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ON THE EXACTNESS OF ORDINARY PARTS OVER A LOCAL FIELD OF CHARACTERISTIC \( p \)

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Abstract. Let \( G \) be a connected reductive group over a non-archimedean local field \( F \) of residue characteristic \( p \). \( P \) be a parabolic subgroup of \( G \), and \( R \) be a commutative ring. When \( R \) is artinian, \( p \) is nilpotent in \( R \), and \( \text{char}(F) = p \), we prove that the ordinary part functor \( \text{Ord}_P \) is exact on the category of admissible smooth \( R \)-representations of \( G \). We derive some results on Yoneda extensions between admissible smooth \( R \)-representations of \( G \).

1. Results

Let \( F \) be a non-archimedean local field of residue characteristic \( p \). Let \( G \) be a connected reductive algebraic \( F \)-group and \( G \) denote the topological group \( G(F) \). We let \( P = MN \) be a parabolic subgroup of \( G \). We write \( P = MN \) for the opposite parabolic subgroup.

Let \( R \) be a commutative ring. We write \( \text{Mod}^\infty_R(G) \) for the category of smooth \( R \)-representations of \( G \) (i.e. \( R[G] \)-modules \( \pi \) such that for all \( v \in \pi \) the stabiliser of \( v \) is open in \( G \) and \( R[G] \)-linear maps). It is an \( R \)-linear abelian category. When \( R \) is noetherian, we write \( \text{Mod}^{\text{adm}}_R(G) \) for the full subcategory of \( \text{Mod}^\infty_R(G) \) consisting of admissible representations (i.e. those representations \( \pi \) such that \( \pi^H \) is finitely generated over \( R \) for any open subgroup \( H \) of \( G \)). It is closed under passing to subrepresentations and extensions, thus it is an \( R \)-linear exact subcategory, but quotients of admissible representations may not be admissible when \( \text{char}(F) = p \) (see \cite[Example 4.4]{AHV17}).

Recall the smooth parabolic induction functor \( \text{Ind}^G_P : \text{Mod}^\infty_M(G) \rightarrow \text{Mod}^\infty_R(G) \), defined on any smooth \( R \)-representation \( \sigma \) of \( M \) as the \( R \)-module \( \text{Ind}^G_P(\sigma) \) of locally constant functions \( f : G \rightarrow \sigma \) satisfying \( f(m \overline{a}g) = m \cdot f(g) \) for all \( m \in M \), \( \overline{a} \in \overline{N} \), and \( g \in G \), endowed with the smooth action of \( G \) by right translation. It is \( R \)-linear, exact, and commutes with small direct sums. In the other direction, there is the ordinary part functor \( \text{Ord}_P : \text{Mod}^\infty_R(G) \rightarrow \text{Mod}^\infty_M(G) \) \( \left( \text{Eme10a} \right. \left. \text{Vig16} \right) \). It is \( R \)-linear and left exact. When \( R \) is noetherian, \( \text{Ord}_P \) also commutes with small inductive limits, both functors respect admissibility, and the restriction of \( \text{Ord}_P \) to \( \text{Mod}^{\text{adm}}_R(G) \) is right adjoint to the restriction of \( \text{Ind}^G_P \) to \( \text{Mod}^{\text{adm}}_M(G) \).

Theorem 1. If \( R \) is artinian, \( p \) is nilpotent in \( R \), and \( \text{char}(F) = p \), then \( \text{Ord}_P \) is exact on \( \text{Mod}^{\text{adm}}_R(G) \).

Thus the situation is very different from the case \( \text{char}(F) = 0 \) (see \cite{Eme10b}). On the other hand if \( R \) is artinian and \( p \) is invertible in \( R \), then \( \text{Ord}_P \) is isomorphic on \( \text{Mod}^{\text{adm}}_R(G) \) to the Jacquet functor with respect to \( P \) (i.e. the \( N \)-coinvariants) twisted by the inverse of the modulus character \( \delta_P \) of \( P \) \( \left( \text{AHV17} \right. \left. \text{Corollary 4.19} \right) \), so that it is exact on \( \text{Mod}^{\text{adm}}_R(G) \) without any assumption on \( \text{char}(F) \).

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Remark. Without any assumption on $R$, $\text{Ind}_P^G : \text{Mod}_{\text{adm}}^G(R) \to \text{Mod}_{\text{adm}}^G(R)$ admits a left adjoint $L_P^G : \text{Mod}_{\text{adm}}^G(R) \to \text{Mod}_{\text{adm}}^G(R)$ (the Jacquet functor with respect to $P$) and a right adjoint $R_P^G : \text{Mod}_{\text{adm}}^G(R) \to \text{Mod}_{\text{adm}}^G(R)$ (see [NSW08, § I.5]). If $R$ is noetherian and $p$ is nilpotent in $R$, then $R_P^G$ is isomorphic to $\text{Ord}_P$ on $\text{Mod}_{\text{adm}}^G(R)$ ([AHV17 Corollary 4.13]). Thus under the assumptions of Theorem 1, $R_P^G$ is exact on $\text{Mod}_{\text{adm}}^G(R)$. On the other hand if $R$ is noetherian and $p$ is invertible in $R$, then $R_P^G$ is expected to be isomorphic to $\delta\rho L_P^G$ (‘second adjointness’), and this is proved in the following cases: when $R$ is the field of complex numbers ([Ber07]) or an algebraically closed field of characteristic $\ell \neq p$ ([Vig96 II.3.8 2]); when $G$ is a Levi subgroup of a general linear group or a classical group with $p \neq 2$ ([Dat09 Théorème 1.5]); when $P$ is a minimal parabolic subgroup of $G$ (see also [Dat09]). In particular, $L_P^G$ and $R_P^G$ are exact in all these cases.

