Native Banach spaces for splines and variational inverse problems

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Abstract

We propose a systematic construction of native Banach spaces for general spline-admissible operators \( L \). In short, the native space for \( L \) and the (dual) norm \( \| \cdot \|_{\mathcal{X}'} \) is the largest space of functions \( f : \mathbb{R}^d \to \mathbb{R} \) such that \( \| Lf \|_{\mathcal{X}'} < \infty \), subject to the constraint that the growth-restricted null space of \( L \) be finite-dimensional. This space, denoted by \( \mathcal{X}_L' \), is specified as the dual of the pre-native space \( \mathcal{X}_L \), which is itself obtained through a suitable completion process. The main difference with prior constructions (e.g., reproducing kernel Hilbert spaces) is that our approach involves test functions rather than sums of atoms (e.g., kernels), which makes it applicable to a much broader class of norms, including total variation. Under specific admissibility and compatibility hypotheses, we lay out the direct-sum topology of \( \mathcal{X}_L \) and \( \mathcal{X}_L' \), and identify the whole family of equivalent norms. Our construction ensures that the native space and its pre-dual are endowed with a fundamental Schwartz-Banach property. In practical terms, this means that \( \mathcal{X}_L' \) is rich enough to reproduce any function with an arbitrary degree of precision.

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1 Introduction

Given a series of data points \((x_m, y_m) \in \mathbb{R}^d \times \mathbb{R}\), the basic interpolation problem is to find a function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) such that \(f(x_m) = y_m\) for \(m = 1, \ldots, M\). In order to make the problem well posed, one needs to impose additional constraints on \(f\); for instance that \(f\) is in the linear span of a finite number of known basis functions (standard regression problem), or that the desired function minimizes some energy functional. The variational formulation of splines follows the latter strategy and ensures the existence and unicity of the solution for energy functionals of the form \(\|Lf\|_2^2\) with \(L\) a suitable differential operator, the prototypical example being \(L = D^n\) with \(D\) the derivative operator [30, 24, 10].

We adopt in this paper an abstract formulation that encompasses the great majority of variational theories of splines that have been considered in the literature such as [9, 5] to give some notable examples. Given a suitable \(^1\) pair of Banach (or Hilbert) spaces \((\mathcal{X}', \mathcal{X}_L')\) whose elements are functions on \(\mathbb{R}^d\), a regularization operator \(L : \mathcal{X}_L' \overset{\approx}{\rightarrow} \mathcal{X}'\), and a vector-valued measurement functional \(\boldsymbol{\nu} : \mathcal{X}_L' \rightarrow \mathbb{R}^M, f \mapsto \boldsymbol{\nu}(f) = (\langle \nu_1, f \rangle, \ldots, \langle \nu_M, f \rangle)\), we define the generalized spline interpolant \(f_{\text{int}} : \mathbb{R}^d \rightarrow \mathbb{R}\) as the solution of

\(^1\)Our notation reflects the property that the two spaces are the topological duals of two primary spaces \((\mathcal{X}, \mathcal{X}_L)\). \(\mathcal{X}\) is a classical function space such as \(L_p(\mathbb{R}^d)\), while \(\mathcal{X}_L'\) is the corresponding native space for \(L\); that is, the largest Banach space such that \(\|Lf\|_{\mathcal{X}'}\) is well-defined.
the variational linear inverse problem

\[ f_{\text{int}} = \arg \min_{f \in \mathcal{X}'} \| Lf \|_{\mathcal{V}} \quad \text{s.t.} \quad \nu(f) = (y_1, \ldots, y_M). \quad (1) \]

The simplest case occurs when there is an isomorphism between \( \mathcal{X}' \) and \( \mathcal{X}'_L \), meaning that the regularization operator \( L \) has a stable inverse \( L^{-1} : \mathcal{X}' \rightarrow \mathcal{X}'_L \). In particular, when \( \nu_m = \delta(\cdot - x_m) : f \mapsto f(x_m) \) and \( \mathcal{H} = \mathcal{X}'_L \) is a reproducing-kernel Hilbert space (RKHS) such that \( \langle f, g \rangle_{\mathcal{H}} = \langle Lf, Lg \rangle = \langle L^*Lf, g \rangle \) for all \( f, g \in \mathcal{H} \), then the solution of (1) with \( \mathcal{X}' = L_2(\mathbb{R}^d) \) is expressible as

\[ f_{\text{int}} = \sum_{m=1}^{M} a_m h(\cdot, x_m), \quad (2) \]

where

\[ h(\cdot, x_m) = (L^*L)^{-1}\{\delta(\cdot - x_m)\} \quad (3) \]

is the (unique) reproducing kernel of \( \mathcal{H} \). The bottom line is that (2) is a linear model parametrized by \( a \in \mathbb{R}^M \). In addition, we have that \( \| f_{\text{int}} \|_{\mathcal{H}}^2 = a^T G a \), where \( G \in \mathbb{R}^{M \times M} \) is the positive-definite Gram matrix whose entries are given by \( [G]_{m,n} = h(x_m, x_n) \); this then yields the solution \( a = G^{-1} y \) of our initial interpolation problem with \( y = (y_1, \ldots, y_M) \).

The theory of RHKS also works the other way around in the sense that, instead of the operator \( L \), one can specify a positive-definite kernel \( h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \). One then constructs the native space \( \mathcal{X}'_L = \mathcal{H} \) by considering the closure of the vector space specified by (2) with varying \( M \in \mathbb{N} \) and \( x_m \in \mathbb{R}^d \), i.e., \( \mathcal{H} = \text{span}\{h(\cdot, x)\}_{x \in \mathbb{R}^d} \).

This kind of result is extendable to the scenario where \( L \) has a finite-dimensional null space \( \mathcal{N}_L = \text{span}\{p_n\}_{n=1}^{N_0} \), in which case the solution takes the generic parametric form

\[ f_{\text{int}} = \sum_{m=1}^{M} a_m h(\cdot, x_m) + \sum_{n=1}^{N_0} b_n p_n \]

with expansion coefficients \( a = (a_1, \ldots, a_M) \in \mathbb{R}^M \) and \( b = (b_1, \ldots, b_{N_0}) \). The delicate point there is to properly define the corresponding native space \( \mathcal{H} \) since the underlying kernel \( h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \), which is still given by (3), is no longer positive-definite. The underlying native space then has the structure of a semi-RKHS, a concept that was already present implicitly in the early works on variational splines.
While the aforementioned results with $X' = L_2(\mathbb{R}^d)$ are classical, there has been recent interest in a variant of Problem (1) with $X' = M(\mathbb{R}^d)$ (the space of Radon measures on $\mathbb{R}^d$) \cite{34,16,7}. It turns out that the latter is much better suited to compressed sensing and to the resolution of inverse problems in general \cite{18}. In particular, when $L$ is shift-invariant, it has been shown \cite{34,Theorem 2} that the minimum of $\|Lf\|_M$ is achieved by an adaptive $L$-spline\footnote{A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be an $L$-spline with knots $\tau_1, \ldots, \tau_K \in \mathbb{R}^d$ if $L(f) = \sum_{k=1}^{K} a_k \delta(\cdot - \tau_k)$ with $a_1, \ldots, a_K \in \mathbb{R}$. For instance, if $d = 1$ and $L = D$ (resp. $L = D^2$), then $f$ is piecewise-constant (resp. piecewise-linear and continuous) with jumps (resp. breaks) at the $\tau_m$.} of the form

$$f = \sum_{k=1}^{K} a_k \rho_L(\cdot - \tau_k) + \sum_{n=1}^{N_0} b_n p_n,$$

where $\rho_L = L^{-1}\{\delta\}$, with the twist that the intrinsic parameters of the spline—the number $K$ of knots and their locations $\tau_k$—are adjustable with $K \leq (M - N_0)$. Remarkably, the generic form (4) of the solution is preserved for arbitrary continuous linear measurement functionals $\nu : X'_L \to \mathbb{R}^M$, far beyond the pure sampling framework of RKHS, which makes the result applicable to linear inverse problems. While there have been attempts to generalize the $M$-norm (or total variation) variant of the reconstruction problem \cite{1,16,7,6}, the part that has remained elusive is the specification of the corresponding native space in the multidimensional scenario ($d > 1$) when the null space of $L$ is nontrivial. The difficulty there is essentially the same as the one that was encountered initially by Duchon with $L = (-\Delta)^{\gamma}$ (fractional Laplacian) and $X = L_2(\mathbb{R}^d)$ \cite{12}; namely, the need to properly restrict the native space in order to exclude the harmonic components of the null space that grow faster than the underlying Green's function. A systematic approach for specifying native spaces was later given by Schaback \cite{28,29}, but it is restricted to the RKHS framework and to kernels that are conditionally positive-definite.

In this paper, we develop an alternative Banach-space formulation that is applicable to a whole range of norms $\|\cdot\|_{X'}$ and operators $L$. In a nutshell, we are proposing a systematic approach for constructing the largest Banach space $X'_L$ that ensures that $\|Lf\|_{X'}$ is well defined, subject to the constraint that the null space of $L$ should be finite-dimensional. The main benefits of our new formulation are as follows:

- The extension of the notion of generalized spline via (1) through the appropriate pairing of a regularization operator $L$ and a space $X'$ that is
the continuous dual of a primary Banach space $\mathcal{X}$. The two aforementioned theories with $R(f) = ||Lf||_{L_2}^2$ (RKHS or Tikhonov regularization) and $R(f) = ||Lf||_\mathcal{M}$ (generalized TV regularization) are covered by taking $\mathcal{X} = L_2(\mathbb{R}^d)$ and $\mathcal{X} = C_0(\mathbb{R}^d)$ (the pre-dual of $\mathcal{M}(\mathbb{R}^d)$), but our framework is considerably more general.

- The precise identification of the class of spline-admissible operators $L$ (see Definition 2). In essence, these are differential-like operators that are injective on $\mathcal{S}(\mathbb{R}^d)$ (Schwartz’ class of test functions) and that admit a well-defined Green’s function, which is symbolically denoted by $(x, y) \mapsto g_L(x, y) = L^{-1}\{\delta(\cdot - y)\}(x)$, where $\delta(\cdot - y)$ is the Dirac impulse at the location $y \in \mathbb{R}^d$.

- The proposal of a Banach counterpart to the notion of conditional positive definiteness from the theory of semi-RKHS: The critical hypothesis here is the continuity of some pseudo-inverse operator $L^{-1}_\phi$ (see Definition 4). In a companion paper, we shall demonstrate the usefulness of this criterion and how it can be readily tested in practice [14].

- A systematic way of constructing the native space for $(L, \mathcal{X}')$, denoted by $\mathcal{X}'_L$, via a completion process that involves test functions and operators rather than linear combinations of kernels. This is a significant extension of the usual approach as it applies to a much larger family of primary spaces, including non-reflexive Banach spaces such as $\mathcal{M}(\mathbb{R}^d)$.

- The guarantee of universality: The native space $\mathcal{X}'_L$ specified by Theorem 4 is rich enough to represent any function with an arbitrary degree of precision. It is also sufficiently restrictive for the null space of $L$ to be finite-dimensional, which is non-obvious for $d > 1$ since the “unrestricted” null space of partial differential operators is either trivial or infinite-dimensional [19]. The proposed approach is a convenient alternative to the more traditional use of Beppo-Levi spaces that involve composite norms with partial derivatives [3, 11, 4, 38].

- The explicit characterization of $\mathcal{X}_L$ (the pre-dual of the native space $\mathcal{X}'_L$), as given in Corollary 2. The practical significance of this result is that it precisely delineates the domain of validity of representer theorems for the solutions of Problem (1) or variants thereof. Indeed, a

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3 This is due to $\mathcal{X}'_L$ being dense in $\mathcal{S}'(\mathbb{R}^d)$.
sufficient condition for existence is the weak* continuity of the measurement functional $\nu : X'_L \to \mathbb{R}^M$, as in [34]. This is equivalent to $\nu_m \in X_L$ for $m = 1, \ldots, M$ [25]. For instance, the requirement for the well-posedness of a regularized interpolation problem is $\delta(\cdot - x_m) \in X_L$ for any $x_m \in \mathbb{R}^d$. The latter condition is automatically satisfied when $X'_L$ is a RKHS. However, it can fail when switching to non-euclidean norms such as, for example, the critical configuration $(L, X') = (D, M(\mathbb{R}))$ which corresponds to the popular total-variation regularization with $R(f) = TV(f) = \|Df\|_M$ [26, 8]. On the other hand, we have that $\delta(\cdot - x_m) \in X_L$ for $(L, X') = (D^2, M(\mathbb{R}))$, which justifies the use of second-order total variation for the regularization of deep neural networks [31].

The paper is organized as follows: In Section 2, we lay out the functional context while introducing the notion of Schwartz-Banach space, which is fundamental to our approach. In Section 3, we give the mathematical conditions for $L$ to be spline-admissible and describe an effective way to stabilize its inverse via the use of a biorthogonal system of $N_L$ (the null space of $L$). The final ingredient is given by some norm-compatibility conditions that enable the specification of the pre-Banach space $P_L$. The completion of $P_L$ in the $\|\cdot\|_{X'_L}$-norm yields our pre-native space $X_L$, whose properties are revealed in the first part of Section 4 (Theorem 3). This then allows us to characterize the native space $X'_L$ (Theorem 4) and to establish its embedding properties. We also identify the complete family of equivalent norms, which yields a complete understanding of the underlying direct-sum topology. Finally, in Section 5, we illustrate the compatibility of our extended formulation with the specification of many classical spaces; in particular, RKHS and $L_p$-type Sobolev spaces.

2 Preliminaries

2.1 Schwartz-Banach spaces

The Banach spaces that we shall consider are embedded in Schwartz’ space of tempered distributions denoted by $\mathcal{S}'(\mathbb{R}^d)$. Formally, an element $f \in \mathcal{S}'(\mathbb{R}^d)$ is a continuous linear functional $f : \varphi \mapsto \langle f, \varphi \rangle$ that associates a real number denoted by $\langle f, \varphi \rangle$ to each test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz’ space of smooth and rapidly decaying functions). For instance, the Dirac impulse at location $x_0 \in \mathbb{R}^d$ is the generalized function $\delta(\cdot - x_0) \in \mathcal{S}'(\mathbb{R}^d)$ defined as

$$\varphi \mapsto \langle \delta(\cdot - x_0), \varphi \rangle = \varphi(x_0).$$
Similarly, any slowly increasing and locally integrable function \( f : \mathbb{R}^d \to \mathbb{R} \) specifies a distribution by way of the integral \( \langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx \).

Our construction relies on the pairing of an operator \( L \) and a Banach space \( \mathcal{X}' \). The latter is the dual of a primary space \( \mathcal{X} \) that is appropriately sandwiched between \( S(\mathbb{R}^d) \) and \( S'(\mathbb{R}^d) \), in accordance with Definition 1.

**Definition 1** (Schwartz-Banach space). A normed vector space \( (\mathcal{X}, \| \cdot \|_\mathcal{X}) \subseteq S'(\mathbb{R}^d) \) is said to be a Schwartz-Banach space if it can be specified as the completion of \( S(\mathbb{R}^d) \) in the \( \| \cdot \|_\mathcal{X} \)-norm or, equivalently, if \( \mathcal{X} \) is a Banach space with the property that \( S(\mathbb{R}^d) \xrightarrow{\text{a}} \mathcal{X} \xrightarrow{\text{a}} S'(\mathbb{R}^d) \) (continuous and dense embeddings).

The reader is referred to Appendix A for the precise definition of the underlying notions of embeddings and a review of supporting mathematical results.

Prominent examples of Schwartz-Banach spaces are \( L^p(\mathbb{R}^d) \) with \( p \in [1, \infty) \) and \( C_0(\mathbb{R}^d) \) (the class of functions that are continuous, uniformly bounded and decaying at infinity). By contrast, the Schwartz-Banach property holds neither for \( L^\infty(\mathbb{R}^d) \) nor for \( \mathcal{M}(\mathbb{R}^d) \) (the space of bounded Radon measures on \( \mathbb{R}^d \)). These two spaces, however, retain relevance for our purpose as duals of (non-reflexive) Schwartz-Banach spaces; i.e., \( L^\infty(\mathbb{R}^d) = (L^1(\mathbb{R}^d))' \) and \( \mathcal{M}(\mathbb{R}^d) = (C_0(\mathbb{R}^d))' \).

**Proposition 1.** The dual \( \mathcal{X}' \) of a Schwartz-Banach space \( \mathcal{X} \) is a Banach space with the following properties:

- \( S(\mathbb{R}^d) \hookrightarrow \mathcal{X}' \xrightarrow{\text{a}} S'(\mathbb{R}^d) \)

- \( \mathcal{X}' = \{ w \in S'(\mathbb{R}^d) : \sup_{\varphi \in S(\mathbb{R}^d) \setminus \{0\}} \frac{\langle w, \varphi \rangle}{\| \varphi \|_{\mathcal{X}}} = \| w \|_{\mathcal{X}'} < +\infty \} \).

- The duality product for \( (\mathcal{X}', \mathcal{X}) \) is compatible with that of \( (S'(\mathbb{R}^d), S(\mathbb{R}^d)) \), which allows us to write

\[
\langle f, g \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle f, g \rangle
\]  

for any \( (f, g) \in \mathcal{X}' \times \mathcal{X} \).

If, in addition, \( \mathcal{X} \) is reflexive, then \( \mathcal{X}' \) is itself a Schwartz-Banach space, which then also yields

\[
\mathcal{X} = \mathcal{X}'' = \{ g \in S'(\mathbb{R}^d) : \| g \|_{\mathcal{X}} = \sup_{\varphi \in S(\mathbb{R}^d) \setminus \{0\}} \frac{\langle g, \varphi \rangle}{\| \varphi \|_{\mathcal{X}'}} < +\infty \}.
\]
The prototypical example that meets the last statement is $\mathcal{X}' = L_q(\mathbb{R}^d) = (L_p(\mathbb{R}^d))'$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$. The property that is lost when $\mathcal{X}$ is not reflexive (e.g., for $(p, q) = (1, \infty)$) is the denseness of the continuous embedding $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{X}'$.

**Proof.** The embedding $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{X}' \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ follows from the Schwartz-Banach property, Theorem 6 on dual embeddings, and the reflexivity of $\mathcal{S}(\mathbb{R}^d)$. When $\mathcal{X}$ is reflexive, we also get that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{X}'$, which proves that $\mathcal{X}'$ is a Schwartz-Banach space. By definition, $\mathcal{X}'$ is the Banach space associated with the dual norm $\|w\|_{\mathcal{X}'} \triangleq \sup_{\|\varphi\|_{\mathcal{X}} \leq 1} \langle w, \varphi \rangle_{\mathcal{X}' \times \mathcal{X}} < +\infty$. Now, the fundamental observation is that $\langle w, \varphi \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle w, \varphi \rangle$ for any $(w, \varphi) \in (\mathcal{X}', \mathcal{S}(\mathbb{R}^d))$ since $\mathcal{X}' \subseteq \mathcal{S}'(\mathbb{R}^d)$, while the determination of the supremum can be restricted to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ in reason of the denseness of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{X}$. This allows us to rewrite the dual norm as

$$\|w\|_{\mathcal{X}'} = \sup_{\|\varphi\|_{\mathcal{X}} \leq 1} \langle w, \varphi \rangle = \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}} \frac{\langle w, \varphi \rangle}{\|\varphi\|_{\mathcal{X}' \times \mathcal{X}}}.$$  \hspace{1cm} (6)

Since $\mathcal{X}' \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ and the expression on the right-hand side of (6) is well-defined for any $w \in \mathcal{S}'(\mathbb{R}^d)$, we have that $\mathcal{X}' \subseteq \mathcal{W} = \{w \in \mathcal{S}'(\mathbb{R}^d) : \|w\|_{\mathcal{X}'} < \infty\}$. To establish the converse inclusion—and, hence, $\mathcal{X}' = \mathcal{W}$—we use (6) to infer that, for any $w \in \mathcal{W} \subseteq \mathcal{S}'(\mathbb{R}^d)$, the linear functional $w : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is bounded by

$$|\langle w, \varphi \rangle| \leq \|w\|_{\mathcal{X}'} \|\varphi\|_{\mathcal{X}}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We then invoke Theorem 1 below with $\mathcal{Y} = \mathbb{R}$ to deduce that $w \in \mathcal{W}$ has a unique continuous extension to the completed space $\mathcal{X} = (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{\mathcal{X}'})$, which proves that $\mathcal{W} \subseteq \mathcal{X}'$. The same argument also gives a precise meaning to the right-hand side of (5). Finally, in the reflexive case, we reapply the first part of Proposition 1 to $\mathcal{X}'$ to obtain the announced characterization of $\mathcal{X}'' = \mathcal{X}$.

