REFINED TWO WEIGHT ESTIMATES FOR THE BERGMAN PROJECTION

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ABSTRACT. We prove sufficient conditions for the two-weight boundedness of the Bergman projection on the unit ball. The first condition is in terms of Orlicz averages of the weights, while the second condition is in terms of the mixed $B_\infty - B_2$ characteristics.

1. Introduction

Let $\mathbb{B}^d$ be the complex unit ball in $\mathbb{C}^d$. The Bergman projection from $L^2(\mathbb{B}^d)$ onto the subspace of holomorphic functions $A^2(\mathbb{B}^d)$ is the integral operator

$$Pf(z) := \int_{\mathbb{B}^d} \frac{f(\zeta)}{(1 - z \bar{\zeta})^{d+1}} \, d\nu(\zeta)$$

where $d\nu$ is the normalised measure on $\mathbb{B}^d$. It is a classical result [ZJ64; FR74] that the Bergman projection $P$ extends to a bounded operator on $L^p(\mathbb{B}^d)$ for all $p \in (1, \infty)$.

Weighted estimates were brought in this context by Békollé and Bonami [BB78; Bek82], where the projection $P$ is seen as a singular integral operator with respect to a particular pseudo-metric on the unit ball. See also [McN94] where this approach has been extended to convex domains of finite type. Békollé and Bonami proved that given a real-valued, positive function $w \in L^1_{\text{loc}}(\mathbb{B}^d)$, the Bergman projection $P$ is bounded on the weighted space $L^p(\mathbb{B}^d, w \, d\nu)$ if and only if the weight $w$ satisfies the following condition

$$[w]_{B_p} := \sup_B \left( \frac{1}{|B|} \int_B w \, d\nu \left( \frac{1}{|B|} \int_B w^{1-p'} \, d\nu \right)^{p-1} \right) < \infty \quad (B_p)$$

where the supremum is taken over all balls $B \subset \mathbb{B}^d$ with respect to the pseudo-metric mentioned above, and $|B| := \nu(B)$ denotes the measure of $B$.

The optimal dependence of the norm $\|P\|_{L^p(w) \to L^p(w)}$ on the Békollé–Bonami characteristic $[w]_{B_p}$ has been obtained a few decades later: in dimension $d = 1$ by Pott and Reguera [PR13], and in higher dimension by Rahm, Tchoundja, and Wick [RTW17]. They proved that

$$\|P\|_{L^2(w) \to L^2(w)} \leq C [w]_{B_2} \quad (1.1)$$

where $C$ is a constant independent on the weight. Estimate (1.1) is analogous to the celebrated $A_2$-theorem [Hyt12] from which optimal weighted $L^p$ estimates for $1 < p < \infty$ can be extrapolated, by mean of the sharp extrapolation theorem [Dra+05]:

$$\|P\|_{L^p(w) \to L^p(w)} \leq C_p \max \left\{ \frac{1}{p-1}, 1 \right\}.$$

In the spirit of [HP13], bound (1.1) has been refined in [APR17] by using a dyadic version of the Békollé–Bonami class and the $B_\infty$ characteristic introduced in [APR17, §5].

We now define the joint Békollé–Bonami characteristic, where the pseudo-balls in $(B_p)$ are replaced with dyadic tents.
Definition 1.1 (Dyadic Békollé–Bonami weights). For two weights \( w, \sigma \) on \( \mathbb{B}^d \) and \( p \in (1, \infty) \) their joint \( B_p \) characteristic is

\[
[w, \sigma]_{B_p} := \sup_{K \in \mathcal{T}} \frac{1}{|K|} \int_K w \, d\nu \left( \frac{1}{|K|} \int_K \sigma \, d\nu \right)^{p-1}
\]

where the supremum is taken over the collection of all dyadic tents \( \mathcal{T} \), whose construction is postponed to §2.6. The quantity \( |K| := \nu(K) \) denotes the volume of the dyadic tent \( K \). When \( \sigma = w^{1-p'} \) is the dual weight of \( w \), the quantity \( [w]_{B_p} := [w, w^{1-p'}]_{B_p} \) is the \( B_p \) characteristic of \( w \). We write \( w \in B_p \) if \( [w]_{B_p} \) is finite.

The \( B_\infty \) characteristic is defined as the Fujii–Wilson characteristic [Wil87; Wil08] for the Muckenhoupt \( A_\infty \) class, by mean of the maximal operator

\[
Mf(z) := \sup_{K \in \mathcal{T}} \left( \frac{1}{|K|} \int_K |f| \, d\nu \right) \mathbb{1}_K(z)
\]

where \( \mathbb{1}_K \) is the indicator function on \( K \).

Definition 1.2 (\( B_\infty \) class). A weight \( \sigma \) belongs to the class \( B_\infty \) if the quantity

\[
[w]_{B_\infty} := \sup_{K \in \mathcal{T}} \frac{1}{\sigma(K)} \int_K M(\mathbb{1}_K) \, d\nu
\]

is finite. We refer to \( [\sigma]_{B_\infty} \) as the \( B_\infty \) characteristic of the weight \( \sigma \).

The class \( B_\infty \) has been further studied in [APR19]. Weights in this class are in general not doubling, see [DMO16, Counterexample 1 and Remark 5.1] and [Kos22]. Nevertheless, the subclass of \( B_\infty \) studied in [APR19] enjoys similar properties to the Muckenhoupt \( A_\infty \) class.

In [APR17, Theorem 5.7 and Corollary 5.9] we have the following refinement of estimate (1.1) in \( d = 1 \).

Theorem A (Aleman, Pott and Reguera 2017). Let \( w \in B_2 \) be a weight on \( \mathbb{B}^1 \). The Bergman projection \( P : L^2(\mathbb{B}^1, w) \to A^2(\mathbb{B}^1, w) \) and the following estimate holds

\[
\|P\|_{L^2(w) \to L^2(w)} \leq C \left( [w]_{B_2}^{1/2} + [w^{-1}]_{B_\infty}^{1/2} \right).
\]

In this note we address the following question:

What are the sufficient conditions on two weight \( u, \omega \) for the boundedness of the Bergman projection \( P : L^2(u) \to A^2(\omega) \)?

