WHICH POWERS OF HOLOMORPHIC FUNCTIONS ARE INTEGRABLE?

JÁNOS KOLLÁR

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The aim of this lecture is to investigate the following, rather elementary, problem:

Question 1. Let \( f(z_1, \ldots, z_n) \) be a holomorphic function on an open set \( U \subset \mathbb{C}^n \). For which \( t \in \mathbb{R} \) is \( |f|^t \) locally integrable?

The positive values of \( t \) pose no problems, for these \( |f|^t \) is even continuous. If \( f \) is nowhere zero on \( U \) then again \( |f|^t \) is continuous for any \( t \in \mathbb{R} \). Thus the question is only interesting near the zeros of \( f \) and for negative values of \( t \). More generally, if \( h \) is an invertible function then \( |f|^t \) locally integrable iff \( |fh|^t \) is locally integrable. Thus the answer to the question depends only on the hypersurface \( (f = 0) \) but not on the actual equation. (A hypersurface \( (f = 0) \) is not just the set where \( f \) vanishes. One must also remember the vanishing multiplicity for each irreducible component.)

It is traditional to change the question a little and work with \( s = -t/2 \) instead. Thus we fix a point \( p \in U \) and study the values \( s \) such that \( |f|^{-s} \) is \( L^2 \) in a neighborhood of \( p \). It is not hard to see that there is a largest value \( s_0 \) (depending on \( f \) and \( p \)) such that \( |f|^{-s} \) is \( L^2 \) in a neighborhood of \( p \) for \( s < s_0 \) but not \( L^2 \) for \( s > s_0 \). Our aim is to study this “critical value” \( s_0 \).

Definition 2. Let \( f \) be a holomorphic function in a neighborhood of a point \( p \in \mathbb{C}^n \). The log canonical threshold or complex singularity exponent of \( f \) at \( p \) is the number \( c_p(f) \) such that

- \( |f|^{-s} \) is \( L^2 \) in a neighborhood of \( p \) for \( s < c_p(f) \), and
- \( |f|^{-s} \) is not \( L^2 \) in any neighborhood of \( p \) for \( s > c_p(f) \).

It is convenient to set \( c_p(0) = 0 \).

The name “log canonical threshold” comes from algebraic geometry. I don’t know who studied these numbers first. The concept is probably too natural to have a well defined inventor. It appears in the works of Schwartz, Hörmander, Lojasiewicz and Gel’fand as the “division problem for distributions”; see [Sch50, Hör58, Loj58, GˇS58]. The general question is considered by Atiyah [Ati70] and Bernstein [Ber71]. The connections with singularity theory were explored by the Arnold school and summarized in [AGZV85]. See [Kol97] for another survey and for further connections.

Algebraic geometers became very much interested in log canonical thresholds when Shokurov [Sho88] discovered that some subtle properties of log canonical thresholds, especially his conjecture (5), are connected with the general MMP (=minimal model program). These connections were systematized and further developed in [Kol92, Secs.17–18]. In this general framework, considering only smooth
complex spaces is not natural. In fact, the inductive theorems require the consideration of cases when \( f \) is holomorphic on a singular complex space. At the end, the singular versions of the ACC (=ascending chain condition) Conjecture [1] and the Accumulation Conjecture [2] emerged as the main open problems.

A novel approach to log canonical thresholds on manifolds was proposed by de Fernex and Mustață in [dFM07]. They rely on non-standard methods (ultraproducts etc.) and the formula for log canonical thresholds using arc-spaces [Mus02]. The end result is the proof of the smooth version of the Accumulation Conjecture for decreasing limits.

The aim of this lecture is three-fold. First, I give an elementary introduction to log canonical thresholds. The second part is a presentation of the proof in [dFM07] using “traditional” methods and the original definition of log canonical thresholds relying on discrepancies of divisors as in (11.4). At its heart, however, the proof in (29) is the same as in [dFM07]. Third, I show how to use the existence of minimal models [BCHM06] to establish a part of the ACC conjecture. This in turn is enough to complete the proof of the smooth version of the Accumulation Conjecture.

There are three, quite distinct, approaches to log canonical thresholds.

- Study the relationship of \( c_p(f) \) and the singularity \( p \in (f = 0) \).
- Study the function \( f \mapsto c_0(f) \) on the space of all holomorphic functions.
- Study the set of all possible values \( c_0(f) \in \mathbb{R} \).

The log canonical threshold is related to other invariants of singularities in many ways; see [Kol97] for a survey. However, I will not say anything about these here, mainly because in higher dimensions these connections have not yet proved useful in the study of the other two problems.

These notes start at an elementary level. I tried hard to avoid the algebraic methods and terminology. However, starting with Section 6, I switch to the language of divisors since it is better suited to handle the general singular case.

### 1. Main conjectures

It was Shokurov in [Sho88] who first proposed to look at all possible values of log canonical thresholds in a fixed dimension and suggested that these sets, though rather complicated, have remarkable properties. The original questions were extended and further developed in [Kol92, Sec.18].

**Definition 3.** Let \( \mathcal{H}T_n \) be the set of log canonical thresholds of all possible \( n \)-variable holomorphic functions. That is,

\[
\mathcal{H}T_n := \left\{ c_0(f) : f \in \mathcal{O}_{0,\mathbb{C}^n} \right\}.
\]

The notation suggests that we are talking about hypersurface thresholds. As we see in Section 5, we get the same set if instead we let \( f \) run through all polynomials or all formal power series over any algebraically closed field of characteristic 0. As far as I know, the answer could be the same if we look at polynomials over any field (e.g. \( \mathbb{Q} \) or even \( \mathbb{F}_p \)).

The sets \( \mathcal{H}T_n \) are different from the sets \( T_n \) used in [Kol97] and in [dFM07] (which are also different from each other).

The paper [dFM07] considers log canonical thresholds when a single holomorphic function \( f \) is replaced by \( \max\{|f_1|, \ldots, |f_r|\} \) where the \( f_i \) are holomorphic. It is easy to rework the results of this note in their more general setting.
The lectures [Kol97] consider log canonical thresholds for functions on singular complex spaces. The present methods apply to that case if there is an a priori bound on the appearing singularities. For instance, the proofs work if we assume that $X$ has only hypersurface singularities. Using the standard covering and partial resolutions tricks (for instance, as in [Kol94, Sec.5]), this is a manageable limitation in dimensions $\leq 3$. Unfortunately, this is a rather unnatural restriction in connection with the higher dimensional MMP.

Note that $|z|^{-s}$ is $L^2$ iff $s < 1$. From this we conclude that, for a 1-variable holomorphic function $f(z)$,

$$c_p(f(z)) = \frac{1}{\mult_p f}.$$ 

In particular,

$$\mathcal{H}T_1 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}.$$ 

The 2-variable case is already quite subtle, but we see in (16.5) that

$$\mathcal{H}T_2 = \left\{ \frac{c_1 + c_2}{c_1c_2 + a_1c_2 + a_2c_1} : a_i + c_i \geq \max\{2, a_{3-i}\} \right\} \cup \{0\}.$$ 

Although the sets $\mathcal{H}T_n$ are not known for $n \geq 3$, and a complete listing as in (3.2) may not even be interesting, they are conjectured to have remarkable properties. The following are the basic results and questions:

**Proposition 4.** All log canonical thresholds are rational and lie between 0 and 1. That is,

$$\mathcal{H}T_n \subset \mathbb{Q} \cap [0, 1].$$

This is proved in (10) and (11).

The key question in this area is the following:

**Conjecture 5** (ACC conjecture, smooth version). For any $n$ there is no infinite increasing subsequence in $\mathcal{H}T_n$.

Note by contrast, that by (16.1), any rational number between 0 and 1 is the log canonical threshold of some function for some $n$.

There are many decreasing sequences of log canonical thresholds, and the following conjecture [Kol97, 8.21.2] describes their limit points:

**Conjecture 6** (Accumulation conjecture, smooth version). The set of accumulation points of $\mathcal{H}T_n$ is $\mathcal{H}T_{n-1} \setminus \{1\}$.

It is easy to see (16.2) that the accumulation points of $\mathcal{H}T_n$ contain $\mathcal{H}T_{n-1} \setminus \{1\}$. The main result of this note is to prove that Conjecture 6 almost holds:

**Theorem 7.** The set of accumulation points of $\mathcal{H}T_n$ is either $\mathcal{H}T_{n-1} \setminus \{1\}$ or $\mathcal{H}T_{n-1}$.

