Velocity Differences as a Probe of Non–Gaussian Density Fields

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Summary

We examine the multi–point velocity field for non–Gaussian models as a probe of non–Gaussian behavior. The two–point velocity correlation is not a useful indicator of a non–Gaussian density field, since it depends only on the power spectrum, even for non–Gaussian models. However, we show that the distribution of velocity differences $v_1 - v_2$, where $v_1$ and $v_2$ are measured at the points $r_1$ and $r_2$, respectively, is a good probe of non–Gaussian behavior, in that $p(v_1 - v_2)$ tends to be non–Gaussian whenever the density field is non–Gaussian. As an example, we examine the behavior of $p(v_1 - v_2)$ for non–Gaussian seed models, in which the density field is the convolution of a distribution of points with a set of density profiles. We apply these results to the global texture model.

Key Words: Cosmology – dark matter – galaxies: clustering – galaxies – large–scale Structure of the Universe.
1 Introduction

A fundamental problem in modern cosmology is to understand how the large scale structures in the observed Universe arose from the primordial energy density fluctuations. The simplest hypothesis is that an inflationary de Sitter phase produced Gaussian, scale–invariant adiabatic fluctuations leading to the generation of large scale clustered structure grew due to their gravitational instability. This scenario, in a Universe dominated by cold dark matter (CDM), has been considered the most successful theory for the formation of galaxy or cluster structures in the Universe (Frenk et al. 1988; Davis et al. 1992).

However, there are also notable setbacks for this model from observational evidence. The most serious of these observational challenges have come from the variance in the cell counts of IRAS galaxies (Efstathiou et al. 1990), from the measurement of the skewness in the distribution of QDOT–IRAS galaxies out to $140\,h^{-1}\text{Mpc}$ (Saunders et al. 1991), from the large–scale clustering of radio galaxies (Peacock & Nicholson 1991) and rich clusters of galaxies (Efstathiou et al. 1992), and from the statistical analysis of the APM galaxy distribution on scales beyond $10\,h^{-1}\text{Mpc}$ (Maddox et al. 1990).

Occurrence of high–redshift quasars (Warren et al. 1987), radio sources (Lilly 1988; Chambers, Miley & van Breugel 1990) and protoclusters of galaxies (Usom, Bagri & Cornwell 1991), as well as the Great Bootes Void, Great Wall, Great and Giant Attractor regions (Kirshner et al. 1981; Geller & Huchra 1990; Lynden–Bell et al. 1988; Scaramella et al. 1989) may pose further problems to the “standard” CDM Gaussian model (see the discussion in Davis et al. 1992).

As a consequence, alternative theories of the origin of large scale structure in the Universe continue to be worth investigation. Among the chief rivals to the inflationary model are a class of models in which the initial inhomogeneities are produced by a primordial population of seeds, e.g. cosmic string loops (Zel’ dovich 1980; Vilenkin 1981), global textures (Gooding et al. 1992 and references therein), or generic “seeds” (Villumsen et al. 1991; Gratsias et al. 1993),
and the final linear matter fluctuations are not described by a Gaussian distribution. Non-Gaussian distributed perturbations are the most general statistical framework for computing cosmological observables, such as spatial galaxy correlation functions (Matarrese, Lucchin & Bonometto 1986; Scherrer & Bertschinger 1991), expected size and frequency of high density regions (Catelan, Lucchin & Matarrese 1988), hotspots and coldspots in the cosmic microwave background (CMB) distribution on the sky (Coles & Barrow 1987; Kung 1993) and higher order temperature correlation functions (Luo & Schramm 1993; Gangui et al. 1994). The importance of the signature of the initial distribution of non-Gaussian density fluctuations for the final clustering pattern of the cosmic structures has been recently emphasized, in a series of N-body simulations, by Messina et al. (1990), Moscardini et al. (1991), Matarrese et al. (1991) and Weinberg & Cole (1992). Coles et al. (1993) show that the mass distribution in models with initially non-Gaussian fluctuations exhibits systematic departures from Gaussian behavior on intermediate to large scales.

