On the Mirabolic Trace Formula for $\mathfrak{gl}(n)$

Shuyang Cheng

Abstract

In this paper, a Chaudouard type trace formula is established for the Lie algebra $\mathfrak{gl}(n)$, by integrating the Lie algebra analogue of the Selberg kernel function against a mirabolic Eisenstein series on $\text{GL}(n)$. The result is a combination of zeta functions $\zeta_E(s)$ of extensions over the base field $F$ of degree $[E : F] \leq n$.

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1 Introduction

Let $G$ be a Lie group and $\Gamma \subset G$ a discrete subgroup such that the quotient space $\Gamma \backslash G$ has finite volume. The Selberg trace formula is an identity of the form

$$\sum_{o} I_o = \sum_{\pi} I_{\pi}$$  \hspace{1cm} (1.1)
where \( o \) ranges over all conjugacy classes in \( \Gamma \), \( \pi \) ranges over all irreducible unitary representations of \( G \) in \( L^2(\Gamma \setminus G) \), and \( I_o, I_\pi \) are distributions on \( G \).

The proof of Selberg \cite{S56} starts with the Selberg kernel function

\[
K(g_1, g_2; f) = \sum_{\gamma \in \Gamma} f(g_1^{-1} \gamma g_2) \tag{1.2}
\]

where \( g_1, g_2 \in \Gamma \setminus G \) and \( f \in C_c^\infty(G) \) is a test function, and then derives the Selberg trace formula \( \text{(1.1)} \) by expanding the integral

\[
\int_{\Gamma \setminus G} K(g, g; f) dg = \text{Tr}(R_f, L^2(\Gamma \setminus G)) \tag{1.3}
\]

in two different ways. In many interesting examples the quotient space \( \Gamma \setminus G \) is non-compact. In this case the integral \( \text{(1.3)} \) will diverge and one integrates a truncated kernel function \( K^T(g, g; f) \) over \( \Gamma \setminus G \) instead.

Alternatively, Zagier \cite{Z81} has introduced another method to regularize the divergent integral \( \text{(1.3)} \), in the first interesting example of \( G = \text{SL}_2(\mathbb{R}) \) and \( \Gamma = \text{SL}_2(\mathbb{Z}) \), namely by integrating

\[
\int_{\Gamma \setminus G} K(g, g; f) E(g, s) dg \tag{1.4}
\]

where \( s \in \mathbb{C} \) and \( E(g, s) \) is an Eisenstein series on \( \Gamma \setminus G \) such that the integral \( \text{(1.4)} \) converges on the half-plane \( \text{Re}(s) > 1 \) and continuous to a meromorphic function defined on the entire complex plane. Formally, the integral \( \text{(1.4)} \) will converge to \( \text{(1.3)} \) as \( s \rightarrow 1 \), and the divergent nature of \( \text{(1.3)} \) is reflected by the existence of a pole of \( \text{(1.4)} \) at \( s = 1 \). Then the divergent integral \( \text{(1.3)} \) could be regularized by evaluating the residue of \( \text{(1.4)} \) at \( s = 1 \).

The method of Zagier \cite{Z81} is reminiscent of the method of zeta function regularization \cite{E12} employed by physicists to evaluate divergent integrals. Indeed, one of the most beautiful features of Zagier’s method is that, after expanding \( \text{(1.4)} \) into the identity

\[
\sum_o I_o(s) = \sum_\pi I_\pi(s) \tag{1.5}
\]

which is analogous to \( \text{(1.1)} \), the geometric distributions \( I_o(s) \) are Dedekind zeta functions of quadratic fields, and the spectral distributions \( I_\pi(s) \) are Rankin–Selberg \( L \)-functions of modular forms, or other special functions of a similar nature. The identity \( \text{(1.5)} \) is the first example of a mirabolic trace formula.
In many introductory expositions, the Selberg trace formula is presented as a non-abelian generalization of the Poisson summation formula
\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \quad (1.6) \]
for the pair \( G = \mathbb{R} \) and \( \Gamma = \mathbb{Z} \). However, there is another less well-known but equally interesting relation between (1.6) and (1.1), namely the trace formula for Lie algebras. Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \Lambda \subset \mathfrak{g} \) a lattice which is preserved under the adjoint action of the discrete subgroup \( \Gamma \subset G \). Then analogous to (1.2), one may define the Lie algebra kernel function
\[ K(g; f) = \sum_{\nu \in \Lambda} f(\nu \cdot \text{ad}(g)) \quad (1.7) \]
where \( g \in \Gamma \setminus G \) and \( f \in \mathcal{S}(\mathfrak{g}) \) is a Schwartz function, and formally arrive at a possibly divergent identity of the form
\[ \sum_o I_o = \sum_o \hat{I}_o \quad (1.8) \]
where \( o \) ranges over all orbits in \( \Lambda \) under the adjoint action of \( \Gamma \), and \( I_o, \hat{I}_o \) are tempered distributions on \( \mathfrak{g} \) such that
\[ \langle \hat{I}_o, f \rangle = \langle I_o, \hat{f} \rangle. \quad (1.9) \]
A regularized version of the trace formula for Lie algebras (1.8) has been established by Chaudouard [C02] following the truncation method of Selberg which has been generalized by Arthur [A78], working over the adeles.

The goal of this paper is to regularize the identity (1.8), following Zagier’s method which has been reformulated by Jacquet–Zagier [JZ87] over \( \text{GL}_2(\mathbb{A}) \), for the adelic pairs \( G = \text{GL}_n(\mathbb{A}), \Gamma = \text{GL}_n(\mathbb{F}) \) and \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{A}), \Lambda = \mathfrak{gl}_n(\mathbb{F}) \). The result is a mirabolic trace formula for \( \mathfrak{gl}(n) \)
\[ \sum_o I_o(s) = \sum_o \hat{I}_o(s). \quad (\text{Theorem 4.4}) \]
Following the main references [C02] and [JZ87], the rest of this paper will be written in the language of adelic algebraic groups.

2 Preliminaries and a motivating example

Algebraic preliminaries Let \( G \) denote the algebraic group \( \text{GL}(n) \) and \( \mathfrak{g} \) denote the Lie algebra \( \mathfrak{gl}(n) \). More generally, the Lie algebra of an algebraic
group will be denoted by the same letter in Fraktur font. Denote the right
adjoint action by
\[ X \cdot \text{ad}(g) = g^{-1}Xg \]
where \( X \in \mathfrak{g} \) and \( g \in G \). Let \( V \) denote the vector space consisting of
\( n \)-dimensional row vectors, then \( G \) operates on \( V \) from the right by right
multiplication.

Let \( Z \subset G \) denote the center consisting of scalar matrices, \( T \subset G \) denote
the maximal torus consisting of diagonal matrices, \( B^{\pm} \subset G \) denote the Borel
subgroups consisting of upper or lower triangular matrices, and \( U^{\pm} \subset B^{\pm} \)
denote the unipotent radicals consisting of matrices with ones along the
diagonal.

A standard parabolic subgroup is a subgroup which contains \( B^{+} \). More
generally, a parabolic subgroup is a subgroup which is conjugate to some
standard parabolic subgroup. If \( P \subset G \) is parabolic, let \( U_{P} \subset P \) denote
the unipotent radical and \( M_{P} = P/U_{P} \) denote the Levi component. If \( P \subset G \) is
standard parabolic, then \( P \) consists of block upper triangular matrices, and
\( M_{P} \subset P \) will be identified with the subgroup consisting of block diagonal
matrices. Each parabolic subgroup \( P \subset G \) is associated with a partial flag

\[ V = V_{r} \supset V_{r-1} \supset \cdots \supset V_{1} \supset V_{0} = 0 \]

with successive subquotients \( W_{i} = V_{i}/V_{i-1} \) such that \( M_{P} \simeq \prod_{i=1}^{r} \text{GL}(W_{i}) \).
Let \( \ell(P) = r + 1 \) denote the length of the partial flag associated to \( P \).

A partition of \( n \) consists of positive integers \( n_{1} \geq \cdots \geq n_{r} \) such that
\( n_{1} + \cdots + n_{r} = n \). Due to inconsistent conventions between the theory
of partitions and linear algebra, partitions will be written in reverse order
and denoted by \([n_{r}, \ldots, n_{1}] = \lambda \vdash n \). If \( \lambda \vdash n \) is a partition, let \( P_{\lambda} \subset G \)
denote the standard parabolic subgroup consisting of block upper triangular
matrices such that the \((i,j)\)th block consists of \( n_{i} \times n_{j} \) matrices.

An element \( X \in \mathfrak{g} \) is regular if its centralizer \( G_{X} \subset G \) is \( n \)-dimensional,
or equivalently if there exists \( v^{*} \in V \) such that \( v^{*}, v^{*}X, \ldots, v^{*}X^{n-1} \) form a
basis of \( V \). Each regular element is conjugate to a unique companion matrix
of the form

\[
X_{p} = \begin{bmatrix}
\begin{array}{cccc}
a_{1} & \cdots & a_{n-1} & a_{n} \\
1 & \cdots & 0 & 0 \\
0 & \cdots & 1 & 0
\end{array}
\end{bmatrix}
\]

\]
where \( p(t) = t^n - a_1 t^{n-1} - \cdots - a_n \) is the characteristic polynomial of \( X \).

More generally by the theory of Frobenius normal form, each \( X \in \mathfrak{g} \) is conjugate to a unique block diagonal matrix of the form

\[
X_{p_r, \ldots, p_1} = 
\begin{pmatrix}
X_{p_r} & 0 & \cdots & 0 \\
0 & X_{p_{r-1}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_{p_1}
\end{pmatrix}
\]

where \( p_r | p_{r-1} | \cdots | p_1 \). Let \( n_i = \deg(p_i) \), then \( X \) has Frobenius normal form of partition type \( \lambda = [n_r, \ldots, n_1] \vdash n \), and the polynomials \( p_r, \ldots, p_1 \) are the invariant factors of \( X \).

**Analytic preliminaries**

Let \( F \) be a global field and let \( \mathbb{A} = \prod_v F_v \) denote the ring of adeles of \( F \), where the product ranges over all places \( v \) of \( F \) which is restricted with respect to the local rings of integers \( O_v \subset F_v \) for all \( v < \infty \). Let \( | \cdot |_v : F_v \to \mathbb{R}_{\geq 0} \) and \( | \cdot | : \mathbb{A} \to \mathbb{R}_{\geq 0} \) denote the local and global norms such that \( | \cdot | = \prod_v | \cdot |_v \) and the product formula

\[
\prod_v |x|_v = 1
\]

holds for all \( x \in F^\times \). Fix a non-trivial additive character \( \psi : \mathbb{A} \to \mathbb{C}^\times \) such that

\[
\psi(x) = 1
\]

holds for all \( x \in F \).

Let \( K = \prod_{v < \infty} G(O_v) \times K_\infty \subset G(\mathbb{A}) \) be a maximal compact subgroup such that the Iwasawa decomposition

\[
G(\mathbb{A}) = \mathbb{P}(\mathbb{A}) K
\]

holds for all parabolic subgroups \( P \subset G \). If \( P \subset G \) is standard parabolic, fix compatible Haar measures such that

\[
\int_{G(\mathbb{A})} h(g) dg = \int_K \int_{U_P(\mathbb{A})} \int_{M_P(\mathbb{A})} h(umk)dmdudk
\]

\[
= \int_K \int_{M_P(\mathbb{A})} \int_{U_P(\mathbb{A})} h(umk) \prod_{i<j} \frac{|\det(m_i)|^{n_j}}{|\det(m_j)|^{n_i}} dudmdk
\]
for all $h \in L^1(G(\A))$, where $m = \begin{bmatrix} m_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_1 \end{bmatrix} \in M_P(\A)$ is block diagonal with $m_i \in \GL_{n_i}(\A)$.

If $W$ is a finite-dimensional vector space, let $\mathcal{S}(W(\A)) = \bigotimes'_v \mathcal{S}(W(F_v))$ denote the vector space of Schwartz functions, where the tensor product is restricted with respect to the characteristic functions $1_{W(O_v)}$ for all $v < \infty$. Let $\langle \cdot, \cdot \rangle$ be a non-degenerate bilinear form on $W$, then $W(\A)$ admits a unique Haar measure $dw$ which is self dual with respect to $\langle \cdot, \cdot \rangle$ and $\psi$ such that the Fourier transform $\hat{f} \in \mathcal{S}(W(\A))$ defined by

$$\hat{f}(v) = \int_{W(\A)} f(w)\overline{\psi}\langle v, w \rangle dw$$

satisfies the Fourier inversion formula

$$\hat{\hat{f}}(w) = f(-w)$$

and the Poisson summation formula

$$\sum_{w \in W(F)} f(w) = \sum_{w \in W(F)} \hat{f}(w)$$

for all $f \in \mathcal{S}(W(\A))$.

