Improved QEDPS for Radiative Corrections in $e^+e^-$ Annihilation

T. Munehisa(*), J. Fujimoto, Y. Kurihara and Y. Shimizu

(*) Faculty of Engineering, Yamanashi University

Takeda, Kofu, Yamanashi 400, Japan

National Laboratory for High Energy Physics (KEK)

Oho 1-1 Tsukuba, Ibaraki 305, Japan

ABSTRACT

The generator QEDPS, a parton shower model in QED, developed by the present authors has been improved. By a careful study on the shower algorithm it was found that some finite contributions have been missed in the original version. They are small, but cannot be neglected when the contribution of the soft photons is dominant. A method is presented to correct these finite pieces and then a new generator, improved QEDPS, is proposed, which is able to make more precise prediction for the processes observed in $e^+e^-$ annihilation as far as the initial radiation is concerned.
Section 1. Introduction

In detailed studies of radiative corrections to $e^+e^-$ processes one needs Monte Carlo generators. Several authors have proposed such generators so far\[1\]–\[3\]. We have already published QEDPS for radiative corrections in the $e^+e^-$ annihilation\[2\] and also in the Bhabha scattering\[3\]. These generators are a QED version of the parton shower which has been originally developed for the study of QCD\[4\]. The algorithm is based on the $Q^2$-evolution according to the renormalization group equations, which enable us to sum up all the collinear singularities.

The current version of QEDPS is written in the leading-logarithm(LL) approximation. Recent experiments at $e^+e^-$ colliders, however, can provide very precise data and this in turn demands that the theoretical uncertainty in a generator should be less than 1% level. In order to respond to this requirement we have been studying the complete next-to-leading-logarithm of the QEDPS like QCD\[5\]. During this study we found that some finite contributions had been neglected in the original version. This takes place in the algorithm we adopted. These contributions are small, but actually cannot be ignored for the observables for which soft photons are important.

We explain how these contributions arises and how to do with them in the next section. Their origin is directly connected to the kinematics used in the generation algorithm. First we take the single cascade scheme which means that the electron with the momentum parallel to the axial gauge vector does not make showers. In section 3 the same problem will be discussed for the double cascade scheme. In this case any electron is able to make showers. Then one possible way to correct them is proposed and some results by the improved generator will be presented in section 4. Final section is devoted to summary and discussion.

Section 2. $Q^2$-evolution

The algorithm of the parton showers is based on the Altarelli-Parisi(AP) equation
which governs the $Q^2$-evolution of the structure function of quarks and gluons in QCD. In QED the strong coupling $\alpha_s$ is replaced by that of QED $\alpha$ and the color factors are removed from the AP equations. The equation for non-singlet $Q^2$-evolution is given by

$$\frac{dD(x,Q^2)}{d\ln Q^2} = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} P_+(x/y) D(y,Q^2).$$

(1)

Here $D(x,Q^2)$ is the structure function of the electrons with $x$ being the electron momentum fraction and $P_+(x) = ((1 + x^2)/(1 - x))_+$ is the split function. In the following we shall neglect the running effect of $\alpha$ only for the sake of simplicity. By taking the moments of this equation, one can easily obtain the solution

$$D(n,Q^2) = \int_0^1 dxx^{n-1}D(x,Q^2) = A_n \exp \left( -\frac{\alpha}{2\pi} \gamma_n \log(Q^2/Q_0^2) \right),$$

(2)

where $A_n$ is the integration constant and

$$\gamma_n = \int_0^1 dxx^{n-1}P_+(x) = \frac{3}{2} - \frac{1}{n} - \frac{1}{(n+1)} - 2 \sum_{j=1}^{n-1} \frac{1}{j}.$$  

(3)

is the anomalous dimension. Thus the $Q^2$-evolution of the structure function is determined by

$$D(n,Q_1^2)/D(n,Q_2^2) = \exp \left( \frac{\alpha}{2\pi} \gamma_n \log(Q_1^2/Q_2^2) \right).$$

(4)

Let us show the algorithm how to generate the parton showers. First we have to regard Eq.(1) as an equation which describes the stochastic process of emitting photons. For this we modify the split function as follows[2]:

$$P_+(x) = \theta(1 - \epsilon - x)P(x) - \delta(1 - x) \int_0^{1-\epsilon} dy P(x), \quad P(x) = \frac{1 + x^2}{1 - x}.$$  

(5)

Here $\epsilon$ is a small quantity and $\theta$ is the step function. Then the evolution equation can be cast into the form

$$D(x,Q^2) = \Pi(Q^2,Q_0^2)D(x,Q_0^2)$$

$$+ \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \Pi(Q^2,K^2) \int_x^{1-\epsilon} \frac{dy}{y} P(y) D(x/y, K^2),$$

(6)
where $Q^2_s$ is the minimum of $Q^2$ and $\Pi(Q^2, Q'^2)$ is the non-emission probability defined by
\[
\Pi(Q^2, Q'^2) = \exp\left(-\frac{\alpha}{2\pi} \int_{Q^2}^{Q'^2} \frac{dK^2}{K^2} \int_0^{1-\epsilon} dx P(x)\right).
\] (7)

The evolution equation in the integral form Eq.(6) allows one to take it as that for stochastic process and the algorithm of the photon shower consists of the following steps.

