Induced Luttinger Liquid Behaviour in an Exactly Solvable Model of Stripes

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We study an exactly solvable model describing a stripe consisting of a Toda array of \( N \) anharmonic elastic chains sandwiched between two conducting chains. It is shown that the presence of a charge on one chain generates a gapless excitation branch (Luttinger liquid) on the other and leads to increase in the phonon frequencies.

During recent several years there have been rather intense interest in systems which combine one-dimensional and two- or three-dimensional features. As examples of such systems one can mention domain wall (or stripe) formations in doped Mott insulators \(^1\) and quasi-one-dimensional systems with electron-phonon interactions.

The capital fact about phonon systems is that phonons are never one-dimensional. Thus in both cases electrons residing on one-dimensional chains are immersed in a two- or three-dimensional active environment. This feature makes such problems very challenging from the theoretical point of view because on one hand the low dimensionality leads to the enhancement of interactions thus requiring non-perturbative approach, and on the other hand the multi-chain structure fits rather awkwardly with available non-perturbative techniques. Usually theorists resort to a hybrid treatment incorporating features of exact solution for individual chains and RPA (Random Phase Approximation) for interchain interactions \(^2\).

From the qualitative point of view there are several interesting problems concerning the systems in question. One is how conducting regions affect each other and another is how changes in the conducting region affect the active medium in which it exists.

Here we discuss these problems using as an example an exactly solvable model of a static stripe suggested by Fateev \(^3\). A remarkable property of this model is that it exists in two different representations describing situations with different physics. One representation is fermionic and the other one is bosonic and they are related by a duality transformation. In both its incarnations the model describes two species of (1+1)-dimensional charged particles located on different sides of an insulating stripe and interacting via its elastic modes (see Fig. \(\text{Fig. 1}\)). The latter ones are anharmonic optical phonons described by the so-called Toda array. The Lagrangian density has the following form:

\[
\mathcal{L} (\beta, N) = \mathcal{L}_{\text{Toda}} + \mathcal{L}_{\text{edge}}
\]  

(1)

\[
\mathcal{L}_{\text{Toda}} = \frac{1}{2} \sum_{n=1}^{N} (\partial_{\mu} \phi_{n})^{2} - (m/\beta)^{2} \sum_{n=1}^{N-1} e^{\beta (\phi_{n+1} - \phi_{n})} \quad (2)
\]

In its fermionic incarnation the edge Lagrangian is given by

\[
\mathcal{L}_{\text{edge}}^{(f)} = \sum_{s=1,2} \left[ i \psi \gamma_{\mu} \partial_{\mu} \psi + \frac{\pi g}{2} (\bar{\psi} \gamma_{\mu} \gamma_{5} \psi)^{2} \right] - m \bar{\psi}_{1} \psi_{1} e^{-\beta \phi_{1}} - m \bar{\psi}_{2} \psi_{2} e^{-\beta \phi_{2}} - (m/\beta)^{2} (e^{-2\beta \phi_{1}} + e^{2\beta \phi_{2}}) \quad (3)
\]

where \( g = \beta^{2}/(4\pi + \beta^{2}) \). It can be interpreted as a model of two charge density waves coupled through the Toda array.

\[
\Phi_{1} \quad \cdots \cdots \quad \Phi_{N}
\]

\[
\psi_{+}
\]

\[
\psi_{-}
\]

FIG. 1. The system under consideration: two conducting regions coupled by the Toda array.

The bosonic representation is more intriguing because of its connotations with the hypothetic stripe formations in doped Mott insulators. The edge Lagrangian density is

\[
\mathcal{L}_{\text{edge}}^{(b)} = \frac{1}{2} \sum_{s=1}^{2} \frac{\partial_{\mu} \Delta_{s} \partial_{\mu} \Delta_{s}}{1 + (\beta/2)^{2} \Delta_{s}^{2}} - \frac{m^{2}}{2} (|\Delta_{1}|^{2} + |\Delta_{2}|^{2}) e^{-\beta \phi_{1}} - \frac{m^{2}}{2} (|\Delta_{1}|^{2} + |\Delta_{2}|^{2}) e^{\beta \phi_{N}} \quad (4)
\]

where \( \Delta_{s} \) are complex bosonic fields which can be interpreted as superconducting order parameters residing on the edges of the stripe.

The duality transformation relating fermionic to the bosonic representation is
which also corresponds to $g \to 1 - g$. Therefore the strong coupling limit of one model is related to the weak coupling limit of another one.

