The basis of quantum mechanics’ compatibility with relativity—whose impairment gives rise to the Klein-Gordon and Dirac equations

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Abstract

Solitary-particle quantum mechanics’ inherent compatibility with special relativity is implicit in Schrödinger’s postulated wave-function rule for the operator quantization of the particle’s canonical three-momentum, taken together with his famed time-dependent wave-function equation that analogously treats the operator quantization of its Hamiltonian. The resulting formally four-vector equation system assures proper relativistic covariance for any solitary-particle Hamiltonian operator which, together with its canonical three-momentum operator, is a Lorentz-covariant four-vector operator. This, of course, is always the case for the quantization of the Hamiltonian of a properly relativistic classical theory, so the strong correspondence principle definitely remains valid in the relativistic domain. Klein-Gordon theory impairs this four-vector equation by iterating and contracting it, thereby injecting extraneous negative-energy solutions that are not orthogonal to their positive-energy counterparts of the same momentum, thus destroying the basis of the quantum probability interpretation. Klein-Gordon theory, which thus depends on the square of the Hamiltonian operator, is as well thereby cut adrift from Heisenberg’s equations of motion. Dirac theory confuses the space-time symmetry of the four-vector equation system with such symmetry for its time component alone, which it fatuously imposes, thereby breaching the strong correspondence principle for the free particle and imposing the starkly unphysical momentum-independence of velocity. Physically sensible alternatives, with external electromagnetic fields, to the Klein-Gordon and Dirac equations are derived, and the simple, elegant symmetry-based approach to antiparticles is pointed out.
Introduction: quantum mechanics’ inherent compatibility with relativity

The inherent compatibility of solitary-particle quantum mechanics with special relativity is a straightforward consequence Schrödinger’s two basic postulates for the wave function [1, 2], i.e., for the quantum state vector in the Schrödinger picture in configuration representation, namely $\langle \mathbf{r} | \psi(t) \rangle$. The first Schrödinger wave-function postulate is his rule for the operator quantization of the particle’s canonical three-momentum,

$$-i\hbar \nabla_{\mathbf{r}}(\langle \mathbf{r} | \psi(t) \rangle) = \langle \mathbf{r} | \mathbf{p} | \psi(t) \rangle, \quad (1a)$$

which is as well, of course, a result of Dirac’s postulated canonical commutation relation [3]. The second Schrödinger wave-function postulate is his famed time-dependent wave equation [1, 3, 2],

$$i\hbar \partial(\langle \mathbf{r} | \psi(t) \rangle)/\partial t = \langle \mathbf{r} | \mathbf{H} | \psi(t) \rangle, \quad (1b)$$

which formally treats the operator quantization of the particle’s Hamiltonian in a manner analogous to the way Eq. (1a) treats the operator quantization of the particle’s canonical three-momentum. The straightforward theoretical physics implication of Eqs. (1a) and (1b) is simply that the operators $\mathbf{p}$ and $\mathbf{H}$ are the generators of the wave function’s infinitesimal space and time translations, respectively. Therefore, in anticipation of the restriction on such generators which special relativity imposes, these two equations are usefully combined into the single formally four-vector Schrödinger equation for the wave function,

$$i\hbar \partial(\langle \mathbf{r} | \psi(t) \rangle)/\partial x_{\mu} = \langle \mathbf{r} | \mathbf{p}^\mu | \psi(t) \rangle, \quad (1c)$$

where the contravariant four-vector space-time partial derivative operator $\partial/\partial x_{\mu}$ is defined as $\partial/\partial x_{\mu} \equiv (c^{-1}\partial/\partial t, -\nabla_{\mathbf{r}})$, and the formal “contravariant four-vector” energy-momentum operator $\mathbf{p}^\mu$ is defined as $\mathbf{p}^\mu \equiv (\mathbf{H}/c, \mathbf{p})$. Since special relativity requires the contravariant space-time partial derivative four-vector operator $\partial/\partial x_{\mu}$ to transform between inertial frames in Lorentz-covariant fashion, it is apparent from Eq. (1c) that the Hamiltonian operator $\mathbf{H}$ will be compatible with special relativity if it is related to the canonical three-momentum operator $\mathbf{p}$ in such a way that also makes the energy-momentum operator $\mathbf{p}^\mu$ a contravariant four-vector which transforms between inertial frames in Lorentz-covariant fashion. This property of the Hamiltonian operator will, of course, be satisfied automatically if it is the quantization of the Hamiltonian of a properly relativistic classical theory. Therefore the strong correspondence principle definitely remains valid in the relativistic domain.

Now for a completely free solitary particle of nonzero mass $m$, the logic of the Lorentz transformation from its rest frame, where it has four-momentum $(mc, 0)$, to a frame where it has velocity $\mathbf{v}$ (where $|\mathbf{v}| < c$) leaves no freedom at all in the choice of its classical Hamiltonian. That Lorentz boost takes this particle’s four-momentum to,

$$(mc(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}, m\mathbf{v}(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}) = (E(\mathbf{v})/c, \mathbf{p}(\mathbf{v})), \quad (2a)$$

which, together with the identity,

$$mc^2(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}} = \sqrt{m^2c^4 + |m\mathbf{v}|^2(1 - |\mathbf{v}|^2/c^2)^{-1}}, \quad (2b)$$

implies that,

$$E(\mathbf{v}) = \sqrt{m^2c^4 + |m\mathbf{v}|^2} = H_{\text{free}}(\mathbf{p}(\mathbf{v})). \quad (2c)$$

Therefore, for the completely free relativistic solitary particle of nonzero mass $m$, theoretically systematic,
conservative adherence to the strong correspondence principle flatly determines the relativistic Hamiltonian operator to be the square-root operator,

\[ \hat{H}_{\text{free}} = \sqrt{m^2c^4 + |\vec{p}|^2}. \]  

(3)

This conclusion even extends to the free spin-\( \frac{1}{2} \) particle of nonzero mass: notwithstanding that spin-\( \frac{1}{2} \) itself is a nonclassical attribute, the nonrelativistic Pauli Hamiltonian operator for such a particle automatically reduces to the usual nonrelativistic purely kinetic-energy Hamiltonian operator in the free-particle limit, and one can always find an inertial frame of reference in which a free particle of nonzero mass is completely nonrelativistic, i.e., for the completely free particle one can always find an inertial frame of reference in which the nonzero-mass relativistic free-particle Hamiltonian operator \( \sqrt{m^2c^4 + |\vec{p}|^2} \) arbitrarily well approximates \( mc^2 + |\vec{p}|^2/(2m) \), which is the Pauli free-particle Hamiltonian operator, offset by the merely constant rest-mass energy term \( mc^2 \).

Since, as we shall shortly see, the Dirac free-particle Hamiltonian operator is very much at odds with the relativistic free-particle square-root Hamiltonian operator of Eq. (3) \( ^2 \), even notwithstanding the complete compatibility of Eq. (3) with the free-particle Pauli theory, it will be imperative to understand exactly how and why the Dirac Hamiltonian operator comes to be in conflict with the fundamental requirement of relativistic quantum mechanics, namely that \( \hat{p}^\mu = (\hat{H}/c, \hat{p}) \) must transform between inertial frames as a Lorentz-covariant four-vector. First, however, we turn to analysis of the Klein-Gordon theory, which rejects the fundamental quantum mechanical Eq. (1a) in that precise form, instead substituting in its place Eq. (1a) together with a once-iterated and Lorentz-contrasted version of Eq. (1c).

