Exact and Heuristic Methods for the Assembly Line Worker Assignment and Balancing Problem

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Abstract

In traditional assembly lines, it is reasonable to assume that task execution times are the same for each worker. However, in sheltered work centres for disabled this assumption is not valid: some workers may execute some tasks considerably slower or even be incapable of executing them. Worker heterogeneity leads to a problem called the assembly line worker assignment and balancing problem (ALWABP). For a fixed number of workers the problem is to maximize the production rate of an assembly line by assigning workers to stations and tasks to workers, while satisfying precedence constraints between the tasks.

This paper introduces new heuristic and exact methods to solve this problem. We present a new MIP model, propose a novel heuristic algorithm based on beam search, as well as a task-oriented branch-and-bound procedure which uses new reduction rules and lower bounds for solving the problem. Extensive computational tests on a large set of instances show that these methods are effective and improve over existing ones.
1 Introduction

The Universal Declaration of Human Rights states that “everyone has the right to work, to free choice of employment, to just and favourable conditions of work and to protection against unemployment” (United Nations, 1948). Despite this, low employment rates still demonstrate the lack of job opportunities for more than 785 million persons with disabilities, including 110 million with a severe deficiency degree, due to factors like prejudices and absence of appropriate technical preparation (Organisation for Economic Co-Operation and Development, 2003). This deficit lead to the creation of programs for the social inclusion of persons with disabilities. Some of them concern their qualification (World Health Organization, 2011), while others ensure opportunities by quota laws (Lobato, 2009). Countries like Spain, Japan and Switzerland merged these two forms by creating Sheltered Work Centres for Disabled (SWDs) (Chaves, 2009), which employ mainly persons with disabilities and provide training and a first job opportunity for them (Miralles et al., 2007). SWDs are not-for-profit industries applying all revenues in improvements for the company and the creation of new jobs.

Miralles et al. (2007) have shown that using assembly lines in SWDs has advantages, because the division of work into smaller tasks can effectively hide the differences among the workers. Furthermore, the execution of repetitive tasks, when properly assigned, can be an excellent therapeutic treatment for workers with disabilities. Traditional approaches to the optimization of assembly lines assume that the workers have similar abilities and are capable of executing the tasks in the same time. The most basic model of this kind is the Simple Assembly Line Balancing Problem (SALBP), which has been extensively studied in the literature (Scholl and Becker, 2006b). Several authors have considered stochastic models of assembly lines, where task times may vary, and remedial actions are taken if the cycle time is exceeded at some station (Silverman and Carter, 1986; Kottas and Lau, 1976; Lyu, 1997). In this paper we are not directly concerned with varying task times of a single worker, but with the case of SWDs, where the workers need different times to execute the tasks, or may even be incapable of executing some of them. To model such a situation, Miralles et al. (2008) proposed the Assembly Line Worker Assignment and Balancing Problem (ALWABP), which assigns tasks to different workers and these workers to the workstations.

1.1 Problem Definition

Let $S$ be a set of stations, $W$ be a set of workers, $|W| = |S|$, and $T$ be a set of tasks. Each workstation $s \in S$ is placed along a conveyor belt and is assigned to exactly one worker $w \in W$, which is responsible for executing a subset of tasks $x_w \subseteq T$. The tasks are partially ordered, and we assume that the partial order is given by a transitively reduced directed acyclic graph $G(T, E)$ on the tasks, such that for an arc $(t, t') \in E$ task $t$ precedes task $t'$. Therefore, the station that executes task $t$ cannot be placed later than that of task $t'$ on the conveyor belt. The execution time of task $t$ for worker $w$ is $p_{tw}$. If a worker $w$ cannot execute a task $t$, $p_{tw}$ is set to $\infty$.

The total execution time of worker $w$ is $D_w = \sum_{t \in x_w} p_{tw}$. The cycle time $C$ of the line is defined by the maximum total execution time $\max_{w \in W} D_w$. In assembly line balancing, a problem of type 1 aims to reduce the number of stations for a given cycle time. Since in SWDs the goal is to include all workers, our problem is of type 2, and aims to minimize the cycle time for a
Figure 1: Example of an ALW ABP instance and an assignment of tasks to workers (in grey). Upper part: precedence constraints among the tasks. Lower part: task execution times.

Figure 1 shows an example of an ALWABP-2 instance. For the assignment given in the figure, we have $D_{w_1} = 5$, $D_{w_2} = 6$, $D_{w_3} = 5$, and a cycle time of $C = \max\{D_{w_1}, D_{w_2}, D_{w_3}\} = 6$.

1.2 Related Work

The majority of the publications on the ALWABP-2 is dedicated to the application of meta-heuristics to find approximate solutions to the problem. Two clustering search methods were proposed by Chaves et al. (2007, 2009), which were outperformed on large instances by a tabu search of Moreira and Costa (2009). Blum and Miralles (2011) proposed an iterated beam search based on the station-oriented branch-and-bound procedure of Miralles et al. (2008). Later, Moreira et al. (2012) used a constructive heuristic with various combinations of priority rules to produce initial solutions for a genetic algorithm (GA). Mutlu et al. (2013) developed an iterated GA that produces valid orders of tasks and applies iterated local search to attribute the tasks in the selected order to the workers.

To the best of our knowledge, the only exact method for the ALWABP-2 is the branch-and-bound procedure of Miralles et al. (2008). It embeds a station-oriented, depth-first branch-and-bound search in a linear lower bound search for the optimal cycle time, and is limited to very small instances.

