The $Q$-index and connectivity of graphs

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Abstract

A connected graph $G$ is said to be $k$-connected if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are deleted. In this paper, for a connected graph $G$ with sufficiently large order, we present a tight sufficient condition for $G$ with fixed minimum degree to be $k$-connected based on the $Q$-index. Our result can be viewed as a spectral counterpart of the corresponding Dirac type condition.

Key words: $Q$-index; Minimum degree; $k$-connected.

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1 Introduction

All graphs considered in this paper are simple connected and undirected. The notations we used are standard. Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. Let $d(v)$ be the degree of a vertex $v$ in $G$, and the minimum degree be $\delta(G) = \delta$. For two vertex-disjoint graphs $G$ and $H$, we denote $G \cup H$ the disjoint union of $G$ and $H$, $G \vee H$ the join of $G$ and $H$, which is a graph obtained by adding all possible edges between $G$ and $H$. Throughout this paper, we use the symbol $i \sim j$ to denote the vertices $i$ and $j$ are adjacent, and $i \nsim j$ otherwise.

A graph $G$ is said to be $k$-connected if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are deleted. In the meantime, a vertex-cut $X$ of $G$ is a subset of $V(G)$ such that $G - X$ is disconnected. The vertex connectivity $\kappa$ is the minimum vertex-cut $X$. We say $G$ is $k$-connected when $\kappa \geq k$, $\kappa = 0$ if $G$ is either trivial or disconnected. In other words, $G$ is $k$-connected if the minimum vertex-cut $X$ satisfies $|X| \geq k$.

The adjacency matrix of $G$ is $A(G) = (a_{ij})_{n \times n}$ with $a_{ij} = 1$ if $i$ and $j$ are adjacent, and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denoted by $\lambda(G)$, is called the spectral radius of $G$. The diagonal matrix of $G$ is $D(G) = (d_{ii})_{n \times n}$, whose diagonal entries $d_{ii}$ satisfy $d_{ii} = d(i)$. The signless Laplacian matrix $Q(G)$ of $G$ is defined as $D(G) + A(G)$. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the $Q$-index (or the signless Laplacian spectral radius) of $G$.

When one talks about spectral graph theory, perhaps one of the most well-known problems is the Brualdi-Solheid problem [3]: Given a set $\mathcal{G}$ of graphs, find a tight upper bound for the spectral radius in $\mathcal{G}$ and characterize the extremal graphs. This problem is well studied in the literature for various classes of graphs, such as graphs with given number of cut vertices or cut edges [11, 21], graphs with given edge chromatic number [4]. For the $Q$-index counterpart of the above problem, Zhang [31] gave the $Q$-index of graphs with given degree sequence, Zhou [32] studied the $Q$-index.

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and Hamiltonicity. Also, from both theoretical and practical viewpoint, the eigenvalues of graphs have been successfully used in many other disciplines, one may refer to [15, 17, 18, 22, 30].

Analogous to the Brualdi–Solheid problem, the following problem was proposed [25]: What is the maximum spectral radius of a graph $G$ on $n$ vertices without a subgraph isomorphic to a given graph $F$? Regarding this problem, Fiedler and Nikiforov [12] obtained tight sufficient conditions for graphs to be hamiltonian or traceable. This motivates further study for such questions. Later, Zhou [32] considered the $Q$-index version of the results in [12]. For further reading in this topic, see [7, 8, 9, 13, 22, 23, 24, 27, 33, 34].

For the connectivity and eigenvalues of graphs, one must mention the classical result from Fiedler [1] which states that the second smallest Laplacian eigenvalue is at most the connectivity for any non-complete graph, which now becomes one of the most attractive research areas. For adjacency eigenvalues, extending the result in [4], Cioabă [5] obtained a simple graph of order $n$ which now becomes one of the most attractive research areas. For adjacency eigenvalues, extending the result in [4], Cioabă [5] obtained $\lambda_n$-index version of the results in [12]. For further reading in this topic, see [15, 17, 18, 29, 30].

