Almost Special Holonomy in Type IIA&M Theory

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ABSTRACT

We consider spaces $M_7$ and $M_8$ of $G_2$ holonomy and Spin(7) holonomy in seven and eight dimensions, with a $U(1)$ isometry. For metrics where the length of the associated circle is everywhere finite and non-zero, one can perform a Kaluza-Klein reduction of supersymmetric M-theory solutions (Minkowski)$^4 \times M_7$ or (Minkowski)$^3 \times M_8$, to give supersymmetric solutions (Minkowski)$^4 \times Y_6$ or (Minkowski)$^3 \times Y_7$ in type IIA string theory with a non-singular dilaton. We study the associated six-dimensional and seven-dimensional spaces $Y_6$ and $Y_7$ perturbatively in the regime where the string coupling is weak but still non-zero, for which the metrics remain Ricci-flat but that they no longer have special holonomy, at the linearised level. In fact they have “almost special holonomy,” which for the case of $Y_6$ means almost Kähler, together with a further condition. For $Y_7$ we are led to introduce the notion of an “almost $G_2$ manifold,” for which the associative 3-form is closed but not co-closed. We obtain explicit classes of non-singular metrics of almost special holonomy, associated with the near Gromov-Hausdorff limits of families of complete non-singular $G_2$ and Spin(7) metrics.

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1 Introduction

Since the seminal work described in [1], the procedures for obtaining four-dimensional physics from the compactification of string theory have been extensively studied. With the proposal of an eleven-dimensional M-theory underlying string theory, it becomes of interest to study the compactifications from the eleven-dimensional standpoint, and to see what further lessons about the relations between string theory and M-theory can be learned by this means.

A key idea in the study of string-theory compactifications is that the six-dimensional internal Calabi-Yau spaces can develop singularities at limiting values of their modulus parameters, at which additional massless four-dimensional states will occur. Since the moduli are themselves interpreted as four-dimensional massless scalar fields, this means that the study of the four-dimensional low-energy effective action requires a proper understanding of the regions in modulus space where the singularities are approached. It can be shown that for the singularities of interest here, the Calabi-Yau metric close to a generic singular point itself is of the form of the conifold, which is the Ricci-flat metric

$$ds_6^2 = dr^2 + r^2 ds_{T^1,1}^2$$

(1)
on the cone over the homogeneous Einstein space $T^{1,1} = (S^3 \times S^3)/U(1)_{\text{diag}}$. The metric is singular at the vertex of the cone, at $r = 0$. As the moduli are moved slightly away from the singular limit, the metric near to the previous conifold point is then a smoothed-out version of (1). It can be shown that there are two possible smoothed-out versions of (1); one is called the “resolved conifold,” with the conifold point blown up to an $S^2$; the other is called the “deformed conifold,” with the conifold point blown up to an $S^3$. They are both asymptotically conical (AC), approaching (1) at large distance.

If Calabi-Yau compactifications are considered in the type IIA string, then an immediate “lift” to M-theory compactifications can be performed, simply by taking the direct product of the Calabi-Yau manifold $Y_6$ and the M-theory circle $S^1$. However, these are not the only compactifications from $D = 11$ that can be considered, and more generally one can choose any compact Ricci-flat 7-manifold $M_7$. In order to have supersymmetry, one requires that $M_7$ have special holonomy, which in the irreducible case is the exceptional group $G_2$. Thus the $G_2$ manifold becomes natural compactification space for the M-theory [3, 4, 5]. The direct product $M_7 = Y_6 \times S^1$ is a degenerate example with $SU(3) \subset G_2$ holonomy. More complicated possibilities, corresponding in the ten-dimensional picture to turning on further fields including the Ramond-Ramond vector and the dilaton in the type IIA theory, can
also arise.

Since an irreducible compact $G_2$ manifold $M_7$ cannot have any continuous symmetries, and hence, in particular, no $U(1)$ action, it follows that there cannot be an M-theory to type IIA reduction in which exclusively massless fields (the Kaluza-Klein vector and the dilaton) are excited. However, if one focuses for now on the structure of the metrics near to singular points, which for the Calabi-Yau six-manifolds are approximated by the resolved and deformed conifolds, then the seven-dimensional lifts of these non-compact “building blocks” can admit circle actions, and hence admit a clean interpretation in terms exclusively of massless Kaluza-Klein field excitations.

Explicit examples of non-compact AC type of $G_2$ and spin(7) manifolds were first construct in [6, 7]. The first ALC manifold were constructed in [8]. Recently, there has been extensive work on constructing new non-compact $G_2$ and Spin(7) manifolds [9, 10, 12, 11, 13, 14, 15, 16, 17, 18, 19, 20]. In some of this, it has been shown that there exist families of non-compact $G_2$ metrics that are asymptotically locally conical (ALC), which take the form of an $S^1$ bundle over an asymptotically conical 6-metric, with the radius of the circle stabilising at large distance. The $G_2$ metrics have a non-trivial parameter (not merely the overall scale) that allows the radius $R_0$ of the circle at infinity to be adjusted while keeping some other measure of the “scale size” fixed. At one extreme of the parameter one has $R_0 \to 0$; this is the “Gromov-Hausdorff limit” in which the metric approaches a direct product of a Calabi-Yau 6-metric and the zero-radius $S^1$. At the other extreme, the radius $R_0$ becomes infinite and then the $G_2$ metric is itself asymptotically conical in a seven-dimensional sense. Interpreting the $S^1$ as the M-theory circle for the reduction to type IIA, we therefore have a ten-dimensional Calabi-Yau reduction in the weak-coupling, or Gromov-Hausdorff limit, whereas in the strong-coupling regime the reduction is intrinsically non-perturbative and eleven-dimensional. A family of $G_2$ metrics denoted by $B_7$, whose weak-coupling limit describes the deformed conifold, was considered in [13], based on previous work in [10, 11]. An analogous family, denoted by $D_7$, yielding the resolved conifold as a weak-coupling limit was recently found in [17]. Subsequently, a larger system of equations for $G_2$ metrics was obtained in [19], which encompasses both of the weak-coupling limits.

In this paper, we shall consider the relation between $G_2$ metrics admitting $U(1)$ actions and their associated circle reductions in more detail. In particular, we shall be concerned with situations such as those described above where there is a non-trivial parameter. This allows one to probe the behaviour near to the Gromov-Hausdorff limit, in which the string coupling is becoming weak but is not yet zero. Of particular interest are those cases where
the radius of the circle that stabilises to $R_0$ at infinity remains non-zero (and finite) everywhere in the manifold, since in such cases the dilaton resulting from the Kaluza-Klein reduction will be everywhere finite. The $\mathbb{D}_7$ metrics found in [17] are examples that exhibit this completely regular behaviour. By contrast, in the $\mathbb{B}_7$ metrics studied in [13], the length of the circle goes to zero on the $S^3$ bolt in the core of the metric. In those cases with an everywhere-regular dilaton, one can study the “near Gromov-Hausdorff” regime of the seven-dimensional $G_2$ metric from a six-dimensional Kaluza-Klein perspective, as a fully non-singular perturbation away from the (Calabi-Yau) $\times S^1$ limiting case.

By this means, we are able to study non-singular perturbations of the resolved conifold metric, which after lifting to $D = 7$ become the $\mathbb{D}_7$ metrics with parameter $R_0$ close to the $R_0 = 0$ Gromov-Hausdorff limit. In a similar vein, we also find non-singular perturbations of Ricci-flat Kähler metrics on a complex line bundle over $S^2 \times S^2$. These metrics, obtained for equal $S^2$ radii in [21, 22], and for unequal radii in [23], have principal orbits $T^{1,1}/\mathbb{Z}_2$. After lifting the perturbed metrics to seven dimensions, we obtain the “near Gromov-Hausdorff regime” of another family of smooth $G_2$ metrics, denoted by $\tilde{C}_7$, which are $\mathbb{R}^2$ bundles over $T^{1,1}$ [19]. These too have the feature that the radius of the circle action is everywhere finite and non-zero.

