ASYMPTOTIC CONES OF LIE GROUPS AND CONE EQUIVALENCES

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ABSTRACT. We introduce cone equivalences between metric spaces. These are maps, more general than quasi-isometries, that induce a bilipschitz homeomorphism between asymptotic cones. Non-trivial examples appear in the context of Lie groups, and we thus prove that the study of asymptotic cones of connected Lie groups can be reduced to that of solvable Lie groups of a special form.

1. Introduction

Let \( X = (X, d) \) be a metric space. The idea of defining an “asymptotic cone” for \( X \), namely a limit for the sequence of metric spaces \( \frac{1}{n}X = (X, \frac{1}{n}d) \) was brought in by Gromov [Gr81] in terms of Gromov-Hausdorff convergence. This definition was satisfactory for the purpose of groups with polynomial growth [Pa1, B], but to generalize the definition to arbitrary metric spaces, it was necessary to drop the hope of getting a limit in a reasonable topological sense, and consider ultralimits, which make use of the choice of an ultrafilter. Following Van der Dries and Wilkie [DW], the limit \( \lim_{\omega} \frac{1}{n}X \), formally defined in Section 2, can be defined when \( \omega \) is a non-principal ultrafilter. This is a metric space, called the asymptotic cone of \( X \) with respect to \( \omega \), and denoted by \( \text{Cone}_\omega(X, d) \) or \( \text{Cone}_\omega(X) \) for short.

Considerable progress in the study of asymptotic cones of groups was then made in Gromov’s seminal book [Gr93], in which the second chapter is entirely devoted to asymptotic cones. Since then, a vast literature appeared on the subject, including the papers [KL, Br, TV, Dr, Ri, KSTT, DrS, Co, BM].

The classification of groups in terms of their asymptotic cones is not as fine as the quasi-isometry classification, but in some case, as connected Lie groups, can look more approachable. Here are a few facts relevant to the general study of connected Lie groups up to quasi-isometry.

Let \( (C_0) \) be the class of triangular Lie groups, i.e. groups isomorphic to a closed connected group of real upper triangular matrices.

- Every connected Lie group is quasi-isometric to a group in the class \( (C_0) \) [Co, Lemma 6.7];
• open question: is it true that any two quasi-isometric groups in the class $(\mathcal{C}_0)$, are isomorphic? In the nilpotent case, this is considered as a major open question in the field.

In the study of the large-scale geometry of a group $G$ in the class $(\mathcal{C}_0)$, a fundamental group is played by the exponential radical $R$ of $G$, which is defined by saying that $G/R$ is the largest nilpotent quotient of $G$.

Define $(\mathcal{C})$ as the class of groups $G$ in $(\mathcal{C}_0)$ having a closed subgroup $H$ (necessary nilpotent) such that

1. $G$ is the semidirect product $R \rtimes H$;
2. the action of $H$ on the Lie algebra of $R$ is $R$-diagonalizable (in particular, $[H,H]$ centralizes $R$).

**Remark 1.1.** Let $G$ be the (topological) unit component of $G^R$, where $G$ is an algebraic $R$-subgroup of the upper triangular matrices and assume that $G$ has no nontrivial homomorphism to the additive one-dimensional group (so that the unipotent radical coincides with the exponential radical). Then $G$ is in the class $(\mathcal{C})$.

In general, and even if $G$ is algebraic, the exact sequence $1 \to R \to G \to G/R \to 1$ can be non-split [Co, Section 4] and there is no such subgroup. Even when a splitting exists, the second condition is not satisfied in general. So the class $(\mathcal{C})$ appears as a class of groups, much smaller than the whole class $(\mathcal{C}_0)$, but in which the large-scale geometry is more likely to be understandable. Our main result is to associate, to every $G$ in the class $(\mathcal{C}_0)$, a “nicer” group $G' \in (\mathcal{C})$, of the same dimension and with the same exponential radical, such that the asymptotic cones of $G$ and $G'$ are the same. To state a precise result, we introduce the following new concept.

We define a map between metric spaces $X, Y$ to be a cone-bilipschitz equivalence if it induces a bilipschitz homeomorphism at the level of asymptotic cones for all ultrafilters, and we say that $X$ and $Y$ are cone-bilipschitz-equivalent if there exists such a map. (The precise definitions will be provided and developed in Section [2]) It is easy and standard that any quasi-isometry between metric spaces is a cone-bilipschitz equivalence, but on the other hand there exist cone-bilipschitz-equivalent metric spaces that are not quasi-isometric (see the example before Corollary [1.3]).

