Direct determination of the Tomonaga-Luttinger parameter $K$ in quasi-one-dimensional spin systems

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We test the analytical formula for the enhancement of the nuclear magnetic resonance rate $T_1^{-1}$ by the critical spin fluctuations, over the simple power-law dependence predicted for purely one-dimensional spin system, recently derived in the random phase approximation [M. Dupont et al., Phys. Rev. B 98, 094403 (2018)]. This new prediction is experimentally confirmed by excellent fits to the published temperature dependence of $T_1^{-1}$ data in the two representative spin compounds, (C$_7$H$_{10}$N)$_2$CuBr$_4$ (DIMPY) and BaCo$_2$V$_2$O$_8$, providing at the same time a direct and convenient experimental determination of the Tomonaga-Luttinger-liquid parameter $K$, very well in agreement with theoretical predictions.

The Tomonaga-Luttinger liquid (TLL) theory provides a general description, including interactions, of low-energy behavior for any gapless one-dimensional (1D) system [1]. Its importance in the description of quasi-1D materials is therefore crucial, and it can be regarded as analogous to what Fermi-liquid description is for three-dimensional (3D) systems. While the main hallmark of the TLL description, namely the power-law dependence of 1D response/correlation functions, had been experimentally well established previously [2, 3], it is only a decade ago that quasi-1D quantum spin compounds have provided the final quantitative verification of the TLL theory [4, 5]. In the spin-ladder compound (C$_7$H$_{12}$N)$_2$CuBr$_4$, also known as BPCB, one could compare the experimental values with the TLL-based predictions for the magnetic field ($B$) dependence of $i)$ the phase boundary $T_c(B)$ of the low-temperature ($T$) ordered phase, $ii)$ the low-$T$ limit of the order parameter of this phase [4, 6], and $iii)$ the nuclear magnetic resonance (NMR) spin-lattice relaxation rate $T_1^{-1}(B)$ in the TLL regime, at $T \gg T_c$ [4, 7]. A successful theoretical description of these data thus confirmed the field induced variations of the two TLL parameters $K$ and $u$ that respectively define the power-law exponents and the renormalized Fermi velocity.

However, the first attempt to directly determine the $K$ parameter from the measured $T_1^{-1}(T)$ dependence, performed in the spin-ladder compound (C$_7$H$_{10}$N)$_2$CuBr$_4$, also known as DIMPY, failed [8]. This was attributed to the enhancement of relaxation by the critical spin fluctuations in very broad vicinity of $T_c$. Indeed, a correct determination of the $K$ value from the power-law exponent of the $T_1^{-1}(T)$ temperature dependence is in practice precluded by the enhancement of relaxation related to the nearby $T_c$ on the low-$T_c$ side, as well as by the inherent limitation of a TLL description to low-energy, and thus low temperature, on the high-$T_c$ side [9, 10].

Recently, this was described theoretically both exactly, by quantum Monte Carlo (QMC) simulations, and analytically, using the random phase approximation (RPA) to describe the effect of fluctuations [11]. The former approach showed that purely 1D (TLL) power-law regime of $T_1^{-1}(T) \propto T^{1/2K-1}$ dependence, observed when the three-dimensional (3D) exchange couplings are 3 or more orders of magnitude smaller than 1D coupling, is rapidly shrank and disappears as soon as 3D couplings strength raises to the level of percent (see Fig. 4 in Ref. [11]). In practice, this means that it is not really expected to be observable in most of experimentally interesting spin systems.

![FIG. 1. The 3D plot of the correction function $\Phi(K, T_c/T)$ defined by Eq. (2) [11]. The dashed and solid contour lines are spaced at intervals of 0.02 and 0.1, respectively.](image)
systems. Furthermore, a closed analytical expression, depending only on $T_c$ and $K$, was derived within the RPA approximation (and checked against QMC) to take into account the fluctuations related to $T_c$ \[11\].

