The (p, q)-analogues of some inequalities for the digamma function

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Abstract. In this paper, we present the (p, q)-analogues of some inequalities concerning the digamma function. Our results generalize some earlier results.

1. Introduction and Preliminaries

The classical Euler’s Gamma function, \( \Gamma(t) \) and the digamma function, \( \psi(t) \) are commonly defined as
\[
\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} \, dx \quad \text{and} \quad \psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.
\]
The \( p \)-analogues of the Gamma and digamma functions are respectively defined as follows.
\[
\Gamma_p(t) = \frac{p!}{t(t+1) \cdots (t+p)} \quad \text{and} \quad \psi_p(t) = \frac{d}{dt} \ln \Gamma_p(t) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0.
\]
where \( \lim_{p \to \infty} \Gamma_p(t) = \Gamma(t) \) and \( \lim_{p \to \infty} \psi_p(t) = \psi(t) \). For some more insights and properties of these functions, see [1], [3] and the references therein.

Similarly, the \( q \)-analogues of the Gamma and digamma functions are respectively defined for \( q \in (0, 1) \) as (see also [1] and [3])
\[
\Gamma_q(t) = (1- q)^{1-t} \prod_{n=1}^\infty \frac{1 - q^n}{1 - q^{t+n}} \quad \text{and} \quad \psi_q(t) = \frac{d}{dt} \ln \Gamma_q(t) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \quad t > 0.
\]
where \( \lim_{q \to 1-} \Gamma_q(t) = \Gamma(t) \) and \( \lim_{q \to 1-} \psi_q(t) = \psi(t) \).

In 2012, Krasniqi [2] defined the \( (p, q) \)-analogue of the Gamma function, \( \Gamma_{p,q}(t) \) as
\[
\Gamma_{p,q}(t) = \frac{[p]^t[p]_q!}{[t]_q[t+1]_q \cdots [t+p]_q}, \quad t > 0, \quad p \in \mathbb{N}, \quad q \in (0, 1).
\]
where \( [p]_q = \frac{1-q^p}{1-q} \). For several properties and characteristics of this function, we refer to [4]

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Similarly, the \((p, q)\)-analogue of the digamma function \(\psi_{p,q}(t)\) is defined as
\[
\psi_{p,q}(t) = \frac{d}{dt} \ln \Gamma_{p,q}(t) = \frac{\Gamma'_{p,q}(t)}{\Gamma_{p,q}(t)}, \quad t > 0, \quad p \in N, \quad q \in (0, 1).
\]
The functions \(\psi(t)\) and \(\psi_{p,q}(t)\) as defined above have the following series representations.
\[
\psi(t) = -\gamma + (t - 1) \sum_{n=0}^{\infty} \frac{1}{(1 + n)(n + t)}, \quad t > 0
\]
\[
\psi_{p,q}(t) = \ln[p]_q + (\ln q)^p \sum_{n=1}^{p} \frac{q^{nt}}{1 - q^n}, \quad t > 0.
\]
where \(\gamma\) is the Euler-Mascheroni’s constant.

By taking the \(m\)-th derivative of these functions, it can easily be shown that the following statements are valid for \(m \in N\).
\[
\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n + t)^{m+1}}, \quad t > 0
\]
\[
\psi^{(m)}_{p,q}(t) = (\ln q)^{m+1} \sum_{n=1}^{p} \frac{n^m q^{nt}}{1 - q^n}, \quad t > 0.
\]

In 2011, Sulaiman \([10]\) presented the following results.
\[
\psi(s + t) \geq \psi(s) + \psi(t) \quad (1.1)
\]
for \(t > 0\) and \(0 < s < 1\).
\[
\psi^{(m)}(s + t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \quad (1.2)
\]
for \(s, t > 0\) and for a positive odd integer \(m\).
\[
\psi^{(m)}(s + t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \quad (1.3)
\]
for \(s, t > 0\) and for a positive even integer \(m\).
\[
\psi^{(m)}(s) \psi^{(m)}(t) \geq \left[\psi^{(m)}(s + t)\right]^2 \quad (1.4)
\]
for \(s, t > 0\) and for a positive odd integer \(m\).

Prior to Sulaiman’s results, Mansour and Shabani by using different techniques established similar inequalities for the function \(\psi_{q}(t)\). These can be found in \([5]\).

Our objective in this paper is to establish that the inequalities \((1.1), (1.2), (1.3)\) and \((1.4)\) still hold true for the \((p, q)\)-analogue of the digamma function.

2. Main Results

We now present the results of this paper.

