ON THE RESIDUAL FINITENESS GROWTHS OF PARTICULAR HYPERBOLIC MANIFOLD GROUPS

PRIYAM PATEL

Abstract. We give a quantification of residual finiteness for compact hyperbolic 3–manifold and 4–manifold groups that virtually embed in specific right-angled Coxeter groups arising as reflection groups for all right polyhedra in $H^3$ and $H^4$, respectively. We show for this class of groups that their residual finiteness growths are at most linear in terms of geodesic length.

1. Introduction

It is well-known that separability properties on groups have deep connections with basic problems in group theory. In the 1940’s, Mal’cev demonstrated that separability properties like residual finiteness and conjugacy separability produce solutions to the word and conjugacy problems for finitely presented groups [13]. In more recent years, separability properties have also played a fundamental role in low dimensional topology. In [17], P. Scott gave an important topological reformulation of subgroup separability when the groups in consideration are the fundamental groups of manifolds; separability allows one to promote an immersed surface to an embedded one in a finite cover. This topological reformulation of subgroup separability played a crucial role in the recent resolutions of Wallhausen’s Virtually Haken Conjecture and Thurston’s Virtually Fibered Conjecture (see [2], [19], [10]).

The simplest separability property, residual finiteness, allows us to separate nontrivial group elements from the identity using finite index subgroups. More precisely, a group $G$ is residually finite if for every nontrivial element $g \in G$, there exists a finite index subgroup $G'$ of $G$ such that $g \notin G'$. Quantifying residual finiteness, a concept first introduced by K. Bou-Rabee in [4], refers to bounding the indexes of the finite index subgroups $G'$ in terms of algebraic data about $G$. In studying separability properties of the fundamental groups of manifolds, the bounds can also be given in terms of geometric data about the manifolds as in [14]. Quantifying residual finiteness informs us on the minimal possible index of a subgroup $G'$ of $G$ that separates $g$ from the identity, and it has been studied for various classes of groups including free groups, surface groups, and virtually special groups (see for instance [4], [5], [6], [8], [9], [12], [14], [16]).

To keep with the notation present in most of the literature on quantifying residual finiteness we introduce the residual finiteness growth, originally defined in [4] as follows. For a group $G$ with a fixed finite generating set $S$, let the divisibility function $D_G : G \setminus \{1\} \to \mathbb{N} \cup \infty$ be defined by

$$D_G(g) = \min\{|G : H| : g \notin H \text{ and } H \leq G\}.$$ 

When $G$ is residually finite, $D_G(g)$ of course takes values in $\mathbb{N}$. We note that compact hyperbolic manifold groups are finitely generated linear groups and are thus known to be residually finite by the work of Mal’cev [13]. Define the residual finiteness function $F_{G,S}(n)$ to be the maximum value of $D_G$ on the set

$$\{g \in G \setminus \{1\} : \|g\|_S \leq n\},$$

Date: December 23, 2014.

2010 Mathematics Subject Classification. Primary: 20E26; Secondary: 57M10, 20F65.

Key words and phrases. Residual finiteness growth, hyperbolic manifolds, right-angled Coexter group.
where \( \| \cdot \|_S \) is the word-length norm with respect to \( S \). The growth of \( F_{G,S} \) is called the \textit{residual finiteness growth}.

In this paper, we are concerned with the residual finiteness growths of the fundamental groups of particular hyperbolic 3–manifolds and 4–manifolds. However, we give all quantifications of residual finiteness in terms of geometric data about the manifolds. We therefore define a new function \( F_{M,\rho} \) for a hyperbolic manifold \((M,\rho)\) by letting \( F_{M,\rho}(n) \) be the maximum value of \( D_{\pi_1(M)} \) on the set

\[ \{ \alpha \in \pi_1(M) - \{1\} : \ell_{\rho}(\alpha) \leq n \}, \]

where \( \ell_{\rho}(\alpha) \) is the length of the unique geodesic representative of \( \alpha \) with respect to the hyperbolic metric \( \rho \) on \( M \). The growth of \( F_{M,\rho} \) is called the \textit{geodesic residual finiteness growth}.

The reasons for quantifying residual finiteness and studying residual finiteness growths are varied. First, residual growths can help distinguish classes of groups. For example, in [7] Bou-Rabee and McReynolds give a characterization of when non-elementary hyperbolic groups are linear in terms of residual growths. Additionally, a quantification of residual finiteness can serve as a foundation on which to build an approach to the quantification of stronger separability properties. In [3], the author presented a quantification of the residual finiteness of hyperbolic surface groups in terms of geodesic length. The result is then used to make effective a theorem of P. Scott [17] that these groups are also \textit{subgroup separable}. The author uses a key insight of Scott that all hyperbolic surface groups embed in a particular right-angled Coxeter group, generated by reflections in the sides of a regular right-angled pentagon in \( \mathbf{H}^2 \). Quantification proofs also usually proceed constructively and can provide insight into how the finite index subgroups/covers associated to residual finiteness and subgroup separability actually arise.

The hyperbolic 3– and 4–manifold groups that we are concerned with are those that embed in particular right-angled Coxeter groups. We begin by letting \( P \) be any compact polyhedron in \( \mathbf{H}^n \), all of whose dihedral angles are \( \pi/2 \), which we will refer to as a compact \textit{all right polyhedron}. These polyhedra serve as the analog of Scott’s right-angled pentagon in [17]. We denote by \( \Gamma_P \) the right-angled Coxeter group of isometries of \( \mathbf{H}^n \) generated by reflections in the faces of \( P \).

Let \( M \) be a compact hyperbolic 3–manifold tiled by copies of \( P \) for some all right polyhedron \( P \subset \mathbf{H}^3 \). That is, \( \pi_1(M) \) embeds in the right-angled Coxeter group \( \Gamma_P \) associated to \( P \). Drawing from the work of Agol, Long, and Reid in [3] and making use of an observation of Agol in [1], we quantify the residual finiteness of \( \pi_1(M) \) in Section 3 with the following theorem:

**Theorem 3.3.** Let \((M,\rho)\) be any compact hyperbolic 3–manifold whose fundamental group embeds in a right-angled Coxeter group \( \Gamma_P \) associated to an all right polyhedron \( P \) in \( \mathbf{H}^3 \). Then for any \( \alpha \in \pi_1(M) - \{1\} \), there exists a subgroup \( \Gamma' \) of \( \pi_1(M) \) such that \( \alpha \notin \Gamma' \). The index of \( \Gamma' \) is bounded above by

\[ \frac{2\pi}{V_P} \sinh^2(\ln(\sqrt{3} + \sqrt{2}) + d_P) \ell_{\rho}(\alpha), \]

where \( \ell_{\rho}(\alpha) \) is the length of the unique geodesic representative of \( \alpha \) and where \( d_P \) and \( V_P \) are the diameter and volume of \( P \), respectively.

In Section 4 we establish the 4–dimensional analog of Theorem 3.3.

