Locally symmetric homogeneous Finsler spaces

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Abstract

Let \((M, F)\) be a connected Finsler space and \(d\) the distance function of \((M, F)\). A Clifford translation is an isometry \(\rho\) of \((M, F)\) of constant displacement, in other words such that \(d(x, \rho(x))\) is a constant function on \(M\). In this paper we consider a connected simply connected symmetric Finsler space and a discrete subgroup \(\Gamma\) of the full group of isometries. We prove that the quotient manifold \((M, F)/\Gamma\) is a homogeneous Finsler space if and only if \(\Gamma\) consists of Clifford translations of \((M, F)\). In the process of the proof of the main theorem, we classify all the Clifford translations of symmetric Finsler spaces.

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1 Introduction

In this paper we give a sufficient and necessary condition for a locally symmetric Finsler space to be globally homogeneous. Specifically, we prove

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Theorem 1.1 Let $\Gamma$ be a properly discontinuous group of isometries of a connected simply connected globally symmetric Finsler space $(M, F)$. Then $(M, F)/\Gamma$ is a homogeneous Finsler space if and only if $\Gamma$ consists of Clifford translations. Further, if $(M, F)/\Gamma$ is homogeneous, and if in the decomposition of $(M, F)$ as the Berwald product of Minkowski space and irreducible symmetric Finsler spaces, none of whose factors is

- a compact Lie group with a bi-invariant Finsler metric,
- an odd-dimensional sphere with standard Riemannian metric,
- a complex projective of odd dimension $> 1$ with standard Riemannian metric,
- $\text{SU}(2n)/\text{Sp}(n)$, $n \geq 2$ with any $\text{SU}(2n)$-invariant Finsler metric (there are non-Riemannian ones), nor
- $\text{SU}(4n + 2)/\text{U}(2n + 1)$, $n \geq 1$, with any $\text{SU}(4n + 2)$-invariant Finsler metric (there are non-Riemannian ones),

then $(M, F)/\Gamma$ is symmetric.

The Riemannian case of Theorem 1.1 was formulated and proved in the second author’s papers [18] for constant sectional curvature and then [19] in general.

Recall that an isometry of a Riemannian manifold is called a Clifford translation if it moves each point of $M$ the same distance. This definition can be generalized in an obvious way to the Finslerian case. Recently, the first author and M. Xu initiated the study of Clifford translations of Finsler spaces in a series papers ([7, 11, 12, 13]). They showed that many important results in the Riemannian case can be generalized to Finsler spaces. Moreover, some new phenomena were found in the Finslerian case. For example, there are some non-Riemannian Finsler spaces which are Clifford homogeneous, in the sense that for any two points $x_1, x_2$ in the manifold, there is a Clifford translation which maps $x_1$ to $x_2$. In view of that result it would be an interesting problem to classify all the Clifford homogeneous Finsler spaces. We note that the Riemannian Clifford homogeneous manifolds were classified by Berestovskii and Nikonorov in [2, 3, 4]. Their list consists of the euclidean spaces, the odd-dimensional spheres with constant curvature, the connected simply connected compact simple Lie groups with bi-invariant Riemannian metrics and the direct products of the above manifolds.

In Section 2, we present preliminaries of Finsler geometry. There we define the notion of orthogonal product of Finsler spaces and deduce a result on the distance function of orthogonal product. It would be an interesting problem to consider whether there is an analogue of the de Rham decomposition of Finsler spaces, as in Riemannian geometry.

The main results of this paper are obtained through the study of Clifford translations of globally symmetric Finsler spaces. In Section 3, the study is reduced
to the case where \((M, F)\) is a Minkowski space, or is of noncompact type, or is of compact type. Clifford translations of Minkowski spaces and symmetric Finsler spaces of noncompact type are also treated in Section 3. That leaves the case where \((M, F)\) is of compact type.

In Section 4 we reduce the proof for compact type to the case where \((M, F)\) is irreducible and of compact type. This is one of the most delicate parts of the proof. In Section 5, the irreducibility allows us to combine results of [19] and [16] with a reduction to the Riemannian case, completing the proof of Theorem 1.1.

Although the main results of this paper are similar to the Riemannian case, the arguments make use of some new ideas. This generalization is important in Finsler geometry, where it is an instance of the principle that the generalization of any important result from Riemannian geometry to Finsler geometry may require a new viewpoint.

2 Preliminaries

In this section, we will recall some definitions and notations in Finsler geometry. In particular, we will survey some results on symmetric Finsler spaces.

2A Finsler spaces

Definition 2.1 Let \(V\) be a \(n\)-dimensional real vector space. A Minkowski norm on \(V\) is a real-valued function \(F\) on \(V\) which is smooth on \(V\setminus\{0\}\) and satisfies the conditions

1. \(F(u) \geq 0, \forall u \in V;\)
2. \(F(\lambda u) = \lambda F(u), \forall \lambda > 0;\) and
3. Given a basis \(u_1, u_2, \ldots, u_n\) of \(V\), write \(F(y) = F(y^1, y^2, \ldots, y^n)\) for \(y = y^1u_1 + y^2u_2 + \cdots + y^nu_n\). Then the Hessian matrix

\[
(g_{ij}) := \left(\frac{1}{2}F^2\right)_{y^iy^j}
\]

is positive-definite at any point of \(V\setminus\{0\}.

The real vector space \(V\) with the Minkowski norm \(F\) is called a Minkowski space, usually denoted as \((V, F)\).
It can be shown that for a Minkowski norm $F$, we have $F(u) > 0$, $\forall u \neq 0$. Furthermore, we have the triangle inequality:

$$F(u_1 + u_2) \leq F(u_1) + F(u_2),$$

where the equality holds if and only if $u_2 = \alpha u_1$ or $u_1 = \alpha u_2$ for some $\alpha \geq 0$. From the triangle inequality one can deduce the fundamental identity:

$$F(w) \geq w^i F_{y^i},$$

with equality holding if and only if there is $\alpha \geq 0$ such that $w = \alpha y$; see [1].