**Theorem 2.1.** Let \(t > 0\), \(0 < s \leq 1\), \(q \in (0, 1)\) and \(p \in N\). Then the following inequality is valid.
\[
\psi_{p,q}(s + t) \geq \psi_{p,q}(s) + \psi_{p,q}(t). \quad (2.1)
\]
Proof. Let $\mu(t) = \psi_{p,q}(s + t) - \psi_{p,q}(s) - \psi_{p,q}(t)$. Then fixing $s$ we have,

$$
\mu'(t) = \psi'_{p,q}(s + t) - \psi'_{p,q}(t) = (\ln q)^2 \sum_{n=1}^{p} \left[ \frac{nq^n(s+t)}{1-q^n} - \frac{nq^nt}{1-q^n} \right] 
$$

That implies $\mu$ is non-increasing. Furthermore,

$$
\lim_{t \to \infty} \mu(t) = \lim_{t \to \infty} [\psi_{p,q}(s + t) - \psi_{p,q}(s) - \psi_{p,q}(t)] 
$$

Therefore $\mu(t) \geq 0$ concluding the proof. \hfill \Box

**Theorem 2.2.** Let $s, t > 0, q \in (0, 1)$ and $p \in N$. Suppose that $m$ is a positive odd integer, then the following inequality is valid.

$$
\psi_{p,q}^{(m)}(s + t) \leq \psi_{p,q}^{(m)}(s) + \psi_{p,q}^{(m)}(t). \tag{2.2}
$$

Proof. Let $\eta(t) = \psi_{p,q}^{(m)}(s + t) - \psi_{p,q}^{(m)}(s) - \psi_{p,q}^{(m)}(t)$. Then fixing $s$ we have,

$$
\eta'(t) = \psi_{p,q}^{(m+1)}(s + t) - \psi_{p,q}^{(m+1)}(t) 
$$

That implies $\eta$ is non-decreasing. Furthermore,

$$
\lim_{t \to \infty} \eta(t) = (\ln q)^{m+1} \lim_{t \to \infty} \sum_{n=1}^{p} \left[ \frac{n^{m+1}q^{n(s+t)}}{1-q^n} - \frac{n^{m+1}q^nt}{1-q^n} \right] 
$$

Therefore $\eta(t) \leq 0$ concluding the proof. \hfill \Box

**Theorem 2.3.** Let $s, t > 0, q \in (0, 1)$ and $p \in N$. Suppose that $m$ is a positive even integer, then the following inequality is valid.

$$
\psi_{p,q}^{(m)}(s + t) \geq \psi_{p,q}^{(m)}(s) + \psi_{p,q}^{(m)}(t). \tag{2.3}
$$
Proof. Let \( \lambda(t) = \psi_p^{(m)}(s + t) - \psi_p^{(m)}(s) - \psi_{p,q}^{(m)}(t) \). Then fixing \( s \) we have,
\[
\lambda'(t) = \psi_p^{(m+1)}(s + t) - \psi_{p,q}^{(m+1)}(t) \\
= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[ \frac{n^{m+1}q^{n(s+t)}}{1-q^n} - \frac{n^{m+1}q^{nt}}{1-q^n} \right] \\
= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[ \frac{n^{m+1}q^{nt}(q^{ns}-1)}{1-q^n} \right] 
\]
That implies \( \lambda \) is non-decreasing. Furthermore,
\[
\lim_{t \to \infty} \lambda(t) = (\ln q)^{m+1} \lim_{t \to \infty} \sum_{n=1}^{\infty} \left[ \frac{n^{m}q^{n(s+t)}}{1-q^n} - \frac{n^{m}q^{ns}}{1-q^n} - \frac{n^{m}q^{nt}}{1-q^n} \right] \\
= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^{m}q^{ns}}{1-q^n} \geq 0. \text{ (since } m \text{ is even)}
\]
Therefore \( \lambda(t) \geq 0 \) concluding the proof. \( \square \)

**Theorem 2.4.** Let \( s, t > 0, \ q \in (0, 1) \text{ and } p \in N. \) Suppose \( m \) is a positive odd integer, then the following inequality holds true.
\[
\psi_p^{(m)}(s)\psi_{p,q}^{(m)}(t) \geq \left[ \psi_p^{(m)}(s + t) \right]^2 \tag{2.4}
\]

**Proof.** We proceed as follows.
\[
\psi_p^{(m)}(s) - \psi_p^{(m)}(s + t) = (\ln q)^{m+1} \sum_{n=1}^{\infty} \left[ \frac{n^{m}q^{ns}}{1-q^n} - \frac{n^{m}q^{ns(s+t)}}{1-q^n} \right] \\
= (\ln q)^{m+1} \sum_{n=1}^{\infty} \left[ \frac{n^{m}q^{ns(1-q^{nt})}}{1-q^n} \right] \geq 0. \text{ (since } m \text{ is odd)}
\]
That implies,
\[
\psi_p^{(m)}(s) \geq \psi_p^{(m)}(s + t) \geq 0.
\]
Similarly we have,
\[
\psi_{p,q}^{(m)}(t) \geq \psi_{p,q}^{(m)}(s + t) \geq 0.
\]
Multiplying these inequalities yields the desired results. Thus,
\[
\psi_p^{(m)}(s)\psi_{p,q}^{(m)}(t) \geq \left[ \psi_p^{(m)}(s + t) \right]^2.
\]
\( \square \)

3. Concluding Remarks

**Remark.** If in inequalities \([2.1], [2.2], [2.3] \text{ and } [2.4] \) we allow \( p \to \infty \text{ as } q \to 1^- \), then we respectively recover the inequalities \([1.1], [1.2], [1.3] \text{ and } [1.4] \). We have thus generalized the earlier results as in \([5] \text{ and } [10] \). The \( k, \ p \text{ and } q \) analogues of \([1.1], [1.2] \text{ and } [1.3] \) can be found in the papers \([7], [8] \text{ and } [9] \). Also, the \((q, k)\)-analogues of \([2.1], [2.2], [2.3] \text{ and } [2.4] \) can be found in \([5] \).

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