Dynamical current correlations in Cooper pair splitters based on proximized quantum dots

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Abstract
Entanglement of electrons is studied by means of current–current correlations in two Cooper pair splitter devices: with one and two proximized quantum dots (1QD and 2QD), in presence of intra- and inter-dot Coulomb interactions, and weakly coupled with metallic electrodes. The 1QD system, where Cooper pairs can be transmitted to the same or split to different normal electrodes, is contrasted with the 2QD device, where double occupancy of a single quantum dot is forbidden and transport is only through an inter-dot singlet due to non-local crossed Andreev reflection processes deep in the superconducting energy gap. Separating the current correlation function into components for partial currents of electrons and holes through various Andreev bound states, one can see bunching and antibunching of split particles: inter-level components between electron and hole currents flowing to different electrodes are positive, while intra-level electron–electron or hole–hole components are negative, respectively. Spectral decomposition of the frequency-dependent current cross-correlation is performed to get better insight into mechanisms of entanglement and dynamics of split Cooper pairs, and to extract various charge fluctuation processes with different relaxation times, related to electron and hole currents flowing through the Andreev bound states. Only low frequency polarization fluctuations are seen in the current cross-correlations, while various negative and positive high frequency (charge fluctuations) components compensate each other in the symmetric system.

1. Introduction
Quantum entanglement [1] is a phenomenon where a pair or a group of particles is perfectly correlated in such way that the information about quantum state of the whole system is larger then sum of the information about quantum states of each individual particle. Therefore, as a result of measurement of a quantum state of an individual particle one determines quantum state of all other entangled particles even if they are far away each other. One of the important realization of the entanglement is the singlet (or the so-called Einstein–Podolski–Rosen) state of two electrons observed in BCS superconductors as a Cooper pair [2, 3]. Hybrid superconductor-quantum dot (QD) devices, as effective sources of entangled electrons resulting from splitting of Cooper pairs, attract a great interest for their potential use as basic components of quantum computers or quantum systems for secure data processing and transmission [4–9].

The novel real-time charge detection techniques are able to count electrons one-by-one as they pass through the nanostructure [10]. Within the help of a full counting statistics (FCS), proposed by Levitov et al [11, 12], one can characterise transmission of charges in the steady state by consecutive moments of the distribution function. FCS has been used to determine current cross-correlations in various systems of QD coupled to superconductor and metallic (normal or ferromagnetic leads) [13–16]. It was found that behaviour of the current cross-correlations is determined by a competition between non-local Crossed
Andreev Reflection (CAR) processes and normal electron electron transfer. Bunching of electrons can be caused by CAR deep inside superconducting energy gap and Andreev-reflection enhanced transmission (process which transfer an electron between normal electrodes with assistance of the superconductor) close to the gap boundary. Both processes can be a source of entangled electrons.

Controlled Cooper pair splitting with near-unity efficiency can be realized by coupling of the BCS superconductor to two normal metal electrodes via two tunable parallel QDs, see e.g. [17–23]. It was predicted that 2QD systems with ferromagnetic electrodes allow to test Bell’s inequalities with spatially separated electrons in maximally entangled spin singlet state [24] by current measurements. The Bell-inequalities were also formulated in terms of current–current correlations in a systems consisting of a superconductor weakly coupled to normal conductors [25, 26]. It was shown that in the system with QDs finite-bias spectroscopy is it shown to identify two processes: non-local CAR (responsible for entanglement) and elastic cotunneling (EC) [19]. The experiments [17, 18] have been modelled by Chevallier et al [27], who found mutual position of the QD energy levels which optimizes the CAR processes or that which favors EC. Entanglement due to CAR processes is seen as positive current cross-correlations while EC give negative contribution. Even weak Coulomb repulsion favors positive current cross-correlations in the antisymmetric case [28]. Cross-correlation was also a direct measure of CAR and EC processes in the diffusive and ballistic regimes but for a hybrid system consisting of two normal metal leads weakly connected to a superconductor [29]. Experimental evidence of the electron entanglement through the shot noise measurements (in particular positive current cross-correlations) in the Cooper pair splitter (CPS) with double QD with strong Coulomb interactions have been shown by Das et al [21]. The positive current cross-correlations have been also found in CPS: in the Kondo regime [30]; in the presence of direct tunneling between QDs and finite intra- and inter-dot Coulomb interactions [31, 32]; for spin–orbit interactions [31]; with two ferromagnetic electrodes [33–36]. Burkard et al [37] showed, that an entangled singlet electron pair gives rise to an enhancement of the current noise (bunching), whereas the triplet pair leads to its suppression (antibunching). In systems with the topologically nontrivial superconducting nanowire, strong current cross-correlations as well as enhanced shot noise are also predicted [38, 39].

The above mentioned studies in the systems with QDs focused mainly on the zero-frequency correlations. Recent measurements [40] of finite-frequency current statistics in a single-electron transistor showed dynamics of tunneling processes and correlations between different spectral components of currents. In our opinion fabrication of a similar hybrid structure and exploration of dynamics of Cooper pair transmission is accessible. For that reason we undertake an issue not previously discussed: dynamical current correlations in the CPS configurations with one or two QDs. In 1QD system Cooper pairs can be transmitted to the same (due to the Direct Andreev Reflection (DAR) processes) or split to different (due to the CAR processes) normal electrodes, while in the 2QD device with large on-dot Coulomb repulsion only non-local CAR processes play a role. We focus our studies on the current cross-correlations, where enhancement of the CAR processes corresponds with a large splitting efficiency [41], which can be a signature of quantum entanglement between electrons tunneling through different branches of the system. In the hybrid device one can expect the anomalous positive components \(\langle \Delta I_e \Delta I_h \rangle\) for the electron–hole (e–h) cross-correlations, apart from the negative components \(\langle \Delta I_e \Delta I_h \rangle\) and \(\langle \Delta I_h \Delta I_h \rangle\) for the electron–electron (e–e) and the hole–hole (h–h) cross-correlations. Therefore, we will divide the cross-correlation function for total current into components to capture correlations between partial electron and hole currents, flowing through the same or different Andreev bound states (ABS), to find which of them are responsible for bunching or antibunching. We will also perform spectral decomposition of the frequency-dependent current cross-correlations and their various (intra- or inter-level) components in order to study dynamics of Cooper pair splitting and to get insight into various charge fluctuation processes and their relaxation times, related to electron and hole transport through ABS. This analysis will show interplay of the various microscopic processes and their role in the total current cross-correlations. Our studies, based on analytical derivations for unidirectional transport, will show how various components of the current cross-correlations depend on the microscopic parameters of the system, which can be useful for experimental investigations and future applications.

The paper is organized as follows. In section 2 we introduce the models and in section 3 describe the method of calculations of currents, frequency-dependent current correlations and all their components. First, the analysis of the zero-frequency current correlations is performed in section 4. Studies of dynamics of split Cooper pairs by means of frequency-dependent current correlations are presented in section 5. Conclusions are included in section 6. To make the presentation of the results more transparent details of calculations are shifted to appendix A and appendix B.
2. Cooper pair splitter models

We consider two CPSs: with 1QD and 2QD coupled to a BCS superconductor (S) and to normal metallic (N) electrodes (left-L and right-R), figures 1(a) and (b). Both devices are described by the Hamiltonian

\[ H = H_{\text{QD}} + H_{\text{el}} + H_{\text{el-QD}}, \]

where \( H_{\text{QD}} \) characterises the QD system with internal interactions, \( H_{\text{el}} \) corresponds to electrons in the N electrodes and coupling with them is given by \( H_{\text{el-QD}} \).

Due to a strong proximity effect from the S electrode the QDs behave like a superconducting grains. To simplify our considerations and focus on Andreev scattering we work in the so called superconducting atomic limit, when the superconducting energy gap \( \Delta \to \infty \), and only subgap Cooper pairs tunneling contribute to transport between the S electrode and the QDs. The others, single quasiparticle tunneling processes, can be neglected.