For a Minkowski norm $F$ on the real vector space $V$ we define:

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$ 

Given $y \neq 0$, we can define two tensors on $V$, namely,

$$g_{y}(u, v) = \sum_{i,j} g_{ij}(y) u^i v^j,$$

$$C_{y}(u, v, w) = \sum_{i,j,k} C_{ijk}(y) u^i v^j w^k.$$ 

They are called the fundamental tensor and the Cartan tensor, respectively. Both the fundamental tensor and the Cartan tensor are symmetric. It is easily seen that

$$C_{y}(y, u, v) = 0, \quad \text{for } y, u, v \in V \text{ with } y \neq 0; \quad (2.2)$$

see [1].

**Definition 2.3** Let $M$ be a (connected) smooth manifold. A Finsler metric on $M$ is a function $F: TM \to [0, \infty)$ such that

1. $F$ is $C^\infty$ on the slit tangent bundle $TM \setminus \{0\}$;

2. The restriction of $F$ to any $T_x M$, $x \in M$ is a Minkowski norm.

Let $(M, F)$ be a Finsler space and $x, y \in M$. For any piecewise smooth curve $\sigma(t)$, $0 \leq t \leq 1$ connecting $x$ and $y$, we define the arc length of the curve by

$$L(\sigma) = \int_0^1 F(\sigma(t), \sigma'(t)) dt.$$
The distance function $d$ of $(M, F)$ is defined by

$$d(x, y) = \inf_{\sigma \in \Gamma(x, y)} L(\sigma),$$

where $\Gamma(x, y)$ denotes the set of all piecewise smooth curves emanating from $x$ to $y$. It can be proved that $d(x, y) \geq 0$ with the equality holding if and only if $x = y$ and $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in M$. However, generically we cannot have $d(x, y) = d(y, x)$. Therefore $d$ is not a metric space distance in the usual sense.

2B The Chern connection

Let $(M, F)$ be a Finsler space and $(x^1, x^2, \cdots, x^n)$ a local coordinate system on an open subset $U$ of $M$. Then $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ form a basis for the tangent space at any point in $U$. Therefore we have the coefficients $g_{ij}$ and $C_{ijk}$. Define

$$C^i_{jk} = g^{is}C_{sjk}.$$

The formal Christoffel symbols of the second kind are

$$\gamma^i_{jk} = g^{is}\frac{1}{2}\left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}\right).$$

They are functions on $TU \setminus \{0\}$. We can also define some other quantities on $TU \setminus \{0\}$ by

$$N^i_j(x, y) := \gamma^i_{jk}y^k - C^i_{jk}y^k y^s,$$

where $y = \sum y^i \frac{\partial}{\partial x^i} \in T_x(M) \setminus \{0\}$.

Now the slit tangent bundle $TM \setminus \{0\}$ is a fiber bundle over the manifold $M$ with the natural projection $\pi$. Since $TM$ is a vector bundle over $M$, we have a pull-back bundle $\pi^*TM$ over $TM \setminus \{0\}$.

Theorem 2.4 (Chern [5]) The pull-back bundle $\pi^*TM$ admits a unique linear connection, which is torsion free and almost $g$-compatible. The coefficients of the connection is

$$\Gamma^i_{jk} = \gamma^i_{jk} - g^{il}\frac{1}{2}(A_{jks}N^s_k - A_{jks}N^s_i + A_{kls}N^s_j).$$

In the literature, the above connection is called the Chern connection. Using the Chern connection, we can define the notions of geodesics, exponential map for Finsler spaces as in Riemannian geometry; see [1] for the details. Note that the Chern connection is invariant under isometries of $(M, F)$.

Definition 2.5 A Finsler space $(M, F)$ is called a Berwald space if in a local standard coordinate system the coefficients $\Gamma^i_{jk}$ are functions of $x \in M$ only. In this case, the coefficients $\Gamma$ define an affine connection on the underlying manifold $M$.
Remark 2.6 It was proved by Z. I. Szabó ([17]) that, if $(M, F)$ is a Berwald space, then there exists a Riemannian metric $Q$ on $M$ whose Levi-Civita connection coincides with the linear connection of $(M, F)$. Then of course $(M, Q)$ is unique up to affine diffeomorphism.

2C Symmetric Finsler spaces

Definition 2.7 A Finsler space $(M, F)$ is called locally symmetric if for any $x \in M$, there exists a neighborhood $U$ of $x$ such that the geodesic symmetry on $U$ is a local isometry. It is called globally symmetric if for any $x \in M$, there is an involutive isometry $\rho$ with $x$ as an isolated fixed point.

It is the main result of [10] that any globally symmetric Finsler space (or a complete locally symmetric Finsler space) must be of the Berwald type. Therefore we usually consider the more generalized class of globally or locally affine symmetric Berwald spaces. Recall that a Berwald space $(M, F)$ is called globally (resp. locally) affine symmetric if its connection is globally (resp. locally) affine symmetric. It is easily seen that a reversible globally affine symmetric Berwald space must be globally symmetric. However, there are many symmetric Berwald spaces that are not reversible.

To study affine symmetric Berwald spaces, we introduced the notion of a Minkowski Lie algebra in [10]. From now on, we shall simplify the term “globally (resp. locally) affine symmetric spaces” as “GASBS (resp. LASBS).”

Definition 2.8 Let $(g, \sigma)$ be a symmetric Lie algebra and let $g = h + m$ denote the canonical decomposition of $g$ with respect to the involution $\sigma$. Let $F$ be a Minkowski norm on $m$ such that

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0,$$

for all $y, u, v \in m$ and $x \in h$ with $y \neq 0$, where $g_y$ and $C_y$ are the fundamental form and Cartan tensor of $F$ at $y$, respectively. Then $(g, \sigma, F)$ is called a Minkowski symmetric Lie algebra.

