Continuity of selected pullback attractors

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Abstract
In this work we obtain theoretical results on continuity of selected pullback attractors for a generalized process and consider the continuity of attractors when the system is asymptotically autonomous. As an example, we apply the theory on a nonautonomous problem with reaction diffusion equations with dynamical boundary conditions, after that, with some additional hypotheses we consider the family of problems converging to autonomous limit problem and show the convergence of their attractors.

Keywords Selected pullback attractors · Continuity of attractors · Reaction–diffusion equations · Dynamic boundary conditions

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1 Introduction

The study of the asymptotic behavior of infinite dimensional evolution problems is connected with the existence of attractors for semigroups (or processes for nonautonomous problems) associated with partial differential equations. Sometimes the uniqueness of solutions of the Initial Value Problem fails or it is not known to hold. In this case one have to use the theory of multivalued semigroups (or multivalued processes for nonautonomous problems).

For some problems it is not possible to guarantee the existence of global attractors or pullback attractors. For this reason, some researchers had introduced different weaker concepts of attractors. Simsen and Gentile [13] had introduced the notion of ϕ-attractor for

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a multivalued semigroup defined by a generalized semiflow. Kapustyan et al. [6, 7] considered 3D-Bénard systems which is a model in hydrodynamics and describes the behavior of the velocity, the pressure and the temperature of an incompressible fluid. These systems, in general, do not have a global attractor because of the lack of good dissipativity estimates for all weak solutions. However, they proved the existence of a $\varphi$-attractor for the 3D-Bénard systems. See also [8, 9].

For the nonautonomous case we can also refer to the theory of trajectory attractors developed by Chepyzhov and Vishik [3] or the theory of generalized process developed by Ball [1], in both these theories we can consider sets with some solutions, not necessarily all solutions, and study their dynamics. In the pioneer work [2], Caraballo et al. introduced the concept of a weak pullback attractor where they consider all solutions and consider an attractor that can attract some solutions but not all. Based in the theory of weak pullback attractor and with slight differences in the approach Samprogna and Simsen [11] introduced the concept of Selected pullback attractors. Despite the difference in the approach to the asymptotic compactness in the theory of Select pullback attractor, it does not have a significant advantage over other theories, but it is a different approach that may help in future research. Therefore, this work is devoted to study some properties of this attractor.

In this paper we develop some abstract results about the Selected pullback attractors and, as example, we apply them to a $p$-Laplacian evolution equation without guarantee of uniqueness of solution. In the Sect. 2, for the completeness of the work, we revise some concepts and results from [11]. Section 3 present new theoretical results on continuity of selected pullback attractors. The last section is devoted to give an application to a $p$-Laplacian reaction–diffusion equation with dynamic boundary conditions.

\section{Selected pullback attractor}

Let $(X, d)$ be a complete metric space. For $x \in X$, $A, B \subset X$ and $\epsilon > 0$ we define

\[ d(x, A) := \inf_{a \in A} \{ d(x, a) \}; \]
\[ \text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} \{ d(a, b) \}; \]
\[ \text{dist}_H(A, B) := \max\{ \text{dist}(A, B), \text{dist}(B, A) \}; \]
\[ \mathcal{O}_\epsilon(A) := \{ z \in X; d(z, A) < \epsilon \}. \]

Denote by $P(X)$, $B(X)$ and $K(X)$ the nonempty, nonempty and bounded and nonempty and compact subsets of $X$, respectively.

\textbf{Definition 1 (Generalized Process)} A \textbf{generalized process} $\mathcal{G} = \{ \mathcal{G}(\tau) \}_{\tau \in \mathbb{R}}$ in $X$ is a family of sets $\mathcal{G}(\tau)$ consisting of functions $\varphi : [\tau, \infty) \rightarrow X$, called solutions, satisfying the following conditions:

(C1) For each $\tau \in \mathbb{R}$ and $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(\tau) = z$;
(C2) If $\varphi \in \mathcal{G}(\tau)$ and $s \geq 0$, then $\varphi^{+s} \in \mathcal{G}(\tau + s)$, where $\varphi^{+s} := \varphi_{[\tau + s, \infty)}$;
(C3) If $\{ \varphi_j \}_{j \in \mathbb{N}} \subset \mathcal{G}(\tau)$ and $\varphi_j(\tau) \rightarrow z$, there is a subsequence $\{ \varphi_{j_k} \}_{k \in \mathbb{N}}$ of $\{ \varphi_j \}_{j \in \mathbb{N}}$ and $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$ and such that $\varphi_{j_k}(t) \rightarrow \varphi(t)$ as $k \rightarrow \infty$, for each $t \geq \tau$.

Whether $\mathcal{G}$ is formed by continuous functions we call this process a continuous process.
Definition 2  A generalized process \( \mathcal{G} = \{ \mathcal{G}(\tau) \}_{\tau \in \mathbb{R}} \) is said to be **locally uniformly upper semicontinuous** (LUUS) if it satisfies the following condition:

(C4) If \( \{ \varphi_j \}_{j \in \mathbb{N}} \) is a sequence such that \( \varphi_j \in \mathcal{G}(\tau) \) and \( \varphi_j(\tau) \to z \), then there is \( \varphi \in \mathcal{G}(\tau) \) with \( \varphi(\tau) = z \) and a subsequence \( \{ \varphi_{j_k} \}_{k \in \mathbb{N}} \) such that \( \varphi_{j_k} \to \varphi \) uniformly on compact subsets of \([\tau, +\infty)\) when \( k \to \infty \).

Definition 3  We say that a generalized process \( \mathcal{G} = \{ \mathcal{G}(\tau) \}_{\tau \in \mathbb{R}} \) is **exact** (or strict) if it satisfies the following condition:

(C5) (Concatenation) Let \( \varphi \in \mathcal{G}(\tau) \) and \( \psi \in \mathcal{G}(\rho) \) such that \( \varphi(s) = \psi(s) \) for some \( s \geq r \geq \tau \). If \( \theta \) is defined by

\[
\theta(t) := \begin{cases} 
\varphi(t), & t \in [\tau, s], \\
\psi(t), & t \in (s, \infty), 
\end{cases}
\]

then \( \theta \in \mathcal{G}(\tau) \).

