Relaxation of stresses in the elements of reinforced concrete structures

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Abstract. The issues of calculating and forecasting the long-term safety of reinforced concrete structures and structures are related to the dynamics of their stress state and lead to relaxation problems to take into account the redistribution of stress from concrete to reinforcement. In this paper, an approach based on the concept of the strength structure of structural materials is proposed for their solution. According to this concept, in contrast to traditional hypotheses, the material is considered as the combination of its fractions (links, layers, fibers) with statistically distributed strengths. The loading generates the loss of the power resistance capacity by a part of the element fractions which entails a redistribution of stresses on its whole fractions. As a result of this process, a nonlinear dependence of the strain on stress calculated under the assumption of equal strength of all fractions, arises. A significant role in the evolution of stresses in the components of a reinforced concrete element is played by creep of the material. In the linear formulation for a material with a uniform strength, the stresses are determined by solving a linear integral equation coupled with its rheological equation of a mechanical state. Based on the concept of the statistical distribution of the strengths of fractions, a modification of L. Boltzmann's superposition principle, well-known in the linear creep theory, is proposed for its applicability and the non-linear dependence of strain on stress. This allows the derivation of a linear integral equation with respect to the so-called structural stress experienced by the part of the element component capable of force resistance. The desired design stress is determined by solving an algebraic equation relating structural and design stresses. The proposed approach greatly simplifies the establishment of stress estimates in the components of structural elements required in the long-term safety forecast of structures.

1. Introduction

The problems considered in this paper are associated with the phenomenon of creep – an increase in the strain $\sigma(t_0)$ generated by stress $\varepsilon(t)$ at $\tau > t_0$. Over time $\tau$ stresses $\sigma(\tau)$ with unchanged deformations $\varepsilon(t_0)$ decrease and this phenomenon is called stress relaxation.

Stress relaxation is a consequence of the development of creep deformations $\varepsilon_H(t,t_0)$ in the material. With a constant total strain $\varepsilon(t_0) = \varepsilon_M(t) + \varepsilon_H(t,t_0)$, the fraction of the instantaneous elastic-plastic deformation $\varepsilon_M(t)$ decreases and the corresponding stress fraction decreases. Note that in real constructions, the phenomena of creep and relaxation occur simultaneously.

1.1. Linear formulation of the problem

In the linear formulation, the stress $\sigma(t) = \sigma(t_0)$ generates a strain

$$\varepsilon(t) = \frac{\sigma(t_0)}{E(t)} + \sigma(t_0)C(t,t_0),$$  \hspace{1cm} (1)

here $E(t)$ is the elastic modulus of instantaneous deformations; $C(t,t_0)$ – a measure of simple creep at the moment $t$ when loading at the moment $t_0$. 

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According to the superposition principle of L. Boltzmann [1], partial increment of creep strain

\[ \varepsilon_{II}(t, \tau_i) = C(t, \tau_i) \cdot \Delta \sigma(\tau_i) \]  

(2)

is determined only by the magnitude and duration \( \Delta \tau_i = \tau - \tau_i \) of the stress \( \sigma(\tau) \) increment \( \Delta \sigma(\tau_i) \) and does not depend neither on the rest of its partial increments \( \Delta \sigma(\tau_j) \), \( (j \neq i) \), nor on their duration.

The interdependence of increments \( \Delta \varepsilon_{II}(t, \tau_i) \) allows the representation of the increment \( \Delta \varepsilon_{II}(t, t_0) \) of creep strain corresponding to the incremental stress

\[ \Delta \sigma(t) = \sum_{i=1}^{n} \Delta \sigma(\tau_i) \]  

(3)

by the sum

\[ \Delta \varepsilon_{II}(t, t_0) = \sum_{i=1}^{n} C(t, \tau_i) \Delta \sigma(\tau_i). \]  

(4)

Passing in the integral sum to the limit, we have

\[ \Delta \varepsilon_{II}(t, t_0) = \int_{t_0}^{t} C(t, \tau) d\sigma(\tau), \]  

(5)

and then, integrating (5) in parts, we get

\[ \Delta \varepsilon_{II}(t, t_0) = C(t, t) \sigma(t) - C(t, t_0) \sigma(t_0) - \int_{t_0}^{t} \sigma(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau, \]  

(6)

here \( C(t, t) \) – the so-called short-term creep.