Progress on this question has been obtained via sparse domination [APR17; Seh21]. Sparse domination is a powerful technique in harmonic analysis to control operators by maximal averages. As a consequence, since weighted estimates for the maximal operator are known, these immediately translate into estimates for the dominated operator. In the case of the unit disc \( \mathbb{B}^1 \), Aleman, Pott and Reguera [APR17] found sufficient and necessary conditions for weights which are modulus of functions in the Bergman space \( A^2(\mathbb{B}^1) \).

This article presents new sufficient conditions for the two-weight boundedness of the Bergman projection on the unit ball. Our first result generalises estimate (1.3) to the two-weight setting and to higher dimension.

Theorem 1. Let \( \sigma, w \) be two weights on \( \mathbb{B}^d \) in the class \( B_\infty \) such that their joint \( B_2 \) characteristic \([w, \sigma]_{B_2}\) is finite. The Bergman projection \( P \) on \( L^2(\mathbb{B}^d) \) satisfies the following bound

\[
\|P(\sigma)\|_{L^2(\sigma) \to L^2(w)} \leq C \left( [w, \sigma]_{B_2}^{1/2} + [\sigma]_{B_\infty}^{1/2} \right)
\]

where \( C \) is a positive constant independent of \( \sigma \) and \( w \).
Remark 1.3. A few remarks are in order.

- The classical $B_2$ characteristic is defined using Carleson tents, which we recall in §2.1. For two weights $w, \sigma$ in $B_\infty$ their classical $B_2$ characteristic and the dyadic one in (1.2) are comparable, see §3.1.
- Theorem 1 extends Theorem A to higher dimension and also to general weights $w, \sigma$ that are not dual to each other.
- For the one-weight theory, the estimate in Theorem 1 improves on the $B_2$ estimates in [RTW17], since $[\sigma]_{B_\infty} \leq [\sigma]_{B_2}$. We recall the bound proved in the original paper [APR17] in Proposition 2.9.

Our second result is a sufficient condition in terms of Orlicz averages of two weights. Orlicz spaces generalise $L^p$ spaces and their norms are defined using Young functions, which we recall in §2.2.

**Definition 1.4.** Given two weights $w, \sigma$ and two Young functions $\Phi, \Psi$, we define the joint Orlicz bump as

$$[w, \sigma]_{\Phi, \Psi} := \sup_{R \in \mathcal{B}} \left( \frac{\langle w \rangle_R}{\langle w^{1/2} \rangle_{\Phi,R}} \right) \left( \frac{\langle \sigma \rangle_R}{\langle \sigma^{1/2} \rangle_{\Psi,R}} \right)$$

where $\langle \cdot \rangle_{\Phi,R}$ denotes the Orlicz average on the dyadic tent $R$ with respect to $\Phi$. See §2.2 for the precise definitions of Orlicz average and the associated maximal function.

We have the following result.

**Theorem 2.** Let $\sigma, w$ be two weights on the unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$ and let $\Phi, \Psi$ be two Young functions such that the associated maximal function defined in (2.1) is bounded on $L^2$. Then the Bergman projection $P$ on $L^2(\mathbb{B}^d)$ satisfies the following bound

$$\|P(\sigma \cdot)\|_{L^2(\sigma) \to L^2(w)} \leq C [w, \sigma]_{\Phi, \Psi}$$

where $C$ is a positive constant independent of $\sigma, w$.

The condition in Theorem 2 is known as bump condition, as the averages of the weights have been “bumped up” in the scale of Orlicz spaces. Theorem 2 is deduced from a sparse operator dominating $P$. In particular, it follows by combining the domination in [RTW17] with the known estimates for sparse forms in [Li17]. Nevertheless, to the best of our knowledge, these estimates have not appeared in the context of Bergman spaces on the ball. Previous results involving fractional Bergman operators on the upper-half plane can be found in [Seh18]. Recently Sehba also deduced two-weight estimates for the Bergman projection on the upper-half plane $\mathbb{R}^2_+$ in terms of Sawyer testing conditions as the one in §2.4 for sparse operators, see [Seh21, Theorem 2].

**Remark 1.5.** Fundamental to our approach is the possibility to approximate Carleson tents with dyadic tents. As in [RTW17], we exploit the available dyadic structure on $\mathbb{B}^d$ developed by Arcozzi, Rochberg, and Sawyer in [ARS02]. We explain how this structure is constructed in §2.6.

The construction of the dyadic family is independent of the two weights. If one could construct a family of subsets which is sparse with respect to the given weights, one could dispense of the $B_\infty$ condition in Theorem 1. Indeed, under this assumption the joint Berezin condition is a necessary and sufficient condition for the two-weight boundedness of $P$, see [FW15].

It is possible to construct a similar dyadic structure also on convex domains of finite type via the dyadic flow tents [GHK22]. This generalises the construction in §2.6 for the ball. The resulting collection of dyadic flow tents is sparse, and it produces weighted estimates for the Bergman projection on convex domains of finite type. Since our bump condition implies the boundedness of a sparse operator, the same condition implies the
boundedness of the Bergman projection on convex domains of finite type by the pointwise control in [GHK22, Lemma 4.1]. Similar results on classes of pseudoconvex domains can also be found in [HWW21].

2. Preliminaries

We recall a few classical definitions and we introduce some notations. For two positive quantities $X$ and $Y$, we write $X \lesssim Y$ if there exists a constant $C > 0$ such that $X \leq CY$. In particular, this constant is independent of the weights. We write $X \simeq Y$ if $X \lesssim Y$ and $Y \lesssim X$ holds, possibly with different constants.

Given a Borel set $E \subset \mathbb{B}^d$ and a locally integrable function $f$, we denote the average of $f$ on $E$ with respect to the measure $d\nu$ by

$$\langle f \rangle_E := \frac{1}{\nu(E)} \int_E f \, d\nu.$$ 

We remind the reader that, although some of the results below are for general $L^p$ spaces, only $L^2$ results are needed for our argument.