In this letter we apply this RPA correction to the published $T_{1}^{-1}(T)$ NMR data in two very different, representative, quasi-1D spin systems [8,12], and find that it provides a remarkable fit to the data. These fits present the first direct experimental determination of the $K$ values that confirms the theoretically predicted values. They also provide a convenient means of experimental characterization of a quasi-1D system, independent of its complete theoretical description that requires the knowledge of the Hamiltonian and of numerical techniques [QMC, density-matrix renormalization group (DMRG)]. Finally, the fit covers the data quite close to $T_c$ and can also provide an independent estimate of the $T_c$ value. Altogether, it constitutes a reference for the normal quasi-1D behavior, which can be used to reveal non-standard cases.

In the following, we first discuss the analytical RPA correction to TLL prediction $T_{1}^{-1}(T)$, which was cast to a multiplicative correction function $\Phi(K, T_c/T)$ \[11\]:

\[
T_{1}^{-1}(T, B) = T_{1\text{TLL}}^{-1}(T, B) \times \Phi(K(B), T_c/T) = a T^{1/2 K-1} \Phi(K, T_c/T),
\]

\[
\Phi(K, T_c/T) = \frac{1}{N(K)} \int_{-\infty}^{\infty} \frac{d\xi}{\sin^2(\frac{\pi \xi}{8 K}) + \sinh^2(\pi \xi)} \frac{\Gamma\left(\frac{1}{2} + i \xi\right)}{\Gamma\left(1 - \frac{1}{2 K} + i \xi\right)} \left(\frac{1}{4 K}\right)^{2} \cos\left(\frac{\pi}{4 K}\right) B\left(\frac{1}{4 K}, 1 - \frac{1}{2 K}\right)
\]

and $N(K) = 2\Gamma^2\left(\frac{1}{4 K}\right) \cos\left(\frac{\pi}{4 K}\right) B\left(\frac{1}{4 K}, 1 - \frac{1}{2 K}\right)$.  

Here, $E(x)$ is the complete elliptic integral of the second kind, $\Gamma(x)$ and $B(x, y)$ are respectively the Euler Gamma and Beta functions, and $a$ is the amplitude whose magnitude is normalized to 1, $\Phi(K)$, $T_c$ are taken e.g. in the temperature interval between 2$T_c$ and 3$T_c$, as in the previous analysis of DIMPY data \[8\], is significantly distorted. For the DIMPY data, this effect is quantified in Fig. 2, indeed, the $K$ values corresponding to the apparent power-law fit are systematically higher than the ones using the RPA+TLL fit defined by Eq. (2). For these latter fits, $T_c$ has been determined independently for each field value from the onset of building up of the order parameter, observed through the splitting of the NMR lines (see Fig. 1(b) in Ref. [8]). Only two parameters are then determined by the least square fit to the data, the amplitude $a$ and the $K$ value. Thus obtained $K$ values nicely follow the theoretical prediction, with the exception of the data taken at 3.5 T. We remark that this lowest field value is relatively close to the critical field $B_{c1} = 2.9$ T, in the vicinity of which the TLL description is not applicable. Finally, the error bars of the RPA+TLL fits are smaller because the temperature interval of these fits is much wider (on the logarithmic scale), which stabilizes the fit.

While the $\Phi(K, T_c/T)$ function (Fig. 1) is weakly dependent on $K$, it clearly diverges as $T$ decreases towards $T_c$ [11]. Therefore, $T_c$ can be taken as the third free parameter of the fit, in addition to $K$ and $a$ in Eq. (1), in order to provide an independent estimate of its value, predicted (extrapolated) from the spin dynamics observed above $T_c$. We present such 3-parameter fits on the example of the published $T_{1}^{-1}$ data in BaCo$_2$V$_2$O$_8$, an Ising-like $S = 1/2$ spin chain [12]. For these fits we note that the correction function $\Phi(K, T_c/T)$ has been calculated \[11\] for systems, such as Heisenberg $S = 1/2$ spin ladders, where the dominant spin fluctuations are the antiferromagnetic (AF) transverse ones, which is
FIG. 2. (left panels) Comparison of the new RPA+TLL fit (red lines and text) with the previous apparent power-law (TLL) fit (blue lines and text) to the DIMPY data, as given in Ref. [8]. Solid data points denote values taken into account in the former least square fit. Pure uncorrected TLL contribution to this fit, \( T_{TLL}^{-1}(T, K) = a T^{1/2 K - 1} \), is given by red dotted lines, to show how much it differs from the apparent power-law fit. The temperature scale of each sub-panel starts with the \( T_c \) value. (right panel) The field dependence of thus determined \( K \) values, compared to the theoretical prediction [8].

