Lorentz violating electrodynamics

Belinka González¹, Santiago A Martínez², Rafael Montemayor² and Luis F Urrutia¹

¹ Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, 04510 México D.F., México
² Instituto Balseiro and CAB, Universidad Nacional de Cuyo and CNEA, 8400 Bariloche, Argentina

E-mail: ¹urrutia@nucleares.unam.mx, ²montemay@cab.cnea.gov.ar

Abstract. After summarizing the most interesting results in the calculation of synchrotron radiation in the Myers-Pospelov effective model for Lorentz invariance violating (LIV) electrodynamics, we present a general unified way of describing the radiation regime of LIV electrodynamics which include the following three different models: Gambini-Pullin, Ellis et al. and Myers-Pospelov. Such unification reduces to the standard approach of radiation in a dispersive and absorptive (in general) medium with a given index of refraction. The formulation is presented up to second order in the LIV parameter and it is explicitly applied to the synchrotron radiation case.

1. Introduction

The appearance of detectable low energy effects arising from possible quantum gravity corrections to particle dynamics [1] has been recently the subject of intense scrutiny [2, 3]. The most direct interpretation of such effects, though not the only one [4], is related to the breaking of observer Lorentz covariance. In this way this subject becomes directly tied to the long time honored investigations, both theoretical and experimental, based upon the Standard Model Extension [5] concerning Lorentz and CPT violations [6]. Heuristic derivations of such effects from loop quantum gravity have been presented [7, 8], which make clear that a better understanding of the corresponding semiclassical limit is required [9]. String theory has also provided models for explaining such quantum gravity induced corrections [10]. Also, effective field theory models to describe such effects have been constructed [11].

The observation of 100 MeV synchrotron radiation from the Crab Nebula has recently been used to impose very stringent limits upon the parameters describing a modified photon dynamics, embodied in Maxwell equations that get correction terms which are linear in the Planck length [3]. Such bounds are based on a set of very reasonable assumptions on how some of the standard results of synchrotron radiation extend to the Lorentz non-invariant situation. This certainly implies some dynamical assumptions, which have been recently tested by two of us (RM and LFU) [12] in the framework of the classical version of the Myers-Pospelov (MP) effective theory, which parameterizes LIV using dimension five operators, as described in the first paper of Ref. [11]. Ref. [12] summarizes a complete calculation of synchrotron radiation in the context of this model. This constitutes an interesting problem on its own whose resolution will subsequently allow the use of additional observational information to put bounds upon
the correction parameters. For example we have in mind the polarization measurements from
cosmological sources. The case of gamma ray bursts has recently become increasingly relevant
[13], although it is still at a controversial stage [14].

This paper is organized as follows. In Section 2, we provide some highlights of the results
obtained in Ref. [12], and from there onwards we develop a formalism for a Lorentz violating
electrodynamics based on extended constitutive relations for the $D$ and $H$. In Section 3 we
state the general structure of this theory, in Section 4 we introduce a parameterization for the
constitutive relations, and in Sections 5 and 6 we summarize the main expressions for the fields,
energy spectrum, power spectrum and polarization in terms of effective refraction indices. We
close with a short summary in section 7.

2. Synchrotron radiation in the Myers-Pospelov model

This model parameterizes LIV using dimension five operators both in the matter and
electromagnetic sectors. There is also a preferred frame four velocity $V^\mu$, which is not a
dynamical field. We choose to work in the rest frame $V^\mu = (1, \mathbf{0})$. As usual the model possesses
observer Lorentz covariance, which means that the fields and the four-velocity $V^\mu$ transform as
tensors when going from one observer frame to another. On the other hand, in each frame we
violate particle Lorentz transformations; that is to say we have a collection of non-dynamical
physical fields in the action, in analogy to the physics going on in the presence of an external
magnetic field which violates particle rotation invariance, for example.

In the rest frame, the modified Maxwell equations are ($c = 1$)

\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\
\nabla \cdot \mathbf{E} &= 4\pi \rho, \\
-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} + \xi \partial_0 (-\nabla \times \mathbf{E} + \partial_0 \mathbf{B}) &= 4\pi \mathbf{j},
\end{align*}

(1)

where the LIV parameter $\tilde{\xi}$ is usually written as $\tilde{\xi} = \xi/M$ with $\xi$ being a dimensionless
number and $M$ being a mass scale characterizing the Lorentz symmetry breaking, which is usually,
but not necessarily, identified with the Plank mass. For the particular case of synchrotron radiation
the modified dynamics for a particle of charge $q$ in an external constant magnetic field in the
$z$-direction produces the standard helicoidal orbits with a modified Larmor frequency $\omega_0$

\[ \vec{r} = \frac{q}{E} \left(1 - \frac{3}{2} \tilde{\eta}E + \frac{9}{4} \tilde{\eta}^2 E^2\right) \left(\mathbf{v} \times \mathbf{B}\right) \equiv \omega_0 \left(\mathbf{v} \times \mathbf{B}\right), \]