By studying the above perturbations of the original Ricci-flat Kähler six-dimensional spaces, we are able to determine what properties the perturbed metrics should have in order that there exist a lift to give $G_2$ holonomy in seven dimensions. We find that at the level of linearised perturbations away from the Calabi-Yau limit, the six-dimensional metrics continue to be Ricci-flat, but they are no longer Kähler. In fact they still satisfy the almost-Kähler condition, (namely that there is an almost complex structure and $dJ = 0$), together with certain additional conditions that are, nonetheless, weaker than the full Kähler condition. (There is a holomorphic 3-form whose real part is closed (in a suitable choice of phase) but whose imaginary part is not.) One interesting aspect of this, therefore, is that at least for non-compact spaces of the type we are considering here, one can have smooth Ricci-flat perturbations that take the metric away from being Kähler.

After giving a general discussion of the perturbative scheme in section 2, we then derive the details of the regular perturbations of the resolved conifold and the complex line bundle over $S^2 \times S^2$ in section 3.

In section 4 we consider a higher-dimensional analogue of the previous discussion, in which eight-dimensional metrics of Spin(7) holonomy, admitting circle actions, are reduced to seven dimensions. In cases where there is a Gromov-Hausdorff limit, with the asymptotic
radius of the circle tending to zero, such a metric approaches a seven-dimensional metric of $G_2$ holonomy times the zero-radius circle. We can again consider the regime in the vicinity of the Gromov-Hausdorff limit, and examine the associated perturbations of the seven-dimensional $G_2$ metric. We find that here too, at the linearised level the seven-dimensional metric remains Ricci-flat, but it no longer has $G_2$-holonomy. In fact the associative 3-form $\Phi_{(3)}$ that satisfied $d\Phi_{(3)} = 0 = d^*\Phi_{(3)}$ in the unperturbed $G_2$ limit still satisfies $d\Phi_{(3)} = 0$, but now $d^*\Phi_{(3)} \neq 0$. By analogy with the case of Kähler manifolds we propose to call such a seven-dimensional metric an “almost $G_2$ metric.”

A concrete example where we can obtain regular perturbations of a $G_2$ metric is provided by the smooth asymptotically conical metric $G_2$ metric with $SU(3)/(U(1) \times U(1))$ principal orbits. We obtain a class of linearised perturbations under which the metric becomes “almost $G_2$,” while remaining Ricci-flat. Upon lifting to eight dimensions, this yields the near Gromov-Hausdorff limit of a class of smooth ALC Spin(7) metrics whose principal orbits are $N(1,-1) = SU(3)/U(1)_{(1,-1)}$, which were found in [13].

In section 5 we consider some further examples, in higher dimensions, of Ricci-flat perturbations of Calabi-Yau metrics that are almost Kähler but not Kähler. Our starting points are the Ricci-flat Kähler metrics on complex line bundles over $SO(n+2)/(SO(n) \times SO(2))$. We show that one can obtain regular perturbations in general, although now one no longer has a natural interpretation involving a lift to one dimension higher.

2 Kaluza-Klein reduction of $G_2$ metrics

2.1 General discussion

In this section, we examine how a seven-dimensional metric of $G_2$ holonomy that has a $U(1)$ isometry reduces to $D = 6$. In subsequent sections we shall be interested in cases where the $G_2$ metric is asymptotically locally conical (ALC), locally approaching the product of a circle of asymptotically constant radius (associated with the above $U(1)$ action) and an asymptotically conical (AC) six-dimensional metric. Furthermore, our principal focus will be on examples where the length of the circle remains non-vanishing (and finite) everywhere in the $G_2$ manifold. This additional assumption that the $U(1)$ Killing vector has a bounded length is not, however, required in the immediate discussion in this subsection.

The seven-dimensional metric with a $U(1)$ isometry can be cast in the adapted form of a Kaluza-Klein reduction ansatz. In general, in a reduction from $(D+1)$ to $D$ dimensions,
one has
\[ ds_{D+1}^2 = e^{-2\alpha \phi} ds_D^2 + e^{2\alpha (D-2)\phi} (dz + A_{(1)})^2, \quad (2) \]
where \( \alpha = [2(D-1)(D-2)]^{-1/2} \). Using the tangent-space frame \( \hat{e}^a = e^{-\alpha \phi} e^a, \hat{e}^z = e^{2\alpha (D-2)\phi} (dz + A_{(1)}) \), with \( 0 \leq a \leq D-1 \) and \( e^a \) the vielbein for \( ds_D^2 \), the spin connection is given by
\begin{align*}
\hat{\omega}_{ab} &= \omega_{ab} - \alpha e^{\alpha \phi} \left( \partial_b \phi \hat{e}^a - \partial_a \phi \hat{e}^b \right) - \frac{1}{2} F_{ab} e^{\alpha D \phi} \hat{e}^z, \\
\hat{\omega}_{az} &= -\alpha (D-2) e^{\alpha \phi} \partial_a \phi \hat{e}^z - \frac{1}{2} F_{ab} e^{\alpha D \phi} \hat{e}^b. \quad (3)
\end{align*}
The condition of Ricci-flatness in \( (D+1) \) dimensions reduces to the equations of an Einstein-Maxwell-scalar system in \( D \) dimensions,
\begin{align*}
R_{ab} &= \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{2} e^{2\alpha D \phi} \left( F_{ab}^2 - \frac{1}{2(D-2)} F_{ab}^2 g_{ab} \right), \\
d* d\phi &= \alpha D (-1)^{D-1} e^{2\alpha D \phi} * F_{(2)} \wedge F_{(2)}, \\
d (e^{2\alpha D \phi} * F_{(2)}) &= 0, \quad (4, 5)
\end{align*}
which are described by the Lagrangian
\[ L_D = R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{2\alpha D \phi} * F_{(2)} \wedge F_{(2)}. \quad (6) \]
The \( G_2 \) metric admits an “associative 3-form” \( \hat{\Phi}_{(3)} \), whose tangent-space components \( \hat{\Phi}_{ABC} \) give the multiplication rules of the seven imaginary unit octonions \( \iota_A \):
\[ \iota_A \iota_B = -\delta_{AB} + \hat{\Phi}_{ABC} \iota_C. \quad (7) \]
This implies that
\[ \hat{\Phi}_{ABE} \hat{\Phi}_{CDE} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + \frac{1}{6} \epsilon_{ABCDE123} \hat{\Phi}_{123}. \quad (8) \]
The \( G_2 \) holonomy then follows from the conditions
\[ d\hat{\Phi}_{(3)} = 0, \quad d* \hat{\Phi}_{(3)} = 0, \quad (9) \]
where \( * \) denotes the seven-dimensional Hodge dual. The conditions (8) and (9) imply that \( \Phi_{(3)} \) is covariantly constant [24].

The dimensional reduction of \( \Phi_{(3)} \) to \( D = 6 \) is given by
\[ \hat{\Phi}_{(3)} = \Psi_{(2)} \wedge (dz + A_{(1)}) + \Psi_{(3)}. \quad (10) \]
With the seven-dimensional indices decomposed as $A = (a, 6)$, where $0 \leq a \leq 5$, then in particular we can read off from (8), by setting two of the free indices to “6,” that

$$\hat{\Phi}_{ab6} \hat{\Phi}_{bc6} = -\delta_{ab},$$

(11)

implying that $\hat{J}_{ab} \equiv \hat{\Phi}_{ab6}$ has the property of being an almost complex structure in $D = 6$:

$$\hat{J}_{ab} \hat{J}_{bc} = -\delta_{ac}.$$  

(12)

We can also then read off that

$$\hat{J}_{ae} \hat{\Phi}_{bce} = \frac{1}{6} \epsilon_{abcd1d2d3} \hat{\Phi}_{d1d2d3}, \quad \hat{\Phi}_{abe} \hat{\Phi}_{cde} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} - \hat{J}_{ac} \hat{J}_{bd} + \hat{J}_{ac} \hat{J}_{bd}. $$

(13)

These conditions imply that $\hat{\Phi}_{abc}$ can be expressed as

$$\hat{\Phi}_{abc} = \Re(e^{i\gamma} \epsilon_{abc}),$$

(14)

where $\gamma$ is an arbitrary phase angle and the antisymmetric tensor $\epsilon_{abc}$ is holomorphic with respect to $\hat{J}_{ab}$.

