Dominator Coloring Parameterized by Cluster Vertex Deletion Number

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Abstract

The Dominator Coloring problem borrows the properties of two classical problems in graph theory - Graph Coloring and Dominating Set. A dominator coloring $\chi_d$ of a graph $G$ is a proper coloring of its vertices such that each vertex dominates a color class - that is, for each $v \in V(G)$, there exists a color $c$ such that $\emptyset \subset \chi_d^{-1}(c) \subseteq N_G[v]$. Given a graph $G$ and a natural number $\ell$, the Dominator Coloring problem asks if there is a dominator coloring of $G$ which uses at most $\ell$-many colors. The problem, which was first described in 2006 and studied in several papers since then, still hosts several important open questions. While it is known that Dominator Coloring is FPT (Fixed-Parameter Tractable) when parameterized by $(t, \ell)$ where $t$ is the number of colors used and $t$ the treewidth of $G$, the structural parameterized landscape of the problem remains unexplored.

Our first result in this paper is a randomized $O^*(c^k)$ algorithm for the problem where $c$ is some constant and $k$ is the size of a graph’s Clique Modulator, a set of vertices whose deletion results in a clique. This algorithm is obtained by a non-trivial adaptation of the recent work by Gutin et al. for List Coloring parameterized by the clique modulator that uses an inclusion-exclusion based polynomial sieving technique, and in addition uses a dynamic programming based exact algorithm we develop for Dominator Coloring.

Later, we go on to prove the main result of the paper that Dominator Coloring is FPT when parameterized by the size of a graph’s Cluster Vertex Deletion (CVD) set, in contrast to the W[1]-hardness result for List Coloring parameterized by the CVD set size. En route, we design a simpler and faster deterministic FPT algorithm when the problem is parameterized by the size of a graph’s Twin Cover. We believe that this algorithm’s approach, which uses a relationship between Dominator Coloring and List Coloring that we establish, is of independent interest.

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1 Introduction

In this paper we address the complexity of a variant of Graph Coloring called Dominator Coloring in the realm of parameterized complexity. A coloring of a graph $G$ is a function $\chi : V(G) \rightarrow C$, where $C$ is a set of colors. A proper coloring of $G$ is a coloring of $G$ such that $\chi(u) \neq \chi(v)$ for all $(u, v) \in E(G)$. The set of all vertices which are colored $c$, for a $c \in C$, is called the color class $c$. We sometimes refer to the color $c$ itself as a color class. We let $|\chi|$ denote $|\text{Im}(\chi)|$, the size of the image of $\chi$.

A vertex $v \in V(G)$ dominates $S \subseteq V(G)$ if $S \subseteq N_G[v]$. A dominator coloring $\chi_d$ of $G$ is a proper coloring of $G$ such that for all $v \in V(G)$, $v$ dominates a color class $c \in \text{Im}(\chi_d)$. Analogous to the classical Graph Coloring problem, we define Dominator Coloring, which we shorten to DomCol, below.

**DomCol**

**Input:** A graph $G$; an integer $\ell$

**Question:** Does there exist a dominator coloring $\chi_d$ of $G$ with $|\chi_d| \leq \ell$?

1.1 Notations

Most notations (for concepts in graph theory and fixed-parameter tractability) used in this paper are standard and are referred from textbooks such as [12] [11] [13].

**Graph Notations.** Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and edge of $G$, respectively. Throughout the paper we use $n$ and $m$ to denote $|V(G)|$ and $|E(G)|$, respectively. For a vertex $v$, we use $N_G(v)$ to denote the set of its neighbors and $N_G[v]$ is defined to be $N_G(v) \cup \{v\}$. For any graph $G$ and a set of vertices $M \subseteq V(G)$, we denote the subgraph of $G$ induced by $M$ by $G[M]$. We let $G - M$, for a $M \subseteq V(G)$, denote $G[V(G) \setminus M]$. Most of the symbols and notations used for graph theoretical concepts are standard and taken from [12].

**Parameterized Complexity and Algorithms.** The goal of parameterized complexity is to find ways of solving NP-hard problems more efficiently than brute force: here the aim is to restrict the combinatorial explosion to a parameter that is hopefully much smaller than the input size. Formally, a parameterization of a problem is assigning a positive integer parameter $k$ to each input instance and we say that a parameterized problem is fixed-parameter tractable (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and $f$ is an arbitrary computable function that depends only on the parameter $k$. We use $O^{*}(f(k))$ to denote the running time of such an algorithm. Such an algorithm is called an FPT algorithm and such a running time is called FPT running time. There is also an accompanying theory of parameterized intractability using which one can identify parameterized problems that are unlikely to admit FPT algorithms. These are essentially proved by showing that the problem is W[3]-hard. We refer the interested readers to books such as [11] [18] for an introduction to the theory of parameterized algorithms.

1.2 Related Work and Motivation

DomCol was introduced by Gera et al. in 2006 [21]. This paper proved that DomCol is NP-hard (even for a fixed $\ell \geq 4$) by a simple reduction from 3-Graph Coloring – given a graph $G$ construct a graph $G'$ by adding an extra vertex to $G$ which is adjacent to all vertices in $V(G)$. Then $G'$ has a dominator coloring of size 4 if, and only if, $G$ is 3-colorable.
Unlike Graph Coloring, DomCol can be solved in polytime when \( \ell = 3 \) [5]. The problem, which marries two of the most well-studied problems in graph theory – Graph Coloring and Dominating Set, has been studied in several papers in the last 15 years. Results in these papers can be broadly categorized into two.

First, there have been several crucial results which establish lower and upper bounds on the size of an optimal dominator coloring of graphs belonging to special graph classes. For example, refer papers [20, 1, 27, 7] and the following survey [24]. The second (seemingly more sparse) are algorithmic results on DomCol. Even for simple graph classes such as trees, algorithmic results have been hard to obtain – indeed, after Gera et al. showed that DomCol can be solved in constant-time for paths in [21], it took close to a decade and incremental works [5, 6] before a polynomial time algorithm was developed for trees in [26]! It is still unknown if DomCol restricted to forests is polynomial time solvable. This inherent hardness of DomCol led to work on its parameterized complexity.

The parameterized complexity of the problem was first explored by Arumugam et al. in 2011 [1]. The authors expressed DomCol in Monodic Second-Order Logic (MSOL) and used a theorem due to Courcelle and Mosbah [8] to prove that DomCol parameterized by \((t, \ell)\), where \(t\) is the treewidth of the input graph, is FPT. Their expression of DomCol in MSOL immediately also shows (by [9]) that DomCol parameterized by \((w, \ell)\), where \(w\) is the clique-width of the input graph, is FPT. However, the problem has remained unexplored when viewed through the lens of structural parameters only – that is, in the case when the only restrictions are on the structure of the graph. Structural parameters are parameters that measure the distance (commonly vertex deletion) from a tractable graph class. They have become increasing popular in the world of parameterized algorithms since they are usually small in practice. We refer the interested reader to the following survey by Fellows et al. for an overview of structural parameterization [15] and [25, 22] for its use in studying Graph Coloring and Dominating Set. Our paper initiates the study of structural parameterization of DomCol. We modify the notation first used by [4] to talk about the structural parameterization of vertex coloring problems in our context.

| DomCol-M \( \mathcal{F} \) | Parameter: \( k \) |
|-----------------------------|-------------------|
| **Input:** A graph \( G \), a set of vertices \( M \) of cardinality \( k \) such that \( G - M \in \mathcal{F} \); an integer \( \ell \) |
| **Question:** Does there exist a dominator coloring \( \chi_d \) of \( G \) with \( |\chi_d| \leq \ell \)? |

### 1.3 Our Results

As a first observation, we note that Dominator Coloring parameterized by the size of the input graph’s vertex cover (an extremely popular structural parameter) is FPT. We use the following result (which was stated in preceding paragraph) of [1]: DomCol parameterized by \((t, \ell)\) is FPT. Let \( M \) be a vertex cover of size \( k \) of the input graph \( G \). First, note that every isolated vertex of \( G \) must get its own color in a valid dominator coloring of \( G \). Thus, removing all isolated vertices from \( G \) and reducing \( \ell \) by the number of such vertices gives us an equivalent instance of the problem. If \( \ell > k \) in a connected graph \( G \) then this is a yes instance of the problem [21, 24]. Moreover, since \( t \leq k \) [25, 11], by the result referred to above, we have an algorithm which solves DomCol in \( f(k) \cdot |V(G)|^{O(1)} \)-time for some computable \( f \). This proves that DomCol parameterized by the vertex cover number is FPT.

While the above result is encouraging, vertex cover is a very restrictive parameter. For example, even for simple graphs such as cliques, this parameter is not all that useful. However,
it is easy to see that DomCol is trivial to solve when \( G \) is a clique – report yes if \( \ell \geq |V(G)| \) and no otherwise! This small observation guides us to the parameter of importance in this paper – the Cluster Vertex Deletion (CVD) number. A CVD set of a graph is a subset of its vertices whose deletion results in a cluster graph (where every component is a clique). We prove that DomCol is FPT when parameterized by this parameter. Since this algorithm’s runtime is large, we look for faster algorithms for larger parameters.

In this regard, we develop FPT algorithms for two well-studied parameters. First, we obtain a simpler algorithm when DomCol is parameterized by the size of a graph’s Twin Cover – a larger parameter than the CVD number but a smaller one than the vertex cover number. We believe that our method to produce this algorithm, which uses a relationship between DomCol and List Coloring that we establish, is of independent interest and can be used to prove the fixed-parameter (in)tractability of DomCol parameterized by other parameters.