(2)

depending upon another LIV parameter $\tilde{\eta} = -\eta/M$ and with $E$ being the energy of the particle.
The minus sign is chosen to make contact with the notation of Ref. [3]. We restrict ourselves
to circular orbits in the $x-y$ plane with Larmor radius $R = \beta/\omega_0$ and we use the standard
notation $\beta = |\mathbf{v}|$ and $\gamma = (1 - \beta^2)^{-1/2}$. When rewriting $\beta$ in terms of the particle energy we
obtain the modified expression

\[ 1 - \beta^2(E) = \frac{\mu^2}{E^2} \left[1 + 2 \frac{\tilde{\eta}E^3}{\mu^2} - \frac{15}{4} \frac{\tilde{\eta}^2 E^4}{\mu^2} + O(\tilde{\eta}^3)\right]. \]

(3)

Due to the admixture of vector and axial vectors in the last Eq. (1) the problem presents
birefringence and the two independent modes of propagation correspond to those of definite
circular polarization $\lambda = \pm$, with refraction indices $n_\lambda(z)$

\[ n_\lambda(z) = \sqrt{1 + z^2 + \lambda z}, \quad z = \tilde{\xi} \omega. \]

(4)
Such modes are identified via the corresponding Green functions, which are most easily calculated in the radiation gauge. It is interesting to observe that Eqs. (11) provide the exact expression

$$4\pi S = \mathbf{E} \times \mathbf{B} - \xi\mathbf{E} \times \partial\mathbf{E}/\partial t,$$

(5)

for the Poynting vector. Also, the magnetic and electric field satisfy the relation

$$\mathbf{B}(\omega, \hat{n}) = \sqrt{1 + z^2} \hat{n} \times \mathbf{E}(\omega, \hat{n}) - iz \mathbf{E}(\omega, \hat{n}).$$

(6)

in the radiation approximation, with \( \hat{n} = r/r \) being the direction of observation. These two effects conspire to produce a positive definite expression for the power flux yielding the following expression for the average angular distribution of the radiated power spectrum

$$\left\langle \frac{d^2 P(\Omega)}{d\omega d\Omega} \right\rangle = \sum_{\lambda=\pm} \sum_{m=0}^{\infty} \delta(\omega - \omega_m) \frac{dP_{m,\lambda}}{d\Omega}, \quad \omega_m = m\omega_0, \quad z_m = \tilde{z}\omega_m,$$

(7)

$$\frac{dP_{m,\lambda}}{d\Omega} = \frac{\omega_0^2 q^2}{4\pi} \frac{1}{\sqrt{1 + z_m^2}} \left[ \lambda \beta n_\lambda(z_m) J'_m(W_{\lambda m}) + \cot \theta J_m(W_{\lambda m}) \right]^2.$$  

(8)

Here, the average \( \langle \rangle \) is taken with respect to the macroscopic time \( T \) and \( J_m, J'_m \) denote the Bessel functions and their derivatives respectively. The argument of the Bessel functions is \( W_{\lambda m} = m n_\lambda(z_m) \beta \sin \theta \). We have also calculated the total averaged and integrated power radiated into the \( m \)-th harmonic

$$P_m = \frac{q^2 \beta^2 \omega_m}{R \sqrt{1 + z_m^2}} \sum_{\lambda=\pm} n_\lambda(z_m) \left[ J'_m(2m n_\lambda(z_m) \beta) - \frac{1 - \beta^2 n_\lambda^2(z_m)}{2\beta^2 n_\lambda^2(z_m)} \int_0^{2m \lambda(z_m) \beta} dx J_m(x) \right],$$

(9)

which clearly indicates the contribution of each polarization \( P_{\lambda m} \). The above result is exact in \( z_m \) and the parity-violating contribution has vanished after the angular integration.

