Nonadditivity of quantum and classical capacities for entanglement breaking multiple-access channels and butterfly network

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We analyze quantum network primitives which are entanglement-breaking. We show superadditivity of quantum and classical capacity regions for quantum multiple-access channel and quantum butterfly network. Since the effects are especially visible at high noise they suggest that quantum information effects may be particularly helpful in the case of the networks with occasional high noise rates. To our knowledge the present effects provide the first qualitative borderline between superadditivities of bipartite and multipartite systems.

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Introduction. Fundamental discoveries of quantum cryptography without\textsuperscript{1} and with entanglement\textsuperscript{2} , quantum dense coding\textsuperscript{3} and quantum teleportation\textsuperscript{4} constitute cornerstones of the domain called quantum channel theory\textsuperscript{5,6}. Very important, purely quantum phenomena are superadditivities of capacities in multipartite variants of quantum capacity $Q$ with classical side-channel\textsuperscript{7} (cf.\textsuperscript{8}). One of the newly observed effects was nonadditivity of classical capacity $C$ of multiple-access channel with no side resource\textsuperscript{9} (see\textsuperscript{10} for continuous variables analog). Recently, a fundamental, most striking superadditivity in bipartite scenario for quantum capacity $Q$ with no side resources was discovered\textsuperscript{11} and followed by an announcement of another surprising phenomenon of breaking additivity of secret key capacity $K$\textsuperscript{12} which can be refined to extreme cases\textsuperscript{13} (cf.\textsuperscript{14}). Very challenging open problems is additivity of classical capacity $C$ in bipartite scenario. The conjecture of additivity of so called Holevo capacity $\chi(\Lambda)$ has been disproved recently in an impressive way\textsuperscript{15} where superadditivity for two channels was proven. The problem of additivity of the capacity $C(\Lambda)$ is still open since the latter is an asymptotic quantity. During the research on that fascinating issue it has been shown in particular that bipartite channels which are entanglement-breaking\textsuperscript{16} (i.e., channels which cannot create entanglement between sender and receiver) cannot contribute to such superadditivity phenomena\textsuperscript{17,18}.

In the present paper we address the question whether superadditivity of capacity of entanglement breaking channels is valid in multipartite scenarios. We find, quite surprisingly, that it is not true: both $Q$ and $C$ (i.e., quantum and classical capacities without side resources) in the case of two-access entanglement breaking channels may exhibit superadditivity when supplied with a highly entangling channel. We show also strong nonadditivity of capacity in the quantum butterfly network\textsuperscript{19}. None of the present effects can have an analog in bipartite scenarios. In this way, our result provides for the first time the superadditivity effects sharply discriminating between bipartite scheme and that with more than two users.

For classical capacities our quantum networks violate a special rule which is valid for all discrete classical networks and follows immediately from the additivity theorem provided in\textsuperscript{19}: in any classical multiple-access network primitive it is impossible to improve the transfer rate of one sender by adding resources to another sender. Here we shall call it the locality rule (LR) of data transfer.

Multiple-access entanglement breaking channels and superadditivity. Let us present a pair of channels for which one has superadditivity of quantum capacity. The first channel is presented in Fig. 1. Alice and Bob have $d$ dimensional inputs, while Charlie has $d$ dimensional output. The channel performs the Bell measurement on two qudits and sends a result of the measurement to Charlie. Formally our channel can be written as a completely positive trace preserving linear map

$$\Lambda(\rho_{AB}) = \sum_i \mathrm{Tr}_B (|\Psi^i\rangle\langle\Psi^i|_{AB} |\Psi^i\rangle_{AB} |\Psi^i\rangle_{CB} |\Psi^i\rangle_{C}, \quad (1)$$

where $|\Psi^i\rangle_{AB}$ are $d^2$ orthogonal Bell states. Because $|\Psi^i\rangle_{AB}$ are Kraus operators of rank one, the channel is entanglement breaking. Hence, the quantum capacity region of this channel is given by $R_A = 0$ and $R_B = 0$. The second channel is the identity qudit channel from Bob to Charlie. Its quantum capacity region is given by $R_A = 0$ and $R_B \leq \log d$.

