\textbf{$E_8$-singularity, invariant theory and modular forms}

Lei Yang

\textbf{Abstract}

As an algebraic surface, the equation of $E_8$-singularity $x^5 + y^3 + z^2 = 0$ can be obtained from a quotient $C_Y/\text{SL}(2,13)$ over the modular curve $X(13)$, where $Y \subset \mathbb{CP}^5$ is an algebraic curve given by a system of $\text{SL}(2,13)$-invariant polynomials and $C_Y$ is a cone over $Y$. It is different from the Kleinian singularity $\mathbb{C}^2/\Gamma$, where $\Gamma$ is the binary icosahedral group. This gives a negative answer to Arnol’d and Brieskorn’s questions about the mysterious relation between the icosahedron and $E_8$, i.e., the $E_8$-singularity is not necessarily the Kleinian icosahedral singularity. In particular, the equation of $E_8$-singularity possesses infinitely many kinds of distinct modular parametrizations, and there are infinitely many kinds of distinct constructions of the $E_8$-singularity.

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\textbf{1. Introduction}

By a rational double point or a simple singularity we understand the singularity of the quotient of $\mathbb{C}^2$ by the action of a finite subgroup of $\text{SL}(2,\mathbb{C})$ (see [27]). Let $\Gamma$ be a finite subgroup of $\text{SL}(2,\mathbb{C})$. Then $\Gamma$ is one of the following: a cyclic group of order $\ell \geq 1$ ($A_\ell$), a binary dihedral group of order $4(\ell - 2)$, $\ell \geq 4$ ($D_\ell$), the binary tetrahedral group ($E_6$), the binary octahedral group ($E_7$), or the binary icosahedral group ($E_8$). In 1874, Klein...
showed that the ring of polynomials in two variables which are invariant under $\Gamma$ is generated by three elements $x, y$ and $z$, which satisfy the following relation

\[ A_{\ell \geq 1} \quad x^{\ell+1} + y^2 + z^2 = 0, \]

\[ D_{\ell \geq 4} \quad x^{\ell-1} + xy^2 + z^2 = 0, \]

\[ E_6 \quad x^4 + y^3 + z^2 = 0, \]

\[ E_7 \quad x^3y + y^3 + z^2 = 0, \]

\[ E_8 \quad x^5 + y^3 + z^2 = 0. \]

These results of Klein on the invariant theory of the binary polyhedral groups were a starting point for later developments. In the minimal resolution of such a singularity an intersection configuration of the components of the exceptional divisor appears which can be described in a simple way by a Dynkin diagram of type $A_\ell$, $D_\ell$, $E_6$, $E_7$ or $E_8$. Up to analytic isomorphism, these diagrams classify the corresponding singularities. In other words, the $ADE$ singularities are the Kleinian singularities, i.e., the quotient singularities of $\mathbb{C}^2$ by a finite subgroup of SL$(2, \mathbb{C})$. In particular, the $E_8$-singularity is the icosahedral singularity $\mathbb{C}^2/\Gamma$, where $\Gamma$ is the binary icosahedral group.

In the present paper, we will show that the $E_8$-singularity can be obtained from a quotient $C_Y/\text{SL}(2,13)$ over the modular curve $X(13)$, where $Y \subset \mathbb{C}P^5$ is an algebraic curve given by a system of $\text{SL}(2,13)$-invariant polynomials and $C_Y$ is a cone over $Y$. This gives an infinitely many kinds of distinct constructions of the $E_8$-singularity which are different from the icosahedral singularity. Our construction is based on the invariant theory for the group $\text{SL}(2,13)$. We obtain some invariants $\Phi_{12}, \Phi_{20}$ and $\Phi_{30}$ for $\text{SL}(2,13)$. Furthermore, over the modular curve $X(13)$, these invariants are modular forms which satisfy the equation of the $E_8$-singularity. Hence, we obtain an homomorphism from the ring of invariants $[\mathbb{C}[z_1, z_2, z_3, z_4, z_5, z_6]/I]^{\text{SL}(2,13)}$ of the group $\text{SL}(2,13)$ acting on the cone $C_Y \subset \mathbb{C}^6$ to the ring $\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{30}^3 - \Phi_{20}^2 - 1728\Phi_{12}^5)$ over the modular curve $X(13)$, where $I$ is an ideal generated by a system of algebraic relations satisfied by the variables $z_1, \ldots, z_6$, $Y$ is the algebraic curve corresponding the ideal $I$.

Let us begin with the invariant theory for $\text{SL}(2,13)$. The representation of $\text{SL}(2,13)$ we will consider is the unique six-dimensional irreducible complex representation for which the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the $\exp(a 2\pi i/13)$ for $a$ a non-square mod 13. We will give an explicit realization of this representation. This explicit realization will play a major role for giving
a complete system of invariants associated to $\text{SL}(2, 13)$. Recall that the six-dimensional representation of the finite group $\text{SL}(2, 13)$ of order 2184, which acts on the five-dimensional projective space $\mathbb{C}P^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}$. This representation is defined over the cyclotomic field $\mathbb{Q}(e^{2\pi i/13})$. Put

$$
S = -\frac{1}{\sqrt{13}} \begin{pmatrix}
\zeta^{12} - \zeta \\
\zeta^{10} - \zeta^3 \\
\zeta^6 - \zeta^7 \\
\zeta^4 - \zeta^9 \\
\zeta^5 - \zeta^8 \\
\zeta^2 - \zeta^{11}
\end{pmatrix}
$$

and

$$
T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5)
$$

where $\zeta = \exp(2\pi i/13)$. We have

$$
S^2 = -I, \quad T^{13} = (ST)^3 = I. \quad (1.1)
$$

Let $G = \langle S, T \rangle$, then $G \cong \text{SL}(2, 13)$. We construct some $G$-invariant polynomials in six variables $z_1, \ldots, z_6$. Let

$$
w_\infty = 13 \mathbf{A}_0^2, \quad w_\nu = (\mathbf{A}_0 + \zeta^\nu \mathbf{A}_1 + \zeta^{4\nu} \mathbf{A}_2 + \zeta^{9\nu} \mathbf{A}_3 + \zeta^{3\nu} \mathbf{A}_4 + \zeta^{12\nu} \mathbf{A}_5 + \zeta^{10\nu} \mathbf{A}_6)^2
$$

for $\nu = 0, 1, \ldots, 12$, where the senary quadratic forms (quadratic forms in six variables) $\mathbf{A}_j$ ($j = 0, 1, \ldots, 6$) are given by

$$
\begin{align*}
\mathbf{A}_0 &= z_1 z_4 + z_2 z_5 + z_3 z_6, \\
\mathbf{A}_1 &= z_1^2 - 2z_2 z_4, \\
\mathbf{A}_2 &= -z_5^2 - 2z_2 z_4, \\
\mathbf{A}_3 &= z_2^2 - 2z_1 z_5, \\
\mathbf{A}_4 &= z_3^2 - 2z_2 z_6, \\
\mathbf{A}_5 &= -z_4^2 - 2z_1 z_6, \\
\mathbf{A}_6 &= -z_6^2 - 2z_3 z_5.
\end{align*} \quad (1.3)
$$

Then $w_\infty$, $w_\nu$ for $\nu = 0, \ldots, 12$ are the roots of a polynomial of degree fourteen. The corresponding equation is just the Jacobian equation of degree fourteen (see [17], pp.161-162). On the other hand, set

$$
\delta_\infty = 13^2 G_0, \quad \delta_\nu = -13G_0 + \zeta^\nu G_1 + \zeta^{2\nu} G_2 + \cdots + \zeta^{12\nu} G_{12} \quad (1.4)
$$

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for $\nu = 0, 1, \ldots, 12$, where the senary sextic forms (i.e., sextic forms in six variables) $G_j$ ($j = 0, 1, \ldots, 12$) are given by

\[
\begin{cases}
G_0 &= D_0^2 + D_\infty^2, \\
G_1 &= -D_7^2 + 2D_0D_1 + 10D_\infty D_1 + 2D_2D_{12} + \\
& - 2D_7D_{11} - 4D_4D_{10} - 2D_6D_5, \\
G_2 &= -2D_1^2 - 4D_0D_2 + 6D_\infty D_2 - 2D_4D_{11} + \\
& + 2D_5D_{10} - 2D_0D_9 - 2D_7D_8, \\
G_3 &= -D_8^2 + 2D_0D_3 + 10D_\infty D_3 + 2D_6D_{10} + \\
& - 2D_0D_7 - 4D_1D_4 - 2D_2D_2, \\
G_4 &= -D_2^2 + 10D_0D_4 - 2D_\infty D_4 + 2D_3D_{12} + \\
& - 2D_6D_8 - 4D_1D_3 - 2D_0D_7, \\
G_5 &= -2D_9^2 - 4D_0D_5 + 6D_\infty D_5 - 2D_{10}D_8 + \\
& + 2D_6D_{12} - 2D_2D_3 - 2D_1D_7, \\
G_6 &= -2D_3^2 - 4D_0D_6 + 6D_\infty D_6 - 2D_{12}D_7 + \\
& + 2D_3D_4 - 2D_5D_1 - 2D_8D_{11}, \\
G_7 &= -2D_{10}^2 + 6D_0D_7 + 4D_\infty D_7 - 2D_1D_{10} + \\
& - 2D_2D_5 - 4D_8D_{12} - 2D_9D_{11}, \\
G_8 &= -2D_4^2 + 6D_0D_8 + 4D_\infty D_8 - 2D_3D_5 + \\
& - 2D_6D_2 - 2D_1D_{10} - 2D_7D_7, \\
G_9 &= -D_{11}^2 + 2D_0D_9 + 10D_\infty D_9 + 2D_5D_4 + \\
& - 2D_1D_8 - 4D_{10}D_{12} - 2D_9D_6, \\
G_{10} &= -D_5^2 + 10D_0D_{10} - 2D_\infty D_{10} + 2D_6D_4 + \\
& - 2D_3D_7 - 4D_9D_1 - 2D_{12}D_{11}, \\
G_{11} &= -2D_{12}^2 + 6D_0D_{11} + 4D_\infty D_{11} - 2D_6D_2 + \\
& - 2D_3D_6 - 2D_7D_4 - 2D_3D_8, \\
G_{12} &= -D_6^2 + 10D_0D_{12} - 2D_\infty D_{12} + 2D_2D_{10} + \\
& - 2D_1D_{11} - 4D_3D_9 - 2D_4D_8.
\end{cases}
\]

Here, the senary cubic forms (cubic forms in six variables) $D_j$ ($j = 0, 1, \ldots, 12$).
Then $\delta_m$ with degree $4$ are given as follows:

\[
\begin{align*}
D_0 &= z_1 z_2 z_3, \\
D_1 &= 2z_2 z_3^2 + z_2^2 z_6 - z_4^2 z_5 + z_1 z_5 z_6, \\
D_2 &= -z_6^3 + z_2^2 z_4 - 2z_2 z_5^2 + z_1 z_4 z_5 + 3z_3 z_5 z_6, \\
D_3 &= 2z_1 z_2^2 + z_1^2 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\
D_4 &= -z_2^2 z_3 + z_1 z_6^2 - 2z_4^2 z_6 - z_1 z_3 z_5, \\
D_5 &= -z_4^3 + z_3^2 z_6 - 2z_3 z_6^2 + z_2 z_5 z_6 + 3z_1 z_4 z_6, \\
D_6 &= -z_5^3 + z_1^2 z_6 - 2z_1 z_4^2 + z_3 z_4 z_6 + 3z_2 z_4 z_5, \\
D_7 &= -z_2^3 + z_3 z_4^2 - z_1 z_3 z_6 - 3z_1 z_2 z_5 + 2z_1^2 z_4, \\
D_8 &= -z_1^3 + z_2 z_6^2 - z_2 z_3 z_6 - 3z_1 z_3 z_4 + 2z_3 z_6, \\
D_9 &= 2z_1^2 z_3 + z_3^2 z_4 - z_2^2 z_6 + z_2 z_4 z_6, \\
D_{10} &= -z_1 z_3^2 + z_2 z_4^2 - 2z_4 z_5^2 - z_1 z_2 z_6, \\
D_{11} &= -z_3^3 + z_1 z_5^2 - z_1 z_2 z_4 - 3z_2 z_3 z_6 + 2z_2^2 z_5, \\
D_{12} &= -z_1^2 z_2 + z_3 z_5^2 - 2z_5 z_6^2 - z_2 z_3 z_4, \\
D_{\infty} &= z_4 z_5 z_6.
\end{align*}
\]

Then $\delta_{\infty}, \delta_{\nu}$ for $\nu = 0, \ldots, 12$ are the roots of a polynomial of degree fourteen. The corresponding equation is not the Jacobian equation. The invariant homogeneous polynomials $\Phi_{m,n}$ are given by

\[
\Phi_{m,n} = w_0^m \delta_0^n + w_1^m \delta_1^n + \cdots + w_{12}^m \delta_{12}^n + w_{\infty}^m \delta_{\infty}^n, \tag{1.7}
\]

with degree $4m + 6n$, where $0 \leq m \leq 14$, $0 \leq n \leq 14$. 

