Quantum Coherence and $W_\infty \times SU(2)$ Algebra
in Bilayer Quantum Hall Systems

Z. F. Ezawa

Department of Physics, Tohoku University, Sendai 980, Japan

Abstract

We analyze the bilayer quantum Hall (QH) system by mapping it to the monolayer QH system with spin degrees of freedom. By this mapping the tunneling interaction term is identified with the Zeeman term. We clarify the mechanism of a spontaneous development of quantum coherence based on the Chern-Simons gauge theory with the lowest-Landau-Level projection taken into account. The symmetry group is found to be $W_\infty \times SU(2)$, which says that the spin rotation affects the total electron density nearby. Using it extensively we construct the Landau-Ginzburg theory of the coherent mode. Skyrmion excitations are topological solitons in this coherent mode. We point out that they are detectable by measuring the Hall current distribution.

PACS: 73.40.Hm, 73.20.Dx, 73.40.-c, 75.10.-b

Keywords: quantum Hall effect, quantum coherence, bilayer electron system, Skyrmions
I. INTRODUCTION:

In the 2-dimensional space entirely new phenomena can occur due to its intrinsic topological structure. For instance, an electron may be transmuted into a boson by making a charge-flux composite in external magnetic field, which is called composite boson. As a result electrons may condense without making Cooper pairs. The fractional quantum Hall (QH) state is such a condensate of composite bosons [1]. When the spin degrees of freedom are taken into account, a quantum coherence develops spontaneously and turns the QH system into a quantum Hall ferromagnet. Skyrmions [2,3] are new topological solitons in this mode, which have been observed experimentally [4]. In a certain bilayer QH system an interlayer coherence develops spontaneously [5] and Josephson-like phenomena are expected to occur [5,6]. Some characteristic properties of the mode have already been observed experimentally [7]. The aim of this paper is to clarify the mechanism of the spontaneous development of quantum coherence in these two systems in a unified manner.

To make a consistent theory of the fractional QH effect it is necessary to make the lowest-Landau-level (LLL) projection [8], which has so far been used only within the single-mode approximation (SMA) [8,9]. In this paper, proposing a bosonic Chern-Simons (CS) gauge theory with the LLL projection, we apply it to the study of two-component QH systems. We map explicitly the bilayer system to the monolayer system with spin degrees of freedom. This allows us to analyze both systems in a unified way. This also helps us to understand the proper roles of the capacitance and tunneling terms in the bilayer system. In particular, the tunneling interaction term corresponds to the Zeeman term.

It is one of our main results that the U(1) symmetry is not broken spontaneously in spite of bose condensation. On the other hand, the SU(2) symmetry is found to be broken spontaneously, yielding a coherent mode. After the LLL projection the dynamics is governed by the \( \mathfrak{w}_\infty \times \text{SU}(2) \) algebra, which says that the spin rotation affects the total electron density. Namely, when we make a spin rotation the total electron density nearby is modulated. Using this property, we derive the Landau-Ginzburg (LG) theory of the coherent mode. Skyrmions [2] are collective excitations in this mode [3]. We show that they are detectable by measuring the Hall current distribution. We also present a systematic method to calculate the current and static correlation functions. Our field-theoretical analysis confirms some of the results made in the SMA [3]. In this paper we use the natural unit \( c = \hbar = 1 \) and take the length unit to be the magnetic radius \( \ell_B = \sqrt{\hbar c/eB} \) with \( B \) the magnetic field. The Landau-level energy gap is given by \( \hbar \omega_c = 1/M \). Hence, the LLL projection is associated with limit \( M \to 0 \).
II. SU(2) SPIN STRUCTURE

We consider a bilayer electron system in strong magnetic field. We denote the electron field at the layer $\alpha (=1,2)$ by $\psi_\alpha$. The Hamiltonian is

$$H = \frac{1}{2M} \sum_\alpha \int d^2x \psi_\alpha^\dagger(x)(P_x^2 + P_y^2)\psi_\alpha(x) + \frac{1}{2} \sum_{\alpha,\beta} \int d^2x d^2y V_{\alpha\beta}(x - y)\psi_\alpha^\dagger(x)\psi_\beta^\dagger(y),$$  

(1)

where $P_k$ is the covariant derivative, $P_k = -i\partial_k + eA_k^{\text{ext}}$ with $A_k^{\text{ext}} = \frac{1}{2}\varepsilon_{kj}x_j B$, and $V_{\alpha\beta}(x - y)$ the Coulomb interaction. It is convenient to introduce the symmetric field ($\psi^\dagger$) and antisymmetric field ($\psi^\dagger$) by

$$\psi^\dagger = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \psi^\dagger = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2),$$  

(2)

and to construct a two-component field $\Psi = (\psi^\dagger, \psi^\dagger)$. We may rewrite the Hamiltonian (1) as $H = H_K + H_C^+ + H_C^-$ with

$$H_K = \frac{1}{2M} \int d^2x \Psi^\dagger(x)(P_x^2 - iP_y)(P_x + iP_y)\Psi(x) + \frac{N}{2}\hbar\omega_c,$$  

(3)

$$H_C^+ = \frac{1}{2} \int d^2x d^2y V_+(x - y)\rho(x)\rho(y),$$  

(4)

$$H_C^- = \frac{1}{2} \int d^2x d^2y V_-(x - y)S^1(x)S^1(y),$$  

(5)

where $V_\pm = V_1 \pm V_2$ and $N$ is the total electron number in the system. We have introduced the SU(2) generator $S^a(x) = \frac{1}{2}\Psi^\dagger \tau^a \Psi$ with $\tau^a$ the Pauli matrices. The total density is $\rho(x) = \Psi^\dagger \Psi = \rho_1 + \rho_2 = \rho^\dagger + \rho$. Note that $2S^1(x) = \rho_1 - \rho_2$ and $2S^3(x) = \rho^\dagger - \rho$. The antisymmetric Coulomb term $H_C^-$ describes the capacitance energy stored between the two layers; see (40).

