THE MULTIDIMENSIONAL TRUNCATED MOMENT PROBLEM:
ATOMS, DETERMINACY, AND CORE VARIETY

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Abstract. This paper is about the moment problem on a finite-dimensional vector space of continuous functions. We investigate the structure of the convex cone of moment functionals (supporting hyperplanes, exposed faces, inner points) and treat various important special topics on moment functionals (determinacy, set of atoms of representing measures, core variety).

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1. Introduction

Let \( \mathbb{N} \) be a finite subset of \( \mathbb{N}_0 \), \( n \in \mathbb{N} \), and \( A = \{ x^\alpha : \alpha \in \mathbb{N} \} \), \( A = \text{Lin} A \) the span of associated monomials, \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \). Suppose that \( K \) is a closed subset of \( \mathbb{R}^n \). Let \( s = (s_\alpha)_{\alpha \in \mathbb{N}} \) be a real sequence and let \( L_s \) denote the corresponding Riesz functional on \( A \) defined by \( L_s(x^\alpha) = s_\alpha \), \( \alpha \in \mathbb{N} \).

The truncated moment problem asks: When does there exist a (positive) Radon measure \( \mu \) on \( K \) such that \( x^\alpha \) is \( \mu \)-integrable and

\[
(1) \quad s_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu \quad \text{for all} \quad \alpha \in \mathbb{N}?
\]

Clearly, (1) is equivalent to

\[
(2) \quad L_s(f) = \int_K f(x) \, d\mu \quad \text{for} \quad f \in A.
\]

The Richter–Tchakaloff theorem (Proposition \ref{richter-tchakaloff}) implies that in the affirmative case there is always a finitely atomic measure \( \mu \) satisfying (1) and (2).

The multidimensional truncated moment problem was first studied in the unpublished Thesis of J. Matzke \cite{matzke} and by R. Curto and L. Fialkow \cite{curto-fialkow1}, \cite{curto-fialkow2}, see \cite{fialkow} for a nice survey. The one-dimensional case is treated in the monographs \cite{fialkow}, \cite{curto-fialkow}.

In the present paper we consider the truncated moment problem in a more general setting. That is, we study moment functionals on a finite-dimensional vector space \( E \) of real-valued continuous functions on a locally compact topological Hausdorff space \( X \). The bridge to the truncated \( K \)-moment problem for polynomials as formulated above is obtained by letting \( E = \{ f \mid K \} \) be the vector space of restrictions of functions \( f \in A \) to \( X := K \). In this manner the results of this paper give new results concerning the truncated \( K \)-moment problem for polynomials.

Let us briefly describe the structure and the contents of this paper. In Section \ref{section:notation} we recall basic notation, definitions and facts on moment sequences and moment functionals. Let \( L \) be a moment functional on \( E \). The set \( W(L) \) of possible atoms of representing measures of \( L \) is investigated in Section \ref{section:atoms}. In Section \ref{section:determinacy} we characterize the determinacy of \( L \) in terms of the set \( W(L) \) (Theorem \ref{determinacy}). Three other important notions associated with \( L \) are studied in Sections \ref{section:nonnegative} and \ref{section:core}. These are the cone \( N_+(L) \) of nonnegative functions of \( E \) which are annihilated by \( L \), the zero set \( V_+(L) \) of \( N_+(L) \) and the core variety \( V(L) \) introduced by L. Fialkow \cite{fialkow}. It is easily seen that
\(W(L) \subseteq V_+(L)\). Equality holds if and only if the moment sequence of \(L\) lies in the relative interior of an exposed face of the moment cone (Theorem 40). It is proved that the set \(W(L)\) is equal to the core variety \(V(L)\) (Theorem 52). In the last Section 4 we assume that \(X = \mathbb{R}^n\) and \(E \subseteq C^1(\mathbb{R}^n; \mathbb{R})\). Then the total derivative of the moment map is used to analyze the structure of the moment cone. A number of characterizations of inner points of the moment cone are given (Theorem 49).

2. Moment sequences and moment functionals

Throughout this paper, we will suppose the following:

- \(X\) is a locally compact topological Hausdorff space,
- \(E\) is a finite-dimensional vector space of real continuous functions on \(X\),
- \(F := \{f_1, \ldots, f_m\}\) is a fixed vector space basis of \(E\).

For a real sequence \(s = (s_j)_{j=1}^m\), the Riesz functionals \(L_s\) is the linear functional \(L_s\) on \(E\) defined by \(L_s(f) = s_j, j = 1, \ldots, m\). This one-to-one correspondence between real sequences and real linear functionals on \(E\) is often used in what follows.

Let \(M_+(X)\) denote the set of Radon measures on \(X\). A measure \(\mu : \mathcal{B}(X) \to [0, +\infty]\) on the Borel \(\sigma\)-algebra \(\mathcal{B}(X)\) such that

\[\mu(M) = \sup \{\mu(K) : K \subseteq M, K \text{ compact}\} \quad \text{for} \quad M \in \mathcal{B}(X).\]

Note that in our terminology Radon measures are always nonnegative!

For \(\mu \in M_+(X)\), let \(L^1(X, \mu)\) denote the real-valued \(\mu\)-integrable Borel functions on \(X\). For \(x \in X\), let \(\delta_x \in M_+(X)\) be defined by \(\delta_x(M) = 1\) if \(x \in M\) and \(\delta_x(M) = 0\) if \(x \notin M\). A measure \(\mu \in M_+(X)\) such that \(\text{supp } \mu = k\) is called \(k\)-atomic; this means that there are \(k\) pairwise different points \(x_1, \ldots, x_k\) of \(X\) and positive numbers \(c_1, \ldots, c_k\) such that \(\mu = \sum_{j=1}^k c_j \delta_{x_j}\). We consider the zero measure as \(0\)-atomic measure. For \(f \in C(X; \mathbb{R})\) we set \(Z(f) := \{x \in X : f(x) = 0\}\).

**Definition 1.** We say that a real sequence \(s = (s_j)_{j=1}^m\) is a moment sequence and the linear functional \(L_s\) is a moment functional if there exists a measure \(\mu \in M_+(X)\) such that \(E \subseteq L^1(X, \mu)\) and

\[s_j = \int_X f_j(x) \, d\mu \quad \text{for} \quad j = 1, \ldots, m,
\]
or equivalently,

\[L_s(f) = \int_X f(x) \, d\mu, \quad \text{for} \quad f \in E.
\]

Any such measure \(\mu\) is called a representing measure of \(s\) resp. \(L_s\). The set of all representing measures of \(s\) resp. \(L_s\) is denoted by \(M_s = M_{L_s}\).

The moment cone \(S\) is the set of all moment sequences. The set of all moment functionals is denoted by \(\mathcal{L}\).

Clearly, \(S\) is a cone in \(\mathbb{R}^m\) and \(\mathcal{L}\) is a cone in the dual space of \(E\). The map \(s \mapsto L_s\) is a bijection of \(S\) to \(\mathcal{L}\).

Thus, we have a one-to-one correspondence between moment sequences \(s\) and moment functionals \(L_s\). At some places we prefer to work with moment sequences, while at others moment functionals are more convenient. Let us adopt the following notational convention: If we introduce a set depending on a general moment sequence \(s\) (or moment functional \(L_s\)), we will take the same set for the moment functional \(L_s\) (or moment sequence \(s\)). That is, for the sets introduced in what follows we define \(N_s(s) = N_+(L_s), V_+(s) = V_+(L_s), W(s) = W(L_s), V(s) = V(L_s)\).

**Remark 2.** Let us discuss briefly how the results on moment functionals on \(E\) apply to the truncated \(K\)-moment problem on \(A\) stated in the introduction. We
set $X = K$ and consider the subspace $E := A[X]$ of $C(X; \mathbb{R})$. Let $L$ be a linear functional on $A$. If

$$L(f) = 0 \text{ for } f \in A \text{ with } f|K = 0,$$

then there exists a well-defined (!) linear functional $\tilde{L}$ on $E$ given by

$$\tilde{L}(f[K]) := L(f), \quad f \in A,$$

and the results on moment functionals on $E$ can be applied to $\tilde{L}$. There are two important cases where (3) is satisfied. First, if $f|K = 0$ implies $f = 0$; this happens (for instance) if $K$ has a nonempty interior in $\mathbb{R}^n$. Secondly, if $L(f) \geq 0$ for all $f \in A$ such that $f \geq 0$ on $K$. Then (3) holds. (Indeed, if $f|K = 0$, then $\pm f \geq 0$ on $K$, hence $L(\pm f) \geq 0$, so that $L(f) = 0$.) This second case is valid if $L$ is a moment functional which has representing measure supported on $K$.

The following well-known fact will be often used.

**Lemma 3.** Let $f \in C(X; \mathbb{R})$ and $\mu \in M_+(X)$. Suppose that $f(x) \geq 0$ for $x \in X$ and $\int f(x) \, d\mu = 0$. Then

$$\text{supp } \mu \subseteq Z(f) \equiv \{x \in X : f(x) = 0\}.$$

**Proof.** Let $x_0 \in X$. Suppose that $x_0 \notin Z(f)$. Then $f(x_0) > 0$. Since $f$ is continuous, there exist an open neighborhood $U$ of $x_0$ and a number $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ on $U$. Then

$$0 = \int_X f(x) \, d\mu \geq \int_U f(x) \, d\mu \geq \varepsilon \mu(U) \geq 0,$$

so that $\mu(U) = 0$. Therefore, since $U$ is an open set containing $x_0$, it follows at once from the definition of the support that $x_0 \notin \text{supp } \mu$. \qed

A crucial result is the following **Richter–Tchakaloff theorem**: it was proved in full generality by H. Richter \[12\] and in the compact case by V. Tchakaloff \[15\].

