Latin Transversals of Rectangular Arrays
by
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Let $m$ and $n$ be integers, $2 \leq m \leq n$. An $m$ by $n$ array consists of $mn$ cells, arranged in $m$ rows and $n$ columns, and each cell contains exactly one symbol. A transversal of an array consists of $m$ cells, one from each row and no two from the same column. A latin transversal is a transversal in which no symbol appears more than once. We investigate $L(m, n)$, the largest integer such that if each symbol in an $m$ by $n$ array appears at most $L(m, n)$ times, then the array must have a latin transversal. We will obtain upper and lower bounds on $L(m, n)$ and also determine $L(2, n)$ and $L(3, n)$.

Note that $L(m, n) \leq L(m + 1, n)$ and that $L(m, n) \leq L(m, n + 1)$. The function $L$ satisfies two more inequalities, stated in Theorems 1 and 2.

**Theorem 1.** For $n \leq 2m - 2$, $L(m, n) \leq n - 1$.

The proof depends on a general construction due to E. T. Parker [6], illustrated for the cases $(m, n) = (4, 4), (4, 5)$, and $(4, 6)$:

1 1 4 4  
2 2 1 1  
3 3 2 2  
4 4 3 3  

1 1 1 4 4  
2 2 2 1 1  
3 3 3 2 2  
4 4 4 3 3  

1 1 1 4 4 4  
2 2 2 1 1 1  
3 3 3 2 2 2  
4 4 4 3 3 3  

In each case an attempt to construct a latin transversal might as well begin with a 1 in the top row. The 2 must then be selected from the 2’s in the second row, and the 3 from the 3’s in the third row. Such choices do not extend to a latin transversal.

**Theorem 2.** For $n \geq 2m - 1$, $L(m, n) \leq (mn - 1)/(m - 1)$.
This theorem follows from the fact that if only \( m - 1 \) distinct symbols appear in an \( m \) by \( n \) array, the array cannot have a latin transversal. In detail, if each of \( m - 1 \) symbols appears at most \( k \) times and \((m - 1)k\) is at least \( mn - 1\), the symbols may fill all the cells. Hence the inequality stated in Theorem 2 holds.

Though Theorem 2 is valid for all \( n \), in view of Theorem 1, it is of interest only for \( n \geq 2m - 1 \).

It is quite easy to determine \( L(2, n) \). A moment’s thought shows that \( L(2, 2) = 1 \) and that \( L(2, n) = 2n - 1 \) for \( n \geq 3 \). This means that for \( m = 2 \) the inequalities in Theorems 1 and 2 become equalities. The case \( m = 3 \) is similar, for it turns out that \( L(3, n) \) equals \( n - 1 \) for \( n = 3, 4 \) and is the largest integer less than or equal to \((3n - 1)/2\) for \( n \geq 5 \). The following two lemmas are used in the proof of the second assertion.

In each case \( x \) stands for 1 or 2.

**Lemma 1.** Assume that in a 3 by \( n \) array, \( n \geq 4 \), the following configuration is present. If there is no latin transversal the cell containing \( y \) must be 1 or 2.

\[
\begin{array}{cccc}
  y & \cdots \\
  x & \cdots \\
  1 & 2 & \cdots \\
\end{array}
\]

The proof is immediate.

**Lemma 2.** Assume that in a 3 by \( n \) array, \( n \geq 4 \), some symbol occurs at most three times. Then, if there is no latin transversal some symbol occurs at least \( 2n - 2 \) times, hence at least \( 3n/2 \) times.

**Proof.** We regard two arrangements of symbols in cells as equivalent if one can be obtained from the other by a permutation of rows, a permutation of columns, and a relabeling. There are seven inequivalent configurations of cells occupied by a symbol that occurs at most three times. We illustrate them by treating the case when 1 appears twice, in one row, as in the following diagram:

\[
\begin{array}{cccc}
  1 & 1 & \cdots \\
  b & b & 2 & b & b & \cdots \\
  a & a & c & a & a & \cdots \\
\end{array}
\]

We may assume the 2 occurs as indicated. It follows that the cells marked \( a \) are filled with 2’s. This implies that all the cells marked \( b \) are also filled with 2’s, and finally that the cell marked \( c \) also contains a 2. Hence the symbol 2 appears at least \( 2n \) times.

