Extended instantons generated on the lattice

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We have been able to observe directly extended instantons on the lattice, with a new method that does not require dislocations to measure them, and where we do not perform cooling. We showed, based on the simple Abelian Higgs model in $1+1$ dim., that one can extract the instanton and anti-instanton density and their size, by measuring the topological charge, $Q_v$, on sub-volumes $v$ larger than the instanton sizes, but smaller than the periodic lattice of size $V$. We are working on the generalization for non-abelian models.

1. Introduction

Instantons \cite{1,2} are expected to play an important role in such diverse phenomena as net Baryon number generation, the $U(1)$ problem or even quark confinement. In order to extract their Quantum contribution to the path integral it is desirable to use a lattice formulation. We will propose a method that enables the observation of these topological non trivial configurations, even for relatively smooth configurations that don’t present lattice artifacts called dislocations. (The topological charge is usually only measured on the whole volume $V$ where, as a consequence of periodic boundary conditions (PBC), we can only see a net charge if we have dislocations). Furthermore, we will show that one can extract information on the instanton density $\rho_I$ and size $L_I$, without the need of cooling \cite{3} to suppress the quantum fluctuations. Note that one will still need improved actions \cite{4} or cooling, for $4$ dimensional gauge theories, in order to avoid small instantons comparable with the lattice spacing $a$. If eventually “perfect actions” are found that would avoid dislocations, our method could be very useful to enable the extraction of topological information.

The method \cite{5}, which we present here, consists in measuring the topological charge on sub-volumes $v$ (of $V$), larger than or of the order of the relevant instantons ($I$, $AI$). Let us first assume that one can reach a coupling regime where the gauge configurations are smooth enough so that there are no individual sites where the topological density gets even close to $\pm 1/2$. This can be reached for the abelian Higgs model in $1+1$ dimensions at weak coupling. Then one has safely configurations without dislocations and if we take PBC the total geometric topological charge is exactly zero. This just means that the charge for a number of $I$ and $AI$ has to add to zero.

We will use as our main observable, the probability distribution to get some topological charge $Q$ in sub-volumes $v$, $P_v(Q)$. For very large $v$, this distribution will have lumps peaked at integer values, representing a net number of $I$ minus $AI$ in $v$. As we reduce the volume $v$ to sizes smaller than the instanton, we can only see a distribution peaked at $Q = 0$ as we have just pieces of instantons in $v$ or quantum fluctuations. Interestingly, these are $Q$ fluctuations in regions of stable topological vacua and therefore tend to cancel out over fairly short distances. Due to their topological character the total dispersion $\sigma$, that they produce in $Q$, only grows as $\sqrt{s}$ with $s$ being the free surface. For larger volumes, of the order or somewhat larger than the instantons, we start getting relevant contributions to $P_v(Q)$ from instantons, where the charge adds coherently to $\pm 1$ (if they are fully in $v$) in contrast to the fluctuations.

The idea is then to look at scales $v$, where:

$$a^D \ll v_I < v < V$$

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so that we have mainly suppressed the fluctuations, and can study the ensemble of I and AI with effectively almost free spatial BC.

For the abelian Higgs model on fine lattices, we have found clearly identifiable extended instantons over several lattice spacings, while the charge over the whole V stays always zero. Assuming an almost free ensemble of I and AI, we have been able to extract an instanton average density \( \rho_I \) and size \( L_I \) (also as function of temperature, \( T = 1/N_t \)). The density \( \rho_I \) which can also be extracted from the second moment of \( P_\nu(Q) \) for large \( \nu \), falls steeply at higher \( T \) as the instantons get squeezed in the time direction. We have also studied \([5]\) the evolution of the position of the instantons in Monte Carlo time and found that they get created locally as I - AI pairs and annihilate later when they encounter another pairing possibility.

