DIFFERENTIAL HARNACK ESTIMATES FOR TIME-DEPENDENT
HEAT EQUATIONS WITH POTENTIALS

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Abstract. In this paper, we prove a differential Harnack inequality for positive
solutions of time-dependent heat equations with potentials. We also prove a gradient
estimate for the positive solution of the time-dependent heat equation.

1. Introduction

In this paper, we will study time-dependent heat equations with potentials on
closed Riemannian manifolds evolving by the Ricci flow
\[
\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.
\]
We will derive differential Harnack inequalities (also known as Li-Yau type Harnack
estimates) for positive solutions of parabolic equations of the type
\[
\frac{\partial f}{\partial t} = \Delta_{g(t)} f + Rf,
\]
where \(\Delta_{g(t)}\) depends on time \(t\), \(R\) is the scalar curvature of \(g(t)\).

The study of differential Harnack estimates for parabolic equations originated in
P. Li and S.-T. Yau’s paper [LY86], in which they proved a differential Harnack
inequality for positive solutions of the heat equation on Riemannian manifolds with a
fixed metric. Namely they proved that, if \(f\) is a positive solution to the heat equation
\[
\frac{\partial f}{\partial t} = \Delta f
\]
on a Riemannian manifold with nonnegative Ricci curvature, then
\[
\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \frac{n}{2t} = \Delta \ln f + \frac{n}{2t} \geq 0.
\]
The idea was later brought to study general geometric evolution equations by the sec-
ond author. The differential Harnack estimates have become an important technique
in the studies of geometric flows.

In [Ham93a], the second author proved a Harnack estimate for the Ricci flow on
Riemannian manifolds with weakly positive curvature operator, its trace version,
\[
(1.1) \quad \frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla R \cdot V + 2Rc(V, V) \geq 0
\]

Date: May 8th, 2008.
2000 Mathematics Subject Classification. Primary 53C44.
∗ Research partially supported by the Jeffrey Sean Lehman Fund from Cornell University.
(here $V$ is any vector field), will be needed in our proof. B. Chow and S.-C. Chu [CC95] gave a nice geometric interpretation by showing that the Harnack quantity is the curvature of a degenerate metric in space-time. In the case of surface, Harnack estimates have been proved by the second author in [Ham88] with positive scalar curvature and by B. Chow in [Cho91b] with arbitrary curvature, respectively. B. Chow and the second author generalized their results for the heat equation and for the Ricci flow on surfaces in [CH97].

The second author proved a matrix Harnack estimate for the heat equation in [Ham93b]. The fundamental solution and Harnack inequality of time-dependent heat equation have also been studied by C. Guenther [Gue02]. In [CN05], H.-D. Cao and L. Ni proved a matrix Harnack estimate for the heat equation on Kähler manifolds. Harnack inequalities have also been discovered for other geometric flows. The second author proved a Harnack estimate for the mean curvature flow in [Ham95]. B. Chow proved Harnack estimates for Gaussian curvature flow in [Cho91a] and for Yamabe flow in [Cho92]. For the Kähler-Ricci flow, H.-D. Cao proved a Harnack estimate in [Cao92] and L. Ni proved a matrix Harnack estimate in [Ni07]. In [And94], B. Andrews obtained Harnack inequalities for various evolving hypersurfaces. In [Per02], G. Perelman proved a Harnack estimate for the fundamental solution of the conjugate heat equation under the Ricci flow. More precisely, let $(M, g(t)), \ t \in [0, T]$, be a solution to the Ricci flow on a closed manifold, $f$ be the positive fundamental solution to the conjugate heat equation

$$\frac{\partial}{\partial t} f = -\Delta f + Rf,$$

$\tau = T - t$ and $u = -\ln f - \frac{n}{2} \ln(4\pi\tau)$. Then on $(0, T)$, G. Perelman proved that

$$2\Delta u - |\nabla u|^2 + R + \frac{u}{\tau} - \frac{n}{\tau} \leq 0$$

(see [Ni06] or [CCG+07, Chapter 16] for a detailed proof).

In the present paper, let $(M, g(t)), \ t \in [0, T]$, be a solution to the Ricci flow on a closed manifold, $f$ be a positive solution of the time-dependent heat equation with potential, i.e.,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

$$\frac{\partial f}{\partial t} = \Delta_{g(t)} f + Rf. \tag{1.3}$$

Notice that under the Ricci flow, we have

$$\frac{d}{dt} \int_M f \mu = 0. \tag{1.4}$$

We also assume that $g(0)$ has weakly positive curvature operator, this property is preserved by the Ricci flow (see [Ham86]). Our first main theorem is the following,