There are also several related results regarding the edge-connectivity and eigenvalues of graphs, one must mention the classical result from Fiedler [1] which states that the second smallest Laplacian eigenvalue is at most the connectivity for any non-complete graph, which now becomes one of the most attractive research areas. For adjacency eigenvalues, extending the result in [4], Cioabă [5] obtained $\lambda_n$-index version of the results in [12]. For further reading in this topic, see [15, 17, 18, 29, 30].

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One of the classical problems of graph theory is to obtain sufficient conditions for a graph possessing certain properties. It is known that [2] Page 4], if $G$ is a simple graph of order $n \geq k+1$, and if $\delta \geq \frac{1}{2}(n+k-2)$, then $G$ is $k$-connected. In this paper, borrowing ideas from [19, 20], by utilizing the $Q$-index, we will establish a new sufficient condition for graphs with fixed minimum degree to be $k$-connected, for sufficiently large order (and therefore for relatively small $\delta$). Such results may be of independent interest. For any $k > 1$ and $n > 2k+1$, let

$$M_k(n) = K_k \lor (K_{n-2k} \cup K_k).$$

For any $k \geq 1$ and $n \geq k+2$, let

$$L_k(n) = K_1 \lor (K_{n-k-1} \cup K_k).$$

In light of the result of Li and Ning [16], Nikiforov [20] proved the following theorem.

**Theorem 1.2** Let $k > 1$, $n \geq k^3 + k + 4$, and let $G$ be a graph of order $n$ with minimum degree $\delta(G) \geq k$. If $\lambda(G) \geq n - k - 1$, then $G$ has a Hamiltonian cycle unless $G = M_k(n)$ or $G = L_k(n)$.

We define

$$\mathcal{M}_1(n,k) = \left\{ G \subseteq M_k(n) - E', \text{ where } E' \subseteq E_1(M_k(n)) \text{ with } |E'| \leq \left\lfloor \frac{k^3}{2} \right\rfloor \right\},$$

$$\mathcal{L}_1(n,k) = \left\{ G \subseteq L_k(n) - E', \text{ where } E' \subseteq E_1(L_k(n)) \text{ with } |E'| \leq \left\lfloor \frac{k^2}{2} \right\rfloor \right\}.$$

Li, Liu and Peng [19] recently obtained the $Q$-index counterpart of Theorem 1.2.

**Theorem 1.3** Let $k > 1$, $n \geq k^4 + k^3 + 4k^2 + k + 6$. Let $G$ be a connected graph with $n$ vertices and minimum degree $\delta(G) \geq k$. If $q(G) \geq 2(n - k - 1)$, then $G$ has a Hamilton cycle unless $G \in \mathcal{M}_1(n,k)$ or $G \in \mathcal{L}_1(n,k)$. 

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For convenience, for the rest of this paper, we denote
\[ A(n, k, \delta) := K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}). \]
Obviously, for any integers \( k > 1, \delta \geq 1 \) and \( n > \delta + 1 \), \( A(n, k, \delta) \) is not \( k \)-connected. For the graph \( A(n, k, \delta) \), let
\[
X := \{ v \in V(A(n, k, \delta)) : d(v) = \delta \}, \quad Y := \{ v \in V(A(n, k, \delta)) : d(v) = n - 1 \},
\]
\[
Z := \{ v \in V(A(n, k, \delta)) : d(v) = n - \delta + k - 3 \}.
\]
Let \( E' \) denote the edge set of \( E(A(n, k, \delta)) \) whose endpoints are both from \( Y \cup Z \). We define
\[
A_1(n, k, \delta) := \left\{ G \subseteq A(n, k, \delta) - E', \text{where } E' \subset E(A(n, k, \delta)) \text{ with } |E'| \leq \left\lfloor \frac{(\delta-k+2)(k-1)}{2} \right\rfloor \right\},
\]
\[
A_2(n, k, \delta) := \left\{ G \subseteq A(n, k, \delta) - E', \text{where } E' \subset E(A(n, k, \delta)) \text{ with } |E'| = \left\lfloor \frac{(\delta-k+2)(k-1)}{2} \right\rfloor + 1 \right\},
\]
\[
F(k, \delta) := (k^2 + 2k - 3)\delta^2 - (2k^3 - k^2 - 17k + 8)\delta + k^4 - 3k^3 - 8k^2 + 23k + 4.
\]
In [8], using the adjacency spectral radius, it is obtained that

