EQUIVALENCE BETWEEN ALMOST-GREEDY AND SEMI-GREEDY BASES

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ABSTRACT. In [3] it was proved that almost-greedy and semi-greedy bases are equivalent in the context of Banach spaces with finite cotype. In this paper we show this equivalence for general Banach spaces.

1. INTRODUCTION

Let \((\mathbb{X}, \| \cdot \|)\) be a Banach space over \(\mathbb{F}\) (\(\mathbb{F}\) denotes the real field \(\mathbb{R}\) or the complex field \(\mathbb{C}\)) and let \(\mathcal{B} = (e_n)_{n=1}^{\infty}\) be a semi-normalized Schauder basis of \(\mathbb{X}\) with constant \(K_B\) and with biorthogonal functionals \((e_n^*)_{n=1}^{\infty}\), i.e., \(0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty\) and \(K_B = \sup_{x} \|S_{N}(x)\|/\|x\| < \infty\) \(\forall x \in \mathbb{X}\), where \(S_{N}(x) = \sum_{j=1}^{N} e_j^*(x)e_j\) denotes the algorithm of the partial sums.

As usual \(\text{supp} (x) = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}\), given a finite set \(A \subset \mathbb{N}\), \(|A|\) denotes the cardinality of the set \(A\), \(P_A\) is the projection operator, that is, \(P_A(\sum_j a_j e_j) = \sum_{j \in A} a_j e_j\), \(P_A^c = I_{\mathbb{X}} - P_A\), \(I_{EA} = \sum_{n \in A} e_n e_n\) with \(|e_n| = 1\) (where \(e_n\) could be real or complex), \(1_A = \sum_{n \in A} e_n\) and for \(A, B \subset \mathbb{N}\), we write \(A < B\) if \(\max_{j \in A} i < \min_{j \in B} j\).

In 1999, S. V. Konyagin and V. N. Temlyakov introduced the Thresholding Greedy Algorithm (TGA) (see [7]): given \(x = \sum_{i=1}^{\infty} e_i^*(x)e_i \in \mathbb{X}\), we define the natural greedy ordering for \(x\) as the map \(\rho : \mathbb{N} \rightarrow \mathbb{N}\) such that \(\text{supp}(x) \subset \rho(\mathbb{N})\) and so that if \(j < k\) then either \(|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|\) or \(|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|\) and \(\rho(j) < \rho(k)\). The \(m\)-th greedy sum of \(x\) is

\[\mathcal{G}_m(x) = \sum_{j=1}^{m} e_{\rho(j)}^*(x)e_{\rho(j)},\]

and the sequence of maps \((\mathcal{G}_m)_{m=1}^{\infty}\) is known as the Thresholding Greedy Algorithm associated to \(\mathcal{B}\) in \(\mathbb{X}\). Alternatively we can write \(\mathcal{G}_m(x) = \sum_{k \in A_m(x)} e_k^*(x)e_k\), where \(A_m(x) = \{\rho(n) : n \leq m\}\) is the greedy set of \(x\): \(\min_{k \in A_m(x)} |e_k^*(x)| \geq \max_{k \in A_m(x)} |e_k^*(x)|\).

To study the efficiency of the TGA, S. V. Konyagin and V. N. Temlyakov introduced in [7] the so called greedy bases.

Definition 1.1. We say that \(\mathcal{B}\) is greedy if there exists a constant \(C \geq 1\) such that

\[\|x - \mathcal{G}_m(x)\| \leq C \sigma_m(x), \forall x \in \mathbb{X}, \forall m \in \mathbb{N},\]

where \(\sigma_m(x)\) is the \(m\)-th error of approximation with respect to \(\mathcal{B}\), and it is defined as

\[\sigma_m(x, \mathcal{B})_{\mathbb{X}} = \sigma_m(x) := \inf \left\{ \|x - \sum_{n \in C} a_n e_n\| : |C| = m, a_n \in \mathbb{F} \right\}.\]

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Also, S. V. Konyagin and V. N. Temlyakov characterized greedy bases in terms of unconditional bases with the additional property of being democratic, i.e., \( \|1_A\| \leq C \|1_B\| \) for any pair of finite sets \( A, B \) with \( |A| \leq |B| \). Recall that a basis \( \mathcal{B} \) in \( X \) is called unconditional if any rearrangement of the series \( \sum_{n=1}^{\infty} c_n(x)e_n \) converges in norm to \( x \) for any \( x \in X \). This turns out to be equivalent the fact that the projections \( P_A \) are uniformly bounded on all finite sets \( A \), i.e. there exists a constant \( C \geq 1 \) such that \( \|P_A(x)\| \leq C \|x\|, \ \forall x \in X \) and \( \forall A \subset \mathbb{N} \).

