Duality in $N=2$ nonlinear sigma–models

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ABSTRACT

We consider $N = 2$ supersymmetric nonlinear sigma–models in two dimensions defined in terms of the nonminimal scalar multiplet. We compute in superspace the one–loop beta function and show that the classical duality between these models and the standard ones defined in terms of chiral superfields is maintained at the quantum one–loop level. Our result provides an explicit application of the recently proposed quantization of the nonminimal scalar multiplet via the Batalin–Vilkovisky procedure.

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1 Introduction

Since the seminal paper by Seiberg and Witten [1] who computed exactly the low energy effective action for $N = 2$ supersymmetric Yang–Mills theories by exploiting a duality symmetry of the moduli space, duality has been playing a central role in the current research in quantum field theory and strings. In a string theory context two dimensional nonlinear sigma–models appear as classical string vacua. The study of dualities between different classes of nonlinear sigma–models may give insights on the relations between different geometries underlying string compactifications.

Here we consider $N = 2$ nonlinear sigma–models defined in terms of complex linear superfields [2, 3]. It is well known [3] that they are classically dual to the standard nonlinear sigma–models defined in terms of chiral superfields. The duality which relates these two sets of models does not require the existence of a target space isometry, in contradistinction with the case of $N = 2$ chiral, twisted–chiral and semi–chiral sigma–models where an abelian isometry is needed in order to realize a duality symmetry (see Grisaru, Massar, Sevrin, Troost’s contribution to these proceedings for an exhaustive review and Ref. [4]). While at the classical level dual models represent a different parametrization of the same theory, the question is whether this property survives after quantization as a property of the full quantum actions. In other words, the question is whether the duality transformations commute with the renormalization procedure. Since the renormalization properties of sigma–models are encoded in their beta functions, we have computed the one–loop beta function for the sigma model defined in terms of complex linear superfields and compared it with the standard beta function for the chiral case [5, 6]. As a result we have obtained that at the one–loop level, classical duality is not affected by renormalization. The calculation has been performed by using the recently proposed quantization of the complex linear multiplet. The linearity constraints are solved in terms of unconstrained gauge superfields and, consequently, an infinite number of gauge invariances appear. They can be consistently gauge–fixed by using the Batalin–Vilkovisky approach [8]. This procedure leads to a gauge–fixed action which contains an infinite tower of ghosts interacting among themselves and with the physical fields. However we will show that, by choosing gauge–fixing functions independent of the physical background, the infinite tower of ghosts decouple in the process of diagonalizing the quadratic action. Eventually we obtain a simple expression for the propagators of the fields that enter the perturbative calculations.

The paper is organized as follows: in Section 2 we review some basic notions concerning the chiral and complex linear superfields and the classical duality between sigma–models defined in terms of chiral superfields and complex linear ones. The quantization of a sigma
model with complex linear superfields is presented in Section 3 where we explain how to deal with the infinite tower of ghosts appearing in the gauge–fixed action. Moreover the propagator for the physical fields is obtained. The one–loop beta function is computed in Section 4 following Ref. [9]. The comparison with the standard one–loop result for the chiral sigma–model is then discussed. Finally, Section 5 contains our conclusions.

2 Chiral and complex linear multiplets: the classical duality

The standard description of $N = 2$ supersymmetric scalar matter in two dimensions is in terms of a minimal multiplet, the chiral multiplet, defined by the constraints $\bar{D} \dot{\alpha} \Phi = 0, \ D^\alpha \bar{\Phi} = 0$. The free action

$$ S = \int d^2 x d^2 \theta d^2 \bar{\theta} \bar{\Phi} \Phi = \int d^2 x \left[ \bar{\phi} \Box \phi - \bar{\psi}^\dot{\alpha} i \partial_{\dot{\alpha}} \psi^\alpha + \bar{F} F \right] $$

