SHIFTED EDGE LABELED TABLEAUX AND LOCALIZATIONS

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ABSTRACT. We prove cases of a conjectural rule of H. Yadav, A. Yong, and the author for structure coefficients of the D. Anderson-W. Fulton ring. In particular, we give a combinatorial description for certain localization coefficients of this ring, which is related to the equivariant cohomology of isotropic Grassmannians.

1. INTRODUCTION

This paper concerns a bijective combinatorics question arising from the author’s previous work [6]. We begin with some Schubert calculus motivation for the problem.

The symplectic group \( G = \text{Sp}_{2n}(\mathbb{C}) \) is the automorphism group of a non-degenerate skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^{2n} \). Let \( Z = \text{LG}(n, 2n) \) be the Lagrangian Grassmannian of \( n \)-dimensional isotropic subspaces of \( \mathbb{C}^{2n} \), where \( V \subseteq \mathbb{C}^{2n} \) is isotropic if \( \langle v_1, v_2 \rangle = 0 \) for all \( v_1, v_2 \in V \). The opposite Borel subgroup \( B_- \leq G \) are those lower triangular matrices in \( G \). Schubert cells are the orbits of under the action of \( B_- \) on \( Z \), of which there are finitely many. The Schubert cells and their Zariski closures, the Schubert varieties \( Z_\lambda = \overline{Z_\lambda} \), are indexed by strict partitions inside the shifted staircase partition \( \rho_n = (n, n-1, n-2, \ldots, 3, 2, 1) \).

A strict partition is an integer partition \( \lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_\ell) \in \mathbb{Z}_{>0}^\ell \). Let \( \mathcal{SP}_n := \{ \lambda \subseteq \rho_n \mid \lambda \text{ a strict partition} \} \).

We identify \( \lambda \) with its shifted shape, the Young diagram with the \( i \)-th northmost row indented \( i-1 \) units from the west.

Example 1.1. We identify \( \lambda = (5, 3, 2) \in \mathcal{SP}_5 \) with the diagram below.

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\end{array}
\]

The maximal torus \( T \) are the diagonal matrices in \( G \). The equivariant Schubert classes \([Z_\lambda]_T\) form a \( H^*_T(pt) \)-module basis of \( H^*_T(Z) \). Define the structure coefficients by

\[
[Z_\lambda]_T \cdot [Z_\mu]_T = \sum_{\nu \subseteq \rho_n} L^\nu_{\lambda, \mu}[Z_\nu]_T,
\]

where \( L^\nu_{\lambda, \mu} \in H^*_T(pt) := \mathbb{Z}[t_1, \ldots, t_n] \). Using a theorem of Graham [2],

\[
L^\nu_{\lambda, \mu} \in \mathbb{Z}_{\geq 0}[\alpha_1, \alpha_2, \ldots, \alpha_n],
\]

where \( \alpha_1 = 2t_1, \alpha_2 = t_2 - t_1, \ldots, \alpha_n = t_n - t_{n-1} \).

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When $|\lambda| + |\mu| = |\nu|$, $L^\nu_{\lambda,\mu}$ recover the ordinary structure coefficients $l^\nu_{\lambda,\mu}$ of $H^*(Z)$. As determined by P. Pragacz [4], these $l^\nu_{\lambda,\mu}$ are the structure coefficients for the multiplication of $Q$-Schur polynomials of I. Schur [8]. Combinatorial rules for $l^\nu_{\lambda,\mu}$ are given by D. Worley [10], B. Sagan [7], and J. Stembridge [9]. For a more in-depth discussion of this story and its relation to the ordinary and maximal Grassmannian settings, see [6].

**Problem 1.2.** Give a combinatorial rule for $L^\nu_{\lambda,\mu}$.

For $\lambda \in SP_n$, let $\sigma_\lambda = \text{Pf}(c_{\lambda_i,\lambda_j})$ be the Pfaffian where

$$c_{p,q} = \sum_{0 \leq a \leq b \leq q} (-1)^b \binom{b}{a} \binom{b-1}{a} z^ac_{p+b-a}c_{q-b}.$$

If $\ell = \ell(\lambda)$ is odd, take $\lambda_{\ell+1} = 0$. In [11] D. Anderson-W. Fulton study the $\mathbb{Z}[z]$-algebra $P = \mathbb{Z}[z, c_1, c_2, \ldots]/(c_{p,p} = 0, \forall p > 0)$ with basis $\{\sigma_\lambda\}_{\lambda \in SP_n}$ over $\mathbb{Z}[z]$. Define structure coefficients $D^\nu_{\lambda,\mu}$ by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \in SP_n} D^\nu_{\lambda,\mu} \sigma_\nu.$$

