THE STOCHASTIC GRAVITATIONAL–WAVE BACKGROUND PRODUCED BY NON–LINEAR COSMOLOGICAL PERTURBATIONS

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Abstract

The cosmological stochastic gravitational-wave background produced by the mildly non-linear evolution of density fluctuations is analyzed, in the frame of an Einstein-de Sitter model, by means of a fully relativistic perturbation expansion up to second order. The form of these gravitational-instability-induced gravitational waves is gauge-dependent. In the synchronous gauge, where the second-order expansion is most easily carried out, the transverse and traceless tensor modes which are produced also contain a Newtonian and post-Newtonian piece, whose interpretation as gravitational waves is non-trivial. A more direct physical understanding of this background is obtained in the so-called Poisson gauge, where it is seen to consist of a constant term plus an oscillating piece whose amplitude decays inside the Hubble radius.

1 Introduction

In these notes we will review some recent results concerning the generation of gravitational waves from the gravitational instability of density perturbations in a cosmological framework. The mixing of different modes (scalars, vectors and tensors) is a consequence of non-linearity, which already arises in second-order perturbation theory. To this aim we will first consider the exact relativistic dynamics of irrotational dust in the Einstein-de Sitter background (i.e. spatially flat Robertson-Walker geometry, with vanishing cosmological constant), in a synchronous and comoving frame. We will then expand the exact equations in this gauge, up to second order in the amplitude of deviations from the background solution. A second-order gauge transformation is then performed in order to obtain the metric perturbations in the so-called Poisson gauge, where the tensor modes are most easily interpreted as gravitational radiation.

Throughout these notes we use signature +2 for the metric; Greek indices take values from 0 to 3, and Latin ones from 1 to 3.

The line-element will be written in the form

\[ ds^2 = a^2(\tau)g_{\mu\nu}dx^\mu dx^\nu , \]

where \( a(\tau) \) is the scale-factor of the background flat Robertson-Walker universe and \( \tau \) is the conformal time. We deal with a pressureless fluid, with vanishing cosmological constant, so
the background Einstein-de Sitter model has \( a(\tau) \propto \tau^2 \). The components of the perturbed conformal metric tensor can be written as

\[
g_{00} = - \left[ 1 + 2 \sum_{r=1}^{\infty} \frac{1}{r!} \psi^{(r)} \right],
\]

\[
g_{0i} = \sum_{r=1}^{\infty} \frac{1}{r!} \omega_i^{(r)},
\]

\[
g_{ij} = \left[ 1 - 2 \left( \sum_{r=1}^{\infty} \frac{1}{r!} \phi^{(r)} \right) \delta_{ij} + \sum_{r=1}^{\infty} \frac{1}{r!} \chi_{ij}^{(r)} \right],
\]

where \( \chi_{ij}^{(r)} = 0 \). Indices are raised and lowered by the Kronecker symbols \( \delta_{ij} \) and \( \delta^{ij} \), respectively. The functions \( \psi^{(r)} \), \( \omega_i^{(r)} \), \( \phi^{(r)} \), and \( \chi_{ij}^{(r)} \) represent the \( r \)-th order perturbations of the metric.

We can split the perturbations into scalar (i.e. longitudinal), vector (i.e. transverse) and tensor (i.e. transverse and traceless) modes. In particular, \( \omega_i^{(r)} \) can be decomposed as

\[
\omega_i^{(r)} = \partial_i \omega^{(r)\parallel} + \omega_i^{(r)\perp},
\]

where \( \omega_i^{(r)\perp} \) is the vector component, i.e., \( \partial^i \omega_i^{(r)\perp} = 0 \). Similarly, the traceless part of the spatial metric can be decomposed as

\[
\chi_{ij}^{(r)} = D_{ij} \chi^{(r)\parallel} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)\perp},
\]

where \( \chi^{(r)\parallel} \) is a scalar potential, \( \chi_i^{(r)\perp} \) are vector modes, and \( \chi_{ij}^{(r)\perp} \) tensor modes (\( \partial^i \chi_{ij}^{(r)\perp} = 0 \)); in the equation above we introduced the symbol

\[
D_{ij} \equiv \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2.
\]

It is also useful to introduce similar expansions for the density contrast

\[
\delta \equiv \frac{\varrho}{\varrho_b} - 1 = \sum_{r=1}^{\infty} \frac{1}{r!} \delta^{(r)},
\]

with \( \varrho(x, \tau) \) the matter density and \( \varrho_b(\tau) = 3/2\pi G a^2(\tau) \tau^2 \) its background mean value, and the fluid four-velocity \( u^\mu \),

\[
u^\mu = \frac{1}{a} \left( \delta_0^\mu + \sum_{r=1}^{\infty} \frac{1}{r!} v_i^{\mu(r)} \right).
\]

The four-vector \( u^\mu \) is subject to the normalization condition \( u^\mu u_\mu = -1 \); therefore at any order the time component \( v_0^{(r)} \) is related to the perturbation \( \psi^{(r)} \). For the first and second-order perturbations one obtains

\[
v_0^{(1)} = -\psi^{(1)},
\]

\[
v_0^{(2)} = -\psi^{(2)} + 3\psi^{(1)}_0 + 2\omega_i^{(1)} v_i^{(1)} + v_i^{(1)} v_i^{(1)}.
\]

The velocity perturbation \( v_i^{(r)} \) can also be split into scalar and vector parts

\[
v_i^{(r)} = \partial^j v_j^{(r)} + v_i^{(r)\perp}.
\]
2 Relativistic dynamics of irrotational dust in Lagrangian coordinates

We start by writing the Einstein’s equations for a perfect fluid of irrotational dust in the synchronous ($g_{00} = -1, g_{0i} = 0$) and comoving ($u^\mu = \delta^\mu_0/a$) gauge. The formalism outlined in this section is discussed in greater detail in Ref. [2].