**Theorem 1.1.** Let $(M, g(t)), \ t \in [0, T]$, be a solution to the Ricci flow (1.2) on a closed manifold, and suppose that $g(t)$ has weakly positive curvature operator. Let $f$
be a positive solution to the heat equation (1.3), \( u = -\ln f \) and
\[
H = 2\Delta u - |\nabla u|^2 - 3R - 2\frac{n}{t}.
\]
Then for all time \( t \in (0, T) \)
\[
H \leq 0.
\]

We shall also prove the following theorem,

**Theorem 1.2.** Let \((M, g(t)), t \in [0, T)\), be a solution to the Ricci flow (1.2) on a closed manifold, and suppose that \( g(t) \) has weakly positive curvature operator. Let \( f \) be a positive solution to the heat equation (1.3), \( v = -\ln f - \frac{n}{2} \ln(4\pi t) \) and \( P = 2\Delta v - |\nabla v|^2 - 3R + \frac{v}{t} - \frac{d}{t} \)
where \( d \) is any constant. Then for all time \( t \in (0, T) \), we have
\[
\frac{\partial}{\partial t}(tP) = \Delta(tP) - 2\nabla(tP) \cdot \nabla v - 2t|v| - R_{ij} - 1 \frac{1}{2t} g_{ij}^2 - 2t(\Delta R + 2|Rc|^2 + R + 2\nabla R \cdot \nabla v + 2R_{ij}v_i v_j).
\]
Moreover, \( \max tP \) is non-increasing.

**Remark 1.1.** In [Cao08], the first author uses a similar method and proves a differential Harnack inequality for all positive solutions of the conjugate heat equation under the Ricci flow, notice that Perelman’s Harnack inequality is only valid for the fundamental solution, while the estimate in [Cao08] has no such restriction (but requires nonnegative scalar curvature). Such Harnack inequality for the conjugate heat equation has also been proved by S. Kuang and Q. Zhang [KZ06] recently.

As one can see that our equation (1.3) is corresponding to the heat equation under the Ricci flow, in the sense of (1.4). Li-Yau’s Harnack inequality gives an estimate on the heat kernel. The Harnack estimate of the Ricci flow on surfaces gives a control on curvature growth. The general Harnack estimate of the Ricci flow allows one to classify the ancient solutions of nonnegative curvature operators. Perelman’s Harnack inequality proves noncollapsing under the Ricci flow. We expect our Harnack estimate to have similar applications in the study of the Ricci flow. There are geometric quantities which satisfy equation (1.3), for example, the scalar curvature on surfaces. Hence our Harnack estimate should lead more information of the blow-up. We also expect this estimate will be useful in understanding the Ricci solitons, as the soliton potential satisfies a heat-type equation. More importantly, the method used here will help us searching for more interesting Harnack inequalities in the Ricci flow as well as in other geometric flows (for example, see [Cao08]).

The rest of this paper is organized as follows. In section 2, we will first derive a general evolution formula for function \( H \), then we give the proof of Theorem 1.1. We will prove a integral version of the Harnack inequality (Theorem 2.3). Finally, we reprove some early results on surfaces. In section 3, We will derive the general evolution formula for function \( P \), then we prove Theorem 1.2. In section 4, we will
define two functionals which are associated to the above two Harnack quantities, and show that they are monotone under the Ricci flow. Finally, in section 5, as a special case of the general evolution formula, we will prove a gradient estimate for positive and bounded solutions to the heat equation under the Ricci flow.

Acknowledgement: The first author thanks Laurent Saloff-Coste, for many helpful discussion on various aspects of Harnack inequalities, and for pointing out the reference [Stu96] to him.

2. Proof of Theorem 1.1

Let us consider positive solutions of
\[
\frac{\partial}{\partial t} f = \triangle f - cRf
\]
for all constant \(c\) which we will fix later. Let \(f = e^{-u}\), then \(\ln f = -u\). We have
\[
\frac{\partial}{\partial t} \ln f = -\frac{\partial}{\partial t} u,
\]
and
\[
\nabla \ln f = -\nabla u, \quad \triangle \ln f = -\triangle u.
\]
Hence \(u\) satisfies the following equation,
\[
(2.1) \quad \frac{\partial}{\partial t} u = \triangle u - |\nabla u|^2 + cR.
\]

