ON ASYMPTOTIC SOLVABILITY
OF RANDOM GRAPH’S LAPLACIANS

A. Khorunzhy*,# and V. Vengerovsky*

*Institute for Low Temperature Physics, Kharkov, UKRAINE
#Faculté de Mathématiques, Université Paris-VII, FRANCE

Abstract We observe that the Laplacian of a random graph $G_N$ on $N$ vertices represents an explicitly solvable model in the limit of infinitely increasing $N$. Namely, we derive recurrent relations for the limiting averaged moments of the adjacency matrix of $G_N$ when $N \to \infty$. These relations allow one to study the corresponding eigenvalue distribution function; we show that its density has a infinite support in contrast to the case of the ordinary discrete Laplacian.

Key words: graph Laplacian, random graphs, random matrices, eigenvalue distribution, walks on trees

The spectral theory of graphs attracts more and more attention in mathematical physics (see e.g. [1, 2]). It is widely believed that the spectral properties of graphs are related with their geometry that is an intriguing assertion from various points of view. However in general, without the knowledge of the graph structure, one can prove just abstract theorems with not large number of concrete results. In this connection, the study of the spectrum of random graphs, as the typical in certain sense objects, can be useful. We confirm this assumption by showing that the models related with large random graphs are explicitly solvable.

Given a set $V_N$ of $N$ vertices $v_1, \ldots, v_N$, one can assume that some pairs of them are either joined by one undirected edge or are not joined. Then $V_N$ and the set $E_N$ of existing edges determines the graph $\Gamma_N = (V_N, E_N)$. In this case the graph Laplacian $\Delta_N$ acting on functions $f(v), v \in V_N$ is determined by the formula

$$\Delta_N f(v_i) = \sum_{j: v_i \sim v_j} [f(v_i) - f(v_j)],$$

where the sum runs over the vertices joined with $v_i$.

It is clear that $\Delta_N$ has $N \times N$ real symmetric matrix and

$$\Delta_N = V^{(N)} - A^{(N)},$$

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where $A^{(N)}$ is the adjacency matrix of the graph $\Gamma_N$

$$A_{ij}^{(N)} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise}, \end{cases}$$

(2)

and $V^{(N)}$ is the diagonal matrix such that $V_{ii}^{(N)} = \text{deg}(v_i)$ is the number of edges attached to $v_i$. Note that $A_{ii}^{(N)} = 0$ and that

$$\Delta_N = P_N[A^{(N)}]^{2}P_N - A^{(N)},$$

(3)

where $P_N$ is the orthogonal projection onto the diagonal. Often the set of eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of $A_N$ is called the graph’s spectrum [3]. Firstly we concentrate on this problem and turn to the spectrum of $\Delta_N$ at the end of this article.

Our principal goal is to describe results obtained for the adjacency matrix $A^{(N,p)}$ of a random graph $G_{N}^{(p)} = (V_N, E_{N}^{(p)})$. We are related with the well-known model (see e.g. [4]), where $\{G_{N}^{(p)}\}$ represents the ensemble of graphs where each edge, independently from the others, is either present or absent with probabilities $p/N$, $0 < p \leq N$ and $1 - p/N$, respectively. Thus we get a real symmetric matrix $A^{(N,p)}$ with the entries $\{A_{ij}^{(N,p)}, i \leq j\}$ determined as a family of jointly independent random variables such that

$$A_{ii}^{(N,p)} = 0 \text{ and } A_{ij}^{(N,p)} = \begin{cases} 1, & \text{with probability } p/N, \\ 0, & \text{with probability } 1 - p/N. \end{cases}$$

(4)

We are interested in asymptotic behaviour in the limit $N \to \infty$ of the moments

$$M_s^{(N,p)} = \mathbb{E}\{N^{-1} \text{ Tr } A^s\}, \quad A \equiv A^{(N,p)},$$

(5a)

where

$$N^{-1} \text{ Tr } [A^{(N,p)}]^s = \frac{1}{N} \sum_{x_i=1}^{N} A_{x_1x_2}A_{x_2x_3} \cdots A_{x_{s-1}x_1}.$$  

(5b)

