The subordinated processes controlled by a family of subordinators and corresponding Fokker–Planck type equations

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Received 17 September 2014
Accepted for publication 29 October 2014
Published 4 December 2014
Online at stacks.iop.org/JSTAT/2014/P12002
doi:10.1088/1742-5468/2014/12/P12002

Abstract. In this work, we consider subordinated processes controlled by a family of subordinators which consist of a power function of a time variable and a negative power function of an $\alpha$-stable random variable. The effect of parameters in the subordinators on the subordinated process is discussed. By suitable variable substitutions and the Laplace transform technique, the corresponding fractional Fokker–Planck-type equations are derived. We also compute their mean square displacements in a free force field. By choosing suitable ranges of parameters, the resulting subordinated processes may be subdiffusive, normal diffusive or superdiffusive.

Keywords: diffusion
1. Introduction

Anomalous diffusion is one of the most universal phenomena in nature [1–3]. In 1D, it is characterized by a mean square displacement (MSD) of the form
\[ \langle (\Delta x)^2 \rangle \sim D t^\alpha \] (1)
with \( \alpha \neq 1 \), deviating from the linear dependence on time found for normal diffusion [1]. The coefficient \( D \) is generalized diffusion constant. It is called subdiffusion for \( 0 < \alpha < 1 \) and superdiffusion for \( \alpha > 1 \) [4].

One of the simplest models to describe anomalous diffusion is continuous time random walk (CTRW), introduced by Montroll and Weiss [5]. Let \( W(x,t) \) be the probability density function (PDF) of finding a particle (with continuous distribution of step length) at point \( x \) at time \( t \), one has [5]
\[ W(x,t) = \sum_n f(x,n)g_n(t), \] (2)
where \( f(x,n) \) is the PDF to find the particle at point \( x \) after \( n \) steps and \( g_n(t) \) is the probability to make exactly \( n \) steps up to time \( t \). Here, the random number of steps \( n(t) \) plays the role of operational time governing the evolution of the system.

A continuous realization of equation (2) is
\[ W(x,t) = \int_0^\infty f(x,\tau)g(\tau,t)d\tau, \] (3)
which has been considered by Fogedby in [6] in terms of a coupled system of Itô stochastic differential equations for position \( x \) and time \( t \)
\[ \frac{dx(\tau)}{d\tau} = F(x) + \Gamma(\tau), \quad \frac{dt(\tau)}{d\tau} = \eta(\tau), \] (4)
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where $F(x)$ is an external force, $\Gamma(\tau)$ and $\eta(\tau)$ are random noise sources that are assumed to be independent. In this frame, the position $x$ and the physical time $t$ are parameterized in terms of the continuous variable $\tau$ which may be considered as operational time. The random walk $Y(t)$ in physical time $t$ is then given by the combined process $Y(t) = x(s(t))$, where $\tau = s(t)$ is the inverse process to $t(\tau)$ defined as

$$s(t) = \inf\{\tau > 0 : t(\tau) > t\},$$

a continuous analogue of the random number of the steps $n(t)$.

In formula (3) $W(x, t)$, $f(x, \tau)$ and $g(\tau, t)$ are the PDFs of the processes $Y(t)$, $x(\tau)$ and $s(t)$, respectively. From the opinion of subordination, which originated in the work of Bochner [7], the combined process $Y(t)$ can be said to be subordinated to parent process $x(\tau)$ using the operational time $\tau = s(t)$. The process $s(t)$ is also called subordinator. It is worth noting that the continuous time random processes in phase (position-velocity) space based on the subordination were discussed in [8–10].

In system (4), the most popular choice on random noise sources $\Gamma(\tau)$, $\eta(\tau)$ is taken $\Gamma(\tau)$ as a Gaussian noise and $\eta(\tau)$ as a fully skew $\alpha$-stable Lévy noise (i.e. $s(t)$ is an inverse $\alpha$-stable subordinator). In recent years, the system of coupled Langevin equations (4) with inverse $\alpha$-stable subordinator has become a hot topic [11–25]. The inverse tempered $\alpha$-stable subordinator and the infinitely divisible inverse subordinators were also considered [26–34].

