COHOMOLOGICAL ASPECTS OF GAUGE INVARIANCE IN THE CAUSAL APPROACH

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Received August 19, 2009

Abstract
Quantum theory of the gauge models in the causal approach leads to some cohomology problems. We investigate these problems in detail.

1 INTRODUCTION

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [1], [5], [4], [11]; for every set of Wick monomials $W_1(x_1), \ldots, W_n(x_n)$ acting in some Fock space $H$ one associates the operator $T^{W_1,\ldots,W_n}(x_1,\ldots, x_n)$; all these expressions are in fact distribution-valued operators called chronological products. It will be convenient to use another notation: $T(W_1(x_1),\ldots,W_n(x_n))$. The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [5], [6] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [14] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If the arbitrariness does not grow with $n$ we have a renormalizable theory. An equivalent point of view uses retarded products [18].

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizability using ghost fields. Such theories are defined in a Fock space $H$ with indefinite metric, generated by physical and un-physical fields (called ghost fields). One selects the physical states assuming the existence of an operator $Q$ called gauge charge which verifies $Q^2 = 0$ and such that the physical Hilbert space is by definition $H_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$. The space $H$ is endowed with a grading (usually called ghost number) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in $H$ is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating $H$. The graded commutator $d_Q$ of the gauge charge with any operator $A$ of fixed ghost number

$$d_Q A = [Q, A]$$

is raising the ghost number by a unit. It means that $d_Q$ is a co-chain operator in the space of Wick polynomials. From now on $[\cdot, \cdot]$ denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called the interaction Lagrangian such that

$$[Q, T] = i \partial_\mu T^n$$

Rom. Journ. Phys., Vol. 55, Nos. 3-4, P. 386-438, Bucharest, 2010
for some other Wick polynomials $T^\mu$. This relation means that the expression $T$ leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f) \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}}$$  \hspace{1cm} (1.3)$$

up to terms which can be made as small as desired (making the test function $f$ flatter and flatter). In all known models one finds out that there exist a chain of Wick polynomials $T^\mu$, $T^{\mu\nu}$, $T^{\mu\nu\rho}$, \ldots such that:

$$[Q, T] = i \partial_\mu T^\mu, \quad [Q, T^\mu] = i \partial_\rho T^{\mu\rho}, \quad [Q, T^{\mu\nu}] = i \partial_\rho T^{\mu\nu\rho}, \ldots$$  \hspace{1cm} (1.4)$$

It so happens that for all these models the expressions $T^{\mu\nu}$, $T^{\mu\nu\rho}$, \ldots are completely antisymmetric in all indices; it follows that the chain of relation stops at the step 4 (if we work in four dimensions). We can also use a compact notation $T^I$ where $I$ is a collection of indices $I = [\nu_1, \ldots, \nu_p]$ ($p = 0, 1, \ldots$) and the brackets emphasize the complete antisymmetry in these indices. All these polynomials have the same canonical dimension

$$\omega(T^I) = \omega_0, \quad \forall I$$  \hspace{1cm} (1.5)$$

and because the ghost number of $T \equiv T^\emptyset$ is supposed null, then we also have:

$$gh(T^I) = |I|$$.  \hspace{1cm} (1.6)$$

One can write compactly the relations (1.4) as follows:

$$d_Q T^I = i \partial_\mu T^{I\mu}.$$  \hspace{1cm} (1.7)$$

For concrete models the equations (1.4) can stop earlier: for instance in the Yang-Mills case we have $T^{\mu\nu\rho} = 0$ and in the case of gravity $T^{\mu\nu\rho\sigma} = 0$.

Now we can construct the chronological products

$$T^{I_1, \ldots, I_n}(x_1, \ldots, x_n) \equiv T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n))$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.7):

$$d_Q T^{I_1, \ldots, I_n} = i \sum_{l=1}^n (-1)^s_l \frac{\partial}{\partial x^\mu_{I_l}} T^{I_1, \ldots, I_{I_l}, \ldots, I_n}$$  \hspace{1cm} (1.8)$$

are true for all $n \in \mathbb{N}$ and all $I_1, \ldots, I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I|_j$$  \hspace{1cm} (1.9)$$

(see also [3]). In particular, the case $I_1 = \ldots = I_n = \emptyset$ it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (1.3).

Such identities can be usually broken by anomalies i.e. expressions of the type $A^{I_1, \ldots, I_n}$ which are quasi-local and might appear in the right-hand side of the relation (1.8). These expressions verify some consistency conditions - the so-called Wess-Zumino equations. One can use these equations in the attempt to eliminate the anomalies by redefining the chronological products. All these operations can be proven to be of cohomological nature and naturally lead to descent equations of the same type as (1.7) but for different ghost number and canonical dimension.
If one can choose the chronological products such that gauge invariance is true then there is still some freedom left for redefining them. To be able to decide if the theory is renormalizable one needs the general form of such arbitrariness. Again, one can reduce the study of the arbitrariness to descent equations of the type as (1.7).

Such type of cohomology problems have been extensively studied in the more popular approach to quantum gauge theory based on functional methods (following from some path integration method). In this setting the co-chain operator (the BRST operator) is non-linear and a priori makes sense only for classical field theories. On the contrary, in the causal approach the co-chain operator is linear so the cohomology problem makes sense directly in the Hilbert space of the model. One needs however a classical field theory machinery to analyze the descent equations more easily. There are approaches which give sense to the non-linear BRST operator in the Epstein-Glaser framework. In our approach we use an on-shell formalism but one can present the causal method in an off-shell formalism also. One can even extend the formalism to curved spacetime manifolds. For an extensive bibliography on these subjects see [13]. There are some differences between these various approaches but it would be a mistake at this stage of development to establish an hierarchy between them. All these methods have their strong and weak points and only future investigations will show which approach will be able to lead to a complete understanding of quantum gauge models.

In this paper we want to give a general description of these cohomology methods in our preferred formalism which is the Epstein-Glaser (causal) approach minimally modified to accommodate unphysical fields needed for the description of quantum gauge models. We will apply them for Yang-Mills models.

In Section 3 we give some general results about the structure of the anomalies and reduce the proof of (1.8) to descent equations. In Section 4 we provide a convenient geometric setting for our problem. We will prove an algebraic form of the Poincaré lemma valid for on-shell fields (The usual Poincaré cannot be applied because the homotopy operator of de Rham does not leave invariant the space of on-shell polynomials.) In Section 5 we determine the cohomology of the operator \( d_Q \) for Yang-Mills models. Using this cohomology and the algebraic Poincaré lemma we can solve the descent equations in various ghost numbers in Section 6. We make some comments about higher orders of perturbation theory in Section 7. For the case of quantum electro-dynamics we give the shortest proof of gauge invariance in all orders.

The present paper includes the results of some previous papers [8], [9], [10], [11] but many the proofs are new and use in an optimal way various cohomological structures. In [15] and [16] one can find similar results but the cohomological methods are not used for the proofs.

## 2 GENERAL GAUGE THEORIES

We give here the essential ingredients of perturbation theory.

### 2.1 Bogoliubov Axioms

Suppose that the Wick monomials \( W_1, \ldots, W_n \) are self-adjoint: \( W_j^\dagger = W_j, \forall j = 1, \ldots, n \). The chronological products \( T(W_1(x_1), \ldots, W_n(x_n)) \) \( n = 1, 2, \ldots \) are verifying the following set of axioms:

- Skew-symmetry in all arguments \( W_1(x_1), \ldots, W_n(x_n) \) :

\[
T(\ldots, W_i(x_i), W_{i+1}(x_{i+1}), \ldots) = (-1)^{f_i f_{i+1}} T(\ldots, W_{i+1}(x_{i+1}), W_i(x_i), \ldots) \tag{2.1}
\]
where $f_i$ is the number of Fermi fields appearing in the Wick monomial $W_i$.

- Poincaré invariance: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $(a, A) \in \text{in}SL(2, \mathbb{C})$ we have:

$$U_{a, A}T(W_1(x_1), \ldots, W_n(x_n))U_{a, A}^{-1} = T(A \cdot W_1(A \cdot x_1 + a), \ldots, A \cdot W_n(A \cdot x_n + a));$$

(2.2)

Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- Causality: if $x_i \geq x_j, \forall i \leq k, j \geq k + 1$ then we have:

$$T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k)) \cdot T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n));$$

(2.3)

- Unitarity: We define the anti-chronological products according to

$$(−1)^n \tilde{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \ldots, I_r \in \text{Part}(\{1, \ldots, n\})} \epsilon \cdot T_{I_1}(X_1) \cdots T_{I_r}(X_r);$$

where the we have used the notation:

$$T_{\{i_1, \ldots, i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k}))$$

(2.5)

and the sign $\epsilon$ counts the permutations of the Fermi factors. Then the unitarity axiom is:

$$\tilde{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_n(x_n))^\dagger.$$  

(2.6)

- The “initial condition”

$$T(W(x)) = W(x).$$

(2.7)

It can be proved that this system of axioms can be supplemented with

$$\sum_\epsilon \langle \Omega, T(W'_1(x_1), \ldots, W'_n(x_n)) \Omega \rangle > : W''_1(x_1), \ldots, W''_n(x_n) : = T(W_1(x_1), \ldots, W_n(x_n));$$

(2.8)

where $W'_i$ and $W''_i$ are Wick submonomials of $W_i$ such that $W_i = : W'_i W''_i :$ and the sign $\epsilon$ takes care of the permutation of the Fermi fields; here $\Omega$ is the vacuum state. This is called the Wick expansion property.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials $W_1, \ldots, W_n$: explicitly:

$$\omega(\langle \Omega, T^{W_1 \ldots W_n}(X) \Omega \rangle) \leq \sum_{l=1}^{n} \omega(W_l) - 4(n - 1)$$

(2.9)

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution $d$ and by $\omega(W)$ we mean the canonical dimension of the Wick monomial $W$; in particular this means that we have

$$T(W_1(x_1), \ldots, W_n(x_n)) = \sum_g t_g(x_1, \ldots, x_n) \cdot W_g(x_1, \ldots, x_n)$$

(2.10)
where $W_g$ are Wick polynomials of fixed canonical dimension and $t_g$ are distributions in $n - 1$ variables (because of translation invariance) with the order of singularity bounded by the power counting theorem [5]:

$$\omega(t_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \quad (2.11)$$

and the sum over $g$ is essentially a sum over Feynman graphs. Up to now, we have defined the chronological products only for self-adjoint Wick monomials $W_1, \ldots, W_n$ but we can extend the definition for Wick polynomials by linearity.

One can modify the chronological products without destroying the basic property of causality iff one can make

$$T(W_1(x_1), \ldots, W_n(x_n)) \rightarrow T(W_1(x_1), \ldots, W_n(x_n)) + R_{W_1, \ldots, W_n}(x_1, \ldots, x_n) \quad (2.12)$$

where $R$ are quasi-local expressions; by a quasi-local expression we mean an expression of the form

$$R_{W_1, \ldots, W_n}(x_1, \ldots, x_n) = \sum_g [P_g(\partial)\delta(X)] W_g(x_1, \ldots, x_n) \quad (2.13)$$

with $P_g$ monomials in the partial derivatives and $W_g$ are Wick polynomials; here $\delta(X)$ is the $n$-dimensional delta distribution $\delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n)$. Because of the delta function we can consider that $P_g$ is a monomial only in the derivatives with respect to, say $x_2, \ldots, x_n$. If we want to preserve (2.9) we impose the restriction

$$\deg(P_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \quad (2.14)$$

and some other restrictions are following from the preservation of Lorentz covariance and unitarity.

The redefinitions of the type (2.12) are the so-called finite renormalizations. Let us note that this arbitrariness, described by the number of independent coefficients of the polynomials $P_g$ can grow with $n$ and in this case the theory is called non-renormalizable. This can happen if some of the Wick monomials $W_j, j = 1, \ldots, n$ have canonical dimension greater than 4. If all the monomials have canonical dimension less of equal to 4 then the arbitrariness is bounded independently of $n$ and the theory is called renormalizable. However, even in the non-renormalizable case if the theory verifies some additional symmetry properties it could happen that the number of arbitrary coefficients from $P_g$ is finite. This seems to be the case for quantum gravity. We will analyze this case in another paper.

It is not hard to prove that any finite renormalization can be rewritten in the form

$$R(x_1, \ldots, x_n) = \delta(X) W(x_1) + \sum_{j=1}^{n} \frac{\partial}{\partial x_1^\mu} R_l(X) \quad (2.15)$$

where the expressions $R_l(X)$ are also quasi-local. But it is clear that the sum in the above expression is null in the adiabatic limit. This means that we can postulate that the finite renormalizations have a much simpler form, namely

$$R(x_1, \ldots, x_n) = \delta(X) W(x_1) \quad (2.16)$$

where the Wick polynomial $W$ is constrained by

$$\omega(W) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1). \quad (2.17)$$
2.2 Gauge Theories and Anomalies

From now on we consider that we work in the four-dimensional Minkowski space and we have the Wick polynomials $T^I$ such that the descent equations (1.7) are true and we also have

$$T^I(x_1)T^J(x_2) = (-1)^{|I||J|} T^I(x_2)T^J(x_1), \quad \forall x_1 \sim x_2$$

(2.18)

i.e. for $x_1 - x_2$ space-like these expressions causally commute in the graded sense.

The equations (1.7) are called a relative cohomology problem. The co-boundaries for this problem are of the type

$$T^I = dQ B^I + i \partial_\mu B^{I\mu}.$$  

(2.19)

Next we construct the associated chronological products

$$T^{I_1,\ldots,I_n}(x_1,\ldots,x_n) = T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)).$$

We will impose the graded symmetry property:

$$T(\ldots,T^{I_k}(x_k),T^{I_{k+1}}(x_{k+1}),\ldots) = (-1)^{|I_k||I_{k+1}|} T(\ldots,T^{I_{k+1}}(x_{k+1}),T^{I_k}(x_k),\ldots).$$  

(2.20)

We also have

$$gh(T^{I_1,\ldots,I_n}) = \sum_{l=1}^n |I_l|.$$  

(2.21)

In the case of a gauge theory there are renormalizations of the type (2.13) which call trivial, namely those of the type

$$R^\cdots(X) = dQ B^\cdots(X) + i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} B^{I_1,\ldots,I_n}(X).$$  

(2.22)

Indeed, as it was remarked above, any co-boundary operator induces the null operator on the physical Hilbert space. Also any total divergence gives a null contribution in the adiabatic limit.

We now write the gauge invariance condition (1.8) in a compact form. We consider the space $\mathcal{C}_n^I$ of co-chains of the form $C_{I_1,\ldots,I_n}(X)$ ($I \equiv |I_1| + \ldots + |I_n|$) which are distribution-valued operators in the Hilbert space with antisymmetry in all indices from every $I_j$, $(j = 1, \ldots, n)$ and also verifying:

$$C_{\ldots,I_k,I_{k+1},\ldots}(\ldots,x_k,x_{k+1},\ldots) = (-1)^{|I_k||I_{k+1}|} C_{\ldots,I_{k+1},I_k,\ldots}(\ldots,x_{k+1},x_k,\ldots).$$  

(2.23)

Then we can define the operator $\delta : \mathcal{C}_n^I \rightarrow \mathcal{C}_n^{I-1}$ according to:

$$\delta C_{I_1,\ldots,I_n} = \sum_{l=1}^n (-1)^{|I_l|} \frac{\partial}{\partial x_l^\mu} C_{I_1,\ldots,I_l\mu,\ldots,I_n}. $$

(2.24)

It is easy to prove that we have:

$$\delta^2 = 0;$$  

(2.25)

we also note that $\delta$ commutes with $dQ$. One can now write the equation (1.8) in a more compact way:

$$dQ T^{I_1,\ldots,I_n} = i\delta T^{I_1,\ldots,I_n}. $$

(2.26)
We now determine the obstructions for the gauge invariance relations (2.26). These relations are true for \( n = 1 \) according to (1.7). If we suppose that they are true up to order \( n - 1 \) then it follows easily that in order \( n \) we must have:

\[
d_Q T^{I_1,\ldots,I_n} = i \delta T^{I_1,\ldots,I_n} + A^{I_1,\ldots,I_n} \tag{2.27}
\]

where the expressions \( A^{I_1,\ldots,I_n}(x_1,\ldots,x_n) \) are quasi-local operators and are called anomalies. It is clear that we have from (2.20) a similar symmetry for the anomalies: namely we have complete antisymmetry in all indices from every \( I_j \), \((j = 1,\ldots,n)\) and

\[
A^{\ldots,I_{k+1},\ldots}(x_k, x_{k+1}, \ldots) = (-1)^{|I_k||I_{k+1}|} \times A^{\ldots,I_{k+1},I_k,\ldots}(x_k+1, x_{k+1}, x_k, \ldots). \tag{2.28}
\]

i.e. \( A^{I_1,\ldots,I_n}(x_1,\ldots,x_n) \) are also co-chains. We also have

\[
gh(A^{I_1,\ldots,I_n}) = \sum_{l=1}^{n} |I_l| + 1. \tag{2.29}
\]

We remind that \( \omega_0 \equiv \omega(T) \); then one has:

\[
A^{I_1,\ldots,I_n}(X) = 0 \text{ iff } \sum_{l=1}^{n} |I_l| > n(\omega_0 - 4) + 4 \tag{2.30}
\]

We can write a more precise form for the anomalies, namely:

\[
A^{I_1,\ldots,I_n}(x_1,\ldots,x_n) = \sum_k \sum_{i_1,\ldots,i_k > 1} [\partial_{\rho_1}^{i_1} \cdots \partial_{\rho_k}^{i_k} \delta(X)] W^{I_1,\ldots,I_n;\rho_1,\ldots,\rho_k}(x_1) \tag{2.31}
\]

and in this expression the Wick polynomials \( W^{I_1,\ldots,I_n;\rho_1,\ldots,\rho_k} \) are uniquely defined. Now from (2.11) we have

\[
\omega(W^{I_1,\ldots,I_n;\rho_1,\ldots,\rho_k}) \leq n(\omega_0 - 4) + 5 - k \tag{2.32}
\]

which gives a bound on \( k \) in the previous sum. We also have some consistency conditions verified by the anomalies. If one applies the operator \( d_Q \) to (2.27) one obtains the so-called Wess-Zumino consistency conditions:

\[
d_Q A^{I_1,\ldots,I_n} = -i \delta A^{I_1,\ldots,I_n}. \tag{2.33}
\]

Let us note that we can suppose, as for the finite renormalizations - see (2.16) that all anomalies which are total divergences are trivial because they spoil gauge invariance with terms which can be made as small as one wished, i.e. we can take the form:

\[
A(x_1,\ldots,x_n) = \delta(X) W(x_1). \tag{2.34}
\]

It is however interesting that in some cases one can prove that the anomalies can be put in this form by suitable redefinitions of the chronological products. This is the case of the Yang-Mills models which we will analyse in the next Sections.

