Statistical theory of self-similarly distributed fields

Alexander Olemskoi\textsuperscript{a,\,*}, Irina Shuda\textsuperscript{b}

\textsuperscript{a} Institute of Applied Physics, National Academy of Sciences of Ukraine, 58, Petropavlovskaya St., 40030 Sumy, Ukraine
\textsuperscript{b} Sumy State University, 2, Rimsky-Korsakov St., 40007 Sumy, Ukraine

\begin{abstract}
A field theory is built for self-similar statistical systems with both generating functional being the Mellin transform of the Tsallis exponential and generator of the scale transformation that is reduced to the Jackson derivative. With such a choice, the role of a fluctuating order parameter is shown to play deformed logarithm of the amplitude of a hydrodynamic mode. Within the harmonic approach, deformed partition function and moments of the order parameter of lower powers are found. A set of equations for the generating functional is obtained to take into account constraints and symmetry of the statistical system.
\end{abstract}

\section{Introduction}

A formal basis of the statistical theory, using quantum field methods, is known to be a generating functional which presents the Fourier–Laplace transform of the partition function from the dependence on the fluctuating distribution of an order parameter to an auxiliary field \cite{1}. Due to the exponential character of this transform, determination of correlators of the order parameter is provided by ordinary differentiation of the generating functional over auxiliary field.

Above scheme becomes inconsistent with passage from simple systems to complex ones because the phase space gets forbidden regions and the phase flow does not ensure statistical mixing \cite{2}. As is known from the theory of critical phenomena, analytical description of complex systems is achieved in the presence of the scaling invariance only \cite{3}. Because in this case the role of a basic function plays the power-law function instead of the exponential one, we need in use of the Mellin transform at constructing of the generating functional. Moreover, one should introduce the Jackson derivative as a generator of the scaling transformation instead of the ordinary derivation operator.

This Letter is devoted to building a field-theoretical scheme based on the use of both Mellin transform and Jackson derivative. The work is organized as follows. In Section 2 we adduce necessary information from the theory of quantum calculus. Section 3 is devoted to construction of the generating functional and finding its connection with related correlators. As the simplest example, the harmonic approach is studied in Section 4 to obtain the partition function and the order parameter moments of the first and second powers in dependence of the deformation parameter. In Section 5 we introduce pair of additive functional whose expansion into deformed series yields both Green functions and proper vertices. Moreover, we find here formal equations governing by the generating functional of systems possessing a symmetry with respect to a field variation and being subjected to an arbitrary constrain. Section 6 concludes our consideration.

\section{Preliminaries}

We begin by citing an information from the quantum calculus \cite{4,5} that will be needed below. A basis of this calculus is the dilatation operator $D^{\lambda}_x := \lambda^\lambda x^\lambda$ being determined by the deformation parameter $\lambda$ and the differentiation operator $\partial_x \equiv \partial/\partial x$. Expanding formally the operator $D^{\lambda}_x$ into the Taylor series, it is easily to define its action onto the power-law function: $D^{\lambda}_x x^\lambda = (\lambda x)^\lambda$. Similarly, the expansion of an analytical function $f(x)$ shows that, in correspon-
dence with the denomination, the operator $D^\lambda \chi$ arrives at the dilata-
lation $\lambda$ of the argument of this function: $D^\lambda \chi(x) = f(\lambda x)$. A set
of eigen-functions of the dilatation operator is reduced to a homo-
geneous functions $h(x)$ defined by the equality $D^\lambda h(x) = \chi h(x)$ with
the self-similarity degree $q$. In general case, this function takes the form $h(x) = A_i(x)x^q$ where a factor $A_i(x)$ is obeyed the invariance condition $A_i(\lambda x) = \lambda A_i(x)$. It is convenient to pass from the dilatation operator to the Jackson derivative
\begin{equation}
D^\lambda \chi := \frac{D^\chi - 1}{(\lambda - 1)x}
\end{equation}
in accordance with the commutation rule $[D^\lambda \chi, x] = D^\lambda \chi$. The action
the Jackson derivative onto the homogeneous function is given by the
relations:
\begin{equation}
(x D^\lambda \chi)(x) = [q]_\lambda h(x), \quad [q]_\lambda \equiv \frac{\lambda^q - 1}{\lambda - 1},
\end{equation}
Basic deformed number $[q]_\lambda$ represents a generalization of the exp-
ponent of the homogeneous function.

