ON THE POLYMATROIDAL PROPERTY OF MONOMIAL IDEALS WITH A VIEW TOWARDS ORDERINGS OF MINIMAL GENERATORS

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Abstract. We prove that a monomial ideal $I$ generated in a single degree, is polymatroidal if and only if it has linear quotients with respect to the lexicographical ordering of the minimal generators induced by every ordering of variables. We also conjecture that the polymatroidal ideals can be characterized with linear quotients property with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables. We prove our conjecture in many special cases.

1. Introduction

A monomial ideal is called polymatroidal, if its monomial generators correspond to the bases of a discrete polymatroid, see [6]. Since the set of bases of a discrete polymatroid is characterized by the so-called exchange property, it follows that a polymatroidal ideal may as well be characterized as follows: let $I \subseteq S = K[x_1, \ldots, x_n]$ be a monomial ideal generated in a single degree and $G(I)$ be the unique minimal set of monomial generators of $I$. Then $I$ is said to be polymatroidal, if for any two elements $u, v \in G(I)$ such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ there exists an index $j$ with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in I$. A squarefree polymatroidal ideal is called matroidal. They have many attractive structural properties which have been considered by many mathematicians in recent years, see for example [2, 4, 5, 7]. Bandari and Herzog [2] conjectured that a monomial ideal is polymatroidal if and only if all its monomial localizations have a linear resolution. In this paper we try to characterize polymatroidal ideals with linear quotients property. Björner in [3, Theorem 7.3.4] showed that a simplicial complex $\Delta$ is a matroid complex if and only if $\Delta$ is pure and every ordering of the vertices induces a shelling. Hence a monomial ideal is matroidal if and only if it is generated in a single degree and it also has linear quotients with respect to every ordering of generators. Herzog and Takayama in [8] showed that a polymatroidal ideal has linear quotients with respect to the reverse lexicographical order of the minimal generators induced by every ordering of variables. A natural question arises whether we can generalize Björner’s result to polymatroidal ideals. In Proposition 2.2, we show that a polymatroidal ideal also has linear quotients with respect to the lexicographical order of the minimal generators. Then in Theorem 2.4, we prove that for a monomial ideal $I$ generated in a single degree, $I$ is polymatroidal if and only if $I$ has linear

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quotients with respect to the lexicographical ordering of the minimal generators induced by every ordering of variables.

It is natural to ask whether we have similar result with the reverse lexicographical order assumption. Due to computational evidence we are lead to conjecture that the monomial ideals generated in a single degree which have linear quotients with respect to the reverse lexicographical order of the generators induced by every ordering of variables are precisely the polymatroidal ideals. We discuss several special cases which support this conjecture. In fact we give a positive answer to the conjecture in the following cases: 1. $I$ is generated in degree 2 (Proposition 2.6), 2. $I$ contains at least $n - 1$ pure powers (Proposition 2.8), 3. $I$ is monomial ideal in at most 3 variables (Remark 2.5 and Proposition 2.9), 4. $I = (u \in M_d \mid u \geq_{lex} v)$ for some $v \in M_d$ (Proposition 2.10), 5. $I$ is a completely lexsegment ideal (Proposition 2.12).

2. The polymatroidality and orderings induced by variables on minimal generators

Throughout $S = K[x_1, \ldots, x_n]$ is the polynomial ring over a field $K$, $>_{lex}$ is the lexicographic order and $>$ is the reverse lexicographic order on monomials of $S$. In the following, we recall some preliminary concepts.

Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial of $S$. We define $\deg(u) = \sum_{i=1}^{n} a_i$, $\deg_{x_i}(u) = a_i$ and $\text{supp}(u) = \{ x_i \mid 1 \leq i \leq n, \ a_i > 0 \}$.

Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$. We define the total order $>_\text{lex}$ on monomials of $S$ by setting $u >_{\text{lex}} v$ if either (i) $\deg(u) > \deg(v)$ or (ii) $\deg(u) = \deg(v)$ and there exists an integer $i$ with $a_i > b_i$ and $a_k = b_k$ for $k = 1, \ldots, i - 1$. It follows that $>_\text{lex}$ is a monomial order on $S$, which is called the lexicographic order on $S$ induced by the ordering $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots >_{\text{lex}} x_n$.

