Towards Relativistic Atomic Physics. II. Collective and Relative Relativistic Variables for a System of Charged Particles plus the Electro-Magnetic Field

David Alba

Dipartimento di Fisica
Universita' di Firenze
Polo Scientifico, via Sansone 1
50019 Sesto Fiorentino, Italy
E-mail ALBA@FI.INFN.IT

Horace W. Crater

The University of Tennessee Space Institute
Tullahoma, TN 37388 USA
E-mail: hcrater@utsi.edu

Luca Lusanna

Sezione INFN di Firenze
Polo Scientifico
Via Sansone 1
50019 Sesto Fiorentino (FI), Italy
E-mail: lusanna@fi.infn.it

Abstract

In this second paper we complete the classical description of an isolated system of "charged positive-energy particles, with Grassmann-valued electric charges and mutual Coulomb interaction, plus a transverse electro-magnetic field" in the rest-frame instant form of dynamics.

In particular we show how to determine a collective variable associated with the internal 3-center of mass on the instantaneous 3-spaces, to be eliminated with the constraints $\vec{K}_{(int)} \approx 0$. Here $\vec{K}_{(int)}$ is the Lorentz boost generator in the unfaithful internal realization of the Poincare' group and its vanishing is the gauge fixing to the rest-frame conditions $\vec{P}_{(int)} \approx 0$. We show how to find this collective variable for the following isolated systems: a) charged particles with a Coulomb plus Darwin mutual interaction; b) transverse radiation field; c) charged particles with a mutual Coulomb interaction plus a transverse electro-magnetic field.

Then we define the Dixon multipolar expansion for the open particle subsystem. We also define the relativistic electric dipole approximation of atomic physics in the rest-frame instant form and we find the a possible relativistic generalization of the electric dipole representation.
November 6, 2008
I. INTRODUCTION

In Ref. [1] we gave the description of the isolated system "N charged scalar positive-energy particles with Coulomb mutual interaction plus the electro-magnetic field in the radiation gauge" in the rest-frame instant form of dynamics. Grassmann-valued electric charges were used to regularize the Coulomb self-energies. In that paper there was the determination of the regularized Lienard-Wiechert electro-magnetic fields in the absence of an incoming radiation field and of the complete potential acting among Coulomb-dressed charged particles. This effective potential turned out to be the Coulomb potential plus the full relativistic expression of the Darwin potential, which at the lowest order in $1/c^2$ reduces at the known form of the Darwin potential, till now obtainable only from the Bethe-Salpeter approach to relativistic bound states in QFT. This full Darwin plus Coulomb potential can be regarded as the classical analogue of the complete transverse as well longitudinal effects of the single photon exchange.

In Ref. [2], quoted as I with its formulas denoted by (I.2.5), we extended the approach of Ref. [1]. In particular in I we relaxed the assumption of no incoming radiation field and we were able to show that, at least at the classical level, there exists a canonical transformation from an initial situation with a free IN transverse radiation field and decoupled particles with a Coulomb plus Darwin interaction, into an interpolating non-radiation transverse electromagnetic field interacting with charged particles having only a Coulomb mutual interaction. At later times we can again canonically transform the system to a free OUT radiation field plus charged particles with Coulomb plus Darwin mutual interaction.

These results can be obtained within the formalism of the rest-frame instant form of dynamics starting from the approach based on parametrized Minkowski theories of relativistic mechanics [3]. The first part of paper I contains a complete updated review of this formalism with a resolution of all the ambiguities connected with the relativistic center-of-mass notion. As shown in paper I, in the inertial rest frame centered on the Fokker-Pryce covariant non-canonical 4-center of inertia $Y^\mu(\tau)$ of the isolated system, the instantaneous 3-spaces are the Wigner space-like hyper-planes orthogonal to the conserved 4-momentum $P^\mu = Mc h^\mu = Mc (\sqrt{1+h^2}; \vec{\hbar})$ of the isolated system, whose invariant mass is $M$. On the Wigner hyper-plane, described by the embedding $z^\mu_W(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon^\mu_\tau(\vec{\hbar}) \sigma^r$ the particles are identified by (Wigner spin-1) 3-vectors $\vec{\eta}_i(\tau)$ determined by the intersection of the particle world-line $x^\mu_i(\tau) = Y^\mu(\tau) + \epsilon^\mu_\tau(\vec{\hbar}) \eta^r_i(\tau)$ with the Wigner hyperplane.

In the rest-frame instant form every isolated system is described as an external decoupled canonical non-covariant 4-center of mass $\tilde{x}^\mu(\tau) = (\tilde{x}_o^\tau; \vec{z}/Mc - \vec{P}/Mc \tilde{x}_o^\tau(\vec{\hbar}))$ parametrized (like also $Y^\mu(\tau)$) in terms of $\tau$ and of the 6 non-evolving Jacobi data $\vec{z}$, $\vec{\hbar}$. This decoupled non-covariant free "particle" carries the internal mass $M$ and the rest spin $\vec{S}$ of the isolated system. In the case of charged particles plus the electro-magnetic field they are

1 As shown in I, $(\tau, \vec{\sigma})$ are observer-dependent 4-radar coordinates with $T = \tau/c$ being the Lorentz-scalar rest time and $\epsilon^\mu_\tau(\vec{\hbar}) = (-h_r; \delta_i - \frac{h_i h_r}{1+h^2})$ are 3 space-like 4-vectors forming together with $h^\mu = \epsilon^\mu_\tau(\vec{\hbar})$ the columns of the standard Wigner boost sending $P^\mu$ to its rest-frame form $Mc (1; 0)$.

2 $\vec{x}_{NW} = \vec{z}/Mc$ is the classical Newton-Wigner position of the 3-center of mass and we have $\tilde{x}^\mu(\tau) = (\tilde{x}_o^\tau; \vec{x}_{NW} + \vec{P}/Mc \tilde{x}_o^\tau(\vec{\hbar}))$. 

3
functions of the 3-coordinates \( \vec{r}_i(\tau) \), their conjugate momenta \( \vec{p}_i(\tau) \) and of the transverse electro-magnetic fields \( \vec{A}_\perp(\tau, \vec{\sigma}) \), \( \vec{\pi}_\perp(\tau, \vec{\sigma}) = \vec{E}_\perp(\tau, \vec{\sigma}) \). There is an external realization of the Poincare’ group, whose generators depend upon \( \bar{z}, \bar{h}, M \) and \( \vec{S} \).

Inside the Wigner hyper-plane there is an internal realization of the Poincare’ group, whose generators \( P_{(int)} = Mc, \vec{P}_{(int)}, J_{(int)} = \vec{S}, \vec{K}_{(int)} \), are evaluated from the energy-momentum tensor of the isolated system obtained from the action of the parametrized Minkowski theory describing the isolated system in arbitrary non-inertial frames. This realization is unfaithful because the rest-frame instant form centered on the Fokker-Pryce inertial observer implies (see paper I) the following second class constraints

\[
\vec{P}_{(int)} \approx 0, \quad \vec{K}_{(int)} \approx 0. \tag{1.1}
\]

While the first 3 constraints are the rest-frame conditions (each Wigner hyper-plane is a rest frame), the other 3 constraints eliminate the internal 3-center of mass avoiding a double counting of the center of mass. Therefore Eqs.(1.1) imply that the dynamics inside the Wigner hyper-planes depends only on relative variables satisfying Hamilton equations having \( Mc \) as Hamiltonian. All the interaction potentials appear only in \( Mc \) and \( K_{(int)} \) and not in \( J_{(int)} = \vec{S} \), since we are in an instant form of dynamics.

This complete understanding of the relativistic kinematics of the rest-frame instant form of relativistic mechanics was obtained only recently in Ref.[4] and in paper I. The important point is to have a Lagrangian description of the isolated system. This will allow the determination of its energy-momentum tensor. Only in this way can we find the interaction-dependent internal generators \( Mc \) and \( K_{(int)} \). In most of the other approaches to relativistic mechanics the Lagrangian formulation is unknown, so that we can find only \( Mc \) but not \( K_{(int)} \). Therefore we lack the interaction-dependent equations which eliminate the internal 3-center mass, the gauge variable conjugate to the rest-frame conditions and we are able to reconstruct neither the orbits nor the particle world-lines in the rest frame.

In Ref.[4], even in absence of a Lagrangian, we were able to guess the form of \( K_{(int)} \), associated with a given class of invariant masses for a two-body system, so that the internal generators satisfy the Poincare’ algebra in the rest frame. In terms of the 3-vectors \( \vec{r}_i(\tau), \vec{\kappa}_i(\tau), i = 1, 2 \), these generators are (\( \vec{p} = \vec{p}_1 - \vec{p}_2 \))\(^3\)

\[
\begin{align*}
Mc &= \sqrt{m_1^2 c^2 + \vec{r}_1^2(\tau) + \Phi(\vec{p}^2(\tau))} + \sqrt{m_2^2 c^2 + \vec{r}_2^2(\tau) + \Phi(\vec{p}^2(\tau))}, \\
\vec{P}_{(int)} &= \vec{r}_1(\tau) + \vec{r}_2(\tau) \approx 0, \\
\vec{J}_{(int)} &= \vec{S} = \vec{p}_1(\tau) \times \vec{r}_1(\tau) + \vec{p}_2(\tau) \times \vec{r}_2(\tau), \\
\vec{K}_{(int)} &= -\vec{p}_1(\tau) \sqrt{m_1^2 c^2 + \vec{r}_1^2(\tau) + \Phi(\vec{p}^2(\tau))} - \vec{p}_2(\tau) \sqrt{m_2^2 c^2 + \vec{r}_2^2(\tau) + \Phi(\vec{p}^2(\tau))} \approx 0,
\end{align*}
\tag{1.2}
\]

\(^3\) Since we have \( \{\eta_i(\tau), \kappa_{js}(\tau)\} = \delta_{ij} \delta_{js} \), we use the following vector notation: \( \vec{r}_i(\tau) = \left( \eta_i^j(\tau) \right), \vec{\kappa}_i(\tau) = \left( \kappa_{ir}(\tau) = -\kappa_{ir}^j(\tau) \right). \)
where $\Phi(\vec{\rho}^2(\tau))$ is an arbitrary potential 4.

If $\vec{\pi}(\tau) = \frac{1}{2} [\vec{\kappa}_1(\tau) - \vec{\kappa}_2(\tau)]$ is the momentum conjugate to $\vec{\rho}(\tau)$, we have

$$M c \approx M_{(\text{int})} c = \sqrt{m_1^2 c^2 + \vec{\pi}^2(\tau) + \Phi(\vec{\rho}^2(\tau))} + \sqrt{m_2^2 c^2 + \vec{\pi}^2(\tau) + \Phi(\vec{\rho}^2(\tau))}.$$  \hspace{1cm} (1.3)

In Ref.[4], after having solved the equations $\vec{K}_{(\text{int})} \approx 0$ and eliminated the internal 3-center of mass, we obtained the following orbit and world-line reconstruction

$$\vec{\eta}_i(\tau) \approx \frac{1}{2} \left( (-1)^{i+1} - \frac{m_1^2 - m_2^2}{M_{(\text{int})}^2 [\vec{\rho}^2(\tau), \vec{\pi}^2(\tau)]} \right) \vec{\rho}(\tau),$$

$$x_\mu^i(\tau) \approx Y_\mu(\tau) + \frac{1}{2} \left( (-1)^{i+1} - \frac{m_1^2 - m_2^2}{M_{(\text{int})}^2 [\vec{\rho}^2(\tau), \vec{\pi}^2(\tau)]} \right) \epsilon_\tau^\mu (\vec{h}) \rho^r(\tau).$$ \hspace{1cm} (1.4)

The relative variables $\vec{\rho}(\tau)$ and $\vec{\pi}(\tau)$ are to be found as solutions of the Hamilton equations with Hamiltonian $M_{(\text{int})} c$.

In this paper we complete the classical description of the system of charged particles plus a transverse electro-magnetic field.

In Section II we define the internal collective and relative variables for the isolated system of $N$ charged particles interacting through the Coulomb plus Darwin potential of Ref.[1]. Then we will perform the elimination of the internal 3-center of mass and the orbit reconstruction in the $N=2$ case.

In Section III we define the internal collective and relative variables for the free radiation field. Here we rely on the results of Refs. [5] and [6], where for the first time there was the study of collective and relative variables for the Klein-Gordon field in the massive and massless cases respectively. The adaptation of this results to the rest-frame instant form is done in Appendix A. This allows one to eliminate the internal 3-center of mass of the transverse radiation field.

Then in Section IV we will use the results of Sections II and III and the canonical transformation of paper I to find the collective and relative variables of the isolated system "$N$ charged particles with mutual Coulomb interaction plus the transverse electro-magnetic field" defined in paper I and to perform the elimination of the internal 3-center of mass also in this case.

In Section V we review the multipolar expansion of the energy-momentum tensor of the particle open subsystem and how to get its (non-canonical) pole-dipole approximation, in which the subsystem is viewed as an effective world-line (the 4-center of motion) carrying the effective spin of the subsystem. This pole-dipole structure must not be confused with the

---

4 The internal boost $\vec{K}_{(\text{int})}$ satisfying $\{\mathcal{P}^i, \mathcal{K}^j\} = \delta_{ij} M$ in the case of the pure Coulomb interaction $Mc = \sqrt{m_1^2 c^2 + \vec{\kappa}^2} + \sqrt{m_2^2 c^2 + \vec{\kappa}^2} + \frac{Q_1 Q_2}{4 \pi (\vec{h}_1 - \vec{h}_2)}$ are not known. Only in the case of Coulomb plus Darwin potential, see next Section, is the internal boost known.
pole-dipole structure carried by the non-covariant (Newton-Wigner) external center of mass of the isolated system “charged particles with mutual Coulomb interaction plus transverse electro-magnetic field”.

While Section VI is devoted to finding the relativistic version of the electric dipole approximation of atomic physics [7, 8, 9], in Section VII we face the problem of defining a relativistic extension of the electric dipole representation of the isolated system. We study various point canonical transformations suggested by standard atomic physics and we find that all of them generate singular terms when the electro-magnetic field is considered dynamical and not an external prescribed field: they are induced by the dipole approximation. We then propose a relativistic representation in which the singular terms are replaced by contact interactions among the particles: the price for this regularization is that in the semi-relativistic limit it is different from the standard electric dipole representation. In Appendix B there is the identification of the generating functions of these point canonical transformations by the introduction of relativistic Lagrangians for the rest-frame instant form description of the isolated system.

In the final Section there are some concluding remarks and an introduction to the subsequent paper III [10].
II. RELATIVISTIC MECHANICS: THE ISOLATED SYSTEM OF N CHARGED PARTICLES INTERACTING THROUGH THE COULOMB PLUS DARWIN POTENTIAL

Let us consider an isolated system of N charged positive-energy scalar particles either free or interacting through the Coulomb + Darwin potential (I-4.5) (see I and Ref.[1]).

On the Wigner hyper-planes \( \Sigma_\tau \) each particle is described by two canonically conjugate Wigner spin-1 3-vectors: the 3-position \( \vec{\eta}_i(\tau) \) and the 3-momentum \( \vec{\kappa}_i(\tau), i = 1, \ldots, N \), restricted by the rest-frame conditions \( \vec{P}_i(\text{int}) \approx 0 \) (vanishing of the internal 3-momentum) and by the conditions eliminating the internal 3-center of mass (vanishing of the internal Lorentz boosts \( \vec{K}_{i(\text{int})} \approx 0 \)).

Therefore we must introduce collective and relative variables on the Wigner hyper-planes and eliminate the collective ones. We consider mostly the \( N = 2 \) case, but everything is well defined for arbitrary \( N \). There are various possibilities and we must find the one which allows one to make the elimination explicitly.

A. Collective and Relative Variables for N Particles

In previous papers [11] the problem of replacing the 3-coordinates \( \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau) \) inside a Wigner hyper-plane with internal collective and relative canonical variables in the rest-frame instant form was solved in two different ways:

a) We may introduce naive collective variables \( \vec{\eta} + = \frac{1}{N} \sum_{i=1}^{N} \vec{\eta}_i \) (independent of the particle masses), \( \vec{\kappa} + = \vec{P}_i(\text{int}) = \sum_{i=1}^{N} \vec{\kappa}_i \approx 0 \) and then completing them with either naive relative variables \( \vec{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \vec{\eta}_i, \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_{ai} \vec{\kappa}_i, a = 1, \ldots, N - 1 \), or with the relative variables in the so-called spin bases [11, 12] \(^5\). This naive canonical basis is obtained with a linear canonical transformation point both in the positions and in the momenta.

b) We may find the canonical basis whose collective variables are the internal 3-center of mass \( \vec{q} + \) \(^6\) and \( \vec{K} + = \vec{P}_i(\text{int}) = \sum_{i=1}^{N} \vec{K}_i \approx 0 \). To these collective variables are then associated relative variables \( \vec{\rho}_{qa}, \vec{\pi}_{qa}, a = 1, \ldots, N - 1 \). However this non-linear canonical transformation depends upon the interactions present among the particles (through the internal Poincare’ generators), so that it is known only for free particles [4, 11]: even in this case it is point only in the momenta. The advantage of this canonical transformation would be to allow one to write the invariant mass in the form \( M_c = \sqrt{M^2 c^2 + \vec{K}_+^2} \approx M \) \(^7\), explicitly showing that in the rest-frame the internal mass depends only on relative variables.

---

\(^5\) Both sets can be used to find the expression of the Dixon multipoles (see Section V) for a two-particle open subsystem in terms of canonical c.o.m and relative canonical variables.

\(^6\) Due to the rest-frame condition \( \vec{P}_i(\text{int}) \approx 0 \), we have \( \vec{q} + \approx \vec{R} + \approx \vec{y} + \), where \( \vec{q} + \) is the internal canonical 3-center of mass (the internal Newton-Wigner position), \( \vec{y} + \) is the internal Fokker-Pryce 3-center of inertia and \( \vec{R} + \) is the internal Möller 3-center of energy [4]. These global quantities are defined in terms of the internal Poincare’ generators [11].

\(^7\) For two free particles we have \( M_c = \sqrt{m_1^2 c^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 c^2 + \vec{\pi}_q^2} = \sqrt{M^2 c^2 + \vec{K}_+^2} \approx M_c = \sqrt{m_1^2 c^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 c^2 + \vec{\pi}_q^2} \).
Since it is more convenient to use the naive linear canonical transformation we will use the following collective and relative variables which, written in terms of the masses of the particles, make it easier to evaluate the non-relativistic limit \( m = \sum_{i=1}^{N} m_i \)

\[
\vec{\eta}_+ = \sum_{i=1}^{N} \frac{m_i}{m} \vec{\eta}_i, \quad \vec{\kappa}_+ = \vec{P}_{(int)} = \sum_{i=1}^{N} \vec{\kappa}_i, \\
\vec{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \vec{\eta}_i, \quad \vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Gamma_{ai} \vec{\kappa}_i, \quad a = 1, \ldots, N - 1, \\
\vec{\eta}_i = \vec{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \Gamma_{ai} \vec{\rho}_a, \\
\vec{\kappa}_i = \frac{m_i}{m} \vec{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a,
\]

with the following canonicity conditions \(^8\)

\[
\sum_{i=1}^{N} \gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N}, \\
\Gamma_{ai} = \gamma_{ai} - \sum_{k=1}^{N} \frac{m_k}{m} \gamma_{ak}, \quad \gamma_{ai} = \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^{N} \Gamma_{ak}, \\
\sum_{i=1}^{N} \frac{m_i}{m} \Gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \Gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \Gamma_{aj} = \delta_{ij} - \frac{m_i}{m}.
\]

(2.2)

For \( N = 2 \) we have \( \gamma_{11} = -\gamma_{12} = \frac{1}{\sqrt{2}}, \Gamma_{11} = \sqrt{2} \frac{m_2}{m}, \Gamma_{12} = -\sqrt{2} \frac{m_1}{m}. \)

Therefore in the two-body case, by introducing the notation \( \vec{\eta}_{12} = \vec{\eta}_+, \vec{\kappa}_{12} = \vec{\kappa}_+ = \vec{P}_{(int)} \), we have the following collective and relative variables

\[
\vec{\eta}_{12} = \frac{m_1}{m} \vec{\eta}_1 + \frac{m_2}{m} \vec{\eta}_2, \quad \vec{\rho}_{12} = \vec{\eta}_1 - \vec{\eta}_2, \\
\vec{\kappa}_{12} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, \quad \vec{\pi}_{12} = \frac{m_2}{m} \vec{\kappa}_1 - \frac{m_1}{m} \vec{\kappa}_2, \\
\vec{\eta}_1 = \vec{\eta}_{12} + \frac{m_2}{m} \vec{\rho}_{12}, \quad \vec{\eta}_2 = \vec{\eta}_{12} - \frac{m_1}{m} \vec{\rho}_{12}, \\
\vec{\kappa}_1 = \frac{m_1}{m} \vec{\kappa}_{12} + \vec{\pi}_{12}, \quad \vec{\kappa}_2 = \frac{m_2}{m} \vec{\kappa}_{12} - \vec{\pi}_{12}.
\]

(2.3)

\(^8\) Eqs.(2.1) describe a family of canonical transformations, because the \( \gamma_{ai} \)'s depend on \( \frac{1}{2}(N - 1)(N - 2) \) free independent parameters.
We use the notation \( m = m_1 + m_2 = m_1 (1 + \frac{m_2}{m_1}) , \mu = \frac{m_1 m_2}{m} = m_2 (1 + \frac{m_1}{m_2})^{-1} \). For \( m_2 >> m_1 \) we have \( \vec{\eta}_2 \approx \vec{\eta}_1 - \frac{m_1}{m_2} \vec{\rho}_2, \vec{\eta}_1 \approx \vec{\eta}_1 + \vec{\rho}_2 \).

The collective variable \( \vec{\eta}_{12}(\tau) \) has to be determined in terms of \( \vec{\rho}_{12}(\tau) \) and \( \vec{\pi}_{12}(\tau) \) by means of the gauge fixings \( \vec{K}_{(int)} \overset{\text{def}}{=} -M \vec{R}_+ \approx 0 \).

B. Two Charged Particles interacting through the Coulomb plus Darwin Potential

From Ref.[1] we get the following internal Poincare’ generators for \( N = 2 \) \(^9\)

\[
\mathcal{E}_{(int)} = Mc^2 = c \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2} + \frac{Q_1 Q_2}{4\pi |\vec{\eta}_1 - \vec{\eta}_2|} + V_{DARWIN}(\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau); \vec{\kappa}_i(\tau)),
\]

\[
\vec{P}_{(int)} = \tilde{\kappa}_1 + \tilde{\kappa}_2 \approx 0,
\]

\[
\vec{J}_{(int)} = \sum_{i=1}^{2} \vec{\eta}_i \times \vec{\kappa}_i,
\]

\[
\vec{K}_{(int)} = -\vec{\eta}_{12} \left[ \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2} + \right.
\]

\[\left. + \frac{Q_1 Q_2}{c} \left( \tilde{\kappa}_1 \cdot \left[ \frac{1}{2} \vec{\partial}_{\tilde{\rho}_{12}} \vec{K}_{12}(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\rho}_{12}) - 2 \vec{A}_{1,LS2}(\tilde{\kappa}_2, \tilde{\rho}_{12}) \right] \right) + \right.
\]

\[\left. + \tilde{\kappa}_2 \cdot \left[ \frac{1}{2} \vec{\partial}_{\tilde{\rho}_{12}} \vec{K}_{12}(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\rho}_{12}) - 2 \vec{A}_{1,LS1}(\tilde{\kappa}_1, \tilde{\rho}_{12}) \right] \right) - \]

\[\left. - \tilde{\rho}_{12} \left( \frac{m_2}{m} \sqrt{m_1^2 c^2 + \tilde{\kappa}_2^2} - \frac{m_1}{m} \sqrt{m_2^2 c^2 + \tilde{\kappa}_2^2} + \right. \right.
\]

\[\left. + \frac{Q_1 Q_2}{c} \left[ \frac{m_2 \tilde{\kappa}_1 \cdot \left[ \frac{1}{2} \vec{\partial}_{\tilde{\rho}_{12}} \vec{K}_{12}(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\rho}_{12}) - 2 \vec{A}_{1,LS2}(\tilde{\kappa}_2, \tilde{\rho}_{12}) \right]}{2 m \sqrt{m_1^2 c^2 + \tilde{\kappa}_1^2}} \right] - \right.
\]

\[\left. - \frac{m_1 \tilde{\kappa}_2 \cdot \left[ \frac{1}{2} \vec{\partial}_{\tilde{\rho}_{12}} \vec{K}_{12}(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\rho}_{12}) - 2 \vec{A}_{1,LS1}(\tilde{\kappa}_1, \tilde{\rho}_{12}) \right]}{2 m \sqrt{m_2^2 c^2 + \tilde{\kappa}_2^2}} \right) \right].
\]

\(^9\) \( V_{DARWIN} = V_D(\vec{\eta}_1 - \vec{\eta}_2, \vec{\kappa}_1, \vec{\kappa}_2) \) is given in Eq.(6.19) of Ref.[1] and in Eq.(I.4.5). \( \vec{J}_{(int)} \) is given in Eq.(6.40) and \( \vec{K}_{(int)} \) in Eq.(6.46) of Ref.[1].
\[
- \frac{1}{2c} Q_1 Q_2 \left( \sqrt{m_2^2 c^2 + \vec{k}_1 \partial \vec{\eta}_1} + \sqrt{m_2^2 c^2 + \vec{k}_2 \partial \vec{\eta}_2} \right) \vec{K}_{12}(\vec{k}_1, \vec{k}_2, \vec{\rho}_{12}) - \\
- \frac{Q_1 Q_2}{4\pi c} \int d^3\sigma \left( \frac{\vec{\pi}_{\perp S_1}(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}, \vec{k}_1)}{|\vec{\sigma} + \frac{m_2}{m} \vec{\rho}_{12}|} + \frac{\vec{\pi}_{\perp S_2}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, \vec{k}_2)}{|\vec{\sigma} - \frac{m_1}{m} \vec{\rho}_{12}|} \right) - \\
- \frac{Q_1 Q_2}{c} \int d^3\sigma \left[ \vec{\pi}_{\perp S_1}(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}, \vec{k}_1) \cdot \vec{\pi}_{\perp S_2}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, \vec{k}_2) + \\
+ \vec{B}_{S_1}(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}, \vec{k}_1) \cdot \vec{B}_{S_2}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, \vec{k}_2) \right] = \\
def \quad -M c \vec{R}_+ \approx 0, \tag{2.4}
\]

where we have used Eqs. (2.3) in the final expression of the internal boost.

The above forms for the generators were found in Ref.[1] by first eliminating the electromagnetic degrees of freedom by forcing them to coincide with the semiclassical phase space Lienard-Wiechert solution by means of second class constraints (no ingoing radiation field). Then having gone to Dirac brackets with respect to these constraints, we found new (Coulomb-dressed) canonical variables \(\vec{\eta}_i, \vec{\kappa}_i\), for the particles leading finally to a reduced phase space containing only particles with mutual instantaneous action-at-a-distance interactions.

These new canonical variables \(\vec{\eta}_i, \vec{\kappa}_i\), given in Eqs.(5.51) of Ref.[1], have the following expression

\[
\vec{\eta}_i(\tau) = \eta_i(\tau) + (-)^{i+1} \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \mathcal{K}_{12}(\vec{\kappa}_1, \vec{\kappa}_2, \vec{\eta}_1 - \vec{\eta}_2)}{\partial \vec{\kappa}_j}, \\
\vec{\kappa}_i(\tau) = \kappa_i(\tau) - (-)^{i+1} \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \mathcal{K}_{12}(\vec{\kappa}_1, \vec{\kappa}_2, \vec{\eta}_1 - \vec{\eta}_2)}{\partial \vec{\eta}_j}, \\
\mathcal{K}_{12}(\tau) = -\mathcal{K}_{21}(\tau) = \int d^3\sigma \left[ \vec{A}_{\perp S_1} \cdot \vec{\pi}_{\perp S_2} - \vec{\pi}_{\perp S_1} \cdot \vec{A}_{\perp S_2} \right](\tau, \vec{\sigma}). \tag{2.5}
\]

The a-dimensional quantity \(\mathcal{K}_{12}(\tau)\) is given in Eq.(5.35) of Ref. [1] and in Eq.(I-3.5), while the Lienard-Wiechert fields \(\vec{A}_{\perp S_i}, \vec{\pi}_{\perp S_i}\) are given in Eqs. (I-2.50), (I-2.51).

Therefore \(\vec{K}_{\text{int}}(\tau) \approx 0\) can be solved to get \(\vec{\eta}_{12} \approx \vec{\eta}_{12}[\vec{\kappa}_1, \vec{\kappa}_2, \vec{\rho}_{12}]\), so that, by taking into account the rest-frame conditions \(\vec{\kappa}_{12} \approx 0\), from Eq.(2.4) we get

\[
\vec{\eta}_1 \approx \vec{\eta}_{12}[\vec{\rho}_{12}, \vec{\pi}_{12}] + \frac{m_2}{m} \vec{\rho}_{12}, \quad \vec{\eta}_2 \approx \vec{\eta}_{12}[\vec{\rho}_{12}, \vec{\pi}_{12}] - \frac{m_1}{m} \vec{\rho}_{12}, \quad \vec{\kappa}_1 \approx -\vec{\kappa}_2 \approx \vec{\pi}_{12}, \tag{2.6}
\]

with \(\vec{\rho}_{12}(\tau)\) and \(\vec{\pi}_{12}(\tau)\) solutions of the Hamilton equations with \(M c\) as Hamiltonian.

Then the inverse of the canonical transformation (2.5) allows one to get
\[ \vec{\eta}_i(\tau) = \vec{\eta}_i(\tau) - (-)^{i+1} \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \tilde{K}_{12}(\tilde{\vec{\kappa}}_1(\tau), \tilde{\vec{\kappa}}_2(\tau), \tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau))}{\partial \tilde{\kappa}_j} \approx \]

\[ \approx \tilde{\eta}_{12}[^{\mu}_{\rho_12}, \tilde{\pi}_{12}] + (-)^{i+1} \frac{m_i}{m} \tilde{\rho}_{12} - (-)^i \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \tilde{K}_{12}(\tilde{\pi}_{12}(\tau), -\tilde{\pi}_{12}(\tau), \tilde{\rho}_{12}(\tau))}{\partial \tilde{\pi}_{12}}, \]

\[ \tilde{\kappa}_i(\tau) = \tilde{\kappa}_i(\tau) + \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \tilde{K}_{12}(\tilde{\vec{\kappa}}_1(\tau), \tilde{\vec{\kappa}}_2(\tau), \tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau))}{\partial \tilde{\rho}_{12}}, \tag{2.7} \]

with \( \tilde{K}_{12} \) being the same function of its tilded arguments as \( K_{12} \) and its derivatives with respect to \( \tilde{\pi}_{12} \) are taken on the first argument.