The line–element is now written in the form

$$ds^2 = a^2(\tau)[ - d\tau^2 + g_{ij}(x, \tau)dx^i dx^j] ,$$

with the spatial coordinates $x$ representing Lagrangian coordinates for the fluid elements.

By subtracting the isotropic Hubble–flow, one introduces the extrinsic curvature of constant $\tau$ hypersurfaces,

$$\vartheta^i_j = \frac{1}{2} g^{ik} g^j_k ,$$

with a prime denoting differentiation with respect to the conformal time $\tau$.

One can then write the Einstein’s equations in a cosmologically convenient form. The energy constraint reads

$$\vartheta^2 - \vartheta^i_j \vartheta^j_i + \frac{8}{\tau} \vartheta + \mathcal{R} = \frac{24}{\tau^2} \delta ,$$

where $\mathcal{R}^i_j(g)$ is the intrinsic curvature of constant time hypersurfaces, i.e. the conformal Ricci curvature of the three–space with metric $g_{ij}$ and $\mathcal{R} = \mathcal{R}^i_i$.

The momentum constraint reads

$$\vartheta^i_{j||} = \vartheta_{,j} ,$$

where the vertical bar indicates a covariant derivative in the three–space with metric $g_{ij}$.

Finally, after replacing the density from the energy constraint and subtracting the background contribution, the evolution equation for the extrinsic curvature reads

$$\vartheta^i_j + \frac{4}{\tau} \vartheta^i_j + \vartheta \vartheta^i_j + \frac{1}{4} \left( \vartheta^k_i \vartheta^j_k - \vartheta^2 \right) \delta^i_j + \mathcal{R}^i_j - \frac{1}{4} \mathcal{R} \delta^i_j = 0 .$$

Also useful is the Raychaudhuri equation for the evolution of the peculiar volume expansion scalar $\vartheta$, namely

$$\vartheta' + \frac{2}{\tau} \vartheta + \vartheta^i_j \vartheta^j_i + \frac{6}{\tau^2} \delta = 0 .$$

An advantage of this gauge is that there are only geometric quantities in the equations, namely the spatial metric tensor with its time and space derivatives. The only remaining variable, the density contrast, can indeed be rewritten in terms of $g_{ij}$, by solving the continuity equation. We obtain

$$\delta(x, \tau) = (1 + \delta_0(x))[g(x, \tau)/g_0(x)]^{-1/2} - 1 ,$$

with $g \equiv \det g_{ij}$.

3 Perturbative approach in the synchronous gauge

In this section we will obtain the perturbations of the Einstein-de Sitter cosmological model up to second order in the synchronous and comoving gauge. Different approaches to this problem have been used so far. The first solution of the second-order relativistic equations has been obtained, in this gauge, in a pioneering work by Tomita [3]. Matarrese, Pantano & Saez [4, 5]...
obtained the leading order terms of the expansion, using a different method, based on the so-called fluid-flow approach (e.g. Ref. [8]). Salopek, Stewart and Croudace [7] used a gradient expansion technique to obtain second-order metric perturbations; an intrinsic limitation of their method is, however, that non-local terms, such as the non-linear tensor modes, are lost. Russ et al. [8] recently rederived the metric perturbations to second order in the synchronous gauge, using a tetrad formalism.

We start by expanding the covariant conformal spatial metric tensor up to second order in the form

\[ g_{ij} = \delta_{ij} + g^{(1)}_{ij} + \frac{1}{2} g^{(2)}_{ij}. \]  

(20)

The contravariant metric takes the form

\[ g^{ij} = \delta^{ij} - g^{(1)ij} - \frac{1}{2} g^{(2)ij} + g^{(1)ik} g^{(1)j}_k. \]  

(21)

The extrinsic curvature tensor \( \vartheta^i_j \) up to second order reads

\[ \vartheta^i_j = \frac{1}{2} \left( g^{(1)i'}_j + \frac{1}{2} g^{(2)i'}_j - g^{(1)ik} g^{(1)j}_k \right). \]  

(22)

The square root of the metric determinant is

\[ g^{1/2} = 1 + \frac{1}{2} g^{(1)i} + \frac{1}{4} g^{(2)i} + \frac{1}{8} (g^{(1)i})^2 - \frac{1}{4} g^{(1)ij} g^{(1)j}, \]  

(23)

with inverse

\[ g^{-1/2} = 1 - \frac{1}{2} g^{(1)i} - \frac{1}{4} g^{(2)i} + \frac{1}{8} (g^{(1)i})^2 + \frac{1}{4} g^{(1)ij} g^{(1)j}, \]  

(24)

so that the density contrast reads

\[ \delta = -\frac{1}{2} g^{(1)i} + \frac{1}{2} g^{(1)i}_0 + \delta_0 - \frac{1}{2} g^{(2)i} + \frac{1}{4} g^{(2)i}_0 + \frac{1}{8} (g^{(1)i})^2 + \frac{1}{4} (g^{(1)i})^2 - \frac{1}{4} g^{(1)ij} g^{(1)j} \]

\[ + \frac{1}{4} g^{(1)ij} g^{(1)j}_0 - \frac{1}{4} g^{(1)i} g^{(1)j}_0 - \frac{1}{2} g^{(1)ij} \delta_0 + \frac{1}{2} g^{(1)i}_0 \delta_0, \]  

(25)

having assumed as initial conditions \( g^{(0)ij}_0 = 0 \) and \( \delta^{(0)}_0 = 0 \) (i.e. \( \delta_0 = \delta^{(0)}_0 \)).

