Double cosets for $\text{SU}(2) \times \cdots \times \text{SU}(2)$ and outer automorphisms of free groups

YURI A. NERETIN

Consider the space of double cosets of the product of $n$ copies of $\text{SU}(2)$ with respect to the diagonal subgroup. We get a parametrization of this space, the radial part of the Haar measure, and explicit formulas for the actions of the group of outer automorphisms of the free group $F_{n-1}$ and of the braid group of $n-1$ strings.

1 Introduction

1.1. The group $\text{SU}(2)$. Denote by $\text{SU}(2)$ the group of unitary $2 \times 2$-matrices with determinant $= 2$. A matrix $g \in \text{SU}(2)$ has the form

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad \text{where } |a|^2 + |b|^2 = 1.$$ 

Therefore, we can identify the manifold $\text{SU}(2)$ with the 3-dimensional sphere $S^3$ in $\mathbb{C}^2 \subset \mathbb{R}^4$.

1.2. Double cosets. Denote by $\text{SU}(2)$ the group of unitary $2 \times 2$ matrices with determinant $= 1$. Denote by $G(n)$ the product of $n$ copies of $\text{SU}(2)$. Elements of $G(n)$ are $n$-tuples

$$(g_1, g_2, \ldots, g_n), \quad \text{where } g_j \in \text{SU}(2). \quad (1.1)$$

Denote by $K = K(n) \simeq \text{SU}(2)$ the diagonal subgroup in $G(n)$; elements of $K$ have the form

$$(h, \ldots, h), \quad \text{where } h \in \text{SU}(2).$$

The object of the paper is the space of double cosets

$$\Pi(n) := K \setminus G/K.$$ 

In other words, we consider $n$-tuples up to the equivalence

$$(g_1, g_2, \ldots, g_n) \sim (hg_1q, \ldots, hg_nq), \quad \text{where } h, q \in \text{SU}(2).$$

1.3. Conjugacy classes.

Observation 1.1 There is a canonical one-to-one correspondence between $\Pi(n)$ and the space of conjugacy classes of $G(n-1)$ with respect to the subgroup $K(n-1)$.

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Indeed,
\[(g_1, g_2, \ldots, g_n) \sim (1, g_1^{-1} g_2, \ldots, g_1^{-1} g_n)\]
Next,
\[(1, r_1, \ldots, r_{n-1}) \sim (1, hr_1 h^{-1}, \ldots, hr_{n-1} h^{-1}).\]

1.4. Closed polygonal curves on the sphere. Consider the 3-dimensional sphere $S^3$ endowed with the usual angular distance $d(\cdot, \cdot)$. Fix positive numbers $\theta_1, \ldots, \theta_{n-1}$. Consider a closed polygonal curve $A_1 A_2 \ldots A_{n-1} A_1$ in $S^3$ such that $d(A_j, A_{j+1}) = \theta_j$, $d(A_{n-1}, A_1) = \theta_{n-1}$. Denote by $\mathcal{X}(\theta)$ the set of all such curves defined up to proper rotations of the sphere.

Observation 1.2 There is a one-to-one correspondence between $\mathcal{X}(\theta)$ and the set of $(n-1)$-tuples
\[(1, r_1, \ldots, r_{n-1}), \quad \text{where } r_j \in \text{SU}(2) \quad (1.2)\]
defined up to a simultaneous conjugation and satisfying the conditions:
— the eigenvalues of $r_k$ are $e^{\pm i\theta_k}$;
— $r_1 r_2 \ldots r_{k-1} = 1$.

Indeed, SU(2) can be considered as a 3-dimensional sphere. To a tuple (1.2), we assign the polygonal curve
\[1, r_1, r_1 r_2, \ldots, r_1 r_2 \ldots r_{n-1} = 1.\]

The space $\mathcal{X}(\theta)$ (and its analog for $\mathbb{R}^3$ and the Lobachevsky 3-space) became a subject of investigations after Klyachko’s work [12], see e.g. [11], [?]. Relations of the present work with this literature is not quite clear for the author. Some other related works are Fock, Rosly[7], Fock [6], Goldman [9], Dynnikov [5].

1.5. Spectral forms. For a point of $\Pi(n)$, we write the spectral form
\[Q(\lambda) := \det \left( \sum_j \lambda_j g_j \right) =: \sum_{i,j} s_{ij} \lambda_i \lambda_j.\]

We describe the set $\Xi(n)$ of possible spectral forms. Namely, they satisfy the conditions:
— $Q(\lambda) \geq 0$;
— $\text{rk} Q \leq 4$;
— $s_{jj} = 1$.