**Theorem 1.2.** Let $G$ be any connected Lie group. Then $G$ is cone-bilipschitz equivalent to a group $G_1$ in the class $(\mathcal{C})$. More precisely, if $G$ is triangulable with exponential radical $R$, then there is a split exact sequence

$$1 \to R \to G_1 \to G/R \to 1,$$

in which $R$ embeds as the exponential radical of $G_1$. 

If \( g \) is a Lie algebra, denote by \((g^i)\) its descending central series \((g^1 = g; g^{i+1} = [g, g^i])\); we have
\[
[g^i, g^j] \subset g^{i+j}
\]
for all \( i, j \), so the bracket induces a bilinear operation
\[
[g^i/g^{i+1}, g^j/g^{j+1}] \to g^{i+j}/g^{i+j+1}.
\]
This defines a Lie algebra structure on
\[
g_{\text{grad}} = \bigoplus_{i \geq 1} g^i/g^{i+1},
\]
called the associated graded Lie algebra. The Lie algebra \( g \) is called gradable (or, more commonly but slightly ambiguously, graded) if it is isomorphic to its associated graded Lie algebra. A simply connected nilpotent Lie group is called gradable if its Lie algebra is gradable, and its associated graded Lie group is defined through the Lie algebra. Pansu proved in \cite{Pa2} that any two quasi-isometric gradable Lie groups are actually isomorphic.

The description of asymptotic cones of simply connected nilpotent Lie groups, due to Pansu and Breuillard \cite{Pa1, B}, can be understood with the help of the notion of cone equivalences. Namely, \cite[Theorem 6.2]{B} implies that any simply connected nilpotent Lie group \( G \) is cone-bilipschitz equivalent to the associated graded simply connected nilpotent Lie group \( G_1 \) (on the other hand, \( G \) and \( G_1 \) may be non-quasi-isometric, see Benoist-Shalom’s example in \cite[Section 4.1]{Sh}). This is explicit: if both groups are identified to their Lie algebra through the exponential map, the cone-equivalence there is just the identity, the associated graded Lie algebra having the same underlying subspace as the original one. Moreover, for a suitable choice of metrics, the multiplicative constants in the definition of cone-equivalence is one, so they induce isometries between the two asymptotic cones.

**Corollary 1.3.** Any connected Lie group \( G \) is cone-bilipschitz-equivalent to a group in the class \((\mathcal{C})\) for which moreover the nilpotent subgroup \( H \) (as in the definition of the class \((\mathcal{C})\)) is graded.

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## 2. Cone maps

In this section, \((X, d)\) is a non-empty metric space and \( x_0 \) is a given point in \( X \). We denote, for \( x \in X, |x| = d(x, x_0). \)
2.1. Asymptotic cones. The asymptotic cone $\text{Cone}_\omega(X, d)$ is defined as the ultralimit with respect to the ultrafilter $\omega$ of the sequence of pointed metric spaces

$$\left( X, \frac{1}{n}d, x \right)$$

for some $x_0 \in X$. This means that $\text{Cone}_\omega(X, d)$ is defined as follows: define $\text{Precone}(X, d)$ as the set of sequences $(x_n)_{n\geq 1}$ in $X$ such that $(d(x_n, x)/n)$ is bounded (this set does not depend on $x_0$). Endow it with the pseudo distance

$$d_\omega((x_n), (y_n)) = \lim_\omega \frac{1}{n}d(x_n, y_n),$$

and obtain $\text{Cone}_\omega(X, d)$ as the metric space obtained from $(\text{Precone}(X, d), d_\omega)$ by identifying points at distance zero. After these identifications, the constant sequence $(x)$ does not depend on $x \in X$ and appears as a natural base point for $\text{Cone}_\omega(X, d)$. See Drutu’s article [Dr] for a more detailed construction and a survey on asymptotic cones.

Example 2.1. The metric space $\mathbb{R}$ with the Euclidean distance, is isometric to any of its asymptotic cones, through the map $x \mapsto (nx)$, the reciprocal map being given by $(x_n) \mapsto \lim_\omega \frac{xn}{n}$.

2.2. Cone-defined maps. For real-valued functions, write $u \leq v$ if $u \leq Av + B$ for some constants $A, B > 0$. When applicable, $u \ll v$ means $u(g)/v(g) \to 0$ when $g \to \infty$.