Question. Are $L_P^G$ and $R_P^G$ exact when $R$ is noetherian, $p$ is nilpotent in $R$, and $\text{char}(R) = p$?

We derive from Theorem 1 some results on Yoneda extensions between admissible $R$-representations of $G$. We compute the $R$-modules $\text{Ext}_G^n$ in $\text{Mod}_{\text{adm}}^G(R)$.

Corollary 2. Assume $R$ artinian, $p$ nilpotent in $R$, and $\text{char}(R) = p$. Let $\sigma$ and $\pi$ be admissible $R$-representations of $M$ and $G$ respectively. For all $n \geq 0$, there is a natural $R$-linear isomorphism

$$\text{Ext}_G^n(\sigma, \text{Ord}_P(\pi)) \overset{\sim}{\rightarrow} \text{Ext}_G^n(\text{Ind}_P^G(\sigma), \pi).$$

This is in contrast with the case $\text{char}(R) = 0$ (see [Han10b]). A direct consequence of Corollary 2 is that under the same assumptions, $\text{Ind}_P^G$ induces an isomorphism between the $\text{Ext}^n$ for all $n \geq 0$ (Corollary 3). When $R = \mathbb{C}$ is an algebraically closed field of characteristic $p$ and $\text{char}(R) = p$, we determine the extensions between certain irreducible admissible $C$-representations of $G$ using the classification of [AHV17] (Proposition 6). In particular, we prove that there exists no non-split extension of an irreducible admissible $C$-representation $\pi$ of $G$ by a supersingular $C$-representation of $G$ when $\pi$ is not the extension to $G$ of a supersingular representation of a Levi subgroup of $G$ (Corollary 4). When $G = \text{GL}_2$, this was first proved by Hu ([Hu17 Theorem A.2]).

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2. Proofs

2.1. Hecke action. In this subsection, $M$ denotes a linear algebraic $F$-group and $N$ denotes a split unipotent algebraic $F$-group (see [CGP15 Appendix B]) endowed with an action of $M$ that we identify with the conjugation in $M \rtimes N$. We fix an open submonoid $M^+$ of $M$ and a compact open subgroup $N_0$ of $N$ stable under conjugation by $M^+$.

If $\pi$ is a smooth $R$-representation of $M^+ \rtimes N_0$, then the $R$-modules $H^\bullet(N_0, \pi)$, computed using the homogeneous cochain complex $C^\bullet(N_0, \pi)$ (see [NSW08 § I.2]), are naturally endowed with the Hecke action of $M^+$, defined as the composite

$$H^\bullet(N_0, \pi) \overset{m}{\rightarrow} H^\bullet(mN_0m^{-1}, \pi) \overset{\text{cont}}{\rightarrow} H^\bullet(N_0, \pi)$$

for all $m \in M^+$. At the level of cochains, this action is explicitly given as follows (see [NSW08 § I.5]). We fix a set of representatives $\bar{N}/mN_0m^{-1} \subseteq N_0$ of the left cosets $N_0/mN_0m^{-1}$ and we write $n \mapsto \bar{n}$ for the projection $N_0 \rightarrow \bar{N}/mN_0m^{-1}$. 


For $\phi \in C^k(N_0, \pi)$, we have

\[(m \cdot \phi)(n_0, \ldots, n_k) = \sum_{\tilde{n} \in N_0/mN_0m^{-1}} \tilde{n}m \cdot \phi(m^{-1}\tilde{n}^{-1}n_0\tilde{n}^{-1}m, \ldots, m^{-1}\tilde{n}^{-1}n_k\tilde{n}^{-1}m)\]

for all $(n_0, \ldots, n_k) \in N_0^{k+1}$.

**Lemma 3.** Assume $p$ nilpotent in $R$ and $\text{char}(F) = p$. Let $\pi$ be a smooth $R$-representation of $M^+ \ltimes N_0$ and $m \in M^+$. If the Hecke action $h_{N_0,m}$ of $m$ on $\pi^{N_0}$ is locally nilpotent (i.e. for all $v \in \pi^{N_0}$ there exists $r \geq 0$ such that $h_{N_0,m}^r(v) = 0$), then the Hecke action of $m$ on $H^k(N_0, \pi)$ is locally nilpotent for all $k \geq 0$.

**Proof.** First, we prove the lemma when $pR = 0$, i.e. $R$ is a commutative $\mathbb{F}_p$-algebra. We assume that the Hecke action of $m$ on $\pi^{N_0}$ is locally nilpotent and we prove the result together with the following fact: there exists a set of representatives $\overline{N_0/mN_0m^{-1}} \subseteq N_0$ of the left cosets $N_0/mN_0m^{-1}$ such that the action of $S := \sum_{\tilde{n} \in \overline{N_0/mN_0m^{-1}}} \tilde{n}m \in \mathbb{F}_p[M^+ \ltimes N_0]$ on $\pi$ is locally nilpotent.

We proceed by induction on the dimension of $N$ (recall that $N$ is split so that it is smooth and connected). If $N = 1$, then the (Hecke) action of $m$ on $\pi^{N_0} = \pi$ is locally nilpotent by assumption, so that the result and the fact are trivially true. Assume $N \neq 1$ and that the result and the fact are true for groups of smaller dimension. Since $N$ is split, it admits a non-trivial central subgroup isomorphic to the additive group. We let $N' \subseteq N$ be the subgroup of $N$ generated by all such subgroups. It is a non-trivial vector group (i.e. isomorphic to a direct product of copies of the additive group) which is central (hence normal) in $N$ and stable under conjugation by $M$ (since it is a characteristic subgroup of $N$). We set $N'' := N/N'$. It is a split unipotent $F$-group endowed with the induced action of $M$ and $\dim(N'') < \dim(N)$.