D. Anderson-W. Fulton make the following connection to $H^*_T(Z)$:

$$L^\nu_{\lambda,\mu}(\alpha_1 \mapsto z, \alpha_2 \mapsto 0, \ldots, \alpha_n \mapsto 0) = D^\nu_{\lambda,\mu}.$$

Taking $\Delta(\nu; \lambda, \mu) := |\lambda| + |\mu| - |\nu|$ and $L(\nu; \lambda, \mu) := \ell(\lambda) + \ell(\mu) - \ell(\nu)$, let

$$d^\nu_{\lambda,\mu} := D^\nu_{\lambda,\mu}/\Delta(\lambda;\lambda,\mu)\Delta(\lambda;\lambda,\mu).$$

In fact, $d^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$. With H. Yadav and A. Yong in [6], the author proposes a rule for $d^\nu_{\lambda,\mu}$ in terms of shifted edge labeled tableaux, with $d^\nu_{\lambda,\mu}$ denoting the number of tableaux satisfying this rule. By Equation (2), this is a conjectural rule for a specialization of Problem 1.2.

**Conjecture 1.3.** [6, Conjecture 10.1] $d^\nu_{\lambda,\mu} = d^\nu_{\lambda,\mu}$.

We contribute to the partial results of [6] for Conjecture 10.1. For $0 \leq m \leq n \in \mathbb{Z}_{>0}$, let $\rho_{n,m} = (n, n-1, \ldots, n-m+1)$.

The following generalizes [6, Theorem 10.6], for which only a proof sketch was included.

**Theorem 1.4.** Suppose $\ell(\mu) = m \leq \ell(\lambda) = n$ such that for $\rho_{n,m} \subseteq \mu$. Then

$$d^\lambda_{\lambda,\mu} = d^\mu_{\lambda,\mu},$$

where $d^\mu_{\lambda,\mu} = 2 \binom{n}{2} - \binom{n-m}{2}$ when $\mu = \rho_{n,m}$ and 0 otherwise.

In the case $\mu = \rho_{n,m}$, Theorem 1.4 highlights an intriguingly simple enumeration. We focus on this case in particular as an enumerative combinatorics problem.

**Example 1.5.** When $\mu = \rho_{n,m}$, Theorem 1.4 shows that $d^\lambda_{\lambda,\mu}$ is computed by subsets of boxes in the first $n-1$ columns of $\mu$. Further, Theorem 1.4 states that these subsets are
in bijection with certain shifted edge labeled tableaux. Below is the shifted edge labeled tableau determined by the given shaded blue subset for $\lambda = (5, 4, 3, 1)$ and $\mu = \rho_{4,2}$.

![Shifted Edge Labeled Tableau Example](image)

2. Combinatorial Background

We first recall the shifted edge labeled tableaux defined in [6].

2.1. Shifted edge labeled tableaux. For $\lambda \subseteq \nu$, the skew shape $\nu/\lambda$ is those boxes of $\nu$ not in $\lambda$. A diagonal edge of $\nu/\lambda$ is the southern edge of a diagonal box of $\nu$, a box in matrix position $(i, i)$. When $\lambda = \emptyset$ we say $\nu = \nu/\lambda$ is a straight shape.

Example 2.1. For $\nu = (5, 3, 2)$ and $\lambda = (3, 2)$, the skew shape $\nu/\lambda$ consists of the five unshaded boxes shown below. The three diagonal edges of $\nu/\lambda$ are bolded in blue.

![Example 2.1 Diagram](image)

A shifted edge labeled tableau of shape $\nu/\lambda$ is a filling of the boxes and diagonal edges of $\nu/\lambda$ with labels $[n] := \{1, 2, 3, \ldots, n\}$ such that:

(i) Each box of $\nu/\lambda$ contains exactly one label.
(ii) A diagonal edge of $\nu/\lambda$ contains a (possibly empty) set of labels.
(iii) Each $i \in [n]$ appears exactly once.
(iv) Labels strictly increase west to east across rows and down columns. Each label on a diagonal edge is strictly larger than any label directly north of it.

Example 2.2. For $\nu$ and $\lambda$ as in Example 2.1, only the leftmost tableau below is a shifted edge labeled tableau of shape $\nu/\lambda$. Reading left to right, the remaining tableaux violate (ii), (iii), and (iv), respectively.

![Example 2.2 Diagram](image)

Let $\text{SELT}(\nu/\lambda, n)$ be the set of all such tableaux. Restricting to tableaux satisfying only (i), (iii) and (iv) results in shifted standard Young tableaux, as described in [10].

