ON RIEMANN SUMS AND MAXIMAL FUNCTIONS IN $\mathbb{R}^n$

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Abstract. In this paper we investigate problems on almost everywhere convergence of subsequences of Riemann sums

$$R_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f \left( x + \frac{k}{n} \right), \quad x \in \mathbb{T}.$$ 

We establish a relevant connection between Riemann and ordinary maximal functions, which allows to use techniques and results of the theory of differentiations of integrals in $\mathbb{R}^n$ in mentioned problems. In particular, we prove that for a definite sequence of infinite dimension $n_k$ Riemann sums $R_{n_k} f(x)$ converge almost everywhere for any $f \in L^p$ with $p > 1$.

1. Introduction

We consider the Riemann sums operators

$$(1.1) \quad R_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f \left( x + \frac{k}{n} \right), \quad x \in \mathbb{T},$$

for the functions defined on the torus $\mathbb{T} = [0, 1] = \mathbb{R}/\mathbb{Z}$. It is not hard to observe that if $f$ is continuous then these sums converge to the integral of $f$ uniformly and they converge in $L^1(\mathbb{T})$ while $f$ is Lebesgue integrable. In this paper we investigate certain problems concerning the almost everywhere convergence of subsequences of Riemann operators. B. Jessen’s classical theorem in [9] is the first result in this concern.

Theorem A (Jessen). Let $\{n_k\}$ be an increasing sequence of positive integers such that $n_k$ divides $n_{k+1}$. Then

$$(1.2) \quad \lim_{k \to \infty} R_{n_k} f(x) = \int_0^1 f(t) dt, \quad a.e.$$ 

for any function $f \in L^1(\mathbb{T})$. Moreover

$$(1.3) \quad |\{x \in \mathbb{T} : \sup_k R_{n_k} |f(x)| > \lambda\}| \leq \frac{1}{\lambda} \|f\|_{L^1}, \quad \lambda > 0.$$ 

The next fundamental result in this direction due W. Rudin [22]. He has constructed an example of a bounded function with divergent Riemann sums. Moreover it was proved

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Theorem B (W. Rudin). Let $D$ be a sequence of positive integers which contains the sets $D_n$ ($n = 1, 2, \ldots$), each consisting of $n$ terms, such that no member of $D_n$ divides the least common multiple of the other members of $D_n$. Then for every $\varepsilon > 0$ there exists a bounded measurable function $f$, such that $0 \leq f \leq 1$, and such that

$$\limsup_{n \to \infty, n \in D} R_n f(x) \geq \frac{1}{2}$$

for all $x$, although $\int f < \varepsilon$.

For example, $D$ could be any sequence of primes. Using the Dirichlet’s theorem on primes in arithmetic progressions W. Rudin in [22] has constructed a sequence $\{n_k\}$ which satisfies the hypothesis of Jessen’s theorem such that $\{1 + n_k\}$ is a sequence of primes. Thus $R_{n_k} f(x)$ converges a.e., although $R_{1+n_k} f(x)$ need not do so. This observation shows that in a.e. convergence of operators $R_{n_k} f(x)$ arithmetic properties of $\{n_k\}$ are crucial.

Following L. Dubins and J. Pitman [6], we define a chain to be an increasing sequence of natural numbers $\{n_k\}$ for which $n_k$ divides $n_{k+1}$. For families of natural numbers $S_1, S_2, \ldots, S_d$ we denote by $[S_1, S_2, \ldots, S_d]$ the set of all naturals which are least common multiple of some numbers $n_1 \in S_1, n_2 \in S_2, \ldots, n_d \in S_d$. We will say a set $S$ has dimension $d$, if $d$ is the least possible integer such that $S$ is the subset of $[S_1, S_2, \ldots, S_d]$ for some chains $S_1, S_2, \ldots, S_d$. An example of a set of dimension $d$ is the set of integers having the factorization

$$(1.4) \quad p_1^{k_1} p_2^{k_2} \cdots p_d^{k_d}, \quad k_1, k_2, \ldots, k_d \in \mathbb{N},$$

for fixed different primes $p_1, p_2, \ldots, p_d$. L. Dubins and J. Pitman in [6] extended the Jessen’s theorem proving

Theorem C. If the set of positive integers has dimension $d$ and $f \in L \log^{d-1} L(T)$ then

$$\lim_{n \to \infty, n \in S} R_n f(x) = \int_0^1 f(x) dx \text{ a.e.}$$

and moreover

$$(1.5) \quad m\{x \in T : \sup_{n \in S} |R_n f(x)| > \lambda\} \leq \frac{C_d}{\lambda} \int_0^1 |f| \log^{d-1}(1 + |f|).$$

In the original proof of this theorem the martingale theory was used. There is a rather elementary and short proof of (1.5) given by an unknown referee of the article Y. Bugeaud and M. Weber [5]. More precisely, the maximal operator in (1.5) is estimated by $d$ iterations of the operator in (1.3). Then the inequality (1.5) is derived by using an interpolation theorem ([23], chap. 12, theorem 4.34). Another elementary proof of this theorem has also suggested by R. Nair in [15]. Y. Bugeaud and M. Weber in [5] proved that Theorem C is nearly sharp.

Theorem D. For any integer $d \geq 2$ and for any real number $\varepsilon > 0$ with $0 < \varepsilon < 1$, there exist a sequence $n_k$ of dimension $d$ and a function $f \in L \log^{d-\varepsilon} L(T)$ such that $R_{n_k} f(x)$ is divergent almost everywhere.

The proof of this theorem is based on the method of R. C. Baker [3], where author has proved a weaker version of this theorem. As it is mentioned in [5] Theorem D does not answer precisely whether the class $L \log^{d-1} L(T)$ in the theorem is optimal
or not. In Theorem 1 we prove that this class in fact is exact and divergence can be everywhere.

In present paper we establish a direct connection between Riemann maximal functions and ordinary maximal functions in Euclidian spaces $\mathbb{R}^d$. Moreover it turns out, that Riemann maximal function corresponding to a given finite set of indexes $D$ is equivalent to a maximal function in Euclidian spaces $\mathbb{R}^d$ with respect to certain $d$-dimensional rectangles which is the content of Theorem 4 in Section 3. Theorem 4 makes possible to use many results and methods of maximal functions in this theory. Many constructions used for Riemann sums get rather simple geometric interpretation in $\mathbb{R}^d$. As applications of Theorem 4 we obtain below solutions of some problems on Riemann sums. To figure out the key point of our observation in Section 3 we display an alternative proof to Jessen’s theorem using a covering property of some sets associated with Riemann sums. We will see a resemblance between this proof and the proof of Hardy-Littlewood maximal inequality where a covering lemma for intervals is used. In the last section we deduce Rudin’s theorem from Theorem 4 using a simple geometry of multidimensional rectangles. In the same section we prove that for a general class of operator sequences the strong sweeping out and $\delta$-sweeping out properties are equivalent.

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Denote by $L^\Phi(T)$ the class of functions $f$ on $T$ with $\Phi(|f|) \in L^1(T)$. If $\Phi$ satisfies $\Delta_2$-condition $\Phi(2x) \leq k\Phi(x)$ then $L^\Phi$ is Banach space with the norm $\|f\|_{L^\Phi} = \|f\|_{\Phi}$ to be the least $c > 0$ for which the inequality

$$\int_T \Phi \left( \frac{|f|}{c} \right) \leq 1$$

holds. The following theorem makes correction in the last theorem and shows that the class $L \log^{d-1} L$ in Theorem C is exact.

**Theorem 1.** Let $n_k$ be the increasing sequence formed the numbers (1.4) with fixed different primes $p_1, p_2, \ldots, p_d$. If an increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition

$$\lim_{x \to \infty} \frac{\phi(x)}{x \ln^{d-1} x} = 0,$$

then there exists a function $f(x) \in L^\phi$ such that the sequence $R_{n_k} f(x)$ is everywhere divergent.

According to the Theorem C Riemann sums corresponding to a set of finite dimension converge a.e. in $L^p$ classes with $p > 1$. As for the sets of infinite dimension it was a problem whether there exists a sequence of infinite dimension $\{n_k\}$ such that $R_{n_k} f(x)$ converges for any function $f \in L^p(T)$ with $p > 1$. In [5] Y. Bugeaud and M. Weber discussed a particular sequence of infinite dimension $E$ consist of all integers defined

$$E = \{p_1 \ldots p_j \partial p_j p_{j+1} \ldots p_k : \ k = 2, 3, \ldots, 1 \leq j \leq k\}$$

where $p_1 < p_2 < \ldots$ is the sequence of primes and the symbol $\partial$ means $p_i$ must be excluded in the product. As it is proved in [6] $E$ has infinite dimension. In [5] (see also [17] p. 18) it is proved the almost everywhere convergence of Riemann sums $R_{n_k} f(x)$ where $\{n_k\} = E$ for the functions $f \in L^2(T)$ with Fourier coefficients...
satisfying
\[ \sum_{n>3} a_n^2 \left( \frac{\ln n}{\ln \ln n} \right) < \infty. \]

It is proved also

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} R_{n_k} f(x) = \int_{0}^{1} f(x) dx \text{ a.e.} \tag{1.8} \]

for any \( f \in L^2(\mathbb{T}) \). We proved that Riemann sums associated to the set \( E \) converge a.e. in any \( L^p, p > 1 \). Moreover, a.e. convergence holds in the Orlicz class \( L^\Phi \) corresponding to the function

\[ \Phi(x) = \frac{x \ln(1+x)}{\ln \ln(3+x)}, \quad x \geq 0, \tag{1.9} \]

and this class is the optimal one for the set \( E \). So we prove the following theorems.

**Theorem 2.** Let \( E \) be the set defined in (1.7) and \( \Phi(x) \) is the function (1.9). Then for any \( f \in L^\Phi \) we have

\[ \lim_{n \to \infty, n \in E} R_n f(x) = \int_{0}^{1} f(x) \text{ a.e.}. \tag{1.10} \]

Moreover

\[ |\{ x \in \mathbb{T} : \sup_{n \in E} |R_n f(x)| > \lambda \}| \leq \int_{0}^{1} \Phi \left( \frac{|f|}{\lambda} \right), \quad \lambda > 0, \tag{1.11} \]

where \( c > 0 \) is an absolute constant.

This theorem immediately implies

**Corollary.** There exists an infinite set \( E \subset \mathbb{N} \) such that for any \( f \in L^p(\mathbb{T}) \) with \( p > 1 \) Riemann sums \( R_n f(x), n \in E \) converge a.e.

**Theorem 3.** If the sequence \( n_1 < n_2 < \ldots \) consists of all the integers of the set \( E \) and the increasing function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfies the condition

\[ \lim_{x \to \infty} \frac{\phi(x) \ln \ln x}{x \ln x} = 0, \tag{1.12} \]

then there exists a function \( f(x) \in L^\phi \) such that the sequence \( R_{n_k} f(x) \) is everywhere divergent.