Since $N'$ is split, we have $N'' = N/N'$. We write $N'_0$ and $N''_0$ for the compact open subgroups $N' \cap N_0$ and $N_0/N''_0$ of $N'$ and $N''$ respectively. They are stable under conjugation by $M^+$. We fix a set-theoretic section $[-] : N''_0 \twoheadrightarrow N_0$.

Since $N'$ is commutative and $p$-torsion, $N''_0$ is a compact $\mathbb{F}_p$-vector space. Thus for any open subgroup $N'_0$ of $N''_0$, the short exact sequence of compact $\mathbb{F}_p$-vector spaces

$$0 \to N'_0 \to N''_0 \to N''_0/N'_0 \to 0$$

splits. Indeed, it admits an $\mathbb{F}_p$-linear splitting (since $\mathbb{F}_p$ is a field) which is automatically continuous (since $N''_0/N'_0$ is discrete). In particular with $N''_0/N'_0 = mN''_0m^{-1}$, we may and do fix a section $N''_0/mN''_0m^{-1} \to N'_0$. We write $N''_0/mN''_0m^{-1}$ for its image, so that $N'_0 = N''_0/mN''_0m^{-1} \times mN''_0m^{-1}$, and $n' \mapsto \tilde{n}'$ for the projection $N''_0 \twoheadrightarrow N''_0/mN''_0m^{-1}$. We set

$$S' := \sum_{\tilde{n}' \in \overline{N''_0/mN''_0m^{-1}}} \tilde{n}'m \in \mathbb{F}_p[M^+ \ltimes N''_0].$$

For all $n'_0 \in N'_0$, we have $n'_0 = \tilde{n}'_0(\tilde{n}'^{-1}n'_0)$ with $\tilde{n}'^{-1}n'_0 \in mN''_0m^{-1}$, thus

$$n'_0S' = \sum_{\tilde{n}' \in \overline{N''_0/mN''_0m^{-1}}} (\tilde{n}'\tilde{n}')m(m^{-1}(\tilde{n}'^{-1}n'_0)m) = S'(m^{-1}(\tilde{n}'^{-1}n'_0)m)$$

with $m^{-1}(\tilde{n}'^{-1}n'_0)m \in N'_0$ (in the first equality we use the fact that $N'_0$ is commutative and in the second one we use the fact that $N''_0/mN''_0m^{-1}$ is a group). Therefore, there is an inclusion $\mathbb{F}_p[N'_0]S' \subseteq S'\mathbb{F}_p[N''_0]$. 

The $R$-module $\pi N_0^\prime$, endowed with the induced action of $N_0^\prime$ and the Hecke action of $M^+$ with respect to $N_0^\prime$, is a smooth $R$-representation of $M^+ \rtimes N_0^\prime$ (see the proof of [Hau16a, Lemme 3.2.1] in degree $0$). On $\pi N_0^\prime$, the Hecke action of $m$ with respect to $N_0$ coincides with the action of $S'$ by definition. On $(\pi N_0^\prime)N_0^\prime = \pi N_0^\prime$, the Hecke action of $m$ with respect to $N_0^\prime$ coincides with the Hecke action of $m$ with respect to $N_0$ (see the proof of [Hau16a, Lemme 3.2.2]) which is locally nilpotent by assumption. Thus by the induction hypothesis, there exists a set of representatives $N_0^\prime/mN_0^\prime m^{-1} \subseteq N_0^\prime$ of the left cosets $N_0^\prime/mN_0^\prime m^{-1}$ such that the action of
\[
S := \sum_{\tilde{n}^\prime \in N_0^\prime/mN_0^\prime m^{-1}} [\tilde{n}^\prime]S' \in F_p[M^+ \rtimes N_0]
\]
on $\pi N_0^\prime$ is locally nilpotent. Moreover, there is an inclusion $F_p[N_0]S \subseteq S F_p[N_0]$ (because $N_0^\prime$ is central in $N_0$ and $F_p[N_0]S' \subseteq S' F_p[N_0]$).

We prove the fact. By [Hau16c, Lemme 2.1],
\[
N_0^\prime/mN_0^\prime m^{-1} := \{[\tilde{n}^\prime] \cdot \tilde{n}' : \tilde{n}^\prime \in N_0^\prime/mN_0^\prime m^{-1}, \tilde{n}' \in N_0/mN_0 m^{-1} \} \subseteq N_0
\]
is a set of representatives of the left cosets $N_0/mN_0 m^{-1}$, and by definition
\[
S = \sum_{\tilde{n} \in N_0/mN_0 m^{-1}} \tilde{n}m.
\]
We prove that the action of $S$ on $\pi$ is locally nilpotent. We proceed as in the proof of [Hu12, Théorème 5.1 (i)]. Let $v \in \pi$ and set $\pi_r := F_p[N_0] \cdot (S^r \cdot v)$ for all $r \geq 0$. Since $F_p[N_0]S \subseteq S F_p[N_0]$, we have $\pi_{r+1} \subseteq S \cdot \pi_r$ for all $r \geq 0$. Since $N_0$ is compact, we have $\dim_{F_p}(\pi_r) < \infty$ for all $r \geq 0$. If $S^r \cdot v \neq 0$, i.e. $\pi_r \neq 0$, for some $r \geq 0$, then $\pi_0^\prime \neq 0$ (because $N_0^\prime$ is a pro-$p$ group and $\pi_r$ is a non-zero $F_p$-vector space) so that $\dim_{F_p}(S \cdot \pi_r) < \dim_{F_p}(\pi_r)$ (because the action of $S$ on $\pi_0^\prime$ is locally nilpotent).

