Cooperative two-phonon phenomena in superconductivity

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Abstract. The problem of the two-photon coherent generation of entanglement photon pairs in quantum optics has been intensively studied over the passing years. It is important to note that the two-quanta cooperative effects also play a main role in other fields of physics. One example is superconductivity, where the Cooper pairs are created due to the simultaneous two-phonon exchange between electrons. It occurs when the one-phonon exchange integral between the band electrons is smaller than that of the two-phonon exchange. This is possible in many-band superconducting materials, in which the two-phonon exchange integral arises through the virtual bands of the material. Some estimates of the two-phonon superconductivity have already been given.

A more realistic model which takes into account the specificities of the many-band aspects of superconductor materials will be proposed. In two-phonon processes, a more complicated temperature dependence on the order parameter is expected. A rigorous study of this anomalous temperature dependence on the order parameter of superconductors is presented. One expects that the two-phonon exchange effects can amplify the superconductivity in a way similar to the way the thermal field amplifies the two-photon super-radiance in a microcavity.

1. Introduction

The collective processes in condensed matter have many analogous properties with cooperative radiation effects in quantum optics [1]. Recently the two-quantum generation of entanglement photons and its applications in informatics and communication have been intensively studied [2, 3].
Figure 1. The two-photon dipole forbidden transitions in the hydrogen-like atomic system. The full curves represent the photons of external coherent EMF and the wavy curves represent the spontaneously generated photons.

The new cooperative emission phenomenon for dipole-forbidden transitions of an inverted system of radiators can be observed in two-photon spontaneous emission processes [4]–[6]. The condition under which a sample of inverted radiators goes into the regime of the two-photon cooperative emission was found. In the two-photon emission process the hydrogen- or helium-like atoms collectively generate pairs of entanglement photons (biphotons) relative to dipole-forbidden transitions $|2S⟩−|1S⟩$. This cooperative two-quantum effect occurs through the virtual third state $|P⟩$ (see figure 1). The second-order correlation function describes the intensity of coherent photon pairs generated in these processes.

Let us study the electronic transitions between the states $|k⟩$ and $|k′⟩$ in the low band of a two-band superconductor with absorption or radiation of phonons (see figure 2). In order to neglect the one-phonon transitions between these states, let us consider that the second band is situated at a distance energetically larger than $\kappa T$ ($\kappa$ is the Boltzmann constant, $T$ is the temperature) on the Fermi level of the first band. It is clear that the one-phonon transition between these two states of the bands is described by the matrix element $g_{21}(q)$ which is proportional to

$$p_{mn} = -i\hbar e_q \int_{\mathbb{R}^3} e^{iqr} U_{2,k+q}^* \nabla U_{1,k},$$

when $m = 2$ and $n = 1$. Here $U_{m,k}(r)$ is the Bloch amplitude of the electronic wavefunction for the $m$ band and $e_q$ is the vector of polarization of the phonon.

In the case when the one-phonon transition between the first and the second bands $g_{21}(q)$ is larger than the matrix element of the one-phonon transition in the intrinsic first band $g_{11}(q)$, the two-phonon transition between the states $|k⟩$ and $|k′⟩$ becomes larger than the one-phonon transition. This becomes possible when the first and the second bands are formed from the local atomic levels with different symmetries. For example, if the first band provided by the local atomic level of the $S$-symmetry crystal and the second band comes from the level of $P$-symmetry, it is clear that in the case when $k \sim k'$ the intrinsic matrix element $g_{11}(q)$ is less than the matrix...
element between the first and the second bands \( g_{21}(q) \). In this case the two-phonon transitions between the states \( |k\rangle \) and \( |k'\rangle \) of the first band is similar to the two-photon transitions between the forbidden states \( |1S\rangle \) and \( |2S\rangle \) of atomic systems. The second band in such two-phonon transitions plays the role of a virtual state \( |P\rangle \).

The role of two-phonon superconductivity in semiconductors and other materials has been studied periodically in the literature [7]–[9]. In many cases these effects were studied decomposing the interaction Hamiltonian according to the electron–phonon interaction constant \([7,10]\). In this case the two-phonon cooperative effects depend on the higher order of the matrix element \( g_{11}(q) \). In our case the two-phonon exchange between the electrons through the virtual bands is similar to many photon effects in quantum optics and in our opinion has a more realistic diagrammatic interpretation. The anomalous behaviour of the order parameter as a function of temperature is observed due to the fact that the two-phonon exchange constant between the electrons increases as a function of temperature.