There exists a correspondence between affine symmetric Berwald spaces and Minkowski symmetric Lie algebras, similar to the orthogonal involutive Lie algebra correspondence of the Riemannian case. Each Minkowski Lie algebra must be an orthogonal symmetric Lie algebra with respect to some inner product. Hence any Minkowski symmetric Lie algebra can be decomposed into the direct sum of an abelian ideal, a Minkowski symmetric Lie algebra of compact type and a Minkowski symmetric Lie algebra of noncompact type. Furthermore, a Minkowski
symmetric Lie algebra of compact or noncompact type can be decomposed into the direct sum of irreducible ones (see [10]). From this we deduce the following.

**Theorem 2.9** Let $(M, F)$ be a connected simply connected GASBS. Then $(M, F)$ can be decomposed into the product of a Minkowski space, a GASBS of compact type and a GASBS of noncompact type. Moreover, every simply connected GASBS of compact or noncompact type can be decomposed into the product of irreducible GASBS’s. The decomposition is unique as manifolds but in general is not unique as Finsler spaces.

In the following, we usually say that $(M, F)$ is the Berwald product of a Minkowski space and the irreducible GASBS’s.

### 3 Product decompositions of Clifford translations

In this section and the next, we see how product decompositions of Theorem 2.9 leads to a corresponding product decomposition of Clifford translations. This is much more delicate than the comparable decompositions in the Riemannian case. Here in Section 3 we develop the basics of these decompositions and their specialization to Minkowski spaces and Finsler symmetric spaces of noncompact type. In Section 4 we take a close look at the situation for Finsler symmetric spaces of compact type. Then in Section 5 we will use these product decompositions to complete the proof of Theorem 1.1.

We first adjust some definitions from [16] to fit the Finsler framework.

**Definition 3.1** Let $\sigma$ be a map from a smooth manifold $M$ into itself and $\gamma : \mathbb{R} \to M$ a curve in $M$. We say that $\sigma$ preserves $\gamma$ if there is a constant $c \in \mathbb{R}$ such that $\sigma(\gamma(t)) = \gamma(t + c)$, for all $t \in \mathbb{R}$.

The following lemma extends a result on Riemannian manifolds; see [16].

**Lemma 3.2** Let $(M, F)$ be a Finsler space, $\sigma$ an isometry of $(M, F)$, $\gamma : \mathbb{R} \to M$ a geodesic of constant speed, and $c \in \mathbb{R}$ a constant. The following are equivalent:

1. $\sigma(\gamma(t)) = \gamma(t + c)$ for all $t$, in other words $\sigma$ preserves $\gamma$;
2. $\sigma_* (\dot{\gamma}(t)) = \dot{\gamma}(t + c)$ for all $t$;
3. $\sigma_* (\dot{\gamma}(t)) = \dot{\gamma}(t + a)^\perp$ and $g_T(\sigma_*(\dot{\gamma}(t)), \dot{\gamma}(t + c)) \geq 0$, where $g_T$ is the fundamental tensor, $T$ is the tangent vector field of $\gamma$, and $\perp$ is taken with respect to $g_T$. 

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For the proof, note that a Finsler space isometry sends geodesics to geodesics.

**Theorem 3.3** Let \( \sigma \) be an isometry of a complete Finsler space \((M, F)\). Then the following are equivalent:

1. \( \sigma \) is a Clifford translation of \((M, F)\);
2. If \( x \in M \) there is a minimal geodesic \( \gamma \) from \( x \) to \( \gamma(x) \) preserved by \( \sigma \);
3. If \( x \in M \) and \( \gamma \) is a minimal geodesic from \( x \) to \( \sigma(x) \), then \( \sigma \) preserves \( \gamma \).

The proof of this theorem is similar to Ozols’ proof for the Riemannian case in [16]. Here one just need use \( g_T \), where \( T \) is the tangent vector field of the curves, to replace the inner product, and use the first variation formula in Finsler geometry (see [1], page 123). We omit the details here; see [16].

Now we turn to the product decompositions. We start by defining a new notion of product for Finsler spaces.

**Definition 3.4** Let \((M_i, F_i)\), \( i = 1, \ldots, s \) be Finsler spaces. We say that \((M, F)\) is the orthogonal product of \((M_i, F_i)\) if the following conditions are satisfied:

1. The manifold \( M \) is the product of the manifolds \( M_i \): \( M = M_1 \times M_2 \times \cdots \times M_m \), and the metrics satisfy \( F_i(y) = F(y) \) for all \( i \) and all \( y \in T(M_i) \setminus \{0\} \).
2. Let \( p_i \) be the projection of \( M \) onto \( M_i \). Then a smooth curve \( \gamma(t) \), \( a < t < b \), is a geodesic of \((M, F)\) if and only if the curve \( p_i(\gamma) \) is a geodesic of \((M_i, F_i)\), for all \( i \).
3. Let \( g \) be the fundamental tensor of \( M \) and \( x = (x_1, x_2, \cdots, x_s) \in M \). If \( y \in T_x(M_i) \setminus \{0\} \) and \( v \in T_x(M_j) \) with \( i \neq j \) then \( g_y(y, v) = 0 \).

Notice that geodesics under our consideration are always of constant speed. The product of Riemannian manifolds is a typical example of an orthogonal product. We will prove in the following that the Berwald product of irreducible globally affine Berwald spaces is orthogonal. Notice that the orthogonal product of the given Finsler spaces is generically not unique, as one can see from the Berwald product of GASBS’s (see [10]). We now start to construct these product decompositions.

