Gapless kinetic theory beyond the Popov approximation

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(Dated: December 17, 2002, submitted to PRA)

We present a unified kinetic theory that describes the finite-temperature, non-equilibrium dynamics of a Bose-Einstein condensed gas interacting with a thermal cloud. This theory includes binary interactions to second order in the interaction potential and reduces to a diagonal quantum Boltzmann equation for Bogoliubov quasiparticles. The Hartree-Fock-Bogoliubov interactions include the pairing field and are expressed as many-body $T$ matrices to second order. The interactions thus include the correct renormalized scattering physics. This renormalized theory is automatically gapless. Thus, the excited Bogoliubov modes are naturally orthogonal to the condensate ground state.

PACS numbers: 03.75.Kk, 05.70.Ln

I. INTRODUCTION

Finite-temperature theories of Bose-Einstein condensation (BEC) have been a very active field of study. The goal is a unified description of a dilute, atomic gas of bosons in a harmonic trap in terms of a condensate mean field interacting with a thermal cloud. The success of the zero-temperature Gross-Pitaevskii (GP) theory in describing BEC experiments spurred interest in effects not contained in this framework. Examples are damping of collective excitations and condensate growth through collisional redistributions of thermal atoms. This distribution cannot be treated in theories that are still of first order in the binary interaction, such as, for example, Hartree-Fock-Bogoliubov (HFB) theories.

We present a kinetic theory of second order in the interaction, which is formulated in terms of Bogoliubov quasiparticles and contains collisional terms beyond the HFB approximation. The HFB interactions are expressed as many-body $T$ matrices to second order in the binary potential and thus include the correct renormalized scattering physics. This theory thus contains no ultraviolet divergences and has a gapless energy spectrum. We extend the papers of Walser et al. [1, 2], which are the basis of the present results, by these two essential aspects.

We go beyond theoretical approaches that drop the anomalous pair matrix $\tilde{m} = \langle \tilde{a} \tilde{a} \rangle$ in the Popov approximation [3–6]. Monte Carlo simulations based on the semi-classical Zaremba-Nikuni-Griffin theory [3] show very good agreement for experimentally observed damping rates and response frequencies [7, 8]. However, recent experiments [9] and their theoretical explanations [10, 11] have shown that the pairing field plays an important role in Bose gases with resonance interactions.

Keeping the pair matrix $\tilde{m}$ in this theory would cause ultraviolet divergences if we replaced the bare interaction potentials with the $s$-wave scattering length $a_s$ in the contact-potential approximation. The vacuum part of the pairing field’s self interaction, for example, will diverge, because the delta-function potential contains unphysically high energy contributions. These divergences can be resolved by writing the interactions in terms of scattering $T$ matrices, which subsume the divergent sums over intermediate scattering states and correctly reduce to the scattering length $a_s$ in the zero-energy and -momentum limit [12–16]. Using these scattering $T$ matrices, we can consistently eliminate all divergences to second order in the interaction.

We thus obtain a renormalized HFB operator, which has a gapless spectrum. The zero-energy eigenspace is spanned by the condensate, and excitations with nonvanishing energy are thus automatically orthogonal to the condensate. This is similar to the renormalized gapless HFB equations proposed in Ref. [14]. Other approaches have to explicitly project the excitations orthogonal to the condensate [12, 17]. Using the condensate as the ground state of this adiabatic basis also simplifies the complicated representation of a condensate band in the Gardiner-Zoller master equation formulation [18].

The Monte Carlo simulations of Jackson and Zaremba [7, 8] show that considering dynamic population-exchange between the condensate and the thermal cloud and within the thermal cloud leads to good agreement with the observed response spectra. We include these kinetic effects, thus going beyond collisionless descriptions [19, 20].

Our presentation builds on the papers by Walser et al. [1, 2] and we will begin by summarizing the kinetic equations derived in these papers in Sec. II. In Sec. III, we examine the first-order HFB propagator and rewrite it in terms of second-order $T$ matrices, by including second-order energy shifts and adiabatically eliminating the pairing field $\tilde{m}$. This shows that the theory is explicitly gapless and renormalized. We then make use of the gaplessness and write the kinetic equations in terms of Bogoliubov quasiparticles, which are orthogonal to the condensate by construction. The results in Sec. IV show that the complicated collision terms presented in the Walser et al. papers can be simplified dramatically by a basis transformation. Practical calculations of the quantum
Boltzmann equation then will require only diagonal elements of the quasiparticle population matrix.

II. SINGLE-PARTICLE KINETIC EQUATIONS

We present the Walser et al. formulation of kinetic theory [2], which was originally derived using a statistical operator approach [1]. The authors of this paper also recently showed a derivation of the same equations [21] from the Kadanoff-Baym [22, 23] theory of non-equilibrium Green functions. They used the gapless second-order Beliaev approximation [4, 24] and showed equivalence to the work of Walser et al. under the Markov approximation. Recently, another independent derivation [25, 26] connected this approach to kinetic theories by Morgan and Proukakis.

Neglecting three-body and higher interactions, we can describe the weakly interacting, dilute gas by the following Hamiltonian

$$\hat{H} = H^{(0)} + V_{\text{ext}}(\mathbf{x})$$

where external harmonic potential $V_{\text{ext}}$. The bosonic creation operator $\hat{a}^\dagger_{1'}$ creates a particle in the state $\left| 1' \right\rangle$, where $1'$ stands for a complete set of quantum numbers, which label a constant, single-particle energy basis, such as, for example, harmonic oscillator states or eigenstates of the GP equation. We use the summation convention for these abbreviated indices and indicate the single-particle basis with primes.

The two-particle matrix elements of the binary interaction potential $V_{\text{bin}}(x_1, x_2)$ are defined by

$$\phi^{1'2'3'4'} = \int dx_1 dx_2 \langle 1' | x_1 \rangle \langle 2' | x_2 \rangle V_{\text{bin}}(x_1, x_2) \times \left\{ \langle x_1 | 3' \rangle \langle x_2 | 4' \rangle + \langle x_1 | 4' \rangle \langle x_2 | 3' \rangle \right\}.$$ (3)

These matrix elements are symmetric in the first and last two indices:

$$\phi^{1'2'3'4'} = \phi^{2'1'3'4'} = \phi^{1'2'4'3'}.$$ (4)

To determine the measurable quantities we want to calculate in this theory, we first define the condensate mean field $\alpha$ as the expectation value of the destruction operator

$$\alpha = \langle \hat{a}_{1'} \rangle \equiv \langle 1' | \hat{a}_{1'} \rangle.$$ (5)

The total density matrix $f$ is defined by

$$f = \langle \hat{a}_{1'}^\dagger \hat{a}_{1'} \rangle \equiv \langle 1' | \hat{1'} \rangle \otimes \langle 2' |.$$ (6)

Subtracting the condensate density matrix

$$f^c = \alpha \otimes \alpha^* \equiv \alpha_{1'} \alpha_{1'}^* \otimes \langle 1' \rangle \otimes \langle 2' \rangle,$$ (7)

we obtain the density matrix of thermal atoms $\hat{f} = f - f^c$. The anomalous average $m$ is split analogously

$$m = \langle \hat{a}_{1'} \hat{a}_{1'} \rangle \otimes \langle 2' \rangle = m^c + \hat{m},$$ (8)

in the condensate part $m^c \equiv \alpha \otimes \alpha$ and fluctuations $\hat{m}$. This set of variables contains all possible combinations of up to two field operators. The course-grained statistical operator parametrized by these variables is thus Gaussian, and we can use Wick’s theorem to truncate the coupling to higher-order correlation functions, i.e., expectation values of more than two field operators. This approximation is valid because of the diluteness of the condensed gas. In a dilute gas the duration of a collision event is short compared to the essentially free evolution between collisions, which allows higher-order correlations to damp.