The closure and co-closure of $\hat{\Phi}_{(3)}$ imply

$$d\Psi_{(2)} = 0, \quad d\Psi_{(3)} = -\Psi_{(2)} \wedge dA_{(1)},$$

$$d(e^{\frac{1}{\sqrt{10}}\phi} \star \Psi_{(3)}) = 0, \quad d(e^{-\frac{1}{\sqrt{10}}\phi} \star \Psi_{(2)}) = e^{\frac{2}{\sqrt{10}}\phi} \star \Psi_{(3)} \wedge dA_{(1)}. $$

(15)

2.2 Gromov-Hausdorff limit

We shall now specialise to the case where the $G_2$ metric is ALC, with the $U(1)$ isometry acting on the circle of stabilised length at infinity.

It is worth remarking that in any Ricci-flat metric of cohomogeneity one with an asymptotic region, in any dimension, the length of any Killing vector $K$ increases monotonically with radius, and thus it is bounded above by its length at infinity. To see this, let $f \equiv |K|^2$ be the squared length of $K$, which, of course, satisfies $\nabla^b \nabla_b K_a + R_{ab} K^b = 0$. Then we have

$$\nabla^a \nabla_a f = 2|\nabla_a K_b|^2 + 2K^a \nabla^b \nabla_b K_a = 2|\nabla_a K_b|^2,$$

(16)

the last step following from the fact that the metric is Ricci-flat. Hence, integrating over $M(r_0)$, the manifold interior to radius $r_0$, we have

$$0 \leq \int_{M(r_0)} |\nabla_a K_b|^2 \sqrt{g} d^n x = \int_{\partial M(r_0)} \nabla_a f d\Sigma^n = \left. \frac{df}{dr} \right|_{r=r_0} \text{Vol}(\partial M(r_0)),$$

(17)
which shows that $df/dr \geq 0$, proving the result. This implies that the string coupling constant always decreases as one moves into the interior, i.e. towards the infra-red. More generally, when the Ricci-flat metric does not necessarily have cohomogeneity one, (16) implies that the quantity $f$ can have no interior maximum.

It is instructive first to consider the Gromov-Hausdorff limit, in which the radius of the circle at infinity is sent to zero, implying that the Kaluza-Klein vector $A_{(1)}$ also goes to zero, and hence from (11) the dilaton $\phi$ becomes a constant. This can always be achieved by using the trivial parameter in any Ricci-flat metric that characterises the overall scale of the metric. Typically, this will lead to a conical singularity at the apex of the cone. Of greater interest, therefore, are cases where the ALC $G_2$ metric has a non-trivial parameter that allows one to send the radius of the circle at infinity to zero while keeping the scale-size of the bolt that resolves the apex of the cone non-vanishing. In the Gromov-Hausdorff limit the seven-dimensional $G_2$ metric becomes just a direct sum of a Ricci-flat six-metric and a circle,

$$\hat{s}^2_7 = dz^2 + ds^2_6,$$

where the circle coordinate $z$ has been appropriately rescaled as the limit is taken, so that it remains of non-vanishing period. In the cases where $\hat{s}^2_7$ has a non-trivial parameter, the metric $ds^2_6$ can be a non-trivial smooth AC 6-metric.

The special holonomy in $D = 7$ requires that $ds^2_6$ must have special holonomy; the most generic possibility is for the Ricci-flat metric $ds^2_6$ to be Kähler, implying that it has $SU(3)$ holonomy. Of course the Ricci-flat seven-metric in (18) also then has $SU(3)$ holonomy, and so it is a degenerate example of a $G_2$ metric. The associative 3-form $\hat{\Phi}_{(3)}$ in $D = 7$ is then given by

$$\hat{\Phi}_{(3)} = \Psi_{(2)} \wedge dz + \Psi_{(3)},$$

where

$$\Psi_{(2)} = J, \quad \Psi_{(3)} = \Re(e^{i\gamma}\epsilon_{(3)}),$$

where $J$ is the Kähler form on $ds^2_6$, $\epsilon_{(3)}$ is the closed holomorphic 3-form on $ds^2_6$ and $\gamma$ is an arbitrary constant phase angle. It is easily seen that $\Psi_{(2)}$ and $\Psi_{(3)}$ satisfy (15) with $A_{(1)} = 0$. Conversely, we may express the Kähler form and holomorphic 3-form in terms of $\Psi_{(2)}$ and

1The four-dimensional Taub-NUT metric is ALC, with only a trivial scale parameter. Nonetheless, in this case the Gromov-Hausdorff limit does give $S^1$ times a smooth 3-metric, namely $\mathbb{R}^3$ viewed as the cone over $S^2$. The exceptional nature of this example arises because the cone metric is itself non-singular here, on account of the base being a round sphere.
$\Psi_{(3)}$ as

$$J = \Psi_{(2)}, \quad \epsilon_{(3)} = e^{-i\gamma} (\Psi_{(3)} - i \Psi_{(3)}). \quad (21)$$

One can of course reverse the argument; if one is given a six-dimensional Ricci-flat Kähler metric $ds^2_6$, with Kähler form $J$ and holomorphic 3-form $\epsilon_{(3)}$, then $ds^2_7$ given in (18) will be a degenerate example of a seven-dimensional $G_2$ metric, with holonomy $SU(3) \subset G_2$, and with associative 3-form given by (13) and (20).

### 2.3 Linearised perturbation around the Gromov-Hausdorff limit

We may next consider a deformation away from the Gromov-Hausdorff limit, in a linearised approximation, in which the Kaluza-Klein vector $A_{(1)}$ is assumed to be small, of order $\varepsilon$, and only quantities up to linear order in $\varepsilon$ are retained.

At this linearised level, it is evident from (4) that the dilaton remains a constant, which without loss of generality we shall take to be zero, and so the full set of equations governing the six-dimensional fields become

$$R_{\mu\nu} = 0, \quad dF_{(2)} = 0 = d\Psi_{(2)},$$
$$d\Psi_{(2)} = 0, \quad d\Psi_{(2)} - *\Psi_{(3)} \wedge F_{(2)} = 0,$$
$$d\Psi_{(3)} + \Psi_{(2)} \wedge F_{(2)} = 0, \quad d*\Psi_{(3)} = 0, \quad (22)$$

plus terms of order $\varepsilon^2$. Note that although the six-dimensional metric is still Ricci-flat in this linearised perturbation around the Gromov-Hausdorff limit, it will no longer have special holonomy, since $\Psi_{(3)}$ is not closed and so we can no longer construct a closed holomorphic 3-form $\epsilon_{(3)}$ as in (21).

Since the six-dimensional metric ceases to have special holonomy once the linearised perturbation away from the Gromov-Hausdorff limit is made, even though it remains Ricci-flat, this means that it will no longer be Kähler. It does however still remain, to order $\varepsilon$, almost Kähler, for which it is necessary only that there exist an Hermitian almost-complex structure $J$ which, after lowering the upper index to give a 2-form, satisfies the closure condition $dJ = 0$. (Co-closure is a consequence of these properties, since $*J = \pm \frac{1}{2} J \wedge J$.)

The fact that the metric is no longer Kähler can also be seen from the fact that $\Psi_{(3)}$, which was given by (21) before the perturbation, now satisfies $d\Psi_{(3)} + \Psi_{2} \wedge F_{(2)} = 0$, as given in (22).

Note, however, that we shall still have $d*\Psi_{(3)} = 0$.

2In principle one might think that the metric could still be Kähler with respect to a different choice of almost complex structure. This possibility can be excluded by considering an explicit example, and verifying
The above discussion suggests that there can be a small perturbation of certain Ricci-flat Kähler manifolds which remain Ricci-flat, but where the metric is no longer Kähler, together with the further conditions discussed above. Furthermore, we can then lift the six-dimensional metric, by making use of Kaluza-Klein ansatz (2), and thereby obtain a $G_2$ metric in seven dimensions. An important point in following this “inverse” procedure is that we can derive an expression for the Kaluza-Klein vector that must be used for the lifting, using only the already-known quantities in the perturbed six-dimensional almost-Kähler metric. To do this, we note from (22) that the almost-Kähler form $J = \Psi_{(2)}$, and the 3-form $\Psi_{(3)}$ obey the relation $d\Psi_{(3)} + J \wedge F_{(2)} = 0$, where $F_{(2)} = dA_{(1)}$ and $A_{(1)}$ is the Kaluza-Klein vector. It is straightforward to show that in all complex dimensions greater than 2, if one has $J \wedge \omega_{(2)} = 0$ where $\omega_{(2)}$ is any 2-form and $J$ is constructed from an almost complex structure, then it must be that $\omega_{(2)} = 0$. Thus the equation

$$d\Psi_{(3)} + J \wedge F_{(2)} = 0 \quad (23)$$

can be solved uniquely for $F_{(2)}$. If $dF_{(2)} = 0$ and $d*F_{(2)} = 0$, then writing $F_{(2)} = dA_{(1)}$ we have the Kaluza-Klein vector required for lifting the metric to seven dimensions.

