Indefinite Kähler-Einstein Metrics on Compact Complex Surfaces

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Abstract
Indefinite Kähler solutions of the Einstein equations are studied, and it is almost completely determined which compact complex surfaces admit such metrics.

1 Introduction
A pseudo-Riemannian metric on a smooth manifold is called Einstein if the Ricci tensor of the Levi-Civita connection equals a scalar multiple of the metric. This equation first appeared as a particular case (vacuum) of the Einstein field equation 'with cosmological constant'. This field equation was introduced by Einstein in order to describe the influence of matter and electromagnetism on spacetime; solutions of the Einstein equations would then describe 'vacuum spacetimes'. In this physical context, one naturally considers Lorentzian type metrics. But mathematicians became mainly interested in the Riemannian case, and positive-definite solutions of the Einstein equations have been intensively studied in the last decades. While we still do not know much about general Riemannian solutions, strong results have been obtained about the existence of (positive definite) Kähler-Einstein metrics (see [1], [15]). In [3] the reader can find a detailed discussion of these topics and an extensive list of references.
In this paper we will consider indefinite Kähler-Einstein metrics. Our main result will be a classification of the compact complex surfaces which admit such metrics.

Let us begin by considering a compact complex manifold $(M^{2n}, J)$. Here $M$ is a $2n$-dimensional smooth compact manifold and $J$ is an integrable almost complex structure on $M$. If $n = 2$, $M$ is called a (compact) complex surface.

A pseudo-Riemannian metric $g$ on $M^{2n}$ is said to be Hermitian (or $J$-compatible) if $g(x, y) = g(Jx, Jy)$ for all $x, y$. At any point of the manifold one can choose an orthogonal basis of the tangent space of the form $\{x_1, Jx_1, \ldots, x_n, Jx_n\}$; so, if $g$ is Hermitian, its signature is of the form $(2k, 2l)$. In particular if $M$ is a complex surface, any indefinite Hermitian metric on $M$ has signature $(2, 2)$.

If $g$ is a Hermitian pseudo-Riemannian metric then $\omega(x, y) = g(Jx, y)$ is a 2-form, called the Kähler form of $g$.

Definition 1 A Hermitian pseudo-Riemannian metric $g$ is called Kähler if its Kähler form is closed. In particular, if $g$ is not positive or negative definite, it is called an indefinite Kähler metric.

Consider now the Levi-Civita connection $\nabla$ of $g$ on $M$. Assume that $g$ is Kähler; then $J$ is parallel with respect to $\nabla$. This is usually stated only in the Riemannian case, but it is not difficult to check that it is also valid in the indefinite case (by exactly the same proof). Let $\text{Ric}$ be the Ricci tensor of $\nabla$. Then $\text{Ric}$ is $J$-invariant and hence $\rho(x, y) = \text{Ric}(Jx, y)$ is a 2-form. It is called the Ricci form of $g$. It is also true in the indefinite case that $-i\rho$ is the curvature of the canonical line bundle of $M$ (the bundle of holomorphic 2-forms); the proof is the same as in the Riemannian case. In particular $\rho$ is closed and the de Rham class $[\rho/2\pi]$ is equal to the first Chern class of $M$ in cohomology with real coefficients.

Definition 2 An indefinite Kähler metric $g$ on $M$ is called indefinite Kähler-Einstein if there exists $\lambda \in \mathbb{R}$ such that $\text{Ric} = \lambda g$ (or $\rho = \lambda \omega$). In this case $\lambda$ is called the Einstein constant.

If $g$ is an indefinite Kähler-Einstein metric on $M$ and $k \in \mathbb{R}$, then $\hat{g} = kg$ is also an indefinite Kähler-Einstein metric (even if $k < 0$). The Kähler form
of \( \hat{g} \) is \( \hat{\omega} = k \omega \) while the Ricci form is \( \hat{\rho} = \rho \). If \( \omega = \lambda \rho \), then \( \hat{\omega} = \lambda k \hat{\rho} \). Without loss of generality, we may therefore assume that \( \lambda \) is either 0 or 1.

Indefinite Kähler-Einstein metrics on compact complex surfaces is the object of study of this paper. Let us begin by constructing the simplest examples.