The tunneling interaction is described by

$$H_Z = -\frac{1}{2}\Delta_{\text{SAS}} \int d^2x (\psi^\dagger_1 \psi_2 + \psi^\dagger_2 \psi_1) = -\frac{1}{2}\Delta_{\text{SAS}} \int d^2x (\psi^\dagger \psi^\dagger - \psi \psi^\dagger),$$  

(6)

with $\Delta_{\text{SAS}}$ the symmetric-antisymmetric energy gap. We may rewrite this as

$$H_Z = -\lambda \int d^2x S^3(x),$$  

(7)

where we have set $\lambda \equiv \Delta_{\text{SAS}}$. We start with the regime where the Coulomb term $H_C^+$ dominates the dynamics so that we can treat the capacitance term $H_C^-$ and the tunneling term $H_Z$ as a perturbation. In this case the Halperin $(m,m,m)$ phase [10] is realized and an interlayer coherence develops spontaneously [3]. The capacitance term is made small when the interlayer distance $d$ is made small compared with the magnetic length $\ell_B$: In particular, $H_C^- \to 0$ as $d \to 0$ and $H_C^- \to H_C^+$ as $d \to \infty$. The unperturbed system has the symmetry group SU(2).
We now consider the monolayer system with spin degrees of freedom. The electron is described by a two-component field \( \Psi = (\psi^\uparrow, \psi^\downarrow) \) with up and down spins. It is obvious that the Hamiltonian is given by

\[
H = H_K + H_C^+ + H_Z
\]

with the kinetic term (3), the Coulomb term (4) and the Zeeman term (7): Here, \( \lambda = g \mu_B B \) with \( g \) is the gyromagnetic factor and \( \mu_B \) the Bohr magneton. We also start with the regime where the Coulomb term \( H_C^+ \) dominates the dynamics so that we can include the Zeeman term \( H_Z \) as a perturbation. In this way the bilayer system is mapped to the monolayer system with spins except for the capacitance term \( H_C^- \).

The spin operator generates a local SU(2) transformation, \( e^{-iO} \), with

\[
O = \sum_{a=1}^3 \int d^2 x f^a(x) S^a(x), \tag{8}
\]

where \( f^a(x) \) is a real function. It acts on the SU(2) field as

\[
\Psi(x) \rightarrow e^{-iO} \Psi(x) e^{iO} = \exp[i \sum f^a(x) \tau^a_2] \Psi(x). \tag{9}
\]

It generates a spin texture on the ground state \( |g\rangle, |\Phi\rangle = e^{iO} |g\rangle \). The spin texture is an excited state since the system does not possess the local SU(2) symmetry.

**III. COMPOSITE BOSONS**

We analyze the unperturbed system consisting of the kinetic term \( H_K \) and the Coulomb term \( H_C^+ \). To show a spontaneous development of quantum coherence it is most convenient to use the composite bosons. The composite boson field \( \phi_\alpha \) is defined [5] by an operator phase transformation with a common phase \( \Theta \), \( \phi_\alpha(x) = e^{i\Theta(x)} \psi_\alpha(x) \). By this transformation the covariant derivative is modified as,

\[
P_k \rightarrow \hat{P}_k \equiv -i\partial_k + eA^\text{ext}_k + C_k, \tag{10}
\]

where the field \( C_k(x) \equiv \partial_k \Theta(x) \) is the CS gauge field to be determined by the CS constraint,

\[
\varepsilon_{jk}\partial_j C_k = 2\pi m\rho, \tag{11}
\]

in terms of the total density \( \rho \). Here, \( m \) is an odd integer which makes the field \( \phi_\alpha \) bosonic. The spin operators and the total density are the same, \( S_\alpha = \frac{1}{2} \Phi^\dagger \tau^a_2 \Phi \) and \( \rho = \Phi^\dagger \Phi \) where \( \Phi = (\phi^\uparrow, \phi^\downarrow) \) in terms of the symmetric and antisymmetric fields, as those in the original electron theory.

From the Hamiltonian with the covariant derivative \( \hat{P}_k \) the mean-field ground state is found to be

\[
\langle \phi^\alpha \rangle = e^{i\phi^\alpha_0} \sqrt{\rho_0^\alpha}, \quad C_k + eA^\text{ext}_k = 0, \quad (\alpha = \uparrow\downarrow) \tag{12}
\]
with arbitrary constants $\phi_0^\alpha$ and $\rho_0^\alpha$ subject to $\rho_0^\uparrow + \rho_0^\downarrow = 2\rho_0$; here $2\rho_0$ stands for the homogeneous background charge. Substituting it into the CS constraint (14) we find this ground state to realize only at the filling factor $\nu \equiv 4\pi\rho_0\hbar c/eB = 1/m$. We have defined the Landau-level filling factor so that it yields $\nu = 1$ when all the up-spin electron sites are filled.