Definition 4  We say that there exists a **complete orbit** through \( x \in X \) at \( \tau \in \mathbb{R} \) if there is a map \( \psi : \mathbb{R} \to X \) with \( \psi(\tau) = x \) and for all \( s \in \mathbb{R} \), \( \psi|_{[\tau+s, \infty)} \in \mathcal{G}(\tau+s) \).

We refer the reader to [12, 14] for more details on generalized process theory.

Definition 5  Let \( \mathcal{A} = \{ \mathcal{A}(t) \}_{t \in \mathbb{R}} \) be a family of sets. We say that \( \mathcal{A} \) **select pullback attracts** an element \( x \in X \) at time \( t \in \mathbb{R} \) if given \( \varepsilon > 0 \) there is \( \tau_0 \leq t \) such that for all \( \tau \leq \tau_0 \) there is \( \varphi_{\tau} \in \mathcal{G}(\tau) \) with \( \varphi_{\tau}(\tau) = x \) and

\[
\varphi_{\tau}(t) \in \mathcal{O}_{\varepsilon}(\mathcal{A}(t)).
\]

The family \( \mathcal{A} = \{ \mathcal{A}(t) \}_{t \in \mathbb{R}} \) is a **global selected pullback attractor** if for each \( t \in \mathbb{R} \), the set \( \mathcal{A}(t) \) select pullback attracts all \( x \in X \) at time \( t \), moreover, each \( \mathcal{A}(t) \) is a compact set.

Definition 6  Let \( \mathcal{A} = \{ \mathcal{A}(t) \}_{t \in \mathbb{R}} \) be a family of subsets of \( X \). We say that \( \mathcal{A} \) is **quasi-invariant** if for each \( z \in \mathcal{A}(\tau) \) for some \( \tau \in \mathbb{R} \), there exists a complete orbit \( \psi \) through \( z \) at \( \tau \) with \( \psi(t) \in \mathcal{A}(t) \) for all \( t \in \mathbb{R} \).

Remark 1  Quasi-invariance is named as weak invariance in [2].

Definition 7  We say that a generalized process \( \mathcal{G} \) has the **uniform selected K-property** if there exists a family of compact sets \( \{ K(t) \}_{t \in \mathbb{R}} \) such that given \( x \in X \) for each \( \tau \in \mathbb{R} \) there is \( \varphi_{\tau} \in \mathcal{G}(\tau) \) with \( \varphi_{\tau}(\tau) = x \), and this family of solutions \( \{ \varphi_{\tau} \}_{\tau \in \mathbb{R}} \) has the following property: for each \( t \in \mathbb{R} \) there is \( \tau_0 \leq t \) and sequence \( \{ \varepsilon_{\tau} \}_{\tau \leq \tau_0} \) such that

\[
\varphi_{\tau}(t) \in \mathcal{O}_{\varepsilon_{\tau}}(K(t)), \quad \tau \leq \tau_0,
\]

with \( \varepsilon_{\tau} \to 0 \) as \( \tau \to -\infty \).

Theorem 1  [11, Theorem 25] If the generalized process \( \mathcal{G} \) possess the uniform selected K-property, then it have a quasi-invariant global selected pullback attractor \( \mathcal{A} = \{ \mathcal{A}(t) \}_{t \in \mathbb{R}} \).
3 Continuity of selected pullback attractors

In this section we present abstract new results.

**Theorem 2** Let $\mathcal{G}$ a continuous and LUUS generalized process with the uniform selected K-property and the family $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is the global selected pullback attractor. Then the setvalued mapping $t \to \mathcal{A}(t)$ is continuous, i.e.,

$$\lim_{s \to t} \text{dist}_H(\mathcal{A}(s), \mathcal{A}(t)) = 0, \quad \forall \ t \in \mathbb{R}.$$ 

**Proof** Fixed $t \in \mathbb{R}$, firstly, consider the limit $\text{dist}(\mathcal{A}(t), \mathcal{A}(s))$ as $s \to t$. Suppose that $\lim_{s \to t} \text{dist}(\mathcal{A}(t), \mathcal{A}(s))$ is not zero, then there would exist $\delta > 0$ and a sequence $s_n \to t$ such that

$$\delta \leq \text{dist}(\mathcal{A}(t), \mathcal{A}(s_n)), \quad \forall \ n \in \mathbb{N}.$$ 

Since $\mathcal{A}(t)$ is compact, for each $n \in \mathbb{N}$ there exists $a_n \in \mathcal{A}(t)$ with

$$\text{dist}(\mathcal{A}(t), \mathcal{A}(s_n)) = \text{dist}(a_n, \mathcal{A}(s_n)) \leq \text{dist}(a_n, \mathcal{A}(s_n))$$

for all $a_n \in \mathcal{A}(s_n)$.

From Theorem 1, for each $a_n \in \mathcal{A}(t)$ there exists a complete trajectory $\varphi_n \in \mathcal{G}(t)$ such that $\varphi_n(s) \in \mathcal{A}(s)$ for all $s \in \mathbb{R}$ with $\varphi_n(t) = a_n$. As $\mathcal{A}(t)$ is a compact set there is $a \in \mathcal{A}(t)$ such that, up to subsequence, $a_n \to a$.

From the LUUS property, there exists $\varphi_t \in \mathcal{G}(t)$ such that $\varphi_n \to \varphi_t$ uniformly on compact subsets of $[t, +\infty)$. However, we have $b_n := \varphi_n(t-1) \in \mathcal{A}(t-1)$ for each $n \in \mathbb{N}$, then, analogous to the above, there exists $b \in \mathcal{A}(t-1) \cap \mathcal{G}(t-1)$ such that, up to subsequence, $b_n \to b$ and \( \varphi_{t-1}(t-1) = b \) with $\varphi_n \to \varphi_{t-1}$ uniformly on compact subsets of $[t-1, +\infty)$. Moreover, $\varphi_{t-1}(s) = \varphi_t(s)$ for each $s \in [t, +\infty)$.

Given $s \in [t-1, \infty)$, define $\varphi(s)$ as the common value of $\varphi_{t-1}(s)$ and $\varphi_t(s)$ if $s \in [t, \infty)$ or just $\varphi_{t-1}(s)$ if $s \in [t-1, t)$. Then $\varphi_n \to \varphi$ uniformly on compact subsets of $[t-1, +\infty)$.

