Partial regularity for parabolic systems with VMO-coefficients

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Abstract. We establish a partial Hölder continuity for vector-valued solutions $u : \Omega_T \to \mathbb{R}^N$ to parabolic systems of the type:

$$u_t - \text{div}(A(x, t, u, Du)) = H(x, t, u, Du) \quad \text{in } \Omega \times (-T, 0),$$

where the coefficients $A : \Omega \times (-T, 0) \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ are possibly discontinuous with respect to $(x, t)$. More precisely, we assume a VMO-condition with respect to $(x, t)$ and continuity with respect to $u$ and prove Hölder continuity of the solutions outside of singular sets.

Keywords. Nonlinear parabolic systems, Partial regularity, VMO-coefficients, $A$-caloric approximation.

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1 Introduction

In this paper, we establish a partial regularity result of weak solutions to second order nonlinear parabolic systems of the following type:

$$u_t - \text{div}(A(z, u, Du)) = H(z, u, Du), \quad z = (x, t) \in \Omega \times (-T, 0) =: \Omega_T, \quad (1.1)$$

where $\Omega$ denotes a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, $u$ takes values in $\mathbb{R}^N$, $N \geq 1$, and the vector field $A : \Omega_T \times \mathbb{R}^N \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ fulfills the $p$-growth condition, $p \geq 2$, and the VMO-condition. More precisely, we assume that the partial mapping $z \mapsto A(z, u, w)/(1 + |w|)^{p-1}$ has vanishing mean oscillation (VMO), uniformly in $(u, w)$. This means that $A$ satisfies the estimate

$$|A(z, u, w) - A(z, u, w_0)| \leq V_{z_0}(z, \rho)(1 + |w|)^{p-1},$$

where $V_{z_0} : \mathbb{R}^{n+1} \times [0, \rho_0] \to [0, 2L]$ are bounded functions with

$$\lim_{\rho \to 0} V(\rho) = 0, \quad V(\rho) := \sup_{z_0} \sup_{0 < r < \rho} \int_{Q_r(z_0) \cap \Omega_T} V_{z_0}(z, r)dz.$$

The vector field $A$ also satisfies the $p$-growth condition such as

$$|A(z, u, w)| + (1 + |w|) |\partial_u A(z, u, w)| \leq L(1 + |w|)^{p-1}$$

for all $z \in \Omega_T$, $u \in \mathbb{R}^N$ and $w \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Moreover $A$ is continuous with respect to $u$. Roughly speaking, under the above assumptions, we prove that the bounded weak solutions of (1.1) are Hölder continuous on some open set $\Omega_u \subset \Omega_T$, i.e., $u \in C^{\alpha, \alpha/2}(\Omega_u, \mathbb{R}^N)$ (see Theorem 2.2).
Regularity problem of weak solutions to parabolic systems are already proved for nonlinear systems with $p = 2$ by Duzaar-Mingione [13], for $p \geq 2$ by Duzaar-Mingione-Steffen [14], for $1 < p < 2$ by Scheven [20] and even on the boundary by Bögelein-Duzaar-Mingione [31]. These previous results are based on the technique so called “A-caloric approximation” (see Lemma 2.2) and proved under the condition that the vector field $A(z, u, w)$ are Hölder continuous with respect to $(z, u)$, i.e., there exists a non-decreasing function $K: [0, \infty) \to [1, \infty)$ and $\beta \in (0, 1)$ such that the inequality

$$|A(z, u, w) - A(z_0, u_0, w)| \leq K(|u|)(|x - x_0| + \sqrt{|t - t_0|} + |u - u_0|)^\beta(1 + |w|^{p-1})$$

holds for every $z = (x, t), z_0 = (x_0, t_0) \in \Omega_T, u, u_0 \in \mathbb{R}^N$ and for all $w \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

The $A$-caloric approximation technique has its origin in the classical harmonic approximation lemma of De Giorgi in version of Simon [10, 21]. It was first applied to nonlinear elliptic systems with quadratic growth condition ($p = 2$) by Duzaar-Grotowski [12], namely “$A$-harmonic approximation”. Using this method, we could obtain the optimal regularity result without the reverse Hölder inequalities, i.e., if the “coefficients” $A(x, u, w)$ are Hölder continuous in $(x, u)$ with some Hölder exponent $\beta \in (0, 1)$ then $Du$ is Hölder continuous with the same exponent $\beta$ on some open set $\Omega_u$.

Then the $A$-caloric approximation technique has been used to prove the regularity result for elliptic systems with super-quadratic growth ($p \geq 2$) and for the case of sub-quadratic growth $(1 < p < 2)$ by Chen-Tan [7, 8]. The $A$-harmonic approximation technique also works for boundary regularity which was proved by Grotowski [18]. Moreover, a relation between the regularity of weak solutions and the smoothness of coefficients is studied. Duzaar-Gastel [11] proved that weak solutions has $C^1$-regularity if the coefficients satisfies Dini-type condition (which is weaker assumption than Hölder continuity condition). The continuous coefficients would not ensure the continuity (and not even boundedness) of the gradient $Du$ but Foss-Mingione [15] showed that we could still except the local Hölder continuity of the solution $u$ itself. The Hölder continuity for the solution $u$ can also be guaranteed under discontinuous coefficients such as the VMO-condition in elliptic setting, which was proved for homogeneous systems by Bögelein-Duzaar-Habermann-Scheven [2] and for inhomogeneous systems by author [19].

On the other hand, $A$-harmonic approximation technique is adapted to parabolic systems, renamed as “$A$-caloric approximation” [13, 14], and it lead us to the partial regularity result for weak solutions in parabolic setting with Hölder continuous coefficients. Dini-type condition and the condition under continuous coefficients are also proved by Baroni [1], Bögelein-Duzaar-Mingione [5] and Foss-Geisbauer [15]. However, as far as we know, no one has been proved regularity result under discontinuous coefficients in parabolic systems. In this paper, we proved the regularity result under the VMO-condition which is the parabolic version of [19] (see Theorem 2.2).

2 Statement of the results

Before we start setting the structure conditions, let us collect some notations which we will use throughout the paper. As mentioned above, we consider a cylindrical domain $\Omega_T = \Omega \times (-T, 0)$ where $\Omega$ is a bounded domain in $\mathbb{R}^n, n \geq 2$, and $T > 0$. $u$ maps from $\Omega_T$ to $\mathbb{R}^N, N \geq 1$, and $Du$ denotes the gradient with respect to the special variables $x$, i.e., $Du(x, t) \equiv D_x u(x, t)$. We write $B_\rho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ and $Q_{\rho}(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ where $z_0 = (x_0, t_0) \in \Omega_T$. The parabolic metric $d_{par}$ is given by

$$d_{par}(z, z_0) = \max\{|x - x_0|, \sqrt{|t - t_0|}\} \quad \text{for} \quad z = (x, t), z_0 = (x_0, t_0) \in \Omega_T, \quad (2.1)$$

and for a given set $X$ we denote by $\mathcal{H}_{par}^{n+2}(X)$ the $(n+2)$-dimensional parabolic Hausdorff measure which is defined by

$$\mathcal{H}_{par}^{n+2}(X) = \sup_{\delta > 0} \mathcal{H}_{par}^{n+2, \delta}(X),$$
where
\[ \mathcal{H}^{n+2,\delta}_\text{par}(X) = \inf \left\{ \sum_{i=1}^{\infty} R_i^{n+2} : X \subset \bigcup_{i=1}^{\infty} Q_{R_i}(z_i), R_i \leq \delta \right\}. \]

Note that \( \mathcal{H}^{n+2}_\text{par} \) is equivalent to the Lebesgue measure in \( \mathbb{R}^{n+1} \), \( \mathcal{L}^{n+1} \). For a bounded set \( X \subset \mathbb{R}^{n+1} \) with \( \mathcal{L}^{n+1}(X) > 0 \), we denote the average of a given function \( g \in L^1(X, \mathbb{R}^n) \) by \( \frac{1}{\mathcal{L}^{n+1}(X)} \int_X g \, dz \), that is, \( f_X \, gdz = \frac{1}{\mathcal{L}^{n+1}(X)} \int_X g \, dz \). In particular, we write \( g_{\Omega, \rho} = \frac{1}{\mathcal{L}^{n+1}(\Omega)} \int_{\Omega} g \, dz \). We write \( \text{Bil}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)) \) for the space of bilinear forms on the space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \) of linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^N \). We denote \( c \) a positive constant, possibly varying from line by line. Special occurrences will be denoted by capital letters \( K, C_1, C_2 \) or the like.