To summarise, we have the following general result. Starting from a Ricci-flat Kähler metric $ds_6^2$, we make a linearised perturbation that preserves the Ricci-flatness, while taking the metric from Kähler to almost-Kähler. If this perturbed metric satisfies two further conditions, namely that for some choice of phase angle $\gamma$ we have $d\Re(e^{i\gamma}\epsilon_{(3)}) = 0$, and that $F_{(2)}$ defined by $d\Im(e^{i\gamma}\epsilon_{(3)}) + J \wedge F_{(2)} = 0$ is closed, then we can lift it to give a $G_2$ metric in $D = 7$. Note that by a theorem of Sekigawa [23], any complete, non-singular, compact, Ricci-flat, almost Kähler manifold must in fact be Kähler. It is not difficult to see, using identities derived in [26], that this result generalises to the non-compact cases we are considering here, as long as we insist on non-singularity of the metric. This means that at higher order, the Ricci-flat almost Kähler conditions cease to hold. (This accords with the discussion in section 2.1, since the dimensional reduction of the smooth $G_2$ metrics away from the Gromov-Hausdorff limit will obey (14) and (15), with $R_{ab}$ becoming non-zero beyond the linearised level.)

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that $R_{abcd}F^{cd}$, the integrability condition for a parallel spinor, has no zero eigenvalues once the perturbation is turned on. We have checked this explicitly for the Ricci-flat deformations obtained in section 3.4.
3 Ricci-flat perturbations of Calabi-Yau 6-metrics

In this section we shall study linearised perturbations around certain classes of Ricci-flat Kähler 6-metrics, and make use of the previous discussion to relate them to \(G_2\) metrics in seven dimensions. Specifically, we shall consider perturbations of the metric on the resolved conifold, with principal orbits \(T^{1,1} = (S^3 \times S^3)/U(1)\), and of the smooth metrics on a complex line bundle over \(S^2 \times S^2\), with principal orbits \(T^{1,1}/Z_2\). In the process, we shall obtain at the linearised level classes of smooth Ricci-flat almost-Kähler metrics. After lifting to \(D = 7\), they turn out to correspond to previously-encountered smooth \(G_2\) metrics near to their Gromov-Hausdorff limits. In fact since the existence of the \(G_2\) metrics has until now been established by numerical integration, our new results provide additional analytic evidence for their regularity.

3.1 Metric ansatz

Our starting point is the class of cohomogeneity one metrics with \(SU(2) \times SU(2) \times U(1)\) isometries, where the principal orbits are \(T^{1,1}\) or \(T^{1,1}/Z_2\). These metrics encompass the deformed and resolved conifolds \([2,3]\); and the complex line bundle over \(S^2 \times S^2\) \([21,22]\) and its generalisation \([23]\). (The generalisation allows the radii of the two \(S^2\) factors to be specified independently, while they are equal in the original example in \([21,22]\).) The metric ansatz will be taken to be

\[
ds_6^2 = dt^2 + a^2 [(\Sigma_1 + g \sigma_1)^2 + (\Sigma_2 + g \sigma_2)^2] + b^2 [(\Sigma_1 - g \sigma_1)^2 + (\Sigma_2 - g \sigma_2)^2]
+ c^2 (\Sigma_3 - \sigma_3)^2.
\]

where \(a, b, c\) and \(g\) are functions only of the radial variable \(t\), and \(\sigma_i\) and \(\Sigma_i\) are left-invariant 1-forms of \(SU(2) \times SU(2)\). They can be expressed in terms of Euler angles as\(^3\)

\[
\sigma_1 + i \sigma_2 = e^{-i\psi} (d\theta + i \sin \theta \, d\phi), \quad \sigma_3 = d\psi + \cos \theta \, d\phi,
\]

\[
\Sigma_1 + i \Sigma_2 = e^{-i\bar{\psi}} (d\bar{\theta} + i \sin \bar{\theta} \, d\bar{\phi}), \quad \Sigma_3 = d\bar{\psi} + \cos \bar{\theta} \, d\bar{\phi},
\]

and they satisfy

\[
d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k, \quad d\Sigma_i = -\frac{1}{2} \epsilon_{ijk} \Sigma_j \wedge \Sigma_k.
\]

\(^3\)Note that although there are ostensibly six coordinates here, when one substitutes into (24) \(\psi\) and \(\bar{\psi}\) appear only through the combination \(\psi - \bar{\psi}\). The parameterisation is a useful one for what follows later when we lift to seven dimensions, where the remaining combination \(\psi + \bar{\psi}\) will form the fibre coordinate.
We shall take the vielbein basis to be

\[
e^0 = dt, \quad e^1 = a (\Sigma_1 + g \sigma_1), \quad e^2 = a (\Sigma_2 + g \sigma_2), \quad e^3 = c (\Sigma_3 - \sigma_3), \quad e^4 = b (\Sigma_1 - g \sigma_1), \quad e^5 = b (\Sigma_2 - g \sigma_2).
\]

(27)

An invariant choice for an almost complex structure is

\[
J = e^0 \wedge e^3 + e^1 \wedge e^5 - e^2 \wedge e^4,
\]

(28)

from which it follows that the following can be chosen as a holomorphic complex vielbein:

\[
e^0 = e^0 + i e^3, \quad e^1 = e^1 + i e^5, \quad e^2 = e^2 - i e^4,
\]

(29)

in terms of which we have \( J = \frac{i}{2} (e^0 \wedge \bar{e}^0 + e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2) \). We then define the holomorphic 3-form

\[
\epsilon^{(3)} \equiv e^0 \wedge e^1 \wedge e^2 = \epsilon_R^{(3)} + i \epsilon_I^{(3)},
\]

(30)

with the real and imaginary parts given by

\[
\epsilon_R^{(3)} = e^0 \wedge e^1 \wedge e^2 - e^0 \wedge e^4 \wedge e^5 + e^1 \wedge e^3 \wedge e^4 + e^3 \wedge e^2 \wedge e^5,
\]

\[
\epsilon_I^{(3)} = e^3 \wedge e^1 \wedge e^2 - e^3 \wedge e^4 \wedge e^5 - e^0 \wedge e^1 \wedge e^4 - e^0 \wedge e^2 \wedge e^5.
\]

(31)

Note that since \(*\epsilon^{(3)} = i \epsilon^{(3)}\), we shall have \(*\epsilon_R^{(3)} = -\epsilon_I^{(3)}\) and \(*\epsilon_I^{(3)} = \epsilon_R^{(3)}\).

3.2 Conditions for SU(3) holonomy

Imposing the closure of \( J \), \( \epsilon_R^{(3)} \) and \( \epsilon_I^{(3)} \) gives

\[
dJ = 0 : \quad 2 (ab)^\cdot + c = 0, \quad 2 (a b g^2)^\cdot + c = 0,
\]

(32)

\[
d\epsilon_R^{(3)} = 0 : \quad 2 (a b c g)^\cdot + (a^2 + b^2) g = 0,
\]

(33)

\[
d\epsilon_I^{(3)} = 0 : \quad [(a^2 - b^2) c]^\cdot = 0, \quad [(a^2 - b^2) c g^2]^\cdot = 0,
\]

\[
\quad [(a^2 + b^2) c g]^\cdot + 2 a b g = 0, \quad c (a^2 - b^2) (1 - g^2) = 0.
\]

(34)

