Abstract

The peak algebra is originally introduced by Stembridge using enriched $P$-partitions. Using the character theory by Aguiar-Bergeron-Sottile, the peak algebra is also the image of $\Theta$, the universal morphism between certain combinatorial Hopf algebras. We extend the notion of peak algebras and theta maps to shuffle, tensor, and symmetric algebras. As examples, we study the peak algebras of symmetric functions in non-commuting variables and the graded associated Hopf algebra on permutations. We also introduce a new shuffle basis of quasi-symmetric functions that its elements are the eigenfunctions of $\Theta$. Using this new basis, we show that the peak algebra is the space spanned by the set of shuffle basis elements indexed by compositions whose all parts are odd.

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1 Introduction

A special family of symmetric functions, known as the Schur’s $Q$ functions, indexed by odd partitions, are introduced in [27] to study the projective representations of symmetric and alternating groups. Combinatorially, the Schur’s $Q$ functions are equipped with a theory of shifted tableaux, including RSK correspondence, Littlewood-Richardson rule and jeu de taquin [25, 28, 32]. Moreover, the subspace spanned by Schur’s $Q$ functions, $\Omega$, forms a self-dual Hopf algebra [20].

Using the combinatorics of enriched $P$-partitions, the Schur’s $Q$ functions are generalized to the peak algebra $\Pi$, indexed by odd compositions, in the fundamental paper [29]. The peak algebra and its graded dual have triggered a variety amount of studies. For instance, it is shown that the peak algebra corresponds to the representation of 0-Hecke-Clifford algebra. Also, the dual peak algebra is a Hopf ideal in noncommutative symmetric functions with respect to the internal product. See [6, 7, 8, 10, 15, 22] for a small portion.

A natural question arises: is it possible to define analogous peak algebra for other Hopf algebras? There are already many trials to generalize peak algebra in the literature. Most of these generalizations are combinatorial through enriched $P$-partition and peak sets. Notably, in [17] the peak algebra is generalized to Poirier-Reutenauer Hopf algebra of standard Young tableaux, which is introduced in [23]. Other generalizations can be found in [2, 5, 16].

In this paper, we try to give an answer to this question from an algebraic angle. As observed in [1], there is another description of $\Omega$ and $\Pi$, given by character theory of combinatorial Hopf algebras. The Hopf algebra of quasi-symmetric functions $\mathsf{QSym}$ and the canonical character $\zeta_{\mathsf{QSym}} : \mathsf{QSym} \rightarrow \mathbb{C} f(x_1, x_2, \ldots) \mapsto f(1, 0, 0, \ldots)$ give the terminal object in the category of combinatorial Hopf algebras. The theta map $\Theta_{\mathsf{QSym}}$ is the unique Hopf morphism that makes the following diagram commute.

The image of $\Theta_{\mathsf{QSym}}$ is exactly $\Pi$. Similarly, $\Omega$ can be constructed as the image of $\Theta_{\mathsf{Sym}}$.

For any combinatorial Hopf algebra $\mathcal{H}$ with morphism $\Phi : \mathcal{H} \rightarrow \mathsf{QSym}$, we define its theta map to be a graded Hopf morphism making the following diagram commute.

The image $\Theta(\mathcal{H})$ is called the peak algebra for $\mathcal{H}$. In order to find the theta maps, we introduce a shuffle basis $\{ S_\alpha \}$ of $\mathsf{QSym}$, indexed by compositions. The shuffle basis satisfy two properties
1. They give a shuffle-like Hopf structure i.e. shuffle product and deconcatenation coproduct,
2. They are eigenfunctions of $\Theta_{\text{QSym}}$.

Using this new basis and its dual basis, we are able to find explicit theta maps and peak algebras for shuffle algebras, tensor algebras and symmetric algebras. The symmetric functions in non-commuting variables, NCSym, can be viewed as a tensor algebra, as a consequence of [18] which finds a set of free generators that are also primitive. Our construction results in a peak algebra indexed by set partitions whose parts all have odd size. It is worthy to mention that in a forthcoming paper, the authors also construct peak algebra of NCSym combinatorially through non-commutative enriched $P$-partitions.

There is another large family of shuffle and tensor algebras that our construction can be applied to. As shown in [3], for any Hopf algebra $H$ that is graded as a coalgebra, the associated graded Hopf algebra, $\text{gr}(H)$, is a shuffle algebra. The graded dual of $\text{gr}(H)$ is a tensor algebra. We use the graded associated Hopf algebra on permutations, also defined in [19, 30], as example and give explicit theta map and peak algebra.

This paper is organized as follows. Section 2 gives necessary background information on the Hopf algebras symmetric functions and quasi-symmetric functions, together with character theory of combinatorial Hopf algebras and precise definitions of theta maps. The shuffle basis is introduced and studied in section 3. In section 4, we revisit the shuffle, tensor and symmetric algebra. We then define candidates of theta maps and give explicit formula. The Hopf subalgebras of odd elements are introduced as the peak algebras under theta maps. And lastly, section 5 gives examples of theta maps and peak algebras, including the symmetric functions in non-commuting variables, and the graded associated Hopf algebra on permutations.

2 Preliminaries

In this section we present the Hopf algebras $\text{Sym}$, $\text{QSym}$, $\text{NSym}$, together with the character theory of combinatorial Hopf algebras. For more details about Hopf algebras we refer the readers to [13]. The base field is set to be $\mathbb{C}$ for convenience.

2.1 The Hopf algebras $\text{QSym}$, $\text{Sym}$ and $\text{NSym}$

A composition $\alpha$ of a positive integer $n$, written $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \ldots, \alpha_\ell)$ where $\alpha_1 + \cdots + \alpha_\ell = n$. We let $\ell(\alpha) = \ell$ be the number of parts of $\alpha$ and it is called the length of $\alpha$. When there is no confusion, we sometimes write $\alpha$ as a word $\alpha_1\alpha_2\cdots\alpha_{\ell(\alpha)}$.

The concatenation of $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}) \models n$ and $\beta = (\beta_1, \ldots, \beta_{\ell(\beta)}) \models m$ is the composition $\alpha\beta = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \ldots, \beta_{\ell(\beta)})$.

For a positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. There is a one-to-one correspondence $\mathcal{I}$ between the set of all compositions of $n$ and the set of all subsets of $[n-1]$ such that for $\alpha \models n$,

$$\mathcal{I}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)-1}\}.$$ 

Let $\alpha, \beta \models n$. We write $\beta \leq \alpha$, or $\alpha$ refines $\beta$ if $\mathcal{I}(\beta) \subseteq \mathcal{I}(\alpha)$.
Let $S_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$. Let $\text{Sh}_{n, m}$ be the following subset of the permutation group $S_{n+m}$,

$$\text{Sh}_{n, m} = \{\sigma \in S_{n+m} : \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n); \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m)\}.$$  

For compositions $\alpha$ and $\beta$ with lengths $n$ and $m$, respectively, we define $\alpha \sqcup \beta$ to be the multi-set

$$\{(\gamma_\sigma(1), \gamma_\sigma(2), \ldots, \gamma_\sigma(n+m)) : \sigma \in \text{Sh}_{n, m}\}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n+m})$ is the concatenation $\alpha \beta$.

The quasi-shuffle $\alpha \sqcup \beta$ is the multi-set of compositions $(\delta_1, \ldots, \delta_\ell)$ such that

1. $(\delta_1, \ldots, \delta_\ell) \leq (\gamma_1, \ldots, \gamma_{n+m})$ for some $(\gamma_1, \ldots, \gamma_{n+m}) \in \alpha \sqcup \beta$,
2. either $\delta_i = \gamma_j$ for some $j$, or $\delta_i = \gamma_j + \gamma_{j+1}$ where $\gamma_j$ is a part of $\alpha$ and $\gamma_{j+1}$ is a part of $\beta$.

For a composition $\alpha$ of $n$, the monomial quasisymmetric function $M_\alpha$ is

$$M_\alpha := \sum_{i_1 < \cdots < i_\ell(\alpha)} x_{i_1}^{\alpha_1} \cdots x_{i_\ell(\alpha)}^{\alpha_\ell(\alpha)}$$

which is an element of the algebra of bounded-degree formal power series in commutative variables $\{x_i\}_{i \geq 1}$. By convention, $M() = 1$ where () is the unique composition of 0. The vector space of quasisymmetric functions is denoted by $\text{QSym}$ and is defined as

$$\text{QSym} = \bigoplus_{n \geq 0} \text{QSym}_n$$

where

$$\text{QSym}_n = \mathbb{C}\text{-span}\{M_\alpha : \alpha \models n\}.$$  

The space of quasisymmetric functions is indeed a Hopf algebra with product inherited from the ring of bounded-degree formal power series. Its product is given by quasi-shuffle

$$M_\alpha \cdot M_\beta = \sum_{\gamma \in \alpha \sqcup \beta} M_\gamma.$$  

Its coproduct is given by deconcatenation

$$\Delta(M_\alpha) = \sum_{\beta \gamma = \alpha} M_\beta \otimes M_\gamma.$$  

A special subspace of $\text{QSym}$ is the Hopf algebra of symmetric functions, $\text{Sym}$. The space $\text{Sym}$ is (commutatively) freely generated by the scaled power sum functions defined as

$$p_n = \frac{1}{n} M_n.$$  

The scaled power sum functions are primitive elements i.e. for $n \geq 1$, $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$. For a composition $\alpha$, we write $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_\ell(\alpha)}$. In particular, $\text{Sym}$ is both commutative and cocommutative.
The complete homogeneous functions of Sym are defined as
\[ h_n = \sum_{\alpha | n} M_\alpha \]
and we write \( h_\alpha = h_{\alpha_1} \cdots h_{\alpha_{\ell(\alpha)}} \).