12, $\infty$) are given as follows:
Let $x_i(z) = \eta(z) a_i(z)$ ($1 \leq i \leq 6$), where

\[
\begin{align*}
a_1(z) &:= e^{-\frac{11\pi i}{26} \theta} \begin{bmatrix} \frac{11}{13} \\ 1 \end{bmatrix} (0, 13z), \\
a_2(z) &:= e^{-\frac{7\pi i}{26} \theta} \begin{bmatrix} \frac{7}{13} \\ 1 \end{bmatrix} (0, 13z), \\
a_3(z) &:= e^{-\frac{5\pi i}{26} \theta} \begin{bmatrix} \frac{5}{13} \\ 1 \end{bmatrix} (0, 13z), \\
a_4(z) &:= -e^{-\frac{3\pi i}{26} \theta} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0, 13z), \\
a_5(z) &:= e^{-\frac{9\pi i}{26} \theta} \begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix} (0, 13z), \\
a_6(z) &:= e^{-\frac{\pi i}{26} \theta} \begin{bmatrix} \frac{1}{13} \\ 1 \end{bmatrix} (0, 13z)
\end{align*}
\]

are theta constants of order 13 and $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi iz}$ is the Dedekind eta function which are all defined in the upper-half plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. In fact, the weight of $x_i(z)$ is 1 and the parabolic modular forms $a_i(z)$ of weight $\frac{1}{2}$ given by (1.8) form a multiplier-system in the sense of the following (see (3.16) in Proposition 3.2):

\[
A(z + 1) = e^{-\frac{3\pi i}{4} T} A(z), \quad A \left( -\frac{1}{z} \right) = e^{\frac{\pi i}{4}} \sqrt{z} S A(z),
\]

where $S$ and $T$ are given as above, $0 < \arg \sqrt{z} \leq \pi/2$ and

\[
A(z) := (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^T.
\]

We will show that there is a morphism

\[
\Phi : X \to Y \subset \mathbb{CP}^5
\]

with $\Phi(z) = (x_1(z), \ldots, x_6(z))$, where $X = X(13)$ is the modular curve $\Gamma(13) \backslash \mathbb{H}$ and $Y$ is an algebraic curve given by a family of $G$-invariant polynomials

\[
\begin{align*}
\Phi_4(z_1, \ldots, z_6) &= 0, \\
\Phi_8(z_1, \ldots, z_6) &= 0, \\
\Phi_{10}(z_1, \ldots, z_6) &= 0, \\
\Phi_{14}(z_1, \ldots, z_6) &= 0,
\end{align*}
\]
where
\[ \Phi_4 = \Phi_{1,0}, \quad \Phi_8 = \Phi_{2,0}, \quad \Phi_{10} = \Phi_{1,1}, \quad \Phi_{14} = \Phi_{2,1}. \] (1.13)

Each \( \Phi_i \) \( (i = 4, 8, 10, 14) \) corresponds to a unique \( \Phi_{m,n} \) with degree \( i = 4m + 6n \). The significance of the algebraic curve \( Y \) is that the finite group \( G \) acts linearly on \( \mathbb{C}^6 \) and on \( \mathbb{C}\mathbb{P}^5 \) leaving invariant \( Y \subset \mathbb{C}\mathbb{P}^5 \) and the cone \( C_Y \subset \mathbb{C}^6 \). Moreover, it is (1.9) that gives an explicit realization of the isomorphism between the unique sub-representation of parabolic modular forms of weight \( \frac{1}{2} \) on \( X(13) \) and the above six-dimensional complex representation of \( \text{SL}(2,13) \) generated by \( S \) and \( T \). Our main theorems are the following:

**Theorem 1.1.** (Main Theorem 1) *The equation of \( E_8 \)-singularity*

\[ \Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5 = 0 \]

possesses an infinitely many kinds of distinct modular parametrizations (with the cardinality of the continuum in ZFC set theory)

\[ (\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_{12}^\lambda, \Phi_{20}^\mu, \Phi_{30}^\gamma) \] (1.14)

over the modular curve \( X \) as follows:

\[
\begin{align*}
\Phi_{12}^\lambda &= \lambda \Phi_{3,0} + (1 - \lambda)\Phi_{0,2} \quad \text{modulo } I, \\
\Phi_{20}^\mu &= \mu \Phi_{5,0} + (1 - \mu)\Phi_{2,2} \quad \text{modulo } I, \\
\Phi_{30}^\gamma &= \gamma_1 \Phi_{0,5} + \gamma_2 \Phi_{3,3} + (1 - \gamma_1 - \gamma_2)\Phi_{6,1} \quad \text{modulo } (I, \Phi_{3,0}, \Phi_{0,2}),
\end{align*}
\] (1.15)

where the ideal
\[ I = (\Phi_4, \Phi_8, \Phi_{10}, \Phi_{14}) \] (1.16)

and the parameter space \( \{ (\lambda, \mu, \gamma) \} \cong \mathbb{C}^4 \).

**Theorem 1.2.** (Main Theorem 2) *There is a morphism of schemes*

\[ f : \text{Spec} \left( \mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5) \right) \to \mathbb{C}Y/G \] (1.17)

over the modular curve \( X \) which is a closed immersion. The map \( f \) is a homeomorphism of \( \text{Spec}(\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5)) \) onto a closed subset of \( \mathbb{C}Y/G \) over \( X \). In particular, there are infinitely many kinds of such triples \( (\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_{12}^\lambda, \Phi_{20}^\mu, \Phi_{30}^\gamma) \) whose parameter space \( \{ (\lambda, \mu, \gamma) \} \cong \mathbb{C}^4 \).

Theorem 1.1 and Theorem 1.2 show that there exist infinitely many kinds of distinct constructions of the \( E_8 \)-singularity: one and only one is given
by the Kleinian singularity $\mathbb{C}^2/\text{SL}(2, 5)$ (see [17]), i.e., the icosahedral singularity, the other infinitely many kinds of constructions are given from the quotient $C_Y/\text{SL}(2, 13)$ over the modular curve $X$. Hence, the equation of $E_8$-singularity possesses infinitely many kinds of distinct modular parametrizations.

In his talk at ICM 1970 [4], Brieskorn showed how to construct the singularity of type $ADE$ directly from the simple complex Lie group of the same type. Namely, assume that $G$ is of type $ADE$, Brieskorn proved a conjecture made by Grothendieck that the intersection of a transversal slice to the sub-regular unipotent orbit with the unipotent variety has a simple surface singularity of the same type as $G$. A fuller treatment was given by Slodowy (see [4] and [27]). A clarification of the occurrence of the polyhedral groups in Brieskorn’s construction (see [11] and [12]), and thus a direct relationship between the simple Lie groups and the finite subgroups of $\text{SL}(2, \mathbb{C})$, was achieved by Kronheimer (see [22] and [23]) using differential geometric methods. His construction starts directly from the finite subgroups of $\text{SL}(2, \mathbb{C})$ and uses hyper-Kähler quotient constructions. Kronheimer also gave an algebraic approach using McKay correspondence. However, Brieskorn had still written at the end of [4]: “Thus we see that there is a relation between exotic spheres, the icosahedron and $E_8$. But I still do not understand why the regular polyhedra come in.” (see also [10], [11], [12] and [5]). On the other hand, Arnol’d pointed out that the theory of singularities is even linked (in a quite mysterious way) to the classification of regular polyhedra in three-dimensional Euclidean space (see [2], p. 43). In his survey article on Platonic solids, Kleinian singularities and Lie groups [28], Slodowy found that the objects of these different classifications are related to each other by mathematical constructions. However, up to now, these constructions do not explain why the different classifications should be related at all.

As a consequence, Theorem 1.1 and Theorem 1.2 show that the $E_8$-singularity is not necessarily the Kleinian icosahedral singularity. That is, the icosahedron does not necessarily appear in the triple (exotic spheres, icosahedron, $E_8$) of Brieskorn [4]. The group $\text{SL}(2, 13)$ can take its place and there are infinitely many kinds of the other triples (exotic spheres, $\text{SL}(2, 13)$, $E_8$).

The link of these infinitely many kinds of distinct constructions of the $E_8$-singularity: $\mathbb{C}^2/\text{SL}(2, 5)$ and $\text{Spec}(\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5)) \rightarrow C_Y/\text{SL}(2, 13)$ over the modular curve $X$ gives the same Poincaré homology
3-sphere, whose higher dimensional lifting:

\[ z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad \sum_{i=1}^{5} z_i \bar{z}_i = 1, \quad z_i \in \mathbb{C} \quad (1 \leq i \leq 5) \quad (1.18) \]

gives the Milnor’s standard generator of \( \Theta_7 \). Hence, this gives a negative answer to Arnol’d and Brieskorn’s questions about the mysterious relation between the icosahedron and \( E_8 \), and the relation between Platonic solids, Kleinian singularities and Lie groups appearing in Slodowy’s survey [28] can be replaced by the relation between \( SL(2,13) \),

\[
\text{Spec } (\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5)) \to \mathbb{C}Y/SL(2,13)
\]

over the modular curve \( X \) and \( E_8 \).

In fact, Klein had noticed the similarity between the relation of the equation \( x^5 + y^3 + z^2 = 0 \) to the icosahedral group \( PSL(2,5) \) and the relation of the equation \( x^7 + y^3 + z^2 = 0 \) to the group \( PSL(2,7) \) (see [18], [19], [20] and [21]). This is the starting point of the work of Dolgachev (see [7]) to which Arnol’d was referring when he spoke about the wonderful coincidences with Lobatchevsky triangles and automorphic functions (see [1]). The normal form of Arnol’d for the quasi-homogeneous singularity \( E_{12} \) in three variables is \( x^7 + y^3 + z^2 \), which can be realized as the quotient conical singularity as follows (see [6] and [7]): The canonical model \( Y \) of the modular curve \( X(7) \) in \( \mathbb{CP}^2 \) is the Klein quartic given by the homogeneous equation \( z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1 = 0 \). The finite group \( PSL(2,7) \) acts linearly on \( \mathbb{C}^3 \) and on \( \mathbb{CP}^2 \) leaving invariant \( Y \subset \mathbb{CP}^2 \) and the cone \( C_Y \subset \mathbb{C}^3 \). Calculations of invariants by Klein and Gordan imply:

\[
[\mathbb{C}[z_1, z_2, z_3]/(z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1)]^{PSL(2,7)} \cong \mathbb{C}[x, y, z]/(x^7 + y^3 + z^2). \quad (1.19)
\]

This algebraic result can be interpreted geometrically as follows: The affine algebraic surface defined by the equation \( x^7 + y^3 + z^2 = 0 \) is the quotient of the cone \( C_Y \) by the group \( PSL(2,7) \) over the modular curve \( X(7) \), where \( C_Y \) is the cone over \( Y \). Similarly, Klein also obtained the structure of the \( \mathbb{C} \)-algebra of \( \mathbb{C}[z_1, z_2]^{SL(2,5)} \) of \( SL(2,5) \)-invariant polynomials on \( \mathbb{C}^2 \):

\[
\mathbb{C}[z_1, z_2]^{SL(2,5)} \cong \mathbb{C}[x, y, z]/(x^5 + y^3 + z^2). \quad (1.20)
\]

This algebraic result can also be interpreted geometrically as follows: The affine algebraic surface defined by the equation \( x^5 + y^3 + z^2 = 0 \) is the
quotient of the cone $C_Y$ by the group $SL(2, 5)$ over the modular curve $X(5)$, where $Y = \mathbb{CP}^1$ is the canonical model of the modular curve $X(5)$ and $C_Y$ is a cone over $Y$. Therefore, (1.14), (1.15), (1.17), (1.19) and (1.20) give a complete and unified description for the structure of rings of $G$-invariant polynomials and the associated singularities corresponding to the genus zero modular curves $X_0(N)$, where $N = 5, 7, 13$ and $G = SL(2, 5), PSL(2, 7)$ and $SL(2, 13)$, respectively:

\[
\begin{array}{ccc}
C_Y/SL(2, 5) & C_Y/PSL(2, 7) & C_Y/SL(2, 13) \\
\downarrow & \downarrow & \downarrow \\
X(5) & X(7) & X(13) \\
E_8\text{-singularity} & E_{12}\text{-singularity} & E_8\text{-singularity}
\end{array}
\]  

(1.21)

Here, $Y = \mathbb{CP}^1$, Klein quartic curve and our curve $Y$ given by (1.12), respectively.

Finally, recall that there is a decomposition formula of the elliptic modular function $j$ in terms of the icosahedral invariants $f$, $H$ and $T$ of degrees 12, 20 and 30 over the modular curve $X(5)$ (see section two, in particular (2.8) for the details):

\[
j(z) : j(z) - 1728 : 1 = H(x_1(z), x_2(z))^3 : -T(x_1(z), x_2(z))^2 : f(x_1(z), x_2(z))^5,
\]

(1.22)

which was discovered by Klein (see [17], [20] and [21]) and later by Ramanujan (see [8]). In contrast with (1.22), we have the following infinitely many kinds of distinct decomposition formulas of the elliptic modular function $j$ in terms of the invariants $\Phi_{12}$, $\Phi_{20}$ and $\Phi_{30}$ over the modular curve $X$:

\[
j(z) : j(z) - 1728 : 1 = \Phi_{20}^3 : \Phi_{20}^2 : \Phi_{30}^5,
\]

(1.23)

where $(\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_{12}^3, \Phi_{20}^2, \Phi_{30}^5)$ are given by (1.15). These infinitely many kinds of distinct decompositions (1.23) and (1.22) have the same form, i.e., the degrees of the invariant polynomials are 12, 20 and 30, respectively. However, they have the different geometric interpretation: one and only one is over the modular curve $X(5)$, the other infinitely many kinds of decompositions are over the modular curve $X(13)$. They also have the different algebraic interpretation: one and only one is invariant under the group $SL(2, 5)$, the other infinitely many kinds of decompositions are invariant under the group $SL(2, 13)$.  

