Another look on tense and related operators

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Abstract. Motivated by the classical work of Halmos on functional monadic Boolean algebras, we derive three basic sup-semilattice constructions, among other things, the so-called powersets and powerset operators. Such constructions are extremely useful and can be found in almost all branches of modern mathematics, including algebra, logic, and topology. Our three constructions give rise to four covariant and two contravariant functors and constitute three adjoint situations we illustrate in simple examples.

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1. Introduction

The paper considers certain sup-semilattice constructions, which encompass a range of known constructions, among other things, the so-called powersets and powerset operators. Such constructions are extremely useful and can be found in almost all branches of modern mathematics, including algebra, logic, and topology.

Apart from classical sources of applications for powersets and powerset operators coming, e.g., from representation theorems for distributive lattices, Boolean algebras, and Boolean algebras with operators (see [15,16], [11], and [7]), we have to mention the work of Zadeh [18] who defined $I^X$ as a new powerset object instead of $P(X)$ (here $I = [0,1]$ is the real unit interval and

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X a set) and introduced, for every map $f : X \rightarrow Y$ between sets $X$ and $Y$, new powerset operators $f^\leftarrow : I^Y \rightarrow I^X$ and $f^\rightarrow : I^X \rightarrow I^Y$, such that for $a \in I^X, b \in I^Y, y \in Y$,

$$f^\leftarrow (a)(y) = \bigvee \{a(x) \mid f(x) = y, x \in X\} \quad \text{and} \quad f^\rightarrow (b) = b \circ f.$$  

This approach was expanded and investigated from different angles of view and further generalizations followed, e.g., [4, 5, 10, 17]. For illustrative examples of possible applications, see, e.g., the introductory part of the paper [14]. In this work, we concentrate on the representation and approximation of sup-semilattices with unary operations motivated by the work of Halmos on functional monadic Boolean algebras [7].

First, we derive three basic constructions. Namely, we construct

(i) a sup-semilattice $L^J$ with a unary operation $F$ (called shortly an $F$-sup-semilattice) from a sup-semilattice $L$ and a relation $J$ (called a frame),

(ii) a sup-semilattice $J \otimes H$ from a frame $J$ and an $F$-sup-semilattice $H$, and

(iii) a frame $J[H, L]$ from an $F$-sup-semilattice $H$ and a sup-semilattice $L$.

Second, these constructions give rise to four covariant and two contravariant functors. In other words, let $S, F\rightarrow S_\leq$, and $J$ denote the categories of sup-semilattices, $F$-sup-semilattices, and frames, respectively. We show that, for arbitrary sup-semilattice $L$, frame $J$ and $F$-sup-semilattice $H$,

(1) $-^J : S \rightarrow F\rightarrow S_\leq$, $- \otimes H : J \rightarrow S, J \otimes - : F\rightarrow S_\leq \rightarrow S, J[H, -] : S \rightarrow J$ are covariant functors, and

(2) $L^- : J \rightarrow F\rightarrow S_\leq$ and $J[-, L_\downarrow] : F\rightarrow S_\leq \rightarrow J$ are contravariant functors.

Finally, we obtain three adjoint situations

(I) $(\eta, \varepsilon) : (J \otimes -) \dashv (-)^J : S \rightarrow F\rightarrow S_\leq$,

(II) $(\varphi, \psi) : (- \otimes H) \dashv J[H, -] : S \rightarrow J$, and

(III) $(\nu, \mu) : J[-, L_\downarrow] \dashv L^- : J \rightarrow F\rightarrow S_\leq^{\text{op}}$.

This new approach presented inherits the approach by Halmos. Namely, let’s fix a sup-semilattice $L$ such that we can guarantee that $\mu_H$ will be an embedding of $F$-sup-semilattices for all $H \in F\rightarrow S_\leq$. We obtain a variant of the classical statement that every monadic Boolean algebra is isomorphic to a functional monadic Boolean algebra [6, Theorem 12].

The paper is structured as follows. After this introduction, in Section 2, we provide some notions and notations that will be used in the paper. We also describe in detail the factorization process in $S$ and $F\rightarrow S_\leq$. In Section 3, we provide the background on our three constructions $(-)(-): S \times J \rightarrow F\rightarrow S_\leq$, $(-) \otimes (-): J \times F\rightarrow S_\leq \rightarrow S$, and $J([-), (-)]: F\rightarrow S_\leq \times S \rightarrow J$. We present the induced adjoint situations $(\eta, \varepsilon), (\varphi, \psi)$, and $(\nu, \mu)$ in Section 4. To illustrate these adjoint situations, we give in Section 5 three examples. We then conclude in Section 6.

For notions and concepts concerned but not explained, please refer to [1], [9], and [19]. The reader should be aware that although we tried to make the paper as self-contained as possible, the lack of space still compelled us to leave some details for his/her own perusal.
2. Preliminaries

In this section we first recall several known but valuable concepts.

2.1. Notation

We use the category-theoretic notation for the composition of maps, that is, for maps $f: A \to B$ and $g: B \to C$, we denote their composition by $g \circ f: A \to C$, so that $(g \circ f)(a) = g(f(a))$ for all $a \in A$. We denote the set of all maps from the $A$ to $B$ by the usual $B^A$. For a map $f: A \to B$ and a set $I$, we write $f^I: A^I \to B^I$ for the map defined by $f^I(x)(i) = f(x(i))$.

As usual, an order-preserving mapping on a poset is called an operator. We say that it is a preclosure operator if it is order-preserving. A preclosure operator is called a closure operator if it is idempotent.

A poset $(A, \leq_A)$ equipped with an operator $F_A$ on $A$ is abbreviated by a posos, denoted by $(A, \leq_A, F_A)$. Note that a posos is an example of an ordered algebra [19].

A homomorphism of posos $A = (A, \leq_A, F_A)$ and $B = (B, \leq_B, F_B)$ is an order-preserving mapping $f: A \to B$ which satisfies $F_B(f(a)) = f(F_A(a))$ for any $a \in A$. We denote by PwoSCO the category of posos with homomorphisms.

A monotone mapping $f: A \to B$ is called a lax morphism if

$$F_B(f(a)) \leq f(F_A(a))$$

for any $a \in A$. The category of posos with lax morphisms as morphisms is denoted by PwoSCO$_\leq$. Clearly, every homomorphism is a lax morphism, so PwoSCO is a subcategory of PwoSCO$_\leq$ which is not necessarily full.

Recall that an order-embedding from a poset $(A, \leq_A)$ to a poset $(B, \leq_B)$ is a mapping $h: A \to B$ such that $a \leq_A a'$ iff $h(a) \leq_B h(a')$, for all $a, a' \in A$. Every order embedding is necessarily an injective mapping. We denote by $\mathcal{E}$ the class of order-embeddings that are homomorphisms. We also denote by $\mathcal{E}_{\leq}$ the class of lax morphisms $h: (A, \leq_A, F_A) \to (B, \leq_B, F_B)$ of posos that are order-embeddings and satisfy the following condition: $\forall a, a' \in A$,

$$F_B(h(a)) \leq h(a') \implies F_A(a) \leq a'.$$  \hfill (2.1)

Evidently, $\mathcal{E}_{\leq}$ is closed under the composition of lax morphisms and $\mathcal{E} \subseteq \mathcal{E}_{\leq}$.

Definition 2.1. We denote by $\mathcal{S}$ a category where the objects are sup-semilattices $G = (G, \lor)$ (posets which have all joins) and morphisms are mappings between them preserving arbitrary joins.

$\mathcal{S}$ is given by an algebraic theory, so it is an equationally presented category. It is a monadic category (over the category of sets) because it has free objects. $\mathcal{S}$ is also a monoidal category, even a star-autonomous category. The dual of a sup-semilattice is its opposite poset, which is also a sup-semilattice since every sup-semilattice also has arbitrary meets. $\mathcal{S}$ is both a complete and cocomplete category. The categorical product $\prod_{i \in I} G_i$ coincides with both the cartesian product and the categorical sum. For a detailed account on $\mathcal{S}$ see [8].
Definition 2.2. We say that a pair \((G, F)\) is an \(F\)-sup-semilattice if \(G\) is a sup-semilattice and \(F\) is a map \(F : G \to G\) satisfying \(F(\bigvee X) = \bigvee F(X)\) for any \(X \subseteq G\). A homomorphism of \(F\)-semilattices is a morphism \(f : G_1 \to G_2\) in \(\mathcal{S}\) which satisfies \(F_2(f(x)) = f(F_1(x))\) for any \(x \in G_1\).

Note that any \(F\)-sup-semilattice is a \(\text{pwos}\), and we may identify \((G, \text{id}_G)\) with \(G\) for every sup-semilattice \(G\). A lax morphism of \(F\)-sup-semilattices is a morphism \(f : G_1 \to G_2\) in \(\mathcal{S}\) which is also a lax morphism in \(\text{PwoSCO}_\leq\).

It is transparent that the class of all \(F\)-sup-semilattices and all homomorphisms between them forms a category. We denote this category as \(F-\mathcal{S}\). Similarly, we represent by \(F-\mathcal{S}_{\leq}\) the category of \(F\)-sup-semilattices and lax morphisms between them. Then \(F-\mathcal{S}\) is a subcategory of \(F-\mathcal{S}_{\leq}\), \(F-\mathcal{S}_{\leq}\) is a subcategory of \(\text{PwoSCO}_\leq\), and \(F-\mathcal{S}\) is a subcategory of \(\text{PwoSCO}\).

Recall also that \(F\)-sup-semilattices are sup-algebras (see [19]).

Remark 2.3. For any sup-semilattice \(G\), we will denote by \(\mathcal{Q}(G)\) the sup-semilattice of all sup-semilattice endomorphisms \(F : G \to G\) (with the pointwise ordering \(F_1 \leq F_2\) iff \(F_1(x) \leq F_2(x)\) for all \(x \in G\)). Hence, there is a one-to-one correspondence (for a fixed \(G\)) between elements of \(\mathcal{Q}(G)\) and \(F\)-sup-semilattices over \(G\).

The sup-semilattice \(\mathcal{Q}(G)\) can be described through a tensor product of sup-semilattices

\[\mathcal{Q}(G) \cong (G \otimes G^{\text{op}})^{\text{op}}\]

(see [8]). This isomorphism is given by

\[F \mapsto \bigvee_{F(t) \leq s} t \otimes s.\]

Similarly, as for sup-semilattices, we can introduce the category \(\mathcal{M}\) of inf-semilattices and infimum preserving mappings between them, the category \(F-\mathcal{M}\) of \(F\)-inf-semilattices and all homomorphisms between them, and the category \(F-\mathcal{M}_{\leq}\) of \(F\)-inf-semilattices and all lax morphisms between them, respectively. Then \(\mathcal{S}^{\text{op}}\) and \(\mathcal{M}, F-\mathcal{S}^{\text{op}}\) and \(F-\mathcal{M}\), and \(F-\mathcal{S}_{\leq}^{\text{op}}\) and \(F-\mathcal{M}_{\leq}\) are isomorphic, respectively. The isomorphism is given by the identity on objects (since every sup-semilattice has arbitrary meets) and prescribing to a sup-preserving mapping its right adjoint on morphisms and operators, respectively.