An inner corner $c$ of $\nu/\lambda$ is a maximally southeast box of $\lambda$. For $T \in \text{SELT}(\nu/\lambda, n)$, we define a jeu de taquin slide $\text{jdt}_c(T)$, by the following. First place $\bullet$ in $c$, and apply one of the following slides determined by the labels around $c$:

(1) $\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} b \\ a \end{bmatrix}$ (if $b < a$, or $a$ does not exist)
(2) \[ \bullet a \rightarrow \begin{array}{c} a \\ b \end{array} \quad (\text{if } a < b, \text{ or } b \text{ does not exist}) \]

(3) \[ \begin{array}{c} a \\ \sigma \end{array} \rightarrow \begin{array}{c} \sigma \\ a \end{array} \quad (\text{if } a < \min(S) \text{ or } S = \emptyset) \]

(4) \[ \begin{array}{c} \sigma \\ a \end{array} \rightarrow \begin{array}{c} \sigma \\ a \end{array} \quad (\text{if } s := \min(S) < a, \text{ or } a \text{ does not exist, where } S' := S \setminus \{s\}) \]

Repeat this process on each new box occupied by \( \bullet \) until \( \bullet \) arrives at a diagonal edge of \( \lambda \) (i.e. (4) is used) or a box that has no labels directly south or east of it. Then obtain \( \text{jdt}_c(T) \) by removing \( \bullet \) from the resulting tableau.

**Example 2.3.** Below is the computation of \( \text{jdt}_{(1,2)}(T) \).

\[
T : \begin{array}{ccc}
\bullet & 2 & \\
1 & 3 & \\
4 & 5 & 7
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & \\
\bullet & 3 & \\
4 & 5 & 7
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & \\
3 & \bullet & \\
4 & 5 & 7
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & \\
3 & 5 & \\
4 & 7 & 1
\end{array}
\]

The row rectification \( \text{Rect}(T) \) of \( T \in \text{SELT}(\upsilon/\lambda, n) \) is defined iteratively: Choose the southmost inner corner \( c_0 \) of \( \upsilon/\lambda \) and compute \( T_1 := \text{jdt}_{c_0}(T) \) with shape \( \nu^{(1)}/\lambda^{(1)} \). Now let \( c_1 \) be a southmost inner corner of \( \nu^{(1)}/\lambda^{(1)} \) and compute \( T_2 := \text{jdt}_{c_1}(T_1) \). Repeat \(|\lambda| \) times, arriving at a straight shape tableau \( \text{Rect}(T) \).

Let \( S_\mu \) be the superstandard tableau of \( \mu \), obtained by filling the boxes of \( \mu \) in English reading order with \([n] \), where \( n = |\mu| \). Define

\[
d_{\nu}^{\upsilon,\lambda,\mu} := \# \{ T \in \text{SELT}(\nu/\lambda, |\mu|) : \text{Rect}(T) = S_\mu \}.
\]

**Example 2.4.** Suppose \( \lambda = (2,1), \mu = (3,2), \nu = (3,2) \). Below are the only shifted edge labeled tableaux that rectify to \( S_\mu \), so \( d_{\nu}^{\upsilon,\lambda,\mu} = 2 \).

\[ \begin{array}{ccc}
1 & 3 & \\
\bullet & 5 & \\
24 & 5
\end{array} \rightarrow \begin{array}{ccc}
1 & \bullet & 3 \\
2 & 5 & \\
4
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 3 \\
\bullet & 5 & \\
4
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}
\]

\[ \begin{array}{ccc}
3 & \\
\bullet & 5 & \\
124 & 5
\end{array} \rightarrow \begin{array}{ccc}
\bullet & 3 & \\
2 & 5 & \\
1
\end{array} \rightarrow \begin{array}{ccc}
\bullet & 1 & 3 \\
2 & 5 & \\
4
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}
\]

### 2.2. Excited Young diagrams.

For \( \lambda, \mu \in SP_n \), place +’s in the shape of \( \lambda \) inside \( \mu \). Call this the *initial diagram* of \( \lambda \) in \( \mu \). Define the following local move on the +’s in \( \mu \):

\[
\begin{array}{c}
+ \\
\end{array} \rightarrow \begin{array}{c}
\end{array}
\]

where the shaded box either does not exist in \( \mu \) or is a box in \( \mu \) unoccupied by a +. An *excited Young diagram* (EYD) of \( \lambda \) in \( \mu \) is a configuration of +’s formed by successive applications of the above local move on the initial diagram \( \lambda \) in \( \mu \). Let \( \mathcal{E}_\mu(\lambda) \) denote the set of all EYDs of \( \lambda \) in \( \mu \). If \( \lambda \not\subseteq \mu \), then \( \mathcal{E}_\mu(\lambda) = \emptyset \).
Example 2.5. For \( \lambda = (2, 1), \mu = (5, 3, 2), \) below are the EYDs in \( E_\mu(\lambda), \) where the leftmost is the initial diagram:

\[
\begin{array}{cccc}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{array}
\begin{array}{cccc}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{array}
\begin{array}{cccc}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{array}
\begin{array}{cccc}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{array}
\]

The following is derived from results of T. Ikeda-H. Naruse \[3\] using Equation (1).

