CONVERGENCE OF THE SASAKI-RICCI FLOW ON SASAKIAN 5-MANIFOLDS OF GENERAL TYPE

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Abstract. In this paper, we show that the uniform $L^4$-bound of the transverse Ricci curvature along the Sasaki-Ricci flow on a compact quasi-regular Sasakian $(2n+1)$-manifold $M$ of general type. As an application, any solution of the normalized Sasaki-Ricci flow converges in the Cheeger-Gromov sense to the unique singular Sasaki $\eta$-Einstein metric on the transverse canonical model $M_{\text{can}}$ of $M$ if $n \leq 3$. In particular for $n = 2$, $M_{\text{can}}$ is a $S^1$-orbibundle over the unique Kähler-Einstein orbifold surface $(Z_{\text{can}}, \omega_{KE})$ with finite point orbifold singularities. The floating foliation $(-2)$-curves in $M$ will be contracted to orbifold points by the Sasaki-Ricci flow as $t \to \infty$.

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1. Introduction

Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian \((2n+1)\)-manifold. Then by the first structure theorem (Proposition \[6\]), \(M\) is a principal \(S^1\)-orbibundle (\(V\)-bundle) over \(Z\) which is also a \(Q\)-factorial, polarized, normal projective orbifold variety such that there is an orbifold Riemannian submersion

\[ \pi : (M, g, \omega) \to (Z, h, \omega_h) \]

with \(\omega = \pi^*(\omega_h)\).

If the orbifold structure \((Z, \Delta)\) of the leave space \(Z\) is well-formed, then its orbifold singular locus and algebro-geometric singular locus coincide, equivalently \(Z\) has no branch divisors with \(\Delta = \emptyset\).

In the case of \(n = 2\), \(Z\) has isolated singularities of a finite cyclic quotient of type \(\frac{1}{r}(1, a)\). In particular, it is Kawamata log terminal singularities. The corresponding singularities in \((M, \eta, \xi, \Phi, g)\) is the foliation cyclic quotient singularities of type \(\frac{1}{r}(1, a)\) at a singular fibre \(S^1\) in \(M\) (Theorem \[7\]). The orbifold canonical divisor \(K^\text{orb}_Z\) and canonical divisor \(K_Z\) are the same and then

\[ K^T_M = \pi^*(\varphi^*K_Z). \]

Note that the class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism due to Smale-Barden (\[S\], \[B\]). Then, in this paper, it is our goal to focus on a classification of compact quasi-regular Sasakian 5-manifolds according to the global properties of the Reeb \(U(1)\)-fibration.

More precisely, there is the Sasaki analogue of Mori’s minimal model program with respect to \(K_Z\) in a such compact quasi-regular Sasakian 5-manifold. In other word, find a finite sequence of basic transverse birational maps \(f_1, \ldots, f_k\) and Sasakian 5-manifolds \(M_1, \ldots, M_k\) with

\[ M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_3} M_2 \xrightarrow{f_4} \cdots \xrightarrow{f_k} M_k \]

so that either \(M_k\) is transverse minimal model or Mori fibre space. That is, to find \(f_i\) which ”remove” \(K^T_{M_i}\)-negative foliation curves \(V\) with \(K^T_{M_i} \cdot V < 0\). In the paper of \[CLW\], we proved that there exists a finite sequence of foliation extremal ray contractions

\[ f_i : M_{i-1} \to M_i, \ i = 1, \ldots, k \]

such that every \(M_i\) is a Sasakian manifold having at worst foliation cyclic quotient singularities and for every \(f_i\) one of the following holds:

(A) Foliation divisorial contraction (The locus of the foliation extremal ray is an irreducible basic divisor): \(f_i\) is a foliation divisorial contraction of a foliation curve \(V\) with \(V^2 < 0\), and the Picard number satisfies \(\rho(Z_i) = \rho(Z_{i-1}) - 1\); or

(B) Foliation fibre contraction (transverse Mori fibre space) (The locus of the foliation extremal ray is \(M_{i-1}\)): \(f_i\) is a singular fibration such that either

(i) there is a map

\[ f : M_k \to pt, \]
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then $K^T_{M_k} < 0$ and thus $M_k$ is transverse minimal Fano and the leave space $Z_k$ is minimal log del Pezzo surface of $\frac{1}{l}(1, a)$-type singularities and Picard number one, or

$$ f : M_k \rightarrow \Sigma_h, $$

then $M_k$ is an $S^1$-orbibundle of a rule surface over Riemann surfaces $\Sigma_h$ of genus $h$, or

(ii) $M_k$ is nef: $f = f_k$, and $M_k$ has at worst foliation cyclic quotient singularities and has no foliation $K^T_M$-negative curves.

The Mori’s minimal model program in birational geometry can be viewed as the complex analogue of Thurston’s geometrization conjecture which was proved via Hamilton’s Ricci flow with surgeries on 3-dimensional Riemannian manifolds by Perelman [P1], [P2], [P3]. Likewise, there is a conjecture picture by Song–Tian [ST] that the Kähler-Ricci flow should carry out an analytic minimal model program with scaling on projective varieties. Recently, Song and Weinkove [SW1] established the above conjecture on a projective algebraic surface.

On the other hand, Sasakian manifolds can be view as an odd-dimensional analogues of Kähler manifolds and the Sasaki-Ricci flow can be viewed as a Sasaki analogue of Cao’s result ([Cao]) for the Kähler-Ricci flow. In the paper of Smoczyk-Wang-Zhang [SWZ], they introduced such a flow and proved that the flow has the longtime solution and asymptotic converges to a Sasaki $\eta$-Einstein metric when the basic first Chern class is null ($c^B(M) = 0$) or negative ($c^B(M) < 0$). The latter case is equivalent to the condition of the transverse canonical line bundle $K^T_M$ is ample.

In this paper, we consider the case where $K^T_M$ is not necessarily ample, but nef and big. Such a Sasakian manifold is known as a smooth transverse minimal model of general type. It is served as an odd-dimensional counterpart of the Kähler-Ricci flow on Kähler surfaces of general type as in [TZ2] and [GSW] via the Sasaki-Ricci flow on a compact quasi-regular Sasakian 5-manifold.

More precisely, by applying the Sasaki analogue of arguments in [TZ2], we show that the $L^4$-norm of the transverse Ricci curvature is uniformly bounded along the normalized Sasaki-Ricci flow on any transverse minimal model of general type and derive the following results in the paper.

**Theorem 1.** Let $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and its leave space $Z$ of the characteristic foliation be well-formed. Suppose that $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ is a smooth transverse minimal model of general type with dimension $n \leq 3$ and $\omega(t)$ is a solution to the normalized Sasaki-Ricci flow

$$ \frac{\partial}{\partial t} \omega(t) = -\text{Ric}^T_{\omega(t)} - \omega(t), \quad \omega(0) = \omega_0. $$

Then $(M, \omega(t))$ converges in the Cheeger-Gromov sense to the unique singular $\eta$-Einstein metric $\omega_\infty$ on the transverse canonical model $M_{\text{can}}$ of $M$ which is a $S^1$-orbibundle over the unique singular Kähler-Einstein normal projective variety $(Z_{\text{can}}, \omega_{KE})$. Here $Z_{\text{can}}$ is the canonical model of $Z$. 

Remark 1. 1. Note that the same as in the Kähler-Ricci flow, we have the same transversal holomorphic foliation ($\xi$ is fixed) but with the new transverse Kähler structure under the Sasaki-Ricci flow.

2. By the second structure theorem on a Sasakian manifold, if $M$ admits an irregular Sasakian structure, it admits many locally free circle actions which is our starting point for a quasi-regular case.

3. If $c_B^1(M) \leq 0$, then $K_T^\tau M$ is nef and semi-ample. Moreover, $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ will be a compact quasi-regular Sasakian 5-manifold ([BG] Theorem 8.1.14). Thus $K_T^\tau M$ is nef and big can be replaced by $c_B^1(M) \leq 0$ and $(c_B^1(M))^2 > 0$ in a compact Sasakian 5-manifold without the assumption of quasi-regularity.

4. In this paper, we assume that $M$ is a compact quasi-regular transverse Sasakian manifold and the space $Z$ of leaves is well-formed which means its orbifold singular locus and algebro-geometric singular locus coincide, equivalently $Z$ has no branch divisors. However when its leave space $Z$ is not well-formed, we conjecture that there is a Sasaki analogue of analytic Log minimal model program with respect to $K_Z + [\Delta]$ via the conical Sasaki-Ricci flow. This is the odd dimensional counterpart of the conical Kähler-Ricci flow ([LZ], [Shen]). We hope to address this issue in the near future.

For $n = 2$, one can show that the limit is a smooth Kähler-Einstein orbifold $Z_{\text{can}}$ with finite orbifold points by a classical argument of removing isolated singularities due to Anderson [A], Bando-Kasue-Nakajima [BKN], and Tian [T]. More precisely, if $M$ is a compact quasi-regular Sasakian 5-manifold ($n = 2$), then the singular set $S \subset Z_\infty$ is dimension 0 and $h_\infty$ will be an orbifold Kähler-Einstein metric on $Z_\infty$. More precisely, the solution $\omega(t)$ of the normalized Sasaki-Ricci flow on $M$ starting with any initial Sasakian metric $\omega_0$ is continuous through finitely many contraction surgeries ([CLW]) in the Gromov-Hausdorff topology for $t \in [0, \infty)$ and converges in the Cheeger-Gromov sense to the unique Sasaki $\eta$-Einstein orbifold metric on the canonical model $M_{\text{can}}$ of $M$ which is a $S^1$-orbibundle over the unique Kähler-Einstein orbifold surface $(Z_{\text{can}}, \omega_{KE})$ with finite point orbifold singularities.

Corollary 1. Let $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $Z$ of the characteristic foliation be well-formed. Suppose that $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ is a smooth transverse minimal model of general type and $\omega(t)$ is a solution to the normalized Sasaki-Ricci flow ([L]). Then $(M, \omega(t))$ converges in the Cheeger-Gromov sense to the unique Sasaki $\eta$-Einstein orbifold metric $\omega_\infty$ on the transverse canonical model $M_{\text{can}}$ with finite orbifold foliation singularities at a singular fibre $S^4$ on $M$. In particular, the floating foliation ($-2$)-curves in $M$ will be contracted to orbifold points by the Sasaki-Ricci flow as $t \to \infty$.

Note that in our proof of the $L^4$-norm bound of the transverse Ricci curvature on a compact quasi-regular Sasakian $(2n + 1)$-manifold, all the integrands are only involved with the transverse Kähler structure $\omega(t)$ and basic sections. Then one expects that, under the Sasaki–Ricci flow, the expressions involved behave essentially the same as in the Kähler-Ricci flow when one applies the Weitzenböck type formulae and integration by parts.
More precisely, our proof relies on the Cheeger-Colding-Tian [CCT] regularity theorem for Kähler orbifolds and the uniform $L^4$-bound of transverse Ricci curvature under the Sasaki-Ricci flow (1.1) on a compact quasi-regular Sasakian manifold where the transverse canonical line bundle is nef and big. In the last section, we will add some remark about the Sasaki analogue of Guo-Song-Weinkove [GSW] arguments for the contraction on the floating foliation $(-2)$-curves.

In section 3 and section 4, we give some fundamental estimates for the Sasaki-Ricci flow. In section 5, we derive the estimate on the $L^4$-bound of the transverse Ricci curvature under the normalized Sasaki-Ricci flow. In the last section, we give a proof of our main theorem by applying the Cheeger-Colding-Tian structure theory for Kähler orbifolds ([CCT], [TZ1] and [TZ2, Theorem 2.3]) to study the structure of desired limit spaces. At the end, we will add some remark about the Sasaki analogue of Guo-Song-Weinkove [GSW] arguments for the contraction on foliation $(-2)$-curves.

For a completeness, we give some preliminaries on structures theorems for Sasakian structures, foliation normal local coordinates, basic transverse holomorphic line bundles and its associated basic divisors, and the type of singularities in Sasakian manifolds in the section 2 and Appendix.

2. Preliminaries

We will address the preliminary notions on the foliation normal coordinate and basic cohomology and Type II deformation in a Sasakian manifold. We refer to [BG], [FOW], and references therein for some details. We will also address on the Sasakian structure, the leave space and its foliation singularities, basic holomorphic line bundles and basic divisors over Sasakian manifolds in the appendix for the completeness.

2.1. Sasakian Manifolds. Let $(M, g, \nabla)$ be a Riemannian $(2n+1)$-manifold. We say $(M, g)$ is called Sasaki if the cone

$$(C(M), \bar{g}) := (\mathbb{R}^+ \times M, dr^2 + r^2 g)$$

such that $(C(M), \bar{g}, J, \overline{\omega})$ is Kähler with

$$\overline{\omega} = \frac{1}{2} \sqrt{-1} \partial \overline{\partial} r^2.$$

The function $\frac{1}{2} r^2$ is hence a global Kähler potential for the cone metric. As $\{r = 1\} = \{1\} \times M \subset C(M)$, we may define

$$\overline{\xi} = J(r \frac{\partial}{\partial r})$$

and the Reeb vector field $\xi$ on $M$

$$\xi = J(\frac{\partial}{\partial r}).$$

Also

$$\overline{\eta}(\cdot) = \frac{1}{2} \bar{g}(\xi, \cdot)$$

and the contact 1-form $\eta$ on $TM$

$$\eta(\cdot) = g(\xi, \cdot).$$
Then $\xi$ is the killing vector field with unit length such that
\[ \eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0. \]
In fact, the tensor field of type $(1,1)$, defined by $\Phi(Y) = \nabla_Y \xi$, satisfies the condition
\[ (\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi \]
for any pair of vector fields $X$ and $Y$ on $M$. Then such a triple $(\eta, \xi, \Phi)$ is called a Sasakian structure on a Sasakian manifold $(M, g)$. Note that the Riemannian curvature satisfying the following
\[ R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi \]
for any pair of vector fields $X$ and $Y$ on $M$. In particular, the sectional curvature of every section containing $\xi$ equals one.

