The Painlevé transcendents with solvable monodromy

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Abstract: We will study special solutions of the fourth, fifth and sixth Painlevé equations with generic values of parameters whose linear monodromy can be calculated explicitly. We will show the relation between Umemura’s classical solutions and our solutions.

1 Introduction

The Painlevé equation can be represented by an isomonodromic deformation of a linear equation

\[ \frac{\partial \Psi}{\partial x} = A(x, y(t), t) \Psi, \]
\[ \frac{\partial \Psi}{\partial t} = B(x, y(t), t) \Psi, \tag{1.1} \tag{1.2} \]

where \( A(x, y, t) \) and \( B(x, y, t) \) are \( 2 \times 2 \) matrices. The integrability condition of (1.1) and (1.2) gives the Painlevé equation for \( y(t) \). We call the linear equation (1.1) the linearization of the Painlevé equation. We call the monodromy data of the linear equation (1.1) a linear monodromy of the Painlevé function \( y(t) \). The linear monodromy cannot be calculated except for special cases. One exceptional case is Umemura’s classical solutions. Umemura showed that there exist two kinds of special solutions for the Painlevé equations, algebraic solutions and the Riccati solutions [41], which are called classical solutions of the Painlevé equations. For most of all Umemura’s classical solutions, the linear monodromy can be calculated, but there exist some Painlevé functions which are not included in Umemura’s classical solutions, such that the linear monodromy can be calculated. If we can determine the linear monodromy of a Painlevé function exactly, we call the Painlevé function monodromy solvable.

It was R. Fuchs who found a monodromy solvable solution at first, which is not included in Umemura’s classical solutions [8]. He calculated the linear monodromy of so-called Picard’s solutions, which satisfies the sixth Painlevé equation with a special parameter. This result was rediscovered recently [28], [29].

The first, second and fourth Painlevé equations have the following simple symmetries which do not change the parameters in the Painlevé equations.

\[ P_I \quad y \rightarrow \zeta^3 y, \quad t \rightarrow \zeta t, \quad (\zeta^3 = 1) \]
\[ P_{II} \quad y \rightarrow \omega y, \quad t \rightarrow \omega^2 t, \quad (\omega^3 = 1) \]
\[ P_{IV} \quad y \rightarrow -y, \quad t \rightarrow -t, \]
There exist finite number of solutions which are invariant under the simple symmetries above. We call such a solution as a symmetric solution. A. V. Kitaev showed that the symmetric solutions for the first and the second Painlevé equation are monodromy solvable [26]. We remark that Kitaev’s symmetric solution for the second Painlevé equation exists for any parameter of the equation.

We will show that the symmetric solutions for the fourth Painlevé equation is monodromy solvable [22] in section 3. Umemura’s special solutions exist only for special values of parameters but our new special solution exists for any value of parameters and the associated linear equation can be reduced to the Whittaker equation for the special initial condition at \( t = 0 \). We describe the relations between the symmetric solution and Umemura’s classical solutions. The symmetric solution includes the rational solution \( y = -2t/3 \) for parameters \((\alpha, \beta) = (0, -2/9)\). The symmetric solution also includes one of the Riccati solutions.

We will study special solutions which are meromorphic at the origin of the fifth and the sixth Painlevé equations in sections 4 and 5. In general the Painlevé transcendent has an essential singularity at the fixed singular points. But it is known that there exist three meromorphic solutions at \( t = 0 \) for the fifth Painlevé equation (see §37 in [13]). There exist four meromorphic solutions at \( t = 0 \) for the sixth Painlevé equation. For the sixth Painlevé equation, we also have four meromorphic solutions at \( t = 1 \) and \( t = \infty \), respectively [24]. We remark that any meromorphic solution \( y(t) \) of the fifth and the sixth Painlevé equations at \( t = 0 \) becomes holomorphic.

We can represent the linear monodromy by using asymptotic expansions of generic Painlevé functions. For the fifth Painlevé equation, such correspondence was given by Andreev and Kitaev [1], [2] using WKB analysis. Although the connection formula by Andreev and Kitaev are very complicated, we determine the monodromy data for special solutions which are analytic at the origin by an elementary method. In section 4 we will show that the Stokes multiplier of the linear monodromy of one of such solutions is zero since linearization is reduced to the Gauss hypergeometric equation at \( t = 0 \).

Umemura’s special solutions exist only for special values of parameters but our solutions exist for generic value of parameters. One of our special solutions includes the algebraic solution \( y \equiv -1 \) for special parameters \((\alpha + \beta = 0, \gamma = 0)\) and also includes one point of the Riccati solution. We will transform Miwa-Jimbo’s linearization to a simple equation without the deformation parameter for a rational solution of the fifth Painlevé equation. The idea to calculate such a transformation of the independent variable is due to K. Okamoto.

For algebraic solutions, we can take a suitable transformation \( z = z(x, t) \) and a gauge transformation \( \Psi = R(z, t) \Psi \) such that (1.1) is transformed to

\[
\frac{\partial \tilde{\Psi}}{\partial z} = \tilde{A}(z) \tilde{\Psi},
\]

which does not contain the deformation parameter \( t \). This fact is observed by R. Fuchs [8] at first. He gave such transformation for some algebraic solutions of the sixth Painlevé equation. We can take a similar transformation for most of all algebraic solutions of Painlevé equations [34]. For the sixth Painlevé equation, we do not know all of algebraic solutions, but many algebraic solutions are constructed by such transformations by Kitaev [28]. In [34], Ohyama and Okumura constructed such transformations for the first to the fifth Painlevé equations.
The meromorphic solution around the origin of the fifth Painlevé equation appeared in [21]. They study a special fifth Painlevé equation with the parameter $\alpha = 1/2, \beta = -1/2, \gamma = -2i, \delta = 2$ (See the equation (2.20) in [21]). This solution is not algebraic. We think that our solutions may have application to mathematical physics although they are special.

For the sixth Painlevé equation Jimbo [20] gave a correspondence between the linear monodromy and a local expansion of the Painlevé function. But if the Painlevé function is meromorphic around a fixed singularity, we can calculate the linear monodromy easily. We will give twelve sets of meromorphic solutions around a fixed singularity for the sixth Painlevé equation. For these meromorphic solutions, we can consider confluence of singularities of (1.1). If a sixth Painlevé function $y(t)$ is holomorphic at $t = 0$, we can take a limit $t \to 0$ in (1.1). The equation (1.1) is still Fuchsian after we take the limit and is reduced to the Gauss hypergeometric equation. We can take another limit $x = 1$ to $x = \infty$. In this case (1.1) is reduced to the Heun equation, which is solved by elementary functions. Therefore we can determine the monodromy of both reduced equations and we can also determine the monodromy of (1.1).

A. D. Bryuno and I. V. Goryuchkina construct asymptotic solutions around the fixed singularity [4] and D. Guzzetti presents the leading term of the critical behavior at the fixed singularity [14] [15] [16]. Another type of local behavior for the sixth Painlevé equation are studied by K. Takano and S. Shimomura [40], [38], [39].

R. Fuchs showed that Picard’s solution is monodromy solvable [8]. Picard’s solution is expressed in terms of Weierstrass’ $\wp$ function:

$$y(t) = 4\wp(c_1\omega_1(t) + c_2\omega_2(t) | g_2, g_3) + \frac{t + 1}{3}, \quad \tag{1.4}$$

$$g_2 = \frac{1}{12}(t^2 - t + 1), \quad g_3 = \frac{1}{432}(t + 1)(2t - 1)(t - 2),$$

where $c_1$ and $c_2$ are arbitrary constants and $\omega_1$ and $\omega_2$ are a pair of fundamental period [36]. This satisfies the sixth Painlevé equation with special parameters: $\alpha = \beta = \gamma = 0$ and $\delta = 1/2$. For the rational numbers $c_1$ and $c_2$, R. Fuchs determined the monodromy invariant:

$$p_{0t} = -2\cos 2c_2\pi, \quad p_{1t} = -2\cos 2c_1\pi, \quad p_{01} = -2\cos 2(c_1 - c_2)\pi. \quad \tag{1.5}$$

Here $p_{ij} = \text{tr}M_iM_j$ and $M_j$ is a monodromy matrix (See section 5 and [20]). He showed that Picard’s solution (1.4) is expanded at $t = 0$ as

$$y(t) = -4\frac{e^{2\pi ik/n}}{2\pi i/n} t^{2k/n} + a_1t^{2k+1}/n + \cdots \quad \tag{1.6}$$

for $c_1 = \frac{k}{n}, c_2 = \frac{l}{n}$ and $\frac{k}{n} < \frac{l}{n} < \frac{1}{2}$. In the case $\frac{k}{n} > \frac{1}{2}$, we have similar expansion of $y(t)$. Therefore we can take a limit $t \to 0$ in (1.1) and (1.1) is also reduced to the Gauss hypergeometric equation. Since (1.4) has a similar expansion at $t = 1$, we can take another limit $t \to 1$ in (1.1). Because the limit of (1.1) is reduced to the Gauss hypergeometric equation again, we obtain the monodromy invariant (1.5).

The paper [8] was completely forgotten for long years. The author thanks Professor Y. Ohyama who introduced him the paper [8].

We will list up all of known monodromy solvable solutions:
• Umemura’s classical solutions \[27], \[34]\n• Symmetric solutions (\[26], \[25], section \[3]\n• meromorphic solutions around fixed singularities (\[23], \[24], sections \[4] and \[5]\n• Picard’s solution \[8]\n
We do not have a rigorous definition of monodromy solvability. One may think any Painlevé function is monodromy solvable. For example, Jimbo \[20\] gave a correspondence between local expansion of generic solutions of the sixth Painlevé equation at \(t = 0\) and linear monodromy. In this sense, a generic sixth Painlevé function is monodromy solvable. Similarly, there exist a correspondence between local expansion of Painlevé functions and linear monodromy for other types of Painlevé equations \[2\]. But our monodromy solvable solutions listed above are more special, since we can determine the linear monodromy by reducing classical special functions, such as the Gauss hypergeometric function or the Kummer confluent hypergeometric function.

In section \[2\] we review the Painlevé equations. In subsection \[2.1\] we list the ”Lax form” of the Painlevé equations. In subsection \[2.2\] we review the Bäcklund transformation groups of the Painlevé equations.

In section \[3\] we show that the symmetric solution of the fourth Painlevé equation is monodromy solvable. This section is based on the paper \[22\]. In section \[4\] we show that meromorphic solutions at \(t = 0\) of the fifth Painlevé equation are monodromy solvable. This section is based on the paper \[23\]. In section \[5\] we show that meromorphic solutions at \(t = 0\) of the sixth Painlevé equation are monodromy solvable. This section is based on the paper \[24\].

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2 The Painlevé equations

The Painlevé equations was found by Paul Painlevé about one hundred years ago \[35\]. He and his pupil Gambier classified second order nonlinear equations without movable singularities \[10\]. After they removed equations which can be solved by known functions, the following six equations are remained.
Dynkin diagrams corresponding to Okamoto’s initial value spaces. By suitable scale transform, and we exclude this case from the Painlevé family.

\[ P_I \quad y'' = 6y^2 + t, \]
\[ P_{II} \quad y'' = 2y^3 + ty + \alpha, \]
\[ P_{III} \quad y'' = \frac{1}{y} y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \]
\[ P_{IV} \quad y'' = \frac{1}{2y} y'^2 + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \]
\[ P_V \quad y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y'^2 - \frac{1}{t} y' + \left( \frac{y-1}{t^2} \right)^2 \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \]
\[ P_{VI} \quad y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) y'^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\
+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right]. \]  

Here \( \alpha, \beta, \gamma \) and \( \delta \) are complex parameters. They are called the Painlevé equations. It is known that generic solutions of these six equations are transcendental functions and they are called the Painlevé transcendents.

We may use a different type of the third Painlevé equation \( P'_{III} \)

\[ q'' = \frac{q^2}{q} \frac{q'}{x} + \frac{\alpha q^2}{4x^2} + \frac{\beta}{4x} + \frac{\gamma q^3}{4x^2} + \frac{\delta}{4q} \]

instead of \( P_{III} \), since it is easy to study isomonodromic deformation for \( P'_{III} \). \( P'_{III} \) is equivalent to \( P_{III} \) by

\[ x = t^2, \quad y = tq. \]

The third Painlevé equation is divided into three types:

- \( D_8^{(1)} \) if \( \alpha \neq 0, \beta \neq 0, \gamma = 0, \delta = 0, \)
- \( D_7^{(1)} \) if \( \delta = 0, \beta \neq 0 \) or \( \gamma = 0, \alpha \neq 0, \)
- \( D_6^{(1)} \) if \( \gamma \delta \neq 0. \)

In the case \( \beta = 0, \delta = 0 \) (or \( \alpha = 0, \gamma = 0 \)), the third Painlevé equation is a quadrature, and we exclude this case from the Painlevé family. \( D_j^{(1)} (j = 6, 7, 8) \) mean the affine Dynkin diagrams corresponding to Okamoto’s initial value spaces. By suitable scale transformations \( t \to ct, y \to dt \), we may fix \( \gamma = 4, \delta = -4 \) for \( D_6^{(1)} \), and \( \gamma = 2 \) for \( D_7^{(1)} \).

For the fifth Painlevé equation, we assume that \( \delta \neq 0 \). When \( \delta = 0, \gamma \neq 0 \), the fifth equation is equivalent to the third equation of the \( D_6^{(1)} \) type. When \( \delta = 0, \gamma = 0 \), the fifth equation is quadrature and we exclude this case from the Painlevé family. By a suitable scale transformation \( t \to ct \), we can fix \( \delta = -1/2 \) for the fifth equation.

### 2.1 Isomonodromic deformation equations

In 1905, R. Fuchs showed that the sixth Painlevé equation is an isomonodromic deformation equation of a second order Fuchsian linear differential equation \[6, 7\]. Later Garnier
showed that other Painlevé equations are also isomonodromic deformation equations of a second order linear differential equation with irregular singularities [11].

We will list up the isomonodromic deformation equations for all Painlevé equations. We use Miwa-Jimbo’s form [19], which is isomonodromic deformation of $2 \times 2$ matrix type linear equations

$$
\frac{\partial Y}{\partial x} = A(x,t)Y,
\frac{\partial Y}{\partial t} = B(x,t)Y.
$$

(2.7)

For a suitable pair $A$ and $B$, the integrability condition

$$
\frac{\partial A}{\partial t}(x,t) - \frac{\partial B}{\partial x}(x,t) + [A(x,t), B(x,t)] = 0
$$

(2.8)

gives the Painlevé equations.

### 2.1.1 The first Painlevé equation

We take

$$
A(x,t) = \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ \end{pmatrix} x^2 + \begin{pmatrix} 0 & y \\ 4 & 0 \\ \end{pmatrix} x + \begin{pmatrix} -z & y^2 + t/2 \\ -4y & z \\ \end{pmatrix},
B(x,t) = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \\ \end{pmatrix} x + \begin{pmatrix} 0 & y \\ 2 & 0 \\ \end{pmatrix}.
$$

(2.9)

The integrability condition (2.8) is

$$
\frac{dy}{dt} = z, \quad \frac{dz}{dt} = 6y^2 + t,
$$

and we obtain the first Painlevé equation

$$
\frac{d^2y}{dt^2} = 6y^2 + t.
$$

By a transformation

$$
x = \zeta^2, \quad Y(x) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta^{-1} \\ \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ \end{pmatrix} Z(\zeta),
$$

(2.7) is transformed into

$$
\frac{dZ}{d\zeta} = A_0(\zeta,t)Z,
\frac{dZ}{dt} = B_0(\zeta,t)Z,
$$

(2.10)

where

$$
A_0 = \begin{pmatrix} 4 & 0 \\ 0 & -4 \\ \end{pmatrix} \zeta^4 + \begin{pmatrix} 0 & -4y \\ 4y & 0 \\ \end{pmatrix} \zeta^2 + \begin{pmatrix} 0 & -2z \\ 2z & 0 \\ \end{pmatrix} \zeta + \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ \end{pmatrix} (2y^2 + t) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ \end{pmatrix} \frac{1}{2\zeta},
$$

$$
B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ \end{pmatrix} \zeta + \begin{pmatrix} y & -y \\ y & -y \end{pmatrix} \frac{1}{\zeta}.
$$
The first equation in (2.10) has a regular singular point at $\zeta = 0$, where the local exponents are 0 and 1. Since solutions have no logarithmic terms at $\zeta = 0$, $\zeta = 0$ is an apparent singularity. A formal solution is given by

$$Z(\zeta) = \left(1 + \frac{Z_1}{\zeta} + \frac{Z_2}{\zeta^2} + \cdots \right) e^{\mathcal{T}(\zeta)},$$

$$\mathcal{T}(\zeta) = \frac{4}{5} \left(1 \begin{array}{c} 0 \\ 0 \end{array} \right) \zeta^5 + \left(t \begin{array}{c} 0 \\ 0 \end{array} \right) \zeta + \frac{1}{2} \log \zeta, \quad Z_1 = \begin{pmatrix} -H_I & 0 \\ 0 & -H_I \end{pmatrix},$$

where

$$H_I = \frac{1}{2} z^2 - (2y^3 + ty)$$

is a Hamiltonian of the first Painlevé equation.

### 2.1.2 The second Painlevé equation

We take

$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x^2 + \begin{pmatrix} 0 & u \\ -2u^{-1}z & 0 \end{pmatrix} x + \begin{pmatrix} -z + t/2 & -uy \\ -2u^{-1}(\theta + yz) & -z - t/2 \end{pmatrix},$$

$$B(x,t) = \frac{1}{2} \left(1 \begin{array}{c} 0 \\ 0 \end{array} \right) x + \frac{1}{2} \left(0 \begin{array}{c} u \\ -2u^{-1}z \end{array} \right).$$

Here $\alpha = \frac{1}{2} - \theta$.

