Entanglement types for two-qubit states with real amplitudes

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Abstract
We study the set of two-qubit pure states with real amplitudes and their geometrical representation in the three-dimensional sphere. In this representation, we show that the maximally entangled states—those locally equivalent to the Bell states—form two disjoint circles perpendicular to each other. We also show that taking the natural Riemannian metric on the sphere, the set of states connected by local gates are equidistant to this pair of circles. Moreover, the unentangled or so-called product states are π/4 units away to the maximally entangled states. This is, the unentangled states are the farthest away to the maximally entangled states. In this way, if we define two states to be equivalent if they are connected by local gates, we have that there are as many equivalent classes as points in the interval [0, π/4] with the point 0 corresponding to the maximally entangled states. The point π/4 corresponds to the unentangled states which geometrically are described by a torus. Finally, for every 0 < d < π/4 the point d corresponds to a disjoint pair of torus. Finally, we also show how this geometrical interpretation allows to clearly see that any pair of two-qubit states with real amplitudes can be connected with a circuit that only has single-qubit gates and one controlled-Z gate.

Keywords Quantum state preparation · Geometry of entanglement · Quantum entanglement
1 Introduction

Entanglement is one of the fascinating properties of multipartite quantum systems and one of the cornerstones toward the existence of many quantum technologies, from the design of quantum communication protocols (e.g., quantum teleportation and quantum cryptography) to the success of many key quantum computer algorithms. The geometry of entanglement, and in general, the geometry of quantum states, has been invaluable and extensively studied by many authors in the past decades [1–6]. In this work, we geometrically describe how the space of pure states of 2-qubit systems with real amplitudes partitions when we group them by their entanglement. Quantum states with real amplitudes might play a role in the design of quantum circuits to solve probabilistic modeling tasks with near-term quantum devices, within the framework of the so-called quantum circuit born machine [7]. In particular, there has been an efficient experimental implementation of such QCBM models where the gate selection was restricted to orthogonal transformations, therefore targeting the generation of quantum states with real amplitudes [8].

In order to study the set of 2-qubit states with real amplitudes, we organize them into “orbits” using the following condition: two 2-qubit states are in the same orbit if they are connected by local gates with real entries, this is, if they are connected by a circuit that uses only the set of gates $L^R = \{ R_y(\theta), X \}$ with

\[
R_y(\theta) = \begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
\quad \text{and} \quad
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We will refer to gates generated by $L^R$ as real local gates. Since $L^R$ agrees with the orthogonal group $O(2)$, we will also call this set, the group of local orthogonal gates. The whole set of 2-qubits with real amplitudes is described by the three-dimensional unit sphere and, for any 2-qubit state, the orbit that it belongs to is either a pair of disjoint circles, a pair of disjoint tori or a single torus. We will also show that, with respect to the natural Riemannian metric of the three-dimensional sphere, these orbits are described as the set of points that are at the same distance $d$ from the only orbit that consist of two disjoint circles.

The equivalent relation considered here is not new. For the general case of $n$-qubits with complex amplitudes, the authors [9–11] have considered the equivalence relation: Two $n$-qubit states are equivalent if they are connected by local gates. Since they are considering $n$-qubits with complex amplitudes, there are not restrictions on the local gates, i.e., gates of the form $U_1 \otimes \cdots \otimes U_n$ where each $U_i$ is a $2 \times 2$ unitary matrix. Each orbit is called an “entanglement type,” and the quotient space is called ”the space of entanglement types”; the latter denoted as $E_n$. In [9], the author shows that $E_2$ is a closed interval and the orbits are either a four-dimensional manifold that contains the unentangled 2-qubit (product) states or, a three-dimensional manifold that contains the EPR (maximally entangled) states or, five-dimensional manifolds for all other cases with intermediate entanglement strength. Placing our result in the same perspective,
let us denote by $E^\mathbb{R}_2$ the quotient space of 2-qubit states with real amplitudes where two 2-qubit states are identified if they can be connected with real local gates. The space $E^\mathbb{R}_2$ is also a closed interval, the interval $[0, \pi/4]$, and the orbits are either: (i) a connected two-dimensional manifold—a torus—that contains all the unentangled 2-qubit product states with real amplitudes, (ii) a pair of disjoint circles (one-dimensional manifolds) that contains the EPR maximally entangled states, or (iii) a pair of tori (two-dimensional manifold) representing all other states with intermediate entanglement values.

