Constructing squeezed states of light with associated Hermite polynomials

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Abstract A new class of states of light is introduced that is complementary to the well-known squeezed states. The construction is based on the general solution of the three-term recurrence relation that arises from the saturation of the Schrödinger inequality for the quadratures of a single-mode quantized electromagnetic field. The new squeezed states are found to be linear superpositions of the photon-number states whose coefficients are determined by the associated Hermite polynomials. These results do not seem to have been noticed before in the literature. As an example, the new class of squeezed states includes superpositions characterized by odd-photon number states only, so they represent the counterpart of the prototypical squeezed-vacuum state which consists entirely of even-photon number states.

1 Introduction

The uncertainty principle represents one of the most distinctive features of quantum mechanics. In contrast to classical theories, this principle denies the possibility of having exact values for simultaneous measurements of canonically conjugated physical variables. Discovered by Heisenberg in 1927 for the position and momentum of an electron “observed” under an idealized microscope [1], this quantum law has undergone improvements, while intense discussion has been developed about its interpretation and meaning [2–5]. Heisenberg’s discovery implies that exact knowledge of the electron position produces a wide spread of its momentum, “the more precisely the position is determined, the less precisely the momentum is known” [1], and vice versa. Although the initial glimpse of this result may be traced back to Dirac and Jordan [3], it was Heisenberg who opened the general question about what can be and what cannot be measured in quantum theory [6]. After some refinements made independently by Kennard [7], Condon [8], and Robertson [9], a more precise mathematical formulation was obtained by Schrödinger in terms of the Schwartz inequality [10].

The correct determination of the minimum values associated with the uncertainty principle makes sense because the quantum formulation is strongly linked to experimental measure-
ments: the observable quantities define what can be included into the theory (Heisenberg) and, at the same time, they are decided by the theory itself (Einstein), see, e.g.,[11].

On the other hand, the trend of looking for minimum uncertainty states embraces a large number of relevant works throughout different stages of modern quantum theory [12]. Pioneering results include the Schrödinger’s wave packets of constant width [13] (representing a mathematical antecedent of the coherent states introduced independently by Sudarshan [14] and Glauber [15] in quantum optics [16,17]), followed almost immediately by the Kennard’s wave packets of oscillating time-dependent width [7] (representing the first example of what are nowadays called squeezed states [18]).

The celebrated Glauber states [17] started the practical applications of quantum theory in optics by extending the notion of optical coherence to the description of quantized electromagnetic fields [12]. According to Glauber, the fully coherent states of the quantized electromagnetic radiation tolerate a description in terms of the Maxwell theory, so they may be considered ‘classical’. These fascinating quantum states are eigenvectors of the boson-annihilation operator with complex eigenvalue [17], and minimize both the Heisenberg–Kennard and Schrödinger inequalities. In position representation, the Glauber states acquire the form of the Schrödinger wave packets, a fact that is usually interpreted in favor of Schrödinger as the precursor of coherent states (see, however, the discussion about the different nature of the Glauber and Schrödinger results in [12]).

On the other hand, a squeezed state of light has less noise in one of its quadratures than the quantum noise limit dictates [19–21]. That is, the variance of such quadrature is squeezed at the expense of increasing the variance of the other quadrature. Although the term ‘squeezed state’ was coined in the field of gravitational-wave detection [18] (see also [22,23]), the first notions can be traced back to the Kennard’s paper of 1927 [7]. The states of light bearing such a name find immediate applications in interferometry [24], quantum communication [25], and quantum cryptography [26]. In contrast to the Glauber states, the squeezed states of light admit no description in terms of the Maxwell theory, so they are nonclassical [12]. (A very comprehensive account of the evolution in the study of nonclassical states can be found in the collection of reviews edited by Dodonov and Man’ko [27].) Moreover, the mathematical structure underlying squeezed states may be extended to account for subjects like higher-order quadratures [28], noncommutative spaces [29], and non-Hermitian structures [30].

In this work, we introduce a new class of squeezed states of light that can be expressed as a linear superposition of photon-number states where the vacuum state is absent. Such a class is complementary to the already known family of squeezed states that includes the squeezed-vacuum as a distinguished example. In fact, while the squeezed-vacuum consists entirely of even-photon number states, the new class includes a counterpart consisting on odd-photon number states only. The striking profile of the number-state superpositions giving rise to the new squeezed states is that the related probability amplitudes are characterized by the associated Hermite polynomials [31] (see also [32]). Such polynomials satisfy a three-term recurrence relation quite similar to that fulfilled by the classical Hermite polynomials but presenting a slight alteration in the labeling of the recurrence coefficients, which modifies the corresponding initial values. The new class of squeezed states is therefore defined by a

\[ \sigma_{A,B}^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2 \]

1 Heisenberg [1], Kennard [7], Condon [8] and Robertson [9] take into account only two of the three quadratic moments that can be associated with two variables. Schrödinger [10] was the first to notice that the covariance \( \sigma_{A,B} \) must be considered together with the variances \( (\Delta A)^2 \) and \( (\Delta B)^2 \) in order to better define the lower bound of the uncertainty principle for \( A \) and \( B \). Nevertheless, it is a common mistake to quote the main result of the Schrödinger paper [10] as the Schrödinger–Robertson inequality. Throughout this work, we opt by quoting \( (\Delta A)^2 (\Delta B)^2 \geq \sigma_{A,B}^2 + \frac{1}{4} |\langle [A, B] \rangle|^2 \) as the Schrödinger inequality.
set of associated Hermite polynomials parameterized by two complex numbers, which also characterize the nonclassical properties of such states.

In Sect. 2, we revisit some generalities of the minimum uncertainty states in order to obtain the three-term recurrence relation we are interested in. Then, a general approach is proposed where the recurrence relations are treated as finite difference equations, the general solution of which is obtained before considering the initial conditions. The squeezed-vacuum states are recovered as immediate example, while a new class of squeezed states consisting entirely of odd-photon number states is introduced. Section 3 is devoted to the construction of squeezed states in terms of the associated Hermite polynomials, hereafter referred to as associated squeezed states. The conventional squeezed states, expressed in terms of the classical Hermite polynomials, are also recovered to show the applicability of the method. Section 4 is addressed to the study and discussion of the nonclassical properties of the associated squeezed states. In Sect. 5, the main results of the work are remarked. For self-consistency, we have included three appendices with detailed information concerning the main points of our approach as well as the generalities of the orthogonal and associated polynomials.

2 Single-mode minimum uncertainty states

The Schrödinger inequality

\[(\Delta A)^2(\Delta B)^2 \geq \sigma_{A,B}^2 + \frac{1}{4}|\langle C \rangle|^2, \quad \sigma_{A,B} = \frac{1}{2}\langle AB + BA \rangle - \langle A \rangle\langle B \rangle,\]  

with \(C = [A, B] = AB - BA\), assumes that the expectation values are evaluated with regular (normalizable) vectors in the Hilbert space \(\mathcal{H}\) where the observables \(A\) and \(B\) are represented by self-adjoint operators. For \(A = B\) one has \(\sigma_{A,A} = \langle A^2 \rangle - \langle A \rangle^2 = (\Delta A)^2\), so the variance is a particular case of covariance. Besides, whereas the expectation value \(\langle C \rangle\) is attainable to the commutation properties of observables \(A\) and \(B\), the covariance \(\sigma_{A,B}^2\) is mainly determined by the statistical independence of such variables. Thus, the Schrödinger inequality takes into account both the statistical properties of the observables to be measured and the states used in the measurement.

We may identify at least three classes of quantum states: (i) States producing null covariance \(\sigma_{A,B} = 0\). These are such that the Condon-Robertson \([8,9]\) and Schrödinger inequalities are equivalent \([33]\), so the lower bound of the uncertainty principle is unambiguously established. The Glauber states are prototypical of this class for the field quadratures: (ii) States for which neither \(\sigma_{A,B}\) nor \(\langle C \rangle\) is equal to zero. This class of states satisfies the uncertainty principle in terms of the Schrödinger inequality. Most of the squeezed states reported in the literature find place in this class for the field quadratures: (iii) States for which \(\langle C \rangle = 0\). By necessity, this class requires the Schrödinger inequality to define the lower bound of uncertainties because the Condon–Robertson inequality becomes trivial \([33]\).

We would like to emphasize that even the Schrödinger inequality (1) could fail in determining the lower bound of the uncertainties for the third group mentioned above. For if \(\langle C \rangle = 0\), then \(\langle AB \rangle = \langle BA \rangle\), and \(\sigma_{A,B} = \langle AB \rangle - \langle A \rangle\langle B \rangle\). If now \(\sigma_{A,B} = 0\), we immediately get \(\langle AB \rangle = \langle A \rangle\langle B \rangle\), which makes redundant the inequality (1) when both \(\Delta A\) and \(\Delta B\) are different from zero \([33]\). Nevertheless, a form of saturating (1) with zeros at both sides would be to consider eigenstates of either \(A\) or \(B\). In such case the identity \(\langle AB \rangle = \langle A \rangle\langle B \rangle\) is automatically satisfied and the variances of \(A\) and \(B\) are both cancelled. The latter means statistical independence between the observables \(A\) and \(B\).
Avoiding the redundant cases, and paying attention to states for which \( A \) and \( B \) are not statistically independent, it may be shown that inequality (1) is saturated by the solutions of the eigenvalue equation [34–36]

\[
(A - i \lambda B)|\beta\rangle = \beta|\beta\rangle, \quad \lambda \in \mathbb{C}, \quad \beta = \langle A \rangle - i \lambda \langle B \rangle.
\] (2)

2.1 Eigenvalue equation

We are interested in solving Eq. (2) for \( A \) and \( B \) the field quadratures, \( x = \frac{1}{\sqrt{2}}(a^\dagger + a) \) and \( p = \frac{i}{\sqrt{2}}(a^\dagger - a) \), respectively. Hereafter, \( a \) and \( a^\dagger \) stand for the boson-ladder operators, which satisfy \( [a, a^\dagger] = 1 \) and act on the Fock (photon-number) states \(|n\rangle\) as follows:

\[
a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad n = 0, 1, \ldots
\] (3)

The kets \(|n\rangle\) form an orthonormal basis for the space of states \( \mathcal{H} = \text{span}\{|n\rangle\}_{n=0}^\infty \). From (2) we have \( x - i\lambda p = \frac{1}{\sqrt{2}}[(1 - \lambda)a + (1 + \lambda)a^\dagger] \), so that we write

\[
(a + \xi a^\dagger)|\alpha, \xi\rangle = \alpha|\alpha, \xi\rangle, \quad \alpha, \xi \in \mathbb{C},
\] (4)

with \( \xi = \frac{1+\lambda}{1-\lambda}, \alpha = \frac{\beta \sqrt{2}}{1-\lambda}, \) and \( \lambda \neq 1 \) [37–42].