Therefore $\pi_r = 0$, i.e. $S^r \cdot v = 0$, for all $r \geq \dim_{F_p}(\pi_0)$. We prove the result. The $R$-modules $H^\bullet(N_0^\prime, \pi)$, endowed with the induced action of $N_0^\prime$ and the Hecke action of $M^+$, are smooth $R$-representations of $M^+ \rtimes N_0^\prime$ (see the proof of [Hau16a, Lemme 3.2.1]). At the level of cochains, the actions of $n^\prime \in N_0^\prime$ and $m$ are explicitly given as follows. For $\phi \in C^i(N_0^\prime, \pi)$, we have
\[
\begin{align*}
(n')^i \cdot \phi(n_0^\prime, \ldots, n_j^\prime) &= [n''^i] \cdot \phi(n_0^\prime, \ldots, n_j^\prime) \\
(m \cdot \phi)(n_0^\prime, \ldots, n_j^\prime) &= S' \cdot \phi(m^{-1} n_0^\prime \tilde{n}^{-1}_0 m, \ldots, m^{-1} n_j^\prime \tilde{n}^{-1}_j m)
\end{align*}
\]
for all $(n_0^\prime, \ldots, n_j^\prime) \in N_0^{j+1}$ (for $\bar{2}$) we use the fact that $N_0^\prime$ is central in $N_0$, for $\bar{3}$ we use $\bar{1}$ and the fact that $n' \mapsto \tilde{n}'$ is a group homomorphism $N_0^\prime \to N_0/mN_0 m^{-1}$.

Using $\bar{2}$ and $\bar{3}$, we can give explicitly the Hecke action of $m$ on $H^\bullet(N_0^\prime, \pi)^{N_0^\prime}$ at the level of cochains as follows. For $\phi \in C^i(N_0^\prime, \pi)$, we have
\[
(m \cdot \phi)(n_0^\prime, \ldots, n_j^\prime) = S \cdot \phi(m^{-1} n_0^\prime \tilde{n}^{-1}_0 m, \ldots, m^{-1} n_j^\prime \tilde{n}^{-1}_j m)
\]
for all $(n_0^\prime, \ldots, n_j^\prime) \in N_0^{j+1}$. Since the action of $S$ on $\pi$ is locally nilpotent and the image of a locally constant cochain is finite by compactness of $N_0$, we deduce that the Hecke action of $m$ on $H^j(N_0^\prime, \pi)^{N_0^\prime}$ is locally nilpotent for all $j \geq 0$. Thus the Hecke action of $m$ on $H^j(N_0^\prime, H^i(N_0^\prime, \pi))$ is locally nilpotent for all $i,j \geq 0$ by the induction hypothesis. We conclude using the spectral sequence of smooth $R$-representations of $M^+$
\[
H^i(N_0^\prime, H^j(N_0^\prime, \pi)) \Rightarrow H^{i+j}(N_0, \pi)
\]
(see the proof of [Hau16a, Proposition 3.2.3] and footnote $\bar{1}$).

$\bar{1}$We do not know whether [Eme10, Proposition 2.1.11] holds true when $\text{char}(F) = p$, but [Hau16a, Lemme 3.1.1] does and any injective object of $\text{Mod}^\infty_{M^+ \rtimes N_0^\prime}(R)$ is still $N_0$-acyclic.
Now, we prove the lemma without assuming \( pR = 0 \). We proceed by induction on the degree of nilpotency \( r \) of \( p \) in \( R \). If \( r \leq 1 \), then the lemma is already proved. We assume \( r > 1 \) and that we know the lemma for rings in which the degree of nilpotency of \( p \) is \( r - 1 \). There is a short exact sequence of smooth \( R \)-representations of \( M^+ \times N_0 \)

\[
0 \to p\pi \to \pi \to \pi/p\pi \to 0.
\]

Taking the \( N_0 \)-cohomology yields a long exact sequence of smooth \( R \)-representations of \( M^+ \)

\[
0 \to (p\pi)^{N_0} \to \pi^{N_0} \to (\pi/p\pi)^{N_0} \to H^1(N_0, p\pi) \to \cdots.
\]

If the Hecke action of \( m \) on \( \pi^{N_0} \) is locally nilpotent, then the Hecke action of \( m \) on \( (p\pi)^{N_0} \) is also locally nilpotent so that the Hecke action of \( m \) on \( H^k(N_0, p\pi) \) is locally nilpotent for all \( k \geq 0 \) by the induction hypothesis (since \( p\pi \) is an \( R/p^{r-1}R \)-module). Using (4), we deduce that the Hecke action of \( m \) on \( (\pi/p\pi)^{N_0} \) is also locally nilpotent so that the Hecke action of \( m \) on \( H^k(N_0, \pi/p\pi) \) is locally nilpotent for all \( k \geq 0 \) (since \( \pi/p\pi \) is an \( \mathbb{F}_p \)-vector space). Using again (4), we conclude that the Hecke action of \( m \) on \( H^k(N_0, \pi) \) is locally nilpotent for all \( k \geq 0 \).

2.2. Proof of the main result. We fix a compact open subgroup \( N_0 \) of \( N \) and we let \( M^+ \) be the open submonoid of \( M \) consisting of those elements \( m \) contracting \( N_0 \) (i.e. \( mN_0m^{-1} \subseteq N_0 \)). We let \( Z_M \) denote the centre of \( M \) and we set \( Z_M^+ := Z_M \cap M^+ \). We fix an element \( z \in Z_M^+ \) strictly contracting \( N_0 \) (i.e. \( \cap_{r \geq 0} z^rN_0z^{-r} = 1 \)).