We would like to point out that our result is not a consequence of Walck’s [9] result due to the fact that both the spaces and the equivalent relations involved are different. In fact, one can easily show with arguments based on the dimension of the manifolds involved that the entanglement spaces $E^\mathbb{R}_n$ and $E^\mathbb{R}_n$ are different for all $n > 2$. An example related to this difference is given in [12], where it is shown a pair of 3-qubit states with real amplitudes that are part of the same orbit in $E_3$ but that are in two different orbits in $E^\mathbb{R}_3$.

There are two interesting results regarding the set of $n$-qubit states with real amplitudes. On the one hand, Wootters at [13] have shown that the density matrix of these states cannot, in general, be determined by using only local gates. On the other hand, Hardy and Wootters at [14] have shown that these states can be determined by the statistic measurements on pair of components by doing "bilocal tomography."

Recall that $|\phi\rangle$ and $|\psi\rangle$ are LOCC equivalent if not only $|\phi\rangle$ transform to $|\psi\rangle$ using local operations and classical communications (LOCC) but also, we have that $|\psi\rangle$ transform to $|\phi\rangle$ using LOCC. It is known [15–17] that if two $n$-qubit states $|\phi\rangle$ and $|\psi\rangle$ share the same entanglement type, this is, if $|\phi\rangle$ and $|\psi\rangle$ differ by a local gates, then these two states are LOCC-equivalent. If we wonder how to classify states where we have LOCC going in only one direction, we have that $|\psi\rangle$ transform to $|\phi\rangle$ using LOCC if and only if the vector or eigenvalues of $\rho_\psi = |\psi\rangle\langle\psi|$ is majorized by $\lambda_\phi$ [18].

In this paper, we give a geometrical interpretation of the orbits on the space of 2-qubit states under an equivalence relation using real orthogonal gates. As we mentioned above, for general pure $n$-qubit states, LOCC equivalence is characterized by an equivalence relation using local gates (unitary matrices) and SLOCC is characterized by the equivalence relation using a bigger group of matrices, the group of invertible local operation [15].

Finally, we show that when we apply the controlled-$Z$ gate (CZ) to a particular equivalent class, we get a set containing points from all the equivalent classes. As a consequence, this provides a geometrical proof that every pair of real 2-qubit states can be connected by a circuit that contains local gates and only one CZ gate. That only one entangling gate is necessary for arbitrary 2-qubit states is a known result and it fits within the general theory and research devoted to efficiently prepare $n$-qubit states (see, e.g., [19–21], and for a subroutine in Rigetti’s Quil platform, see [22]).

Section 2 introduces a geometrical interpretation of the orbits coming for the partition induced by $E^\mathbb{R}_3$ and how they relate to each other. Section 3 presents the proof of our main result, and finally, Sect. 4 shows the conclusions.
2 Geometrical interpretation

To describe the geometry of the orbits of 2-qubit states with real amplitudes, we will start by describing a similar situation one dimension lower. Let us assume that the Earth is a perfect sphere with radius 1, defined by \( S^2 = \{ x \in \mathbb{R}^3 : ||x|| = 1 \} \) (with \( || \cdot || \) as the Euclidean norm). In this case, the equatorial line is a unit circle. The name equatorial line turns out to be appropriate because unit circles on the sphere are geodesics; this means if we consider two points on the equator, the shortest path connecting these points along the surface of the Earth must be part of the equatorial line. Now, let us think of the points on Earth that are at a spherical distance \( d \) from the equatorial line and let us denote this set as \( \Sigma_d \). For \( d = \pi/4 \), the set \( \Sigma_d \) consists of all those points on the Earth that are on a latitude 45 degrees north and 45 degrees south. It is not difficult to see that \( \Sigma_d \), for \( 0 < d < \pi/2 \), is the union of two disjoint circles of Euclidean radius \( \cos(d) \). When \( d = \pi/2 \), \( \Sigma_d \) reduces to only two points: the south and north poles.

