Matrix Characterization of Knots: 
A Simple Statistical Mechanics Application

R. Kariotis 
Department of Physics 
University of Wisconsin 
Madison, Wisconsin 53706 
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Abstract:

In this note, I describe a formalism for treating knots as geometric spaces, and make an application to a simple statistical mechanics computation. The motivation for this study is the natural visual symmetry of the knot, and I describe how this might be carried out. The direct approach, however, fails due to limits of the visual symmetry, but by recasting the problem in terms of the geometry of contours of the knot, the resulting permutation operators provide a better analytical tool. At present, I will be limited to a description of the proposed program, rather than complete results.

In the process of setting up the formalism, an interesting aspect, the use of fractional permutation operators, turned up, and I will present a simple statistical mechanical application of these objects. The application described suggests a relation to anyons, the key element in topological quantum computing, which in turn is related to the association of a knot - in its braid form - with a quantum circuit.
Introduction

Knot theory [1,2] has become an important area of investigation in theoretical physics in recent years due to applications in such diverse areas as quantum gravity, polymer physics and quantum computation [3,4]. (By far the best introduction to this subject is Kauffman’s book Knots and Physics [5].) One important element in the theory which has yet to reach maturity might be described as the lack of a clear analytic means of characterizing knots, in particular, of being able to unambiguously distinguish one knot from another. Polynomial invariants offer a means for dealing with this need, but are not unique; the Burau and Artin representations are insufficiently descriptive. (However, it should be mentioned that computer graphics have been used (HFK theory[8])). The incident of the ”Perko Pair” [6] is a fair indication of the complications involved.

In this note I will outline a possible means for dealing with this need by setting up a formalism that associates each path/contour around a knot with a point in an abstract space, \( C \), the space of contours. Successive contours are related by a displacement operator, and the set of such operators then describes a geometry on \( C \). Interpolating between points in \( C \) suggests assigning metrical meaning to the calculation of fractional permutation operators. I will make a special example of this by applying these matrices to a simple statistical mechanics problem.

Motivation

The obvious visual symmetry in the simplest knots suggests an underlying mathematical symmetry, for example as is apparent in the 3.1 knot of the figure below. A single arc, \( \zeta \), can be used to generate the entire diagram by applying appropriate displacements (labeled \( \gamma \)) and rotations (labeled \( \Gamma \)). Graphics software that has the appropriate underlying algebraic facilities, such as METAPOST, can be very effective in carrying out these operations and has the additional advantage of emphasizing the geometric nature of the diagram [9a,b].

This approach, however, quickly becomes impractical as the knots increase in complexity. In the next figures, the 9.2 and 7.2 knots are shown with suggested basis arcs, but additional pieces would be needed to complete the job, simple as these diagrams are.

Thus, we follow a different geometric approach described in the next section. As it happens, this method is not unlike the geometric approach once considered by Polish and British cryptographers during WWII. In their scheme, a line of encoded text was treated as a set of permutations of the original text, so the operation becomes one of a displacement in the space of text strings, from one line to another. In the context of knots, one contour along the knot is related to another by a permutation matrix, a displacement in the space of contours.

Geometry

Starting at a particular point, a contour is defined as the sequence of numbered crossings, a positive integer for cross-over, negative for cross-under. An example is shown in figure for the knot 9.2 [7]. The starting point for the contour is arbitrary so what we’ll want is the complete set of numbered contours.

There are two lines of interest here:

1) by investigating the set of permutation operators \( \{\gamma_i\} \) in the space \( C \) of contours, it is possible to obtain a generalized set of skein relations, constructed by matrix operations on the space \( \{\gamma_i\} \);
2) it is useful also to make a geometric interpretation of the space of operators, defining displacement, angle and curvature; in this view, the odd-numbered knots, 3.1, 5.1, 7.1, ... are equivalent to a Euclidean geometry, where the distance from one contour to the next is uniform along a trajectory in \( C \); all other knots are non-uniform and it will be useful to take the radical of products - e.g. \( \sqrt{\gamma_1 \gamma_2} \), which sometimes results in a complex operator - in order to calculate curvature.