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This paper consists of four sections. In section two, we revisit the standard construction of the $E_8$-singularity as the well-known Kleinian icosahedral singularity. In section three, we study the invariant theory and modular forms for $SL(2, 13)$. In particular, we construct a system of invariants $\Phi_{m,n}$ for $SL(2, 13)$. These invariants are modular forms over the modular curve $X$. We find the generators $\Phi_{12}$, $\Phi_{20}$ and $\Phi_{30}$ among those modular forms. They satisfy the equation of $E_8$-singularity. Thus, we obtain the ring homomorphism from the ring of invariants $\mathbb{C}[z_1, z_2, z_3, z_4, z_5, z_6]/I^\text{SL}(2, 13)$ to $\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}]/(\Phi_{20}^2 - \Phi_{30}^2 - 1728\Phi_{12}^5)$, where $I$ is an ideal generated by a system of algebraic relations satisfied by the variables $z_1, \ldots, z_6$ and $Y$ is the algebraic curve corresponding to the ideal $I$. In section four, we give a different construction of the $E_8$-singularity coming from a quotient $C_Y/SL(2, 13)$ over the modular curve $X$, where $C_Y$ is the cone over the algebraic curve $Y$.

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2. **Standard construction: $E_8$-singularity as the icosahedral singularity**

Let us recall some classical result on the relation between the icosahedron and the $E_8$-singularity (see [24]). Starting with the polynomial invariants of the finite subgroup of $SL(2, \mathbb{C})$, a surface is defined from the single syzygy which relates the three polynomials in two variables. This surface has a singularity at the origin; the singularity can be resolved by constructing a smooth surface which is isomorphic to the original one except for a set of component curves which form the pre-image of the origin. The components form a Dynkin curve and the matrix of their intersections is the negative of the Cartan matrix for the appropriate Lie algebra. The Dynkin curve is the dual of the Dynkin graph. For example, if $\Gamma$ is the binary icosahedral group, the corresponding Dynkin curve is that of $E_8$, and $\mathbb{C}^2/\Gamma \subset \mathbb{C}^3$ is the set of zeros of the equation

$$x^5 + y^3 + z^2 = 0. \quad (2.1)$$

The link of this $E_8$-singularity, the Poincaré homology 3-sphere (see [16]), has a higher dimensional lifting:

$$z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad \sum_{i=1}^{5} z_i \bar{z}_i = 1, \quad z_i \in \mathbb{C} \quad (1 \leq i \leq 5), \quad (2.2)$$
which is the Brieskorn description of one of Milnor’s exotic 7-dimensional spheres. In fact, it is an exotic 7-sphere representing Milnor’s standard generator of $\Theta_7$ (see [3], [4] and [14]).

In his celebrated book [17], Klein gave a parametric solution of the above singularity (2.1) by homogeneous polynomials $T$, $H$, $f$ in two variables of degrees 30, 20, 12 with integral coefficients, where

$$f = z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}),$$

$$H = \frac{1}{121} \left| \frac{\partial^2 f}{\partial z_1^2} \frac{\partial^2 f}{\partial z_2^2} - \frac{\partial^2 f}{\partial z_1 \partial z_2} \right| = -(z_1^{20} + z_2^{20}) + 228(z_1^{15} z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10},$$

$$T = -\frac{1}{20} \left| \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} - \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} \right| = (z_1^{30} + z_2^{30}) + 522(z_1^{25} z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{20} z_2^{10} + z_1^{10} z_2^{20}).$$

They satisfy the famous (binary) icosahedral equation

$$T^2 + H^3 = 1728 f^5. \quad (2.3)$$

In fact, $f$, $H$ and $T$ are invariant polynomials under the action of the binary icosahedral group. The above equation (2.3) is closely related to Hermite’s celebrated work (see [13]) on the resolution of the quintic equations by elliptic modular functions of order five. Essentially the same relation had been found a few years earlier by Schwarz (see [26]), who considered three polynomials $\varphi_{12}$, $\varphi_{20}$ and $\varphi_{30}$ whose roots correspond to the vertices, the midpoints of the faces and the midpoints of the edges of an icosahedron inscribed in the Riemann sphere. He obtained the identity $\varphi_{30}^3 - 1728 \varphi_{12}^5 = \varphi_{30}^2$. We see this identity as well as (2.3) as the defining relation between three generators $f$, $H$ and $T$ of the ring of invariants $\mathbb{C}[z_1, z_2]^{\Gamma}$ of the binary icosahedral group $\Gamma$ acting on $\mathbb{C}^2$, and we identify this ring with the ring of functions on the affine variety $\mathbb{C}^2/\Gamma$ embedded in $\mathbb{C}^3$ and given by such an equation (see [5]). Namely,

$$\mathbb{C}[z_1, z_2]^{\Gamma} \cong \mathbb{C}[f, H, T] / (T^2 + H^3 - 1728 f^5). \quad (2.4)$$

Thus we see that from the very beginning there was a close relation between the $E_8$-singularity and the icosahedron. Moreover, the icosahedral equation (2.3) can be interpreted in terms of modular forms which was also known by Klein (see [20], p. 631). Let $x_1(z) = \eta(z) a(z)$ and $x_2(z) = \eta(z) b(z)$, where

$$a(z) = e^{-\frac{2\pi i}{3}} \theta \left[ \frac{3}{1}, 0, 5z \right], \quad b(z) = e^{-\frac{2\pi i}{3}} \theta \left[ \frac{5}{1}, 0, 5z \right].$$

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are theta constants of order five and \( \eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) with \( q = e^{2\pi i z} \) is the Dedekind eta function which are all defined in the upper-half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). Then

\[
\begin{align*}
  f(x_1(z), x_2(z)) &= -\Delta(z), \\
  H(x_1(z), x_2(z)) &= -\eta(z)^8 \Delta(z) E_4(z), \\
  T(x_1(z), x_2(z)) &= \Delta(z)^2 E_6(z),
\end{align*}
\]

where

\[
E_4(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^4}, \quad E_6(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^6}
\]

are Eisenstein series of weight 4 and 6, and \( \Delta(z) = \eta(z)^{24} \) is the discriminant. The relations

\[
\begin{align*}
  j(z) &= \frac{E_4(z)^3}{\Delta(z)} = \frac{H(x_1(z), x_2(z))^3}{f(x_1(z), x_2(z))^5}, \\
  j(z) - 1728 &= \frac{E_6(z)^2}{\Delta(z)} = -\frac{T(x_1(z), x_2(z))^2}{f(x_1(z), x_2(z))^5}
\end{align*}
\]

give the icosahedral equation (2.3) in terms of theta constants of order five. Hence, we have the following decomposition formula of the elliptic modular function \( j \) in terms of the icosahedral invariants \( f, H \) and \( T \) over the modular curve \( X(5) \):

\[
\begin{align*}
  j(z) : j(z) - 1728 : 1 \\
  = H(x_1(z), x_2(z))^3 : -T(x_1(z), x_2(z))^2 : f(x_1(z), x_2(z))^5.
\end{align*}
\]

3. Invariant theory and modular forms for \( \text{SL}(2,13) \)

The representation of \( \text{SL}(2,13) \) which we will consider is the unique six-dimensional irreducible complex representation for which the eigenvalues of

\[
\begin{pmatrix}
  1 & 1 \\
  0 & 1
\end{pmatrix}
\]

are the \( \exp(a2\pi i/13) \) for \( a \) a non-square mod 13. We will give an explicit realization of this representation. This explicit realization will play a major role for giving a complete system of invariants associated to \( \text{SL}(2,13) \). At first, we will study the six-dimensional representation of the finite group \( \text{SL}(2,13) \) of order 2184, which acts on the five-dimensional projective space.
\[ \mathbb{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}. \] This representation is defined over the cyclotomic field \( \mathbb{Q}(e^{2\pi i/13}) \). Put

\[
S = -\frac{1}{\sqrt{13}} \begin{pmatrix}
\zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\
\zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\
\zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^{10} - \zeta \\
\zeta^5 - \zeta^8 & \zeta^{11} - \zeta & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} & \zeta^9 & \zeta^9 - \zeta \\
\zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 & \zeta^8 + \zeta^{11} \\
\zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10}
\end{pmatrix}
\]

and

\[
T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5),
\]

where \( \zeta = \exp(2\pi i/13) \). We have

\[
S^2 = -I, \quad T^{13} = (ST)^3 = I.
\]

In [29], we put \( P = ST^{-1}S \) and \( Q = ST^3 \). Then \( (Q^3P^4)^3 = -I \) (see [29], the proof of Theorem 3.1). Let \( G = \langle S, T \rangle \), then \( G \cong \text{SL}(2, 13) \).

Put \( \theta_1 = \zeta + \zeta^3 + \zeta^9 \), \( \theta_2 = \zeta^2 + \zeta^6 + \zeta^5 \), \( \theta_3 = \zeta^4 + \zeta^{12} + \zeta^{10} \), and \( \theta_4 = \zeta^8 + \zeta^{11} + \zeta^7 \). We find that

\[
\begin{aligned}
\theta_1 + \theta_2 + \theta_3 + \theta_4 &= -1, \\
\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 &= 2, \\
\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 &= 4, \\
\theta_1\theta_2\theta_3\theta_4 &= 3.
\end{aligned}
\]

Hence, \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \) satisfy the quartic equation \( z^4 + z^3 + 2z^2 - 4z + 3 = 0 \), which can be decomposed as two quadratic equations

\[
\left( z^2 + \frac{1 + \sqrt{13}}{2}z + \frac{5 + \sqrt{13}}{2} \right) \left( z^2 + \frac{1 - \sqrt{13}}{2}z + \frac{5 - \sqrt{13}}{2} \right) = 0
\]

over the real quadratic field \( \mathbb{Q}(\sqrt{13}) \). Therefore, the four roots are given as
follows:

\[
\begin{aligned}
\theta_1 &= \frac{1}{4} \left( -1 + \sqrt{13} + \sqrt{-26 + 6\sqrt{13}} \right), \\
\theta_2 &= \frac{1}{4} \left( -1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}} \right), \\
\theta_3 &= \frac{1}{4} \left( -1 + \sqrt{13} - \sqrt{-26 + 6\sqrt{13}} \right), \\
\theta_4 &= \frac{1}{4} \left( -1 - \sqrt{13} - \sqrt{-26 - 6\sqrt{13}} \right).
\end{aligned}
\]

Moreover, we find that

\[
\begin{aligned}
\theta_1 + \theta_3 + \theta_2 + \theta_4 &= -1, \\
\theta_1 + \theta_3 - \theta_2 - \theta_4 &= \sqrt{13}, \\
\theta_1 - \theta_3 + \theta_2 + \theta_4 &= -\sqrt{-13 + 2\sqrt{13}}, \\
\theta_1 - \theta_3 - \theta_2 - \theta_4 &= \sqrt{-13 - 2\sqrt{13}}.
\end{aligned}
\]

Let us study the action of $ST^\nu$ on $\mathbb{P}^5$, where $\nu = 0, 1, \ldots, 12$. Put

\[
\alpha = \zeta + \zeta^{12} - \zeta^5 - \zeta^8, \quad \beta = \zeta^3 + \zeta^{10} - \zeta^2 - \zeta^{11}, \quad \gamma = \zeta^9 + \zeta^4 - \zeta^6 - \zeta^7.
\]

We find that

\[
13ST^\nu(z_1) \cdot ST^\nu(z_4) = \beta z_1 z_4 + \gamma z_2 z_5 + \alpha z_3 z_6 + \]

\[
+ \gamma \zeta^2 z_1 + \alpha \zeta^2 z_2 + \beta \zeta^2 z_3 - \gamma \zeta^{12} z_4 + \alpha \zeta^2 z_5 - \beta \zeta^6 z_6 + \]

\[
+ (\alpha - \beta)\zeta^{10} z_1 z_2 + (\beta - \gamma)\zeta^6 z_2 z_3 + (\gamma - \alpha)\zeta z_1 z_3 + \]

\[
+ (\beta - \alpha)\zeta^8 z_4 z_5 + (\gamma - \beta)\zeta^7 z_5 z_6 + (\alpha - \gamma)\zeta^{11} z_4 z_6 + \]

\[
- (\alpha + \beta)\zeta^{12} z_3 z_4 - (\beta + \gamma)\zeta^9 z_1 z_5 - (\gamma + \alpha)\zeta^{3} z_2 z_6 + \]

\[
- (\alpha + \beta)\zeta^{10} z_1 z_6 - (\beta + \gamma)\zeta^4 z_2 z_4 - (\gamma + \alpha)\zeta^{10} z_3 z_5.
\]
This leads us to define the following senary quadratic forms (quadratic forms

\(\nu = 0, 1, \ldots, 12\). Then

\[
\varphi_\nu = (z_1 z_4 + z_2 z_5 + z_3 z_6) + \zeta^\nu (z_1^2 + 2 z_3 z_4) + \zeta^{4\nu} (z_2^2 - 2 z_1 z_5) + \zeta^{9\nu} (z_3^2 - 2 z_2 z_6) + \zeta^{12\nu} (z_4^2 - 2 z_1 z_6) + \zeta^{10\nu} (z_5^2 - 2 z_3 z_5).
\]