In the case of synchrotron radiation from Crab nebula, as well as from other objects like Mkn 501 and GRB021206, one can estimate that the \( m \) corresponding to the observed radiation are very large, let us say in the range \( 10^{15} - 10^{30} \), with the respective values of \( m/\gamma \) varying within \( 10^{10} - 10^{20} \). These properties motivate the large \( m \) and \( m/\gamma \) expansion of the Bessel functions involved in the spectrum. In a similar way to the standard case [15], a first consequence of this approximation is the appearance of the cutoff frequency \( \omega_{\lambda c} = \tilde{m}_{\lambda c} \omega_0 \) with

$$\tilde{m}_{\lambda c} = \frac{3}{2} \left[ 1 - \beta^2(E) n_\lambda^2(z_m) \right]^{-3/2}.$$  

(10)

This means that for \( m > \tilde{m}_{\lambda c} \) the total power decreases as \( P_{\lambda m} \approx \exp(-m/\tilde{m}_{\lambda c}) \). Within the same large-\( m \) approximation, the integrated power in the \( m \)-th harmonic can be expanded to second order in \( \xi \) yielding

$$P_m = \frac{q^2 \omega}{\sqrt{3} \pi R \gamma^2 \kappa} \left[ \frac{m_c}{m} \kappa \left( \frac{m}{m_c} \right) - \frac{2}{\gamma^2} K_{2/3} \left( \frac{m}{m_c} \right) + 2 \left( \frac{\tilde{\xi} \omega \beta}{\gamma} \right)^2 K_{2/3} \left( \frac{m}{m_c} \right) \right],$$

(11)

where \( m_c = 3 \gamma^3/2 \) and \( \kappa(x) = x \int_x^\infty dy K_{5/3}(y) \) is the so called bremsstrahlung function. Here \( K_{p/q} \) denote the Bessel functions of fractional order. Let us notice the appearance of the combination \( \tilde{\xi} \omega m/\gamma = \xi(\omega/M_p)(m/\gamma) \) as the expansion parameter in (11). This is not necessarily a small number, which signals the possibility that such corrections might be relevant in setting bounds upon \( \tilde{\xi} \). This rather unexpected effect is due to the amplifying factor \( m/\gamma \).
Similar results have been obtained in calculations of the synchrotron radiation spectra in the context of non-commutative electrodynamics\cite{10}.

Another possibility for observable effects due to $\tilde{\xi}$ is to look at the averaged degree of circular polarization

$$\Pi_\odot = \frac{\langle P_+\omega - P_-\omega \rangle}{\langle P_+\omega + P_-\omega \rangle},$$  

where $P_\lambda(\omega)$ is the total power distribution per unit frequency and polarization $\lambda$, so that $P_\lambda(\omega) = P_{m\lambda}/\omega_0$. The average here is calculated with respect to an energy distribution of the relativistic electrons, which we take to be $N(E)dE = CE\gamma^{-p}dE$, in some energy range $E_1 < E < E_2$, chosen as $E_1 = 0$ and $E_2 \to \infty$ for simplicity. The result is

$$\Pi_\odot = \tilde{\xi}\omega (\frac{\mu \omega}{qB}) \Pi(p),$$

$$\Pi(p) = \frac{(p - 3)(3p - 1)}{3(3p - 7)} \frac{(p + 1)\Gamma \left(\frac{p}{3} + \frac{13}{12}\right)\Gamma \left(\frac{p}{3} + \frac{5}{12}\right)}{(p - 1)\Gamma \left(\frac{p}{3} + \frac{19}{12}\right)\Gamma \left(\frac{p}{3} + \frac{11}{12}\right)}, \quad p > 7/3.$$

Again, we have the presence of an amplifying factor in Eq. (13), given by $(\mu \omega/qB)$, which is independent of the form of $\Pi(p)$ and not necessarily a small number. An estimation of this factor in the zeroth-order approximation ($\tilde{\xi} = 0 = \tilde{\eta}$), which is appropriate in (13), yields $(\mu \omega/qB) = \omega/(\omega_0\gamma) = m/\gamma$. The expression (13) is analogous to the well-known average of the degree of linear polarization $\Pi_{LIN} = (p + 1)/(p + 7/3)$, under the same energy distribution for the electrons.

3. General structure of Lorentz violating electrodynamics

Three paradigmatic examples of Lorentz violating electrodynamics are given by the effective theories proposed by Gambini and Pullin (GP)\cite{7}, Ellis et al. (EMN)\cite{10}, and Myers and Pospelov (MP)\cite{11}. They can be written in the general form of Maxwell equations

$$\nabla \cdot D = 4\pi \rho, \quad \nabla \cdot B = 0,$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = \frac{\partial D}{\partial t} + 4\pi j,$$

with corresponding constitutive relations

$$D = D(E, B), \quad H = H(E, B),$$

which we next write in detail for each case, after reviewing the corresponding equations. Let us recall that the above equations (15) and (16) imply charge conservation $\partial \rho/\partial t + \nabla \cdot j = 0$, independently of the constitutive equations (17). In an abuse of notation have denoted by $\xi$ the electromagnetic LIV parameter for all models in the sequel.