Principle peculiarity of self-similar statistical systems is that their cons-
ideration is based on the use of the deformed logarithm and exponen-
tial [2]
\begin{equation}
\ln_q x := \frac{x^{1-q} - 1}{1-q}, \quad \exp_q x := \left[1 + (1-q)x\right]^{\frac{1}{1-q}},
\end{equation}
where $[y]_+ \equiv \max(0, y)$. Moreover, one needs to deform the sum
and product as follows:
\begin{equation}
x \oplus_q y = x + y + (1-q)xy,
\end{equation}
\begin{equation}
x \otimes_q y = \left[x^{1-q} + y^{1-q} - 1\right]^\frac{1}{1-q};
\end{equation}
here, in the last equality $x, y > 0$. Respectively, the deformed pro-
duct of $n > 1$ identical multipliers gives the expression of the de-
formed power-law function:
\begin{equation}
x \otimes_q x \otimes_q \cdots \otimes_q x = \left[nx^{1-q} - (n-1)\right]\frac{1}{n};
\end{equation}
Making use of the rules (4) shows that functions (3) are obeyed the
conditions
\begin{equation}
\ln_q (x \otimes_q y) = \ln_q x + \ln_q y, \quad \ln_q (xy) = \ln_q x \otimes_q \ln_q y;
\exp_q (x + y) = \exp_q (x) \otimes_q \exp_q (y),
\exp_q (x \otimes_q y) = \exp_q (x) \exp_q (y).
\end{equation}

3. Generating functional

As mentioned above, the generating functional of self-similar
systems is defined by the Mellin transform
\begin{equation}
Z_q[J] := \int Z_q[\phi] \phi^{f-1}(d\phi),
\end{equation}
\begin{equation}
\phi^{f-1}(d\phi) \equiv \prod_{i=1}^N \phi_i^{f-1} d\phi_i,
\end{equation}
where index $i$ runs over lattice sites with number $N \rightarrow \infty$.\footnote{To escape complications related to continuum space [1] we use the lattice model.} According to Ref. [2], the partition functional $Z_q[\phi] = \exp_q(-S[\phi])$ is reduced to the deformed exponential with the exponent being
inverse action $S = S[\phi]$ determined by the order parameter distri-
bution. Functional $Z_q[J]$ can be presented by the deformed series
\begin{equation}
Z_q[J] = \int \exp_q[-S(J/D_1)] \phi^{f-1}(d\phi)
\end{equation}
of the generating functional
\begin{equation}
Z_q[J] := \int \exp_q[-S(\phi)] \phi^{f-1}(d\phi)
\end{equation}
in terms of the bare part $Z^{(0)}(J)$ related to the action $S_0$. The principle difference of the equality (14) from corresponding expression for simple systems consists in deformation of both exponential operator of perturbation and its action onto the bare functional. Explicit expression of the generating functional (14) yields expansion into the deformed series (8) with the coefficients (9) determining the correlators (13).

Within the framework of the harmonic approximation, when the action $S_0 = \frac{1}{2N} \sum_{i=1}^{N} \phi_i^2$ is reduced to the sum of the $N$ independent constituents, the functional $Z_q^{(0)}(J)$ is expressed throughout the site functions $Z_q^{(0)}(j_i)$ by means of the equality $Z_q^{(0)}(J) = Z_q^{(0)}(J_1) \otimes Z_q^{(0)}(J_2) \otimes \cdots \otimes Z_q^{(0)}(J_N)$. Within the mean field approach, all multipliers coincide so that the use of Eq. (5) arrives at the expression of the generating functional (15) in terms of the site functions:

$$\ln_q[Z_q^{(0)}(J)] \simeq N \ln_q[Z_q^{(0)}(J_i)].$$

As expected, the deformed logarithm of the partition functional proves to be additive value whose magnitude is determined by the site constituent

$$z_q^{(0)}(J) = \frac{1}{2} \left( \frac{2\Delta^2}{1-q} \right)^{1/2} B \left( Q, \frac{1}{2} \right), \quad Q = \frac{2-q}{1-q}. \tag{17}$$

