**Gauge anomaly with vector and axial-vector fields in six dimensional curved space**

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**Abstract.** Imposing the conservation equation of the vector current for a fermion of spin $\frac{1}{2}$ at the quantum level, a gauge anomaly for the fermion coupling with non-Abelian vector and axial-vector fields in six-dimensional curved space is expressed in tensorial form. The anomaly consists of terms that resemble the chiral U(1) anomaly and the commutator terms that disappear if the axial-vector field is Abelian.

1. Introduction

The gauge anomaly breaks down the unitarity of the quantum field theory, and then one cannot calculate higher-order quantum corrections in a consistent manner. The cancellation of the anomaly is a stringent condition on the fermion multiplets allowed in the consistent model. It is meaningful to clarify the form of the anomaly in general non-Abelian gauge theory.

The gauge anomaly arises from the breaking of a certain local gauge symmetry. The concrete form of the anomaly is obtained by calculating some one-loop diagrams of fermions in four dimensions [1, 2, 3], and is also derived from the chiral gauge transformation of the path integral measure for fermions interacting with boson fields [4, 5, 6, 7]. The formal expression of the anomaly in Gaussian cut-off regularization is described by the heat kernel [8, 9].

We consider the action $\tilde{S}$ in the model in which the fermion $\psi$ of spin $\frac{1}{2}$ interacts with the (polar-)vector field $\tilde{V}_\mu$ and the axial-vector field $\tilde{A}_\mu$ in even $2n$-dimensional curved space,

$$\tilde{S} = \int d^{2n}x \, h \bar{\psi} i \gamma^\mu (\nabla_\mu - i \tilde{V}_\mu - i \gamma_{2n+1} \tilde{A}_\mu) \psi,$$

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega^{kl}_\mu \gamma_{kl} \psi, \quad \tilde{V}_\mu = \tilde{V}_\mu^a T^a, \quad \tilde{A}_\mu = \tilde{A}_\mu^a T^a,$$

$$\gamma_{k_1\cdots k_j} = \gamma[k_1 \cdots \gamma_{k_j}], \quad \gamma_{2n+1} = i^n \gamma^1 \gamma^2 \cdots \gamma^{2n},$$

(1)

where $\omega^{kl}_\mu$ is the spin connection, and $h = \det h^k_\mu$, in which $h^k_\mu$ is a vielbein in curved space. Note that the Euclidean metric tensor is $g_{\mu\nu} = h^k_\mu h^{l}_\nu \eta_{kl}$ with $\eta_{kl} = -\delta_{kl}$ in flat space.
tangent space. Moreover, $T^a$ denotes the Hermitian representation matrix of the Lie algebra of a non-Abelian gauge group. Both components $\tilde{V}_\mu^a$ and $\tilde{A}_\mu^a$ of the boson fields are real. The action $\tilde{S}$ is invariant under the infinitesimal local gauge transformation,

$$\delta \psi(x) = (i\alpha(x) + i\beta(x)\gamma_{2n+1})\psi(x), \quad \delta \tilde{\psi}(x) = \tilde{\psi}(x)(-i\alpha(x) + i\beta(x)\gamma_{2n+1}),$$

$$\delta \tilde{V}_\mu(x) = \partial_\mu \alpha(x) + i[\alpha(x), \tilde{V}_\mu(x)] + i[\beta(x), \tilde{A}_\mu(x)],$$

$$\delta \tilde{A}_\mu(x) = \partial_\mu \beta(x) + i[\beta(x), \tilde{V}_\mu(x)] + i[\alpha(x), \tilde{A}_\mu(x)],$$

where $\alpha(x) = \alpha^a(x)T^a$ and $\beta(x) = \beta^a(x)T^a$, in which $\alpha^a(x)$ and $\beta^a(x)$ are real parameters. The Dirac operator $\gamma^\mu(\nabla_\mu - i\tilde{V}_\mu + \gamma_{2n+1}A_\mu)$ in the action $\tilde{S}$ is not Hermitian. If one rotates the axial-vector $\tilde{A}_\mu$ to an imaginary field $iA_\mu$ in which $A_\mu^a$ is real, then the Dirac operator $\mathcal{D} \equiv \gamma^\mu(\nabla_\mu - i\tilde{V}_\mu + \gamma_{2n+1}A_\mu)$ becomes Hermitian. However, the rotation of $\tilde{A}_\mu$ spoils the axial-part of the gauge transformation, and the gauge symmetry for the axial-vector gauge field parametrized by $\beta\gamma_{2n+1}$ breaks in the path integral. For simplicity, rewriting the vector field as $-i\tilde{V}_\mu^a \equiv V_\mu^a$, which is purely imaginary, the action is replaced by

$$S = \int d^{2n}x \ h \bar{\psi} i\mathcal{D} \psi, \quad \mathcal{D} = \gamma^\mu \nabla_\mu + Y \equiv \gamma^\mu D_\mu, \quad Y = \gamma^\mu V_\mu + \gamma_{2n+1}\gamma^\mu A_\mu, \quad V_\mu = V_\mu^a T^a, \quad A_\mu = A_\mu^a T^a.$$

In supergravity coupled with super Yang-Mills theory [10, 11], the Lagrangian contains four-fermion interactions, which are regarded as some two-fermion interactions with bosonic background fields expressed by odd-order tensors. The completely antisymmetric part of the highest order tensor should be rewritten as an axial-vector by contracting its tensor with the Levi-Civita symbol. The vector and the axial-vector parts in the two-fermion interactions can be absorbed in the vector and the axial-vector gauge fields. (Other order tensors may be treated in future work.) The concrete form of the gauge anomaly in the model may be directly calculated by using the heat kernel.