We define the total order $>$ on monomials of $S$ by setting $u > v$ if either (i) $\deg(u) > \deg(v)$ or (ii) $\deg(u) = \deg(v)$ and there exists an integer $i$ with $b_i > a_i$ and $a_k = b_k$ for $k = i + 1, \ldots, n$. It follows that $>$ is a monomial order on $S$, which is called the reverse lexicographic order on $S$ induced by the ordering $x_1 > x_2 > \cdots > x_n$.

For a monomial ideal $I$ and a monomial $v$ of $S$, $\{ u / \gcd(u, v) \mid u \in G(I) \}$ is a set of generators of $I : v$.

**Definition 2.1.** The monomial ideal $I \subset S$ has linear quotients whenever there is an ordering $u_1, \ldots, u_r$ on the minimal generators of $I$ such that for $j = 2, \ldots, r$, the minimal generators of the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$ are variables.

By [8, Lemma 1.3] and [4, Lemma 4.1], a polymatroidal ideal $I$ has linear quotients with respect to the reverse lexicographical order of the minimal generators and so it has a linear resolution. In the following, we show that we have similar result with the lexicographical order assumption.

**Proposition 2.2.** Let $I$ be a polymatroidal ideal. Then $I$ has linear quotients with respect to the lexicographical order of the minimal generators.

**Proof.** Let $u \in G(I)$ and $J = \{ v \in G(I) \mid v >_{\text{lex}} u \}$. In order to prove that $J : u$ is generated by monomials of degree 1, we have to show that for each $v \in G(I)$ such that $v >_{\text{lex}} u$, there exists $x_i \in J : u$ such that $x_i | v : u$. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$. Since $v >_{\text{lex}} u$, there exists an integer $i$ with $b_i > a_i$ and $a_k = b_k$ for
for $k = 1, \ldots , i - 1$. Now since $I$ is a polymatroidal ideal, it follows by [5, Theorem 3.1] that there exists an integer $j$ with $b_j < a_j$ such that $u' = (u/x_j)x_i \in G(I)$. Since $j > i$, we have that $u' \in J$. Hence $x_iu = x_ju' \in J$, so $x_i \in J : u$. On the other hand since $\deg_{x_i}(v : u) = b_i - \min\{b_i, a_i\} = b_i - a_i > 0$, we have $x_i|v : u$. □

For proof of the next theorem, we need the following result.

**Lemma 2.3.** Let $I \subset S$ be a monomial ideal generated in a single degree and $u, v \in G(I)$ such that supp$(u : v) = \{x_1\}$. Suppose $I$ has linear quotients with respect to the lexicographical order $\succ_{\text{lex}}$, induced by $x_1 \succ_{\text{lex}} \cdots \succ_{\text{lex}} x_n$. Then we have the following exchange property:

$$(u/x_1)x_i \in I \text{ for some } i \text{ with } \deg_{x_i}(v) > \deg_{x_i}(u).$$

**Proof.** Let $u : v = x_1^m$ for some $m \geq 1$. By induction on $m$, we show that we have the exchange property. The assertion is trivial, if $m = 1$. Now, let $m \geq 2$. Since $u \succ_{\text{lex}} v$, it follows by linear quotient property that there exists a monomial $w \in G(I)$, such that $w \succ_{\text{lex}} v$ and $w : v = x_1$. Indeed $w = x_1z$ and $v = x_i z$ for some monomial $z$ and $t \neq 1$. Now since $u : v = x_1^{m-1}$, our induction hypothesis implies that $(u/x_1)x_i \in I$, for some $i$ with $\deg_{x_i}(w) > \deg_{x_i}(u)$. Note that $i \neq 1$, that implies $\deg_{x_i}(u) < \deg_{x_i}(v) = \deg_{x_i}(z) \leq \deg_{x_i}(v)$. □

**Theorem 2.4.** Let $I \subset S$ be a monomial ideal generated in a single degree. Then the following conditions are equivalent:

(a) $I$ is polymatroidal.

(b) $I$ has linear quotients with respect to the lexicographical ordering of the minimal generators induced by every ordering of variables.