2. The effective Hamiltonian of the electron–electron interaction

The two-band electronic system, which interacts with the phononic field, is proposed for theoretical discussion. For simplicity of the problem the coulombic interaction between electrons is not taken into consideration. Therefore the Hamiltonian of such a system can be represented in the following form:

\[
H = H_{el} + H_{ph} + H_{int},
\]

where

\[
H_{el} = \sum_{m=1}^{2} \sum_{k} \varepsilon_{m}(k) a_{m,k}^\dagger a_{m,k};
\]

\[
H_{ph} = \sum_{q} \hbar \omega b_{q}^\dagger b_{q};
\]

\[
H_{int} = \frac{1}{\sqrt{V}} \sum_{m,n} \sum_{k,q} (g_{m,n}(q) a_{m,k+q}^\dagger a_{n,k} b_{q} + \text{h.c.}).
\]

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Here $V$ is the volume of the crystal, $b_{q}^{\dagger}$ ($b_{q}$) is the creation (annihilation) operator of the phonon with $q$ denoting the wavevector and $a_{n,k}^{\dagger}$ ($a_{n,k}$) the electron creation (annihilation) operator in the $m$-band ($m = 1, 2$) with the quasi-wavevector $k$ and spin $\sigma$. Therefore the sums on $k$ in Hamiltonian (1) take into account the wavevector and the spin of electrons in the bands $k = (k, \sigma)$. The operators obey the following commutation relations:

$$[b_{q}; b_{q}^{\dagger}] = \delta_{qq}; \quad [b_{q}; b_{q}^{\dagger}] = 0;$$

$$[a_{m,k}; a_{m,k}^{\dagger}] = \delta_{mm_1} \delta_{kk_1} \delta_{\sigma \sigma_1}; \quad [a_{m,k}; a_{m,k}^{\dagger}] = 0.$$

The ‘minus’ and ‘plus’ indexes indicate the commutation and anticommutation operations; $\epsilon_{m}(k)$ is the energy of the electron, which can be calculated from the level of the chemical potential; the matrix elements of the electron–phonon interaction are known [10]:

$$g_{11}(q) = iM_{11}\sqrt{q}, \quad g_{21}(q) = \frac{iM_{21}}{\sqrt{q}}$$

$$M_{11} = \sqrt{\frac{\hbar q}{2\rho \omega_q}} E_{11}, \quad E_{11} \approx \epsilon_F$$

$$M_{21} = \sqrt{\frac{q}{2\hbar \rho \omega_q}} E_{21} p_{21}, \quad p_{21} \approx \sqrt{2m^* \epsilon_{21}}$$

where $\rho$ is the density of the material, $\epsilon_F$ is the energy of the Fermi level, $\epsilon_{21}$ is the energetic distance between bands one and two (see figure 2) and $m^*$ is the effective mass.

Taking into account the proprieties of the local symmetry of Bloch wavefunctions for the lower and upper bands, it is not difficult to demonstrate that in many cases when the second and the first bands arise from the atomic local levels with different symmetries (for example, P and S atomic states) the interband matrix element of the electron–phonon interaction $g_{12}$ is larger than the intraband matrix elements $g_{11}, g_{22} (g_{12} \gg g_{11})$.

In this paper the two-phonon cooperative processes between the electrons of the lowest band through the virtual state of the second band are discussed. Thus, it is necessary first to eliminate the operators of the second band from Hamiltonian (1). If the second band is situated at an energetic distance larger than $\kappa T$, we can eliminate the electronic operators of this band from Hamiltonian (1). In order to solve this problem let us consider the operator $A(t)$ which does not depend on the operators of the second band.