**Theorem 4.3.** Let \((M,\rho)\) be any compact hyperbolic 4–manifold whose fundamental group embeds in a right-angled Coxeter group \( \Gamma_P \) associated to an all right polyhedron \( P \) in \( \mathbf{H}^4 \). Then for any \( \alpha \in \pi_1(M) - \{1\} \), there exists a subgroup \( \Gamma' \) of \( \pi_1(M) \) such that \( \alpha \notin \Gamma' \). The index of \( \Gamma' \) is bounded above by

\[ \frac{8\pi}{3V_P} \sinh^3(\ln(2 + \sqrt{3}) + d_P) \ell_{\rho}(\alpha), \]

where \( \ell_{\rho}(\alpha) \) is the length of the unique geodesic representative of \( \alpha \) and where \( d_P \) and \( V_P \) are the diameter and volume of \( P \), respectively.
In order to state our results in the language of the growth functions, we introduce the following standard notation: For functions $f$ and $g$, we write $f \preceq g$ if there exists $C > 0$ such that $f(n) \leq C \cdot g(Cn)$. Further, we write $f \simeq g$ if $f \preceq g$ and $g \preceq f$. We then have the following corollary:

**Corollary 4.4.** Let $(M, \rho)$ be a compact hyperbolic 3–manifold or 4–manifold with an embedding $\pi_1(M) \to \Gamma_P$ for some all right polyhedron $P$ in $\mathbb{H}^3$ or $\mathbb{H}^4$, respectively. The geodesic residual finiteness growth is at most linear. That is to say, $F_{M,\rho} \preceq n$.

The next lemma allows us to extend Corollary 4.4 to hyperbolic 3–manifold and 4–manifold groups that virtually embed in these right-angled Coxeter groups. A group $G$ (resp. topological space $X$) virtually has property “$\mathcal{X}$” if there exists a finite index subgroup of $G$ (resp. finite sheeted cover of $X$) with property “$\mathcal{X}$”.

**Lemma 5.1.** Let $(M, \rho)$ be a compact hyperbolic $n$–manifold and let $K \leq \pi_1(M)$ be a finite index subgroup with $[\pi_1(M) : K] = C$. Let $(M', \rho')$ be the cover of $M$ of index $C$ corresponding to the subgroup $K$. Then the geodesic residual finiteness function for $(M, \rho)$ is bounded by that of $(M', \rho')$. That is to say, $F_{M,\rho} \preceq C \cdot F_{M',\rho'}$ and hence $F_{M,\rho} \preceq F_{M',\rho'}$.

We then have the following main result:

**Corollary 5.2.** If $(M, \rho)$ be a compact hyperbolic 3–manifold or 4–manifold whose fundamental group virtually embeds in $\Gamma_P$ for some compact right polyhedron $P$ in $\mathbb{H}^3$ or $\mathbb{H}^4$, then $F_{M,\rho} \preceq n$.

We note that in [5], K. Bou-Rabee, M.F. Hagen and the author give a quantification of residual finiteness for right-angled Artin groups (raAgs) in terms of word length. In particular, we prove that the residual finiteness growth of raAgs is at most linear. As suggested by Lemma 5.1, residual finiteness quantifications are essentially preserved under passing to subgroups and finite index extensions. Thus, the quantification for raAgs results in a quantification of residual finiteness for all groups that virtually embed in raAgs, which are called virtually special groups. This class of groups includes hyperbolic 3–manifold groups [2], and in particular contains the class of 3–manifold groups satisfying the hypotheses of Corollary 5.2. However, the proof of Theorem 3.3 does not rely on the canonical completion and retraction methods of [11] for cube complexes. Instead, our proof uses explicit calculations in hyperbolic space inspired by work of Agol–Long–Reid [3] and Agol [1]. Moreover, the methods are analogous to those employed in [14]. Most importantly, the calculations in hyperbolic space lead to the explicit bounds of Theorem 3.3 for 3–manifold groups $\pi_1(M)$ that embed in $\Gamma_P$ for some $P$. Applying the methods of [3] would yield a linear bound on the residual finiteness function whose coefficient depends on the index of the finite index subgroup of $\pi_1(M)$ that embeds in a raAg. Calculating that index should be at least as hard as proving an effective version of Agol’s Virtually Special Theorem of [2].

**Acknowledgements.** Much of the work in this paper originally appeared in the author’s thesis, written under the supervision of Feng Luo, whom the author thanks for his help, support, and encouragement. The author would also like to sincerely thank Ian Agol for sharing his work and ideas, as well as for insightful conversations. Many thanks are also due to Tian Yang for his suggestions regarding the proofs of Lemmas 5.1 and 4.1 to Alan Reid and Nicholas Miller for helpful conversations, and to Ben McReynolds for his encouragement to write this paper. Additionally, the author would like to thank Ben McReynolds and Nicholas Miller for comments and suggestions on an early draft of the paper.

2. Preliminaries

2.1. Notation: Throughout the paper we switch freely between the Poincaré ball ($\mathbb{D}^n$) and half-space ($\mathbb{H}^n$) models of hyperbolic space. In the 3–dimensional Poincaré ball model $\mathbb{D}^3$, we define
the three hyperplanes \( L_x = \{(x, y, u) \in \mathbb{D}^3 : x = 0\} \), \( L_y = \{(x, y, u) \in \mathbb{D}^3 : y = 0\} \), and \( L_u = \{(x, y, u) \in \mathbb{D}^3 : u = 0\} \). In the 3-dimensional half-space model \( \mathbb{H}^3 \) we define the hyperplanes \( L'_x = \{(x, y, u) \in \mathbb{H}^3 : x = 0\} \) and \( L'_y = \{(x, y, u) \in \mathbb{H}^3 : y = 0\} \). Similarly, in \( \mathbb{D}^4 \) we define the four hyperplanes \( L_{x_i} \) for \( i = 1, 2, 3, 4 \) as the hyperplanes obtained by restricting \( x_i \) to be 0. The hyperplanes \( L'_{x_k} \) for \( k = 1, 2, 3 \) in \( \mathbb{H}^4 \) are defined by restricting the \( x_k \) coordinate in \( \mathbb{H}^4 \) to be 0.

2.2. Topological view of residual finiteness and Agol’s Proposition: When \( G \) is the fundamental group of a hyperbolic manifold, we have the following topological formulation of residual finiteness, which is used to prove Theorem 3.3: a manifold group \( \pi \) is residually finite if for every \( \alpha \in \pi_1(M) - \{1\} \) there exists a finite index cover \( \tilde{M} \) of \( M \) where the unique geodesic representative of \( \alpha \) does not lift (i.e. the preimage of \( \alpha \) in \( \tilde{M} \) is a nonclosed path).

We begin by letting \( P \) be an all right polyhedron in \( \mathbb{H}^n \). Then by the Poincaré Polyhedron Theorem, the images of \( P \) under the action of \( \Gamma_P \) will tessellate \( \mathbb{H}^n \).

**Definition 2.1.** The \( P \)-convexification of a compact set \( K \) in \( \mathbb{H}^n \), denoted by \( C_P(K) \), is the smallest closed, convex union of polyhedra in the tessellation of \( \mathbb{H}^n \) that contains \( K \).

Equivalently, we can define the convexification as the intersection of all half spaces bounded by the geodesic hyperplanes in our tessellation of \( \mathbb{H}^n \) that contain \( K \).

If \( M \) is a hyperbolic 3–manifold and \( \pi_1(M) \) embeds in \( \Gamma_P \) for an all right polyhedron \( P \) in \( \mathbb{H}^3 \), then \( M \) and all of its covers are also tiled by copies of \( P \). In [3], Agol, Long, and Reid prove that \( \Gamma_P \) is \( \Phi \)-subgroup separable for all geometrically finite subgroups \( \Phi \subset \Gamma_P \) by showing that the convexification of a compact set in \( \mathbb{H}^3/\Phi \) always involves a finite number of polyhedra. The proof relies on the fact that polyhedra sufficiently far away from the compact set \( K \) cannot lie in its convexification \( C_P(K) \). In [1], Agol proposes an explicit bound for what “sufficiently far” means.