Definition 2.2. Let $Y$ be another non-empty metric space, and also denote by $| \cdot |$ the distance to the given point of $Y$. A map $f : X \to Y$ is cone-defined if for every sequence $(x_n)$ in $X$, $|x_n| \leq n$ implies $|f(x_n)| \leq n$, and moreover for any sequences $(x_n), (x'_n)$ in $X$ with $|x_n|, |x'_n| \leq n$ and $d(x_n, x'_n) \ll n$, we have $d(f(x_n), f(x'_n)) \ll n$.

(Note that this does not depend on the choices of the given points.) This is exactly the condition we need to define, for every ultrafilter $\omega$, the induced map $\text{Cone}_\omega(X) \to \text{Cone}_\omega(Y)$, by mapping the class of a sequence $(x_n)$ to the class of the sequence $(f(x_n))$. The latter map preserves base points of the cones.

Example 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be any function satisfying $\lim_{|x| \to \infty} f(x)/x = 0$. Then the map $x \mapsto x + f(x)$ is cone-defined and induces the identity map at the level of all asymptotic cones.

Proposition 2.4. Let $X, Y$ be metric spaces and $f : X \to Y$ a map. Then $f$ is cone-defined if and only if the two following conditions are satisfied
• \(|f(x)| \leq |x|\);
• \((d(x,y) + 1)/(|x| + |y|) \to 0\) implies \(d(f(x), f(y))/(|x| + |y|) \to 0\).

**Proof.** The conditions are clearly sufficient. Conversely, if the first one fails, for some sequence \((x_i)\) in \(X\), we have \(|x_i| \to \infty\) and \(|f(x_i)|/|x_i| \to \infty\). If \(i(n)\) is the largest \(i\) such that \(|x_i| \leq n\), then the sequence \((x_{i(n)})\) satisfies \(x_{i(n)} \leq n\) and \(f(x_{i(n)}) \not\in Y\). The other part is similar. \(\Box\)

**Proposition 2.5.** Let \(X,Y\) be metric spaces and \(f : X \to Y\) a cone-defined map. Then, for every ultrafilter \(\omega\), the induced map \(\tilde{f} : \text{Cone}_\omega(X) \to \text{Cone}_\omega(Y)\) is continuous.

**Proof.** If \(x_0\) is the base point of the cone, it follows from the first condition of Proposition 2.4 that \(d(\tilde{f}(x), \tilde{f}(x_0)) \leq C d(x, x_0)\) for some constant \(C > 0\), for all \(x \in X\), so \(\tilde{f}\) is continuous at \(x_0\).

Suppose \(x \in \text{Cone}_\omega(X) - \{x_0\}\) and let us check that \(\tilde{f}\) is continuous at \(x\). Write \(x = (x_n)\) and let \(y = (y_n)\). For some function \(u\) tending to 0 at 0, we have \(d(f(x_n), f(y_n))/(|x_n| + |y_n|) \leq u((d(x_n, y_n) + 1)/(|x_n| + |y_n|))\) for all \(n\). We can suppose that \(u \leq C\) and is continuous. We have

\[
\frac{d(f(x_n), f(y_n))}{n} \leq \frac{|x_n| + |y_n|}{n} u \left(\frac{d(x_n, y_n) + 1}{|x_n| + |y_n|}\right);
\]

taking the limit with respect to \(\omega\), we obtain

\[
d(\tilde{f}(x), \tilde{f}(y)) \leq (|x| + |y|) u \left(\frac{d(x, y)}{|x| + |y|}\right);
\]

which tends to zero when \(y\) tends to \(x\). \(\Box\)

**Proposition 2.6.** Let \(f_1, f_2\) be cone-defined maps \(X \to Y\). Then \(f_1\) and \(f_2\) are cone equivalent if and only if for some given point \(x_0 \in X\) and for some sublinear function \(q\), we have, for all \(x \in X\)

\[
d(f_1(x), f_2(x)) \leq q(|x|).
\]

**Proof.** The proof is very similar to the previous one. Suppose the condition is satisfied. Let \((x_n)\) be a linearly bounded sequence in \(X\). Then \(d(f_1(x_n), f_2(x_n))/n \leq q(|x_n|)/n \to 0\) when \(n \to +\infty\), so \((f_1(x_n))\) and \((f_2(x_n))\) coincide in any asymptotic cone of \(Y\).