Recall that the ordinary part of a smooth \( R \)-representation \( \pi \) of \( P \) is the smooth \( R \)-representation of \( M \)

\[
\text{Ord}_P(\pi) := (\text{Ind}^{M^+}_{M^+}(\pi^{N_0}))^{Z_M^{-1}\text{fin}}
\]

where \( \text{Ind}^{M^+}_{M^+}(\pi^{N_0}) \) is defined as the \( R \)-module of functions \( f : M \to \pi^{N_0} \) such that \( f(mm') = m \cdot f(m') \) for all \( m \in M^+ \) and \( m' \in M \), endowed with the action of \( M \) by right translation, and the superscript \( Z_M^{-1}\text{fin} \) denotes the subrepresentation consisting of locally \( Z_M^{-1}\text{fin} \)-finite elements (i.e. those elements \( f \) such that \( R[Z_M] \cdot f \) is contained in a finitely generated \( R \)-submodule). The action of \( M \) on the latter is smooth by [Vig10, Remark 7.6]. If \( R \) is artinian and \( \pi^{N_0} \) is locally \( Z_M^{-1}\text{fin} \)-finite (i.e. it may be written as the union of finitely generated \( Z_M^{-1}\text{fin} \)-invariant \( R \)-submodules), then there is a natural \( R \)-linear isomorphism

\[
\text{Ord}_P(\pi) \cong R[z^{\pm 1}] \otimes_{R[z]} \pi^{N_0}
\]

(cf. [Eme10b, Lemma 3.2.1 (1)], whose proof also works when \( \text{char}(F) = p \) and over any artinian ring).

If \( \sigma \) is a smooth \( R \)-representation of \( M \), then the \( R \)-module \( C_c^\infty(N, \sigma) \) of locally constant functions \( f : N \to \sigma \) with compact support, endowed with the action of \( N \) by right translation and the action of \( M \) given by \( (m \cdot f) : n \mapsto m \cdot f(m^{-1}nm) \) for all \( m \in M \), is a smooth \( R \)-representation of \( P \). Thus we obtain a functor \( C_c^\infty(N, -) : \text{Mod}_M^\infty(R) \to \text{Mod}_P^\infty(R) \). It is \( R \)-linear, exact, and commutes with small direct sums. The results of [Eme10b, § 4.2] hold true when \( \text{char}(F) = p \) and over any ring, thus the functors

\[
C_c^\infty(N, -) : \text{Mod}_M^\infty(R)^{Z_M^{-1}\text{fin}} \to \text{Mod}_P^\infty(R)
\]

\[
\text{Ord}_P : \text{Mod}_P^\infty(R) \to \text{Mod}_M^\infty(R)^{Z_M^{-1}\text{fin}}
\]

are adjoint and the unit of the adjunction is an isomorphism.

**Lemma 4.** Assume \( R \) artinian, \( p \) nilpotent in \( R \), and \( \text{char}(F) = p \). Let \( \pi \) be a smooth \( R \)-representation of \( P \). If \( \pi^{N_0} \) is locally \( Z_M^{-1}\text{fin} \)-finite, then the Hecke action of \( z \) on \( H^k(N_0, \pi) \) is locally nilpotent for all \( k \geq 1 \).
Proof. We set \( \sigma := \text{Ord}_P(\pi) \). The counit of the adjunction between \( C^\infty_c(N, -) \) and \( \text{Ord}_P \) induces a natural morphism of smooth \( R \)-representations of \( P \)

\[
(6) \quad C^\infty_c(N, \sigma) \to \pi.
\]

Taking the \( N_0 \)-invariants yields a morphism of smooth \( R \)-representations of \( M^+ \)

\[
(7) \quad C^\infty_c(N, \sigma)^{N_0} \to \pi^{N_0}.
\]

By definition, \( \sigma \) is locally \( Z_M \)-finite so it may be written as the union of finitely generated \( Z_M \)-invariant \( R \)-submodules \( \langle \sigma_i \rangle_{i \in I} \). Thus \( C^\infty_c(N, \sigma)^{N_0} \) is the union of the finitely generated \( Z_M \)-invariant \( R \)-submodules \( \langle C^\infty_c(\pi^{-1}N_0\sigma^i, \sigma_i)^{N_0} \rangle_{r \geq 0, i \in I} \), so it is locally \( Z_M^+ \)-finite. By assumption, \( \pi^{N_0} \) is also locally \( Z_M^+ \)-finite. Therefore, using \( (6) \) and its analogue with \( C^\infty_c(N, \sigma) \) instead of \( \pi \), the localisation with respect to \( z \) of \( (7) \) is the natural morphism of smooth \( R \)-representations of \( M \)

\[
\text{Ord}_P(C^\infty_c(N, \sigma)) \to \text{Ord}_P(\pi)
\]

induced by applying the functor \( \text{Ord}_P \) to \( (6) \), and it is an isomorphism since the unit of the adjunction between \( C^\infty_c(N, -) \) and \( \text{Ord}_P \) is an isomorphism.

Let \( \kappa \) (resp. \( \iota \)) be the kernel (resp. image) of \( (6) \), hence two short exact sequences of smooth \( R \)-representations of \( P \)

\[
(8) \quad 0 \to \kappa \to C^\infty_c(N, \sigma) \to \iota \to 0
\]

\[
(9) \quad 0 \to \iota \to \pi \to \pi/\iota \to 0
\]

such that the third arrow of \( (8) \) and the second arrow of \( (9) \) fit into a commutative diagram of smooth \( R \)-representations of \( P \)

\[
\begin{tikzcd}
C^\infty_c(N, \sigma) \arrow{r} \arrow{dr}{\iota} & \pi \arrow{d} \\
\iota^{N_0} & \pi^{N_0}
\end{tikzcd}
\]

whose upper arrow is \( (6) \). Taking the \( N_0 \)-invariants yields a commutative diagram of smooth \( R \)-representations of \( M^+ \)

\[
\begin{tikzcd}
C^\infty_c(N, \sigma)^{N_0} \arrow{r} \arrow{dr}{\iota^{N_0}} & \pi^{N_0} \arrow{d}
\end{tikzcd}
\]

whose upper arrow is \( (7) \). Since the localisation with respect to \( z \) of the latter is an isomorphism, the localisation with respect to \( z \) of the injection \( \iota^{N_0} \hookrightarrow \pi^{N_0} \) is surjective, thus it is an isomorphism (as it is also injective by exactness of localisation). Therefore the localisation with respect to \( z \) of the morphism \( C^\infty_c(N, \sigma)^{N_0} \to \iota^{N_0} \) is an isomorphism.

