C-PROJECTIVE COMPACTIFICATION; (QUASI-)KÄHLER METRICS AND CR BOUNDARIES

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Abstract. For complete complex connections on almost complex manifolds we introduce a natural definition of compactification. This is based on almost c-projective geometry, which is the almost complex analogue of projective differential geometry. The boundary at infinity is a (possibly non-integrable) CR structure. The theory applies to almost Hermitian manifolds which admit a complex metric connection of minimal torsion, which means that they are quasi-Kähler in the sense of Gray-Hervella; in particular it applies to Kähler and nearly Kähler manifolds. Via this canonical connection, we obtain a notion of c-projective compactification for quasi-Kähler metrics of any signature. We describe an asymptotic form for metrics that is necessary and sufficient for c-projective compactness. This metric form provides local examples and, in particular, shows that the usual complete Kähler metrics associated to smoothly bounded, strictly pseudoconvex domains in \( \mathbb{C}^n \) are c-projectively compact. For a smooth manifold with boundary and a complete quasi-Kähler metric \( g \) on the interior, we show that if its almost c-projective structure extends smoothly to the boundary then so does its scalar curvature. We prove that in this case (and under some natural assumptions on the extension of \( J \) to the boundary) \( g \) is almost c-projectively compact if and only if this scalar curvature is non-zero on an open dense set of the boundary. In that case it is, along the boundary, locally constant and hence nowhere zero there. Finally we describe the asymptotics of the curvature, showing, in particular, that the canonical connection satisfies an asymptotic Einstein condition. Key to much of the development is a certain real tractor calculus for almost c-projective geometry, and this is developed in the article.

1. Introduction. Consider a smooth manifold \( \overline{M} \) with boundary \( \partial M \), interior \( M \), and a geometric structure on \( M \) which does not admit a smooth extension to the boundary; for example a complete Riemannian metric. It is then natural to ask whether some aspect or weakening of the interior geometry admits a smooth extension to the boundary and gives rise to a geometric structure there. If this works, one can start to relate asymptotics of the interior geometry to the boundary geometry on various levels (geometric objects, equations of geometric origin, and so forth) and also deduce consequences for the spectral theory of suitable operators. Such ideas are fundamental for several areas of mathematics and theoretical physics, including GR, scattering theory, complex analysis, conformal and parabolic geometries, representation theory, and holography see e.g., [1, 3, 17, 18, 19, 21, 24, 27, 28].

The most well-known example of this idea is conformal compactification; a complete metric on \( \overline{M} \) is conformally compact if its conformal structure extends to the boundary \( \partial M \) and it has a suitably uniform asymptotic volume growth toward...
∂M. In this case, the boundary inherits a conformal structure and the resulting structure provides a deep link between this conformal geometry and the interior (pseudo-)Riemannian geometry.

Guided by examples arising from reductions of projective holonomy [10, 11], a notion of compactification linked to projective differential geometry was introduced in [8, 9]. This concept is initially defined for affine connections, so, via the Levi-Civita connection, it automatically applies to pseudo-Riemannian metrics. From the point of view of geometric analysis this projective compactification is motivated by its strong links to the geodesic structure and the fact that many of the natural equations studied in Riemannian geometry and physics have projective invariance, but not conformal invariance. An implicit use of projective compactification is important in recent microlocal analysis advances of Vasy in [30, 31].

In this article, we introduce a concept of compactification that is suitable for complex geometries (and also the more general almost complex geometries); it may be viewed as an analogue of the projective compactification of order two from [8, 9], although the complex situation is considerably more subtle. Hence we make contact with complex analysis, where the study of domains via the geometry of their boundaries is of fundamental importance [15, 17, 23]. The appropriate almost complex version of projective geometry has been classically studied under the name h-projective (or holomorphically projective) geometry. This name is misleading however, since it is not holomorphic in nature. Recently there has been renewed interest in these geometries, see e.g., [25, 26], also because they provide an example of a parabolic geometry. Adopting the latter point of view, a general basic theory of these geometries has been worked out under the name almost c-projective structures in [5], which will be one of our standard references.

An almost c-projective structure can be defined on any almost complex manifold, and is given by an equivalence class of linear connections on the tangent bundle which preserve the almost complex structure and have minimal torsion. Hence in the situation of a manifold M with boundary ∂M and interior M, we will in addition assume that we have given an almost complex structure J on M. The concept of c-projective compactness is then defined for linear connections on TM which preserve J and are minimal in the sense that their torsion is of type (0, 2) (so they are torsion-free if J is integrable). Such a connection V is then called c-projectively compact if certain explicit c-projective modifications of V, constructed from local defining functions for the boundary, admit smooth extensions to the boundary, see Definition 2.

If V is c-projectively compact, then the definition easily implies that the almost c-projective structure defined by V admits a smooth extension to the boundary. In particular, this provides a smooth extension of J to the boundary which endows ∂M with an almost CR structure, which is CR for integrable J. The main case of interest is when this structure is (Levi-) non-degenerate. Moreover, we will always assume that, along the boundary, the Nijenhuis tensor has values tangent to
the boundary, which in particular implies that the boundary structure is partially integrable. This condition emerges naturally from several points of view. Similar to the case of projective compactness, we show that for a complex connection $\nabla$ which preserves a volume form, in addition to requiring that the almost c-projective structure defined by $\nabla$ admits a smooth extension to the boundary, one only has to require a certain uniform rate of volume growth to ensure that $\nabla$ is c-projectively compact, see Proposition 7.

In the integrable case, the Levi-Civita connection $\nabla$ of any (pseudo-)Kähler metric $g$ on $M$ preserves $J$ and is torsion free. In this setting we say $g$ is c-projectively compact if $\nabla$ is c-projectively compact. In the non-integrable situation, a similar concept is defined provided that $g$ is quasi-Kähler which implies that there is a linear connection, which preserves both $g$ and $J$ and is minimal, see Proposition 9. This so-called canonical connection is different from the Levi-Civita connection of $g$ and plays the main role in questions related to c-projective compactness. The main results of this article concern c-projectively compact metrics. In particular, we give two equivalent characterizations of c-projective compactness of quasi-Kähler metrics.

For one of these equivalent characterizations, we assume that the almost complex structure $J$ admits a smooth extension to all of $\overline{M}$, so as discussed above $\partial M$ inherits an almost CR structures. We further assume that this structure is non-degenerate and that the Nijenhuis tensor has asymptotically tangential values. Under these assumptions, we devise a specific asymptotic form for a metric $g$ involving the boundary geometry, see Section 2.6 for details. In Theorem 10 we prove directly that this asymptotic form is sufficient for c-projective compactness. Of course this form may be used to provide local examples. In particular this implies that the standard construction of complete Kähler metrics, from boundary defining functions, on non-degenerate smoothly bounded domains in $\mathbb{C}^n$ always leads to c-projectively compact Kähler metrics; see Proposition 11. This clearly demonstrates the richness and relevance of the class of c-projectively compact metrics.

The second equivalent description of c-projective compactness for metrics is based on a c-projective interpretation of scalar curvature. Having given $\overline{M}$, an almost complex structure $J$ and a quasi-Kähler metric $g$ on $M$ with canonical connection $\nabla$, assume that the almost c-projective structure defined by $\nabla$ admits a smooth extension to $\overline{M}$. (This condition can be easily checked in local frames, see Lemma 19.) After giving a c-projective interpretation of several quantities associated to $g$, in particular the scalar curvature $S$ of $\nabla$, we prove that these quantities admit a smooth extension to $\overline{M}$. The second equivalent condition is then the extendability of the almost c-projective structure together with the requirement that resulting smooth extension of $S$ is nowhere vanishing along $\partial M$.

For the proof of equivalence of these conditions to c-projective compactness we use powerful tools from almost c-projective geometry; these are developed in
Section 3. We construct descriptions of the real tractor bundles $\mathcal{H}$ and $\mathcal{H}^*$ of Hermitian metrics on the c-projective standard tractor bundle $\mathcal{T}$ and on its dual $\mathcal{T}^*$. We describe the canonical tractor connections on these bundles, the associated BGG splitting operators and the induced invariant differential equations. Once one has obtained an extension of the almost c-projective structure to the boundary, all these objects admit extensions to the boundary, which is a major ingredient in our proofs.

The proofs for the equivalence are then carried out in Section 4. The basic extension results for quantities associated to a quasi-Kähler metric on $(M,J)$, for which the almost c-projective structure induced by the canonical connection extends to $\overline{M}$, are proved in Corollary 18. In Theorem 20, it is shown that locally around boundary points for which the extension of $S$ is non-vanishing, $g$ is then necessarily c-projectively compact. The remaining parts of the equivalence are: first a proof that an almost c-projectively compact metric admits a weaker version of the asymptotic form involving the restriction of $S$ to the boundary, see Proposition 21; and then the proof that the boundary value of $S$ is locally constant and finally that the asymptotic form has all required properties, see Theorem 22.

In the last part of the article, we analyze the curvature of c-projectively compact connections and metrics. In particular, we prove that the canonical connection $\nabla$ of a c-projectively compact metric always satisfies an asymptotic version of the Einstein equation and in the integrable case we give a complete description of the curvature of the metric up to terms which admit a smooth extension to the boundary, see Theorem 25. Finally, we show that assuming an asymptotic vanishing condition for the covariant derivative of the Nijenhuis tensor, one may, in many of our results, replace curvature quantities associated to the canonical connection by the corresponding curvature quantities associated to the Levi-Civita connection, see Corollary 28.

It is to be expected that, similar to the projective case (see [9]), one can use the c-projective standard tractor bundle and its canonical tractor connection to obtain a description of the tractors associated to the partially integrable almost CR structure on the boundary. This will be taken up elsewhere.

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2. **c-projective compactness with a real boundary.**

2.1. **Almost c-projective structures.** Almost c-projective structures are the natural almost complex analog of classical projective structures. In the literature, these geometries are often referred to as $h$-projective or holomorphic projective structures, but this is a misleading name. Indeed, there is a holomorphic version of classical projective structures, i.e., one considers complex manifolds with holomorphic linear connections on their tangent bundles which have the same complex geodesics up to parametrization. This, however, is only a special case of
c-projective structures, which moreover is not relevant for many important applications that involve Hermitian metrics. Therefore, following [5], we use the name “(almost) c-projective structures”.

Suppose that \( M \) is a smooth manifold of even dimension \( n = 2m \geq 4 \) endowed with an almost complex structure \( J \) and suppose that \( \nabla \) is a linear connection on \( TM \) such that \( J \) is parallel for the connection induced by \( \nabla \). This reads as \( \nabla_\xi J_\eta = J \nabla_\xi \eta \) for all vector fields \( \xi, \eta \in \mathfrak{X}(M) \). From this it is easy to see that changing \( \nabla \) projectively (in the real sense) will never lead to a connection which again preserves \( J \). The appropriate modification needed to preserve this property is to take the usual definition of a projective change, extend one-forms to complex linear functionals on the tangent space and then use the complex analog of the classical definition. Explicitly, this means that given \( \nabla \) and a one-form \( \Upsilon \in \Omega^1(M) \), one defines a new linear connection \( \hat{\nabla} = \nabla + \Upsilon \) by

\[
\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi) \eta - \Upsilon(J\xi) J\eta + \Upsilon(\eta) \xi - \Upsilon(J\eta) J\xi.
\]

One immediately verifies that \( \nabla J = 0 \) implies \( \hat{\nabla} J = 0 \). One defines two linear connections on \( TM \) which preserve \( J \) to be \emph{c-projectively equivalent} if and only if they are related by (2.1) for some one-form \( \Upsilon \in \Omega^1(M) \).

Let us remark at this point, that while (2.1) is the traditional way to associate a c-projective change of connection to a one-form, it does not agree with the conventions used in [5], see Section 2.1 of that reference. The conventions in [5] are designed to work well in a complexified setting. For the current paper, however, it is of crucial importance to work in a real setting. In this real setting, the traditional conventions lead to formulae which are more closely analogous to the case of projective geometry and thus simplify comparison to the theory of projective compactness developed in [7, 8, 9].

From the definition in (2.1) it is evident that \( \hat{\nabla}_\xi \eta - \nabla_\xi \eta \) is symmetric in \( \xi \) and \( \eta \), which implies \( \nabla \) and \( \hat{\nabla} \) have the same torsion. Now it is well known that for a connection preserving an almost complex structure the \((0,2)\)-component of the torsion (i.e., the part which is conjugate linear in both arguments) is independent of the connection and equals (up to a non-zero factor) the Nijenhuis-tensor of the almost complex structure. On the other hand, the other components of the torsion can be removed by a change of the connection (preserving \( J \)); hence there always are linear connections which preserve \( J \) and whose torsion is of type \((0,2)\). This motivates the following definitions.

**Definition 1.** Let \((\mathcal{M}, J)\) be an almost complex manifold.

(1) A linear connection \( \nabla \) on \( M \) is called \emph{complex} if \( J \) is parallel for \( \nabla \) and it is called \emph{minimal} if its torsion is of type \((0,2)\), i.e., conjugate linear in both arguments.

(2) An \emph{almost c-projective structure} on \( M \) is a c-projective equivalence class \([\nabla]\) of minimal complex linear connections on \( TM \).
(3) The structure is called \textit{c-projective} or \textit{torsion free} if and only if $J$ is integrable or equivalently the connections in the projective class are torsion free.

As in the case of usual projective structures, a linear connection on $TM$ induces linear connections on all natural vector bundles over $M$, i.e., on all vector bundles induced from the complex linear frame bundle of $M$. In particular, one can form real and complex density bundles, which we will frequently need in what follows. Let us fix the conventions we will use. We will denote the complex line bundle $\Lambda^\omega_{\mathbb{C}} TM$ (the highest complex exterior power of the tangent bundle) by $\mathcal{E}(m+1,0)$. We will assume that there exist $(m+1)$st roots of this line bundle, and that a specific root $\mathcal{E}(1,0)$ has been chosen. We then define $\mathcal{E}(-1,0)$, $\mathcal{E}(0,1)$, and $\mathcal{E}(0,-1)$ as the dual, the conjugate, and the conjugate dual bundle to $\mathcal{E}(1,0)$. Forming tensor powers, we thus obtain complex density bundles $\mathcal{E}(k,\ell)$ for $k, \ell \in \mathbb{Z}$.

On the other hand, we can consider the usual real density bundles. As an almost c-projective manifold, $M$ is automatically orientable and the almost complex structure induces an orientation. Thus the bundle of real volume densities naturally includes into $\mathcal{E}(-m-1,-m-1)$, so we denote it by $\mathcal{E}(-2m-2)$. Since this is a trivial real line bundle, we can form arbitrary real roots of this bundle, thus defining $\mathcal{E}(w)$ for all $w \in \mathbb{R}$. Passing to roots, we obtain inclusions $\mathcal{E}(2k) \subset \mathcal{E}(k,k)$ for all $k \in \mathbb{Z}$. This then allows us to define complex density bundles $\mathcal{E}(w,w')$ for all $w, w' \in \mathbb{R}$ provided that $w - w' \in \mathbb{Z}$, and we always get an inclusion of the real line bundle $\mathcal{E}(w)$ into the complex line bundle $\mathcal{E}(\frac{w}{2}, \frac{w}{2})$. We will follow the convention that adding $(w)$ to the name of a real vector bundle indicates a tensor product with $\mathcal{E}(w)$, while for a complex vector bundle, adding $(w, w')$ to the name indicates a complex tensor product with $\mathcal{E}(w, w')$.

Almost c-projective structures with the additional choice of a line bundle $\mathcal{E}(1,0)$ can be equivalently described as (real) normal parabolic geometries of type $(G, P)$, where $G := \text{SL}(m+1, \mathbb{C})$ and $P$ is the stabilizer of a complex line in the standard representation $\mathbb{C}^{m+1}$ of $G$, see [5]. This correspondence is established similarly to the discussion of the real case in Section 4.1.5 of [13], see also Section 4.6 of [6].

This Cartan geometry can be equivalently encoded via the associated bundle corresponding to the standard representation $\mathbb{C}^{m+1}$ of $G$. This is the so-called \textit{standard tractor bundle} $\mathcal{T}$ on which the Cartan connection induces a linear connection $\nabla^\mathcal{T}$, the \textit{standard tractor connection}. This approach is developed in detail (mainly in a complexified picture) in [5].

\textbf{2.2. The notion of c-projective compactness.} We now introduce a notion analogous to projective compactness of order two as introduced in [8, 10], and further studied in [7, 9]. Let $\overline{M}$ be a real smooth manifold of real dimension $n = 2m$ with boundary $\partial M$ and interior $M$. Suppose further that we have given an almost complex structure $J$ on $M$ and a minimal complex linear connection $\nabla$ on $TM$. 
Definition 2. The complex connection $\nabla$ on $TM$ is called \textit{c-projectively compact} if and only if for each $x \in \partial M$, there is a neighborhood $U$ of $x \in M$ and a smooth defining function $\rho : U \to \mathbb{R}_{\geq 0}$ for $U \cap \partial M$ such that the c-projectively equivalent connection $\hat{\nabla} = \nabla + \frac{d\rho}{\rho^2}$ on $U \cap M$ extends smoothly to all of $U$.

Observe first that this condition is independent of the defining function under consideration. Given a defining function $\rho$ on $U$, any other defining function on $U$ can be written as $\hat{\rho} = e^f \rho$ for some smooth function $f : U \to \mathbb{R}$. This immediately implies that $d\hat{\rho} = \hat{\rho} df + e^f d\rho$ and hence $\frac{d\hat{\rho}}{\rho} = df + \frac{1}{\rho} d\rho$. Since $df$ is smooth up to the boundary, this implies that if the connection associated to $\rho$ extends, then so does the one associated to $\hat{\rho}$.

Using this, we can prove a first nice result on c-projectively compact connections.

**Proposition 3.** If the linear connection $\nabla$ on $TM$ is c-projectively compact, then the almost c-projective structure $(J, [\nabla])$ on $M$ naturally extends to all of $\overline{M}$. In particular, the almost complex structure $J$ smoothly extends to $\overline{M}$, which gives rise to a (possibly degenerate) almost CR structure of hypersurface type on $\partial M$. This structure is integrable (CR) if the initial structure is c-projective.

**Proof.** As we have noted in 2.1, the linear connection $\hat{\nabla} = \nabla + \frac{d\rho}{\rho^2}$ on $U \cap M$ satisfies $\hat{\nabla}J = 0$. But since this connection extends smoothly to all of $U$, we can extend $J$ by parallel transport to all of $U$. Since $U \cap M$ is dense in $U$, this extension is uniquely determined by $J$ and satisfies $J \circ J = -\text{id}$ on all of $U$. If the initial almost complex structure $J$ is integrable then the same is true for the extended structure (since its Nijenhuis tensor vanishes on a dense subset).

