ALGEBRAIC $K$-THEORY
OVER THE INFINITE DIHEDRAL GROUP:
AN ALGEBRAIC APPROACH

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Abstract. We prove that the Waldhausen nilpotent class group of an injective index 2 amalgamated free product is isomorphic to the Farrell–Bass nilpotent class group of a twisted polynomial extension. As an application, we show that the Farrell–Jones Conjecture in algebraic $K$-theory can be sharpened from the family of virtually cyclic subgroups to the family of finite-by-cyclic subgroups.

Introduction

The infinite dihedral group is both a free product and an extension of the infinite cyclic group $\mathbb{Z}$ by the cyclic group $\mathbb{Z}_2$ of order 2

$$D_\infty = \mathbb{Z}_2 \ast \mathbb{Z}_2 = \mathbb{Z} \rtimes \mathbb{Z}_2$$

with $\mathbb{Z}_2$ acting on $\mathbb{Z}$ by $-1$. A group $G$ is said to be over $D_\infty$ if it is equipped with an epimorphism $p : G \to D_\infty$. We study the algebraic $K$-theory of $R[G]$, for any ring $R$ and any group $G$ over $D_\infty$. Such a group $G$ inherits from $D_\infty$ an injective amalgamated free product structure $G = G_1 \ast_F G_2$ with $F$ an index 2 subgroup of $G_1$ and $G_2$. Furthermore, there is a canonical index 2 subgroup $G \subset G$ with an injective HNN structure $G = F \rtimes_\alpha \mathbb{Z}$ for an automorphism $\alpha : F \to F$. The various groups fit into a commutative braid of short exact sequences:

The algebraic $K$-theory decomposition theorems of Waldhausen for injective amalgamated free products and HNN extensions give

$$K_\ast(R[G]) = K_\ast(R[F] \to R[G_1] \times R[G_2]) \oplus \widetilde{\text{Nil}}_{\ast-1}(R[F]; R[G_1-F], R[G_2-F])$$

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and
\[ K_*(\mathbb{R}[\mathbb{G}]) = K_*(1 - \alpha : R[F] \to R[F]) \oplus \tilde{\text{Nil}}_{*-1}(R[F], \alpha) \oplus \tilde{\text{Nil}}_{*-1}(R[F], \alpha^{-1}). \]

We establish isomorphisms
\[ \tilde{\text{Nil}}_*(R[F]; R(G_1 - F), R(G_2 - F)) \cong \tilde{\text{Nil}}_*(R[F], \alpha) \cong \tilde{\text{Nil}}_*(R[F], \alpha^{-1}) \]

which we use to prove that the Farrell–Jones isomorphism conjecture in algebraic $K$-theory can be reduced to the family of finite-by-cyclic groups, so that virtually cyclic groups of infinite dihedral type need not be considered.

0.1. **Algebraic semi-splitting.** A homotopy equivalence $f : M \to X = X_1 \cup_Y X_2$ of finite CW complexes is *split along $Y \subset X$* if it is a cellular map and the restriction
\[ g = f|_Y : N = f^{-1}(Y) \to Y \]
is also a homotopy equivalence. The $\tilde{\text{Nil}}$-groups arise as the obstruction groups to splitting homotopy equivalences of finite CW complexes in the case of injective $\pi_1(Y) \to \pi_1(X)$ (Farrell–Hsiang, Waldhausen). In this paper we introduce the considerably weaker notion of a homotopy equivalence being *semi-split*, as defined in §0.2. The following is a special case of our main algebraic result (1.1, 2.7) which shows that there is no obstruction to semi-splitting.

**Theorem 0.1.** Let $G$ be a group over $D_\infty$, with
\[ F = G_1 \cap G_2 = \overline{G} = F \rtimes_{\alpha} \mathbb{Z} \subset G = G_1 \ast_f G_2. \]

(i) For any ring $R$ and $n \in \mathbb{Z}$ the corresponding reduced Nil-groups are naturally isomorphic:
\[ \tilde{\text{Nil}}_*(R[F]; R(G_1 - F), R(G_2 - F)) \cong \tilde{\text{Nil}}_*(R[F], \alpha) \cong \tilde{\text{Nil}}_*(R[F], \alpha^{-1}) \]

(ii) The inclusion $\theta : R[\overline{G}] \to R[G]$ determines induction and transfer maps
\[ \theta_! : K_n(R[\overline{G}]) \to K_n(R[G]), \quad \theta^! : K_n(R[G]) \to K_n(R[\overline{G}]). \]

For all $n \leq 1$, the maps $\theta_!$ and $\theta^!$ restrict to isomorphisms on the $\tilde{\text{Nil}}$-components in the decompositions $\ast$ and $\ast\ast$.

**Proof.** Part (i) is a special case of Theorem 0.5.

Part (ii) follows from Proposition 3.20(ii) (induction) and Proposition 3.22 (transfer). \qed

The $n = 0$ case will be discussed in more detail in §0.3 and §3.1.

**Remark 0.2.** We do not seriously doubt that a more assiduous application of higher $K$-theory would extend Theorem 0.5(ii) to all $n \in \mathbb{Z}$ (see also [DQR]).

As an application of Theorem 0.1, we shall prove the following theorem.

**Theorem 0.3.** Let $\Gamma$ be any group, and let $R$ be any ring. Then the following map of equivariant homology groups with coefficients in the algebraic $K$-theory functor $K_R$ is an isomorphism:
\[ H_*^\Gamma(E_{\text{fbc}} \Gamma; K_R) \longrightarrow H_*^\Gamma(E_{\text{ec}} \Gamma; K_R). \]
In fact, this is a special case of our more general fibered version (Theorem 0.28).

The original reduced Nil-groups $\tilde{\text{Nil}}_*(R) = \tilde{\text{Nil}}_*(R, 1)$ feature in the decompositions of Bass [Bas68] and Quillen [Gra76]:

$$K_*(R[t]) = K_*(R) \oplus \tilde{\text{Nil}}_{*-1}(R),$$

$$K_*(R[Z]) = K_*(R) \oplus K_{*-1}(R) \oplus \tilde{\text{Nil}}_{*-1}(R) \oplus \tilde{\text{Nil}}_{*-1}(R).$$

In §3 we shall compute several examples which require Theorem 0.1

$$K_*(R[Z_2 \ast Z_2]) = \frac{K_*(R[Z_2]) \oplus K_*(R[Z_2])}{K_*(R)} \oplus \tilde{\text{Nil}}_{*-1}(R)$$

$$K_*(R[Z_2 \ast Z_3]) = \frac{K_*(R[Z_2]) \oplus K_*(R[Z_3])}{K_*(R)} \oplus \tilde{\text{Nil}}_{*-1}(R)$$

$$\text{Wh}(G_0 \times Z_2 \ast G_0 \times Z_2) = \frac{\text{Wh}(G_0 \times Z_2) \oplus \text{Wh}(G_0 \times Z_2)}{\text{Wh}(G_0)} \oplus \tilde{\text{Nil}}_0(Z[G_0])$$

where $G_0 = Z_2 \ast Z_2 \ast Z$. The point here is that $\tilde{\text{Nil}}_0(Z[G_0])$ is an infinite torsion abelian group. This provides the first example (Example 3.27) of a non-zero Nil group in the amalgamated product case and hence the first example of a non-zero obstruction to splitting a homotopy equivalence in the two-sided case (A).

0.2. Topological semi-splitting. Let $(X, Y)$ be a separating, codimension 1, finite CW pair, with $X = X_1 \cup_Y X_2$ a union of connected CW complexes such that $\pi_1(Y) \to \pi_1(X)$ is injective. Let $\tilde{X}$ (resp. $\tilde{X}_1, \tilde{X}_2$) be the connected cover of $X$ (resp. $X_1, X_2$) classified by $\pi_1(Y) \subset \pi_1(X)$ (resp. $\pi_1(Y) \subset \pi_1(X_1)$, $\pi_1(Y) \subset \pi_1(X_2)$). Then $X = X^- \cup_Y X^+$ with $\pi_1(X^-) = \pi_1(X^+) = \pi_1(Y)$. Note $\tilde{X}_1 \subset X^- \text{ and } \tilde{X}_2 \subset X^+$ with $\tilde{X}_1 \cap \tilde{X}_2 = Y \subset \tilde{X}$.

A homotopy equivalence $f : M \to X$ from a finite CW complex is semi-split along $Y \subset X$ if $N = f^{-1}(Y) \subset M$ is a subcomplex and the restriction $(f, g) : (M, N) \to (X, Y)$ is a map of pairs such that the relative homology kernel $\mathbb{Z}[\pi_1(Y)]$-modules $K_*(M_2, N)$ vanish. Equivalently, $f$ is semi-split along $Y$ if the following induced $\mathbb{Z}[\pi_1(Y)]$-module morphisms are isomorphisms:

$$\rho_2 : K_*(\tilde{M}^+, N) \to K_*(\tilde{M}^+, M_2) = \mathbb{Z}[\pi_1(X_2) - \pi_1(Y)] \otimes_{\mathbb{Z}[\pi_1(Y)]} K_*(\tilde{M}^-, N).$$

The notation of CW-splitting is explained more in §0.3.

We refer to §3.23 for the definition of an almost-normal subgroup. In particular, finite-index subgroups and normal subgroups are almost-normal.

Theorem 0.4. Let $(X, Y)$ be a separating, codimension 1, finite CW pair, with $X = X_1 \cup_Y X_2$ a union of connected CW complexes such that $\pi_1(Y) \to \pi_1(X)$ is injective. Suppose $\pi_1(Y)$ is an almost-normal subgroup of $\pi_1(X_2)$. Then, for any finite CW complex $M$, any homotopy equivalence $h : M \to X$ is simple homotopic to a semi-split homotopy equivalence $h' : M \to X$ along $Y$.

0.3. Algebraic exposition. For any ring $R$, we establish isomorphisms between two types of codimension 1 splitting obstruction nilpotent class groups. The first type, for separated splitting, arises in the decompositions of the algebraic $K$-theory of the $R$-coefficient group ring $R[G]$ of a group $G$ over $D_\infty$, with an epimorphism $p : G \to D_\infty$ onto the infinite dihedral group $D_\infty$. The second type, for non-separated splitting, arises from the $\alpha$-twisted polynomial ring $R[F]_{\alpha}[t]$, with $F = \ker(p)$ and
\[ \alpha : F \to F \] an automorphism such that
\[ \overline{G} = \ker(\pi \circ p : G \to \mathbb{Z}_2) = F \rtimes_\alpha \mathbb{Z} \]

where \( \pi : D_\infty \to \mathbb{Z}_2 \) is the unique epimorphism with infinite cyclic kernel. Note:

(A) \( D_\infty = \mathbb{Z}_2 \ast \mathbb{Z}_2 \) is the free product of two cyclic groups of order 2, whose generators will be denoted \( t_1, t_2 \).
(B) \( D_\infty = \langle t_1, t_2 \mid t_1^2 = t_2^2 \rangle \) contains the infinite cyclic group \( \mathbb{Z} = \langle t \rangle \) as a subgroup of index 2 with \( t = t_1 t_2 \). In fact there is a short exact sequence with a split epimorphism
\[
\{1\} \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}_2 \longrightarrow \{1\}.
\]

More generally, if \( G \) is a group over \( D_\infty \), with an epimorphism \( p : G \to D_\infty \), then:

(A) \( G = G_1 \ast_F G_2 \) is a free product with amalgamation of two groups
\[ G_1 = \ker(p_1 : G \to \mathbb{Z}_2), \quad G_2 = \ker(p_2 : G \to \mathbb{Z}_2) \subset G \]
amalgamated over their common subgroup \( F = \ker(p) = G_1 \cap G_2 \) of index 2 in both \( G_1 \) and \( G_2 \).
(B) \( G \) has a subgroup \( \overline{G} = \ker(\pi \circ p : G \to \mathbb{Z}_2) \) of index 2 which is an HNN extension \( \overline{G} = F \rtimes_\alpha \mathbb{Z} \) where \( \alpha : F \to F \) is conjugation by an element \( t \in \overline{G} \) with \( p(t) = t_1 t_2 \in D_\infty \).