Expected to be valid when \( K > 0.5 \). It is easy to show that it can also be applied to systems, such as Ising-like chains, where the dominant fluctuations are longitudinal and incommensurate (IC), which is expected to be valid when \( K < 0.5 \). The formulas that describe the relevant spin correlators and spin susceptibilities for these two types of fluctuations, given by Eqs. (6.47), (6.50) and (6.53) in Ref. [14], have identical form up to the \( 1/2 K \leftrightarrow 2K' \) symmetry transformation/correspondence around a so-called “\( \eta \)-inversion” point at \( K = 0.5 \) or \( \eta = 1 (\eta = 1/2K) \) [12, 15, 16], at which the dominant fluctuations change their type. As both the RPA correction function and \( T_c \) are calculated/defined from the dynamic susceptibility, the same symmetry transformation applies to \( \Phi(K, T_c/T) \). Therefore, for the longitudinal IC fluctuations we get:

\[
T_{1}^{-1}(T) \propto T^{2K-1} \Phi(1/4K, T_c/T). \tag{3}
\]

Fig. 3 shows this fit applied to the \( \text{BaCo}_2\text{V}_2\text{O}_8 \) data taken at 4.1 T [12], where both \( K \) and \( T_c \) (and the amplitude) are taken as the fit parameters. Here, the fitted data cover a broad enough temperature interval to well represent both the power-law and the fluctuation-enhanced regime. This is followed by a sharp peak of \( T_{1}^{-1}(T) \), whose maximum reflects the maximum of the critical spin fluctuation and thus precisely defines the \( T_c \) value. The corresponding \( T_c \) value determined from the RPA+TLL fit given by Eq. (3) is only 2% lower, which is within the statistical error as defined by the fit. The equality of these two very different estimates of \( T_c \), one reflecting critical dynamics at \( T_c \) and the other above \( T_c \), constitutes a very strong confirmation for the validity of the employed correction function. Furthermore, the obtained \( K = 0.23(1) \) value is very close to the \( K = 1/4 \) value expected for the nearby critical field value \( B_c = 3.8 \) T. Parenthetically, we observe that the \( 1/2K \leftrightarrow 2K' \) symmetry connects this value to the noninteracting limit \( K' = 1 \).

Fig. 4 presents the fits to the two available \( \text{BaCo}_2\text{V}_2\text{O}_8 \) data sets close to the saturation field \( B_s = 22.8 \) T.
Here, the successful fit is of “mixed” character, \( T_i^{-1}(T) \propto T^{2K-1} \Phi(K, T_c/T) \): the power-law (TLL) contribution is of the same type as at low fields (Fig. 3), corresponding to IC fluctuations, while the correction factor \( \Phi \) corresponds to the dominant transverse AF fluctuations, as expected in the \( \eta \)-inversion scenario [16]. The power-law exponent can then be explained by the nature of the hyperfine coupling in this compound, which filters out the contribution of AF spin fluctuations and thus selects the IC contribution to \( T_i^{-1} \) [12], even when it is sub-dominant. As expected, the obtained values are close to \( K = 1 \) value that is predicted for the nearby saturation field, and they decrease with the field. Therefore, this fit supports the \( \eta \)-inversion scenario at high fields, as also suggested from the neutron data [17], but unlike the previously proposed interpretation of the NMR data, based on the pure TLL description [12]. However, for the new fit, it is not clear why the RPA correction factor of the sub-dominant fluctuations should be the same as for the dominant ones.

