Special class of $G$-Wolfe type fractional symmetric
duality theorems under $G$-pseudoinvexity assumptions

Ramu Dubey$^1$, Arvind Kumar$^2$, Pooja Gupta$^3$, Shubham Jayswal$^4$
and Vishnu Narayan Mishra$^5$,*

$^1,3,4$Department of Mathematics, J. C. Bose University of Science and Technology, YMCA,
Faridabad-121 006, Haryana, India; rdubeyjiya@gmail.com, ramudubey@jcboseust.ac.in,
poojagargdu@gmail.com, shubhamdubey@jcboseust.ac.in

$^2$Department of Mathematics, Dyal Singh College, University of Delhi, Lodhi Road, New
Delhi-110003, India; arvind.ch83@gmail.com

$^5$Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak,
Anuppur, Madhya Pradesh 484887, India; vishnunarayanmishra@gmail.com

*Corresponding author

Abstract. In this article, a pair of $G$-Wolfe-type fractional programming problems is
formulated. For a differentiable function, we consider the definitions of $G$-invexity/$G$-
pseudoinvexity, which extends some kinds of generalized convexity assumptions. In the
next section, we prove the weak, strong and converse duality theorems under $G$-invexity/$G$-
pseudoinvexity assumptions.

keyword: Fractional programming problem, symmetric duality, $G$-invexity, $G$-pseudoinvexity,
$G$-Wolfe model

1. Introduction

Duality assumes a significant part in optimization problems and is extremely valuable both
hypothetically and in real life. If dual of dual is a primal problem, pair of primal and dual is
said to have symmetric property. As opposed to linear programming, the vast majority dual
formulation in nonlinear programming do not have the symmetric property. Various authors
have considered fractional programming problems containing square root of positive semidefi-
nite quadratic forms like Mond [1] and Zhang and Mond [2].

Kim et al. [3] worked a pair of multiobjective symmetric dual problem (cone constraints)
under pseudo-invex functions. Devi [4] formulated a pair of second-order symmetric dual models
and proved duality theorems under bonvex functions. Chen [5] constructed a pair of multiob-
jective higher-order dual nonlinear problem and derived duality results involving higher-order
($F, \alpha, \rho, d$)-convexity assumptions. Also, Chen [6] derived ratio property and established the
optimality conditions with higher-order \((F, \alpha, \rho, d)\)-convexity assumptions. Later on, Wolfe type second-order symmetric duality has been discussed by Yang et al. \[3\] for multiobjective programming problems.

Convexity suspicions are regularly not fulfilled in genuine problems, so there was a need to debilitate them. One of the ways was the presentation of speculation of convexity specifically quasiconvexity and pseudoconvexity. One of the ways was the introduction of generalization of convexity namely quasi/pseudo-convexity. For more data on fractional programming, readers are advised to see \[7, 8, 9, 10, 11, 12\].

In this paper, we consider \(G\)-invexity/pseudoinvexity assumptions. Generalized \(G\)-Wolfe type fractional symmetric dual is proposed and duality results are derived by using the above mentioned functions.

2. Preliminaries and Definitions

Let \(S_1 \subseteq R^n\) and \(S_2 \subseteq R^m\) be open sets and \(f(x, y)\) be real valued differentiable function defined on \(S_1 \times S_2\). Let \(G : R \rightarrow R\) be strictly increasing function in their range \(G : I_f(S_1 \times S_2) \rightarrow R\), where \(I_f(S_1 \times S_2)\) is the range of \(f\), \(\eta_1, \eta_2 : S_1 \times S_2 \rightarrow R^m\).

Definition 2.1. The function \(f(x, y)\) is \(G\)-pseudoinvex in the first variable at \(u \in S_1\) for fixed \(v \in S_2\) with respect to \(\eta_1\), such that for \(x \in S_1\), we have

\[
\eta_1^T(x, u) \left[ G'(f(u, v))\nabla_x f(u, v) \right] \geq 0 \Rightarrow [G(f(x, v)) - G(f(u, v))] \geq 0.
\]

Remark 2.1. If the above inequality sign changes \(\leq\), then the function \(f(x, y)\) is \(G\)-pseudoincave in the first variable at \(u \in S_1\) for fixed \(v \in S_2\) with respect to \(\eta_1\).