2.2. **Foliation Normal Local Coordinate.** Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n + 1)$-manifold with $g(\xi, \xi) = 1$ and the integral curves of $\xi$ are geodesics. For any point $p \in M$, we can construct local coordinates in a neighborhood of $p$ which are simultaneously foliated and Riemann normal coordinates ([GKN]). That is, we can find Riemann normal coordinates $\{x, z^1, z^2, \ldots, z^n\}$ on a neighborhood $U$ of $p$, such that $\frac{\partial}{\partial x} = \xi$ on $U$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of the Sasakian manifold and submersion such that $\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$ is biholomorphic. On each $V_\alpha$, there is a canonical isomorphism
\[ d\pi_\alpha : D_p \to T_{\pi_\alpha(p)}V_\alpha \]
for any $p \in U_\alpha$, where $D = \ker \xi \subset TM$. Since $\xi$ generates isometrics, the restriction of the Sasakian metric $g$ to $D$ gives a well-defined Hermitian metric $g^\alpha T$ on $V_\alpha$. This Hermitian metric in fact is Kähler. More precisely, let $z^1, z^2, \ldots, z^n$ be the local holomorphic coordinates on $V_\alpha$. We pull back these to $U_\alpha$ and still write the same. Let $x$ be the coordinate along the leaves with $\xi = \frac{\partial}{\partial x}$. Then we have the foliation local coordinate $\{x, z^1, z^2, \ldots, z^n\}$ on $U_\alpha$ and $(D \otimes \mathbb{C})$ is spanned by the form
\[ Z_\alpha = \left( \frac{\partial}{\partial x_\alpha} - \theta \left( \frac{\partial}{\partial z^j} \right) \frac{\partial}{\partial x} \right), \ \alpha = 1, 2, \ldots, n. \]
Moreover
\[ \Phi = \sqrt{-1} \left( \frac{\partial}{\partial z^j} + \sqrt{-1} h_j \frac{\partial}{\partial x} \right) \otimes dz^j + \text{conjugate} \]
and
\[ \eta = dx - \sqrt{-1} h_j dz^j + \sqrt{-1} h^j dz^j. \]
Here $h$ is basic: $\frac{\partial h}{\partial x} = 0$ and $h_j = \frac{\partial h}{\partial z^j}$, $h^j = \frac{\partial^2 h}{\partial z^j \partial x}$ with the normal coordinate
\begin{equation}
(2.1) \quad h_j(p) = 0, \ h_j^l(p) = \delta^l_j, \ dh_j^l(p) = 0.
\end{equation}
A frame
\[ \left\{ \frac{\partial}{\partial x}, \ Z_j = \left( \frac{\partial}{\partial z^j} + \sqrt{-1} h_j \frac{\partial}{\partial x} \right), \ j = 1, 2, \ldots, n \right\} \]
and the dual
\[ \{\eta, dz^j, \ j = 1, 2, \ldots, n\} \]
with

\[ [Z_i, Z_j] = [\xi, Z_j] = 0. \]

Since \( i(\xi) d\eta = 0, \)

\[ d\eta(Z_\alpha, Z_\beta) = d\eta(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}). \]

Then the Kähler 2-form \( \omega^T_\alpha \) of the Hermitian metric \( g^T_\alpha \) on \( V_\alpha \), which is the same as the restriction of the Levi form \( d\eta \) to \( \tilde{D}_n^\alpha \), the slice \( \{ x = \text{constant} \} \) in \( U_\alpha \), is closed. The collection of Kähler metrics \( \{ g^T_\alpha \} \) on \( \{ V_\alpha \} \) is so-called a transverse Kähler metric. We often refer to \( d\eta \) as the Kähler form of the transverse Kähler metric \( g^T \) in the leaf space \( D^n \).

The Kähler form \( d\eta \) on \( D \) and the Kähler metric \( g^T \) is define such that

\[ g = g^T + \eta \otimes \eta \]

and

\[ g^T_i = g^T \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right) = d\eta(\frac{\partial}{\partial z^i}, \Phi \frac{\partial}{\partial \bar{z}^j}) = 2h_{ij}. \]

In terms of the normal coordinate, we have

\[ g^T = g^T_\alpha dz^i d\bar{z}^j, \quad \omega = d\eta = 2\sqrt{-1}h_{ij}dz^i \wedge d\bar{z}^j. \]

The transverse Ricci curvature \( \text{Ric}^T \) of the Levi-Civita connection \( \nabla^T \) associated to \( g^T \) is

\[ \text{Ric}^T = \text{Ric} + 2g^T \]

and

\[ R^T = R + 2n. \]

The transverse Ricci form \( \rho^T \)

\[ \rho^T = \text{Ric}^T(J \cdot, \cdot) = -\sqrt{-1}R^T_{ij} dz^i \wedge d\bar{z}^j \]

with

\[ R^T_{ij} = -\frac{\partial}{\partial z^i} \log \det(g^T_\alpha) \]

and it is a closed basic \((1,1)\)-form

\[ \rho^T = \rho + 2d\eta. \]

2.3. Basic Cohomology and Type II Deformation in a Sasakian Manifold.

**Definition 1.** Let \((M, \eta, \xi, \Phi, g)\) be a Sasakian \((2n+1)\)-manifold. Define a \( p \)-form \( \gamma \) is called basic if

\[ i(\xi) \gamma = 0 \] and \( \mathcal{L}_\xi \gamma = 0. \]

Let \( \Lambda^p_B \) be the sheaf of germs of basic \( p \)-forms and \( \Omega^p_B \) be the set of all global sections of \( \Lambda^p_B \). It is easy to check that \( d\gamma \) is basic if \( \gamma \) is basic. Set \( d_B = d|_{\Omega^p_B} \), then

\[ d_B : \Omega^p_B \to \Omega^{p+1}_B. \]

We then have the well-defined operators

\[ d_B := \partial_B + \bar{\partial}_B \]

with

\[ \partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B \]
and
\[
\overline{\partial}_B : \Lambda_{p,q}^B \to \Lambda_{p,q+1}^B.
\]
Then for \( d_c^B := \frac{1}{2} \sqrt{-1}(\overline{\partial}_B - \partial_B) \), we have
\[
d_B d_c^B = \sqrt{-1} \partial_B \overline{\partial}_B, \quad d_c^2 = (d_c^B)^2 = 0.
\]
The basic Laplacian is defined by
\[
\Delta_B := d_B d^*_B + d^*_B d_B.
\]
Then we have the basic de Rham complex \( (\Omega^*_B, d_B) \) and the basic Dolbeault complex \( (\Omega^{p,*}_B, \overline{\partial}_B) \) and its cohomology group \( H^*_B(M, \mathbb{R}) \) \([\text{EKA}]\).

**Definition 2.** (i) We define the basic cohomology of the foliation \( F_\xi \) by
\[
H^*_B(F_\xi) := H^*_B(M, \mathbb{R}).
\]
Then by transverse Hodge decomposition and transverse Serre duality
\[
H^{p,q}_B(F_\xi) \simeq H^{q,p}_B(F_\xi)
\]
and the cohomology of the leaf space \( Z = M/U(1) \) to this basic cohomology of the foliation
\[
H^*_{\text{orb}}(Z, \mathbb{R}) = H^*_B(F_\xi) := H^*_B(M, \mathbb{R}).
\]
(ii) Define the basic first Chern class \( c_1^B(M) \) by
\[
c_1^B = [\rho^T]_B
\]
and the transverse Einstein (Sasaki \( \eta \)-Einstein) equation up to a D-homothetic deformation
\[
[\rho^T]_B = \zeta [d\eta]_B, \quad \zeta = -1, 0, 1.
\]
Basic \( k \)-th Chern class \( c_k^B(M) \) is represented by a closed basic \((k,k)\)-form \( \gamma_k \) which is determined by the formula
\[
det \left( I_n + \frac{\sqrt{-1}}{2\pi} \Omega^T \right) = 1 + \gamma_1 + \cdots + \gamma_k.
\]
Here \( \Omega^T \) is the curvature 2-form of type basic \((1,1)\) with respect to the transverse connection \( \nabla^T \).

**Definition 3.** We define Type II deformations of Sasakian structures \((M, \eta, \xi, \Phi, g)\) by fixing the \( \xi \) and varying \( \eta \). That is, for \( \varphi \in \Omega^0_B \), define
\[
\tilde{\eta} = \eta + d_c^B \varphi,
\]
then
\[
d\tilde{\eta} = d\eta + d_B d_c^B \varphi = d\eta + \sqrt{-1} \partial_B \overline{\partial}_B \varphi
\]
and
\[
\tilde{\omega} = \omega + \sqrt{-1} \partial_B \overline{\partial}_B \varphi.
\]
Note that we have the same transversal holomorphic foliation ($\xi$ is fixed) but with the new Kähler structure on the Kähler cone $C(M)$ and new contact bundle $\tilde{D}$: $\tilde{\omega} = dd^{c}\tilde{r}$, $\tilde{r} = re^{\varphi}$. The same holomorphic structure: $r \frac{\partial}{\partial r} = \tilde{r} \frac{\partial}{\partial r}$, $\xi = J(r \frac{\partial}{\partial r})$ and $\xi + \sqrt{-1} r \frac{\partial}{\partial r} = \xi - \sqrt{-1} \Phi(\xi)$ is the holomorphic vector field on $C(M)$. Moreover, we have

$$\tilde{\Phi} = \Phi - \xi \otimes (d^{c}B \varphi) \circ \Phi,$$

$$\tilde{g} = d\tilde{\eta} \circ (Id \otimes \Phi) + \tilde{\eta} \otimes \tilde{\eta}.$$

and

$$\mathcal{L}_{\xi} \tilde{\Phi} = \mathcal{L}_{\xi} \Phi = 0.$$

3. Asymptotic Convergence of the Sasaki-Ricci Flow

In this section, we will establish the Sasaki analogue of asymptotic convergence of solutions of the Kähler-Ricci flow which is the starting step to prove the main theorem in this paper.

3.1. The Sasaki-Ricci Flow. By a $\partial_{\overline{B}} \overline{B}$-Lemma ([EKA]) in the basic Hodge decomposition, there is a basic function $F : M \rightarrow \mathbb{R}$ such that

$$\rho^{T}(x, t) - \kappa d\eta(x, t) = d_{B}d_{B}^{c}F = \sqrt{-1}d_{B}\overline{B}_{B}F.$$

We focus on finding a new $\eta$-Einstein Sasakian structure $(M, \xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ with

$$\tilde{\eta} = \eta + d^{c}_{B}\varphi, \varphi \in \Omega^{0}_{B}$$

and

$$\tilde{g}^{T} = (g^{T}_{ij} + \varphi_{ij})dz^{i} \wedge d\overline{z}^{j} = 2\sqrt{-1}(h_{ij} + \frac{1}{2} \varphi_{ij})dz^{i} \wedge d\overline{z}^{j}$$

such that

$$\tilde{\rho}^{T} = \kappa d\tilde{\eta}.$$

Hence

$$\tilde{\rho}^{T} - \rho^{T} = \kappa d_{B}d^{c}_{B}\varphi - d_{B}d^{c}_{B}F$$

and it follows that

$$\frac{\det(g^{T}_{\alpha \beta} + \varphi_{\alpha \beta})}{\det(g^{T}_{\alpha \beta})} = e^{-\kappa \varphi + F}.$$ 

This is a Sasakian analogue of the Monge-Ampere equation for the orbifold version of Calabi-Yau Theorem ([EKA]).

Now we consider the Sasaki-Ricci flow on $M \times [0, T)$

$$\frac{d}{dt}g^{T}(x, t) = -\text{Ric}^{T}(x, t) + \kappa g^{T}(x, t)$$

or

$$\frac{d}{dt}d\eta(x, t) = -\rho^{T}(x, t) + \kappa d\eta(x, t).$$

It is equivalent to consider

$$\frac{d}{dt} \varphi = \log \det(g^{T}_{\alpha \beta} + \varphi_{\alpha \beta}) - \log \det(g^{T}_{\alpha \beta}) + \kappa \varphi - F.$$

Note that, for any two Sasakian structures with the fixed Reeb vector field $\xi$, we have

$$\text{Vol}(M, g) = \text{Vol}(M, g').$$
and
\[ \tilde{\omega}^n \wedge \eta = (\sqrt{-1})^n \det(g^T_{\alpha\beta} + \varphi_{\alpha\beta}) dz^1 \wedge \overline{dz^1} \wedge \cdots \wedge dz^n \wedge \overline{dz^n} \wedge dx. \]

### 3.2. Convergence of Solutions of the Sasaki-Ricci Flow

Let \((M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)\) be a compact quasi-regular Sasakian \((2n+1)\)-manifold and its the space \(Z\) of leaves of the characteristic foliation be well-formed. We consider a solution \(\omega(t) = \omega(t)\) of the Sasaki-Ricci flow

\[ \frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{\omega(t)}, \quad \omega(0) = \omega_0. \]

As long as the solution exists, the cohomology class \([\omega(t)]_B\) evolves by

\[ \frac{\partial}{\partial t} [\omega(t)]_B = -c_1^B(M), \quad [\omega(0)]_B = [\omega_0]_B, \]

and solving this ordinary differential equation gives

\[ [\omega(t)]_B = [\omega_0]_B - tc_1^B(M). \]

We see that a necessary condition for the Sasaki-Ricci flow to exist for \(t > 0\) such that

\[ [\omega_0]_B - tc_1^B(M) > 0. \]

This necessary condition is in fact sufficient. In fact we define

\[ T_0 := \sup\{t > 0 \mid [\omega_0]_B - tc_1^B(M) > 0\}. \]

That is to say that

\[ [\omega_0]_B - T_0c_1^B(M) \in C^B_M \]

which is a nef class. Here

\[ C^B_M = \{[\alpha]_B \in H^{1,1}_B(M, \mathbb{R}) \mid \exists \text{ transverse Kähler metric } \omega > 0 \text{ such that } [\omega]_B = [\alpha]_B\}. \]

For a representative \(\chi \in -c_1^B(M)\), we can fix a transverse volume form \(\Omega\) on \((M, \xi_0, \eta_0, \Phi_0, g_0, \omega_0)\) such that

\[ \Omega \wedge \eta_0 = (\sqrt{-1})^n F(z_1, \cdots, z_n) dz_1 \wedge \overline{dz_1} \wedge \cdots \wedge dz_n \wedge \overline{dz_n} \wedge dx. \]

with

\[ \sqrt{-1} \partial_B \overline{\partial_B} \log F = -\text{Ric}^T(\Omega) = \chi, \]

and

\[ \int_M \Omega \wedge \eta_0 = \int_M \omega^n_0 \wedge \eta_0. \]

We choose a reference (transverse) Kähler metric

\[ \tilde{\omega}_t := \omega_0 + t\chi. \]

Then the corresponding transverse parabolic complex Monge-Ampère equation to (3.4) on \(M \times [0, T_0)\) is

\[ \left\{ \begin{array}{l}
\frac{\partial}{\partial t} \varphi(x, t) = \log \frac{(\tilde{\omega}_t + \sqrt{-1} \partial_B \overline{\partial_B} \varphi)^n \wedge \eta_0}{\Omega \wedge \eta_0}, \\
\tilde{\omega}_t = \omega_0 + t\chi,
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
\sqrt{-1} \partial_B \overline{\partial_B} \log \Omega = \chi, \\
\tilde{\omega}_t + \sqrt{-1} \partial_B \overline{\partial_B} \varphi > 0, \\
\varphi(0) = 0.
\end{array} \right. \]
Based on [SW1], [T] and references therein as in the Kähler case, we have the following cohomological characterization for the maximal solution of the Sasaki-Ricci flow (we refer to the proof of Theorem 3 as below):

**Theorem 2.** There exists a unique maximal solution $\omega(t)$ of the Sasaki-Ricci flow (3.4) on $M$ for $t \in [0, T_0)$.