**Proof.** The implication (a) $\implies$ (b) holds by Proposition 2.2.

(b) $\implies$ (a): Let $u \in G(I)$ with $\deg_{x_1}(u) > 0$. We want to show that for any monomial $v \in G(I)$ with $\deg_{x_1}(u) > \deg_{x_1}(v)$ we have the following exchange property:

$$(u/x_1)x_i \in I \text{ for some } i \text{ with } \deg_{x_i}(v) > \deg_{x_i}(u).$$

Let $A$ be the set of those monomials $v \in G(I)$ such that $\deg_{x_1}(v) < \deg_{x_1}(u)$ and $(u/x_1)x_i \notin I$, for all $i$ with $\deg_{x_i}(v) > \deg_{x_i}(u)$. We prove by contradiction that $A$ is an empty set. Assume the opposite that $A$ is not empty. Let $v_1 \in A$ such that $\deg_{x_1}(u : v_1) \geq \deg_{x_1}(u : v)$, for all $v \in A$. Now let

$$B = \{v \in A \mid \deg_{x_1}(u : v) = \deg_{x_1}(u : v_1)\}.$$ 

Consider the lexicographical order $\succ_{\text{lex}}$, induced by $x_1 \succ_{\text{lex}} \cdots \succ_{\text{lex}} x_n$. Let $v \in B$ such that

$$u : v = \min\{u : v \mid v \in B\} = \min\{u : v \mid v \in A , \deg_{x_1}(u : v) = \deg_{x_1}(u : v_1)\}.$$ 

By Lemma 2.3, we know that $|\text{supp}(u : v)| \geq 2$. Let $t + 1 = \min\{i \geq 2 \mid x_i|u : v\}$. Consider the lexicographical order $\succ_{\text{lex}}$, induced by

$$x_{j_1} \succ_{\text{lex}} \cdots \succ_{\text{lex}} x_{j_1} \succ_{\text{lex}} x_{t+1} \succ_{\text{lex}} \cdots \succ_{\text{lex}} x_n \succ_{\text{lex}} x_i \succ_{\text{lex}} \cdots \succ_{\text{lex}} x_{i_1} \succ_{\text{lex}} x_1,$$

where

$$\{x_{i_1}, \ldots , x_{i_t}\} = \{x_l \mid 2 \leq l \leq t , \deg_{x_l}(v) > \deg_{x_l}(u)\},$$

$$\{x_{j_1}, \ldots , x_{j_r}\} = \{x_l \mid 2 \leq l \leq t , \deg_{x_l}(v) = \deg_{x_l}(u)\}.$$
Since \( x_{i+1} | u : v \) and \( \deg_{x_i}(u) = \deg_{x_i}(v) \) for all \( l \in \{j_1, \ldots, j_r\} \), it follows that 
\( u >_{\text{lex}} v \). By linear quotient property, there exists a monomial \( w \in G(I) \), such that 
\( w >_{\text{lex}} v, \; u : w = v : x_i \), and \( x_i | u : v \) for some \( i \). In particular \( w = x_i z \) and \( v = x_i z \) 
for some monomial \( z \) and \( x_i >_{\text{lex}} x_j \). Since \( x_1 \) is the smallest variable with respect to \( >_{\text{lex}} \), we have \( i \neq 1 \). Since \( u : v = u : (x_j z) = (u : z) : x_j \), it follows that either \( u : v = u : z \) if \( x_j \not| u : z \) or \( u : v = \frac{w}{x_j} \) if \( x_j | u : z \). Now since \( u : w = (u : z) : x_i \), it 
follows that either (i) \( u : w = \frac{w}{x_i} \) or (ii) \( u : w = \frac{w}{x_j} \).

Case (i): Let \( u : w = \frac{w}{x_i} \). Then \( u : v >_{\text{lex}} u : w \) and since \( i \neq 1 \), it follows that 
\( \deg_{x_i}(u : w) = \deg_{x_i}(u : v) \). Therefore \( w \not\in A \).