We now find quantum capacity region of the tensor product of these two channels. Let Bob send half of the maximally entangled pair of qudits through the first channel and the other half through the second channel and let Alice send a qudit through the first channel. Because the first channel measures a qudit sent by Alice and a qudit from the maximally entangled state in the Bell basis and sends a result of the measurement to the receiver, it effectively teleports a qudit sent by Alice to
the output of the second channel. Hence, the rate pair
\((R_A, R_B) = (\log d, 0)\) can be achieved. On the other
hand, \(R_A + R_B\) cannot be greater than \(\log d\) because the
first channel performs the complete von Neumann mea-
surement on two qudits. In consequence, the quantum
capacity region of the tensor product of these two chan-
nels is given by
\[
R_A + R_B \leq \log d. \tag{2}
\]

Our channel is an entanglement breaking channel in
contrast to the channel considered in Ref. [9], which
shows nonadditivity of classical capacity regions. One
may wonder if it is possible to show nonadditivity of clas-
sical capacity regions for entanglement breaking channel
and some other channel. Below we demonstrate such a
pair of channels. The first channel is presented in Fig.
2. Alice and Bob have \(d^2\) and \(d\) dimensional inputs,
respectively, while Charlie has \(d\) dimensional output. The
channel transmits a qudit from Bob to Charlie. Depend-
ing on the state of Alice’s qudit, the state of Bob’s qudit
is transformed by one of \(d^2\) unitary operations used in
dense coding protocol. After this transformation, Bob’s
qudit is sent through the depolarizing channel
\[
D_x(q) = (1 - x)\rho + x \frac{I}{d}, \tag{3}
\]
while Alice’s qudit is discarded. For \(x \geq \frac{d}{d^2+1}\), the
depolarizing channel, and hence also our channel, is entan-
glement breaking. The classical capacity region of this
channel is given by \(R_A + R_B \leq C\). \(C\) is Holevo capacity
of the depolarizing channel \(D_x\) and is given by formula
\[
C = \log d - H_d(1 - x \frac{d - 1}{d}), \tag{4}
\]
where \(H_d(x) = -x \log x - (1 - x) \log \frac{1 - x}{d - 1}\). The second
channel is the identity qudit channel from Bob to Charlie.
Its classical capacity region is given by \(R_A = 0\) and \(R_B \leq \log d\).

We now turn our attention to classical capacity re-

gion of the tensor product of these two channels. When
Bob sends half of the maximally entangled pair of qudits
through the first channel and the other half through the
second channel, then Alice can transform the maximally
entangled state to one of \(d^2\) orthogonal states by inputing
to the first channel one of \(d^2\) orthogonal states. Because
the first qudit from the maximally entangled state is sent
through the depolarizing channel and the second qudit is
sent through the identity channel, the parties can achieve
in this way the rate pair \((R_A, R_B) = (C_E, 0)\). \(C_E\) is en-
tanglement assisted classical capacity of the depolarizing
channel \([20]\) and is given by formula
\[
C_E = 2\log d - H_d(1 - x \frac{d^2 - 1}{d^2}). \tag{5}
\]

Alice cannot send more than \(C_E\) bits of information as
she does not control the input to the second channel and
hence the entanglement assisted classical capacity of the
depolarizing channel is the maximal capacity which can
be achieved. On the other hand, \(R_A + R_B \leq \log d + C\) be-
cause it cannot be greater than Holevo capacity of the
tensor product of the depolarizing channel and the
identity qudit channel. Hence, two extreme points of
classical capacity region of the tensor product of these
two channels are given by
\[
(R_A, R_B) = (C_E, 0),
\]
\[
(R_A, R_B) = (0, C + \log d). \tag{6}
\]

These extreme points prove nonadditivity of capacity re-
gions. If \(x \to 1\) then \(C_E/C \to d + 1\) and we can have
arbitrarily large superadditivity of the capacity regions.