(2.1)

describes the dynamics of a physical scalar $\phi = \Phi|$, a spinor $\psi^\alpha = D^\alpha \Phi|$ and an auxiliary field $F = D^2 \Phi|$. A different representation of $N = 2$ supersymmetry is given by the complex linear multiplet which satisfies the constraints $\bar{D}^2 \Sigma = 0, \ D^2 \bar{\Sigma} = 0$. It is a nonminimal reducible representation containing 12 + 12 degrees of freedom [2, 3]. Since this multiplet can be consistently coupled to Yang–Mills it can be used to describe scalar matter. The free action

$$ S = \int d^2 x d^2 \theta d^2 \bar{\theta} \Sigma \bar{\Sigma} $$

$$ = \int d^2 x \left[ \bar{B} \Box B - \bar{\zeta}^\dot{\alpha} i \partial_{\dot{\alpha}} \zeta^\alpha - \bar{H} H + \beta^\alpha \rho_\alpha + \bar{\beta}^\dot{\alpha} \bar{\rho}_\dot{\alpha} + \bar{\rho}^\alpha \rho_\alpha \right] $$

(2.2)

has the same physical content as the action (2.1), $B = \Sigma|, \ z^\alpha = D^\alpha \Sigma|$, but a different auxiliary sector $\rho_\alpha = D^\alpha \Sigma|, \ H = D^2 \Sigma|, \ p_{\alpha \dot{\alpha}} = D^\alpha D^\dot{\alpha} \Sigma|$ and $\bar{\beta}^\dot{\alpha} = \frac{1}{2} D^\alpha D^\dot{\alpha} D^\alpha \Sigma|$

The two actions are classically dual. Writing the first order action

$$ S = \int d^2 x d^2 \theta d^2 \bar{\theta} \left[ \bar{\Phi} \Phi + \Sigma \Phi + \bar{\Sigma} \Phi \right] $$

(2.3)

where $\Sigma$ is complex linear and $\Phi$ unconstrained, it is easy to see that performing the gaussian integral in $\Phi$ we recover the action (2.2), whereas functional integration on $\sigma^\alpha$ and $\bar{\sigma}^\dot{\alpha}$ ($\Sigma = D^\alpha \bar{\sigma}^\dot{\alpha}, \ \bar{\Sigma} = D^\alpha \sigma^\alpha$) leads to the action (2.1). We notice that under duality the chirality constraints $D_\alpha \Phi = 0$ and the equations of motion $D^2 \Phi = 0$ from the
action (2.1) are interchanged with the linearity constraints $D^2 \Sigma = 0$ and the equations of motion $D_{\alpha} \Sigma = 0$ from the action (2.2). This is similar to what happens in the standard electromagnetic duality. Indeed this is not an accident since in four dimensions it has been shown \[10\] that the standard $N = 2$ supersymmetric Yang–Mills theory with chiral matter coupled to an abelian gauge sector is dual to an $N = 2$ abelian Yang–Mills system where the scalar sector is described in terms of a complex linear multiplet. In this contest the $\Phi \rightarrow \Sigma$ duality in the matter sector is accompanied by the electromagnetic duality in the gauge sector.

More generally we consider $N = 2$ two dimensional nonlinear sigma–models in terms of chiral or complex linear superfields. Given a set of superfields $\Phi^{\mu}$, $\bar{\Phi}^{\bar{\mu}}$, $\Sigma^{\mu}$, $\bar{\Sigma}^{\bar{\mu}}$ with $\mu, \bar{\mu} = 1, \ldots, n$ we write the first order action

$$S = \int d^4 x \ d^4 \theta \left[ K(\Phi, \bar{\Phi}) + \Sigma \Phi + \bar{\Sigma} \bar{\Phi} \right]$$

(2.4)

where $\Sigma$ are linear superfields, whereas $\Phi$ are initially unconstrained (we simplify the notations by omitting the indices $\mu, \bar{\mu}$ on the superfields). Performing the functional integral on $\Sigma$ we obtain the chirality constraint on $\Phi$, so that the quadratic terms, being total derivatives, can be dropped and one is left with the standard sigma–model action for chiral superfields with Kähler potential $K$. The dual model is defined as a Legendre transform of the previous theory. One solves the classical equations of motion for $\Phi$

$$\Sigma = -\frac{\partial K}{\partial \Phi} \quad \bar{\Sigma} = -\frac{\partial K}{\partial \bar{\Phi}}$$