As for some models it is hard to avoid dislocations, we have analyzed for our case what happens if one takes rougher lattices where dislocations, we have analyzed for our case what happens if one takes rougher lattices where dislocations start to appear. Only the net number of these dislocations, namely places for which \( |\theta_n| > \pi \), can be suppressed by taking very fine lattices.

The previous discussion has been done using the geometrical definition for the topological density \([1,2]\), but it has the disadvantage of not being a proper topological density, not satisfying exactly the second equality in eq.(2) and so it isn’t strictly 0 in \( V \).

The geometrical charge definition, which is the simplest example of the fibre bundle construction used in the non-abelian case \([3,4]\), is

\[
Q_v^G := \frac{1}{2\pi i} \sum_{n \in \nu} \log(U_n).
\]

This \( Q_v^G \) may still violate eq.(2) by an integer amount in cases where the phases in the links of a plaquette add up to more than \( \pm \pi \), this value being reduced by \( \pm 2\pi \), by the log to its principal branch. These lattice artifacts, called dislocations, are the reason for obtaining net charge on a periodic lattice. These artifacts can be avoided on fine enough lattices at least for this model.

In order to keep track of these dislocations we will directly work with the phases \( \theta_{n,\mu} \) of the links as the fundamental variables. For the plaquettes, \( \theta_n = \theta_{n,\zeta} + \theta_{n+\hat{\xi},\zeta} - \theta_{n+\hat{\xi},\zeta} + \theta_{n,\zeta} \). We then define

\[
Q_v := \frac{1}{2\pi i} \sum_{n \in \nu} [\theta_n],
\]

where the brackets shift \( \theta_n \) by an integer multiple of \( 2\pi \) into the interval \( [-\pi, \pi] \). This definition is clearly equivalent to \([3]\), however, using \( \theta_n \) as the fundamental variable we are now able to locate dislocations, namely places for which \( |\theta_n| \geq \pi \). Note that in our Metropolis algorithm we update the \( \theta_{n,\mu} \) by small changes from the old values, otherwise we could trivially generate dislocations by adding \( 2\pi \) to a link. We find that if we choose parameters corresponding to a fine enough lattice, so that the physical instantons are much larger than \( a \), we can avoid obtaining any dislocations at all over the whole Monte Carlo run. Most simulations have been done then around the “scaling region” \([3]\), \( (\lambda = .2, \kappa = .37, \beta > 7) \).

2. Topological Q for Abelian Higgs model

The lattice formulation of the abelian Higgs model in \( 1+1 \) dim., which is one of the simplest allowing non-trivial topology, has the action \( S_n \):

\[
\lambda (\Phi_n^* \Phi_n - 1)^2 - \beta \text{Re}(U_n) - 2\kappa \text{Re}(\Phi_n^* U_{n,\mu} \Phi_{n+\mu})
\]

Here \( \Phi_n \) is the Higgs field, \( U_{n,\mu} = \exp(i\theta_{n,\mu}) \) are the links and \( U_n = \prod_{\mu} U_{n,\mu} \) the 2-d plaquettes.

In the continuum the topological charge in \( v \) is

\[
Q_v = \frac{1}{4\pi} \int d^2 x \epsilon_{\mu\nu} F_{\mu\nu} = \frac{1}{2\pi} \int ds n_\mu A_\mu.
\]

so that on the whole \( V \) it is always zero for PBC.

On the lattice the field-theoretic definition, giving the right naive continuum limit,

\[
Q_v^F := \frac{1}{2\pi} \sum_{n \in \nu} \text{Im}(U_n),
\]

has the disadvantage of not being a proper topological density, not satisfying exactly the second equality in eq.(2) and so it isn’t strictly 0 in \( V \).

Note that in our Metropolis algorithm we update the \( \theta_n \) in the non-abelian case \([6,7]\), is

\[
Q_v^G := \frac{1}{2\pi i} \sum_{n \in \nu} \log(U_n).
\]

This \( Q_v^G \) may still violate eq.(2) by an integer amount in cases where the phases in the links of a plaquette add up to more than \( \pm \pi \), this value being reduced by \( \pm 2\pi \), by the log to its principal branch. These lattice artifacts, called dislocations, are the reason for obtaining net charge on a periodic lattice. These artifacts can be avoided on fine enough lattices at least for this model.