Lemma 2.1. Let \((M, g(t))\) be a solution to the Ricci flow, and \(u\) satisfies (2.1). Let
\[
H = \alpha \triangle u - \beta |\nabla u|^2 + aR - b\frac{u}{t} - d\frac{n}{t},
\]
where \(\alpha, \beta, a, b\) and \(d\) are constants that we will pick later. Then \(H\) satisfies the following evolution equation,
\[
\frac{\partial}{\partial t} H = \triangle H - 2\nabla H \cdot \nabla u - (2\alpha - 2\beta)u_{ij} - \frac{\alpha}{2\alpha - 2\beta} R_{ij} - \frac{\lambda}{2t} g_{ij} - \frac{2\alpha - 2\beta}{\alpha} \frac{\lambda}{t} H
\]
\[
+ (2\alpha - 2\beta) \frac{n \lambda}{4t^2} - (b + \frac{2\alpha - 2\beta}{\alpha} \lambda \beta) \frac{|\nabla u|^2}{t} + (1 - \frac{2\alpha - 2\beta}{\alpha} \lambda) \frac{b u}{t^2}
\]
\[
+ (1 - \frac{2\alpha - 2\beta}{\alpha} \lambda) d \frac{n}{t^2} + ac\triangle R + (2a + \frac{\alpha^2}{2\alpha - 2\beta}) |Rc|^2
\]
\[
+ (\alpha \lambda + a \frac{2\alpha - 2\beta}{\alpha} \lambda - bc) \frac{R}{t} - 2\alpha R_{ij} u_i u_j + 2(a - \beta c) \nabla R \cdot \nabla u,
\]
where \(\lambda\) is also a constant that we will pick later.

Proof. The proof follows from a direct computation. We first calculate the first two terms in \(H\),
\[
\frac{\partial}{\partial t} (\triangle u) = \triangle (\triangle u) - \triangle (|\nabla u|^2) + c\triangle R + 2R_{ij} u_{ij},
\]
and
\[ \frac{\partial}{\partial t} |\nabla u|^2 = 2\nabla u \cdot \nabla \Delta u + 2Rc(\nabla u, \nabla u) - 2\nabla u \cdot \nabla(2|\nabla u|^2) + 2c
\nabla u \cdot \nabla R \]
\[ = \Delta (|\nabla u|^2) - 2|\nabla \nabla u|^2 - 2\nabla u \cdot \nabla(2|\nabla u|^2) + 2c\nabla u \cdot \nabla R, \]
here we used
\[ \Delta (|\nabla u|^2) = 2\nabla u \cdot \Delta \nabla u + 2|\nabla \nabla u|^2, \]
and
\[ \Delta \nabla u = \nabla \Delta u + Rc(\nabla u, \cdot). \]

Using the evolution equation of \( R \),
\[ \frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2, \]
and \((2.1)\), we have
\[ \frac{\partial}{\partial t} H = \Delta H - \alpha \Delta (|\nabla u|^2) + 2\alpha R_{ij}u_{ij} + 2\beta|\nabla \nabla u|^2 + 2\beta \nabla u \cdot \nabla(|\nabla u|^2) \]
\[ + 2a|Rc|^2 + b\frac{|\nabla u|^2}{t} + d\frac{n}{t^2} + b\frac{u}{t^2} + \alpha c\Delta R - 2\beta c \nabla u \cdot \nabla R - b\frac{cR}{t} \]
\[ = \Delta H - 2\nabla H \cdot \nabla u + (2a - 2\beta)|\nabla \nabla u|^2 + 2\alpha R_{ij}u_{ij} + 2a|Rc|^2 + b\frac{u}{t^2} + d\frac{n}{t^2} + \alpha c\Delta R - b\frac{cR}{t} \]
\[ = \Delta H - 2\nabla H \cdot \nabla u + (2a - 2\beta)|\nabla \nabla u|^2 + 2\alpha R_{ij}u_{ij} + 2a|Rc|^2 + 2\beta c \nabla u \cdot \nabla R - b\frac{cR}{t} \]
\[ - 2(\alpha - \beta)\frac{\lambda}{t} (\Delta u - \frac{\alpha R}{2(\alpha - \beta)}) + (\alpha - \beta)\frac{n\lambda^2}{2t^2} + (2a + \frac{\alpha^2}{2(\alpha - \beta)})|Rc|^2 \]
\[ - 2(\alpha - \beta)\frac{\lambda}{t} u_{ij} + b\frac{u}{t^2} + d\frac{n}{t^2} + \alpha c\Delta R - b\frac{cR}{t} \]
\[ = \Delta H - 2\nabla H \cdot \nabla u + (2a - 2\beta)|\nabla \nabla u|^2 + 2\alpha R_{ij}u_{ij} + 2a|Rc|^2 + 2\beta c \nabla u \cdot \nabla R - b\frac{cR}{t} \]
\[ - 2(\alpha - \beta)\frac{\lambda}{t} H + (\alpha \lambda + a\frac{2(\alpha - \beta)}{\alpha} \lambda - bc)\frac{R}{t} + (\alpha - \beta)\frac{n\lambda^2}{2t^2} \]
\[ + (2a + \frac{\alpha^2}{2(\alpha - \beta)})|Rc|^2 - b + 2(\alpha - \beta)\frac{\lambda}{t^2} + 2\alpha R_{ij}u_{ij} \]
\[ + (1 - 2(\alpha - \beta)\frac{\lambda}{t}) b\frac{u}{t^2} + (1 - 2(\alpha - \beta)\frac{\lambda}{t}) d\frac{n}{t^2} + \alpha c\Delta R. \]