There is a well-known expansion of \( p_n \) in terms of the complete homogeneous functions, cf. \[13\], Ex. 5.43, given by
\[ p_n = \frac{1}{n} \sum_{\alpha | n} (-1)^{\ell(\alpha) - 1} \alpha_{\ell(\alpha)} h_\alpha. \]

The Hopf algebra of non-commutative symmetric functions, NSym, originally defined in \[12\], is the graded dual of QSym. Let \( \{ H_\alpha \} \) be the basis, indexed by compositions, of NSym dual to the basis \( \{ M_\alpha \} \) of QSym. Thus, We have the Hopf pairing \( \langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta} \) where \( \delta_{\alpha,\beta} \) is the Kronecker delta.

Equivalently, NSym = \( \langle H_1, H_2, \cdots \rangle \) is the non-commutative algebra freely generated by the set \( \{ H_n \}_{n \geq 1} \) with coproduct defined, on generators and extended linearly and multiplicatively, as
\[ \Delta(H_n) = \sum_{i+j=n} H_i \otimes H_j. \]

For a composition \( \alpha \), we write \( H_\alpha = H_{\alpha_1} \cdots H_{\alpha_{\ell(\alpha)}} \). The Hopf algebra NSym is graded with \( \text{deg}(H_n) = n \). A linear basis for the homogeneous component \( \text{NSym}_n \) of degree \( n \) is \( \{ H_\alpha : \alpha | n \} \).

It is well-known that Sym is self-dual via the isomorphism defined on generators as \( I : \text{Sym}^* \rightarrow \text{Sym} \)
\[ m_n^* \mapsto h_n. \]

It can be shown, via Cauchy identity, that \( I(p_n^*) = np_n \). Let \( \iota : \text{Sym} \rightarrow \text{QSym} \) be the natural embedding. There is a forgetful projection \( \pi : \text{NSym} \rightarrow \text{Sym} \) defined on generators as \( H_n \mapsto h_n \). The map \( \pi \) is a graded Hopf morphism and is also the map composition \( I^{-1} \circ \iota^* \) where \( \iota^* : \text{NSym} \rightarrow \text{Sym}^* \) is the adjoint map of \( \iota \) i.e. \( \pi = I \circ \iota^* = (\iota \circ I)^* \).

### 2.2 Combinatorial Hopf algebras

A combinatorial Hopf algebra is a graded connected Hopf algebra \( H = \bigoplus_{n \geq 0} H_n \) with a linear and multiplicative map \( \zeta : H \rightarrow \mathbb{C} \). The map \( \zeta \) is called the character. The set of characters \( \text{Hom}(H, \mathbb{C}) \) forms a group under convolution product
\[ \zeta \zeta' = m \circ (\zeta \otimes \zeta') \circ \Delta. \]

The identity element of the group \( \text{Hom}(H, \mathbb{C}) \) is the counit \( \epsilon_H \) and inverse element is given by \( \zeta^{-1} = \zeta \circ S_H \) where \( S_H \) is the antipode of \( H \). We define \( \zeta \) to be the character such that \( \zeta|_{H_n} = (-1)^n \zeta|_{H_n} \).

A morphism of combinatorial Hopf algebras \( \Phi : (H, \zeta_H) \rightarrow (H', \zeta_{H'}) \), is a graded Hopf morphism \( \Phi : H \rightarrow H' \) such that \( \zeta_{H'} \circ \Phi = \zeta_H \). The character theory of Hopf algebras has been studied by Aguiar, Bergeron, and Sottile in [1].

Let \( \zeta_{\text{QSym}} \) be the character of QSym given by
\[ \zeta_{\text{QSym}}(f(x_1, x_2, \ldots)) = f(1, 0, 0, 0, \ldots). \]
Example 2.1. We have $\zeta_{QSym}(p_n) = \frac{1}{n}$ and $\zeta_{QSym}(M_\alpha) = \begin{cases} 1 & \text{if } \ell(\alpha) = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$

And let $\zeta_{Sym}$ be the restricted character $\zeta_{QSym}|_{Sym}$. The characters $\zeta_{QSym}$ and $\zeta_{Sym}$ are canonical characters in the following sense.

Theorem 2.2. [1, Theorem 4.1] For any combinatorial Hopf algebra $(H, \zeta_H)$, there exists a unique morphism of combinatorial Hopf algebras

$$\Phi : (H, \zeta) \to (QSym, \zeta_{QSym}).$$

Moreover, if $H$ is cocommutative, then $\Phi(H) \subseteq Sym$.

The odd and even Hopf subalgebras of a combinatorial Hopf algebra $(H, \zeta_H)$, denoted by $S_-(H, \zeta_H)$ and $S_+(H, \zeta_H)$ respectively, are defined as the largest sub-coalgebra contained in $\ker (\zeta_{QSym}^{-1} - \zeta_H)$ and $\ker (\zeta_H - \zeta_{QSym})$ respectively.

We will be using the following important properties of the odd and even Hopf subalgebras.

Theorem 2.3. [1, Theorem 5.3] The coalgebras $S_-(H, \zeta_H)$ and $S_+(H, \zeta_H)$ are graded Hopf subalgebras of $H$. Moreover, a homogeneous element $h \in H$ belongs to $S_-(H, \zeta_H)$ (or $S_+(H, \zeta_H)$) if and only if $(id \otimes (\zeta_{QSym}^{-1} - \zeta_H) \otimes id) \circ \Delta^{(2)}(h) = 0$ (or $(id \otimes (\zeta_H - \zeta_{QSym}) \otimes id) \circ \Delta^{(2)}(h) = 0$ respectively).

2.3 Peak algebras and the theta maps

Consider the combinatorial Hopf algebra $(QSym, \zeta_{QSym}^{-1} \zeta_{QSym})$. Let $\Theta_{QSym}$ be the unique combinatorial Hopf morphism

$$\Theta_{QSym} : (QSym, \zeta_{QSym}^{-1} \zeta_{QSym}) \to (QSym, \zeta_{QSym}).$$

The map $\Theta_{QSym}$ is originally studied by Stembridge using enriched $P$-partitions [29] and reconstructed in [1]. The image of $\Theta_{QSym}$ is known as the peak algebra of $QSym$, denoted by $\Pi$.

When restricted to $Sym$, let $\Theta_{Sym}$ be the unique combinatorial Hopf morphism

$$\Theta_{Sym} : (Sym, \zeta_{Sym}^{-1} \zeta_{Sym}) \to (Sym, \zeta_{Sym}).$$

The map $\Theta_{Sym}$ maps $h_n$ to $q_n$ where $q_n$ is the Schur’s $Q$ function cf. [20, 27]. It can be shown that $\Theta_{Sym}$ can be equivalently defined on scaled power sum functions, and extended multiplicatively, as

$$\Theta_{Sym}(p_n) = \begin{cases} 2p_n & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

The image of $\Theta_{Sym}$ is known as the space of Schur’s $Q$ functions, and denoted by $\Omega$.

The images $\Pi$ and $\Omega$ are precisely the odd Hopf subalgebras $S_-(QSym, \zeta_{QSym})$ and $S_-(Sym, \zeta_{Sym})$ respectively.

In this paper, we attempt to extend the notion of $\Theta_{QSym}$ and peak algebras to other combinatorial Hopf algebras, in the following sense.
Definition 2.4. Let \((H, \zeta)\) be a combinatorial Hopf algebra, and let \(\Phi : (H, \zeta) \to (\text{QSym}, \zeta_{\text{QSym}})\) be the unique combinatorial Hopf morphism. A theta map for \((H, \zeta)\) is a map \(\Theta : H \to H\) that makes the following diagram commute:

\[
\begin{array}{ccc}
H & \xrightarrow{\Phi} & \text{QSym} \\
\downarrow\Theta & & \downarrow\Theta_{\text{QSym}} \\
\Theta(H) & \xrightarrow{\Phi} & \Pi
\end{array}
\]

The image \(\text{img}(\Theta)\) is called a peak algebra for \((H, \zeta)\).