Lemma 2.6. \[6\] Lemma 10.5 \[\delta^\lambda_{\lambda, \mu} = \# E_\rho(\lambda)(\mu) \times 2^{\mu - \ell(\mu)}.\]

Using this result, \[6\] obtains the following partial result towards Conjecture 1.3.

Theorem 2.7. \[6\] Theorem 10.3 \[\delta^\lambda_{\lambda, (p)} = \binom{\ell(\lambda)}{p} 2^{n-1} = \delta^\lambda_{\lambda, (p)}.\]

3. PROOF OF THEOREM 1.4

For \( T \in \text{SELT}(\nu / \lambda, | \mu |), \) let \( T(i, j) \) be the entry in box \( (i, j) \) (in matrix coordinates) and \( E_i(T) \) be the set of entries on the \( i \)th diagonal edge of \( \nu. \) Additionally, let

\[
\text{col}_k(T) = E_k(T) \cup \{ T(i, k) \mid (i, k) \in \nu / \lambda \text{ where } i \in [k] \}.
\]

Define \( U_{n,m} \in \text{SELT}(\rho_n / \rho_n, | \rho_n, m |) \) by the property \( E_i(U_{n,m}) = \text{col}_i(S_{\rho_n, m}) \) for each \( i \in [n]. \) That is, the labels on the \( i \)th diagonal edge of \( U_{n,m} \) are precisely the labels appearing in the \( i \)th column of \( S_{\rho_n, m}. \) For \( T \in \text{SELT}(\rho_n / \rho_n, | \rho_n, m |) \) and \( I \subseteq E_h(T) \) where \( h \in [n-1], \) define the \( I \)-slide of \( T, \) denoted \( \text{Sl}_I(T), \) by

\[
E_i(\text{Sl}_I(T)) := \begin{cases} 
E_i(T) & \text{if } i \in [n] \setminus \{ h, h+1 \}, \\
E_h(T) \setminus I & \text{if } i = h, \\
E_{h+1}(T) \cup I & \text{if } i = h+1.
\end{cases}
\]

By definition, \( \text{Sl}_I(T) \in \text{SELT}(\rho_n / \rho_n, | \rho_n, m |). \)

Example 3.1. Let \( n = 4 \) and \( m = 3. \) Taking \( I = \{6\} \subseteq E_3(U_{4,3}) = \{3, 6, 8\}, \) below we illustrate \( \text{Sl}_I(U_{4,3}). \)

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & & & \\
\end{array}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & & & \end{array}
\begin{array}{cccc}
1 & & & \end{array}
\]

We say \( T \in \text{SELT}(\rho_n / \lambda, | \rho_n, m |) \) is hopeless if there exists some \( j \in \text{col}_\ell(S_{\rho_n, m}) \cap \text{col}_\ell(T) \) where \( \ell' < \ell. \)

Proposition 3.2. If \( T \in \text{SELT}(\rho_n / \lambda, | \rho_n, m |) \) is hopeless, \( \text{Rect}(T) \neq S_{\rho_n, m}. \)

Proof. This follows by the definition of row rectification. \( \square \)

If \( T \in \text{SELT}(\rho_n / \rho_n, | \rho_n, m |) \) is not hopeless, we say \( T \) is optimistic. Then

\[
T = (\text{Sl}_{I_{n-1}} \circ \ldots \circ \text{Sl}_I)(U_{n,m})
\]
where \( I_\ell \subseteq E_\ell((Sl_{I_{\ell-1}} \circ \ldots \circ Sl_{I_1})(U_{n,m})) \). We call \( I(T) := (I_1, I_2, \ldots, I_{n-1}) \) a slide decomposition of \( T \). For example taking \( T = Sl_I(U_{3,3}) \) from Example 3.1 \( I(T) = (\emptyset, \emptyset, \{6\}) \). Note that by the condition \( I_\ell \subseteq E_\ell((Sl_{I_{\ell-1}} \circ \ldots \circ Sl_{I_1})(U_{n,m})) \), \( I(T) \) is unique.

Suppose \( T \in \text{SELT}(\rho_n/\rho_n, |\rho_{n,m}|) \) is optimistic. For some \( h \in [n-1] \) where \( i > h + 1 \) implies \( I(T)_i = \emptyset \), choose \( j \in I \subseteq E_h(T) \). Define the operator \( \text{shift}_j \) on \( \tilde{T} \in \{ Sl_I(T) \}_{i \in [n+1]} \) such that if \( j \in \text{col}_{h+1}(\tilde{T}) \) and \( \tilde{T}(h,h) < j \):