The \( K \)-theory of type (A). For any ring \( R \) and \( R \)-bimodules \( \mathcal{B}_1, \mathcal{B}_2 \), the Nil-groups \( \widetilde{\text{Nil}}_*(R; \mathcal{B}_1, \mathcal{B}_2) \) are defined to be the algebraic \( K \)-groups \( K_*(\text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2)) \) of the exact category \( \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2) \) with objects quadruples \( (P_1, P_2, \rho_1, \rho_2) \) with \( P_1, P_2 \) f.g. projective \( R \)-modules and
\[ \rho_1 : P_1 \to \mathcal{B}_1 \otimes_R P_2, \quad \rho_2 : P_2 \to \mathcal{B}_2 \otimes_R P_1 \]
\( R \)-module morphisms such that \( \rho_2 \rho_1 : P_1 \to \mathcal{B}_1 \otimes_R \mathcal{B}_2 \otimes_R P_1 \) is nilpotent (or equivalently such that \( \rho_1 \rho_2 : P_2 \to \mathcal{B}_2 \otimes_R \mathcal{B}_1 \otimes_R P_2 \) is nilpotent). The reduced Nil-groups \( \widetilde{\text{Nil}}_* \) are such that
\[ \widetilde{\text{Nil}}_*(R; \mathcal{B}_1, \mathcal{B}_2) = K_*(R) \oplus K_*(R) \oplus \widetilde{\text{Nil}}_*(R; \mathcal{B}_1, \mathcal{B}_2). \]

As already noted above, Waldhausen decomposed the algebraic \( K \)-theory of \( R[G] \) for an injective amalgamated free product \( G = G_1 \ast_F G_2 \) as
\[ K_*(R[G]) = K_*(R[F] \to R[G_1] \times R[G_2]) \oplus \widetilde{\text{Nil}}_{*-1}(R[F]; R[G_1 - F], R[G_2 - F]). \]

In particular, there is defined a split monomorphism
\[ \sigma_A : \widetilde{\text{Nil}}_{*-1}(R[F]; R[G_1 - F], R[G_2 - F]) \to K_*(R[G]), \]
which for \( * = 1 \) is given by
\[ \sigma_A : \widetilde{\text{Nil}}_0(R[F]; R[G_1 - F], R[G_2 - F]) \to K_1(R[G]); \]
\[ [P_1, P_2, \rho_1, \rho_2] \mapsto \left[ R[G] \otimes_R R[F] (P_1 \oplus P_2), \begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix} \right]. \]
The $K$-theory of type (B). Given a ring $R$ and an $R$-bimodule $\mathcal{B}$, let

$$T_R(\mathcal{B}) = R \oplus \mathcal{B} \oplus \mathcal{B} \otimes_R \mathcal{B} \oplus \cdots$$

be the tensor algebra of $\mathcal{B}$ over $R$. The Nil-groups $\text{Nil}_n(R; \mathcal{B})$ are defined to be the algebraic $K$-groups $K_*(\text{Nil}(R; \mathcal{B}))$ of the exact category $\text{Nil}(R; \mathcal{B})$ with objects pairs $(P, \rho)$ with $P$ a f.g. (finitely generated) projective $R$-module and $\rho : P \to \mathcal{B} \otimes_R P$ a $R$-module morphism, nilpotent in the sense that for some $k$, we have

$$\rho^k = 0 : P \to \mathcal{B} \otimes_R P \to \cdots \to \mathcal{B} \otimes_R \cdots \otimes_R \mathcal{B} \otimes_R P.$$

The reduced Nil-groups $\widetilde{\text{Nil}}_*$ are such that

$$\text{Nil}_n(R; \mathcal{B}) = K_*(R) \oplus \widetilde{\text{Nil}}_{n-1}(R; \mathcal{B}) \quad \text{.}$$

Waldhausen [Wal78] proved that if $\mathcal{B}$ is f.g. projective as a left $R$-module and free as a right $R$-module, then

$$K_*(T_R(\mathcal{B})) = K_*(R) \oplus \widetilde{\text{Nil}}_{n-1}(R; \mathcal{B})$$

with a split monomorphism

$$\sigma_B : \widetilde{\text{Nil}}_{n-1}(R; \mathcal{B}) \to K_*(T_R(\mathcal{B})) \quad \text{,}$$

which for $* = 1$ is given by

$$\sigma_B : \widetilde{\text{Nil}}_0(R; \mathcal{B}) \to K_1(T_R(\mathcal{B})); [P, \rho] \mapsto [T_R(\mathcal{B}) \otimes_R P, 1 - \rho].$$

In particular, for $\mathcal{B} = R$

$$\text{Nil}_n(R; R) = \text{Nil}_n(R), \quad \widetilde{\text{Nil}}_n(R; R) = \widetilde{\text{Nil}}_n(R), \quad T_R(\mathcal{B}) = R[t], \quad K_*(R[t]) = K_*(R) \oplus \widetilde{\text{Nil}}_{n-1}(R).$$

Relating the $K$-theory of types (A) and (B). Recall that a category $I$ is filtered if:

- for any pair of objects $\alpha, \alpha'$ in $I$, there exist an object $\beta$ and morphisms $\alpha \to \beta$ and $\alpha' \to \beta$ in $I$, and
- for any pair of morphisms $u, v : \alpha \to \alpha'$ in $I$, there exists an object $\beta$ and morphism $w : \alpha' \to \beta$ such that $w \circ u = w \circ v.$

Note that any directed poset $I$ is a filtered category.

**Theorem 0.5** (General Algebraic Semi-splitting). Let $R$ be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be $R$-bimodules. Suppose that $I$ is a small, filtered category and $\mathcal{B}_2 = \text{colim}_{\alpha \in I} \mathcal{B}_2$, is a direct limit of $R$-bimodules such that each $\mathcal{B}_2^\alpha$ is a f.g. projective left $R$-module. Then, for all $n \in \mathbb{Z}$, the Nil-groups of the triple $(R; \mathcal{B}_1, \mathcal{B}_2)$ are related to the Nil-groups of the pair $(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2)$ by isomorphisms

$$\text{Nil}_n(R; \mathcal{B}_1, \mathcal{B}_2) \cong \text{Nil}_n(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \oplus K_n(R),$$

$$\widetilde{\text{Nil}}_n(R; \mathcal{B}_1, \mathcal{B}_2) \cong \widetilde{\text{Nil}}_n(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2).$$

In particular, for $n = 0$ there are defined inverse isomorphisms

$$i_* : \text{Nil}_0(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \cong \text{Nil}_0(R; \mathcal{B}_1, \mathcal{B}_2);$$

$$(P_1, \rho_{12} : P_1 \to \mathcal{B}_1 \otimes_R \mathcal{B}_2 \otimes_R P_1, [P_2]) \mapsto [P_1, \mathcal{B}_2 \otimes_R P_1 \oplus P_2, \left(\begin{array}{c} \rho_{12} \\ 0 \end{array}\right), (1 \ 0)],$$

$$j_* : \text{Nil}_0(R; \mathcal{B}_1, \mathcal{B}_2) \cong \text{Nil}_0(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \oplus K_0(R);$$

$$[P_1, P_2, \rho_1 : P_1 \to \mathcal{B}_1 \otimes_R P_2, \rho_2 : P_2 \to \mathcal{B}_2 \otimes_R P_1] \mapsto ([P_1, \rho_2 \circ \rho_1], [P_2] - [\mathcal{B}_2 \otimes_R P_1]).$$
The reduced versions are the inverse isomorphisms

\[ i_* : \hat{\text{Nil}}_0(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \xrightarrow{\cong} \hat{\text{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2) ; [P_1, \rho_{12}] \mapsto [P_1, \mathcal{B}_2 \otimes_R P_1, \rho_{12}, 1], \]

\[ j_* : \hat{\text{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2) \xrightarrow{\cong} \hat{\text{Nil}}_0(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) ; [P_1, P_2, \rho_1, \rho_2] \mapsto [P_1, \rho_2 \circ \rho_1]. \]

**Proof.** This follows immediately from Theorems 1.1 and 2.7. \qed

**Remark 0.6.** Theorem 0.5 was already known to Pierre Vogel in 1990 - see [Vog90].

### 0.4. Topological exposition

The proof of Theorem 0.5 is motivated by the obstruction theory of Waldhausen [Wal69] for splitting homotopy equivalences of finite CW complexes $X$ along codimension 1 subcomplexes $Y \subset X$ with $\pi_1(Y) \to \pi_1(X)$ injective, and the subsequent algebraic $K$-theory decomposition theorems of Waldhausen [Wal78].

A codimension 1 pair $(X, Y \subset X)$ is a pair of spaces such that the inclusion $Y = Y \times \{0\} \subset X$ extends to an open embedding $Y \times \mathbb{R} \subset X$. A map of codimension 1 pairs $(f, g) : (M, N) \to (X, Y)$ is a cellular map $f : M \to X$ with $g = f| : N = f^{-1}(Y) \to Y$.

Let $(X, Y)$ be a codimension 1 finite CW pair. A homotopy equivalence $f : M \to X$ from a finite CW complex splits along $Y \subset X$ if there is a map of codimension 1 pairs $(f, g) : (M, N) \to (X, Y)$ so that $g : N \to Y$ is also a homotopy equivalence. A map $f : M \to X$ between finite CW complexes is simple homotopic to a map $f' : M' \to X$ if $M'$ is a finite CW complex, $s : M' \to M$ is a simple homotopy equivalence and $f \circ s$ is homotopic to $f'$. A homotopy equivalence $f : M \to X$ from a finite CW complex is splittable along $Y \subset X$ if $f$ is simple homotopic to a map which splits along $Y \subset X$.

A codimension 1 pair $(X, Y)$ is injective if $X, Y$ are connected and $\pi_1(Y) \to \pi_1(X)$ is injective. Let $\widetilde{X}$ be the universal cover of $X$. As in [0.2] let $\widetilde{X} = \widetilde{X}/\pi_1(Y)$, so that $(\widetilde{X}, Y)$ is a codimension 1 pair with $\widetilde{X} = \widetilde{X}^- \cup \widetilde{X}^+$ for connected subspaces $\widetilde{X}^-, \widetilde{X}^+ \subset \widetilde{X}$ with $\pi_1(\widetilde{X}) = \pi_1(\widetilde{X}^-) = \pi_1(\widetilde{X}^+) = \pi_1(Y)$. As usual, there are two cases, according as to whether $Y$ separates $X$ or not:

(A) $X - Y$ is disconnected, so

\[ X = X_1 \cup_Y X_2 \]

with $X_1, X_2$ connected. By the Seifert-van Kampen theorem

\[ \pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \]

is the amalgamated free product, with $\pi_1(Y) \to \pi_1(X_1), \pi_1(Y) \to \pi_1(X_2)$ injective. The labeling is chosen such that

\[ \widetilde{X}_1 = \widetilde{X}_1/\pi_1(Y) \subset \widetilde{X}^-, \widetilde{X}_2 = \widetilde{X}_2/\pi_1(Y) \subset \widetilde{X}^+, \widetilde{X}_1 \cap \widetilde{X}_2 = \widetilde{Y}. \]

(B) $X - Y$ is connected, so

\[ X = X_1/\{y \sim ty | y \in Y\} \]

for a connected space $X_1$ (a deformation retract of $X - Y$) which contains two disjoint copies $Y, tY \subset X_1$ of $Y$. We shall only consider the case when $\pi_1(Y) \to \pi_1(X_1), \pi_1(Y) \to \pi_1(X_1)$ are isomorphisms, so that

\[ \pi_1(X) = \pi_1(Y) \times_{\alpha} \mathbb{Z} \]
The Whitehead torsion \( \tau \) is stably f.g. free. Hence the projective classes are complementary:

In both cases \((\bar{X}, Y)\) is an injective codimension 1 pair of type \((A)\).

The kernel \(\mathbb{Z}[\pi_1(X)]\)-modules of a map \(f : M \to X\) are the relative homology \(\mathbb{Z}[\pi_1(X)]\)-modules

\[
K_r(M) = H_{r+1}(\tilde{f} : \tilde{M} \to \tilde{X})
\]

with \(\tilde{X}\) the universal cover of \(X\), \(\tilde{M} = f^*\tilde{X}\) the pullback cover of \(M\), and \(\tilde{f} : \tilde{M} \to \tilde{X}\) a \(\pi_1(X)\)-equivariant lift of \(f\). For a map of injective codimension 1 CW pairs \((f, g) : (M, N) \to (X, Y)\) the kernel \(\mathbb{Z}[\pi_1(Y)]\)-modules fit into an exact sequence

\[
\cdots \to K_r(N) \to K_r(M) \to K_r(M^+, N) \oplus K_r(M^-, N) \to K_{r-1}(N) \to \cdots .
\]

If \(f\) is a homotopy equivalence and \(g_\ast : \pi_1(N) \to \pi_1(Y)\) is an isomorphism, then \(g\) is a homotopy equivalence if and only if \(K_*(N) = 0\), which occurs if and only if \(K_*(M^+, N) = K_*(M^-, N) = 0\). In particular, if \(f\) is of type \((A)\) and split, then \(f\) is semi-split.

**Theorem 0.7** (Waldhausen [Wal69] for type \((A)\), Farrell–Hsiang [FH73] for type \((B)\)). Let \((X, Y)\) be an injective, codimension 1, finite CW pair. Suppose \(f' : M' \to X\) is a homotopy equivalence from a finite CW complex.

(i) The homotopy equivalence \(f'\) is simple homotopic to a map of pairs \((f, g) : (M, N) \to (X, Y)\) such that \(g_\ast : \pi_1(N) \to \pi_1(Y)\) is an isomorphism and for some \(n \geq 2\) we have

\[
K_r(N) = 0 \text{ for } r \neq n .
\]

Moreover, the \(\mathbb{Z}[\pi_1(Y)]\)-modules \(K_{n+1}(\bar{M}^\pm, N)\) are f.g. projective, and the direct sum

\[
K_n(N) = K_{n+1}(\bar{M}^-, N) \oplus K_{n+1}(\bar{M}^+, N)
\]

is stably f.g. free. Hence the projective classes are complementary:

\[
[K_{n+1}(\bar{M}^-, N)] = -[K_{n+1}(\bar{M}^+, N)] \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y)]) .
\]

(ii) In the separating case \((A)\) there is defined an exact sequence

\[
\cdots \to \text{Wh}(\pi_1(X_1)) \oplus \text{Wh}(\pi_1(X_2)) \to \text{Wh}(\pi_1(X))
\]

\[
\to \tilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \tilde{N}\tilde{H}_0(\mathbb{Z}[\pi_1(Y)]; \mathcal{B}_1, \mathcal{B}_2) \to \cdots
\]

where

\[
\mathcal{B}_1 = \mathbb{Z}[\pi_1(X_1) - \pi_1(Y)] , \quad \mathcal{B}_2 = \mathbb{Z}[\pi_1(X_2) - \pi_1(Y)].
\]

The Whitehead torsion \(\tau(f) \in \text{Wh}(\pi_1(X))\) has image

\[
[\tau(f)] = ([K_{n+1}(\bar{M}^-, N)], [K_{n+1}(\bar{M}^-, N), K_{n+1}(\bar{M}^+, N), \rho_1, \rho_2])
\]

where

\[
\rho_1 : K_{n+1}(\bar{M}^-, N) \to K_{n+1}(\bar{M}^-, \bar{M}_1) = \mathcal{B}_1 \otimes \mathbb{Z}[\pi_1(Y)] K_{n+1}(\bar{M}^+, N) ,
\]

\[
\rho_2 : K_{n+1}(\bar{M}^+, N) \to K_{n+1}(\bar{M}^+, \bar{M}_2) = \mathcal{B}_2 \otimes \mathbb{Z}[\pi_1(Y)] K_{n+1}(\bar{M}^-, N) .
\]
The homotopy equivalence $f$ is splittable along $Y$ if and only if $|\tau(f)| = 0$.

