CHARACTERS OF THE GROUP $\text{EL}_d(R)$ FOR A COMMUTATIVE NOETHERIAN RING $R$

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Abstract. Let $R$ be a commutative Noetherian ring with unit. We classify the characters of the group $\text{EL}_d(R)$ provided that $d$ is greater than the stable range of the ring $R$. It follows that every character of $\text{EL}_d(R)$ is induced from a finite dimensional representation. Towards our main result we classify $\text{EL}_d(R)$-invariant probability measures on the Pontryagin dual group of $R^d$.

1. Introduction

Let $R$ be a commutative Noetherian ring with unit. Let $\text{EL}_d(R)$ for some $d \in \mathbb{N}$ be the subgroup of the special linear group $\text{SL}_d(R)$ generated by elementary matrices. A trace on the group $\text{EL}_d(R)$ is a positive definite conjugation invariant function $\varphi: \text{EL}_d(R) \to \mathbb{C}$ with $\varphi(e) = 1$. A character of $\text{EL}_d(R)$ is a trace $\varphi$ that cannot be written as a non-trivial convex combination. Our main goal is to understand the set $\text{Ch}(\text{EL}_d(R))$ of characters of the group $\text{EL}_d(R)$.

Normal subgroups. For each normal subgroup $N \trianglelefteq \text{EL}_d(R)$ the characteristic function $1_N$ is a trace on the group $\text{EL}_d(R)$. The function $1_N$ is a character if and only if every non-trivial conjugacy class of the quotient group $\text{EL}_d(R)/N$ is infinite. So our goal depends on understanding the normal subgroup structure of the group $\text{EL}_d(R)$.

Assume that $d \geq 3$. Normal subgroups of the group $\text{EL}_d(R)$ are controlled by ideals in the ring $R$. Let us give a few examples. Consider an ideal $I \trianglelefteq R$. The congruence subgroup $C_d(I)$ is the kernel of the reduction map modulo the ideal $I$

$$\rho_I: \text{EL}_d(R) \to \text{SL}_d(R/I).$$

A closely related normal subgroup $\bar{C}_d(I) \trianglelefteq \text{EL}_d(R)$ is the preimage of the center $Z(\text{SL}_d(R/I))$ via the reduction map $\rho_I$. Another normal subgroup $\text{EL}_d(I) \trianglelefteq \text{EL}_d(R)$ associated to the ideal $I$ is the normal closure of the elementary matrices belonging to $C_d(I)$. By definition $\text{EL}_d(I) \leq C_d(I) \leq \bar{C}_d(I)$.

While these examples are not exhaustive, it is known that

$$\bar{C}_d(I)/\text{EL}_d(I) = Z(\text{EL}_d(R)/\text{EL}_d(I)).$$

In particular every intermediate subgroup $\text{EL}_d(I) \leq H \leq \bar{C}_d(I)$ is normal in $\text{EL}_d(R)$. The converse is also true, namely every normal subgroup $N \trianglelefteq \text{EL}_d(R)$ admits a uniquely determined ideal $J \trianglelefteq R$ such that $\text{EL}_d(J) \leq N \leq \bar{C}_d(J)$.

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1Equivalently $R$ is any quotient of the commutative ring $\mathbb{Z}[x_1, \ldots, x_k]$ for some $k \in \mathbb{N}$. 
The characters of the group $EL_d (R)$ are controlled by ideals in the ring $R$. To state our classification result we introduce the notion of depth ideals.

An ideal $I$ in the ring $R$ is a depth ideal if $|J/I| = \infty$ for every ideal $I \subset J \subset R$. Every ideal $J \subset R$ admits a uniquely determined depth ideal $J^* \subset R$ with $J \subset J^*$ and $|J^*/J| < \infty$. The ring $R$ is called just infinite if $(0)^* = (0)$ and $I^* = R$ for every non-zero ideal $I \subset R$. For example every Dedekind domain is just infinite. On the other hand the ring $\mathbb{Z} [x_1, \ldots, x_k]$ is not just infinite for any $k \in \mathbb{N}$.

**Main results.** Let $\varphi \in Ch (EL_d (R))$ be a character. The kernel of $\varphi$ is

$$\ker \varphi = \{ g \in EL_d (R) : \varphi (g) = 1 \}.$$ 

This is a normal subgroup of $EL_d (R)$ and therefore there is a uniquely determined kernel ideal $K_\varphi < R$ satisfying $EL_d (K_\varphi) \leq \ker \varphi \leq \widetilde{C}_d (K_\varphi)$. The level ideal associated to the character $\varphi$ is the uniquely determined depth ideal $I_\varphi = K_\varphi^*$.

**Theorem 1.1.** Assume that $d > \max \{ sr (R), 2 \}$. Let $\varphi \in EL_d (R)$ be a character with level ideal $I_\varphi$ and kernel ideal $K_\varphi$. Then

- the character $\varphi$ is induced from the normal subgroup $\widetilde{C}_d (I_\varphi)$,
- the subquotient $A_d (I_\varphi, K_\varphi) = \widetilde{C}_d (I_\varphi) / EL_d (K_\varphi)$ is virtually abelian and
- there is a finite essential $EL_d (R)$-orbit $O_\varphi$ in $Ch (A_d (I_\varphi, K_\varphi))$ such that the restriction of the character $\varphi$ to the subgroup $\widetilde{C}_d (I_\varphi)$ is given by

$$\varphi = \frac{1}{|O_\varphi|} \sum_{\psi \in O_\varphi} \psi.$$ 

The notation $sr (R)$ stands for the stable range of the ring $R$. For example $sr (R) \leq 2$ if $R$ is a Dedekind domain and $sr (\mathbb{Z} [x_1, \ldots, x_k]) \leq k + 2$.

It is possible to go in the other direction as well.

**Theorem 1.2.** Assume that $d > \max \{ sr (R), 2 \}$. Let $I, K < R$ be a pair of ideals with $K^* = I$. Then

- the subquotient $A_d (I, K) = \widetilde{C}_d (I) / EL_d (K)$ is virtually abelian, and
- for every finite essential $EL_d (R)$-orbit $O \subset Ch (A_d (I, K))$ there is a uniquely determined character $\varphi \in Ch (EL_d (R))$ with $I_\varphi = I, K_\varphi = K$ and $O_\varphi = O$.

Theorems 1.1 and 1.2 establish a bijective correspondence between $Ch (EL_d (R))$ and the set of all triplets $(I, K, O)$ as above. This amounts to a classification of the characters of $EL_d (R)$ provided that $d > sr (R)$. Arguably the most important parameter of a given character $\varphi$ is its level ideal $I_\varphi < R$.

In fact, the subquotients $A_d (I_\varphi, K_\varphi)$ are virtually central in the quotient group $EL_d (R) / EL_d (K_\varphi)$, i.e. they admit a finite index central subgroup. This property is stronger than being virtually abelian per se, allowing us to deduce the following.

\[\text{A trace on the group } G \text{ is induced from the subgroup } H \text{ if } \varphi (g) = 0 \text{ for all elements } g \in G \setminus H.\]

\[\text{A group is called virtually abelian if it admits a finite index abelian subgroup. A discrete group is virtually abelian if and only if it is type I.}\]

\[\text{The notion of essential orbits is defined in } \cite{10}. \text{ This is a technical non-degeneracy assumption needed to ensure our parametrization is bijective.}\]

\[\text{See } \cite{3} \text{ below for the definition and a discussion of stable range as well as for more information on the normal subgroup structure.}\]
Corollary 1.3. Assume that $d > \max\{sr(R), 2\}$. Let $\varphi \in \text{Ch} (\text{EL}_d (R))$ be a character. Denote $N = C_d (I_\varphi) \triangleleft \text{EL}_d (R)$. Then there is a unitary representation $\pi$ of the group $N$ on a finite dimensional Hilbert space $\mathcal{H}$ such that

$$\varphi (g) = \begin{cases} \frac{\text{tr} (g)}{\dim \mathcal{H}} & g \in N, \\
0 & g \notin N.\end{cases}$$

In other words, assuming $d > sr (R)$ every character of the group $\text{EL}_d (R)$ is induced from a finite dimensional representation of a particular normal subgroup.

The above results generalize and were inspired by Bekka’s paper [Bek07], see the “Scientific Acknowledgements” paragraph below.

Examples. Let us illustrate our Theorems 1.1 and 1.2 with a few examples.

1. If $R$ is a finite field or a product of finite fields then $I^* = R$ for every ideal $I \triangleleft R$. In particular $I_\varphi = R$ and the group $A_d (I_\varphi, K_\varphi)$ is a quotient of the finite group $\text{SL}_d (R)$ for all characters $\varphi \in \text{Ch} (\text{SL}_d (R))$. Our main results are trivially true.

2. Assume that $d > sr (R)$. It follows that $\text{EL}_d (R)$ is a normal subgroup of $\text{SL}_d (R)$. The quotient $\text{SL}_d (R)/\text{EL}_d (R)$ is isomorphic to the $K$-theoretic abelian group $\text{SK}_1 (R)$. Whenever $\text{SK}_1 (R)$ is trivial our results apply equally well to $\text{SL}_d (R)$. The group $\text{SK}_1 (R)$ is known to be trivial if $R$ is a field, a ring of integers $\mathcal{O}_k$ of an algebraic field [BMS67] or $\mathbb{Z} [x_1, \ldots, x_k]$ for some $k \in \mathbb{N} \cup \{0\}$ [BHS68].

3. Let $R$ be the ring $\mathbb{Z}$. The characters of the group $\text{SL}_d (\mathbb{Z})$ for $d > 2$ were classified in [Bek07]. Bekka proved that every character $\varphi \in \text{Ch} (\text{SL}_d (\mathbb{Z}))$ factors through a finite dimensional representation of $\text{SL}_d (\mathbb{Z})$ or is induced from $Z (\text{SL}_d (\mathbb{Z}))$. In our terminology, Bekka’s first and second cases correspond to characters with level ideal $\mathbb{Z}$ and $(0)$ respectively.

4. The same conclusion as in Example 3 applies more generally if $R$ is the ring of integers $\mathcal{O}_k$ in the algebraic number field $k$ or the localization of $\mathcal{O}_k$ at a finite set of primes. This is a Dedekind domain with $sr (R) = 2$. Therefore our work gives a new proof of special cases of [Pet14] and [BH19].

5. The previous Examples 3 and 4 are generalized by the following special case of our main results.

Corollary 1.4. If $R$ is just infinite and $d > sr (R)$ then every character $\varphi \in \text{Ch} (\text{EL}_d (R))$ factors through a finite dimensional representation of $\text{EL}_d (R)$ or is induced from $Z (\text{EL}_d (R))$.

Consider the polynomial ring $\mathbb{F}_p [x]$ for some prime $p \in \mathbb{N}$. It is easy to see that $\mathbb{F}_p [x]$ is just infinite using the fact it is a principal ideal domain. Moreover $sr (\mathbb{F}_p [x]) = 2$ and $\text{SK}_1 (\mathbb{F}_p [x])$ is trivial. Therefore Corollary 1.4 applies to the groups $\text{SL}_d (\mathbb{F}_p [x])$ for all primes $p$ and all $d \geq 3$.

6. The group $\text{SL}_d (\mathbb{Z} [x_1, \ldots, x_k])$ is a called universal lattice. This terminology was introduced by Shalom [Sha99]. The following is a special case of our Corollary 1.3.

Corollary 1.5. If $d > k + 2$ then every character of the universal lattice $\text{SL}_d (\mathbb{Z} [x_1, \ldots, x_k])$ is induced from a finite dimensional representation.
(7) Let $f : \mathcal{R} \to \mathcal{S}$ be a ring epimorphism with $\ker f = \mathcal{I} \triangleleft \mathcal{R}$. Let $\mathcal{O}_\mathcal{S}$ be the group of units of the ring $\mathcal{S}$. The central subquotient $\mathcal{C}_d(\mathcal{I})/\mathcal{C}_d(\mathcal{I})$ is isomorphic to the subgroup $\mathcal{O}_\mathcal{S}^{[d]} = \{ u \in \mathcal{O}_\mathcal{S} : u^d = 1_{\mathcal{S}} \}$. Therefore any multiplicative character of the abelian group $\mathcal{O}_\mathcal{S}^{[d]}$ can be induced via this isomorphism to a character of the group $\text{EL}_d(\mathcal{R})$. We remark that the group $\mathcal{O}_\mathcal{S}^{[d]}$ may in general be infinitely generated.

(8) Let $\mathcal{I} \triangleleft \mathcal{R}$ be an ideal. Assuming that $d > \text{sr} (\mathcal{R})$ the central subquotient $\mathcal{C}_d(\mathcal{I})/\mathcal{C}_d(\mathcal{I})$ is isomorphic to a subgroup of the abelian relative $K$-theoretic group $\text{SK}_1(\mathcal{R} ; \mathcal{I})$. Similarly to Example (7) any multiplicative character of this abelian group can be induced to a character of $\text{EL}_d(\mathcal{R})$.

(9) Let $\mathcal{I} \triangleleft \mathcal{R}$ be a depth ideal and $\mathcal{K} \triangleleft \mathcal{R}$ be an ideal with $\mathcal{K}^* = \mathcal{I}$. The subquotient $\mathcal{C}_d(\mathcal{I})/\mathcal{C}_d(\mathcal{K})$ is a finite group. The group $\text{EL}_d(\mathcal{R})$ is acting on the set of irreducible representations of this finite group. Any finite $\text{EL}_d(\mathcal{R})$-orbit for this action gives rise to a character of the group $\text{EL}_d(\mathcal{R})$.

(10) In the most general case a character $\varphi \in \text{Ch} (\text{EL}_d(\mathcal{R}))$ is a "mixture" of the three constructions presented in Examples (7), (8) and (9).

Remark 1.6. A general Noetherian ring $\mathcal{R}$ may admit infinitely many depth ideals even up to an automorphism. For example the principal ideal $(x^n) \triangleleft \mathbb{Z}[x]$ is a depth ideal for all $n \in \mathbb{N}$.

Measure classification. Consider the "vertical" abelian subgroup $V_1(\mathcal{R})$ of the group $\text{EL}_d(\mathcal{R})$ generated by the elementary subgroups $E_{2,1}(\mathcal{R}), \ldots, E_{d,1}(\mathcal{R})$. The subgroup $V_1(\mathcal{R})$ is isomorphic to the ring $\mathcal{R}^{d-1}$ with its additive structure. The normalizer $N_{\text{EL}_d(\mathcal{R})}(V_1(\mathcal{R}))$ contains a copy of the smaller group $\text{EL}_{d-1}(\mathcal{R})$ acting on the subgroup $V_1(\mathcal{R})$ by conjugation via its natural action on $\mathcal{R}^{d-1}$.

The restriction of any character $\varphi \in \text{Ch} (\text{EL}_d(\mathcal{R}))$ to the abelian subgroup $V_1(\mathcal{R})$ is a positive definite function. This restriction can be expressed as the Fourier transform of a Borel probability measure $\mu_1$ on the Pontryagin dual $\hat{\mathcal{R}}^{d-1}$. Consider the dual action of $\text{EL}_{d-1}(\mathcal{R})$ on the dual group $\hat{\mathcal{R}}^{d-1}$ by group automorphisms. As $\varphi$ is conjugation invariant this action preserves the measure $\mu_1$. Our point of departure towards Theorem 1.7 is a classification of all such $\text{EL}_{d-1}(\mathcal{R})$-invariant probability measures. Such a classification may also be of independent interest.

Theorem 1.7. Assume that $d \geq 2$. Let $\mu$ be an ergodic $\text{EL}_d(\mathcal{R})$-invariant Borel probability measure on $\hat{\mathcal{R}}^d$. Then there exists a uniquely determined depth ideal $\mathcal{I}_\mu \triangleleft \mathcal{R}$ and a finite $\text{EL}_d(\mathcal{R})$-orbit

$$\omega \subset \hat{\mathcal{I}}^d \cong (\hat{\mathcal{R}}/\mathcal{I}_\mu^0)^d$$

such that

$$\mu = \frac{1}{|\omega|} \sum_{x \in \omega} x_* \nu$$

where $\nu$ is the Haar measure on the compact abelian group $(\mathcal{I}_\mu^0)^d$.

Conversely, any such probability measure $\mu_{\mathcal{I}, \omega}$ on $\hat{\mathcal{R}}^d$ determined by a depth ideal $\mathcal{I} \triangleleft \mathcal{R}$ and a finite $\text{EL}_d(\mathcal{R})$-orbit $\omega \subset \hat{\mathcal{I}}^d$ is $\text{EL}_d(\mathcal{R})$-invariant and ergodic.

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6See § and in particular Proposition 5.2 for details.

7To ease our notations we replace $d - 1$ with $d$ in the statement of Theorem 1.7.
The Haar measure \( \nu_K \) of any closed \( \text{EL}_d(\mathbb{R}) \)-invariant subgroup \( K \leq \hat{\mathbb{R}}^d \) is clearly \( \text{EL}_d(\mathbb{R}) \)-invariant. Theorem 1.7 says that, up to finite index, this essentially accounts for all the \( \text{EL}_d(\mathbb{R}) \)-invariant and ergodic probability measures.

In the special case of the ring \( \mathbb{Z} \) of integers, Theorem 1.7 specializes to the well-known classification \[ \text{DK79}, \text{Bur91} \] of ergodic \( \text{SL}_d(\mathbb{Z}) \)-invariant probability measures on the \( d \)-dimensional torus \( \mathbb{R}/\mathbb{Z}^d \). The ring \( \mathbb{Z} \) is just infinite. The depth ideal \( (0) \) corresponds to the Haar measure on the torus and the depth ideal \( \mathbb{Z} \) corresponds to atomic measures supported on rational \( \text{SL}_d(\mathbb{Z}) \)-orbits.

The proof of Theorem 1.7 is inspired by \[ \text{Bur91}, \text{Proposition 9} \] relying on a harmonic analysis principle due to Wiener \[ \text{GM79}, \text{Theorem A.2.2} \]. Our situation is made more complicated by the presence of depth ideals and by the lack of a good description of \( \text{EL}_d(\mathbb{R}) \)-orbits in \( \mathbb{R}^d \).

**Related work.** The study of group characters has a long and fruitful history. Characters of finite groups are intimately related to irreducible representations and were studied extensively by mathematicians such as Frobenius, Schur and others.

Thoma extended the theory of group characters to infinite groups \[ \text{Tho66b} \] and observed the connection to von Neumann algebras and factors. Thoma described the characters of the infinite symmetric group \[ \text{Tho64a} \]. Vershik \[ \text{Ver12} \] revisited the infinite symmetric group and related characters to the study of non-free actions. See e.g. \[ \text{PT16} \] for a more detailed account of the historical development.

Motivated by the Margulis superrigidity \[ \text{Mar77} \] and the Zimmer cocycle superrigidity theorems \[ \text{Zim80} \] Connes conjectured that irreducible lattices in higher rank semisimple Lie groups are *character rigid*. This means that every character of such a lattice \( \Gamma \) factors through a finite dimensional representation of \( \Gamma \) or is centrally induced. This conjecture was outlined by Jones in \[ \text{Jon00} \] as well as e.g. in \[ \text{CP13} \] and \[ \text{Pet14} \].

Taking a first step in the direction of Connes’ question, Bekka showed that the lattices \( \text{SL}_d(\mathbb{Z}) \) are character rigid for all \( d \geq 3 \) \[ \text{Bek07} \]. Peterson and Thom \[ \text{PT10} \] proved that the lattices \( \text{SL}_2(A) \) are character rigid where \( A \) is either an infinite field or a localization of a ring of algebraic integers admitting infinitely many units. Connes’ question was finally settled in the positive by Peterson \[ \text{Pet14} \]. See also the related work of Creutz and Peterson \[ \text{CP13} \] as well as the recent work of Boutonnet and Houdayer \[ \text{BH19} \] on stationary characters.

In as far as classifying characters is a generalization of understanding normal subgroup structure, the above mentioned results as well as the present work follow the line of algebraic investigation of \[ \text{Men65}, \text{Wil72}, \text{Vas81}, \text{BV84} \], as well as that of the Margulis normal subgroup theorem \[ \text{Mar78}, \text{Mar79} \] for lattices and the Stuck–Zimmer theorem \[ \text{SZ94}, \text{BS06} \] for non-free actions.

An important and in a sense the most general class of groups to which our results apply are the so called universal lattices, namely the groups \( \text{SL}_d(\mathbb{Z}[x_1, \ldots, x_k]) \) for some \( d, k \in \mathbb{N} \) (our proof requires \( d > k + 2 \)). Universal lattices are significant in that they surject onto many arithmetic as well as \( S \)-arithmetic lattices. These groups were studied by several authors with relation to representation theory and Kazhdan’s property (T), see e.g. \[ \text{Sha99}, \text{Sha06}, \text{KN06}, \text{She09}, \text{EJZ10} \].

The notion of ”centrally induced” characters was studied in the context of nilpotent groups, see e.g. \[ \text{How77}, \text{CMS84}, \text{Kan06} \] and the references therein.

Other related works dealing with characters of infinite groups in a geometric context include \[ \text{DM13}, \text{DM14}, \text{DG18} \].
**Remark 1.8.** Some authors use the terminology “character” for our “trace”, and “indecomposable character” for our “character”.

**Further questions.** Let us point out a few questions that were left open.

1. We find it intriguing to look for a purely algebraic ring-theoretic characterization of depth ideals (or their classes up to automorphism) in a given commutative Noetherian ring $\mathcal{R}$. For example, this seems like a challenging problem for the ring $\mathbb{Z}[x_1, \ldots, x_k]$.

2. In Bekka’s proof of character rigidity for the group $\text{SL}_d(\mathbb{Z})$ the assumption $d > 2$ is necessary, for the group $\text{SL}_2(\mathbb{Z})$ is virtually free and its character theory is ”wild”. It would be interesting to investigate to what extent our assumption that $d > \text{sr}(\mathcal{R})$ is necessary in the case of the group $\text{EL}_d(\mathcal{R})$.

3. The majority of this work continues to apply with respect to the (perhaps more natural) group $\text{SL}_d(\mathcal{R})$ instead of the group $\text{EL}_d(\mathcal{R})$.

   One key advantage of working with the smaller group $\text{EL}_d(\mathcal{R})$ is that the subquotient $(\overline{\text{SL}}_d(\mathcal{I}) \cap \text{EL}_d(\mathcal{R})) / \text{EL}_d(\mathcal{I})$ is central for an arbitrary ideal $\mathcal{I} \triangleleft \mathcal{R}$ by the theorem of Borevich and Vavilov.

   On the other hand, while the subquotient $\overline{\text{SL}}_d(\mathcal{I}) / \text{EL}_d(\mathcal{I})$ is indeed central in $\text{SL}_d(\mathcal{R}) / \text{EL}_d(\mathcal{I})$ by the relative Whitehead lemma, the larger subquotient $\overline{\text{SL}}_d(\mathcal{I}) / \text{EL}_d(\mathcal{I})$ is to the best of our understanding in general two-step nilpotent.

   This leads to the $K$-theoretic question of studying the two-step nilpotent subquotients $\overline{\text{SL}}_d(\mathcal{I}) / \text{EL}_d(\mathcal{I})$? A definitive answer would allow this work to be extended to deal with the group $\text{SL}_d(\mathcal{R})$.

4. It seems potentially possible but technically challenging to study characters of Chevalley groups over root systems other than $A_n$.

5. The questions studied in this work make sense over the non-commutative free ring $\mathbb{Z} \langle t_1, \ldots, t_k \rangle$ or alternatively over an arbitrary Noetherian non-commutative ring.

**Synopsis.** Depth ideals in Noetherian rings and their basic properties are studied in §2. The whole of §3 is dedicated to the question of classifying $\text{EL}_d(\mathcal{R})$-invariant probability measures on the Pontryagin dual of the additive group of the ring $\mathcal{R}^d$. This includes a careful study of closed $\text{EL}_d(\mathcal{R})$-invariant subgroups and the corresponding Haar measures. It is interesting to note that some ideas that come to the fore in later sections already appear in §3 in rudimentary form.

   The theory of characters is the subject of §4. This section has an expository nature. We cover the GNS construction, the connection to von Neumann algebras and factors, relative and induced characters and Howe’s lemma on the vanishing of characters on the second center. Another important notion discussed in §4 is relative characters of virtually central subgroups.

   The following two §5 and §6 are also for the most part expository and deal with the matrix group $\text{EL}_d(\mathcal{R})$ over a general commutative Noetherian ring and its various subgroups. In §5 we look at normal subgroups, such as the center and congruence subgroups associated to ideals in the ring $\mathcal{R}$. In §6 we consider the “vertical” and “horizontal” abelian subgroups $V_i(\mathcal{R})$ and $H_i(\mathcal{R})$ respectively, and their normalizers $N_i(\mathcal{R})$ in the group $\text{EL}_d(\mathcal{R})$. The assumption that $d > \text{sr}(\mathcal{R})$
plays a role in §6 towards a particular normal form decomposition for conjugates of arbitrary elements, see Proposition 6.8.

The proof of Theorems 1.1 and 1.2 gets going in earnest in §7. This is where we apply our measure classification result Theorem 1.7 to the Fourier transforms of the restriction of a given character \( \varphi \in \text{Ch}(\text{EL}_d(\mathbb{R})) \) to the "vertical" and "horizontal" abelian subgroups. We deduce that all of these probability measures correspond to a uniquely determined depth ideal \( I_\varphi \triangleleft \mathcal{R} \), the so called level ideal associated to the character \( \varphi \).

It is §8 where our proof most diverges from Bekka’s \cite{Bek07} due to complications introduced by the fact that a commutative Noetherian ring will in general admit multiple depth ideals. We consider a uniquely determined kernel ideal \( K_\varphi \triangleleft \mathcal{R} \) associated to the character \( \varphi \in \text{Ch}(\text{EL}_d(\mathbb{R})) \). A very careful analysis in needed to prove that \( K_\varphi = I_\varphi \), or in other words that \( |I_\varphi / K_\varphi| < \infty \).

We show in §9 that any character \( \varphi \in \text{Ch}(\text{EL}_d(\mathbb{R})) \) is induced from the normal subgroup \( \tilde{\text{SL}}_d(I_\varphi) \cap \text{EL}_d(\mathbb{R}) \) where \( I_\varphi \) is the level ideal corresponding to \( \varphi \). In the final §10 we conclude the formal proof of Theorems 1.1 and 1.2 relying on results from the previous sections.

**Scientific acknowledgements.** Our work was tremendously inspired by Bachir Bekka’s paper \cite{Bek07}. We owe Professor Bekka an intellectual debt for his approach to character classification for the group \( \text{SL}_d(\mathbb{Z}) \) with \( d \geq 3 \). This work grew out of an attempt to extend and adapt \cite{Bek07} to general commutative Noetherian rings. We have attempted to explicitly reference \cite{Bek07} when appropriate, however an attentive reader will be able to detect Bekka’s strategy in the scaffolding of Theorems 1.1 and 1.2 and their proofs.

