Blow-up of solutions to semilinear wave equations with a time-dependent strong damping

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Abstract
The paper investigates a class of a semilinear wave equation with time-dependent damping term \((\frac{1}{1+t})^\beta \Delta u_t)\) and a nonlinearity \(|u|^p\). We will show the influence of the parameter \(\beta\) in the blow-up results under some hypothesis on the initial data and the exponent \(p\) by using the test function method. We also study the local existence in time of mild solution in the energy space \(H^1(U^n) \times L^2(U^n)\).

Keywords: Blow-up, local existence, nonlinear wave equations, strong damping.

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1. Introduction
The aim of the paper is to establish a blow-up result for local in time solutions to the Cauchy problem for the following semilinear strong damped wave equation

\[
\begin{align*}
    u_{tt} - \Delta u - \frac{b_0}{(1+t)^\beta} \Delta u_t &= |u|^p, \quad x \in \mathbb{R}^n, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), &\quad x \in \mathbb{R}^n,
\end{align*}
\]

where \(n \geq 1, p > 1, b_0\) is a positive constant, and \(\beta \in \mathbb{R}\). Without loss of generality, we assume that \(b_0 = 1\). Throughout this paper, we assume that

\[
\begin{align*}
    p &\in (1, \infty) \quad \text{for } n = 1, 2, \\
    p &\in (1, \frac{n-1}{n-2}) \quad \text{for } n \geq 3,
\end{align*}
\]

and the initial data are in the energy space

\[
(u_0, u_1) \in H^1(U^n) \times L^2(U^n).
\]

Hereafter, \(\|\cdot\|_q\) and \(\|\cdot\|_{H^l}\) \((1 \leq q \leq \infty)\) stand for the usual \(L^q(U^n)\)-norm and \(H^l(U^n)\)-norm, respectively.

In this paper, we study the blow-up result of solution of (1.1). Before going on, it is necessary to mention that the case \(b_0 = 0\) in (1.1) is the classical semilinear wave equation for which we have the Strauss conjecture. More precisely, this case is characterized by a critical power, denoted by \(p_S\), which is the positive solution of the following quadratic equation \((n - 1)p^2 - (n + 1)p - 2 = 0\), and is given by

\[
p_S = p_S(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}.
\]

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More precisely, if $p \leq p_5$ then there is no global solution for (1.1) under suitable sign assumptions for the initial data, and for $p > p_5$ a global solution exists for small initial data; see e.g. [7, 10, 12, 14] among many other references. A slightly less sharp blow-up result under much weaker assumptions was obtained by Kato [8] with a much easier proof. In particular, Kato pointed out the role of the exponent $(n + 1)/(n - 1) < p_S(n)$, for $n \geq 2$, in order to have more general initial data, but still with compact support.

We take the opportunity to mention here that the test function method, introduced by [13] and used by [5, 6, 9], plays a similar role as of Kato’s method to prove blow-up results. In fact, the test function is effective in the case of parabolic equations which means that it provide us exactly the critical exponent $p_c$. While in the case of hyperbolic equations (cf. [3]) we get the so-called Kato’s exponent $p^*$, i.e. we obtain a blow-up result for $p < p^* < p_c$. This is one of the weakness of the test function method but in general it can be applied to a more general equation and system.

When $\beta = 0$, and $b_0 = 1$, problem (1.1) is reduced to

$$\frac{\partial u}{\partial t} - \Delta u - \Delta u_t = |u|^p, \quad x \in \mathbb{R}^n, t > 0,$$

(1.5)

which is called the viscoelastic damping case. D’Ambrosio and Lucente [3, Theorem 4.2] proved that the solution of (1.5) blows-up in finite time when $1 < p \leq (n + 1)/(n - 1)$, where $\cdot_\ast := \max(0, \cdot)$, by applying the test function method. Similar result has been obtained recently by Fino [4] in the case of an exterior domain. On the other hand, D’Abbicco-Reissig [2] proved that there exists a global solution for (1.5) when $p > 1 + \frac{n}{n - 1} (n \geq 2)$ for sufficiently small initial data. Therefore, the exact value of the critical exponent is still an open question.

