Relational Hypersequent $\mathbf{S4}$ and $\mathbf{B}$ are Cut-Free Hypersequent Incomplete

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1 Introduction

Andrew Parisi’s relational hypersequent systems for standard modal logics $\mathbf{K}$ through to $\mathbf{S5}$ [10, 11] are the first candidate hypersequent systems to meet two commonly cited criteria for “good” proof systems for modal logic: modularity and Došen’s Principle. Parisi’s systems are intended to both provide the basis for an inferentialist account of modality and meet other criteria such as cut-admissibility and adequacy with regard to Kripke frames for these logics. In [10, 11], Parisi provides an indirect proof of sequent completeness for his systems via a translation to sequent systems for the respective modal logics. However, the proofs for his relational hypersequent $\mathbf{S4}$ and $\mathbf{B}$ require treating Cut as a basic rule. Samara Burns and Richard Zach [5] have improved on these results by providing direct cut-free proofs of hypersequent completeness for relational hypersequent $\mathbf{K}$, $\mathbf{T}$ and $\mathbf{D}$, and Restall [14, 15] has done the same for a system equivalent to Parisi’s relational hypersequent $\mathbf{S5}$. The current paper shows that Parisi’s relational hypersequent $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{KB}$, and $\mathbf{B}$ are cut-free hypersequent incomplete, and that the former two are also cut-free sequent and formula incomplete, relative to standard Kripke frames for $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{KB}$ and $\mathbf{B}$ respectively. As a result, the systems fail to meet the criteria of cut-admissibility and adequacy with regard to Kripke frames. This leaves open the question of what hypersequent proof systems can meet Parisi’s intended criteria and also of what kind of models Parisi’s relational hypersequent $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{KB}$, and $\mathbf{B}$ are adequate in regard to.

We begin in Section 2 by providing a brief overview of some of the criteria considered in the literature for “good” proof systems for modal logics. Next

\footnote{There are of course many other kinds of proof systems for modal logics. For example, display logics [17] and labelled sequent systems [8]. See also footnote [4] for tree hypersequent systems.}
in sections 2.1, 2.2 and 2.3 we respectively define the language, models and proof systems that will be studied in the paper. In Section 3 we show that relational hypersequent K4 and S4 are cut-free hypersequent, sequent and formula incomplete relative to Kripke frames for S4 and K4 respectively. In Section 4 we prove that relational hypersequent KB and B are cut-free hypersequent incomplete. We then contrast the two sets of results in Section 5 before Section 6 concludes with a brief discussion of remaining open questions and consequences of these results.

2 Background

Despite its axiomatic origins, contemporary work in modal logic is overwhelmingly model-theoretic. Work in the proof theory of modal logic has been focused on developing proof systems that are both adequate for different classes of models and which have proof-theoretically desirable properties. This work has various motivations, one of which is to provide the basis for an inferentialist account of modality. Inferentialism is a theory of meaning which claims that meaning is determined by norms governing the use of expressions. There is a natural fit between inferentialism and proof-theoretic approaches to semantics, as a proof theory can be interpreted as a formal representation of norms governing the use of expressions in a given language. One way for inferentialists to account for modality is to construct proof systems for modal logics that can be interpreted as determining the meaning of modals expressions like necessity □ and possibility ◊. Parisi [10, 11] uses this to motivate several criteria for proof systems for modal logics, two of which are particularly relevant for the current paper:

- Cut Admissibility: in a sequent calculus or similar setting, the resulting logic from the cut-free calculus is identical to that from the calculus with Cut as a basic rule and

- Došen’s Principle: this principle applies to a set of calculi for modal logics and holds when the operational rules are shared, with calculi only differing in their structural rules.

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2See [1] for an overview of the development of modern modal logic; [4, 16] for an overview of inferentialism; and [9, 13] for accessible overviews of work in the proof theory of modal logics. [10, 11, 12] also contains a discussion of and references to existing literature.

3In a natural deduction setting this would be the requirement that the system normalises.
One way to think about Došen’s Principle is as a proof-theoretic analog of the way in which standard Kripke models for different modal logics share truth conditions for connectives but differ in the restrictions placed on the accessibility relation. A related but different criterion is that of modularity. Burns and Zach state it in a form directly relating proof systems to Kripke models:

- Modularity: “each property of the accessibility relation [of a Kripke model] is captured by a single rule or set of rules.” [5, p.2]

These properties are often considered desirable but are by no means universally endorsed. For example, see [12, Chapter 1.10] for an argument against accepting Došen’s Principle.

2.1 Language

Definition 2.1 ($\mathcal{L}$). $\mathcal{L}$ is the language made up of denumerably many atomic formula $p, q, ...$, the unary connectives $\neg$ and $\Box$, and the binary connectives $\land$ and $\lor$, and whose sentences are all and only those generated recursively from the following rule: all atomic formulas $p$ are sentences and if $\phi$ and $\psi$ are sentences then so are $\neg \phi$, $\Box \phi$, $\phi \land \psi$ and $\phi \lor \psi$.

We will use lower case Greek as sentence variables and upper case for sets of sentences.

Definition 2.2 (Sequents and Hypersequents). A sequent $\Gamma \Rightarrow \Delta$ is an ordered pair of finite sets of sentences, with the turnstile $\Rightarrow$ separating each member of the pair. $\Gamma, \phi$ will be written as shorthand for $\Gamma \cup \{\phi\}$. Instead of writing the empty set $\emptyset$ we simply leave the relevant side of the turnstile blank. $S$, possibly subscripted, is used to represent arbitrary sequents in the metalanguage.

A hypersequent $S_1 / / ... / / S_n$ is a finite sequence (list) of sequents, with $/ /$ separating each member of the sequence. $G$ and $H$, possibly “primed”, are used to represent arbitrary hypersequents in our language.

4 Poggiolesi’s tree hypersequent systems [12] are also motivated by inferentialism. In rejecting Došen’s Principle, Poggiolesi states the principle slightly differently and also appears to be working with a different distinction between operational (logical) and structural rules. For example, the rule $t$ is classified as a structural rule despite it essentially involving $\Box$ in the conclusion sequent [5, §1.10]. This differs from standard structural rules like weakening and contraction, which do not essentially involve any particular vocabulary, a feature that Parisi’s structural rules do have. Došen’s Principle, as stated above, does not hold for Poggiolesi’s systems because one system is obtained from another by varying both the structural and operational rules.
2.2 Models

**Definition 2.3** (Frames and Models). A Kripke frame $\mathfrak{F}$ is a pair $\langle W, R \rangle$ of points $W$ and a binary relation $R$ on $W$.

A Kripke model $\mathfrak{M}$ is a triple $\langle W, R, v, \rangle$ in which $\langle W, R \rangle$ is a Kripke frame and $v$ is a valuation function from members of $W$ and sentences of $\mathcal{L}$ to the truth values $\{1, 0\}$.