Klein-Gordon theory’s impairment of quantum mechanics

Very strongly motivated by considerations of perceived calculational ease, which we briefly discuss in the next section, rather than by those of quantum mechanics, Klein, Gordon and Schrödinger rejected the natural time-dependent Schrödinger equation of Eq. (1b) in favor of a once-iterated and then Lorentz-contrasted Lorentz-scalar version of Eq. (1c) \( ^2 \), which yields,

\[ \partial^2(\langle r|\psi(t)\rangle)/(\partial x^\alpha \partial x_\mu) + (\langle r|\hat{p}_\mu \hat{p}^\mu|\psi(t)\rangle)/\hbar^2 = 0, \]  

(4a)

where, of course,

\[ \hat{p}_\mu \hat{p}^\mu = (\hat{H}^2 - |\hat{p}|^2)/c^2, \]  

(4b)

so that in the special case of the free particle, where \( \hat{H} = \hat{H}_{\text{free}} \), which is given by Eq. (3), it is readily seen that the general Klein-Gordon equation of Eq. (4a) above reduces to,

\[ (\partial^2/(\partial x^\alpha \partial x_\mu) + (mc/\hbar)^2)(\langle r|\psi(t)\rangle) = 0. \]  

(5)

To each stationary eigensolution \( e^{-i\sqrt{m^2c^4 + |\vec{p}|^2} t/\hbar}\langle r|\psi(t)\rangle \) of eigenmomentum \( \vec{p} \) of the natural time-dependent relativistic free solitary-particle Schrödinger equation, which is Eq. (1b) for the case that the Hamiltonian operator \( \hat{H} \) is equal to \( \hat{H}_{\text{free}} \), Eq. (5) adds an extraneous negative-energy partner solution \( e^{+i\sqrt{m^2c^4 + |\vec{p}|^2} t/\hbar}\langle r|\psi(t)\rangle \) of the same momentum, whose sole reason for existing is the entirely gratuitous iteration of Eq. (1c)!: These completely extraneous negative “free solitary-particle” energies, \( -\sqrt{m^2c^4 + |\vec{p}|^2} \), do not correspond to anything that exists in the classical dynamics of a free relativistic solitary particle, and by their negatively unbounded character threaten to spawn unstable runaway phenomena should the free Klein-Gordon equation be sufficiently perturbed (the Klein paradox) \( ^2 \). Due to the fact that the Klein-Gordon equation lacks a corresponding Hamiltonian—it depends on only the square of a Hamiltonian, as is seen from Eq. (4b)—it turns
out, as is easily verified, that the two solutions of the same momentum \( p \) which have opposite-sign energies, i.e., \( \pm \sqrt{m^2c^4 + |p|^2} \), fail to be orthogonal to each other, which outright violates a key property of orthodox quantum mechanics! Without this property the probability interpretation of quantum mechanics cannot be sustained, and the Klein-Gordon equation is unsurprisingly diseased in that regard, yielding, inter alia, negative probabilities [2].

Furthermore, Klein-Gordon theory, which depends on the square of a Hamiltonian operator (see Eq. (4b)) rather than on the Hamiltonian operator itself, is thereby cut adrift from the normal quantum mechanical relationship to the Heisenberg picture, Heisenberg’s equations of motion and the Ehrenfest theorem. In a nutshell, substitution of the gratuitous iteration of Eq. (1c) for the Eq. (1b) part of Eq. (1c), which is exactly what Klein-Gordon theory does, grievously impairs and degrades the unexceptionable quantum-mechanical nature of Eq. (1c) itself! The need to carefully respect all the components of Eq. (1c) rather than to opportunistically try to “bend” aspects of its time component (i.e., Eq. (1b)) “at the edges” unfortunately did not register at all with Dirac ahead of his attempt to “repair” the problems of the Klein-Gordon theory.

**Dirac theory’s fatuous imposition of “space-time coordinate symmetry”**

The probability disease of the Klein-Gordon theory prompted Dirac to sensibly reinstate Eq. (1b), the time-dependent Schrödinger equation. Very unfortunately, being motivated above all else by the same non-physics considerations of perceived calculational ease as Klein, Gordon and Schrödinger [2, 5], Dirac hit upon a completely misguided but nonetheless plausible-sounding “reason”, ostensibly emanating from the “relativistic need” for Eq. (1b) to by itself exhibit “space-time coordinate symmetry” [6, 7, 2], to linearize the square-root Hamiltonian operator \( \hat{H}_{\text{free}} \) of Eq. (3) for the nonzero-mass free particle [8, 6, 5], notwithstanding the fact that Eqs. (2) effectively produce the specifically square-root form of \( \hat{H}_{\text{free}} \) from nothing more than the nonzero-mass free particle’s Lorentz transformation properties!

Unfortunately not being aware of the specifically four-vector form of Eq. (1c), Dirac focused myopically on only Eq. (1b), the time-dependent Schrödinger equation, whose theoretical physics essence is, of course, that the Hamiltonian operator is the generator of infinitesimal time translations of the wave function. This theoretical-physics core essence of Eq. (1b) is, in isolation, supremely indifferent to the typical space-time coordinate symmetry that is such an ubiquitous characteristic of special relativity. It is only on contemplating the whole of the four-vector equation system of Eq. (1c), which Dirac was not aware enough to do, that its global symmetry between space and time coordinates snaps immediately into focus. Very unfortunately deprived of this revelation, and intently focused on just Eq. (1b), Dirac “concluded” that it must of itself be compelled to manifest the “missing” space-time coordinate symmetry! If Dirac had even been fully cognizant of the core theoretical physics role of Eq. (1b), i.e., that of putting the Hamiltonian operator in the driver’s seat of infinitesimal time translations and only time translations, of the wave function, he definitely would have been very much less certain that forcing Eq. (1b) to manifest space-time coordinate symmetry was at all a sensible idea! Such salutary doubts and second thoughts were, however, unfortunately never to occur to Dirac.

So because Eq. (1b) is linear in the time partial derivative operator, Dirac shallowly concluded that “space-time coordinate symmetry” compels it’s relativistic version to also be linear in the space gradient, which implies that the Hamiltonian operator on its right-hand side is likewise linear in the particle three-momentum. Had he been thinking at all about the physical implications of this, Dirac would have quickly noticed a plethora of red flags inherent in any such linearity. In the nonrelativistic limit of small particle three-momentum, it is obvious that the Hamiltonian must become a quadratic function of the particle three-momentum, which is a mathematical impossibility for a Hamiltonian linear in the three-momentum! Note, however, that the relativistic square-root Hamiltonian \( \hat{H}_{\text{free}} \) of Eq. (3), from whose clutches Dirac, like Klein, Gordon, Schrödinger before him, was very strongly resolved, for non-physics reasons of calculational ease, to escape, in fact passes this small three-momentum quadratic-dependence test with flying colors! Furthermore, since the square-root Hamiltonian operator’s classical precursor \( \hat{H}_{\text{free}} \) is established beyond any shadow of a doubt for a nonzero-mass free particle (e.g., see Eqs. (2)), Dirac was effectively boxed into the very fraught position of implicitly disowning the strong correspondence principle in order to cling to his linearized Hamiltonian!
it is an elementary observation from the Heisenberg equation of motion that any Hamiltonian which is linear in the particle three-momentum operator produces a particle velocity operator (and thus also a particle speed operator) which is completely independent of the particle three-momentum operator! This obviously cannot make sense in the free-particle nonrelativistic limit, where it is clear that the particle velocity operator must be proportional to the particle momentum operator, i.e., $\mathbf{d}\mathbf{r}/dt = \mathbf{p}/m$. That free-particle velocity should be independent of free-particle momentum is in fact totally devoid of physical sense in any regime pertaining to the relativistic free particle: it is entirely clear that relativistic free-particle velocity and momentum are always directionally aligned (this even is true for the ultra-relativistic massless photon).

