WEIGHTED HARDY TYPE INTEGRAL INEQUALITIES INVOLVING MANY FUNCTIONS

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Abstract. The object of this paper is to derive some new weighted Hardy type integral inequalities involving many functions and to obtain a classical weighted Hardy type inequality involving many functions.

1. Introduction

In the theory of inequalities one of the well known inequality, due to G. H. Hardy, called classical Hardy inequality, is the following [6, Theorem 330]:

If \( p > 1 \), \( f(x) \geq 0 \) for \( 0 < x < \infty \) and \( R(x) = \frac{1}{x} \int_{0}^{x} f(t) \, dt \), then

\[
\int_{0}^{\infty} R^p(x) \, dx < \left( \frac{p}{p-1} \right)^p \int_{0}^{\infty} f^p(x) \, dx,
\]

(1)

unless \( f \equiv 0 \). The inequality (1) is sharp one, that is, involved constant is the best possible one. It was Hardy who first generalized his own result (1) as:

For \( m \neq 1 \), \( p > 1 \) and \( f : (0,\infty) \to (0,\infty) \), integrable function, the following does hold:

\[
\int_{0}^{\infty} F^p(x) \, dx < \left( \frac{p}{|m-1|} \right)^p \int_{0}^{\infty} x^{p-m} f^p(x) \, dx,
\]

(2)

\[
F(x) = \begin{cases} \int_{0}^{x} f(t) \, dt & m > 1; \\ \int_{0}^{\infty} f(t) \, dt & m < 1, \end{cases}
\]

unless \( f \equiv 0 \). Inequality (2) is also sharp. The inequalities (1) and (2) have great importance in the theory and applications of integral inequalities, and in particular in the analysis of qualitative as well as quantitative properties of solutions of differential and integral equations. Due to this, over the years much efforts and time has been devoted to the improvement and generalization of Hardy’s inequalities (1) and (2). These includes, among others, the works in \( [1, 6, 7, 8, 10, 11, 12, 13, 14] \) and by Hussain, Pečarić in \( [2, 3, 4, 5] \).
The main aim of this paper is to establish new weighted Hardy type integral inequalities involving two or more functions and to derive some weighted classical Hardy type inequalities by means of fairly elementary analysis.

2. Main results

Lemma 1 ([9]). If $C_1, C_2, \ldots, C_n$ are reals and $C_{n+1} = C_1$, then

$$
\sum_{r=1}^{n-k+2} C_r C_{r+1} \cdots C_{r+k-1} \leq \sum_{r=1}^{n} C_r^k, \text{ where } n \geq k - 1. \tag{3}
$$

Theorem 1. For any $1 \leq i \leq n$, let $f_i(x)$ be a non-negative and integrable function on $(0, X)$, $X \in \mathbb{R}_+$ and $w, u_i, z_i : [0, x] \to \mathbb{R}_+$, absolutely continuous with $z_i'$ essentially bounded and positive (a.e). If $u_i$ is increasing and

$$
1 + \frac{u_i(x)w'(x)}{(1-2\alpha)w(x)u'_i(x)} \geq \frac{1}{\gamma_i} > 0 \ (a.e), \text{ for } \alpha > \frac{1}{2} \text{ and } 1 \leq i \leq n, \tag{4}
$$

$$
1 + \frac{u_i(x)w'(x)}{(1-2\alpha)w(x)u'_i(x)} \geq \frac{1}{\delta_i} > 0 \ (a.e), \text{ for } \alpha < \frac{1}{2} \text{ and } 1 \leq i \leq n. \tag{5}
$$

Then,

$$
\sum_{i=1}^{n} \int_{0}^{\infty} w(x) R_i(x) R_{i+1}(x) dx \leq \sum_{i=1}^{n} \left[ \frac{2\lambda_i}{2\alpha - 1} \right]^2 \int_{0}^{\infty} w(x) g_i(x) dx, \tag{6}
$$

where,

$$
g_i(x) \sim \frac{u_i^{4-2\alpha}(x)[z_i'(x)]^2f_i^2(x)}{z_i^2(x) u'_i(x)}; \quad \lambda_i = \max_{1 \leq i \leq n} (\gamma_i, \delta_i) \text{ and }
$$

$$
R_i(x) = \begin{cases} 
\frac{\sqrt{u_i(x)}}{u_i^2(x)} \int_{0}^{x} \frac{u_i(t)z_i'(t)}{z_i(t)} f_i(t) \ dt, & \alpha > \frac{1}{2}, \\
\frac{\sqrt{u_i(x)}}{u_i^2(x)} \int_{x}^{\infty} \frac{u_i(t)z_i'(t)}{z_i(t)} f_i(t) \ dt, & \alpha < \frac{1}{2}.
\end{cases}
$$