Equip the vector space $\mathfrak{g}$ with the non-degenerate bilinear form

$$\langle X, Y \rangle = \text{tr}(XY)$$

which is invariant under the adjoint action and the associated self-dual Haar measure $dX$ on $\mathfrak{g}(\A)$. Then the induced Fourier transform operator on $\mathcal{S}(\mathfrak{g}(\A))$ intertwines with the representation of $G(\A)$ on $\mathcal{S}(\mathfrak{g}(\A))$ via the adjoint action

$$\mathcal{S}(\mathfrak{g}(\A)) \xrightarrow{\hat{f}} \mathcal{S}(\mathfrak{g}(\A))$$

$$\begin{array}{c}
\mathcal{S}(\mathfrak{g}(\A)) \\
\downarrow f(X, \text{ad}(g))
\end{array} \quad \begin{array}{c}
\mathcal{S}(\mathfrak{g}(\A)) \\
\downarrow \hat{f}(X, \text{ad}(g))
\end{array} \quad \begin{array}{c}
\mathcal{S}(\mathfrak{g}(\A)) \\
\downarrow \hat{f}(X, \text{ad}(g))
\end{array}$$

If $P \subset G$ is a parabolic subgroup and $f \in \mathcal{S}(\mathfrak{g}(\A))$, define the parabolic descent $f_P \in \mathcal{S}(m_P(\A))$ by

$$f_P(X) = \int_K \int_{U_P(\A)} f((\tilde{X} + U) \cdot \text{ad}(k)) dU dk$$
where $\tilde{X} \in p(\mathbb{A})$ is a lift of $X \in m_P(\mathbb{A})$. Then the parabolic descent operator intertwines with the Fourier transform operators on $S(g(\mathbb{A}))$ and $S(m_P(\mathbb{A}))$

\[
\begin{align*}
S(g(\mathbb{A})) & \xrightarrow{\hat{\cdot}} S(g(\mathbb{A})) \\
\downarrow f_P & \quad \downarrow \hat{f}_P \\
S(m_P(\mathbb{A})) & \xrightarrow{\hat{\cdot}} S(m_P(\mathbb{A})).
\end{align*}
\]

Equip the vector space $V$ with the non-degenerate bilinear form

\[
\langle u^*, v^* \rangle = u^* (\tau v^*)
\]

where $\tau$ denotes matrix transpose and the associated self-dual Haar measure $dv^*$ on $V(\mathbb{A})$. Then the induced Fourier transform operator on $S(V(\mathbb{A}))$ intertwines with the standard representation of $G(\mathbb{A})$ on $S(V(\mathbb{A}))$ via right multiplication and its contragredient twisted by $|\det|^{-1}$

\[
\begin{align*}
S(V(\mathbb{A})) & \xrightarrow{\hat{\cdot}} S(V(\mathbb{A})) \\
f(v^*g) \downarrow & \quad \downarrow |\det(g)|^{-1} \hat{f}(v^* \tau g^{-1}) \\
S(V(\mathbb{A})) & \xrightarrow{\hat{\cdot}} S(V(\mathbb{A})).
\end{align*}
\]

**The trace formula of Chaudouard** Let $f \in S(g(\mathbb{A}))$, define the kernel function

\[
K(g; f) = \sum_{X \in g(F)} f(X \cdot \ad(g))
\]

and the parabolic kernel functions

\[
K_P(g; f) = \sum_{X \in p(F)} \int_{u_P(\mathbb{A})/u_P(F)} f((X + U) \cdot \ad(g))dU
\]

for $g \in G(F) \backslash G(\mathbb{A})$, then

\[
K(g; f) = K(g; \hat{f}) \quad \text{and} \quad K_P(g; f) = K_P(g; \hat{f})
\]

by the Poisson summation formula.

Following the truncation procedure of Arthur [A78], Chaudouard [C02] has regularized the divergent integral

\[
\int_{G(F) \backslash G(\mathbb{A})} \sum_P (-1)^{\ell(P)} K_P(g; f) dg,
\]

\text{by the Poisson summation formula.}
where \( G(\mathbb{A})^1 \subset G(\mathbb{A}) \) denotes the subgroup consisting of \( g \in G(\mathbb{A}) \) such that \(|\det(g)| = 1\), into an absolutely convergent identity

\[
\sum_{\tilde{o}} J_{\tilde{o}}(f) = \sum_{\tilde{o}} J_{\tilde{o}}(\hat{f})
\]

where the summation \( \sum_{\tilde{o}} \) ranges over all equivalence classes \( \tilde{o} \subset g(F) \) where each \( \tilde{o} \) consists of all \( X \in g(F) \) with a common characteristic polynomial \( p_{\tilde{o}}(t) \in F[t] \).

- If the characteristic polynomial \( p_{\tilde{o}}(t) \) is irreducible over \( F \), then \( \tilde{o} = o_{\text{ell}} \) is an elliptic conjugacy class in \( g(F) \) with centralizer isomorphic to the Weil restriction \( \text{Res}_{E/F} \mathbb{G}_m \) where \( E \simeq F[t]/(p_{o_{\text{ell}}}) \), which is an elliptic torus modulo \( Z \) with \( \text{Res}_{E/F} \mathbb{G}_m(F) \simeq E^\times \).

  In this case the truncation procedure trivializes and

  \[
  J_{o_{\text{ell}}}(f) = \int_{G(F) \backslash G(\mathbb{A})^1} \sum_{X \in o_{\text{ell}}} f(X \cdot \text{ad}(g)) \, dg
  \]

  \[
  = \text{vol}(\mathbb{A}_E^1/E^\times) \int_{G_X(\mathbb{A}) \backslash G(\mathbb{A})} f(X \cdot \text{ad}(g)) \, dg
  \]

  where \( \mathbb{A}_E^1 \subset A_E^\times \) denotes the group of ideles of \( E \) of norm 1.

- More generally if the characteristic polynomial \( p_{\tilde{o}}(t) \) has distinct roots in its splitting field, or equivalently if its discriminant \( \Delta(p_{\tilde{o}}) \neq 0 \), then \( \tilde{o} = o_{\text{rs}} \) is a regular semisimple conjugacy class in \( g(F) \) with centralizer isomorphic to a possibly non-elliptic torus.

  In this case Chaudouard [C02] has obtained a similar expression

  \[
  J_{o_{\text{rs}}}(f) = \text{vol}(Z_{M_1}^X \backslash G_X(\mathbb{A}) \backslash G(\mathbb{A})) \times \\
  \times \int_{G_X(\mathbb{A}) \backslash G(\mathbb{A})} f(X_1 \cdot \text{ad}(g)) v(g, T_0) \, dg
  \]

  as an orbital integral weighted by the weight factor \( v(g, T_0) \) introduced by Arthur [A78].

- Otherwise \( \Delta(p_{\tilde{o}}) = 0 \) and \( \tilde{o} = \bigcup_{[\lambda_i]} o_{[\lambda_i]} \) is a finite union of conjugacy classes indexed by partitions \( \lambda_i \) which correspond to different Jordan block sizes for the \( i \)th distinct eigenvalue.

  In this case the distributions \( J_{\tilde{o}}(f) \) are more difficult to understand. However, for most applications one could impose finitely many local
conditions on the test functions \( f = \otimes_v f_v \) and \( \hat{f} = \otimes_v \hat{f}_v \) to remove all non-regular semisimple or non-elliptic summands and work with the simple trace formula

\[
\sum_{c_{rs}} J_{c_{rs}}(f) = \sum_{c_{rs}} J_{c_{rs}}(\hat{f})
\]

or the very simple trace formula

\[
\sum_{c_{\text{ell}}} J_{c_{\text{ell}}}(f) = \sum_{c_{\text{ell}}} J_{c_{\text{ell}}}(\hat{f}).
\]

**Tate integrals for zeta functions**  Let \( \Phi = \otimes_v \Phi_v \in S(\mathbb{A}) \), then the Tate integral \( I(s; \Phi) \) is an adelic integral of the form

\[
I(s; \Phi) = \int_{\mathbb{A}^\times} \Phi(x)|x|^s \, d^\times x
\]

\[
= \prod_v \int_{F_v^\times} \Phi_v(x_v)|x_v|^s_v \, d^\times x_v
\]

for all \( s \in \mathbb{C} \) such that the integral converges absolutely. In his thesis [T50], Tate has established the following analytic properties of \( I(s; \Phi) \):

- the integral \( I(s; \Phi) \) converges absolutely on the half-plane \( \text{Re}(s) > 1 \) and continuous to a meromorphic function on the entire complex plane, which will also be denoted by \( I(s; \Phi) \);
- there exists a function \( \zeta_F(s) \) which is holomorphic on the half-plane \( \text{Re}(s) > 1 \) such that

\[
I(s; \Phi) = \zeta_F(s) \prod_{v \in S} e_v(s; \Phi_v)
\]

for all \( \Phi \in S(\mathbb{A}) \), where \( S \) is a finite set of places of \( F \) and each \( e_v(s; \Phi_v) \) is an elementary function in \( s \) which continuous holomorphically to the entire complex plane;
- the meromorphic function \( I(s; \Phi) \) satisfies the functional equation

\[
I(s; \Phi) = I(1 - s; \hat{\Phi})
\]

which is another consequence of the Poisson summation formula.
From the analytic properties of $I(s; \Phi)$, Tate [T50] then deduces the classical Euler factorization and functional equation of the zeta function $\zeta_F(s)$.

The group $G$ operates on $V$ with a unique Zariski open orbit denoted by $V' = V - \{0\}$. Let $\Phi \in S(V(\mathbb{A}))$, then the mirabolic Eisenstein series $E(g, s; \Phi)$ is an automorphic function in $g \in Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ defined by

$$E(g, s; \Phi) = |\det(g)|^s \int_{\mathbb{A}^\times/F^\times} \sum_{v^* \in V'(F)} \Phi(zv^* g)|z|^{ns} d^\times z$$

which converges absolutely for $\Re(s) > 1$ and continuous meromorphically to the entire complex plane. As another consequence of the Poisson summation formula, the mirabolic Eisenstein series $E(g, s; \Phi)$ satisfies the functional equation

$$E(g, s; \Phi) = E(\overline{g}^{-1}, 1 - s; \hat{\Phi}).$$

If $X \in \mathfrak{o}_{\text{ell}}$ is an elliptic element in $\mathfrak{g}(F)$ with characteristic polynomial $p_{\mathfrak{o}_{\text{ell}}} \in F[t]$, then the integral

$$I_X(s; \Phi) = \int_{G_X(F)\backslash G_X(\mathbb{A})^1} E(g, s; \Phi) dg$$

is a Tate integral for $\zeta_E(s)$ where $E \simeq F[t]/(p_{\mathfrak{o}_{\text{ell}}})$.

More generally, Tate [T50] has introduced Tate integrals twisted by a multiplicative character $\chi : \mathbb{A}^1/F^\times \to \mathbb{C}^\times$ where $\mathbb{A}^1 \subset \mathbb{A}^\times$ denotes the group of ideles of norm 1, and similarly there exist mirabolic Eisenstein series twisted by the character $\chi \circ \det : G(\mathbb{A}) \to \mathbb{C}^\times$. The results of this paper could be extended to such cases, which would lead to a mirabolic version of the twisted trace formula for $\mathfrak{g}(n)$. However, such an extension will be excluded from this paper for simplicity.

The example of $\mathfrak{gl}(2)$ For the rest of this section let $\mathfrak{g}$ denote $\mathfrak{gl}(2)$, $G$ denote $\text{GL}(2)$, and choose Schwartz functions $f \in S(\mathfrak{g}(\mathbb{A}))$ and $\Phi \in S(\mathbb{A}^2)$. The following have been computed for $\Re(s) > 1$ in [JZS7]:

- If $\mathfrak{o}_{\text{ell}} \subset \mathfrak{g}(F)$ is an elliptic conjugacy class containing $d + \sqrt{\delta} = \left[ \begin{array}{c} d \\ \delta \end{array} \right]$ where $\delta$ is non-square in $F$, then

$$I_{\mathfrak{o}_{\text{ell}}}(s) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{X \in \mathfrak{o}_{\text{ell}}} f(X \cdot \text{ad}(g)) E(g, s; \Phi) dg$$

10
\[ \begin{aligned}
&= \int_{G(F) \setminus G(A)} \sum_{X \in o_{\text{hyp}}} f(X \cdot \text{ad}(g)) \Phi(v^* g) \det(g)^{s} \, dg \\
&= \int_{A[\sqrt{2}] \setminus G(A)} f(\begin{bmatrix} d & \delta \\ 1 & d \end{bmatrix} \cdot \text{ad}(g)) \times \\
&\quad \times \left( \int_{A[\sqrt{2}] \setminus G(A)} \Phi(\alpha g) \det(\alpha g)^{s} \, d^{\times} \alpha \right) \, dg
\end{aligned} \]

is an elliptic orbital integral weighted by the Tate integral
\[ \int_{A[\sqrt{2}] \setminus G(A)} \Phi(\alpha g) \det(\alpha g)^{s} \, d^{\times} \alpha = \zeta_{F(\sqrt{2})}(s) e(g, s; \Phi) \]

where \( e(g, s; \Phi) \) is a finite Euler product of elementary functions in \( s \).

- If \( o_{\text{hyp}} \subset g(F) \) is a hyperbolic conjugacy class containing \( \begin{bmatrix} d_{1} & 0 \\ 0 & d_{2} \end{bmatrix} \) where \( d_{1} \neq d_{2} \), then
\[ \begin{aligned}
I_{o_{\text{hyp}}}(s) &= \int_{Z(A)G(F) \setminus G(A)} \sum_{X \in o_{\text{hyp}}} \left( f(X \cdot \text{ad}(g)) E(g, s; \Phi) + \\
&\quad \text{singular terms arising from all Borel } b \subset g \text{ containing } X \right) \, dg \\
&= \int_{G(F) \setminus G(A)} \sum'_{X \in o_{\text{hyp}}} f(X \cdot \text{ad}(g)) \Phi(v^* g) \det(g)^{s} \, dg
\end{aligned} \]

where the primed summation ranges over all pairs \((X, v^*)\) such that \( v^* \) is not an eigenvector of \( X \), is equal to
\[ \begin{aligned}
&= \int_{(A^\times \times A^\times) \setminus G(A)} f(\begin{bmatrix} d_{1} & 0 \\ 0 & d_{2} \end{bmatrix} \cdot \text{ad}(g)) \times \\
&\quad \times \left( \int_{A^\times} \int_{A^\times} \Phi([a, b] g) |a|^s |b|^s d^{\times} \text{ad}^{\times} b \right) \, dg
\end{aligned} \]

which is a hyperbolic orbital integral weighted by the Tate integral
\[ \int_{A^\times} \int_{A^\times} \Phi([a, b] g) |a|^s |b|^s d^{\times} \text{ad}^{\times} b = \zeta_{F}(s)^2 e(g, s; \Phi) \]

where \( e(g, s; \Phi) \) is a finite Euler product of elementary functions in \( s \).
• If \( o_{\text{par}} \subset g(F) \) is a parabolic conjugacy class containing \([\begin{smallmatrix} d & 1 \\ 0 & d \end{smallmatrix}]\), then

\[
I_{o_{\text{par}}}(s) = \int_{Z(\mathbb{A})G(F) \backslash G(\mathbb{A})} \sum_{X \in o_{\text{par}}} \left( f(X \cdot \text{ad}(g)) E(g, s; \Phi) + \right. \\
- \text{singular terms arising from all Borel } b \subset g \text{ containing } X \left. \right) dg \\
= \int_{G(F) \backslash G(\mathbb{A})} \sum_{X \in o_{\text{par}}} f(X \cdot \text{ad}(g)) \Phi(v^* g) \left| \det(g) \right|^s dg
\]

where the primed summation ranges over all pairs \((X, v^*)\) such that \(v^*\) is not an eigenvector of \(X\), is equal to

\[
= \int_{(\mathbb{A}^\times \times \mathbb{A}) \backslash G(\mathbb{A})} f \left( \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix} \right) \text{ad}(g) \times \\
\times \left( \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \Phi([a, b]g) \frac{db}{|a|^s} |a|^{-s-1} da \right) dg
\]

which is a parabolic orbital integral weighted by the Tate integral

\[
\int_{\mathbb{A}^\times} \left( \int_{\mathbb{A}} \Phi([a, b]g) db \right) |a|^{-s-1} da = \zeta_F(2s - 1)e(g, 2s - 1; \tilde{\Phi})
\]

where \(\tilde{\Phi} \in \mathcal{S}(\mathbb{A})\) is a partial integral of \(\Phi\) along parallel lines \(\mathbb{A} \subset \mathbb{A}^2\).