2.2. Quotients in \(\mathcal{S}\) and \(F-\mathcal{S}_{\leq}\)

Definition 2.4. Let \(G\) be a sup-semilattice. A congruence on the sup-semilattice \(G\) is an equivalence relation \(\theta\) on \(G\) satisfying \(\{(x_i, y_i) \mid i \in I\} \subseteq \theta\) implies \((\bigvee_{i \in I} x_i)\theta(\bigvee_{i \in I} y_i)\). Let us denote the set of all congruences on \(G\) as \(\text{Con}(G)\).

If \(G\) is a sup-semilattice and \(\theta\) a sup-semilattice congruence on \(G\), the factor set \(G/\theta\) is a sup-semilattice again, and the projection \(\pi : G \to G/\theta\) is, therefore, a sup-semilattice morphism. Recall that if \(\theta_i \in \text{Con}(G)\) for all \(i \in I\), then also \(\bigcap_{i \in I} \theta_i \in \text{Con}(G)\).

An \(F\)-congruence on the \(F\)-sup-semilattice \((G, F)\) is a congruence \(\theta\) on \(G\) satisfying \(a \theta b\) implies \(F(a) \theta F(b)\) for all \(a, b \in G\). Note that if \(F = \text{id}_G\), then any congruence on \(G\) is an \(F\)-congruence on \((G, \text{id}_G)\).
For a pwos $\mathcal{A} = (A, \leq_A, F_A)$, a preclosure operator $j$ on $A$ is called a prenucleus if $j$ is a lax morphism (here, one has to recall Banaschewski’s [2] theory of prenuclei). We put

$$A_j = \{a \in A \mid j(a) = a\}, \leq_A = \leq_A \cap (A \times A_j) \text{ and } F_{A_j} = j \circ F_A.$$ 

A prenucleus is said to be a nucleus if it is a closure operator.

**Lemma 2.5.** Let $\mathcal{A} = (A, \leq_A, F_A)$ be a pwos and $j$ a nucleus on $\mathcal{A}$. Then $A_j = (A_j, \leq_A, F_{A_j})$ is a pwos, the inclusion map $i_{A_j}: A_j \to A$ is a lax morphism such that $i_{A_j} \in E_{\leq}$, and the surjection $j: A \to A_j$ is a homomorphism of pwos. Moreover,

1. if $j: A \to A$ is a homomorphism of pwos, then $i_{A_j}: A_j \to A$ is a homomorphism of pwos,
2. if $\mathcal{A}$ is a $F$-sup-semilattice then $A_j$ is a $F$-sup-semilattice.

**Proof.** Clearly, $(A_j, \leq_A)$ is a poset, and $F_{A_j}$ is order-preserving.

We now verify that $i_{A_j}: A_j \to A$ is a lax morphism. Let $a = j(a) \in A_j$. Then

$$F_A(i_{A_j}(a)) = F_A(j(a)) \leq j(F_A(a)) = i_{A_j}(F_{A_j}(a)).$$

Clearly, $i_{A_j}$ is an order-embedding. Now, let $a, a' \in A_j \subseteq A$ and assume that $F_A(a) = F_A(i_{A_j}(a)) \leq i_{A_j}(a') = a'$. We compute:

$$F_{A_j}(a) = j(F_A(a)) \leq j(a') = a'.$$

Let us show that $j: A \to A_j$ is a homomorphism of pwos. Let $a \in A$. Then

$$j(F_A(a)) \leq j(F_A(j(a))) = F_{A_j}(j(a)) \leq j(j(F_A(a))) = j(F_A(a)).$$

Suppose that $j: A \to A$ is a homomorphism of pwos and let $a = j(a) \in A_j$. Then

$$i_{A_j}(F_{A_j}(a)) = j(F_A(a)) = F_A(j(a)) = F_A(i_{A_j}(a)).$$

Assume that $\mathcal{A}$ is a $F$-sup-semilattice. Then $(A_j, \leq_A)$ is a sup-semilattice such that $\bigvee_{A_j} = j \circ \bigvee_{\mathcal{A}}$. Let $M \subseteq A_j$. Then

$$F_{A_j}\left(\bigvee_{A_j} M\right) = \left(j \circ F_A \circ j \circ \bigvee_{\mathcal{A}}\right)(M) \leq \left(j \circ j \circ F_A \circ \bigvee_{\mathcal{A}}\right)(M)$$

$$= \left(j \circ F_A \circ \bigvee_{\mathcal{A}}\right)(M) = j\left(\bigvee_{\mathcal{A}}\{F_A(m) \mid m \in M\}\right)$$

$$\leq j\left(\bigvee_{A_j}\{j(F_A(m)) \mid m \in M\}\right)$$

$$= \bigvee_{A_j}\{F_{A_j}(m) \mid m \in M\} \leq F_{A_j}\left(\bigvee_{A_j} M\right).$$
Similarly, as for sup-semilattices, there is a one-to-one correspondence between nuclei and $F$-congruences on $F$-sup-semilattices, given by

\[ j \mapsto \theta_j; \text{ here } \theta_j = \{(a, b) \in G \times G \mid j(a) = j(b)\}, \]

\[ \theta \mapsto j_\theta; \text{ here } j_\theta(x) = \bigvee \{y \in G \mid x \theta y\}. \]

We omit the straightforward verification of the above fact.

**Lemma 2.6.** Let $(G, F)$ be $F$-sup-semilattice and $j$ a prenucleus on $(G, F)$. Then the poset $(G_j, \leq_{G_j})$ is a closure system on $G$, and the associated closure operator $n(j)$ is given by

\[ n(j)(a) = \bigwedge \{x \in G_j \mid a \leq x\}. \]

Moreover, $n(j)$ is a nucleus on $(G, F)$ and $(G, n(j), \leq_{G_n(j)}) = (G_j, \leq_{G_j})$.

**Proof.** Since $j$ is order-preserving and extensive, we easily obtain that the pair $(G_j, \leq_{G_j})$ is a closure system, and $n(j)$ is a closure operator. It remains to verify that $n(j)$ is a lax morphism. Let $a \in G$. We put

\[ E = \{x \in G \mid a \leq x \leq n(j)(a), F(x) \leq n(j)(F(a))\}. \]

Then $a \in E$, $x \in E$ implies $F(j(x)) \leq j(F(x)) \leq j(n(j)(F(a))) = n(j)(F(a))$ and hence $j(x) \in E$. Also, for any non-empty $X \subseteq E$, we have

\[ F\left(\bigvee X\right) = \bigvee \{F(x) \mid x \in X\} \leq n(j)(F(a)), \]

i.e., $\bigvee X \in E$. Hence $t = \bigvee E \in E$ is the largest element of $E$. Then $j(t) \leq t \leq j(t)$ and $t \in G_j$. Therefore $a \leq t \leq n(j)(a)$ implies $t = n(j)(a) \in E$. This fact says that $F(n(j)(a)) \leq n(j)(F(a))$. The remaining part is evident. \qed

Since $(G, n(j), \leq_{G_n(j)}) = (G_j, \leq_{G_j})$ we will use, in what follows, the shorter version $(G_j, \leq_{G_j})$ for the quotient $(G, n(j), \leq_{G_n(j)})$.

We will need the following.

Let $(G, F)$ be $F$-sup-semilattice, $a \in G$ and $X \subseteq G^2$. We put

\[ X^{-1} = \{(d, c) \mid (c, d) \in X\} \quad \text{and} \]

\[ j[X](a) = a \vee \bigvee \{c \in G \mid d \leq a, (c, d) \in X \cup X^{-1}\}. \]

**Lemma 2.7.** Let $(G, F)$ be $F$-sup-semilattice and $X \subseteq G^2$ such that $(F \times F)(X) \subseteq X$. Then the mapping $j[X]: G \rightarrow G$ is a prenucleus on $(G, F)$. Moreover, for every $F$-sup-semilattice $(H, F_H)$ and every lax morphism of $F$-sup-semilattices $g: G \rightarrow H$ such that $g(c) = g(d)$ for all $(c, d) \in X$ there is a unique lax morphism of $F$-sup-semilattices $\overline{g}: G_{j[X]} \rightarrow H$ such that $g = \overline{g} \circ n(j[X])$. 
Proof. Evidently, \( j[X] \) is order-preserving and extensive. Let us show that it is a lax morphism. Let \( a \in G \). We compute:

\[
F(j[X](a)) = F(a) \lor \bigvee \{F(c) \in G \mid d \leq a, (c,d) \in X \cup X^{-1} \}
\]

\[
\leq F(a) \lor \bigvee \{F(c) \in G \mid F(d) \leq F(a), (c,d) \in X \cup X^{-1} \}
\]

\[
\leq F(a) \lor \bigvee \{u \in G \mid v \leq F(a), (u,v) \in X \cup X^{-1} \}
\]

\[
= j[X](F(a)).
\]

Now, let us put

\[
E = \{x \in G \mid g(x) \leq g(a), a \leq x \}
\]

Then \( a \in E \). Let \( x \in E \). Then \( a \leq x \leq j[X](x) \) and \( g(x) \leq g(a) \). We have

\[
g(j[X](x)) = g(x) \lor \bigvee \{c \in G \mid d \leq x, (c,d) \in X \cup X^{-1} \}
\]

\[
= g(x) \lor \bigvee \{g(c) \mid d \leq x, (c,d) \in X \cup X^{-1} \}
\]

\[
\leq g(x) \lor \bigvee \{g(d) \mid g(d) \leq g(x), d \in G \} \leq g(x) \leq g(a).
\]

Hence \( j[X](x) \in E \). For any non-empty \( Y \subseteq E \), we have

\[
g \left( \bigvee Y \right) = \bigvee \{g(y) \mid y \in Y \} \leq g(a),
\]

i.e., \( \bigvee Y \in E \). Hence \( t = \bigvee E \in E \) is the largest element of \( E \). Then \( j[X](t) \leq t \leq j[X](t) \) and \( t \in G_{j[X]} \). Therefore \( a \leq n(j)(a) \leq t \) implies \( g(a) \leq g(n(j)(a)) \leq g(t) \leq g(a) \). This says that \( \bar{g} \) is correctly defined and \( g = \bar{g} \circ n(j[X]) \). Let us show that \( \bar{g} \) is a lax morphism of \( F \)-sup-semilattices. Clearly, \( \bar{g} \) is order-preserving. Take \( Y \subseteq G_{j[X]} \) and \( a \in G_{j[X]} \). We compute:

\[
\bar{g} \left( \bigvee_{G_{j[X]}} Y \right) = g \left( \bigvee Y \right) = \bigvee \{g(y) \mid y \in Y \} = \bigvee \{\bar{g}(y) \mid y \in Y \}
\]

and

\[
F_H(\bar{g}(a)) = F_H(g(a)) \leq g(F(a)) = g(F(n(j[X])(a)))
\]

\[
\leq g(n(j[X])(F(a))) = \bar{g}(F(n(j[X])(a))).
\]

\[\square\]

3. Basic constructions

Definition 3.1. A frame \( J \) is a pair \((T,S)\), where \( T \) is a set and \( S \) is a binary relation on \( T \), i.e., \( S \subseteq T^2 \). A homomorphism \( f \) between frames \( J_1 \) and \( J_2 \) is a map \( f : T_1 \rightarrow T_2 \) such that whenever the pair \((i,j) \in S_1\), then \((f(i),f(j)) \in S_2\).