Assuming we have at hand the regular solutions of (4), the straightforward calculation shows that the variances of \( x \) and \( p \) can be written in the form:

\[
(\Delta x)^2 = \tau \sqrt{\frac{1}{4} + \frac{\sigma_{x,p}^2}{\tau}}, \quad (\Delta p)^2 = \frac{1}{\tau} \sqrt{\frac{1}{4} + \frac{\sigma_{x,p}^2}{\tau}} \quad \tau = \left| \frac{1-\xi}{1+\xi} \right|,
\] (5)

see App. A for details. Thus, \(|\alpha, \xi\rangle\) saturates the Schrödinger inequality for the field quadratures

\[
(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4} + \sigma_{x,p}^2, \quad \sigma_{x,p} = \frac{1}{2}(xp + px) - \langle x\rangle\langle p\rangle.
\] (6)

Using \( \xi = r e^{i\theta} \), with \( r \in [0, 1) \) and \( \theta \in [-\pi, \pi) \), the \( \tau \)-parameter in (5) can be classified as follows:

(a) \( \tau = 1 \) for either \( r = 0 \) or \( \theta = \pm \frac{1}{2}\pi \) and any allowed value of \( r \).
(b) \( 0 < \tau < 1 \) for \( r \in (0, 1) \) and \( |\theta| < \frac{1}{2}\pi \).
(c) \( \tau > 1 \) for \( r \in (0, 1) \) and \( \frac{1}{2}\pi < |\theta| \leq \pi \).

Figure 1 shows the behavior of \( \tau \) (orange surface) and \( \tau^{-1} \) (blue surface) in terms of \( r \) and \( \theta \). The value \( \tau = 1 \) (green surface) is included as a reference.

The solutions of the eigenvalue equation (4) provide a repository of minimum uncertainty states \(|\alpha, \xi\rangle\) that may produce the squeezing of either \((\Delta x)^2\) or \((\Delta p)^2\) for the appropriate
parameter \( \tau \). For instance, the case (b) implies that the variances \((\Delta x)^2\) and \((\Delta p)^2\) are, respectively, squeezed and stretched. They interchange roles for the parameters defined in (c). The states leading to any of the above cases are called squeezed [18–21]. In turn, case (a) corresponds to the Glauber states for \( \xi = r = 0 \) since \((\Delta x)^2 = (\Delta p)^2 = 1/2\), with \( \sigma^2_{x,p} = 0 \).

On the other hand, fixing \( \theta = \pm \frac{1}{2} \pi \) means \( \xi = \pm ir \), so that \((\Delta x)^2 = (\Delta p)^2\), with \( \sigma^2_{x,p} \) parameterized by \( r \).

2.2 Three-term recurrence relation for squeezing

The construction of regular vectors \( |\alpha, \xi\rangle \in \mathcal{H} \) fulfilling the eigenvalue equation (4) is usually addressed by writing them in terms of the Fock basis \( |n\rangle \), \( n = 0, 1, 2, \ldots \), namely

\[
|\alpha, \xi\rangle = \sum_{n=0}^{\infty} C_n(\alpha, \xi) |n\rangle.
\]

(7)

It is convenient to express the coefficients \( C_n(\alpha, \xi) \) in the form

\[
C_n(\alpha, \xi) = \frac{1}{\mathcal{N}(\alpha, \xi)} \frac{P_n(\alpha, \xi)}{\sqrt{n!}}, \quad n = 0, 1, 2, \ldots,
\]

(8)

where \( \mathcal{N}(\alpha, \xi) \) stands for normalization and \( P_n(\alpha, \xi) \) is to be determined. The term \( \sqrt{n!} \) in the denominator of (8) simplifies the calculations to get \( P_n \). Indeed, introducing (7)–(8) into Eq. (4) gives the three-term recurrence relation

\[
P_{n+2}(\alpha, \xi) - \alpha P_{n+1}(\alpha, \xi) + (n + 1)\xi P_n(\alpha, \xi) = 0, \quad n = 0, 1, 2, \ldots,
\]

(9)

along with the constraint

\[
P_1(\alpha, \xi) - \alpha P_0(\alpha, \xi) = 0.
\]

(10)

Provided the solutions of (9), the normalization factor \( \mathcal{N}(\alpha, \xi) \) may be obtained by demanding the convergence of the series

\[
|\mathcal{N}(\alpha, \xi)|^2 = \sum_{n=0}^{\infty} \frac{|P_n(\alpha, \xi)|^2}{n!}.
\]

(11)

Equation (10) becomes redundant if both \( P_0 \) and \( P_1 \) are equal to zero. Consistently, \( P_0 = 0 \) is forbidden as initial condition since this produces \( P_1 = 0 \) in (10), and leads to the trivial solution of (9). That is, the constraint (10) induces a set \( \{P_n\} \) with \( P_0 \neq 0 \) by necessity. Therefore, if both equations (9) and (10) must be satisfied, the superposition (7) is characterized by the probability amplitude \( C_0 \neq 0 \).

Remark that the nonclassicality of \( |\alpha, \xi\rangle \) may be limited by the classicality of the vacuum state \( |0\rangle \). The latter because \( P_0 \neq 0 \) means that \( |0\rangle \) is always included in the profile of the superposition (7), so the probability density \( |C_0|^2 \neq 0 \) could master the predictions for detecting \( n \)-photons in \( |\alpha, \xi\rangle \).

Then, to enhance the nonclassical properties of the states we are looking for, the vacuum state \( |0\rangle \) may be dropped from the superposition (7), see, for instance, [43–47]. As this means to take \( P_0 = 0 \), the complementary equation (10) is not useful anymore and the eigenvalue equation (4) is not necessarily satisfied. That is, preserving the recurrence relation (9) to define the probability amplitudes (8), but omitting to impose the constraint (10) at the very beginning, we are in position to construct more general superpositions (7). After that, the states \( |\alpha, \xi\rangle \) so constructed would be asked to satisfy the appropriate initial conditions, the
ones that may be ruled by (10) in particular. We shall proceed in this form to investigate new possibilities of constructing nonclassical minimum uncertainty states.

Our approach is based on the calculus of finite differences [48]. Specifically, we identify the three-term recurrence relation (9) with a second-order difference equation and face it by finding the general solution, the initial conditions of which are to be determined. Since this kind of difference equations admits two independent solutions [48], we already know that one of such solutions will reproduce the well-known results associated with the system of equations (9)–(10). The other independent solution is therefore at our disposal to fix a different set of initial conditions such that the superposition (7) is well defined. For details, see “Appendices B and C”. A second step is to determine whether these new states of light minimize a given uncertainty relationship.

We start by reviewing the values of \( \alpha \) and \( \xi \) that reduce the recurrence relation (9) to the two-term case. Well-known minimum uncertainty states of light are recovered at the time that new squeezed states are introduced. Then, we construct the general solution for the three-term case with arbitrary values of \( \alpha \) and \( \xi \).

2.3 Special cases (two-term recurrence relations)

Making either \( \alpha = 0 \) or \( \xi = 0 \), the recurrence relation (9) is reduced to a two-term relationship. We analyze these cases separately.

- **Glauber states** For \( \xi = 0 \) and \( \alpha \in \mathbb{C} \), the eigenvalue problem (4) is reduced to the Glauber’s fully coherence condition \( a|\alpha\rangle = \alpha|\alpha\rangle \). Indeed, the recurrence relation (9) is simplified to the first-order difference equation with constant coefficients:

\[
P_{n+2}(\alpha) - \alpha P_{n+1}(\alpha) = 0, \quad n = 0, 1, \ldots
\]

Equation (12) may be reduced to \( P_{n+1} = \alpha^n P_1 \), and has the general solution \( P_n(\alpha) = \kappa \alpha^n \), with \( \kappa \) a constant fixed by the initial condition \( P_0(\alpha) = \kappa \). Notice that the constraint (10) is automatically satisfied since \( P_1(\alpha) = \kappa \alpha \) holds for any value \( \kappa \) of the initial condition. Without loss of generality, we take \( \kappa = 1 \) in the present case. Therefore, one obtains the Glauber states

\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C}.
\]

Although the coherent states \(|\alpha\rangle\) are classical in the sense introduced by Glauber, unexpected properties were first noticed by Dodonov, Malkin and Man’ko [49]. They realized that the superpositions \(|\alpha_{\pm}\rangle = c_{\pm}(|\alpha\rangle \pm |-\alpha\rangle)\), called even (+) and odd (−) coherent states, exhibit a nonclassical profile. Thus, whereas \(|\alpha\rangle\) are classical states of light, the superpositions \(|\alpha_{\pm}\rangle\) are not classical anymore [49,50].

- **Squeezed-vacuum states** For \( \alpha = 0 \) and \( \xi \in \mathbb{C} \), we get the second-order difference equation with variable coefficients

\[
P_{n+2}(\xi) + \xi (n+1) P_n(\xi) = 0, \quad n = 0, 1, 2, \ldots
\]

The general solution of this equation includes two independent solutions. One of them defines a class of \( P \)-functions that satisfy the constraint (10) as follows:

\[
P_0(\xi) = 1, \quad P_1(\xi) = 0 \quad \Rightarrow \quad P_{2n}(\xi) = \frac{(2n)!}{2^n n!} (-\xi)^n, \quad P_{2n+1}(\xi) = 0,
\]

\( \text{ Springer} \)
As the $P$-functions (15) are labeled by nonnegative integers $2n$, the state (7) consists entirely of even-photon Fock state superpositions. We write

$$\langle \xi; + \rangle = (1 - |\xi|^2)^{1/2} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (-\xi)^n |2n\rangle, \quad |\xi| < 1. \quad (16)$$

The above expression defines the squeezed-vacuum state in non-unitary form, see, for example, [51]. Indeed, as $|n\rangle = (n!)^{-1/2} a^n |0\rangle$, we may use the disentangling formula [51,52]

$$S(\xi) = \exp \left\{ \frac{\arctanh |\xi|}{2|\xi|} (\xi^* a^2 - \xi a^{+2}) \right\} = e^{-\frac{\xi}{2} a^{+2}} (1 - |\xi|^2)^{1/2} (a a^+ + a^+ a) e^{\xi a^2}, \quad (17)$$

to write $|\xi; + \rangle = S(\xi)|0\rangle$, where the invariance of $|0\rangle$ under the action of $\exp(a^2)$ has been used. The proper selection of the parameter $\xi$ yields the single-mode squeezing operator (17) in any of its well-known representations.