Here, the $B$-function decays as $J^{-1}$ near the point $J = 0$ transforming into the power-law dependence $J^{-q}$ in the limit $J \to \infty$. As a result, the bare function (17) decays fast with the $J$ growing in vicinity of the point $J = 0$ and then increases exponentially. At $J = 1$, the dependence (17) gives the deformed partition function per one site

$$z_q^{(0)} = \sqrt{\Delta^2/2(1-q)} B \left( Q, \frac{1}{2} \right). \tag{18}$$

As a result, within the zero approach, the mean order parameter (12) takes the form

$$\mathcal{S}^{(0)} = - \lambda^{-1} \left( 1 - \left\{ \frac{\ln_q[z_q^{(0)}(\lambda)]}{\ln_q[z_q^{(0)}(1)]} \right\}^{1/2} \right). \tag{19}$$

In the information theory, the principle role is played by the Fisher matrix whose elements represent pair correlators of derivatives of the logarithm of the probability distribution function with respect to parameters of this function (in difference of the Tsallis entropy whose value gives a global measure of uncertainty, the Fisher matrix determines a local measure of information stored by the system) [6]. In the absence of a space correlations, such a measure is given by the moment of the second order $\mathcal{C} \equiv \langle [\sigma(\phi)]^2 \rangle$ for which the use of Eqs. (13) and (9) arrives at the expression

$$\mathcal{C}^{(0)} = \lambda^{-2} \left\{ 1 - 2 \left\{ \frac{\ln_q[z_q^{(0)}(\lambda)]}{\ln_q[z_q^{(0)}(1)]} \right\}^{1/2} \right. + \left. \left\{ \frac{\ln_q[z_q^{(0)}(\lambda)]}{\ln_q[z_q^{(0)}(1)]} \right\}^{1/2} \right\}^{1/2}. \tag{20}$$

At determination of the dependence of both moments (19) and (20) on the deformation $\lambda$, one needs to take into account that parameter $q$ is not free because a self-similarity condition restricts its value by the equation [7]

$$\lambda^2 - q - 1 = (\lambda - 1)^2. \tag{21}$$

As a result, variation of the deformation parameter $\lambda$ from 1 to $\infty$ arrives at growing the exponent $q$ since 0.382 to 0.5. According to Figs. 1 and 2, hereby the specific partition function $z_q^{(0)} \sim 1$ slightly increases, while the mean order parameter (19) keeps the zero value before the deformation parameter $\lambda = 6.39$ and then increases monotonically. In contrast to this, the second order moment (20) displays slight maximum in the region $\lambda \geq 1$ that after downward excursion transforms into growing branch. From the point of view of the information theory [6], this means that before the value $\lambda = 6.39$ a global measure of the information uncertainty $\mathcal{S}$ does not appears, while the local measure of information stored by the system $\mathcal{C}$ increases appreciably beginning since dilatations $\lambda \sim 3$.

What about the correlators of the higher orders, in thermodynamic limit $N \to \infty$, they are expressed in terms of the lower moments (19) and (20) with help of different uncouplings. As a result, the use of expression (14) allows one to build up a perturbation theory in analogy with the standard scheme [1].

5. Field theory relations

If a system consists of macroscopically independent parts 1 and 2, then related actions are connected with the additivity condition $S_{1+2} = S_1 + S_2$, whereas the deformed partition functional (15) whose kernel is determined by the expression $\exp_q(-S_{1+2}) = \exp_q(-S_1) \otimes \exp_q(-S_2)$ is obeyed the deformed multiplicativity condition $Z_q^{1+2} = Z_q^1 \otimes Z_q^2$. Therefore, it is convenient to pass to the generating functional

$$\mathcal{G}_q := \ln_q(Z_q) \tag{22}$$

determined on the basis of the deformed logarithm that obeys the rules (6). As a result, the additivity condition $\mathcal{G}_q^{1+2} = \mathcal{G}_q^1 + \mathcal{G}_q^2$ becomes to be satisfied so that the functional $\mathcal{G}_q = \mathcal{G}_q(J)$ can be understood as a thermodynamic potential. Because the latter depends on an auxiliary field $J$, one should use the Legendre transform

$$\Gamma_q(\phi) := \sum_i J_i \sigma_i - \mathcal{G}_q(J), \quad \sigma_i \equiv \ln_{2-q} (\phi_i) \tag{23}$$

to pass to a dependence on the order parameter $\phi$. This transform connects conjugated pair of thermodynamic potentials $\mathcal{G}_q(J)$ and $\Gamma_q(\phi)$ whose using arrives at the state equations

$$\sigma_i = D^\ast_J \mathcal{G}_q \Leftrightarrow J_i = D^\ast_{\mathcal{G}_q} \Gamma_q. \tag{24}$$

