Korn’s inequalities for generalized external cusps

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In this paper, we consider a general class of external cusps defined by linking appropriate collections of John domains. For that class, weighted Korn inequalities are proved by means of rather elementary arguments. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

For a given a domain \( \Omega \subset \mathbb{R}^n \) and a vector field \( u \in W^{1,2}(\Omega) \), Korn’s inequality states

\[
\|Du\|_{L^2(\Omega)}^{1,\infty} \leq C\|\varepsilon(u)\|_{L^2(\Omega)}^{1,\infty},
\]

(1.1)

where \( Du \) is the differential matrix of \( u \) and \( \varepsilon(u) \) its symmetric part

\[
\varepsilon(u)_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\]

Inequality (1.1) introduced in [1, 2] has become a classic subject in the literature of continuum mechanics. In elasticity theory, \( u \) plays the role of the displacement field of an elastic body. In this case, \( \varepsilon(u) \) is called the linearized strain tensor and (1.1) equals to the coercivity of the bilinear form associated to the underlying linear equations.

Non-constant vector fields in the kernel of \( \varepsilon \) cannot obey (1.1), and therefore, Korn’s inequality cannot hold without considering some extra conditions. Two classic cases treated in the seminal works by Korn and called the first and the second case of the inequality state that (1.1) holds if either \( u \) vanishes on the boundary of \( \Omega \) or if

\[
\int_{\Omega} \frac{Du - Du'}{2} = 0,
\]

(1.2)

respectively. The first case can be proved by means of very simple arguments, and as it is well known, it holds for any bounded domain. On the other hand, the second case requires deeper considerations, and actually, it fails for domains with poor regularity.

Inequality (1.1) can be found in different forms involving traditional and weighted spaces. In \( L^p \) norm, it reads

\[
\|Du\|_{L^p(\Omega)}^{1,\infty} \leq C\|\varepsilon(u)\|_{L^p(\Omega)}^{1,\infty},
\]

(1.3)

where \( p \) should be in the range \( 1 < p < \infty \). Another version, sometimes called the general case of Korn's inequality, takes the form

\[
\|Du\|_{L^p(\Omega)}^{1,\infty} \leq C\{\|u\|_{L^p(\Omega)}^{1,\infty} + \|\varepsilon(u)\|_{L^p(\Omega)}^{1,\infty}\},
\]

(1.4)
in which no extra conditions other than \( u \in W^{1,p}(\Omega)^n \) are required. It can be shown, see for example, [3], that (1.3) implies (1.4) for any domain \( \Omega \). On the other hand, for regular domains, (1.3) can be deduced from (1.4) using compactness arguments [4].

Many proofs of Korn's inequality have been given since Korn's original works, and even a short review of this subject would involve a large number of references. Friederichs [5] was unable to reproduce Korn's arguments for the second case and proved the inequality for smooth domains by reducing the second to the first case. Since then, different arguments allowed to treat less regular domains. That is the case of those arguments involving singular integrals (see [4] and the references therein) where typically the derivatives of the vector field \( u \) are written as an average of derivatives of \( \varepsilon(u) \) in a cone. This idea, which follows closely Calderón's extension method, applies naturally for Lipschitz domains because they enjoy the cone property. In this regard, let us recall that the connection of Korn's inequality with extension procedures was also exploited by Nitsche, who used elementary arguments [6] to prove (1.1), in the second case, for Lipschitz domains by modifying appropriately the extension operator due to Stein. The same line of reasoning is applied in [7], where the authors prove that (1.4) stands not only for Lipschitz but for the broader class of uniform domains, using a modification of the extension operator given by Jones in [8]. In spite of these results, let us observe that the second case of (1.3) holds for

\[
\Omega_\infty = \mathcal{B}(0,1) \setminus \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = 0\},
\]

where \( \mathcal{B}(0,1) \) is the unitary ball. However, it is easy to see that \( \Omega_\infty \) is not an extension domain. Therefore, it becomes clear that extension arguments should fail to tackle (1.3) in the more general setting. Actually, (1.3) holds in the second case for a family of domains that contains strictly the set of uniform domains. In this concern, let us recall the class of domains introduced by John in [9], and named John domains after him by Martio and Sarvas [10].

**Definition 1.1**

Let \( 0 < \alpha \leq \beta < \infty \). A domain \( \Omega \subset \mathbb{R}^n \) is called a John domain with parameters \( \alpha \) and \( \beta \) if there is a point \( x_0 \in \Omega \) (the John-center of \( \Omega \)) such that for every \( x \in \Omega \) there is a rectifiable curve with parametrization by arc length \( \gamma : [0,\ell] \to \Omega \) such that \( \gamma(0) = x \) and \( \gamma(\ell) = x_0 \), and

\[
d(\gamma(t), \partial \Omega) \geq \frac{\alpha}{\ell} \quad \forall t \in [0,\ell].
\]

Given \( x \in \Omega \), and its correspondent curve \( \gamma \), the set \( \bigcup_t \mathcal{B}(\gamma(t), \frac{\beta}{\ell} t) \subset \Omega \) can be regarded as a twisted cone with its axis depicted by the curve \( \gamma \). In this sense, we may say that in a John domain \( \Omega \) any \( x \in \Omega \) can be joined with \( x_0 \) through a curve that remains ‘away’ from the boundary. John domains contains star-shaped, Lipschitz, and uniform domains. The inner cusp

\[
\Omega_\alpha = \mathcal{B}(0,1) \setminus \{(x,y) \in \mathbb{R}^2 : |y| < x^\alpha \} \quad \alpha > 1
\]

as well as the limit case (1.5) are examples of John domains. The boundary of a John domain can be very intricate, for instance the Koch snowflake, with fractal boundary is also a John domain. In Figure 1, we show all these examples along with schematic twisted cones. Korn's inequality holds on John domains, with a constant depending only on the parameters \( \alpha \) and \( \beta \). This is shown in [11] where an explicit continuous right inverse of the divergence is constructed. In simple geometries, it is possible to get more information about the constant involved in the inequality. For instance, in a convex domain \( \Omega \), the constant depends linearly on the ratio between the diameter of \( \Omega \) and the diameter of a maximal ball contained in \( \Omega \) [12]. Remarkably, for star-shaped domains, the same bound is valid in dimension two as it is shown in [13]. For star-shaped domains, the dependence of the constant on that ratio in dimension \( n \) is also studied in [12].

Although Korn's inequality holds for very general domains, it fails, as it was early noticed by Friederichs, in domains with external cusps (see [14] for a collection of counterexamples). Roughly speaking, an external cusp is a domain that narrows toward a point (the tip of the cusp) faster than any cone. Observe that this narrowing prevents any external cusp to be a John domain.

The simplest kind of external cusps are given by power type cusps:

\[
\Omega = \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < x_n^\gamma\},
\]

- (a) The inner cusp \( \Omega_\alpha \)
- (b) The limit case \( \Omega_\infty \)
- (c) The Koch snowflake

**Figure 1.** Examples of John domains.
being \( \gamma > 1 \) a real number. This notion is naturally generalized to domains with a profile depicted by a more general function \( \varphi \):

\[
\Omega = \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < \varphi(x_n)\},
\]

(1.10)

where \( \varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is a non-decreasing derivable function such that \( \varphi(0) = 0 \) and \( \varphi'(0) = 0 \), or, more generally, \( \varphi \) is non-decreasing, Lipschitz, and \( \varphi(0) \to 0+ \).

The main precedent that we follow regarding Korn’s inequality on external cusps work with these definitions. In [15], power type cusps are treated, and the existence of a right inverse for the divergence operator in weighted spaces is proved. As a corollary, the following weighted Korn’s inequality is obtained (see [15, Theorem 6.2]):

**Theorem A**

Given \( \Omega \) a domain of the form (1.9), \( 1 < p < \infty \), \( B \subset \Omega \) an open ball and \( \beta \geq 0 \); there exists a constant \( C \), depending only on \( \Omega, B, p, \) and \( \beta \), such that for every \( u \in W^{1,p}_{\varphi^\beta}(\Omega)^p \)

\[
\|Du\|_{L^p(\Omega)^n} \leq C \left\{ \|u\|_{L^p(B)^n} + \|\varphi(u)\|_{L^p(\varphi^\beta+1-p;B)^n} \right\},
\]

where \( d = d(x) \) is the distance to the origin and \( \gamma \) is the exponent of the cusp.

