Optimal networks by mass transportation via points allocation

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Abstract

It is shown that optimal network plans can be obtain, naturally, as a limit of easier problems of points allocations. These point allocation problem are obtained by minimizing the mass transportation on the set of atomic measures of prescribed number of atoms, each can be solved by linear programming.

1 Introduction

In this note I’ll present a result which demonstrate that a solution of a certain class of optimal networks can be obtained as a limit of discrete problems in which we optimize certain functions of finite number of variables. For updated references on optimal networks via mass transportation see [BS, BCM].

Let us start with the definition of the optimal network we have in mind. Consider $N$ points $\{x_1, \ldots, x_N\}$ (sources) in a state space (say, $\mathbb{R}^k$), and another $N$ points $\{y_1, \ldots, y_N\} \subset \mathbb{R}^k$ (sinks). For each source $x_i$ we attribute a certain amount of mass $m_i \geq 0$. Similarly, $m_i^* \geq 0$ is the capacity attributed to the sink $y_i$, while

$$\sum_{i=1}^{N} m_i = \sum_{i=1}^{N} m_i^* > 0.$$  

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We denote this system by an atomic measure \( \lambda := \lambda^+ - \lambda^- \) where

\[
\lambda^+ = \sum_{i=1}^{N} m_i \delta_{x_i} \quad ; \quad \lambda^- = \sum_{i=1}^{N} m_i^* \delta_{y_i}
\]  

(1.1)

where \( \delta_{(\cdot)} \) is the Dirac delta function.

The object is to transport the masses from the sources to sinks in an optimal way, such that the sinks will fill up according to their capacity. For each \( q > 1 \) we consider the cost function for such a transport as \( \hat{W}^{(q)}(\lambda) \) defined below (see [X])

**Definition 1.1.** Let \( \lambda \) be a finite measure satisfying \( \int d\lambda = 0 \). Then

\[
\hat{W}^{(q)}(\lambda) := \min_{m \in H(\lambda)} \int_{\mathbb{R}^k} |m|^{1/q}
\]

where

\[
H(\lambda) := \{ m ; \nabla \cdot m = \lambda \}
\]

where \( \{ m \} \) are the set of \( \mathbb{R}^k \) valued Radon measures in \( \mathbb{R}^k \) and \( \nabla \cdot m = \lambda \) is understood as a distribution, that is

\[
\int \nabla \psi \cdot m = \int \psi d\lambda
\]

for any \( \psi \in C_0^1(\mathbb{R}^k) \).

If \( \lambda \) is atomic as in (1.1) then there is an equivalent, more intuitive definition

**Definition 1.2.**

1. Given \( \lambda \) as in (1.1), an oriented, weighted graph \((\gamma, m)\) associated with \( \lambda \) is a graph \( \gamma \) embedded in \( \mathbb{R}^k \), composed of vertices \( V(\gamma) \) and edges \( E(\gamma) \). The orientation of an edge \( e \in E(\gamma) \) is determined by \( \partial e = v^+_e - v^-_e \) where \( v^+_e \in V(\gamma) \) are the vertices composing the end points of \( e \). The graph \( \gamma \) and the capacity function \( m : E(\gamma) \to \mathbb{R}^+ \cup \{0\} \) satisfy

(a) \( \{x_1, \ldots, x_N, y_1, \ldots, y_N\} \subset V(\gamma) \).
(b) For each \( i \in \{1, N\} \), \( \sum_{\{e, x_i \in \partial^+ e\}} m_e = m_i \) and \( \sum_{\{e, y_i \in \partial^- e\}} m_e = m_i^* \), where \( \partial^\pm e := v_e^\pm \).

(c) For each \( v \in V(\gamma) \setminus \{x_1, \ldots, y_N\} \), \( \sum_{\{e, v \in \partial^+ e\}} m_e = \sum_{\{e, v \in \partial^- e\}} m_e \).