Note that, for $n$ large enough we have $s_n \in [t-1, \infty)$, in this case, from the continuity of the trajectories and the uniform convergence of $\varphi_n \to \varphi$ on compact subsets of $[t-1, +\infty)$, we conclude that

$$\text{dist}(\mathcal{A}(t), \mathcal{A}(s_n)) \leq d(\varphi_n(t), \varphi_n(s_n))$$

$$\leq d(\varphi_n(t), \varphi(t)) + d(\varphi(t), \varphi(s_n)) + d(\varphi(s_n), \varphi_n(s_n)) \to 0,$$

contradicting (1). Therefore, $\lim_{s \to t} \text{dist}(\mathcal{A}(t), \mathcal{A}(s)) = 0$.

On the other hand, now let us show $\lim_{s \to t} \text{dist}(\mathcal{A}(s), \mathcal{A}(t)) = 0$. Suppose that it does not hold, then there would exist $\delta > 0$ and a sequence $s_n \to t$ such that

$$\delta \leq \text{dist}(\mathcal{A}(s_n), \mathcal{A}(t)) \quad \forall \ n \in \mathbb{N}.$$ 

(2)

Since $\mathcal{A}(s_n)$ is compact, there exists $a_n \in \mathcal{A}(s_n)$ such that
\[
\text{dist}(\mathcal{A}(s_n), \mathcal{A}(t)) = \text{dist}(a_n, \mathcal{A}(t)) \leq \text{dist}(a_n, a),
\]

for all \(a \in \mathcal{A}(t)\).

From Theorem 1, for each \(a_n\) there exists a complete trajectory \(\varphi_n\) with \(\varphi_n(s_n) = a_n\) and \(\varphi_n(s) \in \mathcal{A}(s)\) for all \(s \in \mathbb{R}\).

As \(s_n \to t\) there exists \(n_0 \in \mathbb{N}\) such that \(s_n \in [t-1, +\infty)\) for all \(n \geq n_0\). Consider a subsequence starting on \(s_{n_0}\), which we do not relabel.

We have

\[
\varphi_n(t - 1) \in \mathcal{A}(t - 1),
\]

and, provided that \(\mathcal{A}(t - 1)\) is compact, there is \(a_0 \in \mathcal{A}(t - 1)\) such that, up to subsequence, \(\varphi_n(t - 1) \to a_0\) as \(n \to +\infty\).

From the LUUS property there exists \(\varphi \in \mathcal{G}(t - 1)\) with \(\varphi_n \to \varphi\) uniformly on compact subsets of \([t - 1, +\infty)\).

From the continuity of the trajectories and the uniform convergence of \(\varphi_n \to \varphi\) on compact subsets of \([t - 1, +\infty)\), we conclude that

\[
\text{dist}(\mathcal{A}(s_n), \mathcal{A}(t)) \leq d(\varphi_n(s_n), \varphi(t)) \\
\quad \leq d(\varphi_n(s_n), \varphi(s_n)) + d(\varphi(s_n), \varphi(t)) + d(\varphi(t), \varphi_n(t)) \to 0,
\]

contradicting (2).

Remark 2 One version of the above theorem can be found in Proposition 11 of [2], where the authors have used an exact and locally uniform upper semicontinuity in \(t\) for a setvalued process (or multivalued process). Note that, in our case the setvalued process generated by the generalized process \(\mathcal{G}\) on the above theorem do not necessary be an exact setvalued process, and the uniformly upper semicontinuity can be obtained by the LUUS property.

Theorem 3 Let \(\mathcal{A}\) be a forward compact selected pullback attractor, i.e.,

\[
\bigcup_{t \geq t} \mathcal{A}(s)
\]

is a compact set for all \(t \in \mathbb{R}\). Then, we have

\[
\lim_{t \to -\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}(\infty)) = 0,
\]

where \(\mathcal{A}(\infty) := \cap_{t \in \mathbb{R}} \mathcal{A}(r)\).

Proof Note that, since \(\mathcal{A}\) is forward compact the set \(\mathcal{A}(\infty)\) is a compact set.

Suppose that (3) is not true, then there are \(\delta > 0\) and a real sequence \(\{t_n\}_{n \in \mathbb{N}}\) with \(0 < t_n \to +\infty\) such that \(\text{dist}(\mathcal{A}(t_n), \mathcal{A}(\infty)) \geq \delta\) for all \(n \in \mathbb{N}\).

Thus, for each \(n \in \mathbb{N}\) there is \(x_n \in \mathcal{A}(t_n)\) such that

\[
\text{dist}(x_n, \mathcal{A}(\infty)) \geq \delta.
\]

Note that, for some \(t_0 \geq 0\), we have \(\{x_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \geq t_0} \mathcal{A}(s)\) for \(t_n \geq t_0\).

As \(\mathcal{A}\) is a forward compact family we conclude that the sequence \(\{x_n\}_{n \in \mathbb{N}}\) has got a convergent subsequence, which we do not relabel, and let \(x \in \mathcal{X}\) a limit of this subsequence.

From the definition we have \(x \in \mathcal{A}(\infty)\), and then (4) is a contradiction. \(\square\)
Theorem 4  Let $A$ be a backward compact selected pullback attractor, i.e.,

$$\bigcup_{s \leq t} A(s)$$

is a compact set for all $t \in \mathbb{R}$. Then, we have

$$\lim_{t \to -\infty} \text{dist}(A(t), A(-\infty)) = 0,$$

where $A(-\infty) := \bigcap_{t \in \mathbb{R}} \bigcup_{r \leq t} A(r)$.

Proof  Note that, since $A$ is backward compact the set $A(-\infty)$ is a compact set.

Suppose that (5) is not true, then there are $\delta > 0$ and a real sequence $\{t_n\}_{n \in \mathbb{N}}$ with $0 > t_n \to -\infty$ such that $\text{dist}(A(t_n), A(-\infty)) \geq \delta$ for all $n \in \mathbb{N}$.

Thus, for each $n \in \mathbb{N}$ there is $x_n \in A(t_n)$ such that

$$\text{dist}(x_n, A(-\infty)) \geq \delta.$$  \hfill (6)

Note that, for some $t_0 \leq 0$, we have $\{x_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \leq t_0} A(s)$ for $t_n \leq t_0$.