1.3 Structure of the paper

In Section 2 we introduce a new MIP model for the ALWABP-2. In Section 3 we present several lower bounds for the problem. A new heuristic for ALWABP-2 is proposed in Section 4. In Section 5 we present a task-oriented branch-and-bound method for solving the problem exactly. Computational results are presented and analyzed in Section 6. We conclude in Section 7.
Table 1: Notation for ALW ABP-2.

| Symbol  | Description                                      |
|---------|--------------------------------------------------|
| $S$     | set of stations;                                 |
| $W$     | set of workers;                                  |
| $T$     | set of tasks;                                    |
| $G(T,E)$| transitive reduced precedence graph of tasks;    |
| $G^*(T,E^*)$| transitive closure of graph $G(T,E)$;  |
| $p_{tw}$| execution time of task $t$ by worker $w$;        |
| $A_w$   | set of tasks feasible for worker $w$;            |
| $A_t$   | set of workers able to execute task $t$;         |
| $P_t$   | set of direct predecessors of task $t$ in $G$;   |
| $F_t$   | set of direct successors of task $t$ in $G$;     |
| $P_t^*$ | set of all predecessors of task $t$ in $G^*$;    |
| $F_t^*$ | set of all successors of task $t$ in $G^*$;      |
| $C$     | cycle time of a solution.                       |

2 Mathematical formulation

In this section we will present a new mixed-integer model for the ALW ABP-2. Currently, the only model used in the literature, called $M_1$ here, is the one proposed by Miralles et al. (2008). It has $O(|T||W||S|)$ variables, and $O(|T|+|E|+|W||S|)$ constraints. In the following we will use the notation defined in Table 1.

2.1 Formulation with two-index variables

Our formulation is based on the observation that it is sufficient to assign tasks to workers and to guarantee that the directed graph over the workers, induced by the precedences between the tasks, is acyclic. Therefore our model uses variables $x_{wt}$ such that $x_{wt}=1$ if task $t \in T$ has been assigned to worker $w \in W$, and $d_{vw}$ such that $d_{vw}=1$ if worker $v \in W$ must precede worker $w \in W$. In this way, we obtain a model $M_2$ as follows:

\[
\begin{align*}
\text{minimize} & \quad C, \\
\text{subject to} & \quad \sum_{t \in A_w} p_{tw} x_{wt} \leq C, \quad \forall w \in W, \\
& \quad \sum_{w \in A_t} x_{wt} = 1, \quad \forall t \in T, \\
& \quad d_{vw} \geq x_{vt} + x_{wt} - 1, \quad \forall (t, t') \in E, v \in A_t, w \in A_{t'} \setminus \{v\}, \\
& \quad d_{aw} \geq d_{av} + d_{vw} - 1, \quad \forall \{u, v, w\} \subseteq W, |\{u, v, w\}| = 3, \\
& \quad d_{vw} + d_{wv} \leq 1, \quad \forall v \in W, w \in W \setminus \{v\}, \\
& \quad x_{wt} \in \{0, 1\}, \quad \forall w \in W, t \in A_w, \\
& \quad d_{vw} \in \{0, 1\}, \quad \forall v \in W, w \in W \setminus \{v\}, \\
& \quad C \in \mathbb{R}. 
\end{align*}
\]

Constraint (2) defines the cycle time $C$ of the problem. Constraint (3) ensures that every task is executed by exactly one worker. The dependencies between workers are defined by
constraint (4): when a task \( t \) is assigned to worker \( v \) and precedes another task \( t' \) assigned to a different worker \( w \), worker \( v \) must precede worker \( w \). Constraints (5) and (6) enforce transitivity and anti-symmetry of the worker dependencies. As a consequence of these constraints, the workers of a valid solution can always be ordered linearly.

### 2.2 Continuity constraints

We can strengthen the above model by the following observation: if two tasks \( i \) and \( k \) are assigned to the same worker \( w \), then all tasks \( j \) that are simultaneously successors of \( i \) and predecessors of \( k \) should also be assigned to \( w \). These continuity constraints generalize constraints proposed by [Peeters and Degraeve (2006)] for single station loads in the SALBP to several stations:

\[ x_{wj} \geq x_{wt} + x_{wk} - 1, \quad \forall i \in T, j \in F_i^+, k \in F_j^+, w \in A_i \cap A_j \cap A_k. \quad (10) \]

Similarly, if task \( i \) is assigned to worker \( w \), but some successor (predecessor) \( j \) of \( i \) is unfeasible for \( w \), then no successor (predecessor) of \( j \) can be assigned to \( w \). This justifies the constraints

\[ x_{wk} + x_{wi} \leq 1, \quad \forall i \in T, j \in F_i^+, k \in F_j^+, w \in A_i \cap (T \setminus A_j) \cap A_k. \quad (11) \]

Let model \( M_3 \) be model \( M_2 \) with additional constraints (10) and (11). Model \( M_3 \) has \( O(\|W\| + |T| + |W|) \) variables, and \( O(\|E^*\|T |W| + |W|^3 + |E||W|^2) \) constraints, i.e. it has less variables but more constraints than \( M_1 \). As will be seen in Section 6 model \( M_3 \) gives significantly better bounds than \( M_1 \).

### 3 Lower bounds

Lower bounds for ALWABP-2 can be obtained by different relaxations of the problem. In this section we discuss relaxations of the mixed-integer model presented above, as well as relaxations to SALBP-2 and \( R \parallel C_{\text{max}} \).