2. The set of all weighted graphs associated with \( \lambda \) is denoted by \( \Gamma(\lambda) \).

3. Let
\[
\hat{W}^q(\lambda) := \inf_{(\gamma, m) \in \Gamma(\lambda)} \sum_{e \in E(\gamma)} |e|m_e^{1/q} \tag{1.2}
\]

There are two special cases which can be observed. In the limit case \( q = 1 \) the optimal graph satisfies \( V(\gamma) = \{x_1, \ldots, y_N\} \) and \( \hat{W}^{(1)}(\lambda) = W_1(\lambda^+, \lambda^-) \). Here \( W_q(\lambda^+, \lambda^-) \) for \( q \geq 1 \) is the Wasserstein distance between \( \lambda^+ \) to \( \lambda^- \),
\[
W_q(\lambda^+, \lambda^-) := \left( \min_{\{\lambda^{i,j}\}} \sum_{i=1}^N \sum_{j=1}^N |x_i - y_j|^q \lambda^{i,j} \right)^{1/q},
\]
the minimum is taken in the set of \( N \times N \) matrices satisfying
\[
\sum_{i=1}^N \lambda^{i,j} = m_j^* ; \quad \sum_{j=1}^N \lambda^{i,j} = m_i^*.
\]
The second case is the limit \( q = \infty \). This is the celebrated Steiner Tree Problem [HRW]:
\[
\inf_{\gamma \in \Gamma(\lambda)} \sum_{e \in E(\gamma)} |e|,
\]
where, this time, \( \Gamma(\gamma) \) is the set of all graphs satisfying \( \{x_1, \ldots, y_N\} \subset V(\gamma) \) and is, actually, independent of the masses \( m_1 \) and capacities \( m_i^* \).

In [W, Thm 2] it was shown that \( W_1 \) is obtained from \( W_q \) by an asymptotic expression in the limit of infinite mass:

**Theorem 1.1.** If \( \lambda = \lambda^+ - \lambda^- \) is any Borel measure satisfying \( \int d\lambda = 0 \), then
\[
\lim_{M \to \infty} M^{1-1/q} \min_{\mu \in \mathcal{B}^+_M} W_q(\mu + \lambda^+, \mu + \lambda^-) = W_1(\lambda^+, \lambda^-)
\]
where \( \mathcal{B}^+_M \) stands for the set of all positive Borel measures \( \mu \) normalized by \( \int d\mu = M \).
If, in particular, \( \lambda \) is an atomic measure of the form (1.1), than it can be shown that for fixed \( M \) the minimizer of \( W_q(\mu + \lambda^+, \mu + \lambda^-) \) in \( B^+_M \) is an atomic measure of a finite number of atoms as well.

Our main result demonstrates that we can get the network cost \( \hat{W}(q) \) by the same expression, if we replace the total mass \( M \) by the cardinality of the support of the atomic measure \( \mu \).

2 Main results

For each \( n \in \mathbb{N} \), let \( B^{+, n} \) be the set of all atomic, positive measures of at most \( n \) atoms, that is:

\[
B^{+, n} := \left\{ \sum_{j=1}^{n} \alpha_j \delta_{z_j} \; ; \; z_j \in \mathbb{R}^k, \; \alpha_j \geq 0 \right\}
\]

Theorem 2.1. For any \( q > 1 \) and \( \lambda \) as in (1.1)

\[
\lim_{n \to \infty} \inf_{\mu \in B^{+, n}} W_q(\mu + \lambda^+, \mu + \lambda^-) = \hat{W}(q)(\lambda) .
\]

The set \( B^{+, n} \) is, evidently, not a compact one. Still we claim

Lemma 2.1. For each \( n \in \mathbb{N} \), a minimizer \( \mu_n \in B^{+, n} \)

\[
\hat{W}_q(\lambda) := \inf_{\mu \in B^{+, n}} W_q(\mu + \lambda^+, \mu + \lambda^-) = W_q(\mu_n + \lambda^+, \mu_n + \lambda^-)
\]

exists.

Theorem 2.2. Let \( \mu_n \) be a regular\(^2\) minimizer of \( W_q(\mu + \lambda^+, \mu + \lambda^-) \) in \( B^{+, n} \). Then \( \mu \in \lim_{n \to \infty} \mu_n \) if and only if \( \mu \) is supported on an optimal graph \((\gamma, m) \in \Gamma(\lambda) \) of (1.2), \( \mu \) is uniform on each edge \( e \in E(\gamma) \) and \( m_e^{1/q} |e| \) is independent of \( e \in E(\gamma) \).

\(^2\)see Definition 3.1 below
3 Details

It is evident that any $\mu \in B_{+}^{n}$ which satisfies $\mu \geq \mu_{n}$ is a minimizer of (2.2) as well. Let us characterize such a minimizer via a linear program scheme.