**Theorem 1.4** Let \( \delta \geq k \geq 3, n \geq (\delta - k + 2)(k^2 - 2k + 4) + 3 \). Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta(G) \geq \delta \). If
\[
\lambda(G) \geq n - \delta + k - 3,
\]
then \( G \) is \( k \)-connected unless \( G = A(n, k, \delta) \).

Motivated by Theorem 1.3 as the \( Q \)-spectral counterpart of Theorem 1.4, we have the following main result of this paper.

**Theorem 1.5** Let \( G \) be a connected graph of order \( n \) with minimum degree \( \delta(G) = \delta \geq k \geq 3 \). If \( n \geq F(k, \delta) \) and
\[
q(G) \geq 2(n - \delta + k - 3),
\]
then \( G \) is \( k \)-connected unless \( G \in A_1(n, k, \delta) \).

## 2 Preliminaries

In this section, we present some basic notations and lemmas.

Let \( x = (x_1, x_2, \ldots, x_n)^T \neq 0 \), by Rayleigh’s principle, we have
\[
q(G) = \max_x \frac{(Q(G)x, x)}{(x, x)} = \max_x \frac{x^TQ(G)x}{x^Tx}.
\]
By the definition of \( Q(G) \), we have
\[
(Q(G)x, x) = \sum_{i \neq j} (x_i + x_j)^2.
\]
If \( z \) is the corresponding unit positive eigenvector (usually called Perron vector) of \( q(G) \), then
\[
Q(G)z = q(G)z.
\]
According to the Perron-Frobenius theorem, we have \( x_i > 0 \) for each \( i \in V(G) \) if \( G \) is connected. Taking the \( i \)-th entry of both sides and rearranging terms, we have
\[
(q(G) - d(i))z_i = \sum_{i \neq j} z_j. \tag{1}
\]
Let \( N(i) \) denote the set of neighbours of \( i \), \( N[i] = N(i) \cup \{ i \} \). From the above, we have the following lemma.
Lemma 2.1 [10] For any $i, j \in V(G)$, we have
\[
(q(G) - d(i))(z_i - z_j) = (d(i) - d(j))z_j + \sum_{k \in N(i) \setminus N(j)} z_k - \sum_{t \in N(j) \setminus N(i)} z_t.
\] (2)

Lemma 2.2 [10] Let $G$ be a graph of order $n$. Then
\[
q(G) \leq \frac{2m}{n - 1} + n - 2.
\]

Using the ideas in [13], we obtain

Lemma 2.3 Let $G$ be a connected graph of order $n \geq 2\delta - k + 5$, size $m$, minimum degree $\delta(G) = \delta \geq k \geq 2$. If
\[
m > \frac{1}{2}n(n - 1) - (\delta - k + 3)(n - \delta - 2),
\]
then $G$ is $k$-connected unless $G$ is a subgraph of $A(n, k, \delta)$.