However, one cannot obtain directly the characteristics of subordinator $s(t)$ from the definition (5). In [35], Meerschaert and Scheffler showed that the $\alpha$-self-similar process $E(t)$ has the same distribution as the process $t^\alpha Z^{-\alpha}$, where $Z$ is an $\alpha$-stable random variable, i.e. $\langle e^{-uZ} \rangle = e^{-u^\alpha}$. Motivated by this, we assume that the subordinator $s(t)$ has the following three forms $s(t) = t^\alpha Z^{-\alpha}$, $s(t) = t^\beta Z^{-\beta}$ or $s(t) = t^\beta Z^{-\beta}$ and then discuss the resulting subordinated processes $x(s(t))$.

This paper is organized as follows. In section 2, we introduce the subordinated process controlled by subordinator $s(t) = t^\alpha Z^{-\alpha} (0 < \alpha < 1)$ and connect the subordinated process $x(s(t))$ with a time-fractional Fokker–Planck equation (FPE). In section 3, we discuss the subordinated process $x(s(t))$ with $s(t) = t^\beta Z^{-\beta} (0 < \alpha < 1, 0 < \beta < 2)$. The subordinated process $x(s(t))$ with $s(t) = t^\beta Z^{-\beta} (0 < \beta < 2)$ is considered in section 4. The conclusions are presented in section 5.

2. The subordinated process with subordinator $s(t) = t^\alpha Z^{-\alpha}$

Assuming that the parent process $x(\tau)$ satisfies the Langevin equation

$$\frac{dx(\tau)}{d\tau} = F(x(\tau)) + \Gamma(\tau),$$

where $\Gamma(\tau)$ is a white Gaussian noise with $\langle \Gamma(\tau) \rangle = 0$, $\langle \Gamma(\tau) \Gamma(\tau') \rangle = 2\delta(\tau - \tau')$. Thus, the PDF $f(x, \tau)$ obeys the following ordinary FPE [36]:

$$\frac{\partial}{\partial \tau} f(x, \tau) = \left(-\frac{\partial}{\partial x} F(x) + \frac{\partial^2}{\partial x^2} \right) f(x, \tau) \equiv L_{FP} f(x, \tau).$$

Now we assume $s(t) = t^\alpha Z^{-\alpha} (0 < \alpha < 1)$, where $Z$ is an $\alpha$-stable random variable with the PDF $\psi_\alpha(t)$, i.e. the Laplace transform $\langle e^{-uZ} \rangle = \int_0^\infty \psi_\alpha(t) e^{-ut} dt = e^{-u^\alpha}$.
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Since

\[ P(s(t) \leq \tau) = P\left( Z \geq \frac{t}{\tau^{1/\alpha}} \right) = \int_{\frac{t}{\tau^{1/\alpha}}}^{\infty} \psi_{\alpha}(t')dt', \]

we have the PDF \( g(\tau, t) \) of subordinator \( s(t) \) as

\[ g(\tau, t) = \frac{t}{\alpha \tau^{1+1/\alpha}} \psi_{\alpha} \left( \frac{t}{\tau^{1/\alpha}} \right). \]

Recalling the following Laplace transform formulas

\[ \int_0^{\infty} \varphi(at)e^{-ut}dt = \frac{1}{a} \int_0^{\infty} \varphi(t')e^{-\frac{u}{a}t'}dt' = \frac{1}{a} \tilde{\varphi}\left( \frac{u}{a} \right), \quad a > 0, \]

and

\[ \int_0^{\infty} t\varphi(t)e^{-ut}dt = -\frac{d}{du} \int_0^{\infty} \varphi(t)e^{-ut}dt = -\frac{d}{du} \tilde{\varphi}(u), \]

we get the Laplace transform of \( g(\tau, t) \) as

\[ \tilde{g}(\tau, u) = u^{\alpha-1} \exp\{-u^\alpha \tau\}. \]

Substituting \( \tilde{g}(\tau, u) \) into equation (3), we get the PDF \( W(x, t) \) of the subordinated process \( Y(t) = x(s(t)) \) in Laplace space:

\[ \tilde{W}(x, u) = \int_0^{\infty} f(x, \tau) \tilde{g}(\tau, u)d\tau \]
\[ = \int_0^{\infty} f(x, \tau)u^{\alpha-1} \exp\{-u^\alpha \tau\}d\tau \]
\[ = u^{\alpha-1} \tilde{f}(x, u^\alpha). \]