Notice that for \( W^{1,p}(\Omega)^p \) (i.e., \( \beta = 0 \)), the weight on the right-hand side, due to the cuspidal behavior of \( \Omega \) is \( d^{p(1-\gamma)} \). The optimality of Theorem 1 is treated in [14], where the authors work with cusps with a general profile \( \varphi \) and prove the following theorem:

**Theorem B**

Let \( \Omega \) be a cusp with profile \( \varphi \), according to (1.10), \( \beta_1, \beta_2 \in \mathbb{R} \), \( 1 < p < \infty \) and \( B \) a ball compactly contained in \( \Omega \). If there is a constant \( C \) such that

\[
\|DV\|_{L^p(\varphi^\beta;d^p)^n} \leq C \left\{ \|V\|_{L^p(B)^n} + \|\varphi(V)\|_{L^p(\varphi^\beta+1-p;B)^n} \right\},
\]

for every \( v \in W^{1,p}_{\varphi^\beta}(\Omega)^p \), then \( \beta_1 \geq \beta_2 + 1 \).

Observe that for a power type cusp \( \varphi'(t) = \gamma t^{\gamma-1} \), and because inside the cusp one has \( d(x) \sim x_n \), then Korn’s inequality in Theorem 1 corresponds with the case \( \beta_1 = \beta_2 + 1 \) in Theorem 1, which shows that Theorem 1 is sharp.

Other results about external cusps can be found in [16], where weighted anisotropic Korn inequalities for power type cusps are proved in \( \mathbb{R}^3 \). There, the author mentions that more general cases could be treated by using the same arguments developed in that paper.

Let us recall that for a Hölder-\( \alpha \) domain, weighted versions of Korn’s inequality can be found in [17]. Because such a domain may have many ‘cuspidal’ singularities, the weight that naturally arises depends on the distance to the whole boundary. On the other hand, for more general domains, abstract weighted Korn inequalities can be derived from results concerning the existence of a right inverse of the divergence operator, see for instance [18, 19].

The aim of the present paper is to extend Theorem 1 to a wider class of external cusps. Actually, we prove that such a result can be generalized to external cusps whose boundary is locally the one of a John domain. In this context, we show that the function \( \varphi \) does not need to depict the precise profile of the domain but only to give a qualitative description of the narrowing.

Our technique is based on rather simple ideas. Because the behavior of the constant in Korn’s inequality on rectangles is known, we consider chains of rectangles and use a discrete Hardy inequality to pass from one rectangle to another, proving a weighted Korn inequality for the whole chain. The weight is piecewise constant, and it is given by Korn’s constant on each rectangle. This approach depends exclusively on the existence of intermediate rectangles that link each rectangle in the chain with its two neighbors. This approach can be applied to more general subdomains, and in this regard, we introduce the notion of quasirectangle. A quasirectangle is essentially, a domain that contains a rectangle and where Korn and Poincaré inequalities hold with constants similar to those corresponding to this interior rectangle. The boundary of a quasirectangle can be locally as general as the one of a John domain. The results obtained for chains of rectangles are straightforwardly generalized to chains of quasirectangles, yielding weighted Korn’s inequalities for a very general class of domains. Finally, we consider as an example external cusps formed by chains of quasirectangles. These chains are defined by requiring an appropriate narrowing toward a ‘singular’ point. In this way, our generalization of Theorem 1 is a simple corollary of our results for chains of quasirectangles. Counterexamples proposed in [14] can be easily adapted for proving the optimality of the weights obtained in this paper.

Finally, before proceeding, let us mention that the extension procedure for external cusps used in [20] can be appropriately combined with the extension operator for uniform domains constructed in [7]. In that way, it is possible to obtain similar results to those given here. However, by doing that, the notion of ‘locally John’ should be replaced by the notion of ‘locally uniform’ proposed in [20]. This allows to handle a less general class of external cusps than that treated here. This limitation is not surprising in the light of the facts mentioned earlier concerning the use of extension arguments in the context of Korn’s inequality. For that reason, we do not follow this approach.

### 2. Notation and preliminaries

Throughout this article, \( \hat{x}_n \) stands for the \( x_n \) axis, and \( C \) denotes a generic constant that may change from line to line. We say that two positive numbers \( a \) and \( b \) are \( C \)-comparable, and we write \( a \sim b \), if \( \frac{1}{C}a \leq b \leq Ca \). For every collection of sets \( C \), we denote with \( \cup C \) the union of all the sets in \( C \), that is, \( \cup C := \bigcup_{S \in C} S \). Given two sets, we write \( A \equiv B \) if they differ in measure zero.
In this work, we deal with open rectangles \( R \subset \mathbb{R}^n \) with edges \textit{parallel to the coordinate axes}. The size vector of \( R \) is denoted with 
\[ \ell(R) = (\ell_1(R), \ell_2(R), \ldots, \ell_n(R)), \]
where \( \ell_i(R) \) is the length of the \( R \)'s \( i \)-th edge. For a cube \( Q \), we use \( \ell(Q) \) to denote the length of any of its edges, and for a rectangle \( R \), we take 
\[ L_1(R) := \max_{1 \leq i \leq n} \{\ell_i(R)\} \] and 
\[ L_2(R) := \min_{1 \leq i \leq n} \{\ell_i(R)\}. \]
In some point, we restrict our attention to rectangles with \( n - 1 \) short edges of equal size that we denote \( \ell(R) \) and a long edge (the vertical) \( L(R) \). A pair of rectangles \( R_1, R_2 \) are called \textit{C-comparable}, and we write \( R_1 \sim R_2 \), if \( \ell_i(R_1) \sim \ell_i(R_2) \) for \( 1 \leq i \leq n \). For a rectangle \( R \), we denote its center with \( c_R \). If \( c_R = (c_1, \ldots, c_n) \), the upper face \( F^0_2 \) of \( R \) is given by 
\[ F^0_2 = \{ (x_1, \ldots, x_n) \in \mathbb{R} : x_n = c_n + \frac{1}{2} \ell_n(R) \} \] and analogously is defined the lower face \( F^0_1 \).
Given a rectangle \( R \), we denote \( \partial R \) \((a > 1)\), the expanded rectangle centered in \( c_R \) with edges \( \ell_i(aR) = a \ell_i(R) \). With \( z_R \), we denote the last coordinate of points belonging to \( F^0_1 \), that is \( z_R = c_n - \frac{1}{2} \ell_n(R) \). We say that \( R_1 \) and \( R_2 \) are touching rectangles if \( R_1 \cap R_2 = \emptyset \) and \( R_1 \cap R_2 = F \) with \( F \) a face of \( R_1 \) or \( R_2 \).
When we consider external cusps, with tip at the origin, we write, for every \( x \in \mathbb{R}^n \), \( d = d(x) \) for the distance to the origin. For any \( p, 1 < p < \infty, p' \) stands for the conjugate exponent of \( p \), \( \frac{1}{p} + \frac{1}{p'} = 1 \). For a matrix \( A \in \mathbb{R}^{n \times n} \), we denote \( |A|^p = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^p \) and analogously for vectors \( v \in \mathbb{R}^n \).

Let \( \omega : \Omega \longrightarrow \mathbb{R}_{\geq 0} \) be, a measurable function, the weighted \( L^p \) norm of a matrix field \( A : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times n} \) is defined as
\[ \| A \|^p_{L^p(\omega)} := \int_{\Omega} |A|^p \omega = \sum_{i=1}^n \sum_{j=1}^n \| \omega^{\frac{1}{2}} A_{ij} \|^p_{L^p(\Omega)}, \]
and analogously for vector fields. The weighted Sobolev space \( W^{1,p}_\omega(\Omega)^n \) is defined by the set of vector fields \( u : \Omega \longrightarrow \mathbb{R}^n \) for which
\[ \| u \|^p_{W^{1,p}_\omega(\Omega)^n} := \sum_{i=1}^n \int_{\Omega} (|u|^p + |Du(x)|^p)\omega = \sum_{i=1}^n \sum_{|\alpha| \leq 1} \| \omega^{\frac{1}{2}} D^\alpha u \|^p_{L^p(\Omega)}, \]
is finite. In the case \( \omega = 1 \), we drop the subscript and write \( \| u \|_{W^{1,p}(\Omega)^n} \) and \( \| A \|_{W^{1,p}(\Omega)^{n \times n}} \), respectively. The average value of a function \( u \) over a domain \( D \) is written with either of the following notations: \( u_D := \frac{1}{|D|} \int_D u \), where \( |D| \) stands for the measure of \( D \), and analogously for the weighted average, we write \( u_{\omega_D} := \frac{1}{\omega(D)} \int_D u \omega \).