2. Notations

We recall some definitions in measure theory (see [11]). Let \( X \) to be an arbitrary set. A family \( \Omega \) of subsets of \( X \) is called algebra if it is closed with respect to the operations of union, intersection and difference and \( X \in \Omega \). If the algebra is closed also with respect to countable union it is called \( \sigma \)-algebra. The set \( A \) is called atom for the algebra \( \Omega \) if there is no nonempty \( B \in \Omega \) so that \( B \subset A \). We note that if the algebra \( \Omega \) is finite then any set from \( \Omega \) is a union of some atoms of \( \Omega \). If there is also a measure \( \mu \) on \( \Omega \) we denote this measure space by \( (X, \Omega, \mu) \). It is said the measure spaces \( (X, \Omega, \mu) \) and \( (Y, \Delta, \nu) \) are isomorph if there exists a one to one mapping \( \gamma : \Omega \to \Delta \) called isomorphism such that

\[ \gamma(A - B) = \gamma(A) - \gamma(B), \quad \gamma \left( \bigcup_{k=1}^{\infty} A_k \right) = \bigcup_{k=1}^{\infty} \gamma(A_k), \]
and
\[ \nu(\gamma(A)) = \mu(A), \]
for any sets \( A, B \) and \( A_k, k = 1, 2, \ldots, \) from \( \Omega \). If \( \Omega \) is not \( \sigma \)-algebra we suppose in addition \( \bigcup_{k=1}^{\infty} A_k \in \Omega \). We will say \( f : X \rightarrow Y \) is an isomorphism function if the set function \( \gamma(A) = \{ y \in Y : y = f(x), x \in A \} \) determines one to one mapping between \( \Omega \) and \( \Delta \) which is an isomorphism. Suppose the algebras \( \Omega \) and \( \Delta \) are finite and have atoms \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) respectively. It is clear if \( \nu(A_i) = \mu(B_i) \), \( i = 1, 2, \ldots, n \), then the measure spaces \( (\Omega, \mu) \) and \( (\Delta, \nu) \) are isomorph.

We consider the probability space \( (\mathbb{T}^\infty, \lambda) = \prod_{i=1}^{\infty} (\mathbb{T}, \lambda_i) \) where each \( (\mathbb{T}, \lambda_i) \) is the Lebesgue probability space on \( \mathbb{T} \). Remind that measurable sets in \( \mathbb{T}^\infty \) is generated from all the products \( A = \prod_{i=1}^{\infty} A_i \) where each \( A_i \) is Lebesgue measurable set in \( \mathbb{T} \) and only finite number of them differ from \( \mathbb{T} \) (see definition in \[16\], p. 116).

The measure in \( \mathbb{T}^\infty \) is the extension of the measure \( \lambda(A) = \prod_{i=1}^{\infty} |A_i| \). We will use \(|A|\) to indicate measure of \( A \subset \mathbb{T}^\infty \).

Let \( l \in \mathbb{N} \) and \( D \subset \mathbb{N} \) is finite. We will write \( D/l \) if any member of \( D \) divides \( l \). We denote
\[ (2.1) \quad l/D = \{ n \in \mathbb{N} : \frac{l}{n} \in D \}. \]

An important subject in this paper is the relationship between three type of sets. Namely we will consider Riemann sets, integer arithmetic progressions and special rectangles in \( \mathbb{T}^\infty \) having the following descriptions.

\textbf{Riemann sets:} We denote by \( I_l \) the algebra in \( \mathbb{T} \) generated by intervals \([\frac{j}{l}, \frac{j+1}{l})\), \( j = 0, 1, \ldots, l-1 \). Define Riemann sets
\[ (2.2) \quad I_l(n,t) = \bigcup_{i=0}^{n-1} \left[ \frac{t}{l} + \frac{i}{n}, \frac{t}{l} + \frac{i}{n} + \frac{1}{l} \right], \quad t = 0, 1, \ldots, l. \]
where \( n \) divides \( l \). Certainly we have
\[ I_l(n,t) \in I_l, \quad \lambda(I_l(n,t)) = \frac{n}{l}. \]
For fixed \( l \) and \( n \) dividing \( l \) the collection \( \{I_l(n,t)\} \) is a pairwise disjoint partition of \([0,1]\). It is easy to verify if \( x \in [k/l, (k+1)/l) \) then
\[ (2.3) \quad l \int_{k/l}^{(k+1)/l} R_n f(t) dt = \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} f(t) dt, \quad x \in I_l(n,t). \]
Thus, using Lebesgue’s theorem on \( \mathbb{R} \), we get
\[ (2.4) \quad \lim_{l \rightarrow \infty, x \in I_l(n,t)} \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} f(t) dt = R_n f(x) \text{ a.e. , } \quad n = 1, 2, \ldots. \]
For any subset \( D \subset \mathbb{N} \) we define
\[ (2.5) \quad \mathcal{R}^l_D f(x) = \sup_{n \in D : n|l, x \in I_l(n,t)} \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} |f(t)| dt. \]
If \( D_n \) are finite subsets of \( D \) with \( \cup_n D_n = D \) then from \( \{2.4\} \) it follows that
\[ (2.6) \quad \mathcal{R}_D f(x) = \sup_{n \in D} R_n f(x) = \lim_{n \rightarrow \infty} \lim_{l \rightarrow \infty, D_n|l} \mathcal{R}_D^l f(x) \text{ a.e. } . \]
Observe that if \( f(x) \) is \( I_l \)-measurable then
\[ (2.7) \quad \mathcal{R}_D f(x) = \mathcal{R}_D^l f(x). \]
Indeed, since from $l||l'$ follows $\mathcal{A}_l \subset \mathcal{A}_{l'}$, we derive

$$ R_{l'} D f(x) = R_{l} D f(x) \text{ if } l||l', $$

and therefore by (2.6) we have

$$ R_{l} D f(x) = \lim_{l' \to \infty, l||l'} R_{l'} D f(x) = R_{l} D f(x). $$

**Arithmetic progressions:** We shall say a set of integers $A$ is $l$-periodic if $A = l + l'$. We denote by $\mathcal{A}_l$ the family of all $l$-periodic sets of integers and $\mathcal{A} = \bigcup_{l \in \mathbb{N}} \mathcal{A}_l$. It is clear if $l||l'$ then any $l$-periodic set is $l'$-periodic, i.e. $\mathcal{A}_l \subset \mathcal{A}_{l'}$. Observe that $\mathcal{A}$ and each $\mathcal{A}_l$ are algebras. We define the measure of a set $A \in \mathcal{A}$ by

$$ \delta(A) = \lim_{l \to \infty} \frac{\#(A \cap [0, l])}{l}, $$

where $\#B$ denotes the cardinality of the finite set $B$. It is clear that the limit exists and if $A \in \mathcal{A}_l$ then

$$ \delta(A) = \frac{\#(A \cap [0, l])}{l}. $$

Observe that $\delta$ is an additive measure on $\mathcal{A}$. Now consider the arithmetic progressions

(2.8) $$ A_l(t) = \{lj + t, j \in \mathbb{Z}\}, \quad 0 \leq t < l. $$

It is clear

(2.9) $$ A_l(t) \in \mathcal{A}_l, \quad \delta(A_l(t)) = \frac{1}{l}, $$

and any set from $\mathcal{A}_l$ can be written as a finite union of these arithmetic progressions. It means the sets in (2.8) are the atoms of the algebra $\mathcal{A}_l$.

**Rectangles in $\mathbb{T}^\infty$:** We denote $p_1 < p_2 \ldots < p_d < \ldots$ the sequence of all primes. Consider an integer $l$ with factorization

(2.10) $$ l = p_1^{j_1} p_2^{j_2} \ldots p_d^{j_d}. $$

We do not exclude that some of the numbers $l_k$ are zero. Define rectangles in $\mathbb{T}^\infty$ by

(2.11) $$ B_l(j_1, \ldots, j_d) = \left\{ x \in \mathbb{T}^\infty : \frac{j_k}{p_k} \leq x_k < \frac{j_k + 1}{p_k}, k = 1, 2, \ldots, d \right\}, $$

where

$$ 0 \leq j_k < p_k^{j_k}, \quad k = 1, 2, \ldots, d, \quad x = (x_1, x_2, \ldots, x_d). $$

We denote by $B_l$ the algebra generated of all the finite unions of the rectangles (2.11). So the family

$$ \mathcal{B} = \bigcup_{l \in \mathbb{N}} \mathcal{B}_l, $$

is an algebra in $\mathbb{T}^\infty$. We note that $B_l \subset B_{l'}$ while $l||l'$. We shall consider the measure space $(\mathcal{B}, \lambda)$, where $\lambda$ is the Lebesgue’s measure on $\mathbb{T}^\infty$. It is clear

$$ \lambda(B_l(j_1, \ldots, j_d)) = \frac{1}{l}. $$

It is clear that $(\Omega, \lambda)$, $(\mathcal{A}_l, \delta)$ and $(\mathcal{B}_l, \lambda)$ are isomorph, because all have $l$ atoms with with equal measures. In Section 4 we are going to construct a special isomorphism between $\mathcal{A}$ and $\mathcal{B}$ assigning the arithmetic progressions (2.8) to the rectangles (2.11).
3. An alternative proof of Jessen’s theorem

Operators (2.6) play a significant role in the study of a.e. convergence of Riemann sums. To prove Jessen’s theorem it is enough to prove the inequality (1.3), because (1.2) follows from (1.3) by using Banach principle. So we suppose $D = \{m_1, m_2, \ldots, m_d\}$ where $m_k$ divides $m_{k+1}$. We fix a finite subset $U = \{m_1, m_2, \ldots, m_d\} \subset D$ and an integer $l$ divided by $m_d$ and so all $m_k, 1 \leq k \leq d$. It is clear

$$\{R^1_{\mu} f(x) > \lambda\} = \bigcup_j I_j,$$

where $I_j$ are Riemann sets form $\mathcal{I}_l$ with

$$\frac{1}{|I_j|} \int_{I_j} |f(t)| dt > \lambda.$$

We will prove that it may be chosen a subfamily of mutually disjoint sets $\{\tilde{I}_j\}$ such that

$$\bigcup_j I_j = \bigcup_j \tilde{I}_j. \tag{3.1}$$

We define priority $(I)_n$ if $I$ has the form (2.2). It is easy to observe that if priority $(I)$ divides priority $(J)$ and $I \cap J \neq \emptyset$ then we have $I \subseteq J$. We take $\tilde{I}_L$ to be some of $I_j$ with highest priority. Suppose we have chosen $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_m$. We consider all $I_j$’s with $I_j \cap \bigcup_{j=1}^{m} \tilde{I}_j$ and so $I_j \cap \left(\bigcup_{j=1}^{m} \tilde{I}_j\right) = \emptyset$. We take $I_{m+1}$ among these sets having an highest priority. Certainly this process generates a subcollection $\{\tilde{I}_j\}$ of mutually disjoint sets with (3.1). Thus we obtain

$$|\{R^1_B f(x) > \lambda\}| = |\bigcup_j I_j| = \sum_j |\tilde{I}_j| < \frac{1}{\lambda} \sum_j \int_{\tilde{I}_j} |f(t)| dt \leq \frac{\|f\|_{L^1}}{\lambda}.$$

Since the inequality is true for any finite $U \subset D$, applying (2.6) we get (1.3).

