ON ISOSPECTRAL METRIC GRAPHS

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Abstract. A new class of isospectral graphs is presented. These graphs are isospectral with respect to both the normalised Laplacian on the discrete graph and the standard differential Laplacian on the corresponding metric graph. The new class of graphs is obtained by gluing together subgraphs with the Steklov maps possessing special properties. It turns out that isospectrality is related to the degeneracy of the Steklov eigenvalues.

1. Introduction

Spectral geometry of metric graphs is a rapidly developing area of modern analysis joining together spectral theory of ordinary and partial differential equations, theory of almost periodic functions, quasicrystals, analytic functions in several variables with number theory and algebra of multivariate polynomials \cite{8,19,23}. One of the most important directions is understanding how topological and geometrical properties of a metric graph \( \Gamma \) (see precise definitions below) are reflected in the spectral properties of the corresponding differential operators. The first step should be understanding these relations in the case of the standard Laplacian – the second order differential operator uniquely determined by the metric graph. We shall refer to its spectrum as the spectrum of the metric graph \( \Gamma \). Solution of the inverse spectral problem is then identical to the reconstruction of a metric graph from its spectrum. Note that we are interested in what is determined by the spectrum alone without taking into account additional (spectral) data like the response operator, scattering matrix or the spectrum of any other operator associated with the same graph.

The following two results should be taken into account:

- The spectrum determines the metric graph if the edge lengths are rationally independent \cite{13,21}.
- The total length (\( \text{i.e.} \) the sum of edge lengths) and the Euler characteristic (or the number of independent cycles) are uniquely determined by the spectrum \cite{16,17}.

The first result implies that almost all graphs are uniquely determined by their spectra. Looking for isospectral graphs (sometimes called cospectral) – graphs having identical spectra – one should consider the case of rationally dependent edge lengths. In particular one may look at the another extreme case of unilateral graphs (having all edge lengths equal to 1). Explicit examples of isospectral graphs have been constructed adjusting Sunada construction \cite{28} originally developed for partial
differential equations in domains to metric graphs \[2,13,26\]. In this approach two isospectral graphs appear as subgraphs of a certain large symmetric graph. Discussing isospectral graphs one should always remember that the spectra of unilateral metric graphs are determined by the spectra of the corresponding discrete normalised Laplacian matrices:

\[
L_N = I - D^{-1/2}AD^{-1/2},
\]

where \(A\) is the adjacency matrix for the discrete graph \(G\) associated with \(\Gamma\) and \(D\) is the (diagonal) degree matrix for \(G\). Generic (i.e. not equal to \(\pi^2m^2, \ m \in \mathbb{Z}\)) eigenvalues \(\lambda_j\) of an unilateral metric graph \(\Gamma\) and the eigenvalues \(\mu_n\) of the normalised Laplacian matrix \(L_N\) are related as \[5\]

\[
1 - \cos \sqrt{\lambda_j} = \mu_n.
\]

Moreover, two unilateral metric graphs are isospectral if and only if the corresponding normalised Laplacian matrices are isospectral and the graphs have the same number of connected components and independent cycles. In particular isospectrality of normalised Laplacians does not necessarily imply isospectrality of the corresponding unilateral metric graphs.

In the current article we shall look for isospectral unilateral metric graphs. It is not our goal to characterise all such graphs but rather to understand their nature. To the best of our knowledge, the problem has not been solved neither for normalised Laplacian matrices \[11\], nor for usual Laplacian matrices \[14\]. The spectra of unilateral metric graphs on up to 5 vertices have been discussed in \[12\], it was proven that the spectra determine the graphs.

We turned therefore to graphs on 6 vertices and looked for isospectral pairs. We found one such pair (see Section 3) and analysed the corresponding eigenfunctions trying to find any transplantation mapping then to each other. Deep analysis of this pair of graphs led us to interesting observations allowing to construct a wide family of isospectral metric graphs. Surprisingly this family gives not known before normalised-Laplacian-isospectral discrete graphs.

Our methods are based on the analysis of Titchmarsh-Weyl \(M\)-functions associated with the subgraphs and use ideas behind surgery principles in the spectral theory of metric graphs \[6,7,20,22\]. In contrast to the studies modifying Sunada construction derived isospectral families are glued together from the subgraphs having very special spectral properties. In some sense our approach is ideologically close to \[10,11\], but uses a different technique: instead of extensiv use of linear algebra we bring in methods coming from the spectral theory of ordinary differential operators. As a result constructed isospectral families are wider and often contain already known ones. On the other hand it is not always possible to see the equivalence between already existing and newly proposed methods: for example our Method \[2\] reminds of Seidel switching \[10,27\] but is not equivalent. Construction of isospectral graphs from Cayley graphs \[1,25\] can also be generalised introducing symmetries based on \(M\)-functions, the corresponding graphs do not necessarily have any symmetries as metric spaces.

2. LAPLACIANS ON METRIC GRAPHS AND \(M\)-FUNCTIONS

A metric graph \(\Gamma\) is a collection of intervals \(E_n = [x_{2n-1}, x_{2n}] \subset \mathbb{R}, \ n = 1, 2, \ldots, N\) – the edges, – joined together at the vertices \(V_m, \ m = 1, 2, \ldots, M\)
seen as equivalence classes of the set of all interval end points \(\{x_j\}_{j=1}^N\). The intervals belong to different copies of \(\mathbb{R}\). The end points belonging to the same vertex are identified.