Definition 2.2. The function \(f(x, y)\) is \(G\)-pseudoinvex in the second variable at \(v \in S_2\) for fixed \(u \in S_1\) with respect to \(\eta_2\), such that for \(y \in S_2\), we have

\[
\eta_2^T(y, v) \left[ G'(f(u, v))\nabla_y f(u, v) \right] \geq 0 \Rightarrow [G(f(u, y)) - G(f(u, v))] \geq 0.
\]

Remark 2.2. If the above inequality sign changes \(\leq\), then the function \(f(x, y)\) is \(G\)-pseudoincave in the second variable at \(v \in S_2\) for fixed \(u \in S_1\) with respect to \(\eta_2\).

Definition 2.3. The function \(f(x, y)\) is \(G\)-invex in the first variable at \(u \in S_1\) for fixed \(v \in S_2\) with respect to \(\eta_1\) such that for \(x \in S_1\), we have

\[
[G(f(x, v)) - G(f(u, v))] \geq \eta_1^T(x, u) \left[ G'(f(u, v))\nabla_x f(u, v) \right].
\]

Remark 2.3. If the above inequality sign changes \(\leq\), then the function \(f(x, y)\) is \(G\)-incave in the first variable at \(u \in S_1\) for fixed \(v \in S_2\) with respect to \(\eta_1\).

Definition 2.4. The function \(f(x, y)\) is \(G\)-invex in the second variable at \(v \in S_2\) for fixed \(u \in S_1\) with respect to \(\eta_2\), such that for \(y \in S_2\), we have

\[
[G(f(u, y)) - G(f(u, v))] \geq \eta_2^T(y, v) \left[ G'(f(u, v))\nabla_y f(u, v) \right].
\]

Remark 2.4. If the above inequality sign changes \(\leq\), then the function \(f(x, y)\) is \(G\)-incave in the second variable at \(v \in S_2\) for fixed \(u \in S_1\) with respect to \(\eta_2\).
3. G-Wolfe type fractional symmetric pair of primal-dual model

In the following section, we formulate the following pair of G-Wolfe type fractional symmetric dual programming problem:

**Primal Problem (FWP):**

\[
\begin{align*}
\text{Min} & \quad G(f(x,y)) - y^T G'(f(x,y)) \nabla_y f(x,y) \\
\text{Subject to} & \quad \frac{G(g(x,y)) - y^T G'(g(x,y)) \nabla_y g(x,y)}{

-\left[ G(f(x,y)) - y^T G'(f(x,y)) \nabla_y f(x,y) \right] + G'(g(x,y)) \left[ G(f(x,y)) \nabla_y f(x,y) \right] \geq 0, \\
x & \geq 0.
\end{align*}
\]

**Dual Problem (FWD):**

\[
\begin{align*}
\text{Max} & \quad G(f(u,v)) - u^T G'\left(f(u,v)\right) \nabla_x f(u,v) \\
\text{Subject to} & \quad \frac{G(g(u,v)) - u^T G'(g(u,v)) \nabla_x g(u,v)}{

-\left[ G'(f(u,v)) \nabla_x f(u,v) \right] + G'(g(u,v)) \left[ G'(f(u,v)) \nabla_x f(u,v) \right] \geq 0, \\
v & \geq 0,
\end{align*}
\]

where \( f : S_1 \times S_2 \to R \) and \( g : S_1 \times S_2 \to R_+ \setminus \{0\} \) are differentiable functions. The above primal-dual models can be re-written as:

**(EFWP) Min w**

Subject to

\[
\begin{align*}
(G(f(x,y)) - y^T G'(f(x,y)) \nabla_y f(x,y)) - w(G(g(x,y)) - y^T G'(g(x,y)) \nabla_y g(x,y)) &= 0, \\
-\left[ G'(f(x,y)) \nabla_y f(x,y) \right] - w G'(g(x,y)) \nabla_y g(x,y) &\geq 0, \\
x &\geq 0.
\end{align*}
\]

**(EFWD) Min t**

Subject to

\[
\begin{align*}
(G(f(u,v)) - u^T G'(f(u,v)) \nabla_x f(u,v)) - t(G(g(u,v)) - u^T G'(g(u,v)) \nabla_x g(u,v)) &= 0, \\
-\left[ G'(f(u,v)) \nabla_x f(u,v) \right] - t G'(g(u,v)) \nabla_x g(u,v) &\geq 0, \\
v &\geq 0.
\end{align*}
\]

Let \( R^0 \) and \( S^0 \) be the sets of feasible solution of (EFWP) and (EFWD), respectively.