Another important concept in greedy approximation theory is the notion of quasi-greedy bases introduced in [7].

**Definition 1.2.** We say that \( \mathcal{B} \) is quasi-greedy if there exists a constant \( C \geq 1 \) such that
\[
\|x - G_m(x)\| \leq C \|x\|, \ \forall x \in X, \forall m \in \mathbb{N}.
\]
We denote by \( C_q \) the least constant that satisfies (1) and we say that \( \mathcal{B} \) is \( C_q \)-quasi-greedy.

Subsequently, P. Wojtaszczyk proved in [8] that \( \mathcal{B} \) is quasi-greedy in a quasi-Banach space \( X \) if and only if the algorithm converges, that is,
\[
\lim_{m \to \infty} \|x - G_m(x)\| = 0, \ \forall x \in X.
\]

One intermediate concept between greedy and quasi-greedy bases, almost-greedy bases, was introduced by S. J. Dilworth et al. in [5].

**Definition 1.3.** We say that \( \mathcal{B} \) is almost-greedy if there exists a constant \( C \geq 1 \) such that
\[
\|x - G_m(x)\| \leq C \tilde{\sigma}_m(x), \ \forall x \in X, \forall m \in \mathbb{N},
\]
where \( \tilde{\sigma}_m(x, B)_X = \tilde{\sigma}_m(x) := \inf \{\|x - P_A(x)\| : |A| = m \} \). We denote by \( C_{al} \) the least constant that satisfies (2) and we say that \( \mathcal{B} \) is \( C_{al} \)-almost-greedy.

In [5], the authors characterized the almost-greedy bases in terms of quasi-greedy and democratic bases.

**Theorem 1.4.** [5 Theorem 3.3] \( \mathcal{B} \) is almost-greedy if and only if \( \mathcal{B} \) is quasi-greedy and democratic.

We will use the notion of super-democracy instead of democracy. This is a classical concept in this theory.

**Definition 1.5.** We say that \( \mathcal{B} \) is super-democratic if there exists a constant \( C \geq 1 \) such that
\[
\|1_{eA}\| \leq C \|1_B\|,
\]
for any pair of finite sets \( A \) and \( B \) such that \( |A| \leq |B| \) and any choice \( |\varepsilon| = |\eta| = 1 \). We denote by \( C_{sd} \) the least constant that satisfies (3) and we say that \( \mathcal{B} \) is \( C_{sd} \)-super-democratic.

**Remark 1.6.** It is well known that in Theorem 1.4 we can replace democracy by super-democracy (see for instance [1] Theorem 1.3).

On the other hand, S. J. Dilworth, N. J. Kalton and D. Kutzarova introduced in [3] the concept of semi-greedy bases. This concept was born as an enhancement of the TGA to improve the rate of convergence. To study the notion of semi-greediness, we need to define the Thresholding Chebyshev Greedy Algorithm: let \( A_m(x) \) be the greedy set of \( x \) of cardinality...
m. Define the m-th Chebyshev-greedy sum as any element $C \mathcal{G}_m(x) \in \text{span}\{e_i : i \in A_m(x)\}$ such that
\[
\|x - C \mathcal{G}_m(x)\| = \min \left\{ \left\| x - \sum_{n \in A_m(x)} a_n e_n \right\| : a_n \in F \right\}.
\]
The collection \( \{C \mathcal{G}_m\}_{m=1}^\infty \) is the Thresholding Chebyshev Greedy Algorithm.

**Definition 1.7.** We say that \( \mathscr{B} \) is semi-greedy if there exists a constant \( C \geq 1 \) such that
\[
\|x - C \mathcal{G}_m(x)\| \leq C \sigma_m(x), \forall x \in \mathcal{X}, \forall m \in \mathbb{N}.
\]
We denote by \( C_s \) the least constant that satisfies (4) and we say that \( \mathscr{B} \) is \( C_s \)-semi-greedy.

