QUANTUM GROUP OF AUTOMORPHISMS OF A FINITE (QUANTUM) GROUP

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Abstract. A notion of a quantum automorphism group of a finite quantum group, generalising that of a classical automorphism group of a finite group, is proposed and a corresponding existence result proved.

The story of quantum symmetry groups began with the paper [Wan], where S. Wang defined and began to study quantum symmetry groups of finite-dimensional C*-algebras. Soon after that T. Banica, J. Bichon and others expanded this study to quantum symmetry groups of various finite structures, such as (coloured) graphs or metric spaces (for the information on infinite-dimensional extensions, among them the quantum isometry groups of D. Goswami, we refer to [BSK] and references therein). The general idea behind these concepts is based on considering all compact quantum group actions on a given finite quantum space (viewed dually as a finite-dimensional C*-algebra), preserving some extra structure of that space, and looking for a universal object in the resulting category.

In this short note we propose studying in this spirit the quantum group of all quantum automorphisms of a given finite quantum (or classical) group. The starting point of our approach is based on a trivial observation saying that a transformation of a finite abelian group Γ is an automorphism if and only if it induces in a natural way, via the Fourier transform, a transformation of \( \hat{\Gamma} \), the Pontriagin dual of Γ. In particular the automorphism groups of Γ and \( \hat{\Gamma} \) are canonically isomorphic. We thus define a quantum family of maps on a finite quantum group \( G \) to be a quantum family of automorphisms if it induces, via the Fourier transform associated to \( G \), a quantum family of maps on the dual quantum group \( \hat{G} \). We show that a universal quantum family of automorphisms of \( G \) exists, and naturally defines the quantum automorphism group of \( G \) (which is a compact quantum group in the sense of Woronowicz). Its classical version is the group of all automorphisms of \( G \), i.e. these automorphisms of the C*-algebra \( C(G) \) which commute (in a natural sense) with the coproduct of the latter algebra.

It has to be observed that we do not know any example in which the quantum automorphism group as defined in this note is not a classical group. Thus the notion, though apparently natural and satisfactory, needs to be treated as tentative and open to modifications. In particular the main question of interest, i.e. the problem which classical finite groups admit genuinely quantum automorphisms, remains open.

The plan of the article is as follows: in Section 1 we recall the properties of the duality and Fourier transform on finite quantum groups, as defined and studied for example by A. Van Daele ([VD_1], [VD_2]). In Section 2 we introduce the notion of a quantum family of automorphisms of a given finite quantum group \( G \), discuss some equivalent conditions related

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to that definition and use it to prove the existence of the quantum automorphism group of \( \mathbb{G} \), which is canonically isomorphic to that of \( \hat{\mathbb{G}} \). In section 3 we discuss some simplifications and special properties appearing when one considers quantum automorphism groups of classical groups. The symbol \( \otimes \) will denote both the spatial tensor product of \( C^* \)-algebras and the algebraic tensor product of vector spaces (if we want to stress we are using the latter we write \( \otimes_{\text{alg}} \)).

I. Finite quantum groups, duality and the Fourier transform

Let \( \mathbb{G} \) be a finite quantum group. In the following sections we will denote the finite dimensional \( C^* \)-algebra corresponding to \( \mathbb{G} \) by the symbol \( C(\mathbb{G}) \). However, in order to keep the notation lighter, throughout this section the quantum group \( \mathbb{G} \) will be fixed and we will write \( A \) for the algebra \( C(\mathbb{G}) \). Then \( A \) is a finite dimensional Hopf \( C^* \)-algebra whose coproduct, antipode and counit will be denoted by \( \Delta \), \( S \) and \( \varepsilon \) respectively. We will use the symbol \( h \) to denote the Haar state of \( \mathbb{G} \) and occasionally employ the Sweedler notation for the coproduct: \( \Delta(a) := a(1) \otimes a(2) \), \( a \in \mathcal{A} \). The convolution product of two elements \( a, b \in \mathcal{A} \) is given by

\[
\bullet \quad a \ast b = (h \otimes \text{id}) \left( (S \otimes \text{id})\Delta(b) \right) (a \otimes 1),
\]

(1.1)

As \( \mathbb{G} \) is finite, the antipode \( S \) is an involution and \( h \) is a trace. Hence (1.1) coincides with the formula proposed in [VD4, Proposition 2.2], i.e.

\[
\bullet \quad a \ast b = h(S^{-1}(b(1))a)b(2).
\]

Note that a different formula for the convolution product is used in [PoW]. The convolution product is in fact an associative bilinear operation making \( \mathcal{A} \) into an involutive algebra with involution \( \bullet \) defined as

\[
\bullet \quad \mathcal{A} \ni a \mapsto a^* = S(a^*) \in \mathcal{A}.
\]

(1.2)

This involution is referred to as the convolution adjoint.

Let us note here the important relation between the Haar state and antipode: for any \( b, c \in \mathcal{A} \) we have

\[
\bullet \quad S \left( (\text{id} \otimes h) \left( \Delta(b) \otimes c \right) \right) = (\text{id} \otimes h) \left( (1 \otimes b) \Delta(c) \right).
\]

(1.3)

This relation can be found in [VD3, Proof of Proposition 3.11] (see also [VD1, Lemma 5.5]). It is worth mentioning that in the context of Kac algebras [L3] is taken as the defining property of \( h \) ([EnS, Section 2.2]).

In [VD2] Van Daele showed that every finite quantum group \( \mathbb{G} \) possesses a (unique) element \( \eta \in \mathcal{A} \) such that \( \varepsilon(\eta) = 1 \) and

\[
\bullet \quad a\eta = \varepsilon(a)\eta, \quad a \in \mathcal{A}.
\]

For example, when \( \mathbb{G} = G \) for a finite group \( G \), we have \( \eta = \delta_e \) and \( h(\eta) = \frac{1}{|G|} \). On the other hand if \( \mathbb{G} = \hat{G} \) for a finite group \( G \), i.e. \( \mathcal{A} = C[G] \), then \( \eta = \frac{1}{|G|} \sum_{g \in \Gamma} \lambda_g \) but we still have \( h(\eta) = \frac{1}{|G|} \) (see also Subsection 1.5). More generally one can show that \( h(\eta) = \frac{1}{\dim \mathcal{A}} \) (cf. [Wor, Section A.2]).

Let us now briefly describe the dual \( \hat{\mathbb{G}} \) of the quantum group \( \mathbb{G} \). The standard notation for \( C^* \)-algebra corresponding to \( \hat{\mathbb{G}} \) is \( C(\hat{\mathbb{G}}) \), but in this section we will use \( \hat{\mathcal{A}} \) to denote it. As a vector space \( \hat{\mathcal{A}} \) is defined to be the set

\[
\{ h(a) \mid a \in \mathcal{A} \}.
\]
Clearly it is a subspace of the dual space of \( \mathcal{A} \), but thanks to faithfulness of the Haar state \( h \), it is in fact the whole of \( \mathcal{A}^\ast \). The isomorphism of vector spaces
\[
\mathcal{A} \ni a \mapsto h(\cdot a) \in \hat{\mathcal{A}}
\]
is called the Fourier transform and is denoted by \( \mathcal{F} \). The space \( \mathcal{A} \) can be equipped with the structure of a Hopf \( * \)-algebra which we will describe below.

1.1. **Unital \( * \)-algebra structure.** The product in \( \hat{\mathcal{A}} \) is defined as convolution of functionals: for \( \omega_1, \omega_2 \in \hat{\mathcal{A}} \) we define \( \omega_1 \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta \). One easily checks that this is indeed an associative multiplication on \( \hat{\mathcal{A}} \) and the counit of \( \mathcal{A} \) is the unit of \( \hat{\mathcal{A}} \). We will often write \( \hat{1} \) to denote the unit of \( \hat{\mathcal{A}} \). The involution making \( \mathcal{A} \) a \( * \)-algebra is the mapping \( \omega \mapsto \omega^\ast \), where
\[
\omega^\ast(a) = \overline{\omega(S(a))}, \quad a \in \mathcal{A}.
\]

In terms of the Fourier transform the structure described above has the following property (which can be taken as a definition of \( * \)):
\[
\mathcal{F}(a \ast b) = \mathcal{F}(a) \mathcal{F}(b), \quad a, b \in \mathcal{A}.
\]
To see this let us take any \( c \in \mathcal{A} \). Then
\[
\mathcal{F}(a \ast b)(c) = h((a \ast b)c) = h\left((h \otimes \text{id})((S \otimes \text{id})\Delta(b))(a \otimes 1)c\right)
= (h \otimes h)\left((S \otimes \text{id})\Delta(b)(a \otimes c)\right)
= h\left((\text{id} \otimes h)((1 \otimes b)\Delta(c))a\right)
= h(ac(1))h(bc(2)) = (\mathcal{F}(a) \otimes \mathcal{F}(b))\Delta(c) = (\mathcal{F}(a)\mathcal{F}(b))(c),
\]
where we used (1.3) and traciality of \( h \).

