A New Algorithm to Estimate the Parameters of Log-Logistic Distribution Based on the Survival functions

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Abstract. Estimation of the parameters is quite important in the numerous fields for the development of mathematical models. Maximum likelihood estimation is a good method, which is usually used to elaborate on the parameter estimation. The likelihood function formed for the parameter estimation of Log-Logistic is very hard to maximize. Therefore, this paper proposes a new hybrid of Maximum Likelihood Estimator (MLE) and Simplex Downhill Algorithm (SDA) called (MLESDA) to estimate parameters of Log-Logistic distribution based on Survival functions. To compare the suggested method (MLESDA) and classical Maximum Likelihood (MLE) method, simulation is used. The results demonstrate that MLESDA is more efficient than the MLE method.

1. Introduction

Recently, Survival Analysis (SA) is one of the widely used techniques in medical statistics, physics, medicine, epidemiology engineering, economics, biology, and public health [1, 2]. Estimating survival functions that have interested statisticians for numerous years.

The most of the existing books of Survival analysis for Kleinbaum and Klein [4], Allison [5] introduced this topic from a conventional statistical scene instead of a machine learning standpoint. Chung et al. [3] described the statistical methods of survival analysis and their implementation in criminology for predicting the time until recidivism. Recently, Cruz and Wishart [6] and Kourou et al. [7] discussed applications in cancer prediction and used several survival analysis methods. The log-logistic distribution possesses a rather supple functional form [8]. The log-logistic distribution has its own standing as a life testing model; it is viewed as a weighted exponential distribution and also is an increasing failure rate (IFR) model. Due to the importance of this distribution in reliability, it has been used to estimate the estimators to find parameters. This distribution and for the adoption process to assess the estimators of those two parameters have been estimated the survival of this distribution.

The nonlinearity model makes the estimation of parameter and the statistical analysis of parameter estimates more difficult and challenging. Although, SDA algorithm still a good choice for many practitioners in the fields of physical, statistical, medical sciences, and engineering. Since, it is very easy to use and code [9, 10]. However, until now SDA has not been applied in many mathematical problems. Therefore, in this paper, simplex downhill algorithm adopted to estimate the parameters of Log-Logistic distribution based on Survival functions.

The organized paper as follows: section 2 offers some information about Log Logistic distribution. Section 3. is clarifying Maximum likelihood Estimation method. Section 4 describe the proposed
method (MLEDSA). Section 5 Simulation study. Section 6 demonstrates the effectiveness of the proposed method through numerical results. Finally, in Section 6 a conclusion is provided.

2. Log-Logistic Distribution

The probability density function (p. d. f) and the cumulative distribution function (c. d. f) of Log-Logistic distribution are expressed respectively as:

\[ f(x, a, \beta) = \frac{\beta x_a \beta^{-1}}{(1+(x/a)\beta)^2} \]  
\[ F(x, a, \beta) = \frac{1}{1+(x/a)\beta} \]

where \( x \) is a value of random variable, \( \alpha \) is scale parameter, and \( \beta \) is shape parameter and \( a, \beta > 0 \).

\[ S(x) = 1 - F(x, a, \beta) \]  
\[ S(x) = \frac{1}{1+(x/a)\beta} \]

3. Maximum Likelihood Estimation Method (MLE):

Let \( x_1, x_2, \ldots, x_n \) be order random sample of sized \( n \) from a distribution with \( p.d.f \ f(x, a, \beta) \) such that \( (a, \beta) \) are the parameters, then the likelihood function \( L(a, \beta) \) is the joint \( p.d.f \) of the random samples is [11]:

\[ L(x_1, x_2, \ldots, x_n, a, \beta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \left( \frac{\beta x_i a^{-\beta}}{(1+(x_i/a)\beta)^2} \right) \]

\[ = \frac{\beta^n \prod_{i=1}^{n} \left( \frac{x_i}{a} \right)^{\beta-1}}{a^n} \left( \frac{1}{1+(x_i/a)\beta} \right)^n \]