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\[
Q_v := \frac{1}{2\pi i} \sum_{n \in \nu} [\theta_n],
\]

where the brackets shift \( \theta_n \) by an integer multiple of \( 2\pi \) into the interval \( [-\pi, \pi] \). This definition is clearly equivalent to \([3]\), however, using \( \theta_n \) as the fundamental variable we are now able to locate dislocations, namely places for which \( |\theta_n| \geq \pi \). Note that in our Metropolis algorithm we update the \( \theta_{n,\mu} \) by small changes from the old values, otherwise we could trivially generate dislocations by adding \( 2\pi \) to a link. We find that if we choose parameters corresponding to a fine enough lattice, so that the physical instantons are much larger than \( a \), we can avoid obtaining any dislocations at all over the whole Monte Carlo run. Most simulations have been done then around the “scaling region” \([3]\), \( (\lambda = .2, \kappa = .37, \beta > 7) \).
3. Identifying Instantons

In order to be able to identify instantons clearly, we have chosen lattices with spatially very elongated $V$ and with subvolumes $v$ which wrap around the $\tau$ axis in order to suppress the charge fluctuations along that long “boundary”. The finite temperature also serves to get a diluter system of instantons. We took for example $V = 320 \times 6$ and $v = (20 - 160) \times 6$ and could clearly see the creation of $I - A_I$ pairs.

The charge probability distribution $P_{v}(Q)$ is shown in fig. 1, with clear peaks at integer charge. For a quantitative analysis to extract $\rho_I$, $L_I$ and the dispersion in the fluctuations $\sigma$, we developed a simple model of almost noninteracting instantons. Assuming a number density $\rho_I = \rho_{A_I}$, the probability to have $n$ pairs in the total $V$ is

$$p_n(\rho V) = \frac{(\rho V)^{2n}}{n!^2} \frac{1}{N}, \quad (6)$$

where $N$ is the normalization. This $p_n$ has to be folded with the number of ways to get a charge $Q$ in $v$, assuming that the $n$ instantons of size $L_I$ can be located anywhere in $V$. With this distribution $f^v_n(Q)$ in a given sector, we get

$$P_v(Q) = \sum_n p_n f^v_n(Q). \quad (7)$$

This ideal distribution can be smeared by a gaussian with a width $\sigma$ due to quantum fluctuations. The best fit for the parameters in lattice units is $\rho \approx 0.0011$, $L_I \approx 28$ and $\sigma \approx 0.27$. The topological susceptibility can also be modeled for large $V$ being $< Q^2_v > \approx 2 \nu \rho_I$. We have checked compatibility for various $v$, and calculated at some $T$.

4. Beyond optimal conditions

As for some models we cannot reach couplings where we can suppress dislocations, it is interesting to compare for a $\beta = 6.3$, where they just start to appear, the $P_{v_d}(Q)$ for sectors with net number of dislocations (and net instantons) with the $P_v(Q)$ without dislocations with $v = V_d$. The agreement is shown in Table 1, telling us that the net instantons come with the right $S$ and entropy. In this sense the traditional way to measure topology is fine, at least for large instantons.

We also used the field theoretic $Q^F_v$ and found good agreement with $Q^G_v$ for fine lattices. For $SU(2)$ we have observed much smaller fluctuations for $Q^F_v$ if, instead of symmetrizing loops as in Ref.[8], we reorder terms (same $Q$ on full $V$) taking for each hypercube all loops wrapping it.

Table 1

| $n$ | 0     | 1     | 2     | 3     |
|-----|-------|-------|-------|-------|
| with disloc. | .51(4) | .11(1.5) | .005(2) |
| no disloc. | .53(2) | .10(1) | .007(3) |

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