#### 3.1 Relaxation to SALBP-2

If we relax the task processing times to their minimum \( p_t^- = \min\{p_{tw} \mid w \in W\} \), ALWABP-2 reduces to SALBP-2. Therefore, all valid lower bounds for SALBP-2 apply to this relaxation. In particular, we use the lower bounds

\[
LC_1 = \max \left\{ \max\{p_t^- \mid t \in T\}, \left[ \sum_{i \in T} (p_{iw}^-) / |S| \right] \right\} \quad \text{and} \quad \\
LC_2 = \max \left\{ \sum_{0 \leq i \leq k} P_{k[S]}^+ 1 - i \mid 1 \leq k \leq \left\lfloor \frac{|T| - 1}{|W|} \right\rfloor \right\}
\]
(The bound $LC_2$ supposes that the tasks are ordered such that $p_1^{-} \geq \cdots \geq p_T^{-}$.) We further use the SALBP-2 bounds on the earliest and latest possible station of task $t$ for a given cycle time $C$

$$E_t(C) = \left\lceil \frac{\sum_{j \in P_t} p_j^- + p_t^-}{C} \right\rceil$$ and $$L_t(C) = |S| + 1 - \left\lceil \frac{\sum_{j \in F_t} p_j^- + p_t^-}{C} \right\rceil$$

(12) (13) to obtain the lower bound $LC_3$, defined as the smallest cycle time $C$ such that $E_t(C) \leq L_t(C)$ for all $t \in T$. For more details on these bounds we refer the reader to the survey of Scholl and Becker (2006a).

3.2 Relaxation to $R \parallel C_{max}$

By removing the precedence constraints the ALWABP-2 reduces to the problem of minimizing the makespan of the tasks on unrelated parallel machines ($R \parallel C_{max}$), which itself is an NP-hard problem. Several effective lower bounds for $R \parallel C_{max}$ have been proposed by Martello et al. (1997). Their lower bounds $L_1$ and $L_2$ are obtained by Lagrangian relaxation of the cycle time constraints (2) and the assignment constraints (3), respectively. Martello et al. (1997) further propose an additive improvement that can be applied to $L_1$ to obtain a bound $L_{a1} \geq L_1$, as well as an improvement by cuts on disjunctions, that may be applied to $L_{a1}$ and $L_2$ to obtain lower bounds $L_{a1} \geq L_{a1}$ and $L_2 \geq L_2$.

3.3 Linear relaxation of ALWABP-2 models

Bounds obtained from linear relaxations of integer models for the SALBP-2 are usually weaker than the SALBP-2 bounds of Section 3.1. However, the relaxation to minimum task execution times weakens the SALBP-2 bounds considerably. Therefore, the linear relaxations of model $M_3$ provides a useful lower bound for the ALWABP-2 (Moreira et al., 2012).

4 Heuristic search procedure

In this section, we describe a heuristic algorithm IPBS for the ALWABP-2. It systematically searches for a small cycle time by trying to solve the feasibility problem ALWABP-F for different candidate cycle times from an interval ending at the current best upper bound. For each candidate cycle time $C$, a probabilistic beam search tries to find a feasible allocation.

4.1 Probabilistic beam search for the ALWABP-F

The basis for the probabilistic beam search is a station-based assignment procedure, which assigns tasks in a forward manner station by station. For each station it repeatedly selects an available task, until no such task has an execution time less than the idle time of the current station. A task is available if all its predecessors have been assigned already. If there are several
available tasks the highest priority task as defined by a prioritization rule is assigned next. The procedure succeeds if an assignment using at most the available number of stations is found. Station-based procedures can be also applied in a backward manner, assigning tasks whose successors have been assigned already. For this it is sufficient to apply a forward procedure to an instance with reversed dependencies. For the ALWABP we additionally have to decide which worker to assign to the current station. This is accomplished by applying the task assignment procedure to all workers which are not yet assigned, and then choosing the best worker for the current station by a worker prioritization rule.

The probabilistic beam search extends the station-oriented assignment procedure in two aspects. First, when assigning tasks to the current station, it randomly chooses one of the available tasks with a probability proportional to its priority. Second, it applies beam search to find the best assignment of workers and their corresponding tasks.

Beam search is a truncated breadth-first tree search procedure [Lowerre 1976; Ow and Morton 1988]. When applied to the ALWABP-F, it maintains a set of partial solutions called the beam during the station-based assignment. The number of solutions in the beam is called its width $\gamma$. Beam search extends a partial solution by assigning each available worker to the next station, and for each worker, chooses the tasks to execute according to the above probabilistic rule. For each worker this is repeated several times, to select different subsets of tasks. The number of repetitions is the beam’s branching factor $f$. Among all new partial solutions the algorithm selects those of highest worker priority to form the new beam. The number of solutions selected is at most the beam width.

Task and worker prioritization rules are important for the efficacy of station-oriented assignment procedure. Moreira et al. (2012) compared the performance of 16 task priority rules and three worker prioritization rules for the ALWABP-2. We have chosen the task priority rule $\text{MaxPW}^-$ and the worker priority rule $\text{MinRLB}$, which have been found to produce the best results in average for the problem. The task prioritization rule $\text{MaxPW}^-$ gives preference to tasks with larger minimum positional weight $pw_i^- = p_i^- + \sum_{t \in F_i} p_t^-$. The worker prioritization rule $\text{MinRLB}$ gives preference to workers with smaller restricted lower bound $\sum_{t \in T_u} p_t^- (W_u) / |W_u|$, where $p_i^- (W') = \min_{w \in W'} p_{iw}$ with the set $W_u \subseteq W$ corresponding to the unassigned workers and $T_u \subseteq T$ to the set of unassigned tasks of a partial assignment. Before computing $\text{MinRLB}$ we apply to each partial solution the logic of the continuity constraints (10) and (11) to strengthen the bound. If tasks $i$ and $k$ have been assigned already to some worker $w$, we also assign all tasks succeeding $i$ and preceding $k$ to $w$. Similarly, if $i$ has been assigned to $w$ and some successor (predecessor) $j$ of $i$ is infeasible for $w$ we set $p_{kw} = \infty$ for all successors (predecessors) $k$ of $j$. The probabilistic beam search is shown in Algorithm 1.

