The PIT-trap – a general bootstrap procedure for inference about regression models with non-normal response

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S1 FILE. PROOFS OF THEOREMS

We will first prove two lemmas relating to the situation where the cumulative distribution used for the PIT-trap has been misspecified, but in such a way that PIT-residuals remain identically distributed.

Lemma 1. Let \( U = F(Y)Q + F(Y^-)(1 - Q) \) be the probability integral transform but where the cumulative distribution \( F(y) \) may have been misspecified, and the true distribution function is \( G(y) = h \{ F(y) \} \) for some function \( h(\cdot) \).

Then

\[
P(U \leq u) = h(u)
\]

Proof. For simplicity we will consider the continuous case only, the proof follows via a similar method in the discrete case. Let \( u = F(y) \) be the observed value of the probability integral transform residual. Then:

\[
P(U \leq u) = P \{ F(Y) \leq F(y) \} = P(Y \leq y) = G(y) = h \{ F(y) \} = h(u)
\]

Lemma 1 is used directly in the proof of Lemma 2 below.

Lemma 2. Consider a set of \( n \) random variables \( Y_1, \ldots, Y_n \) with distribution function \( G_i(y) \) for \( Y_i \). A PIT-trap sample \( Y_1^*, \ldots, Y_n^* \) is computed using a (possibly misspecified) set of cumulative distributions, denoted \( F_i(y) \) for \( Y_i \).
If \( G_i(y) = h \{ F_i(y) \} \) for some function \( h(\cdot) \), then for each \( i \):

\[
P_*(Y_i^* \leq y) = G_i(y)
\]

Proof.

\[
P_*(Y_i^* \leq y) = P_* \{ F_i(Y^*) \leq F_i(y) \} = P_* \{ U_i^* \leq F_i(y) \} = \sum_{i=1}^{n} \frac{1}{n} P \{ U_i \leq F_i(y) \}
\]

since the bootstrap sample \( U_i^* \) is drawn at random with replacement from the set of observed PIT-residuals.

\[= h \{ F_i(y) \} \text{ from Lemma 1} \]

\[= G_i(y) \]

Lemma 2 shows that if probability integral transform residuals are identically distributed, then the \( Y_i^* \) preserve the marginal distribution of the \( Y_i \).

**Proof of Theorem 1**

We will prove Theorem 1 by showing that asymptotically, the conditions of Lemma 2 are satisfied.

First note that if \( \hat{\theta} \) is \( \sqrt{n} \)-consistent for \( \theta \) then provided that \( F_j(y; \hat{\theta}, x_i) \) is twice differentiable with respect to \( \theta \) then \( F_j(y; \hat{\theta}, x_i) = F_j(y; \theta, x_i) + O_p(n^{-1/2}). \)

Hence, up to a term \( O_p(n^{-1/2}) \), \( F_j(y; \hat{\theta}, x_i) \) satisfies the conditions of Lemma 2 (where \( h(\cdot) \) is the identity function). By Lemma 2, PIT-trap values follow the true cumulative distribution function \( F_j(y; \theta, x_i) \), up to a term no larger than \( O(n^{-1/2}) \).
Note: While this argument uses the result that the $F_j(y; \hat{\theta}, x_i)$ approximate the true distribution $F_j(y; \theta, x_i)$, we can relax this assumption along the lines of Lemma 1 such that there is only the requirement that the PIT-residuals are (asymptotically) identically distributed, $P(U_{ij} \leq u) = h(u)$ for each $(i, j)$. Thus the PIT-trap can preserve the marginal distribution of the data under certain forms of model misspecification.

Proof of Theorem 3

The proof follows via the usual Edgeworth expansion approach in ?.

If $T = g(Y)$ admits an Edgeworth expansion then:

$$P(T \leq t) = \Phi(t) + n^{-1/2}p_1(t)\phi(t) + n^{-1}p_2(t)\phi(t) + O(n^{-3/2})$$

where $p_1(t)$ is an odd polynomial function of the skewness of $T$, $p_2(t)$ is an even polynomial function of the skewness and kurtosis of $T$, and these moments are evaluated with respect to the distribution of the matrix of data $y$, which is characterized by its margins $F(y; \theta, x_i)$, and the correlation between PIT-residuals $var(U_i) = \Sigma$.

If $Y$ is discrete then the same type of expansion applies, but only at continuity-corrected points and not at all $t$ (?).

Under the same assumptions, the distribution of the PIT-trap statistic $T^* = g(Y^*)$ under resampling admits a similar Edgeworth expansion:

$$P_*(T^* \leq t) = \Phi(t) + n^{-1/2}\hat{p}_1(t)\phi(t) + n^{-1}\hat{p}_2(t)\phi(t) + O_p(n^{-3/2})$$

where $\hat{p}_1(t)$ and $\hat{p}_2(t)$ are evaluated with respect to PIT-trapped data $Y^*$ whose marginal distribution is $F(y; \hat{\theta}, x_i)$, where the correlation between PIT-trapped residuals is $var_*(U_i) = \hat{\Sigma}$.

Now from Theorem 1, the cumulative distribution function of a PIT-trap value $Y^*_{ij}$ is $F(y; \theta_j, x_i) + O_p(n^{-1/2})$, and from Theorem 1, $var(U_i^*) = \hat{\Sigma}$ whose entries differ from
those of \( \Sigma \) by \( O_p(n^{-1/2}) \). Since \( F(y; \theta_j, x_i) \) and \( \Sigma \) characterize the joint distribution of the \( Y_i \),

\[
\hat{p}_k(t) = p_k(t) + O_p(n^{-1/2})
\]

for any \( k \) for which the \( k \)th moment of \( Y_{ij} \) is defined.

Hence the coefficients of \( n^{-1/2} \) in the above two Edgeworth expansions match to first order and

\[
P_*(T^* \leq t) = P(T \leq t) + O_p(n^{-1})
\]

As in \( ? \), \( \hat{p}_k(t) \) and \( p_k(t) \) are odd functions for odd \( k \). Hence the odd terms cancel when calculating a two-tailed probability, removing the coefficient of \( n^{-1/2} \) in each expansion, and the coefficients of \( n^{-1} \) match to first order, so

\[
P_*(-t \leq T^* \leq t) = P(-t \leq T \leq t) + O_p(n^{-3/2})
\]

**References**

Hall, P. (1992). *The bootstrap and Edgeworth expansion*. Springer-Verlag, New York.

Kolassa, J. E. and McCullagh, P. (1990). Edgeworth series for lattice distributions. *The Annals of Statistics*, 18(2):981–985.