An important source of inspiration towards the classification of \( \text{EL}_d(\mathbb{R}) \)-invariant probability measures on \( \hat{\mathbb{R}}^d \) in Theorem 1.7 was Marc Burger’s work \cite{Bur91}.

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This work is dedicated to our wives Yael and Neta and to our children. We would like to thank them for their support and patience.
2. Depth in Noetherian rings

Let \( \mathcal{R} \) be a Noetherian commutative unital ring. Let \( x_1, \ldots, x_k \in \mathcal{R} \) with \( k \in \mathbb{N} \) be a fixed set of generators for the ring \( \mathcal{R} \).

**Depth ideals.** Let \( \mathcal{I} \triangleleft \mathcal{R} \) be any ideal. The Noetherianity of \( \mathcal{R} \) implies that there exists a unique ideal \( \mathcal{I}^* \triangleleft \mathcal{R} \) maximal with respect to the two conditions \( \mathcal{I} \subset \mathcal{I}^* \) and \( [\mathcal{I}^*/\mathcal{I}] < \infty \). We will say that the ideal \( \mathcal{I}^* \) is the **depth** of the ideal \( \mathcal{I} \) in the ring \( \mathcal{R} \).

A **depth ideal** is any ideal \( \mathcal{J} \triangleleft \mathcal{R} \) satisfying \( \mathcal{J}^* = \mathcal{J} \). Clearly \( \mathcal{J} \) is a depth ideal if and only if the quotient ring \( \mathcal{R}/\mathcal{J} \) admits no non-zero ideals of finite cardinality. Note that the depth of any ideal is a depth ideal, and any depth ideal is equal to its own depth.

It is clear that the ring \( \mathcal{R} \) satisfies \( |\mathcal{R}| < \infty \) if and only if \( \mathcal{R} \) has a prime generator \( x \). The ring \( \mathcal{R} \) is called **just infinite** if \( \mathcal{R} \) is a depth ideal and every non-zero ideal \( \mathcal{I} \subset \mathcal{R} \) has \( \mathcal{I}^* = \mathcal{R} \).

**Example 2.1.** The only depth ideals in the ring \( \mathbb{Z} \) are \( (0) \) and \( \mathbb{Z} \). If \( \mathbb{Z} \) is any non-zero ideal then \( \mathbb{Z}^* = \mathbb{Z} \). The same situation holds true in any Dedekind domain. In particular every Dedekind domain is just infinite.

The following results are intended to provide a workable criterion for depth ideals.

**Lemma 2.2.** Let \( n, m, N, M \in \mathbb{N} \) be such that \( n < m \) and \( N < M \). Let \( x \in \mathcal{R} \) be any element. If \( N \geq n \) and \( (m-n)|(M-N) \) then \( (x^m - x^n)/(x^M - x^N) \).

**Proof.** Assume that \( M - N = l(m-n) \) for some \( l \in \mathbb{N} \). The fact that \( x^m - x^n \) divides \( x^M - x^N \) in the ring \( \mathcal{R} \) can be seen by considering the following equation

\[
x^M - x^N = x^N (x^{M-N} - 1) = x^{N-n} (x^m - x^n)(x^{(l-1)(m-n)} + \cdots + x^{m-n} + 1).
\]

\( \square \)

**Proposition 2.3.** Let \( D \) be a finitely generated \( \mathcal{R} \)-module. Then \( |D| < \infty \) if and only if

- there exists \( N \in \mathbb{N} \) such that \( N \in \text{Ann}_\mathcal{R}(D) \), and
- for each \( i \in \{1, \ldots, k\} \) there are indices \( n_i, m_i \in \mathbb{N} \) with \( n_i < m_i \) such that \( x_i^{m_i} - x_i^{n_i} \in \text{Ann}_\mathcal{R}(D) \).

**Proof.** Fix a finite generating set \( a_1, \ldots, a_l \) for the \( \mathcal{R} \)-module \( D \) for some \( l \in \mathbb{N} \).

Working in one direction assume that \( |D| < \infty \). In particular \( D \) is a finite abelian group with respect to its additive structure. Therefore \( D \) has a finite exponent \( N \in \mathbb{N} \). In other words \( N \in \text{Ann}_\mathcal{R}(D) \).

Given a generator \( x_i \) of the ring \( \mathcal{R} \) and a generator \( a_j \) of the module \( D \) for \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, l\} \) the elements \( x_i^m a_j D \) cannot be pairwise distinct as \( n \) ranges over \( \mathbb{N} \). Therefore there are some indices \( m_{i,j}, n_{i,j} \in \mathbb{N} \) with \( m_{i,j} > n_{i,j} \) satisfying

\[
x_i^{n_{i,j}} - x_i^{m_{i,j}} \in \text{Ann}_\mathcal{R}(D).
\]

For all \( i \in \{1, \ldots, k\} \) denote

\[
N_i = \max_{j \in \{1, \ldots, l\}} \{n_{i,j}\} \quad \text{and} \quad M_i = N_i + \sum_{j \in \{1, \ldots, l\}} \{m_{i,j} - n_{i,j}\}.
\]

We conclude by Lemma 2.2 that \( x_i^{M_i} - x_i^{N_i} \in \text{Ann}_\mathcal{R}(D) \) for all \( j \in \{1, \ldots, l\} \). This implies that \( x_i^{M_i} - x_i^{N_i} \in \text{Ann}_\mathcal{R}(D) \) for all \( i \in \{1, \ldots, k\} \) as required.
In the other direction, assume that the annihilator $\text{Ann}_R(D)$ contains some integer $N \in \mathbb{N}$ as well as the elements $x_i^{m_i} - x_i^{n_i} \in R$ for each $i \in \{1, \ldots, k\}$ as above. Consider the following countable subset of the module $D$

$$B = \{x_1^{e_1} \cdots x_k^{e_k} a_j : e_i \in \mathbb{N} \cup \{0\}, j \in \{1, \ldots, l\}\}.$$ 

It is clear that $B$ generates $D$ as a $\mathbb{Z}$-module. We may use the elements $x_i^{m_i} - x_i^{n_i}$ of $\text{Ann}_R(D)$ to reduce every exponent $e_i$ below $m_i$. Therefore $D$ admits a finite generating set as a $\mathbb{Z}$-module. Since $N \in \text{Ann}_R(D)$ it follows that $|D| < \infty$. □

This implies that if $D_1$ and $D_2$ are two finitely generated $R$-modules with $\text{Ann}_R(D_1) \subset \text{Ann}_R(D_2)$ and $|D_1| < \infty$ then $|D_2| < \infty$.

**Proposition 2.4.** An ideal $I \triangleleft R$ is a depth ideal if and only if every element $r \in R \setminus I$ satisfies at least one of the following two conditions

1. $nr \notin I$ for all $n \in \mathbb{N}$, or
2. there exists an index $i \in \{1, \ldots, k\}$ such that $(x_i^n - x_i^m)r \notin I$ for all distinct $n, m \in \mathbb{N}$.

**Proof.** Recall that the ideal $I$ is a depth ideal if and only if the quotient ring $R/I$ admits no non-zero ideals of finite cardinality. The last statement is equivalent to the quotient $R/I$ not admitting any non-zero principal ideals of finite cardinality.

Consider any non-trivial element $r + I$ of the quotient ring $R/I$. Let $L = rR + I \triangleleft R/I$ be the principal ideal generated by $r + I$. The ideal $L \triangleleft R/I$ has $|L| = \infty$ if and only if at least one of above two conditions is satisfied, see Proposition 2.3.

**Example 2.5.** Consider the ring $\mathbb{Z}[x_1, \ldots, x_k]$. Any ideal generated by a collection of monomials of the form $x_1^{e_1} \cdots x_k^{e_k}$ with $e_i \in \mathbb{N} \cup \{0\}$ is a depth ideal.

**Example 2.6.** The sum of two depth ideals need not be a depth ideal. This is witnessed by the two depths ideals (2) and (x) in the ring $\mathbb{Z}[x]$.

Let us summarize some further properties of the notion of depth.

**Proposition 2.7.** Let $I, J \triangleleft R$ be a pair of ideals.

1. $(I \cap J)^* = I^* \cap J^*$.
2. If $J \subset I$ then $J^* \subset I^*$.
3. $I^* = J^*$ if and only if $I$ and $J$ are commensurable, i.e. $|I/(I \cap J)| < \infty$ and $|J/(I \cap J)| < \infty$.

**Proof.** We first show that $I^* \cap J^*$ is a depth ideal. Indeed, let $L \triangleleft R$ be any ideal such that $I^* \cap J^* \subset L$ and $|L/(I^* \cap J^*)| < \infty$. Therefore $|(L + I^*)/I^*| < \infty$ as well as $|(L + J^*)/J^*| < \infty$. Since both $I^*$ and $J^*$ are depth ideals it follows that $L \subset I^* \cap J^*$, as required. In other words the quotient ring $R/(I^* \cap J^*)$ admits no non-zero ideals of finite cardinality.

Assume that $J \subset I$. It follows that $J \subset I^*$ and so $J \subset I^* \cap J^* \subset J^*$. The intersection $I^* \cap J^*$ is a depth ideal by the above argument. As $|(I^* \cap J^*)/J| < \infty$ we conclude that $J^* = I^* \cap J^* \subset I^*$. Item (2) follows.

To prove Item (1) it remains to establish that $|(I^* \cap J^*)/(I \cap J)| < \infty$. We may apply Proposition 2.4 to find some integers $N_I, N_J \in \mathbb{N}$ satisfying

$$N_I \in \text{Ann}_R(I^*/I) \quad \text{and} \quad N_J \in \text{Ann}_R(J^*/I).$$
It is easy to verify that
\[ N = \text{lcm}\{N_I, N_J\} \in \text{Ann}_R((I^* \cap J^*)/(I \cap J)). \]
Likewise for each \( i \in \{1, \ldots, k\} \) there are some indices \( m_{I,i}, n_{I,i}, m_{J,i}, n_{J,i} \in \mathbb{N} \)
with \( m_{I,i} > n_{I,i} \) and \( m_{J,i} > n_{J,i} \) satisfying
\[ x_i^{m_{I,i}} - x_i^{n_{I,i}} \in \text{Ann}_R(I^*/I) \quad \text{and} \quad x_i^{m_{J,i}} - x_i^{n_{J,i}} \in \text{Ann}_R(J^*/J). \]
Therefore for each \( i \in \{1, \ldots, k\} \) the integers
\[ n_i = \max\{n_{I,i}, n_{J,i}\} \in \mathbb{N} \quad \text{and} \quad m_i = n_i + \text{lcm}\{m_{I,i} - n_{I,i}, m_{J,i} - n_{J,i}\} \in \mathbb{N} \]
satisfy
\[ x_i^{m_i} - x_i^{n_i} \in \text{Ann}_R((I^* \cap J^*)/(I \cap J)). \]
according to Lemma \ref{app:2.2}. We conclude the proof of Item \ref{app:2.1} relying on the converse
direction of Proposition \ref{app:2.3}.

It remains to prove Item \ref{app:2.3}. In one direction assume that \( I^* = J^* \). It follows
from Item \ref{app:2.1} that \( I^* = J^* \) is the depth of the ideal \( I \cap J \). Therefore \( I \) and \( J \)
are commensurable. The other direction of Item \ref{app:2.3} follows from the definition of
depth.

Item \ref{app:2.1} of Proposition \ref{app:2.7} implies in particular that the intersection of two
depth ideals is again a depth ideal. Moreover, it follows from Item \ref{app:2.3} that having
the same depth is an equivalence relation on the set of ideals of the ring \( R \). The
depth ideals parameterize the equivalence classes.

Remark 2.8. The question of classifying depth ideals in commutative Noetherian
rings seems to be of interest. We were unable to arrive at any definite statement.
See the “Further questions” paragraph of §4.

**Ideal quotient.** Let \( I \) and \( J \) be a pair of ideals in the ring \( R \). The **ideal quotient**
\( J : I \) is discussed e.g. in [AM13, p. 8]. It is given by
\[ J : I = \text{Ann}_R((I + J)/J) = \{ x \in R : xI \subset J \}. \]
Note the the ideal quotient \( J : I \) is an ideal in \( R \). Let \( L \) be another ideal in the
ring \( R \). The following facts are immediate from the above definition.

- \( I(J : I) \subset J \subset J : I \),
- \( (I \cap J) : J = J : I \), and
- if \( J \subset I \) then \( J : L \subset I : L \).

The following is another elementary consequence of the definition.

**Proposition 2.9.** If \( L \lhd R \) then \( L : I : (J : I) \subset L : (J : I)I \).

**Proof.** Consider some element \( z \in (L : I) : (J : I) \). To prove that \( z \in L : (J : I)I \)
we need to verify that all \( y \in (J : I) \) and \( x \in I \) satisfy \( zyx \in L \). But \( zy \in (L : I) \)
so that \( zyx \in (L : I)I \subset L \) as required.

**Proposition 2.10.** Assume that \( J \subset I \). Then \(|I/J| < \infty \) if and only if the ideal
quotient \( J : I \) has finite index in the ring \( R \).

**Proof.** Observe that
\[ \text{Ann}_R(I/J) = J : I = \text{Ann}_R(R/(J : I)). \]
As both \( R \)-modules \( I/J \) and \( R/(J : I) \) are finitely generated, the result follows
from Proposition \ref{app:2.3}. \( \square \)
Proposition 2.11. Assume that $\mathcal{J} \subset \mathcal{I}$ and $|\mathcal{I}/\mathcal{J}| < \infty$. Then $|\mathcal{I}/(\mathcal{J} : \mathcal{I})\mathcal{I}| < \infty$.

Proof. Since $|\mathcal{I}/\mathcal{J}| < \infty$ it follows from Proposition 2.10 that the ideal quotient $\mathcal{J} : \mathcal{I}$ has finite index in $\mathcal{R}$. Regard the ideal $\mathcal{I}$ as a $\mathcal{R}$-module. Then $(\mathcal{J} : \mathcal{I})\mathcal{I}$ is an $\mathcal{R}$-submodule of $\mathcal{I}$. The quotient $\mathcal{R}$-module $\mathcal{I}/(\mathcal{J} : \mathcal{I})\mathcal{I}$ is finitely generated and satisfies

$$\text{Ann}_\mathcal{R}(\mathcal{R}/(\mathcal{J} : \mathcal{I})) = \mathcal{J} : \mathcal{I} \subset \text{Ann}_\mathcal{R}(\mathcal{I}/(\mathcal{J} : \mathcal{I})\mathcal{I}).$$

The conclusion follows from Proposition 2.8 and the remark following it. \(\square\)

Proposition 2.12. Let $\mathcal{L} \triangleleft \mathcal{R}$ be an ideal. Assume that $\mathcal{L} \subset \mathcal{I} \subset \mathcal{R}$. If $|\mathcal{I}/\mathcal{J}| < \infty$ and $|\mathcal{J}/\mathcal{L}| = \infty$ then $|\mathcal{I}/\mathcal{J}|/(\mathcal{L} : \mathcal{I})| = \infty$.

Proof. Note that $|\mathcal{I}/\mathcal{J}|/(\mathcal{L} : \mathcal{I})| = \infty$ if and only if the ideal $(\mathcal{L} : \mathcal{I}) : (\mathcal{J} : \mathcal{I})$ has infinite index in the ring $\mathcal{R}$ by Proposition 2.10. This statement follows provided that the ideal $(\mathcal{L} : (\mathcal{L} : \mathcal{I})\mathcal{I}) = (\mathcal{L} \cap (\mathcal{J} : \mathcal{I})\mathcal{I}) : (\mathcal{J} : \mathcal{I})\mathcal{I}$ has infinite index in the ring $\mathcal{R}$ according to Proposition 2.9. On the one hand, the ideal $(\mathcal{J} : \mathcal{I})\mathcal{I}$ has finite index in $\mathcal{I}$ by Proposition 2.11. One the other hand, the ideal $\mathcal{L} \cap (\mathcal{J} : \mathcal{I})\mathcal{I}$ is contained in $\mathcal{L}$ and in particular has infinite index in $\mathcal{I}$. The result follows from yet another application of Proposition 2.10. \(\square\)

3. Classification of $\text{EL}_d(\mathcal{R})$-invariant probability measures on $\hat{\mathcal{R}}^d$

Let $\mathcal{R}$ be a Noetherian commutative unital ring. Fix some $d \geq 2$. Let $\text{EL}_d(\mathcal{R})$ denote the subgroup of the special linear group $\text{SL}_d(\mathcal{R})$ generated by elementary matrices. These are the matrices $E_{i,j}(r)$ given by

$$(E_{i,j}(r))_{k,l} = \begin{cases} 1 & k = l, \\ r & k = i \text{ and } l = j, \\ 0 & \text{otherwise} \end{cases}$$

for some pair of distinct indices $i, j \in \{1, \ldots, d\}$ and some ring element $r \in \mathcal{R}$.

Consider the discrete abelian group $\Gamma = \mathcal{R}^d$ regarded with its additive ring operation. The matrix group $\text{EL}_d(\mathcal{R})$ is acting on the abelian group $\Gamma$ by group automorphisms via matrix multiplication.

Let $\hat{K}$ denote the Pontryagin dual of the group $\Gamma$. The abelian group $K$ is compact. The matrix group $\text{EL}_d(\mathcal{R})$ is acting on $\Gamma$ by group automorphisms via the dual action, namely

$$g(\chi)(\gamma) = \chi(g^{-1}\gamma) \quad \forall g \in \text{EL}_d(\mathcal{R}), \chi \in K, \gamma \in \Gamma.$$ 

We will use the same notation for both actions of $\text{EL}_d(\mathcal{R})$ on $\Gamma$ and on $K$.

The goal of this section is to classify all $\text{EL}_d(\mathcal{R})$-invariant Borel probability measures on the compact group $K$ in terms of depth ideals in the ring $\mathcal{R}$ and complete the proof of Theorem 1.7. First we would like to restate this classification more precisely using some additional notation.

For a closed subgroup $H \leq K$ we let $H^0 \leq \Gamma$ denote the annihilator of $H$, namely $H^0 = \{ \gamma \in \Gamma : \chi(\gamma) = 1 \, \forall \chi \in H \}$. Similarly, for a subgroup $\Delta \leq \Gamma$ the annihilator $\Delta^0$ is a closed subgroup of $K$.

Given an ideal $\mathcal{I} \triangleleft \mathcal{R}$ denote $\Gamma_\mathcal{I} = \mathcal{I}^d$ and $K_\mathcal{I} = \Gamma_\mathcal{I}^0 = (\mathcal{I}^0)^d$. Every such closed subgroup $K_\mathcal{I}$ is clearly $\text{EL}_d(\mathcal{R})$-invariant.
Theorem 3.1. The set of ergodic \( EL_d(R) \)-invariant probability measures on the compact group \( K \) bijectively corresponds to the set of all pairs \((\mathcal{I}, \omega)\) where \( \mathcal{I} \triangleleft R \) is a depth ideal and \( \omega \) is a finite orbit for the \( EL_d(R) \)-action on the quotient \( K/K_{\mathcal{I}} \). The ergodic probability measure corresponding to the pair \((\mathcal{I}, \omega)\) is given by

\[
\mu_{\mathcal{I}, \omega} = \frac{1}{|\omega|} \sum_{g \in K_{\mathcal{I}} \in \omega} g \nu_{\mathcal{I}}
\]

where \( \nu_{\mathcal{I}} \) is the Haar measure on the compact group \( K_{\mathcal{I}} \).

The above formula for the probability measure \( \mu_{\mathcal{I}, \omega} \) makes sense, as the Haar measure \( \mu_{\mathcal{I}} \) is left-invariant for the action of the group \( K_{\mathcal{I}} \).

Example 3.2. Let \( R = \mathbb{Z} \) so that \( K = (\mathbb{R}/\mathbb{Z})^d \) is the \( d \)-dimensional torus. The ring \( \mathbb{Z} \) is just infinite and admits only two depth ideals, namely \((0)\) and \( \mathbb{Z} \). The case \( K_{(0)} = K \) corresponds to the Haar measure on the torus. The remaining case \( K_{\mathbb{Z}} = \{ e \} \) corresponds to atomic probability measures supported on finite rational orbits. This recovers a well-known measure classification result [DK79, Bur91].

Remark 3.3. We use the same notation \( d \) for the size of the matrix group \( EL_d(R) \) considered in this section as well as elsewhere throughout this paper. However, towards classifying the characters of the group \( EL_d(R) \) we will be applying Theorem 3.1 to the action of the smaller matrix group \( EL_{d-1}(R) \) on the abelian group \( R^{d-1} \).

\( EL_d(R) \)-invariant subgroups. The first step towards proving Theorem 3.1 is to classify all \( EL_d(R) \)-invariant subgroups of \( K \).

Proposition 3.4. A subgroup \( \Delta \) of \( \Gamma \) is \( EL_d(R) \)-invariant if and only if \( \Delta = \Gamma_{\mathcal{I}} \) for some ideal \( \mathcal{I} \triangleleft R \).

Proof. As the group \( EL_d(R) \) is generated by elementary matrices, it is clear that the subgroup \( \Gamma_{\mathcal{I}} = \mathcal{I}^d \) of \( \Gamma \) is \( EL_d(R) \)-invariant for every ideal \( \mathcal{I} \triangleleft R \). This proves one direction of the statement.

To see the converse direction consider an arbitrary \( EL_d(R) \)-invariant subgroup \( \Delta \) of \( \Gamma \). Let \( \mathcal{I} \triangleleft R \) denote the ideal generated by the coordinates of all group elements of \( \Delta \), namely

\[
\mathcal{I} = \sum_{\gamma = (\gamma_1, \ldots, \gamma_d) \in \Gamma} \sum_{i=1}^d R \gamma_i.
\]

Clearly \( \Delta \leq \Gamma_{\mathcal{I}} \). To prove the opposite containment \( \Gamma_{\mathcal{I}} \leq \Delta \) it suffices to show that the subgroup \( \Delta \) contains all elements of the form \((0, \ldots, 0, rs, 0, \ldots, 0)\) where \( r \in R \) is any ring element and \( s \in R \) is any coordinate in some element \( \gamma \in \Delta \). Indeed let \( \gamma \in \Delta \) be a group element whose \( i \)-th coordinate is equal to \( s \in R \). Fix some index \( j \in \{1, \ldots, d\} \setminus \{i\} \). Then for any ring element \( r \in R \)

\[
E_{j,i}(r)x - x = (0, \ldots, 0, rs, 0, \ldots, 0) \in \Delta
\]

with the \( rs \) entry being at the \( j \)-th coordinate. Recall that, up to sign, every two coordinates can be swapped by the \( EL_d(R) \)-action relying on the formula

\[
E_{i,j}(1)E_{j,i}(-1)E_{i,j}(1) = \begin{pmatrix}
* & 0 & +1 \\
0 & +1 & * \\
* & -1 & 0
\end{pmatrix}.
\]
Therefore, replacing the ring element \( r \) by \( -r \) in the above argument if necessary, the element \((0,\ldots,0,rs,0,\ldots,0)\) with the non-zero entry being at an arbitrary coordinate belongs to \( \Delta \). We conclude that \( \Gamma \leq \Delta \) and so \( \Delta = \Gamma \) as required. \( \square \)

A subgroup \( \Delta \) of \( \Gamma \) is \( \text{EL}_d(\mathcal{R}) \)-invariant if and only if its annihilator \( \Delta^0 \) is. The fact that \( \Gamma^0 = K_{\Gamma} \) gives the following.

**Corollary 3.5.** A closed subgroup \( H \) of \( K \) is \( \text{EL}_d(\mathcal{R}) \)-invariant if and only \( H = K_{\mathcal{I}} \) for some ideal \( \mathcal{I} < \mathcal{R} \).

Let \( \mathcal{I} \) and \( \mathcal{J} \) be a pair of ideals in the ring \( \mathcal{R} \). Clearly \( \Gamma_{\mathcal{J}} \leq \Gamma_{\mathcal{I}} \) if and only if \( \mathcal{J} \leq \mathcal{I} \). In that case the index \( [\Gamma_{\mathcal{I}} : \Gamma_{\mathcal{J}}] \) is finite if and only if \( |\mathcal{I} / \mathcal{J}| < \infty \).

These remarks immediately imply that every \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup \( \Delta \leq \Gamma \) is contained in a unique \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup \( \Delta^* \leq \Gamma \) maximal with respect to the condition \( [\Delta^* : \Delta] < \infty \). If \( \Delta = \Gamma_{\mathcal{I}} \) then \( \Delta^* = \Gamma_{\mathcal{I}'} \). There is a dual notion \( H^* \) defined for every closed \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup \( H \) of \( K \).

**Essential subgroups.** The next step towards Theorem 3.1 is to use the ergodicity assumption to understand the support of the Borel probability measure \( \mu \).

For a closed \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup \( H \) of \( K \) denote

\[
\mathcal{O}(H) = H \setminus \bigcup_{H' \leq H} H'
\]

where the union is taken over all \( \text{EL}_d(\mathcal{R}) \)-invariant proper closed subgroups \( H' \) contained in \( H \). There are only countably many \( \text{EL}_d(\mathcal{R}) \)-invariant closed subgroups of \( K \) by Proposition 3.4. Therefore \( \mathcal{O}(H) \) is a Borel subset of \( K \) for every such subgroup \( H \).

The sets \( \mathcal{O}(H) \) form a Borel partition of the compact group \( K \). To see this, note that if \( H_1 \) and \( H_2 \) are two distinct closed \( \text{EL}_d(\mathcal{R}) \)-invariant subgroups of \( K \) then \( H_1 \cap H_2 \) is also a closed \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup of \( K \). Therefore \( \mathcal{O}(H_1) \cap \mathcal{O}(H_2) = \emptyset \). Moreover every element \( g \in K \) belongs to some Borel set \( \mathcal{O}(H) \) where \( H \) is the smallest closed \( \text{EL}_d(\mathcal{R}) \)-invariant subgroup containing \( g \).

Let \( \mu \) be an ergodic \( \text{EL}_d(\mathcal{R}) \)-invariant probability measure on the compact group \( K \). The above discussion shows that there exists a unique ideal \( \mathcal{I}_\mu < \mathcal{R} \) such that \( \text{supp}(\mu) \subset \mathcal{O}(K_{\mathcal{I}_\mu}) \).

We will say that \( \mathcal{I}_\mu \) is the **essential ideal** and \( K_{\mathcal{I}_\mu} \) is the **essential subgroup** associated to the ergodic probability measure \( \mu \). Note that \( \mu(K_{\mathcal{I}_\mu}) = 1 \) and \( \mu(H') = 0 \) for every proper \( \text{EL}_d(\mathcal{R}) \)-invariant closed subgroup \( H' \) of \( K_{\mathcal{I}_\mu} \).