Suppose now that we have fixed the gauge invariance (2.26) and we investigate the renormalizability issue i.e. we make the redefinitions

\[
T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) \rightarrow T(T^{I_1}(x_1),\ldots,T^{I_n}(x_n)) + R^{I_1,\ldots,I_n}(x_1,\ldots,x_n) \tag{2.35}
\]
where $R$ are quasi-local expressions. As before we have

$$R^{I_{k_1}, \ldots, I_{k_{j_k}}, \ldots, I_{k_{j_{k+1}}}, \ldots} = (-1)^{|I_{k_1}|/|I_{k_{j_{k+1}}}|} \times R^{I_{k_{j+1}}, \ldots, I_{k_{j_{k+1}}}, \ldots, x_{k_{j_k+1}}, x_{k_{j_{k+1}}}, \ldots}.$$  \hfill (2.36)

We also have

$$gh(R^{I_{1}, \ldots, I_{n}}) = \sum_{l=1}^{n} |I_l|. \hfill (2.37)$$

and

$$R^{I_{1}, \ldots, I_{n}} = 0 \iff \sum_{l=1}^{n} |I_l| > n(\omega_0 - 1) + 4. \hfill (2.38)$$

If we want to preserve (1.8) it is clear that the quasi-local operators $R^{I_{1}, \ldots, I_{n}}$ should also verify

$$d_Q R^{I_{1}, \ldots, I_{n}} = i \delta R^{I_{1}, \ldots, I_{n}} \hfill (2.39)$$

equations of the type (2.33). In this case we note that we have more structure; according to the previous discussion we can impose the structure (2.16):

$$R^{I_{1}, \ldots, I_{n}}(x_1, \ldots, x_n) = \delta(X) W^{I_{1}, \ldots, I_{n}}(x_1) \hfill (2.40)$$

and we obviously have:

$$gh(W^{I_{1}, \ldots, I_{n}}) = \sum_{l=1}^{n} |I_l| \hfill (2.41)$$

and

$$W^{I_{1}, \ldots, I_{n}} = 0 \iff \sum_{l=1}^{n} |I_l| > n(\omega_0 - 1) + 4. \hfill (2.42)$$

From (2.39) we obtain after some computations that for $n > 2$ there are Wick polynomials $R^{I}$ such that

$$W^{I_{1}, \ldots, I_{n}} = (-1)^s R^{I_{1} \cup \ldots \cup I_{n}}. \hfill (2.43)$$

where

$$s \equiv \sum_{k<l \leq n} |I_k||I_l|. \hfill (2.44)$$

Moreover, we have

$$gh(R^{I}) = |I| \hfill (2.45)$$

and

$$R^{I} = 0, \quad |I| > n(\omega_0 - 1) + 4. \hfill (2.46)$$

Finally, the following descent equations are true:

$$d_Q R^{I} = i \partial_\mu R^{\mu I} \hfill (2.47)$$

and have obtained another relative cohomology problem similar to the one from the Introduction.
3 A PARTICULAR CASE OF THE WESS-ZUMINO CONSISTENCY CONDITIONS

In this Section we consider a particular form of (2.27) and (2.33) namely the case when all polynomials \( T^I \) have canonical dimension \( \omega_0 = 4 \) and \( T^{\mu \nu \rho} = 0 \). In this case (2.30) becomes:

\[
A^{I_1 \ldots I_n}(X) = 0 \quad \text{iff} \quad \sum_{l=1}^{n} |I_l| > 4
\]  

(3.1)

and this means that only a finite number of the equations (2.27) can be anomalous. It is convenient to define

\[
A_1 \equiv A^{0 \ldots 0}, \quad A_2^\mu \equiv A^{[\mu,0 \ldots 0}, \quad A_3^{[\mu [\nu}, \quad A_4^{[\mu [\nu [\rho}, \quad A_5^{[\mu [\nu [\rho [\sigma}, \quad A_6^{[\mu [\nu [\rho [\sigma [\tau}, \quad A_7^{[\mu [\nu [\rho [\sigma [\tau 4}, \quad A_8^{[\mu [\nu [\rho [\sigma [\tau 5}, \quad A_9^{[\mu [\nu [\rho [\sigma [\tau 6}
\]

(3.2)

where we have emphasized the antisymmetry properties with brackets. We have from (2.27) the following anomalous gauge equations:

\[
d_Q T(T(x_1), \ldots, T(x_n)) = i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_{1\ell}} T(T(x_1), \ldots, T^\mu(x_1), \ldots, T(x_n)) + A_1(X)
\]  

(3.3)

\[
d_Q T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) = i \frac{\partial}{\partial x_{1\mu}} T(T^\mu(x_1), T(x_2), \ldots, T(x_n))
\]  

(3.4)

\[
d_Q T(T^{\mu \nu}(x_1), T(x_2), \ldots, T(x_n)) =
\]

\[
i \sum_{l=2}^{n} \frac{\partial}{\partial x_{l\mu}} T(T^{\mu \nu}(x_1), T(x_2), \ldots, T^\rho(x_1), \ldots, T(x_n)) + A_2^\nu(X)
\]

(3.5)

\[
d_Q T(T^{\mu \nu}(x_1), T^\rho(x_2), \ldots, T(x_n)) =
\]

\[
i \sum_{l=2}^{n} \frac{\partial}{\partial x_{l\mu}} T(T^{\mu \nu}(x_1), T^\rho(x_2), \ldots, T^\rho(x_1), \ldots, T(x_n)) + A_3^{[\mu [\nu [\rho}(X)
\]

(3.6)

\[
d_Q T(T^{\mu \nu}(x_1), T^\rho(x_2), \ldots, T(x_n)) =
\]

\[
i \frac{\partial}{\partial x_{1\mu}} T(T^{\mu \nu}(x_1), T^\rho(x_2), \ldots, T(x_n)) - i \frac{\partial}{\partial x_{2\mu}} T(T^{\mu \nu}(x_1), T^\rho(x_2), \ldots, T(x_n))
\]

\[
+ i \sum_{l=3}^{n} \frac{\partial}{\partial x_{l\mu}} T(T^{\mu \nu}(x_1), T^\rho(x_2), T(x_3), \ldots, T^\rho(x_1), \ldots, T(x_n)) + A_4^{\nu \nu}(X)
\]

(3.7)
\[ d_Q T(T^{\mu \nu}(x_1), T^{\rho \sigma}(x_2), T(x_3), \ldots, T(x_n)) = \]
\[ i \sum_{l=3}^{n} \frac{\partial}{\partial x_i^l} T(T^{\mu \nu}(x_1), T^{\rho \sigma}(x_2), T(x_3), \ldots, T^\lambda(x_l), \ldots, T(x_n)) \]
\[ + A_6^{[\mu \nu];[\rho \sigma]}(X) \quad (3.8) \]
\[ d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) = \]
\[ i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) \]
\[ - i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) \]
\[ + i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T(x_n)) \]
\[ - i \sum_{l=4}^{n} \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \ldots, T^\sigma(x_l), \ldots, T(x_n)) \]
\[ + A_7^{[\mu \nu];[\rho \sigma]}(X) \quad (3.9) \]
\[ d_Q T(T^\mu(x_1), T^\nu(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) = \]
\[ i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) \]
\[ - i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) \]
\[ + i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\sigma(x_3), T(x_4), \ldots, T(x_n)) \]
\[ + i \sum_{l=4}^{n} \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\sigma(x_3), T(x_4), \ldots, T^\lambda(x_l), \ldots, T(x_n)) \]
\[ + A_8^{[\mu \nu];[\rho \sigma]}(X) \quad (3.10) \]
\[ d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T(x_n)) = \]
\[ i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T(x_n)) \]
\[ - i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T(x_n)) \]
\[ + i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T(x_n)) \]
\[ - i \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T(x_n)) \]
\[ + i \sum_{l=5}^{n} \frac{\partial}{\partial x_i^l} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \ldots, T^\lambda(x_l), \ldots, T(x_n)) \]
\[ + A_9^{[\mu \nu];[\rho \sigma]}(X) \quad (3.11) \]

where we can assume that:

\[ A_4^{\mu \nu}(X) = 0, \quad A_5^{\mu \nu;\rho} = 0, \quad A_6^{\mu \nu;\rho \sigma} = 0, \quad |X| = 1, \]
\[ A_I^{\mu,\nu} (X) = 0, \quad A_8^{\mu,\nu,\rho,\sigma} = 0, \quad |X| \leq 2, \]
\[ A_9^{\mu,\nu,\rho,\sigma} (X) = 0, \quad |X| \leq 3 \]  (3.12)

without losing generality.

From (2.28) we get the following symmetry properties:

\[ A_1(x_1, \ldots, x_n) \] is symmetric in \( x_1, \ldots, x_n \);  (3.13)
\[ A_2^\nu(x_1, \ldots, x_n) \] is symmetric in \( x_2, \ldots, x_n \);  (3.14)
\[ A_3^{\mu\nu}(x_1, \ldots, x_n) \] is symmetric in \( x_2, \ldots, x_n \);  (3.15)
\[ A_4^{\mu,\nu}(x_1, \ldots, x_n) \] is symmetric in \( x_3, \ldots, x_n \);  (3.16)
\[ A_5^{[\mu\nu]}(x_1, \ldots, x_n) \] is symmetric in \( x_3, \ldots, x_n \);  (3.17)
\[ A_6^{[\mu\nu],[\rho\sigma]}(x_1, \ldots, x_n) \] is symmetric in \( x_3, \ldots, x_n \);  (3.18)
\[ A_7^{\mu\nu,\rho}(x_1, \ldots, x_n) \] is symmetric in \( x_4, \ldots, x_n \);  (3.19)
\[ A_8^{[\mu\nu],[\rho\sigma]}(x_1, \ldots, x_n) \] is symmetric in \( x_4, \ldots, x_n \);  (3.20)
\[ A_9^{\mu,\nu,\rho,\sigma}(x_1, \ldots, x_n) \] is symmetric in \( x_5, \ldots, x_n \);  (3.21)

and we also have:

\[ A_4^{\mu,\nu}(x_1, \ldots, x_n) = -A_4^{\nu,\mu}(x_2, x_1, x_3, \ldots, x_n); \]  (3.22)
\[ A_6^{[\mu\nu],[\rho\sigma]}(x_1, \ldots, x_n) = A_6^{[\rho\sigma],[\mu\nu]}(x_2, x_1, x_3, \ldots, x_n); \]  (3.23)
\[ A_7^{\mu,\nu,\rho}(x_1, \ldots, x_n) = -A_7^{\nu,\mu,\rho}(x_2, x_1, x_3, \ldots, x_n) = -A_7^{\mu,\rho,\nu}(x_1, x_3, x_2, x_4, \ldots, x_n); \]  (3.24)
\[ A_8^{[\mu\nu],[\rho\sigma]}(x_1, x_2, \ldots, x_n) = -A_8^{[\mu\nu],[\sigma\rho]}(x_1, x_3, x_2, x_4, \ldots, x_n); \]  (3.25)
\[ A_9^{\mu,\nu,\rho,\sigma}(x_1, \ldots, x_n) = -A_9^{\mu,\nu,\rho,\sigma}(x_2, x_1, x_3, \ldots, x_n); \]  (3.26)

The Wess-Zumino consistency conditions are in this case:

\[ d_Q A_1(x_1, \ldots, x_n) = -i \sum_{l=1}^{n} \frac{\partial}{\partial x^{\rho}_l} A^{\mu}_2(x_1, x_l, \ldots, x_n) \]  (3.27)
\[ d_Q A_2^{\mu}(x_1, \ldots, x_n) = -i \frac{\partial}{\partial x^{\nu}_1} A_3^{[\nu\mu]}(x_1, \ldots, x_n) + i \sum_{l=2}^{n} \frac{\partial}{\partial x^{\nu}_l} A_4^{\mu\nu}(x_1, x_l, x_2, \ldots, x_n) \]  (3.28)
\[ d_Q A_3^{[\mu\nu]}(x_1, \ldots, x_n) = -i \sum_{l=2}^{n} \frac{\partial}{\partial x^{\nu}_l} A_5^{[\mu\nu]}(x_1, x_l, x_2, \ldots, x_n) \]  (3.29)
\[ d_Q A_4^{\mu,\nu}(x_1, \ldots, x_n) = -i \frac{\partial}{\partial x^{\rho}_1} A_5^{[\rho\mu]\nu}(x_1, \ldots, x_n) + i \frac{\partial}{\partial x^{\rho}_2} A_5^{[\rho\nu]\mu}(x_2, x_1, x_3, \ldots, x_n) \]  (3.30)
Theorem 3.1 One can redefine the chronological products such that

$$A_1(x_1, x_2) = \delta(x_1 - x_2) W(x_1), \quad A_2^\mu(x_1, x_2) = \delta(x_1 - x_2) W^\mu(x_1)$$

$$A_3^{[\mu\nu]}(x_1, x_2) = \delta(x_1 - x_2) W^{[\mu\nu]}(x_1), \quad A_4^{\mu\nu}(x_1, x_2) = -\delta(x_1 - x_2) W^{[\mu\nu]}(x_1),$$

$$A_5^{[\mu\nu][\rho\sigma]}(x_1, x_2) = 0, \quad A_6^{[\mu\nu][\rho\sigma]}(x_1, x_2) = 0.$$
Moreover one has the following descent equations:

\[ d_Q W = -i \partial_\mu W^\mu, \quad d_Q W^\mu = i \partial_\nu W^{[\mu \nu]}, \quad d_Q W^{[\mu \nu]} = 0. \]  

(3.43)

The expressions \( W \) and \( W^\mu \) are relative co-cycles and are determined up to relative co-boundaries. The expression \( W^{[\mu \nu]} \) is a cocycle and it is determined up to a co-boundary.

**Proof:** The symmetry properties are in this case

\[ A_1(x_1, x_n) = A_1(x_2, x_1) \]  

(4.44)

\[ A_4^{\mu \nu}(x_1, x_2) = -A_4^{\nu \mu}(x_2, x_1); \]  

(4.45)

\[ A_6^{[\mu \nu][\rho \sigma]}(x_1, x_2) = A_6^{[\rho \sigma][\mu \nu]}(x_2, x_1) \]  

(4.46)

and the corresponding Wess-Zumino consistency conditions

\[ d_Q A_1(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_2^{\mu}(x_1, x_2) - i \frac{\partial}{\partial x_2^\mu} A_2^{\mu}(x_2, x_1) \]  

(4.47)

\[ d_Q A_3^{\mu \nu}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_3^{\mu \nu}(x_1, x_2) + i \frac{\partial}{\partial x_2^\mu} A_3^{\mu \nu}(x_2, x_1) \]  

(4.48)

\[ d_Q A_5^{[\mu \nu][\rho]}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_5^{[\mu \nu][\rho]}(x_1, x_2) \]  

(4.49)

\[ d_Q A_4^{\mu \nu \rho}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_5^{[\mu \nu][\rho]}(x_1, x_2) + i \frac{\partial}{\partial x_2^\mu} A_5^{[\mu \nu][\rho]}(x_2, x_1) \]  

(4.50)

\[ d_Q A_6^{[\mu \nu][\rho \sigma]}(x_1, x_2) = 0 \]  

(4.52)

will be enough to obtain the result from the statement.

(i) From (2.31) we have:

\[ A_1(x_1, x_2) = \sum_{k=0}^{4} \partial_{\mu_1} \cdots \partial_{\mu_k} \delta(x_2 - x_1) W_1^{\{\mu_1, \ldots, \mu_k\}}(x_1) \]  

(4.53)

where we have emphasized the symmetry properties by curly brackets. We have the restrictions

\[ \omega(W_1^{\{\mu_1, \ldots, \mu_k\}}) \leq 5 - k, \quad gh(W_1^{\{\mu_1, \ldots, \mu_k\}}) = 1 \]  

(4.54)

for all \( k = 0, \ldots, 4 \). We perform the finite renormalization:

\[ T(T^\mu(x_1), T(x_2)) \rightarrow T(T^\mu(x_1), T(x_2)) + \partial_\nu \partial_\rho \partial_\sigma \delta(x_2 - x_1) U_2^{[\nu, \rho, \sigma]}(x_1) \]  

(4.55)

and it is easy to see that if we choose \( U_2^{[\nu, \rho, \sigma]} = -\frac{i}{2} W_1^{[\nu, \rho, \sigma]} \) then we obtain a new expression (3.53) for the anomaly \( A_1 \) where the sum goes only up to \( k = 3 \). (Although the monomials \( W_1^{\{\mu_1, \ldots, \mu_k\}} \) will be changed after this finite renormalization we keep the same notation.) Now we impose the symmetry property (3.44) and consider only the terms with three derivatives on \( \delta \); it easily follows that \( W_1^{\{\mu, \nu, \rho\}} = 0 \) i.e. in the expression (3.53) for the anomaly \( A_1 \) the sum goes only up to \( k = 2 \).
Next we perform the finite renormalization:

\[ T(T^{\mu}(x_1), T(x_2)) \rightarrow T(T^{\mu}(x_1), T(x_2)) + \partial_\nu \delta(x_2 - x_1) U_2^{\mu \nu}(x_1) \]  

(3.56)

and it is easy to see that if we choose \( U_2^{\mu \nu} = -i \frac{1}{2} W_1^{\{\mu, \nu\}} \) then we obtain a new expression (3.53) for the anomaly \( A_1 \) where the sum goes only up to \( k = 1 \). Again we impose the symmetry property (3.44) and consider only the terms with one derivative on \( \delta \); it easily follows that \( W_1^{\mu} = 0 \) i.e. the expression (3.53) has the form from the statement.

(ii) From (2.31) we have:

\[ A_2^{\mu}(x_1, x_2) = \sum_{k \leq 3} \partial_{\rho_1} \ldots \partial_{\rho_k} \delta(x_2 - x_1) W_2^{\mu;\{\rho_1, \ldots, \rho_k\}}(x_1) \]  

(3.57)

and we have the restrictions

\[ \omega(W_2^{\mu;\{\rho_1, \ldots, \rho_k\}}) \leq 5 - k, \quad g h(W_2^{\mu;\{\rho_1, \ldots, \rho_k\}}) = 2 \]  

(3.58)

for all \( k = 0, \ldots, 3 \). We use Wess-Zumino consistency condition (3.47); if we consider only the terms with four derivatives on \( \delta \) we obtain that the completely symmetric part of \( W_2^{\mu;\{\nu, \rho, \sigma\}} \) is null:

\[ W_2^{\mu;\{\nu, \rho, \sigma\}} = 0. \]  

In this case it is easy to prove that one can write \( W_2^{\mu;\{\nu, \rho, \sigma\}} \) in the following form:

\[ W_2^{\mu;\{\nu, \rho, \sigma\}} = \frac{1}{3} (\tilde{W}_2^{[\mu;\{\nu\}} + \tilde{W}_2^{[\mu\rho];\{\nu\}) + \tilde{W}_2^{[\mu\sigma];\{\nu\}}) \]  

(3.59)

with

\[ \tilde{W}_2^{[\mu;\{\nu\}} = \frac{3}{4} W_2^{\mu;\{\nu, \rho, \sigma\}} - (\mu \leftrightarrow \nu). \]  

We perform the finite renormalization

\[ T(T^{[\mu\nu]}(x_1), T(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T(x_2)) + \partial_\nu \partial_\rho \delta(x_2 - x_1) U_3^{[\mu\nu];\{\rho\}}(x_1) \]  

(3.61)

with \( U_3^{[\mu\nu];\{\rho\}} = -i \tilde{W}_2^{[\mu\nu];\{\rho\}} \) and we eliminate the contributions corresponding to \( k = 3 \) from (3.57).

Now we consider the contribution corresponding to \( k = 2 \); again we use the Wess-Zumino consistency condition (3.47); if we consider only the terms with three derivatives on \( \delta \) we obtain that the completely symmetric part of \( W_2^{\mu;\{\nu, \rho\}} \) is null \( W_2^{\mu;\{\nu, \rho\}} = 0 \) and write:

\[ W_2^{\mu;\{\nu, \rho\}} = \frac{1}{2} (\tilde{W}_2^{[\mu;\nu\rho]} + \tilde{W}_2^{[\mu\rho];\nu}) \]  

(3.62)

with

\[ \tilde{W}_2^{[\mu;\nu\rho]} = \frac{2}{3} W_2^{\mu;\{\nu, \rho\}} - (\mu \leftrightarrow \nu). \]  

(3.63)

Now we consider the finite renormalization

\[ T(T^{[\mu\nu]}(x_1), T(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T(x_2)) + \partial_\rho \partial_\sigma \delta(x_2 - x_1) U_3^{[\mu\nu];\{\rho\}}(x_1) \]  

(3.64)

with \( U_3^{[\mu\nu];\{\rho\}} = i \tilde{W}_2^{[\mu\nu];\rho} \) and we get a new expressions (3.57) for which \( W_3^{\mu;\{\nu\}} = 0 \) i.e. the summation in (3.57) goes only up to \( k = 1 \). It is time again to use the Wess-Zumino equation (3.47); if we consider
only the terms with two derivatives on δ we obtain that the completely symmetric part of $W^{\mu\nu}_{2}$ is null i.e. $W^{\mu\nu}_{2} = W^{[\mu\nu]}_{2}$. Now we consider the finite renormalizations

$$
T(T^{[\mu\nu]}(x_1), T(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T(x_2)) + \delta(x_2 - x_1) U^{[\mu\nu]}_{3}(x_1) \tag{3.65}
$$

with $U^{[\mu\nu]}_{3} = -i W^{[\mu\nu]}_{2}$ we will get a new expression (3.57) with only the contributions $k = 0$ i.e. the expression (3.57) has the form from the statement.

It is easy to prove that the Wess-Zumino equation (3.47) is now equivalent to:

$$(d Q W_1 = -i \partial_{\mu} W^\mu_2). \tag{3.66}$$

(iii) From (2.31) we have:

$$A^{[\mu\nu]}_{3}(x_1, x_2) = \sum_{k \leq 2} \partial_{\rho_1} \ldots \partial_{\rho_k} \delta(x_2 - x_1) W^{[\mu\nu];(\rho_1,\ldots,\rho_k]}_{3}(x_1) \tag{3.67}$$

and we have the restrictions

$$\omega(W^{[\mu\nu];(\rho_1,\ldots,\rho_k]}_{3}) \leq 5 - k, \quad gh(W^{[\mu\nu];(\rho_1,\ldots,\rho_k]}_{3}) = 3 \tag{3.68}$$

for all $k = 0, 1, 2$.

We perform the finite renormalization

$$T(T^{[\mu\nu]}(x_1), T^\sigma(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T^\sigma(x_2)) + \partial_{\sigma} \delta(x_2 - x_1) U^{[\mu\nu];[\rho\sigma]}_{5}(x_1) \tag{3.69}$$

with $U^{[\mu\nu];[\rho\sigma]}_{5} = i W^{[\mu\nu];(\rho\sigma]}_{3}$ and we eliminate the contributions corresponding to $k = 2$ from (3.67).

Now we consider the finite renormalization

$$T(T^{[\mu\nu]}(x_1), T^\rho(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T^\rho(x_2)) + \delta(x_2 - x_1) U^{[\mu\nu];[\rho]}_{3}(x_1) \tag{3.70}$$

with $U^{[\mu\nu];[\rho]}_{3} = i W^{[\mu\nu];[\rho]}_{3}$ and we get a new expressions (3.67) with only the contributions $k = 0$ i.e. the expression (3.67) has the form from the statement.

(iv) From (2.31) we have:

$$A^{\mu\nu}_{4}(x_1, x_2) = \sum_{k \leq 2} \partial_{\rho_1} \ldots \partial_{\rho_k} \delta(x_2 - x_1) W^{\mu\nu;\{\rho_1,\ldots,\rho_k\}}_{4}(x_1) \tag{3.71}$$

and we have the restrictions

$$\omega(W^{\mu\nu;\{\rho_1,\ldots,\rho_k\}}_{4}) \leq 5 - k, \quad gh(W^{\mu\nu;\{\rho_1,\ldots,\rho_k\}}_{4}) = 3 \tag{3.72}$$

for all $k = 0, 1, 2$.