Finally, by using Eq.

\[ P^\mu = M c \ h^\mu = M c (\sqrt{1 + \vec{h}^2}; \vec{h}) \] is the total 4-momentum of the 2-body system in an arbitrary inertial frame; \( h^\mu = \epsilon^\mu_r(\vec{h}) \) and \( \epsilon^\mu_r(\vec{h}) \) are the columns of the standard Wigner boost sending \( P^\mu \) to the rest-form \( M c (1; 0) \), we can reconstruct the relativistic orbits and the 4-momenta \( (p^2_i = m_i^2 c^2) \) of the particles.

\[ x^\mu_i(\tau) \approx \dot{Y}^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \left( \tilde{\eta}_{12}[^{\mu}_{\rho_12}, \tilde{\pi}_{12}] + (-)^{i+1} \frac{m_i}{m} \tilde{\rho}_{12} - (-)^i \frac{1}{2} Q_i \sum_{j \neq i} Q_j \frac{\partial \tilde{K}_{12}(\tilde{\pi}_{12}(\tau), -\tilde{\pi}_{12}(\tau), \tilde{\rho}_{12}(\tau))}{\partial \tilde{\pi}_{12}} \right), \]

\[ p^\mu_i(\tau) \approx \sqrt{m_i^2 c^2 + \tilde{\pi}_{12}^2(\tau) \ h^\mu + (-)^{i+1} \epsilon^\mu_r(\vec{h}) \tilde{\pi}_{12}^r(\tau)}. \tag{2.8} \]
III. COLLECTIVE AND RELATIVE VARIABLES FOR THE REST-FRAME DESCRIPTION OF THE RADIATION FIELD IN THE RADIATION GAUGE

Let us now consider the rest-frame instant form of a radiation field in the radiation gauge, which was given in Eqs. (I-2.33)-(I-2.34) of paper I. Given the internal Poincaré’ generators of Eqs. (I-2.35), we must find a canonical basis spanned by collective and relative variables allowing one to eliminate explicitly the internal 3-center of mass as required by the constraints \( \vec{P}_{\text{rad}} \approx 0 \) and \( \vec{K}_{\text{rad}} \approx 0 \). An approach to the determination of such variables has been done in Ref. [5] for the massive Klein-Gordon field and in Ref. [6] for massless fields. The rest-frame instant form of this approach was given in Ref. [13] for the massive Klein-Gordon field. In Appendix A there is review and the adaptation to the rest-frame instant form of the results for the massless scalar field. In this Section we will use these methods for the transverse radiation field.

A. The Radiation Field and its Internal Poincaré’ Generators

As shown in I, Eqs. (I-2.33)-(I-2.34), we have the following rest-frame representation of the radiation field in the radiation gauge on the Wigner hyper-planes

\[
\hat{A}_{\perp \text{rad}}(\tau, \vec{\sigma}) \equiv \int d\vec{k} \sum_{\lambda=1,2} \tilde{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} + a^{*}_\lambda(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right],
\]

\[
\vec{p}_{\perp \text{rad}}(\tau, \vec{\sigma}) = \vec{E}_{\perp \text{rad}}(\tau, \vec{\sigma}) \equiv -\frac{\partial}{\partial \tau} \hat{A}_{\perp \text{rad}}(\tau, \vec{\sigma}) = i \int d\vec{k} \omega(\vec{k}) \sum_{\lambda=1,2} \tilde{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} - a^{*}_\lambda(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right],
\]

\[
\vec{B}_{\text{rad}}(\tau, \vec{\sigma}) = \vec{\partial} \times \hat{A}_{\perp \text{rad}}(\tau, \vec{\sigma}) = i \int d\vec{k} \sum_{\lambda} \vec{k} \times \tilde{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} - a^{*}_\lambda(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right]
\]

\[10 \quad \sigma^A = (\sigma^{\tau}; \sigma^r); \quad k^A = (k^{\tau}; |\vec{k}| = \omega(\vec{k}); k^{r}), \quad k^2 = 0, \quad \text{with } \vec{k} \text{ Wigner spin-1 3-vector and } k^{\tau} \text{ Lorentz scalar; } d\vec{k} = \frac{d^3k}{2\omega(\vec{k})(2\pi)^2}, \quad \Omega(\vec{k}) = 2 \omega(\vec{k}) (2\pi)^3, \quad [d\vec{k}] = [l^{-2}].\]
\[ a_\lambda(\vec{k}) = \int d^3\sigma \, \vec{e}_\lambda(\vec{k}) \cdot \left[ \omega(\vec{k}) \, \vec{A}_\perp(\tau, \vec{\sigma}) - i \, \vec{r}_\perp(\tau, \vec{\sigma}) \right] e^{-i \vec{k} \cdot \vec{\sigma}}, \]

\[ \{ A_\perp^r(\tau, \vec{\sigma}), \pi_\perp^s(\tau, \vec{\sigma}) \} = -c \, P_\perp^{rs}(\vec{\sigma}) \, \delta^3(\vec{\sigma} - \vec{\sigma}_1), \]

\[ \{ a_\lambda(\vec{k}), a_\lambda^*(\vec{k}') \} = -i \, 2 \, \omega(\vec{k}) \, c \, (2\pi)^3 \, \delta_\lambda \lambda' \, \delta^3(\vec{k} - \vec{k}') \overset{\text{def}}{=} -i \, \Omega(\vec{k}) \, c \, \delta_\lambda \lambda' \, \delta^3(\vec{k} - \vec{k}'), \]

\[ \{ a_\lambda(\vec{k}), a_\lambda'(\vec{k}') \} = \{ a_\lambda^*(\vec{k}), a_\lambda^*(\vec{k}') \} = 0, \]

\[ \delta^{ij} = \sum_{\lambda=1,2} \epsilon^i_\lambda(\vec{k}) \, \epsilon^j_\lambda(\vec{k}) + \frac{k^i \, k^j}{|\vec{k}|^2}, \quad \vec{k} \cdot \vec{e}_\lambda(\vec{k}) = 0, \]

\[ \vec{e}_\lambda(\vec{k}) \cdot \vec{e}_\lambda'(\vec{k}) = \delta_\lambda \lambda', \quad \frac{\vec{k}}{|\vec{k}|} \cdot [\vec{e}_1(\vec{k}) \times \vec{e}_2(\vec{k})] = 1, \quad (3.1) \]

and the following expression for the internal Poincare' generators of the radiation field [ \( P_{\text{rad}}^A = (P_{\text{rad}}^r = \mathcal{E}_{\text{rad}} / c = M_{\text{rad}} c; \vec{P}_{\text{rad}}), \mathcal{J}_{\text{rad}}^a = \frac{1}{2} \epsilon^{ars} \mathcal{J}_{\text{rad}}^{rs}, \mathcal{J}_{\text{rad}}^{rs} = \epsilon^{rsu} \mathcal{J}_{\text{rad}}^u \]

\[ M_{\text{rad}} c^2 = \mathcal{E}_{\text{rad}} = c \, P_{\text{rad}}^r = \frac{1}{2} \int d^3\sigma \left[ \pi^2_{\perp} + B_{\perp}^2 \right](\tau, \vec{\sigma}) = \sum_{\lambda=1,2} \int d\vec{k} \, \omega(\vec{k}) \, a_\lambda^*(\vec{k}) \, a_\lambda(\vec{k}), \]

\[ \vec{P}_{\text{rad}} = \frac{1}{c} \int d^3\sigma \left[ \pi_{\perp} \times B_{\perp} \right](\tau, \vec{\sigma}) = \frac{1}{c} \sum_{\lambda=1,2} \int d\vec{k} \, a_\lambda^*(\vec{k}) \, a_\lambda(\vec{k}) \approx 0, \]

\[ \vec{\mathcal{J}}_{\text{rad}} = \vec{S}_{\text{rad}} = \frac{1}{c} \int d^3\sigma \, \vec{\sigma} \times \left( \pi_{\perp} \times \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) = \]

\[ = \frac{i}{c} \sum_{\lambda} \int d\vec{k} \, a_\lambda^*(\vec{k}) \, \vec{k} \times \frac{\partial}{\partial \vec{k}} a_\lambda(\vec{k}) + \]

\[ + \frac{i}{2c} \sum_{\lambda \lambda'} \int d\vec{k} \left[ a_\lambda(\vec{k}) \, a_\lambda^*(\vec{k}) - a_\lambda^*(\vec{k}) \, a_\lambda(\vec{k}) \right] \vec{e}_\lambda(\vec{k}) \cdot \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \vec{e}_\lambda'(\vec{k}) - \]

\[ - \frac{i}{c} \sum_{\lambda \lambda'} \int d\vec{k} \, \vec{e}_\lambda(\vec{k}) \times \vec{e}_\lambda'(\vec{k}) \, a_\lambda^*(\vec{k}) \, a_\lambda(\vec{k}), \]
For the free radiation field the Fourier coefficients and the modulus-phase variables are
dimensions: $|h| [\mathcal{F} = [m^2 t^{-1}].$

For the free radiation field the Fourier coefficients and the modulus-phase variables are $\tau$-independent.
\[ \mathcal{J}_{\text{rad}} = -\frac{1}{c} \sum_{\sigma} \int d\vec{k} I_{\sigma}(\vec{k}) \left( \vec{k} \times \frac{\partial}{\partial k} \right) \phi_{\sigma}(\vec{k}) + \]

\[ + \frac{i}{2c} \int d\vec{k} \left[ I_+(\vec{k}) - I_-(\vec{k}) \right] \left[ \vec{\epsilon}_+(\vec{k}) + \vec{\epsilon}_-(\vec{k}) \right] \cdot \left( \vec{k} \times \frac{\partial}{\partial k} \right) \left[ \vec{\epsilon}_-(\vec{k}) - \vec{\epsilon}_+(\vec{k}) \right] + \]

\[ + \frac{i}{c} \int d\vec{k} \vec{\epsilon}_+^r(\vec{k}) \times \vec{\epsilon}_-^r(\vec{k}) \left[ I_+(\vec{k}) - I_-(\vec{k}) \right]. \]

\[ K_{\text{rad}}^r = -\frac{1}{c} \sum_{\sigma=\pm} \int d\vec{k} I_{\sigma}(\vec{k}) \omega(\vec{k}) \frac{\partial}{\partial k^r} \phi_{\sigma}(\vec{k}) + \]

\[ + \frac{i}{2c} \int d\vec{k} \left[ I_+(\vec{k}) - I_-(\vec{k}) \right] \left[ \vec{\epsilon}_+^r(\vec{k}) + \vec{\epsilon}_-^r(\vec{k}) \right] \cdot \omega(\vec{k}) \frac{\partial}{\partial k^r} \left[ \vec{\epsilon}_-(\vec{k}) - \vec{\epsilon}_+(\vec{k}) \right] \approx 0. \tag{3.4} \]

As shown in Appendix A and Ref.[6], the existence of finite values of these quantities requires [6]:

i) \( |I_{\sigma}(\vec{k})| \to |\vec{k}|^{-3-\delta} \) with \( \delta > 0 \); \( |\phi_{\sigma}(\vec{k})| \to |\vec{k}|^{-1} \) (this is due to the transformation properties under Poincaré transformations; \( \phi'_{\sigma}(\vec{k}) = \phi_{\sigma}(\Lambda^{-1} \vec{k}) + k \cdot a + \sigma \theta(\vec{k}, \Lambda^{-1}) \));

ii) \( |I_{\sigma}(\vec{k})| \to |\vec{k}|^{-2+\epsilon} \) with \( \epsilon > 0 \); \( |\phi_{\sigma}(\vec{k})| \to |\vec{k}|^{\alpha} \) with \( \alpha > -\delta \).

Let us replace the canonical variables \( I_{\sigma}(\vec{k}) \), \( \phi_{\sigma}(\vec{k}) \) with the following new canonical basis \( \Psi_{\text{rad}}(\vec{k}), \Phi_{\text{rad}}(\vec{k}), \lambda_{\text{rad}}(\vec{k}), \rho_{\text{rad}}(\vec{k}) \)\(^{13}\)

\[ \Psi_{\text{rad}}(\vec{k}) = \sum_{\sigma=\pm} I_{\sigma}(\vec{k}), \quad \Phi_{\text{rad}}(\vec{k}) = \frac{1}{2} \sum_{\sigma=\pm} \phi_{\sigma}(\vec{k}), \]

\[ \lambda_{\text{rad}}(\vec{k}) = I_+(\vec{k}) - I_-(\vec{k}), \quad \rho_{\text{rad}}(\vec{k}) = \frac{1}{2} \left[ \phi_+^r(\vec{k}) - \phi_-^r(\vec{k}) \right], \]

\[ \{ \Psi_{\text{rad}}(\vec{k}), \Phi_{\text{rad}}(\vec{k}) \} = \{ \lambda_{\text{rad}}(\vec{k}), \rho_{\text{rad}}(\vec{k}) \} = \Omega(\vec{k}) \delta^3(\vec{k} - \vec{k}'), \]

\[ I_+(\vec{k}) = \frac{1}{2} \left( \Psi_{\text{rad}}(\vec{k}) + \lambda_{\text{rad}}(\vec{k}) \right), \quad I_-(\vec{k}) = \frac{1}{2} \left( \Psi_{\text{rad}}(\vec{k}) - \lambda_{\text{rad}}(\vec{k}) \right), \]

\[ \phi_+(\vec{k}) = \Phi_{\text{rad}}(\vec{k}) + \rho_{\text{rad}}(\vec{k}), \quad \phi_-^r(\vec{k}) = \Phi_{\text{rad}}(\vec{k}) - \rho_{\text{rad}}(\vec{k}), \quad \tag{3.5} \]

in terms of which the helicity and the Poincaré generators become

\(^{13}\) Dimensions: \( [\Psi_{\text{rad}}] = [\lambda_{\text{rad}}] = [I_{\sigma}] = [m t^5 t^{-2}] \).
\[ h_{\text{rad}} = \frac{1}{c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}), \]
\[ P_{\text{rad}}^A = \frac{1}{c} \int d\vec{k} k^A \Psi_{\text{rad}}(\vec{k}) = \left( P_\tau^A = \frac{E_{\text{rad}}}{c} = M_{\text{rad}} c; \bar{P}_{\text{rad}} \approx 0 \right), \]
\[ J_{\text{rad}} = -\frac{1}{c} \int d\vec{k} \Psi_{\text{rad}}(\vec{k}) \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \Phi_{\text{rad}}(\vec{k}) - \]
\[ - \frac{1}{c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \rho_{\text{rad}}(\vec{k}) + \]
\[ + \frac{i}{2c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \left[ \vec{\epsilon}_+(\vec{k}) + \vec{\epsilon}_-(\vec{k}) \right] \cdot \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \left[ \vec{\epsilon}_-(\vec{k}) - \vec{\epsilon}_+(\vec{k}) \right] + \]
\[ + \frac{i}{c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \left[ \vec{\epsilon}_+(\vec{k}) \times \vec{\epsilon}_-(\vec{k}) \right], \]
\[ K_{\tau}^A = \frac{1}{c} \int d\vec{k} \Psi_{\text{rad}}(\vec{k}) \omega(\vec{k}) \frac{\partial}{\partial k_r} \Phi_{\text{rad}}(\vec{k}) + \frac{1}{c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \omega(\vec{k}) \frac{\partial}{\partial k_r} \rho_{\text{rad}}(\vec{k}) + \]
\[ + \frac{i}{2c} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \left[ \vec{\epsilon}_+(\vec{k}) + \vec{\epsilon}_-(\vec{k}) \right] \cdot \omega(\vec{k}) \frac{\partial}{\partial k_r} \left[ \vec{\epsilon}_-(\vec{k}) - \vec{\epsilon}_+(\vec{k}) \right] \approx 0. \tag{3.6} \]

One has:

i) \(|\Psi(\vec{k})| \rightarrow |\vec{k}|^{-3+\epsilon}\) with \(\epsilon > 0\); \(|\Phi(\vec{k})| \rightarrow |\vec{k}|^{-\alpha}\) with \(\alpha > 1 - \epsilon\); \(|\lambda_{\text{rad}}(\vec{k})| \rightarrow |\vec{k}|^{-2-\delta}\) with \(\delta > 0\); \(|\rho_{\text{rad}}(\vec{k})| \rightarrow |\vec{k}|^{-2}\) with \(\beta > -\delta\);

ii) \(|\Psi(\vec{k})| \rightarrow |\vec{k}|^{-3-\gamma}\) with \(\gamma > 0\); \(|\Phi(\vec{k})| \rightarrow |\vec{k}|^{-\gamma}\); \(|\lambda_{\text{rad}}(\vec{k})| \rightarrow |\vec{k}|^{-2-\chi}\) with \(\chi > 0\); \(|\rho_{\text{rad}}(\vec{k})| \rightarrow |\vec{k}|^{-2}\) with \(\beta > \chi\).

**B. Collective and Relative Variables for the Radiation Field**

We can now use the results of Appendix A to replace the canonical variables \(\Psi_{\text{rad}}(\vec{k})\) and \(\Phi_{\text{rad}}(\vec{k})\) with a set of collective and relative canonical variables \(X^A, P^A, H_{\text{rad}}(\vec{k}), K_{\text{rad}}(\vec{k})\). To them must be added the remaining canonical variables \(\lambda_{\text{rad}}(\vec{k})\) and \(\rho_{\text{rad}}(\vec{k})\) describing the difference of the modulus-phase variables with different circular polarization (they are already of the type of relative variables).

By using Eqs.(A6), (A7), (A9) and the function \(F(\vec{k}, P_{\text{rad}}^B)\) of Eq.(A8) with \(\omega(k) = k\) and with the functions \(F_\tau(k), F(k)\) given in Eqs.(A11), the collective and relative variables are
\[ X_{rad}^A = \int d\vec{k} \frac{\partial F(\vec{k}, \mathcal{P}_\phi^B)}{\partial \mathcal{P}_{\phi A}} \Phi_{rad}(\vec{k}), \]

\[ \mathcal{P}_{rad}^A = \frac{1}{c} \int d\vec{k} \, k^A \Psi_{rad}(\vec{k}) = \int d\vec{k} \, k^A \mathcal{F}(\vec{k}, \mathcal{P}_{rad}^B), \]

\[ H_{rad}(\vec{k}) = \int d\vec{q} \, \mathcal{G}(\vec{k}, \vec{q}) \left[ \Psi_{rad}(\vec{q}) - \omega(q) F^r(q) \int d\vec{q}_1 \, \omega(q_1) \Psi_{rad}(\vec{q}_1) + F(q) \vec{q} \cdot \int d\vec{q}_1 \, \vec{q}_1 \Psi_{rad}(\vec{q}_1) \right], \]

\[ K_{rad}(\vec{k}) = \mathcal{D}_k \Phi_{rad}(\vec{k}), \]

\[ \lambda_{rad}(\vec{k}), \quad \rho_{rad}(\vec{k}), \]

\[ \{ X_{rad}^A, \mathcal{P}_{rad}^B \} = -\eta^{AB}, \]

\[ \{ H_{rad}(\vec{k}), K_{rad}(\vec{k}_1) \} = (2\pi)^3 \cdot 2 \omega(k) \delta^3(\vec{k} - \vec{k}_1), \]

\[ \{ \lambda_{rad}(\vec{k}), \rho_{rad}(\vec{k}_1) \} = (2\pi)^3 \cdot 2 \omega(k) \delta^3(\vec{k} - \vec{k}_1), \quad (3.7) \]

with all the other Poisson brackets vanishing. The operator \( \mathcal{D}_\vec{q} \) and its Green function are defined in Ref.[6] and Appendix A. By construction \( X_{rad}^A \) is a 4-center of phase.

The helicity and Poincaré generators become \[ [D^rs = k^r \frac{\partial}{\partial k^s} - k^s \frac{\partial}{\partial k^r}, D^{rr} = \omega(\hat{k}) \frac{\partial}{\partial k^r}] \]

\[ h_{rad} = \frac{1}{c} \int d\vec{k} \lambda_{rad}(\vec{k}), \]

\[ \mathcal{P}_{rad}^A = \left( \mathcal{P}_{rad}^r = \frac{\xi_{rad}}{c} = M_{rad} \right) \quad \bar{\mathcal{P}}_{rad} \approx 0, \]

\[ \mathcal{J}_{rad} = \mathcal{S}_{rad} = \mathcal{X}_{rad} \times \mathcal{P}_{rad} + \bar{\mathcal{S}}_{rad}, \]

\[ \mathcal{S}_{rad} = -\frac{1}{c} \int d\vec{k} \, H_{rad}(\vec{k}) \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) K_{rad}(\vec{k}) - \frac{1}{c} \int d\vec{k} \lambda_{rad}(\vec{k}) \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \rho_{rad}(\vec{k}) + \frac{1}{c} \int d\vec{k} \lambda_{rad}(\vec{k}) \left[ \vec{e}_-(\vec{k}) \times \vec{e}_+(\vec{k}) + \frac{i}{2} \left( \vec{e}_-(\vec{k}) + \vec{e}_+(\vec{k}) \right) \cdot \vec{k} \times \frac{\partial}{\partial \vec{k}} \left( \vec{e}_-(\vec{k}) - \vec{e}_+(\vec{k}) \right) \right], \]

\[ K_{rad}^r = X_{rad}^r \mathcal{P}_{rad}^r - X_{rad}^r M_{rad} - \frac{1}{c} \int d\vec{k} \, H_{rad}(\vec{k}) D^{rr} K_{rad}(\vec{k}) - \frac{1}{c} \int d\vec{k} \lambda_{rad}(\vec{k}) D^{rr} \rho_{rad}(\vec{k}) - \frac{i}{2c} \int d\vec{k} \lambda_{rad}(\vec{k}) \left( \left[ \vec{e}_+(\vec{k}) + \vec{e}_-(\vec{k}) \right] \cdot D^{rr} \left[ \vec{e}_-(\vec{k}) - \vec{e}_+(\vec{k}) \right] \right) \approx 0. \quad (3.8) \]
The gauge fixing $\mathcal{K}_{\text{rad}} \approx 0$ to the rest-frame conditions $\mathcal{P}_{\text{rad}} \approx 0$ leads to the following determination of the 3-center of phase $\vec{X}_{\text{rad}}$

\[ X_{\text{rad}}^\tau \approx - \frac{1}{M_{\text{rad}} c^2} \left[ \int d\vec{k} H_{\text{rad}}(\vec{k}) D^{rr}_{\text{rad}}(\vec{k}) + \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) D^{rr}_{\text{rad}}(\vec{k}) \right. \]
\[ \left. + \frac{i}{2} \int d\vec{k} \lambda_{\text{rad}}(\vec{k}) \left( [\vec{\epsilon}_+^r(\vec{k}) + \vec{\epsilon}_-^r(\vec{k})] \cdot D^{rr} [\vec{\epsilon}_-^r(\vec{k}) - \vec{\epsilon}_+^r(\vec{k})] \right) \right]. \tag{3.9} \]

If $\vec{A}_{\perp\text{rad}}(\tau, \sigma)$ satisfies $\int d\vec{k} w_{lm}(\vec{k}) \left[ \Psi_{\text{rad}}(\vec{k}) - \mathcal{F}(\vec{k}, \mathcal{P}_{\text{rad}}^B) \right] = 0$ for $l \geq 2$ (see Ref.[6] and Appendix A), then we have the following expression of the fields of the transverse radiation field

\[ \vec{A}_{\perp\text{rad}}(\tau, \sigma) = \int d\vec{k} \sum_{\sigma=\pm} \left[ \vec{e}_\sigma(\vec{k}) \sqrt{\frac{1}{2}} \left( \mathcal{F}(\vec{k}, \mathcal{P}_{\text{rad}}^B) + \sigma \lambda_{\text{rad}}(\vec{k}) \right) + D_\vec{k} \mathcal{H}_{\text{rad}}(\vec{k}) \right. \]
\[ \left. \times e^{i[k_\Lambda (X_{\text{rad}}^A - \sigma^A) + \sigma \rho_{\text{rad}}(\vec{k}) + \sigma]} - d_\bar{k}_1 d_\bar{k}_2 K_{\text{rad}}(\bar{k}_1) \mathcal{G}(\bar{k}_1, \bar{k}_2) \triangle(\bar{k}_1, \bar{k}_2) \right] \right] \left. + \text{c.c.} \right] \]

\[ \vec{B}_{\text{rad}}(\tau, \sigma) = i \int d\vec{k} \sum_{\sigma=\pm} \left[ \vec{e}_\sigma(\vec{k}) \sqrt{\frac{1}{2}} \left( \mathcal{F}(\vec{k}, \mathcal{P}_{\text{rad}}^B) + \sigma \lambda_{\text{rad}}(\vec{k}) \right) + D_\vec{k} \mathcal{H}_{\text{rad}}(\vec{k}) \right. \]
\[ \left. \times e^{i[k_\Lambda (X_{\text{rad}}^A - \sigma^A) + \sigma \rho_{\text{rad}}(\vec{k}) + \sigma]} - d_\bar{k}_1 d_\bar{k}_2 K_{\text{rad}}(\bar{k}_1) \mathcal{G}(\bar{k}_1, \bar{k}_2) \triangle(\bar{k}_1, \bar{k}_2) \right] \right] \left. + \text{c.c.} \right], \tag{3.10} \]

so that the radiation fields have the following dependence upon $\tau$ and $\sigma$

\[ \vec{A}_{\perp\text{rad}}(\tau, \sigma) = \vec{\tilde{A}}_{\perp\text{rad}}(\tau - X_{\text{rad}}^\tau, \sigma - \vec{X}_{\text{rad}}), \]
\[ \vec{\pi}_{\perp\text{rad}}(\tau, \sigma) = \vec{\tilde{\pi}}_{\perp\text{rad}}(\tau - X_{\text{rad}}^\tau, \sigma - \vec{X}_{\text{rad}}), \]
\[ \vec{B}_{\text{rad}}(\tau, \sigma) = \vec{\tilde{B}}_{\text{rad}}(\tau - X_{\text{rad}}^\tau, \sigma - \vec{X}_{\text{rad}}), \tag{3.11} \]

The configurations of the radiation field admitting the collective variables $X_{\text{rad}}^A$ have a "monopole" structure, carried by $\vec{X}_{\text{rad}}$ and $\vec{B}_{\text{rad}} \approx 0$, plus the "multipoles" $K_{\text{rad}}(\vec{k})$, $H_{\text{rad}}(\vec{k})$, $\lambda_{\text{rad}}(\vec{k})$, $\rho_{\text{rad}}(\vec{k})$ (these last multipoles describe the helicity structure). Eqs.(3.9) expresses $\vec{X}_{\text{rad}}$ in terms of $M_{\text{rad}}$ and of the higher multipoles.

As in the Klein-Gordon case we have the extra variables $X_{\text{rad}}^\tau$, $\mathcal{P}_{\text{rad}}^\tau = M_{\text{rad}} c$ with a similar interpretation (see Appendix A).

By using Eq.(I-2.25) we have the following representation of the potential of the radiation field: $A_{\text{rad}}^\mu \left( Y^{\alpha}(\tau) + e^\alpha_{\tau}(\vec{h}) \sigma^\alpha \right) = -e^\alpha_{\tau}(\vec{h}) A_{\perp\text{rad}}^\tau(\tau, \sigma)$. 

18
IV. N CHARGED PARTICLES PLUS THE ELECTRO-MAGNETIC FIELD

Let us now consider the problem of identifying the collective and relative variable for the system of two\(^{14}\) positive-energy charged particles with mutual Coulomb interaction plus an arbitrary electro-magnetic field in the radiation gauge (see Ref.\([1]\) and I). This will allow to solve the constraints \(\vec{K}_{\text{(int)}} \approx 0\) and to eliminate the internal 3-center of mass like we made in the previous two Sections.

A. The Internal Poincare’ Generators before the Canonical Transformation

From Eq.(I-2.23) we have the following expression of the internal Poincare’ generators in the original canonical basis \(\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma})\)

\[
\mathcal{E}_{\text{(int)}} = \mathcal{P}^r_{\text{(int)}} c = M c^2 = c \int d^3 \sigma T^{\tau\tau}(\tau, \vec{\sigma}) =
\]

\[
= c \sum_{i=1}^{2} \left( \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \right) - \frac{Q_i}{c} \vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right) +
\]

\[
+ \frac{Q_1 Q_2}{4\pi} \frac{1}{|\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|} + \mathcal{E}_{\text{em}},
\]

\[
\mathcal{E}_{\text{em}} = \mathcal{P}_\text{em}^r = \frac{1}{2} \int d^3 \sigma \left[ \vec{\pi}^2_\perp + \vec{B}^2 \right](\tau, \vec{\sigma}),
\]

\[
\mathcal{P}_{\text{(int)}}^r = \int d^3 \sigma T^{\tau\tau}(\tau, \vec{\sigma}) = \sum_{i=1}^{2} \kappa_i^r(\tau) + \mathcal{P}_{\text{em}}^r \approx 0,
\]

\[
\vec{P}_\text{em}^r = \frac{1}{c} \int d^3 \sigma \left[ \vec{\pi}_\perp \times \vec{B} \right](\tau, \vec{\sigma}),
\]

\[
\mathcal{J}^r_{\text{(int)}} = \vec{S}^r = \frac{1}{2} \epsilon^{r uv} \int d^3 \sigma \sigma^u T^{uv \tau}(\tau, \vec{\sigma}) =
\]

\[
= \sum_{i=1}^{2} \left( \vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau) \right)^r + \mathcal{J}_{\text{em}}^r,
\]

\[
\vec{J}_\text{em}^r = \frac{1}{c} \int d^3 \sigma (\vec{\sigma} \times \left[ \vec{\pi}_\perp \times \vec{B} \right]^r)(\tau, \vec{\sigma}),
\]

\(^{14}\) The case of N particles follows the same scheme.
\[ \mathcal{K}_{(\text{int})} = - \int d^3 \sigma \sigma^r T^{\tau\tau}(\tau, \bar{\sigma}) = \]

\[ = - \sum_{i=1}^{2} \eta_i^r(\tau) \left( \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)} - \frac{Q_i}{c} \frac{\hat{\kappa}_i(\tau) \cdot \hat{A}_{\perp}(\tau, \bar{n}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)}} \right) + \]

\[ + \frac{Q_1 Q_2}{4\pi c} \left[ \eta_1^r(\tau) + \eta_2^r(\tau) \right] - \int \frac{d^3 \sigma}{4\pi} \left( \frac{\sigma^r - \eta_2^r(\tau)}{|\bar{\sigma} - \bar{n}_1(\tau)| |\bar{\sigma} - \bar{n}_2(\tau)|^2} + \frac{\sigma^r - \eta_1^r(\tau)}{|\bar{\sigma} - \bar{n}_2(\tau)| |\bar{\sigma} - \bar{n}_1(\tau)|^2} \right) - \]

\[ - \sum_{i=1}^{2} \frac{Q_i}{4\pi c} \int d^3 \sigma \frac{\pi_i^r(\tau, \bar{\sigma})}{|\bar{\sigma} - \bar{n}_i(\tau)|} + \mathcal{K}_{\text{em}}^r \approx 0, \]

\[ \mathcal{K}_{\text{em}}^r = - \frac{1}{2c} \int d^3 \sigma \sigma^r (\bar{\pi}_1^2 + \bar{B}^2)(\tau, \bar{\sigma}). \quad (4.1) \]

For an electro-magnetic field not of the radiation type the quantities \( \mathcal{E}_{\text{em}}, \mathcal{P}_{\text{em}}, \mathcal{J}_{\text{em}}, \mathcal{K}_{\text{em}} \) are not conserved and do not satisfy a Lorentz algebra.

