The new wave equations of relativistic and non-relativistic quantum mechanics

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Abstract

In this work, we give the wave equations of relativistic and non-relativistic quantum mechanics which are different from the Schrödinger and Klein-Gordon equation, and we also give the new relativistic wave equation of a charged particle in an electromagnetic field.

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We know De Broglie suggested that not only does light have a dual nature but material particles also require a wave-particle description during 1922-23, and he further noticed correspondences between the classical theory of light and the classical theory of mechanics. He thought that we may obtain the wave equation of material particles by comparing the classical theory of light with the classical theory of material particle [1]. Schrödinger used De Broglie’s idea to obtain the wave equation of material particles, i.e., Schrödinger equation [2]. In the following, we also apply De Broglie’s suggestion to research the wave equation of material particles [3].

From the classical electromagnetic theory, the wave equation of electromagnetic wave which frequency is $\nu$ is:

$$\nabla^2 \psi(\vec{r}) + \frac{4\pi^2 n^2 \nu^2}{c^2} \psi(\vec{r}) = 0,$$

(1)

where $n$ is refracting power, $c$ is light velocity and $\psi(\vec{r})$ is a component of electromagnetic field $E$ and $B$. The Eq. (1) describes the wave nature of light such as interference and diffraction phenomenon of light. When light transmits at straight line it is described by the geometrical optics and the geometrical optics is a limiting case of wave theory of light. Fermat had reduced the laws of geometrical optics to the principles of 'least-time'. That is, a light ray follows the path requiring the least time. The Fermat principle is

$$\delta \int nds = 0,$$

(2)

For a material particle, when it moves in potential energy $V(r)$ it can be described by the minimum action principle of Jacobi

$$\delta \int \sqrt{2m(E - V(r))ds} = 0,$$

(3)

where the total energy $E$ is the sum of kinetic $T$ and potential energy $V(r)$. When $n$ is replaced with $\sqrt{2m(E - V(r))}$, the Eq. (2) is the same as Eq. (3). So, we can think that material particle wave equation is similar to Eq. (1) and the wave equation can be written as follows:

$$\nabla^2 \psi(\vec{r}) + Cm(E - V(r))\psi(\vec{r}) = 0,$$

(4)

where $C$ is a constant and it can be obtained in the following. For a free material particle, its potential
energy $V(r) = 0$ and total energy $E = \frac{p^2}{2m}$, and it is associated with a plane wave

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i(\vec{p} \cdot \vec{r} - Et)},$$  \hspace{1cm} (5)

Substitution of Eq. (5) into Eq. (4) gives

$$\left(\frac{i}{\hbar}\vec{p}\right)^2 \psi(\vec{r}) + CmE \psi(\vec{r}) = 0,$$  \hspace{1cm} (6)

The constant $C$ is

$$C = \frac{2}{\hbar^2},$$  \hspace{1cm} (7)

From Eq. (4) and Eq. (7), we obtain

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r)\right] \psi(\vec{r}) = E \psi(\vec{r}).$$  \hspace{1cm} (8)

The Eq. (8) is known as Schrödinger’s time-independent wave equation and gives the states of constant energy [2]. It is corresponding Eq. (1) which describes the electromagnetic wave of constant energy. The time-dependent wave equation of material particle can be obtained by time-dependent wave equation of electromagnetic wave. It is

$$\nabla^2 \psi(\vec{r}, t) - \frac{n^2}{c^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0,$$  \hspace{1cm} (9)

As the same process in the above, we can write down the time-dependent wave equation of material particle

$$\nabla^2 \psi(\vec{r}, t) + Cm(E - V(r)) \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0,$$  \hspace{1cm} (10)

For a free material particle, its potential energy $V(r) = 0$ and total energy $E = \frac{p^2}{2m}$. Substitute for the plane wave equation (5) to Eq. (10), we have

$$\left(\frac{i}{\hbar}\vec{p}\right)^2 \psi(\vec{r}) + CmE \left(\frac{i}{\hbar}E\right)^2 \psi(\vec{r}) = 0,$$  \hspace{1cm} (11)