The essential point is that there are many ground states (12) degenerate one another even at $\nu = 1$: They are indexed by $(\phi_0^\alpha, \rho_0^\alpha)$. It is necessary to choose one of them as the ground state $|g_0\rangle$ upon which to build the Hilbert space. This breaks the SU(2) symmetry spontaneously and develops a quantum coherence. Equivalently, the direction of the spin polarization $S^a$ is spontaneously chosen, $\rho_0 s_0^a \equiv \langle g|S^a(x)|g\rangle = \text{constant}$, where $\sum_{a=1}^{3} (s_0^a)^2 = 1$. This is why the system is called a quantum Hall ferromagnet. It is important that various densities are observable on the state $|g\rangle$, $\langle g|\rho_{1(2)}|g\rangle = \rho_0 (1 \pm s_0^1)$ and $\langle g|\rho_{3(1)}|g\rangle = (1 \pm s_0^3)$. In the monolayer system the Zeeman term fixes the polarization to be

$$s_0^a = \frac{1}{\rho_0} \langle g_0|S^a(x)|g_0\rangle = \delta_0^a,$$

however small it may be ($\lambda \approx 0$). In the bilayer system the capacitance term is minimized by the choice of $s_1^1 = 0$ and the tunneling term is minimized by $s_3^3 = 1$: Hence, the resulting polarization is again given by (13). This is the unique ground state of the total system, where all electrons are in the symmetric state $(\psi^\uparrow)$. Because of this reason it remains to be the ground state even if the Zeeman term (capacitance and tunneling terms) is made large provided that it is not too large.

**IV. COHERENT STATE**

To analyze the spin texture, we decompose the composite boson field $\phi^\alpha$ into the two fields $\phi$ and $n^\alpha$,

$$\phi^\alpha(x) = \phi(x)n^\alpha(x), \quad \phi(x) = e^{ix(x)}\sqrt{\rho(x)}.$$

We substitute (14) into the density operator, obtaining $\rho(x) = \phi^\dagger(x)\phi(x)$ and $n^\dagger(x)n(x) = 1$, where $n(x) = (n^\uparrow, n^\downarrow)$. The spin generator is expressed as $S^a(x) = \rho(x)n^\dagger(x)\tau^a n(x)$. We count the number of the real fields in the decomposition (14). The composite boson $\phi^\alpha$ has four real fields in total, and the U(1) field $\phi$ has two real fields. Hence, the two-component complex field $n^\alpha$ has only two real fields. Such a field is the SU(2) complex projective field and abbreviated to the CP$^1$ field. (In general the SU(N) complex projective field is abbreviated to the CP$^{N-1}$ field.)

The ground state $|g_0\rangle$ satisfying (13) is a coherent state of the CP$^1$ field,

$$n(x)|g_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}|g_0\rangle = \frac{1}{\sqrt{2}}T\begin{pmatrix} 1 \\ 1 \end{pmatrix}|g_0\rangle.$$
where $T$ transforms the two-component electron field $(\psi_1, \psi_2)$ into $(\psi^\uparrow, \psi^\downarrow)$ as in (2),

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T^\dagger T = 1. \quad (16)$$

The basic properties are $T^\dagger \tau^1 T = \tau^3$, $T^\dagger \tau^2 T = -\tau^2$ and $T^\dagger \tau^3 T = \tau^1$. A spin texture is given by performing an SU(2) transformation, $|\Phi\rangle = e^{iO}|g_0\rangle$. It is a coherent state of the CP\(^1\) field. We may parametrize it as

$$|\Phi\rangle = \frac{1}{\sqrt{2}} T \begin{pmatrix} e^{i\varphi(x)/2} \sqrt{1 + \sigma(x)} \\ e^{-i\varphi(x)/2} \sqrt{1 - \sigma(x)} \end{pmatrix} |\Phi\rangle. \quad (17)$$

In terms of the sigma field we have

$$s^1(x) \equiv \frac{1}{\rho_0} \langle \Phi | S^1(x) | \Phi \rangle = \sigma(x)$$

$$s^2(x) \equiv \frac{1}{\rho_0} \langle \Phi | S^2(x) | \Phi \rangle = -\sqrt{1 - \sigma^2(x)} \sin \varphi(x),$$

$$s^3(x) \equiv \frac{1}{\rho_0} \langle \Phi | S^3(x) | \Phi \rangle = \sqrt{1 - \sigma^2(x)} \cos \varphi(x). \quad (18)$$

It is a characteristic feature of the coherent state that both the density ($\sigma$) and its conjugate phase ($\varphi$) have the classical fields. A generic ground state $|g\rangle$ of the unperturbed Hamiltonian is given by the choice of $\sigma(x) = \sigma_0$ and $\varphi(x) = \varphi_0$: In particular, the state $|g_0\rangle$ is by $\sigma_0 = \varphi_0 = 0$.

Using the SU(2) transformation (8) explicitly we may express $s^a$ in terms of $f^a$,

$$s^a(x) = \frac{1}{\rho_0} \langle \Phi | S^a(x) | \Phi \rangle = s^a_0 - \varepsilon^{abc} f^b(x) s^c_0 + \cdots, \quad (19)$$

with (13), where the dots $\cdots$ denote higher order terms in $f^a$. Comparing this with (18), we can relate the functions $f^a$ to the fields $\sigma$ and $\varphi$. We have $s^1(x) = \sigma = -f^2$, $s^2(x) = -\varphi = f^1$ and $s^3(x) = 1$, up to the first order in $f^a$. Hence, the perturbative expansion around the mean-field ground state ($\sigma = \varphi = 0$) corresponds to the expansion in $f^a$.

The spin texture is classified topologically by the Pontryagin number. The topological current, $Q_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial_\nu n^\lambda \partial_\mu n$, conserves trivially, $\partial_\mu Q_\mu = 0$. For the state (18) it yields

$$\langle Q_0(x) \rangle = \frac{1}{8\pi} \varepsilon_{abc} \varepsilon_{ij} s^i_0 \partial^j s^j_0 s^c_0 = \frac{1}{4\pi} \varepsilon_{ij} \sigma \partial^j \varphi. \quad (20)$$

The topological charge $Q = \int d^2x Q_0(x)$ is the Pontryagin number, and topological excitations are Skyrmions [2]. The classical Skyrmion minimizes the nonlinear $\sigma$-model Hamiltonian (40) in the SU(2)-invariant limit.
V. LLL PROJECTION AND $W_\infty \times SU(2)$ ALGEBRA

When the magnetic field is strong enough, the magnetic energy greatly exceeds thermal and potential energies. It is reasonable to assume that electrons are confined within the lowest Landau level. To make a consistent theory it is necessary to make the LLL projection by quenching the kinetic term [8].