In the canonical basis \( \hat{A}_{\perp}(\tau, \bar{\sigma}), \hat{n}_1(\tau), \hat{n}_2(\tau), \) it is not clear which are the collective variables to be eliminated by means of the constraints \( \mathcal{P}_{\text{(int)}} \approx 0 \) and \( \mathcal{K}_{\text{(int)}} \approx 0 \), because we do not know how to define these variables for an electro-magnetic field not of the radiation type. To treat this type of field we now use results obtained in paper I.

**B. The Internal Poincare’ Generators after the Canonical Transformation**

After the canonical transformation defined in I, the system is described by a transverse radiation field \( \hat{A}_{\perp}^\text{rad}(\tau, \bar{\sigma}), \hat{n}_1^\text{rad}(\tau, \bar{\sigma}), \) and by Coulomb-dressed particle variables \( \hat{n}_i(\tau), \hat{\kappa}_i(\tau), \) see Eqs. (I-3.6) and (I-3.10).

From Eqs. (I-4.4), (I-4.1), (I-4.2), (I-4.6) [and (I-3.9) for \( \hat{T}_i \) and \( \hat{K}_{ij} \)] we have that each internal Poincare’ generator becomes the direct sum of the radiation field one of Eq.(3.2) plus the particle one of Eq.(2.4) [see Eq.(6.46) of Ref.[1]]

\[ \mathcal{E}_{\text{(int)}} = \mathcal{P}_{\text{(int)}}^r c = M c^2 = c \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)} + \frac{Q_1 Q_2}{4\pi |\hat{n}_1(\tau) - \hat{n}_2(\tau)|} + \]

\[ + V_{\text{DARWIN}}(\hat{n}_1(\tau), \hat{n}_2(\tau), \hat{n}_1(\tau) - \hat{n}_2(\tau)) + \]

\[ + \frac{1}{2} \int d^3 \sigma \left( \frac{\bar{\pi}_1^2}{|\bar{\sigma} - \bar{\pi}_1(\tau)|} + \bar{B}_\text{rad}^2 \right)(\tau, \bar{\sigma}) = \]

\[ = \left( \mathcal{P}_{\text{matter}}^r + \mathcal{P}_{\text{rad}}^r \right) c, \]

20
$\bar{P}_{(\text{int})} = \sum_{i=1}^{2} \hat{\kappa}_i(\tau) + \frac{1}{c} \int d^3 \sigma \left(\hat{\pi}_{\perp \text{rad}} \times \hat{B}_{\text{rad}}\right)(\tau, \hat{\sigma}) = $

$= \bar{P}_{\text{matter}} + \bar{P}_{\text{rad}} \approx 0,$

$\bar{J}_{(\text{int})} = \sum_i \hat{\eta}_i \times \hat{\kappa}_i + \frac{1}{c} \int d^3 \sigma \hat{\sigma} \times \left(\hat{\pi}_{\perp \text{rad}} \times \hat{B}_{\text{rad}}\right)(\tau, \hat{\sigma}) =$

$= \bar{J}_{\text{matter}} + \bar{J}_{\text{rad}},$

$K^2_{(\text{int})} = -\sum_{i=1}^{2} \hat{\eta}_i \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} -$

$- \frac{1}{2} \frac{Q_1 Q_2}{c} \left[\hat{\kappa}_1 \cdot \left(\frac{\partial \hat{K}_{12}(\hat{k}_1, \hat{\rho}_{12})}{\partial \hat{\kappa}_1} - 2 \hat{A}_{1\perp S2}(\hat{k}_2, \hat{\rho}_{12})\right) + \right.$

$+ \left. \hat{\eta}_2 \cdot \left(\frac{\partial \hat{K}_{12}(\hat{k}_2, \hat{\rho}_{12})}{\partial \hat{\kappa}_2} - 2 \hat{A}_{1\perp S1}(\hat{k}_1, \hat{\rho}_{12})\right)\right] -$

$- \frac{1}{2} \frac{Q_1 Q_2}{c} \left(\sqrt{m_1^2 c^2 + \hat{\kappa}_1^2} \frac{\partial}{\partial \hat{\kappa}_1} + \sqrt{m_2^2 c^2 + \hat{\kappa}_2^2} \frac{\partial}{\partial \hat{\kappa}_2}\right) \hat{K}_{12}(\hat{k}_1, \hat{k}_2, \hat{\rho}_{12}) -$

$- \frac{Q_1 Q_2}{4\pi c} \int d^3 \sigma \left(\hat{\pi}_{\perp S1}(\hat{\sigma} - \hat{\eta}_1, \hat{\kappa}_1) + \hat{\pi}_{\perp S2}(\hat{\sigma} - \hat{\eta}_2, \hat{\kappa}_2)\right) -$

$- \frac{Q_1 Q_2}{c} \int d^3 \sigma \hat{\sigma} \left(\hat{\pi}_{\perp S1}(\hat{\sigma} - \hat{\eta}_1, \hat{\kappa}_1) \cdot \hat{\pi}_{\perp S2}(\hat{\sigma} - \hat{\eta}_2, \hat{\kappa}_2) + \right.$

$+ \hat{B}_{S1}(\hat{\sigma} - \hat{\eta}_1, \hat{\kappa}_1) \cdot \hat{B}_{S2}(\hat{\sigma} - \hat{\eta}_2, \hat{\kappa}_2)\right) -$

$- \frac{1}{2 c} \int d^3 \sigma \hat{\sigma} \left(\hat{\pi}_{\perp \text{rad}}^2 + \hat{B}_{\text{rad}}^2\right)(\tau, \hat{\sigma}) =$

$= \bar{K}_{\text{matter}} + \bar{K}_{\text{rad}} =$

$\overset{\text{def}}{=} -\frac{1}{c} \mathcal{E}_{(\text{int})} \bar{R}_+ = -\frac{1}{c} \left(\mathcal{E}_{\text{matter}} + \mathcal{E}_{\text{rad}}\right) \bar{R}_+ \approx 0. \quad (4.2)$

Let us remark that in $K^2_{(\text{int})} \approx 0$ of Eq. (4.2) all the particles terms depend on $\hat{\rho}_{12}(\tau) = \hat{\eta}_1(\tau) - \hat{\eta}_2(\tau)$ or $\hat{\sigma} - \hat{\eta}_i(\tau).$
We also added the connection of $\mathbf{K}^\text{(int)}_r$ with the Møller internal 3-center of energy $\mathbf{R}_+$\textsuperscript{15}.

Even if all the internal generators are the direct sum of the generators of the two subsystems, the two subsystems are connected by the rest-frame conditions and by the vanishing of the internal boosts, i.e. by the necessity of eliminating the position and momentum of the internal overall 3-center of mass.

C. Semi-Relativistic Expansions of $\mathcal{E}^\text{(int)}$ and $\mathcal{K}^\text{(int)}_r$ before and after the Canonical Transformation

The semi-relativistic limit of the generators $\mathcal{E}^\text{(int)}$ and $\mathcal{K}^\text{(int)}_r$ of Eqs.\textsuperscript{(4.1)} is

\[
\mathcal{E}^\text{(int)} \to c^{-\infty} \left( \sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} \frac{\kappa_i^2(\tau)}{2m_i} + \frac{Q_1 Q_2}{4\pi |\bar{\eta}_1(\tau) - \bar{\eta}_2(\tau)|} - \frac{1}{c} \sum_{i=1}^{2} Q_i \frac{\kappa_i^2(\tau)}{m_i} \cdot \bar{A}_\perp(\tau, \bar{\eta}_i(\tau)) - \frac{1}{c^2} \sum_{i=1}^{2} \frac{\kappa_i^4(\tau)}{8m_i^2} \right) + \frac{1}{c^3} \sum_{i=1}^{2} Q_i \frac{\kappa_i^2(\tau)}{2m_i^2} \frac{\kappa_i(\tau)}{m_i} \cdot \bar{A}_\perp(\tau, \bar{\eta}_i(\tau)) + O(c^{-4}) + \frac{1}{2} \int d^3\sigma [\vec{\pi}^2 + \vec{B}^2](\tau, \bar{\sigma}),
\]

\[
\mathcal{K}^r(\text{int}) \to c^{-\infty} c \mathcal{K}^r_{\text{Galilei}} + O(c^{-1}) \approx 0,
\]

\[
\mathcal{K}_{\text{Galilei}} = -\sum_{i=1}^{N} m_i \bar{\eta}_i(\tau) = -\left( \sum_{i=1}^{N} m_i \right) \bar{x}_n(\tau) \approx 0,
\]

where $\bar{x}_n(\tau)$ is the non-relativistic center of mass, which emerges as the limit of the Møller internal 3-center of energy $\mathbf{R}_+ = -c \mathcal{K}^r(\text{int})/\mathcal{E}^\text{(int)}$. See Section G of paper I for the non-relativistic limit of the relativistic variables $\bar{\eta}_i(\tau), \bar{\kappa}_i(\tau)$ and of the relativistic external center of mass.

After the canonical transformation $\mathcal{E}^\text{(int)}$ is given by Eq.\textsuperscript{(4.2)}. The semi-relativistic limit of its matter part is

\textsuperscript{15} We have $\bar{R}_+ \approx \bar{q}_+ \approx \bar{y}_+$ due to the rest-frame condition $\mathbf{P}^\text{(int)}(\tau) \approx 0$, where $\bar{q}_+$ is the internal 3-center of mass and $\bar{y}_+$ the internal Fokker-Pryce 3-center of inertia.
\[ P_{\text{matter}}^\tau = \sum_{i=1}^{2} \left( m_i c^2 + \frac{\kappa_i^2(\tau)}{2m_i} + \frac{Q_1 Q_2}{4\pi |\hat{\eta}_1(\tau) - \hat{\eta}_2(\tau)|} - \frac{1}{c^2} \frac{\kappa_i^4(\tau)}{8m_i^3} + \frac{Q_1 Q_2}{m_1 m_2 c^2} \frac{\kappa_i^2(\tau)}{16\pi |\hat{\eta}_1(\tau) - \hat{\eta}_2(\tau)|^2} \right) + O(c^{-4}). \quad (4.4) \]

By using Eqs.(I-2.51) - (I-2.53) for the Lienard-Wiechert fields we get \( \tilde{K}_{12} = O(c^{-3}) \), so that the semi-relativistic limit of \( \tilde{K}_{\text{int}} \) of Eq.(4.2) is

\[ \tilde{K}_{\text{int}} = -\sum_{i=1}^{2} \hat{\eta}_i(\tau) \left( m_i c + \frac{\kappa_i^2}{2m_i c} \right) + O(c^{-2}) - \frac{1}{2c} \int d^3\sigma \, \tilde{\sigma} \left( \tilde{\pi}_{\perp \text{rad}}^2 + \tilde{\vec{B}}_{\text{rad}}^2 \right)(\tau, \tilde{\sigma}) = \tilde{K}_{\text{matter}} + \tilde{K}_{\text{rad}} = c \tilde{K}_{\text{Galilei}} + O\left( \frac{1}{c} \right) \approx 0, \]

\[ \tilde{K}_{\text{Galilei}} = -(m_1 + m_2) \tilde{x}_{(u)} = -\sum_{i=1}^{2} m_i \hat{\eta}_i \approx 0 \quad (\text{in the rest frame}). \quad (4.5) \]

D. The Collective Variables after the Canonical Transformation

The results of Section II and III allow us to find the collective and relative variables of the matter and transverse radiation field subsystems in the canonical basis \( \hat{\eta}_i(\tau), \tilde{\kappa}_i(\tau), \tilde{A}_{\perp \text{rad}}(\tau, \tilde{\sigma}), \tilde{\pi}_{\perp \text{rad}}(\tau, \tilde{\sigma}) \).

A) The field quantities \( \tilde{A}_{\perp \text{rad}}(\tau, \tilde{\sigma}), \tilde{\pi}_{\perp \text{rad}}(\tau, \tilde{\sigma}) \) are canonically equivalent to the canonical basis \( X_{\text{rad}}^\tau, P_{\text{rad}}^\tau = M_{\text{rad}} c = \tilde{\xi}_{\text{rad}} c, \tilde{X}_{\text{rad}}, \tilde{P}_{\text{rad}}, \lambda_{\text{rad}}(\tilde{k}), p_{\text{rad}}(\tilde{k}), H_{\text{rad}}(\tilde{k}), K_{\text{rad}}(\tilde{k}) \), given in Eqs.(3.7), which contains the collective variables for the radiation field.

B) For the particle subsystem we make the canonical transformation (2.3) on the Coulomb-dressed particle variables

\[ \hat{\eta}_{12} = \frac{m_1}{m} \hat{\eta}_1 + \frac{m_2}{m} \hat{\eta}_2, \quad \hat{\rho}_{12} = \hat{\eta}_1 - \hat{\eta}_2, \]

\[ \tilde{\kappa}_{12} = \tilde{\kappa}_1 + \tilde{\kappa}_2 \approx 0, \quad \tilde{\pi}_{12} = \frac{m_2}{m} \tilde{\kappa}_1 - \frac{m_1}{m} \tilde{\kappa}_2, \]

\[ \hat{\eta}_1 = \hat{\eta}_{12} + \frac{m_2}{m} \hat{\rho}_{12}, \quad \hat{\eta}_2 = \hat{\eta}_{12} - \frac{m_1}{m} \hat{\rho}_{12}, \]

\[ \hat{\kappa}_1 = \frac{m_1}{m} \hat{\kappa}_{12} + \hat{\pi}_{12}, \quad \hat{\kappa}_2 = \frac{m_2}{m} \hat{\kappa}_{12} - \hat{\pi}_{12}. \quad (4.6) \]
Eqs. (4.2) imply $\vec{P}_{\text{matter}} = \hat{\vec{\kappa}}_{12} \approx -\vec{P}_{\text{rad}} = -\int d^3 \sigma \left( \vec{\pi}_{\perp \text{rad}} \times \vec{B}_{\text{rad}} \right)(\tau, \sigma)$.

C) Let us now put $\hat{\vec{\eta}}_3 = \vec{X}_{\text{rad}}, \hat{\vec{\kappa}}_3 = \vec{P}_{\text{rad}}$, with $\vec{X}_{\text{rad}}$ and $\vec{P}_{\text{rad}}$ given by Eqs. (3.9) and (3.8) respectively, and let us combine it with $\hat{\vec{\eta}}_{12}$ and $\hat{\vec{\kappa}}_{12}$ of Eqs.(4.6) to get the canonical transformation

$$\begin{array}{c|c|c|c|c|c}
\hat{\vec{\eta}}_{12} & \hat{\vec{\rho}}_{12} & \hat{\vec{\eta}}_3 & \hat{\vec{\kappa}}_{12} & \hat{\vec{\kappa}}_3 \\
\hline
\hat{\vec{\kappa}}_{12} & \hat{\vec{P}}_{12} & \hat{\vec{\eta}}_3 & \hat{\vec{\kappa}}_3 & \hat{\vec{\pi}}_{(12)3} \\
\end{array}$$

$$\hat{\vec{\eta}} = \frac{1}{2}(\hat{\vec{\eta}}_{12} + \hat{\vec{\eta}}_3), \quad \hat{\vec{\kappa}} = \hat{\vec{\kappa}}_{12} + \hat{\vec{\kappa}}_3 = \vec{P}_{(\text{int})} \approx 0,$$

$$\hat{\vec{\rho}}_{(12)3} = \hat{\vec{\eta}}_{12} - \hat{\vec{\eta}}_3, \quad \hat{\vec{\pi}}_{(12)3} = \frac{1}{2}(\hat{\vec{\kappa}}_{12} - \hat{\vec{\kappa}}_3),$$

$$\hat{\vec{\eta}}_{12} = \hat{\vec{\eta}} + \frac{1}{2} \hat{\vec{\rho}}_{(12)3}, \quad \hat{\vec{\kappa}}_{12} = \frac{1}{2} \hat{\vec{\kappa}} + \hat{\vec{\pi}}_{(12)3} \approx \hat{\vec{\pi}}_{(12)3},$$

$$\hat{\vec{\eta}}_3 = \hat{\vec{\eta}} - \frac{1}{2} \hat{\vec{\rho}}_{(12)3}, \quad \hat{\vec{\kappa}}_3 = \frac{1}{2} \hat{\vec{\kappa}} - \hat{\vec{\pi}}_{(12)3} \approx \hat{\vec{\pi}}_{(12)3},$$

so that we get

$$\hat{\vec{\eta}}_1 = \hat{\vec{\eta}} + \frac{1}{2} \hat{\vec{\rho}}_{(12)3} + \frac{m_2}{m} \hat{\vec{\rho}}_{12},$$

$$\hat{\vec{\eta}}_2 = \hat{\vec{\eta}} + \frac{1}{2} \hat{\vec{\rho}}_{(12)3} - \frac{m_1}{m} \hat{\vec{\rho}}_{12},$$

$$\vec{X}_{\text{rad}} = \hat{\vec{\eta}}_3 = \hat{\vec{\eta}} - \frac{1}{2} \hat{\vec{\rho}}_{(12)3},$$

$$\hat{\vec{\kappa}}_1 = \frac{m_1}{m} \left( \frac{1}{2} \hat{\vec{\kappa}} + \hat{\vec{\pi}}_{(12)3} \right) + \hat{\vec{\pi}}_{12} \approx \frac{m_1}{m} \hat{\vec{\pi}}_{(12)3} + \hat{\vec{\pi}}_{12},$$

$$\hat{\vec{\kappa}}_2 = \frac{m_2}{m} \frac{1}{2} \hat{\vec{\kappa}} + \hat{\vec{\pi}}_{(12)3} - \hat{\vec{\pi}}_{12} \approx \frac{m_2}{m} \hat{\vec{\pi}}_{(12)3} - \hat{\vec{\pi}}_{12},$$

$$\vec{P}_{\text{rad}} = \hat{\vec{\kappa}}_3 = \frac{1}{2} \hat{\vec{\kappa}} - \hat{\vec{\pi}}_{(12)3} \approx -\hat{\vec{\pi}}_{(12)3}.$$  (4.8)

Therefore our final canonical basis contains the collective variables $\hat{\vec{\eta}}, \hat{\vec{\kappa}} \approx 0$ and the relative variables $\hat{\vec{\rho}}_{12}, \hat{\vec{\pi}}_{12}$ (depending only upon the particles), $K_{\text{rad}}(k), H_{\text{rad}}(k), \lambda_{\text{rad}}(k)$, $\rho_{\text{rad}}(k)$ (depending only upon the "multipoles" the transverse radiation field) and $\hat{\vec{\rho}}_{(12)3}, \hat{\vec{\pi}}_{(12)3}$ (depending on the relative motion of the particle collective variables with respect to the radiation field collective variables, i.e. its "monopole" aspect).

The overall collective variable $\hat{\vec{\eta}}$ is the natural variable for the solution of the gauge fixings $\vec{K}_{(\text{int})} \approx 0$.  

The overall collective variable $\hat{\vec{\eta}}$ is the natural variable for the solution of the gauge fixings $\vec{K}_{(\text{int})} \approx 0$.  

24
E. The Internal 3-Center of Mass $\hat{\eta}$ from the vanishing of the Internal Boosts after the Canonical Transformation.

We have to find $\hat{\eta}$ of Eq.(4.7) from the vanishing of the internal boosts $\vec{K}_{(int)} \approx 0$ in the form (4.2), put it into Eqs.(4.8) and make the inverse canonical transformation (I-3.10) to find the original $\hat{\eta}_{i}(\tau)$ and the particle world-lines $x_{i}^{\mu}(\tau)$ like we made in Eqs.(2.8).

In the boosts in the form (4.2) the first four lines of the particle terms depend on $\hat{\rho}_{12} = \hat{\eta}_{1} - \hat{\eta}_{2}$ and $\hat{\pi}_{12} = \frac{m_{2}}{m} \hat{\kappa}_{1} - \frac{m_{1}}{m} \hat{\kappa}_{2}$, but they also get a dependence on $\hat{\pi}_{(12)3} = \frac{1}{2} (\hat{\kappa}_{1} + \hat{\kappa}_{2} - \hat{P}_{rad})$ through the dependence on the particle momenta. In the next two particle terms we must shift the integration variable to reabsorb the quantity $\hat{\eta} + \frac{1}{2} \hat{\rho}_{(12)3}$ present in $\vec{\sigma} - \hat{\eta}_{i}$.

Finally the last line of the boosts in Eqs.(4.2) is just $\vec{K}_{rad}$ of Eqs.(3.8).

Therefore the internal boosts in the form (4.2) has the following form in the new canonical basis ($\hat{\kappa}_{i}(\tau)$ are given by Eqs.(4.8))

$$
\vec{K}_{(int)} = -\left(\hat{\eta} + \frac{1}{2} \hat{\rho}_{(12)3}\right) \left(\sum_{i=1}^{2} \sqrt{m_{i}^{2} c^{2} + \hat{\kappa}_{i}^{2}} + \frac{1}{2} Q_{1} Q_{2} c \right) \times \\
\times \left[ \frac{\hat{\kappa}_{1} \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\kappa}_{1}, \hat{\kappa}_{2}, \hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \vec{A}_{\perp S_{2}}(\hat{\kappa}_{2}, \hat{\rho}_{12}) \right)}{\sqrt{m_{1}^{2} c^{2} + \hat{\kappa}_{1}^{2}}} + \right. \\
\left. + \frac{\hat{\kappa}_{2} \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\kappa}_{1}, \hat{\kappa}_{2}, \hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \vec{A}_{\perp S_{1}}(\hat{\kappa}_{1}, \hat{\rho}_{12}) \right)}{\sqrt{m_{2}^{2} c^{2} + \hat{\kappa}_{2}^{2}}} \right] \\
- \hat{\rho}_{12} \left( \frac{m_{2}}{m} \right) \sqrt{m_{1}^{2} c^{2} + \hat{\kappa}_{1}^{2}} - \frac{m_{1}}{m} \sqrt{m_{2}^{2} c^{2} + \hat{\kappa}_{2}^{2}} + \\
+ \frac{1}{2} Q_{1} Q_{2} c \left[ \frac{m_{2}}{m} \hat{\kappa}_{1} \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\kappa}_{1}, \hat{\kappa}_{2}, \hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \vec{A}_{\perp S_{2}}(\hat{\kappa}_{2}, \hat{\rho}_{12}) \right) \right]
$$
where we have introduced the following definitions

\[
\begin{align*}
&\hat{\eta}_2 \approx \left( \frac{1}{2} \frac{\partial \hat{\kappa}_{12}(\hat{\sigma}_{12},\hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{12}(\hat{\rho}_{12}) \right) \quad - \\
&- \frac{1}{2} \frac{Q_1 \cdot Q_2}{c} \left( \sqrt{m_1^2 c^2 + \hat{\kappa}_1^2} \frac{\partial}{\partial \hat{\kappa}_1} + \sqrt{m_2^2 c^2 + \hat{\kappa}_2^2} \frac{\partial}{\partial \hat{\kappa}_2} \right) \hat{\kappa}_{12}(\hat{\sigma}_{12},\hat{\rho}_{12}) - \\
&- \frac{Q_1 \cdot Q_2}{4 \pi c} \int d^3 \sigma \left( \frac{\hat{\pi}_{12}(\hat{\sigma} - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1)}{\left| \hat{\sigma} + \frac{m_1}{m} \hat{\rho}_{12} \right|} + \frac{\hat{\pi}_{12}(\hat{\sigma} + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2)}{\left| \hat{\sigma} - \frac{m_2}{m} \hat{\rho}_{12} \right|} \right) - \\
&- \frac{Q_1 \cdot Q_2}{c} \int d^3 \sigma \left[ \hat{\eta} + \hat{\eta} \right] \left( \hat{\pi}_{12}(\hat{\sigma} - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{\pi}_{12}(\hat{\sigma} + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) + \\
&+ \hat{B}_{12}(\hat{\sigma} - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{B}_{12}(\hat{\sigma} + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) \right) - \\
&- X_{rad}^r \hat{\eta}_{(12)3} - M_{rad} c \left( \hat{\eta} - \frac{1}{2} \hat{\rho}_{(12)3} \right) + \hat{D}(H_{rad}, K_{rad}, \lambda_{rad}, \rho_{rad}) \approx 0,
\end{align*}
\]

(4.9)

where the term \( \hat{D}(H_{rad}, K_{rad}, \lambda_{rad}, \rho_{rad}) \) has the following expression

\[
\begin{align*}
D^r(H_{rad}, K_{rad}, \lambda_{rad}, \rho_{rad}) &= -\frac{1}{c} \int d \kappa H_{rad}(\kappa) D^r K_{rad}(\kappa) - \frac{1}{c} \int d \kappa \lambda_{rad}(\kappa) D^r \rho_{rad}(\kappa) - \\
&- \frac{i}{2c} \int d \kappa \lambda_{rad}(\kappa) \left[ \hat{e}_+(\kappa) + \hat{e}_-(\kappa) \right] \cdot D^r \left[ \hat{e}_-(\kappa) - \hat{e}_+(\kappa) \right].
\end{align*}
\]

(4.10)

The solution of Eq.(4.9) is

\[
\hat{\eta} \approx \frac{1}{A + M_{rad} c} \left( - \frac{1}{2} (A - M_{rad} c) \hat{\rho}_{(12)3} - \\
- \left[ \frac{m_2}{m} \sqrt{m_2^2 c^2 + \left( \hat{\pi}_{12} + \frac{m_1}{m} \hat{\pi}_{(12)3} \right)^2} - \frac{m_1}{m} \sqrt{m_2^2 c^2 + \left( \hat{\pi}_{12} - \frac{m_2}{m} \hat{\pi}_{(12)3} \right)^2} + B \right] \hat{\rho}_{12} - C + \\
+ X_{rad}^r \hat{\pi}_{(12)3} + \hat{D} \right),
\]

(4.11)

where we have introduced the following definitions \((\hat{\kappa}_i(\tau) \) are given by Eqs.(4.8))
\[ A(\hat{\rho}_{12}, \hat{\pi}_{12}, \hat{\pi}_{(12)3}) = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} + \frac{1}{2} \frac{Q_1 Q_2}{c} \times \]

\[ \times \left[ \frac{\hat{\kappa}_1 \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\rho}_{12}, \hat{\pi}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S2}(\hat{\pi}_{12}, \hat{\rho}_{12}) \right)}{\sqrt{m_1^2 c^2 + \hat{\kappa}_1^2}} \right. \]

\[ \left. + \frac{\hat{\kappa}_2 \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\rho}_{12}, \hat{\pi}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S1}(\hat{\rho}_{12}, \hat{\pi}_{12}) \right)}{\sqrt{m_2^2 c^2 + \hat{\kappa}_2^2}} \right] - \]

\[ - \frac{Q_1 Q_2}{c} \int d^3 \sigma \left( \hat{\pi}_{\perp S1}(\sigma - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{\pi}_{\perp S2}(\sigma + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) + \right. \]

\[ + \hat{B}_{S1}(\sigma - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{B}_{S2}(\sigma + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) \right). \]

\[ B(\hat{\rho}_{12}, \hat{\pi}_{12}, \hat{\pi}_{(12)3}) = \frac{1}{2} \frac{Q_1 Q_2}{c} \left[ \frac{m_2}{m} \hat{\kappa}_1 \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\rho}_{12}, \hat{\pi}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S2}(\hat{\pi}_{12}, \hat{\rho}_{12}) \right) \right. \]

\[ - \frac{m_1}{m} \hat{\kappa}_2 \cdot \left( \frac{1}{2} \frac{\partial \hat{K}_{12}(\hat{\rho}_{12}, \hat{\pi}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S1}(\hat{\rho}_{12}, \hat{\pi}_{12}) \right) \right]. \]

\[ \mathcal{C}(\hat{\rho}_{12}, \hat{\pi}_{12}, \hat{\pi}_{(12)3}) = \frac{1}{2} \frac{Q_1 Q_2}{c} \left( \sqrt{m_1^2 c^2 + \hat{\kappa}_1^2} \frac{\partial}{\partial \hat{\kappa}_1} + \sqrt{m_2^2 c^2 + \hat{\kappa}_2^2} \frac{\partial}{\partial \hat{\kappa}_2} \right) \hat{K}_{12}(\hat{\rho}_{12}, \hat{\kappa}_1, \hat{\kappa}_2) - \]

\[ - \frac{Q_1 Q_2}{4 \pi c} \int d^3 \sigma \left( \hat{\pi}_{\perp S1}(\sigma - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \right|_{\sigma = \frac{m_1}{m} \hat{\rho}_{12}} \right. \]

\[ + \hat{\pi}_{\perp S2}(\sigma + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) \right) - \]

\[ - \frac{Q_1 Q_2}{c} \int d^3 \sigma \left( \hat{\pi}_{\perp S1}(\sigma - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{\pi}_{\perp S2}(\sigma + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) + \right. \]

\[ + \hat{B}_{S1}(\sigma - \frac{m_2}{m} \hat{\rho}_{12}, \hat{\kappa}_1) \cdot \hat{B}_{S2}(\sigma + \frac{m_1}{m} \hat{\rho}_{12}, \hat{\kappa}_2) \right) \right). \hspace{1cm} (4.12) \]

Since, as shown in Eqs.(4.2), we have \( \vec{K}_{\text{int}} = c \vec{K}_{\text{Galilei}} + O(\frac{1}{c}) \approx 0 \) we have \( \vec{\eta} = \vec{x}_{(n)} + O(c^{-2}) \approx 0 \) in the non-relativistic limit, namely we get the non-relativistic rest frame \( \vec{K}_{\text{Galilei}} = -m \vec{x}_{(n)} = -\sum_{i=1}^{2} m_i \vec{\eta}_i \approx 0 \).