We will have to consider the (anti)symmetry (3.45). From the terms with two derivatives on delta we obtain that $W^{\mu\nu;\{\rho,\sigma\}}_{4}$ is antisymmetric in the first two indices i.e. we have the writing $W^{\mu\nu;\{\rho,\sigma\}}_{4} = W^{[\mu\nu];(\rho,\sigma]}_{4}$.

Next we consider the Wess-Zumino consistency condition (3.48). From the terms with three derivatives on delta we obtain

$$W^{[\mu\nu];(\rho\sigma]}_{4} + W^{[\mu\rho];(\sigma\nu]}_{4} + W^{[\mu\sigma];(\nu\rho]}_{4} = 0. \tag{3.73}$$
We note now that in the finite renormalization (3.69) we have used only the expression $U_5^{[\mu\nu];[\rho\sigma]}$ i.e. $U_5^{[\mu\nu];[\rho\sigma]}$ is still available. It is not so complicated to prove (using the preceding relation) that the choice: $U_5^{[\mu\nu];[\rho\sigma]} = \frac{1}{4} (W_1^{[\mu\nu];[\rho\sigma]} - W_1^{[\nu\rho];[\mu\sigma]} - W_1^{[\nu\rho];[\mu\sigma]} + W_1^{[\mu\nu];[\rho\sigma]})$ is possible i.e. it verifies the (anti)symmetry properties; moreover after this finite renormalization we get a new expression (3.71) for which the term corresponding to $k = 2$ is absent. We can enforce now the (anti)symmetry property (3.45): it is equivalent to:

$$W_4^{\mu\nu\rho\sigma} = W_4^{\nu\rho\mu\sigma}$$

$$W_4^{\mu\nu\rho\sigma} + W_4^{\nu\rho\mu\sigma} + \partial_\rho W_4^{\nu\rho\mu\sigma} = 0. \quad (3.74)$$

We also make explicit the Wess-Zumino consistency condition (3.48); it is:

$$d_Q W_2^\mu = i \partial_\nu W_3^{[\mu\nu]}$$
$$W_4^{\mu\nu} = -W_3^{[\mu\nu]}$$
$$W_4^{\mu\nu\rho\sigma} = -W_4^{\mu\nu\rho\sigma}. \quad (3.75)$$

We note immediately that we have $W_4^{\mu\nu\rho\sigma} = 0$ i.e. the expression (3.71) has the form from the statement. We are left from (3.48) only with

$$d_Q W_2^\mu = i \partial_\nu W_3^{[\mu\nu]} \quad (3.76)$$

(v) From (2.31) we have:

$$A_5^{[\mu\nu];[\rho\sigma]}(x_1, x_2) = \delta(x_2 - x_1) W_3^{[\mu\nu]}(x_1) + \partial_\sigma \delta(x_2 - x_1) W_5^{[\mu\nu];[\rho\sigma]}(x_1) \quad (3.77)$$

and we have the restrictions

$$\omega(W_5^{\mu\nu};[\rho\sigma]) \leq 5, \quad \omega(W_5^{[\mu\nu];[\rho\sigma]}) \leq 4$$
$$gh(W_5^{\mu\nu};[\rho\sigma]) = gh(W_5^{[\mu\nu];[\rho\sigma]}) = 4. \quad (3.78)$$

We consider the Wess-Zumino consistency conditions (3.49). From the terms with two derivatives on delta we obtain:

$$W_5^{[\mu\nu];[\rho\sigma]} = -W_5^{[\mu\nu];[\rho\sigma]} \quad (3.79)$$

i.e. we have the writing $W_5^{[\mu\nu];[\rho\sigma]} = W_5^{[\mu\nu];[\rho\sigma]}$. From the Wess-Zumino consistency conditions (3.50) we consider again the terms with two derivatives on delta and we obtain after some computations:

$$W_5^{[\mu\nu];[\rho\sigma]} = W_5^{[\rho\sigma];[\mu\nu]} \quad (3.80)$$

We now make the finite renormalization:

$$T(T^{[\mu\nu]}(x_1), T^{[\rho\sigma]}(x_2)) \rightarrow T(T^{[\mu\nu]}(x_1), T^{[\rho\sigma]}(x_2)) + \delta(x_1 - x_2) U_6^{[\mu\nu];[\rho\sigma]}(x_1) \quad (3.81)$$

with $U_6^{[\mu\nu];[\rho\sigma]} = i W_5^{[\mu\nu];[\rho\sigma]}$ and we eliminate the second contributions from (3.77). The Wess-Zumino consistency conditions (3.49) becomes equivalent to

$$d_Q W_3^{[\mu\nu]} = 0$$
$$W_3^{[\mu\nu];[\rho\sigma]} = 0. \quad (3.82)$$
In particular we have
\[ A_5^{[\mu\nu],[\rho]} = 0. \] (3.83)
and from (3.49) we are left with:
\[ d_Q W_3^{[\mu\nu]} = 0. \] (3.84)

The Wess-Zumino consistency conditions (3.50) is equivalent to
\[ d_Q W_4^{[\mu\nu]} = 0 \] (3.85)
which follows from the preceding relation if we remember the connection between \( W_3^{[\mu\nu]} \) and \( W_4^{[\mu\nu]} \) obtained at (iv).

(vi) From (2.31) we have:
\[ A_6^{[\mu\nu],[\rho\sigma]}(x_1, x_2) = \delta(x_1 - x_2) W_6^{[\mu\nu],[\rho\sigma]}(x_1) \] (3.86)
and we have the restrictions
\[ \omega(W_6^{[\mu\nu],[\rho\sigma]}) \leq 5 \quad gh(W_6^{[\mu\nu],[\rho\sigma]}) = 5. \] (3.87)

From the symmetry property (3.46) we also have
\[ W_6^{[\mu\nu],[\rho\sigma]} = W_6^{[\rho\sigma],[\mu\nu]}. \] (3.88)

However from the Wess-Zumino consistency condition (3.52) we have
\[ W_6^{[\mu\nu],[\rho\sigma]} = 0 \] (3.89)
so in fact:
\[ A_6^{[\mu\nu],[\rho\sigma]} = 0. \] (3.90)

(vii) Finally we observe that we can make some redefinitions of the chronological products without changing the structure of the anomalies. Indeed we have
\[ T(T(x_1), T(x_2)) \rightarrow T(T(x_1), T(x_2)) + \delta(x_1 - x_2) B(x_1) \] (3.91)
which makes
\[ W \rightarrow W + d_Q B \] (3.92)
and
\[ T(T^\mu(x_1), T(x_2)) \rightarrow T(T^\mu(x_1), T(x_2)) + \delta(x_1 - x_2) B^\mu(x_1) \] (3.93)
which makes
\[ W \rightarrow W + i \partial_\mu B^\mu, \quad W^\mu \rightarrow W^\mu + d_Q B^\mu. \] (3.94)
We also observe that we can consider the finite renormalizations (3.65) and
\[ T(T^\mu(x_1), T^\nu(x_2)) \rightarrow T(T^\mu(x_1), T^\nu(x_2)) + \delta(x_2 - x_1) U_4^{[\mu\nu]}(x_1) \] (3.95)
such that the we have the (anti)symmetry property (2.20). If we take
\[ U_3^{[\mu\nu]} = B^{[\mu\nu]}, \quad U_4^{[\mu\nu]} = -B^{[\mu\nu]} \] (3.96)
we have the redefinitions
\[ W^\mu \rightarrow W^\mu + i \partial_\nu B^{[\mu\nu]}, \quad W^{[\mu\nu]} \rightarrow W^{[\mu\nu]} + d_Q B^{[\mu\nu]} \] (3.97)

All these redefinitions do not modify the form of the anomalies from the statement and we have obtained the last assertion of the theorem. \( \square \)

As we can see one can simplify considerably the form of the anomalies if one makes convenient redefinitions of the chronological products. Moreover, the result is of purely cohomological nature i.e. we did not use the explicit form of the expressions \( T, T^\mu, T^{[\mu\nu]} \). The main difficulty of the proof is to find a convenient way of using Wess-Zumino equations, the (anti)symmetry properties and a succession of finite renormalizations. It is a remarkable fact that the preceding result stays true for arbitrary order of the perturbation theory i.e. we have:

**Theorem 3.2** Suppose that we have gauge invariance up to the order \( n-1 \) of the perturbation theory. Then, by convenient redefinitions of the chronological products, the anomalies from the equations (3.3) - (3.11) can be taken of the form:

\[
A_1(X) = \delta(X) W(x_1), \quad A_2^\mu(X) = \delta(X) W^\mu(x_1)
\]
\[
A_3^{[\mu\nu]}(X) = \delta(X) W^{[\mu\nu]}(x_1), \quad A_4^{\mu\nu}(X) = -\delta(X) W^{[\mu\nu]}(x_1),
\]
\[ A_j^\nu(x) = 0, \quad j = 5, \ldots, 9. \] (3.98)

The expressions \( W, W^\mu \) and \( W^{[\mu\nu]} \) are relative cocycles and are determined up to relative co-boundaries.

**Proof:** For the sake of completeness we provide a minimum number of details for the first anomaly \( A_1 \). From (2.31) we have

\[
A_1(X) = \sum_{2 \leq l \leq n} \partial^l_\mu \partial^l_\nu \partial^l_\rho \partial^l_\sigma \delta(X) W_1^{(\mu\nu\rho\sigma)}(x_1) + \sum_{2 \leq k \neq l \leq n} \partial^k_\mu \partial^k_\nu \partial^k_\rho \partial^k_\sigma \delta(X) W_1^{(\mu\nu\rho\sigma)}(x_1)
\]
\[ + \sum_{2 \leq k \neq l \leq n} \partial^k_\mu \partial^k_\nu \partial^k_\rho \partial^k_\sigma \delta(X) W_1^{(\mu\nu\rho\sigma)}(x_1) + \cdots \] (3.99)

where by \( \cdots \) we mean the terms with three or less derivatives on the delta function and the symmetry property (3.13) is true if we put some supplementary restrictions on the preceding expression. We perform the finite renormalization:

\[
T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \rightarrow T(T^\mu(x_1), T(x_2), \ldots, T(x_n))
\]
\[ + \sum_{2 \leq l \leq n} \partial^l_\mu \partial^l_\nu \partial^l_\rho \partial^l_\sigma \delta(X) U_2^{(\mu\nu\rho\sigma)}(x_1) + \sum_{2 \leq k \neq l \leq n} \partial^k_\mu \partial^k_\nu \partial^k_\rho \partial^k_\sigma \delta(X) U_2^{(\mu\nu\rho\sigma)}(x_1) \] (3.100)

and if we choose it conveniently we can obtain a new expression (3.53) for the anomaly \( A_1 \) without terms with four derivatives on delta, i.e.

\[
A_1(X) = \sum_{2 \leq l \leq n} \partial^l_\mu \partial^l_\nu \partial^l_\rho \partial^l_\sigma \delta(X) W_1^{(\mu\nu\rho\sigma)}(x_1) + \sum_{2 \leq k \neq l \leq n} \partial^k_\mu \partial^k_\nu \partial^k_\rho \partial^k_\sigma \delta(X) W_1^{(\mu\nu\rho\sigma)}(x_1)
\] (3.101)

where by \( \cdots \) we mean the terms with two or less derivatives on the delta function. We impose the symmetry property (3.13) and we can perform a finite renormalization:

\[
T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \rightarrow T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) + \sum_{2 \leq l \leq n} \partial^l_\mu \partial^l_\nu \partial^l_\rho \partial^l_\sigma \delta(X) U_2^{(\mu\nu\rho\sigma)}(x_1) \] (3.102)
such that we eliminate the terms with three derivatives on delta, i.e.

$$A_1(X) = \sum_{2 \leq l \leq n} \partial_{\mu}^l \partial_{\nu}^l \delta(X) W_1^{\mu\nu}(x_1) + \sum_{2 \leq k \neq l \leq n} \partial_{\mu}^k \partial_{\nu}^k \delta(X) W_1^{\mu\nu}(x_1) + \cdots$$

(3.103)

where by $\cdots$ we mean the terms with one or no derivatives on the delta function.

Finally we perform a convenient finite renormalization:

$$T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) \to T(T^\mu(x_1), T(x_2), \ldots, T(x_n)) + \sum_{l=2}^{n} \partial^l \delta(X) U_l^{\mu\nu}(x_1)$$

(3.104)

and we get an expression for $A_1$ as in the statement of the theorem. Proceeding in the same way we arrive after some non-trivial combinatorics at the result from the statement for all anomalies. □

We have proved that renormalization of gauge theories leads to some descent equations. We have the expressions $T^I$ and $R^I$ (with ghost numbers $gh(T^I) = gh(R^I) = |I|$ and canonical dimension $\leq 4$) for the interaction Lagrangian and the finite renormalizations compatible with gauge invariance; we also have the expressions $W^I$ (with ghost numbers $gh(W^I) = |I| + 1$ and canonical dimension $\leq 5$) for the anomalies. In the next Sections we give the most simpler way to solve in general such type of problems.

4 A GEOMETRIC SETTING FOR THE GAUGE INVARIANCE PROBLEM

The cohomology of the operator $d_Q$ can be reformulated in the language of classical field theory (with Grassmann variables).

The kinematical structure of a classical field theory is based on fibered bundle structures. Let $\pi : Y \to X$ be fiber bundle, where $X$ and $Y$ are differentiable manifolds of dimensions $\dim(X) = n, \dim(Y) = m+n$ and $\pi$ is the canonical projection of the fibration. Usually $X$ is interpreted as the “space-time” manifold and the fibers of $Y$ as the field variables. An adapted chart to the fiber bundle structure is a couple $(V, \psi)$ where $V$ is an open subset of $Y$ and $\psi : V \to \mathbb{R}^n \times \mathbb{R}^m$ is the so-called chart map, usually written as $\psi = (x^\mu, y^\alpha)$ ($\mu = 1, \ldots, n; \alpha = 1, \ldots, m$) such that $(\pi(V), \phi)$ where $\phi = (x^\mu)$ ($\mu = 1, \ldots, n$) is a chart on $X$ and the canonical projection has the following expression: $\pi(x^\mu, y^\alpha) = (x^\mu)$. If $p \in Y$ then the real numbers $x^\mu(p), \quad y^\alpha(p)$ are called the (fibered) coordinates of $p$. For simplicity we will give up the attribute adapted in the following. Also we will refer frequently to the first entry $V$ of $(V, \psi)$ as a chart.

Next, one considers the $r$-jet bundle extensions $J^r_n Y \to X$ ($r \in \mathbb{N}$). The construction is the following (see for instance [7]).

**Theorem 4.1** Let $x \in X$, and $y \in \pi^{-1}(x)$. We denote by $\Gamma_{(x,y)}$ the set of sections $\gamma : U \to Y$ such that: (i) $U$ is a neighborhood of $x$; (ii) $\gamma(x) = y$. We define on $\Gamma_{(x,y)}$ the relationship “$\gamma \sim \delta$” iff there exists a chart $(V, \psi)$ on $Y$ such that $\gamma$ and $\delta$ have the same partial derivatives up to order $r$ in the given chart i.e.

$$\frac{\partial^k}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}} \psi \circ \gamma \circ \phi^{-1}(\phi(x)) = \frac{\partial^k}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}} \psi \circ \delta \circ \phi^{-1}(\phi(x)), \quad k \leq r.$$  

(4.1)

Then this relationship is chart independent and it is an equivalence relation.
A \textit{r-order jet with source }x\textit{ and target }y\textit{ is, by definition, the equivalence class of some section }\gamma\textit{ with respect to the equivalence relationship defined above and it is denoted by }j^r_x\gamma.\textit{

Let us define }J^r_{(x,y)} = \Gamma_{(x,y)} \sim \textit{Then the }r\textit{-order jet bundle extension is, set theoretically }J^rY = \bigcup_x J^r_{(x,y)}\pi.\textit{ Let } (V, \psi), \quad \psi = (x^\mu, y^\gamma) \textit{ be a chart on }Y. \textit{ Then we define the couple } (V^r, \psi^r), \textit{ where: }

\[ V^r = (\pi^{r,0})^{-1}(V) \textit{ and } \]

\[ \psi = (x^\mu, y^\gamma, y^\alpha_{\mu_1,\ldots,\mu_k}, \ldots, y^\alpha_{\mu_1,\ldots,\mu_n}), \quad j_1 \leq j_2 \leq \cdots \leq j_k, \quad k = 1, \ldots, r \quad (4.2) \]

\textit{where }

\[ y^\alpha_{\mu_1,\ldots,\mu_k}(j^r_x\gamma) = \frac{\partial^k}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}} y^\alpha \circ \gamma \circ \phi^{-1} \bigg|_{\phi(x)}, \quad k = 1, \ldots, r \]

\[ x^\mu(j^r_x\gamma) = x^\mu(x), \quad y^\alpha(j^r_x\gamma) = y^\alpha(\gamma(x)). \quad (4.3) \]

\textit{Then } (V^r, \psi^r) \textit{ is a chart on } J^rY \textit{ called the associated chart of } (V, \psi).\textit{

Remark 4.2 \textit{The expressions } y^\alpha_{\mu_1,\ldots,\mu_k}(j^r_x\gamma) \textit{ are defined for all indices } \mu_1, \ldots, \mu_k = 1, \ldots, n, \textit{ and the restrictions } j_1 \leq j_2 \leq \cdots \leq j_k \textit{ in the definition of the charts are in order to avoid over-counting and are a result of the obvious symmetry property:}

\[ y^\alpha_{\mu_{P(1)},\ldots,\mu_{P(k)}}(j^r_x\gamma) = y^\alpha_{\mu_1,\ldots,\mu_k}(j^r_x\gamma), \quad (4.4) \]

\textit{for any permutation } P \in \mathcal{P}_k, \quad k = 2, \ldots, r.\textit{

Now we have the following result.}

\textbf{Theorem 4.3} \textit{If a collection of (adapted) charts } (V, \psi) \textit{ are the elements of a differentiable atlas on } Y \textit{ then } (V^r, \psi^r) \textit{ are the elements of a differentiable atlas on } J^r_n(Y) \textit{ which admits a fiber bundle structure over } Y.\textit{

To be able to use the summation convention over the dummy indices we consider } y^\alpha_{\mu_1,\ldots,\mu_k} \textit{ for all values of the indices } \mu_1, \ldots, \mu_k \in \{1, \ldots, n\} \textit{ as smooth functions on the chart } V^r \textit{ defined in terms of the independent variables } y^\alpha_{\mu_1,\ldots,\mu_k}, \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k = 1, 2, \ldots, r \textit{ according to the formula (4.4) and we make a similar convention for the partial derivatives } \frac{\partial}{\partial y^\alpha_{\mu_1,\ldots,\mu_k}}\textit{.}

\textit{Then we define on the chart } V^r \textit{ the following vector fields:}

\[ \partial^\alpha_{\mu_1,\ldots,\mu_k} = \frac{r_1! \cdots r_n!}{k!} \frac{\partial}{\partial y^\alpha_{\mu_1,\ldots,\mu_k}}, \quad k = 1, \ldots, r \quad (4.5) \]

\textit{for all values of the indices } \mu_1, \ldots, \mu_k \in \{1, \ldots, n\}. \textit{ Here } r_l, \quad l = 1, \ldots, n \textit{ is the number of times the index } l \textit{ enters into the set } \{\mu_1, \ldots, \mu_k\} \textit{.}

\textit{One can easily verify the following formulas:}

\[ \partial^\alpha_{\beta_1,\ldots,\beta_k} y^\alpha_{\nu_1,\ldots,\nu_l} = 0, \quad (k \neq l) \quad (4.6) \]

\[ \partial^\alpha_{\beta_1,\ldots,\beta_k} y^\alpha_{\nu_1,\ldots,\nu_k} = \delta^\alpha_\beta S^+_{\mu_1,\ldots,\mu_k} \delta^\mu_{\nu_1} \cdots \delta^\mu_{\nu_k} \quad (4.7) \]

\textit{where } S^+_{j_1,\ldots,j_k} \textit{ is the symmetrization projector operator in the indices } \mu_1, \ldots, \mu_k.\textit{\textbf{\ }}
Also we have for any smooth function $f$ on the chart $V^r$:

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu + \sum_{k=0}^r (\partial^{\mu_1,\ldots,\mu_k}_\alpha f) dy^\alpha_{\mu_1,\ldots,\mu_k} = \frac{\partial f}{\partial x^\mu} dx^\mu + \sum_{|J| \leq r} (\partial_J^f f) dy^J. \quad (4.8)$$

In the last formula we have introduced the multi-index notations in an obvious way. This formula also shows that the coefficients appearing in the definition (4.5) are exactly what is needed to use the summation convention over the dummy indices without over-counting.

We now define the expressions

$$d^r_\rho \equiv \frac{\partial}{\partial x^\rho} + \sum_{k=0}^{r-1} y^\alpha_{\rho,\mu_1,\ldots,\mu_k} \partial^{\mu_1,\ldots,\mu_k}_\alpha \quad (4.9)$$

called formal derivatives. When it is no danger of confusion we denote simply $d_\mu = d^r_\mu$.