Let us state the following discrete weighted inequality of Hardy type [21, page 52]:

**Lemma 2.1**

Let \( \{u_i\} \) and \( \{v_i\} \) be sequences of non-negative weights, and let \( 1 < p \leq q < \infty \). Then, the inequality
\[ \left( \sum_{j=1}^\infty u_j \left( \sum_{i=1}^j b_i \right)^\frac{q}{p} \right)^\frac{p}{q} \leq c \left( \sum_{i=1}^\infty v_i b_i^p \right)^{\frac{1}{p}} \]
holds for every non-negative sequences \( \{b_i\} \) if
\[ A = \sup_{k \to 0} \left( \sum_{j=k}^\infty u_j \right)^{\frac{1}{q}} \left( \sum_{i=0}^k v_i b_i^{1-p} \right)^{\frac{1}{p}} < \infty \]
The constant \( c \) is \( c = MA \), where \( M \) depends only on \( p \) and \( q \).

Throughout this paper, we make extensive use of the following lemma.

**Lemma 2.2**

Let \( \{r_j\} \) and \( a = \{a_j\} \) be sequences such that \( \{r_j\} > 0 \), and \( \sum_{j} r_j = r < \infty \). Let us denote
\[ \tilde{a} = \frac{1}{r} \sum_{j} a_j r_j. \]
Then, the inequality
\[ \left( \sum_{j=1}^\infty |a_j - \tilde{a}|^p r_j \right)^\frac{1}{p} \leq c \left( \sum_{j=1}^\infty |a_{j+1} - a_j|^p r_{j+1} \right)^\frac{1}{p} \]
holds if
\[ A = \sup_{k \to 0} \left( \sum_{j=k}^\infty r_j \right)^{\frac{1}{p}} \left( \sum_{i=0}^k r_i^{1-p} \right)^{\frac{1}{p}} < \infty \]
The constant \( c \) is \( c = MA \) where \( M \) depends only on \( p \).
Proof
Let us define the norm
\[ \|a\|_p = \left( \sum_i |a_i|^p r_i \right) \frac{1}{p}. \]
From Hölder’s inequality, it holds \( |a| r \leq \|a\|_p r^\frac{1}{p} \) and then \( \|a - \tilde{a}\|_p \leq 2 \|a\|_p. \) Applying this last inequality with \( a \) replaced by \( a - a_0 \), we obtain
\[ \|a - \tilde{a}\|_p \leq 2 \|a - a_0\|_p. \]
Therefore,
\[ \sum_i |a_i - \tilde{a}|^p r_i \leq 2^p \sum_i |a_i - a_0|^p r_i \leq 2^p \sum_i \left( \sum_{j=1}^{i} |a_j - a_{j-1}| \right) r_i \]
and we conclude applying Lemma 2.1 with \( u_i = v_i = r_i, q = p \) and \( b_i = |a_i - a_{i-1}| \). \( \square \)

Remark 2.3
Observe that if in Lemma 2.2, \( \{r_i\}_{i \leq N} \) is finite and \( r_i \sim r \) for any \( i \), then
\[ A = \max_{N \geq k > 0} \left( \sum_{j=k}^{N} r_j \right) \frac{1}{p} \leq CN. \]

Proof
In fact,
\[ A = \max_{N \geq k > 0} \left( \sum_{j=k}^{N} r_j \right) \frac{1}{p} \leq C \max_{N \geq k > 0} \left( \sum_{j=k}^{N} r_j \right) \frac{1}{p} = \max_{N \geq k > 0} (N - k)^\frac{1}{p} \frac{k^1 - p^1}{1} \leq CN. \]

\( \square \)

3. Korn and Poincaré inequalities for chains of rectangles

Definition 3.1
A (finite or countable) collection of rectangles \( C = \{R_i\} \) for which \( \sum_j |R_j| < \infty \) is called a chain of rectangles if (i) \( R_i \cap R_j = \emptyset \) for \( |i - j| > 1 \), (ii) for any \( i, R_i, \) and \( R_{i+1} \) are touching, and (iii) there exists a constant \( C \) such that \( R_i \sim R_{i+1}, \) for any \( i \).

Remark 3.1
Given a chain of rectangles \( C = \{R_i\} \), we have the following elementary facts:

- because \( R_i \) and \( R_{i+1} \) are touching and \( C \) is comparable, there exists a rectangle \( R_{i+1} \subset R_i \cup R_{i+1} \) and a constant \( \tilde{C} \) depending only on \( C \), such that
  \[ R_{i+1} \sim \tilde{C} (R_{i+1} \cap R_i) \sim \tilde{C} (R_{i+1} \cap R_{i+1}) \sim \tilde{C} R_{i+1} \]
- thanks to the previous item
  \[ |R_{i+1}| \sim \tilde{C} |(R_{i+1} \cap R_i)| \sim \tilde{C} |(R_{i+1} \cap R_{i+1})| \sim \tilde{C} |R_{i+1}| \]
with \( \tilde{C} \) depending only on \( C \).
- using that \( R_i, R_{i+1} \), and \( R_{i+1} \) are \( \tilde{C} \)-comparable, we get (for instance by changing variables) that there exists a constant \( \tilde{C} \), depending only on \( \tilde{C} \) (therefore only on \( C \)) such that
  \[ C_i \tilde{C} C_{i+1} \sim C_{i+1} \]
being \( C_i, C_{i+1}, C_{i+1} \) the constants in the second case of Korn’s inequality for \( R_i, R_{i+1}, \) and \( R_{i+1} \), respectively.

Definition 3.2
Given a chain of rectangles \( C \), any given collection of intermediate rectangles \( R_{i+1} \) enjoying properties like those mentioned in Remark 3.1 is denoted with \( C_i = \{R_{i+1}\} \).
Definition 3.3
Given a chain of rectangles \( C_i \), we call \( \Omega = \cup (C_i \cup C_j) \) a \( C \)-linked domain.

Remark 3.2
Observe that if \( \Omega \subset \Omega_2 \) and \( |\Omega_2 \setminus \Omega| = 0 \), then \( K_{\Omega_2} \geq K_{\Omega_2} \) and \( P_{\Omega_2} \geq P_{\Omega_2} \) where \( K_{\Omega_2} \) and \( P_{\Omega_2} \), are the constants for the second case of Korn's inequality and of Poincaré's inequality on \( \Omega_j \), respectively. Consequently, the results that we prove for \( C \)-linked domains hold for every \( \Omega \) such that \( \cup (C_i \cup C_j) \subset \Omega \) and \( \Omega \equiv \cup (C_i \cup C_j) \).

Theorem 3.3 (Second case of Korn's inequality for chains of rectangles)
Let \( C = \{R_j\} \) be a chain of rectangles, and let \( C_i \) be the constants for the second case of Korn's inequality on \( R_i \), Then for any \( C \)-linked domain \( \Omega \), and any \( u \in W^1_p(\Omega)^n \) such that \( \int_{\Omega} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} = 0 \), we have
\[
\|Du\|_{L^p(\Omega)}^{\infty} \leq C \left( 1 + A_{i} \right) \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty},
\]
where \( A_{i} \) is defined in (2.2) with \( r_j = |R_j| \) and the weight \( \sigma \) is constant on each \( R_i \), being \( \sigma |_{\Omega} = C i \).

Proof
Let
\[
A_i = \frac{1}{2|R_i|} \int_{R_i} (Du - Du')^2.
\]
Then,
\[
\|Du\|_{L^p(\Omega)}^{\infty} = \sum_i \|Du\|_{L^p(\Omega)}^{\infty} \leq C \sum_i \|Du - A_i\|_{L^p(\Omega)}^{\infty} + C \sum_i \|A_i\|_{L^p(\Omega)}^{\infty}
\]

I leads to
\[
I \leq C \sum_i C_i \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty} \leq C \sum_i \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty} = C \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty}.
\]

For II, apply inequality (2.1) with \( r_i = |R_i| \). Let us observe that \( \sum_i |R_i| A_i = 0 \), therefore taking
\[
A = \sup_{k \geq 0} \left( \sum_{i \leq k} |R_i| \right)^{\frac{1}{2}} \left( \sum_{i \leq k} |R_i|^{1-p} \right)^{\frac{1}{p}},
\]
we have
\[
II = C \sum_i |A_i| \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty} \leq CA \sum_i |A_i| \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty} = C \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty}.
\]

where, in the last inequality, we use that for each \( R_i, R_j \cap R_j \neq \emptyset \) if \( |i - j| > 1 \). Therefore,
\[
II \leq CA \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty}.
\]
and the theorem follows.