(1) Set $x_b = 1$, where $x_b$ becomes the fraction of the light-cone momentum of the electron after the end of the shower or just before the annihilation.

(2) If a given random number $\eta$ is smaller than $\Pi(Q^2, Q'^2_s)$, the evolution stops. If not, one finds the virtuality $K^2$ that satisfies $\eta = \Pi(K^2, Q'^2_s)$ with which a branching is made.

(3) Fix $x$ according to the probability $P(x)$ between 0 and $1 - \epsilon$. Then $x_b$ is replaced by $x_b x$. One should go to (2) by substituting $K^2$ into $Q'^2_s$ and repeat until it stops.

According to this algorithm the probability that no photons is emitted during the evolution from $Q'^2_s$ to $Q'^2$ is $\Pi(Q^2, Q'^2_s)$. The probability for the single photon emission is
\[
\int_{Q'^2_s}^{Q'^2} dK^2 \Pi(K^2, Q'^2_s) \frac{\alpha}{2\pi} \frac{1}{K^2} \int_0^{1-\epsilon} dx P(x) \Pi(Q^2, K^2)
= \Pi(Q^2, Q'^2_s) \int_{Q'^2_s}^{Q'^2} dK^2 \frac{\alpha}{2\pi} \frac{1}{K^2} \int_0^{1-\epsilon} dx P(x),
\] (8)

and for the double photons it is
\[
\int_{Q'^2_s}^{Q'^2} dK^2 \int_{Q'^2_s}^{K^2} dK'_1 \Pi(K^2_1, Q'^2_s) \frac{\alpha}{2\pi} \frac{1}{K^2_1} \int_0^{1-\epsilon} dx_1 P(x_1) \Pi(K^2_2, K^2_1)
\times \frac{\alpha}{2\pi} \frac{1}{K^2_2} \int_0^{1-\epsilon} dx_2 P(x_2) \Pi(Q^2, K^2_2)
\]
\[
= \Pi(Q^2_s, Q'^2_s) \int_{Q'^2_s}^{Q'^2} dK^2 \alpha \frac{1}{2\pi K^2} \int_0^{1-\epsilon} dx_2 P(x_2) \int_{Q'^2_s}^{K^2} dK^2_1 \alpha \frac{1}{2\pi K^2_1} \int_0^{1-\epsilon} dx_1 P(x_1).
\] (9)

For the emission of any number of photons, we have similar expressions.
When one wants to see the distributions on \( x_b \), the \( \delta \)-functions are inserted into the integrands,

\[
\delta(x_b - x_1 x_2 \ldots x_N).
\] (10)

If one takes the moments, one gets a compact expression.

\[
D(n, Q^2) = \exp \left( \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \int_0^{1-\epsilon_0} dx P(x)(x^{n-1} - 1) \right),
\] (11) by noting that

\[
\int_0^1 dx_b x_b^{n-1} \delta(x_b - x_1 x_2 \cdots x_N) = (x_1 x_2 \cdots x_N)^{n-1},
\] (12)

\[
\int_{Q_0^2}^{Q^2} dK_N^2 \int_{Q_0^2}^{K_N^2} dK_{N-1}^2 \cdots \int_{Q_0^2}^{K_1^2} dK_1^2 = \frac{1}{N!} \prod_{i=1}^N \int_{Q_0^2}^{Q^2} dK_i^2.
\] (13)

If we assume a very small constant value \( \epsilon_0 \) for the parameter \( \epsilon \), we reproduce Eqs.(4) because the corresponding structure function becomes

\[
D_0(n, Q^2) = \exp \left( \frac{\alpha}{2\pi} \log(Q^2/Q_0^2) \int_0^{1-\epsilon_0} dx P(x)(x^{n-1} - 1) \right)
\] (14)

\[
= \exp \left( \frac{\alpha}{2\pi} \log(Q^2/Q_0^2) \gamma_n + O(\epsilon_0) \right).
\] (15)

No problem arises, if \( \epsilon_0 \) is a constant. In the parton shower algorithm to generate events, however, we have \( K^2 \)-dependent \( \epsilon \)(see ref.\([2]\)). In QEDPS it is given by

\[
\epsilon_K = Q_0^2/K^2,
\] (16)

which comes from the fact that the kinematical boundary restricts the fraction \( x \) of the light-cone variable. Here \( Q_0^2 \) is a cutoff, a fictitious mass of the emitted photon introduced to regulate the infrared divergence.

