ABSTRACT

This paper investigates the model reference adaptive control (MRAC) problem of switched systems with nonlinear matched uncertainties based on mode-dependent average dwell time method. A novel switched MRAC scheme is proposed, which simultaneously uses both current and recorded data from the switched system, to study the convergence of the state tracking error and the weight error without depending on the persistency of excitation. It is critical to establish the relationship among the nonlinear matched uncertainties, the switched concurrent learning (CL) algorithm, the average dwell time of each subsystem, and the convergence domains of the state tracking error and the weight error. First, we find a class of switching signals characterized by mode-dependent average dwell time conditions such that the switched reference model is bounded-input bounded-state stable. Second, in the case of time-invariant nonlinear uncertainties, we design a switched CL adaptive law that ensures that the tracking error converges to zero, and the adaptive weights exponentially converge to their ideal values. Subsequently, in the case of nonlinear time-varying uncertainties, we estimate the convergence rates of the state tracking error and the weight error of each subsystem outside a ball, and thus obtain sufficient conditions that ensure the convergence of the trajectories of the state tracking error and the weight error to a ball centered at the origin with a given radius. Finally, the proposed approach is applied to an electro-hydraulic system, and its effectiveness is demonstrated.

INDEX TERMS

Switched systems, concurrent learning, mode-dependent average dwell time, model reference adaptive control.

I. INTRODUCTION

As a special class of hybrid systems, switched systems consist of a finite number of subsystems described by continuous or discrete-time dynamics and an associated switching signal determining the switching among the subsystems [1]–[4]. Switched systems have been attracting considerable attention in recent years, primarily because they can be used to effectively model various physical systems that exhibit switching characteristics, such as robotic systems and switching power converters [5]–[7]. Due to the hybrid nature of a switched system, the properties of each individual subsystem are not inherited by the switched system. For example, the stability of a switched system is not guaranteed with rapidly switching despite each of its subsystems being asymptotically stable. Therefore, traditional control theories for non-switched systems may not be directly applicable to switched systems. To date, various approaches have been applied to analyze the stability of switched systems [8]–[11]. Among these approaches, as a time-dependent switching law, the mode-dependent average dwell time (MDADT) logic has been widely used to study the control problem of switched systems wherein each subsystem is assigned its own average dwell time (ADT) by exploiting the information of each mode [12]–[14].

However, uncertainties present ubiquitous problems in several actual systems because of modeling errors or the external environment [15]–[18]. As a complement to robust control, adaptive control is a valid methodology used to handle these problems such that systems achieve their desired behavior.
For uncertain non-switched systems, model reference adaptive control (MRAC) has been extensively investigated in numerous practical control systems to deal with uncertainties [19]–[23]. A model reference adaptive controller was designed to allow the uncertain system to track its dynamics which would aid it in achieving desirable performance. Although the boundedness of all signals in the closed-loop system can be ensured, weight estimates cannot be guaranteed to converge to their true values [24]. In fact, the weight convergence leads to exponential state tracking error convergence and improves the exponentially bounded transient performance. If the system uncertainties are uniformly canceled, the system tracks the reference model exponentially [25]. Moreover, the convergence of the weight estimates to their true values is beneficial for most practical systems, particularly for unmanned aerial vehicle systems [26]. Consequently, the study of weight convergence is of utmost importance. For adaptive laws using only instantaneous data, a persistency of excitation (PE) condition is necessary for the exponential convergence of the weight estimates [27]–[28]. However, the PE condition is restrictive and impractical in several applications that require high precision or smooth operation. A novel adaptive update scheme known as concurrent learning (CL) was recently developed to relax the PE condition [29]–[31]. Unlike traditional adaptive control laws, the CL algorithm concurrently uses both current and recorded data of the adaptive control with an easily verifiable condition of the linear independence of the recorded data. The CL adaptive control has been proven to ensure the convergence of the state tracking error to zero and the convergence of the adaptive weights to their ideal values exponentially without relying on PE [32]–[34].

Naturally, a variety of uncertainties exist in switched systems, which makes the subject more complex. Thus, several switched MRAC schemes have been developed in recent years. Among these schemes, CL algorithms for time-invariant uncertain switched systems are increasingly used to guarantee that the dynamics of the state tracking error and the weight error exponentially converge to zero. For example, [35] gave a sufficient condition to solve the finite-time parameter estimation problem of uncertain switched systems with arbitrary switching signals. Switched nonlinear systems with linear uncertain parameters were studied in [36], and a class of switching signals was found to ensure the exponential convergence of the state tracking error and weight error. Parameter identifiers and CL algorithms for piecewise affine systems were studied in [37], where the exponential parameter convergence was ensured by using pre-given switching signals. The authors of [38] designed switched CL adaptive laws under arbitrary switching signals to solve the MRAC tracking problem of a class of uncertain switched systems. In [39], parameter identifiers and CL were used to propose recursive subsystem estimation in piecewise affine systems.

The aforementioned works were predominantly concerned with parametric uncertainties in switched systems. However, for practical switched systems, nonlinear matched uncertainties cannot be ignored since they usually reduce the stability or induce a decline in the performance of the system. To the best of our knowledge, nonlinear matched uncertainties have been rarely explored for the MRAC tracking problem of switched systems. Therefore, ensuring the stability of the state tracking error and the weight error dynamics without depending on PE for switched systems with nonlinear matched uncertainties poses a challenge. The use of CL method to develop a new adaptive law and determine a class of switching laws for such systems to ensure the convergence of the state tracking error and the weight error needs investigation. This is a challenging problem that has been addressed neither in the context of switched systems nor in the field adaptive controls. Consequently, this is the motivation of the present work.

Thus, this work investigates the MRAC tracking problem of switched systems that involve nonlinear matched uncertainties using the CL method. The crucial obstacle in the study is the complexity arising from the interaction among the switching signals, the nonlinear uncertainties, and the convergence rates of the state tracking error and weight error. Hence, two main challenges are addressed by the present work. First, for non-switched systems, the CL-MRAC scheme is directly applied to deal with these uncertainties without considering switching. However, for switched systems, the traditional non-switched CL-MRAC scheme is inappropriate since the dwell time of each subsystem affects the stability of the state tracking error and weight error dynamics. Therefore, the first difficulty we faced was the design of a switched CL-MRAC strategy and the determination of a class of switching signals that solves the tracking problem. Second, for switched systems with parametric uncertainties, conventional switching methods utilize multiple Lyapunov functions to estimate the convergent rates of the state tracking error and weight error. However, the use of multiple Lyapunov functions introduces nonlinear matched uncertainties, and hence, it cannot be applied to directly estimate those rates. Furthermore, nonlinear matched uncertainties coupled to the control channel increases the design complexity of the switched adaptive controllers. Thus, the design of the switched adaptive controllers and the estimation of the convergent rates of the state tracking error and the weight error is the second challenge addressed by the present work.

To overcome these difficulties, we propose a switched MRAC scheme based on CL for switched systems with nonlinear matched uncertainties by using the MDADT technique. The novelties presented by our work are three-fold: (i) We developed a novel switched MRAC scheme concurrently using both current and recorded data for switched systems with nonlinear matched uncertainties. Unlike most existing switched adaptive control schemes, where the weight estimates of the inactive subsystems are frozen, the proposed switched adaptive control scheme enables the update of the weight estimates of the currently inactive subsystems, which improves the convergence rate of the weight error.
Thus, is a set of nonnegative integers. When \( N \) in which the switching times when the state tracking error and the weight error of each subsystem and develop solvable conditions such that the trajectories of the state tracking error and the weight error converge to a ball centered at the origin with a given radius. (iii) We reveal the interaction among nonlinear uncertainties, the average dwell time of each subsystem, the switched CL algorithm, and the convergence domains of the state tracking error and the weight error.

Notations: we use \( \lambda_{\text{max}}(A) \) (\( \lambda_{\text{min}}(A) \)) for the largest (smallest) eigenvalue of a matrix \( A \); \( \Delta(A) = \sqrt{\lambda(A^T A)} \) for the singular value of a matrix \( A \); \( \| \cdot \| \) for the Frobenius matrix norm; \( \| \cdot \|_2 \) for the matrix norm induced by the Euclidean vector norm, and \( \| \cdot \|_F \) for the Frobenius matrix norm.

