Construction of solutions of the classical field equation for a massless Klein-Gordon field coupled to a static source

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Abstract In this paper, we consider a system of a massless Klein-Gordon field coupled to a static source. The total Hamiltonian is a self-adjoint operator on a boson Fock space. We consider annihilation operators in the Heisenberg picture and define a sesquilinear form. Under infrared regularity conditions, it is proven that the sesquilinear form is a solution of the classical field equation.

MSC 2010 : 81Q10, 47B25
key words : Quantum field theory, Fock space, Self-adjoint operator

1 Introduction

In this paper, we consider a system of a massless Klein-Gordon field coupled to a static source. Let \( \phi_{\text{cl}} = \phi_{\text{cl}}(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^d \), be the classical field function. The Lagrangian density for the classical field is given by

\[
L_{\text{cl}} = \frac{1}{2} \left( \frac{\partial_t \phi_{\text{cl}}}{2} - \frac{1}{2} |\nabla \phi_{\text{cl}}|^2 + \rho(x) \phi_{\text{cl}} \right),
\]

where \( \nabla = \partial_{x_j} \) and \( \rho = \rho(x) \) is the density function of the static source. The Euler-Lagrange equation yields that

\[
(\partial_t^2 - \nabla^2) \phi_{\text{cl}}(t, x) = \rho(x),
\]

where \( \Delta = \sum_{j=1}^d \partial_{x_j}^2 \). The main purpose in this paper is to construct solutions of the above classical field equation from the quantized field. The Hilbert space for the system is defined by a boson Fock space. The quantized total Hamiltonian is given by

\[
H = \int_{\mathbb{R}^d} \omega(k) a^\dagger(k) a(k) d k - \int_{\mathbb{R}^d} \frac{1}{\sqrt{2 \omega(k)}} \left( \hat{\rho}(-k) a(k) + \hat{\rho}(k) a^\dagger(k) \right) d k.
\] (1)

Here \( \omega(k) = |k| \) denotes the dispersion relation, \( a^\dagger(k) \) the creation operator, \( a(k) \) the annihilation operator and \( \hat{\rho}(k) \) the Fourier transform of \( \rho(x) \). Under momentum cutoff conditions for \( \hat{\rho}(k) \), \( H \) is a self-adjoint operator on the boson Fock space. The type of the above Hamiltonian is called van Hove Hamiltonian.

Let us define a sesquilinear form by

\[
\phi_{\text{cl}}, \Phi, \Psi(t, x) = \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2 \omega(k)}} \left( F_{\Phi, \Psi}(t, k) e^{ikx} + F_{\Psi, \phi}(t, k)^* e^{-ikx} \right) d k,
\] (2)

where \( F_{\Phi, \Psi}(t, k) = (\Phi, e^{itH} a(k) e^{-itH} \Psi). \) Note that we consider the annihilation operator \( e^{itH} a(k) e^{-itH} \) in the Heisenberg picture. We additionally assume momentum cutoff conditions for \( \hat{\rho}(k) \), which
include infrared regularity conditions, and introduce a unitary operator $U$. Then the unitary transformation of $H$ is a sum of the free Hamiltonian and a constant number. In this sense, van Hove model is called exactly solvable model. Let $d \geq 3$. Using the unitary transformation and vectors of the form $\Theta_U = U \Theta$, it is proven that

\[
(\partial_t^2 - \triangle) \phi_{\text{cl}, \psi_U}(t, x) = \rho(x).
\] (3)

We may think that the result in this paper shows a relation between a classical field and the quantized field. It also can be regarded as a justifiable feature of field quantizations. Quantum field theory has divergent problems in itself, however, mathematically rigorous results have been obtained by many researchers. For the recent research on van Hove models, the classification of ultraviolet and infrared divergent properties and scattering theory were investigated in [2]. Various mathematical features of interacting quantum fields, which include van Hove models, were considered in [3].

This paper is organized as follows. In Section 2, basic properties of boson Fock spaces and their operators are reviewed, the definition of the total Hamiltonian is given, and the main theorem is stated. In section 3, the proof of the main theorem is given.

