Chiral Anomalies in the Spectral Action

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Abstract

The definition of the spectral action involves the trace operator over states in the physical Hilbert space. We show that in the presence of chiral fermions there are consistency conditions on the fermionic representations. These conditions are identical to the conditions for absence of gauge and gravitational anomalies obtained in the path integral formalism.

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At present we have a great deal of information about the particle physics spectrum at low-energies. Using the tools of noncommutative geometry it is now possible to use the particle physics spectrum to study the geometry of space-time [1, 2, 3, 4, 5]. This program has been advocated in [6] where it is shown that a real spectral triple corresponding to a product of a continuous space by a discrete space serves as a good starting point. In [6] a new principle is proposed, the spectral action principle. This states that the action for the dynamical degrees of freedom contained in the metric is given by some function of the perturbed Dirac operator. The results do not depend crucially on the form of the function used in the spectral action, but mainly on the spectrum of the Dirac operator. In other words, the spectrum contains all the information about geometric invariants and should help in indicating the relevant geometry of space-time.

It is well known that fermions in the standard model are chiral, and that masses are generated when the Higgs field acquires a vev so that the Higgs-fermi-fermi interaction produces a fermionic mass term. The trace operator is defined by summing over eigenstates of the Dirac operator. In the case of chiral fermions, the eigenvalue problem for the Dirac operator cannot be defined without adding fermions with opposite chirality. This is the familiar problem encountered in the path integral formalism, and there one assumes the spinors and their conjugates to be independent. The trace operator, unlike the fermionic kinetic energy, is not invariant under chiral rotations. The invariance is spoiled by the appearance of a phase factor which must be made to vanish for the action to be consistent.

First, I shall briefly review the ingredients that enter the derivation of the spectral action for the standard model, and then show how this action transforms under chiral rotations. The subtleties involved in working with Euclidean signature, as well as with Minkowski signature, are also discussed [7, 8].

A noncommutative space is defined by a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) where \(\mathcal{A}\) is an algebra of operators, \(\mathcal{H}\) a Hilbert space of states, \(D\) an unbounded operator in \(\mathcal{H}\). The space is supplied with a real structure \(J\) satisfying

\[
J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J;
\]

where the value of \(\epsilon, \epsilon', \epsilon''\) is determined by \(n\) modulo 8. The algebra \(\mathcal{A}\) is taken to be

\[
\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F,
\]

where the algebra \(\mathcal{A}_F\) is finite dimensional

\[
\mathcal{A}_F = C \oplus H \oplus M_3(C),
\]

and \(H\) is the algebra of quaternions

\[
H = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) ; \alpha, \beta \in C \right\}.
\]
Similarly,

\[ H = L^2(M, S) \otimes H_F, \quad \text{(5)} \]
\[ D = D_M \otimes 1 + \gamma_5 \otimes D_F, \quad \text{(6)} \]

where \((A_F, D_F)\) is a spectral geometry on \(A_F\), while \(L^2(M, S)\) is the Hilbert space of \(L^2\) spinors and \(D_M\) is the Dirac operator of the Levi-Civita connection. The list of elementary fermions provide a natural candidate for \(H_F\). One lets \(H_F\) be the Hilbert space with basis labeled by elementary leptons and quarks. The \(\mathbb{Z}_2\) grading \(\gamma_F\) is given by +1 for left handed particles and -1 for right handed ones. The involution \(J\) is such that \(Jf = \bar{f}^*\) for any \(f\) in the basis. One has \(J^2 = 1\) and \(J\gamma = \gamma J\). The fermions are represented by

\[ \Psi = \left( \begin{array}{c} f \\ \bar{f}^* \end{array} \right), \quad \text{(7)} \]

where \(\bar{f} \in \overline{H}, \overline{H}\) being the conjugate Hilbert space. In this representation the action of \(J\) is given by

\[ J\Psi = \left( \begin{array}{c} f^* \\ \bar{f} \end{array} \right), \quad \text{(8)} \]

while \(D_F\) is given by

\[ D_F = \left( \begin{array}{cc} Y & 0 \\ 0 & \overline{Y} \end{array} \right), \quad \text{(9)} \]

where \(Y\) is the Yukawa coupling matrix. The group \(\text{Aut}(A)\) of automorphisms of the involutive algebra \(A\) plays the role of diffeomorphisms of the noncommutative geometry. It has a normal subgroup \(\text{Int}(A)\) where an automorphism \(\alpha\) is inner iff there exists a unitary operator \(u \in A, (uu^* = u^*u = 1)\) such that

\[ \alpha(a) = uau^*, \quad \forall a \in A. \quad \text{(10)} \]

The group \(\text{Aut}(A)\) of diffeomorphisms falls in equivalence classes under the normal subgroup of inner automorphisms. In the same way the space of metrics has a natural foliation into equivalence classes. The internal fluctuations of a given metric are given by the formula

\[ D = D_0 + A + JAJ^{-1}, \quad \text{(11)} \]
\[ A = \sum_i a_i [D_0, b_i], \quad \text{(12)} \]

where \(a_i, b_i \in A\) and \(A = A^*\). Starting with \((A, H, D_0)\) one leaves the representation of \(A\) in \(H\) and perturbs the operator \(D_0\).