**Lemma 3.5** Let \((M, F)\) be the orthogonal product of \((M_i, F_i)\), \( i = 1, \ldots, s \). Then \( F(v_1 + v_2 + \cdots + v_s) \geq F(v_1) \), for any \( v_1 \in T(M_1) \setminus \{0\} \) and \( v_k \in T(M_j) \), \( k = 2, \ldots, s \). The equality holds if and only if \( v_k = 0 \), for \( k = 2, \ldots, s \).
Proof. Fix \( x = (x_1, \ldots, x_s) \in M \). Choose a local coordinate system

\[
(x_1^1, \ldots, x_1^{n_1}, x_2^1, \ldots, x_2^{n_2}, \ldots, x_s^1, \ldots, x_s^{n_s})
\]

(3.6)
of \( M \) around \( x \) such that \( (x_1^1, \ldots, x_s^{n_s}) \) is a local coordinate system of \( M_i \) on a neighborhood of \( x_i \). For the convenience of the following we relabel the coordinate system (3.6) as

\[
(z^1, z^2, \ldots, z^{n_1+n_2+\ldots+n_s}).
\]

Denote \( w = v_1 + v_2 + \cdots + v_s \) and \( w_1 = v_2 + \cdots + v_s \). By the fundamental equality we have

\[
F(w) \geq w^j [F_{y^j}(y)]_{y=v_1},
\]

where \( w^j \) is determined by

\[
w = \sum_{j=1}^{n_1+n_2+\ldots+n_s} w^j \frac{\partial}{\partial z^j},
\]

and the automated summation is taken over the range from 1 to \( n_1 + \ldots n_s \). Moreover, the equality holds if and only if there is \( \alpha \geq 0 \) such that \( w = \alpha v_1 \), in other words if and only if \( v_2 = \cdots v_s = 0 \). From this we get that

\[
F(w) \geq (v_1)^j [F_{y^j}(y)]_{y=v_1} + (w_1)^j [F_{y^j}(y)]_{y=v_1} = F(v_1) + (w_1)^j [F_{y^j}(y)]_{y=v_1},
\]

where we have used the Euler’s theorem on homogeneous functions (see [1]). Now using again the Euler’s theorem, we have \( F_{y^j}(y) y^j = 0 \). Thus

\[
g_{v_1}(v_1, w_1) = g_{ij}(v_1)^i(v_1)^j = [F_{y^j}(y) F(y)]_{y=v_1} (v_1)^i(w_1)^j + [F_{y^j}(y)]_{y=v_1} (v_1)^i (w_1)^j
\]

\[
= F(v_1) [F_{y^j}(y)]_{y=v_1} (v_1)^i(w_1)^j + [F_{y^j}(y)]_{y=v_1} (v_1)^i [F_{y^j}(y)]_{y=v_1} (w_1)^j
\]

\[
= F(v_1) (w_1)^j [F_{y^j}(y)]_{y=v_1}.
\]

Since \( (M, F) \) is an orthogonal product of \( (M_i, F_i) \) we have \( g_{v_1}(v_1, w_1) = 0 \). Since \( F(v_1) \neq 0 \) the assertion follows.

Lemma 3.7 Let \( (M, F) \) be a complete Finsler space which is the orthogonal product of the Finsler spaces \( (M_i, F_i) \), \( i = 1, \ldots, s \). For any \( x_j \in M_j \), and \( x_j' \in M_j \), \( j = 1, 2, \ldots, s \), we have

\[
d((x_1, x_2, \ldots, x_s), (x_1', x_2', \ldots, x_s')) \geq d_1(x_1, x_1'),
\]

(3.8)

where \( d, d_1 \) are the distance functions of \( (M, F) \) and \( (M_1, F_1) \), respectively. The equality holds if and only if \( x_j' = x_j \) for \( j = 2, \ldots, s \).
Proof. For any curve \( \gamma \) in \( M \) connecting \( x = (x_1, \ldots, x_s) \) and \( x' = (x'_1, \ldots, x'_s) \), \( p_1(\gamma) \) is a curve in \( M_1 \) connecting \( x_1 \) and \( x'_1 \). By Lemma 3.5 the arc length of \( p(\gamma) \) is less than or equal to \( \gamma \). Taking the infimum we prove the inequality (3.8). Now we prove the second assertion. Select a minimal geodesic \( \gamma_1(t) : a < t < b \) in \( M_1 \) connecting \( x_1 \) and \( x'_1 \). Then by the condition (2) of Definition 3.4, the curve \( t \to (\gamma_1(t), x_2, \ldots, x_s) \) is a geodesic of \((M, F)\), which has the same length of \( \gamma \). Therefore we have

\[
d((x_1, x_2, \ldots, x_s), (x'_1, x'_2, \ldots, x'_s)) \leq d_1(x_1, x'_1).
\]

This proves the “if” part of the assertion. On the other hand, suppose there is one \( x_k \) which is not equal to \( x'_k \), say \( x_2 \neq x'_2 \). Let \( \zeta(t) \), \( a \leq t \leq b \) be a minimal geodesic in \( M \) connecting \((x_1, x_2, \ldots, x_s)\) and \((x'_1, x'_2, \ldots, x'_s)\). Then \( p_1(\zeta) \) is a geodesic in \( M_1 \) connecting \( x_1 \) and \( x'_1 \) and \( p_2(\zeta) \) is a geodesic in \( M_k \) connecting \( x_j \) and \( x'_j \). By the assumption, the tangent vector of \( p_2(\zeta) \) is everywhere nonzero. Then by Lemma 3.5 the length of \( \zeta \), which is just

\[
d((x_1, \ldots, x_s), (x'_1, \ldots, x'_s)),
\]

is larger than that of \( p_{1}(\zeta) \). Therefore we have

\[
d((x_1, \ldots, x_s), (x'_1, \ldots, x'_s)) > d_1(x_1, x'_1).
\]

This completes the proof of the lemma. \( \square \)

We also need a result on conjugate loci and cut loci of GASBS’s. The notions and the fundamental properties of conjugate points and cut points of a general Finsler space can be found in [11]. Suppose \((M, F)\) is a Berwald space and \( x \in M \). As the Chern connection of \((M, F)\) is an affine connection, we can define the notion of conjugate points of \( x \) in exactly the same way as in Riemannian case, through the exponential map. On the other hand, suppose \( x \in M \) and \( \gamma \) is a geodesic emanating from \( x \). A point \( p \) along \( \gamma \) is called a cut point along \( \gamma \) if \( \gamma \) minimizes arc length up to \( p \) but no further. Accordingly, one can define the notions of the conjugate loci and cut loci, as well as the first conjugate locus.