The (differential) Laplace operator \(L\) is defined in the Hilbert space \(L^2(\Gamma) = \bigoplus_{n=1}^N L^2(E_n)\) on the set of functions from \(u \in \bigoplus_{n=1}^N W^2(E_n)\) satisfying at each vertex \(V_m\) standard conditions

\[
\begin{cases}
  x_i, x_j \in V_m \Rightarrow u(x_i) = u(x_j); \\
  \partial u(V_m) := \sum_{x_j \in V_m} \partial u(x_j) = 0,
\end{cases}
\]

where the oriented derivatives are defined in accordance with

\[
\partial u(x_{2n-1}) = \lim_{x \to x_{2n-1}} u'(x), \quad \partial u(x_{2n}) = -\lim_{x \to x_{2n}} u'(x).
\]

The extra sign for the right end points is required in order to treat all edges independently of their parametrisation. The first condition in (2) implies that any function from the domain of the standard Laplacian is not only continuous inside the edges (as a function from \(W^2\)) but even on the whole graph \(\Gamma\). The second condition, sometimes called Kirchhoff, ensures that the operator is self-adjoint. It is not our goal to describe all possible vertex conditions (see [8, 15, 19]). Additionally we shall only use Dirichlet condition \(u(V_m) = 0\) implying that the function is zero at all end-points joined at \(V_m\).

On any metric graph \(\Gamma\) we choose one or several vertices to be contact vertices denoted by \(\partial \Gamma\). Note that every inner point on an edge can be seen as a degree two vertex. For arbitrary nonreal \(\lambda\), \(\Im \lambda \neq 0\) consider solutions of the differential equation

\[
-\dot{u''}(x) = \lambda u(x)
\]

satisfying standard vertex conditions at all (inner) vertices \(V_m \notin \partial \Gamma\) and continuous on \(\partial \Gamma\) (and hence on the whole \(\Gamma\)). Every such solution is uniquely determined by its values on the contact set \(u(V_m)\), \(V_m \in \partial \Gamma\). Otherwise the Laplacian with Dirichlet conditions on \(\partial \Gamma\) (and standard conditions at all other vertices) would have non-real eigenvalues. We introduce the Titchmarsh-Weyl (matrix valued) \(M\)-function associated with the graph \(\Gamma\) and the contact set \(\partial \Gamma\):

\[
M_\Gamma(\lambda) : \{u(V_m)\}_{V_m \in \partial \Gamma} \mapsto \{\partial u(V_m)\}_{V_m \in \partial \Gamma},
\]

where \(\partial u(V_m) = \sum_{x_j \in V_m} \partial u(x_j)\). \(M_\Gamma(\lambda)\) so defined is a matrix-valued Herglotz-Nevanlinna function and encodes information about all not-common eigenfunctions of the standard and Dirichlet Laplacians on \(\Gamma\) [19].

For example the \(M\)-function for the interval \(E\) of length 1 with the contact set formed by both end-points is given by

\[
M_E(\lambda) = \begin{pmatrix}
-k \cot k & k \\
-k \cot k & \sin k \\
\sin k & -k \cot k
\end{pmatrix},
\]

The following two formulas express the \(M\)-functions through the traces of the standard (satisfying standard vertex conditions on \(\partial \Gamma\) and Dirichlet (satisfying Dirichlet
conditions on $\partial \Gamma$ [18,19,22]

\[
M_\Gamma(\lambda) = -\left(\sum_{n=1}^{\infty} \frac{\langle \psi_{st}^n | \partial \Gamma, \cdot \rangle_{L^2(\partial \Gamma)} \psi_{st}^n \mid \partial \Gamma \rangle}{\lambda_{st}^n - \lambda}\right)^{-1},
\]

\[
M_\Gamma(\lambda) - M_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_{st}^n - \lambda)(\lambda_{D}^n - \lambda')} \langle \partial \psi_{D}^n \mid \partial \Gamma, \cdot \rangle_{C^B \partial \psi_{D}^n \mid \partial \Gamma},
\]

where $\lambda_{st}^n$ and $\lambda_{D}^n$ are the corresponding eigenvalues and $\lambda' \in \mathbb{R}$ is an arbitrary reference point not belonging to the spectra. The Dirichlet eigenvalues $\lambda_{D}^n$ determine the singularities of $M_\Gamma(\lambda)$, while $\lambda_{st}^n$ correspond to generalised zeroes of $M_\Gamma(\lambda)$. It is clear that only the eigenfunctions with non-zero traces on the contact set contribute to the $M$-function. Such eigenfunctions are called detectable and the corresponding eigenvalues form detectable spectrum. All other eigenfunctions and the corresponding spectrum are called non-detectable or invisible. Non-detectable eigenfunctions satisfy both standard and Dirichlet conditions at contact vertices. Note that we treat the spectrum as a multiset so that the same real number could be a detectable and a non-detectable eigenvalue at the same time if it is a multiple eigenvalue with some standard eigenfunctions satisfying Dirichlet conditions on $\partial \Gamma$ and some not.

The $M$-functions are Hermitian for almost all real values of $\lambda$ as can be seen from the formulas (5) and (6). The eigenvalues of $M_\Gamma(\lambda)$ will be called Steklov eigenvalues by analogy with the Poincaré-Steklov problem for elliptic partial differential equation in a domain. Note that so-defined Steklov eigenvalues depend on the spectral parameter $\lambda$ as well as the corresponding eigensubspaces, which we simply call Steklov subspaces. For regular $\lambda$ Steklov eigenvalues and subspaces depend continuously on the spectral parameter $\lambda$.

The graphs with equal $M$-functions will be called Steklov-equivalent. Such graphs are also known as isoscattering, isopolar, or isophasal graphs [3] as attaching infinite edges to contact points leads to the scattering matrix, which is essentially a Cayley transform of the $M$-function.