The important point here is that the two (dual) formulas for the norms in Proposition 1 are valid for any tempered distribution $w \in \mathcal{S}'(\mathbb{R}^d)$. In effect, this provides us with a simple criterion for space membership (resp. exclusion): $\|w\|_{\mathcal{X}'} < \infty \Leftrightarrow w \in \mathcal{X}'$ (resp. $w \notin \mathcal{X}' \Leftrightarrow \|w\|_{\mathcal{X}'} = \infty$). Likewise, we have that $g \in \mathcal{X} = \mathcal{X}'' \Leftrightarrow \|g\|_{\mathcal{X}} < \infty$, although the equivalence holds true in the reflexive case only. This explains the greater difficulty in developing
a general theory for non-reflexive native spaces—as opposed to, say, the traditional RKHS that go hand-in-hand with $X = L_2(\mathbb{R}^d)$.

A significant advantage of working with Schwartz-Banach spaces is the compatibility property expressed by (5). In addition, when $f \in X'$ and $g \in X$ are both ordinary functions, the duality product has an explicit transcription as the integral

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)g(x)dx.$$  \hfill (7)

In order to control the algebraic rate of growth/decay of such functions, we rely on weighted Banach spaces such as $L_\infty, \alpha(\mathbb{R}^d) = \{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable s.t. } \| f \|_{\infty, \alpha} < +\infty \}$ with norm

$$\| f \|_{\infty, \alpha} \triangleq \| (1 + \| \cdot \|)^{-\alpha} f \|_{\infty} = \operatorname{ess sup}_{x \in \mathbb{R}^d} \left( (1 + \| x \|)^{-\alpha} | f(x) | \right),$$

where the order $\alpha \in \mathbb{R}^+$ puts a cap on the rate of growth of $f$ at infinity. In conformity with the relation $L_\infty(\mathbb{R}^d) = \left( L_1(\mathbb{R}^d) \right)'$, $L_\infty, \alpha(\mathbb{R}^d)$ is the continuous dual of $L_{1,-\alpha}(\mathbb{R}^d)$: the Schwartz-Banach space associated with the weighted $L_1$ norm

$$\| g \|_{1,-\alpha} \triangleq \| (1 + \| \cdot \|)^{\alpha} g \|_1 = \int_{\mathbb{R}^d} (1 + \| x \|)^\alpha g(x)dx$$

where the switch to a negative exponent $-\alpha \leq 0$ now demands that $g$ decays at infinity; e.g., a sufficient requirement for $g \in L_{1,-\alpha}(\mathbb{R}^d)$ is $|g(x)| \leq \frac{C}{(1 + \| x \|)^{\alpha + \epsilon}}$ with $\epsilon > 0$.

Finally, since our primary interest is with ordinary functions that are well-defined pointwise, we also consider the Banach space

$$C_{b,\alpha}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ continuous and s.t. } \| f \|_{\infty, \alpha} < +\infty \}$$

$\text{4For instance, the non-reflexive spaces } C_0(\mathbb{R}^d) \xrightarrow{\text{iso.}} C_b(\mathbb{R}^d) \xrightarrow{\text{iso.}} L_\infty(\mathbb{R}^d) \text{ are all equipped with the same } \| \cdot \|_\infty \text{-norm. The condition that } g \text{ is bounded is not sufficient to ensure that } g \in C_0(\mathbb{R}^d) \text{ (the only Schwartz-Banach space in the chain); it is also required that } g \text{ be continuous and decaying at infinity. The boundedness criterion for space membership applies to } L_\infty(\mathbb{R}^d) \text{ alone because it is the dual of the Schwartz-Banach space } L_1(\mathbb{R}^d).$
that shares the same norm as $L_{\infty, \alpha}(\mathbb{R}^d)$, but whose elements (functions) are constrained to be continuous. The hierarchy between these various spaces is described by the embedding relations

$$C_{b, \alpha}(\mathbb{R}^d) \overset{\text{iso.}}{\hookrightarrow} L_{\infty, \alpha}(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d)$$
$$S(\mathbb{R}^d) \overset{\iota}{\hookrightarrow} L_{1, -\alpha}(\mathbb{R}^d) \overset{\text{iso.}}{\hookrightarrow} (L_{\infty, \alpha}(\mathbb{R}^d))'$$

with additional explanations given in Appendix A.

### 2.2 Characterization of linear operators and their adjoint

A linear operator is represented by an upright capital letter such as $L$ or $G$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological vector spaces with continuous duals $\mathcal{X}'$ and $\mathcal{Y}'$, respectively. Then, the notation $G : \mathcal{X} \overset{\iota}{\rightarrow} \mathcal{Y}$ indicates that $G$ continuously maps $\mathcal{X} \rightarrow \mathcal{Y}$. The adjoint of $G$ is the unique operator $G^* : \mathcal{Y}' \overset{\iota}{\rightarrow} \mathcal{X}'$ such that $\langle Gx, y' \rangle_{\mathcal{X} \times \mathcal{Y}'} = \langle x, G^*y' \rangle_{\mathcal{X} \times \mathcal{X}'}$ for any $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are both Banach spaces, then the norm of the operator is preserved \[27\], as expressed by

$$\|G\|_{\mathcal{X} \rightarrow \mathcal{Y}} \overset{\triangle}{=} \sup_{\varphi \in \mathcal{X} \setminus \{0\}} \frac{\|G\varphi\|_{\mathcal{Y}}}{\|\varphi\|_{\mathcal{X}}} = \|G^*\|_{\mathcal{Y}' \rightarrow \mathcal{X}'}.$$

The attractiveness of the Schwartz-Banach setting with $S(\mathbb{R}^d) \overset{\iota}{\rightarrow} \mathcal{X}$ is the convenience of being able to specify the operator in the more constrained—but mathematically foolproof—scenario $G : S(\mathbb{R}^d) \overset{\iota}{\rightarrow} \mathcal{Y} \hookrightarrow S'(\mathbb{R}^d)$. The critical property, then, is the existence of a bound of the form

$$\|G\varphi\|_{\mathcal{Y}} \leq C\|\varphi\|_{\mathcal{X}} \quad \text{for all} \quad \varphi \in S(\mathbb{R}^d),$$

which allows one to extend the domain of the operator to the complete space $\mathcal{X}'$ by continuity; i.e., $G : \mathcal{X}' \overset{\iota}{\rightarrow} \mathcal{Y}$ with $\|G\|_{\mathcal{X}' \rightarrow \mathcal{Y}} \leq C$. This is a powerful extension principle that relies on the bounded-linear-transformation (B.L.T.) theorem.

**Theorem 1** (\[25\] Theorem I.7, p. 9)). Let $G$ be a bounded linear transformation from a normed space $(\mathcal{Z}, \|\cdot\|_z)$ to a complete normed space $(\mathcal{Y}, \|\cdot\|_y)$. Then, $G$ has a unique extension to a bounded linear transformation (with the same bound) from the completion of $(\mathcal{Z}, \|\cdot\|_z)$ to $(\mathcal{Y}, \|\cdot\|_y)$.

The other foundational result that supports the present construction is Schwartz’ kernel theorem \[17\], which states that the application of the operator $G : S(\mathbb{R}^d) \overset{\iota}{\rightarrow} S'(\mathbb{R}^d)$ to $\varphi$ yields a tempered distribution $G\{\varphi\} : S'(\mathbb{R}^d) \rightarrow \mathbb{C}$.
\[ S(\mathbb{R}^d) \to \mathbb{R} \] specified by
\[ \psi \mapsto \langle G\{\varphi\}, \psi \rangle = \langle g(\cdot, \cdot), \psi \otimes \varphi \rangle, \tag{9} \]
where \( g(\cdot, \cdot) \in S'(\mathbb{R}^d \times \mathbb{R}^d) \) is the kernel of the operator and \( \psi \otimes \varphi \in S(\mathbb{R}^d \times \mathbb{R}^d) \) with \( (\psi \otimes \varphi)(x, y) \triangleq \psi(x)\varphi(y) \). This property is often symbolized by the formal “integral” equation
\[ G : \varphi \mapsto \int_{\mathbb{R}^d} g(\cdot, y)\varphi(y)dy \tag{10} \]
with a slight abuse of notation. Conversely, the right-hand side of (10) defines a continuous operator \( S(\mathbb{R}^d) \xrightarrow{\text{c.}} S'(\mathbb{R}^d) \) for any \( g(\cdot, \cdot) \in S'(\mathbb{R}^d \times \mathbb{R}^d) \).

The bottom line is that the consideration of a Schwartz-Banach space allows for a concrete and unambiguous description of linear functionals and linear operators in terms of generalized functions and generalized kernels, respectively.

**Proposition 2** (Representation of functionals and operators). Let \( X \) be a Schwartz-Banach space. Then, any continuous linear functional \( g : X \to \mathbb{R} \) is uniquely characterized by a single element \( g|_{S(\mathbb{R}^d)} \in S'(\mathbb{R}^d) \): the restriction of \( g \) to \( S(\mathbb{R}^d) \to \mathbb{R} \). Likewise, any continuous linear operator \( G : X \to Y \) where \( Y \) is a Banach subspace of \( S'(\mathbb{R}^d) \) is uniquely characterized by a single element \( g(\cdot, \cdot) \in S'(\mathbb{R}^d \times \mathbb{R}^d) \), which is the Schwartz kernel of the restriction \( G|_{S(\mathbb{R}^d)} : S(\mathbb{R}^d) \to Y \hookrightarrow S'(\mathbb{R}^d) \).

**Proof.** The Schwartz-Banach property implies that \( S(\mathbb{R}^d) \xrightarrow{\text{d.}} X \). Hence, both \( g \) and \( G \) are fully characterized by their restriction on \( S(\mathbb{R}^d) \). As such, \( g \) is identified as an element of \( S'(\mathbb{R}^d) \), while \( G \) is uniquely characterized by its Schwartz kernel \( g(\cdot, \cdot) \in S'(\mathbb{R}^d \times \mathbb{R}^d) \) when seen as an operator from \( S(\mathbb{R}^d) \) to \( S'(\mathbb{R}^d) \).

---

5The kernel is also called the generalized impulse response of the operator. Formally, this is indicated as \( g(\cdot, y) = G\{\delta(\cdot - y)\} \) with a slight abuse of notation when \( \delta(\cdot - y) \notin X \).
The common practice is to indicate the correspondences in Proposition 2 by (7) and (10), respectively. One should keep in mind, however, that the rigorous interpretation of these relations involves the limit/extension process:

\[
\langle g, f \rangle = \int_{\mathbb{R}^d} g(y) f(y) \, dy \triangleq \lim_i \langle g, f_i \rangle
\]

\[
\int_{\mathbb{R}^d} g(y) f(y) \, dy \triangleq \lim_i \int_{\mathbb{R}^d} g(y) f_i(y) \, dy
\]

for any sequence \((f_i)\) in \(S(\mathbb{R}^d)\) such that \(\|f - f_i\|_X \to 0\) as \(i \to \infty\), with the implicit understanding that the underlying “integrals” are the symbolic representations of continuous linear functionals acting on \(f\) (resp. \(f_i\)).

### 3 Spline-admissible operators

The identification of the native space for an admissible operator \(L\) essentially boils down to the characterization of the solutions of the linear differential equation \(Lf = w\) for a suitable class of excitations \(w \in X'\). This requires that \(L\) be invertible in an appropriate sense. As a minimum, we ask that \(L\) be injective (with a well-defined inverse \(L^{-1}\)) when we restrict its domain to \(S(\mathbb{R}^d)\).

**Definition 2 (Admissible operator).** A linear operator \(L : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)\) is called spline-admissible if there exist an order \(\alpha \in \mathbb{R}\) of algebraic growth, an inverse operator \(L^{-1}\), and a finite-dimensional space \(N_p = \text{span}\{p_n\}_{n=1}^{N_0}\) such that

1. \(L\) and \(L^*\) continuously map \(S(\mathbb{R}^d) \to L_{1,-\alpha}(\mathbb{R}^d)\). (The continuity of the adjoint is required for the extended version of the operator \(L : L_{\infty,\alpha}(\mathbb{R}^d) \to S'(\mathbb{R}^d)\) in Condition 4 to be well defined.)
2. \(L^{-1}\) and \(L^{-1}\) continuously map \(L_{1,-\alpha}(\mathbb{R}^d) \to L_{\infty,\alpha}(\mathbb{R}^d)\). (The assumption \(L^{-1} : L_{1,-\alpha}(\mathbb{R}^d) \to L_{\infty,\alpha}(\mathbb{R}^d)\) is actually sufficient as it implies that \(L^{-1} : L_{1,-\alpha}(\mathbb{R}^d) \to L_{\infty,\alpha}(\mathbb{R}^d)\) by duality.)
3. Invertibility: \(L^{-1}L^* \varphi = \varphi\) and \(L^{-1}L \varphi = \varphi\) for all \(\varphi \in S(\mathbb{R}^d)\).
4. Restricted null space: \(N_p = N_L = \{p \in L_{\infty,\alpha}(\mathbb{R}^d) : L\{p\} = 0\}\).

A preferred scenario is when \(L\) and \(L^{-1}\) are both linear-shift-invariant—that is, convolution operators—with respective frequency response \(\hat{L}(\omega)\) and
$1/L(\omega)$. Then, $\varphi \mapsto L^{-1}\varphi = \rho_L * \varphi \triangleq \int_{\mathbb{R}^d} \rho_L(y)\varphi(\cdot - y)dy$, where

$$\rho_L(x) = \mathcal{F}^{-1}\left\{ \frac{1}{L(\omega)} \right\}(x),$$

which is the (generalized) inverse Fourier transform of $1/L \in \mathcal{S}'(\mathbb{R}^d)$ under the implicit assumption that the latter is a well-defined tempered distribution. In that case, we refer to $\rho_L$, which satisfies the formal property $L\{\rho_L\} = \delta$, as the canonical Green’s function of $L$.

**Example 1.** The derivative operator $D = \frac{d}{dx}$ with $D^* = -D$ is spline-admissible with $N_0 = 1$ and $\alpha = 0$. Indeed, $D : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}) \hookrightarrow L_1(\mathbb{R})$, while its null space is $N_D = \text{span}\{p_1\} \subset C_b(\mathbb{R}) \subseteq L_{\infty,0}(\mathbb{R}) = L_{\infty}(\mathbb{R})$ with $p_1(x) = 1$. Its canonical inverse is given by $D^{-1} : \varphi \mapsto \rho_D * \varphi$ with $ho_D(x) = \frac{1}{2}\text{sign}(x)$ and, hence, such that $D^{-1} : L_1(\mathbb{R}) \xrightarrow{\sim} L_{\infty}(\mathbb{R})$.

Let us now briefly comment on the assumptions in Definition 2. Conditions 1 and 2 ensures that the composition of operators in Condition 3 (i.e., $L^{-1}L^* : \mathcal{S}(\mathbb{R}^d) \xrightarrow{\sim} L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\sim} \mathcal{S}'(\mathbb{R}^d)$ and $L^{-1}L : \mathcal{S}(\mathbb{R}^d) \xrightarrow{\sim} L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\sim} \mathcal{S}'(\mathbb{R}^d)$) are legitimate. Condition 1 limits the framework to operators that do not drastically affect decay. However, it does not penalize a loss of regularity. In particular, it is met by all constant-coefficient (partial) differential operators, which happen to be continuous $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow L_{1,-\alpha}(\mathbb{R}^d)$. The condition with an adequate $\alpha$ is also satisfied by the fractional derivative operators $D^\gamma$ or $(-\Delta)^{\gamma/2}$ (fractional Laplacian) whose impulse responses decay like $1/||x||^{\gamma+d}$. This decay property can be used to show that the fractional derivatives $D^\gamma$ with $\gamma > 0$ are continuous $\mathcal{S}(\mathbb{R}^d) \rightarrow L_{1,-(\gamma+d)}(\mathbb{R}^d)$ [32], while their Green’s functions are included in $L_{\infty,\alpha}(\mathbb{R}^d)$ with $\alpha = (\gamma - d)$, which is a favourable state of affairs in the context of Condition 1.

An important observation is that the left-invertibility of $L^*$ in Condition 3 is equivalent to $\mathcal{L}L^{-1}\varphi = \varphi$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $L^{-1}\varphi \in L_{\infty,\alpha}(\mathbb{R}^d)$ and $\mathcal{L} : L_{\infty,\alpha}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ extended accordingly by duality. Indeed, for any $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$\langle \varphi, \tilde{\varphi} \rangle = \langle \varphi, L^{-1}L^*\tilde{\varphi} \rangle = \langle LL^{-1}\varphi, \tilde{\varphi} \rangle, \quad (11)$$

where the factorization is legitimate, in reason of the compatible range and domain of the underlying operators. We then invoke the denseness of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{S}'(\mathbb{R}^d)$ (see Proposition 6) to extend the validity of (11) for all $\tilde{\varphi} \in \mathcal{S}'(\mathbb{R}^d)$, which yields the desired result.
While Condition 3 tells us that $L$ is invertible on $\mathcal{S}(\mathbb{R}^d)$, it does not guarantee that the property still holds when the domain is extended to $L_{\infty,\alpha}(\mathbb{R}^d)$ because the corresponding null space $\mathcal{N}_L = \mathcal{N}_p$ in Condition 3 may be non-trivial. We shall resolve this ambiguity by factoring out the null-space components. Our approach is based on the construction of a projection operator $\text{Proj}_{\mathcal{N}_p} : L_{\infty,\alpha}(\mathbb{R}^d) \rightarrow \mathcal{N}_p$, as described next. The latter is then used in Section 3.2 to specify a stable pseudo-inverse operator that is a corrected (or regularized) version of $L^{-1}$.