\[
\text{shift}_j(\tilde{T}(r,c)) := \begin{cases} 
\tilde{T}(r,c) & \text{if } c \neq h + 1, \\
\tilde{T}(r,c) & \text{if } c = h + 1 \text{ and } \tilde{T}(r,c) < j, \\
\tilde{T}(r+1,c) & \text{if } c = h + 1, \tilde{T}(r,c) \geq j, \text{ and } r \leq h, \\
\min\{E_{h+1}(\tilde{T})\} & \text{if } c = h + 1, \tilde{T}(r,c) \geq j, \text{ and } r = h + 1.
\end{cases}
\]

\[
\text{shift}_j(E_i(\tilde{T})) := \begin{cases} 
E_i(\tilde{T}) & \text{if } i \in [n] \setminus \{h, h + 1\}, \\
E_h(\tilde{T}) \cup \{j\} & \text{if } i = h, \\
E_{h+1}(\tilde{T}) \setminus \min_{a \geq j}\{a \in E_{h+1}(\tilde{T})\} & \text{if } i = h + 1.
\end{cases}
\]

Otherwise, \( \text{shift}_j \) acts trivially. In short, when nontrivial, \( \text{shift}_j \) moves \( j \) from column \( h + 1 \) to \( E_h(T) \) and moves labels in column \( h + 1 \) up accordingly. For \( J = \{ j_1 < \ldots < j_k \} \subseteq I \) define

\[
\text{shift}_j(\tilde{T}) := \text{shift}_{j_k} \circ \ldots \circ \text{shift}_{j_1}(\tilde{T}).
\]

By construction, \( \text{shift}_J(\tilde{T}) \in \text{SELT}(\rho_n/\lambda, |\rho_{n,m}|) \) for some \( \lambda \subseteq \rho_n \).

**Example 3.3.** Taking \( T \) as below and \( \tilde{T} = Sl_{\{8\}}(T)_7 \), we have

\[
T = \begin{array}{cccc}
25 & 1 & 2 & 4 \\
19 & 36 & 7 & 8 \\
1 & 6 & 8 & 9 \\
25 & 1 & 6 & 10 \\
\end{array}
\]

\[
\tilde{T} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 6 & 7 \\
1 & 6 & 8 & 9 \\
1 & 2 & 6 & 10 \\
\end{array}
\]

\[
\text{shift}_{\{8\}}(\tilde{T}) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 6 & 7 \\
1 & 6 & 8 & 9 \\
1 & 2 & 6 & 10 \\
\end{array}
\]

**Proposition 3.4.** Suppose \( T \in \text{SELT}(\rho_n/\rho_n, |\rho_{n,m}|) \) is optimistic and \( h \in [n-1] \) such that for \( i \geq h \), \( I(T)_i = \emptyset \). Let \( J \subseteq I \subseteq E_h(T) \), where \( \min I \in J \). Then

\[
\text{Sl}_{I \setminus J}(T)_{k'} = \text{shift}_J(\text{Sl}_I(T)_{k'}),
\]

for \( k' < k = \min\{i \mid \text{Sl}_{I \setminus J}(T)_i(h,h) = \min I\} \).

**Proof.** We proceed by induction on \( k' \). Let \( \min I = j \) and

\[
M = \min\{i \in E_{h+1}(\text{Sl}_{I \setminus J}(T)) \mid i > j\}.
\]

For \( k' = 0 \), the result is trivial by the definitions of \( \text{shift}_J \) and \( \text{Sl}_I \). Suppose the statement holds for some \( k' - 1 \) where \( k' < k \). Let \( \text{path}(\text{Sl}_{I \setminus J}(T)_{k'-1}) \) be the sequence of boxes that \( \bullet \) occupies in computing \( \text{Sl}_{I \setminus J}(T)_{k'} \) from \( \text{Sl}_{I \setminus J}(T)_{k'-1} \). By the inductive assumption and definition of \( \text{shift}_J \), \( \text{Sl}_{I \setminus J}(T)_{k'-1} \) and \( \text{Sl}_I(T)_{k'-1} \) may only differ at edge labels in column \( h \) and labels weakly below \( M \) in column \( h + 1 \). Thus we need only consider when \( \text{path}(\text{Sl}_{I \setminus J}(T)_{k'-1}) \) intersects columns \( h \) and \( h + 1 \).
By the definitions of \( k \) and \( M \), if \( M \in \text{Sl}_{I \setminus J}(T)_{k' - 1}(h + 1, h + 1) \) or \( M \in E_{h+1}(\text{Sl}_{I \setminus J}(T)_{k' - 1}) \) the result follows by the inductive assumption. Thus assume \( M \in \text{Sl}_{I \setminus J}(T)_{k' - 1}(r, h + 1) \) for some \( r \in [h] \). Let
\[
y = \min\{r' \mid (r', h + 1) \in \text{path}(\text{Sl}_{I \setminus J}(T)_{k' - 1})\}.
\]

Case 1: (\( y \) does not exist or \( y \geq r \)). If \( y \) does not exist, the result follows by the inductive assumption and definition of \( \tau_j \). Otherwise this implies that \( \text{Sl}_{I \setminus J}(T)_{k'}(y, h) = m' \), where \( m' \geq M > j \) since \( y \geq r \). Since \( k' < k \), \( j \in E_h(\text{Sl}_{I \setminus J}(T)_{k'}) \). Thus \( \text{Sl}_{I \setminus J}(T)_{k'} \) is not increasing down columns, so we cannot have \( y \geq r \).