(iii) In the non-separating case (B) there is defined an exact sequence

$$
\cdots \longrightarrow \text{Wh}(\pi_1(Y)) \xrightarrow{1-\alpha} \text{Wh}(\pi_1(Y)) \longrightarrow \text{Wh}(\pi_1(X)) \\
\longrightarrow \bar{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \tilde{N}_0(\mathbb{Z}[\pi_1(Y)]), \alpha) \oplus \tilde{N}_0(\mathbb{Z}[\pi_1(Y)], \alpha^{-1}) \longrightarrow \cdots 
$$

The Whitehead torsion $\tau(f) \in \text{Wh}(\pi_1(X))$ has image

$$
[\tau(f)] = ([K_{n+1}(\tilde{M}^+, N)], [K_{n+1}(\tilde{M}^+, N), \rho_1], [K_{n+1}(\tilde{M}^-, N), \rho_2])
$$

where

$$
\rho_1 : K_{n+1}(\tilde{M}^-, N) \longrightarrow K_{n+1}(\tilde{M}^-, \tilde{M}_1) = t^{-1}K_{n+1}(\tilde{M}^-, N), \\
\rho_2 : K_{n+1}(\tilde{M}^+, N) \longrightarrow K_{n+1}(\tilde{M}^+, t\tilde{M}_1) = tK_{n+1}(\tilde{M}^+, N).
$$

The homotopy equivalence $f$ is splittable along $Y$ if and only if $|\tau(f)| = 0$.

Proof of Theorem [0.7] (outline). The proof of [Wal69, Theorem 0.7(i)] was based on a one-one correspondence between the elementary operations in the algebraic $K$-theory of the nilpotent categories and the elementary operations (‘surgeries’ or cell-exchanges) for maps of injective codimension 1 pairs. The proof of our Theorem [0.5] shows that there is no algebraic obstruction to making a homotopy equivalence semi-split by elementary operations, and hence there is no geometric obstruction. \hfill $\square$

1. Higher Nil-groups

In this section, we shall prove Theorem [0.5] for non-negative degrees.

Quillen [Qui73] defined the $K$-theory space $K\mathcal{E} := \Omega BQ(\mathcal{E})$ of an exact category $\mathcal{E}$. The space $BQ(\mathcal{E})$ is the geometric realization of the simplicial set $N\bullet Q(\mathcal{E})$, which is the nerve of a certain category $Q(\mathcal{E})$ associated to $\mathcal{E}$. The algebraic $K$-groups of $\mathcal{E}$ are defined for $* \in \mathbb{Z}$

$$
K_*(\mathcal{E}) := \pi_*(K\mathcal{E})
$$

using a nonconnective delooping for $* \leq -1$. In particular, the algebraic $K$-groups of a f.g. projective $R$-modules. The Nil-categories defined in the Introduction all have the structure of exact categories.

**Theorem 1.1.** Let $R$ be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be $R$-bimodules. Suppose that $I$ is a filtered category and $\mathcal{B}_2 = \text{colim}_{\alpha \in I} \mathcal{B}_2^\alpha$ is a direct limit of $R$-bimodules $\mathcal{B}_2^\alpha$, each of which is a f. g. projective left $R$-module. Let $i$ be the exact functor of exact categories of projective nil-objects:

$$
i : \text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \longrightarrow \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2) : (Q, \sigma : Q \rightarrow \mathcal{B}_1 \mathcal{B}_2^\alpha Q) \mapsto (Q, \mathcal{B}_2^\alpha Q, \sigma, 1).
$$

Then the induced map of $K$-theory spaces is a homotopy equivalence:

$$
\tilde{K}i : K\text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \longrightarrow K\text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2)/(0 \times K(R)).
$$

In particular, for all $n \in \mathbb{N}$, there is an induced isomorphism of abelian groups:

$$
i_* : \text{Nil}_n(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \oplus K_n(R) \longrightarrow \text{Nil}_n(R; \mathcal{B}_1, \mathcal{B}_2).$$
The exact functor

\[ j : \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2) \longrightarrow \text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) ; \quad (P_1, P_2, \rho_1, \rho_2) \mapsto (P_1, \rho_2 \circ \rho_1) \]

satisfies \( j \circ i = 1 \).

Proof. It is straightforward to show that tensor product commutes with colimits over a category. Moreover, for any object \( x = (P_1, P_2, \rho_1 : P_1 \to \mathcal{B}_1 P_2, \rho_2 : P_2 \to \mathcal{B}_2 P_1) \), since \( P_2 \) is finitely generated, there exists \( \alpha \in I \) such that \( \rho_2 \) factors through a map \( P_2 \to \mathcal{B}_2' P_1 \), and similarly for short exact sequences of nil-objects. We thus obtain induced isomorphisms of exact categories:

\[
\begin{align*}
\operatorname{colim} \text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2') & \longrightarrow \text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \\
\operatorname{colim} \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2') & \longrightarrow \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2).
\end{align*}
\]

So, by Quillen’s colimit observation [Qui73, Equation (9), page 20], we obtain induced weak homotopy equivalences of \( K \)-theory spaces:

\[
\begin{align*}
\operatorname{colim} K\text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2') & \longrightarrow K\text{Nil}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \\
\operatorname{colim} K\text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2') & \longrightarrow K\text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2).
\end{align*}
\]

Therefore, for each \( \alpha \in I \), it suffices to show that the restriction \( \tilde{K}i^\alpha \) is a homotopy equivalence.

Our setting is the exact category \( \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2') \). By assumption, we may consider objects

\[
\begin{align*}
x & := (P_1, P_2, \rho_1, \rho_2) \\
x' & := (P_1, \mathcal{B}_2^\alpha P_1 \oplus P_2, \begin{pmatrix} 0 \\ \rho_1 \end{pmatrix}, (1, \rho_2)) \\
x'' & := (P_1, \mathcal{B}_2^\alpha P_1, \rho_2 \circ \rho_1, 1) \\
a & := (0, P_2, 0, 0) \\
a' & := (0, \mathcal{B}_2^\alpha P_1, 0, 0).
\end{align*}
\]

Define morphisms

\[
\begin{align*}
f & := (1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) : x \longrightarrow x' \\
f' & := (1, \begin{pmatrix} 1 \\ \rho_2 \end{pmatrix}) : x' \longrightarrow x'' \\
g & := (0, \begin{pmatrix} -\rho_2 \\ 1 \end{pmatrix}) : a \longrightarrow x' \\
g' & := (0, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) : x' \longrightarrow a' \\
h & := (0, \rho_2) : a \longrightarrow a'.
\end{align*}
\]

There are exact sequences

\[
\begin{align*}
0 \longrightarrow x \oplus a & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} x' \oplus a \xrightarrow{\begin{pmatrix} g' \\ h \end{pmatrix}} a' \longrightarrow 0 \\
0 \longrightarrow a & \xrightarrow{g} x' \xrightarrow{f'} x'' \longrightarrow 0.
\end{align*}
\]

Define functors \( F', F'', G, G' : \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2') \longrightarrow \text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2') \) by

\[
F'(x) = x', \quad F''(x) = x'', \quad G(x) = a, \quad G'(x) = a'.
\]
Thus we have two exact sequences of exact functors
\[
\begin{align*}
0 & \longrightarrow 1 \oplus G \longrightarrow F' \oplus G \longrightarrow G' \longrightarrow 0 \\
0 & \longrightarrow G \longrightarrow F' \longrightarrow F'' \longrightarrow 0.
\end{align*}
\]
Recall \( j \circ i = 1 \), and note \( i \circ j = F'' \). By Quillen’s Additivity Theorem [Qui73, p. 98, Cor. 1], we obtain homotopies \( KF' \simeq 1 + KG' \) and \( KF'' \simeq KG + KF'' \). Then
\[
Ki \circ Kj = KF'' \simeq 1 + (KG' - KG),
\]
where the subtraction uses the loop space structure. Observe \( G, G' : \text{Nil}(R; \mathscr{B}_1, \mathscr{B}_2) \to 0 \times \text{Proj}(R) \). Therefore the functor \( i^\alpha \) induces a homotopy equivalence
\[
K i^\alpha : \text{KNil}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2^0) \longrightarrow \text{KNil}(R; \mathscr{B}_1, \mathscr{B}_2)/(0 \times K(R)).
\]

\( \square \)

**Remark 1.2.** The proof of Theorem 1.1 is best understood in terms of finite chain complexes \( x = (P_1, P_2, \rho_1, \rho_2) \) in the category \( \text{Nil}(R; \mathscr{B}_1, \mathscr{B}_2) \), assuming that \( \mathscr{B}_2 \) is a f. g. projective left \( R \)-module. Any such \( x \) represents a class
\[
[x] = \sum_{r=0}^{\infty} (-1)^r [(P_1)_r, (P_2)_r, \rho_1, \rho_2] \in \text{Nil}_0(R; \mathscr{B}_1, \mathscr{B}_2).
\]
The key observation is that \( x \) determines a finite chain complex \( x' = (P'_1, P'_2, \rho'_1, \rho'_2) \) in \( \text{Nil}(R; \mathscr{B}_1, \mathscr{B}_2) \) such that \( \rho'_2 : P'_2 \to \mathscr{B}_2 \otimes_R P_1 \) is a chain equivalence and
\[
[x] = [x'] \in \text{Nil}_0(R; \mathscr{B}_1, \mathscr{B}_2).
\]
Specifically, let \( P'_1 = P_1, P'_2 = \mathcal{M}(\rho_2) \), the algebraic mapping cylinder of the chain map \( \rho_2 : P_2 \to \mathscr{B}_2 \otimes_R P_1 \), and let
\[
\rho'_1 = \begin{pmatrix} 0 \\ 0 \\ \rho_1 \end{pmatrix} : P'_1 = P_1 \longrightarrow \mathscr{B}_1 \otimes_R P'_2 = \mathcal{M}(1_{\mathscr{B}_1} \otimes \rho_2),
\]
\[
\rho'_2 = \begin{pmatrix} 1 & 0 & \rho_2 \end{pmatrix} : P'_2 = \mathcal{M}(\rho_2) \longrightarrow \mathscr{B}_2 \otimes_R P_1,
\]
so that \( P'_2/P_2 = \mathcal{C}(\rho_2) \) is the algebraic mapping cone of \( \rho_2 \). Moreover, the proof of \((*)\) is sufficiently functorial to establish not only that the following maps of the reduced nilpotent class groups are inverse isomorphisms:
\[
i : \tilde{\text{Nil}}_0(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \overset{\simeq}{\longrightarrow} \tilde{\text{Nil}}_0(R; \mathscr{B}_1, \mathscr{B}_2) : (P, \rho) \mapsto (P, \mathscr{B}_2 \otimes_R P, \rho, 1),
\]
\[
j : \tilde{\text{Nil}}_0(R; \mathscr{B}_1, \mathscr{B}_2) \overset{\simeq}{\longrightarrow} \tilde{\text{Nil}}_0(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) : [x] \mapsto [x'],
\]
but also that there exist isomorphisms of \( \tilde{\text{Nil}}_n \) for all higher dimensions \( n > 0 \), as shown above. In order to prove equation \((*)\), note that \( x \) fits into the sequence
\[
0 \longrightarrow x \overset{(1,u)}{\longrightarrow} x' \overset{(0,v)}{\longrightarrow} y \longrightarrow 0 \quad \text{(**)}
\]
with
\[
y = (0, \mathcal{C}(\rho_2), 0, 0),
\]
\[
u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : P_2 \to P_2' = \mathcal{M}(\rho_2),
\]
\[
v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : P_2' = \mathcal{M}(\rho_2) \to \mathcal{C}(\rho_2)
\]
and
\[
[y] = \sum_{r=0}^{\infty} (-1)^r [0, (\mathcal{B}_2 \otimes_R P_1)_{r-1} \oplus (P_2)_r, 0, 0] = 0 \in \widetilde{\text{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2).
\]

The projection \(\mathcal{M}(\rho_2) \to \mathcal{B}_2 \otimes_R P_1\) defines a chain equivalence
\[
x' \simeq (P_1, \mathcal{B}_2 \otimes_R P_1, \rho_2 \circ \rho_1, 1) = ij(x)
\]
so that
\[
[x] = [x'] - [y] = [P_1, \mathcal{B}_2 \otimes_R P_1, \rho_2 \circ \rho_1, 1] = ij[x] \in \widetilde{\text{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2).
\]

Now suppose that \(x\) is a 0-dimensional chain complex in \(\text{Nil}(R; \mathcal{B}_1, \mathcal{B}_2)\), that is, an object exactly as in the proof of Theorem 1.1. Let \(x', x'', a, a', f, f', g, g', h\) be as defined there. The exact sequence of \((**\)) can be written as the short exact sequence of chain complexes
\[
\begin{array}{ccccccccc}
0 & \to & x & \xrightarrow{f} & x' \xrightarrow{g} & a' & \xrightarrow{h} & 0.
\end{array}
\]

The first admissible exact sequence of the proof of Theorem 1.1 is now immediate:
\[
\begin{array}{ccccccccc}
0 & \to & x \oplus a & \xrightarrow{(f \ g)} & x' \oplus a & \xrightarrow{(g' \ h)} & a' & \to & 0.
\end{array}
\]

The second admissible exact sequence is self-evident:
\[
\begin{array}{ccccccccc}
0 & \to & a & \xrightarrow{g} & x' & \xrightarrow{f'} & x'' & \to & 0.
\end{array}
\]

2. Lower Nil-groups

2.1. Cone and suspension rings. Let us recall some additional structures on the tensor product of modules.

Originating from ideas of Karoubi–Villamayor \([KV69]\), the following concept was studied independently by S.M. Gersten \([Ger72]\) and J.B. Wagoner \([Wag72]\) in the construction of the non-connective \(K\)-theory spectrum of a ring.