Proof. Let us define for $\alpha > \frac{1}{2}$, $0 \leq x \leq X$ and $0 < a < b < \infty$:

$$
R_{ia}(x) = \frac{\sqrt{u_i'(x)}}{u_i^2(x)} \int_{a}^{x} \frac{u_i(t)z_i'(t)}{z_i(t)} f_i(t) \ dt, \ 1 \leq i \leq n,
$$

with $R_{i0}(x) = R_i(x)$.

Using the inequality (3) with $k = 2$ for $C_i = R_{ia}(x)$:

$$
\sum_{i=1}^{n} R_{ia}(x) R_{i+1a}(x) \leq \sum_{i=1}^{n} R_{ia}^2(x).
$$
By Hölder’s inequality:

\[ \sum_{i=1}^{n} \int_{a}^{b} w(x) R_{ia}(x) R_{i+1a}(x) \, dx \leq \sum_{i=1}^{n} \int_{a}^{b} w(x) R_{ia}^{2}(x) \, dx. \]  

(7)

Now,

\[ I = \int_{a}^{b} w(x) R_{ia}^{2}(x) \, dx = \int_{a}^{b} w(x) \left[ \sqrt{u_{i}'(x)} \int_{a}^{x} \frac{u_{i}(t) z_{i}'(t)}{z_{i}(t)} f_{i}(t) \, dt \right]^{2} \, dx. \]

Integrating by parts we have

\[
I = \left[ \frac{u_{i}^{1-2\alpha}(x)}{1-2\alpha} \left( \sqrt{w(x)} \int_{a}^{x} \frac{u_{i}(t) z_{i}'(t)}{z_{i}(t)} f_{i}(t) \, dt \right) \right]_{a}^{b} \\
- \int_{a}^{b} \frac{u_{i}^{1-2\alpha}(x)}{1-2\alpha} \frac{d}{dx} \left( \sqrt{w(x)} \int_{a}^{x} \frac{u_{i}(t) z_{i}'(t)}{z_{i}(t)} f_{i}(t) \, dt \right) \, dx \\
= \frac{w(b) u_{i}^{1-2\alpha}(b)}{1-2\alpha} \left[ \int_{a}^{b} \frac{u_{i}(t) z_{i}'(t)}{z_{i}(t)} f_{i}(t) \, dt \right]^{2} - \frac{1}{1-2\alpha} \int_{a}^{b} \frac{w(x) R_{ia}^{2}(x)}{u_{i}'(x)} \, dx \\
+ \frac{2}{1-2\alpha} \int_{a}^{b} \frac{u_{i}^{2-\alpha}(x) w(x) z_{i}'(x) f_{i}(x) R_{ia}(x)}{z_{i}(x) \sqrt{u_{i}'(x)}} \, dx,
\]

i.e.

\[
\int_{a}^{b} w(x) R_{ia}^{2}(x) \left[ 1 + \frac{u_{i}(x) w'(x)}{(1-2\alpha) w(x) u_{i}'(x)} \right] \, dx \leq \frac{2}{2\alpha - 1} \int_{a}^{b} \frac{u_{i}^{2-\alpha}(x) w(x) z_{i}'(x) f_{i}(x) R_{ia}(x)}{z_{i}(x) \sqrt{u_{i}'(x)}} \, dx
\]

(8)

From (4) and (8) we have

\[
\int_{a}^{b} w(x) R_{ia}^{2}(x) \, dx \leq \frac{2\gamma_{i}}{2\alpha - 1} \int_{a}^{b} \frac{u_{i}^{2-\alpha}(x) w(x) z_{i}'(x) f_{i}(x) R_{ia}(x)}{z_{i}(x) \sqrt{u_{i}'(x)}} \, dx
\]