When $\beta \neq 0$, and $b_0 = 1$, we give an intuitive observation for understanding the influence of the damping term $\left(\frac{1}{(1 + t)^{\beta}} \Delta u_t\right)$ by scaling argument. Let $u(t, x)$ be a solution of the linear strong damped wave equation

$$u_{tt}(t, x) - \Delta u(t, x) - \frac{1}{(1 + t)^{\beta}} \Delta u_t(t, x) = 0.$$  

(1.6)

When $\beta \geq -1$, we put

$$v(t, x) = u(\lambda(1 + t), \lambda x), \quad \lambda(t + 1) = s, \lambda x = y,$$

(1.7)

with a parameter $\lambda > 0$, we have

$$v_{ss}(s, y) - \Delta v(s, y) - \frac{1}{(1 + s)^{\beta}} \Delta v_s(s, y) = 0.$$  

Thus, when $\beta = -1$ we notice that the equation (1.6) is invariant, while when $\beta > -1$, letting $\lambda \to \infty$, we obtain the wave equation without damping

$$v_{ss}(s, y) - \Delta v(s, y) = 0.$$

We note that $\lambda \to \infty$ is corresponding to $t \to +\infty$.

On the other hand, when $\beta < -1$, we put

$$v(t, x) = u(\lambda t^{-\frac{1}{\beta}}(1 + t), \lambda x), \quad \lambda(t + 1) = s, \lambda x = y,$$

with a parameter $\lambda > 0$, we have

$$v_{ss}(s, y) - \frac{1}{s^\beta} \Delta v_s(s, y) - \lambda \frac{2 + \beta}{\beta + 1} \Delta v(s, y) = 0.$$  

In this case, letting $\lambda \to \infty$, we obtain the hyperbolic equation

$$v_{ss}(s, y) - \frac{1}{s^\beta} \Delta v_s(s, y) = 0.$$

In this paper, our goal is to generalize Kato’s exponent and give sufficient conditions for finite time blow-up of a new type of class of equations (1.1) for $b_0 > 0$, $\beta \in \mathbb{R}$. Let us mention that our blow-up results and initial conditions are similar to that of Kato.

This paper is organized as follows. We start in Section 2 by introducing the mild solution of (1.1). Then, we state the main theorem of our work. In Section 3 we study the local existence of the solutions of equation (1.1) in the energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Finally, in Section 4 we prove the blow-up theorem (Theorem 1).
2. Main results

This section is aimed to state our main results. For that purpose, we first start by giving the definition of the mild solution of (1.1).

**Definition 1.** (Mild solution)
Let \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). We say that a function
\[ u \in C([0, T]; H^1(\mathbb{R}^n)) \cap C^1([0, T]; L^2(\mathbb{R}^n)) \]
is a mild solution of (1.1) if and has the initial data \(u(0) = u_0, \ u_t(0) = u_1\) and satisfies the integral equation
\[ u(t, x) = R(t, 0)(u_0, u_1) + \int_0^t S(t, s)|u(s)|^p \, ds \] (2.8)
in the sense of \(H^1(\mathbb{R}^n)\), where the operators \(R\) and \(S\) are defined below. Moreover, if \(T > 0\) can be arbitrary chosen, then \(u\) is called a global mild solution (1.1).

Here is the statement of our main theorem in this paper.

**Theorem 1 (Blow-up).** We assume that
\[ (u_0, u_1) \in (L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)) \times (L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) \]
satisfying the following condition:
\[ \int_{\mathbb{R}^n} u_1(x) \, dx > 0. \] (2.9)
If
\[ \begin{align*}
1 < p & \leq \frac{n+1}{(n-1)}, & \text{if } \beta \geq -1, \\
1 < p & \leq \frac{m(1-\beta)+2}{(m-2)(1-\beta)}, & \text{if } \beta \leq -1,
\end{align*} \] (2.10)
where \((\cdot)_+ := \max\{0, \cdot\}\), then the mild solution of (1.1) blows-up in finite time.

**Remark 1.** We stress that the exponent \(\frac{n+1}{(n-1)}\) appearing in (2.10) was introduced first in [8] to prove the nonexistence of global solutions to the semilinear wave equation with the nonlinearity \(|u|^p\), for small initial data with compact support.

**Remark 2.** Theorem 1 asserts that if \(\beta \geq -1\), then the critical exponent for \(p\), is greater than or equal to \(\frac{n+1}{(n-1)}\).
Therefore, the blow-up region obtained in the present work in the case \(\beta \geq -1\) constitute somehow an extension to the results related to the blow-up region of the solution of the equation (1.5) obtained in [3].