The value \( C(t, t) \) is experimentally indefinable and, as in most works on the theory of concrete creep, we assume \( C(t, t) = 0 \). Adding to \( \Delta \varepsilon_{II}(t, t_0) \) the stress-induced \( \sigma(t_0) \) creep strain \( C(t, t_0) \sigma(t_0) \) over the gap \([t, t_0]\), we have

\[ \Delta \varepsilon_{II}(t, t_0) = -\int_{t_0}^{t} \sigma(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau. \]  

(7)

Adding the creep deformation \( \varepsilon_{II}(t, t_0) \) with instantaneous deformation \( \varepsilon_M(t) = \frac{\sigma(t)}{E(t)} \) we arrive at the equality

\[ \varepsilon(t) = \frac{\sigma(t)}{E(t)} - \int_{t_0}^{t} \sigma(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau, \]  

(8)

representing a linear rheological equation of the mechanical state of concrete under prolonged loading of structural elements (beam, column).

According to (8), we obtain the equation

\[ \sigma(t) = \varepsilon(t) E(t) + E(t) \int_{t_0}^{t} \sigma(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau. \]  

(9)

For given deformations \( \varepsilon(t) \) and parameters \( E(t) \) and \( C(t, \tau) \), the stress \( \sigma(t) \) is determined by solving a linear integral equation by known methods, for example, using the iteration method.

For mature and old concrete, it is usually assumed that \( C(t, \tau) = C_0 \left[ 1 - e^{-\gamma (t-\tau)} \right] \), and if aging \( C(t, \tau) = \theta(\tau) \left[ 1 - e^{-\gamma (t-\tau)} \right] \) is taken into account, then equation (9) reduces to a linear differential
equation of the first and second orders, respectively. Here \( \theta(\tau) \) is the aging function, for example [2], \( \theta(\tau) = \frac{A}{\tau} + C_0 \), and \( \gamma \) is a parameter reflecting the measure of the influence of an open surface on the process of concrete formation. Thus, in the linear formulation of the problem of stress relaxation in a concrete element with its axial loading, they are solved rather simply [3].

Consider for illustration a reinforced concrete beam, which is subject to axial load.

\[
N(\tau) = \begin{cases} N_1; \quad 2kT \leq \tau < (2k + 1)T \\ N_2; \quad (2k + 1)T \leq \tau < 2(k + 1)T \end{cases}
\]

Here \( T > 0; \quad N_1 < N_2; \quad k = 0,1,2,.... \)

Efforts \( N_1 \) and \( N_2 \) generate in concrete in moments \( t_1 = 2kT \) and \( t_2 = (2k + 1)T \) stresses equal to

\[
\sigma_1 = \frac{N_1}{A_b(1 + m\mu)}; \quad \sigma_2 = \frac{N_2}{A_b(1 + m\mu)}
\]

respectively; \( m = \frac{E_a}{E_b}; \quad \mu = \frac{A_a}{A_b} \).

The area \( A(\tau) \) of the normal section of the whole \( \sigma(\tau) = \sigma_1 \) with fractions at destruction does not change, and therefore

\[
S^0(\tau) = 1 + V \left[ \frac{\sigma_1}{R(\tau)} \right].
\]

For mature and old concrete it is believed \( C(t, \tau) = C_0 \left[ 1 - e^{-\gamma(t-\tau)} \right] \), and the strength \( R(\tau) \) is equal to its long-term strength \( R \).

At each time interval we have a linear integral equation

\[
\sigma_e(\tau) = \sigma_{ic} + \lambda \int_{t_i}^{\tau} \sigma_{ic}e^{-\gamma(t-\tau)} d\tau, \quad i = 1,2.
\]

\[
\lambda = \frac{mE_aC_0}{1 + m\mu}; \quad t_1 = 2kT; \quad t_2 = (2k + 1)T; \quad k = 0,1,2,....
\]

reduced to a differential equation

\[
\frac{d\sigma_{bc}(t)}{dt} + \delta\sigma_{bc}(t) = \sigma_i; \quad \delta = \gamma + \lambda; \quad i = 1,2.
\]