Since \(\text{img}(\Theta_{\text{QSym}}) = \Pi\), we sometimes use the alternative diagram:

\[
\begin{array}{ccc}
H & \xrightarrow{\Phi} & \text{QSym} \\
\downarrow\Theta & & \downarrow\Theta_{\text{QSym}} \\
\Theta(H) & \xrightarrow{\Phi} & \Pi
\end{array}
\]

When \(H\) is cocommutative, by Theorem 2.2, we have \(\Phi(H) \subseteq \text{Sym}\) and hence:

\[
\begin{array}{ccc}
H & \xrightarrow{\Phi} & \text{Sym} \\
\downarrow\Theta & & \downarrow\Theta_{\text{Sym}} \\
\Theta(H) & \xrightarrow{\Phi} & \Omega
\end{array}
\]

In particular, \(\Pi\) and \(\Omega\) are peak algebras for \((\text{QSym}, \zeta_{\text{QSym}})\) and \((\text{Sym}, \zeta_{\text{Sym}})\) respectively. Additionally, \(\Theta_{\text{QSym}}\) and \(\Theta_{\text{Sym}}\) are theta maps for \((\text{QSym}, \zeta_{\text{QSym}})\) and \((\text{Sym}, \zeta_{\text{Sym}})\) respectively.

3 The shuffle basis of QSym

In this section, we define a new basis of QSym and study its properties.

Definition 3.1. For a composition \(\alpha = (\alpha_1, \ldots, \alpha_\ell)\),

- its odd-to-even set, \(\text{OtE}(\alpha)\) is defined to be \(\{i : \alpha_i \text{ is odd}, \alpha_{i+1} \text{ is even}\}\),
- its even-to-odd set, \(\text{EtO}(\alpha)\) is defined to be \(\{i : \alpha_i \text{ is even}, \alpha_{i+1} \text{ is odd}\}\),
- if \(\text{OtE}(\alpha) = \{i_1, i_2, \ldots, i_k\}\) with \(i_1 < i_2 < \cdots < i_k\), the odd-min composition of \(\alpha\) is defined to be
  \[
m_o(\alpha) = (\alpha_1 + \cdots + \alpha_{i_1}, \alpha_{i_1+1} + \cdots + \alpha_{i_2}, \ldots, \alpha_{i_k+1} + \cdots + \alpha_\ell),
\]
• given a composition \( \beta = (\beta_1, \ldots, \beta_p) \) such that \( m_o(\alpha) \leq \beta \leq \alpha \), then each \( \beta_i \) is a sum of consecutive parts of \( \alpha \). Assume \( \beta_i = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{j+k} \) for some \( j, k \). Let \( \text{O}_o(i) \) and \( \text{E}_o(i) \) be the number of odd and even parts in \( (\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+k}) \) respectively. We define the coefficients

\[
c^\beta_\alpha(i) = \frac{1}{\text{O}_o(i)\text{E}_o(i)!}
\]

and

\[
c^\beta_\alpha = \prod_i c^\beta_\alpha(i).
\]

**Example 3.2.** If \( \alpha = 34421332 \), then \( \text{OtE}(\alpha) = \{1, 7\} \), \( \text{EtO}(\alpha) = \{4\} \) and \( m_o(\alpha) = (3, 17, 2) \). Suppose \( \beta = (3, 8, 9, 2) = (3, 4 + 4, 2 + 1 + 3, 3, 2) \), then

- \( \text{O}_o(1) = 1, \text{E}_o(1) = 0 \) and \( c^\beta_\alpha(1) = 1 \),
- \( \text{O}_o(2) = 0, \text{E}_o(2) = 2 \) and \( c^\beta_\alpha(2) = 1/2 \),
- \( \text{O}_o(3) = 3, \text{E}_o(3) = 1 \) and \( c^\beta_\alpha(3) = 1/6 \),
- \( \text{O}_o(4) = 0, \text{E}_o(4) = 1 \) and \( c^\beta_\alpha(4) = 1 \),

and overall, \( c^\beta_\alpha = 1/12 \).

**Definition 3.3.** The shuffle function \( S_\alpha \), indexed by a composition \( \alpha \models n \), is defined as

\[
S_\alpha = \sum_{m_o(\alpha) \leq \beta \leq \alpha} c^\beta_\alpha M_\beta.
\]

**Proposition 3.4.** The set \( \{S_\alpha : \alpha \models n\} \) of shuffle functions gives a basis of \( \text{QSym}_n \).

**Proof.** It is clear from definition that \( c^\alpha_\alpha = 1 \) for all composition \( \alpha \). Then the proposition follows from triangularity of the transition matrix from shuffle functions to monomial basis.

**Theorem 3.5.** The product of shuffle basis is given by shuffle of compositions i.e.

\[
S_\alpha \cdot S_\beta = \sum_{\gamma \in \alpha \uplus \beta} S_\gamma.
\]

**Proof.** For each composition \( \gamma \in \alpha \uplus \beta \), and \( m_o(\gamma) \leq \delta \leq \gamma \), we construct a pair of compositions \( \hat{\gamma}, \tilde{\gamma} \) and \( \epsilon \in \hat{\alpha} \bar{\beta} \) as follows. By definition, \( \hat{\delta}_i = \alpha_{p_i} + \alpha_{p_i+1} + \cdots + \alpha_{p_i+q} + \beta_{r_i} + \beta_{r_i+1} + \cdots + \beta_{r_i+t} \) for some \( p_i, q, r_i, t \) such that the multi-set \( \{\alpha_{p_i}, \ldots, \alpha_{p_i+q}, \beta_{r_i}, \beta_{r_i+1}, \ldots, \beta_{r_i+t}\} \) is equal to \( \{\gamma_{s_i}, \gamma_{s_i+1}, \ldots, \gamma_{s_i+t}\} \) for some \( s_i \) and \( \text{OtE}(\gamma_{s_i}, \ldots, \gamma_{s_i+t}) = \emptyset \).

We set \( a_i = \alpha_{p_i} + \cdots + \alpha_{p_i+q} \) and \( b_i = \beta_{r_i} + \cdots + \beta_{r_i+t} \). Then, \( \hat{\alpha} \) and \( \tilde{\beta} \) are the compositions obtained from \( (a_1, \ldots, a_{\ell(\delta)}) \) and \( (b_1, \ldots, b_{\ell(\delta)}) \) respectively by removing zeros. Since \( \text{OtE}(\alpha_{p_i}, \ldots, \alpha_{p_i+q}) = \text{OtE}(\beta_{r_i}, \ldots, \beta_{r_i+t}) = \emptyset \), we have \( m_o(\alpha) \leq \hat{\alpha} \leq \alpha \) and \( m_o(\beta) \leq \tilde{\beta} \leq \beta \). Furthermore, \( \epsilon \) is obtained from the shuffle \( (a_1, b_1, \ldots, a_{\ell(\delta)}, b_{\ell(\delta)}) \), removing possible zeros, then merged to \( (a_1 + b_1, \ldots, a_{\ell(\delta)} + b_{\ell(\delta)}) \). As compositions, we have \( \epsilon = \delta \).

This process gives a map

\[
Y : \{(\gamma, \delta) : \gamma \in \alpha \uplus \beta, m_o(\gamma) \leq \delta \leq \gamma\} \to \{((\hat{\alpha}, \tilde{\beta}, \epsilon) : m_o(\alpha) \leq \hat{\alpha} \leq \alpha, m_o(\beta) \leq \tilde{\beta} \leq \beta, \epsilon \in \hat{\alpha} \bar{\beta}\}.
\]
For example, for \( \alpha = 2113, \beta = 122, \gamma = 1222113 \) and \( \delta = 1623 \), then \( a_1 = 0, a_2 = 2, a_3 = 2, a_4 = 3, \) \( b_1 = 1, b_2 = 4, b_3 = 0, b_4 = 0, \hat{\alpha} = 223, \hat{\beta} = 14 \) and \( \epsilon = 1623 \).

The map \( Y \) is surjective. From \( \hat{\alpha}, \hat{\beta} \) and \( \epsilon \), we can recover \( a_i \) and \( b_i \) from the way of quasi-shuffle. Suppose \( a_i = \alpha_{p_i} + \cdots + \alpha_{p_i+q}, b_i = \beta_{r_i} + \cdots + \beta_{r_i+t} \) and \( \text{EtO}(\alpha_{p_i}, \ldots, \alpha_{p_i+q}) = \{ x \} \), \( \text{EtO}(\beta_{r_i}, \ldots, \beta_{r_i+t}) = \{ y \} \) (take \( x = q \) and \( y = t \) when non-existing), then one pre-image is
\[
\gamma = (\ldots, \alpha_{p_i}, \ldots, \alpha_{p_i+x}, \beta_{r_i}, \ldots, \beta_{r_i+y}, \alpha_{p_i+x+1}, \ldots, \alpha_{p_i+q}, \beta_{r_i+y+1}, \ldots, \beta_{r_i+t}, \ldots).
\]
In general, let \( d_\epsilon(i) = \binom{x+t-y}{q} \binom{q+t-x-y}{p} \) and let \( d_\epsilon = \prod_i d_\epsilon(i) \). Then the number of pre-images is \( d_\epsilon \).