(2.5)

obtaining $\Phi = \Phi(\Sigma, \bar{\Sigma})$, $\bar{\Phi} = \bar{\Phi}(\Sigma, \bar{\Sigma})$. The dual action is then given by the action (2.4) once these solutions are inserted

$$S = \int d^4 x \ d^4 \theta \tilde{K}(\Sigma, \bar{\Sigma})$$

(2.6)

where

$$\tilde{K}(\Sigma, \bar{\Sigma}) = [K(\Phi, \bar{\Phi}) + \Sigma \Phi + \bar{\Sigma} \bar{\Phi}]|_{\Phi = \Phi(\Sigma, \bar{\Sigma}), \bar{\Phi} = \bar{\Phi}(\Sigma, \bar{\Sigma})}$$

(2.7)

The classical duality of these theories is well understood in a superspace formulation where all the fields are off-shell. Since the two multiplets have different auxiliary fields, going to components and eliminating the auxiliary fields through their equations of motion might lead to quite different results in the two cases \[11\]. Thus, when investigating the duality properties at the quantum level, it is convenient to proceed in a completely off–shell, superspace formulation.
Going from one model to its dual amounts to replace the potential $K$ with its Legendre transform $\tilde{K}$ in eq. (2.7). If we define the second derivatives matrix for $K$

$$G = \begin{pmatrix}
\frac{\partial^2 K}{\partial \Phi \partial \bar{\Phi}} & \frac{\partial^2 K}{\partial \Phi \partial \Phi} \\
\frac{\partial^2 K}{\partial \bar{\Phi} \partial \Phi} & \frac{\partial^2 K}{\partial \bar{\Phi} \partial \bar{\Phi}}
\end{pmatrix}$$

(2.8)

and a similar matrix for $\tilde{K}$, it is easy to show that under duality transformation

$$G(\Phi, \bar{\Phi}) \rightarrow -\tilde{G}(\Sigma, \bar{\Sigma}) = G(\Phi, \bar{\Phi})^{-1}$$

(2.9)

In order to investigate the issue of quantum duality we will study whether the relation (2.9) is maintained at the one-loop order.

### 3 The quantum approach

Since the duality transformation corresponds to performing functional integrals over the fields in two different orders, in principle it could be affected by the quantization procedure. In a perturbative approach the obvious question to ask is whether the duality properties are maintained order by order under renormalization. This can be established by studying the relation (2.9) beyond tree level. The simplest object which can be computed perturbatively and is a function of the matrix elements in (2.8) is the $\beta$-function.

In the case of a standard $N = 2$ sigma–model the one–loop divergent contribution to the Kähler potential and the corresponding $\beta$–function are given respectively by the well known results

$$K^{(1)} = \frac{1}{\epsilon} \text{tr} \log K_{\mu\bar{\nu}}$$

(3.1)

$$\beta_{\mu\bar{\mu}} = -\partial_\mu \partial_{\bar{\mu}} \text{tr} \log K_{\nu\bar{\nu}} = -R_{\mu\bar{\mu}}$$

(3.2)

We concentrate on the calculation of the first order correction to the potential $\tilde{K}$ in the case of a sigma–model defined in terms of complex linear superfields. Following the standard procedure for the chiral case we perform the calculation by using a quantum–background splitting $\Sigma \rightarrow \Sigma_0 + \Sigma$, $\bar{\Sigma} \rightarrow \bar{\Sigma}_0 + \bar{\Sigma}$ and compute one–loop diagrams with external background lines with no derivatives acting on them. From the expansion of the action (2.6) around the classical background

$$S = \int d^2x d^4\theta \left\{-\Sigma^\mu \bar{\Sigma}^\nu \delta_{\mu\bar{\nu}} + [\tilde{K}_{\mu\bar{\nu}}(\Sigma_0, \bar{\Sigma}_0) + \delta_{\mu\bar{\nu}}] \Sigma^\mu \bar{\Sigma}^\nu + \frac{1}{2} \tilde{K}_{\mu\nu}(\Sigma_0, \bar{\Sigma}_0) \Sigma^\mu \Sigma^\nu \right. \right.

$$+ \left. \frac{1}{2} \tilde{K}_{\mu\bar{\nu}}(\Sigma_0, \bar{\Sigma}_0) \Sigma^\mu \bar{\Sigma}^\nu + \cdots \right\}$$

(3.3)
we can read the vertices to be used in one–loop Feynman diagrams. However, in order to perform the calculation we need to quantize consistently the constrained \( \Sigma \) and \( \bar{\Sigma} \) superfields.