Case (ii): Let \( u : w = \frac{w}{x_j} \). If \( j = 1 \), then \( \deg_{x_1}(u : w) = \deg_{x_1}(u : v) + 1 > \deg_{x_j}(u : v) \). Hence \( w \not\in A \). Now let \( j \geq 2 \). Since \( x_i | u : v \) and \( i \neq 1 \), it follow that \( i \in \{t + 1, \ldots, n\} \). On the other hand, since \( x_j | u : w \), it follows that 
\( \deg_{x_j}(u) > \deg_{x_j}(w) = \deg_{x_j}(z) \), so \( \deg_{x_j}(u) \geq \deg_{x_j}(z) + 1 = \deg_{x_j}(v) \). Hence, 
since \( x_i >_{\text{lex}} x_j \) and \( i \in \{t + 1, \ldots, n\} \), we have \( j \in \{t + 1, \ldots, n\} \) and \( x_i >_{\text{lex}} x_j \) 
Therefore \( u : v > u : w \). Now since \( \deg_{x_i}(u : w) = \deg_{x_i}(u : v) \), it follows that 
\( w \not\in A \).

In both cases (i) and (ii), we show \( w \not\in A \). On the other hand \( \deg_{x_1}(u) > \deg_{x_1}(v) \geq \deg_{x_1}(z) = \deg_{x_j}(w) \). Therefore \( (u/x_1)x_s \in I \), for some \( s \) with 
\( \deg_{x_1}(w) > \deg_{x_s}(w) \). Since \( \deg_{x_s}(w) = \deg_{x_s}(z) + 1 = \deg_{x_s}(v) + 1 \leq \deg_{x_1}(u) \), it 
follows that \( i \neq s \). So \( \deg_{x_s}(v) \geq \deg_{x_s}(z) = \deg_{x_s}(w) > \deg_{x_s}(u) \). It contradicts our assumption \( v \not\in A \).

Replacing \( x_1 \) with \( x_i \), the same argument proves the exchange property for \( I \).

Remark 2.5. Let \( I \subset K[x_1, x_2] \) be a monomial ideal generated in a single degree. Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.
(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.
(c) \( I \) has a linear resolution.

Proposition 2.6. Let \( I \subset S \) be a monomial ideal generated in degree 2. Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.
(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Proof. The implication (a) \( \Rightarrow \) (b) is known.

(b) \( \Rightarrow \) (a): Let \( u, v \in G(I) \) with \( \deg_{x_1}(u) > \deg_{x_1}(v) \). If \( x_i | v \), there is nothing 

Case (i): Let \( u = x_i^2 \) and \( v = x_i x_j \) (it can be \( j = l \), so \( u : v = x_i^2 \). Assume that 
\( x_i > x_d \) for each \( d \neq i \), so \( u > v \). Hence by the assumption there exists \( w \in G(I) \), 
\( w > v \) and \( w : v = x_i \). Hence either \( w = x_i x_j = (u/x_i)x_j \in G(I) \) or 
\( u = x_i x_j = (u/x_i)x_i \in G(I) \).

Case (ii): Let \( u = x_i x_k \) and \( v = x_j x_l \) (it can be \( j = l \)). If either \( t = j \) or \( t = l \), 
there is nothing to prove. Otherwise, assume that \( x_i > x_l > x_k \) and \( x_j > x_i > x_k \) 
So \( v > u \) and \( u : x_kx_i \). Hence by the assumption there exists \( w \in G(I) \), such
that \( w > u \) and we have either \( w : u = x_j \) or \( w : u = x_l \). If \( w : u = x_j \), then \( w = x_j x_j = (u/x_i)x_j \in G(I) \). If \( w : u = x_l \), then \( w = x_l x_l = (u/x_i)x_l \in G(I) \). □

In the following we recall the monomial localization concept, which we will use it in the next proposition.

**Definition 2.7.** Let \( P \) be a monomial prime ideal of \( S = K[x_1, \ldots, x_n] \). Then \( P = P_C \) for some subset \( C \subseteq [n] \), where \( P_C = \{(x_i) \in C \} \) and \( IS_P = JS_P \) where \( J \) is the monomial ideal obtained from \( I \) by the substitution \( x_i \mapsto 1 \) for all \( i \in C \).

We call \( J \) the monomial localization of \( I \) with respect to \( P \) and denote it by \( I(P) \).

**Proposition 2.8.** Let \( I \subseteq K[x_1, \ldots, x_n] \) be a monomial ideal generated in degree \( d \) and suppose that \( I \) contains at least \( n - 1 \) pure powers of the variables, say \( x_1^d, \ldots, x_{n-1}^d \). Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.