Case 2: (\( y < r \)). By assumption for \( i \geq h \), \( \mathcal{I}(T)_i = \emptyset \), so \( E_i(T) = E_i(U_{n,m}) \) for \( i \geq h \). Thus this rectification simply moves up those entries in column \( h + 1 \) below row \( y \). The result follows by the inductive assumption and definition of \( \tau_j \).

We say \( T \in \text{SELT}(\rho_n/\rho_n; |\rho_n,m|) \) is \( r \)-compatible if for each \( j = S_{\rho_n,m}(r', c) \) with \( r' \in \mathcal{I}(T) \)
\[
\text{row}_r(S_{\rho_n,m}) := \{ \text{entries in row } r \text{ of } S_{\rho_n,m} \} = \{ j + \sum_{i=0}^{r-2} n - i \mid j \in [n - r + 1] \}.
\]

For \( T \in \text{SELT}(\rho_n/\rho_n; |\rho_n,m|) \) optimistic and \( k \in [n - 1] \), we say \( \mathcal{I}(T)_k \) is slidable if for
\[
T^{(k)} := (\text{Sl}_{\mathcal{I}(T)_{k-1}} \circ \ldots \circ \text{Sl}_{\mathcal{I}(T)_1})(U_{n,m}),
\]
\[
\mathcal{I}(T)_k \subseteq \bigcup_{i=1}^{3} \{ \min(E_k(T^{(k)}) \cap \text{row}_r(S_{\rho_n,m})) \}.
\]

If \( \mathcal{I}(T)_k \) is slidable for each \( k \in [n - 1] \), we say \( \mathcal{I}(T) \) is slidable.

**Example 3.5.** For \( T \) as below, we compute the right hand side of Equation (4) for \( k = 3 \):
\[
\bigcup_{k=1}^{3} \{ \min(E_3(T) \cap \text{row}_k(S_{\rho_4,4})) \} = \bigcup_{k=1}^{3} \{ \min(\{1, 3, 5, 6, 8\} \cap \text{row}_k(S_{\rho_4,4})) \} = \{1, 5, 8\}.
\]

To the right we have \( S_{\rho_4,4} \) with the entries of \( E_3(T) \) in shaded where the slidable entries \( \{1, 5, 8\} \) are shaded lighter. By Equation (4), any subset \( I \) of the lighter shaded fillings is slidable, so \( \{1, 8\} \) is slidable but \( \{1, 3\} \) is not.

**Proposition 3.6.** Suppose \( T \in \text{SELT}(\rho_n/\rho_n; |\rho_n,m|) \) is optimistic and \( h \in [n - 1] \) such that for \( i \geq h \), \( \mathcal{I}(T)_i = \emptyset \). Let \( J \subseteq I \subseteq E_h(T) \) denote the minimal \# \( J \) elements of \( I \). Then if \( J \) is slidable and \( T \) is \( r - 1 \)-compatible where \( \max J \in \text{row}_r(S_{\rho_n,m}) \):

(1) \( \text{Sl}_I(T)_k = \text{Sl}_{I \setminus J}(T)_k \), where \( k = \min \{ i \mid \max J = \text{Sl}_{I \setminus J}(T)_i(h, h) \} \), and
(2) \( \text{Rect}(\text{Sl}_I(T)) = \text{Rect}(\text{Sl}_{I \setminus J}(T)) \).
Proof. We proceed by induction on \( \#J = \ell \). If \( J = \emptyset \), the result is trivial. Consider \( \#J = \ell \geq 1 \) and suppose the statement holds for \( \#J < \ell \). Let \( j = \max J \). Using the inductive assumption, it suffices to prove the statement when \( J = \{ j \} \). By Proposition 3.6 and the definition of \( k \), it follows that \( \text{SL}_T(T)_{k-1}(n-s,n) = j \) for some \( s \geq 1 \). We will first show \( s = 1 \).