2 Definitions and Main Result

2.1 Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space. The inner product is denoted by $(\Phi, \Psi)$ which is linear in the second argument and the norm by $\|\Psi\|$. For a linear operator $X$ on $\mathcal{H}$, the domain of $X$ is denoted by $D(X)$, the adjoint by $X^*$, and its closure by $\overline{X}$ if $X$ is a closable.

2.2 Fock Spaces

In this subsection, we review basic properties of Fock spaces. (refer to e.g., [1], [4], [5])

[i] Boson Fock Space

Let $d \in \mathbb{N}$ be a spatial dimension. The Hilbert space for symmetric $n$ particles in $\mathbb{R}^d$ is given by $L^2_{\text{sym}}(\mathbb{R}^{dn})$ which consists of all vectors $\Psi^{(n)} \in L^2(\mathbb{R}^{dn})$ such that for all $\sigma \in \mathfrak{S}_n$, $\Psi^{(n)}(k_1, \cdots, k_n) = \Psi^{(n)}(\sigma(k_1), \cdots, \sigma(k_n))$, a.e. $(k_1, \cdots, k_n) \in \mathbb{R}^{dn}$, where $\mathfrak{S}_n$ denotes the symmetric group of degree $n$. The boson Fock space over $L^2(\mathbb{R}^d)$ is defined by

\[
\mathcal{F}_b = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^{dn})
\]

with $L^2_{\text{sym}}(\mathbb{R}^0) = \mathbb{C}$. Let $\Phi = \{\Phi^{(n)}\}_{n=0}^{\infty}$, $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b$. The inner product of $\mathcal{F}_b$ is given by

\[
(\Phi, \Psi) = \sum_{n=0}^{\infty} (\Phi^{(n)}, \Psi^{(n)}).
\]

The finite particle subspace $\mathcal{F}_{b, \text{fin}}$ is the set of all vectors $\Psi \in \mathcal{F}_b$ which satisfy that there exists $N \in \mathbb{N}$ such that for all $n > N$, $\Psi^{(n)} = 0$. Unless otherwise specified, the domain of a linear operator
$X$ on $\mathcal{F}_b$ is defined by

$$ \mathcal{D}(X) = \left\{ \Psi \in \mathcal{F}_b \middle| \sum_{n=0}^{\infty} \| (X\Psi)^{(n)} \|^2 < \infty \right\}. $$

[ii] **Annihilation and Creation Operators**

The annihilation operator $a(f)$ smeared with $f \in L^2(\mathbb{R}^d)$ is defined by

$$ (a(f)\Psi)^{(n)}(k_1, \cdots, k_n) = \frac{1}{n!} \int_{\mathbb{R}^d} f(k) \Psi^{(n+1)}(k, k_1, \cdots, k_n) dk, \quad n \geq 0, $$

where $z^*$ denotes the complex conjugate of $z \in \mathbb{C}$. The creation operator $a^\dagger(g)$ smeared with $g \in L^2(\mathbb{R}^d)$ is defined by

$$ (a^\dagger(g)\Psi)^{(n)}(k_1, \cdots, k_n) = \frac{1}{n!} \sum_{j=1}^n g(k_j) \Psi^{(n-1)}(k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n), \quad n \geq 1, $$

and $(a^\dagger(g)\Psi)^{(0)} = 0$. It holds that $a(f) = (a^\dagger(f))^*$. The creation and annihilation operators satisfy the canonical commutation relations

$$ [a(f), a^\dagger(g)] = (f, g), $$

$$ [a(f), a(g)] = 0, $$

on $\mathcal{F}_{b,\text{fin}}$. The Segal field operators and their conjugate operators are defined by

$$ \phi_s(f) = \frac{1}{\sqrt{2}} (a(f) + a^\dagger(f)), \quad f \in L^2(\mathbb{R}^d), $$

$$ \pi_s(g) = \frac{i}{\sqrt{2}} (-a(g) + a^\dagger(g)), \quad g \in L^2(\mathbb{R}^d). $$