By (11), $1 \in \mathbb{R}$ can not be a limit of a decreasing sequence of log canonical thresholds. As a special case of the ACC conjecture, 1 can not be a limit of an increasing sequence of log canonical thresholds either. Equivalently, for a fixed dimension, no log canonical threshold lies in an interval $(1 - \epsilon_n, 1)$ for some $\epsilon_n > 0$. I call this special case the Gap conjecture [Kol97, 8.16].
Conjecture 8 (Gap conjecture, smooth version). For every $n$ there is an $\epsilon_n > 0$ such that for any $f(z_1, \ldots, z_n)$ that is holomorphic on the closed unit ball $B$,

$$
\int_B \frac{1}{|f|^1 - \epsilon_n} \, dV < \infty \quad \Rightarrow \quad \int_B \frac{1}{|f|^1} \, dV < \infty \quad \forall \ 0 < \epsilon < \epsilon_n.
$$

Various forms of the Gap conjecture are important in the construction of Einstein metrics as in [BGK05]. Ultimately, a gap conjecture type result lies behind the stabilization theorems in [Kol07, Kol05].

As noted in [dFM07], the Gap conjecture and (7) imply the ACC conjecture. However, to obtain the ACC conjecture in a fixed dimension, one needs the Gap conjecture in all dimensions.

There is even a conjecture about the precise value of the optimal $\epsilon_n$.

Consider the sequence defined recursively by

$$
c_{k+1} = c_1 \cdots c_k + 1 \text{ starting with } c_1 = 2.
$$

It starts as

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$$

It is easy to see that

$$
\sum_{i=1}^n \frac{1}{c_i} = 1 - \frac{1}{c_{n+1} - 1} = 1 - \frac{1}{c_1 \cdots c_n}.
$$

In particular, by (16.1),

$$
c_0 \left( z_1^{c_1} + \cdots + z_n^{c_n} \right) = 1 - \frac{1}{c_{n+1} - 1}.
$$

It is conjectured that this is the worst example, that is, the optimal value for $\epsilon_n$ in (8) is

$$
\epsilon_n = \frac{1}{c_{n+1} - 1}. \tag{8.1}
$$

9 (Known special cases). As we noted, $\mathcal{HT}_1$ and $\mathcal{HT}_2$ are known. From these one can read off all the above conjectures for $n \leq 2$. In particular, we get that $\epsilon_1 = \frac{1}{2}$, $\epsilon_2 = \frac{1}{6}$. The value $\epsilon_3 = \frac{1}{12}$ is computed in [Ko94 5.5.7], essentially through a classification of the possible normal forms of singularities with log canonical threshold near 1.

The set $\mathcal{HT}_3$ is still not known, but [Kuw99] determined $\mathcal{HT}_3 \cap [\frac{5}{6}, 1]$ and [Pro02] computed all accumulation points of $\mathcal{HT}_3$ that lie in $[\frac{5}{6}, 1]$.

The ACC conjecture for $\mathcal{HT}_3$ was proved by [Ale93] and the Accumulation conjecture by [MP04]. Both of these papers deal with the general singular case and rely heavily on the MMP in dimension 3. The relevant parts of the MMP are now known in all dimensions [BCHM06]. A missing ingredient in higher dimensions is the Alexeev-Borisov-Borisov conjecture [Ale94]. Even stating it would lead us quite far. The toric cases are treated in [BB92].

[Sou05] proved that, with $\epsilon_n$ as in (8.1), $c_0(z_1^{a_1} + \cdots + z_n^{a_n})$ cannot lie in $(1 - \epsilon_n, 1)$ for any $a_1, \ldots, a_n$. That is, if

$$
\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 1 \quad \text{then} \quad \frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq 1 - \epsilon_n.
$$
2. Computing and estimating \( c_0(f) \)

In this section we discuss how to determine or bound the log canonical threshold. The basic result \([\text{11}]\), first observed by Atiyah, gives a formula for \( c_0(f) \) in terms of an embedded resolution of the hypersurface \((f = 0)\). It is not easy to construct embedded resolutions, but even simple-minded partial resolutions frequently give good upper bounds for \( c_0(f) \). Estimates using the Newton polygon are especially easy to obtain and to use. It is much harder to get good lower bounds.

**Lemma 10.** If \( f(p) \neq 0 \) then \( c_p(f) = +\infty \). If \( f(p) = 0 \) then \( 0 \leq c_p(f) \leq 1 \).

Proof. The first claim is clear. Thus assume that \( f(p) = 0 \). As we noted in \([\text{8}]\), for a 1-variable holomorphic function \( f(z) \) we have \( c_p(f(z)) = \frac{1}{\text{mult}_p f} \).

In the several variable case, pick a smooth point \( p' \) near \( p \) on the hypersurface \((f = 0)\). We can choose local coordinates near \( p' \) such that \( f = (\text{unit})z_1^m \) for some \( m \). By Fubini this shows that \( c_p(f) \leq c_{p'}(f) = 1/m \). \( \square \)

As we see in \([\text{11} \text{5}]\) and \([\text{20} \text{1}]\), in several variables one can only get inequalities relating \( c_0(f) \) and the multiplicity:

\[
\frac{1}{\text{mult}_p f} \leq c_p(f(z_1, \ldots, z_n)) \leq \frac{n}{\text{mult}_p f}.
\]

**11 (Computing the log canonical threshold).** \([\text{Ati70}]\) Set \( \omega = dz_1 \wedge \cdots \wedge dz_n \). Then \(|f|^{-s} \) is locally \( L^2 \) iff, on any compact \( K \subset U \), the integral

\[
\int_K (f \bar{f})^{-s} \omega \wedge \bar{\omega} \text{ is finite.} \tag{11.1}
\]

(We can ignore the power of \( \sqrt{-1} \) that makes this integral real.) Let \( \pi : X \rightarrow U \) be a proper bimeromorphic morphism. We can rewrite the above integral as

\[
\int_K (f \bar{f})^{-s} \omega \wedge \bar{\omega} = \int_{\pi^{-1}(K)} ((f \circ \pi)(\bar{f} \circ \bar{\pi}))^{-s} \pi^* \omega \wedge \bar{\pi}^* \bar{\omega}. \tag{11.2}
\]

The aim now is to choose \( \pi \) such that the local structure of \( f \circ \pi \) and of \( \pi^* \omega \) becomes simple. The best one can do is to take an embedded resolution of singularities for \((f = 0)\). This is a proper bimeromorphic morphism \( \pi : X \rightarrow U \) such that \( X \) is a smooth complex manifold and the zero set of \( f \circ \pi \) plus the exceptional set of \( \pi \) is a normal crossing divisor. That is, at any point \( q \in X \) we can choose local coordinates \( x_1, \ldots, x_n \) such that

\[
f \circ \pi = (\text{invertible}) \prod x_i^{a(i,q)} \text{ and } \pi^* \omega = (\text{invertible}) \prod x_i^{e(i,q)} \cdot dx_1 \wedge \cdots \wedge dx_n,
\]

where \( a(i,q) = \text{mult}_{(x_i=0)}(f \circ \pi) \) and \( e(i,q) = \text{mult}_{(x_i=0)} \text{Jac } \pi \). Here \( \text{Jac} \) denotes the complex Jacobian

\[
\text{Jac } \pi = \det \left( \frac{\partial z_i}{\partial x_j} \right).
\]

Thus the integral \([11.2]\) is finite near \( q \in X \) iff

\[
\int \cdots \int (x_1 \bar{x}_1)^{e(i,q)-s \cdot a(i,q)} \ dV = \pm \prod_i \int (x_i \bar{x}_i)^{e(i,q)-s \cdot a(i,q)} \ dx_i \wedge d\bar{x}_i \tag{11.3}
\]
is finite. This holds iff 
\( e(i, q) - s \cdot a(i, q) > -1 \) for every \( i \), that is, when \( s < (e(i, q) + 1)/a(i, q) \). This gives the formula for the log canonical threshold:

\[
c_p(f) = \min \left\{ 1 + \frac{\text{mult}_E \text{Jac} \pi}{\text{mult}_E(f \circ \pi)} : \text{for those } E \text{ such that } p \in \pi(E) \right\}.
\]

(11.4)

In principle we can take the minimum over all divisors \( E \subset X \) such that \( p \in \pi(E) \). However, only the exceptional divisors of \( \pi \) and the (birational transforms of) irreducible components of \( (f = 0) \) are interesting. For all other \( E \), \( \text{mult}_E \text{Jac} \pi = \text{mult}_E(f \circ \pi) = 0 \) and their contribution to (11.4) is \( 1/0 = +\infty \).