**Lemma 3.9** Let \((M, F)\) be a connected and simply connected GASBS. Then for any \( x \in M \), the first conjugate locus of \( x \) coincides with the cut locus of \( x \).

**Proof.** The proof for the Riemannian case of this result was provided by Crittenden in [6]. As in Remark 2.6, given a GASBS \((M, F)\) there is a Riemannian metric \( Q \) on \( M \) whose Levi-Civit\`a connection coincides with the Chern connection of \((M, F)\). Then \((M, Q)\) is a Riemannian symmetric space. Hence the conjugate points of \( x \) in

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(M, F) and in (M, Q) are the same sets. Based on this observation and Corollary 8.2.2 in [1], which asserts that if \( x' \) is the cut point to \( x \) along a geodesic \( \gamma \), then either \( x' \) is the first conjugate point along \( \gamma \) or there exists at least two distinct geodesics with the same arc length from \( x \) to \( x' \), the proof of Theorem 4 of [6] applies also to the compact Berwaldian case without any change. Thus the lemma is true when the full group of isometries of \((M, F)\) is compact semi-simple. For the general case, notice the following facts:

1. The globally affine symmetric space \((M, F)\) can be decomposed into the product

\[
(M, F) = (M_0, F_0) \times (M_1, F_1) \times (M_2, F_2).
\]

where \((M_0, F_0)\) is a Minkowski space and \((M_1, F_1)\) (resp. \((M_2, F_2)\)) is the products of non-compact (resp. compact) irreducible globally affine symmetric Berwald spaces.

2. The Minkowski space \((M_0, F_0)\) contributes nothing either to the first conjugate loci or cut loci of \( M \).

3. It is proved in [9] that the non-compact \((M_1, F_1)\) has flag curvature everywhere \( \leq 0 \). Hence by the Cartan-Hadamard theorem ([1]), \((M_1, F_1)\) contributes nothing to either the first conjugate loci or the cut loci of \((M, F)\).

Now if \( x' = (x'_0, x'_1, x'_2) \) is the first conjugate point of \( x = (x_0, x_1, x_2) \) along a geodesic \( \gamma \). Then we have \( x'_0 = x_0 \) and \( x'_1 = x_1 \). Thus \( x'_2 \) must be the first conjugate point to \( x_2 \) in \((M_2, F_2)\) along \( p_2(\gamma) \). By the above argument, \( x'_2 \) must be the cut point of \( x_2 \) along \( p_2(\gamma) \) (in \((M_2, F_2)\)). By Lemma [3.7] \( x' \) must be the cut point of \( x \) along \( \gamma \). The converse assertion is obvious.

Here is the next step in reducing the proof of Theorem 1.1 to the flat and the irreducible cases.

**Theorem 3.10** Let \((M, F)\) be a connected simply connected GASBS and suppose that \((M, F)\) is decomposed into the Berwald product

\[
(M, F) = (M_0, F_0) \times (M_1, F_1) \times (M_2, F_2),
\]

where \((M_0, F_0)\) is a Minkowski space and \((M_1, F_1)\) (resp. \((M_2, F_2)\)) is a GASBS of noncompact type (resp. compact type). Suppose \( \sigma \) is a Clifford translation of \((M, F)\). Then there are Clifford translations of \( \sigma_i \) of \((M_i, F_i)\), \( i = 0, 1, 2 \), such that

\[
\sigma = \sigma_0 \times \sigma_1 \times \sigma_2.
\]

Moreover, \( \sigma_0 \) must be an ordinary translation of the affine space underlying \( M_0 \), and \( \sigma_1 \) must be trivial.
Proof. Suppose $\sigma$ is a Clifford translation of $(M, F)$. Then as an isometry of $(M, F)$, $\sigma$ must keep each of the $M_0$, $M_1$ and $M_2$ invariant. Thus there are isometries $\sigma_i$ of $M_i$, $i = 0, 1, 2$, such that $\sigma = \sigma_0 \times \sigma_1 \times \sigma_2$. We now prove that $\sigma_i$, $i = 0, 1, 2$, is a Clifford translation of $(M_i, F_i)$. Let $x_i \in M_i$, $i = 0, 1, 2$, and suppose $\gamma$ is a minimal geodesic of unit speed in $(M, F)$ connecting $x = (x_0, x_1, x_2)$ and $\sigma(x)$. We assert that $\gamma_i = p_i(\gamma)$, $i = 0, 1, 2$, is a minimal geodesic of $(M_i, F_i)$ connecting $x_i$ and $\sigma_i(x_i)$. For $i = 0$ or $1$ this is obvious, since any geodesic must be minimal in $M_0$ or $M_1$. Suppose conversely that $p_2(\gamma)$ is not a minimal geodesic between $x_2$ and $\sigma_2(x_2)$. Suppose $\sigma_2(x_2) = \gamma_2(t_0)$, $t_0 > 0$. Then there is $t_1 \in (0, t_0)$ such that $\gamma_2(t_1)$ is a cut point of $x_2$ along $\gamma_2$. By Lemma 3.3, $\gamma_2(t_1)$ must be the first conjugate point of $x_2$ along the geodesic $\gamma_2$. But then $\gamma(t_1) = (\gamma_0(t_1), \gamma_1(t_1), \gamma_2(t_1))$ must be a conjugate point of $x$ along the geodesic $\gamma$, since the connection of $(M, F)$ is the product of that of the $(M_i, F_i)$, $i = 0, 1, 2$. This implies that $\gamma$ cannot be minimal between $x$ and $\gamma(t)$, for any $t > t_1$. That is a contradiction. Hence $\gamma_i$ are all minimal between $x_i$ and $\gamma_i(x_i)$, $i = 0, 1, 2$. Now by Lemma 3.3 there is a constant $a \in \mathbb{R}$ such that $\sigma(\gamma(t)) = \gamma(t + a)$. Then $\sigma_i(\gamma_i(t)) = \gamma_i(t + a)$, $i = 0, 1, 2$. Using Lemma 3.3 again, we conclude that $\sigma_i$ is a Clifford translation of $(M_i, F_i)$, for $i = 0, 1, 2$.