From (4) and (5), it follows that

$$ [\phi_s(f), \pi_s(g)] = i \text{Re}(f, g), $$

$$ [\phi_s(f), \phi_s(g)] = [\pi_s(f), \pi_s(g)] = 0, $$

on $\mathcal{F}_{b,\text{fin}}$. It holds that $\phi_s(f)$ and $\pi_s(g)$ are essentially self-adjoint. From (4) and (5), it follows that

$$ e^{i \pi_s(g)} a(f) e^{-i \pi_s(g)} = a(f) + \frac{1}{\sqrt{2}} (f, g). $$

[iii] **Second Quantization**

Let $T = T(k)$ be a non-negative and Borel measurable function. The second quantization $d\Gamma_b(T)$ of $T$ is defined by

$$ (d\Gamma_b(T)\Psi)^{(n)}(k_1, \cdots, k_n) = \sum_{j=1}^n T(k_j) \Psi^{(n)}(k_1, \cdots, k_n), \quad n \geq 1, $$

and $(d\Gamma_b(T)\Psi)^{(0)} = 0$. Let $f \in L^2(\mathbb{R}^d)$ with $T^{-1/2}f \in L^2(\mathbb{R}^d)$. It holds that for all $\Psi \in \mathcal{D}(d\Gamma_b(T)^{1/2})$,

$$ \|a(f)\Psi\| \leq \|T^{-1/2}f\| \|d\Gamma_b(T)^{1/2}\Psi\|, $$

$$ \|a^\dagger(f)\Psi\| \leq \|T^{-1/2}f\| \|d\Gamma_b(T)^{1/2}\Psi\| + \|f\| \|\Psi\|. $$

3
For all $f \in L^2(\mathbb{R}^d)$ satisfying $Tf \in L^2(\mathbb{R}^d)$, it holds that

\[ [d\Gamma_b(T), a(f)] = -a(Tf), \quad \text{(11)} \]
\[ [d\Gamma_b(T), a^\dagger(f)] = a^\dagger(Tf), \quad \text{(12)} \]
on $\mathcal{F}$. By (11) and (12), it follows that

\[ e^{it d\Gamma_b(T)} a(f) e^{-it d\Gamma_b(T)} = a(e^{itf}) \quad \text{(13)} \]

and

\[ e^{i\pi_0(g)} d\Gamma_b(T) e^{-i\pi_0(g)} = d\Gamma_b(T) + \Phi_b(Tg) + \frac{1}{2}(g, Tg). \quad \text{(14)} \]

### 2.3 Hamiltonian and Main Theorem

The total Hamiltonian $H$ on $\mathcal{F}$ is given by

\[ H = H_0 + H_1, \quad \text{(15)} \]

where $H_0 = d\Gamma_b(\omega)$ with $\omega(k) = |k|$ and $H_1 = -\Phi_b(f_i)$ with $f_i(k) = \frac{\delta(k)}{\sqrt{\omega(k)}}$.

Suppose the condition below.

\[ (\text{A.1}) \; \rho \in L^1(\mathbb{R}^d) \text{ and } \frac{\hat{\rho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d), \; l = 1, 2. \]

We quickly check the self-adjointness of $H$. By (9) and (10), we have

\[ \|\Phi_b(f_i)\Psi\| \leq \sqrt{2} \left\| \frac{f_i}{\sqrt{\omega}} \right\| \|H_0^{1/2}\Psi\| + \frac{1}{\sqrt{2}} \|f_i\| \|\Psi\|. \]

From spectral decomposition theorem, we obtain $\|H_0^{1/2}\Psi\| \leq \varepsilon \|H_0\Psi\| + \frac{1}{\sqrt{2}} \|\Psi\|$, $\varepsilon > 0$. Therefore,

\[ \|H_1\Psi\| \leq c_1 \varepsilon \|H_0\Psi\| + d_1(\varepsilon) \|\Psi\|, \]

where $c_1 = \sqrt{2} \left\| \frac{f_i}{\sqrt{\omega}} \right\|$ and $d_1(\varepsilon) = \frac{1}{\sqrt{2}\varepsilon} \left\| \frac{f_i}{\sqrt{\omega}} \right\| + \frac{1}{\sqrt{2}} \|f_i\|$. Taking $\varepsilon > 0$ such that $\varepsilon < \frac{1}{c_1}$, the Kato-Rellich theorem yields that $H$ is self-adjoint on $\mathcal{D}(H_0)$.

