Ricci Flat Black Holes and Hawking-Page Phase Transition in Gauss-Bonnet Gravity and Dilaton Gravity

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Abstract

It is well-known that there exists a Hawking-Page phase transition between a spherical AdS black hole and a thermal AdS space. The phase transition does not happen between a Ricci flat AdS black hole whose horizon is a Ricci flat space and a thermal AdS space in the Poincare coordinates. However, the Hawking-Page phase transition occurs between a Ricci flat AdS black hole and an AdS soliton if at least one of horizon coordinates for the Ricci flat black hole is compact. We show a similar phase transition between the Ricci flat black holes and deformed AdS solitons in the Gauss-Bonnet gravity and the dilaton gravity with a Liouville-type potential including the gauged supergravity coming from the spherical reduction of Dp-branes in type II supergravity. In contrast to Einstein gravity, we find that the high temperature phase can be dominated either by black holes or deformed AdS solitons depending on parameters.

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1 Introduction

Since the AdS/CFT correspondence was proposed [1], a lot of attention has been focused on the black holes in AdS space, and various properties of black holes in AdS space have been studied. In the spirit of the AdS/CFT correspondence, Witten [2] has argued that the thermodynamics of black holes in AdS space can be identified with that of dual strong coupling CFTs in high temperature limit. Therefore one can discuss the thermodynamics and phase structure of strong coupling CFTs by studying the thermodynamics of various kinds of black holes in AdS space. Indeed, it is well-known that there exists a phase transition between the Schwarzschild-AdS black hole and thermal AdS space, the so-called Hawking-Page phase transition [3]: the black hole phase dominates the partition function in a high temperature limit, while the thermal AdS space dominates in a low temperature limit. That is, in the AdS space, thermal gas will collapse to form a stable black hole when temperature increases. This phase transition is a first order one, and is interpreted as the confinement/deconfinement phase transition in the dual CFTs [2].

One of remarkable properties of black holes in AdS space is that the black hole horizon is not necessarily a sphere [4]. The case of black hole horizon being a Ricci flat surface was first discussed in [5]. The black hole horizon can also be a negative constant curvature surface [6]. These so-called topological black holes have been investigated in higher dimensions [7,8,9,10] and in dilaton gravity [11,12]. It was found that the Hawking-Page phase transition, which happens for spherical AdS black holes, does not occur for Ricci flat and negative curvature AdS black holes, the latter two being not only locally stable (heat capacity is always positive), but also globally stable (see for example, [7]). Note that to see whether a black hole is globally stable and a phase transition happens, one has to calculate the Euclidean action of the black hole. As is well-known, the gravitational action always diverges due to an infinite space. To get a finite result, one usually takes two different approaches: one is the surface counterterm method, in which some surface terms are added to result in a finite action; the other is called the background subtraction method in which a suitable reference background is chosen so that the solution under study can be asymptotically embedded into the reference background.

For the Ricci flat AdS black hole, which is the main topic of this paper, the conclusion that the solution is globally stable and no Hawking-Page phase transition happens, is drawn by choosing the AdS space in the Poincare coordinates as a reference background (the surface counterterm method is equivalent to choosing the reference background in this case). That is, one views the AdS space in the Poincare coordinates, a reference vacuum, as the lowest energy state. Remarkably, as Horowitz and Myers [13] showed that there does
exist another kind of gravitational configuration, which has lower energy than the AdS space in the Poincare coordinates, but with the same boundary topology as the Ricci flat black hole and the AdS space in the Poincare coordinates. The new configuration is called the AdS soliton. Regarding the AdS soliton as a reference background, Surya, Schleich and Witt [14] found that a phase transition will happen between the Ricci flat AdS black hole and the thermal AdS soliton if at least one of black hole horizon coordinates is compact (see also [15, 16]). The latter is required in order that the black hole can be asymptotically embedded into the background. The compact direction of the horizon plays a crucial role in the Hawking-Page phase transition. The possibility of quasi-normal modes as a probe to the Hawking-Page phase transition has been discussed more recently [17].

In this paper, we discuss the Hawking-Page phase transition between Ricci flat black holes and deformed AdS soliton in the Gauss-Bonnet gravity and dilaton gravity with a Louville-type potential including the gauged supergravity comings from the spherical reduction of Dp-branes in type II supergravity. The effect of Gauss-Bonnet term on the Hawking-Page phase transition of spherical AdS black holes has been studied in [9]. In the next section, we first discuss the Ricci flat black holes and deformed AdS soliton in the Gauss-Bonnet gravity, and see the effect of the Gauss-Bonnet term on the phase transition between the Ricci flat black hole and thermal AdS soliton. In Sec. 3, we consider the Ricci flat black holes and associated deformed AdS soliton in dilaton gravity. The conclusion is given in Sec. 4.

