Analytical $\pi\pi$ scattering amplitude and the light scalars-II

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Abstract

In the paper Phys. Rev. D83, 054008 (2011) we constructed the $\pi\pi$ scattering amplitude $T_0^0$ with regular analytical properties in the $s$ complex plane, describing both experimental data and the results based on chiral expansion and Roy equations. Now the results obtained during development of our work are presented. We dwell on questions dealing with the low $\sigma - f_0$ mixing, inelasticity description and the kaon loop model for $\phi \to \gamma(\sigma + f_0)$ reaction, and show a number of new fits. In particular, we show that the minimization of the $\sigma - f_0$ mixing results in the four-quark scenario for light scalars: the $\sigma(600)$ coupling with the $K\bar{K}$ channel is suppressed relatively to the coupling with the $\pi\pi$ channel, and the $f_0(980)$ coupling with the $\pi\pi$ channel is suppressed relatively to the coupling with the $K\bar{K}$ channel.

The correct analytical properties of the $\pi\pi$ scattering amplitude are reached with the help of rather complicated background function. We also suggest much more simple background parameterization, practically preserving the resonance features, which is comfortable for experimental data analysis, but allows to describe the results based on chiral expansion and Roy equations only on the real $s$ axis.

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I. INTRODUCTION

Study of light scalar resonances is one of the central problems of nonperturbative QCD, it is important for understanding the chiral symmetry realization way resulting from the confinement physics.

In Refs. [1] we described the high-statistical KLOE data on the $\phi \to \pi^0\pi^0\gamma$ decay in the frame of the kaon loop model $\phi \to K^+K^- \to \gamma(f_0 + \sigma) \to \gamma\pi^0\pi^0$ simultaneously with the data on the $\pi\pi$ scattering and the $\pi\pi \to K\bar{K}$ reaction. The description was carried out taking into account the chiral shielding of the $\sigma(600)$ meson [8, 9] and its mixing with the $f_0(980)$ meson, the data yielded evidence in favor of the four-quark nature of the $\sigma(600)$ and $f_0(980)$ mesons.

At the same time it was calculated in Ref. [10] the $\pi\pi$ scattering amplitude in the $s$ complex plane, basing on chiral expansion, dispersion relations, and Roy equations. In particular, the pole was obtained at $s = M_{\sigma}^2 = (6.2 - 12.3i) m_{\pi}^2$, where

$$M_{\sigma} = 441_{-8}^{+16} - i272_{-12.5}^{+9} \text{ MeV},$$

which was assigned to the $\sigma$ resonance. Aiming the comparison of the results of Refs. [1] and [10], we built up the S-wave $\pi\pi$ scattering amplitude $T_{00}^0$ with $I = 0$ with correct analytical properties in the complex $s$ plane [11]. Remain that in our model the $S$ matrix of the $\pi\pi$ scattering is the product of the ”resonance” and ”elastic background” parts:

$$S_{00}^0 = S_{00}^{0\text{back}} S_{00}^{0\text{res}},$$

and we introduced the special $S_{00}^{0\text{back}}$ parametrization to obtain the correct $T_{00}^0$ analytical properties ($S_{00}^{0\text{res}}$ had correct analytical properties in Refs. [1] already). In Ref. [11] we successfully described the experimental data and the Ref. [10] results on the real $s$ axis using the constructed $\pi\pi$ amplitude, while the $\sigma$ pole was located rather far from the Ref. [10] result. We assumed that this deviation is caused by approximate character of the Roy equations, that take into account only the $\pi\pi$ decay channel. This question will be discussed below in more details.

In this paper we present the enlarged data analysis. We dwell on the minimization of the $\sigma - f_0$ mixing, that leads to the four-quark scenario for light scalars: the $\sigma(600)$ coupling with the $KK$ channel is suppressed relatively to the coupling with the $\pi\pi$ channel, and
the \( f_0(980) \) coupling with the \( \pi\pi \) channel is suppressed relatively to the coupling with the \( KK \) channel \[12\]. Inelasticity is also crucial for the analysis, here we describe the peculiar behavior of the data up to 1.2 GeV.

In Refs. \[1, 11\] we used the factor \( P_K \), caused by the elastic \( K\bar{K} \) background phase, that allows to correct the kaon loop model, suggested in Ref. \[3\], under the \( K\bar{K} \) threshold. Now we investigate how small this correction may be.

The set of new fits and tables is presented in Sec. II. The residues of the \( \pi\pi \) scattering amplitude and its resonance part in resonance poles are presented for the first time.

As the analytical background \( S_0^{\text{back}} \) is a rather complicated function, in Sec. III we suggest much more simple background parameterization, practically preserving the resonance features, which is comfortable for experimental data analysis, though allows to describe the results of Ref. \[10\] only on the real \( s \) axis.

The conclusion is in Sec. IV.

Note that the \( S_0^{\text{res}} \) parameterization and the complicated background parameterization are the same as in Ref. \[11\]. The modification of the \( K\bar{K} \) background phase is described in Sec. II.

II. DATA ANALYSIS, BACKGROUND WITH THE CORRECT ANALYTICAL PROPERTIES ("COMPLICATED" BACKGROUND)

The measure of the \( \sigma - f_0 \) mixing intensity is the deviation from the ideal picture, when the \( \pi\pi \) scattering phase \( \delta_0^\sigma \) is equal to 90° at the \( \sigma(600) \) mass \( m_\sigma \), and equal to 270° at the \( f_0(980) \) mass \( m_{f_0} \). We require these phases, \( \delta_0^\sigma(m_\sigma) \) and \( \delta_0^0(m_{f_0}) \), to be close to their "ideal" values.