In [3], the following theorem is proved:

**Theorem 1.8.** [3, Theorem 3.2] Every almost-greedy basis in a Banach space is semi-greedy.

In this paper we study the converse of this theorem. In [3], the authors established the following "converse" theorem:

**Theorem 1.9.** [3, Theorem 3.6] Assume that \( \mathscr{B} \) is a semi-greedy basis in a Banach space \( \mathcal{X} \) which has finite cotype. Then, \( \mathscr{B} \) is almost-greedy.

The objective here is to show that the condition of the finite cotype in the last theorem is not necessary. The main result is the following:

**Theorem 1.10.** Assume that \( \mathscr{B} \) is a Schauder basis in a Banach space \( \mathcal{X} \).

a) If \( \mathscr{B} \) is \( C_q \)-quasi-greedy and \( C_{sd} \)-super-democratic, then \( \mathscr{B} \) is \( C_s \)-semi-greedy with constant \( C_s \leq C_q + 4C_q C_{sd} \).

b) If \( \mathscr{B} \) is \( C_s \)-semi-greedy, then \( \mathscr{B} \) is \( C_{sd} \)-super-democratic with constant \( C_{sd} \leq 2(C_s K_b)^2 \) and \( C_q \)-quasi-greedy with constant \( C_q \leq K_b (2 + 3(K_b C_s)^2) \).

**Remark 1.11.** S. J. Dilworth et al. ([3]) proved the item a) with the bound \( C_s = O(C_q^2 C_d) \), where \( C_d \) is the democracy constant. Here, we slightly relax this bound proving that \( C_s = O(C_q C_{sd}) \).

**Corollary 1.12.** If \( \mathscr{B} \) is a Schauder basis in \( \mathcal{X} \), \( \mathscr{B} \) is almost-greedy if and only if \( \mathscr{B} \) is semi-greedy.

2. Preliminary results

To prove Theorem 1.10, we need the following technical results that we can find in [1] and [5].

2.1. Convexity lemma.

**Lemma 2.1.** [1, Lemma 2.7] For every finite set \( A \subset \mathbb{N} \), we have
\[
co\{1_{\varepsilon A} : |\varepsilon| = 1\} = \left\{ \sum_{n \in A} z_n e_n : |z_n| \leq 1 \right\},
\]
where \( coS = \{ \sum_{j=1}^n \alpha_j x_j : x_j \in S, 0 \leq \alpha_j \leq 1, \sum_{j=1}^n \alpha_j = 1, n \in \mathbb{N} \} \).
As a consequence, for any finite sequence \((z_n)_{n \in A}\) with \(z_n \in \mathbb{F}\) for all \(n \in A\),

\[
\left\| \sum_{n \in A} z_n e_n \right\| \leq \max_{n \in A} |z_n| \varphi(|A|),
\]

where \(\varphi(m) = \sup_{|A|=m,|\varepsilon|=1} \| \mathbf{1}_A \varepsilon \|\).

2.2. The truncation operator. For each \(\alpha > 0\), we define the truncation function of \(z \in \mathbb{F}\) as

\[
T_\alpha(z) = \alpha \text{sgn}(z), \quad |z| > \alpha, \quad T_\alpha(z) = z, \quad |z| \leq \alpha.
\]

We can extend \(T_\alpha\) to an operator in \(\mathbb{X}\) by

\[
T_\alpha(x) = \sum_{i=1}^{\infty} T_\alpha(e^*_i(x)) e_i = \alpha \mathbf{1}_\varepsilon T_\alpha \mathbf{1}_\varepsilon = \alpha \mathbf{1}_\varepsilon \Gamma_\alpha + P_{\Gamma_\alpha}(x),
\]

where \(\Gamma_\alpha = \{ n : |e^*_n(x)| > \alpha \}\) and \(e_j = \text{sgn}(e^*_j(x))\) with \(j \in \Gamma_\alpha\). Hence, this is a well-defined operator for all \(x \in \mathbb{X}\) since \(\Gamma_\alpha\) is a finite set.

This operator was introduced in [3] to prove Theorem [1.8] showing that for quasi-greedy bases, this operator is uniformly bounded. A slight improvement of the boundedness constant was given in [11].

