Abstract. In [Cho09], Choi studied congruences of coefficients (modulo $T^q - T$) for Drinfeld modular forms of level $\Gamma_0(T)$, trivial type and the linear relations between the initial coefficients of those. In this article, we generalize these results for level $\Gamma_0(T)$, arbitrary type.

1. Introduction

The study of congruences of Fourier coefficients of modular forms is an interesting area of research in number theory. There were several important works in the literature on the congruences of Fourier coefficients (cf. [Rib84], [Hid85]), but we are particularly focusing on [CKO05], where the authors studied the $p$-divisibility properties of Fourier coefficients of modular forms for $SL_2(\mathbb{Z})$. As an application, they have retrieved the results of Hatada, Hida on non-ordinary primes. Later, these results were generalized by El-Guindy for cusp forms of level 2 modular forms of trivial type. In [Kaz08], Kazalicki also obtained similar results for $GL_2$ type for $GL_2$.

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It is of real interest to answer similar questions for the coefficients in the $u$-series expansion of Drinfeld modular forms. Such a study has been initiated by Choi. In [Cho08], she studied congruences of coefficients (modulo $T^q - T$)-divisibility properties of the coefficients of Drinfeld modular forms of any weight, any type for $GL_2(A)$, and determined all the linear relations between the initial coefficients of Drinfeld modular forms of trivial type. In [Kaz08], Kazalicki also obtained similar results for $GL_2(A)$.

For the level $\Gamma_0(T)$, these results are only known for Drinfeld modular forms of any weight, trivial type. In [Cho09], Choi studied congruences of coefficients (modulo $T^q - T$)-divisibility properties of the coefficient of Drinfeld modular forms of any weight, trivial type for $\Gamma_0(T)$. In loc. cit., she determined all the linear relations between the initial coefficients in the $u$-series expansion of Drinfeld modular forms of trivial type, level $\Gamma_0(T)$.

In this article, we continue this study for $\Gamma_0(T)$ by generalizing the results of [Cho09] for Drinfeld modular forms of any weight, arbitrary type, level $\Gamma_0(T)$. Throughout the article, we fix to use the following notations:

- $p$ is an odd prime number and $q = p^r$ for some $r \in \mathbb{N}$.
- $k \in \mathbb{N}$ and $l \in \mathbb{Z}/(q - 1)\mathbb{Z}$ such that $k \equiv 2l \pmod{q - 1}$. Let $0 \leq l \leq q - 2$ be a lift of $l \in \mathbb{Z}/(q - 1)\mathbb{Z}$. By abuse of notation, we continue to write $l$ for the integer as well as its class. Then, we define $r_{k,l} := \frac{k - 2l}{q - 1}$ and $r_{k,l,N} := r_{k,l} + N + 1$, where $N$ is a non-negative integer. Recall that $\dim M_{k,l}(\Gamma_0(T)) = 1 + r_{k,l}$ (cf. [DK] Proposition 4.1).

1.1. An overview of the article. The article is organized as follows. In §2 we recall the basic theory of Drinfeld modular forms. In §3 we study congruences of coefficients and generalize [Cho09, Theorem 3.4] for Drinfeld modular forms of any weight, arbitrary type, level $\Gamma_0(T)$. Finally, in §4 we determine all the linear relations between the initial coefficients in $u$-series expansion of Drinfeld modular forms of any weight, arbitrary type for $\Gamma_0(T)$ which generalizes [Cho09, Theorem 4.1].
2. Basic theory of Drinfeld modular forms

In this section, we shall recall some basic theory of Drinfeld modular forms (see [Gos80], [Gos80a], [Gek88], [GR96] for more details).