The model has $U(1) \times U(1)$ symmetry and the corresponding charges are given by

$$Q_s = \frac{i}{\beta} \int dx \psi^*_s \gamma_0 \psi_s = -\frac{i}{2} \int dr \frac{(\Delta^+_s \delta_0 \Delta^-_s - \Delta^-_s \delta_0 \Delta^+_s)}{[1 + (\beta/2)^2 |\Delta_s|^2]}$$

One can modify the Hamiltonian introducing chemical potentials coupled to the charges $H \to H + h^+_Q - h^- Q_-$ which would correspond to populating the edges of the stripe. In this publication we shall be particularly interested in the situation when one chemical potential is much greater than another.

In [3] Fateev gave a proof of integrability of the above models and suggested exact two-body $S$-matrices. At zero chemical potentials all excitations have gaps. The particles living on the edges may be considered as fundamental ones, excitations of the Toda array being their bound states. There are $N - 1$ bound states with the masses

$$M_j = 2 M \sin(\pi j / 2 \lambda), \quad (j = 1, 2, \ldots, N - 1), \quad \lambda = N - g \quad (7)$$

where $M = M(m, \beta)$ is the mass of the fundamental particle. All bound states are neutral. The case $N = 1$ where there are no bound states has been thoroughly studied in [3].

Each fundamental particle is labeled by two indices corresponding to the two $U(1)$ groups (it is either soliton or antisoliton and it belongs to one chain or another). The two-body scattering matrix of the fundamental particles is given by a tensor product of two sine-Gordon $S$-matrices multiplied by a CDD-factor responsible for calculation of double poles. Since the Bethe ansatz equations for the sine-Gordon model is well known, it is quite straightforward to write down such equations for the model in question:

$$e^{-iM \sin\theta_j L} = \prod_{k \neq j} S_0(\theta_j - \theta_k) \times \prod_{a=1}^m e_1(\theta_j - u_a) \prod_{b=1}^m e_1(\theta_j - v_b) \prod_{j=1}^n e_1(\theta_j - u_a) \prod_{b=1}^m e_2(u_b - u_a) \prod_{j=1}^n e_1(\theta_j - v_b) \prod_{b=1}^m e_2(v_b - v_a) \quad (8)$$

$$\prod_{j=1}^n e_1(\theta_j - u_a) = \prod_{b=1}^m e_2(u_b - u_a) \quad (9)$$

$$\prod_{j=1}^n e_1(\theta_j - v_b) = \prod_{b=1}^m e_2(v_b - v_a) \quad (10)$$

where $e_n(x) = \sinh \lambda (x + i n \pi / 2) / \sinh \lambda (x - i n \pi / 2)$, the numbers $m_{1,2} = n/2 - Q_{1,2}$ and

$$S_0(\theta) = e^{2i\delta_0(\theta)} \prod_{a=1}^{N-1} \frac{\sinh \theta - i \sin(\pi a / \lambda)}{\sinh \theta + i \sin(\pi a / \lambda)} \quad \text{with}$$

$$\delta_0(\theta) = \int_0^\infty d\omega \frac{\sin(\omega \theta)}{\omega} \sinh[\pi \omega(\lambda - 1)/2] \cosh(\pi \omega / 2) \sinh(\pi \omega / 2 \lambda)$$

We claim that that creating a charge on one shore of the stripe one generates a band of soft collective modes on the other shore. This statement can be understood without detailed calculations which we leave for the extended publication, just using a qualitative analysis of the Bethe ansatz equations [3, 4, 4] and will be illustrated on figure [3]. As a matter of fact, similar effect may occur in all integrable systems where the $S$-matrix is a tensor product (here Principal Chiral Fields may serve as a good example - see, for example, [4]). In all these cases the chemical potential coupled to one of the conserved charges generates not only its own charge, but also induces a finite susceptibility with respect to the other chemical potential. Thus the second subsystem becomes gapless. We call this effect the induced Luttinger liquid.

Notice that Eqs. (9,10) for $u_a, v_b$ look very similar to BA equations for the spin-1/2 XXZ spin chain (see, for example, [3]) with the anisotropy related to the parameter $g$. In the presence of a large chemical potential $h_\pm \ll M$ applied on the left shore, the density of particles in the Fermi sea $n/L$ is finite and the real rapidities $\theta$ in Eqs. (8,9) with energy $E_0$ are densely distributed over some finite interval $[-B, B]$. The excitation energies $\varepsilon_{-n}$ of the $u$-particles are pushed up by the chemical potential. The excitation energies $\varepsilon_n$ of the $v$-particles have their minima at $|v| \to \infty$ and therefore at $T \ll h_-$ (the Fermi energy of the $\theta$-particles) they are located very far away in the momentum space from the Fermi momenta of the $\theta$-particles. As all energies $\varepsilon_{-n}$ are gapped, the rapidities $\theta$ drop out of consideration and in the equations for $v$ one can neglect a feedback between $v$’s and $\theta$’s and replace $\theta$’s by their ground state values. Furthermore, at large values of $|v| \ll B$ which are important at low energies in these equations one can neglect $\theta$’s in comparison with $v$’s. Then Eqs. (4,5) become exactly like the Bethe ansatz equations for the spin-1/2 XXZ spin chain with the total number of sites $n$ equal to the number of $\theta$’s in the ground state:

$$H = J \sum_j [S^{x}_j S^{x}_{j+1} + S^{y}_j S^{y}_{j+1} + \cos(\pi g) S^z_j S^z_{j+1}]$$

where we set $0 < g < 1$ using the fact that the region $g < 0$ is mapped onto $g > 0$ by the transformation $1 - g \to -g$. Since all these arguments work only for small energies where the spin chain is equivalent to the Luttinger liquid, one can say that the $v$-subsystem describes a spinless Luttinger liquid with the Luttinger parameter

$$K = [2(1 - g)]^{-1}$$

We emphasise that this value of $K$ is independent of number of chains or the number of particles in the Fermi sea.
The corresponding velocity \( V_s \) however, does depend on \( N \) and the number of particles. One can show that \( V_s \) vanishes at \( g \to 0, 1 \) indicating that the collective band described here is generated by the interactions.

Now we shall briefly outline the technical details leading to the above conclusions. Following the standard procedure we can derive the Thermodynamic Bethe Ansatz (TBA) equations corresponding to Eqs. (8,9,10). To make the problem simpler we use the fact that the above Bethe ansatz equations are invariant under the transformation

\[
\begin{align*}
-g &= 1 - g', N = N' - 1 \\
u_a \lambda &= \tilde{u}_a \lambda + i\pi/2, \quad u_a \lambda &= \tilde{v}_a \lambda + i\pi/2
\end{align*}
\]

Due to the periodicity of the phase factors in Eqs. (8,9,10) one can make a replacement

\[
\frac{\sinh \lambda(u + i\pi/2)}{\sinh \lambda(u - i\pi/2)} \to \frac{\sinh(\lambda u - i\pi n g/2)}{\sinh(\lambda u + i\pi n g/2)}
\]

These two facts allow us to restrict our consideration by the region \( 1 > |g| > 1/2 \). To further simplify our presentation we shall give explicit expressions only for the case \( g = 1/\nu > 0 \) with \( \nu \geq 2 \) being an integer number where classification of solutions is simpler. The free energy is then given by

\[
F/L = -\frac{1}{2\pi} \sum_{j=0}^{N-1} M_j \int d\theta \cosh \theta \ln[1 + e^{-E_j(\theta)/T}]
\]

where \( M_0 = M \) and \( M_j \) are given by Eq.(6).

The TBA equations for the bound states are

\[
T \ln[1 + e^{E_j(\theta)/T}] - G_{jk} \star T \ln[1 + e^{E_k(\theta)/T}] = M_j \cosh \theta + G_{j0} \star T \ln[1 + e^{-E_0(\theta)/T}], \quad (j, k = 1, ..., N - 1)
\]

where the star denotes convolution

\[
f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')
\]

In this publication we do not need explicit expressions for the kernels \( G_{jk}, G_{0j} \); the important features of Eqs.(13) being that \( G_{j0} > 0 \) and the energies \( E_j \) are directly coupled only to \( E_0 \). The latter property is related to the fact that bound states are neutral with respect to both U(1) groups.

When the chemical potentials \( h_- \) and \( h_+ \) are applied respectively to the left and right edges, the full system of TBA is given by

\[
T \ln[1 + e^{E_0(\theta)/T}] - K \star T \ln[1 + e^{-E_0(\theta)/T}]
\]

\[
= M \cosh \theta - (h_+ + h_-)/2 + G_{0j} \star T \ln[1 + e^{-E_j(\theta)/T}]
\]

\[
- T \sum_{n=1}^{\nu-1} a_n \star \ln[1 + e^{-t_n(\theta)/T}] - T a_{\nu-1} \star \ln[1 + e^{-e_{-n}(\theta)/T}]
\]

\[
- T s \star \ln[1 + e^{s'(\theta)/T}]
\]

\[
e_{\pm n} = h_\pm \nu/2 - s \pm T \ln[1 + e^{\nu(\nu-2)/T}]
\]

\[
e_{\pm n} = \delta_{n,\nu-1} h_\pm \nu/2 + s \pm T \ln[1 + e^{\nu(\nu-1)/T}]
\]

\[
+ \delta_{\nu-2,\nu-1} s \pm T \ln[1 + e^{-\nu+1}/T]
\]

\[
- \delta_{n,1} s \pm T \ln[1 + e^{-E_0}/T]
\]

where

\[
a_n(\omega) = \frac{\sinh[\pi(1 - n/\nu)\omega/2\lambda]}{\sinh[\pi\omega/2\lambda]}, \quad s(\omega) = [2 \cosh(\pi\omega/2\lambda\nu)]^{-1}
\]

\[
K(\omega) = \frac{\sinh[\pi\omega(1 - g)/2\lambda]\cosh[\pi\omega(\lambda + 2g)/2\lambda]}{2 \cosh(\pi\omega/2\lambda) \sinh(\pi\omega/2\lambda)}
\]