**Remark 3.** It is interesting to recall that thanks to the transformation (1.7) the asymptotic behavior of the solution to (1.6) in the case \(\beta \geq -1\), is given by the free wave equation. Unfortunately, the time-dependent damping term \((-\frac{1}{(1+tf)} \Delta u_t)\) makes the problem parabolic and loses the property that the speed of propagation is finite. For this reason, we obtain the blow-up region given by (2.10) which is included in \((1, p_3(n)]\). We think it is likely possible to extend the blow-up region to a larger interval \((1, p_c]\) where \(\frac{n+1}{(n-1)}, < p_c \leq p_3(n)\) at least for \(\beta\) large enough.

3. Local existence

To prove that the Cauchy problem for (1.1) is locally well-posed in the space \(H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), it is a natural to start by studying the linear homogeneous case.
3.1. Linear homogeneous case

We consider the linear homogeneous equation

\[
\begin{cases}
  u_t - \Delta u - \frac{1}{(1 + t)^d} \Delta u_t = 0, & t > 0, x \in \mathbb{R}^n, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^n,
\end{cases}
\]  

(3.11)

Definition 2. (Strong solution)

Let \((u_0, u_1) \in (H^2(\mathbb{R}^n))^2\). A function \(u\) is said to be a strong solution of (3.11) if

\[ u \in C^1([0, \infty); H^2(\mathbb{R}^n)) \cap C^2([0, \infty); L^2(\mathbb{R}^n)), \]

and \(u\) has the initial data \(u(0) = u_0, u_t(0) = u_1\) and satisfies the equation (3.11) in the sense of \(L^2(\mathbb{R}^n)\).

Proposition 1. For each \((u_0, u_1) \in (H^2(\mathbb{R}^n))^2\), there exists a unique strong solution \(u\) of problem (3.11) that satisfies the following energy estimates

\[
\int_{\mathbb{R}^n} (u_t^2(t, x) + |\nabla u(t, x)|^2) \, dx \leq \int_{\mathbb{R}^n} (u_0^2(x) + |\nabla u_0(x)|^2) \, dx,
\]

(3.12)

\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + T \|(u_1, \nabla u_0)\|_{L^2 \times L^2},
\]

(3.13)

for any \(T > 0\), and all \(0 \leq t \leq T\).

Proof. The existence of the strong solution can be done easily by the semigroup theory (cf. [1]). We now focus to the proof of estimates (3.12) and (3.13). By multiplying (3.11) by \(u_t\), integrating over \(\mathbb{R}^n\) and performing some integration by parts in space, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (u_t^2 + |\nabla u_t|^2) \, dx + \frac{1}{(1 + t)^d} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = 0, \quad \forall t \geq 0.
\]

(3.14)

By integrating in time between 0 and \(t\) the equality (3.14), we get

\[
\int_{\mathbb{R}^n} (u_t^2 + |\nabla u_t|^2) \, dx + 2 \int_0^t \frac{1}{(1 + s)^d} \int_{\mathbb{R}^n} |\nabla u_s|^2 \, dx \, ds = \int_{\mathbb{R}^n} (u_0^2 + |\nabla u_0|^2) \, dx, \quad \forall t \geq 0,
\]

which implies the estimate (3.12). Next, we prove (3.13). Thanks to the basic identity \(u_t = u_0 + \int_0^t u_s(s) \, ds\), we conclude

\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|u_s(s)\|_{L^2} \, ds \quad \forall t \geq 0.
\]

(3.15)

By combining (3.12) and (3.15), we infer

\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|(u_1, \nabla u_0)\|_{L^2 \times L^2} \, ds \leq \|u_0\|_{L^2} + T \|(u_1, \nabla u_0)\|_{L^2 \times L^2} \quad \forall t \geq 0.
\]

This follows (3.13) and we complete the proof of Proposition 1.

Let us denote by \(R(t)\) the operator which maps the initial data \((u_0, u_1) \in (H^2(\mathbb{R}^n))^2\) to the strong solution \(u(t) \in H^2(\mathbb{R}^n)\) at the time \(t \geq 0\), i.e. the solution \(u\) of (3.11) is defined by \(u(t) = R(t)(u_0, u_1)\).