We restrict $v$ as follows:

- $v(\neg \phi, x) = 1$ iff $v(\phi, x) = 0$;
- $v(\phi \land \psi, x) = 1$ off $v(\phi, x) = 1$ and $v(\psi, x) = 1$;
- $v(\phi \lor \psi, x) = 1$ off $v(\phi, x) = 1$ or $v(\psi, x) = 1$; and
- $v(\Box \phi, x) = 1$ iff for all $y$, if $xRy$ then $v(\phi, y) = 1$.

The conditions for $\neg$, $\land$ and $\lor$ are as in Boolean valuations but relative to a point.

We can obtain classes of Kripke frames (models) for various modal logics by placing restrictions on $R$.

**Definition 2.4** (Branch of points). A *branch of points* $w_1, ..., w_n$ is a sequence of points in a frame $\mathfrak{F}$ such that $w_iRw_{i+1}$ for all $1 \leq i \leq n - 1$.

**Definition 2.5** (Countermodel). Sequents: A model $\mathfrak{M}$ is a *countermodel* to a sequent $\Gamma \Rightarrow \Delta$ at a point $w$ iff for all $\phi \in \Gamma$, $v(\phi, w) = 1$ and for all $\psi \in \Delta$, $v(\psi, w) = 0$.

Hypersequents: A model $\mathfrak{M}$ is a *countermodel* to a hypersequent $G$ iff there is a branch of points $w_1, ..., w_n$ in $\mathfrak{M}$ such that $\mathfrak{M}$ is a countermodel to each sequent $S_i \in H$ at $w_i$ for all $1 \leq i \leq n$.

We write $\not\models_X H$ to mean that a particular hypersequent $H$ has a countermodel in the class of $X$ frames, and $\models_X H$ to mean that a particular hypersequent $H$ has no countermodel, i.e. is valid, in the class of $X$ frames.

2.3 Proofs

**Definition 2.6** (The Hypersequent Calculus RK). A derivation in RK is a tree all of whose leaves are instances of the axiom Id and each non-leaf node is obtained from the nodes above via one of the rules of RK (see Figure [1]).
Additional systems are obtained from RK by the addition of further structural rules from Figure 2 as set out in Figure 3.

Figure 2: Additional Structural Rules

\[
\frac{G \parallel \Gamma \Rightarrow \Delta \parallel H}{G \parallel \Gamma \Rightarrow \Delta \parallel H} \text{ EC} \quad \frac{\Gamma_1 \Rightarrow \Delta_1 \parallel \cdots \parallel \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \Rightarrow \Delta_n \parallel \cdots \parallel \Gamma_1 \Rightarrow \Delta_1} \text{ Sym} \quad \frac{G \parallel H}{G \parallel \parallel H} \text{ EW} \\
\frac{G \parallel \Gamma \Rightarrow \Delta \parallel \Sigma \Rightarrow \Lambda \parallel H}{G \parallel \Sigma \Rightarrow \Lambda \parallel \Gamma \Rightarrow \Delta \parallel H} \text{ EE} \quad \frac{G \parallel \parallel G}{G \parallel \parallel \text{ Drop}}
\]
Figure 3: Hypersequent Systems, Logics and Frame Conditions

| System | Additional Rules | Intended Logic. | Intended Frame Conditions |
|--------|------------------|-----------------|---------------------------|
| RD     | Drop             | D               | Seriality                 |
| RT     | EC               | T               | Reflexivity               |
| RKB    | Sym              | KB              | Symmetry                  |
| RK4    | EW               | K4              | Transitivity              |
| RB     | EC and Sym       | B               | Reflexivity and Symmetry  |
| RS4    | EC and EW        | S4              | Reflexivity and Transitivity |
| RS5    | EC, EW and EE    | S5              | Reflexivity, Symmetry and Transitivity |

To say that a particular hypersequent $H$ has a derivation in a particular system $RX$, we write $\vdash_{RX} H$. $\not\vdash_{RX} H$ means that the hypersequent $H$ has no derivation in $RX$. We write $\vdash_{RX_{cf}} H$ to say that $H$ has a cut-free derivation in $RX$.

Burns and Zach identify Parisi’s relational hypersequent systems as “the first candidates for hypersequent calculi for modal logics that are both modular and conform to Došen’s principle” [5, p.2]. However, as they note, the systems are not completely modular as RS5 is obtained by adding $EE$ to RS4 rather than adding $Sym$. The fact that RS4 and RB turn out to be cut-free hypersequent incomplete is plausibly connected to this lack of modularity.

2.4 Completeness

Parisi’s systems are sound relative to standard Kripke frames, in the sense that whenever a hypersequent is provable, there is no counterexample [11]. The converse of this, completeness, is the focus of the paper.

Definition 2.7 (Completeness). A hypersequent calculus $RX$ is $Y$-Complete relative to a class of frames $S$ iff whenever $\vDash_{S} Y$ then $\vdash_{RX} Y$. When $Y$ stands in for: arbitrary hypersequents we say that $RX$ is Hypersequent-Complete (H-Complete); hypersequents of the form $\Gamma \Rightarrow \Delta$ that $RX$ is

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5The systems presented in this paper are strictly speaking Burns and Zach’s. Parisi’s lack the EWR rule that is needed for hypersequent completeness and Parisi refers to his systems with the prefix ‘H’ rather than ‘R’. Burns and Zach, and this paper, also use Lellman’s notation from [5]. Disjunction $\vee$ and its corresponding rules have been added to make $C$, the counterexample to RK4 and RS4’s completeness, more perspicuous. This is only needed to make the proof simpler, as disjunction can be defined using conjunction and negation as usual.

6Interestingly, this lack of modularity in the move from $S4$ to $S5$ is shared by Poggi-olesi’s tree hypersequent systems [12] p.125-6].
Sequent-Complete (S-Complete); and hypersequents of the form \( \Rightarrow \phi \) that 
RX is Formula-Complete (F-Complete).

A hypersequent calculus RX is Cut-Free Y-Complete (CF Y-Complete) relative to a class of frames \( S \) iff whenever \( \models_S Y \) then \( \vdash_{RXCF} Y \).

Figure 4: State of Play

| System | S-Com | H-Com | CF F-Com | CF S-Com | CF H-Com |
|--------|-------|-------|----------|----------|----------|
| RK     | Y (P) | Y (B&Z) | Y (P) | Y (P) | Y (B&Z) |
| RD     | Y (P) | Y (B&Z) | Y (P) | Y (P) | Y (B&Z) |
| RT     | Y (P) | Y (B&Z) | Y (P) | Y (P) | Y (B&Z) |
| RKB    | Y (P) | ?     | ?     | ?     | N        |
| RB     | Y (P) | ?     | ?     | ?     | N        |
| RK4    | Y (P) | Y     | N     | N     | N        |
| RS4    | Y (P) | Y     | N     | N     | N        |
| RS5    | Y (P) | Y (P) | Y (P) | Y (P) | Y(P)     |

Figure 4 contains the current state of play when it comes to completeness results for Parisi’s relational hypersequent systems for modal logic, leaving out Formula-Completeness as a distinct category. Parisi has proved sequent-completeness for all his systems, in some cases cut-free and in others only for the system with Cut as a basic rule. Burns and Zach have shown direct cut-free hypersequent completeness for RK, RD and RT (from this it follows that these systems are cut-free sequent complete and hypersequent complete). The cells marked N are answered in the negative in the current paper: RK4, RS4, RKB and RB are all cut-free hypersequent incomplete, and the former two are also cut-free sequent and formula incomplete. Sequent and formula completeness remains open for RKB and RB. The cells marked Y are answered positively in Appendix A: RK4 and RS4 with Cut as a basic rule are not only sequent complete, but also hypersequent complete.

