Local monodromy of $p$-adic differential equations: an overview

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Abstract
This primarily expository article collects together some facts from the literature about the monodromy of differential equations on a $p$-adic (rigid analytic) annulus, though often with simpler proofs. These include Matsuda’s classification of quasi-unipotent $\nabla$-modules, the Christol-Mebkhout construction of the ramification filtration, and the Christol-Dwork Frobenius antecedent theorem. We also briefly discuss the $p$-adic local monodromy theorem without proof.

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1 Introduction

This paper is the result of an attempt to collect in one place, and present in a uniform fashion, some disparate results about the local monodromy of $p$-adic differential equations. It was initiated as part of a project to establish “semistable reduction” for overconvergent $F$-isocrystals [23]; however, we have decided to separate the paper from this project, as it may have independent interest. This interest would arise from the fact that the local monodromy of $p$-adic differential equations intervenes both in the study of $p$-adic (rigid) cohomology, largely via the work of Crew (e.g., see [14]), and in $p$-adic Hodge theory, largely via the work of Berger (e.g., see [3], [4]).

The purpose of this paper is mainly expository: it is intended to provide an easy entry point into the literature on $p$-adic differential equations for the reader familiar with rigid analytic geometry on a rather basic level (for the most part, the only spaces being considered are annuli). Results exposed here include:

- a classification of quasi-unipotent modules with connection (due to Matsuda);
- a relationship between ramification in the monodromy representation and generic radii of convergence (based on work of Christol-Mebkhout, Crew, Matsuda, Tsuzuki);
- the Frobenius antecedent theorem (due to Christol-Dwork);
• without proof, the $p$-adic local monodromy theorem (due to André, Mebkhout, and the present author).

In the remainder of this introduction, we explain a bit about what it means for a $p$-adic differential equation to have meaningful monodromy, then outline the structure of the paper.

1.1 $p$-adic differential equations and their monodromy

Let $K$ be a field of characteristic zero complete with respect to a nonarchimedean absolute value, whose residue field $k$ has characteristic $p > 0$. Throughout this paper, we will be considering the following situation. We are given a rigid analytic annulus over $K$ and a “differential equation” on the annulus, i.e., a module equipped with a connection (which is automatically integrable because we are in a one-dimensional setting). We now wish to define the “monodromy around the puncture” of this connection, despite not having recourse to the analytic continuation we would use in the analogous classical setting.

We can define a monodromy representation associated to a connection if we can find enough horizontal sections “somewhere”. We will only be looking for horizontal sections on certain étale covers of the annulus (what we call “formally étale covers”). Specifically, we will consider connections which on some such cover become unipotent (filtered by submodules with trivial successive quotients), and define a monodromy representation for these; this will give an equivalence of categories between such “quasi-unipotent” modules with connection and a certain representation category. (Beware that if the field $k$ is not algebraically closed, these representations will only be semilinear.)

In order for such an equivalence to be useful, we need to be able to establish conditions under which a module with connection is forced to be quasi-unipotent. This has been done in case $K$ is discretely valued, by the $p$-adic local monodromy theorem ($p$LMT) of André [1], Mebkhout [27] and this author [21]. The sufficient condition in this theorem is a so-called “Frobenius structure” on the connection; like its complex-analytic analogue (variation of Hodge structure), this extra structure arises naturally in geometric settings, and can also be found (following examples of Dwork) in settings where the geometric origin is a little less clear.

1.2 Structure of the paper

We conclude this introduction with a rundown of the contents of the various chapters of the paper.

In Chapter 2, we recall a bit of the theory of (one-dimensional) local fields, mostly but not exclusively in the classical case of perfect residue field. This will be needed later to talk about monodromy representations.

In Chapter 3, we introduce rigid annuli, and verify (after Gruson) that over a spherically complete coefficient field, any coherent locally free sheaf on a one-dimensional rigid annulus is freely generated by local sections. This will provide a crucial link back to the literature on $p$-adic differential equations, which is mostly phrased in terms of modules over rings of
suitably convergent power series. We also describe the “formally étale covers” of rigid annuli that will intervene in the study of monodromy of $\nabla$-modules.

In Chapter 3 we define quasi-constant and quasi-unipotent modules with connection on a one-dimensional rigid annulus, and construct monodromy representations corresponding to such objects.

In Chapter 5 we explain how the ramification of the monodromy of a quasi-unipotent connection is controlled by certain “radius of convergence” data; this reprises results of Christol-Dwork, Christol-Mebkhout, Matsuda, Crew, and Tsuzuki.

In Chapter 6 we introduce Frobenius structures, state the $p$-adic local monodromy theorem (roughly, every module with connection admitting a compatible Frobenius structure is quasi-unipotent), and verify the Frobenius antecedent theorem of Christol-Dwork.

2 Ramification in one dimension

We start with a quick review of ramification theory for local fields, mostly following Serre [29].

Convention 2.0.1. When speaking of a “discretely valued field”, we insist that the valuation be nontrivial.

2.1 Ramification filtrations

We recall some definitions and results from [29, Chapter IV].

Definition 2.1.1. For $F$ a complete discretely valued field, let $\mathfrak{o}_F$ be the ring of integers of $F$, let $\mathfrak{m}_F$ be the maximal ideal of $\mathfrak{o}_F$, let $\overline{F} = \mathfrak{o}_F/\mathfrak{m}_F$ be the residue field of $F$, and let $v_F : F^* \to \mathbb{Z}$ be the valuation on $F$.

Hypothesis 2.1.2. For the remainder of this section, let $F$ be a complete discretely valued field such that $\text{char}(F) = p > 0$, and let $E/F$ be a separable algebraic extension. Then $E$ admits a unique valuation extending the valuation on $F$; moreover, if $E/F$ is finite, then $E$ is complete for its valuation [29, Proposition II.3].

Definition 2.1.3. Assume that $E/F$ is finite Galois and that $\overline{E}/\overline{F}$ is separable (hence also Galois). For $i \geq -1$, let $G_i$ be the subgroup of $G = \text{Gal}(E/F)$ consisting of those $g$ for which $v_E(a^g - a) \geq i + 1$ for all $a \in \mathfrak{o}_E$; the decreasing filtration $\{G_i\}$ is called the lower numbering filtration of $G$ [29, §IV.1] It can be shown [29, §IV.2] that $G_{-1}/G_0 \cong \text{Gal}(\overline{E}/\overline{F})$, that $G_0/G_1$ is cyclic of order prime to $p$, and that $G_i/G_{i+1}$ is an elementary abelian $p$-group for $i \geq 1$.

Definition 2.1.4. With notation as in Definition 2.1.3 define the function

$$\phi_{E/F}(u) = \int_0^u \frac{dt}{[G_0 : G_1]}.$$
Then $\phi_{E/F}$ is a homeomorphism of $[-1, \infty)$ with itself; let $\psi_{E/F}$ denote the inverse function. Define the upper numbering filtration of $G$ by $G^i = G_{\psi_{E/F}(i)}$ [29 §IV.3]; its key property ("Herbrand’s theorem") is that it commutes with formation of quotients, in that if $E'/F$ is a Galois subextension of $E/F$ with $H = \text{Gal}(E'/F)$, then the image of $G^i$ in $H$ is precisely $H^i$ [29 Proposition IV.14].

**Remark 2.1.5.** With notation as in Definition 2.1.4, the functions $\phi_{E/F}$ and $\psi_{E/F}$ have the following transitivity property [29 Proposition IV.15]: for $E'/F$ a Galois subextension of $E/F$,

$$\phi_{E/F} = \phi_{E'/F} \circ \phi_{E'/F}$$

and

$$\psi_{E/F} = \psi_{E'/F} \circ \psi_{E'/F}.$$

**Definition 2.1.6.** Let $E$ be a Galois extension of $F$ (not necessarily finite) such that $\overline{E}/F$ is separable, and again put $G = \text{Gal}(E/F)$. (In particular, if $F$ is perfect, we may take $E = F^{\text{sep}}$.) Put $G^i = \varprojlim \text{Gal}(E'/F)^i$, where $E'$ runs over all finite Galois subextensions of $E/F$. By [29 Proposition IV.14], $\text{Gal}(E'/F)^i$ is the image of $G^i$ in $\text{Gal}(E'/F)$. Again, we call the resulting filtration the upper numbering filtration of $G$; it satisfies the left continuity property $G^i = \bigcap_{j<i} G^j$ [29 Remark 1, p. 75]. On the other hand, the upper numbering filtration is not right continuous; define $G^i_1 = \bigcup_{j>i} G^j$, which is a closed subgroup of $G^i$ (for the profinite topology) which may be strictly smaller than $G^i$.

**Definition 2.1.7.** With notation as in Definition 2.1.6 we say that $i \geq 0$ is a break of $E/F$ if $G^i \neq G^{i+}$. (The term “break” is short for “ramification break”; the term “jump”, for “ramification jump”, is also used.) If $E/F$ is finite, then there are finitely many breaks; we refer to the largest one of them (or 0 if there are no breaks) as the highest break of $E/F$, and denote it by $b(E/F)$. By Herbrand’s theorem, if $E/F$ is the compositum of $E_1/F$ and $E_2/F$, then

$$b(E/F) = \max\{b(E_1/F), b(E_2/F)\}.$$ 

The highest break is always rational (because it is the image of an integer under $\phi_{E/F}$, which is a piecewise linear function with rational slopes, breaks, and $y$-intercept) but is not necessarily an integer. However, if $E/F$ is abelian, then the Hasse-Arf theorem asserts that $b(E/F)$ is an integer [29 Theorem V.1].

**Definition 2.1.8.** Let $E/F$ be a finite but not necessarily Galois extension, and let $E'/F$ be a finite Galois extension containing $E$. From the transitivity of the $\psi$ and $\phi$ functions (Remark 2.1.4), it follows (as in [29 Remark 2, p. 75]) that the function $\phi_{E'/F} \circ \psi_{E'/F}$ depends only on $E$ and $F$ and not on $E'$. We call this function $\phi_{E/F}$; it has the property that for $m > b(E/F)$ and $E'/F$ finite Galois containing $E$, $b(E'/E) = m$ if and only if $b(E'/F) = \phi_{E/F}(m)$.

**Remark 2.1.9.** If $E$ is a finite separable but not necessarily Galois extension of $F$, we can define the highest break of $E$ to be the highest break of its Galois closure. If $E$ is not a field but only a finite étale $K$-algebra, then it decomposes as a product of finite field extensions of $K$, and we can define the highest break $b(E/F)$ of $E$ to be the maximum of the highest
breaks of any component of \( E \). With this convention, one has the rules

\[
b(E_1 \oplus E_2/F) = \max\{b(E_1/F), b(E_2/F)\}\\
b(E_1 \otimes E_2/F) = \max\{b(E_1/F), b(E_2/F)\}.
\]

**Example 2.1.10.** For \( k \) a field of characteristic \( p > 0 \), \( F = k((t)) \), and \( E = k((t))[z]/(z^p - z - P(t^{-1})) \), where \( P \) is a polynomial over \( k \) whose degree \( d \) is not divisible by \( p \), a straightforward calculation [29, Exercise IV.2.5] shows that \( E/F \) has exactly one break, which is equal to \( d \). It follows that

\[
\phi_{E/F}(m) = \begin{cases} 
m & m \leq d \\
\frac{m - d}{p} & m > d. 
\end{cases}
\]

**Remark 2.1.11.** Note that for any finite separable extension \( E \) of \( F \), \( \phi_{E/F} \) is monotone: this follows for \( E/F \) Galois by Definition 2.1.4 and for \( E/F \) general by Definition 2.1.8.

### 2.2 Unramified and tame extensions

We next recall some more facts about extensions of local fields from [29], and extend a few definitions to the case of an inseparable residue field extension. We retain all definitions and notations from the previous section; we also continue to assume that \( E/F \) is an extension of complete discretely valued fields.

See [29, III.5] for all results implicit in the following definitions.

**Definition 2.2.1.** If \( E/F \) is finite, we say \( E/F \) is unramified if \( m_E = m_F \sigma_E \) and \( E/F \) is separable. Every subextension of an unramified extension is unramified, so we may extend the definition to \( E/F \) infinite by saying that \( E/F \) is unramified if every finite subextension of \( E/F \) is unramified in the previous sense.

**Definition 2.2.2.** The compositum of unramified extensions is again unramified, so any separable algebraic extension \( E/F \) admits a maximal unramified subextension \( U \); we say \( E/F \) is totally ramified if \( U = F \). By Hensel’s lemma, \( U \) is the maximal separable subextension of \( E/F \). In particular, an unramified extension is uniquely determined by its (separable) residue field extension; if \( F = k((t)) \), this means that any finite unramified extension has the form \( k'((t)) \) for some finite separable extension \( k'/k \). By [29, Proposition IV.2], if \( E/F \) is Galois with group \( G \) and \( E/F \) is separable, then the maximal unramified subextension of \( E/F \) is the fixed field of \( G_0 = G^0 \).

Oddly, the following quite standard definition does not occur in [29].

**Definition 2.2.3.** If \( E/F \) is finite Galois, we say \( E/F \) is tamely ramified (or simply tame) if \( \text{Gal}(E/U) \) has order coprime to \( p \), where \( U \) is the maximal unramified subextension of \( E/F \). Any subextension of a tame extension is tame, so we may extend the definition to \( E/F \) infinite by saying that \( E/F \) is tame if each of its finite subextensions is tame. Also, the compositum of tame extensions is tame, so any Galois algebraic extension \( E/F \) has a maximal tame subextension \( T \). If \( E/F \) is separable, then by the properties of the ramification
filtration stated in Definition 2.1.3, \( T \) is the fixed field of \( G^{0+} \). (Note that Remark 2.2.4 below implies that if \( E/F \) is itself tame, then \( \overline{E}/\overline{F} \) is always separable.) We say \( E/F \) is totally wildly ramified if \( T = F \).

**Remark 2.2.4.** If \( E/F \) is finite Galois, totally ramified, and tame of degree \( d \), and moreover \( F \) contains a primitive \( d \)-th root of unity \( \zeta \), then by Kummer theory \([20, \text{X.3}]\), \( T = F(\pi^{1/d}) \) for some generator \( \pi \) of \( m_F \). It follows in this case that \( \overline{E} = \overline{F} \). If on the other hand \( \zeta \notin F \), then \( E(\zeta) \) and \( F(\zeta) \) have the same residue field, so \( \overline{E}/\overline{F} \) is at least separable; since \( E/F \) was assumed to be totally ramified, we must have \( \overline{E} = \overline{F} \). Finally, if \( E/F \) is finite Galois and tame, and \( U \) is the maximal unramified subextension of \( E/F \), then the previous argument shows that \( \overline{E} = \overline{U} \), so \( \overline{E}/\overline{F} \) is separable. As noted in Definition 2.1.3 it then follows that \( \text{Gal}(E/U) = G_1/G_0 \) is cyclic.

**Definition 2.2.5.** If \( E/F \) is finite Galois with maximal unramified subextension \( U \) and maximal tame subextension \( U \), we define the *tame degree* of \( E/F \) to be the degree of \( T \) over \( U \), and the *wild degree* of \( E/F \) to be the degree of \( E \) over \( T \). Then the tame degree is coprime to \( p \), and the wild degree is a power of \( p \).

### 2.3 Break decompositions

We now recall some terminology regarding representations of the absolute Galois group of a local field, following \([20, \text{Chapter 1}]\).

**Hypothesis 2.3.1.** Throughout this section, let \( F \) be a complete discretely valued field whose residue field \( \overline{F} \) is perfect of characteristic \( p > 0 \), put \( G = \text{Gal}(F^{\text{sep}}/F) \), and put \( P = G^{0+} \). Then \( P \) is the \( p \)-Sylow subgroup of \( G \) (in the sense of profinite groups).

**Convention 2.3.2.** When a group \( G \) acts on a set \( M \), let \( M^G \) denote the fixed set of \( M \) under \( G \).

**Definition 2.3.3.** Let \( M \) be a \( \mathbb{Z}[1/p] \)-module on which \( P \) acts via a finite discrete quotient. Then by \([20, \text{Proposition 1.1}]\), there is a unique direct sum decomposition \( M = \bigoplus_{i \geq 0} M(i) \) of \( M \) into \( P \)-stable submodules such that

\[
M(0) = M^P \\
M(i)^G = 0 \quad (i > 0) \\
M(i)^G = M(i) \quad (j > i).
\]

This decomposition is called the *break decomposition* of \( M \); the associated descending filtration \( M_i = \bigoplus_{j \geq i} M(j) \) is called the *break filtration* of \( M \). There are finitely many \( i \geq 0 \) for which \( M(i) \neq 0 \), and they are all rational numbers; they are called the *breaks* of \( M \). If \( M \) is nonzero, there must be at least one break; the largest one is called the *highest break* of \( M \), and is denoted \( b_F(M) \).
Remark 2.3.4. If $E$ is the fixed field of the kernel of the action of $P$ on $M$, then the highest break of $M$ coincides with the highest break of $E/F$.

Often instead of the full break decomposition, we consider some of its numerical invariants.

Definition 2.3.5. With notation as in Definition 2.3.3, suppose that $M$ is a free module over some ring $A$ and that $P$ acts $A$-linearly. Then for each $i \geq 0$, $M(i)$ is projective of some finite rank; that rank is called the multiplicity of $i$ (as a break of $M$). Define the Hasse-Arf polygon of $M$, denoted $P(M)$, as the polygon with left endpoint $(0,0)$ consisting of $n$ segments of horizontal width 1, the $i$-th of which has slope equal to the $i$-th smallest break of $\rho$, counting multiplicities.