The integrability condition (2.8) is

$$\frac{dy}{dt} = y^2 + z + \frac{t}{2}, \quad \frac{dz}{dt} = -2yz - \theta, \quad \frac{du}{dt} = -uy,$$

and we obtain the second Painlevé equation

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \left(\frac{1}{2} - \theta\right).$$

At $x = \infty$, a formal solution is given by

$$Y(x) = \left(1 + \frac{Y_1}{x} + \frac{Y_1}{x^2} + \cdots \right) e^{\mathcal{T}(x)},$$

$$\mathcal{T}(x) = \left(1 \begin{array}{c} 0 \\ 0 \end{array} \right) \frac{x^3}{3} + \left(t \begin{array}{c} 0 \\ 0 \end{array} \right) \frac{x}{2} + \left(\theta \begin{array}{c} 0 \\ 0 \end{array} \right) \log(\frac{1}{x}),$$

$$Y_1 = \begin{pmatrix} -H_{II} & -u/2 \\ -z/u & H_{II} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} H_{II}^2/2 + (z - t\theta)/4 & uy/2 - uH_{II}/2 \\ -z/2 & -z/2 \end{pmatrix},$$

where

$$H_{II} = \frac{1}{2} z^2 + (y^2 + \frac{t}{2})z + \theta y$$

is a Hamiltonian of the second Painlevé equation.
2.1.3 The third Painlevé equation of type $D_{8}^{(1)}$

We take
\[
A(x,t) = \begin{pmatrix} -t^2 & z & x - y \\ \frac{yz^2 + z - (1/4)}{x^2} & -z^2 & -\frac{2}{x} - z \end{pmatrix},
\]
\[
B(x,t) = \begin{pmatrix} \frac{2yz}{t} & \frac{2yx}{t} \\ s & -\frac{2yz^2 + tz'}{tx} & -\frac{2yz}{t} \end{pmatrix}.
\]
(2.12)

Then the integrability condition (2.8) is
\[
ty' = 4y^2z + 2y, \quad tz' = -4yz^2 - 2z - \frac{t^2}{2y^2} + \frac{1}{2}.
\]
(2.13)

This is a Hamiltonian system with the Hamiltonian
\[
tH = 2y^2z^2 + 2yz - \frac{y}{2} - \frac{t^2}{2y}.
\]

From (2.13) we obtain
\[
q'' = \frac{(q')^2}{q} - \frac{q'}{s} + \frac{2q^2}{s^2} - 2.
\]

Changing the variable $s = t^2$, we have $P_{III}'(\alpha = 2, \beta = -2, \gamma = 0, \delta = 0)$
\[
q'' = \frac{(q')^2}{q} - \frac{q'}{s} + \frac{q^2}{2s^2} - \frac{1}{2s}.
\]

By a transformation
\[
x = \zeta^2, \quad Y = \begin{pmatrix} 2\zeta^3 & 2\zeta^3 \\ -2z\zeta + 1 & -2z\zeta - 1 \end{pmatrix} Z,
\]
the Lax form is changed into
\[
\frac{dZ}{d\zeta} = \left( A_0 + \frac{1}{\zeta} A_1 + \frac{1}{\zeta^2} A_2 \right) Z, \quad \frac{\partial Z}{\partial t} = -\frac{A_2}{t\zeta} Z,
\]
where
\[
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 = \frac{1}{2} \begin{pmatrix} -7 & 4yz + 1 \\ 4yz + 1 & -7 \end{pmatrix}, \quad A_2 = \frac{1}{2y} \begin{pmatrix} -y^2 - t^2 & y^2 - t^2 \\ -y^2 + t^2 & y^2 + t^2 \end{pmatrix}.
\]

2.1.4 The third Painlevé equation of type $D_{7}^{(1)}$

We take
\[
A(x,t) = \begin{pmatrix} -yz^2 + \theta_0 z + 1/4 & 0 \\ 0 & -t \end{pmatrix} \frac{1}{x^2} + \begin{pmatrix} 0 & 0 \\ -z^2 & t \end{pmatrix} \frac{1}{x} + \begin{pmatrix} z & -y \\ 0 & -z \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x,
\]
\[
B(x,t) = \begin{pmatrix} \frac{w}{x} & \frac{w}{t} \\ -\frac{z^2q + tz'}{tx} & \frac{w}{t} \end{pmatrix},
\]
(2.14)
Then the integrability condition (2.8) is

$$ty' = 2y^2z - \theta_0y + t, \quad t' = 2y^2z - \theta_0y + \frac{1}{4}. \quad (2.15)$$

This is a Hamiltonian system with the Hamiltonian

$$tH = y^2z^2 + (-\theta_0y + t)z - \frac{y}{4}.$$  

From (2.15) we obtain $P''_{III}(\alpha = 2, \beta = 4(\theta_0 + 1), \gamma = 2, \delta = 0)$

$$q'' = \left(\frac{q'}{q}\right)^2 - \frac{t}{q} + \frac{q^2}{2t^2} - \frac{1}{q} + \frac{1 + \theta_0}{t}.$$  

By a transformation

$$x = \zeta^2, \quad Y(x) = \begin{pmatrix} \zeta^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} Z(\zeta),$$

(2.7) is transformed into

$$\frac{dZ}{d\zeta} = A_0(\zeta, t)Z, \quad \frac{dZ}{dt} = B_0(\zeta, t)Z, \quad (2.16)$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2\theta_0 - 5 & 4yz - 2\theta_0 - 1 \\ 4yz - 2\theta_0 - 1 & 2\theta_0 - 5 \end{pmatrix} \frac{1}{\zeta}$$

$$+ \frac{1}{2} \begin{pmatrix} 4zt - y & 4zt + y \end{pmatrix} \frac{1}{\zeta^2} + \begin{pmatrix} t & t \\ -t & -t \end{pmatrix} \frac{1}{\zeta^3},$$

$$B_0 = \begin{pmatrix} -z + y/4t & -z - y/4t \\ z + y/4t & z - y/4t \end{pmatrix} \frac{1}{\zeta} + \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \frac{1}{\zeta^2}.$$

2.1.5 The third Painlevé equation of type $D_6^{(1)}$

We take

$$A(x, t) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} + \frac{1}{x} \begin{pmatrix} -\theta_\infty/2 & u \\ v & \theta_\infty/2 \end{pmatrix} + \frac{1}{2x^2} G \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} G^{-1},$$

$$B(x, t) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \frac{1}{t} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} + \frac{1}{2x} G \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G^{-1}, \quad (2.17)$$

where $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$. We set

$$\frac{1}{2} G \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} G^{-1} = \begin{pmatrix} z - t/2 & -wz \\ w^{-1}(z - t) & -z + t/2 \end{pmatrix},$$

$$G^{-1} \begin{pmatrix} -\theta_\infty/2 & u \\ v & \theta_\infty/2 \end{pmatrix} G = \begin{pmatrix} \theta_0/2 & \bar{u} \\ \bar{v} & -\theta_0/2 \end{pmatrix}. \quad (2.18)$$
These parameters satisfy the following constraints:

\[ \begin{align*}
ad - 1 & = bc = -z/t, & \quad ab = -wz/t, & \quad cd = -(z - t)/tw, \\
(\theta_0 + \theta_\infty)a - 2cu + 2b\bar{v} & = 0, & \quad (\theta_0 - \theta_\infty)b - 2au + 2du = 0, & \quad (\theta_0 + \theta_\infty)c - 2av + 2d\bar{v} = 0.
\end{align*} \]

The integrability condition (2.3) is

\[ \begin{align*}
t \frac{dG}{dt} & = \left( \begin{array}{cc} 0 & u \\ v & 0 \end{array} \right) G + G \left( \begin{array}{c} 0 \\ \bar{u} \end{array} \right), \\
du & = \frac{\theta_\infty}{t} u + 2tab, & \quad dv & = \frac{\theta_\infty}{t} v + 2tdc, \\
d\bar{u} & = \frac{\theta_0}{t} \bar{u} + 2tbd, & \quad d\bar{v} & = \frac{\theta_0}{t} \bar{v} + 2tc.
\end{align*} \]

We set \( y = -u/zw \). Then we have

\[ \begin{align*}
t \frac{dy}{dt} & = 4zy^2 - 2ty^2 + (2\theta_\infty - 1)y + 2t, \\
t \frac{dz}{dt} & = -4yz^2 + (4ty - 2\theta_\infty + 1)z + (\theta_0 + \theta_\infty)t, \\
t \frac{d}{dt} \log w & = \frac{(\theta_0 + \theta_\infty)t}{z} - 2ty + \theta_\infty.
\end{align*} \]

We obtain the third Painlevé equation

\[ y'' = \frac{1}{y} y' - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \delta, \]

\[ \alpha = 4\theta_0, \quad \beta = 4(1 - \theta_\infty), \quad \gamma = 4, \quad \delta = -4. \]

At \( x = \infty \), a formal solution is

\[ Y(x) = \left( 1 + \frac{Y_1}{x} + \frac{Y_2}{x^2} + \cdots \right) e^T(x), \]

\[ T(x) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} x + \frac{1}{2} \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix} \log \left( \frac{1}{x} \right), \]

\[ Y_1 = \begin{pmatrix} -uv/t - z + t/2 \\ -u/t \\ v/t \\ uv/t + z - t/2 \end{pmatrix}. \]

At \( x = 0 \), a formal solution is

\[ \bar{Y}(x) = \left( 1 + \frac{\bar{Y}_1}{x} + \frac{\bar{Y}_2}{x^2} + \cdots \right) e^{\bar{T}(x)}, \]

\[ \bar{T}(x) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \frac{1}{x} + \frac{1}{2} \begin{pmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{pmatrix} \log x, \]

\[ \bar{Y}_1 = \begin{pmatrix} -\bar{u}\bar{v}/t - z + t/2 \\ -\bar{u}/t \\ \bar{v}/t \\ \bar{u}\bar{v}/t + z - t/2 \end{pmatrix}. \]

The Hamiltonian of the third Painlevé equation is

\[ tH_{III} = 2y^2z^2 + 2(-ty^2 + \theta_\infty y + t)z - (\theta_0 + \theta_\infty)ty - t^2 - \frac{\theta_0^2 - \theta_\infty^2}{4}, \]

and

\[ 2H_{III} = \text{Tr} \left( Y_1 + \bar{Y}_1 \right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]
2.1.6 The fourth Painlevé equation

We take

\[
A(x, t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} \frac{2}{u}(z - \theta_0 - \theta_\infty) \\ \frac{2}{u}(z - \theta_0 - \theta_\infty) \end{pmatrix} \frac{u}{x} + \begin{pmatrix} -z + \theta_0 \\ z - \theta_0 \end{pmatrix},
\]

\[
B(x, t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{2}{u}(z - \theta_0 - \theta_\infty) \end{pmatrix} \frac{u}{x},
\]

where \( y, z \) and \( u \) are functions of \( t \), and \( \theta_0 \) and \( \theta_\infty \) are constants

\[
\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2.
\]

Setting \( w = z/y \), the integrability condition (2.8) gives

\[
\frac{dy}{dt} = -4yw + y^2 + 2ty + 4\theta_0,
\]

\[
\frac{dw}{dt} = 2w^2 - 2yw - 2tw + (\theta_0 + \theta_\infty),
\]

\[
\frac{d\log u}{dt} = -y - 2t.
\]

The system (2.20) is the Hamiltonian system with the polynomial Hamiltonian \( H_4 \):

\[
H_4 = -2yw^2 + y^2w + 2tyw + 4\theta_0w - (\theta_0 + \theta_\infty)y.
\]

The function \( u \) can be obtained from (2.20) by a quadrature.

2.1.7 The fifth Painlevé equation

We take

\[
A(x, t) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} x + \frac{1}{x} \begin{pmatrix} z + \frac{\theta_0}{2} \\ -u(z + \theta_0) \end{pmatrix} + \frac{1}{x - 1} \begin{pmatrix} -z - \frac{\theta_0 + \theta_\infty}{2} \\ \frac{u}{uy}(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}) \end{pmatrix} + \begin{pmatrix} -\frac{\theta_0}{2} \end{pmatrix},
\]

\[
B(x, t) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x
\]

\[
+ \frac{1}{t} \begin{pmatrix} 0 \\ \frac{1}{u} [z - \frac{\theta_0 + \theta_1 + \theta_\infty}{2}] \end{pmatrix} - u \left[ z + \theta_0 - y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \right],
\]

where \( y, z \) and \( u \) are functions of \( t \), and \( \theta_0, \theta_1 \) and \( \theta_\infty \) are parameters. From the integrability condition (2.8), we have

\[
\frac{dy}{dt} = ty - 2z(y - 1)^2 - (y - 1) \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2}y - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} \right),
\]

\[
\frac{dz}{dt} = yz \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - z + \theta_0 \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right),
\]

\[
\frac{d\log u}{dt} = -2z - \theta_0 + y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) + \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right).
\]
Eliminating $z$, we have the fifth Painlevé equation for
\[ \alpha = \frac{1}{2} \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right)^2, \beta = -\frac{1}{2} \left( \frac{\theta_0 - \theta_1 - \theta_\infty}{2} \right)^2, \gamma = 1 - \theta_0 - \theta_1, \delta = -\frac{1}{2}. \]

Putting
\[ w = \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right), \]
we have
\begin{align*}
\frac{dy}{dt} & = ty - 2y^3 w + y^2 \left( \frac{\theta_0 + 3\theta_1 + \theta_\infty}{2} \right) \\
& \quad + 4y^2 w - y(2\theta_1 + \theta_\infty) - 2yw - \frac{\theta_0 - \theta_1 - \theta_\infty}{2}, \quad (2.24) \\
\frac{dw}{dt} & = 3y^2 w^2 - yw(\theta_0 + 3\theta_1 + \theta_\infty) + w^2 - wt - 4yw^2 \\
& \quad + w(2\theta_1 + \theta_\infty) + \frac{\theta_1(\theta_0 + \theta_1 + \theta_\infty)}{2}, \quad (2.25) \\
\frac{d \log u}{dt} & = -2yw + \theta_1 + \theta_\infty + y(yw - \theta_1) + w. \quad (2.26)
\end{align*}

The system (2.24) and (2.25) are the Hamiltonian system with the polynomial Hamiltonian $H_5$ as shown below:
\begin{align*}
\text{t}H_5 & = -y(y - 1)^2 w^2 + \left[ \left( \frac{-\theta_0 + \theta_1 + \theta_\infty}{2} \right) (y - 1)^2 + (\theta_0 + \theta_1) y(y - 1) + ty + (\theta_0 + \theta_1) \right] w \\
& \quad - \frac{\theta_1(\theta_0 + \theta_1 + \theta_\infty)}{2} y.
\end{align*}

The function $u$ can be obtained from (2.26) by a quadrature.

### 2.1.8 The sixth Painlevé equation

We take
\[ A(x, t) = \sum_{j=0,1,t} A_j \frac{A_j}{x - j} = \begin{pmatrix} a_{11}(x, t) & a_{12}(x, t) \\ a_{21}(x, t) & a_{22}(x, t) \end{pmatrix}, \quad B(x, t) = -\frac{A_t}{x - t}, \]
where
\[ A_j = \begin{pmatrix} z_j + \theta_j & -u_j z_j \\ u_j^{-1}(z_j + \theta_j) & -z_j \end{pmatrix} \quad (j = 0, 1, t). \]

We define $A_\infty$, $y$ and $z$ as follows:
\begin{align*}
A_\infty & = -\sum_{j=0,1,t} A_j = \left( \frac{1}{2}(\theta_\infty - \sum_{j=0,1,t} \theta_j) \right) - \frac{1}{2}(\theta_\infty + \sum_{j=0,1,t} \theta_j), \\
a_{12}(x, t) & = -\sum_{j=0,1,t} \frac{u_j z_j}{x - j} = \frac{k(x - y)}{x(x - 1)(x - t)}, \\
z & = -a_{11}(y, t) = \sum_{j=0,1,t} \frac{z_j + \theta_j}{y - j}.
\end{align*}
where $y, z, z_j, u_j$ and $k$ are functions of $t$ and $\theta_j$ ($j = 0, 1, t, \infty$) are parameters.

We then have

$$\sum_{j=0,1,t} z_j = -\frac{1}{2} \left( \sum_{i=0,1,t,\infty} \theta_i \right), \quad \sum_{j=0,1,t} u_j z_j = 0,$$

$$\sum_{j=0,1,t} u_j^{-1}(z_j + \theta_j) = 0, \quad (t + 1)u_0z_0 + tu_1z_1 + u_tz_t = k.$$

In what follows, instead of $\theta_j$, we mainly use the parameters $\alpha_j$ ($j = 0, 1, 2, 3, 4$) defined by the following relations:

$$\theta_0 = \alpha_4, \quad \theta_1 = \alpha_3, \quad \theta_t = \alpha_0, \quad \theta_\infty = 1 - \alpha_1 \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \quad (2.27)$$

From the integrability condition $t(t-1)\frac{dy}{dt} = 2zy(y-1)(y-t) - \alpha_4(y-1)(y-t) - \alpha_3y(y-t) - (\alpha_0-1)y(y-1), \quad (2.28)$

and $t(t-1)\frac{dz}{dt} = (-3y^2 + 2(1+t)y-t)z^2 + [(2y-1-t)\alpha_4 + (2y-t)\alpha_3 + (2y-1)(\alpha_0-1)]z - \alpha_2(\alpha_1 + \alpha_2). \quad (2.29)$

Eliminating $z$, we have the sixth Painlevé equation with

$$\alpha = \frac{\alpha_1^2}{2} = \frac{(1-\theta_\infty)^2}{2}, \quad \beta = -\frac{\alpha_2^2}{2} = -\frac{1}{2} \theta_0^2, \quad \gamma = \frac{\alpha_3^2}{2} = \frac{1}{2} \theta_1^2, \quad \delta = -\frac{1}{2} \frac{\alpha_4^2}{2} = \frac{1-\theta_\infty^2}{2}. \quad (2.30)$$

The system $(2.28)$ and $(2.29)$ can be written as a Hamiltonian system with the polynomial Hamiltonian $H_{VI}$ given by

$$t(t-1)H_{VI} = y(y-1)(y-t)z^2 - \left[ \alpha_4(y-1)(y-t) + \alpha_3y(y-t) + (\alpha_0-1)y(y-1) \right]z + \alpha_2(\alpha_1 + \alpha_2)(y-t). \quad (2.31)$$

**Remark 1** This polynomial Hamiltonian system is the same as Garnier-Okamoto’s Hamiltonian system. Putting $Y = t(\psi_1, \psi_2)$ and eliminating $\psi_2$ from $(2.7)$, we have the same second order single equation as Garnier-Okamoto’s equation $[33]$.

