Simple Regular Skew Group Rings

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Abstract

Given a group \( G \) acting on a ring \( R \) via \( \alpha : G \to \text{Aut}(R) \), one can construct the skew group ring \( R \star_{\alpha} G \). Skew group rings have been studied in depth, but necessary and sufficient conditions for the simplicity of a general skew group ring are not known. In this paper, such conditions are given for certain types of skew group rings, with an emphasis on Von Neumann regular skew group rings. Next the results of the first section are used to construct a class of simple skew group rings. In particular, we obtain more efficient proofs of the simplicity of certain rings constructed by H. Kambara and J. Trlifaj.

1 Introduction

Let \( \alpha : G \to \text{Aut}(R) \) be an action of a group \( G \) on a ring \( R \). Denote \( \alpha(g)(r) \) by \( ^{g}r \) for all \( g \in G \) and \( r \in R \). If the identity is the only element of \( G \) that maps to an inner automorphism then the action is said to be outer. Additively, the skew group ring \( R \star_{\alpha} G \) is the free left \( R \)-module with basis \( G \). Thus elements of \( R \star_{\alpha} G \) are finite sums, \( \sum r_{g}g \) where \( r_{g} \in R \) and \( g \in G \). Multiplication in \( R \star_{\alpha} G \) is given by the multiplication in \( R \) and in \( G \), and by \( gr = ^{g}rg \), for \( r \in R \) and \( g \in G \) and then extending linearly. The support of an element is \( \text{supp}(\sum r_{g}g) = \{ g \in G \mid r_{g} \neq 0 \} \). The length of \( \sum r_{g}g \) is the number of elements in \( \text{supp}(\sum r_{g}g) \), and is denoted \( \text{len}(\sum r_{g}g) \).

Suppose \( I \) is an ideal of \( R \). Then \( I \) is \( G \)-invariant if \( ^{g}I \subseteq I \) for every \( g \in G \). Notice that this is equivalent to \( ^{g}I = I \) for every \( g \in G \) because \( ^{g}I \subseteq I \) and \( ^{g^{-1}}I \subseteq I \) together imply that \( ^{g}I = I \). Define \( R \) to be \( G \)-simple if \( R \) is nonzero and the only \( G \)-invariant ideals of \( R \) are 0 and \( R \).

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The following proposition relating $G$-simplicity of $R$ and simplicity of $R*_{\alpha}G$ is well known. We give the short proof for the reader’s convenience.

**Proposition 1.1** Let $\alpha : G \rightarrow \text{Aut}(R)$ be an action of a group $G$ on a ring $R$. If $R*_{\alpha}G$ is simple, then $R$ is $G$-simple.

Proof: Suppose $I$ is a nonzero $G$-invariant ideal of $R$. Then $I*_{\alpha}G$ is a nonzero ideal of $R*_{\alpha}G$. Since $R*_{\alpha}G$ is simple, $I*_{\alpha}G = R*_{\alpha}G$. Thus $R \subseteq I*_{\alpha}G$, and so we have $R \subseteq I$. Therefore $R = I$ and hence $R$ is $G$-simple. \hfill \square

Thus the $G$-simplicity of $R$ is a necessary condition for $R*_{\alpha}G$ to be simple. While this is not a sufficient condition, it is in certain cases. In section 2 we give various conditions on the ring $R$, the group $G$, and the action $\alpha$ which will then result in the equivalence of $G$-simplicity of $R$ and simplicity of $R*_{\alpha}G$. In particular, we consider $X$-outer actions and how to determine whether an automorphism is $X$-inner.

In the following section we look at examples that arise from letting a group act on a topological space and considering a ring of functions on that space. We use the results of section 2 to determine the simplicity of particular skew group rings constructed this way. Finally we note that rings constructed by Kambara and Trlifaj are isomorphic to rings of this type and then give more efficient proofs of the simplicity of these rings.

## 2 Simplicity of skew group rings

The goal in this section is to find cases where $G$-simplicity of $R$ implies simplicity of the skew group ring. First we will recall a lemma which is useful when showing that a given skew group ring is simple.

**Lemma 2.1** If $R$ is $G$-simple then no proper ideal of $R*_{\alpha}G$ intersects $R$ nontrivially.

Proof: Suppose $R$ is $G$-simple and $I$ is an ideal of $R*_{\alpha}G$ which intersects $R$ nontrivially. Let $J = I \cap R$. If $x$ is a nonzero element of $J$, then $g^{-1}x = g^{-1}x^{-1} \in J$ for every $g \in G$. So $J$ is a $G$-invariant ideal of $R$ and hence $J = R$. Thus $1 \in I$ so $R*_{\alpha}G = I$. \hfill \square
The following proposition is probably well known, but we were unable to locate a reference in the literature.

**Proposition 2.2** If $G$ is an abelian group with outer action $\alpha$ on $R$, then $R \star_\alpha G$ is simple if and only if $R$ is $G$-simple.