Let \( \mathbb{F}_q \) denote the finite field of order \( q \). Then, we set \( A = \mathbb{F}_q[T] \) and \( K = \mathbb{F}_q(T) \). Let \( K_\infty = \mathbb{F}_q((\frac{1}{T})) \) be the completion of \( K \) with respect to the infinite place \( \infty \) (corresponding to \( \frac{1}{T} \)-adic valuation) and denote by \( C \) the completion of an algebraic closure of \( K_\infty \). Let \( L = \tilde{\pi} A \subseteq C \) be the \( A \)-lattice of rank 1, corresponding to the rank 1 Drinfeld module given by \( \rho_T = TX + X^q \), where \( \tilde{\pi} \in K_\infty \) ( \( \frac{1}{T} \) is defined up to \( (q-1) \)-th root of unity. The Drinfeld upper half-plane \( \Omega = C - K_\infty \) has a rigid analytic structure. The group \( \text{GL}_2(K_\infty) \) acts on \( \Omega \) via fractional linear transformations. Any \( x \in K_\infty^\times \) has the unique expression \( x = \zeta_x \tilde{\pi}^{-v_\infty(x)} u_x \), where \( \zeta_x \in \mathbb{F}_q^\times \), and \( v_\infty(u_x - 1) > 0 \) \( (v_\infty \text{ is the valuation at } \infty) \). For any \( k \in \mathbb{N}, l \in \mathbb{Z}/(q-1)\mathbb{Z} \), \( \gamma = (\frac{a}{c} \frac{b}{d}) \in \text{GL}_2(K_\infty) \), and \( f : \Omega \rightarrow C \), we define \( f|_{k,l}\gamma := \zeta_{\det(\gamma)} \left( \frac{\det(\gamma)}{v_{\infty}(\gamma)} \right)^{k/2} (cz + d)^{-k} f(\gamma z) \). For an ideal \( \mathfrak{n} \subseteq A \), let \( \Gamma_0(n) \) denote the congruence subgroup \( \Gamma_0(n) := \{ (\frac{a}{c} \frac{b}{d}) \in \text{GL}_2(A) : c \in \mathfrak{n} \} \).

**Definition 2.1.** A rigid holomorphic (resp., meromorphic) function \( f : \Omega \rightarrow C \) is said to be a holomorphic (resp., meromorphic) Drinfeld modular form for \( \Gamma_0(n) \) of weight \( k \) and type \( l \) if

1. \( f|_{k,l}\gamma = f, \forall \gamma \in \Gamma_0(n) \),
2. \( f \) is holomorphic (resp., meromorphic) at the cusps of \( \Gamma_0(n) \).

Every holomorphic (resp., meromorphic) Drinfeld modular form has an unique expansion at the cusp \( \infty \) w.r.t the parameter \( u(z) := \frac{1}{e_L(z)}, e_L(z) := z \prod_{\gamma \neq \lambda \in L}(1 - \gamma) \). Let \( \mathcal{M}_{k,l}(\Gamma_0(n)) \) denote the \( C \)-vector space of holomorphic Drinfeld modular forms of weight \( k \) and type \( l \) for \( \Gamma_0(n) \). If \( f \in \mathcal{M}_{k,l}(\Gamma_0(n)) \) vanishes at the cusps of \( \Gamma_0(n) \), then we say that \( f \) is a Drinfeld cusp form. Any \( f \in \mathcal{M}_{k,l}(\Gamma_0(n)) \) has \( u \)-series expansion at \( \infty \) of the form \( \sum_{0 \leq i \equiv l \mod (q-1)} a_f(i)u^i \) (similarly, for meromorphic function \( f \), the index \( i \) starts from a negative integer).

By definition, if \( k \neq 2l \mod (q-1) \) then \( \mathcal{M}_{k,l}(\Gamma_0(n)) = \{ 0 \} \). Hence, we always with the vector space \( \mathcal{M}_{k,l}(\Gamma_0(n)) \) where \( k \in \mathbb{N} \) and \( l \in \mathbb{Z}/(q-1)\mathbb{Z} \) such that \( k \neq 2l \mod (q-1) \).

We now give some useful examples of Drinfeld modular forms.