**Proposition 4.** Suppose that $(X, \mu)$ is a measure space and $V$ is a finite-dimensional real subspace of $L^1(X, \mu)$. Let $L^\mu$ be the linear functional on $V$ defined by $L^\mu(f) = \int f \, d\mu$, $f \in V$. Then there is a $k$-atomic measure $\nu = \sum_{j=1}^k m_j \delta_{x_j} \in M_+(X)$, where $k \leq \dim V$, such that $L^\mu = L^\nu$, that is,

$$\int_X f \, d\mu = \int_X f \, d\nu \equiv \sum_{j=1}^k m_j f(x_j), \quad f \in V.$$

An immediate consequence of Proposition 4 is the following.

**Corollary 5.** Each moment functional on $E$ of has a $k$-atomic representing measure, where $k \leq \dim E$.

For $C \subseteq E$, a functional $L$ on $E$ is called $C$-positive if $L(f) \geq 0$ for $f \in C$. Put

$$E_+ := \{f \in E : f(x) \geq 0 \text{ for } x \in X\}.$$

Obviously, each moment functional is $E_+$-positive.

The dual cone of the cone $E_+$ is the cone in the dual space $E^*$ of $E$ defined by

$$(E_+)^\circ = \{L \in E^* : L(f) \geq 0 \text{ for } f \in E_+\}.$$

**Definition 6.** A linear functional $L$ on $E$ is called strictly $E_+$-positive if

$$L(f) > 0 \text{ for all } f \in E_+, f \neq 0.$$

Note that $E_+ = \{0\}$ is possible and then every $L$ is strictly $E_+$-positive.
Lemma 7. Let \( \| \cdot \| \) be a norm on \( E \). For a linear functional \( L \) on \( E \) the following are equivalent:

(i) \( L \) is strictly \( E_+ \)-positive.

(ii) There exists a number \( c > 0 \) such that

\[
L(f) \geq c\|f\| \quad \text{for } f \in E_+.
\]

(iii) \( L \) is an interior point of the cone \( (E_+)^\wedge \) in \( E^* \).

Proof. If \( E_+ = \{0\} \), then all assertions are trivially true. So we assume \( E_+ \neq \{0\} \).

(i)\(\rightarrow\)(ii): Consider the set \( U_+ = \{ f \in E_+: \|f\| = 1 \} \). Since each point evaluation \( l_x, x \in X \), is continuous on the finite-dimensional normed space \((E, \| \cdot \|)\), \( E_+ \) is closed in \( E \). Hence, \( U_+ \) is a bounded closed, hence compact, subset of \((E, \| \cdot \|)\).

Therefore, since the functional \( L \) is also continuous on \((E, \| \cdot \|)\), the infimum of \( L(f) \) on \( U_+ \) is attained, say at \( f_0 \in U_+ \). Then \( f_0 \neq 0 \) and \( f \in E_+ \), so that \( c := L(f_0) > 0 \) by (i). Hence \( L(f) \geq c \) for \( f \in U_+ \). By scaling this yields (ii).

(ii)\(\rightarrow\)(iii): We equip \( E^* \) with the dual norm of \( \| \cdot \| \). Suppose that \( L_1 \in E^* \) and \( \| L - L_1 \| < c \). Then (ii) implies that \( L_1(f) \geq 0 \) for \( f \in E_+ \), that is, \( L_1 \in (E_+)^\wedge \). This shows that \( L \) is an interior point of the cone \((E_+)^\wedge \).

(iii)\(\rightarrow\)(i): Let \( f \in E_+, f \neq 0 \). Then there exists \( x \in X \) such that \( f(x) > 0 \). Since the point evaluation \( l_x \) at \( x \) is in \((E_+)^\wedge \) and \( L \) is an inner point of \((E_+)^\wedge \), there exists a number \( \varepsilon > 0 \) such that \((L - \varepsilon l_x) \in (E_+)^\wedge \). Hence \( L(f) \geq \varepsilon f(x) > 0 \). \(\square\)

Let \( \overline{\mathcal{L}} \) denote the closure of the cone \( \mathcal{L} \) in the norm topology of \( E^* \).

Lemma 8. \((E_+)^\wedge = \overline{\mathcal{L}}\).

Proof. Clearly, if \( L \in \mathcal{L} \) and \( p \in E_+ \), then \( L(p) \geq 0 \). Thus, \( \mathcal{L} \subseteq (E_+)^\wedge \). Therefore, since \((E_+)^\wedge \) is obviously closed, \( \overline{\mathcal{L}} \subseteq (E_+)^\wedge \).

Now we prove the converse inclusion \((E_+)^\wedge \subseteq \overline{\mathcal{L}}\). Assume to the contrary that there exists a functional \( L_0 \in (E_+)^\wedge \) such that \( L_0 \notin \overline{\mathcal{L}} \). Then, by the separation theorem for convex sets applied to the closed cone \( \overline{\mathcal{L}} \) in \( E^* \), there is a linear functional \( F \) on \( E^* \) such that \( F(L_0) < 0 \) and \( F(L) \geq 0 \) for \( L \in \mathcal{L} \). Since \( E \) is finite-dimensional, there is a (unique) element \( f \in E \) such that \( F(L) = L(f) \) for all \( L \in E^* \). Let \( x \in X \). Then the point evaluation \( l_x \) at \( x \) is \( L \), so that \( F(l_x) = l_x(f) = f(x) \geq 0 \). Hence \( f \in E_+ \). Therefore, since \( L_0 \in (E_+)^\wedge \), we get \( F(L_0) = L_0(f) \geq 0 \) which is a contradiction. Thus \((E_+)^\wedge \subseteq \overline{\mathcal{L}}\). \(\square\)

The next proposition is of similar spirit as a result proved in [7].

Proposition 9. Each strictly \( E_+ \)-positive linear functional on \( E \) is a moment functional.

Proof. Let \( L \) be a strictly \( E_+ \)-positive functional on \( E \). Then \( L \) is an inner point of \((E_+)^\wedge \) by Lemma [7] and hence of \( \overline{\mathcal{L}} \) by Lemma [8]. Since the convex set \( \mathcal{L} \) and its closure \( \overline{\mathcal{L}} \) have the same inner points, \( L \) is also an inner point of \( \mathcal{L} \). In particular, \( L \) belongs to \( \mathcal{L} \), that is, \( L \) is a moment functional. \(\square\)

3. The set \( \mathcal{W}(L) \) of atoms

In this subsection, we assume that the following condition is satisfied:

\[(7) \quad \text{For each } x \in X \text{ there exists a function } f_x \in E_+ \text{ such that } f_x(x) > 0.\]

The following important concepts appeared already in [11] and [14].
**Definition 10.** For a moment functional $L$ on $E$ we define

\[ (8) \quad \mathcal{N}_+(L) = \{ f \in E_+ : L(f) = 0 \}, \]

\[ (9) \quad \mathcal{V}_+(L) = \{ x \in \mathcal{X} : f(x) = 0 \text{ for all } f \in \mathcal{N}_+(L) \}, \]

\[ (10) \quad \mathcal{W}(L) = \{ x \in \mathcal{X} : \mu(\{x\}) > 0 \text{ for some } \mu \in \mathcal{M} \}. \]

Thus, $\mathcal{W}(L)$ is the set of points $x \in \mathcal{X}$ which are atoms of some representing measure $\mu$ of $L$. In the important special case $E = \mathcal{A}$, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{N}_+(L)$ consists of real polynomials and $\mathcal{V}_+(L)$ is a real algebraic set. The sets $\mathcal{V}_+(L)$ and $\mathcal{W}(L)$ are fundamental notions in the theory of the truncated moment problem.

**Lemma 11.** Let $L$ be a moment functional on $E$.

(i) $\mathcal{W}(L) \subseteq \mathcal{V}_+(L)$.

(ii) If $L = 0$ and $\mu \in \mathcal{M}$, then $\mu = 0$.

(iii) The set $\mathcal{W}(L)$ is not empty if and only if $L \neq 0$.

**Proof.**

(i): Let $x \in \mathcal{W}(L)$. By (10) there is a measure $\mu \in \mathcal{M}$ such that $\mu(\{x\}) > 0$. For $f \in \mathcal{N}_+(L)$, we obtain

\[ 0 = L(f) = \int f(y)\,d\mu \geq f(x)\mu(\{x\}) > 0. \]

Since $\mu(\{x\}) > 0$, it follows that $f(x) = 0$. Thus $x \in \mathcal{V}_+(L)$.

(ii): Let $x \in \mathcal{X}$ and let $f_x \in E_+$ be the function from condition (7). Since $L = 0$, we have $L(f_x) = 0$. Hence $\text{supp } \mu \subseteq \mathcal{Z}(f_x)$ by Lemma 3. Therefore, $\text{supp } \mu \subseteq \cap_{x \in \mathcal{X}} \mathcal{Z}(f_x)$. Since the latter set is empty by (7), $\mu = 0$.

(iii): By Corollary 5, $\mathcal{L}$ has a finitely atomic representing measure $\mu$. If $L \neq 0$, then $\mu \neq 0$, so $\mathcal{W}(L)$ is not empty. If $L = 0$, then $\mu = 0$ by (i), so $\mathcal{W}(L)$ is empty. \hfill \square

A natural and important question is whether or not there is equality in Lemma 11(i). The following examples show that this is not true in general, but it holds for the one dimensional truncated moment problem on $[0, 1]$.

**Example 12.** Let $\mathcal{X}$ be the subspace of $\mathbb{R}^2$ consisting of the three points $(-1, 0)$, $(0, 0)$, $(1, 0)$ and the two lines $\{(t, 1) : t \in \mathbb{R}\}$, $\{(t, -1) : t \in \mathbb{R}\}$. Let $E$ be the restriction to $\mathcal{X}$ of the polynomials $\mathbb{R}[x_1, x_2]_2$ of degree at most 2. We easily verify that the restriction map $f \mapsto f\lceil \mathcal{X}$ on $\mathbb{R}[x_1, x_2]_2$ is injective; for simplicity we write $f$ instead of $f\lceil \mathcal{X}$ for $f \in \mathbb{R}[x_1, x_2]_2$.