In utter contrast to this miasma of impending physically senseless “predictions” irrespective of what the coefficients of the three-momentum operator in the linearized Dirac Hamiltonian are ultimately chosen to be, the Heisenberg equation of motion in conjunction with the free-particle relativistic square-root $\hat{H}_{\text{free}}$ Hamiltonian operator of Eq. (3) yields only physically impeccable relativistic free-particle results, including, in particular, the result for the relativistic velocity operator! We shall, of course, be adhering to Dirac’s choice of coefficients for the components of the three-momentum in his linearized Hamiltonian, but that particular choice doesn’t alleviate any of the unphysical results just pointed out; in fact, it turns out to sharpen them, with the momentum-independent particle speed coming out to be a universal $c$-number whose value is more than 70% greater than that of light! As if this were not enough, the Dirac choice of coefficients (which turns out to involve mutually anticommuting matrices whose squares are unity) results in a staggering violation of Newton’s first law of motion, with the free electron predicted to have a spontaneous “Compton acceleration” whose minimum magnitude, of order $m_e c^3/\hbar$, is around $10^{28} g$, where $g$ is the acceleration of gravity at the earth surface! Almost needless to say, the relativistic free-particle square-root Hamiltonian operator $\hat{H}_{\text{free}}$ of Eq. (3), in conjunction with Heisenberg’s equation of motion yields, in flawless contrast, utterly strict adherence to Newton’s first law of motion!

In fact, the only obvious “success” that the Dirac theory’s completely misdirected imposition of space-time coordinate symmetry on Eq. (1b) can claim is the non-physics one of calculational ease, which, as we have pointed out, was its overriding motivation from the very start. In configuration representation, the physically sensible relativistic free-particle square root Hamiltonian operator $\hat{H}_{\text{free}}$ of Eq. (3) is a non-local integral operator, which certainly quashes any hoped-for “separation of variables for partial differential equations” technology for solving the relativistic version of the hydrogen atom. Contemplation of this fact of physics seems to have thoroughly rattled Klein, Gordon, Schrödinger and, in due course, Dirac. Quite desperately wanting that standard partial differential equation technology to still be applicable in the relativistic domain, they first invented the quantum-mechanically deficient Klein-Gordon theory and then Dirac’s astonishingly unphysical linearized Hamiltonian. Schrödinger, the inventor of bound-state perturbation theory, ought to have realized that his perturbation method is very well suited to the patently small relativistic corrections to the nonrelativistic model of the hydrogen atom, and called off the jointly shared slighting of theoretical-physics best practice which was tacit in the above-noted quite desperate efforts to preserve the applicability of standard partial differential equation solution technologies in the relativistic domain. To be sure, the very calculational ease that is the key aspect of Dirac’s unphysical linearized Hamiltonian also makes it a relatively easy target to pick apart in detail!

We turn now to Dirac’s choice of coefficients for his linearized relativistic mass-$m$ free-particle Hamiltonian operator $\hat{H}_D$ and the consequences of that choice. Having rejected the strong correspondence principle in favor of imposing “space-time coordinate symmetry” on Eq. (1b) and thus linearity in the components of the three-momentum operator on $\hat{H}_D$, Dirac adopted a severely weakened correspondence principle for $\hat{H}_D$ in relation to the relativistic free-particle square-root Hamiltonian operator $\hat{H}_{\text{free}}$ of Eq. (3), namely,

$$(\hat{H}_D)^2 = (\hat{H}_{\text{free}})^2 = m^2 c^4 + c^2 |\mathbf{p}|^2,$$  \hspace{1cm} (6a)

which ensures that any solutions of the time-dependent Schrödinger equation, i.e., Eq. (1b), with Hamiltonian operator $\hat{H}$ equal to Dirac’s relativistic free-particle $\hat{H}_D$ will also be solutions of the free-particle Klein-Gordon operator.
equation, namely of Eq. (5). Now expressing the linearized $\hat{H}_D$ in terms of four dimensionless Hermitian matrix coefficients $(\beta, \vec{\alpha})$, where $\vec{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \alpha_3)$, in the form,

$$\hat{H}_D = c\vec{\alpha} \cdot \hat{p} + \beta mc^2,$$  \hspace{1cm} (6b)

it is seen that the weak correspondence principle of Eq. (6a) implies that $\beta$ and all the components of $\vec{\alpha}$ mutually *anticommute* and have squares equal to unity. Their anticommutation relations imply that they are traceless \[^2\text{2}\], and therefore $\hat{H}_D$ is traceless as well, and thus has a *negative* energy eigenvalue to match every positive one. So the eigenenergies of Dirac’s linearized $\hat{H}_D$ turn out to *include* all the extraneous negative energies of the Klein-Gordon equation. While the negative-energy eigenstates of $\hat{H}_D$ are properly *orthogonal* to their positive-energy counterparts, the *other* inherent issues which the presence of these negative-energy solutions raise in the context of a free solitary particle, such as total lack of classical correspondence and the Klein paradox remain unresolved \[^4\]. Upon applying the Heisenberg equations of motion to the free-particle Dirac Hamiltonian operator $\hat{H}_D$, as given by Eq. (6b), we obtain the velocity operator,

$$\hat{\nu} = d\vec{r}/dt = c\vec{\alpha},$$  \hspace{1cm} (7a)

which we see, as pointed out above, is *completely independent* of the three-momentum operator $\hat{p}$. Since the three Hermitian matrix components of $\vec{\alpha}$ all have squares equal to unity, we obtain for the free Dirac particle speed operator the *universal* c-number value,

$$|\hat{\nu}| = c\sqrt{3},$$  \hspace{1cm} (7b)

which *exceeds* $c$, the speed of light, by 73%! Also, notwithstanding its nonzero mass $m$, the free Dirac particle *has no inertial rest frame!* These results *show conclusively* that the imposition on Eq. (1b) of “space-time coordinate symmetry” with its resulting *linearized* Hamiltonian operator, that is *crystallized* as the free-particle Dirac Hamiltonian operator $\hat{H}_D$ of Eq. (6b) upon imposition of the severely weakened correspondence principle of Eq. (6a), *violates* special relativity!

To better grasp this point, let us note that if we use the relativistic square-root Hamiltonian operator $\hat{H}_{\text{free}}$ *instead* of $\hat{H}_D$ in the Heisenberg equations of motion, we obtain,

$$\nu = c^2 \hat{p}/(m^2 c^4 + |c\hat{p}|^2)^{\frac{1}{2}},$$

which is a momentum-dependent velocity operator whose magnitude is always strictly smaller than the speed of light $c$, and which will have the value zero for an eigenstate of $\hat{p}$ which has three-momentum eigenvalue equal to 0. So the particle rest frame certainly exists. We are actually in a position to now repeat the Eqs. (2) Lorentz boost of a nonzero-mass free particle’s four-momentum from its rest frame to an inertial frame in which it has nonvanishing velocity, but this time *using for that velocity the operator value $\nu$ that arises from $\hat{H}_{\text{free}}$, which operator $\nu$ is displayed just above, instead of using for that velocity the c-number value $\nu$ of Eqs. (2).* If we simply follow the steps of Eqs. (2), albeit using this particular operator $\nu$, we end up with the *operator four-momentum* $((m^2 c^2 + |\vec{p}|^2)^{\frac{1}{2}}, \vec{p})$, which *accords perfectly* with the three-momentum operator $\hat{p}$ and the *Hamiltonian operator* $(m^2 c^4 + |c\vec{p}|^2)^{\frac{1}{2}}$.

