Fast, Convexified Stochastic Optimal Open-Loop Control For Linear Systems Using Empirical Characteristic Functions

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Abstract—We consider the problem of stochastic optimal control in the presence of an unknown disturbance. We characterize the disturbance via empirical characteristic functions, and employ a chance constrained approach. By exploiting properties of characteristic functions and underapproximating cumulative distribution functions, we can reformulate a non-convex problem by a conic, convex under-approximation. This results in extremely fast solutions that are assured to maintain probabilistic constraints. We construct algorithms for optimal open-loop control using piecewise linear approximations of the empirical characteristic function, and demonstrate our approach on two examples.

I. INTRODUCTION

Stochastic optimal control typically presumes accurate models of the underlying dynamics and stochastic processes [1]–[3]. However, in many circumstances, accurate characterization of uncertainty is difficult. Further, inaccurate characterization of stochastic processes may have unexpected impacts [4], [5], as optimal control actions are typically dependent upon the first and second moments of the stochastic processes [3]. Such inaccuracies could be particularly problematic when the unknown stochastic processes is asymmetric, multimodal, or heavy-tailed. For example, in hypersonic vehicles, excessive turbulence makes aerodynamic processes difficult to model accurately, and their fast time-scale means that erroneous control actions (such as those based upon e.g., a Gaussian stochastic process, may not accurately model turbulence) could result in catastrophic failure.

We consider the case in which the dynamics are known, but the noise process is not known, and focus on the problem of data-driven stochastic optimal control in a chance constrained setting, in which probabilistic constraints must be satisfied with at least a desired likelihood. Approaches such as distributional stochastic optimal control seek robustness to ill-defined distributions with finite samples [5], [6]. Other approaches construct piecewise-affine over-approximations of value functions by solving a chance-constrained problem [7]. Researchers have also employed kernel density estimation [8], [9] to approximate individual chance constraints in nonlinear optimization problems.

One tool to characterize uncertainty through observed data is the empirical characteristic function [10], which is often employed in economics and statistics to characterize models where maximum-likelihood estimation can struggle. The empirical characteristic function generates an approximation of the true characteristic function, and has known convergence properties [11], [12]. The advantage of this approach is that it enables direct, closed-form approximation of the cumulative density function and the moments of the underlying stochastic process [10], both of which are typically necessary for stochastic optimal control problems. However, the main challenge then becomes one of finding computationally efficient under-approximations of the resulting cumulative density function, which may be non-convex.

We propose to employ empirical characteristic functions to characterize unknown disturbance processes in a linear, time-invariant dynamical system with a quadratic cost function. We construct a conic, convex reformulation of the resulting stochastic optimal control problem, that ensures computational tractability [13]. Our approach employs a piecewise under-approximation of the cumulative density function, in which the user can specify the desired trade-off between accuracy and the number of piecewise elements. We use confidence intervals on the approximate cumulative density function to provide probabilistic bounds on the solution to the data-driven stochastic optimal control problem. The main contribution of the this paper is the construction of a convex, conic reformulation of a stochastic optimal control problem in the presence of an unknown, additive disturbance, via empirical characteristic functions, with confidence bounds on the optimal solution.

The outline of the paper is as follows. We first formulate the problem in Section II. Section III presents algorithms to convexify the problem and proofs of its convergence properties. In Section IV we demonstrate our approach on two examples. We provide concluding remarks in section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

We use the following notation throughout the paper. We denote real-valued vectors with lowercase \( w \in \mathbb{R}^n \), matrices with upper case \( V \in \mathbb{R}^{n \times m} \), and random variables via bold-face \( w \). Concatenated vectors or matrices are indicated by an overline, \( \overline{w} = \begin{bmatrix} w[1]^T & w[2]^T & \cdots & w[N-1]^T \end{bmatrix}^T \in \mathbb{R}^{p(N-1)} \). We denote intervals using \( [a, b] \) where \( a, b \in \mathbb{N}, a < b \).