### 2.1.9 Normalized form of the sixth Painlevé equation

In this section, we give the normalized Jimbo-Miwa’s isomonodromic deformation equations whose linear monodromy belongs to $SL(2, \mathbb{C})$. We use this system for the calculation of the monodromy data and the asymptotic expansion of $\tau$-function. Put

$$\bar{Y} = x^{-\frac{\alpha_1}{2}}(x-1)^{-\frac{\alpha_2}{2}}(x-t)^{-\frac{\alpha_3}{2}}Y, \quad (2.32)$$

in $(2.27)$. Then we have

$$\frac{\partial \bar{Y}(x, t)}{\partial x} = \bar{A}(x, t)\bar{Y}(x, t), \quad \bar{A}(x, t) = \sum_{j=0,1,t} \frac{\bar{A}_j}{x-j} = \begin{pmatrix} \bar{a}_{11}(x, t) & \bar{a}_{12}(x, t) \\ \bar{a}_{21}(x, t) & \bar{a}_{22}(x, t) \end{pmatrix}. \quad (2.33)$$
\[ \tilde{A}_j = \begin{pmatrix} \bar{z}_j + \frac{\theta_j}{2} & -\bar{u}_j \bar{z}_j \\ \bar{u}_j^{-1} (\bar{z}_j + \theta_j) & -\bar{z}_j - \frac{\theta_j}{2} \end{pmatrix} \quad (j = 0, 1, t), \]

\[ \frac{\partial \tilde{Y}(x, t)}{\partial t} = B(x, t) \tilde{Y}(x, t), \quad \tilde{B}(x, t) = -\frac{\tilde{A}_1}{x - t}. \quad (2.34) \]

We define \( \tilde{A}_\infty, \tilde{z}, \bar{z} \) as follows:

\[ \tilde{A}_\infty = - \sum_{j=0,1,t} \tilde{A}_j = \begin{pmatrix} \theta_\infty \cdot 0 & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix}, \quad \tilde{a}_{12}(x, t) = \frac{\tilde{k}(x - y)}{x(x - 1)(x - t)}, \quad (2.35) \]

\[ \bar{z} = \bar{a}_{11}(y, t) = \sum_{j=0,1,t} \frac{\bar{z}_j + \frac{\theta_j}{2}}{y - j}, \quad (2.36) \]

where \( y, \bar{z}, \bar{z}_j, \bar{u}_j, \tilde{k} \) are functions of \( t \) and \( \theta_j, \theta_\infty \) are parameters.

We then have

\[ \sum_{j=0,1,t} \bar{z}_j = - \frac{1}{2} \left( \sum_{i=0,1,t,\infty} \theta_i \right), \quad \sum_{j=0,1,t} \bar{u}_j \bar{z}_j = 0, \quad (2.37) \]

\[ \sum_{j=0,1,t} \bar{u}_j^{-1} (\bar{z}_j + \theta_j) = 0, \quad (t + 1) \bar{u}_0 \bar{z}_0 + t \bar{u}_1 \bar{z}_1 + \bar{u}_t \bar{z}_t = \tilde{k}. \quad (2.38) \]

We can solve as follows:

\[ \bar{u}_0 = \frac{\bar{k}y}{t \bar{z}_0}, \quad \bar{u}_1 = -\frac{\bar{k}(y - 1)}{(t - 1) \bar{z}_1}, \quad \bar{u}_t = -\frac{\bar{k}(y - t)}{t(t - 1) \bar{z}_t}, \quad (2.39) \]

\[ \bar{z}_0 = \frac{1}{t \theta_\infty} \left[ y^2 (y - 1)(y - t) \bar{z}^2 + \theta_\infty y(y - 1)(y - t) \bar{z} + \frac{\theta_\infty^2}{4} (y - 1)(y - t) \right] \]

\[ -\frac{1}{4} (\theta_\infty + \theta_0)^2 t + \frac{\theta_0^2}{4} \cdot \frac{y}{y - 1} (t - 1) - \frac{\theta_0^2}{4} t(t - 1) \frac{y}{y - t}, \quad (2.40) \]

\[ \bar{z}_1 = \frac{-1}{(t - 1) \theta_\infty} \left[ y(y - 1)^2 (y - t) \bar{z}^2 + \theta_\infty y(y - 1)(y - t) \bar{z} ight. \]

\[ + \frac{\theta_\infty^2}{4} y(y - t) + \frac{1}{4} (\theta_\infty + \theta_1)^2 t(t - 1) - \frac{\theta_1^2}{4} \cdot \frac{y - 1}{y} t - \frac{\theta_1^2}{4} t(t - 1) \frac{y - 1}{y - t} \], \quad (2.41) \]

\[ \bar{z}_t = \frac{1}{t(t - 1) \theta_\infty} \left[ y(y - 1)(y - t)^2 \bar{z}^2 + \theta_\infty y(y - 1)(y - t) \bar{z} \right. \]

\[ + \frac{\theta_\infty^2}{4} y(y - 1) - \frac{1}{4} (\theta_\infty + \theta_1)^2 t(t - 1) - \frac{\theta_1^2}{4} \cdot \frac{y - t}{y} t - \frac{\theta_1^2}{4} t(t - 1) \frac{y - t}{y - 1} \]. \quad (2.42) \]

Hereinafter we use \( \alpha_j \) which are defined by (2.27). From the integrability condition of (2.33) and (2.34), we have

\[ t(t - 1) \frac{dy}{dt} = 2y(y - 1)(y - t) \bar{z} + y(y - 1), \quad (2.43) \]

\[ t(t - 1) \frac{dz}{dt} = [ -3y^2 + 2(1 + t)y - t ] \bar{z}^2 - (2y - 1) \bar{z} \]

\[ + \left[ \frac{1}{4} - \frac{\alpha_1^2}{4} \cdot \frac{t}{y^2} + \frac{\alpha_3^2}{4} \cdot \frac{t - 1}{y - 1} - \frac{\alpha_0^2}{4} \cdot \frac{t(t - 1)}{(y - t)^2} \right]. \quad (2.44) \]
Eliminating \( \bar{z} \), we again obtain the sixth Painlevé equation with (2.30). The system of equations (2.43) and (2.44) is a rational Hamiltonian system with the Hamiltonian \( H_{VI} \) defined by

\[
t(t - 1)H_{VI} = y(y - 1)(y - t)\bar{z}^2 + y(y - 1)\bar{z} - \left[ \frac{1 - \alpha_1^2}{4} y + \frac{\alpha_3^2}{4} \cdot \frac{t}{y} - \frac{\alpha_3^2}{4} \cdot \frac{t - 1}{(y - 1)} + \frac{\alpha_0}{4} \cdot \frac{t(t - 1)}{(y - t)} \right].
\]

From (2.28) and (2.43), we have

\[
2(z - \bar{z}) = \frac{\alpha_4}{y} + \frac{\alpha_3}{y - 1} + \frac{\alpha_0}{y - t}.
\]

Remark 2 The transformation (2.46) gives the following canonical transformation between two Hamiltonian systems (2.28), (2.29) and (2.43), (2.44) which keeps \( y \) invariant:

\[
dz \wedge dy - dH_{VI} \wedge dt = d\bar{z} \wedge dy - d\bar{H}_{VI} \wedge dt.
\]

2.2 The Bäcklund transformation groups

There exist rational transformations which change a Painlevé equation to another Painlevé equation of the same type with different parameters. The transformation group of each type of Painlevé equations is called the Bäcklund transformation group. The Bäcklund transformation group is isomorphic to an affine Weyl group.

For a classical root system \( R \), we denote the Weyl group by \( W(R) \). We denote by \( P \) and \( Q \) the weight lattice and the root lattice of \( R \), respectively. It is known that the affine Weyl group \( W(R^{(1)}) \cong Q \rtimes W(R) \). We set \( \tilde{W}(R^{(1)}) = P \rtimes W(R) \). Let \( G \) be the Dynkin automorphism group of the extended Dynkin diagram. The quotient \( P/Q \) is contained in \( G \). We denote the extended affine Weyl group by \( \tilde{W}(R) \cong G \rtimes W(R^{(1)}) \).

Since the first Painlevé equation has no parameter, it does not have any Bäcklund transformation. We will list up all of the Bäcklund transformations for the Painlevé equation from the second to the sixth.

2.2.1 Simple symmetry

For the first, second and fourth Painlevé equations, there exist simple transformations which keep the parameters.

\[
\begin{align*}
P_I & \quad y \rightarrow \zeta^3 y, \quad t \rightarrow \zeta t, \quad (\zeta^5 = 1) \\
P_{II} & \quad y \rightarrow \omega y, \quad t \rightarrow \omega^2 t, \quad (\omega^3 = 1) \\
P_{IV} & \quad y \rightarrow -y, \quad t \rightarrow -t,
\end{align*}
\]

They are not contained in the Bäcklund transformation groups. We will use these symmetry to define symmetric solutions of the Painlevé equations.
2.2.2 The second Painlevé equation

The Hamiltonian is

\[ H_{II} = \frac{1}{2}p^2 - \left( q^2 + \frac{t}{2} \right) p - \alpha_1 q. \]  (2.48)

The equation for \( y = q \) is the second Painlevé equation:

\[ \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \]  (2.49)

where \( \alpha = \alpha_1 - \frac{1}{2} \).

The Bäcklund transformation is

\[ \tilde{W}(A_1^{(1)}) = G \ltimes W(A_1^{(1)}) = \langle s_1, \pi \rangle, \]
\[ W(A_1^{(1)}) = \langle s_0, s_1 \rangle, \]
\[ G = P/Q = \text{Aut}(E_7^{(1)}) = \text{Aut}(A_1^{(1)}) = \langle \pi \rangle \cong \mathbb{Z}_2. \]

The birational transformations are given by:

|   | \( \alpha_0 \) | \( \alpha_1 \) | \( q \) | \( p \) | \( t \) |
|---|---------------|---------------|-----|-----|------|
| 0 | \(-\alpha_0\)  | \(\alpha_1 + 2\alpha_0\) | \(q + \frac{2\alpha_0}{f}\) | \(p + \frac{4\alpha_0}{f} + \frac{2\alpha_1}{f^2}\) | \(t\) |
| 1 | \(\alpha_0 + 2\alpha_1\) | \(-\alpha_1\) | \(q + \frac{2\alpha_1}{p}\) | \(p\) | \(t\) |
| 2 | \(\pi\) | \(\alpha_0\) | \(-q\) | \(-f\) | \(t\) |

where \( \alpha_0 = 1 - \alpha_1 \) and \( f = p - 2q^2 - t \).

2.2.3 The third Painlevé equation of \( D_{8}^{(1)} \) type

The Hamiltonian is:

\[ tH_{D_8} = q^2p^2 + qp - \frac{1}{2} \left( q + \frac{t}{q} \right). \]  (2.50)

The equation for \( y = q/\tau, \ t = \tau^2 \) is the special case of the third Painlevé equation:

\[ \frac{d^2y}{d\tau^2} = \frac{1}{y} \left( \frac{dy}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dy}{d\tau} + \frac{4}{\tau} (y^2 - 1) + 4y^3 - \frac{4}{y}. \]  (2.51)

The symmetry of the equation is:

\[ G = \langle \pi \rangle \cong \mathbb{Z}_2. \]

The birational transformations are given by:

|   | \( q \) | \( p \) | \( t \) |
|---|-----|-----|------|
| \( \pi \) | \( \frac{t}{q} \) | \( -\frac{q(2qp + 1)}{2t} \) | \( t \) |
2.2.4 The third Painlevé equation of $D^{(1)}_7$ type

The Hamiltonian is
\[ tH_{D_7} = q^2p^2 + \alpha_1 qp + tp + q. \] (2.52)

The equation for $y = q/\tau, \ t = \tau^2$ is the special case of the third Painlevé equation:
\[ \frac{d^2y}{d\tau^2} = \frac{1}{y} \left( \frac{dy}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dy}{d\tau} + \frac{1}{\tau} (-8y^2 + \beta) - \frac{4}{y}, \] (2.53)

with
\[ \beta = 4(1 - \alpha_1). \] (2.54)

The symmetry of the equation is:
\[ \bar{W}(A^{(1)}_1) = \langle s_1, \sigma \rangle, \]
\[ G = \langle \pi \rangle \cong \mathbb{Z}, \]

where $\pi = \sigma \circ s_1$. The birational transformations are given by:

|   | $\alpha_0$ | $\alpha_1$ | $q$ | $p$ | $t$ |
|---|---|---|---|---|---|
| $s_0$ | $-\alpha_0$ | $\alpha_1 + 2\alpha_0$ | $q$ | $p + \frac{q}{q} - \frac{t}{q}$ | $-t$ |
| $s_1$ | $\alpha_0 + 2\alpha_1$ | $-\alpha_1$ | $-q + \frac{\alpha_1}{p}$ + $\frac{4}{p^2}$ | $-p$ | $-t$ |
| $\sigma$ | $\alpha_1$ | $\alpha_0$ | $tp$ | $-\frac{q}{t}$ | $-t$ |

where $\alpha_0 = 1 - \alpha_1$.

Any element of $G$ has no fixed value of parameters.

2.2.5 The third Painlevé equation of $D^{(1)}_6$ type

The Hamiltonian is
\[ tH_{D_6} = q^2p^2 - (q^2 - (\alpha_1 + \beta_1)q - t)p - \alpha_1 q. \] (2.56)

The equation for $y = q/\tau, \ t = \tau^2$ is the third Painlevé equation:
\[ \frac{d^2y}{d\tau^2} = \frac{1}{y} \left( \frac{dy}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dy}{d\tau} + \frac{1}{\tau} (\alpha y^2 + \beta) + 4y^3 - \frac{4}{y}, \] (2.57)

with
\[ \alpha = 4(\alpha_1 - \beta_1), \ \beta = -4(\alpha_1 + \beta_1 - 1). \] (2.58)

The symmetry of the equation is:
\[ \bar{W}(2A^{(1)}_1) = G \ltimes W((2A^{(1)}_1)) = \langle s_0, s_1, s'_0, s'_1, \pi_1, \pi_2, \sigma \rangle, \]
\[ \bar{W}(2A^{(1)}_1) = \langle s_0, s_1, s'_0, s'_1, \pi_1, \pi_2 \rangle, \]
\[ W((2A^{(1)}_1)) = \langle s_0, s_1, s'_0, s'_1 \rangle, \]
\[ G = \text{Aut}(D^{(1)}_6) = \text{Aut}((2A^{(1)}_1)) = \langle \pi_1, \pi_2, \sigma \rangle \cong D_8, \]
\[ P/Q = \langle \pi_1, \pi_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \]

Since the table of birational transformations are too long, we split into two parts:
\[
\begin{array}{c|cccc}
\alpha_0 & \alpha_1 & \beta_0 & \beta_1 \\
\hline
s_0 & -\alpha_0 & \alpha_1 + 2\alpha_0 & \beta_0 & \beta_1 \\
s_1 & \alpha_0 + 2\alpha_1 & -\alpha_1 & \beta_0 & \beta_1 \\
s_0' & \alpha_0 & \alpha_1 & -\beta_0 & \beta_1 + 2\beta_0 \\
s_1' & \alpha_0 & \alpha_1 & \beta_0 + 2\beta_1 & -\beta_1 \\
\hline
\pi_1 & \alpha_1 & \alpha_0 & \beta_0 & \beta_1 \\
\pi_2 & \alpha_0 & \alpha_1 & \beta_1 & \beta_0 \\
\sigma_1 & \beta_0 & \beta_1 & \alpha_0 & \alpha_1 \\
\sigma_2 & \beta_1 & \beta_0 & \alpha_1 & \alpha_0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\alpha_0 & \alpha_1 & \beta_0 & \beta_1 \\
\hline
s_0 & \alpha_0 & \alpha_1 + 2\alpha_0 & \beta_0 & \beta_1 \\
s_1 & \alpha_0 + 2\alpha_1 & -\alpha_1 & \beta_0 & \beta_1 \\
s_0' & \alpha_0 & \alpha_1 & -\beta_0 & \beta_1 + 2\beta_0 \\
s_1' & \alpha_0 & \alpha_1 & \beta_0 + 2\beta_1 & -\beta_1 \\
\hline
\pi_1 & \alpha_1 & \alpha_0 & \beta_0 & \beta_1 \\
\pi_2 & \alpha_0 & \alpha_1 & \beta_1 & \beta_0 \\
\sigma_1 & \beta_0 & \beta_1 & \alpha_0 & \alpha_1 \\
\sigma_2 & \beta_1 & \beta_0 & \alpha_1 & \alpha_0 \\
\end{array}
\]

where \( \alpha_0 = 1 - \alpha_1, \beta_0 = 1 - \beta_1 \), \( f_1 = \beta_1 q + (p - 1)q^2 + t \) and \( f_2 = \alpha_1 q + pq^2 + t \).

### 2.2.6 The fourth Painlevé equation

The Hamiltonian is
\[
H_{IV} = (p - q - 2t)pq - 2\alpha_1 p - 2\alpha_2 q.
\]  \hspace{1cm} (2.59)

The equation for \( y = q \) is the fourth Painlevé equation:
\[
\frac{d^2 y}{dt^2} = \frac{1}{2y} \left( \frac{dy}{dt} \right)^2 + \frac{3}{2} y^3 + 4t y^2 + 2(t^2 - \alpha) y + \frac{\beta}{y},
\]  \hspace{1cm} (2.60)

where
\[
\alpha = 2\theta_\infty - 1 = \alpha_0 - \alpha_2, \quad \beta = -8\theta_0^2 = -2\alpha_1^2.
\]  \hspace{1cm} (2.61)

The symmetry of the equation is:
\[
\tilde{W}(A_2^{(1)}) = G \ltimes W(A_2^{(1)}) = \langle s_0, s_1, s_2, \sigma_1, \sigma_2 \rangle,
\]
\[
\tilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle,
\]
\[
W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle,
\]
\[
G = \text{Aut}(E_6^{(1)}) = \text{Aut}(A_2^{(1)}) = \langle \sigma_1, \sigma_2 \rangle \cong S_3,
\]
\[
P/Q = \langle \pi \rangle \cong \mathbb{Z}_3.
\]
The birational transformations are:

|     | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $q$ | $p$ | $t$ |
|-----|------------|------------|------------|-----|-----|-----|
| $s_0$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2 + \alpha_0$ | $q + \frac{\alpha_0}{f}$ | $p + \frac{\alpha_0}{q}$ | $t$ |
| $s_1$ | $\alpha_0 + \alpha_1$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $q$ | $p - \frac{\alpha_1}{q}$ | $t$ |
| $s_2$ | $\alpha_0 + \alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $q + \frac{\alpha_2}{p}$ | $p$ | $t$ |
| $\pi$ | $\alpha_1$ | $\alpha_2$ | $\alpha_0$ | $-p$ | $-f$ | $t$ |
| $\sigma_1$ | $\alpha_0$ | $\alpha_2$ | $\alpha_1$ | $-\sqrt{-1}p$ | $-\sqrt{-1}q$ | $\sqrt{-1}t$ |
| $\sigma_2$ | $\alpha_2$ | $\alpha_1$ | $\alpha_0$ | $\sqrt{-1}f$ | $\sqrt{-1}p$ | $\sqrt{-1}t$ |

where $\alpha_0 = 1 - \alpha_1 - \alpha_2$ and $f = p - q - 2t$.