Conversely, if the condition is not satisfied, there exists a sequence \((\xi_i)\) tending to infinity in \(X\) and \(\varepsilon > 0\) such that \(d(f_1(\xi_i), f_2(\xi_i)) \geq \varepsilon|\xi_i|\) for all \(i\). For every \(n\), define \(x_n = \xi_i\), where \(i\) is chosen maximal so that \(|\xi_i| < n + 1\) (this is valid as \((|\xi_i|)_{i \to \infty}\) goes to infinity). Let \(I\) be the set of \(n\) for which \(x_n \neq x_{n-1}\). This means that \(x_n = \xi_i\), where \(n \leq |\xi_i| < n + 1\); as \((|\xi_i|)_{i \to \infty}\) is unbounded, \(I\) is infinite. Let \(\omega\) be a non-principal ultrafilter containing \(I\). Then for \(n \in I\), we have

\[
d(f_1(x_n), f_2(x_n)) \geq \varepsilon|x_n| \geq \varepsilon n,
\]

so \(\lim_{\omega} d(f_1(x_n), f_2(x_n))/n > 0\). \(\Box\)
2.3. Cone Lipschitz maps.

**Definition 2.7.** A cone-defined map \( f : X \to Y \) is called a \( C \)-cone-Lipschitz map if the induced map \( \text{Cone}_\omega(X) \to \text{Cone}_\omega(Y) \) is \( C \)-Lipschitz for all ultrafilters \( \omega \).

In particular, such a map naturally induces a \( C \)-Lipschitz map \( \text{Cone}_\omega(X) \to \text{Cone}_\omega(Y) \). Say that \( f \) is a \( \omega \)-cone Lipschitz map (or \( \omega \)-cone map for short) if it is \( \omega \)-cone \( C \)-Lipschitz for some \( C \in [0, +\infty[ \). Thus \( \text{Cone}_\omega \) is a functor from the category of non-empty metric spaces with \( \omega \)-cone Lipschitz maps to the category of pointed metric spaces with Lipschitz maps preserving base-points.

Say that \( \omega \)-cone maps \( f_1, f_2 : X \to Y \) are \( \omega \)-cone equivalent if they induce the constant map on \( \omega \)-cones. If \( f_2 \) is a constant map, we then say that \( f_1 \) is \( \omega \)-cone null. Say that a \( \omega \)-cone map \( f : X \to Y \) is \( \omega \)-cone equivalence if for some \( \omega \)-cone map \( g : Y \to X \), \( g \circ f \) and \( f \circ g \) are \( \omega \)-cone equivalent to \( \text{Id}_X \) and \( \text{Id}_Y \). Finally, say that a map \( X \to Y \) is a cone map if it is an \( \omega \)-cone map for every non-principal ultrafilter \( \omega \). Similarly, define cone equivalent maps, and cone equivalences.

**Proposition 2.8.** Let \( f \) be a map from \( X \) to \( Y \). Then \( f \) is a cone \( C \)-Lipschitz map if and only if for some sublinear function \( q \), we have, for all \( x, y \in X \)

\[
d(f(x), f(y)) \leq Cd(x, y) + q(|x| + |y|).
\]

**Proof.** Suppose that the condition is satisfied. Let \( (x_n), (y_n) \) be linearly bounded sequences in \( X \). Then

\[
d(f(x_n), f(y_n))/n - Cd(x_n, y_n)/n \leq q(|x_n| + |y_n|))/n,
\]

which tends to zero. So \( \lim_\omega d(f(x_n), f(y_n))/n - Cd(x_n, y_n)/n \leq 0 \) for all \( \omega \).

Conversely, if the condition is not satisfied, there exists sequences \( (\xi_i), (v_i) \) with \( (|\xi_i| + |v_i|) \) tending to infinity, and \( \varepsilon > 0 \) such that

\[
d(f(\xi_i), f(v_i)) - Cd(\xi_i, v_i) \geq \varepsilon(|\xi_i| + |v_i|).
\]

For every \( n \), define \( x_n = \xi_i \) and \( y_n = v_i \), where \( i = i(n) \) is chosen maximal so that

\[
|\xi_i| + |v_i| < n + 1
\]

(this is valid as \( (|\xi_i| + |v_i|) \), tends to infinity). It follows that \( (x_n) \) and \( (y_n) \) are linearly bounded. Let \( I \) be the set of \( n \) for which \( i(n) \neq i(n - 1) \). This means that for some \( i \) (which can be chosen as \( i = i(n) \)), we have \( n \leq |\xi_i| + |v_i| < n + 1 \); as the sequence \( (|\xi_i| + |v_i|) \), is unbounded, \( I \) is infinite. Let \( \omega \) be a non-principal ultrafilter containing \( I \). Then for \( n \in I \), setting \( i = i(n) \), we have

\[
d(f(x_n), f(y_n)) - Cd(x_n, y_n) = d(f(\xi_i), f(v_i)) - Cd(\xi_i, v_i)
\]

\[
\geq \varepsilon|\xi_i| + |v_i| \geq \varepsilon n. \quad \square
\]