4. An isomorphism between arithmetic progressions and rectangles

Let $l$ be an integer with factorization (2.10) and

$$m = p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d}, \quad 0 \leq m_k \leq l_k, \quad k = 1, 2, \ldots, d. \tag{4.1}$$

From the definition (2.11) it follows that

$$B_m(t_1, \ldots, t_d) = \bigcup_{p_k^{l_k-m_k} t_k \leq s_k < p_k^{l_k-m_k} (t_k+1)} B_l(s_1, \ldots, s_d). \tag{4.2}$$

For a fixed integer $t$ we consider the set of integer vectors

$$S_t = \{(s_1, s_2, \ldots, s_d) : 0 \leq s_k < p_k^{l_k}, \quad s_k = t \mod p_k^{m_k}, \quad k = 1, 2, \ldots, d, \}$$

In fact $S_t$ depends also on $l$ and $m$.

**Lemma 1.** There exists a one to one correspondence from

$$U = \{0, 1, \ldots, \frac{l}{m} - 1\}$$

to the set of vectors (4.3), such that the vector $(s_1, s_2, \ldots, s_d) \in S_t$ assigned to $u \in U$ satisfies

$$s_k = (mu + t) \mod p_k^{l_k}, \quad k = 1, 2, \ldots, d. \tag{4.5}$$
Proof. We note that there are \( p_k^{l_k-m_k} \) number of \( s_k \)'s satisfying
\[
0 \leq s_k < p_k^{l_k}, \quad s_k = t \mod p_k^{m_k}.
\]
So we have
\[
\# S_t = \prod_{k=1}^{d} p_k^{l_k-m_k} = \frac{l}{m} = \# U.
\]
Thus, it is enough to prove that for any \( u \in U \) there exists a vector \((s_1, \ldots, s_d) \in S \) with \(4.3\), and the images of different \( u \)'s are different. To determine the vector \((s_1, \ldots, s_d) \) corresponding to \( u \) we define \( s_k \) to be the remainder when \( mu + t \) is divided by \( p_k^{l_k} \). Certainly \((s_1, \ldots, s_d) \) satisfies (4.3) and \(0 \leq s_k < p_k^{l_k} \). Since \( p_k^{m_k} | m \) we get \( s_k = t \mod p_k^{m_k} \). So \((s_1, \ldots, s_d) \in S_t \). Now we suppose \((s_1, s_2, \ldots, s_d) \in S_t \) is assigned to different \( u_1, u_2 \in U \). The numbers \( u_1 \) and \( u_2 \) satisfy the relation (4.5). Hence, we get
\[
m(u_1 - u_2) = 0 \mod p_k^{l_k}, \quad k = 1, 2, \ldots, d.
\]
Since \(0 \leq u_1 - u_2 < \frac{1}{m} \), using (2.10) and (4.1), we conclude \( u_1 - u_2 = 0 \). \( \square \)

Let \( p \geq 2 \) be an integer. Any nonnegative integer \( a \) has \( p \)-adic decomposition
\[
a = a_0 p^k + a_1 p^{k-1} + \cdots + a_k, \quad 0 \leq a_i < p.
\]
We denote
\[
(a)_p = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_0,
\]
the integer with reverse arrangement of \( p \)-digits of \( a \). We shall say that \( (a)_p \) is the \( p \)-reverse of \( a \). We note this action defines a one to one mapping of the set of integers \( \{0, 1, \ldots, p^k - 1\} \) into itself. Notice if
\[
s = p^i v + t, \quad 0 \leq v < p^{i-i}, \quad 0 \leq t < p^i, \ i \leq j,
\]
then
\[
\bar{s} = p^{j-i} \bar{t} + \bar{v}.
\]
It is easy to observe that for a fixed \( t \) the correspondence \( s \to \bar{s} \) is a one to one mapping between the sets
\[
\{s: s = p^i v + t, 0 \leq v < p^{i-i}\} \text{ and } \{s: s = p^{j-i} \bar{t} + \bar{v}, 0 \leq \bar{v} < p^{j-i}\}.
\]

**Lemma 2.** There exists an isomorphism \( \alpha \) from the measure space \((A, \delta)\) to \((B, \lambda)\) assigning any progression \((2.8)\) to a rectangle \((2.11)\).

**Proof.** At first we define \( \alpha \) on the progressions \((2.8)\). We take an arbitrary \( A_l(t) \). Suppose
\[
t_k = t \mod p_k^{l_k}, \quad 0 \leq t_k < p_k^{l_k}, \quad k = 1, 2, \ldots, d,
\]
and denote by \( \bar{t}_k \) the \( p_k \)-reverse of the integer \( t_k \). We have \( 0 \leq \bar{t}_k < p_k^{l_k} \). We define
\[
\alpha(A_l(t)) = B_l(\bar{t}_1, \cdots, \bar{t}_d).
\]
According the definition \((2.8)\) for a given arithmetic progression \( A_m(t) \) we have
\[
A_m(t) = \bigcup_{u=0}^{l/m-1} A_l(mu + t).
\]
We shall prove that
\[ \alpha(A_m(t)) = \bigcup_{u=0}^{t/m-1} \alpha(A_l(mu + t)). \]

According to Lemma 2 there exists a one to one mapping between the sets \( U \) and \( S_l \) defined in (4.13) and (4.14). In addition, if \((s_1, s_2, \cdots, s_d) \in S_l\) is assigned to a given \( u \in U \) then it satisfies the condition (4.5) and therefore by (4.8) we have
\[ \alpha(A_l(mu + t)) = B_l(s_1, \ldots, s_d). \]

Now let \( t_k \) be the remainder when \( t \) is divided by \( p_k^{m_k} \), i.e.
\[ t_k = t \mod p_k^{m_k}, \quad 0 \leq t_k < p_k^{m_k}. \]

From (4.8) we get
\[ \alpha(A_m(t)) = B_m(\bar{t}_1, \cdots, \bar{t}_d). \]

From (4.3) it follows that \( p_k^{m_k} \) divides \( s_k - t \) and therefore by (4.12) it divides also \( s_k - t_k \). So we have
\[ s_k = p_k^{m_k}v_k + t_k, \quad 0 \leq v_k < p_k^{l_k-m_k}, \quad 0 \leq t_k < p_k^{m_k}. \]

Thus, according to (4.6), for the \( p_k \)-reverses \( \bar{s}_k \) of the integer \( s_k \) we have
\[ \bar{s}_k = p_k^{l_k-m_k}\bar{t}_k + \bar{v}_k, \quad 0 \leq \bar{v}_k < p_k^{l_k-m_k}, \quad 0 \leq \bar{t}_k < p_k^{m_k}, \]

where \( \bar{v}_k \) and \( \bar{t}_k \) are the \( p_k \)-reverses of \( v_k \) and \( t_k \) respectively. Hence for any \( u \in U \) may be determined \((\bar{s}_1, \cdots, \bar{s}_d)\) with
\[ \alpha(A_m(t)) = B_m(\bar{t}_1, \cdots, \bar{t}_d) \]
\[ = \bigcup_{\bar{s}_k = p_k^{l_k-m_k}\bar{t}_k \leq \bar{s}_k < p_k^{l_k-m_k}(\bar{t}_k + 1)} \bigcup_{u=0}^{t/m-1} \alpha(A_l(mu + t)). \]

So (4.10) is true. Now take an arbitrary set \( A \in \mathcal{A} \). We have \( A \in \mathcal{A}_l \) for some \( l \in \mathbb{N} \). Since (2.8) are the atoms of \( \mathcal{A}_l \), the set \( A \) is a union of some mutually disjoint atoms, i.e.
\[ A = \bigcup_{i \in I} A_l(i). \]

We define
\[ \alpha(A) = \bigcup_{i \in I} \alpha(A_l(i)). \]

Since \( A \) belongs to different algebras \( \mathcal{A}_l \), there are different representations (4.15) corresponding to different \( l \)'s. However, using (4.10), it is easy to verify that the right side of (4.16) does not depend on the representation (4.15). On the other hand \( \alpha \) is measure preserving, because \( \delta(A_l(i)) = \lambda(\alpha(A_l(i))) = 1/l \) by (4.3). So we conclude that \( \alpha \) is an isomorphism from \( \mathcal{A} \) to \( \mathcal{B} \). In addition, according to (4.8)
it assigns any progression (2.8) to a rectangle (2.11). The proof of Lemma 2 is complete.

For any \(l\)-periodic set of integers \(A \in \mathcal{A}_l\) we define

\[
\beta_l(A) = \bigcup_{k \in A} \left[ \frac{k}{l}, \frac{k + 1}{l} \right].
\]

It is easy to check that \(\beta_l\) determines an isomorphism from the probability space \((\mathcal{A}_l, \lambda)\) to \((\mathcal{I}_l, \delta)\). Moreover

\[
\beta((A_l/n(t)) = I_l(n, t).
\]

Thus, the composition of \(\alpha \circ \beta_l^{-1}\) where \(\alpha\) is from Lemma 2 is an isomorphism from \((\mathcal{I}_l, \lambda)\) to \((\mathcal{B}_l, \lambda)\). Moreover the following lemma is true.

**Lemma 3.** For any \(l \in \mathbb{N}\) there exists a one to one mapping \(\gamma_l : \mathbb{T} \to \mathbb{T}^\infty\) such that

1. \(\gamma_l\) is measure preserving, i.e. \(|\gamma_l(A)| = |A|\) for any Lebesgue measurable \(A \subset \mathbb{T}\),
2. \(\gamma_l\) is an isomorphism function between \((\mathbb{T}, \mathcal{I}_l, \lambda)\) and \((\mathbb{T}^\infty, \mathcal{B}_l, \lambda)\),
3. for any \(I_l(n, t)\) from (2.2) the set \(\gamma_l(I_l(n, t))\) is a rectangle of the form \(B_m(i_1, \ldots, j_d)\) with \(m = \frac{1}{l}\).

**Remark.** The existence of a mapping with the conditions (1) and (2) is trivial. The important part of the lemma is the fact that \(\gamma_l(I_l(n, t))\) is a certain rectangle in \(\mathbb{T}^\infty\).