**Proposition 2.2.** [11] Lemma 2.5] Assume that \(\mathcal{B}\) is \(C_q\)-quasi-greedy basis in a Banach space \(\mathbb{X}\). Then, for every \(\alpha > 0\),

\[
\| T_\alpha(x) \| \leq C_q \|x\|, \forall x \in \mathbb{X}.
\]

We shall also use the following known inequality from [5].

**Lemma 2.3.** [5] Lemma 2.2] If \(\mathcal{B}\) is a \(C_q\)-quasi-greedy basis in \(\mathbb{X}\),

\[
\min_{j \in G} |e^*_j(x)| \| \mathbf{1}_{\varepsilon G} \| \leq 2C_q \|x\|, \forall x \in \mathbb{X}, \forall G \text{ greedy set of } x,
\]

(5)

with \(\varepsilon = \{ \text{sgn}(e^*_j(x)) \}\).

3. PROOF OF THE MAIN RESULT

Using the lemmas of Section 2, we prove Theorem 1.10.

**Proof of Theorem 1.10** First, we show the proof of a). Suppose that \(\mathcal{B}\) is \(C_q\)-quasi-greedy and \(C_{sd}\)-super-democratic. To show the semi-greediness, we will follow the same procedure as in the proof of [4] Theorem 4.1 and [3] Theorem 3.2. Take \(x \in \mathbb{X}\) and \(z = \sum_{i \in B} a_i e_i\) with \(|B| = m\) such that \(\|x - z\| < \sigma_m(x) + \delta\), for \(\delta > 0\). Let \(A_m(x)\) the greedy set of \(x\) of cardinality \(m\). We write \(x - z = \sum_{i=1}^{\infty} y_i e_i\), where \(y_i = e^*_i(x) - a_i\) for \(i \in B\) and \(y_i = e^*_i(x)\) for \(i \notin B\). To prove that \(\mathcal{B}\) is semi-greedy we only have to show that there exists \(w \in \mathbb{X}\) so that \(\text{supp}(x - w) \subset A_m(x)\) and \(\|w\| \leq c \|x - z\|\) for some positive constant \(c\). If \(\alpha = \max_{j \notin A_m(x)} |e^*_j(x)|\), we take the element \(w\) as is defined in [3]:

\[
w := \sum_{i \in A_m(x)} T_\alpha(y_i) e_i + P_{A_m(x)}(x) = \sum_{i=1}^{\infty} T_\alpha(y_i) e_i + \sum_{i \notin B \setminus A_m(x)} (e^*_i(x) - T_\alpha(y_i)) e_i.
\]

Of course, \(w\) satisfies that \(\text{supp}(x - w) \subset A_m(x)\) and we will prove that \(\|w\| \leq (C_q + 4C_qC_3) \|x - z\|\). To obtain this bound, using Proposition 2.2

\[
\| \sum_{i=1}^{\infty} T_\alpha(y_i) e_i \| \leq C_q \|x - z\|.
\]

(6)
Taking into account that \( |e_i^*(x) - T_A(y_i)| \leq 2\alpha \) for \( i \in B \setminus A_m(x) \), using Lemma 2.1,
\[
\left\| \sum_{i \in B \setminus A_m(x)} (e_i^*(x) - T_A(y_i)) e_i \right\| \leq 2\alpha \varphi(|B \setminus A_m(x)|) \leq 2 \min_{j \in A_m(x) \setminus B} \left| e_j^*(x - z) \right| \varphi(\|A_m(x) \setminus B\|). \tag{7}
\]

To improve the bound of \( C_s \) as we have commented in the Remark 1.11 based on (6, Lemma 2.1), we can find a greedy set \( \Gamma \) of \( x - z \) with the following conditions:

\[ \begin{align*}
& |\Gamma| = |B \setminus A_m(x)|, \\
& \min_{j \in A_m(x) \setminus B} \left| e_j^*(x - z) \right| \leq \min_{j \in \Gamma} \left| e_j^*(x - z) \right|.
\end{align*} \]

Thus, using (6), (7), (8), the basis is \( C_s \)-semi-greedy with constant \( C_s \leq (C_q + 4C_qC_{sd}) \).

Now, we prove \( b \). Assume that \( \mathcal{B} \) is \( C_s \)-semi-greedy.