By solving this equation with the initial condition \( \sigma(0) = \sigma_i; \quad i = 1,2 \) is the function

\[
\sigma^*_{bc}(t) = \sigma_{ic} \left[ \frac{\gamma}{\delta} + \frac{\lambda e^{-\delta(t-nT)}}{\delta} \right].
\]

\[
\sigma_i = \begin{cases} \sigma_1; \quad n = 2k \\ \sigma_2; \quad n = 2k + 1 \end{cases}
\]

The calculated stress in the concrete component is determined from the equation

\[
1 + V \left[ \frac{\sigma_{bc}(t)}{R} \right]^m \left[ \frac{\gamma}{\delta} + \frac{\lambda e^{-\delta(t-nT)}}{\delta} \right] = \sigma^*_{bc}(t).
\]

According to (18), the stress in the reinforcement is estimated at \( t \to \infty \).

1.2. The relaxation of stresses
In a reinforced concrete element, the reduction of stresses \( \sigma_b(\tau) \) in the concrete component is accompanied by an increase in the stresses \( \sigma_s(\tau) \) in the reinforcement.

It should be noted that over time, the increase \( \sigma_s(\tau) \) is significant and can lead to rupture of reinforcement and, as a result, to serious consequences in critical structures (in reinforced concrete shells of reactors, conduits). In this regard, in the long-term forecast of the safety of structures and structures, the problem arises of estimating the maximum possible stresses in the reinforcement.

Consider, for example, the relaxation of stresses in a single-reinforced reinforced concrete bar that is bent by the moment \( M(t) \). At a distance \( h_a \) from the neutral axis \( Ox \) we have

\[
\sigma_a(t) = \frac{1}{\mu n_0} \left[ \frac{M(t)h_a}{J_b} - \sigma_b(t, h_a) \right],
\]

here \( \mu = \frac{A_a}{A_b} \); \( A_a \) and \( A_b \) – the area of normal sections of reinforcement and the concrete part of the timber; \( n_0 = \frac{J}{J_b} \), where \( J \) and \( J_b \) are the moments of inertia of the concrete component and the reduced normal section about the axis \( Ox \); \( \sigma_b(t, h_a) \) – normal stress in the concrete layer in contact with the reinforcement.

According to the condition of compatibility deformation at \( h_a \)

\[
\sigma_b(t, h_a) = \bar{\sigma}_b(t, h_a) + \lambda(t) \int_0^t \sigma_b(\tau, h_a) \frac{\partial C(t, \tau)}{\partial \tau} d\tau,
\]

where \( \bar{\sigma}_b(t, h_a) = \frac{M(t)h_a}{J_b(\mu n_0 m(t) + 1)} \) is the instantaneous elastic stress; \( \lambda(t) = \frac{M(t)E_a(t)}{\mu n_0 m(t) + 1} \);

\( m(t) = \frac{E_b(t)}{E_a(t)} \); \( E_a(t) \) – modulus of elastic deformations of reinforcement.

The integral equation is solved by simple iterations with a zero approximation \( \sigma_{b_0}(t, h_a) = \bar{\sigma}_b(t, h_a) \).

In the calculations, it is assumed that \( E_b(t) = E_b \), \( E_a(t) = E_a \), \( C_b(t, \tau) = C_0_b \left[ 1 - e^{-\gamma(t-\tau)} \right] \) and equation (11) is reduced to a differential equation

\[
\frac{d\sigma_b(t, h_a)}{dt} + (\lambda + \gamma)\sigma_b(t, h_a) = \gamma \bar{\sigma}_b(t, h_a) + \frac{d\hat{\sigma}_b(t, h_a)}{dt},
\]

and at a constant bending moment \( M \) to the equation

\[
\frac{d\sigma_b(t, h_a)}{dt} + (\lambda + \gamma)\sigma_b(t, h_a) = \gamma \bar{\sigma}_b(t, h_a).
\]

According to the solutions \( \sigma_b(0, h_a) = \bar{\sigma}_b(0, h_a) \) of these equations with the initial condition, the creep of concrete leads to the relaxation of its stresses and, for sufficiently large ones \( t \), are respectively \( \sigma_b(t, h_a) \)

\[
\sigma_b(t, h_a) = \frac{\gamma \bar{\sigma}_b(t, h_a) + \gamma \hat{\sigma}_b(t, h_a)}{\lambda + \gamma},
\]

\[
\sigma_b(\infty, h_a) = \frac{\gamma}{\lambda + \gamma} \bar{\sigma}_b(0, h_a).
\]
As a result of a prolonged redistribution of stresses $\hat{\sigma}_b(0, h_a)$ from concrete to reinforcement, its initial stress $\sigma_a(0) = \frac{M h_a n_0 m}{J(\mu n_0 m + 1)}$ increases to

$$\sigma_a(\infty) = \frac{1}{\mu n_0} \left[ \frac{M h_a}{J_0} \frac{1}{\gamma + \lambda} \hat{\sigma}_b(0, h_a) \right].$$