For a geometrically finite subgroup \( \Phi \) of \( \Gamma_P \), we let \( \mathcal{K} \) be a compact subset of \( \mathbb{H}^3/\Phi \). We denote by \( N_R(Y) \) the \( R \)-neighborhood of \( Y \), where \( Y \) is the convex hull of \( \mathcal{K} \) in \( \mathbb{H}^3/\Phi \). Letting \( \tilde{Y} \) be the set of all lifts of \( Y \) under the covering map \( \mathbb{H}^3 \to \mathbb{H}^3/\Phi \), the convexification \( C_P(Y) \) maps down to a convex suborbifold \( C \) of \( \mathbb{H}^3/\Phi \), which is a union of polyhedra in the tessellation of \( \mathbb{H}^3/\Phi \). In fact, \( C \) is the smallest, closed convex union of polyhedra in the tessellation of \( \mathbb{H}^3/\Phi \) containing \( Y \). Agol proposed the following in [1]:

**Proposition 2.2** (Agol). *If a polyhedron in our tessellation of \( \mathbb{H}^3/\Phi \) is in \( \mathcal{C} \), then it must intersect \( N_R(Y) \), where \( R = \ln(\sqrt{3} + \sqrt{2}) \).*

We provide a calculation of Agol’s bound in case where \( \mathcal{K} \) is a geodesic line of \( \mathbb{H}^3 \) in the next section and then apply the analog of the author’s technique in [14] for quantifying residual finiteness of hyperbolic surface groups to obtain Theorem 3.3. We then prove Theorem 4.3 (the 4–dimensional generalization of Theorem 3.3) in Section 4.

3. The Proof of Theorem 3.3

Let \( M \) be a hyperbolic 3–manifold whose fundamental group embeds in \( \Gamma_P \) for some compact all right polyhedron in \( \mathbb{H}^3 \), and set \( \rho \) to be the hyperbolic metric on \( M \) induced by this embedding. For \( \alpha \in \pi_1(M) - \{1\} \), \( \Phi = \langle \alpha \rangle \) will denote the cyclic subgroup of \( \pi_1(M) \) generated by \( \alpha \). With \( X = \mathbb{H}^3/\Phi \), the cover of \( M \) corresponding to the subgroup \( \Phi = \langle \alpha \rangle \), we let \( \overline{\alpha} \) be the unique simple closed geodesic lift of \( \alpha \) to \( X \). In the setup of Agol’s Lemma, we take \( \overline{\alpha} \) to be our compact subset \( \mathcal{K} \) of \( X = \mathbb{H}^3/\Phi \) so that \( Y \), the convex hull of \( \alpha \), is equal to \( \overline{\alpha} \) itself.

Let \( \tilde{Y} \) be the geodesic axis in \( \mathbb{H}^3 \) for \( \alpha \in \pi_1(M) \). We note that \( \tilde{Y} \) is invariant under the action of \( \langle \alpha \rangle \) on \( \mathbb{D}^3 \) and consists of all lifts of \( \overline{\alpha} \) under the covering map \( \mathbb{H}^3 \to X \). The image of \( C_P(\tilde{Y}) \), denoted by \( \mathcal{C} \), in \( X \) is the smallest, closed, connected, convex union of polyhedra in the tessellation of \( X \) containing \( \overline{\alpha} \). We refer to \( \mathcal{C} \) as the convexification of \( \overline{\alpha} \) in \( X \) and have the following:
Lemma 3.1. Let $\mathcal{C}$ be the union of polyhedra forming the convexification of $\pi$ via the procedure mentioned above. Then any polyhedron $P_i \in \mathcal{C}$ must intersect $N = N_R(\tilde{\alpha})$ where $R = \ln(\sqrt{3} + \sqrt{2})$.

Proof. We define $\tilde{N}$ to be the set of all lifts of $N$ to $\mathbb{D}^3$. Thus, $\tilde{N}$ forms an $R$-neighborhood around $\tilde{\gamma}$. Suppose $P$ is a polyhedron in our tessellation of $\mathbb{D}^3$ which does not intersect $\tilde{N}$. We aim to show that a hyperplane in $\mathbb{D}^3$ containing one of the faces of $P$ must separate $P$ from $\tilde{\gamma}$, demonstrating that $P \notin \mathcal{C} = C_P(\tilde{\gamma})$ and proving the lemma.

Using Agol’s notation, we let $e$ be the cell of $P$ which is closest to $\tilde{\gamma}$ and set $k = \text{codim}(e)$. Note that $k$ is also the number of faces of $P$ that intersect $e$. Take a shortest geodesic $\tilde{\gamma}e$ from $e$ to $\tilde{\gamma}$ that intersects $\tilde{\gamma}$ at a point $y$ and $e$ at a point $p$. Then $\tilde{\gamma}e$ is an orthogeodesic with $\ell(\tilde{\gamma}e) > R$ since $P$ does not intersect the $R$-neighborhood of $\tilde{\gamma}$. We let $j$ be a hyperplane through $y$ that is perpendicular to $\tilde{\gamma}e$, which separates $e$ from $\tilde{\gamma}$. We note that $j$ necessarily contains $\tilde{\gamma}$.

Case $k = 3$: We first consider the case where $e$ is a vertex of $P$ and $k = 3$. We begin by sending $e = p$ to the origin of $\mathbb{D}^3$ via isometries. Since $P$ is an all right polyhedron, the three faces of $P$ that intersect $e$ necessarily lie in three hyperplanes $L_1$, $L_2$, and $L_3$ that are the isometric image of $L_x$, $L_y$, and $L_u$. Letting $\partial L_i = \partial\mathbb{D}^3 \cap L_i$, we see that the connected components of $\partial\mathbb{D}^3 - \{\partial L_1 \cup \partial L_2 \cup \partial L_3\}$ are right angled spherical triangles in $\partial\mathbb{D}^3$ with edge length $\frac{\pi}{2}$.

We note that we can assume that $\tilde{\gamma}$ passes through the north pole of the hyperplane $j$ (this will be important for our calculation of $R$). We apply isometries to $\mathbb{D}^3$ until $y_i$ and therefore $\tilde{\gamma}e = \tilde{\gamma}y_i$ lies on the line formed by $L_u \cap L_x$ and $\tilde{\gamma}$ lies in $L_x$. The distance $d(P, \tilde{\gamma})$ and the radius, $r$, of $j \cap \partial\mathbb{D}^3 : = \partial j$ are inversely proportional. Indeed, Figures 1a and 1b show that as the distance between $P$ and $\tilde{\gamma}$ (the length of $\tilde{\gamma}y_i$) grows, the radius of $\partial j$ becomes small.

![Figure 1](image)

We are therefore interested in the threshold $R$ of the distance between $P$ and $\tilde{\gamma}$ so that $\partial j$ can be inscribed in a right angled spherical triangle formed by the boundaries of three pairwise orthogonal hyperplanes. Then if $d(P, \tilde{\gamma}) > R$, at least one of the three circles $\partial L_i$ cannot intersect $\partial j$, and thus, one of the three hyperplanes $L_i$ must separate $P$ from $\tilde{\gamma}$.

We begin by calculating the radius of a circle inscribed in such a spherical triangle, whose edge length is $\frac{\pi}{2}$. In Figure 2a, $A$, $B$ and $C$ are the midpoints of the three edges in our spherical triangle, which are also the points of tangency for the inscribed circle. Since the triangle is formed by the intersection of three pairwise orthogonal hyperplanes in $\mathbb{D}^3$ with $\partial\mathbb{D}^3$, the unit sphere in $\mathbb{R}^3$, the length of the edge $DE$ is equal to $\frac{\pi}{2}$, and the length of $DB$ is $\frac{\pi}{4}$.
To calculate the radius $r$ of the inscribed circle we apply the following Spherical Law of Cosines. Let $T$ be a spherical triangle with angles $\alpha$, $\beta$ and $\gamma$, and with edges of lengths $a$, $b$ and $c$ opposite the angles $\alpha$, $\beta$ and $\gamma$, respectively. Then,

$$
\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.
$$

For the triangle in Figure 2b, we have $\cos \frac{\pi}{4} = \sin \frac{\pi}{2} \sin \frac{\pi}{3} \cos r$, and therefore, $r = \cos^{-1} \left( \frac{\sqrt{2}}{\sqrt{3}} \right)$.