Now let us try to *emulate* the above successful “Lorentz boost of particle four-momentum out of the particle rest frame using the operator $\nu$ that arises from $\hat{H}_{\text{free}}$ exercise by instead using the simple operator $\nu$ that arises from the *Dirac* $\hat{H}_D$, which of course is $\nu = c\vec{\alpha}$. For $\vec{p} = 0$, we have that $\hat{H}_D = \beta mc^2$. If we now try to follow the steps of Eqs. (2) from this starting point by using the Dirac $\nu = c\vec{\alpha}$, we end up with the quite senseless operator four-momentum $(\beta mc(-i/\sqrt{2}), \beta \hat{m}c(-i/\sqrt{2}))$ that bears little resemblance to the desired operator four-momentum result $(\vec{\alpha} \cdot \hat{p} + \beta mc, \vec{p})$. In other words, for the Dirac theory Hamiltonian operator $\hat{H}_D$, the attempt to show Lorentz covariance has ended in disaster. The Dirac theory contravenes special relativity just as the Klein-Gordon theory contravenes quantum mechanics.
We have pointed out that for the free-particle nonrelativistic Pauli theory the Hamiltonian operator has only the kinetic energy term $|\hat{p}|^2/(2m)$, which implies that orbital angular momentum $\hat{r} \times \hat{p}$ is exactly conserved. In contrast, the free-particle Dirac theory has a very strong spin-orbit coupling which almost certainly is not physically sensible. From Eq. (7a) it is clear that in the free-particle Dirac theory,

$$d(\hat{r} \times \hat{p})/dt = c\hat{a} \times \hat{p},$$

and therefore the magnitude of the spin-orbit torque is,

$$|d(\hat{r} \times \hat{p})/dt| = c|\hat{p}|\sqrt{\sigma},$$

Now the particle’s kinetic energy is $((m^2c^4 + |\hat{p}|^2)^{1/2} - mc^2)$. If we take the dimensionless ratio of the particle’s spin-orbit torque magnitude to its kinetic energy, we obtain $((1 + (mc/|\hat{p}|)^2 + (mc/|\hat{p}|))\sqrt{2}$, which increases monotonically without bound as $|\hat{p}|$ decreases. This is, of course, not consistent with the free-particle Pauli theory, where this ratio always vanishes identically. So the Dirac theory does not reduce to the Pauli theory merely by going to small values of momentum. This was already clear, of course, from the fact that the Dirac particle’s speed always has the value $c\sqrt{3}$ irrespective of its momentum, which doesn’t accord with the free-particle Pauli theory at all.

We now present the details of the free Dirac particle’s staggering violation of Newton’s first law of motion. Using Heisenberg’s equations of motion, the Dirac free-particle Hamiltonian $\hat{H}_D$ operator of Eq. (6b), and the simple Dirac free particle velocity operator $\hat{v} = c\hat{a}$, we obtain the Dirac free-particle spontaneous acceleration operator,

$$\hat{a} = d\hat{v}/dt = (2mc^3/\hbar)(i\beta\hat{a} + (\hat{p} \times \hat{a})/(mc)),$$

where,

$$\hat{\sigma} \overset{\text{def}}{=} (-i/2)(\hat{a} \times \hat{a}),$$

which means,

$$\sigma_i \overset{\text{def}}{=} (-i/2)\epsilon_{ijk}\alpha_j\alpha_k.$$  

Since $|i\beta\hat{a}|^2 = 3$, $|\hat{p} \times \hat{\sigma}|^2 = 2|\hat{p}|^2$, and $(i\beta\hat{a}) \cdot (\hat{p} \times \hat{\sigma}) = -(\hat{p} \times \hat{\sigma}) \cdot (i\beta\hat{a})$, we obtain for the magnitude of the spontaneous acceleration,

$$|\hat{a}| = (2\sqrt{3} mc^3/\hbar)(1 + (2/3)(|\hat{p}|/(mc))^2)^{1/2},$$

whose minimum value, $(2\sqrt{3} mc^3/\hbar)$, is, for the case of the electron, well in excess of $10^{28} g$, where $g$ is the acceleration of gravity at the earth’s surface. This dumbfounding spontaneous acceleration certainly drives home the point that the “free” Dirac particle has no inertial rest frame. Note that matters don’t even improve as the Dirac particle is made more massive; the minimum spontaneous acceleration is proportional to the particle mass! The systematics of this unphysical absurdity show just how profoundly the Dirac theory bungles special relativity. As previously mentioned, the relativistic free-particle square-root Hamiltonian operator $\hat{H}_{\text{free}}$ strictly adheres to Newton’s first law of motion, yielding $\hat{a} = 0$.

Finally, it is a built-in property of special relativity that the character of the physics it describes becomes Galilean/Newtonian if the speed of light $c$ is taken to be asymptotically large. For example, if we subtract from the relativistic free-particle square-root Hamiltonian operator $\hat{H}_{\text{free}}$ of Eq. (3) its value at $\hat{p} = 0$, which is $mc^2$, the limit of that difference as $c \to \infty$ is, of course, the nonrelativistic free-particle kinetic energy operator.
\[ \hat{p}^2/(2m). \] But if we carry out exactly same steps with the Dirac Hamiltonian operator \( \hat{H}_D \), we face formal divergence as \( c \to \infty \)! Likewise, if we take the particular velocity operator \( \hat{v} \) discussed in the paragraph which follows the one in which Eq. (7b) occurs, which is obtained from the Heisenberg equation of motion in conjunction with \( \hat{H}_{\text{free}} \), and is explicitly given by,

\[
\hat{v} = \frac{c^2 \hat{p}}{(m^2 c^4 + |\hat{p}|^2)^{\frac{3}{2}}},
\]

it is clear that as \( c \to \infty, \hat{v} \to \hat{p}/m \), i.e., in this limit \( \hat{v} \) becomes the nonrelativistic free-particle velocity operator. On the other hand, for the Dirac free-particle velocity operator \( \hat{v} \) of Eq. (7a) we again face formal divergence as \( c \to \infty \! \). The clear lesson from these instances is that a linearized Hamiltonian operator such as the Dirac \( \hat{H}_D \) is inherently incapable of incorporating the physics of special relativity. It is obvious the mathematical presence of a square root is essential to correctly capturing the the properties of relativistic physics. Even in the case of the Klein-Gordon equation, which “squares out” the mathematical presence of the square root, it is impossible to show that the “physics” it describes becomes properly nonrelativistic as \( c \to \infty \). Looking in detail at the solution space of the Klein-Gordon equation, we readily see that this is due the gratuitous presence of of the extraneous negative-energy solutions, which are, of course, unphysical detritus.

**Correct relativistic quantum mechanics with an external electromagnetic field**

It is clear that, for the nonzero-mass solitary relativistic free particle, the Klein-Gordon theory, which disrupts its quantum mechanics, and the Dirac theory, which dumbfoundingly abolishes its rest frame, must both be abandoned in favor of the straightforward square-root Hamiltonian operator \( \hat{H}_{\text{free}} \) of Eq. (3). For the free photon, which is massless, the time-dependent Schrödinger equation (i.e, Eq. (1b)) in conjunction with the zero-mass case of \( \hat{H}_{\text{free}} \) turns out to be already implicit in the source-free case of Maxwell’s equations.

We shall now develop the extension of \( \hat{H}_{\text{free}} \) to the case of a particle of charge \( e \) and nonzero mass \( m \) when an external electromagnetic potential \( A^\mu(r,t) \) is present, first for a spin-0 particle and then for a spin-1 particle. For additional background and detail concerning the derivation of those two Hamiltonians see reference 5.