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Parisi’s proves sequent completeness for RK, RD and RT without using Cut (hence cut-free). The proof of sequent completeness of RS5 does use Cut, as does that for RKB and RS4. These proofs work by showing that his systems are sequent equivalent to a sequent system for K, D, T and S5 respectively, which are already known to be sequent complete. For RS5, he then shows that Cut is an admissible rule in RS5 by showing that it is hypersequent equivalent to a cut-free S5 hypersequent system of Restall’s. Cut-free sequent completeness then follows from cut-free hypersequent completeness.
3 RK4 and RS4 Cut-Free Incompleteness

In this section we prove that RK4 and RS4 are cut-free hypersequent, sequent and formula incomplete. The outline of the proof is as follows: first, we identify a hypersequent $C$ of the form $\Rightarrow \phi$ that is $K4$ and $S4$ valid; second, we define a class of models, PS4 models, relative to which $C$ is invalid; third, we show that cut-free RK4 and RS4 are sound relative to PS4 models, resulting in both of RK4 and RS4 being cut-free formula incomplete; fourth, as an immediate consequence, cut-free RK4 and RS4 are both also sequent and hypersequent incomplete.

To begin the proof, we show that $C = \Rightarrow \neg \square \neg \square (p \land q) \lor \square (\neg \square p \lor \square \neg \square q)$ is $K4$ and $S4$ valid.

**Lemma 3.1.** $\models_{K4} C = \Rightarrow \neg \square \neg \square (p \land q) \lor \square (\neg \square p \lor \square \neg \square q)$

**Proof.** The proof is a *reductio* of the assumption that there is a counter-model. Once the $\lor$s and $\square$s have been decomposed, for there to be a countermodel, we need three points, call them $i$, $j$ and $k$, where $v(\neg \square (p \land q), i) = 1 = v(\square p, j) = v(\square q, k) = 1$, $iRj$ and $jRk$. It follows that $iRk$ and $v(\neg \square (p \land q), k) = 1$. So there must be a point $m$, where $kRm$, $v(\neg (p \land q), m) = 1 = v(q, m)$. But then $jRm$ also, and $v(p, m) = 1 = v(p, m) \neq 0 = v(p \land q, m)$.

**Lemma 3.2.** $\models_{S4} C = \Rightarrow \neg \square \neg \square (p \land q) \lor \square (\neg \square p \lor \square \neg \square q)$
Figure 6: Additional Restrictions on $R$ (straight lines) and $S$ (zigzag lines)

(a) Pseudo-Transitivity

(b) Forth

(c) Back

Proof. Every S4 model is also a K4 model. So, if there were an S4 countermodel to $C$, there would be a K4 countermodel. From Lemma 3.1, there is no K4 countermodel. Hence, there isn’t an S4 one either. 

We next define a new class of models, PS4 models; show that both RK4 and RS4 are sound relative to these models; and that there is a PS4 model that is a countermodel to $C$.

Definition 3.1 (PS4 Frames). A Pseudo S4 (PS4) frame is a triple $\langle W, R, S \rangle$ where $W$ is a non-empty set of points and both $R$ and $S$ are binary relations on $W$.

We set the following restrictions on $R$ and $S$, displayed in Figure 6:

1. $S$ Reflexivity: For all points $x$, $xSx$;
2. $R$ Reflexivity: For all points $x$, $xRx$;
3. Pseudo-Transitivity: For all points $x$, $y$, and $z$: if $xRy$ and $yRz$, then there is a $w$: $xRw$ and $zSw$;
4. Forth: For all points $x$, $y$, and $z$: if $xRy$ and $xSz$, then there is a point $w$: $zRw$ and $ySw$;
5. *Back*: For all points \( x, z, \) and \( w \): if \( xSz \) and \( zRw \), then there is a point \( y \): \( xRy \) and \( ySw \).

Having defined PS4 frames, we will use the following definition of an information order on points in a model to then define PS4 models in Definition 3.3.

**Definition 3.2 (Information Order).** Given two points in a model \( x \) and \( y \), \( x \sqsubseteq y \) iff for all atomics \( p \), if \( v(x, p) \in \{1, 0\} \), then \( v(y, p) = v(x, p) \). When \( x \sqsubseteq y \) we say that \( x \) is earlier than \( y \) in the information order.

**Definition 3.3 (PS4 Models).** A Pseudo-S4 (PS4) model is a quadruple \( \langle W, R, S, v \rangle \) where \( \langle W, R, S \rangle \) is a PS4 frame and \( v \) is a valuation function from pairs of a point and a formula to \( \langle 1, *, 0 \rangle \). We set the following restrictions on \( v \):

1. **Strong Kleene:** \( v \) uses standard modal Strong Kleene truth conditions:
   
   - \( ¬1: v(¬\phi, x) = 1 \) iff \( v(\phi, x) = 0 \);
   - \( ¬0: v(¬\phi, x) = 0 \) iff \( v(\phi, x) = 1 \);
   - \( ∧1: v(\phi ∧ \psi, x) = 1 \) iff \( v(\phi, x) = 1 \) and \( v(x, \psi) = 1 \);
   - \( ∧0: v(\phi ∧ \psi, x) = 0 \) iff \( v(\phi, x) = 0 \) or \( v(x, \psi) = 0 \);
   - \( ∨1: v(\phi ∨ \psi, x) = 1 \) iff \( v(\phi, x) = 1 \) or \( v(\psi, x) = 1 \);
   - \( ∨0: v(\phi ∨ \psi, x) = 0 \) iff \( v(\phi, x) = 0 \) and \( v(\psi, x) = 0 \);
   - \( □1: v(□\phi, x) = 1 \) iff for all \( y \): if \( xRy \) then \( v(\phi, y) = 1 \);
   - \( □0: v(□\phi, x) = 0 \) iff there is a \( y \): \( xRy \) and \( v(\phi, y) = 0 \).

   Note that these match “classical” Kripke models from Definition 2.3 for 1 and 0, but leave a “gap” for *.

2. **S Information Preservation (S_\sqsubseteq):** For all points \( x, \) and \( y \): if \( xSy \), then \( x \sqsubseteq y \).

