Off-critical correlations in the Ashkin-Teller model

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Abstract
We use the exact scattering description of the scaling Ashkin-Teller model in two dimensions to compute the two-particle form factors of the relevant operators. These provide an approximation for the correlation functions whose accuracy is tested against exact sum rules.
1. The computation of correlation functions is a central and notoriously difficult problem of quantum field theory and statistical mechanics. The integrable models of two-dimensional quantum field theory provide an unique framework where such a problem can be studied in a non-perturbative way. Although even in this context the determination of exact analytic expressions for the correlations functions remains a challenging task, it is remarkable that an extremely accurate evaluation of the correlators is nowadays possible and involves a relatively small amount of technical work. The method is quite general and relies on the solution of the integrable model in terms of an exact scattering theory [1]. The natural way to express correlators is then as spectral series over a complete set of intermediate asymptotic states. The computation of the matrix elements of the operators (form factors) entering the spectral series is the crucial intermediate step needed to make contact with the space of local operators starting from the purely on-shell solution of the model [2, 3]. From the physical point of view, a particularly interesting point in the problem of the computation of form factors lies in the identification of the solution of the form factor equations corresponding to a specific operator. In Refs. [4, 5, 6], a set of operator-dependent constraints for the form factors was identified which allows the solution of this identification problem. The need for approximation in this program for the computation of correlators arises only from our present inability to resum exactly the spectral series, what forces us to rely on partial sums.

In this note we briefly illustrate how this set of ideas applies to the scaling limit of the Ashkin-Teller (AT) model. The richness of the spectrum of relevant operators, the presence of different locality sectors and the relation with the Sine-Gordon model are among the features which make this example interesting from the theoretical point of view.

The two-dimensional AT model [7] can be formulated in terms of two planar Ising models coupled by a local four-spin interaction. The lattice Hamiltonian reads

\[ H_{AT} = \sum_{(m,n)} [J(\sigma_1^m \sigma_1^n + \sigma_2^m \sigma_2^n) + K \sigma_1^m \sigma_2^n \sigma_2^m \sigma_2^n] , \]

where \( \sigma_{1,2}^m = \pm 1 \) are the two Ising spins at the site \( m \), the sum is over nearest neighbours and the same coupling \( J \) has been chosen for the Ising models (isotropic case). The model is known to exhibit a line of second order phase transition in the space of the lattice couplings \( J \) and \( K \) along which the critical exponents vary continously [8, 9]. In this note we will consider the scaling region around this critical line. In such a region the correlation length is much larger than the lattice spacing and the model admits a continous description in terms of the Hamiltonian

\[ H_{AT} = H_0^1 + H_0^2 + \tau \int d^2 x (\varepsilon_1(x) + \varepsilon_2(x)) + \rho \int d^2 x \varepsilon_1(x) \varepsilon_2(x) , \]
where $H_i^0$ and $\varepsilon_i(x)$ denote the fixed point Hamiltonian and the energy density operator of the $i$-th Ising model, respectively. The meaning of the last expression is that the scaling limit of the AT model can be regarded as a conformal field theory with twice the central charge of the Ising model (i.e. $c = 1/2 + 1/2 = 1$) perturbed by the two operators $E \equiv \varepsilon_1 + \varepsilon_2$ and $\varepsilon_1\varepsilon_2$.

Consider at first the two critical, non-interacting Ising models, namely the Hamiltonian (2) with $\tau = \rho = 0$. The operator $\varepsilon_1\varepsilon_2$ has twice the scaling dimension of the Ising energy density, namely $x_{\varepsilon_1\varepsilon_2} = 2$. Then it is marginal and taking $\rho \neq 0$ in (2) does not spoil criticality. This means that the coupling $\rho$ parameterises the critical line of the AT model. A critical line with central charge $c = 1$ can be described in terms of a free massless boson $\varphi$ (Gaussian model). The correspondence between the critical AT model and the Gaussian model was established in Ref. [9] (see also [10]) where, in particular, a series of operator identifications was obtained. These identifications are summarised in the first two columns of the Table.