Now we calculate the distance $R = \text{length of } \overrightarrow{y} = d(P, \tilde{Y})$ so that the radius $r$ of $\partial j$ is $\cos^{-1} \left( \frac{\sqrt{2}}{\sqrt{3}} \right)$.

Consider the cross sectional view formed by the intersection of the hyperplane $L_x$ with our setup in Figures 1a, 1b above. This view is represented in Figure 3a.
Since $\partial D^3 \cap \partial L_x$ is a unit circle we know that $\theta = r = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Therefore, the point $x$ in the figure is $\cos \theta = \cos r = \frac{\sqrt{2}}{\sqrt{3}}$. The point $c$ is the center of the circular completion of our geodesic $\tilde{Y}$ and with $q$ the radius of this circle, we see that $y = c - q$.

Using the similar triangles in Figure 3b we calculate that $c = \frac{\sqrt{3}}{\sqrt{2}}$ and, thus, that $q = \frac{1}{\sqrt{2}}$. It follows that $y = \frac{\sqrt{3}-1}{\sqrt{2}}$ and $R = d(0, \tilde{Y}) = \ln\left(\frac{1+y}{1-y}\right)$, which by a simple calculation gives us $R = \ln(\sqrt{2} + \sqrt{3})$.

**Case** $k = 2$: The case where $e$ is an edge of $P$ can be handled in a similar way. In this case, we show that $R = \ln(\sqrt{2} + 1)$.

We assume the same setup as in the previous case where the point $p$ on the edge $e$ that is closest to $\tilde{Y}$ is at the origin of $D^4$ and $y$ lying on $L_u \cap L_x$. The extensions of the two faces of $P$ that intersect $e$ form a pair of orthogonal hyperplanes, $L_1$ and $L_2$, in $D^3$. Their boundaries, $\partial L_1$ and $\partial L_2$, form a spherical bi-disk with angles $\frac{\pi}{2}$. We are looking for the threshold $R$ such that $\partial j$, and thus $\tilde{Y}$, is tangent to such a spherical bi-disk at the endpoints of $\tilde{Y}$. Then, if $d(0, \tilde{Y}) > R$, one of the $\partial L_i$ cannot intersect $\partial j$, and one of the hyperplanes $L_i$ must therefore separate $P$ from $\tilde{Y}$.

A cross sectional view of this situation is shown in Figure 4 below. Given the triangle in the figure, we know that $c = \sqrt{2}$ so that $y = \sqrt{2} - 1$. Thus, $R = \ln\left(\frac{1+\sqrt{2}-1}{1-\sqrt{2}+1}\right) = \ln(\sqrt{2} + 1)$.

![Figure 4](image)

**Case** $k = 1$: Lastly, we consider the case where $e$ is a face of $P$, that is the case where $k = 1$. Then $\overline{pj}$ is an orthogeodesic between the hyperplane containing $e$, which we also call $e$ for notational simplicity, and the hyperplane $j$. Taking any $R > 0$ is sufficient in this case since $e$ itself is a face of $P$ whose hyperplane extension separates $P$ from $\tilde{Y}$ (see Figure 3 below).

We take $R$ to be the largest of the values in the three cases, i.e. $R = \ln(\sqrt{2} + \sqrt{3})$. If $P$ does not intersect $\overline{N} = N_R(\tilde{Y})$, then there is a face of $P$ whose hyperplane extension separates $P$ from $\tilde{Y}$ so that $P \notin \tilde{C}$. Therefore, any polyhedron $P_i$ in $X = H^3/\langle \alpha \rangle$ that is in the convexification $\overline{C}$ of $\overline{\alpha}$ must intersect the $\ln(\sqrt{2} + \sqrt{3})$-neighborhood $N$ of $\overline{\alpha}$.

Given the above lemma, we have that $\overline{C} \subset N_{R+d_P}(\overline{\alpha})$, where $d_P$ is the diameter of $P$. The following lemma allows us to calculate the volume of $N_{R+d_P}(\overline{\alpha})$. 

7
Lemma 3.2. We let $\Omega$ be the solid tubular neighborhood of a geodesic segment in $H^3$ between the points $(0,0,R_0)$ and $(0,0,r_0)$ on the $u$–axis of $H^3$ as shown in Figure 5. Then $\text{Vol}(\Omega) = \pi \sinh^2(b) \ell$, where $\ell = \ln(R_0/r_0)$ is the length of the geodesic between the points $(0,0,R_0)$ and $(0,0,r_0)$ in $H^3$.

Proof. For this volume calculation we find it convenient to use spherical coordinates. The volume form on $H^3$, $\frac{dx\wedge dy\wedge du}{u^3}$, becomes $\frac{1}{r} \tan \phi \sec^2 \phi \, dr \wedge d\phi \wedge d\theta$. Let $\pi(b)$ be the angle of parallelism in the figure formed by the intersection of $\Omega$ with the plane $L_y'$. A simple calculation tells us that the range of values for $\phi$ in $\Omega$ is then $[0, \pi/2 - \pi(b)]$. Therefore,

$$\text{Vol}(\Omega) = \iiint_{\Omega} \frac{1}{r} \tan \phi \sec^2 \phi \, dr \wedge d\phi \wedge d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2} - \pi(b)} \int_{r_0}^{R_0} \frac{1}{r} \tan \phi \sec^2 \phi \, dr \wedge d\phi \wedge d\theta$$

$$= \ln \left( \frac{R_0}{r_0} \right) \int_0^{2\pi} \int_0^{\frac{\pi}{2} - \pi(b)} \tan \phi \sec^2 \phi \, d\phi \wedge d\theta = \ell \left[ \frac{\tan^2 \phi}{2} \right]_0^{\frac{\pi}{2} - \pi(b)} \int_0^{2\pi} d\theta$$

$$= \frac{\pi}{\tan \pi(b)} \ell = \pi \sinh^2(b) \ell,$$

with the last equality coming from the angle of parallelism laws in hyperbolic space.
We can lift \( N_{R+d_P}(\alpha) \) isometrically to \( \mathbb{H}^3 \) so that it forms a region like \( \Omega \) from the previous lemma where \( b = R + d_P \). Lemma 3.2 then implies that \( \text{Vol}(C) < \text{Vol}(N_{R+d_P}(\alpha)) = \pi \sinh^2(R + d_P) \ell_P(\alpha) \). Thus, we have the following theorem:

**Theorem 3.3.** Let \((M, \rho)\) be any compact hyperbolic 3–manifold whose fundamental group embeds in a right-angled Coxeter group \( \Gamma_P \), associated to an all right polyhedron \( P \) in \( \mathbb{H}^3 \). Then for any \( \alpha \in \pi_1(M) - \{1\} \), there exists a subgroup \( H' \) of \( \pi_1(M) \) such that \( \alpha \not\in H' \). The index of \( H' \) is bounded above by

\[
\frac{2\pi}{V_P} \sinh^2(\ln(\sqrt{3} + \sqrt{2}) + d_P) \ell_P(\alpha),
\]

where \( \ell_P(\alpha) \) is the length of the unique geodesic representative of \( \alpha \) and where \( d_P \) and \( V_P \) are the diameter and volume of \( P \), respectively.