**Remark 4.4** The formal derivatives are not vector fields on $J^r Y$.

Next one immediately sees that

$$d_\mu y^\alpha_{\nu_1,\ldots,\nu_k} = y^\alpha_{\mu,\nu_1,\ldots,\nu_k}, \quad k = 0,\ldots, r - 1. \quad (4.10)$$

From the definition of the formal derivatives it easily follows by direct computation that:

$$[\partial^{\mu_1,\ldots,\mu_k}_\alpha, d_\rho] = \frac{1}{k} \sum_{l=1}^k \delta^\mu_\rho \partial^{\mu_1,\ldots,\hat{\mu}_l,\ldots,\mu_k}_\alpha, \quad k = 0,\ldots, r \quad (4.11)$$

where we use Bourbaki conventions $\sum_0 \equiv 0, \quad \prod_0 \equiv 1$.

The formalism presented above extends easily to the Grassmann case. We denote by $\epsilon_\alpha$ the Grassmann parity of the variable $y^\alpha$. We only have to replace commutators with graded commutators and distinguish between left and right derivatives; we will consider here only left derivatives. Then we can interpret equation

$$d_Q R = 0 \quad (4.12)$$

as an equation in classical field theory where we also suppose that the polynomials are restricted to the mass shell and we replace the derivative $\partial^\mu$ by $d^\mu$.

A final word about the notations. Because $y^\alpha_{\mu_1,\ldots,\mu_n} = d_{\mu_1} \cdots d_{\mu_n} y^\alpha$ we freely use both notations. When the index $\alpha$ are downstairs we write $y^{\alpha}_{\mu_1,\ldots,\mu_n}$.

We now prove a sort of Poincaré lemma adapted to our conditions. There are two obstacles in applying the usual Poincaré lemma: first our co-cycles are polynomials and second we are working on the mass shell. If only the first obstacle would be present then we could apply the so-called algebraic Poincaré lemma [2], but unfortunately this nice result breaks down if we work on shell. We make the assumption that we are on the mass shell because the Epstein-Glaser construction is done from the very beginning in a Fock space of some free particles. We will prove below that the obstacles to the Poincaré lemma are easy to describe. Basically we want to find the general solution of equations of the type:

$$d_\mu S^I_{\mu} = 0. \quad (4.13)$$
There are some trivial solutions of this equation namely of this equation namely of the type
\[ S^{I;\mu} = d_\nu S^{I;\mu\nu} \]  
(4.14)
where the expression \( S^{I;\mu\nu} \) is antisymmetric in the last two indices. We will be able to describe the obstruction relevant to this equation i.e. solutions which are not trivial. We start first with:

**Proposition 4.5** Let the expression \( S^{I;\mu} \) be of canonical dimension \( \omega(S^{I;\mu}) = 2 \) and verifying the relation (4.13). Then it is of the form
\[ S^{I;\mu} = c^I_\alpha d^\mu y^\alpha + d_\nu S^{I;\mu\nu} \]  
(4.15)
with the expression \( S^{I;\mu\nu} \) antisymmetric in the last two indices.

**Proof:** The generic form for \( S^{I;\mu} \) is:
\[ S^{I;\mu} = \frac{1}{2} \sum_{\alpha,\beta} c^{I;\mu}_{\alpha\beta} y^\alpha y^\beta + \text{total divergence} \]  
(4.16)
where the expressions \( c^{I;\mu}_{\alpha\beta} \) are constants and we note that the second contribution is linear in the fields. Also we can impose \( c^{I;\mu}_{\alpha\beta} = \epsilon_\alpha \epsilon_\beta c^{I;\mu}_{\beta\alpha} \).

Now it is easy to prove that the condition (4.13) gives \( c^{I;\mu}_{\alpha\beta} = 0 \) so we have \( S^{I;\mu} = d_\nu S^{I;\mu\nu} \) with \( \omega(S^{I;\mu\nu}) = 1 \). We split now the expression \( S^{I;\mu\nu} \) in the symmetric and the antisymmetric part in the indices \( \mu \) and \( \nu \) denoted by \( S^{I;\mu\nu}_\pm \). The condition (4.13) gives \( d_\mu d_\nu S^{I;\mu\nu}_\pm = 0 \) so we necessarily have \( S^{I;\mu\nu}_\pm = \eta^{\mu\nu} A^I \); obviously we must have \( A^I = c^I_\alpha y^\alpha \) and we obtain the expression from the statement. \( \square \)

The case \( \omega = 3 \) is harder.

**Proposition 4.6** Let the expression \( S^{I;\mu} \) be of canonical dimension \( \omega(S^{I;\mu}) = 3 \) and verifying the relation (4.13). Then it is of the form
\[ S^{I;\mu} = \sum_{\alpha,\beta} c^{I;\mu}_{\alpha\beta} y^\alpha d^\mu y^\beta + \sum_\alpha c^{I;\nu}_{\alpha} d^\mu d_\nu y^\alpha + d_\nu S^{I;\mu\nu} \]  
(4.17)
with \( c^{I;\mu}_{\alpha\beta}, c^{I;\nu}_{\alpha} \) some constants, one has \( c^{I;\mu}_{\alpha\beta} = -\epsilon_\alpha \epsilon_\beta c^{I;\mu}_{\beta\alpha} \) and \( S^{I;\mu\nu} \) is antisymmetric in the last two indices.

**Proof:** From the equation (4.13) we get with (4.11):
\[ d_\mu \frac{\partial S^{I;\mu}}{\partial y^\alpha} = 0 \]  
(4.18)
for any \( y^\alpha \). So we can use the preceding proposition and find out
\[ \frac{\partial S^{I;\mu}}{\partial y^\alpha} = \sum_\beta c^{I;\mu}_{\alpha\beta} d^\mu y^\beta + d_\nu S^{I;\mu\nu}_\alpha \]  
(4.19)
with the last expression antisymmetric in \( \mu \) and \( \nu \). Here \( c^{I;\mu}_{\alpha\beta} \) are constants and \( S^{I;\mu\nu}_\alpha \) have canonical dimension \( \omega = 1 \) so we have the generic form:
\[ S^{I;\mu\nu}_\alpha = \sum_\beta s^{I;\mu\nu}_{\alpha\beta} y^\beta \]  
(4.20)
where $s^{I;\mu\nu}_{\alpha\beta}$ are constants and we have antisymmetry in $\mu$ and $\nu$. So we have:

$$\frac{\partial S^{I;\mu}}{\partial y^\alpha} = \sum_\beta c^{I}_{\alpha\beta} d^\mu y^\beta + \sum_\beta s^{I;\mu\nu}_{\alpha\beta} d_\nu y^\beta$$  \hspace{1cm} (4.21)

which can be integrated:

$$S^{I;\mu} = \sum_{\alpha,\beta} c^{I}_{\alpha\beta} y^\alpha d^\mu y^\beta + \sum_{\alpha,\beta} s^{I;\mu\nu}_{\alpha\beta} y^\alpha d_\nu y^\beta + S^{I;\mu}_1$$  \hspace{1cm} (4.22)

where $S^{I;\mu}_1$ depends only on derivatives i.e. is of the form:

$$S^{I;\mu}_1 = \sum_\alpha c^{I;\mu\rho}_{\alpha} d_\rho y^\alpha$$  \hspace{1cm} (4.23)

with $c^{I;\mu\rho}_{\alpha}$ some constants with symmetry in $\nu$ and $\rho$. Now we obtain from (4.13) the following equations:

$$c^{I}_{\alpha\beta} = -\epsilon_\alpha \epsilon_\beta c^{I}_{\beta\alpha}$$

$$s^{I;\mu\nu}_{\alpha\beta} = \epsilon_\alpha \epsilon_\beta s^{I;\mu\nu}_{\beta\alpha}$$

$$c^{I;\mu\rho}_{\alpha} = a_1 \eta^{\mu\rho} a^{I;\mu}_{\alpha} + \frac{1}{2} a_2 (\eta^{\mu\rho} a^{I;\mu}_{\alpha} + \eta^{\nu\rho} a^{I;\nu}_{\alpha})$$  \hspace{1cm} (4.24)

and we easily obtain the expression from the statement. □

Now we give the main result of this Section.

**Theorem 4.7** Let $S^{I;\mu}$ be of canonical dimension $\omega(S^{I;\mu}) \geq 4$ at least tri-linear in the fields (and derivatives) fulfilling the relation (4.13). Then it is of the following generic form:

$$S^{I;\mu} = d_\nu S^{I;\mu}_{\nu}$$  \hspace{1cm} (4.25)

where the expression $S^{I;\mu}_{\nu}$ is antisymmetric in $\mu$, $\nu$ i.e. it gives a trivial contribution.

**Proof:** (i) We first consider the case $\omega(S^{I;\mu}) = 4$ and we have from (4.13)

$$d_\mu \left( \frac{\partial S^{I;\mu}}{\partial y^\alpha} \right) = 0;$$  \hspace{1cm} (4.26)

but the expression $\frac{\partial S^{I;\mu}}{\partial y^\alpha}$ has the canonical dimension 3 so we can apply the preceding proposition and obtain:

$$\frac{\partial S^{I;\mu}}{\partial y^\alpha} = \sum_{\beta,\gamma} c^{I}_{\alpha\beta\gamma} y^\beta d^\mu y^\gamma + d_\nu S^{I;\mu}_{\alpha\beta\gamma}$$  \hspace{1cm} (4.27)

with the expressions $S^{I;\mu}_{\alpha\beta\gamma}$ antisymmetric in $\mu$, $\nu$; the term $\sim d^\mu d^\nu y^\alpha$ does not appear because we have supposed the expression $S^{I;\mu}$ at least tri-linear in the fields. We also have the generic form:

$$S^{I;\mu}_{\alpha\beta\gamma} = \frac{1}{2} \sum_{\beta,\gamma} s^{I;\mu\nu}_{\alpha\beta\gamma} y^\beta y^\gamma$$  \hspace{1cm} (4.28)
with \( s^{I\mu\nu}_{\alpha\beta\gamma} \) some constants and

\[
\begin{align*}
  c^I_{\alpha\beta\gamma} &= -\epsilon_{\beta\gamma} c^I_{\alpha\gamma\beta}, \\
  s^{I\mu\nu}_{\alpha\beta\gamma} &= -s^{I\nu\mu}_{\alpha\beta\gamma}, \\
  s^{I\mu\nu}_{\alpha\beta\gamma} &= \epsilon_{\beta\gamma} s^{I\mu\nu}_{\alpha\gamma\beta}.
\end{align*}
\]

(4.29)

It follows that

\[
\frac{\partial S^{I\mu}}{\partial y^\alpha} = \sum_{\beta,\gamma} c^I_{\alpha\beta\gamma} y^\beta d^\mu y^\gamma + \sum_{\beta,\gamma} s^{I\mu\nu}_{\alpha\beta\gamma} y^\beta d^\nu y^\gamma
\]

(4.30)

We impose the condition

\[
\frac{\partial^2 S^{I\mu}}{\partial y^\gamma \partial y^\alpha} = \epsilon_{\alpha\beta} \frac{\partial^2 S^{I\mu}}{\partial y^\alpha \partial y^\beta}
\]

(4.31)

and obtain:

\[
\begin{align*}
  c^I_{\alpha\beta\gamma} &= \epsilon_{\alpha\beta} c^I_{\beta\alpha\gamma}, \\
  s^{I\mu\nu}_{\alpha\beta\gamma} &= \epsilon_{\alpha\beta} s^{I\mu\nu}_{\beta\alpha\gamma}.
\end{align*}
\]

(4.32)

From the first relations of (4.29) and (4.32) we obtain

\[
c^I_{\alpha\beta\gamma} = 0.
\]

(4.33)

Using the second relation (4.32) we can integrate (4.30) and get:

\[
\frac{\partial S^{I\mu}}{\partial y^\alpha} = \frac{1}{2} \sum_{\alpha,\beta,\gamma} s^{I\mu\nu}_{\alpha\beta\gamma} y^\alpha y^\beta d^\nu y^\gamma + S^I_{1}\mu
\]

(4.34)

where \( S^I_{1}\mu \) depends only on derivatives so it is null (because it must be trilinear). Now we have from (4.29) and (4.32) that the expression \( s^{I\mu\nu}_{\alpha\beta\gamma} \) is completely symmetric (in the graded sense) in the indices \( \alpha, \beta, \gamma \) so we can integrate the preceding relation:

\[
S^{I\mu} = \frac{1}{6} \sum_{\alpha,\beta,\gamma} s^{I\mu\nu}_{\alpha\beta\gamma} d_\nu (y^\alpha y^\beta y^\gamma)
\]

(4.35)

i.e. we have the expression from the statement with

\[
S^{I\mu\nu} = \frac{1}{6} \sum_{\alpha,\beta,\gamma} s^{I\mu\nu}_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma.
\]

(4.36)

(ii) Now we consider the statement of the theorem valid for \( \omega(S^{I\mu}) = 4, \ldots, N \) \( (N \geq 4) \) and we have from (4.13)

\[
d_\mu \left( \frac{\partial S^{I\mu}}{\partial y^\alpha} \right) = 0;
\]

(4.37)

we can apply the induction hypothesis and get

\[
\frac{\partial S^{I\mu}}{\partial y^\alpha} = d_\nu S^{I\mu\nu}_{\alpha}.
\]

(4.38)
the expression $S^I_{\alpha}^{\mu\nu}$ is of maximal degree $N - 1$ in $y^\alpha$ so we have the generic form

$$S^I_{\alpha}^{\mu\nu} = \sum_{k=0}^{n} \frac{1}{k!} s^I_{\alpha_0...\alpha_k} y^{\alpha_1} ... y^{\alpha_n}$$  \hfill (4.39)

where the expression $s^I_{\alpha_0...\alpha_k}$ do not depend on $y^\beta$ are antisymmetric in $\mu$, $\nu$ and (graded) antisymmetric in $\alpha_1, ..., \alpha_n$; moreover $n \leq N - 1$ is the maximal degree in $y^\beta$ and $\omega(s^I_{\alpha_0...\alpha_k}) = N - 1 - k$. Let us also note that we must have $s^I_{\alpha_0...\alpha_{k-1}} = 0$ because this expression has canonical dimension 1 according to the preceding formula but it must have at least a factor $d^\alpha y^\beta$ which has canonical dimension greater than 2. We have two cases:

(a) $n = N - 1$.

In this case the expression $s^I_{\alpha_0...\alpha_n}$ are in fact constants. It is easy to prove from Frobenius condition of integrability that this expression is completely antisymmetric (in the graded sense) in all indices $\alpha_0, ..., \alpha_n$; now we can integrate (4.38) with respect to the variables $y^\beta$ and we have

$$S^I_{\mu} = \frac{1}{(N - 1)!} s^I_{\alpha_0...\alpha_{N-1}} y^{\alpha_0} ... y^{\alpha_{N-2}} d_\nu y^{\alpha_{N-1}} + ...$$  \hfill (4.40)

where by $\cdots$ we mean terms of degree $< N - 1$ in $y^\beta$. From here

$$S^I_{\mu} = \frac{1}{N!} d_\nu (s^I_{\alpha_0...\alpha_{N-1}} y^{\alpha_0} ... y^{\alpha_{N-1}}) + ...$$  \hfill (4.41)

The first term is a trivial solution and can be eliminated. The new $S^I_{\mu}$ will be of degree $< N - 1$ in the variables $y^\beta$; the new $S^I_{\mu}$ verifies again (4.38) and (4.39) with $n = N - 3$.

(b) $n \leq N - 3$.

In this case Frobenius condition of integrability shows that the expression $d_\nu s^I_{\alpha_0...\alpha_n}$ is completely antisymmetric (in the graded sense) in all indices $\alpha_0, ..., \alpha_n$; again we can integrate the system (4.38) and get

$$S^I_{\mu} = \frac{1}{(n + 1)!} (d_\nu s^I_{\alpha_0...\alpha_n}) y^{\alpha_0} ... y^{\alpha_n} + ...$$  \hfill (4.42)

where by $\cdots$ we mean terms of degree $< n - 1$ in $y^\beta$. From here

$$S^I_{\mu} = \frac{1}{(n + 1)!} d_\nu (s^I_{\alpha_0...\alpha_n} y^{\alpha_0} ... y^{\alpha_n}) + ...$$  \hfill (4.43)

The first term is a trivial solution and can be eliminated. The new $S^I_{\mu}$ will again verify (4.38) and (4.39). Because $s^I_{\alpha_0...\alpha_{n-1}} = 0$ we will now obtain from Frobenius condition of integrability that the expression $s^I_{\alpha_0...\alpha_n}$ is completely antisymmetric (in the graded sense) in all indices $\alpha_0, ..., \alpha_n$ and we can repeat the argument from case (a). As a result we obtain a new $S^I_{\mu}$ verifying (4.38) and (4.39) with $n \to n - 1$.

(iii) By recursion we end up with an expressions $S^I_{\alpha}^{\mu}$ and $s^I_{\alpha}^{\mu\nu}$ independent of the variables $y^\beta$. Because the expressions are at least tri-linear in the fields they can be non-zero only for $N \geq 2.3 = 6$. We can repeat the line of argument with $y^\alpha \to y^\mu_i$ because $\omega(\frac{\partial s^I_{\alpha}^{\mu\nu}}{\partial y^\mu_i}) = N - 2 \geq 4$ and we will eliminate the dependence on the first order derivatives. After a finite number of steps we get $S^I_{\mu} = 0$. □

Let us denote by $y^A$ any of the variables $y^\alpha$ and their derivatives. We also denote by $\epsilon_A$ the Grassmann parity of $y^A$. Then we have the following simple corollary:
Corollary 4.8 Suppose that in the preceding theorem we renounce at the hypothesis of tri-linearity. Then the solutions of the equation (4.13) are of the form:

\[ S^{I, \mu} = \sum_{A, B} c_{AB}^I y^A d^\nu y^B + \sum_A c_A^I d^\nu y^A + d_\nu S^{I, \nu} \] (4.44)

where \( c_{AB}^I, c_A^I \) are constants verifying

\[ c_{AB}^I = -\epsilon_A \epsilon_B c_{BA}^I \] (4.45)

and the last contribution is the trivial solution.

5 THE COHOMOLOGY OF THE GAUGE CHARGE OPERATOR

We consider a vector space \( \mathcal{H} \) of Fock type generated (in the sense of Borchers theorem) by the vector field \( v_\mu \) (with Bose statistics) and the scalar fields \( u, \tilde{u} \) (with Fermi statistics). The Fermi fields are usually called ghost fields. We suppose that all these (quantum) fields are of null mass. Let \( \Omega \) be the vacuum state in \( \mathcal{H} \). In this vector space we can define a sesquilinear form \( \langle \cdot, \cdot \rangle \) in the following way:

\[
\begin{align*}
\langle \Omega, v_\mu(x_1)v_\mu(x_2)\Omega \rangle &= i \eta_{\mu\nu} D_0^{(+)}(x_1 - x_2), \\
\langle \Omega, u(x_1)\tilde{u}(x_2)\Omega \rangle &= -i D_0^{(+)}(x_1 - x_2), \\
\langle \Omega, \tilde{u}(x_1)u(x_2)\Omega \rangle &= -i D_0^{(+)}(x_1 - x_2)
\end{align*}
\] (5.1)

and the \( n \)-point functions are generated according to Wick theorem. Here \( \eta_{\mu\nu} \) is the Minkowski metrics (with diagonal \( 1, -1, -1, -1 \)) and \( D_0^{(+)} \) is the positive frequency part of the Pauli-Jordan distribution \( D_0 \) of null mass. To extend the sesquilinear form to \( \mathcal{H} \) we define the conjugation by

\[ v_\mu^\dagger = v_\mu, \quad u^\dagger = u, \quad \tilde{u}^\dagger = -\tilde{u}. \] (5.2)

Now we can define in \( \mathcal{H} \) the operator \( Q \) according to the following formulas:

\[
\begin{align*}
[Q, v_\mu] &= i \partial_\mu u, \quad [Q, u] = 0, \quad [Q, \tilde{u}] = -i \partial_\mu v^\mu, \\
Q\Omega &= 0
\end{align*}
\] (5.3)

where by \([\cdot, \cdot]\) we mean the graded commutator. One can prove that \( Q \) is well defined. Indeed, we have the causal commutation relations

\[
\begin{align*}
[v_\mu(x_1), v_\mu(x_2)] &= i \eta_{\mu\nu} D_0(x_1 - x_2) \cdot I, \\
[u(x_1), \tilde{u}(x_2)] &= -i D_0(x_1 - x_2) \cdot I
\end{align*}
\] (5.4)

and the other commutators are null. The operator \( Q \) should leave invariant these relations, in particular

\[
[Q, [v_\mu(x_1), \tilde{u}(x_2)]] + \text{cyclic permutations} = 0
\] (5.5)

which is true according to (5.3). It is useful to introduce a grading in \( \mathcal{H} \) as follows: every state which is generated by an even (odd) number of ghost fields and an arbitrary number of vector fields is even (resp. odd). We denote by \( |f| \) the ghost number of the state \( f \). We notice that the operator \( Q \) raises the ghost number of a state (of fixed ghost number) by an unit. The usefulness of this construction follows from:
Theorem 5.1 The operator $Q$ verifies $Q^2 = 0$. The factor space $\text{Ker}(Q)/\text{Ran}(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons).