It is customary to view \( \text{mult}_E(f \circ \pi) \) as a valuation on functions on \( \mathbb{C}^n \) and drop \( \pi \) from the notation. Thus we write

\[
\text{mult}_E f \quad \text{instead of} \quad \text{mult}_E(f \circ \pi).
\]

First of all, the formula (11.4) shows that the log canonical threshold of \( f \) is always a rational number, completing the proof of (4).

Second, it gives us ways to compute or at least estimate \( c_0(f) \).

In many cases it is not hard to guess which exceptional divisor computes the log canonical threshold (that is, achieves equality in (11.4)), and to write down a bimeromorphic morphism \( \pi : X \to U \) where this divisor appears. This way we can get upper bounds for \( c_0(f) \). Note that we do not need to arrange that \( X \) be smooth or that \( \pi \) be proper. Any bimeromorphic morphism \( \pi_1 : X_1 \to U \) can be completed to a proper bimeromorphic morphism \( \pi_2 : X_2 \to U \) and then, by resolution of singularities, to a proper bimeromorphic morphism \( \pi_3 : X_3 \to U \) which is an embedded resolution as required for (11.4).

For instance, let \( \pi : B_0 \mathbb{C}^n \to \mathbb{C}^n \) be the blow up of the origin with exceptional divisor \( E \cong \mathbb{P}^{n-1} \). Then \( \text{mult}_E \text{Jac} \pi = n - 1 \) and \( \text{mult}_E(f \circ \pi) = \text{mult}_0 f \). This gives the simple estimate

\[
c_0(f(z_1, \ldots, z_n)) \leq \frac{n}{\text{mult}_0 f}.
\]

(11.5)

12 (Formal power series). The formula (11.4) makes it possible to define the log canonical threshold for a formal power series \( f \in k[[z_1, \ldots, z_n]] \) over any field \( k \) of characteristic 0. Indeed, resolution of singularities is known for complete local rings \([\text{Tem07}]\) and then (11.4) makes sense. It is easy to see that the resulting \( c_0(f) \) is independent of the resolution.

An alternative definition of \( c_0(f) \) using arc spaces is given in \([\text{Mus02}]\).

An especially convenient estimate is obtained using the Newton polygon.

**Definition 13** (Newton polygon). Let \( F = \sum a_I x^I \) be a polynomial or power series in \( n \)-variables. The **Newton polygon** of \( f \) (in the chosen coordinates \( x_1, \ldots, x_n \)) is obtained as follows.

In \( \mathbb{R}^n \) we mark the point \( I = (i_1, \ldots, i_n) \) with a big dot if \( a_I \neq 0 \). Any other monomial \( x^{I'} \) with \( I' \geq I \) coordinatewise will not be of “lowest order” in any sense, so we also mark these. (In the figure below these markings are invisible.)
The Newton polygon is the boundary of the convex hull of the resulting infinite set of marked points.

![Newton polygon](image)

The Newton polygon of

\[ y^7 + y^3x^2 + y^5x^3 + yx^4 + x^6. \]

14 (Estimating \( c_0(f) \) using Newton(\( f \))). Let \( \sum_i a_i x_i = d \) be the equation of a face of Newton(\( f \)). We can assume that the \( a_i \) are relatively prime positive integers.

Then \( a_1, \ldots, a_n \) can be the first column of an \( n \times n \) invertible integral matrix \( M = (a_{ij}) \). Consider the map

\[ \pi : \mathbb{C} \times (\mathbb{C}^*)^{n-1} \rightarrow \mathbb{C}^n \]

given by \( z_i = \prod_j x_j^{a_{ij}} \).

The inverse of \( M \) defines the inverse of \( \pi \) on the open subset \((\mathbb{C}^*)^n \subset \mathbb{C}^n \). We concentrate on the exceptional divisor \( E := \{ x_1 = 0 \} \).

Note that \( \pi^*d\pi = (\prod_j x_j^{a_{ij}}) \left( \sum_j a_{ij} \frac{dx_j}{x_j} \right) \), hence

\[ \pi^*(d\pi_1 \wedge \cdots \wedge d\pi_n) = (\prod_j x_j^{a_{ij}}) \cdot \det M \cdot (x_1 \cdots x_n)^{-1} \cdot (dx_1 \wedge \cdots \wedge dx_n). \]

Thus the Jacobian of \( \pi \) vanishes along \( E \) with multiplicity \(-1 + \sum_i a_{i1} = -1 + \sum_i a_i \).

If \( \prod_i z_i^{b_i} \) is any monomial occurring in \( f \) then \( \sum_i a_i b_i \geq d \) since \((b_1, \ldots, b_n)\) lies above \( \sum_i a_i x_i = d \). On the other hand

\[ (\prod_i z_i^{b_i}) \circ \pi = \prod_j x_j^{A_{ij}} \]

where \( A_j = \sum_i b_i a_{ij} \)

and so it vanishes along \( E \) with multiplicity \( A_1 = \sum_i b_i a_{i1} = \sum_i b_i a_i \geq d \). Thus we conclude that

\[ c_0(f) \leq \frac{\sum_i a_i}{d}. \]

This inequality is equivalent to the first part of the next theorem. For the proof of the second part see [Kou76] and for the third [Var76] or [KSC04, 6.40].

**Theorem 15.** Let \( f(z_1, \ldots, z_n) \) be a holomorphic function near the origin \( 0 \in \mathbb{C}^n \).

Let Newton(\( f \)) be the Newton polygon of \( f \):

1. The vector \((1/c_0(f), \cdots, 1/c_0(f))\) is on or above Newton(\( f \)).
2. Fix Newton(\( f \)) and assume that the coefficients of the monomials in \( f \) are general. Then \((1/c_0(f), \cdots, 1/c_0(f))\) is on Newton(\( f \)).
3. If \( n = 2 \) then one can choose local analytic coordinates \((x, y)\) such that \((1/c_0(f), 1/c_0(f))\) is on Newton(\( f \)).

This gives an easy way to construct many different log canonical thresholds.

Take \( m \leq n \) linearly independent nonnegative vectors \( a_i = (a_{ij}) \) such that their convex hull contains a vector of the form \((1/c, \cdots, 1/c)\). Then, for general \( b_i \in \mathbb{C} \),

\[ c_0\left( \sum_i b_i \prod_j x_j^{a_{ij}} \right) = c. \]

After a suitable change of coordinates we can even assume that all \( b_i = 1 \). These are the log canonical thresholds that can be computed as in (11) using a resolution.
$X \to \mathbb{C}^n$ which is toric, that is, equivariant with respect to the standard $\left(\mathbb{C}^*\right)^n$-action on $\mathbb{C}^n$. The values produced by (15.4) give a large subset of $\mathcal{HT}_n$. It is possible that in fact these values give all of $\mathcal{HT}_n$, but I know of no reasons why this should be true. Note, however, that (15.3) definitely fails already for $n = 3$; see [KSC04, 6.45] for an example.

Computing the case when $(1/c_0(f), 1/c_0(f))$ is on the edge of the Newton polygon between the points $(a_1, a_2 + c_2)$ and $(a_1 + c_1, a_2)$ shows that any 2-variable log canonical threshold can be written as

$$c_0(f) = \frac{c_1 + c_2}{c_1 c_2 + a_1 c_2 + a_2 c_1} \quad (15.5)$$

where $a_1 + c_1 \geq \max\{2, a_2\}$ and $a_2 + c_2 \geq \max\{2, a_1\}$, or it is 0 or 1.

3. Basic properties

In this section we collect the known important properties of log canonical thresholds. I state everything for formal power series, as needed for our proofs. See (23) for comments on the proofs in this setting.

The next result is proved for holomorphic functions with isolated critical points in [AGZV85, II.13.3.5]. The proof given in [Kol97, 8.21] works in general.

Proposition 16. Let $f(x)$ and $g(y)$ be power series in disjoint sets of variables. Then

$$c_0(f(x) \oplus g(y)) = \min\{1, c_0(f) + c_0(g)\},$$

where $\oplus$ denotes the sum in disjoint sets of variables.

As a corollary we obtain that

$$c_0(z_1^{a_1} + \cdots + z_n^{a_n}) = \min\left\{1, \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right\}, \quad (16.1)$$

and if $c_0(f(x)) < 1$ and $m \gg 1$ then

$$c_0(f(x) + g^m) = c_0(f(x)) + \frac{1}{m}. \quad (16.2)$$

A simple but important property is that the log canonical threshold gives a metric on the space of power series that vanish at the origin, [DK01, Thm.2.9] or [Kol97, 8.19].

