The existence of Hamiltonian stationary Lagrangian tori in Kähler manifolds of any dimension

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Abstract

Hamiltonian stationary Lagrangians are Lagrangian submanifolds that are critical points of the volume functional under Hamiltonian deformations. They are natural generalizations of special Lagrangians or Lagrangian and minimal submanifolds. In this paper, we obtain a local condition that gives the existence of a smooth family of Hamiltonian stationary Lagrangian tori in Kähler manifolds. This criterion involves a weighted sum of holomorphic sectional curvatures. It can be considered as a complex analogue of the scalar curvature when the weighting are the same. The problem is also studied by Butscher and Corvino in [5].

1 Introduction

Hamiltonian stationary (or H-minimal) Lagrangians were defined and studied by Oh [17, 18] in a Kähler manifold $(M, g)$. These objects have stationary volume amongst Hamiltonian equivalent Lagrangians. The Euler–Lagrange equation for a Hamiltonian stationary Lagrangian $L$ is $d^* \alpha_H = 0$, where $H$ is the mean curvature vector on $L$, $\alpha_H$ the 1-form on $L$ defined by $\alpha_H(\cdot) = \omega(H, \cdot)$, and $d^*$ the Hodge dual of the exterior derivative $d$.

Special Lagrangians/ Lagrangian and minimal submanifolds are critical points of the volume functional of all variations, and Hamiltonian stationary Lagrangians can be considered as their generalizations. Hamiltonian stationary Lagrangians are related models for incompressible elasticity theory and are closely related to the study of special Lagrangians/ Lagrangian and minimal submanifolds. Although there are no compact special Lagrangians in $\mathbb{C}^n$, there are compact Hamiltonian stationary Lagrangians in $\mathbb{C}^n$. Oh proves in [18, Th. IV] that for $a_1, \ldots, a_n > 0$, the torus $T_{a_1, \ldots, a_n}^n$ in $\mathbb{C}^n$ given by

$$T_{a_1, \ldots, a_n}^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = a_j, \ j = 1, \ldots, n \right\}$$

(1)
is a stable, rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. A Hamiltonian stationary Lagrangian is called stable (or $H$-stable) if the second variational formula of the volume functional among Hamiltonian deformations is nonnegative. A Hamiltonian stationary Lagrangian in $\mathbb{C}^n$ is called rigid (or $H$-rigid) if the Jacobi vector fields for Hamiltonian variations consist only those from the $U(n) \ltimes \mathbb{C}^n$ actions on $\mathbb{C}^n$ (see [10, §2.3]). Other compact stable, rigid, Hamiltonian stationary Lagrangians in $\mathbb{C}^n$ are given in [1].

Hélein and Romon use the theory from integrable system to find a Weierstrass-type representation for all Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$ and $\mathbb{C}P^2$ [7, 8]. Schoen and Wolfson construct singular Hamiltonian stationary Lagrangians in symplectic manifolds by minimizing volume among Lagrangian cycles representing a given homology (or homotopy) class [16]. Very little is known on the existence of smooth Hamiltonian stationary Lagrangians in general. In [10], Joyce, Schoen and the author use the perturbation method to study the problem, and obtain families of compact smooth embedded Hamiltonian stationary Lagrangians in every compact symplectic manifold $(M, \omega)$ with a compatible metric $g$. The result is:

**Theorem [10]** Suppose that $(M, \omega)$ is a compact symplectic $2n$-manifold, $g$ is a Riemannian metric on $M$ compatible with $\omega$, and $L$ is a compact, smooth, embedded, Hamiltonian rigid, Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. Then there exist compact, smooth, embedded, Hamiltonian stationary Lagrangians $L'$ in $M$ which are diffeomorphic to $L$, such that $L'$ is contained in a small ball about some point $p \in M$, and identifying $M$ near $p$ with $\mathbb{C}^n$ near 0 in Darboux geodesic normal coordinates, $L'$ is a small deformation of $tL$ for small $t > 0$. If $L$ is also Hamiltonian stable, we can take $L'$ to be Hamiltonian stable.

The method used in [10] is first to find Darboux coordinates at each point which also admit a nice expression on the metric, and then put a scaled compact Hamiltonian stationary Lagrangian from $\mathbb{C}^n$ in the Darboux coordinates at each point. These submanifolds are Lagrangian in $(M, \omega, g)$, but not Hamiltonian stationary yet. One then tries to perturb these approximate examples in Hamiltonian equivalent class to Hamiltonian stationary. This involves solving a highly nonlinear equation whose linearized equation has approximate kernel. Thus the goal cannot be achieved in general. In [10], the authors first solve the equation perpendicular to the approximate kernel for all initial approximate examples. They then show that the problem of finding Hamiltonian stationary Lagrangians, which are critical points of the volume functional on an infinite dimensional space, can be reduced to finding critical points of a smooth function on a finite dimensional compact space when the model from $\mathbb{C}^n$ is Hamiltonian rigid and $M$ is compact. The existence will follow from the simple fact that every continuous function has critical points on a compact set.

The advantage of the above argument is that it only requires compactness and works in very general cases. But the disadvantage is that we do not know where the Hamiltonian stationary Lagrangian locates in $(M, \omega, g)$. As a consequence, the examples obtained may be far apart for different $t$. In this paper, we take a different approach in the second step and resolve this deficit when $M$
is a Kähler manifold and $L = T^{n}_{a_1,\ldots,a_n}$. More precisely, we show that

**Theorem 4.4** Suppose that $(M,\omega,g)$ is an $n$-dimensional Kähler manifold, and $U$ is the $U(n)$ frame bundle of $M$. The subgroup of diagonal matrices $T^n \subset U(n)$ acts on $U$. Define $F_{a_1,\ldots,a_n} : U/T^n \to \mathbb{R}$ by $F_{a_1,\ldots,a_n}(p,[v]) = \sum_{i=1}^{n} a_i^2 R_{ii}(p)$ for any given $a_i > 0$, $i = 1,\ldots,n$, where $p \in M$ and $v \in U(n)$. The holomorphic sectional curvature $R_{ii}(p)$ is computed w.r.t. the unitary frame $v$ at $p$ whose value is independent of the representative $v$ of $[v]$. Assume that $(p_0,[v_0]) \in U/T^n$ is a non-degenerate critical point of $F_{a_1,\ldots,a_n}$, then for $t$ small there exist a smooth family $(p(t),[v(t)]) \in U/T^n$ satisfying $(p(0),[v(0)]) = (p_0,[v_0])$ and a smooth family of smooth embedded Hamiltonian stationary Lagrangian tori with radii $(t a_1,\ldots,t a_n)$ center at $p(t)$ which are invariant under local $T^n$ action and are posited w.r.t any representative of $[v(t)]$. Moreover, the distance between $(p(t),[v(t)])$ and $(p_0,[v_0])$ in $U/T^n$ is bounded by $ct^2$. The family of embedded Hamiltonian stationary Lagrangian tori do not intersect each other when $t$ is small.

The proof of the theorem is along the same line as in [10] with the following differences:

- On Kähler manifolds, we have a better expression of the metric for Darboux coordinates. And when $L = T^{n}_{a_1,\ldots,a_n}$, we can explicitly compute the leading terms of related estimates.
- In the last step, instead of using compactness to prove the existence, we analyze directly the conditions needed for perturbing approximate examples to Hamiltonian stationary. This is done by deriving explicit expressions up to certain orders in all related estimates. We do not require $M$ being compact in the proof (see Remark 2.4), and thus the result also holds for noncompact Kähler manifolds.

Two different proofs for the last step are given in this paper. The first approach has been used in studying many different problems. We explain the ideas in the following. After solving the equation perpendicular to the approximate kernel for all initial approximate examples, we analyze their images in the approximate kernel by applying Taylor expansion and then try to find examples with this part vanishing as well. The coefficients of the leading terms of the image in the approximate kernel are computed. We then use a non-degenerate critical point condition and inverse function theorem to achieve the goal. From Lemma 3.3, one can see that it is not enough to have the expansion just up to the second order because the leading terms do not involve a whole basis of the approximate kernel. It is only after we compute the expansion up to the third order, that we find out the correct form of $F_{a_1,\ldots,a_n}$. Although the computation here is still complicated and involving, we remark that our argument is in fact already simplified from standard approach by working in a fixed ball in $C^n$ and fixed function spaces, and varying the metric only.

Our second approach is more in the spirit of [10], which solves the problem by finding critical points of a smooth function on a finite dimensional space.
Taylor expansion of this function is derived to analyze the existence of critical points. This approach is more geometrical and the computation is much easier. Although we still need the expansion of the function up to the third order, a simple computation shows that in the tori case all the coefficients of the third order terms vanish. Hence the coefficients of the third order term in the expansion of the metric in fact will not play a role in the tori case. This method also indicates a way of generalization the result to Hamiltonian stationary Lagrangians other than tori.

Our result is an analogue to the case of constant mean curvature (CMC) hypersurfaces in a Riemannian manifold $M$. Ye in [20] showed that near a non-degenerate critical point $p$ of the scalar curvature function on $M$, there exist CMC sphere foliation near $p$. The problem of finding a corresponding condition for Hamiltonian stationary Lagrangian tori on a Kähler manifold is proposed by Schoen, and is the starting point of our project in this direction including [10].

Butscher and Corvino also study the same problem in [5]. Their method is along the same line as our first approach, but uses holomorphic normal coordinates instead of Darboux coordinates. An additional difficulty arising from their approach is that their initial approximate solutions are not necessary Lagrangian. Hence they need to take care of both Lagrangian and Hamiltonian stationary conditions during the perturbation, and the deformation involves vector fields of $n$ components. The argument becomes much more complicated. And also because Butscher and Corvino split the deformation vector field into a gradient and a curl component to make related equations strictly elliptic, only the $n = 2$ case is discussed in [5]. Our advantage on using Darboux coordinates is that the approximate solutions obtained are Lagrangian and will stay Lagrangian under Hamiltonian deformation which involves a function only. The problem is thus reduced to solve a nonlinear 4-th order elliptic PDE. We should emphasize that our approach also starts with holomorphic normal coordinates, but with an additional step that constructs a diffeomorphism with suitable control to transfer the coordinates into Darboux coordinates (Proposition 2.3). Such techniques were used by the author before in [11]. Butscher and Corvino had a different criterion in their 2008 preprint. But now the condition already agrees to ours in their recent 2011 revision. The analysis for deforming an almost Lagrangian and almost Hamiltonian stationary torus to a Hamiltonian stationary one may be useful for studying other problems.