**Complex Tori:** Let \( M = \mathbb{C}^2/\Lambda \) be a complex 2-dimensional torus. Let \( z_1, z_2 \) be the standard coordinates on \( \mathbb{C}^2 \). The 1-forms \( dz_1, dz_2, d\bar{z}_1, d\bar{z}_2 \) then descend to \( M \). If \( A = (a_{jk}) \) is a \( 2 \times 2 \) (constant) Hermitian non-degenerate matrix, then \( \omega = \sum a_{jk} dz_j \wedge d\bar{z}_k \) defines a closed, real, \((1,1)\)-form on \( M \). So \( \omega \) is the Kähler form of a Kähler metric \( g \). Moreover, this pseudo-metric is flat. If we choose \( A \) to be indefinite, then \( g \) is an indefinite Kähler-Einstein metric on \( M \) with Einstein constant 0.

**Minimal Ruled Surfaces:** Let \( S \) be a Riemann surface of genus \( g \geq 2 \). Choose a Riemannian metric \( h_1 \) on \( S \) with constant scalar curvature -1. Fixing an orientation on \( S \) there is an almost complex structure on \( S \) (giving the orientation) for which \( h_1 \) is Hermitian. Then \( h_1 \) is a Kähler-Einstein metric on \( S \) with Einstein constant -1. In the same way construct a Kähler-Einstein metric \( h_2 \) on \( \mathbb{CP}^1 \) with Einstein constant 1. Then \( h_2 - h_1 \) is a well defined indefinite Kähler-Einstein metric on \( M = \mathbb{CP}^1 \times S \) with Einstein constant 1.

The general ruled surface is of the form \( \mathbb{P}(E) \), where \( E \) is a 2-dimensional complex vector bundle over a Riemann surface \( S \). We will later construct indefinite Kähler-Einstein metrics on ‘most’ of these twisted products (assuming always that the genus of \( S \) is greater than 1).

Now we can state the main result of this paper:

**Theorem 1** Let \( M \) be a compact complex surface. If \( M \) admits an indefinite Kähler-Einstein metric, then \( M \) is one of the following:

a) a Complex Torus;

b) a Hyperelliptic surface;

c) a Primary Kodaira surface;

d) a minimal ruled surface over a curve of genus \( g \geq 2 \); or

e) a minimal surface of class \( VII_0 \) with no global spherical shell, and
with second Betti number even and positive.

Remarks: No surface of type (e) is known, and it has been conjectured that they simply do not exist (see [9], section 5). We will display indefinite Kähler-Einstein metrics with Einstein constant 0 on the surfaces (a), (b) and (c) (we have already done it for (a)). We will also display indefinite Kähler-Einstein metrics with Einstein constant 1 on ‘most’ surfaces in (d); but it is not known if every surface described by (d) admits such a metric.

2 Indefinite Kähler Metrics

In this section we will find obstructions to the existence of indefinite Kähler metrics on a compact complex surface \( M \).

The first thing to note is that the Kähler form \( \omega \) of such a metric is a symplectic form on \( M \). Since the metric is indefinite Hermitian we can find an orthogonal basis of \( T_pM \) of the form \( \{ x, Jx, y, Jy \} \) such that \( \omega(x, Jx) > 0 \) and \( \omega(y, Jy) < 0 \). Hence \( \omega \wedge \omega \) defines the non-standard orientation of \( M \).

Moreover, given our symplectic form \( \omega \), there exists a compatible positive almost complex structure \( J_\omega \) on \( M \) (see for instance [13], pages 40, 56). Such \( J_\omega \) is then an almost complex structure giving the non-standard orientation of \( M \).

Notation: As usual \( b_k(M) \) will denote the \( k \)-th Betti number of the 4-dimensional manifold \( M \) and \( b^+(M) \) (\( b^-(M) \)) will denote the dimension of a maximal subspace of \( H^2(M^4, \mathbb{R}) \) where the intersection form is positive (negative) definite. So \( b^+ + b^- = b_2 \) and \( b^+ - b^- = \tau \), the signature of \( M \).

The Todd genus \( \text{Todd}(M) = 1/2(1 - b_1 + b^+) \) of an almost complex manifold \( M \) (of real dimension 4) is an integer. So \( b_1(M) - b^+(M) \) is odd. Assume that \( M \) also admits an almost complex structure compatible with the opposite orientation; then \( b^+(M) - b^-(M) = \tau \) is even. We have proved

**Proposition 1** Let \( M \) be a compact complex surface that admits an indefinite Kähler metric. Then there is an almost complex structure giving the non-standard orientation of \( M \). In particular, \( \tau(M) \) is even.

