UNIVERSAL QUANTUM COMPUTING BASED ON MONODROMY REPRESENTATIONS

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Abstract. A model of quantum computing is presented, based on properties of connections with a prescribed monodromy group on holomorphic vector bundles over bases with nontrivial topology. Such connections with required properties appear in the WZW-models, in which moreover the corresponding \( n \)-point correlation functions are sections of appropriate bundles which are holomorphic with respect to the connection.

Logical gates for a quantum computer. Quantum mechanical processes give rise to new types of computation. Computational networks built out of quantum-mechanical gates provide a natural framework for constructing quantum computers. The computing capacity of such computers drastically exceeds that of traditional computer \( \overline{1} \).\( \overline{2} \).

The quantum analogue of the classical bit is the quantum bit or qubit. Just as the classical bit is represented by a system which can adopt one of two distinct states ‘0’ and ‘1’, one can define a quantum bit as follows:

Definition 1. A qubit is a quantum system whose state can be fully described by a superposition of two orthogonal eigenstates labeled \( |0\rangle \) and \( |1\rangle \).

The space of quantum states is a Hilbert space, which we denote by \( \mathcal{H} \). Therefore, a qubit is a normalized state in the two-dimensional Hilbert space \( \mathbb{C}^2 \). A general state \( |\Psi\rangle \in \mathcal{H} \) of the qubit is given by \( |\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \) with \( |\alpha|^2 + |\beta|^2 = 1 \).

The value of the qubit is the observable \( N \) with the Hermitian operator \( N|i\rangle = i|i\rangle \) on the Hilbert space \( \mathcal{H} \cong \mathbb{C}^2 \), or, in the matrix representation

\[
N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The expectation value of \( N \) is given by

\[
\langle N \rangle = \langle \Psi|N|\Psi \rangle = (\alpha^* \beta^*) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\beta|^2
\]

Thus, \( \langle N \rangle \) gives the probability to find the system in state \( |1\rangle \) if a measurement is performed on the qubit.

While the state of a classical computer can be given as the collection of distinct states of all bits in the memory and processor registers, the “state of a qubit” is a meaningless term, if the machine state is the combined state of more than one system.

Definition 2. The machine state \( |\Psi\rangle \) of an \( n \)-qubit quantum computer is given by \( |\Psi\rangle = \sum_{(d_0, \ldots, d_{n-1})} c_{d_0, \ldots, d_{n-1}} |d_0, \ldots, d_{n-1}\rangle \) with \( \sum |c_{d_0, \ldots, d_{n-1}}|^2 = 1 \).
The quantum state space $\mathcal{H}$ is thus the tensor product of $n$ single qubit Hilbert spaces $\mathcal{H}_j \cong \mathbb{C}^2$, i.e.

\[(1)\] $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n \cong (\mathbb{C}^2)^{\otimes (n)}$

One of the fundamental facts of the classical theory of computing is that there exist finite sets of simple functions, called fundamental or universal gates, such that any function $f : \mathcal{B}^n \rightarrow \mathcal{B}^m$, where $\mathcal{B}$ is the Boolean algebra \{0, 1\}, can be constructed in a simple explicit way from them. For example the set \{NOT, OR, AND\} is such a basis of classical computation.

Similarly, one considers quantum gates on $k$ qubits (or $k$-gates) — unitary $2^k \times 2^k$-matrices acting on the quantum state space of $k$ qubits. A fundamental problem in quantum computing is to find basis of gates, which is “universal”. (More precisely, one distinguishes between universal bases and more restrictive exactly universal ones; a basis $B$ is called exactly universal if, for each $k \geq 2$, every unitary $k$-qubit operator can be obtained exactly by a circuit made up of the $k$-qubit gates produced from the elements of the basis $B$. See [3])

In real world computation, important rôle play devices independent from the environment noise. Thus two of the main requirements on error-free operations are to have a set of gates that is both universal for quantum computing and that can operate in noise-producing environment, i.e. is fault-tolerant (see [3]). For gates involving irrational multiples of $\pi$, called non-elementary, fault-tolerant realization is impossible. Thus, presence of this property makes an elementary gate inappropriate for physical realization.

It is feasible that any quantum system which one would consider potentially useful for quantum computing should contain a generating set of gates, which forms a basis in the above sense.