Note that the full set of equations implies that \(\{24\}\) will be Ricci-flat Kähler, since \(d\epsilon_R^{(3)} = 0\) and \(d\epsilon_I^{(3)} = 0\) are together equivalent to \(d\epsilon^{(3)} = 0\). It is evident from the final equation resulting from \(d\epsilon_I^{(3)} = 0\), which is algebraic, that there is a bifurcation into two non-singular possibilities, namely \(a = b\) or \(g = 1\) (other choice for the signs are equivalent, up to orientation). The case \(a = b\) gives rise to the first-order equations governing the resolved conifold and a class of complex line bundles over \(S^2 \times S^2\), whilst the case \(g = 1\) gives rise to the first-order equations governing the deformed conifold. The special case \(21, 22\) of the
complex line bundle over $S^2 \times S^2$ with equal radii for the 2-spheres is a solution of both systems of first-order equations. For future purposes, where we shall be taking these various Ricci-flat Kähler backgrounds as starting points for linearised perturbative analyses, it is convenient to present the first-order equations as follows:

$$a = b = A, \ c = C, \ g = G :$$

$$\dot{A} = -\frac{C}{4A}, \ \dot{C} = -1 + \frac{C^2(1 + G^2)}{4A^2G^2}, \ \dot{G} = -\frac{C(1 - G^2)}{4A^2G} ,$$

(35)

$$a = A, \ b = B, \ c = C, \ g = 1 :$$

$$\dot{A} = \frac{(A^2 - B^2 - C^2)}{4BC}, \ \dot{B} = \frac{(B^2 - A^2 - C^2)}{4AC}, \ \dot{C} = \frac{(C^2 - A^2 - B^2)}{2AB}.$$  

(36)

3.3 Linearised perturbations

We shall focus our attention on perturbations away from the Ricci-flat Kähler conditions (35). The perturbed metrics will be taken to lie within the same general class (24), but no longer satisfying all three of the conditions (32), (33) and (34) for $SU(3)$ holonomy. We shall still require, however, that the perturbed metrics be Ricci-flat. In the light of our discussion in section 2.3, we shall also require that up to the linearised order in the perturbation the metrics be almost Kähler, and that we can find some complexion angle $\gamma$ such that $d\Re(e^{\gamma} \epsilon_{(3)}) = 0$ at the linearised order.

By inspection of (32) and (33), it is evident that if we consider a perturbation taking the form

$$a = A + \varepsilon u, \quad b = A - \varepsilon u, \quad c = C, \quad g = G,$$

(37)

where the upper-case functions satisfy the unperturbed Ricci-flat Kähler conditions (35), then we have

$$dJ = 0 + O(\varepsilon^2), \quad d\epsilon^R_{(3)} = 0 + O(\varepsilon^2).$$

(38)

On the other hand, from (34) we shall have $d\epsilon_{(3)} = O(\varepsilon)$. Taking the complexion angle to be $\gamma = 0$, we therefore satisfy all the requirements for being able to make a $G_2$ lift, provided that the perturbed metric is Ricci-flat and that $dF_{(2)} = 0$ and $d*F_{(2)} = 0$ at order $\varepsilon$. Substituting (37) into the Ricci tensor for (24), we find that it vanishes at $O(\varepsilon)$ if

$$\ddot{u} - \dot{u} \left( \frac{1}{C} + \frac{C(1 - G^2)}{4A^2G^2} \right) - u \left( \frac{1}{C^2} - \frac{1}{2A^2G^2} + \frac{3C^2(1 - 2G^2)}{16A^4G^4} \right) = 0.$$ 

(39)
By taking $\Psi_{(3)} = \epsilon_{(3)}$, we can solve for $F_{(2)}$ from equation (23), giving

$$F_{(2)} = \varepsilon \left( -\frac{2u}{A} + \frac{2u}{AC} \right) (e^{1} \wedge e^{2} + e^{4} \wedge e^{5}) - \frac{\varepsilon u C}{2A^{3}} \frac{(1 - G^{2})}{2A^{3} G^{2}} (2e^{0} \wedge e^{3} - e^{1} \wedge e^{5} + e^{2} \wedge e^{4}) \right).$$

(40)

Calculating $dF_{(2)}$, we find that it vanishes at leading order in $\varepsilon$ if the condition (39) for Ricci-flatness is satisfied. We can also verify that $d*F_{(2)} = 0$ at this order.

Using the first-order equations (35) satisfied by $A$, $C$ and $G$, we can find a first integral of (39), giving

$$\dot{u} = \frac{u C}{G} + \frac{u C (1 - 3G^{2})}{4A^{2} (1 + G^{2}) G^{2}} + \frac{k_{2}}{4A (1 + G^{2})},$$

(41)

where $k_{2}$ is an arbitrary integration constant. This can be integrated once more, again using (35), to give the general solution of (39),

$$u = \frac{k_{1} (1 + G^{2})}{AC G^{2}} + \frac{A k_{2}}{4C G^{2}},$$

(42)

where $k_{1}$ is another constant of integration.

Using (11) and (12), we can show that $F_{(2)}$ can be written as $F_{(2)} = d(\alpha \Sigma_{(3)} + \beta \sigma_{3})$, where the functions $\alpha$ and $\beta$ are given by

$$\alpha = -\frac{\varepsilon k_{2}}{2G^{2}} - 2\varepsilon \frac{k_{1} (1 - G^{2})}{A^{2} G^{2}}, \quad \beta = \frac{\varepsilon k_{2} (1 - 2G^{2})}{2G^{2}} + 2\varepsilon \frac{k_{1} (1 - G^{2})}{A^{2} G^{2}}.$$

(43)

This implies that $(\alpha \Sigma_{3} + \beta \sigma_{3}) = \varepsilon k_{2} (dz + B_{(1)})$, where $A_{(1)} = \varepsilon k_{2} B_{(1)}$ is the Kaluza-Klein vector and

$$B_{(1)} = -\frac{(1 - G^{2}) (4k_{1} + k_{2} A^{2})}{k_{2} A^{2} G^{2}} dy + \left[ \frac{(1 - 2G^{2})}{2G^{2}} + \frac{2k_{1} (1 - G^{2})}{k_{2} A^{2} G^{2}} \right] \cos \theta \ d\phi$$

$$- \left[ \frac{1}{2G^{2}} + \frac{2k_{1} (1 - G^{2})}{k_{2} A^{2} G^{2}} \right] \cos \tilde{\theta} \ d\tilde{\phi}.$$

(44)

Here $y = \frac{1}{2} (\psi - \tilde{\psi})$, and neither $A_{(1)}$ nor the six-dimensional metric (24) involves the coordinate $z = \frac{1}{2} (\psi + \tilde{\psi})$.

We are now in a position to lift the perturbed Ricci-flat metric to seven dimensions, using Kaluza-Klein formalism discussed in section 2. We therefore obtain

$$ds_{7}^{2} = dt^{2} + a^{2} [(\Sigma_{1} + g \sigma_{1})^{2} + (\Sigma_{2} + g \sigma_{2})^{2}] + b^{2} [(\Sigma_{1} - g \sigma_{1})^{2} + (\Sigma_{2} - g \sigma_{2})^{2}]$$

$$+ c^{2} (\Sigma_{3} - \sigma_{3})^{2} + \varepsilon^{2} \lambda^{2} (dz + A_{(1)})^{2},$$

$$= dt^{2} + a^{2} [(\Sigma_{1} + g \sigma_{1})^{2} + (\Sigma_{2} + g \sigma_{2})^{2}] + b^{2} [(\Sigma_{1} - g \sigma_{1})^{2} + (\Sigma_{2} - g \sigma_{2})^{2}]$$

$$+ c^{2} (\Sigma_{3} - \sigma_{3})^{2} + f^{2} (\Sigma_{3} + g_{3} \sigma_{3})^{2},$$

(45)
where \( f = \alpha \) and \( g_3 = \beta/\alpha \). Defining \( e^6 \equiv f (\Sigma_3 + g_3 \sigma_3) \), we see that the almost-Kähler structure \( J = \Psi_2 \) and the 3-form \( \Psi_{(3)} = \epsilon_3^I \) in six dimensions lift, using (13), to give

\[
\hat{\Phi}_{(3)} = e^0 \wedge (e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6) + (e^1 \wedge e^2 - e^4 \wedge e^5) \wedge e^3 + (e^1 \wedge e^5 - e^2 \wedge e^4) \wedge e^6,
\]

(46)

We have obtained a class of metrics in seven dimensions which, by construction, have \( G_2 \) holonomy at the linearised order in \( \epsilon \). In fact, we have arrived at a linearised version of the general description \( G_2 \) metrics with \( S^3 \times S^3 \) principal orbits that was obtained in [19], where a system of five first-order equations for the metric ansatz (15) was obtained.