For the study of two-qubit quantum states with real amplitudes, let us consider the three-dimensional sphere defined by \( S^3 = \{ x \in \mathbb{R}^4 : ||x|| = 1 \} \). Given an orthonormal basis \( v_1, v_2, v_3, v_4 \) for \( \mathbb{R}^4 \), let us define the set \( E(v_3, v_4) = \{ \cos(\theta)v_3 + \sin(\theta)v_4, \theta \in \mathbb{R} \} \). Clearly, \( E \) is a unit circle which is a geodesic in \( S^3 \). For any positive \( d < \pi/2 \), let us consider the set \( \Sigma_d \) of points in \( S^3 \) that are exactly at a distance \( d \) from the circle \( E \). This time the set \( \Sigma_d \) is not the disjoint union of two circles like in the case of the 2-dimensional sphere, it is just one torus. More precisely, if \( x = \sum_{i=1}^{4} x_i v_i \),

\[
\Sigma_d(v_3, v_4) = \left\{ x \in S^3 : x_1^2 + x_2^2 = \sin^2(d), \quad x_3^2 + x_4^2 = \cos^2(d) \right\},
\]

(2)

A direct computation shows that, viewing \( \Sigma_d(v_3, v_4) \) as a surface of \( S^3 \), its Gauss curvature is zero and the principal curvatures are \( \tan(d) \) and \( -\cot(d) \). The surface \( \Sigma_{\pi/4} \) played an important role in the study of minimal surface on \( S^3 \) since it was conjectured by Lawson in 1970 [23] that this surface was the only embedded minimal torus on the sphere. The conjecture was finally solved by Brendle in 2013 [24]. When \( d = \pi/2 \), the set \( \Sigma_d \) reduces to a circle of radius 1, \( \Sigma_{\pi/2}(v_3, v_4) = \{ \cos(\theta)v_1 + \sin(\theta)v_2, \theta \in \mathbb{R} \} \). Notice that \( \Sigma_{\pi/2}(v_3, v_4) = E(v_1, v_2) \).

As shown in detail in Sect. 2, a remarkably simple geometrical representation emerges when we perform the analysis above on the appropriate orthogonal basis \( v_1, v_2, v_3, v_4 \).

As part of the geometric interpretation that we are providing, we need to identify 2-qubit states with real amplitudes with either points in \( S^3 \) or with vectors in \( \mathbb{R}^4 \). For this reason, we may say “the vector \( \frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle \)” referring to the vector \( \frac{1}{\sqrt{2}} (1, 0, 0, -1) \). In general, we make the qubit \( w_1|00\rangle + w_2|01\rangle + w_3|10\rangle + w_4|11\rangle \) indistinguishable from \( (w_1, w_2, w_3, w_4) \). Selecting the following orthonormal basis
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Fig. 1  Half of the three-dimensional sphere projects into the unit ball in $\mathbb{R}^3$. This figure shows the projection of the typical orbits on this ball. a Projection of the orbit that contains the unentangled or product states. The whole orbit is a torus, and this image shows half of it. b Projection of a generic orbit; here we show the one associated with $d = \pi/6$. For any state associated with $d$ between 0 (Bell states) and $\pi/4$ (product states), its entanglement entropy $S(d) = 1 - \log_2 \sqrt{\frac{(1+\sin 2d)(1+\sin 2d)}{(1-\sin 2d)(1+\sin 2d)}}$. c Projection of the orbit that contains the Bell states. This orbit is the union of two circles. Here, we show one of the circles and half of the other circle.

$$|v_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |v_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|v_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |v_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

will allow us to describe the orbits that we obtain by placing together 2-qubits that can be connected by local gate with real entries. To start with, the Bell states (3) are all in the same orbit. This orbit is the union of the two circles $E(v_3, v_4)$ and $E(v_1, v_2)$. For any $0 < d < \pi/4$, the orbit is the union of the two tori $\Sigma_d(v_1, v_2)$ and $\Sigma_d(v_3, v_4)$ that make up an orbit. And when $d = \pi/4$, we have that the tori $\Sigma_d(v_1, v_2)$ and $\Sigma_d(v_3, v_4)$ are the same. This torus $\Sigma = \Sigma_d(v_1, v_2) = \Sigma_d(v_3, v_4)$ makes up an orbit, and it contains all the unentangle states. We can visualize half of the three-dimensional sphere using the parametrization