We remark that unlike the CMC case, the family of Hamiltonian stationary Lagrangian tori with radii $(ta_1, \ldots, ta_n)$ will not form a foliation because the total dimension $(n + 1)$ of the family of tori does not match with the ambient dimension $2n$. Never the less, they still have the nice property that the torus in the family does not intersect with each other. A possible way to obtain a foliation in the Lagrangian case is to allow $a_1, \ldots, a_n$ changing. However, interesting existence and foliation property are not available for this situation. We finish this introduction by proposing some questions for further investigation.

**Question 1.1.** Does there exist a non-degenerate critical point of $F_{a_1, \ldots, a_n}(p, [v])$ for generic Kähler metrics and complex structures on $M$? We can also allow
changing.

**Question 1.2.** Can one generalize the construction in this paper to find Hamiltonian stationary Lagrangian tori near a closed geodesic as the CMC case by Mazzeo and Pacard in [13], or more generally near an isotropic minimal submanifold as [12]?

**Question 1.3.** Is there any other geometric implication of \( \sum_{i=1}^{n} R_{i\bar{i}i}(p) \), or more generally \( \sum_{i=1}^{n} a_i^2 R_{i\bar{i}i}(p) \)?

The function \( \sum_{i=1}^{n} a_i^2 R_{i\bar{i}i}(p) \) depends not only on the point \( p \in M \), but also on the unitary frame at \( p \). The non-degenerate critical point condition in our Theorem indicates that there are preferred frames. Similar idea was used before. By studying the maximum of \( R_{i\bar{i}i}(p) \) among all points and unit directions, Goldberg and Kobayashi proved that a compact connected Kähler-Einstein manifold of positive holomorphic bisectional curvature must be isometric to \( \mathbb{C}P^n \) with the Fubini-Study metric [6]. This method was first introduced by Berger in [2].

This paper is organized as follows. In §2 we give basic definitions and derive new Darboux coordinates that will play an important role. Some crucial and involving estimates are given in §3. Section 4 consists of the set up of the perturbation method and the first proof for the main theorem. A different proof for the theorem using the second approach is presented in the last section. We also provide a supplement in the Appendix for one estimate used in §4.

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## 2 Notation and Darboux coordinates

### 2.1 Lagrangian and Hamiltonian stationary

We will assume \((M, \omega, g)\) to be a Kähler manifold through this paper, and refer to §2 in [10] for more detailed discussions on the background material.

**Definition 2.1.** A submanifold \( L \) in \((M, \omega)\) is called Lagrangian if \( \dim L = n = \frac{1}{2} \dim M \) and \( \omega|_L \equiv 0 \). It follows that the image of the tangent bundle \( TL \) under the complex structure \( J \) is equal to the normal bundle \( T^\perp L \).

Let \( F : M \rightarrow \mathbb{R} \) be a smooth function on \( M \). The Hamiltonian vector field \( v_F \) of \( F \) is the unique vector field satisfying \( v_F \cdot \omega = \text{d}F \). The Lie derivative satisfies
$$\mathcal{L}_{v_F} \omega = v_F \cdot d\omega + d(v_F \cdot \omega) = 0,$$
so the trajectory of $v_F$ gives a 1-parameter family of diffeomorphisms $\text{Exp}(sv_F) : M \rightarrow M$ for $s \in \mathbb{R}$ which preserve $\omega$. It is called the Hamiltonian flow of $F$. If $L$ is a compact Lagrangian in $M$ then $\text{Exp}(sv_F) L$ is also a compact Lagrangian in $M$.

**Definition 2.2.** A compact Lagrangian submanifold $L$ in $(M, \omega, g)$ is called Hamiltonian stationary, or $H$-minimal, if it is a critical point of the volume functional among Hamiltonian deformations. That is, $L$ is Hamiltonian stationary if

$$\frac{d}{ds} \text{Vol}_g(\text{Exp}(sv_F)L)|_{s=0} = 0$$

for all smooth $F : M \rightarrow \mathbb{R}$. By Oh [18, Th. I], (2) is equivalent to the Euler–Lagrange equation

$$d^* \alpha_H = 0,$$

where $H$ is the mean curvature vector of $L$, and $\alpha_H = (H \cdot \omega)|_L$ is the associated 1-form of $H$ on $L$, and $d^*$ is the Hodge dual of the exterior derivative $d$ on $L$, computed using the metric $h = g|_L$.

When $(M, \omega, g)$ is a Calabi-Yau manifold, one can choose a holomorphic $(n, 0)$-form $\Omega$ on $M$ with $\nabla \Omega = 0$, normalized so that

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}.$$
Here $L$ does not need to be Hamiltonian stationary.

Oh proves in [18, Th. IV] that the torus $T_{a_1,\ldots,a_n}^n$ in $\mathbb{C}^n$ given by (1) is Hamiltonian stationary with (4) nonnegative definite (Hamiltonian stable), and $\text{Ker} \ L$ at $T_{a_1,\ldots,a_n}^n$ consist of functions of the following form

$$Q(z_1,\ldots,z_n) = a + \sum_{j=1}^n (b_j z_j + \bar{b}_j \bar{z}_j) + \sum_{j \neq k} c_{jk} z_j \bar{z}_k$$

restricted on $T_{a_1,\ldots,a_n}^n$, where $a \in \mathbb{R}$, $b_j, c_{jk} \in \mathbb{C}$, and $c_{jk} = \bar{c}_{kj}$ (Hamiltonian rigid, see [10]). If we write $T_{a_1,\ldots,a_n}^n$ in polar coordinates

$$T_{a_1,\ldots,a_n}^n = \{(a_1 e^{\sqrt{-1} \theta_1},\ldots,a_n e^{\sqrt{-1} \theta_n}) \in \mathbb{C}_n : \theta_i \in [0,2\pi), i = 1,\ldots,n\},$$

then $\text{Ker} \ L$ is spanned by

$$1, \cos \theta_i, \sin \theta_i, \cos(\theta_i - \theta_j), \sin(\theta_i - \theta_j)$$

for $i, j = 1,\ldots,n$ and $i \neq j$ [18].

### 2.2 New Darboux coordinates

The convention for curvature operator used in this paper is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$

and

$$R_{ijkl} = \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle.$$

We use the same notion for complex curvature operator and denote

$$R_{ijkl} = \langle R(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}), \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^l} \rangle.$$

The basic definitions and properties for curvature of Kähler metrics can be found in [19].

Denote $\mathbb{C}^n$ with complex coordinates $(z_1,\ldots,z_n)$, where $z_j = x_j + \sqrt{-1}y_j$. Define the standard Euclidean metric $g_0$, Kähler form $\omega_0$, and complex structure $J_0$ on $\mathbb{C}^n$ by

$$g_0 = \sum_{j=1}^n |dz_j|^2 = \sum_{j=1}^n (dx_j^2 + dy_j^2),$$

$$\omega_0 = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d\bar{z}_j = \sum_{j=1}^n dx_j \wedge dy_j, \quad \text{and}$$

$$J_0 = \sum_{j=1}^n \left(\sqrt{-1} dx_j \otimes \frac{\partial}{\partial \bar{z}_j} - \sqrt{-1} dy_j \otimes \frac{\partial}{\partial \bar{z}_j} \right) = \sum_{j=1}^n \left(dx_j \otimes \frac{\partial}{\partial y_j} - dy_j \otimes \frac{\partial}{\partial x_j} \right),$$

noting that $dz_j = dx_j + \sqrt{-1} dy_j$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} (\frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial y_j}).$

Darboux’s Theorem says that we can find local coordinates near any point on a symplectic manifold such that the symplectic structure is like $\omega_0$ in $\mathbb{C}^n$ in these coordinates, which will be called Darboux coordinates. Because we need a good control on the metric as well, we will modify the proof to find better Darboux
coordinates. We first start with holomorphic normal coordinates at points in a Kähler manifold, and proceed as in [10, Prop. 3.2] to convert them into Darboux coordinates. To meet our need, we will not only derive the leading coefficients of the metric in this new Darboux coordinates, but also the coefficients of the next order. More precisely, we have

**Proposition 2.3.** Let \((M, \omega, g)\) be a compact \(n\)-dimensional Kähler manifold with associate Kähler form \(\omega\) and let \(U\) be the \(U(n)\) frame bundle of \(M\). Then for small \(\epsilon > 0\) we can choose a family of embeddings \(\Upsilon_{p,v} : B_\epsilon \to M\) depending smoothly on \((p, v) \in U\), where \(B_\epsilon\) is the ball of radius \(\epsilon\) about 0 in \(\mathbb{C}^n\), such that for all \((p, v) \in U\) we have:

1. \(\Upsilon_{p,v}(0) = p \) and \(d\Upsilon_{p,v}|_0 = v : \mathbb{C}^n \to T_pM\);
2. \(\Upsilon_{p,v} \circ \gamma = \Upsilon_{p,v}\) for all \(\gamma \in U(n)\);
3. \(\Upsilon_{p,v}(\omega) = \omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j\); and
4. \(\Upsilon_{p,v}^*(\omega) = g_0 + \frac{1}{2} \sum Re(R_{ij\bar{k}\ell}(p) z^i \bar{z}^j d\bar{z}^k d\bar{z}^\ell) + \frac{1}{5} \sum Re(R_{ij\bar{k}\ell;m}(p) z^i \bar{z}^j \bar{z}^m d\bar{z}^k d\bar{z}^\ell) + O(|z|^4)\).