A powerful method to probe the matter distribution for non-Gaussian behavior is to examine the large-scale bulk velocity field, which is, in the linear regime (i.e. large scales), directly related to the density field (Peebles 1980). The problem has been addressed by Scherrer (1992). Scherrer, noting that the RMS velocities are the same function of the power spectrum for both Gaussian and non-Gaussian density fields, examined the distribution of velocities for non-Gaussian density fields. For density fields in which the gravitational potential field is non-Gaussian by construction (“local” non-Gaussian fields) the velocity field can be strongly non-Gaussian. However, Scherrer found that the one-point distribution of linear velocities for “seed” models, in which the density field is the convolution of a discrete set of points with a set of density profiles, is highly Gaussian even when the density field is strongly non-Gaussian; this result has also been seen in numerical simulations of the texture model (Gooding, et al. 1992). The reason for this is the central limit theorem: the linear velocity field is an integral over the density field; if the density field is sufficiently uncorrelated, the sum of uncorrelated densities produces a Gaussian velocity field. Of course, the linear velocity field in this case cannot be
an exact multivariate Gaussian, as the divergence of such a field (which is proportional to the
density field) would then have to be exactly a Gaussian.

In this paper, we exploit this fact to derive a velocity statistic which is sensitive to non–
Gaussian density fields. We extend the analysis of Scherrer (1992) to derive the two–point
velocity distribution for seed models. We show that the distribution of velocity differences is
an effective probe of non–Gaussian behavior in the density field.

In the next section, we review the general properties of linear cosmological velocity fields,
valid for any mass density distribution, Gaussian or non–Gaussian. In §3, after reviewing the
statistical properties of seed models, we work out exact analytical expressions for the 1–point
and 2–point velocity distributions using the partition function for randomly distributed seed–
objects. These results are used to derive an expression for the distribution of the difference
between velocities measured at two different points. We apply our results first to a toy seed
model to illustrate some of our general arguments about the usefulness of this method, and
then to the global texture model with cold dark matter (Gooding et al. 1992). Our results are
discussed in Section 4.

2 Cosmological Velocity Fields

During the linear regime, when the amplitude of matter fluctuations is very small ($\delta \ll 1$),
the peculiar velocity $v$ is related to the matter fluctuation $\delta$ by the equation (see, e.g., Peebles
1980)

$$\nabla \cdot v(r, t) = -H \Omega^{0.6} a \delta(r, t) ,$$

where $H$ is the Hubble parameter, $a$ is the scale factor, and $\Omega$ is the mean density in units of
the critical density. Any transverse mode, defined by the condition $\nabla \cdot v = 0$ and corresponding
to the rotational part of the velocity field, decays with the expansion (namely, the vorticity
decays with rate $\propto a^{-2}$ in an Einstein–de Sitter universe). Therefore, the integral expression
of the solution of eq. (1) may be written as (Peebles 1980)
\[ \mathbf{v}(\mathbf{r}, t) = \frac{H \Omega^0 a}{4\pi} \nabla \int d^3\mathbf{r}' \frac{\delta(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} = -\frac{2 \Omega^{-0.4}}{3 \dot{H} a} \nabla \phi(\mathbf{r}, t), \] (2)
which is manifestly irrotational; \( \phi \) is the peculiar gravitational potential. We see that in the linear regime, both density and velocity fields may be derived from a unique potential \( \phi \). In eqs. (1)–(2), we consider only the growing modes for density and velocity. In what follows, we fix \( t \) equal to the present time, taking \( a = 1 \) and \( H = H_0 \).

Results for the linear velocity field are of interest because the observations, when smoothed on sufficiently large scales, recover this linear velocity field. This smoothing is usually done by filtering on the scale \( R \) the velocity field \( \mathbf{v}(\mathbf{r}) \) by means of a window function \( W_R(x) \), \( x \equiv |x| \),
\[ \mathbf{v}_R(\mathbf{r}) = \int d^3x \mathbf{v}(x) W_R(|x - \mathbf{r}|). \] (3)
The windowing convolution averages the velocity field \( \mathbf{v} \) over the point \( x \) in a volume \( \sim R^3 \), where one has no interest in the substructure. Typical choices are the Gaussian and the top–hat window functions (e.g. Bardeen et al. 1986). Working with the Fourier transforms of the quantities in eq. (3), it is easy to show that the smoothed linear velocity field corresponding to a particular density field is identical to the unsmoothed linear velocity field corresponding to that density field smoothed with the same window function (Scherrer 1992). Thus, the smoothed velocity field can be derived by applying eq. (2) to the smoothed density field.