Proof: Proposition gives the result that if $R \star_\alpha G$ is simple then $R$ is $G$-simple. Now suppose $R$ is $G$-simple and $I$ is a nonzero ideal of $R \star_\alpha G$. Let $x = \sum_{i=1}^n r_i g_i$ be an element of $I$ with minimal positive length. If $n = 1$, then $x g_i^{-1}$ is a nonzero element of $R \cap I$. Thus the lemma yields $I = R \star_\alpha G$.

Now suppose $n > 1$. Moreover, without loss of generality assume each $r_i \neq 0$, the $g_i$ are distinct, and $g_1 = 1$.

Since $R$ is $G$-simple and $r_1 \neq 0$, the set $\{g r_1 \mid g \in G\}$ generates $R$ as an ideal. Hence $1 = \sum_j \sum_k a_{kj} g_j r_1 b_{kj}$, for some $a_{kj}, b_{kj} \in R$ and $g_j \in G$. Let

$$y = \sum_j \sum_k a_{kj} g_j x g_j^{-1} b_{kj} = 1 + \sum_j \sum_k a_{kj} \left( \sum_{i=2}^n g_j r_i g_i \right) b_{kj}$$

$$= 1 + \sum_{i=2}^n \left( \sum_j \sum_k a_{kj} g_j r_i g_i b_{kj} \right) g_i.$$

Thus $\text{len}(y) \leq \text{len}(x)$, so $y$ is an element of $I$ of minimal length. Hence, we may assume that $x = 1 + \sum_{i=2}^n r_i g_i$.

Let $r \in R$. Then $rx - xr$ is in $I$ and has length strictly less than the length of $x$. Since $x$ was chosen to have minimal positive length, $rx - xr = 0$. Thus we have $rr_i = r_i g_i r$ for each $r \in R$ and $1 \leq i \leq n$.

If $h \in G$, $\text{len}(hx - xh) < \text{len}(x)$ and hence $hx - xh = 0$. So $r_i^h = r_i$ for each $h \in G$ and each $1 \leq i \leq n$. Thus for every $i$ between 1 and $n$, $R r_i R = R r_i = r_i R$ is $G$-invariant. Hence $R r_i = R = r_i R$, so $r_i$ is a unit, and $g_i r = r_i^{-1} r_i g_i$ for every $r \in R$. Thus if $n > 1$, $\alpha(g_i)$ is inner, which contradicts the action being outer. Therefore the length of $x$ is 1, so $x \in R$ and hence $I = R \star_\alpha G$ as noted above. $\square$

If $G$ is not abelian, the type of actions allowed must be further restricted. Not only do we eliminate conjugation by a unit of $R$, but also group elements that act as conjugation by elements of a quotient ring of $R$. An automorphism $f$ of a semiprime ring $R$ is $X$-inner if there exists a nonzero element $u$ in the left Martindale quotient ring of $R$ so that $f(r)u = ur$ for every $r \in R$.
The action $\alpha$ is said to be $X$-outer if the only element of $G$ that maps to an $X$-inner automorphism is the identity.

Montgomery proved [3, Lemma 3.16] that if $R$ is a semiprime ring and $\alpha : G \to \text{Aut}(R)$ is an $X$-outer action, then every nonzero ideal of $R \ast_\alpha G$ intersects $R$ nontrivially. This is exactly what is needed to prove $R \ast_\alpha G$ is simple. Montgomery proved an analog of the following theorem [3, Corollary 3.18] assuming that $R$ is simple. The same proof works with the condition on $R$ relaxed to $G$-simplicity.

**Theorem 2.3** If the action $\alpha$ of a group $G$ on a semiprime ring $R$ is $X$-outer then $R$ is $G$-simple if and only if $R \ast_\alpha G$ is simple.

Proof: Because of Proposition 1.1, we only need to prove the "only if" statement. So assume that $R$ is $G$-simple and let $I$ be a nonzero ideal of $R \ast_\alpha G$. Then by Montgomery's result $I \cap R$ is a nontrivial ideal of $R$. So by Lemma 2.1 $I = R \ast_\alpha G$ and hence $R \ast_\alpha G$ is simple. 

**Corollary 2.4** If $R$ is a commutative domain and $\alpha$ is a faithful action of $G$, then $R$ is $G$-simple if and only if $R \ast_\alpha G$ is simple.

Proof: If $R$ is a commutative domain, then its Martindale quotient ring is a field. Thus any $X$-inner automorphism must be the identity. Because $\alpha$ is faithful, 1 is the only element of $G$ which $\alpha$ maps to the identity. Therefore the action is $X$-outer and applying the theorem yields the desired result. 

Because showing an action is $X$-outer will help determine if the skew group ring is simple, it is useful to have ways of checking whether or not an automorphism is $X$-inner. To do this, we need to introduce a new concept.

**Definition 2.5** An automorphism $f$ of the ring $R$ is called corner-inner if there exists a nonzero idempotent $e \in R$ so that for $e' = f^{-1}(e)$ there exist elements $u \in eRe'$ and $v \in e'Re$ such that the following conditions hold:

- (a) $uv = e$ and $vu = e'$
- (b) $f(x) = u xv$ for $x \in e' Re'$
- (c) $f^{-1}(y) = vy u$ for $y \in eRe$. 