The constant $C$ is

$$C = -2/E,$$  \hspace{1cm} (12)

We can obtain the time-dependent wave equation of material particle

$$\nabla^2 \psi(\vec{r}, t) - \frac{2m(E - V(r))}{E^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0,$$  \hspace{1cm} (13)
When the potential energy of the system is not a function of time, the time-dependent wave equation (13) has variable separable solutions of the form

$$\psi(\vec{r}, t) = \psi(\vec{r}) f(t)$$  \hspace{1cm} (14)

Substitute (14) into (13) and divide through by $\psi(\vec{r}, t)$

$$\frac{1}{E - V(r)} \frac{1}{\psi(\vec{r})} \nabla^2 \psi(\vec{r}) = \frac{2m}{E^2} \frac{1}{f(t)} \frac{d^2 f(t)}{dt^2} = C,$$  \hspace{1cm} (15)

$C$ is a separation constant independent of $\vec{r}$ and $t$. We have two equations as follows:

$$[\nabla^2 + CV(r)] \psi(\vec{r}) = CE \psi(\vec{r}),$$  \hspace{1cm} (16)

$$\frac{d^2 f(t)}{dt^2} = \frac{2mC}{E^2} f(t),$$  \hspace{1cm} (17)

Obviously, the Eq. (8) is the same as Eq. (16). Comparing their coefficient, we have

$$C = -2m/\hbar^2,$$  \hspace{1cm} (18)

Substitute for $C$ to Eq. (16)-(17) and obtain

$$[-\frac{\hbar^2}{2m} \nabla^2 + V(r)] \psi(\vec{r}) = E \psi(\vec{r}),$$  \hspace{1cm} (19)

$$\frac{d^2 f(t)}{dt^2} + \frac{4m^2}{E^2 \hbar^2} f(t) = 0,$$  \hspace{1cm} (20)

The solution of Eq. (20) is

$$f(t) = Ae^{i\frac{2\pi}{\hbar}t} + Be^{-i\frac{2\pi}{\hbar}t},$$  \hspace{1cm} (21)

The complete solution of Eq. (13) is

$$\psi(\vec{r}, t) = \psi(\vec{r})(Ae^{i\frac{2\pi}{\hbar}t} + Be^{-i\frac{2\pi}{\hbar}t}).$$  \hspace{1cm} (22)

Obviously, Eq. (13) is different from Schrödinger’s time-dependent wave equation [2]

$$[-\frac{\hbar^2}{2m} \nabla^2 + V(r)] \psi(\vec{r}, t) = i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t},$$  \hspace{1cm} (23)

The Schrödinger’s time-dependent wave equation (23) can be obtained by making classical momenta $\vec{p}$ and energy $E$ into operators

$$\vec{p} \to -i\hbar \nabla, \quad E \to i\hbar \frac{\partial}{\partial t}$$  \hspace{1cm} (24)
In Eq. (13), the momenta \( \vec{p} \) can be become operator, i.e., \( \vec{p} \rightarrow -i\hbar \nabla \), but the energy \( E \) cannot be become operator \( i\hbar \frac{\partial}{\partial t} \), since there isn’t unity time \( t \) in relativistic many-particle system. So, it is reasonable that the \( E \) is a number in Eq. (13).

