Hexagonal Structure of Baby Skyrmion Lattices

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We study the zero-temperature crystalline structure of baby Skyrmions by applying a full-field numerical minimization algorithm to baby Skyrmions placed inside different parallelogramic unit-cells and imposing periodic boundary conditions. We find that within this setup, the minimal energy is obtained for the hexagonal lattice, and that in the resulting configuration the Skyrmion splits into quarter-Skyrmions. In particular, we find that the energy in the hexagonal case is lower than the one obtained on the well-studied rectangular lattice, in which splitting into half-Skyrmions is observed.

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I. INTRODUCTION

The Skyrme model [1] is a non-linear theory for pions in (3+1) dimensions with topological soliton solutions called Skyrmions. The model can be used to describe, with due care [2, 3, 4], systems of a few nucleons, and has also been applied to nuclear and quark matter. One of the most complicated aspects of the physics of hadrons is the behavior of the phase diagram of hadronic matter at finite density at low or even zero temperature. Particularly, the properties of zero-temperature Skyrmions on a lattice are of interest, since the behavior of nuclear matter at high densities is now a focus of considerable interest. Within the standard zero-temperature Skyrme model description, a crystal of nucleons turns into a crystal of half nucleons at finite density [5, 6, 7].

To study Skyrmion crystals one imposes periodic boundary conditions on the Skyrme field and works within a unit cell (equivalently, 3-torus) [10], p. 382). The first attempted construction of a crystal was by Klebanov [5], using a simple cubic lattice of Skyrmions whose symmetries maximize the attraction between nearest neighbors. Other symmetries were proposed which lead to slightly lower, but not minimal, energy crystals [6, 7]. It is now understood that it is best to arrange the Skyrmions initially as a face-centered cubic lattice, with their orientations chosen symmetrically to give maximal attraction between all nearest neighbors [8, 9].

The Skyrme model has an analogue in (2+1) dimensions known as the baby Skyrme model, also admitting stable field configurations of a solitonic nature characterized by integral topological charges [11, 12]. Due to its lower dimension, it serves as a simplification of the original model, but nonetheless it has a physical relevance in its own right, as a variant of the model arises in ferromagnetic quantum Hall systems [13, 14, 15, 16, 17]. This effective theory is obtained when the excitations relative to ferromagnetic quantum Hall states are described, in terms of a gradient expansion in the spin density, a field with properties analogous to the pion field in the 3D case [18].

The baby Skyrme model has also been studied in connection with baby Skyrmion lattices under various settings [19, 20, 21, 22, 23] and in fact, it is known that the baby Skyrmions also split into half-Skyrmions when placed inside a rectangular lattice [21]. However, to our knowledge, to date it hasn’t been known if the rectangular periodic boundary conditions yield the true minimal energy configurations over all possible lattice types. Conversely, if there are other non-rectangular baby Skyrmion lattice configurations which have lower energy. Finding the answers to these questions is thus of particular importance both because of their relevance to quantum Hall systems in two-dimensions, and also because they may be used to conjecture the crystalline structure of nucleons in three-dimensions.

In two dimensions there are five lattice types as given by the crystallographic restriction theorem [24], in all of which the fundamental unit cell is a certain type of a parallelogram. To find the crystalline structure of the baby Skyrmions, we place them inside different parallelograms with periodic boundary conditions and find the minimal energy configurations over all parallelograms of fixed area (thus keeping the Skyrmion density fixed). As we show later, there is a certain type of parallelogram, namely the hexagonal, which yields the minimal energy configuration. In particular, its energy is lower than the known ‘square-cell’ configurations in which the Skyrmion splits into half-Skyrmions. As will be pointed out later, the hexagonal structure revealed here is not unique to the present model, but also arises in other solitonic models, such as Ginzburg-Landau vortices [25], quantum Hall systems [14, 15], and even in the context of 3D Skyrmions [26]. The reason for this will be discussed in the concluding section.

In the following section we review the setup of our numerical computations, introducing a systematic way for the detection of the minimal energy crystalline structure of baby Skyrmions. In subsequent sections we outline the

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numerical procedure through which the full field minimal configurations are obtained. In section VI, we present the main results of our study and in section VII a somewhat more analytical analysis of the problem is presented. In the last section we make some remarks with regards to future research.