Next if we assume that $K^T_M$ is nef, it follows from (A.2) and (3.5) that $T_0 = \infty$. On the other hand if $K^T_M$ is big also

$$\int_M (c_1^B(K^T_M))^n \wedge \eta_0 > 0.$$ 

Now by Sasaki analogue of Kawamata base-point free theorem ([CLW]), we obtain that $K^T_M$ is semi-ample, then there exists a $S^1$-equivariant basic base-point free transverse holomorphic map

$$\Psi : M \to (\mathbb{C}P^N, \omega_{FS})$$

defined by the basic transverse holomorphic section $\{s_0, s_1, \cdots, s_N\}$ of $H^0(M, (K^T_M)^m)$ which is $S^1$-equivariant with respect to the weighted $\mathbb{C}^*$ action. Here $N = \dim H^0(M, (K^T_M)^m) - 1$ for a large positive integer $m$ and

$$\frac{1}{m} \Psi^*(\omega_{FS}) = \tilde{\omega}_\infty \geq 0.$$

For the asymptotic behavior, we need to rescale the Sasaki-Ricci flow (3.4) to the normalized Sasaki-Ricci flow on $M \times [0, \infty)$ as following:

$$\begin{align*}
\frac{\partial}{\partial t} \varphi(x, t) &= \log (\tilde{\omega}_t + \sqrt{-1} \partial B \bar{\partial}_B \varphi)^n \wedge \eta_0 - \varphi, \\
\tilde{\omega}_t &= e^{-t} \omega_0 + (1 - e^{-t}) \tilde{\omega}_\infty, \\
\sqrt{-1} \partial B \bar{\partial}_B \log \Omega &= \tilde{\omega}_\infty, \\
\tilde{\omega}_t + \sqrt{-1} \partial B \bar{\partial}_B \varphi &> 0, \\
\varphi(0) &= 0.
\end{align*}$$

We observe that $\chi = \tilde{\omega}_\infty \in -c_1^B$ is a nonnegative $(1, 1)$-current Kähler metric. The starting point to show Theorem 3 we must be able to derive the following basic result which was established for the Kähler-Ricci flow due to [Tsu], [T], and [TZ3].

**Theorem 3.** Let $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and its space $Z$ of leaves of the characteristic foliation be well-formed. Suppose that $-c_1^B(M) \in \overline{C^B_M}$ and

$$\int_M (-c_1^B(M))^n \wedge \eta_0 > 0.$$ 

Then there exists a unique solution $\omega(t)$ of the Sasaki-Ricci flow (3.4) on $M \times [0, \infty)$. Furthermore, there exists an $\eta$-Einstein metric $\omega_\infty$ on $M \setminus \text{Null}(-c_1^B(M))$ which satisfies

$$\text{Ric}^T_{\omega_\infty} = -\omega_\infty$$

such that for any initial transverse Kähler metric $\omega_0$, the rescaled metrics $\frac{\omega(t)}{t}$ converge smoothly on compact subsets of $M \setminus \text{Null}(-c_1^B(M))$ to $\omega_\infty$ as $t \to \infty$. 
We first state the Sasaki analogue of Kodaira lemma for the further application. By the first structure theorem (Proposition 6), $M$ is a principal $S^1$-orbibundle over a Hodge orbifold $Z$ which is also a $Q$-factorial, polarized, normal projective variety such that there is a Riemannian submersion

$$
\pi : (M, g_0, \omega_0) \rightarrow (Z, h_0, \omega_{h_0})
$$

such that $\omega_0 = \pi^*(\omega_{h_0})$. We define

$$
\text{Null}(\alpha) = \bigcup \int_{V} \alpha \dim V - 1 \wedge \eta_0 = 0
$$

which is the union over all positive-dimensional invariant $(2m - 1)$-submanifolds $V \subset M$ such that $\pi(V) = E \subset Z$ and $\pi^*(\hat{\alpha}) = \alpha$ so that

$$
\int_E \hat{\alpha} \dim E - 1 = 0.
$$

By applying the arguments as Collins-Tosatti [CT] and Demailly-Paun [DP] (also [D]) to a Hodge orbifold $Z$ which is a normal projective variety and lifting to $M$ via the Riemannian submersion $\pi$, it follows that

**Proposition 1.** Let $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and $\alpha$ a basic closed real $(1, 1)$-form whose class $[\alpha]_B$ is nef and big

$$
\int_M \alpha^n \wedge \eta_0 > 0.
$$

Then there exists an upper semicontinuous $L^1$-function $\phi : M \rightarrow \mathbb{R} \cup \{-\infty\}$, with $\sup_M \phi = 0$ which is basic and equals $-\infty$ on $\text{Null}(\alpha)$ and is finite, smooth on $M \setminus \text{Null}(\alpha)$ such that

$$
\alpha + \sqrt{-1} \partial_B \bar{\partial}_B \phi \geq \varepsilon \omega_0
$$

on $M \setminus \text{Null}(\alpha)$, for some $\varepsilon > 0$.

We first show the following uniform estimate.

**Lemma 1.** There exists $C > 0$ such that on $M \times [0, \infty)$, we have

$$
|\varphi| + |\partial_t \varphi|^2 \leq C.
$$

**Proof.** Let $\omega = \omega(t)$ be the solution to the normalized Sasaki-Ricci flow (3.7). First, we show that

$$
\varphi(t) \leq C
$$

on $M \times [0, \infty)$. This is a simple consequence of the maximum principle since at any maximum point of $\varphi$ (for $t > 0$) we have

$$
0 \leq \partial_t \varphi = \log \frac{\omega(t) + \sqrt{-1} \partial_B \bar{\partial}_B \varphi(t) \omega(t) \wedge \eta_0}{\omega \wedge \eta_0} - \varphi(t) \leq \log \frac{\omega(t) \wedge \eta_0}{\omega \wedge \eta_0} - \varphi(t) \leq C - \varphi(t),
$$

using that at a maximum point $\hat{\omega}_t \geq \hat{\omega}_t + \sqrt{-1} \partial_B \bar{\partial}_B \varphi(t) > 0$, and we are done. Next, we show that

$$
\frac{\partial \varphi}{\partial t} = \varphi'(t) \leq C(1 + t)e^{-t},
$$

on $M \times [0, \infty)$. Indeed we compute

$$
(\partial_t - \Delta_B) \varphi(t) = \varphi'(t) - n + tr_\omega \hat{\omega}_t,
$$

$$
(\partial_t - \Delta_B) \varphi'(t) = -\varphi'(t) - e^{-t} tr_\omega \omega_0 + e^{-t} tr_\omega \hat{\omega}_\infty,
$$

and conclude that

$$
\varphi'(t) \leq C(1 + t)e^{-t}.
$$

This is a simple consequence of the maximum principle since at any maximum point of $\varphi$ (for $t > 0$) we have

$$
0 \leq \partial_t \varphi = \log \frac{\omega(t) + \sqrt{-1} \partial_B \bar{\partial}_B \varphi(t) \omega(t) \wedge \eta_0}{\omega \wedge \eta_0} - \varphi(t) \leq \log \frac{\omega(t) \wedge \eta_0}{\omega \wedge \eta_0} - \varphi(t) \leq C - \varphi(t),
$$

using that at a maximum point $\hat{\omega}_t \geq \hat{\omega}_t + \sqrt{-1} \partial_B \bar{\partial}_B \varphi(t) > 0$, and we are done. Next, we show that

$$
\frac{\partial \varphi}{\partial t} = \varphi'(t) \leq C(1 + t)e^{-t},
$$

on $M \times [0, \infty)$. Indeed we compute

$$
(\partial_t - \Delta_B) \varphi(t) = \varphi'(t) - n + tr_\omega \hat{\omega}_t,
$$

$$
(\partial_t - \Delta_B) \varphi'(t) = -\varphi'(t) - e^{-t} tr_\omega \omega_0 + e^{-t} tr_\omega \hat{\omega}_\infty,
$$

and conclude that

$$
\varphi'(t) \leq C(1 + t)e^{-t}.
$$
(\frac{\partial}{\partial t} - \Delta_B)((e^t - 1)\varphi'(t) - \varphi(t) - nt) = -tr_\omega \omega_0 < 0,
and so the maximum principle gives

(e^t - 1)\varphi'(t) - \varphi(t) - nt \leq 0

which together with (3.9) gives (3.10) for t > 1. On the other hand, it is clear that (3.10) holds for 0 \leq t \leq 1.

Next, we show that there is a constant C > 0 such that

(3.11) \varphi'(t) + \varphi(t) \geq \phi - C,
on M \times [0, \infty), here we apply Proposition 1 to obtain an upper semicontinuous L^1 function \phi : M \rightarrow \mathbb{R} \cup \{-\infty\}, with \sup_M \phi = 0 and equals \infty on Null(-c^B_1(M)), which is finite and smooth on M\setminus Null(-c^B_1(M)) such that

(3.12) \chi + \sqrt{-1}(\partial B \bar{\partial} B \phi) \geq \varepsilon \omega_0,
on M\setminus Null(-c^B_1(M)), for some \varepsilon > 0. Consider the quantity

Q = \varphi'(t) + \varphi(t) - \phi.
The function Q is lower semicontinuous and it approaches \infty as we approach Null(-c^B_1(M)), and so it achieves a minimum at (x,t), for some t > 0 and x \not\in Null(-c^B_1(M)), and at this point we have

0 \geq (\frac{\partial}{\partial t} - \Delta_B)Q = tr_\omega (\bar{\omega}_\infty + \sqrt{-1}(\partial B \bar{\partial} B \phi) - n \geq \varepsilon tr_\omega \omega_0 - n
\geq n\varepsilon \left(\frac{\omega_\infty^\wedge \omega_0}{\omega_\infty^\wedge \omega_0}\right)^{\frac{1}{n}} - n \geq C^{-1} e^{-\frac{\varphi'(t) + \varphi(t)}{n}} - n,
and so

\varphi'(t) + \varphi(t) \geq -C,
which implies that

Q \geq -C
since \phi \leq 0, this shows (3.11). \square

In the following, we prove Theorem 3.

Proof. The proof is similar to the Kähler-Ricci flow case. Since the convergence is for the rescaled metrics \frac{\omega(t)}{t}, it is convenient to renormalize the Sasaki-Ricci flow as follows:

(3.13) \frac{\partial}{\partial t}\omega(t) = -\text{Ric}_{\omega(t)}^T - \omega(t), \quad \omega(0) = \omega_0.
Then the goal is to show that the solution \omega(t) of (3.13) satisfies

(3.14) \omega(t) \rightarrow \omega_\infty,
in C^\infty_{\text{loc}}(M\setminus Null(-c^B_1(M)) as t \rightarrow \infty and that the limit \omega_\infty is transverse Kähler-Einstein.

Note that (3.13) is equivalent to (3.7) and recall that, from (3.8), we have the uniform bounded estimates for \varphi(t) and \varphi'(t) on M \times [0, \infty). First we show that there is a constant C > 0 such that

(3.15) tr_\omega \omega(t) \leq C e^{-C \phi},
on $M \times [0, \infty)$, here $\phi$ is the function in Proposition\textsuperscript{[1]} for $\alpha = \chi$. Like the Kähler case, we can obtain

$$(\frac{\partial}{\partial t} - \Delta_B) \log tr_{\omega_0} \omega(t) \leq C tr_{\omega(t)} \omega_0,$$

where $-C$ is a lower bound for the transverse bisectional curvature with respect to $\omega_0$. Using this inequality and (3.12), we compute

$$(\frac{\partial}{\partial t} - \Delta_B)(\log tr_{\omega_0} \omega(t) - A(\varphi'(t) + \varphi(t) - \phi))$$

$$\leq C tr_{\omega(t)} \omega_0 + An - Atr_{\omega(t)}(\chi + \sqrt{-1} \partial_B \bar{\partial}_B \phi)$$

$$\leq -tr_{\omega(t)} \omega_0 + C,$$

on $M \setminus \text{Null}(-c_1^B(M))$, if we choose a positive $A$ large enough such that $C \leq A\varepsilon - 1$. Therefore at a maximum point $(x, t)$ of this quantity for some $t > 0$ with $x \notin \text{Null}(-c_1^B(M))$, we have

$$tr_{\omega(t)} \omega_0 \leq C.$$

By applying the inequality

$$tr_{\omega_0} \omega(t) \leq \frac{(tr_{\omega_0} \omega_0)^{n-1} \omega(t) \wedge \eta_0}{(n-1)!},$$

we conclude that at the maximum point $(x, t)$ we have

$$tr_{\omega_0} \omega(t) \leq C \frac{\omega(t) \wedge \eta_0}{\omega_0^n \wedge \eta_0} = Ce^{\varphi'(t) + \varphi(t)} \frac{\Omega \wedge \eta_0}{\omega_0^n \wedge \eta_0} \leq C,$$

used the estimate (3.8). Combining this with (3.11) it implies

$$\log tr_{\omega_0} \omega(t) - A(\varphi'(t) + \varphi(t) - \phi) \leq C,$$

at the maximum and hence everywhere, and thus yields (3.13). Also note that

$$\frac{\omega(t) \wedge \eta_0}{\omega_0^n \wedge \eta_0} \geq C^{-1} e^{\varphi'(t) + \varphi(t)} \geq C^{-1} e^{-\phi},$$

from (3.11) and so given any compact subset $K \subset M \setminus \text{Null}(-c_1^B(M))$ there exists a constant $C_K$ such that