Consider the linear time-invariant dynamical system

\[
\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{u}[k] + G\mathbf{w}[k]
\]
with state $x \in \mathbb{R}^n$, controlled input $u \in \mathbb{R}^m$, disturbance input $w \in \mathbb{R}^p$, and matrices $A,B,G$ of the appropriate dimensions. For a known horizon $k \in [0,N]$ and initial condition $x_0$, we rewrite the dynamics in concatenated form

$$\mathbf{x} = \overline{A}x_0 + \overline{B}u + \overline{G}w$$

(2)

with state $\mathbf{x} \in \mathbb{R}^{nN}$, input $\pi \in \mathcal{U}^{N−1} = [u_{\min}, u_{\max}]^{N−1}$, disturbance $w \in \mathbb{R}^{p(N−1)}$, and concatenated matrices $\overline{A}, \overline{B}, \overline{G}$, as in [14], [15].

We presume $w$ is a stationary, independent stochastic process, that is the concatenation of a sequence of samples, $\{w_j\}_{j=1}^{N}$, drawn from the probability space $\Omega$. The probability space is defined by $(\Omega, \mathcal{B}((\mathbb{R}^{p(N−1)}))$, with $\mathbb{P}$ as the induced probability distribution of $\mathbb{P}$, [16, Prop. 2.1].

**Problem 1.** We seek to solve the optimization program:

$$\min_{\pi} \quad \mathbb{E}\left((\mathbf{x} - \pi_s)^T Q(\mathbf{x} - \pi_s) + \pi^T R\pi\right)$$

(3a)

s.t. \quad $\mathbb{P}\{\mathbf{x} \notin S\} \leq \Delta$

(3b)

subject to the dynamics in (2), for some positive definite matrices $Q \in \mathbb{R}^{nN \times nN}$ and $R \in \mathbb{R}^{n(N−1) \times n(N−1)}$, some polytopic constraint set $S \subseteq \mathbb{R}^{nN}$ that is closed and bounded, and some constraint violation threshold $\Delta \in [0,1]$, but with no direct knowledge of the cumulative distribution function or moments of $w$. We instead presume that we have observed $N_s$ samples $\{w_j\}_{j=1}^{N_s}$.

The standard approach to solving (3) when the disturbance process is well characterized is to tighten the joint chance constraint (3b) via individual chance constraints [14], [15]. However, two main challenges then arise: 1) reliance of (3a) and (3b) upon moments and the cumulative density function, respectively, of the unknown noise process, and 2) nonconvexity of the individual chance constraints. The former can be seen from expanding (3a):

$$\mathbb{E}\left[J(\mathbf{x}, \pi_s, \pi)\right] = \mathbb{E}[\mathbf{x}] - \mathbf{x}_d^T Q((\mathbb{E}[\mathbf{x}] - \mathbf{x}_d) + \mathcal{U}^T R\mathcal{U}) + \text{tr}(Q \mathcal{G} \text{diag}((\mathcal{G} \mathbf{C}_w)^T))$$

(4)

with $\mathbb{E}[\mathbf{x}] = \overline{A}x_0 + \overline{B}u + \overline{G}\mathbb{E}[w]$, $\mathbb{C}_w = \mathbb{E}[w^2] - \mathbb{E}[w]^2$.

Characteristic functions provide a means to obtain moments as well as the cumulative density function.

**Definition 1.** The characteristic function of a random vector $w \in \mathbb{R}^p$ is

$$\Phi_w(t) = \mathbb{E}[\exp(it^T w)] = \int_{\mathbb{R}^p} \exp(it^T w) \ d\Phi(w)$$

(5)

which is the Riemann-Stieltjes integral of $\exp(it^T w)$ over the frequency variable $t \in \mathbb{R}^p$ with respect to the cumulative distribution function, $\Phi(w)$.

**Theorem 1 (Gil-Pileaz Inversion Theorem, [17]).** The cumulative density function of a random variable $y \in \mathbb{R}$ can be written in terms of the characteristic function as

$$\Phi_y(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \text{Im}\left(\frac{\exp(-it_y x)}{t_y} \Phi_y(t_y)\right) \ dt$$

(6)

where $t_y \in \mathbb{R}$ and $\Phi_y(t_y)$ is the characteristic function of the random variable $y$.