The involution of \( \hat{\mathcal{A}} \) is also easily expressed with help of the Fourier transform. Indeed, using the fact that \( h \) is positive and \( S \)-invariant, we obtain
\[
(\mathcal{F}(b))^\ast(a) = \overline{\mathcal{F}(S(a)b)} = h(S(a)^\ast b) = h(b^\ast S(a)) = h(aS(b^\ast)) = \mathcal{F}(S(b^\ast))(a), \quad a, b \in \mathcal{A}.
\]
In other words
\[
\mathcal{F}(b)^\ast = \mathcal{F}(S(b^\ast)) = \mathcal{F}(b^\ast), \quad b \in \mathcal{A}.
\]

Finally, the unit \( \hat{1} \) of \( \hat{\mathcal{A}} \) can be written \( \mathcal{F}\left(\frac{1}{h(1) \eta}\right) \) (this follows immediately from the properties of \( \eta \)).

1.2. **The coproduct.** We identify (canonically) \( \hat{\mathcal{A}} \otimes \hat{\mathcal{A}} \) with \( (\mathcal{A} \otimes \mathcal{A})^\ast \). Now, given \( \omega \in \hat{\mathcal{A}} \) we define \( \hat{\Delta}(\omega) \) as the element of \( \hat{\mathcal{A}} \otimes \hat{\mathcal{A}} \) for which we have
\[
(\hat{\Delta}(\omega))(c_1 \otimes c_2) = \omega(c_1 c_2), \quad c_1, c_2 \in \mathcal{A}.
\]
This defines a coassociative coproduct \( \hat{\Delta}: \hat{\mathcal{A}} \to \hat{\mathcal{A}} \otimes \hat{\mathcal{A}} \).

In terms of the Fourier transform we have \( \hat{\Delta}(\mathcal{F}(a)) = \sum_i \mathcal{F}(a_i) \otimes \mathcal{F}(b_i) \) (for \( a \in \mathcal{A} \)) if and only if
\[
\sum_i h(c_1 a_i)h(c_2 b_i) = h(c_1 c_2 a)
\]
for all \( c_1, c_2 \in \mathcal{A} \).
1.3. The counit, the antipode and the Haar measure. The counit \( \hat{\varepsilon} \) of \( \mathcal{A} \) is the functional given by evaluating \( \omega \in \mathcal{A}^* \) in the unit \( 1 \) of \( \mathcal{A} \) (this is clear from (1.6)). In terms of the Fourier transform we have

\[
\hat{\varepsilon}(\hat{\mathcal{F}}(a)) = h(a), \quad a \in \mathcal{A}.
\]

The antipode is defined directly via the duality of vector spaces, i.e. \( \hat{S}(\omega) = \omega \circ S \) for \( \omega \in \mathcal{A}^* \approx \mathcal{A}^\vee \). Using the fact that \( h \) is \( S \)-invariant and a trace we can write \( \hat{S} \) in terms of the Fourier transform as follows \((a, b \in \mathcal{A})\):

\[
\hat{S}(\hat{\mathcal{F}}(b))(a) = \hat{\mathcal{F}}(b)(S(a)) = h(S(a)b) = h(S(b)a) = \hat{\mathcal{F}}(S(b))(a),
\]

so that

\[
\hat{S} \circ \hat{\mathcal{F}} = \hat{\mathcal{F}} \circ S. \tag{1.7}
\]

The Haar measure of \( \mathcal{A} \) is defined as

\[
\hat{\mathcal{h}}(\hat{\mathcal{F}}(a)) = h(\eta)\varepsilon(a), \quad a \in \mathcal{A}.
\]

This definition is different from the one given by Van Daele because he did not require \( \hat{\mathcal{h}} \) to be a state (the original formula did not incorporate the constant \( h(\eta) \)).

1.4. The Haar element, the convolution product and the convolution adjoint. It can be easily checked that the Haar element of the dual quantum group, \( \hat{\eta} \), is equal to \( \hat{\mathcal{F}}(1) \). Indeed, for any \( a \in \mathcal{A} \) we have \( 1 \ast a = a \ast 1 = h(a)1 \), so

\[
\hat{\mathcal{F}}(1) \hat{\mathcal{F}}(a) = \hat{\mathcal{F}}(1 \ast a) = h(a)\hat{\mathcal{F}}(1) = \hat{\varepsilon}(\hat{\mathcal{F}}(a))\hat{\mathcal{F}}(1)
\]

and the other required equality follows similarly. Furthermore, by (1.7) and (1.5) we have

\[
\hat{\mathcal{F}}(a)^\ast = \hat{S}(\hat{\mathcal{F}}(a))^\ast = \hat{\mathcal{F}}(S(a))^\ast = \hat{\mathcal{F}}(S(a)\ast) = \hat{\mathcal{F}}(a^\ast), \quad a \in \mathcal{A}.
\]

Note also another useful formula which holds for all \( a, b \in \mathcal{A} \):

\[
\varepsilon(a \ast b) = \varepsilon((\hat{\mathcal{F}} \circ \varepsilon)((S \circ \varepsilon)\mathcal{D}(b)(a \circ 1))) = h((\hat{\mathcal{F}} \circ \varepsilon)((S \circ \varepsilon)\mathcal{D}(b)(a \circ 1))) = h(S(b)a).
\]

Now let us examine the convolution product on \( \mathcal{A} \) in terms of the Fourier transform. We have

\[
h(\eta)\hat{\mathcal{F}}(ab) = \hat{\mathcal{F}}(b) \ast \hat{\mathcal{F}}(a), \quad a, b \in \mathcal{A}. \tag{1.8}
\]

Indeed, write \( \hat{\Delta}(\hat{\mathcal{F}}(a)) = \sum_i \hat{\mathcal{F}}(c_i) \otimes \hat{\mathcal{F}}(d_i) \) and compute

\[
\hat{\mathcal{F}}(b) \ast \hat{\mathcal{F}}(a) = (\hat{\mathcal{F}}(b) \otimes 1) \hat{\Delta}(\hat{\mathcal{F}}(a)) = \sum_i (\hat{\mathcal{F}}(c_i) \otimes \hat{\mathcal{F}}(d_i)) (\hat{\mathcal{F}}(b) \otimes 1)
\]

\[
= \sum_i (\hat{\mathcal{F}}(c_i) \otimes \hat{\mathcal{F}}(d_i)) (\hat{\mathcal{F}}(b) \otimes 1)
\]

\[
= \sum_i (\hat{\mathcal{F}}(S(c_i) \otimes d_i)) (\hat{\mathcal{F}}(b) \otimes 1) = \sum_i \hat{\mathcal{F}}(S(c_i) \otimes b) \hat{\mathcal{F}}(d_i)
\]

\[
= h(\eta) \sum_i \varepsilon(S(c_i) \otimes b) \hat{\mathcal{F}}(d_i) = h(\eta) \sum_i h(S(b)S(c_i)) \hat{\mathcal{F}}(d_i)
\]

\[
= h(\eta) \sum_i h(c_i b) \hat{\mathcal{F}}(d_i).
\]
It remains to see that \( \sum_i h(c_i b) \mathcal{F}(d_i) = \mathcal{F}(ab) \). To that end note that for any \( x \in \mathcal{A} \)

\[
\sum_i h(c_i b) \mathcal{F}(d_i)(x) = \sum_i h(c_i b) h(d_i x) = \sum_i h(b c_i) h(x d_i) = h(b x a) = h(ab x) = \mathcal{F}(ab)(x),
\]

where the third equality is a consequence of the definition of \( \hat{\Delta} \) (cf. Section 1.2). This proves formula (1.8).