We denote by $\sigma_i(x)$ and $\mu_i(x)$ ($i = 1, 2$) the order (spin) and disorder operators of the two Ising models. It is a characteristic feature of the AT model that these operators preserve along the whole critical line the basic properties they have in the pure Ising model [8]. In particular, they retain the same scaling dimension $x_\sigma = x_\mu = 1/8$, and produce the neutral fermions $\psi_i$ under operator product expansion: $\sigma_i \times \mu_i \sim \psi_i$.

The parameter $\beta$, equivalent to $\rho$, identifies the different points along the critical line in the Gaussian model. The field $\tilde{\varphi}$, dual to $\varphi$, is defined by $i \partial_\alpha \tilde{\varphi} = \varepsilon_{\alpha\beta} \partial_\beta \varphi$, with $\alpha = 1, 2$ labelling the two directions on the plane. Within the normalisation of the bosonic field $\varphi$ we adopt in this paper, the vertex operators $e^{i\alpha \varphi}$ and $e^{i\alpha \tilde{\varphi}}$ have scaling dimension $\alpha^2/4\pi$, what completes the third column of the Table. The point $\rho = 0$ at which the two Ising models are decoupled corresponds to $\beta = \sqrt{4\pi}$. It was argued in [9] that the AT critical line corresponds to the range $\sqrt{2\pi} \leq \beta \leq \sqrt{8\pi}$.

When $\tau \neq 0$ in the Hamiltonian (2) the system is moved away from criticality and develops a finite correlation length. In view of the identification $E \sim \cos \beta \varphi$, we see that the bosonic equivalent of the resulting massive theory is the Sine-Gordon model defined by the action

$$A_{SG} = \int d^2x \left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \mu \cos \beta \varphi \right),$$

(3)

where $\mu$ is a dimensional parameter proportional to $\tau$. The equivalence of the models (2) and (3) (for the range of $\beta$ specified above) immediately enables us to extend to the scaling AT model the two basic results known for Sine-Gordon, i.e. its integrability and its exact scattering description [1].

The exact particle spectrum of the Sine-Gordon model contains two elementary excitations of mass $m$, the soliton $A_+$ and the antisoliton $A_-$. If from the bosonic point of view
they are topological excitations, they can also be interpreted as the particles associated to the charged elementary fermion of the equivalent massive Thirring model \[1\]. If we write \( A_\pm = A_1 \pm iA_2 \), then the neutral excitations \( A_1 \) and \( A_2 \) are naturally associated to the neutral fermions \( \psi_1 \) and \( \psi_2 \) of the AT model.

Due to integrability, the scattering of the particles \( A_1 \) and \( A_2 \) is completely elastic and factorised. It is entirely specified by the Faddev-Zamolodchikov algebra

\[
A_i(\theta_1)A_j(\theta_2) = \sum_{k,l=1,2} \sigma_{ij}^{kl}(\theta_1 - \theta_2)A_k(\theta_1)A_l(\theta_2),
\]  

The non-zero scattering amplitudes are given by \[1\]

\[
\begin{align*}
2\sigma_{11}^{11}(\theta) &= 2\sigma_{22}^{22}(\theta) = S(\theta) + S_+(\theta), \\
2\sigma_{11}^{22}(\theta) &= 2\sigma_{22}^{11}(\theta) = S_+(\theta) - S(\theta), \\
2\sigma_{12}^{12}(\theta) &= 2\sigma_{21}^{12}(\theta) = S(\theta) + S_-(\theta), \\
2\sigma_{12}^{21}(\theta) &= 2\sigma_{21}^{21}(\theta) = S(\theta) - S_-(\theta),
\end{align*}
\]

where

\[
S_+(\theta) = -\frac{\sinh \frac{\pi}{2(\theta + i\pi)} S(\theta)}{\sinh \frac{\pi}{2(\theta - i\pi)}} S(\theta), \quad S_-(\theta) = -\frac{\cosh \frac{\pi}{2(\theta + i\pi)} S(\theta)}{\cosh \frac{\pi}{2(\theta - i\pi)}} S(\theta),
\]

\[
S(\theta) = -\exp \left\{ -i \int_0^\infty dx \frac{1}{x} \frac{1 - \xi/\pi}{\sinh x \cosh x} \sin \frac{\theta x}{\pi} \right\},
\]

\[
\xi = \frac{\pi\beta^2}{8\pi - \beta^2}.
\]

The interaction among the elementary particles is attractive (repulsive) below (above) the free fermion point \( \beta = \sqrt{4\pi} \). In the attractive regime, the elementary excitations form bound states (breathers). Our subsequent analysis can be easily extended to include breathers. For the sake of simplicity, however, we will not go into the details of the breather sector in this note.