The strong form of the Hasse-Arf theorem [20, Proposition 1.9] yields the following integrality property of the Hasse-Arf polygon.

Proposition 2.3.6. With notation as in Definition 2.3.5, the Hasse-Arf polygon has integer vertices. In particular, the breaks of any abelian extension of $F$ are all integers.

Remark 2.3.7. While the Hasse-Arf polygon looks like a Newton polygon of the sort one associates to a polynomial over a local field, its formalism is quite different. For instance, the highest break of the tensor product of two modules is at most the maximum of the highest breaks of the tensorands, whereas the highest slope of the tensor product of two polynomials (that is, the polynomial whose roots are the products of roots, one from each tensorand) is the sum of the highest slopes of the tensorands. For a thorough development of the formalism of Hasse-Arf polygons (in which it is shown that any class of filtrations which “look enough like” ramification filtrations actually are ramification filtrations), see [1].

Using the Hasse-Arf theorem, we can obtain the following finiteness result about representations of $P$. First, recall Jordan’s theorem on finite linear groups.

Proposition 2.3.8. For any positive integer $n$, there exists an integer $f(n)$ such that for any field $K$ of characteristic zero, any finite subgroup $G$ of $\text{GL}_n(K)$ contains a commutative normal subgroup $H$ of index at most $f(n)$.

Proof. Any such $G$ can be embedded into $\text{GL}_n(\mathbb{Q}_{\text{alg}})$, and hence into $\text{GL}_n(\mathbb{C})$. For the result in this case, see [18].

Proposition 2.3.9. Given a residual characteristic $p$, a positive integer $n$ and a nonnegative real number $\ell$, there exists an integer $N = N(p, n, \ell)$ such that every representation of $P$ of dimension $n$ over a field of characteristic zero, with finite discrete image and highest break $\leq \ell$, has image of order at most $N$.

Proof. There is no loss of generality in working with representations over $\mathbb{C}$ (or even over the algebraic closure of $\mathbb{Q}$). Let $\rho : P \to \text{GL}_n(\mathbb{C})$ be a representation with image $G \subset \text{GL}_n(\mathbb{C})$, and let $E$ be the fixed field of the kernel of $\rho$. Suppose first that $G$ is abelian; then by
Proposition 2.3.6: all of the breaks of $E/F$ are integers, so the number of them is bounded by $\ell$. Moreover, an elementary abelian $p$-subgroup of $\text{GL}_n(\mathbb{C})$ can have at most $p^n$ elements since the matrices in such a subgroup must be simultaneously diagonalizable. Thus $G$ has order at most $p^{n\ell}$ in this case.

Now suppose that $G$ is arbitrary. Apply Jordan’s theorem (Proposition 2.3.8) to choose a commutative normal subgroup $H$ of $G$ of index bounded as a function of $n$, and let $E$ be the fixed field of $H$; then the highest break of the restriction of $\rho$ to $P \cap \text{Gal}(E^{\text{sep}}/E)$ is bounded by a function of $p, n, \ell$. This restriction is abelian, so as before, the order of $H$ is bounded by a function of $p, n, \ell$. This yields the desired result. 

3 Rigid annuli

We now introduce the rigid analytic spaces we will be working with, which are certain one-dimensional annuli. In particular, we verify that over a spherically complete coefficient field, every coherent locally free sheaf on a one-dimensional rigid annulus is freely generated by global sections (Theorem 3.4.3). This provides a bridge between our setup and the existing literature on $p$-adic differential equations, which is mostly conducted in ring-theoretic terms. We also produce a special class of finite étale covers of rigid annuli corresponding to finite étale extensions of $k((t))$; these will be used to discuss the monodromy of $p$-adic differential equations.

We will freely use the language of rigid analytic geometry using the original foundations of Tate et al; see [16] (particularly Chapter 2) for an introduction. If one prefers the Berkovich foundations, as in [5] (see also [6] for an overview), one should in principle have no trouble converting the discussion into those terms, since the rigid spaces under consideration are quasi-separated and admit affinoid coverings of finite type. However, one will probably encounter some subtleties; see for instance Remark 5.1.5.

3.1 Notations

Before proceeding, we set a few notational conventions.

Convention 3.1.1. Let $K$ be a field complete with respect to a nonarchimedean absolute value $| \cdot | : K^* \to \mathbb{R}^+$. Let $\mathfrak{o}_K$ be the subring of $x \in K$ with $|x| \leq 1$, let $\mathfrak{m}_K$ be the ideal of $x \in \mathfrak{o}_K$ with $|x| < 1$, and let $k$ denote the residue field $\mathfrak{o}_K/\mathfrak{m}_K$. Let $\Gamma^*$ denote the divisible closure of the image of $| \cdot |$.

Convention 3.1.2. On any rigid analytic space over $K$, let $\mathcal{O}$ denote the structure sheaf, and let $\mathfrak{o}$ denote the subsheaf of the structure sheaf consisting of functions bounded in absolute value by 1 everywhere (i.e., the “integral subsheaf” of $\mathcal{O}$).

Convention 3.1.3. In case the notation for an object includes an explicit mention of the coefficient field $K$, we will routinely suppress that $K$ from the notation when the choice of $K$ is to be understood (as it almost always will be, except when we need to compare distinct choices). For example, in Definition 3.2.1 we typically abbreviate $\mathcal{R}_{I,K}$ to $\mathcal{R}_I$. 

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**Convention 3.1.4.** When we write the absolute value of a matrix, we mean the maximum of the absolute values of its entries, rather than any sort of operator norm. (Contrast this convention with Definition 5.1.1.) Also, we let $I_n$ denote the $n \times n$ identity matrix over any ring.

### 3.2 Rigid annuli

**Definition 3.2.1.** We say a subinterval $I$ of $[0,1)$ is *aligned* if any nonzero endpoint at which it is closed is contained in $\Gamma^*$. For $I$ aligned, we define the annulus $A(I) = A_K(I)$ as the admissible open subspace of the rigid affine $t$-line given by

$$\{t \in \mathbb{A}_K^1 : \vert t \vert \in I\}.$$ 

If $I$ is given with explicit endpoints, the enclosing parentheses are omitted, so that we write for instance $A([0,1))$ instead of $A([0,1))$.

Let $R_I = R_{I,K}$ denote the ring $\Gamma(\mathcal{O}, A(I))$ of rigid analytic functions on $A(I)$. The elements of $R_I$ can be described as formal Laurent series $\sum_{i \in \mathbb{Z}} c_i t^i$ with each $c_i \in K$; for $r \in I \cap \Gamma^*$, the spectral seminorm on the subspace $\vert t \vert = r$ of $A(I)$, restricted to $R_I$, is equal to the norm $\vert \cdot \vert_r$ given by the formula

$$\left\vert \sum c_i t^i \right\vert_r = \sup \{ \vert c_i \vert r^i \}.$$ 

One has analogues of the maximum modulus principle and the Hadamard three circles theorem for $\vert \cdot \vert_r$.

**Lemma 3.2.2.**

(a) For $x \in R_{[0,b]}$ and $r \in [0,b]$, $\vert x \vert_r \leq \vert x \vert_b$.

(b) For $x \in R_I$, $a,b \in I$, and $c \in [0,1]$, put $r = a^c b^{1-c}$; then $\vert x \vert_r \leq \vert x \vert^c_a \vert x \vert^{1-c}_b$.

**Proof.** (a) If $x = \sum c_j t^j \in R_{[0,b]}$, then $c_j = 0$ for $j < 0$. Hence if $r \in [0,b]$, then

$$\vert c_j \vert r^j \leq \vert c_j \vert b^j;$$

taking suprema yields $\vert x \vert_r \leq \vert x \vert_b$.

(b) Note that the desired inequality holds with equality if $x = c_j t^j$. For a general $x = \sum c_j t^j$, we then have

$$\vert x \vert_r = \sup_j \{ \vert c_j t^j \vert_r \} \leq \sup_j \{ \vert c_j t^j \vert^c_a \vert c_j t^j \vert^{1-c}_b \} \leq \sup_j \{ \vert c_j t^j \vert^c_a \} \sup_j \{ \vert c_j t^j \vert_b \}^{1-c} = \vert x \vert^c_a \vert x \vert^{1-c}_b.$$

as desired.
A result of Lazard \cite{Lazard} Proposition 4] yields the following.

**Proposition 3.2.3.** Let $I$ be a closed aligned subinterval of $[0,1)$. Then every ideal of $\mathcal{R}_I$ is generated by an element of $K[t]$. In particular, $\mathcal{R}_I$ is a principal ideal domain.

**Remark 3.2.4.** The alignedness restriction can be dumped if one allows affinoid spaces to be closed analytic subspaces of polydiscs of arbitrary radii, not just radius 1. This permission is made in Berkovich’s foundations of rigid geometry.

### 3.3 A matrix approximation lemma

We will need a matrix approximation lemma in the spirit of \cite[Lemma 6.2]{Berkovich}. We start with an analogue of \cite[Lemma 6.1]{Berkovich}.

**Lemma 3.3.1.** Put $S = K[t]$ or $K[t,t^{-1}]$, and fix $x, y \in S$. Then there exists $c > 0$ such that for any $\lambda \in K$ with $|\lambda| < c$, $x$ and $y + \lambda$ generate the unit ideal in $S$.

**Proof.** For each $\lambda \in K$, the ideal generated by $x$ and $y + \lambda$ in $S$ is generated by some monic polynomial $e_\lambda$. Note that this limits $e_\lambda$ to a finite set, namely the monic factors of $x$. Each value of $e_\lambda$ not equal to 1 can only occur for one value of $\lambda$: if $e_\lambda = e_\nu$, then $e_\lambda$ divides $(y + \lambda) - (y + \lambda') = \lambda - \lambda' \in K$, contradiction. In particular, for $c > 0$ sufficiently small, $e_\lambda = 1$ for all $\lambda \in K$ with $|\lambda| < c$, as desired.

The following is analogous to \cite[Lemma 6.2]{Berkovich}, but with some slight simplifications because we are taking $I$ to be closed, and because we are only working with power series rings instead of the more general “analytic rings” of \cite{Berkovich}.

**Lemma 3.3.2.** Let $I$ be a closed aligned subinterval of $[0,1)$ which does (resp. does not) contain 0, and let $M$ be an invertible $n \times n$ matrix over $\mathcal{R}_I$. Then there exists an invertible $n \times n$ matrix $U$ over $K[t]$ (resp. over $K[t,t^{-1}]$) such that $|MU - I_n|_r < 1$ for $r \in I$. Moreover, if $|\det(M) - 1|_r < 1$, we can ensure that $\det(U) = 1$.

**Proof.** We proceed by induction on $n$, the case $n = 1$ being vacuously true. By multiplying some row of $M$ by the inverse of $\det(M)$, we may reduce to the case where $\det(M) = 1$. Let $C_i$ denote the cofactor of $M_{ni}$ in $M$, so that $\det(M) = \sum_{i=1}^n C_i M_{ni}$, and in fact $C_i = (M^{-1})_{in} \det(M)$. Thus $C_1, \ldots, C_n$ generate the unit ideal in $\mathcal{R}_I$, so we can find $\alpha_1, \ldots, \alpha_n \in \mathcal{R}_I$ such that $\sum_{i=1}^n \alpha_i C_i = 1$.

For brevity, write $S = K[t]$ if $0 \notin I$ and $S = K[t,t^{-1}]$ if $0 \notin I$. Choose $\beta_1, \ldots, \beta_{n-1}, \beta_n' \in S$ such that for $r \in I$,

$$|\beta_i - \alpha_i|_r < \min_j \{|C_j|_r^{-1}\} \quad (i = 1, \ldots, n-1), \quad |\beta_n' - \alpha_n|_r < \min_j \{|C_j|_r^{-1}\}.$$  

By Lemma 3.3.1 for $\lambda \in K$ of sufficiently small absolute value, $\beta_n = \beta_n' + \lambda$ has the properties that $|\beta_n - \alpha_n|_r < \min_j \{|C_j|_r^{-1}\}$ for $r \in I$, and that $\beta_1, \ldots, \beta_n$ generate the unit ideal in $S$. Since $S$ is a principal ideal domain, we can find a matrix $A$ over $S$ of determinant 1 such
that $A_{ni} = \beta_i$ for $i = 1, \ldots, n$. Put $M' = MA^{-1}$, and let $C'_n$ be the cofactor of $M'_{nn}$ in $M'$. Then

$$C'_n = (M')_{nn}^{-1} \det(M') = (AM^{-1})_{nn} \det(M) = \sum_{i=1}^{n} A_{ni}M^{-1}_{in} \det(M) = \sum_{i=1}^{n} \beta_i C_i,$$

so that

$$C'_n = 1 + \sum_{i=1}^{n} (\beta_i - \alpha_i) C_i$$

and so $|C'_n - 1|_r < 1$ for $r \in I$. In particular, $C'_n$ is a unit in $\mathcal{R}_I$.

Apply the induction hypothesis to the upper left $(n-1) \times (n-1)$ submatrix of $M'$, and extend the resulting matrix $V$ to an $n \times n$ matrix by setting $V_{ni} = V_{in} = 0$ for $i = 1, \ldots, n-1$ and $V_{nn} = 1$. Then we have $\det(M'V) = 1$ and

$$|(M'V - I_n)_{ij}|_r < 1 \quad (i = 1, \ldots, n-1; j = 1, \ldots, n-1; r \in I).$$

We now perform an “approximate Gaussian elimination” over $\mathcal{R}_I$ to transform $M'V$ into a new matrix $N$ with $|N - I_n|_r < 1$ for $r \in I$. First, define a sequence of matrices $\{X^{(h)}\}_{h=0}^{\infty}$ by $X^{(0)} = M'V$ and

$$X^{(h+1)}_{ij} = \begin{cases} X^{(h)}_{ij} & i < n \\ X^{(h)}_{nj} - \sum_{m=1}^{n-1} X^{(h)}_{nm} X^{(h)}_{mj} & i = n; \end{cases}$$

note that $X^{(h+1)}_{ij}$ is obtained from $X^{(h)}_{ij}$ by subtracting $X^{(h)}_{nm}$ times the $m$-th row from the $n$-th for $m = 1, \ldots, n-1$ in succession. At each step, for each $r \in I$, $\max_{1 \leq j \leq n-1} \{|X^{(h)}_{nj}|_r\}$ gets multiplied by a factor no larger than $\max_{1 \leq i,j \leq n-1} \{|(M'V - I_n)_{ij}|_r\}$; since $I$ is closed and $|\cdot|_r$ is a continuous function of $r$, these factors are bounded strictly below 1. Thus for $h$ sufficiently large, we have

$$|X^{(h)}_{nj}|_r < \min\{1, \min_{1 \leq i \leq n-1} \{|X^{(h)}_{im}|^{-1}\}\} \quad (r \in I; j = 1, \ldots, n-1).$$

Pick such an $h$ and set $X = X^{(h)}$; note that $\det(X) = \det(M'V) = 1$. Then for $r \in I$,

$$|(X - I_n)_{ij}|_r < 1 \quad (i = 1, \ldots, n; j = 1, \ldots, n-1)$$

and hence also $|X_{nn} - 1|_r < 1$.

Next, define a sequence of matrices $\{W^{(h)}\}_{h=0}^{\infty}$ by setting $W^{(0)} = X$ and

$$W^{(h+1)}_{ij} = \begin{cases} W^{(h)}_{ij} - W^{(h)}_{in} W^{(h)}_{nj} & i < n \\ W^{(h)}_{ij} & i = n; \end{cases}$$
note that $W^{(h+1)}$ is obtained from $W^{(h)}$ by subtracting $W_{in}^{(h)}$ times the $n$-th row from the $i$-th row for $i = 1, \ldots, n - 1$. At each step, for $r \in I$, $|W_{in}^{(h)}|_r$ gets multiplied by a factor no larger than $|X_{mn}^{(h)} - 1|_r$; again, these factors are bounded strictly below 1 because $I$ is closed. Thus for $h$ sufficiently large,

$$|W_{in}^{(h)}|_r < 1 \quad (r \in I; 1 \leq i \leq n - 1).$$

Pick such an $h$ and set $W = W_h$; then $|W - I_n|_r < 1$ for $r \in I$. (Note that the inequality $|X_{in}X_{nj}|_r < 1$ for $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n - 1$ ensures that the second set of row operations does not disturb the fact that $|W_{ij}^{(h)}|_r < 1$ for $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n - 1$.)

To conclude, note that by construction, $(M'V)^{-1}W$ is a product of elementary matrices over $R_f$, each consisting of the diagonal matrix plus one off-diagonal entry. By suitably approximating the off-diagonal entry of each matrix in the product by an element of $S$, we get an invertible matrix $X$ over $S$ such that $|M'VX - I_n|_r < 1$ for $r \in I$. We may thus take $U = A^{-1}VX$ to obtain the desired result.

**Remark 3.3.3.** It is tempting to believe that one can improve the conclusion of Lemma 3.3.2 to yield that for any given $c > 0$, we can choose $U$ so that $|MU - I_n|_r < c$ for $r \in I$, as in [21] Lemma 6.3). However, this is only possible if we assume $|\det(M) - 1|_r < c$ for $r \in I$, or else the base case $n = 1$ fails. Indeed, this failure is inevitable, since Theorem 3.4.3 does not hold for arbitrary $K$; see Remark 3.4.4.