Our second result is a (faster) randomized algorithm for DomCol parameterized by the size of a Clique Modulator, which is a subset of the vertex set whose deletion results in a clique. This algorithm uses an inclusion-exclusion based polynomial sieving technique introduced in a recent paper \[23\] in addition to an exact dynamic programming single-exponential algorithm to solve DomCol that we develop. We believe that our modification of the results in \[23\] is encouraging – can this powerful technique be used in other graph problems? Finally, we prove some loose lower bounds for DomCol with respect to these parameters.

1.4 Organization of the Paper

We begin by presenting our randomized \( O^*(c^k) \) algorithm for DomCol when \( k \) is the size of a clique modulator and \( c \) is some constant in Section 2. Next, in Subsection 3.1 we establish a relationship between List Coloring and DomCol using Theorem 3.2 and Lemma 3.3. We showcase the use of this construct in the following section where we prove that DomCol parameterized by the size of twin cover admits a \( O^*(2^{O(k \log k)}) \) algorithm. We look at the smallest parameter, the size of CVD set, in Section 4 and prove that it is FPT by obtaining a \( O^*(2^{O(2^k)}) \) algorithm for this parameter. Finally, in Section 5 we establish some lower bounds for DomCol.

2 Parameterized by Clique Modulator Size

Consider a graph \( G \). A subset \( M \) of \( V(G) \) is a Clique Modulator if \( G - M \) is a clique. The set \( M \) is also called a modulator. In this section, we will focus on the DomCol-M_\( \mathcal{F} \) problem where \( \mathcal{F} \) is the collection of complete graphs. We denote this problem by DomCol-CLQ and let \( (G, M, \ell) \) denote an instance of it. We also use \( Q \) to denote the clique \( G - M \) in such an instance. The goal of this section is to prove the following theorem.

\begin{theorem}
There exists a randomized algorithm which solves DomCol-CLQ in \( O^*(c^k) \)-time for some constant \( c \).
\end{theorem}

To prove this theorem, we first present a simple dynamic programming algorithm for DomCol which solves an instance of the problem in \( O(6^n) \)-time. Let \( (G, \ell) \) be an instance of DomCol. We design a dynamic programming algorithm DomCol_DP with input parameters \( (G, V_1, V_2) \) where \( V_1 \) and \( V_2 \) are subsets of \( V(G) \). Given this input, DomCol_DP will compute the minimum number of colors required to color \( G[V_1] \) such that, for all \( v \in V_2 \), there exists a color class \( c \) in this coloring that \( v \) dominates. To solve the instance \( (G, \ell) \) of DomCol, we
need to pass the triple \((G, V(G), V(G))\) to \(\text{DomCol}_{\text{DP}}\) as input and check if the resulting dominator coloring is of size at most \(\ell\).

For an instance \((G, V_1, V_2)\), let \(I(V_1)\) denote the set of non-empty independent sets of \(V_1\) and \(V_2(I)\), for an \(I \in I(V_1)\), denote the set \(\{v \in V_2 \mid N_G[v] \supseteq I\}\). \(\text{DomCol}_{\text{DP}}\) solves \((G, V_1, V_2)\), for \(V_1 \neq \emptyset\), by returning the following value:

\[
\text{DomCol}_{\text{DP}}(G, V_1, V_2) = 1 + \min_{I \in I(V_1)} \text{DomCol}_{\text{DP}}(G, V_1 \setminus I, V_2 \setminus V_2(I))
\]

For \(V_1 = \emptyset\), \(\text{DomCol}_{\text{DP}}\) returns 0 if \(V_2 = \emptyset\) and \(+\infty\) otherwise. The proof of correctness of \(\text{DomCol}_{\text{DP}}\) and its running time analysis is shown below.

\[\textbf{Theorem 2.2.}\] \(\text{DomCol}\) can be solved in \(O(6^n)\)-time.

\[\textbf{Proof.}\] The correctness of \(\text{DomCol}_{\text{DP}}\) follows from the simple observation that all vertices in an independent set can be colored with the same color. Since we are minimizing the value calculated over all independent sets of \(V_1\), \(\text{DomCol}_{\text{DP}}\) correctly computes the minimum number of colors required to color \(V_1\) such that all vertices in \(V_2\) dominate a color class. Since there are at most \(2^{|V_1|}\)-many artificial colors. If \(\text{DomCol}_{\text{DP}}(G, V_1, V_2)\) that are called, \(\text{DomCol}_{\text{DP}}\) returns 0 if \(V_2 = \emptyset\) and \(+\infty\) otherwise. The proof of correctness of \(\text{DomCol}_{\text{DP}}\) and its running time analysis is shown below.

We expect a better analysis or a more involved algorithm to give a better running time bound and leave this as an interesting open question for further research. For now, our interest was to show that \(\text{DomCol}\) can be solved in a \(O(c^n)\)-time for some constant \(c\).

Let \((G, M, \ell)\) be an instance of \(\text{DomCol}_{\text{CLQ}}\) where \(Q = G - M\) is a clique of size at most \(k\). Since \(|M| = k\), \(|V(G)| \leq 2k\) and therefore, by the proof of \[\text{Theorem 2.2}\], \((G, M, \ell)\) can be solved in \(O(36^k)\)-time. Now, consider the case where the size of clique is at least \(k + 1\). Then, in any proper coloring of \(G\), there exists a color which is used (exactly once) in \(Q\) but not in \(M\). An important observation ensues.

\[\textbf{Observation 2.3.}\] Let \((G, M, \ell)\) be an instance of \(\text{DomCol}_{\text{CLQ}}\) such that \(|V(Q)| > k\). Then, every vertex in \(Q\) dominates a color class in any proper coloring of \(G\).

Our algorithm follows a similar randomized strategy using polynomial sieving and Schwartz-Zippel Lemma, as that of [23] for list coloring parameterized by clique modulator. In [23], each \(v \in M\) is associated with a variable \(x_v\). The authors design a \textit{weighted Edmonds matrix} \(A\) whose entries are polynomials over a large field of characteristic 2 such that \(G\) admits a list coloring if, and only if, \(\prod_{v \in M} x_v\) divides \(\det A\). To ensure that each vertex in the modulator dominates a color class (domination for the clique vertices is ensured by \[\text{Observation 2.3}\]), we introduce another variable \(y_v\) corresponding to each \(v \in M\) (and modify \(A\) slightly) whose existence in \(\det A\) will mean that \(v\) dominates some color class in this coloring. Hence, in addition to the \(\prod_{v \in M} x_v\) dividing the determinant, we require that \(\prod_{v \in M} y_v\) divides \(\det A\). We describe the construction below.

Let \(C = \{c_1, c_2, \ldots, c_\ell\}\) denote a set of \(\ell\)-many colors. Moreover, let \(C' = \{c'_v \mid v \in M\}\) be a set of \(k\)-many artificial colors. If \(|V(G)| > |C \cup C'|\), this is a \textit{NO} instance of \(\text{DomCol}_{\text{CLQ}}\) as vertices in \(Q\) must get different colors. Pad \(V(G)\) with \((|C \cup C'| - |V(G)|)\)-many artificial vertices. Let the set of these vertices be \(V'(G)\). We construct a bipartite graph \(B\) with bipartition \((V(G) \cup V'(G), C \cup C')\) by defining its edges as follows. Every vertex in \(V'(G)\) is connected all vertices in \(C\). In addition, \(v \in M\) is also connected to \(c'_v \in C'\). Finally, every vertex in \(V'(G)\) is connected to all vertices in \(C \cup C'\). Each edge \((v, c) \in E(B)\) is associated with a \(S(\nu, c) \subseteq P(M^2)\) by the following relation:
If \( (v, c) \in V(Q) \times C \), \( S_{(v, c)} \) is the collection of \( S = (S_1, S_2) \subseteq M^2 \) where \( S_1 \cup \{v\} \) is an independent set of \( G \) and \( S_2 = \{u \in M \mid NG[u] \supseteq S_1 \cup \{v\} \} \).

If \( (v, c) \in M \times C \), \( S_{(v, c)} \) is the collection of \( S = (S_1, S_2) \subseteq M^2 \) where \( S_1 \) is an independent set of \( G \) which contains \( v \) and \( S_2 = \{u \in M \mid NG[u] \supseteq S_1 \} \).

If \( v \) or \( c \) is an artificial vertex or color, \( S_{(v, c)} = \{0 \times 0\} \).

We now define a matrix \( A \) with dimensions \( |V(G) \cup V'(G)| \times |C \cup C'| \), with rows labeled by \( V(G) \cup V'(G) \) and columns by \( C \cup C' \), whose entries are polynomials over a large field of characteristic 2. Let \( X = \{x_v \mid v \in M\} \) and \( Y = \{y_c \mid v \in M\} \) be sets of variables indexed by vertices in \( M \) and \( Z = \{(z_{(v, c)} \mid (v, c) \in E(B)) \) be a set of variables indexed by edges in \( E(B) \). For each \( (v, c) \in E(B) \), we define a polynomial \( P(v, c) \) as follows: \( P(v, c) = \sum_{u \in S_{(v, c)}} (\prod_{x \in S_1} x, \prod_{y \in S_2} y) \). Here, we assume that the empty product equals 1. We let \( A(v, c) \), for a \( (v, c) \in E(B) \), be \( z_{(v, c)} \cdot P(v, c) \). All other entries of this matrix are 0. We now prove a crucial theorem which has a similar flavor as Lemma 3.2 of [23].