An equivalent manner to describe the velocity field is in terms of the Fourier components of the density field, \( \hat{\delta}(\mathbf{k}) \). An important cosmological observable, the rms peculiar velocity \( \sigma_v \), can be expressed very simply in terms of the power spectrum \( P(k) \):
\[ \sigma_v^2 \equiv \langle \mathbf{v} \cdot \mathbf{v} \rangle = H^2_0 \Omega^1.2 \int_0^\infty 4\pi P(k) \, dk, \] (4)
where \( P(k) \) is defined by \( \langle \hat{\delta}(\mathbf{k}_1) \hat{\delta}(\mathbf{k}_2) \rangle \equiv \delta_D(\mathbf{k}_1 + \mathbf{k}_2) P(k_1) \). The relation between the rms velocity \( \sigma_v \) and the power spectrum \( P(k) \) is valid for any type of density field, Gaussian or non–Gaussian (Scherrer 1992). Thus \( \sigma_v \) cannot be used to distinguish Gaussian from non–Gaussian models. For this reason Scherrer (1992) was motivated to examine the distribution
of \( \mathbf{v} \), which is different for Gaussian and non–Gaussian models. For example, in local non–Gaussian models, in which the potential field is local nonlinear function of a Gaussian field (Kofman et al. 1989; Moscardini et al. 1991) the one–point distribution of a single component of \( \mathbf{v} \) is strongly non–Gaussian (Scherrer 1992). However, for seed models, in which the density field is the convolution of a density profile with a random distribution of points, the distribution of a single component of \( \mathbf{v} \) is nearly Gaussian.

We may look for better tracers of a non–Gaussian density field among the multi–point velocity distribution. An obvious choice is the two–point velocity correlation function

\[
\langle \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}((\mathbf{x} + \mathbf{r})) \rangle = H_0^2\Omega^{1.2} \int \frac{1}{k^2} P(k) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 \mathbf{k}.
\] (5)

We see that just as in the case of the rms velocity, the two–point velocity correlation depends only on the power spectrum. Thus, it is incapable of distinguishing a Gaussian density field from a non–Gaussian field having the same power spectrum.

We argue here that a better test for non–Gaussian behavior is the distribution of velocity differences \( p(\mathbf{v}_1 - \mathbf{v}_2) \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are the velocities measured at the points \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively. In the usual way, we define \( v_{||} \) and \( v_{\perp} \) to be the components of the velocity parallel and perpendicular to \( \mathbf{r}_1 - \mathbf{r}_2 \), respectively. Then it is plausible that the distribution of \( v_{1||} - v_{2||} \) will become non–Gaussian for a non–Gaussian density field in the limit where \( |\mathbf{r}_1 - \mathbf{r}_2| \to 0 \). [We measure \( v_{1||} - v_{2||} \) relative to a coordinate system in which the direction of increasing distance points from \( \mathbf{r}_2 \) to \( \mathbf{r}_1 \); this insures that \( v_{1||} - v_{2||} \) has the same sign as \( \partial v_x / \partial x \) in the limit \( |\mathbf{r}_1 - \mathbf{r}_2| \to 0 \).] From eq. (1), we know that the distributions of \( \nabla \cdot \mathbf{v} \) and \( \delta \) must be the same (up to a multiplicative constant). For an isotropic density field, \( \partial v_x / \partial x \), \( \partial v_y / \partial y \), and \( \partial v_z / \partial z \) must all have the same distribution. There is no simple relationship between the distribution of \( \partial v_x / \partial x \) and the distribution of \( \delta \), since the various partial derivatives of \( \mathbf{v} \) are correlated. However, it would require a very contrived density field for \( \partial v_x / \partial x \), \( \partial v_y / \partial y \) and \( \partial v_z / \partial z \) to all have a nearly Gaussian one–point distribution, while \( \delta \) was highly non–Gaussian. Furthermore, \( v_{1||} - v_{2||} \) is proportional to \( \partial v_x / \partial x \) in the limit where the separation between \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) goes
to zero. Thus, we expect $p(v_1 - v_2)$ to be non-Gaussian for sufficiently small separations whenever $p(\delta)$ is non-Gaussian. This is our motivation for examining $p(v_1 - v_2)$ as a probe of non-Gaussian behavior. In essence, the taking of velocity differences is a poor man’s derivative. Integrating over the density field to derive the velocity field can drive $p(v)$ to a Gaussian because of the central limit theorem; taking velocity difference restores the non-Gaussian behavior lost through integration. Of course, a major issue is the maximum separation over which non-Gaussian behavior can be detected in the distribution of $v_1 - v_2$; if $p(v)$ is Gaussian, and if velocity correlations go to zero at sufficiently large distances, then the $p(v_1 - v_2)$ must also go to a Gaussian over those distances.