**Definition 2.1.** We say \( u \in C^0(-T, 0; L^2(\Omega, \mathbb{R}^N)) \cap L^p(-T, 0; W^{1,p}(\Omega, \mathbb{R}^N)), p \geq 2 \) is a weak solution of (1.1) if \( u \) satisfies
\[ \int_{\Omega_T} \left( \langle u, \varphi_t \rangle - \langle A(z, u, Du), D\varphi \rangle \right) \, dz = \int_{\Omega_T} \langle H, \varphi \rangle \, dz \tag{2.2} \]
for all \( \varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N) \), where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product on \( \mathbb{R}^N \) or \( \mathbb{R}^{nN} \).

We assume the following structure conditions.

**(H1)** \( A(z, u, w) \) is differentiable in \( w \) with continuous derivatives, that is, there exists \( L \geq 1 \) such that
\[ |A(z, u, w)| + (1 + |w|) |\partial_w A(z, u, w)| \leq L (1 + |w|)^{p-1} \tag{2.3} \]
for all \( z \in \Omega_T, u \in \mathbb{R}^N \) and \( w \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \). Moreover, from this we deduce the modulus of continuity function \( \mu : [0, \infty) \to [0, \infty) \) such that \( \mu \) is bounded, concave, non-decreasing and we have
\[ |\partial_w A(z, u, w) - \partial_w A(z, u, w_0)| \leq L \mu \left( \frac{|w - w_0|}{1 + |w| + |w_0|} \right) (1 + |w| + |w_0|)^{p-2} \tag{2.4} \]
for all \( z \in \Omega_T, u \in \mathbb{R}^N, w, w_0 \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \). Without loss of generality, we may assume \( \mu \leq 1 \).

**(H2)** \( A(z, u, w) \) is uniformly strongly elliptic, that is, for some \( \lambda > 0 \) we have
\[ \left( \partial_w A(z, u, w) \bar{w}, \bar{w} \right) := \sum_{1 \leq i < j \leq N} \sum_{1 \leq \alpha < \beta \leq n} \partial_{w_i w_j} A_{\alpha \beta}^i (z, u, w) \bar{w}_i^\alpha \bar{w}_j^\beta \geq \lambda |\bar{w}|^2 (1 + |w|)^{p-2}/2 \tag{2.5} \]
for all \( z \in \Omega_T, u \in \mathbb{R}^N, w, \bar{w} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \).

**(H3)** \( A(z, u, w) \) is continuous with respect to \( w \). There exists a bounded, concave and non-decreasing function \( \omega : [0, \infty) \to [0, \infty) \) satisfying
\[ |A(z, u, w) - A(z, u_0, w)| \leq L \omega \left( |u - u_0|^2 \right) (1 + |w|)^{p-1} \tag{2.6} \]
for all \( z \in \Omega_T, u, u_0 \in \mathbb{R}^N \), \( w \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \). Without loss of generality, we may assume \( \omega \leq 1 \).

**(H4)** \( z \mapsto A(z, u, w)/(1 + |w|)^{p-1} \) fulfills the following VMO-condition uniformly in \( u \) and \( w \):
\[ |A(z, u, w) - A(z, u, w_0)|_{z \in Q_{\rho}(z_0)} \leq V_{z_0}(z, \rho)(1 + |w|)^{p-1}, \quad \text{for all } z \in Q_{\rho}(z_0) \]
whenever \( z_0 \in \Omega_T, 0 < \rho < \rho_0, u \in \mathbb{R}^N \) and \( w \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \), where \( \rho_0 > 0 \) and \( V_{z_0} : \mathbb{R}^n \times [0, \rho_0] \to [0, 2L] \) are bounded functions satisfying
\[ \lim_{\rho \searrow 0} V(\rho) = 0, \quad V(\rho) := \sup_{z_0 \in \Omega_T} \sup_{0 < r \leq \rho} \int_{Q_{\rho}(z_0) \cap \Omega} V_{z_0}(z, r) \, dz. \tag{2.7} \]
Lemma 3.2 with respect to the parabolic metric given in (2.1). In other word, exponent there exists a function satisfying \( u \) ∈ \( \mathbb{R}^n \) with \( |u| \leq M \) and \( u \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N) \).

Under these structure conditions, we proved the following theorem.

Theorem 2.2. Let \( u \in C^0_b((-T, 0); L^2(\Omega, \mathbb{R}^N)) \cap L^p((-T, 0); W^{1, p}(\Omega, \mathbb{R}^N)) \) be a bounded weak solution of the parabolic system \( \text{(H4)} \) under the structure condition (H1), (H2), (H3), (H4) and (H5) with satisfying \( ||u||_{\infty} \leq M \) and \( \frac{2(10^{-9p})^2}{M} > a(M)M \). Then there exists an open set \( \Omega_u \subset \Omega_T \) such that \( u \in C^{\alpha, \alpha/2}(\Omega_u, \mathbb{R}^N) \) with \( H^{n+2}_{\text{par}}(\Omega_T \setminus \Omega_u) = 0 \) for every \( \alpha \in (0, 1) \). Moreover, \( \Omega_T \setminus \Omega_u \subset \Sigma_1 \cup \Sigma_2 \) and

\[
\Sigma_1 := \left\{ z_0 \in \Omega_T : \liminf_{\rho \searrow 0} \int_{Q_\rho(z_0)} |Du - (Du)_{z_0, \rho}|^pdz > 0 \right\},
\]

\[
\Sigma_2 := \left\{ z_0 \in \Omega_T : \limsup_{\rho \searrow 0} |(Du)_{z_0, \rho}| = \infty \right\}.
\]

The previous result means that the weak solution \( u \) is Hölder continuous in \( \Omega_u \) with exponent \( \alpha \) with respect to the parabolic metric given in (2.1). In other word, \( u \) is Hölder continuous in \( \Omega_u \) with exponent \( \alpha \) with respect to space variable \( x \) and with exponent \( \alpha/2 \) with respect to the time variable \( t \).

## 3 Preliminaries

In this section we present the \( A \)-caloric approximation lemma and recall some standard estimates for the proof of our main theorem, (Theorem 2.2).

First we state the definition of \( A \)-caloric function and recall the \( A \)-caloric approximation lemma as below.