2 Deformed AdS Soliton and Ricci flat black holes in Lovelock gravity

In this section we study Ricci flat black hole and AdS soliton in Gauss-Bonnet gravity in $p + 2$-dimensions, whose action is given by

$$S = \frac{1}{16\pi G} \int d^{p+2}x \sqrt{-g} \left( R + \frac{p(p+1)}{l^2} + \alpha R_{GB} \right),$$

(2.1)

where $R_{GB} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$ is the Gauss-Bonnet term, $\alpha$ is called the Gauss-Bonnet coefficient with dimension $(\text{length})^2$, and $l^{-2}$ is related to the cosmological constant $\Lambda = -p(p+1)/2l^2$. Varying the action yields the equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{p(p+1)}{2l^2} g_{\mu\nu} + \alpha \left( \frac{1}{2} g_{\mu\nu} (R_{\gamma\delta\lambda\sigma} R^{\gamma\delta\lambda\sigma} - 4 R_{\gamma\delta} R^{\gamma\delta} + R^2) 
- 2 RR_{\mu\nu} + 4 R_{\mu\gamma} R_{\gamma\nu} + 4 R_{\gamma\delta} R_{\mu\nu}^{\gamma\delta} - 2 R_{\mu\gamma\delta\lambda} R_{\nu}^{\gamma\delta\lambda} \right).$$

(2.2)
The Ricci flat black hole in the Gauss-Bonnet gravity is \[ ds^2 = -V_b(r)dt^2 + V_b(r)^{-1}dr^2 + r^2(dx^2 + h_{ij}dx^i dx^j), \] (2.3)

where \[ V_b(r) = \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 + \frac{64\pi G\tilde{\alpha}M_b}{p\eta_b\Sigma r^{p-1}} - \frac{4\tilde{\alpha}}{l^2}} \right), \] (2.4)

\[ \tilde{\alpha} = (p-1)(p-2)\alpha, \] \[ M_b \] is an integration constant and is related to the mass of the black hole, \[ \eta_b \] is the period of the coordinate \[ x \] (suppose it is compact), and \[ \Sigma \] is the volume of the \( (p-1) \)-dimensional Ricci flat space described by \( h_{ij}dx^i dx^j \) with topology \( \mathcal{M}_{p-1} = R^{p-1}/\Gamma \), where \( \Gamma \) is a finite discrete group. The dual CFT resides on a manifold with topology \( R \times S^1 \times \mathcal{M}_{p-1} \), where \( S^1 \) represents the period of the coordinate \( x \). Taking the limit \( \tilde{\alpha} \to 0 \), we obtain Ricci flat AdS black holes in general relativity

\[ V_b(r) = \frac{r^2}{l^2} - \frac{16\pi G M_b}{p\eta_b \Sigma r^{p-1}}. \] (2.5)

When \( M_b = 0 \), the solution reduces to

\[ ds^2 = -V_r(r)dt^2 + V_r(r)^{-1}dr^2 + r^2(dx^2 + h_{ij}dx^i dx^j), \]

\[ V_r(r) = \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2}} \right). \] (2.6)

This is an AdS space in the Poincare coordinates with the effective cosmological constant radius \( l_{\text{eff}}^2 = 2\tilde{\alpha}/(1 - \sqrt{1 - 4\tilde{\alpha}/l^2}) \). In addition, we see that the Gauss-Bonnet coefficient must satisfy the condition \( 4\tilde{\alpha}/l^2 \leq 1 \), otherwise the theory is not well-defined.

The black hole horizon \( r_+ \) is determined by \( V_b(r_+) = 0 \). Then the black hole mass can be expressed in terms of the horizon \( r_+ \) as

\[ M_b = \frac{p\eta_b \Sigma r_+^{p+1}}{16\pi G l^2}. \] (2.7)

The Hawking temperature of the black hole can be obtained by Wick rotating the black hole solution (2.3) to its Euclidean sector

\[ ds^2 = V_b(r)d\tau^2 + V_b(r)^{-1}dr^2 + r^2(dx^2 + h_{ij}dx^i dx^j). \] (2.8)

To remove the conical singularity at \( r_+ \) in the plane \( (\tau, r) \), the Euclidean time \( \tau \) must have a period \( \beta_b \),

\[ \beta_b \equiv 1/T_b = \frac{4\pi l^2}{(p+1)r_+}, \] (2.9)

\[ \text{In fact, there are two branches for the solution. One branch is argued to be unstable, so we will not discuss that branch}. \]
which is just the inverse Hawking temperature $1/T_b$ of the black hole. To calculate the Euclidean action of the black hole (2.4) and to regularize the action, we choose the vacuum solution (2.6) as a reference background. In order that the Euclidean black hole solution (2.8) can be self-consistently embedded to the reference background, the Euclidean time of the vacuum solution must have a period $\beta_r$, which obeys

$$\beta_r \sqrt{V_r(r_b)} = \beta_b \sqrt{V_b(r_b)}, \quad (2.10)$$

where $r_b$ is the radial radius for a time-like hypersurface ($r = r_b > r_+$), which acts as the boundary of the system. At the end of calculations, we will take the limit $r_b \to \infty$. Then the difference of the two Euclidean actions is

$$I_b = I_b - I_r = \frac{-\eta_b \Sigma \beta_b}{16\pi G} \int_{r_+}^{r_b} dr \, r^p \left( R + \frac{p(p+1)}{l^2} + \alpha R_{GB} \right) + \frac{\eta_b \Sigma \beta_b}{16\pi G} \int_{0}^{r_b} dr \, r^p \left( R + \frac{p(p+1)}{l^2} + \alpha R_{GB} \right). \quad (2.11)$$