**Proof.** For \((p, v) \in U\), we can find holomorphic normal coordinates that is an embedding \(\Upsilon_{p,v}' : B_{\epsilon'} \to M\) satisfying (i),(ii), and

\[
(\Upsilon_{p,v})^*(g) = g_0 - \sum_{i,j,k,l} R_{ij\bar{k}\ell}(p) z^k \bar{z}^l d\bar{z}^k d\bar{z}^\ell + O(|z|^4)
- \frac{1}{2} \sum_{i,j,k,l,m} (R_{ij\bar{k}\ell;m}(p) z^k \bar{z}^l z^m + R_{ij\bar{k}\ell;m}(p) z^k \bar{z}^l \bar{z}^m) d\bar{z}^k d\bar{z}^\ell.
\]

The pull back of the Kähler form has a corresponding similar expression, and \(\Upsilon_{p,v}'\) depends smoothly on \(p, v\). As in the proof of [10, Prop. 3.2], we can use Moser’s method in proving Darboux’ Theorem [15] to modify the maps \(\Upsilon_{p,v}'\) to \(\Upsilon_{p,v}\) with \(\Upsilon_{p,v}(\omega) = \omega_0\). Define closed 2-forms \(\omega_{p,v}^s\) on \(B_{\epsilon'}\) for \((p, v) \in U\) and \(s \in [0,1]\) by \(\omega_{p,v}^s = (1-s)\omega_0 + s(\Upsilon_{p,v}')^*(\omega)\). Then there exist 1-forms \(\zeta_{p,v}\) on \(B_{\epsilon'}\) satisfying \(\frac{\sqrt{-1}}{2} d\zeta_{p,v} = \omega_0 - (\Upsilon_{p,v}')^*(\omega)\), which can be chosen as

\[
\zeta_{p,v} = \frac{1}{4} R_{ij\bar{k}\ell}(p) z^k \bar{z}^\ell (-\bar{z}^j dz^i + z^i d\bar{z}^j) + \frac{1}{10} R_{ij\bar{k}\ell;m}(p) z^k \bar{z}^\ell z^m (-\bar{z}^j dz^i + z^i d\bar{z}^j)
+ \frac{1}{10} R_{ij\bar{k}\ell;m}(p) z^k \bar{z}^\ell \bar{z}^m (-\bar{z}^j dz^i + z^i d\bar{z}^j) + O(|z|^5).
\]

We use the convention that repeated indices stand for a summation whenever there is no confusion. The first term of \(d\zeta_{p,v}\) is computed in the following to
demonstrate the argument, 
\[ d\left(\frac{1}{4} R_{ijkl}(p) z^k \bar{z}^l \left(-\bar{z}^j d\bar{z}^i + z^i d\bar{z}^j\right)\right) \]
\[ = -\frac{1}{4} R_{ijkl}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j - \frac{1}{4} R_{ij\bar{k}\bar{l}}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j + \frac{1}{4} R_{i\bar{j}\bar{k}l}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j + \frac{1}{4} R_{i\bar{j}l\bar{k}}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j \]
\[ = + \frac{1}{4} R_{ikl}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j + \frac{1}{4} R_{ik\bar{l}}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j + \frac{1}{4} R_{i\bar{k}l}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j + \frac{1}{4} R_{i\bar{k}\bar{l}}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j \]
\[ = R_{ijkl}(p) z^k \bar{z}^l d\bar{z}^i \wedge dz^j. \]

In the second equality we use \( R_{ijkl}(p) = R_{kjl}(p) = R_{iklj}(p) \) which is implied by
the Kähler condition, and the last equality follows from changing the indices.
The other terms can be computed similarly. Noting that in the second term of
\( \zeta \), the coefficient of \( dz^i \) has two \( \bar{z} \) and the coefficient of \( d\bar{z}^j \)
has three \( z \), it thus changes the constant factor from \( \frac{1}{10} \) to \( \frac{1}{2} \) when we take the exterior derivative.

Now let \( v^*_{p,v} \) be the unique vector field on \( B_\epsilon \) with \( v^*_{p,v} \cdot \zeta_{p,v} = \frac{\sqrt{2\pi}}{2} \zeta_{p,v} \). If we
denote \( v^*_{p,v} = 2 \Re \sum_j a^{s,j}_{p,v} \frac{\partial}{\partial x^j} = \sum_j \Re(a^{s,j}_{p,v}) \frac{\partial}{\partial x^j} + \Im(a^{s,j}_{p,v}) \frac{\partial}{\partial y^j} \), the coefficient
\( a^{s,j}_{p,v} \) will be
\[ \frac{1}{4} R_{ijkl}(p) z^k \bar{z}^l z^j + \frac{1}{10} R_{ij\bar{k}\bar{l}}(p) z^k \bar{z}^l z^j \]
\[ + \frac{1}{10} R_{ij\bar{k}l}(p) z^k \bar{z}^l z^j + \frac{1}{10} R_{i\bar{k}l\bar{j}}(p) z^k \bar{z}^l z^j + O(|z|^5). \]

For \( 0 < \epsilon \leq \epsilon' \) we construct a family of embeddings \( \varphi_{p,v} : B_\epsilon \rightarrow B_\epsilon \) with \( \varphi_{p,v}^0 = \text{id} : B_\epsilon \rightarrow B_\epsilon \subset B_\epsilon \) by solving the system \( \frac{d}{dx} \varphi_{p,v}^s = v^s_{p,v} \circ \varphi_{p,v}^s \). By compactness
of \([0,1] \times U\), this is possible provided \( \epsilon > 0 \) is small enough. Then \( (\varphi_{p,v}^*)^\ast(\omega_{p,v}) = \omega_0 \) for all \( s \), so that \( (\varphi_{p,v}^1)^\ast((\gamma_{p,v}^{'})^\ast(\omega)) = \omega_0 \). The \( j \)-th component of \( \gamma_{p,v}^{'}, \gamma_{p,v} \) in \( z \) coordinates is
\[ z^j + \frac{1}{4} R_{ijkl}(p) z^k \bar{z}^l z^j + \frac{1}{10} R_{ij\bar{k}\bar{l}}(p) z^k \bar{z}^l z^j + \frac{1}{10} R_{ij\bar{k}l}(p) z^k \bar{z}^l z^j + \frac{1}{10} R_{i\bar{k}l\bar{j}}(p) z^k \bar{z}^l z^j + O(|z|^5). \]

Define \( \gamma_{p,v}^* = \gamma_{p,v}^{'}, \gamma_{p,v} \). Then \( \gamma_{p,v}^{'}, \gamma_{p,v} \) depends smoothly on \( p, v \). Direct computa-
tions give
\[ \gamma_{p,v}^*(g) = g_0 + \frac{1}{2} \sum_{i,j,k,l} \Re(R_{ijkl}(p) z^i \bar{z}^j d\bar{z}^l dz^l) \]
\[ + \frac{1}{5} \sum_{i,j,k,l,m} \Re(R_{ij\bar{k}\bar{l}}(p) z^i \bar{z}^j \bar{z}^m dz^l dz^l) \]
\[ + \frac{2}{5} \sum_{i,j,k,l,m} \Re(R_{ij\bar{k}l}(p) z^i \bar{z}^j \bar{z}^m dz^l dz^l) + O(|z|^4). \]
The different coefficients $\frac{1}{5}$ and $\frac{2}{5}$ in (10) comes from the fact that their corresponding terms in (9) respectively have one $\bar{\varepsilon}$ and two $\bar{\varepsilon}$. The rest of the proof is the same as [10, Prop. 3.2], and we refer to the proof there for details. □

Remark 2.4. The Kähler manifold $M$ does not need to be compact if we allow $\epsilon$ depending on points.

3 Approximate examples with estimates

For $0 < t \leq R^{-1}\epsilon$, consider the dilation map $t : B_R \to B_e$ mapping $t : (z_1, \ldots, z_n) \mapsto (t z_1, \ldots, t z_n)$. Then $\Upsilon_{p,v} \circ t$ is an embedding $B_R \to M$, so we can consider the pullbacks $(\Upsilon_{p,v} \circ t)^*(\omega)$ and $(\Upsilon_{p,v} \circ t)^*(g)$. Define a Riemannian metric $g^t_{p,v}$ on $B_R$ by $g^t_{p,v} = t^{-2}(\Upsilon_{p,v} \circ t)^*(g)$. It depends smoothly on $t \in (0, R^{-1}\epsilon]$ and $(p, v) \in U$, and satisfies

$$g^t_{p,v} = g_0 + \frac{t^2}{2} \sum_{i,j,k,l} \text{Re}(R_{ijkl}(p) z^i \bar{z}^j d\bar{z}^k d\bar{z}^l) + \frac{t^3}{5} \sum_{i,j,k,l,m} \text{Re}(R_{ijklm}(p) z^i \bar{z}^j \bar{z}^k \bar{z}^l d\bar{z}^m d\bar{z}^l) + \frac{2t^3}{5} \sum_{i,j,k,l,m} \text{Re}(R_{ijklm}(p) z^i \bar{z}^j \bar{z}^k \bar{z}^l d\bar{z}^m d\bar{z}^l) + O(t^4|z|^4). \quad (11)$$

Since $t^{-2}(\Upsilon_{p,v} \circ t)^*(g)$ is compatible with $t^{-2}(\Upsilon_{p,v} \circ t)^*(\omega)$, we have that $g^t_{p,v}$ is compatible with the fixed symplectic form $\omega_0$ on $B_R$ for all $t, p, v$. Moreover, there are uniform estimates on these metrics, which are summarized in the following proposition.

Proposition 3.1. There exist positive constants $C_0, C_1, C_2, \ldots$, such that for all $t \in (0, \frac{1}{2}R^{-1}\epsilon]$ and $(p, v) \in U$, the metric $g^t_{p,v} = t^{-2}(\Upsilon_{p,v} \circ t)^*(g)$ on $B_R$ satisfies the estimates

$$\|g^t_{p,v} - g_0\|_{C^0} \leq C_0 t^2 \quad \text{and} \quad \|\partial^k g^t_{p,v}\|_{C^0} \leq C_k t^{k+1} \quad \text{for } k = 1, 2, \ldots, \quad (12)$$

where norms are taken w.r.t. $g_0$, and $\partial$ is the Levi-Civita connection of $g_0$.