Given $Z = (z_{1}, \ldots, z_{n}) \in (\mathbb{R}^k)^n$, define

$$F_{q}(Z; \lambda) := \min \left\{ \sum_{1}^{n} \sum_{1}^{n} \lambda_{i,j}|z_{i} - z_{j}|^{q} + \sum_{i=n+1}^{N+n} \sum_{j=1}^{n} \lambda_{i,j}|x_{i} - z_{j}|^{q} + \sum_{i=1}^{n} \sum_{j=n+1}^{N+n} \lambda_{i,j}|z_{i} - y_{j}|^{q} + \sum_{i=n+1}^{N+n} \sum_{j=n+1}^{N+n} |x_{i} - y_{j}|^{q} \right\}$$

(3.1)

where $\{\lambda_{i,j}\}$ satisfy

$$\lambda_{i,j} \geq 0 \quad n+1 \leq k \leq N+n \implies \sum_{i=1}^{n+N} \lambda_{k,i} = m_k, \quad \sum_{i=1}^{n+N} \lambda_{i,k} = m^*_k$$

$$\sum_{i=1}^{n+N} \lambda_{i,j} = \sum_{i=1}^{n+N} \lambda_{j,i} \text{ for any } n+1 \leq j \leq n+N.$$  

(3.2)

Lemma 3.1. For $\lambda$ as in (1.1),

$$\overline{W}_{q}(\lambda) = \min_{Z \in (\mathbb{R}^k)^n} F(Z; \lambda).$$

where $\overline{W}_{q}$ as given in (2.2).

Definition 3.1. Let $\{\lambda_{i,j}\}$ be a transport plan of $n$ points for a given $\lambda \in (1.1)$. It is called a regular plan if it satisfies the following for any $1 \leq i, j \leq n + N$:

(a) $\lambda_{i,i} = 0$.

(b) $\lambda_{i,j}\lambda_{j,i} = 0$.

(c) There exists at most one $k$ and a chain $i_1 = i, i_2, \ldots, i_k = j$ such that

$$\Pi_{i=1}^{k-1} \lambda_{i}\lambda_{i_{i+1}} > 0.$$
(d) If \( j > n + 1 \) there exists a chain \( i_1 = i, i_2, \ldots i_k = j \) and a chain (of, possibly, different length \( k' \)) \( j_1 = j, j_2, \ldots j_{k'} = i \) such that \( \Pi_{l=1}^{k-1} \lambda_{i_l} \lambda_{i_{l+1}} > 0 \) and \( \Pi_{l=1}^{k' - 1} \lambda_{j_l} \lambda_{j_{l+1}} > 0 \).

If \( \{\lambda_{i,j}\} \) is a regular plan, then \( \mu \in \mathcal{B}^{+,n} \) is called a regular measure if for each \( i \in \{1, \ldots n\} \) there exists \( z_i \in \mathbb{R}^k \) where \( \mu(\{z_i\}) = \sum_{j=1}^{n+N} \lambda_{i,j} \).

**Lemma 3.2.** For each \( Z \in (\mathbb{R}^k)^n \) there exists a regular optimal plan of (3.1). In particular, for any \( n \in \mathbb{N} \) there exists a minimizer \( \mu_n \in \mathcal{B}^{+,n} \) of (2.2) which is regular.

Let now \( \{\lambda_{i,j}\} \) be an optimal regular plan corresponding to a minimizer \( \mu_n \in \mathcal{B}^{+,n} \) of (2.2). We may associate a weighed, directed graph with this plan as follows:

(i) If \( Z = \{z_1, \ldots, z_n\} \in (\mathbb{R}^n)^k \) is a minimizer of \( F_q(\cdot; \lambda) \), then \( V(\gamma) = \{z_1, \ldots, z_n, x_1, \ldots, y_N\} := \{\zeta_1, \ldots, \zeta_{n+2N}\} \).

(ii) \( E(\gamma) \) is given by the set of segments \( e_{k,l} := [\zeta_k, \zeta_l] \) for which \( \lambda_{k,l} > 0 \), while \( \partial e_{k,l} = \zeta_k - \zeta_l \).

(iii) \( m_{e_{k,l}} := \lambda_{k,l} \).