If $\text{rk} Q = 4$, its preimage $\in \Pi(n)$ is a two-point set; we have a branching along the surface $\text{rk} Q = 3$.

Note, that points of the surface $\text{rk} Q = 3$ corresponds to smooth points of the quotient space $\Pi(n) = K \backslash G/K$; the singular locus of $\Pi(n)$ corresponds to the surface $\text{rk} Q \leq 2$.

1.6. Radial part of Haar measure. The group $G(n)$ is endowed with the Haar measure. We consider its pushforward to the space $\Xi(n)$. For $n = 3$
we get the usual Lebesgue measure $ds = ds_{12} ds_{13} ds_{23}$ on $\Xi(3)$, see [14]. For $n = 4$ the measure is given by

$$\det(Q(s))^{-1/2} ds,$$

where $ds$ is the Lebesgue measure. For $n \geq 5$ the description of the measure is given in Theorem 3.5.

1.7. The group $\text{Out}(F_k)$, (see [2], [1]). Consider the free group $F_k$ with $k$ generators $c_1, \ldots, c_k$. Denote by $\text{Aut}(F_k)$ the group of automorphisms of $F_k$. Each automorphism $\varphi$ is determined by images of the generators:

$$c_j \mapsto \varphi(c_j) = c_{j_1}^{\varepsilon_{j_1}} c_{j_2}^{\varepsilon_{j_2}} \ldots,$$

where $\varepsilon_j = \pm 1$. Certainly, the collections $\{\varphi(c_j)\}$ are not arbitrary (generally, a formula of the type (1.3) determines a non-surjective and non-injective map $F_k \rightarrow F_k$).

By the Nielsen theorem (see [13]), the group $\text{Aut}(F_k)$ is generated by the following transformations of the set of generators:

a) permutations of generators;

b) the map

$$c_1 \mapsto c_1^{-1}, \quad c_2 \mapsto c_2, \quad c_3 \mapsto c_3,$$

c) the map

$$c_1 \mapsto c_1, \quad c_2 \mapsto c_1 c_2, \quad c_3 \mapsto c_3, \quad c_4 \mapsto c_4,$$

The group $F_k$ acts on itself by interior automorphisms, it is a normal subgroup in $\text{Aut}(F_k)$. We denote by

$$\text{Out}(F_k) := \text{Aut}(F_k)/F_k$$

the group of outer automorphisms of the free group.

1.8. The action of $\text{Out}(F_{n-1})$ on $\Xi(n)$. For a transformation (1.3) we write the following transformation of $\Xi(n)$:

$$\tilde{r}_j = r_{j_1}^{\varepsilon_{j_1}} r_{j_2}^{\varepsilon_{j_2}} \ldots$$

In Section 4 we obtain explicit formulas for the Nielsen generators.

1.9. Braid groups. See an introduction in [10], and [3], [4], [8]. Denote by $\text{Br}_k$ the Artin braid group. It has generators $\sigma_1, \ldots, \sigma_{k-1}$ and relations

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \quad (1.4)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1. \quad (1.5)$$

There is the following Artin embedding $\text{Br}_k \rightarrow \text{Aut}(F_k)$ (see [10]). The element $h_j$ corresponds to the transformation

$$c_j \mapsto c_j c_{j+1} c_j^{-1}, \quad c_{j+1} \mapsto c_{j+1},$$
other generators are fixed. There is a characterization (the Artin theorem) of the image of $\text{Br}_k$ in $\text{Aut}(F_k)$. Namely, $\kappa \in \text{Out}(F_k)$ is contained in $\text{Br}_k$ if

1) $\kappa$ sends each generator $c_j$ to

$$A_j^{-1}c_{\xi(j)}A_j$$

where $A_j \in F_k$ and $\xi$ is a permutation of generators.

2) $\kappa$ sends $c_1 \ldots c_k$ to itself.

In particular, we get the map $\text{Br}_k \to \text{Out}(F_k)$. It is not injective (see, e.g., [8]), the kernel is generated by

$$\left((\sigma_1\sigma_2\ldots\sigma_{k-1})(\sigma_1\sigma_2\ldots\sigma_{k-2})\ldots\sigma_1\right)^2.$$

In Section 4 we get explicit formulas for the action of generators of the braid group in the terms of spectral forms.