Theorem 17. Let $f(x), g(x) \in k[[x_1, \ldots, x_n]]$ be power series. Then

$$c_0(f + g) \leq c_0(f) + c_0(g).$$

Applying this to the Taylor polynomials $t_m(f)$ of $f$ and to $f - t_m(f)$, and using (11.5), we get the following uniform approximation result.

Corollary 18. Let $f(x) \in k[[x_1, \ldots, x_n]]$ be a power series and $t_m(f)$ its degree $m$ Taylor polynomial. Then

$$\left| c_0(f) - c_0(t_m(f)) \right| \leq \frac{n}{m + 1}. \quad \Box$$

Notation 19. In order to indicate the change from the complex to the algebraic case, I replace $\mathbb{C}^n$ with the affine $n$-space $\mathbb{A}^n$ defined over some field $k$. Its completion at the origin is denoted by $\hat{\mathbb{A}}^n$. It is also $\text{Spec}_k k[[x_1, \ldots, x_n]]$. 

The next result is known as the Ohsawa-Takegoshi extension theorem in complex analysis \textsuperscript{[OT87]} and as the (weak version of) inversion of adjunction in algebraic geometry \textsuperscript{[Kol92, Sec.17]}.

**Theorem 20.** Let \( f(x) \in k[[x_1, \ldots, x_n]] \) be a power series and \( L \subset \mathbb{A}^n \) a smooth submanifold. Then
\[
c_0(f|_L) \leq c_0(f).
\]

For instance, if \( L \subset \mathbb{C}^n \) is a general line through the origin then \( \text{mult}_0(f|_L) = \text{mult}_0 f \) and so we conclude that
\[
c_0(f) \geq \frac{1}{\text{mult}_0 f}. \tag{20.1}
\]

(Thresholds in families). Let \( f_x := \sum a_I(x)z^I \) be polynomials in \( z \) whose coefficients \( a_I(x) \) are rational functions on an algebraic variety \( X \). What can we say about the log canonical thresholds \( c_0(f_x) \) as a function of \( x \in X \)?

Pick a generic point \( x_g \in X \) and take a resolution \( \pi_{x_g} : Y_{x_g} \to X \). Since \( \pi_{x_g}, Y_{x_g} \) and the exceptional divisors \( E_i \) are defined over the generic point of \( X \), there is a Zariski open subset \( X_0 \subset X \) such that, by specialization, for every \( x \in X_0 \) we obtain \( \pi_x : Y_x \to X \) with exceptional divisors \( E_i \). Moreover, we may assume that, for every \( i \), the multiplicities of Jac \( \pi_x \) and of \( f_x \circ \pi_x \) along \( E_i \) do not depend on \( x \in X_0 \). In particular, \( c_0(f_x) \) is also independent of \( x \in X_0 \).

Repeating the argument with \( X \) replaced by \( X \setminus X_0 \), we conclude that \( c_0(f_x) \) is a constructible function of \( x \in X \). That is, its level sets are finite unions of locally closed subvarieties.

It is also easy to see that \( c_0(f_x) \) is a lower semi continuous function of \( x \in X \), cf. \textsuperscript{[Var79]}.

In the complex analytic case, a more precise version of lower semi continuity is proved in \textsuperscript{[DK01, 0.2]}:

**Theorem 22.** Assume that \( f_t(z) \) converges uniformly to \( F(z) \) in a compact neighborhood \( B \) of \( 0 \). Fix \( s < c_0(F) \). Then
\[
\frac{1}{|f_t|^s} \quad \text{converges to} \quad \frac{1}{|F|^s} \quad \text{in} \ L^2(B).
\]

(Comments on the formal power series case). All these results were proved in the algebraic and analytic settings. The methods are either analytic, or, as the proofs of inversion of adjunction in \textsuperscript{[Kol92, Sec.17]} and \textsuperscript{[Kaw07]}, rely ultimately on a relative version of the Kodaira vanishing theorem. This vanishing is known for birational maps between varieties and for bimeromorphic maps between complex spaces. Unfortunately, we would need it in case the base is a formal power series ring.

While the result is no doubt true in this case, the usual proofs of the Kodaira-type vanishing theorems rely on some topological/analytic arguments. Thus, a genuinely new proof may be needed.

Here I go around this difficulty by a reduction to the algebraic case, see Section \textsuperscript{[7]} This, however, should be viewed as but a temporary patch. It is high time to work out the whole MMP over an arbitrary base scheme, especially over complete local rings.
Note that the formal versions of (20) and (17) both follow from the algebraic case once we know that for any formal power series \( f \), the log canonical thresholds of its Taylor approximations converge to \( c_0(f) \). That is, if
\[
\lim_{m \to \infty} c_0(t_m(f)) = c_0(f) \tag{24.1}
\]
The argument in (48) easily yields the inequality
\[
\limsup_{m \to \infty} c_0(t_m(f)) \leq c_0(f),
\]
but the other direction relies on inversion of adjunction (in larger dimensions), creating a vicious circle.

The first complete proof of (24.1) is in [dFM07, 2.5] using arc-space techniques.

24 (Proof of (16) \& (20) \Rightarrow (17)). Create disjoint sets of variables for \( f(x) \) and \( g(y) \). Then, by (16),
\[
c_0(f \oplus g) \leq c_0(f) + c_0(g).
\]
Note that \( f(x) + g(x) \) is naturally isomorphic to \( f(x) \oplus g(y) \) restricted to the diagonal \( L := (x_1 - y_1 = \cdots = x_n - y_n = 0) \). Thus, by (20),
\[
c_0(f + g) \leq c_0(f \oplus g) \leq c_0(f) + c_0(g). \qed
\]

4. Generic limits of power series

25. Consider a sequence of holomorphic functions \( f_i \) defined in a neighborhood of \( 0 \in \mathbb{C}^n \). Assume that the sequence of log canonical thresholds converges to a limit \( c := \lim_i c_0(f_i) \). Can we write down a holomorphic function \( f \) such that \( c_0(f) = c \) and, in some sense, \( f \) is the limit of the functions \( f_i \)?

At first sight the answer is no. Even in some very simple cases when the \( f_i \) do converge to a limit, the log canonical threshold usually jumps. For instance, take \( f_i(z) = z^2 + \frac{1}{i} z \) and \( f(z) = z^2 \). Then \( f_i \to f \) uniformly on any compact set, yet \( c_0(f_i) = 1 \) and \( c_0(f) = \frac{1}{2} \).

We get a different insight from the log canonical threshold formula using exceptional divisors (14). Let \( \pi : X \to \mathbb{C}^n \) be a bimeromorphic map and \( E \subset X \) a divisor such that \( \pi(E) = 0 \). Choose local coordinates \( x_1, \ldots, x_n \) at a general point of \( E \) such that \( E = (x_1 = 0) \). If \( \pi_i \) are the coordinate functions of \( \pi \), then, expanding \( f \circ \pi \) by powers of \( x \), write
\[
f(\pi_1(x), \ldots, \pi_n(x)) = \sum_I P_I(a, b)x^I,
\]
where the \( P_I \) are polynomials, the \( a \) are the coefficients of \( f \) and the \( b \) are the coefficients of \( \pi \).

Note that \( f \circ \pi \) vanishes along \( E \) with multiplicity \( m \) iff \( x_1^m \) divides \( f \circ \pi \). Equivalently, when \( P_I(a, b) = 0 \) whenever the first coordinate of \( I = (i_1, \ldots, i_n) \) is less than \( m \).

This suggests that we should focus on the polynomial relations between the coefficients \( a_I \). This is a key idea that de Fernex and Mustaţă use to study limits of log canonical thresholds.

An interesting feature of the proof is that even if we start with a sequence of functions \( f_i \) that are holomorphic on a fixed open set \( U \), their limit is only a formal power series \( F(z_1, \ldots, z_n) \). Furthermore, the construction naturally yields a power
series \( F \) whose coefficients are not in \( \mathbb{C} \) but in an algebraically closed field \( K \) of countably infinite transcendence degree over \( \mathbb{C} \).

Any such field \( K \) is isomorphic to \( \mathbb{C} \), so at the end we can replace \( F \) with a formal power series \( F^* (z_1, \ldots, z_n) \in \mathbb{C}[[z_1, \ldots, z_n]] \) and, using (32), even with a polynomial \( P (z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n] \), but these steps are rather artificial from the point of view of the proof. It is more natural to work with formal power series over an arbitrary field \( k \).