The procedure followed by Walser et al. [1] is then to write the Heisenberg equations of motion for the variables above and expand the expectation values using Wick’s theorem. Walser et al. thus obtain the equations of motion given in the following two sections [2].

A. Mean-field equations

Since the anomalous fluctuations $\hat{m}$ couple the mean field $\alpha$ to its conjugate $\alpha^* = \alpha_{1'}^* \langle 1' \rangle$, it is convenient to write the generalized Gross-Pitaevskii (GP) equation in a two-by-two matrix form

$$\frac{d}{dt} \chi = (-i \Pi + \gamma \gamma^* - \gamma^\gamma) \chi,$$ (9)

where the two-component state vector

$$\chi = \left( \begin{array}{c} \alpha \\ \alpha^* \end{array} \right)$$ (10)

is defined in terms of $\alpha = \langle \hat{a}_{1'} \rangle \langle 1' \rangle$ and also contains the time-reversed mean field $\alpha^*$.

The generalized GP propagator representing the reversible evolution of $\chi$ is defined as

$$\Pi = \left( \begin{array}{cc} \Pi_N & \Pi_A \\ -\Pi^*_A & -\Pi_N \end{array} \right).$$ (11)

This symplectic propagator consists of the normal Hermitian Hamiltonian

$$\Pi_N = \tilde{H}^{(0)} + U + 2U_f - \mu,$$ (12)

where $\mu$ removes rapid oscillations of the mean field, and the symmetric anomalous coupling

$$\Pi_A = V \hat{m}.$$ (13)
The energy shifts due to both the mean field and the normal fluctuations are given by the matrices
\[
U_f = 2 \phi \, \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 3 \gamma 4} \mid 1' \rangle \otimes \langle 4' \rangle,
\]
whereas the first-order anomalous coupling-strength is given by
\[
V_m = 2 \phi \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma m_{\gamma 3 \gamma 4} \mid 1' \rangle \otimes \langle 2' \rangle.
\]

The second-order irreversible evolution, consisting of damping rates and energy shifts, is given by the collision operator
\[
\Upsilon < = \left( \begin{array}{cc} \Upsilon_N^< & \Upsilon_A^< \\ \Upsilon_A^< & \Upsilon_N^< \end{array} \right)
\]

\[
\Upsilon_N^< = \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 3 \gamma 4} f_{\gamma 2 \gamma 4} f_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle,
\]
\[
\Upsilon_A^< = \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 2 \gamma 4} m_{\gamma 3 \gamma 4} m_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle.
\]

\[
\Gamma_{ff} = 8 \phi \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 3 \gamma 4} f_{\gamma 2 \gamma 4} f_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle,
\]
\[
\Gamma_{fm} = 8 \phi \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 2 \gamma 4} m_{\gamma 3 \gamma 4} f_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle,
\]
\[
\Gamma_{mm} = 8 \phi \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 2 \gamma 4} m_{\gamma 3 \gamma 4} m_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle.
\]

FIG. 1: The diagrams corresponding to the second-order terms \( \Upsilon < \) in the GP equation (9). The dashed potential lines correspond to the symmetrized binary potential \( \phi \) in the single-particle energy basis. The directed propagators represent the normal density \( \tilde{f} \), the remaining ones the anomalous average \( \tilde{m} \) and its conjugate. Note that all diagrams are topologically equivalent, and only propagators are exchanged.

The in-rates of the collision operator \( \Upsilon \) are depicted in Fig. 1.

In this theory, collisional interactions are considered to second order. The effect of higher order terms, which lead to a finite duration of a collision, can be modeled by introducing a parameter \( \eta \), such that every second-order collision operator contains dispersive as well as dissipative parts from the complex-valued matrix element
\[
\phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma \phi \eta \, \gamma 1 \gamma 2 \gamma 3 \gamma 4 \gamma f_{\gamma 3 \gamma 4} f_{\gamma 2 \gamma 4} f_{\gamma 2 \gamma 4} \mid 1' \rangle \otimes \langle 3' \rangle,
\]
where the energy difference \( \Delta \epsilon = -(\epsilon_{1'} + \epsilon_{2'}) + \epsilon_{3'} + \epsilon_{4'} \) has to be smaller than the energy uncertainty \( \eta \) to get a sizable contribution. Note that the papers [1, 2] contain a sign error in the definition of \( \Delta \epsilon \). For small \( \eta \) we obtain
\[
\frac{1}{\eta - i \Delta \epsilon} \xrightarrow{\eta \to 0} \pi \delta_\eta (\Delta \epsilon) + i \mathcal{P}_\eta \frac{1}{\Delta \epsilon},
\]
where \( \mathcal{P} \) indicates that the Cauchy principal value has to be taken upon integration. The parameter \( \eta \) thus represents off-the-energy-shell propagation after a collision. Most off-the-energy-shell coherences decay during subsequent propagation, but, due to the finite time between collisions \( \tau_c \), energy cannot be conserved exactly, because \( \eta \) has to be larger than the collision rate \( 1/\tau_c \).

**B. Equations for normal densities and anomalous fluctuations**

The equations of motion for the fluctuation densities \( \tilde{f} \) and \( \tilde{m} \) are coupled and can also conveniently be written in terms of two-by-two matrices. To achieve this, we define the generalized single-time fluctuation-density matrix \( \tilde{G} < \) as
\[
\tilde{G} < = \left( \begin{array}{cc} \tilde{f} & \tilde{m} \\ \tilde{m}^* (1 + \tilde{f})^* \end{array} \right),
\]
where \( \tilde{f} = f_{12'} \mid 1' \rangle \otimes (2') \) and \( \tilde{m} = m_{12'} \mid 1' \rangle \otimes (2') \) are the matrix representations of the master variables. In Ref. [21], we use the property that this density matrix is the single-time limit of the time-ordered
two-time Green’s function. This showed that the other time ordering is given by
\[
\tilde{G}^> = \left( \frac{1 + \tilde{f}}{\tilde{m}^*} \frac{\tilde{m}}{\tilde{f}^*} \right) = \sigma_1 \tilde{G}^{<*} \sigma_1 = \sigma_3 + \tilde{G}^{<}. \tag{22}
\]
Here, we use the third Pauli matrix \(\sigma_3 = \text{diag}(1,-1)\). Note that our naming of the fluctuation-density matrices \(\tilde{G}^{<}\) and \(\tilde{G}^{>}\) is consistent with the two-time formalism in Ref. [21], but disagrees with Ref. [2].