The orbital integral \(I_{o_{\text{par}}}(s)\) could be further evaluated as

\[
\zeta_F(2s - 1) \int_K \left( \int_{\mathbb{A}^\times} f \left( \begin{bmatrix} d & c \\ 0 & d \end{bmatrix} \right) \text{ad}(k) \right) |c|^{-s} dc e(g, 2s - 1; \tilde{\Phi}) dk \\
= \zeta_F(s) \zeta_F(2s - 1) \int_K e(d, k; s; f) e(g, 2s - 1; \tilde{\Phi}) dk
\]

where \(e(d, k; s; f)\) and \(e(g, 2s - 1; \tilde{\Phi})\) are both finite Euler products of elementary functions in \(s\).

• If \( o_z \subset g(F) \) is a central conjugacy class consisting of \([\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}]\), then

\[
I_{o_z}(s) = \int_{G(F) \backslash G(\mathbb{A})} \Phi(v^* g) \sum_{X \in \mathfrak{b}_z^\times(F)} \left( f(X \cdot \text{ad}(g)) + \\
- \int_{\mathfrak{u}_z^\times(F)} f \left( (X + U) \cdot \text{ad}(g) \right) dU \right) \left| \det(g) \right|^s dg
\]
where $h_{v^*} \subset g$ denotes the Borel subalgebra which stabilizes the line in $F^2$ spanned by $v^*$ and $u_{v^*}$ denotes the nilpotent radical of $h_{v^*}$, is equal to

$$= \int_k \left( \int_{k^*} \tilde{f}^k(d, c^*)|c^*|^s d^* c^* \right) \times$$

$$\times \left( \int_{k^*} \Phi([0, a]k)|d|^{2s} d a \right) dk$$

$$= \zeta_F(s)\zeta_F(2s) \int_k e(d, k, s; \tilde{f}^k) e(k, 2s; \Phi) dk$$

where $f^k$ denotes the composite function $f \circ \text{ad}(k)$ and $\tilde{f}$ denotes the partial Fourier transform of $f$ defined by

$$\tilde{f}(d, c^*) = \int_k f \left( \begin{bmatrix} d & u \\ 0 & d \end{bmatrix} \right) \psi(c^* u) du.$$

**The mirabolic trace formula for $\mathfrak{gl}(2)$** Combining the previous results with the absolute convergence of the summation over all conjugacy classes which will be established for all $\mathfrak{gl}(n)$ later, it follows that the integral

$$\int_{G(F)\backslash G(k)} \sum_{u^* \in F_{2r}} \left( K(g; f) - K_{B,v^*}(g; f) \right) \Phi(v^* g)|\det(g)|^s dg$$

$$= \sum_{\ell} I_{\ell}(s) + \sum_{\ell_{hyp}} I_{\ell}(s) + \sum_{\ell_{par}} I_{\ell}(s) + \sum_{\ell_z} I_{\ell}(s) \quad (2.1)$$

converges absolutely for $\text{Re}(s) > 1$ and continues meromorphically to the entire complex plane. Since $K(g; f) = K(g; \tilde{f})$ and $K_{B,v^*}(g; f) = K_{B,v^*}(g; \tilde{f})$ by the Poisson summation formula, it follows that the distribution defined by (2.1) is invariant under the Fourier transform $f \leftrightarrow \tilde{f}$ on $\mathcal{S}(g(\mathbb{A}))$.

### 3 Contributions from the regular locus

**The regular locus of the mirabolic action** For the rest of this paper let $g$ denote $\mathfrak{gl}(n)$, $G$ denote $\text{GL}(n)$, and choose Schwartz functions $f \in \mathcal{S}(g(\mathbb{A}))$ and $\Phi \in \mathcal{S}(V(\mathbb{A}))$. Motivated by the mirabolic integral

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K(g; f) E(g, s; \Phi) dg$$

$$\int_{G(F)\backslash G(\mathbb{A})} \sum_{X \in g(F)} f(X \cdot \text{ad}(g)) \sum_{v^* \in V^*(F)} \Phi(v^* g)|\det(g)|^s dg,$$
consider the diagonal mirabolic action of $G$ on $\mathfrak{g} \times V'$. The affine quotient of $\mathfrak{g} \times V'$ by $G$ factors through the projection onto the first factor

$$
\begin{array}{ccc}
\mathfrak{g} \times V' & \longrightarrow & (\mathfrak{g} \times V')//G \\
\downarrow & & \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{g}//G
\end{array}
$$

and defines a $G$-torsor over the open subvariety $(\mathfrak{g} \times V')_{\text{reg}} \subset \mathfrak{g} \times V'$ consisting of all pairs $(X, v^*)$ such that $v^*, v^*X, \ldots, v^*X^{n-1}$ form a basis of $V$.

**Proposition 3.1.** The integral

$$
\int_{G(F)\setminus G(\mathbb{A})} \sum_{(X, v^*) \in (\mathfrak{g} \times V')_{\text{reg}(F)}} f(X \cdot \text{ad}(g)) \Phi(v^*g) \left| \det(g) \right|^s dg \quad (3.1)
$$

\[
= \sum_{\mathfrak{o}_{\text{reg}}} \left( \int_{G(F)\setminus G(\mathbb{A})} \sum_{X \in \mathfrak{o}_{\text{reg}}} f(X \cdot \text{ad}(g)) \Phi(v^*g) \left| \det(g) \right|^s dg \right)
\]

converges absolutely if $\text{Re}(s) > 1$, and continues meromorphically over the entire complex plane, where the summation $\sum_{\mathfrak{o}_{\text{reg}}}$ ranges over all regular conjugacy classes $\mathfrak{o}_{\text{reg}} \subset \mathfrak{g}(F)$.

**Proof.** The action of $G(F)$ on $(\mathfrak{g} \times V')_{\text{reg}(F)}$ is faithful. By the theory of companion matrices, each orbit contains a unique representative of the form

$$
\begin{bmatrix}
a_1 & \cdots & a_{n-1} & a_n \\
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0
\end{bmatrix}, \quad e_n^* = [0, \ldots, 0, 1]
$$

where $t^n - a_1 t^{n-1} - \cdots - a_n \in F[t]$ is the characteristic polynomial of $\mathfrak{o}_{\text{reg}}$. Hence the integral (3.1) is equal to

$$
\int_{G(\mathbb{A})} \sum_{a_1, \ldots, a_n \in F} f(X_{a_1, \ldots, a_n} \cdot \text{ad}(g)) \Phi(e_n^*g) \left| \det(g) \right|^s dg \quad (3.2)
$$

\[
= \int_K \left( \int_{U(\mathbb{A})} \int_{(\mathfrak{h} \times)^{n-1}} \right. \\
\times \sum_{a_1, \ldots, a_n \in F} f(X_{a_1, \ldots, a_n} \cdot \text{ad}([t \; 0 \; 0 \; uk]) \left| \det(t) \right|^s dt du \bigg) \times
\]

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\[ \times \left( \int_{Z(\mathbb{A})} \Phi(e_n^*zk) |\det(z)|^s \, dz \right) \, dk \]

by the Iwasawa decomposition \( g = z \left[ \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right] uk \) where \( z \in Z(\mathbb{A}) \), \( t \in \text{GL}_{n-1}(\mathbb{A}) \) is diagonal, \( u \in U(\mathbb{A}) \) and \( k \in K \).

Choosing coordinates

\[ t = \left[ \begin{array}{cc} t^o & 0 \\ 0 & 1 \end{array} \right] w \]

where \( t^o \in \text{GL}_{n-2}(\mathbb{A}) \) is diagonal and \( w \in \mathbb{A}^\times \), and

\[ u = \left[ \begin{array}{cc} u^o & 0 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} I_{n-1} & v \\ 0 & 1 \end{array} \right] \]

where \( u^o \in U_{n-1}(\mathbb{A}) \), \( I_i \) denotes the \( i \times i \) identity matrix and \( v \in \mathbb{A}^{n-1} \), then

\[
X_{a_1,\ldots,a_n} \cdot \text{ad}(\left[ \begin{array}{c} t^o \\ 0 \end{array} \right] u) = \left[ X_{a_1,\ldots,a_{n-1}} \cdot \text{ad}(\left[ \begin{array}{c} w^o \\ 0 \\ 0 \\ 1 \end{array} \right]) \left[ \begin{array}{c} u^o \\ 0 \\ 1 \end{array} \right] \left[ \begin{array}{c} I_{n-1} \\ 0 \\ 1 \end{array} \right] \right] \\
= \left[ X_{a_1,\ldots,a_{n-1}} \cdot \text{ad}(\left[ \begin{array}{c} t^o \\ 0 \\ 1 \end{array} \right] u^o) - wwe^*_{n-1} \\
we^*_{n-1} \right]
\]

where the last column is equal to the transpose of

\[
\left[ \begin{array}{cccccc}
a_n & t^o_1v_1 & t^o_2v_2 & \cdots & t^o_{n-2}v_{n-2} & t^o_{n-1}v_{n-1} \\
w & t^o_1 & t^o_2 & \cdots & t^o_{n-2} & t^o_{n-1} \\
\end{array} \right]
\]

upto a unimodular change of variables of \( v_1, \ldots, v_{n-1} \), where \( v_i \) denotes the \( i \)th entry of \( v \) for \( 1 \leq i \leq n-2 \) and \( t^o_i \) denotes the \( j \)th diagonal entry of \( t^o \) for \( 1 \leq j \leq n-1 \). Hence the first factor of the inner integral in (3.2) is equal to

\[
\int_{U(\mathbb{A})} \int_{(\mathbb{A}^\times)^{n-1}} \sum_{a_1,\ldots,a_n \in F} f(X_{a_1,\ldots,a_{n-1}} \cdot \text{ad}(\left[ \begin{array}{c} t^o \\ 0 \\ 1 \end{array} \right] uk)) \, |\det(t)|^s \, dt \, du
\]

\[
= \sum_{a_1,\ldots,a_{n-1} \in F} \left( \int_{U_{n-1}(\mathbb{A})} \int_{(\mathbb{A}^\times)^{n-2}} \sum_{a_1,\ldots,a_{n-1} \in F} f\left( X_{a_1,\ldots,a_{n-1}} \cdot \text{ad}(\left[ \begin{array}{c} t^o \\ 0 \\ 1 \end{array} \right] u^o - wwe^*_{n-1} \right) \cdot \text{ad}(k) \right) \times
\]

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hence it suffices to show that the analogous integral

\[ I \]

where

\[ \prod_{i=1}^{n-1} dv_i \]

which converges absolutely for Re(s) > 1 and continues meromorphically over \( \mathbb{C} \) by induction on \( n \).

The second factor of the inner integral in (3.2) is equal to

\[ \int_{A^k} \Phi([0, \ldots, 0, z]) |z|^{-ns} dz \]

which converges absolutely for Re(s) > 1 and continues meromorphically over \( \mathbb{C} \), hence so does the entire integral (3.1). \( \square \)

**Proposition 3.2.** If \( \mathfrak{o}_{\text{reg}} \) has characteristic polynomial \( p(t) = \prod_i q_i(t)^{m_i} \) where \( q_i(t) \in F[t] \) are all distinct and irreducible, then

\[ I_{\mathfrak{o}_{\text{reg}}}(s) = \int_{G(F), G(\mathbb{A})} \sum_{X,v^* \in (g \times V')_{\text{reg}}(F)} f(X \cdot \text{ad}(g)) \Phi(v^*) \left| \det(g) \right|^s dg \]

is the product of \( \prod_i \zeta_{E_i}(s) \zeta_{E_i}(2s - 1) \ldots \zeta_{E_i}(m_is - m_i + 1) \) by an entire function in \( s \), where \( E_i \simeq F[t]/(q_i) \).