II. PRELIMINARIES

Consider a class of uncertain switched systems given by

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + \Lambda_{\sigma(t)}(t, x(t)),
\]

where \( x(t) \in \mathbb{R}^n \) is the state; \( u(t) \in \mathbb{R}^q \) is the control input; \( \sigma(t) : [0, +\infty) \to \Xi = \{1, \ldots, M\} \) is a switching signal which is a piecewise constant function of time; \( A_i \in \mathbb{R}^{n \times n} \) \( (i \in \Xi) \) and \( B_i \in \mathbb{R}^{n \times q} \) are known constant matrices of the \( i \)-th subsystem; \( B_i \) have full column rank; and \( \Lambda_i(t, x(t)) \in \mathbb{R}^q \) are nonlinear time-varying matched uncertainties.

In the case of nonlinear time-invariant matched uncertainties, the switched system (1) is replaced with the following switched system

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + \Lambda_{\sigma(t)}(t, x(t)),
\]

where \( \Lambda_i(t, x(t)) \in \mathbb{R}^q \) are nonlinear time-invariant matched uncertainties.

A switching signal \( \sigma(t) \) can be characterized by a switching sequence

\[
\Sigma = \{x(t_0) : (i_0, t_0), \ldots, (i_n, t_0), \ldots | i_n \in \Xi, n \in \mathbb{N}\},
\]

in which \( t_0 \) is the initial time, \( x(t_0) \) is the initial state, and \( N \) is a set of nonnegative integers. When \( t \in [t_h, t_{h+1}) \), \( \sigma = i_h \), i.e., the \( i_h \)-th subsystem is active. For any \( j \in \Xi \), let \( \Sigma [j = \{t_{j_1}, t_{j_2}, \ldots, t_{j_n}, \ldots | t_{j_p} = j, p \in \mathbb{N}\} \) be the sequence of the switching times when the \( j \)-th subsystem is switched on. Thus, \( \{t_{j+1}, \ldots, t_{j_k+1}, \ldots | t_{j_p} = j, p \in \mathbb{N}\} \) is the sequence of the switching times when the \( j \)-th subsystem is switched off.

A chosen switching reference mode is given by

\[
\dot{x}_m(t) = A_{\sigma_{\text{ref}}(t)}x_m(t) + B_{\sigma_{\text{ref}}(t)}r_m(t),
\]

where \( A_{mi} \in \mathbb{R}^{n \times n} \) \( (i \in \Xi) \) and \( B_{mi} \in \mathbb{R}^{n \times q} \) are constant matrices satisfying

\[
A_{mi}'P_i + P_iA_{mi} + 2\lambda_{mi}P_i \leq 0,
\]

where \( P_i > 0 \) and \( \lambda_{mi} > 0 \). \( x_m(t) \) is desirable for \( x(t) \) to track, and \( r(t) \in \mathbb{R}^q \) is a bounded reference input signal. \( e(t) = x(t) - x_m(t) \) is defined as the state tracking error.

The following assumptions are needed in our subsequent sections.

**Assumption 1:** The nonlinear time-invariant matched uncertainties in (2) can be linearly parameterized as

\[
\Lambda_i(x(t)) = \Theta_i^T(x(t)),
\]

where \( \Theta_i \in \mathbb{R}^{r \times q} \) are unknown constant weight matrices, \( \Psi_i(x(t)) \in \mathbb{R}^p \) are known basis functions of the form \( \Psi_i(x(t)) = [\Psi_{i1}(x(t)), \Psi_{i2}(x(t)), \ldots, \Psi_{ip}(x(t))]^T \in \mathbb{R}^p \) and their components are Locally Lipschitz in \( x \).

**Assumption 2:** The time-varying nonlinear matched uncertainties in (1) can be linearly parameterized as

\[
\Lambda_i(t, x(t)) = \Theta_i^T(t)\Psi_i(x(t)),
\]

where \( \Theta_i(t) \in \mathbb{R}^{r \times q} \) are the unknown time-varying weight matrices that satisfy \( \|\Theta_i(t)\| \leq \sigma \), \( \|\Theta_i^T(t)\| \leq w \).

**Assumption 3:** There exist constant matrices \( K_{ri} \in \mathbb{R}^{n \times q} \) and \( K_{ri} \in \mathbb{R}^{q \times q} \) such that \( A_{mi} = A_i + B_iK_{ri}^T \), and \( B_{mi} = B_iK_{ri} \) hold.

The weight error is denoted by

\[
\hat{\Theta}_i(t) = \Theta_i^T(t) - \hat{\Theta}_i(t),
\]

where \( \hat{\Theta}_i(t) \) are estimations of the unknown matrices \( \Theta_i^T(t) \).

In this work, our objective is to solve the MRAC problem of the switched system (1) in the presence of nonlinear matched uncertainties without depending on PE, namely, to develop switched adaptive laws and determine a set of switching signals such that:

(i) for the switched system (1) with time-invariant nonlinear matched uncertainties, i.e., the switched system (2), the zero solution \( e(t), \hat{\Theta}_i(t) \equiv 0 \) is globally exponentially stable;

(ii) for the switched system (1) with time-varying nonlinear matched uncertainties, trajectories of the state tracking error \( e(t) \) and weight error \( \hat{\Theta}_i(t) \) converge to a ball centered at the origin with a given radius.

III. STABILITY OF THE SWITCHED REFERENCE MODEL

In the MRAC problem, the boundedness of the desired state trajectory with a bounded reference input \( r(t) \) is a prerequisite. Thus, it is necessary to analyze the stability of the switched reference model (3). According to [11] and [21], the exponential stability of the following homogeneous system

\[
\dot{x}_m(t) = A_{\sigma_{\text{ref}}(t)}x_m(t)
\]
ensures the bounded-input bounded-state stability of the switched reference model (3) for identical switching sequences.

To find a sufficient condition that ensures the exponential stability of (8), we define the following Lyapunov function candidate

$$V_m(x_m(t)) = V_{mσ(t)}(x_m(t)) = x_m^T(t)P_{σ(t)}x_m(t).$$

(9)

For $a_i = \min \{λ_{\min}(P_i)\}$, $b_i = \max \{λ_{\max}(P_i)\}$, we then have

$$a_i \|x_m(t)\|^2 \leq V_m(x_m(t)) \leq b_i \|x_m(t)\|^2,$$

(10)

which implies that

$$V_{mσ}(x_m(t)) \leq μ_{mσ}V_{m}(x_m(t)), \forall i, j ∈ Σ, i \neq j,$$

(11)

where $μ_{mσ} = \frac{b_i}{a_j}$.

Differentiating (9) along the trajectory of the corresponding subsystem of (8) leads to

$$\dot{V}_m(t) = x_m^T(t)(A_{σ(t)}^mp_i + P_{σ(t)}A_{mσ})x_m(t) \leq -2λ_{mi}V_m(x_m(t)).$$

(12)

According to (10)-(12) and the results presented in [40], if the switching signal $σ(t)$ satisfies the MDADT

$$τ_{mi} ≥ τ_{mi}^* = \frac{\ln μ_{mσ}}{2λ_{mi}}, \quad t > t_0,$$

(13)

then, the homogeneous system (8) is exponentially stable, i.e.,

$$\|x_m(t)\| \leq \frac{b_i}{a_j} \exp \left( \frac{1}{2} \sum_{i=1}^{M} N_{0i} \ln μ_{mσ} \right) \times \exp \left( \max_{i ∈ Σ} \left\{ -\bar{λ}_{mi}(t - t_0) \right\} \right) \|x_m(t_0)\|,$$

where $\bar{λ}_{mi} ∈ [0, λ_{mi})$. Consequently, the boundedness of the reference trajectory can be guaranteed by the switching signal (13).

**Remark 1:** If $A_{mσ} = A_{nσ}$ hold for $i \neq j$ or $A_{mσ}^T P + PA_{mσ} + 2λ_{mi}P < 0$ hold for a common positive-definite matrix $P > 0$, the homogeneous system (8) is exponentially stable under any arbitrary switching signal, which ensures the stability of (3).