The reason for the extra flip and the normalizing factor can be seen from Lemma 1.1 below (if one remembers that \( S \) is anti-homomorphic).

1.5. **Iteration of the Fourier transform.**

**Lemma 1.1.** We have \( \hat{\mathcal{F}} \circ \mathcal{F} = h(\eta) S \). In particular, the Fourier transform, when suitably rescaled, is a “transformation of order 4”.

**Proof.** It is an explicit calculation. Let \( a \in \mathcal{A} \). Put \( b = \hat{\mathcal{F}}(\mathcal{F}(a)) \in ((\mathcal{A})^*)^* \). We will show that for any \( \omega \in \mathcal{A}^* \) we have \( \omega(S(a)) = b(\omega) \). We can assume that \( \omega = \mathcal{F}(c) \) for an element \( c \in \mathcal{A} \). Then

\[
b(\omega) = \hat{h}(\omega \mathcal{F}(a)) \\
= \hat{h}(\mathcal{F}(c) \mathcal{F}(a)) \\
= \hat{h}(\mathcal{F}(c \ast a)) \\
= h(\eta) c(\ast a) \\
= h(\eta)(h \otimes \epsilon)((S \otimes \text{id})\Delta(a)(c \otimes 1)) \\
= h(\eta) h(S(\text{id} \otimes \epsilon)(\Delta(a)) c) \\
= h(\eta) h(S(a)c) \\
= h(\eta) \mathcal{F}(c)(S(a)) = \omega(h(\eta) S(a)).
\]

This ends the proof. \( \square \)

1.6. **Fundamental examples.**

1.6.1. **Algebra of functions on a finite group.** Let \( G \) be a finite group and let \( \mathcal{A} = \text{Fun}(G) \) with the standard pointwise structure. The coproduct can be written in the basis \( \{ \delta_g \}_{g \in G} \) as

\[
\Delta(\delta_g) = \sum_{ab = g} \delta_a \otimes \delta_b, \quad g \in G.
\]

Note that further we have

\[
\delta_g^* = \delta_{g^{-1}}, \quad \text{and} \quad \delta_g \ast \delta_h = \frac{1}{|G|} \delta_{gh}, \quad g, h \in G.
\]

The product \( \mathcal{F}(\delta_{g_1}) \mathcal{F}(\delta_{g_2}) \) of elements of \( \hat{\mathcal{A}} \) is the functional

\[
\mathcal{A} \ni a \mapsto (h(\cdot \delta_{g_1}) \otimes h(\cdot \delta_{g_2})) \Delta(a).
\]

For \( a = \delta_g \) we find the relevant value to be

\[
(h \otimes h) \sum_{ab = g} \delta_a \delta_{g_1} \otimes \delta_b \delta_{g_2} = \begin{cases} \frac{1}{|G|^2}, & g g_1 g_2 = g, \\
0, & \text{else.} \end{cases}
\]
This means that for all $g_1, g_2 \in G$
\[ \mathcal{F}(\delta_{g_1}) \mathcal{F}(\delta_{g_2}) = \frac{1}{|G|} \mathcal{F}(\delta_{g_1g_2}). \]

Similarly, using the description of $\hat{\Delta}$ given in the Subsection 1.2 we find that for all $g \in G$
\[ \hat{\Delta}(\mathcal{F}(\delta_g)) = \frac{1}{|G|} \mathcal{F}(\delta_g) \otimes \mathcal{F}(\delta_g). \]

Thus the map
\[ \mathcal{F} \ni \mathcal{F}(\delta_g) \mapsto \frac{1}{|G|} \lambda_g \in \mathbb{C}[G] \]
is an isomorphism of vector spaces. Note that $\frac{1}{|G|} = h(\eta)$.

### 1.6.2. Group ring of a finite group.
Let $G$ be a finite group and let $\mathcal{A}$ denote this time the group ring of $G$, $\mathbb{C}[G]$: we view it now naturally as the algebra of functions on the dual quantum group $\hat{G}$. The coproduct on $\mathcal{A}$ is given by the formula
\[ \Delta(\lambda_g) = \lambda_g \otimes \lambda_g, \quad g \in G, \]
and further
\[ \lambda_g^* = \lambda_g, \quad \text{and} \quad \lambda_g * \lambda_h = \delta_{g,h} \lambda_h, \quad g, h \in G. \]

This time the Fourier transform $\mathcal{F}$ maps $\mathbb{C}[G]$ onto $\text{Fun}(G)$, and the calculations similar to these above (or an application of Lemma 1.1) yield in this picture the following formula:
\[ \mathcal{F}(\lambda_g) = \delta_{g^{-1}}, \quad g \in G. \]

### 2. Quantum Automorphisms of a Finite Quantum Group

Throughout this section $\mathcal{G}$ will denote a finite quantum group. The corresponding finite dimensional $C^*$-algebra playing the role of the algebra of functions on $\mathcal{G}$ will be denoted by the symbol $C(\mathcal{G})$. If $\mathcal{B}$ is a unital $C^*$-algebra and $\alpha: C(\mathcal{G}) \to C(\mathcal{G}) \otimes \mathcal{B}$ a linear map, we define another linear map $\hat{\alpha}: C(\hat{\mathcal{G}}) \to C(\hat{\mathcal{G}}) \otimes \mathcal{B}$ by the formula
\[ \hat{\alpha} = \frac{1}{h(\eta)}(\mathcal{F} \otimes \text{id}_\mathcal{B}) \circ \alpha \circ \mathcal{F} \circ \hat{S}. \]  
(2.1)

Note that $\mathcal{F}$ denotes here (as in the previous section) simply the Fourier transform associated to the dual quantum group $\hat{\mathcal{G}}$.

Following $[So1]$, we say that $\alpha$ as above represents a quantum family of maps on $\mathcal{G}$ if $\alpha$ is a unital $*$-homomorphism. We say that it represents a quantum family of invertible maps on $\mathcal{G}$ if in addition the Podleś condition holds: $\alpha(C(\mathcal{G})) (1 \otimes \mathcal{B})$ spans $C(\mathcal{G}) \otimes \mathcal{B}$.

Due to finite dimensionality of $C(\mathcal{G})$, the last condition is purely vector space theoretic: it means that the set of elements of the form $\{ (id \otimes \omega)(\alpha(a)) \mid a \in C(\mathcal{G}), \omega \in \mathcal{B}^* \}$ spans $C(\mathcal{G})$.

It may be noteworthy that the formula for $\hat{\alpha}$ can be expressed by another formula which involves the Fourier transform on $\mathcal{G}$ alone. More precisely, we have:

**Proposition 2.1.** $\hat{\alpha} = (\mathcal{F} \otimes \text{id}) \circ \alpha \circ \mathcal{F}^{-1}$.

**Proof.** Using Lemma 1.1 we have $\hat{\mathcal{F}} \circ \hat{S}^{-1} = \hat{h}(\eta) \mathcal{F}^{-1}$, i.e. $\hat{\mathcal{F}} \circ \hat{S} = \hat{h}(\eta) \mathcal{F}^{-1}$. Thus, $\hat{\alpha} = \frac{1}{h(\eta)}(\mathcal{F} \otimes \text{id}_\mathcal{B}) \circ \alpha \circ \mathcal{F} \circ \hat{S} = \frac{\hat{h}(\eta)}{h(\eta)}(\mathcal{F} \otimes \text{id}_\mathcal{B}) \circ \alpha \circ \mathcal{F}^{-1} = (\mathcal{F} \otimes \text{id}) \circ \alpha \circ \mathcal{F}^{-1}$, since $\hat{h}(\eta) = h(\eta)$. □
Before we formulate the next lemma we need another piece of terminology: we say that $\alpha$ as above preserves the convolution product if $\alpha$ is a homomorphism from $(C(G), \ast)$ (i.e. $C(G)$ equipped with the convolution product) to $C(G) \otimes B$ with convolution product on the first factor and the given product of $B$ on the second (so if $\mu: C(G) \otimes C(G) \to C(G)$ denotes the convolution product and $m: B \otimes_{\text{alg}} B \to B$ the product on $B$ then the condition on $\alpha$ is that it is a homomorphism from $C(G)$ with convolution product to $C(G) \otimes B$ with the product $(\mu \otimes m)(\text{id}_{C(G)} \otimes \sigma \otimes \text{id}_B)$, where $\sigma$ is the flip $B \otimes C(G) \to C(G) \otimes B$.