For any set of integers \(D \subset \mathbb{N}\) we define the maximal function

\[
\mathcal{M}_D g(x) = \sup_{x \in B_m(i_1, \ldots, j_d)} \frac{1}{|B_m(i_1, \ldots, j_d)|} \int_{B_m(i_1, \ldots, j_d)} |g(t)| dt
\]

where \(g \in L^1(\mathbb{T}^\infty)\). We note that if \(l\) is a multiple for the numbers from \(D\) then the rectangles in (4.18) are in \(\mathcal{B}_l\). This implies that for the conditional expectation \(E_{\mathcal{B}_l} g(x)\) of \(g(x)\) with respect to the algebra \(\mathcal{B}_l\) we have

\[
\mathcal{M}_D g(x) = \mathcal{M}_D E_{\mathcal{B}_l} g(x).
\]

The following theorem clearly follows from Lemma 3. It creates an equivalency between Riemann maximal function \(R_D^l f(x)\) defined in (2.3) and \(\mathcal{M}_l / D g(x)\), where \(l/D\) is defined in (4.1). It clearly follows from Lemma 3

**Theorem 4.** For any \(l \in \mathbb{N}\) there exists a measure preserving mapping \(\eta_l : \mathbb{T} \to \mathbb{T}^\infty\) such that if \(f(x) \in L^1(\mathbb{T})\) and \(g(x) = f(\tau_l^{-1}(x))\) then

\[
|\{x \in \mathbb{T} : R_D^l f(x) > \lambda\}| = |\{x \in \mathbb{T}^\infty : \mathcal{M}_l / D g(x) > \lambda\}|, \quad \lambda > 0.
\]

**Corollary.** Let \(D\) be a set of indexes and \(\Phi : \mathbb{R}^+ \to \mathbb{R}^+\) be an increasing convex function. Then

\[
\sup_{\|f\|_\infty \leq 1} \left\{ x \in \mathbb{T} : R_D f(x) > \lambda \right\} = \sup_{B \subset D, \|g\|_\infty \leq 1} \left\{ x \in \mathbb{T}^\infty : \mathcal{M}_l / B g(x) > \lambda \right\},
\]

for any \(\lambda > 0\), where in sup finite sets \(B\) are considered.
From (2.7) we have:

\[(4.22) \quad \{x \in \mathbb{T} : \mathcal{R}_B^l f(x) > \lambda \} = \{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \},\]

and \(\|f\|_\Phi = \|g_l\|_\Phi\) since \(\tau_l\) is measure preserving. Taking into account (2.6) we obtain:

\[\left\{ x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda \right\} \leq \sup_{B \subset D, t \in \mathbb{N}} \left\{ x \in \mathbb{T} : \mathcal{R}_B^l f(x) > \lambda \right\} = \sup_{B \subset D, t \in \mathbb{N}} \left\{ x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \right\}.\]

Since \(f \in L^\Phi\) is arbitrary and \(\|f\|_\Phi = \|g_l\|_\Phi\) we get:

\[\sup_{\|f\|_\Phi \leq 1} \left\{ x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda \right\} \leq \sup_{B \subset D, t \in \mathbb{N}, \|g_l\|_\Phi \leq 1} \left\{ x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \right\}.\]

Now suppose \(g \in L^\Phi(\mathbb{T}^\infty), B \subset D\) is finite and \(l \geq 2\) is arbitrary integer. According to (4.11) there exists \(A_l\)-measurable function \(g_l\) such that:

\[(4.23) \quad \mathcal{M}_{l/B} g_l(x) = \mathcal{M}_{l/B} g_l(\tau_l(x)).\]

According to Theorem (4.10) for \(f_l(x) = g_l(\tau_l(x))\) we have:

\[(4.24) \quad \{x \in \mathbb{T} : \mathcal{R}_B^l f_l(x) > \lambda \} = \{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \}.\]

From (2.7) we have:

\[\mathcal{R}_B f_l(x) = \mathcal{R}_B^l f_l(x).\]

So, using also (4.23), (4.24) and relation \(B \subset D\), we get:

\[\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \} = \{x \in \mathbb{T} : \mathcal{R}_B^l f_l(x) > \lambda \} = \{x \in \mathbb{T} : \mathcal{R}_D f_l(x) > \lambda \}\]

and therefore:

\[\sup_{B \subset D, t \in \mathbb{N}, \|g_l\|_\Phi \leq 1} \left\{ x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda \right\} \leq \sup_{\|f_l\|_\Phi \leq 1} \left\{ x \in \mathbb{T} : \mathcal{R}_D f_l(x) > \lambda \right\} .\]

\[\square\]

5. A Covering Lemma

The covering lemma we establish in this section is needed to prove Theorem (4.2).

We consider the function:

\[(5.1) \quad \alpha(x) = \begin{cases} x^{x-1}, & \text{if } x > 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}\]

This is an increasing continuous function from \(\mathbb{R}^+\) to \(\mathbb{R}^+\). It is easy to observe its inverse satisfies the condition:

\[(5.2) \quad \lim_{x \to \infty} \frac{\alpha^{-1}(x) \ln x}{\ln \ln x} = 1.\]

Define the functions:

\[(5.3) \quad \Psi(x) = \int_0^{x} \alpha(t) dt, \quad \Phi(x) = \int_0^{x} \alpha^{-1}(t) dt, \quad x \in \mathbb{R}.\]
These are complementary $N$-functions (see definition in [13], p. 14). Performing simple estimations we get

$$\frac{x \ln(x/2)}{2 \ln \ln(x/2)} < \Phi(x) < \frac{x \ln x}{\ln \ln x}, \quad x > \gamma,$$

where $\gamma$ is an absolute constant. According to the Young’s inequality ([13], (2.6)) we have

$$uv \leq \Phi(u) + \Psi(v), \quad u > 0, v > 0.$$

Everywhere below we will use notation $a \lesssim b$ for the inequality $a \leq c \cdot b$ with an absolute constant $c > 0$. The following lemma is a variant of the lemma 3 from [12].

**Lemma 4.** If $A_1, A_2, \ldots, A_n$ and $A$ are independent sets in some probability space and $\sum_{k=1}^{n} |A_k| \leq 1/2$ then

$$\int_E \Psi \left( \frac{1}{3} \cdot \left( 1 + \sum_{k=1}^{n} I_{A_k}(x) \right) \right) \lesssim |E|,$$

where

$$E = A \cap \left( \bigcup_{k=1}^{n} A_k \right).$$

**Proof.** To prove (5.6) it is enough to get

$$m(\lambda) = \left| \left\{ x \in A : 1 + \sum_{k=1}^{n} I_{A_k}(x) > \lambda \right\} \right| \lesssim |E| \cdot \left( \frac{2}{\lambda - 1} \right)^{\frac{\lambda + 1}{2}} \quad \lambda > 3.$$

Indeed, using the relation $\Psi'(x) = \alpha(x)$, $x > 0$, as a consequence of (5.5), combined with (5.1), we obtain

$$\int_E \Psi \left( \frac{1}{3} \cdot \left( 1 + \sum_{k=1}^{n} I_{A_k}(x) \right) \right) dx = \frac{1}{3} \int_0^{\infty} \Psi' \left( \frac{\lambda}{3} \right) m(\lambda) d\lambda$$

$$= \frac{1}{3} \int_0^{\infty} \alpha \left( \frac{\lambda}{3} \right) m(\lambda) d\lambda \lesssim |A| \int_0^{\infty} \alpha \left( \frac{\lambda}{3} \right) \left( \frac{2}{\lambda} \right)^{\frac{\lambda + 1}{2}} \lesssim |A|.$$  

Putting $\delta_k = |A_k|$, we have $\sum_{k=1}^{n} \delta_k < 1/2$. Then using the independence, we get

$$|E| = |A \cap A_1| + |A \cap (A_2 \setminus A_1)| + \ldots + |A \cap (A_n \setminus \bigcup_{k=1}^{n-1} A_k)|$$

$$= \delta_1 |A| + \delta_2 |A|(1 - |A_1|) + \ldots + \delta_n |A|(1 - |\bigcup_{k=1}^{n-1} A_k|)$$

$$\geq \delta_1 |A| + \frac{\delta_2}{2} |A| + \ldots + \frac{\delta_n}{2} |A| \geq \frac{1}{2} |A| (\delta_1 + \delta_2 + \ldots + \delta_n).$$
We assume \( \lambda > 3 \). Hence

\[
m(\lambda) = \sum_{k=1}^{n} \sum_{i_1 \prec \cdots \prec i_k} |A \cap A_{i_1} \cap \cdots \cap A_{i_k} \cap \left( \bigcap_{j \notin \{i_1, \ldots, i_k\}} (A_j)^c \right)|
\]

\[
= \sum_{k=1}^{n} \sum_{i_1 \prec \cdots \prec i_k} |A| \cdot |A_{i_1}| \cdot \cdots \cdot |A_{i_k}| \prod_{j \notin \{i_1, \ldots, i_k\}} (1 - |A_j|)
\]

\[
= |A| \sum_{k=1}^{n} \sum_{i_1 \prec \cdots \prec i_k} \delta_{i_1} \cdots \delta_{i_k} \prod_{j \notin \{i_1, \ldots, i_k\}} (1 - \delta_j)
\]

\[
\leq |A| \sum_{k=1}^{n} \sum_{i_1 \prec \cdots \prec i_k} \delta_{i_1} \cdots \delta_{i_k} \leq |A| \sum_{k=1}^{\infty} \left( \delta_1 + \cdots + \delta_n \right)^k / k!
\]

\[
< |A| (\delta_1 + \cdots + \delta_n) \sum_{k=1}^{\infty} \frac{1}{k!} \leq 2 |E| \sum_{k=1}^{\infty} \frac{1}{(\lambda - 1) k!} \leq |E| \left( \frac{2^\lambda - 1}{(\lambda - 1)} \right) .
\]

The proof is complete. \( \square \)

For a set of indexes \( S \subseteq \mathbb{N} \) we denote by \( \mathcal{R}(S) \) the algebra generated by the rectangles \( (2.11) \) with \( l_i = 0, i \notin S \). For any set \( R \subseteq \mathcal{R} \) we define its spectrum \( \text{sp}(R) \) to be the smallest set of indexes \( S \) for which \( R \subseteq \mathcal{R}(S) \). That is

\[
\text{sp}(R) = \bigcap_{S : R \subseteq \mathcal{R}(S)} S .
\]

It is easy to observe

\[
\text{(5.9)} \quad \text{if } \text{sp}(B_1), \ldots, \text{sp}(B_k) \text{ are mutually disjoint, then } B_1, \ldots, B_k \text{ are independent,}
\]

\[
\text{(5.10)} \quad \text{if } \text{sp}(R) \subseteq \text{sp}(Q) \text{ and } Q \not\subseteq R \text{ then } R \cap Q = \emptyset .
\]

We denote

\[
\text{(5.11)} \quad l = l_d = p_1 p_2 \cdots p_d ,
\]

\[
\text{(5.12)} \quad E_d = \{ m \in \mathbb{N} : m = p_\nu p_{\mu+1} \cdots p_d, 1 \leq \nu < \mu \} .
\]

Let \( \mathcal{F}_d \) be the family of all rectangles from \( \mathcal{B}_l \) defined

\[
\text{(5.13)} \quad \mathcal{F}_d = \{ B_m(j_1, \ldots, j_d) : m \in E_d \} .
\]