In the following, we will give the relativistic wave equations of material particle. Firstly, we should extend the Fermat’s principle to covariance from

\[
\delta \int nds = 0, \tag{25}
\]

where the \( ds \) is four-dimension differential interval and it is Lorentz-invariant. It is important to find a relativistic variational equation of material particle. We can do it from Hamilton’s principle [4]

\[
\delta \int L dt = 0, \tag{26}
\]

where \( L \) is Lagrange function. The Eq. (26) can be written by \( dt = \gamma d\tau \) and \( ds = cd\tau \) as

\[
\delta \int \frac{\gamma}{c} L ds = 0, \tag{27}
\]

where \( \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \), the \( u \) is particle velocity, the \( c \) is light velocity. Obviously, the Eq. (27) is covariance when the \( \gamma L \) is Lorentz-invariant. For a free particle, the \( L \) can be taken as [5]:

\[
L = -m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}, \tag{28}
\]

From Eqs. (25)-(28), with the same method in the above, we can obtain the time-independent relativistic wave equations of material particle

\[
[E_0^2 - c^2 \hbar^2 \nabla^2] \psi(\vec{r}) = E^2 \psi(\vec{r}), \tag{29}
\]

where \( E_0 = m_0 c^2 \). The equation of classical mass-energy is

\[
E^2 = E_0^2 + c^2 p^2, \tag{30}
\]

Obviously, the Eq. (29) also can be obtained by quantization to Eq. (30), i.e., \( \vec{p} \rightarrow -i\hbar \nabla \). However, the time-dependent relativistic wave equations of material particle can be obtained easily. It is obtained similarly to Eq. (13)

\[
\nabla^2 \psi(\vec{r}, t) - \frac{E^2 - E_0^2}{E^2 c^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0, \tag{31}
\]
where the $E$ is total energy. The Eq. (31) is different from the Klein-Gordon equation

$$\nabla^2 \psi(\vec{r}, t) - \frac{m_0^2 c^2}{\hbar^2} \psi(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0,$$

(32)

and the Eq. (31) has variable separable solutions of the form

$$\psi(\vec{r}, t) = \psi(\vec{r}) f(t),$$

(33)

substituting Eq. (33) into (31), we can obtain two equations as follows:

$$[E_0^2 - e^2 \hbar^2 \nabla^2] \psi(\vec{r}) = E^2 \psi(\vec{r}),$$

(34)

$$\frac{d^2 f(t)}{dt^2} + \frac{E^2}{\hbar^2} f(t) = 0,$$

(35)

The Eq. (34) is the same as (29). The complete solution of Eq. (31) is

$$\psi(\vec{r}, t) = (A e^{i \Phi t} + B e^{-i \Phi t}) \psi(\vec{r}).$$

(36)

In the following, we consider a charged particle in an electromagnetic field. In Eq. (27), the $L$ can be taken as [5]:

$$L = -\frac{m_0 c^2}{\gamma} - \frac{e}{\gamma c} U_\mu A^\mu = -\frac{m_0 c^2}{\gamma} + \frac{e}{c} \vec{u} \cdot \vec{A} - e \Phi,$$

(37)

where the $U_\mu$ is four-velocity, the $A^\mu$ is electromagnetic four-vector. The momentum $\vec{P}$ conjugate to $\vec{X}$ can be defined as follows:

$$P_i = \frac{\partial L}{\partial u_i} = \gamma m u_i + \gamma \frac{e}{c} A_i,$$

or

$$\vec{P} = \vec{p} + \frac{e}{c} \vec{A} = \gamma m \vec{u} + \frac{e}{c} \vec{A},$$

(39)

where $\vec{p} = \gamma m \vec{u}$. From Eq. (39), we have

$$\vec{u} = \frac{c \vec{P} - e \vec{A}}{\sqrt{(\vec{P} - e \vec{A})^2 + m_0^2 c^2}}$$

(40)

$$\gamma = \frac{\sqrt{(\vec{P} - e \vec{A})^2 + m_0^2 c^2}}{m_0 c}.$$  

(41)

The total energy $E$ is

$$E = \sqrt{(c \vec{P} - e \vec{A})^2 + m_0^2 c^4} + e \Phi,$$

(42)
From Eqs. (37)-(42), we obtain

$$\gamma L = -m_0c^2 + \frac{e}{c} \vec{u} \cdot \vec{A} - \gamma e\Phi = -m_0c^2 + \frac{ec}{E_0} \vec{p} \cdot \vec{A} - \frac{E - e\Phi}{E_0} e\Phi,$$