**IV. SWITCHED CL-MRAC STRATEGY IN THE PRESENCE OF NONLINEAR TIME-INVARIANT MATCHED UNCERTAINTIES**

In this section, we focus on the switched system (1) with nonlinear time-invariant matched uncertainties, i.e., the switched system (2). To study the dynamics of the state tracking error and the weight error of the switched system (2), we establish an error switched system and propose switched CL adaptive laws. We then analyze the stability of the error switched system and give a sufficient condition that ensures the zero solution (e(t), $\hat{θ}_{i}(t)$) $≡ 0$ is globally exponentially stable.

**A. SWITCHED CL ADAPTIVE LAWS**

For the switched system (2), the following controller structure is introduced:

$$u(t) = K_{xσ(t)}^T x(t) + K_{rσ(t)}^T r(t) − \hat{θ}_{σ(t)}^T Ψ_{σ(t)}(x(t)).$$

(14)

where $K_{xσ} ∈ R^{n×q}$ and $K_{rσ} ∈ R^{m×q}$ are nominal controller gains, and $\hat{θ}_{σ(t)} = \hat{θ}_{σ}^T − \hat{θ}_{σ}(t)$.

Based on (2) and (14), the following closed-loop switched system can be obtained:

$$\dot{x}(t) = A_{σσ}(x(t) + B_{mσ}r(t) + B_{σσ}(\hat{θ}_{σ(t)}^T Ψ_{σ(t)}(x(t)))).$$

(15)

From (3) and (15), we deduce the following error switching system:

$$\dot{e}(t) = A_{σσ}e(t) + B_{σσ}(\hat{θ}_{σ(t)}^T Ψ_{σ(t)}(x(t))).$$

(16)

Therefore, the MRAC problem can be addressed by designing switched adaptive control laws and determining a class of switching signals that guarantee the exponential convergence of the state tracking error and weight error of the error switched system (16).

As aforementioned, the PE of the state is a necessary condition to achieve the exponential convergence with adaptive controllers using only instantaneous data. However, the PE conditions are restrictive and are often not feasible for implementation or online monitor. As the sufficiency of rich data can be easily determined by online verifiable rank condition on recorded data, we propose a switched CL algorithm for determining the adaptive laws such that the MRAC problem is solved without requiring the persistence excitation of the state. Different from the traditional adaptive controller, the switched CL adaptive controller concurrently uses both recorded and current data for determining the adaption laws.

When the i-th error subsystem of (16) is active for $t ∈ [t_i, t_{i+1}]$, the error variable $E_{i,d}(t) ∈ R^d$ for each recorded data point is defined as follows:

$$E_{i,d}(t) = (B_{i,d}^T B_i)^{-1}B_{i,d}^T (\hat{x}_{i,d} - A_{mσ}x_{i,d} - B_{mσ}r_{i,d}),$$

(17)

where $x_{i,d}, r_{i,d}, Ψ_i(x_{i,d})$ and $\hat{x}_{i,d}$ denote the d-th recorded data point of the i-th subsystem. Due to the limited of data memory, the maximum number of data points is denoted as $d ≥ d$, and the number of recorded basis $Ψ_i(x_{i,d})$ satisfies $d ≥ s$. Four history stacks $X_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,d})$, $R_i = (r_{i,1}, r_{i,2}, \cdots, r_{i,d})$, $Π_i = (Ψ_i(x_{i,1}), Ψ_i(x_{i,2}), \cdots, Ψ_i(x_{i,d}))$, and $X_i = (\hat{x}_{i,1}, \hat{x}_{i,2}, \cdots, \hat{x}_{i,d})$ are reserved for storing these recorded data for each subsystem.

**Remark 2:** In (17), because $x_{i,d}, r_{i,d}$ are known, we only require the estimation of $\hat{x}_{i,d}$, which is the state derivative of a recorded data point in the past without requiring the current information of $\hat{x}(t)$. If the measurement of $\hat{x}_{i,d}$ is unavailable, it can be estimated by using dynamic state-derivative estimators or a fixed point optimal smoother [41].
By using recorded data, we propose the following switched CL adaptive laws
\[
\dot{\theta}_i(t) = \Gamma_i \Psi_i(x(t)) e^T(t) P_i B_i \\
+ \Gamma_i \sum_{d=1}^{\bar{d}} \Psi_i(x_{i,d}) E^T_{i,d}(t), \quad \sigma = i;
\]
\[
\dot{\theta}_i(t) = \Gamma_i \sum_{d=1}^{\bar{d}} \Psi_i(x_{i,d}) E^T_{i,d}(t), \quad \sigma \neq i, \quad (18)
\]
where \( \Gamma_i \) are adaptive gain matrices.

**Remark 3:** As opposed to the traditional switched adaptive laws, whose adaptive weight estimates of the inactive subsystems are frozen, the adaptive weight estimates of all subsystems update simultaneously by the switched CL adaptive laws (18). Thus, the weight estimates of the currently inactive subsystems still converge to their true values. This behavior exhibited by the switched CL adaptive laws improves the convergent rate of the weight error of each subsystem.

Combining (2), (3), (5), and (17) yields
\[
E^T_{i,d}(t) = \Lambda^T_i (x_{i,d}) - \Psi^T_i (x_{i,d}) \hat{\theta}_i(t) \\
= \Psi^T_i (x_{i,d}) \hat{\theta}_i(t). \quad (19)
\]

Combining (18), (19) and \( \dot{\theta}_i = \dot{\theta}_i \) yields the weight error dynamics
\[
\dot{\hat{\theta}}_i(t) = -\Gamma_i \Psi_i(x(t)) e^T(t) P_i B_i \\
- \Gamma_i \sum_{d=1}^{\bar{d}} \Psi_i(x_{i,d}) \Psi^T_i (x_{i,d}) \dot{\hat{\theta}}_i(t), \quad \sigma = i;
\]
\[
\dot{\hat{\theta}}_i(t) = -\Gamma_i \sum_{d=1}^{\bar{d}} \Psi_i(x_{i,d}) \Psi^T_i (x_{i,d}) \dot{\hat{\theta}}_i(t), \quad \sigma \neq i. \quad (20)
\]

**Remark 4:** In (20), because the weight error \( \dot{\hat{\theta}}_i(t) \) is unavailable, we use (20) to analyze the stability of the error switched system (16) instead of updating the weight error directly.

To fill history stacks and update the data point of the switched adaptive laws (18), we propose the switched CL algorithm (Algorithm 1). Initially, the history stacks \( X_i, R_i, \Pi_i, \) and \( \bar{X}_i \) are empty. The data point is added to the empty slot to fill the stacks of the active subsystem. Let \( t_{h_k} \in [t_{h_k}, t_{h_k+1}) \), \((k \in N)\) denote the updated instants when the instantaneous data replaces the oldest data point. Consequently, the history stacks of the i-th subsystem are updated at \( t = t_{h_k} \). Due to the discrete nature of the CL algorithm, it follows that \( t_h \leq t_0 \leq \cdots \leq t_{h_k} \leq t_{h_k+1} \leq t_{h+1} \). Since the history stacks can be obtained by open-loop experiments and simulation testing [31], [32], we can pre-populate the history stacks \( X_i, R_i, \) and \( \bar{X}_i, \) and fill \( \Pi_i \) with \( \bar{d} \) linearly independent columns. For convenience, we suppose that these history stacks are full of appropriate data at \( t = t_0 \) by using a filling history stacks