1.10. Pure braids. The relations

$$\sigma_j^2 = 1$$

together with (7)–(??) determine the symmetric group $S_n$. Therefore we get the homomorphism of $\text{Br}_n \to S_n$. The kernel is called the group of pure braids. The group of pure braids acts on $F_n$ by transformations of the form

$$c_j \mapsto A_j c_j A_j^{-1}$$

recall that $c_1 \ldots c_n \mapsto 1 \ldots c_n$. This implies the following observation

**Observation 1.3** The group of pure braids acts on the space $X(\theta)$ of closed polygonal curves.

1.11. The structure of the paper. In Section 2, we get the characterization of spectral forms. Section 3 contains evaluation of the radial part of the Haar measure. In Section 4, we write out actions of the the groups $\text{Out}(F_{n-1})$ and $\text{Br}_{n-1}$.

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2 Spectral forms

2.1. Spectral forms and the map $\zeta$. For any element of $\Pi(n)$ we write out the quadratic form

$$Q(\lambda_1, \ldots, \lambda_n) = \det(\sum \lambda_j g_j) = \det \left( \sum \lambda_j \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} \right)$$

We denote by $\zeta$ the map from $\Pi(n)$ to the space of quadratic forms.
Proposition 2.1  a) We get a well-defined map from the space $\Pi(n)$ to the space of quadratic forms.

b) Coefficients of $Q$ are real, coefficients in the front of $\lambda_j^2$ are 1.

c) $Q$ is positive semidefinite.

d) The rank of $Q$ is $\leq 4$.

Proof. Indeed, for real $\lambda$,

$$Q(\lambda) = \left(\sum \lambda_j a_j\right) \left(\sum \lambda_j \overline{a_j}\right) + \left(\sum \lambda_j b_j\right) \left(\sum \lambda_j \overline{b_j}\right) = \left|\sum \lambda_j a_j\right|^2 + \left|\sum \lambda_j b_j\right|^2 = \sum \lambda_j^2 + 2 \sum_{i<j} \text{Re}(a_j \overline{a_j} + b_i \overline{b_j})$$

and all the statements become obvious. \(\square\)

2.2. A description of $\Pi(n)$. Denote by $\Xi = \Xi(n)$ the set of all quadratic forms $Q$ satisfying the conditions of the previous statement.

Obviously,

$$\zeta(g_1^t, \ldots, g_n^t) = \zeta(g_1, \ldots, g_n),$$

where $^t$ denotes the transposed matrix.

Theorem 2.2  a) The map $\zeta : \Pi(n) \to \Xi(n)$ is surjective.

b) The $\zeta$-preimage of a point $Q \in \Xi$ consists of two points if $\text{rk} Q = 4$ and of one point if $\text{rk} Q \leq 3$.

c) Moreover, $\text{rk} Q \leq 3$ iff $(g_1^t, \ldots, g_n^t)$ and $(g_1, \ldots, g_n)$ represent one point of $\Pi$.

Proof. For a positive semi-definite quadratic form

$$Q(\lambda) = \sum_{kl} s_{kl} \lambda_k \lambda_l$$

on $\mathbb{R}^n$ there is a collection (a configuration) of vectors $v_j$ in a Euclidean space such that

$$\langle v_k, v_j \rangle = s_{kj}$$

Since $\text{rk} Q \leq 4$, this configuration can be realized in $\mathbb{R}^4$. Since $s_{jj} = 1$, these vectors lie on the unit sphere.

Moreover such a configuration $v_j \in \mathbb{R}^4$ is unique up to the action of the orthogonal group $O(4)$.

Recall that $\text{SU}(2)$ can be identified with the 3-dimensional sphere $S^3$, then $s_{ij}$ are inner products of points of the sphere. Recall that $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)/\{\pm 1\}$. In other words, proper isometries of the sphere $S^3$ correspond to the left-right action of $\text{SU}(2) \times \text{SU}(2)$ on $\text{SU}(2)$ (see, e.g. [15]).

If $\text{rk} Q = 4$, then $v_j$ are not contained in a 3-dimensional hyperplane. Therefore an improper isometry of the sphere gives a non-equivalent configuration in $\Pi(n)$.

If $\text{rk} Q \leq 3$, then the point configuration $v_j$ is contained in a hyperplane. The reflection with respect to the hyperplane fix this configuration. \(\square\)
3 The radial part of the Haar measure

3.1. A reduction. As we noted above, any element of the double coset space $\Pi(n)$ can reduced to the form $(1, g_2, \ldots, g_n)$ and $g_j$ are determined up to a simultaneous conjugation.