### 3.4 Locally free sheaves on rigid annuli

Recall that a theorem of Kiehl [16] Theorem 4.5.2] asserts that a coherent sheaf on an affinoid space is generated by finitely many global sections. This is typically not true on a nonaffinoid space, such an as open rigid annulus, but for coherent locally free sheaves, it turns out we can salvage something. First, however, we must restrict the field of coefficients.

**Definition 3.4.1.** The field $K$, which we are supposing to be complete for a nonarchimedean absolute value, is said to be **spherically complete**, or maximally complete, if every decreasing sequence of closed balls has nonempty intersection. For instance, every discretely valued field is spherically complete, but the field $\mathbb{C}_p$, the completed algebraic closure of $\mathbb{Q}_p$, is not spherically complete.

**Proposition 3.4.2.** Suppose that $K$ is spherically complete. Fix a sequence $(r_1, r_2, \ldots)$ of positive real numbers, and let $c_0$ be the set of sequences $x = (x_1, x_2, \ldots)$ over $K$ with $|x_i| \leq r_i$ for all $i$. For $i = 0, 1, \ldots$, let $f_i$ be an affine functional on $c_0$, that is,

$$f_i(x_1, x_2, \ldots) = a_{i,0} + \sum_{j=1}^{\infty} a_{i,j}x_j$$

for some sequence $a_{i,0}, a_{i,1}, \ldots$ converging to 0 in $K$. Let $S_i$ be the subset of $x \in c_0$ on which $|f_h(x)| < 1$ for $h = 0, \ldots, i$. If $S_i \neq \emptyset$ for each $i$, then $\cap_i S_i \neq \emptyset$. 


Proof. Suppose that \( y_1, y_2, \ldots, y_l \in K \) have been chosen so that \( |y_j| \leq r_j \) for \( j = 1, \ldots, l \), and for each \( i \), the set \( S_{i,t} \) of \( x \in c_0 \) with \( x_1 = y_1, \ldots, x_t = y_t \) and \( |f_h(x)| < 1 \) for \( h = 0, \ldots, i \) is nonempty. Let \( T_{i,t} \) be the set of possible values of \( x_{t+1} \) for a sequence \( x \in S_{i,t} \). Then \( T_{i,t} \) is necessarily an open ball, and the sequence \( T_{0,t}, T_{1,t}, \ldots \) is decreasing, so has a nonempty intersection since \( K \) is spherically complete. We may then pick any \( y_{l+1} \) in that intersection to continue the construction.

The hypothesis in the previous paragraph holds vacuously with \( l = 0 \). We may thus construct \( y_1, y_2, \ldots \) as above, and the resulting sequence belongs to \( \cap_i S_i \).

\[ \square \]

**Theorem 3.4.3.** Let \( I \) be an aligned subinterval of \([0,1]\) and let \( \mathcal{E} \) be a coherent locally free sheaf of rank \( n \) on \( A(I) \). Then there exist sections \( v_1, \ldots, v_n \in \Gamma(\mathcal{E}, A(I)) \) which freely generate \( \mathcal{E} \).

**Proof.** Let \( J_1 \subseteq J_2 \subseteq \cdots \) be a weakly increasing sequence of aligned closed subintervals of \( I \) whose union is all of \( I \), with the property that if \( 0 \in I \), then \( 0 \in J_i \) for all \( i \). Put \( \mathcal{R}_i = \mathcal{R}_{J_i} \) and \( E_i = \Gamma(\mathcal{E}, A(J_i)) \); by Proposition 3.2.3, each \( E_i \) is free of rank \( n \) over \( \mathcal{R}_i \).

Choose a basis \( v_{1,1}, \ldots, v_{1,n} \) of \( E_{i,1} \). Given a basis \( v_{i,1}, \ldots, v_{i,n} \) of \( E_{i,i} \), we choose a basis \( v_{i+1,1}, \ldots, v_{i+1,n} \) of \( E_{i+1} \) as follows. Pick any basis \( e_1, \ldots, e_n \) of \( E_{i+1} \), and define an invertible \( n \times n \) matrix \( M_i \) over \( \mathcal{R}_i \) by writing \( e_t = \sum_j (M_i)_{jt} v_{i,j} \). Apply Lemma 3.3.2 to produce an invertible \( n \times n \) matrix \( U \) over \( S \), where \( S = K[t] \) if \( 0 \in I \) and \( S = K[t, t^{-1}] \) if \( 0 \notin I \), such that \( |M_i U - I_n|_r < 1 \) for \( r \in J_i \). Put \( V_i = M_i U \), and define the basis \( v_{i+1,1}, \ldots, v_{i+1,n} \) of \( E_{i+1} \) by \( v_{i+1,l} = \sum_j (V_i)_{jl} v_{i,j} \).

Now suppose \( e_1, \ldots, e_n \) is a basis of \( E_{1,1} \). Define the invertible \( n \times n \) matrix \( W \) over \( \mathcal{R}_1 \) by

\[
e_t = \sum_j W_{jt} v_{1,j}.
\]

Put \( W_i = (V_1 \cdots V_{i-1})^{-1} W \); then

\[
e_t = \sum_j (W_i)_{jt} v_{i,j}.
\]

Hence \( e_1, \ldots, e_n \) forms a basis of \( E_i \) if and only if \( W_i \) is an invertible matrix over \( \mathcal{R}_i \).

Let \( B_i \) denote the set of \( n \times n \) matrices \( W \) over \( \mathcal{R}_1 \) with

\[
|(V_1 \cdots V_{i-1})^{-1} W - I_n|_r < 1 \quad (r \in J_i);
\]

since \( |V_i - I_n|_r < 1 \) whenever \( r \in J_i \), we have \( B_{i+1} \subseteq B_i \). We may apply Proposition 3.4.2 by fixing some \( s \in J_1 \) and identifying \( c_0 \) with the set of \( n \times n \) matrices \( W \), with \( W_{ij} = \sum_l W_{ij,l} t^l \), satisfying \( |W_{ij,l}|_s \leq 1 \) for all \( i,j,l \); by doing so, we see that \( \cap_i B_i \neq \emptyset \). If \( W \in \cap_i B_i \), we may put \( e_t = \sum_j W_{jt} v_{1,j} \) to obtain a basis \( e_1, \ldots, e_n \) of \( E_1 \) that extends to a basis of \( E_i \) for each \( i \). Thus the \( e_t \) are global sections of \( \mathcal{E} \) which freely generate \( \mathcal{E} \), as desired.

\[ \square \]

**Remark 3.4.4.** Theorem 3.4.3 is false for any field \( K \) which is not spherically complete, even for line bundles. More precisely, Lazard [25, Théorème 2] showed that on an open disc
over $K$, every coherent locally free sheaf of rank 1 is generated by a global section (and hence trivial) if and only if $K$ is spherically complete. Thus Theorem 3.4.3 may be viewed as a higher-rank generalization of Lazard’s result. (For $I = [0, 1)$, i.e., for locally free sheaves on an open disc, this generalization was already given by Gruson \cite[Proposition 2]{17}.) It may also be viewed as an explication of a special case of the comment made in the introduction of \cite{32}, to the effect that the proof given there that the sheaf $\mathcal{O}^*$ has no higher cohomology on any rational subset of the projective line can be carried over to establish the vanishing of $H^1(\text{GL}_n)$.

3.5 Robba rings and formally étale covers

**Definition 3.5.1.** Let $\mathcal{R} = \mathcal{R}_K$ denote the direct limit of the rings $\mathcal{R}_{(a, 1)}$ over all $a \in (0, 1)$; the ring $\mathcal{R}$ is called the Robba ring over $K$. The elements of $\mathcal{R}$ can be viewed as formal Laurent series $\sum_i c_i t^i$, with $c_i \in K$, which converge on some unspecified open annulus with outer radius 1. Let $\mathcal{R}^\text{int} = \mathcal{R}_K^\text{int}$ denote the subring of $\mathcal{R}$ consisting of series with $c_i \in \mathfrak{o}_K$ for all $i \in \mathbb{Z}$; let $m_\mathcal{R} = m_{\mathcal{R}_K}$ denote the ideal of $\mathcal{R}^\text{int}$ consisting of series with $c_i \in m_K$ for all $i \in \mathbb{Z}$.

**Lemma 3.5.2.** The ring $\mathcal{R}^\text{int}$ is a local ring with maximal ideal $m_\mathcal{R}$ and residue field $k((t))$.

**Proof.** For any $x \in m_\mathcal{R}$, we have $|x|_a < 1$ for $a$ sufficiently close to 1, so $1 + x$ is invertible in $\mathcal{R}^\text{int}$. Hence $m_\mathcal{R}$ is contained in the Jacobson radical of $\mathcal{R}^\text{int}$, but the Jacobson radical is the intersection of all maximal ideals. Hence $m_\mathcal{R}$ is the unique maximal ideal of $\mathcal{R}^\text{int}$, proving the claim.

**Remark 3.5.3.** If $K$ is discretely valued, then $\mathcal{R}^\text{int}$ is a discrete valuation ring with corresponding absolute value $|\sum c_i t^i| = \sup\{ |c_i| \}$. On the other hand, if $K$ is not discretely valued, then one can construct $\sum c_i t^i \in m_\mathcal{R}$ with $\sup\{ |c_i| \} = 1$, so we cannot view $\mathcal{R}^\text{int}$ as a valuation ring in this fashion. See \cite[Remark 14]{9} for an example of how readily this discrepancy can crop up if one does not take pains to avoid it.

**Definition 3.5.4.** A pair $(R, I)$, in which $R$ is a ring and $I$ is an ideal of $R$, is henselian if each finite étale extension of $R/I$ lifts uniquely to a finite étale extension of $R$.

**Proposition 3.5.5.** The pair $(\mathcal{R}^\text{int}, m_\mathcal{R})$ is henselian.

**Proof.** By \cite[43.2]{28}, it suffices to show that for any monic polynomial $P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ over $\mathcal{R}^\text{int}$ with the property that $c_{n-1} + 1 \in m_\mathcal{R}$ and $c_i \in m_\mathcal{R}$ for $i = 0, \ldots, n - 2$, there exists a root $z \in \mathcal{R}^\text{int}$ of $P(x)$ such that $z - 1 \in m_\mathcal{R}$. We construct $z$ using a Newton iteration as follows.

Put $z_0 = 1$; given $z_i \in \mathcal{R}^\text{int}$ such that $z_i - 1 \in m_\mathcal{R}$, note that

$$P'(z_i) - 1 = nz_i^{n-1} + (n - 1)c_{n-1}z_i^{n-2} + \cdots + c_1 \equiv n + (n - 1)c_{n-1} \equiv 1 \pmod{m_\mathcal{R}},$$

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so $P'(z_1)$ is a unit in $\mathcal{R}^{\text{int}}$. We may then put

$$z_{i+1} = z_i - \frac{P(z_i)}{P'(z_i)};$$

since $P(z_i) \equiv z_i^n + c_{n-1}z_i^{n-1} \equiv 0 \pmod{\mathfrak{m}_\mathcal{R}}$, we have $z_{i+1} - 1 \in \mathfrak{m}_\mathcal{R}$ and the iteration continues.

Choose $a \in (0, 1) \cap \Gamma^*$ such that for $r \in [a, 1)$, $|c_{n-1} + 1|_r < 1$ and $|c_i|_r < 1$ for $i = 0, \ldots, n-2$. Then by induction, for $r \in [a, 1)$ we have $|z_i - 1|_r < 1$. Moreover, for $r \in [a, 1)$, we have

$$|P(z_{i+1})|_r = \left| \sum_{j=2}^n \frac{P(j)(z_i)}{j!}(z_{i+1} - z_i)^j \right|_r \leq |z_{j+1} - z_j|^2,$$

so $|P(z)|_r \to 0$ and hence $|z_{i+1} - z_i|_r \to 0$. It follows that the $z_i$ converge to a limit $z$ satisfying $|z - i|_r < 1$ for $r \in [a, 1)$ and $P(z) = 0$, as desired. □

**Definition 3.5.6.** We say a finite cover $X$ of $A[a, 1)$ is formally étale if it induces a finite étale extension of $\mathcal{R}^{\text{int}}$; we refer to the induced finite étale extension of $\mathcal{R}^{\text{int}}/\mathfrak{m}_\mathcal{R} \cong k((t))$ as the reduction of $X$. By Proposition 3.5.5, any two formally étale covers of $A[a, 1)$ with isomorphic reductions become themselves isomorphic over $A[b, 1)$ for some $b \in [a, 1) \cap \Gamma^*$.

**Remark 3.5.7.** If the reduction of a formally étale cover $X \to A[a, 1)$ induces a separable residue field extension of $k$, then $X$ itself is isomorphic to an annulus over some finite extension of $K$ which induces an étale extension of $\mathfrak{o}_K$.

### 4 Monodromy of differential equations

We next collect some facts about $p$-adic differential equations on rigid annuli, specifically isolating those with quasi-unipotent monodromy. Our treatment follows somewhat that of Matsuda [26], though we simplify his presentation a bit by making more systematic use of local duality (as in [14]).

**Convention 4.0.8.** Retain the notations introduced in Section 3.1 but now assume further that char($K$) = 0, that char($k$) = $p > 0$, and that the absolute value $|\cdot|$ on $K$ is normalized so that $|p| = p^{-1}$.

**4.1 $\nabla$-modules on annuli: generalities**

To begin with, we define the category in which we will be working.

**Definition 4.1.1.** For $r \in (0, 1) \cap \Gamma^*$, let $\mathcal{M}_r = \mathcal{M}_{r,K}$ denote the category of $\nabla$-modules, i.e., coherent locally free sheaves $\mathcal{E}$ equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$, on the annulus $A[r, 1)$. Then there are natural restriction functors $\mathcal{M}_r \to \mathcal{M}_s$ whenever $s \in [r, 1) \cap \Gamma^*$; using these, we may define the direct limit of the $\mathcal{M}_r$, which we denote by $\mathcal{M} = \mathcal{M}_K$. 

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Convention 4.1.2. Unless otherwise specified, any $\nabla$-module defined over an annulus $A[r, 1)$ will be interpreted as an object of $\mathcal{M}$; that is, a “morphism” between two such objects need only be defined on some possibly smaller annulus $A[s, 1)$, and so on.

Remark 4.1.3. On higher-dimensional spaces, one would ordinarily require the connection $\nabla$ to be integrable, i.e., the composition of $\nabla$ with the induced map $\mathcal{E} \otimes \Omega^1 \to \mathcal{E} \otimes \Omega^2$ should vanish. This is superfluous here because we are working on a one-dimensional space, so $\Omega^1$ is freely generated by $dt$ and $\Omega^2$ vanishes.

Remark 4.1.4. To specify a connection on a sheaf $\mathcal{E}$, it is enough to specify the action on $\mathcal{E}$ of any one differential operator of $A(I)$, e.g., $\frac{\partial}{\partial t}$ or $t \frac{\partial}{\partial t}$. Again, this is because the underlying space is one-dimensional; on a $d$-dimensional space, one must specify the actions of $d$ operators, and integrability is equivalent to the fact that these actions commute.

Remark 4.1.5. Note that given a $\nabla$-module $\mathcal{E}$ on $A[r, 1)$, any $\nabla$-submodule $\mathcal{F}$ is a locally free subsheaf; that is, the quotient $\mathcal{E}/\mathcal{F}$ is torsion-free. This is a standard property of $\nabla$-modules on any smooth rigid analytic space; see for instance [7, Proposition 2.2.3].

We will occasionally need to enlarge the coefficient field $K$, as follows.

Definition 4.1.6. By an “unramified (algebraic) extension of $K$”, we will mean an algebraic extension $K'$ of $K$ such that the integral closure $\sigma_{K'}$ of $\sigma_K$ in $K'$ is étale over $\sigma_K$. This is inconsistent with scheme-theoretic terminology, but is consistent with the established terminology for local fields (Definition 2.2.1) in case $K$ is discretely valued. Let $K^{unr}$ denote the maximal unramified extension of $K$ (within $K^{sep}$).

Definition 4.1.7. We say that $\mathcal{E} \in \mathcal{M}$ is said to be overconvergent if for any $\eta \in (0, 1)$, there exists $r \in (0, 1)$ such that for any closed aligned subinterval $I$ of $[r, 1)$ and any $v \in \Gamma(\mathcal{E}, A(I))$, the sequence

$$\left\{ \frac{1 \otimes \partial^j}{j! \otimes dt} v \right\}_{j=0}^{\infty}$$

is $\eta$-null; that is, in terms of some basis, the norms of the coefficients of the $j$-th term of the series, multiplied by $\eta^j$, converge to zero. It suffices to check this condition for some set of intervals $I$ covering $[r, 1)$, as then the corresponding subannuli form an admissible cover of $A(I)$. Let $\mathcal{M}^{conv} = \mathcal{M}^{conv}_{K^{sep}}$ be the subcategory of $\mathcal{M}$ consisting of overconvergent objects; this category is an abelian subcategory of $\mathcal{M}$.

Remark 4.1.8. This notion of “overconvergent” coincides with the notion of “solvable at 1” in the terminology of [11, 4.1-1]. The property of overconvergence can be shown by a direct calculation to be invariant under automorphisms of $A[0, 1)$.