In the above lemma, let us take \( \alpha = 2, \beta = 1, a = -3, c = -1, \lambda = 2, b = 0, d = 2. \)

As a consequence of the above lemma, we have
**Corollary 2.2.** Let \((M, g(t))\) be a solution to the Ricci flow, \(f\) be a positive solution of
\[
\frac{\partial}{\partial t} f = \Delta f + Rf,
\]
let \(u = -\ln f\) and
\[
H = 2\Delta u - |\nabla u|^2 - 3R - \frac{2n}{t}.
\]
Then we have
\[
\frac{\partial}{\partial t} H = \Delta H - 2\nabla H \cdot \nabla u - 2|u|_{ij} - R_{ij} - \frac{1}{t} g_{ij}|^2 - \frac{2}{t} H - \frac{2}{t} |\nabla u|^2
\]
\[
- 2\Delta R - 4|Rc|^2 - \frac{2}{t} R - 4\nabla R \cdot \nabla u - 4R_{ij} u_i u_j
\]
\[
= \Delta H - 2\nabla H \cdot \nabla u - 2|u|_{ij} - R_{ij} - \frac{1}{t} g_{ij}|^2 - \frac{2}{t} H - \frac{2}{t} |\nabla u|^2
\]
\[
- 2\left(\frac{\partial}{\partial t} R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij} u_i u_j\right).
\]

Now we can finish the proof of Theorem 1.1.

**Proof.** (Proof of Theorem 1.1) It is easy to see that for \(t\) small enough that \(H(t) < 0\).

Since \(g_{ij}\) has weakly positive curvature operator, by the trace Harnack inequality for the Ricci flow proved by the second author in \cite{Ham93a},
\[
\frac{\partial}{\partial t} R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij} u_i u_j \geq 0.
\]

It follows from (2.3) and maximum principle that
\[
H \leq 0
\]
for all time \(t\). \(\square\)

**Remark 2.1.** The theorem is also true on complete non-compact Riemannian manifolds when we can apply maximum principle. For example, if we assume that \(R\) is uniformly bounded and \(\Delta u\) has a upper bound for all time \(t\).

We now can integrate along a space-time path, and we have the following,

**Theorem 2.3.** Let \((M, g(t)), t \in [0, T]\), be a solution to the Ricci flow on a closed manifold, and suppose that \(g(t)\) has weakly positive curvature operator. Let \(f\) be a positive solution to the heat equation
\[
\frac{\partial}{\partial t} f = \Delta f + Rf.
\]
Assume that \((x_1, t_1)\) and \((x_2, t_2), 0 < t_1 < t_2\), are two points in \(M \times (0, T)\). Let
\[
\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + R)dt,
\]
where $\gamma$ is any space-time path joining $(x_1, t_1)$ and $(x_2, t_2)$. Then we have

$$f(x_1, t_1) \leq f(x_2, t_2)(\frac{t_2}{t_1})^n \exp^{\Gamma/2}.$$  

**Proof.** Since $H \leq 0$ and $u$ satisfies

$$\frac{\partial}{\partial t} u = \Delta u - |\nabla u|^2 - R,$$

we have

$$2\frac{\partial}{\partial t} u + |\nabla u|^2 - R - \frac{2n}{t} \leq 0.$$  

If we pick a space-time path $\gamma(x, t)$ joining $(x_1, t_1)$ and $(x_2, t_2)$ with $t_2 > t_1 > 0$, then along $\gamma$, we have

$$\frac{d}{dt} u = \frac{\partial}{\partial t} u + \nabla u \cdot \dot{\gamma} \leq -\frac{1}{2} |\nabla u|^2 + \frac{R}{2} + \frac{n}{t} + \nabla u \cdot \dot{\gamma} \leq \frac{1}{2} (|\dot{\gamma}|^2 + R) + \frac{n}{t}.$$  

Hence

$$u(x_2, t_2) - u(x_1, t_1) \leq \frac{1}{2} \inf_{\gamma} \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + R) dt + n \ln(\frac{t_2}{t_1}).$$  

If we denote $\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + R) dt$, then we have

$$f(x_1, t_1) \leq f(x_2, t_2)(\frac{t_2}{t_1})^n \exp^{\Gamma/2}.$$  

This finishes the proof. \[\Box\]