The Christoffel symbols up to second order read

\[ \Gamma^i_{jk} = \frac{1}{2} \left( g^{(1)j,k}_i + g^{(1)k,j}_i - g^{(1)j}_i \right) + \frac{1}{4} \left( g^{(2)j,k}_i + g^{(2)k,j}_i - g^{(2)j}_i \right) - \frac{1}{2} g^{(1)j}_i \left( g^{(1)k}_k + g^{(1)k}_j - g^{(1)j}_j \right), \]  

(26)

from which, after a lengthy but straightforward calculation, the conformal Ricci tensor of the spatial hypersurfaces is obtained

\[ \mathcal{R}^i_j = \frac{1}{2} \left( g^{(1)j,k}_k - g^{(1)j,k}_j - \nabla^2 g^{(1)j}_i - g^{(1)j,k}_i \right) + \frac{1}{4} g^{(2)j,k}_k - g^{(2)j,k}_j - \nabla^2 g^{(2)j}_i - g^{(2)j,k}_i \]

\[ + \frac{1}{2} \left( g^{(1)j,k}_k - g^{(1)j,k}_j - g^{(1)j,k}_j - g^{(1)j,k}_j \right) + \frac{1}{4} \left( g^{(2)j,k}_k - g^{(2)j,k}_j - g^{(2)j,k}_j - g^{(2)j,k}_j \right) \]

\[ - g^{(1)i,j}_j + g^{(1)i,j}_j - g^{(1)i,j}_j + g^{(1)i,j}_j \]

\[ + \frac{1}{2} g^{(1)i,j}_j - g^{(1)i,j}_j + \frac{1}{2} g^{(1)i,j}_j + g^{(1)i,j}_j - g^{(1)i,j}_j, \]  

(27)

its trace reads

\[ \mathcal{R} = g^{(1)j,k} - g^{(1)j,k}_j + \frac{1}{2} g^{(2)j,k} - g^{(2)j,k}_j + g^{(1)j,k} \left( \nabla^2 g^{(1)j} + g^{(1)j} - 2 g^{(1)j} \right) \]

\[ + g^{(1)j,k} \left( g^{(1)j} - g^{(1)j} \right) + \frac{3}{4} g^{(1)j,k} - \frac{1}{2} g^{(1)j,k} - \frac{1}{2} g^{(1)j,k} - \frac{1}{2} g^{(1)j,k}. \]  

(28)
3.1 First-order perturbations

We are now ready to deal with the equations above at the linear level. Let us then write the conformal spatial metric tensor in the form

\[ g_{ij} = \delta_{ij} + g^{(1)}_{ij}, \]

where, from now on, we use the subscript \( S \) to label synchronous-gauge perturbations.

We then write

\[ g^{(1)}_{ij} = -2\phi^{(1)}_S \delta_{ij} + D_{ij} \chi^{(1)\|}_S + \partial_i \chi^{(1)\perp}_S + \partial_j \chi^{(1)\perp}_S + \chi^{(1)\top}_{ij}, \]

with

\[ \partial_i \chi^{(1)\perp}_S = \chi^{(1)\top}_i = \partial^i \chi^{(1)\top}_S = 0, \]

indices being raised by \( \delta^{ij} \). Remind that at first order the tensor modes \( \chi^{(1)\top}_{ij} \) are gauge-invariant.

As well known, in linear theory, scalar, vector and tensor modes are independent. The equation of motion for the tensor modes is obtained by linearizing the traceless part of the \( \vartheta_{ij} \) evolution equation. One has

\[ \chi^{(1)\top}_{ij}'' + \frac{4}{\tau} \chi^{(1)\top}_{ij}' - \nabla^2 \chi^{(1)\top}_{ij} = 0, \]

which is the equation for the free propagation of gravitational waves in the Einstein-de Sitter universe. The general solution of this equation is

\[ \chi^{(1)\top}_{ij}(x, \tau) = \frac{1}{(2\pi)^3} \int d^3k \exp(ik \cdot x) \chi^{(1)\top}_\sigma(k, \tau)e^\sigma_{ij}(\hat{k}), \]

where \( e^\sigma_{ij}(\hat{k}) \) is the polarization tensor, with \( \sigma \) ranging over the polarization components \( +, \times \), and \( \chi^{(1)\top}_\sigma(k, \tau) \) are the amplitudes of the two polarization states, whose time evolution can be represented as

\[ \chi^{(1)}_\sigma(k, \tau) = A(k)a_\sigma(k) \left( \frac{3j_1(k\tau)}{k\tau} \right), \]

with \( j_1 \) the spherical Bessel function of order 1 and \( a_\sigma(k) \) a zero mean random variable with auto-correlation function

\[ \langle a_\sigma(k)a_{\sigma'}(k') \rangle = (2\pi)^3k^{-3}\delta^3(k + k')\delta_{\sigma\sigma'}. \]

The spectrum of the gravitational-wave background depends on the processes by which it was generated, and for example in most inflationary models, \( A(k) \) is nearly scale invariant and proportional to the Hubble constant during inflation.

In the irrotational case the linear vector perturbations represent gauge modes which can be set to zero: \( \chi^{(1)\perp}_i = 0 \).

The two scalar modes are linked together via the momentum constraint, leading to the condition

\[ \phi^{(1)}_S + \frac{1}{6}\nabla^2 \chi^{(1)\|}_S = \phi^{(1)}_{S0} + \frac{1}{6}\nabla^2 \chi^{(1)\|}_{S0}. \]

The energy constraint gives

\[ \nabla^2 \left[ \frac{2}{\tau}\chi^{(1)\|} + \frac{6}{\tau^2}(\chi^{(1)\|} - \chi^{(1)\|}_{S0}) + 2\phi^{(1)}_{S0} + \frac{1}{3}\nabla^2 \chi^{(1)\|}_{S0} \right] = \frac{12}{\tau^2}\delta_0, \]
having consistently assumed $\delta_0 \ll 1$.