3.1. Gambini-Pullin Electrodynamics

The Maxwell equations for this case are

$$\nabla \cdot B = 0, \quad \nabla \times \left(\mathcal{E} + 2\tilde{\xi}\nabla \times \mathcal{E}\right) + \frac{\partial B}{\partial t} = 0,$$

$$\nabla \cdot \mathcal{E} = 4\pi \rho, \quad \nabla \times \left(\mathcal{B} + 2\tilde{\xi}\nabla \times \mathcal{B}\right) - \frac{\partial \mathcal{E}}{\partial t} = 4\pi j.$$
where the electric and magnetic fields are identified from the homogeneous equation as

\[ E = \mathcal{E} + 2\tilde{\xi}\nabla \times \mathcal{E}, \quad B = \mathcal{B}. \]  

(20)

From the inhomogeneous equations we obtain

\[ D = \mathcal{E}, \quad H = B + 2\tilde{\xi}\nabla \times B, \]  

(21)

which together with the constitutive relations

\[ D + 2\tilde{\xi}\nabla \times D = E, \quad H = B + 2\tilde{\xi}\nabla \times B, \]  

(22)

leave the equations in the required form. These equations define the corresponding functions stated in (17). In momentum space

\[ D = \frac{1}{1 + 4\xi^2 k^2} \left( E - 2i\tilde{\xi}k \times E + 4\tilde{\xi}^2 (k \cdot E) k \right), \quad H = B + 2i\tilde{\xi}k \times B. \]  

(23)

The admixture of vectors and axial vectors in the constitutive relations precludes the parity violation exhibited by the model, together with the presence of birefringence.

3.2. Ellis et. al. Electrodynamics

In this case the modified Maxwell equations are

\[ \nabla \cdot B = 0, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \]  

(24)

\[ \nabla \cdot E + u \cdot \frac{\partial E}{\partial t} = 4\pi \rho_{\text{eff}} = 4\pi (\rho - u \cdot j), \]  

(25)

\[ \nabla \times B - \left( 1 - u^2 \right) \frac{\partial E}{\partial t} + u \times \frac{\partial B}{\partial t} + (u \cdot \nabla) E = 4\pi j_{\text{eff}} = 4\pi (j + u (\rho - u \cdot j)). \]  

(26)

which, following the approach of Ref. [10] and assuming that in momentum space \( u = f(\omega)k \), can be written in the form [15] [16] via constitutive relations which read

\[ H = B - f(\omega)k \times E, \quad D = \left( 1 - f^2(\omega)k^2 \right) E + f^2(\omega)k (k \cdot E) - f(\omega)k \times B. \]  

(27)

Taking \( u \) as a vector, this model conserves parity and shows no birefringence.

3.3. Myers-Pospelov Electrodynamics

This case corresponds to the equations [11]. From the last one we can infer the constitutive relations

\[ H = B - \tilde{\xi} \partial_0 E, \quad D = E - \tilde{\xi} \partial_0 B, \]  

(28)

which produce

\[ \nabla \cdot E = \mathbf{v} \cdot D, \]  

(29)

leaving the third Eq. [11] in desired form. Similarly to the GP case, this model violates parity.

In momentum space Eqs. (28) become

\[ H = B + i\tilde{\xi} \omega E, \quad D = E + i\tilde{\xi} \omega B. \]  

(30)
The above constitutive relations in the three representative models involve linear relations among the fields and can be summarized in momentum space as local relations

\[
D_i (\omega, \mathbf{k}) = \alpha_{ij} (\omega, \mathbf{k}) E_j (\omega, \mathbf{k}) + \rho_{ij} (\omega, \mathbf{k}) B_j (\omega, \mathbf{k}),
\]
\[
H_i (\omega, \mathbf{k}) = \beta_{ij} (\omega, \mathbf{k}) B_j (\omega, \mathbf{k}) + \sigma_{ij} (\omega, \mathbf{k}) E_j (\omega, \mathbf{k}),
\]
where the corresponding momentum dependent coefficients can be read from the equations (23), (27), and (30). Equations (31) are the most general expressions in which any pair of linear constitutive relations can be ultimately written, because one must be able to solve the fields \( \mathbf{D}, \mathbf{H} \) in terms of \( \mathbf{E}, \mathbf{B} \).