4.2 The interval search method IPBS

An upper bound search starts from a known feasible cycle time and tries to reduce it iteratively. A common strategy is to reduce it successively by one and to try to find a better feasible solution by some heuristic algorithm. However, it is well known that heuristic assignment procedures are not monotone, i.e., they may find a feasible solution for some cycle time but not for larger cycle times. To overcome this, we propose to modify the upper bound search to examine an interval of cycle times ending at the current best upper bound. If the current lower and upper
Algorithm 1: Probabilistic beam search

**input** : A set of stations $S$, a candidate cycle time $C$, a beam width $\gamma$ and a beam factor $f$.  
**output**: A valid assignment or “failed” if no valid assignment could be found.

1. $B \leftarrow \{\emptyset\}$; /* current set of partial assignments */
2. for $k \in S$ do
3.   $B' \leftarrow \emptyset$;
4.   for $s \in B$ do
5.     for $f$ times do
6.       for all unassigned workers $w \in W$ do
7.         $s' \leftarrow s$ concatenated with a new empty station $k$;
8.       while there are available tasks $P$ that do not overload the current station do
9.         select a task $t \in P$ with probability proportional to $\text{MaxPW}^{-1}(t)$;
10.        assign $t$ to station $k$ in $s'$;
11.       if all tasks in $T$ are assigned in $s'$ then return Solution $s'$;
12.     else if $|B'| < \gamma$ then $B' \leftarrow B' \cup \{s'\}$;
13.     else
14.         $o \leftarrow \arg\min \{\text{MinRLB}(o') | o' \in B'\};$
15.         if $\text{MinRLB}(s') > \text{MinRLB}(o)$ then $B' \leftarrow B' \cup \{s'\} \setminus \{o\}$;
16.     end
17.   end
18.   $B \leftarrow B'$;
19. return “failed”;

bounds on the cycle time are $C$ and $\overline{C}$, the upper bound search will try to find a feasible solution for all cycle times between $\max\{C, \lfloor pC \rfloor\}$ and $\overline{C} - 1$ for a given factor $p \in (0, 1)$ and update $\overline{C}$ to the best cycle time found, if any. Otherwise, the upper bound search continues with the same interval. Since the beam search is probabilistic this may produce a feasible solution in a later trial. The interval search depends on three parameters: the minimum search time $t_{\text{min}}$, the maximum search time $t_{\text{max}}$ and the maximum number of repetitions $r$. The search terminates if the cycle time found equals the lower bound, or if the maximum time or the maximum number of repetitions are exceeded, but not before the minimum search time has passed.

Initially, the value of $C$ is set to the best of all lower bounds presented in Section 3. The initial upper bound $\overline{C}$ is determined by an single run of the beam search with a beam factor of one.

4.3 Improvement by local search

A local search is applied to the results found by interval search method. It focuses on critical stations whose load equals the cycle time of the current assignment. It tries to remove tasks from a critical station in order to reduce the cycle time. Since there can be multiple critical stations, a move is considered successful if it reduces the number of critical stations. The local search applies the following four types of moves, until the assignment cannot be improved any more.

1. A shift of a task from a critical station to another station.
2. A swap of two tasks. At least one of the tasks must be on a critical station.
3. A sequence of two shift moves. Here the first shift move is allowed to produce a worse result than the initial assignment.
4. A swap of workers between two stations without reassigning the tasks.
5 Task-oriented branch-and-bound algorithm

In this section we propose a branch-and-bound algorithm for ALWABP-2 using the bounds and the heuristic presented in the previous sections.

The algorithm first computes a heuristic solution by running the probabilistic beam search. It also applies the lower bounds $LC_1, LC_2, LC_3, M_3, \mathcal{T}_1, \mathcal{T}_2$ at the root node to obtain an initial lower bound.

If the solution cannot be proven optimal at the root node, the algorithm proceeds with a depth-first search. In branch-and-bound algorithms for assembly line balancing two branching strategies are common. The station-oriented method proceeds by stations and branches on all feasible maximal loads for the current station, while the task-oriented method, proceeds by tasks and branches on all possible stations for the current task. The most effective methods for SALBP use station-oriented branching. However, for the ALWABP the additional worker selection substantially increases the branching factor of the station-oriented approach. A worker-oriented strategy, on the other hand, has to consider much more station loads, since all subsets of unassigned tasks which satisfy the continuity constraints are candidates to be assigned to a worker. Therefore, we use a task-oriented branching strategy.

The proposed task-oriented method executes the recursive procedure shown in Algorithm 2. At each new node it applies the lower bounds $LC_1, LC_2, LC_3, \mathcal{T}_1$ (line 7), since the lower bounds $M_3$ and $\mathcal{T}_2$ are too slow to be applied during the search, although they obtain the best bounds. When a complete solution has been found, the algorithm updates the incumbent (line 2). Otherwise, it selects an unassigned task $t$ (line 4) and assigns it to all feasible workers (loop in lines 5-11).

Algorithm 2: branchTasks(llb, A)

| Line | Description |
|------|-------------|
| 1    | if $A = T$ then |
| 2    | if $llb < gub$ then $gub \leftarrow llb$; |
| 3    | return |
| 4    | select a task $t \in T \setminus A$; |
| 5    | foreach $w \in W : \text{assignmentIsValid}(t,w)$ do |
| 6    | apply reduction rules; |
| 7    | newllb $\leftarrow$ lower bound with new assignment $(t,w)$; |
| 8    | if newllb $< gub$ then |
| 9    | setAssignment$(t,w)$; |
| 10   | branchTasks(newllb, A $\cup \{t\}$); |
| 11   | unsetAssignment$(t,w)$; |

For branching, the task with the largest number of infeasible workers is chosen in line 4. A worker is considered infeasible, if the allocation of the task to the worker creates an immediate cyclic worker dependency or the lower bound $LC_1$ after the assignment is at least the value of the incumbent. In case of ties, the task with the largest lower bound is chosen, where the lower bound of a task is the smallest lower bound $LC_1$ over its feasible workers. This rule gives preference to tasks that tighten the lower bound early. Any remaining tie is broken by the task
after the task has been chosen, a branch is created for each valid worker. The branches are visited in order of non-decreasing lower bounds. Again, ties are broken by the worker index.