Taking the example above, there are totally three \( \gamma \) that yields the same result, namely \( 122113, 1222113 \) and \( 1222113 \).

It is not hard to see that \( \delta = d_\epsilon c_\alpha \delta c_\beta \). Therefore, the statement of this theorem follows from the identities
\[
\sum_{\gamma \in \omega \beta} S_\gamma = \sum_{\gamma \in \omega \beta} \sum_{m_\alpha(\gamma) \leq \delta \leq \gamma} c_\delta M_\delta = \sum_{m_\alpha(\alpha) \leq \delta \leq \alpha} \sum_{\gamma \in \alpha \beta} c_\alpha \delta c_\beta M_\delta = \sum_{m_\alpha(\alpha) \leq \delta \leq \alpha} c_\alpha \delta M_\delta \cdot M_\delta = S_\alpha \cdot S_\beta.
\]

**Theorem 3.6.** The coproduct of shuffle basis is given by deconcatenation of composition i.e.
\[
\Delta(S_\alpha) = \sum_{\beta \gamma = \alpha} S_\beta \otimes S_\gamma.
\]

**Proof.** Fix a composition \( \alpha \models n \). Since QSym is graded, it suffices to show that for each \( i \),
\[
\Delta_{i,n-i}(S_\alpha) = \sum_{\beta = \alpha} S_\beta \otimes S_\gamma.
\]
For simplicity, we use \( \Delta_i \) to denote \( \Delta_{i,n-i} \).

Fix an \( i \). If there is no \( p \) such that \( \alpha_1 + \cdots + \alpha_p = i \), then for any \( \beta \leq \alpha \), there is no \( q \) such that \( \beta_1 + \cdots + \beta_q = i \), which implies \( \Delta_i(M_\beta) = 0 \). In this case,
\[
\Delta_i(S_\alpha) = \sum_{m_\alpha(\alpha) \leq \beta \leq \alpha} c_\beta \Delta_i(M_\beta) = 0.
\]
Assume that \( \alpha_1 + \cdots + \alpha_p = i \) for some \( p \), let \( i \alpha = (\alpha_1, \ldots, \alpha_p) \) and \( \alpha^i = (\alpha_{p+1}, \ldots, \alpha_{\ell(\alpha)}) \), then
\[
\Delta_i(S_\alpha) = \sum_{m_\alpha(\alpha) \leq \beta \leq \alpha} c_\beta \Delta_i(M_\beta)
\]
\[
= \sum_{m_\alpha(\alpha) \leq \beta \leq \alpha} c_\beta \alpha M_{(\beta_1, \ldots, \beta_q)} \otimes M_{(\beta_1^+, \ldots, \beta_q^+)}
\]
\[
= \sum_{m_\alpha(\alpha) \leq \beta \leq \alpha} \left( M_{(\beta_1, \ldots, \beta_q)} \prod_{i=1}^q c_\beta(i) \right) \otimes \left( M_{(\beta_1^+, \ldots, \beta_q^+)} \prod_{i=q+1}^{\ell(\beta)} c_\alpha(i) \right)
\]
\[
= \sum_{m_\alpha(\alpha) \leq \gamma \leq i \alpha} c_\gamma M_\gamma \otimes (c_\alpha M_\mu).
\]
The last equality follows from the fact that $m_\alpha(\alpha) \leq \beta \leq \alpha$ and $\beta_1 + \cdots + \beta_q = i$ if and only if $m_\alpha(i\alpha) \leq i\beta \leq i\alpha$ and $m_\alpha(i\alpha) \leq i\beta \leq i\alpha$. Moreover, it is straightforward from definition that $c^i_\alpha = \prod_{i=1}^q c^{i\beta}(i)$ and $c^{i\beta}_\alpha = \prod_{i=q+1}^{l(\beta)} c^{\beta}(i)$.

By the Hopf structure given above, the shuffle basis makes $\text{QSym}$ a shuffle algebra. And the antipode formula for shuffle basis follows from the well-known antipode formula of shuffle algebra $[11]$.

**Corollary 3.7.** Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. The antipode on shuffle basis is given by

$$S(S_\alpha) = (-1)^\ell S_{\alpha_\ell, \ldots, \alpha_1}.$$  

Recall that a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is odd if each $\alpha_i$ is an odd number.

**Theorem 3.8.** The map $\Theta_{\text{QSym}}$ acts on shuffle basis as

$$\Theta_{\text{QSym}}(S_\alpha) = \begin{cases} 2^{l(\alpha)} S_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** Let $\Phi : \text{QSym} \to \text{QSym}$ be the linear map defined on shuffle basis as

$$\Phi(S_\alpha) = \begin{cases} 2^{l(\alpha)} S_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that $\Phi$ is a Hopf algebra morphism. Since $\Theta_{\text{QSym}}$ is the unique combinatorial Hopf morphism from $(\text{QSym}, \zeta^{-1}_{\text{QSym}} \zeta_{\text{QSym}})$ to $(\text{QSym}, \zeta_{\text{QSym}})$, it suffices to show that $\zeta^{-1}_{\text{QSym}} \zeta_{\text{QSym}} = \zeta_{\text{QSym}} \circ \Phi$.

Recall that

$$\zeta_{\text{QSym}}(M_\alpha) = \begin{cases} 1 & \text{if } \ell(\alpha) = 0 \text{ or } 1 \\ 0 & \text{otherwise}. \end{cases}$$

It follows from the definition of shuffle functions that,

$$\zeta_{\text{QSym}}(S_\alpha) = \begin{cases} \frac{1}{p!(\ell(\alpha) - p)!} & \text{if } \text{EtO}(\alpha) = \{p\} \text{ and } \text{OtE}(\alpha) = \emptyset \\ \frac{1}{\ell(\alpha)!} & \text{if } \text{EtO}(\alpha) = \text{OtE}(\alpha) = \emptyset \\ 0 & \text{otherwise}. \end{cases}$$

Hence, we obtain the following formula for $\zeta_{\text{QSym}} \circ \Phi$

$$\zeta_{\text{QSym}} \circ \Phi(S_\alpha) = \begin{cases} \frac{2^{l(\alpha)}}{\ell(\alpha)!} & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}$$

Combining with the antipode formula yields

$$\zeta^{-1}_{\text{QSym}}(S_\alpha) = \begin{cases} (-1)^{|\alpha| + \ell(\alpha)} \frac{1}{q!(\ell(\alpha) - q)!} & \text{if } \text{EtO}(\alpha) = \emptyset \text{ and } \text{OtE}(\alpha) = \{q\} \\ (-1)^{|\alpha| + \ell(\alpha)} \frac{1}{\ell(\alpha)!} & \text{if } \text{EtO}(\alpha) = \text{OtE}(\alpha) = \emptyset \\ 0 & \text{otherwise}. \end{cases}$$
To sum up,

\[ \zeta_{QSym}(S_\alpha) = \sum_{i=0}^{\ell(\alpha)} \frac{(-1)^i}{i!(\ell(\alpha) - i)!} (\ell(\alpha) - p)! \]

\[ \sum_{i=q}^{p} \frac{(-1)^{q+i} \prod_{i=q}^{p} (1 - i)!}{(\ell(\alpha) - i)!} i!\] if \( \text{EtO}(\alpha) = \{p\}, \text{OtE}(\alpha) = \{q\} \) and \( q < p \)

\[ \sum_{i=q}^{p} \frac{(-1)^{q+i} \prod_{i=q}^{p} (1 - i)!}{(\ell(\alpha) - i)!} i!\] if \( \text{EtO}(\alpha) = \{p\} \) and \( \text{OtE}(\alpha) = \emptyset \)

\[ \sum_{i=q}^{p} \frac{(-1)^{q+i} \prod_{i=q}^{p} (1 - i)!}{(\ell(\alpha) - i)!} i!\] if \( \text{EtO}(\alpha) = \emptyset \) and \( \text{OtE}(\alpha) = \{q\} \)

\[ \sum_{i=q}^{p} \frac{(-1)^{q+i} \prod_{i=q}^{p} (1 - i)!}{(\ell(\alpha) - i)!} i!\] if \( \text{EtO}(\alpha) = \text{OtE}(\alpha) = \emptyset \)

otherwise.

In the first case, the summation is alternating in sign because \( \alpha_i \) are even for \( q < i \leq p \). Therefore, the summation can be simplified to

\[ \frac{(-1)^q}{q!(\ell(\alpha) - p)!} \sum_{i=q}^{p} \frac{(-1)^i}{i!(p - i)!} \] if \( \text{EtO}(\alpha) = \{p\} \) and \( \text{OtE}(\alpha) = \{q\} \)

\[ \frac{(-1)^q}{q!(\ell(\alpha) - p)!} \sum_{i=q}^{p} \frac{(-1)^i}{i!(p - i)!} \] if \( \text{EtO}(\alpha) = \{p\} \) and \( \text{OtE}(\alpha) = \emptyset \)

\[ \frac{(-1)^q}{q!(\ell(\alpha) - p)!} \sum_{i=q}^{p} \frac{(-1)^i}{i!(p - i)!} \] if \( \text{EtO}(\alpha) = \emptyset \) and \( \text{OtE}(\alpha) = \{q\} \)

\[ \frac{(-1)^q}{q!(\ell(\alpha) - p)!} \sum_{i=q}^{p} \frac{(-1)^i}{i!(p - i)!} \] if \( \text{EtO}(\alpha) = \text{OtE}(\alpha) = \emptyset \)

It is easy to see that this summation must vanish, e.g. multiply it by \( (p - q)! \) and use the binomial identity.