We use the recently proposed \[7\] quantization of the complex linear multiplet expressed in terms of two unconstrained spinor prepotentials

\[
\Sigma = \bar{D}_\alpha \bar{\sigma}^{\dot{\alpha}} \quad \bar{\Sigma} = D_\alpha \sigma^{\dot{\alpha}} \quad (3.4)
\]

The quadratic action

\[
S = - \int d^2 x d^4 \theta \bar{\sigma}^{\dot{\alpha}} \bar{D}_\alpha \bar{D}_\alpha \sigma^{\dot{\alpha}} \quad (3.5)
\]

is infinitely reducible and the Batalin–Vilkovisky gauge–fixing procedure introduces an infinite number of ghosts. We choose gauge–fixing functions independent of the background so that the ghosts couple to the quantum fields \( \sigma^{\dot{\alpha}} \) and \( \bar{\sigma}^{\dot{\alpha}} \), but not to the physical background. Starting from the action \( (3.5) \) and performing the gauge–fixing as explained in \[4, 5\] one obtains a final quadratic gauge–fixed action with all the diagonal terms invertible and an infinite number of nondiagonal terms mixing physical fields and ghosts. In order to perform perturbative calculations we need to compute the propagators for the prepotentials in \( (3.4) \) and these can be read from the gauge–fixed action once the diagonalization is performed. Since we are dealing with an infinite number of terms this looks like a very hard task. However in \[9\] it has been shown that a partial diagonalization is sufficient to disentangle the physical fields \( \sigma^{\dot{\alpha}} \) and \( \bar{\sigma}^{\dot{\alpha}} \) from the rest of the ghost action. Therefore, the ghosts, still interacting among themselves, in a highly nontrivial way, completely decouple from the physical sector. This procedure modifies the diagonal part of the action for \( \sigma^{\dot{\alpha}} \) and \( \bar{\sigma}^{\dot{\alpha}} \). From its explicit expression (see Ref. \[9\]) one obtains the following propagator

\[
< \sigma^{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}} > = (\tilde{W}^{-1})_{\dot{\alpha} \dot{\alpha}} = -\frac{i \partial^{\dot{\alpha}}}{\Box} + \frac{3(kk_1')^2 + 4 - 2k_1'^2}{4(kk_1')^2} \frac{i \partial^{\dot{\alpha}}}{\Box} \frac{D^2 \tilde{D}^2}{\Box} + \frac{3k^2 - 2}{4k^2} \frac{i \partial^{\dot{\alpha}}}{\Box} \frac{D_\beta \tilde{D}^2 D^\beta}{\Box} + \frac{2 - k^2}{4k^2} \frac{i \partial^{\dot{\alpha}}}{\Box} \frac{i \partial^{\beta_\dot{\alpha}}}{\Box} \frac{D_\beta \tilde{D}^\beta}{\Box} \quad (3.6)
\]

where \( k \) and \( k_1' \) are gauge parameters introduced in the gauge–fixing procedure.

4 One–loop beta function

Now we have collected all the ingredients which are necessary for the computation of the one-loop \( \beta \)-function for the linear multiplet sigma–model. As shown in the action
the superfields $\sigma^{\alpha}$ and $\bar{\sigma}^{\dot{\alpha}}$ always couple to the external background through their field strengths $\Sigma$, $\bar{\Sigma}$. Therefore, only the $< \Sigma \Sigma >$ propagator enters in the perturbative calculations. From the propagator in (3.6) we obtain

$$< \Sigma \Sigma > = D_{\alpha} < \sigma^{\alpha} \bar{\sigma}^{\dot{\alpha}} > \bar{D}_{\dot{\alpha}} = \frac{D^{2} \bar{D}^{2}}{\Box} + \frac{D_{\alpha} \bar{D}^{2} D^{\alpha}}{\Box} \equiv \Pi$$  \hspace{1cm} (4.1)