   The S Information Preservation condition in Definition 3.3 means that \( S \) preserves the information order. We now show that \( S \) preserves the truth and falsity of formulae in general, rather than just atomics. This fact will be used in the proof of the soundness of RK4 and RS4 relative to PS4 models.

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8 *Forth and Back are standard bisimulation conditions, sometimes known as Zig and Zag respectively. See [2] §2.2 for an accessible introduction to bisimulation in modal logic.*
Lemma 3.3. If \( xSy \), then for all formulae \( \phi \), if \( v(\phi, x) \in \{1,0\} \) then \( v(\phi, y) = v(\phi, x) \)

Proof. We prove this by induction on the complexity of \( \phi \). For atomic sentences, it follows immediately from the \( S \) preservation condition on \( v \).

The extensional connectives are fairly simple.\(^9\) We work through conjunction, leaving negation and disjunction to the reader. We have two subcases. Let \( \phi = \psi \land \delta \):

1. \( v(\psi \land \delta, x) = 1 \). It follows that \( v(\psi, x) = 1 = v(\delta, x) \). So, by the induction hypothesis \( v(\psi, y) = 1 = v(\delta, y) \). Hence, \( v(\psi \land \delta, y) = 1 \) also.

2. \( v(\psi \land \delta, x) = 0 \). It follows that either \( v(\psi, x) = 0 \) or \( v(\delta, x) = 0 \). So, by the induction hypothesis either \( v(\psi, y) = 0 \) or \( v(\delta, y) = 0 \) respectively. Hence, \( v(\psi \land \delta, y) = 0 \) also.

Necessity \( \Box \) is the trickier case and here the bisimulation conditions play a role. We have two subcases. Let \( \phi = \Box \psi \):

1. \( v(\Box \psi, x) = 1 \). By assumption \( xSy \). We need to show \( v(\Box \psi, y) = 1 \). For this, we need to first show that for any \( z \), if \( yRz \) then \( v(\psi, z) = 1 \). Suppose there is some such \( z \). By \textit{Back}, it follows that there is a \( w \) such that \( xRw \) and \( wSz \). \( v(\psi, w) = 1 \) and therefore by the induction hypothesis, \( v(\psi, z) = 1 \) also. Hence, \( v(\Box \psi, y) = 1 \). The condition holds.

2. \( v(\Box \psi, x) = 0 \). Therefore, there is a \( w \) such that \( xRw \) and \( v(\psi, w) = 0 \). By assumption \( xSy \). We need to show \( v(\Box \psi, y) = 0 \). For this, we need to show that there is a \( z \) where \( yRz \) and \( v(\psi, z) = 0 \). By \textit{Forth}, there is a \( z \) where \( yRz \) and \( wSz \). It follows from the induction hypothesis that \( v(\psi, z) = 0 \). Hence, \( v(\Box \psi, y) = 0 \). The condition holds.

The following lemma will also be used in the proof of RK4 and RS4’s soundness relative to PS4 models, specifically for \( EW \).

Lemma 3.4. If in a model there is a branch of points \( w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_n \) then in the same model there is a branch \( w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n \), where for all \( j, i + 1 \leq j \leq n, w_j S w_j \).

\(^9\)See [3, §1.2, §6.2] and [7, p.49] for discussion.
Proof. Suppose there is a branch \( w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_n \). By Pseudo-Transitivity there must be a point \( w_{i+1}' \) such that \( w_{i-1}Rw_{i+1}' \) and \( w_{i+1}Sw_{i+1}' \). By Forth, for all \( j, i + 2 \leq j \leq n \) such that \( w_{j-1}Rw_j \) and \( w_{j-1}Sw_{j-1} \), then there is a \( w_j' \) such that \( w_{j-1}'Rw_j' \) and \( w_jSw_j' \). \( i + 1 \) iterations of this consequence of Forth will result in the desired branch.

What Lemma 3.4 tells us is that whenever Pseudo-Transitivity requires that we make an informational “copy” of a point \( z \), we also make an informational “copy” of each branch of points from \( z \) onwards. This is will be essential for the soundness of the EW rule in the following lemma 3.5.

Lemma 3.5. If \( \vdash_{RS4_{CP}} H \) then \( \models_{PS4} H \)

Proof. The proof proceeds by induction on the length of derivations. Much of this proof is routine. We only explicitly consider EW. The rest are the same as in Parisi [11].

\[
\begin{array}{c}
\vdash G \parallel H \\
\vdash G \parallel \Rightarrow \parallel H \\
\vdash G \parallel H \quad \text{EW}
\end{array}
\]

Suppose there was a countermodel to the endhypersequent. This would be a model with a branch \( w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_n \), where \( w_i \) countermodels \( \Rightarrow \). By Lemma 3.4, there is a branch \( w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n \), where for all \( j, i + 1 \leq j \leq n, w_jSw_j' \). By Lemma 3.3, for all \( j, i + 1 \leq j \leq n \), for all \( \phi \), if \( v(\phi, w_j) \in \{1, 0\} \) then \( v(\phi, w_j') = v(\phi, w_j) \). So, the branch \( w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n \) is a countermodel to the premise hypersequent.

Lemma 3.6. If \( \vdash_{RK4_{CP}} H \) then \( \models_{PS4} H \)

Proof. RK4 differs from RS4 only in lacking the external structural rule EC. So the proof proceeds as for Lemma 3.5 above but without the EC case.

Given lemmas 3.5 and 3.6, if there is a PS4 counterexample to a hypersequent, it will be unprovable in each of RS4 and RK4. We now show that there is a PS4 counterexample to \( C \).

Lemma 3.7. \( \not\models_{PS4} C \implies \not\models \neg \square \neg \square (p \land q) \lor \square (\neg \square p \lor \square \neg \square q) \)

Proof. The model in Figure 7 is a countermodel to \( C \). We have a PS4 frame with:

- six points \( i, j, k, m, n, l \);
• in addition to $R$ being reflexive, we have $iRk, jRk, kRm, iRn, nRl, iRl$;
• in addition to $S$ being reflexive, we have $mSk, mSn, mSl, kSn$.

We set the valuation function $v$ such that:

- $v(q, i) = v(p, i) = v(q, j) = v(p, l) = 0$;
- $v(p, j) = v(p, k) = v(q, k) = v(q, m) = v(p, n) = v(q, n) = v(q, l) = 1$;
- $v(p, m) = *$.

Verification that this is indeed both a PS4 frame and a PS4 model, and that it is a countermodel to $C$ at the branch made up of the single point $i$ is left to the reader. For the verification that it is a countermodel, first identify that $v(□¬□(p ∧ q), i) = 1 = v(□p, j) = v(□q, k)$. The countermodel works by having $iRn$ instead of $iRk$, where $k ⊑ n$, and $v(□(p ∧ q), n) = 0$. The branch $i, n, l$ doesn’t contain $j$ and so we can have $v(□p, n) = 1$ but $v(□q, n) = 0$. It is helpful to compare this with the reasoning in the proof of Lemma 3.1.