It is shown in Sect. 2 that the gauge anomaly stems from the Jacobian for the functional measure of the fermion in the path integral under chiral transformation. The heat kernel is introduced in Sect. 3, in order to give the explicit form of the anomaly in four and six dimensions in Sects. 4 and 5, respectively. Section 6 is devoted to the discussion.

2. Gauge anomaly

The path integral, in which $V_\mu$ and $A_\mu$ are regarded as background fields, is given by

$$W(V_\mu, A_\mu) = \ln \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi \exp S(V_\mu, A_\mu, \tilde{\psi}, \psi).$$

The Ward–Takahashi identity in the path integral can be expressed in the following form, due to the replacement of the fermion corresponding to the infinitesimal transformation of the fermion in (2); $\psi' = (1 + i\alpha + i\beta\gamma_{2n+1})\psi$, $\tilde{\psi}' = \tilde{\psi}(1 - i\alpha + i\beta\gamma_{2n+1}),$

$$0 = \ln \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi' \exp S(V_\mu, A_\mu, \tilde{\psi}', \psi') - \ln \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp S(V_\mu, A_\mu, \bar{\psi}, \psi)$$

$$= \ln \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi \left[ \ln J_{\tilde{\psi}} + \ln J_\psi - \int d^{2n}x \ h \bar{\psi} \gamma^\mu D_\mu(\alpha(x) + \beta(x)\gamma_{2n+1}) \psi \right] e^S,$$
where $J_\psi$ and $J_{\bar{\psi}}$ are the Jacobians for the transformation of $D\psi$ and $\bar{D}\bar{\psi}$. To analyze the Jacobians, we use the complete set of eigenfunctions $\{\varphi_n\}$ of the Hermitian $\bar{D}$:

$$\bar{D}\varphi_n(x) = \lambda_n \varphi_n(x), \quad \int d^2x \ h(x) \varphi^\dagger_m(x) \varphi_n(x) = \delta_{mn}. \quad (6)$$

Expanding the fields $\bar{\psi}$ and $\psi$ by $\{\varphi_n\}$ as

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \bar{\psi} = \sum_n \bar{b}_n \varphi^\dagger_n(x), \quad (7)$$

with Grassmann number coefficients $a_n$ and $\bar{b}_n$, we define the path integral measures by

$$D\bar{\psi}D\psi = \prod_n d\bar{b}_n \, da_n. \quad (8)$$

Then the Jacobian factors $\ln J_\psi + \ln J_{\bar{\psi}}$ in (5) are expressed as

$$\ln J_\psi + \ln J_{\bar{\psi}} = -2i \sum_n \int d^2x \ \beta^a \varphi^\dagger_n(x) T^a \gamma_{2n+1} \varphi_n(x). \quad (9)$$

The relation (5) is rewritten by separating terms containing the real parameters $\alpha^a$ and $\beta^a:

$$D_\mu \langle \bar{\psi} T^a \gamma^\mu \psi \rangle(x) = 0, \quad D_\mu \langle \bar{\psi} T^a \gamma_{2n+1} \gamma^\mu \psi \rangle(x) = G^{(2n)a}(x),$$

$$G^{(2n)a}(x) = -2i \sum_n \varphi^\dagger_n(x) T^a \gamma_{2n+1} \varphi_n(x). \quad (10)$$

Note that the gauge anomaly $G^{(2n)a}$ does not break the conservation law of the vector current, which is related to Noether’s theorem with respect to the gauge transformation parametrized by $\alpha(x)$.

### 3. Heat kernel

Since the expression (10) of the anomaly is ill-defined, it should be regularized in order to calculate concretely in the tensorial form. For this purpose, the Gaussian cut-off regularization is adopted [12]:

$$G^{(2n)a}(x) = -2i \lim_{t \to 0} \sum_n \varphi^\dagger_n(x) T^a \gamma_{2n+1} e^{-t \bar{D}^2} \varphi_n(x)$$

$$= -2i \lim_{t \to 0} \lim_{x' \to x} \Tr \left( T^a \gamma_{2n+1} e^{-t \bar{D}^2} [h(x)]^{-\frac{1}{2}} [h(x')]^{-\frac{1}{2}} \delta^{(2n)}(x, x') \right)$$

$$\equiv -2i \lim_{t \to 0} \lim_{x' \to x} \Tr \left( T^a \gamma_{2n+1} K^{(2n)}(x, x'; t) \right), \quad (11)$$

where $\Tr$ denotes the trace over both spinor indices of $\gamma$-matrices and internal indices of $T^a$. The heat kernel $K^{(2n)}(x, x'; t)$ introduced in (11) satisfies the heat equation and the boundary condition with respect to $t$ [8] [9]:

$$\frac{\partial}{\partial t} K^{(2n)}(x, x'; t) = -\bar{D}^2 K^{(2n)}(x, x'; t), \quad (12)$$

$$K^{(2n)}(x, x'; 0) = 1 [h(x)]^{-\frac{1}{2}} [h(x')]^{-\frac{1}{2}} \delta^{(2n)}(x, x'), \quad (13)$$
where $1$ is a unit matrix not only acting on the spinor but also acting on $T^a$, and $\delta^{(d)}(x, x')$ means the invariant $\delta$ function. Moreover, the square of $\mathcal{D}$ for $\psi$ is rewritten as

$$
\mathcal{D}^2 = D_\mu D^\mu + 2Q^\mu D_\mu + Z = \tilde{D}_\mu \tilde{D}^\mu + X,
$$

$$
Q_\mu = \frac{1}{2} \{ \gamma_\mu, Y \}, \quad Z = \frac{1}{2} \gamma^{\mu\nu} [\nabla_\mu, \nabla_\nu] + \gamma_\mu \nabla_\mu Y + Y^2,
$$

$$
\tilde{D}_\mu = \nabla_\mu + Q_\mu, \quad X = Z - \nabla_\mu Q^\mu - Q_\mu Q^\mu, \quad [\tilde{D}_\mu, \tilde{D}_\nu] \psi = \Lambda_{\mu\nu} \psi. \quad (14)
$$