**Definition 3.1 (\( A \)-caloric function, [14 DEF 3.1]).** Let \( A \) be a bilinear form with constant coefficients satisfying

\[
\lambda |\bar{w}|^2 \leq A(w, \bar{w}), \quad A(w, \bar{w}) \leq L|w||\bar{w}| \quad \text{for all } w, \bar{w} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N).
\]

A function \( h \in L^2(t_0 - \rho^2, t_0; W^{1, 2}(B_{\rho/2}(x_0), \mathbb{R}^N)) \) is called \( A \)-caloric in the cylinder \( Q_\rho(z_0) \) iff it satisfies

\[
\int_{Q_\rho(z_0)} (h \phi_t - A(Dh, D\phi))dz = 0 \quad \text{for all } \phi \in C_0^\infty(Q_\rho(z_0), \mathbb{R}^N).
\]

**Lemma 3.2 (\( A \)-caloric approximation lemma, [14 LEMMA 3.2]).** Given \( \varepsilon > 0 \), \( 0 < \lambda < L \) and \( p \geq 2 \) there exists \( \delta = \delta(n, N, p, \lambda, L, \varepsilon) \geq 1 \) with the following property: Whenever \( A \) is a bilinear form on \( \mathbb{R}^{nN} \) satisfying (3.1), \( \gamma \in (0, 1) \), and whenever

\[
w \in L^p(t_0 - (\rho/2)^2, t_0; W^{1, 2}(B_{\rho/2}(x_0), \mathbb{R}^N))
\]

is a function satisfying

\[
\int_{Q_{\rho/2}(z_0)} \left( \frac{w}{\rho/2}^2 + \gamma p^{-2} \frac{|w|}{\rho/2}^p \right)dz + \int_{Q_{\rho/2}(z_0)} \left( |Dw|^2 + \gamma p^{-2} |Dw|^p \right)dz \leq 1
\]

and

\[
\left| \int_{Q_{\rho/2}(z_0)} (w, \phi_t - A(Dw, D\phi))dz \right| \leq \delta \sup_{Q_{\rho/2}(z_0)} |D\phi|
\]

is a function satisfying
for every $\varphi \in C_0^\infty(Q_{\rho/2}(z_0), \mathbb{R}^N)$ then there exists a function

$$h \in L^p(t_0 - (\rho/4)^2, t_0; W^{1,2}(B_{\rho/4}(x_0), \mathbb{R}^N))$$

which is $\mathcal{A}$-caloric on $Q_{\rho/4}(z_0)$ such that

$$\int_{Q_{\rho/4}(z_0)} \left( \frac{|h|}{\rho/4} + \gamma^{p-2} \frac{|h|}{\rho/4} \right) dz + \int_{Q_{\rho/4}(z_0)} (|Dh|^2 + \gamma^{p-2}|Dh|^p) dz \leq 2 \cdot 2^{n+2+2p}$$

(3.4)

and

$$\int_{Q_{\rho/4}(z_0)} \left( \frac{|w - h|}{\rho/4} + \gamma^{p-2} \frac{|w - h|}{\rho/4} \right) dz \leq \varepsilon.$$  

(3.5)

The next lemma features a standard estimate for $\mathcal{A}$-caloric functions.

**Lemma 3.3** ([LEMMA 4.7]). Let $h \in L^2(t_0 - (\rho/4)^2, t_0; W^{1,2}(B_{\rho/4}(x_0), \mathbb{R}^N))$ be $\mathcal{A}$-caloric function in $Q_{\rho/4}(z_0)$ with $\mathcal{A}$ satisfying (5.1). Then $h$ is smooth in $B_{\rho/4}(x_0) \times (t_0 - (\rho/4)^2, t_0]$ and for any $s \geq 1$ there exists a constant $c_2 = c_2(n, N, L/\lambda, s) \geq 1$ such that for any affine function $\ell : \mathbb{R}^n \to \mathbb{R}$ there holds

$$\int_{Q_{\rho/4}(z_0)} \left| \frac{h - \ell}{\theta \rho} \right|^s dz \leq c_2 \varepsilon^2 \int_{Q_{\rho/4}(z_0)} \left| \frac{h - \ell}{\rho/4} \right|^p dz$$

for every $0 < \theta \leq 1/4$.

For given $u \in L^2(Q_{\rho}(z_0), \mathbb{R}^N)$ we denote by $\ell_{z_0, \rho}$ the unique affine function minimizing

$$\ell \mapsto \int_{Q_{\rho}(z_0)} |u - \ell|^2 dz$$

(3.6)

among all affine functions $\ell(z) = \ell(x)$ which are independent of $t$. An elementary calculation yield that $\ell_{z_0, \rho}$ takes the form

$$\ell_{z_0, \rho}(x) = \ell_{z_0, \rho}(x_0) + D\ell_{z_0, \rho}(x - x_0),$$

where

$$\ell_{z_0, \rho}(x_0) = u_{z_0, \rho}, \quad \text{and} \quad D\ell_{z_0, \rho} = -\frac{n + 2}{\rho^2} \int_{Q_{\rho}(z_0)} u \otimes (x - x_0) dz.$$

Using the Cauchy-Schwarz inequality we have the following lemma.

**Lemma 3.4** ([LEMMA 2.1]). Let $u \in L^2(Q_{\rho}(z_0), \mathbb{R}^N)$, $0 < \theta < 1$ and

$$\ell_{z_0, \rho}(x) = \xi_{z_0, \rho} + D\ell_{z_0, \rho}(x - x_0), \quad \ell_{z_0, \rho}(x) = \xi_{z_0, \rho} + D\ell_{z_0, \rho}(x - x_0)$$

be the unique affine function that minimize

$$\ell \mapsto \int_{Q_{\rho}(z_0)} |u - \ell|^2 dz \quad \text{and} \quad \ell \mapsto \int_{Q_{\rho}(z_0)} |u - \ell|^2 dz$$

among all affine functions $\ell(z) = \ell(x)$ which are independent of $t$, respectively. Then there holds

$$|D\ell_{z_0, \rho} - D\ell_{z_0, \rho}|^2 \leq \frac{n(n + 2)}{(\theta \rho)^2} \int_{Q_{\rho}(z_0)} |u - \ell_{z_0, \rho}|^2 dz.$$  

(3.7)

Moreover, for any $D\ell \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ we have

$$|D\ell_{z_0, \rho} - D\ell|^2 \leq \frac{n(n + 2)}{\rho^2} \int_{Q_{\rho}(z_0)} |u - u_{z_0, \rho} - D\ell(x - x_0)|^2 dz.$$  

(3.8)
Moreover, any affine functions $\ell_q > \text{lowings, we define the parabolic system}\] with the constant $K$.

Let $\rho > 0$ be a constant and $\rho > a(M)$. For any $x_0 = (x_0, t_0) \in \Omega_T$ and $\rho \leq 1$ with $Q_\rho(x_0) \subseteq \Omega_T$, and any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$ with $|\ell(x_0)| \leq M$, we have the estimate

$$\sup_{t_n - (\rho/2)^2 < t < t_n} \int_{B_{\rho/2}(x_0)} \frac{|u - \ell|^2}{\rho^2 (1 + |D\ell|^2)} \, dx + \int_{Q_{\rho/2}(x_0)} \left\{ \frac{|D(Du - D\ell)|^2}{(1 + |D\ell|^2)^2} + \frac{|Du - D\ell|^p}{(1 + |D\ell|^p)} \right\} \, dz$$

$$\leq C_1 \left\{ \int_{Q_\rho(x_0)} \frac{|u - \ell|^2}{\rho^2 (1 + |D\ell|^2)} \, dz + \omega \left( \int_{Q_\rho(x_0)} |u - \ell(x_0)|^2 \, dz \right) + V(\rho) + (a^q |D\ell|^q + b^q) \rho^q \right\},$$

with the constant $C_1 = C_1(\lambda, \mu, L, a(M), M) \geq 1$.