Proof. This is the same as [10, Prop. 3.4]. But since we have a better estimate on the metric from Proposition 2.3, we can increase the order on $t$ by 1. □

We can assume $\sum_{j=1}^n a_j^2 = 1$ for simplicity. The image $(\Upsilon_{p,v} \circ t)(T_{a_1, \ldots, a_n})$ is a Lagrangian contained in a $B_{2t}$ ball at $p$ in $M$. Since the geometry of $B_{2t}(p)$ in $(M, \omega, g)$ is the same as $(B_2, \omega_0, g^t_{p,v})$ in $\mathbb{C}^n$, we will have all computations and discussions in $(B_2, \omega_0, g^t_{p,v})$ instead for simplicity. In the coordinates $z^j = r_j e^{\sqrt{-1}\theta_j}, \ j = 1, \ldots, n$, the metric $g^t_{p,v}$ becomes

$$g^t_{p,v} = \sum (dr_j^2 + r_j^2 d\theta_j^2) + \sum (t^2 \text{Re} A_{ij} + t^3 \text{Re} C_{ij})(dr_i dr_j - r_i r_j d\theta_i d\theta_j) + \sum (t^2 \text{Im} A_{ij} + t^3 \text{Im} C_{ij})(r_i d\theta_i dr_j + r_j dr_i d\theta_j) + O(t^4|z|^4), \quad (13)$$
where
\[
A_{ij} = A_{ji} = \frac{1}{2} \sum_{p,q} R_{p+i\bar{j}}(p) r_p r_q e^{\sqrt{-1}(\theta_p + \theta_q - \theta_i - \theta_j)},
\]
\[
C_{ij} = C_{ji} = \frac{1}{5} \sum_{p,q,m} R_{p+i\bar{j},m}(p) r_p r_q r_m e^{\sqrt{-1}(\theta_p + \theta_q - \theta_i - \theta_j + \theta_m)}
+ \frac{2}{5} \sum_{p,q,m} R_{p+i\bar{j},m}(p) r_p r_q r_m e^{\sqrt{-1}(\theta_p + \theta_q - \theta_i - \theta_j - \theta_m)}.
\] (14)

The restriction of \( g^k_{p,v} \) on \( T^n_{a_1,\ldots,a_n} \) is
\[
h^k_{p,v} = \sum a_i^2 d\theta^2_i - \sum a_i a_j (t^2 \text{ Re } A_{ij} + t^3 \text{ Re } C_{ij}) d\theta_i d\theta_j + O(t^4),
\] (15)
noting that \(|z|\) is assumed to be 1 on \( T^n_{a_1,\ldots,a_n} \). For simplicity, we omit the restriction of \( A_{ij} \) and \( C_{ij} \) on \( T^n_{a_1,\ldots,a_n} \) in (15), and will denote \( g^k_{p,v} \) by \( g_t \) and \( h^k_{p,v} \) by \( h_t \) when there is no confusion. A direct computation yields
\[
h_t^{ij} = \frac{1}{a_i^2} \delta_{ij} + \frac{t^2}{a_i a_j} \text{ Re } A_{ij} + \frac{t^3}{a_i a_j} \text{ Re } C_{ij} + O(t^4),
\]
\[
g_t^{\theta_i \theta_j} = \delta_{ij} - t^2 \text{ Re } A_{ij} - t^3 \text{ Re } C_{ij} + O(t^4 |z|^4),
\]
\[
g_t^{r_1 r_j} = \frac{1}{r_i r_j} \delta_{ij} + \frac{t^2}{r_i r_j} \text{ Re } A_{ij} + \frac{t^3}{r_i r_j} \text{ Re } C_{ij} + O(t^4 |z|^4),
\]
\[
g_t^{r_1 \theta_j} = -\frac{t^2}{r_j} \text{ Im } A_{ij} - \frac{t^3}{r_j} \text{ Im } C_{ij} + O(t^4 |z|^4).\] (16)

Now we are ready to compute the corresponding \( \mathbb{d}^* \alpha_t \) of the initial Lagrangian \( T^n_{a_1,\ldots,a_n} \subset (B_2, \omega_0, g^k_{p,v}) \), and estimate how far it is away from 0.

**Lemma 3.2.** Denote the mean curvature vector of \( T^n_{a_1,\ldots,a_n} \) with respect to \( g_t \) by \( H_t \) and let \( \alpha_t = H_t \cdot \omega_0 = \sum a_i^2 d\theta_i \).Then
\[
\mathbb{d}^* \alpha_t = -\sum_{i,j} \left( \frac{1}{a_i a_j} \frac{\partial}{\partial \theta_i} \text{ Im } (t^2 A_{ij} + t^3 C_{ij}) \right) + \frac{1}{a_i^2} \frac{\partial}{\partial \theta_i} \text{ Re } (t^2 A_{ij} + t^3 C_{ij})
+ \frac{1}{a_i^2} \frac{\partial}{\partial \theta_i} \text{ Re } (t^2 A_{ij} + t^3 C_{ij}) + O(t^4),\] (17)

where \( A_{ij} \) and \( C_{ij} \) are as defined in (14).

**Proof.** Because \( \omega_0 = \sum r_k d r_k \wedge d\theta_k \) and \( T^n_{a_1,\ldots,a_n} \) is Lagrangian, it follows that
\[
\alpha_t = a_k \sum_{i,j} h_{ij}^k (\Gamma^k_{il})_{i,j}^a, \text{ where } (\Gamma^k_{il})_{i,j}^a \text{ is the Christoffel symbol w.r.t. } g_t. \text{ A direct computation gives}
\]
\[
\alpha_t = -1 - \text{ Re } (t^2 A_{kk} + t^3 C_{kk}) + \sum a_k a_i (\text{ Re } (t^2 A_{ik} + t^3 C_{ik}) + \frac{\partial}{\partial \theta_i} \text{ Im } (t^2 A_{ik} + t^3 C_{ik}))
+ \frac{a_k}{2} \sum_i \frac{\partial r_i^2}{\partial \theta_i} \text{ Re } (t^2 A_{ii} + t^3 C_{ii}) + O(t^4).\]
Further computation shows that the $t^2$ and $t^3$ terms of $\alpha^k_t$ are

$$B_k = \sum_i a_k \frac{\partial}{\partial \theta_i} \text{Im}(t^2 A_{ik} + t^3 C_{ik}) + \frac{a_k}{2} \sum_i \frac{\partial}{\partial \theta_i} \text{Re}(t^2 A_{ii} + t^3 C_{ii})$$

$$+ \sum_i \frac{a_k}{a_i} \text{Re}(t^2 A_{ik} + t^3 C_{ik}).$$

(18)

Recall that

$$d^* \alpha_t = -\sum_i \frac{\partial h_{ij}}{\partial \theta_i} \alpha_t^j - \sum_i h_{ij} \frac{\partial \alpha_t^i}{\partial \theta_i} - \frac{1}{2} \sum_i h_{ij} \alpha_t^i \frac{\partial}{\partial \theta_i} \left( \ln \det \left( (h_t)_{ij} \right) \right).$$

Therefore,

$$d^* \alpha_t = \sum_{i,j} \frac{\partial}{\partial \theta_i} \left( \frac{1}{a_i a_j} \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right) - \sum_i \frac{1}{a_i^2} \frac{\partial}{\partial \theta_i} \left( \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right)$$

$$- \sum_{i,j} \frac{1}{2a_i} \frac{\partial}{\partial \theta_i} \left( \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right) + O(t^4).$$

(19)

From (18), we have

$$\sum_{i,j} \frac{1}{a_i} \frac{\partial}{\partial \theta_i} B_i = \sum_{i,j} \frac{1}{a_i a_j} \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \text{Im}(t^2 A_{ij} + t^3 C_{ij}) + \frac{\partial}{\partial \theta_i} \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right)$$

$$+ \frac{1}{2a_i} \frac{\partial^2}{\partial \theta_i \partial r_i} \text{Re}(t^2 A_{ij} + t^3 C_{ij})$$

(20)

Combining (19) and (20), we get

$$d^* \alpha_t = -\sum_{i,j} \left( \frac{1}{a_i a_j} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \text{Im}(t^2 A_{ij} + t^3 C_{ij}) + \frac{1}{2a_i} \frac{\partial^2}{\partial \theta_i \partial r_i} \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right)$$

$$+ \frac{1}{2a_i^2} \frac{\partial}{\partial \theta_i} \left( \text{Re}(t^2 A_{ij} + t^3 C_{ij}) \right) + O(t^4).$$

We next compute the orthogonal projection of $d^* \alpha_t$ to $\text{Ker} \mathcal{L}$. Recall that $\text{Ker} \mathcal{L}$ of $T_{a_1,\ldots,a_n}^n$ in $\mathbb{C}^n$ is spanned by $1, \cos \theta_i, \sin \theta_i, \cos(\theta_i - \theta_j), \sin(\theta_i - \theta_j)$ for $i, j = 1, \ldots, n$ and $i \neq j$. From (14), it follows that only $A_{ij}$ in the leading terms can project to $\cos(\theta_i - \theta_j)$ and $\sin(\theta_i - \theta_j)$, and only $C_{ij}$ in the leading terms can project to $\cos \theta_i$ and $\sin \theta_i$. The result is summarized in the following lemma.

**Lemma 3.3.** Denote the orthogonal projection from $L^2(T_{a_1,\ldots,a_n}^n)$ w.r.t. $g_0$ to
Ker $\mathcal{L}$ by $P$. Then

$$P \frac{d^* \alpha_t}{dt} = 2t^2 \sum_{j>i} \text{Im} \left( -a_j^2 R_{ji}^j(p) + a_i^2 R_{i}^{iii}(p) \right) \frac{\cos (\theta_j - \theta_i)}{a_i a_j}$$

$$+ 2t^2 \sum_{j>i} \text{Re} \left( -a_j^2 R_{ji}^j(p) + a_i^2 R_{i}^{iii}(p) \right) \frac{\sin (\theta_j - \theta_i)}{a_i a_j}$$

$$+ t^3 \sum_{ij} \left( \text{Im} a_i^2 R_{i}^{iii}(p) \frac{\cos \theta_j}{a_j} + \text{Re} a_i^2 R_{i}^{iii}(p) \frac{\sin \theta_j}{a_j} \right) + O(t^4). \quad (25)$$

**Proof.** From the expression of $A_{ij}$ in (14) and the basis of Ker $\mathcal{L}$ in (8), we have

$$PA_{ii} = \frac{1}{2} a_i^2 R_{i}^{iii}(p) + \sum_{j \neq i} a_i a_j R_{ij}^j(p) e^{\sqrt{-1}(\theta_j - \theta_i)}, \quad \text{and}$$

$$PA_{ij} = \sum_{q \neq i,j} a_i a_q R_{iq}^q(p) e^{\sqrt{-1}(\theta_q - \theta_i)} + \sum_{q \neq i,j} a_j a_q R_{qij}^q(p) e^{\sqrt{-1}(\theta_q - \theta_i)}$$

$$+ a_i a_j R_{ij}^j(p) + \frac{1}{2} a_i^2 R_{i}^{iii}(p) e^{\sqrt{-1}(\theta_i - \theta_j)} + \frac{1}{2} a_j^2 R_{i}^{iii}(p) e^{\sqrt{-1}(\theta_j - \theta_i)}$$

for $i \neq j$. Therefore,

$$-P \sum_{ij} \frac{1}{a_i a_j} \frac{\partial^2 \text{Im} A_{ij}}{\partial \theta_i \partial \theta_j} = -P \sum_i \frac{1}{a_i^2} \frac{\partial^2 \text{Im} A_{ii}}{\partial \theta_i^2} - P \sum_{i \neq j} \frac{1}{a_i a_j} \frac{\partial^2 \text{Im} A_{ij}}{\partial \theta_i \partial \theta_j}$$

$$= \sum_{j \neq i} \frac{a_j}{a_i} \text{Im} (R_{i}^{iii}(p) - R_{j}^{iii}(p)) e^{\sqrt{-1}(\theta_j - \theta_i)}. \quad (22)$$