As $A$ is a backward compact family we conclude that the sequence $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence, which we do not relabel, and let $x \in X$ a limit of this subsequence.

From the definition we have $x \in A(-\infty)$, and then (6) is a contradiction. \hfill $\Box$

Definition 8  Let $A = \{A(t)\}$ be a quasi-invariant global selected pullback attractor of a generalized process $G$. We say that the generalized process $G$ is asymptotically autonomous for $A$ if for each sequence $\{x_s\} \subset X$ such that $x_s \in A(s)$ and $x_s \to x_0$ as $s \to -\infty$, let $\varphi_s$ be a complete trajectory with $\varphi_s(t) = x_s$ and $\varphi_s(s) \in A(s)$ for all $s \in \mathbb{R}$, we have there exists a solution $\varphi$ of an autonomous problem such that

$$\varphi_s(t + \tau) \to \varphi(t) \text{ as } \tau \to -\infty \text{ and } \varphi(0) = x_0.$$  \hfill (8)

Theorem 5  Let $G$ be an asymptotically autonomous generalized process for a selected pullback attractor $A$ and let $A_\infty$ be the global attractor of the corresponding autonomous problem. If the selected pullback attractor $A$ is forward compact, then

$$\lim_{t \to -\infty} \text{dist}(A(t), A_\infty) = 0.$$

Proof  Let $A$ forward compact and suppose (7) is not true. Then there would exist an $\varepsilon_0 > 0$ and a real sequence $\{\tau_n\}$ with $0 < \tau_n \to +\infty$ such that $\text{dist}(A(\tau_n), A_\infty) \geq 3\varepsilon_0$ for all $n \in \mathbb{N}$. Since the sets $A(\tau_n)$ are compact, there exists $a_n \in A(\tau_n)$ such that

$$\text{dist}(a_n, A_\infty) = \text{dist}(A(\tau_n), A_\infty) \geq 3\varepsilon_0,$$

for each $n \in \mathbb{N}$.

Let $C = \bigcup_{s \geq 0} A(s)$ a compact set, there is $n_0 > 0$ such that
with $G(t, \cdot)$ is the multivalued semigroup of the corresponding autonomous problem.

Note that the sequence $0 < \tau_n - \tau_{n_0} \to \infty$ for all $n \geq n_0$, and there are a sequence of complete orbits $\varphi_{\tau_n - \tau_{n_0}}$ and a sequence $\{b_n\}_{n \geq n_0}$ with $\varphi_{\tau_n - \tau_{n_0}}(\tau_n - \tau_{n_0}) = b_n \in \mathcal{A}(\tau_n - \tau_{n_0})$ and

$$a_n = \varphi_{\tau_n - \tau_{n_0}}(\tau_{n_0}).$$

We have $\{b_n\} \subset C$, then there is $b$ such that, up to a subsequence, $b_n \to b$.

We have,

$$a_n = \varphi_{\tau_n - \tau_{n_0}}(\tau_n + \tau_n - \tau_{n_0}) \to \varphi(\tau_{n_0})$$

with $\varphi(0) = b$, and then $\varphi(\tau_{n_0}) \in G(\tau_{n_0}, C)$.

Therefore, for $n$ large enough, we have

$$\text{dist}(a_n, \mathcal{A}_\infty) \leq \|a_n - \varphi(\tau_{n_0})\| + \text{dist}(\varphi(\tau_{n_0}), \mathcal{A}_\infty) \leq 2\varepsilon_0$$

which contradicts (8). \hfill $\square$

**Theorem 6** Let $G$ be a LUUS process composed by continuous functions and asymptotically autonomous for $A$ with $\mathcal{A}_\infty$ be the global attractor of the corresponding autonomous problem. If

$$\lim_{t \to \infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) \to 0,$$

then the selected pullback attractor $\mathcal{A}$ is forward compact.

**Proof** Suppose that (9) is true. For fixed $t \in \mathbb{R}$ let $\{x_n\} \subset \bigcup_{r \leq t} \mathcal{A}(r)$. For each $n \in \mathbb{N}$ there is $r_n$ such that $x_n \in \mathcal{A}(r_n)$. Thus we have two cases:

**Case 1:** $r_0 := \sup_n r_n < \infty$.

In this case there is a sequence $\{\varphi_n\}$ with $\varphi_n(r_n) = x_n$ and $\varphi_n(t) \in \mathcal{A}(t)$, as the set $\mathcal{A}(t)$ is compact there is $b \in X$ such that, up to a subsequence, $\varphi_n(t) \to b$. From the LUUS property of $G$ there exists $\varphi \in G(t)$ such that $\varphi_n \to \varphi$ on compact subsets of $[t, \infty)$.

As $\{r_n\} \subset [t, r_0]$ there is $r' \in [t, r_0]$ such that, up to a subsequence, $r_n \to r'$. Given $\varepsilon > 0$ and from the continuity of the solutions and uniform convergence of $\varphi_n$ in $[t, r_0]$, for $n$ large enough, we get

$$\|\varphi_n(r_n) - \varphi(r')\| \leq \|\varphi_n(r_n) - \varphi(r_n)\| + \|\varphi(r_n) - \varphi(r')\| < \varepsilon.$$

**Case 2:** $\sup_n r_n = \infty$.

In this case, up to a subsequence, we may assume $r_n \to \infty$. We have

$$\text{dist}(x_n, \mathcal{A}_\infty) \leq \text{dist}(\mathcal{A}(r_n), \mathcal{A}_\infty) \to 0.$$ 

We can choose $y_n \in \mathcal{A}_\infty$ such that

$$d(x_n, y_n) \leq \text{dist}(x_n, \mathcal{A}_\infty) + \frac{1}{n}.$$

There is $y \in \mathcal{A}_\infty$ such that, up to a subsequence, $y_n \to y$, which implies that $x_n \to y$. \hfill $\square$
Definition 9 Let $A = \{A(t)\}$ be a quasi-invariant global selected pullback attractor of a generalized process $\mathcal{G}$. We say that the generalized process $\mathcal{G}$ is **asymptotically backward autonomous** for $A$ if for each sequence $\{x_i\} \subseteq X$ such that $x_i \in A(\tau)$ and $x_i \to x_0$ when $\tau \to -\infty$, let $\varphi_i$ be a complete trajectory with $\varphi_i(\tau) = x_i$ and $\varphi_i(s) \in A(s)$ for all $s \in \mathbb{R}$, we have there exists a solution $\varphi$ of an autonomous problem such that

$$\varphi_i(t + \tau) \to \varphi(t)$$

with $\tau \to -\infty$ and $\varphi(0) = x_0$.