Frames with frame homomorphisms obviously form a category, which we denote as \( \mathcal{J} \).
Note that one of our motivations for this work comes from the modal-logic approach introduced by Arthur Prior in the 1950’s (see [12]). Prior’s tense logic was subsequently developed by logicians and computer scientists. The relation $S$ is considered as a time preference, i.e., $x S y$ expresses “$x$ is before $y$” and ”$y$ is after $x$”.

**Definition 3.2.** Let $L = (L, \lor)$ be a sup-semilattice and $J = (T, S)$ a frame. Let us define an $F$-sup-semilattice $L_J$ as $L_J = (L_T, F_J)$, where

$$(F_J(x))(i) = \bigvee \{ x(k) | (i, k) \in S \}$$

for all $x \in L_T$. $F_J$ will be called an operator on $L_T$ constructed by means of the frame $J$.

It is evident that Definition 3.2 is correct. Namely, $F_J \in Q(L_T)$ since

$$\left( F_J \left( \bigvee X \right) \right)(i) = \bigvee \{ \left( \bigvee X \right)(k) | (i, k) \in S \} = \bigvee \{ \pi(k) | (i, k) \in S, \pi \in X \}$$

$$= \bigvee \{ \bigvee \{ \pi(k) | (i, k) \in S \} | \pi \in X \}$$

$$= \bigvee \{ (F_J(\pi))(i) | \pi \in X \} = \left( \bigvee \{ F_J(\pi) | \pi \in X \} \right)(i)$$

for all $i \in T$ and all $X \subseteq L_T$.

Recall that $F_J(x)$ can be interpreted as “a case in future of an element $x$” (see [3, Theorem 6.2]).

**Theorem 3.3.** Let $L_1$ and $L_2$ be sup-semilattices, let $f: L_1 \rightarrow L_2$ be a homomorphism, and let $J = (T, S)$ be a frame. Then there exists a homomorphism $f_J: L_1^J \rightarrow L_2^J$ in $S_F$ such that, for every $x \in L_1^T$ and every $i \in T$, it holds $(f_J(x))(i) = f(x(i))$. Moreover, $-J$ is a functor from $S$ to $F^{-S_{\leq}}$.

**Proof.** Since $S$ has arbitrary products, we know that $f_J$ is a morphism in $S$. It remains to check that $f_J$ is a homomorphism. We compute:

$$(F_J(f_J(x)))(i) = \bigvee_{(i,k) \in S} f_J(x)(k) = \bigvee_{(i,k) \in S} f(x(k))$$

$$= f \left( \bigvee_{(i,k) \in S} x(k) \right) = f(F_J(x)(i)) = (f_J(F_J(x)))(i).$$

The functoriality of $-J$ follows from the functoriality of the product construction in $S$. \qed

**Theorem 3.4.** Let $J_1$ and $J_2$ be frames, let $t: J_1 \rightarrow J_2$ be a homomorphism of frames, and let $L$ be a sup-semilattice. Then there exists a lax morphism $L^t: L_2^J \rightarrow L_1^J$ of $F$-sup-semilattices such that, for every $x \in L_2^T$ and every $i \in T_1$, it holds $(L^t(x))(i) = x(t(i))$. Moreover, $L^-$ is a contravariant functor from $J$ to $F^{-S_{\leq}}$. 

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Proof. Evidently, \( L' \) is a homomorphism in \( S \). Moreover, we compute:

\[
(F^{J_1}(L'(x))(i)) = \bigvee_{(i,k) \in J_1} L'(x)(k) = \bigvee_{(i,k) \in S_1} x(t(k)) \leq \bigvee_{(t(i),l) \in S_2} x(l) = (F^{J_2}((x))(t(i))) = (L'(F^{J_2}(x))(i))
\]

for all \( x \in L^{T_2} \) and \( i \in T_1 \). Now, let \( s: J_0 \rightarrow J_1 \) be a homomorphism of frames and let \( x \in L^{T_2} \) and every \( i \in T_0 \). We compute:

\[
L^{t_0s}(x)(i) = x(t(s(i))) = L^t(x)(s(i)) = L^s(L^t(x))(i) = ((L^s \circ L^t)(x))(i).
\]

Clearly, \( L^{id_{T_2}}(x)(i) = x(i) = id_{L^{T_2}}(x)(i) \) for all \( x \in L^{T_2} \) and \( i \in T_2 \). Hence \( L^{-} \) is really a contravariant functor from \( J \) to \( F-S_\leq \).

Recall that \( L' \) taken as a homomorphism in \( S \) is nothing else as the \textit{backward powerset operator} \( t^{-}: L^{T_2} \rightarrow L^{T_1} \) first introduced by L. A. Zadeh [18,14] for the real unit interval \([0,1]\).

Let \( L \) be a sup-semilattice and \( J = (T,S) \) a frame. Then, for arbitrary \( x \in L \) and \( i \in T \), we denote \( x_{iS} \) as an element of \( L^T \) such that \( x_{iS}(j) = x \) if and only if \( (i,j) \in S \) and otherwise we put \( x_{iS}(j) = 0 \). In particular, when \( S \) will be the identity relation on \( T \), we denote \( x_{iS} \) by \( x_{i=} \). Hence,

\[
x_{i=}(j) = \begin{cases} x & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}
\]

Recall that \( \bigvee \{x_{k=} | (i,k) \in S\} = x_{iS} \) and \( \bigvee \{(\pi)(k)= | k \in T\} = \pi \) for every \( x \in L \), \( i \in T \) and \( \pi \in L^T \).

Let \( H = (G,F) \) be an \( F \)-sup-semilattice and \( J = (T,S) \) a frame. We put

\[
[J,H] = \{(x_{iS} \lor F(x)_{i=}, F(x)_{i=}) | x \in G, i \in T \}.
\]

Using \([J,H]\), we try to encode the unary operation \( F \) within an ordinary sup-semilattice \( J \otimes H \) that will be a quotient of \( G^T \) via the premnucleus \( j[J,H] \).

**Definition 3.5.** Let \( J = (T,S) \) be a frame and \( H = (G,F) \) an \( F \)-sup-semilattice. We then define a sup-semilattice \( J \otimes H \) as follows:

\[
J \otimes H = G^T_{j[J,H]}.
\]

Let \( J_1 = (T_1,S_1) \) and \( J_2 = (T_2,S_2) \) be frames, \( L \) a sup-semilattice and \( t: T_1 \rightarrow T_2 \) a map. We define a homomorphism \( t^{-}: L^{T_1} \rightarrow L^{T_2} \) such that for all \( x \in L^{T_1} \) and \( k \in T_2 \) we put \((t^{-}(x))(k) = \bigvee \{x(i) | t(i) = k\}\). In this case, \( t^{-} \) is usually called the \textit{forward powerset operator} [18,14], and \( t^{-} \) is left adjoint to \( t^{-} \). Moreover, \( P: J \rightarrow S \) defined by \( P(J) = L^{T} \) and \( P(t) = t^{-} \) is a functor (this follows immediately from [13, Theorem 2.9]).

**Theorem 3.6.** Let \( J_1 = (T_1,S_1) \) and \( J_2 = (T_2,S_2) \) be frames, \( t: J_1 \rightarrow J_2 \) a homomorphism of frames and \( H = (G,F) \) an \( F \)-sup-semilattice. Then there exists a unique morphism \( t \otimes H: J_1 \otimes H \rightarrow J_2 \otimes H \) of sup-semilattices such that the following diagram commutes:
Moreover, then $- \otimes H$ is a functor from $\mathbb{J}$ to $\mathbb{S}$.

**Proof.** Let $x \in G$ and $T_1 \in I$. It is enough to check that

$$n(j[J_2, H])(t^-(F(x)_{i=})) = n(j[J_2, H])(t^-(x_{iS_1} \lor F(x)_{i=})).$$

We get, for an arbitrary $l \in T_2$:

$$t^-(x_{iS_1})(l) = \bigvee \{x_{iS_1}(k) \mid t(k) = l\} \leq x_{t(i)S_2}(l)$$

and

$$t^-(F(x)_{i=})(l) = \bigvee \{F(x)_{i=}(k) \mid t(k) = l\} = F(x)_{t(i)=}(l).$$

Therefore it holds

$$n(j[J_2, H])(t^-(x_{iS_1} \lor F(x)_{i=})) \leq n(j[J_2, H])(x_{t(i)S_2})$$

$$\leq n(j[J_2, H])(x_{t(i)S_2} \lor F(x)_{t(i)=})$$

$$= n(j[J_2, H])(F(x)_{t(i)=})$$

$$= n(j[J_2, H])(t^-(F(x)_{i=})).$$

We conclude

$$n(j[J_2, H])(t^-(x_{iS_1} \lor F(x)_{i=}))$$

$$\leq n(j[J_2, H])(t^-(x_{iS_1}) \lor n(j[J_2, H])(t^-(F(x)_{i=}))$$

$$\leq n(j[J_2, H])(t^-(F(x)_{i=})) \lor n(j[J_2, H])(t^-(F(x)_{i=}))$$

$$= n(j[J_2, H])(t^-(F(x)_{i=})).$$

Hence we obtain

$$n(j[J_2, H])(t^-(x_{iS_1} \lor F(x_{i})) = n(j[J_2, H])(F(x)_{t(i)=}).$$

From Lemma 2.7 we conclude that there is a unique morphism $t \otimes H$ of sup-semilattices such that

$$n(j[J_2, H]) \circ t^- = (t \otimes H) \circ n(j[J_1, H]).$$

Let us show that $- \otimes H$ is a functor. Evidently, for $J = (T, S)$ and $\text{id}_T$, we have that $\text{id}_{T^*} = \text{id}_{G \times T}$. Hence $\text{id}_T \otimes H = \text{id}_{J \otimes H}$. Now, let $t : J_1 \to J_2$ and $s : J_2 \to J_3$ be homomorphisms of frames. Then $(s \circ t)^- = s^- \circ t^-$. From the uniqueness property of $t \otimes H$, $s \otimes H$, and $(s \circ t) \otimes H$, and a little bit of diagram chasing, we obtain that $(s \circ t) \otimes H = (s \otimes H) \circ (t \otimes H).$ \hfill \square

**Theorem 3.7.** Let $H_1 = (G_1, F_1), H_2 = (G_2, F_2)$ be $F$-sup-semilattices, $f : H_1 \to H_2$ a lax morphism of $F$-sup-semilattices and $J = (T, S)$ a frame. Then there is a unique morphism $J \otimes f : J \otimes H_1 \to J \otimes H_2$ of sup-semilattices such that the following diagram commutes:

\[
\begin{array}{ccc}
G^{T_1} & \to & J_1 \otimes H \\
\downarrow t^- & & \downarrow t \otimes H \\
G^{T_2} & \to & J_2 \otimes H \\
\end{array}
\]
Moreover, $J \otimes -$ is a functor from $F \mathcal{S}$ to $\mathcal{S}$.