The general structure of the TBA can be illustrated by the incident diagram drawn on Fig. 3. From these equations it is manifest that at \( h_+ = 0 \) and \( h_- > M \) when \( E_0 \) has a negative part, all \( e_{-n} \) are positive except, possibly \( e_{-1} \) (see Fig. 3).
From Eq.\([16]\) we see that at \(h_ - > M\) the energy \(E_0\) is negative on some interval \([-B, B]\). From Eq.\([13]\) it is clear that \(E_j\)’s are pushed further up. As follows from Eq.\([13]\) at \(T = 0\)

\[
E_0 \approx K^{-1} \ast (M \cosh \theta_ - - \theta_ - /2) \quad (19)
\]

Substituting this into Eq.\([13]\) we find that the mass gaps for bound states are of order of

\[
\Delta_{ph} \sim M(h_ - /M)^\eta, \quad \eta = g/(N + g) \quad (20)
\]

One can also check that all \(\epsilon_ n\) except of \(\epsilon_ - 1\) still have a gap of order of \(h_ - \) and \(\epsilon_ - 1\) has a much smaller gap \(\sim \Delta_{ph}\). Thus at \(h_ - \gg M\) there are two temperature scales in the problem: \(h_ - \) itself which play a role of the Fermi energy; at temperatures \(T \ll h_ - \) one can omit \(\epsilon_ - n\)’s from Eq.\([16]\). The other scale is the much smaller one - it is the phonon frequency scale \(\Delta_{ph}\). It also follows from Eqs.\([18]\) that all \(\epsilon_ n\)’s with \(n > 1\) are positive and of order of \(T\) and \(\epsilon_ 1\) is purely negative in the limit \(T = 0\). Therefore the term with \(\epsilon_ 1\) in the right hand side of Eq.\([16]\) gives no contribution at \(T = 0\).

\[\text{FIG. 3. The incident diagram for the TBA equations given in the text. The circles correspond to energies and links to nonvanishing kernels. The circles on the vertical line correspond to the bound states.}\]

After this substitution, the TBA equations \([13, 18]\) coincide with the low temperature limit of the TBA equations for the anisotropic spin-1/2 Heisenberg magnet (see Eq.\([11]\) above).

In order to obtain the free energy at low \(T\) we isolate contributions which vanish at \(T = 0\). This is achieved using the standard TBA machinery; the free energy is represented as a sum of two parts corresponding to contributions from two gapless modes:

\[
F/L = f_1 + f_2 \quad \text{with}
\]

\[
f_1 = - T \int_{-\infty}^{\infty} [\sigma(\theta) + \sigma(-\theta)] \ln(1 + e^{-|E_0(\theta)|/T}) d\theta
\]

\[
\approx - \pi^2 T^2/3V_c \quad \text{and}
\]

\[
f_2 = - T \int_{-\theta}^{\theta} d\theta' K(\theta - \theta') \sigma(-\theta') M/2\pi \cosh \theta
\]

This describes independent gapless modes on the left edge \((f_1)\) as well as the induced mode on the right shore \((f_2)\). The function \(\sigma(\theta)\) satisfies

\[
\sigma(\theta) + \int_{-\theta}^{\theta} d\theta' K(\theta - \theta') \sigma(-\theta') = M/2\pi \cosh \theta
\]

where \(\sigma^{(+)}(\theta) = 0\) at \(|\theta| < B\) and \(\sigma^{(-)}(\theta) = 0\) at \(|\theta| > B\). The latter function is the ground state density of real \(\theta\)’s.

For very large chemical potential \(h_ - (B \to \infty)\), we find the velocities \(V_c = 1\) and the velocity of the inducted mode \(V_\sigma = g/(N - g)\). Therefore the inducted gapless band collapses at \(g \to 0\) as it should. The similar thing happens at \(g \to 1\).

In conclusion, charging of one edge of the stripe creates a band of gapless excitations on the other edge and stiffens the phonon modes inside. Their energies, however, remain much smaller than the Fermi energy which means that the system in question has an intermediate energy scales where one should expect nontrivial crossovers.

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