10.161

\[ \zeta_1 \gamma_1 \Gamma_{12} \rightarrow U_{12} \leftarrow \bar{\Gamma}_{12} \]

For example, I show here a scheme where the set of operators relating the two Perko knots[6]; these were originally called 10.161 and 10.162 in the Rolfsen scheme until Ken Perko recognized that they were the same knot.

In general, the relation between two successive displacements, and their composite is given by

\[ \zeta_1 \gamma_1 = \zeta_2 \quad \zeta_1 \Gamma_1 = \bar{\zeta}_1 \quad \bar{\zeta}_1 \bar{\gamma}_1 = \bar{\zeta}_2 \]

\[ \zeta_i \gamma_i = \zeta_{i+1} \]

so each operator acts to connect one contour to the next, is such that all \( \gamma_i \) are equal; in all other knots this operator varies from one point in the function space to the next. The diagram here shows the 3.1 knot, indicating how one might go about constructing the full image from a single arc that can be rotated and displaced in successive moves. Also, below that is shown the 9.2 knot, with indications of which arcs would be needed to construct a full image; in fact, considerable effort was needed in the 7.2 knot and the result is not entirely convincing.

In other words, the visual symmetry is a good indication of the basic topology, but is not very practical as a computational device.

**Euclidean metric:** In this case, displacement, defined generally

**Non-Euclidean metric:** The 5.2 diagram shown below, and the set of paths around its contour suggest the topology of the knot: the operation that carries one path into the next varies as one moves along a trajectory in contour space \( C \). A measure of the non-uniformity is obtained by considering the extent to which products of operators can be represented by a product of identical operators. This requires that we be able to take, for example

\[ \sqrt{\gamma_1 \gamma_2} = Q_1 \]
in general, however, the resulting radical is complex with multiple roots. 

*Curvature:* In order to determine what we mean by curvature, consider first the knots with constant $\gamma_i$, i.e. 3.1, 5.1, 7.1, ... . In this case, the displacement of two next near neighbors, defined as $\gamma_i \gamma_{i+1}$ is simply $\gamma_i^2$ so the space has uniform metrical properties. All other knots are non-uniform and the separation of next near neighbors must be defined as $\sqrt{\gamma_i \gamma_{i+1}}$, which in general is complex.
[7.2] construction using two arcs

$\zeta_1 = \{1, -2, 3, -4, 5, -1, 4, -3, 2, -5\}$
$\zeta_2 = \{5, -1, 2, -3, 4, -5, 3, -2, 1, -4\}$
$\zeta_3 = \{4, -5, 1, -2, 3, -4, 2, -1, 5, -3\}$
$\zeta_4 = \{3, -5, 4, -1, 2, -3, 1, -4, 5, -2\}$
$\zeta_5 = \{2, -5, 4, -3, 1, -2, 3, -4, 5, -1\}$
$\zeta_6 = \{1, -4, 3, -2, 5, -1, 2, -3, 4, -5\}$
$\zeta_7 = \{5, -3, 2, -1, 4, -5, 1, -2, 3, -4\}$
$\zeta_8 = \{4, -2, 1, -5, 3, -4, 5, -1, 2, -3\}$
$\zeta_9 = \{3, -1, 4, -5, 2, -3, 5, -4, 1, -2\}$
$\zeta_0 = \{2, -3, 4, -5, 1, -2, 5, -4, 3, -1\}$

[5.2] typical contour numbering
curvature: normal vector $\hat{n}$

Roots of Permutation Operators To see how this comes about, first note that permutation matrices can always be put into the form of a set of cycles, for example, $(561432) \rightarrow (153)(26)(4)$ so that by rearranging the indices the matrix is put in block form

