METASTABILITY FOR THE DISSIPATIVE QUASI-GEOSTROPHIC EQUATION AND THE NON-LOCAL ENHANCEMENT

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Abstract. In this paper, we study the metastability for the 2-D linearized dissipative quasi-geostrophic equation with small viscosity \( \nu \) around the quasi steady state \( \theta_{\sin} = e^{-\nu t} \sin y \). We proved the linear enhanced dissipation and obtained the dissipation rate. Moreover, the new non-local enhancement phenomenon was discovered and discussed. Precisely we showed that the non-local term re-enhances the enhanced diffusion effect by the shear-diffusion mechanism.

1. Introduction

In this paper, we study the 2D dissipative quasi-geostrophic (QG) equation on the torus \( T_\delta^2 = \{(x, y) : x \in T_{2\pi \delta}, y \in T_{2\pi}\} \) with \( 0 < \delta < 1 \):

\[
\begin{aligned}
&\partial_t \Theta + \nu(-\Delta)^s \Theta + V \cdot \nabla \Theta = 0, \\
&V = (-R_2, R_1) \Theta,
\end{aligned}
\]

Here \( s \in (0, 1], \nu > 0 \) is the dissipative coefficient, \( \Theta : T_\delta^2 \to \mathbb{R} \) is a real-valued scalar function represents the potential temperature, \( V \) is the fluid velocity and \( R_1 = \partial_x(-\Delta)^{\frac{1}{2}} \) and \( R_2 = \partial_y(-\Delta)^{\frac{1}{2}} \) are the Riesz transforms in \( T_\delta^2 \). The cases \( s > \frac{1}{2}, s = \frac{1}{2}, \text{ and } s < \frac{1}{2} \) are called sub-critical, critical, and super-critical respectively.

It is easy to see that the system (1.1) has a quasi steady state

\[
\theta_{\sin} = e^{-\nu t} \sin y, \quad V_{\sin} = (e^{-\nu t} \cos y, 0).
\]

To understand the long time behavior of the solution with initial data \( \Theta_{in} \) close to \( \sin y \), it is natural to introduce the perturbation \( \Theta = \theta_{\sin} + \theta \) and \( V = V_{\sin} + U \). Then \((\theta, U)\) solves the following system:

\[
\begin{aligned}
&\partial_t \theta + L_{\nu,s}(t) \theta = -U \cdot \nabla \theta \\
&U = (-R_2, R_1) \Theta,
\end{aligned}
\]

where

\[
L_{\nu,s}(t) = \nu(-\Delta)^s + e^{-\nu t} \cos y \partial_x(1 - (-\Delta)^{-\frac{1}{2}}).
\]

From which one can derive the linearized QG equation:

\[
\begin{aligned}
&\partial_t \theta + L_{\nu,s}(t) \theta = 0 \\
&\theta|_{t=0} = \theta_{in}.
\end{aligned}
\]

Taking Fourier transform in \( x \) of (1.2), we get that for \( \alpha \neq 0 \)

\[
\begin{aligned}
&\partial_t \hat{\theta} + \nu(-\Delta_\alpha)^s \hat{\theta} + i\alpha e^{-\nu t} \cos y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \hat{\theta} = 0, \\
&\hat{\theta}|_{t=0} = \hat{\theta}_{in}.
\end{aligned}
\]
Here $|\alpha| \geq c_0 > 1$ is the wave number and $\Delta_\alpha = \partial_{yy} - \alpha^2$. We also denote the linear operator by

$$L_{\alpha, \nu, s} = \nu (-\Delta_\alpha)^s + i \alpha e^{-\nu t} \cos y \left( 1 - (-\Delta_\alpha)^{-\frac{1}{2}} \right).$$

In this paper, we mainly study the long time behavior of the solutions to \((1.3)\).

1.1. **Historical comments.** The 2-D dissipative quasi-geostrophic (QG) equation is derived from the study of geophysical fluid dynamics [29] and attracted a lot of attention from mathematicians as it serves as a lower dimensional model of the 3-D Navier-Stokes equations [10]. The existence of global regular solutions for the sub-critical QG equation was given by Constantin-Wu [11] and Resnick [30]. For the critical case, the global regularity was proved independently by Kiselev-Nazarov-Volberg [22] and Caffarelli-Vasseur [6]. While for the super-critical case, whether solutions stay regular or can blow up is still open, and Dabkowski-Kiselev-Silvestre-Vicol [15] proved global regularity for the slightly super-critical QG equation. For more results about QG equation, we refer to [9, 21, 23].

Metastability is common in viscous fluid, which is a quasi-stable state of a dynamical system other than the system’s state of least energy. A famous example is the Kolmogorov flow for the 2-D incompressible Navier-Stokes equations. The previous studies [3, 14, 25, 28, 34] have shown that the solution with initial data closed to $(\cos y, 0)$ will approach to the so called bar states in a much shorter time than the diffusive time $O(\frac{1}{\nu})$. Then they dominate the dynamics for very long time intervals. Afterwards they decay on the diffusive time scale. See [2] for a similar phenomena for Burgers equation with small viscosity.

A reasons for metastability is the so called enhanced dissipation. It is sometimes referred to by modern authors as the ‘shear-diffusion mechanism’. This decay rate is much faster than the diffusive decay of $e^{-\nu t}$. The mechanism leading to the enhanced dissipation is due to the mixing. Generally speaking, the sheared velocity sends information to higher frequency, the diffusion term ‘kill’ the information in the higher frequency. The degeneracy rate of sheared velocity is corresponding to the lowest speed how the information moves to higher frequency, which leads to the different enhanced dissipation rates [1, 4, 12, 32]. Also the stronger diffusion gives stronger enhanced dissipation [19]. We refer to some recent results for the enhanced dissipation phenomena of different flows in the fluid motions: Couette flow [5, 8, 26, 27], Poiseuille flow [13, 16] and other flows [18, 24].

In this paper we proved the enhanced dissipation for the linearized dissipative QG equations. Moreover we discovered and studied the non-local enhancement phenomenon, which means that, the non-local term re-enhances the enhanced dissipation.

1.2. **Main results.** Our first result is the linear enhanced dissipation for the critical case:

**Theorem 1.1 (Critical case).** Suppose that $\theta$ solves \((1.2)\) with $s = \frac{1}{2}$. Then there exists $\nu_0 \leq 1$, $c > 0$, $C > 1$ such that for $0 < \nu < \nu_0$ and $0 \leq t \leq C^{-1}\nu^{-1}$, it holds that

$$\|\theta(t)\|_{L^2(\mathbb{T}_2^d)} \leq C e^{-c\nu^2 t} \|\theta_{in}\|_{L^2(\mathbb{T}_2^d)}$$

where $\theta_{\neq}(t, x, y) = \theta(t, x, y) - \frac{1}{2\pi\delta} \int_{\mathbb{T}_2^{2\delta}} \theta(t, x, y) dy$.

In order to show the non-local enhancement phenomenon, we also studied a toy model and proved the following result.

**Theorem 1.2 (Toy model).** Suppose that $\theta$ solves

$$\begin{cases}
\partial_t \theta + \cos y \partial_x \theta + \nu(-\Delta)^{\frac{1}{2}} \theta = 0 \\
\theta|_{t=0} = \theta_{in}.
\end{cases}$$
Then there exists \( \nu_0 \leq 1, c > 0, C > 1 \) such that for \( 0 < \nu < \nu_0 \), it holds that

\[
\| \theta_{\neq}(t) \|_{L^2(T^2_\nu)} \leq C e^{-c_\nu \frac{1}{2} t} \| \theta_{in} \|_{L^2(T^2_\nu)}
\]

where \( \theta_{\neq}(t, x, y) = \theta(t, x, y) - \frac{1}{\sqrt{2\pi}} \int_{T^2_\nu} \theta(t, x, y) dy \).

We also study the linearized subcritical QG which is similar to the linearized Navier-Stokes equation. We find that the different non-local terms will affect the enhanced dissipation rate.

**Theorem 1.3** (Subcritical case). Suppose that \( \theta \) solves (1.2) with \( s = 1 \). Then there exists \( \nu_0 \leq 1, c > 0, C > 1 \) such that for \( 0 < \nu < \nu_0 \) and \( 0 \leq t \leq C^{-1} \nu^{-1} \), it holds that

\[
\| \theta_{\neq}(t) \|_{L^2(T^2_\nu)} \leq C e^{-c_\nu \frac{1}{2} t} \| \theta_{in} \|_{L^2(T^2_\nu)}
\]

where \( \theta_{\neq}(t, x, y) = \theta(t, x, y) - \frac{1}{\sqrt{2\pi}} \int_{T^2_\nu} \theta(t, x, y) dy \).

1.3. Discussion. Let us first introduce the generalized linear operator

\[
L_{\alpha, \nu, s, \tilde{s}} = \nu (-\Delta)^s + i\alpha \cos y (1 - (-\Delta)^{\tilde{s}})
\]

and

\[
L_{\alpha, \nu, s, \tilde{s}}^S = \nu (-\Delta)^s + i\alpha \cos y
\]

with \( s, \tilde{s} > 0 \). Let

\[
B_{\alpha, \tilde{s}} = \cos y (1 - (-\Delta)^{\tilde{s}}), \quad B^S = \cos y, \quad A_{\alpha, \tilde{s}} = -[\partial_y, B_{\alpha, \tilde{s}}] = \sin y (1 - (-\Delta)^{\tilde{s}}), \quad A^S = -[\partial_y, B^S] = \sin y.
\]

We summarize the results in the following table.

| Operator | Parameters | Dissipation rate | Equations | Reference |
|----------|------------|-----------------|-----------|-----------|
| (1) \( L_{\alpha, \nu, s, \tilde{s}} \) | \( s = \tilde{s} = 1 \) | \( e^{-c_\nu \frac{1}{2} t} \) | Linearized Navier-Stokes | \[20, 33, 34\] |
| (2) \( L_{\alpha, \nu, s}^S \) | \( s = 1 \) | \( e^{-c_\nu \frac{1}{2} t} \) | Transport diffusion | \[3, 19\] |
| (3) \( L_{\alpha, \nu, s, \tilde{s}} \) | \( s = 1, \tilde{s} = \frac{1}{2} \) | \( e^{-c_\nu \frac{1}{2} t} \) | Linearized sub-critical QG | this paper |
| (4) \( L_{\alpha, \nu, s, \tilde{s}} \) | \( s = \tilde{s} = \frac{1}{2} \) | \( e^{-c_\nu \frac{1}{2} t} \) | Linearized critical QG | this paper |
| (5) \( L_{\alpha, \nu, s}^S \) | \( s = \frac{1}{2} \) | \( e^{-c_\nu \frac{1}{2} t} \) | Transport fractional diffusion | this paper |

Comparing the results listed above, we have the following observations:

- The non-local term in Case (1) does not affect the dissipation rate.
- The result (3) in comparison with result (1) and (2), indicates that the non-local term accelerates the dissipation rate. Similar phenomenon happens in the fractional diffusion case, which can be found in result (4) and (5). Actually the non-local term is a compact perturbation to the non-self-adjoint operator \( L_{\alpha, \nu, s} \). However, the non-self-adjoint operator is sensitive to the compact perturbations.
- The stronger diffusion term leads to the faster decay rate. Indeed by modifying our proof, one can obtain that for \( 0 < s \leq 1 \),

\[
\left\| e^{-t L_{\alpha, \nu, s}} \left( \frac{1}{2} \right) \right\|_{L^2 \to L^2} \lesssim e^{-c_\nu \frac{1}{3} \alpha t}, \quad |\alpha| > 1,
\]

and

\[
\left\| e^{-t L_{\alpha, \nu, s}^S} \right\|_{L^2 \to L^2} \lesssim e^{-c_\nu \frac{1}{3+\alpha} t}, \quad \alpha \neq 0.
\]
As for the second result in comparison with which in [19], we can remove the logarithm loss by using suitable time weight.

- Our method works well for the transport fractional diffusion equation with general sheared velocity \( w(y), 0 \):
  \[
  \partial_t \theta + w(y) \partial_x \theta + \nu(-\Delta)^s \theta = 0, \quad 0 < s \leq 1.
  \]

One may easily modify our proof and obtain the following result:

Suppose that \( w \in C^4(\mathbb{T}) \), then it holds that

\[
\|\theta(t)\|_{L^2} \leq C e^{-c^2 \nu^2 t} \|\theta_0\|_{L^2}.
\]

The paper is organized as follows: In section 2, we give a formal explanation of the re-enhancing effect of the non-local term and present the main idea of the proof for the linearized QG equation in critical case. In section 3, we introduce some important auxiliary lemmas and give the energy estimate. In section 4, we establish the decay estimate and complete the proof of Theorem 1.1. In section 5, we discuss the sub-critical QG and show the influence of different diffusion term. In section 6, we study the toy model and clarify the non-local enhancement.