Remain that the background phase of the \( K\bar{K} \) scattering, \( \delta_B^{K\bar{K}} \), changes the modulus of the \( K\bar{K} \rightarrow \pi^0\pi^0 \) amplitude under the \( K\bar{K} \) threshold, at \( m < 2m_K \), in the amplitude \( \phi \rightarrow K\bar{K} \rightarrow \pi^0\pi^0\gamma \[13\]. In Ref. \[11\] we define

\[
P_K = \begin{cases} 
  e^{i\delta_B^{K\bar{K}}} & m \geq 2m_K; \\
  \text{analytical continuation of } e^{i\delta_B^{K\bar{K}}} & m < 2m_K. 
\end{cases}
\]

In the present paper we investigate the influence of \( P_K \) on the \( \phi \rightarrow (f_0 + \sigma)\gamma \) amplitude in the \( f_0(980) \) region, \( m > 850 \) MeV. We upgrade the parametrization of the \( \delta^{K\bar{K}}(m) \), used
in Refs. [1,11], now the $\delta_{B}^{K\bar{K}}$ is parametrized in the following way:

$$e^{2i\delta_{B}^{K\bar{K}}} = \frac{1 + i2p_{K}f_{K}(m^{2})}{1 - i2p_{K}f_{K}(m^{2})}, \quad p_{K} = \frac{1}{2}\sqrt{m^{2} - 4m_{K}^{2}},$$

$$f_{K}(m^{2}) = \frac{(m - m_{2})^{2}/\Lambda_{2}^{2}}{1 + (m - m_{2})^{2}/\Lambda_{2}^{2}}\arctan\left(\frac{m^{2} - m_{1}^{2}}{\Lambda_{1}^{2}}\right) - \phi_{0}.$$  \quad (4)

Note that the $P_{K}$ also provides pole absence in the analytical continuation of the $\phi \rightarrow (f_{0} + \sigma)\gamma$ amplitude under the $\pi\pi$ threshold, see Ref. [1].

The experimental data on the inelasticity $\eta_{0}^{0}$, see Fig. 4, favor the low value near 1.01 GeV and sharp growth up to 1.2 GeV. Below it is shown that it is possible to reach such a behaviour.

Our results for Fits 1-5 are shown in Tables I-VI and Figs. 1-24. Fits 1-5 show that the allowed range of $\sigma(600)$ and $f_{0}(980)$ parameters is rather wide. For example, $g_{f_{0}K^{+}K^{-}}^{2}/4\pi$ is 1 GeV$^{2}$ in Fit 1 and more than 4 GeV$^{2}$ in Fit 5. This result may be important for coordination of the $g_{f_{0}K^{+}K^{-}}^{2}/4\pi$ and $g_{a_{0}K^{+}K^{-}}^{2}/4\pi$ [14].

Note that in Fit 4 the $\sigma(600)$ and $f_{0}(980)$ are coupled only with the $\pi\pi$ channel and the $K\bar{K}$ channel ($xf_{0} = x_{\sigma} = 0$). As seen from Table I and Figs. 9-16, Fit 4 is in excellent agreement with the data and the [10] results.

We introduce 56 parameters, but for restrictions (expresses 5 parameters through others) and parameters (or their combinations), that go to the bound of the permitted range (9 effective links), the effective number of free parameters is reduced to 42. But it is significant that fits describe as the experimental data (65 points), as well as the $\pi\pi$ amplitude from the [10] in the range $-5m_{\pi}^{2} < s < 0.64$ GeV$^{2}$ which is treated along with experimental data.

As in [11] we show resonance poles of the $T_{0}^{0}$ on some unphysical sheets of its Riemannian surface, depending on sheets of the polarization operators $\Pi_{R}^{ab}(s)$. For this choice of sheets the imaginary parts of pole positions $M_{R}$ would be connected to the full widths of the resonances ($2\text{Im}M_{R} = \Gamma_{R} = \sum_{ab}\Gamma(R \rightarrow ab)$) in case of metastable states, decaying to several channels. Note also that we do not show poles for Fit 2.

One can see that the obtained $\sigma(600)$ pole positions lie far from Eq. (1). In Ref. [11] we noted, that one of the possible reasons is the approximate character of the Roy equations, that take into account only the $\pi\pi$ channel.
Table I. Properties of the resonances and main characteristics are shown. The resonance masses $m_R$ and widths $\Gamma_R(m_R)$ (which may be called ”Breit-Wigner” masses and widths) are parameters in the resonance propagators, see Ref. [11]. They have clear physical meaning in contrast to the resonance poles in the complex plane.