- **Odd-photon squeezed states** The second independent solution of Eq. (14) satisfies initial conditions that are not ruled by the constraint (10). The new family of $P$-functions is defined as follows:

$$P_0(\xi) = 0, \quad P_1(\xi) = 1 \quad \Rightarrow \quad P_{2n+1}(\xi) = 2^n n! (-\xi)^n, \quad P_{2n}(\xi) = 0. \quad (18)$$

Then, the superpositions (7)–(8) include odd-photon Fock states only. Explicitly,

$$|\xi; - \rangle = \frac{1}{\mathcal{N}(\xi; -)} \sum_{n=0}^{\infty} \frac{2^n n!}{\sqrt{(2n+1)!}} (-\xi)^n |2n + 1\rangle, \quad \mathcal{N}(\xi; -) \geq 0. \quad (19)$$

with

$$|\mathcal{N}(\xi; -)|^2 = \, _2F_1 \left( \frac{1}{2}, 1; \frac{3}{2}; |\xi|^2 \right) = \frac{\arcsin(|\xi|)}{|\xi| (1 - |\xi|^2)^{1/2}}. \quad (20)$$

As the hypergeometric function $_2F_1(a, b; c; z)$ converges for $|z| < 1$ in the complex $z$-plane [53], we get

$$|\xi; - \rangle = (1 - |\xi|^2)^{1/4} \left[ \frac{|\xi|}{\arcsin(|\xi|)} \right]^{1/2} \sum_{n=0}^{\infty} \frac{2^n n!}{\sqrt{(2n+1)!}} (-\xi)^n |2n + 1\rangle, \quad |\xi| < 1. \quad (21)$$

Proceeding in a similar way to the previous case, one arrives at the expression

$$|\xi; - \rangle = \frac{1}{\mathcal{N}(\xi; -)} \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} (-2\xi a^{+2})^n \, |1\rangle, \quad (22)$$

where we have retrieved the symbolic form of the normalization factor for simplicity. The straightforward calculation shows that the above result admits further simplification

$$|\xi; - \rangle = \frac{1}{\mathcal{N}(\xi; -)} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \xi a^2 \right)^n}{n!(2n+1)!} \exp \left( -\frac{\xi a^{+2}}{2} \right) \, |1\rangle. \quad (23)$$

As the series in square brackets is a confluent hypergeometric function, we formally write

$$|\xi; - \rangle = \mathcal{N}^{-1}(\xi; -) \, _1F_1 \left( \frac{3}{2}, \frac{1}{2} \xi a^{+2} \right) S(\xi)|1\rangle, \quad (24)$$

where (17) and the invariance of $|1\rangle$ under the action of $\exp(a^2)$ have been used.
Fig. 2 Wigner function defined by the odd-photon squeezed state $|\xi; -\rangle$ introduced in (21) for the indicated values of the squeezing parameter $\xi$. The function is plotted on the complex $z$-plane characterizing the Glauber state $|z\rangle$ of Eq. (25). As $\xi \in \mathbb{R}$, the position variable is squeezed, while the momentum variable is stretched, see Fig. 1.

Following the terminology for the so-called Hermite polynomial squeezed states [54,55], Laguerre polynomial states [56], and E-exponential states [46], the odd-photon squeezed solutions (21)–(24) may also be referred to as squeezed confluent-hypergeometric states (the term suggested by the relation $1F_1\left(\frac{1}{2}, \frac{3}{2}, x^2\right) = \frac{\sqrt{\pi}}{2x} \text{Erfi}(x^2)$ sounds a little bizarre).

Quite interestingly, the states defined by (21)–(24) are non-finite superpositions of the Plebánski squeezed number states $|n, \xi\rangle_{\text{Pleb}} = S(\xi)|n\rangle$ [57–59], the latter including the squeezed-vacuum $|\xi; +\rangle = S(\xi)|0\rangle$ as particular case. On the other hand, $|\xi; -\rangle$ is far from being an excitation of $|\xi; +\rangle$ because $(a + \xi a^\dagger)|\xi; +\rangle = (1 - |\xi|^2)S(\xi)|1\rangle$.

To analyze the properties of the odd-photon squeezed states $|\xi; -\rangle$, we may use the Wigner function [60], which enables the detection of squeezed states [61] and permits identifying nonclassicality in the quantum states [62]. Negative values of the Wigner function betray the presence of non-Gaussian properties, which signifies quantumness [60,62]. Non-Gaussian properties and negativity of the Wigner function also reveal nonclassical features such as contextuality [63,64]. For our purposes, we use the most general expression [43]

$$W(z) = e^{|z|^2} \int_C \frac{d^2 \beta}{\pi} (-\beta|\rho|\beta) e^{2(z\beta^* - z^*\beta)},$$

(25)

with $\rho$ the density operator of the state under consideration, and the integration is being performed over the complex $\beta$-plane. The complex parameters $\beta$ and $z$ define, respectively, the Glauber coherent states $|\beta\rangle$ and $|z\rangle$. Our algorithm for depicting the Wigner function follows [40,47] with the pure state $\rho_- = |\alpha, \xi; -\rangle \langle \alpha, \xi; -|$. We illustrate the Wigner function of state $|\xi; -\rangle$ in Fig. 2 for three different real values of the parameter $\xi$. Before starting our analysis, we would like to emphasize the result

$$\lim_{|\xi| \to 0} |\xi; -\rangle = |1\rangle,$$

(26)

which is easily verified by inspecting Eq. (21). That is, cancelling the squeezing parameter $\xi$ in the odd-photon squeezed state $|\xi; -\rangle$ we arrive at the one-photon Fock state $|1\rangle$. Recalling that $|\xi; -\rangle$ is defined on the open disk of unit radius $|\xi| < 1$ in the complex $\xi$-plane, we see that the deformation suffered by the Wigner function signifies a transition of the quantum profile of $|\xi; -\rangle$ that never arrives at the classical limit. The latter is in clear contraposition to the one-photon added coherent state $|\alpha, 1\rangle_{\text{add}}$ for which the quantum-to-classical transition can be measured and characterized by quantum tomography [44]. Indeed, at $\xi = 0$ the Wigner function for $|\xi; -\rangle$ is just the same as that for $|1\rangle$, so the quantumness of $|1\rangle$ is also
Fig. 3 Wigner function defined by the odd-photon squeezed state $|\xi; -\rangle$ introduced in (21) for the indicated values of the complex parameter $\xi = |\xi| \exp(i\theta_\xi)$. Compare with Fig. 2.

associated with $|\xi = 0; -\rangle$, see Fig. 3. Nevertheless, as soon as $\xi$ is different from zero, we find squeezing in the $x$ quadrature. The higher the value of $\xi \in \mathbb{R}$, the stronger the squeezing of $x$. In other words, the state $|\xi; -\rangle$ is never classical for $\xi \in \mathbb{R}$, but it may be either squeezed ($\xi \neq 0$) or not squeezed ($\xi = 0$). In the more general case $\xi \in \mathbb{C}$, the Wigner functions configured with real $\xi$ are affected just by a rotation.

Up to our knowledge, the states $|\xi; -\rangle$ have not been reported anywhere before the present work. Whereas the vectors $|\xi; +\rangle$ were brought to light due to practical necessities in the interferometry of gravitational-waves, the odd-photon squeezed states $|\xi; -\rangle$ seem to be even missing in theoretical approaches dealing with minimum uncertainty states. We think that the lack of results on this subject is because most of the approaches omit the deep revision of the corresponding recurrence relations.

The above note does not apply to the derivation of the Glauber states (13) since the recurrence relationship is a first-order difference equation, so it admits only one independent solution. The even and odd coherent states introduced by Dodonov, Malkin and Man’ko [49] are in this respect different expressions of the same solution.

3 Solutions to the three-term recurrence relation

In this section, we provide solutions to the recurrence equation (9) by following the comparison method introduced in “Appendix B”. We proceed by finding a first solution of Eq. (9), which will reproduce results already reported in the literature after considering the constraint (10). Then, using the formulation included in “Appendix C,” we find a second independent solution for (9) and show that this does not fulfill the constraint (10). Nevertheless, such a solution is well defined and gives rise to new forms of squeezed states.

3.1 Conventional two-parameter squeezed states

A first solution of the three-term recurrence relation (9) is easily achieved and leads to the family (see details in “Appendix B”):

$$P_n(\alpha, \xi; +) = \left(\frac{\xi}{\sqrt{2}}\right)^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{2}\xi}\right) = \xi^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\xi}}\right),$$

with $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ and $H_n(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right)$ the Hermite and scaled Hermite polynomials [53], respectively.

Strikingly, $P_n(\alpha, \xi; +)$ behaves like the scaled Hermite polynomial $H_n(\alpha)$ in the parameter $\alpha$, see Table 1. In particular, from (27) we find $P_n(\alpha, \xi = 1; +) = H_n(\alpha)$, so that...
Table 1 First five elements of the Hermite polynomials $H_n(x)$ and $H_{en}(x)$

| n   | $H_n(x)$ | $H_{en}(x)$ | $P_n(\alpha, \xi; +)$ |
|-----|----------|-------------|------------------------|
| 0   | 1        | 1           | 1                      |
| 1   | $2x$     | $x$         | $\alpha$              |
| 2   | $4x^2 - 2$ | $x^2 - 1$   | $\alpha^2 - \xi$      |
| 3   | $8x^3 - 12x$ | $x^3 - 3x$ | $\alpha^3 - 3\xi\alpha$ |
| 4   | $16x^4 - 48x^2 + 12$ | $x^4 - 6x^2 + 3$ | $\alpha^4 - 6\xi\alpha^2 + 3\xi^2$ |

They are compared with the two-parametric Hermite polynomials $P_n(\alpha, \xi; +)$ introduced in Eq. (27).