Notations: Given a function \( f(x, y) \), we denote its Fourier transform in \( x \)-variable as

\[
\hat{f}(\alpha, y) = \frac{1}{2\pi\delta} \int_{\mathbb{T} \times \mathbb{S}} e^{-i\alpha x} f(x, y) dx,
\]
and its Fourier transform in \((x, y)\) as

\[
\tilde{f}(\alpha, \beta) = \mathcal{F}_y(\hat{f}(\alpha, \cdot)) = \frac{1}{4\pi\delta} \int_{\mathbb{T}\times \mathbb{S}} e^{-i\alpha x - iy\beta} f(x, y) dx dy,
\]
where \( \alpha \) and \( \beta \) are the wave numbers. When no confusion can arise, we will short \( \hat{f}(\alpha, y) \) and \( \tilde{f}(\alpha, \beta) \) to \( \hat{f}(y) \) and \( \tilde{f}(\beta) \).

2. Hypocoercivity and non-local enhancement

In this section, we give the main idea of the proof for the linearized QG equation in critical case. By taking \( s = \frac{1}{2} \) of (1.3), we get that

\[
\partial_t \hat{\theta} + \nu(-\Delta)^{\frac{1}{2}} \hat{\theta} + i\alpha e^{-\nu t} \cos y(1 - (-\Delta)^{\frac{1}{2}}) \hat{\theta} = 0.
\]

Let

\[
A = \sin y(1 - (-\Delta)^{-\frac{1}{2}}), \quad B = \cos y(1 - (-\Delta)^{-\frac{1}{2}}), \quad \gamma(t) = \alpha e^{-\nu t},
\]
and

\[
\mathcal{L}_\nu(\alpha, t) = \nu(-\Delta)^{\frac{1}{2}} + i\gamma(t) B.
\]

Then (2.1) can be written as

\[
\partial_t \hat{\theta} + \mathcal{L}_\nu \hat{\theta} = 0.
\]

The non-self-adjoint operator \( \mathcal{L}_\nu \) can be regarded as a sum of a self-adjoint operator \( \nu(-\Delta)^{\frac{1}{2}} \) and a anti-self-adjoint operator \( i\gamma(t) B \) defining on a weighted Hilbert space equipped with a special inner product

\[
\langle u, w \rangle_s = \langle u, w - (-\Delta)^{-\frac{1}{2}} w \rangle.
\]
where
\[
\langle u, w \rangle = \int_\pi u(y) w(y) dy, \quad \|u\|_{L^2} = \langle u, u \rangle^{\frac{1}{2}},
\]
are the ordinary \(L^2\) inner product and \(L^2\) norm.

It is easy to check that the operators \(A\) and \(B\) are symmetric under this inner product,
\[
\langle u, Aw \rangle_s = \langle Au, w \rangle_s, \quad \langle u, Bw \rangle_s = \langle Bu, w \rangle_s.
\]
Let us also introduce the weighted \(L^2\) norm \(\|u\|_s = \langle u, u \rangle_s^{\frac{1}{2}}\). For \(|\alpha| > 1\), it is clear that
\[
(1 - |\alpha|^{-1})\|u\|_{L^2}^2 \leq \|u\|_s^2 \leq \|u\|_{L^2}^2.
\]

2.1. Hypocoercivity. It is natural to introduce energy functional based on the hypocoercivity method \cite{LL}:
\[
e(t) = \|\hat{\theta}\|_s^2 + \lambda_1 \|(-\Delta_\alpha) \hat{\theta}\|_s^2 + \lambda_2 \langle [(-\Delta_\alpha) \hat{\theta}, B] \hat{\theta}, (-\Delta_\alpha) \hat{\theta} \rangle_s + \lambda_3 \|((-\Delta_\alpha) \hat{\theta}, B] \hat{\theta}\|_s^2.
\]
However the energy functional does not work well in our problem.

Let us introduce the modified energy function:
\[
\Phi(t) = E_0 + a_1 \nu^2 t^2 E_1 + a_2 \nu^2 t^3 E_1 + a_3 \nu^2 t^4 E_2,
\]
where
\[
E_0 = \|\hat{\theta}\|_s^2, \quad E_1 = \|\partial_y \hat{\theta}\|_s^2, \quad E_1 = -\frac{\alpha}{|\alpha|} \mathcal{R}\langle iA \hat{\theta}, \partial_y \hat{\theta} \rangle_s, \quad E_2 = \|\hat{\theta}\|_s^2 - \|B \hat{\theta}\|_s^2.
\]

The key modifications are as follows. First of all, we introduce the time weighted norm which reduces the required regularity of the initial data and is widely used in studying the wellposedness theory in low regularity space \cite{LL, Z}. There are two types of terms in the energy functional. Heuristically, one may say that the ‘pure’ terms \(E_0\) and \(E_1\) will mainly feel the influence of the symmetric part in \(L_\nu\), but that the ‘mixed’ term \(E_1\) will mainly feel the influence of the antisymmetric part in \(L_\nu\). Here we modified \(E_1\) and the corresponding ‘mixed’ term \(E_1\), so that technically we can avoid treating the commutator of two non-local operators. The last modification is replacing the corresponding ‘pure’ term \(\|A \hat{\theta}\|_s\), by \(E_2\) which is from the idea in \cite{LL}. Actually \(E_2\) behaves as \(\|A \hat{\theta}\|_s^2\) and it is conserved for inviscid case \((\nu = 0)\).

2.2. Formal deduction. Next, we would like to give a formal mathematical explanation why the non-local term re-enhances the enhanced dissipation by the ‘shear-diffusion mechanism’. Let us simplify the energy functional by removing the time weight and define
\[
E(t) = E_0 + a_1 \nu^2 b_1 E_1 + a_2 \nu^2 b_2 E_1 + a_3 \nu^2 b_3 E_2.
\]
Note that \(\|\cdot\|_s\) behaves as the \(L^2\) norm and \(\|A \hat{\theta}\|_s^2\) (see Lemma \ref{lem3.2}).

The commutator \(A_{\alpha, \frac{1}{2}} = -[\partial_y, B_{\alpha, \frac{1}{2}}] = \sin y (1 - (\Delta_\alpha)^{-\frac{1}{2}})\) plays an important role in the enhanced dissipation rate. Heuristically, due to the existence of nonlocal operator \((\Delta_\alpha)^{-\frac{1}{2}}\) in \(A_{\alpha, \frac{1}{2}}\), we have \(A_{\alpha, \frac{1}{2}} \approx \|D\|^{-\frac{1}{2}}\) and more generally \(A_{\alpha, \tilde{s}} \approx \|D\|^{-\tilde{s}}\). While without the nonlocal term, formally \(A^S \approx \|D\|^{-1}\). One may regard Lemma \ref{lem3.2} and Lemma \ref{lem6.1} as the evidences. For a more precise description of the commutator \(A_{\alpha, \tilde{s}}\) or \(A^S\), we conjecture as follows:

**Conjecture:** Let \(\lambda\) be the eigenvalue of \(L_{\alpha, \nu, s, \tilde{s}}\) with largest real part and \(\hat{\theta}_\lambda\) be the corresponding eigenfunction, i.e. \(L_{\alpha, \nu, s, \tilde{s}} \hat{\theta}_\lambda = \lambda \hat{\theta}_\lambda\). Then,
\[
C_\alpha^{-1} \|\hat{\theta}_\lambda\|_{H^{-\tilde{s}}} \leq \|A_{\alpha, \tilde{s}} \hat{\theta}_\lambda\|_{L^2} \leq C_\alpha \|\hat{\theta}_\lambda\|_{H^{-\tilde{s}}},
\]
Let \( \lambda \) be the eigenvalue of \( L^S_{\alpha, \nu, s} \) with largest real part and \( \hat{\theta}_\lambda \) be the corresponding eigenfunction, i.e. \( L^S_{\alpha, \nu, s} \hat{\theta}_\lambda = \lambda \theta \). Then,

\[
C_\alpha^{-1} \| \hat{\theta}_\lambda \|_{H^{-1}} \leq \| A^S \hat{\theta}_\lambda \|_{L^2} \leq C_\alpha \| \hat{\theta}_\lambda \|_{H^{-1}}.
\]

Where \( C_\alpha \geq 1 \) is independent of \( \nu \).

Note that if \( s = 1 \), both commutator \( A_{\alpha, 1} \) and \( A^S \) behave similarly, which also explain the same enhanced dissipation rate for the linearized Navier-Stokes equation and its toy model.

With the above formal argument, by the definitions, formally we regard \( E_0 \), \( E_1 \), \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) as \( \| \hat{\theta} \|_{L^2}^2 \), \( \| \hat{\theta} \|_{H^1}^2 \), \( \| \hat{\theta} \|_{H^{1/2}}^2 \) and \( \| \hat{\theta} \|_{H^{-1/2}}^2 \) respectively. To balance each part in the energy, we may formally rewrite

\[
E(t) \approx \| \hat{\theta} \|_{L^2}^2 + a_1 \nu^{2k} \| \hat{\theta} \|_{H^1}^2 + a_2 \nu^{k} \| \hat{\theta} \|_{H^{1/2}}^2 + a_3 \nu^{-k} \| \hat{\theta} \|_{H^{-1/2}}^2,
\]

and in the sense of pairing,

\[
E_0 \approx \nu^{2k} E_1 \approx \nu^{k} \mathcal{E}_1 \approx \nu^{-k} \mathcal{E}_2 \approx \nu^k \| \hat{\theta} \|_{H^{1/2}}^2.
\]

The time evolution of \( E_0 \) and the ‘mixed’ term \( \mathcal{E}_1 \) give us that

\[
\frac{d}{dt} E_0 = -2\nu \| \hat{\theta} \|_{H^{1/2}}^2 (\approx \nu^{k} E_0), \quad \frac{d}{dt} \nu^{k} \mathcal{E}_1 = \nu^{k} \mathcal{E}_2 (\approx \nu^{2k} E_0) + l.o.t.,
\]

Thus we may expect that \( 1 - k = \frac{3k}{2} \) which means \( k = \frac{2}{5} \). As a result, we finally get

\[
\frac{d}{dt} E(t) \leq -\nu^{\frac{3}{5}} E(t),
\]

which gives the enhanced dissipation rate.

While for the transport fractional diffusion with operator \( L^S_{\alpha, \nu, \frac{1}{2}} \), a similar argument gives that

\[
\frac{d}{dt} E^S(t) \leq -\nu^{\frac{3}{5}} E^S(t),
\]

which gives the enhanced dissipation rate. Here

\[
E^S(t) = \| \hat{\theta} \|_{L^2}^2 + b_1 \nu^{\frac{3}{5}} \| \partial_y \hat{\theta} \|_{L^2}^2 + b_2 \Re \langle i A^S \hat{\theta}, \partial_y \hat{\theta} \rangle + b_3 \nu^{-\frac{2}{5}} \| A^S \hat{\theta} \|_{L^2}^2
\]

is the modified energy functional. The corresponding time weighted energy functional can be found in section 6.

### 3. Energy estimates

In this section, we study the linearized equation (2.1) and establish the energy estimates which will be used to prove Theorem 1.1.

Recall that

\[
A = \sin y (1 - (-\Delta)\frac{\alpha}{2}), \quad B = \cos y (1 - (-\Delta)^{-\frac{\alpha}{2}}),
\]

and

\[
E_0 = \| \hat{\theta} \|_*^2, \quad E_1 = \| \partial_y \hat{\theta} \|_*^2, \quad \mathcal{E}_1 = -\frac{\alpha}{|\alpha|} \Re \langle i A \hat{\theta}, \partial_y \hat{\theta} \rangle, \quad \mathcal{E}_2 = \| \hat{\theta} \|_*^2 - \| B \hat{\theta} \|_*^2.
\]

We are aiming to prove the following proposition.
Proposition 3.1. Suppose that $|\alpha| \geq c_0 > 1$. Then it holds that
\[
\frac{d}{dt} E_0 = -2\nu E_2^\frac{1}{2},
\]
\[
\frac{d}{dt} E_1 = -2\nu E_2^\frac{1}{2} - 2|\gamma|E_1,
\]
\[
\frac{d}{dt} E_1 \leq 2\nu \|(-\Delta_\alpha)^\frac{1}{4} A\hat{\theta}\|_{L^2}^2 \frac{1}{2} + c_2 \nu E_1 - |\gamma|\|A\hat{\theta}\|_{L^2}^2 - c_3 |\gamma| \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2}{(\alpha^2 + \beta^2)^\frac{1}{2}} |\hat{\psi}(\beta)|^2,
\]
\[
\frac{d}{dt} E_2 \leq -2\nu E_0 - \nu \|(-\Delta_\alpha)^\frac{1}{4} A\hat{\theta}\|_{L^2}^2 + 2c_4 \nu \|A\hat{\theta}\|_{L^2}^2 E_0^\frac{1}{2},
\]
where
\[
E_1 = 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^\frac{1}{2} - 1)|\hat{\psi}(\beta)|^2, \quad E_2 = 2\pi \sum_{\beta \in \mathbb{Z}} \beta^2 ((\alpha^2 + \beta^2)^\frac{1}{2} - 1)|\hat{\psi}(\beta)|^2,
\]
and $c_2, c_4$ are positive constants depending only on $c_0$, and $0 < c_3 \leq \frac{1}{2}$ is an independent constant.