The above examples show that the RPA+TLL fit can be successfully applied to cover different types of fluctuations spanning the complete theoretical range of \( K \) values in various quasi-1D spin systems. The fit provides the \( K \) value that experimentally characterizes a quasi-1D spin system independently of the availability of a theoretical description. The latter can be unavailable because the microscopic Hamiltonian is only partially defined/known, e.g., when the phase diagram extends up to very high magnetic field values that are beyond the current experimental reach. The temperature range successfully covered by the fit typically goes quite close to \( T_c \), down to about 1.2\( T_c \). This strong extension of the applicable range makes the fit more stable and possible even for systems farther away from the 1D limit. Below this temperature, we expect that the nature of the critical fluctuations changes from the 1D-based one, taken into account by \( \Phi \), to the usual 3D fluctuations, whose typical extension in temperature is of the order of 10%. Additionally, real compounds often present some sort of disorder, leading to a distribution of \( T_c \) values and the corresponding broadening of the peak of the measured \( T_i^{-1}(T) \) data that reflect the critical fluctuations.

We remark that the \( \Phi(K, T_c/T) \) function in principle depends on the geometry of 3D couplings, and that its analytical expression given by Eq. (2) has been calculated for the system of tetragonal symmetry [11], see Supplemental Material (SM) for further details [18]. We have also tested how its form is modified as a function of growing orthorhombic asymmetry (Fig. 5 in [18]). It turns out that this modification can be neglected up to approximately \( J_x/J_y \approx 2 \), a point at which the asymmetry-induced enhancement of the function can be compensated by the effective/fictive increase of the fitted \( T_c \) by only 2.6%. In general, when both geometry and size of the 3D couplings are known, and their \( q_z \) dependence is not frustrated, we can easily compute the exact corresponding \( \Phi \) function [18]. However, for the most of real compounds the size of 3D couplings is not known, and we can thus use Eq. (2) as a suitable proxy for systems that are not strongly anisotropic, and its generalization to the orthorhombic symmetry given by Eq. (2) in SM [18] to describe other systems. Finally, in SM we also discuss how the \( \Phi(K, T_c/T) \) function is evaluated and used in non-linear fits to \( T_i^{-1}(T) \) data in practice, and provide a simple example of the *Wolfram Mathematica* code we used in our fits [18].

In conclusion, we performed the first direct comparison between experimentally determined and theoretically predicted values of the parameter \( K \) that characterizes the power-law dependences predicted by the TLL description of quasi-1D systems. Using the recently proposed RPA-based correction factor that accounts for the enhancement of the NMR \( T_i^{-1} \) rate induced by critical fluctuations [11], we successfully fitted the observed \( T_i^{-1}(T) \) dependence in two quasi-1D spin systems, DIMPY and BaCo\(_2\)V\(_2\)O\(_8\), covering very different regimes of \( K \) values. This analysis establishes a simple reference procedure for characterization of quasi-1D systems. It thus enables us to recognize such systems in compounds whose effective dimension is not evident/known. In particular, it provides a basis to distinguish between quasi-1D and quasi-2D spin systems, whose spin dynamics remains to be characterized. Finally, the RPA correction has been discussed here for the \( T_i^{-1} \) data, but it is expected to be relevant to other observables, such as e.g. specific heat, for which its effect/size remains to be investigated.

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Dupont, Nicolas Laflorencie, and Mihael Grbić.

[1] Thierry Giamarchi, *Quantum Physics in One Dimension* (Clarendon Press Oxford, 2003).

[2] A. Schwartz, M. Dressel, G. Grüner, V. Vescoli, L. Degiorgi, and T. Giamarchi, “On-chain electrodynamics of metallic (TMTSF)$_2$X salts: Observation of Tomonaga-Luttinger liquid response,” *Phys. Rev. B* 58, 1261 (1998).

[3] H. Ishii, H. Kataura, H. Shiozawa, H. Yoshioka, H. Otsubo, Y. Takayama, T. Miyahara, S. Suzuki, Y. Achiba, M. Nakatake, T. Narimura, M. Higashiguchi, K. Shimada, H. Namatame, and M. Taniguchi, “Direct observation of Tomonaga-Luttinger-liquid state in carbon nanotubes at low temperatures. Nature 426, 540 (2003).