2.1. Cooper pair splitter with 1QD

The effective Hamiltonian of the 1QD system can be expressed as [42]

\[ H_{\text{1QD}}^{\text{eff}} = \sum_{\sigma = \uparrow, \downarrow} \epsilon d_{\sigma}^\dagger d_\sigma + Un_\uparrow n_\downarrow - \frac{\Gamma_S}{2} (d_\uparrow^\dagger d_\downarrow^\dagger + \text{h.c.}), \]

where the first term describes a single level QD with an energy \( \epsilon \), \( d_{\sigma}^\dagger \) (\( d_\sigma \)) stands for electron creation (annihilation) on the dot; the second term is an intra-dot Coulomb interaction \( U \) and the particle number operator \( n_\sigma = d_{\sigma}^\dagger d_\sigma \), and the third one describes the proximity effect which leads to intra-dot pairing

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**Figure 1.** Schematic view of the (a) 1QD and (b) 2QD system in the CPS configuration for \( \mu \equiv \mu_L = \mu_R < \mu_S = 0 \). (c) Excitation energies related to ABS: \( E_{ij}^A \text{ or } E_{\alpha ij}^A \) as a function of a relative position (detuning) of the QD level \( \delta = 2\epsilon + U \) (dashed curves) or \( \delta_{LR} = \epsilon_L + \epsilon_R + U_{LR} \) (solid curves) for 1QD or 2QD, respectively. Horizontal dashed lines denote position of the chemical potential \( \mu = -0.5 \) and \( \mu = -1.5 \) for small and large bias voltage, respectively. (d) Scheme of a Cooper pair splitting (see magenta or blue arrows) in the device with 2QD for small bias voltage \( \mu = -0.5 \) when only inner ABS \( E_{ij}^A, E_{\alpha ij}^A \) are in transport window (yellow region). The parameters are: \( \epsilon_L = \epsilon_R \) (identical QDs), \( \Gamma_{SLR} = \Gamma_S = 0.2 \). All energies are measured in units of the intra-dot Coulomb interaction \( U \) or the inter-dot Coulomb interaction \( U_{LR} \) for 1QD or 2QD, respectively. In the latter case, \( U_L = U_R = \infty \), i.e. the intra-dot singlets are neglected. Note that \( E_{ij}^A, E_{\alpha ij}^A > \mu_S = 0 \) and \( E_{ij}^A, E_{\alpha ij}^A < \mu_S \) are ABS for electrons and holes, respectively.
controlled by the coupling $\Gamma_5$ [34]. The Hamiltonian (2) operates in two subspaces: the singlet (even) states and the single electron (odd) states. A coherent superposition of the empty $|0\rangle$ and the doubly occupied $|\uparrow\downarrow\rangle$ states gives the singlet eigenstates $|\pm\rangle = \alpha_+|0\rangle \mp \alpha_-|\uparrow\downarrow\rangle$ with eigenenergies $\epsilon_\pm = \pm \sqrt{\delta^2 + \epsilon_\alpha^2}/2$, where $\epsilon_\alpha = 1/2(\delta + \sqrt{\delta^2 + \epsilon_\alpha^2})$ is a splitting between the states $|+\rangle$ and $|-\rangle$, $\delta = 2eU + U$ is a detuning parameter between the states $|0\rangle$ and $|\uparrow\downarrow\rangle$, and coefficients $\alpha_\pm = \pm(2 \pm \delta/\epsilon_\alpha)^{1/2}$ [43–46]. The single electron states are $|\uparrow\rangle$ and $|\downarrow\rangle$ with the eigenenergy $\epsilon$. The excitation energies related to change charge transfer between the even and the odd subspaces define four ABS: $E_A^{++} \equiv \epsilon_+ - \epsilon = \epsilon_A + \frac{1}{2}U$, $E_A^{--} \equiv \epsilon - \epsilon = -\epsilon_A + \frac{1}{2}U$, $E_A^{-+} \equiv \epsilon - \epsilon = -\epsilon_A - \frac{1}{2}U$, and $E_A^{+-} \equiv \epsilon_+ - \epsilon = -\epsilon_A - \frac{1}{2}U$ [47]. They are presented in figure 1(c) as dashed curves.

2.2. Cooper pair splitter with 2QD

Similarly, one can write the Hamiltonian for the 2QD splitter [42]

$$H_{2QD}^{\alpha} = \sum_{\alpha=L,R} \sum_{\sigma=L,R} c_{\alpha \sigma} d_{\alpha \sigma}^\dagger d_{\sigma \alpha} + U_{LR} \sum_{\sigma=L,R} n_{\alpha \sigma} n_{\sigma \alpha} + \frac{\Gamma_{SLR}}{2} (d_{LR}^\dagger d_{LR}^\dagger d_{\alpha \sigma}^\dagger d_{\sigma \alpha} + \text{h.c.}),$$

(3)

where $\alpha \in \{L, R\}$ stands for the $\alpha$ dot, the inter-dot Coulomb interaction is $U_{LR}$ and $\Gamma_{SLR}$ denotes the inter-dot coupling through the S electrode. Now, the singlet eigenstates are a coherent superposition $|\pm\rangle = \alpha_2|00\rangle \mp \alpha_2|S\rangle$, where $|00\rangle$ is the empty state and $|S\rangle = 2^{-1/2}(d_{LR}^\dagger d_{LR} - d_{LR}^\dagger d_{LR}^\dagger)|00\rangle$ is the inter-dot singlet. Their eigenvalues are $\epsilon_{\alpha 2} = \frac{1}{2}U_{LR} \pm \epsilon_{\alpha 2a}$, where $\epsilon_{\alpha 2a} = \frac{1}{2}(\epsilon_{LR}^2 + 2\epsilon_{SLR}^2)^{1/2}$, $\delta_{LR} = \epsilon_{LR} - \epsilon_\alpha$ + $U_{LR}$ and $\alpha_{\sigma 2} = \frac{1}{2}(2 \pm \delta_{LR}/\epsilon_{\alpha 2a})^{1/2}$. For the symmetric case (with the dot energies $\epsilon_{LR} = \epsilon_{\alpha 2a}$), the excitation energies related to the ABS: $E_A^{++} \equiv \epsilon_2 - \epsilon = \epsilon_2 + \frac{1}{2}U_{LR}$, $E_A^{--} \equiv \epsilon - \epsilon = -\epsilon_2 + \frac{1}{2}U_{LR}$, $E_A^{-+} \equiv \epsilon - \epsilon = -\epsilon_2 - \frac{1}{2}U_{LR}$, and $E_A^{+-} \equiv \epsilon_2 - \epsilon = -\epsilon_2 + \frac{1}{2}U_{LR}$. They are plotted as solid curves in figure 1(c). Here, we confine our considerations to the inter-dot singlet, ignoring inter-dot triplets and much higher in energy intra-dot pairing (i.e. the intra-dot Coulomb interactions are assumed as $U_{LR}, U_{R} \to \infty$). The inter-dot triplet states are important only for $\mu > (\delta_{LR} + U_{LR})/2$, when two electrons with the same spin can independently enter the QDs from the normal leads [42]. In the CPS configuration shown in figure 1(b) such tunneling processes are forbidden.

One can notice that in the 1QD (2QD) system for $|\delta| = 1$ (|$|\delta_{LR}| = 1$) the ABS $E_{A}^{++} (E_{A}^{--})$ and $E_{A}^{-+} (E_{A}^{+-})$ cross each other, indicating on the quantum phase transition [47]. In the rest of the paper we will focus on $|\delta| < 1$ region, with electron $E_{\alpha}^{++}, E_{\alpha}^{--} (E_{\alpha}^{+-}, E_{\alpha}^{-+})$ and hole $E_{\alpha}^{-+}, E_{\alpha}^{+-} (E_{\alpha}^{++}, E_{\alpha}^{--})$ ABS, see figure 1(c).

The last two terms of the Hamiltonian (1) describe the normal electrodes $H_d = \sum_{\alpha,L,R} (\epsilon_{\alpha k} - \mu_0) c_{\alpha k}^\dagger c_{\alpha k}$, and their coupling with 1QD or 2QD systems, respectively $H_{d-1QD} = \sum_{\alpha,\sigma} (\epsilon_{\alpha k} c_{\alpha k \sigma} d_{\sigma \alpha} + \text{h.c.})$ or $H_{d-2QD} = \sum_{\alpha,\sigma} (\epsilon_{\alpha k} c_{\alpha k \sigma} d_{\sigma \alpha} + \text{h.c.})$. $c_{\alpha k}^\dagger c_{\alpha k}$ creates (annihilates) an itinerant electron with momentum $k$ and spin $\sigma = \{\uparrow, \downarrow\}$ in the electrode $\alpha \in \{L, R\}$, $\mu_0$ denotes chemical potential and $\epsilon_{\alpha k}$ is the hopping integral between QD and $\alpha$ electrode. Assuming the wide band approximation we introduce an effective tunneling rates $\Gamma_{\alpha} = 2\pi |\epsilon_{\alpha k}|^2 \sum k \delta(E - \epsilon_{\alpha k}) = 2\pi |\epsilon_{\alpha k}|^2 \rho_0$, which describe an electron (or hole) transfer between QD and the $\alpha$ electrode, where $\rho_0$ is a density of states in the normal state.

3. Derivations of current and noise

In this section we present the method of calculation of the currents and their correlations. The method is known and the reader familiar with it can skip to the next section with the results.