**Definition 2.1** (Gersten, Wagoner). Let the cone ring \(\Lambda \mathbb{Z}\) be the ring of \((N \times N)\)-matrices over \(\mathbb{Z}\) such that each row and column has only a finite number of non-zero entries. Let the suspension ring \(\Sigma \mathbb{Z}\) be the quotient ring of \(\Lambda \mathbb{Z}\) by the two-sided ideal of matrices with only a finite number of non-zero entries. For each \(n \in \mathbb{N}\), write \(\Sigma^n \mathbb{Z} := \underbrace{\Sigma \mathbb{Z} \otimes \cdots \otimes \Sigma \mathbb{Z}}_{n \text{ copies}} \) with \(\Sigma^0 \mathbb{Z} = \mathbb{Z}\). For a ring \(R\) and for \(n \in \mathbb{N}\), define the ring \(\Sigma^n R := \Sigma^n \mathbb{Z} \otimes \mathbb{Z} R\).
Roughly speaking, the suspension should be regarded as the ring of “bounded modulo compact operators.” Gersten and Wagoner showed that $K_i(\Sigma^n R)$ is naturally isomorphic to $K_{i-n}(R)$ for all $i, n \in \mathbb{Z}$, in the sense of Quillen when the subscript is positive, in the sense of Grothendieck when the subscript is zero, and in the sense of Bass when the subscript is negative.

For an $R$-bimodule $\mathcal{B}$, define the $\Sigma^n R$-bimodule $\Sigma^n \mathcal{B} := \Sigma^n \mathbb{Z} \otimes \mathcal{B}$.

**Lemma 2.2.** Let $R$ be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be $R$-bimodules. Then, for each $n \in \mathbb{N}$, there is a natural isomorphism of $\Sigma^n R$-bimodules:

$$t_n : \Sigma^n(\mathcal{B}_1 \otimes_R \mathcal{B}_2) \longrightarrow \Sigma^n \mathcal{B}_1 \otimes_{\Sigma^2 R} \Sigma^n \mathcal{B}_2 : s \otimes (b_1 \otimes b_2) \longmapsto (s \otimes b_1) \otimes (\sum_n b_2).$$

**Proof.** By transposition of the middle two factors, note that

for $\Sigma\mathcal{B}_1 \otimes \Sigma^2 \mathcal{B}_2$ is isomorphic to $\Sigma^n \mathcal{B}_1 \otimes (\Sigma^n \mathcal{B}_2) \otimes \Sigma^n \mathcal{B}_2$.

The next two theorems follow from the definitions and [Wal78, Theorems 1, 3].

**Definition 2.3.** Let $R$ be a ring. Let $\mathcal{B}$ be an $R$-bimodule. For all $n \in \mathbb{N}$, define

$$\text{Nil}_{-n}(R; \mathcal{B}) := \text{Nil}_0(\Sigma^n R; \Sigma^n \mathcal{B})$$
$$\text{Nil}_{-n}(R; \mathcal{B}) := \text{Nil}_0(\Sigma^n R; \Sigma^n \mathcal{B}).$$

**Definition 2.4.** Let $R$ be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be $R$-bimodules. For all $n \in \mathbb{N}$, define

$$\text{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) := \text{Nil}_0(\Sigma^n R; \Sigma^n \mathcal{B}_1, \Sigma^n \mathcal{B}_2)$$
$$\text{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) := \text{Nil}_0(\Sigma^n R; \Sigma^n \mathcal{B}_1, \Sigma^n \mathcal{B}_2)).$$

The next two theorems follow from the definitions and [Wal78, Theorems 1, 3].

**Theorem 2.5** (Waldhausen). Let $R$ be a ring and $\mathcal{B}$ be an $R$-bimodule. Consider the tensor ring

$$T_R(\mathcal{B}) = R \oplus \mathcal{B} \oplus (\mathcal{B} \otimes_R \mathcal{B}) \oplus (\mathcal{B} \otimes_R \mathcal{B} \otimes_R \mathcal{B}) \oplus \cdots .$$

Suppose $\mathcal{B}$ is finitely generated projective as a left $R$-module and free as a right $R$-module. Then, for all $n \in \mathbb{N}$, there is a split monomorphism

$$\sigma_B : \text{Nil}_{-n}(R; \mathcal{B}) \longrightarrow K_{1-n}(T_R(\mathcal{B}))$$
given for $n = 0$ by the map

$$\sigma_B : \text{Nil}_0(R; \mathcal{B}) \longrightarrow K_1(T_R(\mathcal{B})) ; [P, \rho] \longmapsto [T_R(\mathcal{B}) \otimes_R P, 1 - \tilde{\rho}] ,$$

where $\tilde{\rho}$ is defined using $\rho$ and multiplication in $T_R(\mathcal{B})$.

Furthermore, there is a natural decomposition

$$K_{1-n}(T_R(\mathcal{B})) = K_{1-n}(R) \oplus \text{Nil}_{-n}(R; \mathcal{B}).$$
For example, the last assertion of the theorem follows from the equations:
\[
K_{1-n}(T_R(\mathcal{B})) = K_1(\Sigma^n T_R(\mathcal{B})) \\
= K_1(T_{\Sigma^n R}(\Sigma^n \mathcal{B})) \\
= K_1(\Sigma^n R) \oplus \tilde{\text{Nil}}_0(\Sigma^n R; \Sigma^n \mathcal{B}) \\
= K_{1-n}(R) \oplus \tilde{\text{Nil}}_{-n}(R; \mathcal{B}).
\]

**Theorem 2.6** (Waldhausen). Let \( R, A_1, A_2 \) be rings. Let \( R \to A_i \) be ring monomorphisms such that \( A_i = R \oplus \mathcal{B}_i \) for some \( R \)-bimodule \( \mathcal{B}_i \). Consider the pushout of rings

\[
A = A_1 \ast_R A_2 = R \oplus (\mathcal{B}_1 \oplus \mathcal{B}_2) \oplus (\mathcal{B}_1 \otimes \mathcal{B}_2 \oplus \mathcal{B}_2 \otimes \mathcal{B}_1) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oplus (\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \oplus \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2) \oplus \cdots.
\]

Suppose each \( \mathcal{B}_i \) is free as a right \( R \)-module. Then, for all \( n \in \mathbb{N} \), there is a split monomorphism

\[
\sigma_A : \tilde{\text{Nil}}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) \longrightarrow K_{1-n}(A),
\]

given for \( n = 0 \) by the map

\[
\tilde{\text{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2) \longrightarrow K_1(A) ; \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad [(P_1, P_2, \rho_1, \rho_2)] \longmapsto \left[ (A \otimes_R P_1) \oplus (A \otimes_R P_2), \begin{pmatrix} 1 & \hat{\rho}_2 \\ \hat{\rho}_1 & 1 \end{pmatrix} \right],
\]

where \( \hat{\rho}_i \) is defined using \( \rho_i \) and multiplication in \( A_i \) for \( i = 1, 2 \).

Furthermore, there is a natural Mayer–Vietoris type exact sequence

\[
\cdots \xrightarrow{\partial} K_{1-n}(R) \longrightarrow K_{1-n}(A_1) \oplus K_{1-n}(A_2) \longrightarrow K_{1-n}(A) \xrightarrow{\partial} K_{-n}(R) \longrightarrow \cdots.
\]

### 2.3. The isomorphism for lower \( \text{Nil} \)-groups.

**Theorem 2.7.** Let \( R \) be a ring. Let \( \mathcal{B}_1, \mathcal{B}_2 \) be \( R \)-bimodules. Suppose that \( I \) is a filtered category and \( \mathcal{B}_2 = \text{colim}_{\alpha \in I} \mathcal{B}_2^\alpha \) is a direct limit of \( R \)-bimodules \( \mathcal{B}_2^\alpha \), each of which is a f.g. projective left \( R \)-module. Then, for all \( n \in \mathbb{N} \), there is an induced isomorphism:

\[
\text{Nil}_{-n}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \oplus K_{-n}(R) \longrightarrow \text{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2).
\]

**Proof.** Let \( n \in \mathbb{N} \). By Lemma 2.2 and Theorem 1.1 there are induced isomorphisms:

\[
\text{Nil}_{-n}(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) \oplus K_{-n}(R) = \text{Nil}_0(\Sigma^n R; \Sigma^n (\mathcal{B}_1 \otimes_R \mathcal{B}_2)) \oplus K_0 \Sigma^n(R) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \longrightarrow \text{Nil}_0(\Sigma^n R; \Sigma^n (\mathcal{B}_1 \otimes_R \Sigma^n \mathcal{B}_2)) \oplus K_0 \Sigma^n(R) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \longrightarrow \text{Nil}_0(\Sigma^n R; \Sigma^n \mathcal{B}_1, \Sigma^n \mathcal{B}_2) = \text{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2).
\]
3. Applications

We indicate some applications of our main theorem (Theorem 0.5). In §3.1 we prove Theorem 0.1(ii), which describes the restrictions of the maps

\[ \theta_! : K_*({\mathbb{R}[G]}) \to K_*({\mathbb{R}[G]}) \]

\[ \theta^! : K_*({\mathbb{R}[G]}) \to K_*({\mathbb{R}[G]}) \]

to the \( \tilde{\text{Nil}} \)-terms, with \( \theta : G \to G \) the inclusion of the canonical index 2 subgroup \( G \) for any group \( G \) over \( D_\infty \). In §3.2 we give the first known example of a non-zero Nil group occurring in the \( K \)-theory of an integral group ring of an amalgamated free product. In §3.3 we sharpen the Farrell–Jones Conjecture in \( K \)-theory, replacing the family of virtually cyclic groups by the smaller family of finite-by-cyclic groups. In §3.4 we compute the \( K_*({\mathbb{R}[\Gamma]}) \) for the modular group \( \Gamma = \text{PSL}_2(\mathbb{Z}) \).

3.1. Algebraic \( K \)-theory over \( D_\infty \). The overall goal of this section is to show that the abstract isomorphisms \( i_* \) and \( j_* \) coincide with the restrictions of the induction and transfer maps \( \theta_! \) and \( \theta^! \) in the group ring setting.

3.1.1. Twisting. We start by recalling the algebraic \( K \)-theory of twisted polynomial rings.

**Statement 3.1.** Consider any (unital, associative) ring \( R \) and any ring automorphism \( \alpha : R \to R \). Let \( t \) be an indeterminate over \( R \) such that

\[ rt = t\alpha(r) \quad (r \in R) \]

For any \( R \)-module \( P \), let \( tP := \{tx \mid x \in P\} \) be the set with left \( R \)-module structure

\[ tx + ty = t(x + y) , \quad r(tx) = t(\alpha(r)x) \in tP . \]

Further endow the left \( R \)-module \( tR \) with the \( R \)-bimodule structure

\[ R \times tR \times R \to tR ; \quad (q, tr, s) \mapsto t\alpha(q)rs . \]

The \( \tilde{\text{Nil}} \)-category of \( R \) with respect to \( \alpha \) is the exact category defined by

\[ \text{Nil}(R, \alpha) := \text{Nil}(R; tR) . \]

The objects \( (P, \rho) \) consist of any f.g. projective \( R \)-module \( P \) and any nilpotent morphism \( \rho : P \to tP = tR \otimes_R P \). The \( \tilde{\text{Nil}} \)-groups are written

\[ \tilde{\text{Nil}}_* (R, \alpha) := \text{Nil}_* (R; tR) , \quad \tilde{\text{Nil}}_* (R; tR) := \text{Nil}_* (R; tR) , \]

so that

\[ \text{Nil}_* (R, \alpha) = K_* (R) \oplus \tilde{\text{Nil}}_* (R, \alpha) . \]

**Statement 3.2.** The tensor algebra on \( tR \) is the \( \alpha \)-twisted polynomial extension of \( R \)

\[ T_R(tR) = R_\alpha[t] = \sum_{k=0}^{\infty} t^k R . \]

Given an \( R \)-module \( P \) there is induced an \( R_\alpha[t] \)-module

\[ R_\alpha[t] \otimes_R P = P_\alpha[t] \]
whose elements are finite linear combinations \( \sum_{j=0}^{\infty} t^j x_j \) \((x_j \in P)\). Given \(R\)-modules \(P, Q\) and an \(R\)-module morphism \(\rho : P \to tQ\), define its extension as the \(R_0[t]\)-module morphism

\[
\tilde{\rho} = t \rho : P_0[t] \to Q_0[t] ; \quad \sum_{j=0}^{\infty} t^j x_j \mapsto \sum_{j=0}^{\infty} t^j \rho(x_j).
\]