Suppose $G$ is the full group of isometries of the Minkowski space $(M_0, F_0)$. Let $L$ be the isotropy subgroup of $G$ at the origin 0. Then by Theorem 3.1 of [8], there exists $G$-invariant Riemannian metric $Q$ on $M_0$. Since $G$ must contain all the parallel translations of the linear vector space, $Q$ is a euclidean metric and $G$ is the semidirect product of its translation subgroup with $L$. It follows (see [18]) that a Clifford translation of $(M_0, F_0)$ must be an ordinary translation.

Now we consider $(M_1, F_1)$, which is a GASBS of noncompact type. If the Clifford translation $\sigma_1$ is nontrivial, then for any $x_1 \in M_1$ there is a unique geodesic $\xi_{x_1}$ of unit speed connecting $x_1$ and $\sigma_1(x_1)$. Denote the initial tangent vector of $\xi_{x_1}$ by $X_{x_1}$. We obtain a smooth vector field of $X$ on $(M_1, F_1)$ which has length 1 everywhere. By the above mentioned result of Szabó, there is a Riemannian metric $Q_1$ on $M_1$ whose Levi-Civitá connection coincides with the Chern connection of $(M_1, F_1)$. But then $(M_1, Q_1)$ is a Riemannian symmetric space which is the product of irreducible Riemannian symmetric spaces of the non-compact type. Since both $F_1$ and $Q_1$ are homogeneous metrics on $M_1$, the assumption that the vector field $X$ has constant length 1 with respect to $F_1$ implies that it has bounded length with respect to $Q_1$. Now by the main theorem of the second author’s article [19], $X$ must be a nonzero parallel vector field with respect to the Levi-Civitá connection of $Q_1$. This is impossible. Hence any Clifford translation of $(M_1, F_1)$ must be trivial. This completes the proof of the theorem.

■
4 Clifford translations of compact Finsler symmetric spaces

Now we study Clifford translations of connected simply connected GASBS’s of compact type, completing the considerations of Theorem 3.10 and then reducing the analysis of Clifford translations to the case of irreducible compact Riemannian symmetric spaces.

Recall that if \( x \) is a point in a compact Finsler manifold \((M, Q)\), then a point \( x' \) is called an antipodal point to \( x \) if it is of maximal distance from \( x \). We denote the set of antipodal points of \( x \) by \( A_x \) and call it the antipodal set of \( x \).

**Theorem 4.1** Let \((M, F)\) be a GASBS whose full group of isometries is compact and semisimple. Suppose \( \sigma \) is a Clifford translation of \((M, F)\) and \((M, F)\) has a decomposition
\[
(M, F) = (M_1, F_1) \times \cdots \times (M_s, F_s) \tag{4.2}
\]
where each \((M_j, F_j)\) is an irreducible GASBS. Then \((4.2)\) is an orthogonal product of Finsler spaces, and \( \sigma \) decomposes as
\[
\sigma = \sigma_1 \times \cdots \times \sigma_s
\]
where \( \sigma_j \) is a Clifford translation of \((M_j, F_j)\) for \( j = 1, \ldots, s \).

**Proof.** We first prove that \((4.2)\) is an orthogonal product of Finsler spaces. The condition (1) of Definition 3.4 is obviously satisfied. Since the affine connection of \( M \) is the product of the affine connections of \((M_j, F_j)\), (2) is also satisfied. Now we proceed to prove (3). Let \( g = \mathfrak{g} + p \) and \( g_j = \mathfrak{g}_j + p_j \) be the respective Lie algebras of identity component of the full groups of isometries of the Berwald spaces \((M, F)\) and \((M_j, F_j)\). Suppose that \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) and \( \mathfrak{g}_j = \mathfrak{k}_j + \mathfrak{p}_j \) are the canonical decompositions of \( \mathfrak{g} \) and \( \mathfrak{g}_j \). Then with respect to those decompositions, \((\mathfrak{g}, F)\) and \((\mathfrak{g}_j, F_j)\) are Minkowski symmetric Lie algebras. Fixing a designated origin \( x = (x_1, x_2, \ldots, x_s) \) of \( M \), we identify the tangent space \( T_x(M) \) with \( p \), and each \( T_{x_j}(M_j) \) with \( p_j \). Now suppose \( y \in p \setminus \{0\} \) and \( v \in p_j \setminus \{0\} \) with \( \ell \neq j \). By the definition of a Minkowski Lie algebra, we have (taking \( u = y \))
\[
g_y([w, y], v) + g_y(y, [w, v]) + 2C_y([w, y], y, v) = 0
\]
for any \( w \in \mathfrak{k} \) and \( v \in p_j \). By (2.2), the third term in the left side of the equation is equal to 0. Moreover, \( [w, y] = 0 \), for any \( w \in \bigoplus_{i \neq \ell} \mathfrak{k}_i \). Thus we have \( g_y(y, [w, v]) = 0 \), for any \( w \in \bigoplus_{i \neq \ell} \mathfrak{k}_i \) and \( v \in p_j \). Since \((\mathfrak{g}_j, F_j)\) is an irreducible Minkowski symmetric Lie algebra, we have \([\mathfrak{k}_j, p_j] = p_j\). From this we deduce that \( g_j(y, v) = 0 \), for any \( v \in p_j \). This completes the argument that \((4.2)\) is an orthogonal product of Finsler spaces.
Now the action of $\sigma$ can be written as

$$\sigma(x_1, x_2, \ldots, x_s) = (f_1(x_{\tau(1)}), f_2(x_{\tau(2)}), \ldots, f_s(x_{\tau(s)}),$$

where $\tau$ is a permutation of $(1, 2, \ldots, s)$ and $f_j$ is an isometry from $M_{\tau(j)}$ onto $M_j$. Using a similar argument about the minimal geodesics as above, we can prove that if $\tau$ is the trivial permutation, then each $f_j$ is a Clifford translation of $(M_j, F_j)$. Next we consider the case that $\tau$ is nontrivial. Then $\tau$ is the product of disjoint cycles. To obtain a complete understanding, we first consider the case that $\tau = (1, 2, \ldots, s)$.