The generalized Boltzmann equation of motion for this fluctuation-density matrix can be written as
\[
\frac{d}{dt} \tilde{G}^{<} = -i\Sigma \tilde{G}^{<} + \Gamma^{<} \tilde{G}^{>} - \Gamma^{>} \tilde{G}^{<} + \text{H.c.} \tag{23}
\]
This equation has to be solved under the constraints
\[
\alpha^* \tilde{f} = 0 \quad \text{and} \quad \alpha^* \tilde{m} = 0, \tag{24}
\]
which force the fluctuations to be orthogonal to the condensate.

Again, the equation of motion (23) has two parts: The reversible evolution is governed by the Hartree-Fock-Bogoliubov self-energy operator
\[
\Sigma = \left( \begin{array}{cc} \Sigma_N & \Sigma_A \\ -\Sigma_A & -\Sigma_N \end{array} \right), \tag{25}
\]
which in turn consists of the Hermitian Hamiltonian
\[
\Sigma_N = \hat{H}^{(0)} + 2U_{f^e} + 2U_{\bar{f}} - \mu \tag{26}
\]
and the symmetric anomalous coupling
\[
\Sigma_A = V_{m^e} + V_{\bar{m}}. \tag{27}
\]

The irreversible evolution introduced by second-order collisional contributions now consists of the collisional operator
\[
\Gamma^{<} = \left( \begin{array}{cc} \Gamma^{<}_{\tilde{N}} & \Gamma^{<}_{\tilde{A}} \\ -\Gamma^{<}_{\tilde{A}} & -\Gamma^{<}_{\tilde{N}} \end{array} \right), \tag{28}
\]
and its time-reversed counter part \(\Gamma^{>} = -\sigma_1 \Gamma^{<*} \sigma_1\). The diagonal components of the collisional operator are defined as
\[
\Gamma^{<}_{\tilde{N}} = \Gamma_{(f^e+f^e)} f(1+f) + \Gamma_{f^e f^e} f(1+f) + \Gamma_{f f^e} f(1+f) + 2\{\Gamma_{(f^e+f^e)(1+f)f} + \Gamma_{f^e f(1+f)} + \Gamma_{(f^e+f^e)(1+f)f} + 2\{\Gamma_{(f^e+f^e)(1+f)\bar{m}} + \Gamma_{(1+f)m^e\bar{m}} + \Gamma_{(1+f)\bar{m}m^e}\}, \tag{29}
\]
and the off-diagonal, anomalous components as
\[
\Gamma^{<}_{\tilde{A}} = \Gamma_{(\bar{m}+m^e)\bar{m}^e} + \Gamma_{\bar{m}^e m^e} + \Gamma_{\bar{m} m} + 2\{\Gamma_{(f^e+f^e)(1+f)\bar{m}} + \Gamma_{f^e (1+f)\bar{m}} + \Gamma_{f^e f(1+f)} + \Gamma_{f m} + \Gamma_{m^e\bar{m}} + \Gamma_{m \bar{m}^e}\}. \tag{30}
\]

![Fig. 2](image2.png)

**Fig. 2:** The diagrams corresponding to the second-order terms \(\Gamma^{<}\) in the generalized Boltzmann equation (23). The dashed lines depict the symmetrized binary potential \(\phi\) in the single-particle energy basis. The directed propagators represent the normal density \(\bar{f}\), the remaining ones the anomalous average \(\bar{m}\) and its conjugate. The first column of diagrams is identical to those depicted in Fig. 1. The remaining diagrams each have one of the three propagators replaced with an open condensate line.

Both the diagonal and off-diagonal incoming rates are depicted diagrammatically in Fig. 2. For every term that appears in the collisional terms of the generalized GP equation in \(\Gamma^{<}\) [Fig. (1)], we here [Fig. (2)] have three additional terms, where in each of them, one of the three fluctuating contributions is replaced with the co-
responding mean-field quantity. This replacement rule can be seen in the Beliaev collisional self energies presented in Ref. [21] and is a consequence of the fact that the Boltzmann equation (23) can be generated from the \( \Gamma^< \). This replacement rule in Eq. (23), we get terms like \( T \) ultra-violet divergences.

When the collision operator \( \Gamma^< \) is multiplied by \( \bar{G}^> \) as in Eq. (23), we get terms with the propagators \( \Sigma \) and \( \Pi \) to the real parts of many-body grades the bare interaction potentials in the first-order equation [4, 24]. The first-order HFB propagator \( \Sigma \), which appears in the kinetic equations, is, on the other hand, known to exhibit a non-physical energy gap in the long-wavelength, homogeneous limit [28].

We resolve this discrepancy by including second-order collisional energy shifts \( \mathcal{P}\{\Gamma\} \) into the HFB propagator and adiabatically eliminating the anomalous average \( \bar{m} \) in the first-order anomalous potential \( V_{\bar{m}} \) (15). This upgrades the bare interaction potentials in the first-order propagators \( \Sigma \) and \( \Pi \) to the real parts of many-body \( T \) matrices. We can then approximate these matrices by adiabatically eliminating the anomalous average \( \bar{m} \).

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### III. GAPLESSNESS — \( T \) MATRICES

Our goal in this section is to explicitly show that the kinetic equations (23) and (9) are gapless. This should on one hand be obvious, because our previous paper [21] showed them to be equivalent to the Kadanoff-Baym equations [22, 23] in the gapless Beliaev approximation [4, 24].

The first-order HFB propagator \( \Sigma \), which appears in the kinetic equations, is, on the other hand, known to exhibit a non-physical energy gap in the long-wavelength, homogeneous limit [28].

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The upgraded HFB propagator \( \Sigma \), where all binary interactions are written as many-body \( T \) matrices, is explicitly gapless and thus obeys the Hugenholtz-Pines theorem [29]. The propagator thus has zero-energy modes, which are completely specified by the value of the condensate \( \alpha \). If we use the non-zero energy Bogoliubov modes of \( \Sigma \) as a basis for the thermal excitations, the excitations will automatically be orthogonal to the condensate. We follow this idea in Sec. IV.

#### A. Off-diagonal potentials

Here, we update the off-diagonal potentials \( V_{\bar{m}+\bar{m}} \) in the HFB propagator \( \Sigma \) by adiabatically eliminating the pairing field \( \bar{m} \). We integrate the first-order equation of motion for the anomalous average \( \bar{m} \)

\[
\frac{d}{dt} \bar{m} = -i\Sigma_N \bar{m} - i\bar{m} \Sigma_N - i\Sigma_A (1 + \hat{\bar{f}}) - i\hat{\bar{f}} \Sigma_A. \tag{34}
\]

which is obtained by taking the \( \bar{m} \) component of the generalized Boltzmann equation (23) and dropping the second-order terms, because we want to substitute the result for \( \bar{m} \) into the anomalous potential \( V_{\bar{m}} \) and only keep terms up to second order. In stationarity, we solve for \( \bar{m} \) in the eigenbasis of \( \Sigma_N \)

\[
\Sigma_N |\epsilon_{1'}\rangle = (\epsilon_{1'} - \mu) |\epsilon_{1'}\rangle, \tag{35}
\]

and obtain

\[
\bar{m}_{1'2'} = \mathcal{P} \Sigma_{A\mu} (1 + \hat{\bar{f}}) \langle \mu | \bar{m}_{1'2'} \tag{36}
\]

as an adiabatic solution. Adiabatic here means that this solution only includes time-variations with characteristic times long compared to the duration of a collision. We use the Cauchy principal value \( \mathcal{P} \) to indicate omission of the divergent term in an energy integral or sum. This divergent \( \delta \)-function term gives rise to the imaginary part.