The main obstructions to the existence of indefinite Kähler metrics on compact complex surfaces will be obtained using Seiberg-Witten invariants. These were introduced very recently (see [14]) and we will now outline (very
roughly) some of the main facts about them (see [7], [8], [14] for details and more general statements). We will follow [8].

Let $X$ be a 4-dimensional smooth compact oriented manifold. Assume that $X$ admits an almost complex structure $J$ (compatible with the given orientation) and that $b^+(X) \geq 2$. These last conditions are not necessary for the definition of Seiberg-Witten invariants, but will simplify the description and we will not need more general results in this work.

Fix an homotopy class $c$ of almost complex structures on $X$. The Seiberg-Witten invariants will depend only on the class $c$ on $X$, but Riemannian metrics will be involved in the construction.

An almost complex structure $J$ provides $TX$ with the structure of a (2-dimensional) complex vector bundle. Moreover, homotopic almost complex structures produce isomorphic complex vector bundles. Hence the homotopy class $c$ provides $TX$ with a canonical complex structure. Let $T^{1,0}$ be this complex vector bundle and $L = \Lambda^2 T^{1,0}$. The first Chern class of $L$ is an integral lifting of $w_2$ (the second Stiefel-Whitney class of $TX$). Then $L$ is a $Spin^c$ structure on $X$, that depends only on the class $c$.

Not every $Spin^c$ structure on $X$ is obtained in this way, and Seiberg-Witten invariants can be defined for any $Spin^c$ structure. But considering only those coming from almost complex structures will keep things a little simpler.

Any $Spin^c$ structure produces complex 2-dimensional vector bundles $V^+$, $V^-$ over $X$ such that $V^+ \otimes V^{-*} \cong TX \otimes \mathbb{C}$. And choosing a connection $A$ on the line bundle $\det(V^+)$, it is induced a Dirac operator

$$D_A : C^\infty(V^+) \to C^\infty(V^-)$$

If the $Spin^c$ structure is given by the line bundle $L$ coming from the homotopy class $c$, the vector bundles $V^\pm$ can be described more explicitly:

$$
\begin{align*}
V^+ & \cong \mathbb{C} \oplus L \\
V^- & \cong T^{1,0}
\end{align*}
$$

where $\mathbb{C}$ is the trivial line bundle. So $\det(V^+) = L$.

Given a Riemannian metric $g$ on $X$, we can choose $J$ in $c$ such that $g$ is Hermitian with respect to $J$. In this case the metric $g$ induces a Hermitian metric on $L$. 

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The metric $g$ and the orientation of $X$ produce the Hodge operator $\ast : \Lambda^2 T^* X \to \Lambda^2 T^* X$; $\ast^2 = 1$ and considering the $\pm 1$ eigenspaces of $\ast$ we have a splitting $\Lambda^2 T^* X = \Lambda^+ \oplus \Lambda^-$. The sections of $\Lambda^+$ are called self-dual 2-forms and the sections of $\Lambda^-$ anti self-dual.

Fixing a real self-dual 2-form $\varepsilon \in C^\infty(\Lambda^+)$ the (perturbed) Seiberg-Witten equations are

\[
\begin{align*}
D_A \Phi &= 0 \quad (1) \\
iF_A^+ + \sigma(\Phi) &= \varepsilon \quad (2)
\end{align*}
\]

The unknowns are $\Phi \in C^\infty(V^+)$ and $A$, a unitary connection on $L$. $F_A^+$ is the self-dual part of the curvature of $A$ and $\sigma : V^+ \to \Lambda^+$ is given by

\[
\sigma(f, \psi) = (|f|^2 - |\psi|^2)\omega/4 + Im(f\bar{\psi})
\]

where $\omega$ is the Kähler form of $g$.

The Seiberg-Witten equations depend on the pair $(g, \varepsilon)$; the next task is to extract from them invariants independent of $(g, \varepsilon)$.

The ‘gauge group’ of $C^\infty$ maps $u : M \to S^1 \subset \mathbb{C}$ acts on the space of solutions of (1) by $(A, \Phi) \mapsto (A + 2d \log u, u\Phi)$. Let $\mathcal{M}(g, c)$ denote the space of solutions of (1) modulo this action. We can consider (2) as a map $\rho : \mathcal{M}(g, c) \to C^\infty(\Lambda^+)$.