**Definition 2.** The $i^{th}$ moment of $w$ can be written as

$$\mathbb{E}[w^d] = (-i)^d \left[\frac{\partial^d \Phi_w(t)}{\partial t^d}\right]_{t=0}$$

(7)

**Definition 3 (Empirical Characteristic Function [10], [12]).** Let $\{w_j\}_{j=1}^{N_s}$ be the sequence of $N_s$ observations of the random vector $w$. The empirical characteristic function is

$$\hat{\Phi}_w(t) = \sum_{j=1}^{N_s} \alpha_j(w) K_w(t)$$

(8a)

$$K_w(t) = \exp\left(it^T w_j\right) \exp\left(-\frac{1}{2}(t^T \Sigma t)\right)$$

(8b)

for some smoothing parameter matrix $\Sigma \in \mathbb{R}^{p \times p}$, weighting function $\alpha(\cdot) > 0$ with frequency variable $t \in \mathbb{R}^p$.

The smoothing in (8b) is important for ensuring continuity in the cumulative density function [18, Eq. 1.2.1] approximated via Theorem 1 from the empirical characteristic function. A variety of approaches can be used to find a suitable $\Sigma$, to avoid over-smoothing and under-smoothing [19].

Hence to solve Problem 1, we first solve the following.

**Problem 1.a.** Using the empirical characteristic function, 1) construct a concave under-approximation of the approximate cumulative density function $\hat{\Phi}_w(x)$, and 2) approximate the first two moments of $w$.

**Problem 1.b.** Reformulate (3) into a convex, conic stochastic optimal control problem, so that feasible solutions of the convex program are feasible solutions of (3).

### III. METHOD

We first transform the joint chance constraint (3b) into a series of individual chance constraints, each with a risk $\delta_i$. We represent the set $S$ as $S = \{x \in \mathbb{R}^{nN} : P\mathbf{x} \leq q\}$ for some $P \in \mathbb{R}^{nN \times nN}$, $q \in \mathbb{R}^l$. Denoting the $i^{th}$ constraint as $p_i x \leq q_i$, we obtain

$$\mathbb{P}\left\{p_i x \leq q_i - p_i^\top (\overline{A}x_0 + \overline{B}u)\right\} \geq 1 - \delta_i$$

(9a)

$$\Leftrightarrow \Phi_{p_i^\top G w}(q_i - p_i^\top (\overline{A}x_0 + \overline{B}u)) \geq 1 - \delta_i$$

(9b)

$$\sum_{i=1}^l \delta_i \leq \Delta, \delta_i \geq 0, \Delta \in [0,1], \forall i \in \mathbb{N}_{[1,l]}$$

(9c)

for $p_i \in \mathbb{R}^{nN}, q_i \in \mathbb{R}, \delta_i \in [0,1] \subseteq \mathbb{R}$.

Then solutions of the optimization problem

$$\min_{\pi,\delta} \quad \mathbb{E}\left((\mathbf{x} - \pi_s)^T Q(\mathbf{x} - \pi_s) + \pi^T R\pi\right)$$

(10a)

s.t. \quad $\sum_{i=1}^l \delta_i \leq \Delta, \delta_i \geq 0, \Delta \in [0,1]$\quad (10b)

with $\pi \in \mathcal{U}^{N−1}$

(10c)

(10d)

are also feasible solutions of (3). This is because the joint chance constraint (3b) is enforced by (10b) and (10d) with
the additional constraint (10c), which restricts the domain of the \( i \)th constraint by some value \( x_i^{lb} \).

However, several difficulties arise. Note that (10) is non-convex due to (10b). The constraint (10c) ensures a restriction to the concave region of \( \Phi_{p_i}^{\top} \Sigma_w \). For unimodal distributions, the inflection occurs about the mode [20, Def. 1.1], but when the distribution is unknown it is not clear where these inflections occur.

In addition, (10a) is dependent upon the first two moments of \( w \) and (10b) is dependent upon the cumulative density function of \( p_i^T \Sigma_w \), \( \forall i \in \mathbb{N}_{[1,l]} \). Hence we seek approximations to the moments and cumulative density function, as well as a method to reformulate (10b) based on samples \( w_j \).