For this purpose we decompose the composite-boson coordinate $x$ into the center-of-mass coordinate $\hat{X} \equiv (\hat{X}, \hat{Y})$ and the relative coordinate $\hat{R} = (\hat{P}_y, -\hat{P}_x)$, where $x = \hat{X} + \hat{R}$ and $\hat{P}_k$ is given by (11). They satisfy $[\hat{X}, \hat{Y}] = -i$, $[\hat{P}_x, \hat{P}_y] = i$. We use checked quantities for composite-boson variables. Two independent sets of harmonic oscillators are defined, $\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{P}_x + i\hat{P}_y)$ and $\hat{b} \equiv \frac{1}{\sqrt{2}}(\hat{X} - i\hat{Y})$, where $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$.

The LLL projection is to quench the kinetic energy term in the Hamiltonian. In the composite boson theory it is achieved by imposing the LLL condition,

$$\hat{a}\phi_\alpha(x)|\hat{\Phi}\rangle = \frac{1}{\sqrt{2}}(\hat{P}_x + i\hat{P}_y)\phi_\alpha(x)|\hat{\Phi}\rangle = 0,$$

on the state $|\hat{\Phi}\rangle$ in our Hilbert space.

We are concerned about the spin texture $|\Phi\rangle = e^{i\mathcal{O}}|g_0\rangle$, where $\mathcal{O}$ is the SU(2) generator (8). We examine if this state belongs to the lowest Landau level. For this purpose we examine if $\mathcal{O}|\hat{\Phi}\rangle$ belongs to the lowest Landau level when $|\hat{\Phi}\rangle$ does. Since we have

$$a(x)\phi_\alpha(x)\mathcal{O}|\hat{\Phi}\rangle = \frac{1}{2} \sum_b (\tau_b)_{\alpha\beta} a(x) f^b(x) \phi_\beta(x)|\hat{\Phi}\rangle \neq 0,$$

the state $\mathcal{O}|\hat{\Phi}\rangle$ does not belong to the lowest Landau level. We make the LLL projection of the operator $\mathcal{O}$ and the c-number functions $f^a(x)$ as follows.

We do this in a systematic way. We first make a Fourier transformation of $f^a(x)$,

$$f^a(x) = \frac{1}{2\pi} \int d^2q f^a(q) e^{iqx}. $$

The problem of the LLL projection is reduced to that of the plane wave $e^{iqx}$. We make normal ordering with respect to $\hat{a}$ and $\hat{a}^\dagger$ as

$$e^{iqx} = \exp\left[\frac{1}{\sqrt{2}}q\hat{a}^\dagger\right]\exp\left[-\frac{1}{\sqrt{2}}q^*\hat{a}\right]\langle e^{iqx}\rangle, \tag{23}$$

where

$$\langle e^{iqx}\rangle \equiv e^{-(1/4)q^2}\langle e^{i\hat{q}\hat{X}}\rangle,$$ \tag{24}

with $q = q_x + iq_y$. The LLL projection is to quench the operators $\hat{a}$ and $\hat{a}^\dagger$. Hence,

$$\hat{f}^a(x) = \int \frac{d^2q}{(2\pi)^2} f^a(q) \langle e^{iqx}\rangle, \tag{25}$$

and
\[ \hat{O} = \int d^2 x \hat{f}^a(x) S^a(x) = \int d^2 q f^a(-q) \hat{S}^a_q, \quad (26) \]

where

\[ \hat{S}^a_q = \int \frac{d^2 x}{2\pi} S^a(x) e^{-iqx}, \quad (27) \]

is the LLL projected spin operator. The LLL projection of the density operator \( \rho \) is similarly defined, which we denote by \( \hat{\rho}_q \) in the momentum space.

After the LLL projection the Hamiltonian contains no kinetic energy term. Yet, the dynamics arises since the components of the center-of-mass coordinate do not commute, \([\hat{X}, \hat{Y}] = -i\). Using this commutation relation it is easy to verify that the projected operators \( \hat{\rho}_q \) and \( \hat{S}^a_q \) satisfy the \( W_\infty \times SU(2) \) algebra,

\[ \left[ \hat{\rho}_p, \hat{\rho}_q \right] = \frac{i}{\pi} \hat{\rho}_{p+q} \sin \frac{p \wedge q}{2} e^{i(p+q)}, \]
\[ \left[ \hat{S}^a_p, \hat{\rho}_q \right] = \frac{i}{\pi} \hat{S}^a_{p+q} \sin \frac{p \wedge q}{2} e^{i(p+q)}, \]
\[ \left[ \hat{S}^a_p, \hat{S}^b_q \right] = \frac{i}{2\pi} \varepsilon^{abc} \hat{S}^c_{p+q} \cos \frac{p \wedge q}{2} e^{i(p+q)} + \frac{i}{4\pi} \delta^{ab} \hat{\rho}_{p+q} \sin \frac{p \wedge q}{2} e^{i(p+q)}, \quad (28) \]

with \( p \wedge q = \varepsilon_{ij} p_i q_j \). This governs the dynamics. It is a generalization of the \( W_\infty \) algebra characterizing the QH system [3]. We call it the density algebra for simplicity. It is important that the spin operator \( \hat{S}^a \) and the density operator \( \hat{\rho} \) do not commute: Their actions are related in a complicated way. Because of this relation the spin rotation affects the Coulomb term (4) though it involves only the total electron density \( \rho \).