By Hölder’s inequality:

\[
\int_{a}^{b} w(x) R_{ia}^{2}(x) \, dx \leq \frac{2\gamma_{i}}{2\alpha - 1} \sqrt{\int_{a}^{b} \left[ \sqrt{w(x)} R_{ia}(x) \right]^{2} \, dx} \times \sqrt{\int_{a}^{b} \left[ \frac{u_{i}^{2-\alpha}(x) \sqrt{w(x)} z_{i}'(x) f_{i}(x)}{z_{i}(x) \sqrt{u_{i}'(x)}} \right]^{2} \, dx},
\]
i.e.,
\[
\int_{a}^{b} w(x) R_{i,a}^{2}(x) dx \leq 4 \left[ \frac{\gamma_i}{2\alpha - 1} \right]^{2} \int_{0}^{b} \frac{u_{i}^{4-2\alpha}(x) w(x) [z'_{i}(x)]^{2} f^{2}_{i}(x)}{z_{i}^{2}(x) u_{i}'(x)} dx \\
\leq 4 \left[ \frac{\gamma_i}{2\alpha - 1} \right]^{2} \int_{0}^{\infty} \frac{u_{i}^{4-2\alpha}(x) w(x) [z'_{i}(x)]^{2} f^{2}_{i}(x)}{z_{i}^{2}(x) u_{i}'(x)} dx.
\]  
(9)

By letting \( b \to \infty \) and from inequalities (7) and (9) we have
\[
\sum_{i=1}^{n} \int_{0}^{\infty} w(x) R_{i}(x) R_{i+1}(x) dx \leq \sum_{i=1}^{n} \left[ \frac{2\lambda_{i}}{2\alpha - 1} \right]^{2} \int_{0}^{\infty} w(x) g_{i}(x) dx.
\]  
(10)

Let us define for \( \alpha < \frac{1}{2} \) and \( 0 < a < b < \infty \),
\[
R_{i,b}(x) = \sqrt{\frac{u_{i}'(x)}{u_{i}^{2}(x)}} \int_{x}^{b} \frac{u_{i}(t) z'_{i}(t)}{z_{i}(t)} f_{i}(t) dt, \quad 1 \leq i \leq n,
\]
with \( R_{i,\infty}(x) = R_{i}(x) \).

Following the same steps as in the proof of inequality (10) we obtain
\[
\sum_{i=1}^{n} \int_{0}^{\infty} w(x) R_{i}(x) R_{i+1}(x) dx \leq \sum_{i=1}^{n} \left[ \frac{2\lambda_{i}}{1-2\alpha} \right]^{2} \int_{0}^{\infty} w(x) g_{i}(x) dx.
\]  
(11)

Inequalities (10) and (11) are equivalent to (6).

**Corollary 1.** For any \( 1 \leq i \leq n \), let \( f_{i}(x) \) be a non-negative and integrable function on \( 0, X \), \( X \in \mathbb{R}, \lambda_{i} = \max_{1 \leq i \leq n}(\gamma_{i}, \delta_{i}) \) and \( w : [0, X] \to \mathbb{R}^{+}, \) absolutely continuous. Let
\[
1 - \frac{x}{w'(x)} w(x) \geq \frac{1}{\gamma_{i}} > 0, \quad \text{for } \alpha > \frac{1}{2} \text{ and } 1 \leq i \leq n,
\]
\[
1 + \frac{x}{w'(x)} w(x) \geq \frac{1}{\delta_{i}} > 0, \quad \text{for } \alpha < \frac{1}{2} \text{ and } 1 \leq i \leq n.
\]

Then,
\[
\sum_{i=1}^{n} \int_{0}^{\infty} w(x) \left[ \frac{1}{x} \int_{0}^{x} f_{i}(t) dt \right]^{2} dx \leq 4 \sum_{i=1}^{n} \lambda_{i}^{2} \int_{0}^{\infty} w(x) f_{i}^{2}(x) dx.
\]

**Proof.** Follows from Theorem 1 by setting \( z_{i}(t) = u_{i}(t) = t; \alpha = 1 \) and \( f_{i}(t) = f_{i+1}(t) \) for \( 1 \leq i \leq n \).