Remark 3.4
In a recent paper, R. Durán proves that the constant for the second case of Korn’s inequality for any convex domain \( \Omega \) can be bounded taking the quotient between the diameter of \( \Omega \) and the diameter of a maximal ball contained in \( \Omega \) (see [12, Theorem 4.2]). Even when in [12] that result is stated only for \( p = 2 \), the same proof works for \( 1 < p < \infty \). It is important to notice that this implies that given a rectangle \( R \) with edges \( \ell_i(R) \), eventually all different, Korn’s constant in the second case can be taken \( \frac{L(M) + L(B(R))}{\ell_m(R)} \). That estimate is sharp.

Take, for instance, \( n = 2 \) and \( u(x,y) = (-x^2, x^3) \), defined over \( R = (0, L_M) \times (-\frac{1}{2}, \frac{1}{2}) \), we have that

\[
\|Du\|_{L^p(R)_{xx}}^p \sim L_M \|u\|_{L^p(R)}^{p+1}
\]
and therefore,

\[
\frac{\|Du\|_{L^p(R)_{xx}}^p}{\|\varepsilon(u)\|_{L^p(R)_{xx}}^p} \sim \frac{L_M}{L_m} (\frac{L_M}{L_m})^p.
\]

Remark 3.5
Thanks to the previous remark, in Theorem 3.3, \( C_i \) can be taken as follows:

\[
C_i = \frac{L_m}{L_m^n}.
\] (3.1)

Theorem 3.3 can be straightforwardly extended to some weighted spaces.

Definition 3.4
Let \( C = \{R_i\} \), be a chain of rectangles, and \( \omega \) a domain such that \( \Omega = \cup C \). We say that \( \omega \) is an admissible weight in \( \Omega \) if for any \( x \in R_i \),

\[
\omega(x) \sim \omega_{R_i} \sim \omega_{R_{i+1}} \quad \forall i
\]
(3.2)
being \( \omega_{R_i} \), appropriate constants.

The following is an elementary generalization of Theorem 3.3

Theorem 3.6 (Second case of weighted Korn’s inequality for chains of rectangles)
Let \( C = \{R_i\} \) be a chain of rectangles and \( \Omega \) a \( C \)-linked domain. Let \( u \in W^{1,p}_\omega(\Omega)^n \), with \( \omega \) an admissible weight (see (3.2)), be such that

\[
\int_\Omega \frac{Du - Du^i}{2} \omega = 0.
\]

If \( \sum \omega(R_i) = r < \infty \),

\[
\|Du\|_{L^p(\Omega)_{xx}}^p \leq C (1 + A_\omega) \|\varepsilon\|_{L^p(\Omega)_{xx}}^p
\]
where \( A_\omega \) can be taken as in Theorem 3.3, and

\[
A_\omega := \sup_{k \geq 1} \left( \sum_{i \leq k} \omega(R_i) \right)^{\frac{1}{p}} \left( \sum_{i \leq k} \omega(R_i)^{1 - \frac{1}{p}} \right)^{\frac{1}{p'}}.
\] (3.3)

Proof
Let

\[
A'i = \frac{1}{|R_i|} \int_{R_i} \frac{Du - Du^i}{2} \omega \quad \text{and} \quad A'_\omega = \frac{1}{\omega(R_i)} \int_{R_i} \frac{Du - Du^i}{2} \omega.
\]

We take

\[
\|Du\|_{L^p(\Omega)_{xx}}^p = \sum_i \|Du\|_{L^p(R_i)_{xx}}^p \leq C \left\{ \sum_i \|Du - A'_\omega\|_{L^p(R_i)_{xx}}^p + \sum_i \|A'_\omega\|_{L^p(R_i)_{xx}}^p \right\}.
\]

For (a), we write

\[
\|Du - A'_\omega\|_{L^p(R_i)_{xx}}^p \leq \|Du - A'i\|_{L^p(R_i)_{xx}}^p + \|A'i - A'_\omega\|_{L^p(R_i)_{xx}}^p.
\]

\footnote{Observe that aligning \( N \) identical cubes in a rectangle \( R \), we have that Remark 2.3 yields such a constant, written there in terms of \( N = L_M(R)/L_m(R) \).}
and for \( l \), we can take the weight off the norms

\[
P = \|Du - A\|_{L^p_c(\Omega)^{m \times n}} \leq \omega_k \|Du - A\|_{L^p(\Omega)^{m \times n}} \leq C_{\omega_k} \|\varepsilon(u)\|_{L^p_c(\Omega)}^p
\]

On the other hand,

\[
I^p = \|A' - A'_\omega\|_{L^p_c(\Omega)^{m \times n}}^p = \omega(R_i) \left| A' - \frac{1}{\omega(R_i)} \int_{R_i} \frac{Du - Du'}{2} \omega(x) dx \right|^p
\]

\[
\leq C_{\omega(R_i)} \left\{ \int_{R_i} (A' - Du) \omega(x) dx \right\}^p + \left\{ \int_{R_i} (Du - Du') \omega(x) dx \right\}^p
\]

\[
= C_{\omega(R_i)} \left\{ \int_{R_i} (A' - Du) \omega(x)^{1/2} \omega(x)^{1/2} dx \right\}^p + \left\{ \int_{R_i} \varepsilon(u) \omega(x)^{1/2} \omega(x)^{1/2} dx \right\}^p
\]

Applying Hölder inequality in both terms,

\[
I^p \leq C_{\omega(R_i)} \left\{ \int_{R_i} (A' - Du) \omega(x) dx \right\}^p + \left\{ \int_{R_i} \varepsilon(u) \omega(x) \omega(x) dx \right\}^p
\]

\[
\leq C \left\{ F + \|\varepsilon(u)\|_{L^p_c(\Omega)^{m \times n}}^p \right\} \leq C \|\varepsilon(u)\|_{L^p_c(\Omega)}^p
\]

On the other hand, for (b), let us observe that

\[
\sum_i \omega(R_i) A'_\omega = 0
\]

and that

\[
\sum_i \|A'_\omega\|_{L^p_c(\Omega)^{m \times n}}^p = \sum_i \omega(R_i) \|A'_\omega\|^p.
\]

Consequently, Lemma 2.2 with \( a_i = A'_\omega \), and \( r_i = \omega(R_i) \) yields

\[
\sum_i \|A'_\omega\|_{L^p_c(\Omega)^{m \times n}}^p \leq C_{\omega} \sum_{i=1}^{\infty} \|A'_\omega - A'_{\omega+i} - \omega(R_i)\|_{L^p_c(\Omega)^{m \times n}}^p \leq C_{\omega} \sum_{i=1}^{\infty} \|A'_\omega - A'_{\omega+i} - \omega(R_i)\|_{L^p_c(\Omega)^{m \times n}}^p.
\]

Now, we may proceed like in Theorem 3.3, alternating \( A'_\omega \), the weighted average of \( (Du - Du')/2 \) on an overlapping rectangle \( R_{i+1} \), afterwards alternating \( Du \), and finally applying the estimates for (a). We leave the final details to the reader.

Observe that Theorem 3.3 is a corollary of the previous theorem taking \( \omega = 1 \). However, Theorem 3.6 does not provide information unless \( A_\omega < \infty \). A simple way to bound \( A_\omega \) involves a reasonable decay for \( \omega(R_i) \).

**Corollary 3.7**

Under the same hypotheses of Theorem 3.6, assume that for any \( k \),

\[
\omega(R_{i+1}) \leq \omega(R_i) \quad \text{with} \quad 0 \leq \alpha < 1.
\]

Then for any \( u \in W^{1,p}_{\omega}(\Omega)^n \) such that \( \int_{\Omega} \frac{Du - Du'}{2} \omega = 0 \), we have

\[
\|Du\|_{L^p_c(\Omega)^{m \times n}} \leq C \|\varepsilon(u)\|_{L^p_c(\Omega)^{m \times n}},
\]

where the weight \( \sigma \) is constant on each element of \( C \), and can be taken as \( \sigma_i = \left( \frac{\mu_i}{\lambda_i} \right)^{1/p} \).