3 Velocity Differences in Seed Models

To test our velocity–difference statistic, we analyze the large–scale velocity field in a special class of intrinsically non–Gaussian models, in which the initial linear density field can be described as the convolution of a set of density profiles with a random distribution of points (Scherrer & Bertschinger 1991; Scherrer 1992). This set of non–Gaussian models provides a useful test of our method for several reasons. Such distributions are analytically tractable and can describe, for example, the density field produced by textures (Gooding et al. 1992) and the seeded hot dark matter model (Villumsen et al. 1991; Gratsias et al. 1993). Further, such models tend to produce a Gaussian one–point velocity distribution even when the density field is strongly non–Gaussian (Scherrer 1992), so they are useful to determine if the velocity differences provide a better probe of the non–Gaussian behavior.

In these seed models, the linear density field $\delta(r)$ may be written as a convolution of a given density profile $f(m, r)$ with a set of points having mean density $n(m)$, where $m$ is simply a parameter or set of parameters such as mass or formation time:

$$\delta(r) = \sum_h f(m_h, r - x_h)$$

(6)
where the sum is taken over all seeds with positions \( \mathbf{x}_h \). The statistical properties of the density field are entirely determined by \( f(m, \mathbf{r}) \), \( n(m) \), and the spatial correlations of the seeds (see Scherrer & Bertschinger 1991 for more details). Here we follow Scherrer (1992) and consider only randomly distributed seeds for which \( f \) is spherically symmetric: \( f(m, \mathbf{r}) = f(m, r) \). This simpler model is a reasonably good description of the texture model (Gooding et al. 1992) and the seeded hot dark matter model (Villumsen et al. 1991; Gratsias et al. 1993).

In the linear regime, both the density perturbations and the velocities induced by the seeds can be added linearly, so that the perturbation \( \delta(\mathbf{r}) \) in eq. (1) produces at \( \mathbf{r} \) the peculiar velocity \( \mathbf{v}(\mathbf{r}) \) given by

\[
\mathbf{v}(\mathbf{r}) = \sum_h \tilde{v}(m_h, \mathbf{r} - \mathbf{x}_h),
\]

where \( \tilde{v}(m_h, \mathbf{r} - \mathbf{x}) \) is the contribution to \( \mathbf{v} \), measured at the point \( \mathbf{r} \), due to a single \( m \)–seed located at the point \( \mathbf{x} \). From eq. (1), we have

\[
\nabla \cdot \tilde{v}(m, \mathbf{r} - \mathbf{x}) = -H_0 \Omega_0^{0.6} f(m, |\mathbf{r} - \mathbf{x}|) ;
\]

(8)