**Proof.** The orbit \( (\mathfrak{o}_{\text{reg}} \times V')_{\text{reg}} \) contains a representative of the form

\[ \left( X_{\mathfrak{o}_{\text{reg}}} = \begin{bmatrix} X_1 & 0 & \cdots \\ 0 & X_2 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}, v^*_{\mathfrak{o}_{\text{reg}}} = [e_{d_1m_1}, e_{d_2m_2}, \ldots] \right) \]

where \( X_i \) is the \( d_i m_i \times d_i m_i \) companion matrix with characteristic polynomial \( q_i(t)^{m_i} \) with \( \deg(q_i) = d_i \) and \( e_{m_i} = [0, \ldots, 0, 1] \in F^{m_i} \). Since \( G(F) \) acts faithfully, the integral \( I_{\mathfrak{o}_{\text{reg}}}(s) \) is equal to

\[ \int_{G(\mathbb{A})} f(X_{\mathfrak{o}_{\text{reg}}} \cdot \text{ad}(g)) \Phi(v^*_{\mathfrak{o}_{\text{reg}}}) \left| \det(g) \right|^s dg \]

\[ = \prod_{i} \int_{\text{GL}_{d_i m_i}(\mathbb{A}) \backslash G(\mathbb{A})} \left( \int_{\text{GL}_{d_i m_i}(\mathbb{A}) \backslash G(\mathbb{A})} f \left( \begin{bmatrix} X_{\text{ad}(g_1)} & 0 & \cdots \\ 0 & X_{\text{ad}(g_2)} & \cdots \\ 0 & 0 & \cdots \end{bmatrix} \cdot \text{ad}(h) \right) \right. \times \Phi \left( [e_{d_1m_1} g_1, e_{d_2m_2} g_2, \ldots] \right) \prod_i \left| \det(g_i) \right|^s dg_i dh, \]

hence it suffices to show that the analogous integral

\[ I_i(s) = \int_{\text{GL}_{d_i m_i}(\mathbb{A})} f_i(X_i \cdot \text{ad}(g_i)) \Phi_i(e_{d_i m_i} g_i) \left| \det(g_i) \right|^s dg_i, \]

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where \( f_i \) and \( \Phi_i \) are Schwartz functions over \( \mathfrak{g}t_{m_i d_i}(\mathbb{A}) \) and \( \mathbb{A}^{m_i d_i} \) respectively, is an entire multiple of \( \zeta_{E_i}(s)\zeta_{E_i}(2s-1)\cdots\zeta_{E_i}(m_is-m_i+1) \).

The \( d_im_i \times d_im_i \) matrix \( X_i \) is conjugate to the \( m_i \times m_i \) Jordan block matrix

\[
\begin{bmatrix}
X_{q_i} & 0 & \cdots & 0 & 0 \\
I_{d_i} & X_{q_i} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & X_{q_i} & 0 \\
0 & 0 & \cdots & I_{d_i} & X_{q_i}
\end{bmatrix}
\]

where \( X_{q_i} \) is the \( d_i \times d_i \) companion matrix with characteristic polynomial \( q_i(t) \) and \( I_{d_i} \) denotes the \( d_i \times d_i \) identity matrix. The centralizer of \( X_{q_i} \) in \( \text{GL}_{d_i}(F) \) is isomorphic to \( E_i^E \), hence under this identification \( \mathbb{A}^{m_i} \cong \mathbb{A}^{d_im_i} \), \( \text{GL}_{m_i}(\mathbb{A}_{E_i}) \subset \text{GL}_{d_i m_i}(\mathbb{A}_F) \) and choosing a new representative

\[
 \left( X_{E_i} = \begin{bmatrix}
x_{E_i} & 0_{E_i} & \cdots & 0_{E_i} \\
1_{E_i} & x_{E_i} & \cdots & 0_{E_i} \\
\vdots & \ddots & \ddots & \vdots \\
0_{E_i} & 0_{E_i} & \cdots & x_{E_i}
\end{bmatrix}, \epsilon_{E_i}^* = [0_{E_i}, \ldots, 0_{E_i}, 1_{E_i}] \right)
\]

where \( x_{E_i} \in E_i \) corresponds to \( X_{p_i} \) and \( 0_{E_i} \) and \( 1_{E_i} \) denotes the zero and unit element of \( E_i \) respectively, the integral \( I_i(s) \) is equal to

\[
\int_{\text{GL}_{m_i}(\mathbb{A}_{E_i}) \setminus \text{GL}_{d_i m_i}(\mathbb{A}_F)} \left( \int_{\text{GL}_{m_i}(\mathbb{A}_{E_i})} f_i(X_{E_i} \cdot \text{ad}(g_i h_i)) \Phi_i(\epsilon_{E_i}^* g_i h_i) |\det(g_i h_i)| \epsilon_{E_i}^* \text{d}g_i \right) \text{d}h_i.
\]

Hence it suffices to analyze the analogous inner integral defined over \( E_i \), which is treated in the next lemma.

**Lemma 3.3.** Let \( X_d = \begin{bmatrix} d & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d \end{bmatrix} \in G(F) \) be a Jordan block with ones along the subdiagonal and \( \epsilon_n^* = [0, \ldots, 0, 1] \in V(F) \), then the integral

\[
I_d(s) = \int_{G(\mathbb{A})} f(X_d \cdot \text{ad}(g)) \Phi(\epsilon_n^* g) |\det(g)| \epsilon_n^* \text{d}g
\]

is the product of \( \zeta_F(s)\zeta_F(2s-1)\cdots\zeta_F(ns-n+1) \) by an entire function.
Proof. By the Iwasawa decomposition it suffices to evaluate the analogous integral \( I_{d,B^-}(s) \) which is integrated over \( B^-(\mathbb{A}) \) instead of \( G(\mathbb{A}) \). Choosing coordinates

\[
b = \begin{bmatrix} z & 0 & \cdots & 0 & 0 & 0 \\
0 & z & 0 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & z & 0 & 0 \\
0 & 0 & \cdots & u_2 & u_3 & \cdots \\
u_1 & u_2 & \cdots & u_{n-2} & u_{n-1} & z \\
\end{bmatrix} [b^0 \ 0] \\
0 \ 1 ,
\]

where \( z \in \mathbb{A}^\times, u_1, \ldots, u_{n-1} \in \mathbb{A} \) and \( b^0 \in \text{GL}_{n-1}(\mathbb{A}) \) is lower triangular, then the integral \( I_{d,B^-}(s) \) is equal to

\[
\begin{aligned}
&\int_{\mathbb{A}} \Phi([u_1, \ldots, u_{n-1}, z] [b^0 \ 0]) |\det(b^0)|^s |\det(b^0)| \ d^x z \prod_{i=1}^{n-1} \frac{du_i}{|z|} \\
&= \int_{\mathbb{A}} \Phi([v_1, \ldots, v_{n-1}, z]) |\det(b^0)|^s |\det(b^0)| \ d^x z \prod_{i=1}^{n-1} \frac{dv_i}{|z|}
\end{aligned}
\]

where \( X_d^0 \) denotes the analogous \((n-1) \times (n-1)\) lower triangular Jordan block and \([v_1, \ldots, v_{n-1}] = [u_1, \ldots, u_{n-1}]b^0 \). Hence \( I_{d,B^-}(s) \) is the product of \( I_{d,B_{n-1}}(s) \) by a Tate integral equal to an entire multiple of \( \zeta_F(ns - n + 1) \), and the lemma follows by induction on \( n \).

\[\square\]

4 Cancellation of singularities

A mirabolic analogue of the parabolic descent operator Let \( P \subset G \) be the parabolic subgroup associated to a partial flag \( V = V_r \supset \cdots \supset V_0 = 0 \)
with successive subquotients $W_i = V_i/V_{i-1}$. Let $\tilde{f}$ denote the partial Fourier transform of $f$ defined by

$$\tilde{f}(X,U^*_r,\ldots,U^*_2) = \int_{u_P(\mathbb{A})} f(\tilde{X} + U) \prod_{i=2}^r \psi(\text{tr}(U^*_i U)) dU$$

where $U^*_i : W_{i-1}(\mathbb{A}) \rightarrow W_i(\mathbb{A})$ commutes with $X$ and $\tilde{X}$ is a lift of $X$ in $p(\mathbb{A})$. In general $\tilde{f}$ is defined upto a multiplicative constant depending on the choice of the lift $\tilde{X}$. However if $P$ is a standard parabolic subgroup, then $\tilde{f}$ is uniquely defined by choosing $\tilde{X} \in m_P$.

Then the parabolic descent $(f, \Phi)_P \in \mathcal{S}(m_P(\mathbb{A}) \times W_r(\mathbb{A}) \times \cdots \times W_1(\mathbb{A}))$ is defined by

$$(f, \Phi)_P(X,w^*_r,\ldots,w^*_1) = \int_K \int_{u_P(\mathbb{A})} \tilde{f}(X,U^*_r,\ldots,U^*_2) \prod_{i=2}^r dU^*_i \Phi(w^*_1) dk$$

where $f^k$ denotes the composite function $f \circ \text{ad}(k)$ and the inner integral is integrated over the affine space consisting of all $U^*_i \in \text{Hom}(W_{i-1}(\mathbb{A}),W_i(\mathbb{A}))$ such that $w^*_i = w^*_1 U^*_i$. In particular,

- if $X \in m_P(\mathbb{A})$ is regular semisimple which implies that $U^*_i = 0$ for all $i = 2,\ldots,r$, then
  $$(f, \Phi)_P(X,0,\ldots,0,w^*_1) = f_P(X) \Phi(w^*_1);$$

- if $X \in m_P(\mathbb{A})$, $w^*_i \in W_i(\mathbb{A})$ and $(X|_{W_i},w^*_i) \in (\mathfrak{gl}(W_i) \times W_i'_{\text{reg}}$ for all $i = 1,\ldots,r$, then
  $$(f, \Phi)_P(X,w^*_r,\ldots,w^*_1) = \tilde{f}(X,U^*_r,\ldots,U^*_2) \Phi(w^*_1)$$

where $U^*_i$ denotes the unique $\mathbb{A}[X]$-linear map from $W_{i-1}(\mathbb{A})$ to $W_i(\mathbb{A})$ such that $w^*_i = w^*_1 U^*_i$ for all $i = 2,\ldots,r$.

**Definition 4.1.** Define the regularized mirabolic integrals $I(s)$ and $I_{\lambda}(s)$, where $\lambda = [n_r,\ldots,n_1] \vdash n$ and $\mathfrak{o}_\lambda \subset \mathfrak{g}(F)$ is a conjugacy class which has Frobenius normal form of partition type $\lambda$, by

$$I(s) = \int_{G(F) \backslash G(\mathbb{A})} \sum_{P} (-1)^{l(P)} K_P(g; f) \sum_{v^* \in (V^U P)'(F)} \Phi(v^* g) \det(g)^s dg$$
where the summation $\sum_P$ ranges over all parabolic subgroups $P \subset G$ defined over $F$ and $V^{U_P} = V_1 \subset V$ denotes the subspace consisting of $U_P$-invariants, and

$$I_{\sigma}(s) = \int_{M_\lambda(F) \backslash M_\lambda(k)} \sum_{X \in o_\lambda} \sum' \left( f, \Phi \right)_D \left( X \cdot \det(h), w^*_r h, \ldots, w^*_1 h \right) \frac{1}{\det(h)^s} dh$$

where the primed summation ranges over all sequences $(X, w^*_r, \ldots, w^*_1)$ such that $(X|_{W_i}, w^*_i) \in (\mathfrak{gl}(W_i) \times W'_i)_{\text{reg}}$ for all $i = 1, \ldots, r$.

Proposition 4.2. The regularized mirabolic integrals

$$I(s) = \sum_{\lambda \vdash n} \sum_{o_\lambda} I_{\sigma}(s)$$

converge absolutely if $\text{Re}(s) > 1$, and continue meromorphically over the entire complex plane, where the summation $\sum_{o_\lambda}$ ranges over all conjugacy classes $o_\lambda \subset \mathfrak{g}(F)$ which have Frobenius normal form of partition type $\lambda$.

Proof. Define

$$I_\lambda(s) = \int_{M_\lambda(F) \backslash M_\lambda(k)} \sum_{X \in m_\lambda(F)} \sum' \left( f, \Phi \right)_D \left( X \cdot \det(h), w^*_r h, \ldots, w^*_1 h \right) \frac{1}{\det(h)^s} dh$$

where the primed summation ranges over all $(X|_{W_i}, w^*_i) \in (\mathfrak{gl}(W_i) \times W'_i)_{\text{reg}}$, then by Proposition 3.1

$$I_\lambda(s) = \sum_{o_\lambda} I_{o_\lambda}(s)$$

converges absolutely if $\text{Re}(s) > 1$, and continues meromorphically over the entire complex plane. Hence it suffices to show that

$$I(s) = \sum_{\lambda \vdash n} I_\lambda(s)$$

under the assumption that $\text{Re}(s) > 1$ without further convergence issues. Further decomposing

$$I(s) = \sum_{m=1}^n I_m(s)$$
by the dimension \( m = \dim W \) where \( W \subset V \) is the subspace spanned by \( v^*, v^*X, v^*X^2, \ldots \), it suffices to consider a summand

\[
I_m(s) = \int_{\mathbb{A}^{n-m}(\mathbb{A})} \sum_P (-1)^{\ell(P)} \sum_{X \in \mathbb{P}_{[n-m]}(\mathbb{A})} \sum_{\mathbb{V}^m} f((X + U) \cdot \text{det}(g))dU\Phi(w^*g)|\text{det}(g)|^s dg
\]

where the primed summation ranges over all \( (X = \begin{bmatrix} X^c & Y \\ 0 & X_1 \end{bmatrix}, w^* = [0, w_1^*]) \) such that \((X_1, w_1^*) \in (\mathfrak{gl}_m(F) \times F^{m'})_{\text{reg}}\).

Since \((X_1, w_1^*) \) is regular, all parabolic subgroups \( P \) appearing in \((4.1)\) such that \((X, w^*) \in (\mathfrak{p} \times (V^{U_P})'(F)\) must contain \( \text{GL}(m) \) in the lower right corner and are hence either of the form

\[
P^o_+ = \begin{bmatrix} P^o & \text{Hom}(F^{n-m}, F^m) \\ \text{Hom}(F^m, W_2) & \text{GL}(m) \end{bmatrix}
\]

with unipotent radical \( U_{P^o} \simeq U_{P^o} \times \text{Hom}(F^{n-m}/W_2, F^m) \) and partial flag length \( \ell(P^o_+) = \ell(P^o) \), or of the form

\[
P^o_- = \begin{bmatrix} P^o & \text{Hom}(F^{n-m}, F^m) \\ 0 & \text{GL}(m) \end{bmatrix}
\]

with unipotent radical \( U_{P^o} \simeq U_{P^o} \times \text{Hom}(F^{n-m}, F^m) \) and partial flag length \( \ell(P^o) = \ell(P^o) + 1 \), where \( P^o \subset \text{GL}(n - m) \) is a parabolic subgroup which stabilizes the subspace \( W_2 = (F^{n-m})^U_{P^o} \subset F^{n-m} \). Hence \( I_m(s) \) is equal to

\[
\int_{\mathbb{A}^{n-m}(\mathbb{A})} \mathbb{G}(\mathbb{A}) \sum_{X \in \mathbb{P}_{[n-m]}(\mathbb{A})} (-1)^{\ell(P^o)} \sum_{X \in \mathbb{P}_{[n-m]}(\mathbb{A})} \sum_{\mathbb{V}^m} f((X + U^o \cdot \text{det}(g))dU^o +
\]

\[
- \int f(\begin{bmatrix} X^c + U^o & Y + U_+ \\ 0 & X_1 \end{bmatrix} \cdot \text{det}(g))dU_+ +
\]

\[
- \int f(\begin{bmatrix} X^c + U^o & Y + U_+ \\ 0 & X_1 \end{bmatrix} \cdot \text{det}(g))dU_+ +
\]

\[
where \( U_+ \) is integrated over \( \text{Hom}(\mathbb{A}^{n-m}, \mathbb{A}^m)/\text{Hom}(F^{n-m}, F^m) \) and \( U_- \) is integrated over \( \text{Hom}(\mathbb{A}^{n-m}/W_2(\mathbb{A}), \mathbb{A}^m)/\text{Hom}(F^{n-m}/W_2(F), F^m) \).