3.1 Biorthogonal system for the null space of $L$

The fundamental hypothesis here is that the growth-restricted null space $\mathcal{N}_L = \{ f \in L_{\infty,\alpha}(\mathbb{R}^d) : L\{p\} = 0 \}$ admits a basis $p = (p_1, \ldots, p_{N_0})$ of finite dimension $N_0$. The idea then is to select a set of analysis functionals $\phi_1, \ldots, \phi_{N_0} \in \mathcal{Y} \subseteq \mathcal{S}'(\mathbb{R}^d)$ that satisfy the biorthogonality relation

$$\langle \phi_m, p_n \rangle = \delta_{m,n}, \quad m, n \in \{1, \ldots, N_0\}. \quad (12)$$

To give a concrete meaning to the above duality product, we are assuming the existence of a dual pair $(\mathcal{Y}, \mathcal{Y}')$ of subspaces of $\mathcal{S}'(\mathbb{R}^d)$ such that $\phi_1, \ldots, \phi_{N_0} \in \mathcal{Y}$ and $p_1, \ldots, p_{N_0} \in \mathcal{Y}'$. Such a construction is always feasible (see Proposition 3) with the choice of $\phi$ being free. We also understand that there is a whole equivalence class of representations of $\mathcal{N}_L = \mathcal{N}_p = \mathcal{N}_{\tilde{p}}$ with $\tilde{p} = \mathbf{B}p$, under the constraint that the matrix $\mathbf{B} \in \mathbb{R}^{N_0 \times N_0}$ be invertible.

**Definition 3 (Biorthogonal system).** A pair $(\phi, p)$ with $\phi = (\phi_1, \ldots, \phi_{N_0}) \in \mathcal{Y}^{N_0}$ and $p = (p_1, \ldots, p_{N_0}) \in (\mathcal{Y}')^{N_0}$ is called a biorthogonal system for the finite-dimensional subspace $\mathcal{N}_p = \text{span}\{p_n\}_{n=1}^{N_0} \subset \mathcal{Y}' \subseteq \mathcal{S}'(\mathbb{R}^d)$ if any $p \in \mathcal{N}_p$ admits a unique expansion of the form

$$p = \sum_{n=1}^{N_0} \langle \phi_n, p \rangle p_n. \quad (13)$$

The natural norm induced on $\mathcal{N}_p$ by such a system is

$$\|p\|_{\mathcal{N}_p} = \|\phi(p)\|_2 = \left( \sum_{n=1}^{N_0} |\langle \phi_n, p \rangle|^2 \right)^{\frac{1}{2}}.$$

The system is said to be universal if $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ for $n = 1, \ldots, N_0$.

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6In view of the inclusion $\mathcal{N}_L \subset L_{\infty,\alpha}(\mathbb{R}^d)$, the natural choice is to take $(\mathcal{Y}, \mathcal{Y}') = (L_{1,\infty}(\mathbb{R}^d), L_{\infty,\alpha}(\mathbb{R}^d))$. We shall see that this is extendable to $(\mathcal{Y}, \mathcal{Y'}) = (\mathcal{X}_L, \mathcal{X}_L')$. 

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Table 1: Complete set of operators associated with the biorthogonal system \((\phi, p)\) with \(\mathcal{N}_p = \text{span}\{p_n\}_{n=1}^{N_0}\) and \(\mathcal{N}_\phi = \text{span}\{\phi_n\}_{n=1}^{N_0}\).

| Description     | Operator | Kernel |
|-----------------|----------|--------|
| Riesz map \(\mathcal{N}_\phi \to \mathcal{N}_p = \mathcal{N}_\phi'\) | \(R_p\) | \(\sum_{n=1}^{N_0} p_n(x)p_n(y)\) |
| Riesz map \(\mathcal{N}_p \to \mathcal{N}_\phi = \mathcal{N}_p'\) | \(R_\phi\) | \(\sum_{n=1}^{N_0} \phi_n(x)\phi_n(y)\) |
| Projector \(\mathcal{X}_L' \subseteq L_{\infty,\alpha}(\mathbb{R}^d) \to \mathcal{N}_p\) | \(\text{Proj}_{\mathcal{N}_p} = R_pR_\phi\) | \(\sum_{n=1}^{N_0} p_n(x)\phi_n(y)\) |
| Projector \(\mathcal{X}_L \supseteq L_{1,-\alpha}(\mathbb{R}^d) \to \mathcal{N}_\phi\) | \(\text{Proj}_{\mathcal{N}_\phi} = R_\phi R_p\) | \(\sum_{n=1}^{N_0} \phi_n(x)p_n(y)\) |

The unicity of the representation in Definition 3 implies that \(p\) be a basis of \(\mathcal{N}_p\), while the validity of (13) for \(p = p_n\) implies that the underlying functions be biorthogonal, as expressed by (12).

Our next proposition ensures the existence of such systems for any given basis \(p\) under our working hypothesis \(Y \supseteq S(\mathbb{R}^d)\).

**Proposition 3 (Existence of biorthogonal systems).** Let \(\mathcal{N}_p = \text{span}\{p_n\}_{n=1}^{N_0}\) be a \(N_0\)-dimensional subspace of \(S'(\mathbb{R}^d)\). Then, there always exists some (universal) biorthogonal set of functionals \(\phi_1, \ldots, \phi_{N_0} \in S(\mathbb{R}^d)\).

**Proof.** This result is deduced from the following variant of the Hahn-Banach theorem with \(V = S'(\mathbb{R}^d)\).

**Theorem 2 ([27, Theorem 3.5]).** Let \(Z\) be a linear subspace of a topological vector space \(V\), and \(v_0\) be an element of \(V\). If \(v_0\) is not in the closure of \(Z\), then there exists a continuous linear functional \(\phi\) on \(V\) such that \(\langle \phi, v_0 \rangle = 1\) but \(\langle \phi, z \rangle = 0\) for every \(z \in Z\).

We then proceed by successive exclusion of \(v_0 = p_n\) with \(\phi = \phi_n\) and \(Z = \text{span}_{m \neq n}\{p_m\}\), while the finite dimensionality of \(Z\) and the linear independence of the \(p_m\) ensures that \(p_n \notin \overline{Z} = Z\).

**Example 2.** We recall that the derivative operator \(D = \frac{d}{dx}\) from Example 1 is spline-admissible with \(\alpha = 0\), \(N_0 = 1\), and \(\mathcal{N}_D = \text{span}\{p_1\} \subset L_{\infty}(\mathbb{R})\) where \(p_1(x) = 1\). As complementary analysis functional, we may choose any
function $\phi_1 \in L^1(\mathbb{R})$ (or $\phi_1 \in \mathcal{S}(\mathbb{R})$ if we are aiming at a universal solution) such that $\int_{\mathbb{R}} \phi_1(x)dx = \langle \phi_1, 1 \rangle = 1$.

**Example 3.** The $N_0$th-order derivative operator $L = D^{N_0} : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R})$ is spline-admissible with $L^{-1} : \varphi \mapsto \rho_{D^{N_0}} \ast \varphi$ where $\rho_{D^{N_0}}(x) = \frac{1}{2} \text{sign}(x) \frac{2^{N_0-1}}{(N_0-1)!}$, $\alpha = (N_0 - 1)$, and $N_{D^{N_0}} = \text{span}\{p_n\}_{n=1}^{N_0} \subset L_{\infty, N_0-1}(\mathbb{R})$ with $p_n(x) = x^n - 1$. A possible choice of universal biorthogonal system is $(\tilde{\phi}, \tilde{p})$ with $\phi_n(x) = p_n(x)e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$ and $\tilde{p}_1, \ldots, \tilde{p}_{N_0}$ the normalized Hermite polynomials of degree 0 to $(N_0 - 1)$. By definition, the latter form a basis of $N_{D^{N_0}}$—the space of polynomials of degree $(N_0 - 1)$—that fulfills the (bi)-orthogonality relation $\langle e^{-x^2/2}\tilde{p}_n, \tilde{p}_m \rangle = \delta[m-n]$.

Given such a system, we then specify $N_p$ and $N_{\phi} = \text{span}\{\phi_n\}_{n=1}^{N_0}$ as a dual pair of $N_0$-dimensional Hilbert spaces equipped with the inner products $\langle p, q \rangle_{N_p} = \langle R_p\{p\}, q \rangle$ and $\langle \phi, \varphi \rangle_{N_{\phi}} = \langle R_p\{\phi\}, \varphi \rangle$, respectively, where

$$R_p\{\phi\} = \sum_{n=1}^{N_0} \langle p_n, \phi \rangle p_n$$

$$R_{\phi}\{p\} = \sum_{n=1}^{N_0} \langle \phi_n, p \rangle \phi_n.$$

Indeed, $\phi_n = R_{\phi}\{p_n\}$ and $p_n = R_p\{\phi_n\}$, which allows us to identify $R_{\phi}$ (resp. $R_p$) as the Riesz map $N_{\phi} \rightarrow N_{\phi'} = N_p'$ (resp. $R_p : N_{\phi} \rightarrow N_p = N_{\phi'}$). For the proper interpretation of this (Riesz) pairing, we recall that the dual space $N_p'$ is an abstract entity that is formed by the linear functionals that are continuous on $N_p$; strictly speaking, each functional is an equivalence class of tempered distributions. The notation $N_p' = N_{\phi}$ indicates that $N_{\phi}'$ is isometrically isomorphic to $N_{\phi}$, meaning that each (abstract) member of $N_p'$ has a unique representative in $N_{\phi}$, which then serves as our concrete descriptor. The generic elements of these spaces are denoted by $\phi \in N_{\phi}$ with $\|\phi\|_{N_{\phi}} = \|\phi(p)\|_2$, where $\phi(p) = (\langle \phi_1, p \rangle, \ldots, \langle \phi_{N_0}, p \rangle) \in \mathbb{R}^{N_0}$ and $\phi \in N_{\phi}$ with $\|\phi\|_{N_{\phi}} = \|p(\phi)\|_2$.

Under the working assumptions that $N_{\phi} \subseteq \mathcal{Y}$ and $N_p \subseteq \mathcal{Y}'$, where $\mathcal{Y}$ is a suitable Schwartz-Banach space, the domain of continuity of these operators is extendable to $R_p : \mathcal{Y} \xrightarrow{\sim} N_p$ and $R_{\phi} : \mathcal{Y}' \xrightarrow{\sim} N_{\phi}$. The corresponding
Projection operators are
\[
\text{Proj} \mathcal{N}_\phi \{g\} = \sum_{n=1}^{N_0} (p_n, g) \phi_n = R_\phi R_p \{g\},
\]
\[
\text{Proj} \mathcal{N}_p \{f\} = \sum_{n=1}^{N_0} (\phi_n, f) p_n = R_p R_\phi \{f\}
\]
(14)
for any \(g \in \mathcal{Y}\) and \(f \in \mathcal{Y}'\) with the property that \(\text{Proj} \mathcal{N}_\phi \{\phi\} = \phi\) for all \(\phi \in \mathcal{N}_\phi\) and \(\text{Proj} \mathcal{N}_p \{p\} = p\) for all \(p \in \mathcal{N}_p\). We also note that \(\text{Proj}^* \mathcal{N}_\phi = \text{Proj} \mathcal{N}_p\), which emphasizes the symmetry of the construction.

We can also rely on the generic duality bound \(|(p_n, g)| \leq \|p_n\|_{\mathcal{Y}'} \|g\|_{\mathcal{Y}}\) to get a handle on the continuity properties of these operators. Specifically, based on (14), we find that
\[
\left\| p (\text{Proj} \mathcal{N}_\phi \{g\}) \right\|_2 \leq \left( \sum_{n=1}^{N_0} \|p_n\|^2_{\mathcal{Y}'} \right)^{1/2} \left( \sum_{n=1}^{N_0} \|\phi_n\|^2_{\mathcal{Y}} \right)^{1/2} \|g\|_{\mathcal{Y}}
\]
(16)
where the leading constant on the right-hand side provides an upper bound on the norm of the operator \(\text{Proj} \mathcal{N}_\phi : \mathcal{Y} \xrightarrow{c} \mathcal{N}_\phi \subset \mathcal{Y}\). Likewise, we have that \(\text{Proj} \mathcal{N}_p : \mathcal{Y}' \xrightarrow{c} \mathcal{N}_p \subset \mathcal{Y}'\) with \(\|\text{Proj} \mathcal{N}_p\|_{\mathcal{Y}' \to \mathcal{N}_p} \leq \left( \sum_{n=1}^{N_0} \|\phi_n\|^2_{\mathcal{Y}} \right)^{1/2} \).

The whole setup is summarized in Table 1 for the choice of spaces \(\mathcal{Y} = L_{1,-\alpha}(\mathbb{R}^d)\) and \(\mathcal{Y}' = L_{\infty,\alpha}(\mathbb{R}^d)\), which is adopted for the sequel.

### 3.2 Specification of a suitable pseudo-inverse operator

Under our hypotheses, the projection operator \(\text{Proj} \mathcal{N}_p\) defined by (14) is continuous \(\mathcal{Y}' = L_{\infty,\alpha}(\mathbb{R}^d) \xrightarrow{c} L_{\infty,\alpha}(\mathbb{R}^d)\). This, in turn, ensures the continuity of
\[
L_{\phi}^{-1} \triangleq (\text{Id} - \text{Proj} \mathcal{N}_p) \circ L^{-1} : L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{c} L_{\infty,\alpha}(\mathbb{R}^d) \xrightarrow{c} L_{\infty,\alpha}(\mathbb{R}^d).
\]
By observing that \((\text{Id} - \text{Proj} \mathcal{N}_p)^* = (\text{Id} - \text{Proj} \mathcal{N}_\phi)^*\), we then readily deduce that the adjoint of \(L_{\phi}^{-1}\) is such that
\[
L_{\phi}^{-1*} = L^{-1*} (\text{Id} - \text{Proj} \mathcal{N}_\phi) : L_{1,-\alpha}(\mathbb{R}^d) \xleftarrow{c} L_{\infty,\alpha}(\mathbb{R}^d),
\]
(17)
owing to the property that $L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\text{iso.}} (L_{1,-\alpha}(\mathbb{R}^d))'' = (L_{\infty,\alpha}(\mathbb{R}^d))'$.

While (17) ensures that $L_{\phi}^{-1*}$ is well-defined on $L_{1,-\alpha}(\mathbb{R}^d)$, we also want to make sure that the range of this operator can be restricted to the primary space $\mathcal{X}$ (the pre-dual of $\mathcal{X}'$ mentioned in the introduction), which calls for some additional compatibility hypotheses.

**Definition 4** (Compatibility of $(L, \mathcal{Y})$). Let $L$ be a linear operator and $\mathcal{Y}$ a Banach subspace of $S'(\mathbb{R}^d)$. We say that the pair $(L, \mathcal{Y})$ is compatible if

1. There exists a Schwartz-Banach space $\mathcal{X}$ (see Definition 1) such that $\mathcal{Y} = \mathcal{X}'$;
2. the operator $L$ is spline-admissible with order $\alpha$ (see Definition 2);
3. the adjoint operator $L^*$ is continuous $\mathcal{X} \xrightarrow{c.} S'(\mathbb{R}^d)$ and injective;
4. there exists a universal biorthogonal system $(\phi, p)$ of the null space of $L$ such that $L_{\phi}^{-1*} : L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{c.} \mathcal{X}$ where $L_{\phi}^{-1*}$ is specified by (17).

This condition is called $\mathcal{X}$-stability.

Conditions 1 and 2 are explicit and hardly constraining. The injectivity of $L^*$ is equivalent to the intersection between the extended null space of $L^*$ and $\mathcal{X}$ being trivial. The most constraining requirement is Condition 4, which needs to be checked on a case-by-case basis. Interestingly, we shall see that, if the condition holds for one particular biorthogonal system $(\phi, p)$ with $\mathcal{N}_\phi \subset S(\mathbb{R}^d)$ (universality property), then it also holds for any other admissible biorthogonal system $(\tilde{\phi}, \tilde{p})$ as long as $\mathcal{N}_{\tilde{\phi}} \subset \mathcal{X}_L$, where the pre-dual space $\mathcal{X}_L$ is characterized in Theorem 3.

The operator $L_{\phi}^{-1*}$ in (17) is fundamental to our construction. The key will be to extend its domain to make it surjective over $\mathcal{X}$. To that end, we now introduce a suitable pre-Banach space that will then be completed to yield our pre-dual space $\mathcal{X}_L$. 

**Definition 5** (Pre-Banach space for $(L, \mathcal{X}')$). Under the compatibility hypotheses of Definition 4, we specify the pre-Banach space for $(L, \mathcal{X}')$ as the vector space $P_L = \{L^* \varphi + \phi : \varphi \in S(\mathbb{R}^d), \phi \in \mathcal{N}_\phi \} \subseteq L_{1,-\alpha}(\mathbb{R}^d)$. (18)

**Proposition 4.** Under the compatibility hypotheses of Definition 4, for any $g \in P_L$, there is a unique pair $(\varphi = L_{\phi}^{-1*} g, \phi = \text{Proj}_{\mathcal{N}_\phi \{g\}}) \in S(\mathbb{R}^d) \times \mathcal{N}_\phi$ such that $g = L^* \varphi + \phi$, where the two underlying operators are defined by (17) and (14), respectively. In particular, this implies the following:
1. Null-space property: \( L_\phi^{-1*}\phi = 0 \) for all \( \phi \in \mathcal{N}_\phi \subset \mathcal{P}_L \).

2. Left invertibility of \( L^* \): \( L_\phi^{-1*}L^*\varphi = L^{-1*}L^*\varphi = \varphi \) for all \( \varphi \in \mathcal{S}(\mathbb{R}^d) \).

3. Pseudo-right-invertibility: \( L^*L_\phi^{-1*}g = (\text{Id} - \text{Proj}_{\mathcal{N}_\phi})\{g\} \) for all \( g \in \mathcal{P}_L \).

4. \( \mathcal{P}_L \) has a direct sum decomposition as \( \mathcal{P}_L = L^*(\mathcal{S}(\mathbb{R}^d)) \oplus \mathcal{N}_\phi \).

5. \( \mathcal{P}_L \), equipped with the composite norm

\[
\|g\|_{X_L} \triangleq \max(\|L_\phi^{-1*}g\|_X, \|\text{Proj}_{\mathcal{N}_\phi}\{g\}\|_2),
\]

(19)

is a normed subspace of \( L_{1,-\alpha}(\mathbb{R}^d) \).

6. \( L_\phi^{-1*} \) is bounded \( \mathcal{P}_L \to \mathcal{X} \) with \( \|L_\phi^{-1*}g\|_X \leq \|g\|_{X_L} \) for all \( g \in \mathcal{P}_L \).

7. \( \text{Proj}_{\mathcal{N}_\phi} \) is bounded \( \mathcal{P}_L \to \mathcal{N}_\phi \) with \( \|\text{Proj}_{\mathcal{N}_\phi}g\|_{X_L} \leq \|g\|_{X_L} \) for all \( g \in \mathcal{P}_L \).

Before moving to the proof, we observe that every one of the operators

\( \text{Proj}_{\mathcal{N}_\phi} : L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\sim} \mathcal{N}_\phi, \text{L}_\phi^{-1*} : L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\sim} \mathcal{X}, \) and \( L^* \circ L_\phi^{-1*} : L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\sim} \mathcal{X} \xrightarrow{\sim} \mathcal{S}(\mathbb{R}^d) \)

is well-defined on \( g \in \mathcal{P}_L \) because \( \mathcal{P}_L \subseteq L_{1,-\alpha}(\mathbb{R}^d) \) by construction.