Thus, \( f(m, |\mathbf{r} - \mathbf{x}|) \) acts exactly as a density perturbation source for the single seed–velocity contribution \( \tilde{v}(m, \mathbf{r} - \mathbf{x}) \), and the total velocity \( \mathbf{v}(\mathbf{r}) \) is the superposition, for the entire population of seeds, of the single seed velocities. To determine \( \tilde{v}(m, \mathbf{r} - \mathbf{x}) \), we integrate the previous equation on the sphere \( V_x \) centered in \( \mathbf{x} \). Gauss’s law gives

\[
\tilde{v}(m, \mathbf{r} - \mathbf{x}) = -\beta \tilde{f}(m, |\mathbf{r} - \mathbf{x}|) \frac{\mathbf{r} - \mathbf{x}}{|\mathbf{r} - \mathbf{x}|},
\]

(9)

where \( \beta \equiv H_0 \Omega_0^{0.6} \): we see that the \( m \)–seed at \( \mathbf{x} \) produces a velocity component \( \tilde{v}(m, \mathbf{r} - \mathbf{x}) \) at \( \mathbf{r} \) which points toward the seed location, i.e., each seed tends to create a spherical velocity field around it. In eq. (9), we have defined the quantity

\[
\tilde{f}(m, |\mathbf{r} - \mathbf{x}|) \equiv \frac{1}{|\mathbf{r} - \mathbf{x}|^2} \int_0^{\text{max}} dr' r'^2 f(m, r') ,
\]

(10)

which represents the integrated density perturbation due to the seed at \( \mathbf{x} \), enclosed in a sphere of radius \( |\mathbf{r} - \mathbf{x}| \) centered on the seed. From Eqs.(7) and (9) we obtain the total velocity \( \mathbf{v} \) in
term of the accretion pattern,

\[ v(r) = -\beta \sum_h \bar{f}(m_h, |r - x_h|) \frac{r - x_h}{|r - x_h|}. \]  

(11)

Equations (7)–(11) are somewhat more general than those given by Scherrer (1992), in that the location of the seed is arbitrarily chosen.

We now analyze the statistical properties of the velocity field \( v \) for these seed models. If the seeds are randomly distributed, the probability distribution of the total velocity \( v \) may be obtained by applying the Poisson model (Peebles 1980; Fry 1985). To do this, we introduce the stochastic Poisson variables \( \epsilon_h \) in such a way that

\[ \epsilon_h = \begin{cases} 
1 & \text{if a seed } \in dx_h \\
0 & \text{otherwise ,}
\end{cases} \]  

(12)

with mean

\[ \langle \epsilon_h \rangle = n(m_h) dm_h dx_h . \]  

(13)

From eq. (7), we know that \( v(r) \) is the superposition of all the single seed contributions, and to remember the stochastic information, i.e., presence or absence of a seed in \( dx_h \), we define the variable \( \bar{v}(r) \equiv \sum_h \bar{v}(m_h, r - x_h) \epsilon_h \).

The probability distribution function \( p(v) \) may be calculated according to the definition (see, e.g., Ma 1985)

\[ p(v) = \langle \delta_D[\bar{v}(r) - v(r)] \rangle = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3t \ e^{-it \cdot v(r)} \prod_h \phi_{\epsilon_h}(t) , \]  

(14)

where \( \phi_{\epsilon_h}(t) \) is the characteristic (or moment generating) function for the single seed discrete process (Peebles 1980; Fry 1985)

\[ \phi_{\epsilon_h}(t) \equiv \langle e^{-it \bar{v}(m_h, r - x_h) \epsilon_h} \rangle = \exp \left[ n(m_h) dm_h d^3x_h \left( e^{it \bar{v}(m_h, r - x_h)} - 1 \right) \right] \]  

(15)

Therefore, the characteristic function for the total velocity is the product of the individual \( \phi_{\epsilon_h} \)

\[ \phi(t) = \prod_h \phi_{\epsilon_h}(t) \longrightarrow \exp \int n(m) dm d^3x \left( e^{it \bar{v}(m, r - x)} - 1 \right) \]  

(16)
in the continuum limit $dm_h \to 0$ and $d\mathbf{x}_h \to 0$. Finally, the velocity pdf $p(\mathbf{v})$ may be written as

$$p(\mathbf{v}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3t \ e^{-i\mathbf{v} \cdot \mathbf{t}} \exp \int n(\mathbf{m}) \ dm \ d^3\mathbf{x} \left(e^{it \cdot \widehat{\mathbf{v}}(\mathbf{m}, \mathbf{r} - \mathbf{x})} - 1\right),$$

which is a generalization of the result given by Scherrer (1992).