Let \( V \in \text{Hom}(\mathbb{A}^{n-m}, \mathbb{A}^m) \), then

\[
\begin{bmatrix} X^c + U^o & (Y + U_+) \\ 0 & X_1 \end{bmatrix} \cdot \text{det}(\begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix})
\]

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\[
\begin{bmatrix}
X^0 + U^0 & (Y + U_\pm) + (X^0 + U^0)V - VX_1
\end{bmatrix},
\]
hence
\[
\int \hom(\mathbb{A}^{n-m}, \mathbb{A}^m) \sum_{Y \in \hom(F^{n-m}, F^m)} \times \left( \int f\left( \begin{bmatrix}
X^0 + U^0 & Y + U_+ \\
0 & X_1
\end{bmatrix} \right) \cdot \text{ad}(\begin{bmatrix}
1 \\
0
\end{bmatrix}) \right) dU_+ + \\
- \int f\left( \begin{bmatrix}
X^0 + U^0 & Y + U_- \\
0 & X_1
\end{bmatrix} \right) \cdot \text{ad}(\begin{bmatrix}
1 \\
0
\end{bmatrix}) dU_- \right) dV
\]
\[
= \sum_{Y \in \hom(W_2(F), F^m)_V} f_1\left( \begin{bmatrix}
X^0 + U^0 & Y \\
0 & X_1
\end{bmatrix} \right) + \\
- \int \hom(W_2(\mathbb{A}), \mathbb{A}^m)_V f_1\left( \begin{bmatrix}
X^0 + U^0 & 0 \\
0 & 0
\end{bmatrix} \right) dY
\]
where \(\hom(W_2, F^m)_V\) denotes the quotient of \(\hom(W_2, F^m)\) consisting of \(\text{ad}(\begin{bmatrix}
0 & Y \\
1 & 0
\end{bmatrix})\)-coinvariants and \(f_1\) denotes the fiberwise integral of \(f\) along the fibers of the composite map

\[
\hom(\mathbb{A}^{n-m}, \mathbb{A}^m) \rightarrow \hom(\mathbb{A}^{n-m}, \mathbb{A}^m) / \hom(\mathbb{A}^{n-m}/W_2(\mathbb{A}), \mathbb{A}^m) \\
\parallel \\
n\ 
\hom(W_2(\mathbb{A}), \mathbb{A}^m) \\
\downarrow \\
\hom(W_2(\mathbb{A}), \mathbb{A}^m)_V,
\]
which is equal to
\[
\sum_{Y^* \in \hom(F^m, W_2(F))} Y^* \in \hom(\mathbb{A}^m, W_2(\mathbb{A})) \, Y^* = Y^* \text{ ad}(X^0)_{|W_2}
\]
by Poisson summation where \(\tilde{f}\) denotes the partial Fourier transform of \(f_1\) defined by
\[
\tilde{f}\left( \begin{bmatrix}
X^0 + U^0 & 0 \\
0 & X_1
\end{bmatrix}, Y^* \right) = \int_{\hom(W_2(\mathbb{A}), \mathbb{A}^m)_V} f_1\left( \begin{bmatrix}
X^0 + U^0 & Y \\
0 & 0
\end{bmatrix} \right) \psi(\text{tr}(Y^* Y)) dY
\]
where \(Y^* \in \hom(\mathbb{A}^m, W_2(\mathbb{A})) \, Y^* \text{ intertwines the actions of } X_1 \text{ and } X^0.\)
Hence by the partial Iwasawa decomposition $g = \begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g^0 & 0 \\ 0 & g_1 \end{bmatrix}$, where $V \in \text{Hom}(\hat{\mathbb{A}}^{n-m}, \mathbb{A}_m)/\text{Hom}(F^{n-m}, F^m)$, $g^0 \in \text{GL}_{n-m}(F) \setminus \text{GL}_{n-m}(\mathbb{A})$, and $g_1 \in \text{GL}_m(F) \setminus \text{GL}_m(\mathbb{A})$, $k \in K$, the integral $I_m(s)$ is equal to

$$\int_{\text{GL}_m(F) \setminus \text{GL}_m(\mathbb{A})} \sum_{(X_1, w_1^*) \in (\text{gl}_m(F) \times F^m), \text{reg}} \int_K$$

$$\times \left( \int_{\text{GL}_{n-m}(F) \setminus \text{GL}_{n-m}(\mathbb{A})} \sum_{P^0 \in \text{GL}(n-m)} (-1)^{(I_{P^0}) \times \text{u}_{P^0}(\mathbb{A})/\text{u}_{P^0}(F)} \times \sum_{w_2^* \in W_2(F)} \widetilde{f}_k \left( \begin{bmatrix} X^0 & U^0 \end{bmatrix} \text{ad}(g^0) \begin{bmatrix} 0 \\ X_1 \text{ad}(g_1) \end{bmatrix}, w_2^*g^0 \right) \frac{|\text{det}(g^0)|^m}{|\text{det}(g_1)|^{n-m}} \times$$

$$\times dU^0 |\text{det}(g^0)|^s dg^0 \Phi([0, w_1^* g_1] k) dk \frac{|\text{det}(g_1)|^s}{|\text{det}(g^0)|^m} dg_1,$$

where $Y^*$ denotes the unique $F[X]$-linear map from $F^m$ to $W_2(F)$ such that $w_2^* = w_1^* Y^*$, which is equal to

$$\int_{\text{GL}_m(F) \setminus \text{GL}_m(\mathbb{A})} \sum_{(X_1, w_1^*) \in (\text{gl}_m(F) \times F^m), \text{reg}} \left( \sum_{\lambda \in (n-m)} \int_{\text{M}_\lambda(F) \setminus \text{M}_\lambda(\mathbb{A})} \right.$$

$$\times \sum_{X^0 \in \text{M}_\lambda(F)} \sum_{w_1^* \in W_1(F)} (f, \Phi) P_{\lambda, m, l} \left( \begin{bmatrix} X^0 & 0 \end{bmatrix}, w_1^* g^0, \ldots, w_1^* g_1 \right) \times$$

$$\times |\text{det}(h^0)|^s dh^0 |\text{det}(g_1)|^s dg_1$$

$$= \sum_{\lambda \in \mathbb{A}} I_{\lambda}(s)$$

by induction on $n$. \qed

**Proposition 4.3.** If $\mathfrak{o}_\lambda$ has invariant factors $p_r \mid p_{r-1} \mid \cdots \mid p_1$ where $p_j(t) = \prod_i q_i(t)^{m_{ij}}$ and $q_i(t) \in F[t]$ are all distinct and irreducible, then the regularized mirabolic integral $I_{\mathfrak{o}_\lambda}(s)$ is the product of

$$\prod_i \prod_j \prod_{k=1}^{m_{ij}} \zeta_{E_i}(h_{\lambda}(j, k)s - m_{ij} + k)$$

by an entire function in $s$, where $E_i \simeq F[t]/(q_i)$ and

$$h_{\lambda}(j, k) = m_{ij} - k + \{l \geq j \mid m_{il} \geq k\}$$

where $|S|$ denotes the cardinality of a finite set $S$. 

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Proof. By the same argument as in the proof of Proposition 3.2, it suffices to consider the special case when \( f_j = (t - d)^{n_j} \) where \( d \in F \) is the unique eigenvalue of \( \sigma_\lambda \), and proceed as in the proof of Lemma 3.3.

Choosing the representative

\[
\left( X = \begin{bmatrix}
X_r & 0 & \cdots & 0 \\
0 & X_{r-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_1
\end{bmatrix}, \ w_r^* = e_{n_r}^*, \ldots, \ w_1^* = e_{n_1}^*
\right)
\]

where \( X_j = \begin{bmatrix}
d & 0 & \cdots \\
0 & d & \cdots \\
\vdots & \ddots & \ddots \\
\end{bmatrix} \in \text{GL}_{n_j}(F) \) is a Jordan block with ones along the subdiagonal and \( e_{n_j}^* = [0, \ldots, 0, 1] \in W'_j(F) \simeq F^{n_j} \), it suffices to evaluate the analogous integral

\[
I_{\sigma_\lambda, B^-}(s) = \int_{B_{n_1}^-(\mathbb{A})} \cdots \int_{B_{n_r}^-(\mathbb{A})} \times (f, \Phi)_{P_{\lambda}} \left( \begin{bmatrix}
X_r \cdot \text{ad}(b_r) & 0 & \cdots & 0 \\
0 & \cdots & \ddots & \vdots \\
0 & X_1 \cdot \text{ad}(b_1) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}, \ w_r^* b_r, \ldots, \ w_1^* b_1 \right) \times \\
\prod_{j=1}^r |\det(b_j)|^s \ \text{db}_j
\]

integrated over \( B_{n_j}^-(\mathbb{A}) \subset \text{GL}_{n_j}(\mathbb{A}) \).

Choosing coordinates

\[
b_j = b_{j,n_j} \begin{bmatrix}
b_{j,n_j-1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & I_{n_j-2} & \cdots & I_{n_j-1}
\end{bmatrix}
\]

as in the proof of Lemma 3.3 where each \( b_{j,k} \in B_{k}^-(\mathbb{A}) \) is of the form

\[
b_{j,k} = \begin{bmatrix}
z & 0 & \cdots & 0 & 0 \\
0 & z & \cdots & 0 & 0 \\
u_{k-1} & 0 & \cdots & 0 & 0 \\
0 & u_{k-2} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
u_2 & u_3 & \cdots & u_{k-1} & z \\
u_1 & u_2 & \cdots & u_{k-2} & u_{k-1}
\end{bmatrix}
\]

where \( z \in \mathbb{A}^x, \ u_1, \ldots, u_{k-1} \in \mathbb{A} \) and \( I_l \) denotes the \( l \times l \) identity matrix, then by a similar computation, the \( db_{1,k} \) integral produces a Tate integral of the form

\[
\int \cdots \int_{\mathbb{A}^x} \int (f, \Phi)_{P_{\lambda}} (\ldots, X_2 \cdot \text{ad}(b_2 b_{1,k}^0), X_{1,k} \cdot \text{ad}(b_{1,k}), \ldots) \times
\]
\[
\times |\det(b_2b_{1,k}^\circ)|^s \left| z \right|^{k_2s-k_1+1} d \times \prod_{i=1}^{k-1} dv_i
\]

where \(b_{1,k}^\circ \in \text{GL}_{n_2}(\mathbb{A})\) denotes the action of \(b_{1,k}\) on \(W_2(\mathbb{A})\) induced from the unique \(\mathbb{A}[X]\)-linear map from \(W_1(\mathbb{A})\) to \(W_2(\mathbb{A})\) which maps \(e_{n_1}^*\) to \(e_{n_2}^*\).

There are two cases:

1. If \(k \leq n_1 - n_2\), then \(b_{1,k}^\circ = I_{n_2}\) and the Tate integral \((4.2)\) is an entire multiple of

\[
\zeta_F(k_2s - k_1 + 1) = \zeta_F(h_{\lambda}(1,l)s - n_1 + l)
\]

where \(l = n_1 - k + 1\);

2. If \(k > n_1 - n_2\), then

\[
b_{1,k}^\circ = \begin{bmatrix}
  z & \cdots & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n_1-n_2+1} & \cdots & z & 0 \\
  0 & \cdots & 0 & I_{n_1-k}
\end{bmatrix}
\]

and the Tate integral \((4.2)\) is an entire multiple of

\[
\zeta_F(h_{\lambda}(2,l)s + k_2s - k_1 + 1) = \zeta_F(h_{\lambda}(1,l)s - n_1 + l)
\]

by induction, where \(l = n_1 - k + 1\).

**Theorem 4.4** (Mirabolic trace formula for \(\mathfrak{gl}(n)\)). Let \(\Phi \in \mathcal{S}(V(\mathbb{A}))\), then the distribution \(I(s) \in \mathcal{S}'(\mathfrak{g}(\mathbb{A}))\) defined by

\[
I(s) = \sum_{\lambda \vdash n} \sum_{\phi_{\lambda}} I_{\phi_{\lambda}}(s)
\]

where the summation \(\sum_{\phi_{\lambda}}\) ranges over all conjugacy classes \(\phi_{\lambda} \subset \mathfrak{g}(F)\) which have Frobenius normal form of partition type \(\lambda\)

1. converges absolutely if \(\text{Re}(s) > 1\) and continues meromorphically to a sum of entire multiples of zeta functions, and

2. is invariant under the Fourier transform \(f \leftrightarrow \hat{f}\) on \(\mathcal{S}(\mathfrak{g}(\mathbb{A}))\).

