Co-accelerated particles in the C-metric

V. Pravdǎ*, A. Pravdovǎ†
Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic
(November 4, 2018)

With appropriately chosen parameters, the C-metric represents two uniformly accelerated black holes moving in the opposite directions on the axis of the axial symmetry (the $z$-axis). The acceleration is caused by nodal singularities located on the $z$-axis.

In the present paper, geodesics in the C-metric are examined. In general there exist three types of timelike or null geodesics in the C-metric: geodesics describing particles 1) falling under the black hole horizon; 2) crossing the acceleration horizon; and 3) orbiting around the $z$-axis and co-accelerating with the black holes.

Using an effective potential, it can be shown that there exist stable timelike geodesics of the third type if the product of the parameters of the C-metric, $mA$, is smaller than a certain critical value. Null geodesics of the third type are always unstable. Special timelike and null geodesics of the third type are also found in an analytical form.

PACS: 04.20.-q, 04.20.Jb, 04.25.-g

I. INTRODUCTION AND SUMMARY

The C-metric is a vacuum solution of the Einstein equations of the Petrov type D. Kinnersley and Walker \[1\] showed that it represents black holes uniformly accelerated by nodal singularities in opposite directions along the axis of the axial symmetry. In coordinates $\{x, y, p, q\}$ adapted to its algebraical structure, the C-metric reads as follows:

$$ds^2 = \frac{1}{A^2(x + y)^2} \left( G^{-1} dx^2 + F^{-1} dy^2 + G dp^2 - F dq^2 \right), \tag{1}$$

where the functions $F, G$ are the cubic polynomials

$$F = -1 + y^2 - 2mA y^3, \tag{2}$$

$$G = 1 - x^2 - 2mA x^3, \tag{3}$$

with $m$ and $A$ being constant. As we are choosing the signature $+2$, we take $G > 0$.

Although this form of the C-metric \[1\] is simple, it is not suitable for physical interpretation of the solution. Since the metric \[1\] has two Killing vectors $\partial/\partial p, \partial/\partial q$ it is possible to transform it in its static regions given by

$$G > 0, \quad F > 0 \tag{4}$$

to the Weyl form \[2\] (see Sec. IV). By further transformation \[3\] one can show that the C-metric is in fact a radiative boost-rotation symmetric spacetime \[2,4\] (see Sec. IV). The class of boost-rotation symmetric spacetimes is the only class of exact radiative solutions of the full nonlinear Einstein equations that are known in an analytical form, describe moving objects, and are asymptotically flat (see \[4,5,3\] for general treatise). Several generalizations of the C-metric are known, let us mention the charged C-metric \[6\] and the spinning C-metric \[7,9\].

Farhoosh and Zimmermann \[10\] studied a special class of geodesics in the C-metric – test particles moving on the symmetry axis. We examine general geodesics starting in the most physical static region of the C-metric (the region $B$, see Fig. 1) with the help of an effective potential. It turns out that there exist three types of timelike (null) geodesics: 1) geodesics describing particles falling under the black hole horizon and then on the curvature singularity; 2) those ones describing particles crossing the acceleration horizon and reaching future timelike (null) infinity – they are not co-accelerated with the black holes; and 3) geodesics describing particles spinning around the axis of the axial symmetry.

*E-mail: pravda@math.cas.cz
†E-mail: pravdova@math.cas.cz
(the $z$-axis), co-accelerating with the black holes along this axis and reaching future null infinity. We investigate stability of timelike and null geodesics of the third type using the effective potential in the coordinates \( \{x, y, p, q\} \) (Sect. II) and in the Weyl coordinates (Sec. III) in which it is easy to see that the stability of timelike geodesics does not depend on the distribution of conical singularities located on the $z$-axis. We show that null geodesics of this type are always unstable and that there exist stable timelike geodesics of the considered type if the product of the parameters of the C-metric, $m_A$, is smaller than a certain critical value \((22)\). This result indicates that a black hole or a star, having satellites in the equatorial plane, which starts to accelerate in the direction perpendicular to this equatorial plane can retain some of its satellites (which is in fact very not surprising) only if the acceleration is sufficiently small.