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & \ddots \end{pmatrix}$$

and each block has the form (taking the 4x4 for example)

$$A_j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(or, for nxn $(n12345...n-1)$) and matrices of this form are easily diagonalized; a typical eigenvector is

$$\psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix}$$

where the $m^{th}$ eigenvalue is

$$\lambda_m = e^{\phi_m} \quad \phi_m = m\frac{2\pi i}{n}$$

For example, the three element matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} e^{\phi} & e^{2\phi} & e^{3\phi} \\ e^{2\phi} & e^{4\phi} & e^{6\phi} \\ e^{3\phi} & e^{6\phi} & e^{9\phi} \end{pmatrix} \quad \phi = \frac{i2\pi}{3}$$
in general, for integral \( Y \), there will be \( Y \) roots for each of the diagonal elements of \( A^\frac{1}{Y} \), and which one to apply must be dealt with in context. If \( Y \) is rational, the number of roots is finite; if it is irrational the number of roots is infinite, resulting in the so called differential permutation group, far outside the scope of this work. For example: if \( Y = 11 \), there will be 11 roots, located at \( \phi = n\frac{2\pi}{11} \) where \( 1 \leq n \leq 11 \); for \( Y = 11.1 \) there are 111 roots at \( \phi = 10n\frac{2\pi}{11} \) and \( 1 \leq n \leq 111 \); for \( Y = 11.11 \), \( \phi = 100n\frac{2\pi}{111} \) with \( 1 \leq n \leq 1111 \). And so on, as suggested in the diagram. More on this in the next section where a statistical mechanics application is described (see also the Appendix).

\[ \mathbf{\lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

and generally, all roots are sums of the group elements

\[ T^\frac{1}{Y} = \mathfrak{N} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathfrak{J} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \mathfrak{L} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \cdots \]

where \( \mathfrak{N}, \mathfrak{J}, \mathfrak{L}, \ldots \) are numerical constants, which will simplify the computation considerably. The coefficients are readily determined from the eigenvalues and eigenvectors; for example

\[ \mathfrak{N} = \lambda_1 \sqrt{\lambda_1 \lambda_2^2 + \lambda_2 \lambda_3^2 + \lambda_3 \lambda_1^2} \]

where for \( C_3 \), \( \lambda_1 = e^\phi \) and \( \phi = \frac{i2\pi}{3} \). The only complication is that there are \( 2^N \) roots that must be considered.
This formalism is somewhat similar to the versors of geometric algebra [10] where the \((\frac{1}{\pi})^{th}\) root of a rotation operator is most easily handled in Clifford algebra formalism.

**Spin Net Representation**

Each displacement operator \(\Gamma_i\) can be expressed as a direct sum of cycles as described above

\[ \Gamma_i = C_1 \oplus C_2 \oplus C_3 \oplus \cdots \]

unless the geometry of the knot is simple (as in 3.1, 5.1, 7.1, ...), and each cycle acts as an independent particle state. In going from one displacement to the next the independent cycles are transformed: for example in the image below, spin net diagrams the first set of cycles \(\{C_i\}\), on the left, is transformed to the outgoing set \(\{\bar{C}_i\}\).

A diagram of this behavior is shown in Fig. A, and suggests the structure sometimes found in spin nets [11]. What is lacking at this point is the operation that takes

\[ C_1 \oplus C_2 \oplus C_3 \rightarrow \bar{C}_1 \oplus \bar{C}_2 \]

That is, that we need is something like that described by Dorst et al in their book *Geometric Algebra and Computer Science* [10], incrementally transforming \(C \rightarrow \bar{C}\). In his example [sect.1.2.6], a rotation \(V\), expressed in versors, is implemented in \(N\) steps of \(V^{\frac{1}{N}}\).

**Partition Function/Path Integral**

Several models are suggested by the previous discussion.