We consider the moment functional $L$ defined by

\[ (11) \quad L(f) = f(-1, 0) + f(1, 0), f \in E. \]

We show that $\mathcal{N}_+(L) = \{ x_2(bx_2 + c) : |c| \leq b, b, c \in \mathbb{R} \}$. It is obvious that these polynomials are in $\mathcal{N}_+(L)$. Conversely, let $f \in \mathcal{N}_+(L)$. Then $f(1, 0) = f(-1, 0) = 0$, so that $f = x_2(ax_1 + bx_2 + c) + d(1-x_1^2)$, with $a, b, c, d \in \mathbb{R}$. Further, $d = f(0, 0) \geq 0$. From $f(t, \pm 1) \geq 0$ for all $t \in \mathbb{R}$ we conclude that $d = 0$ and $|c| \leq b$.

The zero set $\mathcal{V}_+(L)$ of $\mathcal{N}_+(L)$ is the intersection of $\mathcal{X}$ with the $x_1$-axis, that is, $\mathcal{V}_+(L) = \{(-1, 0), (0, 0), (1, 0)\}$. Let $\mu$ be an arbitrary representing measure of $L$. Then, since $\mu$ is supporting on $\mathcal{V}_+(L)$, there are numbers $\alpha, \beta, \gamma \geq 0$ such that $\mu = \alpha \delta_{(-1,0)} + \beta \delta_{(0,0)} + \gamma \delta_{(1,0)}$. By (7), we have $L(x_1) = 0 = \int x_1\,d\mu = -\alpha + \gamma$ and $L(x_1^2) = 2 = \int x_1^2\,d\mu = \alpha + \gamma$, which implies that $\alpha = \gamma = 1$. Therefore, since $L(1) = 2 = \int 1\,d\mu = \alpha + \beta + \gamma$, it follows that $\beta = 0$. Hence, $\mu(\{(0,0)\}) = 0$, so that $(0, 0) \notin \mathcal{W}(L)$. Thus, $\mathcal{W}(L) \neq \mathcal{V}_+(L)$.

The preceding proof shows that $L$ has a unique representing measure. \hfill \diamond

**Example 13.** Let $\mathcal{A} := \{1, x, \ldots, x^m\}$, and $\mathcal{X} := [0, 1]$. Then we have $\mathcal{W}(L) = \mathcal{V}_+(L)$ for each moment functional on $E$. Indeed, if the corresponding moment
sequence \( s \) is an inner point of the moment cone, then \( \mathcal{N}_s(L) = \{0\} \) and each point of \([0,1]\) is an atom of a representing measure \([8, \text{Corollary II.3.2}]\). If \( s \) is a boundary point of the moment cone, then \( s \) has a unique representing measure \( \mu \) \([8, \text{Theorem II.2.1}]\). In the first case \( \mathcal{V}_+(L) = \mathcal{W}(L) = [0,1] \), while \( \mathcal{V}_+(L) = \mathcal{W}(L) = \text{supp} \mu \) in the second case.

**Lemma 14.** Suppose that \( L \) is a moment functional on \( E \).

(i) If \( \mu \in \mathcal{M}_L \) and \( M \subseteq X \) be a Borel set containing \( \mathcal{W}(L) \), then \( \mu(X \setminus M) = 0 \).

(ii) If \( \mathcal{W}(L) \) is finite, there exists a \( \mu \in \mathcal{M}_L \) such that \( \text{supp} \mu = \mathcal{W}(L) \).

(iii) If \( \mathcal{W}(L) \) is infinite, then for any \( n \in \mathbb{N} \) there exists a measure \( \mu \in \mathcal{M}_L \) such that \( |\text{supp} \mu| \geq n \).

**Proof.** The proofs of all three assertions use Proposition \([8, \text{Theorem II.2.1}]\).

(i): Assume to the contrary that \( \mu(X \setminus M) > 0 \) and define linear functionals \( L_1 \) and \( L_2 \) on \( E \) by

\[
L_1(f) = \int_M f(x) \, d\mu \quad \text{and} \quad L_2(f) = \int_{X \setminus M} f(x) \, d\mu.
\]

Applying Proposition \([8, \text{Theorem II.2.1}]\) to the functionals \( L_1 \) and \( L_2 \) and the measure spaces \( M \) and \( X \setminus M \), respectively, with measures induced from \( \mu \), we conclude that \( L_1 \) and \( L_2 \) have finitely atomic representing measures \( \mu_1 \) and \( \mu_2 \) with atoms in \( M \) and \( X \setminus M \), respectively. Since \( \mu \in \mathcal{M}_L \), we have \( L = L_1 + L_2 \) and hence \( \tilde{\mu} := (\mu_1 + \mu_2) \in \mathcal{M}_L \).

From \( \mu(X \setminus M) > 0 \) and Lemma \([8, \text{Theorem II.2.1}] \) it follows that \( L_2 \neq 0 \). Hence \( \mu_2 \neq 0 \). Therefore, if \( x_0 \in X \setminus M \) is an atom of \( \mu \), then \( \tilde{\mu}(\{x_0\}) = \mu_2(\{x_0\}) > 0 \), so that \( x_0 \in \mathcal{W}(L) \subseteq M \) which contradicts \( x_0 \in X \setminus M \).

(ii): By the definition of \( \mathcal{W}(L) \), for each \( x \in \mathcal{W}(L) \) there is a measure \( \mu_x \in \mathcal{M}_L \) such that \( x \in \text{supp} \mu_x \). Then

\[
\mu := \frac{1}{|\mathcal{W}(L)|} \sum_{x \in \mathcal{W}(L)} \mu_x \in \mathcal{M}_L
\]

and \( \mathcal{W}(L) \subseteq \text{supp} \mu \). (i) implies that \( \text{supp} \mu \subseteq \mathcal{W}(L) \). Thus, \( \text{supp} \mu = \mathcal{W}(L) \).

(iii) is proved by a similar reasoning as (ii). \( \square \)

**Theorem 15.** Each strictly \( E_+ \)-positive linear functional \( L \) on \( E \) is a moment functional such that

\[
\mathcal{W}(L) = X.
\]

**Proof.** That \( L \) is a moment functional follows from Proposition \([8, \text{Theorem II.2.1}]\).

We fix a norm \( \| \cdot \| \) on \( E \). Let \( c \) be the corresponding positive number appearing in the inequality \([8, \text{Lemma IV.4}]\) of Lemma \([8, \text{Lemma IV.4}]\). Suppose that \( x \in X \). Since the point evaluation \( l_x \) at \( x \) is continuous, there is \( C_x > 0 \) such that \( |l_x(f)| = |f(x)| \leq C_x ||f|| \) for \( f \in E \). Fix \( \varepsilon \) such that \( 0 < \varepsilon C_x < c \). Let \( f \in E_+, f \neq 0 \). Using \([8, \text{Lemma IV.4}]\) we derive

\[
(L - \varepsilon l_x)(f) \geq c ||f|| - \varepsilon f(x) \geq (c - \varepsilon C_x) ||f|| > 0.
\]

Therefore, by Lemma \([8, \text{Lemma IV.4}]\) \( L - \varepsilon l_x \) is also strictly \( E_+ \)-positive and hence a moment functional by Proposition \([8, \text{Proposition II.2.1}]\). If \( \nu \) is a representing measure of \( L - \varepsilon l_x \), then \( \mu := \nu + \varepsilon \delta_x \) is a representing measure of \( L \) and \( \mu(\{x\}) \geq \varepsilon > 0 \). Thus, \( x \in \mathcal{W}(L) \). \( \square \)

**Corollary 16.** Let \( L \) be a moment functional on \( E \). Suppose that there exist a closed subset \( U \) of \( X \) and a measure \( \mu \in \mathcal{M}_L \) such that \( \text{supp} \mu \subseteq U \) and the following holds: If \( f(x) \geq 0 \) on \( U \) and \( L(f) = 0 \) for some \( f \in E \), then \( f = 0 \) on \( U \).

Then each \( x \in U \) is atom of some finitely atomic representing measure of \( L \).

**Proof.** Being a closed subset of \( X \), \( U \) is a locally compact Hausdorff space. Since \( \text{supp} \mu \subseteq U \), there is a well-defined (!) moment functional \( \tilde{L} \) on the linear subspace
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\[ E := E \cup \mathcal{U} \text{ of } C(\mathcal{U}; \mathbb{R}) \text{ given by } \hat{L}(f| \mathcal{U}) = L(f), f \in E. \]  
In particular, \( \hat{L} \) is \((\hat{E})_+\)-positive on \( \hat{E} \). The condition on \( \mathcal{U} \) implies that \( \hat{L} \) is strictly positive. Hence it follows from Theorem 15(ii), applied to \( \mathcal{L} \) and \( \hat{E} \subseteq C(\mathcal{U}, \mathbb{R}) \), that \( W(\hat{L}) = \mathcal{U} \). Thus each \( x \in \mathcal{U} \) is atom of some representing measure of \( \hat{L} \) and hence of \( L \). Corollary 5 implies that this measure can be chosen finitely atomic. \( \square \)

4. Determinacy of moment functionals

**Definition 17.** A moment functional \( L \) on \( E \) is called determinate if it has a unique representing measure, or equivalently, if the set \( \mathcal{M}_L \) is a singleton.

The following theorem is the main result of this section. It characterizes determinacy in terms of the size of the set \( W(L) \).

For \( x \in X \) we define

\[
(12) \quad s_F(x) := (f_1(x), \ldots, f_m(x))^T \in \mathbb{R}^m.
\]

Clearly, \( s_F \) the moment vector of the delta measure \( \delta_x \).

**Theorem 18.** For each moment functional \( L \) on \( E \) the following are equivalent:

(i) \( L \) is not determinate.

(ii) The set \( \{s_F(x) : x \in W(L)\} \) is linearly dependent in \( \mathbb{R}^m \).

(iii) \( |W(L)| > \dim(E|W(L)) \).

(iv) \( L \) has a representing measure \( \mu \) such that \( |\text{supp } \mu| > \dim(E|W(L)) \).