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Notice that part (c) of the definition follows from parts (a) and (b). Thus when showing that an automorphism is corner-inner, it is sufficient to only prove conditions (a) and (b).

**Proposition 2.6** Let $R$ be a semiprime ring and $f$ an automorphism of $R$.

(a) If $R$ is commutative then the following are equivalent:

(i) $f$ is $X$-inner.

(ii) $\text{ann}_R((\text{id} - f)(R)) \neq 0$.

(iii) There is a nonzero ideal $I$ in $R$ so that $f$ is the identity on $I$.

(b) If $R$ is commutative and regular, then $f$ is $X$-inner if and only if there exists a nonzero idempotent $e \in R$ so that $f$ is the identity on $eR$.

(c) If $R$ is regular and $f$ is $X$-inner, then $f$ is corner-inner.

Proof: Denote the Martindale quotient ring of $R$ by $Q$.

(a) Let $R$ be a commutative ring.

(i)$\Rightarrow$(ii) Suppose $f$ is $X$-inner. Then there is a nonzero element $x \in Q$ so that $f(r)x = xr$ for every $r \in R$. Because $R$ is commutative, $Q$ is also commutative. Thus $f(r)x = rx$, and hence $(r - f(r))x = 0$ for every $r \in R$. Therefore $(\text{id} - f)(R)x = 0$.

Since $x$ is a nonzero element of the Martindale quotient ring, there is an ideal $A$ of $R$ so that $0 \neq xA \subseteq R$. Because $xA \subseteq \text{ann}_R((\text{id} - f)(R))$ and $xA$ is nonzero, $\text{ann}_R((\text{id} - f)(R)) \neq 0$.

(ii)$\Rightarrow$(iii) Suppose $x$ is a nonzero element of $\text{ann}_R((\text{id} - f)(R))$. Then $xr = xf(r)$ for every $r \in R$. Since $R$ is semiprime and commutative, $x^2 \neq 0$. Now $xf^{-1}(x) = x^2$ so $f(x)x = f(x^2)$. Since $f(x)x = x^2$, we have $x^2 = f(x^2)$. Let $I$ be the ideal generated by $x^2$. Then because

$$f(x^2r) = f(x^2)f(r) = x^2f(r) = x(xr) = x^2r$$

for every $r \in R$, $f$ is the identity on $I$.

(iii)$\Rightarrow$(i) Suppose $f$ is the identity on $I$ and $x$ is a nonzero element of $I$. Then for $r \in R$, we have $f(r)x = f(rx) = xr$ and hence $f$ is $X$-inner.

(b) Assume $R$ is a commutative regular ring.

Clearly if $f$ is the identity on $eR$ for some nonzero idempotent $e$, then by part (a) $f$ is $X$-inner.
Conversely, assume that \( f \) is \( X \)-inner. Then there is a nonzero ideal \( I \) in \( R \) so that \( f(r) = qr \) for every \( r \in R \). Since \( q \) is in \( Q \), there is an essential ideal \( A \) of \( R \) so that \( qA \) is a nonzero right ideal of \( R \). Now \( A \) and \( f(A) \) are essential, so \( A \cap f(A) \cap qA \) is a nonzero right ideal of \( R \) and hence contains a nonzero idempotent, \( e \). Because \( e \in qA \), there exists \( a \in A \) so that \( e = qa \). Then \( e' = f^{-1}(e) \). Now \( eq = f(e')q = qe' \). Thus

\[
f(ae \cdot eq) = f(aeqe') = f(aeq)e = f(aeq)qa = qaeqa = e,
\]

so \( ae \cdot eq = e' \).

Let \( u = eq \) and \( v = e'ae \). Then \( u = qe' \) and hence \( u \in eRe' \). Also, \( v \in e' Re \), while \( uv = e \) and \( vu = e' \). Moreover, for \( x \in e' Re' \),

\[
f(x) = ef(xe')e = ef(xe')qae = efx'e \cdot ae = uxv.
\]

Therefore on \( e' Re' \), \( f(x) = uxv \), so \( f \) is corner-inner. \( \square \)

Combining parts (a) and (c) of this result with Theorem 2.3 we get two new theorems equating \( G \)-simplicity of \( R \) with simplicity of the skew group ring. Because the proof of Theorem 2.3 uses nontrivial results about the Martindale quotient ring, we will give direct proofs of the theorems.

**Theorem 2.7** Suppose \( R \) is a commutative semiprime ring and \( \alpha \) is an action of a group \( G \) on \( R \). Assume that \( 1 \) is the only element of \( G \) whose image under \( \alpha \) is the identity on some nonzero ideal of \( R \). Then \( R \ast \alpha G \) is simple if and only if \( R \) is \( G \)-simple.