2.1. Classical Békollé–Bonami weights. We recall the definition of a Békollé–Bonami weight on the unit ball. These weights satisfy an $A_p$ condition where the role of cubes is played by Carleson tents.

Definition 2.1 (Carleson tent on the unit ball). Given a point $z \in \mathbb{B}^d \setminus \{0\}$, consider the following set

$$T_z := \{\zeta \in \mathbb{B}^d : |1 - \langle \zeta, \frac{z}{|z|}\rangle| \leq 1 - |z|\}.$$ 

When $d = 1$ the set $T_z$ is the intersection of $\mathbb{B}^1$ with the disc centred at $z/|z|$ with radius $1 - |z|$, whose boundary contains the point $z$. When $z = 0$, we set $T_0 = \mathbb{B}^d$.

Definition 2.2 (Békollé–Bonami weights). Given $p \in (1, \infty)$ and two weights $w, \sigma$ on $\mathbb{B}^d$, we define their joint $B_p$ characteristic:

$$[w, \sigma]_{B_p} := \sup_{z \in \mathbb{B}^d} \langle w \rangle_{T_z} \langle \sigma \rangle_{T_z}^p - 1,$$

where $\langle w \rangle_{T_z} := |T_z|^{-1} \int_{T_z} w$, and $|T_z|$ denotes the volume of the tent $T_z$. As before, we denote by $[w]_{B_p} := [w, w^{1-p'}]_{B_p}$. We say that $w \in B_p$ if $[w]_{B_p}$ is finite.

Remark 2.3. This classical definition is quantitatively equivalent to the dyadic one in Definition 1.1, see §3.1.

2.2. Orlicz average. We recall the definition of Orlicz averages used in the statement of Theorem 2. We first need to introduce Young functions.

Definition 2.4 (Young function). Let $\Phi : [0, \infty) \to [0, \infty)$ be a continuous, convex, strictly increasing function such that

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty.$$ 

Given a set $Q$ and a Young function $\Phi$ we define the Orlicz average via the Luxembourg norm

$$\langle f \rangle_{\Phi, Q} := \inf \{\lambda > 0 : \langle \Phi(f/\lambda) \rangle_Q \leq 1\}.$$ 

For example, when $\Phi(t) = t^p$ with $1 < p < \infty$ the Orlicz average corresponds to the usual $L^p$ average on the set $Q$. Given a Young function $\Phi$ and a collection of sets, one can consider the maximal function

$$M_{\Phi} f := \sup_Q \langle |f| \rangle_{\Phi, Q} 1_Q$$ (2.1)
where the supremum is taken over all sets in the collection. In the context of this paper, the sets $Q$ are the dyadic tents $K$ whose construction can be found in §2.6.

In [Pér95, Theorem 1.7] Pérez characterised the Young functions for which the associated maximal function is bounded on $L^p$.

**Theorem B** (Pérez 1995). *Given a Young function $\Phi$, the associated maximal function $M_\Phi$ maps $L^p$ to $L^p$, for $1 < p < \infty$, if and only if
\[
\int_1^\infty \frac{\Phi(t)}{t^{1/p'}} \frac{dt}{t} < +\infty.
\]

Note that the operator $M_\Phi$ is also bounded on $L^\infty$ [And15, Lemma 3.2]. We say that a Young function $\Phi$ belongs to $B_p$ if the condition $(B_p)$ holds. We shall not confuse the class of weights $B_p$ and $B_p$ with the one of Young functions in $B_p$. To help the reader, we will always denote Young functions by capital Greek letters (Φ or Ψ), whilst we keep the lower case notation $w, v, u, \sigma$ for weights.

The proof of Theorem 1 exploits testing conditions for sparse operators, which we now introduce.

### 2.3. Sparse families and sparse operators

We start by recalling the definition of a sparse family:

**Definition 2.5** (Sparse collection). A collection of sets $\mathcal{S}$ is $\frac{1}{\tau}$-sparse, for $\tau \geq 1$, if for any $Q \in \mathcal{S}$ there exists a subset $E_Q \subseteq Q$ such that $\{E_Q \}_{Q \in \mathcal{S}}$ are pairwise disjoint and $|Q| \leq \tau |E_Q|$.

**Remark 2.6.** The definition above uses the Lebesgue measure, and it can be applied to other measures as well, although these notions are not, in general, equivalent.

Nevertheless, if $\mathcal{S}$ is a sparse collection with respect to the measure $d\nu$, and $\sigma$ is a $B_p$ weight, then $\mathcal{S}$ is also sparse with respect to the measure $\sigma d\nu$. See Remark 2.8.

Let $\mathcal{S}$ be a sparse collection. We denote by $\Lambda_\mathcal{S}$ the corresponding sparse operator
\[
\Lambda_\mathcal{S} f := \sum_{Q \in \mathcal{S}} (f) Q \mathbb{1}_Q.
\]

### 2.4. Sawyer testing conditions

The weights for which $\Lambda_\mathcal{S}(\sigma \cdot) : L^p(\sigma) \to L^p(\sigma)$ holds, as well as the equivalent dual formulation $\Lambda_\mathcal{S}(\cdot w) : L^{p'}(w) \to L^{p'}(w)$, have been characterised by Sawyer in terms of following testing conditions:

\[
\|\Lambda_\mathcal{S}(\sigma \mathbb{1}_Q)\|_{L^p(w)} \leq \mathcal{T}_\sigma(Q), \quad \forall Q \in \mathcal{S}
\]

\[
\|\Lambda_\mathcal{S}(w \mathbb{1}_Q)\|_{L^{p'}(\sigma)} \leq \mathcal{T}' w(Q), \quad \forall Q \in \mathcal{S}
\]

where the optimal testing constants are the finite quantities

\[
\mathcal{T} := \mathcal{T}_\mu(w, \sigma) := \sup_Q \frac{\|\mathbb{1}_Q \Lambda_\mathcal{S}(\mathbb{1}_Q \sigma)\|_{L^p(w)}}{\sigma(Q)},
\]

\[
\mathcal{T}' := \mathcal{T}'_\mu(w, \sigma) := \sup_Q \frac{\|\mathbb{1}_Q \Lambda_\mathcal{S}(\mathbb{1}_Q w)\|_{L^{p'}(\sigma)}}{w(Q)}.
\]

These conditions are named after Sawyer, who first derived them for maximal operators [Saw82] and for fractional and Poisson integrals [Saw88]. For sparse operators they have been proved in [LSU09].