In order to see whether the Monte Carlo simulation with the above mentioned algorithm can reproduce the moment given by an analytic expression Eq.(13) we make a numerical comparison. We take \( Q_0^2 = Q_s^2 = 10^{-6}\text{GeV}^2, Q^2 = 10^2\text{GeV}^2 \) and \( \alpha = 1/20 \). The last value is assumed only to magnify the finite contributions. Figure 1 shows the both results from the simulation(points with diamond) together
with those from the analytic formula (solid curve). Apparently there is a sizeable discrepancy. However this does not imply that the cutoff $\epsilon_K$ is wrong.

This can be understood by looking at the ratio of the moments $D(n, Q^2)/D(n, Q^2_1)$. This agrees completely with the analytic results. Hence one concludes that the structure function with the cutoff $\epsilon_K = Q^2_0/K^2$ should give some finite contributions. As long as the $Q^2$-evolution (the ratio at two different $Q^2$'s) is concerned, everything is correct. However if one is interested in the absolute value of the structure function, the algorithm is not sufficiently accurate and should be improved. The generated $D(n, Q^2)$ must be able to reproduce the results of perturbative calculation which have been known in the literature as mentioned in Section 4. Thus one can see that in the actual applications the finite contribution does not cause any problem in QCD, but in QED more careful study is required.

Let us now evaluate the finite contribution. First we define a structure function $D_s(n, Q^2)$ with the cutoff $\epsilon_K$ as

$$D_s(n, Q^2) \equiv \exp\left(\frac{\alpha}{2\pi} \int_{Q^2_0}^{Q^2} \frac{dK^2}{K^2} \int_0^{1-\epsilon_K} dx P(x)(x^{n-1} - 1)\right),$$

(Here we have attached a subscript $s$ to the structure function to stress that it is defined in the single cascade scheme whose meaning will become clear in the next section.) When $Q^2_0 \ll Q^2$, we can modify

$$D_s(n, Q^2) = \exp\left(\frac{\alpha}{2\pi} \int_{\epsilon_Q}^{1} \frac{dt}{t} \int_{\epsilon_K}^{1} dz P(1 - z)[(1 - z)^{n-1} - 1]\right)$$

$$= \exp\left(\frac{\alpha}{2\pi} \int_{\epsilon_Q}^{1} dz \log(\frac{z}{\epsilon_Q}) P(1 - z)[(1 - z)^{n-1} - 1]\right),$$

(18)

with $\epsilon_Q = Q^2_0/Q^2$. From the last expression we find

$$D_s(n, Q^2) = D(n, Q^2) D_{sf}(n) + O(\epsilon_Q),$$

$$D_{sf}(n) = \exp\left(\frac{\alpha}{2\pi} \int_0^{1} dx P(x) \log(1 - x)(x^{n-1} - 1)\right),$$

(19)

where $D(n, Q^2)$ is given by Eq.(11) with $\epsilon = 0$ and $D_{sf}(n)$ is the finite contribution we have looked for. In $x$-space, the equation (19) becomes a convolution integral
with respect to $x$ as

$$D_s(x, Q^2) = D(x, Q^2) \otimes D_{sf}(x).$$  \hfill (20)

In figure 1 one can see that $D_s(n, Q^2)$ is completely reproduced by the Monte Carlo simulation when $Q_s^2 = Q_0^2$. If $Q_0^2 \ll Q_s^2 \ll Q^2$, however, the finite contributions vanish. This implies that $D(n, Q^2)$ should be coincides with the simulation(points with circle). This is also demonstrated in the figure.

Section 3. The double cascade scheme

The algorithm discussed in the previous section is for the case of the single cascade scheme in $e^+e^-$ annihilation. The characteristic of this scheme is that the vector to define the axial gauge is set parallel to the positron and as a result the shower develops only on the electron while the positron stays inactive.

In the program for the actual generator, however, we adopt the double cascade scheme\footnote{\textsuperscript{6}}. This is another formulation which is more suitable than the single cascade in order to assure the symmetric treatment of the radiation from $e^+$ and $e^-$ as both of these are able to develop showers. In this section we discuss this scheme in some details and look at what finite contribution comes out.