This leads us to define the following senary quadratic forms (quadratic forms
in six variables):
\[
\begin{align*}
A_0 &= z_1z_4 + z_2z_5 + z_3z_6, \\
A_1 &= z_1^2 - 2z_3z_4, \\
A_2 &= -z_5^2 - 2z_2z_4, \\
A_3 &= z_2^2 - 2z_1z_5, \\
A_4 &= z_3^2 - 2z_2z_6, \\
A_5 &= -z_4^2 - 2z_1z_6, \\
A_6 &= -z_6^2 - 2z_3z_5.
\end{align*}
\tag{3.7}
\]

Hence,
\[
\sqrt{13}ST^\nu(A_0) = A_0 + \zeta^\nu A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6. \tag{3.8}
\]

Let \( H := Q^5P^2 \cdot P^2Q^6P^8 \cdot Q^5P^2 \cdot P^3Q \) where \( P = ST^{-1}S \) and \( Q = ST^3 \). Then (see [30], p.27)
\[
H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{3.9}
\]

Note that \( H^6 = 1 \) and \( H^{-1}TH = -T^4 \). Thus, \( \langle H, T \rangle \cong \mathbb{Z}_{13} \rtimes \mathbb{Z}_6 \). Hence, it is a maximal subgroup of order 78 of \( \text{PSL}(2, 13) \) with index 14. We find that \( \varphi_\infty^2 \) is invariant under the action of the maximal subgroup \( \langle H, T \rangle \). Note that
\[
\varphi_\infty = \sqrt{13}A_0, \quad \varphi_\nu = A_0 + \zeta^\nu A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6
\]
for \( \nu = 0, 1, \ldots, 12 \). Let \( w = \varphi^2, w_\infty = \varphi_\infty^2 \) and \( w_\nu = \varphi_\nu^2 \). Then \( w_\infty, w_\nu \) for \( \nu = 0, \ldots, 12 \) form an algebraic equation of degree fourteen, which is just the Jacobian equation of degree fourteen (see [17], pp.161-162), whose roots are these \( w_\nu \) and \( w_\infty \):
\[
w_{14}^2 + a_{13}w_{13} + \cdots + a_{13}w + a_{14} = 0.
\]

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On the other hand, we have

\[-13\sqrt{13}ST^\nu(z_1) \cdot ST^\nu(z_2) \cdot ST^\nu(z_3) \]

\[-r_4(\zeta^{8\nu} z_1^3 + \zeta^{7\nu} z_2^3 + \zeta^{11\nu} z_3^3) - r_2(\zeta^{6\nu} z_4^3 + \zeta^{2\nu} z_5^3 + \zeta^{3\nu} z_6^3)\]

\[-r_3(\zeta^{12\nu} z_1^2 z_2 + \zeta^{\nu} z_2^2 z_3 + \zeta^{10\nu} z_2^3 z_1) - r_1(\zeta^{9\nu} z_5^2 z_6 + \zeta^{3\nu} z_6^2 z_4)\]

\[+ 2r_1(\zeta^{3\nu} z_1^2 z_2 + \zeta^{\nu} z_2^2 z_3 + \zeta^{9\nu} z_3 z_1^2) - 2r_3(\zeta^{10\nu} z_4 z_5^2 + \zeta^{12\nu} z_5 z_6^2 + \zeta^{4\nu} z_6 z_4^2)\]

\[+ 2r_4(\zeta^{\nu} z_4^2 z_5 + \zeta^{11\nu} z_5^2 z_6 + \zeta^{8\nu} z_3^2 z_4) - 2r_2(\zeta^{6\nu} z_1 z_4^2 + \zeta^{2\nu} z_2 z_3^2 + \zeta^{5\nu} z_3 z_2)\]

\[+ r_1(\zeta^{3\nu} z_1^2 z_5 + \zeta^{\nu} z_2 z_6 + \zeta^{9\nu} z_3 z_4) + r_3(\zeta^{10\nu} z_2 z_4^2 + \zeta^{12\nu} z_3 z_5^2 + \zeta^{4\nu} z_5 z_6^2)\]

\[+ r_2(\zeta^{6\nu} z_1^2 z_6 + \zeta^{2\nu} z_2^2 z_4 + \zeta^{5\nu} z_3 z_5) + r_4(\zeta^{7\nu} z_3 z_4^2 + \zeta^{11\nu} z_1 z_5^2 + \zeta^{8\nu} z_2 z_6^2)\]

\[+ r_0 z_1 z_2 z_3 + r_\infty z_4 z_5 z_6 + r_1(\zeta^{3\nu} z_3 z_4 z_5 + \zeta^{8\nu} z_2 z_3 z_5 + \zeta^{7\nu} z_1 z_3 z_6)\]

\[+ r_2(\zeta^{2\nu} z_1 z_4 z_5 + \zeta^{5\nu} z_2 z_5 z_6 + \zeta^{6\nu} z_3 z_4 z_6)\]

\[+ r_3(\zeta^{10\nu} z_1 z_2 z_6 + \zeta^{14\nu} z_1 z_2 z_5 + \zeta^{12\nu} z_2 z_3 z_4)\]

where

\[r_0 = 2(\theta_1 - \theta_3) - 3(\theta_2 - \theta_4), \quad r_\infty = 2(\theta_4 - \theta_2) - 3(\theta_1 - \theta_3),\]

\[r_1 = \sqrt{-13 - 2\sqrt{13}}, \quad r_2 = \sqrt{-13 + 3\sqrt{13}}, \quad r_3 = \sqrt{-13 + 2\sqrt{13}}, \quad r_4 = \sqrt{-13 - 3\sqrt{13}}.\]

This leads us to define the following senary cubic forms (cubic forms in six
variables):

\[
\begin{align*}
D_0 &= z_1 z_2 z_3, \\
D_1 &= 2z_2 z_3^2 + z_2^2 z_6 - z_1^2 z_5 + z_1 z_5 z_6, \\
D_2 &= -z_6^3 + z_2 z_4^2 - 2z_2 z_5^2 + z_1 z_4 z_5 + 3z_3 z_5 z_6, \\
D_3 &= 2z_1 z_2^2 + z_1^2 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\
D_4 &= -z_2^2 z_3 + z_1 z_6^2 - 2z_4 z_5^2 - z_1 z_5 z_6, \\
D_5 &= -z_4^3 + z_4 z_6^2 - z_3 z_6^2 + z_2 z_5 z_6 + 3z_1 z_4 z_5, \\
D_6 &= -z_3^2 + z_1 z_6^2 - 2z_1 z_4^2 + z_3 z_4 z_6 + z_2 z_4 z_5, \\
D_7 &= -z_3^2 + z_3 z_4^2 - z_1 z_3 z_6 - 3z_1 z_2 z_5 + 2z_1^2 z_4, \\
D_8 &= -z_1^3 + z_2 z_6^2 - z_2 z_3 z_5 - 3z_1 z_3 z_4 + 2z_3 z_6, \\
D_9 &= 2z_1^2 z_3 + z_2 z_4 - z_5 z_6 + z_2 z_4 z_6, \\
D_{10} &= -z_1 z_3^2 + z_3 z_4^2 - 2z_4 z_5 - z_1 z_2 z_6, \\
D_{11} &= -z_3^2 + z_1 z_6 - z_1 z_2 z_4 - 3z_2 z_3 z_6 + 2z_2 z_5, \\
D_{12} &= -z_1^2 z_2 + z_3 z_5^2 - 2z_5 z_6^2 - z_2 z_3 z_4, \\
D_{\infty} &= z_4 z_5 z_6.
\end{align*}
\]

Then

\[
-13\sqrt{13ST''(D_0)} = r_0 D_0 + r_1 \zeta' D_1 + r_2 \zeta^{2\nu} D_2 + r_1 \zeta^{3\nu} D_3 + r_3 \zeta^{4\nu} D_4 + r_2 \zeta^{5\nu} D_5 + r_2 \zeta^{6\nu} D_6 + r_4 \zeta^{7\nu} D_7 + r_4 \zeta^{8\nu} D_8 + r_1 \zeta^{9\nu} D_9 + r_3 \zeta^{10\nu} D_{10} + r_4 \zeta^{11\nu} D_{11} + r_3 \zeta^{12\nu} D_{12} + r_\infty D_{\infty}.
\]

\[
-13\sqrt{13ST''(D_{\infty})} = r_\infty D_0 - r_3 \zeta^{\nu} D_1 - r_4 \zeta^{2\nu} D_2 - r_3 \zeta^{3\nu} D_3 + r_1 \zeta^{4\nu} D_4 + r_4 \zeta^{5\nu} D_5 - r_4 \zeta^{6\nu} D_6 + r_2 \zeta^{7\nu} D_7 + r_2 \zeta^{8\nu} D_8 + r_3 \zeta^{9\nu} D_9 + r_1 \zeta^{10\nu} D_{10} + r_2 \zeta^{11\nu} D_{11} + r_1 \zeta^{12\nu} D_{12} - r_0 D_{\infty}.
\]

Let

\[
\delta_\infty(z_1, z_2, z_3, z_4, z_5, z_6) = 13^2 (z_1^2 z_2^2 z_3 + z_4^2 z_5 z_6)
\]

and

\[
\delta_\nu(z_1, z_2, z_3, z_4, z_5, z_6) = \delta_\infty(ST''(z_1, z_2, z_3, z_4, z_5, z_6))
\]

for \(\nu = 0, 1, \ldots, 12\). Then

\[
\delta_\nu = 13^2 ST''(G_0) = -13 G_0 + \zeta G_1 + \zeta^{2\nu} G_2 + \cdots + \zeta^{12\nu} G_{12},
\]

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We have that
\[ G_0 = D_0^2 + D^2, \]
\[ G_1 = -D_7^2 + 2D_0D_1 + 10D_\infty D_1 + 2D_2D_{12} + 2D_3D_{11} - 4D_4D_{10} - 2D_5D_5, \]
\[ G_2 = -2D_1^2 - 4D_6D_2 + 6D_\infty D_2 - 2D_4D_{11} + 2D_6D_{10} - 2D_8D_9 - 2D_7D_8, \]
\[ G_3 = -D_8^2 + 2D_0D_3 + 10D_\infty D_3 + 2D_6D_{10} - 2D_9D_7 - 4D_1D_4 - 2D_1D_2, \]
\[ G_4 = -D_2^2 + 10D_0D_4 - 2D_\infty D_4 + 2D_5D_{12} + 2D_6D_8 - 4D_1D_3 - 2D_1D_7, \]
\[ G_5 = -2D_9^2 - 4D_6D_5 + 6D_\infty D_5 - 2D_10D_8 + 2D_6D_{12} - 2D_2D_3 - 2D_11D_7, \]
\[ G_6 = -2D_3^2 - 4D_0D_6 + 6D_\infty D_6 - 2D_{12}7 + 2D_2D_4 - 2D_5D_4 - 2D_8D_{11}, \]
\[ G_7 = -2D_2^2 + 4D_0D_7 + 4D_\infty D_7 - 2D_1D_6 + 2D_5D_5 - 2D_8D_{12} - 2D_9D_{11}, \]
\[ G_8 = -2D_4^2 + 4D_0D_8 + 4D_\infty D_8 - 2D_3D_5 + 2D_6D_2 - 2D_1D_{10} - 2D_1D_7, \]
\[ G_9 = -D_{11}^2 + 2D_0D_9 + 10D_\infty D_9 + 2D_3D_4 + 2D_1D_8 - 4D_10D_{12} - 2D_3D_6, \]
\[ G_{10} = -D_5^2 + 10D_0D_{10} - 2D_\infty D_{10} + 2D_6D_4 + 2D_3D_7 - 4D_0D_1 - 2D_12D_{11}, \]
\[ G_{11} = -2D_2^2 + 6D_0D_{11} + 4D_\infty D_{11} - 2D_9D_2 + 2D_3D_6 - 2D_7D_4 - 2D_3D_8, \]
\[ G_{12} = -D_6^2 + 10D_0D_{12} - 2D_\infty D_{12} + 2D_2D_{10} - 2D_1D_{11} - 4D_3D_9 - 2D_4D_8. \]

We have that \( G_0 \) is invariant under the action of \( \langle H, T \rangle \), a maximal subgroup of order 78 of PSL(2,13) with index 14. Note that \( \delta_\infty, \delta_\nu \) for \( \nu = 0, \ldots, 12 \) form an algebraic equation of degree fourteen. However, we have \( \delta_\infty + \sum_{\nu=0}^{12} \delta_\nu = 0 \). Hence, it is not the Jacobian equation of degree fourteen.

Recall that the theta functions with characteristic \( \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] \in \mathbb{R}^2 \) is defined
by the following series which converges uniformly and absolutely on compact subsets of \( \mathbb{C} \times \mathbb{H} \) (see [9], p. 73):

\[
\theta \left[ \frac{e}{e'} \right] (z, \tau) = \sum_{n \in \mathbb{Z}} \exp \left\{ 2\pi i \left[ \frac{1}{2} (n + \frac{e}{2})^2 \tau + (n + \frac{e}{2}) \left( z + \frac{e'}{2} \right) \right] \right\}.
\]

The modified theta constants (see [9], p. 215) \( \varphi_l(\tau) := \theta[\chi_l](0, k\tau) \), where the characteristic \( \chi_l = \left[ \frac{2l+1}{k} \right] \), \( l = 0, \ldots, \frac{k-3}{2} \), for odd \( k \) and \( \chi_l = \left[ \frac{2l}{k} \right] \), \( l = 0, \ldots, \frac{k}{2} \), for even \( k \). We have the following:

**Proposition 3.1.** (see [9], p. 236). For each odd integer \( k \geq 5 \), the map \( \Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \ldots, \varphi_{\frac{k-1}{2}}(\tau), \varphi_{\frac{k-3}{2}}(\tau)) \) from \( \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \) to \( \mathbb{C}^{k-1/2} \), defines a holomorphic mapping from \( \mathbb{H}/\Gamma(k) \) into \( \mathbb{C}^{k-1/2} \).