In the weak coupling limit with normal electrodes $\Gamma_{L,R} \ll k_B T \ll \Gamma_5$ transport properties of our nanostructure are dominated by sequential tunneling processes through the proximized QD. Calculations are carried out using a diagonalized master equation (DME) in the basis of eigenstates of the effective Hamiltonian (2) and (3). In the literature the generalized master equation (GME) is often used to studies of current correlations, see e.g. [43, 44, 48, 49]. GME assumes that frequencies of internal and external transfers are comparable and the method works in the local basis of site states. The equation of motion contains the term describing coherent evolution of the density matrix, which leads to the Rabi oscillations. In principle, a choice of the basis of states should not play a role, in particular for the stationary transport. Dissipation in the both approaches is given by the Lindblad form of the couplings with the electrodes, which are different [50]. We have checked that for $U = 0$ DME gives correct results with our earlier calculations [51] by means of non-equilibrium Green functions for coherent transport in the hybrid system. We claim that for the considered system of the strongly proximized QD ($\Gamma_5 \gg \Gamma_{L,R}$) with the well established (long-lived) ABS, the DME approach gives better insight into transport and its dynamics.
Tunneling processes through the considered systems are described by the DME

\[ \mathbf{p} = \mathbf{M}_p. \]  

(4)

For the 1QD splitter the probabilities \( \mathbf{p} = [p_1, p_-, p_+]^T \) refer to the single electron occupancy of QD, \( p_1 = p_1^+ + p_1^- \), while probabilities of the eigenstates \( |\rangle \) and \( |+\rangle \) are denoted by \( p_- \) and \( p_+ \), respectively. The evolution matrix \( \mathbf{M} \) has the form

\[
\mathbf{M} = \begin{pmatrix}
-\Gamma_{1-} - \Gamma_{1+} & 2\Gamma_{1-} & 2\Gamma_{1+} \\
\Gamma_{1-} & -2\Gamma_{1-} & 0 \\
\Gamma_{1+} & 0 & -2\Gamma_{1+}
\end{pmatrix},
\]

(5)

where \( \Gamma_{1-} = \sum_{\alpha=L,R} (\Gamma^+_{\alpha+} + \Gamma^-_{\alpha-}), \Gamma_{1+} = \sum_{\alpha=L,R} (\Gamma^+_{\alpha+} + \Gamma^+_{\alpha-}), \Gamma_{1-} = \sum_{\alpha=L,R} (\Gamma^+_{\alpha+} + \Gamma^-_{\alpha-}) \) and \( \Gamma_{1+} = \sum_{\alpha=L,R} (\Gamma^+_{\alpha+} + \Gamma^+_{\alpha-}) \). The effective tunneling rates which account for transfer of one electron \( (E^+_{\alpha} > \mu_0) \) or hole \( (E^-_{\alpha} < \mu_0) \) to \( (+) \) or from \( (-) \) the QD through \( \alpha \in \{L, R\} \) tunnel junction are denoted as \( \Gamma^+_{\alpha+} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0), \Gamma^-_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^-_{\alpha} - \mu_0) \) and \( \Gamma_{\alpha+} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0) \), where \( \sum_{\alpha=L,R} \Gamma_{\alpha} \) is the net tunneling rate and \( f(E) = [1 + \exp (E/k_B T)]^{-1} \) is the Fermi–Dirac distribution function.

Contributions to current \( I_{\alpha} \) from the ABS can be expressed as

\[
J_{\alpha}^+ = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0), \]  

(6)

\[
J_{\alpha}^- = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^-_{\alpha} - \mu_0), \]  

(7)

\[
J_{\alpha}^0 = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0), \]  

(8)

\[
J_{\alpha}^\pm = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^\pm_{\alpha} - \mu_0), \]  

(9)

where \( \mathbf{p}^0 = [p^0_1, p^0_-, p^0_+]^T \) denotes probability in the stationary state. These currents describe transport through the electron \( (E^+_{\alpha} > \mu_0) \) or hole \( (E^-_{\alpha} < \mu_0) \) ABS, see figure 1(c).

Similarly, for the 2QD splitter, \( \mathbf{p} = [p_1, p_R, p_-, p_+]^T \), and \( p_1 = p_{1L} + p_{1R} (p_R = p_{R_L} + p_{R_R}) \) denote single electron occupancy of the L(R) QD, while \( p_- \) and \( p_+ \) stand for probabilities of the eigenstates \( |\rangle \) and \( |+\rangle \). The corresponding evolution matrix \( \mathbf{M} \) has the form

\[
\mathbf{M} = \begin{pmatrix}
-\Gamma_{1-} - \Gamma_{1+} & 0 & 2\Gamma_{1-} & 2\Gamma_{1+} \\
0 & -\Gamma_{1-} - \Gamma_{1+} & 2\Gamma_{1-} & 2\Gamma_{1+} \\
\Gamma_{1-} & \Gamma_{1+} & -2\Gamma_{1-} - 2\Gamma_{1+} & 0 \\
\Gamma_{1+} & \Gamma_{1-} & 0 & -2\Gamma_{1-} - 2\Gamma_{1+}
\end{pmatrix},
\]

(10)

where \( \Gamma_{1-} = \Gamma_{1+} = \Gamma_{1-} + \Gamma_{1+} = \Gamma\), \( \Gamma_{L} = \Gamma_{L} + \Gamma_{L} + \Gamma_{L} + \Gamma_{L} = \Gamma\), \( \Gamma_{R} = \Gamma_{R} + \Gamma_{R} + \Gamma_{R} + \Gamma_{R} = \Gamma\), \( \Gamma_{L} = \Gamma_{L} + \Gamma_{L} + \Gamma_{L} + \Gamma_{L} = \Gamma\), \( \Gamma_{L} = \Gamma_{L} + \Gamma_{L} + \Gamma_{L} + \Gamma_{L} = \Gamma\), \( \Gamma_{R} = \Gamma_{R} + \Gamma_{R} + \Gamma_{R} + \Gamma_{R} = \Gamma\). The superscript \( \alpha \), where \( \alpha = \{L, R\} \) and \( i = \{|+, -+, ++, ++, -\} \) is related to charge transfer between the eigenstates \( |\rangle \), \( |+\rangle \), and the single electron states \( |\uparrow\rangle \), \( |\downarrow\rangle \) on \( \alpha \) QD. The effective tunneling rates which account for transfer of one electron \( (E^+_{\alpha} > \mu_0) \) or hole \( (E^-_{\alpha} < \mu_0) \) to \( (+) \) or from \( (-) \) the QD through L or R tunnel junction are denoted as \( \Gamma^+_{\alpha+} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0), \Gamma^+_{\alpha+} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^-_{\alpha} - \mu_0) \), \( \Gamma^+_{\alpha+} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0) \), where \( \alpha \neq \beta \).

One can derive the electron and hole components of the current \( I_{\alpha} \) through the ABS as

\[
J_{\alpha}^+ = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^+_{\alpha} - \mu_0), \]  

(11)

\[
J_{\alpha}^- = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^-_{\alpha} - \mu_0), \]  

(12)

\[
J_{\alpha}^0 = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^\pm_{\alpha} - \mu_0), \]  

(13)

\[
J_{\alpha}^\pm = \Gamma^+_{\alpha+} - \Gamma^+_{\alpha-} = \alpha^2 \Gamma_{\alpha} \Gamma_{\alpha} f(\pm E^\pm_{\alpha} - \mu_0), \]  

(14)

where \( \mathbf{p}^0 = [p^0_1, p^0_R, p^0_-, p^0_+]^T \) stands for probability in the stationary state.

We are interested in dynamical fluctuations of the current from its averaged value, \( \Delta J_{\alpha}(t) = J_{\alpha}(t) - \langle J_{\alpha}(t) \rangle \). To describe the current fluctuations in the contact \( \alpha \) and \( \beta \) we use the time correlation function [52]

\[
S_{\alpha\beta}(t,t') \equiv \frac{1}{2} \langle \Delta J_{\alpha}(t) \Delta J_{\beta}(t') + \Delta J_{\beta}(t') \Delta J_{\alpha}(t) \rangle.
\]

(15)
For the case with time-independent external fields the correlation function is a function of \( \tau = t' - t \), and its Fourier transform can be expressed as

\[
S_{\alpha\beta}(\omega) = 2 \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} S_{\alpha\beta}(\tau),
\]

which is dubbed noise power.

To determine the current correlation functions for the considered hybrid system we use the generation-recombination approach [53], which applies the regression theorem [54] and the method developed for spinless electron noise in a single electron transistor [55], adapting it to our case with the multi-channel and multi-charge tunneling processes. According to this procedure, the correlation function between the currents flowing through \( \alpha \) and \( \beta \) junctions can be expressed as

\[
S_{\alpha\beta}(\omega) = \sum_{i<j} S_{\alpha j}^{ij}(\omega) S_{\gamma i}^{\beta j}(\omega).
\]

Here, we specified the contributions \( S_{\alpha j}^{ij}(\omega) \) to the correlation function originating from the currents \( j_j^\alpha \) and \( j_j^\beta \) through various ABS. They are formally defined by

\[
S_{\alpha j}^{ij}(\omega) = \delta_{\alpha j} S_{\alpha j}^{\text{Sch}}(\omega) + S_{\alpha j}^{ij}(\omega),
\]

where \( S_{\alpha j}^{\text{Sch}} = 2e^{i(\epsilon_j^\alpha + L_j^\alpha - \epsilon_j^\alpha)} \) is the frequency independent term (calculated in the high frequency limit \( \omega \to \infty \)) of the shot noise, dubbed the Schottky noise. The frequency-dependent part is given by

\[
S_{\alpha j}^{ij}(\omega) = 2e^2 \sum_{m,n} \left[ M_{mn}^{\alpha j} G_{mn}(\omega) M_{mn}^{i j} p_{m}^0 + M_{mn}^{\alpha j} G_{mn}(-\omega) M_{mn}^{i j} p_{m}^0 \right],
\]

where \( G_{mn}(\omega) = (i\omega \mathbf{1} - \mathbf{M})^{-1}_{mn} - p_{m}^0/\omega \) is the matrix Green’s function of the proximized QD. For a compact presentation we have introduced the counting fields \( \lambda_{ij} \) into the off-diagonal elements \( M_{mn}^{\alpha j} \) and use in (19) their derivatives \( \lambda_{ij}^{\alpha j} = (\partial M_{mn}^{\alpha j} / \partial \lambda_{ij})|_{\lambda_{ij}=0} \), which are related to the currents contributed through the \( \alpha \) junction and the \( i \) ABS (see also [56–58] where the frequency dependent FCS was presented and which is equivalent to the shot noise calculations [55]).

