Shannon entropy-based approach for calculating values of WABL parameters

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ABSTRACT
In the application phase of the fuzzy theory, it is an obvious advantage to have a valuable defuzzification. The defuzzification method that we deal with in this work is a flexible, adaptable and multi-purpose method. In this study, we will introduce a new concept to obtain the parameter values of the defuzzification method called WABL. The new concept is based on maximizing the entropy of the level sets weights of the method. We develop two versions for the concept. In the first one, we suppose that we have one decision-maker to supervise a fuzzy process. In the second version, we assume that we have a group of decision-makers to collectively administrate a fuzzy process. For each version, we construct a constrained optimization problem and we solve each problem analytically. The working results of the versions are demonstrated by numerical examples.

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WABL; defuzzification; entropy; fuzzy number; nonlinear optimization

Nomenclature

\( R \) set of real numbers
\( \mu_A(x) \) membership function of a fuzzy set \( A \)
\( \text{supp } A \) support set of a fuzzy number \( A \)
\( F_c(R) \) the class of fuzzy numbers
\( A_\alpha \) \( \alpha \) level-set of a fuzzy number \( A \)
\( L_A(\alpha) \) left side of a fuzzy number \( A \)
\( R_A(\alpha) \) right side of a fuzzy number \( A \)
WABL weighted averaging based on levels method
\( I(A) \) defuzzified value of a fuzzy number \( A \) calculated with WABL
\( c_L \) WABL method weight for the left side of a fuzzy number \( A \)
\( c_R \) WABL method weight for the right side of a fuzzy number \( A \)
\( p(\alpha) \) Distribution Function of WABL method to weight level-sets of a fuzzy number \( A \)
\( p(\alpha_i) \) discrete version of Distribution Function of WABL method
\( H_n(p_1, p_2, \ldots, p_n) \) Shannon entropy
\( L(x_1, \ldots, x_n) \) Lagrange function

1. Introduction

Defuzzification is among the most valuable and studied topics when looking at the application field of fuzzy logic. The reason of this interest is that defuzzification methods are easily adapted to various applied areas such as; decision-making in a fuzzy environment, fuzzy inference system, optimization problems with fuzzy parameters, statistical analysis for fuzzy data, etc. Different scientists have proposed or employed a variety of defuzzification methods with respect to their specific requirements. Goala and Dutta [1] determined the area of a city which may be potentially under threat of terrorist attack by employing the fuzzy multi-criteria decision-making approach. In order to rank the areas with respect to criteria, they utilized centre of area defuzzification method. Vahidi [2] suggested a new approach to solve a fuzzy mathematical problem. The approach was constructed on the defuzzification method that obtains crisp value whose membership value is greater than a pre-determined membership level. Verstraete [3] extended the constrained defuzzification concept for more than one fuzzy numbers. In the approach, he aimed at obtaining defuzzified value that satisfies the constraint and has membership value as high as possible. To overcome the limitations of Chen’s ranking method and include decision maker’s point of view, Peddi [4] proposed a ranking method based on centroid defuzzification method. After disintegrating a trapezoidal fuzzy numbers into three parts (2 triangles and a rectangle), Peddi calculated utility value for the trapezoidal fuzzy number based on centroids of three parts and decision maker’s optimism degree. Lopez et.al. [5] dealt with the problem of creating and controlling rule bases, which are created from big databases, for fuzzy regression. They evaluated the problem in two phases: to create an adaptation of a classic data-driven
method and to suggest a novel scalable strategy. For the second phase, they proposed determining parameters of adaptive defuzzification method by using an evolutionary procedure. Torra and Alfaro [6] suggested a novel defuzzification method for fuzzy rule-based systems. They proposed to employ Choquet integral in the defuzzification method to evaluate some properties of the domain set of the output variable of fuzzy rule-based system. Saneifard and Saneifard [7] suggested a new approach to solve Fuzzy Linear Programming Problem excluding non-negativity restrictions of fuzzy elements of the problem. In the approach, to rank fuzzy numbers, they employed a defuzzification method that benefits from the radius of gyration point of a fuzzy number.

Some of the defuzzification methods are easy to use formulas such as Centre of Area (COA), Mean of Maxima (MOM), etc. But, this type of methods inserts just part of the information provided by fuzzy numbers in concern to the defuzzification process. On the other hand; in literature, there are various defuzzification methods that may employ all of the information abstracted from fuzzy numbers in concern. Because of the advantage, various scientists showed their interests in this topic and proposed valuable defuzzification methods of the mentioned type. Nasibov [8] proposed a novel defuzzification procedure called Weighted Averaging Based on Levels (WABL) which consists of \( p(\alpha) \) – Distribution Function, \( c_L \) – weight for the left side of a fuzzy number and \( c_R \) – weight for the right side of a fuzzy number. Ibanez and Munoz [9] defined a new method to compare fuzzy numbers. The method, called Average Index, is very similar to the WABL method mathematically. Voxman [10] considered different defuzzification like approaches to assign distances between fuzzy numbers. Ma et al. [11] suggested a defuzzification like approach for a general fuzzy quantity. In the approach, they obtained the nearest symmetrical triangular fuzzy number to which a fuzzy quantity is related by using the distance formula. Similarly; by using distance formula, Abbasbandy and Asady [12] obtained the nearest trapezoidal fuzzy number which is related to a fuzzy quantity.