**Theorem 2.** Let the conditions of Theorem 1 be satisfied and \( p_{i} > 1, k \in \mathbb{N} \) are such that \( q_{i} = \frac{p_{i}}{kp_{i} - 1} \) for \( 1 \leq i \leq n \). Let
\[
1 + \frac{u_{i}(x) w'(x)}{(1 - \alpha k p_{i}) w(x) u_{i}'(x)} \geq \frac{1}{\gamma_{i}} > 0 (a.e.), \quad \text{for } \alpha > \frac{1}{k p_{i}} \text{ and } 1 \leq i \leq n,
\]  
(12)
Then,
\[
\sum_{i=1}^{n-k+2} \int_0^\infty w(x) \left[ \prod_{j=i}^{i+k-1} R_j^{p_i}(x) \right] dx \leq \sum_{i=1}^n \left[ \frac{k p_i \lambda_i}{|a k p_i - 1|} \right]^{k p_i} \int_0^\infty w(x) g_i(x) dx,
\]
where,
\[
g_i(x) \to \frac{u_i^{2-a}(x) z_i'(x) f_i(x)}{z_i(x) k p_i u_i'(x)}; \quad \lambda_i = \max (\gamma_i, \delta_i), \quad n \geq k - 1 \text{ and }
\]
\[
R_i(x) = \begin{cases} \\
\frac{k p_i u_i'(x)}{u_i^n(x)} \int_0^x u_i(t) z_i'(t) \frac{f_i(t)}{z_i(t)} dt, \quad \alpha > \frac{1}{k p_i}, \\
\frac{k p_i u_i'(x)}{u_i^n(x)} \int_x^\infty u_i(t) z_i'(t) \frac{f_i(t)}{z_i(t)} dt, \quad \alpha < \frac{1}{k p_i}.
\end{cases}
\]

\textbf{Proof.} Let us define for \( \alpha > \frac{1}{k p_i}, 1 \leq i \leq n \) \( 0 \leq x \leq X \) and \( 0 < a < b < \infty \):
\[
R_{ia}(x) = \frac{k p_i}{u_i^n(x)} \int_a^x u_i(t) \frac{z_i'(t)}{z_i(t)} f_i(t) dt, \quad 1 \leq i \leq n.
\]
By using the inequality (3) for \( C_i = R_{ia}(x) \)
\[
\sum_{i=1}^{n-k+2} R_{ia}^{p_i}(x) R_{i(i+1)a}^{p_i+1}(...R_{i(k-1)a}^{p_i+1}R_{ia}^{p_i}(x)) \leq \sum_{i=1}^n R_{ia}^{k p_i}(x),
\]
i.e.,
\[
\sum_{i=1}^{n-k+2} \int_a^b w(x) \left[ \prod_{j=i}^{i+k-1} R_j^{p_i}(x) \right] dx \leq \sum_{i=1}^n \int_a^b w(x) R_{ia}^{k p_i}(x) dx.
\]
Now consider,
\[
I = \int_a^b w(x) R_{ia}^{k p_i}(x) dx
\]
\[
= \int_a^b u_i^{-akp_i}(x) u_i'(x) \left[ \frac{k p_i u_i(t) z_i'(t)}{z_i(t)} f_i(t) dt \right]^{k p_i} dx.
\]
Integrating by parts, we have
\[
I = \left[ u_i^{1-kp_i a}(x) \left[ \frac{k p_i u_i(t) z_i'(t)}{z_i(t)} f_i(t) dt \right]^{k p_i} \right]^b_a
\]
\[
- \int_a^b u_i^{1-kp_i a}(x) \frac{d}{dx} \left( \frac{k p_i u_i(t) z_i'(t)}{z_i(t)} f_i(t) dt \right)^{k p_i} dx.
\]
\[
\begin{align*}
&= w(b) \frac{u_i^{1-kp_i \alpha}(b)}{1 - kp_i \alpha} \left[ \int_a^b u_i(t) z_i'(t) f_i(t) \, dt \right]^{kp_i} - \frac{1}{1 - kp_i \alpha} \int_a^b u_i(x) w'(x) R_i^{kp_i}(x) u'_i(x) \, dx \\
&\quad - \frac{kp_i}{1 - kp_i \alpha} \int_a^b u_i^{2-\alpha}(x) w(x) z_i'(x) f_i(x) R_i^{kp_i-1}(x) z_i(x) \frac{1}{u'_i(x)} \, dx.
\end{align*}
\]

