Dark matter halos: consistency between rotational curves and gravitational lenses

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Abstract. The main point of this work is to emphasize the fact that the estimation of the mass content of a dark matter halo must be done in a consistent way with the nature of the dark matter. In general relativity, depending on how dark matter nature is modelled, different expressions arise for the computation of the mass and the deflection of light. Here we present the comparison of the mass estimated by assuming two different dark matter models, the first one in which dark matter is modelled by a perfect fluid and the second in which it is modelled by a scalar field, but assuming they both reproduce the same rotation curve and discuss the consequences for the lensing observations.

1. Introduction
It is well known that the ΛCDM cosmology has been very successful in describing the large scale universe but at small scales there is still discrepancies when comparing the results of $N$-body simulations with the observations. A related issue is the fact that the inferred mass from different approaches, as dynamical analysis and gravitational lensing, presents some discrepancies [1], possibly related with the nature of dark matter. The ΛCDM paradigm usually assumes that the dark matter is a perfect fluid of negligible pressure. As dark matter nature is not known yet this assumption could be not correct. For example, in some models of scalar field dark matter pressure can play an important role [2]. So that, the formalisms used to study dark matter, e.g. gravitational lensing modelling, could be incomplete. On the other hand, it has been proposed that combined observations of rotational curves and gravitational lensing could constraint the equation of state of the dark matter [3]. This is a good approach as far as complete expressions for the determination of the gravitational potential, mass function, and lensing observables quantities are used. In the following sections we review the relation between the metric functions with the observable quantities and the validity of the Newtonian limit for dark matter models. Finally we give an example for the calculation of the mass inferred if we consider that dark matter is a perfect fluid or a scalar field.

2. Static and spherical system in general relativity
The most general static and spherically symmetric metric that can be used to describe a dark matter halo is given by

$$ds^2 = -e^{2\phi/c^2} dt^2 + \frac{dr^2}{1 - \frac{2Gm}{c^2 r}} + r^2 d\Omega^2$$

(1)
where, \( d\Omega = d\theta^2 + \sin^2 \theta \, d\phi^2 \). This metric is completely determined by the two metric functions \( m \) and \( \phi \) which depends only on \( r \); a prime stands for derivative respect to \( r \). The dynamics of a test particle moving in a space-time described by this metric is given by the geodesic equations. One of the equations leads to the relation

\[
\phi' = \frac{v^2}{c^2}.
\]  

(2)

for the case of massive test particles in circular orbit [4]. It is important to note that this relation is exactly the same as in Newtonian dynamics [5], but at this point we have not made any assumption on the magnitude of these quantities. Because the potential is given in terms of the circular velocity of the halo, equation (2), and this is very small compared to the speed of light it is well justified to take the weak field limit from now on, that is \( \phi(r)/c^2 \ll 1 \) and \( c^2\phi/c^2 \approx (1 + 2\phi/c^2) \).

On the other hand, we can consider unbound trajectories for photons as test particles. These trajectories are given by

\[
d\phi/dr = \frac{L}{r^2 \sqrt{\left(1 - \frac{2Gm}{rc^2}\right) \left(E^2 e^{-2\phi/c^2} - \frac{L^2}{r^2}\right)}},
\]

(3)

where \( L \) and \( E \) are constants of motion associated to the metric [6]. If the trajectory satisfy the conditions \( \phi = \phi(r_m) \), \( r = r_m \) and \( dr/d\phi = 0 \) at the moment of the closest approach, the trajectory suffers a deflection. Using these conditions the equation for the trajectory can be simplified. In principle, if somehow we are able to identify the trajectory of photons, \( d\phi/dr \), directly to an observable quantity, then we can recover the mass function by solving equation (3). Unfortunately the trajectory of photons is not an observable quantity.

What is closely related to an observable quantity is the deflection angle of the light rays, which is given by the relation \( \alpha(r) = 2(\phi_{\infty} - \phi_m) - \pi \). So, by integrating the simplified equation (3) we obtain the deflection angle, in terms of the mass function, \( m \), and the impact parameter, \( r_m \)

\[
\phi_{\infty} - \phi_m = -\int_{r_m}^{r_{\infty}} \frac{r_m \, dr}{r^2 \sqrt{\left(1 - \frac{2Gm}{rc^2}\right) \left(1 + 2\phi(r_m)/c^2 - \frac{r_m^2}{c^2}\right)}}.
\]  

(4)

The deflection angle has to satisfy the lens equation \( \beta = \theta - \frac{D_{OL}}{D_{OS}} \alpha(D_{LS}\theta) \), where \( \beta \) is the actual position of the source, \( \theta = r_m/D_{OL} \) is the observed apparent position of the image, \( D_{OL}, D_{OS} \) and \( D_{LS} \) are the distances between the lens, the source and the observed are given by the lens configuration. Up to now we have just given the relations between the metric potentials in terms of two types of observations, the rotational curves and gravitational lensing. To see the influence of the dark matter on the determination of these quantities we will consider Einstein’s equations for two types of dark matter modelling, the perfect fluid and the scalar field.

2.1. Einstein equations

For a perfect fluid (PF) the stress energy tensor is a function the density, \( \rho \), and the pressure, \( p \), of the fluid. While the scalar field (SF) stress energy tensor is a function of the potential, \( V(\chi) \), and the scalar field \( \chi \). These stress energy tensors has been used to model the dark matter halos.