Say that a cone-defined map \( f : X \to Y \) \( M \)-cone-expansive if, for every ultrafilter \( \omega \), the induced map \( \tilde{f} : \text{Cone}_\omega(X) \to \text{Cone}_\omega(Y) \) is \( M \)-expansive, i.e. satisfies

\[
d(\tilde{f}(x), \tilde{f}(y)) \geq Md(x, y)
\]
for all \(x, y\). Call it cone-expansive if it is \(M\)-cone-expansive for some \(M > 0\). Call a map \((C, M)\)-cone-bilipschitz if it is \(C\)-cone-Lipschitz and \(M\)-cone-expansive, and cone-bilipschitz if this holds for some positive reals \(M, C\). Also, call \(f\) cone-surjective if \(f\) is surjective for every ultrafilter.

The following proposition is proved in the same lines as the previous ones.

**Proposition 2.9.** Let \(f\) be a cone map from \(X\) to \(Y\). Then \(f\) is a cone \(M\)-expansive map if and only if for some sublinear function \(\kappa\), we have, for all \(x, y \in X\)
\[
d(f(x), f(y)) \geq Md(x, y) - \kappa(|x| + |y|)].
\]

\(\square\)

**Proposition 2.10.** Let \(f\) be a cone bilipschitz map from \(X\) to \(Y\) and \(y_0\) a basepoint in \(Y\). We have the equivalences

(i) \(f\) is cone surjective;
(ii) for some sublinear function \(c\), we have, for all \(y \in Y\), the inequality \(d(y, f(x)) \leq c(|y|)\);
(iii) \(f\) is a cone equivalence.

**Proof.** Clearly (iii) implies (i).

Suppose (ii) and let us prove (iii). Set \(c' = c + 1\). For any \(y \in Y\), choose \(x = g(y)\) with \(d(f(x), y) \leq c'(|y|)\). We claim that \(g\) is a cone map. Indeed,
\[
Md(g(y_1), g(y_2)) \leq d(f \circ g(y_1), f \circ g(y_2)) + \kappa(|g(y_1)| + |g(y_2)|)
\]
\[
\leq d(y_1, y_2) + c'(|y_1|) + c'(|y_2|) + \kappa(|g(y_1)| + |g(y_2)|).
\]

For some constant \(m\), \(\kappa(t) \leq Mt/2\) for all \(t\). Specifying to \(y_1 = y\) and \(y_2 = y_0\), we get, for all \(y \in Y\)
\[
M|g(y)| \leq M|g(y_0)| + |y| + c'(|y|) + c'(0) + M|g(y)|/2 + M|g(y_0)|/2,
\]
which gives a linear control of \(|g(y)|\) by \(|y|\). Thus for some suitable sublinear function \(\kappa'\), we have, for all \(y_1, y_2\)
\[
Md(g(y_1), g(y_2)) \leq d(f \circ g(y_1), f \circ g(y_2)) + \kappa'(|y_1| + |y_2|),
\]
so that \(g\) is cone Lipschitz.

By construction, \(f \circ g\) and \(\text{Id}_Y\) are cone equivalent. It follows that \(f \circ g \circ f\) and \(f\) are cone equivalent. Using that \(f\) is cone bilipschitz, it follows that \(g \circ f\) and \(\text{Id}_X\) are cone equivalent.

Finally suppose (ii) does not hold and let us prove the negation of (i). So there exist \(v_i\) in \(Y\) and \(\varepsilon > 0\) with \(d(v_i, f(X)) \geq \varepsilon|y_i|\). We can find, by the usual argument, a sequence \((y_n)\) in \(Y\) with \(|y_n| \leq n + 1\) for all \(n\) and an infinite subset \(I\) of integers so that for \(n \in I\), \(y_n = v_i\) for some \(i\) and \(|v_i| \geq n\). Pick a non-principal ultrafilter containing \(I\). Suppose by contradiction that the sequence \((y_n)\) is image of a sequence \((x_n)\) for the induced map \(\text{Cone}_\omega(X) \to \text{Cone}_\omega(Y)\). Then \(\lim_\omega d(f(x_n), y_n)/n = 0\). But for \(n \in I\), we have \(d(f(x_n), y_n) = d(f(x_n), v_i) \geq \varepsilon n\) and we get a contradiction.  \(\square\)
Remark 2.11. The cone Lipschitz (and even large scale Lipschitz map) \( f : \mathbb{R} \to \mathbb{R} \) mapping \( x \) to \( x^{1/3} \) is surjective, however the induced map on the cones is constant, so \( f \) is not cone surjective.