The guiding concept is that accurate understanding of the physics experienced by a nonzero-mass solitary particle in an inertial frame where it is instantaneously traveling arbitrarily slowly translates, via a continuous sequence of Lorentz transformations, into the accurate understanding of that physics in an arbitrary inertial frame. Therefore a tested and trusted nonrelativistic theory of a nonzero-mass solitary particle’s behavior ought to always be reasonably straightforwardly upgradable to a correct relativistic one that explicitly reduces to the underlying nonrelativistic one in any inertial frame in which that solitary particle happens to be instantaneously moving at nonrelativistic speed. The approach to attempting to carry out such a program which is followed here is to try to associate individual terms of the solitary particle’s nonrelativistic Hamiltonian with fully Lorentz-covariant energy-momentum four-vectors whose time components reduce to those individual nonrelativistic Hamiltonian terms in inertial frames where that particle is traveling at arbitrarily slow speed. The individual Lorentz-covariant energy-momentum four-vectors so determined are then, of course, summed to produce the solitary particle’s Lorentz-covariant relativistic total energy-momentum four-vector. The resulting solitary-particle relativistic total three-momentum is obviously identified as the generator of the solitary particle’s space translations and thus as the solitary particle’s relativistic canonical three-momentum. Of course the solitary particle’s relativistic total energy, when expressed as function of its relativistic canonical three-momentum, the time, and that particle’s three space coordinates comprises that particle’s relativistic Hamiltonian. Initially, of course, the individual terms contributing to the solitary particle’s relativistic total energy-momentum four-vector will be couched in the language of its three space coordinates, the time, and that particle’s relativistic kinetic three-momentum. Upon identification of the particle’s relativistic canonical (i.e., total) three-momentum, it is then necessary to solve for its relativistic kinetic three-momentum as a function of that relativistic canonical three-momentum in order to be able to reexpress its total relativistic energy as its Hamiltonian. Regrettably, a very conceivable “fly in the ointment” is that there is no guarantee that the particle’s relativistic kinetic three-momentum can be obtained as a function of its relativistic canonical
three-momentum in closed form. Thus the solitary particle’s relativistic Hamiltonian itself may conceivably only be available as a sequence of approximations. Such a state of affairs is obviously not what one would desire, but unlike Klein, Gordon, Schrödinger and Dirac, the physically-motivated, conservative theorist is, like Einstein, obliged to coexist with whatever undesirable calculational consequences that physically-based, conservative theory happens to carry with it. The art of the appropriate approximation, one carefully attuned to a particular problem at hand, is surely yet another vital skill the theoretical physicist is called upon to hone.

Let us now apply the above program to a spin-0 solitary particle of mass \( m \) and charge \( e \) in the presence of an external electromagnetic potential \( A^\mu(\mathbf{r}, t) \). It will be recalled that all magnetic effects of such a potential on the particle’s motion vanish entirely in the particle’s rest frame, and are, more generally, of order \( O(1/c) \), whereas in nonrelativistic physics the speed of light \( c \) is regarded as an asymptotically large parameter. Thus the strictly nonrelativistic Hamiltonian operator for this particle involves only the electromagnetic potential’s time component \( A^0(\mathbf{r}, t) \),

\[
\hat{H}^{(\text{NR})}_{\text{EM}, 0} = |\mathbf{p}|^2/(2m) + eA^0(\mathbf{r}, t). \tag{10a}
\]

Because of the technical issue regarding the choice of ordering of noncommuting operators (whose resolution we allude to below), it will be convenient to develop the relativistic energy-momentum four-vector as a function of classical \((\mathbf{r}, \mathbf{p})\) phase space rather than as a function of the already quantized \((\mathbf{r}, \hat{\mathbf{p}})\) phase space of Eq. (10a). The solitary particle’s nonrelativistic kinetic energy \(|\mathbf{p}|^2/(2m)\), plus its rest mass energy \( mc^2 \), is well-known to correspond to \( c \) times its Lorentz-covariant free-particle kinetic energy-momentum four-vector \( p^\mu \),

\[
p^\mu \overset{\text{def}}{=} ((m^2c^2 + |\mathbf{p}|^2)/c, \mathbf{p}),
\]

where, of course, \( \mathbf{p} \) is the particle’s relativistic kinetic three-momentum, which was distinguished in the above discussion from its relativistic total (i.e., canonical) three-momentum. It is apparent that in the nonrelativistic limit \(|\mathbf{p}| \ll mc\), the time component times \( c \) of \( p^\mu \) does indeed, as just mentioned, behave as,

\[
cp^0 \approx mc^2 + |\mathbf{p}|^2/(2m).
\]

The potential energy term \( eA^0(\mathbf{r}, t) \) of \( \hat{H}^{(\text{NR})}_{\text{EM}, 0} \), divided by \( c \), is obviously the time component of the Lorentz-covariant energy-momentum four-vector \( eA^\mu(\mathbf{r}, t)/c \). Therefore adding \( eA^\mu/c \) to \( p^\mu \) produces a fully Lorentz-covariant total energy-momentum four-vector whose time component times \( c \) reduces, in any inertial frame in which the nonzero-mass charged spin-0 solitary particle instantaneously has arbitrarily small speed, to that particle’s nonrelativistic classical Hamiltonian \( H^{(\text{NR})}_{\text{EM}, 0} \) (which corresponds to the quantized Hamiltonian \( \hat{H}^{(\text{NR})}_{\text{EM}, 0} \) of Eq. (10a)) plus that particle’s rest mass energy \( mc^2 \). We therefore regard,

\[
P^\mu \overset{\text{def}}{=} p^\mu + eA^\mu/c, \tag{10b}
\]

as that solitary particle’s fully relativistic energy-momentum four-vector. Thus the particle’s relativistic total three-momentum is,

\[
\mathbf{P} = \mathbf{p} + e\mathbf{A}(\mathbf{r}, t)/c, \tag{10c}
\]

and its relativistic total energy is,

\[
E(\mathbf{r}, \mathbf{p}, t) = cp^0 = (m^2c^4 + |\mathbf{p}|^2)/c^2 + eA^0(\mathbf{r}, t). \tag{10d}
\]

Here we are in the fortunate position of being able to solve Eq. (10c) for the particle’s relativistic kinetic three-momentum \( \mathbf{p} \) as a function of its relativistic total (i.e., canonical) three-momentum \( \mathbf{P} \) in closed form,
\[ p(P) = P - e A(r, t)/c, \] (10e)

which result for \( p(P) \), i.e., \( p \) as a function of \( P \), we must now substitute into Eq. (10d) for the relativistic total energy in order to reexpress that total energy as the relativistic Hamiltonian \( H_{EM;0}^{(REL)}(r, P, t) \), i.e.,

\[ H_{EM;0}^{(REL)}(r, P, t) \overset{\text{def}}{=} E(r, p(P), t). \]

Therefore Eqs. (10d) and (10e) yield the following fully relativistic classical Hamiltonian \( H_{EM;0}^{(REL)}(r, P, t) \), which corresponds to our original nonrelativistic Hamiltonian operator \( \hat{H}_{EM;0}^{(NR)} \) of Eq. (10a),

\[ H_{EM;0}^{(REL)}(r, P, t) = (mc^2 + |\mathbf{P} - e \mathbf{A}(r, t)|^2)^{1/2} + eA^0(r, t). \] (10f)

Because of the presence of the square root in Eq. (10f) for \( H_{EM;0}^{(REL)}(r, P, t) \), there could conceivably be an issue regarding the ordering of the mutually noncommuting operators \( \mathbf{r} \) and \( \mathbf{P} \) when one attempts quantize this classical Hamiltonian \( H_{EM;0}^{(REL)}(r, P, t) \) to become the Hamiltonian operator \( \hat{H}_{EM;0}^{(REL)} \). Use of the Hamiltonian phase-space path integral [9] with \( H_{EM;0}^{(REL)}(r, P, t) \) in its classical form as given by Eq. (10f) provides one definitive solution to any such operator-ordering issue. Another completely equivalent solution to this issue lies with a natural slight strengthening of Dirac’s canonical commutation rule such that it remains self-consistent [11]. From both of these approaches the resulting unambiguous operator-ordering rule turns out to be the one of Born and Jordan [11].

It is well worth noting that the relativistic classical Hamiltonian \( H_{EM;0}^{(REL)}(r, P, t) \) of Eq. (10f) for the solitary spin-0 charged particle, when inserted into Hamilton’s classical equations of motion, yields, upon taking proper account of Eq. (10c), the fully relativistic version of the Lorentz-force law. In other words, the Hamiltonian \( H_{EM;0}^{(REL)}(r, P, t) \) embodies nothing more or less than the well-known classical relativistic physics of the charged particle developed by H. A. Lorentz [12]. It is truly a lamentable pity that Klein, Gordon and Schrödinger went off on their tangent which plays such havoc with quantum mechanics instead of simply going forward in workmanlike fashion with the quantization of Lorentz’ utterly transparent relativistic legacy, namely the \( H_{EM;0}^{(REL)}(r, P, t) \) of Eq. (10f) above. Of course, upon taking the particle charge \( e \) to zero, \( H_{EM;0}^{(REL)}(r, P, t) \) simply becomes the \( H_{free} \) of Eq. (2c), as it indeed must.