In the rest of this section, we will restrict ourselves on surfaces. We will reprove some early results of B. Chow and the second author in the case of $n = 2$. Take $\alpha = 1, \beta = 0, a = c = -1, b = d = 0$ and $\lambda = 0$, let

$$H = \Delta u - R$$

and

$$H_{ij} = u_{ij} - \frac{1}{2} Rg_{ij}.$$  

On surfaces,

$$\frac{\partial}{\partial t} R = \Delta R + R^2,$$

hence we have

$$\frac{\partial}{\partial t} H = \Delta H - 2\nabla H \cdot \nabla u - 2\nabla R \cdot \nabla u - 2|u_{ij} - \frac{1}{2} R_{ij}|^2 - \frac{3}{2} |Rc|^2 - 2R_{ij} u_i u_j - \Delta R$$

$$= \Delta H - 2|H_{ij}|^2 - 2\nabla H \cdot \nabla u - RH - R^2 - 2\nabla R \cdot \nabla u - R|\nabla u|^2 - \Delta R$$

(2.4)

$$= \Delta H - 2|H_{ij}|^2 - 2\nabla H \cdot \nabla u - RH - R|\nabla u + \nabla \ln R|^2 - R(\frac{\partial \ln R}{\partial t} - |\nabla \ln R|^2).$$
If we further let \( f = R \) and \( u = -\ln R \), we have

\[
\frac{\partial}{\partial t} H = \triangle H - 2|H_{ij}|^2 + 2\nabla H \cdot \nabla \ln R.
\]

It follows that \( H - \frac{1}{t} \leq 0 \). As a consequence, we have

**Corollary 2.4.** (Hamilton [Ham88]) If \((M^2, g(t))\) is a solution to the Ricci flow on a closed surface with \( R > 0 \). The scalar curvature \( R \) satisfies

\[
\frac{\partial}{\partial t} R = \triangle R + R^2.
\]

Then

\[
\frac{\partial}{\partial t} \ln R - |\nabla \ln R|^2 + \frac{1}{t} = \triangle \ln R + R + \frac{1}{t} \geq 0.
\]

Using the above Corollary 2.4, plugging into (2.4), we have

**Corollary 2.5.** (Chow-Hamilton [CH97]) If \((M^2, g(t))\) is a solution to the Ricci flow on a closed surface with \( R > 0 \), and \( f \) is a positive solution to

\[
\frac{\partial}{\partial t} f = \triangle f + R f.
\]

Then

\[
\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \frac{1}{t} = \triangle \ln f + R + \frac{1}{t} \geq 0.
\]

**Remark 2.2.** When \( n = 2 \), (2.5) or (2.6) implies our Harnack estimate in Theorem 1.1.

### 3. Proof of Theorem 1.2

In this section, let \( f = (4\pi t)^{-n/2} e^{-v} \), then \( \ln f = -\frac{n}{2} \ln(4\pi t) - v \). We have

\[
\frac{\partial}{\partial t} \ln f = -\frac{\partial}{\partial t} v - \frac{n}{2t},
\]

and

\[
\nabla \ln f = -\nabla v, \quad \triangle \ln f = -\triangle v.
\]

Hence \( v \) satisfies the following equation,

\[
\frac{\partial}{\partial t} v = \triangle v - |\nabla v|^2 + cR - \frac{n}{2t}.
\]

**Lemma 3.1.** Let \((M, g(t))\) be a solution to the Ricci flow, and \( v \) satisfies (3.1). Let

\[
P = \alpha \triangle v - |\nabla v|^2 + aR - \frac{b}{t} - \frac{c}{t}.
\]
where $\alpha$, $a$, $b$ and $d$ are constants that we will pick later. Then $P$ satisfies

$$
\frac{\partial}{\partial t} P = \Delta P - 2\nabla P \cdot \nabla v + 2(a - c)\nabla R \cdot \nabla v - (2\alpha - 2)|v_{ij} - \frac{\alpha}{2\alpha - 2}R_{ij} - \frac{\lambda}{2t}g_{ij}|^2 \\
- \frac{2\alpha - 2\lambda}{\alpha} t P + (\alpha \lambda + a \frac{2\alpha - 2\lambda - b c}{\alpha}) \frac{R}{t} + (\alpha - 1) \frac{n \lambda^2}{2t^2} + (2a + \frac{\alpha^2}{2\alpha - 2}) |Rc|^2 \\
- (b + \frac{2\alpha - 2\lambda}{\alpha} \frac{|\nabla v|^2}{t}) - 2\alpha R_{ij}v_i v_j + (1 - \frac{2\alpha - 2\lambda}{\alpha} b) \frac{v_i}{t} + b \frac{n}{2t^2} \\
+ (1 - \frac{2\alpha - 2\lambda}{\alpha} d \frac{n}{t^2} + \alpha c \Delta R,
$$

where $\lambda$ is also a constant that we will pick later.