Proof: (i) The fact that $Q$ squares to zero follows easily from (5.3): the operator $Q^2 = 0$ commutes with all field operators and gives zero when acting on the vacuum.

(ii) The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[ \int f_\mu(x) v^\mu(x) + \int g_1(x) u(x) + \int g_2(x) \bar{u}(x) \right] \Omega$$

with test functions $f_\mu, g_1, g_2$ verifying the wave equation equation. We impose the condition $\Psi \in \text{Ker}(Q) \iff Q\Psi = 0$; we obtain $\partial^\mu f_\mu = 0$ and $g_2 = 0$ i.e. the generic element $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is

$$\Psi = \left[ \int f_\mu(x) v^\mu(x) + \int g(x) u(x) \right] \Omega$$

with $g$ arbitrary and $f_\mu$ constrained by the transversality condition $\partial^\mu f_\mu = 0$; so the elements of $\mathcal{H}^{(1)} \cap \text{Ker}(Q)$ are in one-one correspondence with couples of test functions $(f_\mu, g)$ with the transversality condition on the first entry. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$\Psi' = Q\Phi = \left[ \int \partial_\mu g'(x) v^\mu(x) - \int \partial^\mu f'_\mu(x) u(x) \right] \Omega$$

so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $(f_\mu, g)$ then $\Psi + \Psi'$ is indexed by the couple $(f_\mu + \partial_\mu g', g - \partial^\mu f'_\mu)$. If we take $f'_\mu$ conveniently we can make $g = 0$. We introduce the equivalence relation $f_\mu^{(1)} \sim f_\mu^{(2)} \iff f_\mu^{(1)} - f_\mu^{(2)} = \partial_\mu g'$ and it follows that the equivalence classes from $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ are indexed by equivalence classes of wave functions $[f_\mu]$; it remains to prove that the sesquilinear form $\langle \cdot, \cdot \rangle$ induces a positively defined form on $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ and we have obtained the usual one-particle Hilbert space for the photon.

(iii) We go now to the 2-particle space. We borrow an argument from the proof of K"unneth formula [2]. Any 2-particle state is generated by states of the form:

$$\Psi = \sum_{j=1}^n f_j \otimes g_j$$

with $f_j, g_j$ one-particle states. We impose the condition $\Psi \in \text{Ker}(Q)$ and observe that it is sufficient to take $f_j, g_j$ states of fixed ghost number. Moreover, we can take $f_j$ such that their span does not intersect $\text{Ran}(Q)$. Indeed if we have constants $\beta_j$ not all null such that $\sum_{j=1}^n \beta_j f_j \in \text{Ran}(Q)$ then by a redefinition of the vectors $f_j$ we can arrange such that $f_1 = \sum_{j=2}^n \beta_j f_j + Qh$. We substitute this in the formula for $\Psi$ and get: $\Psi = \sum_{j=2}^n f_j \otimes (\beta_j' g_1 + g_j) + Q(h \otimes g_1) - (-1)^{|h|} h \otimes Qg_1$ so if we eliminate the co-boundary we can replace the state $\Psi$ by an equivalent one in which $f_1 \to h$. In this way we replace the expression (5.9) by an equivalent expression for which $\sum_{j=1}^n |f_j|$ decreases by an unit. Recursively we obtain another expression (5.9) modulo $\text{Ran}(Q)$ for which $\text{Span}(f_1) \cap \text{Ran}(Q) = \{0\}$. Now the condition $Q\Psi = 0$ writes $\sum_{j=1}^n (Qf_j \otimes g_j + (-1)^{|f_j|} f_j \otimes Qg_j) = 0$ and it easily follows that both sums must be separately null i.e. we must have $Qg_j = 0$ and $Qf_j = 0$ for all $j = 1, \ldots, n$. It means that we have the canonical isomorphism $(\mathcal{H}^{(2)} \cap \text{Ker}(Q))/(\mathcal{H}^{(2)} \cap \text{Ran}(Q)) \cong (\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q)) \otimes (\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$.

Now we can proceed by induction to the general $n$-particle states. □
We see that the condition \([Q,T] = i \partial_\mu T^\mu\) means that the expression \(T\) leaves invariant the physical Hilbert space (at least in the adiabatic limit).

Now we have the physical justification for solving another cohomology problem namely to determine the cohomology of the operator \(d_Q = [Q, \cdot]\) induced by \(Q\) in the space of Wick polynomials. To solve this problem it is convenient to use the formalism from the preceding Section. We consider that the (classical) fields \(y^\alpha\) are \(v_\mu, u, \tilde{u}\) of null mass and we consider the set \(\mathcal{P}\) of polynomials in these fields and their derivatives. We note that on \(\mathcal{P}\) we have a natural grading. We introduce by convenience the notation:

\[ B \equiv d_\mu v^\mu \] (5.10)

and define the graded derivation \(d_Q\) on \(\mathcal{P}\) according to

\[ d_Q v_\mu = id_\mu u, \quad d_Q u = 0, \quad d_Q \tilde{u} = -i B \quad [d_Q, d_\mu] = 0. \] (5.11)

Then one can easily prove that \(d_Q^2 = 0\) and the cohomology of this operator is isomorphic to the cohomology of the preceding operator (denoted also by \(d_Q\)) and acting in the space of Wick polynomials. The operator \(d_Q\) raises the grading and the canonical dimension by an unit. To determine the cohomology of \(d_Q\) it is convenient to introduce the field strength

\[ F_{\mu\nu} \equiv d_\mu v_\nu - d_\nu v_\mu = v_{\nu;\mu} - v_{\mu;\nu} \] (5.12)

and observe that

\[ d_Q F_{\mu\nu} = 0, \quad d_\nu F_{\mu\nu} = d^\mu B, \quad F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0; \] (5.13)

the last relation is called Bianchi identity. Next we prove that the tensor

\[ F^{(0)}_{\mu\nu;\rho_1,\ldots,\rho_n} \equiv F_{\mu\nu;\rho_1,\ldots,\rho_n} + \frac{1}{n+2} \sum_{l=1}^{n} [\eta_{\mu\rho_l} B_{\nu\rho_1,\ldots,\hat{\rho}_l,\ldots,\rho_n} - (\mu \leftrightarrow \nu)] \] (5.14)

is traceless in all indices and the expressions \(F^{(0)}_{\mu\nu;\rho}\) also verify the Bianchi identities. Now we define

\[ g_{\mu_1,\ldots,\mu_n} = \frac{1}{n} \sum_{l=1}^{n} v_{\mu_1;\mu_1,\ldots,\hat{\mu}_l,\ldots,\mu_n} \] (5.15)

which is the completely symmetric part of the derivative \(v_{\mu_1;\mu_2,\ldots,\mu_n}\) and prove that

\[ v_{\mu_1;\mu_2,\ldots,\mu_n} = g_{\mu_1,\ldots,\mu_n} + \frac{1}{n} \sum_{l=2}^{n} d_{\mu_2} \ldots d_{\mu_l} F_{\mu_1;\mu_l}. \] (5.16)

Finally we define

\[ g^{(0)}_{\mu_1,\ldots,\mu_n} = g_{\mu_1,\ldots,\mu_n} - \frac{2}{n(2n+1)} \sum_{1 \leq p < q \leq n} \eta_{\mu_p\mu_q} B_{\mu_1,\ldots,\hat{\mu}_p,\ldots,\hat{\mu}_q,\ldots,\mu_n} \] (5.17)

which is completely symmetric and traceless.

We will use repeatedly the Künneth theorem:
Theorem 5.2 Let \( P \) be a graded space of polynomials and \( d \) an operator verifying \( d^2 = 0 \) and raising the grading by an unit. Let us suppose that \( P \) is generated by two subspaces \( P_1, P_2 \) such that \( P_1 \cap P_2 = \{0\} \) and \( dP_j \subset P_j, j = 1, 2. \) We define by \( d_j \) the restriction of \( d \) to \( P_j. \) Then there exists the canonical isomorphism \( H(d) \cong H(d_1) \times H(d_2) \) of the associated cohomology spaces.

The proof goes in a similar way to the preceding theorem (see [2]). Now we can prove an important result describing the cohomology of the operator \( d_Q; \) we denote by \( Z_Q \) and \( B_Q \) the co-cycles and the co-boundaries of this operator.

Theorem 5.3 Let \( p \in Z_Q. \) Then \( p \) is cohomologous to a polynomial in \( u \) and \( F^{(0)}_{\mu \nu; \rho_1, \ldots, \rho_n}. \) If we factorize the space \( P_0 \subset P \) of such polynomials to the Bianchi identities we obtain a space which is isomorphic to the cohomology space \( H_Q \) of \( d_Q. \)

Proof: (i) The idea is to define conveniently two subspaces \( P_1, P_2 \) and apply K"unneth theorem. First we use on \( P \) new variables. We eliminate the variables \( v_{\mu_1, \mu_2, \ldots, \mu_n} (n \geq 2) \) in terms of \( g_{\mu_1, \ldots, \mu_n} (n \geq 2) \) and \( F_{\mu \nu; \rho_1, \ldots, \rho_{n-2}} \) using (5.16). Next we eliminate \( F_{\mu \nu; \rho_1, \ldots, \rho_{n-2}} \) in terms of \( F^{(0)}_{\mu \nu; \rho_1, \ldots, \rho_{n-2}} \) and \( B_{\rho_1, \ldots, \rho_{n-2}} \) using (5.14). Finally we eliminate \( g_{\mu_1, \ldots, \mu_n} (n \geq 2) \) in terms of \( g^{(0)}_{\mu_1, \ldots, \mu_n} (n \geq 2) \) and \( B_{\rho_1, \ldots, \rho_{n-2}} \) according to (5.17).

(ii) Now we can take in K"unneth theorem \( P_1 = P_0 \) from the statement and \( P_2 \) the subspace generated by the variables \( B_{\mu_1, \ldots, \mu_n} (n \geq 0), \ g^{(0)}_{\mu_1, \ldots, \mu_n} (n \geq 2), \ u_{\mu_1, \ldots, \mu_n} (n \geq 0), \ v_{\mu_1, \ldots, \mu_n} (n > 0) \) and \( v_\mu. \) We have \( d_Q P_1 = \{0\} \) and

\[
\begin{align*}
d_Q u_{\mu_1, \ldots, \mu_n} &= 0 \\
d_Q g^{(0)}_{\mu_1, \ldots, \mu_n} &= u_{\mu_1, \ldots, \mu_n} \\
d_Q u_{\mu_1, \ldots, \mu_n} &= -i B_{\mu_1, \ldots, \mu_n} \\
dx_Q B_{\mu_1, \ldots, \mu_n} &= 0 \\
d_Q v_\mu &= i u_\mu
\end{align*}
\]

so we meet the conditions of K"unneth theorem. Let us define in \( P_2 \) the graded derivation \( h \) by:

\[
\begin{align*}
h u_\mu &= -i v_\mu \\
h g^{(0)}_{\mu_1, \ldots, \mu_n} &= -i g^{(0)}_{\mu_1, \ldots, \mu_n} (n \geq 2) \\
h B_{\mu_1, \ldots, \mu_n} &= i u_{\mu_1, \ldots, \mu_n} (n \geq 0)
\end{align*}
\]

and zero on the other variables from \( P_2. \) It is easy to prove that \( h \) is well defined: the condition of tracelessness is essential to avoid conflict with the equations of motion. Then one can prove that

\[
[d_Q, h] = Id
\]

on polynomials of degree one in the fields and because the left hand side is a derivation operator we have

\[
[d_Q, h] = n \cdot Id
\]

on polynomials of degree \( n \) in the fields. It means that \( h \) is a homotopy for \( d_Q \) restricted to \( P_2 \) so the the corresponding cohomology is trivial: indeed, if \( p \in P_2 \) is a co-cycle of degree \( n \) in the fields then it is a co-boundary \( p = \frac{1}{n} d_Q hp. \)
According to Künneth formula if $p$ is an arbitrary cocycle from $\mathcal{P}$ it can be replaced by a cohomologous polynomial from $\mathcal{P}_0$; The description of $H_Q$ follows from $\mathcal{P}_0 \cap B_Q = \emptyset$ and this proves the theorem. □

We repeat the whole argument for the case of massive photons i.e. particles of spin 1 and positive mass.

We consider a vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the vector field $v_\mu$, the scalar field $\Phi$ (with Bose statistics) and the scalar fields $u, \tilde{u}$ (with Fermi statistics). We suppose that all these (quantum) fields are of mass $m > 0$. In this vector space we can define a sesquilinear form $< \cdot, \cdot >$ in the following way: the (non-zero) 2-point functions are by definition:

$$\begin{align*}
< \Omega, v_\mu(x_1)v_\mu(x_2)\Omega> &= i \eta_{\mu\nu} D_m^{(+)}(x_1 - x_2), \\
< \Omega, \Phi(x_1)\Phi(x_2)\Omega> &= -i D_m^{(+)}(x_1 - x_2) \\
< \Omega, u(x_1)\tilde{u}(x_2)\Omega> &= -i D_m^{(+)}(x_1 - x_2), \\
< \Omega, \tilde{u}(x_1)u(x_2)\Omega> &= -i D_m^{(+)}(x_1 - x_2)
\end{align*}$$

and the $n$-point functions are generated according to Wick theorem. Here $D_m^{(+)}$ is the positive frequency part of the Pauli-Jordan distribution $D_m$ of mass $m$. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$\begin{align*}
v_\mu^\dagger = v_\mu, & \quad u^\dagger = u, & \quad \tilde{u}^\dagger = -\tilde{u}, & \quad \Phi^\dagger = \Phi.
\end{align*}$$

Now we can define in $\mathcal{H}$ the operator $Q$ according to the following formulas:

$$\begin{align*}
[Q, v_\mu] &= i \partial_\mu u, & \quad [Q, u] = 0, & \quad [Q, \tilde{u}] = -i (\partial_\mu v^\mu + m \Phi) & \quad [Q, \Phi] &= i m u, \\
Q \Omega &= 0.
\end{align*}$$

One can prove that $Q$ is well defined. We have a result similar to the first theorem of this Section:

**Theorem 5.4** The operator $Q$ verifies $Q^2 = 0$. The factor space $\text{Ker}(Q)/\text{Ran}(Q)$ is isomorphic to the Fock space of particles of mass $m$ and spin 1 (massive photons).

**Proof:** (i) The fact that $Q$ squares to zero follows easily from (5.24).

(ii) The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[ \int f_\mu(x)v^\mu(x) + \int g_1(x)u(x) + \int g_2(x)\tilde{u}(x) + \int h(x)\Phi(x) \right] \Omega$$

with test functions $f_\mu, g_1, g_2, h$ verifying the wave equation equation. We impose the condition $\Psi \in \text{Ker}(Q) \iff Q\Psi = 0$; we obtain $h = \frac{1}{m} \partial^\mu f_\mu$ and $g_2 = 0$ i.e. the generic element $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is

$$\Psi = \left[ \int f_\mu(x)v^\mu(x) + \int g(x)u(x) + \frac{1}{m} \int \partial^\mu f_\mu(x)\Phi(x) \right] \Omega$$

with $g$ arbitrary and $f_\mu$ so the elements of $\mathcal{H}^{(1)} \cap \text{Ker}(Q)$ are in one-one correspondence with couples of test functions $(f_\mu, g)$. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$\Psi' = Q\Phi = \left\{ \int \partial_\mu g'(x)v^\mu(x) + \int [m h'(x) - \partial^\mu f'_\mu(x)] u(x) - mg'(x)\Phi(x) \right\} \Omega$$

so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $(f_\mu, g)$ then $\Psi + \Psi'$ is indexed by the couple $(f_\mu + \partial_\mu g', g + m h' - \partial^\mu f'_\mu)$. If we take $h'$ conveniently we can make $g = 0$ and if we take $g'$ conveniently we can
make $f^\mu$ of null divergence; it follows that the equivalence classes from $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ are indexed by wave functions $f_\mu$ constrained by the transversality condition $\partial^\mu f_\mu = 0$; it remains to prove that the sesquilinear form $<\cdot, \cdot>$ induces a positively defined form on $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ and we have obtained the usual one-particle Hilbert space for the massive photon.

(iii) We go now to the $n$-particle space as in the first theorem. □

Now we determine the cohomology of the operator $d_Q = [Q, \cdot]$ induced by $Q$ in the space of Wick polynomials. As before, it is convenient to use the formalism from the preceding Section. We consider that the (classical) fields $g^\mu$ are of mass $m$ and we consider the set $\mathcal{P}$ of polynomials in these fields and their derivatives. We introduce by convenience the notation:

$$C \equiv d_\mu v^\mu + m\Phi$$

and define the graded derivation $d_Q$ on $\mathcal{P}$ according to

$$d_Q v_\mu = id_\mu u, \quad d_Q u = 0, \quad d_Q \tilde{u} = -i C, \quad d_Q \Phi = i m u, \quad [d_Q, d_\mu] = 0. \quad (5.29)$$

Then one can prove that $d_Q^2 = 0$ and the cohomology of this operator is isomorphic to the cohomology of the preceding operator (denoted also by $d_Q$) and acting in the space of Wick monomials. To determine the cohomology of $d_Q$ it is convenient to introduce the field strength $F_{\mu\nu}$ as before and also

$$\phi_\mu \equiv d_\mu \Phi - m v_\mu, \quad \phi_{\mu_1, \ldots, \mu_n} \equiv d_{\mu_1} \ldots d_{\mu_n} \Phi - m g_{\mu_1, \ldots, \mu_n} \quad (n \geq 2). \quad (5.30)$$

Observe that we have

$$d_Q F_{\mu\nu} = 0, \quad d^\rho F_{\mu\nu} = d_\rho C - m \phi_\mu, \quad F_{\mu\nu, \rho} + F_{\nu\rho, \mu} + F_{\rho\mu, \nu} = 0, \quad d_Q \phi_{\mu_1, \ldots, \mu_n} = 0, \quad d^\nu \phi_\mu = -m C = i m d_Q \tilde{u}. \quad (5.31)$$

In the massive case we do not have explicit formulas for the traceless parts of the various tensors; we even do not know if such a traceless parts do exists! However, due to a theorem proved in the Appendix, such traceless parts $F_{\mu\nu, \rho_1, \ldots, \rho_n}^{(0)}, \phi_{\mu_1, \ldots, \mu_n}^{(0)}$ and $g_{\mu_1, \ldots, \mu_n}^{(0)}$ do exists; moreover they are linear combinations of $F_{\mu\nu, \rho_1, \ldots, \rho_n}, \phi_{\mu_1, \ldots, \mu_n}$ and $g_{\mu_1, \ldots, \mu_n}$ and traces of these tensors respectively. Now we can describe the cohomology of the operator $d_Q$ in the massive case.

**Theorem 5.5** Let $p \in \mathbb{Z}_Q$. Then $p$ is cohomologous to a polynomial in $F_{\mu\nu, \rho_1, \ldots, \rho_n}^{(0)}$ and $\phi_{\mu_1, \ldots, \mu_n}^{(0)}$. If we factorize the space $\mathcal{P}_0 \subset \mathcal{P}$ of such polynomials to the Bianchi identities we obtain a space which is isomorphic to the cohomology space $H_Q$ of $d_Q$.

**Proof:** (i) As before, we use on $\mathcal{P}$ new variables. In the first step, we eliminate the variables $v_{\mu_1, \ldots, \mu_n}$ and $F_{\mu\nu, \rho_1, \ldots, \rho_{n-2}}$ and we eliminate the variables $\Phi_{\mu_1, \ldots, \mu_n}$ in terms of $\phi_{\mu_1, \ldots, \mu_n}$ and $g_{\mu_1, \ldots, \mu_n}$.

In the second step we eliminate $F_{\mu\nu, \rho_1, \ldots, \rho_n}$ in terms of $F_{\mu\nu, \rho_1, \ldots, \rho_n}^{(0)}, C_{\rho_1, \ldots, \rho_n}$ and we eliminate $g_{\mu_1, \ldots, \mu_n}$ in terms of $g_{\mu_1, \ldots, \mu_n}^{(0)}, C_{\mu_1, \ldots, \mu_n}$ and $\phi_{\mu_1, \ldots, \mu_n}$.
In the final step we note that the traces of \( u_{\mu_1,\ldots,\mu_n}, \tilde{u}_{\mu_1,\ldots,\mu_n}, C_{\mu_1,\ldots,\mu_n} \) and \( \phi_{\mu_1,\ldots,\mu_n} \) are functions of derivatives of lower order so they can be recursively expressed in terms of the traceless variables: 

\[
\begin{align*}
(u_{\mu_1,\ldots,\mu_n})^{(0)} &= 0, \\
(\tilde{u}_{\mu_1,\ldots,\mu_n})^{(0)} &= i v_{\mu_1,\ldots,\mu_n}, \\
(C_{\mu_1,\ldots,\mu_n})^{(0)} &= -i C^{(0)}_{\mu_1,\ldots,\mu_n}, \\
(\phi_{\mu_1,\ldots,\mu_n})^{(0)} &= 0,
\end{align*}
\]

so we meet the conditions of Künneth theorem. Let us define in \( \mathcal{P}_2 \) the graded derivation \( h \) by:

\[
\begin{align*}
h u &= -\frac{i}{m} \Phi, \\
h v_{\mu} &= -i v_{\mu}, \\
h (u_{\mu_1,\ldots,\mu_n})^{(0)} &= -i (u_{\mu_1,\ldots,\mu_n})^{(0)} (n \geq 2), \\
h C^{(0)}_{\mu_1,\ldots,\mu_n} &= i (C_{\mu_1,\ldots,\mu_n})^{(0)}
\end{align*}
\]

and zero on the other variables from \( \mathcal{P}_2 \). It is easy to prove that \( h \) is well defined due to the condition of tracelessness. Then one can prove as before that we have

\[
[d_Q, h] = n \cdot Id
\]

on polynomials of degree \( n \) in the fields. It means that \( h \) is a homotopy for \( d_Q \) restricted to \( \mathcal{P}_2 \) so the corresponding cohomology is trivial.