First we give a review on the double cascade scheme, where photons are radiated parallel to its parent electron or positron but not anti-parallel to it. To make the argument clear, we take the simple case that only a single soft photon is emitted. The momentum of the photon is denoted as $k$ and those of two (on-shell)electrons are $P_1$ and $P_2$, respectively. For simplicity we neglect the electron mass, that is $P_1^2 = P_2^2 = 0$. The soft photon contribution $I$ is then given by

$$I = \frac{e^2}{(2\pi)^3} \int \frac{d^3k}{2k_0} \frac{2(P_1P_2)}{2(P_1k) 2(P_2k)},$$ \hfill (21)

where $(ab) = a_0b_0 - \mathbf{a} \cdot \mathbf{b}$. We use the light-cone momentum defined as

$$P_\pm = (E \pm P_z)/\sqrt{2},$$ \hfill (22)
and choose a frame in which we have

\[ P_{1\mu} = (P_{1+}, 0_-, 0_T), \quad P_{2\mu} = (0_+, P_{2-}, 0_T), \quad 2(P_1P_2) = Q^2. \]  

(23)

Now two new variables \( z \) and \( t \), the fractions of the light-cone momentum of electrons, are introduced by

\[ z = \frac{(kP_2)}{(P_1P_2)} = \frac{k_+}{P_{1+}}, \quad t = \frac{(kP_1)}{(P_1P_2)} = \frac{k_-}{P_{2-}}. \]  

(24)

In terms of these variables the soft photon contribution \( I \) is rewritten as

\[ I = \frac{\alpha}{4\pi} \int_0^1 \frac{dz}{z} \int_0^1 \frac{dt}{t}. \]  

(25)

This integral is divergent so that a cutoff is needed. For this we impose the lower limit to the transverse momentum squared of the photon, \( Q_0^2 \).

\[ k_T^2 = z2(kP_1) = ztQ^2 > Q_0^2, \]

\[ I(Q_0^2) = \frac{\alpha}{4\pi} \int_0^1 \frac{dz}{z} \int_0^1 \frac{dt}{t} \theta(zt - Q_0^2/Q^2). \]  

(26)

The events can be generated using these \( z, t \) in the full phase space. This generation method is the single cascade scheme.

In the C.M. frame we have \( P_{1+} = P_{2-} = \sqrt{Q^2/2} \) so that the \( z \)-component of the photon momentum is given by

\[ k_z = \frac{k_+ - k_-}{\sqrt{2}} = \frac{\sqrt{Q^2}}{2}(z - t). \]  

(27)

This shows that the photon with \( z > t \) is emitted parallel to the electron, while that with \( t > z \) parallel to the positron. Consequently it would be understood that we can select the parent of a shower by looking at which of \( z \) or \( t \) is greater than the other. Thus a shower can develop from any electron \textit{independently} and emitted photons are parallel to the electron or positron from which it has branched if one imposes the restriction \( z > t \). This is the double cascade scheme.

The parton showers are based on the renormalization group equations by which some physical quantity is calculated. It depends on the process considered and for the case of the deep inelastic scattering, it is the distribution on the Bjorken variable.
The parton showers must give the same distribution which can be calculated in the analytic approach.

In the single cascade scheme the fraction of the light-cone variables is equal to the Bjorken variable. In the double cascade scheme this is not valid and we must find what combination of which variables generated in parton showers is equal to the Bjorken variable.

Using the example of the single soft photon emission, we will look at the problem in details. Then the variable $x_{Bj}$ can be introduced by the following way:

$$P_1 + q = P_2 + k, \quad 2(P_1q) + q^2 = 2(P_2k),$$

$$\frac{1}{x_{Bj}} = \frac{2(P_1q)}{-q^2} = 1 + \frac{(P_2k)}{-q^2}. \quad (28)$$

Since $-q^2 = 2(P_1P_2)(1 + t - z)$, we find

$$x_{Bj} \equiv 1 - z/(1 + t) = 1 - z + O(zt). \quad (29)$$

For small $t$ the Bjorken variable is then determined by only $z$. This kinematics corresponds to the single cascade scheme,

$$D(x_{Bj}) = \delta(x_{Bj} - 1)(1 - I(Q_0^2))$$

$$+ \frac{\alpha}{4\pi} \int_0^1 \frac{dz_1}{z} \int_0^1 \frac{dt_1}{t} \theta(z_1t_1 - Q_0^2/Q^2)\theta(z_1 - t_1)\delta(x_{Bj} - 1 + z_1). \quad (30)$$

In the double cascade scheme, on the other hand, the inequality between $z$ and $t$ directly connected to the choice that either of which electrons emits the photon. In other words, when $z < t$ or $(P_2k) < (P_1k)$, the role of $z$ and $t$ should be interchanged. Hence in our example of the single soft photon emission, the distribution over $x_{Bj}$ becomes

$$D(x_{Bj}) = \delta(x_{Bj} - 1)(1 - I(Q_0^2))$$

$$+ \frac{\alpha}{4\pi} \int_0^1 \frac{dz_2}{z_2} \int_0^1 \frac{dt_2}{t_2} \theta(z_2t_2 - Q_0^2/Q^2)\theta(z_2 - t_2)\delta(x_{Bj} - 1 + t_2). \quad (31)$$
The first term would be easily understood if one notices that the sum of contributions from real photon emissions and loop diagrams is finite (no infrared and no collinear singularity). In this expression the subscript 1(2) designates the electron with momentum $P_1 (P_2)$ and each contribution corresponds to the radiation from them. It is clear then by superposing two showers defined with the constraint $z > t$, we get all the contributions in the full phase space. Also the radiated photons are parallel to the parent electrons. It is this constraint that allows us to draw a picture of jets, a cluster of electrons and photons flowing in the same direction. One should notice that in Eq. (32), $x_{Bj}$ can be replaced by $(1 - z_1)(1 - t_2)$.