4. Parameterization of the constitutive relations

Let us consider Maxwell equations in momentum space

\[
\mathbf{k} \cdot \mathbf{B} (\omega, \mathbf{k}) = 0,
\]
\[
\mathbf{k} \times \mathbf{E} (\omega, \mathbf{k}) = \omega \mathbf{B} (\omega, \mathbf{k}),
\]
\[
\mathbf{i} \mathbf{k} \cdot \mathbf{D} (\omega, \mathbf{k}) = 4\pi \rho (\omega, \mathbf{k}),
\]
\[
\mathbf{i} \mathbf{k} \times \mathbf{H} (\omega, \mathbf{k}) = -i \omega \mathbf{D} (\omega, \mathbf{k}) + 4\pi \mathbf{j} (\omega, \mathbf{k}).
\]

Here we discuss the vacuum situation where the non trivial constitutive relations arise because of LIV effects. Let us take into account corrections up to second order in the LIV parameter \( \xi \) and let us assume that we are in a Lorentz frame where we demand invariance under rotations. This would correspond to the rest frame \( V^\mu = (1, 0) \) in the MP model, for example. We can always go to an arbitrary frame by means of an observer Lorentz transformation. In this way we have the general expressions

\[
\alpha_{ij} = \alpha_0 \delta_{ij} + i \alpha_1 \xi \epsilon_{irj} k_r + \alpha_2 \xi^2 k_i k_j, \quad \rho_{ij} = \rho_0 \delta_{ij} + i \rho_1 \xi \epsilon_{irj} k_r + \rho_2 \xi^2 k_i k_j,
\]
\[
\beta_{ij} = \beta_0 \delta_{ij} + i \beta_1 \xi \epsilon_{irj} k_r + \beta_2 \xi^2 k_i k_j, \quad \sigma_{ij} = \sigma_0 \delta_{ij} + i \sigma_1 \xi \epsilon_{irj} k_r + \sigma_2 \xi^2 k_i k_j,
\]
where \( \alpha_A, \beta_A, \rho_A, \sigma_A, A = 0, 1, 2, \) are scalar functions depending only upon \( \omega, k = |\mathbf{k}|, \) and \( \xi \). The property \( \mathbf{k} \cdot \mathbf{B} = 0 \) sets \( \beta_2 = \rho_2 = 0 \) effectively. In vector notation we then have

\[
\mathbf{D} = \left( \alpha_0 + \alpha_2 k^2 \xi^2 \right) \mathbf{E} + \left( \rho_0 + i \alpha_1 \omega \xi \right) \mathbf{B} + i \left( \rho_1 - i \alpha_2 \omega \xi \right) \mathbf{k} \times \mathbf{B},
\]
\[
\mathbf{H} = \left( \sigma_0 + \sigma_2 k^2 \xi^2 \right) \mathbf{E} + \left( \beta_0 + i \sigma_1 \omega \xi \right) \mathbf{B} + i \left( \beta_1 - i \sigma_2 \omega \xi \right) \mathbf{k} \times \mathbf{B},
\]
where we have used the second Eq. (52) together with \( (\mathbf{k} \cdot \mathbf{E}) \mathbf{k} = \omega (\mathbf{k} \times \mathbf{B}) + k^2 \mathbf{E} \). Next we substitute in Eqs. (33) to obtain the corresponding equations for \( \mathbf{E} \) and \( \mathbf{B} \). The result is

\[
i \left( \alpha_0 + \alpha_2 k^2 \xi^2 \right) (\mathbf{k} \cdot \mathbf{E}) = 4\pi \rho,
\]
\[
i \left( \alpha_0 + \alpha_2 k^2 \xi^2 \right) \omega \mathbf{E} + i \left[ \beta_0 + i (\sigma_1 + \rho_1) \omega \xi + \alpha_2 \xi^2 \omega^2 \right] \mathbf{k} \times \mathbf{B}
\]
\[
+ i \left[ (\sigma_0 + \rho_0) \omega + i \left( \alpha_1 \omega^2 - \beta_1 k^2 \right) \xi \right] \mathbf{B} = 4\pi \mathbf{j} (\omega, \mathbf{k}).
\]

Let us rewrite the inhomogeneous equations in the compact form

\[
i P (\mathbf{k} \cdot \mathbf{E}) = 4\pi \rho,
\]
\[
i \omega P \mathbf{E} + i Q \mathbf{k} \times \mathbf{B} + R \mathbf{B} = 4\pi \mathbf{J} (\omega, \mathbf{k}),
\]
by defining

\[
P = \alpha_0 + \alpha_2 \xi^2 k^2, \quad Q = \beta_0 + i (\sigma_1 + \rho_1) \omega \xi + \alpha_2 \xi^2 \omega^2, \quad R = \left( \beta_1 k^2 - \alpha_1 \omega^2 \right) \xi + i (\sigma_0 + \rho_0) \omega.
\]

Now we have only three independent functions which depend on \( \omega \) and \( k \).
5. Potentials and fields
Starting from (32), (40) and (41) we introduce the standard potentials $\Phi$, $A$

$$B = i k \times A, \quad E = i \omega A - i k \Phi.$$  \hfill (43)

Substituting in the inhomogeneous equations we have

$$- \omega k \cdot A + k^2 \Phi = 4 \pi \rho / P,$$  \hfill (44)

$$\left( k^2 Q - \omega^2 P \right) A + i R \left( k \times A \right) - \left[ Q \left( k \cdot A \right) - P \omega \Phi \right] k = 4 \pi j.$$  \hfill (45)

We can easily fix two convenient gauges as follows.