**Proof.** It remains to establish the invariance under the Fourier transform, which follows from the alternative definition of \(I(s)\) as the alternating sum

\[
\int_{G(F) \setminus G(\mathbb{A})} \sum_P (-1)^{|\ell(P)|} K_P(g; f) \sum_{v^* \in (V^{\mu_P})^*(F)} \Phi(v^* g) |\det(g)|^s dg,
\]

together with the observation that \(K_P(g; f) = K_P(g; \hat{f})\). \(\square\)
5 Miscellaneous remarks

Comparison with Shintani zeta functions  From the perspective of invariant theory, the regular locus \((\mathfrak{g} \times V')_{\text{reg}} \subset \mathfrak{g} \times V'\) could be characterized by the property that the fibered product

\[
(\mathfrak{d} \times V')_{\text{reg}} \longrightarrow (\mathfrak{g} \times V')_{\text{reg}}
\]

\[
\mathfrak{d} \times V' \xrightarrow{\text{fiber}} \mathfrak{g} \times V' \longrightarrow (\mathfrak{g} \times V')/G
\]

where \(\mathfrak{d} \times V' \subset \mathfrak{g} \times V'\) is a fiber of the affine quotient of \(\mathfrak{g} \times V'\) with respect to the mirabolic action, is the unique Zariski open orbit \((\mathfrak{d} \times V')_{\text{reg}} \subset \mathfrak{d} \times V'\).

Here \(\mathfrak{d} \subset \mathfrak{g}\) denotes the equivalence class consisting of all matrices with a common characteristic polynomial, and \((\mathfrak{d} \times V')_{\text{reg}} \subset \mathfrak{o}_{\text{reg}} \times V'\) where \(\mathfrak{o}_{\text{reg}}\) is the unique Zariski open orbit contained in \(\mathfrak{d}\). However, unlike each \(\mathfrak{o}_{\text{reg}} \subset \mathfrak{d}\) which has a complement of codimension \(\geq 3\), the mirabolic open subsets \((\mathfrak{d} \times V')_{\text{reg}} \subset \mathfrak{o}_{\text{reg}} \times V'\) and \((\mathfrak{g} \times V')_{\text{reg}} \subset \mathfrak{g} \times V'\) are all principal open with a common defining polynomial

\[
\det \begin{bmatrix}
v^* \\
\vdots \\
v^* X^{n-1}
\end{bmatrix} \neq 0
\]

which is a relative invariant transforming under the character \(\det : G \to \mathbb{G}_m\).

The situation is reminiscent of the theory of Shintani zeta functions of prehomogeneous vector spaces [SS74]:

- Let \(W\) be a prehomogeneous vector space with respect to an algebraic group \(H \subset \text{GL}(W)\) such that the Zariski open \(H\)-orbit \(W_f \subset W\) is principal open with a relatively invariant defining polynomial \(f \neq 0\) which transforms as

\[
f(w^*h) = \chi(h)f(w^*)
\]

under an algebraic character \(\chi : H \to \mathbb{G}_m\).

- Let \(\Phi \in \mathcal{S}(W(A))\), then under appropriate convergence assumptions the zeta integral

\[
Z(s; \Phi) = \int_{H(F) \backslash H(A)} \sum_{w^* \in W_f(F)} \Phi(w^*h)|\chi(h)|^s dh
\]
continues meromorphically to the entire complex plane and satisfies the functional equation
\[ Z(s; \Phi) = Z^* \left( \frac{\dim(W)}{\deg(f)} - s; \tilde{\Phi} \right) \]

where \( Z^* \) denotes the analogous zeta integral of the contragredient prehomogeneous vector space \( W^* \).

See [M16] for another complementary approach of extending the theory of Shintani zeta functions to the adjoint representation of \( \text{GL}(n) \) on \( \mathfrak{gl}(n) \) via Chaudouard’s trace formula.

Parallel to the Shintani zeta integral, the part of the mirabolic integral over the regular locus
\[ I_{\text{reg}}(s) = \int_{G(F) \setminus G(A)} \sum_{(X, v^* g) \in (g \times V')_{\text{reg}}(F)} f(X \cdot \text{ad}(g)) \Phi(v^* g) \left| \det(g) \right|^s dg \]
may be regarded as a non-prehomogeneous zeta integral
\[ I_{\text{reg}}(s) = Z_{\text{reg}}(s; f, \Phi) \]
which decomposes into
\[ Z_{\text{reg}}(s; f, \Phi) = \sum_{\sigma_{\text{reg}}} I_{\sigma_{\text{reg}}}(s), \]
where \( I_{\sigma_{\text{reg}}}(s) \) is an entire multiple of \( \prod_i \zeta_{E_i}(s) \zeta_{E_i}(2s-1) \ldots \zeta_{E_i}(m_is-m_i+1) \) for each regular conjugacy class \( \sigma_{\text{reg}} \subset g(F) \) with characteristic polynomial \( p(t) = \prod_i q_i(t)^{m_i} \) where \( q_i(t) \in F[t] \) are all distinct and irreducible, and \( E_i \simeq F[t]/(q_i) \).

However, unlike the prehomogeneous zeta integral \( Z(s; \Phi) \), the non-prehomogeneous zeta integral \( Z_{\text{reg}}(s; f, \Phi) \) does not satisfy any functional equation unless all exponents \( m_i = 1 \). This leads to another zeta integral
\[ I_{\text{rs}}(s) = \int_{G(F) \setminus G(A)} \sum_{(X, v^* g) \in (g \times V')_{\text{reg}}(F)} f(X \cdot \text{ad}(g)) \Phi(v^* g) \left| \det(g) \right|^s dg \]
\[ = Z_{\text{rs}}(s; f, \Phi) \]
over the regular semisimple locus \( (g \times V')_{\text{rs}} \subset g \times V' \) which is the principal open subset with defining polynomial
\[ \Delta(X) \det \begin{bmatrix} v^* \\ \vdots \\ v^*X^{n-1} \end{bmatrix} \neq 0 \]
where \( \Delta(X) \) denotes the discriminant of the characteristic polynomial of \( X \), which also transforms under the character \( \det : G \to \mathbb{G}_m \). Then the regular semisimple zeta integral satisfies the functional equation

\[
Z_{rs}(s; f, \Phi) = Z_{rs}(1 - s; f^\top, \hat{\Phi}).
\]

The semisimple locus of the mirabolic action  

At the other extreme, each fiber \( \delta \times V' \subset g \times V' \) also contains a unique Zariski closed orbit \( \tilde{o} \times V'_{ss} \subset g \times V' \) consisting of the pairs \( (X, v^*) \) such that the restriction of \( X \) to the subspace \( W \subset V \) spanned by \( v^*, v^*X, \ldots, v^*X^{n-1} \) is semisimple, in other words the discriminant \( \Delta(X|W) \neq 0 \).

The part of the regularized mirabolic integral

\[
I_{ss}(s) = \int_{G(F) \backslash G(k)} \prod P (-1)^{\ell(P)} \sum_{X \in p(F), v^* \in (V', V')_{ss}} \sum_{(X, v^*) \in (g \times V')_{ss}} f((X + U) \cdot \text{ad}(g)) dU \Phi(v^* g) \det(g)^s dg
\]

over the semisimple locus decomposes into

\[
I_{ss}(s) = \sum_{ss} I_{\tilde{o}_{ss}}(s),
\]

where for each semisimple conjugacy class \( \tilde{o}_{ss} \subset g(F) \) with characteristic polynomial \( p(t) = \prod_i q_i(t)^{m_i} \) where \( q_i(t) \in F[t] \) are distinct and irreducible, the summand \( I_{\tilde{o}_{ss}}(s) \) is an entire multiple of \( \prod_i \zeta_{E_i}(s) \zeta_{E_i}(2s) \ldots \zeta_{E_i}(m_i s) \) where \( E_i \simeq F[t]/(q_i) \).

Hence the regular and semisimple parts \( I_{reg}(s) \leftrightarrow I_{ss}(s) \) of the regularized mirabolic integral are interchanged under the transpose operator \( f \leftrightarrow f^\top \) on \( S(g(k)) \), the Fourier transform \( \Phi \leftrightarrow \hat{\Phi} \) on \( S(V(k)) \), and the substitution \( s \leftrightarrow 1 - s \) on \( \mathbb{C} \), all performed simultaneously. This leaves the intersection \( I_{rs}(s) \) invariant and leads to the functional equation of \( Z_{rs}(s; f, \Phi) \).

The functional equation of \( I(s) \) via Young diagrams  

Let \( o \subset g(F) \) be a conjugacy class with characteristic polynomial \( p(t) = \prod_i q_i(t)^{m_i} \) and invariant factors \( p_r \mid \cdots \mid p_1 \) where \( p_j(t) = \prod_i q_i(t)^{m_{ij}} \) and \( q_i(t) \in F[t] \) are all distinct and irreducible, then each irreducible factor \( q_i \) determines a partition \( \lambda_i = [\ldots, m_{ij}, \ldots, m_{i1}] = m_i \). In particular
• $o = o_{\text{reg}}$ is regular if and only if all $\lambda_i = [m_i]$ with Young diagram
  \[
  \begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
  \end{array},
  \]

• $o = o_{\text{ss}}$ is semisimple if and only if all $\lambda_i = [1, \ldots, 1]$ with Young diagram
  \[
  \begin{array}{cccc}
  & & & \\
  & & & \\
  \end{array},
  \]

• $o = o_{\text{rs}}$ is regular semisimple if and only if all $m_i = 1$ and $\lambda_i = [1]$ with Young diagram
  \[
  \begin{array}{c}
  \end{array}.
  \]

In general, the regularized mirabolic integral satisfies the functional equation

\[ I(s) = I(1 - s) \]

under the transformations $f \leftrightarrow f^{\top}$ and $\Phi \leftrightarrow \hat{\Phi}$, which interchanges each partition $\lambda_i \vdash m_i$ with the conjugate partition $\lambda_i \vdash m_i$ which is obtained by reflecting the Young diagram along the main diagonal.

More precisely, Proposition 4.3 could be reformulated in terms of Young diagrams which states that the regularized mirabolic integral $I_o(s)$ is an entire multiple of

\[ \prod_i \prod_{j,k} \zeta_{E_i} (h_{\lambda_i}(j,k)s - a_{\lambda_i}(j,k)) , \]

where $E_i \simeq F[t]/(q_i)$ and $h_{\lambda}(j,k), a_{\lambda}(j,k)$ denote the hook length and arm length of the $(j,k)$th box of the Young diagram of a partition $\lambda \vdash m$.
The mirabolic trace formula in the group case  Proposition 4.2 could be regarded as the geometric side of a mirabolic trace formula for GL(n).

Let $f \in C^\infty_c(G(\mathbb{A}))$ be a test function, then the group analogues of the kernel functions $K(g; f)$ and $K_P(g; f)$ for $g \in G(F) \setminus G(\mathbb{A})$ have been defined by Arthur [A78], and Proposition 4.2

$$I(s) = \int_{G(F) \setminus G(\mathbb{A})} \sum_P (-1)^{l(P)} K_P(g; f) \sum_{v^* \in (V^U)^\prime(F)} \Phi(v^* g) \left| \det(g) \right|^s dg$$

$$= \sum_{\lambda \vdash n} \sum_{\phi_{\lambda}} I_{\phi_{\lambda}}(s)$$

generalizes to the group case with analogous definitions of $I_{\phi_{\lambda}}(s)$. In fact, the absolute convergence in the group case is easier to establish since there are only finitely many conjugacy classes which intersect the support of $f$.

On the spectral side, the contributions from the cuspidal spectrum

$$I_{\text{cusp}}(s) = \int_{Z(\mathbb{A})G(F) \setminus G(\mathbb{A})} \sum_{\phi} R_{\text{cusp}, f} \phi(g) \overline{\phi(g)} E(g, s; \Phi) \left| \det(g) \right|^s dg,$$

where the summation $\sum_{\phi}$ ranges over an orthonormal basis of the cuspidal spectrum $L^2(G(F) \setminus G(\mathbb{A}))_{\text{cusp}}$ and $R_{\text{cusp}}$ denotes the regular representation of $C^\infty_c(G(\mathbb{A}))$ on $L^2(G(F) \setminus G(\mathbb{A}))_{\text{cusp}}$, decomposes into

$$I_{\text{cusp}}(s) = \sum_{\pi \text{ cuspidal}} \sum_{\phi_\pi} I(s; R_{\text{cusp}, f} \phi_\pi, \overline{\phi_\pi}, \Phi)$$

where each summand is a zeta integral equal to an entire multiple of the Rankin–Selberg $L$-function $L(s, \pi \times \pi^*)$ following [JS81].

At the other extreme opposite to the cuspidal spectrum, for Eisenstein series induced from the Borel subgroup $B \subset G$, the mirabolic double flag variety contains a unique Zariski open orbit

$$(B \setminus G \times B \setminus G \times V')_{\text{reg}} \subset B \setminus G \times B \setminus G \times V'$$
on which the diagonal action of $G$ is faithful. Then the contribution from the regular locus decomposes into

$$\int_{G(F) \setminus G(\mathbb{A})} \int_\mathbb{A}^* \sum_{\chi} \sum_{(\delta_1, \delta_2, v^*) \in (B \setminus G \times B \setminus G \times V')_{\text{reg}}(F)} I'B.f \left( \chi(\delta_1 g) e^{(\xi + \rho)H_B(\delta_1 g)} \frac{\overline{\chi(\delta_2 g)} e^{(\xi + \rho)H_B(\delta_2 g)}}{\chi(\delta_1 g) e^{(\xi + \rho)H_B(\delta_1 g)}} \right.$$
\[ \times \Phi(v^*g) \left| \det(g) \right|^s \mathrm{d}g \]

where the summation \( \sum_\chi \) ranges over all characters \( \chi : T(F) \backslash T(\mathbb{A}) \rightarrow \mathbb{C}^\times \)
and each summand is a zeta integral equal to an entire multiple of a quotient of products of automorphic \( L \)-functions which could be computed following the Langlands–Shahidi method [S81].

More generally, contributions from the regular locus \( (P \backslash G \times P \backslash G \times V')_{\text{reg}} \)
of all parabolic subgroups \( P \subset G \) constitute the regular part of the spectral side. On the other hand the cancellation of singularities on the spectral side and further details will be treated in a forthcoming paper [C20].

A Appendix: The local theory

Local preliminaries For this appendix let \( F_v \) be a non-archimedean local field and let \( X_v \) denote the totally disconnected locally compact space \( X(F_v) \) consisting of the \( F_v \)-valued points of an algebraic variety \( X \).