Let us apply the above argument to showers with indefinite number of photons. Here we will repeat the main points to be argued. In the double cascade scheme, parton showers develop with constraint $(1 - x) \geq t$, where $x$ is the fraction for the electron, not for photons. Then what is the physical quantity that can be calculated by the renormalization group equation and what is the combination of the variables that equal to this physical quantity?

For the deep inelastic scattering it is obvious that the answer for the first question is the Bjorken variable. Then we study the second question. For this we specify the process as

$$e(P_1) + \gamma(q) \rightarrow e(p_1) + X + \gamma(q) \rightarrow e'(p_2) + X \rightarrow X'. \quad (33)$$

Here $X$ denotes anything. A virtual electron with $p_1$ collides with the virtual photon with $q$ and turns into another virtual electron with $p_2$. The momentum conservation gives

$$p_1^2 + 2(p_1q) + q^2 = p_2^2. \quad (34)$$

In the frame

$$P_{1-} = 0, \quad q_+ = -p_{1+}, \quad q_T = 0, \quad x_b = \frac{p_{1+}}{P_{1+}}, \quad (35)$$

we have

$$x_{Bj} \equiv \frac{-q^2}{2(P_1q)} = x_b - \frac{p_2^2}{2(P_1q)} - \frac{P_1^2}{2(P_1q)}, \quad (36)$$
by noting
\[
\frac{2(p_1 q)}{2(P_1 q)} = x_b - \frac{p_2^2}{2(P_1 q)}.
\]  
(37)

Since \(|p_1| = O(1 - x_b)p_1^2\), it can be neglected as well as the product of \(x_b\) and \(p_2^2/(2(P_1 q))\). We find
\[
x_{Bj} \simeq x_b \left(1 - \frac{p_2^2}{2(P_1 q)}\right) = x_b(1 - t_2).
\]  
(38)

In the double cascade scheme \(p_2^2 > 0\). This means that the electron \(p_2\) has a potential to emit photons and we must include contributions from showers of the scattered electron.

Next we have to prove that the distribution over \(x_b(1 - t_2)\) in fact agree with the solution of the renormalization group equation. In the course of the proof one will see that this combination of variables is suitable for our discussion. Also after the proof, one will find the finite contribution we are looking for.

The \(x_b\)-distribution is nothing but the distribution in the single cascade but with the constraint \((1 - x) \geq t\) inside of the integral in exponential.

\[
D_x(n, Q^2) = \exp \left(\frac{\alpha}{2\pi} \int_{Q_1^2}^{Q^2} \frac{dK^2}{K^2} \int_0^{1-cK} P(x)(x^{n-1} - 1)\theta(1 - x - t)\right). 
\]  
(39)

The expansion of the distribution over \(t_2\) is
\[
D_t(t_2, Q^2) = \Pi_c(Q^2, Q_0^2)\delta(t_2)
+ \frac{\alpha}{2\pi} \int_0^1 \frac{dt}{t} \int_0^1 dx P(x)\theta(1 - x - t)\theta((1 - x)t - Q_0^2/Q^2)\Pi_c(Q^2, tQ^2)\delta(t - t_2).
\]  
(40)

Here \(\Pi_c(Q^2, K^2)\) is the probability of non emission with the constraint, i.e. the probability in the double cascade scheme.

\[
\Pi_c(Q^2, K^2) = \exp \left(-\frac{\alpha}{2\pi} \int_{K^2/Q^2}^1 \frac{dt}{t} \int_0^1 dx P(x)\theta(1 - x - t)\theta((1 - x)t - Q_0^2/Q^2)\right).
\]  
(41)

The moment with respect to \(t_2\) is defined by
\[
D_t(n, Q^2) = \int_0^1 dt_2(1 - t_2)^{n-1}D_t(t_2, Q^2).
\]  
(42)
Noting that \((1 - t)^{n-1} \simeq \theta(1/n - t)\) for large \(n\), we find

\[
D_t(n, Q^2) = \exp \left( \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \int_0^{x^n-1} dx P(x)((1 - t)^{n-1} - 1) \theta(1 - x - t) \right) ,
\]

\[\epsilon_Q = Q_0^2/Q^2, \quad t = K^2/Q^2.\]  (43)