2.2.7 The fifth Painlevé equation

The Hamiltonian is

$$t\tilde{H}_V = p(p + t)q(q - 1) + \alpha_2qt - \alpha_3pq - \alpha_1p(q - 1).$$

The Hamiltonian system

$$\tilde{H}_V : \begin{cases} tq' = q(2pq - 2p + tq - t - \alpha_1 - \alpha_3) + \alpha_1, \\
      tp' = -p(2pq - p + 2tq - t - \alpha_1 - \alpha_3) - \alpha_2t,
\end{cases}$$

is equivalent to the fifth Painlevé equation by $y = 1 - 1/q$ for

$$\alpha = \frac{\alpha_1^2}{2}, \quad \beta = -\frac{\alpha_3^2}{2}, \quad \gamma = \alpha_0 - \alpha_2, \quad \delta = -\frac{1}{2},$$

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1.$$
The birational transformations are

| $x$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $q$ | $p$ | $t$ |
|-----|-------------|-------------|-------------|-------------|-----|-----|-----|
| $s_0(x)$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2$ | $\alpha_3 + \alpha_0$ | $q + \dfrac{\alpha_1}{p+1}$ | $p$ | $t$ |
| $s_1(x)$ | $\alpha_0 + \alpha_1$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $\alpha_3$ | $q$ | $p - \dfrac{\alpha_1}{q}$ | $t$ |
| $s_2(x)$ | $\alpha_0$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $\alpha_3 + \alpha_2$ | $q + \dfrac{\alpha_2}{p}$ | $p$ | $t$ |
| $s_3(x)$ | $\alpha_0 + \alpha_3$ | $\alpha_1$ | $\alpha_2 + \alpha_3$ | $-\alpha_3$ | $q$ | $p - \dfrac{\alpha_3}{q-1}$ | $t$ |
| $\pi(x)$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_0$ | $-\frac{t}{q}$ | $t(q-1)$ | $t$ |
| $\sigma(x)$ | $\alpha_0$ | $\alpha_3$ | $\alpha_2$ | $\alpha_1$ | $1 - q$ | $-p$ | $-t$ |

2.2.8 The sixth Painlevé equation

The Hamiltonian is

$$t(t-1)H_{VI} = q(q-1)(q-t)p^2 - [\alpha_4(q-1)(q-t) + \alpha_3q(q-t)] + (\alpha_0 - 1)q(q-1)]p + \alpha_2(\alpha_1 + \alpha_2)(q-t).$$

(2.62)

The equation for $y = q$ is the sixth Painlevé equation:

$$\frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt}$$

$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right],$$

where

$$\alpha = \frac{\alpha_1^2}{2}, \quad \beta = -\frac{\alpha_2^2}{2}, \quad \gamma = \frac{\alpha_3^2}{2}, \quad \delta = -\frac{\alpha_0^2 - 1}{2},$$

(2.63)

and $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$.

The symmetry of the equation is described as follows:

$$\tilde{W}(D_4^{(1)}) = G \ltimes W(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, \sigma_1, \sigma_2, \sigma_3 \rangle,$$

$$\tilde{W}(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, \pi_1, \pi_2 \rangle,$$

$$W(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4 \rangle,$$

$$G = \text{Aut}(D_4^{(1)}) = \mathfrak{S}_4 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle,$$

$$P/Q = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \pi_1, \pi_2 \rangle.$$
Here \( \pi_1 = \sigma_2\sigma_1\sigma_3\sigma_1 \) and \( \pi_2 = \sigma_1\sigma_2\sigma_3\sigma_2 \).

3  Symmetric solutions of the fourth Painlevé equation

In this section, we will determine the linear monodromy of symmetric solutions of the fourth Painlevé equation. This section is based on the paper \([22]\). In order to determine the linear monodromy, it is sufficient to calculate the monodromy data of the linearization for a special variable \( t \), since the Painlevé equation is given by isomonodromic deformation. For the symmetric solutions, we fix the variable \( t = 0 \). Then the linearization of the fourth Painlevé equation is reduced to the Whittaker confluent hypergeometric equation. Therefore we can determine the linear monodromy for the symmetric solution.

For special parameters, the symmetric solutions become Umemura’s classical solutions. We will compare the symmetric solutions with classical solutions.

3.1 Symmetric solutions

The first, the second and the fourth Painlevé equations have simple symmetry explained in subsection 2.2.1. We call a solution of the Painlevé equation which is invariant under the action of the simple symmetry as a symmetric solution. We list all of symmetric solutions. The symmetric solutions of the first and the second Painlevé equations was found by A. V. Kitaev \([26]\).

**Proposition 3**

1) For \( P_I \), we have two symmetric solutions

\[
\begin{align*}
y_1 &= \frac{1}{6} t^3 + \frac{1}{336} t^8 + \frac{1}{26208} t^{13} + \frac{95}{22550144} t^{18} + \cdots, \\
y_2 &= t^{-2} - \frac{1}{6} t^3 + \frac{1}{264} t^8 - \frac{1}{19008} t^{13} + \cdots.
\end{align*}
\]

2) For \( P_{II}(\alpha) \), we have three symmetric solutions

\[
\begin{align*}
y_1 &= \frac{\alpha^2}{2} t^2 + \frac{\alpha}{4} t^5 + \frac{10\alpha^3 + \alpha}{2280} t^8 + \cdots, \\
y_2 &= t^{-1} - \frac{\alpha + 1}{4} t^2 + \frac{(\alpha + 1)(3\alpha + 1)}{112} t^5 + \cdots, \\
y_3 &= -t^{-1} - \frac{\alpha - 1}{4} t^2 - \frac{(\alpha - 1)(3\alpha - 1)}{112} t^5 + \cdots.
\end{align*}
\]

They are equivalent to each other by the Bäcklund transformations.

3) For \( P_{IV}(\alpha, -8\theta_0^2) \), we have four symmetric solutions

\[
\begin{align*}
y_1 &= \pm \theta_0 \left( t - \frac{2\alpha}{3} t^3 + \frac{2}{15} (\alpha^2 + 12\theta_0^2 \pm 8\theta_0 + 1) t^5 + \cdots \right), \\
y_2 &= \pm t^{-1} + \frac{2}{3} (\pm \alpha - 2) t + \frac{2}{45} (-7\alpha^2 \pm 16\alpha + 36\theta_0^2 - 4) t^3 + \cdots.
\end{align*}
\]

They are equivalent to each other by the Bäcklund transformations.
A. V. Kitaev showed that symmetric solutions are monodromy solvable for the first and the second Painlevé equations [26]. From Proposition 3 the solutions of (2.20) with initial data \( y(0) = 0 \) and \( w(0) = 0 \) are expanded as follows:

\[
y = 4\theta_0 t \sum_{k=0}^{\infty} a_k t^{2k}, \tag{3.1}
\]

\[
a_0 = 1, \quad a_1 = \frac{2}{3} (2\theta_\infty - 1), \quad a_2 = \frac{1}{30} \{4(2\theta_\infty - 1)^2 + 3(4\theta_0)^2 + 8(4\theta_0) + 4\}, \ldots,
\]

\[
w = (\theta_0 + \theta_\infty) t \sum_{k=0}^{\infty} b_k t^{2k}, \tag{3.2}
\]

\[
b_0 = 1, \quad b_1 = \frac{2}{3} (\theta_\infty - 3\theta_0 - 1), \quad b_2 = \frac{4}{15} \{ (\theta_\infty - 3\theta_0 - 1)^2 + 4\theta_0(2\theta_\infty - 1)\}, \ldots.
\]

We will determine the linear monodromy of the above solution.

### 3.2 Transformation of the linear equation

The linearization of the fourth Painlevé equation is given by

\[
\frac{\partial Y}{\partial x} = A(x, t) Y, \tag{3.3}
\]

where

\[
A(x, t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} t & u \\ u w - \theta_0 - \theta_\infty & -t \end{pmatrix} + \frac{1}{x} \begin{pmatrix} -y w + \theta_0 & -\frac{u w}{2} \\ \frac{2w \theta_\infty}{u} (y w - 2\theta_0) & y w - \theta_0 \end{pmatrix}.
\]

By putting \( t = 0, y = 0 \) and \( w = 0 \), we have

\[
\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x + \frac{\theta_0}{u} & u \\ -2(\theta_\infty + \theta_0) & -x - \frac{\theta_0}{x} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.4}
\]

By the transformation \( x^2 = \xi \) and \( y_i = \xi^{-\frac{1}{2}} v_i, (i = 1, 2) \), we have the Whittaker equations:

\[
\frac{d^2 v_1}{d\xi^2} + \left[ \frac{-1}{4} + \frac{k}{\xi} + \frac{1}{\xi^2} - m^2 \right] v_1 = 0, \tag{3.5}
\]

\[
\frac{d^2 v_2}{d\xi^2} + \left[ \frac{-1}{4} + \frac{k + \frac{1}{2}}{\xi} + \frac{1}{\xi^2} - (m + \frac{1}{2})^2 \right] v_2 = 0, \tag{3.6}
\]

\[
k = \frac{2\theta_\infty - 1}{4}, \quad m = \frac{2\theta_0 - 1}{4}. \tag{3.7}
\]

Therefore we have
Theorem 4 The symmetric solution \(3.1\) and \(3.2\) of the fourth Painlevé equation is monodromy solvable. For \(3.1\) and \(3.2\), \(3.4\) is reduced to the Whittaker equation when \(t = 0\). The solution of \(3.4\) is given by

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} L_{k,m}(x) \\ \frac{-2k-2m+1}{u(2m+1)}L_{k+\frac{1}{2},m+\frac{1}{2}}(x) \end{pmatrix} \begin{pmatrix} L_{k,-m}(x) \\ \frac{-4m}{u}L_{k+\frac{1}{2},-m-\frac{1}{2}}(x) \end{pmatrix},
\]

where

\[
L_{k,m}(x) = x^{2m+\frac{1}{2}}e^{-\frac{x^2}{2}}\sum_{n=0}^{\infty} \frac{\Gamma(2m+1)\Gamma(m-k+\frac{1}{2}+n)x^{2m}}{\Gamma(2m+1+n)\Gamma(m-k+\frac{1}{2})n!}.
\]

3.3 The linear monodromy

The equation \(3.3\) has a regular singular point \(x = 0\) and an irregular singular point \(x = \infty\) with the Poincaré rank 2. We will define the linear monodromy \(\{M_0, \Gamma, G_1, G_2, G_3, G_4, e^{2\pi iT_\infty}\}\) of \(3.3\) [5], [19].

1) At the regular singularity \(x = 0\), the local behavior of \(Y(x)\) is given by

\[
Y^{(0)}(x) = (1 + O(x)) x^{T_0},
\]

where \(T_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{pmatrix}\).

The local monodromy of \(Y^{(0)}(x)\) around \(x = 0\) is

\[
M_0 = e^{2\pi iT_0}.
\]

2) At the irregular singularity \(x = \infty\), a formal solution is given by

\[
Y^{(\infty)} = \left(1 + \frac{Y_1}{x} + \cdots\right) e^{T(x)},
\]

\[
T(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{x^2}{2} + \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} x + \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix} \log \frac{1}{x},
\]

\[
Y_1 = \frac{1}{2} \left( -H_{IV} \right),
\]

where

\[
H_{IV} = \frac{2}{y} z^2 - \left( y + 2t + \frac{4}{y} \theta_0 \right) z + (\theta_0 + \theta_\infty)(y + 2t).
\]

Since \(x = \infty\) is an irregular singularity, the actual asymptotic behavior of \(Y(x)\) changes the form in the Stokes region of the complex \(x\)-plane:

\[
S_j = \left\{ x \left| \frac{\pi}{2}(j - 1) - \epsilon < \arg x < \frac{\pi}{2}j + \epsilon, \ |x| > R \right. \right\}, (j = 1, 2, 3, 4, 5),
\]
where $\epsilon$ is sufficiently small and $R$ is sufficiently large.

We denote $Y^{(j)}$ is a holomorphic solution in $S_j$. According to the Stokes phenomenon, if $Y^{(j)} \sim Y^{(\infty)}(x)$ as $x \to \infty$ in $S_j$, then $Y^{(j+1)} = Y^{(j)}G_j$ and $Y^{(5)} = Y^{(1)}e^{2\pi i T_\infty}$, where the matrices $G_j$ ($1 \leq j \leq 4$) are called the Stokes matrices and $e^{2\pi i T_\infty}$ is a formal monodromy around $x = \infty$.

3) Connection matrix $\Gamma$
Since both $Y^{(0)}$ and $Y^{(1)}$ satisfy (3.3), they are related by the connection matrix:

$$Y^{(1)} = Y^{(0)} \Gamma.$$  (3.19)

4) We have

$$\Gamma^{-1} M_0 G_1 G_2 G_3 G_4 e^{2\pi i T_\infty} = I_2.  \tag{3.20}$$

Generally, we cannot calculate $G_i$ and $\Gamma$. By the isomonodromic condition, the linear monodromy is invariant for any $t$. For the symmetric solution of the fourth Painlevé equation we can calculate the linear monodromy, because (3.3) is reduced to the Whittaker equation when $t = 0$.

**Theorem 5** For the symmetric solution (3.1) and (3.2) of the fourth Painlevé equation, the linear monodromy is

$$M_0 = \begin{pmatrix} e^{2\pi i \theta_0} & 0 \\ 0 & e^{2\pi i (1-\theta_0)} \end{pmatrix} = \begin{pmatrix} -e^{4\pi i \mu} & 0 \\ 0 & -e^{-4\pi i \mu} \end{pmatrix}, \tag{3.21}$$

$$\Gamma = \begin{pmatrix} \Gamma(-2m) & \Gamma(-2m)e^{-i\pi(k+m+\frac{1}{2})} \\ \Gamma(\frac{1}{2}-m-k) & \Gamma(\frac{1}{2}-m-k) \end{pmatrix}, \tag{3.22}$$

$$G_1 = \begin{pmatrix} 1 & 0 \\ 2\pi e^{i\pi(\frac{1}{2}+2k)} & 1 \end{pmatrix}, \tag{3.23}$$

$$G_2 = \begin{pmatrix} 1 & 0 \\ 2\pi e^{i\pi(\frac{1}{2}-4k)} & 1 \end{pmatrix}, \tag{3.24}$$

$$G_3 = \begin{pmatrix} 1 & 0 \\ 2\pi e^{i\pi(\frac{1}{2}+4k)} & 1 \end{pmatrix}, \tag{3.25}$$

$$G_4 = \begin{pmatrix} 1 & 0 \\ 2\pi e^{i\pi(\frac{1}{2}-8k)} & 1 \end{pmatrix}, \tag{3.26}$$

$$e^{2\pi i T_\infty} = \begin{pmatrix} e^{2\pi i (1-\theta_\infty)} & 0 \\ 0 & e^{2\pi i \theta_\infty} \end{pmatrix} = \begin{pmatrix} -e^{-4\pi i \mu} & 0 \\ 0 & -e^{4\pi i \mu} \end{pmatrix}. \tag{3.27}$$

**Proof.** 1) Two fundamental solutions $X_{k,m}(x)$ and $X_{-k,m}(xe^{2\pi i \tau})$ in the Stokes region $S_j$ are expressed in the linear combination of $L_{k,m}(x)$ and $L_{k,-m}(x) \[17\].
For \( r, s, t \in \mathbb{Z} \),
\[
X_{k,m}(xe^{ir\pi}) = \frac{\Gamma(-2m)e^{ir\theta_0}L_{k,m}(x)}{\Gamma(\frac{r}{2} - m - k)} + \frac{\Gamma(2m)e^{ir(1-\theta_0)}L_{k,-m}(x)}{\Gamma(\frac{r}{2} + m - k)},
\]
\[\tag{3.28}\]
\[
X_{k,m}(xe^{is\pi}) = \frac{\Gamma(-2m)e^{is\theta_0}L_{k,m}(x)}{\Gamma(\frac{r}{2} - m - k)} + \frac{\Gamma(2m)e^{is(1-\theta_0)}L_{k,-m}(x)}{\Gamma(\frac{r}{2} + m - k)},
\]
\[\tag{3.29}\]
\[
X_{-k,m}(xe^{i\pi - \frac{r}{4}i}) = \frac{\Gamma(-2m)e^{i(\theta_0 - \frac{1}{4})}L_{k,m}(x)}{\Gamma(\frac{r}{2} - m + k)} + \frac{\Gamma(2m)e^{i(1-\theta_0 - \frac{1}{4})}L_{k,-m}(x)}{\Gamma(\frac{r}{2} + m + k)},
\]
\[\tag{3.30}\]
hold. Eliminating \( L_{k,m}, L_{k,-m} \), and putting \( s = 0, t = 0 \) and \( x \to xe^{-r\pi} \), then we have
\[
X_{k,m}(x) \sim C_r e^{\frac{r^2}{4}x\theta_\infty - 1} + D_r e^{\frac{r^2}{4}x - \theta_\infty},
\]
\[\tag{3.31}\]
\[
(r - \frac{1}{4})\pi < \arg x < (r + \frac{3}{4})\pi, \quad (r = 0, 1, 2, \cdots).
\]
Similarly, we have
\[
X_{-k,m}(xe^{-i\pi}) \sim E_r e^{\frac{r^2}{4}x\theta_\infty - 1} + F_r e^{\frac{r^2}{4}x - \theta_\infty},
\]
\[\tag{3.32}\]
\[
(r - \frac{1}{4})\pi < \arg x < (r + \frac{3}{4})\pi, \quad (r = 0, 1, 2, \cdots),
\]
where
\[
C_r = e^{r(1-\theta_\infty)i} e^{\frac{r^2}{4}} \left[ \frac{\sin 2(r + 1)m\pi}{\sin 2m\pi} + e^{-2k_\pi} \sin 2rm\pi \right],
\]
\[\tag{3.33}\]
\[
D_r = e^{(r+\frac{1}{2})\theta_\infty} e^{\frac{r^2}{4}} \left[ -2\pi e^{\frac{r^2}{2}(r+1)} e^{-k_\pi} e^{-\frac{r^2}{4}} \sin 2rm\pi \right],
\]
\[\tag{3.34}\]
\[
E_r = e^{\frac{r^2}{4}x\theta_\infty} e^{\frac{r^2}{4}} \left[ e^{2ir} e^{-\frac{r^2}{4}} \sin 2rm\pi \right],
\]
\[\tag{3.35}\]
\[
F_r = -e^{r\theta_\infty} e^{\frac{r^2}{4}} \left[ \frac{\sin 2(r - 1)m\pi}{\sin 2m\pi} + e^{-2k_\pi} \sin 2rm\pi \right].
\]
\[\tag{3.36}\]