For $x \in \partial M$, we define $H_x := T_x \partial M \cap J(T_x \partial M)$. Clearly, this defines a smooth distribution $H \subset T\partial M$ of corank one and the boundary value of $J$ defines an almost complex structure on this distribution. Hence we have obtained an almost CR structure of hypersurface type on $\partial M$. It is well known that this structure is CR if $J$ is integrable.

As we have seen above, any other defining function $\hat{\rho}$ can be written as $\hat{\rho} = e^f \rho$ for a function $f$ which is smooth up to the boundary and then $\frac{d\hat{\rho}}{\rho^2} = \frac{d\rho}{\rho^2} + \frac{1}{2} df$. Hence the connections associated to $\hat{\rho}$ and $\rho$ are c-projectively equivalent on all of $U$. Together with the extension of $J$, we thus get a well-defined almost c-projective structure on $\overline{M}$. \qed

Observe that to obtain the induced almost CR structure on the boundary, one only needs the almost complex structure to extend, the c-projective equivalence class of connections is not really used at this point.

**Remark 4.** As in the case of projective compactness of order two, a c-projectively compact connection $\nabla$ automatically has completeness properties which force the boundary to be at infinity. Since this feature will not be used in the rest of the article, we just outline here the argument involved.
c-projective structures can be defined as a family of curves which is invariant under reparametrizations, but in contrast to the projective case, this class is larger than that which arises from the reparametrization of geodesics. Indeed, given an almost complex manifold \((N,J)\) and a complex connection \(\nabla\) on \(TN\), one defines a smooth curve \(c\) to be \(J\)-planar for \(\nabla\) if, for each parameter \(t\), the vector \(\nabla_{c'} c'(t)\) lies in the span of \(c'(t)\) and \(Jc'(t)\). Evidently any reparametrization of a \(J\)-planar curve is \(J\)-planar and it is a classical result that two minimal complex connections are c-projectively equivalent iff they have the same \(J\)-planar curves, see Proposition 2.1 of [5].

Fixing \(\nabla\), there is a family of distinguished parametrizations of any curve \(c\) which is \(J\)-planar for \(\nabla\). To see this one easily shows that there is a reparametrization \(\hat{c}\) of \(c\) such that \(\nabla_{\hat{c}' \hat{c}'}(t)\) lies in the real span of \(J\hat{c}'(t)\) for each \(t\), and this is unique up to an affine change of parameter.

In the case of \(\overline{M} = M \cup \partial M\) and a c-projectively compact connection \(\nabla\) on \(M\), we now look at any \(J\)-planar curve which meets the boundary \(\partial M\) transversally in a point \(x_0\). Proceeding as in the proof of Proposition 2.4 of [8], we take a local defining function \(\rho\) for \(\partial M\) defined on a neighborhood of \(x_0\) and the corresponding c-projective modification \(\hat{\nabla}\) of \(\nabla\) which is smooth up to the boundary. Then we take a distinguished parametrization of this curve for \(\hat{\nabla}\) that starts from \(x_0\) into \(M\) and run through this curve backwards. This gives a smooth \(J\)-planar curve \(\hat{c} : [0,t_0] \to \overline{M}\) with \(\hat{c}([0,t_0]) \subset M\) and \(\hat{c}(t_0) = x_0\) and the parametrization is distinguished for \(\hat{\nabla}\). This curve can be reparametrized in a way distinguished for \(\nabla\) and the reparametrization can be analyzed exactly as in the proof of Proposition 2.4 of [8]. As there, the result has the form \(c : [0,\infty) \to M\) with \(\lim_{t \to \infty} c(t) = x_0\). Since the geodesics of \(\nabla\) are \(J\)-planar curves this shows that the relevant geodesics are defined for all times and that the boundary is at infinity for those.

### 2.3. On the boundary geometry.

In the case of an integrable complex structure, we always obtain a CR structure on the boundary \(\partial M\). From the point of view of CR geometry, it is natural to assume in addition that the structure is (Levi-) non-degenerate. If the initial almost complex structure \(J\) is not integrable, then, apart from non-degeneracy, some assumptions on the asymptotics of the almost complex structure must be made. A natural assumption would be partial integrability of the induced CR geometry, but it turns out that for our purposes a slightly stronger assumption will be suitable.

To formulate the necessary definitions, recall first that the Lie bracket of vector fields induces a bilinear operator \(\Gamma(H) \times \Gamma(H) \to \Gamma(T\partial M/H)\). This operator is immediately seen to be bilinear over smooth functions and thus it is induced by a bundle map \(L : H \times H \to T\partial M/H\), which is called the Levi-bracket.

**Definition 5.** Let \(\overline{M}\) be a smooth manifold with boundary \(\partial M\) and interior \(M\) and let \(J\) be an almost complex structure on \(\overline{M}\). Let \(H \subset T\partial M\) be the induced almost CR structure and \(L\) its Levi-bracket.
(1) The almost CR structure $H \subset T\partial M$ is called non-degenerate if the value of $\mathcal{L}$ at any point is a non-degenerate bilinear map.

(2) The structure is called partially integrable if $\mathcal{L}$ is Hermitian in the sense that $\mathcal{L}(\xi,\eta) = \mathcal{L}(J\xi, J\eta)$ for all $\xi, \eta \in H$.

(3) We say that the Nijenhuis tensor $\mathcal{N}$ of $J$ has asymptotically tangential values if, along $\partial M$, $\mathcal{N}$ has values in $T\partial M \subset TM|_{\partial M}$.

Now we can easily characterize these conditions in terms of local defining functions.

**Lemma 6.** Let $(\overline{M}, J)$ be an almost complex manifold with boundary $\partial M$ and interior $M$, and let $\mathcal{N}$ be the Nijenhuis tensor of $J$. For a local defining function $\rho$ for the boundary, put $\theta = -d\rho \circ J$.

(1) The fact that $\mathcal{N}$ has asymptotically tangential values is equivalent to either of the following two conditions on any local defining function $\rho$.

- The exterior derivative $d\theta$ is Hermitian on $T\overline{M}|_{\partial M}$.
- For any minimal complex connection $\nabla$ on $M$, the $(0,2)$-tensor field $\nabla d\rho$ is symmetric on $T\overline{M}|_{\partial M}$.

(2) The induced almost CR structure on $\partial M$ is non-degenerate if and only if for any local defining function $\rho$ the one-form $\theta$ restricts to a contact form on $\partial M$. The structure is partially integrable if and only if the restriction of $d\theta$ to $H \subset T\partial M$ is Hermitian.

**Proof.** By construction, for each $x \in \partial M$, the kernel of $\theta(x)|_{T_x\partial M} : T_x\partial M \to \mathbb{R}$ coincides with $H_x$. For sections $\xi, \eta \in \Gamma(H)$ we thus get $d\theta(\xi, \eta) = -\theta([\xi, \eta])$, so the restriction of $d\theta$ to $H \times H$ represents the negative of the Levi-bracket $\mathcal{L}$. Since non-degeneracy of the restriction of $d\theta$ to $H \subset T\partial M$ is equivalent to $\theta$ being a contact form, this implies (2).

(1) Since the Nijenhuis tensor is conjugate linear in both arguments, its values at each point $x \in \overline{M}$ form a complex subspace of $T_x\overline{M}$. Hence the fact that $\mathcal{N}$ has asymptotically tangential values is equivalent to $\mathcal{N}$ having values in $H$ along the boundary. This is in turn equivalent to the insertion of $\mathcal{N}$ into $d\rho$ or into $\theta$ vanishing along $\partial M$.

Now if $\nabla$ is a complex connection on $T\overline{M}$, then we can compute $0 = dd\rho$ as the sum of the alternation of $\nabla d\rho$ and a term in which the torsion of $\nabla$ is inserted into $d\rho$. If $\nabla$ is minimal, this torsion is a non-zero multiple of $\mathcal{N}$, which implies the equivalence to the second condition.

To complete the proof, we show that for all $\xi, \eta \in \mathcal{X}(\overline{M})$ we get

$$d\theta(J\xi, \eta) + d\theta(\xi, J\eta) = d\rho(\mathcal{N}(\xi, \eta)).$$

Since both sides are bilinear over smooth functions, we may without loss of generality assume that $d\rho(\xi) = \theta(J\xi)$ and $\theta(\xi) = -d\rho(J\xi)$ are constant and likewise
for \( \eta \). Assuming this, we get \( d\theta(J\xi,\eta) = -\theta([J\xi,\eta]) = d\rho(J[J\xi,\eta]) \) and similarly for the second term in the left-hand side. On the other hand, we also see that 
\[
d\rho([\xi,\eta]) = -dd\rho(\xi,\eta) = 0 \quad \text{and likewise} \quad d\rho([J\xi,\eta]) = 0,
\]
so the claim follows from the definition of the Nijenhuis tensor. \( \square \)

### 2.4. Volume asymptotics.

We next analyze the effect of a c-projective change of connection on the induced connections on density bundles. From the interpretation of the c-projective change law via complex linear extensions of one-forms, one easily concludes that on the top exterior power \( \Lambda^m CTM \), the change of connection is given by

\[
\hat{\nabla}_\xi s = \nabla_\xi s + (m+1)(\Upsilon(\xi) - i\Upsilon(J\xi))s,
\]

which also explains the conventions for density bundles we have chosen. This immediately implies that for \( w, w' \in \mathbb{R} \) with \( w - w' \in \mathbb{Z} \) and \( s \in \Gamma(E(w,w')) \) we have

\[
(2.2) \quad \hat{\nabla}_\xi s = \nabla_\xi s + ((w + w')\Upsilon(\xi) - (w - w')i\Upsilon(J\xi))s,
\]

while for \( \sigma \in \Gamma(E(w)) \) with \( w \in \mathbb{R} \) we obtain

\[
(2.3) \quad \hat{\nabla}_\xi \sigma = \nabla_\xi \sigma + w\Upsilon(\xi)\sigma.
\]

Having these results at hand, the relation between c-projective compactness and volume asymptotics can be analyzed as it is done for projective compactness in [8]. We call a connection \( \nabla \) special if and only if there is a non-vanishing section \( \sigma \) of some (or equivalently any) real density bundle \( E(w) \) with \( w \neq 0 \), which is parallel for \( \nabla \). The notion of volume asymptotics as introduced in Definition 2.2 of [8] can be used in our setting. Also, the relation to (real) defining densities continues to hold:

**Proposition 7.** Let \( \bar{M} \) be a smooth manifold of real dimension \( 2m \) with boundary \( \partial M \) and interior \( M \), and let \( \nabla \) be a special complex linear connection on \( TM \).

1. If \( \nabla \) is c-projectively compact, then it has volume asymptotics of order \( m + 1 \) in the sense of Section 2.2 of [8]. Moreover, any non-zero section of \( E(2) \) which is parallel for \( \nabla \) extends by zero to a defining density for \( \partial M \).

2. Conversely, assume that the almost c-projective structure on \( M \) defined by \( \nabla \) admits a smooth extension to \( \bar{M} \) and that there is a defining density \( \tau \in \Gamma(E(2)) \) for \( \partial M \) such that \( \tau|_M \) is parallel for \( \nabla \). Then \( \nabla \) is c-projectively compact.

**Proof.** With some trivial modifications, the proof of Proposition 2.3 of [8] applies. \( \square \)
2.5. **c-projective compactness for metrics.** In the setting of projective compactness, one always deals with torsion-free connections, so projective compactness of a pseudo-Riemannian metric can be defined as projective compactness of its Levi-Civita connection. Looking for a c-projective analog of this concept, we will only consider pseudo-Riemannian metrics $g$ which are Hermitian with respect to the given almost complex structure, i.e., such that $g(J\xi, J\eta) = g(\xi, \eta)$ for all $\xi$ and $\eta$. In order to get a sensible concept, we have to associate to $g$ a connection which is complex and minimal. The latter two conditions specify the torsion of the connection, so if the connection also is required to preserve $g$, it is uniquely determined by these properties (if it exists). This motivates the following definitions.

**Definition 8.** (1) For an almost complex manifold $(N, J)$, a pseudo-Riemannian metric $g$ which is Hermitian for $J$ is called **admissible** if and only if there is a linear connection $\nabla$ on $TN$ which is minimal and preserves both $J$ and $g$. If such a connection exists then we know from above that it is uniquely determined and we call it the **canonical connection** associated to $g$.

(2) Consider a smooth manifold $M$ with boundary $\partial M$ and interior $\mathring{M}$, and an almost complex structure $J$ on $M$. An admissible Hermitian metric $g$ on $(M, J)$ is called **c-projectively compact** if and only if its canonical connection is c-projectively compact in the sense of Definition 2.

Now we can give a description of admissible metrics in terms of the Gray-Hervella classification (see [20]) of almost Hermitian structures.

**Proposition 9.** Let $g$ be a Hermitian pseudo-Riemannian metric on an almost complex manifold $(N, J)$. Then $g$ is admissible if and only if it is quasi-Kähler (i.e., of class $W_1 \oplus W_2$) in the sense of [20].

In particular, nearly Kähler metrics (of arbitrary signature) are admissible, and if $J$ is integrable, then $g$ is admissible if and only if it is (pseudo-)Kähler.

**Proof.** Denoting by $\omega$ the fundamental two-form of $g$, so $\omega(\xi, \eta) := -g(\xi, J\eta)$, the Hervella-Gray classification is based on $\nabla^g \omega$, where $\nabla^g$ denotes the Levi-Civita connection of $g$. Now it is well known that there is a connection $\nabla$ on $N$ for which $\nabla g = 0$ and $\nabla J = 0$ and thus $\nabla \omega = 0$. For such a connection, consider the contorsion, i.e., the $(\frac{1}{2})$-tensor field $A$ defined by $\nabla_\xi \eta = \nabla^g_\xi \eta + A(\xi, \eta)$. Since both $\nabla^g$ and $\nabla$ preserve $g$, we conclude that $g(A(\xi, \eta), \zeta) = -g(A(\xi, \zeta), \eta)$ for all $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

Moreover, if we add a $(\frac{1}{2})$-tensor field to $A$ which has the same skew symmetry property and in addition is complex linear in the second variable, this will also lead to a connection which preserves both $g$ and $J$. This means that we can subtract the complex linear part $A_c(\xi, \eta) := \frac{1}{2}(A(\xi, \eta) - JA(\xi, J\eta))$ from $A$, without losing the property that the resulting connection $\nabla$ preserves both $g$ and $J$. Hence from now on we assume without loss of generality that $\nabla$ is chosen in such a way that $A(\xi, J\eta) = -JA(\xi, \eta)$. 

Now we compute $\nabla^g \omega(\xi,\eta,\zeta)$ as

$$-\xi \cdot g(\eta,J\zeta) + g(\nabla^g_\xi \eta,J\zeta) + g(\eta,J\nabla^g_\xi \zeta).$$

Rewriting the last summand as $-g(J\eta,\nabla^g_\xi \zeta)$ we can express each $\nabla^g$ as $\nabla + A$ and using that $\nabla J = 0$ and $\nabla g = 0$, we obtain

$$\nabla^g \omega(\xi,\eta,\zeta) = g(A(\xi,\eta),J\zeta) - g(J\eta,A(\xi,\zeta)).$$

The last term (including the sign) can be written as $-g(J\eta,\nabla^g_\xi \zeta)$ we can express each $\nabla^g$ as $\nabla + A$ and using that $\nabla J = 0$ and $\nabla g = 0$, we obtain

$$\nabla^g \omega(\xi,\eta,\zeta) = g(A(\xi,\eta),J\zeta) - g(J\eta,A(\xi,\zeta)).$$

According to Theorem 3.1 in [20], $g$ is quasi-Kähler if and only if

$$\nabla^g \omega(\xi,\eta,\zeta) + \nabla^g \omega(J\xi,J\eta,\zeta) = 0$$

for all $\xi,\eta,\zeta$. This is clearly equivalent to $A(\xi,\eta) + A(J\xi,J\eta) = 0$ for all $\xi,\eta \in \mathfrak{X}(N)$. But this implies that $A(J\xi,\eta) = A(\xi,J\eta) = -J A(\xi,\eta)$, so $A$ must be conjugate linear in both variables. Hence $T(\xi,\eta) = A(\xi,\eta) - A(\eta,\xi)$ is also conjugate linear in both variables, and of course, $T$ is the torsion of $\nabla$. Hence $T$ must coincide with $-\frac{1}{4}N$. This shows that any quasi-Kähler metric is admissible and that in the integrable case, $\nabla = \nabla^g$, so $g$ is Kähler. Conversely, if $g$ is admissible and $\nabla$ is its canonical connection with torsion $T = -\frac{1}{4}N$, then one can explicitly compute the contorsion tensor $A$ via

$$g(A(\xi,\eta),J\zeta) = \frac{1}{2} \left( -g(T(\xi,J\zeta),\eta) + g(T(\xi,\eta),J\zeta) - g(T(\eta,J\zeta),\xi) \right).$$

Using that $T$ is conjugate linear in both arguments, one immediately verifies that the right-hand side changes sign if one either replaces $(\xi,\eta)$ by $(J\xi,J\eta)$ or $(\eta,\zeta)$ by $(J\eta,J\zeta)$. Of course, this implies that $A(\xi,\eta) + A(J\xi,J\eta) = 0$ and that $A$ is conjugate linear in the second variable, so $g$ is quasi-Kähler.

### 2.6. A sufficient condition for c-projective compactness.

Motivated by the results for projective compactness in [8, 9], we describe an asymptotic form for a Hermitian metric which is sufficient for c-projective compactness. We start with a manifold $\mathcal{M} = M \cup \partial M$ with boundary, which is endowed with an almost complex structure $J$. As observed in 2.2, this induces an almost CR structure on $\partial M$, which is assumed to be non-degenerate. Moreover, we assume that the Nijenhuis tensor $N$ has asymptotically tangential values, which by Lemma 6 implies that the almost CR structure on $\partial M$ is partially integrable.