2.4. An example of a cone-defined map which is not cone Lipschitz. Let \( v \) be a map of \( \mathbb{N} \) onto \( \mathbb{N}_{>0} \) with infinite fibers. Consider the following metric subspace of \( \mathbb{R} \)

\[
X = \bigcup_{n \in \mathbb{N}} \{2^{2n}, 2^{2n}(1 + v(n)^{-2})\}.
\]

Let \( i \) be the embedding of \( X \) into \( \mathbb{R} \) defined as follows

\[
i(2^{2n}) = 2^{2n}; \quad i(2^{2n}(1 + v(n)^{-2})) = 2^{2n}(1 + v(n)^{-1}).
\]

Then \( i \) is a cone map but is not cone \( C \)-Lipschitz for any \( C \). This can be seen directly from Propositions 2.4 and 2.8, where, in the second one, we take \( x_n = 2^{2n}, y_n = 2^{2n}(1 + v(n)^{-2}) \), when \( n \) ranges over an infinite fiber \( v^{-1}(c) \) for some \( c > C \). This can also be seen by making the asymptotic cone of \( X \) explicit. All the verifications of the facts below are easy but somewhat tedious to write down, and are left to the reader.

As the inclusion of \( X \) into \( \mathbb{R} \) is an isometry, we can view \( \text{Cone}_\omega(X) \) as a subset of \( \mathbb{R} \) through the embedding \( (x_n) \mapsto \lim \omega x_n \).

Set \( Y = \{2^{2n} | n \in \mathbb{N}\} \). Denote by \( u \) the “projection” of \( \mathbb{R}_{>0} \) onto \( Y \) defined as follows: let \( U(x) \) be the set of points \( y \) in \( Y \) that minimize the “log-distance” \( |\log(y/x)| \); it consists of one or two points; we define \( u(x) \) to be the largest point in \( U(x) \). Similarly define \( u' \) as the “projection” to the subset \( X - Y \) of \( X \).

Define

\[
\ell = \lim_\omega u(n)/n, \quad \ell' = \lim_\omega u'(n)/n \in [0, \infty].
\]

Then \( \text{Cone}_\omega(X) = \{0, \ell, \ell'\} \cap [0, \infty) \). Let us be more precise.

Then either

\[
\begin{align*}
&\bullet \ell, \ell' \in \{0, \infty\}, \quad \text{Cone}_\omega(X) = \{0\} \text{ (Case 1).} \\
&\bullet \ell, \ell' \notin \{0, \infty\}, \quad \ell = \ell', \quad \text{Cone}_\omega(X) = \{0, \ell\} \text{ (Case 2);} \\
&\bullet \ell, \ell' \notin \{0, \infty\}, \quad \text{Cone}_\omega(X) = \{0, \ell, \ell'\} \text{ (Case 3).}
\end{align*}
\]

Set \( \lambda = \lim_\omega v(n) \in [1, \infty] \), so \( \lambda = \infty \) in Case 2 and \( \lambda < \infty \) in Case 3, in which case we have \( \ell' = (1 + \lambda^{-2})\ell \).

The map \( \tilde{i} \) is given as follows:

\[
\begin{align*}
&\bullet \text{Case 1: } 0 \mapsto 0 \\
&\bullet \text{Case 2: } 0 \mapsto 0, \ell \mapsto \ell \\
&\bullet \text{Case 3: } 0 \mapsto 0, \ell \mapsto \ell, (1 + \lambda^{-2})\ell \mapsto (1 + \lambda^{-1})\ell.
\end{align*}
\]

Its best Lipschitz constant is 0 is Case 1, 1 in Case 2, and \( \lambda \) in Case 3. As \( \lambda \) can be arbitrary large and finite (because \( v \) is surjective with infinite fibers), this best Lipschitz constant is unbounded when \( \omega \) ranges over ultrafilters.
3. Cone equivalences between Lie groups

3.1. Reduction to the split case with semisimple action. Let $G$ be a triangulable group. We shamelessly use the identification between the Lie algebra and the Lie group through the exponential map. Precisely, $G$ has two laws, the group multiplication, and the addition $g + h$, which formally denotes $\exp(\log(x) + \log(y))$, using that the exponential function is duly a homeomorphism for a triangulable group [Di].

Let $R$ be the exponential radical of $G$, $H$ a Cartan subgroup, $W = H \cap R$, and $V$ a complement subspace of $W$ in $H$.