**Algorithm 1** The switched CL algorithm

- **Step 1.** Filling history stacks algorithm
  - if \( \sigma = i \)
  - if new data is available then
    - if \( d < \bar{d} \) then
      - if \( \left\| \Psi_i(x_{i,d}) - \Psi_i(x_{i,d}) \right\| \geq \varepsilon \) then
        - Determine whether to record data \( \Psi_i(x_{i,d}) \) in \( \Pi_i \)
        - \( d = d + 1; \Pi_i(x_{i,d}) = \Psi_i(x_{i,d}) \)
        - Store \( X_i, R_i, \bar{X}_i, \Pi_i \)
    - end if
  - end if
- **Step 2.** Updating data point algorithm
  - if \( d \geq \bar{d} \) then
    - \( MEMORY = \Pi_i \)
    - \( S = \min \Delta (\Pi_i) \) //Calculate minimum singular value of \( \Pi_i \)
  - end if
  - Function (column)
    //determine the corresponding column index
    Function (\( \Pi_i \))
    //according to the Maximum of Minimum Singular Value of \( \Pi_i \)
    //update the data stack
  - else
    - \( X_i, \Pi_i, R_i, \) and \( \bar{X}_i \) remain unchanged
  - end if
  - Function (column)
    column = 0
    for \( k = 1 \) to \( d \);
    \( \Pi_i(x_{i,d}) = \Psi_i(x_{i,d}) \)
    // k-th history data point of \( \Psi_i \) is
    // replaced by the current data point
    \( S(k) = \min \Delta (\Pi_i) \)
    if \( S(k) > S \) then
    //find the Maximum of Minimum Singular Value of \( \Pi_i \)
    //and the corresponding column index \( l \)
    \( S = S(k); \) column = \( k \)
    return (column)
    end if
    \( \Pi_i = MEMORY \)
  - end for

  - Function (\( \Pi_i \))
    if column > 0
    \( \Pi_i(x_{i,d}) = \Psi_i(x_{i,d}) \)
    //column-th history data point of \( \Pi_i \) is
    // replaced by the current data point
    Store \( X_i, R_i, \bar{X}_i, \Pi_i; d = d - 1 \)
  - else
    \( d = d - 1; \Pi_i = MEMORY \)
  - end if
algorithm (Step 1). Moreover, we use the proposed updating data point algorithm (Step 2) to update the history stacks \( \Pi_i \).

**B. STABILITY ANALYSIS OF THE ERROR SWITCHED SYSTEM**

In this section, we analyze the stability of the error switched system (16) by utilizing the MDADT method. A class of switching signals is then found such that the state tracking error and the weight error converge to zero exponentially.

We begin by defining the following Lyapunov function candidate:

\[
V(z(t)) = e^T(t)P_{\sigma(t)}e(t) + \sum_{l=1}^{M} \text{tr}[\tilde{\Theta}_l^T(t)\Gamma^{-1}_l\tilde{\Theta}_l(t)],
\]

where

\[
z(t) = [e^T(t), \tilde{\Theta}_1^T(t), \ldots, \tilde{\Theta}_l^T(t), \tilde{\Theta}_{l+1}^T(t), \ldots, \tilde{\Theta}_M^T(t)]^T.
\]

When the \( i \)-th error subsystem of (16) is active over the interval \( t \in [t_h, t_{h+1}] \), we have

\[
\alpha_i \|z(t)\|^2 \leq V_i(z(t)) \leq \beta_i \|z(t)\|^2,
\]

where \( \alpha_i = \min \left\{ \lambda_{\min}(P_l), \min_{i \in \Xi} \{ \lambda_{\min}( \Gamma^{-1}_l) \} \right\}, \beta_i = \max \left\{ \lambda_{\max}(P_l), \max_{i \in \Xi} \{ \lambda_{\max}(\Gamma^{-1}_l) \} \right\} \).

This leads to

\[
V_i(z(t)) \leq \mu_i V_j(z(t)), \quad \forall i, j \in \Xi, \quad i \neq j,
\]

where \( \mu_i = \frac{\beta_i}{\alpha_j} \).

When the \( i \)-th error subsystem of (16) is active, differentiating \( V(z(t)) \) along the solutions of the error switched system (16) and the switched CL adaptive laws (18) yields

\[
\dot{V}_i(z(t)) = e^T(t)(A_{tm} P_l + P_l A_{m})e(t)
- 2\sum_{l=1}^{M} \text{tr}[\tilde{\Theta}_l^T(t) \sum_{d=1}^{d} \Psi_l(x_i,d,h_k)\Psi_l^T(x_d,d,h_k)\tilde{\Theta}_l(t)].
\]

Let

\[
\Phi_{i,h_k} = \sum_{d=1}^{d} \Psi_l(x_i,d,h_k)\Psi_l^T(x_i,d,h_k).
\]

Substituting (4) and (25) into (24), we get

\[
\dot{V}_i(z(t)) \leq -2\lambda_{mi} \lambda_{\min}(P_l) \|e(t)\|^2
- 2\sum_{l=1}^{M} \left[ \lambda_{\min}(\Phi_{i,h_k}) \|\tilde{\Theta}_l(t)\|^2 \right].
\]

Thus, for \( t \in [t_h, t_{h+1}] \), we have

\[
\dot{V}_i(z(t)) \leq -2\lambda_{i,h_k} V_i(z(t)),
\]

where

\[
\lambda_{i,h_k} = \frac{\min \left\{ \lambda_{mi} \lambda_{\min}(P_l), \min_{i \in \Xi} \{ \lambda_{\min}(\Phi_{i,h_k}) \} \right\}}{\max \left\{ \lambda_{\max}(P_l), \max_{i \in \Xi} \{ \lambda_{\max}(\Gamma^{-1}_l) \} \right\}}.
\]

Because the history stacks \( \Pi_i \) are pre-populated and the number of recorded basis \( \Psi(x_i,d) \) satisfies \( d \geq s \), we have \( \Phi_{i,h_k} > 0 \). Furthermore, except for \( \Phi_{i,h_k} \) that is updated by Step 2, all parameters in (28) are time-invariant over the interval \( [t_h, t_{h+1}] \). Since the algorithm guarantees that \( \lambda_{\min}(\Phi_{i,h_k}) \) is monotonically increasing, we let

\[
\lambda_i = \frac{\lambda_{i,h_k}}{\min_{h_k \in \mathbb{N}} \{ \lambda_{i,h_k} \}}.
\]

Thus, applying (29) to (27) leads to

\[
\dot{V}_i(z(t)) \leq -2\lambda_i V_i(z(t)), \quad t \in [t_h, t_{h+1}].
\]

**Theorem 1:** Consider the error switched system (16) with switched CL adaptive laws (18). Assume that the \( i \)-th subsystem history stacks \( \Pi_i \) contain linearly independent columns. If the history stacks are updated according to the switched CL algorithm, then, the zero solution (\( \dot{e}(t), \tilde{\Theta}(t) = 0 \)) is globally exponentially stable for any switching law \( \sigma(t) \) satisfying the MDADT

\[
\tau_{ai} \geq \tau_{ai}^* = \frac{\ln \mu_i}{2\lambda_i}, \quad i \in \Xi,
\]

and any chattering bound \( N_{th} > 0 \).

**Proof:** Denote \( N_\sigma(t, t_0) = \sum_{i=1}^{M} N_{ai}(t, t_0) \) for \( t \geq t_0 \).

By employing (22), (23), (30), and the results presented in [12], we have

\[
V_{\sigma_i}(z(t)) \leq V_{\sigma_i}(z(t_0)) \times \exp \left( \sum_{i=1}^{M} [N_{ai} \ln \mu_i] \right)
\]

\[
\times \exp \left( \max_{i \in \Xi} \left\{ (\ln \mu_i/\tau_{ai} - 2\lambda_i)(t - t_0) \right\} \right),
\]

(32)

If the switching signal \( \sigma(t) \) satisfies \( \tau_{ai} \geq \tau_{ai}^* = \frac{\ln \mu_i}{2\lambda_i} \), we have

\[
V_{\sigma_i}(z(t)) \leq V_{\sigma_i}(z(t_0)) \times \exp \left( \sum_{i=1}^{M} [N_{ai} \ln \mu_i] \right)
\]

\[
\times \exp \left( \max_{i \in \Xi} \left\{ -2\lambda_i \right\} (t - t_0) \right),
\]

(33)

where \( \tilde{\lambda}_i \in [0, \lambda_i) \).