**Proof.** The proof again follows from direct computation. Recall that we have

$$
\frac{\partial}{\partial t} (\Delta v) = \Delta (\Delta v) - \Delta (|\nabla v|^2) + c\Delta R + 2R_{ij}v_{ij},
$$

and

$$
\frac{\partial}{\partial t} |\nabla v|^2 = 2\nabla v \cdot \nabla \Delta v + 2Rc(\nabla v, \nabla v) - 2\nabla v \cdot \nabla (|\nabla v|^2) + 2c\nabla v \cdot \nabla R \\
= \Delta (|\nabla v|^2) - 2|\nabla v|^2 - 2\nabla v \cdot \nabla (|\nabla v|^2) + 2c\nabla v \cdot \nabla R,
$$

here we used

$$
\Delta (|\nabla v|^2) = 2\nabla v \cdot \Delta \nabla v + 2|\nabla \nabla v|^2,
$$

and

$$
\Delta \nabla v = \nabla \Delta v + Rc(\nabla v, \cdot).
$$

Using the evolution equation of $R$,

$$
\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2,
$$
and (3.1), we arrive at
\[
\frac{\partial}{\partial t} P = \Delta P - \alpha \Delta (|\nabla v|^2) + 2\alpha R_{ij}v_{ij} + 2|\nabla \nabla v|^2 + 2\nabla v \cdot \nabla (|\nabla v|^2)
\]
\[
+ 2a|Rc|^2 + b\frac{|\nabla v|^2}{t} + b\frac{n}{2t^2} + d\frac{n}{t^2} + b\frac{v}{t^2} + \alpha c\Delta R - 2c\nabla v \cdot \nabla R - b\frac{cR}{t}
\]
\[
= \Delta P - 2\nabla P \cdot \nabla v + 2(a - c)\nabla R \cdot \nabla v - b\frac{|\nabla v|^2}{t} - 2\alpha R_{ij}v_{ij} - (2\alpha - 2)|\nabla \nabla v|^2
\]
\[
+ 2\alpha R_{ij}v_{ij} + 2a|Rc|^2 + b\frac{v}{t^2} + b\frac{n}{2t^2} + d\frac{n}{t^2} + \alpha c\Delta R - b\frac{cR}{t}
\]
\[
= \Delta P - 2\nabla P \cdot \nabla v + 2(a - c)\nabla R \cdot \nabla v - (2\alpha - 2)|v_{ij} - \frac{\alpha}{2\alpha - 2}R_{ij} - \frac{\lambda}{2t}g_{ij}|^2
\]
\[
- (2\alpha - 2)\frac{\lambda}{2a - 2}(\Delta v - \frac{\alpha}{2a - 2}R) + (2\alpha - 2)\frac{n}{4t^2}\lambda^2 + (2a + \frac{\alpha^2}{2\alpha - 2})|Rc|^2
\]
\[
- b\frac{|\nabla v|^2}{t} - 2\alpha R_{ij}v_{ij} + b\frac{v}{t^2} + b\frac{n}{2t^2} + d\frac{n}{t^2} + \alpha c\Delta R - b\frac{cR}{t}
\]
\[
= \Delta P - 2\nabla P \cdot \nabla v + 2(a - c)\nabla R \cdot \nabla v - (2\alpha - 2)|v_{ij} - \frac{\alpha}{2\alpha - 2}R_{ij} - \frac{\lambda}{2t}g_{ij}|^2
\]
\[
- \frac{2\alpha - 2\lambda}{\alpha}P + (\alpha\lambda + \frac{2\alpha - 2}{\alpha}\lambda - bc)\frac{R}{t} + (\alpha - 1)\frac{n\lambda}{2t^2} + (2a + \frac{\alpha^2}{2\alpha - 2})|Rc|^2
\]
\[
- (b + \frac{2\alpha - 2}{\alpha})\frac{|\nabla v|^2}{t} - 2\alpha R_{ij}v_{ij} + (1 - \frac{2\alpha - 2}{\alpha}\lambda)b\frac{v}{t^2} + b\frac{n}{2t^2}
\]
\[
+ (1 - \frac{2\alpha - 2}{\alpha}\lambda)d\frac{n}{t^2} + \alpha c\Delta R.
\]

\[
\square
\]

In the above lemma, let take \(\alpha = 2, a = -3, b = -1, c = -1, \lambda = 1\). Then we have

**Corollary 3.2.** Let \((M, g(t))\) be a solution to the Ricci flow, \(f\) be a positive solution of
\[
\frac{\partial}{\partial t} f = \Delta f + Rf,
\]
let \(v = -\ln f - \frac{a}{t}\ln(4\pi t)\) and
\[
P = 2\Delta v - |\nabla v|^2 - 3R + \frac{v}{t} - \frac{d}{t}.
\]
Then we have
\[
\frac{\partial}{\partial t} P = \Delta P - 2\nabla P \cdot \nabla v - 2|v_{ij} - R_{ij} - \frac{1}{2t}g_{ij}|^2 - \frac{1}{t}P
\]
\[
- 2(\Delta R + 2|Rc|^2 + \frac{R}{t} + 2\nabla R \cdot \nabla v + 2R_{ij}v_{ij}),
\]
\[
\frac{\partial}{\partial t}(tP) = \Delta(tP) - 2\nabla(tP) \cdot \nabla v - 2t|v_{ij} - R_{ij} - \frac{1}{2t}g_{ij}|^2 \\
- 2t(\Delta R + 2|Rc|^2 + \frac{R}{t} + 2\nabla R \cdot \nabla v + 2R_{ij}v_i v_j).
\]

Now we are ready to prove Theorem 1.2.

