Quantum superposition of localized and delocalized phases of photons

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Based on a variant of 2-site Jaynes-Cummings-Hubbard model, which is constructed using superconducting circuits, we propose a method to coherently superpose the localized and delocalized phases of photons. In our model, two nonlinear superconducting stripline resonators are coupled to an interfacial circuit composed of parallel combination of a superconducting qubit and a capacitor, which plays the role of a quantum knob for the photon hopping rate: with the knob qubit in its ground/excited state, the injected photons tend to be localized/delocalized in the resonators. We show that, by applying a microwave field with appropriate frequency on the knob qubit, we could demonstrate Rabi oscillation between photonic localized phase and delocalized phase. Furthermore, this set-up offers advantages (e.g. infinite on/off ratio) over other proposals for the realization of scalable quantum computation with superconducting qubits.

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Introduction.—Coupled arrays of nonlinear resonators have recently been shown to be suitable candidates for exploring quantum many-body phenomena of light [1]. So far, various strongly correlated effects and exotic phases have been studied using these effective structures. Examples include effective photon-photon repulsion [2], the Mott insulator-superfluid quantum phase transition of light [3], photonic Josephson effect [4], and time-reversal-symmetry breaking [5]. Compared to other strongly interacting many-particle systems, like Josephson junction arrays [6] and optical lattices [7], coupled resonator arrays have the advantage of accessing individual sites experimentally.

Much recent work has focused on using polaritons in an array of coupled nonlinear resonators, described by the Jaynes-Cummings-Hubbard model (JCHM), to simulate the famous Bose-Hubbard model [8, 9]. In a coupled resonator array, the strong atom-photon coupling inside the resonator leads to an effective polariton repulsion, and the photon hopping between neighboring resonators favors delocalization of the polaritons. As the ratio of the hopping term relative to the on-site repulsion is varied through the quantum critical point, the ground state of the system undergoes a transition from a product of localized states of definite polariton number to a delocalized state with large fluctuations in the polariton number per site.

According to the linear superposition principle of quantum mechanics, any linear combination of two allowed states of a system is also an allowed state. The quantum phases are some special states of many-particle systems. It is thus natural to ask, can we produce the superposition involving distinct quantum phases at will? The purpose of this letter is to explore the possibility of superposing the localized and delocalized phases of polaritons in a coupled resonator array. In order to keep the complexity of the system to a manageable level, we mainly consider the simplest possible case, i.e. two coupled resonators containing a total of two polaritons. In this case, it should be understood that in our usage the term “phase” refers to a certain state of a small finite system, not true phase in the thermodynamic sense.

In our proposed model, two nonlinear stripline resonators are coupled by an interfacial circuit playing the role of a quantum knob, which is composed of parallel combination of a superconducting qubit and a capacitor. Corresponding to the two basis states of the knob qubit, polaritons in the resonators are governed by different effective Hamiltonians, which favor localization and delocalization of the polaritons, respectively. We show that, by applying spectroscopic techniques to the knob qubit, we can demonstrate Rabi oscillation between photonic localized phase and delocalized phase. This opens up a way to make photons or polaritons enter novel quantum states and enables new investigations of many-body physics in coupled resonator arrays. In addition, this architecture can be used to solve the annoying on/off ratio problem of conventional proposals for scalable quantum computation with superconducting qubits.

Model and Hamiltonian.—A sketch of our model is shown in Fig. 1. Two microwave stripline resonators,

![Diagram of two microwave stripline resonators](image)

FIG. 1. (Color online) Two microwave stripline resonators, each containing a superconducting qubit, are coupled by a quantum knob composed of parallel combination of a qubit and a capacitor. The knob qubit is capacitively coupled to the resonators and can be driven by applying a microwave field on the control line.