**Orbits for the \( \text{EL}_d(\mathcal{R}) \)-action on \( \mathcal{R}^d \).** The Euclidean algorithm can be used to prove that the orbits of the \( \text{SL}_d(\mathbb{Z}) \)-action on the abelian group \( \mathbb{Z}^d \) are parametrized by the greatest common divisor of the coordinates of a given point \( x \in \mathbb{Z}^d \). We were unable to give such a precise characterization of orbits in the general case. Still, we show that in certain situations orbits can be assumed to be fairly large, in the following precise sense.

**Proposition 3.6.** Let \( \mu \) be an ergodic \( \text{EL}_d(\mathcal{R}) \)-invariant probability measure on the compact group \( K \) with essential ideal \( \mathcal{I}_\mu \). Let \( \gamma \in \Gamma \setminus \Gamma_{\mathcal{I}_\mu}^* \). Then there exists a subgroup \( \Lambda \) of \( \Gamma \) such that \( \gamma + \Lambda \leq \text{EL}_d(\mathcal{R}) \gamma \) and \( \mu(\Lambda^0) = 0 \).

The notation \( \Gamma_{\mathcal{I}_\mu}^* \) in the statement of Proposition 3.6 stands for the maximal subgroup of \( \Gamma \) containing \( \Gamma_{\mathcal{I}_\mu} \) with respect to the condition that \( \left[ \Gamma_{\mathcal{I}_\mu}^* : \Gamma_{\mathcal{I}_\mu} \right] < \infty \).
In other words $\Gamma_\mu^* = \Gamma I_\mu$, where $I_\mu^*$ is the depth of the essential ideal $I_\mu$. Note that the subgroup $\Lambda$ will not be $\text{EL}_d(\mathcal{R})$-invariant in general.

We first prove a straightforward but technical Lemma, to be used in the proof of Proposition 3.6 below.

**Lemma 3.7.** Let $I \subset \mathcal{R}$ be any ideal. For every element $s \in \mathcal{R}$ consider the subgroup $\Sigma_s \leq \Gamma$ given by

$$
\Sigma_s = \{(\gamma_1, \gamma_2, \ldots, \gamma_d) \in \Gamma : \gamma_1 = s\gamma_2\}.
$$

Then every pair of elements $s, s' \in \mathcal{R}$ satisfy $\Gamma \leq \Sigma_s + \Sigma_{s'}$.

**Proof.** Let $s, s' \in \mathcal{R}$ be a fixed pair of elements. Then

$$(sr, r, 0, \ldots, 0) - (s'r, r, 0, \ldots, 0) = ((s - s')r, 0, 0, \ldots, 0) \in \Sigma_s + \Sigma_{s'}$$

for every element $r \in I$. It is clear that $(0, 0, \ldots, 0, I, 0, 0, \ldots, 0) \leq \Sigma_s + \Sigma_{s'}$ holds true starting from the third coordinate and onwards. This concludes the proof. □

**Proof of Proposition 3.6.** Let $I_\mu$ be the essential ideal and $K_{I_\mu}^*$ be the essential subgroup associated to the ergodic $\text{EL}_d(\mathcal{R})$-invariant probability measure $\mu$.

Write $\gamma = (\gamma_1, \ldots, \gamma_d) \in \Gamma \setminus \Gamma_{I_\mu}^*$ and assume without loss of generality that $\gamma_1 \notin I_{\mu}^*$. Consider the subgroup $\Lambda \leq \Gamma$ given by

$$
\Lambda = \{(\delta_1, \ldots, \delta_d) \in \Gamma : \delta_1 = 0, \delta_2, \ldots, \delta_d \in \mathcal{R}\gamma_1\}.
$$

Applying the product of the commuting elementary matrices $E_{2,1}(r_2), \ldots, E_{d,1}(r_d)$ for some ring elements $r_2, \ldots, r_d \in \mathcal{R}$ to the element $\gamma$ gives

$$
E_{2,1}(r_2) \cdots E_{d,1}(r_d) \gamma = \gamma + (0, r_2\gamma_1, \ldots, r_d\gamma_1).
$$

It follows that $\gamma + \Lambda \subset \text{EL}_d(\mathcal{R})\gamma$. We claim that the subgroup $\Lambda^0$ of $\Lambda$ admits countably many $\text{EL}_d(\mathcal{R})$-translates whose pairwise intersections are $\mu$-null. Since the probability measure $\mu$ is $\text{EL}_d(\mathcal{R})$-invariant, this claim implies that $\mu(\Lambda^0) = 0$ as required.

Since the essential ideal $I_{\mu}^*$ is a depth ideal and $\gamma_1 \notin I_{\mu}^*$ it follows that at least one of the two conditions in Proposition 3.4 is satisfied. In either case it is possible to find a sequence of elements $s_n \in \mathcal{R}$ such that $(s_n - s_m)\gamma_1 \notin I_{\mu}^*$ for every pair of distinct indices $n, m \in \mathbb{N}$.

Consider the elementary matrices $g_n = E_{1,2}(s_n) \in \text{EL}_d(\mathcal{R})$ for all $n \in \mathbb{N}$. Fix a pair of distinct indices $n, m \in \mathbb{N}$. It follows from Lemma 3.7 that

$$
\Gamma_{(s_m - s_n)\gamma_1} \leq g_n\Lambda + g_m\Lambda.
$$

Consider the ideal $\mathcal{L} = I_\mu + ((s_n - s_m)\gamma_1) \subset \mathcal{R}$. We get

$$
\Gamma_{\mathcal{L}} \leq g_n\Lambda + g_m\Lambda + \Gamma_{I_\mu}.
$$

Passing to annihilators reverses the direction of the inclusion, namely

$$
g_n\Lambda^0 \cap g_m\Lambda^0 \cap K_{I_\mu} \leq K_{\mathcal{L}}.
$$

However $(s_n - s_m)\gamma_1 \notin I_{\mu}^*$ and therefore $I_{\mu}^* \neq \mathcal{L}^*$. In particular $I_{\mu} \leq \mathcal{L}$ and so $K_{\mathcal{L}} \leq K_{I_{\mu}^*}$. Since $K_{I_{\mu}^*}$ is the essential subgroup with respect to the probability measure $\mu$ we have that

$$
\mu(g_n\Lambda^0 \cap g_m\Lambda^0) = \mu(g_n\Lambda^0 \cap g_m\Lambda^0 \cap K_{I_\mu}) \leq \mu(K_{\mathcal{L}}) = 0.
$$
This concludes the proof of the above claim.

The next result concerning $EL_d(\mathbb{R})$-orbits is used to show that the probability measures that appear in Theorem 3.1 are indeed ergodic. It is a generalization of a classical proof for the ergodicity of the Haar measure on the $d$-dimensional torus ($\mathbb{R}/\mathbb{Z})^d$ with respect to the natural $SL_d(\mathbb{Z})$-action, see e.g. [Zim84, Example 2.1.4].

**Proposition 3.8.** Let $I \triangleleft \mathbb{R}$ be a depth ideal. Then the $EL_d(\mathbb{R})$-orbit of every non-trivial element of the quotient $\Gamma/\Gamma_I$ is infinite.

**Proof.** Let $\gamma + \Gamma_I \in \Gamma/\Gamma_I$ be any non-trivial element, namely $\gamma / \in \Gamma_I$. In particular there is an index $i \in \{1, \ldots, d\}$ such that the $i$-th coordinate $\gamma_i$ of the element $\gamma$ satisfies $\gamma_i / \notin \mathcal{I}$. As $\mathcal{I}$ is a depth ideal, one of the two conditions in Proposition 2.4 is satisfied with respect to the element $\gamma_i$. In either case we may find a sequence $s_n \in \mathbb{R}$ such that the elements $s_n \gamma_i \in \mathbb{R}$ are pairwise distinct modulo the ideal $\mathcal{I}$.

Fix an arbitrary index $j \in \{1, \ldots, d\} \setminus \{i\}$. The elements $E_{j,i}(s_n)\gamma$ all belong to the $EL_d(\mathbb{R})$-orbit of the element $\gamma$ and are pairwise distinct when considered in the quotient $\Gamma/\Gamma_I$. □

**Haar measure and the Fourier transform.** Let $T$ be an arbitrary locally compact abelian group. Let $A$ be the Pontryagin dual of $T$. We briefly recall a few generalities concerning the group $T$ and its dual $A$ to be used below.

The Fourier transform of a complex Radon measure $\nu$ on the group $T$ is the continuous function $\hat{\nu} : A \to \mathbb{C}$ given by

$$\hat{\nu}(a) = \int_T a(t) \, d\nu(t) \quad \forall a \in A.$$ 

**Proposition 3.9.** Let $H$ be a closed subgroup of $T$. Let $\lambda$ be the Haar measure of $H$ regarded as a probability measure on the group $T$. For every element $t \in T$ the Fourier transform of the translate $t_\ast \lambda$ is given by

$$t_\ast \hat{\lambda}(a) = \begin{cases} t(a) & a \in H^0 \\ 0 & a \notin H^0. \end{cases}$$

**Proof.** Let $a \in A = \hat{T}$ be any element with $a \notin H^0$. We will show that $\hat{\lambda}(a) = 0$. There is some element $h_0 \in H$ with $a(h_0) \neq 1$. Therefore

$$\hat{\lambda}(a) = \int_H a(h) \, d\lambda(h) = \int_H a(h_0 \lambda) \, d\lambda(h) = a(h_0) \hat{\lambda}(a).$$

It follows that $\hat{\lambda}(a) = 0$ for the particular element $a \notin H^0$ as above. Fix any element $t \in T$. It is clear that $t_\ast \hat{\lambda} = t \hat{\lambda}$. The conclusion follows. □

In particular, the Fourier transform of the Haar measure of the subgroup $H$ regarded as a probability measure on the group $T$ is equal to the characteristic function of the annihilator $H^0$.

The following elementary result concerns atomic probability measures on the group $T$.

**Proposition 3.10.** Let $\nu_1$ and $\nu_2$ be a pair of probability measures on the group $T$. The convolution $\nu_1 * \nu_2$ admits atoms if and only if both $\nu_1$ and $\nu_2$ admit atoms.
Proof. It follows from the definition of the convolution $\nu_1 * \nu_2$ that

$$(\nu_1 * \nu_2)(\{t\}) = (\nu_1 \times \nu_2)(\{(t_1, t_2) \in T \times T : t_1 t_2 = t\})$$

for every element $t \in T$. Therefore if $\nu_1(\{t_1\}) > 0$ and $\nu_2(\{t_2\}) > 0$ for some pair of elements $t_1, t_2 \in T$ then $(\nu_1 * \nu_2)(\{t_1 t_2\}) > 0$ as well. Conversely, assume that $(\nu_1 * \nu_2)(\{t\}) > 0$ for some element $t \in T$. According to Fubini’s theorem

$$(\nu_1 * \nu_2)(\{t\}) = \int_T \nu_2(\{t_2 \in T : t_1 t_2 = t\}) \, d\nu_1(t_1) = \int_T \nu_2(\{t_1^{-1} t\}) \, d\nu_1(t_1).$$

This means that $\nu_2(\{t_1^{-1} t\}) > 0$ for $t_1 \in T$ belonging to a set of positive $\nu_1$-measure. In particular both $\nu_1$ and $\nu_2$ admit atoms. \hfill \Box

Proof of the classification theorem. We conclude the classification of $EL_d(\mathcal{R})$-invariant probability measures on the compact group $K$.

**Proposition 3.11.** Let $\mu$ be an ergodic $EL_d(\mathcal{R})$-invariant probability measure on the compact group $K$. Let $K_I$ be the essential subgroup of $\mu$ and $\Gamma_I$ be its annihilator. Then $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \Gamma_I^\times$.

Proof. Assume towards contradiction that $\hat{\mu}(\gamma_0) = z \neq 0$ for some $\gamma_0 \in \Gamma \setminus \Gamma_I^\times$. Making use of Proposition 3.6 we find a subgroup $\Lambda$ of the discrete group $\Gamma$ satisfying both $\gamma_0 + \Lambda \subset EL_d(\mathcal{R})$ and $\mu(\Lambda^0) = 0$.

Let $\nu$ be the complex Radon measure on the compact dual group $K$ given by the formula $d\nu = \gamma_0 d\mu$. The Fourier transform $\hat{\nu}$ of the complex measure $\nu$ satisfies $\hat{\nu}(\gamma) = \hat{\mu}(\gamma + \gamma_0)$ for all elements $\gamma \in \Gamma$. In particular $\hat{\nu}|_{\Lambda} = z$. The two measures $\mu$ and $\nu$ are equivalent so that $\nu(\Lambda^0) = 0$.

Consider the quotient group $K = K/\Lambda^0$ and the quotient map $\pi : K \to K$. Let $\sigma$ denote the push-forward complex Radon measure $\pi_*\nu$ on $K$. The Pontryagin dual of $K$ is isomorphic to the subgroup $\Lambda$. In particular, the Fourier transform of $\pi$ coincides with the restriction of $\hat{\nu}$ to the subgroup $\Lambda$ and is therefore a constant function. The map taking a complex Radon measure to its Fourier transform is injective [Fol94 Proposition 4.17]. It follows that $\sigma = z\delta_{\Lambda^0}$ where $\delta_{\Lambda^0}$ is the atomic probability measure supported at the trivial coset $\Lambda^0$ of the quotient $K$. As $z \neq 0$ this contradicts the assumption that $\nu(\Lambda^0) = 0$. \hfill \Box

The following is inspired by the classical proof of the corresponding fact for the $SL_n(\mathbb{Z})$-action on the $n$-dimensional torus $(\mathbb{R}/\mathbb{Z})^d$, see e.g. [Zim84] Example 2.1.4.

**Proposition 3.12.** Let $\mathcal{I} \subset \mathcal{R}$ be a depth ideal and $\omega$ be a finite orbit for the $EL_d(\mathcal{R})$-action on the quotient $K/K_T$. Let $\nu_\mathcal{I}$ denote the Haar measure of the subgroup group $K_T$ regarded as a probability measure on compact group $K$. Then the probability measure $\mu_{\mathcal{I}, \omega}$ given by

$$\mu_{\mathcal{I}, \omega} = \frac{1}{|\omega|} \sum_{gK_T \in \omega} g_*\nu_\mathcal{I}$$

is $EL_d(\mathcal{R})$-invariant and ergodic.

Proof. Assume to begin with that the orbit $\omega$ consists of the single coset $K_T$ so that $|\omega| = 1$. In this case $\mu_{\mathcal{I}, \omega}$ is simply the Haar measure $\nu_\mathcal{I}$ of the compact group $K_T$. The $EL_d(\mathcal{R})$-action on the group $K_T$ is by automorphisms and hence preserves the measure $\nu_\mathcal{I}$. It remains to show that the measure $\nu_\mathcal{I}$ is ergodic for this action. Consider any $EL_d(\mathcal{R})$-invariant function $f \in L^2(K_T, \nu_\mathcal{I})$. The Fourier
transform \( \hat{f} \in L^2(\Gamma/\Gamma_T) \) of the function \( f \) is therefore \( \text{EL}_d(\mathcal{R}) \)-invariant as well. It follows from Proposition \([3.8]\) that \( \hat{f}(\gamma) = 0 \) for every non-trivial element \( \gamma \in \Gamma/\Gamma_T \). Therefore the function \( f \) is \( \nu_T \)-almost surely constant, as required.

Consider the general case so that \( \omega \subset K/K_T \) is an arbitrary finite \( \text{EL}_d(\mathcal{R}) \)-orbit. The invariance of the probability measure \( \nu_{T,\omega} \) follows from the invariance of the Haar measure \( \nu_T \) established in the above paragraph. Let \( A \) denote the kernel of the \( \text{EL}_d(\mathcal{R}) \)-action on the finite orbit \( \omega \) so that \( A \) is a finite index subgroup. To see that the probability measure \( \nu_{T,\omega} \) is \( \text{EL}_d(\mathcal{R}) \)-ergodic it will suffice to show that \( A \) is ergodic on \( g, \nu_T \) for any coset \( gK_T \in \omega \).

Fix a coset \( gK_T \in \omega \) and an element \( k \in gK_T \). Let \( T : K \to K \) denote the translation \( T : g \mapsto g + k \). We see that
\[
ag = a(g - k) + ak = TaT^{-1} + b(a)
\]
for all elements \( a \in A \) and \( g \in K \), where the map \( b : A \to K_T \) is given by
\[
b(a) = ak - k.
\]
The translation \( T \) can be used to identify the trivial coset \( K_T \) with the coset \( gK_T \). This will make the \( A \)-action on the coset \( gK_T \) correspond to the \( A \)-action on the coset \( K_T \) followed by a translation. Note that Fourier transform takes translation to multiplication (by a complex number of modulus one). Since the non-trivial \( A \)-orbits on the group \( \Gamma/\Gamma_T \) are infinite by Proposition \([3.8]\) the above argument goes through to show that the \( A \)-action on the coset \( gK_T \) is ergodic. \( \square \)

The following gives the forward direction of the classification theorem.

**Proposition 3.13.** Let \( \mu \) be an ergodic \( \text{EL}_d(\mathcal{R}) \)-invariant probability measure on the compact group \( K \) with essential ideal \( \mathcal{I}_\mu \). Then there is a finite orbit \( \omega \) for the \( \text{EL}_d(\mathcal{R}) \)-action on the quotient \( K/K_{\mathcal{I}_\mu}^* \) such that
\[
\mu = \frac{1}{|\omega|} \sum_{g \in \omega} g_\ast \nu
\]
where \( \nu \) is the Haar measure on the compact group \( K_{\mathcal{I}_\mu}^* \).

**Proof.** Consider the essential subgroup \( K_{\mathcal{I}_\mu} \) of the compact group \( K \) associated to the probability measure \( \mu \). In particular \( \mu(K_{\mathcal{I}_\mu}) = 1 \).

The annihilator of the essential subgroup \( K_{\mathcal{I}_\mu} \) is given by the subgroup \( \Gamma_{\mathcal{I}_\mu} \). Therefore every character \( \chi \in \Gamma_{\mathcal{I}_\mu} \) is \( \mu \)-almost surely trivial on the group \( K \). In particular
\[
\hat{\mu}|_{\Gamma_{\mathcal{I}_\mu}} = 1
\]
where \( \hat{\mu} : \Gamma \to \mathbb{C} \) is the Fourier transform of the probability measure \( \mu \). In fact \( \gamma + \chi = \gamma \) holds \( \mu \)-almost surely on \( K \) for every other character \( \gamma \in \Gamma \). Therefore the Fourier transform \( \hat{\mu} \) is constant on cosets of the subgroup \( \Gamma_{\mathcal{I}_\mu} \). On the other hand
\[
\hat{\mu}|_{(\Gamma_{\mathcal{I}_\mu})^*} = 0.
\]
as follows from Proposition \([3.11]\)

Consider the finite abelian quotient group \( F = \Gamma_{\mathcal{I}_\mu}^*/\Gamma_{\mathcal{I}_\mu} \). The above observations show that the restriction of the Fourier transform \( \hat{\mu} \) to the subgroup \( \Gamma_{\mathcal{I}_\mu}^* \) descends to a well-defined positive definite function \( \psi \) on the finite group \( F \). Bochner’s theorem \([\text{Fol94, Theorem 4.18}]\) implies that the function \( \psi \) is the Fourier transform of a uniquely determined probability measure \( \theta \) on the Pontryagin dual \( \hat{F} \cong K_{\mathcal{I}_\mu}^*/K_{\mathcal{I}_\mu}^* \).
It follows that \( \theta \) must be a uniform probability measure supported on some finite orbit \( \omega \subset K_{T_x}/K_{T_z} \subset K/K_{T_z} \).

Let \( \nu_{I,\omega} \) be the ergodic \( EL_d(\mathbb{R}) \)-invariant probability measure corresponding to the depth ideal \( I \) and to the finite orbit \( \omega \). The previous paragraph combined with Proposition 6.13 and the injectivity of the Fourier transform for probability measures show that \( \mu = \nu_{I,\omega} \). □

The two Propositions 4.12 and 4.13 establish the two directions of the correspondence between ergodic \( EL_d(\mathbb{R}) \)-invariant probability measures on the compact group \( K \) and pairs \((I,\omega)\) as stated in the classification Theorem 3.1. It is easy to verify that this correspondence is bijective. The proof of Theorem 1.7 of the introduction is complete.

4. Theory of characters

Let \( G \) be a countable group. A function \( \varphi : G \to \mathbb{C} \) is positive definite if for every \( n \in \mathbb{N} \) and every choice of elements \( g_1, \ldots, g_n \in G \) the \( n \times n \) matrix with entries given by \( \varphi(g_j^{-1} g_i) \) is positive definite.

A trace on the group \( G \) is a positive definite conjugation invariant function \( \varphi : G \to \mathbb{C} \) satisfying \( \varphi(e) = 1 \). A trace \( \varphi \) is said to be irreducible if \( \varphi \) cannot be written as a non-trivial convex combination of traces. A character on the group \( G \) is an irreducible trace.

Let \( \text{Tr} (G) \) and \( \text{Ch} (G) \) respectively denote the set of all traces and characters on the group \( G \). The set \( \text{Tr} (G) \) is a weak-* closed convex subset of \( L^\infty (G) \) [BdLHV08, Lemma C.5.4]. The subset \( \text{Ch} (G) \) can be identified with the set of extreme points of \( \text{Tr} (G) \). Choquet’s theorem [Phe11, §10] says that every trace \( \varphi \in \text{Tr} (G) \) can be written as an integral

\[
\varphi = \int_{\text{Ch}(G)} \psi \, d\mu_{\varphi} (\psi)
\]

for some uniquely determined Borel probability measure \( \mu_{\varphi} \) on \( \text{Ch}(G) \). Conversely, any Borel probability measure on the set \( \text{Ch}(G) \) of extreme points determines a trace of \( G \) via the above formula.

The GNS construction. Positive definite functions on the group \( G \) are closely related to cyclic unitary representations of \( G \).

Theorem 4.1 (Gelfand–Naimark–Segal). Let \( \varphi \) be a positive definite function on the group \( G \) with \( \varphi(e) = 1 \). Then there is a unitary representation \( \pi_{\varphi} : G \to U(\mathcal{H}_{\varphi}) \) on a Hilbert space \( \mathcal{H}_{\varphi} \) admitting a cyclic unit vector \( v_\varphi \in \mathcal{H}_{\varphi} \) such that

\[
\varphi(g) = \langle \pi_{\varphi}(g)v_\varphi, v_\varphi \rangle
\]

for every element \( g \in G \). The triplet \( (\mathcal{H}_{\varphi}, \pi_{\varphi}, v_\varphi) \) is uniquely determined up to a unitary equivalence.

The triplet \( (\mathcal{H}_{\varphi}, \pi_{\varphi}, v_\varphi) \) appearing in the above theorem is called the Gelfand–Naimark–Segal (GNS) construction associated the positive definite function \( \varphi \) on the group \( G \).

The representation \( \pi_{\varphi} \) on the Hilbert space \( \mathcal{H}_{\varphi} \) is associated to a certain “left action” in the GNS construction. In fact, since \( \varphi \) is conjugation invariant, there exists another unitary representation \( \rho_{\varphi} \) on the same Hilbert space \( \mathcal{H}_{\varphi} \) arising from a “right action” in the GNS construction. The representation \( \rho_{\varphi} \) commutes with \( \pi_{\varphi} \) and likewise satisfies \( \varphi(g) = \langle \rho_{\varphi}(g)v_\varphi, v_\varphi \rangle \) for all elements \( g \in G \).
Proposition 4.2. Let $\varphi \in \text{Tr}(G)$ be a trace with GNS triplet $(\mathcal{H}_\varphi, \pi_\varphi, \psi_\varphi)$. If the Hilbert space $\mathcal{H}_\varphi$ is finite dimensional then

$$\varphi = \frac{\text{tr} \pi_\varphi}{\dim_{\mathbb{C}} \mathcal{H}_\varphi}.$$ 

Proof. The conjugation invariance of the character $\varphi$ implies that

$$\varphi(g) = \langle \pi_\varphi(g)\pi_\varphi(h)v_\varphi, \pi_\varphi(h)v_\varphi \rangle$$

holds true for all elements $g, h \in G$. Denote $d = \dim_{\mathbb{C}} \mathcal{H}_\varphi$. Let $h_1, \ldots, h_d \in G$ be group elements such that the vectors

$$\pi_\varphi(h_1)v_\varphi, \ldots, \pi_\varphi(h_d)v_\varphi \in \mathcal{H}_\varphi$$

form a basis for the Hilbert space $\mathcal{H}_\varphi$. Then

$$\varphi(g) = \frac{1}{d} \sum_{i=1}^{d} \langle \pi_\varphi(g)\pi_\varphi(h_i)v_\varphi, \pi_\varphi(h_i)v_\varphi \rangle = \frac{\text{tr} \pi_\varphi(g)}{\dim_{\mathbb{C}} \mathcal{H}_\varphi}$$

for all elements $g \in G$ as required. \hfill \Box

The converse to Proposition 4.2 is also true, namely if $\pi : G \to U(\mathcal{H})$ is any unitary representation in a finite dimensional Hilbert space then the function

$$\varphi = \frac{\text{tr} \pi}{\dim_{\mathbb{C}} \mathcal{H}}$$

is a trace on the group $G$.

Proposition 4.3. If $\varphi \in \text{Tr}(G)$ then $|\varphi|^2 \in \text{Tr}(G)$.

This is a well-known fact, see e.g. [BdLHV08, Proposition C.1.6].

Proof of Proposition 4.3. Let $\varphi \in \text{Tr}(G)$ be a trace on the group $G$. It is clear that the function $|\varphi|^2 : G \to \mathbb{C}$ is conjugation invariant and satisfies $|\varphi|^2(e) = 1$. To see that $|\varphi|^2$ is positive definite consider the GNS construction $(\pi_\varphi, \mathcal{H}_\varphi, \psi_\varphi)$ associated to the trace $\varphi$. Let $\pi_\varphi^*$ be the contragredient representation associated to the unitary representation $\pi_\varphi$. Then

$$|\varphi|^2(g) = \langle \pi_\varphi(g)v_\varphi \otimes \pi_\varphi^*(g)v_\varphi, v_\varphi \otimes v_\varphi \rangle \quad \forall g \in G.$$ 

Therefore $|\varphi|^2$ is positive definite. We conclude that $|\varphi|^2 \in \text{Tr}(G)$. \hfill \Box

Faithful traces. The kernel of a given trace $\varphi \in \text{Tr}(G)$ is

$$\ker \varphi = \{ g \in G : \varphi(g) = 1 \}.$$ 

A trace $\varphi \in \text{Tr}(G)$ is called faithful if $\ker \varphi = \{e\}$.