By using Eqs. (4.11), (3.8), (4.8) and the rest-frame condition \( \vec{P}_{\text{int}} = \vec{\kappa}(\tau) \approx 0 \), we get the following expressions of the internal invariant mass and of the internal angular momentum in Eqs.(4.2)
\[ \mathcal{E}_{(\text{int})} = M c^2 \approx M_{\text{rad}} c^2 + \]
\[ + c \sqrt{m_1^2 c^2 + (\hat{\tilde{\pi}}_{12} + \frac{m_1}{m} \hat{\tilde{\pi}}_{(12)3})^2} + c \sqrt{m_2^2 c^2 + (\hat{\tilde{\pi}}_{12} - \frac{m_2}{m} \hat{\tilde{\pi}}_{(12)3})^2} + \]
\[ + \frac{Q_1 Q_2}{4 \pi |\hat{\rho}_{12}|} + V_{\text{DARWIN}}(\hat{\rho}_{12}; \hat{\tilde{\pi}}_{12} + \frac{m_1}{m} \hat{\tilde{\pi}}_{(12)3}; -\hat{\tilde{\pi}}_{12} + \frac{m_2}{m} \hat{\tilde{\pi}}_{(12)3}), \]
\[ M_{\text{rad}} c^2 = \frac{1}{2} \int d^3 \sigma \left[ \hat{\pi}_{12}^2 + B^2_{\text{rad}} \right] (\tau, \sigma), \quad (4.13) \]

\[ \tilde{\mathcal{J}}_{(\text{int})} = \tilde{\mathcal{S}} \approx \hat{\rho}_{12} \times \hat{\tilde{\pi}}_{12} + \hat{\rho}_{(12)3} \times \hat{\tilde{\pi}}_{(12)3} + \tilde{S}_{\text{rad}}, \]
\[ \tilde{S}_{\text{rad}} = -\frac{1}{c} \int dk H_{\text{rad}}(k) \hat{k} \times \frac{\partial}{\partial k} K_{\text{rad}}(k) - \frac{1}{c} \int dk \lambda_{\text{rad}}(k) \hat{k} \times \frac{\partial}{\partial k} \rho_{\text{rad}}(k) + \]
\[ + \frac{1}{c} \int dk \lambda_{\text{rad}}(k) \left[ \tilde{e}_- (\hat{k}) \times \tilde{e}_+ (\hat{k}) + \frac{i}{2} \left( \tilde{e}_- (\hat{k}) + \tilde{e}_+ (\hat{k}) \right) \cdot \hat{k} \times \frac{\partial}{\partial k} \left( \tilde{e}_- (\hat{k}) - \tilde{e}_+ (\hat{k}) \right) \right]. \quad (4.14) \]

Eqs.(4.7), (4.8) and (4.9) imply

\[ \hat{\eta}_{12} = \hat{\eta} + \frac{1}{2} \hat{\rho}_{(12)3} \approx \frac{1}{A + M_{\text{rad}} c} \left( M_{\text{rad}} c \hat{\rho}_{(12)3} - \right. \]
\[ - \left[ \frac{m_2}{m} \sqrt{m_1^2 c^2 + (\hat{\tilde{\pi}}_{12} + \frac{m_1}{m} \hat{\tilde{\pi}}_{(12)3})^2} - \frac{m_1}{m} \sqrt{m_2^2 c^2 + (\hat{\tilde{\pi}}_{12} - \frac{m_2}{m} \hat{\tilde{\pi}}_{(12)3})^2} + B \right] \hat{\rho}_{12} - \]
\[ - \tilde{C} + X_{\text{rad}}^\tau \hat{\tilde{\pi}}_{(12)3} + \tilde{D}, \]
\[ \hat{\eta}_1 \approx \frac{1}{A + M_{\text{rad}} c} \left( M_{\text{rad}} c \hat{\rho}_{(12)3} + \left[ \frac{m_2}{m} \left( A + M_{\text{rad}} c - \sqrt{m_1^2 c^2 + (\hat{\tilde{\pi}}_{12} + \frac{m_1}{m} \hat{\tilde{\pi}}_{(12)3})^2} \right) + \right. \]
\[ + \frac{m_1}{m} \sqrt{m_2^2 c^2 + (\hat{\tilde{\pi}}_{12} - \frac{m_2}{m} \hat{\tilde{\pi}}_{(12)3})^2} + B \right] \hat{\rho}_{12} - \tilde{C} + X_{\text{rad}}^\tau \hat{\tilde{\pi}}_{(12)3} + \tilde{D}, \]
\[ \hat{\eta}_2 \approx \frac{1}{A + M_{\text{rad}} c} \left( M_{\text{rad}} c \hat{\rho}_{(12)3} - \left[ \frac{m_2}{m} \sqrt{m_1^2 c^2 + (\hat{\tilde{\pi}}_{12} + \frac{m_1}{m} \hat{\tilde{\pi}}_{(12)3})^2} + \frac{m_1}{m} \left( A + M_{\text{rad}} c - \right. \right. \]
\[ - \sqrt{m_2^2 c^2 + (\hat{\tilde{\pi}}_{12} - \frac{m_2}{m} \hat{\tilde{\pi}}_{(12)3})^2} + B \right] \hat{\rho}_{12} - \tilde{C} + X_{\text{rad}}^\tau \hat{\tilde{\pi}}_{(12)3} + \tilde{D} \right). \quad (4.15) \]

These equations, together with the inverse canonical transformation (I-3.10) and by using Eqs.(I-2.51), (I-2.52), (I-3.9), (3.8) and (4.8), allow to get the original variables \( \hat{\eta}_i(\tau) \) and the world-lines (I-2.18) of the particles only in terms of relative variables.
\[ x_1'(\tau) = Y^e(\tau) + \epsilon^e(\hat{\eta}) \eta_1', \]
\[ x_2'(\tau) = Y^e(\tau) + \epsilon^e(\hat{\eta}) \eta_2', \]

\[ \eta_1'(\tau) \approx \hat{\eta}_1'(\tau) - \frac{Q_1}{c} \frac{m}{m_1} \frac{\partial}{\partial \hat{\pi}(12)_3} \int d^3\sigma \left[ \hat{\pi}_{1\text{rad}}(\tau - X^r_{\text{rad}}, \hat{\sigma} + \frac{1}{2} \hat{\rho}(12)_3) \right] \cdot 
\cdot \hat{A}_{1.1}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 - \frac{m_2}{m} \hat{\rho}_{12}, \frac{m_1}{m} \hat{\pi}(12)_3 + \hat{\pi}_{12}) - 
\cdot \hat{A}_{1.2}(\tau - X^r_{\text{rad}}, \hat{\sigma} + \frac{1}{2} \hat{\rho}(12)_3) \cdot 
\cdot \hat{\pi}_{1.1}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12} - 
\cdot \hat{\pi}_{1.2}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) \cdot 
\cdot \hat{A}_{1.2}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) , \]

\[ \eta_2'(\tau) \approx \hat{\eta}_2'(\tau) - \frac{Q_2}{c} \frac{m}{m_2} \frac{\partial}{\partial \hat{\pi}(12)_3} \int d^3\sigma \left[ \hat{\pi}_{1\text{rad}}(\tau - X^r_{\text{rad}}, \hat{\sigma} + \frac{1}{2} \hat{\rho}(12)_3) \right] \cdot 
\cdot \hat{A}_{1.2}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) - 
\cdot \hat{A}_{1.1}(\tau - X^r_{\text{rad}}, \hat{\sigma} + \frac{1}{2} \hat{\rho}(12)_3) \cdot 
\cdot \hat{\pi}_{1.2}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) + 
\cdot \hat{\pi}_{1.1}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) \cdot 
\cdot \hat{\pi}_{1.2}(\hat{\sigma} - \frac{1}{2} \hat{\rho}(12)_3 + \frac{m_1}{m} \hat{\rho}_{12}, \frac{m_2}{m} \hat{\pi}(12)_3 - \hat{\pi}_{12}) , \]

(4.16)

F. The Final Relative Canonical Variables

The final independent canonical relative variables are
a) $\hat{\rho}_{12}(\tau), \hat{\pi}_{12}(\tau)$ (relative motion of the two particles);

b) $\hat{\rho}_{(12)3}(\tau), \hat{\pi}_{(12)3}(\tau)$ (relative motion of the ”particle system” with respect to the ”radiation field system”);

c) $X_{\text{rad}}^{\tau}(\tau), P_{\text{rad}}^{\tau}(\tau) = \frac{1}{c} E_{\text{rad}}(\tau) = M_{\text{rad}}(\tau) c$ (the energy of the radiation field and its conjugate temporal variable, the only surviving collective variables);

d) $H_{\text{rad}}(\vec{k}), K_{\text{rad}}(\vec{k})$ (relative multipoles of the radiation field with respect to its monopole-like collective variables);

e) $\lambda_{\text{rad}}(\vec{k}), \rho_{\text{rad}}(\vec{k})$ (relative variables describing the helicity degrees of freedom of the radiation field).

These variables satisfy Hamilton equations having $E_{\text{(int)}}$ of Eq.(4.13) as Hamiltonian.

There are the following constants of motion:

A) The relative 3-momentum $\hat{\pi}_{(12)3} = \frac{1}{2} (\hat{k}_1 + \hat{k}_2 - \hat{P}_{\text{rad}}) \approx -\hat{P}_{\text{rad}} \approx \hat{k}_1 + \hat{k}_2$: 

$$\{\hat{\pi}_{(12)3}, E_{\text{(int)}}\} = 0.$$  

B) The energy of the radiation field: $\{P_{\text{rad}}^{\tau}, E_{\text{(int)}}\} = 0$. Its conjugate variable $X_{\text{rad}}^{\tau}(\tau), \{X_{\text{rad}}^{\tau}, P_{\text{rad}}^{\tau}\} = -1$, satisfies

$$\frac{d X_{\text{rad}}^{\tau}}{d \tau} = \{X_{\text{rad}}^{\tau}, E_{\text{(int)}}\} = -1.$$  

B) The energy of the radiation field: $\{P_{\text{rad}}^{\tau}, E_{\text{(int)}}\} = 0$. Its conjugate variable $X_{\text{rad}}^{\tau}(\tau), \{X_{\text{rad}}^{\tau}, P_{\text{rad}}^{\tau}\} = -1$, satisfies

Like for the Klein-Gordon case of Appendix A, the two second class constraints $P_{\text{rad}}^{\tau} - E_{\text{rad}} \approx 0, X_{\text{rad}}^{\tau} - \tau \approx 0$, eliminating the last two collective variables, select a symplectic sub-manifold of the surface of constant energy $P_{\text{rad}}^{\tau} = E_{\text{rad}}$ of the radiation field.

C) The multipoles $H_{\text{rad}}(\vec{k}), K_{\text{rad}}(\vec{k}), \lambda_{\text{rad}}(\vec{k}), \rho_{\text{rad}}(\vec{k})$.

Therefore the two subsystems (particles and radiation field), although not coupled in the equations of motion due to $E_{\text{(int)}} = E_{\text{matter}} + E_{\text{rad}}$, are nevertheless effectively interacting through the rest-frame constraints and their gauge fixings as clear from Eq.(4.17).
The canonical variables $\hat{q}_+ = \hat{\kappa}_{12}$, $\hat{\kappa}_+ = \hat{\kappa}_{12}$, $\hat{\rho}_q$, $\hat{\pi}_q$, describing the canonical (Coulomb-dressed) Newton-Wigner 3-center of mass and the relative motion of the 2-particle subsystem (see Subsection IIA), could be found by using Eqs.(4.7) and (4.8). Then we could re-express $E_{(int)}$ of Eq.(4.13) and $\hat{J}_{(int)}$ of Eq.(4.14) in terms of them. In particular these variables would allow to write $E_{(int)}$ of Eq.(4.13) in the form

$$E_{(int)} \approx c \sqrt{\hat{M}_o c^2 + \hat{\pi}_q^2 / 3} + \left( \text{function of } \hat{\rho}_q, \hat{\pi}_q, \hat{\pi}_{(12)3} \ldots \right) + P_{rad}^r,$$

$$\hat{M}_o c = \sqrt{m_1^2 c^2 + \hat{\pi}_q^2} + \sqrt{m_2^2 c^2 + \hat{\pi}_q^2}.$$  \hspace{1cm} (4.19)

In this way the 2-particle subsystem is visualized as an effective pseudo-particle (an atom after quantization) of mass $\hat{M}_o$ plus interactions. However a drawback of these variables is that the Coulomb interaction depends upon $\hat{\rho}_{12}$ and not upon $\hat{\rho}_q$, variables which do not coincide for $\hat{\kappa} \neq 0$.

G. Coming back with the Inverse Canonical Transformation

If we use the inverse canonical transformation (I-3.10) of I, Eqs.(4.7)-(4.8) and the consequences (4.11)-(4.15) of $\hat{K}_{(int)} \approx 0$, we can get the expression of $\hat{P}_{em}$, $\hat{J}_{em}$ and $P_{em}^r$ of Eqs.(4.1) in terms of the radiation field and of the Coulomb-dressed particles from the comparison of the internal Poincare’ generators before and after the canonical transformation.

Eqs. (4.1) and (4.2) imply

$$\hat{P}_{em} = \frac{1}{c} \int d^3 \sigma \left( \hat{\pi}_\perp \times \hat{B} \right)(\tau, \vec{\sigma}) \approx$$

$$\approx \hat{P}_{rad} - \sum_{i=1}^{2} \frac{Q_i}{c} \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i} - \frac{1}{2} \frac{Q_1 Q_2}{c} \left( \frac{\partial}{\partial \hat{\eta}_1} - \frac{\partial}{\partial \hat{\eta}_2} \right) \hat{K}_{12}(\tau) =$$

$$= \frac{1}{c} \int d^3 \sigma \left( \hat{\pi} \perp_{rad} \times \hat{B}_{rad} \right)(\tau, \vec{\sigma}) + O(\frac{1}{c}) \approx$$

$$\approx -\hat{\pi}_{(12)3} + O(\frac{1}{c}).$$  \hspace{1cm} (4.20)
\[ \mathcal{J}_{em} = \frac{1}{c} \int d^3 \sigma \, \vec{\sigma} \times (\vec{\pi}_\perp \times \vec{B})(\tau, \vec{\sigma}) \approx \]
\[ \approx \mathcal{S}_{rad} - \left( \hat{\eta} - \frac{1}{2} \hat{\mu}_{(12)3} \right) \times \hat{\pi}_{(12)3} - \]
\[ - \sum_{i=1}^{2} \frac{Q_i}{c} \left( \hat{\eta}_i \times \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i} + \hat{\kappa}_i \times \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\kappa}_i} \right) - \]
\[ - \frac{1}{2} \frac{Q_1 Q_2}{c} \sum_{i=1}^{2} \left( \hat{\eta}_i \times \frac{\partial \hat{K}_{12}(\tau)}{\partial \hat{\eta}_i} + \hat{\kappa}_i \times \frac{\partial \hat{K}_{12}(\tau)}{\partial \hat{\kappa}_i} \right) = \]
\[ = \mathcal{S}_{rad} + O(\frac{1}{c}), \quad (4.21) \]

\[ \mathcal{P}_{em}^\tau = \frac{1}{2c} \int d^3 \sigma \left( \vec{\pi}_\perp^2 + \vec{B}^2 \right)(\tau, \vec{\sigma}) = \]
\[ = \mathcal{P}_{rad}^\tau + \sum_{i=1}^{2} \left( \sqrt{m_i^2 c^2 + \frac{1}{4} \hat{\pi}_{(12)3}^2 + \frac{(-)^{i+1}}{4} \hat{\pi}_{12}^2} \right)^2 - \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} \right) + \]
\[ + \frac{Q_1 Q_2}{4\pi c |\hat{\rho}_{12}|} - \frac{Q_1 Q_2}{4\pi c |\hat{\rho}_{12}|} + \frac{1}{c} V_{Darwin}(\hat{\rho}_{12}, \hat{\pi}_{12}, \hat{\pi}_{(12)3}) + \]
\[ + \sum_{i=1}^{2} \frac{Q_i}{c} \frac{\hat{\kappa}_i \cdot \hat{A}_\perp(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} = \]
\[ = \frac{1}{2c} \int d^3 \sigma \left( \vec{\pi}_\perp^{rad} + \vec{B}_{}^{rad} \right)(\tau, \vec{\sigma}) + O(\frac{1}{c^2}). \quad (4.22) \]

This is in accord with Eq.(1-2.49) whose semi-relativistic limit \( a_{em \lambda}(\tau, \vec{k}) = a_\lambda(\vec{k}) e^{-i \omega(\vec{k}) \tau} + \omega(\vec{k}) \vec{e}_\lambda(\vec{k}) \cdot \sum_{i=1}^{2} \frac{Q_i}{m_i c} e^{-i \vec{k} \cdot \hat{\eta}_i(\tau)} \hat{k} \times \left( \hat{\eta}_i(\tau) \times \hat{k} \right) + O(c^{-3}) \) says the Fourier coefficients of the transverse electro-magnetic field differ from those of a transverse radiation field for particle terms of order \( O(c^{-1}) \). This is the error when we replace the electro-magnetic field with a radiation field, as is often done in atomic physics.
V. THE MULTIPOLAR EXPANSION OF THE PARTICLE ENERGY-MOMENTUM TENSOR

As said in Section I of paper I, in the rest-frame instant form the isolated system of charged particles plus the transverse electro-magnetic field can be described as a decoupled non-covariant center of mass carrying the invariant mass and the spin of the isolated system, i.e. some type of pole-dipole structure. The invariant mass and the spin are evaluated by means of the energy-momentum tensor $T^{AB}(\tau, \vec{\sigma}) = T^{AB}_{\text{matter}}(\tau, \vec{\sigma}) + T^{AB}_{\text{em}}(\tau, \vec{\sigma})$ determined by using the action (I-2.1) of the parametrized Minkowski theory.

Let us now consider the open subsystem composed only by the particles, whose non-conserved energy-momentum tensor is $T^{AB}_{\text{matter}}(\tau, \vec{\sigma})$. The study of its multipolar expansion (see Ref.[11]) allows one to replace the extended subsystem with its multipoles when some analyticity conditions are satisfied and then to define a pole-dipole approximation of the subsystem. As shown in Ref.[11], this can be done also for single strongly bound groups of particles, to be used to describe atoms after quantization, inside the particle subsystem by identifying the effective energy-momentum of each group.

Till now this is the only description of a relativistic (either free or interacting with the environment) composite system like an atom as a collective point-like system endowed with multipolar properties. The lowest approximation of the composite system is the pole-dipole approximation, in which the system is simulated with a point-like particle (monopole) of mass $M_c$ moving along the world-line of an effective 4-center of motion and carrying the spin dipole $\vec{J}_c$ (to be used for the magnetic dipole moment). In the rest-frame instant form the 4-center of motion has the world-line $w^\mu_c(\tau) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \zeta^r_c(\tau)$ and 4-momentum $P^\mu_c = h^\mu M_c c + \epsilon^\mu_r(\vec{h}) P^r_c$.

Let us remark that this pole-dipole approximation is not the “dipole approximation” of the semi-relativistic atomic physics, which will be studied in the next Section.

As shown in Ref.[11], given a non-isolated cluster of particles the main problem is the determination of an effective 4-center of motion described by 3-coordinates $\zeta^r_c(\tau)$ and with world-line $w^\mu_c(\tau) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \zeta^r_c(\tau)$ in the rest frame of the isolated system. The unit 4-velocity of this center of motion is $u^\mu_c(\tau) = \hat{w}^\mu_c(\tau)/\sqrt{1 - \dot{\zeta}^2_c(\tau)}$ with $\hat{w}^\mu_c(\tau) = h^\mu + \epsilon^\mu_r(\vec{h}) \dot{\zeta}^r_c(\tau)$ (from paper I we have $h^\mu = u^\mu(P) = P^\mu/Mc$). By using $\delta z^\mu(\tau, \vec{\sigma}) = \epsilon^\mu_r(\vec{h}) (\sigma^r - \zeta^r(\tau))$ we can define the Dixon multipoles of the cluster with respect to the world-line $w^\mu_c(\tau)$

$$q^{r_1...r_n,AB}_c(\tau) = \int d^3 \sigma \left[ \sigma^{r_1} - \zeta^{r_1}_c(\tau) \right]...\left[ \sigma^{r_n} - \zeta^{r_n}_c(\tau) \right] T^{AB}_{\text{matter}}(\tau, \vec{\sigma}).$$

The mass and momentum monopoles, and the mass, momentum and spin dipoles are, respectively

---

$^{16}$ See Ref. [11] for a discussion of the main choices present in the literature.
\[ q^{rr}_c = M_c, \quad q^{rr}_c = \mathcal{P}_c^r, \]
\[ q^{rr}_c = -\mathcal{K}_c^r - M_c \zeta_c^r(\tau) = M_c (R_c^r(\tau) - \zeta_c^r(\tau)), \quad q^{rr}_c = p^{ru}_c(\tau) - \zeta_c^r(\tau) \mathcal{P}_c^u, \]
\[
\begin{align*}
p^{ru}_c &= \int d^3 \sigma \sigma^r T_{\text{matter}}^{ru}(\tau, \vec{\sigma}) = \sum_{i=1}^2 \eta^r_i(\tau) \kappa^u_i(\tau) - \\
&- \sum_{i=1}^2 Q_i \int d^3 \sigma c(\vec{\sigma} - \vec{\eta}_i(\tau)) \left( \partial^r A^u_\perp + \partial_u A^r_\perp \right)(\tau, \vec{\sigma}), \\
p^{ru}_c + p^{ru}_c &= \sum_{i=1}^2 \left( \eta^r_i(\tau) \kappa^u_i(\tau) + \eta^u_i(\tau) \kappa^r_i(\tau) \right) - \\
&- 2 \sum_{i=1}^2 Q_i \int d^3 \sigma c(\vec{\sigma} - \vec{\eta}_i(\tau)) \left( \partial^r A^u_\perp + \partial_u A^r_\perp \right)(\tau, \vec{\sigma}) \\
p^{ru}_c - p^{ru}_c &\overset{\text{def}}{=} \epsilon^{ruv} \mathcal{J}_c^v, \\
S^{\mu\nu}_c &= \left[ \epsilon^\mu(\vec{h}) h^\nu - \epsilon^\nu(\vec{h}) h^\mu \right] q^{rr}_c + \epsilon^\mu(\vec{h}) \epsilon^\nu(\vec{h}) (q^{ru}_c - q^{ru}_c) = \\
&= \left[ \epsilon^\mu(\vec{h}) h^\nu - \epsilon^\nu(\vec{h}) h^\mu \right] M_c (R^r_c - \zeta_c^r) + \\
&+ \epsilon^\mu(\vec{h}) \epsilon^\nu(\vec{h}) \left[ \epsilon^{ruv} \mathcal{J}_c^v - (\zeta_c^r \mathcal{P}_c^u - \zeta_c^u \mathcal{P}_c^r) \right], \\
&\Rightarrow m^{\mu}_{c(p)} = -S^{\mu\nu}_c h_\nu = -\epsilon^\mu(\vec{h}) q^{rr}_c, \quad (5.2)
\end{align*}
\]

where \( \mathcal{J}_c \) and \( \mathcal{K}_c \) are the angular momentum and the boost associated with the chosen definition of effective center of motion.

As shown in Ref.[11], it is convenient to choose the center of energy \( \vec{R}_c = -\mathcal{K}_c/M_c \) as center of motion. \( \vec{R}_c \) and \( \mathcal{P}_c \) (momentum monopole) are the non-canonical variables describing the monopole, i.e. the collective pseudo-particle of non-conserved mass \( M_c \) (mass monopole).

On the world-line of this collective pseudo-particle there is also a spin dipole (the anti-symmetric part of the momentum dipole) and the symmetric part of the momentum dipole [17]. See Ref.[11], Eqs. (5.23) - (5.26).

Since we are in a Hamiltonian formulation, the constitutive relation between monopole 3-velocity \( \frac{d\vec{e}_c(\tau)}{d\tau} \) and monopole (non conserved) 3-momentum \( \mathcal{P}_c \) must not be added by hand, but can be determined for each choice of collective center of motion [11].

As shown in Ref.[11], the total 4-momentum \( P^{\mu}_c = \epsilon^\mu_A(\vec{h}) q^{Ar}_c = h^\mu M_c + \epsilon^\mu(\vec{h}) \mathcal{P}_c^r \) and the spin dipole tensor \( S^{\mu\nu}_c \) obey the Papapetrou-Dixon-Souriau equations.

---

17 This symmetric part of the momentum dipole depends on the electromagnetic potential and is connected with the electric dipole: it is directed along the associated relative momentum if one introduces a suitable definition of a relative variable.
A. The Multipolar Expansion after the Canonical Transformation.

Let us consider the open subsystem formed by the particles.

From Eqs.(4.13) and (4.14) we get

\[
E_{\text{matter}}(\tau) = c \sqrt{m_1^2 c^2 + (\hat{\kappa}_{12} + \frac{m_1}{m} \hat{\pi}_{(12)3})^2} + c \sqrt{m_2^2 c^2 + (\hat{\kappa}_{12} - \frac{m_2}{m} \hat{\pi}_{(12)3})^2} +
\]
\[
+ \frac{Q_1 Q_2}{4\pi |\hat{\rho}_{12}|} + V_{\text{DARWIN}}(\hat{\rho}_{12}, \hat{\pi}_{12} + \frac{m_1}{m} \hat{\pi}_{(12)3}; -\hat{\pi}_{12} + \frac{m_2}{m} \hat{\pi}_{(12)3}) \overset{\circ}{=} \text{constant},
\]

\[
\vec{P}_{\text{matter}}(\tau) = \sum_{i=1}^{2} \hat{\kappa}_i(\tau) \approx \hat{\pi}_{(12)3} \overset{\circ}{=} \text{constant},
\]

\[
\vec{J}_{\text{matter}}(\tau) = \sum_{i} \hat{\eta}_i(\tau) \times \hat{\kappa}_i(\tau) = \hat{\rho}_{12} \times \hat{\pi}_{12} + \left( \hat{\eta} + \frac{1}{2} \hat{\rho}_{(12)3} \right) \times \hat{\pi}_{(12)3} \overset{\circ}{=} \text{constant},
\]

(5.3)

While from Eq.(4.2) we get

\[
\hat{\vec{K}}_{\text{matter}} = -\sum_{i=1}^{2} \hat{\eta}_i \sqrt{m_1^2 c^2 + \hat{\kappa}_i^2} -
\]
\[
- \frac{1}{2} \frac{Q_1 Q_2}{c} \left[ \hat{\kappa}_1 \cdot \left( \frac{1}{2} \frac{\partial \hat{\kappa}_{12}(\hat{\kappa}_1, \hat{\kappa}_2, \hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S2}(\hat{\kappa}_2, \hat{\rho}_{12}) \right) \right] +
\]
\[
+ \hat{\eta}_2 \cdot \left( \frac{1}{2} \frac{\partial \hat{\kappa}_{12}(\hat{\kappa}_1, \hat{\kappa}_2, \hat{\rho}_{12})}{\partial \hat{\rho}_{12}} - 2 \hat{A}_{\perp S1}(\hat{\kappa}_1, \hat{\rho}_{12}) \right) \right] -
\]
\[
- \frac{1}{2} \frac{Q_1 Q_2}{c} \left( \sqrt{m_1^2 c^2 + \hat{\kappa}_1^2} \frac{\partial}{\partial \hat{\kappa}_1} + \sqrt{m_2^2 c^2 + \hat{\kappa}_2^2} \frac{\partial}{\partial \hat{\kappa}_2} \right) \hat{\kappa}_{12}(\hat{\kappa}_1, \hat{\kappa}_2, \hat{\rho}_{12}) -
\]
\[
- \frac{Q_1 Q_2}{4\pi c} \int d^3 \sigma \left( \frac{\hat{\pi}_{\perp S1}(\hat{\sigma} - \hat{\eta}_1, \hat{\kappa}_1)}{|\hat{\sigma} - \hat{\eta}_2|} + \frac{\hat{\pi}_{\perp S2}(\hat{\sigma} - \hat{\eta}_2, \hat{\kappa}_2)}{|\hat{\sigma} - \hat{\eta}_1|} \right) -
\]
\[
- \frac{Q_1 Q_2}{c} \int d^3 \sigma \hat{B}_{S1}(\hat{\sigma} - \hat{\eta}_1, \hat{\kappa}_1) \cdot \hat{B}_{S2}(\hat{\sigma} - \hat{\eta}_2, \hat{\kappa}_2) +
\]
\[
\overset{\text{def}}{=} -\frac{1}{c} E_{\text{matter}} \hat{R}_c(\tau).
\]

(5.4)

While the natural choice for the effective center of motion is center of energy \( \hat{\zeta}_c = \hat{R}_c = -c \hat{\vec{K}}_{\text{matter}} / E_{\text{matter}} \), Eq.(5.3) suggests the other possibility \( \hat{\zeta}_{\text{matter}}(\tau) = \hat{\eta}_{12} = \hat{\eta} + 1/2 \hat{\rho}_{(12)3} \) with \( \hat{\eta} \) given
in Eq. (4.11): it would imply the relation $\vec{J}_{\text{matter}}(\tau) = \hat{\rho}_{12} \times \hat{\pi}_{12} + \hat{\zeta}_{\text{matter}}(\tau) \times \hat{\pi}_{(12)3}$ and, differently from the choice $\vec{\zeta}_c = \vec{R}_v$, $\hat{\zeta}_{\text{matter}}$ is a canonical variable like $\vec{\eta}_{12}$ (however $\hat{\pi}_{(12)3}$ is not the conjugate variable).
VI. THE DIPOLE APPROXIMATION OF ATOMIC PHYSICS

Let us now look for the relativistic generalization of the electric dipole approximation of semi-relativistic atomic physics. To this end we consider a 2-particle system and we replace the particle 3-positions \( \vec{\eta}_i \) and 3-momenta \( \vec{\kappa}_i \), \( i = 1, 2 \), with the naive center of mass and relative variables of Eqs. (2.3): \( \vec{\eta}_{12}, \vec{\kappa}_{12}, \vec{\rho}_{12}, \vec{\pi}_{12} \).