We plug this result into the off-diagonal potential (15),

\[
V_{\bar{m}} 1'2' \equiv 4 \mathcal{P} \frac{\phi^{1'2'3'4'} (1 + \hat{\bar{f}}) \langle \mu | \bar{m}_{1'2'} \Sigma_A^{2'2'} \epsilon_{1''} \rangle}{2\mu - (\epsilon_{1'} + \epsilon_{1''})} m_{3'4'}. \tag{37}
\]

where we dropped the recursive \( V_{\bar{m}} \) term in the anomalous coupling \( \Sigma_A \) in order to keep Eq. (37) at second order. We will discuss the recursive term in Sec. III D.

We then recognize that we can write the off-diagonal element of the HFB propagator \( \Sigma_A \) as the real part of a many-body \( T \) matrix

\[
\Sigma_A = V_{m^e} + V_{\bar{m}} = T_{m^e}(2\mu), \tag{38}
\]

which is defined by

\[
T_{m^e} 1'2'3'4' (\epsilon) = 2\mathcal{P} \phi^{1'2'3'4'} (1 + \hat{\bar{f}}) \langle \mu | \bar{m}_{1'2'} \Sigma_A^{2'2'} \epsilon_{1''} \rangle \epsilon_{1''} \tag{39}
\]

The energies \( \epsilon^0 \) are dressed by the normal and mean-field shifts, but are not the full quasiparticle energies, because they do not include the effect of the pairing field. Contractions of this \( T \) matrix with anomalous averages are defined by

\[
T_{m^e} 1'2' (\epsilon) = T_{m^e} 1'2'3'4' (\epsilon)m_{3'4'}. \tag{40}
\]

The \( T \) matrix defined in Eq. (39) is a function of energy through its last two indices in the sense that its argument \( \epsilon = \epsilon_{1'}^0 + \epsilon_{1''}^0 \).

#### B. Diagonal Potentials

In this Section, we want to redefine the diagonal potentials \( U_f \) and \( U_j \) as the real parts of \( T \) matrices by...
using the second-order energy shifts $\mathcal{P}\{\Gamma\}$. With $\mathcal{P}\{\Gamma\}$ we here denote the principal-value part of the collisional terms in Eqs. (28) and (16) according to Eq. (20). We will begin by considering the condensate potential.

\[ -i \, 2U_{f^c} + \mathcal{P}\{2\Gamma_{f^c f} f f^c + \Gamma_{f f^c f^c}\} - 2\Gamma_{f^c f^c f f^c} = -i \, 2U_{f^c} - \mathcal{P}\{\Gamma_{1(1+2)f} f f^c\} \]  \hspace{1cm} (41)

We here assume real eigenfunctions for the single-particle energy basis and do not include any $\Gamma$ terms involving the anomalous average $\bar{m}$, because they are at least of order $V_{1\text{fin}}^2/\langle \Delta \epsilon \rangle^2$ according to Eq. (36). The second-order terms that contain only normal fluctuations $\tilde{f}$ are used in Eq. (46) to rewrite the fluctuation potential $U_{f^c}$. The term in Eq. (41) can again be written in terms of a many-body $T$ matrix

\[ U_{f^c} + \frac{1}{2t} \mathcal{P}\{\Gamma_{1(1+2)f} f f^c\} = T_{f^c}, \]  \hspace{1cm} (42)

which is given by

\[ T^{1'2'3'4'}(\epsilon) = 2\phi^{1'2'3'4'} + 4\mathcal{P}\phi^{1'2'3'4'}(1 + 2\tilde{f})_{4'5'6'7'} \phi^{5'6'7'8'}. \]  \hspace{1cm} (43)

The slight difference compared to Eq. (39) is resolved when we assume diagonal quasiparticle populations $P_{12} = P_{\delta} \delta_{11}$ as will be justified in Sec. IV:

\[ \tilde{f}_{1'2'} = U_{1'} f_{1'2'} = \tilde{f}_{2'1'}. \]  \hspace{1cm} (44)

where $U$ is the transformation matrix to the quasiparticle basis. Alternatively, we note that the $T$ matrix is essentially constant for energy differences up to the duration of a collision. Contraction of the $T$ matrix with normal averages are performed according to

\[ T_{f'}^{1'2'3'4'} = T^{1'2'3'4'}(\epsilon_0^{1'2'3'4'} + \epsilon_0^{1'2'3'4'}) f_{3'2'}. \]  \hspace{1cm} (45)

We now consider the fluctuation potential $U_{f^c}$, again include the truly second-order energy shifts, and obtain

\[ -i \, 2U_{f^c} + \mathcal{P}\{\Gamma_{f f^c f} f - \Gamma_{(1+2)f f^c}\} = -i \, 2U_{f^c} - \mathcal{P}\{\Gamma_{1(1+2)f} f f^c\}, \]  \hspace{1cm} (46)

where we get a different $T$ matrix defined by

\[ T^{1'2'3'4'}(\epsilon) = 2\phi^{1'2'3'4'} + 4\mathcal{P}\phi^{1'2'3'4'}(1 + \tilde{f})_{4'5'6'7'} \phi^{5'6'7'8'}. \]  \hspace{1cm} (47)

which does not have a factor of 2 in the intermediate-population term $(1 + \tilde{f})$. This difference is due to the fact that the mean field $\alpha$ is not bosonically enhanced. If we assume diagonal population $\tilde{fi}1'2' = \delta_{i1'} \tilde{f}1'2'$, which is not a good approximation in this basis, we reproduce the results of Ref. [30] for the GP equation to second order in the interaction potential. In particular, the factor of 2 in their many-body $T$ matrix in the term corresponding to Eq. (46) gets canceled with a negative term from adiabatically eliminating their triple average.

### C. Renormalized Propagators

Using the $T$ matrices defined in the previous sections, we can now rewrite the general GP propagator $\Pi$ given in Eq. (11) and the generalized Boltzmann propagator $\Sigma$ given in Eq. (25).

The Hamiltonian of the GP equation is now

\[ \Pi'_{N} = \hat{H}\langle 0 \rangle + 2T_{f^c} + 2T_{f^c} - \mu, \]  \hspace{1cm} (48)

and the anomalous coupling $\Pi'_{A}$ vanishes, because of the identity

\[ V_{m} \alpha = \{ T_{m} \langle 2\mu \rangle - V_{m} \} \alpha = \{ T_{f^c} \langle 2\mu \rangle - U_{f^c} \} \alpha. \]  \hspace{1cm} (49)

This means that the coupling between $\alpha$ and $\alpha^*$ vanishes and they are no longer independent quantities, and that we can without loss of generality treat $\alpha$ as real.