(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

**Proof.** The implication (a) \( \implies \) (b) is known.

(b) \( \implies \) (a): Let \( k = \max \{ \deg\alpha(u) \mid u \in G(I) \} \) and \( w = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^k \in G(I) \). We want to show that \( x_i^d x_n^k \in G(I) \) for all \( 1 \leq i \leq n - 1 \). We fix \( i \). If \( w = x_i^d x_n^k \), then there is nothing to prove. Otherwise, for monomial \( u \), we define \( \deg_A(u) = \sum_{i \in A} \deg_{x_i}(u) \), where \( A := \text{supp}(w) \setminus \{x_i, x_n\} \). We consider the ordering \( x_n > x_i > x_j \) for all \( j \neq i, n \). Since \( x_i^d > w \) and \( x_n^k > w \), it follows by the assumption that there exists \( w_1 \in G(I) \) such that \( w_1 > w \) and \( w_1 : u = x_i \). Hence \( w_1 = (w/x_j) x_i \), where \( j \in A \), \( \deg_{x_j}(w_1) = a_i + 1 \) and \( \deg_A(w_1) = \deg_A(w) - 1 \). Now since \( x_i^d : w_1 = x_i^{d - (a_i + 1)} \), it follows that there exists \( w_2 \in G(I) \) such that \( w_2 > w_1 \) and \( w_2 : w_1 = x_i \). Hence \( w_2 = (w_1/x_j) x_i \), where \( j \in A \), \( \deg_{x_j}(w_2) = a_i + 2 \) and \( \deg_A(w_2) = \deg_A(w) - 2 \). Continuing in the same way, there exists \( w_{n-1} \in G(I) \) such that \( w_{n-1} > w_1 \) and \( w_{n-1} : w_1 = x_i \). Hence \( w_{n-1} = (w_1/x_j) x_i \), where \( j_n \in A \), \( \deg_{x_j}(w_{n-1}) = a_i + d = k \) and \( \deg_A(w_{n-1}) = \deg_A(w) - h = 0 \).

Therefore \( w_n = x_i^{d - k} x_n^k \).

Now, by induction on \( h \) we show \( x_i^{\alpha_1} \cdots x_{ih-1}^{\alpha_{ih-1}} x_{ih}^{\alpha_{ih} + k} x_n^k \in G(I) \), where \( \alpha_i + \cdots + \alpha_{ih-1} + \alpha_{ih} = d - k \). For \( h = 1 \), we have already proved it. Now assume that \( h > 1 \). We set \( u := x_i^{\alpha_1} \cdots x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih}} x_n^k \) and \( v := x_i^{\alpha_i} \cdots x_{ih-2}^{\alpha_{ih-2} + \alpha_{ih}} x_{ih}^{\alpha_{ih} + k} x_n^k \). By induction hypothesis, \( u, v \in G(I) \). Assume that \( x_n > x_{ih-1} > x_{ih-2} > \cdots > x_{ih-1} > x_i \). Since \( u > v \) and \( u : v = x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih}} \), it follows that there exists \( w_1 \in G(I) \) such that \( w_1 > v \) and \( w_1 : v = x_{ih-1} \). Hence \( w_1 = x_i^{\alpha_1} \cdots x_{ih-2}^{\alpha_{ih-2} + \alpha_{ih}} x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih} - 1} x_n^k \).

Now, since \( u : w_1 = x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih}} \), there exists \( w_2 > w_1 \) such that \( w_2 : w_1 = x_{ih-1} \). Hence \( w_2 = x_i^{\alpha_1} \cdots x_{ih-2}^{\alpha_{ih-2} + \alpha_{ih}} x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih} - 2} x_{ih}^{\alpha_{ih} + k} x_n^k \). Continuing in the same way, there exists \( w_{n-1} \in G(I) \) such that \( w_{n-1} : w_{n-1} = x_i^{\alpha_1} \cdots x_{ih-2}^{\alpha_{ih-2} + \alpha_{ih}} x_{ih-1}^{\alpha_{ih-1} + \alpha_{ih} - 1} x_n^k \), as desired.

