Chirped Lambert W-kink solitons of the complex cubic-quintic Ginzburg-Landau equation with intrapulse Raman scattering

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Abstract

In this paper, an exact explicit solution for the complex cubic-quintic Ginzburg-Landau equation is obtained, by using Lambert W function or omega function. More pertinently, we term them as Lambert W-kink-type solitons, begotten under the influence of intrapulse Raman scattering. Parameter domains are delineated in which these optical solitons exit in the ensuing model. We report the effect of model coefficients on the amplitude of Lambert W-kink solitons, which enables us to control efficiently the pulse intensity and hence their subsequent evolution. Also, moving fronts or optical shock-type solitons are obtained as a byproduct of this model. We explicate the mechanism to control the intensity of these fronts, by fine tuning the spectral filtering or gain parameter. It is exhibited that the frequency chirp associated with these optical solitons depends on the intensity of the wave and saturates to a constant value as the retarded time approaches its asymptotic value.

Keywords: Optical solitons, Frequency chirp, Lambert W function, Moving fronts

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1. Introduction

The complex Ginzburg-Landau equation is a canonical model for weakly nonlinear, dissipative systems and one of the most-studied nonlinear equations in the physics community. It can be used to describe a vast variety of nonlinear phenomena such as superconductivity [1], Bose-Einstein condensation [2], superfluidity [3], strings in field theory [4], liquid crystals [5] and lasers [6, 7]. Exact solitary pulse solutions of complex Ginzburg-Landau equation with the cubic nonlinearity are available, but these pulses are stable only in a region where the background is unstable [8]. However, it is possible to create stable solitary pulses, in a region where background is also stable, with the inclusion of an extra term, represents delayed Raman scattering, in the model [9, 10]. The existence and stability of solitary wave solutions for cubic Ginzburg-Landau equation is also studied in the presence of driven term [11, 12, 13]. Recently, lot of attention has been paid to obtain exact analytical solutions for complex systems modeled by nonautonomous partial differential equations [14, 15, 16, 17, 18].

Soliton propagation in fibers with linear and nonlinear gain and spectral filtering [19] or pulse generation in fiber lasers with additive pulse mode-locking or nonlinear polarization rotation [20, 21] have been studied by considering complex cubic-quintic Ginzburg-Landau equation (CQGLE) as model equation. Also Kengne and Vaillancourt have used modified Ginzburg-Landau equation that describes the pulse propagation in a lossy electrical transmission line [22]. The CQGLE supports a class of localized solutions such as stationary solitons, sources, sinks, moving solitons and fronts with fixed velocity [23, 24, 25]. Apart from these solutions, the CQGLE also possesses the solutions with special propagation properties: pulsating, creeping, and erupting solitons [26, 27, 28]. The erupting solitons are those that periodically exhibits explosive instability. These solitons were found numerically [26] and also experimentally in passively mode-locked lasers [29]. The effect of higher-order terms, namely, third-order dispersion, self-steepening and intrapulse Raman scattering has been investigated on
erupting solitons and it is found that the explosions of an erupting soliton can be controlled or even canceled due to the inclusion of one or more higher-order terms [30, 31, 32, 33, 34, 35]. Recently, work has been done to study the transitions of stationary to pulsating solutions [36] and on the selection mechanism of soliton explosions in CQGLE under the influence of higher-order terms [37]. Facão et al. have studied the effect of intrapulse Raman scattering (IRS) on erupting solitons of CQGLE and numerically shown the propagation of stable traveling solitons for a specific range of IRS parameter [32, 33]. Although, CQGLE is a well studied dynamical system, the exact solutions of CQGLE with IRS term have rarely appeared in the literature. However, in Ref. [38], the authors have presented the exact stationary front solutions for CQGLE with Raman term and generalized these solutions into moving fronts using energy and momentum balance equations for particular cases by assuming either the quintic or Raman term to be zero.