5.1 Valid assignments

The algorithm maintains a directed graph \( H \) on the set of workers to verify efficiently if the precedence constraints are satisfied. It contains an edge \((w, w')\) if there is some task \( t \) assigned to \( w \) and another task \( t' \) assigned to \( w' \), such that \((t, t') \in E^*\). The graph \( H \) also contains all resulting transitive edges. For a valid assignment of tasks, \( H \) must be acyclic. If this is the case, any topological sorting defines a valid assignment of workers to stations. The procedure \( \text{assignmentIsValid}(t, w) \) verifies in time \( O(|T| |W|) \) if the assignment of task \( t \) to worker \( w \) would insert an arc into \( H \) whose inverse arc exists already. Before branching to a new node, the procedure \( \text{setAssignment}(t, w) \) inserts such arcs into \( H \) and computes the new transitive closure in time \( O(|T| |W|) \). This is undone by \( \text{unsetAssignment}(t, w) \) when backtracking. To speed up the selection of a task for branching, we do not consider the violation of transitive dependencies in line 4, but only the creation of an immediate cyclic worker dependency, which results from inserting an edge \((w, w')\) for which \((w', w)\) is already present. This can be tested in time \( O(|P_t| + |F_t|) \).

5.2 Reduction rules

After a task \( t \) has been assigned to a worker \( w \), and before branching, we apply several more costly reduction rules to strengthen the lower bounds (line 6). First, we can set \( p_{tw'} = \infty \) for any \( w' \neq w \). Second, we can enforce the continuity constraints (10) and (11). An application of (10) may assign further tasks to \( w \), and the application of (11) may exclude some tasks from being assigned to \( w \) (whose execution time is set to \( p_{tw} = \infty \)). Finally, we can exclude a task-worker assignment \((t', w)\) if the total execution time \( p_{tw} + p_{t'w} + \sum_{u \in i(t, t')} p_{uw} \) of the tasks \( i(t, t') = (P_t^* \cap F_{t'}^*) \cup (F_t^* \cap P_{t'}^*) \) between \( t \) and \( t' \) is more than or equal to the current upper bound. These rules are repeatedly applied until no more tasks can be assigned or excluded.

6 Computational results

All algorithms were implemented in C++ and compiled with the GNU C compiler 4.6.3 with maximum optimization. The MIP models and their linear relaxations were solved using the commercial solver CPLEX 12.3. The experiments were done on a PC with a 2.8 GHz Core i7 930 processor and 12 GB of main memory, running a 64-bit Ubuntu Linux. All tests used only one core. Details of the results reported in this section are available online.\(^1\)

6.1 Test instances

A set of 320 test instances has been proposed by Chaves et al. (2007). They are characterized by five experimental factors: the number of tasks, the order strength (OS)\(^2\), the number of workers

\(^1\)http://www.inf.ufrgs.br/algopt

\(^2\)The order strength is number of precedence relations of the instance in percent of all possible relations \((|T|^2)\).
Table 2: Instance characteristics. The 320 instances are grouped by five two-level experimental factors into 32 groups of 10 instances.

| Factor                  | Low Level       | High Level      |
|-------------------------|-----------------|-----------------|
| Number of tasks $|T|$ | 25 – 28         | 70 – 75         |
| Order strength (OS)    | 22% - 23%       | 59% - 72%       |
| Number of workers $|W|$ | $|T|/7$          | $|T|/4$          |
| Task time variability (Var) | $[1, t_i]$     | $[1, 2t_i]$     |
| Number of infeasibilities (Inf) | 10%       | 20%             |

($|W|$), the task time variability (Var), and the percentage of infeasible task-worker pairs (Inf). All factors take two levels, as shown in Table 2, defining 32 groups of 10 instances. The instances are based on the SALBP instances Heskia (low $|T|$, low OS), Roszieg (low $|T|$, high OS), Wee-mag (high $|T|$, low OS), and Tonge (high $|T|$, high OS). The first worker of each instance executes task $t \in T$ in time $p_t$ of the corresponding SALBP instance, and the remaining workers have an execution time randomly selected in the interval $[1, p_i]$ (low variability) or $[1, 2p_i]$ (high variability).

6.2 Comparison of lower bounds

We first compare the strength of the lower bounds proposed in Section 3. To compute the lower bound $L_1$ we use the ascent direction method of van de Velde (1993). This bound was improved to $L_a 1$, as proposed by Martello et al. (1997). Their method applies a binary search for the best improved bound, which is obtained by solving $|S|$ knapsack problems of capacity $C$ for each trial cycle time $C$. Different from Martello et al. (1997) we solve the all-capacities knapsack problem by dynamic programming only once and use the resulting table during the binary search. The knapsack problems that arise when computing $L_2$ and $L_2$ by subgradient optimization are solved similarly.

Figure 2 shows the average relative deviation in percent from the best known value and the average computation time over all 320 instances. Looking at the models, the lower bound of $M_2$ is significantly better than $M_1$, and the addition of the continuity constraints improves the relative deviation by another 10%, yielding the best lower bound overall. The computation time of the three models is comparable, with $M_3$ being slower than the other two models. The linear relaxation of $R || C_{max}$ is slightly worse that $M_2$, but two orders of magnitude faster. The bounds $L_1$ and $T_1$ achieve about the same quality an order of magnitude faster than $R || C_{max}$. The lower bounds from the relaxation to SALBP are weaker than most of the other lower bounds, except $LC_1$, but another order of magnitude faster.