**Proof.** We know that \( \text{Vol}(C) < \pi \sinh^2(R + d_P) \ell_P(\alpha) \) so if \( C \) consists of \( k \) polyhedra,

\[
k < \frac{\pi}{V_P} \sinh^2(\ln(\sqrt{3} + \sqrt{2}) + d_P) \ell_P(\alpha).
\]

Let \( \tilde{\alpha} \) be one lift of \( \alpha \) to \( \mathbb{H}^3 \). Using the right-angled tiling of \( X \), we can lift \( C \) to \( \mathbb{H}^3 \) so that the result is a connected, convex union of \( k \) polyhedra denoted by \( \mathcal{C} \). The convexity of the lift is crucial since we will want to apply the Poincaré Polyhedron Theorem to prove the result above.

Set \( \tilde{\alpha}_1 \) to be one of the two lifts of \( \tilde{\alpha} \) that share endpoints with \( \tilde{\alpha} \). If \( \mathcal{C}_1 \) be the convex lift of \( C \) containing \( \tilde{\alpha}_1 \), then \( \mathcal{C}' = \mathcal{C} \cup \mathcal{C}_1 \) is a convex union of \( 2k \) polyhedra in \( \mathbb{H}^3 \), such that one endpoint of \( \tilde{\alpha} \) is contained in the interior of \( \mathcal{C}' \).

Denote by \( H \) the group of isometries of \( \mathbb{H}^3 \) generated by reflections in the sides of \( \mathcal{C}' \). Then \( H < \Gamma \), and \( \mathcal{C}' \) is a fundamental domain for the action of \( H \) on \( \mathbb{H}^3 \) by the Poincaré Polyhedron Theorem. Since \( \mathcal{C}' \) contains \( 2k \) polyhedra, \( [\Gamma : H] = 2k \). Letting \( p : \mathbb{H}^3 \rightarrow \mathbb{H}^3/H \) be the covering map, we then have that the restriction of \( b \) to \( \mathcal{C}' \) is a homeomorphism onto its image in \( \mathbb{H}^3/H \). Thus, \( p(\tilde{\alpha}) \) is not a loop in \( \mathbb{H}^3/H \), and \( \alpha \not\in H \).

Now, if \( H' = H \cap \pi_1(M) \), then \( \alpha \not\in H' \) and \( [\pi_1(M) : H'] \leq [\Gamma : H] = 2k \). The result follows.

\[\square\]

4. The Proof of Theorem 4.3

In this section we obtain the analogous results of the previous section for 4–manifold groups that embed in a right-angled Coxeter group, \( \Gamma_P \), corresponding to a compact all right polyhedron \( P \) in \( \mathbb{H}^4 \). We again denote the diameter and volume of \( P \) by \( d_P \) and \( V_P \), respectively. As before, the images of \( P \) under the action of \( \Gamma_P \) tessellate \( \mathbb{H}^4 \). We have the following analog of Lemma 3.2

**Lemma 4.1.** Let \( \Omega' \) be the analog in \( \mathbb{H}^4 \) of \( \Omega \) from Lemma 3.2. That is to say \( \Omega' \) is a tubular \( b \)–neighborhood of the geodesic segment between the points \((0,0,0,R_0)\) and \((0,0,0,r_0)\) lying on the \( x_4 \)-axis in \( \mathbb{H}^4 \). Then \( \text{Vol}(\Omega') = \frac{4}{3} \pi \sinh^3(b) \ell \), where \( \ell = \ln(R_0/r_0) \) is the length of the geodesic between the points \((0,0,0,R_0)\) and \((0,0,0,r_0)\) in \( \mathbb{H}^4 \).

**Proof.** To calculate the volume of \( \Omega' \) we again find it convenient to use generalized spherical coordinates. In \( \mathbb{H}^4 \) spherical coordinates are defined by:

\[
\begin{align*}
x_1 &= r \sin \phi_1 \sin \phi_2 \sin \phi_3 \\
x_2 &= r \sin \phi_1 \sin \phi_2 \cos \phi_3 \\
x_3 &= r \sin \phi_1 \cos \phi_2 \\
x_4 &= r \cos \phi_1,
\end{align*}
\]

where \( \phi_1 \in [0, \frac{\pi}{2}) \), \( \phi_2 \in [0, \pi] \), \( \phi_3 \in [0, 2\pi] \). Thus, the volume form on \( \mathbb{H}^4 \) is
\[
\frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{x_4^4} = \left( \frac{1}{r} \tan^2 \phi_1 \sec^2 \phi_1 \sin \phi_2 \right) dr \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3.
\]

As before, a simple calculation tells us that the range of values for \( \phi_1 \) in \( \Omega' \) is \([0, \pi/2 - \pi(b)]\), where \( \pi(b) \) is the angle of parallelism in the figure formed by \( \Omega' \cap L_{x_1}' \cap L_{x_2}' \), which is the same as the figure formed by \( \Omega \cap L_y \) in the proof of Lemma 3.2. Additionally, the range of values of \( \phi_2 \) and \( \phi_3 \) in \( \Omega' \) are unrestricted so that \( \phi_2 \) takes values in \([0, \pi]\) and \( \phi_3 \) takes values in \([0, 2\pi]\). Thus,

\[
\text{Vol}(\Omega') = \iiint_{\Omega'} \left( \frac{1}{r} \tan^2 \phi_1 \sec^2 \phi_1 \sin \phi_2 \right) dr \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3
\]

\[
= \int_0^{2\pi} \int_0^\pi \int_0^{\frac{\pi}{2} - \pi(b)} \int_{r_0}^{R_0} \left( \frac{1}{r} \tan^2 \phi_1 \sec^2 \phi_1 \sin \phi_2 \right) dr \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3
\]

\[
= \ln \left( \frac{R_0}{r_0} \right) \int_0^{2\pi} \int_0^\pi \int_0^{\frac{\pi}{2} - \pi(b)} \tan^2 \phi_1 \sec^2 \phi_1 \sin \phi_2 \ d\phi_1 \wedge d\phi_2 \wedge d\phi_3
\]

\[
= \ell \left[ \frac{\tan^3 \phi_1}{3} \right]_{0}^{\frac{\pi}{2} - \pi(b)} \int_0^{2\pi} \int_0^\pi \sin \phi_2 \ d\phi_2 \wedge d\phi_3
\]

\[
= \frac{2\pi}{3\tan^3 \pi(b)} \ell \left[ -\cos \phi_2 \right]_{0}^{\pi} = \frac{4\pi}{3\pi \sinh^3 (b)} \ell = \frac{4}{3} \pi \sinh^3 (b) \ell,
\]

with the last equality coming from the angle of parallelism laws in hyperbolic space.

Next we prove the analog of Lemma 3.1 Let \( M \) be a hyperbolic 4-manifold whose fundamental group embeds in \( \Gamma_P \) for some compact all right polyhedron \( P \) in \( \text{H}^4 \), and set \( \rho \) to be the hyperbolic metric on \( M \) induced by this embedding. As in the previous section, \( \Phi = \langle \alpha \rangle \) is the cyclic subgroup generated by \( \alpha \), \( X \) is the cover corresponding to \( \Phi \), and \( Y = \overline{\alpha} \) is the unique simple closed geodesic lift of \( \alpha \) to \( X \). With \( \overline{Y} \) as the geodesic axis for \( \alpha \), consisting of all lifts of \( \overline{\alpha} \), we again denote the convexification of \( \overline{\alpha} \) in \( X \) by \( \mathcal{C} \).