In our case, the map \( \Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau), \varphi_4(\tau), \varphi_5(\tau)) \) gives a holomorphic mapping from the modular curve \( X(13) = \mathbb{H}/\Gamma(13) \) into \( \mathbb{C}^{5} \), which corresponds to our six-dimensional representation, i.e., up to the constants, \( z_1, \ldots, z_6 \) are just modular forms \( \varphi_0(\tau), \ldots, \varphi_5(\tau) \). Let

\[
\begin{align*}
    a_1(z) &:= e^{-\frac{11\pi i}{120} \theta} \left[ \frac{11}{13} \right] (0, 13z) = q^{\frac{104}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+11n), \\
    a_2(z) &:= e^{-\frac{7\pi i}{40} \theta} \left[ \frac{7}{15} \right] (0, 13z) = q^{\frac{40}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+7n), \\
    a_3(z) &:= e^{-\frac{5\pi i}{20} \theta} \left[ \frac{5}{13} \right] (0, 13z) = q^{\frac{20}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+5n), \\
    a_4(z) &:= -e^{-\frac{3\pi i}{26} \theta} \left[ \frac{3}{13} \right] (0, 13z) = -q^{\frac{9}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+3n), \\
    a_5(z) &:= e^{-\frac{9\pi i}{52} \theta} \left[ \frac{9}{13} \right] (0, 13z) = q^{\frac{81}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+9n), \\
    a_6(z) &:= e^{-\frac{7\pi i}{26} \theta} \left[ \frac{1}{13} \right] (0, 13z) = q^{\frac{1}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^\frac{1}{2}(13n^2+n)
\end{align*}
\]

be the theta constants of order 13 and

\[
A(z) := (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^T.
\]

The significance of our six dimensional representation of \( \text{SL}(2, 13) \) comes from the following:
Proposition 3.2 (see [30], Proposition 2.5). If $z \in \mathbb{H}$, then the following relations hold:

$$A(z + 1) = e^{-\frac{3\pi i}{4}}TA(z), \quad A\left(-\frac{1}{z}\right) = e^{\frac{\pi i}{4}}\sqrt{z}SA(z),$$

(3.16)

where $S$ and $T$ are given in (3.1) and (3.2), and $0 < \arg{\sqrt{z}} \leq \pi/2$.

Note that it is (3.16) that gives an explicit realization of the isomorphism between the unique sub-representation of parabolic modular forms of weight $\frac{1}{2}$ on $X(13)$ and the six-dimensional complex representation of $\mathrm{SL}(2,13)$ generated by $S$ and $T$.

Recall that the principal congruence subgroup of level 13 is the normal subgroup $\Gamma(13)$ of $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ defined by the exact sequence $1 \rightarrow \Gamma(13) \rightarrow \Gamma(1) \rightarrow \mathrm{PSL}(2,13) \rightarrow 1$ where $f(\gamma) \equiv \gamma \pmod{13}$ for $\gamma \in \Gamma = \Gamma(1)$. There is a representation $\rho : \Gamma \rightarrow \mathrm{PGL}(6, \mathbb{C})$ with kernel $\Gamma(13)$ defined as follows: if $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\rho(t) = T$ and $\rho(s) = S$. To see that such a representation exists, note that $\Gamma$ is defined by the presentation $\langle s, t; s^2 = (st)^3 = 1 \rangle$ satisfied by $s$ and $t$ and we have proved that $S$ and $T$ satisfy these relations (in the projective coordinates). Moreover, we have proved that $\mathrm{PSL}(2,13)$ is defined by the presentation $\langle S, T; S^2 = T^{13} = (ST)^3 = 1 \rangle$.

Let $p = st^{-1}s$ and $q = st^3$. Then

$$h := q^5p^2 \cdot p^2q^6 \cdot q^5p^2 \cdot p^3q = \begin{pmatrix} 4,428,249 & -10,547,030 \\ -11,594,791 & 27,616,019 \end{pmatrix}$$

satisfies that $\rho(h) = H$. The off-diagonal elements of the matrix $h$, which corresponds to $H$, are congruent to 0 mod 13. The connection to $\Gamma_0(13)$ should be obvious.

Put $x_i(z) = \eta(z)a_i(z)$, $y_i(z) = \eta^3(z)a_i(z)$ and $u_i(z) = \eta^9(z)a_i(z)$ ($1 \leq i \leq 6$). Let

$$\begin{cases} X(z) = (x_1(z), \ldots, x_6(z))^T, \\ Y(z) = (y_1(z), \ldots, y_6(z))^T, \\ U(z) = (u_1(z), \ldots, u_6(z))^T. \end{cases}$$

Then

$$\begin{cases} X(z) = \eta(z)A(z) \\ Y(z) = \eta^3(z)A(z), \\ U(z) = \eta^9(z)A(z). \end{cases}$$
Recall that $\eta(z)$ satisfies the following transformation formulas:

$$
\eta(z + 1) = e^{\frac{2\pi i}{3}} \eta(z) \quad \text{and} \quad \eta(-\frac{1}{z}) = e^{-\frac{4\pi i}{3}} \sqrt{z} \eta(z).
$$

By Proposition 3.2, we have

$$
X(z + 1) = e^{-\frac{2\pi i}{3}} \rho(t) X(z), \quad X \left(-\frac{1}{z}\right) = z \rho(s) X(z),
$$

$$
Y(z + 1) = e^{-\frac{2\pi i}{3}} \rho(t) Y(z), \quad Y \left(-\frac{1}{z}\right) = e^{-\frac{4\pi i}{3}} z^2 \rho(s) Y(z).
$$

$$
U(z + 1) = \rho(t) U(z), \quad U \left(-\frac{1}{z}\right) = z^5 \rho(s) U(z).
$$

Define $j(\gamma, z) := cz + d$ if $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Hence,

$$
\begin{cases}
X(\gamma(z)) = u(\gamma) j(\gamma, z)^{\rho(\gamma)} X(z), \\
Y(\gamma(z)) = v(\gamma) j(\gamma, z)^2 \rho(\gamma) Y(z), \\
U(\gamma(z)) = j(\gamma, z)^5 \rho(\gamma) U(z)
\end{cases}
$$

for $\gamma \in \Gamma(1)$, where $u(\gamma) = 1, \omega$ or $\omega^2$ with $\omega = e^{\frac{2\pi i}{3}}$ and $v(\gamma) = \pm 1$ or $\pm i$. Since $\Gamma(13) = \ker \rho$, we have $X(\gamma(z)) = u(\gamma) j(\gamma, z) X(z)$, $Y(\gamma(z)) = v(\gamma) j(\gamma, z)^2 Y(z)$ and $U(\gamma(z)) = j(\gamma, z)^5 U(z)$ for $\gamma \in \Gamma(13)$. This means that the functions $x_1(z), \ldots, x_6(z)$ are modular forms of weight one for $\Gamma(13)$ with the same multiplier $u(\gamma) = 1, \omega$ or $\omega^2$ and $y_1(z), \ldots, y_6(z)$ are modular forms of weight two for $\Gamma(13)$ with the same multiplier $v(\gamma) = \pm 1$ or $\pm i$.

From now on, we will use the following abbreviation

$$
A_j = A_j(a_1(z), \ldots, a_6(z)) \quad (0 \leq j \leq 6),
$$

$$
D_j = D_j(a_1(z), \ldots, a_6(z)) \quad (j = 0, 1, \ldots, 12, \infty)
$$

and

$$
G_j = G_j(a_1(z), \ldots, a_6(z)) \quad (0 \leq j \leq 12).
$$
We have

\[
\begin{align*}
A_0 &= q^{\frac{7}{12}} (1 + O(q)), \\
A_1 &= q^{\frac{17}{22}} (2 + O(q)), \\
A_2 &= q^{\frac{29}{32}} (2 + O(q)), \\
A_3 &= q^{\frac{3}{22}} (1 + O(q)), \\
A_4 &= q^{\frac{7}{22}} (-1 + O(q)), \\
A_5 &= q^{\frac{9}{22}} (-1 + O(q)), \\
A_6 &= q^{\frac{3}{22}} (-1 + O(q)),
\end{align*}
\]

and

\[
\begin{align*}
D_0 &= q^{\frac{7}{12}} (1 + O(q)), \\
D_\infty &= q^{\frac{7}{12}} (-1 + O(q)), \\
D_1 &= q^{\frac{5}{6}} (2 + O(q)), \\
D_2 &= q^{\frac{1}{6}} (-1 + O(q)), \\
D_3 &= q^{\frac{1}{6}} (1 + O(q)), \\
D_4 &= q^{\frac{5}{6}} (-2 + O(q)), \\
D_5 &= q^{\frac{5}{6}} (-1 + O(q)), \\
D_6 &= q^{\frac{5}{6}} (-1 + O(q)), \\
D_7 &= q^{\frac{41}{104}} (1 + O(q)), \\
D_8 &= q^{\frac{51}{104}} (3 + O(q)), \\
D_9 &= q^{\frac{9}{104}} (-2 + O(q)), \\
D_{10} &= q^{\frac{67}{104}} (1 + O(q)), \\
D_{11} &= q^{\frac{7}{104}} (-4 + O(q)), \\
D_{12} &= q^{\frac{83}{104}} (-1 + O(q)).
\end{align*}
\]

Hence,

\[
\begin{align*}
G_0 &= q^{\frac{7}{12}} (1 + O(q)), \\
G_1 &= q^{\frac{41}{32}} (13 + O(q)), \\
G_2 &= q^{\frac{7}{32}} (-22 + O(q)), \\
G_3 &= q^{\frac{7}{32}} (-21 + O(q)), \\
G_4 &= q^{\frac{3}{32}} (-1 + O(q)), \\
G_5 &= q^{\frac{7}{32}} (2 + O(q)), \\
G_6 &= q^{\frac{11}{32}} (2 + O(q)), \\
G_7 &= q^{\frac{15}{32}} (-2 + O(q)), \\
G_8 &= q^{\frac{15}{32}} (-8 + O(q)), \\
G_9 &= q^{\frac{3}{32}} (6 + O(q)), \\
G_{10} &= q^{\frac{37}{32}} (1 + O(q)), \\
G_{11} &= q^{\frac{37}{32}} (-8 + O(q)), \\
G_{12} &= q^{\frac{37}{32}} (17 + O(q)).
\end{align*}
\]
Note that

\[ w_\nu = (A_0 + \zeta^{2\nu} A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6)^2 \]
\[ = A_0^2 + 2(A_1 A_5 + A_2 A_3 + A_4 A_6) + \]
\[ + 2\zeta^{2\nu}(A_0 A_1 + A_2 A_6) + 2\zeta^{4\nu}(A_0 A_4 + A_2 A_5) + \]
\[ + 2\zeta^{9\nu}(A_0 A_3 + A_5 A_6) + 2\zeta^{12\nu}(A_0 A_5 + A_3 A_4) + \]
\[ + 2\zeta^{10\nu}(A_0 A_6 + A_1 A_3) + 2\zeta^{3\nu}(A_0 A_2 + A_1 A_4) + \]
\[ + \zeta^{2\nu}(A_1^2 + 2A_4 A_5) + \zeta^{5\nu}(A_3^2 + 2A_1 A_2) + \]
\[ + \zeta^{6\nu}(A_2^2 + 2A_3 A_6) + \zeta^{11\nu}(A_5^2 + 2A_1 A_6) + \]
\[ + \zeta^{8\nu}(A_2^2 + 2A_3 A_5) + \zeta^{7\nu}(A_6^2 + 2A_4 A_2), \]

where

\[ A_0^2 + 2(A_1 A_5 + A_2 A_3 + A_4 A_6) = q^\frac{1}{2} (-1 + O(q)), \]

\[
\begin{align*}
A_0 A_1 + A_2 A_6 &= q^{\frac{17}{26}} (-3 + O(q)), \\
A_0 A_4 + A_2 A_5 &= q^{\frac{21}{26}} (-3 + O(q)), \\
A_0 A_3 + A_5 A_6 &= q^{\frac{5}{26}} (1 + O(q)), \\
A_0 A_5 + A_3 A_4 &= q^{\frac{12}{26}} (-1 + O(q)), \\
A_0 A_6 + A_1 A_3 &= q^{\frac{7}{26}} (-1 + O(q)), \\
A_0 A_2 + A_1 A_4 &= q^{\frac{47}{26}} (-1 + O(q)),
\end{align*}
\]

and

\[
\begin{align*}
A_1^2 + 2A_4 A_5 &= q^{\frac{17}{26}} (6 + O(q)), \\
A_2^2 + 2A_1 A_2 &= q^{\frac{23}{26}} (8 + O(q)), \\
A_1^2 + 2A_3 A_6 &= q^{\frac{25}{26}} (-1 + O(q)), \\
A_5^2 + 2A_1 A_6 &= q^{\frac{9}{26}} (-3 + O(q)), \\
A_2^2 + 2A_3 A_5 &= q^{\frac{29}{26}} (2 + O(q)), \\
A_6^2 + 2A_4 A_2 &= q^{\frac{7}{26}} (1 + O(q)).
\end{align*}
\]