The trace of the evolution equation also gives an equation for the scalar modes,

$$\chi_S^{(1)''} + \frac{4}{\tau} \chi_S^{(1)'} + \frac{1}{3} \nabla^2 \chi_S^{(1)} = -2\phi_S^{(1)}. \quad (38)$$

An equation only for the scalar mode $\chi_S^{(1)}$ can be obtained by combining together the evolution equation and the energy constraint,

$$\nabla^2 \left[ \chi_S^{(1)''} + \frac{2}{\tau} \chi_S^{(1)'} - \frac{6}{\tau^2} (\chi_S^{(1)} - \chi_S^{(1)}) \right] = -\frac{2}{\tau^2} \delta_0 . \quad (39)$$

On the other hand, by linearizing the solution of the continuity equation, we obtain

$$\delta_S^{(1)} = \delta_0 - \frac{1}{2} \nabla^2 (\chi_S^{(1)} - \chi_S^{(1)}) , \quad (40)$$

which replaced in the previous expression gives

$$\delta_S^{(1)''} + \frac{a'}{a} \delta_S^{(1)'} - 4\pi G a^2 \varphi_0 \delta_S^{(1)} = 0 . \quad (41)$$

This is the equation for linear density fluctuations (e.g. Ref. [9]) whose general solution is straightforward to obtain.

The equations above have been obtained in whole generality; one could have used instead the well-known residual gauge ambiguity of the synchronous coordinates (e.g. Refs. [2, 8]) to simplify their form. For instance, one could fix $\chi_0^{(1)}$ so that $\nabla^2 \chi_S^{(1)} = -2\delta_0$, so that the $\chi_S^{(1)}$ evolution equation takes the same form as that for $\delta_S^{(1)}$. With such a gauge fixing one obtains

$$\chi_S^{(1)}(\mathbf{x}, \tau) = \chi_+(\mathbf{x}) \tau^2 + \chi_-(\mathbf{x}) \tau^{-3} , \quad (42)$$

where $\chi_\pm$ set the amplitudes of the growing (+) and decaying (−) modes. In what follows, we shall restrict ourselves to the growing mode. The effect of the decaying mode on second-order perturbations has been considered by Tomita [3] and by Russ et al. [8] and will not be studied here. The amplitude of the growing mode is related to the initial peculiar gravitational potential, through $\chi_+ \equiv -\frac{1}{3} \varphi$, where in turn, $\varphi$ is related to $\delta_0$ through the cosmological Poisson equation

$$\nabla^2 \varphi(\mathbf{x}) = \frac{6}{\tau_0^2} \delta_0(\mathbf{x}) . \quad (43)$$

Therefore,

$$D_{ij} \chi_S^{(1)} = -\frac{\tau^2}{3} \left( \varphi_{,ij} - \frac{1}{3} \delta_{ij} \nabla^2 \varphi \right) . \quad (44)$$

The remaining scalar mode reads

$$\phi_S^{(1)}(\mathbf{x}, \tau) = \frac{5}{3} \varphi(\mathbf{x}) + \frac{\tau^2}{18} \nabla^2 \varphi(\mathbf{x}) . \quad (45)$$

In what follows we will also drop the tensor mode $\chi_{ij}^{(1)\top}$ as a possible source for second-order perturbations, i.e. we will restrict ourselves to initial growing-mode scalar perturbations. The effect of these modes on second-order perturbations has been studied by Tomita [10, 11] and Matarrese, Mollerach & Bruni [12].
3.2 Second-order perturbations

Let us start by the definition

$$g_{Sij}^{(2)} = -2\phi_S^{(2)}\delta_{ij} + \chi_{Sij}^{(2)},$$

(46)

with $\chi_{Sij}^{(2)} = 0$.

The technique of second-order perturbation theory is straightforward: with the help of the relations reported at the beginning of Section 3 [Eqs. (20) - (28)], we first substitute the expansion above in our exact fluid-dynamical equations (momentum and energy constraints plus evolution and Raychaudhuri equations) obtaining equations for $g_{Sij}^{(2)}$, with source terms containing quadratic combinations of $g_{Sij}^{(1)}$ plus a few more terms involving $\delta_0$. Next, we have to solve these equations for the modes $\phi_S^{(2)}$ and $\chi_{Sij}^{(2)}$ in terms of the initial peculiar gravitational potential $\varphi$.

Let us now give the equations which govern the evolution of the second-order metric perturbations.

The second-order Raychaudhuri equation reads

$$\phi_S^{(2)\nu} + \frac{2}{\tau}\phi_S^{(2)\nu} - \frac{6}{\tau^2}\phi_S^{(2)} - \frac{1}{4}\chi_S^{(2)ij} = -\frac{1}{6}g_{Sij}^{(1)}\phi_S^{(2)ij\prime} - \frac{1}{2}\left(\frac{1}{4}\chi_S^{(1)ij} - \frac{1}{2}\left(\frac{1}{2}\delta_S^{(1)i} - \frac{1}{2}\delta_S^{(1)i}\right) \right)$$

$$- \frac{2}{\tau^2}\left(\frac{1}{4}\delta_S^{(1)i} - \frac{1}{2}\delta_S^{(1)i}\right) + \frac{3}{4}\chi_S^{(1)ij}g_{Sj\ell}^{(1)} - \frac{3}{4}\delta_S^{(1)i}g_{Sij}^{(1)} + \chi_S^{(2)ij}.$$