| Fit                          | 1       | 2       | 3       | 4       | 5       |
|------------------------------|---------|---------|---------|---------|---------|
| $m_{f_0}$, MeV               | 978.30  | 974.78  | 981.49  | 979.85  | 980.40  |
| $g_{f_0K^+K^-}$, GeV         | 3.54    | 4.34    | 5.01    | 5.01    | 7.33    |
| $g_{f_0K^+K^-}/4\pi$, GeV$^2$| 1       | 1.5     | 2       | 2       | 4.2782  |
| $g_{f_0\pi^+\pi^-}$, GeV    | -1.3924 | -1.6150 | -1.9836 | -1.6455 | -2.5874 |
| $g_{f_0\pi^+\pi^-}/4\pi$, GeV$^2$ | 0.154  | 0.208   | 0.313   | 0.215   | 0.533   |
| $x_{f_0}$                    | 0.6367  | 0.6039  | 1.1701  | 0       | 1.1972  |
| $\Gamma_{f_0}(m_{f_0})$, MeV| 56.7    | 76.6    | 114.8   | 79.1    | 195.5   |
| $m_{\sigma}$, MeV            | 479.40  | 471.89  | 470.87  | 472.87  | 469.94  |
| $g_{\sigma\pi^+\pi^-}$, GeV | 2.6676  | 2.6614  | 2.7190  | 2.7093  | 2.7362  |
| $g_{\sigma\pi^+\pi^-}/4\pi$, GeV$^2$ | 0.564  | 0.569   | 0.588   | 0.584   | 0.596   |
| $g_{\sigma K^+K^-}$, GeV    | 0.553   | 0.101   | 0.279   | 0.274   | 0.149   |
| $g_{\sigma K^+K^-}/4\pi$, GeV$^2$ | 0.001  | 0.048   | 0.006   | 0.006   | 0.002   |
| $x_{\sigma}$                 | 1.1822  | 0.9187  | 1.7336  | 0       | 1.6291  |
| $\Gamma_{\sigma}(m_{\sigma})$, MeV | 362.1  | 363.2   | 379.5   | 376.0   | 384.7   |
| $C$, GeV$^2$                 | 0.05120 | 0.04465 | 0.01307 | 0.00167 | 0.03345 |
| $\delta$, °                 | -64.69  | -58.7   | -64.6   | -55.4   | -44.0   |
| $a_0^0$, $m_\pi^{-1}$        | 0.223   | 0.220   | 0.224   | 0.223   | 0.225   |

Adler zero in $\pi\pi \rightarrow \pi\pi$ \( (93.5 \text{ MeV})^2 \) (85.6 MeV)$^2$ (96.8 MeV)$^2$ (94.6 MeV)$^2$ (92.3 MeV)$^2$

| $\delta_0^{0 \text{ res}}(m_\sigma)$, ° | 91.8    | 94.1    | 91.0    | 90.6    | 92.3    |
| $\delta_0^{0 \text{ res}}(m_{f_0})$, ° | 250.1   | 250.1   | 260.1   | 255.1   | 258.7   |
| $\eta_0^0(1010 \text{ MeV})$          | 0.55    | 0.52    | 0.51    | 0.51    | 0.51    |
| $\chi^2_{\text{phase}}$ (44 points)   | 53.1    | 48.9    | 42.0    | 40.0    | 55.1    |
| $\chi^2_{sp}$ (18 points)             | 21.2    | 20.8    | 21.3    | 17.0    | 12.6    |
Table II. Parameters of the $K\bar{K}$ background phase, $\delta_{K\bar{K}}^{K\bar{K}}$, are shown.

| Fit | 1       | 2       | 3       | 4       | 5       |
|-----|---------|---------|---------|---------|---------|
| $\Lambda_K$, GeV | 0.975   | 1.245   | 1.375   | 1.450   | 1.894   |
| $\Lambda_1$, MeV | 381.56  | 404.49  | 387.56  | 412.43  | 322.93  |
| $\Lambda_2$, MeV | 83.113  | 81.137  | 86.246  | 65.000  | 68.041  |
| $m_1$, MeV | 827.48  | 823.54  | 801.40  | 791.48  | 808.17  |
| $m_2$, MeV | 909.17  | 923.55  | 911.59  | 970.52  | 963.55  |
| $w$, MeV | 0.471   | 0.618   | 0.492   | 0.750   | 0.750   |
| $\phi_0$ | $-0.299$ | $0.021$ | $0.153$ | $0.271$ | $0.622$ |

As it was shown in the SU(2)×SU(2) linear $\sigma$ model, Ref. [9], the residue of the $\sigma$ pole in the amplitude of the $\pi\pi$ scattering can not be connected to coupling constant in the Hermitian (or quasi-Hermitian) Hamiltonian for it has a large imaginary part. Here we calculate the residues of the amplitude $T^0_0$ in the $\sigma(600)$ pole, see Table V, and illustrate this fact in our case. Note that large imaginary part is both in the residues of the full amplitude $T^0_0$ and its resonance part $T^0_0Res$. So, considering the residue of the $\sigma$ pole in $T^0_0$ or $T^0_0Res$ as proportional to the square of its coupling constant to the $\pi\pi$ channel is not a clear guide to understanding the $\sigma$ meson nature. In addition, this pole can not be interpreted as a physical state for its huge width.

One can see from Figs. 8b, 16, and 24a that for Fits 2-5 with $g_{f_0K^+K^-}^2/4\pi \geq 1.5$ GeV$^2$ the maximum of the $|P_K|^2$ is close to 1 (about 1.2), this means that the correction to the kaon-loop model [3] is small. For lower $g_{f_0K^+K^-}^2/4\pi$ the $|P_K|^2$ increases as a compensation, see Fit 1 and Fig. 8a. This results in a model dependence of the constant determination. A precise measurement of the inelasticity $\eta_0^0$ would resolve this problem.

One can see from Table I that for all Fits 1-5 the resonance phase $\delta^{res}(m)$ is close to $90^\circ$ at $m_\sigma$ and to $270^\circ$ at $m_{f_0}$, see also Figs. 3, 11, and 19a.
Table III. Parameters of the first background ($P_{\pi_1}$), see Ref. [11], are shown.