Proposition 4.4. If $\varphi \in \text{Tr}(G)$ then $\ker \varphi \triangleleft G$ and $\overline{\varphi} : g \ker \varphi \mapsto \varphi(g)$ defines a faithful trace on the quotient group $G = G/\ker \varphi$. In fact $\ker \varphi = \ker \pi_\varphi$.

Proof. Consider the GNS construction $(\mathcal{H}_\varphi, \pi_\varphi, \psi_\varphi)$ associated to the trace $\varphi$. An element $g \in G$ satisfies $g \in \ker \varphi$ if and only if $\pi_\varphi(g)v_\varphi = v_\varphi$. It follows that $\ker \varphi$ is a subgroup of $G$. The conjugation invariance of the trace $\varphi$ implies that the subgroup $\ker \varphi$ is normal. The formula $\varphi(g) = \langle \pi_\varphi(g)v_\varphi, v_\varphi \rangle$ shows that $\varphi(g) = \varphi(gn)$ holds true for every pair of elements $g \in G$ and $n \in \ker \varphi$. So the function $\overline{\varphi}$ is well-defined. It is clear that $\overline{\varphi}$ is conjugation invariant, positive definite and satisfies $\overline{\varphi}(\ker \varphi) = 1$. Lastly, as the vector $v_\varphi$ is cyclic and the character $\varphi$ is conjugation invariant it follows in fact that $\ker \varphi = \ker \pi_\varphi$. \hfill \Box
von Neumann algebras and characters. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. A von Neumann algebra $\mathcal{L}$ on the Hilbert space $\mathcal{H}$ is subalgebra of $B(\mathcal{H})$ that contains the identity, is closed in the weak operator topology and under taking adjoint.

The von Neumann bicommutant theorem says that a subalgebra $\mathcal{L}$ of $B(\mathcal{H})$ containing the identity and closed under taking adjoints is a von Neumann algebra if and only if $\mathcal{L}$ is equal to its double commutant $\mathcal{L}''$.

A von Neumann algebra $\mathcal{L}$ on the Hilbert space $\mathcal{H}$ is called a factor if its center $Z(\mathcal{L})$ is equal to $\mathbb{C}$.

Let $\varphi \in \text{Tr}(G)$ be a trace on a group $G$ with GNS construction $(\mathcal{H}_\varphi, \pi_\varphi, \nu_\varphi)$. The trace $\varphi$ defines a von Neumann algebra $\mathcal{L}_\varphi$ on the Hilbert space $\mathcal{H}_\varphi$. The von Neumann algebra $\mathcal{L}_\varphi$ is defined to be the closure of the algebra generated by the unitary operators $\{\pi_\varphi(g) : g \in G\}$ in the weak operator topology. The trace $\varphi$ is a character if and only if the von Neumann algebra $\mathcal{L}_\varphi$ is a factor [Tho64b].

We remark that the commutant $\mathcal{L}_\varphi'$ coincides with the von Neumann algebra generated by $\{\rho_\varphi(g) : g \in G\}$. In particular an operator $x \in B(\mathcal{H}_\varphi)$ belongs to $Z(\mathcal{L}_\varphi)$ if and only if $x$ commutes with $\pi_\varphi(g)$ and $\rho_\varphi(g)$ for all elements $g \in G$.

Consider the kernel $\ker \varphi$ of the trace $\varphi \in \text{Tr}(G)$. Let $\overline{\varphi}$ be the faithful trace on the quotient group $G/\ker \varphi$ as in Proposition 4.3. Since $\ker \varphi = \ker \pi_\varphi$ the two von Neumann algebras $\mathcal{L}_\varphi$ and $\mathcal{L}_{\overline{\varphi}}$ and isomorphic. In particular $\varphi$ is a character if and only if $\overline{\varphi}$ is.

**Lemma 4.5** (Schur’s lemma). If $\varphi \in \text{Ch}(G)$ is a character then the restriction $\varphi|_{Z(G)}$ is a multiplicative character of the center $Z(G)$ and

$$\varphi(gz) = \varphi(g)\varphi(z)$$

for every pair of elements $g \in G$ and $z \in Z(G)$.

**Proof.** Let $\varphi \in \text{Ch}(G)$ be a character with a corresponding GNS construction $(\mathcal{H}_\varphi, \pi_\varphi, \nu_\varphi)$. The von Neumann algebra $\mathcal{L}_\varphi$ on the Hilbert space $\mathcal{H}_\varphi$ is a factor. For every central element $z \in Z(G)$ the operator $\pi_\varphi(z)$ lies in the center $Z(\mathcal{L}_\varphi)$ of the von Neumann algebra $\mathcal{L}_\varphi$. It follows that $\pi_\varphi(z)\nu_\varphi = \chi(z)\nu_\varphi$ for some multiplicative character $\chi : Z(G) \to S^1$ and all elements $z \in Z(G)$, as required. Finally let $g \in G$ and $z \in Z(G)$ be a pair of elements. Then

$$\varphi(gz) = (\pi_\varphi(gz)\nu_\varphi, \nu_\varphi) = (\pi_\varphi(g)\pi_\varphi(z)\nu_\varphi, \nu_\varphi) = \chi(z)(\pi_\varphi(g)\nu_\varphi, \nu_\varphi) = \varphi(z)\varphi(g)$$

as required. $\square$

**Corollary 4.6.** If $\varphi \in \text{Ch}(G)$ then $Z(G) \leq \ker |\varphi|^2$.

**Proof.** Let $\varphi \in \text{Ch}(G)$ be a character. Then $|\varphi|^2 \in \text{Tr}(G)$ by Proposition 4.3 and $|\varphi|^2|_{Z(G)} = 1$ from Lemma 4.5. $\square$

**Relative characters.** Let $N$ be a normal subgroup of the group $G$. The set of relative traces on the subgroup $N$ is given by

$$\text{Tr}_G(N) = \{ \varphi \in \text{Tr}(N) : \varphi(n^g) = \varphi(n) \quad \forall n \in N, g \in G \}.$$ 

The set of relative characters $\text{Ch}_G(N)$ consists of those relative traces that cannot be written as a non-trivial convex combination of relative traces.

The restriction of any character $\varphi \in \text{Ch}(G)$ to the normal subgroup $N$ results in a relative character $\varphi|_N \in \text{Ch}_G(N)$ [Tho64b Lemma 14]. Conversely, given
any relative character $\psi \in \text{Ch}_G(N)$ there exists a character $\varphi \in \text{Ch}(G)$ such that $\varphi|_N = \psi$ [Tho64b, Lemma 16].

Recall that Choquet’s theorem sets up a bijective correspondence between the set $\text{Tr}(N)$ of all traces on the group $N$ and the set $\mathcal{P}(\text{Ch}(N))$ of all Borel probability measures on the set of extreme points $\text{Ch}(N)$. This correspondence is equivariant with respect to the natural $G$-actions on both sets. It follows that the subset $\text{Tr}_G(N)$ of relative traces corresponds to $G$-invariant probability measures, and furthermore the subset $\text{Ch}_G(N)$ of relative characters corresponds to ergodic ones.

**Proposition 4.7.** If $\psi \in \text{Tr}_G(N)$ is a relative trace then the function $\hat{\psi}: G \to \mathbb{C}$ given by

$$
\hat{\psi}(g) = \begin{cases} 
\psi(g) & g \in N \\
0 & g \notin N 
\end{cases}
$$

belongs to $\text{Tr}(G)$.

**Proof.** Consider the GNS construction $(\mathcal{H}_\psi, \pi_\psi, v_\psi)$ corresponding to the relative character $\psi$ so that $\pi_\psi$ is a unitary representation of the normal subgroup $N$. Let $\tilde{\pi}_\psi = \text{Ind}_N^G(\pi_\psi)$ be the induced unitary representation of the group $G$. The Hilbert space $\mathcal{H}_\psi$ on which the representation $\tilde{\pi}_\psi$ acts is given by the orthogonal direct sum

$$
\mathcal{H}_\psi = \bigoplus_{G/N} \mathcal{H}_\psi.
$$

It is easy to see that the function $\hat{\psi}$ coincides with the matrix coefficient of the vector $v_\psi$ in the unitary representation $\tilde{\pi}_\psi$ and as such is positive definite. The function $\hat{\psi}$ is clearly conjugation invariant and satisfies $\hat{\psi}(e) = 1$ as required. □

**Virtually central subgroups.** A subgroup $H$ of $G$ is virtually central in $G$ if $[H : H \cap Z(G)] < \infty$. Let $N$ be a virtually central normal subgroup of $G$. Denote

$$
Z = N \cap Z(G) \quad \text{and} \quad M = N/Z
$$

so that $|M| < \infty$ by definition.

**Proposition 4.8.** Let $\chi$ be a multiplicative character of the central group $Z$. Then

(1) there are only finitely many characters $\psi \in \text{Ch}(N)$ with $\psi|_Z = \chi$, and

(2) the GNS construction associated to every such character $\psi \in \text{Ch}(N)$ is finite dimensional.

The proof is based on well-known facts concerning projective representations. We outline the proof for the reader’s convenience.

**Proof of Proposition 4.8.** Fix an arbitrary section $\lambda: M \to N$ with $\lambda(e_M) = e_N$. The function $c_\chi: M \times M \to \mathbb{C}$ given by

$$
c_\chi(m_1, m_2) = \chi(\lambda(m_1)\lambda(m_2)\lambda(m_1m_2)^{-1}) \quad \forall m_1, m_2 \in M
$$

is a 2-cocycle [Kar85, §2]. Recall that a projective $c_\chi$-representation of the quotient group $M$ on a given vector space $V$ is a map $\tilde{\pi}: M \to \text{GL}(V)$ satisfying

$$
\tilde{\pi}(m_1)\tilde{\pi}(m_2) = c_\chi(m_1, m_2)\tilde{\pi}(m_1m_2)
$$

for all elements $m_1, m_2 \in M$. The set of all representations of the group $N$ restricting to the multiplicative character $\chi$ on the subgroup $Z$ bijectively corresponds
to the set of all projective \(c_\chi\)-representations of the quotient group \(M\). The latter set stands in a bijective correspondence with the set of all modules over the twisted group algebra \(A_\chi = C^\ast M\) [Kar85, Theorem 3.2.5]. As the \(C^\ast\)-algebra \(A_\chi\) is semisimple [Kar85, Theorem 3.2.10] every irreducible projective \(c_\chi\)-representation is finite dimensional and there are only finitely many isomorphism classes of such representations.

Consider a character \(\psi \in \text{Ch}(N)\) with \(\psi|_Z = \chi\) and with GNS triplet \((\mathcal{H}_\psi, \pi_\psi, v_\psi)\). Let \(\bar{\pi}_\psi\) be the projective \(c_\chi\)-representation of the finite quotient group \(M\) corresponding to the representation \(\pi_\psi\) of the group \(N\). Therefore \(\mathcal{H}_\psi\) can be regarded as a cyclic module over the twisted group algebra \(A_\chi\). Since \(A_\chi\) is a \(C^\ast\)- algebra \(A_\chi\) is semisimple \([\text{Kar85, Theorem 3.2.10}]\) every irreducible projective \(c_\chi\)-representation is finite dimensional and there are only finitely many isomorphism classes of such representations.

We remark that, as with finite groups, characters of the group \(N\) restricting to a given multiplicative character \(\chi\) of the central subgroup \(Z = N \cap Z(G)\) correspond bijectively to irreducible projective \(c_\chi\)-representations of the finite group \(M\) over the 2-cocycle \(c_\chi\) corresponding to \(\chi\) [Kar85, Theorem 7.1.11].

**Proposition 4.9.** If \(\psi \in \text{Ch}_G(N)\) is a relative character then there is a finite \(G\)-orbit \(O_\psi \subset \text{Ch}(N)\) such that

\[
\psi = \frac{1}{|O_\psi|} \sum_{\zeta \in O_\psi} \zeta.
\]

**Proof.** The restriction of the character \(\psi\) to the central subgroup \(Z = N \cap Z(G)\) is given by some multiplicative character \(\chi : Z \to \mathbb{C}\), see Lemma 4.5. There is an ergodic \(G\)-invariant probability measure \(\mu_\psi\) on the space \(\text{Ch}(N)\) such that \(\psi = \int \zeta \, d\mu_\psi(\zeta)\). It is clear that the probability measure \(\mu_\psi\) is supported on the Borel subset

\[
\text{Ch}_\chi(N) = \{ \psi \in \text{Ch}(N) : \psi|_Z = \chi \}.
\]

However the subset \(\text{Ch}_\chi(N)\) is finite by Proposition 4.8. Therefore \(\mu_\psi\) must be a uniform probability measure supported on some finite \(G\)-orbit \(O_\chi \subset \text{Ch}_\chi(N)\). The conclusion follows. \(\square\)

**Corollary 4.10.** Let \(\psi \in \text{Ch}_G(N)\) be a relative character. Then there exists a finite dimensional space \(V_\psi\) and a representation \(\pi_\psi : N \to \text{GL}(V)\) satisfying

\[
\psi(n) = \frac{\text{tr}\pi_\psi(n)}{\dim_\mathbb{C} V}
\]

for all elements \(n \in N\).

**Proof.** Let \(O_\psi \subset \text{Ch}(N)\) be the finite \(G\)-orbit given by Proposition 4.9. Let \((\mathcal{H}_\zeta, \pi_\zeta, v_\zeta)\) be the GNS construction associated to each character \(\zeta \in O_\psi\). As \(G\) is transitive on the orbit \(O_\psi\) the dimension \(d = \dim_\mathbb{C} \mathcal{H}_\zeta\) is independent of the choice of the character \(\zeta \in O_\psi\). Moreover \(d < \infty\) according to Proposition 4.8. Lastly

\[
\zeta(n) = \frac{\text{tr}\pi_\zeta(n)}{d}
\]
holds true for all elements \( n \in \mathbb{N} \) and all characters \( \zeta \in O_\psi \), see Proposition \ref{prop:property1}

To conclude the proof consider the representation \( \pi_\psi = \bigoplus_{\zeta \in O_\psi} \pi_\zeta \) on the finite dimensional Hilbert space \( V = \bigoplus_{\zeta \in O_\psi} \mathcal{H}_\zeta \).

\[ \square \]

Remark 4.11. A virtually central group is clearly virtually abelian. However, the center of a virtually abelian group need not be of finite index. For example, the infinite dihedral group is virtually abelian and center free.

A discrete group is type I if and only if it is virtually abelian \cite{Tho64b}. In particular a virtually central subgroup is type I.

**Induced traces.** Let \( H \) and \( N \) be subgroups of \( G \) with \( N \leq H \leq G \). A trace \( \varphi \in \text{Tr} (H) \) is induced from the subgroup \( N \) if \( \varphi (g) = 0 \) for all elements \( g \in H \setminus N \), or what is equivalent\textsuperscript{8}, the corresponding GNS construction \( \pi_\varphi : H \to \mathcal{U}(\mathcal{H}_\varphi) \) is induced in the representation-theoretic sense from a unitary representation of \( N \). Denote

\[ \text{Ind}_G (H; L) = \{ \varphi \in \text{Tr} (G) : \varphi (g) = 0 \quad \forall g \in H \setminus N \} . \]

In other words \( \text{Ind}_G (H; L) \) consists of those traces \( \varphi \in \text{Tr} (G) \) whose restriction \( \varphi |_H \) is induced from the subgroup \( L \).

The following results are based to a large extent on \cite{Bek07}.

**Lemma 4.12.** Let \( \varphi \in \text{Ind}_G (H; N) \) be a trace with GNS construction \((\pi_\varphi, \mathcal{H}_\varphi, v_\varphi)\). If \( h_n \in H \) is a sequence of elements belonging to pairwise distinct left cosets of the subgroup \( N \) then the unit vectors \( \pi_\varphi (h_n) v_\varphi \) weak-\( * \) converge to 0 in \( \mathcal{H}_\varphi \).

**Proof.** We claim that the unit vectors \( \pi_\varphi (h_n) v_\varphi \) are pairwise orthogonal, as can be seen by observing that

\[ (\pi_\varphi (h_n) v_\varphi , \pi_\varphi (h_m) v_\varphi ) = (\pi_\varphi (h_n^{-1} h_m) v_\varphi , v_\varphi ) = \varphi (h_n^{-1} h_m) = 0 \]

for all distinct indices \( n, m \in \mathbb{N} \). The result follows from Bessel's inequality. \( \square \)

Another way to see why Lemma \ref{lemma:induced_traces} is true would be using the fact that the unitary representation \( \pi_\varphi \) is induced from the subgroup \( N \).

**Lemma 4.13.** Let \( \varphi \in \text{Ind}_G (H; N) \) be a trace and \( g \in G \) be an element. If there is a sequence of elements \( x_n \in G \) such that the commutators \( [g, x_n] \) belong to pairwise distinct cosets of \( N \) and such that \( [g, x_n] \in H \) for all \( n \in \mathbb{N} \) then \( \varphi (g) = 0 \).

**Proof.** The conjugation invariance of the trace \( \varphi \) implies that

\[ \varphi (g) = \varphi (x_n^{-1} g x_n) = \varphi ([g, x_n]) = \langle \pi_\varphi ([g, x_n]) v_\varphi, \pi_\varphi (g^{-1} v_\varphi) \rangle \]

for all indices \( n \in \mathbb{N} \). The sequence of vectors \( \pi_\varphi ([g, x_n]) v_\varphi \) weak-\( * \) converges to the zero vector of the Hilbert space \( \mathcal{H}_\varphi \) by Lemma \ref{lemma:induced_traces} We conclude that \( \varphi (g) = 0 \). \( \square \)

**Lemma 4.14.** Let \( \varphi \in \text{Ind}_G (H; N) \) be a trace. Let \( K \) and \( L \) be subgroups of \( G \) and \( g \in KHL \) be an element. If there is a sequence of elements \( x_n \in N_G (H) \) so that

- \( [g, x_n] \) are pairwise distinct modulo \( N \)
- \( [K, x_n] \subset H \) and \( [L, x_n] \subset H \) for every \( n \in \mathbb{N} \)

then \( \varphi (g) = 0 \).

\textsuperscript{8}See Proposition \ref{prop:property2} and its proof for more details on this equivalence.
Proof. Say that \( g = khl \) for some elements \( k \in K, h \in H \) and \( l \in L \). The conjugates \( x_n^{-1}gx_n \) can be written as

\[
x_n^{-1}gx_n = k [k, x_n] \cdot x_n^{-1}hx_n \cdot [x_n, l^{-1}] l.
\]

We denote

\[
y_n = [k, x_n] \cdot x_n^{-1}hx_n \cdot [x_n, l^{-1}]
\]

for every \( n \in \mathbb{N} \). The assumptions imply that \( y_n \in H \). The computation

\[
l^{-1}y_n l = (kl)^{-1}ky_n l = (kl)^{-1}g [g, x_n]
\]

shows that the elements \( y_n \) are pairwise distinct modulo \( N \). Therefore

\[
\varphi(g) = \varphi((x_n^{-1}gx_n l^{-1}) = \varphi(ky_n) = \langle \pi_{\varphi}(y_n) v_\varphi, \pi_{\varphi}((lk)^{-1})v_\varphi \rangle.
\]

The sequence of vectors \( \pi_{\varphi}(y_n)v_\varphi \) converges to the zero vector of the Hilbert space \( \mathcal{H}_\varphi \) in the weak-* topology by Lemma 4.12. Therefore \( \varphi(g) = 0 \). \( \square \)

Remark 4.15. For our purposes it will be enough to use a simpler variant of Lemma 4.14 involving the product of just two subgroups rather than three. We have decided to include the above slightly more general variant nevertheless, as it might be of independent interest.

Characters of two-step nilpotent groups. Let \( Z^2(G) \) denote the second center (i.e. the second term in the ascending central sequence) of the countable group \( G \). This is a characteristic subgroup containing the center \( Z(G) \) and determined by

\[
Z^2(G)/Z(G) = Z(G/Z(G)) \leq G/Z(G).
\]

Lemma 4.16 (Howe [How77]). If \( \varphi \in \text{Ch}(G) \) is a faithful character then

\[
\varphi \in \text{Ind}_G(Z^2(G); Z(G)).
\]

Proof. Consider any element \( g \in Z^2(G) \setminus Z(G) \). There exists an element \( h \in G \) such that \( e \neq [g, h] \in Z(G) \). The conjugation invariance of the character \( \varphi \) combined with Schur’s lemma (reproduced above as Lemma 4.5) shows that

\[
\varphi(g) = \varphi(g^h) = \varphi(g[g, h]) = \varphi(g)[g, h]).
\]

Since \( \varphi \) is faithful \( \varphi([g, h]) \neq 1 \). We conclude that \( \varphi(g) = 0 \). \( \square \)

The following is a variant of Howe’s lemma for relative characters of two-step nilpotent subgroups. Its proof is essentially the same as that of Lemma 4.16.

Lemma 4.17. Let \( N \) be a normal subgroup of \( G \) with \( [N, N] \leq Z(G) \). Then every faithful relative character \( \varphi \in \text{Ch}_G(N) \) is induced from \( Z(N) \).

5. Normal subgroups of the group \( \text{EL}_d(\mathcal{R}) \)

Let \( \mathcal{R} \) be a Noetherian ring. Let \( d \in \mathbb{N} \) be a fixed integer with \( d \geq 3 \).
Stable range. The stable range of a ring is an important invariant in algebraic \( K \)-theory [Bas68, Chapter V.\S 3]. It controls various stability phenomena, such as the results cited in Theorems 5.1 and 5.7 below. We will use the explicit definition of stable range to prove the normal form decomposition in Proposition 6.8 as well as some general stability results.

Stable range is defined as follows. An \( n \)-tuple of elements \( r_1, \ldots, r_n \in \mathcal{R} \) is called **unimodular** if

\[
\mathcal{R}r_1 + \cdots + \mathcal{R}r_n = \mathcal{R}.
\]

We say that an \((n+1)\)-tuple of elements \( r_1, \ldots, r_{n+1} \in \mathcal{R} \) can be shortened if there are elements \( s_1, \ldots, s_n \in \mathcal{R} \) such that the \( n \)-tuple

\[
r_1 + r_1 s_{n+1}, r_2 + r_2 s_{n+1}, \ldots, r_n + r_n s_{n+1}
\]

is unimodular. The **stable range** of the ring \( \mathcal{R} \) is defined to be the smallest integer \( n \in \mathbb{N} \) such that any unimodular \((n+1)\)-tuple of elements can be shortened. It is denoted \( \text{sr}(\mathcal{R}) \). If no such \( n \) exists then \( \text{sr}(\mathcal{R}) = \infty \).

For example, if \( \mathcal{R} \) is a local ring then \( \text{sr}(\mathcal{R}) = 1 \). If \( \mathcal{R} \) is a Dedekind domain then \( \text{sr}(\mathcal{R}) \leq 2 \). If \( \mathcal{R} \) is a Noetherian ring then \( \text{sr}(\mathcal{R}) + 1 \) is bounded above by the Krull dimension of \( \mathcal{R} \). In particular \( \text{sr}(\mathbb{Z}[x_1, \ldots, x_k]) \leq k + 2 \). Finally \( \text{sr}(\mathcal{R}/\mathcal{I}) \leq \text{sr}(\mathcal{R}) \) for any ideal \( \mathcal{I} \subset \mathcal{R} \).

For references see [Bas68, Propositions V.3.2 and V.3.4, Theorem V.3.5]. For the case of Dedekind domains see e.g. [Che11, Example 12.1.14].

**Elementary matrices.** The **elementary matrix** \( E_{i,j}(x) \in \text{SL}_d(\mathcal{R}) \) corresponding to the ring element \( x \in \mathcal{R} \) and the pair of distinct indices \( i, j \in \{1, \ldots, d\} \) is given by

\[
(E_{i,j}(x))_{k,l} = \begin{cases} x & \text{if } i = k \text{ and } j = l, \\ 1 & \text{if } k = l, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that

\[
E_{i,j}(x_1)E_{i,j}(x_2) = E_{i,j}(x_1 + x_2) \quad \forall x_1, x_2 \in \mathcal{R}.
\]

The **elementary subgroup** \( E_{i,j}(\mathcal{I}) \) of the matrix group \( \text{SL}_d(\mathcal{R}) \) corresponding to the ideal \( \mathcal{I} \subset \mathcal{R} \) is given by

\[
E_{i,j}(\mathcal{I}) = \{ E_{i,j}(x) : x \in \mathcal{I} \}.
\]

The group \( E_{i,j}(\mathcal{I}) \) is naturally isomorphic to the additive group of the ideal \( \mathcal{I} \). The elementary groups satisfy the following commutation relations

\[
[E_{i,j}(\mathcal{R}), E_{i',j'}(\mathcal{R})] = \text{Id}_d \quad \text{and} \quad [E_{i,j}(\mathcal{R}), E_{i',j}(\mathcal{R})] = \text{Id}_d
\]

every pair of indices \( i', j' \in \{1, \ldots, d\} \setminus \{i, j\} \).

Let \( \text{EL}_d(\mathcal{R}) \) be the subgroup of the matrix group \( \text{SL}_d(\mathcal{R}) \) generated by the elementary subgroups \( E_{i,j}(\mathcal{R}) \) for all pairs of distinct indices \( i, j \in \{1, \ldots, d\} \).

**Theorem 5.1** [Bas64]. If \( d > \text{sr}(\mathcal{R}) \) then \( \text{EL}_d(\mathcal{R}) \triangleleft \text{SL}_d(\mathcal{R}) \) and the quotient group \( \text{SK}_1(\mathcal{R}) = \text{SL}_d(\mathcal{R})/\text{EL}_d(\mathcal{R}) \) is abelian and independent of \( d \).

Several cases where \( \text{SK}_1(\mathcal{R}) \) is trivial were mentioned in Example 2 of [11].

\[\text{As the ring } \mathcal{R} \text{ is assumed to be commutative, the subgroup } \text{EL}_d(\mathcal{R}) \text{ is in fact normal in } \text{SL}_d(\mathcal{R}) \text{ for all } d \geq 3. \text{[Sus77]}\]
The center of $\SL_d(\mathcal{R})$. Let $\mathcal{O}_\mathcal{R}$ denote the group of units of the ring $\mathcal{R}$. The group $\mathcal{O}_\mathcal{R}$ is clearly abelian. However $\mathcal{O}_\mathcal{R}$ need not be finitely generated, even if $\mathcal{R}$ is. For example, if $\mathcal{R} = \mathbb{F}_2[x, y]/(y^2)$ then every element of the form $1 + yp(x)$ for some polynomial $p \in \mathbb{F}_2[x]$ is a unit.