5.1. Generalized Lorentz gauge
In the standard situation the Lorentz gauge in momentum space is $k \cdot A = \omega \Phi$. Here we choose

$$k \cdot A = (P/Q) \omega \Phi,$$  \hfill (46)

in such a way that Eqs. (45) and (44) reduce to

$$\left( Q k^2 - P \omega^2 \right) A + i R \left( k \times A \right) = 4 \pi j$$  \hfill (47)

$$\left( Q k^2 - P \omega^2 \right) \Phi = 4 \pi \left( Q / P \right) \rho,$$  \hfill (48)

respectively.

5.2. The radiation gauge
As usual we set $k \cdot A = 0$ and we obtain the equations

$$\Phi = 4 \pi \left( k^2 P \right)^{-1} \rho,$$  \hfill (49)

$$\left( Q k^2 - P \omega^2 \right) A + i R \left( k \times A \right) = 4 \pi j - 4 \pi \omega \rho k^{-2} k.$$  \hfill (50)

Eq. (49) exhibits the scalar potential as a static contribution, which subsequently does not contribute to the radiation field. The use of current conservation in (50) allows the introduction of the transversal current, consistently with the chosen gauge. That is to say we have

$$\left( Q k^2 - P \omega^2 \right) A + i k R \left( \hat{k} \times A \right) = 4 \pi \left[ j - (j \cdot \hat{k}) \hat{k} \right] = 4 \pi j_T.$$  \hfill (51)

The presence of birefringence depends on the term proportional to $k R$, where we have a crucial factor of $i$. This makes clear that a possible diagonalization can be obtained by using a complex basis, which is precisely the circular polarization basis. In fact, decomposing the vector potential and the current in such basis

$$A = A^+ + A^-,$$  \hfill (52)

and recalling the basic properties

$$\hat{k} \times A^+ = -i A^+, \quad \hat{k} \times A^- = i A^-,$$  \hfill (53)

we can separate (51) into the uncoupled equations

$$\left[ Q k^2 - P \omega^2 + \lambda k R \right] A^\lambda = 4 \pi j_T^\lambda, \quad \lambda = \pm 1.$$  \hfill (54)
The general form of the polarized Green function is

\[ n \text{ characterize the propagation mode corresponding to each polarization} \]

propagation, which constitute a generalization of the specific models already considered. We characterize the solutions and proceed here on the basis of general properties of light propagation, which can be neglected in our effective theory valid for \( k \ll \xi^{-1} \). We defer for future work a detailed characterization of the solutions and proceed here on the basis of general properties of light propagation, which constitute a generalization of the specific models already considered. We characterize the propagation mode corresponding to each polarization \( \lambda \) by a refraction index \( n_\lambda(\omega) \) to be read from the appropriate dispersion relation in such a way that \( n_\lambda(\omega) = k_\lambda(\omega)/\omega \). The general form of the polarized Green function is

\[ G^\lambda(\omega, r) = \frac{1}{4\pi r} F^\lambda(\omega)e^{i\omega n_\lambda(\omega)r}, \quad (56) \]

where \( \hat{n} = r/r \) is the direction of observation. The generic form for the total Green function in the far-field approximation is

\[ G_{\text{ret}}^{ij}(\omega, r) = \frac{1}{2} \left[ \left( \delta^{ik} - \hat{n}^i \hat{n}^k + i e^{i\omega \hat{n} r} \right) G^+(\omega, r) + \left( \delta^{ik} - \hat{n}^i \hat{n}^k - i e^{i\omega \hat{n} r} \right) G^-(\omega, r) \right]. \quad (57) \]

Note that from the birefringent case we can go to the non-birefringent one by taking \( n_+(\omega) = n_-(\omega) = n(\omega) \), in which case \( F^+(\omega) = F^-(-\omega) = F(\omega) \) and the Green function (57) becomes

\[ G_{\text{ret}}^{ik}(\omega, r) = \frac{1}{4\pi r} \left( \delta^{ik} - \hat{n}^i \hat{n}^k \right) F(\omega) e^{i\omega n(\omega)r}. \quad (58) \]