**Proof.** (i)→(iii): Assume to the contrary that \( |W(L)| \leq \dim(E|W(L)) \) and let \( \mu_1 \) and \( \mu_2 \) be representing measures of \( L \). Then, since \( \dim E \) is finite, so is \( W(L) \), say \( W(L) = \{x_1, \ldots, x_n\} \) with \( n \in \mathbb{N} \). In particular, \( W(L) \) is a Borel set. Hence, from Lemma 13(i), applied to \( M = W(L) \), we deduce that \( \text{supp } \mu_1 \subseteq W(L) \) for \( i = 1, 2 \), so there are numbers \( c_{ij} \geq 0 \) for \( j = 1, \ldots, n, i = 1, 2 \), such that

\[
L(f) = \int f(x) \, d\mu_i = \sum_{j=1}^n f(x_j)c_{ij} \quad \text{for } f \in E.
\]

From the assumption \( |W(L)| \leq \dim(E|W(L)) \) it follows that there are functions \( f_j \in E \) such that \( f_j(x_i) = \delta_{jk} \). Then \( L(f_j) = c_{ij} \) for \( i = 1, 2 \), so that \( c_{1j} = c_{2j} \) for all \( j = 1, \ldots, n \). Hence \( \mu_1 = \mu_2 \), so \( L \) is determinate. This contradicts (i).

(iii)→(ii): Since the cardinality of the set \( \{s_F(x) : x \in W(L)\} \) exceeds the dimension of \( E|W(L) \) by (iii), the set must be linearly dependent.

(ii)→(i): Since the set \( \{s_F(x) : x \in W(L)\} \) is linearly dependent, there are pairwise distinct points \( x_1, \ldots, x_k \in W(L) \) and real numbers \( c_1, \ldots, c_k \), not all zero, such that \( \sum_{i=1}^k c_i s_F(x_i) = 0 \). Then, since \( \{f_1, \ldots, f_m\} \) is a basis of \( E \), we have

\[
(13) \quad \sum_{i=1}^k c_i f(x_i) = 0 \quad \text{for } f \in E.
\]

We choose for \( x_i \in W(L) \) a representing measure \( \mu_i \) of \( s_\mu \) such that \( x_i \in \text{supp } \mu_i \). Clearly, \( \mu := \frac{1}{k} \sum_{i=1}^k \mu_i \) is a representing measure of \( s_\mu \) such that \( \mu(\{x_i\}) > 0 \) for all \( i \). Let \( \varepsilon = \min \{\mu(\{x_i\}) : i = 1, \ldots, k\} \). For each number \( c \in (-\varepsilon, \varepsilon) \),

\[
\mu_c = \mu + c \sum_{i=1}^k c_i \delta_{x_i}
\]

is a positive (!) measure which represents \( L \) by (13). By the choice of \( x_i, c_i \), the signed measure \( \sum_i c_i \delta_{x_i} \) is not the zero measure. Therefore, \( \mu_c \neq \mu_{c'} \) for \( c \neq c' \). This shows that \( L \) is not determinate.

(iii)→(iv): If \( W(L) \) is finite, by Lemma 13(ii) we can choose \( \mu \in \mathcal{M}_L \) such that \( \text{supp } \mu = W(L) \). If \( W(L) \) is infinite, Lemma 13(iii) implies that there exists
\( \mu \in \mathcal{M}_L \) such that \( \text{supp} \mu > \dim(E|W(L)) \). Thus, the equivalence (iii) \( \leftrightarrow \) (iv) is proved in both cases.

An immediate consequence of Theorem 18 is the following.

**Corollary 19.** If \( |W(L)| > \dim E \) or if there is a measure \( \mu \in \mathcal{M}_L \) such that \( \text{supp} \mu > \dim E \), then \( L \) is not determinate. In particular, \( L \) is not determinate if \( W(L) \) is an infinite set or if \( L \) has a representing measure of infinite support.

**Corollary 20.** Suppose that \( L \) is a strictly \( E_+ \)-positive moment functional on \( E \). Then \( L \) is determinate if and only if \( |X| \leq \dim E \).

**Proof.** From Theorem 15 we obtain \( X = W(L) \). Therefore, \( \dim E = \dim (E|W(L)) \). Hence the assertion follows from Theorem 18 (iii) \( \leftrightarrow \) (i).

The following simple results contain useful sufficient criteria for determinacy.

**Proposition 21.** Let \( s \in S \). Suppose that \( W(s) = \{ x_1, \ldots, x_k \} \), where \( k \in \mathbb{N} \). If for each \( j = 1, \ldots, k \) there exists \( p_j \in E_+ \) such that \( p_j(x_i) = \delta_{ij} \), then \( s \) is determinate.

**Proof.** Let \( \mu \) and \( \nu \) be representing measures of \( s \). Then \( \text{supp} \mu \subseteq W(s) \) and \( \text{supp} \nu \subseteq W(s) \) by Proposition 14(i), so \( \mu = \sum_{i=1}^k \delta_i \delta_{x_i} \) and \( \nu = \sum_{i=1}^k d_i \delta_{x_i} \).

Therefore,
\[
c_i = \int_X p_i \, d\mu = L_s(p_i) = \int_X p_i \, d\nu = d_i \quad \forall i = 1, \ldots, k,
\]
so that \( \mu = \nu \). Hence \( s \) is determinate.

The next proposition contains a sufficient criterion for the existence of such polynomials \( p_j \).

**Proposition 22.** Let \( \mathcal{B} = \{ b_1, \ldots, b_k \} \) be a subset of \( \mathcal{C}(X; \mathbb{R}) \). We suppose that \( \mathcal{B}^2 := \{ b_i b_j : i, j = 1, \ldots, k \} \subseteq E \). Let \( s \) be a moment sequence of \( E \) such that \( W(s) = \{ x_1, \ldots, x_l \} \), \( l \leq k \). If the vectors \( s_{b_1}(x_1), \ldots, s_{b_k}(x_1) \in \mathbb{R}^k \) are linearly independent, then there exist functions \( p_j \in E_+ \) such that \( p_j(x_i) = \delta_{ij} \) for \( i, j = 1, \ldots, l \) and \( s \) is determinate.

**Proof.** Recall that \( s_{b_j}(x) = (b_1(x), \ldots, b_j(x))^T \) for \( x \in X \). Set
\[
M := (s_{b_j}(x_i)^T)_{i=1, \ldots, l} \quad \text{and} \quad M_j := (s_{b_j}(x_i)^T)_{i=1, \ldots, j-1, j+1, \ldots, l} \quad \forall j = 1, \ldots, l.
\]
Since \( \{s_{b_1}(x_1), \ldots, s_{b_l}(x_1)\} \) is linearly independent, we have
\[
\text{rank} \, M = l \quad \text{and} \quad \text{rank} \, M_j = l-1 \quad \forall j = 1, \ldots, l.
\]
Hence, for any \( j \) there exists \( q_j \in \ker M_j \setminus \ker M \). Then \( q_j(x) := \langle q_j, s_{b_j}(x) \rangle \in \text{Lin} \mathcal{B} \). We have \( q_j(x_i) = 0 \) for \( i \neq j \) and \( q_j(x_j) \neq 0 \), since otherwise \( q_j \in \ker M \).
Therefore, \( p_j := q_j(x_j)^{-2} q_j^2 \in E_+ \) has the desired properties.

The determinacy of \( s \) follows from Proposition 21.

In the following example the Robinson polynomial is used to develop an application of Proposition 22.

**Example 23.** Let \( \mathcal{A} := \{(x, y, z)^\alpha : |\alpha| = 6\} \) and \( \mathcal{B} := \{(x, y, z)^\alpha : |\alpha| = 3\} \). Then \( \mathcal{B}^2 \subseteq \mathcal{E} \subseteq \mathcal{A} := \text{Lin} \mathcal{A} \). We consider the homogeneous polynomials of \( \mathcal{A} \) and \( \mathcal{B} \) acting as continuous functions on the projective space \( X = \mathbb{P}(\mathbb{R}^3) \). Our aim is to apply Proposition 22. Let
\[
\mathcal{Z} = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1), (1, 1, 0), \}
\[
(1, -1, 0), (1, 0, 1), (1, 0, -1), (0, 1, 1), (0, 1, -1) \}.\]
It is known that the Robinson polynomial
\[ R(x, y, z) = x^6 + y^6 + z^6 - x^3(y^2 + z^2) - x^4(z^2 + y^2) - 3x^2y^2 z^2 \]
is non-negative on \( \mathbb{R}^3 \), hence \( R \in E_+ \), and that \( R \) has the projective zero set \( Z = \{ r_1, \ldots, r_{10} \} \). Then
\[ M := (s_B(r_1), \ldots, s_B(r_{10}))^T \]
is a full rank \( 10 \times 10 \)-matrix and for \( i = 1, \ldots, 10 \) the matrix
\[ M_i := (s_B(r_1), \ldots, s_B(r_{i-1}), s_B(r_{i+1}), \ldots, s_B(r_{10}))^T \]
has rank 9. Hence, by Proposition 22, there exist polynomials \( p_i \in E_+ \) such that \( p_i(r_j) = \delta_{i,j} \). Therefore,
\[ Z(R + p_i) = \{ r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{10} \}. \]

Using polynomials \( R + p_1 + \ldots + p_k \) we find that \( \mathcal{W}_+(s) = \mathcal{V}_+(s) \) for all moment sequences \( s \) with representing measure \( \mu \) such that \( \text{supp} \mu \subseteq Z \).

In the next example we use the Motzkin polynomial and derive the determinacy from Theorem 18.