**Theorem 3.1** (Weak duality theorem). Let \((x,y,w) \in R^0 \) and \((u,v,t) \in S^0 \). Let
(i) \( f(.,v) \) be \( G \)-invex and \( g(.,v) \) be \( G \)-incave at \( u \) for fixed \( v \) with respect to \( \eta_1 \),

(ii) \( f(x,.) \) be \( G \)-incave and \( g(x,.) \) be \( G \)-invex at \( x \) for fixed \( y \) with respect to \( \eta_2 \),

(iii) \( \eta_1(x,u) + u \geq 0 \) and \( \eta_2(v,y) + y \geq 0 \),

(iv) \( G(g(x,v)) > 0 \),

then, \( w \geq t \).

**Proof.** From assumption (i),

\[
G(f(x,v)) - G(f(u,v)) \geq \eta_1^T (x,u)G'(f(u,v))\nabla_x f(u,v)
\]  

(7)

and

\[
-G(g(x,v)) + G(g(u,v)) \geq -\eta_1^T (x,u)G'(g(u,v))\nabla_x g(u,v).
\]  

(8)

Multiplying by \( t \) in inequality (8) and from (7), we obtain

\[
G(f(x,v)) - tG(g(x,v)) - G(f(u,v)) + tG(g(u,v)) \geq \eta_1^T (x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)].
\]  

(9)

Next, by assumption (ii),

\[
-G(f(x,v)) + G(f(x,y)) \geq -\eta_2^T (v,y)G'(f(x,y))\nabla_y f(x,y)
\]  

(10)

and

\[
G(g(x,v)) - G(g(x,y)) \geq \eta_2^T (v,y)[G'(g(x,y))\nabla_y g(x,y)].
\]  

(11)

Multiplying by \( w \) in inequality (11) and together with (10), we obtain

\[
-G(f(x,v)) + wG(g(x,v)) - G(f(x,y)) + wG(g(x,y)) \geq -\eta_2^T (v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].
\]  

(12)

On adding inequalities (9) and (12), we get

\[
G(f(x,v)) - tG(g(x,v)) - G(f(u,v)) + tG(g(u,v)) - G(f(x,v)) + wG(g(x,v)) - G(f(x,y)) + wG(g(x,y)) \geq \eta_1^T (x,u)[G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)]
\]

\[
- \eta_2^T (v,y)[G'(f(x,y))\nabla_y f(x,y) - wG'(g(x,y))\nabla_y g(x,y)].
\]  

(13)

From dual constraint (3) and assumption (iii), we have

\[
(\eta_1(x,u) + u)^T [G'(f(u,v))\nabla_x f(u,v) - tG'(g(u,v))\nabla_x g(u,v)] \geq 0.
\]
or
\[ \eta_1^T(x, u)[G'(f(u, v))\nabla_x f(u, v) - tG'(g(u, v))\nabla_u g(u, v)] \]
\[ \geq -u^T[G'(f(u, v))\nabla_x f(u, v) - tG'(g(u, v))\nabla_u g(u, v)]. \]  (14)

Similarly, from inequality (2) and assumption (iii), we get
\[ -(\eta_2^T(v, y) + y)^T[G'(f(x, y))\nabla_y f(x, y) - wG'(g(x, y))\nabla_y g(x, y)] \geq 0, \]
or
\[ -\eta_2^T(v, y)[G'(f(x, y))\nabla_y f(x, y) - wG'(g(x, y))\nabla_y g(x, y)] \]
\[ \geq y^T[G'(f(x, y))\nabla_y f(x, y) - wG'(g(x, y))\nabla_y g(x, y)]. \]  (15)

From inequalities (13), (14) and (15), we get
\[ G(f(x, v)) - tG(g(x, v)) - G(f(u, v)) + tG(g(u, v)) - G(f(x, v)) + wG(g(x, v)) \]
\[ = G(f(x, v)) + wG(g(x, y)) \geq u^T[G'(f(u, v))\nabla_x f(u, v) - tG'(g(u, v))\nabla_u g(u, v)] \]
\[ + y^T[G'(f(x, y))\nabla_y f(x, y) - wG'(g(x, y))\nabla_y g(x, y)]. \]

Using equations (11) and (14), it follows that
\[ (w - t)G(g(x, v)) \geq 0. \]

From assumption (iv), above inequality gives
\[ w \geq t. \]

Hence, completes the result. \Box

Remark 3.1 Since every invex function is pseudoinvex. Therefore above weak duality can also be obtained under \( G \)-pseudoinvex assumptions.