**Example 2.2** ([Gos80], [Gek88]). Let \( d \in \mathbb{N} \). For \( z \in \Omega \), the function

\[
g_d(z) := (-1)^{d+1} \tilde{\pi}^{-q^d-1} L_d \sum_{\substack{a,b \in \mathbb{F}_q[T] \\ (a,b) \neq (0,0)}} \frac{1}{(az + b)^{q^d-1}}
\]

is a Drinfeld modular form of weight \( q^d - 1 \) and type \( 0 \) for \( \text{GL}_2(A) \), where \( \tilde{\pi} \) is the Carlitz period and \( L_d := (T^d - T) \ldots (T^q - T) \) is the least common multiple of all monics of degree \( d \). We refer \( g_d \) as an Eisenstein series and it does not vanish at \( \infty \).

**Example 2.3** (Poincaré series). In [Gek88], Gekeler defined the Poincaré series as follows: \( h(z) = \sum_{\gamma \in H \backslash \text{GL}_2(A)} \frac{\det(\gamma)}{(cz+d)^q} \), where \( H = \{ (\frac{a}{c} \frac{b}{d}) \in \text{GL}_2(A) \} \) and \( \gamma = (\frac{a}{c} \frac{b}{d}) \in \text{GL}_2(A) \). Then \( h \) is a cusp form of weight \( q + 1 \), type 1 for \( \text{GL}_2(A) \). The \( u \)-series expansion of \( h \) at \( \infty \) is given by \( -u - u^{-1} + \ldots \). By the properties of \( \Delta \)-function in [Gek86a] Page 228, we deduce that \( h \) vanishes exactly once (resp., \( q \)-times) at \( \infty \) (resp., at \( 0 \)) as a Drinfeld modular form of level \( \Gamma_0(T) \).

We end this section by introducing an important function \( E \). In [Gek88], Gekeler defined the function \( E(z) := \frac{1}{2} \sum_{a \in \mathbb{F}_q[T]} \sum_{\gamma \in \text{GL}_2(A)} \frac{1}{(az)^2} \frac{1}{(cz+d)^q} \) which is analogous to the Eisenstein series of weight 2 over \( \mathbb{Q} \). Though \( E \) is not modular but we can use it to construct a Drinfeld modular form

\[
E_T(z) := E(z) - TE(Tz) \in \mathcal{M}_{2,1}(\Gamma_0(T)).
\]

The \( u \)-series expansion of \( E_T \) at \( \infty \) is given by \( u - Tu^q + \ldots \) (cf. [DK21] Proposition 4.3) for a detailed discussion about this function).

3. Congruences for coefficients of Drinfeld Modular forms

In this section, we generalize [Cho09] Theorem 3.4] to Drinfeld modular forms \( f \) of level \( \Gamma_0(T) \), arbitrary type. We start by introducing the modular forms \( \Delta_T \) and \( \Delta_W \). Recall that \( 0 \) and \( \infty \) are the only cusps of the Drinfeld modular curve \( X_0(T) := \Gamma_0(T)/\Omega \) and the operator
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W_T := \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) permutes the cusps. Consider the functions \( \Delta_T(z) := \frac{g_1(Tz) - g_1(z)}{T - z} \) and \( \Delta_W(z) := \frac{g_1(Tz) - g_1(z)}{z - \infty} \) modulo \( W_T \). Note that, \( \Delta_T, \Delta_W \in M_{g-1,0}(\Gamma(0(T))) \) and their \( u \)-expansions are given by \( \Delta_T = u^{g-1} - u^{q(q-1)} + \ldots \in A[[u]] \) and \( \Delta_W = 1 + Tu^{g-1} - Tu^{q(q-1)} + \ldots \in A[[u]] \).

Proposition 3.1. (DK) Proposition 4.3 \ Let \( \Delta_T, \Delta_W \) and \( E_T \) be as defined before.

1. The modular form \( \Delta_T \) (resp., \( \Delta_W \)) vanishes \( q - 1 \) times at \( \infty \) (resp., at 0) and non-zero on \( \Omega \cup \{0\} \) (resp., on \( \Omega \cup \{\infty\} \)). Hence, \( \Delta_T \) and \( \Delta_W \) are algebraically independent.

2. The set \( S := \{\Delta_T^{r_1}E_T^{r_2}, \Delta_W^{r_1}E_T^{r_2}, \ldots, \Delta_W^{r_1}E_T^{r_2}, \Delta_T^{r_1}E_T^{r_2}\} \) forms a basis for the \( C \)-vector space \( M_{k,l}(\Gamma_0(T)) \).