It is obvious that there is a strong relationship between defuzzification methods and distances or metrics of fuzzy numbers. Nasibov [13] suggested a new Fuzzy Least Squares Regression model based on a novel distance ND1 that was designed based on the WABL method. Sinova et al. [14] proposed a generalized metric between fuzzy numbers. The metric considered both location and shape of fuzzy numbers by employing the WABL method. Sinova et al. [15] proposed a novel family of generalized \( L^2 \) metrics called the wabl/ldev/rdev-based \( L^2 \) metrics again based on the WABL defuzzification method. The proposed metrics, like the WABL defuzzification method, take into account all the information provided by fuzzy data. The metrics are characterized with parameters \( \theta \) and \( \varphi \). Sinova et al examined the effect of selecting values of parameters \((\theta \text{ and } \varphi)\) on the performance of the metrics.

The attention to metrics of fuzzy numbers arises from the fact that there is a serious gap in the field of fuzzy statistical techniques. It is possible to define a measure of central tendency and/or a measure of dispersion for fuzzy data by utilizing a suitable distance or metrics of fuzzy numbers. Sinova et al. [16] extended the population and sample median concepts of a crisp valued random variable to a random fuzzy variable. In the definition of both median concepts, they employed the WABL defuzzification method based distance. Sinova et al. [17] reviewed some of the main central tendency measures proposed for the fuzzy data set. They compared performances of Aumann-type mean, 1-norm median and the wabl/ldev/rdev-median which is defined based on the WABL defuzzification method. They investigated the statistical robustness and the empirical “precision” of the measures. The wabl/ldev/rdev median is always more robust for the independent fuzzy data case. Lubiano et al. [18] analysed an inferential approach to analyse data obtained by using the fuzzy rating scale. Actually; they focused on hypothesis testing about means. They followed the procedure called the boot strapped algorithm approximating the one-sample test for the mean of a random fuzzy number. While calculating the measure of dispersion for test statistics, they employed some distances one of which was very similar to WABL based distance. Saa et al. [19] extended some crisp measures of dispersion to a random fuzzy number case. While they realized the extension, they adapted the WABL defuzzification method as a one of metrics. They investigated the robust behaviour of proposed fuzzy measures of dispersions by employing finite sample breakdown point and sensitivity curves methods. Sinova [20] proposed a robust measure of central tendency for random fuzzy number by dealing with the scale equivalence problem of other measures of central tendency. In the first approach; she tuned parameters of frequently utilized loss functions such as Hampel’s loss, Tukey’s loss and Huber’s loss. In the second; she employed WABL based distance method to reach a robust estimate of the unknown variability in the definition of the proposed measure of central tendency. She concluded that her second approach is reacted better for extreme data case. Sinova and Aelst [21] conducted a simulation study to compare performances of various location estimators such as fuzzy trimmed means, fuzzy M-estimators, sample wabl-median which was suggested based on WABL method, etc. Sinova and Van Aelst [22] designed an empirical study to figure out the maximum asymptotic bias of the fuzzy number-valued measures of central tendency that are Aumann-type sample mean, fuzzy trimmed sample mean, fuzzy M-estimator of location for sample data, 1-norm sample median and...
wabl/ldev/rdev-median for sample data. They created both contaminated and non-contaminated trapezoidal fuzzy random numbers to compare performances of mentioned location measures.

In this study, we specifically are interested in the WABL defuzzification method because it is popular and widely used in the fuzzy statistics area. However, fuzzy statistics is not the only field that the WABL method could be adapted. Nasibov and Kinay [23] studied group decision analysis in a fuzzy environment. They proposed an iterative algorithm to evaluate the fuzzy opinions of decision makers. In the method, they used Centre of Area, Mean of Maxima and WABL defuzzification methods. They concluded that the WABL method produced more accurate results than do other methods. Savaş and Nasibov [24] proposed a classification method for linguistically expressed data by combining the Fuzzy ID3 algorithm, Fuzzy c-means and WABL defuzzification method. They tested the performance of their method on various data. Nasiboglu and Erten [25] proposed a novel approach based on k-means clustering and fuzzy inference system to construct a hierarchical structure of wireless sensor networks. In the fuzzy inference system, they employed the WABL defuzzification method.