Using (2.2), we have $\alpha R_{GB} = -(pR + p(p+1)(p+2)/l^2)/(p-2)$. And notice that for the metric (2.3), one has $R = -(r^p V_b)''/r^p$. Calculating (2.11) and taking the limit $r_b \to \infty$, we obtain

$$I_b = -\frac{\eta_b \Sigma \beta_b r_+^{p+1}}{16\pi G} = -\frac{\eta_b \Sigma}{16\pi G l^2} \left( \frac{4\pi l^2}{p+1} \right)^{p+1} \frac{1}{\beta_b^p}. \quad (2.12)$$

The thermal energy can be calculated via the formula

$$E_b = \frac{\partial I_b}{\partial \beta_b} = \frac{p \eta_b \Sigma r_+^{p+1}}{16\pi G l^2} = M_b, \quad (2.13)$$

which gives the black hole mass (2.7). The entropy of the black hole can be obtained via $S = \beta_b E_b - I_b$, and it gives

$$S = \frac{\eta_b \Sigma}{4G} r_+^p = \frac{A}{4G}, \quad (2.14)$$

where $A$ is the horizon area. Very interestingly, although higher order derivatives appear in the gravity action (2.1), the entropy of the Ricci flat black holes still obeys the so-called area formula. This is the feature of Ricci flat horizon. This feature persists for Ricci flat

\footnote{In order to have a well-defined variable principle, there exist some surface terms in the action (2.11). As the case of general relativity, however, those surface terms have no contributions to the difference of Euclidean actions here.}
black holes in more general Lovelock gravity \cite{10}. For black hole horizons with positive or negative constant curvature, the area formula does no longer hold in Gauss-Bonnet gravity. In addition, let us calculate the heat capacity of the black hole

\[ C = \frac{\partial M_b}{\partial T_b} = \frac{p_m \sum p}{4G r^p}. \quad (2.15) \]

The heat capacity is always positive, which indicates that the black hole can make thermal equilibrium with the surrounding thermal bath. The negative definiteness of the Euclidean action (2.12) implies that the black hole is globally stable, and that the dual CFT is in the deconfinement phase. Unlike its spherical black hole counterpart, the Hawking-Page phase transition does not appear here.

In \cite{13}, Horowitz and Myers found that there exists a so-called AdS soliton solution in Einstein gravity with a negative cosmological constant, which has a lower mass than the AdS vacuum. The AdS soliton is obtained through a double-analytical continuation. In our case, naturally we may expect that the AdS vacuum (2.6) is not the lowest mass solution; following \cite{13} we can obtain the AdS soliton in the Gauss-Bonnet gravity via analytically continuing the Ricci flat black hole (2.3) with \( t \to i\phi \) and \( x \to it \). Then we get a new solution

\[ ds^2 = V_s(r) d\phi^2 + V_s(r)^{-1} dr^2 + r^2 (-dt^2 + h_{ij} dx^i dx^j), \quad (2.16) \]

where

\[ V_s(r) = \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{4\alpha r_s^{p+1}}{l^2 r^{p+1}} - \frac{4\alpha}{l^2}} \right). \quad (2.17) \]

This solution is just the AdS soliton counterpart in Gauss-Bonnet gravity. Obviously there does not exist any horizon in the solution, but a conical singularity at \( r = r_s \), which obeys \( V_s(r_s) = 0 \). To remove this singularity, the coordinate \( \phi \) must have a period \( \beta_s \),

\[ \beta_s = \frac{4\pi l^2}{(p+1)r_s}. \quad (2.18) \]

This solution is asymptotically AdS, and dual CFT now resides on manifold with topology \( S^1 \times R \times M_{p-1} \), where \( S^1 \) denotes the period of the coordinate \( \phi \). In addition, let us mention here that the radial coordinate \( r \) for the soliton solution ranges from \( r_s \) to \( \infty \).

For the soliton solution (2.17), a natural reference background is the case with \( r_s = 0 \), namely to replace \( V_s \) in (2.16) by the same \( V_r(r) \) given in (2.6). Again, to match the vacuum background, the period \( \beta_r \) of the coordinate \( \phi \) for the reference background must obey the condition, \( \beta_r \sqrt{V_r(r_b)} = \beta_s \sqrt{V_s(r_b)} \), on the boundary. With the solution \( V_r \) as the reference background, and noticing that the Euclidean time for the AdS soliton (2.16)
and the reference background can have an arbitrary period $\beta$, we find that the Euclidean action for the deformed AdS soliton solution (2.16) is

$$I_s = -\frac{\beta_s \Sigma r^{p+1}}{16\pi G l^2}. \quad (2.19)$$

The corresponding mass is

$$E_s \equiv \frac{\partial I_s}{\partial \beta} = -\frac{\beta_s \Sigma r^{p+1}}{16\pi G l^2}, \quad (2.20)$$

and as expected, the associated entropy vanishes, $S = \beta E_s - I_s = 0$.