**Proof:** Proposition 1.1 implies that if \( R \ast \alpha G \) is simple then \( R \) is \( G \)-simple.

Now suppose \( R \) is \( G \)-simple. Let \( I \) be a nonzero ideal of \( R \ast \alpha G \) and \( x \) a nonzero element of \( I \) with minimal length. If \( \text{len}(x) = 1 \) then \( x \in R \) and we are done by Lemma 2.1. Otherwise we may assume \( x = r_1 + r_2g_2 + \ldots + r_ng_n \) with \( n \geq 2 \), each \( r_i \neq 0 \), and distinct \( g_i \neq 1 \).
For any \( r \in R \),
\[
rx - xr = (rr_1 - r_1r) + (rr_2 - r_2^{g_2}rg_2 + \ldots + (rr_n - r_n^{g_n}rg_n)
\]
\[
= (rr_2 - r_2^{g_2}rg_2 + \ldots + (rr_n - r_n^{g_n}rg_n)g_n.
\]

Thus \( \text{len}(rx - xr) < n \) and since it is an element of \( I \), \( rx - xr = 0 \) for any \( r \in R \). Therefore \( rr_2 = r_2^{g_2}rg_2 \) for any \( r \in R \). Hence \( r_2^2 = r_2^{g_2}rg_2 \) and \( r_2g_2 = r_2^2 \), which implies that \( g_2r_2r_2 = r_2^{g_2}. \) Thus \( r_2^2 = r_2^{g_2}r_2 \). Now for \( r \in R \),
\[
g_2(r_2^2r) = r_2^2g_2r = r_2(r_2^{g_2}r) = r_2^2r.
\]

Therefore \( \alpha(g_2) \) is the identity on the nonzero ideal \( r_2^2R \). Thus \( g_2 = 1. \) But we assumed otherwise and hence have a contradiction to \( \text{len}(x) > 1 \). Therefore \( x \in R \) and \( I = R *_{g_2} G \) as above. Thus \( R *_{g_2} G \) is simple. \( \square \)

**Theorem 2.8** Suppose \( R \) is a regular ring and \( \alpha \) is an action of a group \( G \) on \( R \) so that \( 1 \) is the only element of \( G \) whose image under \( \alpha \) is corner-inner. Then \( R *_{g_2} G \) is simple if and only if \( R \) is \( G \)-simple.

**Proof:** If \( R *_{g_2} G \) is simple, \( G \)-simplicity of \( R \) follows from Proposition [1].

Suppose \( R \) is \( G \)-simple. Let \( I \) be a nonzero proper ideal of \( R *_{g_2} G \), and \( x \) a nonzero element of \( I \) with minimal length. If the length of \( x \) is \( 1 \) then \( xg^{-1} \in R \) for some \( g \in G \), and by an earlier lemma, \( I = R *_{g_2} G \). Now suppose \( \text{len}(x) = n > 1 \). Then \( x = \sum_{i=1}^{n} r_i g_i \), for some \( r_i \in R \) and distinct \( g_i \in G \). Since \( R \) is regular, there exists \( s \in R \) so that \( sr_1 \) is a nonzero idempotent, say \( e = sr_1 \). Then \( esxg_1^{-1} = e + \sum_{i=2}^{n} esr_n g_n g_1^{-1} \in I \) and
\[
1 \leq \text{len}(esxg_1^{-1}) \leq \text{len}(x).
\]
Thus we may assume that \( x = e + \sum_{i=2}^{n} r_i g_i \) and \( er_i = r_i \) for each \( i \).

Since \( \text{len}(x) \geq 2 \), we have \( r_2 \neq 0 \) and \( g_2 \neq 1 \). There exists a nonzero element \( y \in R \) such that \( r_2yr_2 = r_2 \). Now
\[
x - x^{g_2^{-1}}(yr_2) =
\]
\[
(e - e^{g_2^{-1}}(yr_2)) + (r_2 - r_2yr_2)g_2 + \ldots + (r_n - r_n^{g_n}g_2^{-1}(yr_2))g_n =
\]
\[
(e - e^{g_2^{-1}}(yr_2)) + (r_3 - r_3^{g_3}g_2^{-1}(yr_2))g_3 + \ldots + (r_n - r_n^{g_n}g_2^{-1}(yr_2))g_n
\]
is an element of $I$ with length less than $n$, so $x - x^{g_2^{-1}}(yr_2) = 0$ and hence $e = e^{g_2^{-1}}(yr_2)$.

Similarly $x - r_2yx = 0$, so $r_2ye = e$. Also, $xe - x = 0$ and hence $r_2^{g_2}e = r_2$. If $s \in eRe$, then $sx - xs = 0$ and thus $sr_2 = r_2^{g_2}s$.