**Proof.** (Proof of Theorem 1.2) It is easy to see that the proof follows from Corollary 3.2, the trace Harnack inequality (1.1) for the Ricci flow and the maximum principle. \(\square\)

### 4. Entropy Formulas and Monotonicities

In this section, we will define two entropies which are similar to Perelman’s, and we will show that both of them are monotone under the Ricci flow. Let \((M, g(t))\) be a solution to the Ricci flow on a close manifold, and \(f\) be a positive solution of (1.3). Let \(u = -\ln f\) and \(H\) defined as in (2.2), we have

**Theorem 4.1.** Assume that \((M, g(t))\) be a solution to the Ricci flow with weakly positive curvature operator. Let

\[H = 2\Delta u - |\nabla u|^2 - 3R - 2\frac{n}{t}\]

and

\[F = \int_M t^2H e^{-u}d\mu,\]

then \(\forall t \in (0, T),\) we have \(F \leq 0\) and

\[\frac{d}{dt}F \leq 0.\]

**Proof.** The fact that \(F \leq 0\) follows directly from \(H \leq 0.\) We calculate its time derivative, using (2.3) and \(\frac{\partial}{\partial t}d\mu = -Rd\mu,\) we have

\[
\frac{d}{dt}F = \int_M (2tHe^{-u} + t^2e^{-u} \frac{\partial}{\partial t}H + t^2H \frac{\partial}{\partial t}e^{-u} - R^2 H e^{-u})d\mu \\
= \int_M [\Delta(t^2H e^{-u}) - 2t^2e^{-u}|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 - 2te^{-u}|
abla u|^2 \\
- 2t^2e^{-u}(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_i u_j)]d\mu \leq 0.
\]

\(\square\)

**Remark 4.1.** If we consider the system

\[
\begin{cases}
\frac{\partial}{\partial t}g_{ij} = -2(R_{ij} + \nabla_i \nabla_j u), \\
\frac{\partial}{\partial t}u = \Delta u - R,
\end{cases}
\]
then the measure \( dm = e^{-u}d\mu \) is fixed. This system differs from the original system
\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2R_{ij}, \\
\frac{\partial}{\partial t} u &= \Delta u - |\nabla u|^2 - R
\end{align*}
\]
by a diffeomorphism.

Now we consider \( v = -\ln f - \frac{n}{2} \ln(4\pi t) \) and \( P \) be defined as in (3.2), we have

**Theorem 4.2.** Assume that \((M, g(t))\) be a solution to the Ricci flow with weakly positive curvature operator. Let
\[
P = 2\Delta v - |\nabla v|^2 - 3R + \frac{v}{t} - d\frac{n}{t},
\]
where \( d \) is a constant. Let
\[
W = \int_M tP(4\pi t)^{-n/2}e^{-v}d\mu,
\]
then \( \forall t \in (0, T) \), we have
\[
\frac{d}{dt} W \leq 0.
\]

**Proof.** Using (1.3), (3.3) and \( \frac{\partial}{\partial t} d\mu = -Rd\mu \), we have
\[
\frac{d}{dt} W = \int_M [P(4\pi t)^{-n/2}e^{-v} + t(4\pi t)^{-n/2}e^{-v}\frac{\partial}{\partial t} P + tP\frac{\partial}{\partial t}((4\pi t)^{-n/2}e^{-v}) - Rtp(4\pi t)^{-n/2}e^{-v}]d\mu
\]
\[
= \int_M [\Delta(tPe^{-v}) - 2te^{-v}|v_{ij} - R_{ij} - \frac{1}{2t}g_{ij}|^2 - 2te^{-v}(\frac{\partial}{\partial t} R + \frac{R}{t} + 2\nabla R \cdot \nabla v + 2R_{ij}v_iv_j)(4\pi t)^{-n/2}d\mu \leq 0.
\]

\( \square \)

5. A Gradient Estimate for the Heat Equation

In this section, we consider a special case of our general evolution formula in Lemma 2.1. Let consider the positive solution \( f \) to the heat equation
\[
(5.1) \quad \frac{\partial}{\partial t} f = \Delta f,
\]
since the equation is linear, without loss of generality, we can assume that \( 0 < f < 1 \). Let \( f = e^{-u} \), then \( u \) satisfies
\[
(5.2) \quad \frac{\partial}{\partial t} u = \Delta u - |\nabla u|^2,
\]
and \( u > 0 \).