$P_n(\alpha, \xi; +)$ is a polynomial of order $n$ in $\alpha$ for $\xi = 1$. On the other hand, inspecting Table 1 we realize that $P_n(\alpha, \xi; +)$ is a polynomial of degree $\lfloor n/2 \rfloor$ in $\xi$, with $\lfloor z \rfloor$ the floor function of $z$. It is then remarkable that $P_n(\alpha, \xi; +)$ is twice degenerate in the $n$th order of $\xi$.

Hereafter, the $P$-functions (27) will be referred to as two-parametric Hermite polynomials. Notice that this family satisfies $P_0 = 1$ and $P_1 = \alpha$, so the constraint (10) is fulfilled. Therefore,

$$|\alpha, \xi; +\rangle = (1 - |\xi|^2)^{\frac{1}{2}} e^{-\frac{|\alpha|^2}{2(1-|\xi|^2)}} e^{\frac{i(\alpha^2 + \xi^2\alpha^2)}{4(1-|\xi|^2)}} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} H_n \left( \frac{\alpha}{\sqrt{\xi}} \right) |n\rangle, \quad |\xi| < 1. \quad (28)$$

The normalization factor (11) for these vectors has been achieved through the Mehler’s formula [65] (see Sec. 10.13, p. 194).

The two-parametric minimum uncertainty states defined in (28) reproduce the usual expression of the conventional squeezed states [19–21] for the appropriate profile of the $\xi$-parameter. In particular, the squeezed-vacuum $|\xi; +\rangle$ is recovered at the limit $|\alpha| \to 0$.

3.2 Expanding the set of squeezed states

Provided $P_n(\alpha, \xi; +)$, the second independent solution of the recurrence relation (9) yields the family

$$P_{n+1}(\alpha, \xi; -) = \left( \frac{\xi}{2} \right)^{\frac{n}{2}} \sum_{k=0}^{n} \frac{2^k k!}{H_k(y) H_{k+1}(y)} H_n(y), \quad y = \frac{\alpha}{\sqrt{2\xi}}, \quad n = 0, 1, \ldots, \quad (29)$$

see details in “Appendices B and C”. The straightforward calculation shows that (29) can be rewritten in the form

$$P_{n+1}(\alpha, \xi; -) = \left( \frac{\xi}{2} \right)^{\frac{n}{2}} H_n^{\nu=1} \left( \frac{\alpha}{\sqrt{2\xi}} \right) = \xi^{\frac{n}{2}} H_n^{\nu=1} \left( \frac{\alpha}{\sqrt{\xi}} \right), \quad (30)$$

where $H_n^{\nu}(x)$ stands for the associated Hermite polynomials [31,32] as they are revisited in [66,67]. Compare Eq. (30) with Eq. (27).

The $P$-functions (29)–(30) do not satisfy the constraint (10) since their derivation requires the initial value $P_0(\alpha, \xi; -) = 0$. Indeed, they satisfy the constraint

$$P_2(\alpha, \xi; -) - \alpha P_1(\alpha, \xi; -) = 0. \quad (31)$$

Comparing this result with (10) could lead to the wrong conclusion that (31) is a shifted version of the constraint obeyed by the squeezed states $|\alpha, \xi; +\rangle$. However, the main difference is that $P_0(\alpha, \xi; -) = 0$, whereas $P_0(\alpha, \xi; +) = 1$, see Table 2. The remaining $P$-functions
Table 2  First five elements of the classical polynomials $P_n(\alpha, \xi; +) = \left(\frac{\xi}{\sqrt{2}}\right)^n H_n\left(\frac{\alpha}{\sqrt{2}}\right)$ and the associated ones $P_{n+1}(\alpha, \xi; -) = \left(\frac{\xi}{\sqrt{2}}\right)^n H_{n-1}\left(\frac{\alpha}{\sqrt{2}}\right)$, with $P_0(\alpha, \xi; -) = 0$

| n   | $P_n(\alpha, \xi; +)$                          | $P_n(\alpha, \xi; -)$ |
|-----|-----------------------------------------------|------------------------|
| 0   | 1                                             | 0                      |
| 1   | $\alpha$                                      | 1                      |
| 2   | $\alpha^2 - \xi$                             | $\alpha$               |
| 3   | $\alpha^3 - 3\xi\alpha$                      | $\alpha^2 - 2\xi$      |
| 4   | $\alpha^4 - 6\xi\alpha^2 + 3\xi^2$           | $\alpha^3 - 5\xi\alpha$|

Note that $P_n(\alpha, \xi; -)$ is twice degenerate in the zero-order case

Fig. 4  Polynomials $P_n(\alpha, \xi; +)$ and $P_{n+1}(\alpha, \xi; -)$ as a function of $\alpha \in \mathbb{R}$ for the real parameter $\xi = 0.6$, and $n = 1$ (long-dashed-black), $n = 2$ (dotted-blue), $n = 3$ (solid-red) and $n = 4$ (squared-green). See the analytical form of these polynomials in Table 2

are different from zero in both cases, with $P_{n+1}(\alpha, \xi; -)$ and $P_n(\alpha, \xi; +)$ sharing the same order in $\alpha$ and $\xi$. Thus, $P_n(\alpha, \xi; -)$ is twice degenerate in the zero order of $\alpha$ and twice degenerate in the $n$th order of $\xi$.

Next some basic properties of the associated Hermite polynomials $P_n(\alpha, \xi; -)$ are reviewed in comparison with the classical ones $P_n(\alpha, \xi; +)$.

3.2.1 Properties of the associated Hermite polynomials

Both $P$-functions, $P_n(\alpha, \xi; +)$ and $P_{n+1}(\alpha, \xi; -)$, are polynomials of degree $n$ in $\alpha$ and degree $\lfloor n/2 \rfloor$ in $\xi$. Both sets are free of singularities such that their leading coefficient is equal to 1 for any $n$. That is, they are monic polynomials of degree $n$ in $\alpha$.

The pairing of orders between $P_n(\alpha, \xi; +)$ and $P_{n+1}(\alpha, \xi; -)$ means a shift in the distribution of zeros for these families of polynomials, see Fig. 4. In both cases, the interlacing of zeros obeys an oscillation rule that mimics the one satisfied by real-valued polynomials. In the Hermitian case, this means orthogonality and vice versa. Nevertheless, the latter does not apply to the non-Hermitian case since complex-valued polynomials are not orthogonal in the conventional form. The detailed analysis of such a property is out of the scope of the present work. In any case, the Favard’s theorem [68,69] may be useful on that matter.

To continue our analysis, we first rewrite the $P$-functions (29)–(30) as a power series in $\alpha$, and then, we decouple the result into even and odd contributions. After some simplifications (the material included in “Appendix B” is very useful on the matter), we arrive at the expressions
\[ P_{2n+2}(\alpha, \xi; -) = c_n(\alpha, \xi) \sum_{m=0}^{n} \left( \frac{\alpha^2}{\sqrt{\pi}} \right)^m \frac{(-n)_m}{(\frac{1}{2})_m(m+1)!} F_2 \left( \frac{1}{2}, m-n \mid m+2, m+\frac{3}{2} \right) \],

with \( c_n(\alpha, \xi) = \alpha(n+1)(\frac{3}{2})_n(-2\xi)^n \), and

\[ P_{2n+1}(\alpha, \xi; -) = (\frac{3}{2})_n(-2\xi)^n \sum_{m=0}^{n} \left( \frac{\alpha^2}{\sqrt{\pi}} \right)^m \frac{(-n)_m}{(\frac{3}{2})_m(m+1)!} F_2 \left( \frac{1}{2}, m-n \mid m+1, m+\frac{3}{2} \right). \]

Here, \( F_p(\cdots |z) \) is the generalized hypergeometric function, \( (a)_n = \Gamma(a+n)/\Gamma(a) \) the Pochhammer symbol, and \( \Gamma(z) \) the gamma function [53].

With the above results, it is easy to verify the following conclusions:

(i) At the limit \( \alpha \to 0 \) we have \( P_{2n+2}(\alpha, \xi; -)|_{\alpha=0} = 0 \) for any \( n \). In turn, the odd contribution \( P_{2n+1}(\alpha, \xi; -)|_{\alpha=0} = 2^n n!(-\xi)^n \) coincides with the \( P \)-functions defined for the odd-photon squeezed states \( |\xi; - \rangle \) in Eq. (18).

(ii) At the limit \( \xi \to 0 \) both polynomials \( P_{2n+1}(\alpha, \xi; -) \) and \( P_{2n}(\alpha, \xi; -) \) are different from zero. Besides, their dependence on \( \alpha \) is free of singularities and the initial condition \( P_0(\alpha, \xi; -) = 0 \) is preserved.

### 3.2.2 Associated squeezed states

The introduction of (29) into (7) yields

\[ |\alpha, \xi; - \rangle = \frac{1}{N(\alpha, \xi; -)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{\xi}{2} \right)^{n/2} \frac{H_{n+1}(y)}{\sqrt{n!}} H_k(y) H_{k+1}(y) |n+1\rangle, \quad |\xi| < 1. \]  

Equivalentiy, using (30) one obtains

\[ |\alpha, \xi; - \rangle = \frac{1}{N(\alpha, \xi; -)} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{(n+1)!}} He_n^{\nu=1} \left( \frac{\alpha}{\sqrt{\xi}} \right) |n+1\rangle, \quad |\xi| < 1, \]

compare (35) with (28).

Any of the expressions (34) or (35) defines the associated squeezed states, a class of minimum uncertainty states that is complementary to the well-established family of squeezed states \( |\alpha, \xi; + \rangle \). As far as we know, the new set of squeezed states \( |\alpha, \xi; - \rangle \) has not been reported anywhere before the present work.

The statistical and nonclassical properties of the associated squeezed states (34)–(35) are discussed in the next sections. Before that, we specialize on some concrete cases that are immediately obtained from the expression (34).