Note that the mass (2.20) for the AdS soliton is indeed negative, namely the mass of the soliton is less than the reference background AdS space (2.6). On the other hand, let us notice that for the same boundary topology $R \times S^1 \times M_{p-1}$, there exist three kinds of bulk solutions: Ricci flat black hole solution (2.3), AdS space (2.6), and the deformed AdS soliton solution (2.16). From the point of view of the dual CFTs, three kinds of bulk solutions might correspond to three different phases, among which some phase transition may happen like the Hawking-Page phase transition in the bulk for spherical AdS black holes and confinement/deconfinement phase transition for the dual CFTs. From the above calculations, however, we have seen that unlike the spherical AdS black hole, there does not exist a phase transition between the Ricci flat black hole and AdS space, and the same is true in the case between the AdS soliton and AdS space. Furthermore, let us point out that as in general relativity, the AdS soliton has a less energy than the AdS space, therefore it is more natural to consider the AdS soliton as the reference background. It is quite interesting to see whether there does exist or not any phase transition between the Ricci flat black hole and AdS soliton. To see this, let us calculate the Euclidean action of the Ricci flat black hole by viewing the deformed AdS soliton (2.16) as the reference background. In order that the Ricci flat black hole can be embedded into the AdS soliton background at the boundary $r = r_b$, in the Euclidean sector, the Euclidean time period $\beta_r$ for the AdS soliton and the period $\eta_b$ for the coordinate $x$ in the Ricci flat black hole must obey the following conditions

$$r_b \beta_r = \beta_b \sqrt{V_b(r_b)}, \quad r_b \eta_b = \beta_s \sqrt{V_s(r_b)}. \quad (2.21)$$

With the same procedure, we get the Euclidean action of the black hole

$$I_{bs} = -\frac{\Sigma \beta_b \beta_s}{16\pi G l^2 l_{eff}} (r_+^{p+1} - r_s^{p+1}). \quad (2.22)$$

Here $l_{eff} = \sqrt{2\tilde{\alpha}/(1 - \sqrt{1 - 4\tilde{\alpha}/l^2})}$, and its appearance is due to the fact that the coordinate $x$ for the Ricci flat black hole (2.3) and the coordinate $\varphi$ in the deformed AdS
soliton (2.16) have different dimensions. If we rescale $x$ in (2.16) as $x/l_{\text{eff}}$, the factor $1/l_{\text{eff}}$ in (2.22) will disappears.

In this case, the mass of the black hole is

$$E_{bs} = \frac{\partial I_{bs}}{\partial \beta_b} = \frac{\sum \beta_s}{16\pi G l_{\text{eff}}^2} \left( pr_{+}^{p+1} + r_{s}^{p+1} \right).$$ (2.23)

And the associated entropy

$$S = \frac{\sum \beta_s}{4Gl_{\text{eff}}} r_+^p.$$ (2.24)

Note that from (2.21) one has $\eta_b = \beta_s/l_{\text{eff}}$. Compared to (2.13) and (2.14), as expected, we see that the black hole mass depends on the choice of the reference background, but not for the entropy of the black hole. According to the Euclidean action (2.22), we see that when $r_+ > r_s$, it is negative, while it is positive as $r_s > r_+$. That is, when crossing the boundary $r_+ = r_s$, the Euclidean action changes its sign. The change of sign of the Euclidean action is nothing but the indication of a first order phase transition, as in the case of spherical AdS black hole. But, there is a significant difference between the cases of Ricci flat black holes and spherical AdS black holes. For the Hawking-Page phase transition of spherical AdS black holes, the phase transition is determined by the ratio of black hole horizon ($r_+$) and AdS radius $l$: when $r_+ > l$, the black hole phase dominates, while the thermal AdS space dominates as $r_+ < l$. In our case, we can see from (2.22) that the phase transition is now determined by the ratio $r_+/r_s$.

In addition, recalling $\eta_b = \beta_s/l_{\text{eff}}$, let us notice that the Euclidean action (2.22) is the exactly the same as in the case without the Gauss-Bonnet term [14], although the spacetime metrics are changed. As a result, the Gauss-Bonnet term has no effect on the Hawking-Page phase transition. It is expected that the result also holds for the Ricci flat black holes in the more general Lovelock gravity.