| Fit   | 1   | 2   | 3   | 4   | 5   |
|-------|-----|-----|-----|-----|-----|
| $a_1$ | -2.767 | -1.997 | -2.727 | -3.152 | -2.320 |
| $a_2$ | 0.00997 | 0.02824 | 0.01228 | 0.00995 | 0.00987 |
| $a_3$ | 0 | 0 | 0 | 0 | 0 |
| $a_4$ | 2.4774 | 1.1655 | 1.9460 | 3.6119 | 1.9579 |
| $\alpha_1$, GeV$^2$ | 430.875 | -3.472 | 299.566 | 187.438 | 230.647 |
| $\alpha_2$, GeV$^4$ | 1038.375 | 802.006 | 1006.643 | 924.912 | 876.525 |
| $\alpha_3$, GeV$^6$ | 853.500 | 810.211 | 840.573 | 805.455 | 805.900 |
| $\alpha_4$, GeV$^8$ | 237.251 | 239.362 | 232.860 | 225.823 | 233.065 |
| $\alpha_5$, GeV$^{10}$ | 25.3514 | 25.4850 | 24.8635 | 24.9756 | 25.2960 |
| $\alpha_6$, GeV$^{12}$ | 0.248630 | 0.218526 | 0.225182 | 0.240893 | 0.224103 |
| $c_1$, GeV | 504.558 | 680.672 | 543.245 | 499.429 | 557.733 |
| $c_2$, GeV$^3$ | -2745.58 | -2246.19 | -2532.80 | -2395.72 | -2484.57 |
| $c_3$, GeV$^5$ | 132.007 | 176.850 | 226.617 | 256.520 | 191.569 |
| $c_4$, GeV$^7$ | 390.262 | 379.216 | 399.808 | 404.615 | 394.230 |
| $c_5$, GeV$^9$ | 50.6689 | 51.7071 | 50.4545 | 49.8151 | 51.4728 |
| $c_6$, GeV$^{11}$ | -0.612729 | -0.636956 | -0.633709 | -0.711296 | -0.646822 |
| $m_1$, MeV | 921.52 | 766.81 | 1049.46 | 1088.06 | 915.18 |
| $g_1$, MeV | 301.06 | 302.40 | 302.30 | 333.34 | 330.83 |
| $m_2$, MeV | 1395.84 | 937.07 | 970.74 | 1104.06 | 1025.77 |
| $g_2$, MeV | 367.05 | 305.90 | 310.37 | 421.12 | 324.93 |
| $m_3$, MeV | 1208.42 | 1432.57 | 1098.81 | 1125.51 | 1330.56 |
| $g_3$, MeV | 335.62 | 304.46 | 388.57 | 378.42 | 301.39 |
| $m_4$, MeV | 1078.40 | 898.85 | 1053.21 | 1162.64 | 907.71 |
| $g_4$, MeV | 395.12 | 429.54 | 403.00 | 388.93 | 437.65 |
| $m_5$, MeV | 1011.58 | 991.70 | 1017.19 | 1051.17 | 1008.35 |
| $g_5$, MeV | 499.99 | 503.50 | 507.03 | 500.41 | 502.24 |
| $m_6$, MeV | 932.07 | 1240.82 | 1174.80 | 1154.23 | 1264.43 |
| $g_6$, MeV | 535.90 | 616.91 | 574.55 | 542.66 | 629.02 |
Table IV. Parameters of the second background ($P_{\pi 2}$), see Ref. [11], are shown.

| Fit | 1    | 2    | 3    | 4     | 5     |
|-----|------|------|------|-------|-------|
| $\Lambda$, MeV | 88.506 | 78.187 | 73.302 | 70.443 | 68.804 |
| $k_2$ | 0.0295157 | 0.0310661 | 0.0313951 | 0.0415406 | 0.0250125 |
| $\beta$ | 143.367 | 132.832 | 125.729 | 126.711 | 165.179 |
| $\gamma_1$ | 383.759 | 330.693 | 365.564 | 374.814 | 450.415 |
| $\gamma_2$ | 21.5451 | 24.7134 | 22.9274 | 22.1454 | 23.1322 |
| $m_{1A}$, MeV | 687.75 | 668.05 | 721.92 | 837.74 | 607.47 |
| $g_{1A}$, MeV | 301.22 | 303.07 | 305.36 | 392.13 | 343.56 |
| $m_{2A}$, MeV | 495.44 | 554.28 | 492.36 | 487.79 | 515.18 |
| $g_{2A}$, MeV | 504.69 | 488.58 | 457.53 | 454.08 | 494.25 |
| $m_{3A}$, MeV | 631.67 | 637.64 | 625.43 | 599.93 | 675.83 |
| $g_{3A}$, MeV | 117.86 | 101.74 | 101.00 | 101.35 | 100.98 |

Table V. The $\sigma(600)$ poles (in MeV), the residues of $T_0^0$, $\text{Res } T_0^0$, and of the resonance part $T_0^{0\text{ Res}}$, $\text{Res } T_0^{0\text{ Res}}$, (in 0.01 GeV$^2$) in this pole on different sheets of the complex $s$ plane depending on sheets of polarization operators $\Pi^{ab}(s)$ are shown.

| Sheets of $\Pi^{ab}$ | $\Pi^{\pi\pi}$ | $\Pi^{KK}$ | $\Pi^{\eta\eta}$ | $\Pi^{\eta\eta'}$ | $\Pi^{\eta'\eta'}$ | Fit 1 | Fit 2 | Fit 3 | Fit 4 | Fit 5 |
|-----------------------|----------------|------------|-----------------|------------------|------------------|------|------|------|------|------|
| $\Pi^{\pi\pi}$      |                |            |                 |                  |                  | $\sigma$ pole | $\text{Res } T_0^0$ | $\text{Res } T_0^{0\text{ Res}}$ | $\sigma$ pole | $\text{Res } T_0^0$ | $\text{Res } T_0^{0\text{ Res}}$ |
| II                    | I              | I          | I               | I                |                  | 565 - 204$i$   | $-2 + 14$i$   | $-22 - 11$i$   | 566 - 201$i$   | $-1 + 13$i$   | $-20 - 12$i$   |
| II                    | II             | I          | I               | I                |                  | 612 - 346$i$   | $5 + 3$i$    | $-18 + 14$i$   | 569 - 267$i$   | $3 + 10$i$    | $-26 - 1$i$   |
| II                    | II             | II         | I               | I                |                  | 542 - 396$i$   | $-1 + 3$i$    | $-16 - 3$i$    | 522 - 379$i$   | $-2 + 3$i$    | $-14 - 6$i$   |
| II                    | II             | II         | II              | I                |                  | 577 - 522$i$   | $0.2 + 1$i$   | $-15 - 1$i$    | 612 - 626$i$   | $0.3 + 0.4$i$ | $-15 - 4$i$   |
| II                    | II             | II         | II              | II               |                  | 633 - 534$i$   | $1 + 1$i$     | $-23 + 2$i$    | 664 - 651$i$   | $0.5 + 0.3$i$ | $-19 - 4$i$   |