According to \( \text{(5.13)} \) any \( B \in \mathcal{F}_d \) has the form

\[
\text{(5.14)} \quad B = \left\{ x \in \mathbb{T}^\infty : \frac{j_k}{p_k} \leq x_k < \frac{j_k + 1}{p_k}, k \in \{ \nu \} \cup \{ \mu, \mu + 1, \ldots, d \} \right\} ,
\]

where \( 0 \leq j_k < p_k, k = 1, 2, \ldots \). In the case \( \mu = d + 1 \) we understand \( \{ \mu, \mu + 1, \ldots, d \} = \emptyset \). As \( \mu \) and \( \nu \) in \( \text{(5.14)} \) are uniquely determined for a given \( B \in \mathcal{F}_d \)
sometimes we will use $\mu(B)$, $\nu(B)$ for them. We define the base $\text{bs}(B)$ and the tail $\text{tl}(B)$ of $B$ by

\[(5.15) \quad \text{bs}(B) = \left\{ x \in T^\infty : \frac{j_k}{p_k} \leq x_k < \frac{j_k + 1}{p_k}, k \in \{\mu, \mu + 1, \ldots, d\} \right\}, \]

and

\[(5.16) \quad \text{tl}(B) = \left\{ x \in T^\infty : \frac{j_\nu}{p_\nu} \leq x_\nu < \frac{j_\nu + 1}{p_\nu} \right\}. \]

Obviously for any $B \in F_d$ we have

\[(5.17) \quad B = \text{bs}(B) \cap \text{tl}(B). \]

Observe that if $A, B \in F_d$ then

\[(5.18) \quad \text{bs}(A) \cap \text{bs}(B) \neq \emptyset \quad \Rightarrow \quad \text{bs}(A) \subseteq \text{bs}(B) \text{ or } \text{bs}(B) \subseteq \text{bs}(A), \]

\[(5.19) \quad \text{bs}(A) \subset \text{bs}(B) \quad \Rightarrow \quad \mu(A) < \mu(B), \]

\[(5.20) \quad \text{bs}(A) \subset \text{bs}(B), \quad A \not\subseteq B \quad \Rightarrow \quad \text{tl}(A) \neq \text{tl}(B). \]

**Lemma 5.** Any collection of rectangles $\Theta = \{A_n\} \subset F_d$ contains a finite subcollection $\hat{\Theta} = \{\hat{A}_1, \ldots, \hat{A}_m\}$ with

\[(5.21) \quad \bigcup_{j=1}^m \hat{A}_j = \bigcup_{\alpha} A_\alpha, \]

\[(5.22) \quad \int_{T^\infty} \Psi \left( \frac{1}{3} \sum_{j=1}^m \mathbb{I}_{\hat{A}_j}(x) \right) dx \leq 1. \]

**Proof.** Since $F_d$ is finite and $\Theta \subset F_d$ we can assume $\Theta = \{A_1, A_2, \ldots, A_n\}$ and $\mu(A_i) \geq \mu(A_{i+1})$ for any $i$. The subcollection $\hat{\Theta}$ will be chosen from $\{A_1, A_2, \ldots, A_n\}$ as follows. We choose $\hat{A}_1 = A_1$. If the sets $\hat{A}_1 = A_{l_1}, \ldots, \hat{A}_k = A_{l_k-1}$ with $l_1 < \ldots < l_k-1$ have been chosen then we select $\hat{A}_k$ to be the first set among $A_{l_{k-1}+1}, \ldots, A_n$ satisfying the conditions

\[(5.23) \quad \hat{A}_k \not\subseteq \hat{A}_1 \cup \ldots \cup \hat{A}_{k-1}, \]

\[(5.24) \quad \bigg| \bigcup_{j \leq k, \text{tl}(\hat{A}_j) \cap \text{bs}(\hat{A}_k) \neq \emptyset, \text{bs}(\hat{A}_j) \supseteq \text{bs}(\hat{A}_k)} \text{tl}(\hat{A}_j) \bigg| < \frac{3}{4}. \]

This process generates a sequence $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_m$. According to (5.24), for any fixed $k$ we have

\[(5.25) \quad \bigg| \bigcup_{1 \leq j \leq m, \text{tl}(\hat{A}_j) \cap \text{bs}(\hat{A}_k) \neq \emptyset, \text{bs}(\hat{A}_j) \supseteq \text{bs}(\hat{A}_k)} \text{tl}(\hat{A}_j) \bigg| < \frac{3}{4}. \]

We consider a base $U = \text{bs}(\hat{A}_k)$ satisfying the inequality

\[(5.26) \quad \bigg| \bigcup_{\text{tl}(\hat{A}_j) \cap U \neq \emptyset, \text{bs}(\hat{A}_j) \supseteq U} \text{tl}(\hat{A}_j) \bigg| \geq \frac{1}{4}. \]

It is easy to observe that from

\[\text{tl}(\hat{A}_j) \cap U \neq \emptyset, \quad \text{bs}(\hat{A}_j) \supseteq U, \]
it follows that \( \nu(\tilde{A}_j) < \mu(\tilde{A}_k) \). Therefore the sets

\[
U, \bigcup_{l(\tilde{A}_j) \geq U} \text{tl}(\tilde{A}_j)
\]

have disjoint spectrums and so they are independent according to (5.9). Thus, using (5.26) we conclude

\[
(5.27) \quad \left| U \cap \left( \bigcup_{j=1}^m \tilde{A}_j \right) \right| \geq \left| U \cap \left( \bigcup_{l(\tilde{A}_j) \geq U} \text{tl}(\tilde{A}_j) \right) \right| = \left| U \right| \cdot \left| \bigcup_{l(\tilde{A}_j) \geq U} \text{tl}(\tilde{A}_j) \right| \geq \frac{1}{4} \left| U \right|.
\]

We denote by \( U_1, U_2, \ldots, U_\gamma \) the family of all different bases \( U = \text{bs}(\tilde{A}_k) \) satisfying (5.26). It is clear they are mutually disjoint. Thus, using (5.27) we get

\[
(5.28) \quad \left| \bigcup_{i=1}^\gamma U_i \right| \leq 4 \left| \bigcup_{j=1}^m \tilde{A}_j \right|.
\]

Now suppose \( A_t \) is an arbitrary set which is not in the subcollection \( \{ \tilde{A}_k \} \). We have \( l_{k-1} < t < l_k \) for some \( k \). According to the process of the selection we have either

\[
(5.29) \quad A_t \subset \bigcup_{i=1}^{k-1} \tilde{A}_i
\]

or

\[
\left| \text{tl}(\tilde{A}_t) \cup \left( \bigcup_{j<k, \text{tl}(\tilde{A}_j) \cap \text{bs}(\tilde{A}_t) \neq \emptyset, l(\tilde{A}_j) \geq U} \text{tl}(\tilde{A}_j) \right) \right| \geq \frac{3}{4}.
\]

Since \( \text{tl}(\tilde{A}_t) \leq \frac{1}{4} \) we obtain

\[
\left| \bigcup_{j<k, \text{tl}(\tilde{A}_j) \cap \text{bs}(\tilde{A}_t) \neq \emptyset, l(\tilde{A}_j) \geq U} \text{tl}(\tilde{A}_j) \right| \geq \frac{1}{4},
\]

which means \( \text{bs}(A_t) \subseteq U = \text{bs}(\tilde{A}_{k-1}) \) where \( U \) satisfies (5.26). Hence we have either (5.29) or

\[
A_t \subset \bigcup_{i=1}^\gamma U_i,
\]

and therefore, applying (5.28), we get

\[
\left| \bigcup_t A_t \right| \leq \left| \bigcup_{j=1}^m \tilde{A}_j \right| + \left| \bigcup_{i=1}^\gamma U_i \right| \leq 5 \left| \bigcup_{j=1}^m \tilde{A}_j \right|.
\]

which gives (5.21). To prove (5.22) denote

\[
(5.30) \quad B_k = \text{bs}(\tilde{A}_k) \setminus \left( \bigcup_{\text{bs}(\tilde{A}_i) \subset \text{bs}(\tilde{A}_k)} \text{bs}(\tilde{A}_i) \right), \quad k = 1, 2, \ldots, m.
\]
It is clear $B_1, B_2, \ldots, B_m$ are pairwise disjoint. We note some of this sets can be empty. Using (5.17) we have

$$\bigcup_{k=1}^m B_k = \bigcup_{k=1}^m \text{bs} (\hat{A}_k) \supset \bigcup_{k=1}^m \hat{A}_k.$$  

Thus, to obtain (5.22), it is enough to prove

$$I_k = \int_{B_k} \Psi \left( \frac{1}{3} \sum_{j=1}^m I_{\hat{A}_j} (x) \right) dx \lesssim |B_k|.$$  

Observe that

$$I_k = \int_{B_k} \Psi \left( \frac{1}{3} \sum_{bs (\hat{A}_j) \supset \text{bs} (\hat{A}_k)} I_{\hat{A}_j} (x) \right) dx.$$  

Indeed, according to (5.18), any $\hat{A}_j$ satisfies one of the relations

$$bs (\hat{A}_j) \cap bs (\hat{A}_k) = \emptyset,$$

$$bs (\hat{A}_j) \subset bs (\hat{A}_k),$$

$$bs (\hat{A}_j) \supset bs (\hat{A}_k).$$

In the case (5.34) or (5.35), using (5.30), we have $\hat{A}_j \cap B_k = \emptyset$. So the integral (5.32) depends only on the sets $\hat{A}_j$ with (5.36), which implies (5.33). If $bs (\hat{A}_j) \supset bs (\hat{A}_k)$, then by (5.30) $bs (\hat{A}_j) \supset B_k$. Thus, such that $\hat{A}_j = bs (\hat{A}_j) \cap tl (\hat{A}_j)$ (see (5.17)) from (5.33) we get

$$I_k = \int_{B_k} \Psi \left( \frac{1}{3} \sum_{bs (\hat{A}_j) \supset \text{bs} (\hat{A}_k)} I_{\hat{A}_j} (x) \right) dx.$$  

Now denote

$$C_\nu = \bigcup_{j: \nu (\hat{A}_j) = \nu, \text{tl} (\hat{A}_j) \cap \text{bs} (\hat{A}_k) \neq \emptyset, \text{bs} (\hat{A}_j) \supset \text{bs} (\hat{A}_k)} \text{tl} (\hat{A}_j),$$

and consider all nonempty sets $C_{\nu_1}, C_{\nu_2}, \ldots, C_{\nu_p}$, with decreasing numbering $\nu_1 > \nu_2 > \ldots > \nu_p$. From (5.26) it follows that

$$\left| \bigcup_{i=1}^p C_{\nu_i} \right| < \frac{3}{4}.$$  

Observe that if the sets $\hat{A}_j$ and $\hat{A}_i$ satisfy the relations

$$bs (\hat{A}_j) \supset bs (\hat{A}_i) \text{ and } \nu (\hat{A}_j) \geq \mu (\hat{A}_i)$$

then

$$\text{tl} (\hat{A}_j) \cap \text{bs} (\hat{A}_i) = \emptyset.$$  

Indeed, from (5.40) and the definition of the set $\mathcal{F}_d$ in (5.13) it follows that

$$\text{sp} (\hat{A}_j) \subseteq \{\nu (\hat{A}_j), \nu (\hat{A}_j) + 1, \ldots \} \subseteq \{\mu (\hat{A}_i), \mu (\hat{A}_i) + 1, \ldots \} = \text{sp} (\text{bs} (\hat{A}_i)).$$

Thus, using (5.10) we will have either $\hat{A}_j \supset \text{bs} (\hat{A}_i) \supset \hat{A}_i$ or $\hat{A}_j \cap \text{bs} (\hat{A}_i) = \emptyset$. The first inclusion is not possible because of (5.23). So we have $\hat{A}_j \cap \text{bs} (\hat{A}_i) = \emptyset$.  