Super-democracy can be proved using the technique of [3, Proposition 3.3]. Indeed, take \( A \) and \( B \) with \( |A| \leq |B| \) and \( |\mathcal{E}| = |\mathcal{N}| = 1 \). Select now a set \( D \) such that \( |D| = |A|, D > (A \cup B) \) and define \( z := 1_{\mathcal{E}A} + (1 + \delta)1_D \) with \( \delta > 0 \). It is clear that \( \mathcal{G}_D(z) = (1 + \delta)1_D \). Then,
\[
\|z - \mathcal{G}_D(z)\| = \left\| 1_{\mathcal{E}A} + \sum_{i \in D} c_i e_i \right\|,
\]
where the scalars \((c_i)_{i \in D}\) are given by the Chebyshev approximation. Then,
\[
\|1_{\mathcal{E}A}\| \leq K_b\|1_{\mathcal{E}A} + \sum_{i \in D} c_i e_i\| \leq K_b C_s \mathcal{G}_D(z) \leq K_b C_s \|(1 + \delta)1_D\|.
\]

If \( \delta \) goes to 0,
\[
\|1_{\mathcal{E}A}\| \leq C_s K_b \|1_D\|. \tag{9}
\]

The next step is to obtain that \( \|1_D\| \leq 2K_b C_s \|1_B\| \). For that, we take the element \( y := (1 + \delta)1_{\mathcal{E}B} + 1_D \) with \( \delta > 0 \). Then, \( \mathcal{G}_B(y) = (1 + \delta)1_{\mathcal{E}B} \). Hence,
\[
\|y - \mathcal{G}_B(y)\| = \left\| \sum_{i \in B} d_i e_i + 1_D \right\|,
\]
where as before, the scalars \((d_i)_{i \in B}\) are given by the Chebyshev approximation. Using again the semi-greediness,
\[
\|1_D\| \leq 2K_b \sum_{i \in B} d_i e_i + 1_D \| \leq 2C_s K_b \mathcal{G}_B(y) \leq 2C_s K_b \|(1 + \delta)1_{\mathcal{E}B}\|.
\]

Taking \( \delta \to 0 \), we obtain that
\[
\|1_D\| \leq 2C_s K_b \|1_{\mathcal{E}B}\|. \tag{10}
\]

Using (9) and (10),
\[
\|1_{\mathcal{E}A}\| \leq 2(C_s K_b)^2 \|1_{\mathcal{E}B}\|.
\]

Hence, the basis is super-democratic with constant \( C_{sd} \leq 2(C_s K_b)^2 \).

To prove now the quasi-greediness, with constant \( C_{sd} \leq 2(C_s K_b)^2 \).
support and $A_m(x)$ the greedy set of $x$ with cardinality $m$, take $D > \supp(x)$ with $|D| = |A_m(x)| = m$ and define $z := x - \mathcal{G}_m(x) + (\delta + \alpha)1_D$, where $\delta > 0$ and $\alpha = \min_{j \in A_m(x)} |e_j^*(x)|$. Then, since $A_m(z) = D$, 
\[
\|z - \mathcal{G}_m(z)\| = \left\| x - \mathcal{G}_m(x) + \sum_{i \in D} f_i e_i \right\|,
\]
for some scalars $(f_i)_{i \in D}$ given by the Chebyshev approximation. Then, 
\[
\|x - \mathcal{G}_m(x)\| \leq K_h \left\| x - \mathcal{G}_m(x) + \sum_{i \in D} f_i e_i \right\| \leq K_h C_s \sigma_{m}(z) \leq K_h C_s \|x + (\delta + \alpha)1_D\|.
\]
Taking $\delta \to 0$, 
\[
\|x - \mathcal{G}_m(x)\| \leq K_h C_s \|x + \alpha 1_D\| \leq K_h C_s (\|x\| + \|\alpha 1_D\|). \tag{11}
\]
Select now $y := \sum_{j \in A_m(x)} (e_j^*(x) + \delta e_j) e_j + \sum_{j \in A_m(x)} e_j^*(x) e_j + \alpha 1_D$, with $\delta > 0$ and $e_j = \sgn(e_j^*(x))$ for $j \in A_m(x)$. Then, since $\mathcal{G}_m(y) = \sum_{j \in A_m(x)} (e_j^*(x) + \delta e_j) e_j$, using Chebyshev approximation, 
\[
\|y - \mathcal{G}_m(y)\| = \left\| \sum_{j \in A_m(x)} a_j e_i + \sum_{j \in A_m(x)} e_j^*(x) e_j + \alpha 1_D \right\|.
\]
Hence, 
\[
\|\alpha 1_D\| \leq 2K_b \left\| \sum_{j \in A_m(x)} a_j e_i + \sum_{j \in A_m(x)} e_j^*(x) e_j + \alpha 1_D \right\| \leq 2K_b C_s \sigma_{m}(y)
\]
\[
\leq 2K_b C_s \left\| \sum_{j \in A_m(x)} (e_j^*(x) + \delta e_j) e_j + \sum_{j \in A_m(x)} e_j^*(x) e_j \right\|.
\]
Taking $\delta \to 0$, $\|\alpha 1_D\| \leq 2K_b C_s \|x\|$. Using the last inequality and (11), 
\[
\|x - \mathcal{G}_m(x)\| \leq K_h C_s (\|x\| + 2K_b C_s \|x\|) \leq 3(K_b C_s)^2 \|x\|.
\]
Thus, $\|x - \mathcal{G}_m(x)\| \leq 3(K_b C_s)^2 \|x\|$ for any finite $x \in X$ and $m \leq |\supp(x)|$.