**Theorem 2.4.** \((G, M, \ell)\) is a yes instance of DomCol-CLQ if, and only if, det \( A \) contains a monomial divisible by \( \prod_{x \in X} x \cdot \prod_{y \in Y} y \).

**Proof.** As noted in [23], no cancellation happens in det \( A \). Moreover, a perfect matching \( M \) of \( B \) contributes \( \sigma(M) \cdot \sum_{(v, c) \in M} z_{(v, c)} \cdot P(v, c) \) where \( \sigma \) is the sign of \( M \) (when it is considered as a permutation).

Assume that \((G, M, \ell)\) is a yes instance of DomCol-CLQ. Then, there exists a dominator coloring \( \chi_d \) of \( G \) with \( |\chi_d| \leq \ell \). Assume, without loss in generality, that \( C \) is the codomain of \( \chi_d \). Order the vertices in \( V(Q) \) arbitrarily and let this order be \( \{v_1, v_2, \ldots, v_n\} \). Also order the vertices in \( M \) as \( \{v_{n-k+1}, v_{n-k+2}, \ldots, v_n\} \). We define a perfect matching \( M \) of \( B \) using \( \chi_d \) by scanning the vertices of \( G \) using this ordering as follows:

- If \( v_1 \in V(Q) \), let \( M(v_1) = \chi_d(v_1) \).
- If \( v_1 \in M \) and no \( v_j \) with \( j < i \) has been matched with \( \chi_d(v_i) \), let \( M(v_1) = \chi_d(v_i) \).
- If \( v_i \in M \) and there is a \( v_j \) with \( j < i \) matched with \( \chi_d(v_i) \), let \( M(v_i) = c_{v_j} \).
- If \( v_i \notin V(G) \), let \( M(v_i) \) be an arbitrary unmatched color.

Note that this is indeed a perfect matching of \( B \). For a \( c \in C \), let \( M_c \subseteq M \) be the subset of vertices that are colored \( c \) by \( \chi_d \) and \( M_c^{-1} \subseteq M \) be the subset of vertices that dominate the color class \( c \) in \( \chi_d \). Clearly, for a \( (v, c) \in M \) with \( c \in C \), \( (M_c, M_c^{-1}) \in S_{(v, c)} \).

Let \( p_1(v, c) = \prod_{u \in M_c} x_u \) and \( p_2(v, c) = \prod_{u \in M_c^{-1}} y_u \). By construction, \( p_1(v, c) \cdot p_2(v, c) \) is a term of \( P(v, c) \). Therefore,

\[
\alpha \cdot \sigma(M) \cdot \prod_{(v, c) \in M} z_{(v, c)} \cdot p_1(v, c) \cdot p_2(v, c)
\]

is a monomial of det \( A \) for some \( \alpha > 0 \). To complete our proof, we show that \( \prod_{x \in X} x \) divides \( \prod_{(v, c) \in M} p_1(v, c) \) and that \( \prod_{y \in Y} y \) divides \( \prod_{(v, c) \in M} p_2(v, c) \). Fix a \( u \in M \). Let \( c = \chi_d(u) \) and \( \overline{u} \) be a color class that \( u \) dominates in \( \chi_d \). Let \( v = u \) if \( u \in \chi_d(V(Q)) \) and if \( u \in \chi_d(V(Q)) \), let \( v \) be the vertex in \( Q \) that is colored \( c \). By construction, \( (v, c) \in M \) and \( x_u \in p_1(v, c) \).

Moreover, \( \overline{v} \) is matched with a vertex in \( V(G) \), say \( \overline{v} \). Then, \( y_u = p_2(\overline{v}, \overline{v}) \). Thus, each \( x \in X \) and \( y \in Y \) appear in some \( p_1(v, c) \) and \( p_2(v, c) \) respectively. This proves our claim.

Now, assume that det \( A \) contains a monomial \( T \) divisible by \( \prod_{x \in X} x \cdot \prod_{y \in Y} y \). Then, there exists a perfect matching \( M \) of \( B \) which corresponds to this monomial [23]. We can assume that \( T = \alpha \cdot \prod_{(v, c) \in M} z_{(v, c)} p(v, c) \) where \( \alpha \) is some constant and \( p(v, c) \) is a term of \( P(v, c) \) for every \( (v, c) \in M \). We define a mapping \( \chi_d : V(G) \mapsto C \) from this mapping and prove that it is a dominator coloring of \( G \). For \( v \in V(Q) \), define \( \chi_d(v) = M(v) \). For a
v ∈ M we know that x_v must appear as a term of p(v', c) for some (v', c) ∈ M. Construct a bipartite graph with bipartition (M, C) where (v, c) is an edge if x_v appears in p(v', c) for some v' ∈ V(G). Find a matching saturating C in this graph and assign c ∈ C to its matched vertex in M. Color the rest of the as follows: if there exists such an edge (v', c) with v' ∈ V(Q), let χ_d(v) = c and if not, let χ_d(v) = c for some edge (v', c) where v' ∈ M.

First, we prove that χ_d is a proper coloring. Let u and v be two vertices such that c = χ_d(u) = χ_d(v). If both u and v are vertices in M, then x_u and x_v are terms of p(v', c) for some v' ∈ V(G). By construction, u and v are part of an independent set. Now, assume that u ∈ V(Q). Then, v ∈ M as otherwise M would contain two edges incident on c. Hence x_v is in the term p(u, c) which implies that (u, v) /∈ G(V) by construction. This proves that χ_d is a proper coloring.

We complete the proof of this theorem by showing that every vertex of V(G) dominates a color class in χ_d. Since χ_d is a proper coloring of V(G) and |V(Q)| > k, by Observation 2.3 every vertex in V(Q) dominates a color class. Now, consider a vertex v ∈ M. Then, y_v is a term in p(v', c) for some (v', c) ∈ M. Moreover, by construction, v' ∈ V(G) and c ∈ C. Hence, for all u ∈ χ_d^{-1}(c), x_u ∈ p(v', c). Therefore, χ_d^{-1}(c) ⊆ N_G[v]. Since c is used to color some vertex in V(G) due to our matching trick, χ_d^{-1}(c) ≠ ∅. This proves that v dominates the color class c.

We now use the following results from [23] and [29, 28] respectively.

**Theorem 2.5.** Let P(x_1, x_2, ..., x_n) be a polynomial over a field of characteristic two and J ⊆ {1, 2, ..., n}. Then, there exists a polynomial Q(x_1, x_2, ..., x_n) which can be computed in O(2^|J|)-time such that Q ≠ 0 if, and only if, P contains a term divisible by \(\prod_{j \in J} x_j\).

**Theorem 2.6.** Let P(x_1, x_2, ..., x_n) be a non-zero polynomial over a finite field \(\mathbb{F}\) with maximum degree d. Pick \(\{r_1, r_2, ..., r_n\}\) randomly from \(\mathbb{F}\). Then, \(\Pr[P(r_1, r_2, ..., r_n) = 0] \leq \frac{d}{|\mathbb{F}|} \).

By applying Theorem 2.5 for the \(O^*(2^{2k})\)-many evaluations of \(\det A\) and then applying Theorem 2.6 we have a randomized algorithm which runs in \(O^*(16^k)\)-time to solve DomCol-CLQ when restricted to instances where the size of the clique is greater than k. With a more complicated polynomial sieving method as in [24], we can improve this to an algorithm which runs in \(O^*(4^k)\)-time. Combining this result with the discussion preceding Observation 2.3 gives us the proof of Theorem 2.1.

### 3 Parameterized by Twin Cover Size

Consider a graph G. A subset M of V(G) is a Twin Cover if for all \((u, v) \in E(G)\) either (i) \(u \in M\) or \(v \in M\) or (ii) \(N_G[u] = N_G[v]\). This parameter was introduced by Ganian in 2015 [19] and has been used extensively in the world of parameterized complexity since. In this section, we will focus on the DomCol-M problem where \(\mathcal{F} = \text{Cluster}\) and M is a twin cover of the input graph. We denote this restriction of DomCol-M by DomCol-TM. Let \((G, M, k, \ell)\) denote an instance of DomCol-TM. Let \(Q = \{Q_i\}_{i=1}^\ell\) denote the components of the cluster graph G − M in such an instance and \(V_Q\) denote its vertices. The goal of this section is to prove the following theorem.

**Theorem 3.1.** DomCol-TM is FPT. There exists an algorithm that solves DomCol-TM in \(O^*(2^{O(k \log k)})\)-time.

In the subsection that follows, we design a general construct that, we believe, will be useful in proving the (in)tractability of DomCol parameterized by other structural parameters. We use this construct in Subsection 3.2 to prove Theorem 3.1.
3.1 Relating Dominator Coloring and List Coloring

A partial coloring \( \tilde{\chi} : S \mapsto \tilde{C} \) on \( S \subseteq V(G) \) of a graph \( G \) is a proper coloring of \( G[S] \). We define a partial dominator coloring as a triplet \((S, \tilde{\chi}, \delta)\) where \( S \subseteq V(G) \), \( \delta : V(G) \mapsto \tilde{C} \), and \( \tilde{\chi} : S \mapsto \tilde{C} \) is a partial coloring on \( S \) such that for all \( v \in V(G) \), \( v \) dominates the color class \( \delta(v) \in \tilde{C} \). Throughout the section, we denote such a partial dominator coloring by \( \chi^S_d \). A dominator coloring \( \chi_d \) of \( G \) is an extension of a partial dominator coloring \( \chi^S_d \), if \( \chi_d \) agrees with \( \chi^S_d \) on \( S \), and for all \( v \in V(G) \), \( v \) dominates \( \delta(v) \) with respect to \( \chi_d \). Analogous to the well-studied Pre-Coloring Extension problem (first introduced in [3]), where we must decide if there exists a coloring of size \( \ell \) which is an extension of a given partial coloring, we define \( \text{DomCol}-M_{\mathcal{F}}-\text{Ext} \).