**Proof.** For an arbitrary $x \in G_1$ and $i \in S$, we get:

$$n(j[J, H_2])(f^J(x_{iS})) = n(j[J, H_2])(f(x)_{iS}) \leq n(j[J, H_2])(F_2(f(x))_{i=} = n(j[J, H_2])(f^J(F_1(x))_{i=} = n(j[J, H_2])(f^J(F_1(x))_{i=})).$$

Hence

$$n(j[J, H_2])(f^J(x_{iS} \lor F_1(x)_{i=})) = n(j[J, H_2])(f^J(F_1(x)_{i=})).$$

We conclude, as before from Lemma 2.7, that there is a unique morphism $J \otimes -$ is a functor from $\mathcal{S}$ to $\mathcal{S}$. Hence we have that $\text{id}_H = \text{id}_{F}$. Therefore $J \otimes \text{id}_H = \text{id}_{J \otimes H}$.

Now, let $f: H_1 \to H_2$ and $g: H_2 \to H_3$ be lax morphisms of $F$-sup-semilattices. Then $(g \circ f)^J = g^J \circ f^J$. From the uniqueness property of $J \otimes f$, $J \otimes g$, and $J \otimes (g \circ f)$, we conclude that $J \otimes (g \circ f) = (J \otimes g) \circ (J \otimes f)$. \hfill \Box

**Definition 3.8.** Let $H = (G, F)$ be an $F$-sup-semilattice and let $L$ be a sup-semilattice. We define a frame $J[H, L] = (T_{[H,L]}, S_{[H,L]})$ such that $T_{[H,L]} = S(G, L)$, i.e., the elements of $T_{[H,L]}$ are morphisms $\alpha$ of sup-semilattices from $G$ to $L$. The relation $S_{[H,L]}$ is defined for $\alpha, \beta \in T_{[H,L]}$ as follows:

$$\alpha S_{[H,L]} \beta \text{ if and only if for all } x \in G \beta(x) \leq \alpha(F(x)).$$

**Theorem 3.9.** Let $L_1, L_2$ be sup-semilattices, $H = (G, F)$ an $F$-sup-semilattice, and let $f: L_1 \to L_2$ be a morphism of sup-semilattices. Then there exists a homomorphism $J[H, f]: J[H, L_1] \to J[H, L_2]$ of frames such that

$$(J[H, f])(\alpha)(x) = f(\alpha(x))$$

for all $\alpha \in T_{[H,L_1]}$ and all $x \in G$. Moreover, $J[H, -]$ is a functor from $\mathcal{S}$ to $\mathcal{S}$.

**Proof.** Let $\alpha, \beta \in T_{[H,L_1]}$ be such that $\alpha S_{[H,L_1]} \beta$. Then, for any $x \in G$, we have $\beta(x) \leq \alpha(F(x))$. Therefore $(J[H, f])(\alpha)(x) = f(\beta(x)) \leq f(\alpha(F(x))) = (J[H, f])(\alpha)(F(x))$. We obtain $J[H, f](\alpha) S_{[H,L_2]} J[H, f](\beta)$. 

Let us check that $J[H, -]$ is a functor. Let $L$ be a sup-semilattice, $\text{id}_L$ the identity morphism of sup-semilattices on $L$ and $\alpha \in T_{[H,L]}$. We compute:

$$J[H, \text{id}_L](\alpha) = \text{id}_L \circ \alpha = \alpha = \text{id}_{J[H,L]}(\alpha).$$

Now, let $f : L_1 \to L_2$ and $g : L_2 \to L_3$ be morphisms of sup-semilattices. Then

$$J[H,g \circ f](\alpha) = (g \circ f) \circ \alpha = g \circ (f \circ \alpha) = g \circ J[H,f](\alpha)$$

$$= J[H,g](J[H,f](\alpha)) = (J[H,g] \circ J[H,f])(\alpha). \quad \Box$$

**Theorem 3.10.** Let $H_1 = (G_1,F_1), H_2 = (G_2,F_2)$ be $F$-sup-semilattices, $L$ a sup-semilattice and $f : H_1 \to H_2$ a lax morphism of $F$-sup-semilattices. Then there exists a homomorphism $J[f,L] : J[H_2,L] \to J[H_1,L]$ of frames such that

$$(J[f,L](\alpha))(x) = \alpha(f(x)) = (\alpha \circ f)(x)$$

for all $\alpha \in T_{[H_2,L]}$ and all $x \in G_1$. Moreover, $J[-,L]$ is a contravariant functor from $F\text{-}S \leq \mathcal{J}$.

**Proof.** Let $\alpha, \beta \in T_{[H_2,L]}$ be such that $\alpha S_{[H_2,L]} \beta$. Then, for any $x \in G_2$, we have $\beta(x) \leq \alpha(F(x))$. Therefore

$$(J[f,G](\beta))(x) = \beta(f(x)) \leq \alpha(F(f(x)) \leq \alpha(f(F(x)) = (J[f,G](\alpha))(F(x)).$$

We conclude $J[f,G](\alpha) S_{[H_1,L]} J[f,G](\beta)$.

Let us verify that $J[-,L]$ is a contravariant functor. Let $H = (G,F)$ be an $F$-sup-semilattice, $\text{id}_H$ the identity lax morphism of $F$-sup-semilattices on $H$ and $\alpha \in T_{[H,L]}$. We compute:

$$J[\text{id}_H,L](\alpha) = \alpha \circ \text{id}_L = \alpha = \text{id}_{J[H,L]}(\alpha).$$

Now, let $f : H_1 \to H_2$ and $g : H_2 \to H_3$ be lax morphisms of $F$-sup-semilattices. We have:

$$J[g \circ f,L](\alpha) = \alpha \circ (g \circ f) = (\alpha \circ g) \circ f = J[g,L](\alpha) \circ f$$

$$= J[f,L](J[g,L](\alpha)) = (J[f,L] \circ J[g,L])(\alpha). \quad \Box$$

**Remark 3.11.** Let $H = (G,F)$ be a finite $F$-sup-semilattice, $L$ a finite sup-semilattice and $J = (T,S)$ a finite frame. Then evidently, $J \otimes H$ is a finite sup-semilattice, $L^J$ is a finite $F$-sup-semilattice, and $J[H,L]$ is a finite frame.

### 4. Three adjoint situations

In this section, we introduce three induced adjoint situations $(\eta, \varepsilon), (\varphi, \psi), \text{ and } (\nu, \mu)$.

**Theorem 4.1.** Let $J = (T,S)$ be a frame. Then:
(a) For an arbitrary $F$-sup-semilattice $H = (G, F)$, there exists a lax morphism $\eta_H : H \to (J \otimes H)^J$ of $F$-sup-semilattices defined in such a way that

$$(\eta_H(x))(i) = n(j[J, H])(x_{i=}).$$

Moreover, $\eta = (\eta_H : H \to (J \otimes H)^J)_{H \in F-S_\leq}$ is a natural transformation between the identity functor on $F-S_\leq$ and the endofunctor $(J \otimes -)^J$.

(b) For an arbitrary sup-semilattice $L$, there exists a unique sup-semilattice morphism $\varepsilon_L : J \otimes L^J \to L$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(L^T)^T & \xrightarrow{n(j[J, L^J])} & J \otimes L^J \\
\downarrow e_L & & \downarrow \varepsilon_L \\
L & & L
\end{array}
$$

where $e_L : (L^T)^T \to L$ is defined by $e_L(\bar{x}) = \bigvee_{i \in T}(\bar{x}(i))(i)$ for any element $\bar{x} \in (L^T)^T$. Moreover, $\varepsilon = (\varepsilon_L : J \otimes L^J \to L)_{L \in S}$ is a natural transformation between the endofunctor $J \otimes (\cdot)^J$ and the identity functor on $S$.

(c) There exists an adjoint situation $(\eta, \varepsilon) : (J \otimes -)^J \dashv (\cdot)^J : S \to F-S_\leq$.

Proof. (a): Let us consider an arbitrary $F$-sup-semilattice $H = (G, F)$. Evidently, $\eta_H$ preserves arbitrary joins. Assume that $x \in G$ and $i \in T$. We compute:

$$
(F^J(\eta_H(x)))(i) = \bigvee\{(\eta_H(x))(k) \mid (i, k) \in S\} \\
= \bigvee\{n(j[J, H])(x_{k=}) \mid (i, k) \in S\} \\
= n(j[J, H])(\bigvee\{x_{k=} \mid (i, k) \in S\}) = n(j[J, H])(x_{iS}) \\
\leq n(j[J, H])(x_{iS} \lor F(x)_{i=}) = n(j[J, H])(F(x)_{i=}) \\
= (\eta_H(F(x)))(i).
$$

Therefore $F^J \circ \eta_H \leq \eta_H \circ F$ and hence $\eta_H$ is a lax morphism. Now, let us assume that $H_1 = (G_1, F_1)$ and $H_2 = (G_2, F_2)$ are $F$-sup-semilattices, and that $f : H_1 \to H_2$ is a lax morphism of $F$-sup-semilattices. We have to show that the following diagram commutes:
Assume that $x \in G_1$ and $i \in J$. We compute:

$$(((J \otimes f)^J \circ \eta_{H_1})(x))(i) = (J \otimes f)(\eta_{H_1}(x)(i)) = (J \otimes f)(n(j[J,H_1])(x_i=)) = n(j[J,H_2])(f^J(x_i=)) = ((\eta_{H_2} \circ f)(x))(i).$$

(b): Let $L$ be a sup-semilattice. Assume that $x \in L^T$ and $i \in T$. We compute:

$$e(x_iS) = \bigvee_{k \in T} (x_iS(k))(k) = \bigvee_{iSk} x(k) = (F^J(x))(i) = \bigvee_{k \in T} F^J(x)^i=k(k) = e(F^J(x)^i=).$$

Moreover, it is transparent that $e_L$ preserves arbitrary joins.

By Lemma 2.7, there is a unique morphism $\varepsilon_L : J \otimes L^J \to L$ of sup-semilattices such that $e = \varepsilon_L \circ n(j[J,L^J])$.