We present special geodesics of the third type in an analytical form representing particles (or zero-rest-mass particles) orbiting around the axis of the axial symmetry in a constant distance and uniformly accelerating along the $z$-axis (dragged by the black holes). They are given by \( x = \text{const}, \ y = \text{const} \) in the coordinates \( \{x, y, p, q\} \) (see Sec. II), by \( \bar{\rho} = \text{const}, \ \bar{z} = \text{const} \) in the Weyl coordinates (Sec. III) and finally we examine them in the coordinates adapted to the boost-rotation symmetry (\( \rho = \text{const}, \ z^2 - t^2 = \text{const} \)) where their physical interpretation can be easily understood (Sec. IV).

In \cite{11} it was shown that the Schwarzschild metric can be obtained from the C-metric in the Weyl coordinates by the limiting procedure $A \to 0$ (note that one can get the Weyl coordinates used in \cite{11} by multiplying the Weyl coordinates in our paper by the factor $A$). Using this procedure one can find that the unstable null geodesic in the C-metric representing photon-like particles orbiting around the $z$-axis corresponds to the unstable circular photon orbit in the Schwarzschild metric.

## II. GEODESICS IN \( \{x, y, p, q\} \) COORDINATES

First we study geodesics in the coordinates \( \{x, y, p, q\} \) in which the C-metric has the form \((1)\). The polynomials $F$ and $G$ entering the metric have three different real roots iff the condition

\[
27m^2A^2 < 1
\]

holds. Then the C-metric contains four different static regions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (see Fig. I) where the polynomials $F, G$ satisfy \((4)\). The metric \((1)\) in each of these static regions can be transformed to the Weyl form with different metric functions (see \cite{3}).

Curvature invariants diverge at $x \to \pm \infty$ or $y \to \pm \infty$ \cite{8} where curvature singularities are located (see Fig. I). The region $\mathcal{B}$ is the only static region without curvature singularities. In fact, it is the most physical static region describing a uniformly accelerated black hole (see \cite{8} for analysis of the other regions). Acceleration horizons and black hole horizons are at $y = y_i$ where $y_i$ are the roots of the equation $F = 0$ and are denoted in Fig. I by AH and BH, respectively.

In this paper we consider only the region $\mathcal{B}$ which exists iff the condition \((2)\) is satisfied.

---

**Fig. 1.** The character of the regions determined by roots of the polynomials $F$ and $G$ is schematically illustrated. Static regions are shaded, black hole and acceleration horizons are denoted by BH and AH, respectively. Dashed lines represent curvature singularities.
If a metric has a Killing vector $\xi^\alpha$ then there exists a conserved quantity $\xi^\alpha U_\alpha$ for timelike geodesics with a tangent vector $U^\alpha$ and $\xi^\alpha k_\alpha$ for null geodesics with a tangent vector $k^\alpha$. Since the C-metric \( [1] \) has two Killing vectors $\partial/\partial p$ and $\partial/\partial q$, corresponding covariant components of the 4-velocity, $U_p, U_q$, for particles and components $k_p, k_q$ of the wave 4-vector for zero-rest-mass particles are conserved along geodesics and thus

\[
\frac{dp(\tau)}{d\tau} = L A^2 \left( \frac{x(\tau) + y(\tau)}{G(x(\tau))} \right)^2,
\]

\[
\frac{dq(\tau)}{d\tau} = E A^2 \left( \frac{x(\tau) + y(\tau)}{F(y(\tau))} \right)^2,
\]

where $L$ and $E$ are constants of motion, $\tau$ is a proper time for timelike geodesics and an affine parameter for null geodesics.

Let us examine special geodesics $x(\tau) = x_0$, $y(\tau) = y_0$ with $x_0$, $y_0$ being constants. Then substituting \( [6], [7] \) into the geodesic equations we obtain

\[
\frac{1 + m A x_0^3 + x_0 y_0 + 3 m A x_0^2 y_0}{G(x_0)} L^2 - \frac{G(x_0)}{F(y_0)} E^2 = 0,
\]

\[
\frac{F(y_0) L^2}{G(x_0)} - \frac{(-1 + m A y_0^3 - x_0 y_0 + 3 m A x_0 y_0^2) E^2}{F(y_0)} = 0.
\]

A linear combination of these two equations leads to the condition

\[
3 m^2 A^2 x_0^2 y_0^2 + mA (x_0 y_0 + 3) (y_0 - x_0) - 1 = 0.
\]

Points $[x_0, y_0]$ in the region $\mathcal{B}$ satisfying this condition are plotted in Fig. 2.