Another well known problem related to Graph Coloring is List Coloring. Let \( \mathcal{P}(S) \), for any set \( S \), denote the power set of \( S \). Let \( G \) be a graph and \( C = \{c_1, c_2, \ldots, c_l\} \). In addition to these two input parameters, List Coloring has an additional parameter: a list function \( L : V(G) \mapsto \mathcal{P}(C) \) and asks the following question - does there exist a proper coloring \( \chi \) of \( G \) such that \( \chi(v) \in L(v) \) for all \( v \in V(G) \)? For more information on the (parameterized) complexity of List Coloring, we refer the reader to [10] and the references therein.

Consider an instance \((G, M, k, \chi^S_d, \ell)\) of \( \text{DomCol}-M_{\mathcal{F}}-\text{Ext} \) and let \( H = G - M \in \mathcal{F} \). Moreover, assume that List Coloring can be solved in polynomial time when restricted to \( \mathcal{F} \). We can carefully choose a list function \( L \) on \( V(H) \) so that a proper coloring of \( H \) which respects \( L \) can be used to color \( V(G) \setminus S \subseteq V(H) \) and the coloring so obtained is an extension of \( \chi^S_d \). The detailed construction of \( L \) is given below as part of the proof of the theorem.

▶ Theorem 3.2. For any graph class \( \mathcal{F} \) for which List Coloring is polynomial time solvable, \( \text{DomCol}-M_{\mathcal{F}}-\text{Ext} \) is polynomial time solvable.

Proof. Let \( \mathcal{F} \) be a graph class for which List Coloring is polynomial time solvable. Let \((G, M, k, \chi^S_d, \ell)\) be an instance of \( \text{DomCol}-M_{\mathcal{F}}-\text{Ext} \). Then, \( H = G - M \in \mathcal{F} \). If \( C = \{c_1, c_2, \ldots, c_l\} \) (for some \( \kappa \leq \ell \)) is the range of \( \chi^S_d \). We now create a List Coloring instance \((H, L, \ell)\) by defining the list function \( L \). For all \( v \in V(H) \cap S \), we let \( L(v) = \{\chi^S_d(v)\} \). For a color \( c \in \text{Im}(\delta) \), let \( V^c_d \) denote \( \delta^{-1}(c) \), the set of vertices that dominate the color \( c \) with respect to \( \delta \). Define \( N_G[V^c_d] = \cap_{v \in V^c_d} N_G[v] \). Observe that \( N_G[V^c_d] \) is exactly the set of vertices with the following property – if any vertex in \( N_G[V^c_d] \) is colored with \( c \) then all the vertices \( V^c_d \) will continue dominating the color class \( c \). For ease of notation, for a \( c \notin \text{Im}(\delta) \), we let \( N_G[V^c_d] = V(G) \).

For all \( v \in V(H) \setminus S \), let \( L(v) \) include the colors \( \{c_{k+1}, c_{k+2}, \ldots, c_{l}\} \) in \( L(v) \). For any color \( c \in \{c_1, c_2, \ldots, c_k\} \) include \( c \) in \( L(v) \) for \( v \in V(H) \setminus S \) if, and only if, \( c \notin \chi_d(N_G[v]) \) and \( v \in N_G[V^c_d] \).

Claim. If \((H, L, \ell)\) is an instance of List Coloring, \((G, M, k, \chi^S_d, \ell)\) is a yes instance of \( \text{DomCol}-M_{\mathcal{F}}-\text{Ext} \).
Proof. Since \((H, L, \ell)\) is a yes instance, there exists a feasible list coloring \(\chi_L\) of \(H\). We construct a coloring \(\chi_d\) of \(G\) as follows:

\[
\chi_d(v) = \begin{cases} 
\chi^S_d(v) & \text{if } v \in S \\
\chi_L(v) & \text{if } v \in V(H)
\end{cases}
\]

Note that, for all \(v \in S \cap V(H)\), \(\chi_L(v) = \chi^S_d(v)\) since \(L(v) = \{\chi^S_d(v)\}\). Hence, \(\chi_d\) is indeed well-defined. Moreover, \(|\chi_d| \leq \ell\). We prove that \(\chi_d\) is an extension of \(\chi^S_d\). Observe that \(\chi_d\) induces a proper coloring of \(H\) and \(S\). By construction, for \(v \in V(H) \setminus S\), \(c \notin L(v)\) if \(c\) is used to color a vertex in \(N_G[v]\). Thus \(\chi_d\) is a proper coloring. Next we prove that any vertex \(v\) dominates the color class \(\delta(v)\) in \(\chi_d\). Observe by construction the color \(c = \delta(v)\) is only present in the list corresponding to a vertex \(u\) if, and only if, \(u \in N_G[V^e]\) where \(N_G[V^e] \subseteq N_G[v]\). Therefore the claim holds.

\textbf{Claim.} \((H, L, \ell)\) is a yes instance of \textsc{List Coloring} if \((G, M, k, \chi^S_d, \ell)\) is a yes instance of \textsc{DomCol-M}_{\mathcal{F}}-\text{Ext}.

Proof. Since \((G, M, k, \chi^S_d, \ell)\) is a yes instance, there exists an extension \(\chi_d\) of \(\chi^S_d\) with \(|\chi_d| \leq \ell\). We prove that the restriction of \(\chi_d\) on \(H\), which we call \(\chi_d\), is a list coloring of \(H\) which respects the list function \(L\). Let \(v \in V(H)\). Since \(\chi_d\) is a proper coloring of \(G\), \(\chi_d(v)\) is not used to color any vertices in \(N_G[v]\). If \(\chi_d(v) \notin \text{Im}(\delta)\), then \(v \in N_G[V^e]\). Therefore, \(\chi_d(v) \in L(v)\). On the other hand, if \(\chi_d(v) \notin \text{Im}(\delta)\), \(\chi_d(v) \in L(v)\) by definition. This proves that \(\chi_d\) is a proper coloring of \(H\) which respects the list function \(L\) with \(|\chi_d| \leq \ell\). Therefore, \((H, L, \ell)\) is a yes instance of \textsc{List Coloring}.

By these two claims, we have that \((G, M, k, \chi^S_d, \ell)\) is a yes instance of \textsc{DomCol-M}_{\mathcal{F}}-\text{Ext} if, and only if, \((H, \ell)\) is a yes instance of \textsc{List Coloring}. Since \(H \in \mathcal{F}\), \((H, \ell)\), and hence \((G, M, k, \chi^S_d, \ell)\), can be solved in polynomial time.

For a family \(\mathcal{F}\) where \textsc{List Coloring} is polynomial time solvable, while \textbf{Theorem 3.2} implies that \textsc{DomCol-M}_{\mathcal{F}}-\text{Ext} is in \(\mathcal{P}\), no such assertion can be made about \textsc{DomCol-M}_{\mathcal{F}}.

Assume that we have a collection \(\Gamma(\mathbb{I})\) of partial dominator colorings for each instance \(\mathbb{I} = (G, M, k, \ell)\) with the following property: \((G, M, k, \ell)\) is a yes instance of \textsc{DomCol-M}_{\mathcal{F}} if, and only if, there exists a \(\chi^S_d \in \Gamma(\mathbb{I})\) such that \((G, M, k, \chi^S_d, \ell)\) is a yes instance of \textsc{DomCol-M}_{\mathcal{F}}-\text{Ext}. Moreover, assume that \(|\Gamma(\mathbb{I})| \leq f(k)\) for some computable function \(f\) which only depends on \(k\) and that this collection can be constructed in \(O^\ast(f(k))\)-time. Since for each \(\chi^S_d \in \Gamma(\mathbb{I})\) we can correctly decide if \((G, M, k, \chi^S_d, \ell)\) is yes or no in polynomial time (by \textbf{Theorem 3.2}, in \(O^\ast(f(k))\)-time, we can decide correctly if \((G, M, k, \ell)\) is yes or no. In other words, if we can construct such a collection of partial dominator colorings \(\Gamma(\mathbb{I})\) for each instance \(\mathbb{I}\) of \textsc{DomCol-M}_{\mathcal{F}}, then \textsc{DomCol-M}_{\mathcal{F}}\) is FPT.

\textbf{Lemma 3.3.} Let \(\mathcal{F}\) be a graph family for which \textsc{List Coloring} is polynomial time solvable. Moreover, assume that for each instance \(\mathbb{I} = (G, M, k, \ell)\) of \textsc{DomCol-M}_{\mathcal{F}}, there exists a collection \(\Gamma(\mathbb{I})\), with \(|\Gamma(\mathbb{I})| \leq f(k)\) which can be constructed in \(O^\ast(f(k))\)-time for some computable function \(f\), of partial dominator colorings such that \(\mathbb{I}\) is a yes instance if, and only if, there exists a \(\chi^S_d \in \Gamma(\mathbb{I})\) such that \((G, M, k, \chi^S_d, \ell)\) is a yes instance of \textsc{DomCol-M}_{\mathcal{F}}-\text{Ext}. Then, there exists an algorithm that runs in \(O^\ast(f(k))\)-time to solve \textsc{DomCol-M}_{\mathcal{F}}.