Similar computation gives

$$-P \sum_{ij} \frac{1}{2a_i} \frac{\partial^2 \text{Re} A_{ij}}{\partial \theta_i \partial r_i} = \frac{1}{2} \sum_{i \neq j} \left( \frac{a_i}{a_j} - \frac{a_j}{a_i} \right) \text{Im} R_{i}^{iii}(p) e^{\sqrt{-1}(\theta_j - \theta_i)}, \quad (23)$$

and

$$-P \sum_{ij} \frac{1}{2a_i^2} \frac{\partial \text{Re} A_{ij}}{\partial \theta_i} = \frac{1}{2} \sum_{i \neq j} \left( \frac{a_i}{a_j} - \frac{a_j}{a_i} \right) \text{Im} R_{i}^{iii}(p) e^{\sqrt{-1}(\theta_j - \theta_i)}. \quad (24)$$

Combining (22), (23), and (24), we obtain that the coefficient of the $t^2$ term of $d^* \alpha_t$ is

$$\sum_{i \neq j} \text{Im} \left( \left( -\frac{a_j}{a_i} R_{ij}^j(p) + \frac{a_i}{a_j} R_{i}^{iii}(p) \right) e^{\sqrt{-1}(\theta_j - \theta_i)} \right)$$

$$= 2 \sum_{j>i} \text{Im} \left( -a_j^2 R_{ij}^j(p) + a_i^2 R_{i}^{iii}(p) \right) \frac{\cos (\theta_j - \theta_i)}{a_i a_j}$$

$$+ 2 \sum_{j>i} \text{Re} \left( -a_j^2 R_{ij}^j(p) + a_i^2 R_{i}^{iii}(p) \right) \frac{\sin (\theta_j - \theta_i)}{a_i a_j}. \quad (25)$$
We also have

\[ PC_{ii} = \frac{1}{5} a_i^3 R_{iiii,i}(p)e^{\sqrt{-1}\theta_i}, \]

\[ + \frac{2}{5} a_i^3 R_{iiii,i}(p)e^{-\sqrt{-1}\theta_i} + \frac{2}{5} \sum_{j \neq i} a_i^2 a_j R_{iiij,j}(p) e^{-\sqrt{-1}\theta_i}, \]

and

\[ PC_{ij} = \frac{3}{5} a_i^2 a_j R_{iijj,i}(p)e^{\sqrt{-1}\theta_i}, \]

\[ + \frac{3}{5} a_i^2 a_j R_{iijj,j}(p)e^{-\sqrt{-1}\theta_i}, \]

\[ + 6 \sum_{q \neq i,j} a_i a_j a_q R_{iijj,q}(p)e^{\sqrt{-1}\theta_i}, \]

\[ + \frac{4}{5} \sum_{q} a_i a_j a_q R_{iiij,q}(p)e^{-\sqrt{-1}\theta_i}, \]

for \( i \neq j \). Further computation gives

\[ -P \sum_{i,j} \frac{1}{a_i} \frac{\partial^2 \text{Im} C_{ij}}{\partial \theta_i \partial \theta_j} = -P \sum_{i} \frac{1}{a_i^2} \frac{\partial^2 \text{Im} C_{ii}}{\partial \theta_i^2} - P \sum_{i \neq j} \frac{1}{a_i a_j} \frac{\partial^2 \text{Im} C_{ij}}{\partial \theta_i \partial \theta_j} \]

\[ = \sum_{i} \frac{a_i}{5} \text{Im} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i} + \frac{2a_i}{5} \text{Im} R_{iiii,i}(p)e^{-\sqrt{-1}\theta_i} \]

\[ = -\sum_{i} \frac{a_i}{5} \text{Im} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i}. \] (26)

In the last equality, we use \( \text{Im} R_{iiii,i}(p)e^{-\sqrt{-1}\theta_i} = -\text{Im} R_{iiii,i}(p)e^{-\sqrt{-1}\theta_i} \). For other terms, we similarly have

\[ -P \sum_{i,j} \frac{1}{2a_i} \frac{\partial^2 \text{Re} C_{jj}}{\partial \theta_i \partial r_i} \]

\[ = -\sum_{i} \frac{1}{10a_i} \frac{\partial^2}{\partial \theta_i \partial r_i} \left( r_i^3 \text{Re} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i} + 2r_i^3 \text{Re} R_{iiii,i}(p)e^{-\sqrt{-1}\theta_i} \right) \]

\[ - \sum_{i \neq j} \frac{1}{10a_j} \frac{\partial^2}{\partial \theta_j \partial r_j} \left( 3r_i r_j \text{Re} R_{iiij,j}(p)e^{\sqrt{-1}\theta_i} + 2r_i^2 r_j \text{Re} R_{iiij,j}(p)e^{-\sqrt{-1}\theta_j} \right) \]

\[ = \frac{9}{10} \sum_{i} a_i \text{Re} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i} + \frac{1}{2} \sum_{i \neq j} \frac{a_i^2}{a_j} \text{Re} R_{iiij,j}(p)e^{\sqrt{-1}\theta_i}, \] (27)

and

\[ -P \sum_{i,j} \frac{1}{2a_i^2} \frac{\partial \text{Re} C_{jj}}{\partial \theta_i} \]

\[ = -\sum_{i} \frac{3a_i}{10} \frac{\partial \text{Re} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i}}{\partial \theta_i} - \sum_{i \neq j} \frac{a_i^2}{2a_j} \frac{\partial \text{Re} R_{iiii,j}(p)e^{\sqrt{-1}\theta_j}}{\partial \theta_j} \]

\[ = \frac{3}{10} \sum_{i} a_i \text{Re} R_{iiii,i}(p)e^{\sqrt{-1}\theta_i} + \frac{1}{2} \sum_{i \neq j} \frac{a_i^2}{a_j} \text{Re} R_{iiij,j}(p)e^{\sqrt{-1}\theta_j}. \] (28)
Putting (26), (27) and (28) together, we conclude that the coefficient of the $t^3$ term of $d^*\alpha t$ is
\[
\sum_i a_i \text{Im } R_{i\bar{i},i}(p)e^{\sqrt{-1}\theta_i} + \sum_{i\neq j} \frac{a_i^2}{a_j} \text{Im } R_{i\bar{i},j}(p)e^{\sqrt{-1}\theta_j} = \sum_{ij} \text{Im } a_i^2 R_{i\bar{i},j}(p) \frac{\cos \theta_j}{a_j} + \text{Re } a_i^2 R_{i\bar{i},j}(p) \frac{\sin \theta_j}{a_j}. \tag{29}
\]
Thus (25) and (29) yield (21). \qed

4 Perturbation

We will formulate a family of fourth-order nonlinear elliptic partial differential operators $P_{p,v}^t : C^\infty(T^n,\omega) \to C^\infty(T^n,\omega)$ depending on $(p,v) \in U$ and small $t > 0$, such that $C^1$-small $f \in C^\infty(T^n,\omega)$ correspond to Lagrangians $L^t_{p,v}$ in $M$, and $L^t_{p,v}$ is Hamiltonian stationary when $P_{p,v}^t(f) = 0$.

We first set up the problem and introduce some related properties. Let $L$ be a real $n$-manifold. Then its cotangent bundle $T^*L$ has a canonical symplectic form $\omega$, defined as follows. Let $(x_1, \ldots, x_n)$ be local coordinates on $L$. Extend them to local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*L$ such that $(x_1, \ldots, y_n)$ represents the 1-form $y_1dx_1 + \cdots + y_ndx_n$ in $T^*_L$. Then $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Identify $L$ with the zero section in $T^*L$. Then $L$ is a Lagrangian submanifold of $(T^*L, \omega)$. The following theorem shows that any compact Lagrangian submanifold $L$ in a symplectic manifold looks locally like the zero section in $T^*L$.

Lagrangian Neighborhood Theorem [14, Th. 3.33] Let $(M,\omega)$ be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighborhood $T$ of the zero section $L$ in $T^*L$, and an embedding $\Phi : T \to M$ with $\Phi|_L = \text{id} : L \to L$ and $\Phi^*(\omega) = \omega$, where $\omega$ is the canonical symplectic structure on $T^*L$.

We shall call $T, \Phi$ a Lagrangian neighborhood of $L$. Such neighborhoods are useful for parameterizing nearby Lagrangian submanifolds of $M$. Suppose that $L$ is a Lagrangian submanifold of $M$ which is $C^1$-close to $L$. Then $L$ lies in $\Phi(T)$, and is the image $\Phi(\Gamma_\alpha)$ of the graph $\Gamma_\alpha$ of a unique $C^1$-small 1-form $\alpha$ on $L$. As $L$ is Lagrangian and $\Phi^*(\omega) = \omega$ we see that $\omega|_{\Gamma_\alpha} \equiv 0$. But $\omega|_{\Gamma_\alpha} = -\pi^*(d\alpha)$, where $\pi : \Gamma_\alpha \to L$ is the natural projection. Hence $d\alpha = 0$, and $\alpha$ is a closed 1-form. This establishes a 1-1 correspondence between $C^1$-small closed 1-forms on $L$ and Lagrangian submanifolds $L$ close to $L$ in $M$.

Making $T$ smaller if necessary, we can suppose $T$ is of the form
\[
T = \{(p,\alpha) : p \in L, \alpha \in T^*_pL, |\alpha| < \delta\} \tag{30}
\]
for some small $\delta > 0$, where $|\alpha|$ is computed using the metric $g_0|_L$. Now take $L = T^n_{a_1,\ldots,a_n} \subset C^n$. Let $T^n_{a_1,\ldots,a_n} \subset \Phi(T) \subset B_2 \subset C^n$ and $f \in C^\infty(T^n_{a_1,\ldots,a_n})$.
with \( \|df\|_{C^0} < \delta \). Define the graph \( \Gamma_{df} \) of \( df \) to be \( \Gamma_{df} = \{(q, df|_q) : q \in T^n_{a_1, \ldots, a_n}\} \). Then \( \Gamma_{df} \) is an embedded Lagrangian submanifold in \((T^n, \omega)\). Since \( \Phi^*(\omega_0) = \hat{\omega} \), we see that \( \Phi(\Gamma_{df}) \) is Lagrangian in \((B_2, \omega_0)\).