- **Recovering the odd-photon squeezed states** From item (i) of the previous section, we know that the even functions (32) do not contribute to the profile of \( |\alpha, \xi; - \rangle \) at \( \alpha = 0 \). Thus, none of the even-number Fock vectors \( |2n\rangle \) is included in the superposition \( |\alpha = 0, \xi; - \rangle \), as this would be expected. On the other hand, as the polynomials \( P_{2n+1}(\alpha, \xi; -) \) lead to the coefficients of the odd-photon squeezed state \( |\xi; - \rangle \) at \( \alpha = 0 \), we realize that only the odd-number Fock vectors \( |2n+1\rangle \) contribute to the identity \( |\alpha = 0, \xi; - \rangle = |\xi; - \rangle \), as this was already devised.

As it can be seen, the odd-photon squeezed states \( |\xi; - \rangle \) introduced in Eq. (18) are associated squeezed states \( |\alpha, \xi; - \rangle \) with \( \alpha = 0 \).
• Distorted coherent states From item (ii) of the previous section, we know that both functions (32) and (33) contribute to the profile of $|\alpha, \xi; -\rangle$ at $\xi = 0$. Besides, the condition $P_0(\alpha, \xi; -) = 0$ means that the vacuum $|0\rangle$ is excluded from the superposition (34). Let us pay attention to the resulting vector $|\alpha, \xi = 0\rangle \equiv |\alpha; -\rangle = |\alpha|\sqrt{1 - e^{-|\alpha|^2}}/|\alpha|^2|\alpha, 1\rangle_{\text{add}}$, (36) which is a distorted coherent state [70,71] and exhibits nonclassical behavior. To clarify the point let us pay attention to the relationship $a|\alpha; -\rangle = |\alpha|\sqrt{1 - e^{-|\alpha|^2}}|\alpha\rangle$. (37)

Thus, up to a constant factor, the action of the boson-annihilation operator $a$ on the distorted coherent state $|\alpha; -\rangle$ produces the coherent state $|\alpha\rangle$, which is classical. This result resembles what happens when the operator $a$ acts on the 1-photon number state $|1\rangle$, resulting in the vacuum state $|0\rangle$. One may say that $|\alpha; -\rangle$ is nonclassical as compared with the Glauber state $|\alpha\rangle$.

We can go a step further by considering the expression for the one-photon added coherent state [43]:

$$|\alpha, 1\rangle_{\text{add}} = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(n + 1)}{n!} |\alpha|^2 |n + 1\rangle.$$ (38)

It is well known that $|\alpha, 1\rangle_{\text{add}}$ results from the action of the boson-creation operator $a^\dagger$ on the Glauber state $|\alpha\rangle$. To compare (36) with (38) let us apply $a^\dagger$ on Eq. (37), we get

$$\hat{n}|\alpha; -\rangle = \frac{|\alpha|\sqrt{1 + |\alpha|^2}}{\sqrt{1 - e^{-|\alpha|^2}}} |\alpha, 1\rangle_{\text{add}}, \quad \hat{n} = a^\dagger a.$$ (39)

Thus, the one-photon added coherent state $|\alpha, 1\rangle_{\text{add}}$ is the result of applying the number operator $\hat{n}$ on the distorted coherent state $|\alpha; -\rangle$.

Figure 5 shows the Wigner function of the one-photon added coherent state $|\alpha, 1\rangle_{\text{add}}$ and the distorted coherent state $|\alpha; -\rangle$ for two different values of $\alpha$. Although we identify zones with negative values of $W(z)$ for both states, the deformations of $W(z)$ for $|\alpha; -\rangle$ are much stronger than those generated by $|\alpha, 1\rangle_{\text{add}}$. Then, we say that $|\alpha; -\rangle$ is more nonclassical than $|\alpha, 1\rangle_{\text{add}}$. Besides, albeit both vectors represent nonclassical states of light, the relationship
since the boson ladder operators in terms of the parameter which shows that defined for any trace class operator $A$

To get a measure of the distinguishability of the pair $|\alpha, \xi; \pm\rangle$ as $|\alpha| \to 0$, for which $d(\rho_+, \rho_-) \to 1$. The values of $|\alpha|$ and $\xi$ producing $d(\rho_+, \rho_-) = 0$ are such that the vectors $|\alpha, \xi; \pm\rangle$ represent exactly the same squeezed state. Average number of photons $\langle N(\alpha, \xi; \pm)\rangle$

(39) shows that the distorted coherent state $|\alpha; -\rangle$ can be considered the generating function of the one-photon added coherent state $|\alpha, 1\rangle_{\text{add}}$.

An additional result is easily derived from the formula

$$1 + \alpha \frac{d}{d\alpha} \mid |\alpha; -\rangle = \frac{|\alpha|\sqrt{1 + |\alpha|^2}}{\sqrt{1 - e^{-|\alpha|^2}}} |\alpha, 1\rangle_{\text{add}},$$

which shows that $|\alpha; -\rangle$ provides a form to determine the Segal–Bargmann representation for the boson ladder operators in terms of the parameter $\alpha$ (see for instance [71] and references quoted therein).

3.3 General structure of the space of solutions

To get a measure of the distinguishability of the pair $|\alpha, \xi; \pm\rangle$, we may use the trace norm, defined for any trace class operator $A$ as $|A| = \text{tr} |A|$, with $|A| = \sqrt{A^\dagger A}$ the modulus of $A$ [72]. Therefore, $d(\rho_+, \rho_-) = \frac{1}{2} ||\rho_+ -\rho_-||$ satisfies $0 \leq d(\rho_+, \rho_-) \leq 1$, with $d(\rho_+, \rho_-) = 1$ if the states are distinguishable (orthogonal) and $d(\rho_+, \rho_-) = 0$ for $\rho_+ = \rho_-.$

Figure 6a shows the trace distance between the squeezed states $\rho_\pm = |\alpha, \xi; \pm\rangle\langle \alpha, \xi; \pm|$ for real values of $\xi$, and as a function of $|\alpha|$. Clearly, $d(\rho_+, \rho_-) = 1$ at $\alpha = 0$, meaning that the squeezed-vacuum $|\xi; +\rangle$ and the odd-photon squeezed state $|\xi; -\rangle$ are orthogonal, as this was anticipated in the previous sections. On the other hand, for $|\alpha| \neq 0$ the squeezed state $|\alpha, \xi; +\rangle$ and the associated squeezed state $|\alpha, \xi; -\rangle$ are less distinguishable as $|\alpha| \to \infty$ since $d(\rho_+, \rho_-) \to 0$. At the very limit, $d(\rho_+, \rho_-) = 0$ means that both vectors $|\alpha, \xi; \pm\rangle$ represent exactly the same squeezed state. The trace distance is therefore a measure that permits to delimitate the convergence radius at which these vectors may be assumed as undistinguishable.

To associate the parameters $\alpha$ and $\xi$ with measurable quantities of the system, let us calculate the average number of photons $\langle N(\alpha, \xi; \pm)\rangle$ for the squeezed states $|\alpha, \xi; \pm\rangle$. We may proceed either by applying a Bogoliubov transformation on the ladder operators [59] or by calculating $\langle N(\alpha, \xi; \pm)\rangle = \text{Tr}(a^\dagger a \rho_\pm)$ directly. In the latter case, for $\langle N(\alpha, \xi; +)\rangle$ we obtain a very simple expression

$$\langle N(\alpha, \xi; +)\rangle = \frac{|\xi|^2 (1 - |\xi|^2) + |\alpha|^2 (1 + |\xi|^2) - \alpha^2 \xi - \alpha^* \xi}{(1 - |\xi|^2)^2},$$

(41)
Fig. 7 Probability distribution functions of the squeezed states $|\alpha, \xi; \pm\rangle$ for $\xi = 0.6$. The probability of finding $n$-photons decreases as $|\alpha| \to \infty$. The number of local maxima is correlated with the number of photons.

To obtain the above result, Eq. 5.12.2.1 of [73] is useful. Figure 6b shows the behavior of the average (41). In turn, it is not feasible to get a closed expression for $\langle N(\alpha, \xi; -) \rangle$ in general. Nevertheless, two particular cases are of remarkable interest. Namely,

$$\langle N(\alpha, \xi; -) \rangle|_{\alpha = 0} = \frac{|\xi|}{\arcsin(|\xi|)\sqrt{1 - |\xi|^2}} + \frac{|\xi|^2}{1 - |\xi|^2}, \quad |\xi| < 1,$$

(42)

and

$$\langle N(\alpha, \xi; -) \rangle|_{\xi = 0} = \frac{|\alpha|^2}{1 - e^{-|\alpha|^2}}.$$

(43)

Figure 6c shows the numerical calculation of $\langle N(\alpha, \xi; -) \rangle$, the boundaries of which are defined by $\alpha = 0$ and $\xi = 0$, see Eqs. (42) and (42), respectively. Comparing Fig. 6b, c one realizes that a given point $(|\alpha|, \xi)$, with $\xi \in (0, 1)$, yields a different distribution of photons in each case. This means that the trace distance shown in Fig. 6a links states with different averages in the number of photons for every point $(|\alpha|, \xi)$.