The Heisenberg equation for the mean value of this operator $A(t)$ is

$$\left\langle \frac{dA(t)}{dt} \right\rangle = \frac{i}{\hbar} \left\langle [H; A(t)] \right\rangle. \quad (2)$$

As the second band plays the role of virtual states for the transitions between the electron states in the first band, it is possible to eliminate the electronic operators of the second band (see figure 2). In this case the solution of Heisenberg equation for the operator $a_{2,k}(t)$ in the Born–Markoff approximation can be represented through the vacuum and interaction parts

$$a_{2,k}(t) = a_{2,k}(t_0) \exp\left(-\frac{i}{\hbar} \epsilon_{2}(t - t_0)\right)$$

$$- \frac{1}{\sqrt{V}} \sum_{q_1} \left(\frac{g_{21}(q_1) a_{1,k-q_1} b_{q_1}}{\epsilon_{2}(k) - \epsilon_1(k - q_1) - \hbar \omega_{q_1}} + \frac{g_{12}^{*}(q_1) b_{q_1}^{\dagger} a_{1,k+q_1}}{\epsilon_{2}(k) - \epsilon_1(k + q_1) + \hbar \omega_{q_1}}\right). \quad (3)$$

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It is considered that the interaction between the systems takes place in the moment of time \( t_0 \).

After substitution of the operators \( a_{2,k}(t) \) and \( a_{2,k}^\dagger(t) \) into equation (2) one can eliminate the operators of the second virtual band:

\[
\frac{\langle dA \rangle}{dt} = \frac{i}{\hbar} \sum_k \left\{ \left\langle g_{21}(q) g_{21}^\dagger(q_1) \right\rangle \right. \\
+ \frac{\left\langle [a_{k+q+1} a_k b_q; A] \right\rangle}{\varepsilon_2(k + q) - \varepsilon_1(k + q - q_1) - \hbar \omega_{q_1}} - \frac{\left\langle [a_{k+q-1} a_k b_q; A] \right\rangle}{\varepsilon_2(k + q) - \varepsilon_1(k + q + q_1) + \hbar \omega_{q_1}}
\]

In expression (4), for simplicity, label ‘1’ of the first band is omitted considering that \( a_k \) is equivalent to \( a_{1,k} \). Furthermore, it is considered that the operator \( A(t) \) depends only on the operators of the first band. In this case it is possible to eliminate from equation (4) the following operator combinations of the phonon subsystem: \( b_{q_1}^\dagger b_q, b_{q_1} b_q \) and their hermetical conjugate. The two-phonon combination of the operators (called biphonon) \( b_{q_1}^\dagger b_q \) describes the scattering processes with the absorption of one phonon with momentum \( q \) and the generation of another phonon with momentum \( q_1 \). The biphonon combinations \( b_{q_1} b_q \) and \( b_{q_1}^\dagger b_q^\dagger \) describe the interaction of the electrons with the boson field in which the phonon pairs are absorbed or generated (see figure 2). It is important to emphasize that the scattering effects are stimulated by the thermal phonons from the thermostat.

The solutions of the Heisenberg equations for these biphonon operators can be represented in the following Born–Markoff approximation:

\[
b_{q_1}^\dagger(t) b_q(t) = \tilde{b}_{q_1}^\dagger(t_0) \tilde{b}_q(t_0) \exp(i \hbar (\omega_{q_1} - \omega_q)(t - t_0))
\]

where

\[
\tilde{b}_{q_1}^\dagger(t) \tilde{b}_q(t) = \frac{1}{V} \sum_{k_1} g_{21}(q_1) g_{21}^\dagger(q)
\]

In the same way it is possible to represent all the biphonon operator combinations in expression (4). After the substitution of expression (5) into (4) it is necessary to eliminate the free parts of the biphonon operators: \( \tilde{b}_{q_1}^\dagger(t) b_q(t), b_{q_1}(t) b_{q_1}(t) \) and their hermetic conjugate. Taking into account the method of elimination of the free part \( \tilde{b}_q(t) = \exp(-i \omega_q(t - t_0)) b_q(t_0) \) of the operator \( b_q(t) \), proposed by Bogoliubov [11], we can write

\[
\text{Tr}\{b_q(t) U(el, ph, t)\}_{t_0} = (1 + N_q) \text{Tr}\{\tilde{b}_q(t); U(el, ph, t)\}_{t_0}
\]
where $U(el, ph, t)$ is the function of the fermion and boson operators, $N_q = \{\exp(\frac{\hbar \omega_q}{kT}) - 1\}^{-1}$ is the mean number of phonons in the $q$ state. The following steps of elimination of the free part of biboson combinations in expression (4) are proposed in this paper. In the first stage the following expression is obtained:

$$\text{Tr}\{U(el, ph, t)\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t)\}_t = N_q \text{Tr}\{[\hat{b}_q(t); U(el, ph, t)\hat{b}_{q_1}^\dagger(t)]\}_t.$$