By what we have shown, it follows that \( I(P_{[n]}) = (x_1, \ldots, x_{n-1})^{d-k} \). Hence \( I(P_{[n]}) \) has a linear resolution. So by [2, Proposition 2.4], \( I \) is polymatroidal. □

**Proposition 2.9.** Let \( I \subseteq S = K[x_1, x_2, x_3] \) be a monomial ideal generated in a single degree. The following conditions are equivalent:
(a) $I$ is polymatroidal.
(b) $I$ has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Proof. The implication (a) $\implies$ (b) is known.

(b) $\implies$ (a): Let $\deg_{x_1}(u) > \deg_{x_1}(v)$ and $\deg_{x_2}(u) < \deg_{x_2}(v)$. We have two cases:

(i) If $u: v = x_1^r x_3^s$ for integers $r, s > 0$. Then $v: u = x_1^t$ for integer $t > 0$. We consider the ordering $x_3 > x_2 > x_1$, so $v > u$. Hence by the assumption there exists $w \in G(I)$ such that $w > u$ and $w: u = x_2$. So $w = (u/x_1)x_2 \in G(I)$.

(ii) If $u: v = x_1^r$ for integer $r > 0$.
- Suppose $\deg_{x_1}(u) = \deg_{x_1}(v)$. We assume that $x_3 > x_1 > x_2$, so $u > v$. Hence by the assumption there exists $w_1 \in G(I)$ such that $w_1 > v$ and $w_1 : v = x_1$. So $w_1 = (v/x_2)x_1$. Now since $u > w_1$ and $u : w_1 = x_1^{r-1}$, it follows that there exists $w_2 > w_1$ such that $w_2 : w_1 = x_1$. Hence $w_2 = (w_1/x_2)x_1$. So $u > w_2$ and $u : w_2 = x_1^{r-2}$. Continuing in the same way, there exists $w_{r-1} = (w_{r-2}/x_2)x_1 \in G(I)$, such that $u > w_{r-1}$ and $u : w_{r-1} = x_1$. Thus $u = (w_{r-1}/x_2)x_1$. Hence $(u/x_1)x_2 = w_{r-1} \in G(I)$.
- Suppose $\deg_{x_1}(u) < \deg_{x_1}(v)$. We assume that $x_3 > x_1 > x_2$, so $u > v$. Hence by the assumption there exists $w_1 \in G(I)$ such that $w_1 > v$ and $w_1 : v = x_1$. So either $w_1 = (v/x_2)x_1$ or $w_1 = (v/x_3)x_1$. Now since $u > w_1$ and $u : w_1 = x_1^{r-1}$, it follows that there exists $w_2 > w_1$ such that $w_2 : w_1 = x_1$. So either $w_2 = (w_1/x_2)x_1$ or $w_2 = (w_1/x_3)x_1$. Hence $u > w_2$ and $u : w_2 = x_1^{r-2}$. Continuing in the same way, there exists $w_{r-1} \in G(I)$, such that $u > w_{r-1}$ and $u : w_{r-1} = x_1$. Hence either $u = (w_{r-1}/x_2)x_1$ or $u = (w_{r-1}/x_3)x_1$. Therefore either $(u/x_1)x_2 \in G(I)$ or $(u/x_1)x_3 \in G(I)$.

Let $M_d$ denote the set of all monomials of degree $d$ in the polynomial ring $S = k[x_1, \ldots, x_n]$. We order the monomials lexicographically by the ordering $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$.

**Proposition 2.10.** Let $v = x_1^{b_1} \cdots x_n^{b_n} \in M_d$ and $I = \{u \in M_d \mid u \geq_{lex} v\}$ be a monomial ideal in $S = K[x_1, \ldots, x_n]$. Then the following conditions are equivalent:
(a) $I$ is polymatroidal.
(b) $I$ has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Proof. The implication (a) $\implies$ (b) is known.

(b) $\implies$ (a): We consider $x_n > x_j > x_1$ for $j \neq 1, n$. Since $\deg_{x_1}(v) < \deg_{x_1}(x_1^{b_1+1}x_n^{-d(b_1+1)})$, we have $v > x_1^{b_1+1}x_n^{-d(b_1+1)}$. Assume that $n = (x_2, \ldots, x_n)$.

We have the following cases:

**Case 1.** $v : x_1^{b_1+1}x_n^{-d(b_1+1)} = x_n$. Then $v = x_1^{b_1}x_n^{-d-b_1}$. Hence $I = \sum_{i=0}^{d} x_1^i n^{d-i}$, so $I$ is polymatroidal.