From (12) and (16) we have
\[
I = \int_a^b w(x) R_i^{kp_i}(x) \, dx \leq \frac{\gamma_i kp_i}{kp_i \alpha - 1} \int_a^b u_i^{2-\alpha}(x) w(x) z_i'(x) f_i(x) R_i^{kp_i-1}(x) z_i(x) \frac{1}{u'_i(x)} \, dx.
\]

By Hölder’s inequality:
\[
\begin{align*}
I &\leq \left[ \frac{\gamma_i kp_i}{kp_i \alpha - 1} \right]^{kp_i} \int_a^b w(x) \left[ \frac{u_i^{2-\alpha}(x) z_i'(x) f_i(x)}{z_i(x) \sqrt[2-\alpha]{u_i'(x)}} \right]^{kp_i} \, dx \\
&\leq \left[ \frac{\gamma_i kp_i}{kp_i \alpha - 1} \right]^{kp_i} \int_a^b w(x) \left[ \frac{u_i^{2-\alpha}(x) z_i'(x) f_i(x)}{z_i(x) \sqrt[2-\alpha]{u_i'(x)}} \right]^{kp_i} \, dx.
\end{align*}
\]

From (15) and (18) we have
\[
\sum_{i=1}^{n-k+2} \int_a^b w(x) \left[ \prod_{j=i}^{i+k-1} R_j^{p_j}(x) \right] \, dx \leq \sum_{i=1}^{n} \left[ \frac{kp_i \lambda_i}{akp_i - 1} \right]^{kp_i} \int_a^b w(x) g_i(x) \, dx \\
&\leq \sum_{i=1}^{n} \left[ \frac{kp_i \lambda_i}{akp_i - 1} \right]^{kp_i} \int_a^\infty w(x) g_i(x) \, dx.
\]

Let us define for \( \alpha < \frac{1}{kp_i} \) and \( 0 < a < b < \infty \):
\[
R_{ib}(x) = \frac{\sqrt[kp_i - \alpha]{u_i'(x)}}{u_i^{a}(x)} \int_x^b u_i(t) z_i'(t) f_i(t) \, dt, \quad 1 \leq i \leq n.
\]

Following the same steps as in the proof of inequality (19) we obtain
\[
\sum_{i=1}^{n-k+2} \int_a^b w(x) \left[ \prod_{j=i}^{i+k-1} R_j^{p_j}(x) \right] \, dx \leq \sum_{i=1}^{n} \left[ \frac{kp_i \lambda_i}{1 - akp_i} \right]^{kp_i} \int_a^b w(x) g_i(x) \, dx \\
&\leq \sum_{i=1}^{n} \left[ \frac{kp_i \lambda_i}{1 - akp_i} \right]^{kp_i} \int_a^\infty w(x) g_i(x) \, dx.
\]

By letting \( a \to 0 \) and \( b \to \infty \) in inequalities (19) and (20) we get (14).

**Corollary 2.** For any \( 1 \leq i \leq n \), let \( w, h_i : [0, X] \to \mathbb{R}_+ \) absolutely continuous functions for \( X \in \mathbb{R}_+ \) and \( \lambda_i = \max_{1 \leq i \leq n} (\gamma_i, \delta_i) \). Let
\[
1 + \frac{x w'(x)}{(1-2\alpha p_i) w(x)} \geq \frac{1}{\gamma_i} > 0, \quad \text{for } \alpha > \frac{1}{2p_i} \quad \text{and} \quad 1 \leq i \leq n.
\]
1 + \frac{x \, w'(x)}{(2\alpha p_i - 1) \, w(x)} \geq \frac{1}{\delta_i} > 0, \text{ for } \alpha < \frac{1}{2p_i} \text{ and } 1 \leq i \leq n.