In other words

$$\text{Tr}\{\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t)U(el, ph, t)\}_t = \frac{N_q(1+N_q)}{N_q - N_q} \text{Tr}\{[\hat{b}_q(t); U(el, ph, t)\}_t.$$

From the last expression the final relation follows:

$$\text{Tr}\{\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t)U(el, ph, t)\}_t = \frac{N_q(1+N_q)}{N_q - N_q} \text{Tr}\{[\hat{b}_q(t); U(el, ph, t)\}_t.$$

As a result expression (5) gives the representation of the free part of the bibophon operator via the source part $\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t) = \hat{b}_{q_1}^\dagger(t)\hat{b}_q^\dagger(t) - (\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t))^s$, so that operator $U$ depends only on the electronic variables. In the higher order of expansion on the constant of the electron–phonon interaction it is possible to approximate the slow part of the boson operators in the source part of expression (6) by the following mean values: $\langle b_q^\dagger b_q\rangle_t = N_q$ and $\langle b_{q_1}^\dagger b_{q_1}\rangle_t = 1 + N_q$. Introducing this in the right-hand side of relations (8) and taking into account the commutation relation $[\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t); U(el, t)] = 0$, the following expression for the free part of biboson operators are obtained:

$$\text{Tr}\{\hat{b}_{q_1}^\dagger(t)\hat{b}_q(t)U(el, ph, t)\}_t = -\frac{N_q(1+N_q)}{1+N_q + N_q} \text{Tr}\{[\hat{b}_q(t); U(el, ph, t)\}_t.$$

Considering the Born–Markoff approximation and after the elimination of the free parts of relations (9) one can obtain the following equation for operator $A(t)$:

$$\langle \frac{dA(t)}{dt}\rangle = i\hbar (H'_{el}; A(t)) + \frac{i}{\hbar} (H^{eff}; A(t)) \tag{10}$$

where $H'_{el} = \sum_k \varepsilon_k^o(k)\alpha_{k}^\dagger\alpha_{k}$;

$$H^{eff} = -\frac{1}{V^2} \sum_{k,k_1,q,q_1} |g_{21}(q)|^2 |g_{21}(q_1)|^2 (1 + 2N_q) \phi_{1}(k_1, q_1) \phi_{1}(k_1, q, q_1)$$

$$\times \frac{(\hbar \omega_q + \hbar \omega_{q_1})\alpha_{k+q+q_1}^\dagger a_{k_1+q+q_1}^\dagger a_{k} + (\hbar \omega_q + \hbar \omega_{q_1})^{-2} (\varepsilon_{1}(k_1 + q + q_1) - \varepsilon_{1}(k_1))^2}$$

$$- \frac{2}{V^2} \sum_{k,k_1,q,q_1} |g_{21}(q)|^2 |g_{21}(q_1)|^2 (1 + 2N_q) \phi_{2}(k_1, q_1) \phi_{2}(k_1, q, q_1)$$

$$\times \frac{(\hbar \omega_q - \hbar \omega_{q_1})\alpha_{k+q-q_1}^\dagger a_{k_1+q-q_1}^\dagger a_{k} + (\hbar \omega_q - \hbar \omega_{q_1})^{-2} (\varepsilon_{1}(k_1 + q - q_1) - \varepsilon_{1}(k_1))^2}\tag{11}$$
where
\[ \phi_1(k, q, q_1) = \frac{\varepsilon_2(k + q) + \varepsilon_2(k + q_1) - \varepsilon_1(k + q + q_1) - \varepsilon_1(k)}{(\varepsilon_2(k + q) - \varepsilon_1(k + q + q_1) + \hbar \omega_{q_1})(\varepsilon_2(k + q_1) - \varepsilon_1(k) - \hbar \omega_{q_1})}, \]
\[ \phi_2(k, q, q_1) = \frac{\varepsilon_2(k + q) + \varepsilon_2(k - q_1) - \varepsilon_1(k + q - q_1) - \varepsilon_1(k)}{(\varepsilon_2(k + q) - \varepsilon_1(k + q - q_1) - \hbar \omega_{q_1})(\varepsilon_2(k - q_1) - \varepsilon_1(k) + \hbar \omega_{q_1})}. \]

In this section the effective interaction Hamiltonian of the electrons from the first band through the two-phonon interaction is obtained. The cooperative interaction between the electrons through the biphonon field is reduced not only to the simple processes of simultaneous absorption or emission of the two phonon. As a consequence of effective interaction Hamiltonian (11) the transitions between the two states of the first band can take place with one-absorption or emission of the two phonon. This cooperative exchange between the electrons increases with the increase of temperature. In other words the effective Hamiltonian implicitly depends on the temperature through the mean number of phonons \( N_q \). This is one of the main differences between the one-phonon Bardeen–Cooper–Schrieffer (BCS) exchange [12] and the two-phonon exchange between the electrons. Such a temperature dependence is very important for the formation of the superconducting state.