3. For any \( k \in \mathbb{N} \), the mapping \( \eta : M_{k-2r_1,0}(\Gamma(0(T))) \to M_{k,l}(\Gamma_0(T)) \) defined by \( f \mapsto fE_T^{r_2} \) is an isomorphism.

4. We have \( E_T^{r_2} = \Delta_W \Delta_T \). In particular, the function \( E_T \in M_{2,1}(\Gamma_0(T)) \) vanishes exactly once at the cusps 0, \( \infty \) and non-vanishing elsewhere.

5. \( h(z) = -\Delta_W(z)E_T(z) \).

In [Cho09] Theorem 3.4, Choi multiplied \( f \) with \( E \) to get the congruence for the coefficients of \( fE \) modulo \( T^q - T \), which implies the congruences for the coefficients of \( f \) (cf. Corollary [Cho09 Corollary 3.5]). Although the methodology of the proof of the main result is similar, the novelty in our work is to multiply \( f \) with \( E_T^{r_2} \) to get the congruence for the coefficients of \( fE_T^{r_2} \) modulo \( T^q - T \) and deduce the congruences for the coefficients of \( f \).

Before we state our main result of this section, let us recall some important results which are useful in the proof. First, we shall recall the following Theorem (cf. [Hat77 Theorem 7.14.2]).

Theorem 3.2 (Residue Theorem). Let \( n \) be an ideal of \( A \). For any \( 1 \)-form \( \omega \) on \( X_0(n) := \Gamma_0(n)\backslash \Omega \), we have \( \sum_{\gamma \in X_0(n)} \text{Res}_\gamma \omega = 0 \).

Proposition 3.3. \ Let \( G \) be a meromorphic Drinfeld modular form of weight 2, type 1 for \( \Gamma_0(T) \).

1. If \( G \) is holomorphic at \( P \in \Gamma_0(T) \backslash \Omega \), then \( \text{Res}_P G(z)dz = 0 \).

2. If the \( u \)-series expansion of \( G \) at \( \infty \) is given by \( G(z) = \sum_{i \geq -n} a_G(i(q-1)+1)u^{i(q-1)+1} \), then \( \text{Res}_\infty G(z)dz = -\frac{a_G(-1)}{\pi} \), where \( \pi \) is the Carlitz period.

3. If the \( u \)-series expansion of \( G(\frac{1}{1} 0 1 \frac{-1}{0} ) \) at \( \infty \) is given by \( \sum_{i \geq -n} a_G(0)i(q-1)+1)u^{i(q-1)+1} \), then \( \text{Res}_\infty G(z)dz = -\frac{1}{2}a_G(0) \). Moreover, if the order of vanishing of \( G \) at 0 is at least 2, then \( \text{Res}_0 G(z)dz = 0 \).

Proof.

1. By [G1960] §2.10], if \( G \) is holomorphic at \( P \in \Gamma_0(T) \backslash \Omega \), then \( G(z)dz \) is also holomorphic at \( P \). Hence \( \text{Res}_P G(z)dz = 0 \).

2. The parameter at \( \infty \) is given by \( u = \frac{1}{cz+e(z)} \) which implies \( dz = \frac{-1}{cz+e(z)} du \). Thus \( G(z)dz = -\frac{1}{cz+e(z)} \sum_{i \geq -n} a_G(0)i(q-1)+1)u^{i(q-1)+1} du \). The coefficient of \( u^{-1} \) gives the required result.

3. The parameter at 0 is given by \( u(z) := \frac{1}{cz+e(z)} \) which implies \( dz = -\frac{1}{cz+e(z)} du \). Thus \( G(\frac{1}{1} 0 1 \frac{-1}{0} )dz = -\frac{1}{cz+e(z)} \sum_{i \geq -n} a_G(0)i(q-1)+1)u^{i(q-1)+1} du \). Since the matrix \( \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right) \) permutes the cusps 0 and \( \infty \), by comparing the coefficient of \( u_0^{-1} \) on both sides, we get \( \text{Res}_0 G(z)dz = -\frac{1}{cz+e(z)} a_G(0) \). If the order of vanishing of \( G \) at 0 is \( \geq 2 \), then \( G(\frac{1}{1} 0 1 \frac{-1}{0} ) = \sum_{i \geq 1} a_G(0)i(q-1)+1)u^{i(q-1)+1} \), which implies \( \text{Res}_0 G(z)dz = 0 \).