Proposition 3.1 Each element of $\Pi(n)$ has a representative of the form

$$\left\{ \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ -\overline{b_2} & \overline{a_2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{n-1} & b_{n-1} \\ -\overline{b_{n-1}} & \overline{a_{n-1}} \end{pmatrix} \end{array} \right\},$$

where $b_2 \geq 0$ and $0 \leq \varphi \leq \pi$. (3.1)

For an element in a general position such a representative is unique.

Indeed, after a reduction of $g_2$ to a diagonal form, we can conjugate our tuple by diagonal matrices.

3.2. The Haar measure on SU(2). We can regard the group SU(2) as the unit sphere in the Euclidean space $\mathbb{C}^2$. The Haar measure on SU(2) is the usual surface Lebesgue measure on the sphere. We denote this measure by $dg$.

Recall the following simple facts.

Proposition 3.2 a) The image of the Haar measure under the map $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto (a, \theta)$ is the Lebesgue measure $da \, d\theta$ on the circle $|a| \leq 1$.

b) Represent $b$ in the form $b = \rho e^{i\theta}$. Then the image of the Lebesgue measure under the map $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto (a, \theta)$ is $\text{const} \cdot d\theta \, da \, d\overline{a}$.

c) Consider the map taking a matrix $g$ to its collection of eigenvalues $e^{i\varphi}, e^{-i\varphi}$, where $0 \leq \varphi \leq \pi$. The image of the Haar measure under the map $g \mapsto \varphi$ is $\sin^2 \varphi \, d\varphi.$

Corollary 3.3 The pushforward of the Haar measure in the coordinates (3.1) is

$$\sin^2 \varphi \, d\varphi \, da_2 \, d\overline{a}_2 \, dg_3 \ldots \, dg_n$$

3.3. Coordinates. n-tuples of matrices. To be definite, take $n = 5$,

$$\left( g_1, g_2, g_3, g_4, g_5 \right) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ -\overline{b_1} & \overline{a_1} \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ -\overline{b_2} & \overline{a_2} \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ -\overline{b_3} & \overline{a_3} \end{pmatrix}, \right)$$
where $0 \leq \varphi \leq \pi$. We also denote

$$a_1 = x_1 + iy_1 \quad a_2 = x_1 + iy_2 \quad a_3 = x_3 + iy_3,$$

$$b_1 = e^{i\theta_1} \sqrt{1 - x_1^2 - y_1^2} \quad b_2 = e^{i\theta_2} \sqrt{1 - x_2^2 - y_2^2} \quad b_3 = e^{i\theta_3} \sqrt{1 - x_3^2 - y_3^2}.$$  

(3.2) (3.3)

For $(g_1, g_2, g_3, g_4, g_5) \in \Pi(n)$ the numbers

$$\theta_1 - \theta_2, \quad \theta_2 - \theta_3, \quad \theta_1 - \theta_3$$

make sense (but not $\theta_1, \theta_2, \theta_3$ themselves).

### 3.4. Coordinates. Spectral forms.

Consider the spectral form and denote its coefficients in the following way:

$$\det \left( \sum \lambda g_1 + \mu g_2 + \nu_1 g_3 + \nu_2 g_4 + \nu_3 g_5 \right) =:\lambda^2 + \mu^2 + \nu_1 \nu_2 + 2q_1 \lambda \nu_1 + 2q_2 \lambda \nu_2 + 2q_3 \lambda \nu_3 + 2r_1 \mu \nu_1 + 2r_2 \mu \nu_2 + 2r_3 \mu \nu_3 + 2t_{12} \nu_1 \nu_2 + 2t_{13} \nu_1 \nu_3 + 2t_{23} \nu_2 \nu_3.$$  

(3.4)

The matrix of the form is

$$\Delta = \begin{pmatrix} 1 & p & q_1 & q_2 & q_3 \\ p & 1 & r_1 & r_2 & r_3 \\ q_1 & r_1 & 1 & t_{12} & t_{13} \\ q_2 & r_2 & t_{12} & 1 & t_{23} \\ q_3 & r_3 & t_{13} & t_{23} & 1 \end{pmatrix}.$$  

(3.5)

Then

$$p = \cos \varphi$$

$$q_j = x_j$$

$$r_j = x_j \cos \varphi + y_j \sin \varphi$$

$$t_{ij} = x_i x_j + y_i y_j + \sqrt{1 - x_i^2 - y_i^2} \sqrt{1 - x_j^2 - y_j^2} \cos(\theta_i - \theta_j)$$