each containing a superconducting qubit (e.g. phase 9
or charge 10 qubit), are coupled by an interfacial circuit
composed of parallel combination of a qubit $q_n$ and a ca-
pacitor $C_n$, which plays the role of a quantum knob. The
knob qubit $q_n$ is capacitively coupled to the resulators
and can be driven by applying a microwave field on the
control line. The capacitor $C_n$ leads to direct photon
hopping between the resulators 11. Not considering the
driving field now, the total system can be described
by the Hamiltonian (assuming $h = 1$
\begin{equation}
\mathcal{H} = \sum_{i=1,2} \mathcal{H}_{i}^{JC} + \mathcal{H}^c, \tag{1}
\end{equation}
which includes the Jaynes-Cummings (JC) interaction of
the local resonator-qubit system
\begin{equation}
\mathcal{H}_i^{JC} = \epsilon |\epsilon\rangle_i \langle |\epsilon| + \omega a_i^\dagger a_i + g(\sigma_i^+ a_i + \sigma_i^- a_i^\dagger) \tag{2}
\end{equation}
and the knob-mediated interaction between the resonators
\begin{equation}
\mathcal{H}^c = \epsilon_c |\epsilon^c\rangle \langle |\epsilon^c| + g_c (\sigma_c^+ a_1 + \sigma_c^- a_2 + \mathcal{H}^{c},) + \kappa_0 (a_1^\dagger a_2 + a_1 a_2^\dagger). \tag{3}
\end{equation}
Here, $\epsilon$, $\epsilon_c$ and $w$ are the resonance frequencies for $q_i$, $q_c$ and resonator $i$, respectively, $g (g_c)$ is the coupling
strength between $q_i (q_c)$ and resonator $i$, and $\kappa_0$ is the
fixed photon hopping rate between the resonators induced
by the capacitor $C_c$. The states $|\epsilon\rangle_i (|\epsilon^c\rangle)$ and
$|\epsilon\rangle_i (|\epsilon^c\rangle)$ are ground and excited states for $q_i (q_c)$, the
operators $\sigma_i^+ (\sigma_c^+)$ and $\sigma_i^- (\sigma_c^-)$ are qubit raising
and lowering operators for $q_i (q_c)$, and $a_i^\dagger (a_i)$ is the photon
creation (annihilation) operator for resonator $i$.

With $q_c$ working in the dispersive regime, i.e. $\Delta_c = \epsilon_c - w \gg g_c$, the real energy exchanges between $q_c$ and
the resonators are largely suppressed. In this case, we can
perform the unitary transformation $U = \exp[-\frac{i\kappa}{\Delta_c}(a_1^\dagger \sigma_c^+ - a_1^\dagger \sigma_c^- + a_2^\dagger \sigma_c^+ - a_2^\dagger \sigma_c^-)]$ and expand $UHU^\dagger$ to second order in $\frac{\kappa}{\Delta_c}$ to obtain the effective system Hamiltonian 12
\begin{equation}
\mathcal{H}^{eff} = \sum_{i=1,2} \mathcal{H}_i^{JC} - \frac{g_c^2}{\Delta_c} (a_1^\dagger a_2 + a_2^\dagger a_1) |\epsilon^c\rangle \langle |\epsilon^c| + [\epsilon_c + \frac{2g_c^2}{\Delta_c} \frac{g^2}{\Delta_c} (a_1^\dagger a_2 + a_2^\dagger a_1) |\epsilon_c\rangle \langle |\epsilon_c| + (\kappa_0 + \frac{g_c^2}{\Delta_c}) \sigma_c^z (a_1^\dagger a_2 + a_1 a_2^\dagger), \tag{4}
\end{equation}
where $\sigma_c^z = |\epsilon^c\rangle \langle |\epsilon^c| - |\epsilon^c\rangle \langle |\epsilon^c|$. The second and third
terms of $\mathcal{H}^{eff}$ represent the ac Stark shifted frequency
of $q_c$, and the fourth term is the sum of the direct photon
hopping induced by $C_c$ and the qubit-state-dependent
photon hopping mediated by $q_c$.