The matrix-valued quantities $Q_\mu$, $X$, and $\Lambda_{\mu\nu}$ are expressed from (14) in the following form:

$$
Q_\mu = V_\mu - \gamma_{2n+1} \gamma_\mu A^\rho, \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu],
$$

$$
X = -\frac{1}{4} R + 2(n-1) A_\mu A^\mu - \gamma_{2n+1} A_\mu A^\mu + \gamma_{\mu\nu} \left( \frac{1}{4} F_{\mu\nu} + \frac{2n-3}{2} [A_\mu, A_\nu] \right),
$$

$$
\Lambda_{\mu\nu} = \frac{1}{4} \gamma^{\rho\sigma} R_{\rho\sigma\mu\nu} + F_{\mu\nu} - [A_\mu, A_\nu] - 2 \gamma_{\mu\nu} A_\rho A^\rho + 2 \gamma_{[\mu|\nu} \{ A_{|\nu]}, A_\rho \}
$$

$$
+ 2 \gamma_{2n+1} \gamma_{[\mu|\nu} A_{|\nu]} - 2 \gamma_{\mu\nu\rho\sigma} A^\rho A^\sigma,
$$

(15)

where $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor, and the semi-colon ‘;’ means the covariant differentiation preserving the vector gauge and the gravitational symmetries (e.g. $A^\rho;_\nu = \nabla_\rho A^\rho + [V_\rho, A^\rho]$). The totally antisymmetric product $\gamma_{\mu\nu\rho\sigma}$ in the last term of $\Lambda_{\mu\nu}$ is expressed as $-\epsilon_{\mu\nu\rho\sigma\gamma} \gamma_\gamma$ and $-\frac{1}{2} \epsilon_{\mu\nu\rho\sigma\kappa\lambda} \gamma_\gamma \gamma^{\kappa\lambda}$ in four and six dimensions, respectively.

It is difficult to solve the equation (12) of the heat kernel for the fermion interacting with general background fields. In order to perform the concrete calculation, the following ansatz by DeWitt is applied to the heat kernel $\mathcal{D}$, which satisfies the condition (13):

$$
K^{(2n)}(x, x'); t) \sim \frac{\Delta^{1/2}(x, x')}{(4\pi t)^n} \exp \left( \frac{\sigma(x, x')}{2t} \right) \sum_{q=0}^{\infty} a_q(x, x') t^q,
$$

(16)

where $\sigma(x, x')$ is half of the square of the geodesic distance between $x$ and $x'$, $\Delta(x, x') = |h(x)|^{-1} |h(x')|^{-1} \det \sigma_{\mu\nu}$, and $a_q(x, x')$ stand for bispinors. The coincidence limit of $a_n$ appears in the formal expression of the anomaly, and is defined by $\lim_{x' \rightarrow x} a_n(x, x') \equiv [a_n](x)$. In particular, $[\sigma] = [\sigma, \mu] = 0, [\sigma, \mu] = g_{\mu\nu}, [\Delta] = 1$, and $[a_0] = 1$.

4. Gauge anomaly in four dimensions

Substituting the ansatz (16) into (11), the gauge anomaly in $2n$ dimensions is derived from $[a_n]$, (13)

$$
G^{(2n)a}(x) = \frac{-2i}{(4\pi)^n} \text{Tr} \left\{ T^a \gamma_{2n+1} [a_n] \right\} (x).
$$

(17)
If only $T^a$ written visibly in (17) is a unit matrix, then $G^{(2n)a}$ becomes the chiral U(1) anomaly. The concrete form of $[a_2]$ and $[a_3]$ is given as follows:

$$[a_2] = \frac{1}{12} \Lambda_{\mu\nu} \Lambda^{\mu\nu} + \frac{1}{6} X_{\mu} + \frac{1}{2} \left( \frac{1}{6} R + X \right)^2 + \cdots,$$

$$[a_3] = \frac{1}{60} \left( -\frac{1}{3} (X_{\mu} X_{\nu} + X_{\alpha\beta} X_{\gamma\delta}) - \frac{1}{3} \Lambda_{\mu\nu} \Lambda^{\mu\nu} \right)$$

$$- 2 \left\{ \Lambda^{\mu\nu}, \Lambda_{\rho\sigma} \right\} - \frac{10}{3} R^{\mu\nu} \Lambda_{\rho\sigma} R_{\mu\sigma} + \frac{6}{3} \Lambda_{\mu\nu} \Lambda_{\rho\sigma} R_{\mu\sigma}$$

$$+ \frac{1}{12} \left\{ \frac{1}{6} R + X, -\frac{1}{2} \Lambda_{\mu\nu} \Lambda_{\rho\sigma} \right\} - \frac{1}{30} R_{\mu\nu} R_{\rho\sigma} - \frac{1}{30} R_{\mu\sigma} R_{\nu\rho} \right\}$$

$$- \frac{1}{36} [X_{\mu}, \Lambda_{\mu\nu}] - \frac{1}{12} \left\{ \frac{1}{6} R + X \right\} - \frac{1}{12} \left( \frac{1}{6} R + X \right)^3 + \cdots,$$