Now, we are in a position to state and prove the main result of this section.

Theorem 3.4. \ Let \( f \in M_{k,l}(\Gamma_0(T)) \) be a non-zero Drinfeld modular form such that the \( u \)-series expansion at \( \infty \) belongs to \( A[[u]] \). Suppose the \( u \)-series expansion of \( fE_T^{r_2-1} \) at \( \infty \) is given by \( fE_T^{r_2-1} = \sum_{i \geq 0} a_{fE_T^{r_2-1}}(i(q-1)+1)u^{i(q-1)+1} \) where \( E_T \) is as in \( (2.1) \). Fix \( d \in \mathbb{N} \). Let \( b \in \mathbb{N} \) such that \( r_k + 2 + a f E_T^{r_2-1} = p^b \) holds for some integer \( a \geq 0 \). Then the following congruence

\[ a_{fE_T^{r_2-1}}(p^b(q-1)+1) \equiv 0 \pmod{T^q - T} \]

holds. Moreover, if \( a \) can chosen to be 0, then \( a_{fE_T^{r_2-1}}(p^b(q-1)+1) + 1 = 0 \).
Proof. For any integer \( a \geq 0 \), consider the function \( G(a) := \frac{g_d^a}{\Delta_T^{p^a}} \cdot \frac{h^{q^a}}{\Delta_T^{q^a}} \cdot \frac{T^a}{T} \).

- Since \( g_d \) is non-vanishing at \( \infty \) and \( \Delta_T \) vanishes only at \( \infty \), the function \( \frac{g_d^a}{\Delta_T^{p^a}} \) is holomorphic on \( \Omega \cup \{0\} \) and the possible pole is only at \( \infty \).

- The function \( h \) vanishes exactly once (resp., \( q \)-times) at \( \infty \) (resp., at 0) as a Drinfeld modular form for \( \Gamma_0(T) \). The function \( \Delta_T \) vanishes only at \( \infty \) and the order of vanishing at \( \infty \) is \( q-1 \) (cf. Proposition 3.4). This implies that the function \( \frac{h^{q^a}}{\Delta_T^{q^a}} \) is holomorphic on \( \Omega \cup \{0\} \) with order of vanishing at 0 is \( q \) and has a pole only at \( \infty \).

By Proposition 3.8, we get \( \frac{f}{\Delta_T} \in M_{k,l,0}(\Gamma_0(T)) \) is a holomorphic on \( \Omega \cup \{0,\infty\} \).

Using the above properties, we get the function \( G(a) \) is a meromorphic Drinfeld modular form of weight 2 and type 1 for \( \Gamma_0(T) \). Moreover, the function \( G(a) \) is holomorphic on \( \Omega \cup \{0\} \) with order of vanishing at least \( q \) at 0 and has a pole only at \( \infty \). In particular, by Proposition 3.9, we obtain

- \( \text{Res}_P G(a)(z)dz = 0 \) for \( P \in \Gamma_0(T) \setminus \Omega \), \( \text{Res}_0 G(a)(z)dz = 0 \),

- \( \text{Res}_\infty G(a)(z)dz = -\frac{a G(a)(1)}{\Delta_T^{\delta_E a}}, \) where \( a G(a)(1) \) is the coefficient of \( u \) in the \( u \)-series expansion of \( G \) at \( \infty \).