Scherrer (1992) showed that the distribution of the one point velocity, $p(\mathbf{v})$, is a rather poor indicator of non–Gaussian behavior for seed models with randomly distributed seeds. In particular, for the seeded hot dark matter model, a non–Gaussian velocity field requires an extremely low seed density, while the texture model with cold dark matter has a velocity field which is nearly indistinguishable from a Gaussian.

We now consider the two–point distribution of velocities, $p(\mathbf{v}_1, \mathbf{v}_2)$. Taking advantage of our previous results, it is straightforward to show that the the 2–point distribution is given by

$$p(\mathbf{v}_1, \mathbf{v}_2) = \langle \delta_D[\widehat{\mathbf{v}}(\mathbf{r}_1) - \mathbf{v}(\mathbf{r}_1)] \delta_D[\widehat{\mathbf{v}}(\mathbf{r}_2) - \mathbf{v}(\mathbf{r}_2)] \rangle,$$

$$= \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} dt_1 dt_2 \ e^{-i \mathbf{t}_1 \cdot \mathbf{v}_1 - i \mathbf{t}_2 \cdot \mathbf{v}_2} \phi(\mathbf{t}_1, \mathbf{t}_2),$$

where

$$\phi(\mathbf{t}_1, \mathbf{t}_2) \equiv \exp \left[ \int n(\mathbf{m}) \ dm \ d^3\mathbf{x} \left(e^{it_1 \cdot \widehat{\mathbf{v}}(\mathbf{m}, \mathbf{r}_1 - \mathbf{x}) + it_2 \cdot \widehat{\mathbf{v}}(\mathbf{m}, \mathbf{r}_2 - \mathbf{x})} - 1\right) \right],$$

is the characteristic function for the 2–point distribution of velocities; it generates the 2–point velocity correlation tensor component defined by (Groth, Juszkiewicz & Ostriker 1989)

$$\xi_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle v_{\alpha}(\mathbf{r}_1) v_{\beta}(\mathbf{r}_2) \rangle, \ \alpha, \ \beta = 1, 2, 3,$$

through partial differentiation:

$$\xi_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{\partial \phi}{\partial t_{1\alpha}(0)} \frac{\partial \phi}{\partial t_{2\beta}(0)}. \quad (20)$$

The statistical properties of the large–scale peculiar velocity field, as described by the correlation tensor, are extensively examined in Gorsky et al. (1989) and Landy & Szalay (1992).

From eq. (18), one can in particular obtain the probability distribution of the velocity difference

$$p(\mathbf{v}_1 - \mathbf{v}_2) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3t \ e^{-i \mathbf{t} \cdot (\mathbf{v}_1 - \mathbf{v}_2)} \phi(\mathbf{t})$$

(21)
where now

$$\phi(t) = \exp \left[ \int n(m) \, dm \, d^3x \left( e^{it(\bar{v}(m, r_1 - x) - \bar{v}(m, r_2 - x))} - 1 \right) \right]. \tag{22}$$

A convenient measure of the deviation from a Gaussian is given by the cumulants $\kappa_p$, defined by

$$\ln \phi(t) = \sum_{p=1}^{\infty} \frac{\kappa_p (it)^p}{p!} \tag{23}$$

For the distribution of velocity differences, it is clear that

$$\kappa_p = \int n(m) \, dm \, d^3x \left[ \bar{v}_\alpha(m, r_1 - x) - \bar{v}_\alpha(m, r_2 - x) \right]^p \tag{24}$$

where eq. (24) gives the cumulants corresponding to the distribution of the $\alpha$ component of the velocity differences. The normalized cumulants

$$\lambda_p \equiv \kappa_p / \kappa_2^{p/2} \tag{25}$$

give a measure of the deviation of a distribution from a Gaussian; for $\lambda_p \ll 1$ the distribution is nearly Gaussian, while $\lambda_p \gg 1$ indicates a highly non–Gaussian distribution.