### 3.3.1 Probability distribution functions

The probabilities $P_n(\alpha, \xi; \pm) = |C_n(\alpha, \xi; \pm)|^2$ of finding $n$ photons in the states $|\alpha, \xi; \pm\rangle$ exhibit a shift associated with the distribution of zeros of $P_n(\alpha, \xi; \pm)$. To get some insights on the matter, consider the probability distributions of Fig. 7. The value of $\xi$ has been taken real and is fixed; the figures are plotted in terms of $|\alpha|$. Observe that $P_0(\alpha, \xi; -) = 0$, meaning that the classical state $|0\rangle$ does not contribute to the superposition $|\alpha, \xi; -\rangle$. The largest probabilities in the superpositions $|\alpha, \xi; \pm\rangle$ are $P_0(\alpha, \xi; +)$ and $P_1(\alpha, \xi; -)$, corresponding to 0-photon and 1-photon, respectively. The contribution of each photon-number state $|n\rangle$ to the squeezing $|\alpha, \xi; \pm\rangle$ decreases as $|\alpha| \to \infty$, and exhibits $n$ local maxima in $P_n(\alpha, \xi; +)$ and $P_{n+1}(\alpha, \xi; -)$. These characteristics show that $|\alpha, \xi; +\rangle$ and $|\alpha, \xi; -\rangle$ represent strongly different states for $|\alpha|$ within the convergence radius. The differences between these states are less evident as $|\alpha| \to \infty$, which corroborates the results for the trace distance discussed above.
To get detailed information about the probability distributions shown in Fig. 7, we have constructed the histograms exhibited in Fig. 8. There, the probabilities $P_n(\alpha, \xi; +)$ and $P_n(\alpha, \xi; -)$ are contrasted for three concrete real values of $\alpha$ and the value of $\xi$ used in Fig. 7. In all cases, the columns in blue and those in red refer to $|\alpha, \xi; +\rangle$ and $|\alpha, \xi; -\rangle$, respectively. Figure 8a shows the results for $\alpha = 0$. It is highly probable to find zero photons in $|0, \xi; +\rangle$ and one-photon in $|0, \xi; -\rangle$. The contribution of the remaining probabilities is almost negligible for both vectors. It is then reasonable to consider that $|0, \xi; -\rangle$ is more nonclassical than $|0, \xi; +\rangle$. Increasing the value of $\alpha$ in two units the situation changes, see Fig. 8b. Now the probability of finding zero photons in $|0, \xi; +\rangle$ is very small. The relevant probabilities are concentrated on finding either one, two or three photons, where the probability of finding two photons is the largest one for both states. Nevertheless, $P_n(2, \xi; -)$ and $P_n(2, \xi; +)$ are still different. One step of two-units further, for $\alpha = 4$ in Fig. 8c, one finds $P_n(4, \xi; +) \simeq P_n(4, \xi; -)$. The latter result is in agreement with the convergence radius evidenced in Figs. 6 and 7.

3.3.2 Algebraic structure

The properties of the $P$-functions (29) defining the associated squeezed states $|\alpha, \xi; +\rangle$ induce the eigenvalue equation

$$(a_2 + \xi a_2^\dagger)|\alpha, \xi; -\rangle = \alpha|\alpha, \xi; -\rangle,$$

(44)

with $a_2$ and $a_2^\dagger$ a pair of ladder operators arising from supersymmetric quantum mechanics [74], where they are referred to as distorted ladder operators [70,71]. The sub-label “2” means a particular value of the parameter that characterizes the commutation rules obeyed by these operators:

$$[a_2, a_2^\dagger] = \begin{cases} 0, & \mathcal{H}_0 = \text{span}\{0\} \\ 2, & \mathcal{H}_1 = \text{span}\{1\} \\ 1, & \mathcal{H}_s = \text{span}\{n\}, \ n \geq 2 \end{cases}$$

(45)

That is, in the distorted representation the Fock space of number states $\mathcal{H}$ is decoupled into the direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_s$.

The twice degenerate character of the associated polynomials (29) is therefore expressed through the action of $a_2$ and $a_2^\dagger$ on the Fock basis. Namely,

$$a_2|0\rangle = a_2^\dagger|1\rangle = 0, \quad a_2^\dagger|0\rangle = 0,$$

(46)

and

$$a_2|n + 1\rangle = \sqrt{n + 1}|n\rangle, \quad a_2^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle, \quad n = 1, 2, \ldots$$

(47)
Note that both operators $a_2$ and $a_2^\dagger$ annihilate the vacuum state $|0\rangle$, as it is required by the condition $P_0(\alpha, \xi; -) = 0$.

As the associated squeezed states $|\alpha, \xi; -\rangle$ solve the eigenvalue equation (44), they minimize the uncertainties associated with the quadratures $x_2 = \frac{1}{\sqrt{2}}(a_2^\dagger + a_2)$ and $p_2 = \frac{i}{\sqrt{2}}(a_2^\dagger - a_2)$, which satisfy a commutation rule equivalent to (45). That is, the set $|\alpha, \xi; -\rangle$ is a two-parametric family of minimum uncertainty states for the quadratures $x_2$ and $p_2$.

For $\xi = 0$, one has $|\alpha, \xi = 0; -\rangle = |\alpha\rangle_{\text{dist}}$, with $|\alpha\rangle_{\text{dist}}$ the distorted coherent states introduced in [70,71] and extended to the non-Hermitian case in [47,75]. The relevance of these vectors has been remarked in Sect. 3.2.2, where we have shown that they define the generating function of the one-photon added states.

4 Nonclassical properties of the associated squeezed states

By construction, the associated squeezed states $|\alpha, \xi; -\rangle$ introduced in Sect. 3.2.2 are also minimum uncertainty states. To analyze their nonclassical properties, consider first the Wigner functions of Fig. 9. There, $\alpha$ acquires three different real values, while $\xi$ is a fixed real number. These parameters produce the squeezing of the position variable.

Within the convergence radius discussed in Sect. 3.3, the Wigner function of the associated squeezed states $|\alpha, \xi; -\rangle$ is markedly different from that of the conventional squeezed states $|\alpha, \xi; +\rangle$. In the present case, this difference is illustrated in Fig. 9a, b, where the consistency with the results shown in Figs. 6, 7, and 8 is clear. Outside the radius of convergence, the vectors $|\alpha, \xi; \pm\rangle$ tend to represent the same squeezing state since their trace distance $d(\rho_+, \rho_-)$ goes to zero as $|\alpha| \to \infty$, see Fig. 6. This can be appreciated in Fig. 9c where the prototypical behavior of the conventional squeezed states is immediately recognized.

To go a step further in our analysis, we have depicted the behavior of $\Delta x_2$ and $\Delta p_2$ in Fig. 10. These variances have been calculated with the state $|\alpha, \xi; -\rangle$ for two real values of $\xi$, as functions of $|\alpha|$. For $\xi = 0$ both functions coincide, see Fig. 10a, and decay exponentially to the minimum uncertainty value. The latter is evidence of the nonclassical-to-classical transition that is commonly found in the one-photon added coherent states [43], and verifies our conclusions of Sects. 3.2.2 and 3.3.2 with respect to the relationship between $|\alpha, \xi = 0; -\rangle = |\alpha\rangle_{\text{dist}}$ and $|\alpha, 1\rangle_{\text{add}}$. For $\xi \neq 0$ the variances $\Delta x_2$ and $\Delta p_2$ decay exponentially to the values of $\Delta x$ and $\Delta p$ predicted by the squeezed state $|\alpha, \xi; +\rangle$, see Fig. 10b.

On the other hand, the nonclassical properties of the states $\rho_{\pm}$ can be also exhibited with the help of a beam splitter [76] (see also [77]). Indeed, if $\rho_{\text{in}}$ represents a bi-partite state of
Fig. 10 Variances $\Delta x_2$ (square-blue) and $\Delta p_2$ (circle-red) predicted by the odd-photon squeezed state $|\alpha, \xi; -\rangle$ for the indicated values of $\xi$, and as functions of $|\alpha|$. The inset shows the variances $\Delta x$ (solid-red) and $\Delta p$ (dashed-blue) computed with the conventional squeezed state $|\alpha, \xi; +\rangle$. In all cases the minimum uncertainty is shown in dotted-black lines as a reference.

Fig. 11 Linear entropy of the reduced state $\tilde{\rho}_-$ corresponding to the odd-photon squeezed state $|\alpha, \xi; -\rangle$ at the output of a 50:50 beam-splitter. The horizontal axis refers to $|\alpha|$, the plots correspond to $\xi = 0$ (solid-black), $\xi = 0.2$ (dashed-red), $\xi = 0.6$ (dotted-blue), and $\xi = 0.8$ (square-orange). The inset corresponds to the linear entropy of the conventional squeezed state $|\alpha, \xi; +\rangle$.

light entering a 50:50 beam splitter, the off-diagonal elements of the output state $\rho_{\text{out}}$ certify nonclassical correlations if they are different from zero. In such a case, measuring the number of photons at one of the output ports of the beam splitter is affected by the result of detecting photons at the other port and vice versa [46]. Therefore, the nontrivial off-diagonal elements of $\rho_{\text{out}}$ reveal the nonclassicality of $\rho_{\text{in}}$, at least in one of its two channels [76].

A measure of the above notion of nonclassicality is provided by the linear entropy $S(\rho) = 1 - \text{Tr} \rho^2$ [72], which quantifies the purity of any quantum state $\rho$. In general $0 \leq S \leq D^{-1}$, with $D \geq 2$ the order of the square matrix representing the state $\rho$. In the present case, $\rho$ has an infinite-dimensional matrix representation, so that $D \to \infty$. Then, $0 \leq S \leq 1$, with $S = 0$ for $\rho$ a pure state and $S = 1$ for $\rho$ a completely mixed state. The technique exploits the fact that a completely entangled pure state $\rho_{\text{out}}$ is such that its reduced states are completely mixed. Therefore, if $\rho_{\pm}$ is in channel 1 of the bi-partite state $\rho_{\text{in}}$, the linear entropy of the reduced output state $\tilde{\rho}_{\pm} = \text{Tr}_2 \rho_{\text{out}}$ will provide a measurement of the nonclassicality of $\rho_{\pm}$, which is ranked from 0 to 1.

Figure 11 shows the linear entropy $S(\tilde{\rho}_-)$ associated with the odd-photon squeezed state $|\alpha, \xi; -\rangle$ for different values of $\xi \in \mathbb{R}$, and as a function of $|\alpha|$. In all cases the purity of $\tilde{\rho}_-$ is larger at $|\alpha| = 0$. As in the previous measures of nonclassicality, the state $|\alpha, \xi; -\rangle$ becomes more classical as $|\alpha|$ increases. Outside the radius of convergence, the quantumness of the odd-photon squeezed state is bounded from below by the nonclassicality of the conventional squeezed states $|\alpha, \xi; +\rangle$. 

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5 Concluding remarks

The searching of squeezed states of light leads to three-term recurrence relations that must be solved in order to get appropriate superpositions of photon-number states $|n\rangle$, $n = 0, 1, 2, \ldots$. In the conventional case, one also finds an initial condition that makes the coefficient of the vacuum state $|0\rangle$ different from zero. Such a condition forces the 0-photon state $|0\rangle$ to be always present in the superpositions. Consistently, the prototypical squeezed vacuum is entirely composed by even-photon Fock states $|2n\rangle$.