The Steklov eigenvalues between the singular points depend continuously on the spectral parameter $\lambda$ and are given by non-decreasing functions. The detectable eigenvalues of the standard Laplacian on $\Gamma$ are obtained when the Steklov eigenvalues cross the zero line. Note that these points may coincide with the singularities of $M(\lambda)$, therefore one speaks about generalised zeroes – the singularities of the inverse function $-M^{-1}(\lambda)$. Note that Steklov-equivalent graphs are not necessarily isospectral as their non-detectable eigenvalues may be different.

3. Two isospectral graphs from the complete graph $K_5$.

Searching for isospectral graphs we decided to inspect equilateral graphs on few vertices. The graphs on up to 5 vertices have been considered in [12]. We looked at graphs on 6 vertices aiming to find isospectral pairs. There are altogether more than 100 such graphs and it was a tough job to analyse their spectra. We present here just the result of these tedious calculations.
The spectra of unilateral metric graphs are best described by zeroes of secular polynomials [4, 19, 23] determined by

\[ P_G(z) = \det \left( E(z) - S_v \right) , \]  

where

\[ E(z) = \text{diag} \left( \begin{array}{cc} 0 & z \\ z & 0 \end{array} \right) \]

and \( S_v \) is the vertex scattering matrix. The spectrum of the Laplacian on the unilateral graph \( \Gamma \) is obtained by putting \( z = e^{ik} \) as zeros of the trigonometric polynomial

\[ p_\Gamma(k) = P_G(e^{ik}) . \]

Only zeros of the secular polynomials are relevant, hence these polynomials should be treated projectively so that two polynomials different by a multiplicative factor are equal.

For example the secular polynomial for the complete graph \( K_5 \) is

\[ P_{K_5} = (z - 1)^7(z + 1)^5(2z^2 + z + 2)^4 . \]

Order of the zeroes coincides with the multiplicity of the corresponding non-zero eigenvalues. For example \( z_1 = 1 \) is a zero of multiplicity 7 implying that \( \lambda = (2\pi m)^2 \) is an eigenvalue of multiplicity \( 7 = 1 + \beta_1(K_5) \), where \( \beta_1 \) denotes the number of independent cycles in the graphs (the first Betti number). The multiplicity of \( \lambda = 0 \) is just equal to 1 as the graph is connected.

**Figure 1.** Two isospectral graphs via \( K_5 \).
Let us consider the unilateral graphs $\Gamma_1$ and $\Gamma_2$ obtained from $K_5$ by chopping one of the vertices into two. The new vertices have degrees 2, 2 and 1, 3 respectively. The corresponding secular polynomials coincide,

$$P_{\Gamma_1}(z) = P_{\Gamma_2}(z) = (z - 1)^6(z + 1)^4(2z^2 + z + 2)^3(z^4 + z^3 + 2z^2 + z + 2),$$

implying that the two connected graphs are isospectral.

It would be natural to derive isospectrality of the pair $(\Gamma_1, \Gamma_2)$ using symmetries of $K_5$; we provide an alternative explanation looking at the $M$-functions and analysing the corresponding Steklov subspaces. In fact analysing the above explicit example we arrived at several methods allowing construction of non-trivial isospectral graphs. These methods are described in the following sections.

4. Understanding isospectrality looking at Steklov subspaces

In this section we provide an explicit explanation why the graphs $\Gamma_1$ and $\Gamma_2$ are isospectral looking at the Steklov subspaces of the two subgraphs that differ them. In some sense instead of analysing how the complete graph $K_5$ is chopped to get the pair, we shall analyse how to obtain the isospectral pair by extending the graph $K_4$.

Both graphs $\Gamma_1$ and $\Gamma_2$ contain the complete graph $K_4$. Hence each of these graphs can be obtained by attaching a certain equilateral graph $Q_j$ to $K_4$. Let us amend Figure 1 so that our construction will become more transparent.

The spectra of these graphs can be calculated using their $M$-functions associated with the 4 vertices belonging to the $K_4$ subgraph on the left.

All three graphs in Fig. 2 are divided by contact vertices into two or three separate graphs: $K_4$ and star graphs $S_d$ with $d = 1, 2, 3, 4$ edges. The $M$-functions for the composed graphs are equal to the sums of $M$-functions of the subgraphs.

Let us calculate the $M$-functions for the subgraphs appearing in the decomposition. Each time we shall use that the $M$-functions commute with the permutations of the end points implying that the Steklov subspaces are invariant under permutations and can only be equal to

$$\mathcal{L} \mathbf{1} \quad \text{and} \quad (\mathcal{L} \mathbf{1})^\perp,$$

where $\mathbf{1}$ denotes the vector with all coordinates equal to 1 and the size depending on the number of contact vertices in the subgraph. Hence, there are just two Steklov eigenvalues for each subgraph.
Complete graph $K_4$. The $M$-function associated with all vertices in $K_4$ is equal to the sum of $M$-functions for the edges:

$$
M_{K_4}(\lambda) = \begin{pmatrix}
-3k \cot k & k & k & k \\
k & -3k \cot k & k & k \\
k & k & -3k \cot k & k \\
k & k & k & -3k \cot k
\end{pmatrix}.
$$

Of course one needs to pay attention to which contact points are joined by each edge. The Steklov eigenvalues denoted by $\mu^{K_4}_{1,2}$ are

$$
\mu^{K_4}_1 = -3k \cot k + 3\frac{k}{\sin k} = 3k \tan k/2, \quad \mu^{K_4}_2 = -3k \cot k - \frac{k}{\sin k}.
$$