Remark 4.1.9. Let $f_1, f_2 : A[a, 1) \to A[b, 1)$ be two morphisms which induce the same map on $\mathcal{R}^{int}/m_{\mathcal{R}}$. Then there is a natural isomorphism between the pullback functors $f_1^*$ and $f_2^*$ on $\mathcal{M}^{conv}$, given by the Taylor series

$$1 \otimes v \mapsto \sum_{j=0}^{\infty} \frac{(f_2^*(t) - f_1^*(t))^j}{j!} \otimes \frac{\partial^j}{\partial b^j} v;$$
this is the local analogue of [7, Proposition 2.2.17].

**Definition 4.1.10.** For $E \in \mathcal{M}$, write $H^0(E)$ for the direct limit of $\ker(\nabla)$ on $A[a, 1]$ as $a$ approaches 1. Write $H^1(E)$ for the direct limit of $\coker(\nabla)$ on $A[a, 1]$ as $a$ approaches 1. Note that the formation of these commutes with tensoring over $K$ with a finite Galois extension $K'$, since the trace map from $K'$ to $K$ commutes with the action of $\nabla$. Note also that the Yoneda Ext group $Ext^i(E, E')$ in the category $\mathcal{M}$ is equal to $H^i(E \vee \otimes E')$ for $i = 0, 1$. (For an analogous fact in a more global setting, see [8, Proposition 1.1.2].)

**Example 4.1.11.** For $E = O$, we have $H^0(O) = K$ generated by 1, and $H^1(O) = K$ generated by $dt/t$.

### 4.2 Interlude: semilinear Galois representations

Before continuing, we gather up a few easy but not necessarily standard facts about twisted group representations, which we will use in the construction of the monodromy representation of a quasi-unipotent $\nabla$-module.

**Convention 4.2.1.** All group actions on sets will be right actions. That is, if the group $G$ acts on the set $S$, we write $s^g$ for the image of $s \in S$ under $g \in G$, and require the composition law $s^{gh} = (s^g)^h$.

**Definition 4.2.2.** Let $G$ be a group which acts on a field $F$ (compatibly with the field operations). A *semilinear representation* of $G$ over $F$ is a finite dimensional $F$-vector space $V$ equipped with an action of $G$, which is semilinear in the following sense: for $g \in G$, $c \in F$, and $v \in V$, we have $(cv)^g = c^g v^g$. If we define $F[G]$ to be the twisted group algebra, in which $c(g) \cdot d(h) = cd^{g^{-1}}(gh)$, then a semilinear representation can be reinterpreted as a right $F[G]$-module. We say $V$ is *trivial* if it is isomorphic to $F^n$, with the $G$-action acting on each copy of $F$ separately, for some nonnegative integer $n$.

Maschke’s theorem on the complete reducibility of representations of finite groups goes over to twisted representations as follows.

**Lemma 4.2.3 (Maschke property).** Let $G$ be a finite group which acts on a field $F$ of characteristic zero. Then every semilinear representation of $G$ over $F$ is completely reducible, i.e., is a direct sum of irreducible twisted representations.

**Proof.** It suffices to show that if $V$ is indecomposable, then it is also irreducible. Suppose on the contrary that $V$ is indecomposable, but $V$ has an irreducible twisted subrepresentation $W$. Choose any projector $P \in V^\vee \times V$ with image $W$, and put

$$P' = \frac{1}{\#G} \sum_{g \in G} P^g.$$

Then $P'$ is again a projector with image $W$, but $P'$ is $G$-invariant. Hence $1 - P'$ is a $G$-invariant projector whose image is a complementary subrepresentation $W'$ of $W$, contradicting the indecomposability of $V$ and yielding the claim. \qed
We also have a form of Schur’s lemma, with the usual proof.

Lemma 4.2.4 (Schur’s lemma). Let $G$ be a finite group which acts on a field $F$, and let $V$ and $W$ be irreducible semilinear representations of $G$ over $F$. Then any $G$-equivariant linear map $f : V \to W$ is either zero or invertible. In particular, the set of $G$-endomorphisms of $V$ is a division algebra.

Proof. If $f$ is nonzero, then $\ker(f)$ is a proper subrepresentation of $V$ and so must vanish, as must $\text{im}(f)$. □

Remark 4.2.5. Note that semilinear representations may be viewed as 1-cocycles for $\text{GL}_n(F)$. In particular, if $G = \text{Gal}(F/E)$ for $F/E$ finite, then $H^1(G, \text{GL}_n(F))$ vanishes, so any semilinear representation of $G$ over $F$ is trivial.

Definition 4.2.6. Let $G \to H$ be a homomorphism of finite groups acting on the field $F$; this homomorphism induces a homomorphism $F\{G\} \to F\{H\}$ of noncommutative $F$-algebras. Given a semilinear representation $V$ of $G$, we define the induced representation $\text{Ind}^G_H V = V \otimes_{F\{G\}} F\{H\}$; given a semilinear representation $W$ of $H$, we define the restricted representation $\text{Res}^G_H W = W$ viewed as a right $F\{G\}$-module via the homomorphism $F\{G\} \to F\{H\}$.

Since the functors $\text{Ind}^G_H$ and $\text{Res}^G_H$ are left and right adjoints of each other [19 Proposition 3.8], one obtains the Frobenius reciprocity law as in the linear case.

Lemma 4.2.7 (Frobenius reciprocity). Let $G \to H$ be a homomorphism of finite groups acting on the field $F$, let $V$ be a semilinear representation of $H$, and let $W$ be a semilinear representation of $G$. Then there is a natural isomorphism

$$\text{Hom}_H(V, \text{Res}^G_H W) \cong \text{Hom}_G(\text{Ind}^G_H V, W).$$

Definition 4.2.8. Let $G$ be a finite group acting on the field $F$. The regular representation of $G$ is the semilinear representation corresponding to $F\{G\}$ viewed as a right module over itself.

Corollary 4.2.9. Let $G$ be a finite group acting on the field $F$. Then any irreducible semilinear representation of $G$ is isomorphic to a subrepresentation of the regular representation.

Proof. Let $V$ be an irreducible semilinear representation of $G$. By Frobenius reciprocity applied to the trivial group mapping into $H$, $\text{Hom}_F(F, V) \cong \text{Hom}_G(F\{G\}, V)$ is nonzero. Thus the decomposition of the regular representation into irreducibles must include a copy of $V$, or else $\text{Hom}_G(F\{G\}, V)$ would vanish by Schur’s lemma. □
4.3 Quasi-constant connections

Definition 4.3.1. For $E \in M$, we say $E$ is quasi-constant (or étale) if there exists $r \in (0,1) \cap \Gamma^*$ such that $E$ is defined on $A[r,1]$, and there exists a formally étale cover $f : X \to A[r,1]$ such that $f^*E$ is spanned by finitely many horizontal sections; if $R/k((t))$ is the reduction of $X$, we also say that $E$ is/becomes constant over $X$ or over $R$. Let $M^qc = M^qc_K$ denote the subcategory of $M$ consisting of quasi-constant objects; it is closed under formation of direct sums, tensor products, duals, subobjects, and quotients, though not under formation of extensions.

One can give a representation-theoretic description of quasi-constant $\nabla$-modules as follows.

Lemma 4.3.2. Let $R/k((t))$ be a finite Galois extension, and let $f : X \to A[a,1]$ be a formally étale cover with reduction $R$. Let $K'$ be the integral closure of $K$ in $\Gamma(O,X)$. Then the functor $E \mapsto H^0(E,X)$, from the category of $\nabla$-modules on $A[a,1]$ which become constant on $R$, to the category of semilinear representations of $G = \text{Gal}(R/k((t)))$ in finite dimensional $K'$-vector spaces, is an equivalence of categories.

Proof. First note that the functor is fully faithful: given $E, F$, we have

$$H^0(E,X)^{\vee} \otimes H^0(F,X) \cong H^0(E^{\vee} \otimes F,X)$$

since both sides are $K'$-vector spaces of dimension $(\text{rank } E)(\text{rank } F)$ and there is a natural injective map from the left side to the right. Hence any $G$-equivariant homomorphism between $H^0(E,X)$ and $H^0(F,X)$ corresponds to a $G$-equivariant horizontal section of $E^{\vee} \otimes F$ over $X$, and hence to a horizontal section $E^{\vee} \otimes F$ over $A[a,1]$. The latter corresponds to a morphism from $E$ to $F$, yielding the full faithfulness.

Next note that the functor is essentially surjective: put $F = f^* f_* O_X$, so that $H^0(F,X)$ is the regular representation of $G$ over $K'$. Let $V$ be an irreducible representation of $G$. By Corollary [12.2] we know that $V$ is isomorphic to a subrepresentation of $H^0(F,X)$. That is, we may choose a $G$-equivariant projector $P : H^0(F,X) \to H^0(F,X)$ with image isomorphic to $V$ as a $G$-representation. By the full faithfulness assertion, $P$ comes from a projector $F \to F$, whose image $G$ is a $\nabla$-module on $A[a,1]$ with $H^0(G,X) \cong V$ as a $G$-representation. Hence the functor is essentially surjective, and thus an equivalence of categories.

4.4 Local duality

Definition 4.4.1. For any $E \in M$, define the local duality pairing on $E$ to be the $K$-bilinear pairing

$$H^0(E^{\vee}) \times H^1(E) \to H^1(E^{\vee} \otimes E) \to H^1(O);$$

note again (as in Example [1.11]) that the latter may be identified with $K$ via the residue map taking $\sum c_i t^i \, dt$ to $c_{-1}$. We say $E$ is dualizable if the local duality pairing is perfect, i.e., if it induces an isomorphism $H^0(E^{\vee}) \cong H^1(E)^{\vee}$.
Remark 4.4.2. Note that if $E = E_1 \oplus E_2$ in $\mathcal{M}$, then $E$ is dualizable if and only if $E_1$ and $E_2$ are both dualizable, because the formation of $H^0$ and $H^1$ commutes with direct sums. Also, if $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence in $\mathcal{M}$, and $E_1$ and $E_2$ are both dualizable, then $E$ is also dualizable: this follows from applying the snake lemma to the diagram

\[
\begin{array}{ccccccccc}
0 & \to & H^0(\mathcal{E}^\vee) & \to & H^0(\mathcal{E}) & \to & H^0(\mathcal{E}^\vee) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^1(\mathcal{E}_2)^\vee & \to & H^1(\mathcal{E})^\vee & \to & H^1(\mathcal{E}_1)^\vee & \to & 0.
\end{array}
\]

Proposition 4.4.3. Any $E \in \mathcal{M}^\text{qc}$ is dualizable; moreover, any $E \in \mathcal{M}$ which admits a filtration whose successive quotients are quasi-constant is also dualizable.

Proof. The second assertion follows from the first by Remark 4.4.2, so we will stick to considering $E \in \mathcal{M}^\text{qc}$. Choose a formally étale cover $f : X \to A[0, 1)$ over which $E$ becomes constant. As noted in Definition 4.1.10, the formation of $H^0$ and $H^1$ commutes with tensoring over $K$ with a finite Galois extension of $K$, so we may enlarge $K$ as needed to ensure that $X \cong A[0, 1)$ for some $b$.

As in the proof of Lemma 4.3.2, we see that $E$ is a direct summand of a sum of copies of $f_* \mathcal{O}_X$. By Remark 4.1.2, to verify that $E$ is dualizable, it suffices to verify that $f_* \mathcal{O}_X$ is dualizable. But $H^i(f_* \mathcal{O}_X) = H^i(\mathcal{O}_X)$ for $i = 0, 1$, so it suffices to verify that $\mathcal{O}_X$ is dualizable; this in turn follows from the observation of Example 4.1.11 that $H^0(\mathcal{O}) = H^1(\mathcal{O}) = K$. \qed

4.5 Unipotent connections

Definition 4.5.1. For $E \in \mathcal{M}_K$, we say $E$ is unipotent if $E$ admits a filtration whose successive quotients are constant. Let $\mathcal{M}^\text{unip} = \mathcal{M}^\text{unip}_K$ denote the subcategory of $\mathcal{M}$ consisting of unipotent objects; it is closed under formation of direct sums, tensor products, duals, subobjects, quotients, and extensions.

Unipotent $\nabla$-modules can be characterized in terms of logarithmic connections.

Definition 4.5.2. Let $\Omega^1_\log$ be the coherent sheaf on $A[0, 1)$ freely generated by $\frac{dt}{t}$; we view $\Omega^1$, which is freely generated by $dt$, as a subsheaf of $\Omega^1_\log$. Of course the two coincide away from $t = 0$ (i.e., on $A(0, 1)$). A log-$\nabla$-module on $A[0, 1)$ is a coherent locally free sheaf $E$ equipped with a connection $\nabla : E \to E \otimes \Omega^1_\log$. If $E$ is a log-$\nabla$-module, then $t \frac{d}{dt}$ acts linearly on $\Gamma(E, A[0, 0]) = E_0$ (the stalk of $E$ at $t = 0$), which is a finite dimensional vector space over $K$. We call this vector space equipped with a linear transformation the residue of $E$ (or of $\nabla$).

One then has the following characterization (compare [26, Theorem 4.1] and the discussion in [23, Chapter 3]).

Proposition 4.5.3. The following categories are equivalent.
(a) The category of finite dimensional $K$-vector spaces equipped with nilpotent endomorphisms.

(b) The category of log-$\nabla$-modules on $A[0,1)$ with nilpotent residue, whose restrictions to $A(0,1)$ are overconvergent.

(c) The category of unipotent $\nabla$-modules on $A(I)$ for any aligned subinterval $I$ of $(0,1)$.

Proof. We first exhibit the functor from (a) to (b). Given a finite dimensional $K$-vector space $V$, put $M = R_{(0,1)} \otimes_K V$ and let $E$ be the sheaf on $A(0,1)$ associated to $M$. We may define a connection on $M$ by declaring that the action of $t\frac{d}{dt}$ on $V$ is via the given nilpotent endomorphism, then extending via the Leibniz rule. The resulting log-$\nabla$-module is overconvergent because it is a successive extension of trivial $\nabla$-modules, and the property of overconvergence is stable under extensions.

We next exhibit the functor from (b) to (c). Let $E$ be a log-$\nabla$-module on $A[0,1)$ of rank $n$ with nilpotent residue, let $m$ be the index of nilpotency of the residue map, and let $D$ denote the operator induced by $t\frac{d}{dt}$. Let $P_i$ denote the $i$-th binomial polynomial

$$P_i(x) = \frac{x(x-1) \cdots (x-i+1)}{i!};$$

then the $P_i$ form a $\mathbb{Z}$-basis for the set of integral-valued polynomials in $\mathbb{Q}[x]$. Let $Q_i$ denote the polynomial

$$Q_i(x) = x^{m-1} \left( \frac{(1-x) \cdots (i-x)}{i!} \right)^n;$$

then $Q_{i+1}(x) - Q_i(x)$ is integral-valued of degree $(i+1)n + m - 1$ and vanishes at $0, 1, \ldots, i$, so is an integral linear combination of $P_{i+1}, \ldots, P_{(i+1)n+m-1}$.

By computing on formal power series in $t$ (with which we can formally construct a basis of sections killed by $D$), we see that for any $\eta \in (0,1)$, there exists $b \in [0,1) \cap \Gamma^*$ such that for any $c \in (b,1) \cap \Gamma^*$ and any $v \in \Gamma(E, A[b,c])$, the sequence

$$\left\{ \frac{1}{t^l} P_l(D) \right\}_{i=1}^{\infty}$$

is a well-defined operator on $E$. As we saw above, $\frac{1}{t^l} (Q_{i+1} - Q_i)(D)$ is an integer linear combination of $\frac{1}{t^l+1} P_l(D)$ for $l = i+1, \ldots, (i+1)n + m - 1$.

However,

$$\frac{1}{t^l} P_l(D) = \frac{1}{l!} \frac{d^l}{dt^l};$$

by the overconvergence condition, for any $\eta \in (0,1)$, there exists $b \in [0,1) \cap \Gamma^*$ such that for any $c \in (b,1) \cap \Gamma^*$ and any $v \in \Gamma(E, A[b,c])$, the sequence

$$\left\{ \frac{1}{t^l} \frac{d^l}{dt^l} v \right\}_{i=1}^{\infty}$$

is an integer linear combination of $\frac{1}{t^l+1} P_l(D)$ for $l = i+1, \ldots, (i+1)n + m - 1$.
is \(\eta\)-null. It follows that the sequence
\[
\{t^{-i-1}(Q_{i+1} - Q_i)(D)v\}
\]
is also \(\eta\)-null over \(A[b, c]\).

Now choose \(v \in \Gamma(\mathcal{E}, A[0, c])\) with \(c \in (\eta, 1) \cap \Gamma^*\). By Lemma 3.2.2, the sequence (4.5.1) is \(\eta\)-null not just over \(A[b, c]\), but also over \(A[0, c]\). In particular, the sequence (4.5.1) is \(\eta\)-null over \(A[0, \eta]\), and so the sequence \(\{(Q_{i+1} - Q_i)(D)v\}\) is 1-null over \(A[0, \eta]\). That is, the limit
\[
f(v) = \lim_{i \to \infty} Q_i(D)v
\]
exists in \(\Gamma(\mathcal{E}, A[0, \eta])\).