Let \( \alpha = \min_{i \in \Xi} \{ \alpha_i \}, \beta = \max_{i \in \Xi} \{ \beta_i \} \). Considering (22), (23), and (33), we have

\[
\|z(t)\| \leq \sqrt{\frac{\beta}{\alpha}} \exp \left( \frac{1}{2} \sum_{i=1}^{M} [N_{ai} \ln \mu_i] \right)
\]

\[
\times \exp \left( \max_{i \in \Xi} \left\{ -\tilde{\lambda}_i \right\} (t - t_0) \right) \|z(t_0)\|.
\]

Since

\[
\mu_{mi} = \frac{\max \{ \lambda_{\max}(P_i), \min_{i \in \Xi} \{ \lambda_{\min}(\Phi_{i,h_k}) \} \}}{\min_{i \in \Xi} \{ \lambda_{\max}(\Phi_{i,h_k}) \}},
\]

we obtain

\[
\mu_{mi} \leq \mu_i.
\]
From (28), it holds that
\[
\min_{i \in \mathbb{Z}} \lambda_{mi} \lambda_{mi}(P_i), \min_{i \in \mathbb{Z}} \left\{ \lambda_{mi}(\Phi_i, h_i) \right\} \leq \lambda_{mi}.
\]
(35)

Combining (29) and (35) yields
\[
\lambda_i \leq \lambda_{mi},
\]
(36)
then, ones can deduce from (13), (34) and (36) that
\[
\tau^*_{ai} \geq \tau^*_{mi}.
\]
(37)

Because the switched system (2) switches synchronously with the switched reference model (3), (37) ensures the stability of (3) by using the switching signal (31). It is clear that the dynamics of the state tracking error and the weight error of the error switched system (16) with switched CL adaptive laws (18) converge to zero exponentially for the switching signal \( \sigma(t) \) satisfying the MDADT (31).

This completes the proof.

Remark 5: As a special case, when all \( A_{mi} \) are identical or all error subsystems of (14) admit a common Lyapunov function, the zero solution \( (e(t), \hat{\Theta}(t)) \equiv 0 \) is globally exponentially stable for any arbitrary switching signal \( \sigma(t) \).

Remark 6: For the switched system (2), Theorem 1 describes the relationship among nonlinear time-invariant matched uncertainties, the average dwell time of each subsystem, the switched CL algorithm, and the convergence rates of the state tracking error and the weight error.

V. SWITCHED CL-MRAC STRATEGY IN THE PRESENCE OF NONLINEAR TIME-VARYING MATCHED UNCERTAINTIES

In this section, we study the switched system (1) in the presence of time-varying nonlinear matched uncertainties. Our aim is to design an adaptive controller and determine a class of switching signals that ensure the convergence of the trajectories of the state tracking error and the weight error of the error switched system to a ball centered at the origin with a given radius. Because multiple Lyapunov functions involve nonlinear matched uncertainties, the tracking problem for a time-varying case is more complex than that for a time-invariant. Hence, we set up an error switched system with time-varying weights and define global practical stability. A lemma is then presented to give estimates of convergence rates of the state tracking error and the weight error of each subsystem outside the ball. Finally, a class of switching signals is found that ensures the global practical stability of the error switched system with the switched CL adaptive controller.

A. DESIGN OF THE SWITCHED CL ADAPTIVE LAWS

For convenience, we rewrite the switched system (1) as
\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + \Lambda_{\sigma(t)}(t, x(t)).
\]
A switched adaptive controller is given as
\[
u(t) = K^T_{\sigma(t)}(t)x(t) + K_{\sigma(t)}(t)r(t) - \hat{\Theta}^T_{\sigma(t)}(t)\Psi_{\sigma(t)}(x(t)),
\]
(38)
where \( \hat{\Theta}_{i}(t) = \Theta^*_i(t) - \hat{\Theta}_{i}(t) \).

Combining (1) and (38), we can rewrite the closed-loop switched system with time-varying weights as
\[
\dot{x}(t) = A_{nor}x(t) + B_{nor}r(t) + B_{\sigma}(\hat{\Theta}^T_{\sigma(t)}(t)\Psi_{\sigma(t)}(x(t)))
\]
(39)
From (3) and (39), we get the following error switched system with time-varying weights:
\[
\dot{e}(t) = A_{nor}e(t) + B_{\sigma}(\hat{\Theta}^T_{\sigma(t)}(t)\Psi_{\sigma(t)}(x(t)))
\]
(40)
Let \( z(t) = [e^T(t), \hat{\Theta}^T_{11}(t), \ldots, \hat{\Theta}^T_{i_0}(t), \hat{\Theta}^T_{12}(t), \ldots, \hat{\Theta}^T_{mi}(t)]^T \). In order to study the stability of (40), the following definition is required.

Definition 1 [19]: Given a constant \( r^* > 0 \), the error switched system (40) is said to be globally practically stable with respect to \( r^* \) if there exist a switching law \( \sigma(t) \) and a constant \( T^* \) such that for \( \tau(t) \in \mathbb{S}(r^*) \), the trajectories of \( z(t) \) are identical to \( \mathbb{S}(r^*) \) for \( t \geq t_0 + T^* \).

Our objective is to design switching adaptive control laws and determine a class of switching signal such that the error switched system (40) is globally practically stable.

Considering the benefits of the CL adaptive control, we utilize the switched CL algorithm in the following switched CL adaptive laws
\[
\dot{\hat{\Theta}}_{i}(t) = \Gamma_i \Psi_i(x(t))^T(t)P_iB_i,
\]
(41)
and
\[
\dot{\hat{\Theta}}_{i}(t) = \Gamma_i \Psi_i(x(t))^T(t)P_iB_i,
\]
(42)
where
\[
E_{i,d}(t) = (B_i^T B_i)^{-1}B_i^T (x_{i,d} - A_{mi}x_{i,d} - B_{mi}r_{i,d}).
\]
(43)
Using (41) and (42), we have
\[
\dot{E}_{i,d}(t) = \Psi_i^T (x_{i,d})(\Theta^*_i(t) - \hat{\Theta}_i(t)).
\]
(44)
Remark 7: Although $\hat{\Theta}_i^*(t)$, $\hat{\Theta}_j(t)$, and $\Theta_k^*(td)$ are unavailable in (44), we can use them to analyze the stability of the error switched system (40).

**B. STABILITY ANALYSIS OF THE ERROR SWITCHED SYSTEM**

We analyze the stability of the error switched system (40) with time-varying weights and determine a class of switching signals $\sigma(t)$ such that the error switched system (40) with the switched concurrent learning adaptive controller (41) is globally practically stable with respect to $r^*$. Construct the following Lyapunov function:

$$V(z(t)) = e^T(t)P_{\sigma(t)}e(t) + \sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\Gamma_i^{-1}\tilde{\Theta}_i(t)].$$  

(45)

Apparently, (22) and (23) still hold. When the $i$-th error subsystem of (40) is active, the time derivative of $V(z(t))$ along the trajectory of the error switched system (40) and switched CL adaptive laws (41) is

$$\dot{V}_i(t, z(t)) = e^T(t)(A_{m_i}^TP_i + P_iA_{m_i})e(t) + 2\text{tr}[\tilde{\Theta}_i^T(t)\Gamma_i^{-1}\tilde{\Theta}_i(t)]$$

$$-2\sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\dot{\Theta}_i(t) - \Theta_k^*(td)]$$

$$-2\sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\Theta_k^*(td)].$$  

(46)

Substituting $\Phi_i,h_k = \sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)$ into (46), we obtain

$$\dot{V}_i(t, z(t)) \leq -2\lambda_{mi}\lambda_{min}(P_i)\|e(t)\|^2 - 2\sum_{i=1}^{M} \left[\lambda_{min}(\Phi_i,h_k)\|\dot{\Theta}_i(t)\|_F^2\right]$$

$$+ 2\sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\Theta_k^*(td)]$$

$$-2\sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\Theta_k^*(td)]$$

$$+ 2\text{tr}[\tilde{\Theta}_i^T(t)\Gamma_i^{-1}\tilde{\Theta}_i(t)].$$  

(47)