When we write the Boltzmann propagator in terms of $T$ matrices, we can explicitly show that the energy spectrum is gapless; this theory thus fulfills the Hugenholtz-Pines theorem [29]. The diagonal part of the propagator $\Sigma$ is now

\[ \Sigma'_{N} = \hat{H}\langle 0 \rangle + 2T_{f^c} + 2T_{f^c} - \mu, \]  \hspace{1cm} (50)

and the anomalous coupling is

\[ \Sigma'_{A} = T_{m} \langle 2\mu \rangle. \]  \hspace{1cm} (51)

We now show that this renormalized Boltzmann propagator $\Sigma'$ has a zero-energy eigenvector, i.e., its spectrum is gapless. We begin by writing the generalized GP equation for the condensate ground state

\[ \{ \hat{H}\langle 0 \rangle + 2T_{f^c} \langle 2\mu \rangle + 2T_{f^c} \langle 2\mu \rangle \} \alpha = \mu \alpha. \]  \hspace{1cm} (52)

This equation can be written in terms of the renormalized GP Hamiltonian (48) as $\Pi'_{N} \alpha = 0$, where the energies are now measured relative to the adiabatic chemical potential $\mu$. An immediate consequence of Eq. (52) is that the quasiparticle ground state

\[ P_{\alpha} = C \left( \frac{\alpha}{-\alpha^*} \right), \]  \hspace{1cm} (53)

with a normalization constant $C$, is a zero-energy eigenvector of the renormalized $\Sigma'$, because of the identity

\[ T_{m^c} \langle 2\mu \rangle \alpha^* = T_{f^c} \langle 2\mu \rangle \alpha. \]  \hspace{1cm} (54)

These zero-energy eigenvectors of the HFB propagator are proportional to the condensate mean field $\alpha$. All non-zero-energy eigenvectors $W_{E_{i} \neq 0}$ of $\Sigma$ are thus automatically orthogonal to the condensate, and we can use the complete set $\{ \alpha, W_{E_{i} \neq 0} \}$ as a basis to describe the condensate interacting with thermal excitations [31]. Other approaches to finite-temperature theories [12, 17] have to explicitly orthogonalize their bases using projection operators. We discuss this new basis in Sec. IV.
D. Ladder approximation

We can extend the second-order $T$ matrices introduced in the previous sections to include ladder diagrams to all orders by using consistency arguments. We first note that by keeping the recursive $V_n$ term on the right side of Eq. (37), we can extend our definition of the off-diagonal $T$ matrix to

$$T^{1'2'3'4'}(\epsilon) = 2\delta^{1'2'3'4'} + 4\mathcal{P}\delta^{1'2'3'4'}(1 + 2\bar{f}) \delta^{3'2'4'} (\epsilon - \epsilon_0')$$

This is the real part of the many-body $T$ matrix in the ladder approximation.

In the previous section, we showed that the HFB propagator $\Sigma$ is gapless with the ladder approximation.

Theorem tells us that the full theory should again be gapless because the cancellation in Eq. (54) only works if the two $T$s are identical. However, we used the scattering $\Sigma$ to the two order, because the cancellation in Eq. (54) only works if the two $T$s are identical. However, the Hugenholtz-Pines theorem tells us that the full theory should again be gapless. We thus conclude that we have to upgrade the $T$ matrix on the diagonal of $\Sigma$ to the ladder approximation as well.

We would like to finish our discussion of the scattering matrices in terms of the single-particle energy basis with two remarks. First, as the Liouville-space formulation [32] of density-matrix evolution shows, the scattering should really be formulated in terms of Liouville-space scattering $T$ matrices [33], which can be expressed in terms of Hilbert-space $T$ matrices as

$$T = T \otimes 1 + 1 \otimes T^\dagger + T \otimes T^\dagger.$$  

(56)

Since we only consider Hilbert-space $T$ matrices, we thus would miss higher-order terms of the type $T \otimes T^\dagger$, even if we included the full many-body Hilbert-space scattering matrix. The imaginary part of $T$ gives rise to the inelastic rates in the kinetic equations in Sec. IV.

Second, since the asymptotic states in this scattering problem are typically trapped harmonic-oscillator states for dilute, trapped atomic gases, we are strictly speaking dealing with bound-state $R$ matrices [34] instead of $T$ matrices.

IV. QUASIPARTICLE KINETIC EQUATIONS

A. Quasiparticle Basis

We now want to write the kinetic equations (23) in the Bogoliubov quasiparticle basis. In this basis, the complicated and non-linear evolution due to the HFB propagator $\Sigma'$ is replaced with a simple commutator with the eigenergies and a slow basis rotation. As this theory is gapless, the $E \neq 0$ quasiparticle states together with the condensate $\alpha$ form an orthogonal basis, i.e., the thermal fluctuations are by definition orthogonal to the condensate. Another motivation for transforming to a diagonal first-order Hamiltonian can be found in the numerical results of Walser et al.; they find that the reversible first-order evolution leaves the quasiparticle populations constant. Thus, in the quasiparticle basis, only the second-order collisional terms change the populations. A more detailed account of the transformation to the quasiparticle basis can be found in Ref. [35].

Since the quasiparticles consist of the eigenvectors of the self-energy propagator $\Sigma'$, we consider the propagator’s $2n$ by $2n$ ($n$ is the number of single-particle states considered) eigenvector matrix $W$ defined by

$$\Sigma' W = W E$$

(57)

at each time with the diagonal quasiparticle eigenergy matrix $E$, which is labeled with the quasiparticle indices $1 \equiv E_1$. The eigenvalue equations (57) are the Bogoliubov-de-Gennes equations. We decompose their solution $W$ into two $n$ by $2n$ matrices $U$ and $V^*$,

$$W = \begin{pmatrix} U_{1'1} & U_{1'2} \\ V^*_{1} & V^*_{2} \end{pmatrix},$$

(58)

which are in turn split into $n$ by $n$ matrices for positive $(u^+, v^*)$ and negative quasiparticle energies $(u^-, v^*)$. These quasiparticle eigenergies $\epsilon$ are the column indices and the original single-particle eigenvalues $\epsilon'$ the row indices, such that $U = [U_{1'}^{1}]^{1'}_{1}$. Since the propagator $\Sigma'$ is positive semi-definite, the eigenvalues $E$ in Eq. (57) are real and come in positive and negative pairs of equal magnitude [36]. We argued in the previous section that there exists a zero-energy eigenvector $P_0$ defined in Eq. (53). This means that the propagator $\Sigma'$ is degenerate and has a two-dimensional null space. The eigenvalue equation (57), however, does not yield the second linearly independent zero-energy eigenvector. Instead, the associated vector $Q_\alpha$ is given by [36]

$$\Sigma' Q_\alpha = -i P_0 M,$$

(59)

with a positive constant $M$. In our case, we find

$$ Q_\alpha = -i C \begin{pmatrix} \alpha^* \\ \alpha \end{pmatrix},$$

(60)

up to a normalization constant $C$. In order to find a complete set of basis states, we now define the quadrature components of $P_\alpha$ and $Q_\alpha$:

$$W^{+0} = \frac{1}{\sqrt{2}} (P_\alpha + i Q_\alpha) = \sqrt{2} C \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

$$W^{-0} = -\frac{1}{\sqrt{2}} (P_\alpha - i Q_\alpha) = \sqrt{2} C \begin{pmatrix} 0 \\ \alpha^* \end{pmatrix}. $$

(61a)

(61b)