In the following section the temperature dependence on the order parameter in the superconductivity state will be analysed in order to distinguish this effect.

3. The thermodynamical proprieties of the superconductor

The influence of the second virtual band position on the creation of the two-phonon Cooper effect in the first band plays an important role in this model. For example, if the second virtual band is situated near the Fermi level of the first band, the two-phonon scattering effects with the absorption of one phonon from the thermostat and the generation of another phonon can play a more important role as processes with the simultaneous generation or absorption of phonon pairs (see figure 2).

In this paper we will discuss a simpler effect when the second band is situated at a distance larger than the mean value of the phonon energy. As a consequence of classical approximations in superconductivity, only the electrons which have opposite impulses and spins are considered and they are situated in Debay’s layer near the Fermi surface. In this case the interaction corresponds to the attraction effect between the electrons. In this situation in expression (11) it is considered that \( \hbar \omega_q \ll \varepsilon_2(k + q) - \varepsilon_1(k) \) and \( \phi_1(k, q, q_1) \approx \phi_2(k, q, q_1) \approx 2/\varepsilon_{21} \), where \( \varepsilon_{21} \) is the energetic distance from the Fermi surface to the bottom of the second band (see figure 2). After a simple modification of the effective Hamiltonian (11), where in the first term we replaced \( q + q_1 \rightarrow q \), \( k_1 \rightarrow -k_1 \) and considering that \( q = k_1 - k \), but in the second term we replaced \( q - q_1 \rightarrow q \), the following effective interaction Hamiltonian between the electrons is obtained:

\[ H = \sum_{k} \varepsilon'_1(k) a^+_k a_k - \frac{G(T)}{V} \sum_{k, k_1} a^+_k a^+_{-k_1} a_{-k_1} a_k, \]  

where \( k = (k, \frac{1}{2}) \), \( -k = (-k, -\frac{1}{2}) \) and

\[ G(T) = \frac{4}{V} \sum_{q_1} \left| g_21(q - q_1) \right|^2 \left| g_21(q_1) \right|^2 \left\{ \frac{1}{\hbar \omega_{q - q_1}} + 4N_{q_1} \frac{\hbar \omega_{q - q_1}}{(\hbar \omega_{q_1})^2 - (\hbar \omega_{q_1})^2} \right\}. \]
Introducing the order parameter $\Delta(T) = \frac{G(T)}{2V} \sum_{k_1} \langle a^\dagger_{k_1} a_{-k_1} \rangle$, we can approximate the Hamiltonian (12) in the following form:

$$H = \sum_k \varepsilon_k^1(k) a^\dagger_k a_k - \sum_k (\Delta(T) a_{-k} a_k + \text{h.c.}) \tag{14}$$

Let us now pass to expression (13) from the sum on $q_1$ to the integral and considering that the dispersion law is linear $\omega_q = cq$ (the acoustic phonons), thus the expression for $G(T)$ takes the form

$$G(T) = \frac{4}{(2\pi)^3 \hbar c} \frac{|M_{21}|^4}{\varepsilon_{21}^2} \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta \, d\theta \int_0^{q_{\text{max}}} q_1 dq_1 \frac{1}{|q - q_1|} \left\{ \frac{1}{|q - q_1|} + 4N_{q_1} |q - q_1|^{-2} \right\} \tag{15}$$

Here $\theta$ is the angle between the vectors $q$ and $q_1$ and $c$ is the velocity of sound. After the integration of expression (15) on the solid angle the function $G(T)$ becomes

$$G(T) = \frac{|M_{21}|^4}{\pi^2 \varepsilon_{21}^2 \hbar c} \int_{q_{\text{max}}}^\infty dq_1 \left\{ \ln \left| \frac{q + 2q_1}{q - q_1} + q_1 \right| + 2N_{q_1} \ln \left| \frac{q + 2q_1}{q - 2q_1} \right| \right\} \tag{16}$$