We consider for simplicity the case of $p = 1$. Our main result is that

$$\lim_{N \to \infty} M_s^{(N,1)} = \begin{cases} m_k, & \text{if } s = 2k \\ 0, & \text{if } s = 2k + 1 \end{cases}$$

and $m_k$ can be found from equality

$$m_k = \sum_{r=0}^{k} W_k(r)$$

(6a)
where the numbers $W_k(r)$, $k \geq r \geq 0$ are given by the following recurrent relations

$$W_u(v) = \sum_{i=1}^{u} \sum_{j=v-i}^{u-i-j} \sum_{l=0}^{u-i-j} W_{u-i-j}(l) \left( \frac{l + i - 1}{i - 1} \right) \left( \frac{v - 1}{i - 1} \right) W_j(v - i), \quad (6b)$$

supplied with the initial condition $W_j(0) = \delta_{j0}$.

Derivation of (6) essentially uses the fact that $A^{(N,p)} (4)$ is the matrix whose entries are jointly independent (excepting the symmetry condition) random variables. For such class of matrices, the limits of moments (5) were studied first by E. Wigner [5]. The ensemble considered in [5] is somewhat different from (4), and the method proposed there and then developed in [6] is applicable to (4) in the asymptotic regime $p \gg 1$ only.

The principal observation of [5] and [6] is that the sum over the sequences $X_{2k} \equiv (x_1, x_2, \ldots, x_{2k-1}, x_1)$ in (5a) can be replaced by the sum over the equivalence classes of the walks $X_{2k}$; these classes are labelled by the plane rooted trees $\tau \in T_{k-1}$ drawn in the upper half-plane with the help of $k$ edges.

In paper [7] the method of [5] and [6] was further improved up to the stage when $p \sim 1$ can also be included into consideration. Summing up the arguments of [5, 6] and [7], we claim that

$$m_k = \sum_{q=0}^{k-1} \sum_{\tau \in T_{k-q}} U(k; \tau), \quad (7)$$

where $U(k; \tau)$ is the number of all possible different walks of $2k$ steps such that cover the tree $\tau \in T_{k-q}$ according to the following

**Riding rule**: the walk starts and ends at the root and each edge of the tree is passed even positive number of times; when there is a choice where to move further, the walk chooses either one of the edges that has been already passed or the most left edge among those that have not yet been passed.

Let us call the walks of this type the even ordered walks covering the tree.

**Demonstration of (6)**.

It is convenient to imagine that some particle moves in the upper half-plane. It starts at the root $\rho$, then makes $2k$ steps and returns to the root; each step either produces a new vertex or ends at the vertex that already exists and is connected by the present vertex by an edge. The condition is that each step made in one direction is repeated in inverse direction.

It is easy to see that there exists a tree $\tau \in T_{k-q}$ (and the only one) that the
walk of the particle represents one of the coverings of \( \tau \) satisfying the riding rule.

Let us introduce the set \( \mathcal{W}_u(v) \) of all possible walks of \( 2u \) steps that return to the root \( v \) times and denote by \( W_u(v) = |\mathcal{W}_u(v)| \) the number of such walks. With this definition in mind, it is not hard to derive (6) from (7).

Indeed, regarding the set of walks \( \mathcal{W}_u(v) \), let us denote their first step by \((\rho, \alpha)\) and consider those walks that pass this edge \( 2i \geq 2 \) times. It is easy to see that \( \alpha \) by itself can be regarded as the root of a subwalk that starts and ends at \( \alpha \) and never visits \( \rho \). Chronologically, this trajectory consists of possibly several pieces. Let us call their union the first subwalk. Similarly, the second subwalk starts and ends at \( \rho \) and never visits \( \alpha \).

Let us assume that the second subwalk belongs to \( \mathcal{W}_j(v - i) \) and the first subwalk belongs to \( \mathcal{W}_{u-i-j}(l) \). Then (6b) follows, where in the product, the second factor represents the number of possibilities to perform \( i-1 \) steps from \( \alpha \) to \( \rho \) under the condition that we arrive \( l \) times to \( \alpha \) by the first subwalk; the last passage \( \alpha \to \rho \) is made after that the first subwalk is completed. The third factor in (6b) corresponds to \( i-1 \) steps \( \rho \to \alpha \) made after \( v-i \) arrivals to \( \rho \) by the second subwalk.

\[ \blacksquare \]

In the following remarks we briefly discuss the result obtained

1. The first conclusion relates the maximal eigenvalue of \( A^{(N,1)} \). It is clear that the number of non-zero elements in a row is asymptotically Poissonian when \( N \to \infty \). Thus, with non-zero probability, in \((\mathcal{V}_N, E^{(1)}_N)\) there exists a vertex that has large number of edges attached to it. This explains why \( \|A^{(N,1)}\| \to \infty \) (see also [7]) while the spectrum of the ordinary discrete Laplacians (or more generally, of the regular graphs) remains bounded.