Lemma 2.2. Let $B$ be a unital $C^*$-algebra and $\alpha: C(G) \to C(G) \otimes B$ a linear map. Then

1. $\hat{\alpha}$ is a homomorphism from $C(\hat{G})$ to $C(\hat{G}) \otimes B$ (with usual product) if and only if $\alpha$ preserves the convolution product,
2. $\hat{\alpha}$ is $*$-preserving if and only if $\alpha$ preserves the convolution adjoint (i.e. $(\bullet \otimes \ast)\alpha = \alpha \circ \bullet$),
3. $\hat{\alpha}$ is unital if and only if $\alpha$ preserves the Haar state (i.e. $\alpha(\eta) = \eta \otimes 1_B$),
4. $\hat{\alpha}$ preserves the Haar state if and only if $\alpha$ preserves the counit.

Moreover we have the following equality:

$$\hat{\alpha} = (S \otimes \text{id}_B) \circ \alpha \circ S, \quad (2.2)$$

so that if $\beta = \hat{\alpha}$, then $\alpha = \hat{\beta}$.

Proof. All the statements in the lemma follow from the properties of the Fourier transform established in Section I.

Proof of (1): Let us denote the product maps on $C(G)$ and $C(\hat{G})$ by $m$ and $\hat{m}$ respectively. Then let $m_B$ be the product of $B$ and $\hat{m}$ the convolution products on $C(G)$ and $C(\hat{G})$ respectively. We will use the symbol $\sigma$ to denote flip maps on various tensor products. We have

$$(\hat{m} \otimes m_B) \circ (\text{id} \otimes \sigma \otimes \text{id}) \circ (\hat{\alpha} \otimes \hat{\alpha})$$

$$= \frac{1}{h(\eta)^2}(\hat{m} \otimes m_B)(\text{id} \otimes \sigma \otimes \text{id}) \circ (\hat{\mathcal{F}} \otimes \text{id} \otimes \hat{\mathcal{F}} \otimes \text{id}) \circ (\alpha \otimes \alpha) \circ (\hat{\mathcal{F}} \otimes \hat{\mathcal{F}}) \circ (\hat{S} \otimes \hat{S})$$

$$= \frac{1}{h(\eta)^2}(\mathcal{F} \otimes \text{id}) \circ (\mu \otimes m_B)(\text{id} \otimes \sigma \otimes \text{id}) \circ (\alpha \otimes \alpha) \circ (\hat{\mathcal{F}} \otimes \hat{\mathcal{F}}) \circ (\hat{S} \otimes \hat{S}) \quad (2.3)$$

On the other hand using antimultiplicativity of $\hat{S}$, a dual version of formula (1.8) and the fact that $\hat{h}(\hat{\eta}) = h(\eta)$ we find that

$$\hat{\alpha} \circ \hat{m} = \frac{1}{h(\eta)}(\mathcal{F} \otimes \text{id}) \circ \alpha \circ \hat{\mathcal{F}} \circ \hat{S} \circ \hat{m}$$

$$= \frac{1}{h(\eta)}(\mathcal{F} \otimes \text{id}) \circ \alpha \circ \hat{\mathcal{F}} \circ \hat{m} \circ \sigma \circ (\hat{S} \otimes \hat{S}) \quad (2.4)$$

$$= \frac{1}{h(\eta)^2}(\mathcal{F} \otimes \text{id}) \circ \alpha \circ \mu \circ \sigma \circ (\hat{\mathcal{F}} \otimes \hat{\mathcal{F}}) \circ (\hat{S} \otimes \hat{S}).$$

Now since $\mathcal{F}$ and $\hat{\mathcal{F}} \circ \hat{S}$ are linear isomorphisms, we see from comparing (2.3) to (2.4) that

$$\hat{\alpha} \circ \hat{m} = (\hat{m} \otimes m_B) \circ (\text{id} \otimes \sigma \otimes \text{id}) \circ (\hat{\alpha} \otimes \hat{\alpha})$$

if and only if

$$\alpha \circ \mu = (\mu \otimes m_B)(\text{id} \otimes \sigma \otimes \text{id}) \circ (\alpha \otimes \alpha).$$
Proof of (3): From the fact that the unit $\hat{1}$ of $C(\hat{G})$ is $\frac{1}{h(\eta)} F(\eta)$ and Proposition 2.1 we immediately find that

$$\hat{\alpha}(\hat{1}) = \frac{1}{h(\eta)} (F \otimes \text{id})\alpha F^{-1}(\eta) = \frac{1}{h(\eta)} (F \otimes \text{id})\alpha(\eta).$$

This shows that $\hat{\alpha}(\hat{1}) = 1$ if and only if $\alpha(\eta) = \eta \otimes 1$.

Proof of (2): Remembering that antipodes of finite quantum groups are $*$-preserving maps we compute

$$\hat{\alpha} \circ * = \frac{1}{h(\eta)} (F \otimes \text{id}_B) \circ \alpha \circ \hat{F} \circ \hat{S} \circ * = \frac{1}{h(\eta)} (F \otimes \text{id}_B) \circ \alpha \circ \hat{F} \circ \hat{S}$$

and

$$(\cdot \otimes \cdot) \circ \hat{\alpha} = \frac{1}{h(\eta)} (\cdot \otimes \cdot) (F \otimes \text{id}_B) \circ \alpha \circ \hat{F} \circ \hat{S} = (F \otimes \text{id}_B) \circ (\cdot \otimes \cdot) \circ \alpha \circ \hat{F} \circ \hat{S},$$

so, as all the maps are invertible, we see that

$$(\cdot \otimes \cdot) \circ \hat{\alpha} = \hat{\alpha} \circ *$$

if and only if

$$\alpha \circ \cdot = (\cdot \otimes \cdot) \circ \alpha.$$

Proof of (4): We have

$$\hat{h} \otimes \text{id}_B \circ \hat{\alpha} = \frac{1}{h(\eta)} (\hat{h} \circ F) \otimes \text{id}_B \circ \alpha \circ \hat{F} \circ \hat{S} = (\epsilon \otimes \text{id}_B) \circ \alpha \circ \hat{F} \circ \hat{S}$$

and

$$\hat{h}(\cdot) \text{id}_B = (\hat{h} \circ \hat{S})(\cdot) \text{id}_B = (\epsilon \circ \hat{F} \circ \hat{S})(\cdot) \text{id}_B$$

so that

$$(\hat{h} \otimes \text{id}_B) \hat{\alpha} = \hat{h}(\cdot) \text{id}_B$$

if and only if

$$(\epsilon \otimes \text{id}_B) \alpha = \epsilon(\cdot) \text{id}_B.$$

Formula (2.2) is a consequence of Lemma 1.1, Equation (1.6) and the fact that the antipode is involutive.

**Proposition 2.3.** Let $B$ be a unital $C^*$-algebra and $\alpha: C(G) \to C(G) \otimes B$ represent a quantum family of invertible maps on $G$. Then the following conditions are equivalent:

1. $\alpha$ preserves the convolution multiplication, preserves the convolution adjoint and the Haar element (we will also say in short that $\alpha$ preserves the convolution structure);

2. $\hat{\alpha}$ represents a quantum family of invertible maps on $\hat{G}$.

Moreover in that case we have $\hat{\alpha} = \alpha$.

**Proof.** We claim that $\alpha$ satisfies Podleś condition implies that $\hat{\alpha}$ satisfies Podleś condition. Indeed, suppose that given $a \in C(G)$, there exists elements $a_1, \ldots, a_n \in C(G)$ and $q_1, \ldots, q_n \in B$ such that $\sum_{i=1}^n \alpha(S(a_i))(1 \otimes q_i) = a \otimes 1$. Then

$$\sum_{i=1}^n \hat{\alpha}(S F(a_i))(1 \otimes q_i) = \sum_{i=1}^n (F \otimes \text{id})\alpha(S(a_i))(1 \otimes q_i) = F(a) \otimes 1.$$

This proves the claim. Thus, the equivalence is an immediate consequence of Lemma 2.2.
The second statement follows from the fact that if \( \alpha \) preserves both the usual adjoint and the convolution adjoint, it also preserves the antipode (cf. (1.2), in fact, obviously preserving any two of these maps implies the preservation of the third) and formula (2.2).