1. statistical mechanics of one-dimensional systems such as the Ising model, are described
by the partition function

\[ Z = \sum_{\{m\}} \cdots T(m_a, m_b)T(m_c, m_d)T(m_e, m_f) \cdots \]

where the transfer matrix is given by

\[ \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix} \]

which is easily diagonalized.

2. statistical mechanics of a disordered one-dimensional system, where the partition function is similar to the previous example, but the \( J \) coupling is a random variable, which would correspond to a different eigenvalue \( \lambda_i \) at each site.

3. The properties of the root matrices are suggestive of the propagator in the Dirac-Feynman path integral formulation of quantum mechanics [12]. That is, for a 0-dimensional, two-state system, the exchange

\[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

becomes

\[ \begin{pmatrix} a \\ b \end{pmatrix} = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^\frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3536 + i0.8536 & -0.3536 + i0.1464 \\ -0.3536 + 0.1464 & 0.3536 + i0.8536 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

so that there exists an intermediate state that is a superposition of the two pure states. This suggests a correspondence

\[ e^{\frac{i\pi}{2n}} \frac{\hbar^2}{\sigma_k^2} \to \Lambda^\frac{1}{Y} \]

that is,

\[ \hbar^2 \sim \frac{1}{Y} \]

To make the analogy closer, a complete description of this process would require a sum over all the appropriate roots; recall that the fractional roots of the permutation matrices results in manipulation of the eigenvalues

\[ \lambda = \exp[i\frac{2\pi}{N}] \]

and in general the \( Y^{th} \) root of the eigenvalue is obtained from the phase as

\[ \phi' = \frac{\phi}{Y} + \frac{i2\pi}{Y} n \]

where \( n = 1, 2, ..., Y \) is the multiplicity of the roots. A succession of the \( C_n^Y \) each with different roots has the form of a path integral if the summation over the different roots is performed, suggestive of the ZBW effect in quantum mechanics. This can be written in the usual form

\[ Z = \int D[L(x)] \exp[-\int \mathcal{L}] \]
where \( L(x) \) is related to the root of the \( x^{th} \) \( C \). \( \mathcal{L} \) is effectively the log of \( C \) and path-ordering is implied. In this expression, the integral/sum is over all paths in configuration space for a fixed set of roots. To further complete the evaluation a sum over all possible roots is necessary; this is easily accomplished by working in the diagonal representation, the exponent before sum becomes

\[
\left[ \begin{array}{cc} \exp[-\sum_i \Lambda_1(n_i)] & 0 \\ 0 & \exp[-\sum_i \Lambda_2(n_i)] \end{array} \right]
\]

where for the \( Y^{th} \) root

\[
\Lambda_j(n_i) = J \frac{i2\pi}{NY} + \frac{i2\pi}{Y} n_i
\]

and \( n_i = 1, 2, 3, ..., Y \). Since the exponents are all proportional, the evaluation amounts to the need to obtain an expression for the sum \( n_1 + n_2 + n_3 + ... + n_J = M \) subject to the implied restraint. This has the form of the central limit theorem

\[
P(M) = \int dx_1 p(x_1) \int dx_2 p(x_2) \cdots \delta(x_1 + x_2 + x_3 + ... + x_J - M)
\]

which yields a gaussian in the summation restraint variable \( M \).

The resulting evaluation over a single possible configuration is shown in Fig B. In the following figure the complex amplitudes of the two states are displayed in the complex plane: that for \(|10\rangle\) fixes the lower left corner; that for \(|01\rangle\) fixes the upper right corner. The aspect ratio determines the brightness.