Let \( 0 < t \leq \frac{1}{2} \epsilon \). For each \( f \in C^\infty(T^n_{a_1, \ldots, a_n}) \) with \( \|df\|_{C^0} < \delta \), define \( L^{t,f}_{p,v} = \Upsilon_{p,v} \circ t \circ \Phi(\Gamma_{df}) \). Then \( L^{t,f}_{p,v} \) is an embedded submanifold of \( M \) diffeomorphic to \( T^n_{a_1, \ldots, a_n} \), and as \( \Phi(\Gamma_{df}) \) is Lagrangian in \((B_2, \omega_0)\) and \((\Upsilon_{p,v} \circ t)^*(\omega) = t^2 \omega_0\), we see that \( L^{t,f}_{p,v} \) is Lagrangian in \((M, \omega)\). We can further restrict \( \int_{T^n_{a_1, \ldots, a_n}} f \, d\nu = 0 \) because \( f \) and \( f + c \) define the same Lagrangian submanifold. Here \( d\nu \) is the induced volume form on \( T^n_{a_1, \ldots, a_n} \) w.r.t. \( g_0 \). Denote the induced metric of \( g^t_{p,v} \) on \( \Phi(\Gamma_{df}) \) by \( h^{t,f}_{p,v} \) and \( \Phi_f : q \in T^n_{a_1, \ldots, a_n} \mapsto \Phi(q, df|_q) \). We define

\[
P^t_{p,v} : \{ f \in C^\infty(T^n_{a_1, \ldots, a_n}) : \|df\|_{C^0} < \delta \} \rightarrow C^\infty(T^n_{a_1, \ldots, a_n})
\]

to be the Euler–Lagrangian operator of

\[
F^t_{p,v}(f) = \text{Vol}_{h^t_{p,v}} \Phi(\Gamma_{df}) = \int_{\Phi(\Gamma_{df})} dV_{h^t_{p,v}}
\]

\[
= \int_{T^n_{a_1, \ldots, a_n}} (\Phi_f)^*(dV_{h^t_{p,v}})
\]

\[
= \int_{T^n_{a_1, \ldots, a_n}} G^t_{p,v}(q, df|_q, \nabla df|_q) \, d\nu,
\]

where \( \nabla \) is the Levi-Civita connection of the induced metric \( h_0 \) of \( g_0 \) on \( T^n_{a_1, \ldots, a_n} \).

Assume that \( -d^*\alpha_{H_f} \) is computed w.r.t. \( g^t_{p,v} \) at \( \Phi(\Gamma_{df}) \), then

\[
P^t_{p,v}(f) = -\left(\sqrt{\det((\Phi_f)^*(h^t_{p,v}))}/\sqrt{\det(h_0)}\right) d^*\alpha_{H_f} = -J^t_{p,v} d^*\alpha_{H_f}.
\]

Here we identify the function \( d^*\alpha_{H_f} \) on \( \Phi(\Gamma_{df}) \) with its pull back on \( T^n_{a_1, \ldots, a_n} \) for simplicity. Because \( J^t_{p,v} \neq 0 \), it implies that \( P^t_{p,v}(f) \equiv 0 \) if and only if \( d^*\alpha_{H_f} \equiv 0 \). Thus a zero of \( P^t_{p,v} \) will give a Hamiltonian stationary Lagrangian w.r.t. \( g^t_{p,v} \). We choose \( P^t_{p,v}(f) \) instead of \( d^*\alpha_{H_f} \) just for technical reason.

Noting that \( g^0_{p,v} = g_0 \), we define \( F_0(f) = \text{Vol}_{g_0} \Phi(\Gamma_{df}) \). Denote the corresponding operator at \( t = 0 \) by \( P_0 \), and the linearized operators of \( P^t_{p,v} \) and \( P_0 \) at 0 by \( L^t_{p,v} \) and \( L \) respectively. Here \( L \) is the same as (5) w.r.t. \( g_0 \), but \( L^t_{p,v} \) is slightly different from the one in (5) w.r.t. \( g^t_{p,v} \). We then have

**Proposition 4.1.** [10, Prop. 4.1] Let any \( k \geq 0 \), \( \gamma \in (0,1) \), and small \( \delta' > 0 \) and \( \zeta > 0 \) be given. Then if \( t > 0 \) is sufficiently small, for all \( f \in C^{k+4, \gamma}(T^n_{a_1, \ldots, a_n}) \) with \( \|df\|_{C^0} \leq \frac{1}{2} \delta \) and \( \|\nabla df\|_{C^0} \leq \delta' \), and all \( (p,v) \in \bar{U} \) we have

\[
\|P^t_{p,v}(f) - P_0(f)\|_{C^{k,\gamma}} \leq \zeta \quad \text{and} \quad \|L^t_{p,v}(f) - L(f)\|_{C^{k,\gamma}} \leq \zeta \|f\|_{C^{k+4, \gamma}}.
\]

That is, by taking \( t \) small we can suppose \( P^t_{p,v} \) and \( L^t_{p,v} \) are arbitrarily close to \( P_0 \) and \( L \) as operators from \( C^{k+4, \gamma}(T^n_{a_1, \ldots, a_n}) \) to \( C^{k,\gamma}(T^n_{a_1, \ldots, a_n}) \) (on their respective domains) uniformly in \( (p,v) \in \bar{U} \).
Remark 4.2. From Proposition 3.1, it follows that we only need to take $ct^2 \leq \zeta$ for some fixed constant $c$ (also see the Appendix).

Theorem 4.3. [10, Th. 5.1] Suppose that $0 < t \leq \frac{1}{2} \epsilon$ is sufficiently small and fixed. Then for all $(p, v) \in U$, there exists $f^t_{p,v} \in C^\infty(T_{a_1, \ldots, a_n})$ satisfying

$$P^t_{p,v}(f^t_{p,v}) \in \text{Ker } L \quad \text{and} \quad f^t_{p,v} \perp \text{Ker } L,$$

where $f^t_{p,v} \perp \text{Ker } L$ means $f^t_{p,v}$ is $L^2$-orthogonal to $\text{Ker } L$. Furthermore $f^t_{p,v}$ is the unique solution of (35) with $\|f^t_{p,v}\|_{C^4, \gamma}$ small, and $f^t_{p,v}$ depends smoothly on $(p, v) \in U$.

Because $T^a_{a_1, \ldots, a_n}$ is $T^n$ invariant, it induces a $T^n$ action on the cotangent bundle of $T_{a_1, \ldots, a_n}$ and $T$ is $T^n$-invariant. Moreover, we can choose $\Phi$ to be equivariant under the actions of $T^n$ on $T$ and $\mathbb{C}^n$ following the proof of the dilation-equivariant Lagrangian Neighborhood Theorem in [9, Th. 4.3]. Furthermore, the functions $f^t_{p,v}$ in Theorem 4.3 satisfy $f^t_{p,v}\gamma = f^t_{p,v} \circ \gamma$ for $\gamma \in T^n$. Define $L^t_{p,v} = L^t_{p,v} : (p, v) \in U$ and a smooth map $H^t : U \to \text{Ker } L$ by $H^t : (p, v) \mapsto P^t_{p,v}(f^t_{p,v})$. The map $H^t$ is $T^n$-equivariant, and depends on $t$ smoothly as $t$ changes. We refer to [10] for a detailed discussion on the setting and properties.

Now we are ready to state and prove our main result:

Theorem 4.4. Suppose that $(M, \omega, g)$ is an $n$-dimensional Kähler manifold, and $U$ is the $U(n)$ frame bundle of $M$. The subgroup of diagonal matrices $T^n \subset U(n)$ acts on $U$. Define $F_{a_1, \ldots, a_n} : U/T^n \to \mathbb{R}$ by $F_{a_1, \ldots, a_n}(p, [v]) = \sum_{i=1}^n a_i^2 R^i_{i\overline{i}}(p)$ for any given $a_i > 0$, $i = 1, \ldots, n$, where $p \in M$ and $v \in U(n)$. The holomorphic sectional curvature $R^i_{i\overline{i}}(p)$ is computed w.r.t. the unitary frame $v$ at $p$ whose value is independent of the representative $v$ of $[v]$. Assume that $(p_0, [v_0]) \in U/T^n$ is a non-degenerate critical point of $F_{a_1, \ldots, a_n}$, then for $t$ small there exist a smooth family $(p(t), [v(t)]) \in U/T^n$ satisfying $(p(0), [v(0)]) = (p_0, [v_0])$ and a smooth family of smooth embedded Hamiltonian stationary Lagrangian tori with radii $(t a_1, \ldots, t a_n)$ center at $p(t)$ which are invariant under local $T^n$ action and are posited w.r.t any representative of $[v(t)]$. Moreover, the distance between $(p(t), [v(t)])$ and $(p_0, [v_0])$ in $U/T^n$ is bounded by $ct^2$. The family of embedded Hamiltonian stationary Lagrangian tori do not intersect each other when $t$ is small.