We turn now to the spin-1/2 charged particle, whose nonrelativistic Hamiltonian \( H_{EM;0}^{(NR)} \) is the same as \( H_{EM;0}^{(NR)} \) except for an additional interaction energy between the particle’s intrinsic magnetic moment and the external magnetic field, i.e., its Pauli energy. Notwithstanding that this Pauli magnetic dipole energy is customarily formally written as including a factor of \( (1/c) \), it must nonetheless be kept in the nonrelativistic limit because it fails to vanish in the spin-1/2 particle’s rest frame,

\[ H_{EM;0}^{(NR)} = |\mathbf{p}|^2/(2m) + (ge/(mc^2))(\hbar/2)\hat{\sigma} \cdot (\nabla_r \times \mathbf{A}(r, t)) + eA^0(r, t). \] (11a)

The Pauli magnetic dipole energy contribution to \( H_{EM;0}^{(NR)} \) is the only part of this Hamiltonian which is not a multiple of the two-by-two identity matrix in the spin-1/2 two-by-two-matrix degrees of freedom. Now the relativistic energy-momentum four-vectors we shall be developing will of course themselves be two-by-two matrices, but this should not present difficulties so long as their four components all mutually commute. To ensure that this is the case, we shall “quarantine” the non-identity Pauli energy matrix into a Lorentz scalar, which we can furthermore render dimensionless by dividing by \( mc^2 \). If we now multiply this dimensionless Lorentz scalar by the particle’s kinetic energy-momentum four-vector \( p^\mu = ((m^2c^2 + |\mathbf{p}|^2)^{1/2}, \mathbf{p}) \), we will indeed have the desired energy-momentum contribution whose time component times \( c \) reduces to the Pauli energy matrix in the particle rest frame.

There remains, of course, the challenging problem of turning the complicated Pauli energy term into a Lorentz scalar. In relativistic tensor language, the magnetic field axial vector \( (\nabla_r \times \mathbf{A}(r, t)) \) that appears
in the Pauli term comprises a certain three-dimensional part of the four-dimensional relativistic second-rank antisymmetric electromagnetic field tensor $F^\mu\nu(r, t) = \partial^\mu A^\nu(r, t) - \partial^\nu A^\mu(r, t)$. If we can manage to reexpress the three-dimensional axial spin-$\frac{1}{2}$ vector $(\hbar/2)\vec{\sigma}$ as a “matching” three-dimensional part of a four-dimensional relativistic second-rank antisymmetric tensor $s^{\mu\nu}$ as well, hopefully the Pauli energy will end up being proportional to their Lorentz-scalar contraction $s^{\mu\nu}F_{\mu\nu}(r, t)$. As a first step, we define the natural three-dimensional second-rank antisymmetric spin-$\frac{1}{2}$ tensor $S^{ij}$ in terms of the spin-$\frac{1}{2}$ axial vector,

$$S^{ij} \overset{\text{def}}{=} (\hbar/2)e^{ijk}\sigma^k,$$

and take note that the complicated factor in the Pauli energy term neatly reduces to a contraction of $S^{ij}$ with the well-known magnetic-field three-dimensional part $F^{ij}(r, t)$ of $F^\mu\nu(r, t)$,

$$(\hbar/2)\vec{\sigma} \cdot (\nabla \times \vec{A}(r, t)) = (1/2)S^{ij}F_{ij}(r, t),$$

which allows us to reexpress the nonrelativistic Eq. (11a) in the relativistically more suggestive form,

$$mc^2 + F_{\text{EM},(\text{NR})}^{\text{EM}} = mc^2[1 + |p|^2/(2m^2c^2) + (g/2)(e/(m^2c^3))S^{ij}F_{ij}(r, t)] + e\mathcal{A}^0(r, t). \quad (11b)$$

It is now time to work out the nonzero-mass spin-$\frac{1}{2}$ particle’s fully covariant four-dimensional antisymmetric spin tensor $s^{\mu\nu}$. In the particle rest frame, namely in the special inertial frame where the particle kinetic three-momentum $\vec{p}$ vanishes, the nine space-space components of $s^{\mu\nu}$ must clearly be the nine components of $S^{ij}$, and its remaining seven components must be filled out with zeros, i.e.,

$$s^{\mu\nu}(\vec{p} = 0) \overset{\text{def}}{=} \delta^{\mu}_{i}\delta^{\nu}_{j}S^{ij},$$

which ensures that, in the particle rest frame,

$$s^{\mu\nu}(\vec{p} = 0)F_{\mu\nu}(r, t) = S^{ij}F_{ij}(r, t).$$

Once a tensor is fully determined in one inertial frame, it is fully determined in all inertial frames by application of the appropriate Lorentz transformation to its indices. To get from the particle rest frame to the inertial frame where the particle has kinetic three-momentum $\vec{p}$ simply requires the appropriate Lorentz-boost four-dimensional matrix $\Lambda^\mu_\nu(\vec{v}(\vec{p})/c)$ that is characterised by the corresponding dimensionless scaled relativistic particle velocity,

$$v(\vec{p})/c = (\vec{p}/(mc))/\sqrt{1 + |\vec{p}/(mc)|^2},$$

and its accompanying dimensionless time-dilation factor,

$$\gamma(\vec{p}) = (1 + |\vec{p}/(mc)|^2)^{1/2},$$

so that, in general,

$$s^{\mu\nu}(\vec{p}) = \Lambda^\mu_\nu(\vec{v}(\vec{p})/c)\Lambda^\nu_\nu(\vec{v}(\vec{p})/c)S^{ij},$$

which, of course, ensures that $s^{\mu\nu}(\vec{p})F_{\mu\nu}(r, t)$ is a Lorentz scalar that Lorentz-invariantly carries the particle’s rest-frame value of $S^{ij}F_{ij}(r, t)$.

With that, we are in the position to be able to write down the Lorentz-covariant total energy-momentum four-vector $P^\mu$ for the spin-$\frac{1}{2}$ particle that corresponds to its nonrelativistic Eq. (11b) in the same way that the total energy-momentum four-vector of Eq. (10b) for the spin-0 particle corresponds to its nonrelativistic Eq. (10a),

$$P^\mu \overset{\text{def}}{=} p^\mu[1 + (g/2)(e/(m^2c^3))s^{\alpha\beta}(\vec{p})F_{\alpha\beta}(r, t)] + eA^\mu(r, t)/c. \quad (11c)$$

From $P^\mu$ we obtain the particle’s relativistic total energy,
\[ E(\mathbf{r}, \mathbf{p}, t) = cP^0 = (m^2 c^4 + |\mathbf{p}|^2)^{1/2} [1 + (g/2)(e/(m^2 c^3))] s^{\mu\nu}(\mathbf{p}) F_{\mu\nu}(\mathbf{r}, t) + eA^0(\mathbf{r}, t), \] (11d)

and also its relativistic total three-momentum,

\[ \mathbf{P} = \mathbf{p}[1 + (g/2)(e/(m^2 c^3))] s^{\mu\nu}(\mathbf{p}) F_{\mu\nu}(\mathbf{r}, t)] + e\mathbf{A}(\mathbf{r}, t)/c. \] (11e)