Two bases $A$ and $B$ are called equivalent, if the gates in the basis $A$ can be exactly realized using gates in the basis $B$ and vice versa.

Let us introduce the following operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_y = \frac{\sigma_x + \sigma_z}{\sqrt{2}}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\alpha} \end{pmatrix},$$

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices and have the property: every traceless and Hermitian $2 \times 2$ unitary matrix $U$ can be expressed as $U = x\sigma_x + y\sigma_y + z\sigma_z$, where $x, y, z \in \mathbb{R}$ and $x^2 + y^2 + z^2 = 1$.

For every unitary operator $U \in \mathbb{U}(2)$ we define the controlled $U$-operator as

$$\Lambda_k(U) |x_1, \ldots, x_k\rangle \otimes |\xi\rangle = \begin{cases} |x_1, \ldots, x_k\rangle \otimes U|\xi\rangle & \text{if } \bigwedge_{j=1}^k x_j = 1 \\ |x_1, \ldots, x_k\rangle \otimes |\xi\rangle & \text{if } \bigwedge_{j=1}^k x_j = 0 \end{cases}$$

for all $x_1, \ldots, x_k \in \mathcal{B}$. Here $\bigwedge_{j=1}^k x_j$ denotes action of the operator AND on the boolean variables $\{0, 1\}$. For example, if $k = 0$, then $\Lambda_0(\sigma_x) = \text{NOT}_q : H \rightarrow H$, which is called the quantum NOT operator and by definition one has $\sigma_x = \text{NOT}_q$, which acts on the state $|\alpha\rangle + |\beta\rangle \in \mathcal{H}$ via $|\alpha\rangle + \beta |\beta\rangle \rightarrow \beta |\alpha\rangle + |\beta\rangle$. If the quantum state $|\alpha\rangle + |\beta\rangle$ is written in vector form as $(\alpha, \beta)^\top$, then $\text{NOT}_q(\alpha, \beta) = (\beta, \alpha)$. Second important example is the so called controlled-not operator — cNOT: $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$, which acts on the pairs of qubits and carries out bitwise summation $\text{cNOT}(u, v) = |u, v \oplus u\rangle$, where $\oplus$ is addition mod 2. Let us consider one more operator — the controlled controlled NOT operator ccNOT: $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. 

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The classical one and two bit operators OR and AND can be expressed via ccNOT as follows:

\[
\text{ccNOT}|1, 1, a\rangle = |1, 1, (\text{NOT } a)\rangle,
\]
\[
\text{ccNOT}|a, b, 0\rangle = |a, b, (a \text{ AND } b)\rangle,
\]

for \(a, b \in \mathbb{B}\).

On the other hand, every Boolean function can be expressed in the basis \{NOT, OR, AND\} and therefore there exists the unitary operator

\[
U_f: \mathcal{H} \otimes \cdots \otimes \mathcal{H} \to \mathcal{H} \otimes \cdots \otimes \mathcal{H}
\]
such that

\[
U_f|x, 0\rangle = |x, f(x)\rangle.
\]

Two and three qubit operations, respectively, cNOT and ccNOT, are represented by one qubit operations as follows:

\[
c\text{NOT} = |0\rangle \langle 1| \otimes |1\rangle \langle 0|, \quad \text{cc\text{NOT}} = |0\rangle \langle 0| \otimes 1 \otimes |1\rangle \langle 1| \otimes \text{c\text{NOT}}.
\]

This means that if it is possible to find a finite subset of \(U(2)\) which generates a dense subset of \(U(2)\), then we obtain all operators for quantum computing.

The universal sets of gates for computation have been extensively studied. To emphasize importance of such works let us recall some fundamental facts which concern relations between different bases.

The set of gates \(\{\Lambda_2(\sigma_x), \sigma_z^{1/2}, H\}\), where

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\]

the so called Hadamard operator, is a fault-tolerant basis \([5]\). On the other hand the set of gates \(\{\Lambda_1(\sigma_z^{1/2}), H\}\) is universal \([3]\) and this basis is equivalent to the basis \(\{\Lambda_2(\sigma_x), \sigma_z^{1/2}, H\}\). In \([3]\) is considered the basis \(\{\Lambda_1(\sigma_z), \sigma_z^{1/4}, H\}\) and proved that it is not equivalent to the basis \(\{\Lambda_2(\sigma_x), \sigma_z^{1/2}, H\}\).