Theorem 3.2 (Weak duality): Let \((x, y, w) \in R^0\) and \((u, v, t) \in S^0\). Let

(i) \( f(., v) \) be \( G \)-pseudoinvex and \( g(., v) \) be \( G \)-pseudoincave at \( u \) for fixed \( v \) with respect to \( \eta_1 \),

(ii) \( f(x, .) \) be \( G \)-pseudoincave and \( g(x, .) \) be \( G \)-pseudoinvex at \( y \) for fixed \( x \) with respect to \( \eta_2 \),

(iii) \( \eta_1(x, u) + u \geq 0 \) and \( \eta_2(v, y) + y \geq 0 \),

(iv) \( G(g(x, v)) > 0. \)

Then, \( w \geq t. \)

Proof: The proof follows on the same pattern of theorem 3.1.

Theorem 3.3 (Strong duality theorem). Let \( f \) and \( g \) be differentiable functions. Let \((\bar{x}, \bar{y}, \bar{w})\) be an optimal solution of (EFWP). Suppose that
From inequality (16), we find that $\gamma = 0$. From (19), we get

$$G'(g(\bar{x}, \bar{y}))\nabla_y f(\bar{x}, \bar{y}) - \bar{w}G''(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}) = 0.$$  

Next, we have to show that

$$\alpha - \beta G'(g(\bar{x}, \bar{y})) - \bar{y}G'(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}) = 0,$$

Then, $(\bar{x}, \bar{y}, \bar{w}) \in S^0$ and objective values of (EFWP) and (EFWD) are equal. Moreover, if all the hypotheses of weak duality theorem are satisfied, then $(\bar{x}, \bar{y}, \bar{w}, \bar{\mu} = 0)$ is an optimal solution of (EFWD).

**Proof:** Since $(\bar{x}, \bar{y}, \bar{w})$ is an optimal solution of (EFWP), $\alpha \in R, \beta \in R, \gamma \in R^m, \mu \in R^m$ such that the following Fritz John necessary conditions \[13\] are satisfied at $(\bar{x}, \bar{y}, \bar{w})$ :

$$[\beta(G'(f(\bar{x}, \bar{y})))\nabla_x f(\bar{x}, \bar{y}) - \bar{w}G'(g(\bar{x}, \bar{y}))\nabla_x g(\bar{x}, \bar{y})] + (\gamma - \beta \bar{y})G''(f(\bar{x}, \bar{y}))\nabla_y f(\bar{x}, \bar{y})$$

$$\nabla_x f(\bar{x}, \bar{y}) + G'(f(\bar{x}, \bar{y}))\nabla_{xy} f(\bar{x}, \bar{y}) - \bar{w}(G''(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}))\nabla_x g(\bar{x}, \bar{y})$$

$$+ G'(g(\bar{x}, \bar{y}))\nabla_{yx} g(\bar{x}, \bar{y}) - \mu = 0,$$  \hspace{1cm} \text{(16)}

$$(\gamma - \beta \bar{y})G''(f(\bar{x}, \bar{y}))\nabla_y f(\bar{x}, \bar{y})(\nabla_y f(\bar{x}, \bar{y}))^T + G'(f(\bar{x}, \bar{y}))\nabla_{yy} g(\bar{x}, \bar{y})$$

$$- \bar{w}(G''(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y})) = 0,$$  \hspace{1cm} \text{(17)}

$$\gamma G'(f(\bar{x}, \bar{y}))\nabla_y f(\bar{x}, \bar{y}) - \bar{w}G'(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}) = 0,$$  \hspace{1cm} \text{(18)}

$$\alpha - \beta G'(g(\bar{x}, \bar{y})) - \bar{y}G'(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y})] - \gamma G'(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y}) = 0,$$  \hspace{1cm} \text{(19)}

$$\mu^T \bar{x} = 0,$$  \hspace{1cm} \text{(20)}

$$(\alpha, \beta, \gamma, \mu) \neq 0, \quad (\alpha, \beta, \gamma, \mu) \geq 0.$$  \hspace{1cm} \text{(21)}