Testing constants for off-diagonal estimates $\Lambda_\mathcal{S}(\sigma \cdot) : L^p(\sigma) \to L^q(w)$ for $q \neq p$ and more general sparse forms have also been studied, see [Li17, Theorem 1.1]. In particular we have

\[
\|\Lambda_\mathcal{S}(\sigma \cdot)\|_{L^p(\sigma) \to L^q(w)} \approx \left(\mathcal{T}^{1/p} + (\mathcal{T}')^{1/p'}\right).
\]
In the proof of Theorem 1 we estimate the constants $\Xi, \Xi'$ from above with the Béckollé–Bonami characteristic of the weights $w, \sigma$.

2.5. **Program to deduce weighted estimates.** A possible route to prove weighted estimates for $P$ follows these steps, see also [Ler13].

1. **(Control by a positive operator).** The modulus of the Bergman projection is controlled by the maximal Bergman operator:

$$P^+ f(z) := \int_{\mathbb{B}^d} \frac{f(\zeta)}{|1 - z\zeta|^{d+1}} \, dr(\zeta).$$

Namely we have $|Pf(z)| \leq P^+ |f|(z)$. Note that $P^+(\cdot) = (\cdot)$ is a real-valued, positive operator. For positive weights $w, v$ we have

$$\|P\|_{L^2(w) \to L^2(v)} \leq \|P^+(\cdot)\|_{L^2(w) \to L^2(v)}.$$

2. **(Equivalence with a sparse operator).** Once the dyadic structure on $\mathbb{B}^d$ is constructed (see §2.6), the collection of dyadic tents $\mathcal{T}$ is sparse, see Lemma 2.7. The associated sparse operator $\Lambda_{\mathcal{T}}$ is equivalent to the maximal Bergman operator:

$$P^+ |f|(z) \sim_d \Lambda_{\mathcal{T}} |f|(z) := \sum_{K_a \in \mathcal{T}} \|f\|_{K_a} \frac{1}{K_a}.$$ 

See Lemma 3.1, and [RTW17, Lemma 5] for a proof.

3. **(Bumps for the sparse operator).** Two-weight estimates for sparse operators are well understood. For example, they are equivalent to two-weight estimates for the maximal operator $M$, see [Ler13, Theorem 1.2]. Sufficient conditions on $(w, v)$ for the boundedness of

$$\|M\|_{L^2(w) \to L^2(v)} \quad \text{and} \quad \|\Lambda_{\mathcal{T}}\|_{L^2(w) \to L^2(v)}$$

are known in terms of the testing conditions presented in §2.4.

The task of characterising weights $w, v$ for which $\|P\|_{L^2(w) \to L^2(v)}$ is finite is still open.

2.6. **Dyadic structure on the complex unit ball.** We borrow the dyadic structure on the ball developed by Arcozzi, Rochberg, and Sawyer [ARS02, §2.2] and also used in [RTW17, §2]. This structure introduces a collection of sets called “dyadic kubes”, which comes with a tree structure $\mathcal{T}$ called Bergman tree (namely a collection of partially ordered indexes $\{\alpha \in T\}$. The points $\{c_\alpha\}_{\alpha \in \mathcal{T}}$ are the centres of the dyadic kubes).

We explain how the dyadic structure is constructed.

Let $\varphi_z$ be the bi-holomorphic involution of the ball exchanging $z$ and the origin:

$$\varphi_z(w) := \frac{z - \langle w, \frac{z}{|z|^2} \rangle \frac{z}{|z|^2} - \sqrt{1 - |z|^2} (w - \langle w, \frac{z}{|z|^2} \rangle \frac{z}{|z|^2})}{1 - \langle w, z \rangle}.$$ 

The Bergman metric on the unit ball $\mathbb{B}^d$ is defined as

$$\beta(z, w) := \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$ 

In the following, $B(z_0, r) \subset \mathbb{B}^d$ denotes the ball of centre $z_0$ and radius $r$ in the Bergman metric. We also denote by $S_r$ the sphere of radius $r$ centred at the origin, so $S_r = \partial B(0, r)$.

Fix $R, \delta > 0$. For $n \in \mathbb{N}$, there is a collection of points $\{z^n_j\}_{j=1}^n$ and a partition of the sphere $S_{nR}$ in Borel subsets $\{\Omega^n_j\}_{j=1}^n$ such that

(i) $S_{nR} = \bigcup_{j=1}^n \Omega^n_j$;

(ii) $B(z_j, \delta) \cap S_{nR} \subseteq \Omega^n_j \subseteq B(z_j, C\delta) \cap S_{nR}$ for some $C > 0$. 

Let $\pi_{nR}$ denote the radial projection from $\mathbb{B}^d$ onto the sphere $S_{nR}$. The kubes are given by

\[ K_0^n := B(0, R), \]
\[ K_j^n := \{ \zeta \in B(0, (n + 1)R) \setminus B(0, nR) : \pi_{nR}(\zeta) \in \Omega_j^n \}. \]

The centre of the kube $K_j^n$ is $c_j^n := \pi_{n+\frac{1}{2}}R(z_j^n)$. We say that a point $c_j^{n+1}$ is a child of $c_j^n$ if $\pi_{nR}(c_j^{n+1}) \in \Omega_k^n$. Then the centres form a tree structure $T$, which we will refer to as Bergman tree.