The stress of the present article is that there is an infinite number of the eigenvalues of \( A^{(N,p)} \) that tend to infinity as \( N \to \infty \). We derive this conclusion from relations (6) and the representation

\[ M_{s}^{(N,p)} = \mathbb{E} \int \lambda^s d\sigma_{N,p}(\lambda), \tag{8} \]

where \( \sigma_{N,p}(\lambda) = \sigma(\lambda, A^{(N,p)}) \) is the normalized eigenvalue counting function \( \sigma(\lambda) = \# \{ \lambda_j \leq \lambda \} N^{-1} \).

Basing on (6b), one can easily show that \( W_k(r) \leq (C_1k)^{2r} \) where \( C_1 \) is some constant. Then we derive from (6a) the estimate

\[ m_{2k} \leq (C_2k)^{2k} \]

that implies the uniqueness of the limiting measure \( \sigma^{(1)} = \lim_N \sigma_{N,1} \).
From another hand, it is not hard to derive the estimate

$$m_{2k} \geq (k/2)^{k/2}(1 + o(1)), k \to \infty. \quad (9)$$

Indeed, let us take the set $W_k(k)$ and consider the walks that cover one particular tree $\tau_{k/2}'$ of $k/2$ edges, namely that has all $k/2$ edges attached to the root (here we assume $k = 2l$ for simplicity). Among these walks there are those that pass each edge four times. It clear that the number of these walks is $(k/2)!$ that results in (9).

Inequality (9) implies that the measure $d\sigma(1)$ has an infinite support. This means that the mathematical expectation of the number of eigenvalues that go to infinity is of the order $O(N)$ as $N \to \infty$.

2. Let us note that one can easily apply our method to study the limits $m_k^{(p)}$ of moments (8) with arbitrary constant number $p$. In paper [6] the arguments have been presented showing that there exists a critical value $p_c \sim 2$ that separates two different pictures of the limiting spectra of $A^{(N,p)}$. This is supported by the facts from the random graphs theory stating that $p_c' = 1$ is critical for existing of an infinite connected component in $(V_N, E_N^{(p)})$ as $N \to \infty$ (see e.g. [4]). Our preliminary numerical studies of the moments $m_k^{(p)}$ show that $\sigma^{(p)}$ is not sensitive up to the level that reflects the role of $p_c$ or $p_c'$.

3. It should be mentioned that the moments (7) we compute represent a special kind of the matrix integrals. If one considers the matrices $A_N$ (4) as having the Gaussian invariant distribution, then the corresponding moments (5) can be computed explicitly for fixed $N$ and the formulas obtained are related with the number of maps drawn on the surfaces of given genus (see e.g. [4] for an accessible introduction).

In this aspect, our model characterises another class of random matrices with independent entries that have explicitly countable integrals over them (see [10] for related recent results concerning their asymptotic properties).

4. Finally, let us indicate the way to find explicit relations for the moments of the random graph’s Laplacian. Regarding definition (3), we see that one faces the problem of computing the mixed moments

$$E \left\{ A^{r_1}(x_1, x_2)[A^2(x_2, x_2)]^{s_1} \cdots A^{r_j}(x_j, x_1)[A^2(x_1, x_1)]^{s_j} \right\} .$$

One can apply again the method of [5, 6] with the modification that the trees are constructed with the help of edges of two different colours and the edges of one of the class can be used as the leaves only. Certainly, in the case of finite
the computations become more difficult than those of (6), but the general approach of [7] and the present article remains valid.

Note that the case of multiple edges can be also included into consideration by allowing $A_{ij}$ (4) to take positive integer values. Moreover, our method still works for the random graphs with independent random weights on edges; then the adjacency matrix (2) takes the form $B_{ij} = \Xi_{ij}A_{ij}$, where $\Xi_{ij}, i \leq j$ are jointly independent random variables. All these questions will be considered in more details in separate publication.

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