Since \( f(x, \tau) \) satisfies equation (7), one gets in Laplace space

\[ u\tilde{f}(x, u) - f(x, 0) = L_{FP}\tilde{f}(x, u). \]

Assuming that \( W(x, 0) = f(x, 0) = \delta(x) \), from equations (13) and (14), we have

\[ u\tilde{W}(x, u) - W(x, 0) = u^\alpha \tilde{f}(x, u^\alpha) - f(x, 0) \]
\[ = L_{FP}\tilde{W}(x, u) \]
\[ = u^{1-\alpha} L_{FP}W(x, u). \]

Taking Laplace inverse transform for \( u \to t \) in equation (15), we obtain the following time-fractional FPE [4, 37–39]

\[ \frac{\partial W(x, t)}{\partial t} = 0 D_t^{1-\alpha} L_{FP}W(x, t), \quad 0 < \alpha < 1, \]

where \( 0 D_t^{1-\alpha} \) denotes the Riemann–Liouville fractional derivative [40], defined by

\[ 0 D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{\alpha-1} f(s)ds. \]
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Equation (16) can be also derived from a generalised master equation [41], or from continuous time random walks (CTRWs) [42].

We are also interested in the MSD of the subordinated process \( Y(t) \) in free force field. One can compute \( \langle x^2 \rangle(t) \) easily in terms of equation (16) with natural boundary conditions:

\[
\langle x^2 \rangle(t) = \int_{-\infty}^{\infty} x^2 W(x, t) dx = \frac{2}{\Gamma(\alpha + 1)} t^\alpha, \quad 0 < \alpha < 1.
\] (18)

From (18), we conclude that the proposed subordinated process is subdiffusive.

3. The subordinated process with subordinator \( s(t) = t^\beta Z^{-\alpha} \)

In this section, we assume that \( x(\tau) \) satisfies Langevin equation (6) and \( s(t) \) has the form \( s(t) = t^\beta Z^{-\alpha} (0 < \alpha < 1, 0 < \beta < 2) \).

Since

\[
P(s(t) \leq \tau) = P\left(Z \geq \frac{t^{\beta/\alpha}}{\tau^{1/\alpha}}\right) = \int_{\frac{t^{\beta/\alpha}}{\tau^{1/\alpha}}} \psi_{\alpha}(t') dt',
\] (19)

we have

\[
g(\tau, t) = \frac{1}{\alpha} \frac{t^{\beta/\alpha}}{\tau^{1+1/\alpha}} \psi_{\alpha}\left(\frac{t^{\beta/\alpha}}{\tau^{1/\alpha}}\right).
\] (20)

Setting

\[
h(\tau, t) = \frac{1}{\alpha} \frac{t}{\tau^{1+1/\alpha}} \psi_{\alpha}\left(\frac{t}{\tau^{1/\alpha}}\right),
\] (21)

one can easily obtain

\[
g(\tau, t) = h(\tau, t_1)
\] (22)

with \( t_1 = t^{\beta/\alpha} \).

Since

\[
W(x, t) = \int_{0}^{\infty} f(x, \tau) g(\tau, t) d\tau,
\] (23)

we have

\[
\frac{\partial}{\partial t} W(x, t) = \int_{0}^{\infty} f(x, \tau) \frac{\partial}{\partial t} g(\tau, t) d\tau
\]

\[
= \int_{0}^{\infty} f(x, \tau) \frac{\partial h(\tau, t_1)}{\partial t_1} \frac{d t_1}{d t} d\tau
\] (24)

\[
= \frac{\partial}{\partial t_1} \int_{0}^{\infty} f(x, \tau) h(\tau, t_1) d\tau \frac{d t_1}{d t}.
\]

doi:10.1088/1742-5468/2014/12/P12002
Setting $W_1(x,t) = \int_0^\infty f(x,\tau)h(\tau,t)d\tau$, by comparing equation (21) with equation (9), we obtain that $W_1(x,t)$ satisfies
\[
\frac{\partial W_1(x,t)}{\partial t} = 0 \quad D_t^{1-\alpha}L_{FP}W_1(x,t).
\] (25)