26 (Generic power series). Let \( k \) be a field and \( k[[x_1, \ldots, x_n]] \) the ring of power series with coefficients in \( k \). We can view \( k[[x_1, \ldots, x_n]] \) as an infinite dimensional affine space \( \mathbb{A}^\infty \) over \( k \). Thus if \( f_i (x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]] \) are power series, then we get points \( [f_i] \in \mathbb{A}^\infty \). Assume now that there is a power series \( F \in K[[x_1, \ldots, x_n]] \) over a possibly larger field \( K \) such that \( [F] \) is a “generic point” of the “Zariski closure” \( Z \subset \mathbb{A}^\infty \) of the set \( \{ [f_i] : i \in I \} \). The first main result of [dFM07] says, roughly, that

\[
\lim_{j \to \infty} c_0(f_j) = \lim_{j \to \infty} c_0(f_i) \quad \text{for some subsequence} \quad i_1 < i_2 < \cdots .
\]

One needs to be rather careful with “Zariski closure” and “generic point” in an infinite dimensional setting.

The non-standard method in [dFM07] is used to get a correct “generic point.” Here I use a more explicit construction, getting the Taylor polynomials of \( F \) inductively.

Let \( k \) be a field and \( k[[x_1, \ldots, x_n]] \) the ring of formal power series over \( k \). For \( f (x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]] \), let

\[
t_m (f) \in k[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)^{m+1} =: P_m (m)
\]
denote the truncation, mapping a power series to its degree \( m \) Taylor polynomial. We can view \( P_m (m) \) as an affine space over \( k \) with natural truncation maps \( t_{m, m'} : P_m (m) \to P_{m'} (m) \) for every \( m' \geq m \).

The following technical lemma makes it possible to construct the correct limits of power series.

**Lemma 27.** Let \( k \) be a field and \( f_i (x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]] \) power series indexed by an infinite set \( I \). There are (nonunique) infinite subsets \( I \supseteq I_0 \supseteq I_1 \supseteq \cdots \) such that

1. For every \( m \), the Zariski closure \( Z_m \subset P_m (m) \) of \( \{ t_m (f_i) : i \in I_m \} \) is irreducible (over \( k \)),
2. For every Zariski closed \( Y \subset Z_m \) there are only finitely many \( i \in I_m \) such that \( t_m (f_i) \in Y \), and
3. For every \( m' \geq m \) the truncation maps \( t_{m', m} : Z_{m'} \to Z_m \) are dominant.

Proof. Apply (30) to \( \{ f_i : i \in I \} \) as points in \( P_m (0) \) to obtain \( I_0 := I' \).

Assume now that we already have \( Z_j \subset P_n (j) \) and \( I_j \subset I \) for \( j \leq m \) satisfying the properties (27) 1–3.

Apply (30) to \( \{ t_{m+1} (f_i) : i \in I_m \} \) as points in \( P_m (m+1) \) to obtain \( I_{m+1} := (I_m)' \). The properties (27) 1–2 hold by construction. The truncation map \( t_{m+1, m} : Z_{m+1} \to Z_m \) is defined. The closure of its image contains all the points \( t_{m} (f_i) \) for \( i \in I_{m+1} \). Since (27) 2 holds for \( Z_m \), we conclude that \( t_{m+1, m} : Z_{m+1} \to Z_m \) is dominant. \( \square \)
Sec.IV.1, a
generic point algebraically closed extension of infinite transcendence degree. According to [Wei62, p]
closed Y
map the coordinates of X
(Generic points ` a la Weil)

Lemma 30. Let these be \{p_i : i \in I\}. Then there is a (nonunique) subset \(I' \subset I\) such that

1. the Zariski closure \(Z(I')\) of \(\{p_i : i \in I'\}\) is irreducible, and
2. for every Zariski closed \(Y \subset Z(I')\) there are only finitely many \(i \in I'\) such that \(p_i \in Y\).

Theorem 29. With the above notation, \(c_0(F)\) is a (Euclidean) limit point of \(\{c_0(f_i) : i \in I\} \subset \mathbb{R}\).

Proof. By construction, \(t_m(F)\) is a generic point of \(Z_m\), hence, as in [21], there is a Zariski open \(U_m \subset Z_m\) such that \(c_0(h) = c_0(t_m(F))\) for every \(h \in U_m\). Thus there is a \(j \in I_m\) such that \(c_0(t_m(F)) = c_0(t_m(f_j))\). Therefore, using [18] twice, we obtain that

\[
\left|c_0(F) - c_0(f_j)\right| \leq \left|c_0(F) - c_0(t_m(F))\right| + \left|c_0(t_m(f_j)) - c_0(f_j)\right| \leq \frac{2n}{m+1}.
\]

Lemma 30. Let \(X\) be a Noetherian topological space and \(\{p_i : i \in I\}\) an infinite collection of (not necessarily distinct) points. Then there is a (nonunique) subset \(I' \subset I\) such that

1. the Zariski closure \(Z(I')\) of \(\{p_i : i \in I'\}\) is irreducible, and
2. for every Zariski closed \(Y \subset Z(I')\) there are only finitely many \(i \in I'\) such that \(p_i \in Y\).

Proof. The Zariski closure of \(\{p_i : i \in I\}\) has only finitely many irreducible components. Pick any, say \(X_1\), and set \(I_1 := \{i \in I : p_i \in X_1\}\).

If some irreducible closed \(X_2 \subset X_1\) contains infinitely many \(p_i\) for \(i \in I_1\), let these be \(\{p_i : i \in I_2\}\). We construct \(X_3 \subset X_2\) similarly, and so on.

By the Noetherian property, eventually we obtain an infinite subset \(I' := I_r \subset I\) such that \(X_r\), the Zariski closure of \(\{p_i : i \in I_r\}\), is irreducible, and for every Zariski closed \(Y \subset X_r\) there are only finitely many \(i \in I_r\) such that \(p_i \in Y\).

Note that if a point \(p\) appears among the \(p_i\) infinitely many times, then \(Z(I') = \{p\}\) satisfies the requirements.

31 (Generic points à la Weil). Let \(X \subset \mathbb{A}^n_k\) be an irreducible \(k\) variety and \(K \supset k\) an algebraically closed extension of infinite transcendence degree. According to [Wei62 Sec.IV.1], a generic point of \(X\) is a \(K\)-point \(g_X \in X(K)\) such that a polynomial \(p \in k[x_1, \ldots, x_n]\) vanishes on \(g_X\) iff it vanishes on \(X\). Equivalently, the restriction map \(k(X) \to k(g_X)\) is an isomorphism where \(k(g_X) \subset K\) is the field generated by the coordinates of \(g_X\).

It is easy to construct generic points as follows. We may assume that \(\dim X = d\) and the projection to the first \(d\) coordinates \(\pi : X \to \mathbb{A}^d\) is dominant. Pick \((p_1, \ldots, p_d) \in \mathbb{A}^d\) such that the \(p_i \in K\) are algebraically independent over \(k\). Any

\[
(p_1, \ldots, p_d, p_{d+1}, \ldots, p_n) \in \pi^{-1}(p_1, \ldots, p_d) \subset X
\]

is a generic point of \(X\) over \(k\).
Let $f : X \to Y$ be a dominant map of irreducible $k$-varieties. Let $g_Y \in Y$ be a generic point. We can view $f^{-1}(g_Y)$ as a $k(g_Y)$-variety. Any generic point $g_X \in f^{-1}(g_Y)$ as $k(g_Y)$-variety is also a generic point of $X$ as a $k$-variety.

Thus, given a tower of irreducible $k$-varieties

$$Z_1 \leftarrow Z_2 \leftarrow Z_3 \leftarrow \cdots$$

we can get a compatible system of generic points

$$g_1 \leftarrow g_2 \leftarrow g_3 \leftarrow \cdots$$

5. Taylor polynomials and thresholds

Let $f$ be a holomorphic function and assume that $(f = 0)$ defines an isolated singularity at the origin. A result of [Hir65] says that there is an $m > 0$ such that if $h$ is any other holomorphic function that agrees with $f$ up to high order then there is a local biholomorphism $\phi : (0 \in \mathbb{C}^n) \to (0 \in \mathbb{C}^n)$ that takes $f$ to $h$. In particular, the singularities $(f = 0)$ and $(h = 0)$ are analytically isomorphic and all their local analytic invariants are the same.

This result completely fails if $(f = 0)$ does not have an isolated singularity. If $p_m$ is a general degree $m$ homogeneous polynomial then $(f + p_m = 0)$ has only isolated singularities. In particular, $(f = 0)$ and $(f + p_m = 0)$ are not isomorphic.