Denoting by $P_1$ and $P_{1\perp}$ the orthogonal projectors on the Steklov subspaces, the $M$-function can be written as

$$
M_{K_4}(\lambda) = \mu^{K_4}_1(\lambda)P_1 + \mu^{K_4}_2(\lambda)P_{1\perp}.
$$

Star graph $S_d$. The eigenvalues of the $M$-function for the star graph $S_d$ associated with all degree one vertices do not depend on the valency $d$

$$
\mu^S_1(\lambda) = k \tan k, \quad \mu^S_2(\lambda) = -k \cot k.
$$

The corresponding $M$-function also commutes with the permutations and therefore can be written similar to (12):

$$
M_{S_d}(\lambda) = \mu^S_1(\lambda)P_1 + \mu^S_2(\lambda)P_{1\perp}.
$$

The spectrum of $K_5$ associated with the eigenfunctions not identically equal to zero at the contact vertices is given by the equation

$$
\det \left( M_{K_5}(\lambda) + M_{S_4}(\lambda) \right) = 0,
$$

which, taking into account (12) and (13), reduces to

$$
\begin{cases}
\mu^K_1 + \mu^S_1 = 3k \tan k/2 + k \tan k = 0, \\
\mu^K_2 + \mu^S_2 = -4k \cot k - \frac{k}{\sin k} = 0.
\end{cases}
$$

We get scalar equations as the eigensubspaces for $M_{K_4}(\lambda)$ and $M_{S_4}(\lambda)$ coincide. The first equation determines simple eigenvalues, while the second one leads to triple ones. Additional eigenvalues are due to eigenfunctions equal to zero at all contact points. Such eigenvalues are of the form $(2\pi m)^2, m = 1, 2, \ldots$ and have multiplicity $\beta_1(K_5) = 6$. 


Let us turn to the graphs $\Gamma_j$. The $M$-functions for $Q_1$ and $Q_2$ are expressed using the $M$-functions of 1, 2, and 3-star graphs:

\[
\mathbb{M}_{Q_1}(\lambda) = \mu_1^S P_{(1,1,0,0)} + \mu_2^S P_{(1,-1,0,0)} + \mu_1^S P_{(0,0,1,1)} + \mu_2^S P_{(0,0,1,-1)} \\
\mathbb{M}_{Q_2}(\lambda) = \mu_1^S \left( P_{(1,1,1,1)} + P_{(1,1,-1,-1)} \right) + \mu_2^S \left( P_{(1,-1,0,0)} + P_{(0,0,1,-1)} \right),
\]

The main difference to the previous example is that both eigenvalues have multiplicity 2. The corresponding Steklov eigensubspaces are different, also the vectors $I = (1,1,1,1)$ and $(0,0,1,-1)$ are common for $Q_1$ and $Q_2$. The two $M$-functions can be obtained from each other by swapping the Steklov subspaces. The detectable spectrum is given by

\[
\det \left( \mathbb{M}^{K_j}(\lambda) + \mathbb{M}^{Q_j}(\lambda) \right) = 0.
\]

The eigensubspaces corresponding to the four (non-distinct) eigenvalues of $M^{K_4}(\lambda)$ and each $M_{Q_j}(\lambda)$ can be chosen equal, also these subspaces depend on $j = 1, 2$. Therefore the above equation reduces to three scalar equations

\[
\begin{cases}
\mu_1^{K_1} + \mu_1^S = 3k \tan k/2 + k \tan k = 0, \\
\mu_2^{K_1} + \mu_1^S = -3k \cot k - \frac{k}{\sin k} + k \tan k = 0, \\
\mu_2^{K_1} + \mu_2^S = -4k \cot k - \frac{k}{\sin k} = 0.
\end{cases}
\]

The first two equations determine simple eigenvalues while the third one - double. The equations are identical for the two graphs, but the corresponding subspaces are different:

\[
\Gamma_1 : \mathcal{L}\{(1, 1, 1, 1)\}, \quad \mathcal{L}\{(1, 1, 1, 1)\}, \\
\mathcal{L}\{(1, -1, -1, 1)\}, \quad \mathcal{L}\{(3, -1, -1, 1)\}, \\
\mathcal{L}\{(1, -1, 0, 0)\}, \quad \mathcal{L}\{(0, 2, -1, -1)\}; \\
\Gamma_2 : \mathcal{L}\{(1, 1, 1, 1)\}, \quad \mathcal{L}\{(1, 1, 1, 1)\}, \\
\mathcal{L}\{(3, -1, -1, 1)\}, \quad \mathcal{L}\{(0, 2, -1, -1)\}, \quad \mathcal{L}\{(0, 0, 1, -1)\}, \quad \mathcal{L}\{(0, 0, 1, -1)\}.
\]

The invisible spectrum is formed by the eigenvalues $(2\pi m)^2$, $m = 1, 2, \ldots$ with the multiplicity equal to the genus of the two graphs $\beta_1(\Gamma_1) = \beta_1(\Gamma_2) = 5$. Both $M$-functions for $\Gamma_1$ and $\Gamma_2$ are obtained from the $M$-function for $S_4$ by changing the eigenvalue on a certain one-dimensional eigensubspace from $\mu_2^S$ to $\mu_1^S$. Note that the one-dimensional subspaces are different for $\Gamma_1$ and $\Gamma_2$ but this does not affect the spectrum of the composed graph. So the mechanism behind isospectrality of this pair is as follows: the $M$-function for the original graph $(K_5$ in our example) is given by a sum of the $M$-functions for two subgraphs sharing the same Steklov eigensubspace, one having dimension at least 2, the $M$-functions for the isospectral pair are obtained by amending the eigenvalue on a certain one-dimensional subspace of the distinguished Steklov eigensubspace. The key point is that such procedure gives $M$-functions associated with metric graphs obtained by chopping the original graph.