Then,

\[ \sum_{i=1}^{n} \int_{0}^{x} w(x) \left[ \frac{1}{x^\alpha} \int_{a}^{x} t^{\alpha-1} h_i(t) \, dt \right]^{p_i} \, dx \leq \sum_{i=1}^{n} \left[ \frac{2p_i \lambda_i}{2\alpha p_i - 1} \right]^{2p_i} \int_{0}^{\infty} w(x) h_i(x) \, dx. \]

**Proof.** Follows from Theorem 2 by setting \( z_i(t) = u_i(t) = t; \) \( h_i(t) = h_{i+1}(t); \) \( p_i = p_{i+1} \) and \( f_i(t) \rightarrow t^{\alpha-1} h_i(t) \) for \( k = 2 \) and \( 1 \leq i \leq n. \)

**Theorem 3.** Let the conditions of Theorem 1 be satisfied and \( p_i > 1, \) is such that \( q_i = \frac{p_i}{3p_i-1} \) for \( 1 \leq i \leq n. \) Let

\[ 1 + \frac{u_i(x) \, w'(x)}{(1-3\alpha p_i) \, w(x) \, u_i'(x)} \geq \frac{1}{\gamma_i} > 0 \text{ (a.e.), for } \alpha > \frac{1}{3p_i} \text{ and } 1 \leq i \leq n. \]  

Then,

\[ \sum_{i=1}^{n} \int_{0}^{X} w(x) F_i^{p_i}(x) F_{i+1}^{p_{i+1}}(x) F_{i+2}^{p_{i+2}}(x) \, dx \leq \sum_{i=1}^{n} \left[ \frac{3p_i \gamma_i}{3\alpha p_i - 1} \right]^{3p_i} \int_{0}^{X} w(x) g_i(x) \, dx, \]  

where,

\( g_i(x) \left[ \frac{u_i^{2-\alpha}(x) \, z_i'(x) \, f_i(x)}{z_i(x) \, u_i'(x)} - \frac{u_i^{1-\alpha}(x) \, u_i(\frac{x}{2}) \, z_i'(\frac{x}{2}) \, f_i(\frac{x}{2})}{2 \, z_i(\frac{x}{2}) \, u_i'(x)} \right] \left[ u_i'(x) \right]^{3p_i} \) and

\( F_i(x) = \frac{\left[ u_i'(x) \right]^{3p_i}}{u_i^a(x)} \int_{x/2}^{x} \frac{u_i(t) \, z_i'(t)}{z_i(t)} \, f_i(t) \, dt. \)

**Proof.** Let us define for \( \alpha > \frac{1}{3p_i}, 1 \leq i \leq n, 0 \leq x \leq X : \)

\( F_i(x) = \frac{\left[ u_i'(x) \right]^{3p_i}}{u_i^a(x)} \int_{x/2}^{x} \frac{u_i(t) \, z_i'(t)}{z_i(t)} \, f_i(t) \, dt, 1 \leq i \leq n. \)

By using the inequality (3) with \( k = 3 \) for \( C_i = F_i^{p_i}(x) : \)

\( \sum_{i=1}^{n-1} F_i^{p_i}(x) F_{i+1}^{p_{i+1}}(x) F_{i+2}^{p_{i+2}}(x) \leq \sum_{i=1}^{n} F_i^{3p_i}(x), \)

i.e.,

\( \sum_{i=1}^{n-1} \int_{0}^{X} w(x) F_i^{p_i}(x) F_{i+1}^{p_{i+1}}(x) F_{i+2}^{p_{i+2}}(x) \, dx \leq \sum_{i=1}^{n} \int_{0}^{X} w(x) F_i^{3p_i}(x) \, dx. \)  

(23)