What about their log canonical thresholds? In general, the answer is again negative. For instance, if $f(z_1, \ldots, z_{n-1})$ is any holomorphic function then by (16.2)

$$c_0(f(z_1, \ldots, z_{n-1}) + z_n^m) = \min \{c_0(f) + \frac{1}{m}, 1\}.$$  

Note that $(f = 0) \subset \mathbb{C}^n$ is very non-isolated; it is equisingular along the $z_n$-axis.

The relevant analog of the Hironaka theorem should be about holomorphic functions $f$ for which the origin is isolated “as far as the log canonical threshold is concerned.” That is, functions $f$ such that

$$c_0(f) < c_p(f) \quad \text{for every } p \neq 0 \text{ near } 0.$$  

More generally, we consider the case when $c_0(f)$ is computed by a divisor $E$ whose center on $\hat{\mathbb{A}}^n$ is the origin. That is, if there is a birational morphism $\phi : X \to \hat{\mathbb{A}}^n$ and a divisor $E$ on $X$ such that the center of $E$ on $\hat{\mathbb{A}}^n$ is the origin and

$$c_0(f) = \frac{1 + \text{mult}_E \text{Jac}(\phi)}{\text{mult}_E f}.$$  

In this case an easy argument of [dFM07] shows that $c_0(f + p) \leq c_0(f)$ if $\text{mult}_0 p \gg 1$; see (33). Here I prove that, in fact, $c_0(f + p) = c_0(f)$. For the proof of (13) one needs a more general version, when certain perturbations of low degree terms are also allowed. This is considered in Section 6.

Theorem 32. Let $f \in k[[x_1, \ldots, x_n]]$ be a power series such that $c_0(f)$ is computed by a divisor $E$ whose center on $\hat{\mathbb{A}}^n$ is the origin. Let $p \in k[[x_1, \ldots, x_n]]$ be a power series such that $\text{mult}_0 p > \text{mult}_E f$. Then $c_0(f + p) = c_0(f)$.

The proof of (32) relies on rather heavy machinery; we need the full force of the MMP. Before starting it, let us review a simple argument which gives the inequality $c_0(f + p) \leq c_0(f)$. 


33. Take a log resolution $\pi : X \to \mathbb{A}^n$ and let $E \subset X$ be a divisor such that $\pi(E) = 0$. If $\text{mult}_0 p > \text{mult}_E f$ then $p$ vanishes along $E$ with multiplicity $> \text{mult}_E f$, thus

$$\text{mult}_E(f + p) = \text{mult}_E f. \tag{33.1}$$

If we choose $E$ such that

$$c_0(f) = \frac{1 + \text{mult}_E \text{Jac}(\pi)}{\text{mult}_E f},$$

then we obtain that

$$c_0(f + p) \leq \frac{1 + \text{mult}_E \text{Jac}(\pi)}{\text{mult}_E(f + p)} = \frac{1 + \text{mult}_E \text{Jac}(\pi)}{\text{mult}_E f} = c_0(f). \tag{33.2}$$

In this argument, the dependence on $f$ is rather subtle. We need a complete log resolution in order to choose the right divisor $E$. Unfortunately, $X \to \mathbb{A}^n$ is not a log resolution for the perturbed hypersurface $(f + p = 0)$, except in the isolated singularity case. A priori, the log canonical threshold of $f + p$ may be computed by a divisor $E'$ which does not even appear on $X$. The values of $\text{mult}_{E'} \text{Jac}(\pi)$ and $\text{mult}_E f$ may be completely different from $\text{mult}_E \text{Jac}(\pi)$ and $\text{mult}_E f$. Thus we can not just write $f = (f + p) + (-p)$ and obtain the reverse inequality $c_0(f) = c_0((f + p) + (-p)) \leq c_0(f + p)$.

Note, however, that we are on the right track. By the ACC conjecture [5], there are no log canonical thresholds in some interval $(c_0(f) - \epsilon, c_0(f))$, and, by (13), $c_0(f + p) \geq c_0(f) - n/\text{mult}_0 p$. Thus $c_0(f + p) \geq c_0(f)$ if $\text{mult}_0 p > n/\epsilon$.

Remark 34. Assume that $c_0(f)$ is computed by a divisor $E$ whose center $Z(E)$ on $\mathbb{A}^n$ is not the origin. If $p$ vanishes along $Z(E)$ with multiplicity $> \text{mult}_E f$, then $c_0(f + p) = c_0(f)$ should hold. The problem with the proof is that usually $E$ can not be realized on an algebraic variety $X \to \mathbb{A}^n$.

It should be possible to obtain $c_0(f)$ using the MMP and a log resolution of $(\mathbb{A}^n, (f = 0))$. However, the standard references seem to assume that we consider MMP over a base which itself is a variety or an analytic space.

6. Proofs

For the proofs it is more convenient to change to additive notation. If $X$ is a complex manifold and $D_i \subset X$ are divisors with local equations $f_i$ then we say that $(X, \sum c_i D_i)$ is lc (or log canonical) if $(\prod_i |f_i|^{-c_i})^{1-\epsilon}$ is locally $L^2$ for every $\epsilon > 0$.

In the sequel we also need these concepts when $X$ itself is singular. See [KM98 sec.2.3] for a good introduction.

35 (Proof of (33)). By (30), there is a proper birational morphism $\pi : X \to \mathbb{A}^n$ such that $X$ is $\mathbb{Q}$-factorial and $E$ is (birational to) the unique exceptional divisor of $\pi$. Completing over the origin, we obtain a proper birational morphism $\hat{\pi} : \hat{X} \to \mathbb{A}^n$ such that $E$ is (birational to) the unique exceptional divisor $\hat{E}$ of $\hat{\pi}$ and $E$ is $\mathbb{Q}$-Cartier. Thus $(\hat{X}, \hat{E} + c \cdot \hat{\pi}^{-1}(f = 0))$ is lc where $c = c_0(f)$ and $\hat{\pi}^{-1}$ denotes the birational transform of a divisor.

We need to prove that $(\hat{X}, \hat{E} + c \cdot \hat{\pi}^{-1}(f + p = 0))$ is also lc. As a first step, we claim that

$$(\hat{\pi}^{-1}(f = 0)|_E = (\hat{\pi}^{-1}(f + p = 0))|_E. \tag{32.1}$$

In order to prove this, let us remember how we compute $\hat{\pi}^{-1}(f = 0)$. The zero divisor of $f \circ \hat{\pi}$ is $r\hat{E} + \hat{\pi}^{-1}(f = 0)$ for some $r > 0$. It is better to work with Cartier
Lemma 36. Let \( m > 0 \) such that \( mrr \hat{E} \) and \( m \cdot \hat{\pi}^{-1}(f = 0) \) are both Cartier. Thus, if \( y = 0 \) is a local equation of \( mrr \hat{E} \) then the local equation of \( m \cdot \hat{\pi}^{-1}(f = 0) \) is \( (f^m \circ \hat{\pi})/y = 0 \). By assumption, \( \text{mult}_E p \geq \text{mult}_0 p > \text{mult}_E f \), thus every term of

\[
(f + p)^m \circ \hat{\pi} = \sum_i \left( \frac{m}{i} \right) (f^i \cdot p^{n-i}) \circ \hat{\pi}
\]

is divisible by \( y \) and all except \( (f^m \circ \hat{\pi})/y \) vanish along \( \hat{E} \). Thus (32.1) holds.

By the precise version of inversion of adjunction (38) this means that

\[
(\hat{X}, \hat{E} + c \cdot \hat{\pi}^{-1}(f = 0)) \text{ is lc iff } (\hat{X}, \hat{E} + c \cdot \hat{\pi}^{-1}(f + p = 0)) \text{ is.}
\]

Again we have to be careful with the formal case. The easiest is to notice that

\[
\text{mult}_E p \geq \text{mult}_0 p > \text{mult}_E f.
\]

Thus, if \( y > 0 \) such that \( \pi^* \hat{\pi}^{-1} = K_{\hat{E}} + \Delta_{\hat{E}} \) is not yet known to be lc, but, by (18),

\[
(\hat{X}, \hat{E} + c \cdot \hat{\pi}^{-1}(f + p = 0)) \text{ is lc for } m > \text{mult}_E f.
\]

This implies that \( (\hat{X}, \hat{E} + (t_m(f + p) = 0)) \text{ is lc, thus } c_0(t_m(f + p)) \geq c \text{ for } m \gg 1.
\]

By (15) this implies that

\[
c_0(f + p) = \lim_{m \to \infty} c_0(t_m(f + p)) \geq c = c_0(f). \;
\]

Lemma 37. Let \( \hat{D} \subset \hat{k}^n \) be a divisor and \( E \) a divisor over \( \hat{k}^n \) such that the center of \( E \) on \( \hat{k}^n \) is the origin and \( E \) computes the log canonical threshold of \( \hat{D} \).