The standard electric dipole moment is directed along the relative variable \( \vec{\rho}_{12} \). Usually only neutral systems with opposite charges of the particles are considered in applications of the dipole approximation. As a consequence it is convenient to introduce the following notation for the Grassmann-valued charges

\[
Q_1 = Q + \bar{Q}, \quad Q_2 = -Q + \bar{Q}, \quad Q_1^2 = Q_2^2 = 0, \quad Q_1 Q_2 = Q_2 Q_1 \neq 0,
\]

\[
Q = \frac{1}{2} (Q_1 - Q_2), \quad \bar{Q} = \frac{1}{2} (Q_1 + Q_2), \quad Q^2 = -Q_1 Q_2, \quad Q^2 = Q_1 Q_2 \text{ if } Q \neq 0.
\]  

(6.1)

The restriction to opposite charges is done by introducing the constraint \( Q \approx 0 \), which implies \( e = Q \approx Q_1 \approx -Q_2 \). This allows us to discard the terms in \( Q_1 Q_2 \) coming from \( Q_1 Q_2 \) coming from \( Q_2 \): only these terms will produce effects of order \( e^2 \) in a neutral system with Grassmann regularization (it eliminates the \( e^2 \) terms coming from \( Q_2^1 \) and \( Q_2^2 \)).

Strictly speaking it is only after the quantization of the Grassmann-valued charges \( Q_i \), sending each of them in a two-level system with charges \((+e, 0)\) or \((-e, 0)\), that we are really allowed to use \( e = Q_1 = -Q_2 \).

The natural definition of a Grassmann-valued electric dipole moment is

\[
\vec{d}(\tau) = Q \vec{\rho}_{12}(\tau) = \frac{Q_1 - Q_2}{2} \vec{\rho}_{12}(\tau) \rightarrow_{e=Q_1\approx-Q_2} e \vec{\rho}_{12}(\tau).
\]  

(6.2)

The alternative definition

\[
\vec{D}(\tau) = \sum_{i=1}^{2} Q_i \vec{\eta}_i(\tau) = \vec{d}(\tau) + 2 \bar{Q} \left[ \vec{\eta}_{12}(\tau) + \frac{m_2 - m_1}{2m} \vec{\rho}_{12}(\tau) \right] \rightarrow_{Q\approx 0} \vec{d}(\tau),
\]  

(6.3)

is equivalent to Eq. (6.2) for neutral systems.

The electric dipole has not to be confused with the Dixon spin dipole, which is oriented along the spin \( \vec{S} = \vec{\rho}_{12} \times \vec{\pi}_{12} \) and determines the direction of the magnetic dipole moment.

The length \( |\vec{\rho}_{12}(\tau)| \) is of the size of the atom, i.e. a few Bohr radii. For a monochromatic electromagnetic wave, i.e. for \( \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) \approx \vec{a} e^{i \vec{k} \cdot \vec{\eta}_{12}(\tau)} \) with \( |\vec{k}| = \frac{2\pi}{\lambda_{em}} \), we have

\[
\frac{1}{|\vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau))|} \left| \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) \right| \approx \vec{\rho}_{12}(\tau) \cdot \vec{k} \approx 2\pi \frac{\text{size of atom}}{\lambda_{em}}. \]

This ratio is \(< < 1\) in the long wavelength approximation \( \lambda_{em} >> \text{size of atom} \) (i.e. for radio-frequency, infrared, visible or ultraviolet radiation).
To get the dipole approximation [7] we use Eq.(2.3) and we make the following expansion

\[ m_3 \equiv m_1 \]

\[ \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) = \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) + (-)^{i+1} \frac{m_{i+1}}{m} \vec{\rho}_{i2}(\tau) = \]

\[ = \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) + (-)^{i+1} \frac{m_{i+1}}{m} \left( \vec{\rho}_{i2}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{i2}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) + \]

\[ + O([\vec{\rho}_{i2} \cdot \vec{\partial}_{\vec{\eta}_{i2}}]^2 \vec{A}_\perp). \] (6.4)

For the internal Poincare’ energy generator of Eqs.(4.1) we have \((\mu = m_1 m_2/m)\)

\[ \mathcal{E}_{(int)} = M c^2 = \]

\[ = c \left( \sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \vec{k}_{i2}(\tau) + \vec{\pi}_{i2}(\tau) \right)^2} + \sqrt{m_2^2 c^2 + \left( \frac{m_2}{m} \vec{k}_{i2}(\tau) - \vec{\pi}_{i2}(\tau) \right)^2} \right) - \]

\[ \frac{m_1 Q_1}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \vec{k}_{i2}(\tau) + \vec{\pi}_{i2}(\tau) \right)^2}} + \frac{m_2 Q_2}{\sqrt{m_2^2 c^2 + \left( \frac{m_2}{m} \vec{k}_{i2}(\tau) - \vec{\pi}_{i2}(\tau) \right)^2}} \]

\[ \vec{k}_{i2}(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) - \]

\[ \frac{Q_1}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \vec{k}_{i2}(\tau) + \vec{\pi}_{i2}(\tau) \right)^2}} - \frac{Q_2}{\sqrt{m_2^2 c^2 + \left( \frac{m_2}{m} \vec{k}_{i2}(\tau) - \vec{\pi}_{i2}(\tau) \right)^2}} \]

\[ \vec{\pi}_{i2}(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) - \]

\[ - \frac{\mu}{m} \left( \frac{Q_1}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \vec{k}_{i2}(\tau) + \vec{\pi}_{i2}(\tau) \right)^2}} - \frac{Q_2}{\sqrt{m_2^2 c^2 + \left( \frac{m_2}{m} \vec{k}_{i2}(\tau) - \vec{\pi}_{i2}(\tau) \right)^2}} \right) \vec{k}_{i2} \cdot (\vec{\rho}_{i2}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{i2}}) \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) - \]

\[ - \frac{m_2 Q_1}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \vec{k}_{i2}(\tau) + \vec{\pi}_{i2}(\tau) \right)^2}} + \frac{m_1 Q_2}{\sqrt{m_2^2 c^2 + \left( \frac{m_2}{m} \vec{k}_{i2}(\tau) - \vec{\pi}_{i2}(\tau) \right)^2}} \]

\[ \vec{\pi}_{i2}(\tau) \cdot (\vec{\rho}_{i2}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{i2}}) \vec{A}_\perp(\tau, \vec{\eta}_{i2}(\tau)) + \]

\[ + \frac{Q_1 Q_2}{4\pi |\vec{\rho}_{i2}(\tau)|} + \frac{1}{2} \int d^3\sigma [\vec{\pi}_2^2 + \vec{B}^2](\tau, \sigma) + O([\vec{\rho}_{i2} \cdot \vec{\partial}_{\vec{\eta}_{i2}}]^2 \vec{A}_\perp). \] (6.5)
Its semi-relativistic limit with the restriction \( e = Q_1 \approx -Q_2 \) is

\[
\mathcal{E}_{\text{(int)}} \to c^{-\infty}, e = Q_1 = Q_2 \quad mc^2 + \frac{\vec{k}_{12}^2(\tau)}{2m} + \frac{\vec{\pi}_{12}^2(\tau)}{2\mu} - \frac{e^2}{4\pi |\vec{\rho}_{12}(\tau)|} - \\
- \frac{e}{c} \frac{\vec{\rho}_{12}(\tau)}{\mu} \cdot \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) - \\
- \frac{e}{c} \frac{\vec{k}_{12}(\tau)}{m} \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) - \\
- \frac{e m_2 - m_1}{c m} \frac{\vec{\pi}_{12}(\tau)}{\mu} \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) + \\
+ O(c^{-2}) + O([\vec{\rho}_{12} \cdot \vec{\partial}_{\vec{\eta}_{12}}]^2 \vec{A}_{\perp}).
\]

(6.6)

In this way we recover a semi-classical version of Eqs. (L3), (L4) and (14.34) of Ref. [7], without the \( e^2 \) terms corresponding to \( Q_i^2 = 0 \).

If we could evaluate the internal 3-center of mass \( \vec{q}_+ \) (see Subsection IIA) as a function of \( \vec{\eta}_{12}, \vec{k}_{12} = \vec{\kappa} +, \vec{\rho}_{12}, \vec{\pi}_{12} \), then the results of Section IV would allow us to write the internal Poincaré generators (4.1) in the following form (see Subsection IIA for the relative variables \( \vec{\rho}_q, \vec{\pi}_q \))

\[
\mathcal{E}_{\text{(int)}} = M c^2 \approx c \sqrt{M_o^2 c^2 + \vec{\kappa}_+^2} + \left( \text{function of } \vec{q}_+, \vec{\kappa}_+, \vec{\rho}_q, \vec{\pi}_q \right) + \\
+ \mathcal{P}_{\text{em}}^r + O(\vec{\rho}_{12}^2),
\]

\[
\vec{\mathcal{P}}_{\text{(int)}} = \vec{\kappa}_+ + \vec{\mathcal{P}}_{\text{em}} \approx 0,
\]

\[
\vec{\mathcal{J}}_{\text{(int)}} = \vec{\eta}_{12} \times \vec{\kappa}_{12} + \vec{\rho}_{12} \times \vec{\pi}_{12} + \vec{\mathcal{J}}_{\text{em}} = \\
= \vec{q}_+ \times \vec{\kappa}_+ + \vec{\rho}_q \times \vec{\pi}_q + \vec{\mathcal{J}}_{\text{em}},
\]

\[
\vec{\mathcal{K}}_{\text{(int)}} = \vec{\mathcal{K}}[\vec{q}_+, \vec{\kappa}_+, \vec{\rho}_q, \vec{\pi}_q, \vec{A}_{\perp}, \vec{\pi}_{\perp}] \approx 0.
\]

(6.7)

\( \vec{\mathcal{K}}_{\text{(int)}} \approx 0 \) determines \( \vec{q}_+ = \vec{\eta} + .. \) as it was done in Section IV for \( \vec{\eta} \). In this way we would get \( \mathcal{E}_{\text{(int)}} \) as a function only of the relative variables \( \vec{\rho}_q, \vec{\pi}_q \) in a form useful for the electric dipole approximation.

While the external decoupled (canonical non-covariant) 4-center of mass \( \vec{x}^\mu, P^\mu \), has an effective mass \( M = \mathcal{E}_{\text{(int)}}/c^2 \) and a spin \( \vec{S} = \vec{\mathcal{J}}_{\text{(int)}} \), the particle subsystem (the atom) has the effective mass \( M_o \) of Eq.(6.7), a position \( \vec{q}_+ \), a 3-momentum \( \vec{\kappa}_+ \) and a spin given by the matter part of \( \vec{\mathcal{J}}_{\text{(int)}} \). These quantities replace \( M, \vec{C}, \vec{P}, \vec{\mathcal{J}}_c \) of the pole-dipole approximation to the multipolar expansion of the previous Section and give a canonical pole-dipole description of the atom. We have the following replacements.
\[ \vec{\zeta}(\tau) \mapsto \vec{\eta}_{12}(\tau) \text{ or } \vec{g}(\tau), \]
\[ \vec{P}_c(\tau) \mapsto \vec{\kappa}_{12}(\tau), \]
\[ M_c(\tau) \mapsto M_o(\text{relative variables}) + \text{Coulomb potential} + \text{interaction with the electro-magnetic field}, \]
\[ \vec{J}_c(\tau) \mapsto \text{matter part of } \vec{J}_{(\text{int})}. \]

The results of Section IV allow us to eliminate the overall internal center of mass and introduce a dependence on the variables \( \hat{\pi}_{(12)3} = \text{constant of motion} \) and \( \hat{\rho}_{(12)3} \), describing the relative motion of the atom with respect to the collective variables of the electro-magnetic field configuration.

Usually atoms are described not as the quantization of extended open subsystems with an effective 4-center of motion as in the previous Section, but as point-like systems (monopole approximation: \( M_c \mapsto m, \vec{\zeta}_c(\tau) \mapsto \vec{\eta}_{12}, \vec{P}_c(\tau) \mapsto \vec{\kappa}_{12} \)) with some additional structure (higher multipoles) describing the energy levels and the interaction with an electro-magnetic field. After quantization this point-like description of positive-energy atoms leads to the effective Schrödinger equation of Ref.[14] used to describe the external propagation (its de Broglie wave) in atom interferometry. Conceptually this effective Schrödinger equation should be derived by studying the positive-energy sector of solution of some wave equation with a fixed mass and a spin (spin dipole) \(^\dagger\)\(^\dagger\), which couples to the magnetic field. As shown in Ref.[16], the description of charged positive-energy spinning particles in the rest-frame instant form can be made by using Grassmann variables for the spin.

In this approximation with a fixed mass one looses all the internal structure of the atom, described by its energy levels. To remedy it one adds a finite-level structure to the point particle at the quantum level: this allows to consider more realistic approximations for the coupling to electric fields in the electric dipole approximation. The simplest model is the two-level atom approximation [7, 17].

As it will be shown in paper III, we can define a system in the rest-frame instant form, which after quantization leads to a two-level atom whose electric dipole interacts with the electric field after the transition to the electric dipole representation. There will be extra Grassmann degrees of freedom for the description of the two levels.

Therefore, we must now find the relativistic generalization of the electric dipole representation in the limit of the dipole approximation.

\(^\dagger\)\(^\dagger\) For instance the Dirac equation is used for the coupling to external gravitational fields and the study of gravito-inertial effects [14].

40
VII. THE RELATIVISTIC ELECTRIC DIPOLE REPRESENTATION

In atomic physics the electric dipole approximation suggested the introduction of the electric dipole representation, where the interaction term $\vec{A}_\perp \cdot \vec{r}_i$ is replaced with the interaction of the electric field with the electric dipole $\vec{d}$ of Eq.(6.2), $\vec{d} \cdot \vec{\pi}_\perp$. In this way there is no explicit dependence on the transverse electro-magnetic potential: only the transverse electric and magnetic fields appear.

This representation is a particular case of an equivalent formulation of electro-dynamics, as explained in Chapter IV of Ref[9]. Equivalent formulations are obtained

a) by a change of the gauge of the electro-magnetic field (here we are using the radiation gauge, a special case of Coulomb gauge);

b) by a unitary transformation $e^{i\hat{S}}$ corresponding to a classical canonical transformation whose generating function $S$

\[ S_1 = \frac{1}{c} \bar{d}(\tau) \cdot \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) \]

is determined by a change of the particle Lagrangian by means of a total time derivative $\frac{dS}{dt}$, with the electro-magnetic field considered as an external field with a given time dependence: the electric dipole representation is obtained with the Gøppert-Mayer unitary transformation (see p.635 of the Appendix of Ref.[8] and pp. 266-275 of Chapter IV of Ref.[9]), which is useful when $\sum_i Q_i = 0$ and whose classical generating function is $S_o = \frac{1}{c} \bar{d}(\tau) \cdot \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau))$ with $\bar{d}(\tau)$ of Eq.(6.2) (use is done of the dipole approximation)

\[ S_2 = \frac{1}{c} \bar{d}(\tau) \cdot \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) \]

is not determined by a change of Lagrangian but still with the electro-magnetic field considered as an external prescribed field 21;

\[ S_3 = \frac{1}{c} \bar{d}(\tau) \cdot \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) \]

is connected to a change of Lagrangian with a dynamical electro-magnetic field but with the dipole approximation replaced by a description of the localized system of charges by polarization and magnetization densities 22.

---

19 Strictly speaking the Appendix of Ref.[8] (see Eqs.(4) and subsequent one) makes the separation of the matter part from the radiation field in the Coulomb gauge and in the unitary transformation uses only the radiation electric field. Therefore our canonical transformation of I is the analogous separation in the radiation gauge (we get Darwin as an extra bonus!). The final Hamiltonian of Eq.(4.13) has the interaction reintroduced by the vanishing of the internal boosts (for the $1/c$ expansion see Eq.(4.5)). As a consequence a canonical transformation with generating function $\hat{S}'_o = \frac{1}{c} \bar{d}(\tau) \cdot \vec{A}_{\perp rad}(\tau, \vec{\eta}_{12}(\tau))$ should now work like it happened in the Appendix of Ref.[8].

20 A generalization of the Gøppert-Mayer transformation is given in Ref.[9]. Its generating function is $S_Z = \frac{1}{c} \bar{d}(\tau) \cdot \int_0^1 d\lambda \vec{A}_\perp (\tau, \lambda \vec{\eta}_1 (\tau) + (1 - \lambda) \vec{\eta}_2 (\tau)) = \frac{1}{c} \bar{d}(\tau) \cdot \int_0^1 d\lambda \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) + \frac{\Delta m_2 (1 - \lambda)}{m} \bar{\rho}_{12}(\tau) = \frac{1}{c} \bar{d}(\tau) \cdot \int_0^1 d\lambda \left[ \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) - \left( \frac{m_2}{m} \right) \bar{\rho}_{12}(\tau) \right] \vec{A}_\perp (\tau, \vec{\eta}_{12}(\tau)) + \ldots \right]$ with the integral taken along the straight-line joining the two charges. The first term is the Gøppert-Mayer generating function.

21 See the *Hennenberger unitary transformation*, given at pp.275-279 of Ref.[9], changing the positions instead of the momenta: the generating function is $\hat{S}_o = \frac{1}{c} \sum_i \frac{Q_i}{m} \vec{r}_i (\tau) \cdot \vec{Z}(\tau, \vec{0})$ with $\vec{Z}(\tau, \vec{0}) = - \int_{\tau_1}^{\tau} d\tau_1 \vec{A}_{\text{external}}(\tau_1, \vec{0})$. When the electromagnetic field is dynamical and not external, it is considered a quantized radiation field and the Hennenberg unitary transformation leads to the Pauli-Fierz-Kramers transformation.

22 See the *Power-Zienau-Woolley transformation* of pp. 280-297 of Ref.[9] generalizing the Gøppert-Meyer one.
In Appendix L of Ref.[7], the semi-relativistic electric dipole representation is obtained by starting from the particle Hamiltonian of a neutral 2-particle system in the dipole approximation (the electro-magnetic field is considered external), by evaluating the corresponding Lagrangian, by eliminating a suitable total time derivative $\frac{dS}{dt}$ and by reverting to the Hamiltonian.

A. The Semi-Relativistic Electric Dipole Representation with Grassmann-Valued Charges

In Subsection 1 of Appendix B the results of Ref.[7] on the electric dipole representation are revisited in the case of Grassmann-valued electric charges ($Q_i^2 = 0$, $Q_1 Q_2 \neq 0$) in the radiation gauge for the electro-magnetic field, which is treated as an external field, by starting from the Hamiltonian ($\frac{dF}{dt} = c \frac{dF}{d\tau} = \{F, H_c\}$)

$$H_c = \sum_{i=1}^{2} \left( \frac{\kappa_i(\tau) - \frac{Q_i}{c} \tilde{A} \cdot (\tau, \tilde{\eta}_i(\tau))}{2m_i} \right)^2 - \sum_{i=1}^{2} \frac{\kappa_i^2(\tau)}{2m_i} - \sum_{i=1}^{2} \frac{Q_i}{c} \kappa_i(\tau) \cdot \tilde{A}_\perp (\tau, \tilde{\eta}_i(\tau)),$$  \hspace{1cm} (7.1)

As shown in Appendix B, the final Hamiltonian (B6) becomes the Røntgen Hamiltonian (7.2) of the electric dipole representation, given in Eq.(14.37) of Ref.[7], for $2 \approx 0$, $e = Q_1$

$$H_{1c} = \frac{\kappa_{12}^2(\tau)}{2m} + \frac{\tilde{\pi}_{12}^2(\tau)}{2\mu} + V(\tilde{p}_{12}(\tau)) +$$
$$\frac{e}{mc} \tilde{p}_{12}(\tau) \cdot \left[ \tilde{p}_{12}(\tau) \times \tilde{B}(\tau, \tilde{\eta}_{12}(\tau)) \right] -$$
$$\frac{e}{2\mu c} \frac{m_2 - m_1}{m} \tilde{\pi}_{12}(\tau) \cdot \left[ \tilde{p}_{12}(\tau) \times \tilde{B}(\tau, \tilde{\eta}_{12}(\tau)) \right] -$$
$$\frac{e}{c} \tilde{p}_{12}(\tau) \cdot \tilde{\pi}_\perp (\tau, \tilde{\eta}_{12}(\tau)) - \frac{e}{2c} \frac{m_2 - m_1}{m} \left( \tilde{p}_{12}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}_{12}} \right) \tilde{p}_{12}(\tau) \cdot \tilde{\pi}_\perp (\tau, \tilde{\eta}_{12}(\tau)) -$$
$$\frac{e^2}{c} \frac{3\mu}{4m^2} \left[ \tilde{p}_{12}(\tau) \times \tilde{B}(\tau, \tilde{\eta}_{12}(\tau)) \right]^2.$$  \hspace{1cm} (7.2)

Note that the last term has not the coefficient $\frac{e^2}{8\mu c}$ of Eq.(15.37) or (L.14) of Ref.[7], because in our calculation the terms $Q_i^2 = 0$ are missing before imposing the condition $Q_1 + Q_2 \approx 0$.

The resulting generating function $S$ of Eq.(B3) is an extension to the next order of $S_o$, the generating function of the classical Gøppert-Mayer transformation.

B. A Generating Function from a Relativistic Lagrangian with External Electro-Magnetic Field.

Let us now try to define a relativistic electric dipole representation in the framework of the dipole approximation with the same method but starting from a particle Hamiltonian given
by \( \frac{1}{c} \left( \mathcal{E}_{(\text{int})} - \mathcal{E}_{\text{em}} \right) \) in accord with Eq.(4.1). In Subsection 2 of Appendix B we determine a relativistic Lagrangian for the particles in the dipole approximation treating the electromagnetic field as an external one, following the same method as in semi-relativistic atomic physics [7]. This identifies the generator of a canonical transformation, which will be used to find an electric dipole representation.

If we use the dipole approximation for neutral systems, \( Q \approx 0 \), the emerging total time derivative identifies the following generating function \( \tilde{S} \), given in Eq.(B14) and coinciding with \( S \) of Eq.(B3) at the lowest order, (Eqs.(2.3), (6.2) and (6.4) are used)

\[
\tilde{S}_{|Q\approx 0} = \frac{Q_1 - Q_2}{2c} \tilde{p}_{12}(\tau) \cdot \left[ \tilde{A}_\perp(\tau, \tilde{q}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \frac{\partial}{\partial \tilde{q}_{12}} \left( \tilde{p}_{12}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{q}_{12}(\tau)) \right) \right] + O([\tilde{p}_{12} \cdot \tilde{q}_{12}]^2 \tilde{A}_\perp),
\]

\[
S_{|Q\approx 0} = \frac{1}{c} \tilde{d}(\tau) \cdot \left[ \tilde{A}_\perp(\tau, \tilde{q}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \frac{\partial}{\partial \tilde{q}_{12}} \left( \tilde{p}_{12}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{q}_{12}(\tau)) \right) \right] = S_1 + O([\tilde{p}_{12} \cdot \tilde{q}_{12}]^2 \tilde{A}_\perp),
\]

\[
\tilde{S}_1 = \frac{1}{2c} \tilde{d}(\tau) \cdot \left[ \tilde{A}_\perp(\tau, \tilde{q}_1(\tau)) + \tilde{A}_\perp(\tau, \tilde{q}_2(\tau)) \right] + O([\tilde{p}_{12} \cdot \tilde{q}_{12}]^2 \tilde{A}_\perp).
\]

(7.3)

We also find that \( \tilde{S}_{|Q\approx 0} \) and \( S_{|Q\approx 0} \) differ by terms \( O([\tilde{p}_{12} \cdot \tilde{q}_{12}]^2 \tilde{A}_\perp) \) from a generating function \( \tilde{S}_1 = \frac{1}{2c} \tilde{d}(\tau) \cdot \sum_{i=1}^{2} \tilde{A}_\perp(\tau, \tilde{q}_i(\tau)) \), which could have been defined independently by using the dipole approximation.

The inverse Legendre transformation from the resulting Lagrangian (B15) is the Hamiltonian of Eq.(B20) with the \( \tilde{A}_i(\tau) \) given in Eq.(B16). If we add to it \( \mathcal{E}_{\text{em}} \), we get the Hamiltonian \( M_{\text{e.d.r.}} c \) in the dipole approximation for the electric dipole representation replacing \( \mathcal{P}_{(\text{int})}^\tau = M c \) of Eq.(4.1) of the standard representation (\( \tilde{\kappa}_i(\tau) \) are the new momenta of Eqs.(B17))

43
\[ M_{e.d.r} = \frac{1}{c} \mathcal{E}_{e.d.r.} = \sum_{i=1}^{2} \left( \sqrt{m_i^2 c^2 + \vec{k}_i' c^2} - \frac{\vec{\rho}_i' c}{\sqrt{m_i^2 c^2 + \vec{k}_i' c^2}} \right) \left( \frac{\vec{\rho}_i' + \vec{\rho}_i}{2} \cdot \vec{\pi}_i \right) + \frac{Q_1 Q_2}{4 \pi c |\vec{\rho}_{12}(\tau)|} - \frac{Q_1 - Q_2}{2c} \left[ \vec{\rho}_{12}(\tau) \cdot \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) - \frac{m_2 - m_1}{2m} \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) \right) \right] - \frac{Q_1 + Q_2}{2c} \left[ \frac{m_2 - m_1}{m} \vec{\rho}_{12}(\tau) \cdot \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) \right] - \frac{m^2 - 2m_1 m_2}{2m^2} \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) \right) + \frac{1}{c} \mathcal{E}_{em} \]

\[
\rightarrow Q \approx 0 \sum_{i=1}^{2} \left( \sqrt{m_i^2 c^2 + \vec{k}_i' c^2} - \frac{\vec{\rho}_i' c}{\sqrt{m_i^2 c^2 + \vec{k}_i' c^2}} \right) \left( \frac{\vec{\rho}_i' + \vec{\rho}_i}{2} \cdot \vec{\pi}_i \right) + \frac{Q_1 Q_2}{4 \pi c |\vec{\rho}_{12}(\tau)|} - \frac{Q_1 - Q_2}{2c} \left[ \vec{\rho}_{12}(\tau) \cdot \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) - \frac{m_2 - m_1}{2m} \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \vec{\pi}_1(\tau, \vec{\eta}_{12}(\tau)) \right) \right] + \frac{1}{c} \mathcal{E}_{em},
\]

\[ \vec{\mathcal{A}}(\tau) \rightarrow Q \approx 0 \rightarrow \frac{Q}{2c} \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)). \] (7.4)

This Hamiltonian becomes the Röntgen Hamiltonian \( H_1 \) of Eq.(7.2) in the semi-relativistic limit. However it has not the form expected in the electric dipole representation.

### C. The Relativistic Electric Dipole Representations Induced by the Generating Functions \( S \) and \( S_1 \).

To try to find a relativistic electric dipole representation, let us study the classical point canonical transformation generated by \( T_S = e^{i \mathcal{S}} \) with the generating function \( S \) of Eqs.(B3)-(7.3) (coinciding with \( \hat{S} \) in the dipole approximation)\(^\text{23}\) by considering also the electro-magnetic field dynamical (Eq.(7.4) was obtained by considering it as an external field). In the canonical basis \( \vec{\eta}_{12}(\tau), \vec{k}_{12}(\tau), \vec{\rho}_{12}(\tau), \vec{\pi}_{12}(\tau), \vec{A}_\perp(\tau, \vec{\sigma}), \vec{\pi}_\perp(\tau, \vec{\sigma}) \), the variables \( \vec{A}_\perp, \vec{\eta}_{12}, \vec{\rho}_{12} \) remain unchanged. Instead for the momenta and the internal energy of Eq.(4.1) we get

\[^{\text{23}}\] Since \( S = Q_1 S_1 + Q_2 S_2 \) depends only on the coordinates, we get \( \hat{F} = T_S F = F + Q_1 \{ F, S_1 \} + Q_2 \{ F, S_2 \} + Q_1 Q_2 [\{ \{ F, S_1 \}, S_2 \} + \{ \{ F, S_2 \}, S_1 \}] = F + Q_1 \{ F, S_1 \} + Q_2 \{ F, S_2 \}, \) since \( \{ \{ F, S_0 \}, S_0 \} = 0. \)
\[ \begin{align*}
\kappa_{12}(\tau) & \rightarrow \kappa'_{12}(\tau) = \kappa_{12}(\tau) - \frac{Q_1 - Q_2}{2c} \frac{\partial}{\partial \eta_{12}} \left[ \bar{\rho}_{12}(\tau) \cdot \bar{A}_\perp(\tau, \eta_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \bar{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \eta_{12}} \left( \bar{A}_\perp(\tau, \eta_{12}(\tau)) \right) \right) \right], \\
\pi_{12}(\tau) & \rightarrow \pi'_{12}(\tau) = \pi_{12}(\tau) - \frac{Q_1 - Q_2}{2c} \left[ \bar{A}_\perp(\tau, \eta_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \bar{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \eta_{12}} \left( \bar{A}_\perp(\tau, \eta_{12}(\tau)) \right) \right) \right], \\
\kappa'_1(\tau) & = \frac{m_4}{m} \kappa'_{12}(\tau) + (-)^{i+1} \pi'_{12}(\tau), \\
\pi'_\perp(\tau, \sigma) & \rightarrow \pi'_{\perp}(\tau, \sigma) = \pi'_{\perp}(\tau, \sigma) + \frac{Q_1 - Q_2}{2} \rho'_i(\tau) P^r_{\perp}(\sigma) \left[ \delta^3(\sigma - \eta_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \bar{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \eta_{12}} \delta^3(\sigma - \eta_{12}(\tau)) \right) \right].
\end{align*} \]

If we apply \( T_S = e^{\left(-S\right)} \) to \( E_{(\text{int})} \) of Eq.(4.1) we get its expression in the new canonical basis \( \eta_{12}(\tau), \kappa'_{12}(\tau), \bar{\rho}_{12}(\tau), \pi'_{12}(\tau), \bar{A}_\perp(\tau, \sigma), \pi'_{\perp}(\tau, \sigma) \):

\[
\frac{1}{c} E'(\text{int}) = M' c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \kappa_i'^2(\tau)} + \frac{Q_1 Q_2}{4\pi c |\bar{\rho}_{12}(\tau)|} + \\
+ \sum_{i=1}^{2} (-)^{i+1} \frac{Q_i}{c} \frac{\kappa_i(\tau) \cdot \left[ \bar{F}_i(\tau) - (-)^{i+1} \bar{A}_\perp(\tau, \eta_i(\tau)) \right]}{\sqrt{m_i^2 c^2 + \kappa_i'^2(\tau)}} - \\
- \frac{Q_1 + Q_2}{2c} \sum_{i=1}^{2} (-)^{i+1} \frac{\kappa_i(\tau) \cdot \bar{F}_i(\tau)}{\sqrt{m_i^2 c^2 + \kappa_i'^2(\tau)}} - \\
- \frac{Q_1 Q_2}{2c^2} \sum_{i=1}^{2} \left[ \frac{1}{2} \bar{F}_i(\tau) - \bar{A}_\perp(\tau, \eta_i(\tau)) \right] \cdot \left[ \bar{F}_i(\tau) + \kappa_i(\tau) \frac{\kappa_i'(\tau) \cdot \bar{F}_i(\tau)}{m_i^2 c^2 + \kappa_i'^2(\tau)} \right] - \\
- \frac{Q_1 - Q_2}{2c} \left[ \bar{\rho}_{12}(\tau) \cdot \pi'_{\perp}(\tau, \eta_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \bar{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \eta_{12}} \left( \bar{A}_\perp(\tau, \eta_{12}(\tau)) \right) \right) \right] - \\
- \frac{Q_1 Q_2}{2c} I(\tau) + \frac{1}{2c} \int d^3\sigma \left[ \pi'_{\perp}^2 + \bar{B}^2 \right](\tau, \sigma) \]
In this case we have ˜\( \tilde{F}_1(\tau) = \frac{m_i}{m} \frac{\partial}{\partial \tilde{n}_{12}} \left[ \tilde{p}_{12}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{n}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \tilde{p}_{12}(\tau) \cdot \frac{\partial}{\partial \tilde{n}_{12}} \left( \tilde{p}_{12}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{n}_{12}(\tau)) \right) \right] + (-1)^{i+1} \left[ \tilde{A}_\perp(\tau, \tilde{n}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left( \left( \tilde{p}_{12}(\tau) \cdot \frac{\partial}{\partial \tilde{n}_{12}} \right) \tilde{A}_\perp(\tau, \tilde{n}_{12}(\tau)) + \frac{\partial}{\partial \tilde{n}_{12}} \left( \tilde{p}_{12}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{n}_{12}(\tau)) \right) \right) \right] \right) \right] \right) = O(c^{-2}).