With the help of Cauchy-Bunyakowski-Schwarz (CBS) inequality, we obtain $|tr(M^TN)| \leq \|M\|_F\|N\|_F$ for any arbitrary matrices $M$, $N \in \mathbb{R}^{m \times q}$. Thus, we get the following inequalities from (47)

$$\dot{V}_i(t, z(t)) \leq -2\lambda_{mi}\lambda_{min}(P_i)\|e(t)\|^2$$

$$- 2\sum_{i=1}^{M} \left[\lambda_{min}(\Phi_i,h_k)\|\dot{\Theta}_i(t)\|_F^2\right] + 2\Gamma_i^{-1}\|\dot{\Theta}_i(t)\|_F\|\dot{\Theta}_i(t)\|_F$$

$$+ 2\sum_{i=1}^{M} \text{tr}[\tilde{\Theta}_i^T(t)\sum_{d=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\dot{\Theta}_i(t) - \Theta_k^*(td)]$$

$$+ 2\text{tr}[\tilde{\Theta}_i^T(t)\Gamma_i^{-1}\tilde{\Theta}_i(t)].$$  

(48)

According to $\|\Theta^*(t)\|_F \leq \sigma$ and $\|\dot{\Theta}(t)\|_F \leq w$, we have

$$\dot{V}_i(t, z(t)) \leq -2\lambda_{mi}\lambda_{min}(P_i)\|e(t)\|^2$$

$$- 2\sum_{i=1}^{M} \left[\lambda_{min}(\Phi_i,h_k)\|\dot{\Theta}_i(t)\|_F^2\right] + 2\Gamma_i^{-1}w\|\dot{\Theta}_i(t)\|_F$$

$$+ 2\sigma\sum_{i=1}^{M} \|\Phi_i,h_k\|_F+\tilde{d}\|\Phi_i,h_k\|_2\|\dot{\Theta}_i(t)\|_F.$$  

(49)

Based on the results presented in [29], the following inequalities are satisfied:

$$\left\|\sum_{i=1}^{\tilde{d}} \Psi_i(x_i,d,h_k)\Psi_i^T(x_i,d,h_k)\Theta_k^*(td)\right\|_F \leq \sigma\tilde{d}\|\Phi_i,h_k\|_2.$$  

(50)

Substituting (50) into (49), we arrive at

$$\dot{V}_i(t, z(t)) \leq -2\lambda_{mi}\lambda_{min}(P_i)\|e(t)\|^2$$

$$- 2\sum_{i=1}^{M} \left[\lambda_{min}(\Phi_i,h_k)\|\dot{\Theta}_i(t)\|_F^2\right] + 2\Gamma_i^{-1}w\|\dot{\Theta}_i(t)\|_F$$

$$+ 2\sigma\sum_{i=1}^{M} \|\Phi_i,h_k\|_F+\tilde{d}\|\Phi_i,h_k\|_2\|\dot{\Theta}_i(t)\|_F.$$  

(51)

Because of $\|z(t)\|^2 = \|e(t)\|^2 + \sum_{i=1}^{M} \|\dot{\Theta}_i(t)\|_F^2$ and $\|\dot{\Theta}_i(t)\|_F \leq \|z(t)\|$, we have

$$\dot{V}_i(t, z(t)) \leq -2\min_{i \in \Xi} \left\{\lambda_{mi}\lambda_{min}(P_i), \min_{i \in \Xi} \lambda_{min}(\Phi_i,h_k)\right\}\|z(t)\|^2$$

$$+ 2\left(\Gamma_i^{-1}w + \sigma\sum_{i=1}^{M} \|\Phi_i,h_k\|_F+\tilde{d}\|\Phi_i,h_k\|_2\right)\|z(t)\|.$$  

(52)

A simple calculation then yields

$$\dot{V}_i(t, z(t)) \leq -2(1 - \rho)$$

$$\times \min_{i \in \Xi} \left\{\lambda_{mi}\lambda_{min}(P_i), \min_{i \in \Xi} \lambda_{min}(\Phi_i,h_k)\right\}\|z(t)\|^2$$
\[-2\rho \min \left\{ \lambda_m \lambda_{\min}(P_i), \min_{i \in \mathbb{S}} \left\{ \lambda_{\min}(\Phi_{i,h_k}) \right\} \right\} \|z(t)\|^2 + 2 \left\{ \Gamma_i^{-1} w + \sigma \sum_{i=1}^{M} \left[ \|\Phi_{i,h_k}\|_F + \vec{d} \left\| \Phi_{i,h_k} \right\|_2 \right] \right\} \|z(t)\|, \]

where \(0 < \rho < 1\).

Subsequently, let

\[ r_i = \frac{\rho \min \left\{ \lambda_m \lambda_{\min}(P_i), \min_{i \in \mathbb{S}} \left\{ \lambda_{\min}(\Phi_{i,h_k}) \right\} \right\}}{\lambda_{\min}(\Phi_{i,h_k})}, \]

Then, for \(\|z(t)\| \geq r_i\), we have

\[ \dot{V}_i(t, z(t)) \leq -\eta_i h_k \|z(t)\|^2, \]

where

\[ \eta_i h_k = (1 - \rho) \min \left\{ \lambda_m \lambda_{\min}(P_i), \min_{i \in \mathbb{S}} \left\{ \lambda_{\min}(\Phi_{i,h_k}) \right\} \right\}. \]

When the \(i\)-th error subsystem of (40) is active, \(\lambda_{\min}(\Phi_{i,h_k})\) is monotonically increasing according to the switched CL algorithm. We then let \(\eta_i = \eta_i h_k = \min_{h_k \in \mathbb{N}} \{\eta_i h_k\}\). Combining this with (23) and (55), we obtain

\[ \dot{V}_i(t, z(t)) \leq -\lambda'_{i} V_i(z(t)), \]

i.e., \(V_i(z(t)) \leq \exp(-2\lambda'_{i}(t-t_0))V_i(z(t_0))\), where

\[ \lambda'_{i} = \frac{\eta_i}{\beta_i}. \]

Next, we analyze the stability of each error switched subsystem of (40) individually, i.e., the \(i\)-th error subsystem of (40) that is active for \(t \geq t_0\).

For any given \(r_{0i} \geq r_i\), denote \(S(r_{0i}) = \{z(t) \mid \|z\| \leq r_{0i}\}\). If \(z(t_0) \in R^T/S(r_{0i})\), then, the following inequalities hold

\[ V_i(z(t)) \leq \exp(-2\lambda'_{i}(t-t_0))V_i(z(t_0)), \|z(t_0)\| > r_{0i}, \]

where \(\lambda'_{i} = \frac{\eta_i}{\beta_i}\). Using (22), we have

\[ \|z(t)\| \leq \sqrt{\frac{\beta_i}{\alpha_t}} \exp(-\lambda'_{i}(t-t_0)) \|z(t_0)\| \|z(t_0)\| > r_{0i}. \]

Therefore, we have the following properties of the trajectory \(z(t)\):

(i) Attractiveness. When \(z(t_0) \in R^T/S(r_{0i})\), there exists \(T_i = T_i(z(t_0), r_{0i}) \geq 0\) such that \(\|z(t_0 + T_i)\| = r_{0i}\). This property is depicted in Fig.1:

(ii) Boundedness. When \(z(t_0) \in S(r_{0i})\), considering (22) and (57), we have \(z(t; t_0, z(t_0)) \in \Omega(r_{0i}) \triangleq \{z(t) \mid V(z(t)) \leq \beta_i r_{0i}^2 \}, \) i.e., \(z(t; t_0, z(t_0)) \in S(r_{0i}^*)\). Suppose \(z(t) \|z(t)\| \leq r_{0i}^*\) holds for any given \(r_{0i}^* \geq \sqrt{\frac{\beta_i}{\alpha_t}} r_{0i}\). Moreover, when \(z(t_0) \in S(r_{0i}^*)\), \(z(t; t_0, z(t_0)) \in S(r_{0i}^*)\) holds for \(T_i^*(z(t_0), r_{0i}^*) = 0\). This property is depicted in Fig.2:

(iii) Global practical stability. If \(z(t_0) \in R^T\), combining the attractiveness, and \(r_{0i}^* > r_{0i}\), it holds that

there exists a constant \(T_i^* = T_i^*(z(t_0), r_{0i}^*) < T_i(z(t_0), r_{0i})\) such that \(z(t; t_0, z(t_0)) \in S(r_{0i}^*)\) for \(t \geq t_0 + T_i^*(z(t_0), r_{0i}^*)\), i.e., the trajectory \(z(t)\) converges to a ball centered at the origin with a given radius of \(r_{0i}^*\).