[4] M. Klanjšek, H. Mayaffre, C. Berthier, M. Horvatić, B. Chiari, O. Piovesana, P. Bouillot, C. Kollath, E. Orignac, R. Citro, and T. Giamarchi, “Controlling Luttinger Liquid Physics in Spin Ladders under a Magnetic Field,” *Phys. Rev. Lett.* 101, 137207 (2008).

[5] P. Bouillot, C. Kollath, A. M. Läuchli, M. Zvonarev, B. Thielemann, C. Rüegg, E. Orignac, R. Citro, M. Klanjšek, C. Berthier, M. Horvatić, T. Giamarchi, “Statics and dynamics of weakly coupled antiferromagnetic spin-$\frac{1}{2}$ ladders in a magnetic field,” *Phys. Rev. B* 83, 054407 (2011).

[6] R. Blinder, M. Dupont, S. Mukhopadhyay, M. S. Grbić, N. Laflorencie, S. Capponi, H. Mayaffre, C. Berthier, A. Paduan-Filho, and M. Horvatić, “Nuclear magnetic resonance study of the magnetic-field-induced ordered phase in the NiCl$_2$-4SC(NH$_2$)$_2$ compound,” *Phys. Rev. B* 95, 020404(R) (2017).

[7] M. Jeong, D. Schmidiger, H. Mayaffre, M. Klanjšek, C. Berthier, W. Knafo, G. Ballon, B. Vignolle, S. Krmer, A. Zheludev, and M. Horvatić, “Dichotomy between Attractive and Repulsive Tomonaga-Luttinger Liquids in Spin Ladders,” *Phys. Rev. Lett.* 117, 106402 (2016).

[8] M. Jeong, H. Mayaffre, C. Berthier, D. Schmidiger, A. Zheludev, and M. Horvatić, “Attractive Tomonaga-Luttinger Liquid in a Quantum Spin Ladder,” *Phys. Rev. Lett.* 111, 106404 (2013).

[9] E. Corra, P. Barmettler, T. Giamarchi, and C. Kollath, “Temperature dependence of the NMR spin-lattice relaxation rate for spin-$\frac{1}{2}$ chains,” *Phys. Rev. B* 94, 144408 (2016).

[10] M. Dupont, S. Capponi, and N. Laflorencie, “Temperature dependence of the NMR relaxation rate $1/T_1$ for quantum spin chains” *Phys. Rev. B* 94, 144409 (2016).

[11] M. Dupont, S. Capponi, N. Laflorencie, and E. Orignac, “Dynamical response and dimensional crossover for spatially anisotropic antiferromagnets,” *Phys. Rev. B* 98, 094403 (2018).

[12] M. Klanjšek, M. Horvatić, S. Krämer, S. Mukhopadhyay, H. Mayaffre, C. Berthier, E. Canévet, B. Grenier, P. Lejay, and E. Orignac, “Giant magnetic field dependence of the coupling between spin chains in BaCo$_2$V$_2$O$_8$,” *Phys. Rev. B* 92, 060408(R) (2015).

[13] T. Rikihara and A. Furusaki, “Correlation amplitudes for the spin-$\frac{1}{2}$ XXZ chain in a magnetic field,” *Phys. Rev. B* 69, 064427 (2004).

[14] T. Giamarchi and A. M. Tsvelik, “Coupled Ladders in a Magnetic Field,” *Phys. Rev. B* 59, 11398 (1999).

[15] N. Maeshima, K. Okunishi, K. Okamoto, and T. Sakai “Frustration-Induced $\eta$ Inversion in the $S = 1/2$ Bond-Alternating Spin Chain,” *Phys. Rev. Lett.* 93, 127203 (2004).

[16] K. Okunishi and T. Suzuki, “Field-induced incommensurate order for the quasi-one-dimensional XXZ model in a magnetic field,” *Phys. Rev. B* 76, 224411 (2007).

[17] B. Grenier, V. Simonet, B. Canals, P. Lejay, M. Klanjšek, M. Horvatić, C. Berthier, “Neutron diffraction investigation of the $H - T$ phase diagram above the longitudinal incommensurate phase of BaCo$_2$V$_2$O$_8$,” *Phys. Rev. B* 92, 134416 (2015).