Now consider

\[ I = \int_{0}^{X} w(x) F_i^{3p_i}(x) \, dx \]
By Hölder’s inequality:
\[
\int_0^X u_i^{-3p_i}(x) u_i'(x) \left[ 3p_i \sqrt{w(x)} \int_{x/2}^X \frac{u_i(t) z_i'(t)}{z_i(t)} f_i(t) \, dt \right]^{3p_i} \, dx.
\]
Integrating by parts, we have
\[
I = \left[ u_i^{1-3p_i\alpha}(x) \left[ 3p_i \sqrt{w(x)} \int_{x/2}^X \frac{u_i(t) z_i'(t)}{z_i(t)} f_i(t) \, dt \right]^{3p_i} \right]_0^X
- \int_0^X \frac{u_i^{1-3p_i\alpha}(x)}{1-3p_i\alpha} \frac{d}{dx} \left( 3p_i \sqrt{w(x)} \int_{x/2}^X \frac{u_i(t) z_i'(t)}{z_i(t)} f_i(t) \, dt \right)^{3p_i} \, dx
= \frac{w(X) u_i(X) F_i^{3p_i}(X)}{u_i'(X)(1-3p_i\alpha)} - \int_0^X \frac{w'(x) u_i(x) F_i^{3p_i}(x)}{u_i'(x)(1-3p_i\alpha)} \, dx - \frac{3p_i}{1-3p_i\alpha} \int_0^X \frac{w(x)}{u_i'(x)} \left[ u_i^{2-a}(x) z_i'(x) f_i(x) - u_i^{1-a}(x) z_i'(x/2) f_i(x/2) u_i(x/2) \right] \, dx. \tag{24}
\]
From (21) and (24) we have
\[
\int_0^X w(x) F_i^{3p_i}(x) \, dx
\leq \frac{3p_i \gamma_i}{1-3p_i\alpha} \int_0^X w(x) F_i^{3p_i-1}(x) \left[ u_i^{2-a}(x) z_i'(x) f_i(x) - u_i^{1-a}(x) z_i'(x/2) f_i(x/2) u_i(x/2) \right] \, dx. \tag{25}
\]
By Hölder’s inequality:
\[
I \leq \frac{3\gamma_i p_i}{3p_i\alpha - 1} \sqrt{\int_0^X \left[ 3q_i \sqrt{w(x)} F_i^{3p_i-1}(x) \right]^{3q_i} \, dx}
\times \frac{3p_i}{\sqrt{\int_0^X w(x) \left[ u_i^{2-a}(x) z_i'(x) f_i(x) - u_i^{1-a}(x) z_i'(x/2) f_i(x/2) u_i(x/2) \right] \, dx}}^{3p_i}
\leq \left[ \frac{3\gamma_i p_i}{3p_i\alpha - 1} \right]^{3p_i} \int_0^X w(x) \left[ u_i^{2-a}(x) z_i'(x) f_i(x) - u_i^{1-a}(x) z_i'(x/2) f_i(x/2) u_i(x/2) \right] \, dx. \tag{26}
\]
From (23) and (26) we get (22).

**Lemma 2 ([9]).** If $C_1, C_2, \ldots, C_n$ are reals and $C_{n+1} = C_1$ for $k \geq 1$, then
\[
\left[ \sum_{r=1}^n C_r \right]^k \leq n^{k-1} \sum_{r=1}^n C_r^k. \tag{27}
\]
Theorem 4. Let the conditions of Theorem 1 be satisfied and $p > 1$, $k \geq 1$ are such that $q = \frac{p}{kp - 1}$.

If $\alpha > 0$ and
\[
1 + \frac{u_i(x) w'(x)}{(1 + k\alpha p) w(x) u_i'(x)} \geq \frac{1}{\gamma_i} > 0 \ (a.e.), \text{ for } 1 \leq i \leq n.
\]

Then,
\[
\int_a^b w(x) \left[ \sum_{i=1}^n F_{ia}(x) \right]^{kp} dx \leq \sum_{i=1}^n \left[ \frac{kp\gamma_i \sqrt{n}}{k\alpha p + 1} \right]^{kp} \int_a^b w(x) g_i(x) dx,
\]
where,
\[
g_i(x) = \frac{u_i^{kp}(x) [z_i'(x)]^{kp} f_i^{kp}(x)}{[z_i(x)]^{kp} (u_i'(x))^{p/q}} \text{ and } F_i(x) = u_i^q(x) \sqrt{u_i'(x)} \int_0^x z_i'(t) f_i(t) u_i(t) z_i(t) dt.\]

Proof. Let us define for $1 \leq i \leq n$; $0 \leq x \leq X$ and $0 < a < b < \infty$:

\[
F_{ia}(x) = u_i^q(x) \sqrt{u_i'(x)} \int_0^x z_i'(t) f_i(t) u_i(t) z_i(t) dt, \quad 1 \leq i \leq n.
\]