For the branch-and-bound we chose to use the lower bounds from the relaxation to SALBP and $T_1$, since the other bounds are too costly to be applied at every node of the branch-and-bound tree. We include all of the faster bounds, since they yield complementary results. In particular $LC_1$ obtains the best bound in average at the root node, but is less effective during the search.
6.3 Comparison of MIP models

We next compare the performance of the new MIP models with that of model $M_1$. Table 3 shows the average number of nodes and the average computation time needed to solve the instances to optimality for the 16 groups with a low number of tasks. The instance groups Tonge and Wee-Mag with a high number of tasks are not shown, since none of them could be solved to optimality within an hour.

Overall model $M_2$ needs significantly more nodes than $M_1$, and is a factor of about two slower. It executes more nodes per second, and has a better lower bound, but CPLEX is able to apply more cuts for model $M_1$ at the root, such that in average model $M_2$ has no advantage on the tested instances. However, when the continuity constraints are applied, model $M_3$ needs significantly less nodes and time compared to model $M_1$ (confirmed by a Wilcoxon signed rank test with $p < 0.01$). The results show that the continuity constraints are very effective, in particular for a high order strength and for high numbers of workers.

6.4 Results for the IPBS heuristic

We compare IPBS with three state of the art heuristic methods for the ALWABP-2, namely the hybrid genetic algorithm (HGA) of Moreira et al. (2012), the iterated beam search (IBS) of Blum and Miralles (2011), and the iterative genetic algorithm (IGA) of Mutlu et al. (2013). In
Table 3: Comparison of MIP models $M_1$, $M_2$, and $M_3$ on instances Roszieg and Heskia.

| Instance | $W$ | Var Inf | $M_1$ | $M_2$ | $M_3$ |
|----------|-----|---------|-------|-------|-------|
|          |     |         | Nodes | $t$ (s) | Nodes | $t$ (s) | Nodes | $t$ (s) |
| Roszieg  |     |         |       |        |       |        |       |        |
| 4        | L   | 10%     | 56.9  | 0.6  | 2340.4 | 1.1  | 37.8   | 0.7   |
|          |     | 20%     | 1.1   | 0.6  | 936.0  | 0.4  | 11.7   | 0.4   |
|          | H   | 10%     | 156.6 | 0.8  | 2849.5 | 1.3  | 58.6   | 1.5   |
|          |     | 20%     | 82.9  | 0.8  | 3268.4 | 1.6  | 53.8   | 0.8   |
| 6        | L   | 10%     | 2715.0| 12.1 | 47176.3| 52.4 | 249.9  | 4.6   |
|          |     | 20%     | 2601.3| 11.3 | 36555.7| 29.0 | 168.7  | 2.3   |
|          | H   | 10%     | 3467.0| 13.4 | 83900.5| 66.2 | 389.0  | 6.3   |
|          |     | 20%     | 2785.0| 11.8 | 50294.3| 44.2 | 281.5  | 4.5   |
| Heskia   |     |         |       |        |       |        |       |        |
| 4        | L   | 10%     | 0.0   | 0.6  | 105.2  | 0.2  | 29.8   | 0.3   |
|          |     | 20%     | 25.0  | 0.6  | 198.6  | 0.3  | 37.5   | 0.2   |
|          | H   | 10%     | 65.0  | 0.7  | 136.2  | 0.2  | 49.0   | 0.3   |
|          |     | 20%     | 24.3  | 0.7  | 130.5  | 0.3  | 45.5   | 0.2   |
| 7        | L   | 10%     | 1535.9| 13.4 | 1552.2 | 4.6  | 86.8   | 1.0   |
|          |     | 20%     | 1174.1| 11.1 | 940.8  | 2.2  | 102.4  | 1.0   |
|          | H   | 10%     | 1677.8| 12.9 | 735.9  | 2.5  | 115.4  | 1.1   |
|          |     | 20%     | 1344.1| 13.4 | 663.3  | 2.8  | 151.7  | 1.3   |
| Averages |     |         | 1107.0| 6.6  | 14486.5| 13.1 | 122.1  | 1.7   |

Table 4: Parameters of the probabilistic beam search used in the computational experiments.

| Parameter                              | Value |
|----------------------------------------|-------|
| Beam width $w$                         | 125   |
| Branching factor $f$                   | 5     |
| Cycle time reduction for interval search $p$ | 0.95 |
| Minimum search time $t_{\text{min}}$ (s) | 6    |
| Maximum search time $t_{\text{max}}$ (s) | 900  |
| Maximum number of interval searches $r$ | 20   |

Preliminary experiments we determined reasonable parameters for the probabilistic beam search as shown in Table 4. For the HGA and the IBS we compare in Table 5 the relative deviation from the current best known value (Gap) and the computation time ($t$), in average for each group of instances and over 20 replications per instance. We further report the average computation time to find the best value ($t_b$), and the average relative deviation of the best solution of the 20 replications (Gap$_b$). The total computation time of Blum and Miralles (2011) is always 120s more than the time to find the best value, and has been omitted from the table.