(47)

The second-order energy constraint reads

$$\frac{2}{\tau}\phi_S^{(2)\nu} - \frac{1}{2}\chi_S^{(2)ij} = -\frac{1}{6}\phi_S^{(2)}\phi_S^{(2)} - \frac{1}{2}\chi_S^{(2)ij} = -\frac{1}{6}\phi_S^{(2)}\phi_S^{(2)} - \frac{1}{2}\chi_S^{(2)ij}.$$

(48)

The second-order momentum constraint reads

$$2\phi_S^{(2)\nu} + \frac{1}{2}\chi_S^{(2)ij} = \phi_S^{(1)ik}g_{Sjk,i}^{(1)} + \phi_S^{(1)ik}g_{Sjk,i}^{(1)} - \frac{1}{2}\phi_S^{(1)ik}g_{Sjk,i}^{(1)} - \frac{1}{2}\phi_S^{(1)ik}g_{Sjk,i}^{(1)}.$$

(49)

Finally, the second-order evolution equation reads

$$2\phi_S^{(2)\nu} + \frac{1}{2}\chi_S^{(2)ij} = \phi_S^{(1)ik}g_{Sjk,i}^{(1)} + \phi_S^{(1)ik}g_{Sjk,i}^{(1)} - \frac{1}{2}\phi_S^{(1)ik}g_{Sjk,i}^{(1)} - \frac{1}{2}\phi_S^{(1)ik}g_{Sjk,i}^{(1)}.$$
\[ + 2g^{(1)}_{\ell k}(g_{\ell j}^{(1)i} - g_{\ell j}^{(1)i} - g_{\ell j}^{(1)i}) + 2g_{\ell k}^{(1)i} - g_{\ell j}^{(1)i} - g_{\ell j}^{(1)i} + g_{\ell j}^{(1)i} g_{\ell j}^{(1)i} + g_{\ell j}^{(1)i} g_{\ell j}^{(1)i} \]

\[ + g^{(1)}_{\ell k}(g_{\ell j}^{(1)i} + g_{\ell j}^{(1)i} - g_{\ell j}^{(1)i}) - g^{(1)}_{\ell k}(\nabla^2 g_{\ell k} + g_{\ell k}^{(1)m} - 2g_{\ell k}^{(1)m}) \delta^i_j \]

\[ - g_{\ell k}^{(1)} m_{\ell m} g_{\ell m}^{(1)} \delta^i_j - \frac{3}{4} g_{\ell k}^{(1)m} m_{\ell m} \delta^i_j + \frac{1}{4} g_{\ell k}^{(1)m} m_{\ell m} \delta^i_j \]

\[ + \frac{1}{4} g_{\ell k}^{(1)m} m_{\ell m} \delta^i_j . \] (50)

The next step is to solve these equations. In these calculations, we can make the simplifying assumption that the initial conditions are taken at conformal time \( \tau_0 = 0 \). One can start from the Raychaudhuri equation, to obtain the trace of the second-order metric tensor. (Actually, in order to obtain the sub-leading term we also need to use the energy constraint). The resulting expression for \( \phi_S^{(2)} \) is

\[ \phi_S^{(2)} = \frac{\tau^4}{252} \left( -\frac{10}{3} \phi^{ki} \phi_{,ki} + (\nabla^2 \phi)^2 \right) + \frac{5\tau^2}{18} \left( \phi^{k} \phi_{,k}^2 + \frac{4}{3} \phi \nabla^2 \phi \right) . \] (51)

The expression for \( \chi_S^{(2)} \) is obtained by first replacing \( \phi_S^{(2)} \) into the remaining equations and solving first the energy constraint, next the momentum constraint and finally the (traceless part of the) evolution equation. We obtain

\[ \chi_S^{(2)} = \frac{\tau^4}{126} \left( 19 \phi^{k} \phi_{,k}^2 - 12 \phi_{,ij} \nabla^2 \phi + 4(\nabla^2 \phi)^2 \delta_{ij} - \frac{19}{3} \phi^{k} \phi_{,kl} \delta_{ij} \right) \]

\[ + \frac{5\tau^2}{9} \left( -6 \phi_{,i} \phi_{,j} - 4 \phi_{,ij} + 2 \phi^{k} \phi_{,k} \delta_{ij} + \frac{4}{3} \phi \nabla^2 \phi \delta_{ij} \right) + \pi_{ij} . \] (52)

The transverse and traceless contribution \( \pi_{ij} \), which represents the second-order tensor mode \( \chi_{Sij}^{(2)} \) generated by (growing-mode) scalar initial perturbations, is determined by the inhomogeneous wave-equation

\[ \pi_{ij}'' + \frac{4}{\tau} \pi_{ij}' - \nabla^2 \pi_{ij} = -\frac{\tau^4}{21} \nabla^2 S_{ij} , \] (53)

with

\[ S_{ij} = \nabla^2 \Psi_0 \delta_{ij} + \Psi_{0,ij} + 2 \left( \phi_{,ij} \nabla^2 \phi - \phi_{,ik} \phi^{ik} \right) , \] (54)

where

\[ \nabla^2 \Psi_0 = -\frac{1}{2} \left( (\nabla^2 \phi)^2 - \phi_{,ik} \phi^{ik} \right) . \] (55)