In the proof of Lemma 2.1, let take \( \alpha = 0, \beta = -1, a = c = 0, b = 1 \) and \( d = 0 \), then
\[
H = |\nabla u|^2 - \frac{u}{t},
\]
Differential Harnack Estimates for Time-dependent Heat Equations with Potentials

and we have

$$\frac{\partial}{\partial t} H = \triangle H - 2\nabla H \cdot \nabla u - \frac{1}{t} H - 2|\nabla \nabla u|^2,$$

Theorem 5.1. Let \( (M, g(t)), t \in [0, T] \), be a solution to the Ricci flow on a closed manifold. Let \( f(<1) \) be a positive solution to the heat equation (5.1), \( u = -\ln f \) and \( H = |\nabla u|^2 - \frac{u}{t} \).

Then for all time \( t \in (0, T) \)

$$H \leq 0.$$

Hence on \((0, T)\),

$$|\nabla f|^2 \leq -\frac{f^2 \ln f}{t}.$$

Proof. Notice that as \( t \) small enough, \( H < 0 \), now the proof follows from (5.3) and the maximum principle.

Remark 5.1. In this case, we do not need any curvature assumption.

Remark 5.2. Q. Zhang told us that he has proved the same estimate as our Theorem 5.1 in [Zha06], using a similar idea of the second author.

References

[And94] Ben Andrews. Harnack inequalities for evolving hypersurfaces. Math. Z., 217(2):179–197, 1994.

[Cao92] Huai-Dong Cao. On Harnack’s inequalities for the Kähler-Ricci flow. Invent. Math., 109(2):247–263, 1992.

[Cao08] Xiaodong Cao. Differential harnack estimates for backward heat equations with potentials under the Ricci flow. J. Funct. Anal., 2010, doi:10.1016/j.jfa.2008.05.009.

[CC95] Bennett Chow and Sun-Chin Chu. A geometric interpretation of Hamilton’s Harnack inequality for the Ricci flow. Math. Res. Lett., 2(6):701–718, 1995.

[CCG+07] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. The Ricci flow: techniques and applications. Part II, volume 144 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007. Analytic aspects.

[CH97] Bennett Chow and Richard S. Hamilton. Constrained and linear Harnack inequalities for parabolic equations. Invent. Math., 129(2):213–238, 1997.

[Cho91a] Bennett Chow. On Harnack’s inequality and entropy for the Gaussian curvature flow. Comm. Pure Appl. Math., 44(4):469–483, 1991.

[Cho91b] Bennett Chow. The Ricci flow on the 2-sphere. J. Differential Geom., 33(2):325–334, 1991.

[Cho92] Bennett Chow. The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. Comm. Pure Appl. Math., 45(8):1003–1014, 1992.

[CN05] Huai-Dong Cao and Lei Ni. Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds. Math. Ann., 331(4):795–807, 2005.

[Gue02] Christine M. Guenther. The fundamental solution on manifolds with time-dependent metrics. J. Geom. Anal., 12(3):425–436, 2002.

[Ham86] Richard S. Hamilton. Four-manifolds with positive curvature operator. J. Differential Geom., 24(2):153–179, 1986.
[Ham88] Richard S. Hamilton. The Ricci flow on surfaces. In Mathematics and general relativity (Santa Cruz, CA, 1986), pages 237–262. Amer. Math. Soc., Providence, RI, 1988.

[Ham93a] Richard S. Hamilton. The Harnack estimate for the Ricci flow. J. Differential Geom., 37(1):225–243, 1993.

[Ham93b] Richard S. Hamilton. A matrix Harnack estimate for the heat equation. Comm. Anal. Geom., 1(1):113–126, 1993.

[Ham95] Richard S. Hamilton. Harnack estimate for the mean curvature flow. J. Differential Geom., 41(1):215–226, 1995.

[KZ06] Shilong Kuang and Qi Zhang. A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow. Preprint, 2006.

[LY86] Peter Li and Shing Tung Yau. On the parabolic kernel of the Schrödinger operator. Acta Math., 156(3-4):153–201, 1986.

[Ni06] Lei Ni. A note on Perelman’s LYH-type inequality. Comm. Anal. Geom., 14(5):883–905, 2006.

[Ni07] Lei Ni. A matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow. J. Differential Geom., 75(2):303–358, 2007.

[Per02] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications, 2002, arXiv:math.DG/0211159.

[Stu96] K. T. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. J. Math. Pures Appl. (9), 75(3):273–297, 1996.

[Zha06] Qi S. Zhang. Some gradient estimates for the heat equation on domains and for an equation by Perelman. Int. Math. Res. Not., pages Art. ID 92314, 39, 2006.

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