Let $\mathcal{O}_\mathcal{R}^{[d]}$ denote the subgroup of the group of units $\mathcal{O}_\mathcal{R}$ consisting of elements whose order is a divisor of $d$. In other words

$$\mathcal{O}_\mathcal{R}^{[d]} = \{ u \in \mathcal{O}_\mathcal{R} : u^d = 1 \}.$$

The center of the matrix group $\SL_d(\mathcal{R})$ is

$$Z(\SL_d(\mathcal{R})) = \{ u\Id_d : u \in \mathcal{O}_\mathcal{R}^{[d]} \}.$$

It is shown below that $Z(\SL_d(\mathcal{R})) \leq \EL_d(\mathcal{R})$. In fact $Z(\SL_d(\mathcal{R})) = Z(\EL_d(\mathcal{R}))$. See Corollary 6.4.

Let $\mathcal{S}$ be a commutative Noetherian ring and $f : \mathcal{R} \to \mathcal{S}$ be a surjective ring homomorphism. It induces a reduction map $\hat{f} : \SL_d(\mathcal{R}) \to \SL_d(\mathcal{S})$.

**Proposition 5.2.** $Z(\SL_d(\mathcal{S})) \leq \hat{f}(\EL_d(\mathcal{R}))$.

**Proof.** Let $g \in Z(\SL_d(\mathcal{S}))$ be any element. We may write $g = u\Id_d$ for some unit $u \in \mathcal{O}_\mathcal{S}^{[d]}$. Let $x, y \in \mathcal{R}$ be ring elements so that $f(x) = u$ and $f(y) = u^{-1}$. For every pair of distinct indices $i, j \in \{1, \ldots, d\}$ consider the following elements

$$W_{i,j}(x, y) = E_{i,j}(x)E_{j,i}(-y)E_{i,j}(x) \in \EL_d(\mathcal{R})$$

and

$$D_{i,j}(x, y) = W_{i,j}(x, y)W_{i,j}(-1, -1).$$

A straightforward calculation shows that the reduction of the matrix $D_{i,j}(x, y)$ is given by

$$(\hat{f}D_{i,j}(x, y))_{k,l} = \begin{cases} f(x) = u & k = l = i, \\ f(y) = u^{-1} & k = l = j, \\ 1 & k = l \text{ distinct from } i \text{ and } j, \\ 0 & k \neq l. \end{cases} \forall k, l \in \{1, \ldots, d\}$$

In other words $D_{i,j}(u) := \hat{f}D_{i,j}(x, y)$ is a diagonal matrix with $u$ and $u^{-1}$ at the $i$-th and $j$-th respective positions along the diagonal. As $u \in \mathcal{O}_\mathcal{S}^{[d]}$ it follows that

$$D_{i,j}(u)D_{k,l}(u^2)\ldots D_{d-1,d}(u^{d-1}) = u\Id_d.$$

We conclude that $g = u\Id_d \in \hat{f}(\EL_d(\mathcal{R}))$ as required. \hfill $\square$

The above explicit description of the center of the special linear group shows that $\hat{f}(Z(\SL_d(\mathcal{R}))) \leq Z(\SL_d(\mathcal{S}))$. It may certainly be the case in general that $\hat{f}(Z(\SL_d(\mathcal{R})))$ is a proper subgroup of $Z(\SL_d(\mathcal{S}))$.

**Theorem 5.3.** If $|\ker f| < \infty$ then $\left[ Z(\SL_d(\mathcal{S})) : \hat{f}(Z(\SL_d(\mathcal{R}))) \right] < \infty$.

**Proof.** The ring epimorphism $f : \mathcal{R} \to \mathcal{S}$ induces a group homomorphism $\mathcal{O}_f : \mathcal{O}_\mathcal{R} \to \mathcal{O}_\mathcal{S}$ of the corresponding groups of units. The assumption $|\ker f| < \infty$ implies that $\mathcal{O}_f$ is a group epimorphism as shown in [BL17 Theorem 3.8]. It is moreover clear that $|\ker \mathcal{O}_f| \leq |\ker f| < \infty$. 

The conclusion follows.

□

\[ \text{groups } E_{i,j} \text{ denote the normal closure of } F_d. \]

Clearly \( SL_d \) and \( E_{i,j} \). Therefore the map \( SL_d \) that \( \tilde{E} \). Let \( \tilde{E} \) satisfy \( q \). see \([Kap54, \text{ Theorems 1 and 6}]\). The abelian group \( O_{dN}^d \) can likewise be written up to isomorphism as

\[ O_{dN}^d \cong \bigoplus_{q^k|dN} B_{q^k} \]

where each \( B_{q^k} \) is a direct sum of cyclic groups of prime power order \( q^k \) dividing \( dN \). As \( |\ker O_f| < \infty \) there can only be finitely many cyclic direct summands \( B_{q^k} \) satisfying \( q^k \not| d \) overall. The desired conclusion follows.

\[ \text{Normal subgroup structure. Let } I \lhd R \text{ be an ideal. The congruence subgroup } SL_d(I) \text{ corresponding to the ideal } I \text{ is the kernel of the reduction modulo } I \text{ homomorphism} \]

\[ 1 \to SL_d(I) \to SL_d(R) \to SL_d(R/I). \]

Let \( \tilde{SL}_d(I) \) be the kernel of the composition

\[ SL_d(R) \to SL_d(R/I) \to PSL_d(R/I). \]

In other words \( \tilde{SL}_d(I) \) is the preimage in \( SL_d(R) \) of the center of \( SL_d(R/I) \). Clearly \( SL_d(I) \leq \tilde{SL}_d(I) \) and \( \tilde{SL}_d(I)/SL_d(I) \) is central in \( SL_d(R) / SL_d(I) \).

\[ \text{Remark 5.4. Strictly speaking, the group } \tilde{SL}_d(I) \text{ as well as the group } EL_d(I) \text{ to be introduced below both depend on the ring } R \text{ and not only on the ideal } I. \text{ Since in this work the ring } R \text{ is for the most part kept fixed, or is otherwise clear from the context, we will allow ourselves to drop } R \text{ from the notation. The group } SL_d(I) \text{ on the other hand depends only on the ideal } I \text{ considered as a ring without unit.} \]

\[ \text{Lemma 5.5. Let } J \lhd R \text{ be an ideal with } I \subset J. \text{ Then } \tilde{SL}_d(I) \leq \tilde{SL}_d(J). \]

\[ \text{Proof. The reduction map } \tilde{f} : SL_d(R/I) \to SL_d(R/J) \text{ associated to the surjective ring homomorphism } f : R/I \to R/J \text{ satisfies } \tilde{f}(Z(SL_d(R/I))) \leq Z(SL_d(R/J)). \]

Therefore the map \( SL_d(R) \to PSL_d(R/J) \) factors through the group \( PSL_d(R/I) \). The conclusion follows.

Let \( F_d(I) \) denote the subgroup of \( EL_d(R) \) generated by the elementary subgroups \( E_{i,j}(I) \). The subgroup \( F_d(I) \) need not be normal in general. Let \( EL_d(I) \) denote the normal closure of \( F_d(I) \) in the group \( EL_d(R) \). As \( SL_d(I) \lhd EL_d(R) \) and \( E_{i,j}(I) \leq SL_d(I) \) for every pair of distinct indices \( i, j \in \{1, \ldots, d\} \) we have that \( EL_d(I) \leq SL_d(I) \).

\[ \text{Lemma 5.6 (Tits). Assume that } d \geq 3. \text{ Then } EL_d(I^2) \leq F_d(I). \]
This lemma is a special case of [Tits76, Proposition 2]. Tits’ proof is given in the context of Chevalley group schemes. We briefly reproduce the proof in our special case for the reader’s convenience.

**Proof of Lemma 5.7** Let \( F_d^\alpha(J) \) be the subgroup generated by the two elementary subgroups \( E_{i,j}(J) \) and \( E_{j,i}(I) \) for a given ideal \( J \triangleleft R \) and an unordered pair \( \alpha = \{i,j\} \) of distinct indices \( i,j \in \{1, \ldots, d\} \). Let \( EL_d^\alpha(J) \) denote the normal closure of the subgroup \( F_d^\alpha(J) \) in the subgroup \( F_d^\alpha(R) \).

We claim that the subgroup \( EL_d(J) \) is generated by the subgroups \( EL_d^\alpha(J) \) as \( \alpha \) ranges over all subsets of \( \{1, \ldots, d\} \) with \(|\alpha| = 2\). To establish the claim it suffices to show that the subgroup \( E \) generated as above satisfies \( E \triangleleft EL_d(R) \). Observe that \( E_{i,j}(R), EL_d^\alpha(J) \subset E \) for all subsets \( \alpha \) and all pairs of distinct indices \( i,j \) as above. If \( \alpha = \{i,j\} \) then this observation holds true as \( E_{i,j}(R) \) normalizes \( EL_d^\alpha(J) \), and otherwise as \( [E_{i,j}(R), EL_d^\alpha(J)] \subset F_d(J) \). The claim follows.

Showing that \( EL_d^\alpha(J^2) \leq F_d(I) \) for any given subset \( \alpha = \{i,j\} \) with \(|\alpha| = 2 \) will conclude the proof (in light of the above claim). Let \( G_d^\alpha(I) \) denote the subgroup of \( F_d(I) \) generated by all the elementary subgroups \( E_{k,l}(I) \) other than \( E_{i,j}(I) \) and \( E_{j,i}(I) \). As \( d \geq 3 \) the commutation relations imply that \( F_d^\alpha(J^2) \leq G_d^\alpha(I) \). On the other hand \( F_d^\alpha(R) \) normalizes \( G_d^\alpha(I) \). Therefore \( EL_d^\alpha(J^2) \leq G_d^\alpha(I) \leq F_d(I) \). □

**Theorem 5.7** (Borevich–Vavilov). Assume that \( d > sr(R) \). Let \( I \triangleleft R \) be an ideal. Then the quotient group \( SK_1(R, I) = SL_d(I) / EL_d(I) \) is abelian and independent of \( d \).

**Theorem 5.8** (Normal subgroup structure theorem). Assume that \( d \geq 3 \). A subgroup \( H \triangleleft SL_d(R) \) is normalized by \( EL_d(R) \) if and only if there is an ideal \( J \triangleleft R \) so that
\[
EL_d(J) \leq H \leq SL_d(J).
\]
The ideal \( J \) associated to the subgroup \( H \) is uniquely determined.

**Proof.** This theorem is contained in [Wil72, Vas81, BV84]. See also e.g. [Mag02, Theorem 11.15 and Corollary 11.16]. □

**Theorem 5.9** (Borevich–Vavilov). Assume that \( d \geq 3 \). Let \( I \triangleleft R \) be an ideal. Then
\[
Z(EL_d(R) / EL_d(I)) = \tilde{SL}_d(I) / EL_d(I).
\]

**Proof.** It is shown in [BV84, Theorem 4] that \( [EL_d(R), \tilde{SL}_d(I)] = EL_d(I) \). This gives containment in one direction. The containment in the opposite direction follows from the definition of the normal subgroup \( \tilde{SL}_d(I) \). □

**Virtually central subquotients.** Let \( I, K \triangleleft R \) be a pair of ideals with \( K^* = I \).

Consider the subquotient group
\[
A_d(I, K) = (\tilde{SL}_d(I) \cap EL_d(R)) / EL_d(K).
\]
Recall from [41] that a subgroup \( H \) of a given group \( G \) is virtually central if \(|H : H \cap Z(G)| < \infty \). The following fact will play an important role in our work.

**Theorem 5.10.** The subquotient \( A_d(I, K) \) is virtually central in the quotient group \( EL_d(R) / EL_d(K) \).

In particular the subquotient group \( A_d(I, K) \) is virtually abelian. Before proving Theorem 5.10 we need the following observation.
Proposition 5.11. If $K^* = I$ then $[\widetilde{SL}_d (I) : \widetilde{SL}_d (K)] < \infty$.

Proof. Consider the ring epimorphism $f : R/K \to R/I$ and the corresponding reduction map $\hat{f} : SL_d (R/K) \to SL_d (R/I)$. We know from Theorem 5.3 that

$$i := \left[ Z(SL_d (R/I)) : \hat{f}(Z(SL_d (R/K))) \right] < \infty.$$

It follows from the definition of the groups $\widetilde{SL}_d$ that

$$\widetilde{SL}_d (I) : \widetilde{SL}_d (K) \sim (\hat{f})^{-1} Z(SL_d (R/I)) Z(SL_d (R/K)).$$

We may conclude that

$$\left| \frac{\widetilde{SL}_d (I)}{SL_d (K)} \right| = \left| \frac{(\hat{f})^{-1} Z(SL_d (R/I))}{Z(SL_d (R/K))} \right| \leq i \left| \text{ker } \hat{f} \right| < \infty$$

as required. \qed

Proof of Theorem 5.10. Recall that $I, K \vartriangleleft R$ is a pair of ideals with $K^* = I$. Consider the following series of subgroups (see Lemma 5.5)

$$EL_d (K) \leq \widetilde{SL}_d (K) \leq \widetilde{SL}_d (I).$$

On the one hand, the theorem of Borevich and Vavilov (see Theorem 5.9 above) implies that

$$[EL_d (R), \widetilde{SL}_d (K)] = EL_d (K).$$

On the other hand, it was established in Proposition 5.11 that

$$[\widetilde{SL}_d (I) : SL_d (K)] < \infty.$$

The conclusion follows. \qed

6. Abelian subgroups of the group $EL_d (R)$

Let $R$ be a Noetherian ring. Let $d \in \mathbb{N}$ be a fixed integer with $d \geq 3$.

Horizontal and vertical subgroups. Fix an index $i \in \{1, \ldots, d\}$. The horizontal subgroup $H_i (R)$ of the matrix group $SL_d (R)$ is generated by the elementary subgroups $E_{i,j} (R)$ for all indices $j \in \{1, \ldots, d\} \setminus \{i\}$. Likewise, the vertical subgroup $V_i (R)$ is the subgroup generated by the elementary subgroups $E_{j,i} (R)$ for all indices $j \in \{1, \ldots, d\} \setminus \{i\}$. The commutation relations given in $\S 5\S$ imply that the subgroups $H_i (R)$ and $V_i (R)$ are all isomorphic to the additive group of the ring $R^{d-1}$. Observe that

$$H_i (R) \cap V_j (R) = E_{i,j} (R).$$

Proposition 6.1. The elementary groups satisfy the commutation relations

$$[E_{i,j} (R), H_k (R)] = E_{k,j} (R) \quad \text{and} \quad [E_{i,j} (R), V_k (R)] = E_{i,k} (R)$$

for every three distinct indices $i, j, k \in \{1, \ldots, d\}$. 

Proof. Let us prove the commutation relation involving the horizontal group $H_k(\mathcal{R})$. We have that
\[
[E_{i,j}(\mathcal{R}), H_k(\mathcal{R})] = [E_{i,j}(\mathcal{R}), E_{k,i}(\mathcal{R}) \prod_{l \notin \{i,k\}} E_{k,l}(\mathcal{R})] =
\]
\[
= [E_{i,j}(\mathcal{R}), E_{k,i}(\mathcal{R})] [E_{i,j}(\mathcal{R}), \prod_{l \notin \{i,k\}} E_{k,l}(\mathcal{R})]^E_{k,i}(\mathcal{R}) = E_{k,j}(\mathcal{R})
\]
as required. The proof of the second commutation relation involving the vertical group $V_k(\mathcal{R})$ is analogous. \[\square\]

**Normalizers of horizontal and vertical subgroups.** Fix an index $i \in \{1, \ldots, d\}$. The subgroup $N_i(\mathcal{R})$ of the matrix group $\text{SL}_d(\mathcal{R})$ is given by
\[
N_i(\mathcal{R}) = \{ g \in \text{SL}_d(\mathcal{R}) : g_{i,i} = 1 \text{ and } g_{i,j} = g_{j,i} = 0 \text{ for all } j \in \{1, \ldots, d\} \setminus \{i\} \}.
\]
The groups $N_i(\mathcal{R})$ are all isomorphic to the matrix group $\text{SL}_{d-1}(\mathcal{R})$. Observe that $N_i(\mathcal{R})$ normalizes both subgroups $H_i(\mathcal{R})$ and $V_i(\mathcal{R})$ for every index $i \in \{1, \ldots, d\}$.

Identifying the subgroup $N_i(\mathcal{R})$ with the matrix group $\text{SL}_{d-1}(\mathcal{R})$ and the two subgroups $H_i(\mathcal{R})$ and $V_i(\mathcal{R})$ with the abelian group $\mathcal{R}^{d-1}$ in the obvious way, conjugation corresponds to matrix multiplication. More precisely, given an element $n \in N_i(\mathcal{R})$ as well as a pair of elements $u \in H_i(\mathcal{R})$ and $v \in V_i(\mathcal{R})$ we have that
\[
n^{-1}un = n^t \cdot u \quad \text{and} \quad n^{-1}vn = n^{-1} \cdot v
\]
where "." denotes matrix multiplication with respect to the above identification and $n^t$ denotes the matrix transpose of $n$.

The full normalizer of the horizontal and vertical subgroups $H_i(\mathcal{R})$ and $V_i(\mathcal{R})$ is larger than $N_i(\mathcal{R})$ in general. More precisely we have the following.

**Proposition 6.2.** If $i \in \{1, \ldots, d\}$ is an index then
\[
N_{\text{SL}_d(\mathcal{R})}(V_i(\mathcal{R})) = \{ g \in \text{SL}_d(\mathcal{R}) : g_{i,j} = 0 \text{ for all } j \in \{1, \ldots, d\} \setminus \{i\} \}
\]
and
\[
N_{\text{SL}_d(\mathcal{R})}(H_i(\mathcal{R})) = \{ g \in \text{SL}_d(\mathcal{R}) : g_{j,i} = 0 \text{ for all } j \in \{1, \ldots, d\} \setminus \{i\} \}.
\]

Proof. We compute the normalizer of the subgroup $V_i(\mathcal{R})$. The proof for $H_j(\mathcal{R})$ is analogous. Denote $M_i = \{ g \in \text{SL}_d(\mathcal{R}) : g_{i,j} = 0 \text{ for all } j \neq i \}$. It is easy to verify that $M_i \leq N_{\text{SL}_d(\mathcal{R})}(V_i(\mathcal{R}))$. For the opposite inclusion, a direct computation shows that
\[
(g^{-1}E_{k,i}(1)g)_{i,j} = (g^{-1})_{i,k}g_{i,j} \quad \forall j, k \in \{1, \ldots, d\} \setminus \{i\}.
\]
Every element $g \in N_{\text{SL}_d(\mathcal{R})}(V_i(\mathcal{R}))$ must therefore satisfy $(g^{-1})_{i,k}g_{i,j} = 0$. In particular either $g_{i,j} = 0$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$ or $(g^{-1})_{i,k} = 0$ for all $k \in \{1, \ldots, d\} \setminus \{i\}$. The first case implies that $g \in M_i$ by definition. In the second case $g^{-1} \in M_i$ so that $g \in M_i$ as well since $M_i$ is a subgroup. \[\square\]

Note that if either $g \in N_{\text{SL}_d(\mathcal{R})}(V_i(\mathcal{R}))$ or $g \in N_{\text{SL}_d(\mathcal{R})}(H_i(\mathcal{R}))$ then $g_{i,i} \in O_\mathcal{R}$. 

Centralizers of elementary subgroups. We leave the direct verification of the following computation to the reader.

Proposition 6.3. Let \( x \in \mathcal{R} \) be any element. Denote \( \mathcal{I}_x = \text{Ann}_{\mathcal{R}}(x) \triangleleft \mathcal{R} \). If \( i, j \in \{1, \ldots, d\} \) is a pair of distinct indices then

\[
C_{\text{SL}_d(\mathcal{R})}(E_{i,j}(x)) = \left\{ g \in \text{SL}_d(\mathcal{R}) : \begin{array}{l} g_k,i = g_j,l = 0 \quad \forall k, l \in \{1, \ldots, d\}, k \neq i, l \neq j \\ \text{and} \quad g_{i,i} + \mathcal{I}_x = g_{j,j} + \mathcal{I}_x \end{array} \right\}.
\]

If \( g \in C_{\text{SL}_d(\mathcal{R})}(E_{i,j}(x)) \) then the reduction modulo the ideal \( \mathcal{I}_x = \text{Ann}_{\mathcal{R}}(x) \) of the diagonal entries \( g_{i,i} \) and \( g_{j,j} \) coincides and must be a unit in the quotient ring \( \mathcal{R}/\mathcal{I}_x \). In particular, if the element \( g \) centralizes all elementary subgroups \( E_{i,j}(\mathcal{R}) \) then \( g \) must be of the form \( u\text{Id}_d \) for some unit \( u \in \mathcal{O}^{[d]}_{\mathcal{R}} \).

Corollary 6.4. \( Z(\text{EL}_d(\mathcal{R})) = Z(\text{SL}_d(\mathcal{R})). \)

Proof. This follows from the above discussion combined with Proposition 5.2. \( \square \)

Corollary 6.5. The centralizers of the vertical and the horizontal subgroups are given by

\[
C_{\text{SL}_d(\mathcal{R})}(V_i(\mathcal{R})) = C_{\text{EL}_d(\mathcal{R})}(V_i(\mathcal{R})) = Z(\text{SL}_d(\mathcal{R})) \times V_i(\mathcal{R})
\]

and

\[
C_{\text{SL}_d(\mathcal{R})}(H_i(\mathcal{R})) = C_{\text{EL}_d(\mathcal{R})}(H_i(\mathcal{R})) = Z(\text{SL}_d(\mathcal{R})) \times H_i(\mathcal{R})
\]

for all indices \( i \in \{1, \ldots, d\} \).

Proof. For the centralizer in the group \( \text{SL}_d(\mathcal{R}) \) this follows by intersecting the statement of Proposition 6.3 over all values of the index \( j \in \{1, \ldots, d\} \setminus \{i\} \) and all ring elements \( x \in \mathcal{R} \). The centralizer in the group \( \text{EL}_d(\mathcal{R}) \) is the same according to Corollary 6.4. \( \square \)

Let \( \text{M}_d(\mathcal{R}) \) denote the abelian additive group of \( d \)-by-\( d \) matrices with entries in the ring \( \mathcal{R} \).

Proposition 6.6. Assume that the zero ideal \((0) \triangleleft \mathcal{R} \) is a depth ideal. Let \( g \in \text{SL}_d(\mathcal{R}) \setminus C_{\text{SL}_d(\mathcal{R})}(E_{i,j}(1)) \) be any element for some pair of distinct indices \( i, j \in \{1, \ldots, d\} \). Then there is a sequence of elements \( x_n \in E_{i,j}(\mathcal{R}) \) such that

1. the commutators \([g, x_n]\) are pairwise distinct modulo \( Z(\text{SL}_d(\mathcal{R})) \),
2. if either \( g \in N_{\text{SL}_d(\mathcal{R})}(H_i(\mathcal{R})) \) or \( g \in N_{\text{SL}_d(\mathcal{R})}(V_j(\mathcal{R})) \) then the commutators \([g, x_n]\) belong to \( H_i(\mathcal{R}) \) or \( V_j(\mathcal{R}) \) respectively, and
3. if either \( g \in V_k(\mathcal{R}) \) or \( g \in H_k(\mathcal{R}) \) for some index \( k \in \{1, \ldots, d\} \setminus \{i, j\} \) then moreover the commutators \([g, x_{m}^{-1} x_n]\) for all \( n, m \in \mathbb{N} \) with \( n < m \) are pairwise distinct.

Proof. Let \( X \) and \( Y \) be the matrices in \( \text{M}_d(\mathcal{R}) \) given by

\[
X_{k,l} = \begin{cases} \begin{array}{ll} g_{j,l} & k = i, \\ 0 & \text{otherwise} \end{array} \end{cases}
\]

and

\[
Y_{k,l} = \begin{cases} \begin{array}{ll} g_{k,i} & l = j, \\ 0 & \text{otherwise} \end{array} \end{cases}
\]
for all indices $k, l \in \{1, \ldots, d\}$. A direct computation shows that
\[ E_{i,j}(r)gE_{i,j}^{-1}(r) = g + r(X - Y) \]
for all ring elements $r \in \mathcal{R}$. The assumption that $g \notin C_{\text{SL}_d(\mathcal{R})}(E_{i,j}(1))$ implies that $X \neq Y$. In light of Proposition 6.4 there exists a sequence of elements $r_n \in \mathcal{R}$ such that the matrices $r_n(X - Y)$ are pairwise distinct for all $n \in \mathbb{N}$. In fact, as $d \geq 3$ and the matrix $X - Y$ has at most two non-zero rows and columns, the matrices $r_n(X - Y)$ must be pairwise distinct modulo the center $Z(\text{SL}_n(\mathcal{R}))$ as well. Take $x_n = E_{i,j}(r_n)$. This concludes the proof of Item (1).

To establish Item (2) assume that either $g \in N_{\text{SL}_d(\mathcal{R})}(H_i(\mathcal{R}))$ or $g \in N_{\text{SL}_d(\mathcal{R})}(V_j(\mathcal{R}))$. The same proof as above goes through. In addition, the commutators $[g, x_n]$ all belong to the abelian subgroups $H_i(\mathcal{R})$ or $V_j(\mathcal{R})$, respectively.

Finally assume that $g \in V_k(\mathcal{R})$ for some index $k \in \{1, \ldots, d\} \setminus \{i, j\}$. The proof in the case of $g \in H_k(\mathcal{R})$ is analogous. A direct computation shows that
\[ [E_{i,j}(r), g] = E_{i,k}(rg_{j,k} - g_{i,k}) \]
for all ring elements $r \in \mathcal{R}$. The assumption $g \notin C_{\text{SL}_d(\mathcal{R})}(E_{i,j}(1))$ implies that $g_{j,k} \neq 0$. According to Proposition 6.4 there is a sequence of elements $r_n \in \mathcal{R}$ such that the elements $r_ng_{j,k} \in \mathcal{R}$ are pairwise distinct for all $n \in \mathbb{N}$. Up to passing to a subsequence and reindexing, we may assume that the elements $(r_n - r_m)g_{j,k}$ are pairwise distinct for all $n < m$. The elements $x_n = E_{i,j}(r_n)$ satisfy Item (3). \qed

The assumption that (0) is a depth ideal of the ring $\mathcal{R}$ is equivalent to every non-zero ideal $\mathcal{I} < \mathcal{R}$ satisfying $|\mathcal{I}| = \infty$. This in turn is equivalent to the statement that $\mathcal{R} = \mathbb{Z}[x_1, \ldots, x_k]/\mathcal{J}$ for some $k \in \mathbb{N}$ and some depth ideal $\mathcal{J}$.