Now suppose that $\rho$ is a local defining function for $\partial M$, and consider the one-form $\theta := -d\rho \circ J$ as in 2.3, which in our current setting is smooth up to the boundary. By Lemma 6 our assumptions imply that $\theta$ restricts to a contact form on $\partial M$.
and that $d\theta$ is Hermitian on $T\overline{M}|_{\partial M}$ and non-degenerate on $H$. In particular, the restriction of $d\theta$ to $H$ is the imaginary part of a non-degenerate Hermitian form, so it has a well-defined signature $(p,q)$. Now we assume that $g$ is a pseudo-Riemannian metric on the interior $M$ which is Hermitian for $J$ and has signature $(p+1,q)$ or $(p,q+1)$. The asymptotic form we consider is that, locally near the boundary and for some constant $C$, we can write

$$g = C\left(\frac{d\rho^2}{\rho^2} + \frac{\theta^2}{\rho^2}\right) + \frac{h}{\rho}.$$  

Here $h$ is a Hermitian form which admits a smooth extension to the boundary such that, along the boundary, we have $h(\xi,J\zeta) = C d\theta(\xi,\zeta)$, whenever $\zeta$ lies in the CR subspace. In particular, the boundary value of $h$ is non-degenerate on the CR subspace.

Similarly to the case of projective compactness of order 2, this asymptotic form is independent of the defining function. From Section 2.2, we see that for $\hat{\rho} = e^{f} \rho$ we get $\frac{d\hat{\rho}}{\hat{\rho}} = \frac{d\rho}{\rho} + df$ and $\frac{\hat{\theta}}{\hat{\rho}} = \frac{\theta}{\rho} - df \circ J$. Using this, a simple direct computation shows that an asymptotic form as in (2.4) with respect to $\rho$ implies an analogous form with respect to $\hat{\rho}$ with the same constant $C$ and with

$$\hat{h} = e^{f} h + 2C\left(-df \otimes d\rho + (df \circ J) \otimes \hat{\theta}\right) + C\hat{\rho}(df^2 + (-df \circ J)^2).$$

On the other hand, applying the exterior derivative to the formula for $\hat{\theta}$, one gets

$$d\hat{\theta} = e^{f} d\theta + df \wedge \hat{\theta} - d\hat{\rho} \wedge (df \circ J) - \hat{\rho}d(df \circ J).$$

Along the boundary and for $\zeta$ in the CR subspace, we thus get

$$\hat{h}(\xi,J\zeta) = e^{f} h(\xi,J\zeta) - C df(J\zeta)d\hat{\rho}(\xi) - C df(\zeta)d\hat{\theta}(\xi),$$

and this coincides with $C d\hat{\theta}(\xi,\zeta)$.

Now we can prove our first main result, namely that such an asymptotic form is sufficient for c-projective compactness.

**Theorem 10.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $J$ be an almost complex structure on $\overline{M}$, such that $\partial M$ is non-degenerate and the Nijenhuis tensor $N$ of $J$ asymptotically has tangential values. Let $g$ be an admissible pseudo-Riemannian Hermitian metric on $M$.

For a local defining function $\rho$ for the boundary defined on an open subset $U \subset \overline{M}$, put $\theta = -d\rho \circ J$ and, given a constant $C$, define a Hermitian $(0,2)$-tensor field $h_{\rho,C}$ on $U \cap M$ by

$$h_{\rho,C}(\xi,\eta) := \rho g(\xi,\eta) - \frac{C}{\rho}(d\rho(\xi)d\rho(\eta) + \theta(\xi)\theta(\eta)).$$
Suppose that for each \( x \in \partial M \) there are an open neighborhood \( U \) of \( x \) in \( \overline{M} \), a local defining function \( \rho \) defined on \( U \), and a non-zero constant \( C \) such that

- \( h, C \) admits a smooth extension to all of \( U \)
- for all \( \xi, \zeta \in \mathfrak{X}(U) \) with \( d\rho(\zeta) = \theta(\zeta) = 0 \), the function \( h(\xi, J \zeta) \) approaches \( C d\theta(\xi, \zeta) \) at the boundary.

Then \( g \) is c-projectively compact.

**Proof.** This is parallel to the proof for projective compactness in Theorem 2.6 of [8], and we will partly refer to that proof and emphasize the differences.

Let \( \nabla \) be the canonical connection for \( g \) and let \( \hat{\nabla} \) be the c-projectively modified connection corresponding to \( \gamma = \frac{d\rho}{\partial \rho} \). We have to show that for arbitrary vector fields \( \xi \) and \( \eta \) which are smooth on all of \( U \), also \( \hat{\nabla}\xi \eta \) admits a smooth extension from \( U \cap M \) to all of \( U \). In order to prove this, it suffices to do the following. We first show \( d\rho(\hat{\nabla}\xi \eta) \) admits a smooth extension to the boundary. Next, we prove that for any \( \zeta \in \mathfrak{X}(U) \) such that \( d\rho(\zeta) \) and \( \theta(\zeta) \) vanish identically, \( h(\hat{\nabla}\xi \eta, \zeta) \) admits a smooth extension to all of \( U \). Since we can write \( \theta(\hat{\nabla}\xi \eta) \) as \( -d\rho(\hat{\nabla}\xi \eta) \), these two facts imply that the coordinate functions of \( \hat{\nabla}\xi \eta \) with respect to an appropriate frame for \( T\overline{M}|_U \) admit a smooth extension to the boundary and hence the result.

Shrinking \( U \) if necessary, we may assume that, on \( U \), \( d\rho \) is nowhere vanishing and \( h \) is non-degenerate on \( \ker(d\rho) \cap \ker(\theta) \). Extend \( d\rho \) and \( \theta \) to a coframe for \( \overline{M} \) on \( U \) and take the first element \( \zeta_0 \) in the dual frame. Then non-degeneracy of \( h \) on \( \ker(d\rho) \cap \ker(\theta) \) implies that we can add a section of this subbundle to \( \zeta_0 \) to obtain a vector field \( \zeta_0 \in \mathfrak{X}(U) \) such that \( d\rho(\zeta_0) = 1 \), \( \theta(\zeta_0) = 0 \) and \( h(\xi, \zeta_0) = 0 \), whenever \( \xi \in \mathfrak{X}(U) \) satisfies \( d\rho(\xi) = \theta(\xi) = 0 \).

Now an arbitrary vector field \( \xi \in \mathfrak{X}(U) \) can be written as \( \xi = d\rho(\xi)\zeta_0 + \theta(\xi)J \zeta_0 + \xi_H \), where \( d\rho(\xi_H) = \theta(\xi_H) = 0 \). Inserting into the defining equation for \( h \) and multiplying by \( \rho \), we conclude that \( \rho^2 g(\xi, \zeta_0) = d\rho(\xi)(C + \rho h(\zeta_0, \zeta_0)) \). In particular, we see that we can prove that \( d\rho(\hat{\nabla}\xi \eta) \) admits a smooth extension to all of \( U \) by showing that \( \rho^2 g(\hat{\nabla}\xi \eta, \zeta_0) \) admits such an extension.

On the other hand, assume that \( \zeta \in \mathfrak{X}(U) \) has the property that \( d\rho(\zeta) \) and \( \theta(\zeta) \) vanish identically. Then inserting into the defining equation for \( h \) we see that for each \( \xi \in \mathfrak{X}(U) \), we get \( \rho g(\xi, \zeta) = h(\xi, \zeta) \). Hence we can show that for each such \( \zeta \), \( \rho g(\hat{\nabla}\xi \eta, \zeta) \) admits a smooth extension to all of \( U \) in order to verify the second claimed smoothness property.

As in the projective case, we use a modification of the Koszul-formula for the Levi-Civita connection in order to prove existence of smooth extensions. Since we are dealing with a metric connection with torsion here, the Koszul formula becomes a bit more complicated. Recall that to prove the Koszul formula, one uses torsion freeness of the Levi-Civita connection, which brings in the terms involving Lie brackets. In the presence of torsion, one simply has to add, for each Lie bracket term, a term involving the torsion with the same configuration of arguments. Taking into account that the torsion of a minimal complex connection equals \(-\frac{1}{4}N\), we
conclude that for the canonical connection $\nabla$, we get
\begin{equation}
2g(\nabla_\xi \eta, \zeta) = \xi \cdot g(\eta, \zeta) - \zeta \cdot g(\xi, \eta) + \eta \cdot g(\xi, \zeta) \\
+ g([\xi, \eta], \zeta) - g([\xi, \zeta], \eta) - g([\eta, \zeta], \xi) \\
- \frac{1}{4}g(N(\xi, \eta), \zeta) + \frac{1}{4}g(N(\xi, \zeta), \eta) + \frac{1}{4}g(N(\eta, \zeta), \xi).
\end{equation}
\tag{2.5}

To compute $2g(\hat{\nabla}_\xi \eta, \zeta)$ we have to add to this the expression
\begin{equation}
\frac{d\rho(\xi)}{\rho} g(\eta, \zeta) + \frac{d\rho(\eta)}{\rho} g(\xi, \zeta) + \frac{\theta(\xi)}{\rho} g(J_\eta, \zeta) + \frac{\theta(\eta)}{\rho} g(J_\xi, \zeta)
\end{equation}
\tag{2.6}

In the first step, we use this formula to compute $2g(\hat{\nabla}_\xi \eta, \zeta_0)$ for $\xi, \eta \in \mathcal{X}(U)$, and we ignore terms which admit a smooth extension to the boundary after multiplication by $\rho^2$. In particular, this applies to all terms in the second and third line of (2.5). Now from above we know that $g(\eta, \zeta_0) = \frac{d\rho(\eta)}{\rho^2}(C + \rho h(\zeta_0, \zeta_0))$, and, even if we differentiate once more or multiply by $\frac{1}{\rho}$, the second summand will admit a smooth extension after multiplication by $\rho^2$. Hence in the computations, we may replace $g(\eta, \zeta_0)$ by $C \frac{d\rho(\eta)}{\rho^2}$ and $g(J_\eta, \zeta_0)$ by $-C \frac{\theta(\eta)}{\rho^2}$ and likewise for $\xi$. In particular, the contributions from (2.6) simply add up to
\begin{equation}
2C \frac{1}{\rho^3} (d\rho(\xi)d\rho(\eta) - \theta(\xi)\theta(\eta)).
\end{equation}

On the other hand, in the terms from the first line of (2.5), we only have to take into account those parts in which a vector field differentiates the factor $\frac{1}{\rho}$. This immediately shows that the first and third terms each contribute $-2C \frac{1}{\rho^3} d\rho(\xi)d\rho(\eta)$, while from the second term we get
\begin{equation}
+2C \frac{1}{\rho^3} (d\rho(\xi)d\rho(\eta) + \theta(\xi)\theta(\eta)).
\end{equation}

This completes the proof that $\rho^2 g(\hat{\nabla}_\xi \eta, \zeta_0)$ admits a smooth extension to the boundary.

For the second part, we take $\zeta \in \mathcal{X}(U)$ such that $d\rho(\zeta)$ and $\theta(\zeta)$ vanish identically and we can ignore terms which admit a smooth extension to all of $U$ after multiplication by $\rho$. Then $g(\eta, \zeta) = \frac{1}{\rho} h(\eta, \zeta)$ and likewise for $\xi$, so the contribution of the terms from (2.6) reads as
\begin{equation}
(2.7) \quad \frac{1}{\rho^2} \left( d\rho(\xi)h(\eta, \zeta) + d\rho(\eta)h(\xi, \zeta) + \theta(\xi)h(J_\eta, \zeta) + \theta(\eta)h(J_\xi, \zeta) \right).
\end{equation}

The first two summands in (2.7) are immediately seen to cancel with the contributions coming from the first and third summand in the right-hand side of (2.5). The term $g([\xi, \eta], \zeta)$ clearly admits a smooth extension to the boundary after multiplication by $\rho$ and the same holds for all terms in (2.5) which involve $N'$. by the
assumption on asymptotically tangential values. Hence we are left with determining the contribution of

$$-\zeta \cdot g(\xi,\eta) - g([\xi,\zeta],\eta) - g([\eta,\zeta],\xi).$$

By assumption $\zeta \cdot \rho = 0$, so in all terms we only need the $\frac{1}{\rho^2}$-terms in the formula for $g$. Now $d\rho(\zeta) = 0$ implies $-\zeta \cdot d\rho(\xi) - d\rho([\xi,\zeta]) = d\rho(\xi,\zeta) = 0$, and since $\theta(\zeta) = 0$, we similarly get $-\zeta \cdot \theta(\xi) - \theta([\xi,\zeta]) = d\theta(\xi,\zeta)$. This shows that the total contribution from our three remaining terms is given by

$$C \frac{1}{\rho^2} (d\theta(\xi,\zeta)\theta(\eta) + d\theta(\eta,\zeta)\theta(\xi)).$$

By our assumption on the relation between $h$ and $d\theta$, this cancels with the remaining two summands from (2.7), which completes the proof. $\square$

### 2.7. A class of examples.

Using Theorem 10 we can now construct a class of examples of c-projectively compact metrics which includes the classical examples of complete Kähler metrics associated to smoothly bounded domains in $\mathbb{C}^n$.

Consider an almost complex manifold $(\mathcal{M},J)$ and a domain $U \subset \mathcal{M}$ with smooth boundary $\partial U$. Then $\partial U$ inherits an almost CR structure (see 2.2) and we assume that this structure is Levi non-degenerate. Choose a defining function $\rho$ for $\partial U$ defined on a neighborhood of $\bar{U}$ in $\mathcal{M}$. Assuming that $\rho > 0$ on $U$ the function $\log \rho : U \to \mathbb{R}$ is smooth. Now putting $\omega = d(-d\log \rho \circ J) \in \Omega^2(U)$, we define $g_\rho \in \Gamma(S^2T^*U)$ by $g_\rho(\xi,\eta) := \omega(\xi,J\eta)$ for $\xi,\eta \in \mathfrak{X}(U)$.

**Proposition 11.** For a domain $U$ in an almost complex manifold $(\mathcal{M},J)$ with smooth, Levi non-degenerate boundary $\partial U$ consider the section $g_\rho$ associated to a defining function $\rho$ for $\partial U$ which is positive on $U$.

1. There is an open neighborhood $V$ of $\partial U$ in $\bar{U}$ such that $g_\rho$ defines a smooth pseudo-Riemannian metric on $V \cap U$.

2. If $d\rho(N(\xi,\eta)) = 0$ for all $\xi,\eta \in \mathfrak{X}(U)$, then $g_\rho$ is Hermitian with respect to $J$.

3. If in the setting of (2) $g_\rho$ is admissible, then it is c-projectively compact. This is always the case if $J$ is integrable.

**Proof.** Let us use the notation from 2.3 and put $\theta = -d\rho \circ J$. Since $d\log \rho = \frac{d\rho}{\rho}$ our definition implies that $\omega = d\frac{1}{\rho^2} \theta = -\frac{1}{\rho^2} d\rho \wedge \theta + \frac{1}{\rho} d\theta$. The definition of $g_\rho$ then directly implies that

$$g_\rho(\xi,\eta) = -\frac{1}{\rho^2} (d\rho(\xi)d\rho(\eta) + \theta(\xi)\theta(\eta)) + \frac{1}{\rho} d\theta(\xi,J\eta).$$

Since we have assumed that $\partial U$ is Levi-non-degenerate, $N(\xi,\eta) \mapsto d\theta(\xi,J\eta)$ is non-degenerate on $\ker(d\rho) \cap \ker(\theta)$ along the boundary. Hence there is a neighborhood
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V of ∂U in \(\overline{U}\) such that \(d\theta\) is non-degenerate on \(\ker(d\rho) \cap \ker(\theta)\) on all of V. This readily implies that \(g_\rho\) is non-degenerate on \(V \cap U\) so (1) is proved.

(2) The proof of Lemma 6 shows that \(d\theta\) is Hermitian on \(U\), and then formula (2.8) immediately implies that \(g_\rho\) is Hermitian on \(U\), too.

(3) In the setting of (2), \(\omega\) is (up to sign) the fundamental two-form associated to the Hermitian metric \(g_\rho\). But by construction \(\omega\) is exact and thus closed. In the integrable case, this implies that \(g_\rho\) is Kähler and hence admissible by Proposition 9. Knowing that \(g_\rho\) is admissible, we can multiply formula (2.8) by \(\rho\) and bring the first term on the right-hand side to the left-hand side to see that the assumptions of Theorem 10 are satisfied with \(C = -1\) and \(h(\xi, \eta) = d\theta(\xi, J_\eta)\).

\[\square\]

3. c-projective tractors. To prove general results on the existence of asymptotic forms, as in Theorem 10, we use the c-projective version of tractor calculus. A particularly nice instance of this situation is provided by reductions of c-projective holonomy to a unitary group, which, in the integrable case, are related to Kähler-Einstein metrics.

3.1. The c-projective Schouten tensor. As an important ingredient for tractor calculus, we first discuss the c-projective version of the Schouten tensor (or Rho-tensor). Similarly to the well-known cases of conformal and projective structures, this is a curvature quantity which is closely related to the Ricci curvature, but has nicer transformation laws under a c-projective change. It will be very important for us to work in a purely real picture (i.e., without complexifying the tangent bundle). We will also formulate the results mainly in terms of abstract indices.

Let \((N, J)\) be an almost complex manifold, let \(V\) be a complex linear connection on \(TN\) and let \(R\) be its curvature tensor. In abstract index notation, the almost complex structure is denote by \(J^a_i\) via \(J(\xi)^a = J^a_i \xi^i\), while we write the curvature tensor as \(R_{ab}^c_d\) via \(R(\xi, \eta)(\zeta)^c = R_{ij}^c_k \xi^i \eta^j \zeta^k\). From the construction it follows readily that \(R\) has values in complex linear maps, i.e., \(R_{ab}^c_i J^i_d = R_{ab}^i d J^c_i\). The Ricci tensor \(\text{Ric} = \text{Ric}_{ab}\) of \(V\) is then defined as usual by \(\text{Ric}_{ab} := R_{ia}^i b\). Observe that \(\text{Ric}\) is not symmetric in general. Finally, the c-projective Schouten tensor \(P = P_{ab}\) of \(V\) is defined by

\[
P_{ab} := \frac{1}{2(m+1)} \left( \text{Ric}_{ab} + \frac{1}{m-1} \left( \text{Ric}_{(ab)} - J_a^i J_b^j \text{Ric}_{(ij)} \right) \right),
\]

where the real dimension of \(N\) is \(2m\). Observe that this definition immediately implies that \(P\) is symmetric provided that \(\text{Ric}\) is symmetric. Moreover, if \(\text{Ric}\) is Hermitian, i.e., \(\text{Ric}_{ab} = J_a^i J_b^j \text{Ric}_{ij}\), then \(P\) is just a constant multiple of \(\text{Ric}\).

Now we can describe a characterization of the Schouten tensor as well as its transformation law under a c-projective change of connection.
Proposition 12. (1) The Schouten tensor is uniquely characterized by the property that defining \( W_{abcd} \)

\[ R_{abcd} - 2(\delta^e_{[a}P_{bj]d} - P_{[ab]}\delta^e_d - J^i_{[a}P_{bj]i}J^e_d - J^e_{[a}P_{bj]i}J^i_d), \]

one has \( W_{ia}^b = 0 \).