Let

\[ F_{\Phi, a}(t, k) = (\Phi, e^{itH}a(k)e^{-itH}\Psi), \]

where $a(k)$ is the operator kernel of the annihilation operator, which is defined in Section 3.1. Let

\[ \phi_{\alpha, a, \Psi}(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} \left( F_{\alpha, a}(t, k) e^{-ikx} + F_{\alpha, a}^*(t, k) e^{ikx} \right) dk. \]

Suppose the condition below.

\[ (\text{A.2}) \; \frac{\hat{\rho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d), \; \frac{\hat{\rho}}{\sqrt{\omega}} \in L^1(\mathbb{R}^d) \text{ and } \frac{\hat{\rho}}{\sqrt{\omega}} \in L^1(\mathbb{R}^d). \]

\[ \textbf{Remark 2.1} \; \text{If } d = 3, \text{ it holds that } \int_{\mathbb{R}^d} \frac{1}{\omega(k)} dk = \infty. \text{ Hence the condition } \frac{\hat{\rho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d) \text{ is called infrared regularity condition.} \]
By (A.2), we can define a unitary operator

\[ U = e^{-i\pi (\frac{1}{2}) }. \]

From (14), it holds that

\[ U^* H U = H_0 - \frac{1}{2}(f, f). \]  

(16)

Here we state the main theorem.

**Theorem 2.1** Assume (A.1), (A.2) and \( d \geq 3 \). Then for all \( \Phi, \Psi \in \mathcal{D}(H_0^{\frac{d}{2} + 2}) \) with \( (\Phi, \Psi) = 1 \),

\[ (\partial_t^2 - \Delta) \phi_{U, \Psi_U}(t, x) = \rho(x), \]  

(17)

where \( \Phi_U = U \Phi \) and \( \Psi_U = U \Psi \). In particular, if \( \| \Psi \| = 1 \), \( \phi_{U, \Psi_U}(t, x) \) is the solution.

### 3 Proof of Theorem 2.1

#### 3.1 Operator kernel of annihilation operator

The operator kernel of annihilation operator is defined by

\[ (a(k) \Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1} \Psi^{(n+1)}(k, k_1, \ldots, k_n), \quad n \geq 0. \]

Let \( \mathcal{D}_{a,f} = \{ \Psi \in \mathcal{F}_b \mid \int_{\mathbb{R}^d} |f(k)| \|a(k) \Psi\| \, dk < \infty \}, f \in L^2(\mathbb{R}^d) \). It holds that

\[ \int_{\mathbb{R}^d} f(k)^* (\Phi, a(k) \Psi) \, dk = (\Phi, a(f) \Psi), \quad \Phi \in \mathcal{F}_b, \, \Psi \in \mathcal{D}_{a,f} \cap \mathcal{D}(a(f)). \]  

(18)

Let \( T = T(k) \) be a non-negative and Borel measurable function, and \( f \in L^2(\mathbb{R}^d) \) such that \( T^{-1/2} f \in L^2(\mathbb{R}^d) \). Then it follows that for all \( \Phi \in \mathcal{F}_b, \, \Psi \in \mathcal{D}(d \Gamma_b(T)^{1/2}) \),

\[ \int_{\mathbb{R}^d} |f(k)^* (\Phi, a(k) \Psi)\| \, dk \leq \|T^{-1/2} f\| \|\Phi\| \|d \Gamma_b(T)^{1/2} \Psi\|. \]  

(19)

It also holds that

\[ \int_{\mathbb{R}^d} T(k) \|a(k) \Psi\|^2 \, dk = \|d \Gamma_b(T)^{1/2} \Psi\|^2, \quad \Psi \in \mathcal{D}(d \Gamma_b(T)^{1/2}). \]  

(20)

From (20), the next lemma immediately follows.