In this work, we consider the CQGLE in the presence of IRS and report the existence of exact localized solutions in the form of dark and front solitons. The dark solitons are presented by a new kind of kink solution in terms of Lambert W function which we shall refer to as Lambert W-kink solitons. The Lambert W function is an implicitly elementary function, also known as the product logarithm, has rich variety of applications in number of areas of physics, computer science, pure and applied mathematics and ecology, and is defined as the inverse of $f(W) = We^W$ [39, 40]. Several well-known problems in electrostatics and in quantum mechanics can be solved with greater ease using the notation of Lambert W function. Biswas et al. used the notation of Lambert W function to obtain soliton solutions of modified nonlinear Schrödinger equation using variational principle [41]. This function is also used as a step potential for which the one-dimensional stationary Schrödinger equation is exactly solved in terms of the confluent hypergeometric functions [42]. Recently, soliton solution in the form of Lambert W function was obtained for analytically solvable parity-breaking $\phi^6$ model and the results so obtained were compared with kink of $\phi^4$ theory [43]. Apart from Lambert W-kink solitons, we have also explored moving
front solitons for this model. The evolution of optical solitons can be controlled by judicious choice of model parameters. The frequency chirp is found to be directly proportional to the intensity of the wave and saturates at some finite value as \( t \to \pm \infty \). Frequency chirp is a well-known result of the interaction of the group velocity dispersion and the nonlinear self phase modulation. Chirp is very useful in the process of optical pulse compression and found potential applications in optical communication systems \[44, 45, 46\]. A significant work has been done on the existence of chirped solitons in the context of nonlinear optics \[47, 48, 49, 50\].

2. Model Equation

We begin our analysis by considering the complex cubic-quintic Ginzburg-Landau equation (CQGLE) with intrapulse Raman scattering (IRS) term

\[
iU_z + \frac{1}{2} U_{tt} + \gamma |U|^2 U = i\delta U + i\beta U_{tt} + i\epsilon |U|^2 U - \nu |U|^4 U + i\mu |U|^4 U + T_r (|U|^2) U, \tag{1}
\]

where \( U \) is the normalized envelope of the pulse, \( z \) and \( t \) are the normalized propagation distance and retarded time, respectively. For laser system \[21\], the physical meaning of various coefficients is the following: \( \delta \) is a constant gain (or loss if negative), \( \beta \) describes spectral filtering or gain dispersion, \( \epsilon \) represents nonlinear gain (or two-photon absorption if negative), \( \mu \) represents a higher order correction to the nonlinear amplification or absorption, \( \nu \) is a higher order correction term to the nonlinear refractive index, \( T_r \) represents the IRS coefficient and \( \gamma \) represents positive Kerr effect (or negative Kerr effect if negative).

3. Chirped soliton-like solutions

In order to find the exact solution of Eq. \[1\], we choose the following ansatz

\[
U(z, t) = \rho(\xi)e^{i(\phi(\xi) - kz)}, \tag{2}
\]
where ξ = t − uz is the traveling coordinate, ρ and φ are real functions of ξ. Here \( u = \frac{1}{v} \), where \( v \) indicates the group velocity of the pulse envelope. The corresponding intensity of the propagating pulse is given by \(|U(z, t)|^2 = |\rho(\xi)|^2\). The spectral changes introduced across the pulse at any distance \( z \) are a direct consequence of time dependence of nonlinear phase shift. The frequency change across the pulse is the time derivative of phase and is given by \( \delta \omega(z, t) = -\frac{\partial}{\partial t}[\phi(\xi) - kz] = -\phi'(\xi) \). This time dependence of \( \delta \omega \) is referred to as frequency chirping. Now, substituting Eq. (2) into Eq. (1), and separating out the real and imaginary parts of the equation, we obtain the following coupled equations in ρ and φ,

\[
\begin{align*}
upho' + k\rho + \frac{1}{2}\rho'' - \frac{1}{2}\rho\phi'^2 + \gamma\rho^3 & = -2\beta\rho'\phi' - \beta\rho\phi'' - \nu\rho^5 + 2T_r\rho^2\rho', \\
-\rho' + \rho\phi' + \frac{1}{2}\rho\phi'' & = \delta\rho + \beta\rho'' - \beta\rho\phi'^2 + \epsilon\rho^3 + \mu\rho^5.
\end{align*}
\]

Assuming that the qualitative features of frequency chirp depend considerably on the exact pulse shape through the relation \( \delta \omega(z, t) = -\phi'(\xi) = -(A\rho^2 + B) \), where A and B are the nonlinear and constant chirp parameters, respectively, the coupled equations given by Eq. (3) and Eq. (4) reduce to

\[
\begin{align*}
\rho'' + 4(2\beta A - T_r)\rho^2\rho' + 4\beta B\rho' + (2uB - B^2 + 2k)\rho + 2(uA + \gamma - AB)\rho^3 + (2\nu - A^2)\rho^5 & = 0, \\
\rho'' - \frac{2A}{\beta}\rho^2\rho' + \left(\frac{u - B}{\beta}\right)\rho' + \left(\frac{\delta}{\beta} - B^2\right)\rho + \left(\frac{\epsilon}{\beta} - 2AB\right)\rho^3 + \left(\frac{\mu}{\beta} - A^2\right)\rho^5 & = 0,
\end{align*}
\]

for \( \beta \neq 0 \). By assuming the following identifications:

\[
\begin{align*}
M & \equiv \left(8\beta A - 4T_r = \frac{-2A}{\beta}\right), \\
N & \equiv \left(4\beta B = \frac{u - B}{\beta}\right),
\end{align*}
\]