For \( i \in [n] \), let \( a_i = S_{\rho_n,m}(i,h) \), \( m_i = S_{\rho_n,m}(i,h+1) \), and if \( h+2 \leq n \), \( q_i = S_{\rho_n,m}(i,h+2) \). Since \( T \) is \( r-1 \)-compatible, \( a_i \in \text{col}_h(\text{SL}_{\rho_n}(j))_{k} \) for \( i \in [r-1] \) by the definition of \( j \). Since \( \text{I}(T)_i = \emptyset \) for \( i \geq h \), \( m_i \in \text{col}_{h+1}(\text{SL}_{\rho_n}(j))_{k} \) and \( q_i \in \text{col}_{h+2}(\text{SL}_{\rho_n}(j))_{k} \) for \( i \in [n] \). Thus by the minimality of \( j \) in \( I \) and the definition of \( k \), \( \text{SL}_{\rho_n}(j)_{k} \) has the form below, where \( r_i := r-i \). By Proposition 3.4 and the definition of \( k \), showing \( j > c \) is hopeful. Let \( r_i \in \text{row}_{\rho} (S_{\rho_n,m}) \) where \( r' < r \). Since \( a_{r_i} \in \text{col}_h(\text{SL}_{\rho_n}(j))_{k} \) and \( a_{r_i} \leq a_{r_1} < j \), this forces \( b = a_{r_1} \). Since \( m_{r_1} \in \text{col}_{h+1}(\text{SL}_{\rho_n}(j))_{k} \) this implies that \( c = m_{r_1} \), so \( j > c \).

Thus \( \text{SL}_T(T)_{k-1}(n-1,n) = j \). Then (1) follows by Proposition 3.4 applied to \( k-1 \) and the definition of \( jdt \). Lastly (2) follows directly from (1).

Claim 3.7. Suppose \( T \in \text{SELT} (\rho_n/\rho_n, |\rho_n,m|) \) is optimistic. If \( \text{I}(T)_i \cap \bigcup_{k=1}^{r} \text{row}_{k}(S_{\rho_n,m}) \) is slidable for each \( i \in [n-1] \), then \( T \) is \( r \)-compatible.

Proof. Let \( J^{\ast}_i = \text{I}(T)_i \cap \bigcup_{k=1}^{r} \text{row}_{k}(S_{\rho_n,m}) \). We proceed by induction on \( r \). For \( r = 0 \) the result is trivial. Suppose the result holds for some \( r-1 \geq 0 \) and \( J^{\ast}_i \) is slidable for each \( i \in [n-1] \). By the inductive assumption, \( T^{(\ell)} \) is \( r-1 \)-compatible for each \( \ell \in [n-1] \). By inducting again on \( \ell \) and applying Proposition 3.6, no entries in \( J^{\ast}_i \) violate the compatibility condition in \( T^{(\ell)} \), so the result follows.

Proposition 3.8. Suppose \( T \in \text{SELT} (\rho_n/\rho_n, |\rho_n,m|) \). Then \( \text{Rect}(T) = S_{\rho_n,m} \) if and only if \( T \) is optimistic such that \( \text{I}(T) \) is slidable.

Proof. (\( \Leftarrow \)) Using Claim 3.7, Proposition 3.6 for \( \text{I}(T)_i = J \) for \( i \in [n-1] \) gives

\[
\text{Rect}(T) = \text{Rect}(T^{(n-1)}) = \text{Rect}(T^{(n-2)}) = \ldots = \text{Rect}(T^{(1)}) = \text{Rect}(U_{n,m}) = S_{\rho_n,m}.
\]

(\( \Rightarrow \)) If \( T \) is hopeless, \( \text{Rect}(T) \neq S_{\rho_n,m} \) by Proposition 3.2, so assume \( T \) is optimistic. Let \( j = S_{\rho_n,m}(r,c) \) be minimal in \( [(n+1)/2] \) such that \( j \in \text{I}(T)^{i'} \) for some \( i' \in [n-1] \) where \( \{ j \} \) not slidable. By Claim 3.7, Proposition 3.6 and the minimality of \( j \), without loss of generality,
we may assume if \( j' = S_{\rho_n,m}(r', c') \) for \( r' < r, j' \notin \mathcal{I}(T)_i \) for \( i \in [n - 1] \). Thus by the definition of row rectification, it suffices to consider the case when \( r = 1 \). This gives that \( \mathcal{I}(T)_r \cap [n] \) is not slidable.

Consider the set of all possible \( \mathcal{I}(T') \) for optimistic \( T' \in \text{SELT}(\rho_n/\rho_n, |\rho_n,m|) \). For each such \( T' \), let \( J(T') = (J'_1, J'_2, \ldots, J'_{n-1}) \), where \( J'_i := \mathcal{I}(T')_i \cap [n] \). By the uniqueness of \( \mathcal{I}(T') \), distinct \( J(T') \) give distinct labelings of the diagonal edges of \( \rho_n \) with \([n]\).

For each \( i \in [n-1] \), there are 2 ways to choose slidable \( J'_i \) for \( T' \):

\[
J'_i = \{ \min E_i(T'^{(i)}) \} \text{ or } J'_i = \emptyset.
\]

Thus there are \( 2^{n-1} \) possible slidable choices for \( J(T') \).