**Example 24.** We consider the Motzkin polynomial
\[ M(x, y) = 1 - 3x^2 y^2 + x^2 y^4 + x^4 y^2. \]

Its zero set is \( Z(M) = \{ r_1 = (1, 1), r_2 = (1, -1), r_3 = (-1, 1), r_4 = (-1, -1) \} \). Let us set \( X = \mathbb{R}^2, B := \{ 1, x, y, x^3, xy^2 \} \) and \( E = A = \text{Lin} A \), where
\[ A := B^2 = \{ 1, x, x^2, xy, x^3, x^2 y, xy^2, x^4, x^2 y^2, x^3 y, x^2 y^3, x^6, x^4 y, x^3 y^2 \}. \]

Then \( M \in E_+ \). The set \( \{ s_B(r_1), \ldots, s_B(r_4) \} \) is linearly dependent, since
\[ 0 = s_B(1, 1) - s_B(1, -1) + s_B(-1, 1) - s_B(-1, -1). \]

Hence each polynomial in \( \text{Lin} B \) vanishing at three roots of \( M \) vanishes at the fourth as well. Proposition 22 does not apply. But the set \( \{ s_A(r_1), \ldots, s_A(r_4) \} \) is linearly independent. Therefore, the moment sequence of any measure \( \mu = \sum_{i=1}^4 c_i \delta_{r_i} \) is determinate by Theorem 18 (i)+ (ii).

5. EXPOSED FACES OF THE MOMENT CONE

For \( v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m \) we abbreviate
\[ f_v = v_1 f_1 + \cdots + v_m f_m. \]

Let \( \langle , \rangle \) denote the standard scalar product on \( \mathbb{R}^m \). Then
\[ f_v(x) = v_1 f_1(x) + \cdots + v_m f_m(x) = \langle s_F(x), v \rangle, \quad x \in X. \]

Recall that \( F = \{ f_1, \ldots, f_m \} \) is a basis of the vector space \( E \). Hence \( E \) is the set of all functions \( f_v \), where \( v \in \mathbb{R}^m \). By definition,
\[ E_+ = \{ f_v : v \in \mathbb{R}^m, \langle s_F(x), v \rangle \geq 0 \quad \text{for} \quad x \in X \}. \]

Further, we have
\[ L_t(f_v) = \langle v, t \rangle \quad \text{for} \quad v, t \in \mathbb{R}^m. \]

Indeed, since \( \mathbb{R}^m = S - S \), each vector \( t \in \mathbb{R}^m \) is of the form \( t = \sum_{i=1}^k c_i s_F(x_i) \), where \( c_i \in \mathbb{R} \) and \( x_i \in X \). Then we compute
\[ L_t(f_v) = \sum_{i=1}^k c_i f_v(x_i) = \sum_{i=1}^k c_i \langle v, s_F(x_i) \rangle = \langle v, t \rangle \]
which proves (15). From (15) it follows that for each linear functional \( h \) on \( E \) there is a unique vector \( u \in \mathbb{R}^m \) such that

\[
(17) \quad h_u(f) := h(f) = \langle u, v \rangle, \quad v \in \mathbb{R}^m.
\]

For \( u \in \mathbb{R}^m \) we define

\[
h_u(t) := \langle t, u \rangle, \quad t \in \mathbb{R}^m, \quad \text{and} \quad H_u := \{ x \in \mathbb{R}^m : \langle x, u \rangle = 0 \}.
\]

For the set \( \mathcal{N}_+(s) = \mathcal{N}_+(L_s) \) defined by (8) we have the following crucial fact.

**Lemma 25.** Let \( u \in \mathbb{R}^m \) and \( s \in \mathcal{S} \). Then \( f_u \in \mathcal{N}_+(s) \) if and only if \( h_u(s) = 0 \) and \( h_u(t) \geq 0 \) for \( t \in \mathcal{S} \).

**Proof.** As noted above, \( f_u \in E_+ \) if and only if \( f_u(x) = \langle sF(x), u \rangle \geq 0 \) for all \( x \in \mathcal{X} \).

Let \( t \in \mathcal{S} \). We can write \( t = \sum c_i sF(x_i) \) with \( x_i \in \mathcal{X}, c_i \geq 0 \) for all \( i \). Then

\[
L_t(f_u) = \sum_i c_i \langle sF(x_i), u \rangle = \sum_i c_i f_u(x_i) = \langle u, t \rangle = h_u(t)
\]

by (17). Hence \( f_u \in E_+ \) if and only if \( h_u(t) \geq 0 \) for \( t \in \mathcal{S} \). Further, \( L_s(f_u) = 0 \) if and only if \( h_u(s) = 0 \). The two latter facts give the assertion. \( \square \)

Let us recall two basic definition from convex analysis.

**Definition 26.** Let \( u \in \mathbb{R}^m, u \neq 0, \text{and} s \in \mathcal{S} \). We say that \( H_u \) is a supporting hyperplane of \( \mathcal{S} \) at the point \( s \) if

\[
h_u(s) = 0 \quad \text{and} \quad h_u(t) \geq 0 \quad \text{for all} \quad t \in \mathcal{S}.
\]

The set \( H_u \cap \mathcal{S} \) is called a proper exposed face of the cone \( \mathcal{S} \).

Let \( s \in \mathcal{S} \). Combining Lemma 25 and Definition 26 it follows that \( H_u \) is a supporting hyperplane of \( \mathcal{S} \) at \( s \) if and only if \( f_u \in \mathcal{N}_+(s) \) and \( u \neq 0 \). Further, \( H_u \cap \mathcal{S} \) is a proper exposed face of \( \mathcal{S} \) if and only if \( f_u \in \mathcal{N}_+(s) \) and \( u \neq 0 \). All proper exposed faces of \( \mathcal{S} \) are of this form. In the case \( u = 0 \) we have \( H_u \cap \mathcal{S} = \mathcal{S} \).

Note that by definition the sets \( \mathcal{S} \) and \( \emptyset \) are also exposed faces of \( \mathcal{S} \).

**Proposition 27.** Let \( s \in \mathcal{S} \). Then:

(i) \( \mathcal{N}_+(s) \neq \{0\} \) if and only if \( s \) is a boundary point of \( \mathcal{S} \).

(ii) \( \mathcal{N}_+(s) = \{0\} \) if and only if \( s \) is an inner point of \( \mathcal{S} \).

**Proof.** (i): It is well-known from convex analysis that \( s \) is a boundary point of \( \mathcal{S} \) if and only if there is a supporting hyperplane \( H_u \) of \( \mathcal{S} \) at \( s \). By the preceding the latter holds if and only if \( f_u \in \mathcal{N}_+(s) \) and \( f_u \neq 0 \).

(ii) follows from (i) and the obvious fact that \( s \) is an inner point of \( \mathcal{S} \) if and only if \( s \) is not a boundary point. \( \square \)

**Proposition 28.** For each moment sequence \( s \) there exists \( p \in \mathcal{N}_+(s) \) such that

\[
\mathcal{V}_+(s) = \mathcal{Z}(p) := \{ x \in \mathcal{X} : p(x) = 0 \}.
\]

**Proof.** If \( s \) is an inner point of \( \mathcal{S} \), then \( \mathcal{N}_+(s) = \{0\} \) by Proposition 24 hence \( \mathcal{V}_+(s) = \mathcal{X} \); so we can set \( p = 0 \).

Now let \( s \) be a boundary point of \( \mathcal{S} \). Let \( \mathcal{F} \) be the set of vector \( u \in \mathbb{R}^m \) such that \( h_u(s) = 0 \) and \( h_u(t) \geq 0 \) for all \( t \in \mathcal{S} \). Since \( s \) is a boundary point, \( \mathcal{F} \) contains at least one nonzero vector. Then \( \{ H_u \cap \mathcal{S} : u \in \mathcal{F}, u \neq 0 \} \) is the set of exposed faces of \( \mathcal{S} \). Let \( u_1, \ldots, u_k \) be a maximal linearly independent subset of \( \mathcal{F} \). Set \( u := u_1 + \cdots + u_k \). We show that \( p := f_u \) has the desired properties. Obviously, \( u \in \mathcal{F} \). Hence \( f_u \in \mathcal{N}_+(s) \) by Lemma 25 and \( H_u \cap \mathcal{S} \) is an exposed face of \( \mathcal{S} \). Thus \( \mathcal{V}_+(s) \subseteq \mathcal{Z}(p) \) by definition. Suppose \( x \in \mathcal{Z}(p) = \mathcal{Z}(f_u) \). Let \( v \in \mathcal{N}_+(s) \). Then \( v \in \mathcal{F} \) by Lemma 25. Hence \( v \) is linear combination \( v = \sum \lambda_i u_i \) of \( u_1, \ldots, u_k \).

Since \( f_u(x) = f_{u_1}(x) + \cdots + f_{u_k}(x) = 0 \) and \( f_u \geq 0 \), we have \( f_{u_i}(x) = 0 \) for all \( i \).
and therefore \( f_\nu(x) = \sum \lambda_i f_{u_i}(x) = 0 \). Since \( f_\nu \in \mathcal{N}_k(s) \) was arbitrary, we have shown that \( x \in \mathcal{V}_k(s) \).

Note that the element \( p \) in the preceding proposition is not necessarily unique. For inner points of \( S \) we have \( \mathcal{V}(s) = \mathcal{X} \), hence \( \mathcal{V}(s) = \mathcal{V}_k(s) \), by Lemma 7 and Theorem 15. In general, \( \mathcal{V}(s) \neq \mathcal{V}_k(s) \) as we have seen by Example 12.

The next theorem characterizes those boundary points for which \( \mathcal{V}_k(s) = \mathcal{V}_i(s) \).

**Theorem 29.** Let \( s \) be a boundary point of \( S \). Then \( \mathcal{V}_k(s) = \mathcal{V}_i(s) \) if and only if \( s \) lies in the relative interior of an exposed face of the moment cone \( S \).

**Proof.** By Proposition 28 there exists \( p \in \mathcal{N}_k(s) \) such that \( \mathcal{Z}(p) = \mathcal{V}_k(s) \). Then \( p \) is of the form \( p = f_u \) for some \( u \in \mathbb{R}^m \) and \( H_u \) is a finite-dimensional vector space such that \( s_F(x) \in H_u \) for \( x \in \mathcal{Z}(f_u) = \mathcal{V}_k(s) \). Let us choose \( x_i, \ldots, x_k \in \mathcal{Z}(f_u) \) such that the vectors \( s_F(x_i), \ldots, s_F(x_k) \) are linearly independent and span \( H_u \).