According to Künneth formula if \( p \) is an arbitrary cocycle from \( \mathcal{P} \) it can be replaced by a cohomologous polynomial from \( \mathcal{P}_0 \) and this proves the theorem. \( \Box \)

We note that in the case of null mass the operator \( d_Q \) raises the canonical dimension by one unit and this fact is not true anymore in the massive case. We are led to another cohomology group. Let us take as the space of co-chains the space \( \mathcal{P}_0 \) but not as an element of \( \mathcal{P}_2 \).

Theorem 5.6 Let \( p \in Z^{(n)}_Q \), then \( p \) is cohomologous to a polynomial of the form \( p_1 + d_Q p_2 \) where \( p_1 \in \mathcal{P}_0 \) and \( p_2 \in \mathcal{P}^{(n)} \) such that \( \omega(d_Q p_2) \leq n \). If we factorize the space of such polynomials to the Bianchi identities we obtain a space which is isomorphic to the cohomology space \( H^{(n)}_Q \) of \( d_Q \) in \( \mathcal{P}^{(n)} \).

We will call the cocyles of the type \( p_1 \) (respectively \( d_Q p_2 \)) primary (respectively secondary).

The situations described above (of massless and massive photons) are susceptible of the following generalizations. We can consider a system of \( r_1 \) species of particles of null mass and helicity 1 if we use in the first part of this Section \( r_1 \) triplets \((v_\mu^a, u_a, \tilde{u}_a)\), \( a \in I_1 \) of massless fields; here \( I_1 \) is a set of indices of cardinal \( r_1 \). All the relations have to be modified by appending an index \( a \) to all these fields. If we repeatedly apply Künneth theorem we end up with a generalization of theorem 5.3: the space \( \mathcal{P}_0 \) is generated by \( u_a \) and \( F^{(0)}_{\mu_1,\ldots,\mu_n} \).
In the massive case we have to consider \( r_2 \) quadruples \((v^\mu_a, u^a, \tilde{u}^a, \Phi_a), a \in I_2 \) of fields of mass \( m_a \); here \( I_2 \) is a set of indices of cardinal \( r_2 \). We also have a generalization of theorem 5.5: the space \( \mathcal{P}_0 \) is generated \( F^0_{q\mu\nu, p_1, \ldots, p_n} \) and \( \phi^0_{(i)\mu_1, \ldots, \mu_n} \).

We can consider now the most general case involving fields of spin not greater than 1. We take \( I = I_1 \cup I_2 \cup I_3 \) a set of indices and for any index we take a quadruple \((v^\mu_a, u^a, \tilde{u}^a, \Phi_a), a \in I \) of fields with the following conventions: (a) For \( a \in I_1 \) we impose \( \Phi_a = 0 \) and we take the masses to be null \( m_a = 0 \); (b) For \( a \in I_2 \) we take the all the masses strictly positive: \( m_a > 0 \); (c) For \( a \in I_3 \) we take \( v^\mu_a, u^a, \tilde{u}^a \) to be null and the fields \( \Phi_a \equiv \phi^H_a \) of mass \( m_a \equiv m^H_a \geq 0 \). The fields \( \phi^H_a \) are called \textit{Higgs} fields.

If we define \( m_a = 0, \forall a \in I_3 \) then we can define in \( \mathcal{H} \) the operator \( Q \) according to the following formulas for all indices \( a \in I \):

\[
[Q, v^\mu_a] = i \partial^\mu u^a, \quad [Q, u^a] = 0, \quad [Q, \tilde{u}^a] = -i (\partial^\mu v^\mu_a + m_a \Phi_a), \quad [Q, \Phi_a] = i m_a u^a, \quad Q \Omega = 0.
\] (5.35)

Then the space \( \mathcal{P}_0 \) is generated by \( u^a, a \in I_1, F^0_{q\mu\nu, p_1, \ldots, p_n}, a \in I_1 \cup I_2 \) and \( \phi^0_{(i)\mu_1, \ldots, \mu_n}, a \in I_2 \cup I_3 \).

If we consider matter fields also i.e some set of Dirac fields with Fermi statistics: \( \Psi_A, A \in I_4 \) then we impose

\[
d_Q \Psi_A = 0
\] (5.36)
and the space \( \mathcal{P}_0 \) is generated by \( \Psi_A \) and \( \tilde{\Psi}_A \) also.

\section{The Relative Cohomology of the Operator \( d_Q \)}

A polynomial \( p \in \mathcal{P} \) verifying the relation

\[
d_Q p = i \, d_\mu p^\mu
\] (6.1)
for some polynomials \( p^\mu \) is called a \textit{relative cocycle} for \( d_Q \). The expressions of the type

\[
p = d_Q b + i \, d_\mu b^\mu, \quad (b, b^\mu \in \mathcal{P})
\] (6.2)
are relative co-cycles and are called \textit{relative co-boundaries}. We denote by \( Z^0_Q, B^0_Q \) and \( H^0_Q \) the corresponding cohomological spaces. In (6.1) the expressions \( p_a \) are not unique. It is possible to choose them Lorentz covariant? The next proposition gives a positive answer in a quite general case. The proof will illustrate the descent technique.

\textbf{Theorem 6.1} Let us suppose that the relative cocycle \( p \) is at least tri-linear in the fields and Lorentz covariant. Then the expressions \( p^\mu \) from (6.1) can be chosen to be Lorentz covariant also.

\textbf{Proof}: Let us denote by \( \partial_g \) the action of the Lorentz transformation \( g \in G = SL(2, \mathbb{C}) \) in the space \( \mathcal{P}^{(k)} \). It is clear that \( \partial_g \) commutes with \( d_\mu \). If we denote by \( C^n(G, \mathcal{P}^{(k)}) \) \((n \geq 0)\) the space of maps \( p : G^{\times n} \rightarrow \mathcal{P}^{(k)} \) with the convention that for \( n = 0 \) the functions \( p \) are independent of \( g \) then we have the co-chain operator \( d : C^n(G, \mathcal{P}^{(k)}) \rightarrow C^{n+1}(G, \mathcal{P}^{(k)}) \)

\[
(d \cdot p)(g_1, \ldots, g_{n+1}) = \partial_g \cdot p(g_2, \ldots, g_{n+1}) + \sum_{j=1}^n (-1)^j p(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) + (-1)^{n+1} p(g_1, \ldots, g_n).
\] (6.3)
Because \( d^2 = 0 \) we can define the corresponding cohomology spaces \( Z^n(G, \mathcal{P}^{(k)}) \), \( B^n(G, \mathcal{P}^{(k)}) \) and \( H^n(G, \mathcal{P}^{(k)}) \) \[12\]. By hypothesis we have
\[
\delta_g \cdot p = p \tag{6.4}
\]
which can be written as
\[
d \cdot p = 0. \tag{6.5}
\]
Then we have from (6.1):
\[
d_\mu (\delta_g \cdot p^\mu - p^\mu) = 0 \tag{6.6}
\]
so with the Poincaré lemma we have:
\[
\delta_g \cdot p^\mu - p^\mu = d_\nu p^{\mu\nu}(g) \tag{6.7}
\]
for some polynomials \( p^{\mu\nu}(g) \) antisymmetric in \( \mu, \nu \); the preceding identity can be written as
\[
d \cdot p^\mu = d_\nu p^{\mu\nu}. \tag{6.8}
\]
Proceeding in the same way we obtain the expressions \( p^{\mu\nu\rho}(g_1, g_2) \) and \( p^{\mu\nu\rho\sigma}(g_1, g_2, g_3) \) which are completely antisymmetric and we have
\[
d \cdot p^{\mu\nu\rho} = d_\sigma p^{\mu\nu\rho}
\]
\[
d \cdot p^{\mu\nu\rho\sigma} = d_\sigma p^{\mu\nu\rho\sigma}
\]
\[
d \cdot p^{\mu\nu\rho\sigma} = 0. \tag{6.9}
\]
We have obtained that \( p^{\mu\nu\rho\sigma} \in H^3(G, \mathcal{P}^{(k)}) \). But \( G \) is a connected simply connected Lie group and in this case the study of group cohomology can be reduced to the study of the corresponding Lie algebra cohomology. Because \( G \) is also simple we can apply one of the Whitehead lemmas (see \[12\] ch. II, § 11, cor. 11.1) and conclude that \( H^n(\text{Lie}(G), \mathcal{P}^{(k)}) \) are trivial for \( n \geq 0 \); we obtain that \( p^{\mu\nu\rho\sigma} \) is a trivial cocycle i.e. it is of the form:
\[
p^{\mu\nu\rho\sigma} = d \cdot q^{\mu\nu\rho\sigma} \tag{6.10}
\]
where we can take the co-chain \( q^{\mu\nu\rho\sigma} \) to be completely antisymmetric. If we make the redefinition
\[
p^{\mu\nu\rho} \rightarrow p^{\mu\nu\rho} - d_\sigma q^{\mu\nu\rho\sigma} \tag{6.11}
\]
then we have \( d \cdot p^{\mu\nu\rho} = 0 \) i.e. \( p^{\mu\nu\rho} \in H^2(G, \mathcal{P}^{(k)}) \), etc. In the end we can obtain \( d \cdot p^\mu = 0 \) i.e.
\[
\delta_g \cdot p^\mu = p^\mu \tag{6.12}
\]
and this is the invariance property we claimed in the statement. □

Now we consider the framework and notations from the end of the preceding Section. Then we have the following result which describes the most general form of the Yang-Mills interaction. Summation over the dummy indices is used everywhere.

**Theorem 6.2** Let \( T \) be a relative cocycle for \( d_Q \) which is at least tri-linear in the fields and is of canonical dimension \( \omega(T) \leq 4 \) and ghost number \( \text{gh}(T) = 0 \). Then: (i) \( T \) is (relatively) cohomologous to a non-trivial co-cycle of the form:

\[
T = f_{abc} \left( \frac{1}{2} v_{a\mu} v_{b\nu} F_{\mu\nu}^{c} + u_a v^\mu_\nu d_\mu \bar{u}_c \right)
\]

\[
+ f'_{abc} (\Phi_a \Phi^\mu_{b\nu} v_{\nu\mu} + m_b \Phi_a \bar{u}_b u_c)
\]

\[
+ \frac{1}{3!} f''_{abc} \Phi_a \Phi_b \Phi_c + \frac{1}{4!} \sum_{a,b,c,d \in I_3} g_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d + j_{a}^\mu v_{a\mu} + j_a \Phi_a; \tag{6.13}
\]
where we can take the constants \( f_{abc} = 0 \) if one of the indices is in \( I_3 \); also \( f'_{abc} = 0 \) if \( c \in I_3 \) or one of the indices \( a \) and \( b \) are from \( I_1 \); and \( j^a = 0 \) if \( a \in I_3 \); \( j_a = 0 \) if \( a \in I_1 \). Moreover we have:

(a) The constants \( f_{abc} \) are completely antisymmetric

\[
f_{abc} = f_{[abc]}.
\]

(b) The expressions \( f'_{abc} \) are antisymmetric in the indices \( a \) and \( b \):

\[
f'_{abc} = -f'_{bac}
\]

and are connected to \( f_{abc} \) by:

\[
f_{abc} m_c = f'_{cda} m_a - f'_{eba} m_b.
\]

(c) The (completely symmetric) expressions \( f''_{abc} \) verify

\[
f''_{abc} m_c = \begin{cases} 
\frac{1}{m_c} f'_{abc} (m^2_a - m^2_b) & \text{for } a, b \in I_3, c \in I_2 \\
-\frac{1}{m_c} f'_{abc} m^2_b & \text{for } a, c, I_2, b \in I_3.
\end{cases}
\]

(d) the expressions \( j^a \) and \( j_a \) are bilinear in the Fermi matter fields: in tensor notations;

\[
j^a = \sum_\epsilon \bar{\psi}_\epsilon t^\epsilon_a \otimes \gamma^\mu \gamma^\nu \psi, \quad j_a = \sum_\epsilon \bar{\psi}s^\epsilon_a \otimes \gamma^\mu \psi
\]

where for every \( \epsilon = \pm \) we have defined the chiral projectors of the algebra of Dirac matrices \( \gamma_c \equiv \frac{1}{2} (I + \epsilon \gamma_5) \) and \( t^\epsilon_a, s^\epsilon_a \) are \(|I_4| \times |I_4|\) matrices. If \( M \) is the mass matrix \( M_{AB} = \delta_{AB} M_A \) then we must have

\[
d_{\mu} j^a = m_a j_a \quad \Leftrightarrow \quad m_a s^\epsilon_a = i(M t^\epsilon_a - t^{-\epsilon}_a M).
\]

(ii) The relation \( d_Q T = i d_{\mu} T^\mu \) is verified by:

\[
T^\mu = f_{abc} \left( u_a v_b F^\mu^c - \frac{1}{2} u_a u_b d^\mu u_c \right) + f'_{abc} \Phi_\mu^a \phi_b^\mu u_c + j^a_{\mu} u_a
\]

(iii) The relation \( d_Q T^\mu = i d_{\nu} T^{\mu \nu} \) is verified by:

\[
T^{\mu \nu} = \frac{1}{2} f_{abc} u_a u_b F^\mu_{ab}.
\]

The preceding expressions \( T^I \) are self-adjoint if the constants \( f_{abc}, f'_{abc}, f''_{abc}, g_{abcd} \) are real and if the matrices \( t^\epsilon_a, s^\epsilon_a \) are self-adjoint.

**Proof:** (i) By hypothesis we have

\[
d_Q T = i d_{\mu} T^\mu.
\]

If we apply \( d_Q \) we obtain \( d_{\mu} d_Q T^\mu = 0 \) so with the Poincaré lemma there must exist the polynomials \( T^{\mu \nu} \) antisymmetric in \( \mu, \nu \) such that

\[
d_Q T^\mu = i d_{\nu} T^{\mu \nu}.
\]

Continuing in the same way we find \( T^{\mu \nu}, T^{\mu \nu \rho} \) which are completely antisymmetric and we also have

\[
d_Q T^{\mu \nu} = i d_{\rho} T^{\mu \nu \rho}
\]

\[
d_Q T^{\mu \nu \rho} = i d_{\sigma} T^{\mu \nu \rho \sigma}
\]

\[
d_Q T^{\mu \nu \rho \sigma} = 0.
\]
According to the preceding theorem one can choose the expressions $T^I$ to be Lorentz covariant; we also have
\[ gh(T^I) = |I|. \]  
(6.25)

From the last relation we find, using Theorem 5.6 that
\[ T^{\mu\nu\rho\sigma} = d_Q B^{\mu\nu\rho\sigma} + T_0^{\mu\nu\rho\sigma} \]  
(6.26)
with $T_0^{\mu\nu\rho\sigma} \in \mathcal{P}_0^{(4)}$. The generic form of such an expression is:
\[ T_0^{\mu\nu\rho\sigma} = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} f_{abcd} u_a u_b u_c u_d; \]  
(6.27)
the contributions corresponding to $a, b, c, d \in I_1$ are primary co-cyles and the contributions for which at least one of the indices is in $I_2$ are secondary co-cyles.

If we substitute the preceding expression in the second relation (6.24) we find out
\[ d_Q(T^{\mu\nu} - i d_\sigma B^{\mu\nu}) = i d_\sigma T_0^{\mu\nu}. \]  
(6.28)
The right hand side can be written as a co-boundary: we define
\[ B_0^{\mu\nu} = \frac{1}{3!} \epsilon^{\mu\nu} f_{[abcd]} u_a u_b u_c u_d \]  
(6.29)
and we have in fact;
\[ d_Q(T^{\mu\nu} - i d_\sigma B^{\mu\nu} - B_0^{\mu\nu}) = 0. \]  
(6.30)
We apply again Theorem 5.6 and obtain
\[ T^{\mu\nu} = d_Q B^{\mu\nu} + i d_\sigma B^{\mu\nu\sigma} + T_0^{\mu\nu} \]  
(6.31)
where $T_0^{\mu\nu} \in \mathcal{P}_0^{(4)}$.

We substitute the last relation into the first relation (6.24) and obtain
\[ d_Q(T^{\mu\nu} - i d_\rho B^{\mu\rho}) = i d_\rho T_0^{\mu\nu}. \]  
(6.32)
The right hand side must be a co-boundary. But it is not hard to prove that this is not possible, so we have in fact $f_{[abcd]} = 0 \iff T_0^{\mu\nu\rho\sigma} = 0 \iff B_0^{\mu\nu} = 0$ so
\[ T^{\mu\nu} = B^{\mu\nu} + i d_\sigma B^{\mu\nu\sigma} \]  
(6.33)
and
\[ d_Q(T^{\mu\nu} - i d_\rho B^{\mu\rho}) = 0. \]  
(6.34)
(ii) We use again Theorem 5.6 and obtain
\[ T^{\mu\nu} - i d_\rho B^{\mu\rho} = d_Q B^{\mu\nu} + T_0^{\mu\nu} \]  
(6.35)
where $T_0^{\mu\nu} \in \mathcal{P}_0^{(4)}$. The generic form of such an expression is:
\[ T_0^{\mu\nu} = \frac{1}{2} f_{[abc]}^{(1)} u_a u_b F_{\mu\nu}^{c} + \frac{1}{2} f_{[abc]}^{(2)} \epsilon^{\mu\nu\rho\sigma} u_a u_b F_{\rho\sigma}; \]  
(6.36)
the contributions corresponding to $a, b \in I_1$ are primary co-cycles and the contributions for which at least one of the indices $a, b$ is in $I_2$ are secondary co-cycles. We substitute this in (6.23) and get:

$$d_Q(T^\mu - i d_\nu B^{\mu\nu}) = i d_\nu T_0^{\mu\nu}. \quad (6.37)$$

The right hand side must be a co-boundary. But one can easily obtain that

$$d_\nu T_0^{\mu\nu} = -i d_Q B_1^{\mu\nu} - \frac{i}{2} f^{(1)}_{[ab]c} m_c u_a u_b \phi_c^\mu.$$

where

$$B_1^{\mu\nu} = f^{(1)}_{[ab]c} \left( u_a v_b w_c F^{\mu\nu}_c - \frac{1}{2} u_a u_b \phi_c^\mu \right) - f^{(2)}_{[ab]c} \epsilon^{\mu\nu\rho\sigma} u_a v_b v_c F_{c\rho\sigma}. \quad (6.39)$$

The term $u u \phi^\mu$ must be a co-boundary and there is only the possibility:

$$B_2^{\mu\nu} = f'_{cab} \Phi_a \phi_c^\mu u_b \quad (6.40)$$

where we can take $f'_{cab} = 0$ if one of the indices $a, c$ is from $I_1$. Now the relation

$$-\frac{1}{2} f^{(1)}_{[ab]c} m_c u_a u_b \phi_c^\mu = i d_Q B_2^{\mu\nu}. \quad (6.41)$$

gives the restriction:

$$f^{(1)}_{[ab]c} m_c = f'_{cab} m_a - f'_{cba} m_b. \quad (6.42)$$

If this is true then we have

$$i d_\nu T_0^{\mu\nu} = d_Q B_0^{\mu\nu} \quad (6.43)$$

where

$$B_0^{\mu\nu} = B_1^{\mu\nu} - B_2^{\mu\nu} \quad (6.44)$$

and (6.37) becomes:

$$d_Q(T^\mu - i d_\nu B^{\mu\nu} - B_0^{\mu\nu}) = 0. \quad (6.45)$$

(iii) Now it is again time we use Theorem 5.6 and obtain

$$T^\mu - B_0^{\mu\nu} - i d_\nu B^{\mu\nu} = d_Q B^{\mu\nu} + T_0^{\mu\nu} \quad (6.46)$$

where $T_0^{\mu\nu} \in \mathcal{P}_0^{(4)}$. The generic form of such an expression is:

$$T_0^{\mu\nu} = u_a j_a^{\mu\nu} + \sum_{a \in I_3} f_{abc} \Phi_a \phi_c^\mu u_b$$

where $j_a^{\mu\nu}$ has the form from the statement; but the last term can be eliminated if we redefine the expressions $f'_{cab}$ so in fact we can take:

$$T_0^{\mu\nu} = u_a j_a^{\mu\nu}. \quad (6.47)$$

It means that we have

$$T^\mu = d_Q B^{\mu\nu} + i d_\nu B^{\mu\nu} + T_1^{\mu\nu} \quad (6.48)$$

where

$$T_1^{\mu\nu} = B_0^{\mu\nu} + T_0^{\mu\nu}. \quad (6.49)$$
Now we get from (6.22)
\[ d_Q(T - i \, d_\mu B^\mu) = i \, d_\mu T_1^\mu \] (6.50)

The right hand side must be a co-boundary. But one can easily obtain that
\[ d_\nu T_1^{\mu\nu} = -i \, d_Q B_0 - \frac{1}{2} f_{[ab]c}^{(1)} \, u_a \, F_{b\nu} \, F_c^{\mu\nu} - \frac{1}{2} f_{[ab]c}^{(2)} \, \epsilon_{\mu\nu\rho\sigma} \, u_a \, F_b^{\mu\nu} \, F_c^{\rho\sigma} - m_b \, m_c \, f'_{cba} \, u_a \, v_\mu \, v_\eta \, v_\rho \, F_c \, + \sum_{c \in I_3} f'_{cab} \, m_c^2 \, \Phi_a \, \Phi_c \, u_b + u_a \, d_\mu j_\mu^a \] (6.51)

where
\[ B_0 = \frac{1}{2} f_{[ab]c}^{(1)} \left( \frac{1}{2} \, u_{\alpha\beta} \, v_\gamma \, F_{\beta\gamma}^{\alpha\mu} + u_a \, v_\mu \, \epsilon_{\alpha\beta\mu\nu} \right) - f_{[cab]}^{(2)} \left( \Phi_a \, \Phi_b \, \Phi_c \, v_\mu + m_c \, \Phi_a \, \Phi_c \, \epsilon_{\alpha\beta\mu\nu} \right). \] (6.52)

It means that the expression
\[ -\frac{i}{2} \, f_{[ab]c}^{(1)} \, u_a \, F_{b\nu} \, F_c^{\mu\nu} - \frac{i}{2} \, f_{[ab]c}^{(2)} \, \epsilon_{\mu\nu\rho\sigma} \, u_a \, F_b^{\mu\nu} \, F_c^{\rho\sigma} - m_b \, m_c \, f'_{cba} \, u_a \, v_\mu \, v_\eta \, v_\rho \, F_c \, + \sum_{c \in I_3} f'_{cab} \, m_c^2 \, \Phi_a \, \Phi_c \, u_b + u_a \, d_\mu j_\mu^a \] (6.53)

must be a co-boundary. It is easy to argue that the terms \( u FF \) and \( u d\Phi d\Phi \) cannot be written as co-boundaries so we necessarily have
\[ f_{[ab]c}^{(1)} = -f_{[ac]b}^{(1)}, \quad f_{[ab]c}^{(2)} = -f_{[ac]b}^{(2)}, \quad f'_{cab} = -f'_{acb}. \]

It means that the constants \( f_{abc}^{(1)} \) and \( f_{abc}^{(2)} \) are completely antisymmetric and \( f'_{abc} \) are antisymmetric in the first two indices. We are left with the condition:
\[ \sum_{c \in I_3} f'_{cab} \, m_c^2 \, \Phi_a \, \Phi_c \, u_b + u_a \, d_\mu j_\mu^a = -i \, d_Q B_1 \] (6.54)

so necessarily we must have:
\[ B_1 = \Phi_a \, j_a + \frac{1}{3!} f''_{abc} \, \Phi_a \, \Phi_b \, \Phi_c \] (6.55)

with \( j_a \) as in the statement. We easily obtain (6.17) and (6.19) from the statement.