Theorem 7 Let $\mathcal{G}$ be an asymptotically backward autonomous process for a selected pullback attractor $A$ and let $A_\infty$ be the global attractor of the corresponding autonomous problem. If the selected pullback attractor is backwards compact, then

$$\lim_{t \to -\infty} \text{dist}(A(t), A_\infty) = 0.$$

**Proof** Let $A$ backward compact and suppose (10) is not true. Then, there would exist an $\varepsilon_0 > 0$ and a real sequence $\{\tau_n\}$ with $0 < \tau_n \not\to +\infty$ such that $\text{dist}(A(-\tau_n), A_\infty) \geq 3\varepsilon_0$ for all $n \in \mathbb{N}$. Since the sets $A(-\tau_n)$ are compact, there exists $a_n \in A(-\tau_n)$ such that

$$\text{dist}(a_n, A_\infty) = \text{dist}(A(-\tau_n), A_\infty) \geq 3\varepsilon_0,$$

for each $n \in \mathbb{N}$.

Let $C = \bigcup_{s \leq 0} A(s)$ a compact set, there is $T_0 > 0$ such that

$$\text{dist}(G(T_0, C), A_\infty) \leq \varepsilon_0.$$  

with $G(t, \cdot)$ is the multivalued semigroup of the corresponding autonomous problem.

Note that the sequence $0 > -\tau_n - T_0 \not\to -\infty$, and there are a sequence of complete orbits $\{\varphi_{-\tau_n - T_0}\}_{n \in \mathbb{N}}$ and a sequence $\{b_n\}_{n \in \mathbb{N}}$ with $\varphi_{-\tau_n - T_0}(-\tau_n - T_0) = b_n \in A(-\tau_n - T_0)$ and

$$a_n = \varphi_{-\tau_n - T_0}(-\tau_n).$$

We have $\{b_n\} \subseteq C$, then there is $b \in C$ such that, up to a subsequence, $b_n \to b$.

We have then,

$$a_n = \varphi_{-\tau_n - T_0}(T_0 - \tau_n - T_0) \to \varphi(T_0)$$

with $\varphi(0) = b$, and then $\varphi(T_0) \in G(T_0, C)$.

Therefore, for $n$ large enough, we have

$$\text{dist}(a_n, A_\infty) \leq \|a_n - \varphi(T_0)\| + \text{dist}(\varphi(T_0), A_\infty) \leq 2\varepsilon_0$$

which contradicts (11). \hfill $\square$

Theorem 8 Let $\mathcal{G}$ be a LUUS process composed by continuous functions and asymptotically backwards autonomous for $A$ with $A_\infty$ be the global attractor of the corresponding autonomous problem. If
\[
\lim_{t \to +\infty} \text{dist}(A(t), A_\infty) \to 0,
\]  

then the selected pullback attractor \( A \) is backwards compact.

**Proof** Suppose that (12) is true. For fixed \( t \in \mathbb{R} \) let \( \{x_n\} \subset \bigcup_{r \leq t} A(r) \). For each \( n \in \mathbb{N} \) there is \( r_n \) such that \( x_n \in A(r_n) \). Thus we have two cases:

**Case 1:** \( r_0 := \inf_n r_n > -\infty \).

In this case there is a sequence \( \{\varphi_n\} \) with \( \varphi_n(r_n) = x_n \) and \( \varphi_n(r_0) \in A(r_0) \), as the set \( A(r_0) \) is compact there is \( b \in X \) such that, up to a subsequence, \( \varphi_n(r_0) \to b \). From the LUUS property of \( G \) there exists \( \varphi \in G(r_0) \) such that \( \varphi_n \to \varphi \) on compact subsets of \([r_0, \infty)\).

As \( \{r_n\} \subset [r_0, t] \) there is \( r' \in [r_0, t] \) such that, up to a subsequence, \( r_n \to r' \). Given \( \varepsilon > 0 \) and from the continuity of the solutions and uniform convergence of \( \varphi_n \) in \([r_0, t]\), for \( n \) large enough, we get

\[
\|\varphi_n(r_n) - \varphi(r')\| \leq \|\varphi_n(r_n) - \varphi(r_0)\| + \|\varphi(r_0) - \varphi(r')\| < \varepsilon.
\]

**Case 2:** \( \inf_n r_n = -\infty \).

In this case, up to a subsequence, we may assume \( r_n \searrow -\infty \). We have

\[
\text{dist}(x_n, A_\infty) \leq \text{dist}(A(r_n), A_\infty) \to 0.
\]

We can choose \( y_n \in A_\infty \) such that

\[
\text{dist}(x_n, y_n) \leq \text{dist}(x_n, A_\infty) + \frac{1}{n}.
\]

There is \( y \in A_\infty \) such that, up to a subsequence, \( y_n \to y \), which implies that \( x_n \to y \). \( \square \)

### 4 An application to a reaction–diffusion equation with dynamic boundary conditions

In [11] where the authors have introduced the concept of selected pullback attractor, as an application of the theory the authors have considered the following problem with a nonautonomous p-Laplacian equation,

\[
\begin{cases}
    u_t - \Delta_p u + f_1(t, u) = g_1(t, x), & (t, x) \in (\tau, +\infty) \times \Omega, \\
    u_t + |\nabla u|^{p-2} \partial_n u + f_2(t, u) = g_2(t, x), & (t, x) \in (\tau, +\infty) \times \Gamma, \\
    u(\tau) = u_0,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary \( \Gamma = \partial \Omega, N \geq 3 \) and \( \Delta_p \) denotes the \( p \)-Laplacian operator with \( p \in (2, +\infty) \). The hypotheses for the perturbations \( f_i \) are \( f_i \in C(\mathbb{R}^2) \) and satisfies

\[
a_i(t)|s|^\gamma - k_i(t) \leq f_i(t, s)s,
\]

for a.e. \( t \in \mathbb{R} \) and every \( s \in \mathbb{R} \) with \( a_i \in L^1_{\text{loc}}(\mathbb{R}) \) real functions such that \( a_i(t) \geq a_0 > 0 \) for some fixed real number \( a_0 \), and \( k_i \in L^1_{\text{loc}}(\mathbb{R}) \) are positive functions, for \( i = 1, 2 \), with
\[ \begin{align*}
    r_1 & \in \begin{cases} 
        (p,pN/(N-p)], & \text{if } p \in (2N/(N+2),N), \\
        (p,+) & \text{if } p = N, \\
        [p,+) & \text{if } p > N 
    \end{cases} \\
    r_2 & \in \begin{cases} 
        [2,(N-1)p/(N-p)], & \text{if } p \in (2N/(N+2),N), \\
        [2,+) & \text{if } p \geq N. 
    \end{cases}
\end{align*} \
\]