As we mentioned above, the WABL defuzzification method is widely investigated and employed method. Moreover, the method also is of some good properties such as adaptability, additivity, flexibility, integrative, etc. It is also possible to regulate parameters of WABL method according to the type of problem, decision-maker’s defuzzification strategy, etc. To add decision maker’s defuzzification strategy into the defuzzification process, Liu [26] studied on an extension of Centre of Gravity defuzzification method by employing a weighting function. Based on the weighting function, he defined decision-maker’s optimistic degree which was employed to adjust the defuzzification method. In this paper, we propose an approach to design a defuzzification method for a decision-maker or a group of decision-makers by maximizing the entropy of parameter values of the WABL method subject to a decision-maker’s or a group of decision-makers’ defuzzification strategy. In the literature, studies related to weight calculating in the information operation process were frequently conducted concerning OWA (Ordered Weighted Averaging) aggregation method. Some of researchers who prepared these studies focused on maximizing entropy [27–29] subject to pre-determined orness level. Orness is specially defined to measure how much an OWA operator is similar to “OR” operator. In the study, to maximize entropy of parameter values of the WABL method, we construct a non-linear programming problem for one decision-maker version and a nonlinear multi-objective optimization problem for a group of decision-makers version. We transform these problems to equations systems by utilizing the method of Lagrange Multipliers. Equations system approach was adapted various fields such as biology [30], physics [31,32], engineering [33–35]. In these studies, authors employed different non-linear equations system solving techniques. In the study, we prefer using Secant Method and Newton’s Method to solve our non-linear equations systems. Moreover, we present examples for both one decision-maker and a group of decision-makers versions. In this study, we give decision-maker or a group of decision-makers an opportunity to design own defuzzification method by means of the WABL method. Also, with the defuzzification method personalized by employing this novel approach, decision-maker or a group of decision-makers can add all the information provided by fuzzy numbers in concern.

We organize this paper as four sections. In Section 2, we introduce preliminary information about the concepts required in the other sections. In Section 3, we propose a novel concept and give details related to two application versions of the concept. In Section 4, we demonstrate how the versions produce results by a couple of numerical examples. Section 5 contains the conclusion of our study.

2. Preliminaries

In this part of the study, we will mention some mathematical definitions which are required for the remaining of the paper.

Definition 2.1: A fuzzy number A has following properties:

1. A fuzzy number is actually fuzzy subset of $R$ that is the real line.
2. For arbitrary fuzzy number, there is some $x_0 \in R$ such that $\mu_A(x_0) = 1$. It is called normality.
3. For arbitrary fuzzy number, every $x_1, x_2 \in R, \lambda \in [0,1] \lambda \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$ is valid. It is called convexity with respect to membership function $\mu_A(x)$.

Definition 2.2: The support of fuzzy number $A$ (supp $A$) on $R$ is the crisp set of all $x \in R$ such that $0 < \mu_A(x) \leq 1$.

Definition 2.3: $F_\varepsilon(R)$ will denote the class of (bounded) fuzzy numbers which are mappings $\mu_A : X \rightarrow [0,1]$ such that their $\alpha$ level-sets:

$$A^\alpha = \begin{cases} \{ x \in R : \mu_A(x) \geq \alpha \} & \text{if } \alpha \in (0,1) \\ \{ x \in R : \mu_A(x) > 0 \} & \text{if } \alpha = 0 \end{cases},$$

(2.1)

and

$$L_\alpha = \inf A^\alpha,$$

(2.2)

$$R_\alpha = \sup A^\alpha.$$
**Definition 2.4:** With WABL method, a defuzzified value for a fuzzy number $A$ is obtained as follows [36]:

$$l(A) = \int_{0}^{1} (c_L L_A(\alpha) + c_R R_A(\alpha)) p(\alpha) d\alpha, \quad (2.4)$$

where

$$c_L + c_R = 1,$$

$$\int_{0}^{1} p(\alpha) d\alpha = 1,$$

$$p : [0, 1] \rightarrow [0, \infty),$$

$$c_L \geq 0, \quad c_R \geq 0.$$

The $c_L$ and $c_R$ are the weights for left and right sides of the fuzzy number $A$, respectively. $p(\alpha)$ called Distribution Function is utilized to weight level-sets. With these weights, the WABL method becomes an integrative defuzzification method because all of information provided by fuzzy numbers is added to the defuzzification process.

In some cases, it is not possible to work with all of level-sets of a fuzzy number. We may just reach some level-sets $[\alpha_i; i = 1, 2, \ldots, m]$ of a fuzzy numbers in concern. For such a case, we may adapt our WABL method as follows:

$$l(A) = \sum_{i=1}^{m} (c_L L_A(\alpha_i) + c_R R_A(\alpha_i)) p(\alpha_i), \quad (2.5)$$

where

$$L_A(\alpha_i) = \inf A^{\alpha_i}; i = 1, 2, \ldots, m,$$

$$R_A(\alpha_i) = \sup A^{\alpha_i}; i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} p(\alpha_i) = 1,$$

$$c_L + c_R = 1,$$

$$p(\alpha_i) \geq 0; i = 1, 2, \ldots, m,$$

$$c_L \geq 0, c_R \geq 0.$$

In this paper, we will frequently mention the entropy concept. We will also utilize very familiar entropy named Shannon’s entropy in our approach.

**Definition 2.5:** Let $X$ be a discrete random variable such that possible values $\{x_1, x_2, \ldots, x_n\}$ is a finite set and related probabilities are $(p_1, p_2, \ldots, p_n)$ such that

$$\sum_{i=1}^{n} p_i = 1; \quad p_i \geq 0, i = 1, 2, \ldots, n.$$

Shannon’s Entropy is described by [37]:

$$H_n(p_1, p_2, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i. \quad (2.6)$$

In the defuzzification field, the entropy generally is employed to measure how much information provided by fuzzy numbers is added to process. So, the main intend among scientists is to maximize entropy value as much as possible. Along with our concept, we also maximize entropy subject to a decision-maker’s or a group of decision-makers defuzzification strategy.