Proof. Suppose on the contrary that $G$ is not $k$-connected. Let $X$ be a minimum vertex-cut with $1 \leq |X| \leq k - 1$. Assume that $C_1, C_2, \ldots, C_t$ ($t > 1$) are the components of $G - X$, where $|C_1| \leq |C_2| \leq \ldots \leq |C_t|$. Clearly, for $1 \leq i \leq t$, each vertex in $C_i$ is adjacent to at most $|C_i| - 1$ vertices of $C_i$ and $|X|$ vertices of $X$. Thus
\[
\delta|C_i| \leq \sum_{x \in C_i} d(x) \leq (|C_i| - 1 + |X||C_i|),
\]
hence $|C_i| \geq \delta - |X| + 1$, and therefore $\delta - |X| + 1 \leq |C_i| \leq n - |X| - (\delta - |X| + 1)$, which implies
\[
\delta - |X| + 1 \leq |C_i| \leq n - \delta - 1.
\]
Let $S = \cup_{i=2}^{t} C_i$. Then from above, $\delta - |X| + 1 \leq |S| \leq n - \delta - 1$. Since $G - X$ is disconnected, there are no edges between $C_1$ and $S$ in $G$, we obtain
\[
m \leq \frac{1}{2}n(n - 1) - |C_1||S|.
\]

In order to prove that $G$ is a subgraph of $A(n, k, \delta)$, it suffices to show that $|C_1| = \delta - k + 2$.

If $|C_1| \geq \delta - k + 3$, as $|C_1| \leq |C_2| \leq \ldots \leq |C_t|$ and $S = \cup_{i=2}^{t} C_i$, we have $|C_1| \leq \frac{n - |X|}{2}$. Therefore for $\delta - k + 3 \leq |C_1| \leq \frac{n - |X|}{2}$, we have $|C_1||S| = |C_1|(n - |X| - |C_1|) \geq (\delta - k + 3)(n - |X| - (\delta - k + 3))$, the equality is attained when $|C_1| = \delta - k + 3$. Since $|X| \leq k - 1$,
\[
m \leq \frac{1}{2}n(n - 1) - |C_1||S|
\]
\[
\leq \frac{1}{2}n(n - 1) - (\delta - k + 3)(n - |X| - (\delta - k + 3))
\]
\[
\leq \frac{1}{2}n(n - 1) - (\delta - k + 3)(n - \delta - 2).
\]

From the assumption, we get a contradiction. Thus $|C_1| \leq \delta - k + 2$. Combining this with $|C_1| \geq \delta - |X| + 1 \geq \delta - k + 2$, we have $|C_1| = \delta - k + 2$.

Hence, as the minimum degree of $G$ is $\delta$, we have $|X| = k - 1$ and $d_G(i) = \delta$ for each $i \in C_1$, therefore each vertex of $C_1$ is adjacent to each vertex of $X$. We obtain the result. \[\Box\]
3 Proof of the Main Result

To prove Theorem 1.5 we still need to prove the following several lemmas.

**Lemma 3.1** Assume \( \delta \geq k \geq 3 \), let \( G \) be a connected graph of order \( n \geq F(k, \delta) \) and minimum degree \( \delta \geq k \). For each graph \( G \in A_k(n, k, \delta) \), we have \( q(G) \geq 2(n - \delta + k - 3) \).

**Proof.** Let \( G \) be a graph in \( A_k(n, k, \delta) \). We easily get that \( q(K_{n-\delta+k-2} \cup K_{\delta-k+2}) = 2(n - \delta + k - 3) \).

Now we construct a vector \( z \), where \( z_i = 1 \) if \( i \in Y \cup Z \), \( z_j = 0 \) if \( j \in X \). Obviously \( z \) is the corresponding eigenvector to \( q(K_{n-\delta+k-2} \cup K_{\delta-k+2}) \). Then we obtain
\[
\langle Q(G)z, z \rangle - \langle Q(K_{n-\delta+k-2} \cup K_{\delta-k+2})z, z \rangle = (\delta - k + 2)(k - 1) - 4|E'| \geq 0.
\]

By the Rayleigh’s principle, we have \( q(G) \geq 2(n - \delta + k - 3) \). \( \blacksquare \)

**Lemma 3.2** Assume \( \delta \geq k \geq 3 \), let \( G \) be a connected graph of order \( n \geq F(k, \delta) \) and minimum degree \( \delta \geq k \). For each graph \( G \in A_2(n, k, \delta) \), we have \( q(G) \geq 2(n - \delta + k - 3) - 1 \).