VI. GROUND STATE AND SPIN TEXTURE

We make the LLL projection of various terms in the Hamiltonian,

\[ \hat{H}^+_C = \pi \int d^2 q V_+(q) \hat{\rho}_{-q} \hat{\rho}_q, \quad \hat{H}^-_C = 4\pi \int d^2 q V_-(q) \hat{S}^1_{-q} \hat{S}^1_q, \quad \hat{H}_Z = -2\pi \lambda \hat{S}^3_0, \quad (29) \]

where \( V_\pm(q) \) is the Fourier transformation of the potential \( V_\pm(x) \).

The unperturbed Hamiltonian \( \hat{H}^+_C \) is minimized by requiring

\[ \hat{\rho}_q |g\rangle = 4\pi \rho_0 \delta(q) |g\rangle. \quad (30) \]

We can impose this condition because the gapless mode is absent in the total density fluctuation [3], as leads to the incompressibility of the QH system. This agrees with a result of the representation theory [14] that the \( W_\infty \) algebra with no central extension has merely the trivial vacuum sector.

We have already noticed that the ground state is a coherent state of the \( CP^1 \) field. Hence, we impose
\[ s^1_q = \frac{1}{\rho_0} \langle g | \hat{S}^1_q | g \rangle = 2\pi \sigma_0 \delta(q), \]
\[ s^2_q = \frac{1}{\rho_0} \langle g | \hat{S}^2_q | g \rangle = -2\pi \sqrt{1 - \sigma_0^2} \sin \varphi_0 \delta(q), \]
\[ s^3_q = \frac{1}{\rho_0} \langle g | \hat{S}^3_q | g \rangle = 2\pi \sqrt{1 - \sigma_0^2} \cos \varphi_0 \delta(q). \] (31)

All these states are degenerate with respect to the Hamiltonian \( H_C \). The degeneracy is removed by introducing the Zeeman (tunneling) term, which is to choose \( \sigma_0 = \varphi_0 = 0 \), or

\[ \langle g_0 | \hat{s}^a_q | g_0 \rangle = 2\pi \rho_0 \delta^{a3} \delta(q). \] (32)

This corresponds to (13) after taking its Fourier transformation.

We consider the spin texture \( | \Phi \rangle = e^{i\varphi} | g_0 \rangle \). We evaluate \( \langle \Phi | \hat{S}_q | \Phi \rangle \) and denote its Fourier transformation by \( \rho_0 \hat{s}^a(x) \). Using (28), (30) and (31) we obtain

\[ \hat{s}^a(x) = \delta^{a1} - \varepsilon^{1ab} e^{-(1/2)\nabla^2} f^b(x) + \cdots. \] (33)

We find \( \hat{s}^a(x) = s^a(x) \) with (18) for a sufficiently smooth configuration. It is also easy to find

\[ \hat{\rho}(x) = 2\rho_0 + \nu Q_0(x) + \cdots, \] (34)

where \( \hat{\rho}(x) \) is the Fourier transformation of \( \langle \hat{\Phi} | \hat{\rho}_q | \hat{\Phi} \rangle \) and \( Q_0(x) \) is the Pontryagin number density (20). When \( Q = \int d^2 x Q_0(x) \neq 0 \), the spin texture describes Skyrmion excitations. A Skyrmion is a topological excitation realized in the coherent mode. This equation says that the Skyrmion carries electric charge \( -e\nu Q \). It has a fractional charge in general.

**VII. EFFECTIVE HAMILTONIAN**

Relevant correlation functions are those of density operators \( \hat{\rho} \) and \( \hat{S}^a \). Two point functions such as \( \langle \hat{\Phi} | \hat{\rho}_p \hat{\rho}_q | \hat{\Phi} \rangle \) cannot be evaluated by the algebraic relation alone. In so doing we need to deal with \( \langle g_0 | \{ \hat{S}^a_p, \hat{S}^b_q \} | g_0 \rangle \). We use the formula,

\[ \{ \hat{S}^a_p, \hat{S}^b_q \} = -\frac{1}{2\pi} \varepsilon^{abc} \hat{S}^c_{p+q} \sin\left[ \frac{\mathbf{p} \cdot \mathbf{q}}{2} \right] e^{(1/2)pq} + \frac{1}{4\pi} \delta^{ab} \hat{\rho}_{p+q} \cos\left[ \frac{\mathbf{p} \cdot \mathbf{q}}{2} \right] e^{(1/2)pq} + \{ \hat{S}^a_p, \hat{S}^b_q \} \cdot \] (35)

where

\[ \langle g_0 | \hat{S}^a_p \hat{S}^b_q | g_0 \rangle = \frac{1}{4} \delta^{a3} \delta^{b3} \langle g_0 | \hat{\rho}_p \hat{\rho}_q \rangle | g_0 \rangle, \] (36)

since \( | g_0 \rangle \) is a coherent state of the CP^1 field. We next use

\[ \{ \hat{\rho}_p, \hat{\rho}_q \} = \frac{1}{\pi} \hat{\rho}_{p+q} \cos\left[ \frac{\mathbf{p} \cdot \mathbf{q}}{2} \right] e^{(1/2)pq} + \{ \hat{\rho}_p, \hat{\rho}_q \} \cdot \] (37)
Since $|g_0\rangle$ is an eigenstate of $\hat{\rho}_p$ as in (30), this yields
\[
\langle g_0 : \hat{\rho}_p \hat{\rho}_q : |g_0\rangle = 4(2\pi)^2 \rho_0^2 \delta(p) \delta(q) - 2\rho_0 \delta(p + q)e^{-(1/2)p^2}.
\] (38)
Therefore, we obtain
\[
\langle g_0 | \{ \hat{S}^a_p, \hat{S}^b_q \} | g_0 \rangle = \begin{cases} 
\rho_0 \delta(p + q) \exp[-\frac{\rho_0}{2}p^2] & \text{for } a=b=1,2 \\
2(2\pi)^2 \rho_0^2 \delta(p) \delta(q) & \text{for } a=b=3 \\
0 & \text{otherwise}
\end{cases}
\] (39)
Various correlation functions are calculated by using these formulas.