Besides that, there are functions \( C_i \in L^\infty_{\text{loc}}(\mathbb{R}), i = 1,2 \), such that \( |f_i(t,s)| \leq C_i(t)|s|^{r-1} + 1 \) a.e. for \( t \in \mathbb{R} \) and each \( s \in \mathbb{R} \).

The external forces \( g_i \) satisfies \( g_1 \in L^r_{\text{loc}}(\mathbb{R};L^1(\Omega)), g_2 \in L^r_{\text{loc}}(\mathbb{R};L^2(\Gamma)) \) where \( p' \) denotes the conjugate exponent of \( p \).

The initial state \( u_0 \) belongs to the space \( X^2 \), where
\[ X^2 := L^2(\Omega, dx) \times L^2(\Gamma, dS) = \{ F = (f,g); f \in L^2(\Omega) \text{ and } g \in L^2(\Gamma) \}, \]

with the norm
\[ \|F\|_{X^2} = \left( \int_\Omega |f|^2 dx + \int_\Gamma |g|^2 dS \right)^{1/2}. \]

This space can be identified with the space \( L^2(\overline{\Omega}, d\mu) \) where \( d\mu = dx \oplus dS \), i.e., if \( A \subset \Omega \) is \( \mu - \text{measurable} \), then \( \mu(A) = |A \cap \Omega| + S(A \cap \Gamma) \) and \( S \) is the surface measure in boundary \( \Gamma \), see [10] for more details.

The authors in [10] could not ensure the uniqueness of solution with the assumed hypotheses and then they work with a possibility of the existence of others solutions.

With the following additional assumptions \( g_1 \in L^{2r-2}_{\text{loc}}(\mathbb{R};L^{2r-2}(\Omega)), g_2 \in L^{2r-2}_{\text{loc}}(\mathbb{R};L^{2r-2}(\Gamma)) \), \( r = \max\{r_1, r_2\} \) and, for each \( \lambda \in \mathbb{R} \)
\[ \begin{align*}
    \int_{-\infty}^{\lambda} e^{\theta s} \left( \|g_1(s)\|^2_{L^{2r-2}(\Omega)} + \|g_2(s)\|^2_{L^{2r-2}(\Gamma)} \right) ds & < +\infty, \\
    \int_{-\infty}^{\lambda} e^{\theta s} (k_1(s) + k_2(s)) ds & < +\infty \quad \text{and} \quad \int_{-\infty}^{\lambda} e^{\theta s} (k_1(s)^{r-1} + k_2(s)^{r-1}) ds < +\infty,
\end{align*} \]

where \( \theta > 0 \) is a suitable constants, the authors have ensured the existence of a \( \mathcal{D} \)-pullback attractor for a generalized process composed only with solutions which are from Faedo-Galerkin method and attracts the families of sets \( \{D(t) : t \in \mathbb{R}\} \) of nonempty subsets of \( L^{2r-2}(\Omega) \times L^{2r-2}(\Gamma) \subset L^2(\overline{\Omega}, d\mu) \) such that
\[ \lim_{s \to -\infty} e^{\theta s} |D(s)| = 0, \]

where \( |D(s)| = \sup \left\{ \|u\|^2_{L^{2r-2}(\Omega)} + \|v\|^2_{L^{2r-2}(\Gamma)} : (u,v) \in D(s) \right\} \).

The restriction on the generalized process is because of some technicalities to develop the estimates of the solution, and then the attraction is ensured only for solutions from Faedo-Galerkin method. As we do not have the uniqueness we have no guarantees that there exists another solution that it is not coming from a sequence of Faedo-Galerkin method.
In particular, this \( D \)-pullback attractor when we consider a generalized process composed with any solution of the problem is a selected pullback attractor, it was observed in \([11]\).

Here, we will consider some additional assumptions on the perturbations of the operators and external forces, and ensure the continuity of the selected pullback attractor of the Problem \((P)\).

This example is merely illustrative to show how the selected pullback attractor works, in this case the pullback attractor obtained \([10]\) is totally suitable. For the selected pullback attractor theory it would be more interesting to consider a problem with more specific properties that takes explicit existence of solutions with differences in their behavior, but as an example of the theory this problems will fit.

**Assumption A** For \( \tilde{g}_1 \in L^{2r-2}(\Omega) \) and \( \tilde{g}_2 \in L^{2r-2}(\Gamma) \) we have

\[
\lim_{\tau \to +\infty} \int_{\tau}^{+\infty} \| g_1(\tau + s) - \tilde{g}_1 \|^2_{2r-2,\Omega} + \| g_2(\tau + s) - \tilde{g}_2 \|^2_{2r-2,\Gamma} ds = 0.
\]

**Assumption B**

\[
\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{\theta(s-t)} (k_1(s) + k_2(s)) ds < +\infty \tag{14}
\]

and

\[
\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{\theta(s-t)} (k_1(s)^{-1} + k_2(s)^{-1}) ds < +\infty. \tag{15}
\]

Besides that, \( C_i \in L^\infty(\mathbb{R}) \), for each \( i \in \{1, 2\} \).

Theorem 2 can naturally be applied for the Problem \((P)\). The next lemma will be necessary to obtain the asymptotic continuity, i.e., to be possible apply Theorem 3.

**Lemma 1**  The Selected Pullback Attractor \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) of the Problem \((P)\) is forward compact.