\( I(\tau) = \lim_{\bar{\sigma} \to \tilde{n}_{12}(\tau)} \tilde{f}_{12}(\tau) \tilde{f}_{12}(\tau) P^{\sigma}_\perp(\bar{\sigma}) \left[ \delta^3(\bar{\sigma} - \tilde{n}_{12}(\tau)) + \frac{m_2 - m_1}{m} \left( \tilde{p}_{12}(\tau) \cdot \frac{\partial}{\partial \tilde{n}_{12}} \right) \delta^3(\bar{\sigma} - \tilde{n}_{12}(\tau)) + \left( \frac{m_2 - m_1}{2m} \right)^2 \left( \tilde{p}_{12}(\tau) \cdot \frac{\partial}{\partial \tilde{n}_{12}} \right)^2 \delta^3(\bar{\sigma} - \tilde{n}_{12}(\tau)) \right]. \) (7.6)

It can be checked that in the dipole approximation and in the semi-relativistic limit one recovers the Røntgen Hamiltonian \( H_1 \) (7.2) of the semi-relativistic electric dipole representation: \( \mathcal{E}_{(\text{int})} = \mathcal{M} c^2 = m c^2 + H_1 + \mathcal{E}_{\text{em}} + O(c^{-2}). \)

However, at the order \( O(c^{-1}) \) a singular term \( I(\tau) \) appears. It comes from the \( \pi_{12}^2(\tau, \bar{\sigma}) \) term of the original electro-magnetic energy density. Notwithstanding the Grassmann regularization, we get this diverging term as a consequence of the dipole approximation: both \( \tilde{n}_{12}(\tau) \) collapse in the collective variable \( \tilde{n}_{12}(\tau) \) describing the motion of the electric dipole. Moreover the transverse vector potential is still present in Eq.(7.6) like in Eq.(7.4).

The same problem arises if we consider the point canonical transformation \( T_{\tilde{S}_1} = e^{\{ \cdot \tilde{S}_1 \}} \) generated by the function \( \tilde{S}_1 = \frac{1}{2\hbar} \tilde{d}(\tau) \cdot \sum_i \tilde{A}_\perp(\tau, \tilde{n}_i(\tau)) \) defined in Eq.(7.3) \(^{24}\). Also in this case we get expressions of type of Eq.(7.6), which we omit, with a similar singular term.

The use of \( \tilde{S}_1 \) instead of \( S \) or \( \tilde{S}_1 \) does not change the situation. Therefore it is not clear how to define a relativistic electric dipole representation, free from singularities and connected with the semi-relativistic one in the dipole approximation, when the electro-magnetic field is dynamical as it happens in the rest-frame instant form.

**D. A Relativistic Lagrangian with Dynamical Electro-Magnetic Field and the Induced Relativistic Electric Dipole Representation**

However a relativistic representation in which the singular term is replaced by a contact interaction at order \( O(c^{-1}) \) can be defined by defining a Lagrangian for the isolated system "charged particles plus the electro-magnetic field" on the instantaneous Wigner 3-spaces. This is done in Subsection 3 of Appendix B. By modifying this Lagrangian with a total time

\(^{24}\) In this case we have \( \tilde{S}_1 = (Q_1 - Q_2) \tilde{S}_1 \) with \( \tilde{S}_1 = \frac{1}{2\hbar} \tilde{p}_{12}(\tau) \cdot \sum_i \tilde{A}_\perp(\tau, \tilde{n}_i(\tau)) \) depending only on the coordinates. Therefore we get \( \tilde{F} = T_{\tilde{S}_1} \) \( F = F + (Q_1 - Q_2) \{ F, \tilde{S}_1 \} - Q_1 Q_2 \{ \{ F, \tilde{S}_1 \}, \tilde{S}_1 \}. \)
derivative, we can identify a new generating function \( S_2 = \frac{1}{c} \sum_{i=1}^{2} Q_i \bar{\eta}_i(\tau) \cdot \vec{A}_\parallel(\tau, \bar{\eta}_i(\tau)) \), whose dipole approximation is given in Eq.(B28), leading to this new representation.

Again the variables \( \vec{A}_\parallel(\tau, \bar{\sigma}), \bar{\eta}_1(\tau), \bar{\rho}_1(\tau) \) remain unchanged under the point canonical transformation generated by \( T_{S_2} = e^{(\cdot, S_2)} \) with generating function \( S_2 \). Instead for the momenta we get \(^{25}\)

\[
\begin{align*}
\kappa'_{\tau}(\tau) & \xrightarrow{T_{S_2}} \hat{\kappa}'_{\tau}(\tau) = \kappa'_{\tau}(\tau) - \frac{Q_i}{c} \left[ A^r_{\tau}(\tau, \bar{\eta}_i(\tau)) + \bar{\eta}_i(\tau) \cdot \frac{\partial \vec{A}_\parallel(\tau, \bar{\eta}_i(\tau))}{\partial \eta^r_i} \right], \\
\pi'_{\tau}(\tau, \bar{\sigma}) & \xrightarrow{T_{S_2}} \hat{\pi}'_{\tau}(\tau, \bar{\sigma}) = \pi'_{\tau}(\tau, \bar{\sigma}) + \sum_{i=1}^{2} Q_i P^r_s(\bar{\sigma}) \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \eta^s_i(\tau), \\
\kappa'_{\tau}(\tau) & \xrightarrow{T_{S_2}} \hat{\kappa}'_{\tau}(\tau) = \kappa'_{\tau}(\tau) - \frac{Q_i}{c} A^r_{\tau}(\tau, \bar{\eta}_i(\tau)) + \frac{Q_i}{c} \bar{\eta}_i(\tau) \cdot \frac{\partial \vec{A}_\parallel(\tau, \bar{\eta}_i(\tau))}{\partial \eta^r_i}, \\
\int d^3 \sigma \bar{\pi}^2_{\tau}(\tau, \bar{\sigma}) & \xrightarrow{T_{S_2}} \int d^3 \sigma \bar{\pi}^2_{\tau}(\tau, \bar{\sigma}) - 2 \sum_{i=1}^{2} Q_i \bar{\eta}_i(\tau) \cdot \bar{\pi}_{\tau}(\tau, \bar{\eta}_i(\tau)) + 2 Q_1 Q_2 \bar{\eta}_1(\tau) \bar{\eta}_2(\tau) P^r_s(\bar{\eta}_1(\tau)) \delta^3(\bar{\eta}_1(\tau) - \bar{\eta}_2(\tau)).
\end{align*}
\]

The internal energy of Eq. (4.1) is replaced by the following expression

\[
\begin{align*}
\frac{\mathcal{E}_{(int)}}{c} & = M_c \xrightarrow{T_{S_2}} M_c \\
& \rightarrow \frac{\mathcal{E}_{(int)}}{c} = M_c + \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \sum_{r} \left( \hat{\kappa}'_{\tau}(\tau) + \frac{Q_i}{c} \bar{\eta}_i(\tau) \cdot \frac{\partial \vec{A}_\parallel(\tau, \bar{\eta}_i(\tau))}{\partial \eta^r_i} \right)^2} + \\
& + \frac{Q_1 Q_2}{4 \pi c | \bar{\rho}_{12}(\tau)|} + \frac{1}{2c} \int d^3 \sigma \left[ \bar{\pi}^2_{\tau} + \bar{B}^2 \right](\tau, \bar{\sigma}) - \\
& - \sum_{i=1}^{2} Q_i \bar{\eta}_i(\tau) \cdot \bar{\pi}_{\tau}(\tau, \bar{\eta}_i(\tau)) + \\
& + \frac{Q_1 Q_2}{c} \eta_1(\tau) \eta_2(\tau) P^r_s(\eta_1(\tau)) \delta^3(\eta_1(\tau) - \eta_2(\tau)) =
\end{align*}
\]

\(^{25}\) We use \( F^{sr} = -\epsilon^{sr} B^s, \kappa r_{\tau} \frac{\partial A^s_r}{\partial m_i} = \hat{\kappa}'_{\tau} \eta^s_i \left( F^{sr} + \frac{\partial A^s}{\partial m_i} \right) = \hat{\kappa}'_{\tau} \eta^s_i \left( \vec{\bar{\eta}} \times \vec{B} + \hat{\kappa}'_{\tau} \eta^s_i \frac{\partial A^s}{\partial m_i} \right) = -\hat{\eta}_i \times \hat{\kappa}', \vec{B} + \hat{\kappa}' \eta^s_i \frac{\partial A^s_r}{\partial m_i} \).
The Lorentz boost $\vec{K}_{(int)}$ of Eqs.(4.1) is replaced by the following expression

$$
\mathcal{K}'_{(int)} \xrightarrow{T_{\Sigma^2}} \mathcal{K}^r_{(int)} =
$$

$$
= -\sum_{i=1}^{2} \eta_i^r(\tau) \left[ \sqrt{m_i^2 c^2 + \tilde{k}^2_i(\tau)} - \int \frac{d^3 \sigma}{4\pi} \frac{\tilde{\pi}_i^r(\tau, \vec{\sigma})}{|\vec{\sigma} - \eta_i^r(\tau)|} \right] -
$$

$$
- \sum_{i=1}^{2} \frac{Q_i}{c} \frac{\tilde{\eta}_i^r(\tau) \cdot \left( \tilde{k}_i(\tau) \cdot \frac{\partial}{\partial \eta_i} \right) \vec{A}_\perp(\tau, \tilde{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \tilde{k}^2_i(\tau)}} +
$$

$$
+ \frac{Q_1 Q_2}{4\pi c |\vec{\rho}_{12}(\tau)|} \left[ \frac{\eta_1^r(\tau) + \eta_2^r(\tau)}{|\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)|} \right] -
$$

$$
- \int \frac{d^3 \sigma}{4\pi} \left( \frac{\sigma^r - \eta_1^r(\tau)}{|\vec{\sigma} - \tilde{\eta}_1(\tau)|^3} + \frac{\sigma^r - \eta_2^r(\tau)}{|\vec{\sigma} - \tilde{\eta}_2(\tau)|^3} \right) +
$$

$$
\sum_{i=1}^{2} \frac{Q_i}{4\pi c} \int d^3 \sigma \frac{\tilde{\pi}_i^r(\tau, \vec{\sigma})}{|\vec{\sigma} - \tilde{\eta}_i(\tau)|} -
$$

$$
- \frac{Q_1 Q_2}{4\pi c} \sum_s \left[ \eta_1^s(\tau) P^s_{\perp}(\tilde{\eta}_1(\tau)) + \eta_2^s(\tau) P^s_{\perp}(\tilde{\eta}_2(\tau)) \right] \frac{1}{|\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)|} -
$$

$$
- \frac{1}{2c} \int d^3 \sigma \sigma^r (\tilde{\pi}_\perp^2 + \tilde{\vec{B}}^2)(\tau, \vec{\sigma}) -
$$

$$
- \frac{Q_1 Q_2}{2c} \sum_{\mu\nu} \eta_1^\mu(\tau) \eta_2^\nu(\tau) \left[ P^\mu_{\perp}(\tilde{\eta}_1(\tau)) \left( \eta_1^\mu(\tau) P^\nu_{\perp}(\tilde{\eta}_1(\tau)) \delta^3(\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)) \right) +
$$

$$
+ P^\mu_{\perp}(\tilde{\eta}_2(\tau)) \left( \eta_2^\nu(\tau) P^\nu_{\perp}(\tilde{\eta}_2(\tau)) \delta^3(\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)) \right) \right].
$$

The new form of $\vec{P}_{(int)}$ and $\vec{J}_{(int)}$ of Eq.(4.1) after the canonical transformation is
\[ P_{(\text{int})}^i = \sum_{i=1}^2 \left( \vec{r}_i^i(\tau) + \frac{Q_i}{c} A_{i\perp}(\tau, \vec{\eta}_i(\tau)) \right) + \frac{1}{c} \int d^3 \sigma \left( \vec{\pi}_{\perp} \times \vec{B} \right)^r(\tau, \vec{\sigma}), \]
\[ J_{(\text{int})}^i = \sum_{i=1}^2 \vec{\eta}_i(\tau) \times \vec{r}_i(\tau) + \frac{1}{c} \int d^3 \sigma \times \left( \vec{\pi}_{\perp} \times \vec{B} \right)(\tau, \vec{\sigma}). \quad (7.10) \]

In Eqs. (7.8)-(7.10) the transverse potential \( \vec{A}_{\perp}(\tau, \vec{\sigma}) \) has to be re-expressed in terms of the magnetic field \( \vec{B}(\tau, \vec{\sigma}) \).

In Abelian electro-magnetism, where the 2-form \( F_{rs} \, d\sigma^r \, d\sigma^s \) is closed, the vector potential \( \vec{A}_P(\tau, \vec{\sigma}) \) of the radial (or Poincare') gauge, satisfying \( \vec{\sigma} \cdot \vec{A}_P(\tau, \vec{\sigma}) = 0 \), leads to the exact 1-form \( \vec{A}_P(\tau, \vec{\sigma}) \cdot d\vec{\sigma} \), so that the Poincare' lemma implies
\[ \vec{A}_P(\tau, \vec{\eta}_i(\tau)) = -\int_0^1 d\lambda \, \lambda \vec{\eta}_i(\tau) \times \vec{B}(\tau, \lambda \vec{\sigma}_1). \quad (7.11) \]

The transverse potential \( \vec{A}_{\perp}(\tau, \vec{\sigma}) \) of the radiation gauge can be connected to \( \vec{A}_P(\tau, \vec{\sigma}) \) by means of a gauge transformation such that \( \vec{A}_{\perp}(\tau, \vec{\sigma}) = \vec{A}_P(\tau, \vec{\sigma}) + \vec{\partial} \lambda(\tau, \vec{\sigma}) \) with the following consequence of transversality
\[ \triangle \lambda(\tau, \vec{\sigma}) = -\vec{\partial} \cdot \vec{A}_P(\tau, \vec{\sigma}) = -\int_0^1 d\lambda \, \lambda \vec{\eta}_i(\tau) \cdot \left[ \vec{\partial} \times \vec{B}(\tau, \lambda \vec{\sigma}_1) \right]. \quad (7.12) \]

whose solution is (we disregard solutions of the homogeneous equation)
\[ \lambda(\tau, \vec{\sigma}) = \int d^3 \sigma_1 \frac{\int_0^1 d\lambda \, \lambda \vec{\sigma}_1 \cdot \left[ \vec{\partial}_1 \times \vec{B}(\tau, \lambda \vec{\sigma}_1) \right]}{4\pi |\vec{\sigma} - \vec{\sigma}_1|}. \quad (7.13) \]

Therefore we get
\[ \vec{A}_{\perp}(\tau, \vec{\sigma}) = -\int_0^1 d\lambda \, \lambda \vec{\sigma} \times \vec{B}(\tau, \lambda \vec{\sigma}) + 
+ \vec{\partial} \int d^3 \sigma_1 \frac{\int_0^1 d\lambda \, \lambda \vec{\sigma}_1 \cdot \left[ \vec{\partial}_1 \times \vec{B}(\tau, \lambda \vec{\sigma}_1) \right]}{4\pi |\vec{\sigma} - \vec{\sigma}_1|}. \quad (7.14) \]

The term depending on \( \vec{A}_{\perp}(\tau, \vec{\sigma}) \) in the energy (7.8) and in the boost (7.9) becomes (we use the spin notation \( \vec{S}_i = \vec{\eta}_i \times \vec{r}_i \))
\[ \vec{\rho}_i(\tau) \cdot \left( \vec{k}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) = \]

\[ = -\vec{S}_i(\tau) \cdot \int_0^1 d\lambda \lambda \vec{B}(\tau, \vec{\eta}_i(\tau)) + \]

\[ + \left( \vec{\eta}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) \left( \vec{k}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) \int d^3\sigma_1 \int_0^1 d\lambda \lambda \vec{\sigma}_1 \cdot \left[ \vec{\partial}_1 \times \vec{B}(\tau, \lambda \vec{\sigma}_1) \right] \frac{4\pi}{|\vec{\eta}_i(\tau) - \vec{\sigma}_1|}. \] (7.15)

With this equation we can also evaluate \( \vec{k}_i(\tau) \cdot \left( \vec{\eta}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) = \vec{\eta}_i(\tau) \cdot \left( \vec{k}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) + \vec{S}_i(\tau) \cdot \vec{B}(\tau, \vec{\eta}_i(\tau)). \)

As a consequence we get a \textbf{generalized electric dipole representation}: besides interactions of the particles with the transverse electric field at their position and interactions with a delocalized magnetic field (one of them is with the particle angular momentum), there is a contact \( O(\epsilon^{-1}) \) term replacing the divergent term arising from the previous canonical transformations.

If we make the dipole approximation and the semi-relativistic limit of \( \mathcal{E}_{(\text{int})} \) we get

\[ \mathcal{E}_{(\text{int})} \to c^{-\infty} m c^2 + \frac{\kappa_{12}^2(\tau)}{2m} + \frac{\pi_{12}^2(\tau)}{2\mu} - \]

\[ - \frac{1}{c} \left[ \frac{Q_1 + Q_2}{m} \vec{\eta}_{12}(\tau) \times \vec{k}_{12}(\tau) + \left( \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right) \left( \vec{\eta}_{12}(\tau) \times \vec{k}_{12}(\tau) + \frac{\mu}{m} \vec{\rho}_{12}(\tau) \times \vec{k}_{12}(\tau) \right) \right] \cdot \vec{B}(\tau, \vec{\eta}_{12}(\tau)) - \]

\[ + \frac{1}{m} \left( \frac{m_2}{m_1} Q_1 + \frac{m_1}{m_2} Q_2 \right) \vec{\rho}_{12}(\tau) \times \vec{\pi}_{12}(\tau) \right] \cdot \vec{B}(\tau, \vec{\eta}_{12}(\tau)) - \]

\[ - \frac{1}{c} \left[ \frac{\mu}{m} \left( \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right) \vec{\eta}_{12}(\tau) \times \vec{k}_{12}(\tau) + \right] \]

\[ + \frac{1}{m} \left( \frac{m_2}{m_1} Q_1 + \frac{m_1}{m_2} Q_2 \right) \vec{\rho}_{12}(\tau) \times \vec{\pi}_{12}(\tau) \right] \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{B}(\tau, \vec{\eta}_{12}(\tau)) - \]

\[ - \frac{1}{c} \left[ \frac{\mu}{m^2} \left( \frac{m_2}{m_1} Q_1 + \frac{m_1}{m_2} Q_2 \right) \vec{\rho}_{12}(\tau) \times \vec{k}_{12}(\tau) + \right] \]

\[ + \frac{1}{m^2} \left( \frac{m_2}{m_1} Q_1 - \frac{m_1}{m_2} Q_2 \right) \vec{\rho}_{12}(\tau) \times \vec{\pi}_{12}(\tau) \right] \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{B}(\tau, \vec{\eta}_{12}(\tau)) + \]

\[ + \frac{1}{c} \left[ \frac{Q_1 + Q_2}{m} \vec{k}_{12}(\tau) + \left( \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right) \vec{\pi}_{12}(\tau) \right] \cdot \left( \vec{\eta}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \]
\[
\begin{align*}
+ \frac{1}{m c} \left[ m_2 Q_1 - m_1 Q_2 \vec{k}_{12}(\tau) + \left( \frac{m_2}{m_1} Q_1 + \frac{m_1}{m_2} Q_2 \right) \vec{p}_{12}(\tau) \right] \\
\left( \vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) - \\
- \left[ (Q_1 + Q_2) \vec{p}_{12}(\tau) + \frac{m_2 Q_1 - m_1 Q_2}{m} \vec{p}_{12}(\tau) \right] \cdot \vec{p}_\perp(\tau, \vec{\eta}_{12}(\tau)) - \\
- \frac{m_2 Q_1 - m_1 Q_2}{m} \vec{p}_{12}(\tau) \cdot \left( \vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{p}_\perp(\tau, \vec{\eta}_{12}(\tau)) - \\
- \frac{m_2^2 Q_1 + m_2^2 Q_2}{m^2} \vec{p}_{12}(\tau) \cdot \left( \vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{p}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \\
+ \frac{Q_1 Q_2}{4\pi |\vec{p}_{12}(\tau)|} + \frac{1}{2} \int d^3 \sigma \left[ \vec{p}_\perp^2 + \vec{B}_2^2 \right](\tau, \vec{\sigma}) + \\
+ Q_1 Q_2 \sum_{rs} \eta^r_1(\tau) \eta^s_2(\tau) P^r_2(\vec{\eta}_1(\tau)) \delta^3(\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)) + \\
+ O(\left( \vec{p}_{12} \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right)^2 \vec{A}_\perp) + O(c^{-2}).
\end{align*}
\]

(7.16)

For neutral systems, \( e = Q \approx Q_1 \approx -Q_2 \), \( Q \approx 0 \), we get

\[
\vec{E}_{(int)} \rightarrow_{Q \approx 0} mc^2 + \frac{\vec{k}_{12}^2(\tau)}{2m} + \frac{\vec{p}_{12}^2(\tau)}{2\mu} - \\
- \frac{e}{c} \left( \frac{1}{\mu} \vec{p}_{12}(\tau) \times \vec{p}_{12}(\tau) + \frac{1}{m} \vec{p}_{12}(\tau) \times \vec{k}_{12}(\tau) + \\
+ \frac{m_2 - m_1}{\mu m} \vec{p}_{12}(\tau) \times \vec{k}_{12}(\tau) \right) \cdot \vec{B}(\tau, \vec{\eta}_{12}(\tau)) - \\
- \frac{e}{m c} \left[ \vec{p}_{12}(\tau) \times \vec{k}_{12}(\tau) + \frac{m_2 - m_1}{\mu} \vec{p}_{12}(\tau) \times \vec{k}_{12}(\tau) \times \\
\times \vec{k}_{12}(\tau) \right] \cdot \left( \vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{B}(\tau, \vec{\eta}_{12}(\tau)) - 
\]
\[ \begin{align*}
&- \frac{e}{mc} \left[ \frac{m_2 - m_1}{m} \vec{R}_{12}(\tau) \times \vec{K}_{12}(\tau) + \\
&+ \frac{m_3^2 - m_1^3}{m_1 m_2 m} \vec{R}_{12}(\tau) \times \vec{S}_{12}(\tau) \right] \cdot \left( \vec{P}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{R}_{12}(\tau)} \right) \vec{B}(\tau, \vec{R}_{12}(\tau)) + \\
&+ \frac{e}{\mu c} \vec{S}_{12}(\tau) \left( \vec{R}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{R}_{12}(\tau)} \right) \vec{A}_\perp(\tau, \vec{R}_{12}(\tau)) + \\
&+ \frac{e}{mc} \left[ \vec{R}_{12}(\tau) + \frac{m_2 - m_1}{\mu} \vec{S}_{12}(\tau) \right] \\
&\left( \vec{P}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{R}_{12}(\tau)} \right) \vec{A}_\perp(\tau, \vec{R}_{12}(\tau)) - \\
&- e \left[ \vec{P}_{12}(\tau) \cdot \vec{S}_\perp(\tau, \vec{R}_{12}(\tau)) + \vec{R}_{12}(\tau) \cdot \left( \vec{P}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{R}_{12}(\tau)} \right) \vec{S}_\perp(\tau, \vec{R}_{12}(\tau)) \right] - \\
&- e \frac{m_2 - m_1}{m} \vec{P}_{12}(\tau) \cdot \left( \vec{P}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{R}_{12}(\tau)} \right) \vec{S}_\perp(\tau, \vec{R}_{12}(\tau)) - \\
&\frac{e_2}{4\pi |\vec{P}_{12}(\tau)|} + \frac{1}{2} \int d^3\sigma \left[ \vec{P}_\perp^2 + \vec{B}_\perp^2 \right](\tau, \vec{\sigma}) + \\
&+ Q_1 Q_2 \sum_{rs} \eta_1^r(\tau) \eta_2^s(\tau) P_{12}^{rs}(\vec{R}_{12}(\tau)) \delta^3(\vec{R}_{12}(\tau) - \vec{R}_{2}(\tau)) + \\
&+ O\left( (\vec{P}_{12} \cdot \frac{\partial}{\partial \vec{R}_{12}})^2 \vec{A}_\perp \right) + O(e^{-2}). \\
\end{align*} \]

(7.17)

With respect to the Røntgen Hamiltonian (7.2) there are numerical factors \( \frac{1}{2} \) of difference and extra terms containing the transverse vector potential and effects due to the motion of the collective variable \( \vec{R}_{12}(\tau) \).

Eq. (7.8) seems to be the simplest relativistic extension of the semi-relativistic electric dipole representation in the framework of the rest-frame instant form.
VIII. CONCLUSIONS

In this second paper we concluded the classical treatment of the isolated system of charged positive-energy scalar particles with Grassmann-valued electric charges with mutual Coulomb interaction plus the transverse electromagnetic field in the radiation gauge in the rest-frame instant form of dynamics. In the first paper we showed how every isolated system (particles, fields, strings, fluids) is described in the inertial frame centered on its Fokker-Planck external center of inertia: the isolated systems is replaced by a decoupled canonical non-covariant Newton-Wigner center of mass (described by the frozen Jacobi data $\vec{z}$, $\vec{h}$) carrying its internal mass and spin in such a way that there is an associated external realization of the Poincare’ group. The dynamics inside the isolated system is defined by construction in the instantaneous Wigner 3-spaces and is described only by means of relative canonical variables. This happens because the internal realization of the Poincare’ group built from the internal energy momentum tensor of the system is unfaithful: there are the rest frame conditions $\vec{P}^{\text{(int)}} \approx 0$ and its gauge fixing $\vec{K}^{\text{(int)}} \approx 0$ (vanishing of the internal Lorentz boosts), whose role is the elimination of the internal 3-center of mass and of its three momentum.

In this paper we have studied how to find collective variables in the instantaneous Wigner 3-spaces such that it is possible to solve the equations $\vec{K}^{\text{(int)}} \approx 0$ explicitly. While for particle systems there are no problems, the construction of collective variables for the configurations of massive and massless fields is nontrivial and only holds for a subset of the field configurations with finite Poincare’ generators. We adapted to the rest-frame instant form the methods developed by Longhi and collaborators in Refs.[5, 6] and used them to find a solution of the equations $\vec{K}^{\text{(int)}} \approx 0$ for the transverse radiation field. This made possible to solve this equation also for the isolated system of charged particles with mutual Coulomb interaction plus the transverse electro-magnetic field after the canonical transformation of paper I.

The final result in an arbitrary inertial frame is that inside each instantaneous Wigner 3-space, with origin in the external covariant Fokker-Pryce center of inertia, the internal 3-center of mass becomes a function of the relative canonical variables, so that the dynamics is described only in terms of them.

If we consider an isolated 2-particle system (but the same conclusions hold for an arbitrary isolated system) we have the following situation on the instantaneous Wigner 3-space:

A) the particles are described by the two Wigner 3-vectors $\vec{\eta}(\tau) = \vec{\eta}_{12}(\tau) + (-)^{i+1} \frac{m_i}{m} \vec{\rho}_{12}(\tau) \approx f(\vec{\rho}_{12}(\tau), \vec{\pi}_{12}(\tau)) + (-)^{i+1} \frac{m_i}{m} \vec{\rho}_{12}(\tau)$ with $\vec{\rho}_{12}(\tau) = \vec{\eta}(\tau) - \vec{\eta}_{12}(\tau)$;

B) as shown by Eq.(I.2.19) the external non-covariant canonical center of mass, $\vec{x}^\mu(\tau) = Y^\mu(\tau) + \epsilon^\mu_\mu(\vec{h}) \vec{\sigma}^\tau$, is identified by the 3-vector $\vec{\sigma} = \frac{-\vec{S} \times \vec{h}}{Mc(1+\sqrt{1+h^2})}$. We could then try to define two new particle 3-vectors $\vec{\eta}_{(n)}(\tau) = \vec{\sigma} + (-)^{i+1} \frac{m_i}{m} \vec{\rho}_{12}(\tau)$ and describe the dynamics with them and their conjugate momenta rather than with $\vec{z}$, $\vec{h}$, $\vec{\rho}_{12}$, $\vec{\pi}_{12}$: this would be equivalent to the non-relativistic treatment with $\vec{x}_{(n)i}$ and $\vec{p}_{(n)i}$. However the obstruction to do this is the non-covariance of the external canonical center of mass, which implies that $\vec{\sigma}$ is not a Wigner spin-1 3-vector. The presence of interactions in the Lorentz boosts and the associated No-Interaction theorem, which imply the existence of three relativistic collective variables
replacing the unique non-relativistic center of mass and spanning the non-covariance Møller world-tube around the external Fokker-Pryce center of inertia, lead to a description in which the decoupled canonical center of mass breaks manifest Lorentz covariance and is a global quantity which cannot be locally determined.