![Diagram of a Bergman tree with generations 0 and 1.](image)

**Figure 1.** An example of the first generations of kubes and the respective $\Omega_k^n$ in the dyadic structure on $\mathbb{B}^1$. The structure for $\mathbb{B}^d$ is of a similar spirit.

To simplify the notation, let $\alpha$ be an element in $T$. We denote by $K_\alpha$ the unique kube with centre $\alpha$. If $\beta$ is a descendant of $\alpha$ we write $\beta \geq \alpha$. Given a kube $K_\alpha$, the dyadic tent $\hat{K}_\alpha$ is the union of all kubes whose centres are descendant of $\alpha$ in $T$, namely

\[ \hat{K}_\alpha := \bigcup_{\beta \geq \alpha} K_\beta. \]

The volume of $K_\alpha$ and $\hat{K}_\alpha$ are comparable. This was originally proved in [ARS06, Lemma 2.8], see also [RTW17, Lemma 1].

**Lemma 2.7** (Arcozzi, Rochberg, and Sawyer 2006). Let $T$ be a Bergman tree on $\mathbb{B}^d$ with parameters $R, \delta$. There is a universal constant $\tau > 1$, depending only on $R, \delta$ and the dimension $d$, such that $|\hat{K}_\alpha| \leq \tau |K_\alpha|$ for all $\alpha \in T$.

From this lemma, and from the fact that the kubes $\{K_\alpha\}_{\alpha \in T}$ are pairwise disjoint, it follows immediately that the collection of dyadic tents $\mathcal{T} := \{\hat{K}_\alpha\}_{\alpha \in T}$ is $\frac{1}{\tau}$-sparse, in the sense of Definition 2.5.

**Remark 2.8.** Note that if $\sigma \in B_p$, by Hölder’s inequality the collection $\mathcal{T}$ is $(\tau^p[\sigma]_{B_p})^{-1}$-sparse with respect to the measure $\sigma \, d\nu$. 
The characteristic $[\sigma]_{B_\infty}$ is controlled by $[\sigma]_{B_p}$. We recall the proof from [APR17, Proposition 5.6].

**Proposition 2.9** (Aleman, Pott, Reguera 2017). For $1 < p < \infty$, let $w$ be a weight in $B_p$. Then we have

$$[w]_{B_\infty} \leq [w]_{B_p}.$$  

**Proof.** Let $w \in B_p$ and let $\sigma := w^{1-p}$ be the dual weight. By writing $1 = \sigma^{\frac{1}{p'}} \sigma^{\frac{1}{p}-1}$ and using Hölder’s inequality, we have

$$\int_{\hat{K}} M(w\mathbb{1}_{\hat{K}}) \sigma^{\frac{1}{p'}} \sigma^{\frac{1}{p}-1} \leq \left( \int_{\hat{K}} M(w\mathbb{1}_{\hat{K}})^{p'} \sigma \right)^{1/p'} \left( \int_{\hat{K}} \sigma^{1-p} \right)^{1/p} \leq \|M\|_{L^{p'}(\sigma)} \left( \int_{\hat{K}} w^{p'} \sigma \right)^{1/p'} \left( \int_{\hat{K}} \sigma^{1-p} \right)^{1/p} \lesssim_{p,d} [w]_{B_p} \int_{\hat{K}} w$$

where we used that $w^{p'} = w = \sigma^{1-p}$ together with Buckley’s estimate [Buc93, Theorem 2.5] for the Hardy–Littlewood maximal function:

$$\|M\|_{L^{p'}(\sigma) \to L^{p'}(\sigma)} \lesssim_{p,d} [\sigma]_{B_{p'}}^{1/(p'-1)} = [w]_{B_p}.$$  

A simple proof of the bound for the norm of $M$ can also be found in [Ler08]. \qed

### 3. Proof of Theorem 1

We start by noting that the maximal Bergman operator $P^+$ is controlled by the sparse operator $\Lambda_{\mathcal{T}}$ defined in (2.2).

**Lemma 3.1** ([RTW17, Lemma 5]). There exists a finite collection of Bergman trees $\{T_\ell\}_{\ell=1}^N$ such that

$$P^+|f|(z) \approx \Lambda_{\mathcal{T}}|f|(z) = \sum_{\hat{K}_\alpha \in \mathcal{T}} |\langle f \rangle_{\hat{K}_\alpha}| \mathbb{1}_{\hat{K}_\alpha}$$

where $\mathcal{T} := \cup_{\ell=1}^N \{\hat{K}_\alpha : \alpha \in T_\ell\}$ is a sparse collection of dyadic tents.

Then Theorem 1 and Theorem 2 follow from the respective estimates for $\Lambda_{\mathcal{T}}$. In the rest of the paper we prove of these estimates for a sparse operator $\Lambda_{\mathcal{S}}$ associated to a generic sparse collection $\mathcal{S}$.

The testing conditions for the boundedness of $\|\Lambda_{\mathcal{S}}(\sigma \cdot)\|_{L^p(\sigma) \to L^p(w)}$ are

$$\|\mathbb{1}_{\hat{K}_0} \Lambda_{\mathcal{T}}(\sigma \mathbb{1}_{\hat{K}_0})\|_{L^2(w)}^2 \lesssim [w, \sigma]_{B_2} [\sigma]_{B_\infty} \sigma(\hat{K}_0),$$

$$\|\mathbb{1}_{\hat{K}_0} \Lambda_{\mathcal{T}}(w \mathbb{1}_{\hat{K}_0})\|_{L^2(\sigma)}^2 \lesssim [\sigma, w]_{B_2} [w]_{B_\infty} w(\hat{K}_0).$$

By symmetry, it is enough to prove one of the two inequalities. We choose the first one.