II. BABY SKYRMIONS INSIDE A PARALLELOGRAM

The target manifold in the baby Skyrme model is described by a three-dimensional vector $\phi$ with the constraint $\phi \cdot \phi = 1$ (i.e., $\phi \in S_2$) and its Lagrangian density is:

$$L = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{\kappa^2}{2} [\partial_\mu \phi \cdot \partial^\mu \phi]^2 - (\partial_\mu \phi \cdot \partial_\nu \phi) \cdot (\partial^\mu \phi \cdot \partial^\nu \phi) - \mu^2 (1 - \phi_3)$$

consisting of a kinetic term, a 2D Skyrme term, and a potential term. The static solutions of the model are those field configurations which minimize the static energy functional:

$$E = \frac{1}{2} \int_\Lambda dx dy \left( (\partial_x \phi)^2 + (\partial_y \phi)^2 + \kappa^2 (\partial_x \phi \times \partial_y \phi)^2 + 2\mu^2 (1 - \phi_3) \right)$$

within each topological sector, where, in our setup, the integration is over parallelograms denoted here by $\Lambda$:

$$\Lambda = \{ \alpha_1 (L, 0) + \alpha_2 (sL \sin \gamma, sL \cos \gamma) : 0 \leq \alpha_1, \alpha_2 < 1 \}.$$ (3)

Here $L$ is the length of one side of the parallelogram, $sL$ with $0 < s \leq 1$ is the length of its other side and $0 \leq \gamma < \pi/2$ is the angle between the ‘$sL$’ side and the vertical to the ‘$L$’ side (as sketched in Fig. 1). Each parallelogram is thus specified by a set $\{L, s, \gamma\}$ and the Skyrmion density inside a parallelogram is $\rho = B / (sL^2 \cos \gamma)$, where $B$ is the topological charge of the Skyrmion. The periodic boundary conditions are taken into account by identifying each of the two opposite sides of a parallelogram as equivalent:

$$\phi(x) = \phi(x + n_1 (L, 0) + n_2 (sL \sin \gamma, sL \cos \gamma)),$$ (4)

with $(n_1, n_2 \in \mathbb{Z})$. We are interested in static finite-energy solutions, which in the language of differential geometry are $\mathbb{T}_2 \mapsto S_2$ maps. These are partitioned into homotopy sectors parameterized by an invariant integral topological charge $B$, the degree of the map, given by:

$$B = \frac{1}{4\pi} \int_\Lambda \text{d}x \text{d}y (\phi \cdot (\partial_x \phi \times \partial_y \phi)).$$ (5)

The static energy $E$ can be shown to satisfy

$$E \geq 4\pi B$$

with equality possible only in the ‘pure’ $O(3)$ case (i.e., when both the Skyrme and the potential terms are absent) [21]. We note that while in the baby Skyrme model on $\mathbb{R}^2$ with fixed boundary conditions the potential term is necessary to prevent the solitons from expanding indefinitely, in our setup it is not required, due to the periodic boundary conditions [21]. In this work we study the model both with and without the potential term.

The problem in question can be simplified by a linear mapping of the parallelograms $\Lambda$ into the unit-area two-torus $\mathbb{T}_2$ (see Fig. 1). In the new coordinates, the energy functional becomes

$$E = \frac{1}{2s \cos \gamma} \int_{\mathbb{T}_2} \text{d}x \text{d}y \left( (\partial_x \phi)^2 - 2s \sin \gamma \partial_x \phi \partial_y \phi + (\partial_y \phi)^2 \right) + \frac{\kappa^2 \rho}{2B} \int_{\mathbb{T}_2} \text{d}x \text{d}y (\partial_x \phi \times \partial_y \phi)^2 + \frac{\mu^2 B}{\rho} \int_{\mathbb{T}_2} \text{d}x \text{d}y (1 - \phi_3).$$

III. THE NUMERICAL MINIMIZATION PROCEDURE

The Euler-Lagrange equations derived from the energy functional (7) are non-linear PDE’s, so in general one must resort to numerical techniques. The minimal energy configuration of a Skyrmion with charge $B$ and a given parallelogram (defined by a set $\{s, \gamma\}$) with a specified Skyrmion density $\rho$ is found by a full-field relaxation method on a $100 \times 100$ rectangular grid on the two-torus $\mathbb{T}_2$, where at each point a field triplet $\phi(x_m, y_n)$ is defined. For the calculation of the energy and charge densities, we use the procedure devised by [27], in which the
FIG. 1: The parameterization of the fundamental unit-cell parallelogram $\Lambda$ (in black) and the two-torus $\mathbb{T}_2$ into which it is mapped (in gray).

sure that the final configurations are independent of the discretization and of the cooling scheme. As a further check, some of the minimizations were repeated with numerical derivatives of a higher precision, using eight field points for their evaluation. No apparent changes in the results were detected.