**Statement 3.3.** By Bass [Bas68], Farrell–Hsiang [FH73], and Quillen [Gra76], there are decompositions

\[
K_n(R_0[t]) = K_n(R) \oplus \widetilde{\text{Nil}}_{n-1}(R, \alpha),
\]

\[
K_n(R_{0-1}[t^{-1}]) = K_n(R) \oplus \widetilde{\text{Nil}}_{n-1}(R, \alpha^{-1}),
\]

\[
K_n(R_0[t, t^{-1}]) = K_n(1 - \alpha : R \to R) \oplus \widetilde{\text{Nil}}_{n-1}(R, \alpha) \oplus \widetilde{\text{Nil}}_{n-1}(R, \alpha^{-1}).
\]

In particular for \(n = 1\), by Theorem 2.20 there are defined split monomorphisms

\[
\sigma_B^+ : \widetilde{\text{Nil}}_0(R, \alpha) \xrightarrow{\sigma_B^+} K_1(R_0[t]) ; [P, \rho] \mapsto [P_0[t], 1 - t \rho],
\]

\[
\sigma_B^- : \widetilde{\text{Nil}}_0(R, \alpha^{-1}) \xrightarrow{\sigma_B^-} K_1(R_{0-1}[t^{-1}]) ; [P, \rho] \mapsto [P_{0-1}[t^{-1}], 1 - t^{-1} \rho],
\]

\[
\sigma_B = (\psi^+ \sigma_B^+ \psi^- \sigma_B^-) : \widetilde{\text{Nil}}_0(R, \alpha) \oplus \widetilde{\text{Nil}}_0(R, \alpha^{-1}) \xrightarrow{\sigma_B} K_1(R_0[t, t^{-1}]) ; ([P_1, \rho_1], [P_2, \rho_2]) \mapsto \left( (P_1 \oplus P_2)_0[t, t^{-1}], \begin{pmatrix} 1 - t \rho_1 & 0 \\ 0 & 1 - t^{-1} \rho_2 \end{pmatrix} \right).
\]

These definitions extend to all integers \(n \leq 1\) by the suspension isomorphisms of Section 2.

3.1.2. **Scaling.** Next, consider the effect an inner automorphism on \(\alpha\).

**Statement 3.4.** Suppose \(\alpha, \alpha' : R \to R\) are automorphisms satisfying

\[
\alpha'(r) = u \alpha(r) u^{-1} \in R \ (r \in R)
\]

for some unit \(u \in R\), and that \(t'\) is an indeterminate over \(R\) satisfying

\[
rt' = t' \alpha'(r) \ (r \in R).
\]

Denote the canonical inclusions

\[
\psi^+ : R_0[t] \to R_0[t, t^{-1}] \quad \psi^- : R_{0-1}[t^{-1}] \to R_0[t, t^{-1}]
\]

\[
\psi'^+ : R_{0'}[t'] \to R_{0'}[t', t'^{-1}] \quad \psi'^- : R_{0'-1}[t'^{-1}] \to R_{0'}[t', t'^{-1}].
\]

**Statement 3.5.** The various polynomial rings are related by **scaling isomorphisms**

\[
\beta_u^+ : R_0[t] \xrightarrow{\sim} R_{0'}[t'] ; t \mapsto t'u,
\]

\[
\beta_u^- : R_{0-1}[t^{-1}] \xrightarrow{\sim} R_{0'-1}[t'^{-1}] ; t^{-1} \mapsto u^{-1}t'^{-1},
\]

\[
\beta_u : R_0[t, t^{-1}] \xrightarrow{\sim} R_{0'}[t', t'^{-1}] ; t \mapsto t' u
\]

satisfying the equations

\[
\beta_u \circ \psi^+ = \psi'^+ \circ \beta_u^+ : R_0[t] \to R_{0'}[t', t'^{-1}]
\]

\[
\beta_u \circ \psi^- = \psi'^- \circ \beta_u^- : R_{0-1}[t^{-1}] \to R_{0'}[t', t'^{-1}].
\]
Statement 3.6. There are corresponding scaling isomorphisms of exact categories

\[ \beta^+_u : \text{Nil}(R, \alpha) \xrightarrow{\cong} \text{Nil}(R, \alpha') ; (P, \rho) \mapsto (P, t' u \rho t^{-1} : P \to t' P) \]

\[ \beta^-_u : \text{Nil}(R, \alpha^{-1}) \xrightarrow{\cong} \text{Nil}(R, \alpha'^{-1}) ; (P, \rho) \mapsto (P, t'^{-1} u \rho t : P' \to t'^{-1} P') , \]

where we mean

\[(t' u \rho t^{-1})(tx) := t'ux, \quad (t'^{-1} u \rho t)(t^{-1} x) := t'^{-1}ux . \]

Statement 3.7. For all \( n \leq 1 \), the various scaling isomorphisms are related by equations

\[(\beta^+_u)_* \circ \sigma^+_B = \sigma^{t+}_B \circ \beta^+_u : \widetilde{\text{Nil}}_{n-1}(R, \alpha) \to K_n(R, \alpha'[t^n]) \]

\[(\beta^-_u)_* \circ \sigma^-_B = \sigma^{t-}_B \circ \beta^-_u : \widetilde{\text{Nil}}_{n-1}(R, \alpha^{-1}) \to K_{n}(R, \alpha'^{-1}[t'^{-1}]) \]

\[(\beta_u)_* \circ \sigma_B = \sigma_B \circ \left( \begin{array}{cc} 0 & 0 \\ 0 & \beta_u \end{array} \right) : \widetilde{\text{Nil}}_{n-1}(R, \alpha) \oplus \widetilde{\text{Nil}}_{n-1}(R, \alpha^{-1}) \to K_n(R, \alpha'[t^n, t'^{-1}]) . \]

3.1.3. Group rings. We now adapt these isomorphisms to the case of group rings \( R[G] \) of groups \( G \) over the infinite dihedral group \( D_\infty \). In order to prove Propositions 3.20 and 3.22 the overall idea is to transform information about the product \( t_2 t_1 \) arising from the transposition \( \mathcal{B}_2 \otimes \mathcal{B}_1 \) into information about the product \( t_2'^{-1} t_1'^{-1} \) arising in the second Nil-summand of the twisted Bass decomposition. We continue to discuss the ingredients in a sequence of statements.

Statement 3.8. Let \( F \) be a group, and let \( \alpha : F \to F \) be an automorphism. Recall that the injective \( HNN \) extension \( F \rtimes_{\alpha} \mathbb{Z} \) is the set \( F \rtimes \mathbb{Z} \) with group multiplication

\[(x, n)(y, m) := (\alpha^n(x)y, m + n) \in F \rtimes_{\alpha} \mathbb{Z} . \]

Then, for any ring \( R \), writing \( t = (1_F, 1) \) and \( (x, n) = t^n x \in F \rtimes_{\alpha} \mathbb{Z} \), we have

\[ R[F \rtimes_{\alpha} \mathbb{Z}] = R[F \alpha[t, t^{-1}] . \]

Statement 3.9. Consider any group \( G = G_1 \ast_F G_2 \) over \( D_\infty \), where

\[ F = G_1 \cap G_2 \subset \overline{G} = F \rtimes_{\alpha} \mathbb{Z} \subset G = G_1 \ast_F G_2 . \]

Fix elements \( t_1 \in G_1 - F, t_2 \in G_2 - F \), and define elements

\[ t := t_1 t_2 \in \overline{G}, \quad t' := t_2 t_1 \in \overline{G}, \quad u := (t')^{-1} t^{-1} \in F . \]

Define the automorphisms

\[ \alpha_1 : F \to F ; x \mapsto (t_1)^{-1} x t_1 , \]

\[ \alpha_2 : F \to F ; x \mapsto (t_2)^{-1} x t_2 , \]

\[ \alpha := \alpha_2 \circ \alpha_1 : F \to F ; x \mapsto t^{-1} x t , \]

\[ \alpha' := \alpha_1 \circ \alpha_2 : F \to F ; x \mapsto t'^{-1} x t' \]

such that

\[ xt = t \alpha(x) , \quad xt' = t' \alpha'(x) , \quad \alpha'(x) = u \alpha^{-1}(x) u^{-1} (x \in F) . \]

In particular, note that \( \alpha' \) and \( \alpha^{-1} \) (not \( \alpha \)) are related by left inner automorphism by \( u \).
Statement 3.10. Denote the canonical inclusions
\[ \psi^+: R_\alpha[t] \to R_\alpha[t, t^{-1}] \quad \psi^- : R_{\alpha^{-1}}[t^{-1}] \to R_\alpha[t, t^{-1}] \quad \psi^+: R_{\alpha'}[t'] \to R_{\alpha'}[t', t'^{-1}] \quad \psi^{-'} : R_{\alpha'^{-1}}[t'^{-1}] \to R_{\alpha'}[t', t'^{-1}] . \]
The inclusion \( R[F] \to R[G] \) extends to ring monomorphisms
\[ \theta : R[F]_{\alpha}[t, t^{-1}] \to R[G] \quad \theta' : R[F]_{\alpha'}[t', t'^{-1}] \to R[G] \]
such that
\[ \text{im}(\theta) = \text{im}(\theta') = R[G] \subset R[G] = R[G_1] \ast R[F] R[G_2] . \]
Furthermore, the inclusion \( R[F] \to R[G] \) extends to ring monomorphisms
\[ \phi = \theta \circ \psi^+ : R[F]_{\alpha}[t] \to R[G] \quad \phi' = \theta' \circ \psi'^+ : R[F]_{\alpha'}[t'] \to R[G] . \]

Statement 3.11. By Statement 3.5, there are defined scaling isomorphisms of rings
\[ \beta^+_u : R[F]_{\alpha^{-1}}[t^{-1}] \xrightarrow{\cong} R[F]_{\alpha'}[t'] ; \quad t^{-1} \mapsto t'u , \]
\[ \beta^-_u : R[F]_{\alpha}[t] \xrightarrow{\cong} R[F]_{\alpha^{-1}}[t'^{-1}] ; \quad t \mapsto u^{-1}t'^{-1} , \]
\[ \beta_u : R[F]_{\alpha}[t, t^{-1}] \xrightarrow{\cong} R[F]_{\alpha'}[t', t'^{-1}] ; \quad t \mapsto u^{-1}t'^{-1} \]
which satisfy the equations
\[ \beta_u \circ \psi^- = \psi^+ \circ \beta^+_u : R[F]_{\alpha^{-1}}[t^{-1}] \to R[F]_{\alpha'}[t', t'^{-1}] \]
\[ \beta_u \circ \psi^+ = \psi'^- \circ \beta^-_u : R[F]_{\alpha}[t] \to R[F]_{\alpha'}[t', t'^{-1}] \]
\[ \theta = \theta' \circ \beta_u : R[F]_{\alpha}[t, t^{-1}] \to R[G] . \]

Statement 3.12. By Statement 3.6, there are scaling isomorphisms of exact categories
\[ \beta^+_u : \text{Nil}(R[F], \alpha^{-1}) \xrightarrow{\cong} \text{Nil}(R[F], \alpha') ; \quad (P, \rho) \mapsto (P, t'ut\rho) , \]
\[ \beta^-_u : \text{Nil}(R[F], \alpha) \xrightarrow{\cong} \text{Nil}(R[F], \alpha^{-1}) ; \quad (P, \rho) \mapsto (P, t'^{-1}ut^{-1}\rho) . \]

Statement 3.13. By Statement 3.7, for all \( n \leq 1 \), the various scaling isomorphisms are related by the equations
\[ (\beta^+_u)_* \circ \sigma_B^- = \sigma_B^+ \circ \beta^+_u : \hat{\text{Nil}}_{n-1}(R[F], \alpha^{-1}) \to K_*(R[F]_{\alpha'}[t']) \]
\[ (\beta^-_u)_* \circ \sigma_B^+ = \sigma_B'^- \circ \beta^-_u : \hat{\text{Nil}}_{n-1}(R[F], \alpha) \to K_*(R[F]_{\alpha'}[t'^{-1}]) \]
\[ (\beta_u)_* \circ \sigma_B = \sigma_B^+ \circ \begin{pmatrix} 0 & \beta^+_u \\ \beta^-_u & 0 \end{pmatrix} : \hat{\text{Nil}}_{n-1}(R[F], \alpha) \oplus \hat{\text{Nil}}_{n-1}(R[F], \alpha^{-1}) \to K_*(R[F]_{\alpha'}[t', t'^{-1}]). \]

3.1.4. Transposition. Next, we study the effect of transposition of the bimodules \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) in order to relate \( \alpha \) and \( \alpha' \). In particular, there is no mention of \( \alpha^{-1} \)
in this section.