This equation can be solved by the Green’s method; we obtain for \( \pi_{ij} \) that

\[ \pi_{ij}(x, \tau) = \frac{\tau^4}{21} S_{ij}(x) + \frac{4\tau^2}{3} T_{ij}(x) + \bar{\pi}_{ij}(x, \tau) , \] (56)

where \( \nabla^2 T_{ij} = S_{ij} \) and the remaining piece \( \bar{\pi}_{ij} \), containing a term that is constant in time and another one that oscillates with decreasing amplitude just like the linear tensor modes, can be written as

\[ \bar{\pi}_{ij}(x, \tau) = \frac{1}{(2\pi)^3} \int d^3 k \exp(i k \cdot x) \frac{40}{k^2} S_{ij}(k) \left( \frac{1}{3} - \frac{j_1(k\tau)}{k\tau} \right) , \] (57)

with \( S_{ij}(k) = \int d^3 x \exp(-i k \cdot x) S_{ij}(x) \).
4 Perturbative approach in the Poisson gauge

The Poisson gauge, recently discussed by Bertschinger \[13\], is uniquely defined by $\omega_{P_i}^{(r),j} = \chi_{P_{ij}}^{(r),j} = 0$. This gauge generalizes the so-called longitudinal gauge to include vector and tensor modes. The latter gauge, in which $\omega_{Li}^{(r)} = \chi_{Li}^{(r)} = 0$, has been widely used in the literature to investigate the linear evolution of scalar perturbations \[14\]. Since vector and tensor modes are set to zero by hand, the longitudinal gauge cannot be used to study perturbations beyond the linear regime, because in the nonlinear case vector and tensor modes are dynamically generated by the growth of scalar modes.

Instead of writing the perturbed Einstein’s equations directly in this gauge, we will transform the synchronous gauge quantities, by means of the second-order gauge transformations introduced in \[1\]. This problem has been dealt with in detail in Refs. \[1, 12\].

4.1 Transforming from the synchronous to the Poisson gauge

In these notes, we are only interested in calculating first and second-order perturbations of the metric tensor. A gauge transformation of order $r$ can be associated to a coordinate transformation along $r$ independent vector fields $\xi^\nu_{(r)}$, given by

$$\tilde{x}^\mu = x^\mu - \xi^\mu_{(1)} + \frac{1}{2} \left( \xi^\mu_{(1),\nu} \xi^\nu_{(1)} - \xi^\mu_{(2)} \right) + \ldots .$$

(58)

The general gauge transformation rule for a tensor $T$, with background value $T_0$ and fluctuation $\Delta T = \sum_{r=1}^{\infty} (1/r!) \delta^r T$ is

$$\delta \tilde{T} = \delta T + \mathcal{L}_{\xi_{(1)}} T_\theta ,$$

(59)

to first order,

$$\delta^2 \tilde{T} = \delta^2 T + 2 \mathcal{L}_{\xi_{(1)}} \delta T + \mathcal{L}_{\xi_{(2)}} T_\theta + \mathcal{L}_{\xi_{(3)}} T_\theta ,$$

(60)

to second order, etc.. Here $\mathcal{L}_{\xi}$ is the first-order Lie derivative along the vector $\xi^\mu$, $\mathcal{L}_{\xi}^2$ is the second-order one, etc... and we remind that the Lie derivative acting on a covariant tensor $T_{\mu\nu}$ of rank two (which is the case we are interested in here) is

$$\mathcal{L}_{\xi} T_{\mu\nu} = T_{\mu\nu,\sigma} \xi^\sigma + \xi^\sigma \cdot , \mu T_{\sigma\nu} + \xi^\sigma \cdot , \nu T_{\mu\sigma} .$$

(61)

A general $r$-th order gauge transformation is then determined by a set of $r$ vectors $\xi^\nu_{(r)}$, whose components we can also split in scalar and vector modes, as usual

$$\xi^0_{(r)} = \alpha^{(r)} ,$$

(62)

and

$$\xi^i_{(r)} = \partial^i \beta^{(r)} + d^{(r)i} ,$$

(63)

with $\partial_i d^{(r)i} = 0$.

We are here only interested in performing the gauge transformation on the covariant metric tensor $a^2(\tau)g_{\mu\nu}$. In transforming from the synchronous to the Poisson gauge, to first order, one obtains \[1\]

$$\psi_p^{(1)} = \alpha^{(1)} + \frac{2}{\tau} \alpha^{(1)} ,$$

(64)

$$\alpha^{(1)} = \beta^{(1)} ,$$

(65)

$$\omega_{P, i}^{(1)} = d_i^{(1)} ,$$

(66)
perturbed quantities, namely

\[
\phi^{(1)}_p = \phi^{(1)}_s - \frac{1}{3} \nabla^2 \beta^{(1)} - \frac{2}{\tau} \alpha^{(1)},
\]

\[
D_{ij} \left( \chi^{(1)\perp}_s + 2\beta^{(1)} \right) = 0,
\]

\[
\chi^{(1)\perp}_s + d^{(1)}_{i,j} = 0,
\]

\[
\chi^{(1)\top}_p = \chi^{(1)\top}_s.
\]

From these equations, given the first order perturbed metric in the synchronous gauge, we can obtain the parameters of the gauge transformation and the perturbed metric in the Poisson gauge.