The same argument can be applied to all other cases except the sub-case of 4 that \( \alpha \) is an odd composition. When \( \alpha \) is odd, all terms in the sum have positive sign. Hence, we have

\[ \frac{(-1)^q}{q!(\ell(\alpha) - p)!} \sum_{i=q}^{p} \frac{(-1)^i}{i!(p - i)!} \]

Comparing with the formula of \( \zeta_{QSym} \circ \Phi \) completes the proof.

Let \( \{S^*_n\} \) denote the basis of NSym that is dual to shuffle basis of QSym i.e. we have the Hopf pairing \( \langle S^*_\alpha, S_\beta \rangle = \delta_{\alpha,\beta} \). The basis \( \{S^*_n\} \) is called the dual shuffle basis. It is easy to see that NSym is freely generated by \( \{S^*_1, S^*_2, \ldots\} \) and \( S^*_n \) is primitive for all \( n \).

**Theorem 3.9.** The forgetful projection \( \pi \) maps the dual shuffle basis to the scaled power sum basis. More precisely, for all composition \( \alpha \),

\[ \pi(S^*_\alpha) = p_\alpha. \]

**Proof.** Since both \( \{S^*_n\} \) and \( \{p_\alpha\} \) are multiplicative, it suffices to show that \( \pi(S^*_n) = p_n \).

Since \( \pi = (t \circ I)^* \), we have \( \langle p_\alpha, \pi(S^*_n) \rangle = \langle S^*_n, \pi(p^*_\alpha) \rangle = \langle S^*_n, (t \circ I)p^*_\alpha \rangle \).

Recall that \( p_\alpha = \frac{1}{n} M_\alpha = \frac{1}{n} S_\alpha \), hence, \( \langle p^*_\alpha, \pi(S^*_n) \rangle = \langle S^*_n, np_\alpha \rangle = 1 \). Moreover, if \( \alpha \) has more than one parts, then \( \langle S^*_n, p_\alpha \rangle = \frac{1}{\prod_i \alpha_i} \langle S^*_n, S_{\alpha_1} \cdots S_{\alpha_{\ell(\alpha)}} \rangle = 0 \) because the shuffle product of multiple
S basis elements never contains the term $S_n$. Then, we have

$$\langle p_\alpha^*, \pi(S_n^*) \rangle = \begin{cases} 1 & \text{if } \alpha = (n) \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, $\pi(S_n^*) = \sum_\alpha \langle p_\alpha^*, \pi(S_n^*) \rangle p_\alpha = p_n$. \qed

4 The shuffle algebras, tensor algebras and symmetric algebras

In this section, we revisit shuffle algebras, tensor algebras and symmetric algebras and construct their theta maps.

4.1 The shuffle algebra

Let $V$ be a vector space with a countable basis and

$$V^\otimes n = V \otimes V \otimes \cdots \otimes V.$$  

As vector space, the shuffle algebra on $V$ is

$$S(V) = \bigoplus_{n \geq 0} V^\otimes n.$$  

We fix a basis $A = \{v_1, v_2, \ldots \}$ of $V$ throughout this paper. For convenience, we write $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ as $v_1 v_2 \cdots v_n$. Then a basis for $S(V)$ contains all elements of the form $v_{\alpha_1} v_{\alpha_2} \cdots v_{\alpha_\ell}$. For a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we write $v_{\alpha_1} v_{\alpha_2} \cdots v_{\alpha_\ell}$ as $v_\alpha$. Essentially, the elements of $S(V)$ are linear combinations of the words on the alphabet $A$. Note that $V^\otimes 0$ is generated by the empty word.

We now present the Hopf structure of $S(V)$ on the basis elements, and extended linearly.

The product of $S(V)$ is given by the shuffle of the words, that is,

$$v_\alpha \cdot v_\beta = \sum_{\gamma \in \alpha \sqcup \beta} v_\gamma.$$  

The coproduct is given by deconcatenation of words, that is,

$$\Delta(v_\alpha) = \sum_{i=0}^{\ell(\alpha)} v_{(\alpha_1, \ldots, \alpha_i)} \otimes v_{(\alpha_{i+1}, \ldots, \alpha_{\ell(\alpha)})}.$$  

It is well-known, e.g. in [11], that the antipode formula for $S(V)$ is given by

$$S(v_{(\alpha_1, \ldots, \alpha_{\ell(\alpha)})}) = (-1)^{\ell(\alpha)} v_{(\alpha_{\ell(\alpha)}, \ldots, \alpha_1)}.$$  

When $V$ is equipped with a grading such that the basis elements $v_i$ are homogeneous, the degree of $v_{(\alpha_1, \ldots, \alpha_n)}$ in $V^\otimes n$ is

$$\deg(v_{\alpha_1}) + \deg(v_{\alpha_2}) + \cdots + \deg(v_{\alpha_n}).$$
With such a grading, $S(V)$ becomes a graded connected Hopf algebra. In the rest of this paper, we always assume that $V$ has such a grading and moreover, the set $\{\alpha : \text{deg}(v_\alpha) = n\}$ is finite for all $n$.

Consider the morphism of graded Hopf algebras $\Phi_{S(V)} : S(V) \to \text{QSym}$, that makes $S(V)$ a combinatorial Hopf algebra, defined as

$$
\Phi_{S(V)} : S(V) \to \text{QSym}
$$

$$
v_{(\alpha_1, \ldots, \alpha_\ell)} \mapsto S_{(\text{deg}(v_{\alpha_1}), \ldots, \text{deg}(v_{\alpha_\ell}))}.
$$

The fact that $\Phi_{S(V)}$ is a graded Hopf morphism follows directly from the Hopf structure of the shuffle basis in Theorem 3.5 and 3.6.

The canonical character of $S(V)$, denoted by $\zeta_{S(V)}$ is defined to be the map composition

$$
\zeta_{S(V)} = \zeta_{\text{QSym}} \circ \Phi_{S(V)}.
$$

More explicitly, let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $(\beta_1, \ldots, \beta_\ell) = (\text{deg}(v_{\alpha_1}), \ldots, \text{deg}(v_{\alpha_\ell}))$, then

$$
\zeta_{S(V)}(v_\alpha) = \begin{cases} 
1 & \text{if } \text{EtO}(\beta) = \{p\} \text{ and } \text{OtE}(\beta) = \emptyset \\
\frac{1}{p!(\ell - p)!} & \text{if } \text{EtO}(\beta) = \text{OtE}(\beta) = \emptyset \\
\frac{1}{\ell!} & \text{if } \text{EtO}(\beta) = \emptyset \\
0 & \text{otherwise}.
\end{cases}
$$

In particular, $\zeta_{S(V)}(v_i) = 1$ for all $i$.

4.2 The tensor algebra

Fix a graded vector space $V$ with a homogenous basis $\{v_1, v_2, \ldots\}$. As a vector space, the tensor algebra $T(V)$ is the same as $S(V)$. The Hopf structure of $T(V)$ can be expressed, on basis elements and extended linearly, as follows.

The product is given by concatenation of compositions, that is,

$$
v_\alpha \cdot v_\beta = v_{\alpha\beta}.
$$

The coproduct is given by deshuffle of composition, that is,

$$
\Delta(v_{(\alpha_1, \ldots, \alpha_\ell)}) = \sum_{J \subseteq [\ell]} v_{\alpha_J} \otimes v_{\alpha_{[\ell]\setminus J}}
$$

where $[\ell] = \{1, 2, \ldots, \ell\}$, and if $J = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, then $\alpha_J = (\alpha_{i_1}, \ldots, \alpha_{i_k})$.

Equivalently, $v_i$’s are primitive elements that freely generate $T(V)$.

The antipode formula is given by

$$
S(v_{(\alpha_1, \ldots, \alpha_\ell)}) = (-1)^{\ell(\alpha)} v_{(\alpha_{i(\alpha)}), \ldots, \alpha_1)}.
$$

The tensor algebra and symmetric algebra are graded dual of each other under the Hopf pairing $\langle -, - \rangle : S(V) \times T(V) \to \mathbb{C}$ given by $\langle v_\alpha, v_\beta \rangle = \delta_{\alpha, \beta}$ where $\delta_{\alpha, \beta}$ is the Kronecker delta.
There is a natural morphism of graded Hopf algebras \( \Phi_{T(V)} : T(V) \to \text{Sym} \), that makes \( T(V) \) a combinatorial Hopf algebra, defined, on generators and extended linearly and multiplicatively, as
\[
\Phi_{T(V)} : T(V) \to \text{Sym}
\]
\[
v_i \mapsto p_{\deg(v_i)}.
\]
where \( p \) is the scaled power sum symmetric function. The fact that \( \Phi_{T(V)} \) is a graded Hopf morphism follows from the fact that the scaled power sum functions \( p_n \) are both multiplicative and primitive.