In this paper, we are concerned with \(\mathcal{F} \subseteq \text{Cluster}\). Therefore, to use \textbf{Lemma 3.3} we must prove that \textsc{List Coloring} can be solved in polynomial time when restricted to \text{Cluster}. Thankfully, a simple matching trick proves this to be true.
**Lemma 3.4.** List Coloring, restricted to cluster graphs, is polynomial time solvable.

**Proof.** Let \((H, L, \ell)\) be an instance of List Coloring where \(H\) is a cluster graph and \(Q = \{Q_i\}_{i=1}^q\) are the connected components of \(H\). Note that each \(Q_i \in Q\) is a clique. Let \(C = \{c_1, c_2, \ldots, c_l\}\). We create a bipartite graph \(B_i\) on \((C, Q_i)\) as follows: \((c, v) \in E(B_i)\) if, and only if, \(c \in L(v)\). We return yes if, and only if, there exists a matching saturating \(Q_i\) in \(B_i\) for all \(i \in \{1, 2, \ldots, q\}\). Refer to Figure 1 for an illustration of this construction.

If \((H, L, \ell)\) is a yes instance, then, since each \(Q_i \in Q\) is a clique, a coloring that respects the list function \(L\) and uses at most \(\ell\)-many colors induces a matching saturating \(Q_i\) in \(B_i\). Therefore, our procedure will report yes. On the other hand, if our procedure returns a yes, the coloring obtained by assigning the color that each vertex is matched to is clearly one which uses at most \(\ell\)-many colors and respects \(L\). Thus, in \(O(n^3)\)-time, we can solve List Coloring when restricted to cluster graphs.

As a warm up before its use in Subsection 3.2, we first prove that DomCol can be solved in polytime when restricted to cluster graphs using Lemma 3.3. Note that the proof of the above fact from first principles is not too difficult as well. In this regard, we require two small observations.

**Observation 3.5.** Let \(\chi_d\) be a dominator coloring of a graph \(G\) and \(x, y \in V(G)\) such that \(N_G[x] = N_G[y]\). Consider \(\chi_d\) which swaps the colors of \(x\) and \(y\) and agrees with \(\chi_d\) on all other values. Then, \(\chi_d\) is a dominator coloring of \(G\) with \(|\chi_d| = |\chi_d|\).

**Proof.** Since \(\chi_d^S\) agrees with the proper coloring \(\chi_M\) on \(M\), no two adjacent vertices in \(M\) are colored with the same color by \(\chi_d^S\). Note that \(\text{Im}(\chi_d^S) \setminus \text{Im}(\chi_M) = C_D \cup C_Q^1\) and \(\chi_d^S\) uses these colors exactly once while coloring. Moreover, \(\chi_d^S\) does not use any color in \(\text{Im}(\chi_M)\) outside \(M\). Thus, \(\chi_d\) is a partial coloring of \(M\). By construction, every vertex in \(V(G)\) dominates the color class \(\delta(v)\). Therefore, \(\chi_d^S\) is a partial dominator coloring of \(G\).

**Observation 3.6.** Let \(G\) be a cluster graph and \(Q = \{Q_i\}_{i=1}^q\) denote its connected components. Let \(v \in V(Q_i)\) for some \(Q_i \in Q\). Then, \(v\) cannot dominate a color class which is used to color a vertex in \(V(Q_j)\) for some \(i \neq j\) in any dominator coloring of \(G\).

**Lemma 3.7.** DomCol, restricted to cluster graphs, is polynomial time solvable.

**Proof.** Let \(I = (G, \ell)\) be an instance of DomCol where \(G\) is a cluster graph. Let \(I(G) = \{\chi_i^S\}\) where \(\chi_i^S\) is a partial dominator coloring obtained by uniquely coloring an arbitrary vertex in each clique and assigning this color as the one to dominate for all vertices in that clique. The correctness of this construction follows from Observations 3.5 and 3.6. The lemma now follows from Lemma 3.4 and Lemma 3.3.
3.2 DomCol-TC is Fixed-Parameter Tractable

To prove Theorem 3.1 we must construct (by Lemma 3.3) for each instance \( I = (G, M, k, \ell) \) of DomCol-TC, a collection of partial dominator colorings \( \Gamma(I) \) which have properties listed in the lemma (replace \( f(k) \) with \( 2^{\mathcal{O}(k \log k)} \)). For this purpose, we require the following lemma due to Ganian [19].

**Lemma 3.8.** Let \( G \) be a graph. \( M \subseteq V(G) \) is a twin cover of \( G \) if, and only if, \( G - M \) is a cluster graph and, if \( Q = \{ Q_i \}_{i=1}^{q} \) are its connected components, for any \( Q_i \in Q \) and any \( u, v \in V(Q_i) \), \( N_G[u] = N_G[v] \).

![Figure 2](image-url) Illustration of construction of \( \Gamma(\chi_M, C^0, \delta_0) \).

Define \( C_i = \{ c_1, c_2, \ldots, c_i \} \). Let \( \chi_M : M \rightarrow C_k \) be any arbitrary proper coloring of \( M \), \( C^0 \) be any arbitrary subset of \( C_k \), and \( \delta_0 : M(\chi_M, C^0) \mapsto C_{2k} \) be an arbitrary function defined on \( M(\chi_M, C^0) \subseteq M \). We will present the definition of \( M(\chi_M, C^0, \delta_0) \) shortly. For each triple \((\chi_M, C^0, \delta_0)\), we define a partial dominator coloring \( \Gamma(\chi_M, C^0, \delta_0) \) as follows.

By Lemma 3.8 every vertex of a clique \( Q_i \in Q \) has the same neighborhood. Without slight abuse of notation, we use the term "the neighborhood of a clique" to mean the the neighborhood of its vertices in \( M \). Similarly, we say that a clique dominates a color class \( c \) if each of its vertices dominates the color class \( c \). Let \( Q_a \) be the set of cliques that dominate a color class in \( C^0 \) and \( Q_b = Q \setminus Q_a \). Without loss in generality, assume that \( Q_b = \{ Q_1, Q_2, \ldots, Q_b \} \). We arbitrarily color one vertex from \( Q_i \in Q_b \) with color \( c_{2k+i} \) (see Figure 2). Denote the set of vertices that are colored by \( V^1_b \) and the set of colors used by \( C^1_b \).

Now we define the domination function for the vertices in the cluster. For a \( v \in V(Q_i) \), where \( Q_i \in Q_b \), set \( \delta(v) = c_{2k+i} \). For any vertex \( v \in V(Q_i) \), where \( Q_i \in Q_a \), set \( \delta(v) = c_j \) where \( Q_i \) dominates the color class \( c_j \). Let \( M_a \subseteq M \) be the set of vertices which dominate a color class in \( C^0 \). For all \( v \in M_a \), set \( \delta(v) \) to be a color \( c_i \in C^0 \) that \( v \) dominates. Let \( M_b \subseteq M \setminus M_a \) be the set of vertices which neighbor one of the cliques in \( Q_b \). Set, for \( v \in M_b \), \( \delta(v) = c_{2k+i} \), for an arbitrary \( Q_i \in Q_b \) that neighbors \( v \). Let \( M_c = M \setminus (M_a \cup M_b) \).

We define \( M(\chi_M, C^0, \delta_0) = M_c \) and for any vertex \( v \in M_c \) we let \( \delta(v) = \delta_0(v) \). If, for some \( v \in M_c \), \( \delta(v) = c_{i} \) for some \( c_i \in \text{Im}(\chi_M) \), and \( v \) does not dominate \( c_i \), then set \( \Gamma(\chi_M, C^0, \delta_0) = \emptyset \). Let \( C_D = \{ c \in \delta(M_c) \mid \chi_M^{\delta_0}(c) = \emptyset \} \) and \( M_d \) be the set of vertices in \( M \) that dominates a color class in \( C_D \). Without loss in generality, assume that \( C_D = \{ c_{k+1}, c_{k+2}, \ldots, c_{k+2} \} \). In order to ensure that the colors in \( C_D \) is used at least once we do the following. Construct a bipartite graph \( B_D \) with bipartition \( (V_Q, C_D) \) where \((v, c_i) \in E(B_D) \) if, and only if, \( v \in \cap_{u \in \delta^{-1}(c_i)} N_G[u] \). Refer to Figure 3 for a pictorial
representation of the construction of \( B_D \). If there does not exist a matching saturating \( C_D \) in this graph, set \( \Gamma(\chi_M, C^0, \delta_0) = \emptyset \). Otherwise, select an arbitrary matching saturating \( C_D \) and use \( c_j \in C_D \) to color the vertex that it is matched to. Let the set of matched vertices from the cluster be \( V_D^2 \). Let \( \chi^S_D \) be the partial coloring of \( G \) that we have and \( S \) be its domain. We defer the routine checking that \( \chi^S_D \) is a partial dominator coloring of \( G \) to ?? in the appendix. Set \( \Gamma(\chi_M, C^0, \delta_0) = \{ \chi^S_D \} \). We let \( \Gamma(\emptyset) \) be the union of all \( \Gamma(\chi_M, C^0, \delta_0) \). Note that \( |\Gamma(\emptyset)| \in 2^{O(k \log k)} \) and that it can be constructed in \( O^*(2^{O(k \log k)}) \)-time.