Let us now consider a morphism $f : L_1 \to L_2$ of sup-semilattices. We have to prove that the following diagram commutes:

```latex
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$J \otimes L_1^J$};
  \node (B) at (3,0) {$J \otimes L_2^J$};
  \node (C) at (0,-3) {$L_1$};
  \node (D) at (3,-3) {$L_2$};
  \draw[->] (A) -- node[above] {$J \otimes f^J$} (B);
  \draw[->] (A) -- node[left] {$\varepsilon_{L_1}$} (C);
  \draw[->] (B) -- node[right] {$\varepsilon_{L_2}$} (D);
  \draw[->] (C) -- node[below] {$f$} (D);
\end{tikzpicture}
\end{array}
\end{align*}
```
Let $\bar{x} \in (L_1^T)^T$. We compute:

\[
(\varepsilon_{L_2} \circ (J \otimes f^J))( n(j[J, L_1^J])(\bar{x})) = \varepsilon_{L_2}( n(j[J, L_2^J])(f^J(\bar{x}))) = \varepsilon_{L_2}(f^J(\bar{x})) \\
= \bigvee_{i \in T} ((f^J(\bar{x}))(i)) \\
= \bigvee_{i \in T} f(\bar{x}(i)) \\
= f \left( \bigvee_{i \in T} \bar{x}(i) \right) = f(\varepsilon_{L_1}(\bar{x})) \\
= f((\varepsilon_{L_1} \circ n(j[J, L_1^J])(\bar{x})) \\
= (f \circ \varepsilon_{L_1})( n(j[J, L_1^J])(\bar{x})).
\]

(c): Let $H = (G, F)$ be an $F$-sup-semilattice and $L$ a sup-semilattice. We will prove the commutativity of the following diagrams:

\[
\begin{array}{ccc}
J \otimes H & \xrightarrow{J \otimes \eta_H} & J \otimes (J \otimes H)^J \\
\downarrow \text{id}_{J \otimes H} & & \downarrow \varepsilon_{J \otimes H} \\
J \otimes H & & L^J
\end{array}
\quad
\begin{array}{ccc}
J^J & \xrightarrow{\eta_{L^J}} & (J \otimes L^J)^J \\
\downarrow \text{id}_{J^J} & & \downarrow \varepsilon_{L^J} \\
L^J & &
\end{array}
\]

For the first diagram, assume $\bar{x} \in G^T$. From Theorem 3.7, we know that the following diagram commutes (when necessary, we forget that some morphisms are actually from $F-S_{\leq}$, and we work entirely in $S$):

\[
\begin{array}{ccc}
G^T & \xrightarrow{n(j[J, H])} & J \otimes H \\
\downarrow (\eta_H)^J & & \downarrow J \otimes \eta_H \\
((J \otimes H)^T)^n(j[J, (J \otimes H)^J]) & \xrightarrow{\varepsilon_{J \otimes H}} & J \otimes (J \otimes H)^J \\
\downarrow (J \otimes \eta_{H}) & & \downarrow \text{id}_{J \otimes H} \\
J \otimes H & & J \otimes H
\end{array}
\]
We compute:
\[
\varepsilon_{J \otimes H}(\eta H \circ n(J, (J \otimes H)J))((\eta H)J(\varepsilon))
\]
\[
= \varepsilon_{J \otimes H}((\eta H)J(\varepsilon)) = \bigvee_{i \in T} ((\eta H)J(\varepsilon))(i) = \bigvee_{i \in T} (\eta H(\varepsilon(i)))(i)
\]
\[
= \bigvee_{i \in T} n(J[H,H])(i) = n(J[H,H])\left(\bigvee_{i \in T} (\varepsilon(i))\right).
\]

Hence \(\varepsilon_{J \otimes H} \circ (J \otimes \eta H) = \text{id}_{J \otimes H}\).

To show the commutativity of the second diagram, assume that \(x \in L^T\) and \(i \in T\). We compute:
\[
\left(\left(\varepsilon_L\right)^J \circ \eta_{L^*}(\varepsilon)\right)(i) = \left(\left(\varepsilon_L\right)^J(\eta_{L^*}(\varepsilon))(i)\right) = \varepsilon_L(\eta_{L^*}(\varepsilon))(i)
\]
\[
= \varepsilon_L\left(n(J[L,J])(\varepsilon_i)\right) = \varepsilon_L(\varepsilon(i)) = \bigvee_{k \in T} \varepsilon_i(k)(k) = \varepsilon(i).
\]

We conclude that \(\left(\varepsilon_L\right)^J \circ \eta_{L^*} = \text{id}_{L^*}\). \(\square\)

**Theorem 4.2.** Let \(H = (G, F)\) be an \(F\)-sup-semilattice. Then:

(a) For an arbitrary frame \(J = (T, S)\), there exists a unique homomorphism of frames \(\varphi_J : J \to J[H, J \otimes H]\) defined for arbitrary \(x \in G\) and \(i \in T\) in such a way that
\[
(\varphi_J(i))(x) = n(J[H,H])(x_i).
\]

Moreover, \(\varphi = (\varphi_J : J \to J[H, J \otimes H])_{J \in \mathbb{J}}\) is a natural transformation between the identity functor on \(\mathbb{J}\) and the endofunctor \(J[H, -] \otimes H\).

(b) For an arbitrary sup-semilattice \(L\), there exists a unique sup-semilattice morphism \(\psi_L : J[H, L] \otimes H \to L\) such that the following diagram commutes:

\[
\begin{array}{ccc}
G^T[H, L] & \xrightarrow{n(J[H,L], H)} & J[H, L] \otimes H \\
& \searrow f_L & \downarrow \psi_L \\
& & L
\end{array}
\]

where \(f_L : G^T[H, L] \to L\) is defined by \(f_L(x) = \bigvee_{j \in J[H,L]} (\alpha(x))(\alpha)\) for any \(x \in G^T[H, L]\). Moreover, \(\psi = (\psi_L : J[H, L] \otimes H \to L)_{L \in \mathbb{S}}\) is a natural transformation between the endofunctor \(J[H, -] \otimes H\) and the identity functor on \(\mathbb{S}\).

(c) There exists an adjoint situation \((\varphi, \psi) : (- \otimes H) \vdash J[H, -]) : \mathbb{S} \to \mathbb{J}\).
Proof. (a): We have to show that our definition is correct. Assume $X \subseteq G$ and $i \in I$. We have:

$$(\varphi_J(i)) \left( \bigvee X \right) = n(j[J, H]) \left( \left( \bigvee X \right)\right) = n(j[J, H]) \left( \bigvee \{ x_{i=} \mid x \in X \} \right) = \bigvee \{ n(j[J, H]) (x_{i=}) \mid x \in X \} = \bigvee \{(\varphi_J(i))(x) \mid x \in X \}$$

Hence $\varphi_J(i) \in T_{[H,J\otimes H]}$. Now, let $i, k \in T$ such that $i S k$ and $x \in G$. We compute:

$$(\varphi_J(k))(x) = n(j[J, H])(x_{k=}) \leq n(j[J, H])(x_{iS}) \leq n(j[J, H])(x_{iS} \lor F(x)\ell)$$

$$= n(j[J, H])(F(x)\ell) = (\varphi_J(i))(F(x)).$$

Therefore $\varphi_J(i) S_{[H,J\otimes H]} \varphi_J(k)$.

Let $t: J_1 \rightarrow J_2$ be a homomorphism of frames. We have to show that the following diagram commutes:

$$\begin{array}{ccc}
J_1 & \xrightarrow{t} & J_2 \\
\downarrow \varphi_{J_1} & & \downarrow \varphi_{J_2} \\
J[H, J_1 \otimes H] & \rightarrow & J[H, J_2 \otimes H]
\end{array}$$

Let $i \in J$ and $x \in G$. We obtain

$$((J[H, t \otimes H] \circ \varphi_{J_1})(i))(x) = (J[H, t \otimes H](\varphi_{J_1}(i)))(x)$$

$$= ((t \otimes H) \circ (\varphi_{J_1}(i)))(x) = (t \otimes H)((\varphi_{J_1}(i))(x))$$

$$= (t \otimes H)(n(j[J, H])(x_{i=})) = n(j[J_2, H])(x_{t(i=)})$$

$$= (\varphi_{J_2}(t(i)))(x) = ((\varphi_{J_2} \circ t)(i))(x).$$

(b): It is transparent that $f_L$ preserves arbitrary joins.

Let $x \in G$ and $\alpha \in J[H, L]$ be arbitrary. We compute:

$$e_L(x_{\alpha S_{[H,L]}}) = \bigvee \{ \beta(x_{\alpha S_{[H,L]}}(\beta)) \mid \beta \in T_{[H,L]} \}$$

$$= \bigvee \{ \beta(x) \mid \alpha S_{[H,L]} \beta, \beta \in T_{[H,L]} \} \leq \alpha(F(x))$$

$$= \bigvee \{ \beta(F(x)_{\alpha=(\beta)}) \mid \beta \in T_{[H,L]} \} = e_L(F(x)_{\alpha=}).$$

Hence $e_L(x_{\alpha S_{[H,L]} \lor F(x)_{\alpha=}}) = e_L(F(x)_{\alpha=})$, which assures by Lemma 2.7 the existence of $\psi_L$ from the theorem.

Let $g: L_1 \rightarrow L_2$ be a morphism of sup-semilattices. Let us show that the following diagram commutes:
Let $x \in G^{T[H,L_1]}$. We compute:

\[
(\psi_{L_2} \circ (J[H,g] \otimes H) \circ n(j[J[H,L_1],H]))(x) \\
= (\psi_{L_2} \circ n(j[J[H,L_2],H]) \circ J[H,g]^{-1})(x) = (f_{L_2} \circ J[H,g]^{-1})(x) \\
= f_{L_2}\left(\left(\bigvee_{\beta \in T[H,L_2]} \{x(\alpha) \mid g \circ \alpha = \beta, \alpha \in T[H,L_1]\}\right)\right) \\
= \bigvee_{\beta \in T[H,L_2]}\beta\left(\left(\bigvee_{\alpha \in T[H,L_1]} \{x(\alpha) \mid g \circ \alpha = \beta, \alpha \in T[H,L_1]\}\right)\right) \\
= \bigvee_{\alpha \in T[H,L_1]}g(\alpha(x(\alpha))) = g\left(\bigvee_{\alpha \in T[H,L_1]} \{\alpha(x(\alpha)) \mid \alpha \in T[H,L_1]\}\right) \\
= (g \circ f_{L_1})(x) = (g \circ \psi_{L_1} \circ n(j[J[H,L_1],H]))(x).
\]

Since $n(j[J[H,L_1],H])$ is surjective we have $\psi_{L_2} \circ (J[H,g] \otimes H) = g \circ \psi_{L_1}$.