**Depth ideals and centralizers.**

**Proposition 6.7.** Let $\mathcal{I}, \mathcal{J}$ and $\mathcal{L}$ be ideals in the ring $\mathcal{R}$ satisfying
\[ \mathcal{L} \subset \mathcal{J} \subset \mathcal{I}, \quad |\mathcal{I}/\mathcal{J}| < \infty \quad \text{and} \quad |\mathcal{J}/\mathcal{L}| = \infty. \]
Let $g = E_{j,k}(s)$ be any element where $s \in \mathcal{I} \setminus \mathcal{J}$ and $j, k \in \{1, \ldots, d\}$ are distinct indices. Then there are elements $x_n \in \text{EL}_d(\mathcal{R})$ such that the commutators $[g, x_n]$ belong to $\text{SL}_d(\mathcal{J})$ and are pairwise distinct modulo the subgroup $\text{SL}_d(\mathcal{L})$ for all $n \in \mathbb{N}$.

**Proof.** Fix an index $i \in \{1, \ldots, d\} \setminus \{j, k\}$. Consider the ideal $\mathcal{I}_0 = \mathcal{R}s + \mathcal{J}$ in the ring $\mathcal{R}$. In particular $\mathcal{J} \subset \mathcal{I}_0 \subset \mathcal{I}$. The two ideal quotients $\mathcal{J} : \mathcal{I}_0$ and $\mathcal{L} : \mathcal{I}_0$ satisfy
\[ |(\mathcal{J} : \mathcal{I}_0)/(\mathcal{L} : \mathcal{I}_0)| = \infty \]
according to Proposition 2.12. Therefore there exists a sequence $q_n \in \mathcal{J} : \mathcal{I}_0$ of elements pairwise distinct modulo $\mathcal{L} : \mathcal{I}_0$. Up to passing to a further subsequence and reindexing, we may assume that $q_n - q_m$ are pairwise distinct modulo $\mathcal{L} : \mathcal{I}_0$ for all $n < m$ as well. Denote $y_n = E_{i,j}(q_n)$ for all $n \in \mathbb{N}$.

The $\text{SL}_d(\mathcal{R})$-action by conjugation on the quotient group $\text{SL}_d(\mathcal{I})/\text{SL}_d(\mathcal{J})$ has finite orbits. Up to passing to a further subsequence and reindexing, and by the pigeon hole principle, we may assume that
\[ y_n^{-1}g_{n}^{-1}\text{SL}_d(\mathcal{J}) = y_m^{-1}g_{m}^{-1}\text{SL}_d(\mathcal{J}) \]
for all $n, m \in \mathbb{N}$. Consider the elements
\[ r_n = q_n - q_1 \in \mathcal{J} : \mathcal{I}_0 \quad \text{and} \quad x_n = y_ny_1^{-1} = E_{i,j}(r_n) \in E_{i,j}(\mathcal{R}). \]
In particular it follows that \([g, x_n] \in \text{SL}_d(\mathcal{J})\) for all \(n \in \mathbb{N}\).

It remains to show that the commutators \([g, x_n]\) are pairwise distinct modulo the subgroup \(\overline{\text{SL}}_d(\mathcal{L})\). The ring elements \(r_n\) are clearly distinct modulo the ideal quotient \(\mathcal{L} : \mathcal{I}_0\). We claim that the ring elements \(r_n s\) are pairwise distinct modulo the ideal \(\mathcal{L}\). Indeed, if \(r_n s = r_m s\) modulo the ideal \(\mathcal{L}\) for some \(n, m \in \mathbb{N}\) then \((r_n - r_m)\mathcal{I}_0 \subset \mathcal{L}\) and \(r_n - r_m \in \mathcal{L} : \mathcal{I}_0\), which is a contradiction. Finally, let \(X\) and \(Y\) be the two matrices in \(\text{M}_d(\mathcal{R})\) given by

\[
X_{l,m} = \begin{cases} 
g_{j,m} & l = i, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
Y_{l,m} = \begin{cases} 
g_{l,i} & m = j, \\
0 & \text{otherwise}
\end{cases}
\]

for all indices \(l, m \in \{1, \ldots, d\}\). Note that \(s = (X - Y)_{k,j} \in \mathcal{I} \setminus \mathcal{J}\). A direct computation shows that

\([g, x_n] = (g + r_n(X - Y))g^{-1}\)

for all \(n \in \mathbb{N}\), so that these commutators are pairwise distinct modulo \(\overline{\text{SL}}_d(\mathcal{L})\). □

**A normal form decomposition.** The following result is inspired by [Bek07, Proposition 13]. It shows that every element of \(\text{SL}_d(\mathcal{R})\) is conjugate to a product of three matrices of a particularly simple form. The proof makes an essential use of the notion of stable range and in particular of the fact that \(d > \text{sr}(\mathcal{R})\).

**Proposition 6.8.** Assume that \(d > \text{sr}(\mathcal{R})\). Then any element \(g \in \text{SL}_d(\mathcal{R})\) is conjugate by an element of \(\text{EL}_d(\mathcal{R})\) to an element \(g'\) where

\[g' = hvh'v' n\]

for some elements

\(h, h' \in H_1(\mathcal{R}), \ v, v' \in V_1(\mathcal{R})\) and \(n \in N_1(\mathcal{R})\).

Moreover we may assume that \(h' = E_{1,2}(-1)\).

The pair of implicit indices \(i = 1\) and \(j = 2\) in the above statement can be replaced with any fixed pair of distinct indices \(i, j \in \{1, \ldots, d\}\).

**Proof of Proposition 6.8** Let \(g \in \text{SL}_d(\mathcal{R})\) be any element. If \(j \in \{2, \ldots, d\}\) and \(i \in \{1, \ldots, d\} \setminus \{j\}\) is a pair of distinct indices and \(s \in \mathcal{R}\) is any ring element then

\[(E_{i,j}(s)gE_{i,j}(s)^{-1})_{i,1} = g_{i,1} + sg_{j,1} \cdot 1.
\]

Since \(d > \text{sr}(\mathcal{R})\) the element \(g\) can be conjugated by an appropriate element of the vertical group \(V_2(\mathcal{R})\) in so that the first column \(r_1, \ldots, r_d \in \mathcal{R}^d\) of the resulting matrix is such that \(r_1, r_3, \ldots, r_d\) form a unimodular \((d - 1)\)-tuple. It is therefore possible to find elements \(s_1, s_3, \ldots, s_n \in \mathcal{R}\) satisfying

\[s_1 r_1 + s_3 r_3 + \cdots + s_d r_d = 1 - r_1 - r_2.\]

Up to further conjugating by the element \(E_{2,3}(s_3) \cdots E_{2,d}(s_d) \in H_2(\mathcal{R})\) we may assume that the resulting element \(g'\) satisfies

\[(1 + s_1)g'_{1,1} + g'_{2,1} = 1.\]
Consider the elements $v_1 = E_{2,1}(s_1) \in V_1(\mathcal{R})$ and $h_1 = E_{1,2}(1) \in H_1(\mathcal{R})$. Denote
\[ g'' = h_1v_1g' \]
so that $g''_{1,1} = 1$. Therefore there are suitable elements $v_2 \in V_1(\mathcal{R})$ and $h_2 \in H_1(\mathcal{R})$ so that
\[ n = v_2g''h_2 \in N_1(\mathcal{R}). \]
Rearranging the above equations gives
\[ v_1^{-1}h_1^{-1}v_2^{-1}n = g'h_2. \]
The desired statement follows for a suitable choice of the elements $h, v, h', v'$ and up to replacing the element $g'$ by its conjugate $h_2^{-1}g'h_2$. \qed

7. THE LEVEL IDEAL $\mathcal{I}_\varphi$

Let $\mathcal{R}$ be a Noetherian ring and $d \geq 3$ be a fixed integer. We associate to each character $\varphi \in \text{Ch}(EL_d(\mathcal{R}))$ a uniquely determined depth ideal $\mathcal{I}_\varphi \triangleleft \mathcal{R}$ called its level ideal, see Theorem \ref{levelideal}. It is shown in Theorem \ref{levelideal} below that the character $\varphi$ is induced from the normal subgroup $EL_d(\mathcal{R}) \cap \tilde{\mathcal{S}}L_d(\mathcal{I}_\varphi)$ provided $d > \text{sr}(\mathcal{R})$.

Restrictions to abelian subgroups. Let $\varphi \in \text{Tr}(EL_d(\mathcal{R}))$ be a trace. The restrictions $\varphi|_{H_i(\mathcal{R})}$ and $\varphi|_{V_i(\mathcal{R})}$ of the trace $\varphi$ to the horizontal and vertical abelian subgroups $H_i(\mathcal{R})$ and $V_i(\mathcal{R})$ are positive definite functions for all $i \in \{1, \ldots, d\}$. Bochner’s theorem \cite[Theorem 4.18]{Bochner} says that there are uniquely determined probability measures $\mu^{H_i}$ and $\mu^{V_i}$ on the dual compact abelian groups $\widehat{H}_i(\mathcal{R})$ and $\widehat{V}_i(\mathcal{R})$ respectively such that
\[ \varphi^{H_i} = \mu^{\widehat{H}_i} \quad \text{and} \quad \varphi^{V_i} = \mu^{\widehat{V}_i} \]
for all indices $i \in \{1, \ldots, d\}$.

The conjugation invariance of the trace $\varphi$ implies that $\varphi(g) = \varphi(g^n)$ for all elements $g \in EL_d(\mathcal{R})$ and $n \in N_i(\mathcal{R})$. It follows that the probability measures $\mu^{H_i}$ and $\mu^{V_i}$ are invariant under the dual action corresponding to conjugation by the normalizing group $N_i(\mathcal{R})$ for all $i \in \{1, \ldots, d\}$. Both dual groups $\widehat{H}_i(\mathcal{R})$ and $\widehat{V}_i(\mathcal{R})$ can all be identified with the dual group $\widehat{\mathcal{R}}^{d-1}$ with its additive structure. With this identification the dual action of $N_i(\mathcal{R})$ is the natural one via matrix multiplication.

We are now in a position to apply our invariant measure classification result, namely Theorem \ref{invariantmeasure}. We deduce that the two probability measures $\mu^{H_i}$ and $\mu^{V_i}$ can be uniquely written as convex combinations
\[ \mu^{H_i} = \sum_{\mathcal{I} \triangleleft \mathcal{R}, \mathcal{I} = \mathcal{I}} \alpha^{H_i}_{\mathcal{I}} \mu^{\widehat{H}_i}_{\mathcal{I}} \quad \text{and} \quad \mu^{V_i} = \sum_{\mathcal{I} \triangleleft \mathcal{R}, \mathcal{I} = \mathcal{I}} \alpha^{V_i}_{\mathcal{I}} \mu^{\widehat{V}_i}_{\mathcal{I}}, \]
ranging over all depth ideals $\mathcal{I} \triangleleft \mathcal{R}$ and with coefficients $\alpha^{H_i}_{\mathcal{I}} \geq 0$ and $\alpha^{V_i}_{\mathcal{I}} \geq 0$ satisfying $\sum \alpha^{H_i}_{\mathcal{I}} = \sum \alpha^{V_i}_{\mathcal{I}} = 1$ for all $i \in \{1, \ldots, d\}$. For a given depth ideal $\mathcal{I} \triangleleft \mathcal{R}$ each probability measure $\mu^{H_i}_{\mathcal{I}}$ and $\mu^{V_i}_{\mathcal{I}}$ is a convex combination of a countable family of translates of the Haar measure of the compact dual group $(\mathcal{I}^0)^d \leq \widehat{\mathcal{R}}^d$. 
**Projection valued measures.** Let \( \varphi \in \text{Tr}(\text{EL}_d(\mathcal{R})) \) be a trace. Consider the GNS construction \((\pi_\varphi, \mathcal{H}_\varphi, \nu_\varphi)\) associated to the trace \( \varphi \) as in Theorem 4.1. Namely \( \pi_\varphi \) is a unitary representation of the group \( \text{EL}_d(\mathcal{R}) \) on the Hilbert space \( \mathcal{H}_\varphi \) admitting a cyclic vector \( \nu_\varphi \in \mathcal{H}_\varphi \) and satisfying

\[
\varphi(g) = \langle \pi_\varphi(g) \nu_\varphi, \nu_\varphi \rangle \quad \forall g \in \text{EL}_d(\mathcal{R}).
\]

The restriction of the unitary representation \( \pi_\varphi \) to each horizontal and vertical subgroup gives rise to unique projection-valued measures \( P^{H_i} \) and \( P^{V_i} \) on the dual abelian compact groups \( \hat{H}_i(\mathcal{R}) \) and \( \hat{V}_i(\mathcal{R}) \) respectively such that

\[
\pi_\varphi|_{H_i} = \int_{\hat{H}_i(\mathcal{R})} \chi \, dP^{H_i}(\chi) \quad \text{and} \quad \pi_\varphi|_{V_i} = \int_{\hat{V}_i(\mathcal{R})} \chi \, dP^{V_i}(\chi).
\]

To proceed with the analysis of the projection valued measures \( P^{H_i} \) and \( P^{V_i} \), we extend the discussion to the elementary groups \( E_{i,j}(\mathcal{R}) \) defined for each pair of distinct indices \( i, j \in \{1, \ldots, d\} \). Consider the projection valued measures \( P^{E_{i,j}} \) on the dual abelian compact groups \( \hat{E}_{i,j}(\mathcal{R}) \cong \hat{\mathcal{R}} \) satisfying

\[
\pi_\varphi|_{E_{i,j}(\mathcal{R})} = \int_{\hat{E}_{i,j}(\mathcal{R})} \chi \, dP^{E_{i,j}}(\chi).
\]

It is an elementary fact of harmonic analysis that

\[
\hat{H}_i(\mathcal{R})/E_{i,j}(\mathcal{R})^0 \cong \hat{V}_j(\mathcal{R})/E_{i,j}(\mathcal{R})^0 \cong E_{i,j}(\mathcal{R}).
\]

Let \( p^{H_i} : \hat{H}_i(\mathcal{R}) \to E_{i,j}(\mathcal{R}) \) and \( p^{V_j} : \hat{V}_j(\mathcal{R}) \to E_{i,j}(\mathcal{R}) \) denote the resulting quotient maps.

**Proposition 7.1.** \( p^{H_i} P^{H_i} = p^{V_j} P^{V_j} = P^{E_{i,j}} \) as projection-valued measures on the dual abelian compact group \( E_{i,j}(\mathcal{R}) \) for all pairs of distinct indices \( i, j \in \{1, \ldots, d\} \).

**Proof.** Fix a pair of distinct indices \( i, j \in \{1, \ldots, d\} \). We will prove the statement with respect to the quotient map \( p^{H_i} : \hat{H}_i(\mathcal{R}) \to E_{i,j}(\mathcal{R}) \). The proof for vertical subgroups is essentially the same. For every pair of vectors \( u, w \in \mathcal{H}_\varphi \) consider the complex-valued measures \( P^{H_i}_{u,w} \) and \( P^{E_{i,j}}_{u,w} \) respectively on the dual compact groups \( \hat{H}_i(\mathcal{R}) \) and \( \hat{E}_{i,j}(\mathcal{R}) \) given by

\[
P^{H_i}_{u,w}(E) = \langle P^{H_i}(E)u, w \rangle \quad \text{and} \quad P^{E_{i,j}}_{u,w}(F) = \langle P^{E_{i,j}}(F)u, w \rangle
\]

for all Borel subsets \( E \subset \hat{H}_i(\mathcal{R}) \) and \( F \subset \hat{E}_{i,j}(\mathcal{R}) \). The desired conclusion \( p^{H_i} P^{H_i} = P^{E_{i,j}} \) holds if and only if \( P^{H_i}_{u,w} = P^{E_{i,j}}_{u,w} \) for all vectors \( u, v \in \mathcal{H}_\varphi \). This last statement follows from the uniqueness part of Bochner’s theorem [Fol94, Theorem 4.18] combined with the observation that

\[
\int_{E_{i,j}(\mathcal{R})} \langle \chi(g)u, w \rangle \, dP^{E_{i,j}}_{u,w}(\chi) = \langle \pi(g)u, w \rangle = \int_{\hat{H}_i(\mathcal{R})} \langle \chi(g)u, w \rangle \, dP^{H_i}_{u,w}(\chi) = \int_{E_{i,j}(\mathcal{R})} \langle \chi(g)u, w \rangle \, dP^{H_i} P^{H_i}_{u,w}(\chi)
\]

for all elements \( g \in E_{i,j}(\mathcal{R}) \leq H_i(\mathcal{R}) \). \( \square \)
Determination of the level ideal. The following result is inspired by §
Lemma 10.

**Theorem 7.2.** Let \( \varphi \in \text{Ch}(\text{EL}_d(\mathcal{R})) \) be a character. Then there is a unique depth ideal \( \mathcal{I}_\varphi \lhd \mathcal{R} \) such that \( \mu^V_i = \mu^H_i \) and \( \mu^V_i = \mu^V_i \) for all indices \( i \in \{1, \ldots, d\} \).

The ideal \( \mathcal{I}_\varphi \) is the level ideal corresponding to the character \( \varphi \). It plays an important role towards our main classification result Theorem 1.1.

Before giving the proof of Theorem 7.2 let us introduce some useful notation. Let \( \mathcal{I} \lhd \mathcal{R} \) be a depth ideal. Consider the Borel subset

\[
\mathcal{O}_I^H = \left( \bigcup_{J \subset \mathcal{R}, J^* = \mathcal{I}} H_i(J)^0 \right) \setminus \left( \bigcup_{J \subset \mathcal{R}, J \subsetneq J^*} H_i(J)^0 \right)
\]

of the dual compact group \( H_i(\mathcal{R}) \). The Borel subsets \( \mathcal{O}_I^{V_i} \subset V_i(\mathcal{R}) \) and \( \mathcal{O}_I^{E_{i,j}} \subset E_{i,j}(\mathcal{R}) \) are defined analogously for all pairs of distinct indices \( i, j \in \{1, \ldots, d\} \). Observe that

\[
p^H_i(\mathcal{O}_I^H) = p^{V_i}(\mathcal{O}_I^{V_i}) = \mathcal{O}_I^{E_{i,j}}.
\]

The same argument as in the discussion on “essential subgroups” in \( \text{Bek07} \) shows that \( \nu^H_i(\mathcal{O}_I^H) = \nu^{V_i}(\mathcal{O}_I^{V_i}) = 1 \). Moreover the Borel subsets \( \mathcal{O}_I^H \) as well as \( \mathcal{O}_I^{V_i} \) are pairwise disjoint for a fixed index \( i \in \{1, \ldots, d\} \) as the ideal \( \mathcal{I} \) varies over the different depth ideals of the ring \( \mathcal{R} \).

**Proof of Theorem 7.2.** Let \( (\pi_\varphi, \mathcal{H}_\varphi, \mathcal{V}_\varphi) \) be the GNS construction associated to the character \( \varphi \). Let \( \mathcal{P}^H_i, \mathcal{P}^{V_i} \) and \( \mathcal{P}^{E_{i,j}} \) be the projection valued measures on the dual compact abelian groups \( \mathcal{H}_i(\mathcal{R}), \mathcal{V}_i(\mathcal{R}) \) and \( \mathcal{E}_{i,j}(\mathcal{R}) \) respectively introduced above for each pair of distinct indices \( i, j \in \{1, \ldots, d\} \).

Let \( \mathcal{I} \lhd \mathcal{R} \) be any fixed depth ideal. It follows from Proposition 7.1 and from the paragraph preceding this proof that

\[
\mathcal{P}^H_i(\mathcal{O}_I^H) = \mathcal{P}^{V_i}(\mathcal{O}_I^{V_i}) = \mathcal{P}^{E_{i,j}}(\mathcal{O}_I^{E_{i,j}})
\]

for all pairs of distinct indices \( i, j \in \{1, \ldots, d\} \). Repeating this argument with varying \( i \)'s and \( j \)'s shows that the orthogonal projection

\[
\mathcal{P}_I = \mathcal{P}^{E_{i,j}}(\mathcal{O}_I^{E_{i,j}})
\]

is independent of both indices. The representation theory of locally compact abelian groups \( \text{Fol94} \) Theorem 4.44 implies that \( \mathcal{P}_I \in \pi_\varphi(\text{EL}_d(\mathcal{R}))' \) for all pairs of distinct indices \( i, j \in \{1, \ldots, d\} \). As the elementary groups generate the group \( \text{EL}_d(\mathcal{R}) \) we deduce that

\[
\mathcal{P}_I \in \pi_\varphi(\text{EL}_d(\mathcal{R}))'.
\]

The spectral theorem \( \text{Fol94} \) Theorem 1.44 shows that any bounded operator \( B \in \pi_\varphi(\text{EL}_d(\mathcal{R}))' \) necessarily commutes with \( \mathcal{P}^{E_{i,j}}(E) \) for every Borel subset \( E \subset E_{i,j}(\mathcal{R}) \) and every pair of distinct \( i \) and \( j \). In particular

\[
\mathcal{P}_I \in \pi_\varphi(\text{EL}_d(\mathcal{R})).
\]

We conclude that the projection \( \mathcal{P}_I \) lies in the center \( Z(\mathcal{L}_\varphi) \) of the von Neumann algebra \( \mathcal{L}_\varphi \) generated by the unitary representation \( \pi_\varphi \). Since \( \varphi \) is a character the von Neumann algebra \( \mathcal{L}_\varphi \) is a factor. Therefore the projection \( \mathcal{P}_I \) must be either the zero or the identity operator for each given depth ideal \( \mathcal{I} \lhd \mathcal{R} \).
The Borel subsets $O^{E_{i,j}}_I$ form a Borel partition of the dual compact group $E_{i,j}(R)$ as $I$ varies over all depth ideals in the ring $R$ and for each fixed pair of distinct indices $i, j \in \{1, \ldots, d\}$. Therefore there is a unique depth ideal $I_\varphi < R$ such that $P_{I_\varphi}$ is the identity operator on the Hilbert space $H_\varphi$. The conclusion follows. □

**Properties of the level ideal.** Recall the notation $\text{Ind}_{G} (H; N)$ introduced in §4 — a trace $\varphi$ belongs to $\text{Ind}_{G} (H; N)$ if and only if the restriction of $\varphi$ to the subgroup $H$ is induced from the subgroup $N$.

**Proposition 7.3.** If $\varphi \in \text{Ch} (E_{i,j}(R))$ is a character with level ideal $I_\varphi < R$ then

$$\varphi \in \text{Ind}_{\text{EL}_d(R)} (V_i(R); V_i(I_\varphi)) \cap \text{Ind}_{\text{EL}_d(R)} (H_i(R); H_i(I_\varphi))$$

for all indices $i \in \{1, \ldots, d\}$.

**Proof.** According to Theorem 7.2 the level ideal $I_\varphi$ is such that $\varphi|_{H_i} = \hat{\nu}^{H_i}_{I_\varphi}$ and $\varphi|_{V_i} = \nu^{V_i}_{I_\varphi}$. The desired conclusion follows immediately from Proposition 3.9. □

**Proposition 7.4.** Let $J < R$ be an ideal. Let $\varphi \in \text{Tr} (E_{i,j}(R))$ be a trace satisfying

$$\varphi \in \text{Ind}_{\text{EL}_d(R)} (E_{i,j}(R); E_{i,j}(J))$$

for a pair of distinct indices $i, j \in \{1, \ldots, d\}$. Write

$$\varphi = \int_{\text{Ch}(E_{i,j}(R))} \psi d\mu_\varphi(\psi)$$

where $\mu_\varphi$ is a Borel probability measure on $\text{Ch}(E_{i,j}(R))$. If $J$ is contained in the level ideal $I_\psi$ of $\mu_\varphi$-almost every character $\psi$ then $J^* = I_\psi$ holds true $\mu_\varphi$-almost surely.

**Proof.** Let $\mu^{E_{i,j}}$ be the uniquely determined Borel probability measure on the dual compact abelian group $E_{i,j}(R)$ such that $\varphi|_{E_{i,j}} = \hat{\mu}^{E_{i,j}}$. Let $\nu_\varphi$ denote the Haar measure of the annihilator subgroup $I_0$ of a given depth ideal $I < R$ regarded as a Borel probability measure on $E_{i,j}(R)$ $\cong \hat{R}$.

There is a family of atomic positive measures $\eta_\varphi$ on the dual compact abelian group $E_{i,j}(R)$ defined for every depth ideal $I < R$ with $J^* \subset I$ that satisfy $\eta_\varphi (E_{i,j}(R) \setminus O^{E_{i,j}}_I) = 0$ and

$$\mu^{E_{i,j}} = \sum_{I \text{ depth}, J^* \subset I} \eta_\varphi * \nu_\varphi.$$

In particular $\sum \eta_\varphi = 1$ where (as above) the sum is taken over all depth ideals $I < R$ with $J^* \subset I$.

On the other hand, we may consider the Borel probability measure $\theta$ on the dual compact group $E_{i,j}(R)$ given by

$$\theta = \sum_{I \text{ depth}, J^* \subset I} \eta_\varphi * \nu_\varphi.$$

Note that $\hat{\theta}(g) = 0$ for all elements $g \in E_{i,j}(R) \setminus E_{i,j}(J)$ by Proposition 3.9. Additionally

$$\tilde{\theta}(g) = \sum_{I \text{ depth}, J^* \subset I} \int_{E_{i,j}(R)} \chi(g) d\eta_\varphi = \tilde{\mu}^{E_{i,j}}(g) = \varphi(g)$$
for all elements $g \in E_{i,j}(\mathcal{J})$. We deduce that $\hat{\theta} = \mu^{E_{i,j}}$ overall as functions on the group $E_{i,j}(\mathcal{R})$. The uniqueness statement in Bochner’s theorem \cite[Theorem 4.18]{Fol94} implies that $\theta = \mu^{E_{i,j}}$. Therefore $|\eta_{\mathcal{I}_\mathcal{J}}| = 1$ and $\eta_{\mathcal{I}_\mathcal{J}} = 0$ for every other depth ideal $\mathcal{I}$. This is equivalent to the desired conclusion. □

The last result of the current §7 will be used to prove the converse direction of our main result, namely Theorem 1.2 of the introduction.

**Corollary 7.5.** Let $\varphi \in \text{Tr}(EL_d(\mathcal{R}))$ be a trace with

$$\varphi = \int_{\text{Ch}(EL_d(\mathcal{R})))} \psi \, d\mu_\varphi(\psi)$$

for some Borel probability measure $\mu_\varphi$ on $\text{Ch}(EL_d(\mathcal{R}))$. Let $K \triangleleft \mathcal{R}$ be an ideal satisfying $EL_d(K) \leq \ker \varphi \leq SL_d(K)$. If

$$\varphi \in \text{Ind}_{EL_d(\mathcal{R})}(EL_d(\mathcal{R}); SL_d(K^*))$$

then the level ideal $\mathcal{I}_\psi$ of $\mu_\varphi$-almost every character $\psi$ is equal to $K^*$.