In the following we analyze the main consequences of an electrodynamics characterized by a Green function of the type (57). To warrant that the fields are real, it must be

\[ [G^+(\omega, r)]^* = G^-(-\omega, r). \quad (59) \]

This implies the relations

\[ n^*(\omega) = n_-(\omega), \quad \quad [F^+(\omega)]^* = F^-(-\omega). \quad (60) \]

For a birefringent medium the real and imaginary parts of the refraction index for circular polarization components can contain both \( \omega \)-even and \( \omega \)-odd terms, provided that they satisfy Eq. (50). In the case of a non-birefringent medium the real part of the refraction index is even in \( \omega \), and the imaginary part is odd. We can see that the refraction indices for the Myers-Pospelov theory, Eq. (4), satisfy these requirements.

The Green functions must also be consistent with causality, which means that they cannot have poles in the \( \omega \) upper half plane. This leads to generalized Kramers-Kronig relations, which for a dispersive refraction index imply the existence of an imaginary part. This means that these media are necessarily absorptive. If we assume that these theories are valid in a range of frequencies where the effective medium can be considered as transparent, then in the case of birefringent theories we have a real refraction index with both \( \omega \)-even and \( \omega \)-odd terms, while for non-birefringent theories we have only \( \omega \)-even terms.

Next we turn to the calculation of the angular distribution of the power spectrum. The full vector potential is given by the superposition

\[ A(\omega, r) = \frac{1}{r} \sum_{\lambda=\pm1} F^{\lambda}(\omega)e^{i\lambda \omega r} j^{\lambda}(\omega, k_\lambda) = A_+ + A_- . \quad (61) \]
where \( k_\lambda = n_\lambda(\omega) \omega \hat{n} \) and \( \hat{n} = r/r \). From here we can compute the electric and magnetic fields

\[
\mathbf{B}(\omega, r) = \nabla \times \mathbf{A}(\omega, r) = \frac{\omega}{r} \sum_{\lambda = \pm 1} \lambda \eta_\lambda F^\lambda e^{i \lambda \omega r} j^\lambda(\omega, k_\lambda) = \omega (n_+ A_+ - n_- A_-), \tag{62}
\]

\[
\mathbf{E}(\omega, r) = i \omega \mathbf{A}(\omega, r) = \frac{i \omega}{r} \sum_{\lambda = \pm 1} F^\lambda e^{i \lambda \omega r} j^\lambda(\omega, k_\lambda) = i \omega (A_+ + A_-). \tag{63}
\]

The Poynting vector

\[
\mathbf{S}(\omega, r) = \frac{1}{4\pi} \mathbf{E}(-\omega, r) \times \mathbf{H}(\omega, r),
\]

\[
= \frac{\omega^2}{4\pi} \sum_{\lambda = \pm 1} \left\{ i \left[ \beta_0 + i \lambda \omega \xi \right] n_\lambda + i \lambda \sigma_0 \right\} \left[ A_+ (\lambda \omega, r) \times A_- (-\lambda \omega, r) \right]
\]

\[
+ \lambda \beta_1 \xi \omega n_\lambda^2 \left[ A_+ (\lambda \omega, r) \cdot A_- (-\lambda \omega, r) \right] \hat{n}, \tag{64}
\]

allows us to compute the angular distribution of the energy spectrum

\[
\frac{d^2 E}{d\Omega d\omega} = \frac{r^2}{2\pi} \hat{n} \cdot \left[ \mathbf{S}(\omega, r) + \mathbf{S}(-\omega, r) \right],
\]

\[
= \frac{r^2 \omega^2}{8\pi^2} \left[ Z(\omega) A_- (-\omega, r) \cdot A_+ (\omega, r) + Z(-\omega) A_- (\omega, r) \cdot A_+ (-\omega, r) \right], \tag{65}
\]

where

\[
Z(\omega) = \sum_{\lambda = \pm 1} \left\{ i \sigma_0 (\lambda \omega) + \left[ \beta_0 (\lambda \omega) + i \lambda \sigma_1 (\lambda \omega) \omega \xi \right] n_\lambda (\lambda \omega) + \beta_1 (\lambda \omega) \xi \omega n_\lambda^2 (\lambda \omega) \right\}. \tag{66}
\]