$$\xi(u) = \left(u_1, u_2, u_3, \sqrt{1 - u_1^2 - u_2^2 - u_3^2}\right).$$

This parametrization identifies the unit ball in $\mathbb{R}^3$ with half of the sphere. We will show the projection on the unit ball of $\mathbb{R}^3$ of the part of the orbits that lies on the half part of the three-dimensional sphere parametrized by $\xi$. Figure 1a shows the projection of the orbit that contains the unentangled states. Figure 1b shows an orbit for a $d$ between 0 and $\pi/4$, and finally, Fig. 1c shows the orbit that contains the Bell states.
3 Main theorem and proof

Let us recall that the action of the CZ gate on a pair of qubits is given by matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\] (5)

Up to local gates, the CNOT gate and the CZ gate are equivalent. We are now ready to state and proof the main theorem in this paper.

**Theorem 1** Let us consider the following subset of two qubits
\[
RQ_2 = \{ |w⟩ = w_1|00⟩ + w_2|01⟩ + w_3|10⟩ + w_4|11⟩ : w_i \in \mathbb{R} \},
\] (6)

and let us define the Bell basis as
\[
|v_1⟩ = \frac{1}{\sqrt{2}}(|00⟩ - |11⟩), \quad |v_2⟩ = \frac{1}{\sqrt{2}}(|01⟩ + |10⟩),
|v_3⟩ = \frac{1}{\sqrt{2}}(|00⟩ + |11⟩), \quad |v_4⟩ = \frac{1}{\sqrt{2}}(|01⟩ - |10⟩),
\] (7)

and let us call \((x_1, x_2, x_3, x_4)\) the coordinates of \(RQ_2\) with respect to the basis \(|v_1⟩, \ldots, |v_4⟩\). Notice that for the state \(|w⟩ = w_1|00⟩ + w_2|01⟩ + w_3|10⟩ + w_4|11⟩\) the following relation holds,
\[
x_1 = \frac{w_1 - w_4}{\sqrt{2}}, \quad x_2 = \frac{w_2 + w_3}{\sqrt{2}},
\]
\[
x_3 = \frac{w_1 + w_4}{\sqrt{2}}, \quad x_4 = \frac{w_2 - w_3}{\sqrt{2}}.
\] (8)

Let us consider the equivalence relation \(|ψ_1⟩ \sim |ψ_2⟩\) if there is a local gate with real entries that connects them. We will refer to the orbit that contains a given \(|ψ⟩\) to the set of 2-qubit states that are equivalent to \(|ψ⟩\), this is the set of 2-qubit states that can be connected with \(|ψ⟩\) by means of local gates with real entries. Using the notation introduced earlier we have,

1. The orbit that contains the state \(|v_1⟩\) is the union of the two circles \(E(v_3, v_4)\) and \(E(v_1, v_2)\). Therefore, the 2-qubit states \(|v_1⟩, |v_2⟩, |v_1⟩\) and \(|v_4⟩\) are part of the same orbit. This orbit is the only one given by a one-dimensional manifold.
2. The orbit that contains the state \(|00⟩\) is the torus \(Σ_{π/4}(v_3, v_4) = Σ_{π/4}(v_1, v_2)\). This orbit contains all the unentangled states. Notice that this set is also characterized as the set of point that are farthest away from the orbit that contains the Bell states. The distance between these two orbits is \(π/4\).
3. Any other orbit is given by the union of the two tori \( \Omega_d = \Sigma_d(v_1, v_2) \cup \Sigma_d(v_3, v_4) \) where \( d \) lies between 0 and \( \pi/4 \). As a consequence, any pair of states in \( \Omega_d \) are connected by local gates and, if \( 0 \leq d_1 < d_2 \leq \pi/4 \) and \( \ket{\phi_1} \in \Omega_{d_1} \) and \( \ket{\phi_2} \in \Omega_{d_2} \), then, \( \ket{\phi_1} \) and \( \ket{\phi_2} \) are not connected by local gates.