We are ready to evaluate the Hamiltonian on the spin texture $|\hat{\Phi}\rangle$ in a sufficiently smooth configuration, $\Delta E \equiv \langle \hat{\Phi} | (\hat{H}^+ + \hat{H}^- + \hat{H}_Z) | \hat{\Phi}\rangle$. We define the effective Hamiltonian density $\mathcal{H}_{\text{eff}}$ by $\Delta E = \int d^2x \mathcal{H}_{\text{eff}}$. After a straightforward calculation we obtain
\[
\mathcal{H}_{\text{eff}} = \frac{1}{2}\rho_E \sum_{a=1}^{2} [\nabla s^a(x)]^2 + \frac{1}{2}\rho_A [\nabla s^1(x)]^2 + \frac{e^2 \rho_0^2}{2C} s^1(x)s^1(x) - \lambda s^3(x),
\] (40)
with $C$ the capacitance, and $\rho_E \equiv \rho_+ - \rho_-$, $\rho_A \equiv \rho_+ + \rho_-$;
\[
\rho_\pm = \frac{\nu}{16\pi} \int \frac{d^2q}{(2\pi)^2} V_\pm(q)q^2 e^{-(1/2)q^2}.
\] (41)
We have $\rho_E = \rho_A$ in the monolayer system with spins. This gives the LG theory of the coherent mode, where $\rho_A$ and $\rho_E$ describe the spin stiffness. The result agrees with the one found in the SMA. It describes the nonlinear sigma model in the SU(2)-invariant limit, where the Skyrmion solutions are known explicitly. Although their sizes are infinitely large, they are made finite by the Zeeman term (capacitance and tunneling terms).

**VIII. GOLDSTONE MODE**

We approximate the effective Hamiltonian (40) as
\[
\mathcal{H}_{\text{eff}} = \frac{\rho_E}{2} (\nabla \varphi)^2 + \frac{\rho_A}{2} (\nabla \sigma)^2 + \frac{e^2 \rho_0^2}{2C} \sigma^2 + \lambda \rho_0 (\sigma^2 + \varphi^2)
\] (42)
for small fluctuations of $\sigma$ and $\varphi$. The fields $\varphi(x)$ and $\sigma(x)$ are classical fields. However, the commutation relation, $[\rho_0 \sigma(x), \varphi(y)] = i\delta(x - y)$, follows naturally. To derive it we evaluate the equation of motion,
\[
i\frac{d\hat{S}_k}{dt} = [\hat{S}_k, \hat{H}].
\] (43)
We then take its expectation value by the state $|\hat{\Phi}\rangle = e^{i\hat{O}}|g_0\rangle$. The resulting set of equations agree precisely with the Heisenberg equations of motion of the Hamiltonian \( (42) \) provided that the above commutation relation is imposed.

We may diagonalize the Hamiltonian \( (42) \) by way of the Bogoliubov transformation,

\[
H_{\text{eff}} = \int d^2k E(k) \hat{\alpha}_k^\dagger \hat{\alpha}_k, \quad E_k^2 = \left( \frac{\rho_E k^2}{\rho_0} + \lambda \right) \left( \frac{\rho_A k^2}{\rho_0} + \frac{e^2 \rho_0}{C} + \lambda \right).
\]

(44)

When we switch off the Zeeman (tunneling) term by setting $\lambda = 0$, the dispersion relation $E_k$ describes a gapless mode. This is the Goldstone mode associated with the spontaneous polarization of the spins. When $\lambda \neq 0$ it becomes a gapful mode and called a pseudo-Goldstone mode.

The effective Hamiltonian \( (40) \) is valid for any values of $C$ and $\lambda$ as far as the coherent mode persists. In the bilayer system there are some important comments. When $\lambda = 0$ the dispersion relation reads $E_k \approx e\sqrt{\rho_E/C}|k|$ as $k \to 0$ due to the capacitance effect. It has a linear dispersion relation leading to a superfluid mode. Thus, the capacitance term turns the Goldstone mode into a superfluid mode. However, when the capacitance term becomes too large, the Halperin \((m,m,m)\) phase breaks down with the loss of quantum coherence \[5\]. It is taken over by another Halperin phase \[10\], that is the \((m_1,m_2,n)\) phase with $m_1m_2 \neq n^2$.