Let \( \Phi \) denote the restriction of \( T' \) to entries in \([n]\). By the previous direction, if \( \mathcal{I}(T') \) is slidable, \( \text{Rect}(T') = S_{\rho_n,m} \), so \( \text{Rect}(\Phi(T')) = S_{(n)} \). By Theorem 2.7, \( d_{\rho_n,(n)}^{\rho_n} = \binom{(\rho_n)}{n} 2^{n-1} = 2^{n-1} \). Since there are \( 2^{n-1} \) slidable \( J(T') \), these \( J(T') \) precisely construct the labelings of the diagonal edges of \( \rho_n \) with \([n]\) that count \( d_{\rho_n,(n)}^{\rho_n} \). Thus by the pigeonhole principle, since the labelings of \([n]\) determined by these \( 2^{n-1} \) slidable \( J(T') \) are distinct and \( J(T) \) is not slidable, \( \text{Rect}(\Phi(T)) \neq S_{(n)} \), so \( \text{Rect}(T) \neq S_{\rho_n,m} \).

**Example 3.9.** Below are constructions of \( T \) and \( T' \) with \( \mathcal{I}(T)_k \) and \( \mathcal{I}(T')_k \) given above each arrow. Beneath each arrow, entries of \( E_k(T) \) are shaded where those entries in the right hand side of Equation (4), i.e. the slidable entries, shaded lighter.

Thus \( \mathcal{I}(T)_k \), or \( \mathcal{I}(T')_k \), is slidable if and only if all entries of \( \mathcal{I}(T)_k \), or \( \mathcal{I}(T')_k \), are shaded lightly. Here \( \mathcal{I}(T)_1, \mathcal{I}(T')_1, \) and \( \mathcal{I}(T)_2 \) are all slidable, but \( \mathcal{I}(T')_2 \) is not. Therefore by Proposition 3.8 \( \text{Rect}(T) = S_{\rho_3,3} \), but \( \text{Rect}(T') \neq S_{\rho_3,3} \).

**Theorem 3.10.** \( d_{\rho_n,\rho_n,m}^{\rho_n} = 2^\binom{n}{2} - \binom{n-2}{2} \).

**Proof.** By Proposition 3.8 \( d_{\rho_n,\rho_n,m}^{\rho_n} \) counts the slidable \( \mathcal{I}(T) \) over \( T \in \text{SELT}(\rho_n/\rho_n, |\rho_n,m|) \). Let \( J^k = (J^k_1, J^k_2, \ldots, J^k_{n-1}) \) be such that \( J^k = \mathcal{I}(T)_i \cap [\text{row } k(S_{\rho_n,m})] \). Using the same argument presented in the proof of Proposition 3.8 there are \( d_{\rho_n-k,(n-k)}^{\rho_n-k} = 2^{n-k} \) choices for slidable \( J^k \). Summing over \( k \in [m] \), this gives \( 2\binom{\rho_n}{2} - \binom{n-2}{2} \) ways to choose slidable \( \{J^k\}_{k \in [m]} \) and thus \( 2\binom{\rho_n}{2} - \binom{n-2}{2} \) ways to choose slidable \( \mathcal{I}(T) \).

\[ \square \]
Proof of Theorem 1.4 Suppose $\ell(\mu) = m \leq \ell(\lambda) = n$ such that for $\rho_{n,m} \subseteq \mu$. Observe that $|E_{\rho_n}(\mu)| = 1$ if $\mu \subseteq \rho_n$ and 0 otherwise. Thus by Lemma 2.6 when $\mu \subseteq \rho_n$, i.e. when $\mu = \rho_{n,m}$,

$$d_{\lambda,\rho_{n,m}}^\lambda = 2|\rho_{n,m}| - \ell(\rho_{n,m}) = 2|m| - n = 2^{\left(\frac{n-m}{2}\right)}$$

and otherwise $d_{\lambda,\mu}^\lambda = 0$. This proves the rightmost equality.

By the definition of $d_{\lambda,\mu}^\lambda$, if $\mu \not\subseteq \rho_n$, $d_{\lambda,\mu}^\lambda = 0$. Similarly, it is straightforward to see that $d_{\lambda,\rho_{n,m}}^\lambda = d_{\rho_n,\rho_{n,m}}^\rho$ since $m \leq \ell(\lambda)$. Thus the result follows by Theorem 3.10. □

Example 3.11. Below we illustrate the bijection between the shading of $\lambda$ and the tableau $T$ from Example 1.5. The shaded subsets of the first $n-1$ columns of $\rho_{n,m}$ define a choice of slidable $I(T)$, where box $(i, r)$ is shaded if and only if $\min E_i(T[i]) \cap \text{row}_r(S_{\rho_{n,m}}) \in I(T)_i$. Below each arrow, we illustrate how the choice of $I(T)_i$ affects the shading.

![Diagram](image_url)

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