2. The basic physical ingredients entering the computation of form factors are conveniently illustrated on the example of the two-fermion matrix elements

\[
F^\Phi_{ij}(\theta_1 - \theta_2) = \langle 0|\Phi(0)|A_i(\theta_1)A_j(\theta_2)\rangle, \quad i, j = 1, 2.
\]

They satisfy the unitarity and crossing relations

\[
\begin{align*}
F^\Phi_{ij}(\theta) &= \sum_{k,l=1,2} \sigma_{ij}^{kl}(\theta)F^\Phi_{kl}(-\theta), \\
F^\Phi_{kl}(\theta + 2i\pi) &= e^{2i\pi\gamma_{kl}} F^\Phi_{kl}(-\theta),
\end{align*}
\]

1 The on-shell energy and momentum of the particles are parameterised as \((p^0, p^1) = (m \cosh \theta, m \sinh \theta)\).

2 We denote by \( \Phi \) a generic scalar (under Lorentz transformations) operator.
where the factor $e^{2i\pi \gamma_{\Phi,k}}$ in the crossing equation takes into account a possible non-locality among the operator $\Phi$ and the particle $A_k$ \[3, 12, 13\]. For the operators we are concerned with in the AT model we just need to recall that $A_i$ is associated to the fermion $\psi_i$. The latter is non-local with respect to the operators $\sigma_j$ and $\mu_j$ with non-locality index $\gamma_{\sigma_j,i} = \gamma_{\mu_j,i} = \frac{1}{2}\delta_{i,j}$; $\psi_i$ is instead local with respect to itself and $\varepsilon_j = \bar{\psi}_j \psi_j$. For the operators defined as products, $\Phi = \prod_j \Phi_j$, the non-locality index reads $\gamma_{\Phi,i} = \sum_j \gamma_{\Phi_j,i}$. These considerations provide the fourth column of the Table.

It is obvious from the form of the lattice Hamiltonian (1) that the AT model is invariant under the change of sign of $\sigma_1$ and/or $\sigma_2$ ($Z_2 \times Z_2$–symmetry), and under the exchange $1 \leftrightarrow 2$ (exchange symmetry). The fermions $\psi_i$, and then the particles $A_i$, are odd under the $Z_2 \times Z_2$–symmetry. The behaviour of the different operators under this symmetry then determines the asymptotic states on which each operator has non-zero matrix elements. These states are given in the last column of the Table (modulo pairs of the type $A_j A_j$). The symmetry properties of the operators under the exchange symmetry are important to simplify the functional system (10).

The scalar operators $\Phi$ which are not odd under any of the global symmetries of the model may have a nonvanishing vacuum expectation value $\langle \Phi \rangle$. If in addition they are non-local with respect to the fermions, their two-particle form factors exhibit a pole at $\theta = i\pi$ with residue

$$-i \text{Res}_{\theta = i\pi} F_{jj}^\Phi(\theta) = (1 - e^{2i\pi \gamma_{\Phi,j}}) \langle \Phi \rangle .$$

The only additional singularities allowed on the physical strip $0 < \theta < i\pi$ are the poles associated to the bound states. The functional equations (11), together with the required singularity structure, fix the matrix elements (9) up to polynomials of $\cosh \theta$. The asymptotic arguments of Ref. \[4\] can be used to show that, for the relevant ($x_\Phi < 2$) operators listed in the Table, these polynomials are of degree zero. The non-zero two-fermion matrix elements are then uniquely determined to be $(j, l = 1, 2)$

$$F_{jj}^\mu(\theta) = \frac{i\pi \langle \mu \rangle}{2\xi \omega(i\pi)} \frac{F_0(\theta)}{\sinh \frac{\pi}{2\xi}(\theta - i\pi)} \left[ \omega(\theta) + (-1)^{l+j} \omega(2i\pi - \theta) \right] ,$$

$$F_{jj}^E = c_1 \frac{\cosh \frac{\theta}{2}}{\sinh \frac{\pi}{2\xi}(\theta - i\pi)} F_0(\theta) ,$$

$$F_{jj}^C = c_2 (-1)^j F_0(\theta) ,$$

$$F_{12}^E = F_{21}^E = c_3 F_0(\theta) ,$$

$$F_{12}^C = -F_{21}^C = c_4 \frac{\cosh \frac{\theta}{2}}{\cosh \frac{\pi}{2\xi}(\theta - i\pi)} F_0(\theta) ,$$

$$F_{12}^P = -F_{21}^P = c_5 \frac{F_0(\theta)}{\cosh \frac{\pi}{2\xi}(\theta - i\pi)} ,$$