**Proof.** It is a general property of traces that $\|\varphi\|_\infty \leq 1$. Therefore $\ker \varphi \leq \ker \psi$ holds true for $\mu_\varphi$-almost every character $\psi \in \text{Ch}(EL_d(\mathcal{R}))$. It follows that $K$ is contained in the kernel ideal $K_\psi$ and so $K^* \subset K_\psi \subset I_\psi$ holds true $\mu_\varphi$-almost surely. \footnote{We will show in §8 the much more precise statement $K^*_\psi = I_\psi$. This is not needed yet for the current argument.} The statement now follows from Proposition 7.4. □

8. **The kernel ideal $K_\varphi$**

Let $\mathcal{R}$ be a Noetherian ring. Let $d > \max\{2, \text{sr}(\mathcal{R})\}$ be a fixed integer. Let $\varphi \in \text{Tr}(EL_d(\mathcal{R}))$ be a trace. The kernel $\ker \varphi$ of the trace $\varphi$ is given by

$$\ker \varphi = \{g \in EL_d(\mathcal{R}) : \varphi(g) = 1\}.$$

The kernel $\ker \varphi$ is a normal subgroup of the group $EL_d(\mathcal{R})$, see Proposition 4.4. According to the normal subgroup structure theorem (see Theorem 5.8) there is a uniquely determined ideal $K_\varphi \triangleleft \mathcal{R}$ satisfying

$$EL_d(K_\varphi) \leq \ker \varphi \leq SL_d(K_\varphi).$$

We will say that $K_\varphi$ is the kernel ideal associated to the trace $\varphi$.

**Theorem 8.1.** Let $\varphi \in \text{Ch}(EL_d(\mathcal{R}))$ be a character with level ideal $\mathcal{I}_\varphi$ and kernel ideal $K_\varphi$. Then $K^*_\varphi = I_\varphi$.

Theorem 8.1 is main goal of the current §8. Clearly $K_\varphi \subset I_\varphi$ so that $K^*_\varphi = I_\varphi$ is equivalent to the statement $|I_\varphi/K_\varphi| < \infty$. It is proved below in two parts, making use of two additional auxiliary ideals $D_\varphi$ and $N_\varphi$ associated to the character $\varphi$.

**Remark 8.2.** The kernel ideal is well-defined for any trace on the group $EL_d(\mathcal{R})$ but the level ideal is defined only for characters.
The supporting ideal \( N_\varphi \). Let \( \varphi \in \text{Ch}(\text{SL}_d(R)) \) be a character with level ideal \( \mathcal{I}_\varphi \lhd \mathcal{R} \) and kernel ideal \( \mathcal{K}_\varphi \lhd \mathcal{R} \). We define a certain ideal \( N_\varphi \lhd \mathcal{R} \) associated to the character \( \varphi \) and satisfying \( N_\varphi^* = \mathcal{I}_\varphi \). This is an auxiliary notion needed towards our proof of Theorem \[8.1\].

Lemma 8.3 (Bekka). There exists an ideal \( N_\varphi \lhd \mathcal{R} \) with \( N_\varphi^* = \mathcal{I}_\varphi \) and \( N_\varphi^2 \subset \mathcal{K}_\varphi \).

This Lemma and its proof are essentially the same as \[Bek07\, \text{Lemmas 10 and 11}\].

Proof of Lemma 8.3. Let \((\pi_\varphi, \mathcal{H}_\varphi, v_\varphi)\) be the GNS construction corresponding to the character \( \varphi \) as given in Theorem \[1.1\]. Recall that \( \pi_\varphi \) is a unitary representation of the group \( \text{EL}_d(\mathcal{R}) \) acting on the Hilbert space \( \mathcal{H}_\varphi \) with cyclic vector \( v_\varphi \in \mathcal{H}_\varphi \) and such that

\[
\varphi(g) = (\pi_\varphi(g)v_\varphi, v_\varphi) \quad \forall g \in \text{EL}_d(\mathcal{R}).
\]

The argument of \[Bek07\, \text{Lemma 10}\] shows that there is an ideal \( N_\varphi \lhd \mathcal{R} \) satisfying \( N_\varphi^* = \mathcal{I}_\varphi \) and such that the group \( F_d(N_\varphi) \) admits non-zero invariant vectors in the Hilbert space \( \mathcal{H}_\varphi \).

Indeed, Bekka’s proof is given for the ring \( \mathcal{Z} \) and the countable collection of non-zero ideals in \( \mathcal{Z} \). It extends mutatis mutandis to our situation, by considering the level ideal \( \mathcal{I}_\varphi \) and the countable collection of ideals \( \mathcal{J} \lhd \mathcal{R} \) with \( \mathcal{J}^* = \mathcal{I}_\varphi \). One has to use the fact that a pair of ideals \( \mathcal{J}_1, \mathcal{J}_2 \lhd \mathcal{R} \) with \( \mathcal{J}_1^* = \mathcal{J}_2^* = \mathcal{I}_\varphi \) satisfies \( (\mathcal{J}_1 \cap \mathcal{J}_2)^* = \mathcal{I}_\varphi \), see Proposition \[2.7\].

Recall that \( \text{EL}_d(\mathcal{N}_\varphi^2) \leq F_d(N_\varphi) \) according to Lemma \[5.6 \]. Therefore the Hilbert subspace \( \mathcal{H}_0 \leq \mathcal{H}_\varphi \) consisting of the \( \pi_\varphi(\text{EL}_d(\mathcal{N}_\varphi^2))-\)invariant vectors is non-zero. As \( \text{EL}_d(\mathcal{N}_\varphi^2) \) is a normal subgroup of \( \text{EL}_d(\mathcal{R}) \) it follows that \( \mathcal{H}_0 \) is \( \pi_\varphi(\text{EL}_d(\mathcal{R}))-\)invariant where \( \pi_\varphi \) is the “left action” associated to the GNS construction. As \( \pi_\varphi \) and \( \rho_\varphi \) commute the Hilbert subspace \( \mathcal{H}_0 \) is moreover \( \rho_\varphi(\text{EL}_d(\mathcal{R}))-\)invariant for the “right action” \( \pi_\varphi \).

We conclude that the non-zero orthogonal projection to the subspace \( \mathcal{H}_0 \) lies in the center \( Z(\mathcal{L}_\varphi) \) of the von Neumann algebra \( \mathcal{L}_\varphi \). The fact that \( \varphi \) is a character implies that the von Neumann algebra \( \mathcal{L}_\varphi \) is a factor. It follows that \( \mathcal{H}_0 = \mathcal{H}_\varphi \). In other words \( \text{EL}_d(\mathcal{N}_\varphi^2) \leq \ker \varphi \) so that \( \mathcal{N}_\varphi^2 \subset \mathcal{K}_\varphi \) as required.

We will say that \( N_\varphi \) is the supporting ideal of the character \( \varphi \) and continue using the notation \( N_\varphi \) for the remainder of \[8\].

Let us introduce shorthand notations for several normal subgroups of the group \( \text{EL}_d(\mathcal{R}) \) associated with the character \( \varphi \) to be used below. Denote

\[
\mathcal{Z}_\varphi = \text{SL}_d(\mathcal{K}_\varphi) \cap \text{EL}_d(\mathcal{R}) \quad \text{and} \quad \mathcal{N}_\varphi = \text{SL}_d(N_\varphi) \cap \text{EL}_d(\mathcal{R}).
\]

Two-step nilpotent subquotients. Let \( \varphi \in \text{Ch}(\text{EL}_d(\mathcal{R})) \) be a character with supporting ideal \( N_\varphi \) and kernel ideal \( \mathcal{K}_\varphi \).

The normal subgroup \( \text{SL}_d(N_\varphi) \) and its subquotients are best understood in terms of additive groups of matrices. To be precise, consider the quotient ring

\[
\mathcal{R}_\varphi = \mathcal{R}/\mathcal{K}_\varphi.
\]

Let \( \text{M}_d(\mathcal{R}_\varphi) \) denote the abelian group of \( d \)-by-\( d \) matrices with entries in the ring \( \mathcal{R}_\varphi \) and with group operation given by pointwise addition. Let \( \text{M}_d^1(\mathcal{R}_\varphi) \) and \( \text{M}_d^2(\mathcal{R}_\varphi) \)
respectively denote the subgroups of $M_d(\mathcal{R}_\varphi)$ consisting of matrices with zero trace and of matrices whose diagonal entries are all equal to zero. In particular
\[ M_d^{zd}(\mathcal{R}_\varphi) \leq M_d^{zt}(\mathcal{R}_\varphi) \leq M_d(\mathcal{R}_\varphi). \]

The group $\text{EL}_d(\mathcal{R}_\varphi)$ is acting on the abelian group $M_d(\mathcal{R}_\varphi)$ by automorphisms via matrix conjugation preserving the subgroup $M_d^{zd}(\mathcal{R}_{\varphi})$ (but not the subgroup $M_d^{zt}(\mathcal{R}_{\varphi})$).

**Lemma 8.4.** There is an exact sequence of $\text{EL}_d(\mathcal{R})$-equivariant homomorphisms
\[ 1 \to \text{SL}_d(\mathcal{K}_{\varphi}) \to \text{SL}_d(\mathcal{N}_{\varphi}) \xrightarrow{\iota} M_d^{zt}(\mathcal{R}_{\varphi}) \]
where the homomorphism $\iota$ is given by
\[ \iota(g) = (g - \text{Id}_d)M_d(\mathcal{K}_{\varphi}) \quad \forall g \in \text{SL}_d(\mathcal{N}_{\varphi}). \]

**Proof.** We first show that $\iota(g) \in M_d^{zt}(\mathcal{R}_{\varphi})$ for any given element $g \in \text{SL}_d(\mathcal{N}_{\varphi})$. Leibniz formula for determinants implies that
\[ 1 = \det(g) = 1 + \text{tr}(g - \text{Id}_d) + r \]
for some ring element $r \in \mathcal{N}_{\varphi}^2$. Therefore $\text{tr}(g - \text{Id}_d) \in \mathcal{N}_{\varphi}^2 \subseteq \mathcal{K}_{\varphi}$ which means the map $\iota$ is well-defined in the set-theoretic sense.

We now show that $\iota$ is a group homomorphism. Consider a pair of elements $g_1, g_2 \in \text{SL}_d(\mathcal{N}_{\varphi})$ and write $g_i = \text{Id}_d + A_i$ for some matrices $A_i \in M_d(\mathcal{N}_{\varphi})$ and $i \in \{1, 2\}$. Note that $A_1A_2 \in M_d(\mathcal{K}_{\varphi})$. It follows that
\[ \iota(g_1g_2) = \iota((\text{Id}_d + A_1)(\text{Id}_d + A_2)) = (A_1 + A_2)M_d(\mathcal{K}_{\varphi}) = \iota(g_1) + \iota(g_2) \]
as required. It is clear from the definition of $\iota$ that $\ker \iota = \text{SL}_d(\mathcal{K}_{\varphi})$. Finally, the $\text{EL}_d(\mathcal{R})$-equivariance of the homomorphism $\iota$ follows from the computation
\[ \iota(hgh^{-1}) = \iota(h(\text{Id}_d + A)h^{-1}) = \iota(\text{Id}_d + A)M_d(\mathcal{K}_{\varphi}) = \iota(h)M_d(\mathcal{K}_{\varphi}) = hu(g)h^{-1} \]
for any element $g = \text{Id}_d + A \in \text{SL}_d(\mathcal{N}_{\varphi})$ with $A \in M_d(\mathcal{N}_{\varphi})$ and for every element $h \in \text{EL}_d(\mathcal{R})$. \qed

**Corollary 8.5.** The homomorphism $\iota$ restricts to the following exact sequence
\[ 1 \to F_d(\mathcal{K}_{\varphi}) \to F_d(\mathcal{N}_{\varphi}) \xrightarrow{\iota} M_d^{zd}(\mathcal{N}_{\varphi}/\mathcal{K}_{\varphi}) \to 1. \]

**Proof.** The restriction of the map $\iota$ to each elementary subgroup $E_{i,j}(\mathcal{N}_{\varphi})$ for a given pair of distinct indices $i, j \in \{1, \ldots, d\}$ induces the obvious isomorphism of the quotient group $E_{i,j}(\mathcal{N}_{\varphi})/E_{i,j}(\mathcal{K}_{\varphi})$ with the corresponding off-diagonal coordinate of the additive matrix group $M_d^{zd}(\mathcal{N}_{\varphi}/\mathcal{K}_{\varphi})$. The surjectivity claim implicit in the short exact sequence in question follows. Furthermore the image $\iota F_d(\mathcal{N}_{\varphi})$ is isomorphic to the direct sum of the images of the elementary subgroups. The exactness of the sequence is now a consequence of the universal property of direct sums. \qed

**Corollary 8.6.** The map $\iota$ induces an isomorphism of the subquotient $\mathcal{N}_{\varphi}/\mathcal{Z}_{\varphi}$ with a subgroup of the additive matrix group $M_d^{zd}(\mathcal{R}_{\varphi})$.

Note that the subquotient $\mathcal{Z}_{\varphi}$ is central in the quotient group $\text{EL}_d(\mathcal{R})/\ker \varphi$ by Theorem 5.3.\footnote{Note: This is a reference to a theorem or result not explicitly stated in the text.}

**Corollary 8.7.** The subquotient group $\mathcal{N}_{\varphi}/\ker \varphi$ is two-step nilpotent.

**Proof.** This follows from Lemma 8.4 as $[\mathcal{N}_{\varphi}, \mathcal{N}_{\varphi}] \leq \mathcal{Z}_{\varphi} \leq Z(\text{EL}_d(\mathcal{R})/\ker \varphi)$. \qed
The kernel and and degeneracy ideals are commensurable. Let \( \varphi \) be a character of the group \( EL_d(\mathcal{R}) \). Let \( D_\varphi \) be the preimage in the group \( EL_d(\mathcal{R}) \) of the center of the nilpotent subquotient \( N_\varphi / \ker \varphi \). In other words
\[
D_\varphi / \ker \varphi = Z(N_\varphi / \ker \varphi).
\]
It is clear that \( D_\varphi \subset EL_d(\mathcal{R}) \). By the normal subgroup structure theorem (cited here as Theorem 5.8), there is a uniquely determined ideal \( D_\varphi \subset \mathcal{R} \) satisfying
\[
EL_d(D_\varphi) \leq D_\varphi \leq \hat{\mathcal{I}}_d(D_\varphi).
\]
We will say that \( D_\varphi \) is the degeneracy ideal associated to the character \( \varphi \). The ideals in question form an ascending chain as follows
\[
\mathcal{K}_\varphi \subset D_\varphi \subset N_\varphi \subset \mathcal{I}_\varphi \subset \mathcal{R}.
\]
Our goal of showing that \( \mathcal{K}_\varphi^* = \mathcal{I}_\varphi \) is achieved in two steps. Namely, we first show that \( |D_\varphi / \mathcal{K}_\varphi| < \infty \) and then show that \( |N_\varphi / D_\varphi| < \infty \). Putting together these two steps implies that \( \mathcal{K}_\varphi^* = \mathcal{I}_\varphi \) as \( N_\varphi^* = \mathcal{I}_\varphi \) is already known. We turn our attention to the first step.

**Theorem 8.8.** \( |D_\varphi / \mathcal{K}_\varphi| < \infty \).

The proof of Theorem 8.8 involves a careful analysis of Pontryagin dual groups of additive matrix groups over certain ideals. To be precise, denote \( \mathcal{A}_\varphi = D_\varphi / \mathcal{K}_\varphi \). We regard \( \mathcal{A}_\varphi \) as an ideal in the ring \( \mathcal{R} \). Let \( \hat{\mathcal{A}}_\varphi^{\text{Fin}} \) denote the subgroup of the Pontryagin dual group \( \hat{\mathcal{A}}_\varphi \) given by
\[
\hat{\mathcal{A}}_\varphi^{\text{Fin}} = \bigcup_{\mathcal{K}_\varphi \leq J < \mathcal{R}} (J / D_\varphi)^0.
\]
The subgroup \( \hat{\mathcal{A}}_\varphi^{\text{Fin}} \) is a direct limit of countably many finite groups. In particular \( \hat{\mathcal{A}}_\varphi^{\text{Fin}} \) is countable.

The map \( i \) identifies the abelian subquotient \( D_\varphi / \mathcal{Z}_\varphi \) with a subgroup of the matrix group \( M_d^a(\mathcal{A}_\varphi) \) containing \( M_d^a(\mathcal{A}_\varphi) \), see Corollaries 8.5 and 8.6. Therefore \( D_\varphi / \mathcal{Z}_\varphi \) is a quotient of the dual group \( M_d(\mathcal{A}_\varphi) \cong M_d(\hat{\mathcal{A}}_\varphi) \). Let \( \Theta_\varphi \) denote the image of the subgroup \( M_d(\hat{\mathcal{A}}_\varphi^{\text{Fin}}) \) with respect to this quotient map. As the group \( M_d(\hat{\mathcal{A}}_\varphi^{\text{Fin}}) \) is \( EL_d(\mathcal{R}_\varphi) \)-invariant, its quotient \( \Theta_\varphi \) is \( EL_d(\mathcal{R}_\varphi) \)-invariant as well.

**Proposition 8.9.** A character \( \chi \in D_\varphi / \mathcal{Z}_\varphi \) belongs to the subgroup \( \Theta_\varphi \) if and only if \( (g\chi)|_{M_d^a(\mathcal{A}_\varphi)} \in M_d^a(\hat{\mathcal{A}}_\varphi^{\text{Fin}}) \) for every element \( g \in EL_d(\mathcal{R}_\varphi) \).

**Proof.** Consider the two annihilator subgroups
\[
D = M_d^a(\mathcal{A}_\varphi)^0 = \{ \psi \text{Id}_d : \psi \in \hat{\mathcal{A}}_\varphi \}
\]
and \( C = (D_\varphi / \mathcal{Z}_\varphi)^0 \) so that \( D \leq C \leq M_d(\hat{\mathcal{A}}_\varphi) \). We have by definition that
\[
\Theta_\varphi = M_d(\hat{\mathcal{A}}_\varphi^{\text{Fin}}) + C \leq M_d(\hat{\mathcal{A}}_\varphi) / C.
\]
Let \( \overline{\chi} = \chi + C \in D_\varphi / \mathcal{Z}_\varphi \) be given for some character \( \chi \in M_d(\hat{\mathcal{A}}_\varphi) \). If \( \overline{\chi} \in \Theta_\varphi \) then \( g\overline{\chi} \in \Theta_\varphi \) as well for all elements \( g \in EL_d(\mathcal{R}_\varphi) \). The conclusion in the “only if” direction follows.
Arguing in the “if” direction, write $\chi = \chi_1 + \chi_2$ where $\chi_1 \in M^d_d\left(\hat{A}_\phi\right)$ and $\chi_2 = \text{diag}(\psi_1, \ldots, \psi_d) \in M_d\left(\hat{A}_\phi\right)$. The assumption with respect to the trivial element $g = \text{Id}_d \in \text{EL}_d\left(\mathcal{R}_\phi\right)$ immediately implies that $\chi_1 \in M^d_d\left(\hat{A}^\text{Fin}_{\phi}\right) \leq \Theta_\phi$. It remains to show that $\chi_2 + C \in \Theta_\phi$ as well. To see this, note that the character $E_{i,j}(1)\chi_2$ is given by

$$(E_{i,j}(1)\chi_2)_{k,l} = \begin{cases} 
\psi_i - \psi_j & i = k \text{ and } j = l \\
0 & \text{otherwise}
\end{cases}$$

for all pairs of distinct indices $i, j \in \{1, \ldots, d\}$. The assumption gives $\psi_i - \psi_j \in \hat{A}^\text{Fin}_{\phi}$ for all such pairs $i$ and $j$. In other words, all possible differences of entries in the diagonal element $\chi_2$ belong to $\hat{A}^\text{Fin}_{\phi}$. This implies that $\chi_2 \in M_d\left(\hat{A}^\text{Fin}_{\phi}\right) + D$. The desired conclusion follows.

We know that the function $|\varphi|^2 : \text{EL}_d\left(\mathcal{R}\right) \rightarrow \mathbb{R}$ satisfies $|\varphi|^2 \in \text{Tr}\left(\text{EL}_d\left(\mathcal{R}\right)\right)$ by Proposition 1.3. Moreover $Z_\phi \leq \ker|\varphi|^2$ according to Corollary 1.4. Consider the restriction of the trace $|\varphi|^2$ to the subgroup $D_\phi/Z_\phi$ identified with a subgroup of the matrix group $M^d_d\left(\hat{A}_\phi\right)$ in the manner of Corollary 8.6.

**Proposition 8.10.** $|\varphi|^2|_{D_\phi/Z_\phi} = \hat{\varphi}$ some $\text{EL}_d\left(\mathcal{R}_\phi\right)$-invariant Borel probability measure $\varphi_\phi$ supported on the subgroup $\Theta_\phi \leq \hat{D}_\phi/\hat{Z}_\phi$.

**Proof.** Throughout this proof we will be implicitly relying on the isomorphism $F_d\left(D_\phi\right)/F_d\left(K_\phi\right) \cong M^d_d\left(\hat{A}_\phi\right)$ established in Corollary 8.5. In particular, there exists a Borel probability measure $\lambda_\phi$ on the dual group $M^d_d\left(\hat{A}_\phi\right)$ satisfying $\lambda_\phi\left(M^d_d\left(\hat{A}^\text{Fin}_{\phi}\right)\right) = 1$ and $\varphi|_{F_d\left(D_\phi\right)} = \hat{\lambda}_\phi$. This fact is merely a reformulation of Theorem 7.2 in terms of our standing notations. As $M^d_d\left(\hat{A}^\text{Fin}_{\phi}\right)$ is a subgroup of the convolution $\lambda_\phi * \hat{\lambda}_\phi$ satisfies $(\lambda_\phi * \hat{\lambda}_\phi)\left(M^d_d\left(\hat{A}^\text{Fin}_{\phi}\right)\right) = 1$ as well.

Consider the unique $\text{EL}_d\left(\mathcal{R}_\phi\right)$-invariant Borel probability measure $\theta_\phi$ on the dual group $\hat{D}_\phi/\hat{Z}_\phi$ satisfying $|\varphi|^2|_{D_\phi/Z_\phi} = \hat{\theta}_\phi$. As $|\varphi|^2|_{F_d\left(D_\phi\right)} = \lambda_\phi * \hat{\lambda}_\phi$ it follows from the previous paragraph that $g_* \theta_\phi$-almost every character $\chi \in \hat{D}_\phi/\hat{Z}_\phi$ satisfies $\chi|_{M^d_d\left(\hat{A}_\phi\right)} \in \Theta_\phi$ for all elements $g \in \text{EL}_d\left(\mathcal{R}_\phi\right)$. The desired conclusion follows from Proposition 8.9.

**Proposition 8.11.** $\varphi|_{D_\phi} = \tilde{\eta}_\phi$ for some atomic probability measure $\eta_\phi$ supported on a finite $\text{EL}_d\left(\mathcal{R}\right)$-orbit in $\text{Ch}\left(D_\phi\right)$.

**Proof.** It is a consequence of Proposition 8.10 that $|\varphi|^2|_{D_\phi/Z_\phi} = \hat{\theta}_\phi$ for some atomic probability measure $\theta_\phi$ on the dual group $\hat{D}_\phi/\hat{Z}_\phi$. In fact we may regard $\theta_\phi$ as a Borel probability measure on the larger dual abelian group $\hat{D}_\phi/\ker\phi$. As such it is possible to write $\theta_\phi = \eta_\phi * \tilde{\eta}_\phi$ where $\eta_\phi$ is the unique probability measure on the dual group $D_\phi/\ker\phi$ satisfying $\varphi|_{D_\phi} = \tilde{\eta}_\phi$. The probability measure $\eta_\phi$ must be atomic as well by Proposition 3.10. Finally, since the probability measure $\eta_\phi$ is ergodic and

\[13\] Of course $g_* \theta_\phi = \theta_\phi$ for all elements $g \in \text{EL}_d\left(\mathcal{R}_\phi\right)$. We are using the expression $g_* \theta_\phi$ explicitly to make it clear that the assumptions of Proposition 8.9 are satisfied.
Proof of Theorem 8.8. Let \( \varphi \in \text{Ch}(\text{EL}_d(\mathcal{R})) \) be a character with kernel ideal \( \mathcal{K}_\varphi \), degeneracy ideal \( \mathcal{D}_\varphi \) and level ideal \( \mathcal{I}_\varphi \).

The relative character \( \varphi|_{\mathcal{D}_\varphi} \in \text{Ch}_{\text{EL}_d(\mathcal{R})}(\mathcal{D}_\varphi) \) is a convex combination over a finite \( \text{EL}_d(\mathcal{R}) \)-orbit in the space \( \text{Ch}(\mathcal{D}_\varphi) \) by Proposition 8.11. Write

\[
\varphi|_{\mathcal{D}_\varphi} = \frac{1}{N} \sum_{i=1}^{N} \chi_i
\]

for some integer \( N \in \mathbb{N} \) and characters \( \chi_1, \ldots, \chi_N \in \text{Ch}(\mathcal{D}_\varphi) \). The transitivity of the \( \text{EL}_d(\mathcal{R}) \)-action on the characters \( \chi_i \) allows to find an ideal \( \mathcal{J} \subset \mathcal{R} \) satisfying \( \mathcal{K}_\varphi \subset \mathcal{J}, \mathcal{J}^* = \mathcal{I}_\varphi \) and

\[
\chi_i|_{\mathcal{D}_\varphi} \in M^d_{\mathcal{D}_\varphi} ((\mathcal{J} \cap \mathcal{D}_\varphi)^0)
\]

for all indices \( i \in \{1, \ldots, N\} \). The above statement makes an implicit use of the isomorphism \( F_d(\mathcal{D}_\varphi)/F_d(\mathcal{K}_\varphi) \cong M^d_{\mathcal{D}_\varphi}(\mathcal{A}_\varphi) \), see Corollary 8.3.

Consider the intersection ideal \( \mathcal{L} = \mathcal{J} \cap \mathcal{D}_\varphi \subset \mathcal{R} \). Note that \( \mathcal{L}^* = \mathcal{D}_\varphi^* \) from Proposition 2.7. The ideal \( \mathcal{L} \) satisfies

\[
F_d(\mathcal{L}) \leq F_d(\mathcal{D}_\varphi) \cap \bigcap_{i=1}^{N} \ker \chi_i \leq \ker \varphi.
\]

Recall that the subgroup \( \text{EL}_d(\mathcal{L}) \) was defined as the normal closure of \( F_d(\mathcal{L}) \). Therefore \( \text{EL}_d(\mathcal{L}) \leq \ker \varphi \) and \( \mathcal{L} \subset \mathcal{K}_\varphi \). We may conclude that \( \mathcal{K}_\varphi = \mathcal{D}_\varphi^* \) relying once more on Proposition 2.7.