Therefore, since \( \tilde{A}_j = \text{bs}(\tilde{A}_j) \cap \text{tl}(\tilde{A}_j) \) and \( \text{bs}(\tilde{A}_j) \supseteq \text{bs}(\tilde{A}_i) \) (see (5.40)) we get (5.41). Combining (5.41) with (5.38) we get
\[
C_{j_i} \cap \text{bs}(\tilde{A}_i) = \emptyset,
\]
provided
\[
\text{bs}(\tilde{A}_j) \supseteq \text{bs}(\tilde{A}_k) \supseteq \text{bs}(\tilde{A}_i), \quad \mu(\tilde{A}_i) \leq \nu_j, \quad 1 \leq i \leq j.
\]
Therefore by (5.30)
\[
B_k \cap (C_{j_i} \setminus \cup_{s=1}^{j_i-1} C_{j_s})
\]
\[
= \left( \text{bs}(\tilde{A}_k) \setminus \bigcup_{\text{bs}(\tilde{A}_i) \subseteq \text{bs}(\tilde{A}_k), \mu(\tilde{A}_i) > \nu_j} \text{bs}(\tilde{A}_i) \right) \cap (C_{j_i} \setminus \cup_{s=1}^{j_i-1} C_{j_s}).
\]
Since \( \text{sp}(C_{j_s}) = \nu_s, \nu = \nu_{p-1} < \ldots < \nu_1 \) and \( \text{sp} (\text{bs}(\tilde{A}_i)) = \{ \mu(\tilde{A}_i), \mu(\tilde{A}_i) + 1, \ldots, d \} \) (see (5.12)), each set on the right has spectrum in \( \{ \nu_j, \nu_j + 1, \ldots, d \} \). So we have
\[
\text{sp} \left( B_k \cap (C_{j_i} \setminus \cup_{s=1}^{j_i-1} C_{j_s}) \right) \subset \{ \nu_j, \nu_j + 1, \ldots, d \}.
\]
Hence the sets
\[
B_k \cap (C_{j_i} \setminus \cup_{s=1}^{j_i-1} C_{j_s}), C_{j_{i+1}}, \ldots, C_{j_p}
\]
have mutually disjoint spectrums, so they are independent by (5.9). According to (5.39) these sets satisfy the hypothesis of Lemma 6. Hence, applying (5.6), we get
\[
\int_{B_k \setminus (C_{j_i} \cup_{s=1}^{j_i-1} C_{j_s})} \Psi \left( \frac{1}{3} \left( 1 + \sum_{t=i+1}^{p} \mathbb{I}_{C_{j_t}}(x) \right) \right) dx \lesssim |B_k \cap (C_{j_i} \setminus \cup_{s=1}^{j_i-1} C_{j_s})|,
\]
and therefore
\[
I_k = \int_{B_k} \Psi \left( \frac{1}{3} \sum_{t=1}^{p} \mathbb{I}_{C_{j_t}}(x) \right) dx
\]
\[
= \sum_{i=1}^{p} \int_{B_k \cap (C_{j_i} \cup_{s=1}^{j_i-1} C_{j_s})} \Psi \left( \frac{1}{3} \left( 1 + \sum_{t=i+1}^{p} \mathbb{I}_{C_{j_t}}(x) \right) \right) dx
\]
\[
\lesssim \sum_{i=1}^{p} |B_k \cap (C_{j_i} \cup_{s=1}^{j_i-1} C_{j_s})| \leq |B_k|.
\]
where \( C_{j_0} = \emptyset \). Hence the inequality (5.32) and so the lemma is proved. \( \square \)

In the following lemma \( E \subset Z \) is the set defined in (1.7) and \( M_{l/E} f(x) \) is the maximal function from (4.18).

**Lemma 6.** If \( \Phi(t) \) is the function from (5.3) then
\[
(5.44) \quad |\{ x \in \mathbb{T}^\infty : M_{l/E} f(x) > \lambda \}| \lesssim \frac{1}{\lambda} \left( 1 + \int_{\mathbb{T}^\infty} \Phi(f(t)) \, dt \right), \quad \lambda > 0,
\]
for any \( f \in L^\Phi(\mathbb{T}^\infty) \) and \( l \in \mathbb{N} \).

**Proof.** We suppose \( l \) has the factorization (5.11). From (5.12) and (1.7) we get \( l/E = E_d \). So taking into account (5.13) we have
\[
M_{l/E} f(x) = \sup_{F \in F_d : F \ni x} \frac{1}{|F|} \int_{F} |f(t)| \, dt, \quad x \in \mathbb{T}^\infty, \quad f \in L^1(\mathbb{T}^\infty).
\]
Hence, for any $\lambda > 0$ there exists a collection $F = \{F_k\}$ from $\mathcal{F}_d$ such that
\[
\{x \in \mathbb{T}^\infty : M_{l/E}f(x) > \lambda\} = \bigcup_k F_k,
\]
\[
\frac{1}{|F_k|} \int_{F_k} f(t)dt > \lambda.
\]
According to Lemma 5 we can choose a subfamily $\{\tilde{F}_k\}$ such that
\[
\bigcup_k \tilde{F}_k = \bigcup_k F_k,
\]
\[
\int_{\mathbb{T}^\infty} \Psi\left(\sum_k I_{\tilde{F}_k}(x)\right) dx \leq 1.
\]
Thus, applying (5.45), (5.46) and (5.5) we obtain
\[
|\{x \in \mathbb{T}^\infty : M_{l/B}f(x) > \lambda\}| = \frac{1}{\lambda} \left(\int_{\mathbb{T}^\infty} \Phi(f(x))dx + \int_{\mathbb{T}^\infty} \Psi\left(\sum_k I_{\tilde{F}_k}(x)\right) dx\right) \leq \frac{1}{\lambda} \left(1 + \int_{\mathbb{T}^\infty} \Phi(f(t))dt\right).
\]

6. PROOFS OF THEOREMS

To avoid the repetition of the same standard argument in the proofs of the theorems we will use E. M. Stein’s well-known weak type maximal functions principle (see. [19] or [20] p.458, 3.6). Consider a sequence of convolution operators
\[
T_j = f * \mu_j : L^1(\mathbb{T}) \to \{\text{measurable functions on } \mathbb{R}\}
\]
where $\mu_j$ are positive finite measures on $\mathbb{T}$.

**Lemma 7** (E. M. Stein). Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function such that $\Phi(\sqrt{t})$ is concave. Then if for every $f \in \Phi(L)$
\[
Mf(x) = \sup_j |T_j f(x)| < \infty
\]
on a set of positive measure then
\[
|\{x \in \mathbb{R} : Mf(x) > \lambda\}| \leq \int_{\mathbb{R}} \Phi\left(\frac{c|f|}{\lambda}\right), \quad \lambda > 0,
\]
where $c > 0$ is a constant.

**Proof of Theorem 2.** We suppose $B \subset E$ is an arbitrary finite set. If $l$ is a multiple for the members of $B$ then $l/B \subset l/E$, and so by (4.13) we obtain
\[
M_{l/B}f(x) \leq M_{l/E}f(x).
\]
Hence, according to (5.42) we have
\[
|\{x \in \mathbb{T} : M_{l/B}f(x) > \lambda\}| < \frac{c}{\lambda} \left(1 + \int_{\mathbb{T}} \Phi(f(t))dt\right),
\]
for any finite \( B \subset E \) and \( f \in L^\Phi \). Combining this with the corollary after Theorem \ref{thm2} we obtain
\[
|\{x \in T : R_E f(x) > \lambda\}| \leq \frac{c}{\lambda} \left(1 + \int_{\mathbb{T}^\infty} \Phi(f(t))dt\right),
\]
where \( c > 0 \) is an absolute constant. We have each \( B_m f(x) \) is a convolution operator
with the kernel
\[
\mu_m = \frac{1}{m} \sum_{i=1}^{m} \delta_{i/m}
\]
where \( \delta_a \) is the unit measure (Dirac function) concentrated at \( a \). It is easy to check
as well \( \Phi \) satisfies the hypothesis of Stein’s lemma. Therefore applying the Stein’s
principle from (6.4) we get (1.11). The proof is thus complete. \( \square \)

Suppose \( f(x) \in L^1(T) \), \( D \) is a finite set of naturals and \( l \) is a common multiple for
the members of \( D \). Consider the conditional expectation \( E^D f(x) \) of the function
\( f(x) \) with respect the algebra \( I_l \) defined. For any convex function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) we have
\[
\|E^D f(x)\|_\phi \leq \|f\|_\phi.
\]
To deduce everywhere divergence in Theorem \ref{thm1} and Theorem \ref{thm3} we use the following
general lemma.

**Lemma 8.** Let \( D \) be a set of indexes and \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a convex increasing
function. If there exists a function \( f \in L^\phi \) such that \( R_D f(x) = \infty \) on a set
of positive measure, then it can be found a function \( \tilde{f} \in L^\phi \) with \( R_D \tilde{f}(x) = \infty \)
everywhere.