**Proof.** From \([18]\), \( \mathcal{P}_L = L^*(\mathcal{S}(\mathbb{R}^d)) + \mathcal{N}_\phi \). Thanks to the hypotheses \( p_n \in \mathcal{N}_L \subseteq L_{\infty,\alpha}(\mathbb{R}^d) = (L_{1,-\alpha}(\mathbb{R}^d))^\prime \) and \( \psi = L^*\varphi \in L_{1,-\alpha}(\mathbb{R}^d) \), we have that \( \langle p_n, \psi \rangle = \langle p_n, L^*\varphi \rangle = \langle Lp_n, \varphi \rangle = 0 \) or, equivalently, \( \text{Proj}_{\mathcal{N}_\phi} \{\psi\} = 0 \) for all \( \psi \in L^*(\mathcal{S}(\mathbb{R}^d)) \). Likewise, due to the biorthogonality of \( (\phi, p) \), \( \text{Proj}_{\mathcal{N}_\phi} \{\phi\} = \phi \) for any \( \phi \in \mathcal{N}_\phi \). In other words, for \( g \in \mathcal{P}_L \), the condition \( \text{Proj}_{\mathcal{N}_\phi} \{g\} = g \) is equivalent to \( g \in \mathcal{N}_\phi \); that is, \( L^*(\mathcal{S}(\mathbb{R}^d)) \cap \mathcal{N}_\phi = \{0\} \), which establishes the direct-sum property (Item 4).

By applying the definition of \( L_\phi^{-1*} \) in \([17]\) and by invoking the spline-admissibility of \( L \), we then get that

\[
L_\phi^{-1*}\{L^*\varphi + \phi\} = L^{-1*}(\text{Id} - \text{Proj}_{\mathcal{N}_\phi})\{\psi + \phi\} = L^{-1*}\psi = L^{-1*}L^*\varphi = \varphi.
\]

for all \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) and \( \phi \in \mathcal{N}_\phi \). In doing so, we have actually shown that the map \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{N}_\phi \to \mathcal{P}_L : (\varphi, \phi) \mapsto g = L^*\varphi + \phi \) is invertible and that

\[
L_\phi^{-1*}\psi = L^{-1*}\psi \text{ for all } \psi = L^*\varphi \in L^*(\mathcal{S}(\mathbb{R}^d)).
\]

(20)
The properties in Items 1-3 then simply follow from the observation that
\((\text{Id} - \text{Proj}_{\mathcal{N}_\phi})\{\phi\} = 0\) for any \(\phi \in \mathcal{N}_\phi\) which, in light of the previous identities, also yields \(L_{\phi}^{-1}(L^*(\varphi)) = L_{\phi}^{-1}(\varphi) = \varphi\) for all \(\varphi \in \mathcal{S}(\mathbb{R}^d)\).

The property that \(\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{X}\) ensures that \(\|\cdot\|_X\) is a bona fide norm over \(\mathcal{S}(\mathbb{R}^d)\). This allows us to equip \(\mathcal{S}(\mathbb{R}^d) \times \mathcal{N}_\phi\) with the composite norm \((\varphi, \phi) \mapsto \max(\|\varphi\|_X, \|p(\phi)\|_2)\). We then exploit the bijection between \(\mathcal{S}(\mathbb{R}^d) \times \mathcal{N}_\phi\) and \(L^*(\mathcal{S}(\mathbb{R}^d)) \oplus \mathcal{N}_\phi\) to get the expression of the norm given in \([19]\). In effect, this shows that the normed space \((\mathcal{S}(\mathbb{R}^d), \|\cdot\|_X) \times \mathcal{N}_\phi\) is isometrically isomorphic to \(P_L = L^*(\mathcal{S}(\mathbb{R}^d)) \oplus \mathcal{N}_L\) equipped with the \(\|\cdot\|_{X_L}\)-norm, which is the desired result.

To determine the corresponding norm inequalities (Items 5 and 6), we apply the direct-sum property to rewrite the norm of \(g \in P_L\) as
\[
\|g\|_{X_L} = \max(\|(\text{Id} - \text{Proj}_{\mathcal{N}_\phi})\{g\}\|_{X_L}, \|\text{Proj}_{\mathcal{N}_\phi}\{g\}\|_{X_L}),
\]
from which we deduce that \(\|L_{\phi}^{-1}g\|_X \leq \|g\|_{X_L}\) and \(\|\text{Proj}_{\mathcal{N}_\phi}\{g\}\|_{X_L} \leq \|g\|_{X_L}\) for any \(g \in P_L\). These two inequalities are sharp because \(\text{Proj}_{\mathcal{N}_\phi}\) (resp. \(L_{\phi}^{-1}\)) is an isometry over \(\mathcal{N}_\phi\) (resp. an isometry over \(L^*(\mathcal{S}(\mathbb{R}^d))\)).

We note that the projection property in Item 3 is equivalent to
\[
L^*L_{\phi}^{-1}L^* \varphi = L^* \varphi \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d),
\]
which indicates that \(L_{\phi}^{-1}\) is a generalized inverse of \(L^*\).

4 Direct-sum topology of the native space

In this section, we reveal the Banach structure of the native space \(\mathcal{X}_L'\) and of its pre-dual \(\mathcal{X}_L\). We like to think of \(\mathcal{X}_L'\) as the largest set of functions \(f\) such that \(\|Lf\|_{X'} < \infty\) under the constraint of a finite-dimensional null space \(\mathcal{N}_L = \{f \in \mathcal{X}_L' : L\{f\} = 0\} = \mathcal{N}_p\). We also assume that \(\mathcal{X}\) is a Schwartz-Banach space with the cases of interest being
\begin{itemize}
  \item \(L_2(\mathbb{R}^d)\) equipped with the \(\|\cdot\|_{L_2}\) norm and, more generally,
  \item \(L_p(\mathbb{R}^d)\) equipped with the \(\|\cdot\|_{L_p}\) norm for \(p \in [1, \infty)\);
  \item \(C_0(\mathbb{R}^d)\), which can be specified as the closure of \(\mathcal{S}(\mathbb{R}^d)\) in the \(\|\cdot\|_{\infty}\)-norm.
\end{itemize}
4.1 Pre-dual of the native space

**Definition 6.** The pre-native space $\mathcal{X}_L$ is the completion of the pre-Banach space $\mathcal{P}_L$ of Definition 5 for the $\| \cdot \|_{X_L}$-norm defined by (19).

We now identify this space and prove that it is isometrically isomorphic to $\mathcal{X} \times \mathcal{N}_\phi$, which requires the use of Cauchy sequences to extend the properties of $\mathcal{P}_L$ in Proposition 4. We recall that the two normed spaces that underly the specification of $\mathcal{P}_L = \mathcal{W} \oplus \mathcal{N}_\phi$ in Proposition 4 are $\mathcal{W} = (L^*(\mathcal{S}(\mathbb{R}^d)), \| \cdot \|_{\mathcal{W}})$ with

$$\| \psi \|_{\mathcal{W}} = \| L^{-1\ast}_\phi \psi \|_{X},$$

where $L^{-1\ast}_\phi : \mathcal{P}_L = (\mathcal{W} \oplus \mathcal{N}_\phi) \rightarrow \mathcal{X}$ and $(\mathcal{N}_\phi, \| \cdot \|_{\mathcal{N}_\phi})$ with $\| \phi \|_{\mathcal{N}_\phi} = \| p(\phi) \|_2$.

**Theorem 3 (Pre-dual of the native space).** Under the admissibility and compatibility hypotheses of Definition 4, the completion of $(\mathcal{P}_L, \| \cdot \|_{X_L})$ is the Banach space

$$\mathcal{X}_L = L^*(\mathcal{X}) \oplus \mathcal{N}_\phi,$$

which is itself isometrically isomorphic to $\mathcal{X} \times \mathcal{N}_\phi$. Correspondingly, the bounded operators $\text{Proj}_{\mathcal{N}_\phi} : \mathcal{P}_L \rightarrow \mathcal{N}_\phi$ and $L^{-1\ast}_\phi : \mathcal{P}_L \rightarrow \mathcal{X}$ of Proposition 4 have unique continuous extensions $\text{Proj}_{\mathcal{N}_\phi} : \mathcal{X}_L \rightarrow \mathcal{N}_\phi$ and $L^{-1\ast}_\phi : \mathcal{X}_L \rightarrow \mathcal{X}$ with the following properties:

1. Null space of $\text{Proj}_{\mathcal{N}_\phi}$: $\mathcal{U} \triangleq L^*(\mathcal{X}) = \{ g \in \mathcal{X}_L : \text{Proj}_{\mathcal{N}_\phi} \{ g \} = 0 \}$.

2. Null space of $L^{-1\ast}_\phi$: $\mathcal{N}_\phi \triangleq \text{span}\{ \phi_n \}_{n=1}^N = \{ g \in \mathcal{X}_L : L^{-1\ast}_\phi g = 0 \}$.

3. Left inverse of $L^*$: $L^{-1\ast}_\phi L^* v = v$ for any $v \in \mathcal{X}$.

4. Pseudo-right-inverse: $L^* L^{-1\ast}_\phi g = (\text{Id} - \text{Proj}_{\mathcal{N}_\phi}) \{ g \}$ for any $g \in \mathcal{X}_L$.

where $\text{Proj}_{\mathcal{N}_\phi} : g \mapsto \sum_{n=1}^N \langle p_n, g \rangle \phi_n$ with the property that $\langle p_n, u \rangle = 0$ for all $u \in \mathcal{U}$ and $\langle p_m, \phi_n \rangle = \delta[m-n]$. Moreover, we have the hierarchy of continuous and dense embeddings

$$\mathcal{S}(\mathbb{R}^d) \triangleright L_{1,\alpha}(\mathbb{R}^d) \triangleright \mathcal{X}_L \triangleright \mathcal{S}'(\mathbb{R}^d).$$

**Proof.** By definition, we have that $\mathcal{X}_L = \mathcal{P}_L$ which, in view of Theorem 7 in Appendix B, is itself decomposable as $\mathcal{P}_L = \mathcal{W} \oplus \mathcal{N}_\phi = \mathcal{W} \oplus \mathcal{N}_\phi$ (because $\mathcal{N}_\phi$ is finite-dimensional).
\( \mathcal{W} = \mathcal{U} = L^*(\mathcal{X}) = \{ u = L^*v : v \in \mathcal{X} \} \) equipped with the topology inherited from \( \mathcal{X} \).

Since the map \( L^*: \mathcal{X} \xrightarrow{\sim} \mathcal{S}'(\mathbb{R}^d) \) is injective, we have an isometric isomorphism between the Banach space \((\mathcal{X}, \| \cdot \|_{\mathcal{X}})\) and \( \mathcal{U} \), which is itself a Banach space equipped with the norm \( \| u \|_{\mathcal{U}} = \| v \|_{\mathcal{X}} \) where \( v \) is the unique element in \( \mathcal{X} \) such that \( u = L^*v \). In particular, for any \( \psi \in \mathcal{W} \subseteq \mathcal{U} \), we can use Property 2 of Proposition 4 (invertibility) to show that
\[
\| \psi \|_{\mathcal{U}} = \| L^{-1}_\phi^* \psi \|_{\mathcal{X}} = \| \psi \|_{\mathcal{W}}.
\]

Moreover, because the spaces \((\mathcal{S}(\mathbb{R}^d), \| \cdot \|_{\mathcal{X}})\) and \((\mathcal{W}, \| \cdot \|_{\mathcal{U}})\) are isometric, \( \mathcal{W} \) is dense in \( \mathcal{U} \): for any \( u = L^*v \in \mathcal{U} \) and \( \epsilon > 0 \), there exists some \( \psi_\epsilon \in \mathcal{W} \) such that \( \| u - \psi_\epsilon \|_{\mathcal{U}} \leq \epsilon \). Indeed, the denseness of \( \mathcal{S}(\mathbb{R}^d) \) in \( \mathcal{X} \) implies the existence of \( \varphi_\epsilon \in \mathcal{S}(\mathbb{R}^d) \) such that \( \| v - \varphi_\epsilon \|_{\mathcal{X}} = \| L^*v - L^*\varphi_\epsilon \|_{\mathcal{U}} \leq \epsilon \) so that it suffices to take \( \psi_\epsilon = L^*\varphi_\epsilon \in \mathcal{W} \). Since \( \mathcal{U} \) is complete and admits \( \mathcal{W} \) as a dense subset, it can be identified as the completion of \( \mathcal{W} \) equipped with the \( \| \cdot \|_{\mathcal{U}} \)-norm.

(ii) Extension of operators and functionals

For clarity, we mark the extended operators mentioned in the theorem with a tilde. Specifically, the application of Theorem 1 with \( Z = \mathcal{P}_L \) and \( Y = \mathcal{X}, N_\phi \), and \( \mathbb{R} \) allows us to specify the unique extensions

\[
\begin{align*}
&\widetilde{L}_\phi^{-1*}: \mathcal{P}_L \xrightarrow{\sim} \mathcal{X} \text{ with } \| \widetilde{L}_\phi^{-1*} \| = 1 \\
&\widetilde{\text{Proj}}_{N_\phi}: \mathcal{P}_L \xrightarrow{\sim} N_\phi \text{ with } \| \widetilde{\text{Proj}}_{N_\phi} \| = 1 \\
&\widetilde{p}_n: \mathcal{P}_L \rightarrow \mathbb{R} \text{ with } \| \widetilde{p}_n \| \leq 1.
\end{align*}
\]

The relevant bounds for the two first instances are directly deducible from Properties 6 and 7 in Proposition 4, while the explicit definitions of these extensions are given in (21) and (23). As for the functionals \( p_n: \mathcal{P}_L \rightarrow \mathbb{R} \) for \( n = 1, \ldots, N_0 \), we observe that
\[
| \langle p_n, L^*\varphi + \phi \rangle | = | \langle p_n, \phi \rangle | \leq \max(\| \varphi \|_{\mathcal{X}}, \left( \sum_{m=1}^{N_0} | \langle p_m, \phi \rangle |^2 \right)^{1/2}),
\]
for any \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) and \( \phi \in N_\phi \), which yields the supporting bound
\[
| \langle p_n, g \rangle | \leq \| g \|_{\mathcal{X}_L} \quad \text{for all } g \in \mathcal{P}_L.
\]

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(iii) Derivation of Properties 1-4 by continuity

In line with the argumentation in Item (i), for any \( u \in \mathcal{U} = \mathcal{W} \), we have that

\[
\|u\|_\mathcal{U} = \|\widehat{L}_\phi^{-1} u\|_\mathcal{X} \quad \text{with} \quad \widehat{L}_\phi^{-1} u \triangleq \lim_{i \to \infty} L_\phi^{-1} \psi_i, \tag{21}
\]

where \((\psi_i)\) is any Cauchy sequence in \( \mathcal{W} \) such that \( u = \lim_{i \to \infty} \psi_i \in \mathcal{W} = \mathcal{U} \). In fact, the underlying isometric isomorphism ensures that a Cauchy sequence \((\psi_i)\) in \( \mathcal{W} \) maps to a corresponding sequence \((\varphi_i) = L_\phi^{-1} \psi_i\) that is Cauchy in \((\mathcal{S}(\mathbb{R}^d), \| \cdot \|_\mathcal{X})\), and vice versa by taking \( \psi_i = L^* \varphi_i \). In the limit, we have that \( v = \lim_{i \to \infty} \varphi_i = L_\phi^{-1} \{ \lim_{i \to \infty} \psi_i \} = L_\phi^{-1} u \in \mathcal{X} \) and \( u = \lim_{i \to \infty} \psi_i = L^* \{ \lim_{i \to \infty} \varphi_i \} = L^* v \in \mathcal{U} \). The last characterization also yields that

\[
\langle \overline{p}_n, u \rangle = \lim_{i \to \infty} \langle p_n, \psi_i \rangle = \lim_{i \to \infty} \langle L^* p_n, \varphi_i \rangle = 0 \tag{22}
\]

for all \( u \in \mathcal{U} \), which is consistent with the property that \( \overline{\text{Proj}}_{\mathcal{N}_\phi} u = 0 \). The conclusion is that \( \overline{p}(u) = \mathbf{0} \) for all \( u \in \mathcal{U} \) so that the extended projector \( \overline{\text{Proj}}_{\mathcal{N}_\phi} : \mathcal{Y}_L \rightarrow \mathcal{N}_\phi \) retains the same functional form as before as

\[
\overline{\text{Proj}}_{\mathcal{N}_\phi} \{ g \} = \sum_{n=1}^{N_0} \langle \overline{p}_n, g \rangle \phi_n. \tag{23}
\]

Due to the isometric isomorphism between \( \mathcal{U} \) and \( \mathcal{X} \), it is then also acceptable to decompose the extended pseudo-inverse as \( \overline{L}_\phi^{-1} = L^* \{ \text{Id} - \overline{\text{Proj}}_{\mathcal{N}_\phi} \} \), where \( L^* \) denotes the formal inverse of \( L^* \) from \( \mathcal{U} \to \mathcal{X} \). By plugging in the relevant Cauchy sequences and by invoking (21) and (22), it is then possible to seamlessly transfer Properties 1-3 of Proposition 4 to the completed counterpart of these spaces, which yields Items 1-4.

(iv) Embeddings

We simplify the notation by setting \( \mathcal{Y} = L_{1,-\alpha}(\mathbb{R}^d) \) and consider the decomposition \( \mathcal{Y} = \mathcal{Y}_{p^\perp} \oplus \mathcal{N}_\phi \), where \( \mathcal{Y}_{p^\perp} \triangleq \{ \psi \in \mathcal{Y} : p(\psi) = \mathbf{0} \} \). To prove that \( \mathcal{Y}_{p^\perp} \subseteq \mathcal{U} \), we first invoke the continuity of \( L_\phi^{-1} : \mathcal{Y} \to \mathcal{X} \), which ensures that \( v = L_\phi^{-1} \psi \in \mathcal{X} \) for all \( \psi \in \mathcal{Y}_{p^\perp} \). We then use Property 4 (or the injectivity of \( L^* \)) to get that \( L^* v = L^* L_\phi^{-1} \psi = \psi \) (because \( \text{Proj}_{\mathcal{N}_\phi} \{ \psi \} = 0 \) for all \( \psi \in \mathcal{Y}_{p^\perp} \)), which shows that \( \psi \in L^*(\mathcal{X}) = \mathcal{U} \). This, together with the continuity of \( L^* : \mathcal{X} \to \mathcal{U} \), ensures the continuity of the inclusion/identity map \( I = L^* \circ L_\phi^{-1} : \mathcal{Y}_{p^\perp} \to \mathcal{X} \to \mathcal{U} \), which is equivalent to \( \mathcal{Y}_{p^\perp} \to \mathcal{U} \).
Consequently, we have that \( Y = (Y_p \perp \oplus N_\phi) \hookrightarrow (U \oplus N_\phi) = X_L \). Moreover, since \( P_L \subseteq Y = L_{1,\alpha}(\mathbb{R}^d) \) (see Item 5 in Proposition 4) and \( P_L \) is a dense subspace of \( X_L = P_L \) by construction, we readily deduce that the embedding \( Y \hookrightarrow X_L \) is dense. Likewise, since \( S(\mathbb{R}^d) \hookrightarrow Y \), we get that \( S(\mathbb{R}^d) \hookrightarrow X_L \) by transitivity. Finally, the continuity of \( L^* : X \hookrightarrow S'(\mathbb{R}^d) \) implies that \( U = L^*(X) \hookrightarrow S'(\mathbb{R}^d) \) which, together with \( N_\phi \hookrightarrow S'(\mathbb{R}^d) \), yields that \( X_L \hookrightarrow S'(\mathbb{R}^d) \). Here too, the embedding is dense due to the property that \( S(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d) \) (see Propositions 6 and 7 in Appendix A).