2) Stokes matrices \( G_j \)
For \( r\pi < \arg x < (r + \frac{1}{2})\pi \), \((r \in \mathbb{Z})\), we write the coefficient matrix of \( \text{[3.31]} \) and \( \text{[3.32]} \) as
\[
\begin{pmatrix}
C_r & E_r \\
D_r & F_r
\end{pmatrix}.
\]
\[\tag{3.37}\]
For \((r + \frac{1}{2})\pi < \arg x < (r + 1)\pi\), we have
\[
\begin{pmatrix}
C_r & E_r \\
D_{r+1} & F_{r+1}
\end{pmatrix},
\]
\[\tag{3.38}\]
\[
G_{2r+1} \begin{pmatrix}
C_r & E_r \\
D_{r+1} & F_{r+1}
\end{pmatrix} = \begin{pmatrix}
C_r & E_r \\
D_r & F_r
\end{pmatrix},
\]
\[\tag{3.39}\]
where

\[ G_{2r+1} = \begin{pmatrix} 1 & 0 \\ T_{2r+1} & 1 \end{pmatrix}, \] (3.40)

\[ T_{2r+1} = \frac{D_r - D_{r+1}}{C_r} = \frac{F_r - F_{r+1}}{E_r}. \] (3.41)

Substituting (3.33) and (3.34), we have

\[ T_{2r+1} = \frac{2\pi e^{i\pi(\frac{1}{2} + (4r+2)k)}}{\Gamma(\frac{1}{2} + m - k)\Gamma(\frac{1}{2} - m - k)}, \quad (r = 0, 1, 2, \cdots). \] (3.42)

In similar way, we have

\[ G_{2r} = \begin{pmatrix} 1 & T_{2r} \\ 0 & 1 \end{pmatrix}, \] (3.43)

\[ T_{2r} = \frac{2\pi e^{i\pi(\frac{1}{2} - 4rk)}}{\Gamma(\frac{1}{2} + m + k)\Gamma(\frac{1}{2} - m + k)}, \quad (r = 1, 2, \cdots). \] (3.44)

For special parameters, we have

**Remark 6** We set \( 2\theta_\infty - 1 = \alpha_0 - \alpha_2, \ 2\theta_0 = -\alpha_1 \) and \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \).

1) In case of \( \alpha_0 = 0 \), we have \( m + k = -1/2 \) and \( G_2 = G_4 = I_2 \).

2) In case of \( \alpha_2 = 0 \), we have \( m - k = -1/2 \) and \( G_1 = G_3 = I_2 \).

3) In case of \( \alpha_0 = 0 \) and \( \alpha_2 = 0 \), we have \( G_1 = G_2 = G_3 = G_4 = I_2 \).

### 3.4 Comparison with classical solutions

Umemura studied special solutions of the Painlevé equations [41]. Umemura’s classical solutions are either rational solution or the Riccati solution [30],[31],[42]. We show that the symmetric solution of the fourth Painlevé equation includes rational solutions and one point of the Riccati solution of Umemura’s classical solutions.

1) The Riccati solution

We set \( p = y + 2t - 2w \). Then the system (2.20) is equivalent to the following system:

\[ \frac{dy}{dt} = 2yp - y^2 - 2ty + 4\theta_0, \] (3.45)

\[ \frac{dp}{dt} = 2yp - p^2 + 2tp + 2(\theta_0 - \theta_\infty + 1). \] (3.46)

If \( \alpha_2 = 0, \theta_0 - \theta_\infty + 1 = 0 \). \( p = 0 \) is a special solution and \( y \) satisfies the Riccati equation

\[ \frac{dy}{dt} = -y^2 - 2ty + 4\theta_0, \] (3.47)
which is solved by the Weber function. In this case, the linear monodromy is upper triangular matrices by Remark 6 (2). If $y(0) = 0$ in (3.1), the Riccati solution is a symmetric solution. We remark that the Riccati solutions have the same linear monodromy.

2) Rational solutions

2-1) If $\alpha_0 = \alpha_2 = 0, \theta_0 = -1/2$. The Riccati equation is

$$\frac{dy}{dt} = y^2 + 2ty - 2,$$

which has a rational solution $y = -2t$. This solution is reduced to the Hermite polynomial. The solution $(y, w) = (-2t, 0)$ is a symmetric solution of the fourth Painlevé equation. In this case, every Stokes matrix is a unit matrix by Remark 6 (3).

2-2) If $\alpha_0 = \alpha_1 = \alpha_2 = 1/3$, the fourth Painlevé equation has a rational solution:

$$y = -\frac{2t}{3}, \quad w = \frac{t}{3},$$

which is a symmetric solution of the fourth Painlevé equation. Since we have $(k, m) = (0, -1/3), (3.3)$ is reduced to the Airy function.

3.5 Conclusion

1) The symmetric solution of the fourth Painlevé equation exists for any parameter $\alpha$ and $\beta$.

2) There exist rational solutions and the Riccati solutions for the fourth Painlevé equation for special parameters. Only for such special parameters, the symmetric solution coincides with Umemura’s classical solution. In this sense, the symmetric solution is a new special solution beyond Umemura’s class.

3) Two of four Stokes matrices ($G_1$ and $G_3$ or $G_2$ and $G_4$) become unit matrices when $\alpha_0$ or $\alpha_2 = 0$, and every Stokes matrix becomes a unit matrix when $\alpha_0 = \alpha_2 = 0$. Especially when $\alpha_2 = 0$, the linear monodromy become upper triangular matrices. When $\alpha_0 = \alpha_1 = \alpha_2 = 1/3$ and $y = -2t/3$, the solution of the associated linear equation can be solved by the Airy function.

4 The fifth Painlevé equation

This section is based on the paper [23]. In this section, we will give three holomorphic solutions around $t = 0$, which are invariant under the action of the Bäcklund transformation group. We will calculate the linear monodromy for one of these holomorphic solutions at $t = 0$. The linear equation is reduced to the Gauss hypergeometric equation when $t = 0$. The equation (2.22) has an irregular singularity at $x = \infty$ with the Poincaré rank 1, which becomes a regular singularity when $t = 0$. We will show the extension of the isomonodromic deformation to $t = 0$.

We will transform the linearization (2.22) to (1.3) for the rational solution $y(t) \equiv -1$. Therefore R. Fuchs’ observation is valid for $y(t) \equiv -1$. 

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4.1 Meromorphic solutions around $t = 0$

Generic solutions of the fifth Painlevé equation have an essential singularity around $t = 0$. Meromorphic solutions of the fifth equation around $t = 0$ is classified in §37 [13].

**Theorem 7** 1) Under some generic condition, the fifth Painlevé equation (2.5) has a holomorphic solution around $t = 0$:

$$y = \pm \frac{\theta_0 - \theta_1 - \theta_\infty}{\theta_0 - \theta_1 + \theta_\infty} + \sum_{n=1}^{\infty} a_n t^n.$$

2) Assume that $\theta_0 + \theta_1 \notin \mathbb{Z}$. The fifth Painlevé equation (2.5) has a holomorphic solution around $t = 0$:

$$y = 1 + \frac{t}{1 - \theta_0 - \theta_1} + \sum_{n=2}^{\infty} a_n t^n.$$

**Proof.** Putting $t = 0$ in (2.24), (2.25), we have the initial conditions $(y(0), w(0))$ as follows:

$$
\begin{pmatrix}
\theta_0 - \theta_1 - \theta_\infty \\
\theta_0 - \theta_1 + \theta_\infty
\end{pmatrix},

\begin{pmatrix}
\theta_0 - \theta_1 + \theta_\infty \\
\theta_0 - \theta_1 + \theta_\infty
\end{pmatrix},

\frac{\theta_0 + \theta_1 + \theta_\infty}{\theta_0 - \theta_1 + \theta_\infty},

\frac{\theta_1(\theta_0 - \theta_1 + \theta_\infty)}{2(\theta_0 + \theta_1)}.
$$

Therefore, any holomorphic solution $(y(t), w(t))$ has one of the above initial values. Since the system (2.24), (2.25) is the Briot-Bouquet type, all these solutions converge. We will explain the Briot-Bouquet theorem in section 6.

By the Hamiltonian form (2.24), (2.25), the first solution

$$y(t) = \sum_{k=0}^{\infty} a_k t^k, \quad w(t) = \sum_{k=0}^{\infty} b_k t^k,$$

is expanded as

$$y(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 = \frac{\theta_0 - \theta_1 - \theta_\infty}{\theta_0 - \theta_1 + \theta_\infty},$$

$$a_1 = \frac{a_0}{\Delta} \left[4b_0 a_0^2 - a_0(3\theta_1 + \theta_\infty) - 4a_0 b_0 + 2\theta_1 + \theta_\infty - 1\right],$$

$$\vdots$$

$$w(t) = \sum_{k=0}^{\infty} b_k t^k, \quad b_0 = \frac{(\theta_0 - \theta_1 + \theta_\infty)(\theta_0 + \theta_1 + \theta_\infty)}{-4\theta_\infty},$$

$$b_1 = \frac{b_0}{\Delta} \left[1 - 4a_0 b_0 + 2b_0 + 2\theta_1 + \theta_\infty\right], \quad \vdots$$

(4.1)  

(4.2)
where
\[
\Delta = \left[1 + 6a_0^2b_0 - a_0(\theta_0 + 3\theta_1 + \theta_\infty) - 8a_0b_0 + 2\theta_1 + \theta_\infty + 2b_0\right]
\times\left[-1 + 6a_0^2b_0 - a_0(\theta_0 + 3\theta_1 + \theta_\infty) - 8a_0b_0 + 2\theta_1 + \theta_\infty + 2b_0\right]
-2a_0(a_0 - 1)^2b_0\left[6a_0b_0 - 4b_0 - (\theta_0 + 3\theta_1 + \theta_\infty)\right].
\]

We denote this solution as (I).

The second solution is expanded as
\[
y(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 = \frac{-\theta_0 + \theta_1 + \theta_\infty}{\theta_0 - \theta_1 + \theta_\infty},
\]
\[
a_1 = a_0\Delta \left[4b_0a_0^2 - a_0(\theta_0 + 3\theta_1 + \theta_\infty) - 4a_0b_0 + 2\theta_1 + \theta_\infty - 1\right], \quad \ldots,
\]
\[
w(t) = \sum_{k=0}^{\infty} b_k t^k, \quad b_0 = \frac{(\theta_0 - \theta_1 + \theta_\infty)\theta_1}{-2(\theta_0 - \theta_1)},
\]
\[
b_1 = b_0\Delta \left[1 - 4a_0b_0 + 2b_0 + 2\theta_1 + \theta_\infty\right], \quad \ldots.
\]

We denote this solution as (II). We remark that \(\Delta\) is a different function on \(\theta_0, \theta_1, \theta_\infty\) in (I) and (II) although we use the same notation \(\Delta\).

The third solution
\[
y(t) = \sum_{k=0}^{\infty} a_k t^k, \quad w(t) = \sum_{k=0}^{\infty} b_k t^k, \quad (\theta_0 + \theta_1 \notin \mathbb{Z})
\]
is expanded as
\[
a_0 = 1, \quad a_1 = \frac{1}{1 - \theta_0 - \theta_1}, \quad a_2 = \frac{1}{2 - \theta_0 - \theta_1} \left[a_1 - 2a_1^2b_0 + a_1^2(\theta_0 + 3\theta_1 + \theta_\infty)\right], \quad \ldots,
\]
\[
b_0 = \frac{(\theta_0 + \theta_1 + \theta_\infty)\theta_1}{2(\theta_0 + \theta_1)}, \quad b_1 = \frac{b_0}{1 + \theta_0 + \theta_1} \left[2a_1b_0 - a_1(\theta_0 + 3\theta_1 + \theta_\infty) - 1\right],
\]
\[
b_2 = \frac{1}{2 + \theta_0 + \theta_1} \left[3a_1^2b_0^2 + 2a_2b_0^2 - (a_2b_0 + a_1b_1)(\theta_0 + 3\theta_1 + \theta_\infty) - b_1 - 8a_1b_1b_0\right], \quad \ldots.
\]

We denote this solution as (III).

**Theorem 8** The three holomorphic solutions (I), (II) and (III) are invariant under the action of the Bäcklund transformation group.

| \(s_0\) | \(s_1\) | \(s_2\) | \(s_3\) | \(\pi\) | \(\sigma\) |
|---|---|---|---|---|---|
| I | I | I | I | I | I |
| II | III | I | III | I | I |
| III | III | I | III | I | III |

We can prove the above theorem easily.
4.2 The linear equation at $t = 0$

For a locally holomorphic solution (I) around $t = 0$, we may extend the deformation equation to $t = 0$ because $B(x,t)$ in (2.23) is holomorphic at $t = 0$. Therefore we can continue Miwa-Jimbo’s isomonodromic deformation equation to $t = 0$. We describe more detail in subsection 4.6.

After substituting the solution (I) into the equation (2.22), we put $t = 0$. Then we have

$$\frac{\partial \Psi(x,0)}{\partial x} = A(x,0)\Psi(x,0),$$

$$A(x,0) = \left(\begin{array}{c}
\frac{1}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) \\
\frac{1}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right) - \frac{u_0}{x} \left( \theta_0 + \frac{\theta_1}{2} \right)
\end{array}\right)|_{t=0},$$

which is reduced to the hypergeometric equation.

The above discussion proves the following:

**Theorem 9** We can determine the linear monodromy of the special solution (I). For the solution (I), (2.22) is reduced to the hypergeometric equation when $t = 0$.

The fundamental solution matrix is expressed as follows:

$$\Psi = \begin{pmatrix}
\psi_1^{(1)} & \psi_1^{(2)} \\
\psi_2^{(1)} & \psi_2^{(2)}
\end{pmatrix},$$

where

$$\psi_1^{(1)} = x^{-\theta_0} (x-1)^{-\theta_1/2} F_1 \left( \frac{\theta_0 - \theta_1}{2}, 1 - \frac{\theta_0 + \theta_1}{2}, 1 - \theta_0; x \right),$$

$$\psi_1^{(2)} = x^{-\theta_0} (x-1)^{-\theta_1/2} F_1 \left( \frac{\theta_0 + \theta_0 - \theta_1}{2}, 1 - \frac{\theta_0 - \theta_1}{2}, 1 + \theta_0; x \right),$$

$$\psi_2^{(1)} = x^{-\theta_0/2} u_0^{-\theta_1} (x-1)^{-\theta_0/2} F_1 \left( \frac{\theta_0 + \theta_0 + \theta_1}{2}, 1 + \frac{\theta_0 - \theta_1}{2}, 1 - \theta_0; x \right),$$

$$\psi_2^{(2)} = x^{-\theta_0/2} u_0^{-\theta_1} (x-1)^{-\theta_0/2} F_1 \left( \frac{\theta_0 - \theta_0 + \theta_1}{2}, 1 + \frac{\theta_0 + \theta_1}{2}, 1 + \theta_0; x \right).$$

Here $u_0 = u(0)$.

Since Miwa-Jimbo’s isomonodromic deformation equation can be continued to $t = 0$, the linear monodromy is invariant for any $t \in \mathbb{C}$.

4.3 The linear monodromy

4.3.1 Miwa-Jimbo’s linearization

The equation (2.22) has two regular singular points $x = 0$ and $x = 1$, and an irregular singular point $x = \infty$ with the Poincaré rank 1. We will define the linear monodromy \{M_0, M_1, \Gamma_0, \Gamma_1, \Gamma_2, e^{2\pi i T_\infty}\} of (2.22) following [19].
1) At the regular singularity \( x = \nu, (\nu = 0, 1) \), the local behavior of \( \Psi(x) \) is given by
\[
\Psi^{(\nu)}(x) = (1 + O((x - \nu)))x^{T_\nu},
\]
where
\[
T_\nu = \begin{pmatrix}
\frac{\theta_\nu}{2} & 0 \\
0 & -\frac{\theta_\nu}{2}
\end{pmatrix}.
\]
The local monodromy of \( \Psi^{(\nu)}(x) \) around \( x = \nu \) is
\[
M_\nu = e^{2\pi i T_\nu}.
\]

2) At the irregular singularity \( x = \infty \), a formal solution is given by
\[
\Psi^{(\infty)} = \left(1 + \frac{\Psi_1}{x} + \cdots\right)e^{T(x)},
\]
where
\[
T(x) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} x + \frac{1}{2} \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix} \log \frac{1}{x},
\]
\[
\Psi_1 = \begin{pmatrix}
\Psi_1 \\
\frac{1}{i\mu} \left[ z - \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \right] \end{pmatrix},
\]
with
\[
H_\nu = \frac{-1}{t} \left[ z - \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \right] \left[ z + \theta_0 - y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \right] \frac{H_\nu}{2}.
\]
Since \( x = \infty \) is an irregular singularity, the actual asymptotic behavior of \( \Psi(x) \) changes the form in the Stokes region of the complex \( x \)-plane:
\[
S_j = \{ x \mid \pi(j - 1) - \varepsilon < \arg(xt) < \pi j + \varepsilon, |x| > R \} \quad (j = 1, 2, 3),
\]
where \( \varepsilon \) is sufficiently small and \( R \) is sufficiently large.