| Sheets of $\Pi^{ab}$ | $\Pi^{\pi\pi}$ | $\Pi^{KK}$ | $\Pi^{\eta\eta}$ | $\Pi^{\eta\eta'}$ | $\Pi^{\eta'\eta'}$ | Fit 3 | Fit 4 | Fit 5 |
|-----------------------|----------------|------------|-----------------|------------------|------------------|------|------|------|
| $\Pi^{\pi\pi}$      |                |            |                 |                  |                  | $\sigma$ pole | $\text{Res } T_0^0$ | $\text{Res } T_0^{0\text{ Res}}$ | $\sigma$ pole | $\text{Res } T_0^0$ | $\text{Res } T_0^{0\text{ Res}}$ |
| II                    | I              | I          | I               | I                |                  | 572 - 206$i$   | $-2 + 14$i$   | $-21 - 12$i$   | 579 - 216$i$   | $-3 + 15$i$   | $-23 - 12$i$   |
| II                    | II             | I          | I               | I                |                  | 572 - 279$i$   | $3 + 10$i$    | $-26 + 2$i$    | 579 - 273$i$   | $1 + 12$i$    | $-27 - 1$i$   |
| II                    | II             | II         | I               | I                |                  | 526 - 395$i$   | $-2 + 2$i$    | $-12 - 5$i$    |                  |                  |                  |
| II                    | II             | II         | I               | I                |                  | 623 - 651$i$   | $0.3 + 0.3$i$ | $-14 - 3$i$    |                  |                  |                  |
| II                    | II             | II         | II              | II               |                  | 683 - 679$i$   | $1 + 0.1$i$   | $-19 - 4$i$    |                  |                  |                  |
FIG. 1: The $\pi^0\pi^0$ spectrum in the $\phi \rightarrow \pi^0\pi^0\gamma$ decay, theoretical curve, and the KLOE data (points) are shown: a) Fit 1, b) Fit 2.

Table VI. The $f_0(980)$ poles (in MeV), the residues of $T^0_0$, $\text{Res}T^0_0$, and of the resonance part $T^0_{0\text{Res}}$, $\text{Res}T^0_{0\text{Res}}$ (in 0.01 GeV$^2$) in this pole on different sheets of the complex $s$ plane depending on sheets of polarization operators $\Pi^{ab}(s)$ are shown.

| Sheets of $\Pi^{ab}$ | Fit 1 | Fit 5 |
|----------------------|-------|-------|
| $\Pi^{\pi\pi}$ | $\Pi^{K\bar{K}}$ | $\Pi^{\eta\eta}$ | $\Pi^{\eta'\eta'}$ | $f_0$ pole | $\text{Res}T^0_0$ | $\text{Res}T^0_{0\text{Res}}$ | $f_0$ pole | $\text{Res}T^0_0$ | $\text{Res}T^0_{0\text{Res}}$ |
| II | I | I | I | 986 − 26$i$ | 6 − 2$i$ | −7 + 2$i$ | 986 − 21$i$ | 5 − 1$i$ | −6 + 0.1$i$ |
| II | II | I | I | 913 − 302$i$ | 10 + 5$i$ | −19 − 19$i$ | 1575 − 553$i$ | −8 − 4$i$ | −21 − 23$i$ |
| II | II | II | I | 966 − 450$i$ | 3 − 1$i$ | −12 − 12$i$ | 2101 − 1065$i$ | 0.1 + 5$i$ | −28 − 10$i$ |
| II | II | II | II | 962 − 465$i$ | 3 − 0.3$i$ | −12 − 12$i$ | 2173 − 1158$i$ | 1 + 5$i$ | −25 − 11$i$ |
| II | II | II | II | 954 − 586$i$ | 1 + 0.4$i$ | −3 − 14$i$ | 2452 − 1570$i$ | 3 + 3$i$ | −22 − 10$i$ |

| Sheets of $\Pi^{ab}$ | Fit 3 | Fit 4 |
|----------------------|-------|-------|
| $\Pi^{\pi\pi}$ | $\Pi^{K\bar{K}}$ | $\Pi^{\eta\eta}$ | $\Pi^{\eta'\eta'}$ | $f_0$ pole | $\text{Res}T^0_0$ | $\text{Res}T^0_{0\text{Res}}$ | $f_0$ pole | $\text{Res}T^0_0$ | $\text{Res}T^0_{0\text{Res}}$ |
| II | I | I | I | 986 − 23$i$ | 6 − 1$i$ | −6 + 1$i$ | 985 − 20$i$ | 5 − 1$i$ | −5 + 1$i$ |
| II | II | I | I | 1149 − 485$i$ | 3 − 6$i$ | −14 − 16$i$ | 1187 − 618$i$ | 0.5 − 2$i$ | −11 − 9$i$ |
| II | II | II | I | 1441 − 835$i$ | −3 + 0.4$i$ | −20 − 7$i$ | − | − | − |
| II | II | II | II | 1469 − 885$i$ | −2 + 0.4$i$ | −16 − 9$i$ | − | − | − |
| II | II | II | II | 1607 − 1182$i$ | −1 + 1$i$ | −11 − 8$i$ | − | − | − |
FIG. 2: The phase $\delta_0$ of the $\pi\pi$ scattering (degrees) is shown: a) Fit 1, b) Fit 2. The experimental data from Refs. [15–19].