(v) Identification of \( \tilde{p}_n = p_n, \tilde{\text{Proj}}_{N_\phi} = \text{Proj}_{N_\phi} \) and \( \tilde{L}^{-1}_\phi = L^{-1}_\phi \). So far, we have distinguished the extended operators and functionals from the original ones whose initial domain was restricted to \( L_{1,\alpha}(\mathbb{R}^d) \). We now invoke the Schwartz-Banach property of both \( X_L \) (see Item (iv) and Definition 1) and \( L_{1,\alpha}(\mathbb{R}^d) \) to argue that there is a common underlying characterization (see Proposition 2) that is applicable to both instances. Consequently, it is acceptable to write that \( \tilde{p}_n : g \mapsto \langle p_n, g \rangle \), which then gives a concrete and rigorous interpretation of the extended functionals and, by the same token, the extended projector (23). Likewise, from now on, we denote both the original and extended pseudo-inverse operators by \( L^{-1}_\phi \), under the understanding that their underlying Schwartz kernel is the same.

An important observation is that the \( X \)-stability hypothesis (i.e., \( L^{-1}_\phi : L_{1,\alpha}(\mathbb{R}^d) \hookrightarrow X \)) is only required for the proof of the embeddings (last statement of the theorem). This is a fundamental point as it ensures that \( X_L \) is a Schwartz-Banach space, while it also yields a concrete interpretation of the underlying operators. Last but not least, it guarantees that the actual native space \( X_L' \) is a proper Banach subspace of \( S'(\mathbb{R}^d) \) (by Proposition 1).

Implicit in the statement of Theorem 3 (and explicit in the proof) are the following fundamental properties of \( U \): the primary part of \( X_L \) “perpendicular” to \( N_\phi \).

**Corollary 1.** The space \( U = \{ u = L^*v : v \in X \} \) in Theorem 3 has the following properties:

1. For any \( u \in U \), \( L^{-1}_\phi u = L^{-1}_\phi L^*u = L^*L^{-1}_\phi u = u \).
2. \( U \) is a Banach space equipped with the norm \( \| u \|_U = \| L^{-1}_\phi u \|_X \).
3. \( U = L^*(X) \) is isometrically isomorphic to \( X \): For any \( u \in U \) (resp. for any \( v \in X \)), there exists a unique element \( v = L^{-1}_\phi u \in X \) (resp. \( u = L^*v \in U \)) such that \( \| u \|_U = \| v \|_X \).

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4. \( L^* : \mathcal{X} \xrightarrow{\text{iso.}} \mathcal{U} \) (isometry).

5. \( L^{-1*} = L^{-1}_\phi : \mathcal{U} \xrightarrow{\text{iso.}} \mathcal{X} \) (isometry).

6. For any \((p,u) \in (\mathcal{N}_p \times \mathcal{U}), (p,u) = 0\).

7. \( \mathcal{U} \) is the completion of \( S_{p^\perp}(\mathbb{R}^d) = \{\psi \in \mathcal{S}(\mathbb{R}^d) : \mathcal{p}(\psi) = 0\} \) in the \( \|\cdot\|_U\)-norm.

Proof. Items 1-5 are re-statements/re-interpretations of the invertibility Properties 3 and 4 in Theorem 3. The key is that \((\text{Id} - \text{Proj}_{\mathcal{N}_p})\{u\} = u\) for all \(u \in \mathcal{U}\), which then makes the presence of this operator redundant. Item 6 results from the simple manipulation

\[
(p, u) = (p, L^* L^{-1*}_\phi u) = \langle L p, L^{-1*}_\phi u \rangle = 0,
\]

which is legitimate because \(u \in \mathcal{X}_L = \mathcal{U} \oplus \mathcal{N}_\phi\) and \(p \in \mathcal{N}_p = \mathcal{N}'_\phi \subseteq \mathcal{X}'_L = (\mathcal{U} \oplus \mathcal{N}_\phi)'\). As for Item 7, we consider the direct-sum decomposition \(\mathcal{S}(\mathbb{R}^d) = S_{p^\perp}(\mathbb{R}^d) \oplus \mathcal{N}_\phi\), which is valid whenever \(\mathcal{N}_\phi \subset \mathcal{S}(\mathbb{R}^d)\) (universality assumption). We then observe that

\[
(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{\mathcal{X}_L}) = (S_{p^\perp}(\mathbb{R}^d), \|\cdot\|_{\mathcal{U}}) \oplus (\mathcal{N}_\phi, \|\cdot\|_{\mathcal{N}_\phi}).
\]

This equality holds because \(\|\varphi\|_{\mathcal{X}_L} = \|\psi\|_{\mathcal{U}} + \|\mathcal{p}(\phi)\|_2\) for any \(\varphi = \psi + \phi \in \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{X}_L\) with \((\psi, \phi) \in (S_{p^\perp}(\mathbb{R}^d) \times \mathcal{N}_\phi) \subseteq (\mathcal{U} \times \mathcal{N}_\phi)\). The denseness of the embedding \(\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{X}_L\) from Theorem 3 implies that \((\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{\mathcal{X}_L}) = \mathcal{X}_L = \mathcal{U} \oplus \mathcal{N}_\phi\). Finally, by invoking Theorem 7 and the property that \(\mathcal{N}_\phi, \|\cdot\|_{\mathcal{N}_\phi} = \mathcal{N}_\phi\) (because \(\mathcal{N}_\phi\) is finite-dimensional), we deduce that \((S_{p^\perp}(\mathbb{R}^d), \|\cdot\|_{\mathcal{X}_L}) = \mathcal{U}\), which is equivalent to \(S_{p^\perp}(\mathbb{R}^d) \xrightarrow{\text{ct}} \mathcal{U}\).

Item 1 in Corollary 1 indicates that the pseudo-inverse \(L^{-1*}_\phi\) and the canonical adjoint inverse \(L^{-1}\) are undistinguishable on \(\mathcal{U}\). This implies that the topology of \(\mathcal{U}\) does not depend on the choice of biorthogonal system \((\phi, \mathcal{p})\). By contrast, the effect of the two inverse operators is very different on \(\mathcal{N}_\phi\): for any \(\phi \in \mathcal{N}_\phi\), \(L^{-1*}_\phi \{\phi\} = 0\) by design (Property 1), while \(q = L^{-1*} \{\phi\} \in L_{\infty,\alpha}(\mathbb{R}^d)\) is nonzero and, in general, not even included in \(\mathcal{X}\) unless \(\phi = 0\).

We end this section by listing the properties of \(\mathcal{X}_L\) that we believe to be the most relevant to practice. They are directly deducible from Theorem 3, too.
Corollary 2. Under the admissibility and compatibility hypotheses of Definition 4, the pre-dual space $X_L$ has the following properties:

- $X_L$ is the completion of $\mathcal{S}(\mathbb{R}^d)$ in the $\| \cdot \|_{X_L}$-norm, which is specified as $\| \phi \|_{X_L} = \max(\|L^{-1}\phi\|_X, \|p(\phi)\|_2)$.
- Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then, $f \in X_L$ if and only if there exists $(v, \phi) \in (X \times N)$ such that $f = L^* v + \phi$. Moreover, the decomposition is unique with $v = L^{-1}\phi f$ and $\phi = \text{Proj}_{N\phi}f$.
- $X_L$ is a Schwartz-Banach space.

4.2 Native space

As indicated by the notation, the native space $X_L'$ is the continuous dual of $X_L = U \oplus N$, where $U = L^*(X)$. Accordingly, there is a direct correspondence between the properties of $X_L'$ and those of the predual space $X_L$ in Theorem 3. The whole functional picture is summarized in Figure 1.

**Theorem 4 (Native Banach Space).** Under the admissibility and compatibility hypotheses of Definition 4, the continuous dual of $X_L$ in Theorem 3 is
the native Banach space

\[ X'_L = U' \oplus N_p = L^{-1}_\phi(X') \oplus N_p \]
\[ = \{ L^{-1}_\phi w + p : w \in X', p \in N_p \}, \] (24)

which is isometrically isomorphic to \( X' \times N_p \) equipped with the composite norm \( \|w\|_{X'} + \|\phi(p)\|_2 \). In other words, for any \( f \in X'_L \), there is a unique pair \( w = Lf \in X' \) and \( p = \text{Proj}_{N_p}\{f\} = \sum_{n=1}^{N_0} \langle \phi_n, f \rangle p_n \in N_p \), with the finite-dimensional space \( N_p = \text{span}\{p_n\}_{n=1}^{N_0} \) being the null space of the operator \( L : X'_L \rightarrow X' \). This is consistent with the operator \( L^{-1}_\phi : X' \rightarrow X'_L \), which is the adjoint of \( L^{-1}_\phi^* \) in Theorem 3, having the following properties:

1. Effective range: \( U' = L^{-1}_\phi(X') \triangleq \{ s = L^{-1}_\phi w : w \in X' \} \).

2. Annihilator: \( N_\phi = \{ g \in X_L : \langle L^{-1}_\phi w, g \rangle = 0 \text{ for all } w \in X' \} \).

3. Right inverse of \( L \): \( LL^{-1}_\phi w = w \) for any \( w \in X' \).

4. Left pseudo-inverse: \( L^{-1}_\phi Lf = (\text{Id} - \text{Proj}_{N_p})\{f\} \) for any \( f \in X'_L \).

Moreover, we have the hierarchy of continuous (and sometime dense) embeddings described by

\[ S(\mathbb{R}^d) \hookrightarrow X'_L \hookrightarrow L_{\infty,0}(\mathbb{R}^d) \overset{\text{d}}{\hookrightarrow} S'(\mathbb{R}^d). \]

Finally, if \( X \) is reflexive, then \( X'_L \) is reflexive as well and we have the dense embedding \( S(\mathbb{R}^d) \overset{\text{d}}{\hookrightarrow} X'_L \).

Proof. The listed properties are the dual transpositions of the ones in Theorem 3. The key is \( N'_\phi = N_p \) (see explanation in Section 3.1) and \( X'_L = (U \oplus N'_\phi)' = U' \oplus N'_\phi = U' \oplus N_p \) equipped with the dual composite norm

\[ \|f\|_{X'_L} = \|\langle \text{Proj}_U f, \text{Proj}_{N_p} f \rangle\|_1 \]
\[ = \|\text{Proj}_U f\|_{U'} + \|\phi(f)\|_2. \]

For the details, the reader is referred to Appendix B on direct sums and Proposition 8 with \((p,q) = (\infty,1)\). The other fundamental ingredient is the continuity of the adjoint operators \( L : U' \overset{\text{iso}}{\longrightarrow} X' \), \( L^{-1}_\phi : X' \overset{\text{iso}}{\longrightarrow} U' \), \( L^{-1}_\phi : X' \overset{\text{c}}{\hookrightarrow} X'_L \), \( \text{Proj}_{N'_\phi} = \text{Proj}_{N_p} : X'_L \overset{\text{c}}{\hookrightarrow} N_p \overset{\text{iso}}{\hookrightarrow} X'_L \), which follows from the continuity of \( L^* : X \overset{\text{iso}}{\longrightarrow} U \), \( L^{-1}_\phi^* : U \overset{\text{iso}}{\longrightarrow} X \), \( L^{-1}_\phi^* : X_L \overset{\text{c}}{\longrightarrow} X \),
Proj$_{N}\phi : \mathcal{X}_L \xrightarrow{\subset} N_\phi \xrightarrow{\text{iso.}} \mathcal{X}_L$ in Theorem 3 and Corollary 1. The adjoint relation Proj$_{N\phi}^* = \text{Proj}_N^*$ is due to the special form of the underlying kernel (see Table 1).

(i) Identification of $U' = L^{-1}_\phi(\mathcal{X}')$ with $\|s\|_{U'} = \|Ls\|_{\mathcal{X}}$.
Since the mapping between $\mathcal{X}'$ and $U'$ is isometric and bijective, we have that $U' = L^{-1}_\phi(\mathcal{X}')$, with the two spaces being isometrically isomorphic.

By recalling the definition of the dual norm and invoking the isomorphism between $\mathcal{X} = L^*(\mathcal{X}')$ with $v \mapsto u = L^*v$, we then get that

$$\|s\|_{U'} = \sup_{u \in U \setminus \{0\}} \frac{\langle s, u \rangle_{U' \times U}}{\|u\|_{U}} = \sup_{v \in \mathcal{X} \setminus \{0\}} \frac{\langle s, L^*v \rangle_{U' \times U}}{\|v\|_{\mathcal{X}}} = \sup_{v \in \mathcal{X} \setminus \{0\}} \frac{\langle Ls, v \rangle_{\mathcal{X}' \times \mathcal{X}}}{\|v\|_{\mathcal{X}}} = \|Ls\|_{\mathcal{X}'}.$$

(ii) Derivation of Properties 2-4 by duality
The underlying principle is that the weak topology (resp. the weak* topology) separates the points in (resp. the dual of) a locally convex vector space [27]. Specifically, let $(\mathcal{X}', \mathcal{X})$ be any dual pair of Banach spaces. Then, for any $g_1, g_2 \in \mathcal{X}$ and $f_1, f_2 \in \mathcal{X}'$,

$$g_1 = g_2 \iff \langle f, g_1 \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle f, g_2 \rangle_{\mathcal{X}' \times \mathcal{X}} \text{ for all } f \in \mathcal{X'},$$

$$f_1 = f_2 \iff \langle f_1, g \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle f_2, g \rangle_{\mathcal{X}' \times \mathcal{X}} \text{ for all } g \in \mathcal{X}.$$

Consequently, the null-space Property 2 in Theorem 3 is equivalent to

$$g \in N_\phi \iff 0 = \langle w, L^{-1}_\phi g \rangle_{\mathcal{X}' \times \mathcal{X}} = \langle L^{-1}_\phi w, g \rangle_{\mathcal{X}' \times \mathcal{X}_L}, \forall w \in \mathcal{X'},$$

which is the desired result. The same principle applies for the other properties.

(iii) Embeddings
The application of Theorem 6 to the series of continuous and dense embeddings in Theorem 3 yields

$$S(\mathbb{R}^d) \hookrightarrow \mathcal{X}'_L \hookrightarrow (L_{1,-\alpha}(\mathbb{R}^d))' = L_{\infty,\alpha}(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d).$$

The denseness of the embedding $\mathcal{X}'_L \hookrightarrow S'(\mathbb{R}^d)$ follows from $S(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d)$ and Proposition 7.

\[ \square \]
We note that Property 2 has the other equivalent formulation
\[
U' = \{ g \in X'_L : \phi(g) = 0 \} = \{ g \in X'_L : \text{Proj}_{\mathcal{N}_p}(g) = 0 \},
\]
which is consistent with the direct-sum property. Hence the combination of Properties 2-4 implies a perfect isometry between \(U'\) and \(X'\) with \(w = Ls \in X', \ s = L^{-1}_p w \in U', \) and \(\|s\|_{U'} = \|Ls\|_{X'}\). The consideration of the direct-sum decomposition \(f = s + p\), where \(p = \text{Proj}_{\mathcal{N}_p}\{f\} \in \mathcal{N}_p\) and \(s = (f - p) = L^{-1}_p w \in U'\), allows us to identify the norm of \(X'_L\) as
\[
\|f\|_{X'_L} = \|s\|_{U'} + \|\phi(p)\|_2 = \|Ls\|_{X'} + \|\phi(p)\|_2.
\]
where we have made use of the property that \(L\{s + p\} = Ls\) and \(\phi(s) = 0\) for all \((s,p) \in (U' \times \mathcal{N}_p)\). This is the basis for a restatement of the primary properties of \(X'_L\) in Theorem 4 in a form more suitable for practitioners.

**Corollary 3.** Under the admissibility and compatibility hypotheses of Definition 4, the native space of \((L, X')\), denoted by \(X'_L\), has the following properties:

- \(X'_L\) is a Banach space that admits the explicit definition
  \[
  X'_L = \{ f \in S'(R^d) : \|f\|_{X'_L} = \sup_{\|\varphi\|_{X'_L} \leq 1} \langle f, \varphi \rangle < \infty \} \tag{26}
  \]
  or, equivalently,
  \[
  X'_L = \{ f \in L_{\infty, \alpha}(R^d) : \|f\|_{X'_L} < \infty \},
  \]
  where \(\alpha\) is the growth order associated with \(L\). Moreover, \(f \in X'_L \iff \|f\|_{X'_L} = \|Lf\|_{X'} + \|\phi(f)\|_2\). In particular, \(f \in X'_L \Rightarrow \|Lf\|_{X'} < +\infty\).

- Let \(f \in S'(R^d)\). Then, \(f \in X'_L\) if and only if there exists \((w, p) \in (X' \times \mathcal{N}_p)\) such that \(f = L^{-1}_p w + p\). Moreover, the decomposition is unique, with \(w = Lf\) and \(p = \text{Proj}_{\mathcal{N}_p}\{f\}\).

Equation (26) provides a rigorous, self-contained definition of \(X'_L\), but it has the disadvantage of being a bit convoluted. We like to view this equation as the justification for the two alternative forms
\[
X'_L = \left\{ f \in L_{\infty, \alpha}(R^d) : \|f\|_{X'_L} = \|Lf\|_{X'} + \|\phi(f)\|_2 < \infty \right\} \tag{27}
\]
\[
X'_L = \left\{ f \in L_{\infty, \alpha}(R^d) : \|Lf\|_{X'} < \infty \right\} \tag{28}
\]

which are adequate if one implicitly assumes that \( N_\phi \subset L_{1,-\alpha}(\mathbb{R}^d) \) and \( f \notin X'_L \iff \|Lf\|_{X'} + \|\phi(f)\|_2 = \infty \). While \( Lf \) is a priori undefined for \( f \notin X'_L \), one circumvents the formal difficulty by adopting the more permissive dual definition of the underlying semi-norm

\[
\sup_{\|\varphi\|_{X'} \leq 1: \varphi \in S(\mathbb{R}^d)} \langle f, L^* \varphi \rangle = \begin{cases} \|Lf\|_{X'}, & f \in X'_L \\ +\infty, & \text{otherwise}, \end{cases}
\]

which is valid for any \( f \in L_{\infty,\alpha}(\mathbb{R}^d) \) reason of the denseness of \( S(\mathbb{R}^d) \) in \( X \) and the admissibility condition \( L^* \varphi \in L_{1,-\alpha}(\mathbb{R}^d) \). Under this interpretation, (28) is legitimate as well since \( X'_L \hookrightarrow L_{\infty,\alpha}(\mathbb{R}^d) \). Indeed, the hypothesis of spline-admissibility (Definition 2) requires that the growth-restricted null space of \( L \) be finite-dimensional and spanned by some basis \( p = (p_1, \ldots, p_{N_0}) \), while one necessarily has that \( \|\phi(f)\|_2 < \infty \) for all \( f \in L_{\infty,\alpha}(\mathbb{R}^d) \) because \( N_\phi \subset L_{1,-\alpha}(\mathbb{R}^d) \xrightarrow{\text{loc}} (L_{\infty,\alpha}(\mathbb{R}^d))' = (L_{1,-\alpha}(\mathbb{R}^d))^\prime \). The slight disadvantage is that (28) does not fully specify the underlying topology.

4.3 Equivalent topologies and biorthogonal systems

In order to show that the choice of one biorthogonal system over another—say, \((\tilde{\phi}, \tilde{p})\) vs. \((\phi, p)\)—has no direct incidence on the definition of the underlying native space, we start by extending the range of validity of Theorems 3 and 4 to the complete set of admissible systems \((\tilde{\phi}, \tilde{p})\) with \( \tilde{p}_1, \ldots, \tilde{p}_{N_0} \in N_\tilde{L}, \tilde{\phi}_1, \ldots, \tilde{\phi}_{N_0} \in X_L, \) and \([\tilde{\phi}(\tilde{p}_1) \cdots \tilde{\phi}(\tilde{p}_{N_0})] = I\) (biorthogonality). To that end, we rely on the existence of a primary biorthogonal system such that \( N_\phi = \text{span}\{\phi_n\} \subset S(\mathbb{R}^d) \) (universality property), which ensures that the initial pre-dual space \( X_L \) is well-defined.