4. The entanglement entropy of any state in \( \Omega_d \) is

\[
S(d) = 1 - \log_2 \sqrt{\frac{(1 + \sin 2d)^{1+\sin 2d}}{(1 - \sin 2d)^{-1+\sin 2d}}} \tag{9}
\]

5. For any pair of states \( \ket{\phi_1} \) and \( \ket{\phi_2} \) in \( RQ_2 \), there exist angles \( \theta_0, \theta_1, \theta_2 \) and \( \theta_3 \) such that the circuit in Fig. 2a sends \( \ket{\phi_1} \) to \( \ket{\phi_2} \).

6. If \( \ket{w} = w_1 \ket{00} + w_2 \ket{01} + w_3 \ket{10} + w_4 \ket{11} \) then the circuit in Fig. 2b with

\[
\begin{align*}
\theta_0 &= \text{Arg}(w_1 + iw_2) - \text{Arg}(w_3 + iw_4), \\
\theta_1 &= 2 \arccos(\sqrt{w_1^2 + w_2^2}) \text{ and } \\
\theta_2 &= \text{Arg}(w_1 + iw_2) + \text{Arg}(w_3 + iw_4) \text{ prepares } \ket{w}.
\end{align*}
\]

Proof: Part (1) follows from the fact that all the elements in \( E(v_3, v_4) \) are of the form \( (R_y(-2t) \otimes I_2)\ket{v_3} \) and all the elements in \( E(v_1, v_2) \) are of the form \( (R_y(2t) \otimes I_2)\ket{v_1} \). We also have that \( (X \otimes I_2)\ket{v_1} = -\ket{v_4} \). Therefore, all the elements in \( E(v_3, v_4) \cap E(v_1, v_2) \) are part of the same orbit. We have that this union of two circle gives us the whole orbit because this set is closed under local gates with real amplitudes. Let us consider a positive \( d < \pi/2 \) and a state

\[
\ket{\psi_0} = w_1 \ket{00} + w_2 \ket{01} + w_3 \ket{10} + w_4 \ket{11} \in \Sigma_d. \tag{10}
\]

Looking at the definition of \( \Sigma_d(v_3, v_4) \), we conclude that there exists a pair of angles \( a \) and \( b \) such that

\[
\begin{align*}
\frac{w_1 - w_4}{\sqrt{2}} &= \sin(d) \cos(a), \\
\frac{w_2 + w_3}{\sqrt{2}} &= \sin(d) \sin(a), \\
\frac{w_1 + w_4}{\sqrt{2}} &= \cos(d) \cos(b), \\
\frac{w_2 - w_3}{\sqrt{2}} &= \cos(d) \sin(b), \tag{11}
\end{align*}
\]
Fig. 3 Geometrical proof that one CZ and local gates are enough to connect any 2-qubit states with real amplitudes. Here, we show how to connect two states: a $|\psi_0\rangle$ in $\Sigma_{d_0}$ with a state $|\psi_1\rangle$ in $\Sigma_{d_1}$. Since the surface $\Sigma_{d_1}$ (orange) satisfies the equation $x_1^2 + x_2^2 = \sin^2(d_1)$ and the surface $\text{CZ}\Sigma_{d_0}$ (green) satisfies the equation $x_2^2 + x_3^2 = \sin^2(d)$, these two surfaces intercept allowing to connect states from one surface to the other. In the figure, we depict the connection by local gates between $|\psi'_0\rangle = \text{CZ}|\psi_0\rangle$ and $|\psi_1\rangle$ through $|\psi_2\rangle$ (a state in the interception) (Color figure online)

and therefore,

$$
|\psi_0\rangle = \left( (\cos(a) \sin(d) + \cos(b) \cos(d)) |00\rangle + (\sin(a) \sin(d) + \sin(b) \cos(d)) |01\rangle + (\sin(a) \sin(d) - \sin(b) \cos(d)) |10\rangle + (\cos(b) \cos(d) - \cos(a) \sin(d)) |11\rangle \right)/\sqrt{2} \tag{12}
$$