### 3.4 Explicit discussion

The general solution of the first-order equations (35) is given by

\[
A^2 = \frac{1}{2}(r + \ell_1^2), \quad C^2 = \frac{r (2r^2 + 3(\ell_1^2 + \ell_2^2) r + 6\ell_1^2 \ell_2^2)}{3(r + \ell_1^2)(r + \ell_2^2)}, \quad G^2 = \frac{r + \ell_2^2}{r + \ell_1^2},
\]

(47)

where \( r \) is related to \( t \) by \( dr = C \, dt \), and \( \ell_1 \) and \( \ell_2 \) are constants. The radial variable satisfies \( r \geq 0 \), where \( r = 0 \) is the bolt. If \( \ell_1 \) and \( \ell_2 \) are both non-zero the solution corresponds to the one found in [23] with an \( S^2 \times S^2 \) bolt. The special case \( \ell_1 = \ell_2 \) is the metric found in [21, 22]. Regularity at \( r = 0 \) requires that the coordinate \( y = (\psi - \tilde{\psi})/2 \) on the \( U(1) \) fibre over \( S^2 \times S^2 \) have period \( \pi \), and hence the principal orbits are \( T^{1,1}/Z_2 \) [23, 24].

If \( \ell_2 = 0 \), the bolt is instead an \( S^2 \), and the metric is the resolved conifold found in [2]. Regularity at \( r = 0 \) now requires that \( y \) have period \( 2\pi \), and so the principal orbits are \( T^{1,1} \). (Taking \( \ell_1 = 0 \) instead gives the resolved conifold again, with the roles of the \( \Sigma_i \) and \( \sigma_i \) interchanged.)

The explicit expression for the perturbation function \( u \) can be obtained by substituting (47) into (12). We are interested in obtaining a perturbation which is regular for all \( r \). In particular, we see that to obtain regularity at \( r = 0 \) we must choose

\[
k_1 = -\frac{k_2 \ell_1^4}{8(\ell_1^2 + \ell_2^2)}.
\]

(48)

The function \( u \) is then given by

\[
u = \frac{\sqrt{3} k_2 r^{1/2} \left[ r (\ell_1^2 + \ell_2^2) + 2\ell_1^2 \ell_2^2 \sqrt{r + \ell_2^2} \right]}{4(\ell_1^2 + \ell_2^2) \sqrt{r + \ell_2^2} \sqrt{2r^2 + 3(\ell_1^2 + \ell_2^2) r + 6\ell_1^2 \ell_2^2}}.
\]

(49)

It is easily seen that this is indeed regular for all \( r \). Thus we have succeeded in obtaining a perturbation of the resolved conifold and of the Ricci-flat Kähler metrics in [21, 22] and
that remains Ricci-flat at the linearised order, but which is no longer Kähler (although it is still almost-Kähler). The resolved conifold corresponds to setting $\ell_2 = 0$, implying that the original unperturbed metric, for which

$$A^2 = \frac{1}{2}(r + \ell_1^2), \quad C^2 = \frac{r(2r + 3\ell_1^2)}{3(r + \ell_1^2)}, \quad G^2 = \frac{r}{r + \ell_1^2},$$

is perturbed according to (50) with

$$u = \frac{\sqrt{3k_2} r^{1/2}}{4\sqrt{2r + 3\ell_1^2}}.$$

After lifting to seven dimensions the perturbed metrics give regular metrics of $G_2$ holonomy. The perturbed conifold gives a linearisation around the Gromov-Hausdorff limit of the $\mathbb{D}_7$ metrics found in [17], while the perturbed metrics with $\ell_1$ and $\ell_2$ both non-zero give linearisations around the Gromov-Hausdorff limits of the $\tilde{\mathbb{C}}_7$ metrics found in [19]. In particular, this provides an analytic demonstration, to leading non-trivial order in perturbations, of the conclusions reached by numerical analysis in [17] and [19], namely that the first-order equations for $G_2$ holonomy do indeed admit regular solutions of the types $\mathbb{D}_7$ and $\tilde{\mathbb{C}}_7$.

Evidence for another class of ALC $G_2$ metrics, denoted by $\mathbb{B}_7$, was found in [13, 11]. These have the topology $\mathbb{R}^4 \times S^3$, and for the limiting value of a non-trivial parameter one obtains $S^1$ times the deformed conifold as the Gromov-Hausdorff limit. However, in this case the circle action whose length stabilises at large distance collapses to zero length at short distance, and so from a six-dimensional viewpoint the dilaton diverges there. This means that there is no analogous smooth Ricci-flat perturbation of the deformed conifold.

It is interesting to recall that after imposing the condition of $SU(3)$ holonomy on the metric ansatz (24), we found a bifurcation of the first-order equations into two branches, according to whether the algebraic constraint in (34) is solved by taking $a = b$ or $g = 1$. The former includes the resolved conifold and the metrics on the $\mathbb{R}^2$ bundles over $S^2 \times S^2$, whilst the latter includes the deformed conifold. We obtained $G_2$ metrics by considering perturbations of Calabi-Yau metrics in the first branch, and these satisfy linearisations of first-order equations for $G_2$ holonomy in $D = 7$. The full $G_2$ equations in $D = 7$ in fact also encompass the 6-dimensional first-order equations in the second branch, where $g = 1$. In other words the two branches of first-order equations for $SU(3)$ holonomy, which are disjoint in $D = 6$, become unified in $D = 7$ where they can be viewed as two different Gromov-Hausdorff limits of the first-order equations for $G_2$ holonomy obtained in [19].
4 Almost $G_2$ metrics

Many of the ideas discussed above can also be applied to eight-dimensional metrics of Spin(7) holonomy with a circle action. From these considerations, we are led to introduce the notion of an “Almost $G_2$ Metric.”

4.1 Kaluza-Klein reduction of a Spin(7) metric

Given a Spin(7) metric of the form

$$ds_8^2 = e^{-\sqrt{15}\phi}ds_7^2 + e^{\sqrt{15}\phi}(dz + A_{(1)})^2,$$

the calibrating self-dual 4-form $\Phi_{(4)}$ reduces to

$$\hat{\Phi}_{(4)} = \Phi_{(3)} \wedge (dz + A_{(1)}) + e^{-\sqrt{15}\phi}\ast\Phi_{(3)}$$

in seven dimensions, where $\ast$ denotes the seven-dimensional Hodge dual. The algebraic properties of $\hat{\Phi}_{(4)}$ imply that $\Phi_3$ is $G_2$ invariant and, with the appropriate scaling, it satisfies the associativity property (53). The closure condition $d\Phi_{(4)} = 0$ that implies Spin(7) holonomy therefore gives

$$d\Phi_{(3)} = 0, \quad d(e^{-\sqrt{15}\phi}\ast\Phi_{(3)}) = \Phi_{(3)} \wedge F_{(2)},$$

where $F_{(2)} = dA_{(1)}$.

If we consider a situation where $F_{(2)}$ is small, of order $\varepsilon$, then $\phi$ will be a constant up to order $\varepsilon$, leading to

$$ds_8^2 = ds_7^2 + (dz + A_{(1)})^2,$$

where $ds_7^2$ is Ricci-flat up to linear order in $\varepsilon$. The 3-form $\Phi_{(3)}$ satisfies

$$d\Phi_{(3)} = 0, \quad d\ast\Phi_{(3)} = \Phi_{(3)} \wedge F_{(2)},$$

at this linearised order.

It is useful to introduce the notion of an almost $G_2$ metric, namely a 7-metric admitting a $G_2$-invariant associative 3-form $\Phi_{(3)}$ that is closed, but not necessarily co-closed. It is easy to show that if $X_{(2)}$ is any 2-form, then $\Phi_{(3)} \wedge X_{(2)} = 0$ implies $X_{(2)} = 0$, and hence one can solve for $F_{(2)}$ uniquely from (56). Thus if a Ricci-flat 7-metric is almost $G_2$, and if in addition it is such that $dF_{(2)} = 0$ and $d\ast F_{(2)} = 0$, then in a linearisation around its $G_2$ holonomy limit it can be lifted to give a Spin(7) metric. It is natural to conjecture that there is an analogue of Sekigawa’s theorem [25] for almost $G_2$ manifolds that are Ricci-flat;
i.e. if they are complete and non-singular, they must be $G_2$ manifolds in the standard sense. Nonetheless, as we shall see below, at the level of linearised perturbations there can be complete and non-singular almost $G_2$ Ricci-flat deformations. These remain complete and non-singular, but cease to be Ricci-flat, beyond the linear level.