(3.6) (3.7) (3.8)

It is easy to write the inverse map:

$$\varphi = \arccos p$$

$$x_j = q_j$$

$$y_j = \frac{r_j - q_j p}{\sqrt{1 - p^2}}$$

$$\theta_i - \theta_j = \pm \arccos \frac{\det \begin{pmatrix} 1 & p & q_i \\ p & 1 & r_i \\ q_i & r_i & 1 \end{pmatrix}}{\det^{1/2} \begin{pmatrix} 1 & p & q_i \\ p & 1 & r_i \\ q_i & r_i & 1 \end{pmatrix} \det^{1/2} \begin{pmatrix} 1 & p & q_j \\ p & 1 & r_j \\ q_j & r_j & 1 \end{pmatrix}}$$

(3.9) (3.10) (3.11) (3.12)

7
The last formula requires some calculations. For this reason, we present some intermediate formulas:

\[
1 - x_1^2 - y_1^2 = \frac{\det \begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_1 & r_1 & 1 \end{pmatrix}}{1 - p^2}, \tag{3.13}
\]

\[
t - x_1 x_2 - y_1 y_2 = \frac{\det \begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_2 & r_2 & t \end{pmatrix}}{1 - p^2}.
\]

Note, that we can not reconstruct the sign of \(\theta_i - \theta_j\) from the formula (3.12). Recall that the substitution

\[
\theta_1 \mapsto -\theta_1, \quad \theta_2 \mapsto -\theta_2, \quad \theta_3 \mapsto -\theta_3
\]

corresponds to the simultaneous transposition

\[
(g_1, g_2, g_3, g_4, g_5) \mapsto (g_1^t, g_2^t, g_3^t, g_4^t, g_5^t).
\]

### 3.5. What happens if we forget \(t_{23}\)?

Let we know \(p\), all \(q_j\), all \(r_j\), and \(t_{12}, t_{13}\). Then we can reconstruct \(\varphi, x_j, y_j\) and

\[
\cos(\theta_1 - \theta_2), \quad \cos(\theta_1 - \theta_3).
\]

Without loss of a generality, we can assume \(\theta_1 = 0\). Then we know \(\pm \theta_2\) and \(\pm \theta_3\) and there are two possible variants for \(|\theta_2 - \theta_3|\).

It can be readily checked that there exist \(h \in SU(2)\) such that

\[
h^{-1} g_2 h = g_2^t, \quad h^{-1} g_3 h = g_3^t
\]

(recall that \(g_1\) is the unit matrix). Then without \(t_{23}\) we can not distinguish

\[
(g_1, g_2, g_3, g_4, g_5) \quad \text{and} \quad (g_1, g_2, g_3, g_4, hg_5^t h^{-1}). \tag{3.14}
\]

### 3.6. The radial part of the Haar measure. The cases \(n = 3, n = 4\).

**Theorem 3.4**

\(a)\) Let \(n = 3\). The pushforward of the Haar measure under the map \(\zeta : \Pi(3) \to \Xi(3)\) is

\[
\text{const} \cdot dp dq_1 dr_1.
\]

\(b)\) Let \(n = 4\). Then the image of the Haar measure under the map \(\zeta : \Pi(4) \to \Xi(4)\) is

\[
\text{const} \cdot \det \begin{pmatrix} 1 & p & q_1 & q_2 \\ p & 1 & r_1 & r_2 \\ q_1 & r_1 & 1 & t_{12} \\ q_2 & r_2 & t_{12} & 1 \end{pmatrix}^{-1/2} \quad dp dq_1 dr_1 dr_2 dt_{12}.
\]

\footnote{See, also \[9\].}
Proof. Consider the case \( n = 4 \). The radial part of the Haar measure in the coordinates \( \varphi, x_1, y_1, x_2, y_2, \theta \) is given by
\[
\text{const} \cdot \sin^2 \varphi \, d\varphi \, dx_1 \, dy_1 \, dx_2 \, dy_2 \, d\theta
\]
Next, we must write the Jacobian of the map \([3.9]–[3.12]\). Evidently, the Jacobian is
\[
\frac{\partial \varphi}{\partial p} \cdot \frac{\partial \theta}{\partial t},
\]
this can be easily evaluated.