We denote the word length in a group by $|\cdot|$, by default the group is $G$, otherwise we make it explicit, e.g. $|\cdot|_H$ denotes word length in $H$.

If $h \in H$, it can be written in a unique way as $rv$ with $r \in W$ and $v \in V$, we write $r = \delta(h)$ and $v = [h]$.

**Lemma 3.1.** For $v \in V$ we have

$$|v| \simeq |v|_{G/R}.$$ 

This is obtained in the proof of [Co, Theorem 5.1] (this is proved by writing $v = hw$ where $|h| = |v|_{G/R}$, $w \in W$, $h \in H$, using $|w|_H \leq |v|_H + |h|_H$ then $|v| \leq |h| + |w|$ and using Guivarc’h’s estimates on the length of an element compared to its norm).

**Lemma 3.2.** For $v, v' \in V$ we have

$$|\delta(v^{-1}v')| \leq \log(1 + |v|_{G/R}) + \log(1 + |v'|_{G/R}).$$

**Proof.** As $\delta(v^{-1}v') = v^{-1}v'[v^{-1}v']^{-1}$,

$$|\delta(v^{-1}v')|_H \leq |v|_H + |v'|_H + |[v^{-1}v']|_H,$$

so by Lemma 3.1

$$|\delta(v^{-1}v')|_H \leq |v|_{G/R} + |v'|_{G/R} + |v^{-1}v'|_{G/R} \leq |v|_{G/R} + |v'|_{G/R}.$$

Since $\delta(v^{-1}v')$ belongs to the exponential radical,

$$|\delta(v^{-1}v')| \leq \log(1 + |\delta(v^{-1}v')|_H)$$

and we get the conclusion. □

**Lemma 3.3.** For $r \in R, v \in V$,

$$|rv| \simeq |r| + |v|_{G/R}.$$ 

**Proof.** Clearly $|v|_{G/R} \simeq |v|$, so we obtain the inequality $\simeq$.

From the projection $G \to G/R$, we get $|v|_{G/R} \simeq |rv|$. On the other hand $|r| \leq |rv| + |v|$. Now $|v| \leq |v|_{G/R}$ by Lemma 3.1 So $|rv| \geq |r| + |v|_{G/R}$. □
Consider the action of $H$ on $\mathfrak{r}$ (it is wary here to distinguish the Lie group and its Lie algebra), given by restriction of the adjoint representation. It is given by a homomorphism $\alpha$ from $H$ to the automorphism group of the Lie algebra $\mathfrak{r}$, which is an algebraic group. This action is triangulable, and we can write in a natural way, for every $h \in H$, $\alpha(h) = \beta(h)u(h)$ where $\beta$ is the diagonal part and $u(h)$ is the unipotent part; both are Lie algebra automorphisms of $\mathfrak{r}$ (view this by taking the Zariski closure of $\alpha(H)$, which can be written in the form $DU$ with $D$ a maximal split torus and $U$ the unipotent radical), and $\beta$ defines a continuous action of $H$ on $\mathfrak{r}$. Note that this action is trivial on $[H,H]$ and hence on $W$.

As $H$ is nilpotent, we can decompose $\mathfrak{r}$ into characteristic subspaces for the $\alpha$-action; this way we see in particular that $u$ is an action as well (this strongly relies on the fact that $H$ is nilpotent and connected). In such a decomposition, we see that the matrix entries of $u(h)$, for $h \in H$, are polynomially bounded in terms of $|h|$ (more precisely, $\leq C|h|^k$, where $C$ is a constant (depending only on $G$ and the choice once and for all of a basis adapted to the characteristic subspaces) and $k + 1$ is the dimension of $\mathfrak{r}$).

Now define $A(h)$, $B(h)$, resp. $U(h)$, as the automorphism of $R$ whose tangent map is $\alpha(h)$, $\beta(h)$, resp. $u(h)$. Note that $\alpha(h)$ is the left conjugation by $h$ in $R$. Then $B$ provides a new action of $H$ on $R$. We consider the group $R \rtimes H/W$ defined by the action $B$. We write it by abuse of notation $R \rtimes V$, identifying $V$, as a group, to $H/W$.

Consider the map

$$\psi: G = RV \to R \rtimes V, \quad rv \mapsto rv.$$

In general $\psi$ is not a quasi-isometry (and is even not a coarse map, i.e. there exists a sequence of pairs of points at bounded distance mapped to points at distance tending to infinity).