Let $\varepsilon_H$ denote the harmonic part of $\varepsilon \in C^\infty(\Lambda^+)$ and $c_1^+ \in H^2(M, \mathbb{R})$ be the image of the harmonic self-dual 2-form (with respect to $g$).

The pair $(g, \varepsilon)$ is called excellent if $2\pi c_1^+ \neq [\varepsilon_H]$ and $\varepsilon$ is a regular value of $\rho$.

Under the assumptions that $b^+(X) \geq 2$ and that the $Spin^c$ structure comes from an almost complex structure, we have the following facts:

a) There exist excellent pairs $(g, \varepsilon)$ and the set of excellent pairs is path connected.

b) For any excellent pair $(g, \varepsilon)$, the space of solutions of the Seiberg-Witten equations modulo de action of the gauge group is a finite set of points, with a ‘canonical smooth structure’.
c) Given two excellent pairs the spaces of solutions of the corresponding Seiberg-Witten equations (modulo de action of the gauge group) are cobordant.

Given an excellent pair \((g, \varepsilon)\) let

\[ n_c(N, g, \varepsilon) = \#\{ \text{gauge classes of solutions of (1) and (2)} \} \pmod{2} \]

We can now define the simplest version of the Seiberg-Witten invariants:

**Definition 3** Let \(X\) be a smooth, compact, oriented 4-dimensional manifold which admits almost complex structures and such that \(b^+(X) \geq 2\). The \((\mod 2)\) Seiberg-Witten invariant \(n_c(X)\) of \(X\) with respect to the homotopy class \(c\) of almost complex structures is defined to be \(n_c(X, g, \varepsilon)\); where \((g, \varepsilon)\) is an excellent pair.

The next theorem of Taubes [12] will give the strongest tool to show that some compact complex manifolds do not admit indefinite Kähler metrics.

**Theorem 2 (Taubes)** Let \(X\) be a compact, oriented, 4-dimensional manifold with \(b^+ \geq 2\). Let \(\omega\) be a symplectic form on \(X\) with \(\omega \wedge \omega\) giving the orientation. Then the associated homotopy class of almost complex structures \(c\) has \((\mod 2)\) Seiberg-Witten invariant 1.

This shows that if \(M\) is a compact complex surface with \(b^-(M) \geq 2\) and if \(M\) admits an indefinite Kähler metric, then the Seiberg-Witten invariant of the induced class of almost complex structures is 1.

The following are ‘classical’ results in the theory of Seiberg-Witten invariants.

**Proposition 2 ([7])** Let \(g\) be a Riemannian metric on the smooth compact oriented 4-manifold \(X\). Consider the Seiberg-Witten equations for the pair \((g, 0)\). Any solution \((A, \Phi)\) satisfies the \(C^0\) bound

\[ \|\Phi\|^2 \leq \max(0, -s) \]

at the points where \(\|\Phi\|\) is maximum. Here \(s\) is the scalar curvature of the Levi-Civita connection of \(g\).
**Theorem 3** Let $X$ and $Y$ be smooth compact oriented 4-manifolds. Assume that $b^+(X) \geq 1$ and $b^+(Y) \geq 1$. If $c$ is any homotopy class of almost complex structures on the connected sum of $X$ and $Y$ (compatible with the given orientation), then $n_c(X \# Y) = 0$.

Now we can prove

**Proposition 3** Let $M$ be a compact complex surface. If $M$ is obtained by blowing up another surface $N$ at one point and $b^-(N)$ is positive, then $M$ does not admit an indefinite Kähler metric. In particular if $M$ is obtained by blowing up another surface twice, then $M$ does not admit an indefinite Kähler metric.

Proof: If $M$ is the blown up of $N$, then $M$ is diffeomorphic to the connected sum of $N$ with $\overline{\mathbb{CP}^2}$ ($\mathbb{CP}^2$ provided with the non-standard orientation). Reversing the orientations and applying the last theorem we get that the Seiberg-Witten invariant of $M$ with respect to any almost complex structure giving the non-standard orientation is 0. If $M$ admits an indefinite Kähler metric, Theorem 2 tells us that for at least one of those almost complex structures the Seiberg-Witten invariant is 1. Hence there is no such a metric on $M$. $\square$

Now we can study which compact complex surfaces admit indefinite Kähler metrics. Recall that a complex surface $M$ is called *minimal* if it does not contain an embedded $\mathbb{CP}^1$ with self intersection -1 (i.e. $M$ can not be blown-down).