### 3. Evaluation of level-sets weights with maximal entropy

In this paper, we focus on the WABL defuzzification method, because the method has some beneficial properties such as additivity, adaptability, flexibility, integrative, etc. WABL parameters are adjustable that is WABL method is a flexible one. By means of flexibility, the WABL defuzzification method can expand to various defuzzification methods. Moreover; adjusting WABL parameters subject to decision-maker’s or a group of decision-makers defuzzification strategy transforms the WABL defuzzification method into decision-maker’s or a group of decision-makers idealized defuzzification method. However, it is not easy to express mathematically the defuzzification strategy for some decision-makers. Some methods to help decision-makers in overcoming this difficulty were proposed in the literature [38,39].

It will be better if we are able to embed all of the available information provided by fuzzy numbers in the defuzzification process. This could be achieved by merging the entropy concept and optimization mentality in obtaining values of WABL parameters. Along with our concept, for one decision-maker version, we will construct an optimization problem whose objective function will be written with respect to Shannon Entropy measure of WABL parameters. Constraints of the optimization problem will be related to decision-maker’s defuzzification strategy, normality and non-negativity conditions of WABL parameters. For a group of decision-makers version, we will create multi-objective optimization problem whose one objective function again will be written with respect to Shannon Entropy and the other one will be constructed with respect to a group of decision-makers defuzzification strategy. Constraints of the optimization problem will be normality and non-negativity conditions of WABL parameters. To formalize decision-makers strategy, we will want decision-makers to determine a defuzzified value for a fuzzy number.

#### 3.1. One decision-maker version

For the version, we assume that we have just one decision-maker to manage all the fuzzy process. So, we aim at including his/her defuzzification strategy in to the process. We request decision-maker to assign the weight for left side ($c_L$) of the fuzzy number $A$ between 0 and 1 then the value of $c_R = 1 - c_L$ is calculated. Then, we demonstrate a graphic of the fuzzy number to the
By using these level-sets and replies (the defuzzified value \( a \) and values of \( \lambda_1 \) and \( \lambda_2 \)) of the fuzzy number, we employ discretized version of the fuzzy number to make calculations easier. We determine level-sets to be evaluated as:

\[
[a_i]; \ i = 1, 2, \ldots, m. \tag{3.1}
\]

Using these level-sets and replies (the defuzzified value \( a \) and values of \( c_1 \) and \( c_9 \)), the nonlinear programming problem is written as follows:

\[
\begin{aligned}
- \sum_{i=1}^{m} p(a_i) \ln p(a_i) \rightarrow \max \\
\sum_{i=1}^{m} (c_1 L_A(a_i) + c_9 R_A(a_i)) p(a_i) = a \\
\sum_{i=1}^{m} p(a_i) = 1 \\
p(a_i) \geq 0, \ i = 1, 2, \ldots, m
\end{aligned}
\]  \tag{3.2}

The method of Lagrange Multipliers is one of these suitable methods to solve the problem. With employing the method, we shall transform the problem to a polynomial equation which is then solved to determine the values of WABL parameters. Let

**Lemma 3.1:** The solution of the nonlinear programming problem (3.2) which are constructed based on certain level-sets of the fuzzy number and decision-maker’s replies (the defuzzified value of the fuzzy number \( a \)) and values of \( c_1 \) and \( c_9 \) is calculated as follows:

\[
p(a_i) = e^{\lambda_1 u_i} \cdot e^{\lambda_2 - 1}; \ i = 1, 2, \ldots, m, \tag{3.3}
\]

where

\[
\lambda_2 = 1 - \ln \left( \sum_{i=1}^{m} e^{\lambda_1 u_i} \right),
\]

\[
u_i = c_1 L_A(a_i) + c_9 R_A(a_i); \ i = 1, 2, \ldots, m.
\]

For \( \lambda_1 \), it is not possible to obtain a formula. So, we get the value of \( \lambda_1 \) as the solution of the following equation:

\[
\sum_{i=1}^{m} (a - \nu_i) e^{\lambda_1 u_i} = 0. \tag{3.4}
\]

**Proof:** For the nonlinear programming problem (3.2), the Lagrange function is:

\[
L(p(a_1), \ldots, p(a_m); \lambda_1, \lambda_2) = - \sum_{i=1}^{m} p(a_i) \ln p(a_i) + \lambda_1 \left( \sum_{i=1}^{m} u_i p(a_i) - a \right) + \lambda_2 \left( \sum_{i=1}^{m} p(a_i) - 1 \right), \tag{3.5}
\]

where

\[
\lambda_1, \lambda_2 \in \mathbb{R}.
\]

The partial derivatives of the \( L \) are as follows:

\[
\frac{\partial L}{\partial p(a_i)} = - \ln p(a_i) - 1 + \lambda_1 u_i + \lambda_2 = 0; \ i = 1, 2, \ldots, m. \tag{3.6}
\]

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{m} u_i p(a_i) - a = 0. \tag{3.7}
\]

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{m} p(a_i) - 1 = 0. \tag{3.8}
\]

When we arrange \( m \) equations given in (3.6), we can state \( p(a_i); i = 1, 2, \ldots, m \) parameters as in (3.3) by means of \( \lambda_1 \) and \( \lambda_2 \).