4. Zero-frequency current correlation analysis

We consider the CPS configuration when a positive bias voltage (\( V_L = V_R = V \)) is applied to the N electrodes, so the chemical potential \( \mu \equiv \mu_L = \mu_R = eV < 0 \) (where the electron charge \( e < 0 \)), and \( \mu_S = 0 \) in the S (source) electrode. The Cooper pairs can be ejected from the proximized 1QD (2QD) system as entangled pairs of electrons into one of the N (drain) electrodes or can be split between them. The preceding process occurs due to the local DAR and can be seen in correlations between tunneling currents flowing through the same branch of the system (L or R). The latter process is due to the non-local thermal broadening of the Fermi distribution functions is small, backscattering tunneling processes can be neglected and one can only consider electron transport (from the S to the QD and further to the N electrodes). In this situation there are no single electron currents flowing between the N electrodes and

\[
\text{ABS and corresponding charge fluctuations can be omitted as irrelevant for entanglement studies (but they can be important for electron bunching in the } N \text{ normal metallic splitter [49]).}
\]

To avoid influence of thermal energy, we also limit ourselves to low-temperatures \( k_B T \ll \Gamma, \Gamma_{SLR} \) later in the paper. Under this assumption, the Fermi–Dirac distribution functions for tunneling processes from the N electrode to the QD vanish in the regions between ABS, see figure 1(c). As a result, currents are unidirectional and their correlations can be expressed using simple analytical formulas.

We will consider two bias voltage regimes. For small bias voltage \( \mu = -0.5 \) only inner ABS \( E_{A}^{\gamma,\gamma}, E_{A}^{\gamma,\gamma} (E_{A}^{\gamma,\gamma}, E_{A}^{\gamma,\gamma}) \) of the 1QD (2QD) system are in transport window, see figures 1(c) and (d). For a large bias voltage \( \mu = -1.5 \) also outer ABS \( E_{A}^{\gamma,\gamma}, E_{A}^{\gamma,\gamma} (E_{A}^{\gamma,\gamma}, E_{A}^{\gamma,\gamma}) \) are available for split electrons.

4.1. Small bias voltage

We start our analysis from the 1QD system for a small bias, \( \mu_R = \mu_L = eV = -0.5 \), when only two inner ABS \( E_{A}^{\gamma,\gamma}, E_{A}^{\gamma,\gamma} \) are available for transport, see figure 1(c). When the bias voltage does not match ABS and the thermal broadening of the Fermi distribution functions is small, backscattering tunneling processes can be neglected and one can only consider electron transport (from the S to the QD and further to the N electrodes). In this situation there are no single electron currents flowing between the N electrodes and
transport occurs only due to DAR and CAR processes. It means that tunneling to the normal electrode of an electron with a spin $\sigma = \uparrow$ is accompanied by tunneling of a hole with a spin $\sigma = \downarrow$ from the same or different electrodes, but the electron and the hole cannot be backscattered. For this case one finds simple analytical formulas for the total current

$$J_L = \frac{4e\alpha^2 \alpha^2_+ \Gamma}{\alpha^2_+ + 1}, \quad (20)$$

which is a sum of the partial electron and hole currents $J^{\uparrow+}_L = J^{-\downarrow+}_L = 2e\alpha^2 \alpha^2_+ \Gamma / (\alpha^2_+ + 1)$ through the ABS $E_{\downarrow}^{\uparrow-}$ and $E_{\uparrow}^\downarrow$. Similarly, for the 2QD splitter one finds

$$J_L = \frac{2e\alpha^2 \alpha^2_+ \Gamma}{\alpha^2_+ + 1}, \quad (21)$$

with its partial currents $J^{\uparrow+}_L = J^{\downarrow+}_L = e\alpha^2 \alpha^2_+ \Gamma / (\alpha^2_+ + 1)$ through the hole and electron ABS $E^{\uparrow+}_L$ and $E^{\downarrow+}_A$, respectively. The plots of the currents are presented in figures 2(a) and (e). The current maximum is shifted from the e–h symmetry point $\delta = 0$, see also [33, 35, 44, 59]. This effect is related to the occupation probability of the $|-\rangle$ eigenstate, which is an asymmetric function of $\delta$: $p_{-\downarrow}(\delta) = \alpha^2_+ / (\alpha^2_+ + 1)$. Note that for large bias voltages the symmetry around $\delta = 0$ is recovered when the $|+\rangle$ eigenstate is activated because $p_{\downarrow}(\pm \delta) = p_{\downarrow}(-\delta)$ and $p_{\downarrow}(\delta)$. 

Figure 2. (a), (e) Currents $J_L / \Gamma$; (b), (f) zero-frequency autocorrelations $S_{LL} / \Gamma$; (c), (g) cross-correlations $S_{LR} / \Gamma$ with their components; (d), (h) Fano factors $F_{LL} = S_{LL} / (2eJ_L)$, $F_{LR} = S_{LR} / (2|\langle \sqrt{\Gamma} J_L \rangle|)$ for an unidirectional transport in a small bias voltage regime ($\mu = -0.5 < \mu < 0$) plotted as a function of detuning $\delta$ and $\delta_{LR}$ for the 1QD and the 2QD splitter, the left and the right column, respectively. We assumed symmetric couplings $\Gamma_R = \Gamma_L = \Gamma = 0.002$. The other parameters are the same as in figure 1(c).
Figure 2 presents the zero-frequency of the current auto-correlation and the cross-correlation functions, $S_{1L}(0)$ and $S_{1R}(0)$, as well as the Fano factors, $F_{1L} \equiv S_{1L}(0)/\langle I_{1L}^2 \rangle$ and $F_{1R} \equiv S_{1R}(0)/\langle I_{1R}^2 \rangle$. The frequency independent Schottky term $S_{Sch}^{\text{Sch}} = 2eJ_l$ is an important component of the autocorrelation function $S_{1L}(0)$, which describes the shot noise of the uncorrelated transfers of particles with a Poissonian distribution of time intervals between transfer events [52]. Deviation of $S_{1L}(0)$ from the Schottky value indicates on bunching or antibunching of tunneling events, when $S_{1L}(0) > S_{Sch}^{\text{Sch}}$ or $S_{1L}(0) < S_{Sch}^{\text{Sch}}$, respectively. Figure 2(b) shows $S_{1L}(0)$ for the 1QD splitter with its all components for the various ABS. Correlations in transport between electrons or holes flowing through the same ABS ($E_{A}^L$ or $E_{A}^R$) are represented by $S_{1L}^{L++}(0) = S_{1L}^{L++}(0)$. The inter-level e–h correlations $S_{1L}^{L++}(0)$ are positive, what is in contrast to the normal metallic Y-splitter, where inter-level e–e correlations are negative [49]. These inter-level e–h correlations occur due to Andreev reflections, so signature of bunching between tunneling particles are seen as positive $S_{1L}^{L++}(0)$. As a result the current noise is super-Poissonian, which is confirmed by the Fano factor $F_{1L} > 1$, see figure 2(d). For the 2QD splitter the autocorrelation function $S_{1L}(0)$ is different, see figure 2(f). Its intra-level e–e and h–h current correlations $S_{1L}^{L+-\delta}(0) = S_{1L}^{L+-\delta}(0)$ are positive which indicate bunching between tunneling processes, while inter-level correlations between electron and hole currents are antibunched, so $S_{1L}^{L-\delta\delta}(0) = S_{1L}^{L-\delta\delta}(0) < 0$. Such behaviour leads to the reduction of the Fano factor to the Poissonian value, $F_{1L} = 1$, see figure 2(h).