(3.16) \[ C_K^{-1} \omega_0 \leq \omega(t) \leq C_K \omega_0 \]

on $K \times [0, \infty)$. The higher order estimate on $K$ is then given by

$$||\omega(t)||_{C^k(K, \omega_0)} \leq C_{K,k},$$

for all $t \geq 0$, $k \geq 0$, up to shrinking $K$ slightly. These estimates will imply the function

$$\Delta_B \omega_0 \varphi(t) = tr_{\omega_0} \omega(t) - tr_{\omega_0} \widehat{\omega}_t,$$

is uniformly bounded in $C^k(K, \omega_0)$ for all $k \geq 0$. Form (3.8) $\varphi(t)$ is uniformly bounded on $K$ and elliptic estimates

$$||\varphi(t)||_{C^k(K, \omega_0)} \leq C_{K,k},$$

for all $t \geq 0$, $k \geq 0$, up to shrinking $K$ again. Now for $t \geq 1$, (3.10) yields

$$\varphi'(t) \leq Cte^{-t},$$

and so

$$\frac{\partial}{\partial t}(\varphi(t) + Cte^{-t}(1 + t)) = \varphi'(t) - Cte^{-t} \leq 0.$$
Then the function $\varphi(t) + Ce^{-t}(1+t)$ is thus nonincreasing and uniformly bounded from below on compact subsets of $M \setminus \text{Null}(-c_1^B(M))$, and so as $t \to \infty$ the functions $\varphi(t)$ converge pointwise to a function $\varphi_\infty$ on $C^\infty_{\text{loc}}(M \setminus \text{Null}(-c_1^B(M))$, which is smooth on $M \setminus \text{Null}(-c_1^B(M))$. Also (3.16) shows that $\omega_\infty := \omega + \sqrt{-1}\partial_B \bar{\partial}_B \varphi_\infty$ is a smooth transverse Kähler metric on $M \setminus \text{Null}(-c_1^B(M))$. The flow equation (3.7) implies that $\varphi'(t)$ also converges smoothly to some limit function. Moreover, since $\varphi(t)$ converge smoothly to $\varphi_\infty$ on compact subsets of $M \setminus \text{Null}(-c_1^B(M))$, it follows that given any $x \in M \setminus \text{Null}(-c_1^B(M))$ there is a sequence $t_i \to \infty$ such that $\varphi(x, t_i) \to 0$. But since $\varphi'(t)$ converges smoothly on compact sets to some limit function, it yields that $\varphi'(t) \to 0$ in $C^\infty_{\text{loc}}(M \setminus \text{Null}(-c_1^B(M)))$. Taking then the limit as $t \to \infty$ in (3.7) we obtain

$$0 = \log \frac{\omega_\infty \wedge \eta}{\Omega \wedge \eta} - \varphi_\infty,$$

on $M \setminus \text{Null}(-c_1^B(M))$. Taking $\sqrt{-1}\partial_B \bar{\partial}_B$ of this equation, we finally obtain

$$\text{Ric}_{\omega_\infty}^T = -\omega - \sqrt{-1}\partial_B \partial_B \varphi_\infty = -\omega_\infty$$

as described. \qed

4. The Gradient Estimate

In this section we show the transverse gradient estimate and uniformly bounded of the transverse scalar curvature under the normalized Sasaki-Ricci flow (3.7).

We first prove the following parabolic Schwarz lemma.

**Lemma 2.** Let $\omega = \omega(t)$ be the solution to the normalized Sasaki-Ricci flow (3.7). Then there exists $C > 0$ such that on $M \times [0, \infty)$,

$$(\frac{\partial}{\partial t} - \Delta_B) \text{tr}_\omega \omega_\infty \leq \text{tr}_\omega \chi + C(\text{tr}_\omega \chi)^2 - \left| \nabla^T \text{tr}_\omega \chi \right|^2_{g^T},$$

where $\Delta_B$ is the basic Laplace operator associated to the evolving transverse metric $g^T(t)$.

**Proof.** Since the transverse canonical bundle $K^T_M$ is semi-ample. There exists a basic transverse base point free holomorphic map $\Psi : M \to (\mathbb{C}P^n, \omega_{FS})$ such that $\omega_\infty = \frac{1}{m} \Psi^*(\omega_{FS})$.

Then we have

$$\frac{\partial}{\partial t} \text{tr}_\omega \omega_\infty = \langle \text{Ric}^T, \omega_\infty \rangle + \text{tr}_\omega \omega_\infty$$

and

$$\Delta_B \text{tr}_\omega \omega_\infty = \langle \text{Ric}^T, \omega_\infty \rangle + \left| \nabla^T \text{tr}_\omega \omega_\infty \right|^2_{g^T} - g^T g^{T \bar{T}} S_{\alpha \beta \gamma \delta} \psi^\alpha \psi^\beta \psi^\gamma \psi^\delta$$

$$\geq \langle \text{Ric}^T, \omega_\infty \rangle + \left| \nabla^T \text{tr}_\omega \omega_\infty \right|^2_{g^T} - C(\text{tr}_\omega \omega_\infty)^2,$$

where $C$ is a universal constant given by the upper bound of the bisection curvature $S_{\alpha \beta \gamma \delta}$ of $\omega_{FS}$ on $\mathbb{C}P^n$. This implies the inequality (4.1). \qed

**Proposition 2.** There exists $C > 0$ such that on $M \times [0, \infty)$,

$$\text{tr}_\omega \omega_\infty \leq C.$$
Proof. By (4.1), we compute
\[
(\frac{\partial}{\partial t} - \Delta_B) \log(\text{tr}_\omega \hat{\omega}_\infty + 1) = \frac{1}{\text{tr}_\omega \hat{\omega}_\infty + 1}(\frac{\partial}{\partial t} - \Delta^T)\text{tr}_\omega \hat{\omega}_\infty + \frac{|\nabla^T \text{tr}_\omega \hat{\omega}_\infty |^2_{g^T}}{(\text{tr}_\omega \hat{\omega}_\infty + 1)^2} \\
\leq 1 + C(\text{tr}_\omega \chi).
\]
Then from the evolutions of \(\varphi\) and \(\varphi'\), we obtain
\[
(\frac{\partial}{\partial t} - \Delta_B)[\log(\text{tr}_\omega \hat{\omega}_\infty + 1) - A(\varphi + \varphi')] \leq -(A - C)\text{tr}_\omega \hat{\omega}_\infty + An + 1,
\]
for a large positive constant \(A\) which is larger than \(C\). By the maximum principle, \(\text{tr}_\omega \hat{\omega}_\infty\) is uniformly bounded from above on \(M \times [0, \infty)\).

Denote
\[
u = \varphi + \varphi'.
\]
Since both \(\varphi\) and \(\varphi'\) are uniformly bounded from (3.8), there exists a positive constant \(A\) such that
\[
u + A \geq 1
\]
on \(M \times [0, \infty)\).

**Proposition 3.** There exists \(C > 0\) such that on \(M \times [0, \infty)\), we have
\[
(4.2) \quad |\nabla^T u|_{g^T}^2 \leq C
\]
and
\[
(4.3) \quad |\Delta_B u| \leq C.
\]

**Proof.** We compute the evolution equations of \(|\nabla^T u|_{g^T}^2\) and \(\Delta_B u\) as below. First note that
\[
(\frac{\partial}{\partial t} - \Delta_B) u = \text{tr}_\omega \hat{\omega}_\infty - n.
\]
We obtain
\[
\frac{\partial}{\partial t} |\nabla^T u|^2 = \langle \nabla^T u, \nabla^T \Delta_B u \rangle + \langle \nabla^T \Delta_B u, \nabla^T u \rangle + |\nabla^T u|^2 + \text{Ric}^T(\nabla^T u, \nabla^T u) + 2 \text{Re}(\nabla^T \text{tr}_\omega \hat{\omega}_\infty \cdot \nabla^T u).
\]
On the other hand, the Bonchner formula for the transverse Laplacian \(\Delta_B\) gives
\[
\Delta_B |\nabla^T u|^2 = |\nabla^T \nabla^T u|^2 + |\nabla^T \nabla^T u|^2 + \langle \nabla^T u, \nabla^T \Delta_B u \rangle + \langle \nabla^T \Delta_B u, \nabla^T u \rangle + \text{Ric}^T(\nabla^T u, \nabla^T u).
\]
Hence
\[
(4.4) \quad (\frac{\partial}{\partial t} - \Delta_B) |\nabla^T u|^2 = |\nabla^T u|^2 - |\nabla^T \nabla^T u|^2 - |\nabla^T \nabla^T u|^2 + 2 \text{Re}(\nabla^T \text{tr}_\omega \hat{\omega}_\infty \cdot \nabla^T u).
\]
Also
\[
(\frac{\partial}{\partial t} - \Delta_B) \Delta_B u = \Delta_B u + \langle \text{Ric}^T, \partial_B \bar{\partial} B u \rangle + \Delta_B \text{tr}_\omega \hat{\omega}_\infty \]
\[
= -|\nabla^T \nabla^T u|^2 - \langle \chi, \partial_B \bar{\partial} B u \rangle + \Delta_B u + \Delta_B \text{tr}_\omega \hat{\omega}_\infty.
\]
Let
\[
(4.6) \quad H = \frac{|\nabla^T u|^2}{u + A} + \text{tr}_\omega \hat{\omega}_\infty.
\]
Then
\[
H_t = \frac{1}{u + A} \frac{\partial}{\partial t} |\nabla^T u|^2 - \frac{|\nabla^T u|^2}{(u + A)^2} + \frac{\partial}{\partial t} \text{tr}_\omega \hat{\omega}_\infty.
\]
and

\[(4.7)\quad \nabla^T H = \frac{1}{u+A} \nabla^T | \nabla^T u|^2 - \frac{|\nabla^T u|^2}{(u+A)^2} \nabla^T u + \nabla^T \text{tr}_\omega \hat{\omega}_\infty.\]

On the other hand, since \(| \nabla^T u|^2 = (H - \text{tr}_\omega \hat{\omega}_\infty)(u + A)\), we get

\[\Delta_B | \nabla^T u|^2 = (\Delta_B H - \Delta_B \text{tr}_\omega \hat{\omega}_\infty)(u + A) + (H - \text{tr}_\omega \hat{\omega}_\infty) \Delta_B u + 2 \text{Re}(\nabla^T (H - \text{tr}_\omega \chi) \cdot \nabla^T u),\]

or

\[\Delta_B H = \frac{1}{u+A}(\Delta_B | \nabla^T u|^2 - \frac{|\nabla^T u|^2}{(u+A)^2} (\frac{\partial}{\partial t} - \frac{\partial}{\partial t} H) u) + (H - \text{tr}_\omega \hat{\omega}_\infty) \Delta_B u + \frac{2}{u+A} \text{Re}(\nabla^T (H - \text{tr}_\omega \hat{\omega}_\infty) \cdot \nabla^T u).\]

Thus

\[(\frac{\partial}{\partial t} - \Delta_B) H = \frac{1}{u+A}(\frac{\partial}{\partial t} - \Delta_B) | \nabla^T u|^2 - \frac{|\nabla^T u|^2}{(u+A)^2} (\frac{\partial}{\partial t} - \Delta_B) u + (H - \text{tr}_\omega \hat{\omega}_\infty) \Delta_B u + \frac{2}{u+A} \text{Re}(\nabla^T (H - \text{tr}_\omega \hat{\omega}_\infty) \cdot \nabla^T u).\]

Using (4.7) and express

\[\frac{\nabla^T u}{u+A} = (1 - 2\epsilon) \frac{\nabla^T u}{u+A} + 2 \epsilon \Delta_B u + \nabla^T \text{tr}_\omega \hat{\omega}_\infty).\]

Therefore, from

\[\frac{\partial}{\partial t} H = \frac{1}{u+A} \frac{\nabla^T u}{u+A} - \frac{2}{u+A} \frac{\nabla^T u}{u+A} \nabla^T \text{tr}_\omega \hat{\omega}_\infty - \frac{8 \epsilon}{u+A} \frac{\nabla^T u}{u+A} \nabla^T \text{tr}_\omega \hat{\omega}_\infty - \frac{8 \epsilon}{u+A} \frac{\nabla^T u}{u+A} \nabla^T \text{tr}_\omega \hat{\omega}_\infty | \nabla^T u|\]

For any \(T > 0\), suppose \(H\) attains its maximum at \((x_0, t_0)\) on \(M \times [0, T]\), then

\[(4.8)\quad H_t(x_0, t_0) > 0, \quad \nabla^T H(x_0, t_0) = 0, \quad \Delta_B H(x_0, t_0) \leq 0.\]

Thus, by choosing \(\epsilon = 1/8\), we get that \(| \nabla^T u|^4(x_0, t_0) \leq C\) and by the uniform bound of \(\text{tr}_\omega \chi,\)

\[H(x_0, t_0) \leq C.\]

Therefore, since \(T > 0\) is arbitrary, we then arrive at

\[\frac{|\nabla^T u|^2}{u+A} + \text{tr}_\omega \hat{\omega}_\infty \leq C\]

on \(M \times [0, \infty).\) Now we show the inequality (4.3). Let

\[K = 2 \frac{|\nabla^T u|^2}{u+A} - \frac{\Delta u}{u+A},\]
using (1.3) and (4.4), the evolution equation for $K$ is given by
\[
\begin{align*}
(\frac{\partial}{\partial t} - \Delta_B)K &= \frac{1}{u+4}\{ \frac{\partial}{\partial t} - \Delta_B \} [2|\nabla^T u|^2 - \Delta_B u] + K(n - tr_\omega \hat{\omega}_\infty) + 2 \Re(\nabla^T K \cdot \nabla^T u) \\
&= \frac{1}{u+4}[2|\nabla^T u|^2 - \Delta_B u - 2|\nabla^T \nabla^T u|^2 - |\nabla^T \nabla^T u|^2] \\
&\quad + \frac{1}{u+4}(\langle \hat{\omega}_\infty, \partial_B \nabla_B u \rangle - \Delta_B tr_\omega \hat{\omega}_\infty + K(n - tr_\omega \hat{\omega}_\infty)) \\
&\quad + \frac{2}{u+4} \Re(\nabla^T (K + 2tr_\omega \hat{\omega}_\infty) \cdot \nabla^T u).
\end{align*}
\]
From (4.1), Proposition 2 and Ric$^T = -\partial_B \partial_B u - \hat{\omega}_\infty$, we estimate the term $-\Delta_B tr_\omega \hat{\omega}_\infty$ as follows
\[
-\Delta_B tr_\omega \hat{\omega}_\infty = (\frac{\partial}{\partial t} - \Delta_B)tr_\omega \hat{\omega}_\infty - \frac{\partial}{\partial t} tr_\omega \hat{\omega}_\infty \\
\quad \leq C(tr_\omega \hat{\omega}_\infty)^2 - |\nabla^T tr_\omega \hat{\omega}_\infty|^2 + \langle \partial_B \nabla_B u + \hat{\omega}_\infty, \hat{\omega}_\infty \rangle \\
\quad \leq \frac{1}{4}|\nabla^T \nabla^T u|^2 - |\nabla^T tr_\omega \hat{\omega}_\infty|^2 + C.
\]
By combining the above estimate with inequalities (4.2) and $|\nabla^T \nabla^T u|^2 \geq (\Delta_B u)^2/n$ and applying Schwarz inequality
\[
(\frac{\partial}{\partial t} - \Delta_B)K \leq -\frac{1}{n(u+4)}(\Delta_B u)^2 + \frac{2}{u+4} \Re(\nabla^T K \cdot \nabla^T u) + C.
\]
Again for any $T > 0$, suppose $K$ attains its maximum at $(x_0, t_0)$ on $M \times [0, T]$, then the conditions (4.8) holds for $K$, and hence $(\Delta_B u)(x_0, t_0)$ is bounded uniformly. Therefore, by (4.2) again, $(\Delta_B u)(x, t)$ is bounded uniformly on $M \times [0, T]$ for arbitrary $T > 0$.