In the introduction, the invariant homogeneous polynomials $\Phi_{m,n}$ are defined. Now we give the normalization for the following three families of polynomials of degrees $d = 12, 16, 20$ and $30$. For $d = 12$, there are two such
invariant homogeneous polynomials $\Phi_{3,0}$ and $\Phi_{0,2}$:

\[
\begin{align*}
\Phi_{3,0} &:= -\frac{1}{13 \cdot 30} \left( \sum_{\nu=0}^{12} w_\nu^3 + w_\infty^3 \right), \\
\Phi_{0,2} &:= -\frac{1}{13 \cdot 52} \left( \sum_{\nu=0}^{12} \delta_\nu^2 + \delta_\infty^2 \right),
\end{align*}
\tag{3.18}
\]

For $d = 16$, there are two such invariant homogeneous polynomials $\Phi_{4,0}$ and $\Phi_{1,2}$ which need not to be normalized. For $d = 20$, there are two such invariant homogeneous polynomials $\Phi_{5,0}$ and $\Phi_{2,2}$:

\[
\begin{align*}
\Phi_{5,0} &:= \frac{1}{13 \cdot 25} \left( \sum_{\nu=0}^{12} w_\nu^5 + w_\infty^5 \right), \\
\Phi_{2,2} &:= \frac{1}{13 \cdot 8} \left( \sum_{\nu=0}^{12} w_\nu^2 \delta_\nu^2 + w_\infty^2 \delta_\infty^2 \right),
\end{align*}
\tag{3.19}
\]

For $d = 30$, there are three such invariant homogeneous polynomials $\Phi_{0,5}$, $\Phi_{3,3}$ and $\Phi_{6,1}$:

\[
\begin{align*}
\Phi_{0,5} &:= -\frac{1}{13 \cdot 1315} \left( \sum_{\nu=0}^{12} \delta_\nu^5 + \delta_\infty^5 \right), \\
\Phi_{3,3} &:= -\frac{1}{13 \cdot 216} \left( \sum_{\nu=0}^{12} w_\nu^3 \delta_\nu^3 + w_\infty^3 \delta_\infty^3 \right), \\
\Phi_{6,1} &:= -\frac{1}{13 \cdot 171} \left( \sum_{\nu=0}^{12} w_\nu^6 \delta_\nu^2 + w_\infty^6 \delta_\infty^2 \right),
\end{align*}
\tag{3.20}
\]

**Theorem 3.3.** The $G$-invariant homogeneous polynomials $\Phi_{m,n}$ of degrees $d = 4, 8, 10, 12, 14, 16, 20$ and $30$ in $x_1(z), \ldots, x_6(z)$ can be identified
with modular forms as follows:

\[
\begin{align*}
\Phi_4(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_8(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{10}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{3,0}(x_1(z), \ldots, x_6(z)) &= \Delta(z), \\
\Phi_{0,2}(x_1(z), \ldots, x_6(z)) &= \Delta(z), \\
\Phi_{14}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{4,0}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{1,2}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{5,0}(x_1(z), \ldots, x_6(z)) &= \eta(z)^8 \Delta(z) E_4(z), \\
\Phi_{2,2}(x_1(z), \ldots, x_6(z)) &= \eta(z)^8 \Delta(z) E_4(z), \\
\Phi_{0,5}(x_1(z), \ldots, x_6(z)) &= \Delta(z)^2 E_6(z), \\
\Phi_{3,3}(x_1(z), \ldots, x_6(z)) &= \Delta(z)^2 E_6(z), \\
\Phi_{6,1}(x_1(z), \ldots, x_6(z)) &= \Delta(z)^2 E_6(z).
\end{align*}
\] (3.21)

Proof. We divide the proof into four parts (see also [31]). The first part is the calculation of \(\Phi_{5,0} \text{ and } \Phi_{3,0}\). Up to a constant, \(\Phi_{5,0} = w_0^5 + w_1^5 + \cdots + w_{12}^5 + w_{\infty}^5\). As a polynomial in six variables, \(\Phi_{5,0}(z_1, z_2, z_3, z_4, z_5, z_6)\) is a \(G\)-invariant polynomial. Moreover, for \(\gamma \in \Gamma(1)\),

\[
\Phi_{5,0}(Y(\gamma(z))^T) = \Phi_{5,0}(v(\gamma) j(\gamma, z)^2 (\rho(\gamma) Y(z))^T)
\]

\[
= v(\gamma)^20 j(\gamma, z)^{40} \Phi_{5,0}(\rho(\gamma) Y(z))^T = j(\gamma, z)^{40} \Phi_{5,0}(\rho(\gamma) Y(z))^T.
\]

Note that \(\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G\) and \(\Phi_{5,0}\) is a \(G\)-invariant polynomial, we have

\[
\Phi_{5,0}(Y(\gamma(z))^T) = j(\gamma, z)^{40} \Phi_{5,0}(Y(z))^T, \quad \text{for } \gamma \in \Gamma(1).
\]

This implies that \(\Phi_{5,0}(y_1(z), \ldots, y_6(z))\) is a modular form of weight 40 for the full modular group \(\Gamma(1)\). Moreover, we will show that it is a cusp form.
In fact,

\[ \Phi_{5,0}(a_1(z), \ldots, a_6(z)) = 13^5 q^{\frac{13}{25}} (1 + O(q))^5 + \]
\[ + \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (-1 + O(q)) + \]
\[ + 2 \zeta^\nu q^{\frac{13}{25}} (-3 + O(q)) + 2 \zeta^{3\nu} q^{\frac{13}{25}} (-3 + O(q)) + 2 \zeta^{9\nu} q^{\frac{13}{25}} (1 + O(q)) + \]
\[ + 2 \zeta^{12\nu} q^{\frac{13}{25}} (-1 + O(q)) + 2 \zeta^{10\nu} q^{\frac{7}{25}} (-1 + O(q)) + 2 \zeta^{4\nu} q^{\frac{47}{25}} (-1 + O(q)) + \]
\[ + \zeta^{2\nu} q^{\frac{17}{25}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{42}{25}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{22}{25}} (-1 + O(q)) + \]
\[ + \zeta^{11\nu} q^{\frac{9}{25}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{29}{25}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{1}{25}} (1 + O(q)) ]^5. \]

We will calculate the \( q^{\frac{1}{2}} \)-term which is the lowest degree. For the partition \( 13 = 4 \cdot 1 + 9 \), the corresponding term is

\[ \left( \frac{5}{4, 1} \right) (\zeta^{7\nu} q^{\frac{1}{25}})^4 \cdot (-3) \zeta^{11\nu} q^{\frac{2}{25}} = -15 q^{\frac{1}{2}}. \]

For the partition \( 13 = 3 \cdot 1 + 2 \cdot 5 \), the corresponding term is

\[ \left( \frac{5}{3, 2} \right) (\zeta^{7\nu} q^{\frac{1}{25}})^3 \cdot (2 \zeta^{9\nu} q^{\frac{5}{25}})^2 = 40 q^{\frac{1}{2}}. \]

Hence, for \( \Phi_{5,0}(y_1(z), \ldots, y_6(z)) \) which is a modular form for \( \Gamma(1) \) with weight 40, the lowest degree term is given by

\[ (-15 + 40) q^{\frac{1}{2}} \cdot q^{\frac{3}{25} \cdot 20} = 25 q^3. \]

Thus,

\[ \Phi_{5,0}(y_1(z), \ldots, y_6(z)) = q^3 (13 \cdot 25 + O(q)). \]

The leading term of \( \Phi_{5,0}(y_1(z), \ldots, y_6(z)) \) together with its weight 40 suffice to identify this modular form with \( \Phi_{5,0}(y_1(z), \ldots, y_6(z)) = 13 \cdot 25 \Delta(z)^3 E_4(z) \). Consequently,

\[ \Phi_{5,0}(x_1(z), \ldots, x_6(z)) = 13 \cdot 25 \Delta(z)^3 E_4(z) / \eta(z)^{40} = 13 \cdot 25 \eta(z)^8 \Delta(z) E_4(z). \]

Up to a constant, \( \Phi_{3,0} = w_3^3 + w_1^3 + \cdots + w_{12}^3 + w_{\infty}^3 \), The calculation of \( \Phi_{12} \) is similar as that of \( \Phi_{20} \). We find that

\[ \Phi_{12}(x_1(z), \ldots, x_6(z)) = -13 \cdot 30 \Delta(z). \]
The second part is the calculation of $\Phi_4$, $\Phi_8$ and $\Phi_{4,0}$. The calculation of $\Phi_4$ has been done in [30], Theorem 3.1. We will give the calculation of $\Phi_{4,0}$. Up to a constant, $\Phi_{4,0} = w_0^4 + w_1^4 + \cdots + w_{12}^4 + w_\infty^4$. Similar as the above calculation for $\Phi_{5,0}$, we find that $\Phi_{4,0}(y_1(z), \ldots, y_6(z))$ is a modular form of weight 32 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\Phi_{4,0}(a_1(z), \ldots, a_6(z)) = 13^4 q^2 (1 + O(q))^4 +$$

$$+ \sum_{\nu=0}^{12} q^{\frac{12}{3}} (-1 + O(q)) +$$

$$+ 2\zeta^\nu q^{\frac{12}{3}} (-3 + O(q)) + 2\zeta^{3\nu} q^{\frac{12}{3}} (-3 + O(q)) + 2\zeta^{9\nu} q^{\frac{2}{3}} (1 + O(q)) +$$

$$+ 2\zeta^{12\nu} q^{\frac{2}{3}} (-1 + O(q)) + 2\zeta^{10\nu} q^{\frac{2}{3}} (-1 + O(q)) + 2\zeta^{4\nu} q^{\frac{2}{3}} (1 + O(q)) +$$

$$+ \zeta^{2\nu} q^{\frac{17}{3}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{23}{3}} (8 + O(q)) + \zeta^{8\nu} q^{\frac{23}{3}} (1 + O(q)) +$$

$$+ \zeta^{11\nu} q^{\frac{23}{3}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{23}{3}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{23}{3}} (1 + O(q)) |^4.$$

We will calculate the $q$-term which is the lowest degree. For example, consider the partition $26 = 3 \cdot 1 + 23$, the corresponding term is

$$\left( \binom{4}{3,1} \zeta^{7\nu} q^{\frac{23}{3}} \right)^3 \cdot 8\zeta^{5\nu} q^{\frac{23}{3}} = 32q.$$

For the other partitions, the calculation is similar. In conclusion, we find that the coefficients of the $q$-term is an integer. Hence, for $\Phi_{4,0}(y_1(z), \ldots, y_6(z))$ which is a modular form for $\Gamma(1)$ with weight 32, the lowest degree term is given by

some integer $\cdot q \cdot q^{\frac{23}{16}} = $ some integer $\cdot q^3$.

This implies that $\Phi_{4,0}(y_1(z), \ldots, y_6(z))$ has a factor of $\Delta(z)^3$, which is a cusp form of weight 36. Therefore, $\Phi_{4,0}(y_1(z), \ldots, y_6(z)) = 0$. The calculation of $\Phi_8$ is similar as that of $\Phi_{4,0}$.

The third part is the calculation of $\Phi_{0,2}$ and $\Phi_{0,5}$. Up to a constant, $\Phi_{0,2} = \delta_0^2 + \delta_1^2 + \cdots + \delta_{12}^2 + \delta_\infty^2$. As a polynomial in six variables, $\Phi_{0,2}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a $G$-invariant polynomial. Moreover, for $\gamma \in \Gamma(1)$,

$$\Phi_{0,2}(X(\gamma(z))^T) = \Phi_{0,2}(u(\gamma)j(\gamma, z)(\rho(\gamma)X(z))^T)$$

$$= u(\gamma)^{12} j(\gamma, z)^{12} \Phi_{0,2}((\rho(\gamma)X(z))^T) = j(\gamma, z)^{12} \Phi_{0,2}((\rho(\gamma)X(z))^T).$$