We argued in §2 that the distribution of $v_1 - v_2$ should show the same level of non–Gaussian behavior as the distribution of $\delta$; here we test these claims numerically. First consider a toy seed model which consists of randomly–distributed seeds with number density $n_0$, all having the same mass $m_s$, so that $n(m) = n_0 \delta_D(m - m_s)$. We also assume a spherical tophat density profile: $f(m, r) = mg_0$ for $r \leq r_0$, and $f(r) = 0$ for $r > r_0$. [This is not a physically realistic model, but the distributions for $\delta$ and $v$ can be derived analytically, allowing us to confirm some of our general arguments about the usefulness of the velocity difference statistic. Some of the properties of this model were examined by Scherrer & Bertschinger (1991). Alternately, this model gives the velocity field smoothed with a spherical tophat window function for a set of point–like seed masses]. To illustrate the points discussed earlier, we calculate $\lambda_p$ for the distributions of $\delta$, $v_x$ (a single component of the velocity) and $\partial v_x / \partial x$. If we define $\bar{n} = n_0 4\pi r_0^3 / 3$, then the distribution of $\delta$ is just a Poisson distribution with

$$\lambda_p = \bar{n}^{1-p/2} \tag{26}$$
The value of $\lambda_p$ for $v_x$ can be derived using the techniques outlined in Scherrer (1992); we find (for even $p$)

$$
\lambda_p = \bar{n}^{1-p/2} \left( \frac{3}{p+1} \right) \left( \frac{1}{p+3} + \frac{1}{2p-3} \right) \left( \frac{5}{6} \right)^{p/2}.
$$

Finally, consider the distribution of $\partial v_x/\partial x$, which is proportional to $v_{1\parallel} - v_{2\parallel}$ when the separation goes to 0. In this case, eq. (24) with $\bar{v}_\alpha(m, r_1 - x) - \bar{v}_\alpha(m, r_2 - x)$ replaced by $\partial \bar{v}_x/\partial x$ gives

$$
\lambda_p = \bar{n}^{1-p/2} (5/9)^{p/2} \left[ (-1)^p + \frac{1}{2p-2} \int_{-1}^{1} (3x^2 - 1)^p dx \right].
$$

The lowest order non-zero deviation from a Gaussian for $v_x$ is given by $p = 4$. For this case we find that $\lambda_4 = \bar{n}^{-1}$ (distribution of $\delta$), $\lambda_4 = 0.14\bar{n}^{-1}$ (distribution of $v_x$), and $\lambda_4 = 0.45\bar{n}^{-1}$ (distribution of $\partial v_x/\partial x$). Furthermore, in the limit where $p \to \infty$, the asymptotic expressions for $\lambda_p$ are

$$
\lambda_p = \bar{n}^{1-p/2} \frac{9}{2p^2} \left( \frac{5}{6} \right)^{p/2},
$$

for $v_x$, and

$$
\lambda_p = \bar{n}^{1-p/2} \frac{1}{3p^2} \left( \frac{20}{9} \right)^{p/2},
$$

for $\partial v_x/\partial x$. Thus, for an appropriate choice of $\bar{n}$, the distribution of $v_x$ can be nearly Gaussian while the distribution of $\delta$ is highly non-Gaussian, in agreement with the results of Scherrer (1992). However, the distribution of $\partial v_x/\partial x$ displays roughly the same deviation from Gaussianity as does the distribution of $\delta$. Therefore, for sufficiently small separations, the distribution of $v_{1\parallel} - v_{2\parallel}$ will also trace the non-Gaussian behavior of $\delta$.