The transverse scalar curvature $R^T(t)$ along the normalized Sasaki-Ricci flow (3.7) is expressed by
\[
R^T(t) = -\Delta_B u - tr_\omega \hat{\omega}_\infty.
\]
Recall the evolution of the transverse scalar curvature $R^T$,
\[
(\frac{\partial}{\partial t} - \Delta_B)R^T = |\text{Ric}^T|^2 + R^T.
\]
By the maximum principle, $R^T(t)$ is uniformly bounded from below on $M \times [0, \infty)$ and it is also uniformly bounded from above by (4.3) and Proposition 2.

**Proposition 4.** There exists $C > 0$ such that on $M \times [0, \infty)$, we have
\[
|R^T(t)| \leq C.
\]

5. $L^4$-Bound of the Transverse Ricci Curvature

In this section, we show the $L^4$-bound of the transverse Ricci curvature under the normalized Sasaki-Ricci flow (3.7).

**Theorem 4.** Let $(M, \eta_0, \xi_0, \Phi_0, g_0, \omega_0)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and its the space $Z$ of leaves of the characteristic foliation be well-formed. Suppose that $K^T_M$ is nef and big. Then there exists a positive constant $C$ such that
\[
\int_t^{t+1} \int_M |\text{Ric}^T_{\omega(s)}|^4 \omega(s)^n \wedge \eta_0 ds \leq C,
\]
for all \( t \in [0, \infty) \). Moreover, for any \( 0 < p < 4 \), we have

\[
\int_{t}^{t+1} \int_{M} |\text{Ric}_{\omega_{\omega}(s)} + \omega(s)|^{p} \omega(s)^{n} \wedge \eta_{0} ds \to 0 \quad \text{as} \quad t \to \infty.
\]

(5.2)

Since the transverse canonical bundle \( K_{M}^{T} \) is semi-ample. There exists a basic transverse holomorphic map \( \Psi : M \to (\mathbb{C}P^{N}, \omega_{FS}) \) such that \( \omega_{\infty} = \frac{1}{m} \Psi^{*}(\omega_{FS}) \) and

\[
\text{Ric}_{\omega_{\omega}(t)}^{T} + \sqrt{-1} \partial_{B} \overline{\partial}_{B} u(t) = -\omega_{\infty}.
\]

(5.3)

So, by the uniform bound of \( \omega_{\infty} \) in terms of \( \omega(t) \), to prove the \( L^{4} \) bound of transverse Ricci curvature (5.1) under the normalized Sasaki-Ricci flow, it suffices to show that

\[
\int_{t}^{t+1} \int_{M} |\partial_{B} \overline{\partial}_{B} u(s)|^{4} \omega(s)^{n} \wedge \eta_{0} ds \leq C,
\]

(5.4)

for all \( t \geq 0 \) and for some constant \( C \) independent of \( t \). We need the following Lemmas.

**Lemma 3.** There exists a positive constant \( C = C(\omega_{0}, \omega_{\infty}) \) such that

\[
\int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} \wedge \eta_{0} \leq \int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} + (\text{tr}_{\omega} \omega_{\infty})^{2} \omega(t)^{n} \wedge \eta_{0} \leq C,
\]

for all \( t \in [0, \infty) \).

**Proof.** From the relation (5.1)

\[
\int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} \wedge \eta_{0} \leq \int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} + (\text{tr}_{\omega} \omega_{\infty})^{2} \omega(t)^{n} \wedge \eta_{0},
\]

Applying the integration by parts, we have

\[
\int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} \wedge \eta_{0} = \int_{M} (\Delta_{B} u)^{2} \omega(t)^{n} \wedge \eta_{0}
\]

and also

\[
\int_{M} |\text{Ric}_{T}^{2} \omega(t)^{n} \wedge \eta_{0} = \int_{M} (\Delta_{B} u)^{2} \omega(t)^{n} \wedge \eta_{0}.
\]

Moreover, the \( L^{2} \)-bound of the transverse Riemannian curvature tensor follows from (1.11) and the Sasaki analogue of the Chern-Weil theory as in [Zh] Lemma 7.2:

\[
\int_{M} (2\pi)^{2} [2c_{2} - \frac{1}{n+1}(c_{2})^{2}] \wedge \frac{1}{2n-2(n-2)!} \omega(t)^{n-2} \wedge \eta_{0}
\]

\[
= \int_{M} |\text{Rm}_{T}|^{2} - \frac{1}{2n(n+1)} (\text{R}_{T})^{2} - \frac{1}{2n(n+1)} ((\text{R}_{T})^{2} + 2n(n+1)) - \frac{1}{2n!} \omega(t)^{n} \wedge \eta_{0}.
\]

The following integral inequalities hold for any smooth basic function on \( M \).

**Lemma 4.** There exists a universal positive constant \( C = C(n) \) such that

\[
\int_{M} |\nabla_{T} \overline{\nabla}_{T} u|^{4} \omega^{n} \wedge \eta_{0} \leq C \int_{M} |\nabla_{T} u|^{2} |\nabla_{T} \overline{\nabla}_{T} u|^{2} + |\nabla_{T} \overline{\nabla}_{T} \overline{\nabla}_{T} u|^{2} \omega^{n} \wedge \eta_{0}
\]

(5.6)

and

\[
\int_{M} |\nabla_{T} \Delta_{T} u|^{2} + |\nabla_{T} u|^{2} |\text{Rm}_{T}|^{2} \omega^{n} \wedge \eta_{0} \leq C \int_{M} |\nabla_{T} u|^{2} + |\nabla_{T} u|^{2} |\text{Rm}_{T}|^{2} \omega^{n} \wedge \eta_{0}.
\]

(5.7)
Proposition 5. There exists a positive constant $C = C(\omega_0, \widehat{\omega}_\infty)$ such that (5.8)
\[
\int_t^{t+1} \int_M \left| \nabla^T \overline{\nabla}^T u \right|^4 + \left| \nabla^T \nabla^T \nabla^T u \right|^2 + \left| \nabla^T \overline{\nabla}^T \overline{\nabla}^T u \right|^2 |\omega(s)|^n \wedge \eta_0 ds \leq C,
\]
for all $t \in [0, \infty)$.

Proof. By the previous Lemmas 3 and 4 it sufficient to prove a uniform $L^2$ bound $\nabla^T \Delta_B u$. Since
\[
(\frac{\partial}{\partial t} - \Delta_B)\Delta_B u = \Delta_B u + \langle \text{Ric}^T, \partial_B \overline{\partial_B} u \rangle_\omega + \Delta_B tr_\omega \widehat{\omega}_\infty,
\]
thus
\[
\frac{1}{2}(\frac{\partial}{\partial t} - \Delta_B)(\Delta_B u)^2 = (\Delta_B u)^2 - |\nabla^T \Delta_B u|^2 + \Delta_B u [\langle \text{Ric}^T, \partial_B \overline{\partial_B} u \rangle_\omega + \Delta_B tr_\omega \widehat{\omega}_\infty].
\]
Integrating over the manifold gives
\[
\int_M |\nabla^T \Delta_B u|^2 |\omega^n \wedge \eta_0| \leq \int_M [\langle \Delta_B u \rangle^2 + |\nabla^T \text{Ric}^T| \nabla^T \overline{\nabla}^T u - 2 \text{Re} (\nabla^T \Delta_B u \cdot \overline{\nabla}^T tr_\omega \widehat{\omega}_\infty)] |\omega^n \wedge \eta_0| - \frac{1}{2} \int_M \overline{\partial_B}(\Delta_B u)^2 |\omega^n \wedge \eta_0|.
\]
Applying the uniform bound of $\Delta_B u$ and Lemma 3 we then obtain
\[
\int_M |\nabla^T \Delta_B u|^2 |\omega^n \wedge \eta_0| \leq C \int_M \left[1 + |\nabla^T tr_\omega \widehat{\omega}_\infty|^2 |\omega(s)|^n \wedge \eta_0 \right] ds - \frac{d}{dt} \int_M (\Delta_B u)^2 |\omega^n \wedge \eta_0|.
\]
Integrating over the time interval $[t, t + 1]$, we have
\[
\int_t^{t+1} \int_M |\nabla^T \Delta_B u|^2 |\omega(s)|^n \wedge \eta_0 ds \leq C \int_t^{t+1} \int_M \left[1 + |\nabla^T tr_\omega \widehat{\omega}_\infty|^2 |\omega(s)|^n \wedge \eta_0 ds \right] + C,
\]
for all $t \geq 0$. The integral of the term $|\nabla^T tr_\omega \widehat{\omega}_\infty|$ can be estimated by the Schwarz lemma. From the evolution equation of $tr_\omega \widehat{\omega}_\infty$
\[
-\Delta_B tr_\omega \widehat{\omega}_\infty = (\frac{\partial}{\partial t} - \Delta_B)tr_\omega \widehat{\omega}_\infty - \frac{\partial}{\partial t}tr_\omega \widehat{\omega}_\infty \leq C |tr_\omega \widehat{\omega}_\infty|^2 - \langle \text{Ric}^T, \widehat{\omega}_\infty \rangle_\omega + |\nabla^T tr_\omega \widehat{\omega}_\infty| \leq C |tr_\omega \widehat{\omega}_\infty|^2 + \langle |\text{Ric}^T| tr_\omega \widehat{\omega}_\infty \rangle| - |\nabla^T tr_\omega \widehat{\omega}_\infty|,
\]
where $C$ is a universal constant given by the upper bound of the bisection curvature of $\omega_{FS}$ on $\mathbb{CP}^n$. Because $0 < tr_\omega \widehat{\omega}_\infty \leq C$ under the flow, we have
\[
|\nabla^T tr_\omega \widehat{\omega}_\infty| \leq \Delta_B tr_\omega \widehat{\omega}_\infty + C (|\text{Ric}^T| + 1)
\]
and thus
\[
\int_M |\nabla^T tr_\omega \widehat{\omega}_\infty| |\omega(t)|^n \wedge \eta_0 \leq C \int_M (|\text{Ric}^T| + 1) |\omega(t)|^n \wedge \eta_0 \leq C
\]
uniformly. Substituting into (5.9) we obtain the desired estimate. \hfill \qed

In order to prove (5.2) we use the $L^2$ estimate to the traceless transverse Ricci curvature as following.

Lemma 5. Under the Sasaki-Ricci flow,
\[
(5.10) \quad \int_t^{t+1} \int_M \text{Ric}^T \omega(s) + \omega(s)|^2 \omega(s)|^n \wedge \eta_0 ds \to 0 \text{ as } t \to \infty.
\]
Proof. Recall the evolution of the transverse scalar curvature $R^T = tr_ωRic^T$

$$(\frac{∂}{∂t} - Δ_B)R^T = |Ric^T|^2 + R^T = |Ric^T + ω|^2 - (R^T + n).$$

The maximum principle shows that $\frac{d}{dt}\inf R^T \geq -(\inf R^T + n)$, which implies

$$\inf R^T + n \geq e^{-t}\min(\inf R^T(0) + n, 0) \geq -Ce^{-t}$$

for some positive constant $C = C(ω_0)$. Then

$$\int_M |Ric^T_T + ω|^2ω^n ∧ η_0
= \int_M (\frac{∂}{∂t}R^T + R^T + n)ω^n ∧ η_0
= \frac{d}{dt}\int_M R^Tω^n ∧ η_0 + \int_M (R^T + n)(R^T + 1)ω^n ∧ η_0
= \frac{d}{dt}\int_M R^Tω^n ∧ η_0 + \int_M (R^T + n + Ce^{-t})(R^T + 1)ω^n ∧ η_0
-Ce^{-t}\int_M (R^T + 1)ω^n ∧ η_0
\leq \frac{d}{dt}\int_M R^Tω^n ∧ η_0 + C\int_M (R^T + n)ω^n ∧ η_0 + Ce^{-t}$$

where we used the uniform bound of transverse scalar curvature and volume form $ω(t)^n ∧ η_0$. The integration of $R^T + n$ becomes

$$\int_M (R^T + n)ω^n ∧ η_0
= n\int_M (Ric^T + ω) ∧ ω^{n-1} ∧ η_0 + \int_M (-ω_∞ + \hat{ω}) ∧ \hat{ω}^{n-1} ∧ η_0
= ne^{-t}\int_M (ω_0 - \hat{ω}_∞) ∧ \hat{ω}^{n-1} ∧ η_0 \leq Ce^{-t}.$$  

Then

$$\int_0^∞ \int_M |Ric^T_T + ω|^2ω(t)^n ∧ η_0 dt \leq \lim_{t→∞} \int_M R^T(t)ω(t)^n ∧ η_0 - \int_M R^T(0)ω_0^n ∧ η_0 + C \leq C.$$  

This estimate implies the lemma. \ □

The estimate \[5.2\] when 2 ≤ p < 4 then is a direct consequence of the Hölder inequality

$$\int_0^{t+1} \int_M |Ric^T_T + ω|^p \leq \left(\int_0^{t+1} \int_M |Ric^T_T + ω|^4\right)^{\frac{p-2}{4}} \left(\int_0^{t+1} \int_M |Ric^T_T + ω|^2\right)^{\frac{4-p}{2}}.$$  

When 0 < p < 2 the estimate \[5.2\] is obvious.

6. Cheeger-Gromov Convergence

Let $(M, η, ξ, Φ, g)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and its leave space $Z$ of the characteristic foliation be well-formed which means its orbifold singular locus and algebro-geometric singular locus coincide. In this section if we assume that $Ric^T_T$ is nef and big, we will show that the solution of Sasaki-Ricci flow \[3.7\] converge in the Gromov-Hausdorff topology to an η-Einstein metric on the transverse canonical model without any curvature assumption in the case of the dimension less than or equal to 7.