FIG. 3: The resonance phase of the $\pi\pi$ scattering $\delta_0^{\text{res}}$ (degrees) is shown: a) Fit 1, b) Fit 2.

FIG. 4: The inelasticity $\eta^0_0$ is shown: a) Fit 1, b) Fit 2.
FIG. 5: The phase $\delta_0^0$ of the $\pi\pi$ scattering is shown. The solid line is our description, dashed lines mark borders of the corridor [10], and points are the experimental data from Refs. [15–19, 21, 22]: a) Fit 1, b) Fit 2.

FIG. 6: The real and the imaginary parts of the amplitude $T_0^0$ of the $\pi\pi$ scattering are shown. Solid lines show our description, dashed lines mark borders of the real part corridor and the imaginary part for $s < 0$ [10]: a) Fit 1; b) Fit 2.
FIG. 7: The phase $\delta^{\pi K}$ of the $\pi\pi \to K\bar{K}$ scattering is shown: a) Fit 1; b) Fit 2.

FIG. 8: The $|P_K(m)|^2$ is shown, see Eq. (7): a) Fit 1; b) Fit 2.

FIG. 9: The $\pi^0\pi^0$ spectrum in the $\phi \to \pi^0\pi^0\gamma$ decay, theoretical curve, and the KLOE data (points) are shown: a) Fit 3, b) Fit 4.
FIG. 10: The phase $\delta_0$ of the $\pi\pi$ scattering (degrees) is shown: a) Fit 3, b) Fit 4. The experimental data from Refs. [15–19].

FIG. 11: The resonance phase of the $\pi\pi$ scattering $\delta_0^{\text{res}}$ (degrees) is shown: a) Fit 3, b) Fit 4.

FIG. 12: The inelasticity $\eta_0^0$ is shown: a) Fit 3, b) Fit 4.
FIG. 13: The phase $\delta_0^0$ of the $\pi\pi$ scattering is shown. The solid line is our description, dashed lines mark borders of the corridor, and points are the experimental data from Refs. 15, 18, 21, 22: a) Fit 3, b) Fit 4.

FIG. 14: The real and the imaginary parts of the amplitude $T^0_0$ of the $\pi\pi$ scattering are shown. Solid lines show our description, dashed lines mark borders of the real part corridor and the imaginary part for $s < 0$ from Ref. 10: a) Fit 3; b) Fit 4.
FIG. 15: The phase $\delta^{\pi K}$ of the $\pi\pi \rightarrow K\bar{K}$ scattering is shown: a) Fit 3; b) Fit 4.

FIG. 16: The $|P_K(m)|^2$ is shown, see Eq. (7): a) Fit 3; b) Fit 4.

FIG. 17: The $\pi^0\pi^0$ spectrum in the $\phi \rightarrow \pi^0\pi^0\gamma$ decay, theoretical curve, and the KLOE data (points) [2] are shown: a) Fit 5, b) Fit 6.
Table VII. Parameters of the Fit 6 (with simple background Eq. (5)).

| Parameter                        | Value        | Value        | Value        |
|----------------------------------|--------------|--------------|--------------|
| $m_{f_0}$, MeV                   | 981.80 ± 1.8 | $\Lambda_K$, GeV | 0.8803      |
| $g_{f_0K^+K^-}$, GeV             | 7.3612       | $\Lambda_1$, MeV | 490.24      |
| $g_{f_0K^+K^-}/4\pi$, GeV$^2$   | 4.3120 ± 1.0 | $\Lambda_2$, MeV | 154.08      |
| $g_{f_0\pi^+\pi^-}$, GeV        | -2.3865      | $m_1$, MeV   | 754.53      |
| $g_{f_0\pi^+\pi^-}/4\pi$, GeV$^2$ | 0.453        | $m_2$, MeV   | 422.14      |
| $x_{f_0}$                        | 0.9875       | $w$, MeV     | 0.999       |
| $\Gamma_{f_0}(m_{f_0})$, MeV    | 166.1        | $\phi_0$     | 0.787       |
| $m_{\sigma}$, MeV                | 572.25       | $b_0$        | 1.41426     |
| $g_{\sigma\pi^+\pi^-}$, GeV     | 2.91216      | $b_1$        | 0.97324     |
| $g_{\sigma\pi^+\pi^-}/4\pi$, GeV$^2$ | 0.675        | $b_2$        | -1.09477    |
| $g_{\sigmaK^+K^-}$, GeV          | 0.4583       | $b_3$        | -0.21134    |
| $g_{\sigmaK^+K^-}/4\pi$, GeV$^2$ | 0.017        | $c_0$        | 2.48601     |
| $x_{\sigma}$                     | 1.01775      | $c_1$        | 1.02050     |
| $\Gamma_{\sigma}(m_{\sigma})$, MeV | 387.4        | $c_2$        | 0.45705     |
| $C$, GeV$^2$                     | 0.06582      | $c_3$        | 0.12373     |
| $\delta$, °                      | -5.8         | $\Lambda_1^\pi$ | 160.84      |
| $a^0_0$, $m^{-1}_\pi$            | 0.220        | $\Lambda_2^\pi$ | 522.98      |
| Adler zero in $\pi\pi \rightarrow \pi\pi$ ($89.8$ MeV)$^2$ | $\delta^0_{\text{res}}(m_{\sigma}),$ ° | 93.1 |  
| $\chi^2_{\text{phase}}$ (44 points) | 39.4 | $\delta^0_{\text{res}}(m_{f_0}),$ ° | 251.4 |
| $\chi^2_{\text{sp}}$ (18 points) | 13.9 | $\eta^0_0(1010$ MeV) | 0.45 |

III. SIMPLE BACKGROUND

The background function, suggested in Ref. [11] to reach the correct analytical properties of the $\pi\pi$ scattering amplitude and used above, is rather complicated and costly in computation. In this section we suggest much more simple background parameterization, practically preserving the resonance features, which is comfortable for experimental data analysis and allows to describe the results [10] on the real $s$ axis.