Representing the mean number of phonons through the series $N_{q_1} = \sum_{n=1}^\infty \exp(- n \frac{\hbar c q_1}{\varepsilon_{21}})$ and considering $q \approx 2k_F$ (where $k_F$ is the Fermi quasi-wavevector) one can obtain the following expression for $G(T)$:

$$G(T) = \frac{|M_{21}|^4}{2\pi^2 \varepsilon_{21}^2 \hbar c k_F} \left[ k_F \ln \left| \left( \frac{q_{\text{max}}}{k_F} \right)^2 \right| - 1 \right] + n \ln \left| \frac{q_{\text{max}} + k_F}{q_{\text{max}} - k_F} \right| - 2k_F$$

$$+ \frac{|M_{21}|^4}{\pi^2 \varepsilon_{21}^2 \hbar c} \sum_{n=1}^\infty \frac{1}{n} \left\{ \exp(-n\chi)E_i(n\chi) - \exp(n\chi)E_i(-n\chi) \right\} \tag{17}$$

where $\chi = \hbar c k_F/(\kappa T)$.

Let us consider that $q_{\text{max}} > k_F$. This becomes possible when the concentration of particles is small in the superconductor. Expanding expression (17) in the series, we obtain the final expression for the two-phonon exchange interaction

$$G(T) = G^{(0)}(1 + \gamma), \quad G^{(0)} = \frac{|M_{21}|^4}{\pi^2 \varepsilon_{21}^2 \hbar c} \ln \left| \frac{q_{\text{max}}}{k_F} \right| \tag{18}$$

$$\gamma = \left( \chi \ln \left| \frac{q_{\text{max}}}{k_F} \right| \right)^{-1} \sum_{n=1}^\infty \frac{1}{n} \left\{ \exp(-n\chi)E_i(n\chi) - \exp(n\chi)E_i(-n\chi) \right\} \tag{19}$$

In the limit cases (when $\chi \gg 1$ and $\chi \ll 1$) expression (19) takes the following values:

$$\gamma = \begin{cases} \gamma_1, & \text{for } \chi \gg 1, \\ \gamma_2, & \text{for } \chi \ll 1; \end{cases} \tag{20}$$

where $\gamma_1 = \sigma T^2$, $\sigma = \frac{\varepsilon_s^2}{3} \left( \frac{2\kappa}{\hbar c k_F} \right)^2 \left( \ln \left| \frac{q_{\text{max}}}{k_F} \right| \right)^{-1}$; $\gamma_2 = 2\ln \left| \frac{q_{\text{max}}}{k_F} \right|^{-1}$.

Expressions (19) and (20) indicate that the two-phonon electron–electron interaction is proportional to the square temperature at low temperatures, but at high temperature this interaction is constant.
After the diagonalizations of expression (14) using the Bogoliubov transformations [13] and taking into account expression (18) the following relation for the order parameter as a function of temperature is obtained:

\[
\Delta(T) = \frac{G(T)}{2V} \sum_{k} \frac{\Delta(T)}{\sqrt{\eta_k^2 + \Delta^2(T)}} \tanh \frac{\sqrt{\eta_k^2 + \Delta^2(T)}}{2\kappa T},
\]  

(21)

where \( \eta_k = \varepsilon'_k(k) - \mu, \mu \) is the chemical potential (\( \mu \sim \hbar^2k_F^2/(2m^*) \)). Moreover we can pass from the sum on \( k \)-space to the integral on \( \eta \) [13]:

\[
\ln \frac{\Delta(0)}{\Delta(T)} = 2I \left( \frac{\Delta(T)}{\kappa T} \right) - \frac{\gamma}{\lambda(1 + \gamma)},
\]

(22)

where

\[
I(u) = \int_0^\infty \frac{dx}{\sqrt{x^2 + u^2} \left( \exp \sqrt{x^2 + u^2} + 1 \right)}.
\]

(23)

Here, we consider that the \( \Delta(T) \) is constant in the layer \( 2\hbar\omega_{\text{max}} \) in the vicinity of the Fermi surface. Using the integral expression of the Macdonald’s function we can represent the equation (22) in the following form:

\[
\ln \frac{\Delta(0)}{\Delta(T)} = 2 \sum_{n=1}^\infty (-1)^{n+1} K_0 \left( n \frac{\Delta(T)}{\kappa T} \right) - \frac{\gamma}{\lambda(1 + \gamma)},
\]

(24)

where \( K_0(u) \) is the Macdonald’s function; \( \lambda = |G(0)|N_F(0); \Delta(0) = 2\hbar\omega_{\text{max}} \exp(-\frac{1}{\chi}) \) and \( N_F(0) = \frac{\rho k_F}{2\pi^2 \hbar^2} \) is the density of states on the Fermi surface.