Note that from the definition of quasi-regular Sasakian manifolds, there is a natural projection

$$Π : (C(M), g, J, ω) → (Z, h, ω_h)$$

satisfying the orbifold Riemannian submersion $π : (M, g, ω) → (Z, h, ω_h)$ with $ω = π^*(ω_h)$ such that

$$Π|_{(M, g, ω)} = π$$
and the volume form of the Kähler cone metric on the cone $C(M)$
\begin{equation}
\omega^{n+1} = r^{2n+1}(\Pi^*\omega_h)^n \wedge dr \wedge \eta,
\end{equation}
and the volume form of the Sasaki metric on $M$
\begin{equation}
i \frac{\partial}{\partial r}\omega^{n+1} = (\Pi^*\omega_h)^n \wedge \eta = \omega^n \wedge \eta.
\end{equation}
Furthermore, by adapting notions from Definition 5 and [CZ], $G_i$ is the local uniformizing finite group acting on a smooth complex space $\tilde{U}_i$ such that the local uniformizing group injects into $U(1)$ and the map
\[ \varphi_i : U(1) \times \tilde{U}_i \to U_i \]
is exactly $|G_i|$-to-one on the complement of the orbifold locus. Then we can have the following computation
\begin{equation}
\int_Z |\text{Ric}_{\omega_h(t)}|^p \omega_h(t)^n = 
\sum_i \frac{1}{|G_i|} \int_{\tilde{U}_i} \varphi_i |\text{Ric}_{\omega_h(t)}|^p \omega_h(t)^n
= \sum_i \int_{U(1) \times \tilde{U}_i} \pi^* \varphi_i |\text{Ric}_{\omega(t)}|^p \pi^* \omega_h(t)^n \wedge \eta
= \sum_i \int_{U_i} \pi^* \varphi_i |\text{Ric}_{\omega(t)}|^p \pi^* \omega_h(t)^n \wedge \eta
= \int_M |\text{Ric}_{\omega(t)}|^p \frac{2}{\pi^2} \omega^{n+1}
= \int_M |\text{Ric}_{\omega(t)}|^p \omega(t)^n \wedge \eta.
\end{equation}

With (6.3) in mind, we will apply our previous results plus Cheeger-Colding-Tian structure theory for Kähler orbifolds ([CCT], [TZ1] and [TZ2, Theorem 2.3]) to study the structure of desired limit space. Since $(M, \eta, \xi, \Phi, g)$ is a compact quasi-regular Sasakian manifold, by the first structure theorem on Sasakian manifolds, $M$ is a principal $S^1$-orbibundle ($V$-bundle) over $Z$ which is also a $Q$-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion $\pi : (M, g, \omega) \to (Z, h, \omega_h)$ with
\[ g = g^T + \eta \otimes \eta \]
and
\[ g^T = \pi^*(h), \quad \frac{1}{2} d\eta = \pi^*(\omega_h). \]
The orbit $\xi_x$ is compact for any $x \in M$, we then define the transverse distance function as
\[ d^T(x, y) \triangleq d_g(\xi_x, \xi_y), \]
where $d$ is the distance function defined by the Sasaki metric $g$. Then
\[ d^T(x, y) = d_h(\pi(x), \pi(y)). \]
We define a transverse ball $B_{\xi, g}(x, r)$ as follows:
\[ B_{\xi, g}(x, r) = \{ y : d^T(x, y) < r \} = \{ y : d_h(\pi(x), \pi(y)) < r \}. \]
Note that when $r$ small enough, $B_{\xi, g}(x, r)$ is a trivial $S^1$-bundle over the geodesic ball $B_h(\pi(x), r)$.

Based on Perelman’s non-collapsing theorem for a transverse ball along the unnormalizing Sasaki-Ricci flow, it follows that
Lemma 6. ([Col, Proposition 7.2], [Hei, Lemma 6.2], [TZ2, Lemma 3.14]) Let $(M^{2n+1}, \xi, g_0)$ be a compact Sasakian manifold and let $g^T(t)$ be the solution of the unnormalizing Sasaki-Ricci flow with the initial transverse metric $g_0^T$. Then there exists a positive constant $C$ such that for every $x \in M$, if $|R^T| \leq r^{-2}$ on $B_{\xi,g(t)}(x, r)$ for $r \in (0, r_0]$, where $r_0$ is a fixed sufficiently small positive number, then
\[
\Vol(B_{\xi,g(t)}(x, r)) \geq Cr^{2n}.
\]

Moreover, based on the $L^2$-bound of Riemannian curvature and Kähler-Einstein condition on $Z_\infty$, we can say more about the limit singular space $Z_\infty$ and then $M_\infty$. More precisely, once we obtain (6.3), Theorem 4 and Lemma 6, it follows from arguments of Petersen-Wei [PW1], [PW2], Cheeger-Colding-Tian [CCT] and [TZ1] Theorem 2.37) that we have the following structure theorem of limit spaces $Z_\infty$ and $M_\infty$.

Theorem 5. Let $(M_i, \eta_i, \xi, \Phi_i, g_i, \omega_i)$ be a sequence of quasi-regular Sasakian $(2n+1)$-manifolds with Sasaki metrics $g_i = g_i^T + \eta_i \otimes \eta_i$ such that for basic potentials $\varphi_i$
\[
\eta_i = \eta + d^C_B \varphi_i
\]
and
\[
d\eta_i = d\eta + \sqrt{-1} \partial_B \overline{\partial}_B \varphi_i.
\]
We denote that $(Z_i, h_i, J_i, \omega_{h_i})$ are a sequence of well-formed normal projective orbifolds of complex dimension $n$ which are the corresponding foliation leave space with respect to $(M_i, \eta_i, \xi, \Phi_i, g_i, \omega_i)$ such that
\[
\frac{1}{2} d\eta_i = \pi^*(h_i) = \pi^*(\omega_{h_i}), \quad \pi^*(J_i) = \Phi_i.
\]
Suppose that $(M_i, \eta_i, \xi, \Phi_i, g_i, \omega_i)$ is a smooth transverse minimal model of general type satisfying
\[
(6.4) \quad \int_M |\Ric_{g_i}^T + \omega_i|^p \omega_i^n \wedge \eta \to 0,
\]
and
\[
(6.5) \quad \Vol(B_{\xi,g_i^T}(x_i, 1)) \geq \nu
\]
for some $p > n$, $\nu > 0$. Then passing to a subsequence if necessary, $(M_i, \Phi_i, g_i, x_i)$ converges in the Cheeger-Gromov sense to limit length spaces $(M_\infty, \Phi_\infty, d_\infty, x_\infty)$ and then $(Z_i, h_i, J_i, \pi(x_i))$ converges to $(Z_\infty, h_\infty, J_\infty, \pi(x_\infty))$ such that
\begin{enumerate}
  \item for any $r > 0$ and $p_i \in M_i$ with $p_i \to p_\infty \in M_\infty$,
  \[
  \Vol(B_{h_i}(\pi(p_i), r)) \to \mathcal{H}^{2n}(B_{h_\infty}(\pi(p_\infty), r))
  \]
  and
  \[
  \Vol(B_{\xi,g_i^T}(p_i, r)) \to \mathcal{H}^{2n}(B_{\xi,g_\infty^T}(p_\infty, r)).
  \]
\end{enumerate}
Moreover,
\[
\Vol(B(p_\infty, r)) \to \mathcal{H}^{2n+1}(B(p_\infty, r)),
\]
where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.
\begin{enumerate}
  \item $M_\infty$ is a $S^1$-orbibundle over the normal projective variety $Z_\infty := M_\infty/F_\xi$.
  \item $Z_\infty = \mathcal{R} \cup \mathcal{S}$ such that $\mathcal{S}$ is a closed singular set of codimension 4 and $\mathcal{R}$ consists of points whose tangent cones are $\mathbb{R}^{2n}$.
\end{enumerate}
(4) the convergence on the regular part of $M_\infty$ which is a $S^1$-principle bundle over $\mathcal{R}$ in the $(C^\alpha \cap L^{2,p'})$-topology for any $0 < \alpha < 2 - \frac{2n}{p}$.

Proof. Since $\xi$ is fixed and the metrics are under deformation generated by basic potentials $\phi_i$ such that

$$\eta_i = \eta + d_B^C \phi_i$$

and

$$d\eta_i = d\eta + \sqrt{-1} \partial_B \overline{T}_B \phi_i.$$  

By the first structure theorem on Sasakian manifolds, $M$ is a principal $S^1$-orbibundle ($V$-bundle) over $Z$ which is also a $Q$-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion $\pi : (M, g_i, \omega_i) \rightarrow (Z, h_i, \omega_{h_i})$ with

$$g^T_i = \frac{1}{2} d\eta_i = \pi^*(h_i) = \pi^*(\omega_{h_i})$$

and

$$d^T_i(x, y) = d_{h_i}(\pi(x), \pi(y)).$$

Then

$$g_i = \pi^* h_i + \eta_i \otimes \eta_i.$$  

Hence by Cheeger-Colding-Tian structure theory for Kähler orbifolds ([CCT]) that

$$g^T_i = \pi^* h_i \rightarrow \pi^* h_\infty = g^T_\infty$$

and

$$d^T_i = d^T_{g^T_i} \rightarrow d^T_{g^T_\infty} = d^T_\infty$$

as $i \rightarrow \infty$. Thus

$$\eta_i \rightarrow \eta_\infty$$

and

$$g_i \rightarrow g_\infty = g^T_\infty + \eta_\infty \otimes \eta_\infty$$

as $i \rightarrow \infty$. Moreover ([CCT], [Co2]),

$$g^T_i \rightarrow g^T_\infty$$

such that

$$h_i \rightarrow h_\infty$$

with $g^T_i = \pi^*(h_i)$.

Moreover,

$$B_{h_i}(\pi(x_i), r) \rightarrow B_{h_\infty}(\pi(x_\infty), r)$$

and then

$$B_{\xi, g^T_i}(x_i, r) \rightarrow B_{\xi, g^T_\infty}(x_\infty, r)$$

as $i \rightarrow \infty$. Furthermore,

$$\text{Vol}(B_{h_i}(\pi(x_i), r)) \rightarrow \text{Vol}(B_{h_\infty}(\pi(x_\infty), r))$$

and then

$$\text{Vol}(B_{\xi, g^T_i}(x_i, r)) = \int_{B_{\xi, g^T_i}(x_i, r)} \omega_i^n \wedge \eta \rightarrow \int_{B_{\xi, g^T_\infty}(x_\infty, r)} \omega_\infty^n \wedge \eta = \mathcal{H}^{2n}(B_{\xi, g^T_\infty}(x_\infty, r)).$$
Finally
\[ \text{Vol}(B(x_i, r)) = \int_{B(x_i, r)} \omega^n_i \wedge \eta \rightarrow \int_{B(x_{\infty}, r)} \omega^n_{\infty} \wedge \eta = H^{2n+1}(B(x_{\infty}, r)) \]
as \( i \rightarrow \infty \).

(3) and (4) will follow easily from (6.4), (6.5), (6.6) and the arguments as in [TZ1] Theorem 2.37.

Let \((M, \eta, \xi, g)\) be a compact quasi-regular Sasakian \((2n + 1)\)-manifold and be a principal \(S^1\)-orbibundle over \(Z\) which is a well-formed \(Q\)-factorial, polarized, normal projective orbifold. If \(K_M^T\) is nef and big ([CLW]), then it is semi-ample and then there exists a \(S^1\)-equivariant basic base point free holomorphic map
\[ \Psi : M \rightarrow (\mathbb{CP}^N, \omega_{FS}) \]
defined by the basic transverse holomorphic section \(\{s_0, s_1, \ldots, s_N\}\) of \(H^0(M, (K_M^T)^m)\) with \(N = \dim H^0(M, (K_M^T)^m) - 1\) for a large positive integer \(m\). Its image
\[ \Psi(M) = M_{\text{can}} \]
is called the transverse canonical model of \(M\). Note that since \((M, \eta, \xi, g)\) is a compact quasi-regular Sasakian manifold, \(M\) is a principal \(S^1\)-orbibundle (\(V\)-bundle) over \(Z\) which is also a \(Q\)-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion \(\pi : (M, g) \rightarrow (Z, h, \omega)\) with
\[ c_1^B((K_M^T)^{-1}) = \pi^*c_{\text{orb}}^1(Z) = \pi^*c_1(K_Z^{-1}). \]

Then there exists a base point free holomorphic map
\[ \tilde{\Psi} : Z \rightarrow (\mathbb{CP}^N, \omega_{FS}) \]
defined by the holomorphic section \(\{\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_N\}\) of \(H^0(Z, (K_Z)^m)\) and its image
\[ \tilde{\Psi}(Z) = Z_{\text{can}} \]
is the canonical model of \(Z\) with the Kodaira dimension \(\kappa(Z) = n\).

Let \(V \subset M\) be the basic exceptional locus of \(\Psi\) with \(\pi(V) = E \subset Z\) which coincides with non-Kähler locus \(\text{Null}(\omega_{FS}^B(M))\) of the canonical class as in Proposition 1 ([CT], [TZ3]). It follows from the first structure theorem (Proposition 6, Theorem 3, Theorem 5, and the arguments as in [TZ2] that we have

**Theorem 6.** Let \((M, \eta, \xi, g)\) be a compact quasi-regular Sasakian \((2n+1)\)-manifold and be a principal \(S^1\)-orbibundle (\(V\)-bundle) over \(Z\) which is a well-formed \(Q\)-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion \(\pi : (M, g, \omega) \rightarrow (Z, h, \omega_h)\). Suppose that \((M, \eta, \xi, g)\) is a smooth transverse minimal model of general type in the case of the dimension less than or equal to 7 and \(\omega(t)\) be any solution to the Sasaki-Ricci flow \([3.7]\). Then

1. \(\omega(t)\) converges smoothly to a Sasaki-\(\eta\)-Einstein metric \(\omega_{\infty}\) outside the exceptional locus \(V\);
2. the metric completion of \((Z\setminus E, h_{\infty}, \omega_{h_{\infty}})\) is homeomorphic to \(Z_{\text{can}}\) which is a normal projective variety and then the metric completion of \((M\setminus V, g_{\infty}, \omega_{\infty})\) is homeomorphic to \(M_{\text{can}}\), so it is compact.
3. \((Z_{\infty}, d_{\infty}^T)\) is isometric to the metric completion of \((Z_{\infty}\setminus E, h_{\infty})\) and then \((M_{\infty}, d_{\infty})\) is isometric to the metric completion of \((M\setminus V, g_{\infty}, \omega_{\infty})\).
As a consequence, for any sequence \( t_i \to \infty \), \((M, \omega(t_i))\) converges along a subsequence in the Cheeger-Gromov sense to
\[
\text{Ric}_{\omega_\infty}^T = -\omega_\infty
\]
in the limit space \((M_\infty, d_\infty)\) which is the transverse canonical model \(M_{\text{can}}\) of \(M\).

Then our main results in this paper as in Theorem [1] and Corollary [1] follows easily from Theorem [6].