### 3 Deformed AdS Soliton and Ricci flat black holes in dilaton gravity

In this section we will first consider a special kind of dilaton gravity, which comes from spherical reduction of Dp-branes in type II supergravity. In this kind of dilaton gravity, there exists a kind of domain wall solutions, and the five-dimensional AdS space appears as a special case. For this kind of domain wall configurations, the holographic principle of quantum gravity can be nicely illustrated via the so-called domain wall/QFT (quantum field theory) correspondence [18], which generalizes the AdS/CFT correspondence to the case of non-conformal quantum theories.
Let us start from the action of type II supergravity in the string frame

\[
S = \frac{1}{2\pi G_{10}} \int d^{10}x \sqrt{-g} \left( e^{-2\phi} (R + 4(\partial\phi)^2) - \frac{1}{2(8-p)!} F_{p-2}^{2} \right),
\]

(3.1)

where \(G_{10} = 8\pi^6 \alpha'^4\) is the gravitational constant in ten dimensions. The black Dp-brane solution in the string frame has the form

\[
\begin{align*}
 ds_{\text{string}}^2 &= H^{-1/2} (-f dt^2 + dx_p^2) + H^{1/2} (f^{-1} dr^2 + r^2 d\Omega_{8-p}^2), \\
 e^\phi &= g_s H^{(3-p)/4}, \\
 F_{8-p} &= Q \epsilon_{8-p},
\end{align*}
\]

(3.2)

where \(0 \leq p \leq 6\), \(g_s\) is the string coupling constant at infinity, \(\epsilon_{8-p}\) is the volume form of the sphere \(S^{8-p}\), and \(Q\) is the magnetic charge of the Dp-brane. In addition,

\[
H = 1 + \frac{r_0^{7-p} \sinh^2 \alpha}{r^{7-p}}, \quad f = 1 - \frac{r_0^{7-p}}{r^{7-p}}.
\]

(3.3)

In the decoupling limit, \(\alpha' \to 0\), but keeping fixed \(U = r/\alpha'\), \(U_0 = r_0/\alpha'\) and the Yang-Mills coupling constant \(g_{YM}^2\), with \(g_{YM}^2 = g_s(\alpha')^{(p-3)/2}\), the harmonic function reduces to

\[
H = \frac{g_{YM}^2 N}{\alpha'^2 U^{7-p}},
\]

(3.4)

where \(N\) is the number of Dp-branes and we have absorbed a numerical coefficient into the Yang-Mills coupling constant \(g_{YM}^2\). Except for the case of \(p = 3\), the radius of angular part in the string metric (3.2) depends on \(U\). In order to remove the dependence, let us consider the so-called “dual frame” \[18\]

\[
 ds_{\text{dual}}^2 = (N e^\phi)^{2/(p-7)} ds_{\text{string}}^2.
\]

(3.5)

In this frame, the action turns out to be \[19\]

\[
S = \frac{N^2}{16\pi G_{10}} \int d^{10}x \sqrt{-g(N e^\phi)^\lambda} \left( R + \frac{4(p-1)(p-4)(\partial \phi)^2}{(7-p)^2} - \frac{1}{2N^2(8-p)!} F_{8-p}^2 \right),
\]

(3.6)

where \(\lambda = 2(p-3)/(7-p)\). In the decoupling limit, the solution in the dual frame has the form

\[
\begin{align*}
 ds_{\text{dual}}^2 &= \alpha' \left( (g_{YM}^2 N)^{-1} U^{5-p} (-f dt^2 + dx_p^2) + U^{-2} f^{-1} dU^2 + d\Omega_{8-p}^2 \right), \\
 e^\phi &= \frac{1}{N} (g_{YM}^2 N U^{p-3})^{(7-p)/4}, \\
 F_{8-p} &= (7-p) N (\alpha')^{(7-p)/2} \epsilon_{8-p},
\end{align*}
\]

(3.7)
where \( f = 1 - (U_0/U)^{7-p} \). The metric is of the form \( AdS_{p+2} \times S^{8-p} \) for \( p \neq 5 \) and \( E^{(1,6)} \times S^3 \) for \( p = 5 \) in the dual frame. Note that for the case of \( p = 3 \), these two frames are equivalent to each other since the dilaton is a constant in this case. In addition, let us notice that in the dual frame, the radius of the angular part of the metric is a constant. Therefore in this frame we can conveniently do a spherical reduction of the type II supergravity on \( S^{8-p} \), and obtain the effective action of the gauged supergravity in the Einstein frame \([19]\). 

\[
S = \frac{N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+2}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \Phi)^2 + V(\Phi) \right) - \frac{2N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{-h} K, \tag{3.8}
\]

where we have added a Gibbons-Hawking surface term, \( \Omega_{8-p} \) is the volume of the unit sphere \( S^{8-p} \), and

\[
V(\Phi) = \frac{1}{2} (9-p)(7-p)N^{-2\lambda/p} e^{\alpha \Phi}, \\
\Phi = \frac{2\sqrt{2(9-p)}}{\sqrt{p}(7-p)} \phi, \quad a = -\frac{\sqrt{2}(p-3)}{\sqrt{p}(9-p)}. \tag{3.9}
\]

After the reduction, we obtain a Ricci flat black hole (black domain wall) solution

\[
ds^2 = (N e^{\Phi})^{2\lambda/p} \left[ (g_{YM}^2 N)^{-1} U^{5-p} (-f dt^2 + dx_p^2) + U^{-2} f^{-1} dU^2 \right], \\
e^{\phi} = \frac{1}{N} (g_{YM}^2 U^{p-3})^{(7-p)/4}. \tag{3.10}
\]