**Proof.** Suppose for some \( f \geq 0 \) we have
\[
R_D f(x) = \infty, \quad x \in E,
\]
and \( |E| > 0 \). According to Borel-Cantelli lemma (see. \cite{20}, p. 442 or \cite{23}, section
XIII, 1.24) there exists a sequence \( x_k \in \mathbb{T} \) such that \( \sum_k \mathbb{I}_{E+x_k}(x) = \infty \) a.e..
Denoting \( \hat{f}(x) = \sum_k 2^{-k} f(x + x_k) \), we get \( \hat{f} \in L^\phi \) and
\[
R_D \hat{f}(x) = \infty \text{ a.e.}.
\]
Hence by (2.6) there exist a sequence of finite sets \( D_1 \subset D_2 \subset \ldots \) with \( \cup_n D_n = D \)
and a integers \( l_n \) divided by the members of \( D_n \) such that
\[
|\{x \in T : R^D_{l_n} \hat{f}(x) > n^3\}| > 1 - \frac{1}{\phi(n^3)}.
\]
Since \( R^D_{l_n} \hat{f}(x) \) is \( I_{l_n} \)-measurable, so the set
\[
A_n = \{x \in T : R^D_{l_n} \hat{f}(x) > n^3\}
\]
is. Hence we get
\[
|A_n^c| \leq 1/\phi(n^3),
\]
\[
R^D_{l_n} \mathbb{I}_{A_n^c}(x) = 1, \quad x \in A_n^c.
\]
Thus, denoting
\[
f_n(x) = \hat{f}(x) + n^3 \cdot \mathbb{I}_{A_n^c}(x),
\]
we have
\[ \|f_n\|_{\phi} \leq \|f_n\|_{\phi} + \|n^3 \cdot I_n\|_{\phi} = \|f_n\|_{\phi} + 1 = \|f\|_{\phi} + 1, \]
\[ R_{D_n}^i f_n(x) > n^3, \text{ for all } x \in \mathbb{T}^\infty. \]
Now denote
\[ g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot E f_n(x). \]
According to (6.5) we have
\[ \|g\|_{\phi} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \|E f_n\|_{\phi} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \|f_n\|_{\phi} < \infty, \]
and using (2.7) we get
\[ R_D g(x) \geq R_{D_n}^i g(x) \geq \frac{1}{n^2} R_{D_n}^i E f_n(x) = \frac{1}{n^2} R_{D_n}^i f_n(x) > n, \ x \in \mathbb{T}^\infty, \]
for any \( n \in \mathbb{N} \), i.e. \( R_D g(x) = \infty \) everywhere on \( \mathbb{T} \). The proof is complete. \( \square \)

**Proof of Theorem 3.** We consider the rectangles
\[ B^k_i = \left\{ x \in \mathbb{T}^\infty : \frac{i}{p_k} \leq x_k < \frac{i+1}{p_k} \right\}, \quad i = 0, 1, \ldots, p_k - 1. \]
Since \( \text{sp} \ (B^k_i) = \{k\} \) we have \( B^k_i \in \mathcal{F}_{2d} \) if \( 1 \leq k \leq 2d \). Denote
\[ G_k = \bigcup_{0 \leq i < \left\lfloor \frac{p_k}{d} \right\rfloor} B^k_i, \quad k = 1, 2, \ldots, 2d, \]
(6.6)
\[ G = \bigcup_{k=d+1}^{2d} G_k, \quad C = \bigcap_{k=d+1}^{2d} G_k. \]
(6.7)
It is clear \( p_{d+1} > 2d \). Since the number of \( B^k_i \) in the union (6.6) is \( \left\lfloor \frac{p_k}{d} \right\rfloor \) and \( |B^k_i| = 1/p_k \) we conclude
\[ \frac{1}{d} \geq |G_k| = \left[ \frac{p_k}{d} \right] \frac{1}{p_k} > \frac{1}{d} \left( 1 - \frac{d}{p_k} \right) > \frac{1}{2d}, \text{ if } k > d. \]
(6.9)
Because of independence of the sets \( G_k \) we get
\[ |G| = \left| \bigcap_{k=d+1}^{2d} G_k \right| = 1 - \prod_{k=d+1}^{2d} (1 - |G_k|) > 1 - \left( 1 - (2d)^{-1} \right)^d > 1 - \frac{1}{\sqrt{e}} > \frac{1}{3}, \]
(6.10)
\[ |C| = \prod_{k=d+1}^{2d} |G_k| \leq d^{-d}. \]
(6.11)
Choose an arbitrary \( x \in G \). We have \( x \in G_k \) for some \( k \) and therefore \( x \in B^k_i \) for some \( 0 \leq i < \left\lfloor \frac{p_k}{d} \right\rfloor \) and \( d < k \leq 2d \). On the other hand, using (6.9) and the
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independence of the sets $G_j$, $d < j \leq 2d$, $j \neq k$, with $B^k$, we obtain

$$|C \cap B^k| = \left| \bigcap_{d < j \leq 2d, j \neq k} G_k \right| \cap B^k = \prod_{d < j \leq 2d, j \neq k} |G_k| > \frac{|B^k|}{(2d)^{d-1}}.$$ 

From this we get

$$\frac{1}{|B^k|} \int_{B^k} \mathbb{I}_C(x) dx > (2d)^{1-d}.$$ 

So we conclude

(6.12) $\mathcal{M}_{l_{2d}/E} \mathbb{I}_C(x) > (2d)^{1-d}$, $x \in G$,

where $l_{2d}$ is defined in (5.11). Taking into account (1.12) and (6.11), we have

$$\int_{T^\infty} \phi((2d)^{d-1} \mathbb{I}_C(x)) dx = \phi((2d)^{d-1}) |C| < d^{-d} \phi((2d)^{d-1}) \to 0.$$ 

Thus, we may find a sequence $c_d \to \infty$ such that the function

(6.13) $g_d(x) = c_d(2d)^{-1} \mathbb{I}_C(x).$

satisfies

$$\int_{T^\infty} \phi(g_d(x)) dx \leq 1.$$ 

From (6.13) we get

$$\mathcal{M}_{l_{2d}/E} g_d(x) = c_d(2d)^{-1} \mathcal{M}_{l_{2d}/E} \mathbb{I}_C(x)$$

and so, using (6.10) and (6.12), we obtain

$$|\{x \in T^\infty : \mathcal{M}_{l_{2d}/E} g_d(x) > c_d\}| = |\{x \in T^\infty : \mathcal{M}_{l_{2d}/E} \mathbb{I}_C(x) > (2d)^{1-d}\}| \geq |G| > \frac{1}{3}.$$ 

Applying (4.21) we may find sequence of functions $f_d$ on $T$ with

$$\|f_d\|_\Phi = \|g_d\|_\Phi \leq \int_{T^\infty} \phi(g_d(x)) dx \leq 1$$

such that

$$|\{x \in T : R_E f_d(x) > c_d\}| > \frac{1}{3}.$$ 

Hence, according to Stein’s principle there exists a function $f \in L^\Phi(T)$ such $R_E f(x) = \infty$ a.e.. To get everywhere divergence it remains to use Lemma 8. Theorem 3 is proved.

The proof of Theorem 4 is based on some results in the Theory of Differentiation of Integrals in $\mathbb{R}^n$. According to well known Jessen-Marcinkiewicz-Zygmund theorem (see [10] or [7] chapter 2)

(6.14) $\lim_{\text{diam } R \to 0, x \in R} \frac{1}{|R|} \int_R f(t) dt = f(x)$, a.e

for any $f \in L_{\log^{-1}}(\mathbb{R}^n)$, where $R$ are rectangles with sides parallel to the axis. On the other hand S. Saks in [15] has proved that in this theorem the Orlicz class $L_{\log^{-1}}$ is the optimal. Certainly the relation (6.14) is true also if we consider the rectangles (2.11) with fixed $d$ instead of all rectangles in $\mathbb{R}^n$. As for the divergence theorem the proof is not immediate. However there is a generalization of Saks
theorem due A. Stokolos [21] (see also [8]). According to this theorem if \( \phi \) satisfies (1.12) then there exists a function \( f \in L^\phi(\mathbb{R}^n) \) such that

\[
(6.15) \lim_{\text{diam} R \to 0, x \in R} \frac{1}{|R|} \int_R f(t) dt = \infty,
\]

for any \( x \in \mathbb{R}^n \), where \( R \) are the rectangles of the form (2.11) with fixed \( d \). Moreover, it can be taken any integers greater than or equal 2 instead of primes \( p_1, p_2, \ldots, p_d \).

We note that all this theorems can be stated also on \( T^\infty \).

**Proof of Theorem 1.** Suppose \( D \) is the set of all integers of the form

\[
p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d}, \quad m_k \in \mathbb{Z}^+, \ k = 1, 2, \ldots, d.
\]

Consider a sequence of subsets \( D_n \subset D \) defined

\[
D_n = \{ m = p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d} : 0 \leq m_k \leq n, k = 1, 2, \ldots, d \},
\]

and denote \( l_n = p_1^n p_2^n \cdots p_d^n \).

We have \( \cup_n D_n = D \) and \( l_n/D_n = D_n \). Therefore if the function \( f \in L^\phi(T^\infty) \) satisfies the condition (6.15) then

\[
\lim_{k \to \infty} M_{l_n/D_n} f(x) = \infty, \text{ a.e. on } T^\infty.
\]

Applying (4.21), we get \( R_D g_n(x) \to \infty \) a.e. for a sequence of functions \( g_n \) with \( \|g_n\|_\phi \leq 1 \). Using Stein’s principle, we will get a function \( g \) with \( R_D g(x) = \infty \) a.e., and the existence of a function with *everywhere* divergence Riemann sums follows from Lemma [8].

\[\Box\]

7. ON RUDIN’S THEOREM AND SWEPPING OUT PROPERTIES

In this section we establish equivalency between strong sweeping out and \( \delta \)-sweeping out properties of operator sequences, which seems to be interesting in view of the papers [1],[2]. Then we will deduce Rudin’s theorem in general settings from Theorem [4].

Let \((X, m)\) be a probability space. We consider linear operators

\[
(7.1) \quad T : L^1(X, m) \to \{ \text{measurable functions on } X \}.
\]

**Definition 1.** A sequence of linear operators \( T_n \) is said to be strong sweeping out if given \( \varepsilon > 0 \) there is a set \( E \) with \( mE < \varepsilon \) such that

\[
\limsup_{n \to \infty} T_n 1_{E}(x) = 1 \text{ a.e.}
\]

and

\[
\liminf_{n \to \infty} T_n 1_{E}(x) = 0 \text{ a.e.}.
\]

**Definition 2.** Let \( 0 < \delta \leq 1 \). A sequence of linear operators \( T_n \) is said to be \( \delta \)-sweeping out if given \( \varepsilon > 0 \) there is a set \( E \subset X \) with \( mE < \varepsilon \) such that

\[
\limsup_{n \to \infty} T_n 1_{E}(x) \geq \delta \text{ a.e.}.
\]

**Definition 3.** Let \( 0 < \delta \leq 1 \). A sequence of linear operators \( T_n \) is said to be weak \( \delta \)-sweeping out if given \( r > 0 \) there is a set \( E \) such that

\[
m\{x \in X : \sup_{n \in \mathbb{N}} T_n 1_{E}(x) \geq \delta \} > r \cdot mE.
\]

It turns out that these definitions are equivalent for the sequences of linear operators having the following settings

1. if \( f \geq 0 \) then \( Tf \geq 0 \),
2. \( T(1_X) = 1 \),
(3) for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset X$ and $m(E) < \delta$ then

\[(7.2)\quad m\{x \in X : T_{m(E)}(x) > \varepsilon\} < \varepsilon.\]

**Theorem 5.** If the sequence of linear operators $T_n$ satisfying (1)-(3) is $\delta$-sweeping out for any $0 < \delta < 1$ then it is strong sweeping out.