Finally, we add some remark about the Sasaki analogue of Guo-Song-Weinkove [GSW] arguments for the contraction on the foliation \((-2)\)-curve for \(n=2\). Let \((M, \xi)\) be a compact quasi-regular Sasakian 5-manifold. A basic 1-cycle \(V\) on \(M\) is a formal finite sum \(V = \sum a_i V_i\), for \(a_i \in \mathbb{Z}\) and \(V_i\) is the irreducible invariant Sasakian 3-manifold. We denote by \(N_1(M)_\mathbb{Z}\) the space of 1-cycles modulo numerical equivalence. Write
\[
N_1(M)_\mathbb{Q} = N_1(M)_\mathbb{Z} \otimes \mathbb{Q} \quad \text{and} \quad N_1(M)_\mathbb{R} = N_1(M)_\mathbb{Z} \otimes \mathbb{R}.
\]
Then write \(NE(M)\) for the cone of effective elements of \(N_1(M)_\mathbb{R}\) and \(\overline{NE(M)}\) for its closure. Furthermore, a basic divisor \(D^T\) is ample if and only if
\[
D^T \cdot V > 0
\]
for all nonzero \(V \in \overline{NE(M)}\). It is the Kleiman criterion for the ample line bundle.

In view of the cohomological characterization of the maximal solution of the Sasaki-Ricci flow (3.4) (see section 3), we start with a pair \((M, H^T)\), where \(M\) is a Sasakian manifold with an ample basic divisor \(H^T\). Let
\[
T_0 = \sup \{ t > 0 \mid H^T + tK^T_M \text{ is nef} \}.
\]
Denote
\[
L^T_0 := H^T + T_0 K^T_M
\]
which is a basic \(Q\)-line bundle and semi-ample. In fact, it follows from Kleiman criterion that \(mL^T_0 - T_0 K^T_M\) is ample and then nef and big for some sufficiently large \(m\). Then by Kawamata criterion for base point free, we have the semi-ample for \(L^T_0\).

Next we define a subcone
\[
R := \{ V \in \overline{NE(M)} \mid L^T_0 \cdot V = 0 \}
\]
which is a foliation extremal ray \(R\) with the generic choice of \(H^T\). Moreover, we have \(R = \overline{NE(M)}_{K^T_M<0} \cap \overline{(L^T_0)^\perp}\). Then
\[
0 = L^T_0 \cdot V \Rightarrow K^T_M \cdot V = -\frac{1}{T_0}(H^T \cdot V) < 0.
\]
That is the map \(\Psi\) induced from \((L^T_0)^m\) contract all foliation curves whose class lies in the foliation extremal ray \(R\) with
\[
L^T_0 \cdot V = 0 \quad \text{and} \quad K^T_M \cdot V < 0.
\]
The union of all foliation curves is called the locus of the foliation extremal ray \(R\) which is exactly the set of points where the map \(\Psi : M \to N\) is not isomorphism.

We observe that
\[
K^T_M \cdot V = 0
\]
as $T_0 \to \infty$. Then the floating foliation $(-2)$-curves $V$ which is entirely contained in the smooth locus of $M$ with respective to the foliation $\mathcal{F}_\xi$, will be contracted to orbifold points by the Sasaki-Ricci flow as $T_0 \to \infty$. We refer to [CLW] for some details.

**Appendix A.**

In this appendix, for a completeness, we will address the preliminary notions on the Sasakian structure, the leave space and its foliation singularities, basic holomorphic line bundles and basic divisors over Sasakian manifolds. We refer to [BG], [M], and references therein for some details.

**A.1. Sasakian Structures, Leave Spaces and Its Foliation Singularities.**

**Definition 4.** Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n + 1)$-manifold. If the orbits of the Reeb vector field $\xi$ are all closed, and hence circles, then integrates to an isometric $U(1)$ action on $(M, g)$. Since it is nowhere zero this action is locally free; that is, the isotropy group of every point in $M$ is finite. If the $U(1)$ action is in fact free then the Sasakian structure is said to be regular. Otherwise, it is said to be quasi-regular. It is said to be irregular if the orbits of are not all closed. In this case the closure of the $1$-parameter subgroup of the isometry group of $(M, g)$ is isomorphic to a torus $T^k$, for some positive integer $k$ called the rank of the Sasakian structure. In particular, irregular Sasakian manifolds have at least a $T^2$ isometry.

Note that in the regular or quasi-regular case, the leaf space $Z = M/\mathcal{F}_\xi = M/U(1)$ has the structure of a compact manifold or orbifold, respectively. In the latter case the orbifold singularities of $Z$ descend from the points in $M$ with non-trivial isotropy subgroups which finite subgroups of $U(1)$ and will all be isomorphic to cyclic groups. The transverse Kähler structure described above then pushes down to a Kähler structure on $Z$, so that $Z$ is a compact complex manifold or orbifold equipped with a Kähler metric $h$.

The first structure theorem on Sasakian manifolds states that

**Proposition 6.** ([Rui]) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian manifold of dimension $2n + 1$ and $Z$ denote the space of leaves of the characteristic foliation $\mathcal{F}_\xi$ (just as topological space). Then

(i) $Z$ carries the structure of a Hodge orbifold $Z = (Z, \Delta)$ with an orbifold Kähler metric $h$ and Kähler form $\omega$ which defines an integral class $[p^*\omega]$ in $H^2_{orb}(Z, \mathbb{Z})$ in such a way that $\pi : (M, g, \omega) \to (Z, h, \omega_h)$ is an orbifold Riemannian submersion, and a principal $S^1$-orbibundle ($V$-bundle) over $Z$. Furthermore, it satisfies $\frac{1}{2}d\eta = \omega = \pi^*(\omega_h)$. The fibers of $\pi$ are geodesics.

(ii) $Z$ is also a $\mathbb{Q}$-factorial, polarized, normal projective algebraic variety.

(iii) The orbifold $Z$ is Fano if and only if $\text{Ric}_g > -2$. In this case $Z$ as a topological space is simply connected and as an algebraic variety is uniruled with Kodaira dimension $-\infty$.

(iv) $(M, \xi, g, \omega)$ is Sasaki-Einstein if and only if $(Z, h, \omega_h)$ is Kähler-Einstein with scalar curvature $4n(n + 1)$.
(v) If \((M, \eta, \xi, \Phi, g)\) is regular then the orbifold structure is trivial and \(\pi\) is a principal circle bundle over a smooth projective algebraic variety.

(vi) As real cohomology classes, there is a relation between the basic Chern class and orbifold Chern class

\[
c^B_k(M) := c_k(F_\xi) = \pi^*c^\text{orb}_k(Z).
\]

Conversely, for a compact Hodge orbifold \((Z, h)\). Let \(\pi : M \to Z\) be a principal \(U(1)\)-orbibundle over \(Z\) whose first Chern class is an integral class defined by \([\omega_Z]\), and let \(\eta\) be a 1-form on \(M\) with \(\frac{1}{2}d\eta = \pi^*\omega_h\) (is then \(\eta\) proportional to a connection 1-form). Then \((M, \pi^*h + \eta \otimes \eta)\) is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the structure group \(U(1)\), then the total space \(M\) is a smooth manifold.

Note that in the quasi-regular case, the projection is instead a principal \(U(1)\) orbibundle, with \(\omega_Z\) again proportional to a curvature 2-form. The orbifold cohomology group \(H^2_{\text{orb}}(Z, \mathbb{Z})\) classifies isomorphism classes of principal \(U(1)\) orbibundles over an orbifold \(Z\), just as in the regular manifold case the first Chern class in \(H^2(Z, \mathbb{Z})\) classifies principal \(U(1)\) bundles. The Kähler form \(H^2_{\text{orb}}(Z, \mathbb{Z})\) then defines a cohomology class \([\omega_Z]\) in \(H^2(Z, \mathbb{R})\) which is proportional to a class in the image of the natural map

\[
p : H^2_{\text{orb}}(Z, \mathbb{Z}) \to H^2_{\text{orb}}(Z, \mathbb{R}) \to H^2(Z, \mathbb{R}).
\]

On the other hand, the second structure theorem on Sasakian manifolds states that

**Proposition 7.** ([Ru]) Let \((M, g)\) be a compact Sasakian manifold of dimension \(2n + 1\). Any Sasakian structure \((\xi, \eta, \Phi, g)\) on \(M\) is either quasi-regular or there is a sequence of quasi-regular Sasakian structures \((\xi_i, \eta_i, \Phi_i, g_i)\) converging in the compact-open \(C^\infty\)-topology to \((\xi, \eta, \Phi, g)\). In particular, if \(M\) admits an irregular Sasakian structure, it admits many locally free circle actions.

We recall that

**Definition 5.** An orbifold complex manifold is a normal, compact, complex space \(Z\) locally given by charts written as quotients of smooth coordinate charts. That is, \(Z\) can be covered by open charts \(Z = \sqcup U_i\). The orbifold charts on \((Z, U_i, \varphi_i)\) is defined by the local uniformizing systems \((\tilde{U}_i, G_i, \varphi_i)\) centered at the point \(p_i\), where \(G_i\) is the local uniformizing finite group acting on a smooth complex space \(\tilde{U}_i\) such that \(\varphi_i : \tilde{U}_i \to U_i = \tilde{U}_i/G_i\) is the biholomorphic map. A point \(x\) of complex orbifold \(X\) whose isotropy subgroup \(\Gamma_x \neq \text{Id}\) is called a singular point. Those points with \(\Gamma_x = \text{Id}\) are called regular points. The set of singular points is called the orbifold singular locus or orbifold singular set, and is denoted by \(\Sigma_{\text{orb}}(Z)\).

Let \(\Gamma \subset GL(n, \mathbb{C})\) be a finite subgroup. Then the quotient space \(\mathbb{C}^n/\Gamma\) is smooth if and only if \(\Gamma\) is a reflection group which fixes a hyperplane in \(\mathbb{C}^n\). Now let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian manifold of dimension \(2n + 1\). By the first structure theorem, the underlying complex space \(Z = (Z, U_i)\) is a normal, orbifold variety with the algebro-geometric singular set \(\Sigma(Z)\). Then \(\Sigma(Z) \subset \Sigma_{\text{orb}}(Z)\) and it follows that \(\Sigma(Z) = \Sigma_{\text{orb}}(Z)\) if and only if none of the local
uniformizing groups $\Gamma_i$ of the orbifold $Z = (Z, U_i)$ contain a reflection. If some $\Gamma_i$ contains a reflection, then the reflection fixes a hyperplane in $\tilde{U}_i$ giving rise to a ramification divisor on $\tilde{U}_i$ and a branch divisor on $Z$.

**Definition 6.** (i) The branch divisor $\Delta$ of an orbifold $Z = (Z, \Delta)$ is a $Q$-divisor on $Z$ of the form

$$\Delta = \sum_\alpha (1 - \frac{1}{m_\alpha}) D_\alpha,$$

where the sum is taken over all Weil divisors $D_\alpha$ that lie in the orbifold singular locus $\Sigma^{\text{orb}}(Z)$, and $m_\alpha$ is the gcd of the orders of the local uniformizing groups taken over all points of $D_\alpha$ and is called the ramification index of $D_\alpha$.

(ii) The orbifold structure $Z = (Z, \Delta)$ is called well-formed if the fixed point set of every non-trivial isotropy subgroup has codimension at least two. That is, $Z = (Z, \emptyset)$. Then $Z$ is well-formed if and only if its orbifold singular locus and algebro-geometric singular locus coincide, equivalently $Z$ has no branch divisors.

**Example 1.** For instance, the weighted projective $\mathbb{CP}(1; 4; 6)$ has a branch divisor $\frac{1}{2}D_0 = \{z_0 = 0\}$. But $\mathbb{CP}(1; 2; 3)$ is a unramified well-formed orbifold with two singular points, $(0; 1; 0)$ with local uniformizing group the cyclic group $\mathbb{Z}_2$, and $(0; 0; 1)$ with local uniformizing group $\mathbb{Z}_3$.

Note that the orbifold canonical divisor $K_Z^{\text{orb}}$ and canonical divisor $K_Z$ are related by

$$K_Z^{\text{orb}} = \varphi^*(K_Z + [\Delta]).$$

In particular, $K_Z^{\text{orb}} = \varphi^*K_Z$ if and only if there are no branch divisors.

For all previous discussions with the special case for $n = 2$, we have the following result concerning its foliation cyclic quotient singularities.

**Theorem 7.** ([CLW]) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and $Z$ be its leave space of the characteristic foliation. Then $Z$ is a $Q$-factorial normal projective algebraic orbifold surface satisfying

1. if its leave space $(Z, \emptyset)$ has at least codimension two fixed point set of every non-trivial isotropy subgroup. That is to say $Z$ is well-formed, then $Z$ has isolated singularities of a finite cyclic quotient of $\mathbb{C}^2$ and the action is

$$\mu_{Z_r} : (z_1, z_2) \to (\zeta^a z_1, \zeta^b z_2),$$

where $\zeta$ is a primitive $r$-th root of unity. We denote the cyclic quotient singularity by $\frac{1}{r}(a, b)$ with $(a, r) = 1 = (b, r)$. In particular, the action can be rescaled so that every cyclic quotient singularity corresponds to a $\frac{1}{r}(1, a)$-point with $(r, a) = 1$, $\zeta = e^{2\pi i}$. In particular, it is klt (Kawamata log terminal) singularities. More precisely, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called foliation cyclic quotient singularities of type $\frac{1}{r}(1, a)$ at a singular fibre $S_p^1$ in $M$. 
is called the Hopf $S^1$-manifold. Then the action is
\[
\mu_{Z_r} : (z_1, z_2) \mapsto \left( e^{\frac{2\pi i a_1}{r_1}} z_1, e^{\frac{2\pi i a_2}{r_2}} z_2 \right),
\]
for some positive integers $r_1, r_2$ whose least common multiplier is $r$, and $a_i, i = 1, 2$ are integers coprime to $r_i$, $i = 1, 2$. Then the foliation singular set contains some 3-dimensional submanifolds of $M$. More precisely, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called the Hopf $S^1$-orbibundle over a Riemann surface $\Sigma_h$.

**Definition 7.** (i) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $(Z, \emptyset)$ of the characteristic foliation be well-formed. Then the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called foliation cyclic quotient singularities of type $\frac{1}{p}(1, a)$ at a singular fibre $S^1_p$ in $M$. The foliation singular set is discrete, and hence finite. It is klt (Kawamata log terminal) singularities.

(ii) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $(Z, \Delta)$ has the codimension one fixed point set of some non-trivial isotropy subgroup. Then the foliation singular set contains some 3-dimensional submanifolds of $M$. More precisely, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called the Hopf $S^1$-orbibundle over a Riemann surface $\Sigma_g$.