The Ricci flat black hole has a horizon at \( U = U_0 \), where \( f(U) \) vanishes. When \( p \neq 5 \), we can make a further transformation

\[
u^2 = R^2 (g_{YM}^2 N)^{-1} U^{5-p}, \quad R = 2/(5-p), \tag{3.11}
\]

so that the Ricci flat solution takes the form

\[
ds^2 = (N e^{\Phi})^{2\lambda/p} \left[ \frac{u^2}{R^2} \left( -\tilde{f} dt^2 + dx_p^2 \right) + \frac{R^2}{u^2 f} du^2 \right], \\
e^{\phi} = \frac{1}{N} (g_{YM}^2 N)^{(7-p)/2(5-p)} \left( \frac{u}{R} \right)^{(p-7)(p-3)/2(p-5)}, \\
\tilde{f} = 1 - \left( \frac{u_0}{u} \right)^{2(7-p)/(5-p)}, \tag{3.12}
\]

where \( u_0^2 = R^2 (g_{YM}^2 N)^{-1} U_0^{5-p} \). Clearly the Ricci flat black hole is conformal to \( AdS_{p+2} \).

\[^3\]since \( \alpha' \) will be eventually cancelled at the end of calculations, we will set \( \alpha' = 1 \) in the following.
In [19], the stress-energy tensor of dual quantum field theory and the mass of the Ricci flat black hole have been calculated via the surface counterterm method. The counterterm is found to be
\[
S_{ct} = -\frac{2N^2\Omega_{8-p}}{(2\pi)^7} \int d^{p+1}\sqrt{-h} \frac{c_0}{\ell_{\text{eff}}},
\] (3.13)
where
\[
c_0 = \sqrt{\frac{(9-p)p(p+1)}{2(7-p)}}, \quad \frac{1}{\ell_{\text{eff}}} = \sqrt{\frac{V(\Phi)}{p(p+1)}}.
\] (3.14)
The mass of the black hole is
\[
M = \frac{\Omega_{8-p}}{(2\pi)^7 g^4_{YM}} \frac{9-p}{2} U_0^{7-p} V_p,
\] (3.15)
where \( V_p \) is the volume for the Euclidean space described by \( dx_p^2 \). The Euclidean action of the Ricci flat black hole is
\[
\mathcal{I} = -\frac{N^2\Omega_{8-p}}{(2\pi)^7} \int d^{p+2}x \sqrt{g} \left( R - \frac{1}{2}(\partial\Phi)^2 + V(\Phi) \right)
+ \frac{2N^2\Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{h} K
+ \frac{2N^2\Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{h} \frac{c_0}{L_{\text{eff}}},
\]
\[
= -\frac{\Omega_{8-p} V_p U_0^{7-p}}{(2\pi)^7 g^4_{YM} T} \frac{5-p}{2},
\] (3.16)
where \( T \) is the Hawking temperature of the black hole
\[
T = \frac{7-p}{4\pi} \frac{1}{\sqrt{g^2_{YM} N}} U_0^{5-p}. \] (3.17)
According to \( \mathcal{E} = \partial \mathcal{I} / \partial \beta \) and \( S = \beta \mathcal{E} - \mathcal{I} \), it is easy to see that the energy of the black hole is \( \mathcal{E} = M \) and the entropy of the black hole
\[
S = \frac{\Omega_{8-p} V_p}{2^5\pi^6 g^4_{YM}} \sqrt{g^2_{YM} N U_0^{(9-p)/2}}. \] (3.18)
It is easy to show that the surface counterterm method here is equivalent to the background subtraction method if one chooses the solution (3.10) with \( U_0 = 0 \) or (3.12) with \( u_0 = 0 \) as the reference background. We see from (3.16) that the action is always negative for \( 0 \leq p < 5 \), positive for \( p > 5 \), and vanishes for the case of \( p = 5 \). This implies that compared to the thermal dilaton background (3.10) with \( U_0 = 0 \) or (3.12) with \( u_0 = 0 \), the Ricci flat black hole is globally stable and dominates when \( p < 5 \), is marginal when \( p = 5 \), and unstable for the case \( p > 5 \). Notice the fact that the supergravity configuration is dual to the little string theory for the case \( p = 5 \), while gravity does not be decoupled for
the case $p > 5$. Therefore our results are consistent with that observation. Furthermore, we see from the Euclidean action (3.16) that there does not exist any phase transition between the Ricci flat black holes and thermal dilaton background.