**Lemma 1** Assume that the compact complex surface $M$ admits an indefinite Kähler metric. Then $M$ is minimal or is a one-point blow-up of either $\mathbb{CP}^2$ or a fake $\mathbb{CP}^2$.

*Remark:* A fake $\mathbb{CP}^2$ is a compact complex surface $M$ such that $b_1(M) = 0$ and $b_2(M) = 1$. It is known that there are only finitely many of these surfaces (see [3, p.136]).

Proof: Assume that $M$ admits an indefinite Kähler metric. Suppose that $M$ is the blow-up of a surface $N$. Proposition 3 implies that $N$ has positive definite intersection form (i.e. $b^-(N) = 0$) and is minimal.
If \( b_1(N) = b_1(M) \) is odd Proposition 1 implies that \( M \) does not admit an indefinite Kähler metric. So \( N \) is of Kähler type.

Now it is time to check the classification of compact complex surfaces (see [2], for example). If \( \text{Kod}(N) \) is 0 or 1, then \( c_1^2(N) = 0 \) and the formula \( c_1^2 + 8q + b^+ = 10p_g + 9 \) would imply \( b^-(N) > 0 \). If \( \text{Kod}(N) = -\infty \) the only possibility is \( N = \mathbb{CP}^2 \). If \( \text{Kod}(N) = 2 \), then \( 0 < c_1^2 = 2c_2 + 3\tau \leq 3c_2 \). This implies that \( b_1(N) = 0 \) and \( b_2(N) = 1 \).

Lemma 2 If \( b^-(M) = 0 \) then \( M \) admits no indefinite Kähler metric (so this is the case for \( \mathbb{CP}^2 \), fake \( \mathbb{CP}^2 \)'s, secondary Kodaira surfaces and surfaces of class VII with vanishing second Betti number).

Proof: This is simply because if \( \omega \) is the Kähler form of an indefinite Kähler metric on \( M \), then \( \omega \) defines an element in \( H^2_{\text{deRham}}(M) \) with \( [\omega]^2 < 0 \). □

Lemma 3 No indefinite Kähler metric exists on any K3 surface or Enriques surface.

Proof: Enriques surfaces are quotients of K3 surfaces. If \( g \) is an indefinite Kähler metric on an Enriques surface, its pull-back would define an indefinite Kähler metric on a K3 surface. Hence it is enough to prove that K3 surfaces do not admit indefinite Kähler metrics.

Let \( \overline{M} \) be a K3 surface endowed with the non-standard orientation. Then \( c_1^2(\overline{M}) = 96 \) and \( b^+(\overline{M}) = 19 \). The proof given by S.T. Yau to Calabi’s conjecture [13] shows that \( M \) admits a scalar flat metric \( \hat{g} \). Since \( c_1^2(\overline{M}) > 0 \), we have \( c_1^+(\overline{M}) \neq 0 \) (check the discussion on Seiberg-Witten invariants above). This condition assures that no pair of the form \( (A, 0) \) could be a solution of the Seiberg-Witten equations if \( \varepsilon = 0 \). And using Proposition 2 we get that for the pair \( (\hat{g}, 0) \) the Seiberg-Witten equations have no solution at all. Hence the pair \( (\hat{g}, 0) \) is excellent and the Seiberg-Witten invariants of \( \overline{M} \) vanish. Then Theorem 2 implies that \( M \) does not admit indefinite Kähler metrics. □
Lemma 4 If $M$ is a surface of class $VII_0$ with a global spherical shell (see [9]) and $b_2(M) = b^-(M) > 0$, then $M$ does not admit an indefinite Kähler metric.