It is obvious from Eq. (11e) that we cannot solve for \( \mathbf{p}(\mathbf{P}) \) in closed form, but we can write \( \mathbf{p}(\mathbf{P}) \) in “iteration-ready” form as,

\[ \mathbf{p}(\mathbf{P}) = (\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c)[1 + (g/2)(e/(m^2 c^3))] s^{\mu\nu}(\mathbf{P}) F_{\mu\nu}(\mathbf{r}, t)]^{-1}, \] (11f)

and, of course, from \( E(\mathbf{r}, \mathbf{p}(\mathbf{P}), t) \), we also obtain the schematic form of the particle’s relativistic Hamiltonian,

\[ H_{\text{EM;0}}^{(\text{REL})} (\mathbf{r}, \mathbf{P}, t) = (m^2 c^4 + |\mathbf{p}(\mathbf{P})|^2)^{1/2} [1 + (g/2)(e/(m^2 c^3))] s^{\mu\nu}(\mathbf{P}) F_{\mu\nu}(\mathbf{r}, t)] + eA^0(\mathbf{r}, t). \] (11g)

If we take the limit \( g \to 0 \) in Eqs. (11f) and (11g), then \( H_{\text{EM;0}}^{(\text{REL})} (\mathbf{r}, \mathbf{P}, t) \to \tilde{H}_{\text{EM;0}}^{(\text{REL})} (\mathbf{r}, \mathbf{P}, t) \), as is easily checked from Eq. (10f). Of course it is nothing more than the most basic common sense that fully relativistic spin-\( \frac{1}{2} \) theory simply reduces to fully relativistic spin-0 theory when the spin coupling of the single particle to the external field is switched off, but analogous cross-checking between the Dirac and Klein-Gordon theories is never so much as discussed! It is certainly possible to add a term to the Dirac Hamiltonian that cancels out it’s supposed \( g = 2 \) spin coupling to the magnetic field, but the result of doing this bears very little resemblance to the Klein-Gordon equation in the presence of the external electromagnetic potential! Elementary consistency checks are obviously not the strong suit of those two “theories”!

It is unfortunate that Eq. (11f) for \( \mathbf{p}(\mathbf{P}) \) is not amenable to closed-form solution, but if we assume that the spin coupling term, \( (g/2)(e/(m^2 c^3)) s^{\mu\nu}(\mathbf{P}) F_{\mu\nu}(\mathbf{r}, t) \), which is a dimensionless Hermitian two-by-two matrix, effectively has the magnitudes of both of its eigenvalues much smaller than unity (which should be a very safe assumption for atomic physics), then we can approximate \( \mathbf{p}(\mathbf{P}) \) via successive iterations of Eq. (11f), which produces the approximation \( \mathbf{p}(\mathbf{P}) \approx (\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c)[1 + (g/2)(e/(m^2 c^3))] s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c) F_{\mu\nu}(\mathbf{r}, t)]^{-1} \), through first order in the spin coupling. We wish to interject at this point that since \( s^{\mu\nu}(\mathbf{P}) \) is an antisymmetric tensor, the tensor contraction \( s^{\mu\nu}(\mathbf{P}) F_{\mu\nu}(\mathbf{r}, t) \) is equal to \( 2s^{\mu\nu}(\mathbf{P}) \partial_\mu A_\nu(\mathbf{r}, t) \), which is often a more transparent form. Now if we simply use the approximation \( \mathbf{p}(\mathbf{P}) \approx (\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c) \) through zeroth order in the spin coupling for \( \mathbf{p}(\mathbf{P}) \), we obtain the following approximation to \( H_{\text{EM;0}}^{(\text{REL})} \),

\[ H_{\text{EM;0}}^{(\text{REL})} (\mathbf{r}, \mathbf{P}, t) \approx (m^2 c^4 + |\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)|^2)^{1/2} [1 + (ge/(m^2 c^3)) s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c) \partial_\mu A_\nu(\mathbf{r}, t)] + eA^0(\mathbf{r}, t). \] (11h)

Antiparticle partners from field theoretic symmetry

If one were to write down a purely electromagnetic (and thus parity-conserving) quantum field theory that treats only positive-energy electrons and photons, which would be possible via second quantization of \( \tilde{H}_{\text{EM;0}}^{(\text{REL})} \) and quantization of the electromagnetic field, it is clear that one would then have a quantum field theory which exhibits no invariance whatsoever under charge conjugation. Simply imposing charge conjugation invariance on such a theory forces positrons to exist. This is a very familiar picture indeed: the imposition of symmetries on quantum field theories not at all infrequently forces families of particles to exist. Note that the positrons that would be forced into existence by the imposition of charge conjugation invariance would also be purely positive-energy particles, at least when free. “Reinterpretation” of unbounded-below, free-particle nega-
energy spectra plays no role whatsoever in the existence of these positrons: their existence is driven theoretically entirely by the enforcement of the symmetry!

The imposition of charge conjugation invariance—or, more generally, CP invariance—on solitary-particle quantum mechanics that has been taken to the quantum field theory level does not of itself, however, ensure that external fields can mediate the basic physical processes of particle-antiparticle pair production and annihilation \[ \mathcal{E} \], processes which cannot, of course, be accommodated in the context of a strictly solitary-particle idealization. To accommodate these processes, the imposition of CP invariance at the field theory level needs to be accompanied by the additional imposition of invariance under the interchange of particle absorption/emission with, respectively, antiparticle emission/absorption, which we term “CP-equivalence” symmetry. To give one crude, schematic sketch of an imposition of the dual symmetries of CP invariance and “CP-equivalence”, suppose we have in hand the quantum mechanics Hamiltonian operator \( \hat{H} \) for a solitary relativistic charged particle interacting with external fields. The straightforward second quantization of \( \hat{H} \) produces the quantum-field-theoretic schematic quantized Hamiltonian density \( \psi^\dagger \hat{H} \psi \), which has the particle emission field \( \psi^\dagger \) and the particle absorption field \( \psi \), but no antiparticle fields, and therefore is completely CP noninvariant. This is readily remedied by introducing the antiparticle emission field \( \psi^\dagger_{\text{CP}} \) and the antiparticle absorption field \( \psi_{\text{CP}} \), which correspond to the CP-transformed version of \( \hat{H} \) that we denote as \( \hat{H}_{\text{CP}} \) (a prototypical difference between \( \hat{H} \) and \( \hat{H}_{\text{CP}} \) is, of course, in the sign of the solitary particle’s charge). Using these entities, one simple way to impose CP invariance symmetry is to replace the Hamiltonian density \( \psi^\dagger \hat{H} \psi \) by \( \psi^\dagger \hat{H} \psi + \psi^\dagger_{\text{CP}} \hat{H}_{\text{CP}} \psi_{\text{CP}} \), which can permit external fields to mediate both particle and antiparticle scattering, but not yet particle-antiparticle pair production or annihilation. To include the latter physical phenomena we shall additionally impose the “CP-equivalence” symmetry which was pointed out above by also requiring the Hamiltonian density to be invariant under the particle field interchanges \( \psi \leftrightarrow \psi_{\text{CP}} \) and \( \psi^\dagger \leftrightarrow \psi^\dagger_{\text{CP}} \). It is apparent that one straightforward upgrade of the solitary-particle Hamiltonian density \( \psi^\dagger \hat{H} \psi \) which incorporates the dual symmetries of CP invariance and “CP-equivalence” that we are stipulating here is given by,

\[
\psi^\dagger \hat{H} \psi \to (\psi + \psi^\dagger_{\text{CP}})^\dagger \hat{H} (\psi + \psi^\dagger_{\text{CP}}) + (\psi_{\text{CP}} + \psi^\dagger)^\dagger \hat{H}_{\text{CP}} (\psi_{\text{CP}} + \psi^\dagger),
\]

which clearly can permit external fields to mediate pair production and annihilation, as well as particle and antiparticle scattering.