Example 1. The graph $K_4$ can be substituted with the graph $S_4$, since the only property of $K_4$ which was used is that the $M$-function commutes with the rotations. This gives us two isospectral graphs $\Gamma_1'$ and $\Gamma_2'$ presented in Fig. 3. One may draw
these graphs in a slightly better way as presented in Fig. 3, indicating that this is probably the simplest pair of isospectral connected metric graphs.

In what follows we formulate several principles which can be used to construct isospectral graphs.

5. Method 1: Extending Isospectral Graphs

The method below requires that one isospectral pair is already known, but allows one to get an infinite family of isospectral pairs by extension.

**Method 1** (Extending of isospectral pairs). Assume that $R_1$ and $R_2$ are two Steklov-equivalent graphs (with identical $M$-functions), which in addition are isospectral. Then gluing these graphs to an arbitrary graph $K$ yields a new pair of isospectral graphs.

With this observation in mind we shall always try to identify common part in any isospectral pair and check whether there exists a simpler isospectral pair behind. We prove first one Lemma which will be used several times below.

**Lemma 1.** Assume that $M$-functions $M_{\Gamma_1}(\lambda)$ for two metric graphs $\Gamma_1$ and $\Gamma_2$ are unitary equivalent

\[ M_{\Gamma_1}(\lambda) \overset{\text{unitary}}{\sim} M_{\Gamma_2}(\lambda). \]

Then the graphs are isospectral if and only if the graphs have the same non-detectable spectra.
Proof. It is implicit from the statement of the Lemma that the contact sets for $\Gamma_1$ and $\Gamma_2$ have the same dimension. Unitary equivalence of the $M$-functions together with the explicit formula (3) imply that the detectable spectra of $\Gamma_1$ and $\Gamma_2$ coincide. Then isospectrality of the graphs imply is equivalent to the coincidence of the non-detectable spectra. Remember that the spectrum is treated as a multiset.

Proof of Method 1. We denote by $\Gamma_j$, $j = 1, 2$, the graphs obtained by gluing $K$ and $R_j$. To prove that the graphs $\Gamma_j$ are isospectral it is convenient to divide the eigenfunctions and hence the spectra $\Sigma(\Gamma)$ into two sets:

- Localised eigenfunctions – the eigenfunctions supported by the subgraphs $R_j$;
- Delocalised eigenfunctions – the eigenfunctions having support not localised to $R_j$.

If the eigenvalue is simple, then the eigenfunction is unique and it is clear whether it is localised or not. In the case of multiple eigenvalues $\lambda_j$ one should examine whether the corresponding eigenfunctions can be chosen localised to $R_j$ or not. Then the multiplicity $\text{dim}_{\text{loc}}(\lambda_j)$ of the localised eigenvalue is the maximal number of linearly independent localised eigenfunctions and coincides with the multiplicity of non-detectable eigenvalues on $R_j$. The complementing dimension determines the nonlocalised multiplicity $\text{dim}_{\text{n/loc}}(\lambda_j)$:

$$\text{dim}_{\text{n/loc}}(\lambda_j) = \text{dim}(\lambda_j) - \text{dim}_{\text{loc}}(\lambda_j).$$

The invisible eigenfunctions on $R_1$ and $R_2$ have the same eigenvalues including multiplicities by Lemma 1. It follows that the localised spectra of $\Gamma_1$ and $\Gamma_2$ coincide as the corresponding eigenfunctions are just invisible eigenfunctions on $R_1$ and $R_2$ respectively, extended by zero to the rest of $\Gamma_j$.

Let us prove that the spectra corresponding to delocalised eigenfunctions coincide as well. Consider any such eigenfunction $\psi_{\lambda_j}$ on $\Gamma_1$. Then an eigenfunction $\psi'_{\lambda_j}$ on $\Gamma_2$ can be constructed having the same values on $K$:

$$\psi_{\lambda_j}|_K = \psi'_{\lambda_j}|_K.$$ 

The function is extended to $R_2 \subset \Gamma_2$ having the same values on $\partial R_2$ and therefore having the same derivatives on $\partial R_2$ as $R_1$ and $R_2$ have identical $M$-functions. It follows that $\psi'_{\lambda_j}$ constructed in this way satisfies standard vertex conditions everywhere on $\Gamma_2$. It is clear that this map is injective and the roles of $\Gamma_1$ and $\Gamma_2$ can be exchanged, hence even multiplicities of the eigenvalues coincide. □

6. Method 2: Exchanging Steklov-equivalent subgraphs

Method 2 (Exchanging Steklov-equivalent subgraphs). Assume that a metric graph $\Gamma$ contains two subgraphs $R_1$ and $R_2$ with equal $M$-functions:

$$M_{R_1}(\lambda) = M_{R_2}(\lambda).$$

Then exchanging the subgraphs $R_1$ and $R_2$ one obtains a graph $\Gamma'$ isospectral to $\Gamma$.

Note that we do not require that the subgraphs $R_1$ and $R_2$ are isospectral.