First suppose that \( \mathcal{V}_k(s) = \mathcal{W}(s) \). Then \( x_i \in \mathcal{W}(s) \), so there exists an atomic representing measure \( \mu_i \) of \( s \) such that \( x_i \in \text{supp } \mu_i \). Then \( \mu := \frac{1}{k} \sum_{i=1}^k \mu_i \) is also a representing measure of \( s \) and \( x_i \in \text{supp } \mu \) for all \( i \).

We show that \( s \) is an inner point of the exposed face \( H_u \cap S \) of the moment cone \( S \). Let \( v \in H_u \). Since the \( s_F(x_i) \) are linearly independent and span \( H_u \), there are reals \( c_1, \ldots, c_k \) such that \( v = \sum_{i=1}^k c_i s_F(x_i) \). Since the masses of \( \delta_{x_i} \) are positive in \( \mu \), there exists a \( \varepsilon > 0 \) such that \( s + c \cdot v \in H_u \cap S \) for all \( c \in (-\varepsilon, \varepsilon) \), that is, \( s \) is an inner point of the exposed face \( H_u \cap S \).

Conversely, suppose now that \( s \) is an inner point of some exposed face \( F \) of \( S \). Let \( x \in \mathcal{V}_k(s) \). Then \( s_F(x) \in F \). Since \( s \) is an inner point, there is a \( \delta > 0 \) such that \( s + \delta \cdot s_F(x) \in F \). If \( \mu' \) is representing measure \( \mu \) of \( s' = s + \delta \cdot s_F(x) \), then \( \mu = \mu' + c \cdot \delta_{x} \) is a representing measure of \( s \) and \( \mu(\{x\}) \geq c > 0 \), so that \( x \in \mathcal{W}(s) \). Since always \( \mathcal{W}(s) \subseteq \mathcal{V}_k(s) \), we have shown that \( \mathcal{W}(s) = \mathcal{V}_k(s) \).

6. Set of atoms \( \mathcal{W}(L) \) and core variety \( \mathcal{V}(L) \)

Throughout this section, \( L \) is a moment functional on \( E \) such that \( L \neq 0 \).

We define inductively subsets \( \mathcal{N}_k(L), k \in \mathbb{N} \), of \( A \) and subsets \( \mathcal{V}_j(L), j \in \mathbb{N}_0 \), of \( \mathcal{X} \) by \( \mathcal{V}_0(L) = \mathcal{X} \),

\[
\mathcal{N}_k(L) := \{ p \in A : L(p) = 0, \ p(x) \geq 0 \text{ for } x \in \mathcal{V}_{k-1}(L) \}, \quad k \in \mathbb{N},
\]

\[
\mathcal{V}_j(L) := \{ t \in \mathcal{X} : p(t) = 0 \text{ for } p \in \mathcal{N}_j(L) \}, \quad j \in \mathbb{N}.
\]

If \( \mathcal{V}_k(L) \) is empty for some \( k \), we set \( \mathcal{V}_j(L) = \mathcal{V}_k(L) = \emptyset \) for all \( j \geq k, j \in \mathbb{N} \).

For \( k = 1 \) these notions coincide with those defined by \( 3 \) and \( 4 \), that is, \( \mathcal{N}_1(L) = \mathcal{N}_0(L) \equiv \mathcal{N}_k(s) \) and \( \mathcal{V}_1(L) = \mathcal{V}_1(s) \equiv \mathcal{V}_k(s) \), where \( s \) is the moment sequence of \( L \).

The following important concept was defined and studied by L. Fialkow \( 6 \), see also \( 2 \), for arbitrary linear functionals. We will use it only for moment functionals.

**Definition 30.** The core variety \( \mathcal{V}(L) \) of the moment functional \( L \) on \( A \) is

\[
\mathcal{V}(L) := \bigcap_{k=0}^{\infty} \mathcal{V}_k(L).
\]

From the definition it is clear that \( \mathcal{V}_k(L) \subseteq \mathcal{V}_{k-1}(L) \) for \( k \in \mathbb{N} \). Further, if \( \mu \) is representing measure of \( L \), then a repeated application of Lemma 3 yields

\[
supp \mu \subseteq \mathcal{V}(L) \subseteq \mathcal{V}_j(L) \quad \text{for} \quad j \in \mathbb{N}.
\]

**Proposition 31.** There exists \( k \in \mathbb{N}_0, k \leq \dim E \), such that

\[
\mathcal{X} = \mathcal{V}_0(L) \supseteq \mathcal{V}_1(L) \supseteq \ldots \supseteq \mathcal{V}_k(L) = \mathcal{V}_{k+j}(L) = \mathcal{V}(L), \quad j \in \mathbb{N}.
\]
Proof. We fix a representing measure \( \mu \) of \( L \). Let \( j \in \mathbb{N}_0 \). We denote by \( E^{(j)} := E|\mathcal{V}_j(L) \) the vector space of functions \( f|\mathcal{V}_j(L), f \in E \), and by \( \mathcal{L}^{(j)} \) the corresponding cone of moment functionals on \( E^{(j)} \). Note that in general \( \dim E^{(j)} \) is smaller than \( \dim E \). Since \( \text{supp} \mu \subseteq \mathcal{V}_j(L) \) by (13), \( L \) yields a moment functional \( L^{(j)} \in \mathcal{L}^{(j)} \) given by \( \mu \). Clearly, \( E^{(0)} = E, \mathcal{V}_0(L) = X, L = L^{(0)} \). By these definitions, \( \mathcal{N}_{j+1}(L) = \mathcal{N}_j(L^{(j)}) \) and \( V_{j+1}(L) = V_j(L^{(j)}) \). From Proposition 28, applied to the moment sequence of \( L^{(j)} \), we conclude that there exists \( p_{j+1} \in E \) such that \( p_{j+1}|\mathcal{V}_j(L) \in \mathcal{N}_j(L^{(j)}) = \mathcal{N}_{j+1}(L) \)

\[
V_j(L^{(j)}) = V_{j+1}(L) = \mathcal{Z}(p_{j+1}|\mathcal{V}_j(s)) = \{ x \in \mathcal{V}_j(L) : p_{j+1}(x) = 0 \}. 
\]

First suppose that \( L \) is an inner point of \( L \). Then, by Proposition 27 ii) we have \( \mathcal{N}_j(L) = \{0\} \) and hence \( V_j(L) = X \). From the corresponding definitions it follows that \( \mathcal{N}_j(L) = \{0\} \) and \( V_j(L) = X \) for all \( j \in \mathbb{N} \), so the assertion holds with \( k = 0 \).

Now let \( L \) be a boundary point of \( L \). Then \( \mathcal{N}_1(L) \neq \{0\} \) and hence \( V_1(L) \neq X \). Assume that \( r \in \mathbb{N} \) and \( V_0(L) \supseteq \cdots \supseteq V_r(L) \). We show that \( p_1, \ldots, p_r \) are linearly independent. Assume the contrary. Then \( \sum_{j=1}^r \lambda_j p_j = 0 \), where \( \lambda_j \in \mathbb{R} \), not all zero. Let \( n \) be the largest index such that \( \lambda_n \neq 0 \). Then \( p_n(x) = \sum_{j<n} \lambda_j \lambda_j^{-1} p_j \).
(The sum is set zero if \( n = 1 \).) Since \( V_1(L) \subseteq V_j(L) \) if \( j \leq 1 \) and \( p_j \) vanishes on \( \mathcal{V}_j(L) \) by (20), it follows that \( p_n = 0 \) on \( \mathcal{V}_{n-1}(L) \). Hence \( \mathcal{V}_n(L) \subseteq \mathcal{V}_{n-1}(L) \) by (20), a contradiction.

From the preceding two paragraphs it follows that there exists a number \( k \in \mathbb{N}_0, k \leq \dim E \), such that \( V_k(L) = \mathcal{V}_{k+1}(L) \). Then \( \mathcal{N}_{k+1}(L) = \mathcal{N}_{k+2}(L) \) and hence \( \mathcal{V}_{k+1}(L) = \mathcal{V}_{k+2}(L) \). Proceeding by induction we get \( \mathcal{V}_{k+j}(L) = \mathcal{V}_k(L) \) for \( j \in \mathbb{N} \), so that \( \mathcal{V}(L) = \mathcal{V}_k(L) \).

\[ \tag{20} \mathcal{V}_j(L^{(j)}) = \mathcal{V}_{j+1}(L) = \mathcal{Z}(p_{j+1}|\mathcal{V}_j(s)) = \{ x \in \mathcal{V}_j(L) : p_{j+1}(x) = 0 \}. \]

Theorem 32. If \( L \) is a moment functional on \( E \) and \( L \neq 0 \), then \( \mathcal{W}(L) = \mathcal{V}(L) \).

Proof. From (13) it follows at once that \( \mathcal{W}(L) \subseteq \mathcal{V}(L) \).

By Proposition 33 there exists a number \( k \in \mathbb{N}_0 \) such that (31) holds. We show that the set \( U := \mathcal{V}(L) \) fulfills the assumptions of Corollary 18. By (18), \( \text{supp} \mu \subseteq \mathcal{V}(L) \). Further, if \( f \in E \) satisfies \( f(x) \geq 0 \) on \( U = \mathcal{V}_k(L) \) and \( L(f) = 0 \), then \( f \in \mathcal{N}_{k+1}(L) \) and hence \( f(x) = 0 \) on \( \mathcal{V}_{k+1}(L) = \mathcal{V}(L) = U \). Thus Corollary 18 applies and gives the converse inclusion \( U = \mathcal{V}(L) \subseteq \mathcal{W}(L) \).

7. Differential structure of the moment cone

In this section, we set \( X = \mathbb{R}^n \) and assume that \( E \) is a finite-dimensional linear subspace of \( C^4(\mathbb{R}^n; \mathbb{R}) \). We will use the differential structure to develop further tools to study the moment cone and moment sequences.