**Proof.** Let \( z \) be the vector defined in Lemma 3.1. We have
\[
\langle Q(G)z, z \rangle - \langle Q(K_{n-\delta+k-2} \cup K_{\delta-k+2})z, z \rangle = (\delta - k + 2)(k - 1) - 4|E'| \geq -4.
\]

Similarly, we have \( q(G) \geq 2(n - \delta + k - 3) - \frac{4}{\|z\|^2} > 2(n - \delta + k - 3) - 1 \). \( \blacksquare \)

We put our attention to prove \( q(G) < 2(n - \delta + k - 3) \) for \( G \in A_2(n, k, \delta) \) in the following.

Let \( G \) be a graph among \( A_2(n, k, \delta) \) with the largest \( Q \)-index, assume further that the induced subgraph \( G[Y] \) contains the largest number of edges. Then \( |Y| = k - 1 \geq 2 \), i.e., \( k \geq 3 \).

In the following, let \( x \) be the eigenvector corresponding to \( q(G) \). Moreover we may assume \( \max_{i \in V(G)} x_i = 1 \). Following this, we have

**Lemma 3.3** Assume \( G \in A_2(n, k, \delta) \) as defined above. For each \( i \in X \), we have
\[
x_i \leq \frac{k - 1}{q(G) - (2\delta - k + 1)}.
\]

**Proof.** Using equation (1) at vertex \( i \), we have
\[
(q(G) - d(i))x_i = \sum_{j \in X \setminus i} x_j + \sum_{j \in Y} x_j.
\]

As \( d(i) = \delta \), \( x_i \) is the same for all vertex in \( X \), and \( \max_{i \in V(G)} x_i = 1 \), we have
\[
(q(G) - (\delta + (\delta - k + 1))) x_i = \sum_{j \in Y} x_j.
\]

The proof is completed. \( \blacksquare \)

Now we divide \( Y, Z \) into the following two parts, respectively.

\[
Y_1 = \{i \in Y : d(i) = n - 1\}, \quad Y_2 = \{i \in Y : d(i) \leq n - 2\},
\]

\[
Z_1 = \{i \in Z : d(i) = n - \delta + k - 3\}, \quad Z_2 = \{i \in Z : d(i) \leq n - \delta + k - 4\}.
\]

We first declare the following truth: \( Z_1 \neq \emptyset \) since \( n - \delta - 1 > 2 \left( \left\lfloor \frac{(\delta-k+2)(k-1)}{4} \right\rfloor + 1 \right) + 1 \), and \( n \geq F(k, \delta) \).

We already know the upper bound of \( x_i \) for each \( i \in X \) and \( x_i < 1 \). Clearly \( \max_{i \in V(G)} x_i = \max_{i \in Y \cup Z} x_i \).

We also need the following several lemmas.
Lemma 3.4 Let \( G \in \mathcal{A}_2(n, k, \delta) \) as above. If \( Y_2 \neq \emptyset \), then we have \( x_i > x_j \) for all \( i \in Z_1 \) and \( j \in Y_2 \).

**Proof.** By contradiction, assume that there exist some \( i \in Z_1 \) and \( j \in Y_2 \) such that \( x_i \leq x_j \). For \( k \in Y \) and \( j \sim k \), we define a new graph \( G' \in \mathcal{A}_2(n, k, \delta) \) by removing the edge \( ik \) and adding a new edge \( jk \). Since

\[
\langle Q(G')x, x \rangle - \langle Q(G)x, x \rangle = (x_j - x_i)(x_i + x_j + 2x_k) \geq 0,
\]

we get \( q(G') \geq q(G) \) and the induced graph \( G'[Y] \) has more edges than \( G[Y] \), which contradicts the choice of \( G \). The result follows. \( \blacksquare \)

Lemma 3.5 Assume \( G \in \mathcal{A}_2(n, k, \delta) \) as defined above.