The numerical solution of equation (24) is obtained using the following values of the parameters (see, for example, [14]) \( k_F = 2 \times 10^7 \text{ cm}^{-1}, \eta_{\text{max}} = 1 \times 10^8 \text{ cm}^{-1}, c = 1 \times 10^5 \text{ cm s}^{-1}, \rho = 3 \text{ g cm}^{-3}, \varepsilon_{21} = 3 \text{ eV}, m^* \approx 5m_e-10m_e \) (\( m_e \) is the mass of the free electron). The comparison of the dependence of the order parameter as function of temperature \( \Delta(T) \) with the BCS-like model with same \( \Delta(0) \) is given in figure 3. Changing the effective mass of carriers one can influence the parameters \( \lambda \) and \( \Delta(0) \). One observes that for low values of \( \lambda \) or \( \Delta(0) \) the proposed model substantially differs from the BCS-like model with constant exchange integrals between the carriers. In figure 3 the differences between the investigated model and BCS-like model are plotted for three values of \( \Delta(0) \).

The increase in the order parameter at low temperature can be analytical described if we solve equation (24) in the limits \( \Delta(T) > \kappa T \). In this case it is considered that \( K_0(u) \sim \sqrt{\pi/2u} \exp(-u) \) for \( u \to \infty \) and it is enough to take the basic term of the series, corresponding to \( n = 1 \). As the order parameter \( \Delta(T) \) is close to \( \Delta(0) \), we can consider the following approximation: \( \ln \Delta(0)/\Delta(T) \approx [\Delta(0) - \Delta(T)]/\Delta(0) \). In the low-temperature region the dependence of the order parameter as a function of temperature is obtained:

\[
\Delta(T) = \Delta(0) - \sqrt{2\pi\kappa T\Delta(0)} \exp \left( -\frac{\Delta(0)}{\kappa T} \right) + \frac{\Delta(0)}{\lambda} \frac{\sigma T^2}{1 + \sigma T^2}.
\]

(25)

For a small temperature, from expression (25) it follows that the value of the order parameter \( \Delta(T) \) increases with the increase in temperature (see figure 3). This behaviour of the order parameter differs from BCS one-phonon theory. At low temperatures one observes the increase of the order parameter. After achieving the maximal value, the order parameter decreases (see...
Figure 3. The temperature dependences of the superconductivity order parameter $\Delta(T)$. The full curves represent the temperature dependences of the order parameter for the two-phonon superconductivity model and the dotted curves represent the temperature dependences of the order parameter for BCS-like superconductivity model. (a) $m^* = 5m_e; \lambda = 0.7; \sigma = 0.04; \Delta(0) = 37$ K; $T_{cBCS} \approx 21$ K; $T_c = 86.3$ K. (b) $m^* = 8m_e; \lambda = 2.9; \sigma = 0.04; \Delta(0) = 108$ K; $T_{cBCS} \approx 62$ K; $T_c = 86.6$ K. (c) $m^* = 10m_e; \lambda = 5.6; \sigma = 0.04; \Delta(0) = 128$ K; $T_{cBCS} \approx 73$ K; $T_c = 86.7$ K.

This phenomenon is connected with the strong dependence of the exchange integral on temperature.

At the critical point $T = T_c$ the order parameters take the zero value. Considering the condition $\Delta(T_c) = 0$ in equation (22) one can obtain the following equation for the critical temperature:

$$\ln\left(\frac{\pi \kappa T_c}{\Delta(0)}\right) = \frac{1}{\lambda} \frac{\sigma T_c^2}{1 + \sigma T_c^2} + C_0, \quad C_0 = 0.577 \ldots - \text{Euler constant.} \quad (26)$$