Set \( \mathbb{R}_+ := (0, +\infty) \). Let \( C = (c_1, \ldots, c_K) \) and \( X = (x_1, \ldots, x_K) \), where \( c_j > 0 \) and \( x_j \in \mathbb{R}^n \) for \( j = 1, \ldots, K \), and define a \( k \)-atomic measure, where \( k \leq K \), by

\[
\mu(C, X) = \sum_{j=1}^{K} c_j \delta_{x_j}.
\]

Note that \( \mu(C, X) \) is not \( K \)-atomic in general, since we do not require that the points \( x_j \) are pairwise different. By Corollary 14 each moment sequence has a \( k \)-atomic representing measure with \( k \leq m \). Therefore, if \( m \leq K \), the moment sequences of such measures \( \mu(C, X) \) exhausts the whole moment cone \( \mathcal{S} \).

We write \( (C, X) \in \mathcal{M}_{K, s} \) if \( \mu(C, X) \) is a representing measure of \( s \in \mathcal{S} \).

Definition 33. For \( x \in \mathbb{R}^n \), and \( (C, X) \in \mathbb{R}_+^k \times (\mathbb{R}^n)^k \) we define

\[
S_k(C, X) := \sum_{j=1}^{k} c_j s_{\bar{x}}(x_j), \quad \text{where} \quad s_{\bar{x}}(x) := (f_1(x), \ldots, f_m(x))^T
\]
Clearly, $S_k$ is a $C^1$-map of $\mathbb{R}^k_x \times \mathbb{R}^k$ into $\mathbb{R}^m$. Let $DS_k$ denote its total derivative. We write
\begin{equation}
DS_k(C, X) = (\partial_{x_i} S_k, \partial_{x_i} S_k, \ldots, \partial_{x_i}^{(n)} S_k, \partial_{x_i} S_k, \ldots, \partial_{x_i}^{(n)} S_k)
\end{equation}
(22)
\begin{align*}
= (s(x_1), c_1 \partial s|_{x=x_1}, \ldots, c_1 \partial s|_{x=x_1}, s(x_2), \ldots, c_1 \partial s|_{x=x_1}).
\end{align*}

The following is another very simple example for which $W(s) \neq \mathcal{V}_+(s)$.

**Example 34.** Let $F := \{1, x, x^2(x-1)^2\}$, $E = \text{Lin} F$, and $X = \mathbb{R}$. Set $s := s_F(0) = (1, 0, 0)^T$. Then
\begin{align*}
DS_1 &= (s(0), s'(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{align*}

and
\begin{align*}
\ker DS_1^T &= \mathbb{R} \cdot v \quad \text{with} \quad v = (0, 0, 1)^T.
\end{align*}

Set $p(x) := \langle v, s_F(x) \rangle = x^2(x + 1)^2$. One verifies that $N_+(s) = \mathbb{R}_{\geq} \cdot p$. Hence $\mathcal{V}_+(s) = \{0, 1\}$, but $W(s) = \{0\}$. Thus, $W(s) \neq \mathcal{V}_+(s)$.

**Definition 35.** For $s \in S$ we define the image $\mathcal{I}(s)$, the set $I(s)$, and the defect number $d(s)$ by
\begin{align*}
\mathcal{I}(s) &= \bigcup_{k \in \mathbb{N}} \bigcup_{(C, X) \in \mathcal{M}_{k,s}} \text{range } DS_k(C, X), \\
I(s) &= \bigcap_{v \in \mathcal{I}(s)^\perp} Z(g_v), \\
d(s) &= \text{codim } \mathcal{I}(s) = m - \dim \mathcal{I}(s).
\end{align*}

Note that the dimension of $\mathcal{I}(s)$ is well-defined, since $\mathcal{I}(s)$ is a linear subspace of $\mathbb{R}^m$ by the following lemma.

**Lemma 36.** For each $s \in S$ there exist $k \in \mathbb{N}$ and $(C, X) \in \mathcal{M}_{k,s}$ such that $\mathcal{I}(s) = \text{range } DS_k(C, X)$. In particular, $\mathcal{I}(s)$ is a linear subspace of $\mathbb{R}^m$.

**Proof.** Let us prove that $\mathcal{I}(s)$ is a vector space. Obviously, $\text{range } DS_k(C, X)$ is a vector space for any $(C, X)$. Let $v_1, v_2 \in \mathcal{I}(s)$. There are $(C_i, X_i), k_i \in \mathbb{N}$, and $u_i \in \mathbb{R}^{(m+1)k_i}$ such that $v_i = DS_k(C_i, X_i)u_i$. Then
\begin{align*}
\lambda_1 u_1 + \lambda_2 u_2 &= \lambda_1 DS_{k_1}(C_1, X_1)u_1 + \lambda_2 DS_{k_2}(C_2, X_2)u_2 \\
&= \frac{1}{2}(DS_{k_1}(C_1, X_1), DS_{k_2}(C_2, X_2)) \begin{pmatrix} 2\lambda_1 u_1 \\ 2\lambda_2 u_2 \end{pmatrix} \\
&\in \text{range } DS_{k_1+k_2}((C_1/2, C_2/2), (X_1, X_2)) \subseteq \mathcal{I}(s)
\end{align*}

for any $\lambda_1, \lambda_2 \in \mathbb{R}$.

Let $\{v_1, \ldots, v_l\}$ be a basis of the finite-dimensional vector space $\mathcal{I}(s)$. By the definition of the set $\mathcal{I}(s)$ for each $i$ there exists $(C_i, X_i) \in \mathcal{M}_{k_i,s}$ such that $v_i \in \text{range } DS_{k_i}(C_i, X_i)$. Define
\begin{align*}
k = k_1 + \ldots + k_l, \quad C = (C_1/l, \ldots, C_l/l), \quad X = (X_1, \ldots, X_l).
\end{align*}

Clearly, $(C, X) \in \mathcal{M}_{k,s}$. One easily verifies that $v_i \in \text{range } DS_k(C, X)$ for each $i$. Therefore, $\mathcal{I}(s) = \text{span}\{v_1, \ldots, v_l\} \subseteq \text{range } DS_k(C, X)$. The converse inclusion $\text{range } DS_k(C, X) \subseteq \mathcal{I}(s)$ is trivial.

A pair $(C, X) \in \mathcal{M}_{k,s}$ such that $\mathcal{I}(s) = \text{range } DS_k(C, X)$ is called a representing measure of $\mathcal{I}(s)$.

Let $\tilde{g}_1, \ldots, \tilde{g}_d(s) \in \mathbb{R}^m$ be a basis of $\mathcal{I}(s)^\perp$, where $\mathcal{I}(s)^\perp$ denotes the orthogonal complement of $\mathcal{I}(s)$ with respect to the standard scalar product of $\mathbb{R}^m$. Then
\begin{align*}
g_i(\cdot) := \langle s_F(\cdot), \tilde{g}_i \rangle
\end{align*}
are functions of $E$ and

$$I(s) := \bigcap_{i=1}^{d(s)} \mathcal{Z}(g_i).$$

**Lemma 37.**

(i) $\text{Lin} \{\tilde{g}_1, \ldots, \tilde{g}_{d(s)}\} = \ker DS_k(C, X)^T$ for each representing measure $(C, X)$ of $\mathcal{Z}(s)$.

(ii) $\text{Lin} \{g_1, \ldots, g_{d(s)}\} = \{p \in E : p|\mathcal{W}(s) = 0 \text{ and } \partial_jp|\mathcal{W}(s) = 0 \text{ for all } j = 1, \ldots, n\}$.

(iii) $\mathcal{W}(s) \subseteq I(s) \subseteq \mathcal{V}_+(s)$.

(iv) $d(s) = 1 \Rightarrow I(s) = \mathcal{V}_+(s)$.

**Proof.** (i) follows at once from the corresponding definitions.

(ii): For “$\subseteq$” we have $\tilde{g}_i \perp s(x)$ and $\tilde{g}_i \perp \partial_j s(x)$ for all $x \in \mathcal{W}(s), j = 1, \ldots, n$, and $i = 1, \ldots, d(s)$, i.e.,

$$g_i(x) = (\tilde{g}_i, s_F(x)) = 0$$

and

$$\partial_j g_i(x) = (\tilde{g}_i, \partial_j s_F(x)) = 0$$

for all $x \in \mathcal{W}(s), j = 1, \ldots, n$, and $i = 1, \ldots, d(s)$.

For “$\supseteq$” let $p(\cdot) = (\tilde{g}, s_F(\cdot)) \in E$ such that $p|\mathcal{W}(s) = 0$ and $\partial_j p|\mathcal{W}(s) = 0$. Then $\tilde{p} \in \mathcal{Z}(s)^\perp = \text{span} \{\tilde{g}_1, \ldots, \tilde{g}_{d(s)}\}$.

(iii): First we prove that $\mathcal{W}(s) \subseteq I(s)$. Let $x \in \mathcal{W}(s)$, i.e., there is a representing measure $(C, X)$ with $X = (x, x_2, \ldots, x_k)$. Then

$$s_F(x) \in \text{range} \ DS_k(C, X) \Rightarrow s_F(x) \in \mathcal{Z}(s)$$

$$\Rightarrow s_F(x) \perp \tilde{g}_i \forall i = 1, \ldots, d(s)$$

$$\Rightarrow g_i(x) = (s_F(x), \tilde{g}_i) = 0 \forall i = 1, \ldots, d(s)$$

$$\Rightarrow x \in I(s).$$

Now we prove that $I(s) \subseteq \mathcal{V}_+(s)$. From (ii) and the inclusion

$$\mathcal{N}_+(s) \subseteq \{p \in E \mid p|\mathcal{W}(s) = 0 \text{ and } \partial_j p|\mathcal{W}(s) = 0\}$$

it follows that

$$I(s) = \bigcap_{g \in \mathcal{Z}(s)^\perp} \mathcal{Z}(g) \subseteq \bigcap_{p \in \mathcal{N}_+(s)} \mathcal{Z}(p) = \mathcal{V}_+(s).$$

(iv): Since $d(s) = 1 > 0$, $s$ is not an inner point of the moment cone. Hence there exists $p \in \mathcal{N}_+(s), p \neq 0$. Then $p \in \text{Lin}(g_1) = R \cdot g_1$ by (ii), i.e., $p = c \cdot g_1$ for some $c \neq 0$ and $I(s) = \mathcal{Z}(g_1) = \mathcal{V}(s)$.