Then the Berwald spaces $(M_j, F_j)$ are isometric to each other. Thus we can identify each $(M_j, F_j)$ with $(M_1, F_1)$ and suppose

$$\sigma(x_1, x_2, \ldots, x_s) = (f_1(x_s), x_1, \ldots, x_{s-1}), \quad x_i \in M_1.$$  

As $\sigma$ is Clifford,

$$d(x, \sigma(x)) = d((x_1, x_2, \ldots, x_s), (f_1(x_1), x_1, \ldots, x_{s-1}))$$

is a constant $c > 0$. The choice $x = (x_1, x_1, \ldots, x_1)$ would give us

$$c = d((x_1, x_1, \ldots, x_1), (f_1(x_1), x_1, \ldots, x_1)).$$

By Lemma 3.7, we get that $d_1(x_1, f_1(x_1)) = c$, where $d_1$ is the distance function of $(M_1, F_1)$. On the other hand, if $s > 2$, then we can take $x_2 \neq x_1 = x_s$ in $M_1$. Then we also have

$$c = d((x_1, x_2, \ldots, x_s), (f_1(x_1), x_1, \ldots, x_{s-1})) = d_1(x_1, f_1(x_1)).$$

This is a contradiction with Lemma 3.7. Therefore $s \leq 2$.

If $s = 2$, considering the point $(x_1, x_1)$ we get

$$c = d((x_1, x_1), (f_1(x_1), x_1)) = d_1(x_1, f_1(x_1)).$$

On the other hand, suppose $x_2$ is an antipodal point in $M_1$ of maximal distance from $x_1$ and $l = d_1(x_1, x_2)$. Then we also have

$$c = d((x_2, x_1), (f_1(x_2), x_2)) \geq d_1(x_1, x_2) = l.$$

This implies that $c = l$. Moreover, by Lemma 3.7 we must also have $x_1 = f_1(x_2)$. This means that each point $x_1$ of $(M_1, F_1)$ has a unique antipodal point and $f_1(x_1)$ is exactly the antipodal point of $x_1$. In the following, we denote the (unique) antipodal point of $x \in M_1$ by $A_x$. Then $f_1(x) = A_x$, for any $x \in M_1$. In particular, $f_1$ is a Clifford translation of $(M_1, F_1)$.

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In summary, we have shown that $(M, F)$ has a $\sigma$–invariant decomposition

$$(M, F) = (M_1, F_1) \times \cdots \times (M_k, F_k) \times (M'_1, F'_1) \times \cdots \times (M'_s, F'_s),$$

where $(M_j, F_j)$ is an irreducible GASBS for $j = 1, 2, \ldots, k$ and $(M'_j, F'_j)$ is a Berwald product of two copies of an irreducible GASBS $(N_j, L_j)$ for $j = 1, 2, \ldots, t$. Here $s = k + 2t$, but $k$ or $t$ may be 0. The point is that $\sigma$ decomposes as

$$\sigma = \sigma_1 \times \cdots \times \sigma_k \times \sigma'_1 \times \cdots \times \sigma'_t,$$

where $\sigma_j$ is a Clifford translation of $(M_j, F_j)$ for $j = 1, \ldots, k$, and $\sigma'_j$ is a Clifford translation of $(M'_j, F'_j)$ for $j = 1, \ldots, t$.

Suppose $t \neq 0$. Let $Q_j$ be a Riemannian metric on $N_j$ whose Levi-Civita connection coincides the Chern connection of $L_j$. Using a similar argument on minimal geodesics as above, one can easily show that $\sigma'_j$ is also a Clifford translation of the Riemannian product $(N_j, Q_j) \times (N_j, Q_j)$. Moreover, by the arguments above, $\sigma'_j$ must be of the form $\sigma'_j(x_1, x_2) = (A_{x_1}, x_1)$, where $A_{x_2}$ is the unique antipodal point of $x_2$ with respect to $Q_j$. Now we fix $x_1$ and let $\gamma(t), 0 \leq t \leq l_j$ be a minimal geodesic of $(N_j, Q_j)$ (with unit speed) connecting $x_1$ and $A_{x_1}$. Since

$$d_j(x_1, A_{\gamma(t)}) = d_j(A_{x_1}, A_{\gamma(t)}) = d_j(A_{x_1}, \gamma(t)) = l_j - t,$$

where $d_j$ denotes the distance of $(N_j, Q_j)$, we get that, with respect to the distance $d'_j$ of $(N_j, Q_j) \times (N_j, Q_j)$,

$$d'_j((x_1, \gamma(t)), \sigma'_j(x_1, \gamma(t))) = \sqrt{d_j^2(x_1, A_{\gamma(t)}) + d_j^2(\gamma(t), x_1)} = \sqrt{(l_j - t)^2 + t^2},$$

which is not a constant for $0 \leq t \leq l_j$. Therefore $\sigma'_j$ is not a Clifford translation. This is a contradiction. Hence $t$ must be zero and the proof is completed. ■

5 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 by reducing it to the known Riemannian case [19]. The main tools for that reduction are the Szabó construction of Remark 2.6 and our product decompositions of Sections 3 and 4. We also need a result from Kobayashi and Nomizu [15, Chapter VI, Theorem 3.5]:

**Proposition 5.1** Let $(M, Q)$ be a complete irreducible Riemannian manifold with \(\dim M > 1\). Let $\mathcal{Z}(M, Q)$ denote the group of all isometries of $(M, Q)$, and $\mathfrak{A}(M, Q)$ the group of all affine diffeomorphisms of $(M, Q)$. Then $\mathfrak{A}(M, Q) = \mathcal{Z}(M, Q)$. 