1. If \( Z_2 \neq \emptyset \), then we have \( x_i > x_j \) for all \( i \in Z_1 \) and \( j \in Z_2 \).
2. If \( Y_1, Y_2 \neq \emptyset \), then we have \( x_i > x_j \) for any \( i \in Y_1 \) and \( j \in Y_2 \).
3. If \( Y_1 \neq \emptyset \), then we have \( x_i > x_j \) for any \( i \in Y_1 \) and \( j \in Z_1 \).

**Proof.** (1). Using Lemma 2.4, we have

\[
(q(G) - d(i))(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N(j)} x_k - \sum_{l \in N(j) \setminus N(i)} x_l.
\]

For each \( i \in Z_1 \) and \( j \in Z_2 \), note that \( N(j) \setminus \{i\} \subset N(i) \setminus \{j\} \). Rearranging the last equation, we obtain

\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k. \quad (3)
\]

As \( d(i) > d(j) \), the proof is completed.

(2). If \( Y_1, Y_2 \neq \emptyset \), using Lemma 2.4, we have

\[
(q(G) - d(i))(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N(j)} x_k - \sum_{l \in N(j) \setminus N(i)} x_l.
\]

For each \( i \in Y_1 \) and \( j \in Y_2 \), note that \( N(j) \setminus \{i\} \subset N(i) \setminus \{j\} \). Rearranging the last equation, we obtain

\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k.
\]

as \( d(i) > d(j) \), the proof is completed.

(3). If \( Y_1 \neq \emptyset \), applying Lemma 2.4, we have

\[
(q(G) - d(i))(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N(j)} x_k - \sum_{l \in N(j) \setminus N(i)} x_l.
\]

For each \( i \in Y_1 \) and \( j \in Z_1 \), note that \( N(j) \setminus \{i\} \subset N(i) \setminus \{j\} \). Rearranging the last equation, we obtain

\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k.
\]

as \( d(i) > d(j) \), the proof is completed. \( \blacksquare \)

From above, bearing in mind that \( Z_1 \neq \emptyset \), we have
Lemma 3.6 Assume $G \in A_2(n, k, \delta)$ as defined above. We have
\[
\max_{i \in V(G)} x_i - \min_{j \in Y \cup Z} x_j \leq \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

Proof. We distinguish the proof into two cases.

Case 1: $Y_1 = \emptyset$. Notice that $\max_{i \in V(G)} x_i$ is attained at the vertices in $Z_1$. For each $i \in Z_1$, $d(i) = n - \delta + k - 3$, hence the vertex $i$ is adjacent to all other vertices in $Y \cup Z$.

Subcase 1.1: If $j \in Z_2$, we have $N(i) \setminus N[j] = \{k : k \in Y \cup Z$ and $k \sim j\}$. Thus we have $d(i) - d(j) \leq \left[\frac{(\delta - k + 2)(k - 1)}{4}\right] + 1$ and $|N(i) \setminus N[j]| \leq \left[\frac{(\delta - k + 2)(k - 1)}{4}\right] + 1$. Note that $N(j) \setminus \{i\} \subset N(i) \setminus \{j\}$, applying equation (3), we obtain
\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k
\]
\[
\leq \left[ \frac{(\delta - k + 2)(k - 1)}{2} \right] + 2.
\]

Since $i \in Z_1, \delta \geq k$, we have
\[
x_i - x_j \leq \frac{(\delta - k + 2)(k - 1)}{2(q(G) - (n - \delta + k - 3) + 1)} + 2 = \frac{(\delta - k + 2)(k - 1) + 4}{2(q(G) - n + \delta - k + 4)} < \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