**Proof.** Assume $x_0 \in \Omega_T$ and $\rho \leq 1$ satisfy $Q_\rho(x_0) \subseteq \Omega_T$. We take a standard cut-off functions $\chi \in C_0^\infty(B_\rho(x_0))$ and $\zeta \in C^1(\mathbb{R})$. More precisely, let us take $\ell \in (t_0 - \rho^2/4, t_0)$ and $\eta \in (0, \rho^2/4 - \ell)$ and then $\zeta \in C^1(\mathbb{R})$ satisfying

$$\begin{cases}
\zeta \equiv 1, & \text{on } (-\rho^2/4, \ell - \eta), \\
\zeta \equiv 0, & \text{on } (-\infty, -\rho^2) \cup (\ell, \infty), \\
0 \leq \zeta \leq 1, & \text{on } \mathbb{R}, \\
\zeta \equiv -1/\eta, & \text{on } (\ell - \eta, \ell), \\
|\zeta| \leq 1/\rho^2, & \text{on } (-\rho^2, -\rho^2/4).
\end{cases}$$

Moreover, $\chi \in C_0^\infty(B_\rho(x_0))$ satisfies $0 \leq \chi \leq 1, |D\chi| \leq 4/\rho, \chi \equiv 1$ on $B_{\rho/2}(x_0)$. Then $\varphi(x, t) := \chi(t)\chi(x)(u(x, t) - \ell(x))$ is admissible as a test function in (2.2), and we obtain

$$\int_{Q_\rho(x_0)} \zeta \chi^p(A(z, u, Du), Du - D\ell) \, dz$$

$$= -\int_{Q_\rho(x_0)} \langle A(z, u, Du), p\zeta \chi^p - 1 D\chi \otimes (u - \ell) \rangle \, dz$$

$$+ \int_{Q_\rho(x_0)} \langle u, \partial_t \varphi \rangle \, dz + \int_{Q_\rho(x_0)} \langle H, \varphi \rangle \, dz.$$
Furthermore, we have

\[- \int_{Q_\rho(z_0)} \zeta \chi^p \langle A(z, u, D\ell), Du - D\ell \rangle dz \]

\[\begin{align*}
&= \int_{Q_\rho(z_0)} \langle A(z, u, D\ell), p\zeta \chi^{p-1} D\chi \otimes (u - \ell) \rangle dz - \int_{Q_\rho(z_0)} \langle A(z, u, D\ell), D\varphi \rangle dz, \\
&\quad \text{(4.4)}
\end{align*}\]

and

\[\int_{Q_\rho(z_0)} \langle (A(\cdot, \ell(x_0), D\ell))_{z_0, \rho}, D\varphi \rangle dz = 0. \quad \text{(4.5)}\]

Adding (4.3), (4.4) and (4.5), we obtain

\[- \int_{Q_\rho(z_0)} \zeta \chi^p \langle A(z, u, Du) - A(z, u, D\ell), Du - D\ell \rangle dz \]

\[\begin{align*}
&= - \int_{Q_\rho(z_0)} \langle A(z, u, Du) - A(z, u, D\ell), p\zeta \chi^{p-1} D\chi \otimes (u - \ell) \rangle dz \\
&\quad - \int_{Q_\rho(z_0)} \langle A(z, u, D\ell) - A(z, \ell(x_0), D\ell), D\varphi \rangle dz \\
&\quad - \int_{Q_\rho(z_0)} \langle A(z, \ell(x_0), D\ell) - (A(\cdot, \ell(x_0), D\ell))_{z_0, \rho}, D\varphi \rangle dz \\
&\quad + \int_{Q_\rho(z_0)} \langle u - \ell, \partial_t \varphi \rangle dz \\
&\quad + \int_{Q_\rho(z_0)} \langle H, \varphi \rangle dz \\
&=: I + II + III + IV + V. \quad \text{(4.6)}
\end{align*}\]

The terms I, II, III, IV, V are defined above. Using the ellipticity condition (H2) to the left-hand side of (4.6), we get

\[\langle A(z, u, Du) - A(z, u, D\ell), Du - D\ell \rangle \geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1 - s)D\ell|)^{p-2} ds. \quad \text{(4.7)}\]

Then by using (3.10) in Lemma 3.6, we obtain

\[\langle A(z, u, Du) - A(z, u, D\ell), Du - D\ell \rangle \geq \lambda |Du - D\ell|^2 \int_0^1 (1 + |sDu + (1 - s)D\ell|)^{(p-2)/2} ds \]

\[\geq 2^{(12-9p)/2} \lambda \left( (1 + |D\ell|)^{p-2} |Du - D\ell|^2 + |Du - D\ell|^p \right). \quad \text{(4.8)}\]
For \( \varepsilon > 0 \) to be fixed later, using \((H1)\) and Young’s inequality, we have

\[
|I| \leq \int_{Q_\rho(z_0)} p\zeta\chi^{p-1} \left| \int_0^1 \partial_u A(z, u, Du + s(Du - D\ell))(Du - D\ell)ds \right| |D\chi||u - \ell|dz \\
\leq \int_{Q_\rho(z_0)} c(p, L)\zeta\chi^{p-1} \left\{ (1 + |D\ell|)^{p-2} + |Du - D\ell|^{p-2} \right\} |Du - D\ell||D\chi||u - \ell|dz \\
\leq \varepsilon \int_{Q_\rho(z_0)} \zeta\chi^{p} \left\{ (1 + |D\ell|)^{p-2} |Du - D\ell|^2 + |Du - D\ell|^p \right\} dz \\
+ c(p, L, \varepsilon) \int_{Q_\rho(z_0)} \left\{ (1 + |D\ell|)^{p-2} \left| \frac{u - \ell}{\rho} \right|^2 + \left| \frac{u - \ell}{\rho} \right|^p \right\} dz. \tag{4.9}
\]

In order to estimate II, we use \((H3)\), \(D\varphi = \zeta\chi^{p}(Du - D\ell) + p\zeta\chi^{p-1}D\chi \otimes (u - \ell)\), and again Young’s inequality, we get

\[
|II| \leq \varepsilon \int_{Q_\rho(z_0)} \zeta\chi^{p}(1 + |D\ell|)^{p-2} |Du - D\ell|^2 dz + \varepsilon^{-1} \int_{Q_\rho(z_0)} L^2\omega^2 \left| \left( u - \ell(x_0) \right|^2 \right| (1 + |D\ell|)^p dz \\
+ \varepsilon \int_{Q_\rho(z_0)} (1 + |D\ell|)^{p-2} \left| \frac{u - \ell}{\rho} \right|^2 dz + \varepsilon^{-1} \int_{Q_\rho(z_0)} (4Lp)^2\omega^2 \left| \left( u - \ell(x_0) \right|^2 \right| (1 + |D\ell|)^p dz \\
\leq \varepsilon \int_{Q_\rho(z_0)} \zeta\chi^{p}(1 + |D\ell|)^{p-2} |Du - D\ell|^p dz + \varepsilon \int_{Q_\rho(z_0)} (1 + |D\ell|)^{p-2} \left| \frac{u - \ell}{\rho} \right|^p dz \\
+ c(p, L, \varepsilon)(1 + |D\ell|)^p \omega^2 \left( \int_{Q_\rho(z_0)} \left| u - \ell(x_0) \right|^2 \right), \tag{4.10}
\]

where we use Jensen’s inequality in the last inequality. We next estimate III by using the VMO-condition \((H4)\) and Young’s inequality, we have

\[
|III| \leq \frac{\varepsilon}{2p-1} \int_{Q_\rho(z_0)} \left\{ \zeta\chi^{p}|Du - D\ell| + \frac{4p\zeta}{\rho} |u - \ell| \right\} dz + \left( \frac{2p-1}{\varepsilon} \right)^{q/p} \int_{Q_\rho(z_0)} V_\rho^{q}(x, \rho)(1 + |D\ell|)^p dz.
\]