Our conclusions are summarized in the following lemma to describe the trajectory \(z(t)\) of the \(i\)-th error switched subsystem with time-varying weights.

**Lemma 1:** Consider the \(i\)-th error switched subsystem of (40) with switched CL adaptive laws (41). Assume that the \(i\)-th subsystem history stacks \(\Pi_i\) contain linearly independent columns. If the history stacks are updated according to the switched CL algorithm, then, for any \(r_{0i}\) and \(t \geq t_0\), there exists \(r_{0i}^* > 0\) that ensures the global practical stability of the error switched (40) for \(\sigma(t) \equiv i\).

**Remark 8:** Lemma 1 gives estimates of the convergent rates of the state tracking error and the adaptive weight error of each error subsystem outside a ball centered at the origin with a given radius.

Based on Lemma 1, a sufficient condition for the global practical stability of the error switched system (40) is derived in the following theorem.

**Theorem 2:** Consider the error switched system (40) and the switched concurrent CL laws (41). Assume that the \(i\)-th subsystem history stacks \(\Pi_i\) contain linearly independent columns. If the history stacks are updated according to the switched CL algorithm, then, for any \(r_{0i} \geq \max_{i \in \mathbb{S}} \{r_{0i}\}\), there exists \(r_{0i}^* > 0\) such that the error switched system (40) is globally practically stable with respect to \(r_{0i}^*\) under any
switching law \( \sigma(t) \) that satisfies the MDADT
\[
\tau_{a1}^r \geq \tau_{a1}^{r*} = \frac{\ln \mu_i}{2\lambda_i'}, \tag{59}
\]
and any chattering bound \( N_0 \beta > 0 \).

Proof: For any given \( r_0 \geq \max \{r_0(i)\} \) let \( r^* \geq \sqrt{\frac{\beta}{\alpha}} r_0 \), where \( \alpha = \min \{\alpha_i\} \), and \( \beta = \min \{\beta_i\} \). It is notable that (22), (23) and (57) are satisfied for \( z(t) \in R^n/S (r^*) \). We will prove that there exists a constant \( T^* = T^*(z(t_0), r^*) \geq 0 \) for any \( z(t_0) \in R^n \) such that \( z(t_0, z(t_0)) \in S (r^*) \) for \( t \geq t_0 + T^* \) when the switching law \( \sigma(t) \) satisfies the average dwell time (59).

The proof is divided into three parts. In Part 1, we prove the attractiveness, i.e., if the trajectory \( z(t) \) is outside the ball \( S (r_0) \) at \( t = t_h \), it must reach the boundary of the ball within finite time period and stay inside the ball afterward. We then prove the boundedness in part 2, i.e., \( z(t_{h+1}) \in S (r^*) \) holds for \( z(t_h) \in S (r_0) \). Finally, the global practical stability is proved, namely, if \( z(t_0) \in R^n \), there exists \( T^* \geq 0 \) such that \( z(t_0, z(t_0)) \in S (r^*) \) for \( t \geq t_0 + T^* \).

Part 1 (Attractiveness): We will show that if \( z(t_h) \in R^n/S (r_0) \), then, there exists \( T_h = T_h(z(t_h), r_0) \geq 0 \) such that \( \|z(t_h + T_h)\| = r_0 \) for \( z(t) \in R^n/S (r_0) \) and \( t \in [t_h, t_h + T_h] \).

For any \( t > t_h \), denote \( N_0(t, t_h) = \sum_{i=1}^{M} N_{a1}(t, t_h) \). Here, according to Lemma 1 and results of [12], we have
\[
V_{\sigma(t)}(z(t)) \leq \exp \left( \max_{i \in E} \left\{ (\ln \mu_i / \tau_{a1}^r) - 2\lambda_i' \right\} (t - t_h) \right) \times \exp \left( \sum_{i=1}^{M} [N_0 \ln \mu_i] \right) \times V_{\sigma(t_h)}(z(t_h)). \tag{60}
\]

If the switching signal \( \sigma(t) \) satisfies the MDADT (59), for \( \|z(t_h)\| > r_0 \), we have
\[
V_{\sigma(t)}(z(t)) \leq \exp \left( \max_{i \in E} \left\{ -2\lambda_i' \right\} (t - t_h) \right) \times \exp \left( \sum_{i=1}^{M} [N_0 \ln \mu_i] \right) \times V_{\sigma(t_h)}(z(t_h)), \tag{61}
\]
where \( \lambda_i' \in (0, \lambda_i] \).

It follows from (22), (23), and (61) that
\[
\|z(t)\| \leq \sqrt{\frac{\beta}{\alpha}} \exp \left( \frac{1}{2} \sum_{i=1}^{M} [N_0 \ln \mu_i] \right) \times \exp \left( \max_{i \in E} \left\{ -\lambda_i' \right\} (t - t_h) \right) \|z(t_h)\|.
\]

Therefore, for \( r_0 \geq \max \{r_0(i)\} \), when \( z(t_h) \in R^n/S (r_0) \), there exists \( T_h = T_h(z(t_h), r_0) \geq 0 \) such that \( \|z(t_h + T_h)\| = r_0 \). Naturally, \( T_h \) is a decreasing sequence. Thus, \( T_0 = T_0(z(t_h), r_0) \geq T_h \). This property is depicted in Fig.3.

FIGURE 3. Attractiveness of the switched system.

Part 2 (Boundedness): Here, we will prove that when \( z(t_h) \) is entirely inside \( S (r_0) \), there exist a constant \( r^* \) such that \( z(t_{h+1}) \) stays inside \( S (r^*) \).

Based on Lemma 1, it is straightforward that \( z(t_{h+1}) \in \Omega_h \triangleq \{z(t) | V(z(t)) \leq \beta h R_0^2 \} \) holds for \( z(t_h) \in S (r_0) \). According to (22), we have \( z(t_{h+1}) \in \Omega_h \triangleq \{z(t) | V(z(t)) \leq \beta h R_0^2 \} \). Thus, for any \( r^* \geq \sqrt{\frac{\beta}{\alpha}} r_0 \), we have \( z(t_{h+1}) \in S (r^*) \). When \( z(t_h) \in S (r^*) \), according to Lemma 1, \( z(t; t_h, z(t_h)) \in S (r^*) \) holds for \( T^*(z(t_h), r^*) = 0 \). This property is depicted in Fig.4.

FIGURE 4. Boundedness of the switched system.

Part 3 (Global Practical Stability): When \( z(t_0) \in R^n/S (r^*) \), combining Parts 1 and 2 with \( r^* \geq r_0 \), it holds that there exists a constant \( T^* = T^*(z(t_0), r^*) < T_0(z(t_0), r_0) \) such that \( z(t; t_0, z(t_0)) \in S (r^*) \) for \( t \geq t_0 + T^*(z(t_0), r^*) \). When \( z(t_0) \in S (r^*) \), the result remains true for \( T^*(z(t_0), r^*) = 0 \).

Hence, we conclude that \( z(t) \) is bounded. According to \( \beta_i = \max \{\lambda_{max}(P_i), \max_{i \in E} \{\lambda_{max}(G_i^{-1})\} \} \), (22), and (56), it holds that \( \lambda_i' \leq \lambda_{mi} \). Thus, combining the latter with (34), we arrive at \( r_{a1}^{r*} \geq r_{mi}^* \). Apparently, the bounded-input-bounded-state stability of the switched reference model (3) can be ensured for the identical switching sequences as the error switched system (40). Therefore, all the signals in (3) and (39) are bounded for the switching law \( \sigma(t) \) satisfying the MDADT (59).

This completes the proof.