Since \( C^\infty_c(N, \sigma) \cong \bigoplus_{n \in N_0/N_0} C^\infty_c(nN_0, \sigma) \) as a smooth \( R \)-representation of \( N_0 \), it is \( N_0 \)-acyclic (see [NSW05 § I.3]). Thus the long exact sequence of \( N_0 \)-cohomology induced by \( (8) \) yields an exact sequence of smooth \( R \)-representations of \( M^+ \)

\[
(10) \quad 0 \to \kappa^{N_0} \to C^\infty_c(N, \sigma)^{N_0} \to \iota^{N_0} \to H^1(N_0, \kappa) \to 0
\]

and an isomorphism of smooth \( R \)-representations of \( M^+ \)

\[
(11) \quad H^k(N_0, z) \cong H^{k+1}(N_0, \kappa)
\]

for all \( k \geq 1 \). Since the localisation with respect to \( z \) of the third arrow of \( (10) \) is an isomorphism, the Hecke action of \( z \) on \( \kappa^{N_0} \) is locally nilpotent. Thus the Hecke action of \( z \) on \( H^k(N_0, \kappa) \) is locally nilpotent for all \( k \geq 0 \) by Lemma [3]. Using \( (11) \), we deduce that the Hecke action of \( z \) on \( H^k(N_0, \iota) \) is locally nilpotent for all \( k \geq 1 \).
Taking the $N_0$-cohomology of (9) yields a long exact sequence of smooth $R$-representations of $M^+$
\[(12) \quad 0 \to \iota^{N_0} \to \pi^{N_0} \to (\pi/\iota)^{N_0} \to H^1(N_0, \iota) \to \cdots.\]
Since the localisation with respect to $z$ of the second arrow is an isomorphism and the Hecke action of $z$ on $H^1(N_0, \iota)$ is locally nilpotent, the Hecke action of $z$ on $(\pi/\iota)^{N_0}$ is locally nilpotent. Thus the Hecke action of $z$ on $H^k(N_0, \pi/\iota)$ is locally nilpotent for all $k \geq 0$ by Lemma 3. We conclude using (12) and the fact that the Hecke action of $z$ on $H^k(N_0, \iota)$ is locally nilpotent for all $k \geq 1$. □

**Proof of Corollary 2** Assume $R$ artinian, $p$ nilpotent in $R$, and $\text{char}(F) = p$. Let
\[(13) \quad 0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0\]
be a short exact sequence of admissible $R$-representations of $G$. Taking the $N_0$-invariants yields an exact sequence of smooth $R$-representations of $M^+$
\[(14) \quad 0 \to \pi_1^{N_0} \to \pi_2^{N_0} \to \pi_3^{N_0} \to H^1(N_0, \pi_1).\]
The terms $\pi_1^{N_0}, \pi_2^{N_0}, \pi_3^{N_0}$ are locally $Z_M^+$-finite (cf. [Eme10b, Theorem 3.4.7 (1)], whose proof in degree 0 also works when $\text{char}(F) = p$ and over any noetherian ring) and the Hecke action of $z$ on $H^1(N_0, \pi_1)$ is locally nilpotent by Lemma 4. Therefore, using (5), the localisation with respect to $z$ of (14) is the short sequence of admissible $R$-representations of $M$
\[0 \to \text{Ord}_P(\pi_1) \to \text{Ord}_P(\pi_2) \to \text{Ord}_P(\pi_3) \to 0\]
induced by applying the functor $\text{Ord}_P$ to (13), and it is exact by exactness of localisation. □

2.3. Results on extensions. We assume $R$ noetherian. The $R$-linear category $\text{Mod}_{G}^{\text{adm}}(R)$ is not abelian in general, but merely exact in the sense of Quillen ([Qui73]). An exact sequence of admissible $R$-representations of $G$ is an exact sequence of smooth $R$-representations of $G$
\[\cdots \to \pi_{n-1} \to \pi_n \to \pi_{n+1} \to \cdots\]
such that the kernel and the cokernel of every arrow are admissible. In particular, each term of the sequence is also admissible.

For $n \geq 0$ and $\pi, \pi'$ two admissible $R$-representations of $G$, we let $\text{Ext}^n_G(\pi', \pi)$ denote the $R$-module of $n$-fold Yoneda extensions ([Yon60]) of $\pi'$ by $\pi$ in $\text{Mod}_{G}^{\text{adm}}(R)$, defined as equivalence classes of exact sequences
\[0 \to \pi \to \pi_1 \to \cdots \to \pi_n \to \pi' \to 0.\]
We let $D(G)$ denote the derived category of $\text{Mod}_{G}^{\text{adm}}(R)$ ([Nee90] [Kee96] [Buh10]). The results of [Ver96] § III.3.2] on the Yoneda construction carry over to this setting (see e.g. [Pos11] Proposition A.13), hence a natural $R$-linear isomorphism
\[\text{Ext}^n_G(\pi', \pi) \cong \text{Hom}_{D(G)}(\pi', \pi[n]).\]