A direct verification shows that if we define the parametric surface $|\phi(s, t)\rangle$ by

$$
R_y(2s) \otimes R_y(2t)|\psi_0\rangle = \phi_1|00\rangle + \phi_2|01\rangle + \phi_3|10\rangle + \phi_4|11\rangle, \tag{13}
$$

with

$$
\begin{align*}
\phi_1 &= \left( 2 \sin(d) \cos(a + s + t) + \cos(b - d - s + t) \\
&+ \cos(b + d - s + t) \right)/(2\sqrt{2}), \\
\phi_2 &= \left( \cos(a - d + s + t) - \cos(a + d + s + t) + \\
&+ 2 \cos(d) \sin(b - s + t) \right)/(2\sqrt{2}), \\
\phi_3 &= \left( \cos(a - d + s + t) - \cos(a + d + s + t) \\
&- 2 \cos(d) \sin(b - s + t) \right)/(2\sqrt{2}), \\
\phi_4 &= \left( - 2 \sin(d) \cos(a + s + t) + \cos(b - d - s + t) \\
&+ \cos(b + d - s + t) \right)/(2\sqrt{2}),
\end{align*} \tag{14}
$$
then, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is an immersion because the vectors $\partial \phi / \partial s$ and $\partial \phi / \partial t$ are linear independent due to the fact that

$$\left| \frac{\partial \phi}{\partial s} \right|^2 \left| \frac{\partial \phi}{\partial t} \right|^2 - \left( \frac{\partial \phi}{\partial s} \cdot \frac{\partial \phi}{\partial t} \right)^2 = \sin^2(2d) > 0 .$$  \hspace{1cm} (15)

On the other hand, a direct verification shows that

$$\frac{(\phi_1 - \phi_4)^2}{2} + \frac{(\phi_2 + \phi_3)^2}{2} = \sin^2(d) ,$$

$$\frac{(\phi_1 + \phi_4)^2}{2} + \frac{(\phi_2 - \phi_3)^2}{2} = \cos^2(d) ,$$  \hspace{1cm} (16)

The last two equalities show that $\phi(\mathbb{R}^2)$ is contained in $\Sigma_d(v_3, v_4)$. Since $\phi$ is double periodic (on its domain variables, $s$ and $t$), it defines a function on a torus. Being the torus a compact set and the map $\phi$ an immersion, we conclude that $\phi$ is onto. A direct computation also shows that for any state $\phi(s, t)$, the state $(X \otimes I_2)\phi(s, t)$ is in $\Sigma_{2-d}(v_3, v_4) = \Sigma_d(v_1, v_2)$. We therefore conclude that any pair of states in $\Omega_d$ are connected by local gates.

We continue the proof by pointing out that for any state $\phi(s, t)$ the eigenvalues of either one of the two trace matrices are

$$\lambda_{\pm} = \frac{1}{2}(1 \pm \sin(2d))$$  \hspace{1cm} (17)

From the expression of the eigenvalues above, we conclude that the entanglement entropy of the states on $\Omega_d$ is

$$S(d) = 1 - \frac{1}{2} \lambda_+ \log_2 \lambda_+ - \frac{1}{2} \lambda_- \log_2 \lambda_-$$  \hspace{1cm} (18)

We can easily check that $|00\rangle$ is in $\Sigma_{\pi/4}$; therefore, all elements in $\Sigma_{\pi/4}$ are not entangled states. Since the function $S(d)$ is one to one between 0 and $\pi/4$, we conclude that the only unentangled states are those in $\Sigma_{\pi/4}$. Once again using the expression for the entanglement entropy, we deduce that: two states in $\Omega_{d_1}$ and $\Omega_{d_2}$, respectively, are not connected by local gates if $d_1 < d_2 < \pi/4$. Recall that if two states are connected by local gates, then their entanglement entropy is the same. The previous arguments prove parts (2), (3) and (4). In order to prove part (5), we notice that the states of the form

$$CZ\phi(u, v) = \tilde{\phi}_1(u, v)|00\rangle + \tilde{\phi}_2(u, v)|01\rangle + \tilde{\phi}_3(u, v)|10\rangle + \tilde{\phi}_4(u, v)|11\rangle ,$$  \hspace{1cm} (19)
satisfy the equations $x_2^2 + x_3^2 = \sin^2(d)$ and $x_1^2 + x_4^2 = \cos^2(d)$. This is true because it can be easily checked that
\[
\frac{(\tilde{\phi}_1 + \tilde{\phi}_4)^2}{2} + \frac{(\tilde{\phi}_2 + \tilde{\phi}_3)^2}{2} = \sin^2(d), \quad \frac{(\tilde{\phi}_1 - \tilde{\phi}_4)^2}{2} + \frac{(\tilde{\phi}_2 - \tilde{\phi}_3)^2}{2} = \cos^2(d). \tag{20}
\]