(43)

with the same method in the above, we can obtain the time-independent wave equation when \(n\) is replaced with \(\gamma L\) in Eq. (1). It is

$$\nabla^2 \psi(\vec{r}) + B(-E_0 + \frac{ec}{E_0} \vec{p} \cdot \vec{A} - \frac{E - e\Phi}{E_0} e\Phi)^2 \psi(\vec{r}) = 0,$$

(44)

for a free particle

$$\vec{A} = \Phi = 0,$$

(45)

and

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i(\vec{p} \cdot \vec{r} - Et)},$$

(46)

substitution of Eqs. (45)-(46) into (44) gives

$$B = \frac{E^2 - E_0^2}{E_0^2c^2\hbar^2},$$

(47)

where \(E = \gamma m_0c^2\) for a free particle. Substitute for \(\gamma\) to obtain

$$B = \frac{(e\vec{P} - e\vec{A})^2}{E_0^2c^2\hbar^2} = \frac{p^2}{E_0^2\hbar^2},$$

(48)

$$\nabla^2 \psi(\vec{r}) + \frac{p^2}{E_0^2\hbar^2}(-E_0 + \frac{ec}{E_0} \vec{p} \cdot \vec{A} - \frac{E - e\Phi}{E_0} e\Phi)^2 \psi(\vec{r}) = 0,$$

(49)

Substitute for \(\vec{p} \rightarrow -i\hbar\nabla\) to obtain

$$\nabla^2 \psi(\vec{r}) - \frac{1}{E_0^2} \nabla^2(-E_0 - i\hbar \frac{ec}{E_0} \nabla \cdot \vec{A} - \frac{E - e\Phi}{E_0} e\Phi)^2 \psi(\vec{r}) = 0.$$

(50)

The Eq. (50) is time-independent relativistic wave equation of a charged particle in an electromagnetic field. For time-dependent relativistic wave equation, we can obtain when \(n\) is replaced with \(\gamma L\) in Eq. (9). It is

$$\nabla^2 \psi(\vec{r},t) + B(-E_0 + \frac{ec}{E_0} \vec{p} \cdot \vec{A} - \frac{E - e\Phi}{E_0} e\Phi)^2 \frac{\partial^2 \psi(\vec{r},t)}{\partial^2 t} = 0,$$

(51)

We can obtain the constant \(B\) Similarly to Eq. (48)

$$B = -\frac{p^2}{E_0^2(E - e\Phi)^2},$$

(52)
substitution of Eq. (52) into (51) gives

$$\nabla^2 \psi(\vec{r}, t) - \frac{p^2}{E_0^2(E - e\Phi)^2}(-E_0 + \frac{ec}{E_0}\vec{p} \cdot \vec{A} - \frac{E - e\Phi}{E_0}e\Phi)^2 \frac{\partial^2 \psi(\vec{r}, t)}{\partial^2 t} = 0, \quad (53)$$

Substitute for $\vec{p} \to -i\hbar \nabla$ to obtain

$$\nabla^2 \psi(\vec{r}, t) + \frac{\hbar^2}{E_0^2(E - e\Phi)^2} \nabla^2 (-E_0 - i\hbar \frac{ec}{E_0} \nabla \cdot \vec{A} - \frac{E - e\Phi}{E_0}e\Phi)^2 \frac{\partial^2 \psi(\vec{r}, t)}{\partial^2 t} = 0. \quad (54)$$

The Eq. (54) is time-dependent relativistic wave equation of a charged particle in an electromagnetic field.

In our works, the time-independent wave equations are the same as the standard quantum mechanics. So, the energy of system is same in the two theoretical calculation. Since our time-dependent wave equations are different from standard quantum mechanics the systematic transition probability is different in their calculation. Otherwise, the new time-dependent relativistic wave equations may have some effects to quantum field theory.

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