The norm of the four-velocity is

\[
A^2 (x_0 + y_0)^2 \left( \frac{L^2}{G(x_0)} - \frac{E^2}{F(y_0)} \right) = \epsilon,
\]

where $\epsilon = -1, 0$ for timelike and null geodesics, respectively.

From \( [8] \) and \( [11] \) for a given timelike geodesic ($\epsilon = -1$), i.e. given $x_0$, $y_0$, constants $L$ and $E$ read

\[
L^2 = \frac{G(x_0)^2}{A^2 (x_0 + y_0)^3 (1 + 3 m A x_0) x_0},
\]

\[
E^2 = \frac{F(y_0)^2}{A^2 (x_0 + y_0)^3 (1 - 3 m A y_0) y_0}.
\]

In the region $\mathcal{B}$, $L^2$ and $E^2$ are positive only for $x_0 > 0$ where thus timelike geodesics exist (see Fig. 3). Thus all timelike geodesics in the region $\mathcal{B}$ with $x$, $y$ being constant are given by

\[
x(\tau) = x_0,
\]

\[
y(\tau) = y_0,
\]

\[
p(\tau) = L A^2 \frac{(x_0 + y_0)^2}{G(x_0)} \tau,
\]

\[
q(\tau) = E A^2 \frac{(x_0 + y_0)^2}{F(y_0)} \tau,
\]

where $x_0 \in (0, 3)$, $y_0$ is given by \( [10] \), and $L, E$ are given by \( [12] \) and \( [13] \), respectively.

Spacelike geodesics have $x_0 < 0$ and we do not consider them further.

There exists a null geodesic ($\epsilon = 0$, the circle in Fig. 3)

\[
x(\tau) = 0,
\]

\[
y(\tau) = \frac{1}{3 m A},
\]

\[\]
\[ p(\tau) = L \frac{1}{9m^2} \tau , \]  
\[ q(\tau) = E \frac{3A^2}{1 - 27m^2A^2} \tau , \]
\[ L^2 = E^2 \frac{27m^2A^2}{1 - 27m^2A^2} . \]

Notice that \( L^2 \) is positive since the condition (3) is assumed to be satisfied.

Let us now examine stability of general timelike geodesics. For this purpose we first construct an effective potential of a general freely falling particle whose 4-velocity has the norm

\[ -1 = \frac{-1}{A^2(x + y)^2} \left[ \frac{1}{G} \left( \frac{dx}{d\tau} \right)^2 - \frac{1}{F} \left( \frac{dy}{d\tau} \right)^2 - G \left( \frac{dp}{d\tau} \right)^2 + F \left( \frac{dq}{d\tau} \right)^2 \right] . \]

Substituting from (6), (7) into (16) we get the condition

\[ \frac{F}{E^2A^4(x + y)^4} \left[ \frac{1}{G} \left( \frac{dx}{d\tau} \right)^2 + \frac{1}{F} \left( \frac{dy}{d\tau} \right)^2 \right] = \frac{E^2 - V^2}{E^2} , \]

where the effective potential \( V \) has the form

\[ V = \sqrt{F \left( \frac{1}{A^2(x + y)^2} + \frac{L^2}{G} \right)} . \]

From (17) it follows that a freely falling particle with given \( E \) has access only to those regions where \( E > V \). The potential \( V \) goes to infinity for \( x \to x_2 \) or \( x \to x_3 \) and consequently particles cannot reach left and right edges of the square \( B \). Thus there remain only three possibilities: 1) particles leave the square \( B \) across the upper edge, i.e., they fall under the black hole horizon (on which \( V = 0 \)); 2) particles leave the square \( B \) through the lower edge (on which also \( V = 0 \)), i.e., they cross the acceleration horizon; and 3) particles remain in the square \( B \), i.e., they are co-accelerated with the black holes (see Sec. IV).

Let us now study stability of geodesics of the third type. There exist stable geodesics of this type if \( V \) has its local minimum in the region \( B \), i.e., if there exists a point \((x^*, y^*)\) in the region \( B \) where

\[ V_{xx}(x^*, y^*) = V_{yy}(x^*, y^*) = 0 , \]
\[ V_{xx}(x^*, y^*)V_{yy}(x^*, y^*) - V_{xy}^2(x^*, y^*) > 0 , \]
\[ V_{xx}(x^*, y^*) > 0 . \]