Now we consider double-analytically continuing the Ricci flat black holes with $t \to i\varphi$ and $x_1 \to it$, where $x_1$ is one of coordinates $x_p$. In that case, the black hole solution becomes

$$ds_s^2 = (Ne^{\phi})^{2\lambda/p} \left[ (g_{YM}^2 N)^{-1} U^{7-p} (f d\varphi^2 - dt^2 + dx_{p-1}^2) + U^{-2} f^{-1} dU^2 \right]$$

(3.19)

or

$$ds_s^2 = (Ne^{\phi})^{2\lambda/p} \left[ \frac{u^2}{R^2} \left( \tilde{f} d\varphi^2 - dt^2 + dx_{p-1}^2 \right) + \frac{R^2}{u^2 f} du^2 \right],$$

(3.20)

where $U_0 (u_0)$ is replaced by $U_s (u_s)$ and the dilaton field is still given by (3.10) or (3.12). For this solution, the horizon disappears and in order to remove the conical singularity, the coordinate $\varphi$ has to have a period with

$$\beta_s = \frac{4\pi}{I - p} \sqrt{g_{YM}^2 N U_s^{p-5}}.$$  

(3.21)

In addition, let us notice that to keep the signature of the solution, one has to take $U \geq U_s$ in (3.19) or $u \geq u_s$ in (3.20). Note also that the solution (3.20) is just the AdS soliton when $p = 3$. Naturally, we call the solution (3.19) or (3.20) the dilaton deformed AdS soliton. As in the case of the AdS soliton, we expect that the dilaton deformed AdS soliton has a lower mass than the dilaton background (3.10) with $f = \tilde{f} = 1$. To see this, let us calculate the Euclidean action of the Ricci flat black hole by viewing the soliton solution as the reference background, which gives us with

$$I_{bs} = -\frac{\Omega_{8-p} \beta_s \beta V_{p-1}}{(2\pi)^7 g_{YM}^4} \frac{5 - p}{2} \left( U_0^{7-p} - U_s^{7-p} \right),$$

(3.22)

where $\beta = 1/T$ is the inverse Hawking temperature (3.17) of the Ricci flat black hole. From the Euclidean action, we can clearly see that there does exist the Hawking-Page phase transition between the Ricci flat black hole and dilaton deformed AdS soliton. When $1 \leq p < 5$, the black hole phase is globally stable and dominates when $U_0 > U_s$. When $p = 6$, however, the thermal soliton phase is globally stable and dominates when $U_s > U_0$. This difference is caused by the relation between the Hawking temperature and $U_0$ (3.17): when $p < 5$, a larger black hole has a higher temperature, while the converse happens for the case of $p = 6$. When $p = 5$, the situation is subtle, the temperature of the black hole is a constant, and the Euclidean action always vanishes. From the point of

4In that case, one has to have $p \geq 1$, namely we exclude the black D0-brane solution here.
view of dual little string theory, one has to consider higher order corrections [20]. Viewing
the deformed soliton as the reference background, the mass of the black hole is

\[ \mathcal{E}_{bs} = \frac{\Omega_{8-p}}{2(2\pi)^7} g_{YM}^2 \left( (9 - p)U_0^{7-p} + (5 - p)U_s^{7-p} \right), \]  

(3.23)

and associated entropy with the black hole is still given by (3.18).

So far we have considered the dilaton gravity (3.8) with the coupling constant \( a \) given
by (3.9), which comes from the spherical reduction of Dp-brane in type II supergravity. Next we consider a dilaton gravity with a Liouville-type potential with an arbitrary
coupling constant \( a \),

\[ S = \frac{1}{16\pi G} \int d^{p+2}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 + V_0 e^{-a\phi} \right), \]  

(3.24)

where \( G \) is the gravitational constant in \( p + 2 \) dimensions and \( V_0 \) is a constant. The Ricci
flat black hole solution has the form [12]

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + R^2(r)dx_p^2, \]

\[ R(r) = r^n, \]

\[ \phi(r) = \phi_0 + \sqrt{2np(1-n)} \ln r, \]

\[ f(r) = \frac{V_0 e^{-a\phi_0} r^{2n}}{np(n(p+2)-1)} - m_b r^{1-np}, \]  

(3.25)

where \( \phi_0 \) and \( m_b \) are two integration constants, \( dx_p \) describes the line element of \( p \)-
dimensional Ricci flat space, and the constant \( n \) has a relation to the coupling constant \( a \) as

\[ a = \frac{\sqrt{2np(1-n)}}{np}. \]  

(3.26)

Note that when \( n = 1 \), the solution (3.25) reduces to the \( (p + 2) \)-dimensional AdS Ricci
flat black hole with a constant dilaton. Therefore we will not consider the special case
with \( n = 1 \) in what follows. However, the case of \( n = 1 \) will be naturally included
in the following discussions. The black hole horizon \( r_+ \) is determined by the equation
\( f(r)|_{r=r_+} = 0 \). The associated Hawking temperature is

\[ T = 1/\beta = \frac{V_0 e^{-a\phi_0}}{4\pi np} r_+^{2n-1}. \]  

(3.27)

The entropy of the black hole obeys the so-called area formula since we are working in
the Einstein frame, and is

\[ S = \frac{V_p}{4G} r_+^{np}, \]  