Substituting equation (25) into equation (24), we have
\[
\frac{\partial}{\partial t}W(x,t) = \frac{\partial}{\partial t_1}W_1(x,t_1) \cdot \frac{dt_1}{dt}
\]
\[
= \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t_1} \int_0^{t_1} (t_1-s)^{\alpha-1} L_{FP}W_1(x,s)ds \cdot \frac{dt_1}{dt}
\]
\[
= \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^{t_1} (t_1-s)^{\alpha-1} L_{FP}W_1(x,s)ds
\] (26)
\[
= \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^{t^{\beta/\alpha}} (t^{\beta/\alpha}-s)^{\alpha-1} L_{FP}W_1(x,s)ds.
\]

After replacing the variable $s$ by $t^{\beta/\alpha}$ and using the relation $W_1(x,t^{\beta/\alpha}) = W(x,t)$ in equation (26), we obtain
\[
\frac{\partial}{\partial t}W(x,t) = \frac{\beta}{\Gamma(\alpha+1)} \frac{\partial}{\partial t} \int_0^{t^{\beta/\alpha}} (t^{\beta/\alpha}-t^{\beta/\alpha})^{\alpha-1} f(t')dt'
\]
\[
= \Phi_t L_{FP}W(x,t),
\] (27)

where
\[
\Phi_t f(t) = \frac{\beta}{\Gamma(\alpha+1)} \frac{\partial}{\partial t} \int_0^{t^{\beta/\alpha}} (t^{\beta/\alpha}-t^{\beta/\alpha})^{\alpha-1} f(t')dt'.
\] (28)

When $\beta = \alpha$, equation (27) reduces into the time fractional FPE (16). It is interesting to note that Mura et al derived the integral form of equation (27) from a non-Markovian forward drift equation and called it stretched time-fractional diffusion equation [43].

The MSD $\langle x^2 \rangle(t)$ of $Y(t) = x(t^{\beta}Z^{-\alpha})$ in a free force field can be computed by
\[
\frac{d}{dt} \langle x^2 \rangle(t) = \int_{-\infty}^{\infty} x^2 \frac{\partial W(x,t)}{\partial t} dx
\]
\[
= \frac{2\beta}{\Gamma(\alpha+1)} \frac{d}{dt} \int_0^{t^{\beta/\alpha}} (t^{\beta/\alpha}-t^{\beta/\alpha})^{\alpha-1} dt'
\] (29)
\[
= \frac{2\beta}{\Gamma(\alpha+1)} \frac{d}{dt} \int_0^{t^{\beta/\alpha}} (t^{\beta/\alpha}-s)^{\alpha-1} ds
\]
\[
= \frac{2\beta}{\Gamma(\alpha+1)} t^{\beta-1}.
\]
So, we have
\[
\langle x^2 \rangle(t) = \frac{2}{\Gamma(\alpha + 1)} t^\beta, \quad 0 < \alpha < 1, 0 < \beta < 2.
\] (30)

By comparing equation (30) with equation (18), we conclude that the diffusive process
\(Y(t)\) is controlled by parameter \(\beta\) in subordinator \(s(t)\). It is easy to confirm that the
subordinated process \(Y(t)\) is subdiffusive when \(0 < \beta < 1\), normal diffusive when \(\beta = 1\)
and supdiffusive when \(1 < \beta < 2\).

4. The subordinated process with subordinator \(s(t) = t^\beta Z^{-\beta}\)

Here we assume that \(x(\tau)\) satisfies Langevin equation (6) and \(s(t)\) has the form
\(s(t) = t^\beta Z^{-\beta}(0 < \beta < 2)\) where \(Z\) is an \(\alpha\)-stable random variable \((0 < \alpha < 1)\).