**Proposition 3.2.** Let $\sigma, w$ be two weights. Then for any dyadic tent $\hat{K}_0 \in \mathcal{T}$, we have

$$\|\mathbb{1}_{\hat{K}_0} \Lambda_{\mathcal{T}} \sigma\|_{L^2(w)}^2 \lesssim [w, \sigma]_{B_2} [\sigma]_{B_\infty} \sigma(\hat{K}_0).$$

We refer the reader to [HL12, Prop. 5.2] for a version of this result for dyadic shifts. Since we deal with sparse operators, the proof we present here is simpler. It follows the approach in Hytönen’s work [Hyt14, §5.A] and in [APR17, §5].
Proof of Proposition 3.2. For simplicity, we denote by $L_0 \in \mathcal{T}$ a fixed dyadic tent, instead of $\hat{K}_0$. Recall that, since $\mathcal{T}$ is sparse, there is a fixed $\tau \geq 1$ such that for every $L \in \mathcal{T}$ there exists a subset $E_L \subseteq L$ with the property that $|L| \leq \tau |E_L|$ and the sets in $\{ E_L : L \in \mathcal{T} \}$ are pairwise disjoint. Then we have

$$\|1_{L_0}A_{\mathcal{T}} \sigma\|^2_{L^2(w)} = \int_{L_0} \left( \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} 1_{L} \right)^2 w$$

$$\leq 2 \int_{L_0} \sum_{L \in \mathcal{T}} \sum_{L' \subseteq L} \langle \sigma \rangle_{L'} 1_{L'} w$$

$$= 2 \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} \sum_{L' \subseteq L} \langle \sigma \rangle_{L'} |L'|$$

$$\leq 2 \sup_{L' \in \mathcal{T}} \langle \sigma \rangle_{L'} \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} \sum_{L' \subseteq L} |L'|$$

$$\leq 2\tau \sup_{L' \in \mathcal{T}} \langle \sigma \rangle_{L'} \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} |L|$$

$$\lesssim [\sigma, w]_{B_2} \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} |L|.$$  

The remaining sum is controlled by using the maximal function and the sparseness property. We have

$$[\sigma, w]_{B_2} \sum_{L \in \mathcal{T}} \langle \sigma \rangle_{L} |L| \leq [\sigma, w]_{B_2} \sum_{L \in \mathcal{T}} \inf_{L} M(\sigma 1_{L_0}) |L|$$

$$\leq \tau [\sigma, w]_{B_2} \sum_{L \in \mathcal{T}} \int_{E_L} M(\sigma 1_{L_0})$$

$$\leq \tau [\sigma, w]_{B_2} \frac{1}{\sigma(L_0)} \int_{L_0} M(\sigma 1_{L_0}) \sigma(L_0)$$

$$\leq \tau [\sigma, w]_{B_2} \left( \sup_{L_0 \in \mathcal{T}} \frac{1}{\sigma(L_0)} \int_{L_0} M(\sigma 1_{L_0}) \right) \sigma(L_0)$$

$$\lesssim [\sigma, w]_{B_2} [\sigma]_{B_\infty} \sigma(L_0).$$

This concludes the proof of the proposition. \qed

The proof of Theorem 1 follows by combining the sparse domination in Lemma 3.1 with the bound for the sparse operator in Proposition 3.2. This gives the bound

$$\|P(\sigma \cdot)\|_{L^2(\sigma) \rightarrow L^2(w)} \leq C [w, \sigma]_{B_2}^{1/2} (|w|_{B_\infty}^{1/2} + [\sigma]_{B_\infty}^{1/2}).$$

3.1. Comparison of dyadic and classical characteristics. We conclude by comparing the volume of a Carleson tent $T_z$ with the volume of a dyadic tent $\hat{K}_\alpha$. This is the content of the following two lemmas, which use the concept of Bergman tree introduced in §2.6; see [RTW17, Lemma 3] and [HW20, Lemma 2.4].

Lemma 3.3 (Rahn, Tchoundja, and Wick, 2017). There exists a finite collection of Bergman trees $\{ \mathcal{T}_\ell \}_{\ell=1}^N$ such that for any tent $T_z$ there is $\ell \in \{1, \ldots, N\}$ and $\alpha$ in $\mathcal{T}_\ell$ such that $\hat{K}_\alpha \supseteq T_z$ and $|T_z| \approx |\hat{K}_\alpha|$. 


Note that since a finite union of sparse families is sparse, if we denote by
\[ \mathcal{T} := \bigcup_{\ell=1}^{N} \mathcal{T}_\ell \quad \text{where} \quad \mathcal{T}_\ell := \{ \hat{K}_\alpha : \alpha \in \mathcal{T}_\ell \}, \]
then \( \mathcal{T} \) is a sparse collection of sets in the unit ball \( \mathbb{B}^d \).

**Lemma 3.4** (Huo and Wick 2020). For any dyadic tent \( \hat{K}_\beta \in \mathcal{T} \) there exists a Carleson tent \( T_\varepsilon \) such that \( \hat{K}_\beta \subseteq T_\varepsilon \) and \( |\hat{K}_\beta| \approx |T_\varepsilon| \).

This result is proved for the disc [HW20, Lemma 2.4]; the argument can be adapted for \( d \geq 2 \). Then it holds that \( |w, \sigma|_{B_2} \approx |w, \sigma|_{B_2} \). The proof of Theorem 1 is concluded. \( \square \)

### 4. Proof of Theorem 2

We derive a bump condition in \( L^2 \) for two weights \( w, \sigma \) in terms of Orlicz averages.

We follow the approach in [Li17, Theorem 5.2] and [Hyt14, Theorem 6.1].

**Proposition 4.1.** Let \( \Lambda_{\mathcal{S}} \) be the sparse operator defined in (2.2). For two weights \( w, \sigma \) and two Young functions \( \Phi, \Psi \in \mathcal{B}_2 \), it holds
\[ \| \Lambda_{\mathcal{S}}(\sigma \cdot) \|_{L^2(\sigma) \rightarrow L^2(w)} \lessapprox [\sigma, w]_{\Phi, \Psi}. \]

We split the proof of Proposition 4.1 in a few simple steps. We will use the notation \( \langle f \rangle_\tau := \sigma(Q)^{-1} \int_Q f \sigma \) and the following lemmata for \( p = 2 \).