(2) If \( \hat{\nabla} \) is a connection which is c-projectively related to \( \nabla \) via \( \hat{\nabla} = \nabla + \Upsilon \) as in formula (2.1) for a one-form \( \Upsilon = \Upsilon_a \), then the Schouten tensor \( \hat{P} \) of \( \hat{\nabla} \) is given by

\[ \hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - J^i_{a}J^j_{b} \Upsilon_i \Upsilon_j. \]

Proof. Both properties can be established via slightly lengthy but straightforward computations. Alternatively, they can be deduced from the formulae in Section 2.4 of [5] taking into account the slightly different conventions we use.

3.2. The tractor bundle of Hermitian forms. Recall from Section 2.1 that almost c-projective structures of real dimension \( n = 2m \) with a chosen root \( E(1,0) \) of the canonical bundle are equivalent to regular normal Cartan geometries of type \( (G,P) \). Here \( G = SL(m+1,\mathbb{C}) \) and \( P \subset G \) is the stabilizer of a line in the standard representation \( \mathbb{C}^{m+1} \) of \( G \). We have also noted there that the Cartan geometry can be equivalently encoded as the standard tractor bundle \( \mathcal{T} \), which is the associated bundle with respect to the restriction of the representation \( \mathbb{C}^{m+1} \) to \( P \) and the standard tractor connection \( \nabla^T \), the linear connection on \( \mathcal{T} \) induced by the Cartan connection.

We will mainly be interested in the bundle \( \mathcal{H} \) of Hermitian forms on \( \mathcal{T} \) and in its dual \( \mathcal{H}^* \), which can be interpreted as Hermitian forms on \( \mathcal{T}^* \). The bundle \( \mathcal{H} \) is a real subbundle of the tensor product of the complex dual of \( \mathcal{T} \) and its conjugate dual. Since the fact that we deal with sections of the real bundle is of crucial importance for us, we will avoid working with complexifications. In the real picture, the simplest way to view Hermitian forms is as symmetric real valued bilinear forms for which the almost complex structure is orthogonal (and hence also skew symmetric).

Our first task is to deduce the composition structure of \( \mathcal{H} \) induced by the canonical composition structure of \( \mathcal{T} \). The line in \( \mathbb{C}^{m+1} \) stabilized by \( P \) gives rise to a complex line subbundle \( \mathcal{T}^1 \subset \mathcal{T} \). For an almost c-projective structure on an almost complex manifold \( (N,J) \) it turns out that \( \mathcal{T}^1 \) can be naturally identified with the density bundle \( \mathcal{E}(-1,0) \), while \( \mathcal{T}/\mathcal{T}^1 \cong TN(-1,0) \), and we summarize this composition structure as \( \mathcal{T} \cong \mathcal{E}(-1,0) \oplus TN(-1,0) \).

Now since \( \mathcal{E}(-1,0) \) is a complex line bundle, the bundle of Hermitian bilinear forms on \( \mathcal{E}(-1,0) \) is a real line bundle, which naturally includes into \( \mathcal{E}(1,1) \), so this is the real density bundle \( \mathcal{E}(2) \). Given a Hermitian bilinear form on \( \mathcal{T} \), we can restrict it to \( \mathcal{T}^1 \times \mathcal{T}^1 \), thus defining a surjective bundle map \( \mathcal{H} \to \mathcal{E}(2) \), whose
kernel defines a subbundle $H^1 \subset H$ of real corank one. Taking an element of $H^1$ and restricting to $T \times T^1$, the result descends to a bilinear form on $(T/T^1) \times T^1$, and we define $H^2 \subset H^1$ to be the subbundle of those elements for which also this induced form vanishes. Thus we have obtained a filtration $H \supset H^1 \supset H^2$ of the bundle $H$. Now we can describe the composition structure of $H$.

**Proposition 13.** The composition structure for the canonical filtration $H \supset H^1 \supset H^2$ of $H$ has the form

$$H \cong \left( \text{Herm}(TN) \otimes \mathcal{E}(2) \right) \oplus \left( T^* N \otimes \mathcal{E}(2) \right) \oplus \mathcal{E}(2),$$

where all tensor products are over $\mathbb{R}$ and $\text{Herm}(TN)$ denotes the bundle of (real valued) Hermitian forms on the complex vector bundle $TN$.

**Proof.** We have already seen that $H/H^1 \cong \mathcal{E}(2)$ and that $H^1/H^2$ and $H^2$ can be identified with the space of Hermitian bilinear forms on $T/T^1 \times T^1$ and on $T/T^1 \times T/T^1$, respectively. Identifying these spaces of Hermitian forms is a purely pointwise question of linear algebra.

Thus let us assume that $V$ and $\ell$ are complex vector spaces of complex dimension $m$ and $1$, respectively. To complete the proof we have to describe Hermitian bilinear forms on $(V \otimes \mathbb{C} \ell) \times \ell$ and $(V \otimes \mathbb{C} \ell) \times (V \otimes \mathbb{C} \ell)$ in terms of the space $H_\ell$ of Hermitian forms on $\ell$.

Observe first that the complex tensor product $V \otimes \mathbb{C} \ell$ is the quotient of the real tensor product $V \otimes \ell$ by the linear subspace spanned by all elements of the form $v \otimes a + iv \otimes ia$ for $v \in V$ and $a \in \ell$. Given a (real) linear map $\varphi : V \rightarrow H_\ell$ we consider the map $V \times \ell \rightarrow L(\ell, \mathbb{R})$ mapping $(v, a)$ to $\varphi(v)(a, \_)-\varphi(iv)(ia, \_).$ This is evidently bilinear over the reals, so it factorizes to $V \otimes \ell$ and then kills all elements of the form $v \otimes a + iv \otimes ia$, so it induces a bilinear map $(V \otimes \mathbb{C} \ell) \times \ell \rightarrow \mathbb{R}$. Now the complex structure on $V \otimes \mathbb{C} \ell$ is given by multiplication by $i$ on either component. So to verify that our map is indeed Hermitian, it suffices to compare the values of the corresponding trilinear map $V \times \ell \times \ell \rightarrow \mathbb{R}$ on $(v, a, b)$ and $(v, ia, ib)$. But for the latter element we get

$$\varphi(v)(ia, ib) - \varphi(iv)(-a, ib) = \varphi(v)(a, b) - \varphi(iv)(ia, b).$$

Thus we have constructed a linear map from $L_\mathbb{R}(V, H_\ell)$ to the space of Hermitian bilinear forms on $(V \otimes \mathbb{C} \ell) \times \ell$. If $\varphi$ is in the kernel of this map, we in particular get $0 = \varphi(v)(a, a) - \varphi(iv)(ia, a)$ for all $v \in V$ and $a \in \ell$. But any element of $H_\ell$ vanishes on $(ia, a)$ and non-zero elements are non-zero on $(a, a)$ provided that $a \neq 0$. Hence we see that $\varphi(v) = 0$ for all $v \in V$. Thus our map is injective and hence a linear isomorphism by a dimension count.

Next assume that $\psi : V \times V \rightarrow H_\ell$ is symmetric and bilinear (over $\mathbb{R}$) and satisfies $\psi(iv, iw) = \psi(v, w)$ for all $v, w \in V$. Then similar to the above we can
consider the 4-linear map $V \times \ell \times V \times \ell$ defined by
\[
(v, a, w, b) \mapsto \psi(v, w)(a, b) - \psi(iv, w)(ia, b).
\]

One verifies step by step that this map induces a Hermitian bilinear map $(V \otimes_{\mathbb{C}} \ell) \times (V \otimes_{\mathbb{C}} \ell) \to \mathbb{R}$ and that this gives rise to the required isomorphism. \qed

Using this result, we can now give a description of $\mathcal{H}$ in a purely real picture. We start by discussing the explicit description of $T$ following \cite{5} and the general results in Chapter 5 of \cite{13}.

A choice of a connection $\nabla$ in the $c$-projective class gives rise to a splitting $T \cong (TN \otimes_{\mathbb{C}} \mathcal{E}(-1, 0)) \oplus \mathcal{E}(-1, 0)$ of the canonical filtration. We will write elements of this bundle in such a splitting as vectors of the form $(\xi_{\nu} \otimes_{\mathbb{C}} \rho)$ with $\nu, \rho$ in $\mathcal{E}(-1, 0)$ and $\xi \in TM$. In this picture, the behavior under a change of connection is easy to describe. Changing from $\nabla$ to $\hat{\nabla} = \nabla + \Upsilon$ as in (2.1), the change is given by
\[
(3.3) \quad \left(\frac{\xi \otimes_{\mathbb{C}} \rho}{\nu}\right) = \left(\frac{\xi \otimes_{\mathbb{C}} \rho}{\nu} - \Upsilon(\xi)_{\rho} + \Upsilon(J\xi)_{i\rho}\right).
\]

Here $\Upsilon$ is viewed as real valued and as in the proof of Proposition 13 one checks that this expression really makes sense for a tensor product over $\mathbb{C}$.

Now in view of Proposition 13, having chosen a connection $\nabla$ in the $c$-projective class, we represent elements of $\mathcal{H}$ as triples (or vectors) of the form $(\tau, \varphi, \psi)$ were $\tau$ is an element of the space $\mathcal{E}(2)$ of Hermitian forms on $\mathcal{E}(-1, 0)$, $\varphi$ is a linear map $TN \to \mathcal{E}(2)$, and $\psi$ is a Hermitian form $TN \times TN \to \mathcal{E}(2)$. The action of such a triple is then defined as
\[
(3.4) \quad \left(\begin{array}{c}
\tau \\
\varphi \\
\psi
\end{array}\right) \left(\begin{array}{c}
(\xi_{1} \otimes_{\mathbb{C}} \rho_{1})_{\nu_{1}} \\
(\xi_{2} \otimes_{\mathbb{C}} \rho_{2})_{\nu_{2}}
\end{array}\right) := \tau(\nu_{1}, \nu_{2}) + \varphi(\xi_{1})(\rho_{1}, \nu_{2}) - \varphi(J\xi_{1})(i\rho_{1}, \nu_{2}) + \varphi(\xi_{2})(\rho_{2}, \nu_{1}) \]
\[- \varphi(J\xi_{2})(i\rho_{2}, \nu_{1}) + \psi(\xi_{1}, \xi_{2})(\rho_{1}, \rho_{2}) - \psi(J\xi_{1}, \xi_{2})(i\rho_{1}, \rho_{2}).
\]

Changing to a different connection $\hat{\nabla} = \nabla + \Upsilon$ as in (2.1) we know the change of splittings on $T$, and of course the change of splitting on $\mathcal{H}$ is determined by
\[
\left(\begin{array}{c}
\tau \\
\varphi \\
\psi
\end{array}\right) \left(\begin{array}{c}
(\xi_{1} \otimes_{\mathbb{C}} \rho_{1})_{\nu_{1}} \\
(\xi_{2} \otimes_{\mathbb{C}} \rho_{2})_{\nu_{2}}
\end{array}\right) = \left(\begin{array}{c}
\tau \\
\varphi \\
\psi
\end{array}\right) \left(\begin{array}{c}
(\xi_{1} \otimes_{\mathbb{C}} \rho_{1})_{\nu_{1}} \\
(\xi_{2} \otimes_{\mathbb{C}} \rho_{2})_{\nu_{2}}
\end{array}\right).
\]

A lengthy but straightforward computation then shows that (as expected) $\hat{\tau} = \tau$, while $\hat{\varphi}(\xi) = \varphi(\xi) + \Upsilon(\xi)\tau$ and
\[
\hat{\psi}(\xi_{1}, \xi_{2}) = \psi(\xi_{1}, \xi_{2}) + \Upsilon(\xi_{1})\varphi(\xi_{2}) + \Upsilon(J\xi_{1})\varphi(J\xi_{2}) + \Upsilon(\xi_{2})\varphi(\xi_{1}) + \Upsilon(J\xi_{2})\varphi(J\xi_{1}) - \Upsilon(\xi_{1})\Upsilon(\xi_{2})\tau.
\]
This shows that we can nicely formulate things in (real) abstract index notation. Following the usual conventions, we write \( \phi = \varphi_a \) for a section of \( \mathcal{E}_a(2) \) and \( \psi = \psi_{ab} \) a section of \( \mathcal{E}_{ab}(2) \) such that \( \varphi_{ab} = \varphi_{cd}J_c^aJ_d^b \). Then the transformation determined by \( \Upsilon = \Upsilon_a \) (a section of \( \mathcal{E}_a \)) can be expressed as

\[
\begin{pmatrix}
\tau \\
\varphi_a \\
\psi_{bc}
\end{pmatrix} = \begin{pmatrix}
\tau \\
\varphi_a + \Upsilon_a \tau \\
\psi_{bc} + (\delta^i_b \delta^j_c + J_i^b J_j^c) (\Upsilon_i \varphi_j + \Upsilon_j \varphi_i + \Upsilon_i \Upsilon_j \tau)
\end{pmatrix}.
\]

(3.5)

3.3. The tractor connection on \( \mathcal{H} \). The relation between \( \mathcal{H} \) and \( \mathcal{T} \) introduced in 3.2 can be also used to compute the tractor connection \( \nabla^\mathcal{H} \) on \( \mathcal{H} \). The first step towards this to describe the standard tractor connection on \( \mathcal{T} \). This can be taken from [5] or directly deduced from the principles in Section 5.2 of [13] (taking into account different sign conventions for the rho-tensor). In the notation from Section 3.2 for the standard tractor bundle, this can be explicitly described as

\[
\nabla^\mathcal{T}_\eta \left( \xi \otimes_C \rho \right) = \begin{pmatrix}
(\nabla_\eta \xi) \otimes_C \rho + \xi \otimes_C \nabla_\eta \rho + \eta \otimes_C \nu \\
\nabla_\eta \nu - P(\eta, \xi) \rho + P(\eta, J\xi) i \rho
\end{pmatrix},
\]

(3.6)

where \( P \) is the Schouten tensor defined in Section 3.1. As before, we view \( P \) as an ordinary \( (0,2) \)-tensor field on \( N \), i.e., \( P(x) \) is a real valued bilinear form on \( T_x N \). Nonetheless, the combination occurring in the second row is well defined on complex tensor products.

Now we can use the paring defined in (3.4) to compute the tractor connection \( \nabla^\mathcal{H} \) on \( \mathcal{H} \) via

\[
(\nabla^\mathcal{H}_\eta \Phi)(s_1, s_2) = \eta \cdot (\Phi(s_1, s_2)) - \Phi(\nabla^\mathcal{T}_\eta s_1, s_2) - \Phi(s_1, \nabla^\mathcal{T}_\eta s_2)
\]

for \( \Phi \in \Gamma(\mathcal{H}) \) and \( s_1, s_2 \in \Gamma(\mathcal{T}) \). Computing this in the splitting determined by a choice of connection \( \nabla \) from the c-projective class is again straightforward but a bit tedious, and the result is

\[
\nabla^\mathcal{H}_\eta \begin{pmatrix}
\tau \\
\varphi \\
\psi
\end{pmatrix} = \begin{pmatrix}
\nabla_\eta \tau - 2 \varphi(\eta) \\
\nabla_\eta \varphi + P(\eta, \varphi) \psi(\eta, \varphi) \\
\nabla_\eta \psi + A(\eta)
\end{pmatrix},
\]

where \( A \) is defined by

\[
A(\eta)(\xi_1, \xi_2) = P(\eta, \xi_1) \varphi(\xi_2) + P(\eta, \xi_2) \varphi(\xi_1) + P(\eta, J\xi_1) \varphi(J\xi_2) + P(\eta, J\xi_2) \varphi(J\xi_1).
\]
But this shows that we can express the tractor connection on $\mathcal{H}$ nicely in the abstract index notation introduced in Section 3.2 as

$$
\nabla^H_a \begin{pmatrix} \tau \\ \varphi_b \\ \psi_{cd} \end{pmatrix} = 
\begin{pmatrix}
\nabla_a \tau - 2 \varphi_a \\
\nabla_a \varphi_b + P_{ab} \tau - \psi_{ab}
\end{pmatrix} + 
\begin{pmatrix}
\nabla_a \psi_{cd} + P_{ac} \varphi_d + P_{ad} \varphi_c + P_{ai} J_i^j \varphi_j J^i_c + P_{ai} J^i_c \varphi_j J^i_j \\
\nabla a \varphi + P_{ad} \varphi_c + P_{ai} J_i^j \varphi_j J^i_d + P_{ai} J^i_d \varphi_j J^i_j
\end{pmatrix}.
$$

(3.7)

**Remark 14.** Formally, there is no need to use the relation to parabolic geometries in order to obtain the elements of tractor calculus we need here. Following the spirit of [2], one could define the standard tractor bundle $T$ by requiring that any choice of a connection $\nabla$ from the c-projective class defines an isomorphism $T \cong (TN \otimes \mathcal{E}(-1,0)) \oplus \mathcal{E}(-1,0)$ and that the change of isomorphism for a c-projectively related connection $\hat{\nabla}$ is described by (3.3). Using the transformation of the Schouten tensor from Proposition 12 as well as formula (2.2), one then verifies directly that the result of formula (3.6) is the same for all connections in the c-projective class. Hence one can use it to *define* a linear connection $\nabla^T$ on $T$, and then clearly the pair $(T, \nabla^T)$ is canonically associated to the almost c-projective structure (and the choice of $\mathcal{E}(1,0)$). This basically recovers all the input needed for the further developments.

### 3.4. The metricity bundle and its connection.

The *metricity bundle* is the dual bundle $\mathcal{H}^*$ to $\mathcal{H}$, which of course inherits a connection $\nabla^{\mathcal{H}^*}$ from $\nabla^\mathcal{H}$ (and hence from the standard tractor connection). It can be viewed as the bundle of Hermitian forms on $\mathcal{T}^*$, but this point of view is not important at this stage. We can recover the description of the bundle and the connection just by dualizing $\mathcal{H}$.