Let $e' = g_2e$. Then the equations above yield $e' = e'yr_2$ and $r_2e' = r_2$. Now define $u = r_2e'$ and $v = e'ye$. Then $u \in eRe'$ and $v \in e'Re$. We have

$$uv = (r_2e')(e'ye) = r_2e'ye = r_2ye = e,$$

and also

$$vu = (e'ye)(r_2e') = e'yr_2e' = e'e' = e'.$$

Finally, for any $t \in e'Re'$,

$$utv = r_2tye = r_2^{g_2}(e^{g_2^{-1}t}e)ye$$

$$= (e^{g_2^{-1}t}e)r_2ye = g_2^{-1}(e'te')e$$

$$= g_2^{-1}t = \alpha(g_2^{-1})(t).$$

Thus $\alpha(g_2^{-1})$ is corner-inner. Since $g_2^{-1} \neq 1$ we have a contradiction and therefore $\text{len}(x) = 1$ and so $I = R*_{\alpha}G$. Hence $R*_{\alpha}G$ is simple. $\square$

## 3 Examples

An easy way to construct a skew group ring is to let $G$ be a group acting via homeomorphisms on a topological space $X$. If $T$ is the ring of locally constant functions from $X$ to a field $k$, then $G$ acts on $T$ by composition. Thus we get a skew group ring $T*_{\alpha}G$. Since $T$ is Von Neumann regular, as long as $G$ is a locally finite group and no subgroup of $G$ has order divisible by $\text{char}(k)$, the resulting skew group ring is Von Neumann regular \[1\], and hence semiprime. Since our goal is to construct simple skew group rings, it is useful to know what conditions are needed on a group action on a topological space for our construction to yield a simple skew group ring.

Throughout this section, $G$ is a locally finite group of homeomorphisms on a totally disconnected, compact, Hausdorff topological space $X$ and $T$ is the ring of locally constant functions from $X$ to a field $k$ of characteristic 0. A subset $U$ of $X$ is called clopen if it is both open and closed. Denote the
complement of any set \( V \) in \( X \) by \( V^c \). Clearly if \( V \) is clopen then so is \( V^c \).
Since \( X \) is compact, functions in \( T \) have only finitely many values. If \( g \in G \) and \( t \in T \) then \( G \) acts on \( T \) by \( (\alpha g) t)(x) = g^t(x) = t(g^{-1}(x)) \).

The ring \( T \) must be \( G \)-simple if \( T \ast_n G \) is going to be simple. Thus criteria are needed to check if \( T \) is \( G \)-simple. Here separate necessary conditions, and sufficient conditions for \( G \)-simplicity of \( T \) are given.

**Theorem 3.1** If the only open subsets of \( X \) invariant under \( G \) are \( \emptyset \) and \( X \), then \( T \) is \( G \)-simple.

Proof: Let \( I \) be a nonzero \( G \)-invariant ideal of \( T \), and \( f \) a nonzero element of \( I \). If \( f \) is a unit then \( I = T \). So assume otherwise. Because \( f \) is locally constant, there exist nontrivial pairwise disjoint clopen sets \( U_0, U_1, \ldots, U_m \) so that for each \( i \), \( f|_{U_i} \) is constant, \( f(U_0) = 0 \) and \( f(U_i) = \lambda_i \neq 0 \) if \( i > 0 \), and \( \bigcup_{i=0}^{m} U_i = X \). Define \( h : X \to k \) so that \( h(U_0) = 0 \) and \( h(U_i) = \lambda_i^{-1} \) for \( i > 0 \). Then \( f = fh \) is zero on \( U_0 \) and one on \( \bigcup_{i=1}^{m} (U_i) \). Moreover \( f \in I \) so we may assume the image of \( f \) is \( \{0, 1\} \). Thus there is a nontrivial clopen subset \( U \) of \( X \) with \( f(U) = 1 \) and \( f(U^c) = 0 \). Then \( \bigcup_{g \in G} g(U) \) is a nonempty, open, \( G \)-invariant subset of \( X \) and hence equals \( X \). Because \( X \) is compact, there exist distinct \( g_1, g_2, \ldots, g_n \in G \) with \( X = \bigcup_{i=1}^{n} g_i(U) \).

If \( x \) is an element of \( X \), there is a \( g_j \) so that \( x \in g_j(U) \). Thus if \( \varphi = \sum_{i=1}^{n} a_i f_i \), then for any \( x \in X \), \( \varphi(x) \neq 0 \) since \( f(g_j(U)) = 1 \) for some \( j \) and \( f(g_i^{-1}(x)) = 0 \) or \( 1 \) for each \( i \). Therefore \( \varphi \) is a unit of \( T \). But \( I \) is \( G \)-invariant, so \( \varphi \in I \) and hence \( I = T \). \( \Box \)

**Theorem 3.2** If \( T \) is \( G \)-simple, then the only clopen subsets of \( X \) invariant under \( G \) are \( \emptyset \) and \( X \).