3.7. The Haar measure, general case. For an \( n \)-tuple \((g_1, \ldots, g_n)\) consider its spectral form
\[
\det \left( \sum_j \lambda_j g_j \right) =: \sum_j \lambda_j^2 + 2 \sum_{i<j} s_{ij} \lambda_i \lambda_j
\]

Theorem 3.5

a) The coefficients \( s_{12}, s_{13}, s_{23} \) are distributed as
\[
\text{const} \cdot ds_{12} \, ds_{13} \, ds_{23}
\]
in the domain \( \begin{pmatrix} 1 & s_{12} & s_{13} \\ s_{12} & 1 & s_{23} \\ s_{13} & s_{23} & 1 \end{pmatrix} \geq 0 \).

b) For fixed \( s_{12}, s_{13}, s_{23} \) in a general position, a vector \( v_j := (s_{1j} \ s_{2j} \ s_{3j}) \) is distributed as
\[
\text{const} \cdot \det(\Delta_j)^{-1/2} \, ds_{j1} \, ds_{j2} \, ds_{j3},
\]
where
\[
\Delta_j = \begin{pmatrix} 1 & s_{12} & s_{13} \\ s_{12} & 1 & s_{23} \\ s_{13} & s_{23} & 1 \end{pmatrix}.
\]
A vector \((s_{1j} \ s_{2j} \ s_{3j})\) ranges in the domain \( \Delta_j \geq 0 \).

c) The random variables \( v_4, v_5, \ldots, v_n \) are independent.

d) Let us fix \( s_{1j}, s_{2j}, s_{3j} \) for all \( j \). For such a collection in a general position there are 2 equiprobable variants of a choice of \( s_{4j} \). These samplings are independent for \( j = 5, 6, \ldots \).

e) Let us fix \( s_{1j}, s_{2j}, s_{3j} \) for all \( j \) and fix \( s_{4j} \). Then a.s. all other variables \( s_{ij} \) are uniquely determined.

Proof. The statements a)-b) are a rephrasing of Theorem 3.4.

Next, for fixed \( g_1, g_2, g_3 \) the matrices (‘random variables’) \( g_3, g_4, \ldots \) are independent. A matrix \( g_j \) determines a vector \( v_j \), and \( g_j \) is uniquely determined by a vector \( v_j \). This proves c).
Now we write the matrix of a spectral form
\[
\begin{pmatrix}
1 & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & \ldots \\
 s_{12} & 1 & s_{23} & s_{24} & s_{25} & s_{26} & \ldots \\
 s_{13} & s_{23} & 1 & s_{34} & s_{35} & s_{36} & \ldots \\
 s_{14} & s_{24} & s_{34} & 1 & ?_1 & ?_2 & \ldots \\
 s_{15} & s_{25} & s_{35} & ?_1 & 1 & * & \ldots \\
 s_{16} & s_{26} & s_{36} & ?_2 & * & 1 & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The left upper $5 \times 5$ minor is zero. This gives a quadratic equation for $s_{45}$. Both the solutions are admissible, they correspond to collections of matrices given by (3.14). We repeat the same operation for $s_{46}$ etc.

In the terminology of Subsections 3.4–3.5, the knowledge of the first three rows gives us $\pm \theta_1, \pm \theta_2, \pm \theta_3$ etc. All these choices are equiprobable.

Having all $s_{4j}$, we get a unique way to complete the matrix to the matrix of rank 4.

4 Actions of Out($F_{n-1}$) and braid group

4.1. The action of Out($F_{n-1}$) on $\Pi(n)$. We regard $\Pi(n)$ as the set of collections $(1, g_2, \ldots, g_n)$ defined up to a simultaneous conjugation.

The Nielsen transformations act on $\Pi[n]$ in the obvious way. We get:

a) permutations of $g_j$;

b) the transformation $g_2 \mapsto g_2^{-1}$;

c) the map $(g_2, g_3, g_4, \ldots) \rightarrow (g_2, g_2 g_3, g_4, \ldots)$.

To be definite, take $n = 5$. The action of permutations is obvious.