Recall that the only difference between $\text{CZ}\phi$ and $\phi$ is that the $\tilde{\phi}_4 = -\phi_4$, the other coordinates with respect to the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ are the same. Let us consider the state $|\psi_0\rangle$ in $\Sigma_d$ and another state $|\psi_1\rangle$ in $\Sigma_{d_1}$. We will see that the existence of the circuit using only one CZ gate and local gates is equivalent to the existence of point in the intersection $\Lambda = \text{CZ}\phi(R_2) \cap \Sigma_{d_1}$. The reason is that once we have a state $|\phi_2\rangle$ in $\Lambda$, then, since $|\phi_2\rangle$ in $\text{CZ}\phi(\mathbb{R}^2)$ we have that $|\phi_2\rangle = \text{CZ}|\phi_3\rangle$ for some $|\phi_3\rangle$ in $\Sigma_d$ and since $|\phi_3\rangle$ is in $\Sigma_d$ then $|\phi_3\rangle$ and $|\phi_0\rangle$ are connected by local gates. On the other hand, since $|\phi_2\rangle$ is also on $\Sigma_{d_1}$, then $|\phi_2\rangle$ and $|\phi_1\rangle$ are connected by local gates. Therefore, the states $|\phi_0\rangle$ and $|\phi_1\rangle$ are connected by a circuit that only uses one CZ gate and local gates. The reason why the surfaces $\text{CZ}\phi(\mathbb{R}^2)$ and the surface $\Sigma_{d_1}$ intersect is that they are two transversal tori. We can see this algebraically by explicitly solving the equations
\[
x_2^2 + x_3^2 = \sin^2(d), \quad x_1^2 + x_4^2 = \cos^2(d), \quad x_1^2 + x_2^2 = \sin^2(d_1), \quad x_3^2 + x_4^2 = \cos^2(d_1). \tag{21}
\]

Since quantum circuits are reversibles, we can assume that $d > d_1$. A solution of the equation system (21) is given by
\[
x_1 = 0, \\
x_2 = \sin(d_1), \\
x_3 = \sqrt{\sin^2(d) - \sin^2(d_1)}, \\
x_4 = \cos(d). \tag{22}
\]

For the proof of part (6), a direct calculation of the output state $|w\rangle$, from the quantum circuit given in part (6), yields
\[
w_1 = \cos(\theta_1/2) \cos(\theta_3), \\
w_2 = \cos(\theta_1/2) \sin(\theta_3), \\
w_3 = \sin(\theta_1/2), \\
w_4 = \sin(\theta_4). \tag{23}
\]
the above result shows that $\theta_0 = \theta_3 - \theta_4$ and $\theta_2 = \theta_3 + \theta_4$.  

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4 Conclusions

In this work, we described how the space of pure states of 2-qubit state with real amplitudes partitions when we group them by their entanglement. We studied the geometry of each one of these groups when viewed as subsets of the three-dimensional sphere $S^3$. Recall that the 2-qubit states with real amplitudes naturally identify with point in $S^3$.

The group that stands up for being different is the group of 2-qubits with maximal entanglement ($S = 1$). These 2 quits form a pair of disjoint circle of radius 1 in $S^3$. This is the only group represented by a one-dimensional manifold. Any other group is characterized by the property of being within the same distance to this pair of circles. They are either pair of tori when the entanglement entropy is positive but less than 1 and the group of 2-qubit states that have entanglement entropy zero form a torus.

We show that any pair of 2-qubit states taken from the same group can be connected with $R_y$ and $X$ gates.

By considering the geometry of the action of the CZ gate on these groups, we provide a geometrical proof that every pair of 2-qubit states with real amplitudes can be connected with a circuit that contains only one CZ gate and $R_y$ and $X$ gates. For the particular case, when one of these 2-qubit states is the state $|00\rangle$, we explicitly provide the circuit.