Proof: Suppose there is a nontrivial \( G \)-invariant clopen subset \( A \) of \( X \). Then define \( f : X \to k \) by \( f(A) = 0 \) and \( f(A^c) = 1 \). Clearly \( f \in T \). Now \( f \) is nonzero since \( A^c \neq \emptyset \), and \( f \) is not a unit since \( A \neq \emptyset \). Let \( I \) be the ideal of \( T \) generated by \( f \). The set \( A \) is \( G \)-invariant, so \( g f(x) = f(x) \) for every \( x \in X \) and every \( g \in G \). Thus \( g f = f \), so \( I \) is a nontrivial, \( G \)-invariant ideal of \( T \). \( \Box \)

It would be nice to have a necessary and sufficient condition for \( G \)-simplicity of \( T \) along the lines of the previous two theorems. Unfortunately this can not be done. There are spaces with groups acting on them such that the converse of Theorem 3.2 does not hold.
Example 3.3 A totally disconnected, compact, Hausdorff space $X$ with a locally finite group $G$ acting via homeomorphisms so that the ring $T$ of locally constant functions from $X$ to $k$ is not $G$-simple and yet there are no nontrivial clopen subsets of $X$ invariant under $G$.

Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ with the convention that $\infty > n$ for every $n \in \mathbb{N}$. Suppose $T$ is the ring of all locally constant functions from $X$ to $k$. For each $n \in \mathbb{N}$, define $g_n : X \to X$ by

$$g_n(x) = \begin{cases} 
    x + 1 & \text{if } x < n \\
    1 & \text{if } x = n \\
    x & \text{if } x > n
  \end{cases}$$

Then each $g_n$ is a homeomorphism of $X$ and $g_n$ generates a cyclic group of order $n$. Let $G$ be the group generated by all of the $g_n$. Then $G$ is a locally finite group acting on $X$ via homeomorphisms. Suppose $I = \{f \in T \mid f(\infty) = 0\}$. Then $I$ is a nontrivial $G$-invariant ideal of $T$ and hence $T$ is not $G$-simple.

Now suppose $U$ is a proper clopen subset of $X$. If $\infty \in U$ then there exists $m \geq 2$ so that $x \geq m$ implies $x \in U$ and yet $m - 1 \notin U$. There exists $n \in \mathbb{N}$ so that $g_n^{-1}(m) = m - 1$. Thus $(g_n^{-1})(m) \notin U$ so $U$ is not $G$-invariant. If $\infty \notin U$ then $U^c$ is a proper clopen subset of $X$ containing $\infty$ and by the above argument, $U^c$ is not $G$-invariant. If $U^c$ is not $G$-invariant then $U$ is not either. In both cases, the clopen set $U$ is not $G$-invariant and hence no proper clopen subset of $X$ is $G$-invariant.

Thus we have a space $X$ which is compact, Hausdorff, and totally disconnected, with a locally finite group $G$ acting on $X$ via homeomorphisms so that no proper clopen subset of $X$ is $G$-invariant and yet the ring $T$ is not $G$-simple. \hfill \square

Thus we do not have a condition that is both necessary and sufficient for proving the $G$-simplicity of $T$. But the conditions we have are still very useful in proving the $G$-simplicity of certain skew group rings. This method provides an easier way to approach already known examples of simple rings.

Example 3.4

Let $X_n = \{1, 2, \ldots, 2^n\}$ with the discrete topology. Then $X_n$ can be mapped onto $X_{n-1}$ via $\phi_n(i) = \left\lfloor \frac{i+1}{2} \right\rfloor$, where $\lfloor \ldots \rfloor$ denotes the floor function. For
m < n define \( \phi_{n,m} : X_n \to X_m \) by \( \phi_{n,m} = \phi_{m+1} \phi_{m+2} \ldots \phi_{n-1} \phi_n \). Let \( X \) be the inverse limit of this inverse system with projection maps \( \pi_n : X \to X_n \) and \( T \) the ring of all locally constant functions from \( X \) to the field \( k \). Since each \( X_n \) is compact, Hausdorff, and totally disconnected, so is \( X \).

Let \( g_n : X_n \to X_n \) be the homeomorphism that sends \( i \) to \( i + 1 \) for \( i < 2^n \) and sends \( 2^n \) to 1. Notice that \( \phi_m \circ g_m^2 = g_{m-1} \circ \phi_m \) for every \( m \in \mathbb{N} \). We claim that each \( g_n \) extends to a map \( \overline{g_n} : X \to X \) by

\[
\overline{g_n}(y) = (z_m)_{m \in \mathbb{N}} \quad \text{where} \quad z_m = \begin{cases} 
\pi_{m-n}^{-1}(y) & \text{for } m \geq n \\
\phi_{n,m}(g_n \pi_n(y)) & \text{for } m < n .
\end{cases}
\]

To check that \( \overline{g_n}(y) \in X \), we must show that \( \phi_m(z_m) = z_{m-1} \) for every \( m \in \mathbb{N} \). If \( m > n \), then

\[
\phi_m(z_m) = \phi_m(g_m^{2m-n} \pi_m(y)) = g_{m-1}^{2m-1-n} \phi_m \pi_m(y) = g_{m-1}^{2m-1-n} \pi_{m-1}(y) = z_{m-1}.
\]

On the other hand, if \( m < n \), then

\[
\phi_m(z_m) = \phi_m(\phi_{n,m} g_n \pi_n(y)) = \phi_{n,m-1} g_n \pi_n(y) = z_{m-1}.
\]

Now if \( m = n \), we have

\[
\phi_n(z_n) = \phi_n(g_n \pi_n(y)) = \phi_{n,n-1}(g_n \pi_n(y)) = z_{n-1}.
\]

Since in each case \( \phi_m(z_m) = z_{m-1} \), we do have \( \overline{g_n}(y) \in X \) as desired.