(25)

2. Method of solution for nonlinear formulation of the problem

The increasing load on the structural element entails structural changes, giving rise to a progressive process of discontinuity of its material [4, 5]. A material (concrete, steel, wood, plastic) is considered as a combination of its fractions (links, fibers, layers) with statistically distributed strengths $\tau_i$. The concept of statistical variation of the quantity $\tau_i$, which goes back to [6], was developed in [7]. This concept serves as the basis for modifying L. Boltzmann’s superposition principle for its applicability in a nonlinear formulation.

A force $N(\tau)$ increasing to a normal cross section results in the destruction of a part of the fractions, reducing the cross-sectional area $A$ to the area $A(\tau)$ formed by the whole fractions at the moment $\tau$. Magnitude

$$\sigma_c(\tau) = \frac{N(\tau)}{A(\tau)},$$

(26)

associated with structural damage to the material is structural, and the value

$$\sigma(\tau) = \frac{N(\tau)}{A}$$

(27)

calculated normal stress in the element. According to (17) and (18)

$$\sigma_c(\tau) = \frac{A}{A(\tau)} \sigma(\tau),$$

(28)

and the function $s^0(\tau) = \frac{A}{A(\tau)}$ describes the process of destruction of fractions during force deformation. This process is accompanied by a redistribution of loading $N(\tau)$ on the area $A(\tau)$, because only fractions that are whole at this point have a resistance.

In contrast to the linear formulation, which means equal strength of all links, interdependent increments of creep deformations do not correspond to stepwise increment $\Delta \sigma = \sum_{i=1}^{n} \Delta \sigma(t_i)$ – the action $\Delta \sigma(t_i)$ at the moment $\tau = \tau_j$ is enhanced by the action $\Delta \sigma(t_j)$ that destroys part of the whole fractions by this time [8].

Let us mentally select in the concrete (steel) component of an element $V$ its part $V_t$ consisting of integers in between $[0, t]$ fractions. The stress $V_t$ in the current $\tau \in [0, t]$ coincides with the stress in the part $V_\tau$ of the element consisting of integers at the moment $\tau$ of fractions. Thus, the stress in the part $V_t$ is a structural stress $\sigma_c(\tau)$. Since the fractions $V_t$ are equally strong until the partial $\tau = t$ increments $\Delta \varepsilon_{II}(t, t_i) = C(t, t_i) \cdot \Delta \sigma_c(t_i)$ of creep deformations are independent of each other and the stress $\Delta \sigma_c(t) = \sum_{i=1}^{n} \Delta \sigma_c(t_i)$ generates an increment.
\[ \Delta \varepsilon_{II}(t,t) = \sum_{i=1}^{n} C(t, \tau_i) \Delta \sigma_c(\tau) \]  
(29)

creep strain. This allows the repetition of standard operations to derive the rheological equation of the mechanical state in a nonlinear formulation [9], [10].

\[ \varepsilon(t) = \frac{\sigma_c(t)}{E(t)} - \int_{t_0}^{t} \frac{\varepsilon_c(\tau)}{\sigma_c(\tau)} \frac{\partial C(t, \tau)}{\partial \tau} d\tau. \]  
(30)

Since \( \sigma_c(\tau) = S^0(\tau) \), then according to (30)

\[ \varepsilon(t) = \frac{S^0(t) \sigma_c(t)}{E(t)} - \int_{t_0}^{t} S^0(\tau) \sigma_c(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau. \]  
(31)

3. Results

Equality (29) represents a modification of the L. Boltzmann principle of superposition of creep deformations.