3.1. Useful lemmas. Let us first introduce some important lemmas.

Lemma 3.2. If $|\alpha| \geq c_0 > 1$, and $(-\Delta_\alpha)^\frac{1}{2}\hat{\psi} = \hat{\theta}$, then
\[
\|A\hat{\theta}\|_{L^2}^2 + 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^\frac{1}{2} - 1)|\hat{\psi}(\beta)|^2 \leq E_2 \leq 2 \left( \|A\hat{\theta}\|_{L^2}^2 + 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^\frac{1}{2} - 1)|\hat{\psi}(\beta)|^2 \right).
\]

Proof. From the definition, we have
\[
\langle u, w \rangle_* = \langle (1 - (-\Delta_\alpha)^{-\frac{1}{2}})u, (1 - (-\Delta_\alpha)^{-\frac{1}{2}})w \rangle + \langle (-\Delta_\alpha)^{-\frac{1}{2}}u, (1 - (-\Delta_\alpha)^{-\frac{1}{2}})w \rangle,
\]
\[
\langle (1 - (-\Delta_\alpha)^{-\frac{1}{2}})u, (1 - (-\Delta_\alpha)^{-\frac{1}{2}})w \rangle = \langle Au, Aw \rangle + \langle Bu, Bw \rangle,
\]
\[
\langle Bu, Bw \rangle = \langle Bu, Bw \rangle_* + \langle (-\Delta_\alpha)^{-\frac{1}{2}}Bu, Bw \rangle.
\]

It follows that
\[
\langle u, w \rangle_* - \langle Bu, Bw \rangle_* = \langle (-\Delta_\alpha)^{-\frac{1}{2}}u, (1 - (-\Delta_\alpha)^{-\frac{1}{2}})w \rangle + \langle Au, Aw \rangle + \langle (-\Delta_\alpha)^{-\frac{1}{2}}Bu, Bw \rangle.
\]

As a result, it holds that
\[
E_2 = \|\hat{\theta}\|_{L^2}^2 - \|B\hat{\theta}\|_{L^2}^2
\]
\[
= \langle (-\Delta_\alpha)^{-\frac{1}{2}}\hat{\theta}, (1 - (-\Delta_\alpha)^{-\frac{1}{2}})\hat{\theta} \rangle + \langle A\hat{\theta}, A\hat{\theta} \rangle + \langle (-\Delta_\alpha)^{-\frac{1}{2}}B\hat{\theta}, B\hat{\theta} \rangle
\]
\[
\geq \langle A\hat{\theta}, A\hat{\theta} \rangle + \|\hat{\psi}, (-\Delta_\alpha)^{-\frac{1}{2}} - 1\|_{L^2}^2
\]
\[
= \|A\hat{\theta}\|_{L^2}^2 + 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^\frac{1}{2} - 1)|\hat{\psi}(\beta)|^2.
\]

In the last equality, we use the Plancherel theorem.

To prove the upper bound, it suffice to estimate $\langle (-\Delta_\alpha)^{-\frac{1}{2}}B\hat{\theta}, B\hat{\theta} \rangle$.

Let
\[
\hat{u} = (1 - (-\Delta_\alpha)^{-\frac{1}{2}})\hat{\theta},
\]
then, by the fact that
\[
\left( \frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - 1}{\alpha^2 + (\beta - 1)^2} \right)^2 = \frac{\alpha^2 + \beta^2 + 1 - 2(\alpha^2 + \beta^2)^{\frac{1}{2}}}{\alpha^2 + \beta^2 + 1 - 2\beta} < 1,
\]
we deduce that
\[
\langle (-\Delta_\alpha)^{-\frac{1}{2}} B\hat{\theta}, B\hat{\theta} \rangle = \langle (-\Delta_\alpha)^{-\frac{1}{2}} \cos \hat{y}, \cos \hat{y} \rangle
\]
\[
= \int_T (-\Delta_\alpha)^{-\frac{1}{2}} e^{iy} e^{-iy} \hat{u} \cdot \frac{1}{2} e^{iy} e^{-iy} \hat{u} dy
\]
\[
= \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \frac{|\hat{u}(\beta - 1) + \hat{u}(\beta + 1)|^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}
\]
\[
\leq \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \frac{|\hat{u}(\beta - 1)|^2 + |\hat{u}(\beta + 1)|^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}
\]
\[
= \sum_{\beta \in \mathbb{Z}} \frac{(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} - 1)^2 |\tilde{\psi}(\beta - 1)|^2 + ((\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} - 1)^2 |\tilde{\psi}(\beta + 1)|^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}
\]
\[
= 2\pi \sum_{\beta \in \mathbb{Z}} \frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - 1}{(2(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} + \frac{2(\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} - 1}{2(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}}})((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)|\tilde{\psi}(\beta)|^2}
\]
\[
\leq 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)|\tilde{\psi}(\beta)|^2.
\]
This completes the proof of this lemma. \(\square\)

**Lemma 3.3.** Under the same assumptions of Lemma 3.2, it holds that
\[
\mathcal{E}_2 \leq C \left( |\alpha|^{\frac{1}{2}} \|A\hat{\theta}\|_{L^2}^2 + \sum_{\beta \in \mathbb{Z}} \frac{2\pi |\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2 \right),
\]
where \(C\) is a constant depending only on \(c_0\).

**Proof.** As Lemma 3.2 shows that
\[
\mathcal{E}_2 \leq 2 \left( \|A\hat{\theta}\|_{L^2}^2 + 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)|\tilde{\psi}(\alpha, \beta)|^2 \right),
\]
it suffice to prove that
\[
2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)|\tilde{\psi}(\beta)|^2 \leq C \left( |\alpha|^{\frac{1}{2}} \|A\hat{\theta}\|_{L^2}^2 + \sum_{\beta \in \mathbb{Z}} \frac{2\pi |\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2 \right)
\]
for some constant \(C\) depends only on \(c_0\).

For \(|\beta| \geq |\alpha|\), it is clear
\[
((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)|\tilde{\psi}(\beta)|^2 \leq 2 \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2,
\]
so we only need to focus on the case \(|\beta| \leq |\alpha|\).
However, when $|\beta| \leq |\alpha|$, it holds that
\[
\left( \frac{1}{2} - 1 - \frac{|\alpha| \frac{1}{2} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} \right) |\tilde{\psi}(\beta)|^2 = \left( \frac{\alpha^2 + \beta^2 - |\alpha| \frac{1}{2} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} - 1 \right) |\tilde{\psi}(\beta)|^2 \leq (|\alpha| - 1)|\tilde{\psi}(\beta)|^2,
\]
so the problem reduces to the estimate of
\[
\sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} (|\alpha| - 1)|\tilde{\psi}(\beta)|^2.
\]
We start from the case $|\alpha| \geq 100$, for which $|\alpha| - 1 \geq 9|\alpha|^{\frac{1}{2}}$. We define
\[
f(\beta) = \begin{cases} 
1, & |\beta| \leq |\alpha|, \\
2 - |\frac{\beta}{|\alpha|}|, & |\alpha| < |\beta| \leq 2|\alpha|, \\
0, & 2|\alpha| < |\beta|,
\end{cases}
\]
and deduce that
\[
\sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} (|\alpha| - 1)|\tilde{\psi}(\beta)|^2
\leq \sum_{\beta \in \mathbb{Z}} (|\alpha| - 1)f(\beta)|\tilde{\psi}(\beta)|^2
= \frac{1}{2} \sum_{\beta \in \mathbb{Z}} (|\alpha| - 1)\beta f(\beta - 1)\tilde{\psi}(\beta - 1)^2 - (|\alpha| - 1)\beta f(\beta + 1)\tilde{\psi}(\beta + 1)^2
= \frac{1}{2} \sum_{\beta \in \mathbb{Z}} (|\alpha| - 1)\beta (f(\beta - 1)\tilde{\psi}(\beta - 1) - f(\beta + 1)\tilde{\psi}(\beta + 1))
\]
\[
\leq \frac{1}{4} |\alpha|^{\frac{1}{2}} (|\alpha| - 1)^2 \sum_{\beta \in \mathbb{Z}} |f(\beta - 1)\tilde{\psi}(\beta - 1) - f(\beta + 1)\tilde{\psi}(\beta + 1)|^2
+ \frac{1}{4} |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\beta|^2 |f(\beta - 1)\tilde{\psi}(\beta - 1) + f(\beta + 1)\tilde{\psi}(\beta + 1)|^2
\]
def\(I_1 + I_2\).

From the definition of \(f\), one can see that
\[
I_1 = \frac{1}{4} |\alpha|^{\frac{1}{2}} (|\alpha| - 1)^2 \sum_{\beta \in \mathbb{Z}} |f(\beta - 1)\tilde{\psi}(\beta - 1) - f(\beta + 1)\tilde{\psi}(\beta + 1)|^2
= \frac{1}{4} |\alpha|^{\frac{1}{2}} (|\alpha| - 1)^2 \sum_{\beta \in \mathbb{Z}} |f(\beta)\tilde{\psi}(\beta - 1) - f(\beta)\tilde{\psi}(\beta + 1)
+ (f(\beta - 1) - f(\beta))\tilde{\psi}(\beta - 1) - (f(\beta + 1) - f(\beta))\tilde{\psi}(\beta + 1)|^2
\leq \frac{1}{4} |\alpha|^{\frac{1}{2}} (|\alpha| - 1)^2 \sum_{\beta \in \mathbb{Z}} f^2(\beta) |\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1)|^2 + 2\frac{(|\alpha| - 1)^2}{|\alpha|^{3/2}} \sum_{\beta \in \mathbb{Z}} |\tilde{\psi}(\beta)|^2
\leq \frac{2}{\pi} |\alpha|^{\frac{1}{2}} \|A\theta\|_{L^2}^2 + 4|\alpha|^{\frac{1}{2}} \sum_{|\beta| \leq |\alpha|} |\tilde{\psi}(\beta)|^2 + 10 \sum_{\beta \in \mathbb{Z}} \frac{|\alpha| \frac{1}{2} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2.
\]
In the last inequality, we use the fact that

\[(3.3) \quad |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} \left( (\alpha^2 + \beta^2)^{\frac{1}{2}} - 1 \right)^2 |\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1)|^2 \leq \frac{4}{\pi} |\alpha|^{\frac{1}{2}} \|A\tilde{\theta}\|_{L^2}^2 + 8 |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\tilde{\psi}(\beta)|^2.
\]

Indeed, from the definition of $A$, we have

\[
|\alpha|^{\frac{1}{2}} \|A\tilde{\theta}\|_{L^2}^2 = |\alpha|^{\frac{1}{2}} \| \sin \theta((-\Delta)_{\alpha}^{\frac{1}{2}} - 1) \tilde{\psi} \|_{L^2}^2
\]

\[
= |\alpha|^{\frac{1}{2}} \int_T \frac{e^{iy} - e^{-iy}}{2i}((-\Delta)_{\alpha}^{\frac{1}{2}} - 1) \tilde{\psi} : \frac{e^{iy} - e^{-iy}}{2i}((-\Delta)_{\alpha}^{\frac{1}{2}} - 1) \tilde{\psi} dy
\]

\[
= \frac{\pi |\alpha|^{\frac{1}{2}}}{2} \sum_{\beta \in \mathbb{Z}} \left( (\alpha^2 + \beta^2)^{\frac{1}{2}} - 1 \right) (\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1))
\]

\[
\quad + \frac{(1 - 2\beta) \tilde{\psi}(\beta - 1)}{(\alpha^2 + \beta^2)^{\frac{1}{2}} + (\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}}} + \frac{(1 + 2\beta) \tilde{\psi}(\beta + 1)}{(\alpha^2 + \beta^2)^{\frac{1}{2}} + (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}}}
\]

\[
\geq \frac{\pi |\alpha|^{\frac{1}{2}}}{2} \sum_{\beta \in \mathbb{Z}} \left( \frac{1}{2} \left( (\alpha^2 + \beta^2)^{\frac{1}{2}} - 1 \right) (|\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1)|)^2 - (|\tilde{\psi}(\beta - 1)| + |\tilde{\psi}(\beta + 1)|)^2 \right)
\]

\[
\geq \frac{\pi |\alpha|^{\frac{1}{2}}}{4} \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1)^2 |\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1)|^2 - 2\pi |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\tilde{\psi}(\beta)|^2.
\]