Then using the fact that \(V_\rho^{q} = V_\rho^{q-1} \cdot V_\rho \leq (2L)^{q-1}V_\rho \leq 2LV_\rho\), we infer

\[
|III| \leq \varepsilon \int_{Q_\rho(z_0)} \zeta\chi^{p}|Du - D\ell|^p dz + c(p, \varepsilon) \int_{Q_\rho(z_0)} \left| \frac{u - \ell}{\rho} \right|^p dz + c(p, L, \varepsilon)(1 + |D\ell|)^p V_\rho. \tag{4.11}
\]

To estimate IV, recall that \(\zeta\) satisfies \(\zeta = -1/\eta \) on \((\tilde{t} - \eta, \tilde{t})\) and \(|\zeta| \leq 1/\rho^2\) on \((-\rho^2, -\rho^2/4)\). This implies

\[
IV = \int_{Q_\rho(z_0)} \zeta\chi^{p}|u - \ell|^2 dz + \int_{Q_\rho(z_0)} \zeta\chi^{p} \cdot \partial_z \frac{1}{2} |u - \ell|^2 dz \\
= \frac{1}{2} \int_{Q_\rho(z_0)} \zeta\chi^{p}|u - \ell|^2 dz \\
= \frac{1}{2} \frac{1}{|Q_\rho(z_0)|} \int_{t_0 - \rho^2/4}^{t_0 - \rho^2} \int_{B_\rho(x_0)} \chi^{p} \left| \frac{u - \ell}{\rho} \right|^2 dxdt - \frac{1}{2\eta|Q_\rho(z_0)|} \int_{\tilde{t}}^{\tilde{t}} \int_{B_\rho(x_0)} \chi^{p}|u - \ell|^2 dxdt \\
\leq \frac{1}{2} \frac{1}{|Q_\rho(z_0)|} \int_{t_0 - \rho^2}^{t_0 - \rho^2} \int_{B_\rho(x_0)} \chi^{p} \left| \frac{u - \ell}{\rho} \right|^2 dxdt - \frac{1}{2\eta|Q_\rho(z_0)|} \int_{\tilde{t} - \eta}^{\tilde{t} - \eta} \int_{B_\rho(x_0)} \chi^{p}|u - \ell|^2 dxdt. \tag{4.12}
\]
For $\varepsilon' > 0$ to be fixed later, using (H5), Lemma 3.5 and Young’s inequality, we have

\[
|V| \leq \int_{Q_p(z_0)} \left| u - \ell \right| \left| \frac{\partial u}{\partial \ell} \right| (z) dz + \int_{Q_p(z_0)} \left| b \zeta \chi_p \right| \left| \frac{u - \ell}{\rho} \right| (z) dz + \int_{Q_p(z_0)} \left| (1 + \varepsilon') |D u - D \ell| \right| (z) dz
\]

\[
\leq \int_{Q_p(z_0)} \left| u - \ell \right| \left| \frac{\partial u}{\partial \ell} \right| (z) dz + \int_{Q_p(z_0)} \left| b \zeta \chi_p \right| \left| \frac{u - \ell}{\rho} \right| (z) dz + \int_{Q_p(z_0)} \left| (1 + \varepsilon') |D u - D \ell| \right| (z) dz
\]

\[
\leq a \left( 1 + \varepsilon' \right) \left( 2M + |D \ell| |p| \right) \int_{Q_p(z_0)} \left| u - \ell \right| \left| \frac{\partial u}{\partial \ell} \right| (z) dz + \int_{Q_p(z_0)} \left| b \zeta \chi_p \right| \left| \frac{u - \ell}{\rho} \right| (z) dz
\]

\[
+ \varepsilon \left( 1 + |D \ell| |p| \right) \left( a^q K^q |D \ell|^q + b^q \right).
\]

Combining (4.10), (4.8), (4.10), (4.11), (4.12) and (4.13), and set $\lambda' = 2^{(12-9p)/2}\lambda C \Lambda := \lambda' - 3\varepsilon - a(1 + \varepsilon')(2M + |D \ell| |p|)$, this gives

\[
\int_{Q_p(z_0)} \left| u - \ell \right| \left| \frac{\partial u}{\partial \ell} \right| (z) dz \leq \varepsilon \left( 1 + K(p, \varepsilon') \right) \left( a^q K^q |D \ell|^q + b^q \right)
\]

(4.14)

Now choose $\varepsilon = \varepsilon(\lambda, p, a(M), M) > 0$ and $\varepsilon' = \varepsilon'(\lambda, p, a(M), M) > 0$ in a right way (for more precise way of choosing $\varepsilon$ and $\varepsilon'$, we refer to [12, Lemma 4.1]) and taking the limit $\eta \to 0$, we obtain (14.1).\hfill $\square$

To use the $A$-caloric approximation lemma, we need to estimate $\int_{Q_p(z_0)} (u - \ell) \cdot \varphi\dot{\ell} - A(D(u - \ell), D\varphi)) dz$.

**Lemma 4.2.** Assume the same assumptions in Lemma 4.1. Then for any $z_0 = (x_0, t_0) \in \Omega_T$ and $\rho \leq \rho_0$ satisfy $Q_{2\rho}(z_0) \in \Omega_T$, and any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}$ with $|\ell(x_0)| \leq M$, the inequality

\[
\int_{Q_p(z_0)} (v \varphi) - A(Dv, D\varphi)) dz 
\]

\[
\leq C_2 (1 + |D\ell|) \left[ \mu^{1/2} (\Psi(0, 2\rho, \ell) + \Psi(0, 2\rho, \ell) + \rho |D\ell| + b) \right] \sup_{Q_p(z_0)} |D\varphi| (4.15)
\]

holds for all $\varphi \in C^0_0(B_{p}(x_0), \mathbb{R}^N)$ and a constant $C_2 = C_2(u, \lambda, L, p, a(M)) \geq 1$, where

\[
A(Dv, D\varphi) := \frac{1}{(1 + |D\ell|)^p} \left( \partial_{\omega} A(\cdot, (\ell(x_0), D\ell))_{x_0, \rho} Dv, D\varphi \right),
\]

\[
\Phi(z_0, \rho, \ell) := \int_{Q_p(z_0)} \left| \frac{Dv - D\ell}{(1 + |D\ell|)^2} \right| (z) dz + \int_{Q_p(z_0)} \left| \frac{|u - \ell|^p}{(1 + |D\ell|)^p} \right| (z) dz,
\]

\[
\Psi(z_0, \rho, \ell) := \int_{Q_p(z_0)} \left| \frac{|u - \ell|^2}{\rho^2(1 + |D\ell|)^2} \right| (z) dz + \int_{Q_p(z_0)} \left| \frac{|u - \ell|^p}{\rho^p(1 + |D\ell|)^p} \right| (z) dz,
\]

\[
\Psi_*(z_0, \rho, \ell) := \Psi(z_0, \rho, \ell) + \omega \int_{Q_p(z_0)} \left| u - \ell(x_0) \right|^2 d\mu + V(\rho) + (a^q |D\ell|^q + b^q) \rho^q,
\]

\[
v := u - \ell = u - \ell(x_0) - D\ell(x - x_0).
\]
Proof. Assume \( z_0 \in \Omega_T \) and \( \rho \leq 1 \) satisfy \( Q_{2\rho}(z_0) \in \Omega_T \). Without loss of generality we may assume 

\[
\sup_{Q_{\rho}(z_0)} |D\varphi| \leq 1.
\]

Note \( \sup_{Q_{\rho}(z_0)} |\varphi| \leq \rho \leq 1 \). Using the fact that \( \int_{Q_{\rho}(z_0)} A(z_0, \ell(x_0), u) D\varphi dx = 0 \), we deduce

\[
(1 + |D\ell|)^{-1} \int_{Q_{\rho}(z_0)} \left( v \cdot \varphi_t - A(Dv, D\varphi) \right) dz
\]

\[
= \int_{Q_{\rho}(z_0)} \int_0^1 \left\langle \left( \partial_t A(\cdot, \ell(z_0), D\ell) \right)_{z_0, \rho} - \left( \partial_t A(\cdot, \ell(z_0), D\ell + sDv) \right)_{z_0, \rho} \right\rangle Dv, D\varphi \, ds \, dz
\]

\[
+ \int_{Q_{\rho}(z_0)} \left\langle A(z_0, \ell(u), Du) - A(z, u, Du), D\varphi \right\rangle dz
\]

\[
+ \int_{Q_{\rho}(z_0)} \left\langle H, \varphi \right\rangle dz
\]

\[
= : I + II + III + IV
\]  (4.16)

where terms I, II, III, IV are define above.