Choosing $\frac{\kappa_0}{\Delta_c} = \kappa_0$, the parallel combination of $q_c$ and $C_c$ can function as a quantum
knob 13; with $q_c$ in its ground/excited state, the
photon hopping between the resonators will be switched
off/on.

Now, we verify the above results by means of numerical
simulations. Initially, one photon is injected into
resonator 1, with resonator 2 being empty. The parameters
we choose are $g_c = g$, $\kappa_0 = 0.1g$, $\epsilon = 45g$, $\epsilon_c = 50g$, and $w = 40g$, which yield $\frac{g_c^2}{\Delta_c} = \kappa_0$. In Fig. 2, we plot the
polariton numbers of resonator 1 (thick lines) and resonator
2 (thin lines) as functions of time, with the knob qubit in (a) $|\epsilon^c\rangle$, and (b) $|\epsilon^c\rangle$. The solid and dashed lines
represent the dynamics governed by $\mathcal{H}$ and $\mathcal{H}^{eff}$, respectively.

**Fig. 2.** (Color online) The polariton numbers of resonator 1 (thick lines) and resonator 2 (thin lines) as functions of time, with the knob qubit in (a) $|\epsilon^c\rangle$, and (b) $|\epsilon^c\rangle$. The solid and dashed lines represent the dynamics governed by $\mathcal{H}$ and $\mathcal{H}^{eff}$, respectively. The evolution of the system can be described by $\mathcal{H}^{eff}$ to a very good approximation, and we can really switch on and off the photon hopping by engineering the quantum state of $q_c$. **Eigensstates of the effective Hamiltonian $\mathcal{H}^{eff}$.** In the following, we will calculate the eigensstates of the system Hamiltonian and show that, by choosing appropriate parameters, our proposed 2-site JCHM can work in distinct regimes via only changing the internal state of $q_c$. For simplicity, we use the effective system Hamiltonian and restrict our analysis to the case of the resonators containing a total of two polaritons.