**Lemma 3.1** Assume (A.1). Then, for all \( \Phi \in \mathcal{F}_b \) and \( \Psi \in \mathcal{D}(H) \),

\[ \left( \int_{\mathbb{R}^d} \omega(k) |F_{0, \Psi}(t, k)|^2 \, dk \right)^{1/2} \leq \|\Phi\| \|H_0^{1/2} e^{-iH} \Psi\|. \]

Let

\[ F_{0, \Psi}(k) = (\Phi, a(k) \Psi). \]
Proposition 3.2 Assume (A.1) and (A.2). Then it holds that for all $\Phi, \Psi \in \mathcal{D}(H)$,

$$F_{\Phi, \Psi_U}(t, k) = e^{-it\omega(k)}F_{0, \Phi, \Psi}(k) + \frac{1}{\sqrt{2}}\hat{\rho}(k), \quad \text{a.e. } k \in \mathbb{R}^{d}.$$  

(Proof) Let $\Phi, \Psi \in \mathcal{D}(H)$. We set $F_{\omega, \Phi, \Psi_U}(t, k) = \sqrt{\omega(k)}F_{\Phi, \Psi_U}(t, k)$. For all $h \in L^{2}(\mathbb{R}^{d})$ such that $\sqrt{\omega}h \in L^{2}(\mathbb{R}^{d})$, we have

$$\int_{\mathbb{R}^{d}} h(k)^{*}F_{\omega, \Phi, \Psi_U}(t, k)dk = \int_{\mathbb{R}^{d}} h(k)^{*}(U\Phi,e^{itH}a(k)e^{-itH}U\Psi)dk$$

$$= (e^{-itH}U\Phi, a(\sqrt{\omega}h)e^{-itH}U\Psi)$$

$$= (U^{*}e^{-itH}U\Phi, (U^{*}a(\sqrt{\omega}h)U)e^{-itH}U\Psi)$$  

(21)

By (16),

$$U^{*}e^{-itH}U = e^{-itU^{*}H} = e^{\frac{t}{2}(\frac{f_{1}}{\sqrt{\omega}})}e^{-it\hat{H}_{0}}.$$  

(22)

From (8),

$$U^{*}a(\sqrt{\omega}h)U = a(\sqrt{\omega}h) + \frac{1}{\sqrt{2}}(h, \frac{f_{1}}{\sqrt{\omega}}).$$  

(23)

By (22), (23) and $\Phi, \Psi = 1$, we have

$$(U^{*}e^{-itH}U\Phi, (U^{*}a(\sqrt{\omega}h)U)e^{-itH}U\Psi) = (\Phi, e^{it\hat{H}_{0}a(\sqrt{\omega}h)e^{-it\hat{H}_{0}}} + \frac{1}{\sqrt{2}}(h, \frac{f_{1}}{\sqrt{\omega}}))$$

$$= (\Phi, a(\sqrt{\omega}e^{it\hat{H}_{0}})\Psi) + \frac{1}{\sqrt{2}}(h, \frac{f_{1}}{\sqrt{\omega}}).$$

Here we used (13) in the last line. Then we have

$$\int_{\mathbb{R}^{d}} h(k)^{*}F_{\omega, \Phi, \Psi_U}(t, k)dk = \int_{\mathbb{R}^{d}} h(k)^{*}\left(\sqrt{\omega(k)}e^{-it\omega(k)}(\Phi, a(k)\Psi) + \frac{1}{\sqrt{2}}\frac{f_{1}(k)}{\sqrt{\omega(k)}}\right)dk.$$  

Since the set which consists of all vectors $h \in L^{2}(\mathbb{R}^{d})$ such that $\sqrt{\omega}h \in L^{2}(\mathbb{R}^{d})$ is dense in $L^{2}(\mathbb{R}^{d})$ and $\int_{\mathbb{R}^{d}} |F_{\omega, \Phi, \Psi_U}(t, k)|^{2}dk = \int_{\mathbb{R}^{d}} \omega(k)|F_{\Phi, \Psi}(t, k)|^{2}dk < \infty$ from Lemma 3.1 we have

$$F_{\omega, \Phi, \Psi_U}(t, k) = \sqrt{\omega(k)}e^{-it\omega(k)}(\Phi, a(k)\Psi) + \frac{1}{\sqrt{2}}\frac{\hat{\rho}(k)}{\sqrt{\omega(k)}}, \quad \text{a.e. } k \in \mathbb{R}^{d}.$$  