Using the modulus of continuity \( \mu \) from (H1), Jensen’s inequality and Hölder’s inequality, we estimate

\[
|I| \leq c(p, L)(1 + |D\ell|)^{-1} \int_{Q_{\rho}(z_0)} \mu \left( \frac{|Du - D\ell|}{1 + |D\ell|} \right) \left( \frac{|Du - D\ell|}{1 + |D\ell|} \right)^{p-1} dz
\]

\[
\leq c(1 + |D\ell|)^{-1} \left[ \mu^{1/2} \left( \sqrt{\Phi(z_0, \rho, \ell)} \right) + \mu^{1/p} \left( \Phi^{1/2}(z_0, \rho, \ell) - \Phi(z_0, \rho, \ell) \right) \right]
\]

\[
\leq c(1 + |D\ell|)^{-1} \left[ \mu^{1/2} \left( \sqrt{\Phi(z_0, \rho, \ell)} \right) + \Phi(z_0, \rho, \ell) \right].
\]  (4.17)

The last inequality follows from the fact that \( a^{1/p} b^{1/q} = a^{1/p} b^{1/p} (1 - p) \leq a^{1/2} b^{1/2} + b \) holds by Young’s inequality.

By using the VMO-condition, Young’s inequality and the bound \( V_{z_0}(x, \rho) \leq 2L \), the term II can be estimated as

\[
|II| \leq c(p)(1 + |D\ell|)^{-1} \int_{Q_{\rho}(z_0)} \left( V_{z_0}(z_0, \rho) + V_{z_0}(z_0, \rho) \right) \left( \frac{|Du - D\ell|}{1 + |D\ell|} \right)^{p-1} dz
\]

\[
\leq c(1 + |D\ell|)^{-1} \left[ \left( 1 + (2L)^{p-1} \right) V(\rho) + \Phi(z_0, \rho, \ell) \right].
\]  (4.18)

Similarly, we estimate the term III by using the continuity condition (H3), Young’s inequality, the bound \( \omega \leq 1 \) and Jensen’s inequality. This leads us to

\[
|III| \leq L \int_{Q_{\rho}(z_0)} \int_0^1 \left( 1 + |D\ell| + |Du - D\ell| \right)^{p-1} \omega \left( |u - \ell(x_0)|^2 \right) dz
\]

\[
\leq c(p, L)(1 + |D\ell|)^{p-1} \left[ \omega \left( \int_{Q_{\rho}(z_0)} |u - \ell(x_0)|^2 dz \right) + \Phi(z_0, \rho, \ell) \right].
\]  (4.19)

By using the growth condition (H5) and \( \sup_{B_{\rho}(x_0)} |\varphi| \leq \rho \leq 1 \), we have

\[
|IV| \leq \int_{Q_{\rho}(z_0)} \rho(a |Du|^p + b) dz
\]

\[
\leq 2^{p-1} a(1 + |D\ell|)^{p} \Phi(z_0, \rho, \ell) + 2^{p-1} \rho(1 + |D\ell|)^{p-1} (a |D\ell|^p + b).
\]  (4.20)
Therefore combining (4.16), (4.17), (4.18), (4.19) and (4.20), and using Caccioppoli-type inequality (Lemma 4.1), we have

$$\int_{Q_{\rho/2}(z_0)} ((v, \varphi_{t}) - A(Dv, D\varphi)) \, dz \leq 2^{p+1}(1 + |D\ell|)^p(1 + a + (2L)^{p-1}) \times \left[ \mu^{1/2} \left( \sqrt{\Psi(z_0, \rho, \ell)} \right) + \Phi(z_0, \rho, \ell) + \Psi_\ast(z_0, \rho, \ell) + \rho(a|D\ell|^p + b) \right]$$

$$\leq C_2(1 + |D\ell|)^p \left[ \mu^{1/2} \left( \sqrt{\Psi(z_0, 2\rho, \ell)} \right) + \Psi_\ast(z_0, 2\rho, \ell) + \Psi_\ast(z_0, 2\rho, \ell) + \rho(a|D\ell|^p + b) \right],$$

where we set $C_2 := 2^{n+p+3}C_1(1 + a + (2L)^{p-1})$ at the last inequality and this completes the proof. □

From now on, we write $\Phi(\rho) = \Phi(z_0, \rho, \ell_{z_0, \rho}), \Psi(\rho) = \Psi(z_0, \rho, \ell_{z_0, \rho}), \Psi_\ast(\rho) = \Psi_\ast(z_0, \rho, \ell_{z_0, \rho})$ for $z_0 \in \Omega_T$ and $0 < \rho \leq 1$. Here $\ell_{z_0, \rho}$ is a minimizer which we introduce in (3.6).

Now we are ready to establish the excess improvement.

**Lemma 4.3.** Assume the same assumption in Lemma 4.1. Let $\theta \in (0, 1/4]$ be arbitrary and impose the following smallness conditions on the excess:

(i) $\mu^{1/2} \left( \sqrt{\Psi_\ast(\rho)} \right) + \sqrt{\Psi_\ast(\rho)} \leq \frac{\delta}{2}$ with the constant $\delta = \delta(n, N, p, \lambda, L, \theta^{n+p+4})$ from Lemma 3.3.

(ii) $\Psi(\rho) \leq \frac{\theta^{n+4}}{4n(n+2)}$.

(iii) $\gamma(\rho) := [\Psi^{q/2}(\rho) + \delta^{-q}\rho^q(a|D\ell| + b)^q]^{1/q} \leq 1$.

Then there holds the excess improvement estimate

$$\Psi(\theta \rho) \leq C_3 \theta^2 \Psi_\ast(\rho)$$

(4.21)

with a constant $C_3 = C_3(n, \lambda, L, p, a(M)) \geq 1$.