[18] See Supplemental Material in the following pages for the definition and the ways of calculating the $\Phi(K,T_c/T)$ function, as well as an example of the “RPA+TLL” fits.
SUPPLEMENTAL MATERIAL

to “Direct determination of the Tomonaga-Luttinger parameter $K$ in quasi-one-dimensional spin systems”
by M. Horvatić, M. Klanjšek, and E. Orignac

$T^{-1}_1(T)$ in the RPA approximation for a quasi-1D system

We calculate the nuclear spin lattice relaxation rate

$$T^{-1}_1 \propto T \int d^3q \lim_{\omega \to 0} \text{Im} \chi(q, \omega)/\omega,$$

for the dynamic susceptibility given in the random phase approximation (RPA)

$$\chi_{\text{RPA}}(q, \omega) = \frac{\chi_{1D}(q, \omega)}{1 + J_\perp(q)\chi_{1D}(q, 0)} ,$$

where $J_\perp(q)$ is the Fourier transform of the transverse (3D) couplings between the 1D systems. As $\hbar \omega_{\text{NMR}} \ll k_B T$, where $\omega_{\text{NMR}}$ is the NMR resonance frequency, NMR probes the low-energy limit $\omega \to 0$ of the dynamical susceptibility, where $\text{Im} \chi(q, \omega) \propto \omega$. Eq. (4) thus reads

$$T^{-1}_{1\text{RPA}} \propto T \int d^3q \lim_{\omega \to 0} \text{Im} \chi_{1D}(q, \omega)/\omega \frac{1}{1 + J_\perp(q)\chi_{1D}(q, 0)} .$$

For a quasi-1D system having dominant transverse ($xx$) staggered (around the antiferromagnetic wave vector $B\perp$) couplings to the four nearest neighbors we have

$$s = \hbar u(q_x - Q_x)/(4\pi k_B T) , \quad w = \hbar \omega/(4\pi k_B T) ,$$

and $B$ is the Euler Beta function. Inserting Eqs. (7)-(9) into Eq. (6) we find for the enhancement of the relaxation due to the presence of $J_\perp(q)$, that is the ratio

$$T^{-1}_{1\text{RPA}}/T^{-1}_{1\text{TL}} = \Phi,$$

$$\Phi(K, T_c) = \int d^3q \lim_{\omega \to 0} \text{Im} \mathcal{B}(K, q, \omega)/\omega \frac{1}{1 - J_\perp(q)\chi_{1D}(q, 0)} .$$

We have written the denominator of the first integral using ratios relative to the divergence of $\chi_{\text{RPA}}(Q, 0)$ at $T_c$, namely the point at which $1 + J_\perp(Q)\chi_{1D}(Q, 0) = 0$. As the $\mathcal{B}$ function defined by Eq. (8) is strongly localized around $s = 0$ and all the quantities in Eq. (10) appear as ratios, we can remove $u/T$ from the integral over $q_x$, and the integration over $q$ can thus be taken over the scaled variables without units. That is, Eq. (10) is precise if $J_\perp(q)$ does not depend on $q_x$, and it is either a good or bad approximation when the $q_x$-dependence of $J_\perp(q)$ is respectively unfrustrated or frustrated.

For the system of orthorhombic symmetry and the couplings to the four nearest neighbors we have

$$J_\perp(q) = \frac{\cos(q_x) + \alpha \cos(q_y)}{1 + \alpha} ,$$

where $\alpha = J_y/J_x$ measures the anisotropy, and all the integrals of Eq. (11) can be taken over the $[0, \pi]$ interval. Integrating over the $q_x$ and $q_y$ variables we get the formula for the enhancement factor in case of orthorhombic symmetry

$$\phi(K, T) = \left[ \frac{\frac{4\pi}{1 + \alpha} h(K, \xi)}{1 - \frac{(1 - \frac{\pi}{4\xi})^4}{(1 - \frac{\pi}{4\xi})^4} \frac{1}{\pi} h(K, \xi)} \right]^{1/2} \left[ \frac{1}{\pi} \frac{(1 - \frac{\pi}{4\xi})^4}{(1 - \frac{\pi}{4\xi})^4} \frac{\pi}{4\xi} \right]$$

$$\left[ 1 - \left( \frac{4\pi}{1 + \alpha} \right)^4 \left( \frac{\pi}{4\xi} \right)^4 \right] \frac{1}{\pi} h(K, \xi)$$

and $N(K) = 2\Gamma^2 \left( \frac{1}{4K} \right) \cos \left( \frac{\pi}{4K} \right) B \left( \frac{1}{4K}, 1 - \frac{1}{2K} \right)$.