However, in order to make this a non-trivial problem, a non-commuting operator is needed, such as

\[
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

A better way to see this analogy is in the case of translation/diffusion (i.e. classical/quantum) in one-dimension [15].
path of summation for two configurations

Fig. B: \( Y=4 \) and \( Y=16 \)

Increasing \( Y \) is like increasing temperature.
complex coefficients of $|10\rangle$ and $|01\rangle$ mapped to a rectangle

**Discussion**

As stated in the introduction, the purpose of this note is to outline a geometric formulation of knot descriptions that has sufficient depth so that different knots can be distinguished unambiguously. The simplest knots, 3.1, 5.1, 7.1 ... represent the equivalent of Euclidean space, the less simple knots the equivalent of curved spaces, each distinctly described by that curvature. Touching on the interesting, but not immediate subject of the *differential permutation group* we made use of the idea of fractional permutation to determine the normal to the knot at a given point, and then suggested but was not able to carry out, the use of a quantum field theory to characterize the geometry of the knot.

The physical nature of the multiple roots is similar to that of the anyons used in quantum computation, that is, that an incomplete permutation is performed that creates a phase shift in the system.
Appendix: This is not the place, nor the author, to be giving a detailed description of the group properties of the permutation roots, however, the subject is too interesting to leave without further mention. Consider the case of the 3x3 cycle: calculating the square root and then constructing the multiplication table yields a fractal-like pattern. To see this define
\[
\mathbf{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]
then the square root of this in diagonal form is
\[
\mathbf{z}^{\frac{1}{2}} \rightarrow \begin{pmatrix} e^{\phi} & 0 & 0 \\ 0 & e^{2\phi} & 0 \\ 0 & 0 & e^{3\phi} \end{pmatrix}
\]
where \( \phi = \frac{2\pi}{YN} \). Initially there are \( Y^N = 2^3 \) roots, however if they are multiplied out, additional elements of the group are generated. (In our notation, \( Y = 2, 3, 4, \ldots \) for the square, cubic, quartic,\ldots roots of the NxN permutation matrices.) Numbering the elements 1-24 (only the first eight are roots), the multiplication table is:

\[
\begin{array}{cccccccccccc}
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 & 19 & 20 & 18 & 21 & 22 & 23 & 24 & 1 \\
11 & 12 & 9 & 10 & 15 & 16 & 13 & 14 & 20 & 21 & 17 & 18 & 22 & 23 & 24 & 2 \\
12 & 11 & 10 & 9 & 15 & 16 & 13 & 14 & 20 & 21 & 17 & 18 & 22 & 23 & 24 & 3 \\
13 & 14 & 15 & 16 & 9 & 10 & 11 & 12 & 21 & 22 & 23 & 24 & 17 & 18 & 19 & 4 \\
14 & 13 & 16 & 15 & 10 & 9 & 12 & 11 & 22 & 21 & 24 & 23 & 17 & 18 & 19 & 5 \\
15 & 16 & 13 & 14 & 11 & 12 & 9 & 10 & 23 & 24 & 21 & 22 & 19 & 20 & 17 & 6 \\
16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 24 & 23 & 22 & 21 & 19 & 18 & 17 & 7 \\
18 & 17 & 19 & 20 & 21 & 22 & 23 & 24 & 2 & 4 & 3 & 6 & 5 & 8 & 7 & 8 \\
19 & 20 & 17 & 18 & 23 & 24 & 21 & 22 & 3 & 6 & 4 & 7 & 2 & 1 & 4 & 3 \\
20 & 19 & 18 & 17 & 24 & 23 & 21 & 22 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
21 & 22 & 23 & 24 & 17 & 18 & 19 & 20 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
22 & 21 & 24 & 23 & 18 & 17 & 20 & 19 & 6 & 5 & 7 & 8 & 1 & 2 & 3 & 4 \\
23 & 24 & 21 & 22 & 19 & 20 & 17 & 18 & 7 & 8 & 6 & 5 & 4 & 3 & 2 & 1 \\
24 & 23 & 22 & 21 & 19 & 20 & 18 & 17 & 9 & 10 & 12 & 11 & 14 & 13 & 15 & 16 \\
\end{array}
\]

but a better way to get a rough notion of the multiplicative structure is through a color coding as shown in Fig C. What is of particular interest is that the off-diagonal elements exhibit a pattern similar to that of the cycle summation described above.