Taking the natural logarithm for the equation (5), so we get the function:

\[ LnL(x_i, a, \beta) = nLn \left( \frac{\beta}{a} \right) + (\beta - 1) \sum_{i=1}^{n} Ln \left( \frac{x_i}{a} \right) - 2 \sum_{i=1}^{n} Ln \left( 1 + \left( \frac{x_i}{a} \right) \beta \right) \]

\[ LnL(x_i, a, \beta) = nLn(\beta) - nLn(a) + \beta \sum_{i=1}^{n} Ln(\frac{x_i}{a}) - \sum_{i=1}^{n} Ln \left( \frac{x_i}{a} \right) + 2 \sum_{i=1}^{n} Ln \left( 1 + \left( \frac{x_i}{a} \right) \beta \right) \]

The partial derivative for the equation (6) with respect to the unknown parameters \( (\alpha, \beta) \), respectively:
\[ \frac{\partial \ln L(x_i; \alpha, \beta)}{\partial \alpha} = -n \alpha^{-1} - \beta \sum_{i=1}^{n} \frac{1}{\alpha} + \sum_{i=1}^{n} \frac{1}{\alpha} - 2 \sum_{i=1}^{n} \frac{\beta \left( \frac{x_i}{\alpha} \right)^{\beta-1} \left( \frac{x_i}{\alpha} \right)}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} \]

\[ \frac{\partial \ln L(x_i; \alpha, \beta)}{\partial \alpha} = \frac{2 \beta}{\alpha} \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^{\beta} n \beta}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} \]  

(7)

Set equation (7) equal to zero, we get

\[ \frac{2 \beta}{\alpha} \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^{\beta} n \beta}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} \frac{n \beta}{\alpha} = 0 \]  

(8)

\[ \frac{\partial \ln L(x_i; \alpha, \beta)}{\partial \beta} = \frac{n \beta}{\beta} + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\alpha} \right) - 2 \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^{\beta} \ln \left( \frac{x_i}{\alpha} \right) \]  

(9)

After putting equation (9) to zero, then

\[ \frac{n \beta}{\beta} + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\alpha} \right) - 2 \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^{\beta} \ln \left( \frac{x_i}{\alpha} \right) = 0 \]  

(10)

Numerical technique for this method which is Newton Raphson method has been used. From equation (8), we get:

\[ 2 \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^{\beta}}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} n \beta = 0 \]  

(11)

Let \( f_1(\alpha, \beta) \) is define of equation (11), we get:

\[ f_1(\alpha, \beta) = 2 \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^{\beta}}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} - n \]  

(12)

From equation (9), we get:

\[ \frac{n \beta}{\beta} + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\alpha} \right) - 2 \sum_{i=1}^{n} \frac{\ln \left( \frac{x_i}{\alpha} \right)}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} = 0 \]  

(13)

Let \( g_1(\alpha, \beta) \) define of equation (13), we get:

\[ g_1(\alpha, \beta) = \frac{n \beta}{\beta} + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\alpha} \right) - 2 \sum_{i=1}^{n} \frac{\ln \left( \frac{x_i}{\alpha} \right)}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} \]  

(14)

Since the two-nonlinear equations are complicated to be solved, it is impossible to find estimators of parameters \((\alpha, \beta)\). For this reason using the numerical analysis (iterative method) to obtain and estimate \((\alpha, \beta)\) which maximizes the likelihood function. One of these numerical procedures is Newton-Raphson method [12]. It is one of the important methods in numerical analysis because it is very fast and the error
of this iterative method is quadratic approximation. In Newton-Raphson method using the Jacobean matrix $J_i$ which is the first derivatives for each equation of $f_1(\alpha, \beta)$ and $g_1(\alpha, \beta)$ with respect to $\alpha$ and $\beta$.