Remark 4. From Proposition 1 the operator \(R(t)\) can be extended uniquely such that \(R(t) : H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow C([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n))\). Indeed, for any fixed \(T > 0\), due to the energy estimates (3.12)-(3.13), the following estimation

\[
\|R(t)(u_0, u_1)\|_{H^1} + \|\partial_t R(t)(u_0, u_1)\|_{L^2} \leq C(1 + T) \|(u_0, u_1)\|_{H^2 \times L^2},
\]

holds for all \(0 \leq t \leq T\). It follows that the operator \(R(t)\) can be extended uniquely to an operator such that \(R(t) : H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow C([0, T], H^1(\mathbb{R}^n) \cap C^1([0, T], L^2(\mathbb{R}^n))\). Since \(T\) is arbitrary, we conclude the desired extension.
3.2. Linear inhomogeneous case

Let us now consider the linear inhomogeneous equation

\[
\begin{aligned}
&u_t - \Delta u - \frac{1}{(1 + t)^{\beta}} \Delta u_t = F(t, x), \\
&u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \\
&\end{aligned}
\] (3.16)

**Definition 3.** Let \((u_0, u_1) \in H^1 \times L^2\) and \(F \in C([0, \infty); L^2)\). We say that a function \(u\) is a mild solution of (3.16) if \(u \in C([0, \infty); H^1) \cap C^1([0, \infty); L^2)\) and \(u\) has the initial data \(u(0) = u_0, u_t(0) = u_1\) and satisfies the integral equation

\[
\begin{aligned}
&u(t, x) = R(t, 0)(u_0, u_1) + \int_0^t S(t, s)F(s, x)\, ds \\
&\end{aligned}
\] (3.17)

in the sense of \(H^1(\mathbb{R}^n)\), where \(S(t, s)g := R(t, s)(0, g)\) for a function \(g \in H^1(\mathbb{R}^n)\).

By a classical result as in [1,15] or similarly as in [11, Proposition 9.15], we have the following

**Proposition 2.**

Let \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), F \in C([0, \infty); L^2(\mathbb{R}^n))\). Then there exists a unique mild solution \(u\) of (3.16). Moreover, the mild solution \(u\) satisfies the following energy estimates

\[
\| (u_t, \nabla u) \|_{L^2(\times \mathbb{R}^n)} \leq C \| (u_1, \nabla u_0) \|_{L^2(\times \mathbb{R}^n)} + C \int_0^t \| F(s, \cdot) \|_{L^2} \, ds, \\
\| u(t) \|_{L^2} \leq C \| u_0 \|_{L^2} + C t \| (u_1, \nabla u_0) \|_{L^2(\times \mathbb{R}^n)} + \int_0^t \| F(t, \cdot) \|_{L^2} \, dt \, ds. 
\] (3.18) (3.19)

3.3. Semilinear case

Using Gagliardo-Nirenberg’s inequality, Proposition 2 and the Banach fixed point theorem we get the following local existence theorem.

**Proposition 3.** Let \(\beta \in \mathbb{R}\). Under the assumptions (1.2), (1.3), the problem (1.1) admits a unique maximal mild solution \(u\), i.e. satisfies the integral equation (2.8) such that

\[
u \in C([0, T_{\max}); H^1(\mathbb{R}^n)) \cap C^1([0, T_{\max}); L^2(\mathbb{R}^n)), 
\]

where \(0 < T_{\max} \leq \infty\). Moreover, if \(T_{\max} < \infty\), then it follows that

\[
\| u(t) \|_{H^1} + \| u_t(t) \|_2 \to \infty \quad \text{as} \quad t \to T_{\max}. 
\]

4. Proof of Theorem 1

In order to prove Theorem 1 we are going to use the test function method which is rely on the weak solution of (1.1). More precisely, the weak formulation associated with (1.1) reads as follows:

**Definition 4.** (Weak solution)

Let \(T > 0\), and \(u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)\). A function \(u\) is said to be a weak solution of (1.1) if

\[
u \in L^p((0, T); L^p(\mathbb{R}^n)),
\]

and \(u\) satisfies the weak formulation

\[
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u|^p \psi \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} u_t(x)\psi(0, x) \, dx - \int_0^T \int_{\mathbb{R}^n} u_0(x)\Delta \psi(0, x) \, dx - \int_0^T \int_{\mathbb{R}^n} u_0(x)\psi_t(0, x) \, dx \\
&= \int_0^T \int_{\mathbb{R}^n} u \psi_t \, dx \, dt + \int_0^T \frac{1}{(1 + t)^{\beta}} \int_{\mathbb{R}^n} u \Delta \psi \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} u \Delta \psi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \frac{\beta}{(1 + t)^{\beta + 1}} u \, \Delta \psi \, dx \, dt.
\end{aligned}
\]

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for all compactly supported function \( \psi \in C^2([0, T] \times \mathbb{R}^n) \) such that \( \psi(\cdot, T) = 0 \) and \( \psi(\cdot, T) = 0 \). We denote the lifespan for the weak solution by

\[
T_u(u_0) := \sup\{T \in (0, \infty) ; \text{ there exists a unique weak solution } u \text{ to (1.1)}\}.
\]

Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_u(u_0) = \infty \), then \( u \) is called a global weak solution of (1.1).

We also need the following

**Remark** 5. (Mild \( \to \) Weak)

Let \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). Under the assumption (1.2), if \( u \) is a global mild solution of (1.1), then \( u \) is a global weak solution of (1.1).

**Proof of Theorem** 1. Let \( u \) a global mild solution of (1.1). Thanks to Remark 5, we conclude that \( u \) is a global weak solution of (1.1), i.e.

\[
\int_0^T \int_{\mathbb{R}^n} |u|^p \psi_1 \ dx \ dt + \int_0^T \int_{\mathbb{R}^n} u_1(x) \psi(0, x) \ dx - \int_0^T \int_{\mathbb{R}^n} u_0(x) \Delta \psi(0, x) \ dx - \int_0^T \int_{\mathbb{R}^n} u_0(x) \psi(0, x) \ dx
\]

\[
= \int_0^T \int_{\mathbb{R}^n} u \psi_1 \ dx \ dt + \int_0^T \frac{1}{(1 + t)^\beta} \int_{\mathbb{R}^n} u \Delta \psi_1 \ dx \ dt - \int_0^T \int_{\mathbb{R}^n} u \Delta \psi \ dx \ dt - \int_0^T \int_{\mathbb{R}^n} \frac{\beta}{(1 + t)^{\beta + 1}} \int_{\mathbb{R}^n} u \Delta \psi \ dx \ dt,
\]

for all \( T > 0 \), and all compactly supported function \( \psi \in C^2([0, T] \times \mathbb{R}^n) \) such that \( \psi(\cdot, T) = 0 \) and \( \psi(\cdot, T) = 0 \).

Let \( T > 0 \). Now, we introduce the following test function:

\[
\psi(x, t) = \psi_1(x) \psi_2(t)
\]

where

\[
\psi_1(x) := \Phi \left( \frac{|x|}{T^d} \right), \quad \psi_2(t) := \Phi \left( \frac{t}{T} \right).
\]

where \( \ell, \eta \) are sufficiently large constants that will be determined later and \( \Phi \in C^\infty(\mathbb{R}_+) \) be a cut-off non-increasing function such that

\[
\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1/2, \\ \gamma & \text{if } 1/2 \leq r \leq 1, \\ 0 & \text{if } r \geq 1. \end{cases}
\]

The constant \( d > 0 \) in the definition of \( \psi_1 \) is fixed and will be chosen later. In the following, we denote by \( \Omega(T) \) the support of \( \psi_1 \) and by \( \Delta(T) \) the set containing the support of \( \Delta \psi_1 \) which are defined as follows:

\[
\Omega(T) = \{x \in \mathbb{R}^n : |x| \leq 2T^d\}, \quad \Delta(T) = \{x \in \mathbb{R}^n : T^d/2 \leq |x| \leq T^d\}.
\]