IV. RESULTS

Using the minimization procedure presented in the previous section, we have found the minimal energy static Skyrmion configurations over all parallelograms, for various settings: The ‘pure’ $O(3)$ case, in which both $\kappa$, the Skyrme parameter, and $\mu$, the potential coupling, are set to zero, the Skyrmion case for which only $\mu = 0$, and the general case for which neither the Skyrme term nor the potential term vanish.

In each of these settings, the parameter space of parallelograms was scanned, while the Skyrmion density $\rho$ is held fixed, yielding for each set of $(s, \gamma)$ a minimal energy configuration. The choice as to how many Skyrmions are to be placed inside the unit cells was made after some preliminary testing in which Skyrmions of other charges (up to $B = 8$) were also examined. The odd-charge minimal-energy configurations turned out to have substantially higher energies than even-charge ones, where among the latter, the charge-two Skyrmion was found to be the most fundamental, as it was observed that it serves as a ‘building-block’ for the higher-charge ones. In what follows, a summary of the results is presented.

A. The pure $O(3)$ case ($\kappa = \mu = 0$)

The pure $O(3)$ case corresponds to setting both $\kappa$ and $\mu$ to zero. In this case, analytic solutions in terms of Weierstrass elliptic functions may be found [21, 22, 23] and the minimal energy configurations, in all parallelogram settings, saturate the energy bound in (6) giving $E = 4\pi B$. Thus, comparison of our numerical results with the analytic solutions serves as a useful check on the precision of our numerical procedure. The agreement is to 6 significant digits. Contour plots of the charge densities for different parallelogram settings for the charge-two Skyrmions are shown in Fig. 2, all of them of equal energy $E/8\pi = 1$.

B. The Skyrme case ($\kappa \neq 0, \mu = 0$)

As pointed out earlier, in setting $\mu = 0$, the dependence of the energy functional on the Skyrme parameter $\kappa$ is only through $\kappa^2 \rho$, so without loss of generality we vary $\rho$ and fix $\kappa^2 = 0.03$ throughout (this particular choice for $\kappa$ was made for numerical convenience). Minimization of the energy functional (7) over all parallelograms yielded the following. For any fixed density $\rho$, the.
minimal energy was obtained for $s = 1$ and $\gamma = \pi/6$. This set of values corresponds to the ‘hexagonal’ or ‘equilateral triangular’ lattice. In this configuration, any three adjacent zero-energy loci (or ‘holes’) are the vertices of equilateral triangles, and eight distinct high-density peaks are observed (as shown in Fig. 3b). This configuration can thus be interpreted as the splitting of the two-Skyrmion into eight quarter-Skyrmions. This result turned out to be independent of the Skyrmion density $\rho$. In particular, the well-studied square-cell minimal energy configuration (Fig. 3a), in which the two-Skyrmion splits into four half-Skyrmions turned out to have higher energy than the hexagonal case. Figure 3 shows the total energies (divided by $8\pi$) and corresponding contour plots of charge densities of the hexagonal, square and other configurations (for comparison), all of them with $\rho = 2$.

The total energy of the Skyrmions in the hexagonal setting turned out to be linearly proportional to the density of the Skyrmions, reflecting the scale invariance of the model (Fig. 4). In particular, the global minimum of $E = 4\pi B = 8\pi$ is reached when $\rho \to 0$. This is expected since setting $\rho = 0$ is equivalent to setting the Skyrme parameter $\kappa$ to zero, in which case the model is effectively pure $O(3)$ and inequality (6) is saturated.

C. The general case ($\kappa \neq 0$, $\mu \neq 0$)

The hexagonal setting turned out to be the energetically favorable in the general case as well. Since, however, in this case, the Skyrmion possesses a definite size (as can be verified by looking at the $\rho$ dependence in the energy functional), the Skyrmion structure is different at low and at high densities and a phase transition is observed. While at low densities the Skyrmions stand isolated from one another, at high densities they fuse together forming the quarter-Skyrmion crystal as in the Skyrme case reported above. As the density $\rho$ decreases or equivalently the value of $\mu$ increases, the size of the Skyrmions becomes small compared to the cell size. The exact shape of the lattice loses its effects and the differences in energy among the various lattice types become very small. This is illustrated in figures 4 and 5.