Proof: Such a surface is diffeomorphic to the connected sum of $S^1 \times S^3$ with $b_2(M)$ copies of $\mathbb{CP}^2$ (see [4]). The lemma follows from Theorems 2 and 3. \qed

By the classification of compact complex surfaces (see [2]), the previous lemmas prove:

Theorem 4 Suppose $M$ admits an indefinite Kähler metric. Then

i) If $\text{Kod}(M) = -\infty$, then $M$ is a ruled surface or is as in Theorem 1 (e).

ii) If $\text{Kod}(M) = 0$, then $M$ is a torus, an Hyperelliptic surface or a Primary Kodaira surface.

iii) If $\text{Kod}(M) = 1$, then $M$ is minimal.

iv) If $\text{Kod}(M) = 2$, then $M$ is minimal or the blow-up of a fake $\mathbb{CP}^2$.

Remark 1: Every ruled surface $M$ is of the form $\mathbf{P}(E)$; where $\pi : E \to S$ is a 2-dimensional holomorphic vector bundle over a Riemann surface $S$. Given a Hermitian metric on $E$ and a Kähler form $\omega_0$ on $S$, a sign variation on a well known form gives $\omega = \pi^*(\omega_0) - i\partial \bar{\partial} \log ||W||$; which, for small $s$, is the Kähler form of an indefinite Kähler metric on $M$.

Remark 2: The surfaces listed in (ii) do admit indefinite Kähler metrics. We will show later that they actually admit indefinite Kähler-Einstein metrics.

Remark 3: It is not known which surfaces like (iii) and (iv) admit indefinite Kähler metrics. But the product of two Riemann surfaces of genus $g \geq 2$ belongs to (iv) and the product of an elliptic curve and a curve of genus $g \geq 2$ belongs to (iii) and both admit an indefinite Kähler metric.

3 Indefinite Kähler-Einstein Metrics

We will first compute the Kodaira number of a compact complex surface that admits an indefinite Kähler-Einstein metric.
Proposition 4 If $M$ admits an indefinite Kähler-Einstein metric with Einstein constant $\neq 0$, then $\text{Kod}(M) = -\infty$ and $c_1^2 < 0$.

Proof: If $M$ admits such a metric then its Ricci form $\rho = k\omega$ is everywhere non-degenerate and indefinite. If $\gamma \in \mathcal{O}(K^m)$ is not trivial then

$$\rho = \frac{1}{im}\partial\bar{\partial}\log |\gamma|^2$$

would be semi-negative where $|\gamma|$ attains its maximum. Hence, for all $m > 0$, $K^m$ has no non-trivial global section; and $\text{Kod}(M) = -\infty$.

The second assertion follows from the facts that $[\rho] = 2\pi c_1$ and $\omega \wedge \omega < 0$.

Corollary 1 If $M$ admits an indefinite Kähler-Einstein metric with Einstein $\neq 0$, then $M$ is as in (d) or (e) of Theorem 1.

Proposition 5 If $M$ admits an indefinite Kähler-Einstein metric with Einstein constant 0, then $\text{Kod}(M) = 0$ and $c_1(M, R) = 0$.

Proof: Suppose that $M$ admits such a metric $g$. Then $c_1(M, R) = 0$ and $M$ must be minimal. The only surfaces with Kodaira number $-\infty$ and vanishing real first Chern class are the minimal surfaces of class $VII$ with 0 second Betti number; which do not admit indefinite Kähler metrics. So we can assume that there exists $m > 0$ and $\gamma \in \mathcal{O}(K^m_M)$ non-trivial. Let $\tilde{M}$ be the universal covering of $M$. The pull-back of $g$ gives an indefinite Kähler-Einstein metric on $\tilde{M}$ (with Einstein constant 0). Since this metric is Ricci flat, there are holomorphic 2-forms of constant length in a neighborhood of any point (this fact is usually stated only in the Riemannian case, but it is not difficult to check that the proof also works in the indefinite case). Since $\tilde{M}$ is simply connected it then admits a global non trivial holomorphic 2-form $\varphi$ of constant length. The pull back $\tilde{\gamma}$ of $\gamma$ can be written $\tilde{\gamma} = f \varphi^m$ for some holomorphic function $f$ on $\tilde{M}$ of bounded length. Hence $f$ is constant and $||\gamma||$ is constant. Then $\gamma$ is never zero and $K^m_M$ is trivial. It follows that $\text{Kod}(M) = 0$.

\[ \square \]
Corollary 2 If $M$ admits an indefinite Kähler-Einstein metric with Einstein constant 0, then $M$ is as is (a), (b) or (c) of Theorem 1.