Then we clarified how to get the multipolar expansion of the energy-momentum of the particle subsystem and its non-canonical pole-dipole approximation, not to be confused with the canonical pole-dipole structure carried by the external Newton-Wigner center of mass. Thinking to strongly bound clusters of particles as classical atoms, it is found how to define their effective 4-center of motion.

Finally we have studied the relativistic electric dipole approximation. We have shown that it is non trivial to find a relativistic generalization of the electric dipole representation used in atomic physics. When the electro-magnetic field is dynamical, as requested by the rest-frame instant form, the point canonical transformations suggested by the semi-relativistic approach tend to generate singular terms at the order $O(c^{-1})$. They are connected to the use of the dipole approximation. From a study of the Lagrangian on the instantaneous Wigner 3-spaces it is possible to define a relativistic representation in which the singular terms are replaced by contact interactions. In this new relativistic representation, whose semi-relativistic limit is different from the standard electric dipole representation, there are interactions only with the transverse electric and magnetic fields, like in the semi-relativistic case.

In the third paper we will quantize the rest-frame instant form of our formulation of relativistic particle mechanics. This will allow us to consider relativistic bound states of clusters of particles as atoms. The coupling to a quantized transverse electro-magnetic fields will allow us to define relativistic atomic physics and relativistic generalizations of the two-level atom. Future applications will be relevant to study relativistic entanglement and atom interferometry.
APPENDIX A: DEFINITION OF THE COLLECTIVE VARIABLES FOR THE MASSIVE KLEIN-GORDON FIELD.

The first formulation of collective and relative variables for a field was done in Ref.[5] for the free massive Klein-Gordon (KG) field. It was then adapted to the rest-frame instant form in Ref.[13]. Here we give the main results to introduce the methodology to be used to find such variables for the transverse radiation field starting from Ref.[6], where the results of Ref.[5] were extended to a scalar massless field.

1. The Massive Scalar Field

In Ref.[13] the scalar massive field \( \phi(x) \) was reformulated in the rest-frame instant form as the field \( \phi(\tau, \vec{\sigma}) = \phi(z_W(\tau, \vec{\sigma})) \) on the Wigner hyper-planes \( z_W^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + e_\mu^\mu(\vec{h}) \). Its conjugate momentum is \( \pi(\tau, \vec{\sigma}) \), \( \{ \phi(\tau, \vec{\sigma}), \pi(\tau, \vec{\sigma}_1) \} = \delta^3(\vec{\sigma} - \vec{\sigma}_1) \). A first canonical transformation replaces these canonical variables with their Fourier coefficients \( a(\tau, \vec{k}), a^*(\tau, \vec{k}) \), whose non-vanishing Poisson brackets are \( \{a(\tau, \vec{k}), a^*(\tau, \vec{k}_1)\} = -i (2\pi)^3 2 \omega(k) \delta^3(\vec{k} - \vec{k}_1) \), \( \omega(k) = \sqrt{m^2 c^2 + k^2} \).

Then a second canonical transformation replaces the Fourier coefficients with the modulus-phase canonical variables \( I(\tau, \vec{k}) = a^*(\tau, \vec{k}) a(\tau, \vec{k}) \), \( \varphi(\tau, \vec{k}) \), \( \varphi(\tau, \vec{k}) = \frac{1}{2i} \ln \frac{a(\tau, \vec{k})}{a^*(\tau, \vec{k})} \), whose non-vanishing Poisson brackets are \( \{I(\tau, \vec{k}), \varphi(\tau, \vec{k}_1)\} = (2\pi)^3 2 \omega(k) \delta^3(\vec{k} - \vec{k}_1) \). The final expression of the original fields is \( (d\vec{k}) = \frac{d^3k}{(2\pi)^3 2\omega(k)} \)

\[
\begin{align*}
\phi(\tau, \vec{\sigma}) &= \int d\vec{k} \sqrt{I(\tau, \vec{k})} \left( e^{i \varphi(\tau, \vec{k}) - i [\omega(k) \tau - \vec{k} \cdot \vec{\sigma}]} + e^{-i \varphi(\tau, \vec{k}) + i [\omega(k) \tau - \vec{k} \cdot \vec{\sigma}]} \right), \\
\pi(\tau, \vec{\sigma}) &= -i \int d\vec{k} \omega(k) \sqrt{I(\tau, \vec{k})} \left( e^{i \varphi(\tau, \vec{k}) - i [\omega(k) \tau - \vec{k} \cdot \vec{\sigma}]} - e^{-i \varphi(\tau, \vec{k}) + i [\omega(k) \tau - \vec{k} \cdot \vec{\sigma}]} \right). \quad (A1)
\end{align*}
\]

The internal Poincare’ generators are (we must have \( a_{\lambda}(\tau, \vec{k}), \quad \vec{\partial} a_{\lambda}(\tau, \vec{k}) \in L_2(R^3, d^3k) \) for the existence of the following ten integrals)

\[
\begin{align*}
P^\tau_\phi &= M_\phi c = \int d\vec{k} \omega(k) I(\tau, \vec{k}), \quad \vec{P}_\phi = \int d\vec{k} \vec{k} I(\tau, \vec{k}) \approx 0, \\
\vec{J}_\phi &= \vec{S}_\phi = \int d\vec{k} I(\tau, \vec{k}) \vec{k} \times \frac{\partial}{\partial \vec{k}} \varphi(\tau, \vec{k}), \\
\vec{K}_\phi &= -\vec{P}_\phi \tau - \int d\vec{k} \omega(k) I(\tau, \vec{k}) \frac{\partial}{\partial \vec{k}} \varphi(\tau, \vec{k}) \approx 0. \quad (A2)
\end{align*}
\]

\[26\] The quantity \( k_A^{\sigma^A} = \omega(k) \tau - \vec{k} \cdot \vec{\sigma} \) is built with \( \sigma^A = (\tau; \vec{\sigma}) \) and \( k^A = (k^\tau = \omega(k); \vec{k}) \). It is a Lorentz scalar because on the instantaneous Wigner 3-spaces the 3-vectors \( \vec{\sigma} \) and \( \vec{k} \) are Wigner spin-1 3-vectors and \( k^\tau = \omega(k) \) is a Lorentz scalar. For the free KG field the Fourier coefficients and modulus-phase variables are \( \tau \)-independent: in this Appendix we leave a formal \( \tau \)-dependence, because the formalism can be used also when interactions are present as in Ref.[13].
In particular we have

\[ \{ \varphi(\tau, \vec{k}), P^A_\phi \} = -k^A, \]  

(A3)

where \( P^A_\phi = (P^T_\phi; \vec{P}_\phi) \) and \( k^A = (\omega(k); \vec{k}). \)

The collective canonical variables are assumed to have the form

\[
\begin{align*}
X^T_\phi &= \int d\tilde{k} \omega(k) F^T(k) \varphi(\tau, \vec{k}), \quad P^T_\phi = M_\phi c, \\
\vec{X}_\phi &= \int d\tilde{k} \vec{k} F(k) \varphi(\tau, \vec{k}), \quad \vec{P}_\phi \approx 0,
\end{align*}
\]

(A4)

where the two Lorentz-scalar functions \( F^T(k), F(k) \) have to be determined by the canonical conditions \( \{ X^T_\phi, P^T_\phi \} = -1, \{ X^r_\phi, P^s_\phi \} = \delta^{rs}, \{ X^T_\phi, X^r_\phi \} = \{ X^r_\phi, X^s_\phi \} = \{ X^T_\phi, P^r_\phi \} = \{ X^r_\phi, P^s_\phi \} = 0. \) The solution of these conditions is \([13]\)

\[
\begin{align*}
F^T(k) &= \frac{16\pi^2 e^{-4\pi \frac{k^2}{mc^2}}}{mc k^2 \sqrt{m^2 c^2 + k^2}}, \quad F(k) = -48\pi^2 \frac{\sqrt{m^2 c^2 + k^2}}{mc k^4} e^{-4\pi \frac{k^2}{mc^2}}, \\
\int d\tilde{k} \omega^2(k) F^T(k) &= 1, \quad \int d\tilde{k} \omega(k) k^r F^T(k) = 0, \\
\int d\tilde{k} k^r k^s F(k) &= -\delta^{rs}, \quad \int d\tilde{k} \omega(k) k^r F(k) = 0.
\end{align*}
\]

(A5)

This result implements the original construction of Ref.[5] adapted to the rest-frame instant form. The basic idea is to consider a real Lorentz-scalar function \( F(\vec{k}, P^B_\phi) \) normalized in the following way

\[
\int d\tilde{k} k^A F(\vec{k}, P^B_\phi), \quad k^A = (\omega(k); \vec{k}) = P^A_\phi.
\]

(A6)

Then, by using Eq.(A3) and (A6), one can check that the collective variable

\[
X^A_\phi = \int d\tilde{k} \frac{\partial F(\vec{k}, P^B_\phi)}{\partial P^A_\phi} \varphi(\tau, \vec{k}),
\]

(A7)

satisfies \( \{ X^A_\phi, X^B_\phi \} = 0, \{ X^A_\phi, P^B_\phi \} = -\eta^{AB}. \)

Eqs.(A5) imply that the function \( \mathcal{F}(\vec{k}, P^B_\phi) \) has the form

\[
\mathcal{F}(\vec{k}, P^B_\phi) = \omega(k) F^T(k) P^T_\phi - F(k) \vec{k} \cdot \vec{P}_\phi.
\]

(A8)
Since the functions $F^r(k)$ and $F(k)$ are singular at $\vec{k} = 0$, the collective variables exist for field configurations whose phase variables satisfy $\varphi(\tau, \vec{k}) \rightarrow_{k \rightarrow 0} k^\eta$ with $\eta > 0$.

Let us remark that for field configurations $\phi(\tau, \vec{\sigma})$ such that the Fourier transform $\hat{\phi}(\tau, \vec{k})$ has compact support in a sphere centered at $\vec{k} = 0$ of volume $V$, we get $X_\phi = -\frac{1}{V} \int \frac{dk}{\omega(k)} \varphi(\tau, \vec{k})$, $\vec{X}_\phi = \frac{1}{V} \int d^3 k \frac{3 \vec{k}}{k^2} \varphi(\tau, \vec{k})$.

The canonical relative variables, having vanishing Poisson brackets with the collective ones, are [5, 13]

$$
\begin{align*}
\mathbf{H}(\tau, \vec{k}) &= \int d\vec{q} \mathcal{G}(\vec{k}, \vec{q}) \left[ I(\tau, \vec{q}) - \omega(\vec{q}) F^r(\vec{q}) \int d\vec{q}_1 \omega(\vec{q}_1) I(\tau, \vec{q}_1) + F(\vec{q}) \vec{q} \cdot \int d\vec{q}_1 \vec{q}_1 I(\tau, \vec{q}_1) \right], \\
\mathbf{K}(\tau, \vec{k}) &= \mathcal{D}_\vec{k} \varphi(\tau, \vec{k}),
\end{align*}
$$

where $\{ \mathbf{H}(\tau, \vec{k}), \mathbf{K}(\tau, \vec{k}_1) \} = (2\pi)^3 2 \omega(\vec{k}) \delta^3(\vec{k} - \vec{k}_1)$, $\{ \mathbf{H}(\tau, \vec{k}), \mathbf{H}(\tau, \vec{k}_1) \} = \{ \mathbf{K}(\tau, \vec{k}), \mathbf{K}(\tau, \vec{k}_1) \} = 0.$

Here (see Ref.[5]) $\mathcal{G}(\vec{q}, \vec{k})$ is the Green function of the operator $\mathcal{D}_\vec{q} = 3 - m^2 c^2 \Delta_{LB} = 3 - m^2 c^2 \vec{\partial}_q^2 - 2 \vec{q} \cdot \vec{\partial}_q - (\vec{q} \cdot \vec{\partial}_q)^2$ such that the Fourier transform $\mathcal{D}_\vec{q} \mathcal{G}(\vec{q}, \vec{k}) = (2\pi)^3 2 \omega(\vec{k}) \delta^3(\vec{k} - \vec{q})$.

Another needed distribution is $\triangle(\vec{k}, \vec{q}) = (2\pi)^3 2 \omega(\vec{k}) \delta^3(\vec{k} - \vec{q}) - F^r(\vec{k}) \omega(\vec{k}) \omega(\vec{q}) + F(\vec{k}) \vec{k} \cdot \vec{q}$ enjoying the semi-group property $\int d\vec{q} \triangle(\vec{k}, \vec{q}) \triangle(\vec{q}, \vec{k}_1) = \triangle(\vec{k}, \vec{k}_1)$.

The final expression of the original fields in terms of the collective and relative canonical variables is

\begin{align*}
\phi(\tau, \vec{\sigma}) &= 2 \int d\vec{k} A(\tau, \vec{k}) \cos \left( \vec{k} \cdot \vec{\sigma} + B(\tau, \vec{k}) \right), \\
\pi(\tau, \vec{\sigma}) &= -2 \int d\vec{k} \omega(k) A(\tau, \vec{k}) \sin \left( \vec{k} \cdot \vec{\sigma} + B(\tau, \vec{k}) \right), \\
A(\tau, \vec{k}) &= \sqrt{I(\tau, \vec{k})} = \sqrt{F^r(\vec{k}) \omega(k) \mathbb{P}_\phi^r - F(\vec{k}) \vec{k} \cdot \vec{P}_\phi + \mathcal{D}_\vec{k} \mathbf{H}(\tau, \vec{k})}, \\
B(\tau, \vec{k}) &= \varphi(\tau, \vec{k}) - \omega(k) \tau = -\vec{k} \cdot \vec{X}_\phi - \omega(k) (\tau - X_\phi^r) + \int d\vec{k}_1 d\vec{k}_2 \mathbf{K}(\tau, \vec{k}_1) \mathcal{G}(\vec{k}_1, \vec{k}_2) \triangle(\vec{k}_1, \vec{k}_2).
\end{align*}

\[\Delta_{LB}\] is the Laplace-Beltrami operator of the mass-shell sub-manifold $H_3^1$ of the KG field. The operator $\mathcal{D}_\vec{q}$ is scalar, formally self-adjoint with respect to the Lorentz invariant measure for the massive case and has $k^A$ as null eigenvectors, $\mathcal{D}_\vec{q} k^A = 0$. 

\[\text{A10}\]
The classical field configurations which can be described in this way must belong to a function space such that:

a) the internal Poincaré generators (A2) are finite (this is required in every approach to be able to build the Poincaré representation);

b) the collective and relative variables are well defined: as a consequence of the existence of the Poincaré' generators and of the study of the zero modes of the operator $\vec{D}_q$ done in Ref.[5], we must have: $b_1) \mid I(\tau, \vec{k}) \mid \rightarrow k^{-3-\sigma}, \sigma > 0, \mid \varphi(\tau, \vec{k}) \mid \rightarrow k$ for $k \rightarrow \infty$; $b_2) \mid I(\tau, \vec{k}) \mid \rightarrow k^{-3+\epsilon}, \epsilon > 0, \mid \varphi(\tau, \vec{k}) \mid \rightarrow k^0, \eta > 0, \text{ for } k \rightarrow 0$.

The field configurations satisfying these conditions may be described in terms of the following multipolar structure in the rest-frame instant form:

A) An effective particle $X^A_\phi = (X^r_\phi; \vec{X}_\phi)$ whose conjugate momentum $P^A_\phi = (P^r_\phi = M_\phi c; \vec{P}_\phi \approx 0)$ describes the mass and momentum components of the Dixon monopole [13] in the multi-polar analysis of the energy-momentum tensor of the KG field. While $\vec{X}_\phi$ is a 3-center of phase for the field configuration, $X^r_\phi$ is an internal time variable conjugate to $P^r_\phi = M_\phi c$. For particle systems an analogue of $X^r_\phi$ does not exist, because the invariant mass is a function of the particle canonical variables and not an independent function like $M_\phi$. As shown in Ref.[13], the rest-frame Hamilton equations imply that $P^r_\phi = M_\phi c$ is a constant of motion and that $X^r_\phi$ satisfies the equation $\frac{dX^r_\phi}{d\tau} = -1$.

B) A set of canonical relative variables $H(\tau, \vec{k}), K(\tau, \vec{k})$, which are constant of motion for a free field due to the complete integrability (Liouville theorem) of the free KG field. In terms of them we can build canonical multipoles with respect to the Fokker-Pryce center of inertia, origin of the instantaneous Wigner 3-spaces, and then all Dixon multipoles [13].

C) The canonical variables $\vec{X}_\phi, \vec{P}_\phi$ are not independent: we have $\vec{P}_\phi \approx 0$ from the rest-frame conditions and $\vec{X}_\phi \approx -\frac{1}{M_\phi c} \int d\vec{k} \omega(k) H(\tau, \vec{k}) \frac{\partial}{\partial k} K(\tau, \vec{k})$ from the elimination of the internal 3-center of mass implied by $\vec{K}_\phi \approx 0$.

D) The variables $P^r_\phi = M_\phi c$ and $X^r_\phi$ have the following interpretation. If we consider a constant energy surface $\mathcal{E}_E$, i.e. $M_\phi c^2 \approx E$, in the KG phase space, then $\mathcal{E}_E$ is not a symplectic submanifold due to the presence of the extra variable $X^r_\phi$. However, if we add the $\tau$-dependent gauge fixing $X^r_\phi - a \tau \approx 0$ ($X^r_\phi \approx -2 \tau$ from Hamilton equations) to the constraint $M_\phi c^2 - E \approx 0$ and we go to Dirac brackets, then $\mathcal{E}_E$ becomes a phase space spanned by the canonical variables $H(\tau, \vec{k}), K(\tau, \vec{k})$ for every value of $E$. The Dirac Hamiltonian becomes a numerical constant $H_D = E/c$ and, as shown from Eqs.(A6), the fields become $\tau$-independent (they only depend upon constants of motion for the free KG theory).

2. The Massless Scalar Field

In Ref.[6] it was shown that the construction based on Eqs.(A6) and (A7) also holds for the massless scalar field. Therefore we have to find the limit of the function $\mathcal{F}(\vec{k}, P^B_\phi)$ for $m \rightarrow 0$. Now we have $\omega(k) = k = |\vec{k}|$. 

58
By using the definition of the delta function in the form
$$\delta(x) = \lim_{\epsilon \to 0} \left( \frac{1}{\sqrt{\pi} \epsilon} e^{-x^2/\epsilon} \right)$$
with $\epsilon = m^2 c^2 / 4\pi \to 0$, we get that the functions $F^\tau(k)$ and $F(k)$ of Eqs.(A5) become the following distribution in the massless limit
$$F^\tau(k) = \frac{8\pi^2}{k^3} \delta(k), \quad F(k) = -\frac{24\pi^2}{k^3} \delta(k). \quad (A11)$$

It can be checked that the conditions contained in Eqs.(A5) are still satisfied since $\delta(k)$ is an even function of $k$ and we have
$$\int d\bar{k} \omega^2(k) F^\tau(k) = 2 \int_0^\infty dk \delta(k) = \int_{-\infty}^{\infty} dk \delta(k) = 1,$$
$$\int d\bar{k} k^r k^s F(k) = -2 \delta^{rs} \int_0^\infty dk \delta(k) = -\delta^{rs} \int_{-\infty}^{\infty} dk \delta(k) = -\delta^{rs}. \quad (A12)$$

The collective variables (A7) are now well defined for the field configurations whose phase variables behave as $\varphi(\tau, \vec{k}) \to k \to 0$ and have the form
$$X^\tau_\phi = \frac{1}{2\pi} \int \frac{d^3k}{k^3} \delta(k) \varphi(\tau, \vec{k}),$$
$$\bar{X}_\phi = \frac{3}{2\pi} \int \frac{d^3k}{k^3} \delta(k) \varphi(\tau, \vec{k}). \quad (A13)$$

As shown in Ref.[6] the relative canonical variables, having zero Poisson brackets with the collective ones, are again given by Eqs.(A9), but now $\omega(k) = k$ and $\mathcal{G}(\vec{k}, \vec{q})$ is the Green function of the operator $\mathcal{D}_q = 3 - 2 \vec{q} \cdot \vec{\partial}_q - (\vec{q} \cdot \vec{\partial}_q)^2$ \textsuperscript{28}. Also Eqs.(A10) for the reconstruction of the massless KG fields hold.

Now, the field configurations having well defined collective and relative canonical variables are restricted by the requirement of the existence of the Poincare’ generators. This requires \textsuperscript{[6]}: i) $|I(\tau, \vec{k})| \to k^{-3-\delta}$, $\delta > 0$, and $|\varphi(\tau, \vec{k})| \to k$ for $k \to \infty$; ii) $|I(\tau, \vec{k})| \to k^{-2+\epsilon}$, $\epsilon > 0$, and $|\varphi(\tau, \vec{k})| \to k^\alpha$, $\alpha > -\delta$ for $k \to 0$. However an extra infinite set of integral restrictions comes from the study of the zero modes $w_{lm}(\vec{k})$ of the operator $\bar{D}_q$ done in Ref.[6]. If we define the quantities $P_m = \int d\bar{k} w_{lm}(\vec{k}) I(\tau, \vec{k})$, then the restrictions are $P_m \equiv \int d\bar{k} w_{lm}(\vec{k}) \mathcal{F}(\vec{k}, \mathcal{P}_m^\text{B})$ for $l \geq 2$ (the $P_m$ with $l > 2$ are called super-translations, being connected with the generators of the algebra of BMS group studied in this framework in Ref.[15]).

\textsuperscript{28} Like in the massive case this operator is scalar, formally self-adjoint with respect the Lorentz invariant measure for the massless case and has $k^A = (|\vec{k}|; \vec{k})$ as null eigenvectors.
APPENDIX B: RELATIVISTIC LAGRANGIANS FOR THE ELECTRIC DIPOLE APPROXIMATION

1. The Standard Electric Dipole Representation with Grassmann-valued Electric Charges

Let us revisit the standard electric dipole representation starting from the Hamiltonian (7.1) with Grassmann-valued charges. We go to the collective and relative variables (2.3) and we make the dipole approximation (6.2) (i.e. we neglect terms of order $O\left((\vec{p}_{12} \cdot \frac{\partial}{\partial \vec{q}_{12}})^2 \vec{A}_\perp\right)$), we get from the Hamilton equations ($\mu = m_1 m_2/m$, $\dot{a} = \frac{dS}{dt}$, $\tau = ct$, $\int d\tau L(\tau) = \int dt \tilde{L}(t)$)

$$
\vec{p}_{12}(\tau) = mc \vec{A}_{12}(\tau) + \frac{Q}{c} \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + 
+ \frac{Q}{c} \frac{m_2 - m_1}{m} \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)),
$$

$$
\vec{A}_{12}(\tau) = \mu c \vec{A}_{12}(\tau) + \frac{Q}{c} \left[ \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \frac{m_2 - m_1}{m} \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) \right] + 
+ \frac{Q}{c} \left[ \frac{m_2 - m_1}{m} \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \frac{m_2^2 + m_2^2}{m^2} \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) \right], \quad (B1)
$$

where the notation of Eq.(6.1) has been used.

Then by means of the inverse Legendre transformation we get the Lagrangian

$$
L = \frac{mc}{2} \vec{A}_{12}(\tau) + \frac{mc}{2} \vec{A}_{12}(\tau) +
+ \left(\frac{Q}{c} + \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \vec{A}_{12}(\tau) \cdot \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) +
+ \frac{2Q}{c} \vec{A}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \left(\frac{Q}{c} + \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \vec{A}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) +
+ \left(\frac{Q}{c} \frac{m_2 - m_1}{m} + \frac{Q}{c} \frac{m_1^2 + m_2^2}{m^2} \right) \vec{A}_{12}(\tau) \cdot \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)). \quad (B2)
$$

Let us write $L = L_1 + \frac{dS}{dt}$ with the following function $S$

$$
S = \frac{m_2 Q_1 - m_1 Q_2}{2mc} \vec{p}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) +
+ \frac{m_2^2 Q_1 + m_1^2 Q_2}{2m^2} \vec{p}_{12}(\tau) \cdot \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) =
= \frac{Q}{c} \left[ \vec{p}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \vec{p}_{12}(\tau) \cdot \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) \right] +
+ \frac{Q}{c} \left[ \frac{m_2 - m_1}{m} \vec{p}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \frac{m_2^2 + m_2^2}{2m^2} \vec{p}_{12}(\tau) \cdot \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) \right]
\rightarrow Q \rightarrow 0 \quad \frac{Q}{c} \vec{p}_{12}(\tau) \cdot \left[ \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) + \frac{m_2 - m_1}{2m} \left(\vec{p}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{q}_{12}}\right) \vec{A}_\perp(\tau, \vec{q}_{12}(\tau)) \right], \quad (B3)
$$
where we used the notation of Eq.(6.1). For neutral systems, $\mathcal{Q} \approx 0$, $S = S_o + O\left( (\vec{\rho}_{12} \cdot \frac{\partial}{\partial \vec{\eta}_{12}}) \vec{A}_{\perp} \right)$ is the extension to the next order of the classical counterpart $S_o$ of the generator of the Goppert-Mayer unitary transformation.

If we use $\frac{d \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau))}{d \tau} = -\vec{\pi}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) + \left( \vec{\eta}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau))$, $L_1$ takes the following form

$$L_1 = \frac{mc}{2} \dot{\vec{\eta}}_{12}(\tau) + \frac{mc}{2} \ddot{\vec{\rho}}_{12}(\tau) + \frac{2}{c} \frac{\mathcal{Q}}{m} \vec{\eta}_{12}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) +$$

$$+ \left( \frac{Q}{c} + \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \left[ \vec{\rho}_{12}(\tau) \cdot \vec{\pi}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) - \vec{\eta}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right) \right] +$$

$$+ \left( \frac{Q}{c} \frac{m_2 - m_1}{2m} + \frac{Q}{c} \frac{m_1^2 + m_2^2}{2m^2} \right) \left[ \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{\pi}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) -$$

$$- \vec{\dot{\rho}}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right) \right]. \quad (B4)$$

We have used $\vec{B}(\tau, \vec{\eta}_{12}(\tau)) = -\frac{\partial}{\partial \vec{\eta}_{12}} \times \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau))$, $\vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) = \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) - \frac{\partial}{\partial \vec{\eta}_{12}} \left( \vec{\rho}_{12}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) \right)$. The new momenta are

$$\vec{\kappa}_{12}(\tau) = mc \dot{\vec{\eta}}_{12}(\tau) + \frac{2}{c} \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) - \left( \frac{Q}{c} + \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)),$$

$$\vec{\pi}_{12}(\tau) = mc \ddot{\vec{\rho}}_{12}(\tau) - \left( \frac{Q}{c} \frac{m_2 - m_1}{2m} + \frac{Q}{c} \frac{m_1^2 + m_2^2}{2m^2} \right) \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)). \quad (B5)$$

Then by Legendre transformation we get the Hamiltonian of the electric dipole representation in the dipole approximation

$$H_1 = \frac{\vec{\kappa}_{12}^2(\tau)}{2mc} + \frac{\vec{\pi}_{12}^2(\tau)}{2mc} - \left( \frac{Q}{c} + \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \vec{\rho}_{12}(\tau) \cdot \vec{\pi}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) -$$

$$- \left( \frac{Q}{c} \frac{m_2 - m_1}{2m} + \frac{Q}{c} \frac{m_1^2 + m_2^2}{2m^2} \right) \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{\pi}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) +$$

$$+ \frac{1}{mc} \left( \frac{Q}{c} \frac{m_2 - m_1}{m} \right) \vec{\kappa}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right) +$$

$$+ \frac{1}{mc} \left( \frac{Q}{c} \frac{m_2 - m_1}{2m} + \frac{Q}{c} \frac{m_1^2 + m_2^2}{2m^2} \right) \vec{\pi}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right) -$$

$$- Q_1 Q_2 \frac{3 \mu}{4mc^2} \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right)^2 -$$

$$- \frac{2}{mc^2} \left[ \vec{\kappa}_{12}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) + \left( Q + \frac{Q m_2 - m_1}{m} \right) \vec{A}_{\perp}(\tau, \vec{\eta}_{12}(\tau)) \cdot \left( \vec{\rho}_{12}(\tau) \times \vec{B}(\tau, \vec{\eta}_{12}(\tau)) \right) \right] +$$

$$+ \frac{4}{2mc^3} \vec{A}_{\perp}^2(\tau, \vec{\eta}_{12}(\tau)). \quad (B6)$$

For $2 \mathcal{Q} = Q_1 + Q_2 \approx 0$, $e = Q_1$, we get the Røntgen Hamiltonian (7.2) of the electric dipole representation, given in Eq.(14.37) of Ref.[7].
2. A Lagrangian for the Internal Motion of the Particles with an External Electromagnetic Field

Let us now follow the method of Appendix L of Ref.[7] of determining the Lagrangian from the Hamiltonian by considering the electro-magnetic field as an external field.

We begin with the internal energy \( \frac{1}{c} E^{(int)} = P^{(int)} = M c \) of Eqs.(4.1) and we do the inverse Legendre transformation on the particle variables to find an effective Lagrangian for the particles. After the dipole approximation and by using \( e = Q_1 \approx -Q_2 \), a term in the resulting Lagrangian will be a total time derivative identifying the generating function \( S \). Then we will do the Legendre transformation of the Lagrangian without the total time derivative. The semi-relativistic limit of the resulting Hamiltonian turns out to be the Røntgen Hamiltonian (7.2) of Ref.[7].