(c): Let $J = (S,T)$ be a frame and $L$ a sup-semilattice. We will prove the commutativity of the following diagrams:

Let $x \in G^{T}$. According to Theorem 3.6, we know that the following diagram commutes:
We compute:

\[(\psi_{J \otimes H} \circ (\varphi_{J \otimes H}) \circ n(J, H))((x)) = (\psi_{J \otimes H} \circ n(J[H, J \otimes H], H)) \circ \varphi_{J}^{-1}(x)\]

\[= (f_{J \otimes H} \circ \varphi_{J}^{-1}(x)) = f_{J \otimes H} \left( \bigvee \{ x(i) \mid \varphi_{J}(i) = \alpha \} \right)_{\alpha \in T[H, J \otimes H]}\]

\[= \bigvee_{\alpha \in T[H, J \otimes H]} \varphi_{J}(i) = \alpha \in T[H, J \otimes H] \varphi_{J}(i) = \alpha, i \in T\]

\[= \bigvee_{i \in T} \{ n(j[J, H])((x(i)) \mid i \in T) \} = \bigvee_{i \in T} \{ n(j[J, H])(x \in T) \} = n(J[H, H])(x).\]

Hence the first diagram commutes. Now, let \(\alpha \in J[H, L]\) and \(x \in G\). We obtain:

\[((J[H, \psi_{L}] \circ \varphi_{J[H,L]})((\alpha)))(x) = (J[H, \psi_{L}]((\varphi_{J[H,L]}(\alpha))))(x)\]

\[= \psi_{L}((\varphi_{J[H,L]}(\alpha)))(x) = \psi_{L}(\ n(J[H, L], H)](x \alpha = )) = f_{L}(x \alpha = )\]

\[= \bigvee_{\beta(x \alpha = \beta)}(\beta) \mid \beta \in T[H, L]\]

which yields the commutativity of the second diagram. \(\square\)

**Theorem 4.3.** Let \(L\) be a sup-semilattice. Then the following holds:

(a) For an arbitrary frame \(J = (T, S)\), there exists a unique homomorphism of frames \(\nu_{J} : J \rightarrow J[L^{J}, L]\) defined for arbitrary \(x \in L^{T}\) and \(i \in T\) in such a way that

\[\nu_{J}(i)(x) = x(i)\]

Moreover, \(\nu = (\nu_{J} : J \rightarrow J[L^{J}, L])_{J \in \mathbb{I}}\) is a natural transformation between the identity functor on \(\mathbb{I}\) and the endofunctor \(J[L^{-}, L]\).

(b) For an arbitrary \(F\)-sup-semilattice \(H = (G, F)\), there exists a homomorphism \(\mu_{H} : H \rightarrow L^{J[H,L]}\) of \(F\)-sup-semilattices defined for arbitrary \(x \in G\) and \(\alpha \in T[J[H,L]]\) by

\[(\mu_{H}(x))(\alpha) = \alpha(x).\]
Moreover, \( \mu = (\mu_H : H \to L^J[H,L])_{H \in F-S_{\leq}} \) is a natural transformation between the identity functor on \( F-S_{\leq} \) and the endofunctor \( L^J[-,L] \).

(c) There exists an adjoint situation \((\nu, \mu) : [J,-,L] \dashv L^- : \mathbb{J} \to F-S_{\leq}^{op}\).  

Proof. (a): Let \( i, k \in T, x \in L^T \) and \( i \leq k \). We compute:

\[
(\nu_J(k))(x) = x(k) \leq \bigvee \{x(l) \mid i \leq l \in T\} = (F^J(x))(i) = (\nu_J(i))(F^J(x)).
\]

Hence \( \nu_J(i) \circ S_{[L',L]} \nu_J(k) \) and \( \nu_J \) is a frame homomorphism. Assume that \( t : J_1 \to J_2 \) is a frame homomorphism between frames \( J_1 = (T_1, S_1) \) and \( J_2 = (T_2, S_2) \). We have to show that the following diagram commutes:

\[
\begin{array}{ccc}
J_1 & \xrightarrow{t} & J_2 \\
\downarrow{\nu_{J_1}} & & \downarrow{\nu_{J_2}} \\
J[L^{J_1},L] & \xrightarrow{} & J[L^{J_2},L]
\end{array}
\]

Assume that \( i \in T_1 \) and \( x \in L^{T_2} \). We compute:

\[
((J[L',L] \circ \nu_{J_1})(i))(x) = (J[L',L])(\nu_{J_1}(i))(x) = (\nu_{J_1}(i) \circ L^t)(x) \\
= \nu_{J_1}(i)(L^t(x)) = \nu_{J_1}(i)(x \circ t) = (x \circ t)(i) = x(t(i)) \\
= (\nu_{J_2}(t(i)))(x) = ((\nu_{J_2} \circ t)(i))(x).
\]

Hence \( J[L',L] \circ \nu_{J_1} = \nu_{J_2} \circ t \).

(b): Evidently, \( \mu_H \) preserves arbitrary joins. We have to verify that

\[
F^{J[H,L]} \circ \mu_H = \mu_H \circ F.
\]

Firstly, we will show that \( F^{J[H,L]} \circ \mu_H \leq \mu_H \circ F \). Let \( x \in G \) and \( \alpha \in J[H,L] \). We compute:

\[
\left( (F^{J[H,L]} \circ \mu_H)(x) \right)(\alpha) = \left( F^{J[H,L]}(\mu_H(x)) \right)(\alpha) \\
= \bigvee \{(\mu_H(x))(\beta) \mid \beta \in J[H,L], \alpha S_{[H,L]} \beta \} \\
= \bigvee \{\beta(x) \mid \beta \in J[H,L], \alpha S_{[H,L]} \beta \} \\
\leq \alpha(F(x)) = \mu_H(F(x))(\alpha) = ((\mu_H \circ F)(x))(\alpha).
\]

We obtain that \( \mu_H \) is a lax morphism.

Let us now prove the reverse inequality, i.e.,

\[
\mu_H \circ F \leq F^{J[H,L]} \circ \mu_H.
\]

Recall that, for every \( \alpha \in J[H,L] \), we have \( \alpha \circ F = \alpha \circ F \) trivially. But this means that \( \alpha S_{[H,L]} \alpha \circ F \) because of \( \alpha \circ F \in J[H,L] \).
Let $x \in G$ and $\alpha \in J[H, L]$. We compute:

\[ ((\mu_H \circ F)(x))(\alpha) = \mu_H(F(x))(\alpha) = \alpha(F(x)) = (\alpha \circ F)(x) \]

\[ \leq \bigvee \{\beta(x) \mid \beta \in J[H, L], \alpha S[H,L] \beta \} \]

\[ = \bigvee \{((\mu_H(x))(\beta) \mid \beta \in J[H, L], \alpha S[H,L] \beta \} \]

\[ = (F^J[H,L])(\mu_H(x))(\alpha) = \left((F^J[H,L] \circ \mu_H)(x)\right)(\alpha) \]

since $(\alpha \circ F)(x) \in \{\beta(x) \mid \beta \in J[H, L], \alpha S[H,L] \beta \}$.

We conclude that $\mu_H$ is a homomorphism.

Now, let us assume that $H_1 = (G_1, F_1)$ and $H_2 = (G_2, F_2)$ are $F$-sup-semilattices and that $f : H_1 \to H_2$ is a lax morphism of $F$-sup-semilattices. We have to verify that the following diagram commutes:

\begin{equation*}
\begin{array}{cccc}
H_1 & \xrightarrow{f} & H_2 \\
\downarrow{\mu_{H_1}} & & \downarrow{\mu_{H_2}} \\
J^{[H_1, L]} & \xrightarrow{L^J} & J^{[H_2, L]} \\
\end{array}
\end{equation*}

Assume that $x \in G_1$ and $\alpha \in T_{[H_2, L]}$. We compute:

\[ \left((L^J[f,L] \circ \mu_{H_1})(x)\right)(\alpha) = \left(L^J[f,L](\mu_{H_1}(x))\right)(\alpha) = (\mu_{H_1}(x))(J[f, L](\alpha)) \]

\[ = (J[f, L](\alpha))(\alpha \circ f)(x) = \alpha(f(x)) \]

\[ = ((\mu_{H_2} \circ f)(x))(\alpha). \]

hence $L^J[f,L] \circ \mu_{H_1} = \mu_{H_2} \circ f$.

(c): Let $J = (S, T)$ be a frame and $H = (G, F)$ an $F$-sup-semilattice. We will prove the commutativity of the following diagrams:

\begin{equation*}
\begin{array}{ccc}
L^J & \xrightarrow{\mu_{L^J}} & L^J[J^L, L] \\
\downarrow{id_{L^J}} & & \downarrow{L^{\nu_J}} \\
L^J & & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{ccc}
J[H, L] & \xrightarrow{\nu_{[H, L]}} & J[J^L[H, L], L] \\
\downarrow{id_{J[H, L]}} & & \downarrow{J[\mu_H, L]} \\
J[H, L] & & \\
\end{array}
\end{equation*}

Let $x \in L^T$ and $i \in T$. We compute:

\[ ((\nu_J \circ \mu_{L^J})(x))(i) = (L^{\nu_J}(\mu_{L^J}(x)))(i) = (\mu_{L^J}(x))(\nu_J(i)) = (\nu_J(i))(x) \]

\[ = x(i) = (id_{L^J}(x))(i). \]

Hence $L^{\nu_J} \circ \mu_{L^J} = id_{L^J}$ in $F$-$S_{\leq}$. Assume that $x \in G$ and $\alpha \in T_{[H,L]}$. We compute:

\[ ((J[\mu_H, L] \circ \nu_{[H,L]})(\alpha))(x) = (J[\mu_H, L](\nu_{[H,L]}(\alpha)))(x) \]

\[ = (\nu_{[H,L]}(\alpha))(\mu_H(x)) = (\mu_H(x))(\alpha) = \alpha(x) = (id_{J[H,L]}(\alpha))(x). \]
Figure 1. Sup-semilattices $G$ and $L$

Table 1. $S(G, L)$

|   | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|----|----|----|---|
| $f_1$ | 0 | 0 | 0 | 0 | 0 |
| $f_2$ | 0 | 0 | 0 | 1 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 1 |
| $f_4$ | 0 | 1 | 0 | 0 | 1 |
| $f_5$ | 0 | 1 | 1 | 0 | 1 |
| $f_6$ | 0 | 1 | 0 | 1 | 1 |
| $f_7$ | 0 | 0 | 1 | 1 | 1 |
| $f_8$ | 0 | 1 | 1 | 1 | 1 |

Therefore $J_{[\mu_H, L]} \circ \nu_{J[H, L]} = \text{id}_{J[H, L]}$. □

Remark 4.4. Our induced adjoint situations $(\eta, \varepsilon), (\varphi, \psi)$, and $(\nu, \mu)$ evidently restrict to an adjoint situation between the category of finite sup-semilattices and the category of finite $F$-sup-semilattices, between the category of finite sup-semilattices and the category of finite frames, and the category of finite frames and the dual of the category of finite $F$-sup-semilattices, respectively.