**Proof**

From Remark 3.4, we know that \( C_{\sigma} \leq C_{\omega} \) with \( C_{\sigma} \) defined in Theorem 3.3. Therefore, only remains to show that \( \sum_i \omega(R_i) < \infty \) and \( A_\omega < C \). These follow from the bounds \( \omega(R_i) \leq \omega(R_k) \leq \omega(R_k) \) for \( 0 \leq i \leq k \) and \( \omega(R_i) \leq \omega(R_k) \) for \( i \geq k \). Indeed,

\[
A_\omega = \sup_{k>0} \left( \sum_{i=k}^{\infty} \omega(R_i) \right)^{\frac{p}{2}} \left( \sum_{i=0}^{k} \omega(R_i)^{1-p'} \right)^{\frac{p'}{2}} \leq \omega(R_k)^{\frac{p}{2}} \left( \sum_{i=0}^{\infty} \omega(R_i) \right)^{\frac{p}{2}} \omega(R_k)^{\frac{p}{2} - 1} \left( \sum_{i=0}^{k} \omega(R_i)^{p-1} \right)^{\frac{p}{2}}
\]
Applying Theorem 3.8, On the other hand, then \[ A_M \leq \left( \frac{1}{1 - \alpha} \right)^{\frac{1}{n}} \left( \frac{1}{1 - \alpha^{p-1}} \right)^{\frac{1}{p}}, \]
and the corollary follows. □

Everything performed so far for the second case of Korn’s inequality for chains of rectangles can be performed for Poincaré inequality following step by step the arguments given earlier. Because the constant in Poincaré inequality for rectangles (and in general for convex domains) depends only on the diameter of the rectangle, the weight involved in the inequality can be weakened as it is stated as follows.

**Theorem 3.8 (Poincaré inequality for chains of rectangles)**

Let \( 1 < p < \infty \) and \( C = \{R_i\} \) be a chain of rectangles and \( \Omega \) a \( C\)-linked domain. Let \( \omega \) be an admissible weight (see (3.2)), such that for any \( k \), \( \omega(R_{k+1}) = v \omega(R_k) \) with \( 0 \leq v < 1 \). Then if \( u \in W^{1,p}_{\omega}(\Omega)^n \), and \( \int_{\Omega} \omega = 0 \), we have

\[
\|u\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)^{n \times n}},
\]
where the weight \( \omega \) is constant on each \( R_i \) and can be taken as \( \omega|_{R_i} = \ell^p_0 \).

**Remark 3.9**

Let us mention that Theorem 3.8, as well as subsequent results involving Poincaré inequalities, also holds for \( p = 1 \). The key point is that during the proof, Lemma 2.2 is invoked with the choice \( \ell = \omega(R) \) under the decay condition \( \omega(R_{k+1}) \leq \omega(R_k) \). In this case, it is easy to see, by a direct proof of Lemma 2.1 for \( p = 1 \) taking \( u_i = v_i = r_i \), that (2.1) holds for \( p = 1 \) and \( c = \frac{2}{1 - \alpha} \).

The following version will be useful in the sequel.

**Corollary 3.10**

With the same hypotheses of Theorem 3.8, assume that \( B \) is a ball such that \( B \subset \Omega \), and \( B \cap R_j \neq \emptyset \) only for a finite number of rectangles. Then, for every \( u \in W^{1,p}_{\omega}(\Omega)^n \), we have

\[
\|u\|_{L^p(\Omega)^n} \leq C \left\{ \|u\|_{L^p(\Omega)^n} + \|Du\|_{L^p(\Omega)^{n \times n}} \right\},
\]
where the weight \( \omega \) is constant on each \( R_i \) and can be taken as \( \omega|_{R_i} = \ell^p_0 \).

**Proof**

For the sake of clarity, we write the case \( \omega \equiv 1 \).

\[
\|u\|_{L^p(\Omega)^n} \leq \|u - u_B\|_{L^p(\Omega)^n} + \|u_B\|_{L^p(\Omega)^n} \leq \|u - u_\Omega\|_{L^p(\Omega)^n} + \|u_\Omega - u_B\|_{L^p(\Omega)^n} + \|u_B\|_{L^p(\Omega)^n}.
\]

Applying Theorem 3.8,

\[
I \leq C \|Du\|_{L^p(\Omega)^{n \times n}}.
\]

On the other hand,

\[
II^p = \int_{\Omega} \left( \int_B u \right)^p = \left( \frac{\Omega}{|B|^p} \right)^p \left( \int_B u \right)^p \leq \frac{\Omega}{|B|^p} \left( \int_B u \right)^p \leq \frac{\Omega}{|B|^p} \|u\|_{L^p(B)}^p.
\]

For \( II \), applying Hölder inequality,

\[
|u_\Omega - u_B| \leq \frac{1}{|B|} \int_B |u_\Omega - u| \leq \frac{|\Omega|^\frac{1}{p}}{|B|^\frac{1}{p}} \|u - u_\Omega\|_{L^p(B)} \leq \frac{1}{|B|^\frac{1}{p}} \|u - u_\Omega\|_{L^p(\Omega)^n} \leq \frac{C}{|B|^\frac{1}{p}} \|Du\|_{L^p(\Omega)^{n \times n}}
\]
then

\[
II \leq C \frac{|\Omega|^\frac{1}{p}}{|B|^\frac{1}{p}} \|Du\|_{L^p(\Omega)^{n \times n}},
\]
and the lemma follows for \( \omega \equiv 1 \).

The general case follows similarly using (3.2), and taking into account that \( B \) only meets a finite number of rectangles and then \( \|u\|_{L^p(\Omega)^n} \leq \|u\|_{L^p(B)} \).

\[ \tag{4943} \]

We now prove the general case of Korn’s inequality for chains of rectangles. Our proof is a straightforward adaptation of the classic argument given in [3]. Let us notice that we require that \( \ell_0 \leq C \) for any \( i \). That is in order to remove the weight \( \sigma \) from the Poincaré inequality given earlier.
Theorem 3.11 (General case of Korn’s inequality for chains of rectangles)
Let \( C = \{ R_i \} \) be a chain of rectangles and \( \Omega \) a \( C \)-linked domain. Consider a weight \( \omega \) such that (3.2) holds, and assume that \( \omega (R_{k+1}) \leq a \omega (R_k) \) with \( 0 \leq a < 1 \) and that \( L_m < C \), for any. If \( B \) is a ball such that \( B \subset \Omega \), and \( B \) meets only a finite number of rectangles \( R_i \), then for any \( u \in W^{1,p}_\omega (\Omega)^n \), we have
\[
\| Du \|_{\nu (\Omega)}^{p,n} \leq C \left\{ \| u \|_{L^p (B)^n} + \| \varepsilon (u) \|_{\nu, \omega (\Omega)}^{p,n} \right\},
\]
(3.5)
where the weight \( \sigma \) is constant on each element of \( C \) and can be taken as \( \sigma |_{R_i} = \left( \frac{L_m}{L_m} \right)^{p} \).