Again from the formal power series computation, we see that \(f\) acts as the \((m - 1)\)-st power of the residue map modulo \(t\), and that \(Df(v) = 0\) for all \(v\). We can find some \(v \in \Gamma(\mathcal{E}, A[0, c])\) whose fibre at zero is not killed by the \((m - 1)\)-st power of the residue map; then \(f(v) \neq 0\) but \(Df(v) = 0\). That is, the log-\(\nabla\)-submodule of \(\mathcal{E}\) spanned by the kernel of \(\nabla\) on \(A[0, c]\) is nonzero. The rank of that submodule does not increase as \(c\) increases, so it must be constant for \(c \in (0, 1) \cap \Gamma^*\) sufficiently close to 1. Thus these submodules fit together to yield a constant \(\nabla\)-submodule of \(\mathcal{E}\) on \(A[0, 1]\); quotienting by this submodule and repeating the argument, we obtain the desired unipotent filtration.

Finally, we note that the functor from (c) to (a) is straightforward: it suffices to verify that on each open subinterval of \(I\), \(\mathcal{E}\) is spanned by sections killed by \(D^n\) with \(n = \text{rank}(\mathcal{E})\). That in turn follows by induction on rank, using the fact that for \(I\) open, the cokernel of \(\frac{d}{dt}t\) on \(\mathcal{R}_I\) is generated over \(K\) by 1.

**Remark 4.5.4.** This discussion goes over to higher-dimensional polydiscs; see [23, Chapter 3]. We recall also a remark from [23, Chapter 3]: the application of Lemma 3.2.2 must be to a sequence without poles, which necessitates the introduction of the sequence (4.5.1) in lieu of working directly on the sequence \(\{\frac{d}{dt}t\}v\). Proposition 4.5.3 reduces most questions about unipotent \(\nabla\)-modules to linear algebra, as in the following special case of [26, Lemma 7.6].

**Lemma 4.5.5.** If \(\mathcal{U} \in \mathcal{M}^{\text{unip}}\) is nonzero and indecomposable, then \(\dim_K \text{Ext}^i(\mathcal{U}) = 1\) for \(i = 0, 1\).

**Proof.** Note that any element of \(\mathcal{M}^{\text{unip}}\) is dualizable by Proposition 4.4.3, so it suffices to prove the claim for \(i = 0\). By Proposition 4.5.3 we may translate the claim into linear algebraic terms: the result is simply the fact that a nilpotent linear transformation on a finite dimensional vector space is indecomposable if and only if it can be written as a single Jordan block.

We also need the following result on the interaction between quasi-constant and unipotent \(\nabla\)-modules.

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Lemma 4.5.6. For $P \in \mathcal{M}^{qc}$ and $U \in \mathcal{M}^{unip}$, the natural map $H^0(P) \otimes_K H^0(U) \to H^0(P \otimes U)$ is an isomorphism of $K$-vector spaces.

Proof. If $U = \mathcal{O}$, this is obvious; otherwise, we proceed by induction on $\text{rank}(U)$. Choose a short exact sequence $0 \to \mathcal{O} \to U \to U_1 \to 0$, and consider the diagram

$$
\begin{array}{cccccc}
0 & \to & H^0(P) & \to & H^0(P) \otimes H^0(U) & \to & H^0(P) \otimes H^0(U_1) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(P) & \to & H^0(P \otimes U) & \to & H^0(P \otimes U_1) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & & & H^1(P) & & \\
\end{array}
$$

(4.5.2)

in which the first row is obtained by applying the snake lemma to the diagram

$$
\begin{array}{cccccc}
0 & \to & \mathcal{O} & \to & U & \to & U_1 \\
& & \downarrow d & & \downarrow \nabla & & \downarrow \nabla \\
0 & \to & \Omega^1 & \to & U \otimes \Omega^1 & \to & U_1 \otimes \Omega^1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & & & 0 & & \\
\end{array}
$$

(4.5.3)

and then tensoring with $H^0(P)$, whereas the second row is obtained by first tensoring (4.5.3) with $P$ and then applying the snake lemma. In (4.5.2), the first vertical arrow is visibly an isomorphism, and the third is an isomorphism by the induction hypothesis. As for the fourth arrow, note that $H^1(P) \cong H^0(P^\vee)^\vee$ by Proposition 4.4.3, whereas by Lemma 4.3.2, forming $H^0$ of a quasi-constant $\nabla$-module commutes with taking duals (since a semilinear representation is trivial if and only if its dual is trivial). Hence the fourth vertical arrow is also an isomorphism; by the five lemma, the arrow $H^0(P) \otimes H^0(U) \to H^0(P \otimes U)$ is an isomorphism, as desired. 

4.6 Quasi-unipotent connections

Definition 4.6.1. For $E \in \mathcal{M}$, we say $E$ is quasi-unipotent if $E$ admits a filtration whose successive quotients are quasi-constant; if each successive quotient becomes constant over $X$ or over $R$, we say we say $E$ is/becomes unipotent over $X$ or over $R$. Let $\mathcal{M}^{\text{qu}} = \mathcal{M}^{\text{qu}}_K$ denote the subcategory of $\mathcal{M}_K$ consisting of quasi-unipotent objects; it is closed under formation of direct sums, tensor products, duals, subobjects, quotients, and extensions. Note that any element of $\mathcal{M}^{\text{qu}}$ is dualizable by Proposition 4.4.3.

Remark 4.6.2. The definition of quasi-unipotence in [20] is slightly different: there it is required that $E$ admit a unipotent filtration over $X$ for some formally étale cover $f : X \to A[r, 1]$. This clearly follows from the existence of a filtration over $A[r, 1]$ whose successive quotients are quasi-constant, but in fact the reverse is also true: if $E$ admits a unipotent filtration over $X$, then there is a unique such filtration of shortest length, namely the one whose first step is spanned by the full kernel of $\nabla$ over $X$, and so on. By Galois descent, this filtration descends to $A[r, 1]$.

Quasi-unipotent $\nabla$-modules can be canonically decomposed into “isotypical” pieces.
Definition 4.6.3. Suppose $\mathcal{F} \in \mathcal{M}$ is irreducible. We say that $\mathcal{E} \in \mathcal{M}$ is $\mathcal{F}$-typical if $\mathcal{E}$ admits a filtration whose successive quotients are all isomorphic to $\mathcal{F}$.

In this language, we have the following partial analogue of [26, Lemma 7.6].

Lemma 4.6.4. Suppose that $\mathcal{E}_j$ is $\mathcal{F}_j$-typical for $j = 1, 2$, and that $\mathcal{F}_1 \not\cong \mathcal{F}_2$. Then $\text{Ext}^i(\mathcal{E}_1, \mathcal{E}_2) = 0$ for $i = 0, 1$ (in the category $\mathcal{M}$).

Proof. As noted earlier (see Definition 4.1.10), $\text{Ext}^i(\mathcal{E}_1, \mathcal{E}_2) \cong H^i(\mathcal{E}_1^\vee \otimes \mathcal{E}_2)$. By Proposition 4.4.3, it suffices to check the claim for $i = 0$, in which case it follows by induction on rank plus Schur’s lemma.

Proposition 4.6.5. Each $\mathcal{E} \in \mathcal{M}$ admits a unique (up to reordering) decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$, in which each $\mathcal{E}_i$ is $\mathcal{F}_i$-typical for some irreducible $\mathcal{F}_i \in \mathcal{M}$, and no two of the $\mathcal{F}_i$ are isomorphic.

Proof. For existence, we proceed by induction on rank($\mathcal{E}$). If $\mathcal{E}$ is irreducible, there is nothing to check; otherwise, choose an exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}' \to 0$ with $\mathcal{F}$ irreducible. By the induction hypothesis, $\mathcal{E}'$ admits a decomposition $\mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_m$ of the desired form. Let $\mathcal{E}_i$ be the inverse image of $\mathcal{E}'_i$ in $\mathcal{E}$; for all but possibly one index $i$, $\mathcal{E}_i$ is $\mathcal{F}_i$-isotypical for some $\mathcal{F}_i \not\cong \mathcal{F}$, and so the exact sequence $0 \to \mathcal{F} \to \mathcal{E}_i \to \mathcal{E}'_i \to 0$ splits by Lemma 4.6.4. We may thus decompose $\mathcal{E}$ in the desired form.

For uniqueness, note that if $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m \cong \mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_m$, then the isomorphism must carry $\mathcal{E}_i$ into an $\mathcal{F}_i$-typical summand by Lemma 4.6.4, and so on.

One can further refine Proposition 4.6.5; the result is essentially [26, Theorem 7.8].

Theorem 4.6.6 (Matsuda). Every $\mathcal{E} \in \mathcal{M}_{\mathcal{K}}$ admits a canonical decomposition as a direct sum $\sum_i \mathcal{F}_i \otimes \mathcal{U}_i$, where each $\mathcal{F}_i$ is quasi-constant and irreducible, each $\mathcal{U}_i$ is unipotent, and no two of the $\mathcal{F}_i$ are isomorphic.

Proof. By virtue of Proposition 4.6.5, it suffices to check that for $\mathcal{F} \in \mathcal{M}_{\mathcal{K}}$ irreducible, any $\mathcal{F}$-typical element $\mathcal{E}$ of $\mathcal{M}$ has the form $\mathcal{F} \otimes \mathcal{U}$ for some $\mathcal{U} \in \mathcal{M}_{\text{unip}}$. We check this by induction on rank($\mathcal{E}$); we may of course assume $\mathcal{E}$ is indecomposable.

If $\mathcal{E}$ is irreducible, the claim is clear; otherwise, construct a short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{F} \to 0$, and note that $\mathcal{E}_1$ is $\mathcal{F}$-typical. By the induction hypothesis, we have $\mathcal{E}_1 \cong \mathcal{F} \otimes \mathcal{U}_1$ for some $\mathcal{U}_1 \in \mathcal{M}_{\text{unip}}$, necessarily indecomposable. By Proposition 4.4.3 and Lemma 4.5.6

$$\text{Ext}^1(\mathcal{F}, \mathcal{E}_1) = H^1(\mathcal{F}^\vee \otimes \mathcal{E}_1)$$
$$= H^0(\mathcal{E}_1^\vee \otimes \mathcal{F})^\vee$$
$$= H^0(\mathcal{F}^\vee \otimes \mathcal{F} \otimes \mathcal{U}_1^\vee)^\vee$$
$$= H^0(\mathcal{F}^\vee \otimes \mathcal{F})^\vee \otimes H^0(\mathcal{U}_1^\vee)$$
$$= H^0(\mathcal{F}^\vee \otimes \mathcal{F})^\vee \otimes H^1(\mathcal{U}_1)$$
$$= H^0(\mathcal{F}^\vee \otimes \mathcal{F})^\vee \otimes \text{Ext}^1(\mathcal{O}, \mathcal{U}_1).$$
Let $\mathcal{U}_1 = \oplus_i \mathcal{U}_{1,i}$ be the decomposition of $\mathcal{U}_1$ into indecomposables. By Lemma 4.5.5, for each $i$, $\dim_K \text{Ext}^1(\mathcal{O}, \mathcal{U}_{1,i}) = 1$. We may thus adjust each component of $\mathcal{E}_1 \cong \oplus_i (F \otimes \mathcal{U}_{1,i})$ by an automorphism of $F$ to ensure that the element of $\text{Ext}^1(F, \mathcal{E}_1)$ corresponding to the exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to F \to 0$ maps to an element of $H^0(F^\vee \otimes F)^\vee \otimes \text{Ext}^1(\mathcal{O}, \mathcal{U}_1)$ in the image of the map $1 \otimes \text{id}$. In this case, the exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to F \to 0$ is obtained from some exact sequence $0 \to \mathcal{U}_1 \to \mathcal{U} \to \mathcal{O} \to 0$ by tensoring with $F$; in particular, $\mathcal{E} \cong \mathcal{U} \otimes F$, as desired.

Remark 4.6.7. One can deduce from Theorem 4.6.6 that every quasi-unipotent $\nabla$-module is overconvergent, by checking in the unipotent and quasi-constant cases. In the latter case, one can construct a so-called “unit-root” Frobenius structure as in [26, Lemma 5.3] and then apply Lemma 6.1.4 below; alternatively, one can check that the overconvergence property descends down formally étale covers.

4.7 Monodromy representations

Theorem 4.6.6 allows us to describe the category of quasi-unipotent $\nabla$-modules in representation-theoretic terms, as follows.

Definition 4.7.1. Put $G_{\log} = G_{K}' = \text{Gal}(k((t))^{sep}/k((t))) \times K$, where the second factor represents the additive group. A semilinear representation of $G_{\log}$ on a finite dimensional $K$-vector space is permissible if its restriction to some open subgroup of $\text{Gal}(k((t))^{sep}/k((t)))$ is trivial (in the sense of being isomorphic to a product of copies of $K'^{unr}$), and its restriction to $K$ is algebraic (and hence necessarily unipotent).

Definition 4.7.2. Define the sheaf $\mathcal{O}_{\log}$ on $A[a, 1)$ by the formula

$$\Gamma(\mathcal{O}_{\log}, A[b, c]) = \Gamma(\mathcal{O}, A[b, c])[\log(t)],$$

where $\log(t)$ is an indeterminate; define the sheaf $\mathcal{O}_{\log}$ on a finite étale cover of $A[a, 1)$ as the pullback from $A[a, 1)$. Extend $d$ to a $K'$-derivation $\mathcal{O}_{\log} \to \Omega^1_{\log}$ by setting $d(\log(t)) = \frac{dt}{t}$. For $\mathcal{E} \in \mathcal{M}$ and $f : X \to A[a, 1)$ a formally étale cover, let $H^0_{\log}(\mathcal{E}, X)$ denote the kernel of $\nabla$ on $\Gamma(\mathcal{O}_{\log}, X)$.

Remark 4.7.3. Beware that one has a canonical isomorphism $\tau^* \mathcal{O}_{\log} \to \mathcal{O}_{\log}$ (i.e., an action of $\tau$ on $\mathcal{O}_{\log}$) not for an arbitrary automorphism $\tau$ of $A[a, 1)$, but only for those $\tau$ satisfying $|\tau^*(t) - t|_r < 1$ for $r \in [a, 1)$; the point is that these are the $\tau$ for which the series defining $\log(\tau^*(t)/t)$ converges in $\Gamma(a, A[a, 1))$.

Lemma 4.7.4. Let $f : X \to A[a, 1)$ be a formally étale cover over which $\mathcal{E} \in \mathcal{M}^{\text{uni}}$ becomes unipotent, and let $K'$ be the integral closure of $K$ in $\Gamma(\mathcal{O}, X)$. Then $\dim_K H^0_{\log}(\mathcal{E}, X) = \text{rank}(\mathcal{E})$.

Proof. By Theorem 4.6.6, it is enough to verify this for $\mathcal{E}$ quasi-constant, in which case it is evident, and for $\mathcal{E}$ unipotent, in which case it follows from Proposition 4.5.3. \qed

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Definition 4.7.5. For $\mathcal{E} \in \mathcal{M}^{\text{qu}}$, let $f : X \to A[a, 1)$ be a formally étale cover over which $\mathcal{E}$ becomes unipotent, and let $K'$ be the integral closure of $K$ in $\Gamma(\mathcal{O}, X)$. Then we may view $\
abla$ as a semilinear representation of $G^\log$, where the action of $c \in K$ is by the map $\log(x) \mapsto \log(x) + c$. This representation does not depend on the choice of $X$; we call it the monodromy representation of $\mathcal{E}$.

We can now give our representation-theoretic description of $\mathcal{M}^{\text{qu}}$.

Theorem 4.7.6. The monodromy representation, viewed as a functor from $\mathcal{M}^{\text{qu}}$ to the category of permissible semilinear representations of $G^\log$, is an equivalence of categories.

Proof. The full faithfulness of the functor follows as in the proof of Lemma 4.3.2. As for the essential surjectivity, note that any indecomposable representation of the product group $G^\log$ is the tensor product of a permissible representation of each factor. Each permissible representation of $\text{Gal}(k((t))^{\text{sep}}/k((t)))$ occurs in $\mathcal{M}^{\text{qu}}$ by Lemma [4.3.2] each permissible representation of $K$ corresponds to a nilpotent linear transformation, and hence to a unipotent $\nabla$-module by Proposition [4.5.3]. Thus the functor is essentially surjective, completing the proof. \hfill \qed

Remark 4.7.7. For $k$ algebraically closed (and $K$ discretely valued, though this is less critical), Theorem [4.7.6] is essentially due to André [1, Théorème 7.1.1].

5 Radius of convergence and ramification

In this chapter, we recall a method of Christol and Mebkhout that attaches numerical invariants to a quasi-unipotent $\nabla$-module, and a theorem of Matsuda that relates these to the ramification breaks. Note that our derivation of Matsuda’s theorem is direct, and does not depend per se on the theory of Christol-Mebkhout.

We retain all notations and conventions from the preceding chapters, notably Convention [4.0.8]

5.1 Generic radius of convergence

We first recall the notion of “generic radius of convergence” from Christol-Dwork [10].