(12, 13, 14, 15, 16, 17)
\[ F^\star_{j^*} = \frac{i\pi}{\xi} \langle P^\star \rangle F_0(\theta) \frac{\sinh \frac{\pi}{2\xi}(\theta - i\pi)}{\sinh \frac{\pi}{2\xi}(\theta - i\pi)}. \]  

Here the \( c_n \) are normalisation constants, \( \langle \mu \rangle \equiv \langle \mu_1 \rangle = \langle \mu_2 \rangle \), and the functions

\[ \omega(\theta) = \exp \left\{ 2 \int_0^{\infty} \frac{dx}{x} \frac{\sinh \left(1 - \frac{\xi}{\pi} \frac{\theta}{\pi} \frac{x^2}{2\pi} \right)}{\sinh \frac{\xi}{2\pi} \sinh 2x} \right\}, \]  

\[ F_0(\theta) = -i \sinh \frac{\theta}{2} \exp \left\{ \int_0^{\infty} \frac{dx}{x} \frac{\sinh \frac{\xi}{2\pi} \left(1 - \frac{\xi}{\pi} \frac{x^{1/2}}{\pi} \frac{\sin^2 \left(\frac{i(\pi - \theta)x}{2\pi}\right)}{2\pi} \right)}{\sinh x} \right\} \]  

satisfy the equations

\[ \omega(\theta) = \omega(-\theta), \quad \omega(\theta + 2i\pi) = -\frac{\sinh \frac{\pi}{2\xi}(\theta + i\pi)}{\sinh \frac{\pi}{2\xi}(\theta - i\pi)} \omega(\theta - 2i\pi), \]

\[ F_0(\theta) = S(\theta) F_0(-\theta), \quad F_0(\theta + 2i\pi) = F_0(-\theta). \]  

For large values of \(|\theta|\) they behave as \( \omega(\theta) \sim \exp \left[ \left(\frac{\pi}{\xi} - 1 \right) \frac{|\theta|}{4} \right] \) and \( F_0(\theta) \sim \exp \left[ \left(\frac{\pi}{\xi} + 1 \right) \frac{|\theta|}{4} \right] \).

The form factors of the operators \( e^{i\alpha\phi} \) are known for the Sine-Gordon model \([2, 3, 14]\).

Of course, the two-fermion matrix elements for the operators \( E, E^-, P \) and \( P^\star \) determined here coincide with those of the corresponding bosonic operators in Sine-Gordon.

The notation we use in the first column of the Table for the AT operators refers to the high temperature phase of the model. The high and low temperature phases, however, are mapped into one another by the duality transformations \( \sigma_i \leftrightarrow \mu_i, \psi_i \leftrightarrow \bar{\psi}_i \). In particular, this implies that \( \sigma_i \) and \( \mu_i \) represent the same operator in the two different phases. In the form factor approach their relative normalisation is fixed by the factorisation condition

\[ \lim_{|\theta| \to \infty} F^\mu_{ii}(\theta) = \frac{\langle F^\sigma_{ii} \rangle^2}{\langle \mu \rangle}, \]

which determines the one-particle matrix element \( F^\sigma_{ii} = \langle 0|\sigma_i(0)|A_i \rangle \).

Although the generalisation of the functional system \([10]\) to the matrix elements with more than two fermions in the asymptotic state is straightforward, its solution poses a non-trivial mathematical problem \([3]\). For the reasons to be explained in a moment, we do not need to go into this technical part here.