Proof
Again, let us focus first on the case \( \omega = 1 \). Consider the space
\[\mathcal{RM}(\Omega)^n = \{ v \in W^{1,p}(\Omega)^n : \ \varepsilon (v) = 0 \},\]
every function in \( \mathcal{RM}(\Omega)^n \) can be written as
\[v(x) = a + Mx,\]
where \( M \in \mathbb{R}^{n \times n} \) is symmetric. On the other hand, a complement of \( \mathcal{RM} \) in \( W^{1,p} \) can be defined as follows:
\[\hat{W}^{1,p}(\Omega)^n = \left\{ w \in W^{1,p}(\Omega)^n : \ \int_B w = 0, \ \int_{\Omega} \frac{Dw - Dw'}{2} = 0 \right\}.
\]
In fact, given \( u \in W^{1,p}(\Omega)^n \), we can take \( v \in \mathcal{RM}(\Omega)^n \):
\[v = a + M(x - \bar{x})\]
with
\[a = \frac{1}{\mu} \int_B u \quad \text{and} \quad m_{ij} = \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)\]
being \( \bar{x} \) the center of \( B \). Obviously, \( w = u - v \in \hat{W}^{1,p}(\Omega)^n \), and in particular,
\[W^{1,p}(\Omega)^n = \mathcal{RM}(\Omega)^n \oplus \hat{W}^{1,p}(\Omega)^n.
\]
Moreover, it is clear by definition that
\[\| v \|_{W^{1,p}(\Omega)^n} \leq C \| u \|_{W^{1,p}(\Omega)^n}, \quad \| w \|_{W^{1,p}(\Omega)^n} \leq C \| u \|_{W^{1,p}(\Omega)^n}.
\]
If (3.5) does not hold, there is a sequence \( \{ u_n \} \subset W^{1,p}(\Omega)^n \) such that
\[\| Du_n \|_{\nu (\Omega)}^{p,n} = 1\]
but,
\[\| u_n \|_{L^p (B)^n} + \| \varepsilon (u_n) \|_{\nu, \omega (\Omega)}^{p,n} < \frac{1}{n},\]
(3.6)
If we write
\[u_n = v_n + w_n\]
with \( v_n \in \mathcal{RM}(\Omega)^n \) and \( w_n \in \hat{W}^{1,p}(\Omega)^n \), \( w_n \) admits both Poincaré inequality in \( B \) and second case of Korn inequality in \( \Omega \)
\[\| w_n \|_{W^{1,p}(\Omega)} \leq \| w_n \|_{L^p (B)^n} + \| Dw_n \|_{L^p (\Omega)^n} \leq C \left( \| w_n \|_{L^p (B)^n} + \| Dw_n \|_{L^p (\Omega)^n} \right) \leq C \| Dw_n \|_{L^p (\Omega)^n} \leq C \| \varepsilon (w_n) \|_{\nu, \omega (\Omega)^n} \leq \frac{1}{n}.
\]
(3.7)
And then, \( w_n \rightarrow 0 \) in \( W^{1,p} \). On the other hand, \( v_n \) belongs to the finite dimensional space \( \mathcal{RM}(\Omega)^n \) and is bounded on \( \Omega \). Consequently, there is a subsequence, called again \( v_n \), such that \( v_n \rightarrow v \in \mathcal{RM}(\Omega)^n \) strongly in \( W^{1,p}(\Omega)^n \). As \( w_n \rightarrow 0 \), we have that
\[u_n \rightarrow v \in R(B)^n \quad \text{in} \ W^{1,p}(\Omega)^n.
\]
But because of (3.7), \( \| v \|_{L^p (B)^n} = 0 \), and \( v \) is a linear function, so \( v = 0 \) on \( \Omega \), which contradicts (3.6), and the result follows in the case \( \omega = 1 \). The general case can be treated by the same means defining the appropriate weighted versions \( \mathcal{RM}_\omega (\Omega)^n = \{ v \in W^{1,p}_\omega (\Omega)^n : \ \varepsilon (v) = 0 \} \), and \( \hat{W}^{1,p}_\omega (\Omega)^n = \{ \overline{v} \in W^{1,p}_\omega (\Omega)^n : \ \int_B \overline{v} \omega = 0, \ \int_{\Omega} \frac{\partial \overline{v}}{\partial x_j} \omega = 0 \}. \)
4. Korn and Poincaré inequalities for chains of quasirectangles

The job performed for chains of rectangles can be easily generalized to chains of more general sets, all we have to do is to write appropriate hypotheses.

Definition 4.1
Let $\mathcal{V} = \{\Omega_i\}$ be a (finite or countable) collection of disjoint open sets. Assume that there exists a chain of rectangles $\mathcal{C} = \{R_i\}$ (in the sense of Definition 3.1) with $R_i \subset \Omega_i \subset CR_i$ for a fixed constant $C$. Finally, assume that there exist fixed constants $C_k$, $C_p$ such that $C_k \leq C_k \leq C_k$ and $C_p \leq C_p$ being the constants for the Korn's inequality and Poincaré inequality, respectively, for $\Omega_i$. Then, $\mathcal{V} = \{\Omega_i\}$ is called a $C$-chain of quasirectangles associated to the chain of rectangles $\mathcal{C}$. The constant $C$ is taken as $C = \max\{C_k, C_p\}$. Each $\Omega_i$ is called a $C$-quasirectangle associated to $R_i$.

In order to minimize the notation, we use quasirectangle (resp. chain of quasirectangles) instead of $C$-quasirectangle (resp. $C$-chain of quasirectangles).

Definition 4.2
If each $R_i$ in Definition 4.1 is a cube $Q_i$ (instead of a general rectangle), then $\Omega_i$ is called a quasicube associated to the cube $Q_i$, and $C_k \leq C_k \leq C_k$.

Definition 4.3
Let $\mathcal{V}$ be a chain of quasirectangles associated to the chain of rectangles $\mathcal{C}$, and let $C_i$ be a collection of intermediate rectangles associated to $\mathcal{C}$. We say that $\Omega = \bigcup(C_i \cup \mathcal{V})$ is a $\mathcal{V}$-linked domain.

Remark 4.1
Recalling Remark 3.2, the results that we state for $\mathcal{V}$-linked domains hold for every $\Omega$ such that $\bigcup(C_i \cup \mathcal{V}) \subset \Omega$ and $\Omega \equiv \bigcup \mathcal{V}$.

Definition 4.4
Let $\mathcal{V} = \{\Omega_i\}$ be a chain of quasirectangles, and let $\Omega$ be a domain such that $\Omega \equiv \bigcup \mathcal{V}$. We say that $\omega$ is an admissible weight in $\Omega$ if for any $x \in \Omega$, \[ \omega(x) \leq \omega, \quad \forall \Omega_i \]
being $\omega_{\Omega_i}$, appropriate constants.

Remark 4.2
From Definitions 4.1, 4.3, and 4.4, one readily finds that any proof given in previous section for $C$-linked domains can be carried out for $\mathcal{V}$-linked domains. For this reason, Theorems 4.3–4.6 are stated later without further analysis.

Theorem 4.3 (Second case of Korn's inequality for chains of quasirectangles)
Let $\mathcal{V} = \{\Omega_i\}$ be a chain of rectangles, and let $C_k$, be the constants for the second case of Korn's inequality on $\Omega_i$. Then for any $\mathcal{V}$-linked domain $\Omega$, and any $u \in W^{1,p}(\Omega)$ such that $f_{\Omega} \frac{Du - Du}{2} = 0$, we have
\[ \|Du\|_{L^p(\Omega)^{n \times n}} \leq C(1 + A) \|\varepsilon(u)\|_{L^p(\Omega)^{n \times n}}, \]
where $A$ is defined in (2.2) with $f_i = |\Omega_i|$ and the weight $\sigma$ is constant on each $\Omega_i$ being $\sigma|_{\Omega_i} = C_k$.

Theorem 4.4 (Second case of Korn's inequality for chains of quasirectangles: weighted version)
Let $\mathcal{V} = \{\Omega_i\}$ be a chain of quasirectangles and $\Omega$ a $\mathcal{V}$-linked domain. Assume that for any $k$, $\omega(R_{k+1}) \leq \alpha \omega(R_k)$ with $0 < \alpha < 1$. Let $u \in W^{1,p}_0(\Omega)$, with $\omega$ an admissible weight (see (4.1)), be such that $f_{\Omega} \frac{Du - Du}{2} = 0$. Then,
\[ \|Du\|_{L^p(\Omega)^{n \times n}} \leq C\|\varepsilon(u)\|_{L^p(\Omega)^{n \times n}}, \]
where $\sigma|_{\Omega_i}$ can be taken as $\sigma|_{R_i} = \left( \frac{\omega_i}{\omega_i} \right)^p$.

Theorem 4.5 (Poincaré inequality for chains of rectangles)
Let $1 < p < \infty$ and $\mathcal{V} = \{\Omega_i\}$ be a chain of quasirectangles and $\Omega$ a $\mathcal{V}$-linked domain. Let $u \in W^{1,p}_0(\Omega)$, with $\omega$ an admissible weight (see (4.1)), be such that $u \in W^{1,p}_0(\Omega)$, and $f_{\Omega} \omega = 0$. Assume that for any $k$, $\omega(R_{k+1}) \leq \alpha \omega(R_k)$ with $0 < \alpha < 1$, then we have
\[ \|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p_0(\Omega)^{n \times n}}, \]
where the weight $\sigma$ is constant on each $\Omega_i$ and can be taken as $\sigma|_{R_i} = L^p_{L^p}$.

Theorem 4.6 (General case of Korn's inequality for chains of quasirectangles)
Let $\mathcal{V} = \{\Omega_i\}$ be a chain of rectangles, and let $\Omega$ be a $\mathcal{V}$-linked domain. Consider a weight $\omega$ such that (4.1) holds, and assume that $\omega(R_{k+1}) \leq \alpha \omega(R_k)$ with $0 < \alpha < 1$ and that $L^p_{L^p} < C$, for any $i$. If $B$ is a ball such that $B \subset \Omega$, and $B$ meets only a finite number of quasirectangles $\Omega_i$, then for any $u \in W^{1,p}_0(\Omega)$, we have
\[ \|Du\|_{L^p_0(\Omega)^{n \times n}} \leq C \left\{ \|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_0(\Omega)^{n \times n}} \right\}, \]
and respectively. We show how to handle (4.3) because the other one follows similarly. We use Theorem 4.3 applied to chain solvability of the divergence equation \[11\] and the validity of the (improved) Poincaré inequality \[23\].

\[\text{Figure 2. Quasirectangles.}\]

where the weight \(\sigma\) is constant on each element of \(C\), and can be taken as \(\sigma|_{R_i} = \left(\frac{\omega(R_i)}{\ell_i}\right)^p\).