The role of symmetries in accounting for and compelling the existence of energy degeneracies has been a robust and highly fruitful theme of quantum physics since its very earliest days; the names of such pioneers as Wigner and Weyl come readily to mind. The existence of antiparticles is a classic example of energy degeneracy in the context of quantum field theory, and there is absolutely no reason whatsoever that it should not take its rightful place in the theoretical physics pantheon of symmetry-driven phenomena. Unbounded-below negative energies that require “reinterpretation” are a staggeringly bizarre manifestation of undiluted metaphysics that physical science would do extremely well indeed to shed forever, especially in light of the fact that their origin resides in gratuitously-generated entirely extraneous “solutions” (Klein-Gordon theory) or an equally gratuitous weakening of the correspondence principle that springs from extraordinary insistence on, for a physically nonviable reason, dealing with the squares of dynamical variables instead of those dynamical variables themselves (Dirac theory). The “reinterpretation” of the egregiously metaphysical unbounded-below negative-energy spectra of the Klein-Gordon and Dirac theories is a quintessential instance of the syndrome that afflicted Einstein’s knight, “who dyed his whiskers green, and then used a large fan so that they should not be seen.”

The breaking of symmetries has become a very strong theme in the last seven decades, and the breaking of the antiparticle-associated CP invariance has been empirically firmly established. Indeed a glance at the gross particle-antiparticle “nonsymmetry” that makes our galaxy’s existence possible powerfully suggests that there is much that remains to be learned about the breaking of this symmetry. A theory that assigns two independent fields to the description of the particle and its distinguishable antiparticle makes it far more straightforward to model what are no doubt extremely important effects pertaining to that symmetry breaking. With two independent fields one can essay such basics as a slight mass difference between particle and antiparticle, which
is certainly not available if particle and antiparticle are construed as merely somehow reflecting different parts of the same operator’s energy spectrum.

Conclusion

It has been made very clear in the above that Schrödinger’s simple postulates regarding the solitary particle’s wave function specify that the generators of that wave function’s infinitesimal space and time translations are, respectively, the canonical three-momentum operator and the Hamiltonian operator, and that this fact provides an extremely satisfactory basis, which is completely compatible with the strong correspondence principle, for the fully relativistic version of solitary-particle quantum mechanics. From the standpoint of systematic, conservative physical theory, the utterly clear implication of the above observations, in light of the strong correspondence principle, is that the correct Hamiltonian operator for the solitary relativistic free particle is given by the square-root operator of Eq. (3), and that the interaction of a spinless nonzero-mass solitary charged relativistic particle with an external electromagnetic potential in the context of quantum mechanics must be described by the quantization of the Hamiltonian that corresponds to the fully relativistic version of H. A. Lorentz’ electromagnetic force law, specifically the Hamiltonian of Eq. (10f).

Unfortunately, the pellucid implications for the relativistic quantum mechanics of the solitary particle of Schrödinger’s postulates for the wave function never dawned on Schrödinger himself, nor on Klein, Gordon, nor Dirac. That this is the case is made painfully clear by Dirac’s simultaneously superfluous and extremely damaging imposition of “space-time coordinate symmetry” on Schrödinger’s time-dependent wave function equation (Eq. (1b)), notwithstanding that precisely this particular appearance of space-time coordinate symmetry is the most striking feature of Schrödinger’s four-vector wave function equation (Eq. (1c))! Although these pioneers failed to appreciate even the very existence of Eq. (1c), notwithstanding that it is a mere summary of Schrödinger’s wave-function postulates, they did have an appreciation of the correspondence principle, albeit they regarded it as a rather more plastic concept than the definitive strong form which the facts of both the Hamiltonian phase-space path integral and the completed version of Dirac’s canonical commutation rule very clearly reveal. But, extremely unfortunately, because of perceived issues of calculational convenience, these pioneers gave the correspondence principle short shrift indeed, and instead chose ostensibly “relativistic quantum mechanics” routes that terribly distort the quantum mechanical implications and impact of the underlying classical special relativistic mechanics of the solitary particle which had been so ably developed and expounded by Lorentz; the gratuitous injection of totally extraneous unphysical, unbounded-below negative-energy “solutions” is, of course, a prime example of this. For any half-way serious disciple of the correspondence principle, these patently absurd “solutions”, which are at complete loggerheads with the classical relativistic solitary-particle mechanics of Lorentz, would of themselves have been reason enough to immediately call off the attempted tweaking of the time-dependent Schrödinger equation and/or of the ostensibly “relativistic” Hamiltonians being fed to it, and to return forthwith to strictly Lorentzian classical relativistic basics as the correct physical foundation upon which to erect relativistic quantum mechanics.

Notwithstanding subsequent “reinterpretation” of these unbounded-below negative-energy “solutions” to “accommodate” the physical existence of mass-degenerate antiparticles, which is universally acclaimed as a “triumph” 2, it is vastly less conceptually tortuous and equally vastly more in keeping with all of the rest of known quantum physics to account for the existence of mass-degenerate antiparticles as the utterly straightforward consequence of CP invariance of the underlying quantum field theory. Since this invariance is, indeed, a slightly broken one in the manner of so many other symmetries of physics—a fact that would appear to be of critical importance to the very existence of the familiar physical world of our experience—it would seem all the more important to have available the theoretical tool of two completely independent quantum fields (with each having bounded-below energy, of course) for the descriptions, respectively, of a particle and its distinguishable antiparticle, which permits their masses, for example, to be very slightly different, instead of being tied to the “reinterpretated negative energy spectrum” consequence of having certain attributes of particle and antiparticle being irrevocably identical. It is symmetries and not “reinterpretations” of blatantly gratuitous and absurdly unphysical equation “solutions” that are reasonably and plausibly held to be responsible for the existence of
mass-degenerate partner particles in all instances other than that of distinguishable antiparticles. It is surely now well past time for the theoretical physics treatment of mass-degenerate distinguishable antiparticles to be put on a track which is completely parallel to the sensible symmetry-based handling accorded as a matter of standard routine to all other ostensibly “understood” particle mass degeneracies.

Departure from the theoretical physics scene of the physically misbehaving Klein-Gordon and Dirac equations would not only restore physical good sense to relativistic solitary-particle quantum mechanics, it would unveil physics’ harmonious hierarchical integrity: the underlying physical essence of the relativistic Lorentzian Hamiltonian of Eq. (10f) is explicitly its simple nonrelativistic counterpart of Eq. (10a), and the same role is played by the nonrelativistic Pauli theory of Eq. (11a) in relation to its fully relativistic upgrade that is given by Eqs. (11g) and (11f). Likewise, when the upgrade of these solitary-particle theories to fully interacting field theories is made via second quantization and the imposition of the dual symmetries of CP invariance and “CP-equivalence”, the physical essence of the multi-particle interactions is obviously firmly anchored in those of the merely solitary relativistic particle with external fields, which interactions, in turn, as we have just noted, may be no more than the relativistic upgrade of such nonrelativistic basics as a stationary charged particle’s interaction with an electric potential or a stationary magnetic dipole’s interaction with a magnetic field! Likewise, the properties of any quantized Hamiltonian are highly sensitive to those of its classical precursor, as the key stationary phase path for the path integral amply attests. And of course both the presence and absence of symmetries is of crucial importance to the very character of the physics. It is precisely because of the profound underlying linkages inherent in what at first glance may appear to be very diverse aspects of physics that the fatuous mathematical “modifications” unwarily introduced into relativistic quantum mechanics by Klein, Gordon, Schrödinger, and Dirac are so very damaging.

References

[1] L. I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1955).
[2] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).
[3] P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford University Press, London, 1958).
[4] S. K. Kauffmann, arXiv:0909.4025 [physics.gen-ph] (2009).
[5] S. K. Kauffmann, arXiv:1004.1820 [physics.gen-ph] (2010).
[6] P. A. M. Dirac, Proc. Roy. Soc. (London) A117, 610 (1928).
[7] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper & Row, New York, 1961).
[8] N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, New York, 1959).
[9] S. K. Kauffmann, arXiv:0910.2490 [physics.gen-ph] (2009).
[10] S. K. Kauffmann, arXiv:0908.3755 [quant-ph] (2009).
[11] M. Born and P. Jordan, Z. Physik 34, 858 (1925).
[12] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Butterworth-Heinemann, Oxford, 1975).
[13] G. E. Hahne, private correspondence.