### A. Approximating the cumulative distribution function and moments from the empirical characteristic function

Applying Definition [3] we obtain

\[
\Phi_{p_i}^{\top} \Sigma_w(t) = \alpha \sum_{j=1}^{N_s} \exp \left( i t p_i^T \Sigma_w w_j \right) \exp \left( -\frac{1}{2} \left( (p_i^T \Sigma) (p_i^T \Sigma)^{\top} t^2 \right) \right) \tag{11a}
\]

\[
\hat{\Phi}(t) = \alpha \sum_{j=1}^{N_s} \exp(\mathbf{i}^T \mathbf{w}_j) \exp \left( -\frac{1}{2} (\mathbf{T}^\top \Sigma \mathbf{T}) \right) \tag{11b}
\]

where \( \Sigma = \text{diag}(\Sigma_1 \cdots \Sigma_{N_s}) \in \mathbb{R}^{p(N-1) \times p(N-1)} \), \( \mathbf{T} = [t_1 \cdots t_{N_s}]^\top \in \mathbb{R}^{p(N-1)} \) and \( \alpha = 1/N_s \). We use (9) to obtain \( \hat{\Phi}_{p_i}^{\top} \Sigma_w(x) \), and (7) to obtain the approximate moments of \( \Sigma_w \).

### B. Constructing a Convex Restriction for (10b)

We seek to obtain an empirical convexification of (10b) with a restriction for which it is concave [20, Def 1.1]. For a user-defined error, \( \epsilon \), and desired number of affine terms, \( N_{dt} \), we construct a piecewise affine under-approximation,

\[
\hat{\Phi}_{p_i}^{\top} \Sigma_w = \min_{a_{i,j}, c_{i,j}} \quad \forall j \in \mathbb{N}_{[1,z]}
\]

such that

\[
0 \leq \hat{\Phi}_{p_i}^{\top} \Sigma_w(x) - \hat{\Phi}_{p_i}^{\top} \Sigma_w(x) \leq \epsilon \tag{13}
\]

is assured for \( x < x_i^{lb} \), \( \forall i \in \mathbb{N}_{[1,l]} \) over the domain \( \mathcal{D} = [\min(p_i^T \Sigma_w), \infty] \) for the finite set of \( N_s \) observations. We define \( a_{i,j} \) and \( c_{i,j} \) as the slope and intercept terms for the \( j \)th affine term in the under-approximation of \( \Phi_{p_i}^{\top} \Sigma_w(x) \).

We propose the following algorithm to construct the piecewise linear under-approximation of the cumulative density function derived from the empirical characteristic function.

Alg [4] is based on the sandwich algorithm [21], and demonstrated in Figure [4]. At each of \( N_p \) evaluation points, \( \{x_p, \hat{\Phi}_w(x_p)\}_{p=1}^{N_p} \), the algorithm constructs affine terms, and stores the affine terms which result in largest positive error close to \( \epsilon \). This is repeated until the break conditions are met (line 9) with a total of \( z \) piecewise affine terms. We choose an upper bound \( y_{N_p} \) (line 12) as it is unreasonable to infer the probability of an event beyond \( \max(p_i^T \Sigma_w) \), and assures (13) holds on \( \mathcal{D} \). This solves Problem 1.a.

### Algorithm 1 Computing \( \hat{\Phi}_w \) from \( \hat{\Phi}_w \)

**Evaluations of cumulative distribution function**

\[
\{(x_p, \hat{\Phi}_w(x_p))\}_{p=1}^{N_p}, \text{ desired error } \epsilon, \text{ desired number of affine terms } N_{dt}, \text{ Output: affine terms } \hat{\Phi}_w \left\{ (a_j, c_j) \right\}_{j=1}^{z}, \text{ restriction } x_i^{lb}
\]

1: continue \( \leftarrow \) true, \( p \leftarrow N_p 
2: \text{ while } continue = \text{ true do } \forall \in \mathbb{N}_{[1,p-1]}, \forall \in \mathbb{N}_{[1,p]} \)
3: \( a_j \leftarrow \hat{\Phi}_w(x_p) - x_p \)
4: \( c_j \leftarrow \hat{\Phi}_w(x_p) - m_j x_p 
5: y_jk \leftarrow a_j x_k + c_j 
6: \text{else, } \hat{\Phi}_w \leftarrow \{(a_j, c_j)\}_{j=p}, \forall = w 
7: \text{ end if}
8: \text{ end while}