For given \( m, A, \) and \( L \) there is only one point \((x^*, y^*)\) in the region \( B \) satisfying (19). It lies on the curve (10) with \( L \) given by (12), i.e., it corresponds to the geodesic (14). The condition (21) is always satisfied, however, the condition (22) is quite complicated and numerical calculations (see also Fig. 3 and the text bellow) show that the necessary condition for satisfying (22) is

\[ mA \sim 4.54 \times 10^{-3} . \]

**FIG. 2.** The solid curve is given by (10) (the circle represents the null geodesic given by (13), each point on the curve between the circle and the cross represents a timelike geodesic (14) and the remaining points correspond to spacelike geodesics), the dashed curve is given by \( V_{xx}V_{yy} - V_{xy}^2 = 0 \), where \( L \) was substituted from (12) after performing the derivatives for: a) \( m = 1/2, A = 1/3 \) (not satisfying (22)); b) \( m = 0.02, A = 0.05 \) (satisfying (22)) – between the intersections of the two plotted curves the geodesics (14) are stable and the corresponding potential \( V \) has there its local minimum.
Figs. 3a, b illustrate the behaviour of the potential $V$ for parameters $m$, $A$ which do not satisfy and do satisfy the condition (22), respectively.

**Fig. 3.** The function $V$ as a function of $x$, $y$ for $L = 0.084$ and a) $m = 1/2$, $A = 1/3$ (not satisfying (22)); b) $m = 0.02$, $A = 0.05$ (satisfying (22)), where a local minimum exists.

For parameters $m$, $A$ not satisfying the condition (22) there is no local minimum of the potential $V$ and thus a small perturbation causes that a freely falling particle moving along the geodesic (14) falls either under the black hole horizon or under the acceleration horizon.

For parameters $m$, $A$ satisfying the condition (22) and for suitable $L$ (see Fig. 2) there exists a region (the region bounded by a closed curve in Fig. 3a) from which freely falling particle with $E$ lower than a certain critical value cannot escape (see Fig. 3a). In this region considered geodesics are stable. As it will be clear in Sec. IV these trapped particles are co-accelerated with the uniformly accelerated black hole. If the parameter $E$ of these trapped particles is increased over a certain critical value they fall under the acceleration horizon (and thus they are not co-accelerated with the black hole, see Figs. 4b, 4c; 5b, 5c) or under the black hole horizon (see Figs. 4c, 5c).

**Fig. 4.** Curves $V=\text{const}$ for $m = 0.02$, $A = 0.05$, $L = 0.084$: a) $V = \sqrt{372.9}$; b) $V = \sqrt{373}$; c) $V = \sqrt{429}$.
FIG. 5. Curves $V = \text{const}$ as in Fig. 4 and numerically obtained geodesics for $m = 0.02$, $A = 0.05$, $L = 0.084$. All geodesics start at the same point $(x_0, y_0)$ which lies on the curve (10) but have different initial velocities, i.e. different constants of motion $E$: a) $E = \sqrt{372.9}$ – a geodesic of a co-accelerated particle; b) $E = \sqrt{373}$ – a geodesic of a particle crossing the acceleration horizon; c) $E = \sqrt{479}$ – geodesics of particles which fall under the acceleration or the black hole horizon depending on direction of an initial velocity.

Similarly we may derive an effective potential $\Lambda$ for zero-rest-mass test particles substituting (6), (7) into the relation for the norm of the wave 4-vector $k_{\alpha}k^{\alpha} = 0$:

$$\frac{F}{E^2A^4(x+y)^4} \left[ \frac{1}{G} \left( \frac{dx}{d\tau} \right)^2 + \frac{1}{F} \left( \frac{dy}{d\tau} \right)^2 \right] - 1 = \frac{L^2}{E^2} \frac{1}{\Lambda^2},$$

where

$$\Lambda = \sqrt{\frac{G}{F}}.$$  

Thus photon-like particles can reach only regions where $L/E < \Lambda$. In the considered part $B$ of the spacetime, the function $\Lambda$ has vanishing first derivatives at the point $x = 0$, $y = 1/(3mA)$ (the circle in Fig. 2), however, there is not a local extreme and thus the geodesic (15) is unstable.

III. GEODESICS IN THE WEYL COORDINATES

For interpreting the geodesics (13), (15) we transform them into the Weyl coordinates in this section and into the coordinates adapted to the boost and rotation symmetries in the next section.