Now we turn to the estimate of $I_2$ in (3.2). It holds that

\[
I_2 = \frac{1}{4} |\alpha|^{-\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\beta|^2 |f(\beta - 1) \tilde{\psi}(\beta - 1) + f(\beta + 1) \tilde{\psi}(\beta + 1)|^2
\]

\[
= \frac{1}{4} |\alpha|^{-\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} \left( f(\beta - 1) \tilde{\psi}(\beta - 1) - f(\beta + 1) \tilde{\psi}(\beta + 1) \right)
\]

\[
\quad + \left( (\beta - 1)f(\beta - 1) \tilde{\psi}(\beta - 1) + (\beta + 1)f(\beta + 1) \tilde{\psi}(\beta + 1) \right)^2
\]

\[
\leq \frac{1}{2} |\alpha|^{-\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} f^2(\beta) |\tilde{\psi}(\beta - 1) - \tilde{\psi}(\beta + 1)|^2 + 2 |\alpha|^{-\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\beta|^2 f^2(\beta) |\tilde{\psi}(\beta)|^2
\]

\[
\leq \frac{1}{4\pi} |\alpha|^{\frac{1}{2}} \|A\tilde{\theta}\|_{L^2}^2 + \frac{1}{2} |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}} |\tilde{\psi}(\beta)|^2 + 5 \sum_{\beta \in \mathbb{Z}} \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2
\]

\[
\leq \frac{1}{4\pi} |\alpha|^{\frac{1}{2}} \|A\tilde{\theta}\|_{L^2}^2 + \frac{1}{2} |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} |\tilde{\psi}(\beta)|^2 + 6 \sum_{\beta \in \mathbb{Z}} \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2.
\]

In the second last inequality, we used (3.3) and the fact that $f(\beta) = 0$ for $|\beta| \geq 2|\alpha|$. From the above estimates, we have

\[
\sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} (|\alpha| - 1)^2 |\tilde{\psi}(\beta)|^2
\]

\[
\leq \frac{9}{4\pi} |\alpha|^{\frac{1}{2}} \|A\tilde{\theta}\|_{L^2}^2 + \frac{9}{2} |\alpha|^{\frac{1}{2}} \sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} |\tilde{\psi}(\beta)|^2 + 16 \sum_{\beta \in \mathbb{Z}} \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2.
\]
Then, it follows from the fact $|\alpha| \geq 100$ and $|\alpha| - 1 \geq 9|\alpha|^{\frac{1}{2}}$ that
\[
\sum_{\beta \in \mathbb{Z}, |\beta| \leq |\alpha|} (|\alpha| - 1)|\hat{\psi}(\beta)|^2 \leq \frac{9}{2\pi} |\alpha|^{\frac{1}{2}} \|\hat{A}\theta\|_{L^2}^2 + 32 \sum_{\beta \in \mathbb{Z}} \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2.
\]
For $|\alpha| < 100$ and $0 < |\beta| < |\alpha|$, it is obvious that
\[
(|\alpha| - 1)|\hat{\psi}(\beta)|^2 \leq 2000 \frac{|\alpha|^{\frac{1}{2}} \beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2.
\]
The rest is the case $1 < c_0 \leq |\alpha| < 100$ and $|\beta| = 0$. Similar to the proof of (3.3), we have
\[
\|\hat{A}\theta\|_{L^2}^2 = \|\sin y((-\Delta_\alpha)^{\frac{1}{2}} - 1)\psi\|_{L^2}^2 \\
\geq 2\pi |(|\alpha| - 1)\hat{\psi}(0) - ((\alpha^2 + 4)^{\frac{1}{2}} - 1)\hat{\psi}(2)|^2 \\
\geq \pi (|\alpha| - 1)^2 |\hat{\psi}(0)|^2 - 2\pi ((\alpha^2 + 4)^{\frac{1}{2}} - 1)^2 |\hat{\psi}(2)|^2.
\]
It follows that
\[
(|\alpha| - 1)|\hat{\psi}(0)|^2 \leq \frac{1}{\pi(\alpha - 1)} \|\hat{A}\theta\|_{L^2}^2 + \frac{2000 |\alpha|^{\frac{1}{2}}}{\alpha - 1 (\alpha^2 + 4)^{\frac{3}{2}}} |\hat{\psi}(2)|^2.
\]
Combining these results, we arrive at the conclusion of this lemma. \qed

**Lemma 3.4.** Under the same assumptions of Lemma [13], it holds that
\[
\mathbb{R}(\langle BA - AB \rangle \partial_y \theta, \partial_y \hat{\theta}) \leq -\frac{1}{C} \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2,
\]
where $C > 0$ is a independent constant.

**Proof.** By the definitions, it is clear that
\[
BA - AB \\
= \cos y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \sin y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}) - \sin y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \cos y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \\
= \left( -\cos y(-\Delta_\alpha)^{-\frac{1}{2}} \sin y + \sin y(-\Delta_\alpha)^{-\frac{1}{2}} \cos y \right) (1 - (-\Delta_\alpha)^{-\frac{1}{2}}).
\]
A direct calculation shows that
\[
\cos y(-\Delta_\alpha)^{-\frac{1}{2}} \sin y - \sin y(-\Delta_\alpha)^{-\frac{1}{2}} \cos y \\
= \left( (e^{iy} + e^{-iy})(-\Delta_\alpha)^{-\frac{1}{2}} (e^{iy} - e^{-iy}) - (e^{iy} - e^{-iy})(-\Delta_\alpha)^{-\frac{1}{2}} (e^{iy} + e^{-iy}) \right) / (4i) \\
= \left( e^{-iy}(-\Delta_\alpha)^{-\frac{1}{2}} e^{iy} - e^{iy}(-\Delta_\alpha)^{-\frac{1}{2}} e^{-iy} \right) / (2i) \\
= \left( e^{-iy}(-\Delta_\alpha)^{\frac{1}{2}} e^{iy} - 1 - (e^{iy}(-\Delta_\alpha)^{\frac{1}{2}} e^{-iy})^{-1} \right) / (2i) \\
= \left( (-\Delta_{\alpha,1})^{-\frac{1}{2}} - (-\Delta_{\alpha,-1})^{-\frac{1}{2}} \right) / (2i) \\
= \left( (-\Delta_{\alpha,1})^{-\frac{1}{2}} - (-\Delta_{\alpha,-1})^{-\frac{1}{2}} \right) (-\Delta_{\alpha,1})^{-\frac{1}{2}} (-\Delta_{\alpha,-1})^{-\frac{1}{2}} / (2i),
\]
where
\[
(-\Delta_{\alpha,1})^{-\frac{1}{2}} = e^{-iy}(-\Delta_\alpha)^{\frac{1}{2}} e^{iy}, \quad (-\Delta_{\alpha,-1})^{-\frac{1}{2}} = e^{iy}(-\Delta_\alpha)^{\frac{1}{2}} e^{-iy}.
\]
From Fourier point of view, one can see that
\[ \mathcal{F}_y((-\Delta_\alpha)\frac{1}{2} f) (\beta) = (\alpha^2 + (\beta + 1)^2)\frac{1}{2} \hat{f}(\beta). \]

It follows that
\[
\mathcal{F}_y(\cos y(-\Delta_\alpha)^{-\frac{1}{2}} \sin y - \sin y(-\Delta_\alpha)^{-\frac{1}{2}} \cos y) = \left( (\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} - (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} \right) (\alpha^2 + (\beta - 1)^2)^{-\frac{1}{2}} (\alpha^2 + (\beta + 1)^2)^{-\frac{1}{2}} / (2i) = 2i\beta \left( (\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} + (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} \right) - 1(\alpha^2 + (\beta - 1)^2)^{-\frac{1}{2}} (\alpha^2 + (\beta + 1)^2)^{-\frac{1}{2}}.
\]

Thus, by using the fact that \(|\alpha| > 1\), one can deduce that
\[
\Re(\langle BA - AB \rangle \hat{\theta}, \partial_\gamma \hat{\theta})_* = \Re \left( \left( - \cos y(-\Delta_\alpha)^{-\frac{1}{2}} \sin y + \sin y(-\Delta_\alpha)^{-\frac{1}{2}} \cos y \right) (1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \partial_\gamma \hat{\theta} \right)_* \]
\[
= \Re \left( - ((-\Delta_{\alpha - 1})^{\frac{1}{2}} - (-\Delta_{\alpha 1})^{\frac{1}{2}}) (-\Delta_{\alpha - 1})^{-\frac{1}{2}} (-\Delta_{\alpha - 1})^{-\frac{1}{2}} (1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \hat{\theta} (1 - (-\Delta_\alpha)^{-\frac{1}{2}}) \partial_\gamma \hat{\theta} \right) = - \frac{1}{4}\pi \sum_{\beta \in \mathbb{Z}} \frac{(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} - 1}{(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} / (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}}} \beta^2 |\tilde{\psi}(\beta)|^2 \leq - \frac{1}{C} \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\tilde{\psi}(\beta)|^2.
\]

This completes the proof. \( \square \)

3.2. **Energy estimate.** Recall the energy functional
\[ \Phi(t) = E_0 + a_1 \nu^2 t^2 E_1 + a_2 \nu^2 t^3 E_1 + a_3 \nu^2 t^4 E_2, \]
where the coefficients \(a_1, a_2, a_3\) are determined in the proof. It is natural to study the time evolution of the energy functional \(\Phi(t)\):
\[
\frac{d}{dt} \Phi(t) = \frac{d}{dt} E_0 + a_1 \nu^2 t^2 \frac{d}{dt} E_1 + a_2 \nu^2 t^3 \frac{d}{dt} E_1 + a_3 \nu^2 t^4 \frac{d}{dt} E_2 + 2a_1 \nu^2 t E_1 + 3a_2 \nu^2 t^2 E_1 + 4a_3 \nu^2 t^3 E_2.
\]

Now, we are in a position to prove Proposition \ref{prop:3.1} and to estimate the time evolution of \(E_0, E_1, E_2\) and \(E_2\).

**Proof of Proposition \ref{prop:3.1}**. We only prove for the case \(\alpha > 1\), for which \(E_1 = -\Re(iA\hat{\theta}, \partial_\gamma \hat{\theta})_*\) and \(\gamma > 0\). The proof for \(\alpha < -1\) is the same.

Recalling that \(L_\nu(\alpha, t) = \nu(-\Delta_\alpha)^{\frac{1}{2}} + i\gamma(t)B\) and \(E_0 = \|\hat{\theta}\|^2_z = 2\pi \sum_{\beta \in \mathbb{Z}} \frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - 1}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\hat{\theta}(\beta)|^2\), we have
\[
\frac{d}{dt} E_0 = 2\Re(\partial_\gamma \hat{\theta}, \hat{\theta})_* = -2\Re(L_\nu \hat{\theta}, \hat{\theta})_* = -2\nu \Re((-\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \hat{\theta})_* - 2\gamma \Re(iB \hat{\theta}, \hat{\theta})_*.
\]
Here we use the fact that $B$ is symmetric with respect to $\langle \cdot, \cdot \rangle_*$. For $E_1 = \| \partial_y \hat{\theta} \|_*^2 = 2\pi \sum_{\beta \in \mathbb{Z}} \frac{\beta^2((\alpha^2+\beta^2)^\frac{1}{2} - 1)}{(\alpha^2+\beta^2)^\frac{1}{2}} |\hat{\theta}(\beta)|^2$, one can deduce that

\[
dt E_1 = 2\Re(\partial_y \partial_t \hat{\theta}, \partial_y \hat{\theta})_* - 2\Re(\partial_y L_y \hat{\theta}, \hat{\theta})_* = -2\nu E_1^2 - 2\Re(\partial_y \partial_t \hat{\theta}, \partial_y \hat{\theta})_* - 2\gamma \Re(iB \partial_y \hat{\theta}, \partial_y \hat{\theta})_* + 2\gamma \Re(iA \hat{\theta}, \partial_y \hat{\theta})_* + 2\gamma \Re(iA \hat{\theta}, \partial_y \hat{\theta})_* + 2\gamma \Re(iA \hat{\theta}, \partial_y \hat{\theta})_* - 2\nu E_1^2 - 2\gamma E_1.
\]

Next, we turn to $\mathcal{E}_1$ and deduce that

\[
\dt \mathcal{E}_1 = -\Re(iA \partial_t \hat{\theta}, \partial_y \hat{\theta})_* - \Re(iA \hat{\theta}, \partial_y \partial_t \hat{\theta})_*
\]

\[
= \nu \Re(iA(\Delta_\alpha)^{\frac{1}{4}} \hat{\theta}, \partial_y \hat{\theta})_* + \gamma \Re(AB \hat{\theta}, \partial_y \hat{\theta})_* + \nu \Re(iA \hat{\theta}, (\Delta_\alpha)^{\frac{1}{4}} \partial_y \hat{\theta})_* + \gamma \Re(A \hat{\theta}, A \hat{\theta})_*
\]

\[
= \nu \Re(iA(\Delta_\alpha)^{\frac{1}{4}} \hat{\theta}, \partial_y \hat{\theta})_* + \nu \Re(iA \hat{\theta}, (\Delta_\alpha)^{\frac{1}{4}} \partial_y \hat{\theta})_* + \gamma \Re((BA - AB) \hat{\theta}, \partial_y \hat{\theta})_* - \gamma \Re(A \hat{\theta}, A \hat{\theta})_*. \tag{3.4}
\]