From the duality, we can readily derive a canonical composition structure of $\mathcal{H}^*$. The annihilators of $\mathcal{H}_1 \supset \mathcal{H}_2$ define a natural real line subbundle contained in a natural subbundle of real rank $2m + 1$. The corresponding composition series has the form

$$
\mathcal{H}^* = \mathcal{E}(-2) \oplus (TN \otimes \mathcal{E}(-2)) \oplus (\text{Herm}(T^*N) \otimes \mathcal{E}(-2)).
$$

Choosing a connection $\nabla$ in the c-projective class, one obtains a splitting of this filtration, i.e., an isomorphism $\mathcal{H}^* \cong (\text{Herm}(T^*N) \otimes \mathcal{E}(-2)) \oplus (TN \otimes \mathcal{E}(-2)) \oplus \mathcal{E}(-2)$. We denote corresponding elements as column vectors $(\sigma^{ab}, \mu^c, \nu)^t$. We fix the pairing of this with elements in the splitting of $\mathcal{H}$ determined by $\nabla$ from Section 3.2 by

$$
\langle (\sigma^{ab}, \mu^c, \nu)^t, (\tau, \varphi_a, \psi_{cd})^t \rangle := \tau \nu + \varphi_i \mu^i + \frac{1}{2} \psi_{ij} \sigma^{ij}.
$$

(3.8)

In most of what follows, the precise form of this pairing is not important, but at one stage, we have to compute the inverse of a non-degenerate section of $\mathcal{H}^*$. There the factors are important, so let us briefly explain this choice. To determine the inverse, one converts a Hermitian form on $\mathcal{T}$ into a linear map $\mathcal{T} \to \mathcal{T}^*$ and then converts the inverse of this map into a bilinear form on $\mathcal{T}^*$. The natural idea to choose a pairing thus is to convert the Hermitian forms into linear maps, compose them to
obtain an endomorphism of $T$ and define the pairing to be its trace. However, we have to be careful about the difference between real and complex traces in this setting. Taking the pairing of two elements concentrated in the one-dimensional slots as $\tau\nu$ shows that we work with the real part of the complex trace, since these act on a complex line bundle. But then the factor $\psi_{ai}\sigma^{ib}$ is the real matrix representing a complex linear map, so the real part of its complex trace is half the real trace, which motivates putting $\frac{1}{2}\psi_{ij}\sigma^{ij}$ into the pairing. Since we will need inverses only in scales for which the middle slots are trivial, this is all the information we need.

The choice of a factor one for $\phi_{i}\mu^{i}$ is motivated by the fact that, while we take a real trace, $\phi_{b}$ and $\mu_{c}$ each represent one row and one column in a Hermitian matrix, so usually, they would be counted with a factor two.

**Proposition 15.** Changing from $\nabla$ to a c-projectively related connection $\hat{\nabla} = \nabla + \Upsilon$ as in formula (2.1) with a one-form $\Upsilon = \Upsilon_{a}$, the corresponding splittings of the composition structure are related by

$$
\begin{pmatrix}
\sigma^{ab} \\
\mu^{c} \\
\nu
\end{pmatrix}
= \begin{pmatrix}
\sigma^{ab} \\
\mu^{c} - 2\Upsilon_{i}\sigma^{ic} \\
\nu - \Upsilon_{i}\mu^{i} + \Upsilon_{j}\Upsilon^{ij}
\end{pmatrix}
$$

(3.9)

The connection $\nabla^{H^{*}}_{a}$ dual to $\nabla^{H}$ is, in the splitting corresponding to $\nabla$, given by

$$
\nabla^{H^{*}}_{a} \begin{pmatrix}
\sigma^{bc} \\
\mu^{d} \\
\nu
\end{pmatrix}
= \begin{pmatrix}
\nabla_{a}\sigma^{bc} + \delta^{(b}_{a}\mu^{c) + J_{a}^{(b}\Upsilon^{c)}}_{i}\mu^{i} \\
\nabla_{a}\mu^{d} - 2\sigma^{di}P_{ai} + 2\nu\delta^{d}_{a} \\
\nabla_{a}\nu - \mu^{i}P_{ai}
\end{pmatrix}
$$

(3.10)

**Proof.** The change of splittings is determined by the fact that the pairing in formula (3.8) has to be the same in both splittings. The claimed formula then follows from (3.5) by a simple direct computation. Likewise, the dual connections by definition satisfy

$$
\nabla_{a} \begin{pmatrix}
\tau \\
\varphi_{b} \\
\psi_{cd}
\end{pmatrix}
= \nabla^{H}_{a} \begin{pmatrix}
\tau \\
\varphi_{b} \\
\psi_{cd}
\end{pmatrix}
+ \nabla^{H^{*}}_{a} \begin{pmatrix}
\sigma^{ij} \\
\mu^{k} \\
\nu
\end{pmatrix},
$$

and using (3.7), the claimed formula follows by a straightforward computation. $\square$

**3.5. The metricity equation.** Via the general machinery of BGG-sequences, see [4, 14], any tractor bundle gives rise to a sequence of invariant differential operators. The first of these operators is always overdetermined. For the case of $H^{*}$, the corresponding facts can also be verified by rather simple direct computations.

The bundle $T^{*}N \otimes \text{Herm}(T^{*}N)$ can be naturally decomposed a trace part and a trace-free part. In fact the maps showing up in the top line of (3.10) represent
the inclusion of the trace part. In abstract index notation, a section of \( T^*N \otimes \operatorname{Herm}(T^*N) \) has the form \( \psi^{bc}_a \) and this is called trace-free if \( \psi^{ia}_a = 0 \). (Note that by symmetry, there is just one possible trace.) On the other hand, for a section \( \mu^a \) of \( TN \), one immediately verifies that \( \delta^a_i \mu^c + J_a^b J_c^i \mu^i \) is a section of \( T^*N \otimes \operatorname{Herm}(T^*N) \). The trace of this section is \( m \mu^a \), where \( \dim \mathbb{R}(N) = 2m \). Hence we see that we can uniquely decompose any \( \psi^{bc}_a \) into a trace part and a trace-free part, with the latter being given by

\[
\text{tfp} \left( \psi^{bc}_a \right) := \psi^{bc}_a - \frac{1}{m} \left( \delta^a_i \psi^c_i + J_a^b J_c^i \psi^{ij}_j \right).
\]

Of course, this works in the same way if we twist by a real density bundle.

**Proposition 16.** (i) For any section \( \sigma^{bc}_a \) of \( \operatorname{Herm}(T^*N) \otimes \mathcal{E}(-2) \), the trace-free part \( \text{tfp}(\nabla_a \sigma^{bc}_a) \) is independent of the choice of connection \( \nabla \) in the c-projective class. Mapping \( \sigma^{bc}_a \) to \( \text{tfp}(\nabla_a \sigma^{bc}_a) \) is an invariant differential operator.

(ii) There is an invariant differential operator \( L : \Gamma(\operatorname{Herm}(T^*N) \otimes \mathcal{E}(-2)) \to \Gamma(\mathcal{H}^*) \) such that

- the top component of \( L(\sigma^{ab}) \) equals \( \sigma^{ab} \)
- \( \sigma^{bc} \) satisfies \( \text{tfp}(\nabla_a \sigma^{bc}_a) = 0 \) if and only if \( \nabla_a^{\mathcal{H}^*} L(\sigma) \) has vanishing top component (and hence is a one-form with values in annihilator of \( \mathcal{H}^2 \subset \mathcal{H} \)).

(iii) There is a natural linear connection \( \nabla^p \) on the bundle \( \mathcal{H}^* \) such that for a section \( \sigma^{ab}_a \), \( \text{tfp}(\nabla_a \sigma^{bc}_a) = 0 \) is equivalent to \( \nabla^p_a L(\sigma^{bc}_a) = 0 \).

**Proof.** In terms of the theory of BGG sequences, \( L \) is the BGG-splitting operator and \( \sigma^{bc}_a \mapsto \text{tfp}(\nabla_a \sigma^{bc}_a) \) is the corresponding first BGG operator, which implies (i) and the first part of (ii). The second part of (ii) is an explicit version of the fact that \( \sigma \) is a solution of the first BGG operator if and only if \( \nabla^{\mathcal{H}^*} L(\sigma) \) is a section of the subbundle \( \operatorname{im}(\partial^*) \), where \( \partial^* \) is the Kostant-codifferential. Part (iii) follows from the general construction of prolongation connections in [22].

There also is a direct proof of this result along the following lines. Using equation (3.9), one immediately verifies that mapping \( (\sigma^{bc}_a, \mu^a, \nu_a)^t \in T^*N \otimes \mathcal{H}^* \) to \( (0, \sigma^{ia}_a, \frac{1}{2} \mu^i_a)^t \in \mathcal{H}^* \) is independent of the choice of splitting, thus defining a natural bundle map. Given \( \sigma^{ab} \) one verifies that, in a given splitting, there is a unique section \( L(\sigma^{ab}) \in \Gamma(\mathcal{H}^*) \) with top entry equal to \( \sigma^{ab} \) such that \( \nabla_a^{\mathcal{H}^*} L(\sigma^{bc}_a) \) lies in the kernel of this bundle map. Explicitly, this simply means that the top two components of \( \nabla_a^{\mathcal{H}^*} L(\sigma^{bc}_a) \) are trace-free. From this a direct computation shows that, in the splitting determined by \( \nabla \), one gets

\[
(3.11) \quad L(\sigma^{ab}) = \begin{pmatrix}
\sigma^{ab} \\
-\frac{1}{m} \nabla^i \sigma^{ic} \\
\frac{1}{4m^2} \nabla^i \nabla^j \sigma^{ij} + \frac{1}{2m} \sigma^{ij} P_{ij}
\end{pmatrix}.
\]
Since the characterization of $L(\sigma^{bc})$ is independent of all choices we obtain an invariant operator $L$. A part of the above computation shows that the top component of $\nabla^\mathcal{H} a L(\sigma^{bc})$ equals $\operatorname{tfp}(\nabla a \sigma^{bc})$, which implies (i) and (ii). A direct construction for $\nabla^p$ on the complexification of $\mathcal{H}^*$ can be found in Theorem 4.6 of [5].

3.6. Remarks on BGG equations and holonomy reductions. The only result on the metricity equation we will need in what follows is that any admissible Hermitian pseudo-Riemannian metric, whose canonical connection lies in the c-projective class gives rise to a solution to the metricity equation. Given $g_{ab}$, one simply takes the inverse metric $g^{ab}$ and multiplies is by an appropriate power of the volume density of $g$ to obtain a section $\sigma^{ab}$ of the right weight. This is even parallel for the canonical connection, so by projective invariance $\operatorname{tfp}(\tilde{\nabla} a \sigma^{bc}) = 0$ for any connection $\tilde{\nabla}$ in the c-projective class. It is not difficult to show that any solution $\sigma^{ab}$ to the metricity equation which is non-degenerate as a bilinear form is obtained in this way, compare with Section 4.3 of [5].

As we have noted in the proof of Proposition 16, the metricity equation is a so-called first BGG equation. It is known in general for such equation that there is a subspace of so-called normal solutions, which are characterized by the fact that $L(\sigma^{ab})$ is parallel for the tractor connection $\nabla^\mathcal{H} a$ and these exhaust all parallel sections of $\mathcal{H}^*$. We can now easily describe these normal solutions:

As noted above, any solution $\sigma^{ab}$ determines a connection $\nabla a$ in the c-projective class such that $\nabla a \sigma^{bc} = 0$. In the scale determined by this connection, the splitting operator simplifies to $L(\sigma^{ab}) = (\sigma^{ab}, 0, \frac{1}{2m} \sigma^{ij} P_{ij})^t$. This also shows that in this splitting

$$
\nabla^\mathcal{H} a L(\sigma^{bc}) = \begin{pmatrix}
0 \\
\frac{1}{m} \sigma^{ij} P_{ij} \delta_a^b - 2 \sigma^{bi} P_{ai} \\
\frac{1}{2m} \nabla a \sigma^{ij} P_{ij}
\end{pmatrix}.
$$

Non-degeneracy of $\sigma^{ab}$ implies that there is an inverse $\sigma_{ab}$ which is also Hermitian. Then vanishing of the middle slot is equivalent to the fact that $P_{ab}$ is proportional to $\sigma_{ab}$. This is equivalent to the fact that $\sigma$ comes from a Hermitian Einstein metric, see also Proposition 4.8 in [5]. This nicely complements the result in [12] for the projective metricity equation.

The parallel section of a tractor bundle corresponding to a normal solution of the metricity equation defines a holonomy reduction of the canonical Cartan connection associated to the almost c-projective structure to a special unitary group, see [11] for the general theory of these reductions. If the parallel section $L(\sigma^{ab})$ of $\mathcal{H}^*$ is non-degenerate as a bilinear form on $\mathcal{T}^*$ (which essentially means that the metric is Einstein but not Ricci flat), then its inverse defines a parallel section of $\mathcal{H}$. In the latter interpretation, such holonomy reductions are discussed in Section 3.3.
of [11]. Also in this dual picture, there is a parallel story with a first BGG equation and normal solutions of this equation as follows, compare with Sections 4.6 and 4.7 of [5].

To a section $\tau$ of $E(2)$ one naturally associates a section $L(\tau)$ of $\mathcal{H}$ as follows. One requires $L(\tau)$ to have $\tau$ in the top slot, and also requires that $\nabla^H_a L(\tau)$ has vanishing top slot while its middle slot has vanishing symmetric Hermitian part. These two conditions are immediately seen to be independent of the choice of splitting and, using this characterization, one directly computes that, in the splitting corresponding to $\tilde{\nabla}_a$, one gets

$$L(\tau) = \begin{pmatrix} \tau \\ 1/2 \tilde{\nabla}_a \tau \\ 1/2 \left( \delta^{ij}_{(b_c)} + J^i_{(b_c)} J^j_{(c)} \right) \left( 1/2 \tilde{\nabla}_i \tilde{\nabla}_j \tau + \tilde{P}_{ij} \tau \right) \end{pmatrix},$$

where $\tilde{P}_{ab}$ is the c-projective Schouten tensor of $\tilde{\nabla}_a$. This then implies that mapping $\tau$ to the anti-Hermitian symmetric part of $\tilde{\nabla}_a \tilde{\nabla}_b \tau + 2 \tilde{P}_{ab} \tau$ defines an invariant differential operator, which is the first BGG operator associated to $\mathcal{H}$.

If $\tau$ lies in the kernel of this operator, then it is easy to see that $\tau$ is non-vanishing on a dense open subset. On this subset, there is a unique connection $\nabla_a$ in the c-projective class for which $\nabla_a \tau = 0$. Then the c-projective Schouten tensor $P_{ab}$ must be symmetric and since $\tau$ solves the BGG equation, $P_{ab}$ has to be Hermitian. Conversely, it is easy to see that any connection in the c-projective class with symmetric and Hermitian projective Schouten tensor locally gives rise to a solution of the first BGG equation.

In the scale determined by $\nabla_a$, we then have $L(\tau) = (\tau, 0, \tau P_{ab})^t$ and this readily shows that in this splitting $\nabla^H_a L(\tau) = (0, 0, \tau \nabla_a P_{bc})$. Hence normal solutions are exactly those, for which $P_{bc}$ is in addition parallel for $\nabla_a$. If $P_{bc}$ is non-degenerate as a bilinear form, it defines a Hermitian metric, which must be admissible since it is preserved by the minimal complex connection $\nabla_a$, which therefore has to be its canonical connection. Finally, since the c-projective Schouten tensor is a multiple of the Ricci curvature of $\nabla_a$, this metric has to be Einstein.

4. c-projectively compact metrics.

4.1. The c-projective interpretation of scalar curvature. Let $(N, J)$ be an almost complex manifold and let $g$ be an admissible pseudo-Riemannian metric which is Hermitian with respect to $J$. We show that $g$ determines a section of the tractor bundle $\mathcal{H}^\ast$ over $N$, which in turn leads to a c-projective interpretation of the scalar curvature of the canonical connection of $g$.

On the one hand, for $\dim_\mathbb{R}(N) = 2m$ the bundle of real volume densities was defined in 2.1 to be $\mathcal{E}(-2m - 2)$. As a pseudo-Riemannian metric, $g$ determines a
volume density $\text{vol}_g \in \Gamma(\mathcal{E}(-2m-2))$, which is parallel for any linear connection which preserves $g$ and thus, in particular, is nowhere vanishing. One can take any root of this density to obtain a nowhere vanishing section of any real density bundle which is parallel for the canonical connection $\nabla$ of $g$. We will mainly need $\tau := \text{vol}_g^{-1/(m+1)} \in \Gamma(\mathcal{E}(2))$ and its inverse $\tau^{-1} \in \Gamma(\mathcal{E}(-2))$.

On the other hand, a smooth section $H$ of $\mathcal{H}^*$ defines a Hermitian form on the cotractor bundle $\mathcal{T}^*$. To such a form, one may associate a well-defined determinant, which is non-zero if and only if the form is non-degenerate, as follows. Take the complex valued Hermitian extension $\tilde{H}$ of $H$ and for a complex local frame $\{v_1, \ldots, v_{m+1}\}$ for $\mathcal{T}^*$ consider $\det(\tilde{H}(v_i, v_j))$, which is real since the matrix is Hermitian. Changing the local frame corresponds to a smooth function $A$ with values in $GL(m+1, \mathbb{C})$ and the corresponding change of matrix is given by $A(\tilde{H}(v_i, v_j))A^*$, so the determinant changes by multiplication with $|\det_C(A)|^2 = \det_\mathbb{R}(A)$. This shows that

$$v_1 \wedge Jv_1 \wedge \cdots \wedge v_{m+1} \wedge Jv_{m+1} \mapsto \det \left( \tilde{H}(v_i, v_j) \right)$$

induces a well-defined field of linear functionals on the top exterior power $\Lambda_{\mathbb{R}}^{2m+2}\mathcal{T}^*$. Thus we obtain a well-defined section $\det(H) \in \Gamma(\Lambda_{\mathbb{R}}^{2m+2}\mathcal{T})$. Since $\mathcal{T}$ is induced by a principal bundle with structure group $\text{SL}(m+1, \mathbb{C})$ endowed with a canonical connection, the bundle $\Lambda_{\mathbb{R}}^{2m+2}\mathcal{T}$ is naturally trivial. Hence up to an overall, non-zero constant factor, we can view $\det(H)$ as a smooth function on $N$. It is also easy to see that $\det(H)$ can be computed as the square root of the determinant of the real Gram matrix of $H$.

**Proposition 17.** Let $(N, J)$ be an almost complex manifold and $g = g_{ab}$ a pseudo-Riemannian metric on $N$, which is Hermitian for $J$ and admissible. Put $\tau := \text{vol}_g^{1/(m+1)} \in \Gamma(\mathcal{E}(2))$ and let $g^{ab}$ be the inverse metric for $g$. Then for the almost c-projective structure on $N$ determined by the canonical connection $\nabla$ of $g$, we have

(i) The section $\sigma^{ab} := \tau^{-1}g^{ab}$ of $\text{Herm}(\mathcal{T}^*N) \otimes \mathcal{E}(-2)$ is a solution of the metricity equation from part (i) of Proposition 16.