**Connection with singularity as generator of quantum gates.** In \([7]\) is considered the model of quantum computing, called by authors holonomic, based on the well known Berry phase. Main ingredient in holonomic quantum computation is a smooth vector bundle \(E \to X\) with fibre \(C^2 \otimes \cdots \otimes C^2\) and unitary connection \(\Omega\). The encoding space of information is in this case the fibre of the bundle and processing of information is represented by the holonomy operator \(P \exp(\int_\gamma \Omega)\), which acts on the encoding space as \(v \mapsto P \exp(\int_\gamma \Omega)v\), where \(\gamma: [0, 1] \to X\) is a smooth path and \(P\) denotes path-ordered exponential.

In the present section is given the construction of a universal set of gates from the monodromy representation of any system of differential equations of Fuchs type, which can be considered as dynamic equation of a quantum system. By our opinion, sets of gates obtained in this way are expected to be fault-tolerant by reasons similar to ones given in \([8]\): action of a monodromy operator corresponding to a loop is unchanged by small fluctuations of the loop, provided they do not result in crossing any singularities.

At first let us review a simple example, the differential equation with regular singular points (for more details see \([4]\))

\[
df(z) = \frac{\partial f}{\partial z} = \frac{a}{z^a}, \quad a \in \mathbb{C}, \text{ on } \mathbb{C} \setminus \{0\}.
\]

The solution of this equation is the many valued function \(f(z) = z^a\) which by definition is \(z^a = e^{a \log z}\), which under analytic continuation along a path \(\gamma\) looping once counterclockwise around the origin transforms into \(e^{2\pi ia} z^a\). The fundamental group of \(\mathbb{C} \setminus \{0\}\) is isomorphic to \(\mathbb{Z}\), with generator \(\gamma\), and the monodromy representation
in $\text{GL}_2(\mathbb{C})$ is given by $\gamma \mapsto e^{2\pi i \alpha}$. Analogously, let $A$ be a Hermitian $2 \times 2$ constant matrix, then the matrix function $(z - s)^A$ is a many valued function on the domain $|z - s| > 0$ and is the solution of the matrix differential equation $\frac{dz}{dz} = \frac{A}{z}$. The monodromy representation of this system is given by $\gamma \mapsto e^{2\pi i A}$, and as $A$ is Hermitian, $e^{2\pi i A}$ is unitary and therefore we obtain the monodromy representation in the unitary group. In particular, if we can obtain any one-qubit gate, for example $\tau$, it will suffice to choose a Hermitian $2 \times 2$ matrix $\alpha$, such that $e^{2\pi i \alpha} = \tau$, then $e^{2\pi i \alpha}$ acts as locally unitary operator (processing of information) on the fibre (encoding space of information) $\mathbb{C}^2$ of the trivial vector bundle $\mathbb{C}^2 \times \mathbb{C}^* \to \mathbb{C}^*$.

In [10] we consider the case, when the holomorphic vector bundle is given on a punctured compact Riemann surface. In this case, fundamental group of the base of the bundle is a free group and obtaining a universal set of logical gates does not present a difficult problem after application of solution methods of the Riemann-Hilbert problem.

Let $\mathbb{CP}^n$ be the $n$-dimensional complex projective space, and let $s_1, s_2, s_3, s_4 \in \mathbb{CP}^1$ and $s_j \neq \infty$. Denote by $X_4 = \mathbb{CP}^1 - \{s_1, s_2, s_3, s_4\}$ and by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ the generators of $\pi_1(X_4, z_0)$ with relations $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$. Let $M_1, M_2, M_3, M_4$ be matrices such that $E_j = \frac{1}{2\pi i} \ln M_j, j = 1, 2, 3, 4$ and $M_1, M_2, M_3$ generate a basis of the Lie algebra of $SU(2)$ and satisfy the condition $M_1 M_2 M_3 M_4 = 1$. Consider the representation $\rho : \pi_1(X_4, z_0) \to SU(2)$ defined by the map $\gamma_j \mapsto M_j$. Then for $\rho$ there exits a system of differential equations of Fuchs type