In terms of the component of the stress energy tensor Einstein’s equations can be written as

\[
\frac{4\pi G r}{c^4} T^r_r = \left(1 - \frac{2Gm}{rc^2}\right) \frac{v^2}{c^2} - \frac{Gm}{c^2 r^2}.
\]  

(5)
\[ T^r_r + \left( \frac{1}{r} \left( \frac{v_c}{c} \right)^2 + \frac{2a}{r} \right) \left( T^r_r - T^t_t \right) = 0 \]  \hspace{1cm} (6)

and \( m'(r) = -\frac{4\pi r^2}{c^2} T^t_t \) gives the density. The factor \( a \) differentiates between the two stress energy tensors, i.e. \( a = 0 \) for the perfect fluid and \( a = 1 \) for the scalar field. Note we have substituted the potential function in terms of the circular velocity.

In the weak field limit, equation (5) and (6) are combined to get the following relations between the mass and the circular velocity.

\[
\frac{m}{r^3}' = \frac{c^2}{G} \left( \frac{v_c^2}{c^2 r^2} \right)' \hspace{1cm} \text{PF} \hspace{1cm} (7)
\]

\[
(mr)' = -\frac{c^2}{G} r^2 \left( \frac{v_c^2}{c^2} \right)' \hspace{1cm} \text{SF} \hspace{1cm} (8)
\]

From equations (7) and (8) we see that the mass function is also determined by the circular velocity, but it has completely different expressions if the matter is modelled by a perfect fluid or by a scalar field. This means that although the gravitational potential is completely determined by the observation of the circular velocity, the mass inferred from this observation depends on the kind of matter we are assuming. In fact, equation (7), leads to the familiar Newtonian expression \( v_c^2 = \frac{GM(r)}{r} \) [5], but equation (8) do not have a Newtonian counterpart.

3. Comparison between models

To get a better idea of what we are talking about, we will see an example. Rotational curves of spiral galaxies have been fitted from a wide variety of models, some based on the luminosity properties of the galaxies and some others based on the result of \( N \)-body simulations. We consider that there is a set of observations that are well fitted by the expression

\[
v_c^2 = k \left( \frac{r}{r_0} \right) \left( -\frac{r/r_0}{1 + r/r_0} + \ln \left[ 1 + r/r_0 \right] \right) \hspace{1cm} (9)
\]

where \( k \) is a constant associated to the mass of the galaxy and \( r_0 \) is some characteristic radius.

According to equations (7) and (8) the mass function is

\[
m(r) = \frac{k}{G} \left( -\frac{r/r_0}{1 + r/r_0} + \ln \left[ 1 + r/r_0 \right] \right) \hspace{1cm} \text{PF} \hspace{1cm} (10)
\]

\[
m(r) = \frac{k}{Gr/r_0} \left[ \frac{1}{1 + r/r_0} - 3 (1 + r/r_0) \right] + 3 \ln \left[ 1 + r/r_0 \right] + (1 + r/r_0) \ln \left[ 1 + r/r_0 \right] \hspace{1cm} \text{SF} \hspace{1cm} (11)
\]

Note that equation (10) is the mass profile of the Navarro-Frenk-White model \((k = 4\pi G \rho_0 r_0^3)\), not surprising because the rotational curve we chose is exactly the one predicted from the NFW density profile [7]. In the case of the scalar field dark matter, the expression for the mass results different, equation (11). In figure (1.a) we show a graphical comparison of the expressions given above where we can see that the mass can be over (under) estimated if we are not modelling dark matter correctly (dashed line for the perfect fluid and solid line for the scalar field dark matter).
4. Gravitational lensing as a consistency test
Observations of gravitational lensing are used to estimate the mass content of the dark matter distribution that causes the lens. The formalism used consider that the deflection of light caused by an extended mass distribution can be calculated from the superposition of the deflection caused by point masses, this assumptions implies that equation (4) reduces to

$$\alpha = \frac{4 G M(\xi)}{c^2 \xi}$$

(12)

where $M(\xi) = 2\pi \int_0^\xi d\xi' \Sigma(\xi')$ is the mass projected on to a plane defined by the point of closest approach [6], i.e. where deflection occurs, being this expression the one used for study gravitational lensing. In the case of the dark matter is modelled by a perfect fluid there are no differences in using expression (4)(solid curve in figure (1.b)) or equation (12)(dashed line in figure (1.b)), except for the case where the deflection occurs at the inner regions of the halo.

When the dark matter is a scalar field then we have to look for a consistency on the mass estimated by expressions presented above and the ability of this mass distribution to reproduce the observed lensed system. For doing this it is important to compute the lensing quantities using the complete expression, equations (2),(4) and (8).

5. Conclusions
We have presented a simple way to test the nature of dark matter based on the observation of rotational curves of galaxies and its consistency with gravitational lensing observations. In Figure (1.a) should be noted that a simple scaling of the constant $k$ for the scalar field is not enough to reconcile the results. To get good conclusions from this consistency test the most important task is to have observations of the circular velocity and gravitational lensing for the same system, and have a large number of systems so the statistics becomes relevant. Finally fig.(1.b) shows small differences for the deflection angle for deflections at the very inner regions, so that for future observations may be necessary to take into account this small differences.

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