**Proposition 4.1** The transformation $g_2 \mapsto g_2^{-1}$ corresponds to the map $\Xi(5) \rightarrow \Xi(5)$ given by

\[
\begin{align*}
\bar{p} &= p; \\
\bar{q}_j &= q_j; \\
\bar{t}_{ij} &= t_{ij} \\
\bar{r}_j &= -r_j + 2pq_j
\end{align*}
\]

**Proof.** In the notation (3.2)–(3.3), (3.5)–(3.8), we have

\[
\bar{r}_1 = x_1 \cos \varphi - x_1 \sin \varphi = r_1 - 2y_1 \sin \varphi.
\]

On the other hand,

\[
y_1 \sin \varphi = r_1 - x_1 \cos \varphi = r_1 - pq_1.
\]

This completes the calculation. \qed
Theorem 4.2 The transformation

\[(1, g_2, g_3, g_4, g_5) \mapsto (1, g_2g_3, g_4, g_5)\]

corresponds to the map \(\tilde{\Xi}(5) \rightarrow \Xi(5)\) given by

\[
\begin{align*}
\tilde{p} &= p; \\
\tilde{q}_2 &= q_2, \quad \tilde{q}_3 = q_3; \\
\tilde{q}_1 &= -r_1 + 2pq_1; \\
\tilde{r}_2 &= r_2, \quad \tilde{r}_3 = q_3; \\
\tilde{t}_{1j} &= pt_{1j} - q_jr_1 + q_1r_j \mp \det \begin{pmatrix} 1 & p & q_1 & q_j & q_r \\ p & 1 & r_1 & r_j & r_t \\ q_1 & r_1 & 1 & t_{1j} \\ q_j & r_j & t_{1j} & 1 \end{pmatrix}^{1/2}, \text{ where } j = 2, 3. \quad (4.1)
\end{align*}
\]

\[\tilde{t}_{23} = t_{23}\]

The group \(\text{Out}(F_n)\) acts on \(\tilde{\Xi}(5)\) and not on \(\Xi(5)\) and the choice of signs \(\mp\) requires explanations. They are given below.

PROOF. First, we write the coefficients of the spectral form for the transformed collection. Only the variables \(q_1, r_1, t_{12}, t_{13}\) change. We have

\[
\begin{align*}
\tilde{r}_1 &= x_1 \cos \varphi + y_1 \sin \varphi = \\
&= (x_1 \cos \varphi - y_1 \sin \varphi) \cos \varphi + (y_1 \cos \varphi + x_1 \sin \varphi) \sin \varphi = x_1 = q_1.
\end{align*}
\]

and

\[
\begin{align*}
\tilde{q}_1 &= x_1 \cos \varphi - y_1 \sin \varphi = q_1p - (r_1 - q_1p) = -r_1 + 2q_1p. \quad (4.2)
\end{align*}
\]

The evaluation of \(\tilde{t}_{1j}\) is heavier,

\[
\begin{align*}
\tilde{t}_{12} &= \tilde{x}_1\tilde{x}_2 + \tilde{y}_1\tilde{y}_2 + \sqrt{1 - \tilde{x}_1^2 - \tilde{y}_1^2} \sqrt{1 - \tilde{x}_2^2 - \tilde{y}_2^2} \cos(\tilde{\vartheta}_1 - \tilde{\vartheta}_2).
\end{align*}
\]

By construction, \(\tilde{x}_2 = x_2, \tilde{y}_2 = y_2, \tilde{\vartheta}_2 = \vartheta_2\). Next,

\[
\tilde{a}_1 = \tilde{x}_1 + i\tilde{y}_1 = (x_1 + iy_1) e^{i\varphi}
\]

and therefore

\[
1 - \tilde{x}_1^2 - \tilde{y}_1^2 = 1 - x_1^2 - y_1^2.
\]

Also, \(\tilde{\vartheta}_1 = \vartheta_1 + \varphi\). Denote \(\theta := \vartheta_1 - \vartheta_2\). Thus,

\[
\tilde{t}_{12} = \tilde{x}_1x_2 + \tilde{y}_1y_2 + \sqrt{1 - \tilde{x}_1^2 - \tilde{y}_1^2} \sqrt{1 - x_2^2 - y_2^2} (\cos \theta \cos \varphi - \sin \theta \sin \varphi).
\]

The variable \(\tilde{x}_1\) is evaluated in (4.2),

\[
\tilde{y}_1 = x_1 \sin \varphi + y_1 \cos \varphi = \frac{r_1 - q_1p}{\sqrt{1 - p^2}} \cdot p + q \sqrt{1 - p^2}.
\]
We use formula \(3.13\) for square roots and formula \(3.12\) for \(\cos \theta\). After this, we can evaluate \(\sin \theta\),
\[
\sin^2 \theta = \frac{(1 - p^2) \cdot \det \begin{pmatrix} 1 & p & q_1 & q_2 \\ p & 1 & r_1 & r_2 \\ q_1 & r_1 & 1 & t_{12} \\ q_2 & r_2 & t_{12} & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_1 & r_1 & 1 \end{pmatrix} \cdot \det \begin{pmatrix} 1 & p & q_2 \\ p & 1 & r_2 \\ q_2 & r_2 & 1 \end{pmatrix}}.
\]
After this, we get a unexpectedly long chain of cancelations and get the desired formula. \(\Box\).