**Proof.** Set

$$w := \frac{w - \ell_{z_0, \rho}}{C_2(1 + |D\ell|)\gamma(\rho)}.$$

From Lemma 4.2 and the assumption (i) we have

$$\int_{Q_{\rho/2}(z_0)} ((w, \varphi_{t}) - A(Dw, D\varphi)) \, dz \leq \left[ \mu^{1/2} \left( \sqrt{\Psi_\ast(\rho)} \right) + \sqrt{\Psi_\ast(\rho)} + \frac{\delta}{2} \right] \sup_{Q_{\rho/2}(z_0)|D\varphi|} |D\varphi|$$

$$\leq \delta \sup_{Q_{\rho/2}(z_0)} |D\varphi|,$$

for all $\varphi \in C_0^\infty(Q_{\rho/2}(z_0), \mathbb{R}^N)$. Moreover, using Caccioppoli-type inequality and the assumption (iii), we get

$$
\int_{Q_{\rho/2}(z_0)} \left| \frac{w}{\rho/2} \right|^2 + \gamma^{p-2} \left| \frac{w}{\rho/2} \right|^p \, dz + \int_{Q_{\rho/2}(z_0)} |Dw|^2 + \gamma^{p-2}|Dw|^p \, dz \\
\leq \frac{1}{C_2^2} \left\{ 2^{n+p+2} \Psi(\rho) + C_1 \Psi_\ast(\rho) \right\} \\
\leq \frac{\max\{2^{n+p+2}, C_1\}}{C_2^2} \leq 1.
$$

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Therefore the $A$-caloric approximation lemma (Lemma 3.2) implies the existence of 
\[ h \in L^p(t_0 - (\rho/4)^2, t_0; W^{1,2}(B_{\rho/4}(x_0), \mathbb{R}^N)) \]
which is $A$-caloric on $Q_{\rho/4}(z_0)$ and satisfies
\[
\int_{Q_{\rho/4}(z_0)} \left( \frac{h}{\rho/4} \right)^2 + \gamma^{p-2} \left| \frac{h}{\rho/4} \right|^p \, dz + \int_{Q_{\rho/4}(z_0)} \left( |Dh|^2 + \gamma^{p-2} |Dh|^p \right) \, dz \leq 2 \cdot 2^{n+2+2p}.
\]
and
\[
\int_{Q_{\rho/4}(z_0)} \left( \frac{|w - h|}{\rho/4} \right)^2 + \gamma^{p-2} \left| \frac{w - h}{\rho/4} \right|^p \, dz \leq \theta^{n+p+4}. \tag{4.22}
\]
Then from Lemma 3.3, we have for $s = 2$ respectively for $s = p$
\[
\gamma^{s-2} (\theta \rho)^{-s} \int_{Q_{\rho/4}(z_0)} |h - h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0)|^s \, dz \\
\leq c(s) \gamma^{s-2} \theta^s \left( \frac{\rho}{4} \right)^s \int_{Q_{\rho/4}(z_0)} |h - h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0)|^s \, dz \\
\leq 3^{s-1} c(s) \gamma^{s-2} \theta^s \left( \frac{\rho}{4} \right)^s \int_{Q_{\rho/4}(z_0)} |h|^s \, dz + \int_{Q_{\rho/4}(z_0)} |(Dh)_{z_0,\rho/4}|^s \left( \frac{\rho}{4} \right) \, dz \\
\leq 2 \cdot 3^{s-1} c(s) \gamma^{s-2} \theta^s \left( \frac{\rho}{4} \right)^s \left[ \frac{\rho}{4} \int_{Q_{\rho/4}(z_0)} |h|^s \, dz + \int_{Q_{\rho/4}(z_0)} |Dh|^s \, dz \right] \\
\leq 2^{n+4+p} \cdot 3^{s-1} c(s) \theta^s.
\]
Thus, using (4.22) we obtain
\[
\gamma^{s-2} (\theta \rho)^{-s} \int_{Q_{\rho/4}(z_0)} |w - h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0)|^s \, dz \\
\leq 2^{s-1} (\theta \rho)^{-s} \left[ \int_{Q_{\rho/4}(z_0)} \gamma^{s-2} |w - h|^s \, dz + \gamma^{s-2} \int_{Q_{\rho/4}(z_0)} |h - h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0)|^s \, dz \right] \\
\leq 2^{s-1} \left[ 4^{n+2-s} \theta^{-n-2-s} \int_{Q_{\rho/4}(z_0)} \gamma^{s-2} \left| \frac{w - h}{\rho/4} \right|^s \, dz + 3^{s-1} \cdot 2^{n+4+p} c(s) \theta^s \right] \\
\leq 2^{s-1} \left[ 4^{n+2-s} + 3^{s-1} \cdot 2^{n+4+p} c(s) \right] \theta^2.
\]
Scaling back to $u$ we have
\[
(\theta \rho)^{-s} \int_{Q_{\rho/4}(z_0)} |u - \ell_{z_0,\rho}|^s \, dz \\
\leq c(n, s) (\theta \rho)^{-s} \int_{Q_{\rho/4}(z_0)} |u - \ell_{z_0,\rho} - C_2 \gamma (1 + |D\ell_{z_0,\rho}|) (h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0))|^s \, dz \\
= c(n, s) C_2^s \gamma (1 + |D\ell_{z_0,\rho}|) s (\theta \rho)^{-s} \int_{Q_{\rho/4}(z_0)} |w - h_{z_0,\rho/4} - (Dh)_{z_0,\rho/4}(x - x_0)|^s \, dz \\
\leq c(n, s, p, C_2) \gamma^2 (1 + |D\ell_{z_0,\rho}|)^s \theta^2 \\
\leq c(1 + |D\ell_{z_0,\rho}|)^s \theta^2 |\Psi_k^{\gamma/2}(\rho)| + 2^{q/p} \delta^{-q} \Psi_k(\rho)^2/q \\
\leq c(1 + |D\ell_{z_0,\rho}|)^s \theta^2 |\Psi_k(\rho)|.
Here we want to replace the term \((1 + |D\ell_{z_0,\rho}|)\) by \((1 + |D\ell_{z_0,\theta\rho}|)\). To do this, using (3.7) from Lemma 3.4 and the assumption (ii), we have

\[
|D\ell_{z_0,\theta\rho} - D\ell_{z_0,\rho}|^2 \leq \frac{n(n + 2)}{(\theta\rho)^2} \int_{Q_{\rho}(z_0)} |u - \ell_{z_0,\rho}|^2 dz \\
\leq \frac{n(n + 2)}{\theta^{n+4}\rho^2} \int_{Q_{\rho}(z_0)} |u - \ell_{z_0,\rho}|^2 dz \\
\leq \frac{n(n + 2)}{\theta^{n+4}} (1 + |D\ell_{z_0,\rho}|)^2 \Psi(\rho) \leq \frac{1}{4} (1 + |D\ell_{z_0,\rho}|)^2.
\]

This yields

\[
1 + |D\ell_{z_0,\rho}|^2 \leq 2(1 + |D\ell_{z_0,\rho}|).
\]

Thus we have

\[
(\theta\rho)^{-1} \int_{Q_{\rho}(z_0)} |u - \ell_{z_0,\theta\rho}|^2 dz \leq c(1 + |D\ell_{z_0,\theta\rho}|)^{\theta^2} \Psi(\rho),
\]

and this immediately yields the claim.

Let fix an arbitrarily Hölder exponent \(\alpha \in (0, 1)\) and define the Campanato-type excess

\[
C_\alpha(z_0, \rho) := C_\alpha(\rho) = \rho^{-2\alpha} \int_{Q_{\rho}(z_0)} |u - u_{z_0,\rho}|^2 dz.
\]

Here we iterate the excess improvement estimate (4.21) and obtain the boundedness of two excess functionals, \(\Psi(\rho)\) and \(C_\alpha(\rho)\).

**Lemma 4.4.** Assume the same assumption in Lemma 4.1. For every \(\alpha \in (0, 1)\) there exist constants \(\varepsilon_*, \kappa_*, \rho_* > 0\) and \(\theta_* \in (0, 1/8]\) such that the conditions

\[\Psi(\rho) < \varepsilon_* \quad \text{and} \quad C_\alpha(\rho) < \kappa_* \quad (A_0)\]

for all \(0 < \rho < \rho_*\) with \(Q_\rho(z_0) \in \Omega_T\), imply

\[\Psi(\theta_*^k \rho) < \varepsilon_* \quad \text{and} \quad C_\alpha(\theta_*^k \rho) < \kappa_* \quad (A_k)\]

respectively, for every \(k \in \mathbb{N}\).