**Proposition 5.** Let \( \tilde{\phi} = (\tilde{\phi}_n) \) with \( \tilde{\phi}_n \in X_L \) for \( n = 1, \ldots, N_0 \) be such that the matrix \( C = [\tilde{\phi}(p_1) \cdots \tilde{\phi}(p_{N_0})] \in \mathbb{R}^{N_0 \times N_0} \) is invertible. Then, there exists a unique basis \( \tilde{p} \) of \( N_L = N_{\tilde{p}} \) such that Theorems 3, 4 and Corollaries 2, 3 remain valid for the biorthogonal system \((\tilde{\phi}, \tilde{p})\) and define a pair of native and pre-dual spaces \( X'_L = \mathcal{U}' \oplus N_{\tilde{p}} \) and \( \tilde{X}_L = \mathcal{U} \oplus N_{\phi} \), where \( \mathcal{U} = L^*(X) \). This construction then specifies four operators with the following properties:

- \( \text{Proj}_{N_\phi} = \text{Proj}_{N_{\tilde{p}}} : \tilde{X}_L \xrightarrow{\epsilon} N_{\phi} : g \mapsto \sum_{n=1}^{N_0} \tilde{\phi}_n \langle \tilde{p}_n, g \rangle \) such that
  \[
  \forall \phi \in N_{\phi} : \text{Proj}_{N_\phi} \{\phi\} = \phi \\
  \forall u \in \mathcal{U} : \text{Proj}_{N_\phi} \{u\} = 0.
  \]
\[ L^{-1}\phi = L^{-1}(\text{Id} - \text{Proj}_{N_{\phi}}) : L_{1, -\alpha}(\mathbb{R}^d) \hookrightarrow \tilde{X}_L \rightarrow X \text { such that } \]
\[ \forall \phi \in N_{\phi} : \quad L^{-1}\phi = 0 \]
\[ \forall g \in \tilde{X}_L : \quad L^* L^{-1}\phi g = (\text{Id} - \text{Proj}_{N_{\phi}})\{g\} \]
\[ \forall v \in X : \quad L^{-1}\phi^* v = v. \]

- \( \text{Proj}_{N_{\phi}} : \tilde{X}_L \rightarrow N_{\phi} : f \mapsto \sum_{n=1}^{N_0} \tilde{p}_n \langle \tilde{\phi}_n, f \rangle \) such that
\[ \forall p \in N_{\phi} : \quad \text{Proj}_{N_{\phi}}\{p\} = p \]
\[ \forall v \in U : \quad \text{Proj}_{N_{\phi}}\{v\} = 0. \]

- \( L^{-1} : X' \rightarrow \tilde{X}_L \) such that
\[ \forall w \in X' : \quad \tilde{\phi}(L^{-1}\{w\}) = 0 \]
\[ \forall w \in X' : \quad LL^{-1}w = w \]
\[ \forall f \in X'_L : \quad L^{-1}f = (\text{Id} - \text{Proj}_{N_{\phi}})\{f\}. \]

**Proof.** The new basis \( \tilde{p} \) of \( N_{\phi} = N_{\tilde{p}} \) is given by \( \tilde{p} = C^{-1}p \), which can easily be seen to be biorthogonal to \( \tilde{\phi} \). Next, we check that the operators \( \text{Proj}_{N_{\tilde{p}}} \) and \( (\text{Id} - \text{Proj}_{N_{\tilde{p}}}) \) are continuous on \( X_L = L^*(X) \oplus N_{\tilde{p}} \) under the hypothesis that \( \tilde{\phi}_n \in X_L \). Specifically, for any \( g \in X_L \), we have that
\[ \|\text{Proj}_{N_{\tilde{p}}}g\|_{X_L} \leq \sum_{n=1}^{N_0} \|\tilde{p}_n\| \|\tilde{\phi}_n\|_{X_L} \|g\|_{X_L} \quad (\text{by the triangle inequality}) \]
\[ \leq \sum_{n=1}^{N_0} \|\tilde{p}_n\|_{X'_L} \|g\|_{X_L} \|\tilde{\phi}_n\|_{X_L} = C_1 \|g\|_{X_L}, \]

where \( C_1 = \sum_{n=1}^{N_0} \|\tilde{p}_n\|_{X'_L} \|\tilde{\phi}_n\|_{X_L} < \infty \). The property that \( \text{Proj}_{N_{\tilde{p}}}\{g\} \in N_{\tilde{p}} \) (by construction) translates into \( \text{Proj}_{N_{\tilde{p}}} : X_L \rightarrow N_{\tilde{p}} \rightarrow X_L \). It is also obvious from the definition that \( \text{Proj}_{N_{\tilde{p}}} \) is the adjoint of \( \text{Proj}_{N_{\phi}} \) whose Schwartz kernel is \( (x, y) \mapsto \sum_{n=1}^{N_0} \tilde{p}_n(x)\tilde{\phi}_n(y) \).

Likewise, we have that \( \|(\text{Id} - \text{Proj}_{N_{\tilde{p}}})\{g\}\|_{X_L} \leq (1 + C_1)\|g\|_{X_L} \). By invoking the biorthogonality of \((\tilde{\phi}, \tilde{p})\), we readily verify that \( \tilde{p}(g - \text{Proj}_{N_{\tilde{p}}}\{g\}) = 31 \)
\[ 0 \Leftrightarrow p(g - \text{Proj}_{N_{\tilde{\phi}}}(g)) = 0, \] which yields \((\text{Id} - \text{Proj}_{N_{\tilde{\phi}}})\{g\} \in U\), thereby proving that \((\text{Id} - \text{Proj}_{N_{\tilde{\phi}}}) : X_L \xrightarrow{\mathcal{L}} U \xhookrightarrow{} X_L\). Since \(L^{-1*} : U \xhookrightarrow{} X\) (see Corollary 1), we can therefore chain the two operators, which results in \(L^{-1*} \circ (\text{Id} - \text{Proj}_{N_{\tilde{\phi}}}) : X_L \xrightarrow{\mathcal{L}} U \xhookrightarrow{} X\), thereby proving the continuity of \(L^{-1*} : X_L \xrightarrow{\mathcal{L}} X\). Finally, we invoke the continuous embedding \(L_{1,-\alpha}(\mathbb{R}^d) \hookrightarrow X_L\) (see Theorem 3), which ensures that \(L^{-1*} : L_{1,-\alpha}(\mathbb{R}^d) \xhookrightarrow{} X\), in accordance with the last compatibility requirement in Definition 4.

Given that the underlying operators all satisfy the required continuity and annihilation properties, we can then revisit the proofs and constructions in Theorems 3 and 4 to specify the corresponding pair of spaces \(\tilde{X}_L\) and \(\tilde{X}_L'\), which inherit the same embedding properties as \(X_L\) and \(X_L'\).

An important outcome of the proof of Proposition 5 is that the compatibility condition for a single instance \((\phi, \tilde{p})\) is transferred to all admissible biorthogonal systems \((\tilde{\phi}, \tilde{\tilde{p}})\). Theorem 5 describes the effect of such a change of biorthogonal system on the underlying norms, while it ensures that the underlying topologies are equivalent.

**Theorem 5 (Equivalent direct-sum topologies).** Let \(L\) be a spline-admissible operator that is compatible with \(X_L'\) in the sense of Definition 4. Then, for any two biorthogonal systems \((\phi, \tilde{p})\) and \((\tilde{\phi}, \tilde{\tilde{p}})\) with \(N_{\tilde{\phi}} = N_{\tilde{\tilde{p}}} = N_L\) and \(\phi, \tilde{\phi} \in X_L^{N_0}\), we have that

- \(X_L = \tilde{X}_L\) and \(X_L' = \tilde{X}_L'\) as sets;
- the norms \(\| \cdot \|_{X_L}\) and \(\| \cdot \|_{\tilde{X}_L}\) are equivalent on \(X_L\);
- the norms \(\| \cdot \|_{X_L'}\) and \(\| \cdot \|_{\tilde{X}_L'}\) are equivalent on \(X_L'\).

More precisely, with the operator definitions in Proposition 5 and

\[
\| f \|_{\tilde{X}_L'} = \| Lf \|_{X'} + \| \tilde{\phi}(f) \|_2 \\
\| g \|_{\tilde{X}_L} = \| L^{-1*}\tilde{\phi} g \|_X + \| \tilde{\tilde{p}}(g) \|_2,
\]

we have the equivalence relations

\[
\forall f \in X_L' : \| Lf \|_{X'} = \| (\text{Id} - \text{Proj}_{N_{\tilde{\phi}}})\{f\} \|_{\tilde{X}_L'} \quad (29) \\
\forall f \in X_L' : A_1 \| f \|_{X_L'} \leq \| f \|_{\tilde{X}_L'} \leq A_2 \| f \|_{X_L'} \quad (30) \\
\forall p \in N_L = N_{\tilde{\phi}} : B_1 \| \phi(p) \|_2 \leq \| \tilde{\phi}(p) \|_2 \leq B_2 \| \phi(p) \|_2 \quad (31)
\]
\( \forall u \in \mathcal{U} = \mathcal{L}^*(\mathcal{X}) : \|L_{\phi}^{-1}s u\|_{\mathcal{X}} = \| (\text{Id} - \text{Proj}_{\mathcal{N}_{\phi}}) u \|_{\mathcal{X}_L} = \|L_{\phi}^{-1}s u\|_{\mathcal{X}} \) (32)

\( \forall g \in \mathcal{X}_L : A_1 \|g\|_{\mathcal{X}_L} \leq \|g\|_{\mathcal{X}_L} \leq A_2 \|g\|_{\mathcal{X}_L} \) (33)

\( \forall g \in \mathcal{X}_L : B_1 \|p(g)\|_2 \leq \|p(g)\|_2 \leq B_2 \|p(g)\|_2 \) (34)

for some suitable constants \( A_1, A_2, B_1, B_2, A_1', A_2', B_1', B_2' > 0 \).

Note that the fixed parts of the construction are \( \mathcal{N}_L = \mathcal{N}_p = \mathcal{N}_{\phi} \) and \( \mathcal{U} = \mathcal{L}^*(\mathcal{X}) \) (see Figure 1), which are associated with (31) and (32), respectively. On the other hand, we may have that \( \mathcal{N}_{\phi} \neq \mathcal{N}_{\phi} \), although what distinguishes those two spaces needs to be included in \( \mathcal{U} = \mathcal{L}^*(\mathcal{X}) \) in the sense that \( (\text{Id} - \text{Proj}_{\mathcal{N}_{\phi}}) \{\phi\}, (\text{Id} - \text{Proj}_{\mathcal{N}_{\phi}}) \{\phi\} \in \mathcal{U} \) for any \( \phi \in \mathcal{N}_{\phi} \) and \( \tilde{\phi} \in \mathcal{N}_{\phi} \). In particular, when both \( \phi \) and \( \tilde{\phi} \) are biorthogonal to the same \( p = \tilde{p} \), we have that \( p(\phi_n - \phi_n) = 0 \) for \( n = 1, \ldots, N_0 \), so that the condition \( \phi_n \in \mathcal{X}_L \) is equivalent to

\[
\mathcal{L}_{\phi}^{-1}s \{\phi_n - \phi_n\} = \mathcal{L}_{\phi}^{-1}s \{\tilde{\phi}_n - \phi_n\} \in \mathcal{X},
\]

which yields a criterion for admissibility that is convenient since it no longer depends on \( \mathcal{L}_{\phi}^{-1}s \).

**Proof.** Proposition 5 ensures that underlying spaces and operators are well-defined.

(i) **Delineation as sets**

To avoid circularity, we assume once more that \( \mathcal{N}_{\phi} = \text{span}\{\phi_n\} \subset \mathcal{S}(\mathbb{R}^d) \) (universality condition). We then consider the generic members \( f = \mathcal{L}_{\phi}^{-1}s w + p \) and \( \tilde{f} = \mathcal{L}_{\phi}^{-1}s w + p \) of the underlying spaces with \( w \in \mathcal{X} \) and \( p \in \mathcal{N}_p = \mathcal{N}_{\phi} = \text{span}\{\tilde{p}_n\} \). Since \( \mathcal{L}\{\tilde{f} - f\} = (w - w) = 0 \), the two functions can only differ by a components \( \tilde{p} = (\tilde{f} - f) \in \mathcal{N}_p \), which proves that \( \mathcal{X}_L = \tilde{\mathcal{X}}_L \) (as a set). Likewise, on the side of the pre-dual space, we have that \( g = \mathcal{L}*v + \phi \in \mathcal{X}_L \) with \( (v, \phi) \in (\mathcal{X} \times \mathcal{N}_{\phi}) \) and \( \mathcal{N}_{\phi} \subset \mathcal{S}(\mathbb{R}^d) \). Now, if \( \mathcal{N}_{\phi} \subset \mathcal{X}_L \), we obviously also have that \( \tilde{g} = \mathcal{L}*v + \phi \in \mathcal{X}_L \) for any \( \tilde{\phi} \in \mathcal{N}_{\phi} \) and \( v \in \mathcal{X} \), which implies that \( \tilde{\mathcal{X}}_L \subseteq \mathcal{X}_L \). Conversely, since \( \mathcal{S}(\mathbb{R}^d) \hookrightarrow \tilde{\mathcal{X}}_L \), \( g = \mathcal{L}*v + \phi \in \mathcal{X}_L \) for any \( (v, \phi) \in (\mathcal{X} \times \mathcal{N}_{\phi}) \), which yields \( \mathcal{X}_L \subseteq \tilde{\mathcal{X}}_L \).

(ii) **Norm inequalities**

To get (29), we observe that

\[
\| (\text{Id} - \text{Proj}_{\mathcal{N}_{\phi}}) \{f\} \|_{\tilde{\mathcal{X}}_L} = \frac{\| L (f - \text{Proj}_{\mathcal{N}_{\phi}} \{f\}) \|_{\mathcal{X}'} + \| \tilde{\phi}(f - \text{Proj}_{\mathcal{N}_{\phi}} \{f\}) \|_2}{\| L f \|_{\mathcal{X}'}} = 0
\]
because \( \text{Proj}_{N^\phi} \{ f \} \in N^\phi \) is annihilated by \( L \) and \( \tilde{\phi}(f - \text{Proj}_{N^\phi} \{ f \}) = 0 \) by construction, irrespective of the choice of \((\tilde{\phi}, \tilde{p})\). Likewise, the dual relation \( (32) \) is a direct consequence of the form of the adjoint operator \( L^{-1}_\phi = L^{-1} (\text{Id} - \text{Proj}_{N^\phi}) \) and the orthogonality condition \( \text{Proj}_{N^\phi} \{ u \} = 0 \) for all \( u \in \mathcal{U} \).

In order to estimate \( \| f \|_{\tilde{X}_L^\prime} \), we first invoke the duality bound

\[
|\langle \tilde{\phi}_n, f \rangle| \leq \| \tilde{\phi}_n \|_{X_L} \| f \|_{X_L^\prime}
\]

with the role of \( X_L \) and \( \tilde{X}_L \) (resp. \( \tilde{\phi}_n \) and \( \phi_n \)) being interchangeable. This suggests the estimate

\[
\| f \|_{\tilde{X}_L^\prime} = \| Lf \|_{X_L^\prime} + \left( \sum_{n=1}^{N_0} |\langle \tilde{\phi}_n, f \rangle|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \| Lf \|_{X_L^\prime} + \sum_{n=1}^{N_0} \| \tilde{\phi}_n \|_{X_L} \| f \|_{X_L^\prime} \quad \text{(by the triangle inequality)}
\]

\[
\leq \left( 1 + \sum_{n=1}^{N_0} \| \tilde{\phi}_n \|_{X_L} \right) \| f \|_{X_L^\prime}. \quad (36)
\]

Likewise, we have that

\[
\| f \|_{X_L^\prime} \leq \left( 1 + \sum_{n=1}^{N_0} \| \phi_n \|_{X_L} \right) \| f \|_{\tilde{X}_L}
\]

which, when combined with \( (36) \), yields \( (30) \) with

\[
A_1 = \left( 1 + \sum_{n=1}^{N_0} \| \phi_n \|_{X_L} \right)^{-1} \quad \text{and} \quad A_2 = \left( 1 + \sum_{n=1}^{N_0} \| \tilde{\phi}_n \|_{X_L} \right). \quad (37)
\]

We apply a similar technique to derive \( (33) \) by considering the generic
element \( g = L^*v + \phi \in \mathcal{X}_L \) with \( v \in \mathcal{X} \) and \( \phi \in \mathcal{N}_p \). It leads to
\[
\|g\|_{\tilde{\mathcal{X}}_L} = \|L^*v + \phi\|_{\tilde{\mathcal{X}}_L} = \max\left(\|v\|_{\mathcal{X}}, \left(\sum_{n=1}^{N_0} |\langle \tilde{p}_n, \phi \rangle|^2\right)^{\frac{1}{2}}\right)
\]
\[
\leq \max\left(\|v\|_{\mathcal{X}}, \sum_{n=1}^{N_0} \|\tilde{p}_n\|_{\mathcal{X}_L'} \|\phi\|_{\mathcal{X}_L}\right)
\]
\[
\leq \left(1 + \sum_{n=1}^{N_0} \|\tilde{p}_n\|_{\mathcal{X}_L'}\right) \|g\|_{\mathcal{X}_L},
\]
where we have used the property that \( \|g\|_{\mathcal{X}_L} = \max(\|v\|_{\mathcal{X}}, \|\phi\|_{\mathcal{X}_L}) \). Likewise, the complementary bounding constant \( A'_1 \) is obtained by substituting \( \phi_n \) by \( p_n \) and \( \tilde{\mathcal{X}}_L \) by \( \tilde{\mathcal{X}}_L' \) in the first part of (37).

As for the null-space component \( \tilde{\phi} \in \mathcal{N}_p \), we recall that any admissible \( \tilde{\phi} \) must be such that the cross-product matrix
\[
\mathbf{C} = [\tilde{\phi}(p_1) \cdots \tilde{\phi}(p_{N_0})] \in \mathbb{R}^{N_0 \times N_0}
\]
has full rank for any basis \( p \) of \( \mathcal{N}_p \). The biorthogonal basis \( \tilde{p} \) is then given by
\[
\tilde{p} = \mathbf{B}p,
\]
where \( \mathbf{B} = \mathbf{C}^{-1} \). The entries of these matrices are denoted by
\[
b_{m,n} = [\mathbf{B}]_{m,n} = \langle \phi_m, \tilde{p}_n \rangle \quad \text{(38)}
\]
\[
c_{m,n} = [\mathbf{C}]_{m,n} = \langle \tilde{\phi}_m, p_n \rangle, \quad \text{(39)}
\]
respectively, where the right-hand side of (38) follows from the biorthogonality of \( (\phi, p) \). Let us now consider some arbitrary \( p = \sum_{n=1}^{N_0} \langle \phi_n, p \rangle p_n \in \mathcal{N}_p \) whose initial norm is \( \|\phi(p)\|_2 \). As we change the system of coordinates, we get
\[
\|\tilde{\phi}(p)\|_2^2 = \sum_{m=1}^{N_0} \left| \sum_{n=1}^{N_0} \langle \phi_n, p \rangle \langle \tilde{\phi}_m, p_n \rangle \right|^2 \]
\[
\leq \sum_{m=1}^{N_0} \left( \sum_{n=1}^{N_0} |\langle \tilde{\phi}_m, p_n \rangle|^2 \right) \left( \sum_{n=1}^{N_0} |\langle \phi_n, p \rangle|^2 \right) \quad \text{(by Cauchy-Schwarz)}
\]
\[
= \left( \sum_{m=1}^{N_0} \sum_{n=1}^{N_0} c_{m,n}^2 \right) \|\phi(p)\|_2^2.
\]
Likewise, by interchanging the role of $\phi$ and $\tilde{\phi}$, we find that

$$
\|\phi(p)\|_2^2 \leq \left( \sum_{m=1}^{N_0} \sum_{n=1}^{N_0} b_{m,n}^2 \right) \|\tilde{\phi}(p)\|_2^2.
$$

The combination of these two inequalities yields (31) with $B_2 = \|C\|_F$ (the Frobenius norm of the matrix $C$) and $B_1 = 1/\|B\|_F$.