By using the inequality (27) for $C_i = F_{ia}(x)$ and $k \to kp$:

\[
\left[ \sum_{i=1}^n F_{ia}(x) \right]^{kp} \leq n^{pk-1} \sum_{i=1}^n F_{ia}^{kp}(x),
\]

i.e.,

\[
\int_a^b w(x) \left[ \sum_{i=1}^n F_{ia}(x) \right]^{kp} dx \leq n^{pk-1} \sum_{i=1}^n \int_a^b u_i^{kp}(x) u_i'(x) \left[ \sqrt{w(x)} \int_a^x z_i'(t) f_i(t) u_i(t) z_i(t) dt \right]^{kp} dx.
\]

Now consider,

\[
I = \int_a^b u_i^{kp}(x) u_i'(x) \left[ \sqrt{w(x)} \int_a^x z_i'(t) f_i(t) u_i(t) z_i(t) dt \right]^{kp} dx.
\]

Integrating by parts, we have

\[
I = \left. \frac{u_i^{1+kpa}(x)}{1+kpa} w(x) \left[ \int_a^x z_i'(t) f_i(t) u_i(t) z_i(t) dt \right]^{kp} \right|_a^b \\
- \int_a^b \frac{u_i^{1+kpa}(x)}{1+kpa} \frac{d}{dx} \left( \sqrt{w(x)} \int_a^x z_i'(t) f_i(t) u_i(t) z_i(t) dt \right)^{kp} dx \\
= \frac{w(b) u_i(b) F_{ia}^{kp}(b)}{u_i'(b)(1+kpa)} - \int_a^b \frac{w(x) u_i(x) F_{ia}^{kp}(x)}{u_i'(x)(1+kpa)} dx \\
+ \frac{kp}{1+kpa} \int_a^b w(x) \left[ \frac{F_{ia}^{kp-1} u_i^q(x) z_i'(x) f_i(x)}{z_i(x)^{kp} u_i'(x)} \right] dx.
\]
From (28) and (31) we have:

\[
\int_a^b w(x)F_{ia}^{kp}(x)dx \leq \frac{kp\gamma_i}{1+kpa} \int_a^b w(x) \frac{F_{ia}^{kp-1}(x)u_i^a(x)z_i'(x)f_i(x)}{z_i(x)^{1/q}u_i'(x)}dx
\]  

(32)

By Hölder’s inequality:

\[
I \leq \frac{k\gamma_ip}{kpa+1} \left[ \int_a^b \left( \sqrt[k]{w(x)F_{ia}^{kp-1}(x)} \right)^k \right]^{1/k} \left[ \int_a^b w(x) \frac{u_i^a(x)z_i'(x)f_i(x)}{z_i(x)^{1/q}u_i'(x)} \right]^{kp} dx
\]

\[
\leq \left[ \frac{k\gamma_ip}{kpa+1} \right]^{kp} \int_a^b w(x) \frac{u_i^{kp}(x)[z_i'(x)]^{kp}f_i^{kp}(x)}{z_i^{kp}(x)[u_i'(x)]^{p/q}} dx.
\]  

(33)

From (30) and (33) we get (29).

**Corollary 3.** For any \(1 \leq i \leq n\), let \(w, h_i : [a, b] \to \mathbb{R}_+\) be absolutely continuous functions for \(0 < a < b < \infty\) and \(p > 1\). If

\[
1 + \frac{x w'(x)}{(1+kap)w(x)} \geq \frac{1}{\gamma_i} > 0, \text{ for } \alpha > \frac{1}{2p} \text{ and } 1 \leq i \leq n.
\]

Then,

\[
\int_a^b w(x) \left[ \sum_{i=1}^n x^{a} \int_a^x \frac{h_i(t)}{t^2} dt \right]^{kp} dx \leq \sum_{i=1}^n \left[ \frac{k\sqrt{n}kp\gamma_i}{kpa+1} \right]^{2pi} \int_a^b w(x) h_i^{kp}(x)dx.
\]

**Proof.** Follows from Theorem 4 by setting \(z_i(t) = u_i(t) = t \) and \(f_i(t) \mapsto t^{1-a}h_i(t)\) for \(1 \leq i \leq n\).

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