We denote \( \Psi_{\infty}^{(j)} \) is a holomorphic solution in \( S_j \). According to the Stokes phenomenon, if
\[
\Psi_{\infty}^{(j)} \sim \Psi^{(\infty)}(x) \quad \text{as} \quad x \to \infty \quad \text{in} \quad S_j,
\]
\[
\Psi_{\infty}^{(2)} = \Psi_{\infty}^{(1)}G_1, \quad \Psi_{\infty}^{(3)} = \Psi_{\infty}^{(2)}G_2, \quad \Psi_{\infty}^{(3)} = \Psi_{\infty}^{(1)}e^{2\pi i T_{\infty}},
\]
where the matrices \( G_j, (j = 1, 2) \) are called the Stokes matrices and \( e^{2\pi i T_{\infty}} \) is a formal monodromy around \( x = \infty \).

3) Since both \( \Psi^{(0)}, \Psi^{(1)} \) and \( \Psi_{\infty}^{(1)} \) satisfy (2.22), they are related by the connection matrix \( \Gamma_{\nu \infty} \):
\[
\Psi^{(\nu)} = \Psi_{\infty}^{(0)}\Gamma_{\nu \infty}, \quad (\nu = 0, 1).
\]

4) We have
\[
\Gamma_{0 \infty}M_0\Gamma_{0 \infty}^{-1}\Gamma_{1 \infty}M_1\Gamma_{1 \infty}^{-1}G_1G_2e^{2\pi i T_{\infty}} = I_2.
\]
Generally, we cannot calculate \( G_j \) and \( \Gamma_{\nu \infty} \). By the isomonodromic condition, the linear monodromy is invariant for any \( t \). For the solution (I) of the fifth Painlevé equation, we can calculate the linear monodromy, because (2.22) is reduced to the hypergeometric equation when \( t = 0 \).
Theorem 10  For the the solution (I) of the fifth Painlevé equation, the linear monodromy is

\[
M_0 = \begin{pmatrix} e^{-i\pi \theta_0} & 0 \\ 0 & e^{i\pi \theta_0} \end{pmatrix}, \quad M_1 = \begin{pmatrix} e^{-i\pi \theta_1} & 0 \\ 0 & e^{i\pi \theta_1} \end{pmatrix},
\]

\[
\Gamma_{0\infty} = \begin{pmatrix} \frac{e^{i\pi(\theta_\infty-\theta_1-\theta_0)}}{\Gamma(1-\theta_0)\Gamma(1-\theta_\infty)} & \frac{e^{i\pi(\theta_\infty-\theta_1+\theta_0)}}{\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} \\ \frac{e^{i\pi(\theta_\infty-\theta_1+\theta_0)}\Gamma(\theta_\infty-1)}{\Gamma(1-\theta_0)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} & \frac{e^{i\pi(\theta_\infty-\theta_1-\theta_0)}\Gamma(\theta_\infty-1)}{\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} \end{pmatrix},
\]

\[
\Gamma_{1\infty} = \begin{pmatrix} \frac{\Gamma(1-\theta_1)\Gamma(1-\theta_\infty)}{\Gamma(1-\theta_0)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} & \frac{e^{i\pi\theta_1}\Gamma(1+\theta_1)\Gamma(1-\theta_\infty)}{\Gamma(1-\theta_0)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} \\ \frac{e^{i\pi\theta_1}\Gamma(1-\theta_1)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)}{\Gamma(1-\theta_0)\Gamma(1+\theta_1)\Gamma(1-\theta_\infty)} & \frac{\Gamma(1+\theta_1)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)}{\Gamma(1-\theta_0)\Gamma(1+\theta_0)\Gamma(1-\theta_\infty)} \end{pmatrix},
\]

\[
G_1 = G_2 = I_2, \quad e^{2\pi T_\infty} = \begin{pmatrix} e^{i\pi \theta_\infty} & 0 \\ 0 & e^{-i\pi \theta_\infty} \end{pmatrix}.
\]

Remark. While \( x = \infty \) is an irregular singularity with the Poincaré rank 1 in (2.22), \( x = \infty \) becomes the regular singularity when \( t = 0 \). This means that the formal solution around the irregular singularity \( x = \infty \), which is expressed in the form of an asymptotically expanded power series converges for any \( t \) by the isomonodromic condition. Therefore, every Stokes matrix becomes the unit matrix. It is difficult to prove this fact directly but we prove this for the special value of parameters: \( \alpha + \beta = 0, \gamma = 0 (\theta_0 = \theta_1 = 1/2) \), in section five.

Remark. The Stokes multipliers become zero for our solutions, which are analytic around zero. Y. Sibuya studied differential equations whose Stokes multipliers vanish at irregular singular points [31] (Professor Okamoto taught us Sibuya’s paper). Although he did not consider isomonodromic deformations, we think that the isomonodromic deformation equations become simple when Stokes multipliers vanish.

4.4 Comparison with classical solutions

Umemura studied special solutions of the Painlevé equations [31], which are called classical solutions. Umemura’s classical solutions are either algebraic solutions or the Riccati solutions [30], [31], [42].

We show that the new special solution (I) includes an algebraic solution \( y \equiv -1 \) and one point of the Riccati solution. Since (II) and (III) are the Bäcklund transforms of (I), (II) and (III) also contain classical solutions.

We have the following Riccati solutions:

1. In case of \( \theta_0 + \theta_1 + \theta_\infty = 0 \), we have \( w \equiv 0 \) from (2.25) and (4.2), and \( y \) satisfies the Riccati equation.

2. In case of \( \theta_0 - \theta_1 - \theta_\infty = 0 \), we have \( y \equiv 0 \) from (2.23) and (4.1), and \( w \) satisfies the Riccati equation.

3. In case of \( \theta_0 + \theta_1 - \theta_\infty = 0 \), all monodromy data become upper half triangular matrices by Theorem 10.
4. In case of $\theta_0 + \theta_1 - \theta_\infty = -2$, all monodromy data become lower half triangular matrices by Theorem 10.

In every case above, (I) includes one point of the Riccati solution. We remark that the Riccati solutions have the same linear monodromy.

In case of $\alpha + \beta = 0$ and $\gamma = 0$ (i.e. $\theta_0 = \theta_1 = 1/2$), the system (4.24) and (4.25) has a special solution $y \equiv -1$ and $w = \frac{1 + \theta_\infty}{4} + \frac{t}{8}$, which is a rational solution of $P_V$. We will study this rational solution in the next section.

We remark that (III) contains the Riccati solution $y = e^t$ for $\theta_0 = \theta_1 = \theta_\infty = 0$.

4.5 R. Fuchs’ observation for the solution $y \equiv -1$

For an algebraic solution of the Painlevé equation, R. Fuchs observed that the associated linear equation can be transformed by an appropriate variable change to an equation which does not include the deformation parameter $t$. He showed that the linear equation for special Picard’s solutions, which correspond to three, four and six divided points of elliptic curves, can be reduced to the hypergeometric equation [8].

In this section we will show that R. Fuchs’ observation is true for the rational solution $y \equiv -1$ for $\theta_0 = \theta_1 = 1/2$ of the fifth Painlevé equation. This solution is a special case of (I), as we claimed in the previous section. For a generic parameter, we can directly calculate the linear monodromy of (I) only for $t = 0$. But in case of $\theta_0 = \theta_1 = 1/2$, we can directly calculate the linear monodromy of $y \equiv -1$ for generic $t \in \mathbb{C}$. The authors learned the method in this section from Professor Kazuo Okamoto.

We substitute the solution $y \equiv -1$ into Miwa-Jimbo’s isomonodromic deformation equations (4.22) and (4.23):

$$\frac{\partial \Psi(x,t)}{\partial x} = A(x,t) \Psi(x,t), \tag{4.3}$$

$$A(x,t) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} - \frac{1}{x} \left( u^{-1} \left( \frac{1 + \theta_\infty}{4} + \frac{t}{8} \right) \left( \frac{1 - \theta_\infty}{4} - \frac{t}{8} \right) \right)$$

$$- \frac{1}{x - 1} \left( \frac{1}{u} \left( \frac{1 + \theta_\infty}{4} - \frac{t}{8} \right) \left( \frac{1 - \theta_\infty}{4} + \frac{t}{8} \right) \right),$$

$$\frac{\partial \Psi(x,t)}{\partial t} = \left( -\frac{u^{-1}}{4} - \frac{\nu}{x} \right) \Psi(x,t). \tag{4.4}$$

Putting

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$
we have equations for \( \psi_1 \):

\[
\frac{\partial^2 \psi_1}{\partial x^2} + \left[ \frac{1}{x} + \frac{1}{x-1} - \frac{2t}{2tx - t + 2(1 - \theta_\infty)} \right] \frac{\partial \psi_1}{\partial x} - \left[ \frac{t^2}{4} + \frac{1}{16x^2} + \frac{1}{16(x-1)^2} + \frac{4(1 - \theta_\infty)t^2}{(x-1)^2 - 4(1 - \theta_\infty)^2} \cdot \frac{2tx - t + 2(1 - \theta_\infty)}{x - 1} \right] \psi_1 = 0,
\]

(4.5)

The equation (4.5) has three regular singularities; \( x = 0, \ x = 1 \) and \( x = \frac{1}{2} - \frac{1 - \theta_\infty}{t} \) and an irregular singularities at \( x = \infty \) with the Poincaré rank 1. We remark that \( x = \frac{1}{2} - \frac{1 - \theta_\infty}{t} \) is an apparent singularity.

We take new variables:

\[
(\psi_1, x) \rightarrow (\phi_1, \xi) :
\]

\[
\psi_1 = \phi_1 [x(x-1)]^{\frac{1}{4}} \left[ x - \frac{1}{2} + \sqrt{x(x-1)} \right] e^{\frac{t}{4} + \frac{t}{1 - \theta_\infty} \sqrt{x(x-1)}},
\]

\[
\xi = \left( x - \frac{1}{2} + \sqrt{x(x-1)} \right) e^{\frac{t}{1 - \theta_\infty} \sqrt{x(x-1)}}.
\]

Then (4.5) is reduced to

\[
\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{3 \partial \phi_1}{\xi \partial \xi} + \left[ 1 - \left( \frac{1 - \theta_\infty}{2} \right)^2 \right] \frac{\partial \phi_1}{\xi^2} = 0,
\]

(4.6)

which is independent of \( t \). We can solve (4.6) easily:

\[
\phi_1 = c_1 \xi^{\alpha_1} + c_2 \xi^{\alpha_2}, \quad \left( \alpha_1, \alpha_2 = -1 \pm \frac{1 - \theta_\infty}{2} \right).
\]

Therefore solutions of (4.5) are given by

\[
\psi_1 = [x(x-1)e^{-t}]^{\frac{1}{4}} \left[ c_1 \left( \sqrt{x} + \sqrt{x-1} \right)^{1 - \theta_\infty} e^{\frac{t}{4} \sqrt{x(x-1)}} \right.
\]

\[
+ c_2 \left( \sqrt{x} + \sqrt{x-1} \right)^{-1 + \theta_\infty} e^{\frac{t}{4} \sqrt{x(x-1)}}] ,
\]

where \( c_1, c_2 \) are constants.

We notice that if \( \theta_0 = 1/2, \theta_1 = 1/2 \) and \( y = -1 \), we have \( z = -(t + 2\theta_\infty + 2)/8 \) and \( u = -c^{-1}e^{t/2} \) for a constant \( c \). The fundamental solution \( \Psi \) of (4.3) is

\[
\Psi = \left( c e^{\frac{t}{4} + \frac{t}{4} \sqrt{x(x-1)}} \left( \sqrt{x} + \sqrt{x-1} \right)^{1 - \theta_\infty} \right. \left. c e^{\frac{t}{4} + \frac{t}{4} \sqrt{x(x-1)}} \left( \sqrt{x} + \sqrt{x-1} \right)^{1 + \theta_\infty} \right) .
\]

(4.7)
The solution \((4.7)\) has a regular singularity at \(x = \infty\) if we put \(t = 0\). Although \((4.7)\) has irregular singularity at \(x = \infty\) in case of \(t \neq 0\), every Stokes matrix becomes a unit matrix since they give convergent series around \(x = \infty\).

The linear monodromy of the fundamental solution \((4.7)\) is
\[
M_0 = \begin{pmatrix} 0 & ie^{i\pi\theta_{\infty}} \\ ie^{-i\pi\theta_{\infty}} & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},
\]
\[
\Gamma_{0\infty} = \Gamma_{1\infty} = I_2, \quad G_1 = G_2 = I_2,
\]
\[
e^{2\pi i T_{\infty}} = \begin{pmatrix} e^{-i\pi\theta_{\infty}} & 0 \\ 0 & e^{i\pi\theta_{\infty}} \end{pmatrix}.
\]

We have
\[
M_0 M_1 e^{2\pi i \theta_{\infty}} = I_2.
\]

### 4.6 Extension of deformation to \(t = 0\)

In this section, we will show that the fundamental solution \(\Psi(x, t)\) of \((2.22)\) exists for any \(t \in \mathbb{C}\). The equation \((2.22)\) has an irregular singularity at \(x = \infty\) with the Poincaré rank 1, which turns out the regular singularity when \(t = 0\). For the special solution (I), \(B(x, t)\) in \((2.23)\) is holomorphic at \(t = 0\). Therefore we have a fundamental solution \(\Psi(x, t)\) which is analytic on \((x, t)\) and has a branch along \(x = \infty\).

We set the Pauli matrix
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The following theorem assures that the isomonodromic deformation extends to \(t = 0\).

**Theorem 11** For the special solution (I), we have a fundamental solution at \(x = \infty\)
\[
\Psi(x, t) = \left( 1 + \frac{\Psi_1(t)}{x} + \frac{\Psi_2(t)}{x^2} + \cdots \right) e^{T(x)}, \tag{4.8}
\]
where
\[
T(x) = \left( \frac{t}{2} e - \frac{\theta_{\infty}}{2} \log x \right) \sigma_3.
\]

Here \(\Psi_j(t)\) is holomorphic around \(t = 0\) for \(j = 1, 2, 3, \ldots\).

**Proof.** We write equation \((2.23)\) as follows:
\[
\frac{\partial \Psi(x, t)}{\partial t} = \left[ x \sigma_3 + \frac{1}{t} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right] \Psi(x, t),
\]
where
\[
b = -u \left[ z + \theta_0 - y \left( z + \frac{\theta_0 - \theta_1 + \theta_{\infty}}{2} \right) \right],
\]
\[
c = u^{-1} \left[ z - \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_{\infty}}{2} \right) \right].
\]

For the solution (I), we have \(b(0) = c(0) = 0\).
At the irregular singularity $x = \infty$, a formal solution (4.8) exists for $t \neq 0$. Therefore, we have

$$\frac{\partial \Psi^{(\infty)}}{\partial t} = \left[ \left( 1 + \frac{\Psi'_1}{x} + \frac{\Psi'_2}{x^2} + \cdots \right) + \left( 1 + \frac{\Psi_1}{x} + \frac{\Psi_2}{x^2} + \cdots \right) \frac{x}{2 \sigma_3} \right] e^{T(x)}$$

$$= \left[ \frac{x}{2 \sigma_3} + \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \right] \left( 1 + \frac{\Psi_1}{x} + \frac{\Psi_2}{x^2} + \cdots \right) e^{T(x)},$$

where $'$ means a derivation by $t$.

$$\left[ \begin{array}{cc} 1 + \frac{\Psi'_1}{x} + \frac{\Psi'_2}{x^2} + \cdots + \left( 1 + \frac{\Psi_1}{x} + \frac{\Psi_2}{x^2} + \cdots \right) \frac{x}{2 \sigma_3} \end{array} \right]$$

$$= \left[ \frac{x}{2 \sigma_3} + \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \right] \left( 1 + \frac{\Psi_1}{x} + \frac{\Psi_2}{x^2} + \cdots \right).$$

We put $\Psi_n = \left( \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right)$ and compare the coefficients of the equal degree of $x$ in both sides:

1) $x^0$: We have

$$\frac{1}{2} [\Psi_1, \sigma_3] = \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right).$$

Therefore

$$\left( \begin{array}{cc} 0 & -b_1 \\ c_1 & 0 \end{array} \right) = \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right).$$

Since $b(0) = 0, c(0) = 0, b_1$ and $c_1$ are holomorphic around $t = 0$.

2) $x^{-1}$: We have

$$\frac{1}{2} [\Psi_2, \sigma_3] = \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \Psi_1 - \Psi'_1.$$

Therefore

$$\left( \begin{array}{cc} 0 & -b_2 \\ c_2 & 0 \end{array} \right) = \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) - \left( \begin{array}{cc} a'_1 & b'_1 \\ c'_1 & d'_1 \end{array} \right)$$

$$= \left( \begin{array}{cc} c_1 b/t - a'_1 & d_1 b/t - b'_1 \\ a_1 c/t - c'_1 & b_1 c/t - d'_1 \end{array} \right).$$

Compared with the diagonal components, $a'_1$ and $d'_1$ are holomorphic because $b(0) = 0$ and $c(0) = 0$. Therefore $\Psi_1 = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right)$ is holomorphic around $t = 0$.

Compared with the off-diagonal components, $b_2$ and $c_2$ are holomorphic.

3) $x^{-n}$: We have

$$\frac{1}{2} [\Psi_{n+1}, \sigma_3] = \frac{1}{t} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \Psi_n - \Psi'_n.$$
Therefore
\[
\begin{pmatrix}
0 & -b_{n+1} \\
\frac{c_n}{c_{n+1}} & 0
\end{pmatrix}
= \frac{1}{t}
\begin{pmatrix}
0 & b \\
c & 0
\end{pmatrix}
\begin{pmatrix}
a_n & b_n \\
c_n & d_n
\end{pmatrix}
- \begin{pmatrix}
a'_n & b'_n \\
c'_n & d'_n
\end{pmatrix}
= \begin{pmatrix}
\frac{c_n b}{t} - \frac{a'_n}{d'_n} & \frac{d_n b}{t} - \frac{b'_n}{d'_n} \\
\frac{a_n c}{t} - \frac{c'_n}{d'_n} & \frac{b_n c}{t} - \frac{d'_n}{d'_n}
\end{pmatrix}.
\]
In the same way, \(a'_n, d'_n\) and \(b_{n+1}, c_{n+1}\) are holomorphic. Therefore \(\Psi_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\) is holomorphic around \(t = 0\).

For \(\alpha + \beta = 0, \gamma = 0\) we showed that we may put \(t = 0\) in section 4.2.