3. In general, the knowledge of the \( n \)-particle form factors \( \langle 0|\Phi(0)|n \rangle \) allows expressing the correlation functions in the form of the spectral series

\[ \langle \Phi_1(x)\Phi_2(0) \rangle = \sum_{n=0}^{\infty} \langle 0|\Phi_1(0)|n \rangle \langle n|\Phi_2(0)|0 \rangle e^{-E_n|x|}, \]  

where \( E_n \) denotes the total energy of the \( n \)-particle asymptotic state. Although the exact resummation of the spectral series is beyond the present technical possibilities, it is known
that partial sums provide very good approximations (see e.g. Ref. [15] for a more detailed discussion and additional references on this subject). Remarkably, this is already the case for the “two-particle approximation”, which corresponds to neglecting the contribution of all the states with \( n > 2 \) in the spectral series \((24)\). The fermionic matrix elements computed in this note are what is needed to implement the two-particle approximation in the breatherless region \(4\pi \leq \beta^2 \leq 8\pi\). As a quantitative check of the convergence of the spectral series, we evaluate in this approximation the central charge and the scaling dimension \( x_\mu \) through the sum rules \([16, 17, 6]\)

\[
c = \frac{3}{4\pi} \int d^2x |x|^2 \langle \Theta(x)\Theta(0) \rangle_c, \tag{25}
\]

\[
x_\mu = -\frac{1}{2\pi \langle \mu \rangle} \int d^2x \langle \Theta(x)\mu(0) \rangle_c, \tag{26}
\]

where \( \langle \cdot \cdot \cdot \rangle_c \) denotes connected correlators and \( \Theta(x) \) is the trace of the stress-energy tensor. The latter is proportional to the energy operator \( \mathcal{E}(x) \) and its two-fermion form factors are given by \([13]\) with the normalisation fixed by \( F_{jj}^\Theta(i\pi) = 2\pi m^2 \). Numerical integration shows that the two-fermion contribution to the central charge decreases monotonically from 1 at \( \beta^2 = 4\pi \) to 0.987 at \( \beta^2 = 8\pi \). For the two-fermion contribution to \( x_\mu \) one finds a monotonic decrease from 0.125 to 0.110 over the same range. The circumstance that this computation gives the exact answer at \( \beta^2 = 4\pi \) simply follows from the fact that \( \Theta \sim \mathcal{E} = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 \) couples only to the two-particle states at the free fermion point. The fact that the approximation is more accurate for \( c \) than for \( x_\mu \) away from the free point is due to the factor \( |x|^2 \) in the integral \( (24) \) which suppresses the contribution of the short distances. This is the region where our error localises when we truncate the large distance expansion \( (24) \).

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\(^3\)Our normalisation for the asymptotic states is defined by \( \langle A(\theta)|A(\theta') \rangle = 2\pi \delta(\theta - \theta') \).
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Φ
Bosonic form
σi
µi
E = ε1 + ε2
C = ε1 − ε2
E+ = ψ1ψ2 + ψ2ψ1
E− = ψ1ψ2 − ψ2ψ1
P = σ1σ2 − σ2σ1
P* = μ1μ2 + μ2μ1
σ1μ2
µ1σ2
γΦ,j
Z2 × Z2
1 ↔ 2
Fermion sector

|   | Bosonic form | xΦ | γΦ,j | Z2 × Z2 | 1 ↔ 2 | Fermion sector |
|---|--------------|-----|-------|---------|--------|----------------|
| Φ | σi          | 1/8 | 1/2δi,j | (−)i × (−)i+1 | + + + | A_i |
|   | µi          | 1/8 | 1/2δi,j | + + + | + | A_jA_j |
| E = ε1 + ε2 | cos βφ | 0 | 0 | + + + | + | A_jA_j |
| C = ε1 − ε2 | cos 4πβφ | 0 | 0 | + + + | − | A_jA_j |
| E+ = ψ1ψ2 + ψ2ψ1 | sin 4πβφ | 0 | 0 | − × − | + | A_1A_2 |
| E− = ψ1ψ2 − ψ2ψ1 | sin βφ | 0 | 0 | − × − | − | A_1A_2 |
| P = σ1σ2 − σ2σ1 | sin 3/2βφ | 0 | 0 | − × − | − | A_1A_2 |
| P* = μ1μ2 + μ2μ1 | cos 3/2βφ | 0 | 0 | + + + | + | A_jA_j |
| σ1μ2 | cos 2πβφ | 0 | 0 | + + + | − | A_1 |
| µ1σ2 | sin 2πβφ | 0 | 0 | − × − | + | A_2 |

Table