By ignoring the field energy \( E_{em} \) in Eq.(4.1), the inverse Legendre transformation produces the Lagrangian

\[
L(\tau) = \kappa_1 \frac{d\tilde{\eta}_1(\tau)}{d\tau} + \kappa_2 \frac{d\tilde{\eta}_2(\tau)}{d\tau} - M c =
\]

\[
= -m_1 c \sqrt{1 - \left( \frac{d\tilde{\eta}_1(\tau)}{d\tau} \right)^2} - m_2 c \sqrt{1 - \left( \frac{d\tilde{\eta}_2(\tau)}{d\tau} \right)^2} - \frac{Q_1 Q_2}{4 \pi c |\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)|} +
\]

\[
+ \frac{Q_1}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_1(\tau)) \cdot \frac{d\tilde{\eta}_1}{d\tau} + \frac{Q_2}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_2(\tau)) \cdot \frac{d\tilde{\eta}_2}{d\tau}.
\]

(B7)

Switching to the center of mass and relative variables of Eqs.(2.3) (here we put \( \tilde{\eta} = \tilde{\eta}_{12} \) and \( \tilde{\rho} = \tilde{\rho}_{12} \)) we obtain

\[
L(\tau) = -m_1 c \sqrt{1 - \left( \frac{d\tilde{\eta}(\tau)}{d\tau} + \frac{m_2}{m} \frac{d\tilde{\rho}(\tau)}{d\tau} \right)^2} - m_2 c \sqrt{1 - \left( \frac{d\tilde{\eta}(\tau)}{d\tau} - \frac{m_1}{m} \frac{d\tilde{\rho}(\tau)}{d\tau} \right)^2} - \frac{Q_1 Q_2}{4 \pi c |\tilde{\rho}(\tau)|} +
\]

\[
+ \frac{Q_1}{c} \left( \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) + \frac{m_2}{m} \frac{\partial}{\partial \tilde{\eta}} \tilde{\rho}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) \right) \cdot \left( \frac{d\tilde{\eta}(\tau)}{d\tau} + \frac{m_2}{m} \frac{d\tilde{\rho}(\tau)}{d\tau} \right) +
\]

\[
+ \frac{Q_2}{c} \left( \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{m_1}{m} \frac{\partial}{\partial \tilde{\eta}} \tilde{\rho}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) \right) \cdot \left( \frac{d\tilde{\eta}(\tau)}{d\tau} - \frac{m_1}{m} \frac{d\tilde{\rho}(\tau)}{d\tau} \right),
\]

(B8)

in which we have expanded the vector potentials about \( \tilde{\eta} \) by using the dipole approximation. By using the notation \( Q = \frac{1}{2} (Q_1 - Q_2) \), \( \bar{Q} = \frac{1}{2} (Q_1 + Q_2) \), of Eq.(6.1) after some algebra we find \(^{29}\)

\(^{29}\) Note that we divided the \( \frac{d\tilde{\rho}(\tau)}{d\tau} \cdot \frac{\partial}{\partial \tilde{\eta}} \left( \tilde{\rho}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) \right) \) term into separate and equal parts following the treatment of Ref.[7]: this will produce the correct \( \frac{1}{2} \) factors.
\[ L(\tau) = -m_1 c \sqrt{1 - \left(\frac{d\tilde{\eta}(\tau)}{d\tau} + \frac{m_2}{m} \frac{d\tilde{p}(\tau)}{d\tau}\right)^2} - m_2 c \sqrt{1 - \left(\frac{d\tilde{\eta}(\tau)}{d\tau} - \frac{m_1}{m} \frac{d\tilde{p}(\tau)}{d\tau}\right)^2} + \]
\[ + \frac{Q_1 Q_2}{4\pi c |\tilde{p}(\tau)|} \left(\frac{d\tilde{\eta}(\tau)}{d\tau}\right) + \frac{Q}{c} \left(\frac{d\tilde{p}(\tau)}{d\tau} \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) + \frac{\Delta m}{2m} \tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \left(\tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right) - \right) \]
\[ - \frac{d\tilde{\eta}(\tau)}{d\tau} \left[ \frac{\partial}{\partial \tilde{\eta}} \left(\tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right) - \tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \left(\tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right)\right] + \]
\[ + \frac{\Delta m}{2m} \frac{d\tilde{p}(\tau)}{d\tau} \left[ \tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{\partial}{\partial \tilde{\eta}} \left(\tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right)\right] - \]
\[ - \tilde{\eta}(\tau) \cdot \left(\frac{d\tilde{\eta}(\tau)}{d\tau} \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right) + \]
\[ + \frac{\Delta m}{m} \frac{d\tilde{\eta}(\tau)}{d\tau} \left[ \tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{\partial}{\partial \tilde{\eta}} \left(\tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right)\right] - \]
\[ - \tilde{\eta}(\tau) \cdot \left(\frac{d\tilde{\eta}(\tau)}{d\tau} \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right) + \frac{d\tilde{S}}{d\tau} = \]
\[ \overset{\text{def}}{=} L_1(\tau) + \frac{d\tilde{S}}{d\tau}. \]  

in which we have ignored the second order spatial gradient terms of the vector potential (dipole approximation) and we have put \( \Delta m = m_2 - m_1, \mu = \frac{m_1 m_2}{m} \).

The total time derivative terms, not contributing to the equations of motion, identify the generating function \( \tilde{S} \):

\[ \tilde{S} = \frac{Q}{c} \left[ \tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) + \frac{\Delta m}{2m} \tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right] + \]
\[ + \frac{Q}{c} \left[ \tilde{\eta}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) + \frac{\Delta m}{m} \tilde{p}(\tau) \cdot \tilde{A}_\perp(\tau, \tilde{\eta}(\tau)) + \right. \]
\[ + \frac{m^2 - 2\mu m}{2m^2} \tilde{p}(\tau) \cdot \left(\tilde{p}(\tau) \cdot \frac{\partial}{\partial \tilde{\eta}} \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))\right). \]  

Since we are in the radiation gauge we have \( \tilde{E}_\perp(\tau, \tilde{\eta}(\tau)) = -\frac{\partial \tilde{A}_\perp(\tau, \tilde{\eta}(\tau))}{\partial \tau} \). Then by using
\[
\vec{\rho} \times \vec{B} = \frac{\partial}{\partial \vec{\eta}} \left( \vec{\rho} \cdot \vec{A}_{\perp}(\tau, \vec{\eta}) \right) - \vec{\rho} \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}),
\]
\[
\vec{\eta} \cdot \left( \frac{d\vec{\eta}}{d\tau} \cdot \vec{\nabla} \right) \vec{A}_{\perp}(\tau, \vec{\eta}) = \frac{d\vec{\eta}}{d\tau} \cdot \left( \frac{\partial}{\partial \vec{\eta}} \left( \vec{\eta} \cdot \vec{A}_{\perp}(\tau, \vec{\eta}) \right) - \vec{A}_{\perp}(\tau, \vec{\eta}) \right),
\]
\[
\vec{\eta} \times \vec{B} = \vec{A}_{\perp}(\tau, \vec{\eta}) - \frac{\partial}{\partial \vec{\eta}} \vec{\eta} \cdot \vec{A}_{\perp}(\tau, \vec{\eta}) - \vec{\eta} \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}),
\]
\[
\frac{d\vec{\eta}}{d\tau} \cdot \vec{\eta} \times \vec{B} = -\vec{\eta} \cdot \left( \frac{d\vec{\eta}}{d\tau} \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}) - \frac{d\vec{\eta}}{d\tau} \cdot \left( \vec{\eta} \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}) \right) \right),
\]
\]

we obtain

\[
L_1(\tau) = -m_1 c \sqrt{1 - \left( \frac{d\vec{\eta}(\tau)}{d\tau} + \frac{m_2}{m} \frac{d\vec{\rho}(\tau)}{d\tau} \right)^2} - m_2 c \sqrt{1 - \left( \frac{d\vec{\eta}(\tau)}{d\tau} - \frac{m_1}{m} \frac{d\vec{\rho}(\tau)}{d\tau} \right)^2} - 
\]
\[
- \frac{Q_1 Q_2}{4\pi c |\vec{\rho}(\tau)|} + \frac{Q}{c} \left( \vec{\rho}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) + \frac{\Delta m}{2m} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right) \right) - 
\]
\[
- \frac{d\vec{\eta}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) - \frac{\Delta m}{2m} \frac{d\vec{\rho}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) + 
\]
\[
+ \frac{Q}{c} \left( \frac{\Delta m}{m} \vec{\rho}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) + \frac{m^2 - 2\mu m}{2m^2} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right) \right) - 
\]
\[
- \frac{m^2 - 2\mu m}{2m^2} \frac{d\vec{\rho}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) - \frac{\Delta m}{m} \frac{d\vec{\eta}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) + 
\]
\[
+ \frac{\Delta m}{2m} \frac{d\vec{\rho}(\tau)}{d\tau} \cdot \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right). \tag{B12}
\]

In the case \( e = Q \approx Q_1 \approx -Q_2, \ Q \approx 0 \), this becomes

\[
L_1(\tau) = -m_1 c \sqrt{1 - \left( \frac{d\vec{\eta}(\tau)}{d\tau} + \frac{m_2}{m} \frac{d\vec{\rho}(\tau)}{d\tau} \right)^2} - m_2 c \sqrt{1 - \left( \frac{d\vec{\eta}(\tau)}{d\tau} - \frac{m_1}{m} \frac{d\vec{\rho}(\tau)}{d\tau} \right)^2} + 
\]
\[
+ \frac{e^2}{4\pi c |\vec{\rho}(\tau)|} + \frac{e}{c} \vec{\rho}(\tau) \cdot \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) + \frac{\Delta m}{c} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right) - 
\]
\[
- \frac{e}{c} \frac{d\vec{\eta}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) - \frac{\Delta m}{c} \frac{d\vec{\rho}(\tau)}{d\tau} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)). \tag{B13}
\]

and the generating function \( \tilde{S} \) reduces to

\[
\tilde{S} = \frac{e}{c} \vec{\rho}(\tau) \cdot \left[ \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) + \frac{\Delta m}{2m} \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right) \right]. \tag{B14}
\]

In order to obtain the Hamiltonian from Eq.(B12) we revert from the composite variables back to the constituent variables, which mean that we need to rewrite \( L \) in terms of the individual velocities using \( \vec{\eta} = \vec{\eta}_1 + \vec{\eta}_2 \) in \( m_1 \vec{\eta}_1 + m_2 \vec{\eta}_2 \), \( \vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2 \).
We obtain

\[ L_1(\tau) = -m_1 c \sqrt{1 - \left(\frac{d\vec{\eta}_1(\tau)}{d\tau}\right)^2} - m_2 c \sqrt{1 - \left(\frac{d\vec{\eta}_2(\tau)}{d\tau}\right)^2} + \]

\[ + \frac{d\vec{\eta}_1(\tau)}{d\tau} \cdot \vec{A}_1(\tau, \vec{\eta}(\tau)) + \frac{d\vec{\eta}_2(\tau)}{d\tau} \cdot \vec{A}_2(\tau, \vec{\eta}(\tau)) - \]

\[ - \frac{Q_1 Q_2}{4\pi c |\vec{\rho}(\tau)|} + \frac{Q}{c} \vec{\rho}(\tau) \cdot \vec{\pi}_\perp(\tau, \vec{\eta}(\tau)) + \frac{Q}{c} \frac{\Delta m}{2m} \vec{\rho}(\tau) \cdot \left(\vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{\pi}_\perp(\tau, \vec{\eta}(\tau))\right) + \]

\[ + \frac{Q}{c} \frac{\Delta m}{m} \vec{\rho}(\tau) \cdot \vec{\pi}_\perp(\tau, \vec{\eta}(\tau)) + \frac{Q}{c} \frac{m^2 - 2\mu m}{2m^2} \vec{\rho}(\tau) \cdot \left(\vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{\pi}_\perp(\tau, \vec{\eta}(\tau))\right), \quad (B15) \]

where

\[ \vec{A}_i(\tau) = \frac{Q m_i}{cm^2} \vec{\pi}_\perp(\tau, \vec{\eta}(\tau)) - \]

\[ - \left(\frac{Q m_i}{mc} + (-)^{i+1} \frac{Q}{c} \frac{\Delta m}{2m} + \frac{Q}{c} \frac{m^2 - 2\mu m}{2m^2} + \frac{Q}{c} \frac{\Delta m m_i}{m^2}\right) \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)). \quad (B16) \]

By evaluating the new momenta

\[ \vec{\kappa}_i'(\tau) = \frac{\partial L_1(\tau)}{\partial \frac{d\vec{\eta}_i(\tau)}{d\tau}} = \frac{m_i c \frac{d\vec{\eta}_i(\tau)}{d\tau}}{\sqrt{1 - \left(\frac{d\vec{\eta}_i(\tau)}{d\tau}\right)^2}} + \vec{A}_i(\tau). \quad (B17) \]

we obtain after some manipulations the Hamiltonian

\[ H_1 = \vec{\kappa}_1'(\tau) \cdot \frac{d\vec{\eta}_1(\tau)}{d\tau} + \vec{\kappa}_2'(\tau) \cdot \frac{d\vec{\eta}_2(\tau)}{d\tau} - L_1 = \]

\[ = \sqrt{m_1^2 c^2 + (\vec{\kappa}_1'(\tau) - \vec{A}_1(\tau))^2} + \sqrt{m_2^2 c^2 + (\vec{\kappa}_2'(\tau) - \vec{A}_2(\tau))^2} + \]

\[ + \frac{Q_1 Q_2}{4\pi c |\vec{\rho}(\tau)|} - \frac{Q}{c} \vec{\rho}(\tau) \cdot \vec{\pi}_\perp(\tau, \vec{\eta}(\tau)) - \frac{Q}{c} \frac{\Delta m}{2m} \vec{\rho}(\tau) \cdot \left(\vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{\pi}_\perp(\tau, \vec{\eta}(\tau))\right) - \]

\[ - \frac{Q}{c} \frac{\Delta m}{m} \vec{\rho}(\tau) \cdot \vec{\pi}_\perp(\tau, \vec{\eta}(\tau)) - \frac{Q}{c} \frac{m^2 - 2\mu m}{2m^2} \vec{\rho}(\tau) \cdot \left(\vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \vec{\pi}_\perp(\tau, \vec{\eta}(\tau))\right). \quad (B18) \]

This can be brought to desired form using relative variables by substituting for \( \vec{\kappa}_i \) and expanding the vector potential from the square root by using the Grassmann property.

Consider first the non-relativistic limit (in which we keep terms of order \( 1/c \)). First we use

\[ \sqrt{m_i^2 c^2 + (\vec{\kappa}_i'(\tau) - \vec{A}_i(\tau))^2} = \sqrt{m_i^2 c^2 + \vec{\kappa}_i'(\tau)^2} - \frac{\vec{\kappa}_i'(\tau) \cdot \vec{A}_i(\tau)}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i'(\tau)^2}} \quad (B19) \]
Note that we cannot use the Grassmann argument to eliminate the $\tilde{A}^2(\tau)$ terms since they involve the nonzero $Q^2 = -2Q_1Q_2$ terms even with $Q \approx 0$ terms. We shall instead ignore them as being too high an order of $1/c^2$. So we get

$$
H_1 = \sqrt{m_1^2 c^2 + \tilde{k}'_1^2(\tau) - \frac{\tilde{k}'_1(\tau) \cdot \tilde{A}_1(\tau)}{\sqrt{m_1^2 c^2 + \tilde{k}'_1^2(\tau)}}} + \sqrt{m_2^2 c^2 + \tilde{k}'_2^2(\tau) - \frac{\tilde{k}'_2(\tau) \cdot \tilde{A}_2(\tau)}{\sqrt{m_2^2 c^2 + \tilde{k}'_2^2(\tau)}}} + \frac{Q_1Q_2}{4\pi c |\tilde{\rho}(\tau)|} - \frac{Q}{c} \tilde{\rho}(\tau) \cdot \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{Q}{2m} \frac{\Delta m}{c} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{\Delta m}{c} \frac{m^2 - 2\mu m}{2m^2} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)).
$$

In terms of the relative and collective variables

$$
H_1 = \sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \tilde{k}'(\tau) + \tilde{\pi}'(\tau) \right)^2} - \frac{\left( \frac{m_1}{m} \tilde{k}'(\tau) + \tilde{\pi}'(\tau) \right) \cdot \tilde{A}_1(\tau)}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \tilde{k}'(\tau) + \tilde{\pi}'(\tau) \right)^2}} + \frac{Q_1Q_2}{4\pi c |\tilde{\rho}(\tau)|} - \frac{Q}{c} \tilde{\rho}(\tau) \cdot \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{Q}{2m} \frac{\Delta m}{c} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{\Delta m}{c} \frac{m^2 - 2\mu m}{2m^2} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)).
$$

Next, look at the limit as $c \to \infty$ in which

$$
\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \tilde{k}'(\tau) + (-)^{i+1} \tilde{\pi}'(\tau) \right)^2} \to m_i c + \frac{\left( \frac{m_1}{m} \tilde{k}'(\tau) + (-)^{i+1} \tilde{\pi}'(\tau) \right)^2}{2m_i c},
$$

$$
\frac{1}{\sqrt{m_1^2 c^2 + \left( \frac{m_1}{m} \tilde{k}'(\tau) + (-)^{i+1} \tilde{\pi}'(\tau) \right)^2}} \to \frac{1}{m_i c} - \frac{\left( \frac{m_1}{m} \tilde{k}'(\tau) + (-)^{i+1} \tilde{\pi}'(\tau) \right)^2}{2m_i^2 c^3},
$$

and we keep only the terms of $O(1/c)$ and lower obtaining

$$
H_1 c = m c^2 + \frac{\tilde{k}'^2(\tau)}{2m} + \frac{\tilde{\pi}'^2(\tau)}{2\mu} - \frac{\tilde{k}'(\tau)}{m} \cdot \left( \tilde{A}_1(\tau) + \tilde{A}_2(\tau) \right) - \frac{\tilde{\pi}'(\tau)}{m} \cdot \left( \frac{\tilde{A}_1(\tau)}{m_1} - \frac{\tilde{A}_2(\tau)}{m_2} \right) + \frac{Q_1Q_2}{4\pi |\tilde{\rho}(\tau)|} - Q \tilde{\rho}(\tau) \cdot \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{Q}{2m} \frac{\Delta m}{c} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)) - \frac{\Delta m}{c} \frac{m^2 - 2\mu m}{2m^2} \tilde{\rho}(\tau) \cdot \left( \frac{\partial}{\partial \tilde{\eta}} \right) \tilde{\pi}_\perp(\tau, \tilde{\eta}(\tau)).
$$

Now using Eq.(B16) we obtain in the non-relativistic limit
\[ H_1 c = m c^2 + \frac{\vec{\kappa}^2(\tau)}{2m} + \frac{\vec{\pi}^2(\tau)}{2\mu} + \]
\[ + \frac{\vec{\kappa}(\tau)}{m} \cdot \left[ \left( \frac{Q}{c} + \frac{\Delta m}{m} \right) \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) - \frac{Q}{c} \vec{A}_{\perp}(\tau, \vec{\eta}(\tau)) \right] + \]
\[ + \left( \frac{Q}{c} \frac{\Delta m}{2m\mu} + \frac{Q}{c} \frac{m^2 - 2\mu m}{2m^2\mu} \right) \vec{\pi}'(\tau) \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) + \]
\[ + \frac{Q_1 Q_2}{4\pi |\vec{\rho}(\tau)|} - Q \vec{\rho}(\tau) \cdot \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)) - Q \frac{\Delta m}{2m} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \right) \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)) - \]
\[ - Q \frac{\Delta m}{m} \vec{\rho}(\tau) \cdot \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)) - Q \frac{m^2 - 2\mu m}{2m^2} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \right) \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)). \quad (B24) \]

If we use \( Q \approx 0, \quad e = Q \approx Q_1 \approx -Q_2, \) we get

\[ H_1 c = m c^2 + \frac{\vec{\kappa}^2(\tau)}{2m} - \frac{e^2}{4\pi |\vec{\rho}(\tau)|} + \]
\[ + \frac{\vec{\pi}^2(\tau)}{2\mu} + \frac{\vec{\kappa}(\tau)}{m} \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) + \frac{e}{c} \frac{\Delta m}{2m\mu} \vec{\pi}'(\tau) \cdot \vec{\rho}(\tau) \times \vec{B}(\tau, \vec{\eta}(\tau)) - \]
\[ - e \vec{\rho}(\tau) \cdot \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)) - e \frac{\Delta m}{2m} \vec{\rho}(\tau) \cdot \left( \vec{\rho}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}} \right) \vec{\pi}_{\perp}(\tau, \vec{\eta}(\tau)), \]

\[ (B25) \]

which agrees with Eq.(L.14) and (14.37) of Ref.[8] with \( e^2 = 0 \) (one could also find the \( e^2 \) terms of Eq.(7.2)).

3. A Lagrangian for the Internal Motion of the Particles plus the Electro-Magnetic Field

Let us consider the isolated system "charged particles with mutual Coulomb interaction plus the transverse electromagnetic field" inside the instantaneous Wigner 3-spaces. Its motion is determined by the Hamiltonian \( Mc = \frac{1}{2} E_{(int)} \) given in Eqs.(4.1) and then restricted by the rest-frame conditions \( \vec{P}_{(int)} \approx 0 \) and by their gauge fixing \( \vec{K}_{(int)} \approx 0 \). Therefore we can look for the Lagrangian corresponding to the Hamiltonian \( Mc \).

By using the first half of the Hamilton equations with Hamiltonian \( Mc \)
\[
\frac{d \tilde{\eta}_i(\tau)}{d\tau} = \left\{ \tilde{\eta}_i(\tau), Mc \right\} = \frac{\tilde{\kappa}_i(\tau) - \frac{Q_i}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + (\tilde{\kappa}_i(\tau) - \frac{Q_i}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)))^2}}.
\]

\[
\frac{\partial \tilde{A}_\perp(\tau, \bar{\sigma})}{\partial \tau} = \left\{ \tilde{A}_\perp(\tau, \bar{\sigma}), Mc \right\} = -\bar{\pi}_\perp(\tau, \bar{\sigma}),
\]

\[
\downarrow
\]

\[
\tilde{\kappa}_i(\tau) = \frac{m_i c \frac{d \tilde{\eta}_i}{d\tau}}{\sqrt{1 - \left(\frac{d \tilde{\eta}_i}{d\tau}\right)^2}} + \frac{Q_i}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)),
\]

\[
\Rightarrow \quad \sqrt{m_i^2 c^2 + (\tilde{\kappa}_i(\tau) - \frac{Q_i}{c} \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)))^2} = \frac{m_i c}{\sqrt{1 - \left(\frac{d \tilde{\eta}_i}{d\tau}\right)^2}}.
\]

the inverse Legendre transformation leads to the following Lagrangian

\[
\mathcal{L}(\tau) = \int d^3\sigma \left( \sum_i \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)) \left[ \tilde{\kappa}_i(\tau) \cdot \frac{d \tilde{\eta}_i}{d\tau} - \frac{1}{c} \left[ \bar{\pi}_\perp \cdot \frac{\partial \tilde{A}_\perp}{\partial \tau} \right] (\tau, \bar{\sigma}) \right] - \frac{1}{c} \mathcal{E}_{\text{int}} \right) - \\
- \frac{Q_1 Q_2}{8 \pi c |\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)|} + \frac{Q_i}{c} \frac{d \tilde{\eta}_i(\tau)}{d\tau} \cdot \tilde{A}_\perp(\tau, \bar{\sigma}) + \\
+ \frac{1}{2c} \left[ \left( \frac{\partial \tilde{A}_\perp}{\partial \tau} \right)^2 - \vec{B}^2 \right] (\tau, \bar{\sigma}) = \\
= \int d^3\sigma \left( \sum_i \delta^3(\bar{\sigma} - \tilde{\eta}_i(\tau)) \left[ -m_i c \sqrt{1 - \left(\frac{d \tilde{\eta}_i}{d\tau}\right)^2} - \frac{Q_1 Q_2}{8 \pi c |\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)|} - \\
- \frac{Q_i}{c} \tilde{\eta}_i(\tau) \cdot \left( \frac{\partial \tilde{A}_\perp(\tau, \bar{\sigma})}{\partial \tau} + \frac{d \tilde{\eta}_i(\tau)}{d\tau} \cdot \frac{\partial \tilde{A}_\perp(\tau, \bar{\sigma})}{\partial \bar{\sigma}} \right) \right] + \\
+ \frac{1}{2c} \int d^3\sigma \left[ \left( \frac{\partial \tilde{A}_\perp}{\partial \tau} \right)^2 - \vec{B}^2 \right] (\tau, \bar{\sigma}) + \frac{d S_2(\tau)}{d\tau} = \\
= \mathcal{L}_1(\tau) + \frac{d S_2(\tau)}{d\tau}.
\]

In the last lines we have redefined the Lagrangian by isolating a total time derivative. This leads to the identification of the following generating function
\[ S_2 = \frac{1}{c} \sum_i \int d^3\sigma \, \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \, Q_i \vec{\eta}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\sigma}) = \]

\[ = \frac{1}{c} \sum_i Q_i \vec{\eta}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) = \]

\[ = \frac{Q}{c} \left[ \vec{\rho}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \right. \]

\[ + \frac{m_2 - m_1}{m} \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \]

\[ + \vec{\eta}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) \bigg] + \]

\[ + \frac{Q}{c} \sum_{i=1}^{2} \vec{\eta}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) + O\left( (\vec{\rho}_{12} \cdot \frac{\partial}{\partial \vec{\eta}_{12}})^2 \vec{A}_\perp \right) \]

\[ \rightarrow_{Q \approx 0} \frac{Q}{c} \left[ \vec{\rho}_{12}(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \right. \]

\[ + \frac{m_2 - m_1}{m} \vec{\rho}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + \]

\[ + \vec{\eta}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) \bigg] + \]

\[ + \frac{Q}{c} \vec{\eta}_{12}(\tau) \cdot \left( \vec{\rho}_{12}(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_{12}} \right) \vec{A}_\perp(\tau, \vec{\eta}_{12}(\tau)) + O\left( (\vec{\rho}_{12} \cdot \frac{\partial}{\partial \vec{\eta}_{12}})^2 \vec{A}_\perp \right). \quad (B28) \]

In Eq.(B28) we have shown the dipole approximation of \( S_2 \). It contains an electric dipole term similar to the function \( S \) of Eqs.(B3) - (7.3) but with \( \frac{m_2 - m_1}{2m} \) replaced by \( \frac{m_2 - m_1}{m} \). Moreover it contains an extra term connected to the motion of the collective variable \( \vec{\eta}_{12}(\tau) \) of the 2-particle system.
[1] H.W. Crater and L. Lusanna, *The Rest-Frame Darwin Potential from the Lienard-Wiechert Solution in the Radiation Gauge*, Ann.Phys. (N.Y.) **289**, 87 (2001).

[2] D. Alba, H.W. Crater and L. Lusanna, *Towards Relativistic Atom Physics. I. The Rest-Frame Instant Form of Dynamics and a Canonical Transformation for a system of Charged Particles plus the Electro-Magnetic Field* (arXiv: 0806.2383).

[3] L. Lusanna, *The Chrono-Geometrical Structure of Special and General Relativity: A Re-Visitation of Canonical Geometrodynamics*, lectures at 42nd Karpacz Winter School of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, Ladek, Poland, 6-11 Feb 2006, Int.J.Geom.Methods in Mod.Phys. **4**, 79 (2007). (gr-qc/0604120).

[4] D. Alba, H.W. Crater and L. Lusanna, *Hamiltonian Relativistic Two-Body Problem: Center of Mass and Orbit Reconstruction*, J.Phys. **A40**, 9585 (2007) (gr-qc/0610200).

[5] G. Longhi and M. Materassi, *Collective and Relative Variables for a Classical Real Klein-Gordon Field*, Int.J.Mod.Phys. **A14**, 3387 (1999) (hep-th/9809024).

[6] L. Assenza and G. Longhi, *Collective and Relative Variables for Massless Fields*, Int.J.Mod.Phys. **A15**, 4575-4601 (2000).

[7] W.P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001).

[8] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Atom-Photon Interactions. Basic Processes and Applications* (Wiley, New York, 1992).

[9] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms. Introduction to Quantum Electrodynamics* (Wiley, New York, 1989).

[10] D. Alba, H.W. Crater and L. Lusanna, *Towards Relativistic Atom Physics. III. Clock Synchronization and Quantization*, in preparation.

[11] D. Alba, L. Lusanna and M. Pauri, *New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems* (2005), in Atomic and Molecular Clusters: New Research, ed.Y.L.Ping (Nova Science, New York, 2006) (hep-th/0505005).

D. Alba, L. Lusanna and M. Pauri, *Centers of Mass and Rotational Kinematics for the Relativistic N-Body Problem in the Rest-Frame Instant Form*, J.Math.Phys. **43**, 1677-1727 (2002) (hep-th/0102087).

D. Alba, L. Lusanna and M. Pauri, *Multipolar Expansions for Closed and Open Systems of Relativistic Particles*, J. Math.Phys. **46**, 062505, 1-36 (2004) (hep-th/0402181).

[12] D. Alba, L. Lusanna and M. Pauri, *Dynamical Body Frames, Orientation-Shape Variables and Canonical Spin Bases for the Nonrelativistic N-Body Problem* J.Math.Phys. **43**, 373 (2002) (hep-th/0011014).

[13] L. Lusanna and M. Materassi, *A Canonical Decomposition in Collective and Relative Variables of a Klein-Gordon Field in the Rest Frame Wigner Covariant Instant Form*, Int.J.Mod.Phys. **A15**, 2821-2916 (2000) (hep-th/9904202).

[14] C.J. Borde’, *Quantum Theory of Atom-Wave Beam Splitters and Applications to Multidimensional Atomic Gravito-Inertial Sensors*, Gen.Rel.Grav. **36**, 475 (2004); *Atomic Clocks and Inertial Sensors*, Metrologia **39**, 435 (2002).

[15] G. Longhi and M. Materassi, *A Canonical Realization of the BMS Algebra*, J.Math.Phys. **40**, 480 (1999) (hep-th/9803128).

[16] D. Alba and L. Lusanna, *Quantum Mechanics in Noninertial Frames with a Multitemporal
Quantization Scheme: I. Relativistic Particles, Int. J. Mod. Phys. A21, 2781 (2006) (hep-th/0502194).

[17] J. Larson, Dynamics of the Jaynes-Cummings and Rabi models: Old Wine in New Bottles, Phys. Scripta 76, 146 (2007).