It follows from the Hirzebruch Jung continued fraction ([R]) that

**Theorem 8.** ([CLW]) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold $M$ with a foliation cyclic quotient singularity of type $\frac{1}{p}(1, a)$ at a singular fibre $S^1_p$, then the Hirzebruch Jung continued fraction
\[
\frac{z}{a} = [b_1, \ldots, b_l]
\]
gives the information on the foliation minimal resolution $\varphi : \tilde{M} \to M$ of $M$. We denote the exceptional foliation curves $V_i$ of such a resolution. Then the exceptional foliation curves form a chain of $\{V_1, \ldots, V_l\}$ such that each $V_i$ has self intersection $V_i^2 = -b_i$ for every $i = 1, \ldots, l$ and $V_i$ intersects another foliation curve $V_j$ transversely only if $j = i - 1$ or $j = i + 1$. In particular, for a foliation cyclic quotient singularity of type $\frac{1}{k}(1, 1)$, we have $\frac{k}{1} = [k]$ as the foliation $(-k)$-curve in $\tilde{M}$.

A.2. **Basic Holomorphic Line Bundles, Basic Divisors on Sasakian Manifolds.** Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n+1)$-manifold. We define $D = \ker \eta$ to be the holomorphic contact vector bundle of $TM$ such that
\[
TM = D \oplus \langle \xi \rangle = T^{1,0}(M) \oplus T^{0,1}(M) \oplus \langle \xi \rangle.
\]
Then its associated strictly pseudoconvex CR $(2n+1)$-manifold to be denoted by $(M, T^{1,0}(M), \xi, \Phi)$.

**Definition 8.** ([La]) Let $(M, T^{1,0}(M))$ be a strictly pseudoconvex CR $(2n+1)$-manifold and $E \to M$ be a $C^\infty$ complex vector bundle over $M$. A pair $(E, \overline{\partial}_b)$ is a CR-holomorphic vector bundle if the differential operator
\[
\overline{\partial}_b : \Gamma^\infty(E) \to \Gamma^\infty(T^{0,1}(M)^* \otimes E)
\]
is defined by

\[(i)\]
\[\overline{\partial}_Z(fs) = (\overline{\partial}_b f)(\overline{Z}) \otimes s + f \overline{\partial}_Zs,\]

\[(ii)\]
\[\overline{\partial}_Z\overline{W}s - \overline{\partial}_W\overline{Z}s - \overline{\partial}_{\overline{Z}W}s = 0,\]

for any \(f \in C^\infty(M) \otimes \mathbb{C}, s \in \Gamma^\infty(E)\) and \(Z, W \in \Gamma^\infty(T^{1,0}(M))\).

The condition \((ii)\) of the definition means that \(0,2\)-component of the curvature operator \(R(E)\) is vanishing when \(E\) admits a connection \(D\) whose \((0,1)\)-part is the operator \(\overline{\partial}_b\) as in the following Lemma.

**Lemma 7.** Let \((M, T^{1,0}(M), \theta)\) be a strictly pseudoconvex CR \((2n+1)\)-manifold and \((E, \overline{\partial}_b)\) a CR-holomorphic vector bundle over \(M\). Let \(h = \langle \cdot, \cdot \rangle\) be a Hermitian structure in \(E\). Then there exists a unique (Tanaka) connection \(D\) in \(E\) such that

\[(i)\]
\[D\overline{Z}s = (\overline{\partial}_b s)\overline{Z},\]

\[(ii)\]
\[Z\langle s_1, s_2 \rangle = \langle D_Zs_1, s_2 \rangle + \langle s_1, D_Zs_2 \rangle,\]

\[(iii)\] The \((0,2)\)-component of the curvature operator \(\Theta(E)\) is vanishing. Here \(\Theta(E) := D^2s\).

**Definition 9.** Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold. A CR-holomorphic vector bundle \((E, \overline{\partial}_b)\) over \(M\) is a basic transverse holomorphic vector bundle over \((M, T^{1,0}(M))\) if there exists an open cover \(\{U_\alpha\}\) of \(M\) and the trivializing frames on \(U_\alpha\), such that its transition functions are matrix-valued basic CR functions. The trivializing frames is called the basic transverse holomorphic frame.

**Example 2.** Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold. Then, with respect to the trivializing frames

\[\left\{ \frac{\partial}{\partial z_j} : Z_j = \left(\frac{\partial}{\partial z_j} + \sqrt{-1}h_{j\bar{k}}\frac{\partial}{\partial z_k}\right), \quad j = 1, 2, \ldots, n \right\}\]

the transition functions of such frames are basic transverse holomorphic functions, that is \(h\) is basic. Thus \(T^{1,0}(M)\) is a basic transverse holomorphic vector bundle. Moreover, the canonical (determinant) bundle \(K_T^M\) of \(T^{1,0}(M)\) is a basic transverse holomorphic line bundle whose transition functions are given by \(t_{\alpha\beta} = \det(\partial z_j^\alpha / \partial z_j^\beta)\) on \(U_\alpha \cap U_\beta\), where \((x, z_1^\alpha, \ldots, z_n^\alpha)\) is the normal coordinate on \(U_\alpha\).

**Definition 10.** (i) Let \((M, \eta, \xi, \Phi, g)\) be a Sasakian \((2n+1)\)-manifold and \(L\) be a basic transverse holomorphic bundle over \(M\). A basic transverse holomorphic section \(s\) of \(L\) is a collection \(\{s_\alpha\}\) of CR-holomorphic maps \(s_\alpha : U_\alpha \rightarrow \mathbb{C}\) satisfying the transformation rule \(s_\alpha = t_{\alpha\beta}s_\beta\) on \(U_\alpha \cap U_\beta\). The transition function \(t_{\alpha\beta}\) is basic. A basic Hermitian metric \(h\) on \(L\) is a collection \(\{h_\alpha\}\) of smooth positive functions \(h_\alpha : U_\alpha \rightarrow \mathbb{R}\) satisfying the transformation rule

\[h_\alpha = |t_{\beta\alpha}|^2 h_\beta\]
on $U_\alpha \cap U_\beta$. Given a basic transverse holomorphic section $s$ and a Hermitian metric $h$, we can define the pointwise norm squared of $s$ with respect to $h$ by

$$|s|^2_h = h_\alpha s_\alpha \overline{s_\alpha}$$

on $U_\alpha$. The reader can check that $|s|^2_h$ is a well-defined function on $M$.

(ii) A Hermitian metric is called a basic Hermitian metric if $h_\alpha$ is basic. It always exists if $L$ is a basic transverse holomorphic line bundle.

(iii) We define the curvature $R^T_h$ of a basic Hermitian metric $h$ on $L$ to be the basic closed $(1,1)$-form on $M$ given by

$$R^T_h = -\frac{\sqrt{-1}}{2\pi} \partial B \overline{\partial B} \log h_\alpha$$

on $U_\alpha$. This is well-defined. The basic first Chern class $c^B_1(L)$ of $L$ to be the cohomology class $[R^T_h]_B \in H^1_{-1}(M, \mathbb{R})$. Since any two basic Hermitian metrics $h$, $h'$ on $L$ are related by $h' = e^{-\phi}h$ for some smooth basic function $\phi$, we see that $R^T_h = R^T_{h'} + \frac{\sqrt{-1}}{2\pi} \partial B \overline{\partial B} \phi$ and hence $c^B_1(L)$ is well-defined, independent of choice of basic Hermitian metric $h$. We say that $(L, h)$ is positive if the curvature $R^T_h$ is positive definite at every $p \in M$.

Example 3. Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n + 1)$-manifold. If $g^T$ is a transverse Kähler metric on $M$, then $h_\alpha = \det((g_\alpha^T)^T)$ on $U_\alpha$ defines a basic Hermitian metric on the canonical bundle $K^T_M$. The inverse $(K^T_M)^{-1}$ of $K^T_M$ is sometimes called the anti-canonical bundle. Its basic first Chern class $c^B_1((K^T_M)^{-1})$ is called the basic first Chern class of $M$ and is often denoted by $c^B_1(M)$. Then it follows from the previous result that $c_1^B(M) = [\text{Ric}^T(\omega)]_B$ for any transverse Kähler metric $\omega$ on a Sasakian manifold $M$.

Definition 11. (i) Let $(L, h)$ be a basic transverse holomorphic line bundle over a Sasakian manifold $(M, \eta, \xi, \Phi, g)$ with the basic Hermitian metric $h$. We say that $L$ is very ample if for any ordered basis $\underline{s} = (s_0, \cdots, s_N)$ of $H^0_B(M, L)$, the map

$$i_\underline{s} : M \to \mathbb{CP}^N$$

given by

$$i_\underline{s}(x) = [s_0(x), \cdots, s_N(x)]$$

is well-defined and an embedding which is $S^1$-equivariant with respect to the weighted $\mathbb{C}^*$ action in $\mathbb{C}^{N+1}$ as long as not all the $s_i(x)$ vanish. We say that $L$ is ample if there exists a positive integer $m_0$ such that $L^m$ is very ample for all $m \geq m_0$.

(ii) $L$ is a semi-ample basic transverse holomorphic line bundle if there exists a basic Hermitian metric $h$ on $L$ such that $R^T_h$ is a nonnegative $(1,1)$-form. In fact, there exists a $S^1$-equivariant foliation base point free holomorphic map

$$\Psi : M \to (\mathbb{CP}^N, \omega_{FS})$$

defined by the basic transverse holomorphic section $\{s_0, s_1, \cdots, s_N\}$ of $H^0_B(M, L^m)$ which is $S^1$-equivariant with respect to the weighted $\mathbb{C}^*$ action. Here $N = \dim H^0_B(M, L^m) - 1$ for a large positive integer $m$ and

$$0 \leq \frac{1}{m} \Psi^*(\omega_{FS}) = \hat{\omega}_\infty \in c^B_1(L).$$

There is a Sasakian analogue of Kodaira embedding theorem on a compact quasi-regular Sasakian $(2n+1)$-manifold due to [RT], [HLM]:

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Proposition 8. Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian \((2n+1)\)-manifold and \((L, h)\) be a basic transverse holomorphic line bundle over \(M\) with the basic Hermitian metric \(h\). Then \(L\) is ample if and only if \(L\) is positive.

Definition 12. (i) First we say that a subset \(V\) of a (quasi-regular) Sasakian \((2n+1)\)-manifold \((M, \eta, \xi, \Phi, g)\) an invariant (Sasakian) submanifold (with or without singularities) of dimension \(2n-1\) if \(\xi\) is tangent to \(V\) and \(\Phi TV \subset TV\) at all points of \(V\) and is locally given as the zero set \(\{ f = 0 \}\) of a locally defined basic CR holomorphic function \(f\). In general, \(V\) may not be a submanifold. Denote by \(V^{\text{reg}}\) the set of points \(p \in V\) for which \(V\) is a submanifold of \(M\) near \(p\). We say that \(V\) is irreducible if \(V^{\text{reg}}\) is connected. A transverse divisor \(D^{T}\) on \(M\) is a formal finite sum \(\sum a_i V_i\) where \(a_i \in \mathbb{Z}\) and each \(V_i\) is an irreducible invariant submanifold of dimension \(2n-1\). We say that \(D^{T}\) is effective if the \(a_i\) are all nonnegative. The support of \(D^{T}\) is the union of the \(V_i\) for each \(i\) with \(a_i \neq 0\).

(ii) Given a transverse divisor \(D^{T}\), we define an associated line bundle as follows. Suppose that \(D^{T}\) is given by local defining basic functions \(f_\alpha\) (vanishing on \(D^{T}\)) to order 1 over an open cover \(U_\alpha\). Define transition functions \(f_\alpha = t_{\alpha \beta} f_\beta\) on \(U_\alpha \cap U_\beta\). These are basic CR holomorphic and nonvanishing in \(U_\alpha \cap U_\beta\), and satisfy

\[
t_{\alpha \beta} t_{\beta \gamma} = 1; \quad t_{\alpha \beta} t_{\beta \gamma} = t_{\alpha \gamma}.
\]

Write \([D^{T}]\) for the associated basic line bundle, which is well-defined independent of choice of local defining functions.

(iii) One can define

\[
L_M \cdot V = \int_V R^T_h \wedge \eta
\]
for all invariant Sasakian 3-manifold \(V\) in \(M\). Here \(h\) is a basic Hermitian metric on the basic line bundle \(L_M\). From (ii), for a compact Sasakian 5-manifold \(M\), a transverse divisor \(D^{T}\) defines an element of \(H^{1,1}_{\partial B}(M, \mathbb{R})\) by \(D^{T} \rightarrow [R^T_h] \in H^{1,1}_{\partial B}(M, \mathbb{R})\) for a basic Hermitian metric on the associate basic line bundle \([D^{T}]\), and we define

\[
\alpha \cdot \beta = \int_M \alpha \wedge \beta \wedge \eta
\]
for \(\alpha, \beta \in H^{1,1}_{\partial B}(M, \mathbb{R})\). Then for an invariant 3-manifold \(V\) which is both a foliation curve and a transverse divisor, then \(V \cdot V\) is well-defined and we may write \(V^2\) instead of \(V \cdot V\).

Remark 2. (Gei) Note that the Sasakian 3-manifold \(V\) is either canonical, anticanonical or null. \(V\) is up to finite quotient a regular Sasakian 3-manifold, i.e., a circle bundle over a Riemann surface of positive genus. In the positive case, \(V\) is covered by \(S^3\) and its Sasakian structure is a deformation of a standard Sasakian structure.

Definition 13. (i) We say that a basic line bundle \(L\) is nef if \(L \cdot V \geq 0\) for any invariant Sasakian 3-manifold \(V\) in \(M\). In particular if \(M\) is quasi-regular, then \(V\) is the \(S^1\)-oribundle over the curve \(C\) in \(Z\) so that

\[
L_Z \cdot C = \int_C R_{h_Z} \geq 0.
\]
Here $c^B_1(L_M) = \pi^* c^\text{orb}_1(L_Z)$ and $h_Z$ is the Hermitian metric in the corresponding line bundle $L_Z$. Define
\[(A.2) \quad C^B_M = \{ [\alpha]_B \in H^1_B(M, \mathbb{R}) \mid \exists \omega > 0 \text{ such that } [\omega]_B = [\alpha]_B \}. \]

Then we can also define a class $[\alpha]_B$ called nef class if $[\alpha]_B \in C^B_M$ and a class $[\alpha]_B$ called big if
\[ \int_M \alpha^n \wedge \eta > 0. \]

(ii) If the Sasakian $(2n+1)$-manifold $(M, \eta, \xi, \Phi, g)$ has the canonical basic line bundle $K^T_M$ nef, then we say that $M$ is a smooth transverse minimal model. If $M$ has $K^T_M$ big, then we say that $M$ is of general type.

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