From equation (26) it follows that the intersection of $\ln(aT)$ with the function $\sigma T^2/\left(\lambda + \lambda \sigma T^2\right)$ gives us the critical temperature value (here $a = \exp(-C_0\pi \kappa/\Delta(0))$). The main difference between the BCS-like model and the model proposed here consists in the influence of the parameter $\sigma$ on the critical temperature. For numerical comparison, we use BCS-like model with $\sigma = 0$ in equation (24). In this case the well known dependence of critical temperature on the gap is obtained: $\kappa T_{cBCS} = \exp(C_0)\Delta(0)/\pi$. In the model presented in this paper the critical temperature increases with the increase of the parameter $\sigma$. However, this influence of parameter $\sigma$ on the critical temperature in comparison with the BCS-like model can be observed for the small value of parameter $\lambda$ which is equivalent to the small value of $\Delta(0)$. In figures 3(a)–(c)
the difference between the BCS-like model and proposed model (with $\sigma \neq 0$) is represented. As we observe, the behaviour of our model in comparison with BCS-like model is more evident for the small gap.

In order to estimate the power law dependence of order parameter near the critical temperature ($T_c$) one can decompose the integral in equation (22) on the small parameter $\Delta(T)/\kappa T$. Taking into account equation (26) for $T_c$, we obtain the following temperature dependence of the order parameter:

$$\Delta(T) = \frac{4\pi \kappa T}{\sqrt{14\xi(3)}} \left[ \ln \frac{T_c}{T} + \frac{1}{\lambda} \left( \frac{\sigma T^2}{1 + \sigma T^2} - \frac{\sigma T^2}{1 + \sigma T^2} \right) \right]^{1/2},$$

(27)

where $\xi(3)$ is the value of the Riemann zeta-function ($\xi(3) \approx 1.2$). Equation (27) can be modified to a simpler form if one considers that $\ln T_c/T \approx [T_c - T]/T_c$ when $T \to T_c$

$$\Delta(T) = \frac{4\pi \kappa T_c}{\sqrt{14\xi(3)}} \left[ \left( 1 - \frac{T}{T_c} \right) \left( 1 - \frac{2\sigma T^2_c}{\lambda(1 + \sigma T^2_c)} \right) \right]^{1/2}.$$

(28)

The decrease of $\Delta(T)$ as a function of temperature in the proposed model is more slow than in the BCS-like model. This follows from numerical results and the analytic dependencies (27) and (28). Indeed the absolute values of the derivations of the order parameter on temperature is less than in the BCS-like model $|\partial \Delta(T) / \partial T| < |\partial \Delta_{BCS}(T) / \partial T|$. The slow decrease of $\Delta(T)$ give us a higher critical temperature than in traditional superconductivity theory with a constant exchange integral when $\sigma = 0$.

4. Results and discussions

In this paper the formation of Cooper pairs in two-phonon exchange processes between the electrons through the phonon field was discussed. It is shown that the exchange integral between the electrons increases with temperature due to the fact that the two-phonon scattering effect stimulates the coherent formation of Cooper pairs in the superconductor. This effect opens a new concept regarding the formation of Cooper pairs in the processes of multi-phonon exchange between the carriers, and is similar to cooperative ‘phasing’ of radiators in optical two-photon processes [4]–[6]. It is clear that with the increase of temperature, the superconductivity gap increases slower than $\kappa T$ so that the order parameter at low temperature achieves the maximum value and after that decreases until the critical temperature $T_c$ (see figure 3). This anomalous dependence on the temperature of the order parameter is described by a complicated two-phonon exchange between the electron subsystem and the phonon subsystem. In addition, from equation (26) it results that the temperature of the phase transition is higher if the two-phonon interaction is more intensive. The increase in the two-quantum cooperative effects stimulated by the thermal field was observed in the two-photon super-radiance too [15, 16]. This phenomenon is connected with the nature of the two-quanta transitions and was analysed in the paper [15].

Special attention is devoted to studying the cooperative two-phonon processes of interaction between the electrons from the lower band with the contribution of the higher situated virtual bands. The effective model Hamiltonian was obtained in approximation $\varepsilon_2(k) - \varepsilon_1(k_F) \gg \kappa T$ and therefore in this case the small decomposition Hamiltonian parameter as a result of electron–phonon interaction is $\delta = \sqrt{\varepsilon_2(k) - \varepsilon_1(k_F)} < 1$. An interesting situation appears when the virtual bands verge towards the Fermi level. In this case the small parameter $\delta$ increases substantially,
and the summation of the electron–phonon interaction diagrams becomes more complicated. This summation procedure required greater prudence and the situation may be similar to the problem found in the papers [17, 18].

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