From regularity we also obtain the following:

**Lemma 3.3.** If \( \lambda \) as in (1.1), \( \mu_n \in \mathcal{B}^{+,n} \) is a regular minimizer of (2.2) and \( \{\lambda_{i,j}\} \) the associated regular optimal plan, then the associated graph \( (\gamma, m) \) contains no cycles. In addition, there exist some \( K \) depending on \( N \) (and independent of \( n \)) such that the number of vertices in \( V(\gamma) \) of degree larger than 1 is smaller than \( K \), and the maximal degree of all vertices is bounded by \( K \) as well.

In particular, the cardinality \( |E(\gamma)| \) of \( E(\gamma) \) is smaller than \( n + 2N + K^2 \). By definition \( W_q(\mu) = \sum_{e \in E} m_e |e|^q \) and the Holder inequality applied to the regular minimizer \( \mu_n \in \mathcal{B}^{+,n} \) yields

\[
\hat{W}^{(q)}(\lambda) = \sum_{e \in E(\gamma)} m^{1/q}_e |e| \leq \left( \sum_{e \in E(\gamma)} m_e |e|^q \right)^{1/q} |E|^{(q-1)/q} \leq |n + 2N + K^2|^{(q-1)/q}(\gamma)W_q(\mu_n + \lambda^+, \mu_n + \lambda^-)
\]
This implies the inequality
\[ \lim_{n \to \infty} n^{1-1/q} \inf_{\mu \in \mathcal{B}^+} W_q (\mu + \lambda^+, \mu + \lambda^-) \geq \widehat{W}(q)(\lambda). \]

To prove the reverse inequality in (2.1) we consider an optimal weighed graph \((\gamma, m)\) and construct \(\mu_n \in \mathcal{B}^+\) supported on \(\gamma\) which satisfy
\[ \lim_{n \to \infty} n^{1-1/q} W_q (\mu_n + \lambda^+, \mu_n + \lambda^-) = \sum_{e \in E(\gamma)} m_e^{1/q} |e| = \widehat{W}(q)(\lambda). \]

Assume \(n_e\) is the number of points of \(\mu_n\) on the edge \(e\), and any atom of \(\mu_n\) is of weight \(m_e\). The contribution to \(W_q(\mu_n + \lambda^+, \mu_n + \lambda^-)\) from \(e\) is, then
\[ \approx m_e \left( \frac{|e|}{n_e} \right)^q n_e = m_e |e|^q \frac{n_e^{q-1}}{n_e} \approx n^{q-1} \sum_{e \in E(\gamma)} m_e |e|^q \frac{n_e^{q-1}}{n_e}. \]

The constraint on \(n_e\) is given by \(\sum_{e \in E(\gamma)} n_e = n\). Let us rescale \(w_e := n_e/n\). Then we need to minimize
\[ F(w) := \sum_{e \in E(\gamma)} m_e |e|^q \frac{n_e^{q-1}}{w_e^{q-1}} \]
subjected to \(\sum_{e \in E(\gamma)} w_e = 1\). Let \(\alpha\) be the Lagrange multiplier with respect to the constraint \(\sum_{e \in E(\gamma)} w_e\). Since \(F\) is convex in \(w_e\) we get that \(F\) is maximized at
\[ \max_{\alpha} \min_w F(w) + \alpha \left( \sum_{e \in E(\gamma)} w_e - 1 \right). \quad (3.3) \]

So, let
\[ G(\alpha) := \min_w F(w) + \sum_{e \in E(\gamma)} w_e \alpha \]

The minimizer is obtained at
\[ (q - 1) \frac{m_e |e|^q}{w_e^{q-1}} = \alpha \Rightarrow w_e = (q - 1)^{1/q} m_e^{1/q} |e| \alpha^{1/q} \]
so
\[ G(\alpha) = q(q - 1)^{1/q-1} \sum_{e \in E(\gamma)} m_e^{1/q} |e| \alpha^{(q-1)/q}. \]
and the minimum is obtained at

$$\min_{(m,\gamma) \in \Gamma(\lambda)} \max_{\alpha} G(\alpha) - \alpha = \max_{\alpha} q(q-1)^{1/q-1} \tilde{W}^{(q)}(\lambda) \alpha^{(q-1)/q} - \alpha = \left( \tilde{W}^{(q)}(\lambda) \right)^q \quad (3.4)$$

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