### C. Underapproximative, Conic Optimization Problem

We replace the individual chance constraints in (10b) and the lower bounds in (10c) with a conic, convex reformulation, obtained from Alg. [1] so that the reformulated optimization is

\[
\min_{\pi, \delta} \quad \mathbb{E} \left[ (\mathbf{x} - \pi_d)Q(\mathbf{x} - \pi_d) + \pi^T R \pi \right] \tag{14a}
\]

s.t.

\[
\begin{align*}
\forall i \in \mathbb{N}_{[1,l]} & \quad \forall j \in \mathbb{N}_{[1,z]}
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{l} \delta_i \leq \Delta, \quad \delta_i \geq 0, \quad \Delta \in [0,1] \tag{14d}
\end{align*}
\]

where \( \pi \in \mathcal{U}^{N_{dt}} \)

Alg [2] summarizes how the methods described in this section are employed to solve (14).

### Algorithm 2 Underapproximative, conic optimization (14)

**Time horizon N, \( \Delta T \), polytopic set \{P,q\}, samples \( \{w_j\}_{j=1}^{N_p}, \{(a_j, c_j)\}_{j=1}^{z}, \text{ evaluation points } \hat{\Phi}_w, \text{ desired error } \epsilon, \text{ desired number of affine terms } N_{dt}, \text{ smoothing matrix } \Sigma \).**

**Output:** Open loop input \( \pi \), risk allocation \( \delta \)

1: for \( i \in \mathbb{N}_{[1,l]} \) do
2: \( \text{Let } \mathcal{D} = \{ \forall j \in \mathbb{N}_{[1,z]} \} \quad \min_{\pi, \delta} \quad \max_{a_{i,j}, c_{i,j}} (a_{i,j}, c_{i,j}) \}
3: \{ (x_p, \hat{\Phi}_w(x_p))\}_{p=1}^{N_p} \leftarrow \text{From Alg. [1]}
4: \{ (a_j, c_j)\}_{j=1}^{z} \leftarrow \text{From Alg. [1]}
5: \text{end for}
6: \mathbb{E}[\|w\|^2], \mathbb{E}[\|w\|^2] \leftarrow \text{Using (7) and (11b.)}
7: \text{C} \leftarrow \mathbb{E}[\|w\|^2] - \mathbb{E}[\|w\|^2]
8: \{\pi, \delta\} \leftarrow \text{Solve (14)}

### D. Convergence and Confidence Intervals

While (10) is convex and conic, its relationship to (3) is not clear, as it utilizes an under-approximation of the esti-
Given an empirical CDF, \( \hat{\Phi}_N(x) \), from \( N_s \) samples, the probability that the worst deviation is above some \( \epsilon_E \) is

\[
P\left\{ \sup_{x \in \mathbb{R}} \left( |\hat{\Phi}_N^E(x) - \Phi_{p_i}^E(x)| > \epsilon_E \right) \right\} \leq \alpha \tag{15}
\]

for \( \alpha = 2e^{-2N_s\epsilon_E^2} \).

Hence for a desired confidence level \( \alpha \), using \( N_s \) samples, we have \( \epsilon_E = (2N_s)^{-1} \ln(2/\alpha) \)^{1/2}. To make use of (15) for \( \hat{\Phi} \), we make the following assumption.

**Assumption 1.** For \( x \in D \), \( |\hat{\Phi}^{E}_{p_i} - \Phi_{p_i}^E(x)| \leq \epsilon_D \).

Assumption 1 is dependent upon \( \Sigma \) and \( N_s \), and reasonable for \( \Sigma \) chosen to avoid under- or over-smoothing. Both terms converge to \( \Phi_{p_i}^E(x) \) as \( N_s \to \infty \), so their difference tends to 0 [24, Thm. 20.6].