Now consider a physically realistic non-Gaussian model: the $\Omega_0 = 1$ CDM global texture model (Gooding et al. 1992 and references therein). This model can be approximated as a seed model, with a random distribution of seeds (the locations of the texture knots). We take the parameter $m$ to be the conformal time $\tau$ at which the knots unwind; then the number density of knots unwinding at $\tau$ is (Gooding et al. 1992)

$$
 dn = \nu \tau^{-4} d\tau,
$$

(31)
with $\nu = 0.04$. We use an analytic approximation for $f(\tau, r)$ given by Gooding (1992)

$$f(\tau, r) = Ce^{-\lambda r/\tau}(1 - \lambda r/2\tau)f_g, \quad (32)$$

where $\lambda = 1.88$ and the constant $C$ does not affect our calculations. The growth factor $f_g$ is

$$f_g = a_i[\delta_2(a_i)\delta_1(a) - \delta_1(a_i)\delta_2(a)], \quad (33)$$

where $a$ is the scale factor at which the density perturbations are measured, $a_i$ is the scale factor corresponding to the conformal time $\tau$, and $\delta_1$ and $\delta_2$ are the growing and decaying modes in linear theory (see Scherrer 1992 for more details).

Using this approximation for the CDM texture model, we have numerically evaluated the normalized skewness $\lambda_3$ of the distribution of the component of the velocity difference along the separation $\mathbf{r}_1 - \mathbf{r}_2$, $v_{\|}(\mathbf{r}_1) - v_{\|}(\mathbf{r}_2)$, as a function of separation distance $|\mathbf{r}_1 - \mathbf{r}_2|$. In Figure 1, we plot this skewness versus the separation distance of the points at which the velocities are measured (solid curve). We see that for relatively small separations, the difference of velocities is a good tracer of non–Gaussian behavior. Our statistic is useful up to a separation of about $10h^{-2}\text{Mpc}$. The density field for the texture model has positive skewness, so it is not surprising that that $v_{\|}(\mathbf{r}_1) - v_{\|}(\mathbf{r}_2)$ has negative skewness (see eq. 1 and our argument in §2). It is trivial to extend these results to the case of the smoothed velocity field because, as noted in §2, the smoothed linear velocity field is identical to the unsmoothed linear velocity field corresponding to the smoothed density field. For a velocity field smoothed with a Gaussian window function $W(r) = \exp(-r^2/r_0^2)$, the skewness of the distribution of velocity differences along the separation vector is given in Figure 1 for $r_0 = 2h^{-2}\text{Mpc}$ (dashed curve). We find, surprisingly, that the smoothed velocity field actually deviates more strongly from a Gaussian than the unsmoothed field. The density fluctuation produced by a single texture integrated out to a radius $R$ goes to zero in the limit $R \gg \tau$ (Gooding, et al. 1992) so smoothing a texture with a window function of radius $r_0 \gg \tau$ effectively erases the texture, and we are left with a density field consisting of larger textures with larger separations, giving a more non–Gaussian
field. Thus, the enhancement of the non–Gaussian nature of $p(v_{\parallel}(r_1) - v_{\parallel}(r_2))$ with smoothing is probably peculiar to the texture model.

4 Discussion

Our results indicate that the distribution of $v_{\parallel}(r_1) - v_{\parallel}(r_2)$, the component of the velocity difference at two points along the vector separating the observation points, is a useful probe of non–Gaussian behavior. This distribution tends to be non–Gaussian whenever the underlying density field is non–Gaussian, in contrast to the one–point distribution of velocities, which tends to be more Gaussian than the underlying density field (Scherrer 1992). Although we have examined only a single class of non–Gaussian models, our results should be generally applicable to any non–Gaussian model, because of our argument in §2. We note that a related argument for detecting signatures of non–Gaussian density fields in the cosmic microwave background has been given by Moessner, Perivolaropoulos, and Brandenberger (1994). They argue that the temperature gradient of the CMB should provide a better test to distinguish the cosmic string model than the distribution of the temperature field itself.

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Figure Captions

Figure 1. The normalized skewness $\lambda_3$ of the distribution of $v_{\parallel}(r_1) - v_{\parallel}(r_2)$ [the component of the velocity difference parallel to the separation $r_1 - r_2$], as a function of the separation $|r_1 - r_2|$ for the unsmoothed density field (solid curve) and the density field smoothed with a Gaussian window function of radius $2h^{-2}$ Mpc (dashed curve).