4.2 Almost $G_2$ Metrics with $SU(3)/(U(1) \times U(1))$ Principal Orbits

In the notation of [30], we introduce a set of left-invariant 1-forms $(\sigma_1, \sigma_2, \Sigma_1, \Sigma_2, \nu_1, \nu_2, \lambda, Q)$ for $SU(3)$, which satisfy the exterior algebra

\[
\begin{align*}
    d\sigma_1 &= -\frac{1}{2} \lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2} Q \wedge \sigma_2, \\
    d\sigma_2 &= \frac{1}{2} \lambda \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2 + \frac{3}{2} Q \wedge \sigma_1, \\
    d\Sigma_1 &= \frac{1}{2} \lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1 - \frac{3}{2} Q \wedge \Sigma_2, \\
    d\Sigma_2 &= -\frac{1}{2} \lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2 + \frac{3}{2} Q \wedge \Sigma_1, \\
    d\nu_1 &= -\lambda \wedge \nu_2 - \sigma_2 \wedge \Sigma_1 + \sigma_1 \wedge \Sigma_2, \\
    d\nu_2 &= \lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2, \\
    d\lambda &= 2\sigma_1 \wedge \sigma_2 - 2\Sigma_1 \wedge \Sigma_2 + 4\nu_1 \wedge \nu_2, \\
    dQ &= 2\sigma_1 \wedge \sigma_2 + 2\Sigma_1 \wedge \Sigma_2. \tag{57}
\end{align*}
\]

Here $\lambda$ and $Q$ are the 1-forms along the $U(1) \times U(1)$ subalgebra.

We consider 7-metrics of the form

\[
    ds_7^2 = dt^2 + a^2(\sigma_1^2 + \sigma_2^2) + b^2(\Sigma_1^2 + \Sigma_2^2) + c^2(\nu_1^2 + \nu_2^2). \tag{58}
\]

If we define the 3-form $\Phi_{(3)}$ by

\[
\begin{align*}
    \Phi_{(3)} &= e^0 \wedge (e^1 \wedge e^2 - e^3 \wedge e^4 - e^5 \wedge e^6) + e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 \\
    &\quad + e^2 \wedge e^3 \wedge e^6 + e^2 \wedge e^4 \wedge e^5, \tag{59}
\end{align*}
\]

where the vielbein is taken to be

\[
    e^0 = dt, \quad e^1 = a \sigma_1, \quad e^2 = a \sigma_2, \quad e^3 = b \Sigma_1, \quad e^4 = b \Sigma_2, \quad e^5 = c \nu_1, \quad e^6 = c \nu_2, \tag{60}
\]

then we find that closure and co-closure imply

\[
\begin{align*}
    d\Phi_{(3)} &= 0 : \quad (a b c)^\cdot = a^2 + b^2 + c^2, \tag{61} \\
    d\ast \Phi_{(3)} &= 0 : \quad (a^2 b^2)^\cdot = (b^2 c^2)^\cdot = (c^2 a^2)^\cdot = 4a b c. \tag{62}
\end{align*}
\]
Together, these conditions produce the first-order equations for $G_2$ holonomy\textsuperscript{4} that were obtained in \cite{13}:

\[
\begin{align*}
\dot{a} &= \frac{b^2 + c^2 - a^2}{bc}, \\
\dot{b} &= \frac{c^2 + a^2 - b^2}{ca}, \\
\dot{c} &= \frac{a^2 + b^2 - c^2}{ab}.
\end{align*}
\]

(63)

It was shown in \cite{13} that these could be solved completely, leading to $G_2$ metrics that are in general singular, except in the case that any two of the three functions $a$, $b$ and $c$ are set equal. Under those circumstances, the solution then gives rise to the long-known AC metric of $G_2$ holonomy on the $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$, found in \cite{6, 7}. One could say, therefore, that perturbations of the AC metric on the $\mathbb{R}^3$ bundle over $\mathbb{CP}^2$, within the class described by (58), will be singular if one requires that the perturbed metric also have $G_2$ holonomy.

Our goal now will be to consider perturbations, again contained within the class (58), which are Ricci-flat but no longer of $G_2$ holonomy. In fact, we shall seek almost $G_2$ perturbed metrics, with $d\Phi_{(3)} = 0$ but $d^*\Phi_{(3)} \neq 0$. We consider perturbed metrics for which

\[
a = A + \varepsilon u, \quad b = A - \varepsilon u, \quad c = C,
\]

(64)

where $A$ and $C$ satisfy the first-order equations

\[
\dot{A} = \frac{C}{A}, \quad \dot{C} = 2 - \frac{C^2}{A^2}
\]

(65)

that follow from setting $a = b = A$, $c = C$ in (63). We then find that the Ricci tensor of the metric (58) will vanish at order $\varepsilon$ if the perturbing function $u$ satisfies the second-order equation

\[
\ddot{u} + \frac{\dot{u}}{C} + \left( \frac{6}{A^2} + \frac{2C^2}{A^4} - \frac{8}{C^2} \right) u = 0.
\]

(66)

Using (65) we can integrate this once, to give

\[
(AC^2 u) = k_2 C^2,
\]

(67)

where $k_2$ is an arbitrary constant of integration. There is no easy way to integrate this again abstractly, unless one sets $k_2 = 0$. Making this choice implies that the equations (63) for $G_2$ holonomy are satisfied. Since, as was shown in \cite{13}, the associated solutions will necessarily be singular when $u \neq 0$, we therefore wish to keep $k_2$ non-zero here.\textsuperscript{5}

\textsuperscript{4}It happens in this particular case that the equations following from $d^*\Phi_{(3)} = 0$ imply that $d\Phi_{(3)} = 0$, but this is not true in general.

\textsuperscript{5}We have explicitly verified that if $k_2$ is non-zero, then the integrability condition $R_{abcd} \Gamma^{cd}$ has no zero eigenvalues. This proves that the perturbed metric does not have $G_2$ holonomy, and thereby excludes the possibility that there might have existed a perturbed associative 3-form that was still co-closed as well as closed.
The general solution to the unperturbed equations (65) can be written, after eliminating trivial integration constants, as

\[ A^2 = r^2, \quad C^2 = r^2 (1 - r^{-4}), \]  

where \( r \) is a new radial variable defined in terms of \( t \) by \( dr = (1 - r^{-4})^{1/2} dt \). This gives the AC metric on the \( \mathbb{R}^3 \) bundle over \( \mathbb{C}P^2 \) that was found in [6, 7], which has a \( \mathbb{C}P^2 \) bolt at \( r = 1 \). Substituting into (67), we can solve for \( u \), giving

\[ u = k_1 r + \frac{k_2 r^2}{3\sqrt{r^4 - 1}} + \frac{2k_2 r F(\arcsin r^{-1} - 1)}{3(r^4 - 1)}, \]  

where \( F(\phi|m) \) is the elliptic integral of the first kind. The function \( u \) is regular at the bolt at \( r = 1 \) if the constants are chosen so that

\[ k_1 = -\frac{2}{3} k_2 K(-1), \]  

where \( K(m) \) is the complete elliptic integral of the first kind. We find that \( u \) is regular everywhere, with \( u \sim \frac{1}{3} k_2 \sqrt{r-1} \) at short distance, and \( u \sim \frac{1}{3} k_2 \) at large distance. Thus we have a regular Ricci-flat perturbation of the AC metric on the \( \mathbb{R}^3 \) bundle over \( \mathbb{C}P^2 \).