By (4.20), we get that

\[
\int_0^T \int_{\Omega(T)} |u|^p \psi_1 \ dx \ dt + \int_0^T \int_{\Delta(T)} u_1(x) \psi(0, x) \ dx 
\]

\[
\leq \int_0^T \int_{\Omega(T)} |u| \psi_1 \ dx \ dt + \int_0^T \int_{\Omega(T)} |u| \Delta \psi_1 \ dx \ dt + \int_0^T \frac{1}{(1 + t)^\beta} \int_{\Delta(T)} |u| \Delta \psi \ dx \ dt 
\]

\[
+ \int_0^T \frac{\beta}{(1 + t)^{\beta + 1}} \int_{\Delta(T)} |u| \Delta \psi \ dx \ dt + \int_0^T \int_{\Omega(T)} \left( |\Delta \psi(0, x)| + |\psi_1(0, x)| \right) \ dx
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Let \( \varepsilon > 0 \). By applying \( \varepsilon \)-Young’s inequality

\[
AB \leq \varepsilon A^p + C(\varepsilon, p)B^{p'}, \quad A \geq 0, B \geq 0, \quad p + p' = p p', \quad C(\varepsilon, p) = \varepsilon^{-1/(p-1)} p^{-p/(p-1)}(p - 1),
\]

\[
A \leq \varepsilon B^p + C(\varepsilon, p)B^{p'}, \quad A \geq 0, B \geq 0, \quad p + p' = p p', \quad C(\varepsilon, p) = \varepsilon^{-1/(p-1)} p^{-p/(p-1)}(p - 1),
\]

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Therefore,

\[ I_1 \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \psi' \left( \psi_2 \right) dx \quad dt \]

\[ \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \psi_2 \right) x dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.24)

Using \( \left( \psi_2 \right)_h = \eta \left( \psi_2 \right)_h + \eta(\eta - 1) \psi_2 \), the inequality (4.24) becomes

\[ I_1 \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \psi_2 \right) x dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.25)

Using and proceeding similarly as for (4.25) and using the identity \( \Delta (\psi_1') = \ell \psi_1' \Delta \psi_1 + \ell(\ell - 1) \psi_1^2 \Delta |\nabla |^2 \), we easily deduce

\[ I_2 \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \psi_2 \right) x dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.26)

In the same way, thanks to \( \Delta \psi_1 = \Delta (\psi_1' \psi_2) \), we write

\[ I_3 \leq \int_0^T \int_{\Omega(T)} \frac{1}{1 + t} \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \Delta (\psi_1') \right) \left( \psi_2 \right) x dx \quad dt \]

\[ \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \Delta (\psi_1') \right) \left( \psi_2 \right) x dx dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.27)

Clearly,

\[ \frac{1}{1 + t} \leq C T^{-\beta'}, \quad \forall t \in \left( \frac{T}{2}, T \right). \]

Therefore,

\[ I_3 \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \Delta (\psi_1') \right) \left( \psi_2 \right) x dx dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt \]

\[ + C T^{-\beta'} \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.28)

In the same manner,

\[ I_4 \leq C \int_0^T \int_{\Omega(T)} \frac{1}{1 + t} \int_{\Omega(T)} |u||\psi|^{-1/p} \psi_2 \left( \Delta (\psi_1') \right) x dx dt \]

\[ \leq e \int_0^T \int_{\Omega(T)} |u||\psi|^{-1/p} \left( \psi_2 \right) x dx dt + C \int_0^T \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt \]

\[ + C \int_0^T \int_{\Omega(T)} \frac{1}{1 + t} \int_{\Omega(T)} \psi_2 \left( \psi_2 \right) x' \left( \psi_2 \right) x dx dt. \]  

(4.29)

Finally, it remains only to control the term \( I_5 \). Note by exploiting the identities

\[ \Delta \psi(0, x) = \Delta (\psi_1') = \ell \psi_1' \Delta \psi_1 + \ell(\ell - 1) \psi_1^2 \nabla |\nabla |^2 \quad \text{and} \quad \psi(0, x) = \eta \psi_1' \psi_2(0), \]

we infer

\[ I_5 \leq C \int_{\Omega(T)} |u| \left( \psi_1' \Delta \psi_1 + \psi_1^2 \nabla |\nabla |^2 + |\psi_2(0) \psi_1' \right) dx. \]  

(4.30)
we have

Subcritical case (\( \beta \leq -1 \)).

In this case, we choose \( d = 1 \).