We have shown that a thorough search of the solutions of the three-term recurrence relations gives rise to unnoticed squeezed states of light. In fact, one may proceed by solving the recurrence relation without imposing any initial value a priori. Once the general solution is achieved, the second step consists in determining the initial conditions that each of the independent solutions must satisfy in order to be well defined.

Our method produces superpositions of photon-number states whose coefficients are determined by the associated Hermite polynomials. As far as we know, this class of squeezed states has not been reported anywhere before the present work.

The conventional minimum uncertainty states that lead to the squeezing of either of the field quadratures have been recovered as particular cases.

We have also obtained superpositions consisting entirely of odd-photon Fock states $|2n + 1\rangle$. These new squeezed states are the counterpart of the squeezed vacuum and do not seem to have been noticed before in the literature.

The odd-photon squeezed states have been constructed by dropping the vacuum $|0\rangle$ from the corresponding superposition of photon-number states. The latter implied a modification of the initial conditions such that the coefficient of $|0\rangle$ resulted equal to zero. Our purpose was to find a combination of photon-number states with enhanced nonclassical properties, so we suppressed the participation of $|0\rangle$ to avoid its classicality. Consequently, the odd-photon squeezed states can be reduced to the one-photon state $|1\rangle$ for the appropriate values of the parameters involved. Indeed, the corresponding Wigner function exhibits negativity as well as squeezing, so it behaves like a deformed version of the Wigner function for the one-photon state.

To conclude our remarks, notice that squeezed single photon states can be generated from the squeezed vacuum by means of the photon subtraction technique [78,79]. The latter produces states with more robust nonclassical properties than those obtained with the photon added method [43,44,64]. We think that the odd-photon squeezed states may be configured using such a technique, at least in the restricted case of an expansion of the squeezed vacuum up to the first three or four photon-number states [78]. The possible applications of the states introduced in this work are, therefore, in the same field of applicability as conventional squeezed states.

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A Variances associated to the squeezed states

Consider the operators $A = A^\dagger$ and $B = B^\dagger$, which act on $\mathcal{H} = \text{Span}\{|n\rangle\}_{n=0}^\infty$. Let $|\beta\rangle \in \mathcal{H}$ be a solution of the eigenvalue equation

$$ (A - i\lambda B)|\beta\rangle = \beta|\beta\rangle, \quad \beta = \langle A \rangle - i\lambda\langle B \rangle \in \mathbb{C}. \tag{A-1} $$

The variances of $A$ and $B$ are easily computed using $\tilde{A} = A - \langle A \rangle$ and $\tilde{B} = B - \langle B \rangle$ through

$$ \langle \Delta A \rangle^2 = \langle \tilde{A}^2 \rangle, \quad \langle \Delta B \rangle^2 = \langle \tilde{B}^2 \rangle. \tag{A-2} $$

From Eq. (A-1), it follows $\langle \tilde{A}\rangle|\beta\rangle = i\lambda \tilde{B}|\beta\rangle$, so that $\langle \tilde{A}^2 \rangle = i\lambda \langle \tilde{A}\rangle|\beta\rangle = i\lambda \langle \tilde{B}\rangle|\beta\rangle$. Then,

$$ \langle \Delta A \rangle^2 \equiv \langle \tilde{A}^2 \rangle = i\lambda \langle \tilde{A}\rangle|\beta\rangle = i\lambda \langle|\beta\rangle \left( \frac{AB + BA}{2} + \frac{AB - BA}{2} - \langle A \rangle \langle B \rangle \right)|\beta\rangle $$

$$ = i\lambda \left( \sigma_{A,B} + \frac{i}{2} \langle ([A, B]) \rangle \right), \tag{A-3} $$

where we have used $[A, B]^\dagger = -[A, B]$. Considering now the complex number $\lambda = |\lambda|e^{i\theta_{\lambda}}$ and $(\sigma_{A,B} + \frac{i}{2} \langle ([A, B]) \rangle) = (\sigma_{A,B} + \frac{i}{4} \langle ([A, B]) \rangle^2)e^{i\theta_{A,B}}$, we obtain

$$ \langle \Delta A \rangle^2 = |\lambda| \left( \sigma_{A,B}^2 + \frac{1}{4} \langle ([A, B]) \rangle^2 \right)^{1/2} e^{i\left(\frac{\pi}{2} + \phi_{\lambda} + \phi_{A,B} \right)}. \tag{A-4} $$

The complex phase in (A-4) is fixed by recalling $\langle \Delta A \rangle^2 \geq 0$, as $A = A^\dagger$, and the expectation value is computed through regular vectors. Therefore, we find the following relation between the complex phases:

$$ \frac{\pi}{2} + \phi_{\lambda} + \phi_{A,B} = 2n\pi, \quad n = 0, 1, \ldots, \tag{A-5} $$

and consequently, the variance for $A$ is reduced to

$$ \langle \Delta A \rangle^2_{\lambda} = |\lambda| \left( \sigma_{A,B}^2 + \frac{1}{4} \langle ([A, B]) \rangle^2 \right)^{1/2}. \tag{A-6} $$

The calculations for $\langle \Delta B \rangle^2$ are carried out in a similar way. We multiply $\tilde{B}$ to the left of the eigenvalue equation $\tilde{A}|\beta\rangle = i\lambda \tilde{B}|\beta\rangle$, and the straightforward calculation, together with the condition for the complex phases in Eq. (A-5), leads to

$$ \langle \Delta B \rangle^2_{\lambda} = \frac{1}{|\lambda|} \left( \sigma_{A,B}^2 + \frac{1}{4} \langle ([A, B]) \rangle^2 \right)^{1/2}. \tag{A-7} $$

Using $A = x = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $B = p = \frac{i}{\sqrt{2}}(a^\dagger - a)$ in the above results, we recover Eq. (5).

B Solving recurrence relations by the comparison method

In this appendix, we construct the general solution of recurrence relations before considering the initial conditions. Our approach, hereafter referred to as comparison method, is addressed to determine whether the solutions of two recurrence relations can be paired by assuming that one of the recurrence problems is already solved. We focus on three-term recurrence relations, but the procedure is easily adapted to recurrences having a different number of terms.
Consider the recurrences
\[ a_n f_{n+1} + b_n f_n + d_n f_{n-1} = 0, \quad a_n, b_n, d_n \in \mathbb{C}, \quad n = 1, 2, \ldots, \]  
(B-1)

and
\[ A_n F_{n+1} + B_n F_n + D_n F_{n-1} = 0, \quad A_n, B_n, D_n \in \mathbb{C}, \quad n = 1, 2, \ldots \]  
(B-2)

Assuming that the set \( F_n \) defined by (B-2) is already known, we want to determine whether \( f_n \) can be written in terms of \( F_n \). The affirmative answer depends strictly on the profile and properties of \( F_n \).

Let \( h_n \) be a function such that \( f_n = h_n F_n \). Introducing it into Eq. (B-1) and comparing the result with (B-2), we arrive at the relationships
\[ \frac{h_{n+1}}{h_n} = \frac{A_n b_n}{B_n a_n}, \quad \frac{h_{n+1}}{h_n} = \frac{B_{n+1} d_{n+1}}{D_{n+1} b_{n+1}}. \]  
(B-3)

Both equations in (B-3) should lead to the same function \( h_n \), so we impose the compatibility condition
\[ \frac{A_n D_{n+1} b_n b_{n+1}}{B_n B_{n+1} a_n d_{n+1}} = 1, \quad n = 0, 1, 2, \ldots \]  
(B-4)

If (B-4) is fulfilled, the auxiliary function is obtained by solving any of the recurrence relations in (B-3). One finds
\[ h_{n+1} = h_0 \prod_{k=0}^{n} \frac{A_k b_k}{B_k a_k} = h_0 \prod_{k=0}^{n} \frac{B_{k+1} d_{k+1}}{D_{k+1} b_{k+1}}, \quad n = 0, 1, \ldots, \]  
(B-5)

where \( h_0 \) may be determined from the initial conditions through \( f_n = h_n F_n \).

B-2.1 The three-term recurrence relation of Sect. 2.2

We apply the comparison method to solve the recurrence relation (9) of Sect. 2.2. First we show the way in which the conventional results are recovered and then we obtain more general results.

B-2.1.1 Usual solution

Let us rewrite the recurrence relation (9) as follows:
\[ P_{n+1} - \alpha P_n + \xi n P_{n-1} = 0, \quad n = 1, 2, \ldots \]  
(B-6)

To apply the comparison method we use \( f_n = P_n(\alpha, \xi) \), with \( a_n = 1, b_n = \alpha \) and \( d_n = \xi n \).

Exploring the well-known recurrence relations for the classical orthogonal polynomials [80], we find that the recurrence relation for the Hermite polynomials
\[ H_{n+1}(z) - 2z H_n(z) + 2n H_{n-1}(z) = 0, \quad n = 1, 2, \ldots, \]  
(B-7)

is useful in the present case. That is, taking \( F_n = H_n(z) \), with \( A_n = 1, B_n = -2z \), and \( D_n = 2n \), the compatibility condition (B-5) is fulfilled with \( z = \alpha/\sqrt{2\xi} \). Therefore, we obtain \( h_n = h_0 \left( \frac{\xi}{2} \right)^{n/2} \).

