)$ expression for $C_{NS}(A)$, available from \[1\], and the defined within dimensional regularisation \[18\] expression $C_{SI}(A_s) = \overline{C}_{SI}(A_s)/Z_{\beta}^{SI}(A_s)$ \[6\], where $Z_{\beta}^{SI}$ is the finite renormalization constant of the SI Axial-vector current. In order to get the pqQED limit of all functions, contributing to $C_{SI}(A_s)$, in the work \[\] $Z_{\beta}^{SI}(A)$ was determined from the pqQED limit of $Z_{\beta}^{NS}(A_s)$ finite renormalization constant, analytically evaluated in \[1\]. Combining these inputs the validity of the identity of Eq. (11) at level of third order corrections. In the case of $SU(N_c)$ gauge group with fermions the similar approximation of pqQED was demonstrated in the analytical form \[7\]. To fix the $O(A^4)$ pqQED correction to these functions one can use the pqQED expression of the related analytical result from \[2\]. This result coincides with the one, obtained in \[19\] from the classical Crewther relation of Eq. (8), supplemented with the pqQED $O(A^4)$ analytical approximation for $C_{D}^{NS}(A)$, first presented in \[20\]. The $O(A^4)$ pqQED expression for $C_{NS}(A)$ reads

$$C_{NS}(A) = 1 - 3A + \frac{21}{2}A^2 - \frac{3}{2}A^3 - \left(\frac{4823}{8} + 96\zeta_3\right)A^4 + O(A^5) .$$

It should coincide with the pqQED limit of still unknown $O(A^4)$ coefficient of the SI contribution into the Ellis-Jaffe sum rule.

In the case of $SU(N_c)$ gauge group with fermions the similar $O(A^2)$ expression for the NS coefficient functions follows from the results of the work of Ref.\[10\] and reads

$$C_{NS}(A_s) = 1 - 3C_F A_s + \left(\frac{21}{2}C_F^2 - C_F C_A\right)A_s^2 - \left[\frac{3}{2}C_F^3 + 65C_F^2 C_A + \left(\frac{523}{12} - 216\zeta_3\right)C_F C_A^2\right]A_s^3 .$$

It corresponds to the conformal invariant limit of the perturbative result for the $SU(N_c)$ gauge group with fermions and was obtained in Ref.\[10\] using the the Crewther relation of Eq. (8) and the derived in \[21\] $\beta$-expanded expression for $C_{D}^{NS}(A_s)$-function, based on the developed in Ref.\[21\] $\beta$-expanded generalization of the BLM approach, proposed in \[22\]. Here $C_F$ and $C_A$ are the Casimir operators of the $SU(N_c)$ group.

Taking into account the derived by us expression of Eq. (11) we conclude, that the this expression should coincide with the similar approximation of $C_{SI}(A_s)$-contribution into Eq. (2).

Note, that in the conformal limit the ratios of the corresponding perturbative approximations for the Ellis-Jaffe and Bjorken sum rules give us the following relations

$$\frac{EJ_{p}^{(n)}(Q^2)}{BJ_{p}(Q^2)} = \frac{1}{2} + \frac{a_8}{6a_3} + \frac{2\Delta \Sigma}{3a_3} .$$

where $a_8 = 3a_3 - 4D$, $a_3$, $a_8$ and $\Delta \Sigma$ are defined through the polarised parton distributions below Eqs. (2) and $D$ is the hyperon decay constant. These relations coincide with the ones, obtained within massless quark-parton model and can be re-written as

$$\frac{EJ_{p}(Q^2)}{BJ_{p}(Q^2)} = 1 + \frac{2(\Delta \Sigma - D)}{3a_3} ; \quad \frac{EJ_{in}(Q^2)}{BJ_{p}(Q^2)} = \frac{2(\Delta \Sigma - D)}{a_3} .$$

They lead to the standard quark-parton model definition of the Bjorken sum rule through the the Ellis-Jaffe sum rules, namely

$$BJ_{p} = EJ_{p}^{in} .$$

Thus, our considerations are self-consistent.
IV. CONFORMAL SYMMETRY AND THE BJORKEN SUM RULE

It is worth to stress, that the “conformal invariant” expression for the Bjorken sum rule with the perturbative coefficient function defined in Eq. (13) can be used in phenomenological studies of experimental data for the Bjorken sum rule. This can be done with the help of the principle of maximal conformality (PMC), introduced in the works of Ref. [23], Ref. [24] and already applied in the analysis of Tevatron and LHC data in Ref. [25]. Within PMC principle, one should specify in the scale-dependence of the parameter $A_s$ and substitute instead of $A_s$ its scale-dependent definition $A_s(Q^2_{PMC})$ into Eq. (13), leaving the analytical coefficients in the related perturbative approximation identical to those, obtained in the conformal invariant limit of $SU(N_c)$ theory. Note, however, that instead of using new scale in every new order of perturbation theory, as was prescribed in Refs. [24], [25], it may be worth to use the unique scale $Q^2_{PMC}$, which should absorb all non-conformal invariant contributions into the expressions of the $\overline{MS}$-scheme coefficients of $C_{NS}(A_s)$ coefficient function, defined as

$$C_{NS}(A_s) = 1 + \sum_{l \geq 0} c_l A_s^{l+1}(Q^2) .$$

Within the framework of the approach of Ref. [21] the $\overline{MS}$-coefficients should be expanded in powers of the $\beta_i$ coefficients of the renormalization-group $\beta$-function

$$\mu^2 \frac{\partial A_s}{\partial \mu^2} = - \sum_{l \geq 0} \beta_l A_s^{l+1} .$$

as

$$c_2 = \beta_0 c_2[1] + c_2[0]$$
$$c_3 = \beta_0^2 c_3[2] + \beta_1 c_3[0, 1] + \beta_0 c_3[1] + c_3[0]$$
$$c_4 = \beta_0^3 c_4[3] + \beta_1 \beta_0 c_4[1, 1] + \beta_2 c_4[0, 0, 1] + \beta_0^2 c_4[2] + \beta_1 c_4[0, 1] + \beta_0 c_4[1] + c_4[0]$$

The defined in Eq. (18) terms $c_2[1]$, $c_3[2]$, $c_3[0, 1]$ and $c_3[1]$ are known from the studies of Ref. [10]. Notice, that the proposed in Ref. [21], expansion differs a bit from the PMC realization, considered in Ref. [24] by the presence in the expressions for $c_3$ and $c_4$ coefficients of the additional $\beta_0 c_3[1]$ known and still explicitly unknown $\beta_0^2 c_4[2]$, $\beta_1 c_4[0, 1]$ and $\beta_0 c_4[1]$ terms, which disappear in the conformal invariant limit. In view of this the analogs of PMC scales, fixed from the expansion of Eq. (18), will differ from the similar scales, analogous to the ones, fixed in Ref. [24] in the process of the analysis of the perturbative predictions for the $R(e^+e^- \rightarrow hadrons)$. The similar Bjorken sum rules studies, with more detailed discussions of the applications of both realizations $\beta$-expansions, which can be compared within the proposed in Ref. [26] generalization of the original BLM approach [22], will be considered elsewhere [27].

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