Similarly to the partition functional (7), above potentials are presented by the following deformed series:

$$\mathcal{G}_q(J) = \sum_{n=1}^{\infty} \frac{1}{\left[ n \right]_q} \sum_{i_1 \cdots i_n} \mathcal{G}_q^{(n)}(J_{i_1} \cdots J_{i_n}), \tag{25}$$
Within account an arbitrary condition $\delta$ with respect to the field variation $\delta J$, the first of the pointed equations allows one to take into account constraints and symmetry of the statistical field theory based on above pointed distributions which take place in simple systems [1].

As a result, the generating functional takes the elongated form (15). As a result, the generating functional takes the elongated form (15).

Concluding this section, let us show that similarly to simple systems the generating functional (15) obeys a set of formal equations. The first of them is related to the system symmetry with respect to the field variation $\delta \ln(\phi_i) = \epsilon f_i(\phi)$ being proportional to an arbitrary functional $f_i(\phi)$ in the limit $\epsilon \to 0$. Due to this variation the integrand of the functional (15) obtains the multiplier $1 + \epsilon [\delta \ln(\phi_i) + \delta \ln(\phi_i)] = 1 + \epsilon f_i(\phi)$ (the sum over repeated indices is meant). Collecting all multipliers before the factor $\epsilon$, in accordance with the invariance condition of the functional (15) one finds:

$$\left( f_i \left[ \frac{\delta}{\delta J} \right] \exp \left[ -S \left( \frac{\delta}{\delta J} \right) \right] \frac{\partial S}{\partial \ln(\phi_i)} \left[ \frac{\delta}{\delta J} \right] - J_i \right) \gamma_q(J) = 0. \quad (27)$$

At $f_i(\phi) = \text{const}$, this equation is simplified to take the form following immediately from the functional (15) after variation over the integration variable $\ln(\phi_i)$.

The second of the pointed equations allows one to take into account an arbitrary condition $F_i(\ln(\phi)) = 0$ imposed on fields to be found. Accounting this condition is achieved by introducing the $\delta$-functional $\delta F$ into integrand of the last expression (15). As a result, the generating functional takes the elongated form

$$\gamma_q^{(F)}(J) := \int \exp \left[ -S(\phi) \right] \times \exp \left[ J \ln(\phi) + \lambda F \left[ \ln(\phi) \right] \right] d\ln(\phi). \quad (28)$$

Variation of this expression over an auxiliary field $\lambda_i$ arrives at the desired equation

$$F_i \left[ \frac{\delta}{\delta J} \right] \gamma_q^{(F)}(J) = 0. \quad (29)$$

6. Concluding remarks

Following the standard scheme [1], we have considered the field theory of self-similar statistical systems whose states are distributed in accordance with the Tsallis exponential law. Because this distribution is characterized by the power-law tail, we have used the generating functional (7) based on the Mellin transform and the Jackson derivative (1) as the generator of the scaling transformation. Along this line, the role of the order parameter plays the mean value (12) of the deformed logarithm of the amplitude of a hydrodynamic mode. The use of the harmonic approach shows the specific partition function (18) slightly increases, the mean order parameter (19) keeps the zero value before the deformation $\lambda > 6$ and then increases monotonically, while the second order moment (20) displays slight maximum in the region $\lambda > 1$ and after downward excursion transforms into growing branch. Apart from the generating functional (7), we have introduced pair of the additive functional (22) and (23) whose expansions into deformed series (25) and (26) yield both Green functions and proper vertices. To take into account constraints and symmetry of the statistical system we have obtained Eqs. (27) and (29) for the generating functional.

The special peculiarity of our consideration is that the kernel of the Mellin transform (7) is reduced to the Tsallis exponential (3). But it is worthwhile to stress that the Tsallis deformation is not sole of possible procedures to obtain a distribution with power-law behavior. Another possibility is known to be given by the basic deformed distribution [8] that is invariant with respect to action of the Jackson derivative. Moreover, generalized three-parameter deformation procedure have been elaborated recently by Kaniadakis to obtain the whole set of power-law tailed distributions [9]. Building of generalized field theory based on above pointed distributions is in progress.

References

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