The spectral action that reproduces the standard model Lagrangian takes the very simple form

\[ (\Psi |D| \Psi) + Tr F \left( \frac{D}{m} \right). \quad \text{(13)} \]
where $m$ is a cut-off scale, needed to make the argument of the function $F$ dimensionless.

It is shown in [6] that the fermionic action reproduces the fermionic kinetic energies as well as their gauge and Higgs interactions. The bosonic action contains, to lowest orders, a cosmological constant, the curvature scalar, Higgs mass term as well as Weyl gravity coupled conformally to the Higgs and gauge fields. The dependence on the function $F$, to the lowest orders indicated, enters only through three parameters, $f_0$, $f_2$ and $f_4$ which multiply the cosmological constant, the curvature and Higgs terms and the conformal terms respectively, where

\begin{align}
    f_0 &= \int_0^\infty F(u)du, \\
    f_2 &= \int_0^\infty F(u)du, \\
    f_{2(n+2)} &= (-1)^n F^{(n)}(0).
\end{align}

Higher order terms are suppressed by $\frac{1}{m}$ factors.

Fermions that appear in the physical Hilbert space in any realistic model must be chiral. The grading operator $\gamma$ is such that

\begin{align}
    \gamma \psi_\pm &= \pm \psi_\pm, \\
    \gamma D \psi_\pm &= \mp D \gamma \psi_\pm,
\end{align}

which implies that $D \psi_\pm$ have opposite chirality to $\psi_\pm$. Therefore, for chiral fermions with non-zero eigenvalues, one cannot set the eigenvalue problem for $D$ but only for $D^2$. We can write

\begin{equation}
    D^2 \psi_{n\pm} = \lambda_n \psi_{n\pm}, \quad \lambda_n = \lambda_n^*.
\end{equation}

At this point, it is important to mention that the eigenvalue problem is solved for spaces with Euclidean signature. In this case it is also necessary to define the conjugate fermions $\overline{\psi}_{n\pm}$ which are taken to be independent of $\psi_{n\pm}$. Indeed, for Euclidean signature, $\psi_{n\pm}$ and their complex conjugates have the same chirality. This makes an expression like $(\psi_{n\pm} | D | \psi_{n\pm})$, which is relevant for the kinetic energy, vanishes identically. The way out is to define the inner product to contain $\overline{\psi}_{n\pm} D \psi_{n\pm}$ and where one assumes that $\psi_{n\pm}$ is independent of $\psi_{n\pm}$ and is of opposite chirality [4, 8]. This doubling of fermions is the same encountered in the Fujikawa path integral treatment of chiral fermions [10]. It is also clear that this doubling is not necessary in a space with Minkowski signature as the conjugate fermions have opposite chirality. By normalising $(\psi_{n\mp} | \psi_{n\pm})$ to one, we can write $\lambda_n = (\overline{\psi}_{n\mp} | D^2 | \psi_{n\pm})$. Therefore, both in the fermionic kinetic energy term as well as in the bosonic trace one encounters the problem of doubling of fermions. This is again completely parallel to the path integral formalism where doubling is needed to define both the fermionic kinetic energy and the fermionic measure. One cannot resist mentioning that there is one particular case where a Majorana type term involving the

\begin{align*}
    f_0 &= \int_0^\infty F(u)du, \\
    f_2 &= \int_0^\infty F(u)du, \\
    f_{2(n+2)} &= (-1)^n F^{(n)}(0).
\end{align*}
fermions without their complex conjugates could be written. This happens if one chooses the real structure $J$ to have dimension 2 (mod 8) as one can then impose the condition $J\Psi = \Psi$. A realisation of this is provided by the 16 spinor representation of $SO(10)$ where one can fit all quarks and leptons, and in addition a right-handed neutrino. Unfortunately, when this representation is used for the standard model, the only Majorana interaction generated without violating the standard model quantum numbers is for the right-handed neutrino. Therefore in the case of the standard model this property of $J$ does not solve the fermions doubling problem.

Assuming that all fermions in the spectrum have positive chirality (negative chirality fermions could always be written as the charge conjugates of positive chirality fermions). The bosonic spectral action in this case could be written as

$$I_b = Tr \left( F(D^2) \right) = \sum_n \left( \psi_{n-}^* \left| F(D^2 ) \right| \psi_{n+} \right), \quad (19)$$

where $D^2$ could be replaced by $D_- D_+$ when acting on $\psi_{n+}$.