Since
\[
P(s(t) \leq \tau) = P\left(Z \geq \frac{t}{\tau^{1/\beta}}\right) = \int_{\tau^{1/\beta}}^{\infty} \psi_\alpha(t')dt',
\] (31)
we have
\[
g(\tau, t) = \frac{1}{\beta} \frac{t}{\tau^{1/\beta + 1}} \psi_\alpha \left( \frac{t}{\tau^{1/\beta}} \right).
\] (32)

After using the formulas (10) and (11), one can get the Laplace transform of the PDF
\(g(\tau, t)\)
\[
\tilde{g}(\tau, u) = \frac{\alpha}{\beta} t^{\alpha/\beta - 1} u^{\alpha - 1} \exp\{-u^{\alpha} \tau^{\alpha/\beta}\}.
\] (33)

It is worth noting that the subordinated process \(Y(t)\) constructed in this section can be
connected to a CTRW with correlated waiting times (see [21]) in terms of equation (33).
Indeed, in [21], the authors modeled CTRW with correlated waiting times by the following
coupled Langevin equations
\[
\frac{dx(\tau)}{d\tau} = \Gamma(\tau), \quad \frac{dt(\tau)}{d\tau} = \int_0^\tau M(\tau - \tau')\eta(\tau')d\tau',
\] (34)
where \(M(\tau)\) is a non-negative continuous memory function and \(\eta(\tau)\) is a fully skew
\(\alpha\)-stable Lévy noise. The Laplace transform of the PDF \(g(\tau, t)\) of the process \(s(t)\) defined
by equation (5) has the form
\[
\tilde{g}(\tau, u) = u^{\alpha - 1} \phi(\tau) \exp\{-u^\alpha \phi(\tau)\},
\] (35)
where \(\phi(\tau) = \int_0^\tau d\tau' (\int_0^{\tau'} M(\tau'' - \tau')d\tau'')^\alpha\). By comparing equation (35) with equation (33),
we have \(\phi(\tau) = \tau^{\alpha/\beta}\), which gives the connection between the subordinated process \(Y(t)\)
constructed in this section and the CTRW with correlated waiting times introduced in [21].

Some other kinds of correlated CTRW processes were given in [44–48]. The CTRW
with correlated jumps was studied in [44]. The CTRWs with correlated waiting times
defined as \(t(\tau) = \int_0^\tau |L_\alpha(\tau')|d\tau'\) in [45–47] and with heterogeneous memorized waiting
times defined as \(D_{0+}^\mu \mu t_x(\tau) = |x|^\gamma |L_\alpha(\tau)|\) in [48] are different from that introduced
in [21].
Let us turn to the subordinated process \( Y(t) \). In Laplace space, the PDF \( W(x, t) \) can be represented by

\[
\tilde{W}(x, u) = \int_0^\infty f(x, \tau)\tilde{g}(\tau, u)d\tau. \tag{36}
\]

Inserting equation (33) into equation (36), we have

\[
\tilde{W}(x, u) = \int_0^\infty f(x, \tau)\frac{\alpha}{\beta}\tau^{\alpha/\beta-1}u^{\alpha-1}\exp\{-u^\alpha\tau^{\alpha/\beta}\}d\tau
\]

\[
= u^{\alpha-1}\int_0^\infty f(x, s^\beta) e^{-u^\alpha s} ds
\]

\[
= u^{\alpha-1}\int_0^\infty f_1(x, s) e^{-u^\alpha s} ds
\]

\[
= u^{\alpha-1}\tilde{f}_1(x, u^\alpha),
\]

where \( f_1(x, \tau) = f(x, \tau^{\beta/\alpha}) \). Obviously there exists the relation

\[
\frac{\partial}{\partial \tau} f_1(x, \tau) = \frac{\partial f(x, \tau_1)}{\partial \tau_1} \frac{d\tau_1}{d\tau}, \tag{38}
\]

where \( \tau_1 = \tau^{\beta/\alpha} \).

Since the PDF \( f(x, \tau) \) satisfies equation (7), we have

\[
\frac{\partial}{\partial \tau} f_1(x, \tau) = L_{FP} f(x, \tau_1) \frac{d\tau_1}{d\tau}
\]

\[
= L_{FP} f_1(x, \tau) \frac{\beta}{\alpha} \tau^{\beta/\alpha-1}. \tag{39}
\]

We set \( q(\tau) = \frac{\beta}{\alpha} \tau^{\beta/\alpha-1} \), for which the Laplace transform is \( \tilde{q}(u) = \frac{\Gamma(\beta/\alpha+1)}{u^{\beta/\alpha}} \). Taking Laplace transform for variable \( \tau \) in equation (39), one gets

\[
u\tilde{f}_1(x, u) - f_1(x, 0) = \int_0^\infty L_{FP} f_1(x, \tau) q(\tau) e^{-u\tau} d\tau
\]

\[
= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} L_{FP} \tilde{f}_1(x, s) \tilde{q}(u-s) ds, \tag{40}
\]

with \( 0 < a < Re(u) \).