We have obtained closed forms for the moment distributions using some approximation. Since we know that the product of \(x_b\) and \(1 - t_2\) is the Bjorken variable, the moment of \(x_{Bj}\), \(D_d(n, Q^2)\) is the product of the moment \(D_x(n, Q^2)\) with respect to \(x_b\) and \(D_t(n, Q^2)\) to \(t_2\),

\[
D_d(n, Q^2) = D_x(n, Q^2) D_t(n, Q^2).
\]  (44)

Next we compare these moment distributions with those obtained by Monte Carlo simulation, which will confirm our analysis. First the distribution of \(x_b\) will be discussed. The analytic form of the moment distribution can be calculated easily.

\[
D_x(n, Q^2) = \exp \left( \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \int_0^{x^n-1} dx P(x)(x^{n-1} - 1) \theta(1 - x - t) \right),
\]

\[
= \exp \left( \frac{\alpha}{2\pi} \int_0^1 dz \log(\frac{z^2}{\epsilon_Q}) P(1 - z)[(1 - z)^{n-1} - 1] \right).\]  (45)

Figure 2 shows the Monte Carlo results for \(x_b\)-distribution in the double cascade scheme and those obtained from the analytic form Eq.(45). The agreement justifies our formulation.

Now let us calculate the finite contributions in the double cascade scheme. First we notice

\[
\frac{1}{t} P(1 - z) = \frac{1}{z} P(1 - t) + \frac{2 - t}{z^2} \left( -1 + \frac{2 + z}{t} \right).\]  (46)

The moments in the scheme are then

\[
D_d(n, Q^2) = D_x(n, Q^2) D_t(n, Q^2)
\]

\[
= \exp \left( \frac{\alpha}{2\pi} \int_0^1 \frac{dt}{t} \int_{\epsilon_Q/t}^{1} dx P(x)\left( x^{n-1} - 1 \right) \theta(x - \epsilon_Q/t) \theta(z - t) \right)
\]

\[
\times \exp \left( \frac{\alpha}{2\pi} \int_0^1 dt \int_0^1 dz \left( \frac{2 - t}{z^2} - \frac{2 - z}{t} \right) \theta(zt - \epsilon_Q) \theta(z - t) \right)
\]

\[
\times [(1 - t)^{n-1} - 1].\]  (47)
The first exponential on the right-hand side is the same as the moments \( D_s(n, Q^2) \) appearing in the single cascade scheme. In the latter the integration over \( z \) is done and \( O(\epsilon Q) \) terms are neglected. After changing variable \( t \) by \( z \), we get

\[
D_d(n, Q^2) = D_s(n, Q^2) \times \exp \left( \frac{\alpha}{2\pi} \int_0^1 dz \left( (2 + z) \log(z) - \frac{3}{2z} + 2 - \frac{z}{2} \right) [(1 - z)^{n-1} - 1] \right). \tag{48}
\]

Hence we finally find

\[
D_d(n, Q^2) = D(n, Q^2) D_{df}(n), \tag{49}
\]

where the finite term is given by

\[
D_{df}(n) = D_s(n) \exp \left( \frac{\alpha}{2\pi} \int_0^1 dz \left( (2 + z) \log(z) - \frac{3}{2z} + 2 - \frac{z}{2} \right) [(1 - z)^{n-1} - 1] \right),
\]

\[
= \exp \left( \frac{\alpha}{2\pi} \int_0^1 dz \left[ \log(z) P(1 - z) + ((2 + z) \log(z) - \frac{3}{2z} + 2 - \frac{z}{2}) \right] \right) \times [(1 - z)^{n-1} - 1]. \tag{50}
\]

This analytic form can be compared with the Monte Carlo results in Fig.2. The agreement confirms definitely our discussions. Hence to get the solution of the renormalization group equation, \( D(n, Q^2) \), we have to subtract the finite contribution in Eq.(50) or calculate the ratio of distributions with different \( Q^2 \). Thus we have completed the analysis of the deep-inelastic scattering.

In \( e^+e^- \) annihilation, a more complicated situation arises due to its kinematics. It is the fact that the squared mass of the virtual photon, \( q^2 \), is calculated by momentum fractions \( x_1, x_2 \) of the electron and the positron.

\[
q^2 = x_1 x_2 s, \tag{51}
\]

where \( s \) is the square of the energy in the center-of-mass system.