We restitute the equivalence of each \( p(a_i) \) in (3.7) and (3.8) so that we can obtain the values of \( \lambda_1 \) and \( \lambda_2 \).

\[
\sum_{i=1}^{m} u_i e^{\lambda_1 u_i} \cdot e^{\lambda_2 - 1} - a = 0. \tag{3.9}
\]

\[
\sum_{i=1}^{m} e^{\lambda_1 u_i} \cdot e^{\lambda_2 - 1} - 1 = 0. \tag{3.10}
\]

From (3.10), first, we obtain \( e^{\lambda_2 - 1} = (1 / \sum_{i=1}^{m} e^{\lambda_1 u_i}) \) then we put the equivalence in (3.9), so, we reach (3.4). Moreover, from (3.10), we can write \( \lambda_2 = 1 - \ln \left( \sum_{i=1}^{m} e^{\lambda_1 u_i} \right) \).

To obtain \( p(a_i); i = 1, 2, \ldots, m \) values, we should follow a bunch of steps. We first solve the equation (3.4) to obtain the value of \( \lambda_1 \). Although; it is possible to exploit various methods to obtain solutions of the above-mentioned problem, as in these references [33–35], we prefer employing the Secant Method. After getting the value of \( \lambda_1 \) through the Secant Method, we calculate \( \lambda_2 = 1 - \ln \left( \sum_{i=1}^{m} e^{\lambda_1 u_i} \right) \). Finally, from (3.3), we calculate \( p(a_i); i = 1, 2, \ldots, m \) which reflect the decision-maker’s defuzzification strategy and maximize the total entropy. In this phase, we have to prove that the problem (3.2) will have at least one solution.

**Theorem 3.1:** Let \( u_i = c_1 L_A(a_i) + c_9 R_A(a_i); i = 1, 2, \ldots, m \) be calculated based on pre-determined values of \( c_1 \) and \( c_9 \) and the fuzzy number. We denote \( a = \min u_i \) and \( \bar{a} = \max u_i \). As soon as the decision-maker’s reply of the defuzzified value of the fuzzy number (a) within the interval \([a, \bar{a}]\), the equation \( \sum_{i=1}^{m} c_1 L_A(a_i) + c_9 R_A(a_i) p(a_i) = a \) has a solution like \( p(a_i) \geq 0; i = 1, 2, \ldots, m \).
Proof: It is clear that \( f(p) = \sum_{i=1}^{m} p(a_i)u_i \) is a linear function of all \( p(a_i) \). Let \( g \) and \( \tilde{a} \) be two extreme values of the function \( f(p) \). So, an arbitrary value between \( g \) and \( \tilde{a} \) can be stated as a convex combination of \( g \) and \( \tilde{a} \). It is obvious that there are \( p(a_i) \geq 0; i = 1, 2, \ldots, m \) like that
\[
\begin{align*}
g &= f(p) = \sum_{i=1}^{m} p(a_i)u_i \\
\tilde{a} &= f(\tilde{p}) = \sum_{i=1}^{m} p(a_i)u_i.
\end{align*}
\] (3.11)

Thus, we can write the following statement.
\[
\begin{align*}
a &= \beta g + (1 - \beta)\tilde{a} = \beta f(p) + (1 - \beta)f(\tilde{p}) \\
&= f(\beta p) + f((1 - \beta)\tilde{p}) = f(\beta p + (1 - \beta)\tilde{p}),
\end{align*}
\] (3.12)

where \( \beta \in [0, 1] \).

As a result, there are \( p(a_i) = \beta p(a_i) + (1 - \beta)\tilde{p}(a_i); i = 1, 2, \ldots, m \) and these are a solution of the equation given in the theorem. Moreover, because the solution is convex combination of \( p(a_i) \geq 0; i = 1, 2, \ldots, m \) and \( \tilde{p}(a_i) \geq 0; i = 1, 2, \ldots, m \), it is \( p(a_i) \geq 0; i = 1, 2, \ldots, m \).

The process can be algorithmically summarized as follows:

Algorithm:

Step 0: Level-sets to be processed \( \{a_i\}; i = 1, 2, \ldots, m \) are pre-determined, a fuzzy number \( A \) determined. Move to Step 1.

Step 1: The decision-maker assigns the weight for left side \( (c_l) \) of the fuzzy number \( A \) between 0 and 1 then the value of \( c_R = 1 - c_l \) is calculated. The \( u_i = c_lL_A(a_i) + c_RR_A(a_i); i = 1, 2, \ldots, m \) are calculated based on the decision-maker's assignment. \( g = min u_i \) and \( \tilde{a} = max u_i \) are determined.

Step 2: It is requested from the decision-maker to determine a defuzzified value \( (a) \) for the fuzzy number \( A \) between \( g \) and \( \tilde{a} \).