**Theorem 3** (Confidence Interval for \( \hat{\Phi}^E_{p_i} \)). Given Def. 4 and Assumption 7 we have that with probability \( 1 - \alpha \),

\[
|\hat{\Phi}^E_{p_i} - \Phi_{p_i}^E(x)| \leq \epsilon_E + \epsilon_D \tag{16}
\]

**Proof.** For \( x \in D \), by Def. 4 and by the least upper bound property [25, Def. 5.5.5], we have that \( |\hat{\Phi}^E_{p_i} - \Phi_{p_i}^E(x)| \leq \epsilon_E \) is satisfied with probability \( 1 - \alpha \). By the properties of absolute value [25, Prop. 4.3.3],

\[
\hat{\Phi}^E_{p_i}(x) - \epsilon_E \leq \hat{\Phi}^E_{p_i}(x) \leq \hat{\Phi}^E_{p_i}(x) + \epsilon_E \tag{17}
\]

By Assumption 1 and the properties of absolute value,

\[
\hat{\Phi}^E_{p_i}(x) - \epsilon_D \leq \hat{\Phi}^E_{p_i}(x) \leq \hat{\Phi}^E_{p_i}(x) + \epsilon_D \tag{18}
\]

Since \( \Phi_{p_i}^E(x) \), \( \hat{\Phi}^E_{p_i}(x) \), and \( \Phi_{p_i}^E(x) \) are positive, bounded, right-hand continuous functions [24], we combine (17) and (18), so that \( \Phi_{p_i}^E(x) - \epsilon_E - \epsilon_D \leq \Phi_{p_i}^E(x) - \epsilon_E \leq \Phi_{p_i}^E(x) + \epsilon_E + \epsilon_D \). Thus, we have (16) by the properties of absolute value.

**Corollary 1.** Given \( \hat{\Phi}^E_{p_i} \), which underapproximates \( \Phi_{p_i}^E \) according to (15) on \( D \), and the confidence interval \( \epsilon_D + \epsilon_E \) in (16) with likelihood \( 1 - \alpha \), we have \( \Phi_{p_i}^E(x) - \epsilon - \epsilon_E - \epsilon_D \leq \Phi_{p_i}^E(x) \) with likelihood \( 1 - \alpha \).

**Proof.** Follows directly from (15) and (16).
Corollary 1 establishes a worst-case under-approximation to the true cumulative distribution function. A similar approach can be taken for $\mathbb{E}[W]$ and $\mathbb{E}[W^2]$, using results from [26] and [27], respectively. However, because the approximate moments are cheap to compute (i.e., 3.22 seconds for $10^5$ samples), numerical approximations can be quite accurate (Fig. 2). In contrast, the computational cost of sampling is high for the chance constraint under-approximation.

Algorithm 2 and the optimization reformulation (14), along with convergence results and confidence intervals in this section, solve Problem 1.b.

IV. EXAMPLES

We demonstrate our approach on two examples, with code available at [https://github.com/unm-hsc1/vigsv-CSS-L-STOC-ECF](https://github.com/unm-hsc1/vigsv-CSS-L-STOC-ECF). We presume $N_s = 1000$, $N_p = 1000$, $\epsilon = 1 \times 10^{-3}$, $N_{dr} = 20$. We compare our method to a mixed-integer particle control approach [28]. All computations were done in MATLAB with a 3.80GHz Xeon processor and 32GB of RAM. The optimization problems were formulated in CVX [29] and solved with Gurobi [30]. The inversion (6) uses CharFunTool [31] and system formulations are implemented in SReachTools [32]. We use [33], which employs linear diffusion and a plug-in method, to compute $\Sigma$. For both examples, the overall constraint violation threshold is $\Delta = 0.2$.

A. Double Integrator

Consider a double integrator

$$x[k+1] = \begin{bmatrix} 1 & \Delta T \frac{\Delta^2}{2} \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} \Delta^2 T^2 \frac{\Delta^2}{2} \\ 0 \end{bmatrix} u[k] + w[k]$$

with state $x \in \mathbb{R}^2$, disturbance $w \in \mathbb{R}^2$, input $u \in U = [-100, 100] \subset \mathbb{R}$, sampling time $\Delta T = 0.25$, and time horizon $N = 10$. Disturbance samples are drawn independently for each dimension, from a uniform distribution $U[-5, 5]$ on $w_1$, and from a scaled gamma distribution $\text{Gam}(k = 8, \theta = 0.5)$ on $w_2$. The cost function has $Q = 10I_{20 \times 20}$, $R = 10^{-2}I_{9 \times 9}$. The time-varying constraint set is $S = \{ (t, x) \in N_{[1, N]} \times \mathbb{R}^2 : p_1 + q_1 \leq x_1 \leq p_2 + q_2 \}$ with $p_1 = -p_2 = -2$, $q_1 = -q_2 = -50$. The reference trajectory, $x_d = [50 \ 0]^T$, was chosen intentionally to be outside of the constraint set, to test constraint violation.