This expression is independent of any gauge parameter introduced in the gauge-fixing procedure and this provides a consistency check of the methods used in the quantization of the linear multiplet. From the action (3.3) we define

$$V \equiv (\tilde{K}_{\mu \dot{\nu}} + \delta_{\mu \dot{\nu}}), \quad U \equiv \tilde{K}_{\mu \nu}$$  \hspace{1cm} (4.2)

It is easy to see that one–loop divergent contributions to the effective action for the linear multiplet sigma–model are of two types: the ones corresponding to graphs which contain only $\Sigma V \bar{\Sigma}$ interactions and the ones associated to diagrams with both $\Sigma V \bar{\Sigma}$ and $\Sigma U \bar{\Sigma}$ vertices.

In order to compute the divergent contributions we use standard $D$-algebra techniques very similar to the ones used in [3]. The details of the calculations can be found in Ref. [1]. Here we give only the final results. From the first set of diagrams one obtains

$$\tilde{K}^{(1)}_{1} \rightarrow - \frac{1}{\epsilon} \text{tr} \log (1 - V)$$  \hspace{1cm} (4.3)

while the sum of the infinite set of one–loop diagrams containing any number of $V$ vertices and an equal number of $U$ and $\bar{U}$ vertices gives

$$\tilde{K}^{(1)}_{2} \rightarrow - \frac{1}{\epsilon} \text{tr} \log (1 - U \frac{1}{1 - V} \bar{U} \frac{1}{1 - V})$$  \hspace{1cm} (4.4)

Adding the results in (4.3) and (4.4) we obtain the total one-loop divergent contribution

$$\tilde{K}^{(1)} = - \frac{1}{\epsilon} \left[ \text{tr} \log (1 - V) + \text{tr} \log (1 - U \frac{1}{1 - V} \bar{U} \frac{1}{1 - V}) \right]$$  \hspace{1cm} (4.5)

or, in terms of the second derivatives of the potential,

$$\tilde{K}^{(1)} = - \frac{1}{\epsilon} \text{tr} \log [- (\tilde{K}_{\mu \dot{\nu}} - \tilde{K}_{\mu \dot{\nu}} \tilde{K}^{-1}_{\nu \bar{\rho}} \tilde{K}_{\bar{\rho} \hat{\mu})}]$$  \hspace{1cm} (4.6)

Using the definition (2.8) for the second derivative matrix of $\tilde{K}$, it can be rewritten as

$$\tilde{K}^{(1)} = \frac{1}{\epsilon} \text{tr} \log (-\tilde{G})_{\mu \dot{\nu}}$$  \hspace{1cm} (4.7)
Now we compare the above expression with the result in (3.1), where the one-loop divergent contribution to the Kähler potential of the $N = 2$ theory is exhibited. On the basis of the classical correspondence in (2.9), we conclude that the results in (3.1) and (4.7) explicitly maintain the expected duality.

5 Conclusions

We have computed the one–loop divergent contribution to the potential for a nonlinear sigma–model defined in terms of complex linear superfields and proved that classical duality with the standard chiral sigma–model is maintained at one–loop order.

We performed the calculation in the quantization scheme recently proposed in [7]. We have shown that, even if in general that method might be difficult to use in applications due to the presence of an infinite chain of ghosts, in our case an intelligent choice of the gauge–fixing terms allows for the complete decoupling of the ghosts from the physical sector as in standard abelian gauge theories. Further applications in quantum supersymmetric theories containing complex linear superfields (see for instance [10]) might be viable.

Quantum duality at higher loop levels needs to be studied: there duality transformations could receive nontrivial quantum corrections as in the more familiar case of dual bosonic sigma–models [12]. Moreover, our result can be easily extended [13] to mixed nonlinear sigma–models defined in terms of both chiral and complex linear superfields which naturally arise in supersymmetric extensions of QCD low–energy effective action [4, 14].

The geometry underlying a standard $N = 2$ chiral sigma–model is Kähler, while a complete geometric interpretation is still lacking for the dual theory (however see ref. [13]). A detailed discussion of this issue can be found in [13].

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