Up to linear order in \( \varepsilon \), we find that

\[ d\Phi(3) = 0 + O(\varepsilon^2), \quad d^*\Phi(3) = \Phi(3) \wedge F(2), \]  

which, after using using (57), implies \( F(2) = 2\varepsilon k_2 (\sigma_1 \wedge \sigma_2 + \Sigma_1 \wedge \Sigma_2) \). From (57) we therefore have \( F(2) = dA(1) \) with

\[ dz + A(1) = \varepsilon k_2 Q. \]  

It is easily verified that \( d^*F(2) = 0 \), and hence all the requirements for our perturbed 7-metric to admit a lift to an eight-dimensional metric of Spin(7) holonomy are fulfilled. From (53), we obtain the Spin(7) metric

\[ ds_8^2 = dt^2 + a^2 (\sigma_1^2 + \sigma_2^2) + b^2 (\Sigma_1^2 + \Sigma_2^2) + c^2 (\nu_1^2 + \nu_2^2) + f^2 Q^2, \]  

where \( f \) is a small constant, given by \( f = \varepsilon k_2 \). The self-dual calibrating 4-form is given by

\[ \tilde{\Phi}(4) = \Phi(3) \wedge e^7 + *\Phi(3), \]  

where \( e^7 \equiv f Q \).

This system in eight-dimensions can in fact be “non-linearised.” The metric (73) is precisely one that was considered in [13], for Spin(7) metrics whose principle orbits are the
\( N(k, \ell) = SU(3)/U(1)^{k,\ell} \) space with \( k = 1, \ell = -1 \). The first-order equations implying Spin(7) holonomy in this case are

\[
\begin{align*}
\dot{a} &= \frac{b^2 + c^2 - a^2}{bc} - \frac{f}{a}, \\
\dot{b} &= \frac{a^2 + c^2 - b^2}{ca} + \frac{f}{b}, \\
\dot{c} &= \frac{a^2 + b^2 - c^2}{ab}, \\
\dot{f} &= \frac{f^2}{a^2} - \frac{f^2}{b^2}.
\end{align*}
\]

(75)

It was shown in [13], by performing a Taylor expansion, that these equations admit regular short-distance solutions in which just the function \( c \) goes to zero at \( t = 0 \). This corresponds to an \( S^5 \) bolt. By numerical analysis, it was also found in [13] that the metrics are regular at large distance, provided that the initial values for the non-vanishing metric functions on the bolt are chosen appropriately. The metrics are in general ALC, with \( f \) stabilising to a constant radius at infinity. Note that the radius \( f \) of the circle remains finite and non-zero everywhere. A non-trivial parameter adjusts the value of this radius, while keeping the scale-size of the bolt non-zero, and so we can take a Gromov-Hausdorff limit. In this limit, we get the product of a circle and the AC metric on the \( \mathbb{R}^3 \) bundle over \( \mathbb{C}P^2 \).

The perturbative analysis that we performed above has yielded the metrics in seven-dimensions that correspond to the Kaluza-Klein reduction of the regular ALC metrics found in [13], as one moves away from the Gromov-Hausdorff limit.

## 5 Almost-Kähler Ricci-flat metrics

In the section 3, we obtained examples, as linearised perturbations of Calabi-Yau metrics, of six-dimensional metrics that are Ricci-flat and almost-Kähler but not Kähler. In this section we obtain some analogous examples in higher dimensions.

### 5.1 Line bundles over \( SO(n + 2)/(SO(n) \times SO(2)) \)

Cohomogeneity one metrics with \( SO(n + 2)/SO(n) \) principal orbits were discussed in [27]. Let \( L_{AB} \) be the left-invariant 1-forms on the group manifold \( SO(n + 2) \). These satisfy

\[
dL_{AB} = L_{AC} \wedge L_{CB}.
\]

(76)
We consider the $SO(n)$ subgroup, by splitting the index as $A = (1, 2, i)$. The $L_{ij}$ are the left-invariant 1-forms for the $SO(n)$ subgroup. We make the following definitions:

$$\sigma_i \equiv L_{1i}, \quad \tilde{\sigma}_i \equiv L_{2i}, \quad \nu \equiv L_{12}.$$  

(77)

These are the 1-forms in the coset $SO(n+2)/SO(n)$. We have

$$d\sigma_i = \nu \wedge \tilde{\sigma}_i + L_{ij} \wedge \sigma_j, \quad d\tilde{\sigma}_i = -\nu \wedge \sigma_i + L_{ij} \wedge \tilde{\sigma}_j, \quad d\nu = -\sigma_i \wedge \tilde{\sigma}_i.$$  

(78)

Note that the 1-forms $L_{ij}$ lie outside the coset, and so one finds that they do not appear eventually in the expressions for the curvature (see also [28, 29]). The metrics were written as

$$ds^2 = dt^2 + a^2 \sigma_i^2 + b^2 \tilde{\sigma}_i^2 + c^2 \nu^2,$$  

(79)

where $a, b$ and $c$ are functions of the radial coordinate $t$.

The first-order equations that imply $SU(n+1)$ holonomy are [27]

$$\dot{a} = \frac{b^2 + c^2 - a^2}{2bc}, \quad \dot{b} = \frac{a^2 + c^2 - b^2}{2ac}, \quad \dot{c} = \frac{n (a^2 + b^2 - c^2)}{2ab}.$$  

(80)

Here, we shall consider perturbations of the metric (79) around the special case $a = b = A, c = C$, for which we have

$$\dot{A} = \frac{C}{2A}, \quad \dot{C} = n - \frac{n C^2}{2A^2},$$  

(81)

where

$$a = A + \varepsilon u, \quad b = A - \varepsilon u, \quad c = C.$$  

(82)

Note that the perturbations we shall consider will be Ricci-flat, but they will not be required to satisfy the first-order equations (80).

Requiring Ricci-flatness up to linear order in $\varepsilon$, we find that the perturbation $u$ must satisfy the second-order equation

$$\ddot{u} + \left( \frac{n}{C} + \frac{(n-2)C}{2A^2} \right) \dot{u} + \left( \frac{n}{A^2} - \frac{4}{C^2} + \frac{3C^2}{4A^4} \right) u = 0.$$  

(83)

Using (81), we can find a first integral of this, namely

$$(u \frac{C^{2/n} A}{\lambda A^{2n-4}})^n = \lambda C^{4/n-1} A^{2n-4},$$  

(84)

where $\lambda$ is an arbitrary constant. If $\lambda = 0$, this first-order equation follows by substituting (82) into the first-order equations (80) for $SU(n+1)$ holonomy, but if $\lambda \neq 0$ the perturbed Ricci-flat metric is not Kähler. Unlike the previous example in (3.3), here the equation
cannot in general be easily integrated again abstractly, and so we shall proceed to the explicit solution.

The general solution of the first-order system (81) can be written as

\[ A^2 = \frac{nr^2}{2(n+1)}, \quad C^2 = \frac{n^2r^2}{(n+1)^2} \left( 1 - \frac{1}{r^{2n+2}} \right), \tag{85} \]

in terms of a new radial coordinate \( r \) defined by \( dr = (1 - r^{-2n-2}) dt \). In fact the metric corresponding to this solution is precisely in the class constructed in [21, 22], for the special case of the complex line bundle over the Einstein-Kähler metric on the Grassman manifold \( SO(n+2)/(SO(n) \times SO(2)) \). The radial coordinate satisfies \( r \geq 1 \). The general solution for the perturbation \( u \) is then given by

\[ u = r^{-2/n-1} \left( k_1 + k_2 r^{4/n-2n+4} {}_2F_1[a_1, a_2; a_3; r^{-2/(n+1)}] \right) \left( 1 - \frac{1}{r^{2n+2}} \right)^{-1/n}, \tag{86} \]

where

\[ a_1 = \frac{n^2 - 2n - 2}{n(n+1)}, \quad a_2 = \frac{n-2}{n}, \quad a_3 = \frac{2n^2 - n - 2}{n(n+1)}. \tag{87} \]

Regularity of \( u \) on the bolt at \( r = 1 \) requires that the constants \( k_1 \) and \( k_2 \) be chosen such that

\[ k_1 = -\frac{k_2 \Gamma[(n-1)/n] \Gamma[(2n^2 - n - 2)/(n(n+1))] \Gamma[n/(n+1)]}{\Gamma[(2n^2 - n - 2)/(n(n+1))]}. \tag{88} \]

This implies that we have \( u \sim (r - 1)^{1/n} \) near to \( r = 1 \). At large distance we then have

\[ u \sim \begin{cases} r^3, & n = 1 \\ 1, & n = 2 \\ r^{-2/n-1}, & n \geq 3 \end{cases} \tag{89} \]

Thus we see that if \( n \geq 2 \), the perturbation will be regular for all \( r \).

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Note Added

After this work was completed, a paper appeared [31] that overlaps with some of our discussion of fibred \( G_2 \) manifolds.
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