By dividing both sides of the above equation by $\sqrt{\omega(k)}$, the proof is obtained. ■

Lemma 3.3 Let $T = T(k)$ be a non-negative and Borel measurable function. Then, for all $p \in \mathbb{N}$,

$$\|d\Gamma_{b}(T^{p})^{1/2}\Psi\| \leq \|d\Gamma_{b}(T)^{p/2}\Psi\|^{2}, \quad \Psi \in \mathcal{D}(d\Gamma_{b}(T)^{p/2}).$$

(Proof) Let $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{D}(d\Gamma_{b}(T)^{p/2})$. We see that

$$\int_{\mathbb{R}^{dn}} \left(\sum_{j=1}^{n} T(k_{j})^{p} \right) J_{n}(\Psi^{(n)}(k_{1}, \ldots, k_{n})) d\mu(k) \leq \int_{\mathbb{R}^{dn}} \left(\sum_{j=1}^{n} T(k_{j})^{p} \right)^{2} d\mu(k) = \int_{\mathbb{R}^{dn}} \left(\sum_{j=1}^{n} T(k_{j}) \right)^{p} \left|\Psi^{(n)}(k_{1}, \ldots, k_{n})\right|^{2} d\mu(k).$$

From this inequality, it is directly proven that $\|d\Gamma_{b}(T^{p})^{1/2}\Psi\| \leq \|d\Gamma_{b}(T)^{p/2}\Psi\|$. ■

From (20) and Lemma 3.3 next corollary follows.
Corollary 3.4 Let $p \in \mathbb{N}$. Then,
\[
\int_{\mathbb{R}^d} \omega(k)^p \|a(k)\Psi\|^2 dk \leq \|H_{0}^{p/2}\Psi\|^2, \quad \Psi \in \mathcal{D}(H_{0}^{p/2}).
\]

Proposition 3.5 Assume (A.1) and $d \geq 3$. Then for all $l = 0, 1, 2$ and $j = 1, \ldots, d$,
\[
\int_{\mathbb{R}^d} \frac{|k_j|^l}{\sqrt{\omega(k)}} |F_{\omega, \Psi}(k)| dk \leq c_d \|\Phi\| \left( \|H_{0}^{d/2}\Psi\|^2 + \|\Psi\|^2 \right), \quad \Psi \in \mathcal{D}(H_{0}^{d/2}),
\]
where $c_d = \left\| \frac{1}{\omega} \right\|_{L^2(B_d)} + \left\| \frac{1}{\omega^{d/2}} \right\|_{L^2(B_d^l)}$.