We can see that the problem can be considered well solved for a low number of tasks, since all three methods find the optimal solution with a few exceptions in less than ten seconds. In six instance groups the IPBS terminates in less than the minimum search time, since the solution was provably optimal. For instances with a high number of tasks, IBS produces better solutions for more workers, while the HGA is better on less workers. IPBS always achieves better results than both methods (confirmed by a Wilcoxon signed rank test with $p < 0.01$). This holds for the
| Instance | | Var | Inf | HGA | | IBPS | | |
|---|---|---|---|---|---|---|---|
| | | | | t(s) | t_b(s) | Gap | Gap_b(s) |
| Roszieg | 4 | L | 10% | 3.3 | 0.0 | 0.0 | 0.0 |
| | | | | 20% | 4.5 | 0.0 | 0.0 |
| | | H | 10% | 4.0 | 0.0 | 0.0 | 0.1 |
| | | 6 | L | 10% | 3.6 | 0.0 | 0.0 |
| | | | | 20% | 4.0 | 0.1 | 0.0 |
| | | H | 10% | 4.5 | 0.0 | 0.0 | 0.1 |
| | | 7 | L | 10% | 6.9 | 0.2 | 0.0 |
| | | | | 20% | 9.3 | 0.3 | 0.1 |
| | | H | 10% | 9.2 | 0.3 | 0.0 | 0.0 |
| | | 10 | L | 10% | 8.0 | 0.2 | 0.5 |
| | | | | 20% | 7.4 | 0.3 | 0.6 |
| | | H | 10% | 6.6 | 0.2 | 0.3 | 0.0 |
| | | 17 | L | 10% | 205.7 | 34.4 | 5.9 |
| | | | | 20% | 241.2 | 34.9 | 4.2 |
| | | H | 10% | 347.5 | 56.9 | 4.3 | 2.7 |
| | | 19 | L | 10% | 136.9 | 56.9 | 6.6 |
| | | | | 20% | 158.8 | 60.1 | 7.6 |
| | | H | 10% | 248.5 | 115.8 | 8.9 | 3.9 |
| | | 11 | L | 10% | 136.9 | 56.9 | 6.6 |
| | | | | 20% | 158.8 | 60.1 | 7.6 |
| | | H | 10% | 248.5 | 115.8 | 8.9 | 3.9 |
| | | 19 | L | 10% | 283.7 | 97.9 | 15.3 |
| | | | | 20% | 288.1 | 108.9 | 13.0 |
| Total averages | | | | 143.7 | 40.1 | 4.9 | 2.7 |

The table above presents the comparison of the proposed heuristic with a hybrid genetic algorithm (Moreira et al., 2012) and an iterated beam search (Blum and Miralles, 2011). In general, IPBS is more robust and finds solutions that are closer to the best known values compared to the other methods. The execution times vary depending on the instance and the method used. The average computation times are as follows: HGA: 36.4 s, IBS: 8.2 s, and IPBS: 2.5 s. The difference in execution times is due to the different methods used and the hardware conditions.
time to find the best solution.) The faster average computation times are mainly due to the instances with a high number of tasks, for which IPBS scales better. The best solutions are almost always found in less than 30 seconds.

For all heuristics, the computation time is significantly less for a low number of tasks, a low number of workers, and a low order strength. Similarly, the relative deviations are smaller for a low number of tasks and low order strength. However, the relative deviation does not depend significantly on the number of workers, except for the HGA, which produces better solution for a low number of workers. (These findings are confirmed by a Wilcoxon signed rank test at significance level $p < 0.01$.) For IBS and IBPS there is an interaction between the number of workers and the order strength: both produce better solutions for a low number of workers and a high order strength or vice versa.

Since for the IGA no detailed results are available, we compare in Table 6 with the summarized values reported by Mutlu et al. (2013): the average cycle time ($C$), the average cycle time

![Table 6: Comparison of the proposed heuristic with an iterated genetic algorithm (Mutlu et al., 2013).](https://example.com/table6.png)
of the best found solution ($C_b$), and the average computation time to find the best value ($t_b$). The values are again averages for all groups of instances, but over only 10 replications for the IGA. The results for our method are the same as in Table 5 but in absolute values. Note that this evaluation may mask large deviations in instances with low cycle times and overestimate small deviations for high cycle times.

As the other methods, the IGA solves the small instances optimally, but not the larger ones. Compared to our method, its average performance is worse except for three groups of wee-mag with a low number of workers, where the average cycle time is about 0.2 lower. The comparison is similar for the best found values, where the IGA is better by 0.4 in a single group. In average over all large instances our method produces a cycle time of about 1 unit less.

The execution times of the two methods are comparable. The results of Mutlu et al. (2013) have been obtained on a Intel Core 2 Duo T5750 processor running at 2.0 GHz, whose performance is within a factor of three from our machine. Taking this into account, our methods find the best value about 50% faster.

In summary, the results show that IPBS can compete with and often outperforms the other methods in solution quality as well as computation time. The difference to the other methods is smallest for the large instances with a low order strength and a low number of workers. IPBS in general is very robust over the entire set of instances.

### 6.5 Results for the branch-and-bound algorithm

We evaluated the branch-and-bound algorithm on the same 320 test instances. For the tests, IPBS was used to produce an initial heuristic solution. It was made deterministic by fixing a random seed of 42 and configured with a minimum search time of 0s and a maximum search time of $|T||W|/10s$. During the search the number of iterations of the ascent direction method to compute $L_1$ has been limited to 50, and the number of iterations for the subgradient optimization to compute $L_2$ to 20.

The only other branch-and-bound algorithm in the literature proposed by Miralles et al. (2008) for the ALWABP-2 has been found inferior to model $M_1$ by Chaves et al. (2009) in tests with CPLEX (version 10.1). We therefore limit our comparison to the MIP models. We first compare our approach to CPLEX on the best model $M_3$ on the instances with a low number of tasks in Table 7. In Table 8 we then present the results of the branch-and-bound algorithm with a time limit of one hour on the larger instances. CPLEX is not able to solve any of the models on the instances with a high number of tasks within this time limit.

Table 7 shows the average solving time and the average number of nodes in the branch-and-bound tree for all instance groups with a low number of workers. On these instances both methods have a similar performance, solving all instances in a few seconds, and are even competitive with the heuristic methods. In most cases the branch-and-bound algorithm needs fewer nodes than CPLEX, except for five groups with a low number of workers. Computation times are also comparable, although the time of the branch-and-bound algorithm is dominated by the initial heuristic.