As was mentioned earlier the C-metric in each of its static regions can be transformed to the static Weyl form

$$\begin{eqnarray*}
\frac{d^2s}{d\tau^2} &=& e^{-2U} [e^{2\nu}(d\rho^2 + d\bar{z}^2) + \rho^2 d\bar{\phi}^2] - e^{2U}d\bar{t}^2 \\
&=& e^{-2U} [e^{2\nu}(d\rho^2 + m^2) + \rho^2 d\bar{\phi}^2] - e^{2U}d\bar{t}^2
\end{eqnarray*}$$

by transformation (see [3])

$$\bar{z} = \frac{1 + mAxy(x-y) + xy}{A^2(x+y)^2},$$

$$\bar{\rho} = \frac{\sqrt{FG}}{A^2(x+y)^2},$$

$$\bar{\phi} = p,$$

$$\bar{t} = q.$$  

Transforming the C-metric in the region $B$ into the Weyl coordinates we obtain

$$\begin{eqnarray*}
e^{2U} &=& \frac{[R_1 - (\bar{z} - \bar{z}_1)] [R_3 - (\bar{z} - \bar{z}_3)]}{R_2 - (\bar{z} - \bar{z}_2)}, \\
e^{2\nu} &=& \frac{1}{4} \frac{m^2}{A^6(\bar{z}_2 - \bar{z}_1)^2(\bar{z}_3 - \bar{z}_2)^2} \frac{[R_2 R_3 + \bar{\rho}^2 + (\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_3)] [R_1 R_2 + \bar{\rho}^2 + (\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)]}{R_1 R_2 R_3 [R_1 R_3 + \bar{\rho}^2 + (\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)]} e^{2U},
\end{eqnarray*}$$

where functions $R_1$, $R_2$, and $R_3$ are defined by

$$R_i = \sqrt{(\bar{z} - \bar{z}_i)^2 + \bar{\rho}^2},$$

and $\bar{z}_1 < \bar{z}_2 < \bar{z}_3$ are the roots of the equation

$$2A^4\bar{z}_i^3 - A^2\bar{z}_i^2 + m^2 = 0.$$  

As was shown in [2], the C-metric in the Weyl coordinates corresponds to the field of a rod between $\bar{z}_1$ and $\bar{z}_2$, a semi-infinite line mass at $\bar{z} > \bar{z}_3$ and conical singularities for $\bar{z} < \bar{z}_1$ and $\bar{z}_2 < \bar{z} < \bar{z}_3$ keeping them apart. The rod between $\bar{z}_1$ and $\bar{z}_2$ represents the black hole horizon and the semi-infinite line mass at $\bar{z} > \bar{z}_3$ represents the acceleration horizon (see Fig. 3).

The regularity condition of the axis
\[ e^{2\nu}(\bar{\rho} = 0, \bar{z}) = 1 \]  

(31)

is not satisfied at points where nodal singularities appear. Since the metric (25) with metric functions

\[ e^{-2U'} = a e^{-2U}, \]

\[ e^{2\nu'} = b e^{2\nu}, \]  

(32)

\( a \) and \( b \) being constants, also satisfies the vacuum Einstein equations, by choosing the constant \( b \) appropriately we may regularize (i.e. fulfill the condition (31)) either the part of the axis \( \bar{z} < \bar{z}_1 \) or \( \bar{z}_2 < \bar{z} < \bar{z}_3 \) (see [3]).

The conditions (6), (7) for geodesics in the Weyl coordinates (corresponding to the existence of two Killing vectors \( \partial/\partial \bar{t}, \partial/\partial \bar{\phi} \)) read as follows

\[ \frac{d\bar{\phi}(\tau)}{d\tau} = L \frac{e^{2U}}{\bar{\rho}(\tau)^2}, \]

(33)

\[ \frac{d\bar{t}(\tau)}{d\tau} = E e^{-2U}. \]  

(34)

Due to the transformation (26), the geodesics \( x(\tau) = x_0 = \text{const}, y(\tau) = y_0 = \text{const} \), discussed in the previous section, now have the form \( \bar{\rho}(\tau) = \bar{\rho}_0 = \text{const}, \bar{z}(\tau) = \bar{z}_0 = \text{const} \). The condition (10) in terms of the coordinates \( \bar{\rho}, \bar{z} \) reads

\[ R_1 R_3 - R_3 R_2 - R_1 R_2 = 0, \]  

(35)

where \( R_1, R_2, R_3 \) are given by (29) and the corresponding curve is plotted in Fig. 6.