$$dF(z) = \left( \sum_{j=1}^{4} \frac{A_j}{z - s_j} \right) F(z),$$

whose monodromy representation coincides with $\rho [\ref{10}]$. The monodromy representation $\rho$ induces two-dimensional vector bundle $E \to \mathbb{CP}^1$ with meromorphic connection form

$$\omega = \sum_{j=1}^{4} \frac{A_j}{z - s_j} dz$$

and Chern number $c_1(E) = \sum_{j=1}^{4} E_j$. The solution space $\mathcal{H}$ of the system is a two-dimensional vector space. Moreover $\pi_1(X_4, z_0)$ acts on $\mathcal{H}$ and any unitary operator can be obtained in this way.

Therefore, we have proved the following proposition:

**Proposition 1 [10]**. The connection form $\omega$ generates the basis for the computation.

Consider the general case. Let $X$ be a compact Riemann surface of genus $g \geq 2$ with marked point $z_0$. The fundamental group of $X' = X - \{z_0\}$ is generated by $\alpha_1, \beta_1, ..., \alpha_g, \beta_g, \gamma$ with relations $\Pi_{j=1}^g [\alpha_j, \beta_j] = \gamma$, where $\alpha_1, \beta_1, ..., \alpha_g, \beta_g$ are generators of the fundamental group $\pi_1(X)$ and $\gamma$ is a loop going around $z_0$. Consider such homomorphism $\rho : \pi_1(X, z_0) \to SU(2)$ that $\text{Im} \rho$ is a dense subgroup of $SU(2)$. The homomorphism $\rho$ defines a two-dimensional holomorphic vector bundle $E' \to X'$ and there exists a $SU(2)$-system of differential equations $Df = \omega f$ on $X$ which has one regular singular point $z_0$ and the monodromy representation of this
system coincides with $\rho$. The pair $(E', \omega)$ can be extended to a possibly holomorphically nontrivial bundle $E \to X$, for which $\omega$ can be a meromorphic connection. Therefore we proved the following proposition

**Proposition 2** [10]. The connection $\omega$ of the bundle $E \to X$ densely generates all unitary operators $H \to H$.

Below we consider the case when the fundamental group of the base is not free. Suppose $D = \bigcup_{j=1}^{m+1} D_j$ is a divisor, where $D_j$, $j = 1, ..., m+1$ are hyperplanes in $\mathbb{CP}^n$. The main result of this paper is the following theorem.

**Theorem.** There exists a Fuchs type Pfaff system

$$df = \omega f$$

on $\mathbb{CP}^n - D$ whose monodromy representation gives a universal set of quantum gates.

Let us choose a line $L$ which intersects the divisor $D = \bigcup_{j=1}^{m+1} D_j$ at nonsingular points $a_j = L \cap D_j$, $j = 1, 2, ..., m+1$. Consider the fundamental group $\pi_1(L - \{a_1, ..., a_{m+1}\}, z_0)$. Let $[\gamma_1], ..., [\gamma_m]$ be the generators of $\pi_1(L - \{a_1, ..., a_{m+1}\}, z_0)$, where the loops have form $\gamma_j = \sigma_j[\alpha_j]\sigma_j^{-1}$, $\alpha$ is a path from $z_0$ to a neighborhood $V_{a_j}$ and $\sigma_j$ is small a loop in the neighborhood $V_{a_j}$ which generates $\pi_1(V_{a_j} - \{a_j\}) \cong \mathbb{Z}$.

It is known that $[\gamma_1], ..., [\gamma_m]$ too are generators of $\pi_1(\mathbb{CP}^n - D, z_0)$ under some conditions.

Suppose we have the family of representations

$$\rho_\lambda : \pi_1(\mathbb{CP}^n - D, z_0) \to GL_m(\mathbb{C})$$

such that

$$\rho_\lambda([\gamma_j]) = 1 + \lambda M_1^j + \lambda^2 M_2^j + ... + \lambda^k M_k^j + ..., \quad (3)$$

where $M_k^j$ are $m \times m$-matrices. The family of representations $[\mathbf{3}]$ satisfying the condition (3) is called analytic.