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Note that \( \rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G \) and \( \Phi_{0,2} \) is a \( G \)-invariant polynomial, we have
\[
\Phi_{0,2}(X(\gamma(z))^T) = j(\gamma, z)^{12} \Phi_{0,2}(X(z)^T), \quad \text{for } \gamma \in \Gamma(1).
\]
This implies that \( \Phi_{0,2}(x_1(z), \ldots, x_6(z)) \) is a modular form of weight 12 for the full modular group \( \Gamma(1) \). Moreover, we will show that it is a cusp form. In fact,
\[
\Phi_{0,2}(a_1(z), \ldots, a_6(z)) = 13^4 q^\frac{7}{2}(1 + O(q))^2 + \\
+ \sum_{\nu=0}^{12}[-13 q^\frac{7}{2}(1 + O(q)) + \\
+ \zeta^\nu q^\frac{7}{2}(13 + O(q)) + \zeta^{2\nu} q^\frac{7}{2}(-22 + O(q)) + \zeta^{3\nu} q^\frac{7}{2}(-21 + O(q)) + \\
+ \zeta^{4\nu} q^\frac{7}{2}(-1 + O(q)) + \zeta^{5\nu} q^\frac{7}{2}(2 + O(q)) + \zeta^{6\nu} q^\frac{7}{2}(2 + O(q)) + \\
+ \zeta^{7\nu} q^\frac{7}{2}(-2 + O(q)) + \zeta^{8\nu} q^\frac{7}{2}(-8 + O(q)) + \zeta^{9\nu} q^\frac{7}{2}(6 + O(q)) + \\
+ \zeta^{10\nu} q^\frac{7}{2}(1 + O(q)) + \zeta^{11\nu} q^\frac{7}{2}(-8 + O(q)) + \zeta^{12\nu} q^\frac{7}{2}(17 + O(q)))^2.
\]
We will calculate the \( q^{\frac{1}{2}} \)-term which is the lowest degree. For the partition \( 26 = 3 + 23 \), the corresponding term is
\[
\left( \begin{array}{cc} 2 \\ 1, 1 \end{array} \right) \cdot \zeta^{4\nu} q^\frac{7}{2} \cdot (-1) \cdot \zeta^{9\nu} q^\frac{7}{2} \cdot 6 = -12 q^{\frac{1}{2}}.
\]
For the partition \( 26 = 7 + 19 \), the corresponding term is
\[
\left( \begin{array}{cc} 2 \\ 1, 1 \end{array} \right) \cdot \zeta^{5\nu} q^\frac{7}{2} \cdot 2 \cdot \zeta^{8\nu} q^\frac{7}{2} \cdot (-8) = -32 q^{\frac{1}{2}}.
\]
For the partition \( 26 = 11 + 15 \), the corresponding term is
\[
\left( \begin{array}{cc} 2 \\ 1, 1 \end{array} \right) \cdot \zeta^{6\nu} q^\frac{7}{2} \cdot 2 \cdot \zeta^{7\nu} q^\frac{7}{2} \cdot (-2) = -8 q^{\frac{1}{2}}.
\]
Hence, for \( \Phi_{0,2}(x_1(z), \ldots, x_6(z)) \) which is a modular form for \( \Gamma(1) \) with weight 12, the lowest degree term is given by \((-12 - 32 - 8)q^\frac{1}{2} \cdot q^{\frac{7}{2}} = -52q\). Thus,
\[
\Phi_{0,2}(x_1(z), \ldots, x_6(z)) = q(-13 \cdot 52 + O(q)).
\]
The leading term of \( \Phi_{0,2}(x_1(z), \ldots, x_6(z)) \) together with its weight 12 suffice to identify this modular form with
\[
\Phi_{0,2}(x_1(z), \ldots, x_6(z)) = -13 \cdot 52 \Delta(z).
\]
Up to a constant, \( \Phi_{0,5} = \delta_0^5 + \delta_1^5 + \cdots + \delta_{12}^5 + \delta_{\infty}^5 \). As a polynomial in six variables, \( \Phi_{0,5}(z_1, z_2, z_3, z_4, z_5, z_6) \) is a \( G \)-invariant polynomial. Similarly as above, we can show that \( \Phi_{0,5}(x_1(z), \ldots, x_6(z)) \) is a modular form of weight 30 for the full modular group \( \Gamma(1) \). Moreover, we will show that it is a cusp form. In fact,

\[
\Phi_{0,5}(a_1(z), \ldots, a_6(z)) = 13^{10} q^{\frac{34}{4}} (1 + O(q))^5 + \\
+ \sum_{\nu=0}^{12} [-13q^{\frac{\nu}{4}} (1 + O(q)) + \\
+ \zeta^\nu q^{\frac{\nu}{42}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{\nu}{42}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{\nu}{42}} (-21 + O(q)) + \\
+ \zeta^{4\nu} q^{\frac{\nu}{42}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{\nu}{42}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{\nu}{42}} (2 + O(q)) + \\
+ \zeta^{7\nu} q^{\frac{\nu}{42}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{\nu}{42}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{\nu}{42}} (6 + O(q)) + \\
+ \zeta^{10\nu} q^{\frac{\nu}{42}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{\nu}{42}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{\nu}{42}} (17 + O(q))].
\]

We will calculate the \( q^{\frac{3}{4}} \)-term which is the lowest degree. (1) For the partition 39 = 4 \cdot 3 + 27, the corresponding term is

\[
\binom{5}{4,1} (\zeta^{4\nu} q^{\frac{\nu}{42}} \cdot (-1))^4 \cdot \zeta^{10\nu} q^{\frac{\nu}{42}} = 5q^{\frac{3}{4}}.
\]

(2) For the partition 39 = 3 \cdot 3 + 7 + 23, the corresponding term is

\[
\binom{5}{3,1,1} (\zeta^{4\nu} q^{\frac{\nu}{42}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{\nu}{42}} \cdot 2 \cdot \zeta^{9\nu} q^{\frac{\nu}{42}} \cdot 6 = -240q^{\frac{3}{4}}.
\]

(3) For the partition 39 = 3 \cdot 3 + 11 + 19, the corresponding term is

\[
\binom{5}{3,1,1} (\zeta^{4\nu} q^{\frac{\nu}{42}} \cdot (-1))^3 \cdot \zeta^{6\nu} q^{\frac{11\nu}{42}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19\nu}{42}} \cdot (-8) = 320q^{\frac{3}{4}}.
\]

(4) For the partition 39 = 3 \cdot 3 + 2 \cdot 15, the corresponding term is

\[
\binom{5}{3,2} (\zeta^{4\nu} q^{\frac{\nu}{42}} \cdot (-1))^3 \cdot (\zeta^{7\nu} q^{\frac{14\nu}{42}} \cdot (-2))^2 = -40q^{\frac{3}{4}}.
\]

(5) For the partition 39 = 2 \cdot 3 + 3 \cdot 11, the corresponding term is

\[
\binom{5}{2,3} (\zeta^{4\nu} q^{\frac{\nu}{42}} \cdot (-1))^2 \cdot (\zeta^{6\nu} q^{\frac{11\nu}{42}} \cdot 2)^3 = 80q^{\frac{3}{4}}.
\]
(6) For the partition 39 = 2 · 3 + 2 · 7 + 19, the corresponding term is
\[ \left( \frac{5}{2, 2, 1} \right) (\zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{2}} \cdot 2)^2 \cdot \zeta^{8\nu} q^{\frac{10}{2}} \cdot (-8) = -960q^{\frac{4}{3}}. \]

(7) For the partition 39 = 2 · 3 + 7 + 11 + 15, the corresponding term is
\[ \left( \frac{5}{2, 1, 1, 1} \right) (\zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{2}} \cdot 2)^2 \cdot \zeta^{11\nu} q^{\frac{13}{2}} \cdot (-2) = -480q^{\frac{3}{2}}. \]

(8) For the partition 39 = 1 · 3 + 3 · 7 + 15, the corresponding term is
\[ \left( \frac{5}{1, 3, 1} \right) (\zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{2}} \cdot 2)^2 \cdot \zeta^{7\nu} q^{\frac{15}{2}} \cdot (-2) = 320q^{\frac{4}{3}}. \]

(9) For the partition 39 = 1 · 3 + 2 · 7 + 2 · 11, the corresponding term is
\[ \left( \frac{5}{1, 2, 2} \right) (\zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{2}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{2}} \cdot 2)^2 \cdot (\zeta^{7\nu} q^{\frac{15}{2}} \cdot (-2) = -480q^{\frac{4}{3}}. \]

(10) For the partition 39 = 4 · 7 + 11, the corresponding term is
\[ \left( \frac{5}{4, 1} \right) (\zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{2}} \cdot 2)^2 \cdot \zeta^{6\nu} q^{\frac{11}{2}} \cdot 2 = 160q^{\frac{3}{2}}. \]

Hence, for \( \Phi_{0.5}(x_1(z), \ldots, x_6(z)) \) which is a modular form for \( \Gamma(1) \) with weight 30, the lowest degree term is given by
\[(5 - 240 + 320 - 40 + 80 - 960 - 480 + 320 - 480 + 160)q^{\frac{4}{3}} \cdot q^{\frac{30}{2}} = -1315q^2.\]

Thus,
\[ \Phi_{0.5}(x_1(z), \ldots, x_6(z)) = q^2(-13 \cdot 1315 + O(q)). \]

The leading term of \( \Phi_{0.5}(x_1(z), \ldots, x_6(z)) \) together with its weight 30 suffice to identify this modular form with
\[ \Phi_{0.5}(x_1(z), \ldots, x_6(z)) = -13 \cdot 1315\Delta(z)^2 E_6(z). \]

The last part is the calculation of \( \Phi_{10}, \Phi_{14}, \Phi_{1,2}, \Phi_{2,2}, \Phi_{3,3} \) and \( \Phi_{6,1} \). For \( \Phi_{10} = w_0\delta_0 + w_1\delta_1 + \cdots + w_{12}\delta_{12} + w_{\infty}\delta_{\infty} \). As a polynomial in six variables, \( \Phi_{10}(z_1, z_2, z_3, z_4, z_5, z_6) \) is a \( G \)-invariant polynomial. Moreover, for \( \gamma \in \Gamma(1) \),
\[ \Phi_{10}(U(\gamma(z))^T) = \Phi_{10}(j(\gamma, z)^5(\rho(\gamma)U(z))^T) = j(\gamma, z)^5\Phi_{10}((\rho(\gamma)U(z))^T). \]

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Note that $\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G$ and $\Phi_{10}$ is a $G$-invariant polynomial, we have

$$\Phi_{10}(U(z)) = j(\gamma, z)^{50} \Phi_{10}(U(z)^T), \quad \text{for } \gamma \in \Gamma(1).$$

This implies that $\Phi_{10}(u_1(z), \ldots, u_6(z))$ is a modular form of weight 50 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\Phi_{10}(a_1(z), \ldots, a_6(z)) = 13^3 q^{\frac{2}{3}} (1 + O(q))^2 +$$

$$+ \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (1 + O(q)) +$$

$$+ 2 \zeta^{\nu} q^{\frac{11}{26}} (-3 + O(q)) + 2 \zeta^{3\nu} q^{\frac{12}{26}} (-3 + O(q)) + 2 \zeta^{9\nu} q^{\frac{35}{26}} (1 + O(q)) +$$

$$+ 2 \zeta^{12\nu} q^{\frac{31}{26}} (-1 + O(q)) + 2 \zeta^{10\nu} q^{\frac{33}{26}} (-1 + O(q)) + 2 \zeta^{4\nu} q^{\frac{35}{26}} (-1 + O(q)) +$$

$$+ \zeta^{2\nu} q^{\frac{17}{26}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{23}{26}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{29}{26}} (-1 + O(q)) +$$

$$+ \zeta^{11\nu} q^{\frac{33}{26}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{35}{26}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{35}{26}} (1 + O(q))]$$

$$\times [-13q^{\frac{5}{4}} (1 + O(q)) +$$

$$+ \zeta^{\nu} q^{\frac{43}{26}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{45}{26}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{26}} (-21 + O(q)) +$$

$$+ \zeta^{4\nu} q^{\frac{53}{26}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{27}{26}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{12}{26}} (2 + O(q)) +$$

$$+ \zeta^{7\nu} q^{\frac{14}{26}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{23}{26}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{33}{26}} (6 + O(q)) +$$

$$+ \zeta^{10\nu} q^{\frac{27}{26}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{26}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{26}} (17 + O(q))]$$

We will calculate the $q^{\frac{5}{4}}$-term which is the lowest degree:

$$2q^{\frac{5}{26}} \cdot q^{\frac{5}{26}} \cdot (-1) + q^{\frac{1}{26}} \cdot 1 \cdot q^{\frac{11}{26}} \cdot 2 = 0.$$ 

Hence, for $\Phi_{10}(u_1(z), \ldots, u_6(z))$ which is a modular form for $\Gamma(1)$ with weight 50, the lowest degree term is given by

$$\text{some integer} \cdot q^{\frac{5}{4}} \cdot q^{\frac{a}{21}} = \text{some integer} \cdot q^5.$$ 

This implies that $\Phi_{10}(u_1(z), \ldots, u_6(z))$ has a factor of $\Delta(z)^5$, which is a cusp form of weight 60. Therefore, $\Phi_{10}(u_1(z), \ldots, u_6(z)) = 0$. Consequently, $\Phi_{10}(x_1(z), \ldots, x_6(z)) = 0$. The calculation of $\Phi_{14}$ and $\Phi_{1,2}$ is similar as that of $\Phi_{10}$.
For \( \Phi_{2,2} = w_0^2 \delta_0^2 + w_1^2 \delta_1^2 + \cdots + w_{12}^2 \delta_{12}^2 + w_{\infty}^2 \delta_\infty^2 \). As a polynomial in six variables, \( \Phi_{2,2}(z_1, z_2, z_3, z_4, z_5, z_6) \) is a \( G \)-invariant polynomial. Moreover, for \( \gamma \in \Gamma(1) \),

\[
\Phi_{2,2}(U(\gamma(z))^T) = \Phi_{2,2}(j(\gamma, z)^5(\rho(\gamma)U(z))^T) = j(\gamma, z)^{100} \Phi_{2,2}((\rho(\gamma)U(z))^T).
\]

Note that \( \rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G \) and \( \Phi_{2,2} \) is a \( G \)-invariant polynomial, we have

\[
\Phi_{2,2}(U(\gamma(z))^T) = j(\gamma, z)^{100} \Phi_{2,2}(U(z)^T), \quad \text{for } \gamma \in \Gamma(1).
\]