**Proof.** Assume $\{T_n\}$ satisfies the hypothesis of the theorem. Using a standard argument, one can easily choose a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ and measurable sets $E_k \subset X$ such that

\[(7.3)\quad mE_k < \varepsilon 2^{-k},\]

\[(7.4)\quad m\{x \in X : \sup_{n_{k-1} \leq m \leq n_k} T_m(E_k(x) > 1 - 2^{-k})\} > 1 - 2^{-k},\]

\[(7.5)\quad m\{x \in X : \sup_{n_{k-1} \leq m \leq n_k} T_m\left(\sum_{j=k+1}^{\infty} I_{E_j}(x)\right) > 2^{-k}\} < 2^{-k}.\]

The selection of $n_k$ and $E_k$ is realized in this order: $E_1, n_1, E_2, n_2, \ldots$. To avoid big expressions we use the notation $U_k = \sup_{n_{k-1} \leq m \leq n_k} T_m$. Denote

\[\tilde{E}_k = E_k \setminus \bigcup_{j=k+1}^{\infty} E_j,\]

\[E = \bigcup_{j=0}^{\infty} \tilde{E}_{j+1},\]

\[A_k = \left\{x \in X : U_k\left(\sum_{j=k+1}^{\infty} I_{E_j}(x)\right) \leq 2^{-k}\right\},\]

\[B_k = \left\{x \in X : U_k I_{E_k}(x) > 1 - 2^{-k}\right\},\]

\[G = (\liminf_{k \to \infty} A_k) \cap (\liminf_{k \to \infty} B_k).\]

From (7.6) and (7.4) we get $mG = 1$. Given an arbitrary $x \in G$ we have

\[x \in A_k \cap B_k, \quad k > k_0,\]

and consequently

\[(7.6)\quad U_k\left(\sum_{j=k+1}^{\infty} I_{E_j}(x)\right) \leq 2^{-k}, \quad U_k I_{E_k}(x) > 1 - 2^{-k}, \quad k > k_0.\]

Thus

\[U_{2k+1} I_E(x) \geq U_{2k+1} \tilde{E}_{2k+1}(x)\]

\[\geq U_{2k+1} I_{E_{2k+1}}(x) - U_{2k+1}\left(\sum_{j=2k+2}^{\infty} I_{E_j}(x)\right) > 1 - 2^{-(2k+1)} - 2^{-(2k+1)} = 1 - 2^{-2k}.\]

This implies

\[\limsup_{m \to \infty} T_{m(E)}(x) = 1, \quad x \in E.\]

It is easy to observe $E \cap E_{2k} = \emptyset$. So we have $E \subset \tilde{E}_{2k}$ and from (7.5) we derive

\[U_{2k} I_E(x) \leq U_{2k} \tilde{E}_{2k}(x) = 1 - U_{2k} I_{E_{2k}}(x)\]

\[\leq 1 - U_{2k} I_{E_{2k}}(x) + U_{2k}\left(\sum_{j=2k+1}^{\infty} I_{E_j}(x)\right) < 1 - (1 - 2^{-2k}) + 2^{-2k} = 2^{-2k+1}.\]
Hence
\[ \liminf_{m \to \infty} T_m I_E(x) = 0, \quad x \in E, \]
and the proof is complete. \(\square\)

Now suppose \((X, m)\) in \((\mathcal{T}, \lambda)\). In the next theorem we consider translation invariant operators \(T_n\) defined
\begin{equation}
T_n f(x) = T_n f(x + t), \tag{7.7}
\end{equation}
where \(f_x(t) = f(x + t)\).

**Theorem 6.** If the sequence of translation invariant operators \(\{T_n\}\) with (1)-(3) is weak \(\delta\)-sweeping out for any \(0 < \delta < 1\) then it is strong sweeping out.

**Proof.** According to the previous theorem it is enough to prove that \(\{T_n\}\) is \(\delta\)-sweeping out for any \(0 < \delta < 1\). By weak \(\delta\)-sweeping property we may choose measurable sets \(F_k\) such that
\[ |\{x \in X : \sup_{n \geq k} T_n I_{F_k}(x) \geq 1 - \frac{1}{2^k}\}| \to \infty. \]
Taking subsequences of \(F_k\) (with possible repetitions) allows us to find a sequence of sets \(E_k\), a sequences \(\delta_k \searrow 1\), and \(n_k \to \infty\) so that, taking
\[ A_k = \{x \in X : \sup_{n > n_k} T_n I_{E_k}(x) \geq \delta_k\}, \]
we have
\[ \sum_{k=1}^{\infty} |A_k| = \infty, \quad \sum_{k=1}^{\infty} |E_k| < \epsilon. \]
Applying Borel-Cantelli lemma, we can choose a sequence \(x_k\) so that
\[ |\limsup_{k \to \infty} (A_k + x_k)| = 1. \]
Since \(T_n\) are translation invariant operators, denoting
\[ E = \bigcup_{k=1}^{\infty} (E_k + x_k) \]
we get
\[ |\{x \in X : \limsup_{n \to \infty} T_n I_E(x) = 1\}| \geq |\limsup_{k \to \infty} \{x \in X : \sup_{n > n_k} T_n I_{E_k + x_k}(x) \geq \delta_k\}| = |\limsup_{k \to \infty} (A_k + x_k)| = 1, \]
and
\[ |E| \leq \sum_{j=1}^{\infty} |E_k + x_k| = \sum_{j=1}^{\infty} |E_k| < \epsilon. \]

Clearly Riemann sums operators satisfy the conditions (1)-(3). (1) and (2) are clear. Let us verify (3). If for \(E \subset \mathcal{T}\) we have \(|E| < \delta = \epsilon^2\) then
\[ \int_{\mathcal{T}} R_n I_E(x) dx = |E| < \epsilon^2 \]
and therefore, using Chebyshev’s inequality, we get

$$|\{x \in T : R_n I_E(x) > \varepsilon\}| < \varepsilon,$$

which proves (3). Analyzing Rudin’s proof one can easily understand it allows to get

$\delta$-sweeping out property for any $0 < \delta < 1$. Thus applying Theorem 5 we conclude

that if $\{n_k\}$ satisfies the hypothesis of Rudin’s theorem then $R_{n_k}$ is strong sweeping

out. We note that this assertion for Riemann sums was proved by M. Akcoglu, A. Bellow, R. Jones, V. Losert, K. Reinhold-Larsson and M. Wierdl in [1] by using

Rudin’s ideas. Now consider the operators

(7.8)$$\frac{1}{n} \sum_{j=1}^{n} f(jx).$$

J. M. Marstrand in [14], solving Kinchine’s conjecture, has proved this sequence

has 1-sweeping out property. Applying Theorem 5 we get the sequence (7.8) is

strong sweeping out. We note that alternate proofs of Rudin’s and Marstrand’s

theorems follows from Bourgain Entropy Theorem [4] a general tool for investigation

of divergence of certain operator sequences.

Proof of Rudin’s theorem based on Theorem 4. Fix a number $0 < \delta < 1$. According

to the conditions of Rudin’s theorem for any $k \in \mathbb{N}$ there exists a collection $D_k = \{n_1, n_2, \ldots, n_k\} \subset D$ such that no member of $D_k$ divides the least common multiple

of the others. It means we can choose primes $p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_k}$ such that $p_{\nu_j} | n_{\nu_j}$ and $p_{\nu_j} / n_{\nu_i}$ if $i \neq j$. Let $l$ be the least common multiple of the numbers $n_1, n_2, \ldots, n_k$.

Denoting $q_j = l/n_j$ we have

$$l/D_k = \{q_1, q_2, \ldots, q_k\}.$$ In addition

$$q_j = p_{\nu_1}^{m_1(j)} \cdot p_{\nu_2}^{m_2(j)} \cdots p_{\nu_k}^{m_k(j)} \cdot \gamma_j$$

where

$$m_j(j) = 0, \quad m_i(j) > 0, \text{ if } i \neq j.$$ Denote by $Q_j$ the collection of rectangles (2.11) corresponding to $l = q_j$ and suppose $Q = \cup_{j=1}^{k} Q_k$. According to (4.18) we have

$$\mathcal{M}_{l/D_k} f(x) = \sup_{B \in Q : x \in B} \frac{1}{|B|} \int_{B} |f(t)| dt.$$ On the other hand any rectangle of the form

$$\left\{ x \in T^\infty : \left[ \frac{t_i}{p_{\nu_i}}, \frac{t_i + 1}{p_{\nu_i}} \right), 1 \leq j \leq k, j \neq i \right\},$$

$$0 \leq t_i < p_{\nu_i}, \quad 1 \leq j \leq k, j \neq i,$$

can be represented as a disjoint union of rectangles from $Q_i$. Thus the same assertion is true also for the set

$$C_i = \{ x \in T^\infty : 0 \leq x_{\nu_j} < \frac{r_j}{p_{\nu(j)}}, 1 \leq j \leq k, j \neq i \}, \quad r_j = [\delta p_{\nu(j)}] + 1.$$ Denote

$$C = \bigcap_{j=1}^{k} C_j = \{ x \in T^\infty : 0 \leq x_{\nu_j} < \frac{r_j}{p_{\nu(j)}}, j = 1, 2, \ldots, k \}. $$
It is easy to observe if $B \in Q_j$ and $B \subset C_j$ then
\[ |B \cap C_j| = \frac{r_j}{p_{\nu(j)}} |B|. \]
Therefore, since $\frac{p_{\nu(j)}}{p_{\nu(k)}} > \delta$ we obtain
\[ M_{l/D_k} \mathbb{I}_C > \delta, \quad x \in \bigcup_{j=1}^{k} C_j. \]
On the other hand we have
\[ \left| \bigcup_{j=1}^{k} C_j \right| = |C| \left( 1 + \sum_{j=1}^{k} \frac{p_{\nu(j)}}{r_j} \right) > (k + 1)|C|. \]
Thus we get
\[ \left| \left\{ x \in T^\infty : M_{l/D_k} \mathbb{I}_C(x) > \delta \right\} \right| > (k + 1)|C|. \]
According Theorem 4 for some $G \subset T$ we get
\[ \left| \left\{ x \in T : R_{l/D_k} \mathbb{I}_G(x) > \delta \right\} \right| > (k + 1)|G|. \]
In addition, since $C$ is $\mathcal{B}_I$-measurable we have $G$ is $\mathcal{I}_I$-measurable. Thus from (2.7) we conclude
\[ \left| \left\{ x \in T : R_{D_k} \mathbb{I}_G(x) > \delta \right\} \right| \geq \left| \left\{ x \in T : R_{D_k} \mathbb{I}_C(x) > \delta \right\} \right| \geq (k + 1)|G|, \quad k = 1, 2, \ldots. \]
This implies the sequence $R_n f(x)$, $n \in D$, has weak $\delta$ sweeping out property for any $0 < \delta < 1$. Applying Theorem 6 we obtain it has strong sweeping out property. The proof is complete.

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