Using Eqs. (60) and (61), together with the general property \([j^\lambda(\omega, k)]^* = j^{-\lambda}(-\omega, -k^*)\), which allows for complex refraction indexes, we finally get

\[
\frac{d^2 E}{d\Omega d\omega} = \frac{\omega^2}{8\pi^2} \sum_{\lambda = \pm 1} Z(\lambda \omega) \left| F^\lambda(\omega) \right|^2 e^{-2\omega \text{Im}[n_\lambda(\omega)] r} \sum_{\lambda = \pm 1} |j^\lambda(\omega, k_\lambda(\omega))|^2 \cdot j^\lambda(\omega, k_\lambda(\omega)), \tag{67}
\]

where the vectors \( k_\lambda(\omega) \) were defined after Eq. (61). As expected, absorption is present through the non-zero imaginary part of the refraction indexes. Eq. (67) is related with the angular distribution of the radiated power spectrum by

\[
\frac{d^2 E}{d\Omega d\omega} = \int dT \frac{d^2 P(T)}{d\omega d\Omega}, \tag{68}
\]

where \( T \) is a macroscopic time, which leads to

\[
\frac{d^2 P(T)}{d\Omega d\omega} = \frac{\omega^2}{8\pi^2} \sum_{\lambda = \pm 1} Z(\lambda \omega) \left| F^\lambda(\omega) \right|^2 e^{-2\omega \text{Im}[n_\lambda(\omega)] r}
\]

\[
\times \int_{-\infty}^{\infty} d\tau \, e^{-i \omega \tau} j^\lambda_T(T + \tau/2, k_\lambda(\omega)) P^\lambda_{kr} j^\lambda_T(T + \tau/2, k_\lambda(\omega)) \right], \tag{69}
\]

where \( P^\lambda_{kr} = 1/2 \left( \delta_{kr} - \hat{n}_k \cdot \hat{n}_r + i \lambda \epsilon_{kr}, \hat{n}_k, \hat{n}_r \right) \) are the circular polarization projectors.

In the sequel we consider synchrotron radiation in the situation where the vacuum absorption can be neglected so that the refraction indices are real and Eq. (60) reduces to \( n^+(\omega) = n^-(-\omega) \).
Also we assume that $F^\lambda(\omega)$ and $Z^\lambda(\omega)$ are real functions. Using the current corresponding to the circular orbits in (2) we can compute the cycle average over the macroscopic time $T$ obtaining
\[
\left\langle \frac{d^2P(T)}{d\omega d\Omega} \right\rangle = \sum_{\lambda=\pm1} \sum_{m=0}^{\infty} \delta(\omega - m\omega_0) \frac{dP_{m,\lambda}}{d\Omega},
\] (70)
with
\[
\frac{dP_{m,\lambda}}{d\Omega} = \frac{\omega_m q^2}{8\pi} Z(\lambda\omega) \left[ \frac{F^\lambda(\omega)}{n^\lambda(\omega)} \right]^2 \left[ \lambda\beta n(\omega)J'_m(W_{m,\lambda}) + \cot\theta J_m(W_{m,\lambda}) \right]^2,
\] (71)
where $W_{m,\lambda} = mn^\lambda(\omega)\beta\sin\theta$. The total averaged power radiated in the $m^{th}$ harmonic
\[
P_m = \frac{q^2\beta^2\omega_m}{2R} \sum_{\lambda=\pm1} \left\{ Z(\lambda\omega) \left[ \frac{F^\lambda(\omega)}{n^\lambda(\omega)} \right]^2 \left[ J'_m(2m\beta\omega) - \frac{1 - \beta^2n^2\omega^2}{2\beta^2n^2\omega^2} \int_0^{2m\beta\omega} dx J^2_m(x) \right] \right\},
\] (72)
where the contribution of each polarization is exhibited.

In the case of the Myers and Pospelov effective theory the non-null coefficients in the constitutive relations (36-37) are
\[
\alpha_0 = \beta_0 = 1, \quad \sigma_0 = \rho_0 = i\tilde{\xi}\omega,
\] (73)
while the remaining functions are given by
\[
F^\lambda(\omega) = \frac{n^\lambda(\omega)}{\sqrt{1 + (\tilde{\xi}\omega)^2}}, \quad Z(\omega) = 2\sqrt{1 + (\tilde{\xi}\omega)^2}.
\] (74)

From the general expressions obtained in this section we recover the results of Section 2.

7. Final remarks

We have presented a compact review of the most interesting results for synchrotron radiation in the Myers-Pospelov LIV electrodynamics. This is one of several existing models of LIV electrodynamics that also include those proposed by Gambini-Pullin, and Ellis et. al., for example. Motivated by such diversity we have developed a general approach based on generalized constitutive relations, set out in terms of the standard methods for describing radiation in a dispersive and absorptive medium with a given index of refraction. The resulting formalism has been explicitly applied to synchrotron radiation in the case of transparent media.

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