Statement 3.14. The \( R[F] \)-bimodules
\[ \mathcal{B}_1 = R[G_1 - F] = t_1 R[F] \quad \mathcal{B}_2 = R[G_2 - F] = t_2 R[F] \]
are free left and right $R[F]$-modules of rank one. We shall use the $R[F]$-bimodule isomorphisms

$$
\mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2 \to tR[F]; \quad t_1x_1 \otimes t_2x_2 \mapsto t\alpha(x_1)x_2
$$

$$
\mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1 \to t' R[F]; \quad t_2x_2 \otimes t_1x_1 \mapsto t'\alpha_1(x_2)x_1
$$

to make the identifications

$$
\mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2 = tR[F], \quad \text{Nil}(R[F]; \mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2) = \text{Nil}(R[F], \alpha),
$$

$$
\mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1 = t' R[F], \quad \text{Nil}(R[F]; \mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1) = \text{Nil}(R[F], \alpha').
$$

**Statement 3.15.** Theorem 0.5 gives inverse isomorphisms

$$
i_* : \widetilde{\text{Nil}}_*(R[F], \alpha) \xrightarrow{\cong} \widetilde{\text{Nil}}_*(R[F]; \mathcal{B}_1, \mathcal{B}_2),
$$

$$
j_* : \widetilde{\text{Nil}}_*(R[F]; \mathcal{B}_1, \mathcal{B}_2) \xrightarrow{\cong} \widetilde{\text{Nil}}_*(R[F], \alpha)
$$

which for $* = 0$ are given by

$$
i_* : \widetilde{\text{Nil}}_0(R[F], \alpha) \xrightarrow{\cong} \widetilde{\text{Nil}}_0(R[F]; \mathcal{B}_1, \mathcal{B}_2); \quad [P, \rho] \mapsto [P, t_2P, \rho, 1],
$$

$$
j_* : \widetilde{\text{Nil}}_0(R[F]; \mathcal{B}_1, \mathcal{B}_2) \xrightarrow{\cong} \widetilde{\text{Nil}}_0(R[F], \alpha); \quad [P_1, P_2, \rho_1, \rho_2] \mapsto [P_1, \rho_2 \circ \rho_1].
$$

**Statement 3.16.** Similarly, there are defined inverse isomorphisms

$$
i'_* : \widetilde{\text{Nil}}_*'(R[F], \alpha') \xrightarrow{\cong} \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_2, \mathcal{B}_1),
$$

$$
j'_* : \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_2, \mathcal{B}_1) \xrightarrow{\cong} \widetilde{\text{Nil}}_*'(R[F], \alpha')
$$

which for $* = 0$ are given by

$$
i'_* : \widetilde{\text{Nil}}_0'(R[F], \alpha') \xrightarrow{\cong} \widetilde{\text{Nil}}_0'(R[F]; \mathcal{B}_2, \mathcal{B}_1); \quad [P', \rho'] \mapsto [P', t_1P', \rho', 1],
$$

$$
j'_* : \widetilde{\text{Nil}}_0'(R[F]; \mathcal{B}_2, \mathcal{B}_1) \xrightarrow{\cong} \widetilde{\text{Nil}}_0'(R[F], \alpha'); \quad [P_2, P_1, \rho_1, \rho_2] \mapsto [P_2, \rho_1 \circ \rho_2].
$$

**Statement 3.17.** The transposition isomorphism of exact categories

$$
\tau_A : \text{Nil}(R[F]; \mathcal{B}_1, \mathcal{B}_2) \xrightarrow{\cong} \text{Nil}(R[F]; \mathcal{B}_2, \mathcal{B}_1); \quad (P_1, P_2, \rho_1, \rho_2) \mapsto (P_2, P_1, \rho_2, \rho_1)
$$

induces isomorphisms

$$
\tau_A : \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_1, \mathcal{B}_2) \cong \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_2, \mathcal{B}_1),
$$

$$
\tau_A : \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_2, \mathcal{B}_1) \cong \widetilde{\text{Nil}}_*'(R[F]; \mathcal{B}_1, \mathcal{B}_2).
$$

Note, by Theorem 0.5, the composites

$$
\tau_B := j'_* \circ \tau_A \circ i_* : \widetilde{\text{Nil}}_*(R[F], \alpha) \xrightarrow{\cong} \widetilde{\text{Nil}}_*(R[F], \alpha'),
$$

$$
\tau'_B := j_* \circ \tau_A^{-1} \circ i'_* : \widetilde{\text{Nil}}_*'(R[F], \alpha') \xrightarrow{\cong} \widetilde{\text{Nil}}_*'(R[F], \alpha)
$$

are inverse isomorphisms, which for $* = 0$ are given by

$$
\tau_B : \widetilde{\text{Nil}}_0(R[F], \alpha) \xrightarrow{\cong} \widetilde{\text{Nil}}_0(R[F], \alpha'); \quad [P, \rho] \mapsto [t_2P, t_2\rho],
$$

$$
\tau'_B : \widetilde{\text{Nil}}_0(R[F], \alpha') \xrightarrow{\cong} \widetilde{\text{Nil}}_0(R[F], \alpha); \quad [P', \rho'] \mapsto [t_1P', t_1\rho'].
$$
Furthermore, note that the various transpositions are related by the equation
\[ \tau_A \circ i_* = i'_* \circ \tau_B : \tilde{\Nil}_n(R[F]; \alpha) \to \tilde{\Nil}_n(R[F]; \mathcal{B}_2, \mathcal{B}_1). \]

**Statement 3.18.** Recall from Theorem 2.6 that there is a split monomorphism
\[ \sigma_A : \tilde{\Nil}_{n-1}(R[F]; \mathcal{B}_1, \mathcal{B}_2) \to K_n(R[G]) \]
such that the \( n = 1 \) case is given by
\[ \sigma_A : \tilde{\Nil}_0(R[F]; \mathcal{B}_1, \mathcal{B}_2) \to K_1(R[G]): \]
\[ [P_1, P_2, \rho_1, \rho_2] \mapsto \left[ P_1[G] \oplus P_2[G], \begin{pmatrix} 1 & t_2 \rho_2 \\ t_1 \rho_1 & 1 \end{pmatrix} \right]. \]

Elementary row and column operations produce an equivalent representative:
\[ \begin{pmatrix} 1 & -t_2 \rho_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \rho_2 \\ t_1 \rho_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t \rho_2 \rho_1 \\ 0 & 1 \end{pmatrix}. \]

Thus the \( n = 1 \) case satisfies the equations (with a similar argument for the second equality):
\[ \sigma_A[P_1, P_2, \rho_1, \rho_2] = [P_1[G], 1 - t \rho_2 \rho_1] = [P_2[G], 1 - t' \rho_1 \rho_2]. \]

Therefore for all \( n \leq 1 \), the split monomorphism \( \sigma'_A \), associated with the amalgamated free product \( G = G_2 \ast_F G_1 \), satisfies the equation
\[ \sigma_A = \sigma'_A \circ \tau_A : \tilde{\Nil}_{n-1}(R[F]; \mathcal{B}_1, \mathcal{B}_2) \to K_n(R[G]). \]

**3.1.5. Induction.** We consider the effect of induction maps on \( \tilde{\Nil} \)-summands (Prop. 3.20).

**Statement 3.19.** Recall from Theorem 0.5 the isomorphism
\[ i_* : \tilde{\Nil}_{n-1}(R[F], \alpha) = \tilde{\Nil}_{n-1}(R[F]; \mathcal{B}_1 \otimes_R [F] \mathcal{B}_2) \to \tilde{\Nil}_{n-1}(R[F]; \mathcal{B}_1, \mathcal{B}_2); \]
\[ [P, \rho] \mapsto [P, t_2 P, \rho, 1]. \]

Let \( (P, \rho) \) be an object in the exact category \( \Nil(R[F], \alpha) \). By Statement 3.18 note
\[ \sigma_A i_* [P, \rho] = \sigma_A[P, t_2 P, \rho, 1] = [P[G], 1 - t \rho] = \phi \sigma_B^+[P, \rho]. \]

Thus, for all \( n \leq 1 \), we obtain the key equality
\[ \sigma_A \circ i_* = \phi \circ \sigma_B^+ : \tilde{\Nil}_{n-1}(R[F], \alpha) \to K_n(R[G]). \]

**Proposition 3.20.** Let \( n \leq 1 \) be an integer.
(i) The split monomorphisms $\sigma_A, \sigma_A', \sigma_B^+, \sigma_B'^+$ are related by a commutative diagram

\[ \begin{array}{cccc}
\tilde{\text{Nil}}_{n-1}(R[F]; \beta') & \stackrel{\sigma_B^+}{\longrightarrow} & K_n(R[F]_\alpha[t]) \\
\tau_B \cong & & & \phi_1 \\
\tilde{\text{Nil}}_{n-1}(R[F]; \beta, \beta_1) & \stackrel{\tau_A}{\longrightarrow} & K_n(R[G]) & \cong (\beta_u)_1 \\
\tau_A \cong & & & \phi_1' \\
\tilde{\text{Nil}}_{n-1}(R[F]; \beta_2, \beta_1) & \stackrel{\sigma_A'}{\longrightarrow} & K_n(R[F]_{\alpha'[t, t^{-1}]}) \\
\end{array} \]

(ii) The induced map $\theta_1$ is such that there is a commutative diagram

\[ \begin{array}{cccc}
\tilde{\text{Nil}}_{n-1}(R[F], \alpha) \oplus \tilde{\text{Nil}}_{n-1}(R[F], \alpha^{-1}) & \stackrel{\sigma_B}{\longrightarrow} & K_n(R[G]) \\
(i_*, \tau_A^{-1} i'_* \beta_+^*) & & & \phi_1 \\
\tilde{\text{Nil}}_{n-1}(R[F]; \beta_1, \beta_2) & \stackrel{\sigma_A}{\longrightarrow} & K_n(R[G]) \\
\end{array} \]

Proof. Part (i) follows from the following implications:

- Statement 3.10 gives $\phi_1 = \theta_1 \circ \psi_1^+$ and $\phi_1' = \theta_1' \circ \psi_1'^+$
- Statement 3.11 gives $\theta_1 = \theta_1' \circ (\beta_u)_1$
- Statement 3.17 gives $\tau_A \circ i_* = i'_* \circ \tau_B$
- Statement 3.18 gives $\sigma_A = \sigma_A' \circ \tau_A$
- Statement 3.19 gives $\sigma_A \circ i_* = \phi_1 \circ \sigma_B^+$ and $\sigma_A' \circ i'_* = \phi_1' \circ \sigma_B'^+$.

Part (ii) follows from Part (i):

- $\sigma_A \circ i_* = \phi_0 \circ \sigma_B^+ = \theta_1 \circ \psi_1^+ \circ \sigma_B^+ = \theta_1 \circ \sigma_B[\tilde{\text{Nil}}_{n-1}(R[F], \alpha)]$
- Statement 3.13 and Statement 3.11 give $\sigma_A \circ \tau_A^{-1} \circ i'_* \circ \beta_+^* = \sigma_A' \circ i'_* \circ \beta_+^* = \phi_0' \circ \sigma_B^+' \circ \beta_+^* = \theta_1 \circ (\beta_u)_1^{-1} \circ \psi_1'^+ \circ (\beta_+^*)_1 \circ \sigma_B = \theta_1 \circ \psi_1^+ \circ \sigma_B^+ = \theta_1 \circ \sigma_B[\tilde{\text{Nil}}_{n-1}(R[F], \alpha^{-1})]$. 

\[ \square \]

3.1.6. Transfer. We consider the effect of induction maps on $\tilde{\text{Nil}}$-summands (Prop. 3.22).

Statement 3.21. Given an $R[G]$-module $M$, let $M'$ be the $R[G]$-module defined by $M$ with the $R[G]$-action restricted to the subring $R[G]$. The transfer functor of
induces the transfer maps in algebraic $K$-theory

$$\theta^i : K_*(R[G]) \to K_*(R[G]) ; \ M \mapsto M^i$$

The exact functors of Theorem 0.5 combine to an exact functor

$$\Phi : \text{Proj}(R[G]; \mathcal{B}_1, \mathcal{B}_2) \to \text{Nil}(R[F], \alpha) \times \text{Nil}(R[F], \alpha') ; \quad [P_1, P_2, \rho_1, \rho_2] \mapsto ([P_1, \rho_2 \circ \rho_1], [P_2, \rho_1 \circ \rho_2])$$

inducing a map between reduced Nil-groups

$$\tilde{\theta}^i : \tilde{\text{Nil}}_n(R[F]; \mathcal{B}_1, \mathcal{B}_2) \to \tilde{\text{Nil}}_n(R[F], \alpha) \oplus \tilde{\text{Nil}}_n(R[F], \alpha').$$

**Proposition 3.22.** Let $n \leq 1$ be an integer. The transfer map $\theta^i$ restricts to the isomorphism $j_*$ in a commutative diagram

$$\begin{array}{ccc}
\tilde{\text{Nil}}_{n-1}(R[F]; \mathcal{B}_1, \mathcal{B}_2) & \xrightarrow{\sigma_A} & K_*(R[G]) \\
\bigg( j_* \bigg) & \downarrow & \bigg( \theta^i \bigg) \\
\tilde{\text{Nil}}_{n-1}(R[F], \alpha) \oplus \tilde{\text{Nil}}_{n-1}(R[F], \alpha^{-1}) & \xrightarrow{(\psi^+ \sigma_B^+ \beta_u \psi^- \sigma_B^-)} & K_*(R[G])
\end{array}$$

**Proof:** Using the suspension isomorphisms of Section 2 we may assume $n = 1$. Let $(P_1, P_2, \rho_1, \rho_2)$ be an object in $\text{Nil}(R[F]; \mathcal{B}_1, \mathcal{B}_2)$. Define an $R[G]$-module automorphism

$$f := \begin{pmatrix} 1 & t_2 \rho_2 \\ t_1 \rho_1 & 1 \end{pmatrix} : P_1[G] \oplus P_2[G] \to P_1[G] \oplus P_2[G].$$

By Theorem 2.6 we have $[f] = \sigma_A[P_1, P_2, \rho_1, \rho_2] \in K_1(R[G])$. Note the transfer is

$$\theta^i(f) = \begin{pmatrix} 1 & t_2 \rho_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t_1 \rho_1 & 1 & 0 & 0 \\ 0 & 0 & t_2 \rho_2 & 1 \end{pmatrix}$$

as an $R[G]$-module automorphism of $P_1[G] \oplus t_1 P_2[G] \oplus t_2 P_2[G] \oplus t_1 t_2 P_2[G]$. Furthermore, elementary row and column operations produce a diagonal representation:

$$\begin{pmatrix} 1 & -t_2 \rho_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t_1 \rho_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \theta^i(f) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t_2 \rho_2 & 1 \end{pmatrix} = \begin{pmatrix} 1-t'_2 \rho_2 \rho_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-t_1 \rho_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So $\theta^i[f] = [1-t'_2 \rho_2 \rho_1] + [1-t_1 \rho_1 \rho_2]$. Thus we obtain a commutative diagram

$$\begin{array}{ccc}
\tilde{\text{Nil}}_0(R[F]; \mathcal{B}_1, \mathcal{B}_2) & \xrightarrow{\sigma_A} & K_1(R[G]) \\
\bigg( j_* \bigg) & \downarrow & \bigg( \theta^i \bigg) \\
\tilde{\text{Nil}}_0(R[F], \alpha) \oplus \tilde{\text{Nil}}_0(R[F], \alpha') & \xrightarrow{(\psi^+ \sigma_B^+ \beta_u \psi^- \sigma_B^-)} & K_1(R[G])
\end{array}$$
Finally, by Statement 3.13 and Statement 3.11 note
\[ \psi'^+ \circ \sigma'_B \circ \beta'^+ \circ \beta^+ \circ \sigma^- \circ \beta^- = \beta_u \circ \psi^- \circ \sigma_B^- . \]

\[ \square \]

3.2. Waldhausen

Nil. Natural examples of bimodules originate from group rings of amalgamated product of groups.