This background function is an upgrade of the one, used in Ref. [1];
FIG. 18: The phase $\delta^0$ of the $\pi\pi$ scattering (degrees) is shown: a) Fit 5, b) Fit 6. The experimental data from Refs. [15–19].

FIG. 19: The resonance phase of the $\pi\pi$ scattering $\delta^0_{\text{res}}$ (degrees) is shown: a) Fit 5, b) Fit 6.

FIG. 20: The inelasticity $\eta^0_0$ is shown: a) Fit 5, b) Fit 6.
FIG. 21: The phase $\delta_0^{\pi\pi}$ of the $\pi\pi$ scattering is shown. The solid line is our description, dashed lines mark borders of the corridor [10], and points are the experimental data from Refs. [15–19, 21, 22]: a) Fit 5, b) Fit 6.

FIG. 22: The real and the imaginary parts of the amplitude $T_0^{0\pi}$ of the $\pi\pi$ scattering are shown. Solid lines show our description, dashed lines mark borders of the real part corridor and the imaginary part for $s < 0$ from Ref. [10]: a) Fit 5; b) Fit 6.

$$\tan(\delta_{B}^{\pi\pi}) = -\frac{p_{\pi}}{m_{\pi}} \frac{b_0 - b_1 \frac{p_{\pi}^2}{m_{\pi}^2} + b_2 \frac{p_{\pi}^4}{m_{\pi}^2} + b_3 \frac{p_{\pi}^6}{m_{\pi}^2} + \frac{m_{\pi}}{m_{\pi}} \left( c_0 + c_1 \frac{p_{\pi}^2}{m_{\pi}^2} + c_2 \frac{p_{\pi}^4}{m_{\pi}^2} + c_3 \frac{p_{\pi}^6}{m_{\pi}^2} \right)}{(1 + 4p_{\pi}^2/\Lambda_1^2)(1 + 4p_{\pi}^2/\Lambda_2^2)},$$  \hspace{1cm} (5)

here $p_{\pi} = \sqrt{m^2 - 4m_{\pi}^2}/2$. Note that in comparison with Ref. [1] the function (5) has a left cut.
FIG. 23: The phase $\delta_{\pi K}$ of the $\pi\pi \rightarrow K\bar{K}$ scattering is shown: a) Fit 5; b) Fit 6.

FIG. 24: The $|P_K(m)|^2$ is shown, see Eq. (7): a) Fit 5; b) Fit 6.

Let us build the $\chi^2$ function. It may be divided into 3 parts:

$$\chi^2 = \chi^2_{data} + \chi^2_{Roy} + \chi^2_{restr}$$

where the first one is the usual $\chi^2$ function for the experimental data, the second one provides the description of the results [10], and the third one provides the restrictions.

The $\chi^2_{data}$ is constructed with the help of the same data, as in Ref. [11], except the $\delta_0$ data in the region $2m_\pi < m < 800$ MeV, where we use the [10] results. Note that in Table I we show $\chi^2_{phase}$, obtained in the full region $2m_\pi < m < 1200$ MeV with the "old data" [15–19].

The $\chi^2_{Roy}$ caused by the real and imaginary parts of the $T_0^0(m)$ contributions in the region $-5m_\pi^2 < s < 4m_\pi^2$, and the $\delta_0^0$ contribution from the region $4m_\pi^2 < s < (800$ MeV)$^2$. Here
FIG. 25: The comparison of Fit 3 and Fit 7 (with the same resonance parameters, but the background parameterization (5)): a) the phase $\delta^0$, b) the amplitude $T^0$ under the $\pi\pi$ threshold. Solid lines are Fit 7, dashed lines are Fit 3, points are the experimental data. The curves are practically the same.

for $ReT^0$ and $\delta^0$ we used points and errors, kindly sent us by H. Leutwyler, and for $ImT^0$ the approximate curve $ImT^0_0(m) = -0.0327(m/2m_\pi)^2$, obtained using Fig. 1 in Ref. [10], providing us with central values, and the error is assumed to be 25%. Note that for $ImT^0_0$ we used the "reper" points $s = -(30 \text{ MeV})^2, -(50 \text{ MeV})^2, -(100 \text{ MeV})^2, -(150 \text{ MeV})^2, -(200 \text{ MeV})^2, -(250 \text{ MeV})^2, -(280 \text{ MeV})^2, -(308.95 \text{ MeV})^2$, the last is the end of the domain of validity of the Roy equations, connected with the Lehmann-Martin ellipse, see [10].

We impose the following set of restrictions, contributing to $\chi^2_{\text{rest}}$:

1) $85^\circ < \delta^0_{\text{res}}(m_\sigma) < 95^\circ$ and $250^\circ < \delta^0_{\text{res}}(m_f_0) < 290^\circ$ to provide small $\sigma - f_0$ mixing, a kind of diagonalization that results in the four-quark model scenario.