Step 3: The solution of the Equation (3.4) is obtained by means of the Secant method.

Step 4: Values of \( p(a_i); i = 1, 2, \ldots, m \) are calculated by using first \( \lambda_2 = 1 - \ln \left( \sum_{i=1}^{m} \epsilon_i^{1/u_i} \right) \) and then (3.3).

Based on the theorem and the lemma, we claim that the algorithm will always produce a solution if the defuzzified value \( (a) \) that is the reply of the decision-maker will be in \( [g, \tilde{a}] \). The main advantage of the version is that we can obtain the values of WABL parameters even based on decision-maker's one reply. And also, we make more WABL parameters been involved to expose decision-maker’s defuzzification strategy with the aid of the objective function of the nonlinear programming problem.

The case evaluated in this part of the study is designed to deal with the situation that all the fuzzy process is managed by just one decision-maker. However, for some instances, a fuzzy process may be handled by a group of decision-makers. So, it is a need to develop a method for these instances.

3.2. A group of decision-makers version

For a group of decision-makers version, we assume that we have \( k \)-many decision-makers to manage all the fuzzy process collectively. So, we aim at including their collective defuzzification strategy into the fuzzy process. It is obvious that determining WABL parameters that reflect exactly all decision-makers’ defuzzification strategy is not an easy accomplishment. Therefore, in a group of decision-makers version, we will create the objective function that minimizes the total of squared differences between decision-makers’ replies and their estimations that are calculated by the WABL defuzzification method. Moreover, we want to embed all of available information that is provided by fuzzy numbers in the defuzzification process. Therefore, we will create another objective function with respect to Shannon Entropy measure of WABL parameters. That means we will deal with a nonlinear multi-objective optimization problem.

To construct the optimization problem, first; we ask decision-makers to assign the compromise weight for left side \( (c_l) \) between 0 and 1 then the value of \( c_R = 1 - c_l \) is calculated. Then, we demonstrate a fuzzy number to each decision-maker and request all decision-makers to determine a defuzzified value \( (a_j; j = 1, 2, \ldots, k) \) of the shown fuzzy number. Without loss of generality, we employ discretized versions of the fuzzy numbers to make calculations easier. We determine level-sets to be evaluated as \( \{a_i\}; i = 1, 2, \ldots, m \). Using these levels-sets and replies (defuzzified values \( (a_j; j = 1, 2, \ldots, k) \) and values of \( c_l \) and \( c_R \)), the nonlinear multi-objective optimization problem is constructed as follows:

\[
\left\{ \begin{array}{l}
\sum_{i=1}^{m} p(a_i) \ln p(a_i) \rightarrow \max \\
\sum_{j=1}^{k} \left( \sum_{i=1}^{m} (c_lL_A(a_i) + c_RR_A(a_i))p(a_i) - a_j \right)^2 \rightarrow \min \\
p(a_i) = 1 \\
p(a_i) \geq 0; i = 1, 2, \ldots, m
\end{array} \right. 
\] (3.13)

The problem (3.13) can be simplified. It is possible to write the first objective function in the minimization form. So, this enables us to write below the form of the
problem.

\[
\sum_{i=1}^{m} p(\alpha_i) \ln p(\alpha_i)
\]
\[+
\sum_{j=1}^{k} \left( \sum_{i=1}^{m} (c_L L_i(\alpha_i) + c_R R_i(\alpha_i)) p(\alpha_i) - a_j \right)^2 \rightarrow \min
\]

\[
\sum_{i=1}^{m} p(\alpha_i) = 1
\]
\[
p(\alpha_i) \geq 0; \ i = 1, 2, \ldots, m
\]
(3.14)

We may use the method of Lagrange Multipliers to solve this problem. Using the method of Lagrange Multipliers, we shall transform the problem to a nonlinear equations system which is then solved. For the nonlinear programming problem (3.14), the Lagrange function is:

\[
L(p(\alpha_1), \ldots, p(\alpha_m), \lambda) = - \sum_{i=1}^{m} p(\alpha_i) \ln p(\alpha_i) + \sum_{j=1}^{k} \left( \sum_{i=1}^{m} u_{ij} p(\alpha_i) - a_j \right)^2 + \lambda \sum_{i=1}^{m} p(\alpha_i) - 1,
\]

where

\[
\lambda \in \mathbb{R},
\]
\[
u_{ij} = c_L L_i(\alpha_i) + c_R R_i(\alpha_i); \ i = 1, 2, \ldots, m;
\]
\[
j = 1, 2, \ldots, k.
\]

The partial derivatives of the L are as follows:

\[
\frac{\partial L}{p(\alpha_i)} = \ln p(\alpha_i) + 1 + 2 \sum_{j=1}^{k} u_{ij} \left( \sum_{i=1}^{m} u_{ij} p(\alpha_i) - a_j \right)
\]
\[+ \lambda = 0; \ i = 1, 2, \ldots, m.
\]

(3.16)

\[
\frac{\partial L}{\lambda} = \sum_{i=1}^{m} p(\alpha_i) - 1 = 0.
\]

(3.17)

In this system, we have \( m+1 \) nonlinear equations (3.16) – (3.17) and \( m+1 \) variables. The solution set of this system (3.16) – (3.17) is overlapped the nonlinear multi-objective optimization problem (3.13). We prefer utilizing Newton’s Method to solve the system (3.16) – (3.17).