**Proof**  Due to ensure the existence of \( D \)-pullback attractor of the work \([10]\) the authors have showed the existence of a \( D \)-pullback absorbing set, given by

\[
B = \{ B(t) \}_{t \in \mathbb{R}} := \left\{ B_{\mathbb{V}^p} (0, R(t)) \cap K(t) \right\}_{t \in \mathbb{R}}, \tag{16}
\]

where \( \mathbb{V}^p \) is a Banach space compactly embedding in \( \mathbb{X}^2 \). What matters to us is that, assuming the Assumption B, we can write briefly
\[
R(t) := \left[ C_0 + C \left( e^{-\theta(t+1)} \int_{-\infty}^{t+1} e^{\theta s} (\|g_1(s)\|_{2r-2,\Omega}^2 + \|g_2(s)\|_{2r-2,\Gamma}^2) ds \right.ight.
\]
\[
\left. + e^{-\theta t} \int_{-\infty}^{t+1} e^{\theta s} (\|g_1(s)\|_{2\Omega}^2 + \|g_2(s)\|_{2\Gamma}^2) ds \right]^{1/\theta}.
\]

From Assumption A, there is \( N \in \mathbb{N} \) such that
\[
\int_{N}^{\infty} \|g_1(s)\|_{2r-2,\Omega}^2 + \|g_2(s)\|_{2r-2,\Gamma}^2 ds < 1.
\]

Note that,
\[
\sup_{t \geq N} \int_{-\infty}^{t+1} e^{\theta(s-(t+1))} (\|g_1(s)\|_{2r-2,\Omega}^2 + \|g_2(s)\|_{2r-2,\Gamma}^2) ds
\]
\[
\leq \sup_{t \geq N} \int_{-\infty}^{t+1} e^{\theta s} (\|g_1(s)\|_{2r-2,\Omega}^2 + \|g_1(s)\|_{2r-2,\Omega}^2 + \|g_2(s)\|_{2r-2,\Gamma}^2) ds
\]
\[
+ \|\tilde{g}_1\|_{2r-2,\Omega}^2 + \|\tilde{g}_2\|_{2r-2,\Gamma}^2
\]
\[
\leq \sup_{t \geq N} \left( \int_{-\infty}^{N} e^{\theta s} (\|g_1(s)\|_{2r-2,\Omega}^2 + \|g_1(s)\|_{2r-2,\Omega}^2) ds + \int_{N}^{t+1} \|g_1(s)\|_{2r-2,\Omega}^2 ds \right.
\]
\[
+ \int_{-\infty}^{N} e^{\theta s} (\|g_2(s)\|_{2r-2,\Gamma}^2 + \|g_2(s)\|_{2r-2,\Gamma}^2) ds + \int_{N}^{t+1} \|g_2(s)\|_{2r-2,\Gamma}^2 ds
\]
\[
+ \|\tilde{g}_1\|_{2r-2,\Omega}^2 + \|\tilde{g}_2\|_{2r-2,\Gamma}^2
\]
\[
\leq \int_{-\infty}^{N} e^{\theta s} (\|g_1(s)\|_{2r-2,\Omega}^2 + \|g_2(s)\|_{2r-2,\Omega}^2) ds + 1
\]
\[
+ \|\tilde{g}_1\|_{2r-2,\Omega}^2 + \|\tilde{g}_2\|_{2r-2,\Gamma}^2
\]
from the hypotheses for existence of \( \mathcal{D} \)-pullback attractor, see section 5 of [10], this last line of the inequality above is bounded.

From Lemma 2.1 of [10], we get
composed with all solutions of the Problem (\( A \)).

In the proof of the previous Lemma the \( \mathcal{D} \)-pullback absorbing set \( B \) absorbs only the solutions that coming from Faedo–Galerking method, because of this it ensures just the existence of selected pullback attractor when we consider the generalized process \( G \) composed with all solutions of the Problem (\( P \)).

As an immediate consequence of Theorem 3 we obtain

\[
\lim_{t \to -\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}(\infty)) = 0,
\]

where \( \mathcal{A}(\infty) := \cap_{t \in \mathbb{R}} \overline{\mathcal{A}(t)} \).

### 4.1 The asymptotically autonomous case

An autonomous version of the Problem (\( P \)) were considered in [4, 5], where they considered the following problem

\[
\begin{align*}
&u_t - A_p u + \hat{f}_1(u) = \hat{g}_1(x), & (t, x) \in (0, +\infty) \times \Omega, \\
&u_t + |\nabla u|^{p-2} \nabla \cdot u + \hat{f}_2(u) = \hat{g}_2(x), & (t, x) \in (0, +\infty) \times \Gamma, \\
&u(0) = u_0,
\end{align*}
\]

with similar assumptions assumed in [10], but they assumed an additional assumption that the derivatives of the functions \( \hat{f}_i \) are bounded. This additional assumption ensure the uniqueness of solution. The authors of these works have studied the forward asymptotic behavior of solutions, therefore this problem posses a global attractor \( A_{\infty} \) in \( \mathbb{R}^2 \).

Let us now suppose one more condition:
Assumption C. For each $\tau \in \mathbb{R}$ there exists a function $\alpha_\tau : [0, +\infty) \to [0, +\infty)$ such that $\alpha_\tau(t) \to 0$ as $t \to +\infty$ for each $t \in [0, +\infty)$ and
\[
\langle f_1((t + \tau), u(t + \tau)) - \hat{f}_1(v(t)), u(t + \tau) - v(t) \rangle_{L^2(\Omega)} \geq -\alpha_\tau(t)
\]
and
\[
\langle f_2((t + \tau), \gamma(u)(t + \tau)) - \hat{f}_2(\gamma(v)(t)), \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)} \geq -\alpha_\tau(t)
\]
for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and any solutions $U := (u, \gamma(u))$ of the Problem $(P)$ and $V := (v, \gamma(v))$ of the Problem $(Pa)$.

Lemma 2. Suppose that assumptions A and C are satisfied. Then a solution $U = (u, \gamma(u))$ of the Problem $(P)$ with initial condition in $U_\tau \in \mathbb{X}^2$ converges to the solution $V = (v, \gamma(v))$ of Problem $(Pa)$ with initial condition in $V_0 \in \mathbb{X}^2$, in the following sense:
\[
\|U(\tau + t, \tau, U_\tau) - V(t, V_0)\|_{\mathbb{X}^2} \to 0 \text{ as } \tau \to +\infty \text{ for each } t \geq 0,
\]
whenever $\|U_\tau - V_0\|_{\mathbb{X}^2} \to 0$ as $\tau \to +\infty$.