**IX. ELECTRIC CURRENTS:**

We analyze the electric current in external electric field. We first consider the monolayer system. In the unprojected space the current may be defined as follows. We make an infinitesimal local phase transformation, $\psi \to e^{i\int(\omega)\psi}$, with the gauge field fixed. Since this is not a symmetry, the Hamiltonian is modified by $\Delta H_f = (1/e) \int d^2x J_i \partial_i f$. This defines the current $J_i$, which is essentially the Nöther current. By making the LLL projection we obtain the formula

\[
k_i \hat{J}_i(k) = \left. \frac{ie \delta \Delta \hat{H}_f}{\hbar \delta f_{-k}} \right|_{f \to 0}
\]

(45)

in the momentum space. The generator of the local phase transformation is a smeared density operator. It is $\hat{O}_f = \int d^2q f(-q) \hat{\rho}_q$ after the LLL; see \[24\]. The Hamiltonian transforms as

\[
\hat{H} \to e^{-i\hat{O}_f} \hat{H} e^{i\hat{O}_f} \equiv \hat{H} + \Delta \hat{H}_f.
\]

(46)

Since $\hat{O}_f$ is just a c-number on $|g\rangle$ due to the condition \[30\] we obtain $\langle g | \Delta \hat{H}_f | g \rangle = 0$ from \( (46) \) and hence $J_i = 0$ from \( (15) \). Therefore, no currents are induced by the Coulomb and Zeeman terms.
However, we cannot apply this argument to the external electric field term $H_E$. For simplicity we assume a constant field $E_i$, and we choose the potential $A_0 = -x_i E_i$. The term $H_E$ is given by

$$\widetilde{H}_E = e \int d^2 x \tilde{A}_0(x) \rho(x) = e \int d^2 q A_0(q) \tilde{\rho}_{-q},$$

(47)

after the LLL projection. The term is ill defined on the state $|g\rangle$, $\tilde{H}_E |g\rangle = 4 \pi \rho_0 A_0(q = 0) |g\rangle$ since $q_j A_0(q) = 2 \pi i E_j \delta(q)$. Hence, there is no reason that $\langle g | \Delta \tilde{H}_f |g\rangle = 0$ for $\tilde{H}_E$ in (46).

Indeed, we can calculate explicitly $\Delta \tilde{H}_f$ for $\tilde{H}_E$, which reads

$$\Delta \tilde{H}_f = -ie \int d^2 q A_0(q) \int d^2 k f_{-k} \tilde{p}_k \tilde{\rho}_{-q},$$

(48)

Substituting this into the formula (15) and using the density algebra (28) we obtain

$$\tilde{J}_i(k) = e^2 \epsilon_{ij} E_j \tilde{\rho}_k,$$

(49)

for a homogeneous electric field. This is an operator identity. The charge-current conservation reads

$$i \frac{d}{dt} \tilde{J}_0(k) = [\tilde{J}_0(k), \tilde{H}] = -k_i \tilde{J}_i(k),$$

(50)

where $\tilde{J}_0(k) \equiv -e \tilde{\rho}_k$ is the charge density. Eq.(13) gives the familiar Hall current on the ground state $|g\rangle$. Evaluating it on the spin texture, $\langle \tilde{\Phi} | \tilde{J}_i(k) | \tilde{\Phi} \rangle$, and taking its Fourier transformation, we obtain

$$J_i(x) = \frac{e^2}{2 \pi} \epsilon_{ij} E_j [1 + Q_0(x)],$$

(51)

where $Q_0(x)$ is the Skyrmion density (20) and we have used (24). The Skyrmion density is dependent of time in the presence of an electric field. However, Skyrmions would actually be trapped by impurities just as quasiparticles. A Skyrmion excitation is observable as a localized static object by means of measuring the Hall current distribution.

One might ask why we can use the state $|g\rangle$ in evaluating the Hall current (13) when $\langle g | \tilde{H}_E |g\rangle$ is ill defined. Indeed, the ground state is no longer given by $|g\rangle$ in the presence of a homogeneous electric field. It is given by [12]

$$|g_D\rangle = e^{iMO} |g\rangle; \quad O = \epsilon_{ij} E_j \int d^2 x x_i \rho(x),$$

(52)

where $M$ is the electron mass which should be set zero ($M \to 0$) in the large Landau-level gap-energy limit. Since $O$ generates an ordinary local U(1) transformation, $|g_D\rangle$ contains states in various Landau levels: the factor $e^{iMO}$ mixes Landau levels. It produces a nontrivial term in the Hamiltonian from the kinetic term $H_K$ containing the factor $1/M$. Indeed, this factor is determined based on the requirement that the anomalous term in $H_E$ is canceled.
by the term so produced. We should mention that the Hall current is obtainable \[^{12}\] by a
direct evaluation of \(<gD|J_i(x)|gD>\) with \(J_i \approx (e/M)|\psi|^2 P_i \psi\) since it involves the factor \(1/M\). Namely, it is the state \(|gD>\) that supports a drift current. Therefore, we should use \(|gD>\) also in evaluating the Hall current \(^{19}\). However, this formula does not contain the factor \(1/M\). To evaluate such a quantity there exists no distinction between two states \(|g>\) and \(|gD>\) in the limit \(M \rightarrow 0\). This justifies the use of the state \(|g>\). We remark that our formula \(^{14}\) is quite general which we can apply to a heterogeneous electric field and also to any kind of currents. We conclude that the current flows via a Landau-level mixing though the mixing is infinitesimal as \(M \rightarrow 0\). We believe that this is the precise statement for the naive argument given in Ref. \(^{8}\).