(ii) The image $H := L(\sigma^{ab}) \in \Gamma(\mathcal{H}^*)$ of $\sigma^{ab}$ under the splitting operator has the property that, up to an overall non-zero constant, $\det(H)$ coincides with the scalar curvature $S := g^{ij}\text{Ric}_{ij}$ of $\nabla$.

**Proof.** (i) For the canonical connection $\nabla$, we of course have $\nabla_a g^{bc} = 0$ and $\nabla_a \tau^{-1} = 0$. Hence $\nabla_a \sigma^{bc} = 0$ and the claim follows from c-projective invariance of the metricity equation.

(ii) We can compute in the splitting determined by $\nabla$, in which $\nabla_a \sigma^{bc} = 0$. In this splitting, formula (3.11) simplifies to $L(\sigma^{ab}) = (\sigma^{ab}, 0, \frac{1}{2m} \sigma^{ij} P_{ij})^t$, and the last component equals $\frac{1}{2m} \tau^{-1} g^{ij} P_{ij}$. Now since $g_{ab}$ is Hermitian, so is $g^{ab}$, and using this the defining equation (3.1) for the c-projective Schouten tensor shows...
that \( g^{ij} P_{ij} = \frac{1}{2(m+1)} g^{ij} \text{Ric}_{ij} \). Hence we end up with \( H = (\tau^{-1} g^{ab}, 0, C \tau^{-1} S)^t \) for some non-zero constant \( C \) and in the splitting corresponding to \( \nabla \).

To compute the determinant \( \det(H) \), we need to interpret the notation as triples correctly. The choice of \( \nabla \) induces a splitting of the cotractor bundle \( T^* \) into a direct sum of a complex bundle of rank \( m \) and a complex line bundle. The fact that the middle component of the triple is zero says that this splitting is orthogonal for \( H \). The real function \( C \tau^{-1} S \) describes the action on the complex line bundle, whereas \( \tau^{-1} g^{ab} \) is the real matrix associated to the restriction of \( h \) to the complex rank \( m \) bundle. From the discussion of determinants above, we conclude that, up to a non-zero overall constant (which includes \( C \)),

\[
\det(H) = \tau^{-1} S \sqrt{\det(\tau^{-1} g^{ab})} = \tau^{-1} S \sqrt{\tau^{-2m} \det(g^{ab})} = S \tau^{-m-1}(\text{vol}_g)^{-1} = S.
\]

This has immediate consequences for our setting of a manifold with boundary and an admissible Hermitian metric in the interior. Namely, if we assume that the almost c-projective structure determined by the canonical connection \( \nabla \) admits a smooth extension to the boundary, then we can use this structure to prove extendibility of several quantities associated to \( \nabla \).

**Corollary 18.** Let \( \overline{M} \) be a smooth manifold of real dimension \( 2m \) with boundary \( \partial M \) and interior \( M \), let \( J \) and \( g \) be an almost complex structure and a pseudo-Riemannian metric on \( M \), such that \( g \) is Hermitian with respect to \( J \) and admissible. Let \( g^{ab} \) be the inverse of \( g \), \( \nabla \) its canonical connection, \( \text{vol}_g \) its volume density and put \( \tau := (\text{vol}_g)^{-1/(m+1)} \). Suppose further that the c-projective structure determined by \( \nabla \) admits a smooth extension to all of \( \overline{M} \), so that the tractor bundle \( \mathcal{H}^* \) is defined on \( \overline{M} \).

Then the sections \( \sigma^{ab} := \tau^{-1} g^{ab} \in \Gamma(\text{Herm}(T^* \overline{M}) \otimes \mathcal{E}(2)) \) and \( L(\sigma^{ab}) \in \Gamma(\mathcal{H}^*) \) and the scalar curvature \( S \) of \( \nabla \) admit smooth extensions to all of \( \overline{M} \).

**Proof.** Since the almost c-projective structure determined by \( \nabla \) admits a smooth extension to the boundary, the same is true for all bundles and connections naturally associated to an almost c-projective structure. Hence both the tractor connection \( \nabla^{\mathcal{H}^*} \) and the connection \( \nabla^p \) from part (iii) of Proposition 16 are defined and smooth on all of \( \overline{M} \).

Now by part (iii) of Proposition 16, \( L(\sigma^{ab}) \) is a smooth section of \( \mathcal{H}^* \) over \( M \), which is parallel for the connection \( \nabla^p \). Hence it can be smoothly extended to all of \( \overline{M} \) by parallel transport with respect to \( \nabla^p \). Projecting this extension to the quotient bundle \( \text{Herm}(T^* \overline{M}) \otimes \mathcal{E}(2) \), one obtains the required extension of \( \sigma^{ab} \). On the other hand, the extension of \( L(\sigma^{ab}) \) provides a Hermitian form \( H \) on \( T^* \) over all of \( \overline{M} \). Hence \( \det(H) \) is a smooth function on \( \overline{M} \), and by Proposition 17 this coincides with \( S \) over \( M \), up to a non-zero constant. Hence \( S \) admits a smooth extension to the boundary. \( \square \)
4.2. On extendibility of the almost c-projective structure. In our standard situation $\overline{M} = M \cup \partial M$, suppose that we have given an almost complex structure $J$ and a complex linear, minimal connection $\nabla$ on $M$. Then the condition that the almost c-projective structure determined by $\nabla$ admits a smooth extension to the boundary can be checked by local computations. Consider a complex local frame $\{\xi_1, \ldots, \xi_m\}$ for $T\overline{M}$ defined on an open subset $U \subset \overline{M}$. Defining $\xi_{m+i} = J\xi_i$ for $i = 1, \ldots, m$, we obtain a real frame $\xi_1, \ldots, \xi_n$ for $T\overline{M}$ over $U$, with respect to which $J$ is represented by a constant matrix. The connection coefficients $\Phi^i_{jk}$ of $\nabla$ with respect to this frame are defined on $U \cap M$ by

$$\nabla_{\xi_j} \xi_k = \sum_i \Phi^i_{jk} \xi_i.$$ 

Since the connection $\nabla$ is complex, they have the property that $\Phi^i_{jk} = -\Phi^j_{ki}$, where $J^i_j$ is the (constant) coordinate representation of the almost complex structure $J$. Now one defines the (complex) tracefree part of the connection coefficients as

$$\Psi^i_{jk} := \Phi^i_{jk} - \frac{1}{2m+2} \left( \varphi_j \delta^i_k + \varphi_k \delta^i_j - J^i_j \varphi_k J^k_j - J^k_k \varphi_j J^i_j \right),$$

where $\varphi_j := \Phi^k_{jk}$. One immediately verifies that this satisfies $\Psi^i_{jk}, J^k_j = J^i_j \Psi^k_{jk}$ as well as $\Psi^k_{kj} = 0$. By minimality of $\nabla$, the alternation of the $\Phi^i_{jk}$ is conjugate linear in both arguments and hence tracefree, which implies that also $\Psi^i_{kj} = 0$. 

**Lemma 19.** The almost c-projective structure determined by $\nabla$ admits a smooth extension to $\overline{M}$ if and only if for each boundary point $x \in \partial M$, there is a frame $\{\xi_1, \ldots, \xi_n\}$ as above, defined on an open subset $U \subset \overline{M}$ with $x \in U$, such that the tracefree parts $\Psi^i_{jk}$ of the connection coefficients for $\nabla$ with respect to the frame, which are initially defined on $U \cap M$, admit a smooth extension to $U$.

**Proof.** This is completely parallel to the proof of the analogous result for projective structures as treated in Proposition 2 of [7].

By Proposition 3, extendibility of the almost c-projective structure determined by $\nabla$ is a necessary condition for c-projective compactness of $\nabla$. However, the two conditions are of different nature, since only very specific connections in a c-projective class can be c-projectively compact. For example, Proposition 7 shows that c-projective compactness for a connection preserving a volume density implies uniform volume growth of a specific rate (depending on the dimension). Also, by definition a c-projectively compact connection cannot admit a smooth extension to any neighborhood of a boundary point.

4.3. The case of non-zero scalar curvature. Our second main result is that for an admissible pseudo-Riemannian metric, extendibility of the almost c-projective structure determined by the canonical connection $\nabla$ together with a rather weak condition on the scalar curvature of $\nabla$ implies c-projective compactness.
THEOREM 20. Suppose that $\overline{M} = M \cup \partial M$, $J$ and $g$ satisfy the conditions of Corollary 18, so $J$ and $g$ are defined on $M$, $g$ is admissible and the almost $c$-projective structure defined by its canonical connection $\nabla$ admits a smooth extension to $\overline{M}$. Assume further, that the connection $\nabla$ itself does not admit a smooth extension to any open neighborhood of a boundary point.

Suppose that $x \in \partial M$ is such that the smooth extension $S : \overline{M} \to \mathbb{R}$ of the scalar curvature of $\nabla$ provided by Corollary 18 has the property that $S(x) \neq 0$. Then $g$ is $c$-projectively compact on a neighborhood of $x$ in $\overline{M}$.

Proof. We denote by $\text{vol}_g$ the volume density of $g$ and put $\tau := (\text{vol}_g)^{-1/(m+1)} \in \Gamma(\mathcal{E}(2))$. By Proposition 7, we can prove the theorem by showing that, locally around $x$, $\tau$ can be extended by zero to a defining density for $\partial M$.

By restricting to an appropriate neighborhood of $x$ in $\overline{M}$, we may assume that $S$ is nowhere vanishing. Then the section $H := L(\sigma^{ab}) \in \Gamma(\mathcal{H}^*)$, where $\sigma^{ab} = \tau^{-1}g^{ab}$, defines a Hermitian form on the bundle $\mathcal{T}^*$ which by Proposition 17 is everywhere non-degenerate. Hence we can consider its inverse, which is a non-degenerate Hermitian form on $\mathcal{T}$ and therefore defines a smooth section $\Phi \in \Gamma(\mathcal{H})$. Over $M$, we can work in the splitting determined by $\nabla$. From the proof of Proposition 17 we know that, in this splitting, we have $H = (\tau^{-1}g^{ab}, 0, C\tau^{-1}S)^t$ for some non-zero constant $C$. This shows that in the splitting of $\mathcal{H}$ corresponding to $\nabla$, we get $\Phi = (C^{-1}S^{-1}\tau, 0, \tau g^{ab})^t$, compare with Section 3.4.

Now the component $C^{-1}S^{-1}\tau$ is the image of $\Phi$ under the canonical projection $\Gamma(\mathcal{H}) \to \Gamma(\mathcal{E}(2))$, so this is independent of the choice of splitting. Since $\Phi$ and $S$ are smooth up to the boundary, this shows that $\tau$ admits a smooth extension to the boundary. Next we claim that this extension vanishes along the boundary. Indeed, if $\tau(y) \neq 0$ for some $y \in \partial M$, then we can consider a neighborhood $V$ of $y \in \overline{M}$ on which $\tau$ is nowhere vanishing. It is then well known that there is a unique connection $\hat{\nabla}$ in the restriction of the $c$-projective class to $V$ for which $\tau$ is parallel. But then on $V \cap M$, the section $\tau$ is parallel for both $\nabla$ and $\hat{\nabla}$, so the two connections have to agree on $V \cap M$. Thus $\hat{\nabla}$ provides a smooth extension of $\nabla$ to $V$ which contradicts our assumptions.

Knowing that $\tau$ vanishes along the boundary, it suffices to prove that its derivative with respect to any connection which is smooth up to the boundary is nowhere vanishing along the boundary. To prove this, we have to convert the information on $\nabla^*\mathcal{H}$ provided by part (ii) of Proposition 16 into the fact that $\Phi$ satisfies a differential equation. Let us first make the definition of $\Phi$ as the inverse of $H$ more explicit. As a non-degenerate Hermitian form on $\mathcal{T}^*$, the section $H$ gives rise to an isomorphism $A : \mathcal{T}^* \to \mathcal{T}$ of vector bundles. Using the (real) dual pairing between $\mathcal{T}$ and $\mathcal{T}^*$, this is characterized by $\langle A(\xi), \eta \rangle = H(\xi, \eta)$ for $\xi, \eta \in \mathcal{T}^*$. Then one uses the inverse $A^{-1}$ to define $\Phi$ via $\Phi(s, t) := \langle s, A^{-1}(t) \rangle$ for $s, t \in \mathcal{T}$.

Since the connection $\nabla^\mathcal{H}$ is induced by the standard tractor connection on $\mathcal{T}$, also the dual connection $\nabla^\mathcal{H}^*$ is induced by the tractor connection. Together with the above description, this implies that $\nabla^\mathcal{H}_a \Phi(s, t) = -\nabla^\mathcal{H}_a H(A^{-1}(s), A^{-1}(t))$. 
Now we need only rather rough information to conclude the argument. Recall from Section 3.2 that the tractor bundle $T$ contains a natural complex line subbundle $T^1$. Dually, one has the annihilator $(T^1)^\circ \subset T^\ast$, which is a complex subbundle of complex corank one. Now from above we know that the projection of $\Phi$ to the quotient bundle $E(2)$ of $H$ vanishes along the boundary. This implies that, along the boundary, $A^{-1}(T^1) \subset (T^1)^\circ$. On the other hand, part (ii) of Proposition 16 says that $\nabla^H H^\ast a H$ has trivial projection to $\text{Herm}(T^\ast N) \otimes E(-2)$, which exactly means that $\nabla^H H^\ast a H$ vanishes (everywhere) upon insertion of two sections from $(T^1)^\circ \subset T^\ast$. Hence we conclude that inserting two sections of $T^1$ into $\nabla^H a \Phi$, the result vanishes along the boundary.

In terms of splitting into slots, this simply means that the top slot of $\nabla^H a \Phi$ (which actually is independent of the choice of splitting) vanishes along the boundary. Now consider a connection $\hat{\nabla}$ in the c-projective class which is smooth up to the boundary. In the corresponding splitting we must have $\Phi = (C^{-1}S^{-1}\tau, \varphi_a, \psi_{bc})^t$ for some sections $\varphi_a$ and $\psi_{bc}$, which are smooth up to the boundary. Moreover, since $C^{-1}S^{-1}\tau$ vanishes along the boundary, non-degeneracy of $\Phi$ implies that $\varphi_a$ is nowhere vanishing along the boundary (since otherwise $\Phi$ would be degenerate). Using formula (3.7) for $\nabla^H$, we conclude that

$$C^{-1}\tau\hat{\nabla}_aS^{-1} + C^{-1}S^{-1}\hat{\nabla}_a\tau - 2\varphi_a$$

vanishes along the boundary. Since $\tau$ vanishes along the boundary, we conclude that, along the boundary, $\hat{\nabla}_a\tau = 2CS\varphi_a$ and thus is nowhere vanishing. □

4.4. Scalar curvature of c-projectively compact metrics. For a c-projectively compact metric, Corollary 18 shows that the scalar curvature of the canonical connection admits a smooth extension to the boundary. As a first step towards a converse of Theorem 20, we show that the boundary value of this extension is non-zero on a dense open subset of the boundary.

Proposition 21. Let $\overline{M}$ be a smooth manifold of real dimension $2m$ with boundary $\partial M$ and interior $M$ and let $J$ be an almost complex structure on $M$. Consider an admissible pseudo-Riemannian Hermitian metric $g$ on $(M, J)$, which is c-projectively compact. Let $\nabla$ be the canonical connection of $g$ and $S : \overline{M} \to \mathbb{R}$ be the smooth extension of the scalar curvature of $\nabla$ guaranteed by Corollary 18. Then we have.

1. The set $\{x \in \partial M : S(x) \neq 0\}$ is open and dense in $\partial M$.

2. Let $x \in \partial M$ be such that $S(x) \neq 0$, let $\rho$ be a defining function for $\partial M$ on some neighborhood of $x$ and put $\theta := -d\rho \circ J$. Then locally around $x$, $g$ admits an asymptotic form

$$g = -\frac{m}{2}(g^{ij}P_{ij})^{-1}\left(\frac{dp^2}{\rho^2} + \frac{\theta^2}{\rho^2}\right) + \frac{h}{\rho},$$

where $h = O(\rho)$.
where \( P_{ij} \) is the c-projective Schouten tensor of \( \nabla \) and \( h \) is a Hermitian bilinear form, which is smooth up to the boundary.

(3) Suppose that \( x \in \partial M \) is as in (2) and that \( \xi, \eta \in X(M) \) vector fields, which are smooth up to the boundary. Then locally around \( x \), the function \( \rho^2 g(\xi, \eta) \) admits a smooth extension to the boundary. If for at least one of the vector fields the insertion into \( d \rho \) vanishes along the boundary and for at least one the insertion into \( \theta \) vanishes along the boundary, then even \( \rho g(\xi, \eta) \) admits a smooth extension to the boundary.

Proof. As in the proofs of Proposition 17 and of Theorem 20 we consider the tractor \( H := L(\sigma^{ab}) \in \Gamma(H^*) \) determined by the solution \( \sigma^{ab} = \tau^{-1} g^{ab} \) of the metricity equation determined by \( g = g_{ab} \). Knowing that \( g \) is c-projectively compact, we get the additional information that certain specific connections admit a smooth extension to the boundary. Take a point \( x \in \partial M \) and a local defining function \( \rho \) for the boundary defined on a neighborhood \( U \) of \( x \) in \( M \). Consider the c-projective modification \( \hat{\nabla} := \nabla + d \rho \) of \( \nabla \) on \( U \cap M \), which by c-projective compactness admits a smooth extension to all of \( U \).

(1) With a view towards contradiction, we assume that \( S \) vanishes on an open subset of \( \partial M \) and choose \( U \) in such a way that \( S \) vanishes on \( U \cap \partial M \). From the proof of Proposition 17, we know that on \( U \cap M \) and in the scale determined by \( \nabla \), we have \( H = (\tau^{-1} g^{ab}, 0, C \tau^{-1} S)^t \) for some non-zero constant \( C \). Now we can use Proposition 15 to compute \( H \) on \( U \cap M \) in the splitting corresponding to \( \hat{\nabla} \). By formula (3.9), we obtain

\[
H = \left( \begin{array}{c} \tau^{-1} g^{ab} \\ -\frac{1}{\rho} \tau^{-1} \rho_i g^{ic} \\ C \tau^{-1} S + \tau^{-1} \frac{1}{4 \rho^2} \rho_i \rho_j g^{jk} \end{array} \right),
\]

where we write \( \rho_a \) for \( d \rho \). Since \( \hat{\nabla} \) admits a smooth extension to the boundary, all three slots in this expression must have the same property. Now recall from Proposition 7 that \( \tau \) is a defining density for \( \partial M \), and thus of the form \( \rho \hat{\tau} \) for a density \( \hat{\tau} \), which is nowhere-vanishing along \( U \cap \partial M \). Hence \( \hat{\tau}^{-1} \) has the same property and \( \tau^{-1} = \rho^{-1} \hat{\tau}^{-1} \).