This implies that \( \Phi_{2,2}(u_1(z), \ldots, u_6(z)) \) is a modular form of weight 100 for the full modular group \( \Gamma(1) \). Moreover, we will show that it is a cusp form. In fact,

\[
\Phi_{2,2}(a_1(z), \ldots, a_6(z)) = 13^6 q^{\frac{1}{2}} (1 + O(q))^2 + \\
+ \sum_{\nu=0}^{12} [q^\nu (-1 + O(q)) + \\
+ 2 \zeta^\nu q^{\frac{41}{51}} (-3 + O(q)) + 2 \zeta^{3\nu} q^{\frac{10}{51}} (-3 + O(q)) + 2 \zeta^{9\nu} q^{\frac{3}{51}} (1 + O(q)) + \\
+ 2 \zeta^{12\nu} q^{\frac{11}{51}} (-1 + O(q)) + 2 \zeta^{10\nu} q^{\frac{2}{51}} (-1 + O(q)) + 2 \zeta^{4\nu} q^{\frac{40}{51}} (-1 + O(q)) + \\
+ \zeta^{2\nu} q^{\frac{17}{51}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{34}{51}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{25}{51}} (-1 + O(q)) + \\
+ \zeta^{11\nu} q^{\frac{9}{51}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{28}{51}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{11}{51}} (1 + O(q)) ]^2 \\
x \times [-13q^{\frac{7}{2}} (1 + O(q)) + \\
+ \zeta^\nu q^{\frac{44}{51}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{51}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{50}{51}} (-21 + O(q)) + \\
+ \zeta^{4\nu} q^{\frac{34}{51}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{51}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{14}{51}} (2 + O(q)) + \\
+ \zeta^{7\nu} q^{\frac{15}{51}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{10}{51}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{51}} (6 + O(q)) + \\
+ \zeta^{10\nu} q^{\frac{22}{51}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{44}{51}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{51}{51}} (17 + O(q)) ]^2
\]

We will calculate the \( q^{\frac{4}{7}} \)-term which is the lowest degree, there are three such terms:

(1) \[
(2 \zeta^{9\nu} q^{\frac{3}{51}})^2 \cdot (\zeta^{4\nu} q^{\frac{7}{51}} \cdot (-1))^2 = 4 q^{\frac{4}{7}}. 
\]

(2) \[
(\zeta^{7\nu} q^{\frac{9}{51}})^2 \cdot (\zeta^{6\nu} q^{\frac{11}{51}} \cdot 2)^2 = 4 q^{\frac{4}{7}}. 
\]

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2 \cdot 2 \zeta^{9\nu} q^{\frac{5}{26}} \cdot \zeta^{7\nu} q^{\frac{1}{26}} \cdot 1 \times [2 \zeta^{4\nu} q^{\frac{3}{52}} \cdot (1) \cdot \zeta^{5\nu} q^{\frac{11}{52}} \cdot 2 + (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2] = 0.

Hence, for \Phi_{2,2}(u_1(z), \ldots, u_6(z)) which is a modular form for \Gamma(1) with weight 100, the lowest degree term is given by

\[(4 + 4) q^{\frac{1}{2}} \cdot q^{\frac{9}{24}} \cdot 20 = 8 q^8.\]

Thus,

\[\Phi_{2,2}(u_1(z), \ldots, u_6(z)) = q^8(13 \cdot 8 + O(q)).\]

The leading term of \Phi_{2,2}(u_1(z), \ldots, u_6(z)) together with its weight 100 suffice to identify this modular form with \Phi_{2,2}(u_1(z), \ldots, u_6(z)) = 13 \cdot 8\Delta(z)^8 E_4(z).

Consequently,

\[\Phi_{2,2}(x_1(z), \ldots, x_6(z)) = 13 \cdot 8\Delta(z)^8 E_4(z) = 13 \cdot 8\eta(z)^8 \Delta(z) E_4(z).\]

The calculation of \Phi_{3,3} and \Phi_{6,1} is similar as that of \Phi_{2,2}. We have

\[\Phi_{3,3}(x_1(z), \ldots, x_6(z)) = -13 \cdot 216\Delta(z)^2 E_6(z).\]

\[\Phi_{6,1}(x_1(z), \ldots, x_6(z)) = -13 \cdot 171\Delta(z)^2 E_6(z).\]

After normalization, this completes the proof of Theorem 3.3.

As a consequence of Theorem 3.3, we find an explicit construction of the modular curve \(X = X(13)\).

**Theorem 3.4.** There is a morphism

\[\Phi : X \to Y \subset \mathbb{CP}^5\]

with \(\Phi(z) = (x_1(z), \ldots, x_6(z))\), where \(Y\) is an algebraic curve given by a family of \(G\)-invariant polynomials

\[
\begin{align*}
\Phi_4(z_1, \ldots, z_6) &= 0, \\
\Phi_8(z_1, \ldots, z_6) &= 0, \\
\Phi_{10}(z_1, \ldots, z_6) &= 0, \\
\Phi_{14}(z_1, \ldots, z_6) &= 0.
\end{align*}
\]

(3.22)
Proof. Theorem 3.3 implies that
\[
\begin{align*}
\Phi_4(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_8(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{10}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{14}(x_1(z), \ldots, x_6(z)) &= 0.
\end{align*}
\]

\[\Box\]

4. A different construction: \(E_8\)-singularity from \(C_Y/\text{SL}(2,13)\) over \(X(13)\)

In this section, we will give a different construction of the \(E_8\)-singularity from a quotient \(C_Y/G\) over the modular curve \(X\). The significance of the algebraic curve \(Y\) is that the finite group \(G\) acts linearly on \(C^6\) and on \(\mathbb{CP}^5\) leaving invariant \(Y \subset \mathbb{CP}^5\) and the cone \(C_Y \subset C^6\).

**Theorem 4.1.** The equation of \(E_8\)-singularity
\[
\Phi_3^3 - \Phi_5^2 - 1728\Phi_4^{12} = 0
\]
possesses an infinitely many kinds of distinct modular parametrizations (with the cardinality of the continuum in ZFC set theory)
\[
(\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_{12}^\lambda, \Phi_{20}^\mu, \Phi_{30}^\gamma)
\]
over the modular curve \(X\) as follows:
\[
\begin{align*}
\Phi_{12}^\lambda &= \lambda \Phi_{3,0} + (1 - \lambda)\Phi_{0,2} \mod I, \\
\Phi_{20}^\mu &= \mu \Phi_{5,0} + (1 - \mu)\Phi_{2,2} \mod I, \\
\Phi_{30}^\gamma &= \gamma_1\Phi_{0,5} + \gamma_2\Phi_{3,3} + (1 - \gamma_1 - \gamma_2)\Phi_{6,1} \mod (I, \Phi_{3,0}, \Phi_{0,2}),
\end{align*}
\]
where the ideal
\[
I = (\Phi_4, \Phi_8, \Phi_{10}, \Phi_{14})
\]
and the parameter space \(\{(\lambda, \mu, \gamma)\} \cong \mathbb{C}^4\).

**Proof.** By Theorem 3.3, for degree \(d = 12\), the invariant homogeneous polynomials \(\Phi_{3,0}\) and \(\Phi_{0,2}\) form a two-dimensional complex vector space, and \(\Phi_{3,0} = \Phi_{0,2} = \Delta(z)\) over the modular curve \(X\) (after normalization). Hence, \(\Phi_{12}^\lambda = \Delta(z)\) over the modular curve \(X\). For degree \(d = 20\), the invariant homogeneous polynomials \(\Phi_{5,0}\) and \(\Phi_{2,2}\) form a two-dimensional complex
vector space, and \( \Phi_{5,0} = \Phi_{2,2} = \eta(z)^8 \Delta(z) E_4(z) \) over the modular curve \( X \) (after normalization). Hence, \( \Phi_2^{\mu} = \eta(z)^8 \Delta(z) E_4(z) \) over the modular curve \( X \). Finally, for degree \( d = 30 \), the invariant homogeneous polynomials \( \Phi_{0,5}, \Phi_{3,3} \) and \( \Phi_{6,1} \) form a three-dimensional complex vector space, and \( \Phi_{0,5} = \Phi_{3,3} = \Phi_{6,1} = \Delta(z)^2 E_6(z) \) over the modular curve \( X \) (after normalization). Hence, \( \Phi_\gamma^3 = \Delta(z)^2 E_6(z) \) over the modular curve \( X \). This shows that

\[
\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5 = 0
\]

for \( (\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_\lambda^{12}, \Phi_\mu^{20}, \Phi_\gamma^{30}) \) over \( X \).

\[\Box\]

**Theorem 4.2.** There is a morphism of schemes

\[
f : \text{Spec} \left( \mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5) \right) \to C_Y / G
\]

over the modular curve \( X \) which is a closed immersion. The map \( f \) is a homeomorphism of \( \text{Spec}(\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5)) \) onto a closed subset of \( C_Y / G \) over \( X \). In particular, there are infinitely many kinds of such triples \( (\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_\lambda^{12}, \Phi_\mu^{20}, \Phi_\gamma^{30}) \) whose parameter space \( \{(\lambda, \mu, \gamma)\} \cong \mathbb{C}^3 \).

**Proof.** By Theorem 3.3 and Theorem 4.1, we have the following ring homomorphism

\[
[\mathbb{C}[z_1, z_2, z_3, z_4, z_5, z_6] / I]^\text{SL(2,13)} \to \mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5)
\]

over the modular curve \( X \). This induces a morphism of schemes

\[
\text{Spec} \left( \mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5) \right) \to C_Y / G
\]

over \( X \) which is a closed immersion. In fact, the map \( f \) is a homeomorphism of \( \text{Spec}(\mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_2^3 - \Phi_3^2 - 1728 \Phi_{12}^5)) \) onto a closed subset of \( C_Y / G \) over \( X \). This completes the proof of Theorem 4.2.

\[\Box\]

Note that by Theorem 3.3, we have the following relations over the modular curve \( X \):

\[
j(z) = \frac{E_4(z)^3}{\Delta(z)} = \frac{\Phi_2^3}{\Phi_{12}^5}, \quad j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = \frac{\Phi_3^2}{\Phi_{12}^5}.
\]

Hence, we obtain an infinitely many kinds of distinct decomposition formulas of the elliptic modular function \( j \) in terms of the invariants \( \Phi_{12}, \Phi_{20} \) and \( \Phi_{30} \) over the modular curve \( X \):

\[
j(z) : j(z) - 1728 : 1 = \Phi_2^3 : \Phi_3^2 : \Phi_{12}^5.
\]
where \((\Phi_{12}, \Phi_{20}, \Phi_{30}) = (\Phi_{12}^\lambda, \Phi_{20}^\mu, \Phi_{30}^\gamma)\) are given by (4.2). This shows that there are infinitely many kinds of distinct decomposition formulas of the elliptic modular function \(j\) in terms of the invariant polynomials of the same degrees 12, 20 and 30 over the modular curves, one and only one is given by \(\text{SL}(2, 5)\) corresponding to the modular curve \(X(5)\), the other infinitely many kinds of distinct decomposition formulas are given by \(\text{SL}(2, 13)\) corresponding to the modular curve \(X(13)\).

In the end, let us recall some facts about exotic spheres (see [14]). A \(k\)-dimensional compact oriented differentiable manifold is called a \(k\)-sphere if it is homeomorphic to the \(k\)-dimensional standard sphere. A \(k\)-sphere not diffeomorphic to the standard \(k\)-sphere is said to be exotic. The first exotic sphere was discovered by Milnor in 1956 (see [25]). Two \(k\)-spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of \(k\)-spheres constitute for \(k \geq 5\) a finite abelian group \(\Theta_k\) under the connected sum operation. \(\Theta_k\) contains the subgroup \(bP_{k+1}\) of those \(k\)-spheres which bound a parallelizable manifold. \(bP_{4m}\) \((m \geq 2)\) is cyclic of order \(2^{2m-2}(2^{2m-1} - 1)\), where \(B_m\) is the \(m\)-th Bernoulli number. Let \(g_m\) be the Milnor generator of \(bP_{4m}\). If a \((4m - 1)\)-sphere \(\Sigma\) bounds a parallelizable manifold \(B\) of dimension \(4m\), then the signature \(\tau(B)\) of the intersection form of \(B\) is divisible by 8 and \(\Sigma = \frac{\tau(B)}{8} g_m\). For \(m = 2\) we have \(bP_8 = \Theta_7 = \mathbb{Z}/28\mathbb{Z}\). All these results are due to Milnor-Kervaire (see [15]). In particular,

\[
\sum_{i=0}^{2m} z_i^2 - 1 = 1, \quad z_0^3 + z_1^{6k-1} + z_2^2 + \cdots + z_{2m}^2 = 0
\]

is a \((4m-1)\)-sphere embedded in \(S^{4m+1} \subset \mathbb{C}^{2n+1}\) which represents the element \((-1)^m k \cdot g_m \in bP_{4m}\). For \(m = 2\) and \(k = 1, 2, \cdots, 28\) we get the 28 classes of 7-spheres. Theorem 4.1 and Theorem 4.2 shows that the higher dimensional liftings of infinitely many kinds of distinct constructions of the \(E_8\)-singularity: \(\mathbb{C}^2/\text{SL}(2, 5)\) and

\[
\text{Spec} \left( \mathbb{C}[\Phi_{12}, \Phi_{20}, \Phi_{30}] / (\Phi_{20}^3 - \Phi_{30}^2 - 1728\Phi_{12}^5) \right) \to C_Y/\text{SL}(2, 13)
\]

over the modular curve \(X\) give the same Milnor’s standard generator of \(\Theta_7\).
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