Before the proof, note that in this Lemma $U(\tau + t, \tau, U_\tau)$ represents any solution of the Problem $(P)$, even if this solution it is not coming from a Faedo-Galerkin method. Let us go to the proof of the previous Lemma.

Proof. Let $U$ a solution of the Problem $(P)$ and $V$ a solution of the Problem $(Pa)$. From the definition of weak solution of the Problem $(P)$ multiplying by $U(t + \tau) - V(t)$, we have
\[
\langle \partial_t U(t + \tau), U - V \rangle_{\mathbb{X}^2} + \langle \nabla u |^{p-2} \nabla u, \nabla u - v \rangle_{L^2(\Omega)}
\] + \langle f_1(t + \tau, u), u(t + \tau) - v(t) \rangle_{L^2(\Omega)}
\] + \langle f_2(t + \tau, v), \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)}
\]
= \langle g_1(t + \tau), u(t + \tau) - v(t) \rangle_{L^2(\Omega)} + \langle g_2(t + \tau), \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)}
\]
and for the Problem $(Pa)$ with solution $V$, we get
\[
\langle \partial_t V(t), U(t + \tau) - V(t) \rangle_{\mathbb{X}^2} + \langle \nabla v |^{p-2} \nabla v, \nabla u - v \rangle_{L^2(\Omega)}
\] + \langle \hat{f}_1(v(t)), u(t + \tau) - v(t) \rangle_{L^2(\Omega)} + \langle \hat{f}_2(v(t)), \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)}
\]
= \langle \hat{g}_1, u(t + \tau) - v(t) \rangle_{L^2(\Omega)} + \langle \hat{g}_2, \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)}.
\]
Subtracting (18) of (19),
\[ \frac{d}{dt} \| U(t + \tau) - V(t) \|_{X^2}^2 + \langle |\nabla u|^p - 2 \nabla u - |\nabla v|^p - 2 \nabla v, \nabla u - v \rangle_{L^2(\Omega)} \\
+ \langle f_1(t + \tau, u(t + \tau)) - f_1(v(t)), u(t + \tau) - v(t) \rangle_{L^2(\Omega)} \\
+ \langle f_2((t + \tau), u(t + \tau)) - f_2(v(t)), \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)} \\
= \langle g_1(t + \tau) - \tilde{g}_1, u(t + \tau) - v(t) \rangle_{L^2(\Omega)} \\
+ \langle g_2(t + \tau) - \tilde{g}_2, \gamma(u)(t + \tau) - \gamma(v)(t) \rangle_{L^2(\Gamma)}. \]

From Tartar inequality, assumption C and Young’s Inequality

\[ \frac{d}{dt} \| U(t + \tau) - V(t) \|_{X^2}^2 \leq 4\alpha_\tau(t) + c\| U(t + \tau) - V(t) \|_{X^2}^2 \\
+ \| g_1(t + \tau) - \tilde{g}_1 \|_{2, \Omega}^2 + \| g_2(t + \tau) - \tilde{g}_2 \|_{2, \Gamma}^2. \]

The Gronwall Lemma gives

\[
\| U(t + \tau) - V(t) \|_{X^2} \leq e^{\int_{\tau}^{t}} \| U_\tau - V_0 \|_{X^2} \\
+ \int_{\tau}^{t} e^{(t-s)}(4\alpha_\tau(s) + \| g_1(s + \tau) - \tilde{g}_1 \|_{2, \Omega}^2 + \| g_2(s + \tau) - \tilde{g}_2 \|_{2, \Gamma}^2) ds \\
\leq e^{\int_{\tau}^{t}} \left( \| U_\tau - V_0 \|_{X^2} + \frac{4t}{c} \text{esssup}_{s\in[0, +\infty)} \alpha_\tau(s) \\
+ \int_{\tau}^{+\infty} e^{-cs}(\| g_1(s + \tau) - \tilde{g}_1 \|_{2, \Omega}^2 + \| g_2(s + \tau) - \tilde{g}_2 \|_{2, \Gamma}^2) ds \right) \\
\]

and, from Lemma 2.1 of [10], we have

\[
\int_{0}^{+\infty} e^{-cs}(\| g_1(s + \tau) - \tilde{g}_1 \|_{2, \Omega}^2 + \| g_2(s + \tau) - \tilde{g}_2 \|_{2, \Gamma}^2) ds \\
\leq \int_{\tau}^{+\infty} (\| g_1(s) - \tilde{g}_1 \|_{2, \Omega}^2 + \| g_2(s) - \tilde{g}_2 \|_{2, \Gamma}^2) ds \\
\leq \int_{\tau}^{+\infty} (\| g_1(s) - \tilde{g}_1 \|_{2, \Omega}^{2r - 2} + \| g_2(s) - \tilde{g}_2 \|_{2, \Gamma}^{2r - 2} + K ds \\
\]

and

\[
\lim_{\tau \to +\infty} \int_{\tau}^{+\infty} K ds = \lim_{\tau \to +\infty} \left( \lim_{a \to -\infty} Ka - K \tau \right) = \lim_{a \to -\infty} Ka - \lim_{\tau \to -\infty} K \tau = \lim_{a \to -\infty} Ka - Ka = 0. \]

Therefore, when \( \tau \to +\infty \) we have

\[ \| U_\tau - V_0 \|_{X^2} \to 0 \quad \text{and} \quad \frac{4t}{c} \text{esssup}_{s\in[0, +\infty)} \alpha_\tau(s) \to 0, \]

and, from assumption A, we conclude that \( U(t + \tau) \) converges to \( V(t) \) in \( X^2 \) when \( \tau \to \infty \) for each \( t \in \mathbb{R}^+ \).
From Lemma 2 we have the generalized process $\mathcal{G}$ is asymptotically autonomous for $\mathcal{A}$, and from Lemma 1 we conclude that $\mathcal{A}$ is forward compact. Therefore the conditions of Theorem 5 are satisfied, and then we can ensure the following result.

**Theorem 9** Suppose the Assumptions A, B and C are satisfied. Then

$$\lim_{t \to \infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0.$$ 

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