![Figure 3](image)

**Figure 3** Illustration of construction of \( \Gamma(\chi_M, C^0, \delta_0) \).

Let \( \chi \) be any valid dominator coloring of \( G \) with \( \ell \) colors. We prove that there exist a triple \((\chi_M, C^0, \delta_0)\) such that \( \Gamma(\chi_M, C^0, \delta_0) \) can be extended to a proper dominator coloring with at most \( \ell \) colors. Up to renaming of colors, we can assume that \( \chi(M) \subseteq C_k \). Let \( \chi_M \) be that partial coloring of \( M \). Consider the set of colors \( C^0 \) in \( \chi \) which are only used in \( M \). At least one vertex in each clique which does not dominate a color in \( C^0 \) must be colored with a color which will not be used to color any other vertex in \( G \) (follows from Observation 3.5). Up to renaming of colors and from Observation 3.5, we can assume that these vertices form \( V_Q^1 \) and are assigned the colors \( C^1_Q \).

We construct a domination function of \( \chi \) as follows. For any vertex \( v \in V(Q_i) \), with \( Q_i \in Q_n \), set \( \delta(v) = c_j \in C^0 \) where \( Q_i \) dominates the color class \( c_j \). If \( Q_i \in Q_b \), set \( \delta(v) = c_{2k+1} \in C^1_Q \). If \( v \in M \) and \( v \) dominates a color in \( c_j \in C^0 \cup C^1_Q \), set \( \delta(v) = c_j \). Up to renaming colors, the rest of the vertices in \( M \) (which form the set \( M(\chi_M, C^0) = M_c \)) dominate a color class in \( C_{2k} \). Set \( \delta(v) \) to be one such color for each \( v \in M_c \). Clearly, the constructed function \( \delta \) is indeed a domination function of \( \chi \).

Let \( \delta_0 \) be the restriction of \( \delta \) on \( M(\chi_M, C^0) \). \( \Gamma(\chi_M, C^0, \delta_0) \) is non-empty since \( \chi \) induces a matching saturating \( C_D \) in \( B_D \). Moreover, it is clear that \( \Gamma(\chi_M, C^0, \delta_0) = \{ \chi^S_D \} \) - that is, the domination function of the partial dominator coloring so constructed is \( \delta \). We have shown that \( \chi \) agrees with \( \chi^S_D \) on all vertices except possibly on the vertices in \( V_D^2 \).

Next we construct a dominator coloring \( \chi' \) from \( \chi \) such that \( \chi' \) is an extension of \( \chi^S_D \) and uses at most \( \ell \) colors. For a \( Q \in Q \), let \( C_Q \) be the colors assigned to \( Q \) by \( \chi \) and \( C_Q \) be the colors used in \( \chi^S_D \) in \( Q \). For each \( Q \in Q \), we can use the colors in \( C_Q \setminus C_Q \) to color the rest of the vertices of \( Q \). Since \( \chi \) is a dominator coloring of \( G \) and agrees with \( \chi^S_D \) on \( M \) while sharing the same domination function with it, it follows that this new coloring, which we call \( \chi' \), will also be a proper dominator coloring with domination function \( \delta \). Moreover, it is clear that \( |\chi'| \leq |\chi| \leq \ell \) and that \( \chi' \) is an extension of \( \chi^S_D \). This proves our claim. By Lemma 3.3, we have the proof of Theorem 3.1.
4 Parameterized by CVD size

Consider a graph $G$. A subset $M$ of $V(G)$ is a Cluster Vertex Deletion (CVD) set if $G - M$ is a cluster graph. The study of parameterisation by the size of the CVD set was initiated by Doucha and Kratochvíl in 2012 [14]. In this section, we will focus on the DomCol-$M_F$ problem where $F$ is the collection of cluster graphs. We use DomCol-CVD to denote this restricted version of DomCol-$M_F$. We state the problem below as a refresher.

**DomCol-CVD**

**Parameter:** $k$

**Input:** A graph $G$ and a set of vertices $M$ of cardinality $k$ such that $G - M$ is a cluster graph; an integer $\ell$

**Question:** Does there exist a dominator coloring $\chi_d$ of $G$ with $|\chi_d| \leq \ell$?

We let $(G, M, k, \ell)$ denote an instance of DomCol-CVD. We also let $Q = \{Q_i\}_{i=1}^q$ denote the components of the cluster graph $G - M$ in such an instance. Note that each $Q_i \in Q$ is a clique. Consider an arbitrary ordering $\{m_1, m_2, \ldots, m_k\}$ of the vertices of $M$. For $0 \leq j \leq 2^k$, we let $M_j$ denote the subset of $M$ which contains $m_p \in M$ if, and only if, the $p^{th}$ bit in the binary representation of $j$ is 1. We associate each $Q_i \in Q$ to a neighborhood vector $N_i$ of size $2^k$ as follows: $N_i(j) = | \{ v \in V(Q_i) | N_{G}[v] \cap M = M_j \} |$. That is, the entry $N_i(j)$ denotes the number of vertices of $Q_i$ with neighborhood $M_j$ in $M$.

Let $\mathcal{N} = \{N_i | 1 \leq i \leq q\}$. Define an equivalence relation $\sim$ on $\mathcal{N}$ as follows: $N_i \sim N_j$ if $N_i(p) \neq 0$ if, and only if, $N_j(p) \neq 0$. Note that $\sim$ partitions $\mathcal{N}$ into $\lambda \leq 2^k$ many equivalent classes. Let this partition be $\mathcal{E}$. We say a clique $Q_i$ is in $\mathcal{E}_p \in \mathcal{E}$ if $N_i(p) \neq 0$. For each $\mathcal{E}_i \in \mathcal{E}$, we call the elements of the set $\{M_j \subseteq M | N_p(j) \neq 0 \}$ for a $N_p \in \mathcal{E}_i$ its neighborhood classes and we denote it by $\mathcal{N}_i$.

For any proper dominator coloring $\chi$, denote the colors used in coloring the vertices in $M$ by $C^M(\chi)$ and let $\chi_M$ be the partial coloring of $M$. Without loss in generality, we can assume that $C^M(\chi) \subseteq \{c_1, c_2, \ldots, c_k\}$. Let $C^0(\chi) \subseteq C^M(\chi)$ be the set of the colors that are only used in $M$. Consider any arbitrary set of colors $C^U(\chi)$ such that each color in $c \in C^U(\chi)$ is used to color only one vertex of a clique, i.e. $\chi^{-1}(c) = \{v\}$ for some $v \in Q_j$ and no two color of $C^U(\chi)$ is used to color the vertices of the same clique. With slight abuse of notation, we will drop $(\lambda)$ from the terms defined with respect to a coloring $\chi$. The corresponding coloring will be clear from the context.

From Observation 3.6, we know that the vertices in a clique $Q_i \in Q$ either dominate a color in $C^U \cup C^0$ or a color which is used to color vertices only in $V(Q_i) \cup M$ and not anywhere else. Consider a clique $Q_i$, such that no vertex of $Q_i$ is colored with a color in $C^U$. Then, every vertex dominates a color class in $C^M$. For such a clique $Q_i$, let $C_{Q_i} \subseteq C^M \setminus C^0$ denote a minimal set of colors such that every vertex in $Q_i$ dominates a color in $C_{Q_i} \cup C^0$. Let $C^1 = \bigcup_{i=1}^q C_{Q_i}$, and define $C^2 = C^M \setminus (C^0 \cup C^1)$. Let $\alpha_C$ be the number of cliques in which at least one color of $C^1$ was used. Let the partition of $C^1$ among $\alpha_C$ many cliques be $\mathcal{P}^1 = \{P^1_1, P^1_2, \ldots, P^1_{\alpha_C}\}$. Consider the following observation which follows from the definition of the equivalence classes in $\mathcal{E}$.

**Observation 4.1.** Let $\mathcal{E}_p \in \mathcal{E}$ be any equivalent class with neighborhood classes $\mathcal{N}_i$ and $\chi$ be any proper coloring. Then one of the two cases can happen

**Case (i)** For all cliques $Q_i \in \mathcal{E}_p$, every vertex dominates a color class in $C^0$

**Case (ii)** There exists a set of neighborhoods $\mathcal{N}^*_{C'} \subseteq \mathcal{N}_i$ such that for all $M' \in \mathcal{N}^*_{C'}$ in each clique there exist a vertex with neighborhood $M'$ in $M$ which does not dominate a color in $C^0$. 


By Observation 4.1, if in an $\mathcal{E}_i \in \mathcal{E}$, there exists a clique in this equivalence class which contains a vertex which does not dominate any color in $C^0$, every clique in $\mathcal{E}_i$ must be assigned a $C^1_j \in \mathcal{P}^1$ to color its vertices or a $c \in C^U$ to color one of its vertices. For any equivalent class $\mathcal{E}_i$ such that every clique in $\mathcal{E}_i$ has at least one vertex which does not dominate any color in $C^0$, suppose we are given the following set of partitions $\mathcal{P}^1_i \subseteq \mathcal{P}^1(\lambda)$ assigned to $\mathcal{E}_i$; and for all $c \in C^1_j$ in each partition $C^1_j \in \mathcal{P}^1_i$, the neighborhood $\omega(c)$ in $M$ of the vertex that will get the color $c$. Denote such a set of neighborhoods by $N_M(c)$.