**Proof of Corollary 2** Since $\text{Ind}_G^M$ and $\text{Ord}_P$ are exact adjoint functors between $\text{Mod}_{M}^{\text{adm}}(R)$ and $\text{Mod}_{G}^{\text{adm}}(R)$ by Theorem 3, they induce adjoint functors between $D(M)$ and $D(G)$, hence natural $R$-linear isomorphisms
\[\text{Ext}^n_M(\sigma, \text{Ord}_P(\pi)) \cong \text{Hom}_{D(M)}(\sigma, \text{Ord}_P(\pi)[n]) \cong \text{Hom}_{D(G)}(\text{Ind}_G^M(\sigma), \pi[n]) \cong \text{Ext}^n_G(\text{Ind}_G^M(\sigma), \pi)\]
for all $n \geq 0$. □
Remark. We give a more explicit proof of Corollary 2. The exact functor $\text{Ind}_P^G$ and the counit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$ induce an $R$-linear morphism

$$\text{Ext}^n_M(\sigma, \text{Ord}_P(\pi)) \to \text{Ext}^n_M(\text{Ind}_P^G(\sigma), \pi).$$

In the other direction, the exact (by Theorem 1) functor $\text{Ord}_P$ and the unit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$ induce an $R$-linear morphism

$$\text{Ext}^n_M(\text{Ind}_P^G(\sigma), \pi) \to \text{Ext}^n_M(\sigma, \text{Ord}_P(\pi)).$$

We prove that (16) is the inverse of (15). For $n = 0$ this is the unit-counit equations. Assume $n \geq 1$ and let

$$0 \to \text{Ord}_P(\pi) \to \sigma_1 \to \cdots \to \sigma_n \to \sigma \to 0$$

be an exact sequence of admissible $R$-representations of $M$. By [Yon60 § 3], the image of the class of (17) under (15) is the class of any exact sequence of admissible $R$-representations of $G$

$$0 \to \pi \to \pi_1 \to \cdots \to \pi_n \to \text{Ind}_P^G(\sigma) \to 0$$

such that there exists a commutative diagram of admissible $R$-representations of $G$

$$
\begin{array}{cccccccc}
0 & \to & \text{Ind}_P^G(\text{Ord}_P(\pi)) & \to & \text{Ind}_P^G(\pi_1) & \to & \cdots & \to & \text{Ind}_P^G(\pi_n) & \to & \text{Ind}_P^G(\sigma) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \pi & \to & \pi_1 & \to & \cdots & \to & \pi_n & \to & \text{Ind}_P^G(\sigma) & \to & 0
\end{array}
$$

in which the upper row is obtained from (17) by applying the exact functor $\text{Ind}_P^G$, the lower row is (15), and the leftmost vertical arrow is the natural morphism induced by the counit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$. Applying the exact functor $\text{Ord}_P$ to the diagram and using the unit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$ yields a commutative diagram of admissible $R$-representations of $M$

$$
\begin{array}{cccccccc}
0 & \to & \text{Ord}_P(\pi) & \to & \sigma_1 & \to & \cdots & \to & \sigma_n & \to & \sigma & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Ord}_P(\pi) & \to & \text{Ord}_P(\pi_1) & \to & \cdots & \to & \text{Ord}_P(\pi_n) & \to & \text{Ord}_P(\text{Ind}_P^G(\sigma)) & \to & 0
\end{array}
$$

in which the lower row is obtained from (18) by applying the exact functor $\text{Ord}_P$, the upper row is (17), and the rightmost vertical arrow is the natural morphism induced by the unit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$. The leftmost vertical morphism is the identity by the unit-counit equations. Thus the image of the class of (18) under (16) is the class of (17) by [Yon60 § 3]. We have proved that (16) is a left inverse of (15). The proof that it is a right inverse is dual.

**Corollary 5.** Assume $R$ artinian, $p$ nilpotent in $R$, and $\text{char}(F) = p$. Let $\sigma$ and $\sigma'$ be two admissible $R$-representations of $M$. The functor $\text{Ind}_P^G$ induces an $R$-linear isomorphism

$$\text{Ext}^n_M(\sigma', \sigma) \cong \text{Ext}^n_M(\text{Ind}_P^G(\sigma'), \text{Ind}_P^G(\sigma))$$

for all $n \geq 0$.

**Proof.** The isomorphism in the statement is the composite

$$\text{Ext}^n_M(\sigma', \sigma) \cong \text{Ext}^n_M(\sigma', \text{Ord}_P(\text{Ind}_P^G(\sigma))) \cong \text{Ext}^n_M(\text{Ind}_P^G(\sigma'), \text{Ind}_P^G(\sigma))$$

where the first isomorphism is induced by the unit of the adjunction between $\text{Ind}_P^G$ and $\text{Ord}_P$, which is an isomorphism, and the second one is the isomorphism of Corollary 2 with $\sigma'$ and $\text{Ind}_P^G(\sigma)$ instead of $\sigma$ and $\pi$ respectively. 

□
We fix a minimal parabolic subgroup $B \subseteq G$, a maximal split torus $S \subseteq B$, and we write $\Delta$ for the set of simple roots of $S$ in $B$. We say that a parabolic subgroup $P = MN$ of $G$ is standard if $B \subseteq P$ and $S \subseteq M$. In this case, we write $\Delta_P$ for the corresponding subset of $\Delta$, and given $\alpha \in \Delta_P$ (resp. $\alpha \in \Delta \setminus \Delta_P$) we write $P_\alpha = M_\alpha N_\alpha$ for the standard parabolic subgroup corresponding to $\Delta_P \setminus \{\alpha\}$ (resp. $\Delta_P \cup \{\alpha\}$).