Suppose $D_j = \{h_j = 0\}$ and consider the 1-form $\Omega_k = \sum_{j=1}^{m+1} U_k^j \frac{dh_j}{h_j}$, where $U_k^j$ are constant matrices and $\sum_{j=1}^{m+1} U_k^j = 0$. Consider the family of meromorphic 1-forms

$$\Omega(\lambda) = \lambda \Omega_1 + \lambda^2 \Omega_2 + ... + \lambda^k \Omega_k + .... \quad (4)$$

If $U_k^j(\lambda) = \sum_{k=1}^{\infty} \lambda^k U_k^j$ is a converging power series then the family $\Omega(\lambda)$ is called analytic family of Fuchs systems.

It is known that for every $\lambda$ this system satisfies the condition

$$dz \Omega(\lambda) = 0, \quad z \in C.$$  

If the analytic family of Fuchs systems satisfies the condition

$$\Omega(\lambda) \wedge \Omega(\lambda) = 0,$$

then the family of Fuchs systems is called integrable.

Similar terminology will be used for a Pfaff system $df = \Omega(\lambda)f$. 

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The monodromy representation of the integrable family of Pfaff systems

\[ df = \Omega(\lambda)f \]

is an analytic family of representations of the fundamental group \( \pi_1(\mathbb{C}P^n - D, z_0) \) \[1\].

Let \( \Phi \) be the fundamental matrix of the system (5) represented by the Peano series

\[ \Phi = 1 + \int \Omega(\gamma) + \int \Omega(\gamma)\Omega(\gamma) + ... + \int \Omega(\gamma)\Omega(\gamma) + ..., \]

where \( \int \Omega(\gamma)\Omega(\gamma) \) is the Chen iterated integral. If such representation of the fundamental matrix is chosen, then for every \( \gamma \in \pi_1(\mathbb{C}P^n - D, z_0) \) one will have

\[ \rho(\gamma) = 1 + \int \gamma \Omega(\gamma) + \int \gamma \Omega(\gamma)\Omega(\gamma) + ... + \int \gamma \Omega(\gamma)\Omega(\gamma) + .... \]

**Theorem 1** \[1\]. For every analytic family of representations (2), when the parameter is small, there exists a family of Pfaff systems (5) whose monodromy coincides with (2).

**Sketch of proof.** We will show, that there exists a family of Fuchs type Pfaff systems (5), where \( \Omega(\lambda) \) has the form (4), \( \Omega_k = \sum_{j=1}^{m} U_j^k \omega_j, \omega_j = \frac{dh_j}{h_j} - \frac{dh_{j+1}}{h_{j+1}} \), so that its family of representations coincides with (3).

Step 1. By (4) we find \( U_j^k, k = 1, 2, ..., j = 1, 2, ..., m \). Indeed, let \( \rho(\gamma_j) \) be represented as Peano series like (6), and rewrite it in Lappo-Danilevski form \[12\]:

\[ \rho(\gamma_j) = 1 + \sum_{j=1}^{m} \int_{\gamma_j} \omega_j U^j(\lambda) + \sum_{j,k=1}^{m} \int_{\gamma_j} \omega_j \omega_k U^j(\lambda) U^k(\lambda) + ... + \]

\[ + \sum_{j_1,j_2,...,j_k=1}^{m} \int_{\gamma_j} \omega_{j_1}...\omega_{j_k} U^{j_1}(\lambda) U^{j_2}(\lambda) \]

for the generators we have (4) and therefore we obtain:

\[
\lambda M_1^j + \lambda^2 M_2^j + ... + \lambda^k M_k^j + ... = \\
= \lambda \int_{\gamma_j} \Omega_1 + \lambda^2 (\int_{\gamma_j} \Omega_1 \Omega_1 + \int_{\gamma_j} \Omega_2) + ...
\]

... + \lambda^k (\int_{\gamma_j} \Omega_k + \sum_{q=2}^{m} \sum_{k_1+...+k_q=k} \int_{\gamma_j} \Omega_{k_1}...\Omega_{k_q}) + ...