For the first two terms on the right hand side, we have

\[
\Re(iA(\Delta_\alpha)^{\frac{1}{4}} \hat{\theta}, \partial_y \hat{\theta})_* + \Re(iA \hat{\theta}, (\Delta_\alpha)^{\frac{1}{4}} \partial_y \hat{\theta})_* = 2\Re(i(\Delta_\alpha)^{\frac{1}{4}} A \hat{\theta}, (\Delta_\alpha)^{\frac{1}{4}} \partial_y \hat{\theta})_* + \Re \left( i \left( A(\Delta_\alpha)^{\frac{1}{4}} - (\Delta_\alpha)^{\frac{1}{4}} A \right) \hat{\theta}, \partial_y \hat{\theta} \right)_* \leq 2\| (\Delta_\alpha)^{\frac{1}{4}} A \hat{\theta} \|_{L^2} E_{\frac{1}{4}}^\frac{1}{4} + \Re(i \left( A(\Delta_\alpha)^{\frac{1}{4}} - (\Delta_\alpha)^{\frac{1}{4}} A \right) \hat{\theta}, \partial_y \hat{\theta} \right)_*. \]

Due to the fine property of the communicator, the last term is a lower order term. Indeed, recalling that $\hat{u} = (1 - (\Delta_\alpha)^{\frac{1}{4}}) \hat{\theta}$, one can deduce that

\[
\left| \Re \left( i \left( A(\Delta_\alpha)^{\frac{1}{4}} - (\Delta_\alpha)^{\frac{1}{4}} A \right) \hat{\theta}, \partial_y \hat{\theta} \right)_* \right|
\]

\[
= \Re \left( i \left( \sin y(\Delta_\alpha)^{\frac{1}{4}} - (\Delta_\alpha)^{\frac{1}{4}} \sin y \right) \hat{u}, \partial_y \hat{u} \right)
\]

\[
= \Re \left( \frac{1}{2} \int_T \left( \sin y(\Delta_\alpha)^{\frac{1}{4}} - (\Delta_\alpha)^{\frac{1}{4}} \sin y \right) \hat{u} \cdot \partial_y \hat{u} dy \right)
\]

\[
= \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \beta \left( (\alpha^2 + (\beta - 1)^2)^{\frac{1}{4}} - (\alpha^2 + \beta^2)^{\frac{1}{4}} \right) (\hat{u}(\beta) \hat{u}(\beta - 1) - \hat{u}(\beta - 1) \hat{u}(\beta))
\]
Here we emphasize that \(0 < c_0 \leq \frac{1}{4}\) is an independent constant.

By using Lemma 3.3 to estimate the third term on the right hand side of (3.4), we conclude that
\[
\frac{d}{dt} E_1 \leq 2\nu \|(\alpha^2 - \partial_y^2)^{1/4} A\hat{\theta}\|_{L^2}^2 + c_2 \nu E_\frac{1}{2} - \gamma \|A\hat{\theta}\|^2 - c_3 \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2}{(\alpha^2 + \beta^2)^\frac{3}{2}} |\hat{\psi}|^2.
\]

For \(E_2 = \|\hat{\theta}\|^2 - \|B\hat{\theta}\|^2\), by using (3.1) and the fact that \(B\) is symmetric, we have
\[
\frac{d}{dt} E_2 = -2\nu \text{Re}(\partial_t \hat{\theta}, \hat{\theta}) - 2\text{Re}(B\partial_t \hat{\theta}, \hat{\theta})
\]
\[
= -2\nu \text{Re}(\{-\Delta_\alpha \frac{1}{2} \hat{\theta}, \hat{\theta}\}) - 2\nu \text{Re}(\{B(-\Delta_\alpha \frac{1}{2} \hat{\theta}, \hat{\theta}\}) - 2\gamma \text{Re}(\{iB\hat{\theta}, \hat{\theta}\}) + 2\gamma \text{Re}(\{iBB\hat{\theta}, \hat{\theta}\})
\]
\[
= -2\nu \left( \text{Re}(\{-\Delta_\alpha \frac{1}{2} B(-\Delta_\alpha \frac{1}{2} \hat{\theta}, \hat{\theta}) + \text{Re}(\{A(-\Delta_\alpha \frac{1}{2} A\frac{1}{2} \hat{\theta}, A\hat{\theta}) + \text{Re}(\{\hat{\theta}, \hat{\theta} - (-\Delta_\alpha - \frac{1}{2} \hat{\theta})\}) \right) \right).
\]

A direct calculation shows that
\[
\text{Re}(A(-\Delta_\alpha \frac{1}{2} B(-\Delta_\alpha \frac{1}{2} \hat{\theta}, \hat{\theta}) + \text{Re}(\{A(-\Delta_\alpha \frac{1}{2} A\frac{1}{2} \hat{\theta}, A\hat{\theta}) - \text{Re}(\{A(-\Delta_\alpha \frac{1}{2} - (-\Delta_\alpha A \frac{1}{2}) \hat{\theta}\|_{L^2}^2 \|A\hat{\theta}\|_{L^2}^2.
\]

Recalling that \(\hat{u} = (1 - (-\Delta_\alpha - \frac{1}{2}) \hat{\theta}, one can see that
\[
\|A(-\Delta_\alpha \frac{1}{2} \hat{\theta}, A\hat{\theta})\|_{L^2}^2 = \|A(-\Delta_\alpha \frac{1}{2} - (-\Delta_\alpha \frac{1}{2} \hat{\theta}, \hat{\theta})\|_{L^2}^2
\]
\[
\leq 2\pi \sum_{\beta \in \mathbb{Z}} \left| \frac{(1 - 2\beta)\hat{u}(\beta - 1)}{(\alpha^2 + (\beta - 1)^2)^\frac{1}{2} + (\alpha^2 + \beta^2)^\frac{1}{2}} - \frac{(1 + 2\beta)\hat{u}(\beta + 1)}{(\alpha^2 + (\beta + 1)^2)^\frac{1}{2} + (\alpha^2 + \beta^2)^\frac{1}{2}} \right|^2
\]
\[
\leq 2\pi \sum_{\beta \in \mathbb{Z}} |\hat{u}(\beta)|^2 = 2\pi \sum_{\beta \in \mathbb{Z}} \left( \frac{(\alpha^2 + \beta^2)^\frac{1}{2} - 1)^2}{\alpha^2 + \beta^2} \right) |\hat{\theta}(\beta)|^2
\]
\[
\leq c_4 E_0,
\]
where \(c_4\) is a constant depending only on \(c_0\).

Similarly, we deduce that
\[
\text{Re}((-\Delta_\alpha - \frac{1}{2} B(-\Delta_\alpha \frac{1}{2} \hat{\theta}, B\hat{\theta})
\]
As a conclusion, we arrive at

\[ \mathcal{R}(B(-\Delta_a)^{\frac{1}{2}} \hat{\theta}, (-\Delta_a)^{-\frac{1}{2}} B \hat{\theta}) \]
\[ = \frac{1}{2} \langle \cos y(-\Delta_a)^{\frac{1}{2}} \hat{u}, (-\Delta_a)^{-\frac{1}{2}} \cos y \hat{u} \rangle \]
\[ + \frac{1}{2} \langle (-\Delta_a)^{-\frac{1}{2}} \cos y \hat{u}, \cos y(-\Delta_a)^{\frac{1}{2}} \hat{u} \rangle \]
\[ = \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \frac{\left(\alpha^2 + (\beta - 1)^2\right)^{\frac{1}{2}} + \left(\alpha^2 + (\beta + 1)^2\right)^{\frac{1}{2}}}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\bar{u}(\beta - 1)|^2 \]
\[ + \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \frac{\left(\alpha^2 + (\beta - 1)^2\right)^{\frac{1}{2}} + \left(\alpha^2 + (\beta + 1)^2\right)^{\frac{1}{2}}}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} \mathcal{R}(\bar{u}(\beta - 1) \bar{u}(\beta + 1)) \]
\[ = \frac{\pi}{4} \sum_{\beta \in \mathbb{Z}} \frac{\left(\alpha^2 + (\beta - 1)^2\right)^{\frac{1}{2}} + \left(\alpha^2 + (\beta + 1)^2\right)^{\frac{1}{2}}}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\bar{u}(\beta - 1) + \bar{u}(\beta + 1)|^2 \]
\[ - \frac{\pi}{4} \sum_{\beta \in \mathbb{Z}} \frac{\beta (|\bar{u}(\beta - 1)|^2 - |\bar{u}(\beta + 1)|^2)}{(\alpha^2 + \beta^2)^{\frac{1}{2}} ((\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} + (\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}})} \]
\[ \geq - \frac{\pi}{4} \sum_{\beta \in \mathbb{Z}} (\alpha^2 + \beta^2)^{\frac{1}{2}} |\bar{u}(\beta - 1) - \bar{u}(\beta + 1)|^2 \]
\[ = - \frac{1}{2} \|(-\Delta_a)^{\frac{1}{2}} A \hat{\theta}\|_{L^2}^2, \]

which means that

\[ \mathcal{R}((-\Delta_a)^{-\frac{1}{2}} B(-\Delta_a)^{\frac{1}{2}} \hat{\theta}, B \hat{\theta}) + \frac{1}{2} \|(-\Delta_a)^{\frac{1}{2}} A \hat{\theta}\|_{L^2}^2 \geq 0. \]

As a conclusion, we arrive at

\[ \frac{d}{dt} \mathcal{E}_2 \leq -2\nu E_0 - \nu \|(-\Delta_a)^{\frac{1}{2}} A \hat{\theta}\|_{L^2}^2 + 2\epsilon_4 \nu \|A \hat{\theta}\|_{L^2} E_0^{\frac{1}{2}}. \]

This complete the proof. \(\square\)

4. ENHANCED DISSIPATION

In this section, we determine the coefficients \(a_1, a_2, a_3\) in

\[ \Phi(t) = E_0(t) + a_1 \nu^2 t^2 E_1(t) + a_2 \nu^2 t^3 E_1(t) + a_3 \nu^2 t^4 E_2(t), \]

and study the time evolution of \(\Phi(t)\). Precisely, we prove the following lemma.

Lemma 4.1. If we take

\[ a_1 = \frac{|\alpha|^{-1}}{4c_2(6c_4^2 + \frac{12c_4^2}{c_3})^2}, \quad a_2 = \frac{|\alpha|^{-\frac{1}{2}}}{8c_2(6c_4^2 + \frac{12c_4^2}{c_3})^3}, \quad a_3 = \frac{1}{8c_2(6c_4^2 + \frac{12c_4^2}{c_3})^4}, \]
then for $\nu \leq 1$ and $0 \leq t \leq \nu^{-\frac{3}{2}}$, it holds that

$$
\Phi(t) \geq E_0(t), \quad \frac{d}{dt} \Phi(t) \leq -a_3\nu^3t^4E_0(t).
$$

**Proof.** For the same reason to Proposition 3.1, we only prove for the case $\alpha > 1$, for which $\mathcal{E}_1 = -\Re(iA\hat{\theta}, \partial_y\hat{\theta})_s$ and $\gamma > 0$. The proof for $\alpha < 1$ is the same.

With the help of Proposition 3.1, we deduce that

$$
\frac{d}{dt} \Phi(t) \leq -2\nu E_\frac{3}{4}
+ 2a_1\nu^3t^4E_\frac{3}{4} - 2a_1\nu^3t^2\gamma E_1
+ 3a_2\nu^3t^3\|(-\Delta)\frac{1}{4}A\hat{\theta}\|_{L^2}^2 + a_2c_2\nu^3t^3E_{\frac{1}{2}}^\frac{1}{2}
- a_2\gamma\nu^3t^3\|A\hat{\theta}\|_{L^2}^2 - a_2c_3\gamma\nu^3t^3 \sum_{\beta \in \mathbb{Z}} \frac{2\pi\beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2
+ 4a_3\nu^3t^3 \mathcal{E}_2 - 2a_3\nu^3t^4E_0 - a_3\nu^3t^4\|(-\Delta)^\frac{1}{4}A\hat{\theta}\|_{L^2}^2 + 2a_3c_4\nu^3t^4\|A\hat{\theta}\|_{L^2}E_0^\frac{1}{2}
= - (a_2c_2\nu E_\frac{3}{4} - a_2c_2\nu^3t^3E_{\frac{1}{2}})
- (a_1\nu E_\frac{3}{4} - 2a_1\nu^3tE_1 + a_1\nu^3t^2E_\frac{3}{2})
- \left(\frac{3a_2 - 2a_1\gamma}{2a_3}\right)^2 \nu E_\frac{3}{4} - (3a_2 - 2a_1\gamma)\nu^2t^2 \mathcal{E}_1 + \frac{1}{2}a_3\nu^3t^4\|(-\Delta)\frac{1}{4}A\hat{\theta}\|_{L^2}^2
- (a_1\nu^3t^2E_\frac{3}{2} - 2a_2\nu^3t^3\|(-\Delta)^\frac{1}{4}A\hat{\theta}\|_{L^2}E_\frac{3}{2} + \frac{1}{2}a_3\nu^3t^4\|(-\Delta)^\frac{1}{4}A\hat{\theta}\|_{L^2}^2)
- (a_3\nu^3t^4E_0 - 2a_3c_4\nu^3t^4\|A\hat{\theta}\|_{L^2}E_0^\frac{1}{2} + \frac{1}{2}a_2\gamma\nu^3t^3\|A\hat{\theta}\|_{L^2}^2)
- \left(\frac{1}{2}a_2\gamma\nu^3t^3\|A\hat{\theta}\|_{L^2}^2 - 4a_3\nu^3t^3 \mathcal{E}_2 + a_2c_3\nu^3t^3 \sum_{\beta \in \mathbb{Z}} \frac{2\pi\beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2\right)
- (2 - a_2c_2 - a_1 - \frac{3a_2 - 2a_1\gamma}{2a_3})\nu^3t E_\frac{3}{2}
- a_3\nu^3t^4E_0
\text{def} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
$$

Our analysis is based on the above inequality.