Definition 5.1.1. The operator norm $|T|_{\text{op}}$ of a continuous operator $T$ on a normed space $V$ is equal to the smallest nonnegative real number $c$ such that $|Tv| \leq c|v|$ for all $v \in V$. For instance, if $V = \mathbb{R}$ equipped with the norm $|\cdot|_r$ for some $r \in I$, and $T = \frac{d}{dt}$, then $|Tv|_r \leq r^{-1}|v|_r$ with equality for $v = t$, so the operator norm equals $r^{-1}$. 
Definition 5.1.2. The spectral norm $|T|_{sp}$ of a continuous operator $T$ on a normed space $V$ is defined as

$$|T|_{sp} = \limsup_{s \to \infty} |T^s|_\text{op}^{1/s}.$$ 

For instance, if $V = \mathcal{R}_I$ equipped with the norm $| \cdot |_r$ for some $r \in I$, and $T = \frac{d}{dt}$, then $|T^s v|_r \leq r^{-s} |s!||v|_r$ with equality for $v = t^s$, so

$$\left| \frac{d}{dt} \right|_{sp} = \limsup_{s \to \infty} (r^{-s} |s!|)^{1/s} = p^{-1/(p-1)} r^{-1}.$$

Definition 5.1.3. Let $I$ be a closed aligned subinterval of $[0, 1)$, and let $\mathcal{E}$ be a $\nabla$-module over $A(I)$; put $M = \Gamma(\mathcal{E}, A_K(I))$. Since $\mathcal{R}_I$ is a principal ideal domain (Proposition 3.2.3), $M$ is free over $\mathcal{R}_I$; let $e_1, \ldots, e_n$ be a basis. Let $T$ denote the operator induced by $\frac{d}{dt}$ on $M$, and define the matrix $D_s$ by

$$T^s e_j = \sum_i (D_s)_{ij} e_i.$$

For $\rho \in I$, define the $\rho$-spectral norm of $T$ (with respect to $e_1, \ldots, e_n$) to be

$$|T|_{\rho, sp} = \min \{ p^{-1/(p-1)} \rho^{-1}, \limsup_{s \to \infty} \max_{i,j} |(D_s)_{ij}|^{1/s} \};$$

then a short calculation [10, Proposition 1.2] shows that the $\rho$-spectral norm is independent of the choice of basis.

Definition 5.1.4. Let $I$ be an aligned subinterval of $[0, 1)$, and let $\mathcal{E}$ be a $\nabla$-module on $A(I)$. Define the function $\rho \mapsto R(\mathcal{E}, \rho)$ on $I$ as follows:

$$R(\mathcal{E}, \rho) = p^{-1/(p-1)} |T|_{\rho, sp}^{-1},$$

where $T$ is the operator induced by $\frac{d}{dt}$ on $\Gamma(\mathcal{E}, J)$ for any closed aligned subinterval $J$ of $I$ containing $\rho$. (Note that the choice of $J$ does not matter: one can compare the spectral norms obtained for one $J$ and a strictly smaller $J$ by using the same basis to compute them. Indeed, one could even take $J = [\rho, \rho]$.)

Remark 5.1.5. The function $\rho \mapsto R(\mathcal{E}, \rho)$ is log-concave [10, Proposition 2.3], so in particular it is continuous on the interior of $I$. It also turns out to be continuous at the endpoints of $I$, but this is subtler [10, Théorème 2.3]. One can make similar continuity statements using the Berkovich foundations of rigid analytic geometry, but they carry somewhat more content; see [2].

Remark 5.1.6. In terms of the function $R$, the condition that $\mathcal{E}$ be overconvergent in our sense is precisely that $\lim_{\rho \to 1} R(\mathcal{E}, \rho) \rho^{-1} = 1$; this is literally the condition that $\mathcal{E}$ be “soluble at 1” in the terminology of [11, 4.1-1].
5.2 Generic points

The interpretation of $R(\mathcal{E}, \rho)$ as a “generic radius of convergence” is as follows.

**Definition 5.2.1.** Let $\mathbb{C}$ be an algebraically closed extension of $K$ complete for an absolute value extending $| \cdot |$ on $K$, whose residue field is transcendental over that of $K$. For $\rho \in (0, 1) \cap \Gamma^*$, a generic point of radius $\rho$ is an element $t_\rho \in \mathbb{C}$ such that $|t_\rho - x| = \rho$ for any $x \in K_{\text{alg}}$ with $|x| \leq \rho$. (Since this includes $x = 0$, we have in particular that $|t_\rho| = \rho$.)

**Lemma 5.2.2.** Let $I$ be an aligned subinterval of $(0, 1)$, and suppose $\rho \in I \cap \Gamma^*$. For any $x = \sum c_i t^i \in \mathcal{R}_I$ and any generic point $t_\rho \in \mathbb{C}$ of radius $\rho$,

$$|x|_\rho = \left| \sum c_i t^i_\rho \right|.$$

**Proof.** Since the supremum $\sup\{|c_i| \rho^j\}$ defining $|x|_\rho$ is achieved by at least one $i$, it suffices to check the equality for $x \in K[t, t^{-1}]$. Moreover, multiplying $x$ by $t$ multiplies both sides of the proposed equality by $\rho$, so we may reduce to the case where $x \in K[t]$.

Factor $x = c_n \prod_{j=1}^n (t - z_j)$ with each $z_j \in K_{\text{alg}}$. Then $|t_\rho - z_j| = \max\{\rho, |z_j|\} = |t - z_j|_\rho$ for each $j$, because $t_\rho$ is a generic point. Hence

$$|x|_\rho \geq \left| \sum c_i t^i_\rho \right| = |c_n| \prod_{j=1}^n |t_\rho - z_j| \geq |x|_\rho,$$

yielding the desired equality.

**Proposition 5.2.3.** Let $I$ be an aligned subinterval of $[0, 1)$, and let $\mathcal{E}$ be a $\nabla$-module on $A(I)$. For $\rho \in I \cap \Gamma^*$, $t_\rho \in \mathbb{C}$ a generic point of radius $\rho$, and $r \in \mathbb{R}$, the following are equivalent.

(a) $|R(\mathcal{E}, \rho)| \geq r$;

(b) for any $z \in \mathbb{C}$ with $|z| = \rho$, $\mathcal{E}$ admits a basis of horizontal sections in the disc $|t - z| < r$;

(c) $\mathcal{E}$ admits a basis of horizontal sections in the disc $|t - t_\rho| < r$.

**Proof.** Note that for $v \in \mathcal{E}$, the sum

$$f(v) = \sum_{s=0}^{\infty} \frac{(t - z)^s}{s!} \frac{d^s}{dt^s} v,$$
if convergent, yields a horizontal section of $E$ on a disc centered at $z$. Thus given (a), we may evaluate $f(v)$ on any basis of $E$ to obtain horizontal sections in the disc $|t - z| < r$; that is, (a) implies (b). Also, (b) implies (c) trivially. On the other hand, given (c), one deduces $|R(E, \rho)| \geq r$ from Lemma 5.2.2.

Corollary 5.2.4. Given an automorphism $\phi$ of $A[0,1)$, there exists $\rho_0 \in [0,1) \cap \Gamma^*$ such that $R(E, \rho) = R(\phi^* E, \rho)$ for $\rho \in [\rho, 1) \cap \Gamma^*$.

Proof. This follows from Proposition 5.2.3 and the fact that $\phi$ preserves $A[\rho_0, 1)$ for $\rho_0$ sufficiently close to 1.

5.3 Formally étale covers and generic points

We will need to make a few calculations concerning the way discs around generic points transform under formally étale covers.

Lemma 5.3.1. Let $n$ be an integer relatively prime to $p$. For $\rho \in (0, \infty) \cap \Gamma^*$, choose $t_\rho \in \mathbb{C}$ with $|t_\rho| = \rho$, and choose an $n$-th root $t_\rho^{1/n}$ of $t_\rho$. Then for any $r \in (0,1] \cap \Gamma^*$, the map $z \mapsto z^n$ induces an isomorphism between the discs

$$|t - t_\rho^{1/n}| < r\rho^{1/n} \quad \text{and} \quad |t - t_\rho| < r\rho.$$

Proof. If $|z - t_\rho^{1/n}| < \rho^{1/n}$, then

$$|z^n - t_\rho| = \rho \left| \frac{1}{n} \left( (z - t_\rho^{1/n})/t_\rho^{1/n} \right)^n - 1 \right|
= \rho \sum_{i=1}^{\infty} \binom{n}{i} \left( (z - t_\rho^{1/n})/t_\rho^{1/n} \right)^i
= \rho |z - t_\rho^{1/n}|/t_\rho^{1/n}$$

since $n$ is relatively prime to $p$. Thus the map $z \mapsto z^n$ induces a map from the disc $|t - t_\rho^{1/n}| < r\rho^{1/n}$ into the disc $|t - t_\rho| < r\rho$.

We define the inverse map by the binomial series

$$z \mapsto t_\rho^{1/n} \sum_{i=0}^{\infty} \binom{1/n}{i} \left( \frac{z}{t_\rho} - 1 \right)^i.$$

If $|z - t_\rho| < r\rho$, then $|(z/t_\rho) - 1| < r$, so the series converges to a value in the disc $|t - t_\rho^{1/n}| < r\rho^{1/n}$. This yields the desired result.

The situation is a bit different when $n$ is not coprime to $p$; it will be enough for us to consider $n = p$. 


Lemma 5.3.2. For \( \rho \in (0, \infty) \cap \Gamma^* \), choose \( t_\rho \in \mathbb{C} \) with \( |t_\rho| = \rho \), and choose a \( p \)-th root \( t_\rho^{1/p} \) of \( t_\rho \). Then for any \( r \in (0, p^{1/(p-1)}] \cap \Gamma^* \), the map \( z \mapsto z^p \) induces an isomorphism between the discs

\[
|t - t_\rho^{1/p}| < r \rho^{1/p} \quad \text{and} \quad |t - t_\rho| < r \rho;
\]

for \( r \in (p^{-p/(p-1)}, 1) \cap \Gamma^* \), the disc

\[
|t - t_\rho^{1/p}| < r \rho^{1/p} \quad \text{is carried into the disc} \quad |t - t_\rho| < r^p \rho.
\]

Proof. If \( |z - t_\rho^{1/p}| < p^{-p/(p-1)} \rho^{1/p} \), then

\[
|z^p - t_\rho| = \rho |(1 + (z - t_\rho^{1/p})/t_\rho^{1/p})^p - 1|
\]

\[
= \rho \sum_{i=1}^{p} \binom{p}{i} |(z - t_\rho^{1/p})/t_\rho^{1/p}|^i
\]

\[
= \rho p (z - t_\rho^{1/p})/t_\rho^{1/p}.
\]

Thus for \( r \leq p^{-p/(p-1)} \), the map \( x \mapsto x^p \) induces a map from the disc \( |t - t_\rho^{1/p}| < r \rho^{1/p} \) into the disc \( |t - t_\rho| < r \rho \). For \( r > p^{-p/(p-1)} \), the argument breaks down because the dominant term in the binomial expansion is no longer the first term, but the last term; however, we still conclude that the disc \( |t - t_\rho^{1/p}| < r \rho^{1/p} \) maps into the disc \( |t - t_\rho| < r^p \rho \).

For \( r \leq p^{-p/(p-1)} \), we again define the inverse map by the binomial series

\[
z \mapsto t_\rho^{1/p} \sum_{i=0}^{\infty} \frac{1/p}{i} \left( \frac{z}{t_\rho^{1/p}} - 1 \right)^i.
\]

This time, however, if \( |z - t_\rho| < r \rho \), then \( |(z/t_\rho) - 1| < r < p^{-p/(p-1)} \), so the series converges to a value in the disc \( |t - t_\rho^{1/p}| < r \rho^{1/p} \). This yields the desired result. \( \square \)

Finally, we consider a cover corresponding to a wildly ramified cover of \( \text{Spec } k((t)) \).

Lemma 5.3.3. Suppose that \( K \) contains an element \( \pi \) with \( \pi^{p-1} = -p \). For \( \rho \in (p^{-p/(p-1)}, 1) \cap \Gamma^* \), choose \( t_\rho \in \mathbb{C} \) with \( |t_\rho| = \rho \), and choose \( u_\rho \in \mathbb{C} \) such that \((1 + \pi u_\rho)^p = 1 + p \pi t_\rho^{-1} \). Then \( |u_\rho| = \rho^{-1/p} \), and for \( r \in [0, \rho] \), the ring inclusion

\[
\mathcal{R}_{(\rho, 1)} \to \mathcal{R}_{(\rho, 1)}[u]/((1 + \pi u)^p - (1 + p \pi t^{-1}))
\]

induces an isomorphism between the discs

\[
|u^{-1} - u_\rho^{-1}| < r \rho^{(2-p)/p} \quad \text{and} \quad |t - t_\rho| < r \rho.
\]

Proof. The polynomial defining \( u_\rho \) may be rewritten

\[
0 = 1 + \sum_{i=1}^{p} \binom{p}{i} (u_\rho^{1/p})^i (t_\rho^{1/p} \pi^{-1})^{p-i},
\]

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and the resulting polynomial in $u^t_\rho^{1/p}$ has vertices corresponding to the terms of degree 0
and $p$. Hence $|u^t_\rho^{1/p}| = 1$, i.e., $|u^t_\rho| = \rho^{-1/p}$.

By Lemma 5.3.1, the discs $|t - t^\rho| < r\rho$ and $|t^{-1} - t^{-\rho}| < r\rho^{-1}$ are identical, and the discs
$|u^{-1} - u^{-\rho}| < r\rho^{(1-(p-1))/p}$ and $|u - u^\rho| < r\rho^{((-p-1))/p}$ are identical. By Lemma 5.3.2 (and
the fact that $|1 + \pi u| = 1$ because $\rho > p^{-p/(p-1)}$, the ring inclusion induces an isomorphism
between the discs
$$|(1 + p\pi t^{-1}) - (1 + p\pi t^{-\rho})| < rp^{-p/(p-1)\rho^{-1}}$$
and
$$|(1 + \pi u) - (1 + \pi u^\rho)| < rp^{-1/(p-1)\rho^{-1}}.$$

This yields the desired result. $\square$

5.4 Highest breaks and radii of convergence

We now recall a relationship between generic radii of convergence of a quasi-unipotent $\nabla$-
module and the breaks of its monodromy representation (Theorem 5.4.10). For more on the
history of this relationship, see Remark 5.4.13.

**Definition 5.4.1.** Given $E \in M$ and $\beta \in \mathbb{R}_{\geq 0}$, we say that $E$ has highest break $\leq \beta$ (resp. $\geq \beta$) if there exists $\rho_0 \in (0, 1)$ such that

$$R(E, \rho) \geq \rho^{\beta+1} \quad (\text{resp. } R(E, \rho) \leq \rho^{\beta+1})$$

for $\rho \in (\rho_0, 1) \cap \Gamma^*$. We say $E$ has highest break $\beta$ if it has highest break $\leq \beta$ and $\geq \beta$. Note that this definition is stable under pullback by automorphisms of the disc, thanks to Corollary 5.2.4.

**Remark 5.4.2.** Note that Definition 5.4.1 does not guarantee that $E$ has a highest break. However, if $E$ is overconvergent and $K$ is spherically complete, then by [11, Théorème 4.2-1] and [12, Théorème 2.1-2] (and an application of Theorem 3.4.3 to convert $E$ into a module over the Robba ring), $E$ has highest break $\beta$ for some $\beta \in \mathbb{Q} \cap [0, \infty)$. We will not explicitly use this result; however, note that it is implicit in the proofs of the $p$-adic local monodromy theorem given in [11] and [27]. (On the other hand, it plays no role in the proof given in [24].)

**Remark 5.4.3.** Christol and Mebkhout use the term “plus grande pente”, which translates as “greatest slope”, for what we are calling the “highest break”. We have avoided the term “slope”, which Christol and Mebkhout use by analogy with the theory of classical differential modules (in which Newton polygons of certain $t$-adic polynomials play a role), to avoid confusion with the “Frobenius slopes” that also inhabit the theory of $p$-adic differential equations, as in [24]. Instead, we use the term “break” to evoke the connection with ramification breaks, as encapsulated in Theorem 5.4.10.

To relate breaks to monodromy, we start in the unipotent case.

**Lemma 5.4.4.** For $E \in M$ unipotent, $E$ has highest break 0.
Proof. By Proposition 4.5.3 we can find a basis $v_1, \ldots, v_n$ of $E$ such that $t\frac{d}{dt}v_i = \sum_{j<i} c_{ij}v_j$ with $c_{ij} \in K$. Since

$$\frac{dt}{t} = \frac{d(t - t_\rho)}{t_\rho + (t - t_\rho)} = \frac{d(t - t_\rho)}{t_\rho} \sum_{i=0}^\infty \left(\frac{t_\rho - t}{t_\rho}\right)^i,$$

$E$ has a full basis of solutions on any disc of the form $|t - t_\rho| < \rho$, where $|t_\rho| = \rho$. This proves the claim. $\square$

**Lemma 5.4.5.** For $E, F \in M$ and $\alpha, \beta \in \mathbb{R}_{\geq 0}$ with $\alpha < \beta$, if $E$ has highest break $\leq \alpha$ and $F$ has highest break $\beta$, then $E \otimes F$ also has highest break $\beta$.

**Proof.** Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be bases of local horizontal sections of $E$ and $F$, respectively, around a generic point $t_\rho$. Then a basis of local horizontal sections of $E \otimes F$ around $t_\rho$ is given by $v_i \otimes w_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. By Lemma 5.4.4 we have that $v_1, \ldots, v_m$ converge for $|t - t_\rho| < \rho^\alpha$. Hence for $r \geq \alpha + 1$, all of the $w_j$ converge for $|t - t_\rho| < \rho^r$ if and only if all of the $v_i \otimes w_j$ converge there. This yields the desired result. $\square$

We next observe how breaks are affected by a tame extension of $k((t))$.