Observe that for each clique in $\mathcal{E}_i$ which is not assigned a partition of colors from $\mathcal{P}^1_i$, exactly one vertex must be colored with a color from $C^U$. We assume that we also know the neighborhood in $M$, $M^U = \{M^U_1, \ldots, M^U_k\}$ of the vertices in which these colors appear. Let $C = \{c_1, \ldots, c_j\}$ and $C^Q = C \setminus \{C^M \cup C^U\}$. Let $M_d \subseteq M$ be the set of vertices which dominate a color class in $C^U \cup C^1$ and $M_b$ be the set which dominates a color class in $C^U$. Let $M_c = M \setminus (M_a \cup M_b)$. Without loss in generality, we can assume that vertices in $M_c$ dominate a color class in $\{c_1, c_2 \ldots c_k\}$. Let $M_d \subseteq M_c$ denote the vertices that dominate a $c \notin C^M$ and let $C_D \subseteq C^Q$ be the set of colors they dominate. Every color in $C_D$ must appear at least once in $G - M$. For each $c \in C_D$, assume we know the neighborhood in $M$, say $M^D(c)$, of the vertex in which it appears. Thus we can construct a $\delta : M \rightarrow \{c_1, c_2 \ldots c_k\}$ where $\delta(m_a)$ denotes the color that the vertex $m_a \in M$ dominates.

Given all this information, for each clique $Q \in \mathcal{E}_i$ and for any $C^1_j$, we can determine whether it is possible to color the rest of the vertices with $C^2 \cup C^Q$ where, for all $c \in C^1_j$, one vertex with neighborhood $\omega(c)$ gets the color $c$. Similarly, we can compute, for any $Q \in \mathcal{E}_i$ and $M^U_i$, whether it is possible to color the rest of the vertices with $C^2 \cup C^Q$ where one vertex with neighborhood $M^U_i$ gets a color from $C^U$.

Lemma 4.2. For any clique $Q_i$; given the set of colors $C^1_j \subseteq C^1$, and, for all $c \in C^1_j$, the neighborhood $\omega(c)$ in $M$ of the vertex that will get the color $c$, we can determine in polynomial time whether it is possible to color one vertex with neighborhood $\omega(c)$ with color $c$ for all $c \in C^1_j$ and the rest of the vertices with colors from $C^2 \cup C^Q$ such that for any vertex $m_a \in M$, the color $\delta(m_a)$ appears in the neighborhood of $m_a$.

Proof. Assign every color $c \in C^1_j$ to one arbitrary vertex with neighborhood $\omega(c)$. If every vertex in $Q_i$ does not dominate at least one color from $C^0 \cup C^1_j$, report NO. Otherwise, create a bipartite graph $G_i$ with vertex bipartition $V(Q_i) \cup C^1_j \cup C^2 \cup C^Q$. For any color $c \in C^Q$, put an edge between $c$ and every vertex in $Q_i$. For $c \in C^1_j$, let $M_c \subseteq M$ be the set of vertices which are assigned the color $c$ and $M_d \subseteq M$ be the set of vertices $v$ with $\delta(v) = c$. Put an edge between $c$ and $v$ if $v \notin \cup_{m_a \in M_c} N_G(m_a)$ and $v \in \cap_{m_d \in M_d} N_G(m_d)$. Return YES if there exists a matching saturating $V(Q_i)$ in $G_i$.

Observe that there are some colors $C_e \subseteq C_D$ which must appear in the coloring of the vertices of the cliques to satisfy the domination requirement of some vertex in $M$. Assume that for each such color $c$, we know the neighborhood of the vertex, $\mu(c)$ which will be assigned the color. For any subset of colors $C'_e \subseteq C_e$, we can modify the construction of $G_i$ in Lemma 4.2 to determine whether there exists a coloring which satisfies the its premise and, in addition, every color of $C'_e$ is used to color one vertex from $Q_i$. The modification is as follows: for each vertex $v$ with neighborhood $\mu(c)$, delete all edges incident on $v$ except the edge $(v, c)$.

Consider any equivalent class $\mathcal{E}_i$ such that every clique in $\mathcal{E}_i$ has at least one vertex which does not dominate any color in $C^0$. For such a class, assume that we are given the set of partitions $\mathcal{P}^1_i \subseteq \mathcal{P}^1$ assigned to $\mathcal{E}_i$ and, for each partition $C^1_j \in \mathcal{P}^1_i$ for all $c \in C^1_j(\lambda)$, the neighborhood $\omega(c)$ in $M$ of the vertex that will get the color $c$. Denote such a set of
neighborhoods by \( N_M(c) \). We also have a subset \( C'_e \subset C_e \). We would like to decide whether the cliques in \( E_i \) can be colored with \( C^2 \cup C^Q \) when each clique is assigned either a partition from \( P^2_i \) or a color from \( C_i \). Moreover, we require that each color from \( C'_e \) should be used to color at least one vertex. From Lemma 4.2, we know that for an arbitrary clique, an arbitrary part in \( P^2_i \), and any subset \( C'_e \subset C'_e \) we can find out whether there exists a feasible coloring or not. In \( \text{FPT} \) time we can guess the partitioning of \( C'_e \) over different cliques and for each part \( C_x \) in such partitioning, whether it will appear in a part of \( P^2_i \) or if it will appear in a clique which is assigned a color from \( C_i \).

**Lemma 4.3.** For any equivalent class \( E_i \) given a partition \( P^1_i \subseteq P^1 \), a subset \( C_u \subseteq C_i \) of cardinality \( |E_i| = |P^1_i| \), a set of subset of colors \( \nu = \{\nu_1, \ldots, \nu_a\} \) where \( \nu_j \subseteq C_i \) for \( 1 \leq j \leq a \), a function \( \alpha: \nu \to P^1_i \cup C_u \) and for color \( c \in \nu \) a function \( \beta(c) \subseteq M_i \); we can determine, in polynomial time, if there exist a partial dominator coloring of \( G \) such that the vertices in \( E_i \) are colored with colors in \( P^1_i \cup C_u \cup C^2 \cup C^Q \) and

1. each color in \( c \in C_j \in P^1_i \) appears in the same clique in a vertex with neighborhood \( \omega(c) \),
2. each color \( c \in \nu_j \) should appear in the same clique where \( \alpha(c) \) appears in a vertex with neighborhood \( \beta(c) \).

**Proof.** Similar to the proof of Lemma 4.2, we can create a bipartite graph with bipartition \( E_i \) and \( \{P^1_i \cup C_u\} \times \nu \). That is, we create a vertex for each clique on one side. In the other side we have a vertex for each \( (\zeta, \gamma) \) where \( \zeta \in \{P^1_i \cup C_u\} \) and either \( \gamma \) is empty or \( c \in \gamma_1 \in P^1_i \) where \( \alpha(c) = \zeta \). We create an edge between \( Q_\zeta \) and \( (\zeta, \gamma) \) if \( (\zeta, \gamma) \) is feasible with respect to \( Q_\zeta \) (Lemma 4.2). Report YES if there exist a perfect matching in this graph report and NO otherwise.

Therefore by trying all possible values for the following parameters, the rest of the problems can be solved in polynomial time.

\[
\begin{align*}
X_M & : \text{Coloring of the modulator} \\
C^0(\chi) & : \text{Colors that have only appeared in } M \\
C^1(\chi) & : \text{Minimal set of colors } c \text{ which have appeared in } M \text{ and in a clique } Q_i \text{ such that there exist at least one vertex } v \in Q_i \text{ such that } c \text{ is the only color class that } v \text{ dominates} \\
P^1(\chi) & : \text{Partition of } C^1(\chi) \text{ over different equivalent classes} \\
M_\zeta(\chi) & : \text{Neighborhood of the vertex } v \text{ in } G - M \text{ with } \chi(v) = c \text{ for } c \in C^1(\chi) \\
N_\zeta(\chi) & : \text{Neighborhood vector of the equivalent class in which } \zeta_1 \text{ appeared} \\
N_U(\chi) & : \text{Neighborhood classes of } E_i \text{ which are adjacent to a vertex colored } c \in C^U(\chi) \\
\delta(\chi) & : \text{Domination function for vertices which do not dominate a color class in } C^0(\chi) \cup C^1(\chi) \cup C^U(\chi) \\
\gamma_\chi & : \text{Tuple with entries neighborhood of } v \in Q_i \text{ in } M \text{ and the neighborhood vector of } Q_i \text{ where } v \in M \text{ that dominates a color not in } C^M(\chi) \cup C^U(\chi)
\end{align*}
\]

Therefore, we have the following theorem.

**Theorem 4.4.** \( \text{DomCol-CVD} \) is \( \text{FPT} \). There exists an algorithm that solves it in \( O^*(2^{O(k^2)}) \)-time.

## 5 Lower Bounds

In this section, we establish some lower bounds for \( \text{DomCol-CLQ} \), \( \text{DomCol-TC} \), and \( \text{DomCol-CVD} \). Consider a graph \( G \). Construct a graph \( G' \) from \( G \) as follows: \( V(G') = \)
$V(G) \cup \{v_0\}$ for a new vertex $v_0$ and $E(G') = E(G) \cup \{(v_0, v) \mid v \in V(G)\}$. By [21], we have the following.

- **Observation 5.1.** $G$ has a proper coloring using at most $\ell$-many colors if, and only if, $G'$ has a dominator coloring using at most $(\ell + 1)$-many colors.

Note that $V(G)$ is a clique modulator and a twin cover in $G'$. Therefore, if DomCol-CLQ, DomCol-TC, or DomCol-CVD can be solved in $O^{*}(f(n))$-time, where $k$ is the size of the modulator and $f$ is some computable function, then Graph Coloring can be solved in $O(f(n))$-time where $n$ is the number of vertices in an instance of this problem. It is known, under the Exponential-Time Hypothesis (ETH), that Graph Coloring cannot be solved in $2^{o(n)}$-time (refer [17] [11]). Moreover, the question of whether there exists an algorithm that solves Graph Coloring in $O((2 - \epsilon)^n)$-time, for some $\epsilon \in (0, 1)$, is famously open - refer [17]. We state this discussion as a lemma below.