**Definition 2.4.** Let \( B \) be a unital \( C^* \)-algebra and let \( \alpha : C(G) \to C(G) \otimes B \) represent a quantum family of invertible maps. We say that \( \alpha \) represents a quantum family of automorphisms of \( G \) (indexed by \( B \)) if the (equivalent) conditions from Proposition 2.3 hold.

**Corollary 2.5.** Let \( B \) be a unital \( C^* \)-algebra and \( \alpha : C(G) \to C(G) \otimes B \). Then the following conditions are equivalent

1. \( \alpha \) represents a quantum family of automorphisms of \( G \);
2. \( \hat{\alpha} \) represents a quantum family of automorphisms of \( \hat{G} \).

**Proof.** Assume that (1) holds. By Proposition 2.3 \( \hat{\alpha} = \alpha \) is a unital \( * \)-homomorphism satisfying Podleś condition, so in fact \( \hat{\alpha} \) represents a quantum family of automorphisms of \( \hat{G} \).

If (2) holds, then (by the same token as in the argument in the implication (1) \( \Rightarrow \) (2)) the map \( \beta = \hat{\alpha} \) represents a quantum family of automorphisms of \( G \); in particular \( \hat{\beta} = \beta \). But, by Lemma 2.2, \( \hat{\beta} = \alpha \), which completes the proof.

**Corollary 2.6.** Let \( B \) be a unital \( C^* \)-algebra and let \( \alpha : C(G) \to C(G) \otimes B \) represent a quantum family of automorphisms of \( G \). Then \( \alpha \) preserves the counit and the Haar state. Moreover,

\[
\hat{\alpha} = \frac{1}{h(\eta)}(\hat{S} \circ \hat{\mathcal{F}} \otimes \text{id}_B) \circ \alpha \circ \hat{\mathcal{F}}.
\]  

(2.5)

**Proof.** Observe first that the fact that \( \alpha(\eta) = \eta \otimes 1_B \) implies that \( \alpha \) preserves the counit. Indeed, if \( a \in C(G) \) then

\[
\eta \otimes \epsilon(a) 1_B = \alpha(\epsilon(a) \eta) = \alpha(\alpha \eta) = \alpha(a) \alpha(\eta) = \alpha(a) (\eta \otimes 1_B)
\]

\[
= \left(1_{C(G)} \otimes ((\epsilon \otimes 1_B) \alpha(a))\right) (\eta \otimes 1_B) = \eta \otimes ((\epsilon \otimes 1_B) \alpha(a)),
\]

and as \( \eta \neq 0 \) we see that \( \alpha \) preserves the counit. From Corollary 2.5 we deduce that \( \hat{\alpha} \) preserves the counit of \( \hat{G} \). But then a combination of Lemma 2.2 and the equality \( \hat{\alpha} = \alpha \) show that \( \alpha \) preserves the Haar state.

Formula (2.5) follows from the fact that \( \alpha \) preserves the antipode.

**Proposition 2.7.** Let \( \alpha : C(G) \to C(G) \otimes B \) represent a quantum family of automorphisms of \( G \) and assume that \( B \) is commutative and \( X \) is a compact space such that \( B = C(X) \). Then there is a family of Hopf \( * \)-algebra automorphisms \( \{\psi_x\}_{x \in X} \) of \( C(G) \) such that for any \( x \in X \), \( a \in C(G) \),

\[(\text{id} \otimes \xi_x) \alpha(a) = \psi_x(a),
\]

where \( \xi_x \) is the evaluation map \( B \ni f \mapsto f(x) \in \mathbb{C} \). Moreover, for a fixed \( a \in C(G) \) the elements \( \psi_x(a) \) depend continuously on \( x \).

If, in addition, \( G \) is a classical finite group then each \( \psi_x \) is an automorphism of \( G \).

**Proof.** If \( B = C(X) \) then for each \( x \in X \) we define \( \psi_x = (\text{id} \otimes \xi_x) \circ \alpha \). Then \( \psi_x : C(G) \to C(G) \) is a unital \( * \)-homomorphism and the map \( x \mapsto \psi_x(a) \) is continuous for any \( a \in C(G) \) (this is a standard fact).

Moreover, \( \psi_x \) is an automorphism of \( C(G) \). Indeed, if \( \psi_x(a) = 0 \) for some \( a \in C(G) \) then \( \psi_x(a^*a) = 0 \). Then \( h(a^*a) = h(\psi_x(a^*a)) = 0 \) because \( \alpha \) preserves the Haar measure of \( G \), i.e.

\[ (h \otimes \text{id}) \alpha(b) = h(b) 1, \quad (b \in C(G)) \]
As $h$ is faithful, we see that $a = 0$. This shows that $\ker \psi_x = \{0\}$, and so $\psi_x$ is a linear automorphism of the finite dimensional vector space $C(G)$.

The map $\psi_x$ is also a $*$-homomorphism for the convolution product and convolution adjoint on $C(G)$. This shows that $\psi_x \circ S = S \circ \psi_x$ (cf. formula (1.2)).

Now let us see that $\psi_x$ preserves the comultiplication. For any $a, b \in C(G)$ we have

$$a \ast b = \psi_x^{-1}(\psi_x(a) \ast \psi_x(b)).$$

Expanding this according to the definition of the convolution product yields

$$(h \otimes \text{id})\big((S \otimes \text{id})(\Delta(b))(a \otimes 1)\big) = (h \otimes \psi_x^{-1})\big((S \otimes \text{id})(\Delta(\psi_x(b)))(\psi_x(a) \otimes 1)\big).$$

(2.6)

Now since $h \circ \psi_x = h$ we can substitute $h \circ \psi_x^{-1}$ for $h$ on the right hand side and rewrite (2.6) as

$$(h \otimes \text{id})\big((S \otimes \text{id})(\Delta(b))(a \otimes 1)\big) = (h \otimes \text{id})(\psi_x^{-1} \otimes \psi_x^{-1})\big((S \otimes \text{id})(\Delta(\psi_x(b)))(\psi_x(a) \otimes 1)\big)$$

$$= (h \otimes \text{id})\big((S \otimes \text{id})(\psi_x^{-1} \otimes \psi_x^{-1})\Delta(\psi_x(b))(a \otimes 1)\big),$$

as $\psi_x^{-1}$ commutes with $S$. We arrive at

$$(ah \circ S)(\otimes \text{id})\big(\psi_x^{-1} \otimes \psi_x^{-1}\Delta(\psi_x(b))\big) = ((ah \circ S) \otimes \text{id})(\Delta(b)).$$

Taking into account the fact that $S$ is a linear automorphism of $C(G)$ and faithfulness of $h$, we see that as $a$ varies over $C(G)$ the functionals $ah \circ S$ fill the whole space $C(G)^*$. This immediately implies that

$$(\psi_x^{-1} \otimes \psi_x^{-1}) \circ \Delta \circ \psi_x = \Delta,$$

so that $\psi_x$ is a Hopf algebra automorphism of $C(G)$. □

**Definition 2.8.** Let $G$ be a finite quantum group and let $\text{QAUT}(G)$ denote the category of quantum families of automorphisms of $G$: its objects are pairs $(\alpha, B)$, where $B$ is a unital $C^*$-algebra and $\alpha: C(G) \to C(G) \otimes B$ represents a quantum family of automorphisms of $G$ (understood as in Definition 2.4) and a morphism from $(\alpha, B)$ to $(\alpha', B')$ is defined as a unital $*$-homomorphism from $B$ to $B'$ intertwining $\alpha$ and $\alpha'$.

In the next lemma we will use the notion of composition of quantum families of maps introduced in [So1], Section 3. We will only need this notion in the context of quantum families of maps $G \to G$: let $B$ and $C$ be $C^*$-algebras and let $\beta: C(G) \to C(G) \otimes B$ and $\gamma: C(G) \to C(G) \otimes C$ represent quantum families of maps $G \to G$. The composition of $\beta$ and $\gamma$ is by definition the quantum family of maps $G \to G$ represented by

$$(\beta \otimes \text{id}) \circ \gamma: C(G) \to C(G) \otimes B \otimes C = C(G) \otimes (B \otimes C).$$

We will denote the composition of $\beta$ and $\gamma$ by the symbol $\beta \Delta \gamma$.