5 The sixth Painlevé equation

This section is based on the paper [24]. In this section, we will give four meromorphic solutions around each fixed singularity \(t = 0, 1, \infty\), respectively, which are transformed each other by the action of the Bäcklund transformation group. We will calculate the linear monodromy for one of these meromorphic solutions at \(t = 0\) by Jimbo’s method given in [20]. We take two confluenes of singularities of the linear equation. One is the confluence between \(x = 0\) and \(x = t\) and the other is the confluence between \(x = 1\) and \(x = \infty\). For the former, the linear equation is reduced to the Gauss hypergeometric equation and for the latter, it is reduced to a Heun’s type equation whose general solution can be obtained as a linear combination of two monomials. From these two confluenes we obtain the linear monodromy for our solution explicitly. We will give the comparison with Umemura’s classical solutions.

5.1 Meromorphic solutions around the fixed singularities

In this section we will classify all of the meromorphic solutions around a fixed singularity. We consider a solution of (2.43) and (2.44) (and that of (2.28) and (2.29) simultaneously) around \(t = 0\):

\[
y(t) = t^l \sum_{i=0}^{\infty} a_i t^i, \quad \bar{z}(t) = t^k \sum_{i=0}^{\infty} b_i t^i, \quad z(t) = t^n \sum_{i=0}^{\infty} c_i t^i \quad (l, k, n \in \mathbb{Z}). \quad (5.1)
\]

**Theorem 12** For generic values of parameters, the sixth Painlevé equation has the following four meromorphic solutions around \(t = 0\):

\[
(0-I) : \quad y(t) = \frac{\alpha_4 - t}{\alpha_4 - \alpha_0} + \frac{\alpha_0 \alpha_4 \delta^2 - \alpha_0 + \alpha_3^2 + (\alpha_4 - \alpha_0)^2 t^2 + O(t^3)}{2 [1 - (\alpha_4 - \alpha_0) t] (\alpha_4 - \alpha_0)^2}, \quad (5.2)
\]

\[
\bar{z}(t) = \frac{1 - \alpha_4^2 + \alpha_3^2 - (\alpha_4 - \alpha_0)^2}{4 [1 - (\alpha_4 - \alpha_0)^2]} + O(t), \quad (5.3)
\]

\[
z(t) = \frac{\alpha_4 - \alpha_0}{t} + O(t^0), \quad (5.4)
\]
(0-II) : \[ y(t) = \frac{\alpha_4}{\alpha_4 + \alpha_0} t + \frac{-\alpha_0 \alpha_4 [1 + \alpha_4^2 - \alpha_0^2 - (\alpha_4 + \alpha_0)^2]}{2 [1 - (\alpha_4 + \alpha_0)^2]} t^2 + O(t^3), \] (5.5)
\[ z(t) = \frac{1 - \alpha_4^2 + \alpha_3^2 - (\alpha_4 + \alpha_0)^2}{4 [1 - (\alpha_4 + \alpha_0)^2]} + O(t), \] (5.6)
\[ z(t) = \frac{\alpha_2 (\alpha_1 + \alpha_2)}{1 - \alpha_4 - \alpha_0} + O(t), \] (5.7)
(0-III) : \[ y(t) = \frac{\alpha_1 + \alpha_3}{\alpha_1} + \frac{-\alpha_3 [1 + \alpha_4^2 - \alpha_0^2 - (\alpha_1 + \alpha_3)^2]}{2 \alpha_1 [1 - (\alpha_1 + \alpha_3)^2]} t + O(t^2), \] (5.8)
\[ z(t) = \frac{-\alpha_1 \alpha_2}{2 (\alpha_1 + \alpha_3)} + O(t), \quad z(t) = \frac{-\alpha_1 \alpha_2}{\alpha_1 + \alpha_3} + O(t), \] (5.9)
(0-IV) : \[ y(t) = \frac{\alpha_1 - \alpha_3}{\alpha_1} + \frac{\alpha_3 [1 + \alpha_4^2 - \alpha_0^2 - (\alpha_1 - \alpha_3)^2]}{2 \alpha_1 [1 - (\alpha_1 - \alpha_3)^2]} t + O(t^2), \] (5.10)
\[ z(t) = \frac{-\alpha_1}{2 (\alpha_1 - \alpha_3)} + O(t), \quad z(t) = \frac{-\alpha_1 (\alpha_1 + \alpha_2)}{\alpha_1 - \alpha_3} + O(t). \] (5.11)

These solutions satisfy the system \[ (2.28),\quad (2.29) \] and \[ (2.43),\quad (2.44) \] and they are convergent since \[ (2.28) \] and \[ (2.29) \] are of the Briot-Bouquet type at \( t = 0 \) [3]. We gave the proof in section 6. For generic values of parameters, there are no meromorphic solutions around \( t = 0 \) except for these four solutions.

**Remark 13**

(1) These four solutions exist for the following condition:

(0-I) : \( \alpha_1 \neq 0, \quad \alpha_4 - \alpha_0 \notin \mathbb{Z}, \) \( (0-II) : \alpha_1 \neq 0, \quad \alpha_4 + \alpha_0 \notin \mathbb{Z}, \)
(0-III) : \( \alpha_1 \notin \mathbb{Z}, \quad \alpha_1 + \alpha_3 \notin \mathbb{Z}, \) \( (0-IV) : \alpha_1 \notin \mathbb{Z}, \quad \alpha_1 - \alpha_3 \notin \mathbb{Z}. \) (5.12)
(5.13)

(2) In the case of \( \alpha_0 = 0, \) \( y(t) \) of the solution (0-I) coincide with (0-II) and \( y(t) \equiv t. \)
In the case of \( \alpha_3 = 0, \) \( y(t) \) of the solution (0-III) coincide with (0-IV) and \( y(t) \equiv 1. \)
Both \( y(t) \equiv t \) and \( y(t) \equiv 1 \) are Riccati solutions.

(3) In the case of \( \alpha_1 = 0 \) (\( \alpha = 0 \)), the sixth Painlevé equation has the following special solution around \( t = 0: \)

\[ y(t) = t^{\alpha_3} (a_0 + a_1 t + a_2 t^2 + \cdots), \quad z(t) = t^{\alpha_3} (b_0 + b_1 t + b_2 t^2 + \cdots), \]
\[ z(t) = t^{\alpha_3} (c_0 + c_1 t + c_2 t^2 + \cdots) \quad (a_i, b_i, c_i \in \mathbb{C}). \] (5.14)

The Bäcklund transformations for the sixth Painlevé equation are defined in subsection 2.2.8, (where \( y = q, z = p \) [32]. If we let \( \sigma_1 \) and \( \sigma_2 \) act on the solutions (0-I), (0-II), (0-III) and (0-IV), we then obtain the meromorphic solutions of the system \[ (2.28),\quad (2.29) \] and \[ (2.43),\quad (2.44) \] which are meromorphic around \( t = 1 \) and \( t = \infty. \)

**Theorem 14** The sixth Painlevé equation has the following meromorphic solutions around \( t = 1 \) and \( t = \infty. \)
(1) Around $t = 1$:

(1-I):
\[
y(t) = 1 + \frac{\alpha_3}{\alpha_0 - \alpha_3}(1-t) + \frac{\alpha_0 \alpha_3 [-1 + \alpha_0^2 + (\alpha_0 - \alpha_3)^2]}{2 [1 - (\alpha_0 - \alpha_3)^2] (\alpha_0 - \alpha_3)^2} (1-t)^2 + O((1-t)^3),
\]
\[
\tilde{z}(t) = \frac{1}{4} \frac{1 - \alpha_1^2 + \alpha_2^2 - (\alpha_0 - \alpha_3)^2}{1 - (\alpha_0 - \alpha_3)^2} + O((1-t)),
\]
\[
z(t) = \frac{\alpha_0 - \alpha_3}{1-t} + O((1-t)^0),
\]

(1-II):
\[
y(t) = 1 + \frac{-\alpha_3}{\alpha_0 + \alpha_3}(1-t) + \frac{\alpha_0 \alpha_3 [1 + \alpha_1^2 - \alpha_4^2 - (\alpha_0 + \alpha_3)^2]}{2 [1 - (\alpha_0 + \alpha_3)^2] (\alpha_0 + \alpha_3)^2} (1-t)^2 + O((1-t)^3),
\]
\[
\tilde{z}(t) = \frac{1 - \alpha_1^2 + \alpha_4^2 - (\alpha_0 + \alpha_3)^2}{4 [1 - (\alpha_0 + \alpha_3)^2]} + O((1-t)),
\]
\[
z(t) = \frac{\alpha_2 (\alpha_1 + \alpha_2)}{\alpha_0 + \alpha_3 - 1} + O((1-t)),
\]

(1-III):
\[
y(t) = -\frac{\alpha_4}{\alpha_1} + \frac{\alpha_4 [1 + \alpha_2^2 - \alpha_0^2 - (\alpha_4 + \alpha_1)^2]}{2 \alpha_1 [1 - (\alpha_4 + \alpha_1)^2]} (1-t) + O((1-t)^2),
\]
\[
\tilde{z}(t) = \frac{\alpha_1}{2 (\alpha_1 + \alpha_4)} + O((1-t)), \quad z(t) = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_4} + O((1-t)),
\]

(1-IV):
\[
y(t) = \frac{\alpha_4}{\alpha_1} + \frac{-\alpha_4 [1 + \alpha_3^2 - \alpha_2^2 - (\alpha_4 - \alpha_1)^2]}{2 \alpha_1 [1 - (\alpha_4 - \alpha_1)^2]} (1-t) + O((1-t)^2),
\]
\[
\tilde{z}(t) = \frac{-\alpha_1}{2 (\alpha_4 - \alpha_1)} + O((1-t)), \quad z(t) = \frac{\alpha_1 (\alpha_2 + \alpha_4)}{\alpha_1 - \alpha_4} + O((1-t)).
\]

(2) Around $t = \infty$:

(\infty-I): 
\[
y(t) = \frac{\alpha_1 - \alpha_0}{\alpha_1} t + \frac{\alpha_1}{2 \alpha_0 [1 - (\alpha_0 - \alpha_1)^2]} \left[1 + \alpha_3^2 + \alpha_4^2 - \alpha_1^2 - (\alpha_0 - \alpha_1)^2 \right] \left(\frac{\alpha_0}{\alpha_1}\right)^2 + 1 + \alpha_0^2 + O((t^{-1}))
\]
\[
\tilde{z}(t) = -\frac{\alpha_1}{2 (\alpha_1 - \alpha_0)} \frac{1}{t} + O(t^{-2}), \quad z(t) = -\frac{\alpha_1 (\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_0) t} + O(t^{-2}),
\]

(\infty-II):
\[
y(t) = \frac{\alpha_1 + \alpha_0}{\alpha_1} t + \frac{-\alpha_1}{2 \alpha_0 [1 - (\alpha_0 + \alpha_1)^2]} \left[1 + \alpha_3^2 + \alpha_4^2 - \alpha_1^2 - (\alpha_0 + \alpha_1)^2 \right] \left(\frac{\alpha_0}{\alpha_1}\right)^2 + 1 + \alpha_0^2 + O((t^{-1}))
\]
\[
\tilde{z}(t) = -\frac{-\alpha_1}{2 (\alpha_1 + \alpha_0)} \frac{1}{t} + O(t^{-2}), \quad z(t) = -\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_0} \frac{1}{t} + O(t^{-2}).
\]
\[(\infty{-}III): \quad \psi(t) = \frac{\alpha_4}{\alpha_4 + \alpha_3} + \frac{-\alpha_3 \alpha_4 [-1 + \alpha_0^2 - \alpha_1^2 + (\alpha_3 + \alpha_4)^2]}{2 [1 - (\alpha_3 + \alpha_4)^2] (\alpha_3 + \alpha_4)^2} \frac{1}{t} + O(t^{-2}) (5.29)\]

\[\tilde{\psi}(t) = 1 - \frac{\alpha_1^2 + \alpha_3^2 - (\alpha_3 - \alpha_4)^2}{4 [1 - (\alpha_3 - \alpha_4)^2]} \frac{1}{t} + O(t^{-2}) (5.30)\]

\[z(t) = \frac{\alpha_2 (\alpha_1 + \alpha_2)}{1 - \alpha_3 - \alpha_4} \cdot \frac{1}{t} + O(t^{-2}) (5.31)\]

\[(\infty{-}IV): \quad \psi(t) = \frac{\alpha_4}{\alpha_4 - \alpha_3} + \frac{\alpha_3 \alpha_4 [-1 + \alpha_0^2 - \alpha_1^2 + (\alpha_3 - \alpha_4)^2]}{2 [1 - (\alpha_3 - \alpha_4)^2] (\alpha_3 - \alpha_4)^2} \frac{1}{t} + O(t^{-2}) (5.32)\]

\[\tilde{\psi}(t) = 1 - \frac{\alpha_1^2 + \alpha_3^2 - (\alpha_3 - \alpha_4)^2}{4 [1 - (\alpha_3 - \alpha_4)^2]} \frac{1}{t} + O(t^{-2}) (5.33)\]

\[z(t) = \alpha_4 - \alpha_3 + O(t^{-1}) (5.34)\]

**Remark 15** If we assume the meromorphy of a solution around \(t = 0\) and \(t = 1\), \(\psi(t)\) and \(\tilde{\psi}(t)\) inevitably become holomorphic there.

**Theorem 16** These twelve meromorphic solutions are invariant under the action of the Bäcklund transformation group.

\[
\begin{array}{cccc}
(\infty{-}I) & s_1 & (\infty{-}II) & s_2 & (\infty{-}III) & s_3 & (\infty{-}IV) \\
\uparrow \sigma_2 & \uparrow \sigma_2 & \uparrow \sigma_2 & \uparrow \sigma_2 \\
(0{-}I) & s_4 & (0{-}II) & s_2 & (0{-}III) & s_3 & (0{-}IV) \\
\uparrow \sigma_1 & \uparrow \sigma_1 & \uparrow \sigma_1 & \uparrow \sigma_1 \\
(1{-}I) & s_3 & (1{-}II) & s_2 & (1{-}III) & s_4 & (1{-}IV) \\
\end{array}
\]

**Figure 1:** The Bäcklund transformations of the twelve solutions

### 5.2 The linear monodromy for the solution (0-I)

For a solution of the sixth Painlevé equation, let \(M_j (j = 0, t, 1, \infty)\) be the monodromy matrices of the equation \((2.33)\) along the path around \(x = j\) shown in Figure 2.

Note that \(M_j (j = 0, t, 1, \infty)\) satisfy

\[M_\infty M_t M_1 M_0 = I_2. \quad (5.35)\]

We can then calculate the linear monodromy \(\{M_0, M_t, M_1, M_\infty\}\) explicitly for the solution (0-I) by the method given in \([20]\).

### 5.3 The limit of \((2.33)\)

We will take the limit \(t \to 0\) after substituting the solution (0-I) into \((2.33)\).

**5.3.1** Put \(Y = t(\tilde{\psi}^{(1)}, \tilde{\psi}^{(2)})\), then the limit

\[\tilde{\psi}_0^{(1)}(x) = \lim_{t \to 0} \tilde{\psi}^{(1)}(x, t) \quad (5.36)\]
Figure 2: The paths going around regular singular points with the base point \( x_0 \).

The paths going around regular singular points with the base point \( x_0 \).

satisfies
\[
\frac{d^2 \tilde{\psi}_0^{(1)}}{dx^2} + \left( \frac{1}{x} + \frac{1}{x-1} \right) \frac{d\tilde{\psi}_0^{(1)}}{dx} - \left[ \frac{(\alpha_0 - \alpha_4)^2}{4} \frac{1}{x^2} + \frac{\alpha_3}{4} \frac{1}{(x-1)^2} - \frac{1 - \alpha_1^2 + \alpha_2^2 + (\alpha_0 - \alpha_4)^2}{4x(x-1)} \right] \tilde{\psi}_0^{(1)} = 0.
\] (5.37)

The Riemann scheme of (5.37) is
\[
P \begin{pmatrix} x = 0 \cdot t & x = 1 & x = \infty \\ \frac{\alpha_0 - \alpha_4}{2} & \frac{-\alpha_3}{2} & \frac{1}{2} (1 + \alpha_1) \\ -\frac{\alpha_0 - \alpha_4}{2} & \frac{\alpha_3}{2} & \frac{1}{2} (1 - \alpha_1) \end{pmatrix} \begin{pmatrix} x = 0 \cdot t & x = 1 & x = \infty \\ 0 & 0 & \alpha_0 + \alpha_1 + \alpha_2 \\ \alpha_4 - \alpha_0 & \alpha_3 & \alpha_0 + \alpha_2 \end{pmatrix} ; x \right).
\] (5.38)

Therefore a fundamental system of solutions of (5.37) is
\[
\begin{pmatrix} x^{\alpha_0 - \alpha_4} (x - 1)^{-\alpha_3} F_1(\alpha_0 + \alpha_1 + \alpha_2, \alpha_0 + \alpha_2, 1 + \alpha_0 - \alpha_4; x), \\ x^{\alpha_4 - \alpha_0} (x - 1)^{-\alpha_3} F_1(\alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_4, 1 + \alpha_4 - \alpha_0; x) \end{pmatrix}.
\] (5.40)

The linear monodromy of (5.37) is equivalent to \{\(M_t, M_0, M_1, M_\infty\}\).