Since

$$\left(G''(f(\bar{x}, \bar{y}))\nabla_y f(\bar{x}, \bar{y})\nabla_y f(\bar{x}, \bar{y})^T + G'(f(\bar{x}, \bar{y}))\nabla_{yy} f(\bar{x}, \bar{y}) - \bar{w}(G''(g(\bar{x}, \bar{y}))\nabla_y g(\bar{x}, \bar{y})$$

$$(\nabla_y g(\bar{x}, \bar{y}))^T + G'(g(\bar{x}, \bar{y}))\nabla_{yy} g(\bar{x}, \bar{y})) \right)$$

is non-singular, it follows from (17) that

$$\gamma = \beta \bar{y}.$$  \hspace{1cm} \text{(22)}

Next, we have to show that $\beta \neq 0$. If possible, then consider $\beta = 0$, then from (22), we find that $\gamma = 0$. From (19), we get $\alpha = 0$, which contradicts (21). This together with (16), we arrive that $\mu = 0$. Hence, $\beta \neq 0 \implies \beta > 0$. Now, (17) and (22) and in particular, by (22), $\beta > 0$ and since $\gamma \geq 0$, hence we find that $\bar{y} \geq 0$.

From inequality (16), we find that

$$G'(f(\bar{x}, \bar{y}))\nabla_x f(\bar{x}, \bar{y}) - \bar{w}G'(g(\bar{x}, \bar{y}))\nabla_x g(\bar{x}, \bar{y}) = \frac{\mu}{\beta} \geq 0.$$  \hspace{1cm} \text{(23)}
Therefore, \((\bar{x}, \bar{y}, \bar{w}) \in S^0\).

Next, we have to prove that the objective values of the problem are same. For this, it is sufficient to show that
\[
\frac{G(f(\bar{x}, \bar{y})) - \bar{x}^T G'(f(\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y})}{G(g(\bar{x}, \bar{y})) - \bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y})} = \frac{G(f(\bar{x}, \bar{y})) - \bar{y}^T G'(f(\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y})}{G(g(\bar{x}, \bar{y})) - \bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y})}
\]

Now, multiplying (23), by \(\bar{y}\), we have
\[
\frac{\bar{x}^T G'(f(\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y})}{\bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y})} = \bar{w}.
\] (24)

Using (22) and (18), we obtain
\[
\frac{\bar{y}^T G'(f(\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y})}{\bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y})} = \bar{w}.
\] (25)

Equations (24) and (25) give
\[
\frac{\bar{x}^T G'(f(\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y})}{\bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y})} = \frac{\bar{y}^T G'(f(\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y})}{\bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y})}
\]
i.e.
\[
(\bar{x}^T G'((\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y}))(\bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y}))
\]
\[
= (\bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y}))(\bar{y}^T G'((\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y})).
\] (26)

By hypothesis \((ii)\), we get
\[
\bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y}) G(f(\bar{x}, \bar{y})) + \bar{y}^T G'((\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y}) G(g(\bar{x}, \bar{y}))
\]
\[
= \bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y}) G(f(\bar{x}, \bar{y})) + \bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y}) G(g(\bar{x}, \bar{y})).
\] (27)

On subtracting (27) from (26) and after this we adding \(G(f(\bar{x}, \bar{y})) G(g(\bar{x}, \bar{y}))\) of both sides, we have
\[
G(f(\bar{x}, \bar{y})) G(g(\bar{x}, \bar{y})) - G(f(\bar{x}, \bar{y})) \bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y}) - \bar{y}^T G'(g(\bar{x}, \bar{y}))
\]
\[
\nabla_y f(\bar{x}, \bar{y}) G(g(\bar{x}, \bar{y})) + \bar{x}^T G'((\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y}) \bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y})
\]
\[
= G(f(\bar{x}, \bar{y})) G(g(\bar{x}, \bar{y})) - \bar{x}^T G'((\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y}) G(g(\bar{x}, \bar{y})) - \bar{y}^T G'(g(\bar{x}, \bar{y}))
\]
\[
\nabla_y g(\bar{x}, \bar{y}) G(f(\bar{x}, \bar{y})) + \bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y}) \bar{y}^T G'((\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y}).
\]