Note that, as

\[
\int_0^T (1 + t)^{-\frac{n\beta}{(n - 1)\beta}} \, dt \leq C \begin{cases} 
T^{1-\frac{n\beta}{(n - 1)\beta}} & \text{if } \beta p < -1, \\
\ln(T) & \text{if } \beta p = -1, \\
1 & \text{if } \beta p > -1,
\end{cases}
\]

we have \( \int_0^T (1 + t)^{-\frac{n\beta}{(n - 1)\beta}} \, dt \leq CT \), for all \( T > 1 \). Then (4.32) implies

\[
\int_0^T |u|^p \psi \, dx \, dt + \int_0^T u_1(x) \phi_1'(x) \, dx \leq C T^{-2p' + 1 + n} + C T^{-2p' + 1 + n} + C T^{-p'} + C \left(T^{-2} + 1\right) \int_{\mathbb{R}^n} |u_0(x)| \, dx,
\]

for all \( T > 1 \). We use the fact \( \beta \geq -1 \), to conclude that

\[
\int_0^T |u|^p \psi \, dx \, dt + \int_0^T u_1(x) \phi_1'(x) \, dx \leq C T^{-2p' + 1 + n} + C \left(T^{-2} + 1\right) \int_{\mathbb{R}^n} |u_0(x)| \, dx, \quad \forall \, T > 1.
\]

Note that, we can easily see that \(-2p' + 1 + n < 0\), if \( p < \frac{n}{2(n - 1)} \). Letting \( T \to \infty \), and using the Lebesgue dominated convergence theorem, we conclude that

\[
\int_{\mathbb{R}^n} u_1(x) \, dx \leq 0.
\]

This contradicts our assumption (4.22).
Critical case \((p = \frac{n+2}{n-1})\), when \(n \geq 2\).

Let \(n \geq 2\), and

\[
p = \frac{n + 1}{n - 1}.
\]

From the subcritical case (4.35), we can see easily that we have

\[
u \in L^p((0, \infty); L^p(\mathbb{R}^n)).
\]

On the other hand, by applying Hölder’s inequality instead of Young’s inequality, we get

\[
\int_{\Omega(T)} u_1(x) \psi_1'(x) dx \leq C \int_{\Omega(T)} |u|^p \psi dx dt + C \int_0^T \int_{\Omega(T')} |u|^p \psi dx dt + C \left(T^{-2} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)| dx.
\]

Letting \(T \to \infty\) and taking into consideration (4.36) we get a contradiction.

**II. Case of \( \beta < -1 \).**

In this case, we choose \(d = \frac{1-\beta}{p} \).

Note that, as \(p \beta < \beta < -1\), we have

\[
\int_0^T (1 + t)^{-\frac{\beta+1}{p}} dt \leq CT^{1-(\beta+1)p}.
\]

Therefore, (4.32) becomes

\[
\int_0^T \int_{\Omega(T)} |u|^p \psi dx dt + \int_{\Omega(T)} u_1(x) \psi_1'(x) dx \leq C T^{-2p' + 1 + nd} + C T^{-2dp' + 1 + nd} + C T^{-2dp' + 1 + nd} + C \left(T^{-2d} + T^{-k}\right) \int_{\mathbb{R}^n} |u_0(x)| dx.
\]

Now, we distinguish two subcases:

**Subcritical case \( p < \frac{n(1-\beta+2)}{n(1-\beta)-2} \).**

As \( \beta < -1 \), we have

\[
-2dp' + 1 + nd < -(\beta+1)p' - 2dp' + 1 + nd < 0 \quad \text{and} \quad -2p' + 1 + nd < 0,
\]

where we have used

\[
p < \frac{n(1-\beta) + 2}{n(1-\beta) - 2},
\]

for all \(n \geq 1\). So, letting \(T \to \infty\), we get a contradiction.

**Critical case \( p = \frac{n(1-\beta+2)}{n(1-\beta)-2} \).**

Let \(n \geq 1\), and

\[
p = \frac{n(1-\beta) + 2}{n(1-\beta) - 2}.
\]

From the subcritical case, we can see easily that we have

\[
u \in L^p((0, \infty); L^p(\mathbb{R}^n)).
\]

On the other hand, by applying Hölder’s inequality instead of Young’s inequality, we get

\[
\int_{\Omega(T)} u_1(x) \psi(0, x) dx \leq C \int_{\Omega(T)} |u|^p \psi dx dt + C \int_0^T \int_{\Omega(T')} |u|^p \psi dx dt + C \left(T^{-2} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)| dx.
\]

Letting \(T \to \infty\) and taking into consideration (4.39) we get a contradiction.
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