For the top slot we obtain \( \hat{\tau}^{-1} \frac{1}{\rho} g^{ab} \). This implies that for one-forms \( \varphi = \varphi_a \) and \( \psi = \psi_b \), which are smooth up to the boundary, we can write \( g^{ij} \varphi_i \psi_j \) as \( \rho f \) for a function \( f \) which admits a smooth extension to the boundary. For the middle slot we get \( \hat{\tau}^{-1} \frac{1}{\rho} \rho_i g^{ic} \), so \( \frac{1}{\rho} \rho_i g^{ic} \) admits a smooth extension to the boundary. This means that for each one-form \( \varphi = \varphi_a \) which is smooth up to the boundary, \( g^{ij} \rho_i \varphi_j \) is of the form \( \rho^2 f \) for a function \( f \) which is smooth up to the boundary. For the
bottom slot, we get

\[ \hat{\tau}^{-1} \frac{1}{4\rho} \left( 4CS + \frac{1}{\rho^2} \rho_i \rho_j g^{ij} \right). \]

Now both summands in the bracket admit a smooth extension to the boundary, so we conclude that they have to add up to zero along the boundary. Assuming that \( S \) vanishes along the boundary, we conclude that \( \frac{1}{\rho} \rho_i \rho_j g^{ij} \) goes to zero along the boundary, so \( \rho_i \rho_j g^{ij} \) can be written as \( \rho^3 f \) for some function \( f \) which is smooth up to the boundary.

Let us collect the above facts and use them to obtain a description of the Gram matrix of \( g^{ab} \) in terms of an appropriate local frame consisting of one-forms which are smooth up to the boundary. We use a complex frame in a real picture, which has \( \rho_a \) and \( J^i_a \rho_i \) as the first two elements, and then continues with appropriate pairs of the form \( \varphi_a \) and \( J^i_a \varphi_i \). Denoting the resulting (symmetric) matrix of inner products by \( A = (a_{k\ell}) \) we know that all entries vanish along the boundary, the entries in the first two rows and columns are of the form \( \rho^2 f_{k\ell} \) for functions \( f_{k\ell} \) which are smooth up to the boundary, while the top left corner has the form \( \rho^3 (a_0) \) for a function \( a \) which is smooth up to the boundary. Computing \( \det(A) \) we can pull out a factor of \( \rho^2 \) from each of the first two rows and a factor of \( \rho \) from each of the subsequent rows, and then finally one factor of \( \rho \) from each of the first two columns, so \( \det(A) = \rho^{2m+4} f \) for some function \( f \) which is smooth up to the boundary.

On the other hand, computing the volume density of \( g \), we get the product of \( \sqrt{\det(A^{-1})} \) with a nowhere vanishing density which expresses the change from a local coordinate frame to the frame of the tangent bundle which is dual to the frame used above. By definition, the result is \( \tau^{-m-1} = \rho^{-m-1} \hat{\tau}^{-m-1} \). But this implies that \( \rho^{2m+2} \det(A^{-1}) \) admits a smooth extension to the boundary with non-vanishing boundary value \( \tilde{f} \). Together with the above, we get

\[ \rho^{2m+2} = \rho^{2m+2} \det(A^{-1}) \det(A) = \tilde{f} \rho^{2m+4}, \]

for a function \( f \) which is smooth up to the boundary. This leads to \( f = \rho^{-2} \frac{1}{\tilde{f}} \), a contradiction.

(2) Knowing that \( S(x) \neq 0 \), we can choose \( U \) in such a way that \( S \) is nowhere vanishing, so also \( g^{ij} P_{ij} \) is nowhere vanishing. As in the proof of Theorem 20, we can thus form the inverse \( \Phi \in \Gamma(\mathcal{H}) \) of \( H \). On \( U \cap M \) and in the splitting corresponding to \( \nabla \), we know from that proof that \( \Phi = (2m \tau (g^{ij} P_{ij})^{-1}, 0, \tau g_{ab}) \). Computing the expression in the splitting corresponding to the connection \( \hat{\nabla} = \nabla + \frac{\rho}{\partial \rho} \), as in part (1), we again know that all slots in this expression admit smooth extensions to the boundary. Using formula (3.5) we conclude that in the splitting
corresponding to \( \hat{V} \), we get

\[
(4.1) \quad \Phi = \begin{pmatrix}
2m \tau (g^{ij} P_{ij})^{-1} \\
\frac{m \tau (g^{ij} P_{ij})^{-1}}{\rho} \\
\tau g_{bc} + m (g^{ij} P_{ij})^{-1} \frac{1}{2 \rho^2} (\delta^i_b \delta^j_c + J^i_b J^j_c) \rho_i \rho_j \\
\end{pmatrix}.
\]

The first two slots evidently admit a smooth extension to the boundary. On \( U \), we then define \( h_{bc} := \rho g_{bc} + \frac{m}{2 \rho} (g^{ij} P_{ij})^{-1} (\rho_b \rho_c + \theta_b \theta_c) \). Then the bottom slot in (4.1) equals \( \tau h_{bc} \), so \( h_{bc} \) admits a smooth extension to the boundary. But this exactly means that we get the required asymptotic form.

(3) From part (2) we conclude that

\[
\rho^2 g(\xi, \eta) = -\frac{m}{2} (g^{ij} P_{ij})^{-1} (d \rho(\xi) d \rho(\eta) + \theta(\xi) \theta(\eta)) + \rho h(\xi, \eta).
\]

The right-hand side evidently admits a smooth extension to the boundary. Under the additional assumptions on \( \xi \) and \( \eta \), we can write one of the factors \( d \rho \) and one of the factors \( \theta \) as \( \rho \) times a function which admits a smooth extension to the boundary, which implies the second claim.

\[
\square
\]

4.5. Necessity of the asymptotic form. The last step to obtain converses to Theorems 10 and 20 is showing that the scalar curvature of a c-projectively compact metric is asymptotically (locally) constant.

**Theorem 22.** Let \( \overline{M} \) be a smooth manifold with boundary \( \partial M \) and interior \( M \). Let \( J \) be an almost complex structure on \( \overline{M} \) and let \( g \) be an admissible Hermitian pseudo-Riemannian metric on \( (M, J) \) which is c-projectively compact. Suppose that the resulting extension of \( J \) has the property that \( \partial M \) is non-degenerate and that the Nijenhuis tensor has asymptotically tangential values.

Then the boundary value of the smooth extension of the scalar curvature \( S \) of the canonical connection \( \nabla \) of \( g \) guaranteed by Corollary 18 is locally constant, and \( g \) admits an asymptotic form as in formula (2.4) in Section 2.6 satisfying the conditions stated there.

**Proof.** By Proposition 21 the boundary value of \( S \) is non-vanishing on an open dense subset of \( \partial \overline{M} \) and we work locally around a point \( x \) in this subset. Let \( \nabla \) be the canonical connection of \( g \) and for a local defining function \( \rho \) for the boundary let \( \delta^\nabla \) be the corresponding c-projective modification of \( \nabla \) which admits a smooth extension to the boundary. Choose a vector field \( \mu \) along the boundary such that \( d \rho(\mu) \) is identically one on the boundary whereas \( \theta(\mu) \) vanishes along the boundary. As in the proof of Lemma 1 of [9], we can extend \( \mu \) to a vector field defined.
locally around $x$ such that $\rho \nabla_{\mu} \mu = 0$. Moreover, we can use the flow of $\mu$ to identify an open neighborhood of $x$ in $\partial M$ with $V \times [0, \epsilon)$ for an open neighborhood $V$ of $x$ in $\partial M$ and some $\epsilon > 0$.

Now consider the function $\rho^2 g(\mu, \mu)$ which admits a smooth extension to the boundary by part (3) of Proposition 21. By part (2) of that proposition, this is given by

$$
\rho^2 g(\mu, \mu) = \hat{C} S^{-1} (d\rho(\mu)^2 + \theta(\mu)^2) + \rho h(\mu, \mu)
$$

for some constant $\hat{C}$. Along $\partial M$, we have $d\rho(\mu) = 1$ and $\theta(\mu) = 0$, so this approaches $\hat{C} S^{-1}$ at the boundary.

Now consider a vector field $\xi \in \mathfrak{X}(\partial M)$, which is a section of the CR-subbundle. Then we can extend this to a vector field $\zeta$ on a neighborhood of $x$ such that $d\rho(\zeta) = 0$ and $\theta(\zeta) = 0$. The derivative of the boundary value of $\hat{C} S^{-1}$ in direction $\zeta$ can then be computed as the boundary value of $\zeta \cdot (\rho^2 g(\mu, \mu))$.

**Claim 1.** For any vector field $\zeta$ such that $d\rho(\zeta) = 0$ and $\theta(\zeta) = 0$, the function $\zeta \cdot (\rho^2 g(\mu, \mu))$ vanishes along the boundary.

To prove this claim, we first observe that $\zeta \cdot \rho = 0$, so on $M$, we can write $\zeta \cdot (\rho^2 g(\mu, \mu))$ as $2\rho^2 g(\nabla \zeta, \mu)$. Since $g$ is admissible, the torsion of $\nabla$ equals $-\frac{1}{4} \mathcal{N}$, where $\mathcal{N}$ is the Nijenhuis tensor, so the definition of torsion shows that

$$
\nabla_{\zeta} \mu = \nabla_{\mu} \zeta + [\zeta, \mu] - \frac{1}{4} \mathcal{N}(\zeta, \mu).
$$

By assumption on the Nijenhuis tensor, $d \rho(\mathcal{N}(\zeta, \mu))$ and $\theta(\mathcal{N}(\zeta, \mu))$ both vanish along the boundary. Hence part (3) of Proposition 21 shows that $\rho^2 g(-\frac{1}{4} \mathcal{N}(\zeta, \mu), \mu)$ vanishes along the boundary.

Likewise, expanding $0 = dd\rho(\zeta, \mu)$ we obtain $d \rho([\zeta, \mu]) = -\mu \cdot d\rho(\zeta) + \zeta \cdot d\rho(\mu)$. By construction, $d\rho(\zeta) = 0$ and $d\rho(\mu) = 1 + \rho f$ for some smooth function $f$, so since $\zeta \cdot \rho = 0$, we see that $d \rho([\zeta, \mu])$ vanishes along the boundary. Since also $\theta(\mu)$ vanishes along the boundary, we can again use part (3) of Proposition 21 to conclude that $\rho^2 g([\zeta, \mu], \mu)$ vanishes along the boundary.

Finally, over $M$, we can write

$$
\rho^2 g(\nabla_{\mu} \zeta, \mu) = \rho^2 \mu \cdot g(\zeta, \mu) - \rho^2 g(\zeta, \nabla_{\mu} \mu)
$$

$$
= \rho \mu \cdot (\rho g(\zeta, \mu)) - \rho d\rho(\mu) g(\zeta, \mu) - \rho^2 g(\zeta, \nabla_{\mu} \mu).
$$

Now the construction of $\mu$ together with formula (2.1) for a c-projective modification implies that

$$
\nabla_{\mu} = - \frac{d \rho(\mu)}{\rho} \mu + \frac{d \rho(J \mu)}{\rho} J \mu = - \frac{d \rho(\mu)}{\rho} \mu - \frac{\theta(\mu)}{\rho} J \mu.
$$
Inserting this into (4.3), we see that $\rho^2 g(\nabla_{\mu} \zeta, \mu)$ can be written as

$$\rho \mu \cdot (\rho g(\zeta, \mu)) - \theta(\mu) \rho g(\zeta, J\mu).$$

Now by part (3) of Proposition 21, both $\rho g(\zeta, \mu)$ and $\rho g(\zeta, J\mu)$ admit smooth extensions to the boundary, so since $\theta(\mu)$ vanishes along the boundary, we see that also $\rho^2 g(\nabla_{\mu} \zeta, \mu)$ vanishes along the boundary. This completes the proof of Claim 1.

Hence we have verified that the derivative of the boundary value of $\hat{CS}^{-1}$ in any direction lying in the CR subspace of $T\partial M$ vanishes. By non-degeneracy, this implies that this boundary value is locally constant. Denoting by $C$ this locally constant function on the boundary extended constantly along flow lines of $\mu$, we see that $\hat{CS}^{-1} = C + \rho f$ for a function $f$ which is smooth up to the boundary. Hence part (2) of Proposition 21 shows that $g$ admits an asymptotic form as in formula (2.4) from Section 2.6. To complete the proof of the theorem, it thus remains to verify that for vector fields $\xi$ and $\zeta$ such that $d\rho(\zeta)$ and $\theta(\zeta)$ vanish, $h(\xi, J\zeta)$ approaches $Cd\theta(\xi, \zeta)$ at the boundary.

**Claim 2.** Suppose that $\xi$ is a vector field that is smooth up to the boundary such that $d\rho(\xi)$ vanishes along the boundary. Then, for $\zeta$ as above, $C d\rho(\rho \nabla_\xi \zeta)$ approaches $\frac{1}{2} h(\xi, \zeta)$ at the boundary.

To prove this claim, we first observe that by part (3) of Proposition 21 the function $\xi \cdot (\rho g(\zeta, \mu))$ admits a smooth extension to the boundary, and, on $M$ we can compute this as

$$d\rho(\xi) g(\zeta, \mu) + \rho g(\nabla_\xi \zeta, \mu) + \rho g(\zeta, \nabla_\xi \mu).$$

Since $d\rho(\xi)$ vanishes along the boundary, the first term in this sum admits a smooth extension to the boundary. Next, we compute

$$\nabla_\xi \zeta = \rho \nabla_\xi \zeta - \frac{d\rho(\xi)}{2\rho} \zeta - \frac{\theta(\xi)}{2\rho} J\zeta.$$

The coefficient of $\zeta$ in the second summand on the right-hand side admits a smooth extension to the boundary, so hooking this into $\rho g(\zeta, \mu)$ one obtains a function which admits a smooth extension to the boundary. Likewise, we compute

$$\nabla_\xi \mu = \rho \nabla_\xi \mu - \frac{d\rho(\xi)}{2\rho} \mu - \frac{d\rho(\mu)}{2\rho} \xi - \frac{\theta(\xi)}{2\rho} J\mu - \frac{\theta(\mu)}{2\rho} J\xi.$$

Here the first, second, and the last summand in the right-hand side admit smooth extensions to the boundary and hence hooking them into $\rho g(\zeta, \mu)$ one obtains functions which admit a smooth extension to the boundary. The upshot of this is
that
\[
\rho g(\rho \nabla_{\xi} \zeta, \mu) - \frac{1}{2} \theta(\xi) g(J \zeta, \mu) - \frac{1}{2} d\rho(\mu) g(\zeta, \xi) - \frac{1}{2} \theta(\xi) g(\zeta, J \mu) = \rho g(\rho \nabla_{\xi} \zeta, \mu) - \frac{1}{2} d\rho(\mu) g(\zeta, \xi)
\]

admits a smooth extension to the boundary. Hence if we multiply by \(\rho\) we must get a function tending to 0 at the boundary. But since we already know that we have the asymptotic form for \(g\), we can evaluate this directly which readily leads to the statement in Claim 2.

This already suffices to the compute the boundary value of \(Cd\theta(\xi, \zeta)\) in the case that also \(d\rho(\xi)\) and \(\theta(\xi)\) vanish identically. In this case, \(Cd\theta(\xi, \zeta) = -C\theta([\xi, \zeta]) = Cd\rho(\rho J \xi, \zeta)\). Since the torsion of \(\rho \nabla\) has values in the CR subspace along the boundary, we can compute the boundary value of this as the boundary value of

\[
Cd\rho(J(\rho \nabla_{\xi} \zeta - \rho \nabla_{\xi} J \zeta)) = Cd\rho(\rho \nabla_{\xi} J \zeta) - Cd\rho(\rho \nabla_{\zeta} J \xi)
\]

and use Claim 2 to compute both summands in the right-hand side. This leads to
\[
\frac{1}{2} h(J \zeta, \xi) - \frac{1}{2} h(J \xi, \zeta) = h(\xi, J \zeta).
\]

To complete the proof, it suffices to show that we get the right boundary value for \(\xi = J \mu\). Then \(\theta(\xi) = d\rho(\mu) = 1 + \rho f\) for some smooth function \(f\). Since \(d\rho(\zeta) = 0\) we conclude that \(\zeta \cdot \theta(\xi)\) goes to zero at the boundary, so we can again compute \(Cd\theta(\xi, \zeta)\) as
\[
Cd\rho(\rho \nabla_{\xi} J \zeta) - Cd\rho(\rho \nabla_{\zeta} J \xi).
\]

The first summand can be computed using Claim 2, so to complete the proof, it suffices to show that \(Cd\rho(\rho \nabla_{\xi} \mu)\) approaches \(\frac{1}{2} h(\zeta, \mu)\) at the boundary. But in the proof of Claim 1, we have already seen that \(\rho^2 g(\nabla_{\xi} \mu, \mu)\) approaches zero at the boundary, so \(\rho g(\nabla_{\xi} \mu, \mu)\) admits a smooth extension to the boundary. Rewriting
\[
\nabla_{\xi} \mu = \rho \nabla_{\xi} \mu - \frac{d\rho(\mu)}{2\rho} \zeta - \frac{\theta(\mu)}{2\rho} J \zeta
\]

and observing that the coefficient of \(J \zeta\) in the last summand admits a smooth extension to the boundary, this follows in the same way as Claim 2. \(\square\)

4.6. Curvature asymptotics. We next analyze the curvature of \(c\)-projectively compact special affine connections and, more specifically, the canonical connections of \(c\)-projectively compact metrics. As a first step, we describe the asymptotic behavior of the \(c\)-projective Schouten tensor. As we have noted in Section 3.1, the \(c\)-projective Schouten tensor \(P_{ab}\) is not symmetric in general. Let us denote by \(\beta_{ab} := P_{ab}\) its skew symmetric part. Further, the symmetric part \(P_{(ab)}\) is a symmetric bilinear form on the tangent spaces, which are complex vector spaces.
Hence this can be decomposed into a Hermitian part $\hat{P}_{ab}^+ := \frac{1}{2}(P_{(ab)} + J^i_a J^j_b P_{(ij)})$ and an anti-Hermitian part $\hat{P}_{ab}^- := \frac{1}{2}(P_{(ab)} - J^i_a J^j_b P_{(ij)})$.