Furthermore, due to the finite size of the Skyrmion, there is an optimal density for which the energy is minimal among all densities. Figure 6 shows the contour plots of the charge density of the charge-two Skyrmion for several densities with $\kappa^2 = 0.03$ and $\mu^2 = 0.1$. The energy of the Skyrmion is minimal for $\rho \approx 0.14$ (Fig. 4).

V. SEMI-ANALYTICAL APPROACH

The energy functional (7) depends both on the Skyrme field $\phi$ and on the parallelogram parameters $\gamma$ and $s$. Formally, the minimal energy configuration over all parallelograms may be obtained by functional differentiation with respect to $\phi$ and regular differentiation with respect to $\gamma$ and $s$. However, since the resulting equations are very complicated, a direct numerical solution is quite hard. Nonetheless, some analytical progress may be achieved in the following way. As a first step, we differentiate the energy functional (7) only with respect to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Charge-two Skyrmions in the pure $O(3)$ case: Contour plots of the charge densities ranging from violet (low density) to red (high density) for various parallelogram settings, all of which saturate the energy bound $E = 4\pi B = 8\pi$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Charge-two Skyrmions in the Skyrme case with $\kappa^2 = 0.03$ and $\rho = 2$: Contour plots of the charge densities for the hexagonal, square and other fundamental cells ranging from violet (low density) to red (high density). As the captions of the individual subfigures indicate, the hexagonal configuration has the lowest energy.}
\end{figure}
FIG. 4: Total energy $E$ (divided by $8\pi$) of the charge-two Skyrmion in the hexagonal lattice (□ — Skyrme case and ◊ — general case) and in the square lattice ( ■ — Skyrme case and ♦ — general case) as function of the Skyrmion density (in the Skyrme case, $\kappa^2 = 0.03$ and in the general case $\kappa^2 = 0.03$ and $\mu^2 = 0.1$). Note the existence of an optimal density (at $\rho \approx 0.14$) in the general case, for which the energy attains a global minimum. Figure (b) is an enlargement of the lower left corner of figure (a).

$\gamma$ and $s$:

$$\frac{\partial E}{\partial \gamma} = -\frac{1}{2s\cos^2\gamma} (\sin\gamma(E_{yy} + s^2E_{xx}) - 2sE_{xy}) = 0, \quad \frac{\partial E}{\partial s} = -\frac{1}{2s^2\cos\gamma} (E_{yy} - s^2E_{xx}) = 0,$$

where $E_{ij} = \int_{T^2} dx dy (\partial_i\phi \cdot \partial_j\phi)$ and $i,j \in \{x,y\}$. Solving these equations for $\gamma$ and $s$ yields

$$s = \sqrt{\frac{E_{yy}}{E_{xx}}},$$

$$\sin\gamma = \frac{E_{xy}}{\sqrt{E_{xx}E_{yy}}},$$

Substituting these expressions into the energy functional (7), we arrive at a ‘reduced’ functional

$$E = \sqrt{E_{xx}E_{yy} - (E_{xy})^2} + \frac{\kappa^2 \rho^2 E_{sk}}{2B} + \frac{\mu^2 B E_{pot}}{\rho},$$

where $E_{sk} = \int_{T^2} dx dy (\partial_x\phi \times \partial_y\phi)^2$ is the Skyrme energy and $E_{pot} = \int_{T^2} dx dy (1 - \phi^3)$ is the potential energy. Now that both $\gamma$ and $s$ are eliminated from the resultant expression, and the conditions for their optimization are built into the functional, the numerical minimization is carried out. We note here, however, that the procedure presented above should be treated with caution. This is since the ‘minimization’ conditions for $\gamma$ and $s$ in (11) are in fact only extremum conditions, and may turn out to be maximum or saddle-point conditions. Hence, it is important to confirm any results obtained using this method by comparing them with corresponding results obtained from the method described in the previous section.