For the tetragonal symmetry, where $\alpha = 1$, this expression obviously reduces to the one given by Eq. 2 of the main manuscript.

While in principle a numerical evaluation of a 1D integral given by Eq. (12) should be faster than the evaluation of the “original” 3D integral given by the Eq. (10), in practice it turns out that, e.g., Wolfram Mathematica software handles the latter integral fast enough for normal usage. Direct usage of the Eq. (10) is thus preferred, as the corresponding code is simpler to write, and we can as well implement other geometries of the $J_\perp(q)$ couplings.
Finally, when the nuclear site that we use for recording the $T_1^{-1}$ data is coupled to several different electronic spins, its hyperfine coupling becomes $q$-dependent, which introduces the filtering factor $\mathcal{F}(q)$ in the integral over $q$

$$T_1^{-1} \propto T \int d^3q \mathcal{F}(q) \lim_{\omega \to 0} \text{Im} \chi(q, \omega)/\omega.$$ 

In the numerical evaluation of Eq. (10), the corresponding modification can be taken into account, and is expected to be important if $\mathcal{F}(Q) \sim 0$.

**Anisotropy dependence of the RPA correction factor for an orthorhombic system**

Fig. 5(a) shows the anisotropy $\alpha = J_y/J_x$ dependence of the RPA correction factor $\Phi(K=1, T_c/T)$ for the $J_\perp(q)$ coupling defined by Eq. (11). As the $K$ dependence of $\Phi$ is quite weak, the presented results for $K = 1$ are very representative for the whole relevant range of $K$ values. We can see that the growing anisotropy enhances $\Phi$. This is indeed expected, as we are in fact approaching the case of 2D, where the fluctuations should be enhanced. Nevertheless, the effect remains quite small up to $\alpha \approx 1/2$. Apparently, the family of curves shown in Fig. 5(a) can be superposed by scaling their $(\Phi - 1)$ values, and we did that using the least square fit in the [0.0, 0.8] interval, see Fig. 5(b). The overlap is nearly perfect for all the curves, with the exception of the 2D limit, $J_y = 0$. We can thus fit *any* orthorhombic system using simply the $[\sigma(\Phi - 1) + 1]$ scaling based on the $\Phi$ function given by Eq. (2) of the main manuscript, and thereby experimentally determine the anisotropy of the system from the value of the fitted scaling parameter $\sigma$.

As regards the 2D limit ($J_y = 0$), we might doubt its validity, because the 1D susceptibility $\chi_{1D}(q_z, \omega)$ should no longer be a good starting point of the RPA approximation to describe purely 2D fluctuations. Finally, we can equally well superpose the $\Phi$ curves shown in Fig. 5(a) by extending their $T_c/T$ scale (not shown). In this way we find that the corresponding effective/fictive modification of $T_c$ amounts to only $+2.6\%$ for $\alpha = 1/2$, which is comparable to the error bars on $T_c$ determination. This means that a weak anisotropy, i.e., the values $1 \geq \alpha \geq 1/2$, will be practically undetectable by $T_1^{-1}(T)$ data.

**Numerical evaluation of the RPA correction factor**

Both Eq. (12) and Eq. (10) can be literally converted into a very compact, e.g., Wolfram Mathematica code, and the numerical evaluation of the $\Phi$-function values is quite fast. However, when such an integral function is further employed in a non-linear fit, the latter becomes inconveniently slow. We thus find it better to first tabulate the function values over a convenient domain of variables and then use in fitting the function redefined as the polynomial interpolation over the tabulated values.