**Case 2.** $v : x_1^{b_1+1}x_n^{-d(b_1+1)} = x_j$, where $j \neq 1, n$. Hence $v = x_1^{b_1}x_j x_n^{-d(b_1+1)}$. Therefore $I = \sum_{i=b_1+1}^{d} x_1^i n^{d-i} + J$, where

$$J := \{w \in M_d \mid \deg_{x_1}(w) = b_1 \text{ and } w \geq_{lex} v\}.$$

Now we want to prove that $I$ is polymatroidal.
Theorem 1.3, since 

Suppose (i) holds. Since deg \( x \) completely lexsegment is called a completely lexsegment ideal. if all the iterated shadows of \( L \) are again lexsegments. An ideal spanned by a

Proposition 2.12. Let \( I \subset K[x_1, \ldots, x_n] \) be a monomial ideal generated in degree \( d \) and suppose that \( I \) is a completely lexsegment ideal. Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.
(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Proof. The implication (a) \( \implies \) (b) is known.

(b) \( \implies \) (a): Let \( I = (L(u, v)) \), where \( u = x_1^{a_1} \cdots x_n^{a_n} \) and \( v = x_1^{b_1} \cdots x_n^{b_n} \). By [1, Theorem 3.3], since \( I \) has a linear resolution, one of the following conditions holds:

(i) \( u = x_2^{r}x_2^{d-a} \) and \( v = x_2^{r}x_2^{d-a} \) for some \( a, 0 < a < d \).
(ii) \( b_1 \leq a_1 - 1 \).

Suppose (i) holds. Since \( \deg_{x_1}(u) = \deg_{x_1}(v) \), so \( I = x_1^{a_1}(x_2, \ldots, x_n)^{d-a_1} \). Obviously \( I \) is polymatroidal.

Case 3. \( v : x_1^{a_1+1}x_n^{d-(b_1+1)} = x_j, \ldots, x_j, \) where \( l > 1 \) and \( j_r \neq n \) for each \( r = 1, \ldots, l \). Then there exists \( u \in G(I) \) such that \( w > x_1^{b_1+1}x_n^{d-(b_1+1)} \) and \( u : x_1^{b_1+1}x_n^{d-(b_1+1)} = x_j \), for some \( 1 \leq s \leq l \). Now, since we consider the ordering \( x_n > x_j > x_1 \) for \( j \neq 1, n \), it follows that \( w = x_1^{b_1}x_jx_n^{d-(b_1+1)} \). This is a contradiction, since \( w \geq_{\text{lex}} v \). \( \square \)

Definition 2.11. A lexsegment (of degree \( d \)) is a subset of \( M_d \) of the form

\[ \mathcal{L}(u, v) = \{ w \in M_d \mid u \geq_{\text{lex}} w \geq_{\text{lex}} v \} \]

for some \( u, v \in M_d \) with \( u \geq_{\text{lex}} v \). A lexsegment \( L \) is called completely lexsegment if all the iterated shadows of \( L \) are again lexsegments. An ideal spanned by a completely lexsegment is called a completely lexsegment ideal.

We recall that the shadow of a set \( T \) of monomials is the set \( \text{Shad}(T) = \{ v \in T \mid 1 \leq i \leq n \} \}. The i-th shadow is recursively defined as \( \text{Shad}^{i}(T) = \text{Shad}^{i-1}(T) \).

Proposition 2.12. Let \( I \subset K[x_1, \ldots, x_n] \) be a monomial ideal generated in degree \( d \) and suppose that \( I \) is a completely lexsegment ideal. Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.
(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Proof. The implication (a) \( \implies \) (b) is known.

(b) \( \implies \) (a): Let \( I = (\mathcal{L}(u, v)) \), where \( u = x_1^{a_1} \cdots x_n^{a_n} \) and \( v = x_1^{b_1} \cdots x_n^{b_n} \). By [1, Theorem 3.3], since \( I \) has a linear resolution, one of the following conditions holds:

(i) \( u = x_2^{r}x_2^{d-a} \) and \( v = x_2^{r}x_2^{d-a} \) for some \( a, 0 < a < d \).
(ii) \( b_1 \leq a_1 - 1 \).