One of the applications of quantum states with real amplitudes is probabilistic models to assist unsupervised machine learning tasks [7,8]. Understanding the structure of their Hilbert space and the different pathways toward their efficient preparation is one of the main motivations of this work. Natural extensions of this work include the extension to quantum states with a larger number of qubits.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Brody, D.C., Hughston, L.P.: Geometric quantum mechanics. J. Geom. Phys. 38, 19 (2001)
2. Bengtsson, I., Zyczkowski, K.: Geometry of Quantum States: An Introduction to Quantum Entangle- ment. Cambridge University Press, Cambridge (2006)
3. Jevtic, S., Pusey, M., Jennings, D., Rudolph, T.: Quantum steering ellipsoids. Phys. Rev. Lett. 113, 020402 (2014)
4. Milne, A., Jevtic, S., Jennings, D., Wiseman, H., Rudolph, T.: Quantum steering ellipsoids, extremal physical states and monogamy. New J. Phys. 16, 083017 (2014)
5. Boyer, M., Liss, R., Mor, T.: Geometry of entanglement in the Bloch sphere. Phys. Rev. A 95, 032308 (2017)
6. Avron, J., Kenneth, O.: An elementary introduction to the geometry of quantum states with a picture book. arXiv:1901.06688 (2019)
7. Benedetti, M., Garcia-Pintos, D., Perdomo, O., Leyton-Ortega, V., Nam, Y., Perdomo-Ortiz, A.: A generative modeling approach for benchmarking and training shallow quantum circuits. NPJ Quantum Inf. 5, 45 (2019)
8. Leyton-Ortega, V., Perdomo-Ortiz, A., Perdomo, O.: Robust implementation of generative modeling with paratemrized quantum circuits. arXiv:1901.08047 (2019)
9. Walck, S.N., Glasbrenner, J.K., Lochman, M.H., Hilbert, S.A.: Topology of the three-qubit space of entanglement types. Phys. Rev. A 72, 052324 (2005)
10. Znidaric, M., Giraud, O., Georgeot, B.: Optimal number of controlled-NOT gates to generate a three-qubit state. Phys. Rev. A 77, 032320 (2008)
11. Linden, N., Popescu, S.P.: Optimal number of controlled-NOT gates to generate a three-qubit state. Fortschr. Phys. 46, 567 (1998)
12. Perdomo, O.: Clifford 3-qubit states. Int. J. Quantum Inf. 18(3), 2050004 (2020)
13. Zurek, W.H. (ed.): Complexity, Entropy and the Physics of Information. Addison-Wesley, Boston (1990)
14. Hardy, L., Wootters, W.K.: Limited Holism and real-vector-space quantum theory. Theory. Found Phys. 42, 454 (2012)
15. Aulbach, M.: Classification of entanglement in symmetric states. arXiv:quant-ph/1110.5200 (2016)
16. Vidal, G.: Entanglement monotones. J. Mod. Opt. 47, 355 (2000)
17. Bennett, C.H., Popescu, S., Rohrlich, D., Smolin, J.A., Thapliyal, A.V.: Exact and asymptotic measures of multipartite pure-state entanglement. Phys. Rev. A 63, 012302 (2000)
18. Nielsen, M.A.: Conditions for a class of entanglement transformations. Phys. Rev. Lett. 83, 436 (1999)
19. Shende, V.V., Bullock, S.S., Markov, I.L.: Synthesis of quantum-logic circuits. IEEE Trans. Comput. Aided Des. Integr. Circuits Syst. 25, 1000 (2006)
20. Plesch, M., Brukner, Č.: Quantum-state preparation with universal gate decompositions. Phys. Rev. A 83, 032302 (2011)
21. Iten, R., Colbeck, R., Kukuljan, I., Home, J., Christandl, M.: Quantum circuits for isometries. Phys. Rev. A 93, 032318 (2016)
22. Smith, R., Curtis, M.J., Zeng, W.J.: A practical quantum instruction set architecture. arXiv:1608.03355 (2016)
23. Lawson, H.B.: The unknottedness of minimal embeddings. Invent Math. 11, 183 (1970)
24. Brendle, S.: Embedded minimal tori in s3 and the Lawson conjecture. Acta Math. 211, 177 (2013)

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