Then there is a proper birational morphism \( \pi : X \to \hat{k}^n \) with only 1 exceptional divisor, which is (birational to) \( E \).

Proof. The first step is to note that \( E \) is an algebraic divisor. That is, there is a proper birational morphism \( g : Y \to \hat{k}^n \) such that \( g \) is an isomorphism outside the origin, \( E \) is a divisor on \( Y \) and \( \text{Ex}(g) \) is a snc divisor. This follows from the resolution of indeterminacies of maps. See [KSC04, p.113] for an elementary proof.

If \( \hat{D} \) is defined by the power series \( f \), set \( D_m := (t_m f = 0) \). Set \( c = c_0(f) \). Write

\[
c \cdot g^* \hat{D} = K_Y + \Delta + c \cdot g_+^{-1} \hat{D} \quad \text{and} \quad c \cdot g^* D_m = K_Y + \Delta_m + c \cdot g_+^{-1} D_m.
\]

Note that \( \Delta_m = \Delta \) for \( m \gg 1 \), thus \( E \) appears in \( \Delta_m \) with coefficient \( 1 \). The pair \((\hat{k}^n, c \cdot D_m)\) is not yet known to be lc, but, by (15), \( c_0(D_m) \) converges to \( c = c_0(\hat{D}) \).

Thus, for \( c' < c \) and \( m \gg 1 \), \((\hat{k}^n, c' \cdot D_m)\) is klt and if we write

\[
c' \cdot g^* D_m = K_Y + \Delta'_m + c' \cdot g_+^{-1} D_m
\]

then \( E \) appears in \( \Delta'_m \) with coefficient \( > 0 \). Apply (37) to \( E \) and \((\hat{k}^n, c' \cdot D_m)\) to get \( \pi : X \to \hat{k}^n \) as required. \( \square \)

Lemma 37. Let \( X \) be a variety and \( D \) a \( \mathbb{Q} \)-divisor on \( X \) such that \( (X, D) \) is klt. Let \( E \) be a divisor over \( X \) such that \( 0 \leq a(E, X, D) < 1 \). Then there is a proper birational morphism \( \pi : X_E \to X \) with only 1 exceptional divisor, which is (birational to) \( E \).

Proof. Let \( g : Y \to X \) be a log resolution such that \( E \) is a divisor on \( Y \). Write

\[
K_Y + cE + A - B + g_+^{-1} D \sim_\mathbb{Q} g^* (K_X + D),
\]
where \( e = a(E, X, D) \), the \( \mathbb{Q} \)-divisors \( A, B \) are effective and have no common components and do not contain \( E \). For some \( 0 < \eta < 1 \), run the \( (Y, eE + (1+\eta)A) \)-MMP [BCHM06] to obtain \( \pi : X_E \rightarrow X \). Note that

\[
K_Y + eE + (1 + \eta)A \sim_{\mathbb{Q}} \eta A + B + g^*(K_X + D)
\]

and on the right hand side \( \eta A + B \) contains every \( g \)-exceptional divisor with positive coefficient, save \( E \). Thus the restriction of any birational transform of \( K_Y + eE + (1 + \eta)A \) to the (birational transform) of \( E \) is \( \mathbb{Q} \)-linearly equivalent to an effective divisor plus a pulled-back divisor, hence we never contact \( E \). On the other hand, an effective exceptional divisor is never relatively nef, thus we have to contract \( \text{Supp}(\eta A + B) \). Thus \( \pi : X_E \rightarrow X \) has only one exceptional divisor, which is (birational to) \( E \).

38 (Precise inversion of adjunction). As stated in [20], inversion of adjunction is only an inequality. It is possible to make it into an equality. Assume that \( X \) is smooth, \( E \subset X \) is a hypersurface and \( \Delta \) an effective \( \mathbb{Q} \)-divisor which does not contain \( E \). The precise inversion of adjunction says that

\[
(X, E + \Delta) \text{ is lc near } E \iff (E, \Delta|_E) \text{ is lc.} \tag{38.1}
\]

In [38] we use a version where \( X \) is singular. For the statements and proofs see [Kol92, Sec.17] and [Kaw07].

39 (Proof of (7)). It follows from [10] that the set of accumulation points of \( \mathcal{H}_T_n \) contains \( \mathcal{H}_T_{n-1} \setminus \{1\} \). It is thus enough to prove that it is contained in \( \mathcal{H}_T_{n-1} \).

Let \( f_i \in K[[x_1, \ldots, x_n]] \) be power series such that \( c_0(f_i) \) is a nonconstant sequence converging to some \( c \in \mathbb{R} \). By passing to a subsequence we may assume that \( c_0(f_i) \neq c_0(f_j) \) for \( i \neq j \). By [20], we get a power series \( F \in K[[x_1, \ldots, x_n]] \) such that \( c_0(F) = c \).

If \( c_0(F) \) is computed by a divisor whose center \( Z(E) \) is not the origin, then localizing at the generic point of \( Z(E) \) and completing gives a complete, regular, local ring of dimension \( n - \dim Z(E) \) and a power series \( F^* \) such that \( c_0(F^*) = c_0(F) \). Thus \( c_0(F) \in \mathcal{H}_T_{n-1} \) and we are done. Otherwise \( c_0(F) \) is computed by a divisor \( E \) whose center is the origin. By [10] this implies that \( c_0(F) = c_0(f_j) \) for some infinite subsequence \( i_1 < i_2 < \cdots \), a contradiction. \( \square \)

**Proposition 40.** Let \( k \supset k \) be a field extension and \( F(x_1, \ldots, x_n) \in K[[x_1, \ldots, x_n]] \) a power series. Assume that \( c_0(F) \) is computed by a divisor \( E \) whose center on \( \mathbb{A}^n \) is the origin. Let \( Z_m \subset P_n(m) \) denote the \( k \)-Zariski closure of \( t_m(F) \). Then there is an \( m \geq 0 \) and a nonempty open subset \( U_m \subset Z_m \) such that if \( f(x_1, \ldots, x_n) \in K[[x_1, \ldots, x_n]] \) is any power series such that \( t_m(f) \in U_m \) then \( c_0(f) = c_0(F) \).

**Proof.** By [30], there is a proper birational morphism \( \phi : X \rightarrow \mathbb{A}^n \) defined over \( K \) with a unique \( \phi \)-exceptional divisor, which is (birational to) \( E \).

The data \( \phi, X \) and \( E \) are defined over a finitely generated subextension of \( K/k \), hence over \( k(t_m(F)) \) for all \( m \gg m_K \) for some \( m_K \). Set \( m_E := \text{mult}_E(F \circ \phi) \) and choose any \( m > \max\{m_E, m_K\} \).

Since \( \phi, X \) and \( E \) are defined over the generic point of \( Z_m \), there is a Zariski open subset \( U_m \subset Z_m \) such that \( \phi, X \) and \( E \) can be extended to be defined over \( U_m \). Moreover, we may assume that for any \( u \in U_m \), the resulting \( \phi(u) : X(u) \rightarrow \mathbb{A}^n \)
is birational, $E(u) \subset X(u)$ is a divisor with the same discrepancy as $E \subset X$, $F(u)(\phi_1(u), \ldots, \phi_n(u))$ vanishes along $E(u)$ with multiplicity $m_E$.

By shrinking $U_m$ if necessary, we may also assume that if $t_m(f) \in U_m$ then $c_0(t_m(f)) = c_0(t_m(F))$.