FIG. 6. The region \( B \) in the Weyl coordinates: the axis \( \bar{\rho} = 0 \) with the black hole horizon (BH, \( \bar{z} < \bar{z}_1 \)) and the acceleration horizon (AH, \( \bar{z} > \bar{z}_3 \)) and the curve given by Eq. (35) for \( m = 1/2, A = 1/3 \) (the circle corresponds to the null geodesic (37), each point between the circle and the cross represents a timelike geodesic (36) and the other points on the curve correspond to spacelike geodesics).

Timelike geodesics (14) have in the Weyl coordinates the form

\[ \bar{\rho}(\tau) = \bar{\rho}_0, \]

\[ \bar{z}(\tau) = \bar{z}_0, \]

(36)

\[ \bar{\phi}(\tau) = L \frac{e^{2U(\bar{\rho}_0, \bar{z}_0)}}{\bar{\rho}_0^2} \tau, \]

\[ \bar{t}(\tau) = E e^{-2U(\bar{\rho}_0, \bar{z}_0)} \tau, \]

where \( \bar{z}_0, \bar{\rho}_0 \) are constants satisfying (35), and \( E, L \) are constants of motion ((33), (34)).

Similarly the null geodesic (13) in the Weyl coordinates reads (the circle in Fig. 6)

\[ \bar{z}(\tau) = 9m^2, \]

\[ \bar{\rho}(\tau) = \sqrt{3(1 - 27m^2A^2)} \frac{m}{A}, \]

(37)

\[ \bar{\phi}(\tau) = L \frac{1}{9m^2} \tau, \]

\[ \bar{t}(\tau) = E \frac{3A^2}{1 - 27m^2A^2} \tau. \]
Analogously as in Sec. II, (16)–(18), a motion of a freely falling particle with the constants of motion \( E \), \( L \) is restricted to a region where \( E > V \), the effective potential \( V \) being

\[
V = \sqrt{e^{2U} \left( 1 + \frac{L^2 e^{2U}}{\rho^2} \right)} .
\]  

(38)

Notice, that since the potential \( V \) and the condition (35) do not depend on the function \( e^{2\nu} \), after changing the constant \( b \), i.e. changing the distribution of nodal singularities, the geodesics (36) remain geodesics and, moreover, this change does not affect their stability.

In Fig. 7 (analogous to Fig. 4) curves with different values of \( V^2 \) are plotted. There again appear a region bounded by a closed curve from which trapped particles with a given parameter \( E \) cannot escape. Particles with \( E \) higher than a certain critical value fall either under the black hole horizon (the axis between \( w_1, w_2 \) where \( w = |\tilde{z}|^{1/4} \text{sign } \tilde{z} \)) or under the acceleration horizon (the axis between \( w_3 \) and \( \infty \)).

FIG. 7. Curves \( V^2 = \text{const} \) for \( m = 0.02, A = 0.05, L = 0.084 \) (to compactify the picture coordinate \( w = |\tilde{z}|^{1/4} \text{sign } \tilde{z} \) is used instead of \( \tilde{z} \); \( w_i \) correspond to \( \tilde{z}_i \)). The potential \( V \) is infinite on the axis at \( w \in (-\infty, w_1) \) and \( w \in (w_2, w_3) \) and null at the black hole and acceleration horizons at \( w \in (w_1, w_2) \), \( w \in (w_3, \infty) \), respectively.

IV. GEODESICS IN THE CANONICAL COORDINATES ADAPTED TO THE BOOST-ROTATION SYMMETRY

To find an interpretation of geodesics studied in the previous sections we transform the metric (25) by the transformation

\[
\bar{\rho}^2 = \rho^2(z^2 - t^2) ,
\]

\[
\bar{\phi} = \phi ,
\]

\[
\bar{z} - \bar{z}_3 = \frac{1}{2}(t^2 + \rho^2 - z^2) ,
\]

\[
\bar{t} = \arctanh(t/z) ,
\]

(39)

into the form

\[
ds^2 = -e^\lambda d\rho^2 - \rho^2 e^{-\nu} d\phi^2 - \frac{1}{z^2 - t^2} \left[ (e^{\lambda} z^2 - e^{\nu} t^2) dz^2 - 2zt(e^{\lambda} - e^{\nu}) dz dt + (e^{\lambda} t^2 - e^{\nu} z^2) dt^2 \right]
\]