If we now substitute these two states $W^{+0}$ and $W^{-0}$ into the zero-energy columns of $W$, we can normalize all eigenvectors using the symplectic norm

$$ W^\dagger \sigma_3 W = \sigma_3.$$

(62)
The negative-energy states thus have negative norm, and the positive-energy states positive norm. In particular, the zero-energy vector \( W_{+0} \) has positive norm and thus belongs to the positive-energy part of \( W \). This normalization fixes the constant \( C \) for the zero-energy states to

\[
C = \frac{1}{\sqrt{2}a^\dagger a} = \frac{1}{\sqrt{2N_c}}.
\]

where we defined the number of condensate atoms \( N_c \). The completeness relation for the quasiparticle basis also has symplectic structure

\[
W\sigma_3W^\dagger\sigma_3 = \mathbb{I}_{2n},
\]

where \( \mathbb{I}_{2n} \) indicates the \( 2n \)-dimensional unit-matrix. The symplectic structure of the orthonormalization and completeness relations Eqs. (62) and (64), i.e., the appearance of the matrix \( \sigma_3 \) in these expressions, guarantees that the transformation to the quasiparticle basis is canonical [36]. Canonical here means that the new quasiparticle operators obey the boson commutation relations.

To further examine the structure of the quasiparticle states \( W \), we consider the following symmetry of the HFB propagator \( \Sigma' \)

\[
\Sigma' = -\sigma_1\Sigma'^*\sigma_1,
\]

which holds according to its definition in Eq. (25). This symmetry implies [2, IV.B.] the following relation for the quasiparticle states \( W \):

\[
W = \begin{pmatrix} u_+ & u_- \\ v_+^* & v_-^* \end{pmatrix} = \begin{pmatrix} u_+ & v_+ \\ v_+^* & u_+^* \end{pmatrix}.
\]

This relation in Eq. (66) shows that the positive- and negative-energy eigenvectors of \( \Sigma' \) are not independent but related by

\[
W^\dagger = \sigma_1W^{−1*}. \tag{67}
\]

Plugging \( W \) from Eq. (66) into the full completeness relation Eq. (64), we find a completeness relation for the independent elements of \( W \) alone:

\[
\frac{1}{N_c}a\alpha^\dagger + \sum_{i>0}(u_i^\dagger u_i^{1*} - v_i^\dagger v_i^{1*}) = \mathbb{I}_n. \tag{68}
\]

This is the basis we want to use for the kinetic equations. The explicit split in a condensate mode \( \alpha \) and the orthogonal fluctuation modes \( u_i^{1>0} \) and \( v_i^{1>0} \) is similar to the formalism of Ref. [31]. While the condensate mode \( \alpha \) evolves according to Eq. (9) with the updated propagator \( \Pi' \), we have to find an evolution equation for the quasiparticle populations.

We thus write the generalized single-time fluctuation-density matrix \( \tilde{G}^< \) defined in Eq. (21) in terms of quasiparticles as

\[
\tilde{G}^< = WPW^\dagger = \begin{pmatrix} UPU^\dagger & UPV^\dagger \\ V^* PU^\dagger & V^* PV^\dagger \end{pmatrix}, \tag{69}
\]

where \( P \) is the not necessarily diagonal \( 2n \) by \( 2n \) quasiparticle population matrix. The time-reversed density matrix \( \tilde{G}^< \) transforms according to its definition Eq. (22). From the same equation, we can also deduce that the quasiparticle population matrix \( P \) is Hermitian and fulfills a similar identity:

\[
\sigma_3 + P = \sigma_1P^*\sigma_1 = \sigma_1P^\dagger\sigma_1. \tag{70}
\]

This identity implies that \( P \) can be written as

\[
P = \begin{pmatrix} p & q \\ q^* & (1 + p)^* \end{pmatrix}, \tag{71}
\]

in a structure similar to \( \tilde{G}^< \) in Eq. (21). Since the classical mean field \( \alpha \) only undergoes stimulated emission, the ground-state factors \( P_{++0} = P_{--0} \) do not contain enhancement terms. Thus, the zero-energy components of \( \sigma_3 \) in Eq. (70) and similar enhancement terms associated with \( \alpha \) actually have to be zero.

We also represent the condensate (10) by a population matrix \( P_c \) in the quasiparticle basis

\[
\chi\chi^\dagger = WP_cW^\dagger. \tag{72}
\]

In order to maintain orthogonality between the condensate and the thermal excitations, we have to demand an analogue to the constraint (24) in the quasiparticle basis. We here have to distinguish two cases.

In the first case, the system evolves slowly enough that the ground-state of the adiabatic basis given in Eq. (68) is the true condensate state \( \alpha(t) = \alpha \) (real). We can then explicitly give the condensate matrix as

\[
P_c = N_c\epsilon_0\delta_{0^0}, \tag{73}
\]

with a vector \( \delta_{0^0} = \delta_{1^0 \pm 0} \), such that all non-zero-energy components of \( P_c \) vanish. To implement the orthogonality constraint, we then explicitly set the \( E = \pm 0 \) elements of the quasiparticle matrix \( P \) to zero:

\[
P_{01} = P_{10} = 0, \quad \text{for all } \mathbb{I}. \tag{74}
\]

In the second case, the system evolves too fast for the quasiparticle basis to follow. Then, the condensate matrix contains components of non-zero energy. In this case, the more general orthogonality constraint

\[
\langle \alpha(t) | P = \langle \alpha(t) | \mathbb{I} \rangle P_{12} | 2 \rangle = 0 \tag{75}
\]

has to be fulfilled. This constraint is particularly important when the condensate is coherently excited in linear response. For adiabatic evolution, Eq. (75) reduces to Eq. (74). In the next section, we will find the evolution equation for \( P \).

### B. Kinetic Equations

With these ingredients, we can now try to obtain an equation of motion for the occupation number matrix \( P \).
from the kinetic equation (23). We use Eq. (69) to substitute the fluctuation-density matrices $G$ and obtain

$$\frac{d}{dt}P = -i[E, P]_t + \left\{ W^{-1}\frac{dW}{dt}P + \text{H.c.} \right\} + \left\{ W^{-1} \Gamma < W(\sigma_3 + P) - W^{-1} \Gamma > WP + \text{H.c.} \right\}. \tag{76}$$

We are now left with the task of transforming the collisional contributions to the quasiparticle basis. To this end, we define new two-particle matrix elements in the quasiparticle energy basis by

$$\Phi^{1234}_{\pi} \equiv \phi^{1'2'3'4'} V_{1'} V_{2'} U_{3'} U_{4'}^{\dagger}. \tag{77}$$

These quasiparticle matrix elements have the same symmetries as the original ones given in Eq. (4). Furthermore, they fulfill

$$\Phi^{1234}_{\pi} = (\Phi^{1234}_{\pi})^\dagger = \Phi^{3-4-1-2}. \tag{78}$$

The careful examination of symmetries in Ref. [35] shows that the collision operator Eq. (28) reduces to terms containing the completely symmetric matrix elements

$$\psi^{1234}_{\pi} \equiv \frac{1}{\sqrt{3}} \sum_{\pi} \Phi^{\pi(1234)}_{\pi}, \tag{79}$$

which are defined as a sum over all index permutations $\pi$.