The cross-correlations $S_{1L}(0)$ between currents flowing to different electrodes are presented in figures 2(c) and (g). Their dependencies are different for the 1QD and 2QD system. For 1QD the intra-level e–e and h–h correlations $S_{1L}^{L++}(0) = S_{1L}^{L++}(0)$ are negative and compensate the positive inter-level e–h correlations $S_{1L}^{L-\delta\delta}(0)$, leading to $S_{1L}(0) = 0$ at $\delta = 1/\sqrt{2}$ (i.e. for $\alpha^2 = 2\alpha^2$). The crossed tunneling processes are bunched, so the Fano factor $F_{1L}$ is positive. For the 2QD splitter the crossed correlations are exactly equal to the autocorrelations, $S_{1L}(0) = S_{1L}(0)$. This result shows the current conservation for the CAR processes and that displacement currents are absent [60]; thus, the split particles are perfectly entangled. The similar features can be seen for the partial correlations. The components of $S_{1L}(0) = S_{1L}(0)$ are pairwise equal: $S_{1L}^{L++\delta}(0) = S_{1L}^{L++\delta}(0) = S_{1L}^{L+\delta\delta}(0)$, which are negative, and $S_{1L}^{L-\delta\delta}(0) = S_{1L}^{L-\delta\delta}(0) = S_{1L}^{L+\delta\delta}(0)$, which are positive. The Fano factor is in the sub-Poissonian regime and is reduced to $F_{1L} = F_{1R} = 1/2$ at $\delta_{LR} = \Gamma_{LR}/2$, which indicates on strong correlations in the system. Notice that for the total current injected to the N electrodes the power noise $S^{nL}(0) = S_{1L}(0) + 2S_{1R}(0) + S_{RR}(0)$ and the Fano factor is twice larger. These partial correlation functions indicate the crossed Andreev processes, injection of an electron into the L electrode is accompanied by a hole ejection from the R electrode and these reflection processes are highly correlated as it is exhibited in figure 1(d). Similar characteristics will be apparent for higher bias voltages, with all ABS available for transport.

4.2. Large bias voltage

Let us consider the case with a large bias voltage, when all ABS participate in transport. Still the unidirectional currents are assumed. The problem is more complex, but now, there is an e–h symmetry (in contrast to the small voltage regime).

The results are presented in figure 3. For the 1QD splitter the current is expressed as

$$J_l = 4e\alpha^2_2 \alpha^2_s \Gamma,$$  (22)

where its electron and hole components $J_{l+} = J_{l+} = 2e\alpha^2_4 \alpha^2_s \Gamma$ and $J_{l-} = J_{l-} = 2e\alpha^4 \alpha^2_s \Gamma$ are equal for transport through the inner ($E_{A}^L$, $E_{A}^R$) and the outer ($E_{A}^L$, $E_{A}^R$) ABS, respectively. For the 2QD splitter we have

$$J_l = 2e\alpha^2_2 \alpha^2_s \Gamma,$$  (23)

where its electron and hole components are: $J_{l+} = J_{l+} = e\alpha^2_4 \alpha^2_s \Gamma$ and $J_{l-} = J_{l-} = e\alpha^4 \alpha^2_s \Gamma$ for transport through the inner ($E_{A}^L$, $E_{A}^R$) and the outer ($E_{A}^L$, $E_{A}^R$) states.

The auto and the cross-correlation functions, $S_{1L}(0)$ and $S_{1R}(0)$, show the same features as for the small voltage regime. For the 1QD splitter all components of $S_{1L}(0)$, the intra- and the inter-level correlations separately for the inner or the outer ABS, are positive, except for $S_{1L}^{++}(0) = S_{1L}^{++}(0) = S_{1L}^{--}(0) = S_{1L}^{+-\delta}(0) = S_{1L}^{+-\delta}(0) = S_{1L}^{++}(0) < 0$ for some $\delta$. Note that these e–h correlations occur between particles tunneling through inner and outer ABS. The current cross-correlation $S_{1L}(0)$ is always positive, with a minimum $S_{1L}(0) = 0$ at the e–h symmetry point $\delta = 0$. Its inter-level e–h components $S_{1L}^{++}(0) = S_{1L}^{++}(0), S_{1L}^{+-\delta}(0) = S_{1L}^{+-\delta}(0)$ are positive, the intra-level e–e and h–h components $S_{1L}^{++}(0) = S_{1L}^{++}(0), S_{1L}^{+-\delta}(0) = S_{1L}^{+-\delta}(0)$ as well as e–h components between inner and outer ABS $S_{1L}^{++}(0) = S_{1L}^{++}(0), S_{1L}^{+-\delta}(0) = S_{1L}^{+-\delta}(0)$ can be negative. These current correlations show that the Cooper pairs are
injected to the left or to the right electrode and there is bunching between these tunneling events, what is confirmed by the Fano factor.

For the 2QD splitter the auto- and the cross-correlations are equal, $S_{LL}(0) = S_{LR}(0)$, similarly as for the small voltage case. The analysis of all components shows that many of them are equal:

$$
S_{LL}^{-,L++}(0) = S_{LR}^{R--}(0), S_{LR}^{+,R++}(0) = S_{LL}^{-,L--}(0), S_{LL}^{+,R++}(0) = S_{LR}^{R--}(0), S_{LL}^{-,L+-}(0) = S_{LR}^{R++}(0),
$$

and similarly for the cross-correlation components:

$$
S_{LL}^{+,L--}(0) = S_{LR}^{R--}(0), S_{LR}^{+,R++}(0) = S_{LL}^{+,R--}(0), S_{LL}^{-,L++}(0) = S_{LR}^{R--}(0), S_{LL}^{-,L++}(0) = S_{LR}^{R++}(0).
$$

Many of the components cancel each others and one gets

$$
S_{LL}(0) = 2S_{LL}^{-,L++}(0) + 2S_{LL}^{+,R++}(0) + 4S_{LL}^{+,L++}(0) = S_{LR}(0) = 2S_{LR}^{+,L++}(0) + 2S_{LR}^{R++}(0) + 4S_{LR}^{+,R--}(0).
$$

The $e$–$e$ and $h$–$h$ tunneling processes through inner and outer ABS contribute to $S_{LL}(0)$, which are equal to the contributions of the $e$–$h$ tunneling processes to $S_{LR}(0)$. These results show, similarly as in the small bias voltage case, that displacement currents are absent and there is perfect splitting of electrons originating from inter-dot singlet due to CAR processes.

Figure 3. Plots for a large voltage window ($\mu = -1.5 < \mu_s = 0$) for: (a), (e) currents $J_L / \Gamma$; (b), (f) zero-frequency auto correlations $S_{LL} / \Gamma$; (c), (g) cross correlations $S_{LR} / \Gamma$ with their components; (d), (h) Fano factors $F_{LL}$, $F_{LR}$. The other parameters are the same as in figure 2.
The components of the current correlation functions are: $Un$.

We extended the 2QD model and added to the Hamiltonian (3) the term with intra-dot Coulomb interactions: $U_{n1} n_{a1} + U_{n2} n_{a2}$. The current and its correlations were derived for a large voltage when all ABS, with the inter-dot and the intra-dot singlets, are in the transport window. The results are presented in figure 4 as a function of the dot energy $\epsilon \equiv \epsilon_u = \epsilon_r$. The current shows two peaks: at $\epsilon = -U_{LR}/2$ and $\epsilon = -U/2$, when the inter-dot singlet and the intra-dot singlet become activated, respectively. At the first current peak, where the inter-dot singlet plays a role, the autocorrelations $S_{LL}(0)$ and the cross-correlations $S_{LR}(0)$ cover each other. At $\epsilon = -U/2$, the autocorrelation $S_{LL}(0)$ becomes much larger and the cross-correlation $S_{LR}(0)$ shows a negative dip, which indicates the correlated transfer of Cooper pairs through each of the both QDs due to DAR processes. These correlation functions have similar features to those in the 1QD model. In realistic devices, our studies can help distinguish various processes contributing to auto and cross-correlation measurement data and find optimal parameters of the system for efficient splitting.

5. Dynamics in Cooper pair splitter: frequency-dependent studies

Now, we would like to analyse dynamics of the currents and their correlations. In the unidirectional regime one can analytically determine the frequency dependence of the auto and crossed correlation functions $S_{LL}(\omega)$ and $S_{LR}(\omega)$.

5.1. Small bias voltage

For the 1QD splitter case and the small voltage window one gets:

$$S_{LL}(\omega) = \frac{8e^2 \alpha^2 \alpha^2_{+}}{\alpha_{=}^2 + 1} \left[ 1 + \frac{2(2\alpha^2 - \alpha^2_{=})^2 \Gamma^2}{4(\alpha_{=}^2 + 1)^2 \Gamma^2 + \omega^2} \right],$$

$$S_{LR}(\omega) = \frac{8e^2 \alpha^2 \alpha^2_{=} \Gamma}{\alpha_{=}^2 + 1} \left[ \frac{2(2\alpha^2 - \alpha^2_{=})^2 \Gamma^2}{4(\alpha_{=}^2 + 1)^2 \Gamma^2 + \omega^2} \right].$$

The frequency dependent part of $S_{LL}(\omega)$ is exactly equal to $S_{LR}(\omega)$, the difference is due to the Schottky term only. $S_{LL}(\omega) > 0$ at e–h symmetry point, i.e. for $\delta = 0$, which means inter-level bunching between transmitted Cooper pairs. The components of the current correlation functions are:

$$S_{LL}^{+-,-+}(\omega) = S_{LL}^{-+,++}(\omega) = \frac{4e^2 \alpha^2 \alpha^2_{=} \Gamma}{\alpha_{=}^2 + 1} \left[ 1 - \frac{8\alpha^2 \alpha^2_{=} \Gamma^2}{4(\alpha_{=}^2 + 1)^2 \Gamma^2 + \omega^2} \right],$$

$$S_{LR}^{+-,-+}(\omega) = S_{LR}^{-+,++}(\omega) = \frac{4e^2 \alpha^2 \alpha^2_{=} \Gamma}{\alpha_{=}^2 + 1} \left[ \frac{-8\alpha^2 \alpha^2_{=} \Gamma^2}{4(\alpha_{=}^2 + 1)^2 \Gamma^2 + \omega^2} \right].}$$
Equations (27) and (28) describe the intra-level e–e (h–h) correlations, which are typical for a single electron transport, where one can see antibunching (the dynamical part is negative). The e–h correlations, described by the inter-level components (29) are positive, and they are relevant for the total shot noise (25) and (26).