Let \( f \in S(g_v) \), then by the Weyl integration formula

\[
\int_{g_v} f(X) \mathrm{d}X = \sum_{\lambda \vdash n} \sum_{E_v^\times \subset M_{\lambda,v}} \frac{1}{|W(M_{\lambda,v}, E_v^\times)|} \times \nabla \int_{E_v^\times \backslash G_v} f(X \cdot \text{ad}(g)) \mathrm{d}g |\Delta(X)|_v^{-1} \mathrm{d}X
\]

where the summation \( \sum_{E_v^\times} \) ranges over all conjugacy classes of maximal tori \( E_v^\times \subset M_{\lambda,v} \) elliptic modulo the center of \( M_{\lambda,v} \), \( E_v = \mathfrak{e}_v \subset \mathfrak{g}_v \) denotes the Lie algebra of \( E_v^\times \) and \( E_{\text{reg},v} \subset E_v \) denotes the regular locus defined by \( \Delta(X) \neq 0 \) where \( \Delta(X) \) denotes the discriminant of the characteristic polynomial of \( X \), and \( W(M_{\lambda,v}, E_v^\times) \) denotes the relative Weyl group of \( E_v^\times \) with respect to \( M_{\lambda,v} \).

Fix a non-trivial additive character \( \psi_v : F_v \rightarrow \mathbb{C}^\times \). Let \( \langle \cdot, \cdot \rangle_v \) be a non-degenerate bilinear form on a vector space \( W_v \) equipped with the self-dual Haar measure \( \mathrm{d}w \). For all \( f \in S(W_v) \), define its Fourier transform by

\[
\widehat{f}(v) = \int_{W_v} f(w) \overline{\psi_v(\langle v, w \rangle_v)} \mathrm{d}w.
\]

Then the Fourier transform satisfies the inversion formula

\[
\widehat{\widehat{f}}(w) = f_1(-w)
\]
and the Plancherel formula
\[
\int_{W_v} \hat{f}_1(w) f_2(w) \, dw = \int_{W_v} f_1(w) \hat{f}_2(w) \, dw
\]
for all \( f_1, f_2 \in S(W_v) \).

The local trace formula of Waldspurger Let \( f_1, f_2 \in S(g_v) \), define the local kernel function
\[
K_v(g; f_1, f_2) = \int_{g_v} f_1(X) f_2(X \cdot \text{ad}(g)) \, dX
\]
for \( g \in G_v \), then
\[
K_v(g; \hat{f}_1, f_2) = K_v(g; f_1, \hat{f}_2)
\]
by the Plancherel formula.

Applying the results of Arthur [A91], Waldspurger [W95] has determined the asymptotic behavior of the divergent integral
\[
\int_{Z_v \backslash G_v} K_v(g; f_1, f_2) \, dg
\]
over an increasing system of compact subsets of \( Z_v \backslash G_v \). The resultant local trace formula is an identity of the form
\[
J(\hat{f}_1, f_2) = J(f_1, \hat{f}_2)
\]
between absolutely convergent integrals defined by
\[
J(f_1, f_2) = \sum_{\lambda \vdash n} \sum_{E^\times_v \subset M_{\lambda,v}} (-1)^{\ell(P_\lambda)} |W(M_{\lambda,v}, E^\times_v)| \int_{E^\text{reg},v} J_{M_{\lambda,v}}(X; f_1, f_2) \, dX
\]
where the distributions \( J_{M_{\lambda,v}}(X; f_1, f_2) \) are combinations of weighted orbital integrals \( J_{M_{\lambda,v}}^{P_1}(X; (f_1)_{P_1,v}) \) and \( J_{M_{\lambda,v}}^{P_2}(X; (f_2)_{P_2,v}) \) of the parabolic descent of \( f_1 \) and \( f_2 \) along all standard parabolic subgroups \( M_\lambda \subset P_1, P_2 \subset G \).

• If \( \lambda = [n] \) and \( E^\times_v \subset G_v \) is elliptic modulo \( Z_v \) where \( E_v/F_v \) is a field extension of degree \( n \), then
\[
J_{G,v}(X; f_1, f_2) = |\Delta(X)|_{v}^{1/2} \int_{E^\times_v \backslash G_v} f_1(X \cdot \text{ad}(g_1)) \, dg_1 \times
\]
\[
\times |\Delta(X)|^{1/2} \int_{E_v^\times \backslash G_v} f_2(X \cdot \text{ad}(g_2))dg_2
= I_{G,v}^G(X; f_1) I_{G,v}^G(X; f_2)
\]

where
\[
I_{G,v}^G(X; f) = |\Delta(X)|^{1/2} \int_{E_v^\times \backslash G_v} f(X \cdot \text{ad}(g))dg
\]
denotes a normalized elliptic orbital integral.

- Waldspurger [W95] has also introduced another version of the local trace formula

\[
\sum_{\lambda} \sum_{E_v^\times} \frac{(-1)^{t(P_\lambda)}}{|W(M_{\lambda,v}, E_v^\times)|} \int_{E_{\text{reg},v}} I_{M_{\lambda,v}}^G(X; \tilde{f}_1) I_{G,v}^G(X; f_2)dX
= \sum_{\lambda} \sum_{E_v^\times} \frac{(-1)^{t(P_\lambda)}}{|W(M_{\lambda,v}, E_v^\times)|} \int_{E_{\text{reg},v}} I_{G,v}^G(X; f_1) I_{M_{\lambda,v}}^G(X; \tilde{f}_2)dX
\]

where \(I_{M,v}^G(X; f)\) are conjugation invariant analogues of the weighted orbital integrals \(J_{M,v}^G(X; f)\).

**Local Tate integrals** Let \(\Phi \in S(F_v)\), then the local Tate integral \(I_v(s; \Phi)\) is a local integral of the form
\[
I_v(s; \Phi) = \int_{F_v^\times} \Phi(x)|x|^s d^\times x
\]
for all \(s \in \mathbb{C}\) such that the integral converges absolutely. In addition to the analytic properties of the global zeta functions, Tate [T50] has also established the following local analytic properties of \(I_v(s; \Phi)\):

- the integral \(I_v(s; \Phi)\) converges absolutely on the half-plane \(\text{Re}(s) > 0\) and continuous to a meromorphic function on the entire complex plane, which will also be denoted by \(I_v(s, \Phi)\);
- there exists a Laurent polynomial \(p(t) \in \mathbb{C}[t, t^{-1}]\) such that

\[
I_v(s; \Phi) = \frac{p(q_v^{-s})}{1 - q_v^{-s}}
\]

where \(q_v = |F_v/O_v|\) denotes the cardinality of the residue field of \(F_v\);
• the meromorphic functions \( I_v(s; \Phi_1) \) and \( I_v(s; \Phi_2) \) satisfy the local functional equation

\[
I_v(s; \Phi_1)I_v(1 - s; \hat{\Phi}_2) = I_v(1 - s; \hat{\Phi}_1)I_v(s; \Phi_2)
\]

for all \( \Phi_1, \Phi_2 \in S(F_v) \), which follows from the Plancherel formula.

Parallel to the global Tate integrals, Tate [T50] has also introduced local Tate integrals twisted by a local multiplicative character \( \chi_v : F_v^\times \to \mathbb{C}^\times \), which will also be excluded from this paper for simplicity.

**Definition A.1.** Let \( \Phi_1, \Phi_2 \in S(V_v) \), then the local mirabolic Eisenstein integral \( E_v(g, s; \Phi_1, \Phi_2) \) is defined by

\[
E_v(g, s; \Phi_1, \Phi_2) = |\det(g)|_v^s \int_{F_v^\times} \int_{V_v} \Phi_1(v^* g) \Phi_2(z v^* g) dv^* |z|_v^{ns} dz
\]

which converges absolutely for \( \text{Re}(s) > 0 \) and continues meromorphically to the entire complex plane, where \( g \in Z_v \setminus G_v \).

**Proposition A.2.** Let \( f_1, f_2 \in S(g_v) \) and \( \Phi_1, \Phi_2 \in S(V_v) \), then the integral

\[
\int_{Z_v \setminus G_v} K_v(g; f_1, f_2)E(g, s; \Phi_1, \Phi_2) dg
\]

\[
= \sum_{\lambda \vdash n} \sum_{E_v^\times \subset M_{\lambda,v}} \frac{1}{|W(M_{\lambda,v}, E_v^\times)|} \int_{E_{\text{reg},v}} I_v(X, s; f_1, f_2, \Phi_1, \Phi_2) dX
\]

where

\[
I_v(X, s; f_1, f_2, \Phi_1, \Phi_2) = \int_{E_v^\times \setminus (G_v \times G_v)} f_1(X \cdot \text{ad}(g_1)) f_2(X \cdot \text{ad}(g_2)) \times
\]

\[
\times \int_{V_v} \Phi_1(v^* g_1) \Phi_2(v^* g_2) dv^* \times
\]

\[
\times |\det(g_1)|_v^{1-s} |\det(g_2)|_v^s \Delta(X)|_v^s dg_1 dg_2
\]

where the primed integral is over all \( v^* \in V_v \) such that \( (X, v^*) \in (g \times V')_{\text{reg},v} \), converges absolutely if \( 0 < \text{Re}(s) < 1 \) and continuous meromorphically over the entire complex plane.

**Proof.** Restricting to the subset \( (g \times V')_{\text{reg},v} \subset g_v \times V_v \) of full measure, choosing \( e_{E_v^\times}^* \in V_v \) for each \( E_v^\times \) such that the orbit \( e_{E_v^\times}^* E_v^\times \subset V_v \) is Zariski
open, and applying the Weyl integration formula, then the integral (A.1) is equal to

\[
\begin{align*}
&= \int_{G_v} \int_{g_v} f_1(X) f_2(X \cdot \text{ad}(g)) dX \int_{V_v} \Phi_1(v^*) \Phi_2(v^* g) \left| \det(g) \right|^{s}_v dg \\
&= \sum_{\lambda \vdash n} \sum_{E_v^x \subset M_{\lambda, v}} \frac{1}{|W(M_{\lambda, v}, E_v^x)|} \int_{G_v} \int_{E_v^x \setminus G_v} \left( \int_{E_v^x \setminus G_v} \right. \\
&\left. \times f_1(X \cdot \text{ad}(g_1)) f_2(X \cdot \text{ad}(g_1 g)) \int_{V_v} \Phi_1(v^* g_1) \Phi_2(v^* g_1 g) dv^* \times \\
&\left. \times |\Delta(X)|_v^{s} dX \right) dg_1 \left| \det(g) \right|^{s}_v dg \\
&= \sum_{\lambda \vdash n} \sum_{E_v^x \subset M_{\lambda, v}} \frac{1}{|W(M_{\lambda, v}, E_v^x)|} \int_{G_v} \int_{G_v} \\
&\times \left( \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \left. \\
&\left. \times \Phi_1(e_{E_v}^*, g_1) \Phi_2(e_{E_v}^*, g_1 g) \right| \det(g_1) \right|^{1-s}_v dX \times \\
&\left. \times \Phi_1(e_{E_v}^*, g_1) \Phi_2(e_{E_v}^*, g_2) \right| \det(g_1) \right|^{1-s}_v \left| \det(g_2) \right|^{s}_v dg_1 dg_2.
\end{align*}
\]

If \( \lambda = [n_r, \ldots, n_1] \) and \( E_v^x \) elliptic in \( M_{\lambda, v} \), then

\[
E_v^x \simeq \begin{bmatrix} E_{r,v}^x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{1,v}^x \end{bmatrix} \subset \begin{bmatrix} \text{GL}_{n_r,v} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{GL}_{n_1,v} \end{bmatrix} \simeq M_{\lambda, v}
\]

where \( E_{i,v}/F_v \) is a field extension of degree \( n_i \) for \( i = 1, \ldots, r \), and

\[
I_v(g, s; \Phi) = \left| \det(g) \right|^{s}_v \int_{E_{r,v}^x} \cdots \int_{E_{1,v}^x} \Phi([e_{E_v}^*, x_r, \ldots, e_{E_v}^*, x_1]|g) \prod_{i=1}^r \left| \det(x_i) \right|^{s}_v dx_i
\]

is a family of products of local Tate integrals which is locally constant and compactly supported in \( g \in E_v^x \). Hence the integral (A.1) is equal to

\[
\begin{align*}
&= \sum_{\lambda \vdash n} \sum_{E_v^x \subset M_{\lambda, v}} \frac{1}{|W(M_{\lambda, v}, E_v^x)|} \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \left( \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \int_{E_v^x \setminus G_v} \right.
\end{align*}
\]

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\[
\times f_1(X \cdot \text{ad}(g_1)) f_2(X \cdot \text{ad}(g_2)) I_v(g_1, 1 - s, \Phi_1) I_v(g_2, s, \Phi_2) \times \\
\times d g_1 d g_2 |\Delta(X)|_v dX
\]

which converges absolutely on the strip \(0 < \text{Re}(s) < 1\) and continuous meromorphically over the entire complex plane, where

\[
I_v(g_1, 1 - s, \Phi_1) I_v(g_2, s, \Phi_2) = \int_{E_v^\times} \int_{V_v} \Phi_1(v^* x^{-1} g_1) \Phi_2(v^* x g_2) dv^* dx
\]

and the integrand in (A.2) is equal to \(I_v(X, s; f_1, f_2, \Phi_1, \Phi_2)\).

**Corollary A.3.** If \(E_v^\times\) is an elliptic torus in \(M_{\lambda,v}\) where \(\lambda = [n_r, \ldots, n_1]\) and \(X \in E_{\text{reg},v}\), then \(I_v(X, s; f_1, f_2, \Phi_1, \Phi_2)\) is the product of

\[
\zeta_{E_v}(q_v^{-s}) = \prod_{i=1}^{r} \frac{1}{(1 - q_v^{-n_1 + n_i})(1 - q_v^{-n_1})}
\]

by an entire Laurent polynomial of the form \(p(q_v^{-s}) \in \mathbb{C}[q_v^{-s}, q_v^s]\).