We now have what we need to show that RS4 is cut-free incomplete.
Theorem 1. RS4 is cut-free (i) formula, (ii) sequent, and (iii) hypersequent incomplete relative to S4 (transitive and reflexive Kripke) frames.

Proof. (i) follows from lemmas 3.2, 3.5, and 3.7.
From Lemma 3.5 we know that if C were RS4 cut-free provable, then C would be valid in PS4 models. However, from this and Lemma 3.7, we know that C is not cut-free provable in RS4. Yet C is valid in S4 Kripke frames. So, RS4 is cut-free incomplete relative to S4 Kripke frames.
(ii) and (iii) follow immediately from (i).

Theorem 2. RK4 is cut-free (i) formula, (ii) sequent, and (iii) hypersequent incomplete relative to S4 (transitive and reflexive Kripke) frames.

Proof. The reasoning is the same as for Theorem 1 but using lemmas 3.1, 3.6 and 3.7.

4 RKB and RB Cut-Free Incompleteness

The proof in this section has a slightly different structure to that of the previous section. We first identify a hypersequent J that is KB and B valid; second, we define a new proof system RTB and show that J is unprovable in both RKB and RTB; third, we show that the rule EC is admissable in RTB, meaning that anything that is RB provable is also RTB provable. It follows that J is also unprovable in RB, resulting in both RKB and RB being hypersequent cut-free incomplete. The cut-free formula and sequent completeness of the two systems, however, remains open.

Lemma 4.1. $\models_{KB} J \Rightarrow p \Leftrightarrow \square(\neg
\square
\square
p \wedge \neg \square \neg q) \Rightarrow q$.

Proof. The proof is a reductio of the assumption that there is a counter-model. We start with three points, call them i, j and k, where iRj, jRk, $v(p, i) = 0 = v(q, j) = v(\square(\neg \square \square p \wedge \neg \square \neg q), j)$. Because this is a symmetric Kripke frame, jRi and kRj also. Because $v(\square(\neg \square \square p \wedge \neg \square \neg q), j) = 0$ there must be a point, call it m, where jRm, mRj and $v(\neg \square \square p \wedge \neg \square \neg q, m) = 0$. 

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We have two possibilities: one where \( v(\Box\Box p, m) = 1 \) and another where \( v(\Box\Box q, m) = 1 \). In the former, \( v(\Box p, j) = v(i, p) = 1 \neq 0 = v(p, i) \). In the latter, \( v(\Box q, j) = v(q, k) = 1 \neq 0 = v(q, k) \). In each case, a contradiction results. So, there can be no \( \text{KB} \) countermodel.

\[\Box \neg \neg p \implies \neg q\]

\[\neg \neg (\Box p \land \Box q)\]

\[\neg q\]

\[\neg p\]

\[\neg q\]

\[\neg (\Box p \land \Box q)\]

\[m\]

\[\neg (\Box p \land \Box q)\]

Lemma 4.2. \( \models_{\text{B}} p \implies \Box (\neg \neg p \land \neg \neg q) \implies q \).

Proof. Every \( \text{B} \) model is also a \( \text{KB} \) model. So, if there were an \( \text{B} \) countermodel to \( J \), there would be a \( \text{KB} \) countermodel also. From lemma 4.1 there is no \( \text{KB} \) countermodel. Hence, there isn’t a \( \text{B} \) one either.

Figure 8: Reflexivity-like Rules

\[
\frac{G \parallel \Gamma = \Delta \parallel \Gamma \implies \Delta \parallel G'}{G \parallel \Gamma \implies \Delta \parallel G'} \quad_{\text{EC}} \quad \frac{G \parallel \Gamma \implies \Delta \parallel G'}{G \parallel \Gamma \implies \Delta \parallel G'} \quad_{\text{Merge}} \quad \frac{G \parallel \Gamma, \phi \implies \Delta \parallel G'}{G \parallel \Gamma, \Box \phi \implies \Delta \parallel G'} \quad_{\text{T}}
\]

What follows has two parts. One part involves showing that the rule \( EC \) is admissible in RTB, the system that is just like RB except that it has
the rule $T$ as a basic rule instead of $EC$ (see Figure 8). The other part involves showing that $J$ is unprovable in RKB and RTB. We do the latter first and then show that $EC$ is admissible in RTB. It then follows that $J$ is unprovable in RB.

4.1 $J$ is unprovable in RKB and RTB

RKB and RB are known to be sound relative to KB and B Kripke frames respectively [5, 11]. To this we add that RTB is sound relative to B Kripke frames.

Lemma 4.3. If $\vdash_{RTB} H$ then $\models_B H$.

Proof. The proof proceeds by induction on the length of derivations. We only display the $T$ case. We have a derivation $\delta$ of the following form:

\[
\vdots \\
G \parallel \Gamma, \phi \Rightarrow \Delta \parallel G' \\
G \parallel \Gamma, \Box \phi \Rightarrow \Delta \parallel G'^T
\]

Suppose we have a countermodel $\mathfrak{M}$ to the endhypersequent. This is a branch of points $w_1, ..., w_i, ... w_n$ with $w_i$ being a countermodel to the displayed sequent. By the reflexivity condition on B frames, $w_i R w_i$. Hence, by the $\Box$ truth conditions, $v(\phi, w_i) = 1$. This means that our branch of points is also a countermodel to the premise hypersequent.

Lemma 4.4. $\not\vdash_{RTB} J \Rightarrow p \Rightarrow \Box (\neg \Box \Box p \land \neg \Box \Box q) \Rightarrow q$.

Proof. We perform a simple backwards proof search in RTB. For the sake of reductio, suppose we have a proof $\delta$ of $J$. The last rule applied in $\delta$ would need to either be $TR$, with a subcase for each formula, or $Sym$.

Consider the first case, with three subcases. The subproof $\delta'$ of $\delta$ ending immediately before the application of $TR$ would be of one of the following forms:

(i) $\vdots \delta' \\
\Rightarrow \Rightarrow \Box (\neg \Box \Box p \land \neg \Box \Box q) \Rightarrow q$

(ii) $\vdots \delta' \\
\Rightarrow p \Rightarrow \Rightarrow \Rightarrow q$
Each of (i)–(iii) has a simple B countermodel. Hence, via Lemma 4.3 if J were RTB provable, the last step could not be an application of TR. Consider now the second case of Sym. The subproof \( \delta' \) of \( \delta \) ending immediately before the application of Sym would be of the following form ending in the hypersequent \( J' \):

(iv) \[ \vdots \delta' \]

\[ J' \Rightarrow q \Rightarrow \square \neg \square \neg p \land \neg \square \neg q \Rightarrow p \]

The last step of \( \delta' \) could also, like that of \( \delta \), either be TR, with a subcase for each formula, or Sym. The former case is near identical to that of \( \delta \), with simple B countermodels to each possible TR predecessor of \( J' \).