This implies

\[ \int_{\gamma_j} \Omega_1 = M_1^j, j = 1, ..., m \]

\[ \int_{\gamma_j} \Omega_1 \Omega_1 + \int_{\gamma_j} \Omega_2 = M_2^j, j = 1, ..., m, \]

\[ \int_{\gamma_j} \Omega_k + \sum_{q=2}^{m} \sum_{k_1+...+k_q=k} \int_{\gamma_j} \Omega_{k_1}...\Omega_{k_q} = M_k^j, j = 1, ..., m \]
and so on. As \( \int_{\gamma_j} \Omega_k = 2\pi i U^j_k \), for \( U^j_k \), \( j = 1, \ldots, m \), \( k = 1, 2, \ldots \), we obtain

\[
U^j_1 = \frac{1}{2\pi i} M^j_1, \\
U^j_2 = \frac{1}{2\pi i} (M^j_2 - \int_{\gamma_j} \Omega_1 \Omega_1) = \frac{1}{2\pi i} (M^j_2 - \sum_{k_1, k_2 = 1}^{m} \int_{\gamma_j} \omega_{k_1} \omega_{k_2} U^{k_1}_1 U^{k_2}_1), \\
U^j_k = \frac{1}{2\pi i} (M^j_k - \sum_{q=2}^{k} \sum_{k_1 + k_2 + \ldots + k_q = k} \int_{\gamma_j} \Omega_{k_1} \cdots \Omega_{k_q}),
\]

\( j = 1, 2, \ldots, m \). Therefore we obtain a formal family of 1-forms

\[
\Omega(\lambda) = \lambda \Omega_1 + \lambda^2 \Omega_2 + \ldots + \lambda^k \Omega_k + \ldots = \sum_{j=1}^{m} U^j(\lambda)\omega_j,
\]

\[
\Omega_k = \sum_{j=1}^{m} U^j_k \omega_j, \Omega^j(\lambda) = \sum_{k=1}^{\infty} \lambda^k U^j_k.
\]

Step 2. The formal series \( \boxed{3} \) is convergent for small \( \lambda \). \( \boxed{12} \)

Step 3. The family of 1-forms \( \boxed{8} \) is integrable, i.e. the identity \( \Omega(\lambda) \wedge \Omega(\lambda) = 0 \) is satisfied \( \boxed{11} \).

**Theorem 2.** Let

\[
\rho: \pi_1(\mathbb{CP}^n - D, z_0) \to \text{GL}_m(\mathbb{C})
\]

be a representation such that \( \rho([\gamma_j]) \), \( j = 1, \ldots, m \), are close to identity. Then \( \rho \) is realizable as monodromy representation of an integrable Fuchs system \( df = f \Omega \), \( \Omega = \sum_{j=1}^{m} U^j \omega_j \), where \( U^j \) are close to the zero matrix.

We will apply results of this subsection to a special divisor. In particular, suppose \( D = \bigcup_{j<k} D_{ij} \cup D_0 \), where \( D_{ij} = \{(z_0, \ldots, z_n) \in \mathbb{CP}^n | z_i = z_j, i, j \neq 0 \} \), \( D_0 = \{z \in \mathbb{CP}^n | z_0 = 0 \} \). Let \( X_n = \mathbb{CP}^n - D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j, i, j \neq 0 \} \).

The fundamental group of \( X_n \) is called the pure braid group on \( n \) strings, which we will denote by \( P_n \). The symmetric group \( S_n \) acts on \( X_n \) by \( g.(z_1, \ldots, z_n) = (z_{g(1)}, \ldots, z_{g(n)}) \), \( g \in S_n \). The fundamental group of the quotient space \( X_n/S_n \) is called the braid group on \( n \) strings and denoted by \( B_n \). The braid group has \( n - 1 \) generators \( \sigma_1, \ldots, \sigma_{n-1} \) with relations

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n - 2,
\]

\[
\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2.
\]

Choose a point \( z_0 = (1, 2, \ldots, n) \in X_n \) and denote by \( pr: X_n \to X_n/S_n \) the natural projection. We have exact sequence of groups:

\[
1 \to P_n \to B_n \to S_n \to 1
\]

and generators of the pure braid group \( P_n \) are \( \tau_{ij}, 1 \leq i < j \leq n \), where

\[
\tau_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} \sigma_{j}^{2} \sigma_{j+1}^{-1} \cdots \sigma_i^{-1}, 1 \leq i < j \leq n
\]

satisfying the relations

\[
\tau_{rs} \tau_{sk} = \tau_{sk} \tau_{rs} \text{ for } s < i \text{ or } k < s,
\]

\[
\tau_{ks} \tau_{ik} \tau_{ks}^{-1} = \tau_{ik}^{-1} \tau_{ks} \tau_{ik} \text{ for } i < k < s,
\]

\[
\tau_{rk} \tau_{ik} \tau_{rk}^{-1} = \tau_{ik}^{-1} \tau_{ir} \tau_{ik} \text{ for } i < r < k,
\]
\[
\tau_r \tau_k \tau_r^{-1} = \tau_r^{-1} \tau_k \tau_r \tau_k \tau_r^{-1} \tau_k \tau_r \text{ for } i < r < k < s.
\]