In real situations, generally, a group of people make decisions rather than one person. In this kind of situations, a group of decision-makers version looks like suitable to construct a collective defuzzification strategy. Naturally, all decision-makers want to affect the final decision. The use of this methodology makes the final decision of the fuzzy system in concern covered the all decision-makers’ defuzzification strategy because the values of WABL parameters are obtained based on all decision-makers’ replies. For the situation, it is not possible to obtain values of WABL parameters that can mimic exactly all decision-makers’ replies. So, we aim at minimizing total of squared differences between decision-makers’ replies and their estimations that are calculated by the WABL defuzzification method. Moreover, we would like to utilize as much information as possible from fuzzy numbers. We again employ the objective function concerning entropy. Using these two objective functions and normality and non-negativity conditions of WABL parameters, we construct the nonlinear multi-objective optimization problem (3.13). Then we transform (3.13) to the nonlinear equations system (3.16) – (3.17), finally; solve the system (3.16) – (3.17) by employing Newton’s Method.

4. Applications of versions

In this part of the study, we design two examples to show how cases are applied in numerically. In the first example, we obtain one decision-maker’s strategy by collecting just one reply. The first case that is designed for one decision-maker is applied four different decision-makers and results are presented in the first example. In the second example, we aim at determining four decision-makers’ collective defuzzification strategy. The second case is repeated for three different groups of four decision-makers. In both examples, \( \alpha_i = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\} \) level-sets are utilized.

Example 1: In the example, we utilize the trapezoidal fuzzy number \( A = (3, 6, 7, 11) \) to display the decision-maker. Before the decision-maker choose a defuzzified value for the fuzzy number, we want the decision-maker to assign the weight for left side \( c_L \) of the fuzzy number \( A \) between 0 and 1 then value of \( c_R = 1 - c_L \) is calculated. Based on replies and using (3.4), we construct the equation. Applying Secant Method with \( \varepsilon = 0.0001 \), we first obtain \( \lambda_1 \) as the solution of the equation and then \( \lambda_2 \) is calculated by means of \( \lambda_1 \). We put these two results in (3.3) to reach values of \( p(\alpha_i) \); \( i = 1, 2, \ldots, 11 \). We repeat the same procedure for 4 different decision-makers. The resulting values of WABL parameters, \( \lambda_1, \lambda_2 \) and entropy values for each decision-maker are displayed in Table 1.

Example 2: The fuzzy numbers employed in the Example 2 are \( A_1 = (10, 12, 15) \), \( A_2 = (4, 7, 9, 16) \), \( A_3 = (5, 6, 10, 14) \), \( s = 2 \), \( A_4 = (2, 4, 6, 11) \). In the first step, we ask four decision-makers to collectively assign the weight for left side \( c_L \) of the fuzzy number \( A \) between 0 and 1 then value of \( c_R = 1 - c_L \) is calculated. And, we randomly match each decision-maker with a fuzzy number and collect defuzzified values of fuzzy numbers. We construct a nonlinear equations system with 12 equations and 12 variables based on replies of four decision-makers. We solve the system by utilizing Newton’ Method with \( N = 1000 \) (the maximum number
of iteration). We repeat the same procedure for three different groups of decision-makers.

In Table 2, the values of \( p(\alpha_i) \); \( i = 1, 2, \ldots, 11 \), \( \lambda \) and entropy values are displayed. Defuzzified values determined by decision-makers, the estimated defuzzified values by WABL defuzzification method for each fuzzy number and total of squared differences for all groups are illustrated in Table 3.

5. Conclusions

In recent years, various scientists pay attention to fuzzy statistical methods area. Especially, fuzzy extensions of measures of central tendency and dispersion are widely studied. WABL defuzzification method finds a huge place in the fuzzy statistical methods area. The WABL parameters \( c_L \) and \( c_R \) are employed to model weights (determined by a decision-maker or a group of decision-makers) for left and right sides of the fuzzy number. The function \( p(\alpha) \) (or discreet \( p(\alpha_i) \) parameters) is used to weight level-sets of a fuzzy number A. So, it is possible to bend the WABL defuzzification method with respect to the decision-maker’s strategy.