Consider the second Sym case. If the last step of \( \delta' \) were Sym, then the end-hypersequent of the subproof \( \delta'' \) ending immediately before the application of Sym would be \( J \) itself. But the proof couldn’t just be endless iterations of Sym! Eventually there would have to be an application of TR. Yet we’ve just seen that none of the possible TR predecessors of \( J \) nor its converse \( J' \) are RTB provable. Hence, \( J \) is not RTB provable.

\[ \square \]

Lemma 4.5. \( \not\models_{RKB} J \)

Proof. This is an immediate corollary of Lemma 4.3

\[ \square \]

4.2 EC is Admissible in RTB

We now show that EC is admissible in RTB. The first step in this is to show that EC is derivable from the rule Merge.

\[ \vdots \]

\[ G \parallel \Gamma \Rightarrow \Delta \parallel \Gamma \Rightarrow \Delta \parallel G' \]

\[ G \parallel \Gamma \Rightarrow \Delta \parallel G' \]

EC \[ G \parallel \Gamma \Rightarrow \Delta \parallel G' \]

\[ G \parallel \Gamma_1 \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel G' \]

Merge \[ G \parallel \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \parallel G' \]

Lemma 4.6. EC is derivable from Merge

\[ \vdots \]

Proof. \[ G \parallel \Gamma \Rightarrow \Delta \parallel \Gamma \Rightarrow \Delta \parallel G' \]

Merge \[ G \parallel \Gamma, \Gamma \Rightarrow \Delta, \Delta \parallel G' \]

As our sequents are pairs of sets, the conclusion hypersequent is identical to \( G \parallel \Gamma \Rightarrow \Delta \parallel G' \).

\[ \square \]
We next show that Merge is admissible in RTB. For this we use the following modified definitions of a main sentence and main sequent from [10, p.88].

**Definition 4.1 (Main Sentence).** \( \phi \) is the main sentence of TL, TR, \( \neg L \), \( \neg R \), \( \wedge L \), \( \wedge R \), \( \square L \), \( \square R \) or \( T \), if \( \phi \) appears in the conclusion of one of those but not the premise(s). In Id, the main sentence is the only sentence present.

**Definition 4.2 (Main Sequent).** The main sequent of a rule schema is either the sequent containing the main sentence of the rule, or is given by the following list:

- The main sequent of EW\(L\) and EW\(R\) is \( \Rightarrow \); and
- For \( \square L \) and \( \square R \), the left-main sequent is the one containing the main sentence. The right-main sequent is either the one immediately following the left-main sequent or none at all.

**Lemma 4.7.** If \( \Gamma \vdash_{\text{RTB}} \Delta_1 \Rightarrow \Gamma_2 \Rightarrow \Delta_2 \) then \( \Gamma \vdash_{\text{RTB}} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \).

**Proof.** The proof proceeds by induction on the length of derivations. The base case is trivial as the antecedent of the lemma does not hold. For the induction step, there are a lot of cases to check. We display those for the \( \square \) rules and \( \wedge R \). The remainder are left to the reader. In each case we have a derivations \( \delta, \delta' \) of length \( n \) of the premise of the rule, which is extended to a derivation \( \delta'' \) of length \( n + 1 \). We show that if lemma \ref{lemma:admissibility} holds of the conclusion of \( \delta \) then it also holds of the conclusion of \( \delta'' \).

Case 1: \( \square L \). We have four subcases, depending on whether \( \Gamma_1 \Rightarrow \Delta_1 \) or \( \Gamma_2 \Rightarrow \Delta_2 \) are left- or right-main. We display the two subcases where, first \( \Gamma_1 \Rightarrow \Delta_1 \) is left-main and \( \Gamma_2 \Rightarrow \Delta_2 \) is right-main, and second \( \Gamma_1 \Rightarrow \Delta_1 \) is right-main.

Subcase 1: \( \Gamma_1 \Rightarrow \Delta_1 \) is left-main (and \( \Gamma_2 \Rightarrow \Delta_2 \) is right-main). We have a derivation \( \delta' \) of the form

\[
\frac{G \parallel \Gamma_1 \Rightarrow \Delta_1 \parallel \Gamma_2, \phi \Rightarrow \Delta_2 \parallel G'}{G \parallel \Gamma_1, \square \phi \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel G'} \quad \text{\( \square L \)}
\]

\[\text{\footnotesize 10 The order of left and right have been changed from [10] p.88 because of the difference in notation.}\]
We apply the induction hypothesis to $\delta$ and then apply $T$, giving us the desired result
\[ IH\delta \]
\[ G \parallel \Gamma_1, \phi, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \parallel G^t \]
Subcase 2: $\Gamma_1 \Rightarrow \Delta_1$ is right-main. We have a derivation $\delta'$ of the form
\[ \vdots \]
\[ G \parallel \Sigma \Rightarrow \Lambda \parallel \Gamma_1, \phi \Rightarrow \Delta_1, \Delta_2 \parallel G' \]
We apply the induction hypothesis to $\delta$ and then apply $\Box L$, giving us the desired result
\[ \vdots \]
\[ IH\delta \]
\[ G \parallel \Sigma \Rightarrow \Lambda \parallel \Gamma_1, \phi, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \parallel G' \]
The remaining subcases are like Subcase 2 in that $\Box L$ is used after applying the induction hypothesis, rather than $T$.

Case 2: $\Box R$. We have two subcases, one where $\Gamma_2 \Rightarrow \Delta_2$ is left-main and another where neither of $\Gamma_1 \Rightarrow \Delta_1$ or $\Gamma_2 \Rightarrow \Delta_2$ is left-main nor right-main. We display the former and leave the latter to the reader (the reasoning is essentially the same). We have a derivation $\delta'$ of the form
\[ \vdots \]
\[ G \parallel \Sigma \Rightarrow \Lambda \parallel \Gamma_1, \phi \Rightarrow \Delta_1, \Delta_2 \parallel G' \]
\[ G \parallel \Sigma, \Box \phi \Rightarrow \Lambda \parallel \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2 \parallel G' \]
We then assume the induction hypothesis of $\delta$ and apply $\Box R$:
\[ \vdots \]
\[ IH\delta \]
\[ G \parallel \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \Box \phi \]

Case 3: $\land R$. We have three subcases. Two where $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ respectively are main sequents and another where neither are. We display the first.
\[ \vdots \]
\[ G \parallel \Gamma_1 \Rightarrow \Delta_1, \phi \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel G' \]
\[ G \parallel \Gamma_1 \Rightarrow \Delta_1, \phi \land \psi \parallel \Gamma_2 \Rightarrow \Delta_2 \parallel G' \]
We then assume the induction hypothesis to $\delta$ and $\delta^+$ and apply $\land R$:
4.3 Incompleteness

We now put the results of the previous two sections together to prove cut-free hypersequent incompleteness of RB.