By Theorem 3.2, we get \( \text{Res}_\infty G(a)(z)dz = 0 \) and hence \( a G(a)(1) = 0 \). By Proposition 3.10 (3)&(4) we have \( h = -\frac{E_T}{\Delta_T^p} \). Consequently, \( G(a) = -\frac{a G(a)(1)}{\Delta_T^{\delta_E a}} \). Since \( g_d \equiv 1 \) (mod \( T^{q^a} - T \)), we get

\[
G(a) = \frac{g_d^a E_T^{q^a-1} f}{\Delta_T^{r_{k,l} + a} T^{q^a} + 2} \equiv \frac{E_T^{q^a-1} f}{\Delta_T^{r_{k,l} + a} T^{q^a} + 2} \quad (\text{mod } T^{q^a} - T). \tag{3.1}
\]

Now, we begin the proof of the theorem. By hypothesis, let \( b \in \mathbb{N} \) such that \( r_{k,l} + a \frac{p^a}{q^a} + 2 = p^b \) holds for some \( a \geq 0 \). With this choice of \( a \), (3.1) becomes

\[
G(a) = \frac{g_d^a E_T^{q^a-1} f}{\Delta_T^{p^a}} \equiv \frac{E_T^{q^a-1} f}{\Delta_T^{p^a}} \quad (\text{mod } T^{q^a} - T). \tag{3.2}
\]

Since \( \frac{1}{\Delta_T^{p^a}} = u^{-p^a(q-1)}(1-u^{p^a(q-1)^2}+\ldots) \), an easy computation shows that the coefficient of \( u \) in the \( u \)-series expansion of \( \frac{E_T^{q^a-1} f}{\Delta_T^{p^a}} \) at \( \infty \) is \( a f E_T^{q^a-1}(p^b(q-1)+1) \). By (3.2), we obtain \( a f E_T^{q^a-1}(p^b(q-1)+1) \equiv - (\text{mod } T^{q^a} - T) \). Since \( a G(a)(1) = 0 \), we are done.

Remark 3.5. When \( d = 1 \), the existence of \( a \geq 0 \) with \( r_{k,l} + a + 2 = p^b \) is automatic if \( b \gg 0 \).

Theorem 3.4 is a generalization of [Cho09, Theorem 3.4] from trivial type to arbitrary type.

Now, we have a corollary of Theorem 3.4

Corollary 3.6. Let \( f \in M_{k,l}(\Gamma_0(T)) \) be a Drinfeld modular form such that the \( u \)-series expansion at \( \infty \) is given by \( f = \sum_{j \geq 0} a_f(i(q-1) + l)u^{j(q-1) + 1} \in A[[u]] \). If \( p^a \mid l \) for some \( a \in \mathbb{N} \) and \( m \leq \alpha \) be a natural number such that \( p^m > r_{k,l} + 1 \), then the congruence

\[
a_f((p^m - 1)(q-1) + l) \equiv 0 \quad (\text{mod } T^{q^a} - T) \text{ holds.}
\]

Remark 3.7. Corollary [Cho09 Corollary 3.5] is a special case of Corollary 3.6 for \( l = 0 \) and \( m = 1 \). In fact, the assumption \( q \gg p+1 \) in Corollary loc.cit. is redundant.

Proof of Corollary 3.6 Let \( q-1 = p^a x \) for some \( x \in \mathbb{N} \). Now

\[
\sum_{n \geq 0} a_f E_T^{p^a x}(n(q-1) + 1)u^{n(q-1)+1} = f E_T^{p^a x}
\]

\[
= \sum_{j \geq 0} a_f(j(q-1) + l)u^{j(q-1)+1} \cdot \left( \sum_{i \geq 0} (a_E(i(q-1) + 1))^{p^a} u^{p^a(i(q-1)+i+1)} \right)^x.
\]

By comparing the coefficients of \( u^{p^a(q-1)+1} \) on both sides, we get

\[
a_f E_T^{q^a-1}(p^m(q-1) + l) = a_f((p^m - 1)(q-1) + l) a_E(1)^{p^a} = a_f((p^m - 1)(q-1) + l),
\]
since \( a_{E^i}(1) = 1 \). The first equality follows from the fact that \( p^\alpha(q - 1) + p^\beta > p^\gamma(q - 1) + 1 \). Now, the result follows from Theorem 3.3 by taking \( d = 1, b = m \).  