Let $C$ be an algebraically closed field of characteristic $p$. Given a standard parabolic subgroup $P = MN$ and a smooth $C$-representation $\sigma$ of $M$, there exists a largest standard parabolic subgroup $P(\sigma) = M(\sigma)N(\sigma)$ such that the inflation of $\sigma$ to $P$ extends to a smooth $C$-representation $^c\sigma_P$ of $P(\sigma)$, and this extension is unique [AHV17, I.7 Corollary 1]. We say that a smooth $C$-representation of $G$ is supercuspidal if it is irreducible, admissible, and does not appear as a subquotient of $\text{Ind}^G_P(\sigma)$ for any proper parabolic subgroup $P = MN$ of $G$ and any irreducible admissible $C$-representation $\sigma$ of $M$. A supercuspidal standard $C[G]$-triple is a triple $(P, \sigma, Q)$ where $P = MN$ is a standard parabolic subgroup, $\sigma$ is a supercuspidal $C$-representation of $M$, and $Q$ is a parabolic subgroup of $G$ such that $P \subseteq Q \subseteq P(\sigma)$. To such a triple is attached in [AHV17] a smooth $C$-representation of $G$

$$I_G(P, \sigma, Q) := \text{Ind}_P^G(\sigma \otimes \text{St}_Q^P(\sigma))$$

where $\text{St}_Q^P(\sigma) := \text{Ind}_Q^P(\sigma) / \sum_{Q \subseteq Q' \subseteq P(\sigma)} \text{Ind}_Q^P(\sigma')$ (here 1 denotes the trivial $C$-representation) is the inflation to $P(\sigma)$ of the generalised Steinberg representation of $M(\sigma)$ with respect to $M(\sigma) \cap Q$ ([GK13, Ly15]). It is irreducible and admissible (AHV17, 1.3 Theorem 1).

**Proposition 6.** Assume $\text{char}(F) = p$. Let $(P, \sigma, Q)$ and $(P', \sigma', Q')$ be two supercuspidal standard $C[G]$-triples. If $Q \not\subseteq Q'$, then the $C$-vector space

$$\text{Ext}_G^1(I_G(P', \sigma', Q'), I_G(P, \sigma, Q))$$

is non-zero if and only if $P' = P$, $\sigma' \cong \sigma$, and $Q' = Q^\alpha$ for some $\alpha \in \Delta_Q$, in which case it is one-dimensional and the unique (up to isomorphism) non-split extension of $I_G(P', \sigma', Q')$ by $I_G(P, \sigma, Q)$ is the admissible $C$-representation of $G$

$$\text{Ind}_{P(\sigma)}^G(\sigma \otimes (M(\sigma) \cap P, \sigma, M(\sigma)^\alpha \cap Q)).$$

**Proof.** There is a natural short exact sequence of admissible $C$-representations of $G$

$$0 \rightarrow \sum_{Q \subseteq Q' \subseteq P(\sigma')} \text{Ind}_{Q'}^P(\sigma') \rightarrow \text{Ind}_{Q''}^G(\sigma'') \rightarrow I_G(P', \sigma', Q') \rightarrow 0. \quad (19)$$

Note that we can restrict the sum to those $Q''$ that are minimal, i.e. of the form $Q''_{\alpha}$ for some $\alpha \in \Delta_{P(\sigma')} \setminus \Delta_Q$. Moreover, we deduce from [AHV17] Theorem 3.2 that its cosocle is isomorphic to $\bigoplus_{\alpha \in \Delta_{P(\sigma')} \setminus \Delta_Q} I_G(P', \sigma', Q'_\alpha)$. Now if $Q \not\subseteq Q'$, then $\text{Ord}_{Q'}(I_G(P, \sigma, Q)) = 0$ by [AHV17] Theorem 1.1 (ii) and Corollary 4.13 so that using Corollary 2 we see that the long exact sequence of Yoneda extensions obtained by applying the functor $\text{Hom}_G(-, I_G(P, \sigma, Q))$ to (19) yields a natural $C$-linear isomorphism

$$\text{Ext}_G^{n-1}((\sum_{Q \subseteq Q'' \subseteq P(\sigma')} \text{Ind}_{Q''}^G(\sigma''), I_G(P, \sigma, Q)) \rightarrow \text{Ext}_G^n(I_G(P', \sigma', Q'), I_G(P, \sigma, Q))$$

for all $n \geq 1$. In particular, with $n = 1$ and using the identification of the cosocle of the sum and [AHV17] 1.3 Theorem 2], we deduce that the $C$-vector space in the statement is non-zero if and only if $P' = P$, $\sigma' \cong \sigma$, and $Q = Q^\alpha$ for some $\alpha \in \Delta_{P(\sigma')} \setminus \Delta_Q$ (or equivalently $Q' = Q^\alpha$ for some $\alpha \in \Delta_Q$), in which case it is one-dimensional. Finally, using again [AHV17, Theorem 3.2], we see that for all $\alpha \in \Delta_Q$ the admissible $C$-representation of $G$ in the statement is a non-split extension of $I_G(P, \sigma, Q^\alpha)$ by $I_G(P, \sigma, Q)$. \qed
Corollary 7. Assume $\text{char}(F) = p$. Let $\pi$ and $\pi'$ be two irreducible admissible $C$-representations of $G$. If $\pi$ is supercuspidal and $\pi'$ is not the extension to $G$ of a supercuspidal representation of a Levi subgroup of $G$, then $\text{Ext}^1_G(\pi', \pi) = 0$.

Proof. By [AHHV17, L.3 Theorem 3], there exist two supercuspidal standard $C[G]$-triples $(P, \sigma, Q)$ and $(P', \sigma', Q')$ such that $\pi \cong I_G(P, \sigma, Q)$ and $\pi' \cong I_G(P', \sigma', Q')$. The assumptions on $\pi$ and $\pi'$ are equivalent to $P = G$ and $Q' \neq G$. In particular, $Q \not\subseteq Q'$ and $P \neq P'$ so that $\text{Ext}^1_G(\pi', \pi) = 0$ by Proposition 6.

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