Poles of the current correlation functions describe relaxation to the steady state driven by a generation-recombination processes [53]. In the considered 1QD case all tunneling processes to the N electrodes participate in the relaxation time: \( \tau_{\text{rel}}^{-1} = \Gamma_{L}^{-} + \Gamma_{R}^{+} + 2(\Gamma_{L}^{+} + \Gamma_{R}^{-}) \), which simplifies to \( \tau_{\text{rel}}^{-1} = 2(\alpha^{2} + 1)\Gamma \) for the symmetric coupling. There is not the e–h symmetry, because relaxation from the even and the odd subspace depend on \( 4\alpha^{2} \Gamma \) and \( 2\alpha^{2} \Gamma \), respectively. Because \( \tau_{\text{rel}} \) depend on \( \delta \) one can find relevant tunneling processes participating in relaxation of the system for different gate voltages. For \( \delta < 0 \) the main contribution comes from the relaxation processes \( \Gamma_{L}^{+,-} (\Gamma_{L}^{-,+}) \) and \( \Gamma_{R}^{+,-} (\Gamma_{R}^{-,+}) \) while for \( \delta > 0 \) the relaxation processes \( \Gamma_{L}^{+,-} (\Gamma_{L}^{-,+}) \) and \( \Gamma_{R}^{+,-} (\Gamma_{R}^{-,+}) \) play a dominant role. At \( \delta = 0 \) all tunneling rates contribute equally, as mentioned earlier.

For the 2QD splitter the auto and the cross-correlation functions are

\[
S_{LL}(\omega) = \frac{2e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{\alpha_{L}^{2} + 1} \left[ 2 + \frac{2(\alpha_{L}^{2} - \alpha_{R}^{2})^{2}\Gamma^{2}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} - \frac{\alpha_{L}^{4} \Gamma^{2}}{\alpha_{L}^{2} + 1} \right],
\]

\[
S_{LR}(\omega) = \frac{2e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{\alpha_{L}^{2} + 1} \left[ \frac{2(\alpha_{L}^{2} - \alpha_{R}^{2})^{2}\Gamma^{2}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} + \frac{\alpha_{L}^{4} \Gamma^{2}}{\alpha_{L}^{2} + 1} \right],
\]

where their components

\[
S_{LL}^{+,-,-,+}(\omega) = \frac{2e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{\alpha_{L}^{2} + 1} \left[ 1 - \frac{2\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} \right],
\]

\[
S_{LR}^{+,-,-,+}(\omega) = \frac{2e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{\alpha_{L}^{2} + 1} \left[ \frac{2\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} \right],
\]

\[
S_{LL}^{+,-,-,-}(\omega) = \frac{e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{3}}{\alpha_{L}^{2} + 1} \left[ \frac{4\alpha_{L}^{4} + \alpha_{R}^{4}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} - \frac{\alpha_{L}^{4} \Gamma^{2} + \alpha_{R}^{4} \Gamma^{2}}{\alpha_{L}^{2} + 1} \right],
\]

\[
S_{LR}^{+,-,-,-}(\omega) = \frac{e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{3}}{\alpha_{L}^{2} + 1} \left[ \frac{4\alpha_{L}^{4} + \alpha_{R}^{4}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} + \frac{\alpha_{L}^{4} \Gamma^{2} + \alpha_{R}^{4} \Gamma^{2}}{\alpha_{L}^{2} + 1} \right].
\]

Although \( S_{LL}(0) = S_{LR}(0) \), their frequency dependencies (30) and (31) are different. There are two relaxation frequencies \( \tau_{\text{NN}}^{-1} = (\alpha_{L}^{2} + 1)\Gamma \) and \( \tau_{\text{PP}}^{-1} = \alpha_{L}^{2} \Gamma \). The first term is positive and describes bunching. Notice that the frequency terms with \( \tau_{\text{PP}}^{-1} = \alpha_{L}^{2} \Gamma \) cancel each others in the correlation function \( S_{\text{LL}}(\omega) = 2S_{\text{LL}}(\omega) + S_{\text{LR}}(\omega) \) for the total current. Moreover, the functions \( S_{LL}^{+,-,-,+}(\omega) \) (32) describe e–e and h–h correlations with the frequency \( \tau_{\text{NN}}^{-1} = (\alpha_{L}^{2} + 1)\Gamma \). The same negative term occurs in the cross-correlations \( S_{LR}^{+,-,-,-}(\omega) = \frac{2e^{2}\alpha_{L}^{2} \alpha_{R}^{2} \Gamma^{2}}{\alpha_{L}^{2} + 1} \left[ \frac{2(\alpha_{L}^{2} - \alpha_{R}^{2})^{2}\Gamma^{2}}{(\alpha_{L}^{2} + 1)^{2}\Gamma^{2} + \omega^{2}} + \frac{\alpha_{L}^{4} \Gamma^{2}}{\alpha_{L}^{2} + 1} \right] \) (33). These correlation functions describe antibunching between electrons or holes. Similar relations were found for the current correlations in the systems with the normal electrodes [49, 61, 62]. The e–h correlations (34) and (35) have both relaxation processes. The terms with \( \tau_{\text{NN}}^{-1} = (\alpha_{L}^{2} + 1)\Gamma \) are positive and they are responsible for bunching seen in \( S_{LL}(\omega) \) and \( S_{LR}(\omega) \).

To analyse these relaxation processes, we consider dynamical fluctuations of a local charge operator \( \hat{X} \), for which the correlation function is expressed by

\[
S_{XX}(\omega) = 4 \sum_{m,n} X_{m} G_{mn}(\omega) X_{n} p_{m}^{\alpha},
\]

Here, \( X_{m} \) denotes an eigenvalue of \( \hat{X} \) and \( G_{mn}(\omega) \) are the matrix elements of the Green function defined in section 3. For the considered 2QD model we calculate fluctuations of the total charge \( \hat{N} = \sum_{\alpha = \downarrow, \uparrow} (n_{L\alpha} + n_{R\alpha}) \):
\[ S_{\text{NN}}(\omega) = \frac{1}{1+\alpha_2^2} \frac{32\alpha_2^2 \alpha_4^2 \Gamma}{(1+\alpha_2^2)\Gamma^2 + \omega^2} \]  
and fluctuations of the polarization \( \hat{P} = \sum_{\sigma=\uparrow,\downarrow} (\hat{m}_{\sigma} - \bar{n}_{\sigma}) \) of the charge at the L and R QD:
\[ S_{\text{PP}}(\omega) = \frac{1}{1+\alpha_2^2} \frac{32\alpha_2^2 \alpha_4^2 \Gamma}{\alpha_2^2 \Gamma^2 + \omega^2}. \]

Comparing the poles with those in (30) and (31), one can say that the first frequency dependent term in \( S_{\text{LL}}(\omega) \) and \( S_{\text{LR}}(\omega) \) corresponds relaxation due to the charge fluctuations with the characteristic frequency \( \tau_{\text{NN}}^{-1} = (\alpha_2^2 + 1)\Gamma \). The last terms in \( S_{\text{LL}}(\omega) \) and \( S_{\text{LR}}(\omega) \) are equal but have the opposite sign; they are related to the inter-dot polarization fluctuations with the frequency \( \tau_{\text{PP}}^{-1} = \alpha_2^2 \Gamma \).

In this symmetric case one could recognize various charge fluctuations, but in a general situation, with asymmetric couplings \( \Gamma_{\text{L}} \neq \Gamma_{\text{R}} \), local charge accumulation occurs and relaxation processes are mixed, therefore their frequencies are unable to separate.

### 5.2. Large bias voltage

For the large transport window all ABS participate in fluctuations. Fortunately, for the symmetric case one can derive analytically the frequency dependent \( S_{\text{LL}}(\omega) \) and \( S_{\text{LR}}(\omega) \) and their components, which show two relaxation processes related with charge and polarization fluctuations. For the 1QD splitter one gets
\[ S_{\text{LL}}(\omega) = 8e^2 \alpha_2^2 \alpha_4^2 \Gamma \left[ 1 + \frac{2(\alpha_2^2 - \alpha_4^2)^2 \Gamma^2}{4\Gamma^2 + \omega^2} \right], \]
\[ S_{\text{LR}}(\omega) = 8e^2 \alpha_2^2 \alpha_4^2 \Gamma \left[ \frac{2(\alpha_2^2 - \alpha_4^2)^2 \Gamma^2}{4\Gamma^2 + \omega^2} \right]. \]

At the e–h symmetry point \( \delta = 0 \) (i.e. for \( \alpha_1^2 = \alpha_2^2 = \frac{1}{2} \)) one gets \( S_{\text{LR}}(\omega) = 0 \) and \( S_{\text{LL}}(\omega) = 2e^2 \Gamma \), which is only the Schottky contribution. This is a feature characteristic for independent, uncorrelated transport of single particles [52], but in our system the partial currents are highly correlated. Appendix A contains all components of the current correlations, which have two different relaxation terms: with \( \tau_{\text{NN}}^{-1} = 4\Gamma \) and \( \tau_{\text{PP}}^{-1} = 2\Gamma \). One can attribute them with charge fluctuations \( \hat{N} = \hat{n}_{\uparrow} + \hat{n}_{\downarrow} \) on the levels \( |\pm\rangle \) and inter-level polarization fluctuations \( \hat{P} = \hat{n}_{\uparrow} - \hat{n}_{\downarrow} \). The e–e and h–h current correlation components related with \( \tau_{\text{PP}}^{-1} \) are zero, while the e–h components are nonzero but they cancel out each. When the e–h symmetry is broken, \( \delta \neq 0 \), contributions from e–e and h–h as well as e–h components lead to the super-Poissonian noise with bunching.