**Lemma 4.2.** Let \( \mathcal{C} \) be the union of polyhedra forming the convexification of \( \overline{\alpha} \) in \( X \) via the procedure mentioned above. Then any polyhedron \( P_i \in \mathcal{C} \) must intersect \( N = N_R(\overline{\alpha}) \), the \( R \)-neighborhood of \( \overline{\alpha} \), where \( R = \ln (2 + \sqrt{3}) \).

**Proof.** As in the proof of Lemma 3.1 we let \( \overline{N} \) be the set of all lifts of \( N \) to \( \text{D}^4 \), which forms an \( R \)-neighborhood around \( \overline{Y} \). Suppose \( P \) is a polyhedron in our tessellation of \( \text{D}^4 \) which does not intersect \( \overline{N} \). We aim to show that a hyperplane in \( \text{D}^4 \) containing one of the faces of \( P \) must separate \( P \) from \( \overline{Y} \), demonstrating that \( P \not\in \mathcal{C} \) and proving the lemma.

Letting \( e \) be the cell of \( P \) which is closest to \( \overline{Y} \), we set \( k = \text{codim}(e) \). Again, \( k \in [1, 4] \) is also the number of faces of \( P \) that intersect \( e \). Take a shortest geodesic \( \overline{py} \) from \( e \) to \( \overline{Y} \) that intersects \( \overline{Y} \) at a point \( y \) and \( e \) at a point \( p \). Then \( \overline{py} \) is an orthogeodesic with \( \ell(\overline{py}) \geq R \) since \( P \) does not intersect the \( R \)-neighborhood of \( \overline{Y} \). We let \( j \) be a hyperplane through \( y \) that is perpendicular to \( \overline{py} \), which separates \( e \) from \( \overline{Y} \). Again \( j \) necessarily contains \( \overline{Y} \).

We note that the proof in the cases \( k = 3, 2, 1 \) are exactly the proofs for \( k = 3, 2, 1 \), respectively, of Lemma 3.1. We therefore consider the case where \( e \) is a vertex of \( P \) and \( k = 4 \). We begin by sending \( e = p \) to the origin of \( \text{D}^4 \) via isometries. Recall that \( L_{x_i} \) is the hyperplane of \( \text{D}^4 \) obtained
by restricting the $x_i$ coordinate to zero for $i = 1, 2, 3, 4$. Since $P$ is an all right polyhedron, the four faces of $P$ that intersect $e$ necessarily lie in four hyperplanes $L_1, L_2, L_3,$ and $L_4$ that are the isometric image of $L_{x_1}, L_{x_2}, L_{x_3},$ and $L_{x_4}$. Letting $\partial L_i = \partial D^3 \cap L_i$, we see that $\partial L_1, \partial L_2, \partial L_3,$ and $\partial L_4$ form an all right-angled spherical tetrahedron in $\partial D^4$. We are, therefore, interested in the threshold, $R$, of the distance between $P$ and $\tilde{Y}$ so that $\partial j$ can be inscribed in an all right spherical tetrahedron formed by the boundaries of four pairwise orthogonal hyperplanes. Then if $d(P, \tilde{Y}) > R$, at least one of the four two-spheres $\partial L_i$ cannot intersect $\partial j$, and thus, one of the four hyperplanes $L_i$ must separate $P$ from $\tilde{Y}$.

We begin by calculating the radius $r$ of a sphere inscribed in such an all right tetrahedron as shown in Figure 7. As indicated by the figure, we calculate the radius $r$ using a spherical triangle formed by a vertex, $A$ of the tetrahedron, a point of tangency $M$ between the inscribed sphere and the tetrahedron, and the center $O$ of the sphere.

Each face of the tetrahedron is a spherical right-angled triangle. Applying a second Spherical Law of Cosines to Figure 8, we see that the length of the segment $a = \overline{AM} = \arctan \sqrt{2}$ since $\cos \frac{\pi}{2} = \cos^2 a + \sin^2 a \cdot \cos \frac{2\pi}{3}$. Thus, the triangle with vertices $A, M,$ and $O$ forms Figure 9.
Another application of the second Spherical Law of Cosines gives that
\[
\cos \theta = -\cos \frac{\pi}{2} \cdot \cos \frac{\pi}{4} + \sin \frac{\pi}{2} \cdot \sin \frac{\pi}{4} \cdot \cos(\arctan \sqrt{2}) = \frac{1}{\sqrt{6}}.
\]
Then, substituting \( \theta = \cos^{-1} \left( \frac{1}{\sqrt{6}} \right) \) into the first Spherical Law of Cosines
\[
\cos \frac{\pi}{4} = -\cos \frac{\pi}{2} \cdot \cos \theta + \sin \frac{\pi}{2} \cdot \sin \theta \cdot \cos r
\]
gives \( r = \arccos \frac{\sqrt{3}}{2} \).

We now use the same cross sectional picture as in the proof of Lemma 3.1 which is shown in Figures 3a and 3b. The point \( x \) in the figure is now \( \cos r = \frac{\sqrt{3}}{\sqrt{4}} \). The point \( c \) is the center of the circular completion of our geodesic \( \tilde{Y} \) and with \( q \) the radius of this circle, we see that \( y = c - q \). Using the similar triangles in Figure 3a we calculate that \( c = \frac{\sqrt{3}}{\sqrt{3}} \) and, thus, that \( q = \frac{1}{\sqrt{3}} \). It follows that \( y = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \) and \( R = d(0, \tilde{Y}) = \ln \left( \frac{1+q}{1-q} \right) \), which by a simple calculation gives us
\[
R = \ln(2 + \sqrt{3}) = \ln(2 + \sqrt{3}).
\]

Taking the maximum \( R \) over the four cases gives \( R = \ln(2 + \sqrt{3}) \).

The proof of Theorem 4.3 now follows exactly as the proof of Theorem 3.3. Lemma 4.2 tells us that \( \tilde{\mathcal{C}} \subset N_{R+d_P}(\tilde{\alpha}) \), where \( R \) can be taken to be \( \ln(2 + \sqrt{3}) \). We can lift \( N_{R+d_P}(\tilde{\alpha}) \) isometrically to \( H^4 \) so that it forms a region like \( \Omega' \) from Lemma 4.1 where \( b = R + d_P \). The lemma then implies that \( \text{Vol}(\mathcal{C}) < \text{Vol}(N_{R+d_P}(\tilde{\alpha})) = \frac{4\pi}{3} \sinh^3(R + d_P) \ell_P(\alpha) = \frac{4\pi}{3} \sinh^3(\ln(2 + \sqrt{3}) + d_P) \ell_P(\alpha) \).

**Theorem 4.3.** Let \( (M, \rho) \) be any compact hyperbolic 4–manifold whose fundamental group embeds in a right-angled Coxeter group \( \Gamma_P \), associated to an all right polyhedron \( P \) in \( H^4 \). Then for any \( \alpha \in \pi_1(M) - \{1\} \), there exists a subgroup \( H' \) of \( \pi_1(M) \) such that \( \alpha \notin H' \). The index of \( H' \) is bounded above by
\[
\frac{8\pi}{3V_P} \sinh(\ln(2 + \sqrt{3}) + d_P) \ell_P(\alpha),
\]
where \( \ell_P(\alpha) \) is the length of the unique geodesic representative of \( \alpha \) and where \( d_P \) and \( V_P \) are the diameter and volume of \( P \), respectively.