Table 8 shows the results of the branch-and-bound algorithm on the larger instances. We report the number of optimal solutions found (Opt) and the number of solutions proven to be optimal (Prov), the average computation time ($t$), the average relative deviation from the best
Table 7: Comparison of model $M_3$ to the branch-and-bound algorithm on instances with a low number of workers.

| Instance | W | Var | Inf | $t$ (s) | Nodes | B&B | $t$ (s) | Nodes |
|----------|---|-----|-----|--------|-------|-----|--------|-------|
| Roszieg | 4 | L 10% | 0.7 | 37.8 | 0.2 | 34.0 | 0.4 | 11.7 | 0.2 | 16.1 |
|         |   | H 10% | 1.5 | 58.6 | 0.3 | 44.8 | 20% | 53.8 | 0.2 | 37.9 |
|         | 6 | L 10% | 4.6 | 249.9 | 0.7 | 126.5 | 20% | 168.7 | 0.7 | 77.9 |
|         |   | H 10% | 6.3 | 389.0 | 0.8 | 208.1 | 20% | 281.5 | 0.8 | 130.5 |
| Heskia  | 4 | L 10% | 0.3 | 29.8 | 0.6 | 35.3 | 20% | 37.5 | 0.6 | 40.8 |
|         |   | H 20% | 0.3 | 49.0 | 0.7 | 54.6 | 20% | 45.5 | 0.7 | 64.8 |
|         | 7 | L 10% | 1.0 | 86.8 | 2.5 | 20.2 | 20% | 102.4 | 2.9 | 23.2 |
|         |   | H 10% | 1.1 | 115.4 | 2.4 | 13.5 | 20% | 151.7 | 3.1 | 17.3 |
|         |   | Averages | 1.7 | 116.8 | 1.1 | 59.1 |

Table 8: Results of the branch-and-bound algorithm on instances with a high number of workers.

| Instance | W | Var | Inf | Opt. | Prov. | $t$ (s) | Gap | C |
|----------|---|-----|-----|------|-------|--------|-----|--|
| Tonge    | 10 | L 10% | 10 | 10 | 175.7 | 0.0 | 90.6 |
|          |   | H 20% | 10 | 10 | 144.1 | 0.0 | 106.7 |
|          |   | H 10% | 10 | 10 | 406.8 | 0.0 | 159.3 |
|          |   | H 20% | 10 | 10 | 213.4 | 0.0 | 163.9 |
|          | 17 | L 10% | 10 | 10 | 775.5 | 0.0 | 31.6 |
|          |   | H 20% | 10 | 9 | 928.6 | 0.0 | 36.9 |
|          |   | H 10% | 7 | 7 | 1453.0 | 0.7 | 63.5 |
|          |   | H 20% | 10 | 10 | 1211.7 | 0.0 | 61.2 |
| Wee-mag  | 11 | L 10% | 5 | 3 | 3102.1 | 2.2 | 26.7 |
|          |   | H 20% | 5 | 2 | 3504.9 | 2.3 | 31.9 |
|          |   | H 10% | 4 | 2 | 3051.6 | 2.4 | 46.9 |
|          |   | H 20% | 4 | 4 | 2853.2 | 1.9 | 45.2 |
|          | 19 | L 10% | 2 | 0 | 3600.0 | 2.2 | 10.1 |
|          |   | H 20% | 5 | 2 | 3151.5 | 0.9 | 11.4 |
|          |   | H 10% | 4 | 3 | 2732.1 | 3.7 | 17.7 |
|          |   | H 20% | 4 | 3 | 3032.5 | 2.9 | 17.6 |
|         |   | Totals/Averages | 111 | 95 | 1896.0 | 1.20 |
known value (\textit{Gap}), and the average cycle time for each group of instances (C).

In about 70\% of the instances the optimal solution was found, and about 60\% of the solutions could be proven to be optimal within the time limit. All except four instances with a high order strength were solved. The average relative deviation over all 320 instances is 0.60\%, about one third of the average case of the best heuristic.

As expected, the solution times are higher than those of the heuristic methods but for the instances with a high order strength only about an order of magnitude, in average. The solving time depends mainly on the number of tasks, the number of workers, and the order strength (as confirmed by a Kruskal-Wallis test followed by Wilcoxon signed rank post hoc tests at significance level \( p < 0.01 \)). The instances with a high order strength or a low number of workers are easier to solve, because the reduction rules are more effective.

7 Conclusion

We have presented a new MIP model, a heuristic search procedure and an exact algorithm for solving the Assembly Line Worker Assignment and Balancing Problem of type 2. The new MIP model shows the importance of including continuity constraints in this type of problem, and its linear relaxation gives the current best lower bound for the problem. The proposed heuristic IPBS is competitive with the current best methods, often outperforms them in computation time and solution quality, and shows a robust performance over the complete set of 320 test instances. Finally, the branch-and-bound method can solve instances with a low number of tasks in a few seconds, and was able to optimally solve 95 of the 160 instances with a high number of tasks for the first time.

With respect to the problem, constraints that enforce continuity have shown to be the most effective way of strengthening the lower bounds in the models as well as the heuristic and exact algorithm. Besides the size of the instance, the number of workers and the order strength has the strongest influence on the problem difficulty. All methods are able to solve instances with a high order strength better. This also holds for the branch-and-bound algorithm on instances with a low number of workers.

Our results show that assembly lines with heterogeneous workers can be balanced robustly and close to optimal for problems of sizes of about 75 tasks and 20 workers. Problems of this size arise, for example, in Sheltered Work Centers for Disabled, and we hope that these methods will contribute to a better integration of persons with disabilities in the labour market. A very interesting future line of research in this context may be the integration of persons with disabilities into larger assembly lines with regular workers.

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