In the case when a symbol occurs only once, it is not hard to show that some symbol appears at least \( 2n - 2 \) times. In most of the other cases a symbol appears almost \( 3n \).
Theorem 3. (a) $L(3, 3) = 2$ and $L(3, 4) = 3$. (b) For $n \geq 5$, $L(3, n)$ is the greatest integer less than or equal to $(3n - 1)/2$.

Proof of (a). To begin we show that $L(3, 3) = 2$.

By Theorem 1, we know that $L(3, 3)$ is at most 2. All that remains is to show that if each symbol in a 3 by 3 array appears at most twice, then the array has a latin transversal.

First of all, at least five different symbols must appear in the array, hence at least one, say 1, must appear exactly once. Without loss of generality, we may assume that the following configuration occurs in the array:

\[
\begin{array}{ccc}
1 & 4 & \\
2 & 3 & \\
3 & 2 & \\
\end{array}
\]

To avoid the formation of a latin transversal, the two empty cells in the first column must contain the symbol 4, in violation of our assumption that each symbol appears at most twice. Thus $L(3, 3) = 2$.

Next we show that $L(3, 4) = 3$.

We may assume that if there is no latin transversal in a 3 by 4 array that either all cells contain the same symbol or else the following configuration occurs, where $x$ stands for either 1 or 2:

\[
\begin{array}{ccc}
1 & x & \\
2 & & \\
1 & x & \\
\end{array}
\]

This breaks into two cases depending on whether the top $x$ is 1 or 2. In either case, the condition that there is no latin transversal forces another cell to be an $x$.

\[
\begin{array}{ccc}
1 & 1 & \\
x & 2 & \\
1 & x & \\
\end{array}
\]

(1)

\[
\begin{array}{ccc}
1 & 2 & \\
x & 1 & x \\
2 & & \\
\end{array}
\]

(2)

In case (1), since there are already three 1’s, the $x$’s must be 2’s, and the cell labeled $y$ below is part of a latin transversal.
Case (2) is slightly different. Using the lower $x$, we obtain two cases:

\[
\begin{array}{ccc}
1 & x & 2 \\
x & 2 & 1 \\
\end{array}
\quad \text{and} \quad 
\begin{array}{ccc}
1 & 2 \\
x & 2 & x \\
\end{array}
\]

In the first case both $x$'s must be 2's, forcing the presence of more than three 2's. In the second, both $x$'s must be 1's, and there are at least four 1's. Since no symbol is assumed to appear more than 3 times, these contradictions complete the proof.

**Proof of (b).** We show first that $L(3, 5) = (3 \cdot 5 - 1)/2$, that is, $L(3, 5) = 7$.

Consider a 3 by 5 array without a latin transversal. We will show that some symbol occurs at least eight times.

First of all, if all fifteen cells contain the same symbol, a symbol appears at least eight times. We therefore assume that at least two different symbols occur, and therefore can assume that the following configuration is present:

\[
\begin{array}{ccc}
1 & x & x \\
2 & x & x \\
1 & \end{array}
\]

This breaks into three cases, depending on the top two $x$'s.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
(1) & 2 & 1 & \quad & x & x \\
\end{array}
\quad \text{and} \quad 
\begin{array}{ccc}
1 & 1 & 2 \\
(2) & 2 & x & 1 & x & x \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 2 \\
(3) & x & 2 & x & x & 1 & x & x \\
\end{array}
\]

We will analyze case (2); the other two cases are similar. We may fill in three more
cells with $x$’s, in one case using Lemma 1:

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & \\
2 & 1 & x
\end{array}
\]

This case breaks into two cases, depending on whether the lowest cell in the first column is filled with a 1 or a 2. These two cases, including the implied $x$’s in other cells are shown here:

\[
\begin{array}{ccc}
1 & x & 1 \\
2 & 2 & x \\
1 & 1 & x \\
1 & 1 & 2 \\
2 & 2 & x \\
2 & 1 & x 
\end{array}
\]

In both cases two cells remain empty. If both are filled with 1’s or 2’s, then some symbol occurs more than seven times. On the other hand, if some other symbol occurs there, then again, by Lemma 2, some symbol occurs more than seven times.