(18)

where the exclamation mark ‘!’ means a new covariant differentiation $\tilde{D}_\mu$ introduced in (14), and terms without matrices are omitted from (18) and (19). The tensorial form of the gauge anomaly in four dimensions is written as

$$G^{(4)a} = \frac{-2i}{(4\pi)^2} tr T^a \left\{ -\frac{1}{48} R_{\alpha\beta\mu\nu} R_{\gamma\delta}^{\mu\nu} - \frac{1}{2} A_{\alpha\beta} A_{\gamma\delta} - \frac{1}{6} A_{\alpha\beta} A_{\gamma\delta} \right\}$$

$$+ \frac{4}{3} \left\{ V_{\alpha\beta} A_{\gamma} A_{\delta} + A_{\alpha} V_{\beta\gamma} A_{\delta} + A_{\alpha} A_{\beta} V_{\gamma\delta} \right\} - \frac{16}{3} A_{\alpha} A_{\beta} A_{\gamma} A_{\delta}$$

$$- \frac{2}{3} A_{\mu} A_{\nu} + \frac{1}{3} R A_{\mu} - \frac{2}{3} R A_{\mu} + \frac{4}{3} [A_{\mu}, V_{\nu}] + \frac{1}{3} [A_{\mu}, V_{\nu}]$$

$$- \frac{4}{3} \left\{ \left\{ A_{\mu}, A_{\nu} \right\}, A_{\mu\nu} \right\} + \frac{2}{3} \left\{ A_{\mu} A_{\nu}, A_{\mu\nu} \right\} - 4 A_{\mu} A_{\mu\nu} A_{\nu}$$

$$= \frac{-2i}{(4\pi)^2} tr T^a \left\{ -\frac{1}{48} R_{\alpha\beta\mu\nu} R_{\gamma\delta}^{\mu\nu} - \frac{1}{2} A_{\alpha\beta} A_{\gamma\delta} \right\}$$

$$+ \frac{3}{3} R A_{\delta} - 2 [A_{\mu}, F_{\alpha\beta}] - \frac{1}{3} \left\{ \left\{ A_{\alpha\beta}, A_{\gamma} \right\}, A_{\delta} \right\} - \frac{4}{3} \left\{ \left\{ A_{\mu}, A_{\nu} \right\}, A_{\delta} \right\}$$

$$+ \frac{1}{2} \left\{ \left\{ F_{\alpha\beta}, A_{\gamma} \right\}, A_{\delta} \right\} + \frac{2}{3} [A_{\mu}, [A_{\mu}, [A_{\mu}, A_{\nu}]]] - \frac{2}{3} [A_{\mu\nu}, [A_{\mu}, A_{\nu}]]$$

$$+ \frac{1}{3} \epsilon^{\alpha\beta\gamma\delta} \left\{ A_{\alpha}, A_{\beta} A_{\gamma}, A_{\delta} \right\},$$

(20)

where tr runs only over the internal indices of $T^a$, and

$$V_{\mu\nu} \equiv \partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu} + [V_{\mu}, V_{\nu}] + [A_{\mu}, A_{\nu}],$$

$$A_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} [V_{\mu}, A_{\nu}] - [V_{\nu}, A_{\mu}] = A_{\nu\mu} - A_{\mu\nu}.$$

The expression (20) of the gauge anomaly is well known [3, 17, 18, 19, 20, 21]. The leading terms without $A_{\mu}$ in the anomaly are represented by $\epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\mu\nu} R_{\gamma\delta}^{\mu\nu}$ and $\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$, which are shown in the chiral U(1) anomaly. The terms containing $A_{\mu}$ in (21) consist of the total derivative terms and the commutator terms of $F_{\alpha\beta}$, $A_{\mu}$, and their derivatives.

\[ \dagger \] The matrix $T^a$ noted here does not mean that contained in $[a_n]$.

\[ \S \] In $[a_3]$, some terms should be described by a commutator and some anticommutators, because $A_{\mu}$ and $F_{\mu\nu}$ in $X$, $\Lambda_{\mu\nu}$ and their derivatives do not commute with $T^a$ in (17). If $X$ and $\Lambda_{\mu\nu}$ commute with $T^a$, then a commutator disappears, and anticommutators of two quantities double the product of them, in the trace formula.
5. Gauge anomaly in six dimensions

Since the chiral U(1) anomaly in the vector and axial-vector model in six dimensions has been derived [22], we can present the leading and the total derivative terms of the gauge anomaly:

\[
G^{(6)\alpha} = -\frac{2i}{8\pi^3} \text{tr} T^\alpha \left[ -\frac{i}{8} \epsilon^{\alpha\beta\gamma\delta\kappa\lambda} \left( \frac{1}{48} R_{\alpha\beta\rho\sigma} R_{\gamma\delta\rho\sigma} + \frac{1}{6} F_{\alpha\beta} F_{\gamma\delta} \right) F_{\kappa\lambda} 
+ \left( \frac{1}{180} (A^\mu_{\mu\nu} \nu^\rho + A^\mu_{\mu\nu} \nu^\rho \nu^\rho + A^\mu_{\mu\nu} \nu^\rho \nu^\rho) - \frac{1}{120} R A^\mu_{\mu\nu} \nu^\rho - \frac{1}{90} R^{\rho\mu} A_{\mu\nu} \nu^\rho 
+ \frac{2}{45} R_{\mu\nu} A^\mu_{\mu\nu} \nu^\rho - \frac{1}{36} R_{\mu\nu} A_{\mu\nu} \nu^\rho \nu^\rho + \frac{1}{120} (-R_{\mu\nu} A_{\mu\nu} \nu^\rho + R_{\mu\nu} A_{\mu\nu} \nu^\rho + R_{\mu\nu} A_{\mu\nu} \nu^\rho) 
+ \frac{1}{30} (-R_{\mu\nu} A_{\mu\nu} \nu^\rho + R_{\mu\nu} A_{\mu\nu} \nu^\rho) A_{\nu\mu} + \frac{1}{288} R^2 A^\mu - \frac{1}{72} R R_{\mu\nu} A_{\mu\nu} + \frac{1}{90} R_{\mu\nu} R_{\mu\nu} A^\rho 
+ \frac{1}{36} R_{\mu\nu} R_{\mu\nu} A_{\lambda} - \frac{1}{180} R_{\mu\nu} R_{\mu\nu} A_{\lambda} - \frac{7}{1440} R_{\mu\nu} R_{\mu\nu} A_{\lambda} 
- \frac{1}{24} \{ F_{\mu\nu}, \{ F_{\mu\nu}, A^\rho \} \} + \frac{1}{12} \{ F_{\mu\nu}, \{ F_{\mu\nu}, A^\rho \} \} \} - \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta\kappa\lambda} \{ F_{\alpha\beta}, A_{\gamma\delta} A_{\kappa} \} 
+ \frac{1}{30} \{ A_{\mu} A^\mu, A_{\nu} \} \} + \frac{2}{3} \{ A_{\mu} A^\mu, A_{\nu} \} \} - \frac{19}{30} \{ A_{\mu} A^\mu, A_{\nu} \} \} - \frac{1}{20} \{ A^\rho, A^\mu, A_{\nu} \} \} + \frac{3}{20} \{ A^\rho, A^\mu, A_{\nu} \} \} + \frac{1}{60} \{ A^\rho, A^\mu, A_{\nu} \} \} - \frac{11}{60} \{ A^\rho, A^\mu, A_{\nu} \} \} 
+ \frac{1}{60} \{ A_{\mu\nu}, A_{\mu\nu}, A^\rho \} \} + \frac{2}{5} \{ A^\rho, A_{\mu\nu}, A^\rho \} \} 
- \frac{1}{20} \{ A_{\nu\mu}, A_{\nu\mu}, A^\rho \} \} + \frac{1}{30} \{ A_{\nu\mu}, A_{\nu\mu}, A^\rho \} \} \} 
+ \frac{1}{5} \{ A_{\mu\nu}, A_{\mu\nu}, A^\rho \} \} - \frac{1}{15} \{ A_{\mu\nu}, A_{\mu\nu}, A^\rho \} \} 
- \frac{3}{20} \{ F_{\mu\nu}, [A_{\mu} A^\mu, A^\rho] \} - \frac{1}{5} \{ F_{\mu\nu}, [A^\rho, A^\mu, A^\rho] \} \} - \frac{1}{15} \{ F_{\mu\nu}, [A^\rho, A^\mu, A^\rho] \} \} 
- \frac{1}{18} R A_{\mu} A^\mu A^\rho + \frac{32}{45} R_{\mu\nu} A_{\mu\nu} A^\rho - \frac{1}{15} R_{\mu\nu} A_{\mu\nu} A^\rho - \frac{1}{45} R_{\mu\nu} A_{\mu\nu} A^\rho 
- \frac{1}{20} \epsilon^{\alpha\beta\gamma\delta\kappa\lambda} A_{\alpha} A_{\beta} A_{\gamma} A_{\delta} A_{\kappa} + \frac{1}{5} A_{\mu} A^\mu A_{\nu} A_{\nu} A^\rho \] \right] + G_{\text{com}}^{(6)\alpha},
\]

where terms \(G_{\text{com}}^{(6)\alpha}\) containing the commutator are given in the following form:

\[
G_{\text{com}}^{(6)\alpha} = \frac{1}{8\pi^3} \text{tr} T^\alpha \left[ \frac{1}{12} \{ [F_{\mu\nu}, F_{\mu\nu}], A^\rho \} + \frac{1}{36} \{ F_{\mu\nu}, A_{\mu\nu}, A^\rho \} + \frac{1}{15} \{ A_{\mu\nu}, A_{\mu\nu}, A^\rho \} \right]
+ \frac{1}{30} \{ A_{\mu\nu}, A_{\mu\nu}, A^\rho \} \] \right] \]
$$\begin{align*}
&+ i \epsilon^{\alpha \beta \gamma \delta \kappa \lambda} \left( \frac{1}{96} R_{\alpha \beta \rho \sigma} R_{\delta \rho \sigma} \right) [A_\kappa, A_\lambda] + \frac{1}{48} \left[ F_{\alpha \beta}, F_{\gamma \delta} A_\kappa A_\lambda \right] \\
&- \frac{1}{144} \left[ F_{\alpha \beta} A_\kappa F_{\gamma \delta}, A_\lambda \right] - \frac{1}{72} \left[ F_{\alpha \beta} A_\kappa, A_\lambda F_{\gamma \delta} \right] \\
&- \frac{3}{20} [A^\alpha, [A_\mu, A_\rho \rho_{\mu \nu}]] - \frac{7}{45} [A^\mu, [A_\mu, A_\rho \rho_{\nu \mu}]] + \frac{1}{30} [A^\alpha, [A^\beta, A^\rho_{\mu \beta \alpha}]] \\
&- \frac{7}{90} [A^\alpha, [A^\beta, A_\alpha \beta \nu]] + \frac{11}{90} [A^\alpha, [A^\beta, A_\alpha \rho \beta]] - \frac{11}{270} [A^\alpha, [A^\beta, A^\rho_{\alpha \beta \rho}]] \\
&+ \frac{29}{270} [A^\alpha, [A^\beta, A^\rho_{\gamma \beta \rho}]] + \frac{7}{180} [[A^\alpha, A^\beta], A_{\alpha \beta \rho}] - \frac{11}{180} [[A^\alpha, A^\beta], A_{\alpha \beta \rho}] \\
&+ \frac{11}{540} [[A^\alpha, A^\beta], A^\rho_{\alpha \beta \rho}] - \frac{29}{540} [[A^\alpha, A^\beta], A^\rho_{\gamma \beta \rho}] \\
&+ \frac{1}{60} [A^\alpha \beta, [A_\beta \alpha, A^\rho_{\mu \beta}]] + \frac{41}{180} [A^\alpha \beta, [A^\rho_{\mu}, A_{\alpha \beta}]] + \frac{13}{135} [A_\rho, [A^\mu \rho, A^\nu_{\mu \nu}]] \\
&- \frac{2}{45} [A_\rho, [A^\rho_{\mu \nu}, A_{\mu \nu}]] - \frac{13}{18} [A_\rho, [A^\rho_{\mu \nu}, A_{\mu \nu}]] - \frac{4}{45} [A^\rho, [A^\rho_{\mu \nu}, A_{\mu \nu}]] \\
&- \frac{5}{108} [A^\rho, [A^\mu \rho, A_{\mu \nu}]] - \frac{1}{540} [A^\rho, [A^\mu \rho, A_{\mu \nu}]] - \frac{5}{36} [A^\rho, [A^\mu \rho, A_{\mu \nu}]] \\
&+ \frac{1}{135} [A^\rho, [A^\mu \nu, A_{\rho \nu \mu}]] + \frac{1}{9} [A^\rho, [A^\mu \nu, A_{\rho \nu \mu}]] + \frac{2}{45} [A^\rho, [A^\mu \nu, A_{\mu \nu}]] \\
&+ \frac{1}{108} [[A^\rho, A_{\mu \nu}], A_{\nu \rho \mu}] + \frac{11}{108} [[A^\rho, A_{\mu \nu}], A_{\nu \rho \mu}] + \frac{7}{36} [[A^\rho, A_{\mu \nu}], A_{\nu \rho \mu}] \\
&- \frac{1}{36} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] + \frac{13}{45} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] + \frac{2}{45} [[A_\rho, A_{\rho \nu \mu}], A_{\rho \nu \mu}] \\
&- \frac{7}{108} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] + \frac{1}{60} [[A_\rho, A_{\rho \nu \mu}], A_{\rho \nu \mu}] - \frac{11}{540} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] \\
&- \frac{1}{45} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] + \frac{1}{180} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] - \frac{1}{45} [[A_\rho, A_{\mu \nu}], A_{\rho \nu \mu}] \\
&+ \left( \frac{19}{90} [A^\alpha, [A_\alpha, A^\beta_{\beta \rho \beta}]] - \frac{7}{90} [A^\rho, [A^\alpha, A^\beta_{\alpha \beta}]] \right) + \frac{1}{30} \left( [A^\rho, [A^\alpha, A^\beta_{\beta \rho \beta}]] \right)
\end{align*}$$
\[
- \frac{25}{36} [A_{\alpha \beta}, A_\gamma A^\gamma] + \frac{7}{45} [A_\gamma, A_{\alpha \beta} A^\gamma] - \frac{7}{90} [A_\gamma, [A^\gamma, A_{\alpha \beta}]] \\
+ \frac{23}{180} [[A_{\alpha \gamma}, A^\gamma], A_\beta] - \frac{1}{145} [A_\gamma, A_\alpha A_\beta A^\gamma] + \frac{1}{36} [A_{\alpha \gamma}, [A_\beta, A^\gamma]] \\
+ \frac{7}{180} [A^\gamma, A_{\gamma \alpha} A_\beta] + \frac{1}{2} [A^\gamma, A_\alpha A_\beta A_\gamma] + \frac{7}{12} [A_{\gamma \alpha}, A_\beta A^\gamma] \\
- \frac{1}{90} [A_{\gamma \alpha}, A^\gamma A_\beta] + \frac{1}{5} [A_{\gamma \alpha} A^\gamma, A_\beta] - \frac{11}{180} [A^\gamma A_{\gamma \alpha}, A_\beta] \\
+ R^{\alpha \beta \gamma \delta} \left( - \frac{1}{270} [A_\alpha A_\beta, A_{\gamma \delta}, A_\epsilon A_\zeta] + \frac{3}{10} [A_\gamma, A_{\beta \delta} A_\alpha] - \frac{23}{180} [A_\gamma, A_\alpha A_{\beta \delta}] \\