We compared Algorithm 2 with a particle filter approach [28], a mixed-integer optimization which uses disturbance samples (we chose 50) to compute an open-loop controller. We then validated both algorithms under $10^5$ disturbance sequences via Monte-Carlo simulation, by evaluating the percentage of resulting trajectories which meet the desired constraints. While the resultant mean state trajectories are similar (Fig. 3), the constraint satisfaction likelihoods vary considerably: 0.912 for Algorithm 2 and 0.698 for the particle approach (Table I). As expected, because the control from Alg. 2 is designed to ensure a lower bound on the safety probability, the empirically observed safety probability is above the 0.80 threshold. However, the particle filter fails to meet this threshold, because of its inherent undersampling.

![Fig. 3: Mean trajectories for the double integrator.](image)

If the number of samples were increased, the probability would approach the desired constraint satisfaction likelihood through the law of large numbers, but with increased computational cost, which is exponential in the number of particles.

B. One-way Hypersonic Vehicle

Consider a hypersonic vehicle with longitudinal dynamics

$$\dot{h} = V \sin(\theta - \alpha)$$

$$\dot{V} = \frac{1}{m} (T(f, \Phi) \cos \alpha - D(\alpha, \delta_e)) - g \sin(\theta - \alpha)$$

$$\dot{\alpha} = \frac{m}{V} (-T(f, \Phi) \sin \alpha - L) + Q + \frac{\rho}{V} \cos(\theta - \alpha)$$

$$\dot{\theta} = Q$$

$$\dot{Q} = M (\alpha, \delta_e, \Phi) / I_{yy}$$

with state $x = [h V \alpha \theta Q]^T$ and input $u = [\Phi \delta_e]^T$, that includes fuel-to-air ratio $\Phi$ and elevator deflection $\delta_e$. [34]. We linearize (20) about the trim condition, $x_d = [85000 \mathrm{ft}, 7702 \mathrm{ft/s}, 1.52^\circ, 1.52^\circ, 0^\circ/\mathrm{s}]$, which is also the reference trajectory, and $u_d = [0.25, 11.46^\circ]$, and add a disturbance $w \in \mathbb{R}^2$, which affects the first two states only. We discretize in time with $\Delta T = 0.25$. For $N = 10$, the cost function has $Q = 10I_{50 \times 50}$ and $R = 10^{-2}I_{18 \times 18}$. We presume $N_s = 1000$, with $w_1$ drawn from a scaled Weibull distribution, $2 \text{Weib}(k = 5, \theta = 4)$, and $w_2$ drawn from a gamma distribution, $\text{Gam}(k = 5, \theta = 1)$. The safe set, $S = \{ (t, x) \in N_{[1, N]} \times \mathbb{R}^5 : h \in [50000 \mathrm{ft}, 50200 \mathrm{ft}], V \in [7650 \mathrm{ft/s}, 7750 \mathrm{ft/s}] \}$, describes constraints arising from the flight envelope and the operational mode [35]–[37].

We again compare Algorithm 2 to the particle filter approach [28] with 50 particles, through empirical evalua-
Further, the computation time for Algorithm 2 is significantly
of 0.80. As in the previous example, the violation of the
likelihood is 0.889 for Alg. 2, but only 0.629 for particle
respect to the speed constraint. The constraint satisfaction
with the desired likelihood, but under particle control, violate
trajectories (Fig. 4) under Alg. 2 satisfy the constraint set
their feedback and discussions.

We present a conic, convex reformulation of a stochastic
optimal control problem for LTI systems with an uncharac-
terized stochastic disturbance. We use the empirical charac-
teristic function to recover moments and chance constraints,
allowing fast solution while ensuring constraint satisfaction
with a desired likelihood. We demonstrated our approach on
two examples, and compared it to a particle filter approach.

V. CONCLUSION

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