Let now $f$ be any power series with $t_m(f) \in U_m$ and apply (32) first to $t_m(F)$ and $p := F - t_m(F)$ and then to $t_m(f)$ and $p := f - t_m(f)$. We obtain that

$$
c_0(F) = c_0(t_m(F) + (F - t_m(F))) = c_0(t_m(F)), \quad \text{and}
$$

$$
c_0(f) = c_0(t_m(f) + (f - t_m(f))) = c_0(t_m(f)).
$$

Since $c_0(t_m(f)) = c_0(t_m(F))$ by the choice of $U_m$, we conclude that $c_0(f) = c_0(F)$. □

7. Technical comments on inversion of adjunction

41 (Inversion of adjunction: the isolated singularity case). Let us consider inversion of adjunction (20) in case $D := (f = 0)$ has an isolated singularity. Then there is an algebraic hypersurface $D'$ with an isolated singularity at the origin such that a formal change of coordinates transforms $D$ to $D'$. Let $\pi : X \to \mathbb{A}^n$ be an algebraic resolution of $(\mathbb{A}^n, D')$ (in a neighborhood of the origin) that is an isomorphism outside the origin. By completion, we get a log resolution $\hat{\pi} : \hat{X} \to \hat{\mathbb{A}}^n$ of $(\hat{\mathbb{A}}^n, D)$ which is an isomorphism outside the origin. Up to coordinate change we may assume that $L \subset \hat{\mathbb{A}}^n$ is a coordinate subspace that is the completion of a linear subspace $L \subset \mathbb{A}^n$.

Set $D_m := (t_m(f) = 0)$. Then $\pi : X \to \mathbb{A}^n$ is a log resolution of $(\mathbb{A}^n, D_m)$ in a neighborhood of the origin for $m \gg 1$.

The proof of inversion of adjunction given in [Kol92, Sec.17] shows that for each $m \gg 1$ there is an irreducible component $E_m \subset \pi^{-1}_* L \cap \pi^{-1}(0)$ such that $a(E_m, L, c_0(f) \cdot D_m|_L) \geq 1$. Since $\pi^{-1}_* L \cap \pi^{-1}(0)$ has only finitely many irreducible components, by passing to a subsequence we may assume that $E = E_m$ does not depend on $m$. By (33), for divisors with center at the origin the discrepancy stabilizes, hence

$$a(E, L, c_0(f) \cdot D|_L) = a(E, L, c_0(f) \cdot D_m|_L) \text{ for } m \gg 1.$$

Thus $c_0(f|_L) \leq c_0(f)$. □

As in (24), this implies the following special case of (17):

Lemma 42. Let $f, g \in k[[x_1, \ldots, x_n]]$ be formal power series with isolated singularities at the origin. Then $c_0(f + g) \leq c_0(f) + c_0(g)$. □

43. Now we are ready to prove the theorems of Section 3 in the formal case. As noted in (28), we only need to show that

$$\lim_{m \to \infty} \inf c_0(t_m(f)) \geq c_0(f).$$

Let $f \in k[[x_1, \ldots, x_n]]$ be a formal power series with degree $m$ Taylor polynomial $t_m(f)$. Let $h$ be a general degree $m + 1$ homogeneous polynomial. Then both

$$t_m(f) + h \quad \text{and} \quad (f - t_m(f)) - h
$$

have isolated singularities at the origin and so

$$c_0(f) \leq c_0(t_m(f) + h) + c_0(f - t_m(f) - h).$$
On the other hand, by the algebraic version of (17), $c_0(t_m(f) + h) \leq c_0(t_m(f)) + c_0(h)$. Therefore, using (11.5) we get that

$$c_0(f) \leq c_0(t_m(f)) + c_0(h) + c_0(f - t_m(f) - h) \leq c_0(t_m(f)) + \frac{c_0}{m+1} + \frac{c_0}{m+1}.$$  

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References

[AGZV85] V. I. Arnol’d, S. M. Guseĭn-Zade, and A. N. Varchenko, Singularities of differentiable maps. Vols. I–II, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985, Translated from the Russian by Ian Porteous and Mark Reynolds. MR MR777682 (86f:58018)

[Ale03] Valery Alexeev, *Two two-dimensional terminations*, Duke Math. J. **69** (1993), no. 3, 527–545. MR MR1208810 (94a:14013)

[Ale04] Valery Alexeev, *Boundedness and $K^2$ for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810. MR MR1298994 (95k:14048)

[Ati70] M. F. Atiyah, *Resolution of singularities and division of distributions*, Comm. Pure Appl. Math. **23** (1970), 145–150. MR MR0256156 (41 #815)

[BB92] A. A. Borisov and L. A. Borisov, *Singular toric Fano three-folds*, Mat. Sb. **183** (1992), no. 2, 134–141. MR MR1166957 (93i:14034)

[BCHM06] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0610203, 2006.

[Ber71] I. N. Bernšteǐn, *Modules over a ring of differential operators. An investigation of the fundamental solutions of equations with constant coefficients*, Funkcional. Anal. i Priložen. **5** (1971), no. 2, 1–16. MR MR0290097 (44 #7282)

[BGK05] Charles P. Boyer, Krzysztof Galicki, and János Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005), no. 1, 557–580. MR MR2178969 (2006j:53058)

[DK01] Jean-Pierre Demailly and János Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 545–556. MR MR1852009 (2002e:32032)

[dFM07] Tommaso de Fernex and Mircea Mustaţă, *Limits of log canonical thresholds*, http://www.citebase.org/abstract?id=oai:arXiv.org:0710.4978, 2007.

[GKP89] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics*, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989, A foundation for computer science. MR 91f:00001

[GS58] I. M. Gel’fand and G. E. Šilov, *Obozshchennyje funkcii i deistviya iad nimi, Obobščennye funkci, Vypusk 1. Generalized functions, part 1*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. MR MR0097715 (20 #4182)

[Hir65] Heisuke Hironaka, *On the equivalence of singularities. I*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 153–200. MR MR0201433 (34 #1317)

[Hör58] Lars Hörmander, *On the division of distributions by polynomials*, Ark. Mat. **3** (1958), 555–568. MR MR0102143 (23 #A2044)

[Kaw07] Masayuki Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. **167** (2007), no. 1, 129–133. MR MR2264806 (2008a:14025)

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR MR1658959 (2000b:14018)

[Kol92] János Kollár (ed.), *Flips and abundance for algebraic threefolds*, Sociéte Mathématique de France, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR 94f:14013
[Kol94], Log surfaces of general type; some conjectures, Classification of algebraic varieties (L’Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 261–275. MR MR1272703 (95c:14042)

[Kol97], Singularities of pairs, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR MR1492525 (99m:14033)

[Kol05], Einstein metrics on five-dimensional Seifert bundles, J. Geom. Anal. 15 (2005), no. 3, 445–476. MR MR2190241 (2007c:53056)

[Kol07], Einstein metrics on connected sums of $S^2 \times S^3$, J. Differential Geom. 75 (2007), no. 2, 259–272. MR2286822 (2007k:53061)

[Kou76], A. G. Kouchnirenko, Poly` edres de Newton et nombres de Milnor, Invent. Math. 32 (1976), no. 1, 1–31. MR MR0419433 (54 #7454)

[KSC04] János Kollár, Karen E. Smith, and Alessio Corti, Rational and nearly rational varieties, Cambridge Studies in Advanced Mathematics, vol. 92, Cambridge University Press, Cambridge, 2004. MR MR2062787 (2005i:14011)

[Kuw99] Takayasu Kuwata, On log canonical thresholds of surfaces in $\mathbb{C}^3$, Tokyo J. Math. 22 (1999), no. 1, 245–251. MR MR1692033 (2000e:14055)

[Loj58] S. /suppress Lojasiewicz, Division d’une distribution par une fonction analytique de variables r´eelles, C. R. Acad. Sci. Paris 246 (1958), 683–686. MR MR0096120 (20 #2616)

[MP04] James McKernan and Yuri Prokhorov, Threelfold thresholds, Manuscripta Math. 114 (2004), no. 3, 281–304. MR MR2075967 (2005g:14011)

[Mus02] Mireia Mustatǎ, Singularities of pairs via jet schemes, J. Amer. Math. Soc. 15 (2002), no. 3, 599–615 (electronic). MR MR1896234 (2003b:14005)

[OT87] Takao Ohsawa and Kenzo Takegoshi, On the extension of $L^2$ holomorphic functions, Math. Z. 195 (1987), no. 2, 197–204. MR MR892051 (88g:32029)

[Sch50] L. Schwartz, Th´eorie des distributions. Tome I, Actualit´es Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9, Hermann & Cie., Paris, 1950. MR MR0035918 (12,341d)

[Sho88] V. V. Shokurov, Problems about fano varieties, Birational Geometry of Algebraic Varieties, Open Problems, 1988, pp. 30–32.

[Slo03] N. J. A Sloane, The on-line encyclopedia of integer sequences, AT&T Research, http://www.research.att.com/ njas/sequences/, 2003.

[Sou05] K. Soundararajan, Approximating 1 from below using $n$ Egyptian fractions, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0502247, 2005.

[Wei62] André Weil, Foundations of algebraic geometry, American Mathematical Society, Providence, R.I., 1962. MR MR0144898 (26 #2439)