Now, we find the formulas of Jacobean matrix as follows:

$$J = \begin{bmatrix} \frac{\partial f_1(\alpha, \beta)}{\partial \alpha} & \frac{\partial f_1(\alpha, \beta)}{\partial \beta} \\ \frac{\partial g_1(\alpha, \beta)}{\partial \alpha} & \frac{\partial g_1(\alpha, \beta)}{\partial \beta} \end{bmatrix}$$

$$\frac{\partial f_1(\alpha, \beta)}{\partial \alpha} = -2 \frac{\beta}{\alpha} \sum_{i=1}^{n} \left( \frac{1}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right) \left( \frac{\alpha}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right)^{\beta - 1}$$

$$\frac{\partial f_1(\alpha, \beta)}{\partial \beta} = 2 \frac{\beta}{\alpha} \sum_{i=1}^{n} \ln \left( \frac{x_i(\alpha)}{\alpha} \right) \left( \frac{\alpha}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right)^{\beta - 1}$$

and,

$$\frac{\partial g_1(\alpha, \beta)}{\partial \alpha} = \frac{n}{\alpha} + 2 \frac{\alpha}{\alpha} \sum_{i=1}^{n} \left( \frac{1}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right)^2 + 2 \frac{\beta}{\alpha} \sum_{i=1}^{n} \ln \left( \frac{x_i(\alpha)}{\alpha} \right) \left( \frac{\alpha}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right)^{\beta - 1}$$

$$\frac{\partial g_1(\alpha, \beta)}{\partial \beta} = \frac{n}{\beta^2} - 2 \sum_{i=1}^{n} \ln \left( \frac{x_i(\alpha)}{\alpha} \right)^2 \left( \frac{\alpha}{1 + \left( \frac{x_i(\alpha)}{\alpha} \right)^{-\beta} } \right)^{\beta - 1}$$

Thus, the following equations matrixes are applied to estimate the parameters for Log-Logistic distribution by using Newton-Raphson method.

$$\begin{pmatrix} \hat{\alpha}_{MLE} \\ \hat{\beta}_{MLE} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\beta}_0 \end{pmatrix} - J^{-1} \begin{pmatrix} f_1(\alpha, \beta) \\ g_1(\alpha, \beta) \end{pmatrix}$$

Let

$$v_1 = \frac{\partial f_1(\alpha, \beta)}{\partial \alpha}, v_2 = \frac{\partial f_1(\alpha, \beta)}{\partial \beta}, v_3 = f_1(\alpha, \beta)$$

$$v_4 = \frac{\partial g_1(\alpha, \beta)}{\partial \alpha}, v_5 = \frac{\partial g_1(\alpha, \beta)}{\partial \beta}, v_6 = g_1(\alpha, \beta)$$

Now, substitute equation (19) in equation (20), we get:

$$\hat{\alpha}_{MLE} = \alpha_0 + h_1$$

$$\hat{\beta}_{MLE} = \beta_0 + k_1$$

where:

$$h_1 = \frac{v_4v_6 - v_3v_5}{v_1v_5 - v_2v_4}, \quad k_1 = \frac{-v_3 - v_1h_1}{v_4}$$

So, to estimate the survival analyses $\hat{S}(x)$, we substitute equations (21) and (22) in equation (3).

4. Simplex Downhill Algorithm and Maximum Likelihood Estimation Method (MLEDHA)

Simplex downhill algorithm (SDA) was introduced in 1962 [13]. However, Nelder and Mead [14] modified this algorithm in 1965 to the modern form. SDA called Nelder-Mead or Amoeba. Simplex downhill algorithm is a mathematical method that uses geometric relationships to aid in finding approximate solutions to solve complex and optimization problems. The benefit of this method is it does not require an evaluation of the derivative of the function. But only guesses number of solutions for each decision variable. The idea of SDA generates $N + 1$ points (vertex) in an $N$-dimensional space. Then the vertices sorted by ascending order such as: $f(x_1) \leq f(x_2) \leq ... \leq f(x_n) \leq$
\[ f(x_{n+1}), \text{where } x_{n+1} \text{ is worse solution and } x_1 \text{ best solution.} \] The algorithm iteration updates to improve the worst solution by four operations as follows:

**Reflection step:**

compute the reflection point \( x_r \) from \( x_r = m + \lambda (m - x_{n+1}) \)

and evaluate \( f \) for \( x_r \),

where \( m \) is the centroid of the \( N \) best solution in the vertices of the simplex

\[ m = \text{mean}(x((:n))) \text{ and } \lambda = 1. \]

If \( f(x_1) \leq f(x_r) < f(x_n) \), then put the worst solution in reflected point \( x_{n+1} = x_r \).