For the general case, we take $x \in X$ and $A_m(x)$ the greedy set of $x$ with cardinality $m$. We can find a number $N \in \mathbb{N}$ such that $A_m(x) \subset \{1, \ldots, N\}$. Then, since $\mathcal{G}_m(x) = \mathcal{G}_m(S_N(x))$, applying that $\mathcal{B}$ is Schauder and quasi-greedy for elements with finite support, 
\[
\|x - \mathcal{G}_m(x)\| \leq \|x - S_N(x)\| + \|S_N(x) - \mathcal{G}_m(x)\| = \|x - S_N(x)\| + \|S_N(x) - \mathcal{G}_m(S_N(x))\| \leq 2K_b \|x\| + 3(K_b C_s)^2 \|S_N(x)\| \leq K_b (2 + 3(K_b C_s)^2) \|x\|.
\]

This completes the proof. \[ \Box \]

Proof of Corollary 1.12: The proof follows using Theorem 1.10, Theorem 1.4, and Remark 1.6. \[ \Box \]
Remark 3.1. In [2, Section 6-Question 3], the authors ask the following question: if a basis $B$ satisfies Property (A) and the inequality (5), is $B$ semi-greedy? We remind that $B$ satisfies Property (A) if there is a positive constant $C_a$ such that
\[
\|x + \mathbf{1}_A\| \leq C_a\|x + \mathbf{1}_B\|,
\]
for any $x \in X$, $A, B$ such that $|A| = |B| < \infty$, $A \cap B = \emptyset$, $(A \cup B) \cap \text{supp} \,(x) = \emptyset$, $|\varepsilon| = |\eta| = 1$ and $\max \,|e_i^*(x)| \leq 1$. The answer is not due to the example in [1, Subsection 5.5] of a basis $B$ in a Banach space such that $B$ satisfies the Property (A) and (5), but is not quasi-greedy, hence is not almost-greedy and using Theorem 1.10, $B$ is not semi-greedy.

4. Open Questions

As discussed in [8] (see also [4]), one can define the Thresholding Greedy Algorithm and the Thresholding Chebyshev Greedy Algorithm in the context of Markushevich bases, that is, $\{e_i, e_i^*\}$ is a semi-normalized biorthogonal system, $X = \text{span}\{e_i : i \in \mathbb{N}\}$ and $X^* = \text{span}\{e_i^* : i \in \mathbb{N}\}$. In section a) of Theorem 1.10, it is enough to work with Markushevich bases instead of Schauder bases. However, in the item b), seems to be necessarily to use that $B$ is Schauder to prove the result.

**Question 1**: Is it possible to remove the condition to be Schauder in section b) of Theorem 1.10?

Another interesting problem is to establish if almost-greediness implies the condition to be Schauder. Of course, if $B$ is greedy then $B$ is Schauder since greediness implies unconditionality. As far as we know, all of examples of almost-greedy bases in the literature seem to be Schauder bases, but we don’t know if almost-greediness implies that $B$ is Schauder or not.

**Question 2**: If $B$ is almost-greedy, is it necessarily Schauder?

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