(iv) If we denote
\[ T_1 = B_0 + B_1 \] (6.56)

then we have from (6.50)
\[ d_Q(T - i \, d_\mu B^\mu - T_1) = 0 \] (6.57)

so a last use of Theorem 5.6 gives
\[ T - T_1 - i \, d_\mu B^\mu = d_Q B + T_0 \] (6.58)
where \( T_0 \in \mathcal{P}_0^{(4)} \). The generic form of such an expression is:

\[
T_0 = \frac{1}{3!} \sum_{a,b,c \in I_3} f''_{abc} \Phi_a \Phi_b \Phi_c + \frac{1}{4!} \sum_{a,b,c,d \in I_3} g_{(abcd)} \Phi_a \Phi_b \Phi_c \Phi_d
\]

(6.59)

but we can get rid of the first term if we redefine the expressions \( f''_{(abc)} \). It is easy to prove that the expression \( f''_{(abc)} \epsilon_{\mu\nu\rho\sigma} v^\mu_b v^\nu_c F^\rho_{a} \sigma \) from (6.52) is in fact a total divergence so it can be eliminated and we obtain the expression \( T \) from the statement.

(v) We prove now that \( T \) from the statement is not a trivial (relative) cocycle. Indeed, if this would be true i.e. \( T = d_Q B + i d_\mu B^\mu \) then we get \( d_\mu (T^\mu - d_Q B^\mu) = 0 \) so with Poincaré lemma we have \( T^\mu = d_Q B^\mu + i d_\mu B^{[\mu\nu]} \). In the same way we obtain from here: \( T^{[\mu\nu]} = d_Q B^{[\mu\nu]} + i d_\mu B^{[\mu\nu]} \). But it is easy to see that there is no such an expression \( B^{[\mu\nu]} \) with the desired antisymmetry property in ghost number 3 so we have in fact \( T^{[\mu\nu]} = d_Q B^{[\mu\nu]} \). This relation contradicts the fact that \( T^{[\mu\nu]} \) is a non-trivial cocycle for \( d_Q \) as it follows from Theorem 5.3. \( \Box \)

If \( T \) is bilinear in the fields we cannot use the Poincaré lemma but we can make a direct analysis. The result is the following.

**Theorem 6.3** Let \( T \) be a relative cocycle for \( d_Q \) which is bilinear in the fields, of canonical dimension \( \omega(T) \leq 4 \) and ghost number \( gh(T) = 0 \). Then: (i) \( T \) is (relatively) cohomologous to an expression of the form:

\[
T = \sum_{a \in I_1} f_{ab} (v_{a\mu}\phi_b^\mu - m_b u_a \tilde{u}_b) + f'_{(ab)} \phi_{a\mu} \phi_b^\mu + \sum_{a,b \in I_3} f''_{(ab)} \Phi_a \Phi_b
\]

(6.60)

with \( f_{ab} m_b = 0 \), \( \forall b \in I_3 \).

(ii) The relation \( d_Q T = i d_\mu T^\mu \) is verified with

\[
T^\mu = \sum_{a \in I_1} f_{ab} u_a \phi_b^\mu
\]

(6.61)

and we also have \( d_Q T^\mu = 0 \).

There are no linear solutions of this descent problem which are also non-trivial. The first theorem gives us the generic form of the interaction Lagrangian for Yang-Mills models. Both theorems can be used to describe the finite renormalizations \( R^f \) (see the end of Section 2) which preserve gauge invariance. The expression from the first theorem produces a renormalization of the coupling constant and the expression from the second theorem produces renormalization of the propagators (or wave functions).

In the same way one can analyze the descent equations (3.43) and provide the general form of the anomalies for Yang-Mills models. We give only the result.

**Theorem 6.4** Let \( W \) be a relative cocycle for \( d_Q \) which is at least tri-linear in the fields, of canonical dimension \( \omega(W) \leq 5 \) and ghost number \( gh(W) = 1 \). Then: (i) \( W \) is (relatively) cohomologous to a non-trivial co-cycle of the form:

\[
W = \frac{1}{2} f_{abcd} (u_a v_{b\mu} v_{c\nu} F^{'\mu\nu}_d - u_a v_{b\mu} v_{c\nu} \partial_\mu \tilde{u}_d),
- f'_{abcd} (u_a v_{b\mu} \Phi_c \phi_d^\mu - \frac{1}{2} m_d u_a v_{b\mu} \Phi_c \tilde{u}_d)
\]
\[ + \sum_{a,b \in I_1} g_{abc} \left( u_a v_{by} \phi^\mu_{ab} - \frac{1}{2} m_c u_a u_b \tilde{u}_c \right) \]
\[ + \frac{1}{3!} f''_{abcd} u_a \Phi_b \Phi_c \Phi_d + \frac{1}{4!} \sum_{b,c,d,e \in I_3} g_{a\{bcede\}} u_a \Phi_b \Phi_c \Phi_d \Phi_e \]
\[ + f'_{ab} u_a v_{by} + j_{ab} u_a \Phi_b + \sum_{a \in I_1} k_a u_a \]
\[ + h^{(1)}_{a\{bc\}} u_a F^\mu_{b} F_{\epsilon \mu \nu} + h^{(2)}_{a\{bc\}} \epsilon_{\mu \nu \rho \sigma} u_a F^\mu_{b} F^\rho_{c} \]
\[ + h^{(3)}_{a\{bc\}} u_a \phi_{by} \Phi^C_{b} + \sum_{a \in I_1} \sum_{b,c \in I_3} h^{(4)}_{a\{bc\}} u_a \Phi_b \Phi_c. \]  

(6.62)

We can take the constants \( f_{abcd} = 0 \) if one of the indices is in \( I_3 \); we can take \( f'_{abcd} = 0 \) if one of the indices \( a \) and \( b \) is in \( I_3 \) or one of the indices \( c \) and \( d \) are from \( I_1 \); also we can take \( g_{abc} = 0 \) if \( c \in I_3 \) and \( f^{(4)}_{abc} = 0 \) if \( b,c \in I_3 \). Moreover we have:

(a) The constants \( f_{abcd} \) are completely antisymmetric:

\[ f_{abcd} = f_{[abcd]}. \]  

(6.63)

(b) The expressions \( f'_{abcd} \) is antisymmetric in \( a,b \) and in \( c,d \):

\[ f_{abcd} = f'_{[abcd]} \]  

(6.64)

and verifies

\[ f_{abcd} m_d = f'_{abc} m_c + f'_{bca} m_a + f'_{cab} m_b. \]  

(6.65)

(c) For \( a \in I_2 \) we can write \( f''_{abcd} = m_a f_{abcd} \) and eliminate the completely symmetric part \( \tilde{f}_{(abcd)} \); we also have:

\[ f''_{abcd} m_b - f''_{bacd} m_a = \left\{ \begin{array}{ll}
  f'_{abcd} m_d^2 + f'_{abc} m_i^2 & \text{for } c,d \in I_3 \\
  f'_{a\{bcd\} m_b^2} & \text{for } c \in I_2, d \in I_3.
\end{array} \right. \]  

(6.66)

(d) The expressions \( j^\mu_{ab}, j_{ab} \) and \( k_a \) are bilinear in the Fermi matter fields: in tensor notations;

\[ j^\mu_{ab} = \sum_{\epsilon} \bar{\psi} t^\epsilon_{ab} \otimes \gamma^\mu \gamma_{\epsilon} \psi \]
\[ j_{ab} = \sum_{\epsilon} \bar{\psi} s^\epsilon_{ab} \otimes \gamma_{\epsilon} \psi \]
\[ k_a = \sum_{\epsilon} \bar{\psi} k^\epsilon_{a} \otimes \gamma_{\epsilon} \psi, \]  

(6.67)

and we have the relations

\[ m_b s^\epsilon_{ab} - m_a s^\epsilon_{ba} = i(M t^\epsilon_{ab} - t_{\epsilon \lambda}^\mu M). \]  

(6.68)

(ii) The relation \( d_Q W = -i d_q W^\mu \) is verified by:

\[ W^\mu = f_{abcd} \left( \frac{1}{2} u_a u_b v_{ca} F^\mu_{d} + \frac{1}{3!} u_a u_b u_c \tilde{d}^\mu u_d \right) - \frac{1}{2} f'_{abcd} u_a u_b \Phi_c \phi^\mu_{d} \]
\[ + \frac{1}{2} \sum_{a,b \in I_1} g_{abc} u_a u_b \phi^\mu_{d} + \frac{1}{2} j^\mu_{ab} u_a u_b. \]  

(6.69)
(iii) The relation \( d_Q W^\mu = i d_W W^{\mu\nu} \) is verified by:

\[
W^{\mu\nu} = \frac{1}{3!} f_{abcd} u_a u_b u_c F_c^{\mu\nu}. \tag{6.70}
\]

(iv) If we have \( W = 0 \) i.e. the equation (3.3) does not have anomalies, then we also have \( W^\mu = 0 \), \( W^{\mu\nu} = 0 \).

If the expression \( W \) is bilinear in the fields we can make a direct analysis:

**Theorem 6.5** Let \( W \) be a relative cocycle for \( d_Q \) which is bilinear in the fields, of canonical dimension \( \omega(W) \leq 5 \) and ghost number \( gh(W) = 1 \). Then \( W \) is (relatively) cohomologous to an expression of the form:

\[
W = \sum_{a \in I_1, b \in I_3} g_{ab} u_a \Phi_b \tag{6.71}
\]

and we have \( d_Q W = 0 \).

There are linear solutions of this descent problem which are non-trivial, namely \( W = \sum_{a \in I_1} h_a u_a \). As a matter of terminology, if in the generic scheme presented above we have \( I_2 = I_3 = \emptyset \) we say that we have a **pure gauge model**. The physically relevant cases are quantum electro-dynamics and quantum chromo-dynamics. If \( I_2 \neq \emptyset \) we say that the theory is **spontaneously broken**. In this case it can be proved that we must necessarily have \( I_3 \neq \emptyset \); without Higgs fields gauge invariance is not valid already in the second order of perturbation theory. The physically relevant case is the electro-weak interaction (the standard model).

Using Wick expansion property (2.8) one can prove that the tree graphs give anomalies only for \( n = 2, 3 \).

A slightly more general case is to consider that in the Fermi sector the fields are Majorana; indeed one can express a Dirac field in terms of two Majorana fields in a similar way to the decomposition of a complex number in the real and imaginary part. If the fields are Majorana, then \( \psi \) and \( \bar{\psi} \) are not independent and we can replace in the expression of the Lagrangean \( \bar{\psi} \rightarrow \psi^T \) and in this case one can take the matrices \( t_a^\pm, s_a^\pm \) to be antisymmetric.

### 7 YANG-MILLS MODELS IN HIGHER ORDERS OF PERTURBATION THEORY

The theory is gauge invariant in all orders iff we can prove that \( W = 0 \) in an arbitrary order. This is possible in some simple cases like quantum electro-dynamics. We have to take in the generic scheme presented in the preceding Section \(|I_1| = |I_4| = 1, \quad I_2 = I_3 = \emptyset\). So we have a triplet \((v_\mu, u, \tilde{u})\) of null mass fields (\( v_\mu \) is called the electromagnetic potential) and one Dirac field of mass \( M \) with the interaction Lagrangian

\[
T =: v_\mu \bar{\psi} \gamma^\mu \psi : \tag{7.1}
\]

and

\[
T^\mu =: u \bar{\psi} \gamma^\mu \psi : \tag{7.2}
\]

An important observation is the following one. Let us define the so-called **charge conjugation** operator according to

\[
U_c v_\mu U_c^{-1} = -v_\mu, \quad U_c u U_c^{-1} = -u, \quad U_c \tilde{u} U_c^{-1} = -\tilde{u},
\]

\[
U_c \psi U_c^{-1} = -C \gamma_0 \psi^\dagger,
\]

\[
U_c \Omega = 0 \quad (7.3)
\]
where $C$ is the charge conjugation matrix. Then we can easily prove that
\[ U_c\ T\ U_c^{-1} = T, \quad U_c\ T^\mu\ U_c^{-1} = T^\mu. \] (7.4)

The result (sometimes called Furry theorem) is then:

**Theorem 7.1** The chronological products can be chosen such that the theory is gauge invariant in all orders of perturbation theory.

**Proof:** (i) First we can define the chronological products such that they are charge conjugation invariant in all orders of perturbation theory by induction. We suppose that the assertion is true up to order $n - 1$ i.e.
\[ U_c\ T^{I_1,...,I_k}\ U_c^{-1} = T^{I_1,...,I_k}, \quad k < n. \]

If $T^{I_1,...,I_n}$ do not verify this relation we simply replace:
\[ T^{I_1,...,I_n} \to \frac{1}{2} (T^{I_1,...,I_n} + U_c\ T^{I_1,...,I_n}\ U_c^{-1}). \] (7.5)

So we can suppose that we have
\[ U_c\ T^{I_1,...,I_k}\ U_c^{-1} = T^{I_1,...,I_k}, \quad \forall n. \] (7.6)

(ii) Suppose now that the theory is gauge invariant up to order $n - 1$. Then in order $n$ we might have the anomaly $W$. From the preceding relation we have however:
\[ U_c\ W\ U_c^{-1} = W. \] (7.7)

In our particular case (6.71) does not give solutions and the relation (6.62) considerably simplifies:
\[ W = u\ \overline{\psi}\psi + u\ \overline{\psi}\gamma_5\psi + h^{(1)}\ u\ F^{\mu\nu}\ F_{\mu\nu} + h^{(2)}\ \epsilon_{\mu\nu\rho\sigma}\ u\ F^{\mu\nu}\ F^{\rho\sigma}. \] (7.8)

If we substitute this generic expression in the preceding relation we obtain $W = 0$; the same is true for the linear solution $W = h\ u$ and this proves gauge invariance in order $n$. □

In the similar way one can treat other models for which a charge conjugation operator do exists e.g. $SU(n)$ invariant models without spontaneously broken symmetry.

Now we consider again the generic case from the preceding Section. One can compute explicitly the expression of the anomaly $W$ in the second order of the perturbation theory. Imposing $W = 0$ one finds out new restrictions on the various constants. The computations are given in [9], [10] and [11] so we give only the results. Computing $A_3^{[\mu\nu]}$ we find
\[ f_{abcd} = 2i\ (f_{abe}\ f_{cde} + f_{bce}\ f_{ade} + f_{cae}\ f_{bde}) \] (7.9)

so if we impose $f_{abcd} = 0$ we find out that the constants $f_{abc}$ verify Jacobi identities. Computing $A_2^\mu$ we find the same expression for $f_{abcd}$ and moreover
\[ f'_{abcd} = 2i\ (f_{abe}\ f'_{cde} + f'_{bce}\ f_{ade} - f'_{cae}\ f_{bde}) \] (7.10)
\[ t'_{ab} = 2\ (t'_{a}^e t'_{eb} - i\ f_{abc} t'_{c}) \] (7.11)
so the cancellation of this anomaly tells us that \( t'_{ab} \) and \( (T_c)_{ab} = -f'_{abc} \) are representations of the Lie algebra with structure constants \( f_{abc} \).

Finally, computing \( A_1 \) we find the same expressions for \( f_{abcd}, f'_{abcd}, t'_{ab} \) and moreover

\[
s'_{ab} = 2 \left( t_a^- s_b^- + s_b^+ t_a^+ + i f'_{cha} s_c^{-} \right) \quad (7.12)
\]

\[
f''_{abcd} = 2i H_{abcd}, \quad a \in I_1 \quad (7.13)
\]

\[
f''_{abcd} = i m_a (F_{abcd} - F_{(abcd)}), \quad a \in I_2 \quad (7.14)
\]

where

\[
H_{abcd} = f'_{eba} f''_{ecd} + f''_{eca} f'_{ebd} + f'_{eda} f''_{ebc}
\]

and

\[
F_{abcd} \equiv \left\{ \begin{array}{ll}
\frac{2}{ma} H_{abcd} & \text{for } a \in I_2 \\
0 & \text{for } a \in I_1 \cup I_3
\end{array} \right.
\]

We also have

\[
g_{ab1...b4} = 8i S_{a_1...a_4} (f'_{eb_1a} g_{eb_2b_3b_4})
\]

and all other possible pieces of the anomaly (6.62) are null. The explicit expressions for the finite renormalizations which must be used to put \( W \) in such a form are:

\[
T(T^{\mu\nu}(x_1), T(x_2)) \rightarrow T(T^{\mu\nu}(x_1), T(x_2)) + \delta(x_1 - x_2) N^{\mu\nu}(x_1)
\]

\[
T(T^{\mu\nu}(x_1), T^\rho(x_2)) \rightarrow T(T^{\mu\nu}(x_1), T^\rho(x_2)) + \delta(x_1 - x_2) N^{\mu\nu\rho}(x_1)
\]

\[
T(T^\mu(x_1), T^\nu(x_2)) \rightarrow T(T^\mu(x_1), T^\nu(x_2)) + \delta(x_1 - x_2) N^{\mu\nu}(x_1)
\]

\[
T(T^\mu(x_1), T^\nu(x_2)) \rightarrow T(T^\mu(x_1), T^\nu(x_2)) + \delta(x_1 - x_2) N^{\mu\nu}(x_1)
\]

where:

\[
N^{\mu\nu} \equiv \frac{1}{2} f_{abe} f_{cde} u_a u_b v_c^\mu v_d^\nu
\]

\[
N^{\mu\nu\rho} \equiv -\frac{1}{2} f_{abe} f_{cde} [\eta^{\mu\nu} u_a u_b v_c^\nu - (\mu \leftrightarrow \nu)]
\]

\[
N^\mu \equiv f_{abe} f_{cde} u_a v_b^\mu v_c^\nu v_d^{\nu} + f'_{eca} f''_{ebd} u_b v_a^\mu \Phi_c \Phi_d
\]

\[
\tilde{N}^{\mu\nu} \equiv f_{abe} f_{cde} u_b v_a^\nu u_c v_d^\mu
\]

\[
N \equiv \frac{1}{2} f_{abe} f_{cde} u_a v_b^\mu v_c^\nu v_d \sum_{\sum_{a \in I_2}} \frac{1}{ma} f'_{eba} f''_{ecd} v_{au} v_b^\mu \Phi_c \Phi_d + \frac{1}{2} \sum_{a \in I_2} \frac{1}{ma} f'_{eba} f''_{ecd} v_{au} v_b^\mu \Phi_c \Phi_d
\]

(7.19)

If we go to the third order of perturbation theory and use the Wick expansion property (2.8) we obtain a much simpler expression for the generic anomaly:

\[
W = \sum_{a,bc \in I_1} g_{abc} \left( u_a v_b \phi_c^\mu - \frac{1}{2} m_c u_a u_b \bar{u}_c \right) + \sum_{a \in I_1} k_a u_a
\]

\[
+ \frac{1}{3!} f_{a(bed)} u_a \Phi_b \Phi_c \Phi_d + \frac{1}{4!} \sum_{b,c,d,e \in I_3} g_{a(bcde)} u_a \Phi_b \Phi_c \Phi_d \Phi_e
\]

(7.20)
Explicit computations give non-null expressions for $h_{abc}^{(2)}$ (the so-called axial anomaly) and $g_{ab[bcde]}$ which gives the value of the quadri-linear Higgs coupling i.e. a supplementary term in the last relation (7.19).