**Lemma 5.4.6.** Let $f : A(0,1) \to A(0,1)$ be the formally étale cover defined by $f^*(t) = t^n$, for $n$ a positive integer not divisible by $p$. For $E \in M$ and $\beta \in \mathbb{R}_{\geq 0}$, $E$ has highest break $\geq \beta$ (resp. $\leq \beta$) if and only if $f^*E$ has highest break $n\beta$ (resp. $\leq n\beta$) if and only if $f_*E$ has highest break $\beta/n$ (resp. $\leq \beta/n$).

**Proof.** This follows at once from Lemma 5.3.1. $\square$

**Lemma 5.4.7.** Given $a \in k^*$ and $n$ a positive integer not divisible by $p$, let $f : X \to A[p^0,1]$ be a formally étale cover with reduction $k((t))[u]/(u^p - u - at^{-n})$. Suppose $E \in M$ has no trivial submodules, but $f^*E$ is constant. Then $E$ has highest break $n$.

**Proof.** We may assume without loss of generality that $K$ contains an element $\pi$ with $\pi^{p-1} = -p$, and that $E$ is irreducible and nonconstant. Then $E$ has rank 1, and is equal to one of the nontrivial summands of $f_*f^*O$. If we write $\Gamma(O, X) = R_{[p^0,1]}[u]/(1 + p\pi bt^{-n} - (1 + \pi u)^p)$

for $b \in \mathfrak{o}_K$ reducing to $a$ in $k$, then the trivial summand is generated by 1 and the nontrivial summands are generated by $(1 + \pi u)^i$ for $i = 1, \ldots, p - 1$. Since

$$p\frac{d(1 + \pi u)^i}{(1 + \pi u)^i} = \frac{d(1 + p\pi bt^{-n})}{1 + p\pi bt^{-n}},$$

each summand is isomorphic to the rank one $\nabla$-module generated by $v$ with $\nabla v = -\pi nbt^{-n-1}(1 + p\pi bt^{-n})^{-1}v \otimes dt$. 

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However, since $\exp(x)$ converges for $|x| < |\pi|$, this $\nabla$-module is isomorphic to the $\nabla$-module generated by $w$ with

$$\nabla w = -\pi nbt^{-n-1} w \otimes dt$$

over a suitable annulus (namely $A[\rho, 1]$ for $\rho = \max\{\rho_0, p^{-1/n}\}$).

When we expand around a generic point $t_\rho$ with $|t_\rho| = \rho$, we get a local horizontal section given by

$$\exp(-\pi b(t^{-n} - t_\rho^{-n})) w.$$ 

This section converges in the disc $|t - t_\rho| < \rho^{\beta + 1}$ if and only if $|t^{-n} - t_\rho^{-n}| < 1$ throughout the disc. However, by Lemma 5.3.1, the discs $|t^{-n} - t_\rho^{-n}| < 1$ and $|t - t_\rho| < \rho^{n+1}$ are identical. Thus $E$ has highest break $n$, as desired.

**Lemma 5.4.8.** Given $a \in k^*$ and $n$ a positive integer not divisible by $p$, let $f : X \to A[\rho_0, 1]$ be a formally étale cover with reduction $k((t))[u]/(w^p - u - at^{-n})$. For $\beta \geq n$, $E$ has highest break $\leq \beta$ if and only if $f^*E$ has highest break $\leq \beta p - n(p - 1)$.

**Proof.** There is no loss of generality in enlarging $k$ by adjoining an $n$-th root of $a$, and so by changing series parameters, we may reduce to the case $a = 1$. Also, we will not use the precise value of $\rho_0$, so there is no harm in allowing it to increase (by stipulating the existence of certain formally étale covers over $A[\rho_0, 1]$).

In case $n = 1$, the claim follows at once from Lemma 5.3.3. In the general case, let $g : A[\rho_0, 1] \to A[\rho_0^n, 1]$ be the $n$-th power map. Let $f_0 : X' \to A[\rho_0, 1]$ be a formally étale cover with reduction $k((t))[u]/(w^p - u - t^{-1})$; then $f \circ g$ factors as $g_0 \circ f_0$, where $g_0 : X \to X'$ is a formally étale cover whose reduction is totally tamely ramified of degree $n$.

By the $n = 1$ case together with Lemma 5.4.6, the following assertions are equivalent.

- $E$ has highest break $\leq \beta$.
- $g_*E$ has highest break $\leq \beta/n$.
- $f_0^*g_*E$ has highest break $\leq \beta p/n - (p - 1)$.
- $g_0^*f_0^*E \cong f^*g_*E$ has highest break $\leq \beta p - n(p - 1)$.

Moreover, there are maps

$$E \to g^*g_*E \to E$$

whose composition is multiplication by $n$, given by the trace and adjunction maps (for a finite étale ring extension); in particular, $E$ is isomorphic to a subobject of $g^*g_*E$. Thus if $E$ has highest break $\leq \beta$, then $f^*E$ has highest break $\leq \beta p - n(p - 1)$.

On the other hand, the following assertions are also equivalent.

- $f^*E$ has highest break $\leq \beta p - n(p - 1)$.
- $(g_0)_*f^*E$ has highest break $\leq \beta p/n - (p - 1)$.
\[ (f_0)_*(g_0)_* f^* \mathcal{E} \cong g_* f_* f^* \mathcal{E} \] has highest break \( \leq \beta/n \). (We did not directly address \((f_0)_* \) above, but we may apply Lemma 5.3.3 to deduce this.)

- \( g_* f_* f^* \mathcal{E} \) has highest break \( \leq \beta \).

Moreover, there are maps
\[ \mathcal{E} \to f_* f^* \mathcal{E} \to \mathcal{E} \]
whose composition is multiplication by \( p \), given by the adjunction and trace maps (for a finite étale ring extension); in particular, \( \mathcal{E} \) is isomorphic to a subobject of \( f_* f^* \mathcal{E} \), and likewise of \( g_* f_* f^* \mathcal{E} \). Thus if \( f^* \mathcal{E} \) has highest break \( \leq \beta p - n(p - 1) \), then \( \mathcal{E} \) has highest break \( \leq \beta \).

**Lemma 5.4.9.** Given a polynomial \( P(t^{-1}) \) over \( k \) of degree \( n \) coprime to \( p \), let \( f : X \to A[p_0, 1] \) be a formally étale cover with reduction \( k((t))[u]/(u^p - u - P(t^{-1})) \). Suppose that \( \mathcal{E} \in \mathcal{M}_K \) has no trivial submodules, but \( f^* \mathcal{E} \) is constant. Then \( \mathcal{E} \) has highest break \( n \).

**Proof.** As in Lemma 5.3.7, we may assume that \( K \) contains \( \pi \) such that \( \pi^{p-1} = -p \), and that \( \mathcal{E} \) is irreducible and nonconstant; in particular, \( \mathcal{E} \) is of rank 1. We may also assume \( k \) is algebraically closed, and that \( P(t^{-1}) = \sum c_i t^{-i} \) where \( c_i = 0 \) whenever \( i \) is divisible by \( p \).

We proceed by induction on \( n \). Write \( P(t^{-1}) = a_n t^{-n} + Q(t^{-1}) \), where \( Q \) has degree \( d < n \) coprime to \( p \). Then we can write \( \mathcal{E} = \mathcal{F} \otimes \mathcal{G} \), where \( \mathcal{F} \) is nonconstant but becomes constant over \( k((t))[y]/(y^p - y - Q(t^{-1})) \), and \( \mathcal{G} \) is nonconstant but becomes constant over \( k((t))[z]/(z^p - z - a_n t^{-n}) \). By the induction hypothesis, \( \mathcal{F} \) has highest break \( d \); by Lemma 5.4.7 \( \mathcal{G} \) has highest break \( n \). By Lemma 5.4.5 \( \mathcal{E} \) also has highest break \( n \), as desired.

At last, we can relate generic radii of convergence to ramification filtrations.

**Theorem 5.4.10.** Suppose that \( k \) is perfect. For \( \mathcal{E} \in \mathcal{M}^{\text{qu}} \) whose monodromy representation has highest break \( \beta \), \( \mathcal{E} \) has highest break \( \beta \).

**Proof.** Assume without loss of generality that \( k \) is algebraically closed. Let \( \tau : G \to \text{GL}(V) \) be the corresponding representation. By Lemma 5.4.6 we may reduce to the case where the image \( H \) of \( \tau \) is a \( p \)-group. Moreover, thanks to Theorem 4.6.6, Lemma 5.4.4 and Lemma 5.4.5 we may assume that \( \mathcal{E} \) is quasi-constant.

We induct primarily on the order of \( H \); the base case of order \( p \) is handled by Lemma 5.4.9. We induct secondarily on the lowest break \( i \) of the ramification filtration induced on \( H \). Note that by Proposition 2.3.6 \( i \) is an integer.

Assume without loss of generality that \( \mathcal{E} \) is irreducible. Choose a maximal subgroup \( N \) of \( H \) containing \( H^i \). Then \( N \) is normal of index \( p \), so it fixes an Artin-Schreier extension of \( k((t)) \), say \( k((t))[u]/(u^p - u - f) \). We can take \( f \) of the form \( \sum c_j t^{-j} \), with \( c_j \in k \) and \( c_j = 0 \) when \( j \) is divisible by \( p \). (We may omit the term \( c_0 \) because \( k \) is algebraically closed.)

We now pass from \( k((t)) \) to \( E = k((t))[x]/(x^p - x - c_i t^{-i}) \). If \( \mathcal{E} \) becomes constant, then Lemma 5.4.7 implies that \( \mathcal{E} \) has highest break \( i \), completing the induction in this case. Otherwise, let \( \beta \) be the highest ramification break over \( k((t)) \); then the highest ramification break over \( E \) is \( \beta p - (p - 1)i \) (as in Example 2.1.10). If \( \tau \) restricted to \( \text{Gal}(E^{\text{sep}}/E) \) has image
$H$, its lowest break must be strictly less than $i$; otherwise, the image of $\tau$ is a proper subgroup of $H$. Hence the induction hypothesis applies, so that $E$ has highest break $\beta - (p - 1)i$ over $E$. By Lemma 5.4.8, $E$ has highest break $\leq \beta$, with equality as long as $\beta \neq i$. But we cannot have $\beta = i$: otherwise $H$ would be elementary abelian, hence of order $p$ since $E$ is irreducible, but that case is the base case which we handled with Lemma 5.4.9. Thus $\beta > i$, and $E$ indeed has highest break $\beta$. This completes both of the inductions and yields the claim.

\[\square\]

**Corollary 5.4.11.** The equivalence of categories given by Theorem 4.7.6 commutes with the formation of break decompositions.

**Remark 5.4.12.** Given Theorem 5.4.10, the filtration $E_i$ of $E$ corresponding to the break filtration has the following property: the rank of $E_i$ is, for $\rho \in (0, 1)$ sufficiently close to 1, the maximum number of linearly independent horizontal sections of $E$ on the open disc of radius $\rho^{i+1}$ around a generic point $t_\rho$.

**Remark 5.4.13.** In the case of $K$ discretely valued, Theorem 5.4.10 (stated in the equivalent form of Corollary 5.4.11) is due to Matsuda [26, Corollary 8.8]. Substantively similar results (comparing the “irregularity” of the differential equation with the Swan conductor of the monodromy representation) have been given by Tsuzuki [30, Theorem 7.2.2] and Crew [15, Theorem 5.4]. For more discussion, as well as a Tannakian reformulation, see [1, Complement 7.1.2] and subsequent remarks.

**Remark 5.4.14.** Note that our approach to Theorem 5.4.10 is highly revisionist; the original construction by Christol and Mebkhout of the break decomposition of an overconvergent $\nabla$-module is by topological means, inspired by classical techniques for studying ordinary (complex) differential equations. We do not know whether our derivation of Theorem 5.4.10 was known to Christol and Mebkhout, or if so, whether it motivated their construction. However, our argument by “breaking down the module” with successive small extensions is loosely styled after the derivation of the $p$-adic local monodromy theorem given by Mebkhout, specifically the proof of [24, Théorème 5.0-20].

### 6 Frobenius structures on differential equations

In this chapter, we recall the notion of a Frobenius structure on a differential equation, and explain how it interacts with some of the other structures we have introduced.

#### 6.1 Frobenius structures

Recall the notion of a Frobenius structure on a $\nabla$-module.

**Definition 6.1.1.** Let $\sigma_K : K \to K$ be a continuous homomorphism acting modulo $m_K$ as the $q$-th power Frobenius, for some power $q = p^n$ of $p$. A Frobenius lift (of order $n$) extending $\sigma_K$ is a map $\sigma : A[a, 1) \to A[a^n, 1)$ for some $a \in (0, 1) \cap \Gamma^*$ which induces a map on $R^{\text{int}}$.  

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which reduces modulo \( \mathfrak{m}_R \) to the \( q \)-th power map. Note that a Frobenius lift pulls back to a Frobenius lift on any formally étale cover.

**Definition 6.1.2.** Given a Frobenius lift \( \sigma \) of order \( n \), a Frobenius structure (of order \( n \)) on \( \mathcal{E} \in \mathcal{M} \) is an isomorphism \( F : \sigma^* \mathcal{E} \to \mathcal{E} \); we will often view such an \( F \) as a \( \sigma \)-linear map on sections of \( \mathcal{E} \). It will follow from Lemma 6.1.4 below (and Remark 4.1.9) that the choice of a Frobenius structure for a single \( \sigma \) determines a Frobenius structure for every \( \sigma \).

**Lemma 6.1.3.** Suppose \( \mathcal{E} \in \mathcal{M} \) admits a Frobenius structure. Then for any \( \lambda \in (0, 1) \), there exists \( \rho_0 \in (0, 1) \) such that for any \( \rho \in (\rho_0, 1) \), there exists a closed aligned subinterval \( I \) of \( (\rho_0, 1) \) containing \( \rho \) and a basis \( e_1, \ldots, e_n \) of \( \Gamma(\mathcal{E}, A(I)) \), such that the matrix \( N \) over \( R \) defined by

\[
i \frac{d}{dt} e_j = \sum_i N_{ij} e_i
\]

satisfies \( |N|_r < \lambda \) for \( r \in I \).

**Proof.** Fix a choice of a \( \nabla \)-module representing \( \mathcal{E} \), which we will hereafter confound with \( \mathcal{E} \). Choose a closed aligned subinterval \( I_0 = [a, b] \) of \( (0, 1) \) on which \( \mathcal{E} \) is defined, such that \( b > a^{1/q} \), and \( \sigma \) maps \( A[a, 1) \to A[a^q, 1) \). For \( l = 0, 1, \ldots \), put \( I_l = [a^{q^{-l}}, b^{q^{-l}}] \); given a basis \( e_1, \ldots, e_n \) of \( \Gamma(\mathcal{E}, A(I_0)) \), define the matrix \( N_l \) by

\[
t \frac{d}{dt} F^l e_j = \sum_i (N_l)_{ij} F^l e_i.
\]

Then

\[
N_{l+1} = \frac{dt^\sigma/dt}{t^\sigma/t} N_l^\sigma.
\]

Put \( u = \frac{dt^\sigma/dt}{t^\sigma/t} \); then there exists \( c \in (0, 1) \) such that for \( \rho \) sufficiently close to 1, we have \( |N_l|^\rho |_1 = |N_l|_\rho \) and \( |u|_\rho \leq c \). It follows that for \( l \) sufficiently large, \( |N_l|_\rho < \lambda \) for \( \rho \in I_l \). Since the intervals \( I_l, I_{l+1}, \ldots \) overlap (because \( b > a^{1/q} \)), they cover \( [a^{q^{-1}}, 1) \); thus we have the desired result. \( \square \)

As in [7, Théorème 2.5.7], one has the following.

**Lemma 6.1.4.** If \( \mathcal{E} \in \mathcal{M} \) admits a Frobenius structure, then \( \mathcal{E} \) is overconvergent.

**Proof.** We need to show that for any \( \eta \in (0, 1) \), there exists \( \rho \in [0, 1) \) such that for any closed aligned subinterval \( I \) of \( [\rho, 1) \) and any \( v \in \Gamma(\mathcal{E}, A(I)) \), the sequence

\[
\left\{ \frac{1}{j!} \frac{d^j}{dt^j} v \right\}_{j=0}^\infty
\]

is \( \eta \)-null. We can rewrite the \( j \)-th term of this sequence as

\[
t^{-j} \prod_{i=0}^{j-1} \left( t \frac{d}{dt} - i \right) v.
\]

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Suppose that $\Gamma(\mathcal{E}, A(I))$ admits a basis on which $t \frac{d}{dt}$ acts via a matrix $N$ with $|N|_r \leq |(p^m)!|$ for some integer $m$. Then for any integer $n$,

$$\prod_{i=0}^{j-1} \left( t \frac{d}{dt} - (np^m + i) \right)$$

acts on this basis via a matrix $N_n$ with $|N_n|_r \leq |(p^m)!|$. On such a basis, $\frac{t-j}{j} \prod_{i=0}^{j-1} (t \frac{d}{dt} - i)$ acts via a matrix $N_j$ with

$$|N_j|_r \leq r^{-j} \frac{|(p^m)!!| |j/p^m|}{|j|!} \leq r^{-j} p^{-(j/p^m)((p^m-1)/(p-1))} + j/(p-1)
= r^{-j} p^{j/(p^m(p-1))}.$$

Choose $m$ such that $p^{-1/(p^m(p-1))} > \eta$. By Lemma 6.1.3, we can find $\rho \in (0, 1)$ such that we can cover $[\rho, 1)$ with closed aligned subintervals $I$ such that each $\Gamma(\mathcal{E}, A(I))$ admits a basis on which $t \frac{d}{dt}$ acts via a matrix $N$ with $|N|_r \leq |(p^m)!|$ for each $r \in I$. By increasing $\rho$ if needed, we can ensure that $\rho^{-1} p^{1/(p^m(p-1))}\eta < 1$. Then the matrices $N_j$ form an $\eta$-null sequence; it follows that $\mathcal{E}$ is $\eta$- convergent, as desired.