WABL is a valuable defuzzification method because of its parameters’ being adjustable subject to one decision-maker’s or a group of decision-makers defuzzification strategy and its widely usage in fuzzy statistics area. In this paper, we focus on determining the values of WABL parameters. We proposed a novel concept which satisfies two requirements. The first requirement is that values of WABL parameters mirror a particular decision-maker’s or a group of decision-makers’ defuzzification strategy. And the second is that total weights of \( \alpha \) level-sets must be scattered among all level-sets so that the entropy of the level-sets weights

Table 1. \( p(\alpha) \) and entropy values for one decision-maker case.

| \( c_L = 0.15 \) | \( c_R = 0.85 \) | \( \lambda_1 \) | \( \lambda_2 \) | Entropy |
|---|---|---|---|---|
| \( p(0.0) \) | 0.241693 | 0.249392 | 0.008352 | 0.005886 |
| \( p(0.1) \) | 0.186689 | 0.190388 | 0.011984 | 0.008622 |
| \( p(0.2) \) | 0.142427 | 0.145344 | 0.017195 | 0.013223 |
| \( p(0.3) \) | 0.111358 | 0.110956 | 0.019032 | 0.010001 |
| \( p(0.4) \) | 0.086036 | 0.084705 | 0.024672 | 0.029704 |
| \( p(0.5) \) | 0.064546 | 0.064665 | 0.030791 | 0.044520 |
| \( p(0.6) \) | 0.043533 | 0.043634 | 0.036872 | 0.066727 |
| \( p(0.7) \) | 0.030626 | 0.028770 | 0.040312 | 0.019818 |
| \( p(0.8) \) | 0.023657 | 0.021693 | 0.045267 | 0.007858 |
| \( p(0.9) \) | 0.018273 | 0.016767 | 0.050791 | 0.044520 |
| \( p(1.0) \) | 0.013859 | 0.012767 | 0.056212 | 0.033672 |

Table 2. \( p(\alpha_i) \) and entropy values for multiple decision-makers case.

| Group-1 | Group-2 | Group-3 |
|---|---|---|
| \( c_L = 0.10 \) | \( c_R = 0.90 \) | \( c_L = 0.45 \) | \( c_R = 0.55 \) | \( c_L = 0.80 \) | \( c_R = 0.20 \) |
| \( p(0.0) \) | 0.318377 | 0.150468 | 0.200096 |
| \( p(0.1) \) | 0.193252 | 0.094457 | 0.127348 |
| \( p(0.2) \) | 0.129884 | 0.077652 | 0.107456 |
| \( p(0.3) \) | 0.091303 | 0.068866 | 0.094287 |
| \( p(0.4) \) | 0.066845 | 0.064943 | 0.084494 |
| \( p(0.5) \) | 0.050899 | 0.064777 | 0.076812 |
| \( p(0.6) \) | 0.040253 | 0.068156 | 0.070601 |
| \( p(0.7) \) | 0.030699 | 0.075529 | 0.065843 |
| \( p(0.8) \) | 0.028060 | 0.088066 | 0.067124 |
| \( p(0.9) \) | 0.024973 | 0.079700 | 0.057622 |
| \( p(1.0) \) | 0.022949 | 0.139117 | 0.054587 |
| \( \lambda_1 \) | \( 1.4 \times 10^{-6} \) | \( 1.7 \times 10^{-6} \) | \( 2.2 \times 10^{-6} \) |
| Entropy | 2.019558 | 2.352293 | 2.312482 |

Table 3. Decision-makers’ defuzzified values and estimated defuzzified values for multiple decision-makers case.

| Group 1 | Group 2 | Group 3 |
|---|---|---|
| \( A_1 \) | 13.8 | 13.9 | 12.6 | 12.7 | 11.1 | 11.3 | 6.6 | 6.7 | 6.7 |
| \( A_2 \) | 14.2 | 13.36 | 10.0 | 9.34 | 6.6 | 6.7 | 6.7 | 6.7 | 6.7 |
| \( A_3 \) | 12.4 | 12.65 | 8.8 | 9.34 | 6.6 | 6.7 | 6.7 | 6.7 | 6.7 |
| \( A_4 \) | 8.5 | 9.10 | 5.7 | 9.34 | 6.6 | 6.7 | 6.7 | 6.7 | 6.7 |
| Total of Squared Diff. | 1.1381 | 0.9917 | 0.2045 |
can be maximized. That kind of scatter makes available information that is provided by fuzzy numbers added to the defuzzification process as much as possible. Based on the novel concept, we evaluate two different versions. In the first version, we assume that a fuzzy process is managed by one decision-maker, so we aim at including his/her defuzzification strategy into the fuzzy process. In the second version, we suppose that a fuzzy process is controlled by a group of decision-makers. We focus on embedding their collective defuzzification strategy into the fuzzy process.

We state these two versions in detail and give an example for each. From examples, we conclude that approaches perform well. We choose 11 different \( \alpha \) level-sets, so the maximum value of the entropy is 2.3979. For the first version, we reach satisfactory entropy values, and all of the \( p(\alpha_i) \) values are greater than zero as we wish. For the second version, calculated entropy values are satisfactory again. Moreover, Total of Squared Differences values for each group from Table 2 support the claim that there is no huge gap between decision-makers’ choice and our estimated defuzzified values. And again, for each group, all of the \( p(\alpha_i) \) values are greater than zero as we aim at.

Various values of WABL defuzzification parameters definitely will affect the performance of the method. In the future, the effect of WABL parameters on performances of fuzzy statistical methods could be investigated. Also, it is possible to transform fuzzy linear programming problems into crisp versions reflecting decision-maker’s point of view via the WABL defuzzification method. With the aid of a group of decision-makers version given in this paper, it is possible to define more flexible fuzzy multi-criteria decision-making models and to produce solution approaches for them.

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