**Lemma 4.8.** $\not\vdash_{RB} J$

*Proof.* Given $EC$ is admissible in RTB (Lemma 4.6 and lemma 4.7), anything provable in RB is provable in RTB. But we know that $J$ is unprovable in RTB (Lemma 4.4). Hence $J$ is unprovable in RB. □

**Theorem 3.** RB is cut-free hypersequent incomplete relative to $B$ (symmetric, reflexive) Kripke frames.

*Proof.* $\models_B J$ (Lemma 4.2) but $\not\vdash_{RB} J$ (lemma 4.8). □

**Theorem 4.** RKB is cut-free hypersequent incomplete relative to KB (symmetric) Kripke frames.

*Proof.* $\models_{KB} J$ (lemma 4.1) but $\not\vdash_{RKB} J$ (Lemma 4.5). □

It would be nice to know whether RKB and RB are also cut-free formula and sequent incomplete. Unfortunately this will remain open in the current paper.

5 Contrasting the 4 and B results

We have managed to show formula, sequent and hypersequent cut-free incompleteness for RK4 and RS4, whereas we have only managed to show hypersequent cut-free incompleteness for RKB and RB. In the former two, formula, sequent and hypersequent incompleteness directly hang together. For, in both RK4 and RS4, given a hypersequent $H$, there is a formula $I(H)$ such that $H$ is provable iff the hypersequent $\Rightarrow I(H)$ is provable. In RKB and RB, however, it is unclear whether given a hypersequent $H$ there is such an equivalent formula.

In the RK4 and RS4 cases, we use a translation from hypersequents to formulas from Burns and Zach [5, p.6]:
\[ I(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta \]
\[ I(\Gamma \Rightarrow \Delta \parallel H) = (\bigwedge \Gamma \rightarrow \bigvee \Delta) \lor \Box I(H) \]

where \( \phi \rightarrow \psi := \neg \phi \lor \psi \). Burns and Zach show the equivalence of a relational hypersequent \( H \) and its formula translation \( I(H) \) within Kripke frames. The counterexample \( C \) to RK4 and RS4’s cut-free completeness is the formula translation of the hypersequent \( \Box \neg \Box (p \land q) \Rightarrow \Box p \Rightarrow \Box q \Rightarrow \). That this hypersequent was a counterexample to RK4 and RS4’s hypersequent cut-free completeness was found first and then formula (and therefore sequent) cut-free incompleteness was found via the formula translation. For in RK4 and RS4 a hypersequent \( H \) and its formula translation are also equivalent. In the lead up to proving the equivalence, we state the following reduction lemmas.

**Lemma 5.1.** For both RK4 and RS4:

1. If \( \vdash H \parallel \Gamma \Rightarrow \Delta, \phi \lor \psi \parallel G \) then \( \vdash H \parallel \Gamma \Rightarrow \Delta, \phi, \psi \parallel G \);
2. If \( \vdash H \parallel \Gamma, \phi \land \psi \Rightarrow \Delta \parallel G \) then \( \vdash H \parallel \Gamma, \phi, \psi \Rightarrow \Delta \parallel G \);
3. If \( \vdash H \parallel \Gamma \Rightarrow \Delta, \neg \phi \parallel G \) then \( \vdash H \parallel \Gamma, \phi \Rightarrow \Delta \parallel G \);
4. If \( \vdash H \parallel \Gamma \Rightarrow \Delta, \Box \phi \parallel G \) then \( \vdash H \parallel \Gamma \Rightarrow \Delta \parallel \Rightarrow \phi \).

**Proof.** The proofs are routine inductions on the length of derivations. We display the \( \land R \) case for (4) as an example.

\[
\begin{array}{c}
\vdash \delta \\
H \parallel \Gamma \Rightarrow \psi, \Delta, \Box \phi \parallel G \\
\vdash \delta' \\
H \parallel \Gamma \Rightarrow \xi, \Delta, \Box \phi \parallel G \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \text{IH} \delta \\
H \parallel \Gamma \Rightarrow \psi, \Delta \parallel \Rightarrow \phi \\
\vdash \text{IH} \delta' \\
H \parallel \Gamma \Rightarrow \xi, \Delta \parallel \Rightarrow \phi \\
\end{array}
\]

We simply assume IH of \( \delta \) and \( \delta' \) and then apply \( \land R \).}

\[ ^{11} \text{The same kind of reasoning can be used to show that they are equivalent in the PS4 models from Definition 3.3.} \]
Theorem 5. For both RK4 and RS4: \( \vdash H \iff \vdash \Rightarrow I(H) \)

Proof. For the proof we have the two hypersequents:

- \( H = \Gamma_1 \Rightarrow \Delta_1 \parallel ... \parallel \Gamma_n \Rightarrow \Delta_n \); and
- \( I(H) = (\land \Gamma_1 \to \lor \Delta_1) \lor \Box((... \Box (\land \Gamma_n \to \lor \Delta_n)...)) \).

For the left to right direction we assume a derivation of \( H \). It is simply a matter of applying the connective rules to derive \( I(H) \) from \( H \).

For the right to left direction we assume a derivation \( \delta \vdash I(H) \). We proceed by induction on \( n \) as follows: In the base \( n = 1 \) case, where \( H = \Gamma_1 \Rightarrow \Delta_1 \) and \( I(H) = \land \Gamma_1 \to \lor \Delta_1 \), you just apply Lemma 5.1(1)-(3) in whichever order you want to obtain the fact that there is a derivation \( \delta \vdash H = \Gamma_1 \Rightarrow \Delta_1 \).

For the induction step we have the instances:

- \( H = \Gamma_1 \Rightarrow \Delta_1 \parallel H'; \) and
- \( I(H) = (\land \Gamma_1 \to \lor \Delta_1) \lor \Box I(H') \)

where \( H' \) is \( n \) sequents long. Applying first Lemma 5.1(1) and then (4) shows us that there is a derivation \( \delta \vdash (\land \Gamma_1 \to \lor \Delta_1) \parallel I(H') \). We then apply the same reasoning as in the base case to show that there is a derivation \( \delta'' \vdash (\Gamma_1 \Rightarrow \Delta_1) \parallel I(H'). \) From the induction hypothesis we have that there is a derivation \( \delta''' \vdash (\Gamma_1 \Rightarrow \Delta_1) \parallel H' \).