- \frac{5}{36} [A_\alpha A_\gamma, A_{\beta \delta}] - \frac{11}{135} [A_{\beta \delta} A_\gamma, A_\alpha] - \frac{31}{540} [A_\gamma A_{\beta \delta}, A_\alpha] \right) \\
+ i \epsilon^{\alpha \beta \gamma \delta \epsilon \zeta} \left( - \frac{1}{10} [A_{\alpha A_{\beta \gamma}, \gamma \delta \epsilon A_\zeta}] + \frac{1}{12} [A_{\alpha \beta}, A_{\gamma \delta} A_\epsilon A_\zeta] - \frac{1}{15} [A_{\alpha \beta A_{\gamma \delta}, \gamma \delta A_\epsilon}] \right) \\
+ \left( \begin{array}{l}
- \frac{31}{60} [A^\lambda A^\gamma, A_\alpha A_\beta A_\delta] - \frac{19}{16} [A_\alpha A^\lambda A^\gamma, A_\beta A^\delta A_\alpha] + \frac{26}{15} [A_\alpha A^\alpha A^\lambda A^\gamma, A_\beta A^\beta] \\
- \frac{1}{4} [A_\alpha A_\beta A_\gamma A^\lambda A^\alpha] + \frac{5}{18} [A^\lambda A^\lambda A_\alpha, A_\beta A_\gamma A^\gamma] - \frac{13}{12} [A_\alpha A^\lambda A^\lambda A_\gamma, A_\beta A_\beta A^\beta] \\
+ \frac{58}{15} [A_\alpha A^\alpha A^\lambda A_\beta A_\gamma, A^\alpha A_\beta A_\gamma A^\lambda] - \frac{47}{30} [A_\alpha A^\alpha A^\beta A_\beta A_\gamma, A^\lambda A_\alpha A_\gamma] \\
+ \frac{7}{6} [A_\alpha A^\alpha A^\lambda A_\beta A_\gamma, A_\beta A_\gamma A^\beta] - \frac{53}{15} [A_\alpha A^\alpha A^\lambda A_\beta A_\gamma, A_\beta A_\gamma A^\beta] \\
+ \frac{21}{10} [A^\lambda A_\alpha A^\alpha A_\beta A_\gamma, A_\alpha A_\beta A^\alpha A_\gamma] - \frac{31}{30} [A_\alpha A^\lambda A_\beta A_\gamma A^\alpha, A_\alpha A_\beta A^\alpha A_\gamma] \\
+ \frac{5}{12} [A^\alpha A^\lambda A_\beta A_\gamma A_\alpha A_\beta, A_\alpha A_\beta A^\alpha A_\gamma] - \frac{31}{30} [A_\alpha A_\beta A_\gamma A^\lambda A_\alpha, A_\beta A_\gamma A^\lambda A_\alpha] \\
+ \frac{8}{15} [A^\alpha A^\lambda A_\beta A_\gamma A_\alpha A_\beta, A_\alpha A_\beta A^\alpha A_\gamma] - \frac{17}{10} [A_\alpha A^\alpha A^\lambda A_\beta A_\gamma A_\alpha, A_\beta A_\gamma A^\lambda A_\alpha] \\
+ \frac{101}{60} [A^\alpha A_\alpha A^\alpha A_\beta A_\gamma A_\beta, A_\alpha A^\beta A_\gamma A^\lambda A_\alpha] - \frac{97}{60} [A_\alpha A^\alpha A_\beta A_\gamma A_\beta, A_\gamma A^\lambda A_\alpha] \\
+ \frac{13}{12} [A_\alpha A_\alpha A^\lambda A_\beta A_\gamma, A_\gamma A_\alpha A^\beta A_\gamma] - \frac{13}{12} [A_\alpha A^\alpha A_\gamma A_\beta A^\beta A_\gamma] - \frac{1}{4} [A^\lambda A_\gamma A_\alpha A_\beta A^\alpha A^\gamma] \\
+ \frac{62}{45} [A_\alpha A^\alpha A^\lambda A_\beta A_\gamma A^\alpha A^\beta] + \frac{5}{2} [A_\alpha A_\beta A^\gamma A^\lambda A_\alpha A^\alpha A^\beta] + \frac{8}{3} [A_\alpha A_\beta A^\gamma A^\lambda A_\alpha A^\alpha A^\beta] \\
+ \frac{2}{9} [A^\lambda A_\alpha A_\beta A^\alpha A^\gamma A^\beta] + \frac{101}{90} [A_\alpha A_\gamma A^\lambda A_\beta A^\alpha A^\beta] - \frac{43}{15} [A_\alpha A_\gamma A_\beta A^\lambda A^\alpha A^\beta] \\
+ \frac{7}{4} [A_\alpha A_\beta A_\gamma A^\alpha A^\beta A_\lambda A^\lambda A_\alpha A^\alpha A^\beta] + \frac{7}{20} [A_\lambda, A_\alpha A_\beta A_\gamma A^\gamma A^\beta] + \frac{2}{3} [A_\alpha A_\beta A_\gamma A^\lambda A^\alpha A^\beta] \\
+ \frac{12}{5} [A_\alpha A_\beta A_\gamma A^\lambda A^\gamma A^\beta] + \frac{11}{10} [A_\alpha A_\beta A_\gamma A^\lambda A^\beta A_\gamma] + \frac{9}{20} [A_\lambda, A_\alpha A_\beta A^\gamma A^\alpha A^\beta] \\
+ \frac{4}{3} [A_\alpha A_\lambda, A_\beta A^\alpha A^\beta A_\gamma A^\lambda A^\beta] + \frac{11}{12} [A_\alpha A_\beta A_\gamma A^\lambda A^\gamma A^\beta] + \frac{5}{12} [A_\alpha A_\beta A_\gamma A^\gamma A^\alpha A^\beta] 
\right)
\]
Many commutator terms may be rewritten by using the Jacobi identity of the commutator and the commutation relation of the covariant differentiation. These changes do not vary the degree of $A_\mu$. The last term with $A_\mu$ of six degrees forms a commutator factor $\epsilon^{\alpha\beta\gamma\delta\kappa\lambda} A_\alpha A_\beta A_\gamma A_\delta A_\kappa A_\lambda$. 