**Choice of signs.** We use formula \(3.12\) and find
\[
\pm (\theta_1 - \theta_2), \quad \pm (\theta_1 - \theta_3), \quad \pm (\theta_2 - \theta_3)
\]
These numbers must be consistent, in fact only two variants are possible (this corresponds to a choice of a sheet of the covering map \(\tilde{\Xi} \to \Xi\)). Now let we have chosen the signs. Then we take 'minus' in (4.1) if \((\theta_1 - \theta_j) \geq 0\). Otherwise, we take 'plus'.

4.2. The action of the braid group.

**Lemma 4.3** The transformation
\[
(1, g_2, g_3, g_4, g_5) \mapsto (1, g_2 g_3 g_2^{-1}, g_2, g_4, g_5)
\]
corresponds to the map \(\tilde{\Xi}(5) \to \tilde{\Xi}(5)\) given by
\[
\begin{align*}
\tilde{p} &= p \\
\tilde{q}_k &= q_k, \quad \text{where } k = 1, 2, 3; \\
\tilde{r}_k &= r_k, \quad \text{where } k = 1, 2, 3; \\
\tilde{t}_{1j} &= t_{1j} - 2 \cdot \det \begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_j & r_j & t_{1j} \end{pmatrix} + 2p \cdot \det \begin{pmatrix} 1 & p & q_1 & q_j \\ p & 1 & r_1 & r_j \\ q_1 & r_1 & 1 & t_{1j} \\ q_j & r_j & t_{1j} & 1 \end{pmatrix}^{1/2}, \\
\tilde{t}_{23} &= t_{23}.
\end{align*}
\]

**Proof.** We evaluate
\[
\tilde{t}_{12} = \Re(a_1 \overline{a}_2 + b_1 \overline{b}_2 e^{2i\varphi}) = \\
= x_1 x_2 + y_1 y_2 + \sqrt{1 - x_1^2 - y_1^2} \sqrt{1 - x_2^2 - y_2^2} (\cos \theta \cos 2\varphi - \sin \theta \sin 2\varphi)
\]
as in the previous proof. \(\Box\)
Now we can write the action of the braid group. To write formulas for generators, we represent the matrix of the spectral form as

\[
\begin{pmatrix}
1 & p_1 & p_2 & p_3 & \ldots \\
p_1 & 1 & h_{12} & h_{13} & \ldots \\
p_2 & h_{12} & 1 & h_{23} & \ldots \\
p_3 & h_{13} & h_{23} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We also set \( h_{ij} := h_{ji} \). Then the formula for a generator \( \sigma_k \) of the braid group is

\[
\begin{align*}
\bar{h}_{ij} &= h_{ij}, \quad \text{if } i, j \neq k, k+1 \\
\bar{p}_i &= p_i, \quad \text{if } i \neq k, k+1 \\
\bar{p}_k &= p_{k+1} \\
\bar{p}_{k+1} &= p_k \\
\bar{h}_{(k+1)j} &= h_{kj},
\end{align*}
\]

and

\[
\bar{h}_{kj} = h_{(k+1)j} - 2 \cdot \det \begin{pmatrix}
1 & p_k & p_{k+1} \\
p_k & 1 & h_{k(k+1)} \\
p_j & h_{kj} & h_{(k+1)j}
\end{pmatrix} - 2p_k \cdot \det \begin{pmatrix}
1 & p_k & p_{k+1} & p_j & p_{k+1} \\
p_k & 1 & h_{k(k+1)} & h_{kj} & h_{(k+1)j} \\
p_{k+1} & h_{k(k+1)} & 1 & h_{kj} & h_{(k+1)j}
\end{pmatrix}^{1/2} (4.3)
\]

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Math.Dept., University of Vienna, Nordbergstrasse, 15, Vienna, Austria
&
Institute for Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia
&
Mech.Math. Dept., Moscow State University, Vorob’evy Gory, Moscow
e-mail: neretin(at) mccme.ru
URL:www.mat.univie.ac.at/∼neretin
wwwth.itep.ru/∼neretin