Now, we shall give an example satisfying Corollary 3.6

**Example 3.8.** Let \( p = 3, q = 3^2, l = 6 \) and \( k = 12 \), the assumptions of Corollary 3.6 are satisfied and the space \( M_{12,0}(\Gamma_0(T)) \) is generated by \( E^0_7 \). Hence, by Corollary 3.6 we get that \( a_{E^7}(p - 1)(q - 1) + 6) \equiv 0 \pmod {(T^q - T)} \). In fact, a straight forward calculation shows that \( a_{E^7}((p - 1)(q - 1) + pm) = 0 \) for any \( m \in \mathbb{N} \) with \( pm < q - 1 \).

We can also produce infinitely many examples satisfying Corollary 3.6 as follows:

**Example 3.9.** Suppose \( p|l \). Consider the function \( g^{-2}_l \in M_{(q - 1)(p - 1) + 2l, l} \). Since \( p > r_{(q - 1)(p - 1) + 2l, l} + 1 \), we get \( a_{g^{-2}_l}((p - 1)(q - 1) + 1) \equiv 0 \pmod {(T^q - T)} \) by Corollary 3.6

### 4. Linear relations between the initial Fourier coefficients

In this section, we determine all the linear relations between the initial coefficients in \( u \)-series expansion of Drinfeld modular forms of any weight, arbitrary type for \( \Gamma_0(T) \). For any integer \( i \geq 0 \), consider the map

\[
a^*_i : M_{k,l}(\Gamma_0(T)) \to C \quad \text{defined by } f \mapsto a_f(i(q - 1) + 1),
\]

where \( f = \sum_{0 \leq j \leq (\text{mod } q - 1)} a_f(j)u^j \). Then \( a^*_i \in (M_{k,l}(\Gamma_0(T)))^* \), the dual of \( M_{k,l}(\Gamma_0(T)) \). The elements of \( S \) in Proposition 3.12 satisfy \( a^*_i(\Delta_{V_k}^{(l)}) = 0 \) for \( i < j \), which implies

**Proposition 4.1.** The set \( \{a^*_i\}_{i=0}^{\infty} \) forms a basis for the \( C \)-vector space \( (M_{k,l}(\Gamma_0(T)))^* \).

For any integer \( N \geq 0 \), consider the surjective map

\[
\psi_{k,l,N} : C^{r_{k,l,N}+1} \to (M_{k,l}(\Gamma_0(T)))^* \quad \text{defined by } (c_0, c_1, \ldots, c_{r_{k,l,N}}) \mapsto \sum_{i=0}^{r_{k,l,N}} c_i a^*_i.
\]

**Definition 4.2.** For \( k \in \mathbb{N}, l \in \mathbb{Z}/(q-1)\mathbb{Z} \) such that \( k \equiv 2l \pmod {q-1} \), we define \( L_{k,l,N}(\Gamma_0(T)) := \text{Ker}(\psi_{k,l,N}). \) By Proposition 4.1, \( \dim C L_{k,l,N}(\Gamma_0(T)) = N + 1 \).

Now, we state the main result of this section, which is a generalization of [Cho09, Theorem 4.1] from trivial type to arbitrary type. For each Drinfeld modular form \( g \in M_{N(q-1)+2l, l}(\Gamma_0(T)) \), we define the elements \( b(k,l,N,g;i), c(k,l,N,g;i) \in C \) by the \( u \)-series expansion

\[
\sum_{i=0}^{r_{k,l,N}} b(k,l,N,g;i)u^{(q-1)i+1-l} + \sum_{i=1}^\infty c(k,l,N,g;i)u^{(q-1)i+1-l}.
\]

**Theorem 4.3.** The map \( \phi_{k,l,N} : M_{N(q-1)+2l, l}(\Gamma_0(T)) \to L_{k,l,N}(\Gamma_0(T)) \) defined by \( \phi_{k,l,N}(g) = (b(k,l,N,g;0), b(k,l,N,g;1), \ldots, b(k,l,N,g; r_{k,l,N})) \) is an isomorphism of \( C \)-vector spaces.