The current \(^{51}\) is all that we have in the monolayer system. In the bilayer system we can apply a different electric field \(E_j^\alpha\) at each layer \(\alpha (=1,2)\). We introduce currents \(J^\pm = J^1 \pm J^2\) and fields \(E^\pm = \frac{1}{2}(E^1 \pm E^2)\). The current \(J^+\) is the one associated with the local \(U(1)\) phase transformation, while \(J^-\) is associated with the local \(SU(2)\) transformation \(^{1}\) with \(f^1 = f^3 = 0\) and \(f^2 = 2f(x)\). The Hall currents \(\bar{J}^E_i^\pm\) are given by \(^{51}\) with \(\bar{J}_i = \bar{J}_i^E\) and \(E_j = E_j^\pm\). Using the formula \(^{13}\) and the coherent-state condition \(^{31}\), we derive the supercurrent

\[
\bar{J}_i^{C^-}(x) = 2e\rho_0 \partial_i \varphi(x),
\]

from the Coulomb term and we derive the tunneling current

\[
\bar{J}^z(x) \equiv 2e\lambda \rho_0 s^2(x) = -2e\lambda \rho_0 \sqrt{1 - \sigma(x)^2} \sin \varphi(x),
\]

from the tunneling term between the two layers. These are all that we have in the bilayer system. They are derived in an analogous way.

\[\text{X. CONCLUSION}\]

We have analyzed the monolayer QH system with spin degrees of freedom and the bilayer QH system in the \((m,m,m)\) phase. Based on the bosonic CS gauge theory with the LLL projection we have presented a systematic way to investigate the quantum coherence spontaneously developed in these systems. We have mapped the bilayer system to the monolayer system. In particular, the tunneling term is identified with the Zeeman term. The capacitance term \(H_C\) is specific to the bilayer system. As the interlayer distance \(d\) increases its existence becomes crucial, and eventually the \((m,m,m)\) phase breaks down together with the loss of quantum coherence. It is the capacitance term and not the tunneling term that breaks the quantum coherence in the bilayer system. This would explain the experimental fact \(^{1}\) that the quantum coherence has been observed at a strong tunneling interaction such as \(\Delta_{SAS} \approx 8.5\text{K}\).
When the Coulomb interaction term $H_C^+$ dominates the system, these two systems have the SU(2) symmetry which is spontaneously broken. The Zeeman term (the tunneling and capacitance terms) selects the ground state $|g_0\rangle$ out of many degenerate states $|g\rangle$ of the unperturbed system $H_C^-$. It is possible to select another state $|g'\rangle$ as the ground state by making experimental arrangements upon which the QH effect is observed. This could be done by tilting the external magnetic field in the monolayer system and by applying a bias voltage between the two layers in the bilayer system. Indeed, in the presence of a bias voltage the capacitance term $H_C^-$ is not minimized by (13) but by a certain nonzero value of the density difference $2\rho_0 s^1 = \rho_1 - \rho_2$ between the two layers. It can be changed continuously by changing the bias voltage. This freedom exists only in the $(m, m, m)$ phase. Experimental checks of these phenomena will be the easiest way to verify the existence of the coherent mode in the QH system. We would like to propose such experiments.

We have derived the effective Hamiltonian governing the dynamics of the coherent mode in a bilayer quantum Hall system using the $W_\infty \times SU(2)$ algebra extensively. We have already shown elsewhere [6] that such a Hamiltonian leads to quantum coherent phenomena including the Josephson-like effect. We have also shown that Skyrmion excitations are observable by measuring the Hall current distribution. We believe that our field-theoretical method is a powerful tool to analyze various aspects of the QH effect. Detailed calculations of the present work will be reported in a forthcoming paper [13].
REFERENCES

[1] S.M. Girvin and A.H. MacDonald, Phys. Rev. Lett. 58, 1252 (1987); S.C. Zhang, T.H. Hansen and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989); N. Read, Phys. Rev. Lett. 62, 86 (1989); Z.F. Ezawa and A. Iwazaki, Phys. Rev. B 43, 2637 (1991); Z.F. Ezawa, M. Hotta and A. Iwazaki, Phys. Rev. B 46, 7765 (1992); S.C. Zhang, Int. J. Mod. Phys. B 6, 25 (1992).

[2] A.A. Belavin and A.M. Polyakov, JETP Letters 22, 245 (1975); A. D'Adda, A. Luscher and P. DiVecchia, Nucl. Phys. B146, 63 (1978); D.J. Gross, Nucl. Phys. B132, 439 (1978).

[3] D.H. Lee and C.L. Kane, Phys. Rev. Lett. 64, 1313 (1990); S.L. Sondhi, A. Karlhede, S. Kivelson and E.H. Rezayi, Phys. Rev. B 47, 16419 (1993).

[4] S.E. Barrett, G. Dabbagh, L.N. Pfeiffer, K.W. West and R. Tycko, Phys. Rev. Lett. 74, 5112 (1995);

[5] Z.F. Ezawa and A. Iwazaki, Int. J. Mod. Phys. B 6, 3205 (1992); Phys. Rev. B 47, 7295 (1993); Phys. Rev. B 48, 15189 (1993); Int. J. Mod. Phys. B 8, 2111 (1994).

[6] Z.F. Ezawa, Phys. Rev. B 51, 11152 (1995).

[7] S.Q. Murphy, J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 72, 728 (1994).

[8] S.M. Girvin, A.H. MacDonald and P.M. Platzman, Phys. Rev. B 33, 2481 (1986); S.M. Girvin and T. Jach, Phys. Rev. B 29, 5617 (1984).

[9] K. Yang, K. Moon, L. Zheng, A.H. MacDonald, S.M. Girvin, D. Yoshioka and S.C. Zhang, Phys. Rev. Lett. 72, 732 (1994); K. Moon, H. Mori, K. Yang, S.M. Girvin, A.H. MacDonald, L. Zheng, D. Yoshioka and S.C. Zhang, Phys. Rev. B 51, 5138 (1995).

[10] B.I. Halperin, Helv. Phys. Acta 56, 75 (1983).

[11] V. Kac and A. Radul, Comm. Math. Phys. 157, 429 (1993); H. Awata, M. Fukuma, Y. Matsuo and S. Odake, Progr. Theor. Phys. Suppl. 118, 343 (1995).

[12] K. Shizuya, private communication.

[13] Z.F. Ezawa, Phys. Rev. B 55, 7771 (1997).