**Remark 4.7**

All these results can be proved exactly like the ones for chains of rectangles, except for a subtle detail: we impose the decreasing measure condition (3.4) on the rectangles \(R_i\) and not on the subdomains \(\Omega_i\), as it would be natural. This is possible because of the relationship between the measures of \(\Omega_i\) and \(R_i\). Indeed, because \(\omega\) is admissible and \(|R_i| \leq |\Omega_i| \leq C|R_i|\), we have

\[\omega(\Omega_i) \leq C\omega|\Omega_i| \leq C\omega|R_i| \leq C\omega(R_i),\]

and

\[\omega(R_i) \leq C\omega|R_i| \leq C\omega|\Omega_i| \leq C\omega(\Omega_i).\]

And consequently,

\[A_o = \sup_{k > 0} \left(\frac{1}{k} \sum_{j=0}^{\infty} \omega(\Omega_j)\right)^\frac{1}{2} \leq C \sup_{k > 0} \left(\frac{1}{k} \sum_{j=0}^{\infty} \omega(R_j)\right)^\frac{1}{2} \leq \left(\frac{1}{k} \sum_{j=0}^{\infty} \omega(\Omega_j)\right)^\frac{1}{2} = C.\]

So, if the decreasing property (3.4) is imposed on the rectangles \(R_i\), we have that \(A_o\) is finite.

The reader may wonder how a quasirectangle could be. An easy corollary of Theorem 4.3, useful in the next section, shows that some quasirectangles can be obtained from finite union of quasicubes.

**Corollary 4.8**

Let \(W = \{\Omega_i\}_{1 \leq i \leq N}\) be a finite chain of quasicubes associated to a finite chain of cubes \(C = \{Q_i\}_{1 \leq i \leq N}\) with their centers placed along a straight line parallel to an axis. Assume that for \(1 \leq i \leq N, \ell(Q_i) = \ell\). Then, any \(W\)-linked domain \(\Omega\) is a quasirectangle associated to \(R\), being \(R\) the minimal rectangle containing the chain \(C\).

**Proof**

It is enough to show that Korn and Poincaré constants \(C_k\) and \(C_P\) can be bounded as

\[C_k \leq \frac{L_m(R)}{L_m(R)}\]

and

\[C_P \leq CL_m(R)\]

respectively. We show how to handle (4.3) because the other one follows similarly. We use Theorem 4.3 applied to chain \(W\). Because \(|\Omega_i| \sim |Q_i| = \ell^n\), we get from (2.2) and Remark 2.3 that \(A \leq CN\). On the other hand, \(N = \frac{\omega(\Omega)}{\omega(R)}\) and then, (4.3) follows straightforwardly, because by definition of quasirectangle \(C_k(\Omega_i) \leq C\frac{\omega(\Omega)}{\omega(R)} = C\).

Observe that this corollary only provides quasirectangles with interior rectangles having \(n-1\) equal short edges and a long one. We limit our approach to that kind of quasirectangles in the context of external cusps treated later.

Taking into account that we work with quasirectangles that are made of quasicubes, we need now to provide examples of general \(C\)-quasicubes.

A non-trivial quasicube is given, for instance, by any bounded star-shaped domain with respect to a maximal ball \(B\) contained in a cube \(Q\). For such a domain \(D\), calling \(C_B\) a constant for which \(Q \subset D \subset C_B Q\), it is known that second case of Korn and Poincaré inequalities hold. For star-shaped domains, some information about the constants \(C_k\) and \(C_P\) for Korn’s and Poincaré inequality, respectively, is available in the literature ([12, 13, 22]). However, all we need, in order to build quasirectangles, is to find a chain of quasicubes \(W = \{Q_i\}\) for which \(C_k \leq C\) and \(C_P \leq C\ell(Q)\). This behavior of the constants is guaranteed in general for John domains, thanks to the solvability of the divergence equation [11] and the validity of the (improved) Poincaré inequality [23].
In Figure 2(A), we show a John domain that is a quasicube obtained by adding iteratively properly scaled cubes to a central fixed cube. In Figure 2(B), a quasirectangle is given by collecting identical quasicubes like the one in Figure 2(A). Finally, Figure 2(C) shows a quasirectangle formed by a finite union of quasicubes that are not identical but have similar aspect ratio.

Example 4.1 (Locally John quasirectangle)
Let $C = \{Q_i\}, i = 1, \ldots, N$ a chain of cubes with $\ell(Q_i) = \ell$ and with centers $c_{Q_i}$ placed along a straight line. Let $U = \{\Omega_i\}, i = 1, \ldots, N$ a set of disjoint John domains with parameters $\alpha, \beta$ and with centers in $c_{Q_i}$ such that $Q_i \subset \Omega_i \subset C, Q_i$. Then, any $U$-linked domain $\Omega$ is a quasirectangle associated to $R$, being $R$ the minimal rectangle containing $U \cup C$. This sort of quasirectangle is called a locally John quasirectangle.

5. Application to external cusps

External cusps are easily described using a ‘profile’ function $\phi$, as presented in (1.10). In particular, if $\psi(x_n) \sim x_n^\alpha$, the weighted version of Korn’s inequality given in Theorem A was presented in [15, Theorem 6.2]. A more general class of external cusps can be defined as follows:

Definition 5.1
Let $\sigma \subset \mathbb{R}^{n-1}$ be a Lipschitz domain, and assume that $0 \in \sigma$. We say that $\Omega$ is a sectionally Lipschitz external cusp if

$$\Omega \cap U = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x' \in \psi(x_n) \sigma\}$$

(5.1)

for some neighborhood of the origin $U$, and with $\psi$ a non-decreasing $C^1$ function, such that $\psi$ in non-decreasing, $\psi(0) = 0$ and $\psi(t)/t \to 0 (t \to 0^+)$. 

This definition is introduced in [24], in the context of extension of functions in Sobolev spaces. In fact, in [24], the authors drop the requirement $0 \in \sigma$ that here is considered for the sake of simplicity. In Figure 3, we show two external cusps satisfying Maz’ya and Poborchi’s definition. In Figure 3(A), $\sigma$ is an ellipse containing the origin, whereas in Figure 3(B), $\sigma$ is an ellipse that does not contain the origin.

In this work, we consider external cusps of a very general kind given by linking appropriate chains of quasirectangles.

Definition 5.2 (Generalized external cusp)
Let $\mathcal{V} = \{\Omega_i\}$ be a chain of quasirectangles with an associated chain of rectangles $\mathcal{C} = \{R_i\}$. Let us assume that

- each $R_i$ is such that $\ell(R_i) = (\ell_1, \ell_2, \cdots, \ell_n)$, with $\ell_i \leq L_i$,
- the rectangles $R_i$ are placed one above the other, along the $x_n$ axis in such a way that $\overline{R_{i+1}} \cap \overline{R_i} = F^u_{R_{i+1}}$ (the upper face of $R_{i+1}$),
- $z_i \to 0$, where $z_i$ is the $x_n$ coordinate of the points in the floor of $R_i$,
- $|R_{i+1}| \leq a|R_i|$ for some $a < 1$, and
- there exists a nondecreasing $C^1$ function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\psi'$ is nondecreasing, $\psi(0) = \psi'(0) = 0$ and $\psi(z_i) = \ell_i$.

Then, any $\mathcal{V}$-linked domain $\Omega$ and, more generally, any domain $\Omega$ satisfying the requirements of Remark 4.1, is called a generalized external cusp.

It is clear that a generalized external cusp $\Omega$ agrees with previous notions of external cusps. However, $\psi$ does not give the precise profile of $\Omega$ but only a qualitative description of its narrowing toward the origin. On the other hand, the theory presented in the previous section applies straightforwardly for generalized external cusps.