If $\frac{g_c^2}{\Delta_c} = \kappa_0$ and $q_c$ is in $|\epsilon^c\rangle$, the photon hopping terms vanish and the system can be described by Hamiltonian $\mathcal{H}^{eff}_{q_c} = \sum_{i=1,2} \mathcal{H}_i^{JC}$. Here, we have written the ac Stark shifting terms into $\mathcal{H}_i^{JC}$ and the resonator frequency $w$ is replaced by a shifted frequency $w' = w - \frac{g_c^2}{\Delta_c}$. The local JC Hamiltonian $\mathcal{H}_i^{JC}$ can be diagonalized in the basis of polariton states 14. Let $\{n,g\} (n,e)$ represent a resonator that contains $n$ photons and a single qubit in the ground (excited) state, then the upper
and lower $n$-polariton states of resonator $i$ can be given by $|n+\rangle_i = \sin \theta_i |n-1,\rangle_i + \cos \theta_i |n,\rangle_i$, and $|n-\rangle_i = \cos \theta_i |n-1,\rangle_i - \sin \theta_i |n,\rangle_i$, respectively, where $\tan \theta_i = (\frac{g_c}{\Delta_c} + \sqrt{(\frac{g_c}{\Delta_c})^2 + \frac{g^2}{\Delta_c}})/\sqrt{n}$. Specially, the zero-polariton state $|0,\rangle_i = |0,\rangle_i$. In order of increasing energy, the eight eigenstates of $\mathcal{H}^{eff}_{q_c}$ are $\{|1-\rangle_1|1-\rangle_2, \{2-\rangle_1|0-\rangle_2, \{0-\rangle_1|2-\rangle_2, |1-\rangle_1|1+\rangle_2, \{1+\rangle_1|1-\rangle_2, \{2+\rangle_1|0+\rangle_2, \{0+\rangle_1|2+\rangle_2, \{1+\rangle_1|1+\rangle_2\}$. The state $|1-\rangle_1|1-\rangle_2$ is exactly the ground state of 2-site
JCHM in the localized regime.
If $\frac{q^2}{2m} = \kappa_0$ and $q_c$ is in $|e\rangle$, the system can be described by $H^{eff}_e = \sum_{i=1,2} H^{JC}_i + 2\kappa_0(a_1^\dagger a_2 + a_2 a_1^\dagger)$. In this case, the resonator frequency $w$ is replaced by a shifted frequency $w' = w + \frac{q^2}{2m}$ and the two resonators are coupled by a strength $J = 2\kappa_0$. If $J$ is much smaller than the energy splitting between the upper polariton branch and lower polariton branch, then the mixing of different branches is negligible, and the lowest three eigenstates of $H^{eff}_e$ are linear combinations of $|1 \rangle_0|1 \rangle_2$, $|2 \rangle_0|0 \rangle_2$, and $|0 \rangle_1|2 \rangle_2$. If we further require that $J$ dominates over the effective repulsive energy $u$, between two polaritons of lower branch ($u_c$ equals the energy splitting between $|1 \rangle_0|1 \rangle_2$ and $|2 \rangle_0|0 \rangle_2$), then the coupled resonators work in the delocalized regime. In this case, the lowest three eigenstates of $H^{eff}_e$ can be approximated by the states $|0 \rangle^1_e = \frac{1}{\sqrt{3}}|2 \rangle_0|0 \rangle_2 + \frac{1}{\sqrt{3}}|0 \rangle_1|2 \rangle_2 - \frac{1}{\sqrt{3}}|1 \rangle_0|1 \rangle_2$, $|0 \rangle^2_e = \frac{1}{\sqrt{3}}|2 \rangle_0|0 \rangle_2 - \frac{1}{\sqrt{3}}|0 \rangle_0|0 \rangle_2 + \frac{2}{\sqrt{3}}|1 \rangle_1|1 \rangle_2$, and $|0 \rangle^3_e = \frac{1}{\sqrt{3}}|2 \rangle_0|0 \rangle_2 - \frac{1}{\sqrt{3}}|0 \rangle_0|0 \rangle_2 + \frac{2}{\sqrt{3}}|1 \rangle_1|1 \rangle_2$, respectively. These states are the eigenstates of 2-site JCHM in the large hopping limit. Note that, "J dominates over $u_c" does not mean $J \gg u_c$. Following the results of the pure Bose-Hubbard model, the quantum critical point of entering into delocalized regime is $J \sim 0.3u_c$.[12]

In Fig. 3, we plot the spectrum of $H^{eff}_e$ with the resonators containing a total of two polaritons. The parameters we choose are $g_c = g$, $\kappa_0 = 0.1g$, $\epsilon = 41g$, $\epsilon_c = 50g$, and $w = 40g$. The lower eight eigenstates of $|\psi\rangle$ with $l = 8$ are two manifolds corresponding to $q_c$ in $|g\rangle$ and $|e\rangle$, respectively, $E_l$ ($1 \leq l \leq 16$) are the related sixteen eigenvalues, and the eigenstates $|\psi\rangle^m$ with $12 \leq m \leq 16$ all involve upper polariton states of the resonators. With these parameters, we have $u_c = 0.259g$, $J = 0.2g$, and $|9\langle \psi|\psi\rangle^c|/|\psi\rangle^c| = 0.980$, $|10\langle \psi|\psi\rangle^c|/|\psi\rangle^c| = 0.998$, $|11\langle \psi|\psi\rangle^c|/|\psi\rangle^c| = 0.979$. Therefore, by choosing these parameters, we can make this 2-site JCHM stay in localized and delocalized regime via engineering $g_c$ in $|g\rangle$ and $|e\rangle$, respectively.