6. Discussion

In the model in which the fermion interacts with the vector and the axial-vector fields in curved space, imposing the conservation equation of the vector current at the quantum level, the gauge anomalies in four and six dimensions are represented in tensorial form. The gauge anomaly in the model in four-dimensional flat space has already been given [3, 17, 18, 19, 20, 21]. In curved space, the expression (20) of $G^{(4)a}$ contains terms described by contraction of $R^{ab}_{\mu\nu}$ and a covariant derivative of $A_\mu$, and is rewritten as in (21). If only $T^a$ written visibly in (17) is a unit matrix, then the gauge anomaly agrees with the chiral U(1) anomaly. When all $T^a$ do not commute each other, the gauge anomaly should have the chiral U(1) anomaly-like part, which is expressed by the contraction of the curvature 2-form $R^a$, the trace of the strength 2-form $F$ of the vector field $V_\mu$ and the total derivative terms containing $A_\mu$, due to the index theorem [23, 24]. The other part of the gauge anomaly becomes terms containing
the commutator of \( A_\mu \), \( F_\alpha\beta \) and their derivatives, because the commutators should disappear in the absence of the above \( T^a \). The non-derivative term containing \( A_\mu \) of the same degree as the spacetime dimension \( 2n \) is expressed by a commutator factor 
\[
\epsilon^{\mu_1\mu_2\cdots\mu_{2n}} \text{tr} T^a [A_{\mu_1}, A_{\mu_2} \cdots A_{\mu_{2n}}],
\]
because of the contraction of the Levi-Civita symbol with the product of \( A_\mu \). Indeed, \( G^{(6)a} \) consists of the leading terms containing \( R^{ab}_{\mu\nu} \) and \( F_{\mu\nu} \), the total derivative part in (22) and the commutator part \( G^a_{\text{com}} \) in (23).

The Lagrangian in (3) can be rewritten by using right- and left-handed Weyl fermions:
\[
\mathcal{L} = h \left( \bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_L i \not{D} \psi_L \right),
\]
\[
\psi_R = P_+ \psi, \quad \not{D}_R = \gamma^\mu (\nabla_\mu + R_\mu) P_+, \quad R_\mu \equiv V_\mu + A_\mu,
\]
\[
\psi_L = P_- \psi, \quad \not{D}_L = \gamma^\mu (\nabla_\mu + L_\mu) P_-, \quad L_\mu \equiv V_\mu - A_\mu.
\] (24)

Though the Dirac operators \( \not{D}_R \) and \( \not{D}_L \) are not Hermitian, \( \not{D}_L \) and \( \not{D}_R \) in the path integral under the gauge transformation gives a half contribution of the gauge anomaly, which arises from the difference between the heat kernels regulated by \( \not{D}_L \not{D}_L^\dagger \) and \( \not{D}_L^\dagger \not{D}_L \), instead of \( \not{D}_R^2 \) in (11). The Jacobian for \( D\bar{\psi}_R \not{D} \psi_R \) yields the opposite sign of the contribution. Though \( V_\mu \) and \( A_\mu \) of the anomaly in (20)-(23) are replaced by \( R_\mu \) and \( L_\mu \), respectively, the form of \( (R + L)_\mu \) in the anomaly is restricted, since \( V_\mu \) appears only in the field strength \( F_{\mu\nu} \) and the covariant derivatives. However, the form of \( (R - L)_\mu \) is no restriction, because the gauge symmetry for \( A_\mu \) breaks down by the Wick rotation of the axial-gauge field, by which \( \not{D} \) becomes Hermitian. The breaking of the symmetry spoils the gauge transformation of \( R_\mu \) and \( L_\mu \).

If all \( A_\mu \) are Abelian in (21), then \( G^{a(4)} \) corresponds to the gauge anomaly in space with torsion, which is originally expressed by the third-order antisymmetric tensor. The dual vector of the tensor in four dimensions behaves as the axial-vector [27]. Then, there is no commutator term in (21). Note that the dual tensor of torsion in six or higher dimensions is the third- or higher-order antisymmetric tensor. Therefore, the anomaly with the vector and the axial-vector fields in six-dimensional space with torsion will have new terms containing the third-order torsion tensor.

In supergravity [28, 29], the gravitino is described by the Rarita-Schwinger field \( \psi_\mu \) with a suitably fixed gauge [30, 31, 32, 33, 34], and some quantum effects for the gravitino are evaluated by the heat kernel for a fermion of spin \( \frac{3}{2} \). By treating the vector index ‘\( \mu \)’ of \( \psi_\mu \) as the internal index of a representation matrix for the Lie algebra of the special orthogonal group, the heat kernel for the gravitino can be applied like that for a fermion of spin \( \frac{1}{2} \). When the chiral U(1) anomaly for the gravitino is calculated, the strength \( (T^a F^a_{\mu\nu})_{\alpha\beta} \) of the vector gauge field may be replaced by the curvature tensor \( R_{\mu\nu\alpha\beta} \). However, since \( T^a \) cannot exist by itself, there is no gauge anomaly for the gravitino of spin \( \frac{3}{2} \). In supergravity coupled with super Yang-Mills theory, the gauge anomaly for the gaugino of spin \( \frac{1}{2} \) in curved space may be expressed by the vector and the axial-vector bosons, which take the bilinear form of the gravitino and the gaugino, in place of \( V_\mu \) and \( A_\mu \), respectively.
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