**Expansion step:**

If \( f(x_r) < f(x_1) \) then generate a new point \( x_e \) by expansion, from

\[ x_e = x_r + \beta (x_r - m), \text{where } \beta = 2. \]

- If \( f(x_e) < f(x_r) \) then replace \( x_{n+1} \) with \( x_e \).
- else \( x_{n+1} = x_r \).

**Contraction step:**

If \( f(x_{n+1}) \leq f(x_r) \), generate a new solution \( x_c \)

\[ \text{where } x_c = m + \gamma (m - x_{n+1}). \]

- If \( f(x_c) < f(x_r) \) then \( x_{n+1} = x_c \).
- else \( x_{n+1} = x_r \).

The step of shrinkage is used, if the three steps are failing in above.

**Shrinkage Step:**

We keep the best one \( x_1 \) then generate the \( n \) new vertices by using \( x_{sj} = x_1 + \sigma (x_{sj} - x_1), j = \{2 \ldots n+1\} \) and \( \sigma = 0.5 \). The next iteration consists of the simplex vertices as \( x_1, x_2, \ldots, x_{n+1} \).

The objective function of the simplex downhill algorithm in this method was the log likelihood function. By minimizing the log likelihood function, the SDA estimates the parameters of the Log-Logistic Distribution.

### 5. Simulation:

The estimation performance of the proposed method is verified through simulation in this section. In addition, each simulation condition was generated by 1000 replications. The simulation program was written by Matlab 2016. For examining the effect of sample size, various sample sizes are tested: 40, 80, and 100. The simulation steps as follows;

**Step 1:** Generate random samples as \( u_1, u_2, \ldots, u_n \), which are following the continuous uniform distribution defined on the interval \((0,1)\). Then transform it to a random sample follows Log-Logistic distribution using c.d.f. as follow;
\[ F(x, \alpha, \beta) = \frac{1}{1 + \left( \frac{x}{\alpha} \right)^{\beta}} \quad U_i = \frac{1}{1 + \left( \frac{x_i}{\alpha} \right)^{\beta}} \quad x_i = a \left( \frac{U_i}{1-U_i} \right)^{\frac{1}{\beta}} \]

Then, let \( G \) is a vector of all parameters required, such as \( G = [\alpha, \beta] \) and generate \( k+1 \) solutions for \( X \).

Step 2: Recall the S from the equation (4).

Step 3: By (L=1000) replication, we compute \( \hat{S} \) based on MLE using equations (21), and (22), and the best solution from \( (f_1) \) MLEDSA method.

6. Result of Simulation

In order to verify the performance of the MLEDSA method to estimate the parameters, we made simulation by examining various sample sizes (30, 60, 90) . We considered different values of parameters of \( (\alpha, \beta) \) as \((0.5, 2), (2, 4), (3, 1), (1,0.5), (1,5)\) respectively. Based on MSE. Each table explained the result of estimate values of \( \alpha \) and \( \beta \), survival and estimate survival, respectively.

The following tables (1-5) showed that the proposed algorithm offered less MSE for estimate parameters and survival function. Therefore, MLEDSA method provides better results.