Moreover this definition yields a continuous function of \( X \) because \( \pi_m \circ \overline{g_n} = g_m^{2m-n} \circ \pi_m \) or \( \pi_m \circ \overline{g_n} = \phi_{n,m} \circ g_n \circ \pi_n \), both of which are continuous for every \( n, m \in \mathbb{N} \). Since \( \overline{g_n} \) generates a cyclic group of order \( 2^n \), we have \( \overline{g_n}^{2^n-1} = \overline{g_{n-1}} \), which is continuous and hence \( \overline{g_n} \) is a homeomorphism of \( X \).

Let \( G_n \) be the cyclic group of order \( 2^n \) generated by \( \overline{g_n} \). Now \( \pi_m \circ \overline{g_n}^2 = \pi_m \circ \overline{g_{n-1}} \) for every \( m, n \in \mathbb{N} \). So \( \overline{g_n}^2 = \overline{g_{n-1}} \) for \( n > 1 \) and hence \( G_{n-1} \subseteq G_n \) for every \( n > 1 \). Thus if \( G \) is the group generated by all the \( \overline{g_n} \), then \( G \) is abelian, locally finite, and in fact \( G \cong \mathbb{Z}_{2^n} \).

As indicated above, this induces an action \( \alpha \) on the ring \( T \). Since \( T \) is commutative, \( \alpha \) is outer. Because \( G \) is an abelian, locally finite group and \( \text{char}(k) = 0 \), we obtain a regular skew group ring \( T *_{\alpha} G \).

Suppose \( U \) is a nonempty proper open subset of \( X \). Let \( y = (y_m)_{m \in \mathbb{N}} \in U \) and \( z = (z_m)_{m \in \mathbb{N}} \in U^c \). Because \( U \) is open, there is a basic open neighborhood, \( \theta \subseteq U \), of \( y \) so that

\[
\theta = \left( \{y_1\} \times \{y_2\} \times \ldots \times \{y_m\} \times \prod_{i > m} X_i \right) \cap X.
\]
There exists $g^j_{m+1} \in G_m$ so that $g^j_{m+1}(z_{m+1}) = y_{m+1}$. If $\overline{g} = g_{m+1}^j$, then for $i \leq m$
\[
\pi_i\overline{g}(z) = \phi_{m+1,i}(g^j_{m+1} \pi_{m+1}(z)) = \phi_{m+1,i}(y_{m+1}) = y_i.
\]
Thus $\overline{g}(z) \in \theta$. Hence $U$ is not $G$-invariant for any nonempty proper open $U \subseteq X$, so $T$ is $G$-simple by Theorem 3.1. Now $G$ is abelian, $\alpha$ is outer, and $T$ is $G$-simple, so $T *_{\alpha} G$ is simple by Proposition 2.2. \hfill \Box

Example 3.5

Let $T_n$ be a direct product of $2^n$ copies of the field $k$ for each $n$, with primitive idempotents $e_1^{(n)}, \ldots, e_{2^n}^{(n)}$. Let $\tau_n : T_n \rightarrow T_{n+1}$ be the $k$-algebra map so that $\tau_n(e_i^{(n)}) = e_{2i-1}^{(n+1)} + e_{2i}^{(n+1)}$ for each $i$ and let $T'$ be the direct limit of this system of $k$-algebras with $\iota_n$ the injection from $T_n$ to $T'$. For $n \leq j$, define $k$-algebra maps $g_{n,j} : T_j \rightarrow T_j$ by $g_{n,j}(e_i^{(j)}) = e_i^{(j)}(j-n)$ for $i \leq 2^j - 2(j-n)$ and $g_{n,j}(e_i^{(j)}) = e_{i-2^j+2(j-n)}$ for $i > 2^j - 2(j-n)$. Then each $g_{n,j}$ is an automorphism of $T_j$. Moreover $\tau_i g_{n,i} = g_{n,i+1} \tau_i$ for every $i \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists an automorphism $g_n$ of $T'$ so that for every $i \geq n$, $g_n \circ \iota_i = \iota_i \circ g_n$. If $G'$ is the group generated by all of the $g_n$ then $G' \cong \mathbb{Z}_2^{\infty}$ because each $g_n$ generates a cyclic group of order $2^n$ and for each $n$, $g_{n+1}^2 = g_n$. Since $G'$ is a subgroup of $\text{Aut}(T')$, we have a group action $\beta : G' \rightarrow \text{Aut}(T')$. If $\overline{T'}$ is the maximal right quotient ring of $T'$, then every automorphism of $T'$ extends uniquely to an automorphism of $\overline{T'}$. Thus we have a group action $\overline{\beta} : G' \rightarrow \text{Aut}(\overline{T'})$. Since both $T'$ and $\overline{T'}$ are commutative, $\beta$ and $\overline{\beta}$ are outer.