For $\nu \leq 1$ and $t \in [0, \nu^{-3/5}]$, it holds that $\frac{1}{3}|\alpha| \leq \gamma \leq |\alpha|$ and $\nu^3t^3 \leq \nu$. By the definition, it is easy to see that

$$
\mathcal{E}_1 \leq E_\frac{3}{2} \mathcal{E}_\frac{3}{2}^\frac{1}{2}, \quad \mathcal{E}_1 \leq E_\frac{3}{2} \|(-\Delta)^\frac{1}{4}A\hat{\theta}\|_{L^2}.
$$

From Lemma 3.3 and the fact that $c_3 \leq \frac{1}{2}$, we have

$$
\frac{1}{2}a_2\gamma\nu^3t^3\|A\hat{\theta}\|_{L^2}^2 + a_2c_3\gamma\nu^3t^3 \sum_{\beta \in \mathbb{Z}} \frac{2\pi\beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\psi}(\beta)|^2 \geq \frac{c_3}{c_1}|\alpha|^{-\frac{1}{2}}a_2\gamma\nu^3t^3 \mathcal{E}_2.
$$
Therefore, to ensure that all $I_i$ for $i = 1, \ldots, 7$ are negative, we obtain the following constraint conditions:

\[ a_2^2 \leq \frac{1}{2} a_1 a_3, \quad a_3 \leq \frac{a_2 |\alpha|}{6c_4}, \quad a_3 \leq \frac{c_3 a_2 |\alpha|^{\frac{1}{2}}}{12c_1}, \quad a_2 \leq \frac{1}{2c_2}, \quad a_1^2 |\alpha|^2 \leq \frac{a_3}{2}. \]

So we choose

\[ a_1 = \frac{|\alpha|^{-1}}{4c_2(6c_4^2 + 12c_1 c_3)^2}, \quad a_2 = \frac{|\alpha|^{-\frac{1}{2}}}{8c_2(6c_4^2 + 12c_1 c_3)^3}, \quad a_3 = \frac{1}{8c_2(6c_4^2 + 12c_1 c_3)^4} \]

to fulfill the above conditions.

As a result, we have

\[ \frac{d}{dt} \Phi(t) \leq -a_3 \nu^3 t^4 E_0(t). \]

From Lemma 3.2 one can see that

\[ E_1 \leq E_1^\frac{1}{2} \| \hat{A} \|_{L^2} \leq E_1^\frac{1}{2} \| \hat{\theta} \|_{L^2}. \]

As $a_2^2 \leq \frac{1}{2} a_1 a_3$, it follows that $\Phi(t) \geq E_0(t)$. It completes the proof. \[ \square \]

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** From Lemma 4.1 one can see that

\[ \Phi(t) \leq \Phi(0) = E_0(0), \quad \forall t \in [0, \nu^{-3/5}]. \]

Next, we introduce a new energy functional which is defined on $t \in [\nu^{-3/5}, \nu^{-1}]$

\[ \tilde{\Phi}(t) = E_0(t) + a_1 \nu^{4/5} E_1(t) + a_2 \nu^{1/5} E_1(t) + a_3 \nu^{-2/5} E_2(t). \]

It is easy to check that

\[ \frac{d}{dt} \tilde{\Phi}(t) = -2\nu E_1^\frac{1}{2} - 2a_1 \nu^{9/5} E_1^\frac{1}{2} - 2a_1 \nu^{4/5} |\gamma| E_1 + 2a_2 \nu^{6/5} \| (-\Delta_\alpha)^\frac{1}{2} A \hat{\theta} \|_{L^2} E_1^\frac{1}{2} \]

\[ + a_2 \nu^{6/5} E_1^\frac{1}{2} - a_2 |\gamma| \nu^{1/5} \| A \hat{\theta} \|_{L^2}^2 - a_2 c_3 |\gamma| \nu^{1/5} \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\hat{\psi}(\beta)|^2 \]

\[ - 2a_3 \nu^{3/5} E_0 - a_3 \nu^{3/5} \| (-\Delta_\alpha)^\frac{1}{2} A \hat{\theta} \|_{L^2}^2 + 4a_3 c_4 \nu^{3/5} \| A \hat{\theta} \|_{L^2} E_0^\frac{1}{2} \]

\[ \leq -a_3 \nu^{3/5} E_0 - a_1 \nu^{7/5} E_1 - a_2 \nu^{4/5} E_1 - a_3 \nu^{1/5} E_2 \]

\[ \leq -a_3 \nu^{3/5} \tilde{\Phi}(t). \]

Here $a_1, a_2, a_3$ are given in Lemma 4.1.

It is clear that

\[ \tilde{\Phi}(\nu^{-3/5}) = \Phi(\nu^{-3/5}) \leq E_0(0). \]

Then we have for $t \geq \nu^{-3/5}$ that

\[ E_0(t) \leq \tilde{\Phi}(t) \leq e^{-a_3 \nu^{3/5}(t-\nu^{-3/5})} \tilde{\Phi}(\nu^{-3/5}) \leq 2e^{-a_3 \nu^{3/5} t} E_0(0). \]

As $a_3 > 0$ depends only on $c_0$, the result of Theorem 1.1 follows immediately. \[ \square \]
5. Sub-critical QG Equation

In this section, we study the equation \((1.3)\) with \(s = 1\)

\[
\partial_t \hat{\theta} - \nu \Delta_\alpha \hat{\theta} + i\alpha e^{-\nu t} \cos y(1 - (-\Delta_\alpha)^{-\frac{1}{2}})\hat{\theta} = 0,
\]

which is the linearized sub-critical QG equation.

Now let

\[
\mathcal{L}_{\nu,1} = -\nu \Delta_\alpha + i\gamma(t)B.
\]

Then \((5.1)\) can be written as

\[
\partial_t \hat{\theta} + \mathcal{L}_{\nu,1} \hat{\theta} = 0.
\]

Here we still use the notations

\[
A = \sin y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}), \quad B = \cos y(1 - (-\Delta_\alpha)^{-\frac{1}{2}}), \quad \gamma(t) = \alpha e^{-\nu t},
\]

as well as the inner product

\[
\langle u, w \rangle_* = \langle u, w - (-\Delta_\alpha)^{-\frac{1}{2}}w \rangle.
\]

For the sub-critical case, we introduce the new energy functional with different time weight:

\[
\Phi_1(t) = E_0 + a_1 \nu t E_1 + a_2 \nu t^2 E_1 + a_3 \nu t^3 E_2,
\]

where \(E_0, E_1, E_1, E_2\) are consistent with the critical case,

\[
E_0 = \|\hat{\theta}\|^2, \quad E_1 = \|\partial_\gamma \hat{\theta}\|^2, \quad E_1 = -\frac{\alpha}{|\alpha|} \Re \langle iA\hat{\theta}, \partial_\gamma \hat{\theta} \rangle_*, \quad E_2 = \|\hat{\theta}\|^2 - \|B\hat{\theta}\|^2.
\]

We have the following proposition:

**Proposition 5.1.** Suppose that \(|\alpha| \geq c_0 > 1\). Then it holds that

\[
\frac{d}{dt} E_0 = -2\nu E_1 - 2\nu |\alpha|^2 E_0,
\]

\[
\frac{d}{dt} E_1 = -2\nu E_2 - 2\nu |\alpha|^2 E_1 - 2\gamma |E_1|,
\]

\[
\frac{d}{dt} E_1 \leq -\nu (2|\alpha|^2 + 1) E_1 + 2\nu \|A \partial_\gamma \hat{\theta}\|_L^2 E_2 - |\gamma| \|A \hat{\theta}\|^2 |\partial_\gamma \hat{\theta}|^2 - c_3 |\gamma| \sum_{\beta \in \mathbb{Z}} \frac{2\beta^2}{(\alpha^2 + \beta^2)^{\frac{1}{2}}} |\hat{\psi}(\beta)|^2,
\]

\[
\frac{d}{dt} E_2 \leq -2\nu \left( E_2 + \|A \partial_\gamma \hat{\theta}\|^2 \right),
\]

where

\[
E_1 = 2\pi \sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1) |\hat{\theta}(\beta)|^2,
\]

\[
E_2 = \|\partial_\gamma^2 \hat{\theta}\|^2 = 2\pi \sum_{\beta \in \mathbb{Z}} \frac{\beta^4 ((\alpha^2 + \beta^2)^{\frac{1}{2}})}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} |\hat{\theta}(\beta)|^2,
\]

and \(0 < c_3 \leq \frac{1}{2}\) is an independent constant.

**Proof.** We prove the case \(\alpha > 1\). The proof for the case \(\alpha < -1\) is the same.

By using the symmetric property of \(B\) and following the idea in the proof of Proposition 3.1, we obtain that

\[
\frac{d}{dt} E_0 = -2\Re \langle \mathcal{L}_{\nu,1} \hat{\theta}, \hat{\theta} \rangle_* = 2\nu \Re \langle \Delta_\alpha \hat{\theta}, \hat{\theta} \rangle_* - 2\gamma \Re \langle iB\hat{\theta}, \hat{\theta} \rangle_* = -2\nu E_1 - 2\nu |\alpha|^2 E_0,
\]

and

\[
\frac{d}{dt} E_1 = -2\Re \langle \partial_\gamma \mathcal{L}_{\nu,1} \hat{\theta}, \hat{\theta} \rangle_* = 2\nu \Re \langle \Delta_\alpha \partial_\gamma \hat{\theta}, \partial_\gamma \hat{\theta} \rangle_* - 2\gamma \Re \langle i\partial_\gamma (B\hat{\theta}), \partial_\gamma \hat{\theta} \rangle_*
\]
Here we use the fact that

\[
\text{Proof of Theorem 1.3.}
\]

\[
\frac{d}{dt}\mathcal{E}_1 = -\nu\Re(iA\Delta_\alpha\hat{\theta}, \hat{\theta})_* - \gamma\Re(AB\hat{\theta}, \hat{\theta})_*
\]
\[
=-\nu\Re(iA\Delta_\alpha\hat{\theta}, \hat{\theta})_* + \gamma\Re(A\hat{\theta}, \hat{\theta})_*
\]
\[
=\nu\Re(iA\Delta_\alpha\hat{\theta}, \hat{\theta})_* + \gamma\Re(AB\hat{\theta}, \hat{\theta})_*
\]
\[
\leq -\nu(2|\alpha|^2 + 1)\mathcal{E}_1 + 2\nu\Re(iA\partial_y\hat{\theta}, \partial_y\hat{\theta})_* + \gamma\Re(AB\hat{\theta}, \hat{\theta})_*
\]
\[
\leq -\nu(2|\alpha|^2 + 1)\mathcal{E}_1 + 2\nu\|A\partial_y\hat{\theta}\|_{L^2}^2 - \gamma\|A\hat{\theta}\|_{L^2}^2 - c_3\sum_{\beta \in \mathbb{Z}} \frac{2\pi\beta^2}{(\alpha^2 + \beta^2)^{3/2}} |\tilde{\psi}(\beta)|^2.
\]

For \(\mathcal{E}_2 = \|\hat{\theta}\|^2 - \|B\hat{\theta}\|^2\), recalling the identity (3.1), one can deduce that

\[
\frac{d}{dt}\mathcal{E}_2 = 2\nu\Re(A\Delta_\alpha\hat{\theta}, \hat{\theta})_* - 2\nu\Re(B\Delta_\alpha\hat{\theta}, \hat{\theta})_* - 2\gamma\Re(iB\hat{\theta}, \hat{\theta})_* + 2\gamma\Re(iBB\hat{\theta}, \hat{\theta})_*
\]
\[
= 2\nu\Re(A\Delta_\alpha\hat{\theta}, \hat{\theta})_* - 2\nu\Re(B\Delta_\alpha\hat{\theta}, \hat{\theta})_*
\]
\[
= -2\nu\left(\Re((-\Delta_\alpha)^{-1}\hat{\theta}, \hat{\theta}) - \Re((-\Delta_\alpha)^{-1}B\hat{\theta}, \hat{\theta})_* - \Re(A\Delta_\alpha\hat{\theta}, \hat{\theta})_*\right)
\]
\[
\leq -2\nu\left(\|\hat{\theta}\|_{L^2}^2 + \|A\hat{\theta}\|_{L^2}^2 + \|B\hat{\theta}\|_{L^2}^2\right).
\]

Here we use the fact that

\[
\Re(A\Delta_\alpha\hat{\theta}, \hat{\theta}) + \Re((-\Delta_\alpha)^{-1/2}B\Delta_\alpha\hat{\theta}, B\hat{\theta})
\]
\[
= -\alpha^2 \|\sin y\hat{u}\|^2_{L^2} + \frac{1}{2} \int_T \sin^2 y(\hat{\theta}_t \cdot \vec{\partial}_y^2 \hat{\theta} - \hat{\theta}_t \cdot \vec{\partial}_y^2 \hat{\theta}) dy
\]
\[
- \Re((-\Delta_\alpha)^{-1/2} \cos y\hat{u}, \cos y\hat{u}) + \Re((-\Delta_\alpha)^{-1/2} \cos y, \partial_y^2 \hat{\theta})
\]
\[
\leq -\alpha^2 \|\sin y\hat{u}\|^2_{L^2} - (\alpha^2 + 1) \|\sin y\partial_y\hat{u}\|^2_{L^2}
\]
\[
\leq -\alpha^2 \|\sin y\hat{u}\|^2_{L^2} - \|y\partial_y\hat{u}\|^2_{L^2}
\]
\[
\leq -\alpha^2 \|\sin y\hat{u}\|^2_{L^2} - \|y\partial_y\hat{u}\|^2_{L^2} - \|A\hat{\theta}\|^2_{L^2} - \|B\hat{\theta}\|^2_{L^2},
\]

where \(\hat{u} = (1 - (-\Delta_\alpha)^{-1/2})\hat{\theta}\).