**Proof.** We know that \( \text{Vol}(\mathcal{C}) < \frac{4\pi}{3} \sinh^3(\ln(2 + \sqrt{3}) + d_P) \ell_P(\alpha) \) so if \( \mathcal{C} \) consists of \( k \) polyhedra,
\[
k < \frac{4\pi}{3V_P} \sinh^3(\ln(2 + \sqrt{3}) + d_P) \ell_P(\alpha).
\]
We form the convex set \( \mathcal{C}' = \tilde{\mathcal{C}} \cup \tilde{\mathcal{C}}_1 \) as in the proof of Theorem 3.3. Thus, \( C' \) is the convex union of \( 2k \) copies of \( P \) containing one endpoint of a lift \( \tilde{\alpha} \) of \( \alpha \) to \( H^4 \) in its interior.

Let \( H \) be the group of isometries of \( H^4 \) generated by reflections in the sides of \( C' \). Then \( H < \Gamma \), and \( C' \) is a fundamental domain for the action of \( H \) on \( H^4 \) by the Poincaré Polyhedron Theorem. Thus, the image of \( \tilde{\alpha} \) is not a loop in \( H^4/H \), and \( \alpha \notin H \). Now, let \( H' = H \cap \pi_1(M) \). Then, \( \alpha \notin H' \) and \( \left[ \pi_1(M):H' \right] \leq [\Gamma:H] = 2k \). The result follows.

An immediate corollary of Theorems 3.3 and 4.3 is:

**Corollary 4.4.** Let \( (M, \rho) \) be a compact hyperbolic 3–manifold or 4–manifold with an embedding \( \pi_1(M) \hookrightarrow \Gamma_P \) for some all right polyhedron \( P \) in \( H^3 \) or \( H^4 \), respectively. The geodesic residual finiteness growth is at most linear. That is to say, \( F_{M,\rho} \leq n \).
5. Extension to Groups that Virtually Embed in $\Gamma_P$

In this section we extend Corollary 4.4 to all hyperbolic 3–manifold and 4–manifold groups that virtually embed in a right-angled Coxeter group of the form $\Gamma_P$, where $P$ is of the appropriate dimension. The key fact is the following lemma, which is the analog of [5] Lemma 2.2 for geodesic residual finiteness growth functions (rather than the usual residual finiteness growth functions, calculated with respect to word length).

**Lemma 5.1.** Let $(M,\rho)$ be a compact hyperbolic $n$–manifold and let $K \leq \pi_1(M)$ be a finite index subgroup with $[\pi_1(M) : K] = C$. Let $(M',\rho')$ be the cover of $M$ of index $C$ corresponding to the subgroup $K$. Then the geodesic residual finiteness function for $(M,\rho)$ is bounded by that of $(M',\rho')$. That is to say, $F_{M,\rho} \leq C \cdot F_{M',\rho'}$ and hence $F_{M,\rho} \leq F_{M',\rho'}$.

**Proof.** For an element $\alpha \in \pi_1(M)$, we see that $\{ H \leq \pi_1(M) : \alpha \notin H \} \supseteq \{ K' \leq K \leq \pi_1(M) : \alpha \notin K' \}$. Therefore,

$$D_{\pi_1(M)}(\alpha) = \min \{ [\pi_1(M) : H] : \alpha \notin H, H \leq \pi_1(M) \}$$

$$\leq \min \{ [\pi_1(M) : K'] : \alpha \notin K', K' \leq K \leq \pi_1(M) \}$$

$$= C \min \{ [K : K'] : \alpha \notin K', K' \leq K \} = C \cdot D_K(\alpha),$$

where we set $D_K(\alpha) = 1$ if $\alpha \notin K$. The equality above comes from the fact that $[\pi_1(M) : K'] \leq [\pi_1(M) : K][K : K'] = C[K : K]$.

Next, we claim that

$$F_{M,\rho}(n) = \max \{ D_{\pi_1(M)}(\alpha) : \alpha \in \pi_1(M) - \{1\}, \ell_\rho(\alpha) \leq n \}$$

$$\leq \max \{ C \cdot D_K(\beta) : \beta \in K - \{1\}, \ell_{\rho'}(\beta) \leq n \} = C \cdot F_{M',\rho'}.$$

First, observe that $D_{\pi_1(M)}(\alpha) \leq C$ for all $\alpha \notin K$. Thus, if the maximum value for $D_{\pi_1(M)}(\alpha)$ on the set $\{ \alpha \in \pi_1(M) - \{1\} : \ell_\rho(\alpha) \leq n \}$ is achieved by an element $\alpha \notin K$, the above inequality follows from the fact that $F_{M',\rho'} \geq 1$. Additionally, we have that for all $\alpha \in K \leq \pi_1(M)$, $\ell_\rho(\alpha) = \ell_{\rho'}(\alpha)$. If the maximum value of $D_{\pi_1(M)}(\alpha)$ on the set $\{ \alpha \in \pi_1(M) - \{1\} : \ell_\rho(\alpha) \leq n \}$ is achieved by an element $\alpha \in K \leq \pi_1(M)$, then $\alpha \in \{ \beta \in K - \{1\} : \ell_\rho'(\beta) \leq n \}$. Therefore, the maximum value of $C \cdot D_K(\beta)$ on the set $\{ \beta \in K - \{1\} : \ell_\rho'(\beta) \leq n \}$ is at least as big as $C \cdot D_K(\alpha) = D_{\pi_1(M)}(\alpha)$, and the inequality follows.

Corollary 4.4 together with Lemma 5.1 gives the following main result:

**Corollary 5.2.** If $(M,\rho)$ is a compact hyperbolic 3–manifold or 4–manifold whose fundamental group virtually embeds in $\Gamma_P$ for some compact all right polyhedron $P$ in $\mathbb{H}^3$ or $\mathbb{H}^4$, then we have that $F_{M,\rho} \leq n$.

Again, we note that Corollary 5.2 fits into a body of work dedicated to bounding the residual finiteness growths of various classes of groups, which include free groups, surface groups, and the virtually special groups (see for instance [4, 5, 6, 8, 9, 12, 14, 16]).

6. Generalization to Higher Dimensions

Given the generalization to 4–dimensional hyperbolic manifolds obtained in Section 4 it is natural to ask whether Theorem 3.3 could actually be generalized to compact hyperbolic $n$–manifolds whose fundamental groups embed in $\Gamma_P$ for some all right polyhedron in $\mathbb{H}^n$. Discussions with Agol have lead us to believe that there should be a higher dimensional analog of Lemmas 3.1 and 4.2. In particular, given a compact hyperbolic $n$–manifold $M$ whose fundamental group embeds...
on $\Gamma_P$ for some all right polyhedron in $\mathbb{H}^n$ and an element $\alpha \in \pi_1(M) - \{1\}$, Agol proposed the following:

**Conjecture 6.1 (Agol).** Let $\mathcal{C}$ be the union of polyhedra forming the convexification of $\overline{\alpha}$ in $\mathbb{H}^n/\langle \alpha \rangle$ via the procedure mentioned above. Then any polyhedron $P_i \in \mathcal{C}$ must intersect $N = N_R(\overline{\alpha})$, the $R$–neighborhood of $\overline{\alpha}$, where $R = \ln(\sqrt{n} + \sqrt{n - 1})$.

For an $n$–dimensional generalization of Theorem 3.3 we would need the following generalization of Lemma 3.2.

**Lemma 6.2.** Let $\Omega_n$ be the analog in $\mathbb{H}^n$ of $\Omega$ from Lemma 3.2. That is to say $\Omega_n$ is a tubular $b$–neighborhood of the geodesic segment between the points $(0, \ldots, R_0)$ and $(0, \ldots, r_0)$ lying on the $x_n$–axis in $\mathbb{H}^n$. Then $\text{Vol}(\Omega_n) = \text{Vol}(B^{n-1}) \sinh^{n-1}(b) \ell$, where $\ell = \ln(R_0/r_0)$ is the length of the geodesic between the points $(0, \ldots, R_0)$ and $(0, \ldots, r_0)$ in $\mathbb{H}^n$ and $\text{Vol}(B^{n-1})$ is the volume of the unit ball in $\mathbb{R}^{n-1}$.