The presence of a Frobenius structure provides some restriction on the monodromy representation, as follows.

**Proposition 6.1.5.** Given the residual characteristic $p$ of $k$, and positive integers $n$ and $d$, there exists an integer $N = N(n, p, d)$ with the following property. Suppose that $\mathcal{E} \in M^{an}$ has rank $d$, and that $\mathcal{E}$ admits a Frobenius structure $F$ of order $n$. Then for any formally étale cover $f : X \to A[a, 1]$ over which $\mathcal{E}$ becomes unipotent, with reduction $R$, the prime-to-$p$ order of the image of the inertia subgroup of $\text{Gal}(R/k((t)))$ in $\text{Aut}(H^0(\mathcal{E}, X))$ is at most $N$.

**Proof.** Note that the canonical minimal filtration of $\mathcal{E}$ (in which the first step is the maximal quasi-constant submodule, and so on) is preserved by any Frobenius structure, so it suffices to consider the case where $\mathcal{E}$ is quasi-constant, then apply that case to each successive quotient. Also, we may assume without loss of generality that $k$ is algebraically closed, since the desired conclusion is insensitive to changing $K$.

Put $G = \text{Gal}(R/k((t)))$ and $V = H^0(\mathcal{E}, X)$; there is no harm in assuming that $G$ injects into $\text{Aut}(V)$. By Jordan’s theorem (Proposition 2.4.8), we can find a commutative normal subgroup $H$ of $G$ such that $|G/H|$ is bounded as a function of $d$ alone. By passing to the fixed field of $H$, we may reduce to the case where $G$ is abelian. We may also enlarge $K$ so that all characters of $G$ are defined over $K$.

Let $\chi_1, \ldots, \chi_d$ be the characters via which $G$ acts on $V$. The fact that $F$ commutes with $G$ means that the map $\chi \mapsto \chi^a$ permutes the $\chi_i$; in particular, $\chi_i^{q^{a}} = \chi_i$ for each $i$. It follows that $\chi_i$ takes values in the fixed field of $K$ under $\sigma^a_K$; the group of prime-to-$p$ roots of unity in that fixed field is isomorphic to the group of prime-to-$p$ roots of unity in $\F_{q^a}$ via the Teichmüller map. In particular, each $\chi_i$ has prime-to-$p$ order dividing $q^a - 1$, so the prime-to-$p$ order of $G$ is bounded by $(q^a - 1)^d$, as desired. \qed

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Remark 6.1.6. In case $K$ is discretely valued, it can be shown (using the fact that the kernel of $GL_d(\mathfrak{o}_K) \rightarrow GL_d(\mathfrak{o}_K/m_K^i)$ is torsion-free if the exponential map converges on $m_K$) that the entire order of the image of inertia is bounded by a function of $p, n, d$ and the absolute ramification index $e$ of $K$. Indeed, that observation is implicit in Tsuzuki’s proof of the unit-root case of the $p$-adic local monodromy theorem [31]. Alternatively, one can bound the $p$-part of the order of inertia by bounding the ramification breaks of the monodromy representation using the Christol-Mebkhout construction, then applying Proposition 2.3.9.

Remark 6.1.7. Note that one cannot obtain any bound in the spirit of Proposition 6.1.5 in the absence of a Frobenius structure. For instance, the rank one $\nabla$-module given by $\nu \mapsto r\nu \otimes \frac{dt}{t}$ is quasi-constant for any $r \in \mathbb{Q}$ whose denominator $d$ is coprime to $r$, but the cover required has degree $d$.

6.2 The $p$-adic local monodromy theorem

We now give a form of the $p$-adic local monodromy theorem ($p$LMT) of André [1], Mebkhout [27], and the present author [21], which gives a context in which our results concerning quasi-unipotent $\nabla$-modules can be applied, at least when $K$ is discretely valued. Our “proof” is nothing more than a derivation of our particular statement from the form of the $p$LMT given in [21]. We follow up with a series of remarks on potential variant forms of the $p$LMT.

Theorem 6.2.1 ($p$-adic local monodromy theorem). Suppose that $K$ is discretely valued. Then any $E \in \mathcal{M}_K$ which admits a Frobenius structure is quasi-unipotent.

Proof. As per Definition 6.1.2, we may assume the Frobenius structure is with respect to any prescribed $\sigma$. In particular, we may assume that $\sigma$ is a power of a Frobenius lift of order 1 (so that we may apply the results of [21]). For $n$ a nonnegative integer, put $K^n = K^{\sigma^{-n}}$; that is, $K^n$ is a copy of $K$ viewed as a $K$-algebra via $\sigma^{-n}$. Then by [21, Theorem 6.12], there exists a formally étale cover $f^n : X^n \rightarrow A_{K^n}[a,1]$ for some $n$, and some $a \in [0,1)$, such that $E$ becomes unipotent on $X$.

The image of $\Gamma(\mathcal{O},X^n)$ under $\sigma^n$ induces a formally étale cover $f : X \rightarrow A_K[a,1]$ whose pullback along $A_{K^n}[a,1] \rightarrow A_K[a,1]$ coincides with $f^n$, and the images of horizontal elements of $\Gamma(f^n)^*E, X^n)$ under the $n$-th power of the Frobenius structure are horizontal elements of $\Gamma(f^*E, X)$. Thus $f^*E$ admits a nontrivial constant submodule; quotienting and repeating, we deduce that $f^*E$ is actually unipotent over $X$. This completes the argument.

Remark 6.2.2. Note that the references cited above all assume in some fashion that $k$ is perfect (or even algebraically closed); this is so that, in our terminology, the cover $X$ can be identified with an annulus over a finite unramified extension of $K$. While this sort of identification is convenient for making intermediate calculations, it does not intervene in the definition of unipotence that we are using. For a more robust treatment of the $p$LMT (following but simplifying the approach of [21]), see [24].

Remark 6.2.3. One may wonder whether the $p$-adic local monodromy theorem holds for $K$ which are not discretely valued. Proving the $p$LMT over an extension of $K$ proves the
pLMT over $K$ (e.g., by the argument of \cite{21}*Proposition 6.11; see also \cite{23} Chapter 3), so it is enough to consider spherically complete $K$. The discussion of this point naturally bifurcates following the two proof strategies available for proving the pLMT; see the two subsequent remarks.

**Remark 6.2.4.** The generalization of the pLMT to spherically complete $K$ via the “differential” approach used by André \cite{1} and Mebkhout \cite{27} can probably be generalized without much additional effort. This approach is based on the Christol-Mebkhout $p$-adic index theorem \cite{12} Section 6], which is known to hold for spherically complete $K$. Although neither André nor Mebkhout asserts the pLMT for spherically complete $K$, it seems not so difficult to check that their arguments carry over; a key point in André’s argument is the fact that Crew \cite{13} Theorem 4.11] verified the pLMT in rank 1, and this argument carries over essentially unchanged.

**Remark 6.2.5.** The generalization of the pLMT to spherically complete $K$ via the “Frobenius” approach used by the present author \cite{21} can probably be generalized, but a fair bit of additional effort will be required. For one thing, it depends on Tsuzuki’s unit-root $p$-adic local monodromy theorem \cite{31} Theorem 4.2.6], whose proof has only partly been generalized to arbitrary $K$ by Christol \cite{9}; the argument founders on a thorny technical problem that Christol did not see how to resolve, and neither do we \cite{9} Remark 14]. For another thing, the intermediate structural results of \cite{21} would have to be generalized; in some cases this seems tractable (e.g., the Bézout property for “analytic rings” can probably be generalized by imitating the corresponding arguments of Lazard \cite{25}, and the “existence of eigenvectors” should be straightforward), but in some cases it is less clear how to avoid leaning on the discreteness hypothesis (e.g., when “raising the Newton polygon”). This issue is discussed in somewhat more detail throughout \cite{21}.

**Remark 6.2.6.** It is conceivable that a version of the pLMT holds for overconvergent $\nabla$-modules not necessarily admitting a Frobenius structure; its conclusion would assert (by loose analogy with Jordan’s theorem, a/k/a Proposition \cite{23.3.3}) that after pulling back along a suitable formally étale cover, the given $\nabla$-module splits as a direct sum of rank 1 $\nabla$-modules, each of the form $\nabla v = cv \otimes \frac{dt}{t}$ for some $c \in K$. Under an additional technical hypothesis (the property “NLE” in the terminology of \cite{27}), and still assuming that $K$ is discretely valued (though perhaps the argument can be extended to allow $K$ to be spherically complete), this is essentially the “théorème de Turrittin” of Mebkhout \cite{27} Théorème-5.0-20]; the technical hypothesis forces the numbers $c$ to be $p$-adically non-Liouville. Unfortunately, it seems the only way to verify this hypothesis in practice is by exhibiting a Frobenius structure. Moreover, even if the Turrittin theorem held without the non-Liouville hypothesis, it is not clear that one would be able to avoid the use of Frobenius structures elsewhere in the theory, e.g., in the “full faithfulness of overconvergent-to-convergent restriction” \cite{22}. (Thanks to Nobuo Tsuzuki for prompting this remark.)
6.3 Frobenius antecedents

The Frobenius antecedent theorem of Christol-Dwork \[10, \text{Théorème 5.4}\] gives a sufficient condition for a \(\nabla\)-module on an annulus to be a Frobenius pullback of another \(\nabla\)-module. The approach in \[10\] is via explicit calculations with cyclic vectors; we will instead use Taylor series.

**Convention 6.3.1.** Throughout this section, fix a Frobenius lift \(\sigma_K\) of order 1 on \(K\), and let \(\sigma\) denote the standard extension of \(\sigma_K\) to \(R_I\) for each interval \(I\); that is,
\[
\left(\sum c_i t^i\right)^\sigma = \sum_i c_i^{\sigma_K} t^{pi}.
\]

**Theorem 6.3.2.** For \(\rho_0 \in (0,1) \cap \Gamma^*\), let \(E\) be a \(\nabla\)-module on \(A[\rho_0,1)\) with the property that
\[R(E,\rho) > p^{-1/(p-1)} \rho\]
for all \(\rho \in [\rho_0,1)\). Then there exists a unique \(\nabla\)-module \(F\) on \(A[\rho_0,1)\) such that \(E \sim \sigma^* F\) and
\[R(F,\rho^p) = R(E,\rho)^p\]
for all \(\rho \in [\rho_0,1)\).

**Proof.** Thanks to the uniqueness assertion in the claim, it suffices to prove the claim after adjoining to \(K\) a primitive \(p\)-th root of unity \(\zeta\).

For \(i = 0, \ldots, p-1\), let \(g_i : A[0,1) \to A[0,1)\) be the map induced by the ring map \(t \mapsto t\zeta^i\). Then the map \(h_i : g_i^* E \to E\) defined by
\[h_i(v) = \sum_{n=0}^\infty \frac{(\zeta^i - 1)^n t^n}{n!} \otimes \frac{d^n}{dt^n} v\]
is well-defined because \(R(E,\rho) > p^{-1/(p-1)} \rho\), and is in fact an isomorphism. (Note that this is an example of the functoriality of rigid cohomology, as described in \[7\]; see also \[23, \text{Proposition 2.8.1}\].)

We may interpret the \(h_i\) as giving rise to endomorphisms of \(E\) which are semilinear in the sense that \(h_i(tv) = \zeta^i th_i(v)\). As in \[23, \text{Section 5.2}\], we set
\[f_i(v) = t^{-i} \sum_{e=0}^{p-1} \zeta^{-ei} h_e(v),\]
and note that \(f_i(v)\) is fixed by the \(h_i\); we also note that if we apply each \(f_i\) to each of a set of generators of \(E\), the images generate \(E\) over \(O^\sigma\). In other words, the images generate a \(\nabla\)-module \(F\) over \(A[\rho_0^p,1)\) with \(\sigma^* F \cong E\). Moreover, if \(t_\rho \in \mathbb{C}\) is a generic point of radius \(\rho\), then we can apply the \(f_i\) to horizontal sections of \(E\) on the disc \(|t - t_\rho| < \rho^{\beta+1}\) to obtain horizontal sections of \(F\) on the disc \(|t - t_\rho^p| < \rho^{p(\beta+1)}\). We thus have \(R(F,\rho^p) \geq R(E,\rho)^p\); we have the reverse inequality also by Lemma 6.3.2. Thus
\[R(F,\rho^p) = R(E,\rho)^p.\]
To conclude, we verify uniqueness. To do this, it suffices to show that if there exists $F'$ such that $E \cong \sigma^* F'$ and $R(F', \rho^p) = R(E, \rho)^p$ for all $\rho \in [\rho_0, 1) \cap \Gamma^*$, then $F'$ is fixed under each of the $h_i$; once this is known, it follows that the fixed locus of the $h_i$ is precisely $F'$, and hence that $F \cong F'$.

Let $I = [a, b]$ be a closed aligned subinterval of $[\rho_0, 1)$ such that $R(F', \rho^p) \geq p^{-p/(p-1)} b^p$ for any $\rho \in I$, and put $I^p = [a^p, b^p]$. Then the Taylor series map gives an isomorphism $\pi_2^* F' \to \pi_1^* F'$ on the subspace of $A(I^p) \times A(I^p)$, with coordinates $t_1$ and $t_2$, where $|t_1 - t_2| \leq p^{-p/(p-1)} b^p$. This isomorphism pulls back to the Taylor series isomorphism $\pi_2^* E \to \pi_1^* E$ on the subspace of $A(I) \times A(I)$ where $|t_1^p - t_2^p| \leq p^{-p/(p-1)} b^p$, and in particular to the subspace where $|t_1 - t_2| \leq p^{-1/(p-1)} b$. That means we can reconstruct the $h_i$ starting with the trivial action on $F'$. As noted above, this yields that $F \cong F'$, yielding the desired uniqueness. □

Remark 6.3.3. Note that [10, Théorème 5.4] is slightly weaker, as it requires by hypothesis the stronger bound $R(E, \rho) > p^{-1/p} \rho$. By contrast, the condition $R(F, \rho^p) = R(E, \rho)^p$ is quite necessary, as evidenced by the following example [10, 5.1]: if $F$ is the rank 1 $\nabla$-module given by $\nabla v = (pt)^{-1} \otimes dt$, then $F \not\cong \mathcal{O}$ but $\sigma^* F \cong \sigma^* \mathcal{O}$.

### 6.4 Frobenius and ramification

As we have seen, the Christol-Mebkhout characterization of breaks of a quasi-unipotent $\nabla$-module involves inspecting the rate of convergence of horizontal sections near the boundary of a disc. In the presence of a Frobenius structure, its global convergence properties can be used instead.

**Proposition 6.4.1.** Retain Convention 6.3.1. Suppose that for some $\epsilon \in (0, 1) \cap \Gamma^*$, $E$ is a $\nabla$-module over $A(\epsilon, 1)$ equipped with a Frobenius structure given by an isomorphism $F : \sigma^* E \to E$ over $A(\epsilon^{1/p}, 1)$. Suppose further that for any closed aligned subinterval $I$ of $[\epsilon, 1)$, there exists an $\mathfrak{o}$-lattice in $\Gamma(E, A(I))$ stable under $t \frac{\partial}{\partial t}$. Then the highest break $\beta$ of $E$ (in the sense of Theorem 5.4.10) satisfies the inequality

$$\beta \leq \frac{1}{(p-1) \log_p(\epsilon^{-1})}.$$ 

**Proof.** For $\rho \in (\epsilon, 1) \cap \Gamma^*$, let $t_\rho$ be a generic point of radius $\rho$. By hypothesis, over the disc $|t - t_\rho| < \rho$, $E$ admits an $\mathfrak{o}$-lattice stable under $t \frac{\partial}{\partial t}$. Since $(t - t_\rho)/t$ has norm less than 1 throughout this disc, an $\mathfrak{o}$-lattice stable under $t \frac{\partial}{\partial t}$ is also stable under $(t - t_\rho) \frac{\partial}{\partial (t - t_\rho)}$. By a direct calculation, we deduce that

$$R(E, \rho) \geq p^{-1/(p-1)} \rho.$$ 

By applying Frobenius (and Lemma 5.3.2), we have $R(E, \rho^{1/p^m}) \geq p^{-1/(p^{m}(p-1))} \rho^{1/p^m}$. If $\beta$ is the highest break of $E$, for large $m$ (hence for any $m$) one then has the inequality

$$\rho^{\beta/p^m} \geq p^{-1/(p^{m}(p-1))} ;$$

by taking limits, we obtain the same inequality with $\rho = \epsilon$. This yields the desired result. □
Remark 6.4.2. Note that Proposition 6.4.1 shows that given the $p$LMT, one can use explicit convergence information for Frobenius to control the extension needed to find the unipotent basis. It would be interesting to turn this argument on its head, and use Frobenius convergence information to give a more direct proof of the $p$LMT. However, we have no idea how to do this, even in the unit-root case originally treated by Tsuzuki [31]; such an approach would suggest a method for extending Tsuzuki’s arguments to the case of $K$ spherically complete. The Christol-Mebkhout $p$-adic index theorem from [12] does something analogous using connection convergence information, but it does not say anything about tame ramification.

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