Subcase 1.2: If $j \in Y_2$, we have $N(i) \setminus N[j] = \{k : k \in Y \cup Z$ and $k \sim j\}$, and $N(j) \setminus N[i] = X$. Thus $|N(i) \setminus N[j]| \leq \left[\frac{(\delta - k + 2)(k - 1)}{4}\right] + 1$. Meanwhile, note that $|d(i) - d(j)| \leq \left[\frac{(\delta - k + 2)(k - 1)}{4}\right] + \delta - k + 3$. Similarly, we obtain
\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k - \sum_{l \in X} x_l
\]
\[
\leq (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k
\]
\[
\leq \frac{(\delta - k + 2)(k - 1)}{2} + \delta - k + 2 + 2
\]
\[
= \frac{(\delta - k + 2)(k + 1) + 4}{2}.
\]
Since \( i \in Z_1, \delta \geq k \), we easily have
\[
x_i - x_j \leq \frac{(\delta - k + 2)(k + 1) + 2}{q(G) - (n - \delta + k - 3) + 1} = \frac{(\delta - k + 2)(k + 1) + 4}{2(q(G) - n + \delta - k + 4)} < \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

**Case 2:** \( Y_1 \neq \emptyset \). Notice that \( \max_{i \in V(G)} x_i \) is attained at the vertices in \( Y_1 \). When \( i \in Y_1, d(i) = n - 1 \), hence the vertex \( i \) is adjacent to all other vertices in \( V(G) \).

**Subcase 2.1:** If \( j \in Y_2 \), we have \( N(i) \setminus N[j] = \{ k : k \in Y \cup Z \text{ and } k \approx j \} \). Thus we have
\[
d(i) - d(j) \leq \left[ \frac{(\delta - k + 2)(k - 1)}{4} \right] + 1 \quad \text{and} \quad |N(i) \setminus N[j]| \leq \left[ \frac{(\delta - k + 2)(k - 1)}{4} \right] + 1.
\]
Note that \( N(j) \setminus \{ i \} \subset N(i) \setminus \{ j \} \), applying equation (3), we obtain
\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k
\]
\[
\leq \left[ \frac{(\delta - k + 2)(k - 1)}{2} \right] + 2.
\]
Since \( i \in Y_1, \delta \geq k \), we easily have
\[
x_i - x_j \leq \frac{(\delta - k + 2)(k - 1) + 2}{q(G) - (n - 1) + 1} = \frac{(\delta - k + 2)(k - 1) + 4}{2(q(G) - n + 2)} < \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

**Subcase 2.2:** If \( j \in Z_1 \), we have \( N(i) \setminus N[j] = X - \{ j \} \), and \( N(j) \setminus N[i] = \emptyset \). Thus \( |N(i) \setminus N[j]| = (n - 1) - (n - \delta + k - 3 + 1) = \delta - k + 1 \). Note that \( d(i) - d(j) = \delta - k + 2 \), we similarly obtain
\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k
\]
\[
\leq \delta - k + 2 + \delta - k + 1
\]
\[
< 2(\delta - k + 2).
\]
Since \( i \in Y_1, \delta \geq k \), we have
\[
x_i - x_j < \frac{2(\delta - k + 2)}{q(G) - (n - 1) + 1} < \frac{4(\delta - k + 2)}{2(q(G) - n + 1)} < \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

**Subcase 2.3:** If \( j \in Z_2 \), we have \( N(i) \setminus N[j] = \{ k : k \in X \cup Y \cup Z \text{ and } k \approx j \} \), and \( N(j) \setminus N[i] = \emptyset \). Thus we have \( |N(i) \setminus N[j]| \leq \delta - k + 2 + \left[ \frac{(\delta - k + 2)(k - 1)}{4} \right] + 1 \). Meanwhile, note that
\[
d(i) - d(j) \leq \left[ \frac{(\delta - k + 2)(k - 1)}{4} \right] + \delta - k + 3.
\]
Similarly, we obtain
\[
(q(G) - d(i) + 1)(x_i - x_j) = (d(i) - d(j))x_j + \sum_{k \in N(i) \setminus N[j]} x_k
\]
\[
\leq \frac{(\delta - k + 2)(k - 1)}{2} + 2(\delta - k + 2) + 2
\]
\[
= \frac{(\delta - k + 2)(k + 3) + 4}{2}.
\]
Since \( i \in Y_1, \delta \geq k \), we have

\[
x_i - x_j \leq \frac{(\delta - k + 2)(k + 1)}{2} + \frac{2}{q(G) - (n - 1) + 1} = \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 2)} < \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)}.
\]

The proof is completed. \( \blacksquare \)

Now we can prove Lemma 3.7, which is crucial for Theorem 1.5.