5. Three examples

In this section, we will analyze three examples to illustrate adjoint situations $(\eta, \varepsilon), (\varphi, \psi)$, and $(\nu, \mu)$.

Example 5.1. Let $H = (G, F)$ be an $F$-sup-semilattice, $G = (\{0, a, b, c, 1\}, \lor)$ and $0 < a, b, c < 1$. Operator $F$ is given by the prescription $F(0) = 0, F(a) = a, F(b) = c, F(c) = b, F(1) = 1$. Let $L = (\{0, 1\}, \lor)$ be a sup-semilattice where $0 < 1$. Both sup-semilattices are shown in Figure 1:

Let us define a frame $J[H, L] = (S(G, L), S_{[H, L]})$ where $S_{[H, L]}$ is a relation from Definition 3.8. Let us denote $S_{[H, L]}$ as $\rho$. Clearly, $S(G, L)$ has 8 elements, which we will denote $f_i$, where $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ and their description is given by Table 1.

Moreover, $f_1 \leq f_2 \leq f_6, f_7 \leq f_8$ and $f_1 \leq f_3 \leq f_5, f_7 \leq f_8$ and $f_1 \leq f_4 \leq f_5, f_6 \leq f_8$. 

Let us now describe the relation \( \rho \) on \( S(G, L) \). By definition,
\[
f_i \rho f_j \iff \forall x \in G \ f_j(x) \leq f_i(F(x)).
\]

Clearly \( f_i \rho f_i \) and \( f_i \rho f_1 \) for any \( i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \). Furthermore, it is easy to see that \( f_1 \circ F = f_1 \), \( f_2 \circ F = f_3 \), \( f_3 \circ F = f_2 \), \( f_4 \circ F = f_4 \), \( f_5 \circ F = f_6 \), \( f_6 \circ F = f_5 \), \( f_7 \circ F = f_7 \) and \( f_8 \circ F = f_8 \). This means \( f_2 \rho f_3 \), \( f_3 \rho f_2 \), \( f_4 \rho f_4 \), \( f_5 \rho f_6 \), \( f_6 \rho f_5 \), \( f_7 \rho f_7 \). Hence also \( f_5 \rho f_2 \), \( f_5 \rho f_4 \), \( f_6 \rho f_3 \), \( f_6 \rho f_4 \), \( f_7 \rho f_2 \), \( f_7 \rho f_3 \). We obtain that
\[
\rho = \{(f_8, f_1), (f_8, f_2), (f_8, f_3), (f_8, f_4), (f_8, f_5), (f_8, f_6), (f_7, f_1), (f_6, f_1), (f_5, f_1), (f_4, f_1), (f_3, f_1), (f_2, f_1), (f_1, f_1), (f_2, f_3), (f_3, f_2), (f_5, f_6), (f_6, f_5), (f_5, f_2), (f_5, f_4), (f_6, f_3), (f_6, f_4), (f_7, f_2), (f_7, f_3), (f_8, f_7), (f_8, f_8), (f_7, f_7), (f_4, f_4)\}.
\]

By Theorem 4.3, there exists a homomorphism \( \mu_H : H \rightarrow L^{[H, L]} \) of \( F \)-sup-semilattices defined for arbitrary \( x \in G \) and \( f_i \in S(G, L) \) by
\[
(\mu_H(x))(f_i) = f_i(x).
\]

Let us now compute \( \mu_H \) on elements of \( G \). It holds that:
\[
(\mu_H(0))(f_i) = 0 \text{ and } (\mu_H(1))(f_i) = 1 \text{ for any } i \in \{1, 2, 3, 4, 5, 6, 7, 8\},
\]
\[
(\mu_H(a))(f_i) = \begin{cases} 1 & \text{for } i \in \{4, 5, 6, 8\} \\ 0 & \text{otherwise,} \end{cases}
\]
\[
(\mu_H(b))(f_i) = \begin{cases} 1 & \text{for } i \in \{3, 5, 7, 8\} \\ 0 & \text{otherwise,} \end{cases}
\]
\[
(\mu_H(c))(f_i) = \begin{cases} 1 & \text{for } i \in \{2, 6, 7, 8\} \\ 0 & \text{otherwise.} \end{cases}
\]

Evidently, \( \mu_H \) is injective. Recall that the ordering of elements of \( L^{[H, L]} \) is given by:
\[
\alpha \leq \beta \iff \forall i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \ \alpha(f_i) \leq \beta(f_i).
\]

Since \( \mu_H \) preserves arbitrary joins and it is injective, we immediately have that \( \mu_H \) is an order embedding. We conclude that \( \mu_H \) is an element of \( \mathcal{E}_\leq \) since \( \mathcal{E} \subseteq \mathcal{E}_\leq \).

**Example 5.2.** Let \( J = (T, S) \) be a frame where \( T = \{f_2, f_3, f_4\} \), \( f_2, f_3, f_4 \) are mappings from Example 5.1 and \( S \) is a restriction of the relation \( \rho \) on \( T \) from Example 5.1. Hence \( S \) is a relation on \( \{f_2, f_3, f_4\} \) given by \( f_2 \ S \ f_3 \), \( f_3 \ S \ f_2 \) and \( f_4 \ S \ f_4 \). Let \( L = \{\{0, 1\}, \vee\} \) be a sup-semilattice where \( 0 \prec 1 \).

By Theorem 4.1, there exists a unique homomorphism \( \nu_J : J \rightarrow J[L^J, L] \) of frames defined for arbitrary \( x \in L^T \) and \( f_i \in T \) in such a way that
\[
(\nu_J(f_i))(x) = x(f_i).
\]

First, \( L^T \) is an 8-element set, and we denote its elements as \( \alpha_k, k \in \{1, 2, 3, 4, 5, 6, 7, 8\} \) (see Table 2). Let us now compute the induced tense operator \( F^J \) on \( L^J \). By its definition, \( F^J \) is given by
\[
(F^J(\alpha_k))(f_i) = \bigvee \{\alpha_k(f_i) \mid f_i \ S \ f_i\},
\]
for $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $i \in \{2, 3, 4\}$. Since $f_2 \ S \ f_3$, $f_3 \ S \ f_2$ and $f_4 \ S \ f_4$, it is easy to see that $(\mathbf{F}^J(\alpha_k))(f_2) = \alpha_k(f_3)$, $(\mathbf{F}^J(\alpha_k))(f_3) = \alpha_k(f_2)$ and $(\mathbf{F}^J(\alpha_k))(f_4) = \alpha_k(f_4)$. Using this fact, we can evaluate for $\mathbf{F}^J(\alpha_k)$ for all $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $i \in \{2, 3, 4\}$, which is shown in Table 2.

We will now describe the relation $S_{\mathbf{J}[\mathbf{L}^J, \mathbf{L}]}$, which we denote as $\rho'$. Recall that by definition, for two sup-semilattice homomorphisms $\varphi, \psi: \mathbf{L}^T \to \mathbf{L}$ we have

$$\varphi \rho' \psi \iff (\forall k \in \{1, 2, 3, 4, 5, 6, 7, 8\})(\psi(\alpha_k) \leq \varphi(\mathbf{F}^J(\alpha_k))).$$

For $\nu_J: \mathbf{J} \to \mathbf{J}[\mathbf{L}^J, \mathbf{L}]$ this by definition of $\nu_J$ means that $\nu_J(f_i) \rho' \nu_J(f_j)$ if and only if

$$\alpha_k(f_j) = (\nu_J(f_j))(\alpha_k) \leq (\nu_J(f_i))(\mathbf{F}^J(\alpha_k)) = (\mathbf{F}^J(\alpha_k))(f_i),$$

for all $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. This fact allows us to easily describe whether the respective pairs $(\nu_J(f_i), \nu_J(f_k))$, $i, k \in \{2, 3, 4\}$ belong to $\rho'$. We see that $(\nu_J(f_2), \nu_J(f_2))$ and $(\nu_J(f_3), \nu_J(f_3))$ don’t belong to $\rho'$, since $\alpha_2(f_2) = 1 > 0 = (\mathbf{F}^J(\alpha_2))(f_2)$ and $\alpha_3(f_3) = 1 > 0 = (\mathbf{F}^J(\alpha_3))(f_3)$ respectively. Similarly, we can see that $(\nu_J(f_3), \nu_J(f_4))$ doesn’t belong to $\rho'$, since $\alpha_3(f_3) = 1 > 0 = (\mathbf{F}^J(\alpha_3))(f_3)$ and it can be analogically shown that $(\nu_J(f_4), \nu_J(f_3))$ doesn’t belong to $\rho'$ either.

Moreover, since $(\mathbf{F}^J(\alpha_k))(f_2) = \alpha_k(f_3)$, $(\mathbf{F}^J(\alpha_k))(f_3) = \alpha_k(f_2)$ and $(\mathbf{F}^J(\alpha_k))(f_4) = \alpha_k(f_4)$ for all $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, we immediately obtain $(\nu_J(f_2), \nu_J(f_3)) \in \rho'$, $(\nu_J(f_3), \nu_J(f_2)) \in \rho'$ and $(\nu_J(f_3), \nu_J(f_4)) \in \rho'$. In conclusion, this means $f_i \ S \ f_k$ in our original frame $\mathbf{J}$ if and only if $\nu_J(f_i) \rho' \nu_J(f_k)$ in the frame $\mathbf{J}[\mathbf{L}^J, \mathbf{L}]$.