Similarly, we establish the norm inequality (34) by constructing the estimates

$$
\|\tilde{p}(u)\|_2^2 = \sum_{n=1}^{N_0} |\langle \tilde{p}_n, u \rangle|^2 \leq \|B\|_F^2 \|p(u)\|_2^2,
$$

and

$$
\|p(u)\|_2^2 = \sum_{n=1}^{N_0} |\langle p_n, u \rangle|^2 \leq \|C\|_F^2 \|\tilde{p}(u)\|_2^2.
$$

5 Link with classical results

5.1 Operator-based solution of differential equations

The use of the regularized inverse operator $L_{\phi}^{-1}$ has been proposed for the resolution of stochastic partial differential equations of the form (see [35, 13])

$$
Ls = w \quad \text{s.t.} \quad \phi(s) = b,
$$

where $w \in S'(\mathbb{R}^d)$ is a realization of a $p$-admissible white-noise innovation process and $b \in \mathbb{R}^{N_0}$ is a boundary-condition vector that may be deterministic (the typical choice being $b = 0$) or not. Under the assumption that $L_{\phi}^{-1} : S(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d)$ with $p \geq 1$, the solution is then given by $s = L_{\phi}^{-1}w + \sum_{n=1}^{N_0} b_n p_n$. The connection with the present work is that there are corresponding results available on specific choices of $\phi$ that guarantee the continuity of $L_{\phi}^{-1}$ [36, Chap 5]. The scenario most studied in one dimension is $L = D$ with $(\phi_1, p_1) = (\delta, 1)$, as it enables the construction of the whole family of Lévy processes [20]. These can be described as $s = D_{\delta}^{-1}w$, where $w$ is a white Lévy noise, which is compatible with the classical boundary condition $s(0) = \langle \delta, s \rangle = 0$ [36, Section 7.4, pp. 163-166]. Instead of the standard integrator $D^{-1}$, which outputs the primitive of the function, the scheme uses the anti-derivative operator

$$
D_{\delta}^{-1}\{\varphi\}(x) = \int_0^x \varphi(y)dy,
$$

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whose adjoint $D^{-1}_\delta$ is continuous $\mathcal{S}(\mathbb{R}) \xrightarrow{\epsilon} \mathcal{R}(\mathbb{R}) = \cap_{\alpha \in \mathbb{Z}} L_{\infty,\alpha}(\mathbb{R})$ (the Fréchet space of rapidly decreasing functions) [36], Theorem 5.3, p. 100. Since $L_q(\mathbb{R}^d) \hookrightarrow \mathcal{R}(\mathbb{R})$ for all $q \leq 1$ and $\|D^{-1}_\delta \{\varphi\}\|_{L_q} \leq \|\varphi\|_{L_1}$, we can readily extend the domain of continuity of the adjoint pseudo-inverse to $D^{-1}_\delta : L_1(\mathbb{R}) \xrightarrow{\epsilon} L_q(\mathbb{R}^d)$. By taking $\mathcal{X}' = L_p(\mathbb{R}) = (L_q(\mathbb{R}))'$ with $p \geq 1$, this then leads to the native Banach spaces

$$L_{p,D}(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{p,D} \triangleq \|Df\|_p + |f(0)| < \infty\},$$

which are Sobolev spaces of degree 1. Theorem 4 ensures that $\mathcal{S}(\mathbb{R}) \hookrightarrow L_{p,D}(\mathbb{R}) \hookrightarrow L_{\infty}(\mathbb{R})$, which is consistent with the classical embedding properties of Sobolev spaces. In fact, the statement can be refined to $L_{p,D}(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$ for any $p \geq 1$ (see [33]).

### 5.2 Total variation and BV

While the connection in Section 5.1 is enlightening, it does not cover the case $(L, \mathcal{X}') = (D, \mathcal{M}(\mathbb{R}))$ (total variation) with $\mathcal{X}' = C_0(\mathbb{R})$ because $D^{-1}_\delta \{\varphi\}$ does systematically present a discontinuity at the origin when $\langle \varphi, 1 \rangle \neq 0$, even though it is smooth everywhere else (see [36], Figure 5.1, p. 91). The problem is that $\delta \notin C_{0,D}(\mathbb{R})$. This can be fixed by selecting a more regular boundary functional (i.e., any $\phi_1 \in L_1(\mathbb{R})$ with $\langle \phi_1, 1 \rangle = 1$) which then yields a corrected operator that is universal in the sense that $D^{-1}_\delta : \mathcal{S}(\mathbb{R}) \xrightarrow{\epsilon} \mathcal{S}(\mathbb{R})$. It allows us to specify the proper native space

$$\mathcal{M}_D(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{\mathcal{M},D} \triangleq \|Df\|_{\mathcal{M}} + |\langle \phi_1, f \rangle| < \infty\},$$

which extends $BV(\mathbb{R})$ (functions of bounded variations) slightly. In the classical definition of $BV(\mathbb{R})$, the second term in the norm is replaced by $\|f\|_1$. This is more constraining as it makes the null space trivial by excluding constant signals. In contrast with $L_{p,D}(\mathbb{R})$ including the limit scenario $p = 1$, the continuity of the members of $\mathcal{M}_D(\mathbb{R})$ or $BV(\mathbb{R})$ is guaranteed only almost everywhere: in other words, it can happen that the term $|f(0)|$ is not well-defined, which is the fundamental reason why it needs to be replaced by $|\langle \phi_1, f \rangle|$.

### 5.3 Sobolev/Beppo-Levi spaces with $d = 1$

We have seen that the $N_0$th derivative operator $L = D^{N_0}$ is spline-admissible with $\alpha = (N_0 - 1)$ and $\mathcal{N}_{D^{N_0}} = \{x^n\}_{n=0}^{N_0-1}$ (polynomials of degree $N_0 - 1$). Since we already known that $D^{-1}_\delta : L_1(\mathbb{R}) \xrightarrow{\epsilon} L_p(\mathbb{R})$, $p = 1$ included,
we can iterate the operator to construct an admissible pseudo-inverse of \( D^{m*} = (-1)^m D^m \) as

\[
D^{m*}_\phi = (D_\delta^{-1})^m : L_1(\mathbb{R}) \xrightarrow{c} L_p(\mathbb{R}).
\]

with \( N_0 = m \) and \( \phi_n = (-1)^{(n-1)}\delta^{(n-1)} \). Indeed, since \( f = D_\delta^{-1}\{w\} \) imposes the boundary condition \( f(0) = 0 \) and is left-invertible with \( w = Df, g = (D_\delta^{-1})^m\{w\} \) is invertible as well and such that \( g^{(n)}(0) = ((-1)^{(n-1)}\delta^{(n-1)}, g) = 0 = \langle \phi_{n+1}, g \rangle \) for \( n = 0, \ldots, (m - 1) \). By observing that the underlying \( \phi_n \) are biorthogonal to \( \tilde{p}_n(x) = \frac{x^n}{n!} \), we can then safely define the corresponding native spaces as

\[
L_{p,D^m}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{p,D^m} \overset{\triangle}{=} \|D^m f\|_p + \left( \sum_{n=0}^{m-1} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \right)^{1/2} < \infty \right\},
\]

which, as expected, are Sobolev spaces of order \( m \), albeit homogeneous extensions of the classical ones for they also includes the polynomials of degree less than \( m \). For \( p = 2 \), we recover the typical kind of Beppo-Levi space \([2]\) used to specify smoothing splines; i.e., the classical form of variational polynomial splines, which goes back to the pioneering works of Schoenberg and de Boor \([30, 10]\).

### 5.4 Reproducing-kernel Hilbert spaces

The best known examples of native spaces on \( \mathbb{R}^d \) are RKHS \([1, 4, 38, 29]\). They are included in the framework by taking \( X = L_2(\mathbb{R}^d) \). In that case, the stability condition \( L_{-1}^{-1} : L_{1-\alpha}(\mathbb{R}^d) \xrightarrow{c} L_2(\mathbb{R}^d) \) implies that \( L_{-1}^{-1} : L_2(\mathbb{R}^d) \xrightarrow{c} L_{\infty, \alpha}(\mathbb{R}^d) \). This means that the canonical inverse operator \( L_{-1}^{-1} \) has the unique extension \( L_{-1}^{-1} = L_{\phi}^{-1} : U = L^*(L_2(\mathbb{R}^d)) \xrightarrow{c} L_2(\mathbb{R}^d) \) with \( \|u\|_{2,L} = \|L_{-1}^{-1} u\|_{2} = \|L_{\phi}^{-1} u\|_{L_2} \) for any \( u \in U \). This allows us to define the pair of self-adjoint operators \( A = (L_{-1}^{-1}L_{-1}^{-1}) : U \xrightarrow{c} U \) and \( A_\phi = (L_{\phi}^{-1}L_{\phi}^{-1}) : L_{2,L}(\mathbb{R}^d) \xrightarrow{c} L_{2,L}(\mathbb{R}^d) \). Since \( S_{p^\perp}(\mathbb{R}^d) \xrightarrow{d} U \hookrightarrow X_\perp \) (by Corollary \([1]\), we have that

\[
\forall \varphi \in S_{p}(\mathbb{R}^d) \setminus \{0\} : \|\varphi\|_{2,L} = \langle L_{-1}^{-1} \varphi, L_{-1}^{-1} \varphi \rangle = \langle A_\varphi, \varphi \rangle = \langle A_\phi \varphi, \varphi \rangle > 0. \tag{40}
\]

This expresses the (strict) \( p \)-conditional positive definiteness of \( A \) (resp. \( A_\phi \)) which can also be identified as the inverse (resp. the pseudo-inverse) of \( (L^*L) \).

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In the particular case where \( p \) is a basis of the polynomials of degree \( n_0 \), Condition (40) is equivalent to the notion of \( n_0 \)th-order conditional positive definiteness used in approximation theory \([22, 38]\). It is a classical hypothesis in the theory of (semi-)RKHS and is also necessary for our construction. Classically, the strict (\( n_0 \)th-order conditional) positive definiteness of \( A \) (or of its underlying kernel) is known to be sufficient to yield a (semi-)RKHS. This is not quite the case here because we also want the embedding \( \mathcal{S}(\mathbb{R}^d) \hookrightarrow L_{2,1}^*(\mathbb{R}^d) \hookrightarrow L_{\infty,\alpha}(\mathbb{R}^d) \) that controls the growth of the members of the native space. The latter calls for the continuity of \( L_{1,-\alpha}^{-1*} : L_{1,-\alpha}(\mathbb{R}^d) \hookrightarrow L_2(\mathbb{R}^d) \) (\( L_2 \)-stability) or, equivalently, of \( A_{\phi} : L_{1,-\alpha}(\mathbb{R}^d) \hookrightarrow L_{\infty,\alpha}(\mathbb{R}^d) \).

### 5.5 Connections with kernel methods and splines

In \([33]\), we shall identify a simple condition on the kernel of \( A = (LL^*)^{-1} \) that ensures that the stability requirement in Definition 4 for \( \mathcal{X} = L_2(\mathbb{R}^d) \) is met. This will enable us to prove that the combination of spline admissibility in Definition 2 and the classical (conditional-)positivity requirement \([40]\) are necessary and sufficient for the native space of \((\mathcal{X}', L) \) with \( \mathcal{X} = \mathcal{X}' = L_2(\mathbb{R}^d) \) to be a RKHS, with the property that \( \mathcal{S}(\mathbb{R}^d) \hookrightarrow L_{2,1}^*(\mathbb{R}^d) \hookrightarrow C_{b,\alpha}(\mathbb{R}^d) \).

We shall further the argument by reformulating the primary results of the present paper in terms of kernels, rather than operators. This will provide us with explicit criteria for checking that the compatibility conditions in Definition 3 are met for a broad variety of primary spaces \( \mathcal{X}' \). We shall also devote a particular attention to the case \( \mathcal{X}' = \mathcal{M}(\mathbb{R}^d) \), which is central to the theory of \( L \)-splines \([34]\). Examples of applications of our native Banach-space formalism, including the resolution of variational inverse problems and the derivation of representer theorems, will be presented in \([14]\).
Appendix A: Topological embeddings

The notion of embedding for topological vector spaces comes in four gradation: inclusion as a set (symbolized by \( \mathcal{X} \subseteq \mathcal{Y} \)), continuous embedding (\( \mathcal{X} \hookrightarrow \mathcal{Y} \)), isometric embedding (\( \mathcal{X} \overset{\text{iso.}}{\hookrightarrow} \mathcal{Y} \)), and, finally, continuous and dense embedding (\( \mathcal{X} \overset{\text{d.}}{\hookrightarrow} \mathcal{Y} \)).

**Definition 7 (Continuous embedding).** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two locally convex topological vector spaces where \( \mathcal{X} \subseteq \mathcal{Y} \) (as a set). \( \mathcal{X} \) is said to be continuously embedded in \( \mathcal{Y} \), which is denoted by \( \mathcal{X} \hookrightarrow \mathcal{Y} \), if the inclusion/identity map \( I : \mathcal{X} \rightarrow \mathcal{Y} : x \mapsto x \) is continuous.

In particular, if \( \mathcal{X} \) and \( \mathcal{Y} \) are two Banach spaces, then the definition can be restated as: for all \( x \in \mathcal{X} \), \( I\{x\} = x \in \mathcal{Y} \) with \( \|x\|_\mathcal{Y} \leq C_0\|x\|_\mathcal{X} \) for some constant \( C_0 > 0 \). If, in addition, \( \|x\|_\mathcal{X} = \|x\|_\mathcal{Y} \) for all \( x \in \mathcal{X} \subseteq \mathcal{Y} \), then the embedding is isometric, a property that is denoted by \( \mathcal{X} \overset{\text{iso.}}{\hookrightarrow} \mathcal{Y} \). For instance, a classical result is that any Banach space \( \mathcal{X} \) is isometrically embedded in its bidual; i.e., \( \mathcal{X} \overset{\text{iso.}}{\hookrightarrow} \mathcal{X}'' \). In fact, we have that \( \mathcal{X} = \mathcal{X}'' \) (meaning that the two spaces are isometrically isomorphic) if and only if \( \mathcal{X} \) is reflexive.

An example of such embeddings that is relevant to this paper is

\[
S(\mathbb{R}^d) \overset{\text{d.}}{\hookrightarrow} L_{1,-\alpha}(\mathbb{R}^d) \overset{\text{iso.}}{\hookrightarrow} (L_{1,-\alpha}(\mathbb{R}^d))'' = (L_{\infty,\alpha}(\mathbb{R}^d))'.
\]

So far we have emphasized the property of continuity, but there are instances such as \( S(\mathbb{R}^d) \overset{\text{d.}}{\hookrightarrow} L_{1,-\alpha}(\mathbb{R}^d) \) that are more powerful because the embedding also happens to be dense; i.e., \( \mathcal{X} \overset{\text{d.}}{\hookrightarrow} \mathcal{Y} \), where \( \mathcal{Y} \) can be specified as the completion of \( \mathcal{X} \) for \( \| \cdot \|_\mathcal{Y} \).

**Definition 8 (Dense embedding).** Let \( \mathcal{X} \) be a linear subspace of a locally convex topological vector space \( \mathcal{Y} \). Then, \( \mathcal{X} \) is said to be dense in \( \mathcal{Y} \) if it has the ability to separate distinct elements of the dual space \( \mathcal{Y}' \); that is, if, for any \( y' \in \mathcal{Y}' \),

\[
\langle y', x \rangle_{\mathcal{Y}' \times \mathcal{Y}} = 0 \text{ for all } x \in \mathcal{X} \subseteq \mathcal{Y} \iff y' = 0.
\]

In the case where \( \mathcal{Y} \) is a Banach space, the denseness of \( \mathcal{X} \) has another equivalent formulation: for any \( y \in \mathcal{Y} \) and \( \epsilon > 0 \), there exists some \( x_\epsilon \in \mathcal{X} \) such that \( \|y - x_\epsilon\|_\mathcal{Y} < \epsilon \), which means that \( \mathcal{X} \) is rich enough to represent any element of \( \mathcal{Y} \) with an arbitrary degree of precision.
**Theorem 6** (Dual embedding). Let $X$ and $Y$ be two locally convex topological vector spaces such that $X \overset{d}{\hookrightarrow} Y$, where the embedding is continuous and dense. Then, $Y' \hookrightarrow X'$. Moreover, if $X$ is reflexive, then $Y' \overset{d}{\hookrightarrow} X'$.

Likewise, in the case of Banach spaces, $X \overset{iso}{\hookrightarrow} Y$ implies that $Y' \hookrightarrow X'$ with bounding constant one (preservation of norm). Moreover, if there exists a topological vector space $S$ such that $S \overset{d}{\hookrightarrow} X$ and $S \overset{d}{\hookrightarrow} Y$, then $Y' \overset{iso}{\hookrightarrow} X'$.

**Proof.** Since $X \overset{d}{\hookrightarrow} Y$, the linear functionals that are continuous on $Y$ are also continuous on $X$ so that $X' \subseteq Y'$ (as a set). Moreover, the continuity of the identity/inclusion map $I : X \rightarrow Y$ implies the continuity of its adjoint $I^* : Y' \rightarrow X'$, defined as

$$\langle I^*\{y'\}, x \rangle_{X' \times X} = \langle y', I\{x\} \rangle_{Y' \times Y} = \langle y', x \rangle_{Y' \times Y}$$  \hspace{1cm} (41)

for all $y' \in Y'$, $x \in X$. Finally, the denseness of $X$ in $Y$ ensures that $I^*\{y'\} = y'$, which proves that $Y' \hookrightarrow X'$.

**Second part by contradiction:** Suppose that $Y'$ is not dense in $X'$. Then, there is an $x''_0 \in X''$ that is not identically zero such that $\langle x''_0, y' \rangle_{X'' \times X'} = 0$ for all $y' \in Y' \subseteq X'$ (contrapositive of the statement in Definition 8). Moreover, due to the reflexivity of $X$, there is a corresponding $x_0 \in X \hookrightarrow Y$ such that $B\{x_0\} = x''_0$, where $B : X \rightarrow X''$ is the canonical bijective mapping for a reflexive space to its bidual. Therefore,

$$0 = \langle x''_0, y' \rangle_{X'' \times X'} = \langle B\{x_0\}, y' \rangle_{X'' \times X'} = \langle y', x_0 \rangle_{Y' \times Y},$$

for all $y' \in Y'$. Since the topological spaces $Y'$ and $Y$ form a dual pair, the identity $\langle y', x_0 \rangle_{Y' \times Y} = 0$ implies that $x_0 = 0$, which is a contradiction.