This finishes the proof. \qed

Now we give the proof of Theorem 1.3

**Proof of Theorem 1.3.** Recall the energy functional

\[
\Phi_1(t) = E_0 + a_1\nu tE_1 + a_2\nu t^2\mathcal{E}_1 + a_3\nu t^3\mathcal{E}_2.
\]
By Proposition 5.1, we get that
\[
\frac{d}{dt} \Phi_1(t) \leq -2\nu E_1 - 2\nu|\alpha|^2 E_0
\]
\[
+ a_1\nu E_1 - 2a_1\nu^2 tE_2 - 2a_1|\alpha|^2 \nu^2 tE_1 - 2a_1\nu t|\gamma|\mathcal{E}_1
\]
\[
+ 2a_2\nu t\mathcal{E}_1 - a_2\nu^2 t^2 (2|\alpha|^2 + 1)\mathcal{E}_1 + 2a_2\nu^2 t^2 \|A\partial_\gamma \tilde{\theta}\|_2 \frac{1}{2} E_2
\]
\[
- a_2|\nu t^2 \|A\tilde{\theta}\|_2^2 - a_2c_3|\nu t^2 \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2 |\tilde{\psi}(\beta)|^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}
\]
\[
+ 3a_3\nu^2 t^2 \mathcal{E}_2 - 2a_3\nu^2 t^3 \mathcal{E}_1 \frac{1}{2} - 2a_3\nu^2 t^3 |\alpha|^2 \|A\tilde{\theta}\|_2^2 - 2a_3\nu^2 t^2 \|A\partial_\gamma \tilde{\theta}\|_2^2.
\]
It is clear that
\[
\mathcal{E}_1 \leq \|A\tilde{\theta}\|_2 \frac{1}{2} E_1^2.
\]
By using Lemma 3.3, we have
\[
\frac{1}{2} a_2|\nu t^2 \|A\tilde{\theta}\|_2^2 + a_2c_3|\nu t^2 \sum_{\beta \in \mathbb{Z}} \frac{2\pi \beta^2 |\tilde{\psi}(\beta)|^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} \geq \frac{c_3}{c_1} |\alpha|^{-\frac{1}{2}} a_2|\nu t^2 \mathcal{E}_2.
\]
Lemma 3.2 gives us that
\[
\nu^\frac{3}{2} t^\frac{3}{2} E_0 \leq \left(\nu^2 t^3 E_2^2\right) \left(\nu t^2 \left(\sum_{\beta \in \mathbb{Z}} ((\alpha^2 + \beta^2)^{\frac{1}{2}} - 1) |\tilde{\psi}(\beta)|^2 \right)^{\frac{1}{2}}\right) \leq \left(\nu^2 t^3 E_2^2\right) \left(\nu t^2 E_2^2\right).
\]
Therefore, we can choose
\[
a_1 = \frac{|\alpha|^{-\frac{1}{2}}}{\left(\frac{18c_1}{c_3}\right)^2}, \quad a_2 = \frac{|\alpha|^{-\frac{1}{2}}}{2 \left(\frac{18c_1}{c_3}\right)^3}, \quad a_3 = \frac{1}{2 \left(\frac{18c_1}{c_3}\right)^4}
\]
to ensure for \( t \in [0, \nu^{-\frac{1}{2}}] \) that
\[
\Phi_1(t) \geq E_0(t), \quad \frac{d}{dt} \Phi_1(t) \leq -a_3 \nu^\frac{3}{2} t^\frac{3}{2} E_0(t).
\]
Similarly, we introduce a new energy functional defined on \( t \in [\nu^{-\frac{3}{4}}, \nu^{-1}]\),
\[
\tilde{\Phi}_1(t) = E_0(t) + a_1 \nu^\frac{3}{4} E_1(t) + a_2 \nu^\frac{1}{4} \mathcal{E}_1(t) + a_3 \nu^{-\frac{2}{3}} \mathcal{E}_2(t),
\]
which satisfies
\[
\frac{d}{dt} \tilde{\Phi}_1(t) \leq -a_3 \nu^\frac{3}{2} \tilde{\Phi}_1(t),
\]
and
\[
\tilde{\Phi}_1(\nu^{-\frac{3}{4}}) = \tilde{\Phi}_1(\nu^{-\frac{1}{2}}) \leq E_0(0).
\]
Therefore, we conclude for \( t \in [\nu^{-\frac{3}{4}}, \nu^{-1}] \) that
\[
E_0(t) \leq \tilde{\Phi}_1(t) \leq e^{-a_3 \nu^\frac{3}{2} (t-\nu^{-\frac{3}{4}})} \tilde{\Phi}_1(\nu^{-\frac{1}{2}}) \leq 2e^{-a_3 \nu^\frac{3}{2} t} E_0(0).
\]
The assertion of Theorem 1.3 follows immediately.
6. Transport fractional diffusion equation

In this section, we consider the toy model:
\[ \partial_t \hat{\theta}(t, \alpha, y) + L_{\alpha, \nu, \frac{1}{2}}^S \hat{\theta}(t, \alpha, y) = 0, \]
where
\[ L_{\alpha, \nu, \frac{1}{2}}^S = \nu(-\Delta_\alpha)^{\frac{1}{2}} + i\alpha \cos y. \]
We define the energy functional
\[ \Phi^S(t) = E_0^S(t) + a_1 \nu^2 t^2 E_1^S(t) + a_2 \nu^2 t^3 E_2^S(t) + a_3 \nu^2 t^4 E_3^S(t), \]
where
\[ E_0^S = \|\hat{\theta}\|_{L^2}^2, \quad E_1^S = \|\partial_y \hat{\theta}\|_{L^2}^2, \quad E_2^S = -\frac{\alpha}{|\alpha|} \langle i \sin \hat{\theta}, \partial_y \hat{\theta} \rangle, \quad E_3^S = \|\sin y \hat{\theta}\|_{L^2}^2. \]

Let us first introduce a new interpolation lemma.

Lemma 6.1. It holds that
\[ \|\hat{\theta}\|_{L^2}^2 \leq 2 \|\partial_y (\cos y \hat{\theta})\|_{L^2} \|\sin \hat{\theta}\|_{L^2}. \]

Proof. A direct calculation shows that,
\[ -\int_T \sin y \hat{\theta} \partial_y (\cos y \hat{\theta}) + \partial_y (\cos y \hat{\theta}) \sin y \hat{\theta} dy \]
\[ = 2 \int_T \sin^2 y \hat{\theta} dy - \sin y \cos y \hat{\theta} \partial_y \hat{\theta} - \sin y \cos y \partial_y \hat{\theta} dy \]
\[ = 2 \int_T \sin^2 y \hat{\theta} dy - \frac{1}{2} \int_T \sin 2y \partial_y \hat{\theta} dy \]
\[ = 2 \int_T \sin^2 y \hat{\theta} dy + \int_T \cos y \hat{\theta} dy \]
\[ = \int_T \hat{\theta} dy. \]

The results follow immediately. \(\square\)

Now we give the proof for Theorem 1.2.

Proof of Theorem 1.2. We start with some basic estimates, here we still focus only on the case \(\alpha > 1\). It is easy to check that
\[ \frac{d}{dt} E_0^S = -2\nu \langle (-\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \hat{\theta} \rangle - 2\alpha \Re \langle i \cos y \hat{\theta}, \hat{\theta} \rangle = -2\nu E_0^S, \]
and
\[ \frac{d}{dt} E_1^S = -2\nu \langle (-\Delta_\alpha)^{\frac{1}{2}} \partial_y \hat{\theta}, \partial_y \hat{\theta} \rangle - 2\alpha \Re \langle i \sin y \partial_y \hat{\theta}, \partial_y \hat{\theta} \rangle + 2\alpha \Re \langle i \sin y \hat{\theta}, \partial_y \hat{\theta} \rangle \]
\[ = -2\nu E_1^S - 2\alpha E_1^S. \]

Here we use \( E_0^S \) and \( E_1^S \) to denote \( \langle (-\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \hat{\theta} \rangle \) and \( \langle (-\Delta_\alpha)^{\frac{1}{2}} \partial_y \hat{\theta}, (-\Delta_\alpha)^{\frac{1}{2}} \partial_y \hat{\theta} \rangle \) respectively.

For \( E_1^S \), we have
\[ \frac{d}{dt} E_1^S = \nu \Re \langle i \sin y (-\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \partial_y \hat{\theta} \rangle - \alpha \Re \langle i \sin y \cos y \hat{\theta}, \partial_y \hat{\theta} \rangle. \]
\[ + \nu \mathcal{R} (i \sin y \hat{\theta}, -(\Delta_\alpha)^{\frac{1}{2}} \partial_y \hat{\theta}) + \alpha \mathcal{R} (\sin y \hat{\theta}, \cos y \partial_y \hat{\theta}) - \alpha \mathcal{R} (\sin y \hat{\theta}, \sin y \hat{\theta}) \]
\[ = 2 \nu \mathcal{R} (i (-(\Delta_\alpha)^{\frac{1}{2}} \sin y \hat{\theta}, -(\Delta_\alpha)^{\frac{1}{2}} \partial_y \hat{\theta}) - \alpha (\sin y \hat{\theta}, \sin y \hat{\theta}) + \nu \mathcal{R} (i [\sin y, -(\Delta_\alpha)^{\frac{1}{2}}] \hat{\theta}, \partial_y \hat{\theta}). \]

For the last term \( \nu \mathcal{R} (i [\sin y, -(\Delta_\alpha)^{\frac{1}{2}}] \hat{\theta}, \partial_y \hat{\theta}) \), it holds that
\[
\left| \mathcal{R} (i [\sin y, -(\Delta_\alpha)^{\frac{1}{2}}] \hat{\theta}, \partial_y \hat{\theta}) \right|
= \frac{1}{2} \left| \int_T i \left( \sin y (-(\Delta_\alpha)^{\frac{1}{2}} - (\Delta_\alpha)^{\frac{1}{2}} \sin y) \hat{\theta} \cdot \partial_y \hat{\theta} dy \right) \right|
= \frac{\pi}{2} \left| \sum_{\beta \in \mathbb{Z}} i \beta \left( (\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} - (\alpha^2 + \beta^2)^{\frac{1}{2}} \right) \left( \bar{\hat{\theta}} (\beta) \bar{\hat{\theta}} (\beta - 1) - \bar{\hat{\theta}} (\beta - 1) \bar{\hat{\theta}} (\beta) \right) \right|
\leq 4E_1^S.
\]

Therefore, we get
\[
\frac{d}{dt} E_1^S \leq 2\nu \| (-(\Delta_\alpha)^{\frac{1}{2}} \sin y \hat{\theta}\|_{L^2} E_2^S \frac{1}{2} + 4\nu E_1^S - \alpha E_2^S.
\]