Suppose (i) holds. Since \( \deg_{x_1}(u) = \deg_{x_1}(v) \), so \( I = x_1^{a_1}(x_2, \ldots, x_n)^{d-a_1} \). Obviously \( I \) is polymatroidal.
Now let (ii) holds. Assume that \( a_3 + a_4 + \cdots + a_n \neq 0 \). We consider the ordering \( x_1 > x_2 > x_3 > \cdots > x_n \). Suppose \( w = x_1^{a_1-1}x_2^{d-a_1+1} \). Since \( w \in I \), \( w > u \), \( w : u = x_2^{d-a_1-a_2+1} \) and also \( d - a_1 - a_2 + 1 > 1 \), it follows that there exist \( w' \in I \) such that \( w' > u \) and \( w' : u = x_2 \). Hence \( w' = (u/x_1)x_2 \) for some \( 3 \leq i \leq n \). So \( w' > _{lex} u \), which is a contradiction.

Now let \( a_i = 0 \) for \( i = 3, \ldots, n \). Hence \( u = x_1^{a_1}x_2^{a_2} \) and so \( x_1^{b_1}x_2^{d-(b_1+1)} \in I \). We consider \( x_n > x_j > x_1 \) for \( j = 1, n \), hence \( v > x_1^{b_1}x_n^{d-(b_1+1)} \). Assume that \( n = (x_2, \ldots, x_n) \). We have the following cases:

**Case 1.** \( v : x_1^{b_1+1}x_n^{d-(b_1+1)} = x_n \). Then \( v = x_1^{b_1}x_n^{d-b_1} \). Hence \( I = \sum_{i=b_1}^{a_1} x_i^{d-i} \), so \( I \) is polymatroidal.

**Case 2.** \( v : x_1^{b_1}x_n^{d-(b_1+1)} = x_j \), where \( j = 1, n \). Hence \( v = x_1^{b_1}x_jx_n^{d-(b_1+1)} \). Therefore \( I = \sum_{i=b_1}^{a_1} x_1^{d-i} + J \), where

\[
J := (w \in M_d \mid deg_x(w) = b_1 \text{ and } w \geq_{lex} v).
\]

With the same arguments as used in the proof of case 2 of Proposition 2.10, \( I \) is polymatroidal.

**Case 3.** \( v : x_1^{b_1}x_n^{d-(b_1+1)} = x_{j_1} \cdots x_{j_r} \), where \( l > 1 \) and \( j_r \neq n \) for each \( r = 1, \ldots, l \). With the same arguments as used in the proof of case 3 of Proposition 2.10, \( I \) is polymatroidal.

Based on Remark 2.5, Proposition 2.6, Proposition 2.8, Proposition 2.9, Proposition 2.10, Proposition 2.12 and based on experimental evidence we are inclined to make the following:

**Conjecture 1.** Let \( I \subset S \) be a monomial ideal generated in a single degree. Then the following conditions are equivalent:

(a) \( I \) is polymatroidal.

(b) \( I \) has linear quotients with respect to the reverse lexicographical ordering of the minimal generators induced by every ordering of variables.

Let \( I \subset S \) be a monomial ideal minimally generated by \( u_1, \ldots, u_r \). We say that \( I \) has **quotients with linear resolution** with respect to the ordering \( u_1, \ldots, u_r \) whenever \( I \) has a linear resolution and, moreover, for all \( j = 2, \ldots, r \), the colon ideal \( (u_1, \ldots, u_{j-1}) : u_j \) has a linear resolution.

**Remark 2.13.** It is clear that if a monomial ideal generated in a single degree has linear quotients with respect to the ordering \( u_1, \ldots, u_r \) of minimal generators then it also has quotients with linear resolution with respect to the ordering \( u_1, \ldots, u_r \). So polymatroidal ideals have quotients with linear resolution with respect to the (reverse) lexicographical ordering of the minimal generators induced by every ordering of variables, but it is not polymatroidal. We note that \( I \) does not have linear quotients with respect to the lexicographical ordering of the
minimal generators induced by the ordering $x_3 >_{lex} x_2 >_{lex} x_1$ and the reverse lexicographical ordering of the minimal generators induced by the ordering $x_3 > x_2 > x_1$ of variables.

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