Now $G$ is isomorphic to the group $G'$ from the previous example. If we identify $G$ and $G'$, then $T'$ is isomorphic to the ring $T$ from that example via a $G$-equivariant isomorphism. Therefore $T'$ is $G'$-simple. If $I$ is a nonzero $G'$-invariant ideal of $\overline{T'}$ then $I \cap T'$ is a nonzero $G'$-invariant ideal of $T'$. But since $T'$ is $G'$-simple, we have $I \cap T' = T'$ and so $I = \overline{T'}$. Thus $\overline{T'}$ is $G'$-simple, so $\overline{T'} *_{\overline{\beta}} G'$ is simple by Proposition 2.2.

Moreover $\overline{T'} *_{\overline{\beta}} G'$ is isomorphic to a ring that Kambara [2, page 112] constructed. He showed that the right maximal quotient ring of this ring $R$ is directly finite and only one-sided self-injective. Our approach provides an alternate proof of the simplicity of $R$, which is a key ingredient in Kambara’s construction. \hfill \Box

Another nice class of examples arises if $X$ is a topological group and $G$ is a
dense subgroup of $X$ acting by left multiplication. Given suitable conditions on the topology of $X$, we can show that $T \ast \alpha G$ will be simple.

**Proposition 3.6** Suppose $X$ is a compact, Hausdorff, totally disconnected topological group, and $G$ is a dense subgroup of $X$. Let $G$ act on $X$ by left multiplication. Then the skew group ring $T \ast \alpha G$ is simple.

Proof: Suppose $V$ is a nontrivial proper open subset of $X$ and $x \in X \setminus V$. Then right multiplication by $x$ is a homeomorphism of $X$ and thus $Gx$ is a dense subset of $X$. Since $V$ is open, there is an element $g \in G$ so that $gx \in V$. Thus $V$ is not invariant under $G$ so by Theorem 3.1, $T$ is $G$-simple.

Now suppose $I$ is a nonzero ideal of $T$ and $1 \neq g \in G$. Let $t$ be a nonzero element of $I$ and let $U$ be a clopen subset of $X$ so that $t(U) = 0$. Since $G$ is dense, there is an element $h \in U \cap G$. Then $gh \neq h$ so there are disjoint clopen sets $W$ and $\theta$ so that $h \in W \subseteq U$ and $gh \in \theta$. Let $t_0$ be a locally constant function so that $t_0(W) = 1$ and $t_0(\theta) = 0$. Then not only is $t_0$ an element of $I$, but $t_0(gh) = 0$ and $t_0(h) \neq 0$. Thus $g^{-1}(t_0t) \neq t_0t$, so $\alpha(g^{-1})$ is not the identity on $I$. Because $T$ is $G$-simple and $1$ is the only element of $G$ whose image under $\alpha$ is the identity on some nonzero ideal of $T$, by Theorem 2.7 $T \ast \alpha G$ is simple. ☐

**Example 3.7**

Let $G$ be a residually finite group and $\{H_i\}_{i \in I}$ a directed system of normal subgroups of $G$ with finite index so that $i \leq j$ implies $H_j \subseteq H_i$ and $\bigcap H_i = 1$. Then the groups $G/H_i$ together with projection maps $\pi_{i,j} : G/H_i \to G/H_j$ for each $i \geq j$ form an inverse system and we let $X$ be the inverse limit. If we endow each $G/H_i$ with the discrete topology then since each $G/H_i$ is finite and hence compact, $X$ is compact. Moreover $X$ is a totally disconnected, Hausdorff topological group.

The image of the natural embedding of $G$ into $X$ is dense in $X$. Thus if we identify $G$ with this image we obtain a simple skew group ring $T \ast \alpha G$ as in Proposition 3.6.

The ring $D_G$ derived from $KG$ described in [4, page 2241] is isomorphic to the ring $T \ast \alpha G$. Thus we have a different proof of the simplicity of $D_G$ [4, Lemma 1.2]. Trlifaj showed that if $G$ is a locally finite direct product of countably many finite groups, then this ring is a counterexample to the

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conjecture “If $R$ is a regular ring which is not semisimple then for each simple left $R$-module $J$ there exists a nonzero right $R$-module $M$ so that $M \otimes_R J = 0$” [4, Theorem 3.8].

\[\square\]

References

[1] Ricardo Alfaro, Pere Ara, and Angel Del Rio. Regular skew group rings. Journal of the Australian Mathematical Society, 58:167–182, 1995.

[2] Hikoji Kambara. On existence of directly finite, only one-sided self-injective regular rings. Journal of Algebra, 144(1):110–116, 1991.

[3] Susan Montgomery. Fixed Rings of Finite Automorphism Groups of Associative Rings, volume 818 of Lecture Notes in Mathematics. Springer-Verlag, 1980.

[4] Jan Trlifaj. Rings derived from group rings. Communications in Algebra, 20(8):2239–2252, 1992.

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