The degeneracy and the supporting ideals are commensurable. Let \( \varphi \) be a character of the group \( \text{EL}_d(\mathcal{R}) \) with kernel ideal \( \mathcal{K}_\varphi \), degeneracy ideal \( \mathcal{D}_\varphi \), supporting ideal \( \mathcal{N}_\varphi \) and level ideal \( \mathcal{I}_\varphi \).

Proposition 8.12. \( \varphi \in \text{Ind}_{\text{EL}_d(\mathcal{R})}(N_\varphi; \mathcal{D}_\varphi) \).

Proof. The restriction \( \varphi|_{N_\varphi} \) is a faithful relative character of the subquotient group \( N_\varphi/\ker \varphi \). Recall that \( [N_\varphi, N_\varphi] \leq Z_\varphi \leq Z(\text{G}) \) as in Corollary 8.7. Moreover \( \mathcal{D}_\varphi/\ker \varphi = Z(N_\varphi/\ker \varphi) \) by definition. The conclusion follows from Lemma 4.17.

We proceed with Theorem 8.13 which is the second and final step towards proving Theorem 8.1.

Theorem 8.13. \( |N_\varphi/\mathcal{D}_\varphi| < \infty \).

Proof. Recall that \( \varphi \in \text{Ch}(\text{EL}_d(\mathcal{R})) \) is a character with kernel ideal \( \mathcal{K}_\varphi \), degeneracy ideal \( \mathcal{D}_\varphi \), supporting ideal \( \mathcal{N}_\varphi \) and level ideal \( \mathcal{I}_\varphi \). Assume towards contradiction that \( |N_\varphi/\mathcal{D}_\varphi| = \infty \). We claim that

\[
\varphi \in \text{Ind}_{\text{EL}_d(\mathcal{R})}(E_{1,2}(\mathcal{I}_\varphi); E_{1,2}(N_{\mathcal{I}_\varphi}))
\]

Indeed, consider any element \( g \in E_{1,2}(\mathcal{I}_\varphi) \setminus E_{1,2}(N_{\mathcal{I}_\varphi}) \). The assumption \( |N_\varphi/\mathcal{D}_\varphi| = \infty \) allows us to apply Proposition 6.7 with respect to the ideals

\[
\mathcal{L} = \mathcal{D}_\varphi, \quad \mathcal{J} = \mathcal{N}_\varphi \quad \text{and} \quad \mathcal{I} = \mathcal{I}_\varphi
\]
and obtain a sequence elements $x_n \in EL_d(\mathcal{R})$ such that the commutators $[g,x_n]$ belong to $N_\varphi = SL_d(N_\varphi) \cap EL_d(\mathcal{R})$ and are pairwise distinct modulo the normal subgroup $D_\varphi$ for all $n \in \mathbb{N}$. Recall that $\varphi \in \text{Ind}_{EL_d(\mathcal{R})}(N_\varphi; D_\varphi)$ as was established in Proposition 8.12. We conclude that $\varphi(g) = 0$ relying on Lemma 4.13 applied with respect to the subgroups $N = D_\varphi$ and $H = N_\varphi$. The claim follows.

The assumption towards contradiction leads to the conclusion $\varphi(E_{1,2}(r)) = 0$ for all ring elements $r \in R \setminus D_\varphi$. Indeed:

1. if $r \in R \setminus I_\varphi$ then $\varphi(E_{1,2}(r)) = 0$ from the properties of the level ideal $I_\varphi$ (see Proposition 7.3),
2. if $r \in I_\varphi \setminus N_\varphi$ then $\varphi(E_{1,2}(r)) = 0$ from the claim made in the previous paragraph, and
3. if $r \in N_\varphi \setminus D_\varphi$ then $\varphi(E_{1,2}(r)) = 0$ from Proposition 8.12.

As $|N_\varphi/D_\varphi| = \infty$ we arrive at a contradiction to yet another property of the level ideal $I_\varphi$, namely Proposition 7.4. □

Since $|I_\varphi/N_\varphi| < \infty$ essentially by definition (see Lemma 8.3), the two Theorems 8.8 and 8.13 put together imply that $|I_\varphi/K_\varphi| < \infty$. In other words $K_\varphi = I_\varphi$. The proof of Theorem 9.1 is now complete.

9. THE CHARACTER $\varphi$ IS INDUCED FROM $\widetilde{SL}_d(I_\varphi) \cap EL_d(\mathcal{R})$

Let $\mathcal{R}$ be a Noetherian ring and $d \geq 3$ be a fixed integer. Let $\varphi \in \text{Ch}(EL_d(\mathcal{R}))$ be a character with level ideal $I_\varphi$ and kernel ideal $K_\varphi$. These two ideals satisfy $K_\varphi = I_\varphi$ by Theorem 8.4. The main goal of §9 is the following.

**Theorem 9.1.** If $d > sr(\mathcal{R})$ then the character $\varphi$ is induced from the normal subgroup $\widetilde{SL}_d(I_\varphi) \cap EL_d(\mathcal{R})$.

Recall that a character of a group $G$ is said to be induced from the subgroup $H$ if it vanishes on the complement $G \setminus H$.

The statement of Theorem 9.1 is vacuous if $I_\varphi = \mathcal{R}$. For this reason we will assume for the remainder of §9 that the level ideal $I_\varphi$ is a proper ideal.

The proof of Theorem 9.1 is built out of a sequence of several Propositions gradually enlarging the subset of the group $EL_d(\mathcal{R})$ where the character $\varphi$ is known to vanish. We use the notation Ind introduced on §4, page 24.

**Remark 9.2.** Strictly speaking, we are abusing the notation Ind in making the following statements, for the groups $\widetilde{SL}_d(I_\varphi)$ etc. are not necessarily subgroups of $EL_d(\mathcal{R})$. In that case our notation is to be understood in the sense of taking the respective intersections with the group $EL_d(\mathcal{R})$.

**Proposition 9.3.** $\varphi \in \text{Ind}_{EL_d(\mathcal{R})}(\widetilde{SL}_d(K_\varphi)V_1(\mathcal{R}); \widetilde{SL}_d(K_\varphi)V_1(I_\varphi))$.

**Proof.** Recall that $EL_d(K_\varphi) \leq \ker \varphi$ and $\widetilde{SL}_d(K_\varphi) \cap EL_d(\mathcal{R}) \leq Z(EL_d(\mathcal{R})/\ker \varphi)$. Schur’s lemma says that the restriction of the character $\varphi$ to the subgroup $\widetilde{SL}_d(K_\varphi)$ is multiplicative (see Lemma 4.5 for details). This means that $\varphi(g) = 0$ if and only if $\varphi(gh) = 0$ for all elements $g \in EL_d(\mathcal{R})$ and $h \in \widetilde{SL}_d(K_\varphi) \cap EL_d(\mathcal{R})$. On the other hand, the properties of the level ideal $I_\varphi$ and Proposition 7.3 in particular imply that

$\varphi \in \text{Ind}_{EL_d(\mathcal{R})}(\ker \varphi V_1(\mathcal{R}); \ker \varphi V_1(I_\varphi))$.

These two facts conclude the proof. □
Proposition 9.4. \( \varphi \in \text{Ind}_{\text{EL}_{d}(\mathcal{R})} \left( \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) V_{1}(\mathcal{R}); \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) \right) \).

Proof. Let \( g \in \text{EL}_{d}(\mathcal{R}) \cap \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) V_{1}(\mathcal{R}) \) be any element. Assume that \( g \notin \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) \). The element \( g \) can be written as \( g = hv \) for a pair of elements \( h \in \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) \) and \( v \in V_{1}(\mathcal{R}) \setminus V_{1}(\mathcal{I}_{\varphi}) \). In particular there is some index \( j \in \{2, \ldots, d\} \) such that \( v_{j,1} \notin \mathcal{I}_{\varphi} \). Fix an arbitrary index \( k \in \{2, \ldots, d\} \setminus \{j\} \). The reduction modulo the level ideal \( \mathcal{I}_{\varphi} \) of the element \( v \) does not centralize the elementary subgroup \( E_{k,j}(\mathcal{R}_{\varphi}) \subseteq \text{EL}_{d}(\mathcal{R}_{\varphi}) \) by Proposition 6.9. According to Item \( 3 \) of Proposition 6.9 there are elements \( x_{n} \in E_{k,j}(\mathcal{R}) \) such that the commutators

\[
[x_{m}^{-1} x_{n}, v] \in V_{1}(\mathcal{R})
\]

all belong to pairwise distinct cosets of \( V_{1}(\mathcal{I}_{\varphi}) \) for all \( n, m \in \mathbb{N} \) with \( n < m \).

The subquotient group \( \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) / \text{SL}_{d}(\mathcal{K}_{\varphi}) \) is finite according to Proposition 5.11. In particular, the action by conjugation of the group \( \text{EL}_{d}(\mathcal{R}) \) on this subquotient has finite orbits. By the pigeon hole principle and up to passing to a further subsequence, we may assume that \( x_{n}^{-1} h x_{n} = x_{m}^{-1} h x_{m} \) modulo the normal subgroup \( \text{SL}_{d}(\mathcal{K}_{\varphi}) \) and for every \( n < m \). Using standard commutator identities we now deduce that

\[
[x_{m}^{-1} x_{n}, g] = [x_{m}^{-1} x_{n}, h v] = [x_{m}^{-1} x_{n}, v] [x_{m}^{-1} x_{n}, h]^{v}.
\]

Note that \( [x_{m}^{-1} x_{n}, v] \in V_{1}(\mathcal{R}) \) and \( [x_{m}^{-1} x_{n}, h]^{v} \in \text{SL}_{d}(\mathcal{K}_{\varphi}) \).

Consider the trace \( |\varphi|^{2} \in \text{Tr}(\text{EL}_{d}(\mathcal{R})) \), see Proposition 4.4. Clearly \( \varphi(h) = 0 \) if and only if \( |\varphi|^{2}(h) = 0 \) for any element \( h \in \text{EL}_{d}(\mathcal{R}) \). So \( |\varphi|^{2} \in \text{Ind}_{\text{EL}_{d}(\mathcal{R})}(H; N) \) with respect to the normal subgroups

\[
H = \text{SL}_{d}(\mathcal{K}_{\varphi}) V_{1}(\mathcal{R}) \quad \text{and} \quad N = \text{SL}_{d}(\mathcal{K}_{\varphi}) V_{1}(\mathcal{I}_{\varphi})
\]

and according to Proposition 9.3 Moreover \( \text{SL}_{d}(\mathcal{K}_{\varphi}) \subseteq \ker |\varphi|^{2} \) by Corollary 4.6.

As the elements \( [x_{m}^{-1} x_{n}, v] \) of \( V_{1}(\mathcal{R}) \) are all distinct modulo \( V_{1}(\mathcal{I}_{\varphi}) \), we may conclude that \( \varphi(g) = 0 \) from Lemma 4.13 applied with respect to the subgroups \( H \) and \( N \). \( \square \)

Proposition 9.5. \( \varphi \in \text{Ind}_{\text{EL}_{d}(\mathcal{R})} \left( \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) N_{\text{EL}_{d}(\mathcal{R})}(V_{1}(\mathcal{R})); \widetilde{\text{SL}}_{d}(\mathcal{I}_{\varphi}) \right) \).

Proof. Let \( g \in \text{EL}_{d}(\mathcal{R}) \cap \text{SL}_{d}(\mathcal{I}_{\varphi}) N_{\text{EL}_{d}(\mathcal{R})}(V_{1}(\mathcal{R})) \) be any element. We may assume without loss of generality that \( g \notin \text{SL}_{d}(\mathcal{I}_{\varphi}) V_{1}(\mathcal{R}) \) for otherwise the result follows from the previous Proposition 9.4.

The reduction modulo the level ideal \( \mathcal{I}_{\varphi} \) of the element \( g \) does not centralize the vertical group \( V_{1}(\mathcal{R}_{\varphi}) \) in the group \( \text{EL}_{d}(\mathcal{R}_{\varphi}) \), see Corollary 6.15. It follows from Item \( 2 \) of Proposition 6.9 applied with respect to the quotient ring \( \mathcal{R}_{\varphi} \) that there is a sequence of elements \( x_{n} \in V_{1}(\mathcal{R}) \) such that \( [g, x_{n}] \in V_{1}(\mathcal{R}) \) and those commutators belong to pairwise distinct cosets of \( V_{1}(\mathcal{I}_{\varphi}) \). Consider the subgroups

\[
H = \text{SL}_{d}(\mathcal{I}_{\varphi}) V_{1}(\mathcal{R}), \quad K = N_{\text{EL}_{d}(\mathcal{R})}(V_{1}(\mathcal{R})), \quad L = \{e\} \quad \text{and} \quad N = \text{SL}_{d}(\mathcal{I}_{\varphi}).
\]

In particular, it is clear that \( x_{n} \in N_{G}(H) \), \( [K, x_{n}] \subseteq H \) and \( [L, x_{n}] = \{e\} \subseteq H \). We conclude that \( \varphi(g) = 0 \) relying on Lemma 4.14 applied with respect to these subgroups \( H, K, L \) and \( N \). \( \square \)
The previous three propositions are stated for the vertical group $V_1(\mathcal{R})$. Of course, analogous results hold true for all other vertical groups $V_i(\mathcal{R})$ as well as the horizontal groups $H_j(\mathcal{R})$, but this fact is not needed below.

**Corollary 9.6.** Let $g \in EL_d(\mathcal{R}) \setminus \widetilde{SL}_d(\mathcal{I}_\varphi)$ be an element whose reduction modulo the level ideal $\mathcal{I}_\varphi$ in the group $EL_d(\mathcal{R}_\varphi)$ centralizes the elementary group $E_{j,i}(\mathcal{R}_\varphi)$ for a given pair of distinct indices $i, j \in \{1, \ldots, d\}$. Then $\varphi(g) = 0$.

**Proof.** The centralizer of the elementary group $EL_d(\mathcal{R}_\varphi)$ satisfies

$$C_{EL_d(\mathcal{R}_\varphi)}(E_{j,i}(\mathcal{R}_\varphi)) \leq Z(EL_d(\mathcal{R}_\varphi))(N_i(\mathcal{R}_\varphi)V_i(\mathcal{R}_\varphi) \cap N_j(\mathcal{R}_\varphi)H_j(\mathcal{R}_\varphi)).$$

See Proposition 6.3 for details. Combined with the normalizer computations done in Proposition 6.2 this means that

$$g \in \widetilde{SL}_d(\mathcal{I}_\varphi)N_{EL_d(\mathcal{R})}(V_i(\mathcal{R})) \setminus \widetilde{SL}_d(\mathcal{I}_\varphi).$$

The fact that $\varphi(g) = 0$ follows immediately from Proposition 9.6. □

We are ready to establish that the character $\varphi$ is induced from the subgroup $\widetilde{SL}_d(\mathcal{I}_\varphi) \cap EL_d(\mathcal{R})$ corresponding to its level ideal $\mathcal{I}_\varphi$.

**Proof of Theorem 9.1.** Consider an element $g \in EL_d(\mathcal{R}) \setminus \widetilde{SL}_d(\mathcal{I}_\varphi)$. We may without loss of generality replace $g$ by any of its conjugates as the character $\varphi$ is conjugation invariant. Relying on the assumption that $d > sr(\mathcal{R})$ and making use of the normal form decomposition introduced in Proposition 6.3, we may replace $g$ by a suitable conjugate and write $g = hnm$ for some corresponding three elements

$$h \in H_1(\mathcal{R}), \ n \in N_2(\mathcal{R})H_2(\mathcal{R}) \text{ and } m \in N_1(\mathcal{R})V_1(\mathcal{R}).$$

First consider the special case where the element $h$ is trivial. This means that

$$g = nm \in N_2(\mathcal{R})H_2(\mathcal{R})N_1(\mathcal{R})V_1(\mathcal{R}) \setminus \widetilde{SL}_d(\mathcal{I}_\varphi).$$

If the reduction modulo the level ideal $\mathcal{I}_\varphi$ of the element $g$ in $SL_d(\mathcal{R}_\varphi)$ centralizes the elementary group $E_{2,1}(\mathcal{R}_\varphi)$ then $\varphi(g) = 0$ according to Corollary 9.6. Otherwise there are elements $x_n \in E_{2,1}(\mathcal{R}) \leq V_1(\mathcal{R})$ such that the commutators $[g, x_n]$ are pairwise distinct modulo the normal subgroup $\widetilde{SL}_d(\mathcal{I}_\varphi)$ according to Item 1 of Proposition 6.6. Consider the subgroups

$$K = \{e\}, \ H = N_2(\mathcal{R})H_2(\mathcal{R}), \ L = N_1(\mathcal{R})V_1(\mathcal{R}) \text{ and } N = \widetilde{SL}_d(\mathcal{I}_\varphi).$$

Note that $x_n \in H, [x_n, K] \subset H$ and $[x_n, L] \subset V_1(\mathcal{R}) \leq H$. The fact that $\varphi(g) = 0$ follows from Lemma 4.14 applied with respect to these subgroups $K, H, L$ and $N$.

Next consider the general case, namely $g = hnm$ with the element $h \in H_1(\mathcal{R})$ being arbitrary. If the reduction modulo the level ideal $\mathcal{I}_\varphi$ of the element $g$ in $EL_d(\mathcal{R}_\varphi)$ centralizes the elementary group $E_{2,3}(\mathcal{R}_\varphi)$ then $\varphi(g) = 0$ according to Corollary 9.6. Otherwise according to Item 1 of Proposition 6.6 there are elements $y_n \in E_{2,3}(\mathcal{R})$ such that the commutators $[g, y_n]$ are pairwise distinct modulo the normal subgroup $\widetilde{SL}_d(\mathcal{I}_\varphi)$. Consider the subgroups

$$K = H_1(\mathcal{R}), \ H = N_2(\mathcal{R})H_2(\mathcal{R})N_1(\mathcal{R})V_1(\mathcal{R}), \ L = \{e\} \text{ and } N = \widetilde{SL}_d(\mathcal{I}_\varphi).$$

Note that $y_n \in H$ and $[y_n, L] \subset H$. Moreover Proposition 6.11 gives

$$[y_n, K] \subset E_{1,3}(\mathcal{R}) \leq N_2(\mathcal{R}) \leq H.$$
Recall that $\varphi(g) = 0$ for all elements $g \in N_2(\mathcal{R})H_2(\mathcal{R})N_1(\mathcal{R})V_1(\mathcal{R}) \setminus \widetilde{S}_d(I_\varphi)$ as established in the first case of this proof. This implies that $\varphi(g) = 0$ relying on Lemma 4.14 applied with respect to these subgroups $K, H, L$ and $N$. □

10. Classification of characters

Let $\mathcal{R}$ be a commutative Noetherian ring with unit. Fix an integer $d > \max\{sr(\mathcal{R}), 2\}$. The goal of the current section is to complete the classification of the characters of the group $\text{EL}_d(\mathcal{R})$ thereby proving our main results Theorems 1.1 and 1.2.

Consider any fixed character $\varphi \in \text{Ch}(\text{EL}_d(\mathcal{R}))$. Let $I_\varphi \vartriangleleft \mathcal{R}$ be the level ideal associated to the character $\varphi$ as established in Theorem 7.2. Let $K_\varphi \vartriangleleft \mathcal{R}$ be the kernel ideal defined so that $\text{EL}_d(K_\varphi) \leq \ker \varphi \leq \tilde{\text{SL}}_d(K_\varphi)$.

We proved in Theorem 8.1 that $K_\varphi^* = I_\varphi$. Consider the subquotient group $A_d(\varphi) = A_d(I_\varphi, K_\varphi) = \left(\tilde{\text{SL}}_d(I_\varphi) \cap \text{EL}_d(\mathcal{R})\right) / \ker \varphi$.

The group $A_d(\varphi)$ is virtually central in the quotient group $\text{EL}_d(\mathcal{R}) / \ker \varphi$ (and is in particular virtually abelian) according to Theorem 5.10. There is a natural $\text{EL}_d(\mathcal{R})$-action on the set $\text{Ch}(A_d(\varphi))$ given by $(g\psi)(x) = \psi(x^g)$ for all elements $g \in \text{EL}_d(\mathcal{R})$. Given a subset $O \subset \text{Ch}(A_d(\varphi))$ define its annihilator $\text{Ann}(O) = \bigcap_{\psi \in O} \ker \psi \leq A_d(\varphi)$.

If the subset $O$ is $\text{EL}_d(\mathcal{R})$-invariant then $\text{Ann}(O)$ is a normal subgroup of $A_d(\varphi)$. In particular $\text{Ann}(O)$ is associated to some ideal in the ring $\mathcal{R}$ in the sense of Theorem 5.8. We say that the subset $O$ is essential if $\text{Ann}(O) \leq \tilde{\text{SL}}_d(K_\varphi)$.

Proof of Theorem 1.1. The fact that the character $\varphi$ is induced from the normal subgroup $\tilde{\text{SL}}_d(I_\varphi) \cap \text{EL}_d(\mathcal{R})$ is the content of Theorem 9.1.

Recall that the subquotient group $A_d(\varphi)$ is virtually central in the quotient group $\text{EL}_d(\varphi) / \ker \varphi$. It follows from Proposition 4.9 that there is a finite $\text{EL}_d(\mathcal{R})$-orbit $O_\varphi \subset \text{Ch}(A_d(\varphi))$ such that

$$\varphi|_{A_d(\varphi)} = \frac{1}{|O_\varphi|} \sum_{\psi \in O_\varphi} \psi.$$ 

It remains to show that the orbit $O_\varphi$ is essential. Note that

$$\text{Ann}(O_\varphi) = \tilde{\text{SL}}_d(I_\varphi) \cap \ker \varphi \leq \tilde{\text{SL}}_d(K_\varphi).$$

This concludes the proof. □

It is now straightforward to conclude that any character $\varphi$ of the group $\text{EL}_d(\mathcal{R})$ is induced from a finite dimensional representation.

Proof of Corollary 1.3. This follows immediately from Theorem 1.1 combined with Corollary 4.10. □

We are ready to prove the converse direction of the character classification.
Proof of Theorem 1.2. Let $\mathcal{I}, \mathcal{K} \triangleleft \mathcal{R}$ with $\mathcal{K}^* = \mathcal{I}$ be a pair of ideals as in the statement of the Theorem. Denote

$$A_d(\mathcal{I}, \mathcal{K}) = \widetilde{\text{SL}}_d(\mathcal{I}) / \ker \varphi.$$ 

We know that the subquotient $A_d(\mathcal{I}, \mathcal{K})$ is a virtually central subgroup of the quotient $\text{EL}_d(\mathcal{R}) / \ker \varphi$ according to Theorem 5.10. Consider the finite essential $\text{EL}_d(\mathcal{R})$-orbit $O \subset \text{Ch}(A_d(\mathcal{I}, \mathcal{K}))$ given in the statement of the Theorem.

Let $\psi \in \text{Ch}_G \left( \left( \text{SL}_d(\mathcal{I}) \cap \text{EL}_d(\mathcal{R}) \right) \right)$ be the relative character corresponding to the $\text{EL}_d(\mathcal{R})$-orbit $O$. Let $\varphi \in \text{Tr}(G)$ be the trace obtained by inducing $\psi$ from the normal subgroup $\widetilde{\text{SL}}_d(\mathcal{I}) \cap \text{EL}_d(\mathcal{R})$ to the entire group $\text{EL}_d(\mathcal{R})$ as done in Proposition 4.7, namely

$$\varphi(g) = \begin{cases} 
\psi(g) & g \in \widetilde{\text{SL}}_d(\mathcal{I}_\varphi) \\
0 & \text{otherwise}
\end{cases} \quad \forall g \in \text{EL}_d(\mathcal{R}).$$

We claim that the induced trace $\varphi$ is indeed a character, namely that it cannot be written as a non-trivial convex combination. Consider the uniquely determined Borel probability measure $\mu_\varphi$ on the set $\text{Ch}(\text{EL}_d(\mathcal{R}))$ such that $\varphi = \int \zeta \, d\mu_\varphi(\zeta)$ as provided by Choquet’s theorem. The level ideal $\mathcal{I}_\varphi$ of $\mu_\varphi$-almost every character $\zeta \in \text{Ch}(\text{EL}_d(\mathcal{R}))$ coincides with the ideal $\mathcal{K}^* = \mathcal{I}$, see Corollary 7.3. This means that $\zeta(g) = 0$ for all elements $g \notin \text{SL}_d(\mathcal{I})$ and $\mu_\varphi$-almost all characters $\zeta$ by the properties of level ideals, see Theorem 5.11. On the other hand, the restriction of every character $\zeta \in \text{Ch}(\text{EL}_d(\mathcal{R}))$ to the normal subgroup $\widetilde{\text{SL}}_d(\mathcal{I}) \cap \text{EL}_d(\mathcal{R})$ is a relative character. As such these restrictions must $\mu_\varphi$-almost surely coincide with the restriction $\varphi|_{\widetilde{\text{SL}}_d(\mathcal{I}) \cap \text{EL}_d(\mathcal{R})}$ to the same subgroup. We conclude that $\mu_\varphi$ is an atomic probability measure supported on a single point. In other words $\varphi \in \text{Ch}(\text{EL}_d(\mathcal{R}))$ as required.

The fact that the level ideal $\mathcal{I}_\varphi$ of the character $\varphi \in \text{Ch}(\text{EL}_d(\mathcal{R}))$ constructed above coincides with the given ideal $\mathcal{I}$ was established in the course of the previous claim. Since the given orbit $O$ is essential the kernel ideal $\mathcal{K}_\varphi$ of the character $\varphi$ must coincide with the given ideal $\mathcal{K}$. Lastly, it is clear that the associated $\text{EL}_d(\mathcal{R})$-orbit $O_\varphi \subset A_d(\mathcal{I}_\varphi, \mathcal{K}_\varphi)$ coincides with the given one $O$.

It remains to show that the character $\varphi \in \text{Ch}(\text{EL}_d(\mathcal{R}))$ satisfying the three conditions $\mathcal{I}_\varphi = \mathcal{I}, \mathcal{K}_\varphi = \mathcal{K}$ and $O_\varphi = O$ is uniquely determined. Indeed any such character $\varphi$ satisfies $\varphi(g) = 0$ for all elements $g \in \text{EL}_d(\mathcal{R}) \setminus \text{SL}_d(\mathcal{I}_\varphi)$, and its restriction to the normal subgroup $\widetilde{\text{SL}}_d(\mathcal{I}_\varphi) \cap \text{EL}_d(\mathcal{R})$ is determined by the kernel ideal $\mathcal{K}_\varphi$ and the finite essential orbit $O_\varphi$. \hfill \Box

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