In the double cascade scheme we consider the following process allowing both electron and positron to radiate photons,

\[
e^-(P_1) + e^+(P_2) \rightarrow e^-(p_1) + e^+(p_2) + X \rightarrow \gamma(q) + X. \tag{52}
\]
By this equation we mean that the spacelike virtual electron($p_1$) and positron($p_2$) annihilates into the virtual photon($q$). In the frame that $P_{1\mu} = (P_{1+},0_-,0_T)$, $P_{2\mu} = (0_+,P_{2-},0_T)$ a simple algebra gives

$$q^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2x_{b1}x_{b2}(P_{1+}P_{2-}) + \frac{(p_1^2 + p_{12}^2)(p_2^2 + p_{22}^2)}{2x_{b1}x_{b2}(P_{1+}P_{2-})} - 2p_1 \cdot p_2, \quad (53)$$

where

$$x_{b1} = \frac{p_{1+}}{P_{1+}}, \quad x_{b2} = \frac{p_{2-}}{P_{2-}}. \quad (54)$$

If one neglects the terms of order $(1 - x_b)t$, one finds

$$q^2 = x_{b1}x_{b2}(1 - t_1)(1 - t_2)s, \quad (55)$$

where $t_1 = |p_1^2|/s$, $t_2 = |p_2^2|/s$.

In the deep-inelastic scattering one calculates the Bjorken variables in terms of the light-cone fraction of the electron in one shower and the squared virtual mass of the electron in another independent shower. $x_{Bj} = x_{b1}(1 - t_2)$. While in the annihilation process both of the light-cone fraction and the squared virtual mass determined by the generated shower are needed. Hence the moments with respect to the variable $x_b(1 - t)$ are required for one shower. These are given by

$$\Pi(Q^2, Q_0^2) = \int_0^1 \frac{dt}{t} (1 - t)^{n-1} \Pi(Q^2, tQ^2) \times \frac{\alpha}{2\pi} \int_0^1 dx P(x)x^{n-1}\theta(1 - x - t)\theta(1 - x)\theta((1 - x)t - Q_0^2/Q^2) \times \exp \left( \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \int dy P(y)(y^{n-1} - 1)\theta(1 - x - K^2/Q^2)\theta((1 - x)K^2 - Q_0^2) \right). \quad (56)$$

By using $(1 - t)^n \sim \theta(1/n - t)$, it turns out that the moments of $Q^2/s$ is given by the square of $D_a(n, Q^2)$.

$$D_a(n, Q^2) = \int_0^1 dx P(x)x^{n-1}D_a(x, Q^2) \times \exp \left( \frac{\alpha}{2\pi} \int_{Q_0^2}^{Q^2} \frac{dK^2}{K^2} \int_0^1 dx P(x)((x(1 - t))^{n-1} - 1)\theta(1 - x - t) \right),$$

$$\xi = Q^2/s, \quad (57)$$
The finite contributions $D_{af}(n)$ is then given by

$$
D_a(n, Q^2) = D(n, Q^2) D_{af}(n, Q^2)
$$

$$
D_{af}(n, Q^2) = \exp \left( \frac{\alpha}{2\pi} \int_0^1 dx \left[ P(x) \log(1 - \sqrt{x}) - \frac{1 - x}{x} \log \frac{\sqrt{1-x}}{1 + \sqrt{1-x}} 
+ \frac{1 - x}{2} \log(1 - x) \right] (x^{n-1} - 1) \right),
$$

(58)

for $\epsilon_Q \ll 1$.

Section 4. The model

Discussions in the previous sections suggest that it is easy to correct the finite contributions in the moment space. For the annihilation process

$$
D(n, Q^2) = D_a(n, Q^2)/D_{af}(n).
$$

(59)

Since $D_{af}(n)^{-1} < 1$, its inverse transform $D_{af}(x)$ is well defined,

$$
D_{af}(x) = \frac{1}{2\pi i} \int_{i\infty+c}^{i\infty-c} \frac{1}{x^n} D_{af}(n).
$$

(60)

We will show how to take into account these corrections in $e^+e^-$ annihilation process. As a simple analytic form is not easy to get for $D_{af}(x)$, we make the following approximation

$$
D_{af}(x) \simeq D_{af}^A(x) \equiv C \delta(1 - x) + \theta(x_f - x) F(x),
$$

(61)

where the function $F(x)$ is given by

$$
F(x) = \frac{\alpha}{2\pi} \left[ P(x) \ln(1 - \sqrt{x}) - \frac{1 - x}{x} \ln \frac{\sqrt{1-x}}{1 + \sqrt{1-x}} 
+ \frac{1 - x}{2} \ln(1 - x) \right] \exp \left( -\frac{\alpha}{2\pi} \ln^2(1 - x) \right).
$$

(62)

Here $1 - x_f \ll 1$ is assumed and the constant $C$ is fixed by the requirement that

$$
\int_0^1 dx D_{af}^A(x) = 1.
$$

(63)

The parameter $x_f$ is fixed by the condition that a photon with energy fraction $1 - x_f$ cannot be observed in the detectors. In usual experiments the limit of the
measurable energy is in the order of 10 MeV. From this fact in the program we set 
\( x_f = 1 - 10^{-4} \). If \( 1 - x_f \) is extremely small, the approximation of \( D_{af}^A(x) \) for \( D_{af}(x) \) is not justified. Then we add the following algorithm before developing showers.