Now, taking into account the constraint (10), meaning \( P_0 \neq 0 \), we may fix \( h_0 \) by the initial condition \( P_0(\alpha, \xi) = 1 \). Therefore, \( f_n = h_n F_n \) yields the well known result \( P_n(\alpha, \xi) = (\xi/2)^{n/2} H_n(\alpha/\sqrt{2\xi}) \). These roots of the recurrence problem (9)–(10) have been used to recover the expression of the conventional squeezed states \( |\alpha, \xi; + \rangle \) in Eq. (28) of the main text.
B-2.1.2 General solution

Looking for a general solution, one should recall that the confluent hypergeometric function 
\[ _1F_1(a; c; z) = M(a; c; z), \]
with \( a = -n \) and \( c \neq -m \) yields a polynomial of degree \( n \) in \( z \) \[53\] (\( n \) and \( m \) positive integers). Then, we may wonder whether the solutions of (B-6) can be paired with such polynomials. A first insight is obtained by comparing the confluent hypergeometric recurrence relation

\[(c - a) M(a - 1, c; z) + (2a - c + z) M(a, c; z) - a M(a + 1, c; z) = 0, \quad a = -n, c \neq -m,\]

with Eq. (B-6) since it makes clear that the compatibility condition (B-4) cannot be achieved. Nevertheless, decoupling (B-6) into even and odd values of \( n \), we, respectively, have

\[ P_{2n+2}(\alpha, \xi) + (4\xi n + \xi - \alpha^2) P_{2n}(\alpha, \xi) + 4\xi^2 n (n - \frac{1}{2}) P_{2n-2}(\alpha, \xi) = 0, \quad (B-9) \]

and

\[ P_{2n+3}(\alpha, \xi) + (4\xi n + 3\xi - \alpha^2) P_{2n+1}(\alpha, \xi) + 4\xi^2 n (n + \frac{1}{2}) P_{2n-1}(\alpha, \xi) = 0. \quad (B-10) \]

These results are now compatible with (B-8) for either \( c = \frac{1}{2} \) or \( c = \frac{3}{2} \). It is useful to recall the relationship between the confluent hypergeometric function and the Hermite polynomials

\[ M(-n, \frac{1}{2}; \xi) = (-1)^n \frac{n!}{(2n)!} H_{2n}(\xi), \quad M(-n, \frac{3}{2}; \xi) = \frac{(-1)^n n!}{2\xi (2n + 1)!} H_{2n+1}(\xi). \quad (B-11) \]

Thus, in the present case we may consider \( F_n = M(-n, c; z) \) with \( c \) equal to either \( 1/2 \) or \( 3/2 \) in order to get the corresponding auxiliary function (B-5). The latter provides a first solution to the problem. A second solution can be obtained by recalling that the confluent hypergeometric equation admits two linearly independent solutions. Given \( y_1 = M(a; c; z) \), the function \( y_2 = z^{1-c} e^{c} M(1-a, 2-c, -z) \) is such that \( W(y_1, y_2) = (1-c)z^{-c} e^{c} \) \[53\], so that \( y_1 \) and \( y_2 \) are linearly independent if \( c \neq 1 \). Therefore, if \( f_{2n} = h_{2n} M(-n, \frac{1}{2}, z) \) is our first solution, we may write \( \tilde{h}_{2n} M(1 + n, \frac{3}{2}, -z) \) for the second one, with \( \tilde{h}_{2n} \) absorbing the factors \( z^{1/2} e^{z} \) and being to be determined. In this form, the general solution for the even labels \( 2n \) is written as a linear combination of the above functions. The straightforward calculation yields

\[ P_{2n} = (-2\xi)^n \left[ \left( \frac{1}{2} \right)_n \kappa_1 M\left(-n, \frac{1}{2}; \frac{\alpha^2}{2\xi}\right) + n! \tilde{\kappa}_1 M\left(n + 1, \frac{3}{2}; -\frac{\alpha^2}{2\xi}\right) \right], \quad (B-12) \]

where the complex-valued coefficients \( \kappa_1(\alpha, \xi) \) and \( \tilde{\kappa}_1(\alpha, \xi) \) are fixed by the initial conditions. Equivalently, for the odd labels \( 2n + 1 \), we have

\[ P_{2n+1} = (-2\xi)^n \left[ \left( \frac{3}{2} \right)_n \kappa_2 M\left(-n, \frac{3}{2}; \frac{\alpha^2}{2\xi}\right) + n! \tilde{\kappa}_2 M\left(n + 1, \frac{1}{2}; -\frac{\alpha^2}{2\xi}\right) \right], \quad (B-13) \]

with \( \kappa_2(\alpha, \xi) \) and \( \tilde{\kappa}_2(\alpha, \xi) \) defined by the initial conditions.

- Solutions obeying the constraint (10).

  Taking into account the constraint (10), that is \( P_0 \neq 0 \), we may take \( P_0(\alpha, \xi) = 1 \). Then, \( P_1(\alpha, \xi) = \alpha \), and

  \[ \tilde{\kappa}_1(\alpha, \xi) = \tilde{\kappa}_2(\alpha, \xi) = 0, \quad \kappa_1(\alpha, \xi) = 1, \quad \kappa_2(\alpha, \xi) = \alpha. \]

  The above results permit to recover the well-known expression of the squeezed states (28).
• Solutions that do not satisfy the constraint (10).

Making \( P_0^{(2)}(\alpha, \xi) = 0 \) and \( P_1^{(2)}(\alpha, \xi) = 1 \) we find \( \tilde{\kappa}_1(\alpha, \xi) = \tilde{\kappa}_2(\alpha, \xi) = 1 \), and

\[
\kappa_1(\alpha, \xi) = -M\left(1, \frac{3}{2}; -\frac{q^2}{2\xi}\right), \quad \kappa_2(\alpha, \xi) = \frac{q^2}{\xi} M\left(1, \frac{3}{2}; -\frac{q^2}{2\xi}\right).
\]

After some calculations, from the above expressions one arrives at the results presented in Eqs. (32) and (33) of the main text.

C Orthogonal and associated polynomials

Following [81], we consider a set \( \{p_n(x)\} \) of orthogonal polynomials

\[
p_n(x) = \gamma_n x^n + \gamma_{n-1} x^{n-1} + \cdots + \gamma_0, \quad \gamma_0 > 0,
\]

that satisfy the three-term recurrence relation

\[
x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geq 0,
\]

with initial values

\[
p_{-1}(x) = 0, \quad p_0(x) = 1,
\]

and

\[
a_n = \frac{\gamma_{n-1}}{\gamma_n} > 0, \quad b_n = \int x p_n^2(x) d\mu(x) \in \mathbb{R}.
\]

The function \( \mu(x) \) in the recurrence coefficients (C-3) is a probability measure on the real line such that

\[
\int p_n(x) p_m(x) d\mu(x) = \delta_{nm}, \quad m, n \geq 0.
\]

Markedly, except for the classical orthogonal polynomials [53], finding the measure \( \mu \) and the solutions \( p_n(x) \) of the system (C-1)–(C-4) represents a formidable amount of work in general. In this respect the Favard’s theorem [69] (see also [68]) is very useful since it states that for the recurrence problem defined by (C-1)–(C-2), there exists a probability measure \( \mu \) so that the recurrence coefficients acquire the form (C-3) and the orthogonality (C-4) is satisfied [81], and vice versa. Therefore, it is natural to concentrate in solving (C-1)–(C-2) and then to allude the Favard’s theorem to ensure orthogonality.

A slight alteration of the three-term recurrence relation (C-1) produces new results. Namely,

\[
x p_n^{(k)}(x) = a_{n+k+1} P_{n+1}^{(k)}(x) + b_{n+k} p_n^{(k)}(x) + a_{n+k} P_{n-1}^{(k)}(x), \quad n \geq 0,
\]

with

\[
p_{-1}^{(k)}(x) = 0, \quad p_0^{(k)}(x) = 1,
\]

defines the \( k \)th associated orthogonal polynomials \( p_n^{(k)}(x) \) [81,82] (associated with the ones with \( k = 0 \)), also called numerator polynomials [68]. Given \( k \), the set \( \{p_n^{(k)}(x)\} \) defines a solution of the recurrence relation (C-1) with \( \mu^{(k)} \) the corresponding measure. (Guidelines for determining \( \mu^{(k)} \) can be found in [68,83], and references quoted therein.) The associated recurrence problem (C-5)–(C-6) is very useful for the purposes of this work since it permits to avoid the strong restriction \( P_0 \neq 0 \) from the constraint (10).

In the main text, we work with (C-1) rewritten in the form: [68,84]

\[
p_{n+1} = (x - c_n) p_n - \lambda_n p_{n-1}, \quad p_{-1} = 0, \quad p_0 = 1, \quad n = 0, 1, 2, \ldots
\]
where the coefficients $c_n$ and $\lambda_n$ are complex in general.

We identify (C-7) with a second-order difference equation [48], so there are two independent solutions for each value of $n$. If $p_n$ and $g_n$ solve a difference equation, they are independent if

$$\mathcal{C}(p_n, g_n) = \text{Det} \begin{pmatrix} p_n & g_n \\ p_{n+1} & g_{n+1} \end{pmatrix} \neq 0.$$  \hspace{1cm} (C-8)

The above determinant is known as the Casorati function [48] and is the discrete version of the Wronskian in differential equations.

Given a first solution $p_n$, a second solution $g_n$ such that $\mathcal{C}(p_n, g_n) \neq 0$ may be constructed by reducing the order of the corresponding difference equation [85]. The method is summarized with the algorithm

$$\frac{g_{n+1}}{p_{n+1}} = d_0 + d_1 \sum_{m=0}^{n} \frac{\mathcal{R}(m)}{p_m p_{m+1}}, \quad g_0 = 0, \quad \mathcal{R}(m) = \prod_{k=0}^{m-1} \lambda_k,$$  \hspace{1cm} (C-9)

with $d_0$ and $d_1$ arbitrary complex constants. Nevertheless, such a method yields unnecessary complications. We circumvent them by considering the additional difference problem

$$p_{n+1}^{(1)} = (x - c_{n+1})p_n^{(1)} - \lambda_{n+1}p_{n-1}^{(1)}, \quad p_{-1}^{(1)} = 0, \quad p_0^{(1)} = 1, \quad n = 0, 1, \ldots,$$  \hspace{1cm} (C-10)

where $\lambda_n$ and $c_n$ are the same as those defining (C-7). Thus, we pay attention to the associated polynomials $p_n^{(1)}(x)$ of the $p_n(x)$ that solve (C-7).

Making $g_n = p_{n-1}^{(1)}$, and considering the initial condition $p_{-1}^{(1)} = 0$, produces $g_0 = 0$, which is the initial condition considered in the algorithm (C-9). Therefore, the difference problem (C-10) acquires the form

$$g_{n+1} = (x - c_n)g_n - \lambda_n g_{n-1}, \quad g_0 = 0, \quad g_1 = 1.$$  \hspace{1cm} (C-11)

It is now clear that the associated polynomials $g_n$ do not satisfy the constraint (10) since $g_0 = 0$. Besides, the Casorati function between $p_n$ and $g_n$ is $\mathcal{C}(p_n, g_n) = \lambda_1$ [81], so these solutions are independent, provided that $\lambda_1 \neq 0$. That is, the second independent solution $g_n$ given by Eq. (C-9) may be also computed from the recurrence relation (C-11).

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