**Banach Isometries:** From the definition of the dual norm and the property that $\|\varphi\|_X = \|\varphi\|_Y$ when $\varphi \in \mathcal{X}$, for any $y' \in Y' \subseteq X'$, we have that

$$\|y'\|_{Y'} = \sup_{\varphi \in Y'\setminus\{0\}} \frac{\langle y', \varphi \rangle}{\|\varphi\|_Y} \geq \sup_{\varphi \in X'\setminus\{0\}} \frac{\langle y', \varphi \rangle}{\|\varphi\|_X} = \sup_{\varphi \in X'\setminus\{0\}} \|\varphi\|_X \|y'\|_{X'}$$  \hspace{1cm} (42)

where the inequality results from the search space $X$ on the right being a subset of $Y$; hence, $X' \hookrightarrow Y'$. In the second scenario, the two search spaces in (42) can be replaced by the dense subspace $S$, which then yields an equality. However, this setting also implies that $X = Y$ as both spaces are the unique completion of $S$ in the $\|\cdot\|_X = \|\cdot\|_Y$-norm. 

\[\square\]
As demonstration of usage, we now prove the following remarkable result in Schwartz’ theory of distributions:

**Proposition 6.** $\mathcal{S}(\mathbb{R}^d)$ is continuously and densely embedded in $\mathcal{S}'(\mathbb{R}^d)$; i.e., $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

**Proof.** For any $\phi \in \mathcal{S}(\mathbb{R}^d)$, the map $\varphi \mapsto \langle \phi, \varphi \rangle = \int_{\mathbb{R}^d} \phi(x) \varphi(x) dx$ specifies a continuous linear functional over $\mathcal{S}(\mathbb{R}^d)$, which already shows that $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ (as a set). To prove that the embedding is continuous, we invoke the continuity of the identity operator $I : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, which is obvious from the underlying topology. As for the denseness property, the relevant annihilator space is

$$\mathcal{S}^\perp = \{ \varphi \in (\mathcal{S}'(\mathbb{R}^d))' = \mathcal{S}(\mathbb{R}^d) : \langle \varphi, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^d) \}.$$

In particular, if $\varphi \in \mathcal{S}^\perp \subseteq \mathcal{S}(\mathbb{R}^d)$, then $\langle \varphi, \varphi \rangle = 0 \iff \|\varphi\|_{L^2} = 0 \iff \varphi = 0$, which proves that $\mathcal{S}^\perp = \{0\}$. \[\blacksquare\]

In practice, it is often easier to prove that an embedding is continuous than establishing its denseness. Fortunately, it is possible to transfer such properties by taking advantage of functional hierarchies.

**Proposition 7 (Hierarchy of dense embeddings).** Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three locally convex topological vector spaces such that $\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ (continuous embeddings). Then, we have the following implications (dense embeddings).

1. $\mathcal{X} \xrightarrow{d} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{d} \mathcal{Z} \Rightarrow \mathcal{X} \xrightarrow{d} \mathcal{Z}$
2. $\mathcal{X} \xrightarrow{d} \mathcal{Z} \Rightarrow \mathcal{Y} \xrightarrow{d} \mathcal{Z}$.

**Proof.**

Statement 1: $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{Z}$ as sets. Since the closure of $\mathcal{Y}$ in the topology of $\mathcal{Z}$ is $\mathcal{Z}$, the closure of $\mathcal{X}$ in the topology of $\mathcal{Z}$ must also be $\mathcal{Z}$.

Statement 2: The annihilators of $\mathcal{X} \subseteq \mathcal{Z}$ and $\mathcal{Y} \subseteq \mathcal{Z}$ in $\mathcal{Z}'$ are

$$\mathcal{X}^\perp = \{ u \in \mathcal{Z}' : \langle u, x \rangle_{\mathcal{Z}' \times \mathcal{Z}} = 0 \text{ for all } x \in \mathcal{X} \}$$

$$\mathcal{Y}^\perp = \{ u \in \mathcal{Z}' : \langle u, y \rangle_{\mathcal{Z}' \times \mathcal{Z}} = 0 \text{ for all } y \in \mathcal{Y} \}$$

with the property that $\mathcal{Y}^\perp \subseteq \mathcal{X}^\perp$ (as a set) from the definition. Hence, $\mathcal{X}^\perp = \{0\} \Rightarrow \mathcal{Y}^\perp = \{0\}$. \[\blacksquare\]
Appendix B: Direct-sum topology

**Definition 9.** Let $\mathcal{U}$ and $\mathcal{V}$ be two subspaces of a linear space $\mathcal{W}$. Then, the sum space is $\mathcal{U} + \mathcal{V} = \{ f = u + v : (u, v) \in \mathcal{U} \times \mathcal{V} \} \subset \mathcal{W}$. The sum is called direct and is notated $\mathcal{U} \oplus \mathcal{V}$ if $\mathcal{U} \cap \mathcal{V} = \{0\}$.

If $\mathcal{U}$ and $\mathcal{V}$ in the above definition are normed with respective norms $\| \cdot \|_\mathcal{U}$ and $\| \cdot \|_\mathcal{V}$, then $\| (\|u\|_\mathcal{U}, \|v\|_\mathcal{V}) \|_p$ is a norm for both $\mathcal{U} \times \mathcal{V}$ and $\mathcal{U} \oplus \mathcal{V}$. The choice of the exponent $p \geq 1$ in the composite norm is flexible since all finite-dimensional norms are equivalent. One also defines the corresponding linear projection operators $\text{Proj}_U : (\mathcal{U} \oplus \mathcal{V}) \to \mathcal{U}$ and $\text{Proj}_V : (\mathcal{U} \oplus \mathcal{V}) \to \mathcal{V}$. These are such that, for any $(u, v) \in \mathcal{U} \times \mathcal{V}$,

$$\text{Proj}_U \{ u + v \} = u,$$

$$\text{Proj}_V \{ u + v \} = v.$$  

In summary, any $f \in \mathcal{U} \oplus \mathcal{V}$ has a unique decomposition as $f = u + v$ with $\text{Proj}_U \{ f \} = u \in \mathcal{U}$ and $\text{Proj}_V \{ f \} = v \in \mathcal{V}$, while $\|f\|_{\mathcal{U} \oplus \mathcal{V}} = \| (\|\text{Proj}_U \{ f \}\|_\mathcal{U}, \|\text{Proj}_V \{ f \}\|_\mathcal{V}) \|_p$.

The concept is also applicable to Banach spaces, which are often specified explicitly as the completion of some normed space. This normed space is called a pre-Banach space when it is not yet completed.

**Theorem 7** (Completion of a direct sum space). Let $\mathcal{U}_{\text{pre}} \oplus \mathcal{V}_{\text{pre}}$ be the direct sum of two normed spaces $(\mathcal{U}_{\text{pre}}, \| \cdot \|_\mathcal{U})$ and $(\mathcal{V}_{\text{pre}}, \| \cdot \|_\mathcal{V})$. Then, $\mathcal{U}_{\text{pre}} \oplus \mathcal{V}_{\text{pre}} = \mathcal{U} \oplus \mathcal{V}$, where $\mathcal{U} = (\mathcal{U}_{\text{pre}}, \| \cdot \|_\mathcal{U})$ and $\mathcal{V} = (\mathcal{V}_{\text{pre}}, \| \cdot \|_\mathcal{V})$ are the Banach spaces associated with the $\| \cdot \|_\mathcal{U}$-norm and $\| \cdot \|_\mathcal{V}$-norm, respectively. Moreover, the resulting projection operators $\text{Proj}_U : \mathcal{U} \oplus \mathcal{V} \xrightarrow{\sim} \mathcal{U}$ with $\|\text{Proj}_U\| = 1$ and $\text{Proj}_V : \mathcal{U} \oplus \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ with $\|\text{Proj}_V\| = 1$ are the unique continuous extensions of $\text{Proj}_{\mathcal{U}_{\text{pre}}} : \mathcal{U}_{\text{pre}} \oplus \mathcal{V}_{\text{pre}} \to \mathcal{U}$ and $\text{Proj}_{\mathcal{V}_{\text{pre}}} : \mathcal{U}_{\text{pre}} \oplus \mathcal{V}_{\text{pre}} \to \mathcal{V}$, respectively.

**Proof.** Let $\mathcal{G}_{\text{pre}} = \mathcal{U}_{\text{pre}} \oplus \mathcal{V}_{\text{pre}}$, whose completion with respect to the direct-sum norm is the Banach space $\mathcal{G}$. We then pick some Cauchy sequence $(g_i)$ in $\mathcal{G}_{\text{pre}}$ that converges to $g \in \mathcal{G}$. Because the projectors $\text{Proj}_{\mathcal{U}_{\text{pre}}} : \mathcal{P}_{\text{pre}} \to \mathcal{U}_{\text{pre}}$ and $\text{Proj}_{\mathcal{V}_{\text{pre}}} : \mathcal{P}_{\text{pre}} \to \mathcal{V}_{\text{pre}}$ are contractive maps, the transformed sequences $(u_i) = (\text{Proj}_{\mathcal{U}_{\text{pre}}} \{ g_i \})$ and $(v_i) = (\text{Proj}_{\mathcal{V}_{\text{pre}}} \{ g_i \})$ are Cauchy in $\mathcal{U}_{\text{pre}}$ and $\mathcal{V}_{\text{pre}}$, respectively, and converge to some limits $u = \lim_{i \to \infty} u_i \in \mathcal{U} \upharpoonright \mathcal{U}_{\text{pre}}$ and $v = \lim_{i \to \infty} v_i \in \mathcal{V} \upharpoonright \mathcal{V}_{\text{pre}}$. Hence, $u + v \in \mathcal{U} \oplus \mathcal{V}$. Conversely, because of the direct sum property, the sum of any two Cauchy sequences in $\mathcal{U}_{\text{pre}}$ and $\mathcal{V}_{\text{pre}}$ is Cauchy in $\mathcal{G}_{\text{pre}}$ converges to $\lim_{i \to \infty} (u_i + v_i) = \lim_{i \to \infty} u_i + \lim_{i \to \infty} v_i = \ldots$
u + v ∈ \mathcal{G}_{\text{pre}} = \mathcal{G}, which shows that \mathcal{U} + \mathcal{V} ⊆ \mathcal{G}. Hence, we conclude that \mathcal{G} = \mathcal{U} + \mathcal{V}.

Since \mathcal{U}_{\text{pre}} \overset{\text{iso.}}{\longrightarrow} \mathcal{U} and \text{Proj}_{\mathcal{U}_{\text{pre}}} is a projector, it is bounded \mathcal{G}_{\text{pre}} \rightarrow \mathcal{U} with \|\text{Proj}_{\mathcal{U}_{\text{pre}}}\| = 1. By the B.L.T. theorem, it therefore admits a unique continuous extension \text{Proj}_{\mathcal{U}} : \mathcal{G}_{\text{pre}} \overset{\epsilon}{\longrightarrow} \mathcal{U} with \|\text{Proj}_{\mathcal{U}}\| = 1, whose explicit definition is

\text{Proj}_{\mathcal{U}}\{g\} \triangleq \lim_{i \to \infty} \text{Proj}_{\mathcal{U}_{\text{pre}}}\{g_i\}

where \(g_i\) is any Cauchy sequence in \mathcal{G}_{\text{pre}} that converges to \(g \in \mathcal{G}\). The same holds true for \text{Proj}_{\mathcal{V}} : \mathcal{G}_{\text{pre}} \overset{\epsilon}{\longrightarrow} \mathcal{V}, which is such that

\text{Proj}_{\mathcal{V}}\{g\} \triangleq \lim_{i \to \infty} \text{Proj}_{\mathcal{V}_{\text{pre}}}\{g_i\} = \lim_{i \to \infty} v_i = v.

We then use these extended operators to show that the sum \mathcal{U} + \mathcal{V} is direct. Specifically, by invoking the basic properties of \text{Proj}_{\mathcal{U}_{\text{pre}}}, we get

\text{Proj}_{\mathcal{U}}\{g\} = \lim_{i \to \infty} \text{Proj}_{\mathcal{U}_{\text{pre}}}\{\text{Proj}_{\mathcal{U}_{\text{pre}}}\{g_i\} + \text{Proj}_{\mathcal{V}_{\text{pre}}}\{g_i\}\}

= \lim_{i \to \infty} \left( \text{Proj}_{\mathcal{U}_{\text{pre}}}\{u_i\} + \text{Proj}_{\mathcal{V}_{\text{pre}}}\{v_i\} \right) = \lim_{i \to \infty} (u_i + 0) = u \in \mathcal{U},

which is equivalent to \text{Proj}_{\mathcal{U}}\{u + v\} = u for any \((u, v) \in \mathcal{U} \times \mathcal{V}\). Correspondingly, we also obtain that \text{Proj}_{\mathcal{V}}\{g\} = \lim_{i \to \infty} (0 + v_i) = v = (g - u) \in \mathcal{V} with \text{Proj}_{\mathcal{V}}\{u\} = 0 and \text{Proj}_{\mathcal{V}}\{v\} = v, which proves that \mathcal{U} \cap \mathcal{V} = \{0\}.

\[ \square \]

The final element is the identification of the dual space which, as expected, also has a direct-sum structure (see [21, Theorem 1.10.13]), albeit with a suitable adaptation of the composite norm.

**Proposition 8.** Let \((\mathcal{U}, \|\cdot\|_\mathcal{U}) and (\mathcal{V}, \|\cdot\|_\mathcal{V}) be two complementary Banach subspaces of \mathcal{W} and \mathcal{U} \oplus \mathcal{V} the corresponding direct-sum space equipped with the composite norm \(\|(\|u\|_\mathcal{U}, \|v\|_\mathcal{V})\|_p\). Then, the continuous dual of \mathcal{U} \oplus \mathcal{V} is the to direct-sum Banach space \mathcal{U}' \oplus \mathcal{V}' equipped with the dual composite norm \(\|u' + v'\|_{\mathcal{U}' \oplus \mathcal{V}'} = \|(\|u'\|_{\mathcal{U}'}, \|v'\|_{\mathcal{V}'})\|_q\) where \(q = \frac{p}{p-1}\) is the conjugate exponent of \(p \geq 1\).

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References

[1] N. Aronszajn. Theory of reproducing kernels. Transactions of the American Mathematical Society, 68(3):337–404, 1950.

[2] M. Atteia. Hilbertian Kernels and Spline Functions. Elsevier, 1992.

[3] R. K. Beatson, H.-Q. Bui, and J. Levesley. Embeddings of Beppo–Levi spaces in Hölder–Zygmund spaces, and a new method for radial basis function interpolation error estimates. Journal of Approximation Theory, 137(2):166–178, 2005.

[4] A. Berlinet and C. Thomas-Agnan. Reproducing Kernel Hilbert Spaces in Probability and Statistics, volume 3. Kluwer Academic Boston, 2004.

[5] A. Y. Bezhaev and V. A. Vasilenko. Variational Theory of Splines. Kluwer Academic/Plenum Publishers, New York, 2001.

[6] C. Boyer, A. Chambolle, Y. De Castro, V. Duval, F. De Gournay, and P. Weiss. On representer theorems and convex regularization. arXiv preprint arXiv:1806.09810, 2018.

[7] K. Bredies and M. Carioni. Sparsity of solutions for variational inverse problems with finite-dimensional data. arXiv preprint arXiv:1809.05045, 2018.

[8] A. Chambolle. An algorithm for total variation minimization and applications. Journal of Mathematical Imaging and Vision, 20:89–97, 2004.

[9] C. de Boor. On “best” interpolation. Journal of Approximation Theory, 16(1):28–42, 1976.

[10] C. de Boor and R. Lynch. On splines and their minimum properties. Journal of Mathematics and Mechanics, 15(6):953–969, 1966.

[11] J. Deny and J.-L. Lions. Les espaces du type de Beppo Levi. Annales de l’Institut Fourier, 5:305–370, 1954.

[12] J. Duchon. Splines minimizing rotation-invariant semi-norms in Sobolev spaces. In W. Schempp and K. Zeller, editors, Constructive Theory of Functions of Several Variables, pages 85–100. Springer-Verlag, Berlin, 1977.
[13] J. Fageot, A. Amini, and M. Unser. On the continuity of characteristic functionals and sparse stochastic modeling. *The Journal of Fourier Analysis and Applications*, 20(6):1179–1211, December 2014.

[14] J. Fageot, M. Unser, and J. Ward. Native Banach spaces: Concrete criteria for existence. *Under development*.

[15] J. Ferreira and V. Menegatto. Positive definiteness, reproducing kernel Hilbert spaces and beyond. *Annals of Functional Analysis*, 4(1):64–88, 2013.

[16] A. Flinth and P. Weiss. Exact solutions of infinite dimensional total-variation regularized problems. *arXiv preprint arXiv:1708.02157*, 2017.

[17] I. Gelfand and N. Y. Vilenkin. *Generalized Functions. Vol. 4. Applications of Harmonic Analysis*. Academic press, New York, USA, 1964.

[18] H. Gupta, J. Fageot, and M. Unser. Continuous-domain solutions of linear inverse problems with Tikhonov versus generalized TV regularization. *IEEE Transactions on Signal Processing*, 66(17):4670–4684, September 1, 2018.

[19] L. Hörmander. *The analysis of linear partial differential operators. I. Distribution Theory and Fourier Analysis*, volume 256. Springer-Verlag, Berlin, second edition, 1990.

[20] P. Lévy. *Théorie de l’Addition des Variables Aléatoires*. Gauthier-Villars, Paris, 2nd edition, 1954.

[21] R. E. Megginson. *An Introduction to Banach Space Theory*. Springer, 1998.

[22] C. A. Micchelli. Interpolation of scattered data: Distance matrices and conditionally positive definite functions. *Constructive Approximation*, 2(1):11–22, 1986.

[23] A. Mosamam and J. Kent. Semi-reproducing kernel Hilbert spaces, splines and increment kriging. *Journal of Nonparametric Statistics*, 22(6):711–722, 2010.

[24] P. Prenter. *Splines and Variational Methods*. Wiley, New York, 1975.

[25] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. Vol. 1: Functional Analysis*. Academic Press, 1980.
[26] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60(1-4):259–268, 1992.

[27] W. Rudin. *Functional Analysis*. McGraw-Hill, New York, 2nd edition, 1991. McGraw-Hill Series in Higher Mathematics.

[28] R. Schaback. Native Hilbert spaces for radial basis functions I. In *New Developments in Approximation Theory*, pages 255–282. Springer, 1999.

[29] R. Schaback. A unified theory of radial basis functions: Native Hilbert spaces for radial basis functions II. *Journal of Computational and Applied Mathematics*, 121(1-2):165–177, 2000.

[30] I. Schoenberg. Spline functions and the problem of graduation. *Proceedings of the National Academy of Sciences*, 52(4):947–950, October 1964.

[31] M. Unser. A representer theorem for deep neural networks. *arXiv preprint arXiv:1802.09210*, 2018.

[32] M. Unser and T. Blu. Self-similarity: Part I—Splines and operators. *IEEE Transactions on Signal Processing*, 55(4):1352–1363, April 2007.

[33] M. Unser and J. Fageot. Native Banach spaces: From operators to kernels. *Under development*.

[34] M. Unser, J. Fageot, and J. P. Ward. Splines are universal solutions of linear inverse problems with generalized-TV regularization. *SIAM Review*, 59(4):769–793, December 2017.

[35] M. Unser, P. Tafti, and Q. Sun. A unified formulation of Gaussian versus sparse stochastic processes—Part I: Continuous-domain theory. *IEEE Transactions on Information Theory*, 60(3):1945–1962, March 2014.

[36] M. Unser and P. D. Tafti. *An Introduction to Sparse Stochastic Processes*. Cambridge University Press, 2014.

[37] G. Wahba. *Spline Models for Observational Data*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990.

[38] H. Wendland. *Scattered Data Approximations*. Cambridge University Press, 2005.