**Noisy extensions.** It is worth noting that one can
table two natural modifications of the channel which
declare nonadditivity of quantum capacity. (i) The
first one is a mixture of the Bell measurement which hap-
pens with probability \(1 - q\) and classical uniform noise
which happens with probability \(q\). Together with the
identity qudit channel from Bob to Charlie, this channel
can simulate the quantum depolarizing channel \(D_q\) from
Alice to Charlie. In fact, with probability \(1 - q\) Charlie
can completely recover a quantum message while with
probability \(q\) he is left with the completely random noise
coming from part of the singlet state (apart from com-
pletely useless classical uniform noise). Hence, in this
case one can achieve \(R_A = Q(D_q)\). (ii) Suppose that
instead of the just descried channel, we have a mixture

**FIG. 1:** Entanglement breaking multiple-access channel. \(BM\)
stands for Bell measurement.

**FIG. 2:** Entanglement breaking multiple-access channel. \(U_i\)
stands for controlled unitary operation, \(D_x\) stands for de-
polarizing channel.
which may contain many systems at the local sender's system or, more generally, the sender's site. However, each of the states is just the identity qudit channel from Bob to Charlie then one can achieve $R_A = \log d$.

General networks: amplifying swapping transfer and quantum version of the butterfly network. Consider the channel $\Phi$ provided in Fig. 3. Each sender has $d^2$-dimensional classical input and $d$-dimensional quantum one. Since here we deal with quantum channels which have more than one sender, we may also include the common information rate [21], i.e., the rate of the same information that is faithfully transferred to both receivers $\tilde{A}$, $\tilde{B}$. We denote the common information rate by $R_X^{(o)}$, where $X = \{A, B\}$ stands for the single sender’s system or, more generally, the sender’s site which may contain many systems at the local sender disposal. The total rate vector is denoted by $\mathbf{R} = (R_{AA}, R_{AB}, R_{BB}, R_{B\tilde{B}}, R_A^{(o)}, R_{B}^{(o)})$. We must stress here that this description is more detailed than the one usually used (cf. [19]). In fact, one often analyzes only rates $R_{AB}$, $R_{B\tilde{B}}$ for the fixed values of $R_{AA}$, $R_{BB}$ which are assumed to contain also common information which is not counted separately. We keep here all rates since it is more natural taking into account the structure of the channel we consider. From the fact that just before both outputs of the channel we have depolarizing channels $D_x$, it follows that the total capacity region of the channel is contained in the set $\mathcal{S}$ satisfying the following conditions:

$$R_{AA} + R_{B\tilde{A}} + R_A^{(o)} \leq C,$$

$$R_{AB} + R_{B\tilde{B}} + R_B^{(o)} \leq C.$$  

Thus, we have in short $C(\Phi) \subset \mathcal{S}$. Suppose now that we assist the channel with product of two identity channels

$$\Theta_{A'B'\rightarrow \tilde{A'}\tilde{B}'} = I_{A'\rightarrow \tilde{A'}} \otimes I_{B'\rightarrow \tilde{B}}.$$  

This channel has clearly the transmission rate region $C(\Theta)$:

$$R_{A'\tilde{A'}} \leq \log d,$$

$$R_{B'\tilde{B}'} \leq \log d. \quad (8)$$

Consider the special strategy achieving particularly interesting transmission rates for the butterfly network from Fig. 3 assisted by two identity channels (see Fig. 4 for the assistance scheme). Any message $a$ by Alice and $b$ by Bob can be sent down their classical input of the channel and at the same time can be encoded by $((U_{AA}^a) \otimes I_{A'})|\Psi^{+}\rangle_{AA'}$ and $((U_{BB}^b) \otimes I_{B'})|\Psi^{+}\rangle_{BB'}$, where $|\Psi^{+}\rangle_{XX'}$ is $d \otimes d$ maximally entangled state and $U_{AA}^a$ is one of $d^2$ unitary operations which are used in dense coding protocol. The channel will effectively send the first half of the state $(U_{AA}^a \otimes I_{A'})|\Psi^{+}\rangle_{AA'}$ through the depolarizing channel and the second half through the identity channel to Alice’s receiver’s side and at the same time it will effectively send the first half of the state $(U_{BB}^b \otimes I_{B'})|\Psi^{+}\rangle_{BB'}$ through the depolarizing channel and the second half through the identity channel to Bob’s receiver’s site. However, each of the states is just the same as if it was coming out of bipartite entanglement assisted quantum depolarizing channel as in the previous paragraph. Hence, both senders achieve now the cross-transfer rates (here the subscripts denote sides and not the systems which must have been marked by additional $X$ notations):

$$R_{AB} = C_E,$$

$$R_{B\tilde{A}} = C_E.$$  

where $C_E > C$. The other rates in vector $\mathbf{R}$ are equal to zero in the case of these states.

To compare the effect with the classical case, we should prove that it is impossible in a classical network. To show this, consider the part of the network with one of the receivers traced out, for example tracing out Bob’s
receiver’s parts $\tilde{B}$, $\tilde{B}$'. Since the input local messages are independent, a classical analog of such a remaining network primitive (i.e. the one with two senders and one receiver) must obey our locality rule (LR). This says immediately that all what the classical network may offer in bits transmitted from $B$ to $\tilde{A}$ in this case, is $C$ in $\mathcal{E}$ (instead of $C_E$) which is the original bound \cite{Bennett84}. To see it more clearly, let us notice that the additional noiseless $d$-ary forward channel from $A'$ to $\tilde{A'}$ cannot improve the transfer rate $R_{B,\tilde{A}}$ (according to the locality rule applied to this 2-access channel with two senders $A$, $B$, and one receiver $\tilde{A}$), so the latter must remain equal to $C$ as if the new connection $A' \rightarrow \tilde{A'}$ did not exist. The above remark is completely independent of possible internal machinery of the 2-access channel considered as long as it is classical. In particular, it is obeyed by the original XOR gate. This clearly proves that superadditivity of this type cannot happen in classical networks.

Conclusions. Superadditivities of all kinds found so far in quantum scenarios (irrespective of whether they were bipartite or multipartite) required that both channels may create entanglement. For quantum capacities this rule is well understood in the case of bipartite scenario. On the one hand, entanglement breaking channel can be simulated with the help of forward classical communication \cite{Bennett01}. On the other hand, forward classical communication cannot increase quantum capacity of the channel \cite{Bennett03}. For classical capacities of bipartite channels the rule was proven independently in \cite{Bennett04}, where it was shown that Holevo capacity is additive on a tensor product of two channels, when one of the channels is entanglement breaking. It could be expected that the rule could be generalized to multipartite networks. Here we have shown that this is not the case. We have considered two types of primitives for quantum networks: 2-access channels, i.e. one with two senders and one receiver and the butterfly network. We have proven that even if one channel or network is entanglement breaking, the superadditivity effect may still hold for both classical and quantum capacities if other channels have their transmission rates good enough (the identity channels may be perturbed by small noise and still our results hold by simple continuity arguments). Usually one looks for the effects that discriminate between different types of communication resources. For instance, multipartite entanglement is different from bipartite entanglement since there are nonequivalent types of multipartite entanglement (GHZ and W states). We may ask about the qualitative differences between bipartite and multipartite communication. So far it seemed that all superadditivity effects found in the multipartite case had their, much harder to find, but of similar type, analogs in bipartite scenario. The present superadditivity effects for entanglement-breaking channels are the first ones that sharply discriminate between bipartite and multipartite scenarios, i.e., they cannot happen in bipartite scenarios. Finally, we note that the size of the amplification at high noise rates makes it interesting for applications in occasionally very noisy communication systems.

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