Proof. As before we divide the eigenfunctions into localised and delocalised ones. The localised eigenfunctions $\psi_{\lambda_j}$ and $\psi'_{\lambda_j}$ on $\Gamma$ and $\Gamma'$ respectively, are essentially the same – they are mapped in accordance to:

$$\psi'_{\lambda_j}|_{R_i} = \psi_{\lambda_j}|_{R_i}.$$
Note that in the formula above we identified the points belonging to the same subgraphs $R_i$ belonging to $\Gamma$ and $\Gamma'$. The supports of localised eigenfunctions move together with the subgraphs $R_j$.

The delocalised eigenfunctions are identical outside the subgraphs

$$\psi'_{\lambda_j}|_{\Gamma\setminus(R_1\cap R_2)} = \psi_{\lambda_j}|_{\Gamma\setminus(R_1\cap R_2)},$$

where we identified points in $\Gamma$ and $\Gamma'$ outside the subgraphs. The function $\psi'_{\lambda_j}$ is continued inside the subgraphs $R_j$ having the same boundary values at the contact points $\partial R_1, 2$

$$\psi'_{\lambda_j}|_{\partial R_1} = \psi_{\lambda_j}|_{\partial R_2}, \quad \psi'_{\lambda_j}|_{\partial R_2} = \psi_{\lambda_j}|_{\partial R_1}.$$

For each of the subgraphs there is always a solution of (3) having the above values at the contact points. Even if the $M$-function is singular the values at contact points satisfy the conditions ensuring solvability of the problem. As the subgraphs have equal $M$-functions, we have the same relation for the derivatives

$$\partial \psi'_{\lambda_j}|_{\partial R_1} = \partial \psi_{\lambda_j}|_{\partial R_2}, \quad \partial \psi'_{\lambda_j}|_{\partial R_2} = \partial \psi_{\lambda_j}|_{\partial R_1},$$

and the extended function is an eigenfunction on $\Gamma'$.

The mappings of the localised and non-localised eigenfunctions are clearly injective and the roles of $\Gamma$ and $\Gamma'$ can be exchanged, hence these graphs are isospectral.

Steklov-equivalent graphs. Probably the simplest pair of Steklov-equivalent graphs can be obtained by modifying our example of isospectral graphs (see Fig. 5).

One can find a lot of examples of Steklov-equivalent graphs in the literature where they are often called isoscattering graphs (see [3]). Note that most of these examples come from isospectral graphs implying that the graphs are not only Steklov-equivalent, but also share the same non-detectable spectrum.

![Figure 5. Steklov-equivalent metric graphs.](image-url)
Steklov-equivalent graphs via inner symmetries. Following [9] we consider the case where the graph $\mathbf{R}$ has a non-trivial (inner) symmetry $g$ not affecting the contact points:

\begin{equation}
 g \mathbf{R} = \mathbf{R}, \quad g \neq \mathbb{I}_\mathbf{R}, \quad g|_{\partial \mathbf{R}} = \mathbb{I}_{\partial \mathbf{R}}.
\end{equation}

where $\mathbb{I}$ denotes the identity transformations. Since the graph is finite, there exists an integer $n \geq 2$ such that $g^n = \mathbb{I}_\mathbf{R}$. Consider any solution $u$ to the differential equation (3) which is continuous on $\mathbf{R}$ and satisfies standard conditions at non-contact vertices $V_m \notin \partial \mathbf{R}$. It is clear that $gu, g^2u, \ldots, g^n u$ are also solutions. We introduce

\begin{equation}
 u_m(x) := \sum_{i=0}^{n} e^{\frac{2\pi i m}{n}} g^i u(x).
\end{equation}

It is clear that $u_0(x)$ is invariant under $g$

\begin{equation}
 gu_0(x) = u(x)
\end{equation}

while all other functions $u_m(x), m = 1, 2, \ldots, n - 1$ are quasi-invariant:

\begin{equation}
 gu_m(x) = e^{-\frac{2\pi i m}{n}} u_m(x).
\end{equation}

Then $g|_{\partial \mathbf{R}} = \mathbb{I}_{\partial \mathbf{R}}$ implies that

\begin{equation}
 u_m|_{\partial \mathbf{R}} = 0, \quad m = 1, 2, \ldots, n - 1.
\end{equation}

Therefore calculating $M_{\mathbf{R}}(\lambda)$ only the symmetric solution $u_0(x)$ may be used. Consider any point $x_0 \in \mathbf{R}$ not invariant under $g$ and denote by $x_j$ its images:

\begin{equation}
 x_j := g^j x_0, \quad j = 1, 2, \ldots, n - 1.
\end{equation}

The function $u_0$ attains the same values at all $x_j, j = 0, 1, 2, \ldots, n - 1$, hence joining these points together into a vertex $V_0$ will not affect the symmetric solutions: the function $u_0$ is not only continuous at $V_0$ but the sum of oriented derivatives is zero. It does not matter whether the original point $x_0$ was an inner point on an edge, or a vertex. Let us denote the graph obtained from $\mathbf{R}$ by introducing the new vertex $V_0 = \{x_0, x_1, \ldots, x_{n-1}\}$ by $\mathbf{R}'$. Note that the new graph is invariant under the same symmetry $g$ (properly understood). It is clear that the graphs $\mathbf{R}$ and $\mathbf{R}'$ have the same $M$-functions but their non-detectable spectra need not be equal as the quasi-invariant eigenfunctions may fail to satisfy standard conditions at the new vertex $V_0$.

We illustrate this construction with one explicit example appeared already in [24]. We take the cycle graph with two opposite points as contact vertices. The symmetry $g$ is then the transformation exchanging the two edges keeping the vertices. Joining together any two symmetric points inside the edges turns the graph into figure-eight graph with the most remote points as contact vertices. It is clear that the spectrum of the second graph depends on the position of the point $x_0$ on the edges. Hence these two graphs are Steklov-equivalent but not isospectral.