Let us provide as a particular case the standard model of the electro-weak interactions. We have to take in the general scheme: $I_1 = I_{ph} \cup I_g$ where $|I_1| = 1$, $|I_2| = 3$, $|I_3| = 1$; we denote the corresponding indices by 0, 1, 2, 3, $H$ and $j \in I_g$ respectively. The vector fields corresponding to $I_{ph}, I_2$ and $I_g$ are the photon, the heavy Bosons and the gluons. The field $\phi_H$ is called the Higgs field. We also have: $|I_4| = 8N$ where $N$ is called the number of generations. Then the non-zero constants $f_{abc}$ for the values $I_1 \cup I_2$ are:

$$f_{210} = g \sin \theta, \quad f_{321} = g \cos \theta, \quad f_{310} = 0, \quad f_{320} = 0$$ (7.21)

with $\cos \theta > 0$, $g > 0$ and the other constants determined through the anti-symmetry property. The expressions $f_{jkl}, j, k, l \in I_g$ are the structure constants of the Lie algebra $su(3)$ and this means that $|I_g| = 8$.

The Jacobi identity is verified and the corresponding Lie algebra is isomorphic to $u(1) \times su(2) \times su(3)$. The angle $\theta$, determined by the condition $\cos \theta > 0$ is called the Weinberg angle. The masses of the heavy Bosons are constrained by:

$$m_1 = m_2 = m_3 \cos \theta;$$ (7.22)

The non-zero constants $f'_{abc}$ are completely determined by the antisymmetry property in the first two indices and:

$$f'_{H11} = f'_{H22} = \frac{\epsilon g}{2}, \quad f'_{H33} = \frac{\epsilon g}{2 \cos \theta}, \quad f'_{210} = g \sin \theta,$$
$$f'_{321} = -f'_{312} = g \frac{\cos \theta}{2}, \quad f'_{123} = -g \frac{\cos 2\theta}{2 \cos \theta};$$ (7.23)

the rest of them being zero. Here $\epsilon = \pm$ but if $\epsilon = -1$ we can make the redefinition $\phi_H \to -\phi_H$ and make $\epsilon = 1$.

The non-zero constants $f''_{abc}$ are determined by:

$$f''_{H11} = f''_{H22} = f''_{H33} = \frac{9}{2m_1} m_H^2, \quad f''_{HHH} = \frac{3m_H^2}{2}$$ (7.24)

and we also have

$$g_{HHHH} = 0.$$ (7.25)

Moreover, we must have a supplementary term in the last relation from (7.19) such that the known form of the Higgs potential is obtained.

The Dirac fields are considered with values in $\mathbb{C}^2 \otimes \mathbb{C}^{4N}$ so use a matrix notation i.e. we put

$$\psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$$ (7.26)

with $\psi_1, \psi_2 \in \mathbb{C}^{4N}$. Then in the electro-weak sector

$$t_1^+ = \frac{1}{2} g \left( \begin{array}{cc} 0 & C^{-1} \\ C & 0 \end{array} \right), \quad t_2^+ = \frac{1}{2} g \left( \begin{array}{cc} 0 & -iC^{-1} \\ iC & 0 \end{array} \right),$$
$$t_3^+ = \frac{1}{2} \left[ -g \cos \theta \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) + g' \sin \theta \mathbf{1} \right],$$
$$t_0^+ = -\frac{1}{2} \left[ g \sin \theta \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) + g' \cos \theta \mathbf{1} \right]$$ (7.27)
\[ t_1^- = t_2^- = 0, \quad t_3^- = -tg \theta t_0^+, \quad t_0^- = t_0^+ \] (7.28)

with \( C \) a \( 4N \times 4N \) unitary matrix, \( I \) the \( 4N \times 4N \) unit matrix and

\[
g' = g \begin{pmatrix} D & 0 \\ 0 & -I \end{pmatrix} \] (7.29)

with \( D \) a diagonal, traceless \( Tr(D) = 0 \) and Hermitian \( 4N \times 4N \) matrix which commutes with \( C \). The matrix \( C \) is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Because every Fermi fields can be redefined by multiplication with a phase factor without changing the physics (i.e. the Dirac equation and the expression of the propagator) one can use this freedom to put this matrix in a preferred form [16]. It seems that there are only \( \mathcal{N} = 3 \) generations and the corresponding fields \( \psi_{1j}, \psi_{2j}, j = 1, \ldots, 12 \) are

\[
\psi_1 = \nu_e, \nu_\mu, \nu_\tau, u_p, c_p, t_p \\
\psi_2 = e, \mu, \tau, d_p, s_p, b_p. \] (7.30)

Here the Dirac fields \( e, \mu, \tau \) are the leptons (producing the electron and the particles \( \mu \) and \( \tau \)), \( \nu_e, \nu_\mu, \nu_\tau \) the associated neutrinos and the Dirac fields \( u_p, c_p, t_p, d_p, s_p, b_p \) are the quarks (up, charm, top, down, strange, bottom) each with \( p = 1, 2, 3 \) colors. The representations (7.27) and (7.28) must be tensored with the representation \( t_j \) of the algebra \( su(3) \) acting in the color space.

All the preceding conditions are compatible with gauge invariance conditions up to the third order of perturbation theory.

One can introduce the electric charge operator according \( Q_e \) to

\[
Q_e \Omega \\
[Q_e, v_1^\mu] = ie v_2^\mu, \quad [Q_e, v_2^\mu] = -ie v_1^\mu, \\
[Q_e, \Phi_1] = ie \Phi_2, \quad [Q_e, \Phi_2] = -ie \Phi_1, \\
[Q_e, u_1] = ie u_2, \quad [Q_e, u_2] = -ie u_1, \\
[Q_e, \bar{u}_1] = ie u_2, \quad [Q_e, \bar{u}_2] = -ie u_1, \\
[Q_e, \psi] = i t_0^+ \psi
\] (7.31)

and the rest of the fields are commuting with \( Q_e \); here \( e \) is a positive number (the electric charge). Then one can prove that the electric charge is leaving invariant the expressions \( T^I \)

\[
[Q_e, T^I] = 0. \] (7.32)

If one takes the matrix \( D \) from the expression (7.29) to be proportional to \(- tan(\theta)\) in the lepton sector and \( \frac{1}{3} tan(\theta) \) in the quark sector, then we have the condition of tracelessness for \( D \); moreover, the lepton states will have charge \(-e\), the quarks \( u, c, t \) will have charge \( \frac{2e}{3} \) and the quarks \( d, s, b \) will have charge \(-\frac{e}{3} \).

8 CONCLUSIONS

The cohomological methods presented in this paper leads to the most simple understanding of quantum gauge models in lower orders of perturbation theory and extract completely the information from the consistency Wess-Zumino equations. We have illustrate the methods for the case of Yang-Mills models. In a subsequent paper we will consider the same methods for case of gravity considered as a perturbative theory of particles of helicity (spin) 2.
9 APPENDIX

In this Appendix we prove a trace decomposition result:

Theorem 9.1 Let \( t_{\mu_1, \ldots, \mu_n} \) be a Lorentz covariant tensor and also parity invariant. Then one can write this tensor in the following form:

\[
t_{\mu_1, \ldots, \mu_n} = \sum_{P} \eta_{I_1} \cdots \eta_{I_k} \ t_{I_0}^P
\]

where the sum goes over the partitions \( P = \{ I_0, \ldots, I_k \} \) of the set \( \{ \mu_1, \ldots, \mu_n \} \) such that \( |I_1| = \cdots = |I_k| = 2 \) and the tensors \( t_{I_0}^P \) are Lorentz covariant, parity invariant and also traceless. These tensors can be obtained from various traces of the tensor \( t_{\mu_1, \ldots, \mu_n} \).

Proof: (i) As it is usual in such sort of problems it is convenient to consider instead of \( t_{\mu_1, \ldots, \mu_n} \), the associated \( SL(2, \mathbb{C}) \)- covariant tensor:

\[
t_{a_1, \ldots, a_n, \bar{b}_1, \ldots, \bar{b}_n} \equiv \sigma_{a_1 b_1} \cdots \sigma_{a_n b_n} \ t_{\mu_1, \ldots, \mu_n}
\]

Here \( \sigma^\mu = (I, \sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices and we use dotted and undotted Weyl indices \( a, \bar{b} = 1, 2 \). We will use in the following a number of formulas involving Pauli matrices. We find convenient to list them. First we define:

\[
\sigma_{ab}^{\mu \nu} \equiv \frac{i}{4} \left[ \sigma_{ab}^\mu \epsilon^{\nu \delta} \sigma_{\delta c}^\mu - (\mu \leftrightarrow \nu) \right] \quad \bar{\sigma}_{cd}^{\mu \nu} \equiv -\frac{i}{4} \left[ \sigma_{ac}^\mu \epsilon^{\nu b} \sigma_{bd}^\mu - (\mu \leftrightarrow \nu) \right].
\]

The first expression is symmetric in \( a, b \) and the second is symmetric in \( \bar{c}, \bar{d} \). Then:

\[
\sigma_{ab}^\mu \epsilon^{\nu \delta} \sigma_{\delta c}^\mu = \epsilon_{ca} \ g^{\mu \nu} - 2i \sigma_{ac}^{\mu \nu},
\]

\[
\eta_{\rho \sigma}^{\alpha \beta} \sigma_{ab}^{\mu \nu} \sigma_{cd}^{\nu \rho} = 2\epsilon_{ca} \ \epsilon_{bd},
\]

\[
\eta_{\alpha \beta}^{\rho \sigma} \sigma_{ab}^{\mu \nu} \sigma_{cd}^{\mu \nu} = \frac{i}{2} (\epsilon_{ca} \ \sigma_{bd}^{\rho \sigma} + \epsilon_{bc} \ \sigma_{ad}^{\rho \sigma}),
\]

\[
\eta_{\alpha \beta}^{\rho \sigma} \sigma_{ab}^{\mu \nu} \sigma_{cd}^{\mu \nu} = \frac{1}{4} \left( \epsilon_{ac} \ \epsilon_{bd} + \epsilon_{ad} \epsilon_{be} \right) \eta_{\mu \nu} - \frac{i}{2} (\epsilon_{ac} \ \sigma_{bd}^{\mu \nu} + \epsilon_{ad} \ \sigma_{bc}^{\mu \nu} + \epsilon_{bc} \ \sigma_{ad}^{\mu \nu} + \epsilon_{bd} \ \sigma_{ac}^{\mu \nu}),
\]

\[
\eta_{\alpha \beta}^{\rho \sigma} \sigma_{ab}^{\mu \nu} \sigma_{cd}^{\mu \nu} = \frac{1}{4} \left( \epsilon_{ac} \ \epsilon_{bd} + \epsilon_{ad} \epsilon_{be} \right) \eta_{\mu \nu} + \frac{i}{2} (\epsilon_{ac} \ \sigma_{bd}^{\mu \nu} + \epsilon_{ad} \ \sigma_{bc}^{\mu \nu} + \epsilon_{bc} \ \sigma_{ad}^{\mu \nu} + \epsilon_{bd} \ \sigma_{ac}^{\mu \nu}),
\]

\[
\eta_{\alpha \beta}^{\rho \sigma} \sigma_{ab}^{\mu \nu} \sigma_{cd}^{\mu \nu} = -\frac{1}{8} \left[ \sigma_{ac}^{\mu \nu} \epsilon_{bd} + \left( a \leftrightarrow b \right) + \left( \bar{c} \leftrightarrow \bar{d} \right) + \left( a \leftrightarrow b, \bar{c} \leftrightarrow \bar{d} \right) \right],
\]

\[
\sigma_{ab}^{\mu \nu} \epsilon^{\nu \delta} \sigma_{\delta c}^{\mu \beta} = -\frac{1}{4} \epsilon_{ac} \left( \eta^{\mu \beta} \eta^{\nu \alpha} - \eta^{\nu \beta} \eta^{\mu \alpha} + i \epsilon^{\mu \nu \alpha \beta} \right)
\]

\[
+ \frac{i}{4} \left( \eta^{\mu \beta} \sigma_{ac}^{\alpha \nu} - \eta^{\nu \beta} \sigma_{ac}^{\alpha \mu} - \eta^{\nu \alpha} \sigma_{ac}^{\beta \nu} + \eta^{\nu \beta} \sigma_{ac}^{\alpha \nu} \right) + \frac{i}{4} \left( \epsilon^{\mu \nu \rho \sigma} \sigma_{ac}^{\rho \alpha} - \epsilon^{\mu \nu \rho \sigma} \sigma_{ac}^{\rho \beta} \right),
\]

\[
\sigma_{ab}^{\mu \nu} \epsilon^{\nu \delta} \sigma_{\delta c}^{\mu \beta} = \frac{i}{2} \left( \eta^{\mu \alpha} \sigma_{ac}^{\nu \beta} - \eta^{\nu \alpha} \sigma_{ac}^{\mu \beta} - i \epsilon^{\mu \nu \alpha \beta} \sigma_{ac}^{\beta \rho} \right).
\]

(ii) The correspondence between \( t_{\mu_1, \ldots, \mu_n} \) and \( t_{a_1, \ldots, a_n, \bar{b}_1, \ldots, \bar{b}_n} \) is one-one because we have the formulas (9.4) and (9.5). We have:

\[
t_{\mu_1, \ldots, \mu_n} = \frac{1}{2^n} \sigma_{a_1 b_1}^{\mu_1} \cdots \sigma_{a_n b_n}^{\mu_n} t_{a_1, \ldots, a_n, \bar{b}_1, \ldots, \bar{b}_n}^{\mu_1, \ldots, \mu_n}
\]
where the Weyl indices are raised and lowered with the metric $\epsilon_{ab}$ and $\epsilon_{\bar{a}\bar{b}}$ e.g. $t^a = \epsilon^{ab} t_b$.

(iii) We consider an arbitrary tensor $t_{a_1,\ldots,a_n}$ and we decompose it with respect to the first two indices, into the symmetric and antisymmetric part:

$$t_{a_1,\ldots,a_n} = t_{\{a_1,a_2\},a_3,\ldots,a_n} + \epsilon_{a_1a_2} t_{a_3,\ldots,a_n}$$

(9.13)

Now we have by direct computation:

$$t_{\{a_1,a_2\},a_3,\ldots,a_n} = t_{\{a_1,a_2,a_3\},\ldots,a_n} + \frac{1}{3} (t_{\{a_1,a_2\},a_3,\ldots,a_n} - t_{a_1,a_2,a_3,\ldots,a_n} + \frac{1}{3} (t_{\{a_1,a_2\},a_3,\ldots,a_n} - t_{a_2,a_3,\ldots,a_n})$$

(9.14)

and the second (third) term is antisymmetric in $a_2, a_3$ (resp. in $a_1, a_3$). It means that we have in fact a decomposition:

$$t_{a_1,\ldots,a_n} = \epsilon_{a_1a_2} t^{(3)}_{a_3,\ldots,a_n} + \epsilon_{a_2a_3} t^{(1)}_{a_1,a_4,\ldots,a_n} + \epsilon_{a_3a_1} t^{(2)}_{a_2,a_4,\ldots,a_n} + t_{\{a_1,a_2,a_3\},\ldots,a_n}$$

(9.15)

We continue by recursion and we find out

$$t_{a_1,\ldots,a_n} = \sum_P \epsilon_{I_1} t^{(P)}_{I_0} + t_{\{a_1,a_2,a_3,\ldots,a_n\}}$$

(9.16)

where $P = \{I_0, I_1\}$ is a partition of the set $A \equiv \{a_1, \ldots, a_n\}$ such that $|I_1| = 2$. We apply the same argument to every tensor $t^{(P)}_{I_0}$ and at the very end we get the decomposition formula:

$$t_{a_1,\ldots,a_n} = \sum_P \epsilon_{I_1} \cdots \epsilon_{I_k} t^{(P)}_{I_0}$$

(9.17)

where $P = \{I_0, \ldots, I_k\}$ is a partition of the set $A \equiv \{a_1, \ldots, a_n\}$ such that $|I_1| = \ldots = |I_k| = 2$ and the tensors $t^{(P)}_{I_0}$ are completely symmetric. In the same way we have:

$$t_{a_1,\ldots,a_n,b_1,\ldots,b_\bar{n}} = \sum_{P,Q} \epsilon_{I_1} \cdots \epsilon_{I_k} \epsilon_{J_1} \cdots \epsilon_{\bar{J}_\bar{n}} t^{(P,Q)}_{I_0,J_0}$$

(9.18)

where $P = \{I_0, \ldots, I_k\}$ is a partition of the set $A \equiv \{a_1, \ldots, a_n\}$ and $Q = \{\bar{J}_0, \ldots, \bar{J}_\bar{n}\}$ is a partition of the set $B \equiv \{b_1, \ldots, b_\bar{n}\}$ such that $|I_1| = \ldots = |I_k| = |\bar{J}_1| = \ldots = |\bar{J}_\bar{n}| = 2$; the tensors $t^{(P,Q)}_{I_0,J_0}$ are completely symmetric in the dotted and undotted indices. The preceding formula is in fact the decomposition in irreducible tensors: the tensor $t^{(P,Q)}_{I_0,J_0}$ transforms according to the irreducible representation $D^{(|I_0|/2,|J_0|)}$.

(iv) We consider all possible terms from (9.18) and the contributions they are producing in (9.12). The term without $\epsilon$ factors from (9.18) is producing in (9.12) a traceless contribution because of (9.5). We consider a term with at least one factor $\epsilon_f$ and use the formula (9.4) to eliminate all such factors. Because the representation is irreducible only one of the $2^I$ resulting contributions can be non-zero. So we must have either a contribution with at least $\eta$ factor or a contribution only with factors $\sigma^{\mu\nu}$ i.e. of the type:

$$\sigma^{a_1\bar{a}_1} \cdots \sigma^{a_p\bar{a}_p} \sigma^{a_{\bar{p}}_{\bar{a}}_{\bar{p}}} \cdots \sigma^{a_q\bar{a}_q} t_{a_1,a_2,a_3,\ldots,b_1,b_2,\ldots,b_\bar{n},c_1,\ldots,c_{\bar{p}},\bar{d}_1,\ldots,\bar{d}_q}$$

(9.19)

and we must prove that these contributions are producing in (9.12) either traceless terms or terms with one factor $\eta$. We have two cases: if the contribution is without factors $\epsilon_f$ then the tensor
Indeed, we may take $l > k$. 

$\epsilon$ is of the type $k$ and observe that for $n$ of rank $k$ tensor. According to (v) we have the following decomposition for the space of parity invariant tensors with one $\epsilon$ factor. The last contribution must be zero because of parity invariance.

So in the preceding formula it remains to consider the case when we have at least one factor $\epsilon$. Again we have two subcases: if the factor $\epsilon$ is of the type $\epsilon^b_1\epsilon^c_k$ we use the formulas (9.10) and if it is of the type $\epsilon^b_1\epsilon^c_k$ we use the formulas (9.11) to obtain in (9.12) a contribution with a factor $\eta$ or $\epsilon^{\mu\nu\rho\sigma}$. So we still have to consider the case when we have in (9.19) only factors of the type $\epsilon^{\mu\nu\rho\sigma}$. In this case we use the formula (9.4). The first term from (9.4) is giving a null contribution in (9.19) so we get zero i.e. we have (9.24).

Indeed, if this would not be true that we would have a non-null $t \in T^n_k$ such that

$$< t, e^{(k)}_\alpha > = 0, \quad \forall \alpha \quad \iff \quad t \perp T^n_k.$$
If we use (9.24) we find out that $t \in T^n_+$ and because $\langle \cdot, \cdot \rangle$ is non-degenerate it follows that $t = 0$. The contraction proves (9.26).

We write any $t \in T^n_+$ in the form

$$t = \sum_{k, \alpha} t^{(k)}_{\alpha} e^{(k)}_{\alpha}$$

(9.28)

and we have from here

$$\langle t, e^{(k)}_{\alpha} \rangle = \sum_{\beta} \langle e^{(k)}_{\alpha}, e^{(k)}_{\beta} \rangle t^{(k)}_{\beta}.$$  

(9.29)

If we take into account (9.26) it means that we can express the tensors $t^{(k)}_{\beta}$ as linear combinations of $\langle t, e^{(k)}_{\alpha} \rangle$. But it is easy to see that these expressions are some traces of the tensor $t$. This proves the last assertion from the statement. $\square$

Acknowledgement: This paper was partially supported by the contract 454/2009, cod CNCSIS ID-44.

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