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This gives a bridge between Berwald spaces and Riemannian manifolds:

**Corollary 5.2** Let \((M, F)\) be a complete irreducible Berwald space, \(\dim M > 1\). Let \(\mathcal{I}(M, F)\) denote its isometry group. Let \((M, Q)\) be a Riemannian manifold whose Levi-Civit\`{a} connection coincides with the linear connection of \((M, F)\), as in Remark 2.6. Then \(\mathcal{I}(M, F)\) is a closed subgroup of \(\mathcal{I}(M, Q)\).

**Proof.** Since an isometry of a Berwald space preserves its Chern connection, Proposition 5.1 gives us \(\mathcal{I}(M, F) \subset \mathcal{I}(M, Q)\). These Lie groups carry the compact open topology from their action on \(M\), so \(\mathcal{I}(M, F)\) is closed in \(\mathcal{I}(M, Q)\).

The result is stronger in the symmetric case:

**Theorem 5.3** Let \((M, F)\) be a connected, simply connected irreducible GASBS of compact type, and \((M, Q)\) a Riemannian manifold whose Levi-Civit\`{a} connection coincides with the linear connection of \((M, F)\), as in Remark 2.6. Then \(\mathcal{I}(M, F)\) is a subgroup of finite index in \(\mathcal{I}(M, Q)\) which contains the identity component.

**Proof.** Let \(G\) denote the subgroup of \(\mathcal{I}(M, F)\) generated by products of pairs of symmetries, in other words by transvections. Since \((M, Q)\) is an irreducible Riemannian symmetric space, \(G\) is the identity component of \(\mathcal{I}(M, Q)\).

**Corollary 5.4** Let \((M, F)\) be a connected, simply connected irreducible GASBS of compact type, and \((M, Q)\) a Riemannian manifold whose Levi-Civit\`{a} connection coincides with the linear connection of \((M, F)\), as in Remark 2.6. Let \(\Gamma\) be a subgroup of \(\mathcal{I}(M, F)\). Then the centralizer of \(\Gamma\) in \(\mathcal{I}(M, F)\) is transitive on \(M\) if and only if the centralizer of \(\Gamma\) in \(\mathcal{I}(M, Q)\) is transitive on \(M\).

**Proof.** Since \(M\) is connected, the centralizer of \(\Gamma\) in \(\mathcal{I}(M, F)\) is transitive on \(M\) if and only if its identity component is transitive on \(M\), and that happens if and only if the centralizer of \(\Gamma\) in the identity component \(\mathcal{I}(M, F)^0\) is transitive on \(M\). Similarly the centralizer of \(\Gamma\) in \(\mathcal{I}(M, Q)\) is transitive on \(M\) if and only if the centralizer of \(\Gamma\) in \(\mathcal{I}(M, Q)^0\) is transitive on \(M\). Since \(\mathcal{I}(M, F)^0 = \mathcal{I}(M, Q)^0\) by Theorem 5.3, the assertion follows.

**Lemma 5.5** Let \((M, F)\) be a connected, simply connected irreducible GASBS of compact type, and \((M, Q)\) a Riemannian manifold whose Levi-Civit\`{a} connection coincides with the linear connection of \((M, F)\), as in Remark 2.6. If \(g\) is a Clifford translation of \((M, F)\) then \(g\) is a Clifford translation of \((M, Q)\).

**Proof.** We follow the line of argument of [16]. As we noted in Theorem 5.3 if \(x \in M\) then \(g\) preserves every minimizing \((M, F)\)-geodesic from \(x\) to \(g(x)\). Every
such \((M, F)\)-geodesic \(t \mapsto \gamma_x(t)\) also is a \((M, Q)\)-geodesic. Express \(M\) as the coset space \(G/K_x\) where \(G = \mathcal{Z}(M, F) \subset \mathcal{Z}(M, Q)\) and \(K_x\) is its isotropy subgroup at \(x\). Let \((g, \sigma_x)\) denote the Minkowski symmetric Lie algebra for \((M, F)\) centered at \(x\). So \(g = k_x + m_x\) under the symmetry \(\sigma_x\) at \(x\), and the Lie algebra \(k_x\) of \(K_x\) is the fixed point set of \(\sigma_x\) on \(g\). Since \((M, Q)\) is a Riemannian symmetric space we have \(g = \exp(\xi_x)k_x\) where \(\xi_x \in m_x\) and \(k_x \in K_x\), and we can assume the parameterization of \(\gamma_x\) to be such that \(\gamma_x(t) = \exp(t\xi_x)x\) for all \(t\). As \(g\) preserves \(\gamma_x\) in the sense of Definition 3.1 \(\text{Ad}(k_x)\xi_x = \xi_x\). We have proved the analog of [16, Proposition 2.1(i)] for \((M, F)\). There note \(\text{Pres}(g) = M\), and that the analog of [16, Proposition 2.1(ii)] is vacuous in our special situation.

The considerations of [16] following [16, Proposition 2.1] are purely group-theoretic, and somewhat simplified because here \(\text{Pres}(g) = M\), so we have the \((M, F)\)-analog of the remainder of [16] for our Clifford translation \(g\). In particular the centralizer of \(g\) in \(\mathcal{Z}(M, F)\) is transitive on \(M\). Now the centralizer of \(g\) in \(\mathcal{Z}(M, Q)\) is transitive on \(M\), so \(g\) is a Clifford translation of \((M, Q)\).

\textbf{Proof of Theorem 1.1.}\ By Theorem 3.10 the proof of Theorem 1.1 is reduced to the case where \((M, F)\) is irreducible and of compact type, i.e. where \(\mathcal{Z}(M, F)\) is a compact semisimple Lie group. There, Lemma 5.5 says that if the properly discontinuous group \(\Gamma\) consists of Clifford translations of \((M, F)\) then it consists of Clifford translations of \((M, Q)\). By [19, Theorem 6.1] its centralizer in \(\mathcal{Z}(M, Q)\) is transitive on \(M\), and Corollary 5.4 says that its centralizer in \(\mathcal{Z}(M, F)\) is transitive on \(M\), so \((M, F)\)/\(\Gamma\) is homogeneous. The remaining statement follows directly from [19, Theorem 6.2].

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