The resulting kinetic equations for the quasiparticle populations $P$ are

$$\frac{d}{dt}P = -iEP + BW P + C_{PP} + C_{CP} + \text{H.c.}, \tag{80}$$

with the non-adiabatic basis-rotation term

$$BW = W^{-1}\frac{dW}{dt} = \sigma_3 W^\dagger \sigma_3 \frac{dW}{dt}, \tag{81}$$

and the Boltzmann collision rates between fluctuations

$$C_{PP} = C_{PP}^\ominus(\sigma_3 + P) - C_{PP}^\ominus P \tag{(82)}$$

and between fluctuations and the condensate

$$C_{CP} = 3 \Gamma_{PC}(\sigma_3 + P)(\sigma_3 + P) - 3 \Gamma_{PC}(\sigma_3 + P)P. \tag{83}$$

These collision rates are defined in terms of the quasiparticle collision operator

$$\sigma_3 \Gamma_{pp} = \frac{1}{2} \psi^{1234}_{\pi} \psi^{1'2'3'4'} P_{-1-1} P_{-2-2} P_{4+4} |3\rangle \otimes |3'\rangle. \tag{84}$$

In this operator, $P$ can stand for any one of the three possibilities $P_{\pi}$, $(\sigma_3 + P)$, or $P_{\text{c}}$, as needed in Eqs. (82) and (83). The approximately energy conserving matrix element $\psi_{\pi}$ is explicitly given by

$$\psi_{\pi}^{1234} = \psi^{1'2'3'4'} \pi \delta_1 (E_1 + E_2 + E_3 + E_4), \tag{85}$$

where the quasiparticle energies can be positive or negative depending on their index. In an on-shell scattering event, this delta function forces two of the indices to be positive and the remaining two to be negative. This has to be considered in interpreting the collision terms $C_{PP}$ and $C_{CP}$, because there all the sums run over positive and negative indices. The principal-value part in Eq. (20), which appears in the single-particle kinetic equations, is absorbed in the $T$ matrices in Sec. III and thus does not appear anymore in the quasiparticle equations.

To complete the presentation of the quasiparticle kinetic equations, we write the generalized Gross-Pitaevskii equation (9) with the updated GP propagator $\Pi$ and the second-order collisional contributions expressed in terms of quasiparticle populations $P$ and obtain

$$\frac{d}{dt} \chi = -i \Pi \chi + W \sigma_3 (\Gamma_{PP}(\sigma_3 + P) - \Gamma_{(\sigma_3 + P)(\sigma_3 + P)}) W^{-1} \chi \tag{(86)}$$

In Sec. V, we write this equation for $\sigma(t)$ alone. This equation together with Eq. (80) gives a complete description in terms of Bogoliubov quasiparticles of a condensate coupled to a thermal cloud at finite temperatures.

C. Orders of magnitude

We now want to discuss the orders of magnitude of several of the quantities of this theory. This will suggest some approximations to the full quasiparticle kinetic equations (80).

We first consider the basis transformation $W$. The completeness relation for the quasiparticle basis Eq. (64) tells us that

$$W = O(1). \tag{87}$$
For example, for high-energy eigenfunctions, the effect of the condensate becomes small, and the quasiparticle transformation reduces to

\[ u \rightarrow 1 \quad \text{and} \quad v \rightarrow 0, \quad \text{as} \quad E \rightarrow \infty. \quad (88) \]

Now, considering the basis-rotation term defined in Eq. (81) we find

\[ B_W = W^{-1} \frac{dW}{dt} = O(1) = O(\Gamma) < O \left( \frac{1}{\tau_c} \right), \quad (89) \]

because the time-scale for population changes, which change the quasiparticle basis \( W \), is limited by the time between collisions \( dt > \tau_c \). In equilibrium, the populations are constant due to detailed balance of the in and out rates. Thus, the net collision rate \( \Gamma \equiv C_{PP} + C_{\alpha P} + H.c. \) gives a better estimate for population changes \( dt \approx \Gamma^{-1} \). This also confirms that \( B_W = 0 \) in equilibrium, since \( dW/dt = 0 \).

We now show that the stationary solutions \( P \) of the Boltzmann equation (80) are diagonal. Considering the stationary solution \( \frac{d}{dt} P_{13} = 0 \) of Eq. (80) for an off-diagonal element with \( 1 \neq 2 \), we obtain

\[ P_{1 \neq 2} = \frac{1}{i} \frac{B_W P + C_{PP} + C_{\alpha P} + H.c.}{E_2 - E_1} = O \left( \frac{\Gamma}{\Delta \epsilon} \right). \quad (90) \]

This shows that the off-diagonal elements of the quasiparticle population are small compared to the diagonal ones, which are of the order of the number of particles \( N \), and vanish at equilibrium.

D. Quasiparticle \( T \) matrices

The \( T \) matrix in Eq. (39) (or the ladder extension Eq. (55)) has been obtained appropriately to second order. The second-order term with the factor of 1 in Eq. (39) is the divergent part, which is renormalized in Eq. (55) when replaced by the \( T \)-matrix. The term containing \( 2 \dot{f} \) of Eq. (39) is a convergent many-body second-order term. It is thus reasonable to replace the \( T \) of Eq. (55) with

\[ T^{1'2'3'4'}(E) = T_{2B}^{1'2'3'4'}(E + 2\mu) + 8 P \phi^1 2' 3' 4' \tilde{f}_{1'2'3'4'} \Phi_{3'4'} = \frac{E - (E_{3'} + E_{4'})}{E_{2'} + E_{4'}} (91) \]

where we replaced the single-particle energies in the denominator by quasiparticle energies, because the difference is of higher order. We here use the following two-body \( T \) matrix

\[ T_{2B}^{1'2'}(E) = 2 \phi^1 2' 3' 4' + 4 P \phi^1 2' 3' 4' \phi^3 4' \phi^3 4' \frac{E - (E_{3'} + E_{4'})}{\epsilon - (E_{3'} + E_{4'})}, \quad (92) \]

which is given in terms of single-particle energies \( \epsilon^{sp} \) defined by

\[ \hat{H}^{(0)} | \epsilon^{sp}_{1'1} \rangle = \left( \frac{p^2}{2m} + V_{ext}(\vec{x}) \right) | \epsilon^{sp}_{1'1} \rangle = \epsilon^{sp}_{1'1} | \epsilon^{sp}_{1'1} \rangle. \quad (93) \]

The collisional terms of the kinetic equations (80) correspond to the imaginary part of a Liouville-space \( T \)-matrix (56). Examining the argument of the \( \delta_{\eta} \)-function in Eq. (85), which defines \( \Psi_{\eta} \), in comparison with the ladder \( T \) in Eq. (55) shows that (when the energies are correlated with the appropriate elements of \( W \), see [35]) two of the energies in the denominator must be positive and two negative, which, together with the fact that \( P \) is diagonal close to equilibrium, leads to terms in the kinetic equation of the form \( P P (1 + P) (1 + P) \), etc. The kinetic equation (80) then becomes the quantum Boltzmann equation for quasiparticle populations. The equilibrium solution is therefore the expected Bose-Einstein distribution for the quasiparticles, as the steady-state solution of Eqs. (80) and (86) shows [12, Sec. IV]. The interaction matrix elements \( \phi \) in Eq. (77) can also be upgraded to \( T \). These equations in terms of \( T \) are now consistent with an impact approximation treatment (with elastic scattering not contributing when \( T \otimes T' \) terms are considered, cf Eq. (56)).