Quantum superposition of distinct quantum phases of photons.—As shown above, our proposed system has the features of nonlinear spectrum and qubit-state-dependent light phases. These features allow us to demonstrate the Rabi oscillation between the localized and delocalized phase using a spectroscopic technique. Now, we apply a microwave field with frequency $\omega_d$ and strength $\Omega$ to the control line of $g_c$, then the total system Hamiltonian reads

$$H_{tot} = H + \Omega (\sigma_c^+ e^{-i\omega_d t} + \sigma_c^- e^{i\omega_d t}).$$

(5)

If $\omega_d$ is chosen to be equal to the energy splitting between $|\psi\rangle^1$ and $|\psi\rangle^9$, and $\Omega$ is not big enough to induce the off-resonant transitions, then the effective system Hamiltonian in the interaction picture is approximately

$$H_{tot}^{int} = \Omega (|\psi\rangle^1 \otimes |\psi\rangle^9 + |\psi\rangle^9 \otimes |\psi\rangle^1),$$

(6)

where $\Omega = \langle \psi|\Omega\sigma_c^+|\psi\rangle^1$ is the effective transition element. Start with the initial state $|\psi\rangle^1$, the evolution of the system can be described by

$$|\Psi(t)\rangle = \cos(\Omega t)|\psi\rangle^1 + \sin(\Omega t)|\psi\rangle^9,$$

(7)

Because $|\psi\rangle^1$ and $|\psi\rangle^9$ describe a 2-site JCHM in localized and delocalized phase, respectively, Eq. (7) can be seen as the Rabi oscillation between two distinct quantum phases. By choosing $t = \frac{\pi}{2\Omega}$, the system can be engineered into a Schrödinger cat superposition of localized and delocalized phases. Here, the “dead cat” corresponds to the resonators are decoupled and the polaritons are frozen in local sites, and the “living cat” corresponds to the resonators are strongly coupled and the polaritons are delocalized through the sites.

For a practical situation, the errors induced by the off-resonant transitions have to be considered. In Fig. 4(a), we present all the related nonzero transition elements between the eigenstates when we pump the system from $|\psi\rangle^1$ to $|\psi\rangle^9$. The solid arrow represents the designed transition, and dashed arrows represent the unwanted off-resonant transitions. The minimal off resonance can be obtained as $\delta = \min(E_2, E_{10} - E_3 - E_2)$. To suppress the

FIG. 3. (Color online) Spectrum of $H^{eff}_e$ with the resonators containing a total of two polaritons. The parameters we choose are $g_c = g$, $\kappa_0 = 0.1g$, $\epsilon = 41g$, $\epsilon_c = 50g$, and $w = 40g$.

FIG. 4. (Color online) (a) The related nonzero transition elements between the eigenstates when we pump the system from $|\psi\rangle^1$ to $|\psi\rangle^9$. The solid arrow represents the designed transition, and dashed arrows represent the unwanted off-resonant transitions. (b) The dynamics of the system governed by the exact Hamiltonian $H_{tot}$. The solid and dashed lines represent the time-dependent values of $|\langle \psi|\Psi(t)\rangle|^{1}$ and $|\langle \psi|\langle e\rangle|\Psi(t)\rangle|^{1}$, respectively.
off-resonant transitions, the transition element $\Omega'$ is required to be smaller than $\delta$. In Fig. 4(b), we show the dynamics of the system governed by the exact Hamiltonian $H_{\text{tot}}$ in Eq. (5). The parameters we choose are $g_c = g$, $\kappa_0 = 0.1g$, $\epsilon = 41g$, $\epsilon_c = 50g$, $w = 40g$, $w_d = 50.2750g$ and $\Omega = 0.07g$, which yield $\delta = 0.2151g$ and $\Omega = 0.0495g$ [10]. The solid and dashed lines represent the time-dependent values of $|1\rangle\langle 1|\psi(t)|\psi(t)\rangle$ and $|1\rangle\langle 1|\phi(c)|\phi(c)\rangle$, respectively, where $|\psi(t)\rangle$ is the exact system state obtained by numerically integrating $H_{\text{tot}}$. At the time instants marked by small rectangles, the system is engineered into the equal superposition of $|g\rangle^c|1\rangle^1|1\rangle^1_2$ and $|e\rangle^c|1\rangle^1|1\rangle^1_2$.