Note that for all voltages and any \( \omega \), we have the inter-level components for autocorrelations and cross-correlation exactly the same due to the left-right symmetry \( \text{L} = \text{R} \). In turn, the intra-level components for autocorrelations and cross-correlations differ only in the Schottky term.

For the 2QD splitter situation is different than in the 1QD case, because now the CAR processes through the inter-dot singlet state are relevant. One gets
\[ S_{\text{LL}}(\omega) = 4e^2 \alpha_2^2 \alpha_4^2 \Gamma \left[ 1 - \frac{2\alpha_2^2 \alpha_4^2 \Gamma^2}{\Gamma^2 + \omega^2} \right], \]
which is the sub-Poissonian in contrast to the 1QD case (39). Now the antibunching effect is seen which is related with the e–h inter-level correlations between the inner and outer ABS. The main contribution is due to \( S_{\text{LL}}^{\uparrow +, \uparrow +}(\omega) \), see appendix B. The cross-correlation function
\[ S_{\text{LR}}(\omega) = 4e^2 \alpha_2^4 \alpha_4^2 \Gamma^3 \left[ \alpha_2^2 + \alpha_4^2 \right] \frac{\alpha_2^2 + \alpha_4^2}{\Gamma^2 + \omega^2}. \]

is always positive. Notice that although \( S_{\text{LL}}(0) = S_{\text{LR}}(0) \), their frequency dependences are different. For the crossed correlations \( S_{\text{LR}}(\omega) \) the relevant contributions are due to the e–h CAR within inner and outer states given by
\[
S_{\text{LR}}^{\downarrow -, \uparrow +}(\omega) + S_{\text{LR}}^{\uparrow +, \downarrow -}(\omega) + S_{\text{LR}}^{\uparrow -, \downarrow +}(\omega) + S_{\text{LR}}^{\downarrow -, \uparrow +}(\omega)
= 4e^2 \alpha_2^2 \alpha_4^2 \Gamma^3 \left[ \frac{\alpha_2^2 + 6\alpha_2^4 \alpha_4^2 + \alpha_4^2}{\Gamma^2 + \omega^2} + \frac{4\alpha_2^2 \alpha_4^2 (\alpha_2^4 + \alpha_4^4)}{4\Gamma^2 + \omega^2} \right].
\]  

There are relaxation processes with a characteristic frequency \( \tau_{\text{PP}}^{-1} = \Gamma \) and \( \tau_{\text{NN}}^{-1} = 2\Gamma \), which can be assigned to the inter-dot polarization fluctuations and the charge fluctuations, respectively. Transfer of
entangled electrons to the N electrodes due to CAR processes is not instantaneous but occurs with the relaxation time $T_{PW}$ corresponding to the inter-dot e–h polarization fluctuations. The high frequency current correlations, corresponding to charge fluctuations, are cancelled out in $S_{LR}(\omega)$ (for details see appendix B).

### 6. Final remarks

In the paper we thoroughly studied current correlations and their components in symmetric CPS with one or two identical QDs in the presence of Coulomb interactions. In the 1QD system the influence of interplay between DAR and CAR processes determines transport characteristics. In the 2QD device non-local entangled pairs of electrons can be transmitted due to CAR processes. We considered a small bias voltage case, when only inner ABS are activated and a large voltage regime, with all ABS participating in transport. In the former case charge accumulation is relevant while for the latter one transport characteristics show the e–h symmetry. We found that in both CPS devices the total current cross-correlations $S_{LR}(0)$ are positive due to CAR processes. These correlation functions are composition of various components between partial electron or hole currents flowing through the same or different ABS. In the 1QD splitter positive e–h and negative e–e (or h–h) correlations compensate, so the total cross-correlations vanish for the same bias voltages for which the Fano factor $F_{LL} = 1$, which is characteristic for the uncorrelated Poisson statistics of particle transfers [32]. However, in the considered case, electron and hole partial currents are highly correlated and these features are related with entanglement of split particles, which is maximal when $S_{LR}(0) = 0$. For the 2QD splitter the situation is different, because transport is due to the inter-dot singlet state. We found signatures of perfect entanglement in the current correlations caused by CAR processes, i.e. injection of an electron into one N electrode and ejection of a hole from the second N electrode are correlated.

To get insight into dynamics of split Cooper pairs we analytically calculated frequency-dependent current correlation functions with their components and performed spectral decomposition to extract characteristic relaxation processes. For the 1QD splitter and small bias voltages we found that all tunneling processes (DAR as well as CAR) contribute to only one relaxation time. For large bias voltages there are two relaxation times, assigned with high frequency intra-level (e–e or h–h) charge fluctuations and low frequency inter-level (e–h) polarization fluctuations. Negative and positive terms corresponding with charge fluctuations cancel each others, so in the current cross-correlation $S_{LR}(\omega)$ there are only terms corresponding with inter-level e–h polarization fluctuations. For the 2QD splitter CAR processes within inner and outer ABS play a significant role. These processes, assigned to the inter-dot polarization fluctuations, are responsible for entanglement seen in the positive crossed correlation $S_{LR}(\omega)$. This means that transfer of entangled electrons is not instantaneous but occurs with characteristic relaxation time corresponding to inter-dot polarization fluctuations. Transport through the inter-dot singlet state results on the current autocorrelation $S_{LL}(\omega)$, which is sub-Poissonian indicating antibunching—in contrast to the 1QD splitter, where bunching is seen. Summarizing, we studied how entanglement and splitting manifests itself in various frequency-dependent components of the current correlations.

As far as we know, our theoretical studies can be related only to experiments performed by Das et al. [21] who found positive zero-frequency cross-correlations in the CPS based on 2QD (in the normal state) with moderate Coulomb interaction between them, seems to be a bigger challenge. Only in such the device our results concerning CPS with 2QD could be verified in the entire range of transport voltages. One could also try to explore dynamics of Cooper pair tunneling through measurements and analysis of finite-frequency current statistics, which allow to extract correlations between different spectral components of currents, see [40]. In addition, using ultra-fast techniques of time-resolved spectroscopy it is possible to study system properties under non-stationary conditions, e.g. transport through QDs after applying short-time voltage pulses or other rapid changes of system parameters [64, 65]. We are convinced that in the future, our predictions on the current correlations and the dynamics of entangled electrons can be verified experimentally by means of these time-resolved techniques, which allow for real-time observation how the system returns to equilibrium from various excited states.
Appendix A. Current correlations in 1QD splitter for large bias voltages

Here, we present analytical derivations of current correlations for the 1QD splitter in the large bias voltage. The research was financed by National Science Centre (Poland) under the Project No. 2016/21/B/ST3/02160.

The auto and crossed correlations

\begin{align}
S_{LL}(\omega) &= 8\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ 1 + \frac{2(\alpha^2 - \alpha^2 \Gamma^2)}{4\Gamma^2 + \omega^2} \right], \\
S_{LR}(\omega) &= 8\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{2(\alpha^2 - \alpha^2 \Gamma^2)}{4\Gamma^2 + \omega^2} \right],
\end{align}

have components:

\begin{align}
S_{LL}^{++} + (\omega) &= S_{LL}^{-+} - (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ 1 + \frac{8\alpha^2 \alpha^2 \alpha^2 (\alpha^2 - \alpha^2 \Gamma^2)}{4\Gamma^2 + \omega^2} - \frac{16\alpha^4 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LL}^{-+} - (\omega) &= S_{LL}^{++} + (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ 1 - \frac{8\alpha^2 \alpha^2 \alpha^2 (\alpha^2 - \alpha^2 \Gamma^2)}{4\Gamma^2 + \omega^2} + \frac{16\alpha^4 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LL}^{++} - (\omega) &= S_{LL}^{--} + (\omega) = 8\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{\alpha^2 (4\alpha^2 + (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2)}{4\Gamma^2 + \omega^2} + \frac{4\alpha^2 \alpha^2 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LL}^{--} - (\omega) &= S_{LL}^{++} - (\omega) = 8\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{\alpha^2 (4\alpha^2 + (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2)}{4\Gamma^2 + \omega^2} - \frac{4\alpha^2 \alpha^2 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LR}^{++} + (\omega) &= S_{LR}^{-+} - (\omega) = S_{LR}^{--} + (\omega) = S_{LR}^{+-} - (\omega) = 8\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{\alpha^2 (4\alpha^2 + (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2)}{4\Gamma^2 + \omega^2} - \frac{2(\alpha^2 + \alpha^2 \Gamma^2)}{16\Gamma^2 + \omega^2} \right],
\end{align}