Definition 3.23. A subgroup \( H \) of a group \( G \) is almost-normal if \([ H : H \cap xHx^{-1}] < \infty \) for every \( x \in G \). Equivalently, \( H \) is an almost-normal subgroup of \( G \) if every \((H, H)\)-double coset \( HxH \) is a (disjoint) union of finitely many left cosets \( ghH \) and a (disjoint) union of finitely many right cosets \( Hg \).

Remark 3.24. Almost-normal subgroups arise in the Shimura theory of automorphic functions, with \((G, H)\) called a Hecke pair. Here are two sufficient conditions for a subgroup \( H \subset G \) to be almost-normal: if \( H \) is a finite-index subgroup of \( G \), or if \( H \) is a normal subgroup of \( G \). Interesting examples of almost-normal subgroups are given in [Kri90, p. 9].

Here is our reduction for a certain class of group rings, specializing the General Algebraic Semi-splitting of Theorem 0.5.

Corollary 3.25. Let \( R \) be a ring. Let \( G = G_1 *_F G_2 \) be an injective amalgamated product of groups over a common subgroup \( F \) of \( G_1 \) and \( G_2 \). Suppose \( F \) is an almost-normal subgroup of \( G_2 \). Then, for all \( n \in \mathbb{Z} \), there is an isomorphism of abelian groups:
\[ j_* \circ \tilde{\text{Nil}}_n(R[F]; R[G_1 - F], R[G_2 - F]) \rightarrow \tilde{\text{Nil}}_n(R[F]; R(G_1 - F) \otimes_{R[F]} R(G_2 - F)). \]

Proof. Consider the set \( J := (F \backslash G_2/F) - F \) of non-trivial double cosets. Let \( \mathcal{I} \) be the poset of all finite subsets of \( J \), partially ordered by inclusion. Note, as \( R[F] \)-bimodules, that
\[ R[G_2 - F] = \text{colim}_{I \in \mathcal{I}} R[I] \quad \text{where} \quad R[I] := \bigoplus_{F \nmid F' \in I} R[F g F]. \]

Since \( F \) is an almost-normal subgroup of \( G_2 \), each \( R[F] \)-bimodule \( R[I] \) is a finitely generated, free (hence projective) left \( R[F] \)-module. Observe that \( \mathcal{I} \) is a filtered poset: if \( I, I' \in \mathcal{I} \) then \( I \cup I' \in \mathcal{I} \). Therefore we are done by Theorem 0.5. \[ \square \]

The case of \( G = D_\infty = \mathbb{Z}_2 \ast \mathbb{Z}_2 \) has a particularly simple form.

Corollary 3.26. Let \( R \) be a ring and \( n \in \mathbb{Z} \). There are natural isomorphisms:

1. \( \widetilde{\text{Nil}}_n(R; R, R) \cong \widetilde{\text{Nil}}_n(R) \)
2. \( K_n(R[D_\infty]) \cong (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_2]))/K_n(R) \oplus \widetilde{\text{Nil}}_{n-1}(R) \).

Proof. Part (i) follows from Corollary 3.25 with \( F = 1 \) and \( G_i = \mathbb{Z}_2 \). Then Part (ii) follows from Waldhausen’s exact sequence (Thm. 2.6), where the group retraction \( \mathbb{Z}_2 \rightarrow 1 \) induces a splitting of the map \( K_n(R) \rightarrow K_n(R[C_2]) \times K_n(R[C_2]) \). \[ \square \]

Example 3.27. Consider the group \( G = G_0 \times D_\infty \) where \( G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \). Since \( G \) surjects onto the infinite dihedral group, there is a corresponding amalgamated product decomposition
\[ G = (G_0 \times \mathbb{Z}_2) *_{G_0} (G_0 \times \mathbb{Z}_2) \]
with the corresponding index 2 subgroup
\[ \tilde{G} = G_0 \times \mathbb{Z}. \]

Corollary 3.26(1) gives an isomorphism
\[ \tilde{\text{Nil}}_{-1}(\mathbb{Z}[G_0]; \mathbb{Z}[G_0], \mathbb{Z}[G_0]) \cong \tilde{\text{Nil}}_{-1}(\mathbb{Z}[G_0]). \]

On the other hand, Bass showed that the latter group is an infinitely generated, abelian group of exponent a power of two [Bas68 XII, 10.6]. Hence, by Waldhausen’s algebraic K-theory decomposition result, Wh(G) is infinitely generated due to Nil elements. Now construct a codimension one, finite CW pair \((X, Y)\) with \(\pi_1 X = G\) realizing the above amalgamated product decomposition – for example, let \(Y \to Z\) be a map of connected CW complexes inducing the first factor inclusion \(G_0 \to G_0 \times \mathbb{Z}_2\) on the fundamental group and let \(X\) be the double mapping cylinder of \(Z \leftarrow Y \to Z\). Next construct a homotopy equivalence \(f : M \to X\) of finite CW complexes whose torsion \(\tau(f) \in \text{Wh}(G)\) is a non-zero Nil element. Then \(f\) is non-splittable along \(Y\) by Waldhausen [Wal69] (see Theorem 0.7). *This is the first explicit example of a non-zero Waldhausen Nil group and a non-splittable homotopy equivalence in the two-sided case.*

### 3.3. Farrell–Jones Conjecture

The Farrell–Jones Conjecture says that the family of virtually cyclic subgroups is a “generating” family for \(K_n(R[G])\). In this section we apply our main theorem to show that the Farrell–Jones Conjecture holds if and only if the smaller family of finite-by-cyclic subgroups is a generating family for \(K_n(R[G])\).

Let \(\text{Or}_G\) be the orbit category of a group \(G\); objects are \(G\)-sets \(G/H\) where \(H\) is a subgroup of \(G\) and morphisms are \(G\)-maps. Davis–Lück [DL98] defined a functor \(K_R : \text{Or}_G \to \text{Spectra}\) with the key property \(\pi_n K_R(G/H) = K_n(R[H])\). The utility of such a functor is that it allows the definition of an equivariant homology theory, indeed for a \(G\)-CW-complex \(X\), one defines
\[
H_n^G(X; K_R) = \pi_n(\text{map}_G(-, X)_+ \wedge_{\text{Or}_G} K_R(-))
\]
(see [DL98] section 4 and 7 for basic properties). Note that the “coefficients” of the homology theory are given by \(H_n^G(G/H; K_R) = K_n(R[H])\).

A family \(\mathcal{F}\) of subgroups of \(G\) is a nonempty set of subgroups closed under subgroups and conjugation. For such a family, \(E_{\mathcal{F}} G\) is the classifying space for \(G\)-actions with isotropy in \(\mathcal{F}\). It is characterized up to \(G\)-homotopy type as a \(G\)-CW-complex so that \((E_{\mathcal{F}} G)^H\) is contractible for subgroups \(H \in \mathcal{F}\) and is empty for subgroups \(H \notin \mathcal{F}\). Four relevant families are \(\text{fin} \subset \text{fbc} \subset \text{vc} \subset \text{all}\), the families of finite subgroups, finite-by-cyclic, virtually cyclic subgroups and all subgroups respectively. Here
\[
\text{fbc} = \text{fin} \cup \{H < G : H \cong F \rtimes \mathbb{Z} \text{ with } F \text{ finite}\}
\]
\[
\text{vc} = \{H < G : \exists \text{ cyclic } C < H \text{ with finite index}\}.
\]

The Farrell–Jones conjecture in \(K\)-theory for the group \(G\) [FJ93, DL98] states that
\[
H_n^G(E_{\text{vc}} G; K_R) \to H_n^G(E_{\text{all}} G; K_R) = K_n(R[G])
\]
is an isomorphism.

We now state a more general version, the fibered Farrell–Jones conjecture. If \(\varphi : \Gamma \to G\) is a group homomorphism and if \(\mathcal{F}\) is a family of subgroups of \(G\), define
the family of subgroups
\[ \varphi^* \mathcal{F} = \{ H < \Gamma : \varphi(H) \in \mathcal{F} \}. \]

The fibered Farrell–Jones conjecture in K-theory for the group \( G \) states that for every group epimorphism \( \varphi : \Gamma \to G \) and for every ring \( R \), that following induced map is an isomorphism:

\[ H_n^G(E_{\varphi^* \text{vc}}(G)\Gamma; K_R) \to H_n^G(E_{\varphi^* \text{all}}(G)\Gamma; K_R) = K_n(R[\Gamma]). \]

The following theorem was inspired by Frank Quinn. It is proven below.

**Theorem 3.28.** Let \( \varphi : \Gamma \to G \) be an epimorphism of groups. Let \( R \) be any ring. Then the following induced map is an isomorphism:

\[ H_n^\Gamma(E_{\varphi^* \text{fbc}}(G)\Gamma; K_R) \to H_n^\Gamma(E_{\varphi^* \text{vc}}(G)\Gamma; K_R). \]

Hence we have the following conjecture.

**Conjecture 3.29.** Let \( G \) be a discrete group, and let \( R \) be a ring.

1. There is an isomorphism:
   \[ H_n^G(E_{\text{fbc}} G; K_R) \to H_n^G(E_{\text{all}} G; K_R) = K_n(R[G]). \]

2. For any epimorphism \( \varphi : \Gamma \to G \) of groups, there is an isomorphism:
   \[ H_n^\Gamma(E_{\varphi^* \text{fbc}}(G)\Gamma; K_R) \to H_n^\Gamma(E_{\varphi^* \text{all}}(G)\Gamma; K_R) = K_n(R[\Gamma]). \]

We have shown that first statement is equivalent to the Farrell–Jones conjecture in K-theory and the second statement is equivalent to the fibered Farrell–Jones conjecture in K-theory.

The proof of Theorem 3.28 will require three auxiliary results, some of which we quote from other sources. The first is Theorem A.10 of Farrell–Jones [FJ93].

**Transitivity Principle.** Let \( \mathcal{F} \subset \mathcal{G} \) be families of subgroups of a group \( \Gamma \). Let \( E : \text{Or} \Gamma \to \text{Spectra} \) be a functor. Then for every \( H \in \mathcal{G} - \mathcal{F} \), if the assembly map

\[ H_n^H(E_{\mathcal{F} \cap H} H; E) \to H_n^H(E_{\text{all} H} H; E) \]

is an isomorphism, then the following map is an isomorphism:

\[ H_n^\Gamma(E_{\mathcal{F}} \Gamma; E) \to H_n^\Gamma(E_{\text{all}} \Gamma; E). \]

Of course, we shall apply this principle to the families \( \text{fbc} \subset \text{vc} \). The following lemma is well-known (see [SW79, Theorem 5.12]), but we offer an alternative proof.

**Lemma 3.30.** Let \( G \) be a virtually cyclic group. Then either

1. \( G \) is finite.
2. \( G \) admits an epimorphism to \( \mathbb{Z} \), hence \( G = F \rtimes_{\alpha} \mathbb{Z} \) with \( F \) finite.
3. \( G \) admits an epimorphism to \( D_{\infty} \), hence \( G = G_1 *_F G_2 \) with \( |G_1 : F| = 2 \) and \( F \) finite.

**Proof.** Assume \( G \) is an infinite virtually cyclic group. The intersection of the conjugates of a finite index, infinite cyclic subgroup is a normal, finite index, infinite cyclic subgroup \( C \). Let \( Q \) be the finite quotient group. Embed \( C \) as an index \( |Q| \) subgroup of an infinite cyclic subgroup \( C' \). The image of the obstruction cocycle under the map \( H^2(Q; C) \to H^2(Q; C') \) is trivial, so \( G \) embeds as a finite index subgroup of a semidirect product \( G' = C' \rtimes Q \). Note \( G' \) maps epimorphically to \( \mathbb{Z} \) (if \( Q \) acts trivially) or to \( D_{\infty} \) (if \( Q \) acts non-trivially). In either case, \( G \) maps
epimorphically to a subgroup of finite index in $D_\infty$, which must be either infinite cyclic or infinite dihedral.