**Table 1.** MSE of estimate the parameters and survival analyses when \( \alpha = 0.5 \) and \( \beta = 2 \)

| Simple size | Method   | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( S \)   | \( \hat{S} \)   |
|-------------|----------|---------------------|---------------------|--------|---------------------|
| n = 30      | MLE      | 0.47737             | 1.88420             | 0.50000 | 0.47820             |
|             | MLE/SDA  | 0.47806             | 1.92247             | 0.50000 | 0.48644             |
|             | MLE      | 0.56668             | 2.56315             | 0.50000 | 0.57342             |
| n = 60      | MLE/SDA  | 0.56609             | 2.45316             | 0.50000 | 0.57055             |
|             | MLE      | 0.46123             | 1.82234             | 0.50000 | 0.36129             |
| n = 90      | MLE/SDA  | 0.46475             | 2.01930             | 0.50000 | 0.45952             |

**Table 2.** MSE of estimate the parameters and survival analyses when \( \alpha = 2 \) and \( \beta = 4 \)

| Simple size | Method   | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( S \)   | \( \hat{S} \)   |
|-------------|----------|---------------------|---------------------|--------|---------------------|
| n = 30      | MLE      | 1.95162             | 3.83078             | 0.99611 | 0.99460             |
|             | MLE/SDA  | 1.95154             | 3.84493             | 0.99611 | 0.99471             |
|             | MLE      | 2.15225             | 3.76869             | 0.99611 | 0.99593             |
| n = 60      | MLE/SDA  | 2.15427             | 3.80256             | 0.99611 | 0.99614             |
|             | MLE      | 1.85573             | 3.92688             | 0.99611 | 0.99423             |
| n = 90      | MLE/SDA  | 1.85557             | 4.03466             | 0.99611 | 0.99499             |

**Table 3.** MSE of estimate the parameters and survival analyses when \( \alpha = 3 \) and \( \beta = 1 \)

| Simple size | Method   | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( S \)   | \( \hat{S} \)   |
|-------------|----------|---------------------|---------------------|--------|---------------------|
7. Conclusion
In this paper, a new algorithm MLESDA has been recommended to estimate the parameters for log logistic distribution based on survival analyses. Simulation is founded to compare between the suggested method and MLE. The results showed that MLESDA variant more precisely estimate the parameters than the MLE.

References
[1] Rich J, Neely J, Paniello R, Volker Ch, Nussenbaum B and Wang E 2010 A practical guide to understanding Kaplan-Meir curves Otolaryngology–head and neck surgery official journal of American Academy of Otolaryngology-Head and Neck Surgery 143(3) 331–6
[2] Chung Ch-F, Schmidt P and Witte A D 1991 Survival analysis: A survey Journal of Quantitative Criminology 7(1) 59–98
[3] Kleinbaum D G and Klein M 2006 Survival Analysis: A Self-learning Text Springer Science & Business Media
[4] Allison P D 2010 *Survival Analysis Using SAS: A Practical Guide* Sas Institute
[5] Cruz J A and Wishart D S 2006 Applications of machine learning in cancer prediction and prognosis Cancer Informatics
[6] Fotiadis K K, Exarchos Th P, Exarchos K P, Karamouzis M V and I Dimitrios 2015 Machine learning applications in cancer prognosis and prediction *Computational and Structural Biotechnology Journal* 13 8–17
[7] Balakrishnan N, Malik H J and S Puthenpura 1987 Best line run biased Estimation of location and scale parameters of the Log-Logistic distribution *Commun. Statist–Theor. Meth* 16(12) 3477–3495
[8] Abdul Jabbar K, Atiya B and Atiya B 2017 Application of the Downhill Simplex Algorithm for Solving Aggregate Production Planning problems *Sci.Int.(Lahore)* 29(5) 1075-1081
[9] Chelouah R and Siarry P 2005 A hybrid method combining continuous tabu search and nelder[mead simplex algorithms for the global optimization of multimimima functions *European Journal of Operational Research* 161(3) 636-654
[10] AL-Yasseri AY T 2014 Using simulation to estimate two parameters & reliability function for logistic distribution Master thesis college of education, Al-Mustansirah University
[11] Ojo M O and Olapade A K 2003 On the generalized Logistic and Log-Logistic distribution, Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria, Kraguje vac J. Math. 25 65-73
[12] Spendley W, Hext G R and Himsworth F R 1962 Sequential application of simplex designs optimization and evolutionary operation *Technometrics* 4(4) 411-461
[13] Nelder J A and Mead R 1965 A simplex method for function minimation *The computer journal* 7(4) 308-313