Consider the matrix valued 1-form
\[
\Omega = \sum_{1 \leq i < j \leq n} \Omega_{ij} d \log(z_i - z_j)
\]
holomorphic on \(X_n\), where \(\Omega_{ij} \in M_m(C), 1 \leq i < j \leq n\). Let \(E \to X_n\) be a holomorphic rank \(m\) trivial vector bundle with connection \(\nabla\) for which \(\Omega\) is connection form. Holomorphic sections \(f = (f^1, \ldots, f^m)\) of this bundle are solutions of the Fuchs system
\[
df = \Omega f,
\]
where \(\Omega\) has the form (12).

**Proposition 3.** The system (13) is integrable if and only if the following condition is satisfied
\[
[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = [\Omega_{ij} + \Omega_{ik}, \Omega_{ji}], i < j < k,
\]
(14)
\[
[\Omega_{ij}, \Omega_{kl}] = 0, \text{ for distinct } i, j, k, l.
\]
(15)

By theorem 2 we have (3):

**Theorem 3.** If \(\rho : P_n \to \text{GL}_m(C)\) is such representation that \(||\rho(\tau_{ij}) - 1||\) is sufficiently small for each \(1 \leq i < j \leq n\), then there exist matrices \(\Omega_{ij} \in M_m(C), 1 \leq i < j \leq n\) close to 0, which satisfy the conditions (14)-(15) and monodromy representation of the Fuchs system
\[
df = \sum_{1 \leq i < j \leq n} \Omega_{ij} \frac{d(z_i - z_j)}{z_i - z_j} f
\]
(16)

on the trivial \(m\)-dimensional vector bundle \(C^m \times X_n \to X_n\) which gives the action on \(C^m\) as follows: \(C^m \ni v \mapsto \rho(\tau)v\), for every \(\tau \in P_n\).

In this manner we obtain universal quantum logical gates which are based on holomorphic vector bundles over \(X_n\) with integrable connection (16).

**Conformal field theory.** Consider a 1-form \(\Omega\) of special type defined in (12). Let \(V_1, \ldots, V_m\) be \(\mathfrak{sl}_2\)-modules. Put \(V = V_1 \otimes \ldots \otimes V_m\). Let \(\{I_j\}\) be an orthogonal basis of \(\mathfrak{sl}_2\) and \(c = \sum I_j I_j \in U(\mathfrak{sl}_2)\) the Casimir element in the universal enveloping algebra \(U(\mathfrak{sl}_2)\). Let \(\Delta : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)\) be the diagonal homomorphism determined by \(\Delta(x) = x \otimes 1 + 1 \otimes x, x \in \mathfrak{sl}_2\). Set \(\Omega = \frac{1}{2} (\Delta c - c \otimes 1 - 1 \otimes c)\).
Consider a family $\rho_i : \mathfrak{sl}_2 \to \text{End}(V_i)$, $i = 1, ..., n$ of irreducible representations and define the representations
\begin{equation}
(\rho_i \otimes \rho_j : \mathfrak{sl}_2 \to \text{End}(V_i \otimes \cdots \otimes V_n))
\end{equation}
by formulae
\begin{equation}
(\rho_i \otimes \rho_j)(x) = I_1 \otimes \cdots \otimes \rho_i(x) \otimes \cdots \otimes I_j \otimes \cdots \otimes I_n + I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes \rho_j(x) \otimes \cdots \otimes I_n,
\end{equation}
where $I_k$ denotes the identity operator acting on $V_k$.