In order to see how the Farrell–Hsiang and Waldhausen Nil-groups relate to equivariant homology (and hence to the Farrell–Jones Conjecture), we need Lemma 3.1, Theorem 1.5, as follows.

**Lemma 3.31** (Davis–Quinn–Reich). Let $\overline{G}$ be a group of the form $F \rtimes_a \mathbb{Z}$, and let $\mathcal{F}$ be the smallest family of subgroups of $\overline{G}$ containing $F$. Let $G$ be a group of the form $G_1 *_{F'} G_2$ with $|G_1 : F| = 2$, and let $\mathcal{F}$ be the smallest family of subgroups of $G$ containing $G_1$ and $G_2$. Note that $F$ need not be finite.

1. The following exact sequences are split, and hence short exact:
   
   $H^G_n(E_{F'} \mathbb{G}; \mathbb{K}_R) \to H^G_n(E_{all} \overline{G}; \mathbb{K}_R) \to H^G_n(E_{all} \overline{G}, E_{F'} \mathbb{G}; \mathbb{K}_R)$
   
   $H^G_n(E_{F'} \mathbb{G}; \mathbb{K}_R) \to H^G_n(E_{all} \overline{G}; \mathbb{K}_R) \to H^G_n(E_{all} \overline{G}, E_{F'} \mathbb{G}; \mathbb{K}_R)$.

2. Moreover, the relative terms are:
   
   $H^G_n(E_{all} \overline{G}, E_{F'} \mathbb{G}; \mathbb{K}_R) \cong \Nil_{n-1}(R[F], \alpha) \oplus \Nil_{n-1}(R[F], \alpha^{-1})$
   
   $H^G_n(E_{all} \overline{G}; E_{F'} \mathbb{G}; \mathbb{K}_R) \cong \Nil_{n-1}(R[F]; R[G_1 - F], R[G_2 - F])$.

3. Let $p : G \to D_\infty$ be the epimorphism induced the amalgamated product splitting above. Then
   
   $H^G_n(E_{p^* fbc} G, E_{p^* fin} G; \mathbb{K}_R) \cong \Nil_{n-1}(R[F], \alpha)$,
   
   where the ring automorphism $\alpha : R[F] \to R[F]$ is induced by conjugation of an indivisible element of infinite order in $D_\infty$.

Furthermore, it is not difficult to compute $H^G_n(E_{F'} \mathbb{G}; \mathbb{K}_R)$ and $H^G_n(E_{F'} \mathbb{G}; \mathbb{K}_R)$ in terms of a Wang sequence and a Mayer–Vietoris sequence respectively. We shall focus on a particular example in Subsection 3.4.

**Proof of Theorem 3.28** Let $\varphi : \Gamma \to G$ be an epimorphism of groups. We will use the Transitivity Principle applied to the families $\varphi^* fbc \subset \varphi^* vc$. By Lemma 3.30 if $H \in \varphi^* vc - \varphi^* fbc$, then $\varphi(H)$ admits an epimorphism $p : \varphi(H) \to D_\infty$ with finite kernel. Thus $\varphi(H)$ is an injective amalgam $\varphi(H) = G'_1 *_{F'} G'_2$ of finite groups, where $F'$ is an index two subgroup of each $G'_i$. It is not difficult to show

$$p^* \text{fin} = \text{fin}$$

$$p^* \text{fbc} = \text{fbc}.$$ 

For clarity we denote the restriction by $\varphi : H \to \varphi(H)$. The decomposition of $H$ induced by $p \circ \varphi$ is an injective amalgam $H = G_1 *_{F} G_2$ where $F := \varphi^{-1}(F')$ is an index two subgroup of each $G_i := \varphi^{-1}(G'_i)$. Let $\mathcal{F}$ be the smallest family of subgroups of $H$ containing $G_1$ and $G_2$. Using the above two displayed equalities and the fact that every finite subgroup of $G'_1 *_{F'} G'_2$ is conjugate to a subgroup of $G'_1$ or $G'_2$ (see [Ser03 Corollary to Theorem 4.3.8]), one can show

$$\mathcal{F} = (p \circ \varphi)^* \text{fin}$$

$$(\varphi^* \text{fbc}) \cap H = (p \circ \varphi)^* \text{fbc}.$$
By Lemma 3.31 parts (2) and (3), we have
\[ H_n^H(E_{all}H, E_FH; \mathbb{K}_R) = \tilde{\text{Nil}}_{n-1}(R[F]; R(G_1 - F], R(G_2 - F)] \]
\[ H_n^H(E_{(p \circ \varphi)^*fbc} H, E_{(p \circ \varphi)^*fin} H; \mathbb{K}_R) = \tilde{\text{Nil}}_{n-1}(R[F]; \alpha). \]

Then the map
\[ H_n^H(E_{(p \circ \varphi)^*fbc} H, E_{(p \circ \varphi)^*fin} H; \mathbb{K}_R) \to H_n^H(E_{all}H, E_FH; \mathbb{K}_R) \]
induces the isomorphism of Theorem 0.1. Hence, by the exact sequence of a triple, \( H_n^H(E_{all}H, E_FH; \mathbb{K}_R) = 0 \). Therefore, by the Transitivity Principle, we have proven \( H_n^H(E_{(p \circ \varphi)^*c} \Gamma, E_{(p \circ \varphi)^*fbc} \Gamma; \mathbb{K}_R) = 0 \). \( \square \)

3.4. \textit{K}-theory of the modular group. Let \( \Gamma = \mathbb{Z}_2 \ast \mathbb{Z}_3 = \text{PSL}_2(\mathbb{Z}) \). The following theorem follows from applying our main theorem and the recent proof \[ \text{[BLR08]} \] of the Farrell–Jones conjecture in \textit{K}-theory for word hyperbolic groups.

The Cayley graph for \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) with respect to the generating set given by the nonzero elements of \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) has the quasi-isometry type of the usual Bass–Serre tree for the amalgamated product (Figure 1). This is an infinite tree with alternating vertices of valence two and three. The group \( \Gamma \) acts on the tree, with the generator of order two acting by reflection through an valence two vertex and the generator of order three acting by rotation through an adjoining vertex of valence three.

Any geodesic triangle in the Bass–Serre tree has the property that the union of two sides is the union of all three sides. It follows that the Bass–Serre graph is \( \delta \)-hyperbolic for any \( \delta > 0 \), the Cayley graph is \( \delta \)-hyperbolic for some \( \delta > 0 \), and hence \( \Gamma \) is a hyperbolic group.

\textbf{Theorem 3.32.} For any ring \( R \) and integer \( n \),
\[ K_n(R[\Gamma]) = (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]))/K_n(R) \]
\[ \oplus \bigoplus_{\mathcal{M}_C} \tilde{\text{Nil}}_{n-1}(R) \oplus \tilde{\text{Nil}}_{n-1}(R) \oplus \bigoplus_{\mathcal{M}_D} \tilde{\text{Nil}}_{n-1}(R). \]
where $\mathcal{M}_G$ and $\mathcal{M}_D$ are the set of conjugacy classes of maximal infinite cyclic subgroups and maximal infinite dihedral subgroups, respectively. Moreover, all virtually cyclic subgroups of $\Gamma$ are cyclic or infinite dihedral.

**Proof.** By Lemma 3.31, the homology exact sequence of the pair $(E_{\text{att}}\Gamma, E_{\text{fin}}\Gamma)$ is short exact and split:

$$H_n^\Gamma(E_{\text{fin}}\Gamma; K_R) \rightarrow H_n^\Gamma(E_{\text{att}}\Gamma; K_R) \rightarrow H_n^\Gamma(E_{\text{att}}\Gamma, E_{\text{fin}}\Gamma; K_R).$$

Then, by the Farrell–Jones Conjecture [31LR08] for word hyperbolic groups, we obtain

$$K_n(R[\Gamma]) = H_n^\Gamma(E_{\text{fin}}\Gamma; K_R) \oplus H_n^\Gamma(E_{\text{att}}\Gamma, E_{\text{fin}}\Gamma; K_R).$$

Note that $E_{\text{fin}}\Gamma$ is constructed as a pushout of $\Gamma$-spaces

$$\Gamma \sqcup \Gamma \longrightarrow \Gamma/\mathbb{Z}_2 \sqcup \Gamma/\mathbb{Z}_3 \quad \downarrow \quad \downarrow$$

$$\Gamma \times D^1 \longrightarrow E_{\text{fin}}\Gamma.$$  

Then $E_{\text{fin}}\Gamma$ is the Bass–Serre tree for $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$. Note that $H^\Gamma_1(\Gamma/H; K_R) = K_*[R[H]]$. The pushout gives, after canceling a $K_n(R)$ term, a long exact sequence

$$\cdots \rightarrow K_n(R) \rightarrow K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]) \rightarrow H_n^\Gamma(E_{\text{fin}}\Gamma; K_R) \rightarrow K_{n-1}(R) \rightarrow \cdots.$$  

Hence

$$H_n^\Gamma(E_{\text{fin}}\Gamma; K_R) = (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]))/K_n(R).$$

For a word hyperbolic group $G$,

$$H_n^G(E_{\text{vc}}V, E_{\text{fin}}V; K) \cong \bigoplus_{[V] \in \mathcal{M}(G)} H_n^V(E_{\text{vc}}V, E_{\text{fin}}V; K)$$

where $\mathcal{M}(G)$ is the set of conjugacy classes of maximal virtually cyclic subgroups of $G$ (see Lück [Lüc08], Theorem 8.11) and [31PL06]. The geometric interpretation of this result is that $E_{\text{vc}}G$ is obtained by coning off each geodesic in the tree $E_{\text{fin}}G$; then apply excision.

The Kurosh subgroup theorem implies that a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$ is a free product of $\mathbb{Z}_2$’s, $\mathbb{Z}_3$’s, and $\mathbb{Z}$’s. Note that $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 \rangle$, $\mathbb{Z}_3 * \mathbb{Z}_3 = \langle c, d \mid c^3 = 1 = d^3 \rangle$, and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle e, f, g \mid e^2 = f^2 = g^2 = 1 \rangle$ have free subgroups of rank 2, for example $\langle ab, ab^2 \rangle$, $\langle cd, cd^2 \rangle$, and $\langle ef, fg \rangle$. On the other hand the rank 2 free group $F(2)$ is not a virtually cyclic group since its first Betti number $\beta_1(F(2)) = \text{rank } H_1(F(2)) = 2$, while for a virtually cyclic group $V$, transferring to the cyclic subgroup $C \subset V$ of finite index shows that $\beta_1(V)$ is 0 or 1. Subgroups of virtually cyclic groups are also virtually cyclic. Therefore all virtually cyclic subgroups of $\Gamma$ are cyclic or infinite dihedral.

By the fundamental theorem of $K$-theory and Waldhausen’s Theorem (see also Lemma 3.31)

$$H_n^k(E_{\text{vc}}\mathbb{Z}, E_{\text{fin}}\mathbb{Z}; K_R) = \tilde{\text{Nil}}_{n-1}(R) \oplus \tilde{\text{Nil}}_{n-1}(R)$$

$$H_n^D(E_{\text{vc}}D, E_{\text{fin}}D; K_R) = \tilde{\text{Nil}}_{n-1}(R; R, R)$$

Finally, by Corollary 3.26(1), we obtain exactly one type of Nil-group:

$$\text{Nil}_{n-1}(R; R, R) \cong \tilde{\text{Nil}}_{n-1}(R).$$
Remark 3.33. The sets $\mathcal{M}_C$ and $\mathcal{M}_D$ are countably infinite. This can be shown by parameterizing these subsets either: combinatorially (using that elements in $\Gamma$ are words in $a, b, b^2$), geometrically (maximal virtually cyclic subgroups correspond to stabilizers of geodesics in the Bass–Serre tree $E_{fin}\Gamma$, where the geodesic may or may not be invariant under an element of order 2), or number theoretically (using solutions to Pell’s equation and Gauss’ theory of binary quadratic forms [Sar07]).

Let us give an overview and history of some related work. The Farrell–Jones Conjecture and the classification of virtually cyclic groups (see Lemma 3.30) focused attention on the algebraic $K$-theory of groups mapping to the infinite dihedral group. Several years ago James Davis and Bogdan Vajiac outlined a unpublished proof of Theorem 0.1 when $n \leq 0$ using controlled topology and hyperbolic geometry. Lafont and Ortiz [LO08] proved that $\tilde{\text{Nil}}_n(\mathbb{Z}[F]; \mathbb{Z}[V_1 - F], \mathbb{Z}[V_2 - F]) = 0$ if and only if $\tilde{\text{Nil}}_n(\mathbb{Z}[F], \alpha) = 0$ for any virtually cyclic group $V$ with an epimorphism $V \to D_\infty$ and $n = 0, 1$. More recently, Lafont–Ortiz [LO09] have studied the more general case of the $K$-theory $K_n((R[G_1 *_F G_2])$ of an injective amalgam, where $F, G_1, G_2$ are finite groups. Finally, we mentioned the paper [DQR], which was written in parallel with this one. It an alternate proof of Theorem 0.1 assuming a controlled topology result of Frank Quinn. Also, [DQR] provides several auxiliary results used in Subsection 3.3 of this paper.

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