E. Conservation laws

Because of the non-vanishing pairing field, the trace of the quasiparticle density matrix \( P \) is in general not equal to the number of excited particles in the system. Hence, the number of quasi-particles is not conserved. Our single-particle kinetic equations (9) and (23) do, however, conserve the mean total number

\[ \langle N \rangle = \alpha^\dagger \alpha + \text{Tr} \{ f \} = N_c + \text{Tr} \{ f \} = \text{const} \quad (94) \]

this can be proven explicitly by plugging these kinetic equations into \( \frac{d}{dt} \langle N \rangle \) and canceling terms. We thus adopt a self-consistent procedure for the quasiparticle kinetic equations Eqs. (80) and (86), by requiring

\[ \langle N \rangle = N_c + \text{Tr} \{ U P U^\dagger \} = \text{const} \quad (95) \]

This equation self-consistently constrains the number of condensate atoms \( N_c \): in equilibrium at temperature \( \beta^{-1} \), the quasiparticle matrix \( P \) consists of \( q = 0 \) and

\[ p(E) = \frac{1}{e^{\beta (E - \mu)} - 1} (96) \]

according to Eq. (71) with the chemical potential \( \mu \) given by the GP equation (52). As temperature tends to zero, we obtain the usual corrections for the anomalous average and condensate depletion [12]. Note that the number of excited atoms is not given by the trace of \( p \), but has to include the basis transformation \( U \) as discussed above. If we drop the basis-rotation term \( B_W \) in numerical simulations, we will incur number-non-conservation on the order of \( \Gamma \), while away from equilibrium.

Since we use a Markovian collision integral with a damping function of finite width \( \eta \) in order to include
off-the-energy-shell propagation, this theory is not exactly energy conserving. Markovian theories fail to track the decay of initial correlations and thus do not account for the decay of the correlation energy [37]. In our case, with a self-consistent \( \eta \), energy is conserved to order \( \eta \), which is consistent with the order of approximation. For a detailed discussion of these memory effects and how they affect the conservation laws see [38].

\textbf{V. SUMMARY}

We summarize the main results of this paper given in Secs. III and IV. We formulate a kinetic theory in terms of Bogoliubov quasiparticle modes \( W \), which are defined by the eigenvalue equation for the renormalized Hartree-Fock-Bogoliubov propagator \( \Sigma' \)

\[
\left( \hat{H}^{(0)} + 2T_{f^-} + 2T_{f}^+ - \mu \begin{pmatrix} \Sigma'_{\alpha} \\ -\Sigma'_{N} \end{pmatrix} \right) W = WE. \tag{57}
\]

The \( T \) matrices are defined to second order in Eqs. (39) and (47) and Eq. (55) gives an extension to the ladder approximation. The Gross-Pitaevskii equation

\[
\left\{ \hat{H}^{(0)} + T_{f^-}(2\mu) + 2T_{f}^+(2\mu) \right\} \alpha = \mu \alpha \tag{52}
\]

for the ground state \( \alpha \) is contained in Eq. (57), because the renormalized \( \Sigma' \) is gapless. The GP equation defines the value of the chemical potential \( \mu \). To find a complete basis \( W \), we have to find the second zero-energy mode and form quadrature components as discussed in Sec. IV A.

We find the following Boltzmann equation for the thermal excitations in terms of Bogoliubov quasiparticles:

\[
\frac{d}{dt} P = -i[E, P] + \left\{ \sigma_3 W^\dagger \sigma_3 \frac{dW}{dt} P + \text{H.c.} \right\} + \left\{ \Gamma_{PP'(\sigma_3 + P)}(\sigma_3 + P) - \Gamma_{(\sigma_3 + P)P'P}P + \text{H.c.} \right\} + 3 \left\{ \Gamma_{P-(\sigma_3 + P)P'}(\sigma_3 + P) - \Gamma_{P'(\sigma_3 + P)PP}P + \text{H.c.} \right\}.
\tag{80}
\]

The basis-rotation term containing \( \frac{d}{dt} W \) can be dropped for adiabatic evolution. The quasiparticle density matrix \( P \) is diagonal close to equilibrium, and its elements obey an equilibrium Bose-Einstein distribution as the second and third lines show. The collision terms containing the general condensate matrix \( \Sigma' \) defined in Eq. (72) represent population exchange between the thermal cloud and the condensate. They are balanced in the following equation for the condensate

\[
\frac{d}{dt} \alpha(t) = \left\{ \hat{H}^{(0)} + T_{f^-} + 2T_{f}^+ - \mu \right\} \alpha(t) + U \sigma_3 \Gamma_{PP'(\sigma_3 + P)} - \Gamma_{(\sigma_3 + P)P'P}P \sigma_3 \times \left\{ U^\dagger \alpha(t) - V^\dagger \alpha^*(t) \right\}. \tag{86}
\]

In general, \( \alpha(t) \) can be different from the \( \alpha \) used as the ground state of the adiabatic basis. A non-adiabatic change in a driving force, for example, would cause \( \alpha(t) \) to change quickly.

The coupled Eqs. (80) and (86) have to be solved under the orthogonality constraints (75) or (74) depending on whether the evolution is adiabatic or not. Furthermore, the total particle number has to fulfill the self-consistent constraint Eq. (95).

\textbf{VI. CONCLUSIONS}

We have extended the non-equilibrium kinetic theory of Walser et al. [1, 2] in two important respects. First, we write the binary interactions as many-body \( T \) matrices to second order in the interaction by subsuming ultra-violet divergent terms. This procedure removes divergences caused by the anomalous average \( \bar{n} \) and contained in the second-order terms. We can then replace the low-energy limit of the \( T \) matrix with the \( s \)-wave scattering length. This is to our knowledge the only consistent treatment of these difficulties associated with the anomalous average in a theory that includes second-order collisional terms. The updated Hartree-Fock-Bogoliubov propagator \( \Sigma' \) is then gapless, which greatly facilitates a consistent treatment of the condensate coupled to thermal fluctuations.

The second extension of the Walser et al. theory makes use of the gapless HFB propagator by using its quasiparticle eigenmodes to parameterize the thermal fluctuations, which are then automatically orthogonal to the condensate mean field. We find that this basis greatly simplifies the second-order collision terms of the Walser et al. theory. Another important result reported here is that, in equilibrium, the Bogoliubov modes diagonalize the quasiparticle population matrix \( P \). This means that, close to equilibrium, the second-order terms can be evaluated in \( n^4 \) operations, where \( n \) is the number of energy levels considered. This is a vast improvement compared to \( n^8 \) operations for a basis that does not diagonalize the population matrix, such as, for example, the single-particle basis used by Walser et al.

\textbf{Acknowledgments}

J.W. acknowledges financial support from the National Science Foundation. R.W. gratefully acknowledges support from the Austrian Academy of Sciences through an APART grant. M.H. gratefully acknowledges support from the U.S. Department of Energy, Office of Basic Energy Sciences via the Chemical Sciences, Geosciences, and Biosciences Division. J.C. is retired and acknowledges support from TIAA-CREF.
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