**Lemma 2.9.** Let $B$ and $C$ be $C^*$-algebras and let $\beta: C(G) \to C(G) \otimes B$ and $\gamma: C(G) \to C(G) \otimes C$ represent quantum families of automorphisms of $G$. Then the composition $\beta \Delta \gamma: C(G) \to C(G) \otimes (B \otimes C)$ represents a quantum family of automorphisms of $G$.

**Proof.** The fact that a given $*$-homomorphism represents a quantum family of automorphisms of $C(G)$ means that it

- preserves the Haar measure of $G$,
- is a homomorphism with respect to the convolution product,
- is a $*$-map for the convolution adjoint on $C(G)$,
• satisfies the Podleś condition.

If $\beta: C(G) \to C(G) \otimes B$ and $\gamma: C(G) \to C(G) \otimes C$ satisfy the first three of the above three requirements then one can show by quite trivial direct computation that so does $\beta \Delta \gamma$. However even those simple computations can be avoided when we realize that

• invariance of a given state is preserved under composition of quantum families of maps ([So1], Proposition 14),

• $\beta \Delta \gamma$ is a $*$-homomorphism for the convolution multiplication and convolution adjoint on $C(G)$ by definition of the composition of quantum families.

Finally, one can show that composition of quantum families satisfying Podleś condition also satisfies Podleś condition ([SK]). In particular $\beta \Delta \gamma$ satisfies the Podleś condition, and so it represents a quantum family of automorphisms of $C(G)$.

We introduce one more piece of terminology.

Definition 2.10. Let $\mathbb{H}$ be a compact quantum group, $G$ a finite quantum group and $\alpha: C(G) \to C(G) \otimes C(\mathbb{H})$ be an action of $\mathbb{H}$ on $C(G)$: recall this means that $\alpha$ represents a quantum family of invertible maps on $G$ and the action equation holds:

$$(\text{id}_{C(G)} \otimes \Delta_{\mathbb{H}}) \circ \alpha = (\alpha \otimes \text{id}_{\mathbb{H}}) \circ \alpha.$$ 

If in addition $\alpha$ represents a quantum family of automorphisms of $G$ we say that $\alpha$ is an action of $\mathbb{H}$ on $G$ by (quantum) automorphisms.

The next theorem is the key existence result of the note.

Theorem 2.11. Let $G$ be a finite quantum group. The category $\text{qAUT}(G)$ admits a universal final object, $(B_u, \alpha_u)$. Moreover the algebra $B_u$ admits a unique structure of the algebra of continuous functions on a compact quantum group (to be denoted $\text{qAut}(G)$) such that $\alpha_u$ defines an action of $\text{qAut}(G)$ on $G$. The quantum group $\text{qAut}(G)$ will be called the quantum automorphism group of $G$: it is a quantum subgroup of the Wang’s quantum automorphism group of $(C(G), h)$.

Proof. Let $\mathbb{K}$ be Wang’s quantum automorphism group of $(C(G), h)$ – see [Wan] – and let $\beta: C(G) \to C(G) \otimes C(\mathbb{K})$ be its action on $G$. The universal property of $(\mathbb{K}, \beta)$ says that for any unital C*-algebra $B$ and any unital $*$-homomorphism $\beta: C(G) \to C(G) \otimes B$ preserving $h$ and satisfying the Podleś condition there exists a unique $\Lambda: C(\mathbb{K}) \to B$ such that $\alpha = (\text{id} \otimes \Lambda) \circ \beta$.

\footnote{For example take $a, b \in C(G)$ and let $\gamma(a) = \sum_i a_i \otimes x_i$, $\gamma(b) = \sum_j b_j \otimes y_j$ and $\beta(a_i \star b_j) = \sum_k c_{ij}^k \otimes z_k$. We have

$$(\beta \Delta \gamma)(a \star b) = (\beta \otimes \text{id})\gamma(a \star b) = (\beta \otimes \text{id})\gamma(\sum_{i,j} a_i \star b_j \otimes x_i y_j) = \sum_{i,j} \beta(a_i \star b_j) \otimes x_i y_j.$$}

Further, for each $i$ and $j$ let $\beta(a_i) = \sum_p u^i_p \otimes w_p$ and $\beta(b_j) = \sum_q v^j_q \otimes r_q$. As the map $\beta$ represents a quantum family of automorphisms of $G$, we have $\sum_k c_{ij}^k \otimes z_k = \sum_{p,q} u^i_p v^j_q \otimes w_p r_q$ for each $i, j$. Now denoting by $\mu$ and $m$ the convolution multiplication on $C(G)$ and the multiplication on $B \otimes C$ and by $\sigma$ the flip $B \otimes C \otimes C(G) \to C(G) \otimes B \otimes C$ we compute

$$(\mu \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\beta \Delta \gamma)(a) \otimes (\beta \Delta \gamma)(b)) \sum_{i,j, p,q} u^i_p v^j_q \otimes w_p r_q \otimes x_i y_j$$

$$= \sum_{i,j} \left(\sum_{p,q} u^i_p v^j_q \otimes w_p r_q \otimes x_i y_j \right) = \sum_{i,j,k} c_{ij}^k \otimes z_k \otimes x_i y_j = (\beta \Delta \gamma)(a \star b).$$
Denote by $m_{C(K)}$ the multiplication map $C(K) \otimes_{\text{alg}} C(K) \to C(K)$ and by $\mu$ the convolution multiplication $C(G) \otimes_{\text{alg}} C(G) \to C(G)$. Now let $S$ be the quotient of $C(K)$ by the smallest (closed two sided) ideal containing the following three sets:

$$\{(\omega \otimes \text{id})(\mu \otimes m_{C(K)})(\sigma \otimes \text{id})(\beta(a) \otimes \beta(b) - \beta(a \ast b)) \mid a, b \in C(G), \ \omega \in C(G)^* \}$$

$$\{(\omega \otimes \text{id})(\bullet \otimes \ast)(\beta(a) - \beta(a^*)) \mid a \in C(G), \ \omega \in C(G)^* \}$$

$$\{\epsilon \otimes \text{id})(\beta(a) - \epsilon(a)1 \mid a \in C(G) \}$$

where $\sigma$ denotes the flip map $C(K) \otimes C(G) \to C(G) \otimes C(K)$. Then let $\alpha = (\text{id} \otimes \pi) \circ \beta$, with $\pi$ the quotient map $C(K) \to S$. Clearly $\alpha$ represents a quantum family of automorphisms of $G$. Moreover it obviously has the following universal property: if $C$ is a unital $C^*$-algebra and $\gamma: C(G) \to C(G) \otimes C$ represents a quantum family of automorphisms of $G$ then there exists a unique $\Lambda: S \to C$ such that

$$\gamma = (\text{id} \otimes \Lambda) \circ \alpha.$$

Showing that there exists a unique $\Delta: S \to S \otimes S$ such that

$$(\text{id} \otimes \Delta) \circ \alpha = (\alpha \otimes \text{id}) \circ \alpha$$

is now quite standard: the right hand side of (2.7) represents a quantum family of automorphisms of $G$. Therefore it is of the form given by the left hand side of (2.7) for a unique $\Delta: S \to S \otimes S$. Coassociativity of $\Delta$ follows from associativity of the operation of composition of quantum families ([So1, Proposition 5]).

In the same way as in the proofs of [So1, Proposition 12, Theorem 16(6), Theorem 21(6)] one shows that the quotient map $q: C(K) \to S$ satisfies

$$(q \otimes q) \circ \Delta_K = \Delta \circ q.$$ 

Since $q$ is a continuous surjection, the density conditions

$$\Delta(S)(1 \otimes S) \text{ and } (S \otimes 1)\Delta(S)$$

hold and so we can define a compact quantum group $q\text{Aut}(G)$ by putting $C(q\text{Aut}(G)) = S$. We have already shown that $q\text{Aut}(G)$ is the universal object in the category $Q\text{AUT}(G)$. □

**Proposition 2.12.** The compact quantum groups $q\text{Aut}(G)$ and $q\text{Aut}(\hat{G})$ are (canonically) isomorphic.