From equation (37), with the initial condition \( W(x, 0) = f_1(x, 0) = \delta(x) \), we have

\[
u\tilde{W}(x, u) - W(x, 0) = u^\alpha \tilde{f}_1(x, u^\alpha) - f_1(x, 0)
\]

\[
= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} L_{FP} \tilde{f}_1(x, s) \tilde{q}(u^\alpha-s) ds
\]
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\[ W(x, t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} L_{\text{FP}} \tilde{W}(x, s^{1/\alpha}) s^{i-\alpha} \frac{\Gamma(\beta/\alpha + 1)}{(u^\alpha - s)^{\beta/\alpha}} ds \]  

(41)

\[ = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^{i-\alpha} \frac{\Gamma(\beta/\alpha + 1)}{(u^\alpha - s)^{\beta/\alpha}} ds \int_0^\infty L_{\text{FP}} W(x, t') e^{-s^{1/\alpha} t'} dt'. \]

Setting \( \tilde{k}(u, s) = s^{i-\alpha} \frac{\Gamma(\beta/\alpha + 1)}{(u^\alpha - s)^{\beta/\alpha}} \) and taking inverse Laplace transform for \( u \to t \) in equation (41), we obtain

\[ \frac{\partial}{\partial t} W(x, t) = \Phi_t L_{\text{FP}} W(x, t), \]  

(42)

where

\[ \Phi_t h(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} k(t, s) ds \int_0^\infty h(t') e^{-s^{1/\alpha} t'} dt'. \]  

(43)

Next we also compute the MSD \( \langle x^2 \rangle(t) \) of the subordinated process \( Y(t) = x(t^\beta Z^{-\beta}) \) in a free force field.

Since

\[ \langle x^2 \rangle(t) = 2 \int_0^\infty \tau g(\tau, t) d\tau, \]  

(44)

we have

\[ \langle x^2 \rangle(u) = 2 \int_0^\infty \tau g(\tau, u) d\tau \]

\[ = 2 \int_0^\infty \tau^{\alpha/\beta} u^{a-1} \exp\{-u^\alpha \tau^{\alpha/\beta}\} d\tau \]

\[ = 2u^{a-1} \int_0^\infty s^{\beta/\alpha} e^{-u^a s} ds \]  

(45)

\[ = 2\Gamma(\beta\alpha/\alpha) \frac{1}{u^{\beta+1}}. \]

So

\[ \langle x^2 \rangle(t) = \frac{2\Gamma(\beta/\alpha)}{\alpha \Gamma(\beta)} t^\beta. \]  

(46)

From equation (46), we conclude that the subordinated process is subdiffusive when \( 0 < \beta < 1 \), normal diffusive when \( \beta = 1 \) and supdiffusive when \( 1 < \beta < 2 \).

5. Conclusions

In this work, we consider the subordinated process \( Y(t) \) which is controlled by a family of subordinators. For the case of \( s(t) = t^\alpha Z^{-\alpha} \), the PDF \( W(x, t) \) of the subordinated process

doi:10.1088/1742-5468/2014/12/P12002
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$Y(t)$ satisfies a time-fractional Fokker–Planck equation. For the case of $s(t) = t^β Z^{−α}$, the corresponding diffusion equation is a stretched time-fractional Fokker–Planck equation. For the case of $s(t) = t^β Z^{−β}$, we confirm that the Laplace transform of the PDF $g(τ, t)$ is of the form $\tilde{g}(τ, u) = \frac{2}{β} τ^{α/β−1} u^{α−1} \exp\{-u^{α} τ^{α/β}\}$. The resulting process can be connected with CTRW with correlated waiting time.

Acknowledgments

This project was supported by the Natural Science Foundation of China (Grant Nos. 11371016 and 11271311), the Chinese Program for Changjiang Scholars and Innovative Research Team in University (PCSIRT) (Grant No. IRT1179), the Lotus Scholars Program of Hunan province of China.

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