Let us now consider the behavior of this action under chiral transformations. The fermionic action is invariant under chiral rotations (suppressing the subscript $n$),

$$|\psi_+ \rangle \rightarrow e^{i\theta \gamma_5} |\psi_+ \rangle ,$$

$$(\psi_+) \rightarrow (\psi_+) e^{i\theta \gamma_5}, \quad (20)$$

which implies that

$$(\psi_+ \left| D \right| \psi_+) \rightarrow (\psi_+ \left| e^{i\theta \gamma_5} D e^{i\theta \gamma_5} \right| \psi_+) . \quad (21)$$

It is a simple matter to see that

$$e^{i\theta \gamma_5} D e^{i\theta \gamma_5} = D,$$

implying the invariance of the fermionic action.

Under chiral rotations the bosonic action $(19)$ transforms as

$$I_b \rightarrow Tr \left\{ \sum_n \left( e^{-i\theta T^a \gamma_5} \psi_{n-} \left| F(D^2 ) e^{i\theta T^a \gamma_5} \right| \psi_{n+} \right) \right\}$$

$$= Tr \left\{ e^{2i\theta T^a \gamma_5} \left( \psi_{n-} \left| F(D^2 ) \right| \psi_{n+} \right) \right\} , \quad (22)$$

where $T^a$ are matrix representations for the fermions. We now use the heat-kernel expansion for the function $F$ \[13\]

$$F(P) = \sum_{n \geq 0} f_n e_n(P) , \quad (23)$$

where $e_n(P)$ are geometric invariants whose trace gives the Seeley-de Witt coefficients for the operator $P = D^2$. Under an infinitesimal transformation equation $(22)$ simplifies to

$$I_b \rightarrow I_b + \sum_{n \geq 0} f_n Tr \left( 2i\theta T^a \gamma_5 e_n(P) \right) . \quad (24)$$
Because of the presence of $\gamma_5$ in the trace, the first non-vanishing gauge term comes from $e_4 (P)$ and is of the form

$$2i\theta^a Tr \left( \gamma_5 \gamma^\mu \gamma^\rho \gamma^\sigma T^a T^b T^c \right) G^b_{\mu \nu} G^c_{\rho \sigma} = i \epsilon^{\mu \nu \rho \sigma} \theta^a G^b_{\mu \nu} G^c_{\rho \sigma} Tr \left( T^a \left\{ T^b, T^c \right\} \right).$$

(25)

Therefore the bosonic action is non-invariant, except when the condition on the fermionic group representations

$$Tr \left( T^a \left\{ T^b, T^c \right\} \right) = 0,$$

(26)

is satisfied. We can relate the anomaly generating term to non-conservation of currents. Let $\theta = \theta^a (x) T^a$, then

$$(\psi_+ | D | \psi_+) \rightarrow (\psi_+ | D | \psi_+) - i \theta^a D_\mu (\psi_+ | \gamma^\mu T^a | \psi_+),$$

(27)

where we have used the identity

$$e^{i\theta \gamma} D e^{i\theta \gamma} = D + i\gamma^\mu \gamma_5 D_\mu \theta^a T^a,$$

and integrated by parts. Defining the fermionic current

$$j_\mu^a = (\psi_+ | \gamma^\mu T^a | \psi_+),$$

(28)

we then have

$$D_\mu j_\mu^a = -\epsilon^{\mu \nu \rho \sigma} G^b_{\mu \nu} G^c_{\rho \sigma} Tr \left( T^a \left\{ T^b T^c \right\} \right).$$

(29)

There is also one gravitational term that comes from $e_4 (P)$

$$\epsilon^{\mu \nu \rho \sigma} \theta^a Tr \left( T^a \right) F^c_{\mu \nu} F^c_{\rho \sigma},$$

(30)

Therefore gravitational anomalies are absent if one further imposes the condition

$$Tr \left( T^a \right) = 0.$$ 

(31)

One can make similar analysis for anomalies on higher dimensional manifolds. It can be immediately seen that in ten dimensions the first non-vanishing trace in the heat-kernel expansion would result from a term of the form

$$Tr \left( \gamma_1 \gamma_9 \gamma^\mu_0 T^a_1 \cdots T_{a_9} \right) F^a_1 \cdots F^a_9,$$

(32)

which is obviously related to the well known hexagon diagram.

We therefore reach the conclusion that chiral gauge anomalies arise in the path integral formulation because the fermionic measure is not invariant under chiral rotations, while in the spectral action they arise because of the non-invariance of the trace operator. The anomaly cancellation conditions are, however, identical in both cases.
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