In contrast to RK4 and RS4, in RKB and RB a hypersequent and its Burns and Zach formula translation are not always equivalent. The left to right direction of the equivalence does hold – the reasoning simply involves applying the relevant connective rules to the hypersequent \( H \), just as with RK4 and RS4. The equivalence breaks down, however, in the right to left direction. For \( J \) is unprovable, whereas \( \Rightarrow I(J) \) is provable. Consider the following proof of \( I(J) \):
If there is an adequate formula translation of hypersequents for RKB and RB, i.e. a mapping $I'$ such that a hypersequent $H$ is provable iff the hypersequent $\Rightarrow I'(H)$ is provable, then we have a quick route to formula and, therefore also, sequent cut-free incompleteness. Whether there is one remains to be found.\footnote{12}

6 Conclusions and Open Questions

This paper has answered a number of questions that were raised in Section 2.4. We now know that RKB, RB, RK4 and RS4 are cut-free hypersequent incomplete, and that the latter two are also cut-free sequent and formula incomplete. Hence, the cut-free systems are not adequate for the intended Kripke frames. Importantly, as a consequence Cut is not an admissible rule in any of the four systems, causing problems for Parisi’s project of using them as the basis for an inferentialist account of modality. This still leaves open the cut-free sequent and formula completeness, and hence

\footnote{12} We also have a breakdown of the equivalence of $\Box(\phi \land \psi)$ and $\Box\phi \land \Box\psi$. We do have $\vdash_{RKB} \Box\phi \land \Box\psi \Rightarrow \Box(\phi \land \psi)$ and $\vdash_{RKB} \Box(\phi \land \psi) \Rightarrow \Box\phi \land \Box\psi$. Interestingly, however, while $J$ is unprovable, the hypersequent $J' := p/\Rightarrow \Rightarrow \Box\Box p \land \Box\Box q/\Rightarrow q$ is provable (the proof is very similar to that of $I(J)$). This is also a concrete example of Cut failing, because if Cut were admissible, from the proof of $J'$ we would know that there is a proof of $J$.\footnote{23}
adequacy, of RKB and RB. They may turn out to be cut-free formula and sequent complete, even if though they are cut-free hypersequent incomplete. If so, it’s conceivable that someone might be more concerned about the former than the latter. After all, sequent completeness captures the notion of being complete in regards to arguments and formula completeness in regards to theorems. In contrast, there isn’t a pre-existing notion that hypersequent completeness captures, plausibly because hypersequents have been introduced as a tool for obtaining an adequate proof theory. However, if, with Parisi, one accepts that Cut Admissibility is required for an inferentialist account of modality, then Parisi’s systems RK4, RS4, RKB and RB will not do, questions of completeness aside.

A number of other questions remain, some technical, others more philosophical:

- Are RKB and RB cut-free formula and sequent complete?
- Are there adequate hypersequent systems that meet Parisi’s, and Burns and Zach’s criteria, i.e. cut-admissibility, Došen’s principle, and modularity? What is common in the cases discussed in this paper is that the tree structure of standard Kripke frames is not fully captured in Parisi’s relational hypersequents, at least for RK4, RS4, RKB and RB. There might be a way to capture this with relational hypersequents using different rules. Alternatively, a more complex structure like Poggiolesi’s tree hypersequents [12] might be needed.
- What kind of models are Parisi’s cut-free RKB, RB, RK4 and RS4 complete relative to? The latter two may be complete relative to the Pseudo S4 models defined in this paper. Conversely, what is the logic of the Pseudo-Models? While cooked up for the purpose proving incompleteness, they may be worth studying in their own right.

A RK4 and RS4 with Cut are Hypersequent Complete

For clarity, we refer to RK4 and RS4 with Cut as a basic rule as $\text{RK}4_{\text{Cut}}$ and $\text{RS}4_{\text{Cut}}$ respectively.

The following is a modification of Burns and Zach’s cut-free completeness proofs [5]. Rather than reproduce the proof in total, only the modifications are given here, with the reader directed to the relevant parts of [5]. These are all from §3 of [5] onwards.
Definitions 13-15 are left unchanged. We modify Definition 16 [5, p.10, 15] to replace Burns and Zach’s reduction rule for □L with □L′.

\[
\begin{array}{c|c|c|c|}
\square L & \Gamma & & \Gamma' \\hline \\
\square L' & \Gamma' & \phi \in \sigma' & \Delta' \\hline \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|}
\square L & \Gamma & \phi \in \sigma' & \Delta' \\hline \\
\square L' & \Gamma' & \phi \in \sigma' & \Delta' \\hline \\
\end{array}
\]

Note that □L is an instance of □L′.

The proof of Proposition 17 [5, p.11-12] is then modified to show that □L′ preserves unprovability in RK4Cut and RS4Cut. This is shown by the following derivation:

\[
\begin{array}{c|c|c|c|}
G \parallel \Gamma & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & * \\
\hline \\
G \parallel \Gamma, \phi & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & TL \\
\hline \\
G \parallel \Gamma, \phi & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & EWL \\
\hline \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|}
G \parallel \Gamma & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & EWL \\
\hline \\
G \parallel \Gamma, \phi & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & EW \\
\hline \\
G \parallel \Gamma, \phi & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & Cut \\
\hline \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|}
G \parallel \Gamma & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & * \\
\hline \\
G \parallel \Gamma, \phi & G' \parallel \Sigma & \phi \in \Lambda \parallel G'' & Cut \\
\hline \\
\end{array}
\]

* Multiple possible applications of internal and external weakening.

We leave Proposition 18 and its proof unchanged. However, we modify Proposition 19 [5, p.11, 15] to add on further components Proposition 19(6) and (7):

(6) If □φ ∈ Γ, σR^+ τ and τ occurs in H, then φ ∈ Γ(H, τ).

(7) If □φ ∈ Γ, σR^+ τ and τ occurs in H, then φ ∈ Γ(H, τ).

Proof. For (6), suppose that □φ ∈ Γ, σR^+ τ and τ occurs in H. Since Σ(H) is an R^1-branch and σR^+ τ, the component H(τ) occurs to the right of H(σ).

Because □φ ∈ Γ, the hypersequent G \parallel Γ, □φ ⊢ Δ \parallel G' \parallel Σ, φ ⊢ Λ \parallel G'' is a □L′ τ-reduct of G \parallel Γ, □φ ⊢ Δ \parallel G' \parallel Σ ⊢ Λ \parallel G''. Since H is τ-reduced, H is identical to all its □L′ τ-reducts. Therefore, φ ∈ Γ(H, τ).

(7) follows from (5′) and (6). □

The following definitions 20 and 22, and propositions 21 and 23 are left unchanged.

Lastly, we change the definition of a model used in Proposition 24 to use R^+ in the RK4Cut case and R^* in the RS4Cut case, and employ our additions to Proposition 19, (6) and (7) in the proof. For RK4Cut in the proof of Proposition 24, in the case where □φ ∈ Γ(σ) we use Proposition 19(7) instead of Proposition 19(5). For RK4Cut in the case of the case where □φ ∈ Γ(σ), we use Proposition 19(7).
Theorem 6. If $\models_{K4} H$ then $\vdash_{RK4Cut} H$

Proof. This follows from the modified proof of Proposition 24 above, setting the accessibility relation to $R^+$. 

Theorem 7. If $\models_{S4} H$ then $\vdash_{RS4Cut} H$

Proof. This follows from the modified proof of Proposition 24 above, setting the accessibility relation to $R^*$. 

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