As an example of such a fit, in the following page, we provide a copy of the Mathematica code for the fit of BaCo$_2$V$_2$O$_8$ data at 4.1 T that is presented in Fig. 5 of the main manuscript.

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[1] Thierry Giamarchi, *Quantum Physics in One Dimension* (Clarendon Press Oxford, 2003).
(* Skip this cell if you already have the "PhiTab.dat" file in your local folder. If not, edit the definition of the local folder and run the cell to tabulate the very slow correction function numerical integral in order to use the fast cubic interpolation over these points as the fit function. This takes ~18 minutes on my PC. You can then skip the following cell and proceed directly to fits. *)

```
folder = "D:\Work\DOC\19\RPA_corrections\Paper";

phi[K_, t_] := NIntegrate[
  1/
  Sin[π/8 K]^2 + Sinh[π z]^2 Abs[
  Gamma[1/8 K + i z]/
  Gamma[1 - 1/8 K + i z]]^2 EllipticE[t^4/2 Abs[
  Gamma[1 - 1/8 K]/
  Gamma[1 - 1/8 K + i z]]^4] - 1/t^4 Abs[
  Gamma[1 - 1/8 K]/
  Gamma[1 - 1/8 K + i z]]^4],
  (z, -∞, ∞) / (2 Gamma[1/4 K]^3 Cos[π/4 K] Beta[1/4 K, 1 - 1/2 K])];
tabphi = Flatten[Table[{{x, t}, phi[x, t]}, {x, 0.52, 2.5, 0.02}, {t, 0.0, 0.98, 0.02}], 1];
Save[FileNameJoin[{folder, "PhiTab.dat"}], tabphi];
Phi = Interpolation[tabphi];

(* Edit the definition of the local folder, then read the tabulated function from the file. *)
folder = "D:\Work\DOC\19\RPA_corrections\Paper";
Get[FileNameJoin[{folder, "PhiTab.dat"}]];
Phi = Interpolation[tabphi];

(* fit BaCo2V2O8 at 4.1 T *)
data = {{1.4, 0.53191}, {1.45, 0.65359}, {1.56, 1.0929},
    {1.6, 1.38504}, {1.67, 1.8315}, {1.68, 1.98413}, {1.75, 2.2779}, {1.85, 2.1097},
    {1.97, 1.74825}, {2.07, 1.5625}, {2.18, 1.44509}, {2.34, 1.24069}, {2.5, 1.13379},
    {2.7, 1.06496}, {3.04, 0.97087}, {3.5, 0.86207}, {4.31, 0.73529}, {5.5, 0.64935},
    {6.7, 0.57143}, {8.4, 0.50251}, {10.1, 0.45872}, {12.1, 0.42373}};
fitdata = {{2.07, 1.5625}, {2.18, 1.44509}, {2.34, 1.24069}, {2.5, 1.13379},
    {2.7, 1.06496}, {3.04, 0.97087}, {3.5, 0.86207}, {4.31, 0.73529}, {5.5, 0.64935},
    {6.7, 0.57143}, {8.4, 0.50251}, {10.1, 0.45872}, {12.1, 0.42373}};
fit = NonlinearModelFit[fitdata, a T^2 K Abs[Pi/(4 K)] + K Abs[Pi/(4 K)] / (4 K),
  {{a, 1.0}, {K, 0.25}, {Tc, 1.7}}, T];
fit["ParameterTable"]

| Estimate | Standard Error | t-Statistic | P-Value |
|----------|----------------|-------------|---------|
| a        | 1.58134        | 0.0492973   | 32.0777 | 2.04086*10^-11 |
| K        | 0.231737       | 0.00980879  | 23.6254 | 4.18754*10^-10 |
| Tc       | 1.70535        | 0.0178002   | 95.8054 | 3.75866*10^-16 |

Show[ListLogLogPlot[{{{data, fitdata}, PlotStyle -> {Blue, Red}}, LogLogPlot[
  {fit[T], 1.58134*T^2 + 0.231737 - 1}, {T, 1.8, 13.0}, PlotStyle -> {Red, Orange, Dashed}]}]]
```