**Proof.** First we show that the image of \( \phi_{k,l,N} \) belongs to \( L_{k,l,N}(\Gamma_0(T)) \). Let \( g \in M_{N(q-1)+2l, l}(\Gamma_0(T)) \). Then, for any \( f = \sum_{j=0}^{\infty} a_f(j(q - 1) + l)u^{(q-1)i+1} \in M_{k,l}(\Gamma_0(T)) \), consider the function \( G := \frac{hg}{\Delta_{V_k}^{(l)}} \). Then

\[
f = \frac{hg}{\Delta_{V_k}^{(l)}} \frac{f}{E^l}.
\]

- The function \( h \) vanishes exactly once (resp., \( q \)-times) at \( \infty \) (resp., at \( 0 \)) and \( \Delta_{V_k} \) vanishes only at \( \infty \) and the order of vanishing is \( q - 1 \) (Proposition 3.11). This implies that \( \frac{hg}{\Delta_{V_k}^{(l)}} \) is holomorphic on \( \Omega \cup \{0\} \) with order of vanishing \( q - 1 \) at 0 and a pole only at \( \infty \).

- By Proposition 3.13, \( \frac{f}{E^l} \in M_{N(q-1),0}(\Gamma_0(T)) \) and \( \frac{f}{E^l} \in M_{k-2l,0}(\Gamma_0(T)) \)

These properties imply that the function \( G \) is a meromorphic Drinfeld modular form of weight 2 and type 1 for \( \Gamma_0(T) \). Moreover, the function \( G \) is holomorphic on \( \Omega \cup \{0\} \) with order of vanishing at least \( q \) at 0 and has a pole only at \( \infty \). Note that, the coefficient of \( u \) in the \( u \)-series expansion of \( G \) is \( \sum_{i=0}^{r_{k,l,N}} b(k,l,N,g;i)a_f(i(q - 1) + l) \), which is equal to 0 by Theorem 3.2 and Proposition 3.3. Thus the image of \( \phi_{k,l,N} \) belongs to \( L_{k,l,N}(\Gamma_0(T)) \). Clearly, \( \phi_{k,l,N} \) is linear.
Now, we show that φ_{k,l,N} is injective. Suppose that φ_{k,l,N}(g) = 0 for some g ∈ M_{N(q-1)+2l,N}(Γ_0(T)). For any f ∈ M_{k,l}(Γ_0(T)), by (1.1), we get
\[ \frac{hg}{\Delta_{k,l,N}E_T^2} f = \sum_{i=1}^{\infty} c(k,l,N,g;i)a_i^{(q-1)+1-i} + \sum_{j=0}^{\infty} a_f(j-1+l)a_i^{(q-1)+1}. \tag{4.2} \]
Therefore \( \frac{hg}{\Delta_{k,l,N}E_T^2} \) is a doubly cuspidal holomorphic Drinfeld modular form of weight 2, type 1 for Γ_0(T). Since the modular curve X_0(T) has genus 0, the function \( \frac{hg}{\Delta_{k,l,N}E_T^2} \) is identically zero. This forces g = 0. Hence the map φ_{k,l,N} is injective. Since the dimensions of \( L_{k,l,N}(Γ_0(T)) \), \( M_{N(q-1)+2l,N}(Γ_0(T)) \) are equal, the map φ_{k,l,N} is an isomorphism. \( \square \)

Theorem above and Proposition 3.1((2) and (3)), we get

**Corollary 4.4.** The C-vector spaces \( L_{k,l,N}(Γ_0(T)) \) and \( L_{-2l,0,N}(Γ_0(T)) \) are isomorphic.

We conclude the article with a remark that one can extend [Cho08 Theorem 4.1] for GL_2(A) from trivial type to arbitrary type by replacing the function \( \frac{g^{q-\alpha}}{\Delta_{k,l,N}E_T^2} \) in [Cho08 Page 99] by \( \frac{g^{q-\alpha}}{\Delta_{k,l,N}E_T^2} u \) in [Cho08 Page 99] where \( f ∈ M_{N(q^2-1)+l(q+1),N}(GL_2(A)) \) and \( \beta = \frac{a}{q-1} \), for some \( \frac{a}{q-1} \neq 0 \).

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