7. Methods 3 and 4: swapping Steklov subspaces

The third method is inspired by the example described in Section 4. We tried to generalise the idea behind.
Method 3 (Swapping Steklov subspaces 1). Let $K$ be a metric graph with a certain degenerate Steklov eigenvalue $\mu(\lambda)$ with the eigensubspace $V(\lambda)$, $\dim V(\lambda) > 1$. Let $Q_j$ be two isospectral metric graphs such that

\begin{equation}
M_{Q_1}(\lambda)|_{V(\lambda)} \overset{\text{unitary}}{\sim} M_{Q_2}(\lambda)|_{V(\lambda)},
\end{equation}

holds, then the graphs $\Gamma_j$, $j = 1, 2$ obtained by gluing together $K$ and $Q_j$ are isospectral.

**Proof.** Proving that $\Gamma_1$ and $\Gamma_2$ are isospectral we shall again use localised and delocalised eigenfunctions. Our assumptions imply that the graphs $Q_j$ have the same Steklov eigenvalues (also the graphs do not need to be Steklov-equivalent), in other words, the corresponding $M$-functions are unitary-equivalent:

\begin{equation}
M_{Q_1}(\lambda) = U(\lambda)M_{Q_2}(\lambda)U^{-1}(\lambda),
\end{equation}

with

\[U(\lambda)|_{V(\lambda)^\perp} = I_{V(\lambda)^\perp}.
\]

Hence it holds

\[M_K(\lambda) = U(\lambda)M_{Q_2}(\lambda)U^{-1}(\lambda),
\]

as $M_K(\lambda)|_{V(\lambda)} = \mu(\lambda)I_{V(\lambda)}$. It follows that Steklov eigenvalues for $\Gamma_1$ and $\Gamma_2$ coincide between the singular points:

\begin{equation}
\begin{aligned}
\left( M_{\Gamma_1}(\lambda) + M_{\Gamma_2}(\lambda) \right) U^{-1}(\lambda) &= U(\lambda) \left( M_{\Gamma_1}(\lambda) + M_{\Gamma_2}(\lambda) \right) U^{-1}(\lambda),
\end{aligned}
\end{equation}

which in turn imply that the detectable spectra of $\Gamma_1$ and $\Gamma_2$ coincide.

Lemma \[\text{[H]}\] implies that non-detectable spectra coincide as well, since the subgraphs $Q_j$ are isospectral and have the same Steklov spectra and the subgraph $K$ is common. \[\square\]

Unfortunately we do not have any recipe to obtain a wide class of graphs satisfying the assumptions. Our main tool are numerous generalisations of the example from Section \[\text{[H]}\] where the Steklov subspaces are independent of the spectral parameter $\lambda$. These examples are present in the next section.

It is also possible to generalise the last method by relaxing the assumption that the graphs $\Gamma_j$ have a common part (the graph $K$).
Method 4 (Swapping Steklov subspaces 2). Let \((K_1, K_2)\) and \((Q_1, Q_2)\) be two pairs of graphs with the same number of contact points and unitary equivalent \(M\)-functions, such that for a certain invariant for \(M_{K_j}(\lambda)\) and \(M_{Q_j}(\lambda)\) subspace \(V(\lambda)\) it holds
\[
M_{K_1}(\lambda)|_{V(\lambda)\perp} = M_{K_2}(\lambda)|_{V(\lambda)\perp}, \quad M_{Q_1}(\lambda)|_{V(\lambda)\perp} = M_{Q_2}(\lambda)|_{V(\lambda)\perp}.
\]
Denoting by \(\mu_{K_j}^{s}(\lambda)\), \(\mu_{Q_j}^{s}(\lambda)\), \(s = 1, 2, \ldots, S\) the pairwise equal Steklov eigenvalues, let us assume that the corresponding eigensubspaces can be chosen equal:
\[
V_{s}^{K_j}(\lambda) = V_{s}^{Q_j}(\lambda), \quad s = 1, 2, \ldots, S, \quad j = 1, 2.
\]
If in addition the graphs 
\[
K_1 \cup Q_1 \quad \text{and} \quad K_2 \cup Q_2
\]
are isospectral, then the graphs \(\Gamma_1\) and \(\Gamma_2\) obtained by gluing together \(K_j\) and \(Q_j\) are also isospectral.

The proof of this method goes along the same lines.

8. Clarifying example

In this section we construct an example of two isospectral metric graphs using most of the formulated methods. We hope that this example is sufficiently transparent so that it clearly illustrates all possibilities. Good examples sometimes tell more than complete characterisation of all possibilities.

We start by constructing the graph \(K\) with degenerate Steklov eigenvalues. This can be done using symmetries. For example let us take the complete graph \(K_5\) with the vertices \(V_1, \ldots, V_5\), where all edges are substituted with arbitrary Steklov-equivalent graphs \(A\) and \(B\) having two contact points \(M_A(\lambda) = M_B(\lambda)\).

One may use for example the graphs presented in Fig. 6. The \(M\)-function associated with the contact set \(\partial K = \{V_j\}_{j=1}^{5}\) commutes with the permutations, although the graph does not necessarily have any symmetry after the edges are substituted with the graphs \(A\) and \(B\) (see Fig. 7, the edges are marked by the red and magenta colours). To the constructed graph one may attach the star graph on 5 edges. Consider the 5-star graph where degree 1 vertices are denoted by \(V_1, \ldots, V_5\) and with all edges substituted with the same graphs \(C\) having two contact points (these edges are marked by the cyan colour). The degeneracy of the Steklov eigenvalues persists, although their values are affected. Finally to the central vertex \(V_6\) in the star graph we attach arbitrary graph \(E\) (marked by green colour in the figure). The constructed graph \(K\) has a complicated structure, may have no symmetries but the corresponding \(M\)-function associated with \(\partial K\) for almost every \(\lambda\) has degenerate eigenvalues of multiplicity 4 with the corresponding eigensubspace given by \(V(\lambda) = 1^\perp\).