**Lemma 4.2.** Let \( \sigma \) be a weight and let \( \mathcal{S} \) be a \( \frac{1}{\tau} \)-sparse family with respect to the measure \( \sigma \, \text{div} \). For \( 1 < p < \infty \) and a function \( f \) we have
\[ \left( \sum_{F \in \mathcal{S}} \langle (f)^{p}_\tau \rangle^p \sigma(F) \right)^{1/p} \lessapprox_{\tau, p} \| f \|_{L^p(\sigma)} \]
where the implicit constant depends only on the sparse family and on the exponent \( p \).

**Proof.** Since \( \mathcal{S} \) is \( \frac{1}{\tau} \)-sparse, for every \( F \in \mathcal{S} \) there is \( E_F \subseteq F \) with \( \sigma(F) \leq \tau \sigma(E_F) \), and the \( \{ E_F : F \in \mathcal{S} \} \) are disjoint. Let \( M^\sigma \) be the maximal function defined by
\[ M^\sigma f := \sup_{F \in \mathcal{S}} \langle (f)^{p}_\tau \rangle_\tau \mathbb{1}_F. \]
We bound
\[ \sum_{F \in \mathcal{S}} \langle (f)^{p}_\tau \rangle^p \sigma(F) \leq \tau \sum_{F \in \mathcal{S}} \langle \inf_{E_F} M^\sigma f \rangle^p \sigma(E_F) \leq \tau \sum_{F} \int_{E_F} |M^\sigma f|^p \sigma \, d\nu \leq \tau \| M^\sigma \|_{L^p(\sigma) \rightarrow L^p(\sigma)} \| f \|_{L^p(\sigma)}^p. \]
Since the norm of the dyadic maximal function \( \| M^\sigma \|_{L^p(\sigma) \rightarrow L^p(\sigma)} \leq p' \) and does not depend on the weight \( \sigma \), the result follows. The estimate for the maximal function is classical, a proof in our case can be found in [HWW21, Lemma 3.13]. \( \square \)

**Lemma 4.3.** Let \( \mathcal{S} \) be a \( \frac{1}{\tau} \)-sparse family, \( \tau \geq 1 \). For \( 1 < p < \infty \) let \( \Psi \in \mathcal{B}_p \) be a Young function. Then for any \( G \in \mathcal{S} \) the following estimate holds
\[ \sum_{Q \in \mathcal{S}} \langle (w^{1/p})^p_{\Psi, Q} \rangle_\tau |Q| \lessapprox w(G), \]
where the implicit constant depends only on \( \tau \) and \( \| M_\Psi \|_{L^p \rightarrow L^p} \).
Proof. By Theorem B, since \( \Psi \in \mathcal{B}_p \), the maximal function \( M_\Psi \) is bounded on \( L^p \). For \( Q \subseteq G \), we have \( \langle w^{1/p} \rangle_{\Psi, Q} = \langle w^{1/p}1_G \rangle_{\Psi, Q} \). Then \([Q] \leq \tau |E_Q|\) and
\[
\sum_{Q \subseteq G} \langle w^{1/p} \rangle^p_{\Psi, Q} |Q| \leq \tau \int_{E_Q} M_\Psi(w^{1/p}1_G)^p \leq \tau \int_G M_\Psi(w^{1/p}1_G)^p \leq \tau \|M_\Psi\|_{L^p(w)}^p \|w^{1/p}\|_{L^p(G)}.
\]

We are ready to prove Proposition 4.1. We recall the two testing conditions in (2.3):
\[
\|\Lambda_\mathcal{F}(\sigma 1_Q)\|_{L^p(w)}^p \leq T_\sigma(Q), \quad \forall Q \in \mathcal{F}
\]
\[
\|\Lambda_\mathcal{G}(w 1_Q)\|_{L^p(\sigma)}^p \leq T_w(Q), \quad \forall Q \in \mathcal{G}
\]
By symmetry, it is enough to focus on one of the two.

1. Reduction to dyadic form. By duality, the left hand side of the two-weight estimate
\[
\|\Lambda(f\sigma)\|_{L^2(w)} \leq C\|f\|_{L^2(\sigma)}
\]
is the supremum over \( g \in L^2(w) \) of \( |\langle \Lambda(f\sigma), gw \rangle| \). Then it is enough to show that for non-negative functions \( f \) and \( g \) we have
\[
|\langle \Lambda(f\sigma), gw \rangle| = \sum_{Q \in \mathcal{F}} \langle f\sigma \rangle_Q \langle gw \rangle_Q |Q| = \sum_{Q \in \mathcal{F}} \langle f\rangle_Q^\sigma \langle g\rangle_Q^w |Q| \lesssim [\sigma, w]_{\Phi, \Psi} \|f\|_{L^2(\sigma)} \|g\|_{L^2(w)}.
\]

2. Stopping families. We assume that both \( f, g \) are both non-negative and supported on the set \( Q_0 \). Let \( \mathcal{D}(Q_0) \) be the family of dyadic cubes inside \( Q_0 \). We will select special cubes from \( \mathcal{D}(Q_0) \) using the “parallel corona” decomposition. We denote the principal cubes for \( (f, \sigma) \) and \( (g, w) \) by \( \mathcal{F} \) and \( \mathcal{G} \) respectively. These are defined as stopping families for the weighted averages of \( f \) and \( g \):
\[
\mathcal{A}_f^*(Q) = \{ S \in \mathcal{D}(Q), S \text{ maximal} : \langle f \rangle_S^\sigma > 2\langle f \rangle_Q^\sigma \},
\]
\[
\mathcal{A}_g^*(Q) = \{ S \in \mathcal{D}(Q), S \text{ maximal} : \langle g \rangle_S^w > 2\langle g \rangle_Q^w \}.
\]