Cases (1) and (3) are similar. Thus $L(3, 5) = 7$.

We will now prove by induction that for even $n \geq 6$, $L(3, n) = (3n - 2)/2$ and that for odd $n \geq 5$, $L(3, n) = (3n - 1)/2$.

Assume that the induction holds for a particular even $n$, that is, $L(3, n) = (3n - 2)/2$. We will show that it holds for $n + 1$, which is odd, that is, $L(3, n + 1) = (3n + 2)/2$. Note that in this case we would have $L(3, n + 1) = L(3, n) + 2$.

Consider a 3 by $n + 1$ array in which each symbol occurs at most $(3n + 2)/2$ times. If each symbol occurs at most $(3n - 2)/2$ times, delete one column, obtaining a 3 by $n$ array, which has a latin transversal, by the inductive assumption. Hence the original array has, also.

Now assume that there is at least one symbol occurring at least $3n/2$ times. If there are two such symbols, they occupy at least $3n$ cells. Hence some symbol appears at most three times. By Lemma 2, some symbol occurs at least $3(n + 1)/2$ times, which contradicts the assumption that each symbol occurs at most $(3n + 2)/2$ times.

Hence there is only one symbol that occurs at least $3n/2$ times, that is, $3n/2$ or $(3n + 2)/2$ times. There must be a column in which it appears at least twice. Deleting that column, we obtain a 3 by $n$ array in which each symbol occurs at most $(3n - 2)/2$ times. By the inductive assumption, this array has a latin transversal, hence the original array does.
The argument when \( n \) is odd and \( n + 1 \) is even is similar. (It is a bit shorter, since in this case \( L(3, n + 1) = L(3, n) + 1 \).) This completes the proof of the theorem.

The next theorem gives a non-trivial lower bound on \( L(m, n) \).

**Theorem 4.** \( L(m, n) \geq n - m + 1 \).

The proof is an induction on \( m \).

The theorem is true for \( m = 2 \) or \( m = 3 \) and for \( n = m \). We will consider the case \( n \geq m + 1 \). Assuming the theorem is true for \( m - 1 \), we will prove it for any array \( A \) with \( m \) rows. In order to simplify the diagrams and the exposition, we consider the case \( m = 5 \), which illustrates the argument in the general case.

Assuming that \( L(4, n) \) is at least \( n - 3 \), we will show that \( L(5, n) \) is at least \( n - 4 \).

Consider a 5 by \( n \) array \( A \) in which each symbol appears at most \( n - 4 \) times. By the induction assumption, the 4 by \( n \) array consisting of the first four rows of \( A \) has a latin transversal. Assuming that \( A \) does not have a latin transversal, we may conclude that \( A \) contains an equivalent of the following configuration:

\[
\begin{array}{cccccccc}
1 & y & y & y & y & x & x & x & \ldots \\
2 & & & & & & & & \\
3 & & & & & & & & \\
4 & & & & & & & & \\
y & y & y & y & 1 & x & x & x & \ldots \\
\end{array}
\]

An \( x \) stands for 1, 2, 3 or 4 while a \( y \) stands for any symbol in \( A \). There are cells marked \( x \) since we are assuming that \( A \) has no latin transversal.

At this point \( 2(n - 4) \) cells contain \( x \) or 1. Since 1 occurs at most \( n - 4 \) times in \( A \), there must be a 2, 3, or 4 in some cell marked \( x \). It is no loss of generality to take that symbol to be 2.