At \( Q_s^2 \), we choose \( x \) according to \( D_{af}^A(x) \). If \( x \neq 1 \), a photon with the momentum fraction \( 1 - x \) is emitted parallel to the electron and the energy fraction of the electron is \( x \). In the generator \( Q_s^2 \gg Q_0^2 \) is assumed.

Next we will present results by the improved QEDPS and compare them with the analytic results. Since the coupling constant \( \alpha \) is small, we make a comparison on the integrated radiator over small \( 1 - x \), that is,

\[
R_i(x, Q^2) = \int_x^1 dy R(y, Q^2),
\]

\[
R(x, Q^2) = \int_0^1 dx_1 dx_2 D(x_1, Q^2) D(x_2, Q^2) \delta(x - x_1 x_2).
\]

In the Monte Carlo we count the fraction of events with \( s_{\text{effective}} \geq xs \). In the simulation we assumed the following values for the parameters.

\[
\alpha = 1/137, Q_s^2 = 10^{-6}\text{GeV}^2, Q^2 = 10^4\text{GeV}^2, Q_0^2 = 10^{-8}\text{GeV}^2.
\]

The results are shown in figure 3.

In analyzing the experimental data people use very sophisticated radiators, which involve the higher order corrections and others. A review on these will be found in ref.[7]. We call the radiator given there \( R_{\text{Suppl}}(x, Q^2) \). This and \( R(x, Q^2) \) are plotted in figure 3, in which one still finds a small discrepancy on the order of 1%. We should note, however, that

\[
\int_0^1 dx R(x, Q^2) = 1,
\]

while

\[
\int_0^1 dx R_{\text{Suppl}}(x, Q^2) = 1 + \frac{\alpha}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right) = C_a.
\]

We introduce a modified \( R_M(x, Q^2) \) as

\[
R_M(x, Q^2) = R(x, Q^2)C_a.
\]
Then we plot $R_M(x, Q^2)$ and $R_{Suppl}(x, Q^2)C_a$ to find a satisfactory agreement whose accuracy is less than 0.1%. Hence the normalization is fixed by $C_a$ instead of unity in the shower.

Section 5. Summary and discussions

In this paper we present the improved version of QEDPS, a generator for radiative corrections to the processes in $e^+e^-$ annihilation. This new model has been obtained from the old one, that has been proposed by the authors a few years ago, by correcting the finite contributions originated from our shower algorithm. It was shown that these contributions appear when the infra-red cutoff is $K^2$-dependent.

In the present work we did not take account of the running effect for the QED coupling $\alpha$. If one applies this study to improve the finite contributions in QCD showers, the running effects should be considered. We would like to discuss this point in a separated paper.

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Figure Captions

Fig.1 Moments of the structure functions. The solid curve is the correct results obtained by the renormalization group equation. The dashed curve is given by the analytic form of $D_s(n, Q^2)$. Diamond(circle) data points are Monte Carlo results in the single cascade scheme for $Q_s^2 = Q_0^2 (Q_s^2 \gg Q_0^2)$. Parameters are $Q^2 = 10^4 GeV^2, Q_s^2 = 10^{-6} GeV^2, \alpha = 1/20$. For the Monte Carlo data of $Q_s^2 \gg Q_0^2$, we choose $Q_0^2 = 10^{-8} GeV^2$.

Fig.2 Moments of the structure functions in the double cascade scheme. The solid(dashed) curve is calculated analytically by $D_s(n, Q^2) \ (D_x(n, Q^2)D_t(n, Q^2))$. Diamond(circle) data points of the Monte Carlo simulation corresponds to $x_b(x_{Bj} = x_b(1 - t))$. The long-dashed curve presents the results of $D(n, Q^2)$. Parameters are the same as Fig.1.

Fig.3 Differences between the Monte Carlo data and analytic expression for the integrated radiator, $\int_x^1 dy R(y, Q^2)$. Parameters are the same as Fig.2 except $\alpha$. Here $\alpha = 1/137$. One million events are generated.
Fig. 1

- **analytic**
- **M.C. for $Q_s=Q_0$**
- **M.C. for $Q_s>>Q_0$**

$D(n, Q^2)$ vs $(n-1)$
Fig. 2

- analytic: \( D(x, Q^2) \)
- MC for \( x_b \)
- analytic: \( D(d, Q^2) \)
- MC for \( x_B \)
- \( D(n, Q^2) \)
without the overall factor

with the overall factor

Fig. 3