It is therefore reassuring that numerical minimization of the reduced functional (12) gives $\sin\gamma = 0.498$ ($\gamma \approx \pi/6$) and $s = 1$ (both for the Skyrme case and the general

FIG. 5: Energy difference (in %) between the hexagonal setting and the square setting, in the Skyrme case (□) and in the general case (♦), as function of the Skyrmion density. As the density decreases, the energy difference between the two lattice types is reduced.
In the general ($\mu \neq 0$) case, the energy functional may be further differentiated with respect to the Skyrme density $\rho$ to obtain the optimal density for which the Skyrmion energy is minimal. Differentiating with respect to $\rho$, and substituting the obtained expression into the energy functional, results in the functional

$$E = \frac{1}{2} \left( \mathcal{E}_{xx} \mathcal{E}_{yy} - (\mathcal{E}_{xy})^2 \right) + \kappa \mu \sqrt{2 \mathcal{E}_{sk} \mathcal{E}_{pot}}.$$  

(13)

Numerical minimization of the above expression for $\kappa^2 = 0.03$ and various $\mu$ values ($0.1 \leq \mu^2 \leq 10$) yielded the hexagonal setting as in the Skyrme case. In particular, for $\mu^2 = 0.1$ the optimal density turned out to be $\rho \approx 0.14$, in accord with results presented in sub-section IV.C.

VI. SUMMARY AND FURTHER REMARKS

In this paper, we have studied the crystalline structure of baby Skyrmions in two dimensions by finding the minimal energy configurations of baby Skyrmions placed inside parallelogramic fundamental unit cells and imposing periodic boundary conditions. In the pure $O(3)$ case (where both the Skyrme and potential terms are absent), we verified that there are no favorable lattices in which the Skyrmions order themselves, as all parallelogram settings yielded the same minimal energy, saturating the minimal energy bound given by inequality (6).

In the Skyrme case, without the potential term, the results are different. For any fixed Skyrmion density, the parallelogram for which the Skyrmion energy is minimal turns out to be the hexagonal lattice (for which $s = 1$ and $\gamma = \pi/6$). In particular, the hexagonal lattice has lower energy than the four half-Skyrmions configuration on the rectangular lattice. For example, at $\rho = 2$, $E = 1.433$ for the hexagonal lattice vs. $E = 1.446$ for the rectangular lattice.

The hexagonal setting turns out to be the energetically most favorable also in the general case with both the Skyrme term and the potential term. In this case, however, the model is not scale invariant and the Skyrmion has a definite size. This results in the existence of a phase transition as a function of density and the appearance of an optimal density, for which the total energy of the Skyrmion is minimal. This is analogous to the Skyrmion behavior in the 3D Skyrme model, in which the Skyrmions also possess a definite size.

As pointed out in the Introduction, the special role of the hexagonal lattice revealed here is not unique to the baby Skyrme model, but in fact arises in other solitonic models. In the context of Skyrme models, the existence of a hexagonal two-dimensional structure of 3D Skyrmions has been found by [29], where it has already been noted that energetically, the optimum infinite planar structure of 3D Skyrmions is the hexagonal lattice, which resembles the structure of a graphite sheet, the most stable form of carbon thermodynamically ([10], p. 384). Other examples in which the hexagonal structure is revealed are Ginzburg-Landau vortices which are known to have lower energy in a hexagonal configuration than in a square lattice configuration [27]. Thus, it should not come as a surprise that the hexagonal structure is found to be the most favorable in the baby Skyrme model.

As briefly noted in the Introduction, a certain type of baby Skyrmions also arise in quantum Hall systems as low-energy excitations of the ground state near ferromagnetic filling factors (notably 1 and 1/3) [14]. It has been pointed out that this state contains a finite density of Skyrmions [28], and in fact the hexagonal configuration has been suggested as a candidate for their lattice structure [15]. Our results may therefore serve as a supporting evidence in that direction, although a more detailed analysis is in order.

Our results also raise some interesting questions. The first is how the dynamical properties of baby Skyrmions on the hexagonal lattice differ from their behavior in the usual rectangular lattice. Another question has to do with their behavior in non-zero-temperature settings.

One may also wonder whether and how these results can be generalized to the 3D case, once a systematic study like the one reported here is conducted. Is the face-centered cubic lattice indeed the minimal energy
crystalline structure of 3D Skyrmions among all parallelepipeds? If not, what would the minimal energy structure be? and how would these results change when a mass term is present? We hope to answer these questions in future research.

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