Proof. By Theorem 4.3, the problem of finding Hamiltonian stationary Lagrangians is reduced to finding zeros of $H^t$. Because we will change $(p, v) \in U$, we now rewrite $\alpha_i$ in §3 as $\alpha_{p,v}^t$ to indicate its dependency. We have

$$P^t_{p,v}(0) = -\left(\frac{\det(h^t_{p,v})}{\sqrt{\det(h_0)}} \right) \text{d}^* \alpha^t_{p,v} = -\text{d}^* \alpha^t_{p,v} + O(t^4)$$

from (15) and Lemma 3.2. Hence $\|P^t_{p,v}(0)\|_{C^{k, \gamma}} \leq ct^2$ by Lemma 3.2 again. The Implicit Function Theorem used in the proof of Theorem 4.3 will then give
\[ \| f_\gamma \|_{C^{k+4,\gamma}} \leq ct^2. \] The constant \( c \) may need to be modified at different places, but we will still use the same symbol. It follows from the Appendix that

\[ P_{p,v}(f^t_{p,v}) = -P d^* \alpha_{p,v}^t + O(t^4), \quad (36) \]

where \( P \) is the orthogonal projection from \( L^2(T^n_{a_1,\ldots,a_n}) \) w.r.t. \( g_0 \) to \( \text{Ker} \mathcal{L} \). We remark that from the definition of \( P_{p,v}(f) \) in (33) we have

\[ \int_{T^n_{a_1,\ldots,a_n}} P_{p,v}(f^t_{p,v}) \, dV_0 = 0. \]

Now we need to find \( (p(t), v(t)) \in U \) such that the coefficients of \( \cos \theta_i, \sin \theta_i, \cos(\theta_i - \theta_j) \), and \( \sin(\theta_i - \theta_j) \) for \( P_{p,v}(f^t_{p,v}) \) all vanish for \( i, j = 1, \ldots, n \) and \( i \neq j \). Because \( H^t \) is \( T^n \)-equivariant, if \( (p(t), v(t)) \) is a zero of \( H^t \), so is \( (p(t), v(t) \circ \gamma) \) for any \( \gamma \in T^n \). However, they determine the same Hamiltonian stationary Lagrangian torus since \( f^t_{p,v(0)} = f^t_{p,v} \circ \gamma \). That is, the Hamiltonian stationary Lagrangian torus obtained is locally \( T^n \) invariant.

Suppose that \( (p_0, v_0) \in U/T^n \) is a critical point of \( F_{a_1,\ldots,a_n} \). We can also consider \( F_{a_1,\ldots,a_n} \) as a function on \( U \), and \( (p_0, v_0) \in U \) is its critical point. Applying variations which vary \( p \) in \( M \), one concludes that \( \sum_i a_i^2 R_{i\bar{i}i\bar{i}}(p_0) = 0 \) for any \( j \). It follows that the \( t^4 \) terms of \( H^t(p, v) = P_{p,v}(f^t_{p,v}) \) vanish at \( (p_0, v_0) \) by Lemma 3.3 and (36). Suppose \( a_{ij} \in \mathfrak{u}(n), i < j \), satisfy

\[ a_{ij} e_i = e_j, \quad a_{ij} e_j = -e_i, \quad \text{and} \quad a_{ij} e_k = 0 \quad \text{for} \quad k \neq i, j, \]

and \( b_{ij} \in \mathfrak{u}(n), i < j \), satisfy

\[ b_{ij} e_i = -\sqrt{-1} e_j, \quad b_{ij} e_j = -\sqrt{-1} e_i, \quad \text{and} \quad b_{ij} e_k = 0 \quad \text{for} \quad k \neq i, j. \]

Applying a variation along \( a_{ij} \in \mathfrak{u}(n) \) at \( (p_0, v_0) \), it yields

\[
0 = 2a_i^2 R_{j\bar{i}i\bar{j}}(p_0) + 2a_i^2 R_{j\bar{j}i\bar{i}}(p_0) - 2a_j^2 R_{j\bar{i}i\bar{j}}(p_0) - 2a_j^2 R_{j\bar{j}i\bar{i}}(p_0) \\
= 4 \text{Re} \left( a_i^2 R_{j\bar{i}i\bar{j}}(p_0) - a_j^2 R_{j\bar{j}i\bar{i}}(p_0) \right). \quad (37)
\]

where we have used \( R_{j\bar{i}i\bar{j}} = R_{j\bar{j}i\bar{i}} \). Similarly, applying a variation along \( b_{ij} \in \mathfrak{u}(n) \) at \( (p_0, v_0) \), it yields

\[
0 = -2a_i^2 \sqrt{-1} R_{j\bar{i}i\bar{j}}(p_0) + 2a_i^2 \sqrt{-1} R_{j\bar{j}i\bar{i}}(p_0) - 2a_j^2 \sqrt{-1} R_{j\bar{i}i\bar{j}}(p_0) + 2a_j^2 \sqrt{-1} R_{j\bar{j}i\bar{i}}(p_0) \\
= 4 \text{Im} \left( a_i^2 R_{j\bar{i}i\bar{j}}(p_0) - a_j^2 R_{j\bar{j}i\bar{i}}(p_0) \right). \quad (38)
\]

From Lemma 3.3 and (36), the equalities (37) and (38) show that the \( t^4 \) terms of \( H^t(p, v) = P_{p,v}(f^t_{p,v}) \) vanish at \( (p_0, v_0) \).

We denote

\[
P_{p,v}(f^t_{p,v}) = \sum_{j > i} D^t_{ij}(p, v) \frac{\cos(\theta_j - \theta_i)}{a_i a_j} + \sum_{j > i} E^t_{ij}(p, v) \frac{\sin(\theta_j - \theta_i)}{a_i a_j} \\
+ \sum_j F_j(p, v) \frac{\cos \theta_j}{a_j} + \sum_j G_j(p, v) \frac{\sin \theta_j}{a_j},
\]

\[ 18 \]
and for $t \neq 0$ define a new map $G^t : U \rightarrow \{ f \in \text{Ker } L : \int_{T^na_1,\ldots,a_n} f \, dV_0 = 0 \}$ by

$$G^t(p, \upsilon) = -\sum_{j>i} \frac{D^t_{ij}(p, \upsilon)}{t^2} \frac{\cos(\theta_j - \theta_i)}{a_i a_j} - \sum_{j>i} \frac{E^t_{ij}(p, \upsilon)}{t^2} \frac{\sin(\theta_j - \theta_i)}{a_i a_j} + \sum_j \frac{F^t_j(p, \upsilon)}{t^3} \frac{\cos \theta_j}{a_j} - \sum_j \frac{G^t_j(p, \upsilon)}{t^3} \frac{\sin \theta_j}{a_j}.$$ 

From (36), Lemma 3.3 and the above discussions we have

$$G^t(p, \upsilon) = \frac{1}{2} \sum_{j>i} \left( \frac{\partial F_{a_1,\ldots,a_n}^t(p, \upsilon)}{\partial \theta_j} + O(t^2) \right) \frac{\cos(\theta_j - \theta_i)}{a_i a_j} + \frac{1}{2} \sum_{j>i} \left( \frac{\partial F_{a_1,\ldots,a_n}^t(p, \upsilon)}{\partial a_{ij}} + O(t^2) \right) \frac{\sin(\theta_j - \theta_i)}{a_i a_j} + \frac{1}{2} \sum_j \left( \frac{\partial F_{a_1,\ldots,a_n}^t(p, \upsilon)}{\partial y_j} + O(t^2) \right) \frac{\cos \theta_j}{a_j} + \frac{1}{2} \sum_j \left( \frac{\partial F_{a_1,\ldots,a_n}^t(p, \upsilon)}{\partial x_j} + O(t^2) \right) \frac{\sin \theta_j}{a_j}. \quad (39)$$

In above we use the observation that only $t$'s odd order terms can contribute to the coefficients of $\cos \theta_j$ and $\sin \theta_j$ in $F_{a_1,\ldots,a_n}^t(p, \upsilon)$ from the proofs of Proposition 2.3, Lemma 3.2 and Lemma 3.3. We can extend $G^t$ in (39) smoothly to $t = 0$ and consider it as a smooth map $G$ on $[0, \epsilon) \times U$. Because $(p_0, [\upsilon_0])$ is a non-degenerate critical point of $F_{a_1,\ldots,a_n}$ from $U/T^n$ to $\mathbb{R}$, it follows that $G(0, p_0, \upsilon_0) = 0$ and the differential $dG_0|_{(p_0, \upsilon_0)}$ is surjective. The subspace of $u(n)$ which is perpendicular to the Lie algebra of $T^n$ is spanned by $a_{ij}$ and $b_{ij}$. Hence the Implicit Function Theorem implies that there exists a smooth function $(p(t), [\upsilon(t)]) \in U/T^n$ for small $t$ with $(p(0), [\upsilon(0)]) = (p_0, [\upsilon_0])$, such that $G^t(p(t), \upsilon(t)) = 0$ for any representative $\upsilon(t)$ of $[\upsilon(t)]$. For $t \neq 0$ the zeros of $H^t$ and $G^t$ are the same. Hence $H^t(p(t), \upsilon(t)) = 0$. Moreover, the distance between $(p(t), [\upsilon(t)])$ and $(p_0, [\upsilon_0])$ is bounded by $ct^2$ and therefore the Hamiltonian stationary Lagrangian tori with radii $(t_1, \ldots, t_n)$ obtained will not intersect each other for small $t$. This completes the proof of the theorem. \hfill \Box

## 5 A different proof for Theorem 4.4

In [10], we define a smooth function $K^t : U \rightarrow \mathbb{R}$ by

$$K^t(p, \upsilon) = t^{-n} \text{Vol}_\upsilon(I^t_{p,\upsilon}) = \text{Vol}_{g_{p,\upsilon}} \Phi(\Gamma_{d^t_{p,\upsilon}}), \quad (40)$$

and prove that under suitable identification $dK^t|_{(p, \upsilon)}$ is the same as $H^t(p, \upsilon)$ [10, Prop. 6.2]. Hence finding zeros of $H^t(p, \upsilon)$ is equivalent to finding critical points of $K^t$. Noting that $K^t$ is smooth in $t$ as well, we will take the expansion of $K^t$ in $t$ to analyze the critical points of $K^t$. Recall that $F^t_{p,\upsilon}(f) = \text{Vol}_{g_{p,\upsilon}} \Phi(\Gamma_{d^t_f})$ and $F_0(f) = \text{Vol}_{g_0} \Phi(\Gamma_{d_f})$. 

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Proposition 5.1. We have $K^t(p, v) = F^t_{p, v}(0) + O(t^4)$. That is, the leading terms of $K^t(p, v)$ in $t$ are the same as the leading terms of the area of $T^n_{a_1, ..., a_n}$ w.r.t. $g^t_{p, v}$.