**Proof.** First set

\[
\theta_* := \min \left\{ \left( \frac{1}{16n(n + 2)} \right)^{1/(2 - 2\alpha)}, \frac{1}{\sqrt{4C_3}} \right\} < \frac{1}{8},
\]

and take \(\varepsilon_* > 0\) which satisfies

\[
\varepsilon_* \leq \frac{\theta_*^{n+4}}{16n(n + 2)} \quad \text{and} \quad \mu^{1/2} (\sqrt{4\varepsilon_*}) + \sqrt{4\varepsilon_*} \leq \frac{\delta}{2}.
\]

Note that the choice of \(\theta_*\) fixes the constant \(\delta = \delta(n, N, \lambda, L, p, \theta_*^{n+p+4}) > 0\) from Lemma 3.2. Then choose \(\kappa_* > 0\) so small that

\[
\omega(\kappa_*) < \varepsilon_*.
\]

Finally, we take \(\rho_* > 0\) which satisfies

\[
\rho_* \leq \min \{ \rho_0, \kappa_*^{1/(2 - 2\alpha)}, 1 \}, \quad V(\rho_0) < \varepsilon_* \quad \text{and} \quad \left\{ (a \sqrt{n(n + 2)\kappa_*})^q + b^q \right\} \rho_*^{\theta_*^q} < \varepsilon_*.
\]
Thus, we have
\[ |\partial \ell^2_{z0, \theta^*_\rho}|^2 \leq \frac{n(n+2)}{(\theta^k \rho)^2} \int_{Q_{\alpha\rho^k}(z_0)} |u - \ell_{z0, \theta^*_\rho}|^2 dz \]
\[ \leq n(n+2)(\theta^k \rho)^{2-2\alpha} C_{\alpha\rho^k} \]
\[ \leq n(n+2)\rho^{2-2\alpha} \kappa. \]  

(4.23)

Thus, we have
\[ \Psi_*^{(\theta^k \rho)} \leq \Psi(\theta^k \rho) + \omega(C_{\alpha\rho^k}(z_0, \theta^k)) + V(\theta^k \rho) + (a^q |\partial \ell_{z0, \theta^k}|^q + b^q)(\theta^k \rho)^q \]
\[ \leq \varepsilon_* + \omega(\kappa_*) + V(\rho_*) + \left( a \sqrt{n(n+2)\kappa_*} + b \right)^q \rho^q \leq 4\varepsilon_* \]

This implies
\[ \mu^{1/2} \left( \sqrt{\Psi_*^{(\theta^k \rho)}} \right) + \sqrt{\Psi_*^{(\theta^k \rho)}} < \frac{\mu}{\sqrt{\varepsilon_*}} + \sqrt{4\varepsilon_*} \leq \frac{\delta}{2}, \]  

(4.24)

and
\[ \Psi(\theta^k \rho) < \varepsilon_* < \frac{\theta^{n+1}}{4n(n+2)}. \]  

(4.25)

Furthermore, we have
\[ \gamma(\theta^k \rho) = \left[ \Psi_*^{1/2}(\theta^k \rho) + \delta^{-q}(\theta^k \rho)^q(a |\partial \ell_{z0, \theta^k}| + b)^q \right]^{1/q} \leq 1. \]  

(4.26)

To check (4.20), the first term of (4.20) can be estimated by the choice of \( \varepsilon_* \) and the fact \( \Psi_*^{(\theta^k \rho)} < 1 \):
\[ \Psi_*^{1/2}(\theta^k \rho) \leq \Psi_*^{1/2}(\theta^k \rho) < \sqrt{\varepsilon_*} \leq \frac{\delta}{2}. \]

To estimate the second term of (4.20), using (4.23) and the fact \( \rho^{\alpha-1} \geq 1 \), we obtain
\[ \delta^{-q}(\theta^k \rho)(a |\partial \ell_{z0, \theta^k}| + b)^q \leq \delta^{-q} \rho^q \left( a \rho^{\alpha-1} \sqrt{n(n+2)\kappa_*} + b \right)^q \]
\[ \leq 2^{q/\delta-q} \rho^q \varepsilon_* \]
\[ \leq 2^{-q+4/\delta} \delta^{2-q} \leq \frac{\delta}{\delta}. \]

Therefore, we have (4.20) and this allowed us to apply Lemma 4.4 with the radius \( \theta^k \rho \) instead of \( \rho \), which yields
\[ \Psi^{(\theta^k+1 \rho)} \leq C_{\delta\theta^2} \Psi_*^{(\theta^k \rho)} \leq 4C_{\delta\theta^2} \varepsilon_* \leq \varepsilon_* \]

Thus, we have established the first part of the assertion (\( A_{k+1} \)) and it remains to prove the second one, that is, \( C_{\alpha}(z_0, \theta^{k+1} \rho) \). For this aim, we first compute
\[ \frac{1}{(\theta^k \rho)^2} \int_{Q_{2\alpha\rho^k}(z_0)} |u - \ell_{z0, \theta^k \rho}|^2 dz \leq (1 + |\partial \ell_{z0, \theta^k \rho}|)^2 \Psi^{(\theta^k \rho)} \leq 2\varepsilon_* + 2\varepsilon_* |\partial \ell_{z0, \theta^k \rho}|^2 \]
where we used the assumption $[A_1]$ in the last step. Since $\ell_{z_0, \theta^k \rho}(x) = u_{z_0, \theta^k \rho} + D\ell_{z_0, \theta^k \rho}(x - x_0)$, we can estimate

$$C_\alpha(z_0, \theta^{k+1} \rho) \leq (\theta^{k+1}_* \rho)^{-2\alpha} \int_{Q_{\theta^{k+1}_* \rho}(z_0)} |u - u_{z_0, \theta^k \rho}|^2 \, dz$$

$$\leq 2(\theta^{k+1}_* \rho)^{-2\alpha} \left[ \int_{Q_{\theta^{k+1}_* \rho}(z_0)} |u - \ell_{z_0, \theta^k \rho}|^2 \, dz + |D\ell_{z_0, \theta^k \rho}|^2 (\theta^{k+1}_* \rho)^2 \right]$$

$$\leq 2(\theta^{k+1}_* \rho)^{-2\alpha} \left[ \theta^{-n-2}_* \int_{Q_{\theta^k \rho}(z_0)} |u - \ell_{z_0, \theta^k \rho}|^2 \, dz + |D\ell_{z_0, \theta^k \rho}|^2 (\theta^{k+1}_* \rho)^2 \right]$$

$$\leq 4(\theta^k \rho)^{2-2\alpha} \left[ \varepsilon \theta^{-n-2-2\alpha}_* + |D\ell_{z_0, \theta^k \rho}|^2 (\varepsilon \theta^{-n-2-2\alpha}_* + (\theta^k \rho)^{-2\alpha}) \right].$$

Recalling the choice of $\rho_*$, $\varepsilon_*$ and $\theta_*$, we deduce

$$C_\alpha(z_0, \theta^{k+1} \rho) \leq 4\rho_*^{-2\alpha} \left[ \varepsilon \theta^{-n-2-2\alpha}_* + n(n + 2)\kappa_\rho \rho_*^{-2\alpha} (\varepsilon \theta^{-n-2-2\alpha}_* + (\theta^k \rho)^{-2\alpha}) \right]$$

$$\leq \frac{1}{4} \rho_*^{-2\alpha} \theta^{-2\alpha}_* + 8n(n + 2)\kappa_\rho \theta^{-2\alpha}_*$$

$$\leq \frac{1}{4} \kappa_\rho + \frac{1}{2} \kappa_* < \kappa_*.$$

This proves the second part of the assertion ($A_{k+1}$) and we completes the proof.

To obtain the regularity result (Theorem 2.2), it is similar arguments in [2] with using the integral characterization of Hölder continuous functions with respect to the parabolic metric of Campanato-Da Prato [9].

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