5
\[ Q \equiv \left( 2uB + 2k - B^2 = \frac{\delta}{\beta} - B^2 \right), \]  
\[ R \equiv \left( 2Au - 2AB + 2\gamma = \frac{\epsilon}{\beta} - 2AB \right), \]  
\[ S \equiv \left( 2\nu - A^2 = \frac{\mu}{\beta} - A^2 \right), \]

Eqs. (5) and (6) can be mapped into a single equation

\[ \rho'' + M\rho^2 \rho' + N\rho + Q\rho + R\rho^3 + S\rho^5 = 0. \]  

Solving Eqs. (7)-(11), we obtain the constraint conditions as

\[ A = \frac{2\beta T_r}{1 + 4\beta^2}, \quad u = \frac{1}{A} \left( \frac{\epsilon}{2\beta} - \gamma \right), \quad B = \frac{u}{1 + 4\beta^2} \]

\[ k = \frac{\delta}{2\beta} - uB, \quad \mu = 2\beta \nu. \]  

Eq. (12) can be solved to obtain exact localized solution for compatible form of first-order differential equation for the function \( \rho(\xi) \). In this work, we have explored the Lambert W-kink and moving front soliton solutions for this equation.

3.1. Lambert W-kink solitons

In order to explore exact analytical solution of Eq. (12), use shall be made of the differential equation

\[ \rho' = (a^2 - \rho^2) (a - \rho), \]  

(a is a real parameter here), that admits Lambert W-kink solution of the form

\[ \rho(\xi) = a \left( 1 - \frac{2}{1 + W(e^{4a^2\xi + 1})} \right), \]

where \( W \) represents Lambert W function. The corresponding second-order differential equation for \( \rho(\xi) \) reads

\[ \rho'' - 3\rho^5 + 5a\rho^4 + 2a^2\rho^3 - 6a^3\rho^2 + a^4\rho + a^5 = 0. \]
For $\rho'$ given by Eq. (14), Eq. (12) is consistent with the Eq. (16) by the identification of various unknown parameters as $M = -5, N = a^2, Q = 2a^4, R = -4a^2, S = 2$. Solving these conditions along with constraints given by Eq. (13), the model coefficients and solution parameter ‘$a$’ fixed as

$$T_r = \frac{5}{4}(1 + 4\beta^2), \quad \epsilon = \frac{11\beta\gamma}{2(4 + 5\beta^2)}, \quad \nu = 1 + \frac{25}{8}\beta^2,$$

$$\delta = \frac{\gamma^2 (1 + 32\beta^2)}{4\beta (4 + 5\beta^2)^2}, \quad a = \sqrt{-\frac{2\gamma}{4 + 5\beta^2}}. \quad (17)$$

![Figure 1: Curves of model coefficients versus spectral filtering term ‘$\beta$’ for $\gamma = -1$.](image)

It should be noted that $\gamma$ will take only negative values, as solution parameter ‘$a$’ should be real, and $\beta$ can be chosen arbitrarily ($\beta \neq 0$) while the other model coefficients depend on $\beta$ and $\gamma$. In Fig. 1 we have presented the allowed values of the model coefficients with $\beta$ lying in the interval $[0, 1]$ for $\gamma = -1$. In Fig. 2 we have shown the amplitude profile of Lambert W-kink solution, given by Eq. (15), for different values of the spectral filtering term ‘$\beta$’ and $\gamma = -1$. From this plot, one can observe that kink wave has large amplitude and becomes more steep for small values of $\beta$ as solution parameter ‘$a$’ is inversely proportional to $\beta$. It should be noted that, from Fig. 1 these Lambert W-kink solutions are possible only for $\delta > 0, \epsilon < 0$ and $\gamma < 0$.

Using Eq. (15) into the Eq. (2), the intensity expression of Lambert W-kink
Figure 2: Amplitude profiles of Lambert W-kink solution for different values of $\beta$, $\beta = 0.1$ (thick line), $\beta = 0.5$ (dashed line) and $\beta = 0.9$ (dotted line).

solitons, for the model equation Eq. (1), reads

$$I_W(z, t) = a^2 \left( 1 - \frac{2}{1 + W(e^{4a^2\xi + 1})} \right)^2,$$  

(18)

with chirping given by

$$\delta\omega(z, t) = -\left[ \frac{5a^2\beta}{2} \left( 1 - \frac{2}{1 + W(e^{4a^2\xi + 1})} \right)^2 - \frac{\gamma}{2\beta (4 + 5\beta^2)} \right].$$  

(19)

The intensity profile of Lambert W-kink soliton is depicted in the Fig. 3(a,b) for $\gamma = -1$ and different values of coefficient $\beta$, 0.1 and 0.5, respectively. These intensity profiles are similar to dark solitons (albeit asymmetric in nature) and shows relative compression of pulses with the modulation of spectral filtering term ‘$\beta$’. Fig. 3(c) shows the profiles of corresponding frequency chirp $\delta\omega$ across the pulse of Lambert W-kink soliton at $z = 0$. One can observe that frequency chirp saturates to negative value as the retarded time approaches its asymptotic limit and the amplitude of chirp can be controlled for judicious choice of parameter ‘$\beta$’.