Proof. We pull back the induced metric of $g^t_{p, v}$ on $\Phi(\Gamma_{df^t_{p, v}})$ to $T^n_{a_1, ..., a_n}$, and denote it as $h = h_0 + t^2 h_2 + t^3 h_3 + O(t^4)$ from (11), where $h_0$ is the induced metric of $g_0$ on $T^n_{a_1, ..., a_n}$ and consider $h$, $h_0$, $h_2$, $h_3$ as matrices. A direct computation gives

\[
det(h) = det(h_0 + t^2 h_2 + t^3 h_3 + O(t^4)) = det(h_0) det(I + t^2 h_0^{-1} h_2 + t^3 h_0^{-1} h_3 + O(t^4)),
\]

and

\[
\sqrt{\det(h)} = \sqrt{\det(h_0)}(1 + \frac{1}{2} t^2 \text{Tr}(h_0^{-1} h_2) + \frac{1}{2} t^3 \text{Tr}(h_0^{-1} h_3) + O(t^4)),
\]

where $h_0^{-1}$ is the inverse matrix of $h_0$. We thus have

\[
K^t(p, v) = F^t_{p, v}(f^t_{p, v}) = \text{Vol}_{g^t_{p, v}}\Phi(\Gamma_{df^t_{p, v}})
= \int_{T^n_{a_1, ..., a_n}} (1 + \frac{1}{2} t^2 \text{Tr}(h_0^{-1} h_2) + \frac{1}{2} t^3 \text{Tr}(h_0^{-1} h_3) + O(t^4)) dV_0
= F_0(f^t_{p, v}) + F^t_{p, v}(0) - F_0(0) + O(t^4).
\]

In the last equality, we use the observation that by (11) and (49) the contribution of $f^t_{p, v}$ and $g^t_{p, v}$ in $h_2$ and $h_3$ can be completely separated, and then rewrite the formula by the expansions of $F_0(f^t_{p, v})$ and $F^t_{p, v}(0)$. On the other hand, we have

\[F_0(f^t_{p, v}) = F_0(0) + \frac{d}{ds} F_0(s f^t_{p, v})|_{s=0} + O(t^4) = F_0(0) + O(t^4),
\]

where we use (49) in the first equality and $T^n_{a_1, ..., a_n}$ Hamiltonian stationary w.r.t. $g_0$ in the second one. Plugging (43) into (42), we get $K^t(p, v) = F^t_{p, v}(0) + O(t^4)$ as desired.

Remark 5.2. Proposition 5.1 works for any compact Hamiltonian stationary Lagrangian $L$ in $\mathbb{C}^n$. However, we need $L$ Hamiltonian rigid to identify $dK^t$ with $H^t$.

Proposition 5.3. Further expansion gives

\[
K^t(p, v) = (1 - \frac{1}{4} t^2 \sum_{i=1}^n a_i^2 R_{ii}^2(p)) \text{Vol}_{g_0}(T^n_{a_1, ..., a_n}) + O(t^4).
\]

Proof. The induced metric $h^t_{p, v}$ of $g^t_{p, v}$ on $T^n_{a_1, ..., a_n}$ is given in (15) and the inverse matrix $h_0^{-1}$ has entries $h_0^{-1}_{ij} = a_i^{-2} \delta_{ij}$. Combining (15) and a similar argument as in (42), we get

\[
F^t_{p, v}(0) = F_0(0) - \frac{1}{2} \int_{T^n_{a_1, ..., a_n}} \sum_{i=1}^n (t^2 \text{Re} A_{ii} + t^3 \text{Re} C_{ii}) dV_0 + O(t^4),
\]
where \( \text{Re} A_{ii} \) and \( \text{Re} C_{ii} \) are as in (14). Noting that the integration of \( \cos \) and \( \sin \) function on \( T^{n}_{a_{1},...,a_{n}} \) will vanish, only the terms involving \( R_{i_{i}i_{i}}(p) \) remain.

We thus have \( F_{p,v}(0) = (1 - \frac{1}{4} t^{2} \sum_{i=1}^{n} a_{i}^{2} R_{i_{i}i_{i}}(p)) F_{0}(0) + O(t^{4}) \). Combining with Proposition 5.1, it follows that

\[
K^{t}(p,v) = (1 - \frac{1}{4} t^{2} \sum_{i=1}^{n} a_{i}^{2} R_{i_{i}i_{i}}(p)) F_{0}(0) + O(t^{4}).
\]

Replacing \( F_{0}(0) \) by \( \text{Vol}_{p_{0}}(T^{n}_{a_{1},...,a_{n}}) \), it completes the proof. \( \square \)

**Proof of Theorem 4.4.** Both \( K^{t} \) and \( \sum_{i=1}^{n} a_{i}^{2} R_{i_{i}i_{i}}(p) \) are invariant under \( T^{n} \) action, so we can consider \( K^{t} \) as a map from \( U/T^{n} \) to \( \mathbb{R} \). Note that \( K^{0}(p, [v]) \equiv F_{0}(0) \) and every \((p, [v])\) is a critical point of \( K^{0} \). Suppose \((p_{0}, [v_{0}])\) is a non-degenerate critical point of \( F_{a_{1},...,a_{n}}(p, [v]) = \sum_{i=1}^{n} a_{i}^{2} R_{i_{i}i_{i}}(p) \). We in particular consider \( K^{0}(p_{0}, [v_{0}]) \) near \((p_{0}, [v_{0}])\). It follows from the Implicit Function Theorem that there exists a smooth function \((p(t), [v(t)]) \in U/T^{n} \) for small \( t \) with \((p(0), [v(0)]) = (p_{0}, [v_{0}])\), such that \((p(t), [v(t)]) \) is a critical point of \( K^{t} \). Moreover, the distance between \((p(t), [v(t)]) \) and \((p_{0}, [v_{0}])\) in \( U/T^{n} \) is bounded by \( ct^{2} \) from (45). Therefore, the Hamiltonian stationary Lagrangian tori with radii \((ta_{1},...,ta_{n})\) obtained are locally \( T^{n} \) invariant and will not intersect each other for small \( t \). This completes the proof of the theorem. \( \square \)

We remark that similar methods as in this section are also employed in other contexts. For instance, in constructing the blow-up examples of the Yamabe equation in [3], Brendle reduces the problem to finding critical points of a functional \( F_{g} \) and defines an auxiliary function \( F \). He shows that when \( n \geq 52 \), one can find a non-degenerate critical point of \( F \) and when \( g \) is sufficiently close to the Euclidean metric and other parameters are small, \( F_{g} \) has a critical point as well. He and Marques later generalize the construction to \( 25 \leq n \leq 51 \) by the same approach in [4].

**Appendix**

Here we give a supplement to the estimate in (36). Rewrite \( d^{*} \alpha_{H_{j}} \) in (33) as \( d^{*} \alpha(t,p,v,f) \) to indicate its dependency. We have

\[
d^{*} \alpha(t,p,v,f) = d^{*} \alpha(t,p,v,0) + \int_{0}^{1} \frac{d}{ds} d^{*} \alpha(t,p,v,sf) \, ds = d^{*} \alpha(t,p,v,0) + \int_{0}^{1} (d^{*} \alpha)_{\psi}(t,p,v,sf) f \, ds
\]

\[
= d^{*} \alpha(t,p,v,0) + \int_{0}^{1} (-\bar{\mathbb{L}}^{t}_{p,v} f + \int_{0}^{1} \frac{d}{du} ((d^{*} \alpha)_{\psi}(t,p,v,usf) f) \, du) \, ds
\]

\[
= d^{*} \alpha(t,p,v,0) - \bar{\mathbb{L}}^{t}_{p,v} f + \int_{0}^{1} \int_{0}^{1} (d^{*} \alpha)_{\psi\psi}(t,p,v,usf) sf^{2} \, du \, ds,
\]

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where $\mathcal{L}_{p,v}^t f = -(d^*\alpha)_{\psi}(t, p, v, 0)f$ is as (5) computed w.r.t. $g_{p,v}^t$. Denote

$$Q_{p,v}^t(f) = \int_0^1 \int_0^1 (d^*\alpha)_{\psi}(t, p, v, usf) sf^2 du ds. \quad (47)$$

The induced metric of $g_{p,v}^t$ on $T_{a_1, \ldots, a_n}$ is uniformly bounded from Proposition 3.1, and so is $d^*\alpha(t, p, v, 0)$. When the norm of $f$ is small, it will not change the induced metric too much. More precisely, we have $\|d^*\alpha(t, p, v, f)\|_{k, \gamma}$ bounded if $\|f\|_{k+4, \gamma}$ is small. Similarly, we have $\|(d^*\alpha)_{\psi}(t, p, v, usf)\|_{k, \gamma}$ bounded for $0 \leq s \leq 1$ and $0 \leq u \leq 1$ if $\|f\|_{k+4, \gamma}$ is small. Thus (47) gives

$$\|Q_{p,v}^t(f)\|_{k, \gamma} \leq c \|f\|_{k+2, \gamma}^2. \quad (48)$$

By Lemma 3.2, we have $\|d^*\alpha(t, p, v, 0)\|_{C^{k, \gamma}} = O(t^2)$ for any integer $k \geq 0$ and $\gamma \in (0, 1)$. It leads to

$$\|f_{p,v}^t\|_{C^{k+4, \gamma}} \leq ct^2 \quad (49)$$

from the Implicit Function Theory used in its construction in Theorem 4.3. Therefore, $d^*\alpha(t, p, v, f_{p,v}^t) = O(t^2)$ from (46), (5), (48) and (49). By (13) and (49), we then have $J_{p,v}^t = 1 + O(t^2)$ in (33). Hence (33) becomes

$$P_{p,v}^t(f_{p,v}^t) = -d^*\alpha(t, p, v, f_{p,v}^t) + O(t^4) = -P d^*\alpha(t, p, v, f_{p,v}^t) + O(t^4),$$

since $P_{p,v}^t(f_{p,v}^t)$ is in Ker $\mathcal{L}$ by Theorem 4.3.

Combining (48) and (49), it gives $Q_{p,v}^t(f_{p,v}^t) = O(t^4)$. Noting that $\bar{\mathcal{L}}_{p,v}^0 = \mathcal{L}$, by Proposition 3.1 and (5) we have

$$\|\mathcal{L}_{p,v}^t(f) - \mathcal{L}(f)\|_{C^{k, \gamma}} \leq ct^2 \|f\|_{C^{k+4, \gamma}}. \quad (50)$$

Because $\mathcal{L}$ is a self-adjoint operator, it follows that $\mathcal{L}(f_{p,v}^t)$ is perpendicular to Ker $\mathcal{L}$. We then have $P \mathcal{L}_{p,v}^t(f_{p,v}^t) = O(t^4)$ by (49) and (50). Therefore,

$$P_{p,v}^t(f_{p,v}^t) = -P d^*\alpha(t, p, v, 0) + O(t^4),$$

which is (36) by denoting $d^*\alpha(t, p, v, 0)$ as $d^*\alpha_{p,v}^t$.

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