2) $1.2 > |P_K|^2 > 0.8$ for $m > 850 \text{ MeV}$. The maximum is found dynamically (at every calculation of the $\chi^2$ function), the minimum in our situation is at 850 MeV.

3) $-0.1 > \delta > -1.5$, trying to be not far from the result [20].

4) $0.1 < w < 1$; $0.1 \text{ GeV} < m_2 < 1.5 \text{ GeV}$; $0.5 \text{ GeV} < A_1 < 2.2 \text{ GeV}$; $65 \text{ MeV} < A_2$ to provide reasonable form of the $|P_K|^2$.

To provide, for example, the condition $\delta > -0.1$, we add to $\chi^2_{\text{rest}}$ the term

$$T = W(-\delta - 0.1 + |\delta + 0.1|)^2,$$

where $W$ is the big number. So till $\delta > -0.1$ the contribution $T$ is equal to 0, but when
–0.1 > δ, T becomes large, so the minimization procedure can go outside the barrier only on a negligible distance. Our $\chi^2_{\text{restr}}$ is the sum of contributions like Eq. (6).

Using the constructed $\chi^2$ function, we obtain Fit 6. One can see that this Fit perfectly describes the experimental data and the results based on Roy equations on the real $s$ axis, see Table VII and Figs. 17-24. Note that in Table VII the $m_{f_0}$ and $g_{f_0K^+K^-}/4\pi$ errors are adduced.

To illustrate the abilities of the background, we perform Fit 7 with the same resonance parameters as for Fit 3. Fit 7 provides practically the same experimental data description as Fit 3. The theoretical curves for phase $\delta^0_0$ are shown in Fig. 25 (a), they are practically the same. It is obvious that both Fit 3 and Fit 7 provide practically identical mass spectrum in $\phi \rightarrow \pi^0\pi^0\gamma$ decay also. The inelasticity is exactly the same. Additionally, Fit 7 and Fit 3 provide indistinguishable curves for $T^0_0$ at $4m_{\pi}^2 > s > 0$, see Fig. 25 (b).

IV. CONCLUSION

Our investigation shows that the scenario, based on the four-quark model, completely agrees with the current experimental data and theoretical requirements. It is shown that the requirement of the weak $\sigma(600) - f_0(980)$ mixing leads to the $g_{\sigma K^+K^-}$ and $g_{f_0\pi^+\pi^-}$ suppression, that is predicted by the four-quark model, see Table I.

The behaviour of the factor $P_K(m)$, which corrects the kaon loop model, is model dependent. We show that for large enough $g_{f_0K^+K^-}/4\pi$ constant (f.e., 1.5 GeV$^2$) the current data (including the Ref. [10] results) may be well-described with this factor close to 1 at 850 MeV < $m$, but for smaller values of this constant (f.e., 1 GeV$^2$) the correction increases. New precise data on the $\pi\pi \rightarrow K\bar{K}$ reaction and the inelasticity ($\eta^0_0$) of the $\pi\pi$ scattering would give an ability to understand more about this factor and reduce the region of possible values of parameters.

The obtaining of the $\sigma$ pole in Ref. [10] gave the strong argument in favor of the $\sigma(600)$ existence, but followed efforts aiming the precise determination of the pole are not productive. The Roy equations are one-channel, that is, are approximate and even slight discrepancy in the amplitude in the physical region may lead to large changes in the complex plane. Remind that the Riemannian surface of the $\pi\pi$ scattering amplitude has many sheets (strictly speaking, infinite number of sheets), and even for relatively narrow $f_0(980)$ is sometimes a
problem to determine on what sheet should we find the pole, see, for example, Fit 2 in Ref. [11]. But even if we obtained the pole precisely, it would give us practically no information on the resonance nature, because it can not be connected to coupling constant in the Hermitian (or quasi-Hermitian) Hamiltonian, see also Ref. [9], because of large imaginary part. Besides, the residue of the amplitude in the pole is strongly distorted by the background part of the amplitude, see Tables V and VI, that gives essential contribution even for relatively narrow $f_0(980)$.

Let us dwell on the results, presented in Table VI. Remind that for a stable particle with the mass $m_0$ there is the pole in the amplitude

$$T = -\frac{g^2/16\pi s}{s - m_0^2}$$

at $s = m_0^2$, the residue of the amplitude $ResT$ is connected to the coupling constant ($g$) of the stable particle with the $\pi\pi$ channel.

One can see that the real part of the $T_0^0$ residue in the $f_0(980)$ pole is positive, so the coupling constant should be practically pure imaginary, what is physically meaningless. Note that the residue of the amplitude resonance part $T_0^{0\text{Res}}$ is good. That is why the best way for understanding the nature of the light scalars is the investigation of their production mechanisms in physical processes.

The simple background parameterization, suggested in Sec. III, may be used for experimental data analysis and the description of the Ref. [10] results for real $s$. It is shown that the resonance features are practically preserved, moreover, one can see that for even more simple background, used in Ref. [1], they changed not so much, though the Ref. [10] results were not included.

In this investigation we paid more attention to the inelasticity $\eta_0^0$, namely, we tried to reproduce the peculiar behaviour near the threshold, indicated by the experimental data. Unfortunately, the current data have large errors, so the precise measurement of the inelasticity $\eta_0^0$ near 1 GeV in $\pi\pi \rightarrow \pi\pi$ would be very important.

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\[
M_{\text{sig}} = g(m) \left( \phi \epsilon - \frac{(\phi q) (\epsilon p)}{pq} \right) T \left( K^+K^- \to \pi^0\pi^0 \right) \times 16\pi, \tag{7}
\]

where \( g(m) \) is the kaon loop function, \( \phi \) and \( \epsilon \) are polarization vectors of the \( \phi \) meson and photon, see Ref. [1, 11].

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