**Proof.** Follows from Corollary 2.5. □

### 3. Quantum Automorphisms of a Finite Group

Let $\Gamma$ be a finite group. Denote by $\mathcal{A}$ the vector space of all complex valued functions on $\Gamma$. We will use its canonical basis $\{\delta_x\}_{x \in \Gamma}$. In what follows we shall study a linear map $\alpha: \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}$ is a unital $C^*$-algebra. The map $\alpha$ defines a matrix $P \in M_{|\Gamma|}(\mathcal{B})$ by

$$\alpha(\delta_y) = \sum_{x \in \Gamma} \delta_x \otimes p_{x,y}. \quad (3.1)$$

The following three propositions follow from elementary calculations.
Proposition 3.1. The map $\alpha$ is a unital $*$-homomorphism for the pointwise $*$-algebra structure on $\mathcal{A}$ if and only if
\begin{align}
p_{x,y} &= p_{x,y}, \quad x, y, \in \Gamma, \quad (3.2a) \\
p_{x,y}^2 &= p_{x,y}, \quad x, y, \in \Gamma, \quad (3.2b) \\
\sum_y p_{x,y} &= 1, \quad x \in \Gamma. \quad (3.2c)
\end{align}

Proposition 3.2. $\alpha$ preserves the convolution product on $\mathcal{A}$ if and only if
\[ p_{x,yz} = \sum_{u \in \Gamma} p_{u,y} p_{u^{-1}x,z}, \quad x, y, z \in \Gamma. \quad (3.3) \]

Proposition 3.3. $\alpha$ is a $*$-map for the involution $\cdot$ on $\mathcal{A}$ if and only if
\[ p_{x,y}^* = p_{x^{-1},y^{-1}}, \quad x, y \in \Gamma. \]

By results of [Wan] we know that if, in fact, $\mathcal{B} = C(\mathbb{G})$, where $\mathbb{G}$ is a compact quantum group and $\alpha$ is an action of $\mathbb{G}$ on $\Gamma$, then $\alpha$ preserves the state $h$ on $\mathcal{A}$ defined by $h(\delta_x) = \frac{1}{|\mathbb{G}|}$ for all $x \in \Gamma$. It follows that
\[ \sum_x p_{x,y} = 1, \quad y \in \Gamma. \quad (3.4) \]

The next proposition shows that if only the action of a compact quantum group $\mathbb{G}$ on $\Gamma$ is faithful (i.e. the algebra spanned by the action’s ‘coefficients’ is dense in $C(\mathbb{G})$) and preserves the convolution product, it must be an action by automorphisms.

Proposition 3.4. Assume that $\alpha$ is a faithful action of $\mathbb{G}$ on $\Gamma$ and that $\alpha$ preserves the convolution product on $\mathcal{A}$. Then
\[ p_{e,y} = \delta_{y,e} 1, \quad y \in \Gamma, \]
\[ p_{x,e} = \delta_{x,e} 1, \quad x \in \Gamma. \]

Further $\alpha$ is an action of $\mathbb{G}$ on $\Gamma$ by quantum automorphisms.

Proof. The assumption that $\alpha$ is an action of $\mathbb{G}$ on $\Gamma$ means that the matrix $(p_{x,y})_{x,y \in \Gamma}$ is a magic unitary (i.e. a unitary matrix, whose entries are projections); in particular projections in each row and column are mutually orthogonal. Faithfulness means that the set $(p_{x,y})_{x,y \in \Gamma}$ generates $C(\mathbb{G})$ as a C*-algebra.

Note that once this is known, the formula (3.3) is equivalent to the following equality:
\[ p_{u,y} p_{x,yz} = p_{u,y} p_{u^{-1}x,z}, \quad u, x, y, z \in \Gamma. \quad (3.5) \]

Inserting $z = e$ and $x = u$ in the above equality we get $p_{u,y} p_{u,y} = p_{u,y} p_{e,e}$, i.e. $p_{u,y} = p_{u,y} p_{e,e}$. As the projections we consider are self-adjoint, we also have $p_{u,y} = p_{e,e} p_{u,y}$. Putting $u = e$ in this equation and summing over $y$, we have $p_{e,e} = 1$. Then, since $\sum_y p_{e,y} = 1 = \sum y p_{y,e}$, we have $p_{e,x} = p_{x,e} = \delta_{e,x} 1$. Therefore, for each $x \in \Gamma$
\[ p_{e,x} = \delta_{e,x} 1, \]
\[ p_{x,e} = \delta_{e,x} 1. \]

The formulas above imply in particular that $\alpha$ preserves the Haar element of $\mathbb{C}[\Gamma]$ (i.e. $\delta_e$).

Return now to formula (3.5) and put $x = e$, $z = y^{-1}$. This yields $p_{u,y} p_{e,e} = p_{u,y} p_{u^{-1},y^{-1}}$, so also
\[ p_{u,y} = p_{u,y} p_{u^{-1},y^{-1}}, \quad u, y \in \Gamma. \]
Replacing \( u \) by \( u^{-1} \) and \( y \) by \( y^{-1} \) we see that
\[
p_{u^{-1}, y^{-1}} = p_{u^{-1}, y^{-1}} p_{u, y}
\]
and self-adjointness yields
\[
p_{u^{-1}, y^{-1}} = p_{u, y}, \quad u, y \in \Gamma.
\]
This shows that \( \alpha \) is a unital *-homomorphism with respect to the convolution structure on \( \mathcal{A} \). Together with results of Section 2 it ends the proof. \( \square \)

Recall that the counits \( \epsilon \) and \( \hat{\epsilon} \) and Haar measures \( h \) and \( \hat{h} \) of \((\mathcal{A}, \Delta)\) and \((\mathcal{A}, \hat{\Delta})\) respectively are given by
\[
\epsilon(\delta_x) = \hat{h}(\delta_x) = \delta_{x,e}, \quad x \in \Gamma.
\]
We noted above that \( h \) (and thus \( \hat{\epsilon} \)) is invariant for \( \alpha \). It follows from the form of the matrix \( P \) that so is \( \epsilon \) (and thus \( \hat{h} \)).

**Corollary 3.5.** If \( \alpha \) is an action of \( G \) on \( \Gamma \) and a *-homomorphism for the convolution structure on \( \mathcal{A} \) then it leaves \( \epsilon \) invariant.

The next proposition shows that in Proposition 3.4 one can replace the assumption of faithfulness of the action by its unitality with respect to the convolution structure (i.e. the fact that it preserves the Haar element).

**Proposition 3.6.** Assume that \( \alpha \) is an action of \( G \) on \( \Gamma \) and that \( \alpha \) is a unital homomorphism for the convolution structure on \( \mathcal{A} \). Then \( \alpha \) is a *-homomorphism for the convolution structure on \( \mathcal{A} \); in other words, \( \alpha \) is an action of \( G \) on \( \Gamma \) by quantum automorphisms.

**Proof.** The fact that \( \alpha \) is unital for the convolution structure means that in addition to (3.3) we also have
\[
\sum_x \delta_x \otimes p_{x,e} = \delta_e \otimes 1.
\]
It follows that \( p_{x,e} = \delta_{x,e} 1 \). On the other hand (3.3) shows that for any fixed \( z \in \Gamma \)
\[
p_{x,e} = \sum_y p_{y,z} p_{y^{-1},x,z^{-1}}.
\]
Inserting \( x = e \) we obtain
\[
\sum_y p_{y,z} p_{y^{-1},z^{-1}} = 1
\]
and multiplying from the right by \( p_{u,z^{-1}} \) we conclude by (3.2a), (3.2b) and (3.4) that
\[
p_{u^{-1},z} p_{u,z^{-1}} = p_{u,z}.
\]
Thus \( p_{u,z^{-1}} \leq p_{u^{-1},z} \) and substitution of \((u^{-1}, z^{-1})\) for \((u, z)\) shows that
\[
p_{u,z^{-1}} = p_{u^{-1},z}, \quad u, z \in \Gamma.
\]
By Proposition 3.3 \( \alpha \) is a *-homomorphism for the convolution structure on \( \mathcal{A} \). \( \square \)
3.1. Quantum automorphisms and order. In this subsection we show that, as in the classical case, quantum automorphisms in a natural sense preserve the order of elements and show that certain quantum automorphism groups are classical. Recall that the order of an element $x \in \Gamma$ is defined as $\text{ord}(x) = \min\{n \in \mathbb{N} \mid x^n = e\}$.