The canonical character of \( T(V) \), denoted by \( \zeta_{T(V)} \) is defined to be the map composition
\[
\zeta_{T(V)} = \zeta_{\text{Sym}} \circ \Phi_{T(V)}.
\]
Equivalently, \( \zeta_{T(V)} \) can be defined, on generators and extended linearly and multiplicatively, as
\[
\zeta_{T(V)} : T(V) \to \mathbb{C}
\]
\[
v_i \mapsto \frac{1}{\deg(v_i)}.
\]

### 4.3 The symmetric algebra

Let \( I(V) \) be the commutator ideal of \( T(V) \) i.e. \( I(V) = \langle v_iv_j - v_jv_i \rangle \) for all pairs of basis elements. Then, the symmetric algebra on \( V \) is defined as
\[
\text{Sym}(V) = \frac{T(V)}{I(V)} = \mathbb{C}\text{-span}\{v_{\alpha} = v_\alpha + I(V)\}.
\]

We will use \( v_{\alpha} \) to denote the equivalence class of \( v \) throughout this paper.

The quotient space \( \text{Sym}(V) \) is a commutative and cocommutative Hopf algebra. Its antipode \( S \) maps each \( v_\alpha \) to \( (-1)^{\ell(\alpha)}v_\alpha \). Since the ideal \( I(V) \) is homogeneous, \( \text{Sym}(V) \) inherits a grading from \( T(V) \) given by \( \deg(v_\alpha) = \deg(v_\alpha) \).

It is easy to see that \( I(V) \subseteq \ker \Phi_{T(V)} \). We have the induced Hopf morphism defined, on generators and extended linearly and multiplicatively, as
\[
\Phi_{\text{Sym}(V)} : \text{Sym}(V) \to \text{Sym}
\]
\[
v_i \mapsto p_{\deg(v_i)}.
\]

The canonical character of \( \text{Sym}(V) \), denoted by \( \zeta_{\text{Sym}(V)} \) is defined to be the induced character
\[
\zeta_{\text{Sym}(V)} : \text{Sym}(V) \to \mathbb{C}
\]
\[
v_i \mapsto \frac{1}{\deg(v_i)}.
\]

For a composition \( \alpha \), let \( m_i \) be the number of parts in \( \alpha \) that are equal to \( i \) i.e. \( m_i = |\{k : \alpha_k = i\}| \), and let \( z_\alpha = m_1!^{m_1}m_2!^{m_2}m_3!^{m_3} \cdots \).

**Theorem 4.1.** The symmetric algebra \( \text{Sym}(V) \) is self-dual under the Hopf pairing
\[
\left\langle \frac{\deg(v_\alpha)!}{\sqrt{z_\alpha}}, \frac{\deg(v_\beta)!}{\sqrt{z_\beta}} \right\rangle = \delta_{\text{sort}(\alpha), \text{sort}(\beta)}
\]
where \( \deg(v_\alpha)! = \deg(\alpha_1)! \deg(\alpha_2)! \cdots \deg(\alpha_{\ell(\alpha)})! \), \( \deg(v_\beta)! = \deg(\beta_1)! \deg(\beta_2)! \cdots \deg(\beta_{\ell(\beta)})! \) and \( \text{sort}(\alpha) \) is the composition obtained by rearranging parts of \( \alpha \) in decreasing order.
It is not hard to see that the Hopf pairing is well-defined i.e. does not depend on the choice of representative. The theorem above follows from the isomorphism between (ungraded) Hopf algebras

\[
\text{Sym}(V) \rightarrow \text{Sym} \\
\frac{v_{\alpha}}{v_{\alpha}} \mapsto p_{\alpha}
\]

and the well-known Hopf pairing on scaled power sum basis

\[
\left\langle \frac{\prod_i \alpha_i}{\sqrt{z_{\alpha}}} p_{\alpha}, \frac{\prod_i \beta_i}{\sqrt{z_{\beta}}} p_{\beta} \right\rangle = \delta_{\text{sort}(\alpha),\text{sort}(\beta)}.
\]

4.4 The Hopf subalgebras of odd and even elements

We call an element \( v_{(\alpha_1, \ldots, \alpha_n)} \) of \( V^\otimes n \) odd if the degree of each \( v_{\alpha_i} \) is odd, and even if the degree of each \( v_{\alpha_i} \) is even.

Let the subspace of odd elements of \( V \) be

\[ O(V) = \mathbb{C}\text{-span}\{v_i : \deg(v_i) \text{ is odd}\}, \]

and let the subspace of even elements be

\[ E(V) = \mathbb{C}\text{-span}\{v_i : \deg(v_i) \text{ is even}\}. \]

Proposition 4.2. (1) The shuffle algebras

\[ S(O(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is odd}\} \]

and

\[ S(E(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is even}\} \]

are Hopf subalgebras of \( S(V) \).

(2) The tensor algebras

\[ T(O(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is odd}\} \]

and

\[ T(E(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is even}\} \]

are Hopf subalgebras of \( T(V) \).

(3) The symmetric algebras

\[ \text{Sym}(O(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is odd}\} \]

and

\[ \text{Sym}(E(V)) = \mathbb{C}\text{-span}\{v_\alpha : v_\alpha \text{ is even}\} \]

are Hopf subalgebras of \( \text{Sym}(V) \).

Proof. All these subspaces are closed under multiplication and comultiplication operations. \( \square \)
We call $S(O(V))$, $T(O(V))$ and $\text{Sym}(O(V))$ the Hopf subalgebras of odd elements of $S(V)$, $T(V)$ and $\text{Sym}(V)$ respectively. We call $S(E(V))$, $T(E(V))$ and $\text{Sym}(E(V))$ the Hopf subalgebras of even elements of $S(V)$, $T(V)$ and $\text{Sym}(V)$ respectively. In this paper, we mostly study the Hopf subalgebra of odd elements since later we show that they are the images of certain theta maps.

**Corollary 4.3.** The Hopf algebras $S(O(V))$ and $T(O(V))$ are graded dual of each other with Hopf pairing $\langle v_\alpha, v_\beta \rangle = \delta_{\alpha, \beta}$ where $\delta_{\alpha, \beta}$ is the Kronecker delta. Additionally, $\text{Sym}(O(V))$ is a self-dual Hopf algebra under the Hopf pairing $\left( \frac{\text{deg}(v_\alpha)!}{\sqrt{z_\alpha}}, \frac{\text{deg}(v_\beta)!}{\sqrt{z_\beta}} \right) = \delta_{\text{sort}(\alpha), \text{sort}(\beta)}$.

**Proof.** Both $S(O(V))$ and $T(O(V))$ are Hopf subalgebras, their basis have the same index set and the Hopf pairing follows from the duality between $S(V)$ and $T(V)$. Similarly, the self-duality and Hopf pairing of $\text{Sym}(O(V))$ are inherited from $\text{Sym}(V)$, as in Theorem 4.1.

We then present the connection between Hopf subalgebras of odd and even elements with the odd and even Hopf subalgebras.

**Proposition 4.4.** We have the following inclusions

1. $S(O(V)) \subseteq S_-(S(V), \zeta_S(V)), S(E(V)) \subseteq S_+(S(V), \zeta_S(V)),$
2. $T(O(V)) \subseteq S_-(T(V), \zeta_T(V)), T(E(V)) \subseteq S_+(T(V), \zeta_T(V)),$
3. $\text{Sym}(O(V)) \subseteq S_-(\text{Sym}(V), \zeta_{\text{Sym}}(V)), \text{Sym}(E(V)) \subseteq S_+(\text{Sym}(V), \zeta_{\text{Sym}}(V)).$

**Proof.** Recall from Theorem 2.3 that $v_\alpha \in S_-(S(V), \zeta_S(V))$ if and only if

$$(\text{id} \otimes \overline{\zeta_S(V)} - \zeta_S(V) \otimes \text{id}) \circ \Delta^{(2)}(v_\alpha) = 0.$$  

Proof of 1. Fix a word $v_\alpha = v_{\alpha_1} \cdots v_{\alpha_\ell}$ such that $\deg(v_{\alpha_i}) = \beta_i$ is odd for all $i$, let $\beta = (\beta_1, \ldots, \beta_\ell)$. Since the coproduct in $S(V)$ is given by deconcatenation, it suffices to show that

$$\left( \overline{\zeta_S(V)} - \zeta_S(V) \right)(v_\alpha) = 0.$$

By definition, $\zeta_{S(V)} = \zeta_{\text{QSym}} \circ \Phi_{S(V)}$ and $\Phi_{S(V)}(v_\alpha) = S_\beta$, we have

$$\left( \overline{\zeta_{S(V)}} - \zeta_{S(V)} \right)(v_\alpha) = \overline{\zeta_{\text{QSym}}(S_\beta)} - \zeta_{\text{QSym}}(S_\beta) = \frac{(-1)^{\deg(v_\alpha) + \ell}}{\ell !} - \frac{1}{\ell !} = 0$$

where the second to last equality follows from equation 3.1 and 3.2. The even part can be proved using similar arguments.