By now we have already proved Theorem 1. The only thing remaining is to construct the promised examples of indefinite Kähler-Einstein metrics.

Hyperelliptic surfaces: It is shown in [4, p.585] that any hyperelliptic surface $M$ is of the form $M = C \times C/G$, where $F$ and $C$ are elliptic curves and $G$ is finite group of fixed-point-free automorphisms of $F \times C$. Moreover, let $F = C/\Lambda$ with $\Lambda = \langle 1, \tau \rangle$; then $G = \langle \phi, \varphi \rangle$, where $\phi$ is of the form $\phi(z, w) = (z + \tau/m, e^{2k\pi i/m})$ and $\varphi$ is a translation of order $m$. If $z, w$ are the standard holomorphic coordinates in $C^2$ then $dz \wedge d\bar{z} - dw \wedge d\bar{w}$ is the Kähler form of an indefinite Kähler flat metric (on $C^2$). This form is invariant through translations and so projects to a $(1,1)$-form on $F \times C$. A direct computation shows that this form is invariant through $\phi$ and $\varphi$ and hence defines a $(1,1)$ on $M$; this is the Kähler form of an indefinite Kähler-Einstein metric on $M$ with Einstein constant 0.

Primary Kodaira surfaces: As described in [6, p.786] such a surface $M$ is of the form $M = C^2/G$ where $G = \langle g_1, g_2, g_3, g_4 \rangle$, each $g_i$ is an affine automorphism of $C^2$ and $G$ is fixed point free. More precisely each $g_i$ is of the form

$$g_i(w_1, w_2) = (w_1 + \alpha_i, w_2 + \bar{\alpha}_i w_1 + \beta_i) \ , \ \alpha_i, \beta_i \in C$$

Consider the $(1,1)$-form $\gamma = -(w_1 + \bar{w}_1)dw_1 \wedge d\bar{w}_1 + dw_1 \wedge d\bar{w}_2 + dw_2 \wedge d\bar{w}_1$ on $C^2$. Direct computations show that $\gamma$ defines an indefinite Kähler flat metric on $C^2$ and is invariant under the $g_i$’s. Hence $\gamma$ induces an indefinite Kähler-Einstein metric on $M$ with Einstein constant 0.

Minimal irrational ruled surfaces: Now let $M = \mathbf{P}(E)$, where $E$ is a 2-dimensional holomorphic vector bundle over a curve $S$ of genus $g \geq 2$. We will construct indefinite Kähler-Einstein metrics on $M$ when the bundle $E$ is stable or the direct sum of two line bundles of the same degree (see [3, 11]).

Note that given vector bundles $E$ and $\hat{E}$, $\mathbf{P}(E)$ and $\mathbf{P}(\hat{E})$ are isomorphic if and only if $\hat{E} = E \otimes L$ for a line bundle $L$; and that $\hat{E}$ verifies any of the conditions above if and only if $E$ does. So both conditions are really properties of $M$. 

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Consider $M$ as a $\mathbb{CP}^1$-bundle over $S$. Let $(U_i)_{i=1}^N$ be an open cover of $S$ and $g_{ij} : U_i \cap U_j \to GL(2, \mathbb{C})$ be a set of transition functions for $E$. Then $[g_{ij}] : U_i \cap U_j \to PGL(2, \mathbb{C})$ are transition functions for $M$. Under the conditions stated above, Narasimhan and Seshadri proved that $M$ admits constant transition functions in $P(U2)$. Let $g_1$ be the Fubini-Study metric on $\mathbb{CP}^1$; then $g_1$ is a Kähler-Einstein metric on $\mathbb{CP}^1$, invariant through the action of $P(U2)$. Renormalize $g_1$ so that the Einstein constant is 1 and let $g_2$ be a Kähler-Einstein metric on $S$ with Einstein constant -1. Then $g_1 - g_2$ is invariant through the transition functions and so defines an indefinite Kähler-Einstein metric on $M$ with Einstein constant 1.

Remark: In [10, p.395] M.S. Narasimhan and S. Ramanan proved that every vector bundle (over a curve of genus greater than 1) can be ‘approximated’ by stable vector bundles. A little more precisely, every vector bundle is contained in an analytic family of vector bundles for which the set of stable bundles is open and dense.

The cases considered above therefore contain ‘most’ of the minimal ruled surfaces (over curves of genus greater than 1).

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