Then we define
\[
\mathcal{F}_0 := \{Q_0\}, \quad \mathcal{F}_{n+1} := \bigcup_{Q \in \mathcal{F}_n} \mathcal{A}_f^*(Q), \quad \mathcal{F} := \bigcup_{n \in \mathbb{N}} \mathcal{F}_n
\]
and in a similar way for \( \mathcal{G} \). The families \( \mathcal{F} \) and \( \mathcal{G} \) constructed in this way are sparse with respect to the measures \( \sigma d\nu \) and \( w d\nu \), respectively. We denote by \( \pi_\mathcal{F}(Q) \) the minimal cube in \( \mathcal{F} \) containing \( Q \), and similarly for \( \pi_\mathcal{G}(Q) \). Given a pair of cubes \( (F, G) \in \mathcal{F} \times \mathcal{G} \), we consider the collection of cubes such that their projection to \( \mathcal{F} \) and \( \mathcal{G} \) are \( F \) and \( G \) respectively. Such collection is
\[
\{ Q : \pi(Q) = (F, G) \}, \quad \text{where } \pi(Q) := (\pi_\mathcal{F}(Q), \pi_\mathcal{G}(Q)).
\]
Using the stopping families we can write
\[
\sum_{Q \in \mathcal{F}} = \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{G}} \sum_{Q \in \mathcal{F}} \sum_{\pi(Q) = (F, G)}.
\]
Since either \( F \subseteq G \) or \( F \supseteq G \), by symmetry it is enough to study only one case. We focus on the latter. Notice that since \( \pi_{\mathcal{F}}(Q) = G \subseteq F \), then \( F \) is the minimal cube in \( \mathcal{F} \) containing \( G \), namely \( \pi_{\mathcal{F}}(G) = F \). We have
\[
\sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{G}} \sum_{Q \in \mathcal{Q}} \langle f \rangle_{\mathcal{F}}^{\sigma}(g)_{\mathcal{G}}^{w}(\sigma)_{\mathcal{Q}}(w)_{\mathcal{Q}}|Q|
\leq 4 \sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w} \sum_{Q \in \mathcal{Q}} \langle \sigma \rangle_{\mathcal{Q}}(w)_{\mathcal{Q}}|Q|.
\] (4.1)

3. Insert Orlicz bumps. We focus on the last summand in (4.1). We see that
\[
\langle \sigma \rangle_{\mathcal{Q}}(w)_{\mathcal{Q}} = \left( \frac{\langle \sigma \rangle_{\mathcal{Q}}(w)_{\mathcal{Q}}}{\langle \sigma^{1/2} \rangle_{\mathcal{F}, \mathcal{Q}}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}} \right) \langle \sigma^{1/2} \rangle_{\mathcal{F}, \mathcal{Q}}(w_{1/2})_{\mathcal{Q}, \mathcal{Q}}.
\]
The supremum over all dyadic cubes \( Q \) of the quantity in brackets is \( [\sigma, w]_{\mathcal{F}, \mathcal{Q}} \). Then we have
\[
\sum_{Q \in \mathcal{Q}} \langle \sigma \rangle_{\mathcal{Q}}(w)_{\mathcal{Q}}|Q| \leq \sum_{Q \in \mathcal{Q}} (\sigma^{1/2})_{\mathcal{F}, \mathcal{Q}}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}|Q|.
\]
Using the Cauchy–Schwarz inequality and Lemma 4.3 we estimate
\[
\sum_{Q \in \mathcal{Q}} (\sigma^{1/2})_{\mathcal{F}, \mathcal{Q}}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}|Q| \leq \left( \sum_{Q \in \mathcal{Q}} (\sigma^{1/2})_{\mathcal{F}, \mathcal{Q}}^{2}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}^{2}|Q| \right)^{1/2} \left( \sum_{Q \in \mathcal{Q}} (w^{1/2})_{\mathcal{Q}, \mathcal{Q}}^{2}|Q| \right)^{1/2} \leq \left( \sum_{Q \in \mathcal{Q}} (\sigma^{1/2})_{\mathcal{F}, \mathcal{Q}}^{2}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}^{2}|Q| \right)^{1/2} \leq \left( \sum_{Q \in \mathcal{Q}} w(G)^{1/2} \right)^{1/2}.
\]

Putting all the estimates together, and using the Cauchy–Schwarz inequality in \( L^{2} \) in the third and fifth inequality and Lemma 4.3 in the second and the fourth, we obtain
\[
\sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w} \sum_{Q \in \mathcal{Q}} \langle \sigma \rangle_{\mathcal{Q}}(w)_{\mathcal{Q}}|Q|
\leq [\sigma, w]_{\mathcal{F}, \mathcal{Q}} \sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w} \sum_{Q \in \mathcal{Q}} \langle \sigma^{1/2} \rangle_{\mathcal{F}, \mathcal{Q}}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}|Q|
\leq [\sigma, w]_{\mathcal{F}, \mathcal{Q}} \sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \left( \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w}(w(G))^{1/2} \right)^{1/2} \left( \sum_{Q \in \mathcal{Q}} \langle \sigma^{1/2} \rangle_{\mathcal{F}, \mathcal{Q}}^{2}(w^{1/2})_{\mathcal{Q}, \mathcal{Q}}^{2}|Q| \right)^{1/2} \leq [\sigma, w]_{\mathcal{F}, \mathcal{Q}} \left( \sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \left( \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w}(w(G))^{1/2} \right)^{1/2} \sigma(F)^{1/2} \right)^{1/2} \leq [\sigma, w]_{\mathcal{F}, \mathcal{Q}} \left( \sum_{F \in \mathcal{F}} \langle f \rangle_{\mathcal{F}}^{\sigma} \left( \sum_{G \in \mathcal{G}} \langle g \rangle_{\mathcal{G}}^{w}(w(G))^{1/2} \right)^{1/2} \right)^{1/2}.
\[ \|\sigma, w\|_{\mathcal{F}} \|f\|_{L^2(\sigma)} \|g\|_{L^2(w)} \prec \|f\|_{L^2(\sigma)} + \|g\|_{L^2(w)} \]

where the last inequality follows from Lemma 4.2, concluding the proof. □

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