No matter which \( x \) is replaced by 2, there is a unique partial latin transversal consisting of that cell and cells marked 1, 3, and 4. This permits us to fill in the second row with four \( y \)'s, one in each column that meets that transversal, and \( n - 5 \) \( x \)'s. The next diagram illustrates one of the two cases.
There are now 3\((n - 4)\) cells containing x, 1, or 2. Among these cells must be a cell containing either 3 or 4. We may assume, after permuting rows and columns and relabeling, that it is 3. There are essentially five different positions in which that symbol may appear, depending on which of the three rows that have x’s it lies in and where, relative to cells containing 1 or 2, it is situated. (In each case a unique partial latin transversal forms with cells labeled 1, 2, 3, or 4.) The following diagram illustrates one case. The y’s in the third row are again in the columns that contain the partial latin transversal that includes the cell with the new 3, which is not in the third row.

\[
\begin{array}{cccccccc}
1 & y & y & y & y & 2 & x & x & x & \ldots \\
2 & y & y & y & y & x & x & x & x & \ldots \\
3 & y & y & 3 & y & x & x & x & x & \ldots \\
4 & y & y & y & y & 1 & 3 & x & x & x & \ldots \\
\end{array}
\]

There are now at least 4\((n - 4) + 1\) cells occupied by 1, 2, 3, or 4. Since each symbol appears at most \(n - 4\) times, this is a contradiction, and the theorem is proved.

In view of our experience with \(m = 2\) and 3, it is tempting to conjecture that the values for \(L(m, n)\) suggested by Theorems 1 and 2 would be correct even for \(m \geq 4\). In other words, one is tempted to conjecture that for \(m \leq n \leq 2m - 2\), we have \(L(m, n) = n - 1\) and that for \(n \geq 2m - 1\), we have \(L(m, n)\) equal to the greatest integer less than or equal to \((mn - 1)/(m - 1)\). Dean Hickerson has shown that \(L(4, 4) = 3\), in agreement with the first part of the conjecture. However, he also has shown that \(L(4, 7)\) is at most 8, hence is a counterexample to the second part.

A result of Hall [4] lends some support for the conjecture that \(L(n - 1, n) = n - 1\). Consider an abelian group of order \(n\), \(A = \{a_1, a_2, \ldots, a_n\}\) and \(b_1, b_2, \ldots, b_{n-1}\), a sequence of \(n - 1\) elements of \(A\), not necessarily distinct. Construct an \(n - 1\) by \(n\) array by placing \(b_i a_j\) in the cell where row \(i\) meets column \(j\). Hall proved that such an array has a latin transversal.

Stein [9] showed that in an \(n\) by \(n\) array where each element appears exactly \(n\) times there is a transversal with at least approximately \((0.63)n\) distinct elements. Erdős and Spencer [3] showed that an \(n\) by \(n\) array in which each symbol appears at most \((n - 1)/16\)
times has a latin transversal. This gives a lower bound for \( L(n, n) \), namely \((n - 1)/16\). The algorithm in which you try to construct a latin transversal by choosing a cell in the top row, then a cell in the second row, and work down row by row yields a different lower bound. If each symbol appears at most \( k \) times in an \( m \) by \( n \) array, the algorithm is certainly successful if \((m - 1)k \leq n - 1\). This implies \( L(m, n) \geq (n - 1)/(m - 1)\).

Snevily [8] offered a conjecture closely related to Hall’s theorem: Any \( k \) by \( k \) submatrix of the group table of an abelian group of odd order has a latin transversal. (Note that in such a matrix each symbol appears at most \( k \) times.)

In the case of latin squares there are several results concerning transversals that have many distinct elements, cited in [2, 9]. Ryser [7] conjectured that every latin square of odd order has a latin transversal, and, more generally, that the number of latin transversals of a latin square has the same parity as the order of the square. However, E. T. Parker [6] pointed out that many latin squares of order 7 have an even number of latin transversals, for instance (6) and many other cases in [5]. Confirming half of Ryser’s conjecture, Balasubramanian [1] proved that a latin square of even order has an even number of latin transversals.

References

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