In [11], the equation is obtained in a nonlinear formulation

\[ \varepsilon(t) = \frac{S_M}{E(t)} - \int_{t_0}^{t} \frac{S_M}{E(t)} \sigma(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau, \]  
(32)

where \( S_M \) and \( S_H \) are nonlinear functions of stresses, respectively, of instantaneous deformations and creep deformations. Since in the physical aspect both types of deformations are generated by a single force factor, these functions coincide and represent nothing more than structural stress \( \sigma_c(t) \) and \( \sigma_c(\tau) \).

According to \( \sigma_c(\tau) = S^0(\tau) \sigma(\tau) \) a single function of stress is represented as \( S[\sigma(\tau)] = S^0(\tau) \sigma(\tau) \). Non-linear function \( S^0(\tau) \) in applications is given by the equality

\[ S^0(\tau) = 1 + V \left[ \frac{\sigma(\tau)}{R(\tau)} \right]^m, \]  
(33)

where \( V \) and \( m \) – empirical parameters; for concrete it is commonly believed \( m = 4 \).

The function \( S^0(\tau) \) describing the process of loss of power resistance ability by fractions of \( R_i < \sigma(\tau) \) depends on the stress level \( \eta(\tau) = \frac{\sigma(\tau)}{R(\tau)} \).

Similar to equation (8) and equation (30) is reduced to a linear integral equation already with respect to structural stress. After determining its solution \( \sigma^*(t) \), the calculated stress \( \sigma^*(t) \) is found by solving the equation

\[ S^0(\tau) \sigma(\tau) = \sigma^*(\tau). \]  
(34)

In applications [12], according to (33) and (34), the quantity \( \sigma^*(t) \) is the root of the equation

\[ [\sigma(t)]^{m+1} + \frac{[R(t)]^m}{V} \sigma(t) - \frac{[R(t)]^m}{V} \sigma^*(t) = 0. \]  
(35)
Along with (33), the nonlinear function of stresses $\sigma_c(\tau) = a[\sigma(\tau)]^b$ is used [13], where $a$ and $b$ and are empirical parameters. In this case $a[\sigma^*(t)]^b = \sigma_c^*(t)$ and $\sigma_c^*(t) = \left[\frac{\sigma_c(t)}{a}\right]^\frac{1}{b}$.

3.1. Calculation of stresses
The increasing normal force $N(\tau)$ reduces the area $A_b(\tau)$ and $A_a(\tau)$ the normal sections of the components of the element that are capable of force resistance and increases the calculated stresses $\sigma_b(\tau)$ and $\sigma_a(\tau)$ to structural stresses $\sigma_{b_c}(\tau)$ and $\sigma_{a_c}(\tau)$.

Equations (19) - (22) and estimates (23) - (25) also hold for structural stresses $\sigma_{b_c}(\tau)$ and $\sigma_{a_c}(\tau)$, and estimates for the maximum values of design stresses are determined according to (34) and (35).

4. Different approach to the problem
In [19] the approach to the problem is described according to the equation

$$e_b(t) = \frac{\sigma_b(t)}{E_b(t)} + \int_{t_0}^{t} \sigma_b(\tau) \frac{\partial C_p(t, \tau)}{\partial \tau} d\tau + \int_{\sigma(t_0)}^{\max \sigma} f(\sigma(t)) F[T(\sigma, t)] d\sigma,$$

here $f(\sigma(t)) = \xi[\sigma(t)]^n$; $\xi$ - small parameter; $u$ - nonlinearity parameter; $F[T(\sigma, t)] = F_0 \left[1 - e^{-\varphi T(\sigma, t)}\right]$; $T(\sigma, t)$ - total duration of the stress $\sigma$; $F_0$ and $\varphi$ - empirical parameters. Stress $\sigma_b(t)$ is sought in the form

$$\sigma_{b_c}(t) = \sigma_0(t) + \xi \sigma_1(t) + \xi^2 \sigma_2(t) + \ldots$$

Poincaré’s small parameter method involving the Laplace transform. It should be noted that in this case, the procedure for finding stresses $\sigma_{b_c}(t)$ and $\sigma_{a_c}(t)$ is difficult and busy.

Some other approaches to this problem are given in the following papers [15 – 17].

5. Conclusion
1. Estimates of the maximum possible (over a long time) stress values in the reinforcement are important in the long-term forecast because its rupture causes serious consequences.
2. The proposed approach based on the strength structure of structural materials (for example, the distribution of the strengths $R(\tau)$ of element fractions according to the normal law) is much simpler than the known methods for estimating stresses in the components of elements.

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