The following examples show that both inclusions in (iii) can be strict. In fact, $\mathcal{W}(s) \subseteq I(s) = \mathcal{V}_+(s)$ in Example 33 and $\mathcal{W}(s) = I(s) \subseteq \mathcal{V}_+(s)$ in Example 38.

**Example 38.** Let $A := \{1, x^2, x^4, x^5, x^6, x^7, x^8\}$ and $a, b \in R \setminus \{0\}$ s.t. $|a| \neq |b|$. Set $\mu = c_1 \delta_{-a} + c_2 \delta_a + c_3 \delta_b$ with $c_i > 0$ for $i = 1, 2, 3$ and let $s$ be the moment sequence of $\mu$. Then we have $\ker(DS_3)^T = R \cdot v$, where

$$v = (a^4 b^4, -2(a^4 b^2 + a^2 b^4), a^4 + 4a^2 b^2 + b^4, 0, -2(a^2 + b^2), 0, 1)^T,$$

$$p(x) := \langle v, s_F(x) \rangle = (x^2 - a^2)^2(x^2 - b^2)^2.$$

Hence $\mathcal{V}_+(s) = \{a, -a, b, -b\}$. We show that $\mathcal{W}(s) = \{a, -a, b\}$. Clearly, $\{a, -a, b\} \subseteq \mathcal{W}(s) \subseteq \mathcal{V}_+(s)$. Assume to the contrary that $\mathcal{W}(s) = \{a, -a, b, -b\}$. Then $s$ has a representing measure of the form $\mu^* = c_1^* \delta_{-a} + c_2^* \delta_a + c_3^* \delta_b$ and we obtain

$$s = c_1 s_F(-a) + c_2 s_F(a) + c_3 s_F(b) = c_1 s_F(-a) + c_2 s_F(a) + c_3 s_F(b) + c_4 s_F(-b)$$

$$0 = (c_1^* - c_1) s_F(-a) + (c_2^* - c_2) s_F(a) + (c_3^* - c_3) s_F(b) + c_4^* s_F(-b).$$

This implies that $c_4 = 0$ and $c_i = c_i^*$ for $i = 1, 2, 3$. Thus $\mu = \mu^*$ and we have

$$\mathcal{W}(s) \subseteq I(s) = \mathcal{V}_+(s).$$
Example 39. Let $X = \mathbb{R}$, 

$$A := \{x^{\alpha \beta} \mid \alpha, \beta = 0, 1, 2\} = \{1, x^3, x^5, x^6, x^8, x^{10}, x^{11}, x^{12}, x^{16}\}$$

and let $s$ be the moment sequence of $\mu := c_1 \delta_{-1} + c_2 \delta_0 + c_3 \delta_1 + c_4 \delta_2$. Then 

$$\ker DS_2^T = v_1 \mathbb{R} + v_2 \mathbb{R},$$

where 

$$v_1 = (0, 12864, -17152, -14580, 29163, -14584, 4288, 0, 1)^T,$$

$$v_2 = (0, 192, -256, -220, 441, -222, 64, 1, 0)^T,$$

so that 

$$p_1(x) = \langle v_1, sA(x) \rangle$$

$$= (x - 2)^2(x - 1)^2x^3(x + 1)^2(x^7 + 4x^6 + 14x^5 + 40x^4 + 107x^3 + 4556x^2$$

$$+ 3216x + 3216),$$

$$p_2(x) = \langle v_2, sA(x) \rangle$$

$$= (x - 2)^2(x - 1)^2x^3(x + 1)^2(x^3 + 68x^2 + 48x + 48).$$

But neither $p_1$ nor $p_2$ is non-negative. It is not difficult to verify that 

$$p(x) := p_1(x) - 67p_2(x) = (x - 2)^2(x - 1)^2x^6(x + 1)^2(x + 2)^2(x^2 + 10)$$

is, up to a constant factor, the only non-negative element of $A$ such that $L_s(p) = 0$. Therefore, $N_+(s) = \mathbb{R}^+$. $p$ and $V_+(s) = \{-2, -1, 0, 1, 2\}$. Using that the vectors $s_2(-2), s_2(-1), s_2(0), s_2(1), s_2(2)$ are linearly independent, a similar reasoning as in the preceding example shows $W(s) = \{-1, 0, 1, 2\}$. Thus, 

$$W(s) = I(s) \subseteq V_+(s).$$

The next theorem is the main result of this section. It collects various characterizations of inner points of the moment cone.

Theorem 40. For $s \in \mathcal{S}$ the following are equivalent:

(i) $s$ is an inner point of the moment cone $\mathcal{S}$.

(ii) $N_+(s) = \{0\}$.

(iii) $W(s) = \mathbb{R}^n$.

(iv) $I(s) = \mathbb{R}^n$.

(v) $V_+(s) = \mathbb{R}^n$.

(vi) $d(s) = 0$.

(vii) $\exists(s) = \mathbb{R}^m$.

Proof. (i)$\leftrightarrow$(ii) follows from Lemma 25 iii).

(i)$\rightarrow$(iii): Let $x \in \mathbb{R}^n$. Since $s$ is an inner point, so is $s' := s - \varepsilon s(x)$ for some $\varepsilon > 0$. If $\mu'$ is a representing measure of $s'$, then $\mu := \mu' + \varepsilon \delta_x$ represents $s$ and $\mu(\{x\}) \geq \varepsilon > 0$, so that $x \in W(s)$.

(iii)$\rightarrow$(iv)$\rightarrow$(v) follows from Lemma 33 ii).

(v)$\rightarrow$(ii): Let $p \in N_+(s)$. Then, since $\mathbb{R}^n = V_+(s) \subseteq \mathcal{Z}(p)$ by (v), $p = 0$.

(vii)$\rightarrow$(i): By Lemma 33 there exists a representing measure $(C, X)$ of $s$ such that $DS_2(C, X)$ has full rank. But then an open neighborhood of $(C, X)$ is mapped onto an open neighborhood of $s$, i.e., $s$ is an inner point.

(iii)$\rightarrow$(vii): Since $W(s) = \mathbb{R}^n$ and the functions $f_1, \ldots, f_m$ are linearly independent, there are points $x_1, \ldots, x_m \in \mathbb{R}^n = W(s)$ such that the vectors $s_{f_i}(x_1), \ldots, s_{f_i}(x_n)$ are linearly independent. Then for any $x_i$ there is an atomic measure $\mu_i$ such that $x_i \in \text{supp} \mu_i$. Setting $\mu := \frac{1}{m} \sum_{i=1}^m \mu_i$, $\mu$ is a representing measure of $s$ and range $DS_{k_1 + \ldots + k_m}(\mu)$ is $m$-dimensional, so that $\mathbb{R}^m \subseteq \exists(s) \subseteq \mathbb{R}^m$. □
Corollary 41. For each \( s \in \mathcal{S} \) following statements are equivalent:

(i) \( s \) is a boundary point of the moment cone.
(ii) \( N_s(s) \neq \{0\} \).
(iii) \( W(s) \subseteq \mathbb{R}^n \).
(iv) \( I(s) \subseteq \mathbb{R}^m \).
(v) \( V_+(s) \subseteq \mathbb{R}^n \).
(vi) \( d(s) > 0 \).
(vii) \( \exists(s) \subseteq \mathbb{R}^m \).

Corollary 42. Suppose that \( s \) is a boundary point of \( \mathcal{S} \) with representing measure \( (C, X) \). If \( \text{codim range } DS(C, X) = 1 \), then \( \exists(s) = \text{range } DS(C, X) \).

Proof. Since \( 1 = \text{codim range } DS(C, X) \geq d(s) \geq 1 \), it follows that \( (C, X) \) is a representing measure of \( \exists(s) \).

The following proposition collects a number of useful properties of the set \( \mathcal{M}_{k,s} \equiv S_k^{-1}(s) \) of at most \( k \)-atomic representing measures of \( s \).

Proposition 43. Suppose that \( n \in \mathbb{N} \) and \( E \subset C^r(\mathbb{R}^n, \mathbb{R}) \), \( r \geq 0 \). Let \( s \in \mathcal{S} \).

(i) The set \( S_k^{-1}(s) \) of \( k \)-atomic representing measures \( (C, X) \) of \( s \) is closed.
(ii) \( S_k^{-1}(s)|_{c_k+1=0} = S_k^{-1}(s) \times \{0\} \times \mathbb{R}^n \).
(iii) Suppose that \( (C, X) \) is an at most \( k \)-atomic representing measure of \( s \) and \( DS_k(C, X) \) has full rank. In a neighborhood of \( (C, X) \), \( S_k^{-1}(s) \) is a \( C^r \)-manifold of dimension \( k(n+1) - m \) in \( \mathbb{R}^{k(n+1)} \). The tangent space \( T_{(C,X)}S_k^{-1}(s) \) at \( (C, X) \) is

\[
T_{(C,X)}S_k^{-1}(s) = \ker DS_k(C, X).
\]

(iv) If \( s \) is regular, then \( S_k^{-1}(s) \) is a \( C^r \)-manifold and \( \text{(ii)} \) holds at any representing measure \( (C, X) \).
(v) \( W(s) = \{ x \in \mathbb{R}^n : (c_1, \ldots, c_k; x, x_2, \ldots, x_k) \in S_k^{-1}(s), c_1 > 0, \text{ for some } k \geq 1 \} \).

Proof. The continuity of the map \( S_k \) gives (i). (ii) is obvious.

(iii) and (iv) are straightforward applications of the implicit function theorem.

(v) follows easily from the definitions of \( W(s) \) and \( S_k \).

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