Proof of 3. Since $T(V)$ is freely generated by $\{v_i\}$, and by Theorem 2.3, $S_-(T(V), \zeta_T(V))$ is an algebra, it suffices to show that $v_i \in S_-(T(V), \zeta_T(V))$ for all $v_i$ with odd degree. Since $v_i$ is primitive in $T(V)$, we have $S(v_i) = -v_i$ and hence $\overline{\zeta_T(V)}(v_i) = \zeta_T(V)(v_i)$. Therefore, $\overline{\zeta_T(V) - \zeta_T(V)}(v_i) = 0$ and $v_i \in S_-(T(V), \zeta_T(V))$ follows from

$$(\text{id} \otimes \overline{\zeta_T(V)} - \zeta_T(V) \otimes \text{id}) \circ \Delta^{(2)}(v_i) = 0.$$  

The even part can be proved using similar arguments. Since $\text{Sym}(V)$ is generated by $\overline{v_i}$ with odd degree, the last inclusions can be proved using a similar arguments as well.\qed

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4.5 The theta maps

Consider the linear maps defined on basis elements and extended linearly

\[ \Theta_{S(V)} : S(V) \to S(V), \quad v_\alpha \mapsto \begin{cases} 2^{\ell(\alpha)} v_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \]

\[ \Theta_{T(V)} : T(V) \to T(V), \quad v_\alpha \mapsto \begin{cases} 2^{\ell(\alpha)} v_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \]

\[ \Theta_{\text{Sym}(V)} : \text{Sym}(V) \to \text{Sym}(V), \quad \overline{v}_\alpha \mapsto \begin{cases} 2^{\ell(\alpha)} \overline{v}_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \]

It is easy to see that the images of \( \Theta_{S(V)} \), \( \Theta_{T(V)} \) and \( \Theta_{\text{Sym}(V)} \) are \( S(O(V)) \), \( T(O(V)) \) and \( \text{Sym}(O(V)) \) respectively.

Next, we show that \( \Theta_{S(V)} \), \( \Theta_{T(V)} \) and \( \Theta_{\text{Sym}(V)} \) are theta maps for \( (S(V), \zeta_{S(V)}) \), \( (S(V), \zeta_{T(V)}) \) and \( (S(V), \zeta_{\text{Sym}(V)}) \) respectively. As a consequence, \( S(O(V)) \), \( T(O(V)) \) and \( \text{Sym}(O(V)) \) are peak algebras of \( S(V) \), \( T(V) \) and \( \text{Sym}(V) \) respectively.

**Theorem 4.5.** The map \( \Theta_{S(V)} \) is a graded Hopf morphism and moreover, the following diagram commutes.

\[
\begin{array}{ccc}
S(V) & \xrightarrow{\Phi_{S(V)}} & \text{QSym} \\
\downarrow \Theta_{S(V)} & & \downarrow \Theta_{\text{QSym}} \\
S(O(V)) & \xrightarrow{\Phi_{S(V)}} & \Pi
\end{array}
\]

So the map \( \Theta_{S(V)} \) is a theta map for \( (S(V), \zeta_{S(V)}) \).

**Proof.** The map \( \Theta_{S(V)} \) is an algebra morphism since for odd compositions \( \alpha \) and \( \beta \),

\[
\Theta_{S(V)}(v_\alpha) \Theta_{S(V)}(v_\beta) = 2^{\ell(\alpha)} v_\alpha \cdot 2^{\ell(\beta)} v_\beta = 2^{\ell(\alpha) + \ell(\beta)} \sum_{\gamma \in \alpha \sqcup \beta} v_\gamma = \Theta_{S(V)}(v_\alpha \cdot v_\beta)
\]

and both sides vanish if at least one of \( \alpha \) and \( \beta \) is not odd.

The map \( \Theta_{S(V)} \) is a coalgebra morphism since for odd composition \( \alpha \),

\[
\Delta(\Theta_{S(V)}(v_\alpha)) = \Delta(2^{\ell(\alpha)} v_\alpha) = 2^{\ell(\alpha)} \sum_{\beta \gamma = \alpha} v_\beta \otimes v_\gamma = (\Theta_{S(V)} \otimes \Theta_{S(V)}) \circ \Delta(v_\alpha)
\]

and both sides vanish if \( \alpha \) is not odd.

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By Theorem 3.8, we have
\[
Θ_{\text{QSym}} ∘ Φ_{S(V)}(v_α) = Θ_{\text{QSym}}(S_α) = \begin{cases} 2^{ℓ(α)}S_α & \text{if } α \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}
\]

On the other hand,
\[
Φ_{S(V)} ∘ Θ_{S(V)}(v_α) = \begin{cases} Φ_{S(V)}(2^{ℓ(α)}v_α) & \text{if } α \text{ is odd} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2^{ℓ(α)}S_α & \text{if } α \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}
\]

Hence, the diagram commutes.

\[\square\]

**Theorem 4.6.** The map \(Θ_{T(V)}\) is a graded Hopf morphism and moreover, the following diagram commutes.

\[
\begin{array}{ccc}
T(V) & \xrightarrow{Φ_{T(V)}} & \text{Sym} \\
\downarrow Θ_{T(V)} & & \downarrow Θ_{\text{Sym}} \\
T(O(V)) & \xrightarrow{Φ_{T(V)}} & Ω
\end{array}
\]

So the map \(Θ_{T(V)}\) is a theta map for \((T(V), ζ_{T(V)})\).

**Proof.** The map \(Θ_{T(V)}\) is an algebra morphism since for odd compositions \(α\) and \(β\),
\[
Θ_{T(V)}(v_α)Θ_{T(V)}(v_β) = 2^{ℓ(α)}v_α \cdot 2^{ℓ(β)}v_β = 2^{ℓ(α)+ℓ(β)}v_{αβ} = Θ_{T(V)}(v_α \cdot v_β)
\]
and both sides vanish if at least one of \(α\) and \(β\) is not odd.

The map \(Θ_{T(V)}\) is a coalgebra morphism since for odd composition \(α\),
\[
∆(Θ_{T(V)}(v_α)) = ∆(2^{ℓ(α)}v_α) = 2^{ℓ(α)} \sum_{J \subseteq [ℓ(α)]} v_{α,J} v_{α|J}\partial J = (Θ_{T(V)} ⊗ Θ_{T(V)}) ∘ ∆(v_α)
\]
and both sides vanish if \(α\) is not odd.

By the property of \(Θ_{\text{Sym}}\), we have
\[
Θ_{\text{Sym}} ∘ Φ_{T(V)}(v_α) = Θ_{\text{Sym}}(p_α) = \begin{cases} 2^{ℓ(α)}p_α & \text{if } α \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}
\]

On the other hand,
\[
Φ_{T(V)} ∘ Θ_{T(V)}(v_α) = \begin{cases} Φ_{T(V)}(2^{ℓ(α)}v_α) & \text{if } α \text{ is odd} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2^{ℓ(α)}p_α & \text{if } α \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}
\]

Hence, the diagram commutes.
Theorem 4.7. The map $\Theta_{\text{Sym}(V)}$ is a graded Hopf morphism and moreover, the following diagram commutes.

\[
\begin{array}{ccc}
\text{Sym}(V) & \xrightarrow{\Phi_{\text{Sym}(V)}} & \text{Sym} \\
\downarrow{\Theta_{\text{Sym}(V)}} & & \downarrow{\Theta_{\text{Sym}}} \\
\text{Sym}(O(V)) & \xrightarrow{\Phi_{\text{Sym}(V)}} & \Omega
\end{array}
\]

and so the map $\Theta_{\text{Sym}(V)}$ is a theta map for $(\text{Sym}(V), \xi_{\text{Sym}(V)})$.

Proof. The same arguments as in the proof of Theorem 4.6 can be applied. \qed

Theorem 4.8. The maps $\Theta_{S(V)}$ and $\Theta_{T(V)}$ are adjoint maps of each other. Additionally, the map $\Theta_{\text{Sym}(V)}$ is self-adjoint.

Proof. By definition of $\Theta_{S(V)}$ and $\Theta_{T(V)}$ and the duality of $S(V)$ and $T(V)$, we have

\[
\langle \Theta_{S(V)}(v_\alpha), v_\beta \rangle = \begin{cases} 
2\ell(\alpha) & \text{if } \alpha = \beta \text{ and } \alpha \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\langle v_\alpha, \Theta_{T(V)}(v_\beta) \rangle = \begin{cases} 
2\ell(\alpha) & \text{if } \alpha = \beta \text{ and } \alpha \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]

Similar argument works for $\Theta_{\text{Sym}(V)}$. \qed

5 Examples of theta maps and peak algebras

5.1 The classical cases

Let $V$ be a vector space with a distinguished basis $\{v_1, v_2, \ldots\}$ and consider the grading $\deg(v_i) = i$.