**Proof.** To calculate the volume of $\Omega_n$ we use generalized spherical coordinates. In $\mathbb{H}^n$ spherical coordinates are defined by:

$$
x_1 = r \sin \phi_1 \cdots \sin \phi_{n-2} \sin \phi_{n-1}
$$

$$
x_2 = r \sin \phi_1 \cdots \sin \phi_{n-2} \cos \phi_{n-1}
$$

$$
\vdots
$$

$$
x_{n-2} = r \sin \phi_1 \sin \phi_2 \cos \phi_3
$$

$$
x_{n-1} = r \sin \phi_1 \cos \phi_2
$$

$$
x_n = r \cos \phi_1,
$$

where $\phi_1 \in [0, \frac{\pi}{2})$, $\phi_2, \ldots, \phi_{n-2} \in [0, \pi]$, and $\phi_{n-1} \in [0, 2\pi]$. Thus, the volume form on $\mathbb{H}^n$ is

$$
\left(\frac{1}{r} \tan^{n-2} \phi_1 \sec^2 \phi_1 \sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{dr} \wedge d\phi_1 \wedge d\phi_2 \cdots \wedge d\phi_{n-1}.
$$

As before, a simple calculation tells us that the range of values for $\phi_1$ in $\Omega_n$ is $[0, \pi/2 - \pi(b)]$, where $\pi(b)$ is the angle of parallelism for $b$. Thus,

$$
\text{Vol}(\Omega_n) = \int_{\Omega_n} \cdots \int_{\Omega_n} \left(\frac{1}{r} \tan^{n-2} \phi_1 \sec^2 \phi_1 \sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{dr} \wedge d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_{n-1}
$$

$$
= \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_{r_0}^{R_0} \left(\frac{1}{r} \tan^{n-2} \phi_1 \sec^2 \phi_1 \sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{dr} \wedge d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_{n-1}
$$

$$
= \ln \left(\frac{R_0}{r_0}\right) \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^{\pi/2 - \pi(b)} \left(\tan^{n-2} \phi_1 \sec^2 \phi_1 \sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{dr} \wedge d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_{n-1}
$$

$$
= \ell \left[\tan^{n-1} \phi_1 \right]_0^{\pi/2 - \pi(b)} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \left(\sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{d}\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_{n-1}
$$

$$
= \ell \left[\tan^{n-1} \pi(b)\right]_0^{\pi/2 - \pi(b)} \left(\frac{1}{n-1} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \left(\sin^{n-3} \phi_2 \cdots \sin \phi_{n-2}\right) \ \text{d}\phi_2 \wedge \cdots \wedge d\phi_{n-1}\right)
$$

$$
= \text{Vol}(B^{n-1}) \sinh^{n-1}(b) \ell,
$$

where $\ell = \ln(R_0/r_0)$ is the length of the geodesic between the points $(0, \ldots, R_0)$ and $(0, \ldots, r_0)$ in $\mathbb{H}^n$ and $\text{Vol}(B^{n-1})$ is the volume of the unit ball in $\mathbb{R}^{n-1}$. 


with the last equality coming from the angle of parallelism laws in hyperbolic space.

Conjecture 6.1 together with Lemma 6.2 yield the following conjecture:

**Conjecture 6.3.** Let \((M, \rho)\) be any compact hyperbolic n–manifold whose fundamental group embeds in a right-angled Coxeter group \(\Gamma_P\), associated to an all right polyhedron \(P\) in \(H^n\). Then for any \(\alpha \in \pi_1(M) - \{1\}\), there exists a subgroup \(H'\) of \(\pi_1(M)\) such that \(\alpha \notin H'\). The index of \(H'\) is bounded above by

\[
\frac{2\text{Vol}(B^{n-1})}{V_P} \sinh^{n-1} \left( \ln(\sqrt{n} + \sqrt{n-1}) + d_P \right) \ell_\rho(\alpha),
\]

where \(\ell_\rho(\alpha)\) is the length of the unique geodesic representative of \(\alpha\), where \(d_P\) and \(V_P\) are the diameter and volume of \(P\), respectively, and \(B^{n-1}\) is the volume of the unit ball in \(R^{n-1}\).

If Conjecture 6.3 holds, then Lemma 5.1 would tell us that the geodesic residual finiteness growth of a compact hyperbolic n–manifold group that virtually embeds in a right-angled Coxeter group of the form \(\Gamma_P\) is at most linear.

**References**

[1] I. Agol, Geometrically Finite Subgroup Separability for the Figure-8 Knot Group, Preliminary Report/ Private Communication.

[2] I. Agol, The virtual Haken conjecture, with an appendix by I. Agol, D. Groves and J. Manning, Documenta Math. 18 (2013), 1045-1087.

[3] I. Agol, D. D. Long, and A. W. Reid, The Bianchi groups are separable on geometrically finite subgroups, Ann. of Math. (2) 152 (2001), no. 3, 599-621.

[4] K. Bou-Rabee, Quantifying Residual Finiteness, Journal of Algebra 323 (2010), 729-737.

[5] K. Bou-Rabee and M. Hagen and P. Patel, Quantifying residual finiteness of virtually special groups, Math. Z. to appear. arXiv:1402.6974 [math.GR].

[6] K. Bou-Rabee and D. B. McReynolds, Asymptotic growth and least common multiples in groups, Bull. London Math. Soc. 43(6) (2011), 1059-1068.

[7] K. Bou-Rabee and D.B. McReynolds, Characterizing linear groups in terms of growth properties. arXiv:1403.0983.

[8] K. Bou-Rabee and D.B. McReynolds, Extremal behavior of divisibility functions, Geom. Dedicata to appear. arXiv:1211.4727 [math.GR].

[9] N. V. Buskin, Efficient separability in free groups, Siberian Mathematical Journal 50(4) (2009), 603-608.

[10] F. Haglund, Finite index subgroups of graph products, Geom. Dedicata 135 (2008), 167-209.

[11] F. Haglund and D.T. Wise, Special cube complexes, Geom. Funct. Anal. 17 (2008), no. 5, 1551–1620.

[12] M. Kassabov and F. Matucci, Bounding the Residual Finiteness of Free Groups, Proc. Amer. Math. Soc. 139 (July 2011), no. 7, 2281-2286.

[13] A. I. Mal’cev, On the faithful representation of infinite groups by matrices, Mat. Sb. 8(50) (1940), 405-422.

[14] P. Patel, On a Theorem of Peter Scott, Proc. Amer. Math. Soc. 142 (8) (2014), 2891-2906.

[15] P. Patel, Quantifying Algebraic Properties of Surface Groups and 3-Manifold Groups in Terms of Geometric Data, Rutgers University, 2013.

[16] I. Rivin, Geodesics with one self-intersection, and other stories, Adv. Math 231 (5) (2012), 2391-2412.

[17] P. Scott, Subgroups of Surface Groups Are Almost Geometric, J. London Math. Soc. (2) 17 (1978), 555-565.

[18] P. Scott, Correction to ‘Subgroups of Surface Groups Are Almost Geometric’, J. London Math. Soc. (2) 32 (1985), 217-220.

[19] Daniel T. Wise, The structure of groups with a quasiconvex hierarchy, 205 pp. Preprint 2011.