Figure 3. Maz’ya and Poborchi’s cusps.
In Figure 4, we show examples of general external cusps. Figure 4(A) is just a chain of rectangles satisfying $L_{i+1} \sim \frac{1}{\sqrt{z_i^2}}$ and $\ell_i = z_i^2$. Figure 4(B) shows an external cusp with locally smooth boundary away from the origin. The interior chain of rectangles is like the one in Figure 4(A), but leant. On the other hand, Figure 4(C) is a perturbation of Figure 4(B), formed by a chain of locally John quasirectangles. Finally, observe that a domain satisfying (5.1), but taking $\omega$ a John domain with respect to the center of a cube included in $\Omega$, is a general external cusp based on John quasirectangles. We provide a proof of this for a particular case later. In Figure 4(D), we present an example of this situation, taking $\psi(t) = t^2$ and $\sigma$ an inner cusp like $\Omega_{\infty}$.

The unweighted results of the previous section can be immediately applied to general external cusps obtaining Poincaré and Korn's inequalities for them. Moreover, if we take a weight $\omega$ which is a nondecreasing function of $x_n$ (or $|x|$), we have

$$\omega(R_{i+1}) \leq \max_{|x_n|} \varphi \left\{ |R_{i+1}| \leq \alpha \min_{|x_n|} \varphi \right\} |R_i| \leq \alpha \omega(R_i),$$

and the decreasing property (3.4) is fulfilled. In this way, we can consider some particularly interesting weights. For example, being $\varphi'$ non-decreasing, we can take weights of the form $\omega(x) = (\varphi')^{\beta}$ with $\beta \geq 0$. On the other hand, we can also take weights of the form $\omega(x) = x_n^{\beta}$, being $\beta \geq 0$. In this way, we obtain the following theorem where we denote $L(a)$ and $\ell(a)$ the lengths of the edges $L(R)$ and $\ell(R)$, being $R$ the rectangle at height $a$.

**Theorem 5.1**

Let $\Omega$ be a generalized external cusp and $\sigma(x) = \left( \frac{\ell(|x|)}{L(|x|)} \right)^{-p}$. Then, the inequality

$$\omega(R_{i+1}) \leq \max_{|x_n|} \varphi \left\{ |R_{i+1}| \leq \alpha \min_{|x_n|} \varphi \right\} |R_i| \leq \alpha \omega(R_i),$$

and the decreasing property (3.4) is fulfilled. In this way, we can consider some particularly interesting weights. For example, being $\varphi'$ non-decreasing, we can take weights of the form $\omega(x) = (\varphi')^{\beta}$ with $\beta \geq 0$. On the other hand, we can also take weights of the form $\omega(x) = x_n^{\beta}$, being $\beta \geq 0$. In this way, we obtain the following theorem where we denote $L(a)$ and $\ell(a)$ the lengths of the edges $L(R)$ and $\ell(R)$, being $R$ the rectangle at height $a$.

**Theorem 5.1**

Let $\Omega$ be a generalized external cusp and $\sigma(x) = \left( \frac{\ell(|x|)}{L(|x|)} \right)^{-p}$. Then, the inequality
\[ \|Du\|_{L^p(\Omega)}^{\infty} \leq C \left\{ \|u\|_{L^p(\Omega)} + \|\varepsilon(u)\|_{L^p(\Omega)}^{\infty} \right\} \]

holds for weights of the form

(a) \[ \omega(x) = \gamma x_\alpha^\beta, \quad \beta \geq 0 \]

(b) \[ \omega(x) = (\psi')^\beta, \quad \beta \geq 0 \]

It is important to observe that, if \( \varphi \) is such that \( \varphi(z_{i-1}) - \varphi(z_i) \sim \varphi(z) \), then

\[ \frac{\ell_i}{L_i} = \frac{\varphi(z_i)}{z_{i-1} - z_i} \sim \frac{\varphi(z_{i-1}) - \varphi(z_i)}{z_{i-1} - z_i} \sim \varphi'(z). \]

Hence, \( \sigma \sim (\psi')^{-\beta} \) and item (b) in Theorem 5.1 is a generalization of Theorem 1. In fact, as in Theorem 1, the weight on the left-hand side is \( (\psi')^\beta \), whereas the one on the right-hand side is \( (\psi')^{(\beta - 1)} \). Here, \( \psi \) is not forced to be a power function, and it does not depict the precise profile of \( \Omega \) but only provides a qualitative description of its cuspidal behavior, allowing the boundary of \( \Omega \) to be very general considering that it might be based on locally John quasirectangles. It is also noteworthy that the critical case of Theorem 1 is reached.

On the other hand, let us consider a profile cusp satisfying (5.1), but taking \( \sigma \) a John domain (an example can be seen in Figure 4(D)). Moreover, let us suppose \( \varphi(z) = z^\gamma \) for some \( \gamma > 1 \). We show how the rectangles can be chosen in order to prove that such a cusp is a generalized external cusp based on locally John quasirectangles.

Let us take

\[ z_i = \frac{1}{2^i}. \]

The rectangle \( R_i \) is placed at height \( z_i \), and the length of its edges is

\[ \ell_i = \varphi(z_i) = \frac{1}{2^\gamma}, \quad \text{and} \quad L_i = z_{i-1} - z_i = \frac{1}{2^i}. \]

Let us consider a weight of the form

\[ \omega(x) = \left( \frac{\ell_i}{L_i} \right)^\beta = \frac{1}{2^{(\gamma - 1)\beta}} \quad \forall x \in R_i. \]

Then,

\[ \omega(R_{i+1}) = \left( \frac{\ell_{i+1}}{L_{i+1}} \right)^\beta |R_{i+1}| = \frac{1}{2^{(\gamma - 1)\beta}} \frac{1}{2^{(\gamma - 1)\beta}} \frac{1}{2^{(\gamma - 1)\beta}} = \frac{1}{2^{(\gamma - 1)\beta + \gamma(n-1)+1}} \omega(R_i). \]

Hence, the decreasing property (3.4) is satisfied when

\[ \frac{1}{2^{(\gamma - 1)\beta + \gamma(n-1)+1}} < 1, \]

or in other words,

\[ (\gamma - 1)\beta + \gamma(n-1)+1 > 0, \]

which leads us to

\[ \beta > \frac{1 + \gamma(n-1)}{(\gamma - 1)\beta}. \]

Notice that in this case, \( \sigma \sim (\psi')^\beta \), and then, we can express the weight in terms of \( \psi' \).

On the other hand, observe that each \( R_i \) is a locally John quasirectangle. Indeed, each \( R_i \) can be thought as an \( \mathcal{W}_1 \)-linked domain, being \( \mathcal{W}_1 = \{ \Omega_j \}_{1 \leq i \leq N} \). Here, \( N \) is the integer part of \( \frac{\varphi(z)}{\ell} \), and each \( \Omega_j \) has height almost equal to \( \ell_i \). It can be easily seen that any \( \Omega_j \) is a quasicube given by a John domain. Indeed, first show that a cylindrical set \( J \) of the form \( \sigma \times (a,b) \subset \mathbb{R}^n \), with \( b - a = \text{diam}(\Omega) \), is a John domain with constants given by those of \( \sigma \). Then, conclude by observing that \( \Omega_j \) almost agrees with a dilatation of \( J \) (actually it is possible to construct a bi-Lipschitz mapping \( F \) such that \( F(\Omega_j) = \ell_i J \), see [20]).
Therefore, we can now state the following:

**Theorem 5.2**
Let $\Omega$ be an external cusp satisfying (5.1), but taking $\sigma \subset \mathbb{R}^{n-1}$ a John domain, and $\psi(x) = z^\gamma$, with $\gamma > 1$. Then,

$$\|Du\|_{L^p(\Omega)} \leq C \left( \|u\|_{L^p(\Omega)} + \|\varepsilon(u)\|_{L^p(\Omega)} \right),$$

with

$$\sigma(x) = (\psi'(x))^{-p} \quad \text{and} \quad \omega(x) = (\psi'(x))^{p\beta},$$

being $\beta > -\frac{1+\gamma(n-1)}{(\gamma-1)p}$.

This result is also a generalization of Theorem 1. It imposes more restrictions than Theorem 5.1 on the boundary of $\Omega$, but it admits a negative range for the exponent $\beta$. On the other hand, the critical case $\beta_1 = \beta_2 + 1$ in Theorem 1 is once again reached. It is important to notice that the counterexamples proposed in [14] for proving Theorem 1 are given in terms of functions that depend only on the last coordinate and on the profile function $\psi$. Consequently, they are independent of the boundary of the cusp and can be easily adapted for general external cusps based on locally John quasirectangles.

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