**Example 5.3.** Let $\mathbf{H} = (\mathbf{G}, F)$ be the $F$-sup-semilattice from Example 5.1 where $\mathbf{G} = (\{0, a, b, c, 1\}, \lor)$ and $0 < a, b, c < 1$. Operator $F$ is given by the prescription $F(0) = 0, F(a) = a, F(b) = c, F(c) = b, F(1) = 1$. Let $\mathbf{J} = (T, S)$ be the frame from Example 5.2 where $T = \{f_2, f_3, f_4\}$ and $S$ is a relation on $\{f_2, f_3, f_4\}$ given by $f_2 \ S \ f_3$, $f_3 \ S \ f_2$ and $f_4 \ S \ f_4$.  

| $f_2$ | $f_3$ | $f_4$ |
|-------|-------|-------|
| $\alpha_1$ | 0 | 0 | 0 | $\mathbf{F}^J(\alpha_1) = \alpha_1$ | 0 | 0 | 0 |
| $\alpha_2$ | 1 | 0 | 0 | $\mathbf{F}^J(\alpha_2) = \alpha_3$ | 0 | 1 | 0 |
| $\alpha_3$ | 0 | 1 | 0 | $\mathbf{F}^J(\alpha_3) = \alpha_2$ | 1 | 0 | 0 |
| $\alpha_4$ | 0 | 0 | 1 | $\mathbf{F}^J(\alpha_4) = \alpha_4$ | 0 | 0 | 1 |
| $\alpha_5$ | 1 | 1 | 0 | $\mathbf{F}^J(\alpha_5) = \alpha_5$ | 1 | 1 | 0 |
| $\alpha_6$ | 1 | 0 | 1 | $\mathbf{F}^J(\alpha_6) = \alpha_7$ | 0 | 1 | 1 |
| $\alpha_7$ | 0 | 1 | 1 | $\mathbf{F}^J(\alpha_7) = \alpha_6$ | 1 | 0 | 1 |
| $\alpha_8$ | 1 | 1 | 1 | $\mathbf{F}^J(\alpha_8) = \alpha_8$ | 1 | 1 | 1 |
From Theorem 4.1, we obtain a lax morphism $\eta_H : H \to (J \otimes H)^J$ of $F$-sup-semilattices that is defined as

$$(\eta_H(x))(i) = \eta(j[J,H])((x_i =)).$$

Let us first construct $J \otimes H$. To do that, we will need

$$[J, H] = \{(x_i S \lor F(x)_i =, F(x)_i =) \mid x \in G, i \in T\}.$$ By definition, $x_{f,s}, F(x)_{f_i}$ and $x_{f,s} \lor F(x)_{f_i}$ for an arbitrary $x \in G$ are given by Table 3.

One can quickly obtain that $J \otimes H$ has 15 elements and $(J \otimes H)^J$ has $15^3$ elements.

Also, we can compute $\eta_H$ using Table 4:

This notation means that when $((\eta_H(x))(f_i) \in J \otimes H$ is mapped to $[(\alpha, \beta, \gamma)]$, then $((\eta_H(x))(f_i)$ is a class of maps $J \to H$ represented by the map of the form $f_2 \mapsto \alpha, f_3 \mapsto \beta, f_4 \mapsto \gamma$.

Now we will show that the lax morphism $\eta_H$ is a homomorphism of $F$-sup-semilattices. To do that, we will have to check that $(\eta_H(F(x)))$ and $F^J(\eta_H(x))$, where $F^J$ is the induced tense operator $F^J$ on $(J \otimes H)^J$, is the same map for every $x \in G$.

By the definition of $F$, it is clear that $\eta_H(F(x))$ where $x \in G$ is given in Table 5.

Let us now describe the induced tense operator $F^J$. Let $\varphi \in (J \otimes H)^J$ be an arbitrary map. By definition of the relation $\rho$, it is easy to see that for $F^J$, the following holds

$$(F^J(\varphi))(f_i) = \begin{cases} \varphi(f_3) & \text{if } i = 2, \\ \varphi(f_2) & \text{if } i = 3, \\ \varphi(f_4) & \text{if } i = 4. \end{cases}$$

This fact allows us to compute $F^J(\eta_H(x))$ where $x \in G$ and compare it with Table 5. The respective values are given in Table 6.

We conclude that $\eta_H$ is a homomorphism of $F$-sup-semilattices. Since $\eta_H$ is evidently injective, we obtain, as in Example 5.1, that $\eta_H$ is an element of $\mathcal{E}_\leq$.

6. Conclusions

In this paper, we have presented three basic construction methods to construct

(i) an $F$-sup-semilattice from a sup-semilattice and a relation,
(ii) a sup-semilattice from an $F$-sup-semilattice and a relation, and
(iii) a relation from an $F$-sup-semilattice and a sup-semilattice, and obtained three induced adjoint situations between the respective categories. This result gives us a unifying view of recent results about representations of tense operators in different categories of posets and lattices.

For future work, there are a number of open problems that we plan to address. In particular, we can mention the following ones:
Table 3. Maps $x_{f_iS}$, $F(x)_{f_i}=x$ and $x_{f_iS} ∨ F(x)_{f_i}=x$.

|       | $f_2$ | $f_3$ | $f_4$ |       | $f_2$ | $f_3$ | $f_4$ |       | $f_2$ | $f_3$ | $f_4$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $0_{f_2S}$ | 0     | 0     | 0     | $F(0)_{f_2}=0$ | 0     | 0     | 0     | $0_{f_2S} ∨ F(0)_{f_2}=0$ | 0     | 0     | 0     |
| $0_{f_3S}$ | 0     | 0     | 0     | $F(0)_{f_3}=0$ | 0     | 0     | 0     | $0_{f_3S} ∨ F(0)_{f_3}=0$ | 0     | 0     | 0     |
| $0_{f_4S}$ | 0     | 0     | 0     | $F(0)_{f_4}=0$ | 0     | 0     | 0     | $0_{f_4S} ∨ F(0)_{f_4}=0$ | 0     | 0     | 0     |
| $a_{f_2S}$ | 0     | $a$  | 0     | $F(a)_{f_2}=a$ | 0     | 0     | 0     | $a_{f_2S} ∨ F(a)_{f_2}=a$ | $a$  | $a$  | 0     |
| $a_{f_3S}$ | $a$  | 0     | 0     | $F(a)_{f_3}=a$ | 0     | 0     | 0     | $a_{f_3S} ∨ F(a)_{f_3}=a$ | $a$  | $a$  | 0     |
| $a_{f_4S}$ | 0     | 0     | $a$  | $F(a)_{f_4}=a$ | 0     | 0     | 0     | $a_{f_4S} ∨ F(a)_{f_4}=a$ | 0     | 0     | $a$  |
| $b_{f_2S}$ | 0     | $b$  | 0     | $F(b)_{f_2}=b$ | 0     | 0     | 0     | $b_{f_2S} ∨ F(b)_{f_2}=b$ | $c$  | $b$  | 0     |
| $b_{f_3S}$ | $b$  | 0     | 0     | $F(b)_{f_3}=b$ | 0     | 0     | 0     | $b_{f_3S} ∨ F(b)_{f_3}=b$ | $b$  | $c$  | 0     |
| $b_{f_4S}$ | 0     | 0     | $b$  | $F(b)_{f_4}=b$ | 0     | 0     | 0     | $b_{f_4S} ∨ F(b)_{f_4}=b$ | 0     | 0     | 1     |
| $c_{f_2S}$ | 0     | $c$  | 0     | $F(c)_{f_2}=c$ | 0     | 0     | 0     | $c_{f_2S} ∨ F(c)_{f_2}=c$ | $b$  | $c$  | 0     |
| $c_{f_3S}$ | $c$  | 0     | 0     | $F(c)_{f_3}=c$ | 0     | 0     | 0     | $c_{f_3S} ∨ F(c)_{f_3}=c$ | $b$  | $b$  | 0     |
| $c_{f_4S}$ | 0     | 0     | $c$  | $F(c)_{f_4}=c$ | 0     | 0     | 0     | $c_{f_4S} ∨ F(c)_{f_4}=c$ | 0     | 0     | 1     |
| $1_{f_2S}$ | 0     | 1     | 0     | $F(1)_{f_2}=1$ | 0     | 0     | 0     | $1_{f_2S} ∨ F(1)_{f_2}=1$ | 1     | 1     | 0     |
| $1_{f_3S}$ | 1     | 0     | 0     | $F(1)_{f_3}=1$ | 0     | 0     | 0     | $1_{f_3S} ∨ F(1)_{f_3}=1$ | 1     | 1     | 0     |
| $1_{f_4S}$ | 0     | 0     | 1     | $F(1)_{f_4}=1$ | 0     | 0     | 1     | $1_{f_4S} ∨ F(1)_{f_4}=1$ | 0     | 1     | 0     |
Table 4. Map $\eta_H$

|   | $f_2$ | $f_3$ | $f_4$ |
|---|-----|-----|-----|
| $\eta_H(0)$ | $[0,0,0]$ | $[0,0,0]$ | $[0,0,0]$ |
| $\eta_H(a)$ | $[a,a,0]$ | $[a,a,0]$ | $[0,0,a]$ |
| $\eta_H(b)$ | $[c,b,0]$ | $[b,c,0]$ | $[0,0,1]$ |
| $\eta_H(c)$ | $[b,c,0]$ | $[c,b,0]$ | $[0,0,1]$ |
| $\eta_H(1)$ | $[1,1,0]$ | $[1,1,0]$ | $[0,0,1]$ |

Table 5. Map $\eta_H \circ F$

|   | $f_2$ | $f_3$ | $f_4$ |
|---|-----|-----|-----|
| $\eta_H(F(0))$ | $[0,0,0]$ | $[0,0,0]$ | $[0,0,0]$ |
| $\eta_H(F(a))$ | $[a,a,0]$ | $[a,a,0]$ | $[0,0,a]$ |
| $\eta_H(F(b))$ | $[b,c,0]$ | $[c,b,0]$ | $[0,0,1]$ |
| $\eta_H(F(c))$ | $[c,b,0]$ | $[b,c,0]$ | $[0,0,1]$ |
| $\eta_H(F(1))$ | $[1,1,0]$ | $[1,1,0]$ | $[0,0,1]$ |

Table 6. Map $F^J \circ \eta_H$

|   | $f_2$ | $f_3$ | $f_4$ |
|---|-----|-----|-----|
| $F^J(\eta_H(0))$ | $[0,0,0]$ | $[0,0,0]$ | $[0,0,0]$ |
| $F^J(\eta_H(a))$ | $[a,a,0]$ | $[a,a,0]$ | $[0,0,a]$ |
| $F^J(\eta_H(b))$ | $[b,c,0]$ | $[c,b,0]$ | $[0,0,1]$ |
| $F^J(\eta_H(c))$ | $[c,b,0]$ | $[b,c,0]$ | $[0,0,1]$ |
| $F^J(\eta_H(1))$ | $[1,1,0]$ | $[1,1,0]$ | $[0,0,1]$ |

(1) As we have seen in Section 5, the lax morphism $\eta_H$ from Example 5.3 was a homomorphism of $F$-sup-semilattices. We will also look for necessary and sufficient conditions to ensure this in a general case.

(2) We intend to represent or approximate any pwos $\mathcal{A}$ in the following manner for a fixed sup-semilattice $\mathcal{L}$. The choice of $\mathcal{L}$ will, of course, depend on what behaviour we will be looking at (one could choose $\mathcal{L}$ to be a Boolean algebra, an MV-algebra or simply a finite chain). We plan to construct a suitable completion $\mathcal{H}_\mathcal{A}$ of $\mathcal{A}$, and using (iii), to obtain a new frame $\mathcal{J}_\mathcal{A}$. The composition of $\mathcal{A} \rightarrow \mathcal{J}_\mathcal{A}$ and $\eta_{H\mathcal{A}}$ will give us the desired representation or approximation.

(3) We intend to add more structure to our starting category of sup-semilattices. More precisely, we plan to substitute this category with a category of sup-algebras of a given type and the category of $F$-sup-semilattices with a category of $F$-sup-algebras of the same type, and try to answer the same questions as above.
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