In principle, this scheme can be generalized to the more complicated cases (e.g., connecting a larger number of resonators and injecting more polaritons). However, some problems will arise when we deal with a larger system. First, the complicated system has a very large Hilbert space and a very dense energy spectrum, so it is difficult to identify the designed transitions using spectroscopic techniques. Second, the increase in the number of polaritons will lead to a shorter coherence time of the system, so it is difficult to complete the wanted operations before dissipations occur.

Controllable interbit coupling with infinite on/off ratio.— Superconducting circuits are promising candidates for constructing quantum bits because of their potential suitability for large-scale quantum computation [17]. Usually, the coupling and decoupling of the superconducting qubits are implemented by tuning their frequencies in and out of resonance, respectively [12, 18]. The residual interaction that exists when the qubits are detuned from each other, however, limits the accuracy of these proposals. A controllable coupling mechanism, which has infinite on/off ratio, is desirable. Here, we will show that, the circuit proposed in this letter can be used to achieve this goal. Let us use the zero-polariton state and lower 1-polariton state of resonator $i$ to represent the two states of logical qubit $i$, i.e., $|0\rangle^i_L \equiv |0\rangle_i = |0, g\rangle_i$, and $|1\rangle^i_L \equiv |1\rangle_i = \cos\theta_i|0, e\rangle_i - \sin\theta_i|1, g\rangle_i$, where $\tan\theta_i = \frac{\gamma_c}{\gamma_c^*} + \sqrt{\left(\frac{\gamma_c}{\gamma_c^*}\right)^2 + 1}$. We make the knob qubit $q_c$ always stay in its ground state $|g\rangle$, then the two resonators are coupled by a strength $J = \kappa_0 - \frac{\gamma_c^2}{\gamma_c}$. To decouple the logical qubits, we tune $\Delta_c$ to an appropriate value so that $\kappa_0 = \frac{\gamma_c^2}{\gamma_c}$ and $J = 0$. To switch on the coupling, we tune $\Delta_c$ to another value so that $J = \kappa_0 - \frac{\gamma_c^2}{\gamma_c} \neq 0$ but $J$ is much smaller than the effective repulsive energy $w_r$. In this case, if one local resonator has a polariton in it, the strong photon blockade effect will prevent a second polariton from entering it. Finally, one can easily get the state evolution of the system:

$|0\rangle^i_0^0|0\rangle^i_L^2 \rightarrow |0\rangle^i_0^0|0\rangle^i_L^2$,  
$|1\rangle^i_0^1|1\rangle^i_L^2 \rightarrow |1\rangle^i_0^1|1\rangle^i_L^2$,  
$|0\rangle^i_1^1|1\rangle^i_L^2 \rightarrow \cos(Jt)|0\rangle^i_1^1|1\rangle^i_L^2 - i\sin(Jt)|1\rangle^i_1^1|0\rangle^i_L^2$,  

where $J' = (\kappa_0 - \frac{\gamma_c^2}{\gamma_c})\sin^2\theta_1$ is the effective polariton hopping rate. By choosing $t = \frac{\pi}{2J'}$, we can realize the $\sqrt{\text{SWAP}}$ gate of two logical qubits. This architecture is scalable to a large number of logical qubits and may be specially suitable for implementing one-way quantum computation [19].

Conclusion.— In this letter, we propose a method to engineer two microwave resonators into a quantum superposition of being decoupled and strongly coupled (correlated with a knob qubit in ground state and excited state). Using a variant of 2-site Jaynes-Cummings-Hubbard model, we generate entanglement between the distinct quantum phases of the injected polaritons and the internal states of the knob qubit. This architecture can also be used to solve the annoying on/off ratio problem of conventional proposals for scalable quantum computation. Our proposed circuit may play an important role in quantum engineering of novel states of microwave photons and quantum information processing with superconducting qubits.

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