**Lemma 3.7** Let \( G \) be a connected graph of order \( n \geq F(k, \delta) \) and minimum degree \( \delta \geq k \geq 3 \). For each graph \( G \in \mathcal{A}_2(n, k, \delta) \), we have \( q(G) < 2(n - \delta + k - 3) \).

**Proof.** We assume \( G \in \mathcal{A}_2(n, k, \delta) \) such that \( G \) has the largest \( Q \)-index \( q(G) \) among \( \mathcal{A}_2(n, k, \delta) \) and \( G[Y] \) contains the largest number of edges. Let \( x \) be the eigenvector corresponding to \( q(G) \), and \( G'[X] \) be the complete graph \( K_{\delta - k + 2} \) induced by \( X \). Lemmas 3.3 and 3.6 imply that

\[
\langle Q(G)x, x \rangle - \langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle \leq \frac{\langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle}{2} \left( \frac{k - 1}{q(G) - (2\delta - k + 1)} \right)^2 + (k - 1)(\delta - k + 2) \left( 1 + \frac{k - 1}{q(G) - (2\delta - k + 1)} \right)^2 - 4|E'| \left( 1 - \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)} \right)^2.
\]

As \( |E'| = \left\lfloor \frac{(\delta - k + 2)(k - 1)}{4} \right\rfloor + 1 \geq \frac{(\delta - k + 2)(k - 1) + 1}{4} \), we have

\[
\langle Q(G)x, x \rangle - \langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle \leq \frac{\langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle}{2} \left( \frac{k - 1}{q(G) - (2\delta - k + 1)} \right)^2 + (k - 1)(\delta - k + 2) \left( 1 + \frac{k - 1}{q(G) - (2\delta - k + 1)} \right)^2 - ((k - 1)(\delta - k + 2) + 1) \left( 1 - \frac{(\delta - k + 2)(k + 3) + 4}{2(q(G) - n + 1)} \right)^2.
\]

Since \( n \geq F(k, \delta) \), and \( q(G) > 2(n - \delta + k - 3) - 1 \) by Lemma 3.2, we have

\[
\langle Q(G)x, x \rangle - \langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle < 0.
\]

According to the Rayleigh’s principle,

\[
\langle Q(K_{\delta - k + 2} + K_{n-\delta+k-2})x, x \rangle \leq q(K_{\delta - k + 2} + K_{n-\delta+k-2}) = 2(n - \delta + k - 3).
\]

Therefore, we have \( q(G) = \frac{\langle Q(G)x, x \rangle}{\langle x, x \rangle} < 2(n - \delta + k - 3). \) \( \blacksquare \)

Now we are ready to prove Theorem 1.5.

**Proof.** By Lemma 2.2, we have

\[
2(n - \delta + k - 3) \leq q(G) \leq \frac{2m}{n - 1} + n - 2.
\]
Therefore
\[
m \geq \frac{(n - 2\delta + 2k - 4)(n - 1)}{2} = \frac{n(n - 1)}{2} - (\delta - k + 3)(n - \delta - 2) + n - \delta - 2 - (\delta - k + 2)(\delta + 1) > \frac{n(n - 1)}{2} - (\delta - k + 3)(n - \delta - 2),
\]
the last inequality holds as \( n \geq F(k, \delta) \). By Lemma 2.3, \( G \) is \( k \)-connected unless \( G \in A(n, k, \delta) \). Together with Lemmas 3.1 and 3.7, the result follows.

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