3.2. Moving front solitons

To exemplify the existence of moving front solitons as exact solutions of this model, let us consider the differential equation in $\rho$

$$\rho' = c\rho \left( 1 - \frac{\rho^2}{2b} \right),$$  

(20)
Figure 3: (a,b) Intensity profile of Lambert W-kink soliton for \( \gamma = -1 \) and different values of coefficient \( \beta \), 0.1 and 0.5, respectively. (c) The corresponding chirp profiles for \( \beta = 0.1 \) (thick line) and \( \beta = 0.5 \) (dashed line).

where \( b, c \) are real parameters. The explicit moving front soliton is given by

\[
\rho(\xi) = \sqrt{b} (1 + \tanh(c\xi)),
\]  

(21)

for \( b > 0 \). The corresponding second-order differential equation for \( \rho(\xi) \) reads

\[
\rho'' - \frac{3c^2}{4b^2} \rho^5 + \frac{2c^2}{b} \rho^3 - c^2 \rho = 0.
\]

(22)
Substituting Eq. (20) into Eq. (12) and comparing the resultant equation with Eq. (22), the solution parameters found to be

\[ c = \frac{-N \pm \sqrt{N^2 - 4Q}}{2}, \quad b = \frac{(N + 4c)c}{2(R + Mc)}, \]

along with constraint on model coefficient \( \nu = \frac{A^2}{2} + \frac{2M}{36} - \frac{3c^2}{8b^2} \). Here, the parameters \( M, N, Q \) and \( R \) can be obtained from Eqs. (7)-(10), using Eq. (13), for different values of the model coefficients \( \beta, \epsilon, \gamma, \delta \) and \( T_r \). For illustrative purpose, we choose the model coefficients \( \epsilon > 0, \gamma > 0 \) and \( \delta < 0 \), just opposite to the case of Lambert W-kink solution, to depict the evolution of kink solutions.

The amplitude profile of kink solution is shown in Fig. 4 for \( \epsilon = 0.8, \gamma = 1, \delta = -1, T_r = 0.2 \) and different values of \( \beta \). The expression of the intensity for these kink solution can be written as

\[ I_K(z, t) = b \left( 1 + \tanh(c\xi) \right). \]

The corresponding chirping is given by

\[ \delta \omega(z, t) = -\left( \frac{u + 2\beta T_r b (1 + \tanh(c\xi))}{1 + 4\beta^2} \right). \]
Figure 5: (a,b) Intensity profile of front soliton for different values of $\beta$, 0.3 and 0.5, respectively. (c) The corresponding chirp profiles for $\beta = 0.3$ (thick line) and $\beta = 0.5$ (dashed line). The other parameters are same as in Fig. 4.

The intensity and chirp profiles for front solitons are shown in Fig. 5 for different values of $\beta$. For $\beta = 0.3$, the frequency chirp approaches negative value as $t \to \pm \infty$ while it approaches negative and positive value as $t \to +\infty$ and $t \to -\infty$, respectively, for $\beta = 0.5$. 
4. Conclusion

In conclusion, we have shown the existence of exact explicit solution for the complex cubic-quintic Ginzburg-Landau equation in terms of Lambert W function or omega function, under the influence of intrapulse Raman scattering. Parameter domains are delineated in which these optical solitons exit in the ensuing model. It is observed that these optical solitons are possible for negative values of nonlinear gain and Kerr effect and positive value of constant gain. Whereas, no such restrictions are imposed on the moving fronts or optical shock-type solitons that are obtained as a byproduct of this model. We have observed that the intensity of these fronts has been doubled with a slight change in the value of the spectral filtering or gain parameter. The frequency chirp associated with these nonlinear waves has been identified. Furthermore, we have explicated the pivotal role played by this nonlinear chirp on the intensity of these waves. These results may be useful for experimental realization of undistorted transmission of optical waves in optical fibers and further understanding of their optical transmission properties. Finally, we hope that the exact nature of these nonlinear waves presented here may be profitably exploited in designing the optimal Raman fiber laser experiments.

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