To be more specific, write the table as
\[
\begin{pmatrix}
A & B & E \\
B & E & \mathfrak{z}A \\
E & \mathfrak{z}A & \mathfrak{z}B
\end{pmatrix}
\]
where
\[
\mathfrak{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ldots
\]
Fig C: $Y^N = 2^3$ product table

defining the transformations that simplify $Q$ as $U$

$$T_A = \begin{bmatrix}
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2
\end{bmatrix}$$

$$T_B = \begin{bmatrix}
17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
18 & 17 & 20 & 19 & 22 & 21 & 24 & 23 \\
19 & 20 & 17 & 18 & 23 & 24 & 21 & 22 \\
20 & 19 & 18 & 17 & 24 & 23 & 22 & 21 \\
21 & 22 & 23 & 24 & 18 & 17 & 20 & 19 \\
22 & 21 & 24 & 23 & 17 & 18 & 19 & 20 \\
23 & 24 & 21 & 22 & 19 & 20 & 17 & 18 \\
24 & 23 & 22 & 21 & 20 & 19 & 18 & 17
\end{bmatrix}$$

$$T_A - B = \begin{bmatrix}
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8
\end{bmatrix}$$

which is obtained from

$$F = \begin{bmatrix}
A & DA \\
DA & A
\end{bmatrix}$$

and

$$A = \begin{bmatrix}
0 & 8 & 5 & 0 \\
8 & 0 & 0 & 5 \\
5 & 0 & 0 & 8 \\
0 & 5 & 8 & 0
\end{bmatrix}$$
and the displacement operator is

$$\mathcal{D} = exp[-3\partial]$$

in other words, the fractal-like appearance is explainable, at least in this simple case, in terms of a series of matrix operations on a pair of vectors \((1, 1, 1, 1)\) and \((1, -1, -1, 1)\).

In the above we have considered only real fractional powers, but it is also of interest to treat the case where the power is a complex number, e.g. \(S_i^{\dagger}S_i\) where \(Y = Y_r + iY_i\). In this instance however, the finite nature of the group properties is lost; if the eigenvalues acquire a "non-unitary" part, i.e. \(|\lambda| \neq 1\) then the closed nature of the group table is lost.
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Software:

Computation: Most of the numerical work indicated here was done using Linux based, home spun C++ coding, then later checked using either the SAGEmath package [13a], or MATHEMATICA [13b]. As far as I could tell neither SAGE nor MATHEMATICA provided direct access to the multiple roots of the permutation operators, but both were easily coerced to work with the eigenvalues.

Graphics: The outlines of the 5.2 and 9.2 knots were made initially using the KnotTheory package [7], then edited using POSTSCRIPT commands; the 3.1 and 7.2 knots, as well as Fig B, were constructed using the linear algebra facilities available in METAPOST; Fig. A and the matrix plots in the Appendix were drawn using a special METAPOST plotting module [14]; the lattice figure was constructed using the 3D plotting modules in SAGE. This article was written using SAGETEX [10a], a typesetting miracle.

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[9a] Most graphics packages today have some facilities in linear algebra, but few provide the full extent of these properties; however, newer graphics programs implement the Clifford algebra - commonly known as Geometric algebra - formalism which better emphasizes the additive nature of visual images;

[9b] on another issue, object-oriented design used in nearly all modern computer projects may be a powerful tool for the programmer, but is a distraction in use; the need to be able add a line, a disk and a cube, to an existing frame is a geometric demand, easily carried out in the context of a graded algebra, regardless of the internal coding; See, for a differing view and additional references, D. Hildenbrand , D. Fontijne , C. Perwass and L. Dorst, Geometric Algebra and its Application to Computer Graphics http://www.science.uva.nl/ga/
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