Constructing the graph \(K\) we could also attach Steklov-equivalent graphs to the vertices \(V_1, \ldots, V_5\), or combine the two ideas. It is not our aim to describe all possibilities here.

To the graph \(K\) we shall add two different graphs \(Q_j\) constructed by splitting the star graph on five edges. Consider the 5-star graph where degree 1 vertices are denoted by \(V_1, \ldots, V_5\) and with all edges substituted with the graphs \(D\) having two contact points (marked by the back colour in the figure). Then each of the graphs \(Q_j\) is formed by splitting the central vertex in the star graph into the vertices
Figure 7. Isospectral graphs by gluing.
The numbers indicate positions of different vertices.

$V_6$ and $V_7$ having valencies either 2 and 3, or 1 and 4. To each of these vertices $V_j$, $j = 6, 7$, we attach as many copies of an arbitrary graph $F$ (marked by the blue colour in the figure) as the number of graphs $D$ joined at the vertex. The contact set for $Q_j$ is formed by the first five vertices $\partial Q_j = \{V_j\}^5_{j=1}$, but these vertices are naturally divided into two sets corresponding to the two connected components in each of the graphs. For each connected component the vector $1$ (having dimension 1, 2, 3, or 4) is not only a Steklov eigenvector (for almost any $\lambda$) but the corresponding Steklov eigenvalue does not depend on the dimension. To achieve that different number of graphs $F$ were attached. The orthogonal complement is again a Steklov eigensubspace with the eigenvalue equal to the $M$-function for the graph $D$ with Dirichlet condition introduced at the opposite contact point.

Gluing pairwise the contact vertices in $K$ and $Q_j$, $j = 1, 2$, we get two isospectral graphs $\Gamma_1$ and $\Gamma_2$. These graphs may have no symmetries. One may even take the
graphs $A, B, C,$ and $D$ Steklov-equivalent but having different invisible eigenvalues. Then it is possible to exchange these graphs in arbitrary way destroying all remains of possibly existing symmetries. The same argument applies to graphs $E$ and $F$: for example one may take 5 Steklov-equivalent graphs $F_j, j = 1, \ldots, 5$ and attach them in arbitrary manner to the vertices $V_6$ and $V_7$ (of course keeping the number equal to the degree).

Moreover, creating isospectral graphs one may use not the same graphs $K$, but the graphs with exchanged Steklov-equivalent building blocks – the graphs $A, B, C,$ and $D$.

It is clear that although the constructed graphs may formally not possess any symmetry, they are still symmetric if one just looks at the involved $M$-functions for the building blocks used in the construction. One may treat separately the eigenvalues with the eigenfunctions supported by single blocks (invisible eigenvalues) and the rest of the spectrum.

The presented example illustrates the enormous possibilities to construct isospectral graphs provided by our methods. We plan to return back to this question in one of our forthcoming publications.

9. Comparison to alternative constructions

The presented construction determines not only isospectral metric graphs, but also discrete graphs isospectral with respect to the normalised Laplacian – it is enough to reduce our construction to unilateral metric graphs and use von Below formula connecting the spectra of the normalised and differential Laplacians on these graphs. Let us briefly describe connections of our approach to two existing methods to obtain isospectral graphs.

- **Sunada approach** originally suggested in [28], broadly applied to discrete graphs and in particular to metric graphs in [13].

  This approach uses large symmetric graphs to obtains isospectral pairs. Each graph in the pair can be seen as a subgraph of the original large graph. Considering the simplest isospectral pair presented in Fig. 4 we failed to find any explanation via the Sunada’s approach. It might be interesting to prove such impossibility rigorously.

- **Seidel’s switching** following [10, 11, 27].

  This construction starts from two regular graphs, which are joined together in two different ways (see e.g. Fig. 6 and 7 in [10]). They are different from the presented isospectral graphs as regularity does not play any role in our construction.

  Moreover isospectrality of normalised the Laplacians associated with discrete graphs does not necessarily imply isospectrality of the differential Laplacian on the metric graphs obtained by providing unit length to all edges. We have the following:

  **Proposition 1.** Let $\Gamma_1, \Gamma_2$ be two unilateral metric graphs, and let $G_1, G_2$ be the corresponding discrete graphs. For $j = 1, 2$, let $L(\Gamma_j)$ denote the standard differential Laplacian on $\Gamma_j$, and let $L_N(G_j)$ denote the normalised Laplacian on $G_j$. Then

  \[ \sigma(L(\Gamma_1)) = \sigma(L(\Gamma_2)) \quad \iff \quad \begin{cases} \sigma(L_N(G_1)) = \sigma(L_N(G_2)), \\ \beta_1(G_1) = \beta_1(G_2), \end{cases} \]
where \( \sigma \) denotes the spectrum of the corresponding operator and \( \beta_1 \) denotes the first Betti number (the number of the independent cycles).

As a consequence of this proposition, the two graphs \( \Delta_1 \) and \( \Delta_2 \) presented in Section 5 of [11] do not lead to isospectral metric graphs as they have different numbers of edges. They are only isospectral as discrete graphs.

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