The second-order transformation, in the particular case where only scalar modes are present at first order \( (\chi^{(1)\perp}_s = \chi^{(1)\top}_s = v^{(1)}(t), = 0 \), implying \( d^{(1)}_{i,j} = \omega^{(1)}_p = \chi^{(1)\top}_p = 0 \), yields

\[
\psi^{(2)} = \beta'_{(1)} \left[ \beta''_{(1)} + \frac{10}{\tau} \beta''_{(1)} + \frac{6}{\tau^2} \beta''_{(1)} \right] + \beta^i_{(1)} \left( \beta''_{(1)x} + \frac{2}{\tau} \beta''_{(1)x} \right) + 2\beta^i_{(1)} \alpha^{(2)} + \frac{2}{\tau},
\]

\[
\omega^{(2)}_p - i = -2 \left( 2\phi^{(1)} + \beta^{(1)} - \frac{2}{3} \nabla^2 \beta^{(1)} \right) \left( \beta''_{(1)} - 2\beta''_{(1)} \beta^{(1)} + \alpha^{(2)} - 2\beta^i_{(1)} \beta^{(1)} - d^{(2)}_{i,j} \right),
\]

\[
\phi^{(2)}_p = \phi^{(2)}_s + \beta_{(1)} \left[ 2(\phi^{(1)} + \frac{4}{\tau} \phi^{(1)}_s - \frac{6}{\tau^2} \beta_{(1)} + \frac{2}{\tau} \beta''_{(1)} \right] - \frac{1}{3} \left( -4\phi^{(1)} + \beta_{(1)} \partial_0 + \beta^i_{(1)} \partial_i + \frac{8}{\tau} \beta_{(1)} + \frac{4}{3} \nabla^2 \beta^{(1)} \right) \nabla^2 \beta^{(1)}
\]

\[
\chi^{(2)}_{p,ij} = \chi^{(2)}_{s,ij} + 2 \left( 4\nabla^2 \beta^{(1)} - 4\phi^{(1)} - \beta_{(1)} \partial_0 - \beta^k_{(1)} \partial_k \right) D_{ij} \beta^{(1)}
\]

One can, at least implicitly, compute the parameters involved in this second-order transformation in terms of the second order perturbed metric in the synchronous gauge and first order perturbed quantities, namely

\[
\nabla^2 \nabla^2 \beta^{(2)} = -\frac{3}{4} \chi^{(2),ij}_{s,ij} + 6\phi^{(1),ij}_{s,ij} - 2\nabla^2 \phi^{(1)} \nabla^2 \beta^{(1)} + 8\phi^{(1),i}_{s,ij} \nabla^2 \beta^{(1)} + 4\phi^{(1)} \nabla^2 \nabla^2 \beta^{(1)}
\]

\[
+ \frac{4}{3} \nabla^2 \beta^{(1)} \nabla^2 \beta^{(1)} + \frac{1}{6} \nabla^2 \beta^{(1)} \nabla^2 \beta^{(1)} + 5\beta^i_{(1)} \beta^{(1)} + \frac{2}{3} \nabla^2 \beta^{(1)} \nabla^2 \beta^{(1)} + \frac{3}{2} \beta^{(1)} \beta^{(1)} + \beta^{(1)} \beta^{(1)} + \beta^{(1)} \beta^{(1)} + \beta^{(1)} \beta^{(1)}
\]

\[
- \frac{1}{2} \nabla^2 \beta^{(1)} \nabla^2 \beta^{(1)} + 2\beta^{(1)} \nabla^2 \beta^{(1)} + \beta^{(1)} \nabla^2 \beta^{(1)} + \beta^{(1)} \nabla^2 \beta^{(1)}
\]

\[
\nabla^2 \delta^{(2)} = -\frac{4}{3} \nabla^2 \beta^{(2)}_{s,ij} + \chi^{(2),ij}_{s,ij} + 8\phi^{(1),ij}_{s,ij} D_{ij} \beta^{(1)} + \frac{16}{3} \phi^{(1)} \nabla^2 \beta^{(1)} + \frac{2}{3} \nabla^2 \beta^{(1)} \beta^{(1)} + \frac{10}{3} \beta^{(1)} \beta^{(1)}
\]

\[
- \frac{8}{9} \nabla^2 \beta^{(1)} \nabla^2 \beta^{(1)} + 2\beta^{(1)} \nabla^2 \beta^{(1)} + \frac{4}{3} \beta^{(1)} \nabla^2 \beta^{(1)} + \frac{4}{3} \beta^{(1)} \nabla^2 \beta^{(1)}
\]

\[
\nabla^2 \alpha^{(2)} = \nabla^2 \beta^{(2)} - 2 \left( 2\phi^{(1),i}_{s,ij} + \beta^{(1)}_{s,ij} + \frac{1}{3} \nabla^2 \beta^{(1)} \right) \beta^{(1)} + \frac{2}{3} \nabla^2 \beta^{(1)} - 2\beta^{(1)} \beta^{(1)}.
\]
4.2 Poisson gauge results

Using the metric perturbations in the synchronous gauge presented in Section 2, we can now fix the parameters of the first-order gauge transformation as

\[
\alpha^{(1)} = \frac{\tau}{3} \phi, \quad \beta^{(1)} = \frac{\tau^2}{6} \phi, \tag{78}
\]

and \( d^{(1)i} = 0 \), because of the absence of linear vector modes. We can then compute the first-order metric perturbations in the Poisson gauge,

\[
\psi^{(1)}_P = \phi^{(1)}_P = \phi, \tag{80}
\]

\[
\chi^{(1)}_{Pij} = \chi^{(1)}_{ij}. \tag{81}
\]

These equations show the well-known result for scalar perturbations in the longitudinal gauge and the gauge invariance for tensor modes at the linear level.