This can be rewritten as:
\[
\frac{G(f(\bar{x}, \bar{y})) - \bar{x}^T G'(f(\bar{x}, \bar{y})) \nabla_x f(\bar{x}, \bar{y})}{G(g(\bar{x}, \bar{y})) - \bar{x}^T G'(g(\bar{x}, \bar{y})) \nabla_x g(\bar{x}, \bar{y})} = \frac{G(f(\bar{x}, \bar{y})) - \bar{y}^T G'(f(\bar{x}, \bar{y})) \nabla_y f(\bar{x}, \bar{y})}{G(g(\bar{x}, \bar{y})) - \bar{y}^T G'(g(\bar{x}, \bar{y})) \nabla_y g(\bar{x}, \bar{y})}.
\]

Under the weak duality theorem, if \((\bar{x}, \bar{y}, \bar{w})\) is not an optimal solution of (EFWD), then there are other \((u, v, W) \in S^0\) such that \(\bar{w} \geq W\). Since, \((\bar{x}, \bar{y}, \bar{w}) \in R^0\), So, we obtain that \(\bar{w} \geq W\), which is a contradiction. Thus, \((\bar{x}, \bar{y}, \bar{w})\) is an optimal solution of (EFWD). Hence, the result.

**Theorem 3.4** (Strict converse duality). Let \(f\) and \(g\) be differentiable functions. Let \((\bar{v}, \bar{w}, \bar{l})\) be an optimal solution of (EFWD). Suppose that

\[\text{(i)}\]

\[\text{(ii)}\]

\[\text{(iii)}\]
\( (i) \quad \left( G''(f(\bar{v}, \bar{w}))\nabla_x f(\bar{v}, \bar{w}) - G'(f(\bar{v}, \bar{w}))\nabla_x f(\bar{v}, \bar{w}) \right) = 0. \)

Then, there exists \((\bar{v}, \bar{w}, \bar{t}) \in \mathbb{R}^n\) and objective values are equal. Moreover, if all the hypotheses of weak duality theorem are satisfied, then \((\bar{v}, \bar{w}, \bar{t})\) is an optimal solution of (EFWP).

**Proof:** Proof of strict converse duality theorem follows on the lines of Theorem 3, due to symmetric programming problem.

**Conclusion**

In this article, we considered fractional dual symmetric programming problem and derived duality theorems under \(G\)-invexity/\(G\)-pseudoinvexity conditions. The present work can be extended to multiobjective symmetric fractional dual programs. This may be taken as the future task of the researchers.

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**References**

[1] B. Mond, A class of nondifferentiable fractional programming problems. *ZAMM.* 1978, 58, 337-341.
[2] J. Zhang and B. Mond, Duality for a class of nondifferentiable fractional programming problems. *Int. J. Mang. Sys.* 1998, 14, 71-88.
[3] D.S. Kim, Y.B. Yun and W.J. Lee, Multiobjective Symmetric duality with cone constraints. *Eur. J. Oper. Res.* 1998, 107, 686-691.
[4] G. Devi, Symmetric duality for nonlinear programming problem involving \(\eta\)-convex functions. *Eur. J. Oper. Res.* 1998, 107, 615-621.
[5] X. Chen, Sufficient conditions and duality for a class of multiobjective fractional programming problems with higher-order \((F, \alpha, \rho, d)\)-convexity. *J. Appl. Math. Comput.* 2008, 28, 107-121.
[6] X. M. Yang, X. Q. Yang, K. L. Teo and S. H. Hou, Multiobjective second order symmetric duality with \(F\)-convexity. *Eur. J. Oper. Res.* 2005, 165, 585-591.
[7] S. Khurana, Symmetric duality in multiobjective programming involving generalized cone-invex functions. *Eur. J. Oper. Res.* 2005, 165, 592-597.
[8] D. B. Ojha, On second-order symmetric duality for a class of multiobjective fractional programming problem. *Tamkang J. Math.* 2012, 43, 267-279.
[9] I. M. Stancu-Minasian, A seventh bibliography of fractional programming. *Adv. Model. Optim.* 2013, 15, 309-386.
[10] G. Ying, Higher-order symmetric duality for a class of multiobjective fractional programming problems. *Adv. Model. Optim.* 2012, 142.
[11] M. Bhatia, Higher order duality in vector optimization over cones. *Optim. Lett.* 2012, 6, 17-30.
[12] R. Dubey, L. N. Mishra and C. Cesarano, Multiobjective fractional symmetric duality in mathematical programming with \((C, G_r)\)-invexity assumptions. *Axioms* 2019, (8) 97; doi:10.3390/axioms8030097.
[13] M. Schechter, More on subgradient duality. *J. Math. Anal. Appl.* 1979, (71), 251-262.