- **Lemma 5.2.** DomCol-CLQ, DomCol-TC, and DomCol-CVD do not admit $O^{*}(2^{o(k)})$-time algorithms unless ETH fails. Moreover, if any of these problems can be solved in $O^{*}((2 - \epsilon)^k)$-time, for some $\epsilon \in (0, 1)$, then Graph Coloring admits a $O((2 - \epsilon)^n)$-time algorithm for such an $\epsilon$.

We now show that unless NP $\subseteq$ coNP/poly, DomCol-CVD does not admit a polynomial kernel. We also show that under the Set Cover Conjecture [10], DomCol-CVD does not admit a $O^{*}((2 - \epsilon)^k)$-time algorithm. Let $U$ be a set of elements and $F \subseteq \mathcal{P}(U)$. A set $S \subseteq U$ is called a hitting set of $F$ if, for all $F \in F$, $F \cap S \neq \emptyset$.

| Hitting Set | Parameter: $|F|$ |
|-------------|-----------------|
| **Input:** A set of elements $U$ and $F \subseteq \mathcal{P}(U)$; an integer $\kappa$ | **Question:** Does there exist a hitting set $S \subseteq U$ of $F$ with $|S| \leq \kappa$? |

We let $(U, F, \kappa)$ denote an instance of Hitting Set and assume that $|U| = n$. We will construct an instance $(G, M, k, \ell)$ of DomCol-CVD from an instance $(U, F, \kappa)$ of Hitting Set which has the following property: $(U, F, \kappa)$ is a yes instance if, and only if, $(G, M, k, \ell)$ is a yes instance where $\ell = n + 2$.

For each $F \in F$, we have a vertex $m_F \in V(G)$ and for each $u \in U$, we have a corresponding vertex $v_u \in V(G)$. Moreover, in $V(G)$, we have a set $Q_2$ of $(n - \kappa)$-many vertices and two special vertices $m_1$ and $m_2$. We now define the edges of $G$. All the vertices of $Q_1 = \{v_u \mid u \in U\}$ are connected to each other as are all vertices in $Q_2$. The vertices $m_1$ and $m_2$ are adjacent to all vertices in $Q_1 \cup Q_2$ and to each other. For an $F \in F$, $N_G(m_F) = \{v_u \in Q_1 \mid u \in F\}$. Let $M_F = \{m_F \mid F \in F\}$. Refer to Figure 4 for a pictorial representation of $G$.

Note that $M = M_F \cup \{m_1, m_2\}$ is a cluster vertex deletion set of $G$ and $G - M$ consists of the two cliques $Q_1$ and $Q_2$. Let $\ell = n + 2$ and $k = |M| = |F| + 2$. First, note that $G$ cannot be colored with fewer than $(n + 2)$-many colors since $Q_1 \cup \{w_1, w_2\}$ is a clique of size $n + 2$. The main result of this section is the following.

- **Theorem 5.3.** $(U, F, \kappa)$ is a yes instance of Hitting Set if, and only if, $(G, M, k, \ell)$ is a yes instance of DomCol-CVD.

It is known, due to [13], that Hitting Set does not admit a polynomial kernel. Hence, Theorem 5.3 proves that DomCol-CVD does not admit a polynomial kernel. Moreover, this also implies, under the Set Cover Conjecture [10], that DomCol-CVD does not admit any $O^{*}((2 - \epsilon)^k)$-time algorithm. We prove this theorem in two steps: first we show that $(U, F, \kappa)$ is a yes instance of Hitting Set if, and only if, there exists a dominator coloring
Lemma 5.4. \((U,F,\kappa)\) is a yes instance of Hitting Set if, and only if, there exists a dominator coloring \(\chi_d\) of \(G\) with \(|\chi_d| = n + 2\) such that \(\chi_d(m_1) = \chi_d(m_F)\) for all \(m_F \in M_F\).

Proof. Assume that \((U,F,\kappa)\) is a yes instance of Hitting Set. Then, there exists a set \(S \subseteq U\) with \(|S| \leq \kappa\) which hits every set of \(F\). We define a coloring \(\chi_d\) of \(G\) as follows. Color each vertex in \(V(Q_1) \cup \{m_1, m_2\}\) a different color and let \(\chi_d(m_F) = \chi_d(m_1)\) for all \(m_F \in M_F\). Let \(Q_1^S = \{v_u \in V(Q_1) \mid u \in S\}\). Then, \(|Q_1^S| = |S| \leq \kappa\) and hence \(|V(Q_1) \setminus Q_1^S| \geq n - \kappa\).

Color the \((n - \kappa)\)-many vertices in \(Q_2\) using the colors of in \(\chi_d(V(Q_1) \setminus Q_1^S)\).

Note that \(\chi_d\) is indeed a proper coloring of \(G\) - for if \(\chi_d(u) = \chi_d(v)\) for some \(u\) and \(v\), then either \(u, v \in M \setminus \{m_2\}\) or (without loss in generality) \(u \in V(Q_1)\) and \(v \in V(Q_2)\). In either case, \((u, v) \notin E(G)\). Moreover, vertices in \(V(Q_1) \cup V(Q_2) \cup \{m_1, m_2\}\) dominate the color class \(\chi_d(m_2)\). Consider a vertex \(m_F \in M_F\). Since there exists a \(s \in S \cap F\), \(x_F\) is adjacent to \(v_s \in V(Q_1)\). Since \(\chi_d(v_s)\) is not used to color any other vertex in the graph, \(m_F\) dominates this color class. Since \(|\chi_d| = n + 2\) and \(\chi_d(m_1) = \chi_d(m_F)\) for all \(m_F \in M_F\), we have the forward direction of this proof.

Now, let \(\chi_d\) be a dominator coloring of \(G\) with \(|\chi_d| = n + 2\) where \(\chi_d(m_1) = \chi_d(m_F)\) for all \(m_F \in M_F\). Since \(Q_1 \cup \{m_1, m_2\}\) is a clique of \(G\), \(|\chi_d(Q_1 \cup \{m_1, m_2\})| = n + 2\). Moreover, the \((n - \kappa)\)-many vertices in \(Q_2\) cannot be colored with either \(\chi_d(m_1)\) or \(\chi_d(m_2)\) as \(N_G[m_1] = N_G[m_2] \supseteq V(Q_2)\). Thus, \(\chi_d\) must use \((n - \kappa)\)-many colors of \(\chi_d(Q_1)\) to color the vertices of \(Q_2\).

Moreover as every vertex in \(M_F\) is colored with \(\chi_d(m_1)\), they must dominate a color class that is only used in \(Q_1\). By the final sentence of the previous paragraph, there are at most \(\kappa\)-many such vertices in \(Q_1\). Let \(Q_1^S\) denote this set of vertices and \(S = \{u \in U \mid v_u \in Q_1^S\}\). Therefore, each vertex in \(M_F\) must be adjacent to at least one vertex in \(Q_1^S\). By the construction of \(G\) this implies that \(S \cap F \neq \emptyset\) for all \(F \in F\). Since \(|S| \leq \kappa\), \((U,F,\kappa)\) is a yes instance of Hitting Set.
Lemma 5.5. (G, M, k, ℓ) is a yes instance of DomCol-CVD if, and only if, there exists a dominator coloring $\chi_d$ of G with $|\chi_d| = n + 2$ such that $\chi_d(m_1) = \chi_d(m_F)$ for all $m_F \in M_F$.

Proof. Assume that (G, M, k, ℓ) is a yes instance of DomCol-CVD. Then, there exists a dominator coloring $\chi_d$ of G with $|\chi_d| = n + 2$. By construction of G, a vertex $m_F \in M_F$ must either dominate the color class $\chi_d(m_F)$ or $\chi_d(v_u)$ for some $v_u \in V(Q_1)$. Note that, in either case, exactly one vertex is colored with the color $m_F$ dominates. Let $\delta(m_F)$ denote that vertex. Let $M_a = \{m_F \in M_F | \delta(m_F) = m_F\}$. We construct a coloring $\chi_d$ of G from $\chi_d$ as follows: color every vertex in $M_F \setminus \chi_d(m_F)$, for each $m_F \in M_a$ color an arbitrary vertex in $N_G(m_F)$ with $\chi_d(m_F)$ (if $N_G(m_F) = \emptyset$, then $|\chi_d| > n + 2$), and color the rest of the vertices with the same color that was used by $\chi_d$.

Note that the only color that is used multiple times by $\chi_d$ but not by $\chi_d$ is $\chi_d(m_1) = \chi_d(m_1)$. Since $\chi_d^{-1}(m_1) = M_F \cup \{m_1\}$ is an independent set, $\chi_d$ is a proper coloring of G. Moreover, note that every vertex in $V(G) \setminus M_F$ dominates $\chi_d(m_2)$ as $m_2$ is the only vertex colored with $\chi_d(m_2)$ by $\chi_d$. As every vertex $m_F \in M_F$ has a $v \in N_G(m_F)$ which gets a unique color in $\chi_d$, they dominate a color class as well. Hence, $\chi_d$ is a dominator coloring of G with $|\chi_d| = n + 2$ such that $\chi_d(m_1) = \chi_d(m_F)$ for all $m_F \in M_F$. Since the other side of the claim is trivial, this completes the proof of this lemma.

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