(40)

which is adapted to the boost and rotation symmetries (see (38)). The inverse transformation to (39) has the form

\[
\rho = \sqrt{\sqrt{\bar{\rho}^2 + (\bar{z} - \bar{z}_3)^2} + (\bar{z} - \bar{z}_3)} ,
\]

\[
z = \pm \sqrt{\sqrt{\bar{\rho}^2 + (\bar{z} - \bar{z}_3)^2} - (\bar{z} - \bar{z}_3) \cosh \bar{t}} ,
\]

\[
t = \pm \sqrt{\sqrt{\bar{\rho}^2 + (\bar{z} - \bar{z}_3)^2} - (\bar{z} - \bar{z}_3) \sinh \bar{t}} ,
\]

(41)
where either upper or lower signs are valid.

From (41) it follows that geodesics we are interested in, satisfying \( \bar{\rho} = \bar{\rho}_0 = \text{const}, \bar{z} = \bar{z}_0 = \text{const} \) in the Weyl coordinates, in the coordinates \( \{ t, \rho, z, \phi \} \) satisfy \( \rho = \text{const} \) and \( z^2 - \rho^2 = \text{const} \) (the worldline is a hyperbola in the \( (z,t) \)-plane which corresponds to a uniformly accelerated motion along the \( z \)-axis, see Fig. 8). Geodesics of this type (corresponding to (36), (37)) now have the form

\[
\begin{align*}
\rho(\tau) &= K_1 , \\
\phi(\tau) &= c_2 \tau , \\
z(\tau) &= \pm K_2 \cosh c_1 \tau , \\
t(\tau) &= K_2 \sinh c_1 \tau ,
\end{align*}
\]

where the constants \( K_1, K_2, c_1, \) and \( c_2 \) read

\[
\begin{align*}
K_1 &= \sqrt{\bar{\rho}_0^2 + (\bar{z}_0 - \bar{z}_3)^2} , \\
K_2 &= \sqrt{z^2 - t^2} = \sqrt{\bar{\rho}_0^2 + (\bar{z}_0 - \bar{z}_3)^2 - (\bar{z}_0 - \bar{z}_3)} , \\
c_1 &= E e^{-2U(\bar{\rho}_0, \bar{z}_0)} , \\
c_2 &= L e^{2U(\bar{\rho}_0, \bar{z}_0)} .
\end{align*}
\]

These geodesics describe particles orbiting the \( z \)-axis and uniformly accelerating along the \( z \)-axis.

![FIG. 8. Uniformly accelerated black holes (the shaded region) and co-accelerated test particles (dashed lines).](image)

[1] W. Kinnersley and M. Walker, Uniformly Accelerating Charged Mass in General Relativity, Phys. Rev. D 2, 1359 (1970).
[2] W. B. Bonnor, The sources of the vacuum C-metric, Gen. Rel. Grav. 15, 535 (1983).
[3] V. Pravda and A. Pravdová, Boost-rotation symmetric spacetimes - review, Czech. J. Phys. 50, 333 (2000); see also gr-qc/0003067.
[4] J. Bićák and B. G. Schmidt, Asymptotically flat radiative space-times with boost-rotation symmetry: The general structure, Phys. Rev. D 40, 1827 (1989).
[5] J. Bićák, Selected solutions of Einstein’s field equations: their role in general relativity and astrophysics, in “Einstein’s Field Equations and Their Physical Meaning”, ed. B. G. Schmidt, Springer Verlag, Berlin – New York (2000).
[6] W. Kinnersley and M. Walker, Uniformly Accelerating Charged Mass in General Relativity, Phys. Rev. D 2, 1359 (1970).
[7] F. H. J. Cornish and W. J. Uttley, The interpretation of the C metric. The charged case when \( e^2 \leq m^2 \), Gen. Rel. Grav. 27, 735 (1995).
[8] J. F. Plebański and M. Demiański, Rotating, charged and uniformly accelerating mass in general relativity, Ann. Phys. (USA) 98, 98 (1976).
[9] J. Bićák and V. Pravda, Spinning C-metric as a boost-rotation symmetric radiative spacetime, Phys. Rev. D 60, 044004 (1999).
[10] H. Farhoosh and R. L. Zimmerman, *Killing horizons and dragging of the inertial frame about a uniformly accelerating particle*, Phys. Rev. D 21, 317 (1980).

[11] F. H. J. Cornish and W. J. Uttley, *The interpretation of the C metric. The Vacuum case*, Gen. Rel. Grav. 27, 439 (1995).