(Proof) Let $\Psi \in \mathcal{D}(H_{0}^{d/2})$. We see that
\[
\int_{\mathbb{R}^d} \frac{|k_j|^l}{\sqrt{\omega(k)}} |F_{\omega, \Psi}(k)| dk \leq \|\Phi\| \int_{\mathbb{R}^d} \frac{|k_j|^l}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk
\]
Note that $\int_{\mathbb{R}^d} \frac{1}{\omega(k)^{l/2}} dk < \infty$, for all $d \geq 3$. Then,
\[
\int_{B_d} \frac{|k_j|^l}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk \leq \int_{B_d} \frac{1}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk \\
\leq \left\| \frac{1}{\omega} \right\|_{L^2(B_d)} \left( \int_{B_d} \omega(k) \|a(k)\Psi\|^2 dk \right)^{1/2} \\
\leq \left\| \frac{1}{\omega} \right\|_{L^2(B_d)} \|H_{0}^{1/2}\Psi\|. \quad (24)
\]
We also note that $\int_{\mathbb{R}^d \setminus B_d} \frac{1}{\omega(k)^{l/2}} dk < \infty$ for all $d \in \mathbb{N}$. Let $p_{d,l} = d + 2l$, $l = 0, 1, 2$. Then,
\[
\int_{B_d^l} \frac{|k_j|^l}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk \leq \int_{B_d^l} \frac{\omega(k)^l}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk \\
\leq \left\| \frac{1}{\omega} \right\|_{L^2(B_d^l)} \left( \int_{\mathbb{R}^d \setminus B_d} \omega(k)^{p_{d,l}} \|a(k)\Psi\|^2 dk \right)^{1/2} \\
\leq \left\| \frac{1}{\omega} \right\|_{L^2(B_d^l)} \|H_{0}^{p_{d,l}}\Psi\|. \quad (25)
\]
From (24) and (25), we have
\[
\int_{\mathbb{R}^d} \frac{|k_j|^l}{\sqrt{\omega(k)}} \|a(k)\Psi\| dk \leq \left\| \frac{1}{\omega} \right\|_{L^2(B_d)} \|H_{0}^{1/2}\Psi\| + \left\| \frac{1}{\omega^{(d+4)/2}} \right\|_{L^2(B_d^l)} \|H_{0}^{p_{d,l}}\Psi\|. \quad (26)
\]
By spectral decomposition theorem, $\|H_{0}^{1/2}\Psi\| \leq \|H_{0}^{d+4}\Psi\| + \|\Psi\|$ and $\|H_{0}^{p_{d,l}}\Psi\| \leq \|H_{0}^{d+4}\Psi\| + \|\Psi\|$, $l = 0, 1, 2$. Hence we obtain the proof. ■
(Proof of Theorem 2.1)
Let $\Phi, \Psi \in \mathcal{D}(H_0^{d+2})$. From Proposition 3.2, Proposition 3.3 for $l = 0$ and (A.2), we have
\[
\phi_{\Phi, \Psi} (t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} \left( e^{-i\omega(k) + i\omega(k)} F_{0, \Phi, \Psi}(k) + e^{i\omega(k) - i\omega(k)} F_{0, \Psi, \Phi}(k)^* \right) dk
\]
and
\[
\Delta \phi_{\Phi, \Psi} (t, x) = -\frac{1}{\sqrt{2\pi}} \sum_{j=1}^d \left\{ \int_{\mathbb{R}^d} \frac{k_j^2}{\sqrt{2\omega(k)}} \left( e^{-i\omega(k) + i\omega(k)} F_{0, \Phi, \Psi}(k) + e^{i\omega(k) - i\omega(k)} F_{0, \Psi, \Phi}(k)^* \right) dk \right\}
\]
By Proposition 3.5 for $l = 1, 2$, we have
\[
\partial_t^2 \phi_{\Phi, \Psi} (t, x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \frac{\omega(k)^2}{\sqrt{2\omega(k)}} \left( e^{-i\omega(k) + i\omega(k)} F_{0, \Phi, \Psi}(k) + e^{i\omega(k) - i\omega(k)} F_{0, \Psi, \Phi}(k)^* \right) dk
\]
and
\[
\partial_t^2 \phi_{\Phi, \Psi} (t, x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \frac{\omega(k)^2}{\sqrt{2\omega(k)}} \left( e^{-i\omega(k) + i\omega(k)} F_{0, \Phi, \Psi}(k) + e^{i\omega(k) - i\omega(k)} F_{0, \Psi, \Phi}(k)^* \right) dk
\]
Then we have $(\partial_t^2 - \Delta) \phi_{\Phi, \Psi} (t, x) = \rho(x)$. $\blacksquare$

[Concluding remark]
Let us consider massive cases $\omega_m(k) = \sqrt{k^2 + m^2}$, $m > 0$. Similarly, we can construct the solutions of $(\partial_t^2 - \Delta) \phi_{\Phi} (t, x) = \rho(x)$. In these cases we do not need to suppose $d \geq 3$, since $\left\| \frac{1}{\omega_m(k)} \right\|_{L^2(B_d)} < \infty$, $d \in \mathbb{N}$, which is correspond to $\left\| \frac{1}{\omega(k)} \right\|_{L^2(B_d)} < \infty$, $d \geq 3$, in Proposition 3.5.

Acknowledgments This work is supported by JSPS grant 16K17607.

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