Remark 9: Theorem 2 establishes the relationship among time-varying matched uncertainties of each subsystem, the MADT, the convergence domain \( r^* \) of the state tracking
error $e(t)$ and weight error $\tilde{g}_i(t)$, and the switched CL algorithm.

Remark 10: The trajectory $z(t)$ that converges to a small ball with a radius of $r^* \geq \sqrt{r^0}$. $r^*$ is related to the CL adaptive and time-varying weights of each subsystem. $r^*$ can be made arbitrarily small by appropriately choosing the $d$-th recorded data point $\Psi_t(x_i, d, h_i)$ of $\Pi_h$ in the switched CL algorithm.

VI. EXAMPLE

In this section, we apply the proposed switched CL-MRAC scheme to an electro-hydraulic system to illustrate the effectiveness. The system consists of a MOOG 760 torque motor/flapper operated with a four-way double-acting servo valve, a hydraulic pump, two accumulators, a single rod actuator, and a hydraulic arm. The schematic diagram is depicted in Fig.5. A cylinder with a diameter of 32 mm is used, and a stroke of approximately 100 mm is used as the actuator arm. An aircraft control surface is represented by the end of the hydraulic arm connected to an inertial load. The control voltage imposing on the torque motor current amplifier is the input $u$ of the system. A linear variable differential transducer that measures the actuator arm displacement is the output $y$. The system parameters are adjusted by switchable accumulators.

![Diagram of the electro-hydraulic system](image)

We choose two operating conditions with two distinct supply pressures: condition 1 (11.0MPa) and condition 2 (1.4MPa). According to the results of [42], we have the following transfer functions:

**Condition 1**: $G_1(s) = \frac{62.4}{s(s + 4.58)}$.

**Condition 2**: $G_2(s) = \frac{47.2}{s(s + 9.19)}$. 

Set $x(t) = (x_1(t), x_2(t))^T$, $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. The transfer functions above can be rewritten in a control canonical forms with nonlinear time-varying matched uncertainties as follows:

**Condition 1**: 
\[
\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -4.58 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 62.4 \end{pmatrix} (u + \Lambda_1(t, x)),
\]

**Condition 2**: 
\[
\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -9.19 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 47.2 \end{pmatrix} (u + \Lambda_2(t, x)).
\]

(63)

For the subsystems corresponding to Conditions 1 and 2, the reference models are given by

**Condition 1**: 
\[
\dot{x}_m(t) = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix} x_m(t) + \begin{pmatrix} 0 \\ 31.2 \end{pmatrix} r(t),
\]

**Condition 2**: 
\[
\dot{x}_m(t) = \begin{pmatrix} 0 & 1 \\ -27 & -12 \end{pmatrix} x_m(t) + \begin{pmatrix} 0 \\ 23.6 \end{pmatrix} r(t).
\]

(64)

with the reference input of $r(t) = 1$, which is not PE.

To satisfy Assumption 3, we let $K_{r1}^T = [-0.2404 - 0.0548]$, $K_{r2}^T = [-0.5720 - 0.0595]$, $K_{r1} = 0.5$, and $K_{r2} = 0.5$. Selecting $\lambda_{m1} = 0.65$ and $\lambda_{m2} = 0.75$, and solving (4), we have

\[
P_1 = \begin{bmatrix} 1.2565 & 0.0843 \\ 0.0843 & 0.0585 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.2443 & 0.0843 \\ 0.0518 & 0.0326 \end{bmatrix}.
\]

Using (13), (22) and (23), we have $\alpha_1 = 0.0525$, $\alpha_2 = 0.0304$, $\beta_1 = 1.2565$, $\beta_2 = 1.2465$, $\mu_{m1} = 41.3322$, $\mu_{m2} = 23.7428$, $\tau_{m1} = 4.135$, and $\tau_{m2} = 2.112$.

First, we consider the switched system (63) with the following nonlinear time-invariant matched uncertainties

**Condition 1**: 
\[
\Lambda_1(x(t)) = \Theta_1^T \Psi_1(x) = \begin{pmatrix} 1 & 1.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

**Condition 2**: 
\[
\Lambda_2(x(t)) = \Theta_2^T \Psi_2(x) = \begin{pmatrix} 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

(65)

The adaptive gain matrices are selected as $\Gamma_1 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $\Gamma_2 = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$. Let $\bar{d} = 3$ and initialize the parameters of the recording algorithms $\Phi_1 = \begin{bmatrix} 0.5253 & 0.4165 \\ 0.1465 & 0.0440 \end{bmatrix}$, $\Phi_1 = \begin{bmatrix} 2.0083 & 2.3795 \\ 2.3795 & 4.1195 \end{bmatrix}$. According to (31), we have $\tau_{a1} = 35.7457$ and $\tau_{a2} = 27.6490$. When $\tau_{a1} \geq 36$ and $\tau_{a2} \geq 28$, the conditions of Theorem 1 are satisfied, and the MRAC of system (63) with time-invariant matched uncertainties (65) can be solved. Let $\epsilon(t_0) = (2, 1)^T$. The switching signal is illustrated in Fig.6, and we obtain the state tracking error as shown in Fig.7. Moreover, the weight error is given in Figs.8 and 9. From Figs.7-9, we can observe that the tendency of the state tracking error and the weight error...
Then, we turn to consider the switched system (63) with the following nonlinear time-varying matched uncertainties:

**Condition 1:**

\[
\Lambda_1(x(t)) = \Theta_1^T \Psi_1(x) = (0.5 \sin 0.01t \quad 2) \begin{pmatrix} 1 \\ x_2 \end{pmatrix}.
\]

**Condition 2:**

\[
\Lambda_2(x(t)) = \Theta_1^T \Psi_2(x) = (1.5 \sin 0.02t \quad 3) \begin{pmatrix} 1 \\ x_1 \end{pmatrix}.
\]

Based on the calculations, we have \(\alpha = 0.0304, \beta = 1.2565, \lambda_1 = 0.008, \lambda_2 = 0.005\). We choose \(\rho = 0.7, r_{01} = r_1 = 6.5198, r_{02} = r_2 = 5.8797, r_0 = 6.5198, r^* = 41.9159, \tau_{a1}^* = 47.8421\) and \(\tau_{a2}^* = 34.1675\). When \(\tau_{a1} \geq 48\) and \(\tau_{a2} \geq 35\), the conditions of Theorem 2 are satisfied and the MRAC of the system (63) with time-varying uncertainties is convergent. When \(t > 150\) the state tracking error converges to zero, and the weights converge to their true values.
matched uncertainties $e(t_0) = (2, -1)^T$ with $\|e(t_0)\| = 2.24 < r^*$, and $e(t_0) = (33, -30)^T$ with $\|e(t_0)\| = 44.60 > r^*$. The results of our simulations are depicted in Fig.10-16. The switching signal is shown in Fig.10. For $e(t_0) = (2, -1)^T$, Fig.11 depicts the state tracking error and Figs. 12 and 13 present the weight error. For $e(t_0) = (33, -30)^T$, Fig.14 depicts the state tracking error and Figs. 15 and 16 present the weight error. The simulation results indicate that the state tracking error and the weight error converge into a ball with the radius $r^*$ regardless of whether the initial state error is within this ball or not.

VII. CONCLUSION
In this paper, a switched system with nonlinear time-varying uncertainties has been studied by using the MDADT method. To achieve the convergence of the state tracking error and the weight error, we proposed a switched CL-MRAC strategy wherein PE is not required. First, in order to obtain the boundedness of the reference state, the bounded-input bounded-state stability of the switched reference model was studied under the MDADT condition. Second, in the case of nonlinear time-invariant uncertainties, we proposed a switched CL-MRAC scheme, by which the convergence of exponential state tracking error and weight error is guaranteed. Subsequently, in the case of nonlinear time-varying uncertainties, the estimates of the convergence rates of the state tracking error and weight error of each subsystem were given outside a ball. Thus, the global practical stability of the state tracking error and the weight error of the switched system were obtained by using the MDADT method. Finally, a practical example of an electro-hydraulic system was provided, and the effectiveness of the proposed methodology was demonstrated.

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