The parameters of the second-order transformation read \[15\]

\[
\alpha^{(2)} = -\frac{2}{21} \tau^3 \Psi_0 + \tau \left( \frac{10}{9} \phi^2 + 4 \Theta_0 \right), \tag{82}
\]

\[
\beta^{(2)} = \tau^4 \left( \frac{1}{72} \phi^i \phi^j - \frac{1}{42} \Psi_0 \right) + \frac{\tau^2}{3} \left( \frac{7}{2} \phi^2 + 6 \Theta_0 \right), \tag{83}
\]

where

\[
\nabla^2 \Theta_0 = \Psi_0 - \frac{1}{3} \phi^i \phi_i, \tag{84}
\]

and

\[
\nabla^2 d^{(2)}_j = \tau^2 \left( -\frac{4}{3} \phi_j \nabla^2 \phi + \frac{4}{3} \phi^i \phi_{ij} - \frac{8}{3} \Psi_{0,j} \right). \tag{85}
\]

For the second-order metric perturbations, one obtains \[16\]

\[
\psi^{(2)}_P = \tau^2 \left( \frac{1}{6} \phi^i \phi_i - \frac{10}{21} \Psi_0 \right) + \frac{16}{3} \phi^2 + 12 \Theta_0, \tag{86}
\]

\[
\phi^{(2)}_P = \tau^2 \left( \frac{1}{6} \phi^i \phi_i - \frac{10}{21} \Psi_0 \right) + \frac{4}{3} \phi^2 - 8 \Theta_0, \tag{87}
\]

\[
\nabla^2 \omega^{(2)i}_P = -\frac{8}{3} \tau \left( \phi^i \nabla^2 \phi - \phi^{ij} \phi_{ij} + 2 \Psi_0^i \right), \tag{88}
\]

\[
\chi^{(2)}_{Pij} = \tilde{\pi}_{ij}. \tag{89}
\]

Quite interesting is to write down the equation governing the evolution of the second-order tensor modes, \( \chi^{(2)}_{Pij} = \tilde{\pi}_{ij} \), in the Poisson gauge. It reads

\[
\tilde{\pi}''_{ij} + \frac{4}{\tau} \tilde{\pi}'_{ij} - \nabla^2 \tilde{\pi}_{ij} = -\frac{40}{3} T_{ij}, \tag{90}
\]

with the source \( T_{ij} \) defined in Section 3.2 [after Eq. (56)].

Note also that the expressions for \( \psi^{(2)}_P \) and \( \phi^{(2)}_P \) could have been recovered, except for the sub-leading time-independent terms, by taking the weak-field limit of Einstein’s theory (e.g. ref. \[16\]) and then expanding in powers of the perturbation amplitude.
5 Discussion

As we have seen, both in the synchronous and in the Poisson gauge, second-order tensor modes are dynamically generated by the gravitational instability of scalar fluctuations. The form of these modes is however quite different in the two gauges. The synchronous-gauge tensor modes contain four terms: the first one, $\propto \tau^4$, can be easily seen to represent a Newtonian contribution, describing the dynamical tidal induction acting from the environment on the fluid element, then there is a post-Newtonian term, $\propto \tau^2$, a constant post-post-Newtonian term, required by the vanishing initial conditions, having no obvious observational effects, and, finally, a wave-like piece, which has just the usual form as free cosmological gravitational waves. Quite interesting is the fact that the Newtonian and post-Newtonian terms are dropped by the transformation leading to the Poisson gauge, so making the physical meaning of our second-order tensor modes more transparent.

It has been argued [2] that a possible observational evidence for these tensor modes could be in terms of a tensor generalization of the so-called Rees-Sciama effect [17, 18, 19, 20, 21, 22, 23, 24], a secondary anisotropy of the Cosmic Microwave Background. Let us discuss this point in more detail. Mollerach & Matarrese [15] have recently obtained a general formula for the full (scalar, vector and tensor) Rees-Sciama effect. Their expression reads

$$\delta T_{RS} = \frac{1}{2} \int_{\lambda=0}^{\lambda_{\infty}} d\lambda \left( \psi^{(2)'} + \phi^{(2)'} + \omega_{\ell}^{(2)'} e_{\ell} - \frac{1}{2} \chi^{(2)'}_{ij} e_i e_j \right), \quad (91)$$

where $e_i$ represents the unit vector of the unperturbed incoming photon and the integration is along the unperturbed photon path. The first two terms correspond to the scalar contribution to the effect, while the third and fourth ones to the vector and tensor contributions, respectively. This splitting of the contributions depends strongly on the gauge choice. For example, it can be shown that in the Poisson gauge the contributions coming from the vector and tensor pieces are subdominant with respect to the scalar ones, while in the synchronous gauge the scalar and tensor contributions are of the same order of magnitude.

One may then wonder whether there is any hope to detect the cosmological stochastic gravitational-wave background produced by second-order scalar fluctuations, which we have discussed in these notes. The general prospects for detecting a stochastic gravitational-wave background have been nicely reviewed by Bruce Allen at this meeting (see also Ref. [25]). It is, of course, the oscillating part of $\chi_{ij}^{(2)\top}$ which is of relevance for earth or space detectors. The problem for these wave-like modes is that their energy density suffers the usual $a^{-4}$ dilution caused by free-streaming inside the Hubble radius, while on the horizon scale their closure density is already extremely small, $\Omega_{gw} \sim \delta^4_H$ (where $\delta_H$ is the $rms$ density contrast at horizon-crossing), because of their secondary origin. More promising is the possibility that a non-negligible amount of gravitational radiation is produced during the strongly non-linear stages of the collapse of cosmic proto-structures. This issue would however require a non-perturbative approach, which is beyond the available analytical techniques in this field. A first step in the direction of estimating the amplitude of such an effect has been recently made by Matarrese & Terranova [4], who studied the collapse of homogeneous triaxial ellipsoids embedded in an Einstein-de Sitter universe, including post-Newtonian tensor modes, and showed that these post-Newtonian terms tend to dominate over the Newtonian ones, during the late stages of collapse.

Acknowledgements.

This work has been partially supported by the Italian MURST and by the Vicerrectorado
de investigación de la Universidad de Valencia. We would like to thank M. Bruni and S. Sonego for many useful discussions.

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