Then we have the following isomorphisms of graded Hopf algebras,

\[
\begin{align*}
S(V) & \cong \text{QSym} \\
v_\alpha & \mapsto S_\alpha, \\
T(V) & \cong \text{NSym} \\
v_\alpha & \mapsto S_\alpha^*, \\
\text{Sym}(V) & \cong \text{Sym} \\
\overline{v_\alpha} & \mapsto p_\alpha.
\end{align*}
\]

Via these isomorphisms, $\Theta_{S(V)}$ and $\Theta_{\text{Sym}(V)}$ can be identified as the maps $\Theta_{\text{QSym}}$ and $\Theta_{\text{Sym}}$ respectively. Additionally, the Hopf subalgebras of odd elements $S(O(V))$ and $\text{Sym}(O(V))$ can be identified as $\Pi$ and $\Omega$ respectively.

On the other hand, the theta map $\Theta_{T(V)}$ can be identified as the map $\Theta_{\text{NSym}} : \text{NSym} \to \text{NSym}$ that
\[ S^*_\alpha \mapsto \begin{cases} 2S^*_\alpha & \text{if } \alpha \text{ is odd} \\ 0 & \text{otherwise}. \end{cases} \]

The map \( \Theta_{\text{NSym}} \) is also defined in \([22, 26]\) in the group algebra \( \mathbb{C}\mathcal{S} \) using descent and peak sets. The equivalence can be deduced from the fact that \( \Theta_{\text{NSym}} \) is the adjoint map of \( \Theta_{\text{QSym}} \).

In particular, we recover the following results

**Corollary 5.1.** The Hopf algebras \( \Pi \) and \( \Omega \) are shuffle algebra and symmetric algebra respectively. Hence, \( \Omega \) is self-dual. Moreover, \( \text{img}(\Theta_{\text{NSym}}) \cong \Pi^* \).

The relations can be seen more clearly from the following commutative diagram. The rectangle on the left is dual to the rectangle on the right.

5.2 The Hopf algebra \( \mathcal{V} \)

As is shown in \([3]\), the associated graded Hopf algebra for any graded cofree coalgebra is a shuffle algebra. In this subsection we present a co-commutative Hopf structure, \( \mathcal{V} \), on permutations that is the associated graded Hopf algebra to the Malvenuto–Reutenauer Hopf algebra. This Hopf algebra is isomorphic to Grossman-Larson Hopf algebra \([14]\) and also appears in \([19, 30]\).

Fix a permutation \( \sigma \in \mathcal{S}_n \), its global descent set \( \text{GD}(\sigma) \) is defined to be \( \{i : \sigma(a) > \sigma(b) \text{ for all } a \leq i < b\} \). For example, \( \text{GD}(5763421) = \{3, 5, 6\} \). The shifted concatenation of two permutations \( \sigma_1 \cdots \sigma_n \) and \( \tau_1 \cdots \tau_m \) is defined as \( \sigma \circ \tau = (\sigma_1 + m) \cdots (\sigma_n + m)\tau_1 \cdots \tau_m \). For example, \( 132 \circ 3421 = 5763421 \). For each permutation \( \sigma \), there is a unique decomposition

\[ \sigma = \sigma^1 \circ \sigma^2 \circ \cdots \circ \sigma^\ell \]

such that \( \sigma^i \) are permutations and \( \text{GD}(\sigma^i) = \emptyset \). For example, \( 5763421 = 32 \circ 12 \circ 1 \circ 1 \). In this subsection, we view a permutation as a word whose letters are permutations without global descent. The degree of a permutation \( \sigma \), denoted by \( \text{deg}(\sigma) \), is \( n \) if \( \sigma \in \mathcal{S}_n \).

As a graded vector space, \( \mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n \) where \( \mathcal{V}_n = \mathbb{C}\text{-span}\{\sigma \in \mathcal{S}_n\} \). For convenience, \( \mathcal{S}_0 \) contains the unique empty permutation \( \emptyset \). The multiplication is given by shuffle of permutations

\[ \sigma \cdot \tau = \sum_{\eta \in \sigma \circ \tau} \eta, \]

and the comultiplication is given by deconcatenation

\[ \Delta(\sigma) = \sum_{\tau \circ \eta = \sigma} \tau \otimes \eta. \]
For example,
\[
132 \circ 12 \cdot 12 \circ 1 = 132 \circ 12 \circ 12 \circ 1 + 132 \circ 12 \circ 1 + 132 \circ 12 \circ 12 \circ 1 + 132 \circ 12 \circ 12 \circ 1 \circ 12 \\
+ 12 \circ 132 \circ 12 \circ 1 + 12 \circ 132 \circ 12 \circ 12 \circ 1 + 12 \circ 12 \circ 1 \circ 132 \circ 12 \circ 12 \circ 1,
\]
\[
\Delta(132 \circ 12 \circ 1) = \emptyset \circ 132 \circ 12 \circ 1 + 132 \circ 12 \circ 1 + 132 \circ 1 \circ 132 \circ 12 \circ 12 \circ 1.
\]
With this Hopf structure, \( V \) is a shuffle algebra. The linear map defined on basis elements
\[
\Phi_V : V \rightarrow \text{QSym}
\]
\[
\sigma^1 \circ \sigma^2 \circ \cdots \circ \sigma^\ell \mapsto S_{(\deg(\sigma^1), \ldots, \deg(\sigma^\ell))}
\]
makes \( V \) a combinatorial Hopf algebra. Then, Theorem 4.5 describes a theta map for \( V \) as follows.

**Corollary 5.2.** The linear map defined on basis elements as
\[
\Theta_V : V \rightarrow V,
\]
\[
\sigma^1 \circ \sigma^2 \circ \cdots \circ \sigma^\ell \mapsto \begin{cases} 2^\ell (\sigma^1 \circ \sigma^2 \circ \cdots \circ \sigma^\ell) & \text{if } \deg(\sigma^i) \text{ is odd for all } i \\
0 & \text{otherwise.}
\end{cases}
\]
is a theta map for \( V \).

The peak algebra for \( V \) is given by \( \Theta_V(V) = \mathbb{C}\text{-span}\{\sigma^1 \circ \cdots \circ \sigma^\ell : GD(\sigma^i) = \emptyset, \deg(\sigma^i) \text{ are odd}\} \). Its Hilbert series is given by
\[
\text{Hilb}(\Theta_V(V)) = 1 + x + x^2 + 4x^3 + 7x^4 + 81x^5 + 164x^6 + \cdots.
\]

### 5.3 The Hopf algebra \( \text{NCSym} \)

The Hopf algebra of symmetric functions in non-commuting variables \( \text{NCSym} = \bigoplus_{n \geq 0} \text{NCSym}_n \) are originally defined in [31]. For more information about what algebraic structures of \( \text{NCSym} \) has been revealed see [4, Introduction].

A set partition of \([n]\) is a set of disjoint non-empty subsets of \([n]\) whose union is \([n]\). A set partition of \([n]\) is said to be atomic if there exists \( i \) such that for any \( a \leq i < b, a \) and \( b \) are not in the same subset. The bases of \( \text{NCSym}_n \) are indexed by set partitions of \([n]\). It is known that \( \text{NCSym} \) is free as an algebra, whose generators are indexed by atomic set partitions. Lauve and Mastnak [18] discovered that there is a set of primitive elements \( \{p_\pi : \pi \text{ atomic}\} \) that freely generates \( \text{NCSym} \), in particular, \( \text{NCSym} \) is a tensor algebra.

The degree of \( \pi \) is \( n \) if \( \pi \) is a set partition of \([n]\). As a consequence of Theorem 4.6, the linear map defined on basis elements as
\[
\Theta_{\text{NCSym}} : \text{NCSym} \rightarrow \text{NCPeak}
\]
\[
p_{\pi_1} \cdots p_{\pi_\ell} \mapsto \begin{cases} 2p_{\pi_1} \cdots p_{\pi_\ell} & \deg(\pi_i) \text{ are odd,} \\
0 & \text{otherwise.}
\end{cases}
\]
is a theta map for \( \text{NCSym} \).

The peak algebra for \( \text{NCSym} \) is given by \( \Theta_{\text{NCSym}}(\text{NCSym}) = \mathbb{C}\text{-span}\{p_{\pi_1} \cdots p_{\pi_\ell} : \deg(\pi_i) \text{ are odd}\} \). Its Hilbert series is given by
\[
\text{Hilb}(\Theta_{\text{NCSym}}(\text{NCSym})) = 1 + x + x^2 + 3x^3 + 5x^4 + 29x^5 + 57x^6 + \cdots.
\]
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