**Theorem 3.4.** The map $\psi$ is a cone-bilipschitz equivalence.

**Lemma 3.5.** If $a, b \geq 0$ and $c \geq 1$, and if $|\log(a) - \log(b)| \leq \log(c)$, then $|\log(1 + a) - \log(1 + b)| \leq \log(c)$. \[\square\]

**Proof of Theorem 3.4.** On both groups, the word length is equivalent by Lemma 3.3 to $L(rv) = \ell(r) + |v|_{G/R}$, where $\ell$ is a length (subadditive and symmetric) on $R$ with $\ell(r) \simeq \log(1 + \|r\|)$, and we consider the corresponding left-invariant “distances” $d_1$ and $d_2$.

We consider two group laws at the same time on the Cartesian product $R \times V$; in order not to introduce tedious notation, we go on writing both group laws by the empty symbol; on the other hand to avoid ambiguity, we write the length $L$ as $L = L_1 = L_2$ (and $\ell = \ell_1 = \ell_2$) and when we compute a multiplication inside

\[\text{ These are not necessarily distances but are equivalent to the left-invariant word or Riemannian distances.}\]
the symbols $L_1()$ or $d_1(,)$, we mean the multiplication inside $G$, while in $L_2$ and $d_2$ we mean the new multiplication from $R \rtimes V$.

We have, for $r, r' \in R; v, v' \in V$
\[ d_1(rv, r'v') = L_1(v^{-1}r^{-1}r'v') = L_1(A(v^{-1})(r^{-1}r')v^{-1}v') \]
\[
\begin{align*}
\text{We have} \quad d_1(rv, r'v') &= L_1(A(v^{-1})(r^{-1}r')\delta(v^{-1}v')[v^{-1}v']) \\
&= \ell_1(A(v^{-1})(r^{-1}r')\delta(v^{-1}v')) + |v^{-1}v'|_{G/R}
\end{align*}
\]
and
\[
\begin{align*}
\text{and} \quad d_2(rv, r'v') &= \ell_2(B(v^{-1})(r^{-1}r')) + |v^{-1}v'|_{G/R}
\end{align*}
\]
(no $\delta$-term for $d_2$ because $V$ is a subgroup for the second law).

So
\[
\begin{align*}
d_1(rv, r'v') - d_2(rv, r'v') &= \ell_1(A(v^{-1})(r^{-1}r')\delta(v^{-1}v')) - \ell_1(B(v^{-1})(r^{-1}r')).
\end{align*}
\]
Write for short $\rho = B(v^{-1})(r^{-1}r')$ (which defines the same element for the two group laws), so
\[
\begin{align*}
d_1(rv, r'v') - d_2(rv, r'v') &= \ell_1(U(v^{-1})(\rho)\delta(v^{-1}v')) - \ell_2(\rho).
\end{align*}
\]
We then get
\[
|d_1(rv, r'v') - d_2(rv, r'v')| = |\ell(U(v^{-1})(\rho)) - \ell(\rho)| + \ell(|v^{-1}v'|).
\]
(We see that we can write $\ell$ because this does no longer depends on the choice of one of the two laws.)

Let us now work on the manifold $R$, endowed with a Riemannian length $\lambda$. As the tangent map of $U(v^{-1})$ is $C|v|^k$-bilipschitz (i.e. both it and its inverse are $C|v|^k$-Lipschitz), the same is true for $U(v^{-1})$ itself. In particular,
\[
\frac{1}{C|v|^k}\lambda(\rho) \leq \lambda(U(v^{-1})\rho) \leq C(1 + |v|)^k\lambda(\rho).
\]
This means that
\[
|\log(\lambda(\rho)) - \log(U(v^{-1})\rho)| \leq \log(C) + k \log(1 + |v|).
\]
By Lemma 3.5 (we can pick $C \geq 1$),
\[
|\log(1 + \lambda(\rho)) - \log(1 + U(v^{-1})\rho)| \leq \log(C) + k \log(1 + |v|),
\]
thus
\[
|\ell(\rho) - \ell(U(v^{-1})\rho)| \leq \log(1 + |v|);
\]
moreover by Lemma 3.2
\[
\delta(v^{-1}v') \leq \log(1 + |v|) + \log(1 + |v'|).
\]
Accordingly,
\[
|d_1(rv, r'v') - d_2(rv, r'v')| \leq \log(1 + |v|) + \log(1 + |v'|) \ll |rv| + |r'v'|.
\]
Thus the map $\psi$ is a cone equivalence (with constants one). This means that the cones defined from $d_1$ and $d_2$ are isometric; however note that these are equivalent to metrics but are not necessarily metric (they are maybe not subadditive) so the statement obtained for the usual cones (defined with genuine metrics) are that they are bilipschitz. □

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