For \( E_2^S = \| \sin y \hat{\theta}\|_{L^2}^2 \), we have
\[
\frac{d}{dt} E_2^S = -2 \nu \mathcal{R} (\sin y (-(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \sin y \hat{\theta}) - 2 \alpha \mathcal{R} (i \sin y \cos y \hat{\theta}, \sin y \hat{\theta})
= -2 \nu \mathcal{R} (\sin y (-(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \sin y \hat{\theta})
= -2 \nu \mathcal{R} (-(\Delta_\alpha)^{\frac{1}{2}} \sin y \hat{\theta}, -(\Delta_\alpha)^{\frac{1}{2}} \sin y \hat{\theta}) - 2 \nu \mathcal{R} (|\sin y, -(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \sin y \hat{\theta})
\leq -2 \nu \| (-(\Delta_\alpha)^{\frac{1}{2}} \sin y \hat{\theta}\|_{L^2}^2 + 4\nu E_0^S \frac{1}{2} E_2^S \frac{1}{2},
\]
where we use the fact that
\[
\left| \mathcal{R} (|\sin y, -(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \sin y \hat{\theta}) \right| \leq \| |\sin y, -(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}, \sin y \hat{\theta}\|_{L^2} E_2^S \frac{1}{2},
\]
and
\[
\| |\sin y, -(\Delta_\alpha)^{\frac{1}{2}} \hat{\theta}\|_{L^2}^2
= \int_T \left( \sin y (-(\Delta_\alpha)^{\frac{1}{2}} - (\Delta_\alpha)^{\frac{1}{2}} \sin y) \hat{\theta} \cdot \left( \sin y (-(\Delta_\alpha)^{\frac{1}{2}} - (\Delta_\alpha)^{\frac{1}{2}} \sin y) \hat{\theta} dy \right) \right)
= \frac{\pi}{2} \sum_{\beta \in \mathbb{Z}} \left| \frac{(1 - 2\beta) \bar{\hat{\theta}} (\beta - 1)}{(\alpha^2 + (\beta - 1)^2)^{\frac{1}{2}} + (\alpha^2 + \beta^2)^{\frac{1}{2}}} - \frac{(1 + 2\beta) \bar{\hat{\theta}} (\beta + 1)}{(\alpha^2 + (\beta + 1)^2)^{\frac{1}{2}} + (\alpha^2 + \beta^2)^{\frac{1}{2}}} \right|^2
\leq 4E_0^S.
\]
As a result, we have
\[
\frac{d}{dt} \Phi^S(t) \leq -2\nu E^S_\frac{t}{2} + 2a_1 \nu^2 tE^S_1 - 2a_1 \nu^3 t^2 E^S_\frac{t}{2} - 2a_1 \nu^2 t^2 \alpha E^S_1
\]
\[
+ 3a_2 \nu^2 t^2 E^S_1 + 2a_2 \nu^3 t^3 \|(-\Delta)^\frac{1}{3} \sin y\theta\|_{L^2} E^S_\frac{t}{2} + 4a_2 \nu^3 t^3 E^S_2 - a_2 \nu^2 t^3 \alpha E^S_2
\]
\[
+ 4a_3 \nu^2 t^3 E^S_2 - 2a_3 \nu^2 t^4 \|(-\Delta)^\frac{1}{4} \sin y\theta\|_{L^2} + 4a_3 \nu^3 t^4 E^S_0 E^S_\frac{t}{2} E^S_2^\frac{1}{2}.
\]

It is clear that
\[
\nu^2 t^2 E^S_1 \leq \nu E^S_\frac{t}{2} + \nu^3 t^3 \|(-\Delta)^\frac{1}{3} \sin y\theta\|_{L^2}, \quad \nu^2 t E^S_1 \leq \nu E^S_\frac{t}{2} + \nu^3 t^2 E^S_2.
\]

Therefore, by using Lemma 6.1, we deduce that
\[
\nu^2 t^2 E^S_0 \leq 2\nu^2 t^2 E^S_2 + |\alpha|^\frac{1}{2} \nu^2 t^2 E^S_2 + |\alpha|^{-\frac{1}{2}} \nu^2 t E^S_1
\]
\[
\leq 2\nu^2 t^2 E^S_2 + |\alpha|^\frac{1}{2} \nu^2 t^2 E^S_2 + \nu E^S_\frac{t}{2} + |\alpha|^{-1} \nu^3 t^2 E^S_2
\]
\[
\leq 2|\alpha|^\frac{1}{2} \nu^2 t^3 E^S_2 + 2\nu E^S_\frac{t}{2} + |\alpha|^{-1} \nu^3 t^2 E^S_2.
\]

In the last inequality, we use the following facts. When \( t \geq 2 \), then
\[
2\nu^2 t^2 E^S_2 \leq \nu^2 t^3 E^S_2,
\]
when \( t \leq 2 \) and \( \nu \leq \frac{1}{8} \), then
\[
2\nu^2 t^2 E^S_2 \leq \nu E^S_\frac{t}{2}.
\]

By using the same technique, for \( \nu \leq \frac{1}{8} \) and \( t \leq \nu^{-2/3} \), we have
\[
4a_3 \nu^2 t^4 E^S_0^\frac{3}{2} E^S_2^\frac{1}{2} \leq 4a_3 \nu^2 t^2 E^S_2^\frac{3}{2} E^S_2^\frac{1}{2}
\]
\[
\leq 2a_3 \nu^2 t^2 E^S_0 + 2a_3 \nu^2 t^3 E^S_2
\]
\[
\leq 4a_3 \nu^2 t^2 E^S_2 + 2a_3 |\alpha|^{-\frac{1}{2}} \nu^2 t E^S_1 + 4a_3 |\alpha|^{\frac{1}{2}} \nu^2 t^3 E^S_2
\]
\[
\leq 4a_3 \nu^2 t^2 E^S_2 + a_3 \nu E^S_\frac{t}{2} + a_3 |\alpha|^{-1} \nu^3 t^2 E^S_2 + 4a_3 |\alpha|^{\frac{1}{2}} \nu^2 t^3 E^S_2
\]
\[
\leq 2a_3 \nu E^S_\frac{t}{2} + a_3 |\alpha|^{-1} \nu^3 t^2 E^S_2 + 8a_3 |\alpha|^{\frac{1}{2}} \nu^2 t^3 E^S_2.
\]

By choosing
\[
a_1 = 2^{-20} |\alpha|^{-1}, \quad a_2 = 2^{-32} |\alpha|^{-\frac{1}{2}}, \quad a_3 = 2^{-40},
\]
we deduce for \( t \in [1, \nu^{-\frac{2}{3}}] \) that
\[
\Phi^S(t) \geq E^S_0,
\]
and by (6.1),
\[
\frac{d}{dt} \Phi^S(t) \leq -2^{-34} |\alpha|^{\frac{1}{2}} \nu^2 t^3 E^S_2 - 2^{-2} \nu E^S_\frac{t}{2} - 2^{-22} |\alpha|^{-1} \nu^3 t^2 E^S_2
\]
\[
\leq -2^{-35} \nu^2 t^2 E^S_0.
\]

Similar to the proof of Theorem 1.1, we define on \( t \in [\nu^{-2/3}, \nu^{-1}] \) that
\[
\tilde{\Phi}^S(t) = E^S_0 + 2^{-20} |\alpha|^{-1} \nu^2 t^2 E^S_1 + 2^{-32} |\alpha|^{-\frac{1}{2}} E^S_1 + 2^{-40} \nu^{-2/3} E^S_2,
\]
which satisfies
\[
\frac{d}{dt} \tilde{\Phi}^S(t) \leq -\nu E^S_1 - 2^{-19}|\alpha|^{-1} \nu^{5/3} E^S_2 - 2^{-33}|\alpha|^4 E^S_2 \leq -2^{-35} \nu^{2/3} \tilde{\Phi}^S(t),
\]
and
\[
\tilde{\Phi}^S(\nu^{-2/3}) = \Phi^S(\nu^{-2/3}) \leq E^S_0(0).
\]
Then it follows that for \( t \geq \nu^{-2/3} \)
\[
E^S(t) \leq \tilde{\Phi}^S(t) \leq e^{-2^{-35} \nu^{2/3}(t-\nu^{-2/3})} \tilde{\Phi}^S(\nu^{-2/3}) \leq 2e^{-2^{-35} \nu^{2/3} t} E^S_0(0).
\]
The result of Theorem 1.2 follows immediately. \( \Box \)

REFERENCES

[1] D. Albritton, R. Beekie, and M. Novack, Enhanced dissipation and hormander’s hypoellipticity, arXiv:2105.12308, (2021).
[2] M. Beck and C. E. Wayne, Using global invariant manifolds to understand metastability in the Burgers equation with small viscosity, SIAM Rev., 53 (2011), 129–153.
[3] M. Beck and C. E. Wayne, Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier-Stokes equations, Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), 905–927.
[4] J. Bedrossian and M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows, Arch. Ration. Mech. Anal., 224 (2017), 1161–1204.
[5] J. Bedrossian, V. Vicol, and F. Wang, The sobolev stability threshold for 2d shear flows near couette, Journal of Nonlinear Science, 28 (2018), 2051–2075.
[6] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. (2), 171 (2010), 1903–1930.
[7] D. Chen, Z. Zhang, and W. Zhao, Fujita-Kato theorem for the 3-D inhomogeneous Navier-Stokes equations, J. Differential Equations, 261 (2016), 738–761.
[8] Q. Chen, T. Li, D. Wei, and Z. Zhang, Transition threshold for the 2-d couette flow in a finite channel, arXiv preprint arXiv:1808.08736, (2018).
[9] P. Constantin, Energy spectrum of quasigeostrophic turbulence, Physical Review Letters, 89 (2002), https://doi.org/10.1103/PhysRevLett.89.184501 (Copyright: Copyright 2017 Elsevier B.V., All rights reserved).
[10] P. Constantin, A. J. Majda, and E. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity, 7 (1994), 1495–1533.
[11] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal., 30 (1999), 937–948.
[12] M. Coti Zelati, Stable mixing estimates in the infinite Péclet number limit, J. Funct. Anal., 279 (2020), 108562, 25.
[13] M. Coti Zelati, T. M. Elgindi, and K. Widmayer, Enhanced dissipation in the Navier-Stokes equations near the Poiseuille flow, Comm. Math. Phys., 378 (2020), 987–1010.
[14] Y. Couder, Observation expérimentale de la turbulence bidimensionnelle dans un film liquide mince, Comptes-rendus des séances de l’Académie des sciences. Série 2, Mécanique-physique, chimie, sciences de l’univers, sciences de la terre, 297 (1983), 641–645.
[15] M. Dabkowski, A. Kiselev, L. Silvestre, and V. Vicol, Global well-posedness of slightly supercritical active scalar equations, Anal. PDE, 7 (2014), 43–72.
[16] S. Ding and Z. Lin, Enhanced dissipation and transition threshold for the 2-D plane Poiseuille flow via resolvent estimate, arXiv:2008.10057, (2020).
[17] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal., 16 (1964), 269–315.
[18] E. Grenier, T. T. Nguyen, F. Rousset, and A. Soffer, Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method, Journal of Functional Analysis, 278 (2020), 108339. https://doi.org/10.1016/j.jfa.2019.108339
http://www.sciencedirect.com/science/article/pii/S0022123619303337
[19] S. He, Enhanced dissipation, hypoellipticity for passive scalar equations with fractional dissipation, arXiv:2103.07906, (2021).
[20] S. Ibrahim, Y. Maekawa, and N. Masmoudi, *On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows*, Ann. PDE, 5 (2019), Paper No. 14, 84.

[21] N. Ju, *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space*, Comm. Math. Phys., 251 (2004), 365–376.

[22] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math., 167 (2007), 445–453.

[23] A. Kiselev, L. Ryzhik, Y. Yao, and A. Zlatoš, *Finite time singularity for the modified SQG patch equation*, Ann. of Math. (2), 184 (2016), 909–948.

[24] T. Li, D. Wei, and Z. Zhang, *Pseudospectral and spectral bounds for the Oseen vortices operator*, Ann. Sci. Éc. Norm. Supér. (4), 53 (2020), 993–1035.

[25] Z. Lin and M. Xu, *Metastability of Kolmogorov flows and inviscid damping of shear flows*, Arch. Ration. Mech. Anal., 231 (2019), 1811–1852.

[26] N. Masmoudi and W. Zhao, *Stability threshold of the 2D Couette flow in Sobolev spaces*, arXiv:1908.11042, (2019).

[27] N. Masmoudi and W. Zhao, *Enhanced dissipation for the 2D Couette flow in critical space*, Communications in Partial Differential Equations, (2020), 1–20.

[28] W. Matthaeus, W. Stribling, D. Martinez, S. Oughton, and D. Montgomery, *Decaying, two-dimensional, navier-stokes turbulence at very long times*, Physica D: Nonlinear Phenomena, 51 (1991), 531–538.

[29] J. Pedlosky et al., *Geophysical fluid dynamics*, vol. 710, Springer, 1987.

[30] S. G. Resnick, *Dynamical problems in non-linear advective partial differential equations*, ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Chicago.

[31] C. Villani, *Hypocoercivity*, Memoirs of the American Mathematical Society, 202 (2009), 1–140.

[32] D. Wei, *Diffusion and mixing in fluid flow via the resolvent estimate*, Sci. China Math., 64 (2021), 507–518.

[33] D. Wei and Z. Zhang, *Enhanced dissipation for the Kolmogorov flow via the hypocoercivity method*, Sci. China Math., 62 (2019), 1219–1232.

[34] D. Wei, Z. Zhang, and W. Zhao, *Linear inviscid damping and enhanced dissipation for the Kolmogorov flow*, Adv. Math., 362 (2020), 106963, 103.

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