The exponent matrices of (5.37) at \( x = 0, 1 \) and \( \infty \) are given by
\[
T_0 = \begin{pmatrix} \frac{\alpha_0 - \alpha_4}{2} & 0 & -\frac{\alpha_0 - \alpha_4}{2} \\ 0 & \frac{\alpha_3}{2} & 0 \\ \frac{1 + \alpha_1}{2} & 0 & \frac{1 - \alpha_1}{2} \end{pmatrix}, \quad T_1 = \begin{pmatrix} \frac{-\alpha_3}{2} & 0 & \frac{\alpha_3}{2} \\ 0 & \frac{-\alpha_3}{2} & 0 \end{pmatrix}, \quad T_\infty = \begin{pmatrix} \frac{1 + \alpha_1}{2} & 0 & \frac{1 - \alpha_1}{2} \\ 0 & \frac{1 + \alpha_1}{2} & 0 \end{pmatrix}.
\] (5.41)

We may assume
\[
M_t M_0 = e^{2\pi i T_0} = \begin{pmatrix} e^{\pi i (\alpha_0 - \alpha_4)} & 0 & 0 \\ 0 & e^{-\pi i (\alpha_0 - \alpha_4)} & 0 \end{pmatrix}, \quad M_1 = \Gamma_0 \Gamma_1 e^{2\pi i T_1}, \quad M_\infty = \Gamma_\infty \Gamma_0 e^{2\pi i T_\infty} \Gamma_0 \infty.
\] (5.44)
A fundamental system of solutions of (5.48) is

\[ \xi \text{ satisfies } \]

In the following, we consider the confluence of \( x = 1 \) and \( x = \infty \) in (2.33). Put \( x = t\xi \), then the limit

\[ \bar{\psi}_1^{(1)}(\xi) = \lim_{t \to 0} \bar{\psi}_1^{(1)}(t\xi, t) \]  

satisfies

\[ \frac{d^2 \bar{\psi}_1^{(1)}}{d\xi^2} + \left( \frac{1}{\xi} + \frac{1}{\xi - 1} - \frac{1}{\xi - s} \right) \frac{d\bar{\psi}_1^{(1)}}{d\xi} - \frac{\alpha_2^2 \frac{1}{4} + \frac{\alpha_0}{4} \frac{1}{(\xi - 1)^2} + \frac{-\alpha_4^2 - \alpha_0^2 + (\alpha_0 - \alpha_4)^2}{4\xi(\xi - 1)}}{\bar{\psi}_1^{(1)}} = 0, \]  

where

\[ s = \frac{\alpha_4}{\alpha_4 - \alpha_0}. \]  

This is a Heun’s type equation with an apparent singularity at \( \xi = s = \alpha_4/(\alpha_4 - \alpha_0) \). The singularities \( \xi = 0,1, \infty \) correspond to \( x = 0, t \) and \( 1 \cdot \infty \), respectively. The Riemann scheme of (5.48) is

\[ P \begin{pmatrix} \xi = 0 & \xi = 1 & \xi = s & \xi = \infty \\ \frac{-\alpha_0}{2} & \frac{-\alpha_0}{2} & 0 & \frac{-\alpha_4-\alpha_0}{2} \\ \frac{-\alpha_4}{2} & \frac{-\alpha_4}{2} & 2 & \frac{-\alpha_4-\alpha_0}{2} \end{pmatrix}; \xi \]  

A fundamental system of solutions of (5.48) is

\[ \begin{pmatrix} \xi^{-\frac{\alpha_4}{2}}(\xi - 1)^{\frac{\alpha_0}{2}}, \xi^{\frac{\alpha_4}{2}}(\xi - 1)^{-\frac{\alpha_0}{2}} \end{pmatrix}. \]
The linear monodromy \( \{L_0, L_1, L_\infty\} \) of (5.48) is equivalent to \( \{M_0, M_t, M_\infty M_1\} \).

\[
M_0 = P^{-1} L_0 P, \quad M_t = P^{-1} L_1 P, \quad M_\infty M_1 = P^{-1} L_\infty P
\]  

(5.52)

for a matrix \( P \in SL(2,C) \).

The linear monodromy \( \{L_0, L_1, L_\infty\} \) is given by

\[
L_0 = \begin{pmatrix} e^{-\pi i \alpha_0} & 0 \\ 0 & e^{\pi i \alpha_0} \end{pmatrix}, \quad L_1 = \begin{pmatrix} e^{\pi i \alpha_4} & 0 \\ 0 & e^{-\pi i \alpha_4} \end{pmatrix},
\]

(5.53)

\[
L_\infty = \begin{pmatrix} e^{-\pi i (\alpha_0 - \alpha_4)} & 0 \\ 0 & e^{\pi i (\alpha_0 - \alpha_4)} \end{pmatrix}.
\]

(5.54)

Comparing (5.43) and (5.52), (5.53), we have

\[
M_1 M_0 = P^{-1} L_1 L_0 P, \quad M_t M_0 = L_1 L_0 = \begin{pmatrix} e^{\pi i (\alpha_0 - \alpha_4)} & 0 \\ 0 & e^{-\pi i (\alpha_0 - \alpha_4)} \end{pmatrix}.
\]

(5.55)

Therefore \( P \) is a diagonal matrix, since \( \alpha_0 - \alpha_4 \notin \mathbb{Z} \) for the solution (0-I).

**Theorem 17** The linear monodromy of (2.33) for the solution (0-I) is as follows:

\[
M_0 = \begin{pmatrix} e^{-\pi i \alpha_0} & 0 \\ 0 & e^{\pi i \alpha_4} \end{pmatrix}, \quad M_t = \begin{pmatrix} e^{\pi i \alpha_4} & 0 \\ 0 & e^{-\pi i \alpha_0} \end{pmatrix},
\]

(5.56)

\[
M_1 = \Gamma_{01}^{-1} \begin{pmatrix} e^{-\pi i \alpha_3} & 0 \\ 0 & e^{\pi i \alpha_3} \end{pmatrix} \Gamma_{01}, \quad M_\infty = \Gamma_{0\infty}^{-1} \begin{pmatrix} -e^{-\pi i \alpha_1} & 0 \\ 0 & -e^{\pi i \alpha_1} \end{pmatrix} \Gamma_{0\infty}.
\]

(5.57)

where \( \Gamma_{01} \) and \( \Gamma_{0\infty} \) are given in (5.45) and (5.46). We remark that \( \alpha_0 - \alpha_4 \notin \mathbb{Z} \) if the solution (0-I) exists.

In a similar way, we can calculate the linear monodromy explicitly for all of the twelve solutions in Theorem 12 and Theorem 14.

**Theorem 18** The twelve solutions in Theorem 12 and Theorem 14 are all monodromy solvable.
5.4 Comparison with classical solutions

Umemura studied special solutions of the Painlevé equations [31]. Umemura’s classical solutions are either rational solutions or the Riccati solutions. We show that some of our solutions include an algebraic solution and one of the Riccati solutions.

1) In the case of \( \alpha_1 = \alpha_4 \) and \( \alpha_0 = \alpha_3 \) \((\alpha + \beta = 0, \ \gamma + \delta = \frac{1}{2})\), the sixth Painlevé equation has an algebraic solution

\[
y(t) = \sqrt{t} = 1 + \frac{1}{2}(t - 1) + \frac{1}{2!} \cdot \frac{-1}{4}(t - 1)^2 + \cdots ,
\]
\[
z(t) = \frac{1}{4}(2\alpha_3 + 2\alpha_4 - 1) \frac{1}{\sqrt{t}} = \frac{1}{4}(2\alpha_3 + 2\alpha_4 - 1) \left[ 1 - \frac{1}{2}(t - 1) + \frac{1}{2!} \cdot \frac{3}{4}(t - 1)^2 - \cdots \right].
\]  

(5.58)

The solution (5.58) is a special case of the solution (1-II) for \( \alpha_1 = \alpha_4, \alpha_0 = \alpha_3 \).

2) In the case of \( \alpha_0 = 0 \) \((\delta = \frac{1}{2})\), the sixth Painlevé equation has the Riccati solution

\[
\begin{align*}
(1) \quad y(t) &= t, \\
z(t) &= \frac{2F_1(\alpha_2, \alpha_1 + \alpha_2, 1 - \alpha_4; t)}{2F_1(\alpha_2, \alpha_1 + \alpha_2, 1 - \alpha_4; t)} = \frac{\alpha_2(\alpha_1 + \alpha_2)}{1 - \alpha_4} + O(t), \\
(2) \quad y(t) &= t, \\
z(t) &= \frac{t(1 + O(t))}{f_{\alpha_2}(\alpha_2 + \alpha_1 + \alpha_2 + \alpha_4, 1 + \alpha_4; t)} = \alpha_4 \frac{t}{t} (1 + O(t)).
\end{align*}
\]

(5.59)

(5.60)

These are obtained by putting \( \alpha_0 = 0 \) in the solution (0-II) and (0-I), respectively.

3) In the case of \( \alpha_2 = 0 \), the system (2.28) and (2.29) has the Riccati solution

\[
\begin{align*}
(1) \quad z(t) &= 0, \\
y(t) &= \frac{t(t - 1)[(t - 1)^{\alpha_4}F_1(\alpha_4, 1 - \alpha_3, \alpha_0 + \alpha_4; t)]'}{\alpha_1} \\
&= \frac{\alpha_4}{\alpha_0 + \alpha_4} t + O(t^2), \\
(2) \quad z(t) &= 0, \\
y(t) &= \frac{-t(t - 1)[(t - 1)^{\alpha_3}F_1(1 - \alpha_0, 1 + \alpha_0, 1 + \alpha_1 + \alpha_3; t)]'}{\alpha_1} \\
&= \frac{\alpha_1 + \alpha_3}{\alpha_1} + O(t).
\end{align*}
\]

(5.61)

(5.62)

These are obtained by putting \( \alpha_2 = 0 \) in the solution (0-II) and (0-III), respectively.

4) In the case of \( \alpha_3 = 0 \) \((\gamma = 0)\), the system (2.28) and (2.29) has the Riccati solution

\[
\begin{align*}
(1) \quad y(t) &= 1, \\
z(t) &= -t[t^{\alpha_2}F_1(\alpha_2, \alpha_2 + \alpha_4, 1 - \alpha_1; t)]' = -\alpha_2 + O(t), \\
(2) \quad y(t) &= 1, \\
z(t) &= -t[t^{\alpha_1 + \alpha_2}F_1(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_4, 1 + \alpha_1; t)]' \\
&= -(\alpha_1 + \alpha_2) + O(t).
\end{align*}
\]

(5.63)

(5.64)
These are obtained by putting \( \alpha_3 = 0 \) in the solution (0-III) and (0-IV), respectively.

5) In the case of \( \alpha_4 = 0 \) (\( \beta = 0 \)), the system (2.28) and (2.29) has the Riccati solution

\[
\begin{align*}
(1) \quad y(t) & \equiv 0, \\
\quad z(t) & = (t - 1) \frac{(t - 1)^{\alpha_2} F_1(\alpha_2, \alpha_2 + \alpha_3, 1 - \alpha_0; t)}{(t - 1)^{\alpha_2} F_1(\alpha_2, \alpha_2 + \alpha_3, 1 - \alpha_0; t)} \\
& \quad = \frac{\alpha_2(\alpha_1 + \alpha_2)}{1 - \alpha_0} + O(t). \\
(2) \quad y(t) & \equiv 0, \\
\quad z(t) & = (t - 1) \frac{t^{\alpha_0}(t - 1)^{\alpha_2} F_1(\alpha_0 + \alpha_2, \alpha_0 + \alpha_2 + \alpha_3, 1 + \alpha_0; t)}{t^{\alpha_0}(t - 1)^{\alpha_2} F_1(\alpha_0 + \alpha_2, \alpha_0 + \alpha_2 + \alpha_3, 1 + \alpha_0; t)} \\
& \quad = \frac{-\alpha_0}{t} + O(t^0). \\
\end{align*}
\]

These are obtained by putting \( \alpha_4 = 0 \) in the solution (0-II) and (0-I), respectively.

### 6 The Briot-Bouquet theorem

The Briot-Bouquet theorem [3] is well-known but we will explain the Briot-Bouquet theorem for a system and give a brief proof. From the Briot-Bouquet theorem series expansions of solutions of the fifth and the sixth Painlevé equations in Theorem 7, 12 and 14 converge around the fixed singularities. We denote \((f)_0 := f(0)\) for a holomorphic function \(f\).

**Theorem 19 (Briot-Bouquet)** For the simultaneous equations

\[
\begin{align*}
t \frac{du}{dt} &= f(u, v, t), \\
\frac{dv}{dt} &= g(u, v, t),
\end{align*}
\]

where \(f\) and \(g\) are holomorphic functions of \(u, v\) and \(t\) near the origin. We assume that \(f(0, 0, 0) = 0, g(0, 0, 0) = 0\). Then a holomorphic solution with the initial condition \(u(0) = 0, v(0) = 0\) exists if

\[
\Delta_n := \left| \begin{array}{cc}
(n - (f_u)_0 & (-f_v)_0 \\
(-g_u)_0 & n - (g_v)_0
\end{array} \right| \neq 0,
\]

for any non-negative integer \(n\).

**Proof.** At first we will show the existence of a formal solution \(u, v\) for (6.1) and (6.2). Expand \(u, v\) as

\[
\begin{align*}
u = a_1 t + a_2 t^2 + \cdots, \\
v = b_1 t + b_2 t^2 + \cdots.
\end{align*}
\]

From (6.1) and (6.2) we have

\[
\begin{align*}
t \frac{d^2 u}{dt^2} + \frac{du}{dt} &= f_t + f_u \frac{du}{dt} + f_v \frac{dv}{dt}, \\
t \frac{d^2 v}{dt^2} + \frac{dv}{dt} &= g_t + g_u \frac{du}{dt} + g_v \frac{dv}{dt}.
\end{align*}
\]
Putting $t = 0$, we have

$$[1 - (f_u)_0](\frac{du}{dt})_0 - (f_v)_0(\frac{dv}{dt})_0 = (f_t)_0,$$

$$-(g_u)_0(\frac{du}{dt})_0 + [1 - (g_v)_0](\frac{dv}{dt})_0 = (g_t)_0.$$  

Solving the above equations, we obtain

$$a_1 = \left(\frac{du}{dt}\right)_0 = \frac{1}{\Delta_1} \left| (f_t)_0 \quad (-f_v)_0 \right|, \quad b_1 = \left(\frac{dv}{dt}\right)_0 = \frac{1}{\Delta_1} \left| 1 - (f_u)_0 \quad (f_t)_0 \right|.$$  

Here $\Delta_1 \neq 0$ from the assumption.

By differentiating (6.3) and (6.4) with $t$ and putting $t = 0$, $(a_2, b_2)$ is also determined uniquely as follows:

$$a_2 = \frac{1}{2!} \left(\frac{d^2u}{dt^2}\right)_0 = \frac{1}{2! \Delta_2} \left| A_1 \quad (-f_v)_0 \right|, \quad B_1 = 2 - (g_u)_0 - (g_v)_0,$$

$$b_2 = \frac{1}{2!} \left(\frac{d^2v}{dt^2}\right)_0 = \frac{1}{2! \Delta_2} \left| 2 - (f_u)_0 - (f_v)_0 \quad A_1 \right|, \quad (g_u)_0 \quad B_1,$$

where

$$A_1 = \left[ f_{tt} + 2f_{tu} \frac{du}{dt} + f_{uu}(\frac{du}{dt})^2 + 2f_{tv} \frac{dv}{dt} + 2f_{uv} \frac{du}{dt} \frac{dv}{dt} + f_{vv}(\frac{dv}{dt})^2 \right]_{t=0},$$

$$B_1 = \left[ g_{tt} + 2g_{tu} \frac{du}{dt} + g_{uu}(\frac{du}{dt})^2 + 2g_{tv} \frac{dv}{dt} + 2g_{uv} \frac{du}{dt} \frac{dv}{dt} + g_{vv}(\frac{dv}{dt})^2 \right]_{t=0},$$

and so on. Thus coefficients $(a_n, b_n)$ can be uniquely determined.

In the second step, we will show the formal solutions $u = \sum_{k=1}^{\infty} a_k t^k$ and $v = \sum_{k=1}^{\infty} b_k t^k$ are convergent. We prepare the following auxiliary functions $p(t), q(t)$ defined as implicit functions:

$$p = f(t, p, q), \quad q = g(t, p, q).$$

Since $\Delta_1 \neq 0$, the holomorphic functions $p$ and $q$ with $p(0) = 0, q(0) = 0$ exist by the implicit function theorem:

$$p = c_1 t + c_2 t^2 + \cdots, \quad q = d_1 t + d_2 t^2 + \cdots.$$  

In the similar way, we have

$$c_1 = \left(\frac{dp}{dt}\right)_0 = \frac{1}{\Delta'_1} \left| (f_t)_0 \quad (-f_q)_0 \right|, \quad d_1 = \left(\frac{dq}{dt}\right)_0 = \frac{1}{\Delta'_1} \left| 1 - (f_p)_0 \quad (f_t)_0 \right|,$$

where

$$\Delta_1 = \Delta'_1 = \left| 1 - (f_p)_0 \quad (-f_q)_0 \right| \neq 0.$$
and
\[ a_2 = \frac{1}{2!} \left( \frac{d^2 p}{dt^2} \right)_0 = \frac{1}{2!} \Delta_2' \begin{bmatrix} A_1' & (-f_q)_0 \\ B_1' & 1 - (g_p)_0 - (g_q)_0 \end{bmatrix} , \]
\[ d_2 = \frac{1}{2!} \left( \frac{d^2 q}{dt^2} \right)_0 = \frac{1}{2!} \Delta_2' \begin{bmatrix} 1 - (f_p)_0 - (f_q)_0 & A_1' \\ (-g_p)_0 & B_1' \end{bmatrix} , \]
where
\[ \Delta_2' = 1 - (f_p)_0 - (f_q)_0 \begin{bmatrix} 1 - (g_p)_0 - (g_q)_0 & (-f_q)_0 \\ (-g_p)_0 & 1 \end{bmatrix} \neq 0 , \]
\[ A_1' = \begin{bmatrix} f_u + 2f_{tp} \frac{dp}{dt} + f_{pp} \left( \frac{dp}{dt} \right)^2 + 2f_{tq} \frac{dq}{dt} + 2f_{pq} \frac{dp}{dt} \frac{dq}{dt} + f_{qq} \left( \frac{dq}{dt} \right)^2 \end{bmatrix} \bigg|_{t=0} \],
\[ B_1' = \begin{bmatrix} g_u + 2g_{tp} \frac{dp}{dt} + g_{pp} \left( \frac{dp}{dt} \right)^2 + 2g_{tq} \frac{dq}{dt} + 2g_{pq} \frac{dp}{dt} \frac{dq}{dt} + g_{qq} \left( \frac{dq}{dt} \right)^2 \end{bmatrix} \bigg|_{t=0} , \]
and so on.

Comparing \((u, v)\) with \((p, q)\), we have
\[ a_n \Delta_n = c_n \Delta'_n , \quad b_n \Delta_n = d_n \Delta'_n , \quad |\Delta'_n| \leq |\Delta_n| . \]
Therefore
\[ |a_n| \leq |c_n| , \quad |b_n| \leq |d_n| . \]

Since \(p\) and \(q\) are dominant series of \(u\) and \(v\), \(u = \sum_{k=1}^{\infty} a_k t^k\) and \(v = \sum_{k=1}^{\infty} b_k t^k\) are convergent. Thus the theorem is proved. \(\square\)

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