and

\begin{align}
S_{LR}^{++} - (\omega) &= S_{LR}^{--} + (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{8\alpha^2 \alpha^2 \alpha^2 (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2}{4\Gamma^2 + \omega^2} - \frac{16\alpha^4 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LR}^{--} - (\omega) &= S_{LR}^{++} + (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ - \frac{8\alpha^2 \alpha^2 \alpha^2 (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2}{4\Gamma^2 + \omega^2} + \frac{16\alpha^4 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LR}^{+-} + (\omega) &= S_{LR}^{++} - (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{\alpha^2 (4\alpha^2 + (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2)}{4\Gamma^2 + \omega^2} + \frac{4\alpha^2 \alpha^2 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right], \\
S_{LR}^{--} + (\omega) &= S_{LR}^{++} - (\omega) = 4\epsilon^2 \alpha^2 \alpha^4 \Gamma \left[ \frac{\alpha^2 (4\alpha^2 + (\alpha^2 - \alpha^2 \Gamma^2)\Gamma^2)}{4\Gamma^2 + \omega^2} + \frac{4\alpha^2 \alpha^2 \alpha^2 \Gamma^2}{16\Gamma^2 + \omega^2} \right],
\end{align}
The auto-correlation function is expressed as
\[
S_{L+}^{++} = S_{L+}^{-+} = S_{L-}^{-+} = S_{L-}^{++} = \frac{1}{16} \left( 1 - \frac{8\alpha^2 \alpha^6}{4\Gamma^2 + \omega^2} + \frac{8\alpha^2 \alpha^6}{16\Gamma^2 + \omega^2} \right),
\]
(A.13)

The average current is
\[
\alpha^2 \left[ \frac{(\alpha^2 - \alpha^4)^2}{4\Gamma^2 + \omega^2} - \frac{2(\alpha^4 + \alpha^6)}{16\Gamma^2 + \omega^2} \right].
\]
(A.14)

Appendix B. Current correlations in 2QD splitter for large bias voltages

For the 2QD splitter in the large bias voltage regime we assume unidirectional transport and take the transfer rates:
\[
\Gamma_{L-}^{++} = \Gamma_{R-}^{-+} = \frac{\alpha^2 \alpha^2}{4\Gamma^2 + \omega^2} \Gamma, \quad \Gamma_{L-}^{+-} = \Gamma_{R+}^{+-} = \frac{\alpha^2 \alpha^2}{16\Gamma^2 + \omega^2} \Gamma, \quad \Gamma_{L+}^{--} = \Gamma_{R-}^{--} = \frac{\alpha^2 \alpha^2}{\omega^2} \Gamma, \quad \Gamma_{L+}^{++} = \Gamma_{R-}^{-+} = 0 \quad \text{and} \quad \Gamma_{L-}^{--} = \Gamma_{R+}^{++} = \Gamma_{L+}^{++} = \Gamma_{R-}^{-+} = 0.
\]

The average current is
\[
J_L = J_R = 2e\alpha^2 \alpha^2 \Gamma, \quad \text{and its components are:} \quad J_L^{++} = J_R^{-+} = J_R^{-+} = J_R^{++} = 0.
\]

The auto-correlation function is expressed as
\[
S_{L+}(\omega) = 4e^2 \alpha^2 \alpha^2 \Gamma \left( 1 - \frac{2\alpha^2 \alpha^2 \Gamma^2}{\Gamma^2 + \omega^2} \right)
\]
(B.1)

and its components
\[
S_{L+}^{+-}(\omega) = S_{L-}^{-+}(\omega) = 2e^2 \alpha^2 \alpha^2 \Gamma \left[ 1 + \frac{2\alpha^2 \alpha^2 (\alpha^2 - \alpha^4)^2}{\Gamma^2 + \omega^2} - \frac{4\alpha^2 \alpha^2 \Gamma^2}{4\Gamma^2 + \omega^2} \right],
\]
(B.2)
\[
S_{L-}^{+-}(\omega) = S_{L+}^{--}(\omega) = 2e^2 \alpha^2 \alpha^2 \Gamma \left[ 1 - \frac{2\alpha^2 \alpha^2 (\alpha^2 - \alpha^4)^2}{\Gamma^2 + \omega^2} + \frac{4\alpha^2 \alpha^2 \Gamma^2}{4\Gamma^2 + \omega^2} \right],
\]
(B.3)
\[
S_{L+}^{++}(\omega) = S_{L-}^{--}(\omega) = 4e^2 \alpha^2 \alpha^2 \Gamma \left[ \frac{\alpha^2 (\alpha^2 - \alpha^4)^2}{\Gamma^2 + \omega^2} + \frac{\alpha^2 + \alpha^4}{4\Gamma^2 + \omega^2} \right],
\]
(B.4)
\[
S_{L-}^{++}(\omega) = S_{L+}^{--}(\omega) = 4e^2 \alpha^2 \alpha^2 \Gamma \left[ \frac{\alpha^2 (\alpha^2 - \alpha^4)^2}{\Gamma^2 + \omega^2} - \frac{\alpha^2 + \alpha^4}{4\Gamma^2 + \omega^2} \right],
\]
(B.5)
\[
S_{L+}^{+-}(\omega) = S_{L-}^{--}(\omega) = \frac{1}{4\Gamma^2 + \omega^2} + \frac{1}{4\Gamma^2 + \omega^2},
\]
(B.6)
\[
S_{L+}^{+-}(\omega) = S_{L-}^{++}(\omega) = 2e^2 \alpha^2 \alpha^2 \Gamma \left[ \frac{\alpha^2 \alpha^2}{\Gamma^2 + \omega^2} - \frac{2\alpha^2 \alpha^2}{4\Gamma^2 + \omega^2} \right].
\]
(B.7)

Notice, that the components
\[
2S_{L+}^{++}(\omega) + 2S_{L-}^{--}(\omega) + 4S_{L+}^{+-}(\omega) = 0
\]
cancel each other for any \(\omega\).

The cross-correlation function is
\[
S_{LR}(\omega) = 4e^2 \alpha^2 \alpha^2 \alpha^2 \Gamma \alpha^2 \alpha^2 \Gamma \left( \frac{\alpha^2}{\Gamma^2 + \omega^2} \right)
\]
(B.8)

and its components are:
\[
S_{LR}^{RR}(\omega) = S_{LR}^{RL}(\omega) = 4e^2 \alpha^2 \alpha^2 \alpha^2 \Gamma \left( \frac{\alpha^2}{\Gamma^2 + \omega^2} \right),
\]
(B.9)
\[
S_{LR}^{LL}(\omega) = S_{LR}^{RL}(\omega) = 4e^2 \alpha^2 \alpha^2 \alpha^2 \Gamma \left( \frac{\alpha^2}{\Gamma^2 + \omega^2} \right),
\]
(B.10)
\[ S_{LR}^{L-,L-}(\omega) = s_{LR}^{L-,L-}(\omega) = 2e^2\alpha_2^4\alpha_2^4\Gamma_1^3 \left[ \frac{\alpha_2^2(3\alpha_2^2 + \alpha_1^2)}{\Gamma^2 + \omega^2} + \frac{2\omega^2(\alpha_1^2 + \alpha_2^2)}{4\Gamma^2 + \omega^2} \right], \quad (B.11) \]

\[ S_{LR}^{L-,L+}(\omega) = s_{LR}^{L-,L+}(\omega) = 2e^2\alpha_2^4\alpha_2^4\Gamma_1^3 \left[ \frac{\alpha_2^2(\alpha_2^2 + 3\alpha_1^2)}{\Gamma^2 + \omega^2} + \frac{2\omega^2(\alpha_1^2 + \alpha_2^2)}{4\Gamma^2 + \omega^2} \right], \quad (B.12) \]

\[ S_{LR}^{L-,L++}(\omega) = s_{LR}^{L-,L++}(\omega) = 2e^2\alpha_2^4\alpha_2^4\Gamma_1^3 \left[ \frac{(\alpha_2^2 - \alpha_2^2)}{\Gamma^2 + \omega^2} + \frac{2\omega^2(\alpha_1^2 + \alpha_2^2)}{4\Gamma^2 + \omega^2} \right], \quad (B.13) \]

\[ S_{LR}^{L-,L--}(\omega) = s_{LR}^{L-,L--}(\omega) = 2e^2\alpha_2^4\alpha_2^4\Gamma_1^3 \left[ \frac{(\alpha_2^2 - \alpha_2^2)}{\Gamma^2 + \omega^2} - \frac{2\omega^2(\alpha_1^2 + \alpha_2^2)}{4\Gamma^2 + \omega^2} \right]. \quad (B.14) \]

Notice, that the components \(2S_{LR}^{L-,L-}(\omega) + 2S_{LR}^{L-,L+}(\omega) + 4S_{LR}^{L-,L++}(\omega) = 0\) for any \(\omega\).

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