**Proposition 3.7.** Let $G$ be a compact quantum group and let $\alpha: A \to A \otimes C(\hat{\Gamma})$ given by the prescription (3.1) define an action of $G$ by quantum automorphism. Then the following conditions are satisfied:

1. $p_{x,y} = 0$ if $x, y \in \Gamma$, $\text{ord}(x) \neq \text{ord}(y)$;
2. $p_{x,y}p_{x^n,z} = p_{x^n,z}p_{x,y}$, for $x, y, z \in \Gamma$, $n \in \mathbb{N}$;
3. $p_{y,x}p_{z,y} = p_{z,x}p_{y,x}$, for $x, y, z \in \Gamma$, $n \in \mathbb{N}$;
4. $p_{x^n,y} \geq p_{x,y}$ for $x, y \in \Gamma$, $n \in \mathbb{N}$.

**Proof.** We begin by recalling the formula (3.5) satisfied by the elements $\{p_{x,y} \mid x, y \in \Gamma\}$ determining the action $\alpha$. Note first that this can be rewritten as

$$p_{x,y}p_{z,u} = p_{x,y}p_{xz,yu}, \quad u, x, y, z \in \Gamma. \quad (3.7)$$

Note that the last expression is symmetric with respect to the swapping of rows and columns of the matrix $(p_{x,y})_{x,y \in \Gamma}$.

Apply it with $z = x$. This gives $p_{x,y}p_{x,u} = p_{x,y}p_{x^2,yu}$, so we obtain

$$p_{x,y}p_{x^2,yu} = \delta_{yu}p_{x,y}, \quad x, y, u \in \Gamma.$$  

The expression on the left can be further rewritten using (3.7), so that we get

$$p_{x,y}p_{x^3,y^2u} = \delta_{yu}p_{x,y}, \quad x, y, u \in \Gamma,$$

and further inductively

$$p_{x,y}p_{x^k,y^ku} = \delta_{yu}p_{x,y}, \quad x, y, u \in \Gamma, \quad k \in \mathbb{N}.$$  

Note that by selfadjointness the same holds with the projections on the left switching sides (in particular, the two projections featuring on the left commute).

If then say $\text{ord}(x) = l < \text{ord}(y)$ we obtain (putting $y = u$)

$$p_{x,y} = p_{x,y}p_{x^l,y^l} = p_{x,y}p_{e,y^l} = 0.$$  

This proves statement (i) in the proposition. $\square$

**Corollary 3.8.** Let $\Gamma = \mathbb{Z}_p$, where $p$ is prime. Then the quantum automorphism group of $\Gamma$ is the classical group $\mathbb{Z}_p$.

**Proof.** It suffices to note that any non-zero element of $\Gamma$ generates $\Gamma$ and use (ii) in Proposition 3.7. $\square$

**Remark 3.9.** The arguments of the last proposition allow us to show that the quantum automorphism group is the classical automorphism group for many groups $\Gamma$ (for example $\mathbb{Z}_p$ for $p \leq 11$).

3.2. Quantum automorphisms of a dual of a finite group. Consider now another, in a sense converse, approach to the problem studied earlier in this section. Let $\Gamma$ be again a finite group and assume that a compact quantum group $G$ acts on the dual of $\Gamma$. We want to once again identify the weakest conditions on the action, so that in fact in induces the action of $G$ on $\Gamma$ itself (i.e. provide a dual counterpart of Proposition 3.4). Here in fact the situation turns out to be far more satisfactory – already the preservation of the convolution product by a given action on $\hat{\Gamma}$ implies this is an action by automorphisms.
Theorem 3.10. Suppose that $\Gamma$ is a finite group, $\mathbb{G}$ is a compact quantum group and let $\alpha : C[\Gamma] \to C[\Gamma] \otimes C(\mathbb{G})$ be an action of $\mathbb{G}$ on $\hat{\Gamma}$ which is a homomorphism for the convolution product of $C[\Gamma]$. Then $\alpha$ is an action on $\hat{\Gamma}$ by quantum automorphisms.

Proof. Let $\alpha$ as above be given by the formula

$$\alpha(x) = \sum_{y \in \Gamma} y \otimes u_{y,x}, \quad x \in \Gamma.$$ 

By the well-known fact, going back to the thesis of P. Podles (see [So2] for more references), concerning the decomposition of actions of compact quantum groups into isotypical components, it follows that the elements $u_{y,x}$ belong to $\text{Pol}(\mathbb{G})$, the canonical dense Hopf $*$-subalgebra of $C(\mathbb{G})$. Further the application of the counit $\epsilon$ of $\text{Pol}(\mathbb{G})$ yields

$$\epsilon(u_{y,x}) = \delta_{x,y}, \quad x, y \in \Gamma. \quad (3.8)$$

Further by the action equation we also have

$$\Delta_{\mathbb{G}}(u_{x,y}) = \sum_{z \in \Gamma} u_{x,z} \otimes u_{z,y}, \quad x, y \in \Gamma. \quad (3.9)$$

The fact that $\alpha$ is a homomorphism with respect to the convolution product implies that each $u_{x,y}$ is idempotent. Further, an easy norm argument implies that these elements are also contractive, and therefore self-adjoint, which means that $\alpha$ preserves the convolution adjoint. Consider then the fact that each $x \in \Gamma$ is unitary (viewed as an element of $C[\Gamma]$). This implies

$$1 \otimes 1 = \alpha(x^*x) = \alpha(x)^*\alpha(x) = \sum_{y,z \in \Gamma} y^*z \otimes u_{y,x}^* u_{z,x} = \sum_{y,z \in \Gamma} y^*z \otimes u_{y,x} u_{z,x}.$$ 

In particular

$$\sum_{y \in \Gamma} u_{y,x} = \sum_{y \in \Gamma} u_{x,y} u_{y,x} = 1.$$ 

Further, as for $x, z \in \Gamma$, $x \neq z$, we have $x \star z = 0$, it follows that $u_{y,x} u_{y,z} = 0$ for each $y \in \Gamma$. Thus the matrix $(u_{x,y})_{x,y \in \Gamma}$ is a matrix of self-adjoint projections mutually orthogonal in each row and column and such that the sum of entries in each row is equal to 1. It remains to see that the corresponding sum in each column is equal to one (as then it will follow that $\alpha$ preserves the convolution unit). To this end introduce a new map, $\beta : C(\Gamma) \to C(\Gamma) \otimes C(\mathbb{G}^{\text{op}})$, given by the formula:

$$\beta(\delta_x) = \sum_{y \in \Gamma} \delta_y \otimes u_{x,y}, \quad x \in \Gamma,$$

where $C(\mathbb{G}^{\text{op}})$ is as a C*-algebra isomorphic to $C(\mathbb{G})$, but has a ‘tensor flipped’ coproduct. It is easy to check that $\beta$ is a unital $*$-homomorphism and it satisfies the action equation. We only verify the latter (choosing first $x \in \Gamma$):

$$(\text{id}_{C(\Gamma)} \otimes \Delta_{\mathbb{G}^{\text{op}}}) \beta(\delta_x) = \sum_{y \in \Gamma} \delta_y \otimes \Delta_{\mathbb{G}^{\text{op}}}(u_{x,y}) = \sum_{y \in \Gamma} \delta_y \otimes \left( \sum_{z \in \Gamma} u_{z,y} \otimes u_{x,z} \right)$$

$$= \sum_{z \in \Gamma} \beta(\delta_z) \otimes u_{x,z} = (\beta \otimes \text{id}_{\mathbb{G}^{\text{op}}}) \beta(\delta_x),$$

where in the second equality we used (3.9). It remains to notice that the identity map identifies $\text{Pol}(\mathbb{G})$ with $\text{Pol}(\mathbb{G}^{\text{op}})$ and that the counit of $\text{Pol}(\mathbb{G}^{\text{op}})$ coincides with that of $\text{Pol}(\mathbb{G})$. Thus (3.8) implies that we have

$$(\text{id}_{C(\Gamma)} \otimes \epsilon) \circ \beta = \text{id}_{C(\Gamma)}.$$
By Remark 2.3 of [So₂] it follows that $\beta$ is an action of $\mathbb{G}^{\text{op}}$ on $C(\Gamma)$. Thus, by [Wan] (we used this argument already in [3.4]), it must preserve the uniform measure on $\Gamma$ – and this means that $\sum_{x \in \Gamma} u_{y,x} = 1$ and the proof is finished. $\square$

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