**Lemma 23.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $J$ be an almost complex structure on $M$ and let $\nabla$ be a minimal complex linear connection on $TM$ which is c-projectively compact. Suppose further that the smooth extension of $J$ to the boundary has the property that its Nijenhuis tensor has asymptotically tangential values.

Let $\rho$ be a local defining function for $\partial M$, put $\rho_a = d\rho$ and $\theta_a = -J^i_a \rho_i$, and let $\hat{\nabla}$ be the corresponding c-projective modification of $\nabla$. Then for the c-projective Schouten tensor $\hat{P}_{ab}$ of $\hat{\nabla}$,

$$\rho \hat{P}_{ab} + \frac{1}{4\rho}(\rho_a \rho_b + \theta_a \theta_b)$$

admits a smooth extension to the boundary and its boundary value coincides with $\frac{1}{2}\hat{\nabla}_a \rho_b$. In particular, $\beta_{ab}$ and $\rho \hat{P}_{ab}^-$ admit smooth extensions to the boundary.

**Proof.** Denoting by $\hat{\hat{P}}_{ab}$ the c-projective Schouten tensor of $\hat{\nabla}$, formula (3.2) for the change of Schouten tensor shows that

$$\hat{P}_{ab} = \hat{\hat{P}}_{ab} + \hat{\nabla}_a \Upsilon_b + \Upsilon_a \Upsilon_b - J^i_a J^j_b \Upsilon_i \Upsilon_j,$$

where $\Upsilon_a = \frac{1}{2\rho} \rho_a$. Now $\hat{\nabla}_a \Upsilon_b = -\frac{1}{2\rho} \rho_a \rho_b + \frac{1}{2\rho} \hat{\nabla}_a \rho_b$ and adding $\Upsilon_a \Upsilon_b$, we get $-\frac{1}{4\rho^2} \rho_a \rho_b + \frac{1}{2\rho} \hat{\nabla}_a \rho_b$. On the other hand, the last summand in the right-hand side gives $-\frac{1}{4\rho^2} \theta_a \theta_b$. Bringing two terms to the other side and multiplying by $\rho$, we get

$$\rho \hat{P}_{ab} + \frac{1}{4\rho}(\rho_a \rho_b + \theta_a \theta_b) = \frac{1}{2} \hat{\nabla}_a \rho_b + \rho \hat{\hat{P}}_{ab}. \tag{4.5}$$

Since $\hat{\hat{P}}_{ab}$ is smooth up to the boundary, it only remains to prove the claims on $\beta_{ab}$ and $\hat{P}_{ab}^-$. Skew symmetrizing in (4.5) over $a$ and $b$ and using that $\hat{\nabla}_a \rho_b$ is symmetric along the boundary by Lemma 6, we obtain $\rho \beta_{ab} = \rho \hat{\hat{P}}_{ab}$, so $\beta_{ab}$ admits a smooth extension to the boundary.

On the other hand, symmetrizing over $a$ and $b$ in (4.5), does not affect the second summand in the left-hand side, which in addition is Hermitian. Hence forming the symmetric anti-Hermitian part in the left-hand side, we just get $\rho \hat{P}_{ab}^-$, whereas in the right-hand side all terms still admit a smooth extension to the boundary. \qed

Suppose that $\varphi_{ab}$ is a tensor field on an almost complex manifold which is symmetric and Hermitian, i.e., such that $\varphi_{ab} = \varphi_{(ab)} = J^i_a J^j_b \varphi_{ij}$. Then one can associate to $\varphi$ a tensor field $C_{ab}^{c \, d}$ defined by

$$C_{ab}^{c \, d} := 2(\delta^c_{[a} \varphi_{b]d} - J^i_a \varphi_{b]i} J^c_d - J^c_a \varphi_{b]}i J^i_d).$$
This is obviously skew symmetric in $a$ and $b$ and a direct computation shows that $C_{[ab}^{c}d] = 0$, so $C$ has the symmetries of a curvature tensor. Direct computations also show that $C_{ab}i_{d}J_{i}^{c} = C_{ab}^{c}jJ_{d}^{j}$, so $C$ has values in complex linear maps. Finally, one shows that $C_{ab}c^{d} = J_{a}^{d}J_{b}^{j}C_{ij}^{cd}$, so $C$ is of type $(1,1)$. Hence we call $C$ the complex rank one curvature tensor associated to $\varphi$. Using this, we can now describe the curvature asymptotics of a general c-projectively compact connection.

**Proposition 24.** Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $J$ be an almost complex structure on $M$ and let $\nabla$ be a minimal complex linear connection on $TM$ which is c-projectively compact. Suppose further that the resulting smooth extension of $J$ to the boundary has the property that its Nijenhuis tensor has asymptotically tangential values.

Then the curvature tensor $R$ of $\nabla$ has the property that for any local defining function $\rho$ for the boundary, $\rho^{2}R$ admits a smooth extension to the boundary with boundary value $-\frac{1}{4}C_{ab}^{cd}$, where $C_{ab}^{cd}$ is the complex rank one curvature tensor associated to $\rho \partial_{b} + \theta_{a} \partial_{b}$.

If $J$ is integrable, then the boundary value of $\rho R_{\alpha\beta}^{\gamma\delta} + \frac{1}{4\rho}C_{\alpha\beta}^{\gamma\delta}$ is given by

$$
\delta_{\alpha}^{\gamma} \nabla_{\beta} \rho_{\gamma} - J_{[\alpha}^{i} (\nabla_{\beta} \rho_{\alpha]) J_{\gamma}^{i} - J_{[\alpha}^{i} (\nabla_{\beta} \rho_{\alpha]) J_{\gamma}^{i} J_{d}^{i}.
$$

**Proof.** The characterization of the c-projective Schouten tensor $P_{\alpha\beta}$ in part (1) of Proposition 12 can be reformulated as

$$
R_{\alpha\beta}^{\gamma\delta} = W_{\alpha\beta}^{\gamma\delta} + 2(\delta_{\alpha}^{\gamma} P_{\beta\delta} - P_{[\alpha\beta]} \delta_{\delta}^{\gamma} - J_{[\alpha}^{i} P_{\beta]i} J_{\gamma}^{i} - J_{[\alpha}^{i} P_{\beta]i} J_{\gamma}^{i} J_{d}^{i}),
$$

where $W_{\alpha\beta}^{\gamma\delta} = 0$. Now let $\hat{\nabla} = \rho \nabla$ be the c-projective modification of $\nabla$ determined by $\rho$. Then of course the associated Weyl curvature $\hat{W}_{\alpha\beta\gamma\delta}^{ab}$ is smooth up to the boundary. Now the relation between $W_{\alpha\beta}^{\gamma\delta}$ and $\hat{W}_{\alpha\beta}^{\gamma\delta}$ is described in Proposition 2.13 of [5]. This is done in a complexified picture, discussing the $(p,q)$-components of the two-form $W$ with values in $\text{End}_{\mathbb{C}}(TM)$. The components of $(p,q)$-types $(2,0)$ and $(1,1)$ of $W_{\alpha\beta}^{\gamma\delta}$ and $\hat{W}_{\alpha\beta}^{\gamma\delta}$ agree. In type $(0,2)$, the difference of the components of $\hat{W}$ and $W$ is obtained by contracting the torsion of $\nabla$ into a tensor $v_{\alpha\beta}^{cd}$ which is a linear combination of tensor products of one factor $\Upsilon_{\ell}$ with either a Kronecker delta or two copies of $J$. Since $\rho \Upsilon_{\ell} = \frac{1}{2} \rho_{\ell}$ is smooth up to the boundary, we see that $\rho v_{\alpha\beta}^{cd}$ admits a smooth extension to the boundary, so $\rho W_{\alpha\beta}^{cd}$ admits a smooth extension to the boundary.

Multiplying (4.6) by $\rho^{2}$, we thus conclude form Lemma 23 that the right-hand side admits a smooth extension to the boundary. Moreover, the first summand and the term involving $P_{[\alpha\beta]}$ do not contribute to the boundary value, and in the other summands we may replace $P_{\alpha\beta}$ by $-\frac{1}{4}(\rho_{\alpha} \partial_{b} + \theta_{a} \partial_{b})$ without changing the boundary value. Hence the first result follows from the definition of $C_{\alpha\beta}^{cd}$.

If $J$ is integrable, then by Proposition 2.13 of [5], the $(0,2)$-component of $W_{\alpha\beta}^{\gamma\delta}$ vanishes for any connection in the c-projective class. Hence from above we
conclude that in this case $W_{abcd}$ admits a smooth extension to the boundary. So up to terms admitting a smooth extension to the boundary, we can write $R_{abcd}$ as in (4.6) but leaving out the terms containing $W_{abcd}$ or $P_{[ab]}$ in the right-hand side. Then the claimed expression follows directly from Lemma 23. □

4.7. The asymptotic Einstein property. We conclude the article by discussing the curvature asymptotics for c-projectively compact metrics. In particular, we show that the canonical connection associated to such a metric automatically satisfies an asymptotic version of the Einstein equation.

THEOREM 25. Let $\overline{M}$ be a smooth manifold with boundary $\partial M$ and interior $M$. Let $J$ be an almost complex structure on $M$ and let $g$ be an admissible Hermitian metric on $M$, which is c-projectively compact. Suppose further that the Nijenhuis tensor of the resulting smooth extension of $J$ to the boundary has asymptotically tangential values. Let $P_{ab}$ be the Schouten tensor of the canonical connection $\nabla$ of $g$ and $P_{+ab}$ its symmetric Hermitian part.

(1) The tracefree part $P_{ab}^{\circ} := P_{ab}^+ - \frac{1}{2m}g^{ij}P_{ij}g_{ab}$ of $P^+$ admits a smooth extension to the boundary.

(2) If $J$ is integrable, then $P_{ab}^+ = P_{ab}$, so the Kähler metric $g$ satisfies an asymptotic version of the Einstein equation. In this case, let $\rho$ be a local defining function for the boundary, put $\rho_a = d\rho$ and $\theta_a = -J^i_a\rho_i$, let $(d\theta)_{ab}$ be the exterior derivative of $\theta_a$ and let $C_{abcd}$ be the complex rank-one curvature tensor associated to $\rho_a\rho_b + \theta_a\theta_b$. Then up to terms admitting a smooth extension to the boundary, the curvature $R_{abcd}$ of $g$ is given by

$$-\frac{1}{4\rho^2}C_{abcd} + \frac{1}{2\rho} \left( \delta_c^{[a} J^i_{b]}(d\theta)_{di} - (d\theta)_{ab} J^i_{cd} + J^c_{[a} J^i_{b]}(d\theta)_{ij} J^j_{d} \right).$$

Proof. (1) Let $\rho$ be a local defining function for the boundary. Taking the Hermitian part in formula (4.5) from the proof of Lemma 23, we conclude that $\rho P_{abcd}^+ + \frac{1}{4\rho}(\rho_a\rho_b + \theta_a\theta_b)$ admits a smooth extension to the boundary with boundary value the symmetric Hermitian part of $\frac{1}{4}\hat{\nabla}_a\rho_b$. By Lemma 6, $\hat{\nabla}_a\rho_b$ is symmetric along the boundary, so we just have to take the Hermitian part $\frac{1}{4}(\hat{\nabla}_a\rho_b + J^j_{[a} J^i_{b]}(d\theta)_{ij} J^j_{d})$.

On the other hand, we use the asymptotic form for $g$ from formula (2.4) from Section 2.6 provided by Theorem 22. Comparing to part (2) of Proposition 21, we see that that constant $C$ occurring in this asymptotic form equals the boundary value of $-\frac{m}{2}(g^{ij}P_{ij})^{-1}$. Hence we conclude that

$$\rho \frac{g^{ij}P_{ij}}{2m}g_{ab} + \frac{1}{4\rho}(\rho_a\rho_b + \theta_a\theta_b)$$

admits a smooth extension to the boundary with boundary value $\frac{1}{4\rho}h_{ab}$, where $h_{ab}$ is the Hermitian form that occurs in the asymptotic form of $g$. Hence we conclude.
that \( \rho P^a_{ab} \) admits a smooth extension to the boundary with boundary value

\[
\frac{1}{4} \left( \hat{\nabla}_a \rho_b + J^i_a J^j_b \hat{\nabla}_i \rho_j + \frac{1}{C} h_{ab} \right).
\]

Now let us contract \( J \) into the \( b \)-index of the expression in the bracket. Using symmetry and the fact that \( J \) is parallel, the boundary values of the result coincides with the one of

\[
(4.7) \quad \hat{\nabla}_a J^i_b \rho_i - \hat{\nabla}_b J^i_a \rho_i + \frac{1}{C} h_{ai} J^i_b = \frac{1}{C} h_{ai} J^i_b - \hat{\nabla}_a \theta_b + \hat{\nabla}_b \theta_a.
\]

Now since the Nijenhuis tensor also hooks trivially into \( \theta_a \), the boundary values of the last two terms give \( (d\theta)_{ab} \), so (4.7) vanishes by Theorem 22. Hence we see that the smooth extension of \( \rho P^a_{ab} \) to the boundary vanishes along the boundary, so \( P^a_{ab} \) itself admits a smooth extension to the boundary.

(2) If \( J \) is integrable, then by Proposition 9, \( g \) is (pseudo-)Kähler metric, so its Ricci curvature is well known to be Hermitian, which also implies that \( P^a_{ab} \) is Hermitian. By the proof of part (1), we conclude that in this case \( \hat{\nabla}_a \rho_b \) is Hermitian along the boundary and coincides with \( \frac{1}{2} J^i_a (d\theta)_{bi} \) along the boundary. Using this, the form of the curvature follows readily from Proposition 24. \( \square \)

### 4.8. The case of asymptotically parallel Nijenhuis tensor.

In the case of a non-integrable almost complex structure, all the results of Section 4 are based on the canonical connection \( \nabla \) of a quasi-Kähler metric \( g \) and not on its Levi-Civita connection \( \nabla^g \). In particular, all conditions and results on curvature concern the curvature of \( \nabla \). To conclude this article, we study an asymptotic vanishing condition on the covariant derivative of the Nijenhuis tensor, which in particular is satisfied for nearly Kähler metrics. Assuming this condition, we show that in the results of Section 4, one may often use the curvature of \( \nabla^g \) instead of the curvature of \( \nabla \).

**Definition 26.** Let \( \overline{M} \) be a smooth manifold with boundary \( \partial M \) and interior \( M \). Let \( J \) be an almost complex structure on \( M \) with Nijenhuis tensor \( \mathcal{N} \), and let \( \nabla \) be a linear connection on \( TM \) which preserves \( J \). We say that \( \mathcal{N} \) is asymptotically parallel for \( \nabla \) if the tensor field \( \nabla \mathcal{N} \) admits a smooth extension to all of \( \overline{M} \) which vanishes along the boundary.

**Proposition 27.** Let \( \overline{M} \) be a smooth manifold with boundary \( \partial M \) and interior \( M \). Let \( J \) be an almost complex structure on \( \overline{M} \) with Nijenhuis tensor \( \mathcal{N} \), and let \( g \) an admissible Hermitian metric on \((M, J)\) such that

- \( \mathcal{N} \) as asymptotically tangential values
- \( \mathcal{N} \) is asymptotically parallel for the canonical connection \( \nabla \) of \( g \).

Then for the curvature \( R \) of \( \nabla \) and the curvature \( R^g \) of the Levi-Civita connection \( \nabla^g \) of \( g \), the difference \( R - R^g \) admits a smooth extension to all of \( \overline{M} \).
Proof. Consider the contorsion tensor \( A \) defined by \( \nabla^g_\xi \eta = \nabla_\xi \eta - A(\xi, \eta) \). The explicit form of \( A \) has already been used in the proof of Proposition 9. Replacing \( J_\zeta \) by \( \zeta \) in that formula and contracting with an inverse metric, the result reads in abstract index notation as

\[
A^c_{ab} = \frac{1}{2} \left( -g_{ib} T^i_{aj} g^{jc} - g_{ia} T^i_{bj} g^{jc} + T^c_{ab} \right),
\]

where \( T = T^c_{ab} \) is the torsion of \( \nabla \) and hence equals \(-\frac{1}{4} \mathcal{N}\). Since \( \mathcal{N} \) has asymptotically tangential values, part (3) of Proposition 21 implies that \( \rho g_{ib} T^i_{aj} \) admits a smooth extension to the boundary. From the proof of that proposition, we also know that \( \rho^{-1} g^{jc} \) admits a smooth extension to the boundary. Hence we conclude that the contorsion tensor \( A \) admits a smooth extension to the boundary. Computing the covariant derivative \( \nabla A \) using that \( \nabla \) is metric, we see that we only get covariant derivatives hitting \( T \), so we similarly conclude that \( \nabla A \) admits a smooth extension to the boundary. But then from the definition of curvature, one easily concludes that for \( \xi, \eta, \zeta \in \mathfrak{X}(M) \), one may write the difference \( R^g(\xi, \eta)(\zeta) - R(\xi, \eta)(\zeta) \) as

\[
-(\nabla A)(\xi, \eta, \zeta) + (\nabla A)(\eta, \xi, \zeta) - A \left( T(\xi, \eta), \zeta \right) + A \left( \xi, A(\eta, \zeta) \right) - A \left( \eta, A(\xi, \zeta) \right),
\]

which implies the result. \( \square \)

Corollary 28. Under the assumptions of Proposition 27, which are in particular satisfied if \( g \) is a nearly Kähler metric (of any signature), one may replace curvature quantities associated to the canonical connection \( \nabla \) by quantities associated to the Levi-Civita connection \( \nabla^g \) of \( g \) as follows:

- scalar curvature in Theorem 22
- Schouten tensor in Lemma 23 and Theorem 25
- full curvature tensor in Proposition 24.

Proof. First note that on a nearly Kähler manifold, the Nijenhuis tensor is globally parallel for the canonical connection, see Remark 4.5 in [5] and [29], so the assumptions of Proposition 27 are satisfied in this case. Next, in any case where the assumptions of Proposition 27 are satisfied we have that \( R \) and \( R^g \) differ by terms which admit a smooth extension to the boundary. Forming the Ricci contraction, we conclude that the same holds for the difference of the Ricci tensors and hence for the difference of the Schouten tensors of the two connections. Finally, the scalar curvature is obtained by contracting an inverse metric \( g^{ab} \) into the Ricci curvature. As we have noted in the proof of Proposition 27, \( \rho^{-1} g^{ab} \) admits a smooth extension to the boundary. Thus we conclude that the difference of the scalar curvatures of the two connections not only admits a smooth extension to the boundary but this extension also vanishes along the boundary. From this all the claims follow from the statements of the results listed in the corollary. \( \square \)
C-PROJECTIVE COMPACTNESS

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