The representations (17) extend to the universal enveloping algebra $U(\mathfrak{sl}_2)$, these representations we denote again by $\rho_i \otimes \rho_j$; thus we have the representations $\rho_i \otimes \rho_j : U(\mathfrak{sl}_2) \to \text{End}(V_1 \otimes \cdots \otimes V_n)$. Let $\Omega_{ij} = (\rho_i \otimes \rho_j)\Omega$. The linear operators $\Omega_{ij} : V \to V$, $i < j$, act as $\Omega \otimes 1 + 1 \otimes \Omega$ on $V_i \otimes V_j$ and trivially on all of the other factors. The Fuchs type Pfaff system
\begin{equation}
\frac{\partial \Psi(z_1, ..., z_n)}{\partial z_i} = \frac{1}{\lambda} \sum_{j=1, i \neq j}^{n} \frac{\Omega_{ij}}{z_i - z_j} \Psi, \quad i = 1, ..., n,
\end{equation}
where $\Psi(z_1, ..., z_n)$ is a $V$-valued function on $X_n$, is called the Knizhnik-Zamolodchikov equation. Here $\lambda$ is a complex parameter. Solutions of (18) are covariant constant sections of the trivial bundle $X_n \times V \to X_n$ with flat connection
\begin{equation}
\sum_{j=1}^{n} \frac{\Omega_{ij}}{z_i - z_j} d(z_i - z_j).
\end{equation}

It follows from previous subsections that (18) is integrable if and only if the conditions (13)-(15) are satisfied. The monodromy representation of (18) can be extended to a representation of the braid group $B_n$. Let $V_1 = V_2 = \cdots = V_n \cong \mathbb{C}^2$, then we obtain that the braid group $B_n$ acts on $V^\otimes n \cong (\mathbb{C}^2)^\otimes n$. For quantum computing an appropriate choice of this action is necessary (see [8]).

The Knizhnik-Zamolodchikov equation was invented in conformal field theory. Its solutions describe $n + 1$-point correlation functions on the Riemann sphere in the Wess-Zumino-Witten model. This model is uniquely determined by choice of a simple Lie algebra $\mathfrak{g}$ for every $k > 0$, $k \in \mathbb{Z}$. Any model of the conformal field theory has a certain set of primary fields $\{\phi_j\}$. They are in one-to-one correspondence with irreducible representations $\{\rho_j\}$ of the Kac-Moody algebra $\hat{g}$. Let $g = \mathfrak{sl}_2(\mathbb{C})$. For each nonnegative half integer $j$ there exists a unique irreducible $g$-module $V_j$, called that of spin $j$, with highest weight $j\alpha$ and $\dim V_j = 2j + 1$.

**Theorem 4** [13, 4]. The $n$-point function $\Psi = \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle$, where $\phi_j$ are primary fields, satisfies the system of differential equations (18) and the monodromy representation of this system is unitarizable.

For example consider the case $n = 2$ with
\begin{equation}
\Omega = \Omega_{12} = \Omega_{21} = \begin{pmatrix}
1/2 & 0 & 0 & 0 \\
0 & -1/2 & 1 & 0 \\
0 & 1 & -1/2 & 0 \\
0 & 0 & 0 & 1/2
\end{pmatrix},
\end{equation}
then (18) have the form

\[
\begin{align*}
\frac{\partial F}{\partial z_1} &= \frac{1}{\lambda} \frac{\Omega_{12}}{z_1 - z_2} F, \\
\frac{\partial F}{\partial z_2} &= \frac{1}{\lambda} \frac{\Omega_{21}}{z_2 - z_1} F,
\end{align*}
\]

where \( F \) is a function on \( X_2 = \mathbb{C}_2 = \{z_1 = z_2\} \) with values in \( V^{\otimes 2}, \dim \mathbb{C} V = 2 \).

The solutions to the above system are given by \( F(z) = e^{\frac{1}{\lambda} \ln |z_1 - z_2|} \Omega C \), where \( C \) is a constant 4-vector. The image of the generator \( \sigma_1 \in B_2 \) by the monodromy representation of the system (19)-(20) is \( e^{\frac{\pi i}{\lambda} \Omega} \Omega \), which can be considered as a nontrivial 2-qubit gate.

In [15] is given an example of such 4-point function, for which the corresponding monodromy representation is \( \sigma_2 \). In [18] is considered the exact solution and integrability of the reduced BCS model of superconductivity considered from the conformal field theory point of view.

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