(3.28)
where $V_P$ is the volume of the manifold $dx_P^2$. Furthermore, if we choose the solution with $m_b = 0$ as the reference background, the mass of the black hole has the form

$$M = \frac{m_b V_p n p}{16 \pi G} = \frac{V_0 e^{-a \phi_0} V_p r_+^{n(p+2)-1}}{16 \pi G (n(p + 2) - 1)}.$$ (3.29)

The Euclidean action of the black hole can also be calculated by the formula: $\mathcal{I} = \beta \mathcal{F} = \beta M - S$, where $\mathcal{F}$ is the free energy of the black hole, which is equivalent to obtaining the Euclidean action from (3.24). We find

$$\mathcal{I} = -\frac{V_p r_+^{np}}{4G} \frac{2n - 1}{n(p + 2) - 1}.$$ (3.30)

The action is always negative if $n > 1/2$ or $n < 1/(p + 2)$, and positive if $1/(p + 2) < n < 1/2$. Let us notice that in order to have a well-behaved boundary metric, on which the dual QFT resides, we have $n > 1/(p + 2)$ from the solution (3.25). In addition, when $n = 1/2$, the action vanishes like the case of D5-branes.

The dilaton deformed AdS soliton solution for the action (3.24) can be obtained by double analytical continuation from the black hole solution (3.25) via $t \to i \varphi$ and $x_1 \to it$, so that we have

$$ds^2 = f(r) d\varphi^2 + f(r)^{-1} dr^2 + R^2(r) (-dt^2 + dx_{p-1}^2),$$

$$R(r) = r^n,$$

$$\phi(r) = \phi_0 + \sqrt{2np(1 - n)} \ln r,$$

$$f(r) = \frac{V_0 e^{-a \phi_0} r^{2n}}{np(n(p + 2) - 1)} - m_s r^{1-np},$$ (3.31)

To remove the conical singularity at $r = r_s$, which satisfies $f(r_s) = 0$, the coordinate $\varphi$ has to have a period $\beta_s$ obeying

$$\beta_s = \frac{4 \pi np}{V_0 e^{-a \phi_0} r_s^{1-2n}}.$$ (3.32)

Viewing the soliton as the reference background, we obtain the Euclidean action of the black hole

$$\mathcal{I}_{bs} = -\frac{V_0 e^{-a \phi_0} V_{p-1} \eta_b \beta_s}{16 \pi G n p} \frac{2n - 1}{n(p + 2) - 1} \left( r_+^{n(p+2)-1} - r_s^{n(p+2)-1} \right),$$ (3.33)

where $\eta_b$ is the period of the coordinate $x_1$ for the black hole solution (3.25), which has a relation to $\beta_s$ via $\eta_b = \beta_s / l_{\text{eff}}$ with $l_{\text{eff}} = \sqrt{np(n(p+2)-1)}/V_0 e^{-a \phi_0}$. Again, in this background, the energy of the black hole is

$$E_{bs} = \frac{V_0 e^{-a \phi_0} \eta_b V_{p-1}}{16 \pi G np} \frac{1}{n(p + 2) - 1} \left( np r_+^{n(p+2)-1} + (2n - 1)r_s^{n(p+2)-1} \right).$$ (3.34)
And the entropy is still given by (3.28). We see from the action (3.33) that there does exist a phase transition between the Ricci flat black hole and the dilaton deformed AdS soliton when \( n \neq 1/2 \). When \( n > 1/2 \), the black hole phase dominates if \( r_+ > r_s \). When \( n < 1/2 \), the thermal soliton phase dominates if \( r_s > r_+ \). As the Euclidean action changes its sign, a Hawking-Page phase transition happens.

4 Conclusions and Discussions

In this paper we have studied Ricci flat black holes and deformed AdS soliton in Gauss-Bonnet gravity and dilaton gravity with a Louville-type dilaton potential including the gauged supergravity coming from the spherical reduction of Dp-branes in type II supergravity. In Gauss-Bonnet gravity, the black hole solution and AdS soliton are greatly deformed by the Gauss-Bonnet term, but the black hole entropy still obeys the area formula and the Hawking-Page phase transition between the black hole and soliton background is still determined by the black hole temperature through the horizon \( r_+ \) and the compact radius \( \eta_b \) through \( r_s \). The Gauss-Bonnet coefficient \( \alpha \) explicitly disappears in the Euclidean action. As a result, the Gauss-Bonnet term has no effect on the Hawking-Page phase transition, compared to the case without the Gauss-Bonnet term. This is quite different from the case of spherical black holes [9].

In dilaton gravity, the high temperature phase is dominated by black holes in some cases, and in other cases is dominated by deformed thermal solitons, depending on the dilaton coupling constant. For example, see the action (3.22). When \( p < 5 \), the high temperature phase is dominated by the black hole while by the soliton background for \( p = 6 \). The same happens in (3.33): in the high temperature phase, the black hole dominates when \( n > 1/2 \), while the deformed AdS soliton dominates when \( n < 1/2 \). This feature is new, compared to the case of Einstein gravity, where black hole always dominates in the high temperature phase [14].

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