**Proof.** By the analytic properties of local Tate integrals,

\[
I_v(X, s; f_1, f_2, \Phi_1, \Phi_2) = \int_{E_v^\times \setminus G_v} \int_{E_v^\times \setminus G_v} \\
\times f_1(X \cdot \text{ad}(g_1)) f_2(X \cdot \text{ad}(g_2)) |\Delta(X)|_v \times \\
\times I_v(g_1, 1 - s, \Phi_1) I_v(g_2, s, \Phi_2) d g_1 d g_2
\]

could be interpreted as the integral of a vector-valued function in

\[
\mathcal{C}_v^\times \left( E_v^\times \setminus G_v \times E_v^\times \setminus G_v, \zeta_{E_v}(q_v^{-s}) \cdot \mathbb{C}[q_v^{-s}, q_v^s] \right)
\]

where \(\zeta_{E_v}(q_v^{-s}) \cdot \mathbb{C}[q_v^{-s}, q_v^s] \subset \mathbb{C}[q_v^{-s}, q_v^s]\)-fractional ideal generated by \(\zeta_{E_v}(q_v^{-s})\). Since the integral only has finitely many summands, the corollary follows.

**Theorem A.4** (Local mirabolic trace formula). Let \(\Phi_1, \Phi_2 \in S(V_v)\), then the distribution \(I_v(s) \in S'(g_v \times g_v)\) defined by

\[
I_v(s; f_1, f_2) = \sum_{\lambda \in \pi^*} \sum_{E_v^\times \subset M_{\lambda,v}} \frac{1}{|W(M_{\lambda,v}, E_v^\times)|} \times \\
\times \int_{E_v^\times \setminus \text{reg},v} I_v(X, s; f_1, f_2, \Phi_1, \Phi_2) dX
\]

where the summation \(\sum_{E_v^\times}\) ranges over all conjugacy classes of maximal tori \(E_v^\times \subset M_{\lambda,v}\) elliptic modulo the center of \(M_{\lambda,v}\)
- converges absolutely if $0 < \operatorname{Re}(s) < 1$ and continues meromorphically to a sum of entire multiples of products of local zeta functions, and
- satisfies the identity

$$I_v(s; \tilde{f}_1, f_2) = I_v(s; f_1, \tilde{f}_2).$$

**Proof.** It remains to establish the identity under the Fourier transforms, which follows from the alternative definition

$$I_v(s; f_1, f_2) = \int_{Z_v \setminus G_v} K_v(g; f_1, f_2) E(g, s; \Phi_1, \Phi_2) dg$$

together with the identity $K_v(g; \tilde{f}_1, f_2) = K_v(g; f_1, \tilde{f}_2)$. \qed

**Remarks on the geometry of the double mirabolic action** The integral (A.1) could be reformulated in terms of the double mirabolic action of $G \times G$ on the relative diagonal $\Delta_G(g \times V)$ defined by

$$\Delta_G(g \times V) = (g \times V) \times_{g/G} (g \times V) \rightarrow g \times V$$

where the commutative square is Cartesian, and the composite morphism $\Delta_G(g \times V) \rightarrow g/G$ could be identified with the quotient morphism with respect to the double mirabolic action of $G \times G$.

Define the rational function $\delta : \Delta_G(g \times V) \rightarrow \mathbb{P}^1$ by

$$\delta(X_1, X_2, v_1^s, v_2^s) = \det \left( \begin{bmatrix} v_1^s & \vdots & v_2^s \\ v_1^s X_1^{n-1} & \vdots & v_2^s X_2^{n-1} \end{bmatrix} \right),$$

then the regular locus $\Delta_G(g \times V)_{\text{reg}} \subset \Delta_G(g \times V)$ defined by $\delta \neq 0, \infty$ is a $(G \times G)$-torsor with quotient $g_{\text{reg}}/G$. Hence the integral (A.1) is equal to

$$= \int_{\Delta_G(g \times V)_{\text{reg}, v}} f_1(X_1) f_2(X_2) \Phi_1(v_1^s) \Phi_2(v_2^s) \times \left| \delta(X_1, X_2, v_1^s, v_2^s) \right| dX_1 dX_2$$

$$= \int_{(g_{\text{reg}}/G)_v} \left( \int_{G_v \times G_v} f_1(X \cdot \operatorname{ad}(g_1)) f_2(X \cdot \operatorname{ad}(g_2)) \times \right.$$
\[ \times \Phi_1(e^{s g_1}) \Phi_2(e^{s g_2}) \mid \det(g_1) \bigg|_v^{1-s} \mid \det(g_2) \bigg|_v^s dg_1 dg_2 \] 

where \( d\tilde{X} \) denotes the quotient measure on \( (\mathfrak{g}_{\text{reg}}//G)_v \) which corresponds to the Weyl measure \( |\Delta(X)|_v dX \) under the identification \( (\mathfrak{g}_{\text{rs}}//G)_v \simeq \bigcup_{\lambda^v \subset M_{\lambda,v}} \bigcup_{E^{\times} \subset M_{\lambda,v}} E_{\text{reg},v}^{\times} W(M_{\lambda,v}, E^{\times}_v) \)

Analogously, the original untruncated integral of Waldspurger
\[ \int_{Z_v \backslash G_v} K_v(g; f_1, f_2) dg \]
\[ = \int_{(Z_v \times Z_v) G_v \backslash (G_v \times G_v)} f_1(X \cdot \text{ad}(g_1)) f_2(X \cdot \text{ad}(g_2)) dX dg_1 dg_2 \]
\[ = \int_{\Delta_G(\mathfrak{g})_{\text{reg},v}} f_1(X_1) f_2(X_2) \text{vol}(Z_v \backslash G_{X_1,v}) \text{vol}(Z_v \backslash G_{X_2,v}) dX_1 dX_2 \]

admits a similar reformulation, where \( \Delta_G(\mathfrak{g}) = \mathfrak{g} \times_{\mathfrak{g}/G} \mathfrak{g} \subset \mathfrak{g} \times \mathfrak{g} \) denotes the relative diagonal with regular locus \( \Delta_G(\mathfrak{g})_{\text{reg}} \subset \Delta_G(\mathfrak{g}) \) consisting of all \( (X_1, X_2) \in \mathfrak{g}_{\text{reg}} \times \mathfrak{g}_{\text{reg}} \). The integral diverges due to the existence of non-trivial stabilizers of the \((\mathfrak{g} \times \mathfrak{g})\)-action on \( \Delta_G(\mathfrak{g})_{\text{reg}} \).

For the rest of this appendix it is more convenient to work over \( \mathbb{C} \). The situation could be summarized by the commutative diagram
\[ \begin{array}{c}
\Delta_G(\mathfrak{g} \times V)_{\text{reg}} \xrightarrow{j} \Delta_G(\mathfrak{g} \times V) \xrightarrow{i} \Delta_G(\mathfrak{g}) \\
\mathfrak{g}_{\text{reg}}//G \quad \sim \quad \mathfrak{g}//G \quad \sim \quad t//W
\end{array} \]

where \( j \) denotes the open embedding of \( \Delta_G(\mathfrak{g} \times V)_{\text{reg}} \subset \Delta_G(\mathfrak{g} \times V) \) and \( i \) denotes the closed embedding of \( \Delta_G(\mathfrak{g}) \subset \Delta_G(\mathfrak{g} \times V) \) with \( v_1^* = v_2^* = 0 \), which is reminiscent of the construction of the Hilbert scheme of \( n \) points on the plane \([N99]\).

- Let \( \Gamma_{[,\cdot]}(\mathfrak{g}) \subset \mathfrak{g} \times \mathfrak{g} \) denote the commuting subvariety consisting of all \( (X_1, X_2) \in \mathfrak{g} \times \mathfrak{g} \) such that \([X_1, X_2] = 0\).

Then the quotient of \( \Gamma_{[,\cdot]}(\mathfrak{g}) \) under the diagonal adjoint action of \( G \) is isomorphic to \( \Gamma_{[,\cdot]}(\mathfrak{g})//G \simeq (t \times t)//W \), where the Weyl group \( W \) operates on \( t \times t \) diagonally and the quotient \( (t \times t)//W \) has symplectic singularities.
• Let \( \tilde{H} \subset g \times g \times V \times V \) denote the almost commuting subvariety consisting of all \((X_1, X_2, v_1^*, v_2^*) \in g \times g \times V \times V\) such that
\[
[X_1, X_2] + (v_1^*)^\top v_2^* = 0
\]
with the regular locus \( \tilde{H}_{\text{reg}} \subset \tilde{H} \) defined by the stability condition
\[
V = v_1^* \mathbb{C}[X_1, X_2]
\]
where \( v_2^* = 0 \) and \( \mathbb{C}[X_1, X_2] \) denotes the polynomial algebra in the commuting variables \( X_1, X_2 \).

Then the quotient of \( \tilde{H}_{\text{reg}} \) under the diagonal mirabolic action of \( G \) is isomorphic to \( \tilde{H}_{\text{reg}}/\!/G \simeq H \), where \( H \) denotes the Hilbert scheme of \( n \) points on the plane which is a non-singular symplectic variety.

• The Hilbert–Chow morphism \( \pi : H \to \text{Sym}^n(\mathbb{C}^2) \simeq (t \times t)/W \) fits into the commutative diagram
\[
\begin{array}{ccc}
\tilde{H}_{\text{reg}} & \xrightarrow{j} & \tilde{H} \\
\downarrow \text{G-torsor} & & \downarrow \\
H & \xrightarrow{\pi} & \text{Sym}^n(\mathbb{C}^2) \xrightarrow{\sim} (t \times t)/W \\
\end{array}
\]

and defines a symplectic resolution of singularities of \( \text{Sym}^n(\mathbb{C}^2) \).

See [GG06] for a proof of the isomorphism \( \Gamma_{[\cdot, \cdot]}(g)/G \simeq (t \times t)/W \) and an interesting relation between the almost commuting variety \( \tilde{H} \) and certain abstract harmonic analytic objects known as mirabolic \( D \)-modules.

Curiously, almost all known examples of symplectic resolutions fall into two broad types: Hilbert–Chow type or Grothendieck–Springer type, where a typical example of the latter is the Springer resolution \( \pi_0 : T^*B/G \to \mathcal{N} \) of the nilpotent cone \( \mathcal{N} \subset \mathfrak{g} \) by the cotangent bundle of the flag variety \( B/G \), which is the fiber over \( 0 \in t/W \) of the morphism \( \pi : \tilde{g} \to \mathfrak{g} \) in the commutative diagram
\[
\begin{array}{ccc}
\tilde{g} & \xrightarrow{\pi} & \mathfrak{g} \times B/G \\
\downarrow & & \downarrow \\
\mathfrak{g} & \to & t/W \\
\end{array}
\]

which is Cartesian over the regular locus \( \mathfrak{g}_{\text{reg}} \subset \mathfrak{g} \).

Finally, the construction of the mirabolic parabolic descent operator at the beginning of Section 4 suggests the following mirabolic analogue of the Grothendieck–Springer resolution:
• Identify the quasi-affine variety $U \backslash G$ with the space consisting of all triples $(g, \chi, v^*)$ where $g \in B \backslash G$, $\chi$ is a generic character of $g^{-1} U g$, in other words $\chi : V_i / V_{i-1} \to V_{i+1} / V_i$ is non-zero for all $i = 1, \ldots, n - 1$ where $V = V_n \supset \cdots \supset V_0 = 0$ denotes the flag associated with the Borel subgroup $g^{-1} B g \subset G$, and $v^* \in (V_1)'$.

Let $(g \times V')_{\text{reg}} \subset g \times U \backslash G$ be the subvariety consisting of all $(X, g, \chi, v^*)$ such that $X \cdot \text{ad}(g^{-1}) \in \mathfrak{b}$, and $\chi : V_i / V_{i-1} \to V_{i+1} / V_i$ intertwines with $X : V_{j+1} / V_j \to V_{j+1} / V_j$ for all $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n$.

Let $(g \times V')_{\text{eig}} \subset g \times V'$ be the eigen locus defined to be the subvariety consisting of all $(X, v^*) \in g \times V'$ such that $v^*$ is an eigenvector of $X$, and then define the mirabolic Grothendieck–Springer morphism $\pi : (g \times V')_{\text{eig}} \to (g \times V')_{\text{reg}}$ by assigning $(X, v^*)$ to $(X, g, \chi, v^*)$.

• Let $(\tilde{H})_{\text{reg}} \subset \tilde{H}_{\text{reg}} \times B \backslash G$ be the subvariety consisting of all

$$(X_1, X_2, v_1^*, v_2^* = 0, g) \in g \times g \times V' \times V \times B \backslash G$$

such that $X_1 \cdot \text{ad}(g^{-1}) \in \mathfrak{b}$, $X_2 : V_i \to V_{i+1}$ for all $i = 1, \ldots, n$ where $V = V_n \supset \cdots \supset V_0 = 0$ denotes the flag associated with the Borel subgroup $g^{-1} B g \subset G$, and $v_1^* \in (V_1)'$.

Then define the morphisms $(\tilde{H})_{\text{reg}} \to (g \times V')_{\text{eig}}$ and $(\tilde{H})_{\text{reg}} \to H$ by assigning $(X_1, g, \chi_2, v_1^*)$ and $(X_1, X_2, v_1^*, 0)$ to $(X_1, X_2, v_1^*, 0, g)$, where $\chi_2(Y) = \text{tr}(X_2 Y)$ for all $Y \in g^{-1} U g$.

• The Hilbert–Chow morphism $\pi : H \to \text{Sym}^n(\mathbb{C}^2)$ and the mirabolic Grothendieck–Springer morphism $\pi : (g \times V')_{\text{eig}} \to (g \times V')_{\text{reg}}$ fit into the commutative diagram

$$(g \times V')_{\text{eig}} \xleftarrow{\pi} (\tilde{H})_{\text{reg}} \xrightarrow{\pi} H$$

$$(g \times V')_{\text{eig}} \xrightarrow{\pi} \text{Sym}^n(\mathbb{C}^2)$$

where $\text{Sym}^n(\mathbb{C}^2) \simeq (t \times t) / W \to t / W$ denotes the morphism induced from the projection onto the first factor.

• More generally, one may replace $B = P_{1, \ldots, 1}$ by $P_\lambda$ for all $\lambda \vdash n$ in the definition of the mirabolic Grothendieck–Springer resolution.
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