Sign Retrieval in Shift-Invariant Spaces with Totally Positive Generator

José Luis Romero$^{1,2}$

Received: 5 June 2020 / Revised: 15 December 2020 / Accepted: 16 December 2020 / Published online: 16 March 2021
© The Author(s) 2021

Abstract

We show that a real-valued function $f$ in the shift-invariant space generated by a totally positive function of Gaussian type is uniquely determined, up to a sign, by its absolute values $\{|f(\lambda)| : \lambda \in \Lambda\}$ on any set $\Lambda \subseteq \mathbb{R}$ with lower Beurling density $D^-(\Lambda) > 2$.

We consider a totally positive function of Gaussian type, i.e., a function $g \in L^2(\mathbb{R})$ whose Fourier transform factors as

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} dx = C_0 e^{-\gamma \xi^2} \prod_{\nu=1}^{m} (1 + 2\pi i \delta_{\nu} \xi)^{-1}, \quad \xi \in \mathbb{R},$$

(1)

with $\delta_1, \ldots, \delta_m \in \mathbb{R}, C_0, \gamma > 0, m \in \mathbb{N} \cup \{0\}$, and the shift-invariant space

$$V^\infty(g) = \left\{ f = \sum_{k \in \mathbb{Z}} c_k g(\cdot - k) : c \in \ell^\infty(\mathbb{Z}) \right\},$$

generated by its integer shifts within $L^\infty(\mathbb{R})$. As a consequence of (1), each $f \in V^\infty(g)$ is continuous, the defining series converges unconditionally in the weak* topology of $L^\infty$, and the coefficients $c_k$ are unique [6, Theorem 3.5].

Keywords Sign retrieval · Phase retrieval · Shift-invariant space · Totally positive function

Communicated by Afonso Bandeira.

José Luis Romero
jose.luis.romero@univie.ac.at ; jlromero@kfs.oeaw.ac.at

1 Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
2 Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria
The shift-invariant space $V_\infty(g)$ enjoys the following sampling property [4]: every separated set $\Lambda \subseteq \mathbb{R}$ with lower Beurling density
\[
D^-(\Lambda) := \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#[\Lambda \cap [x-r, x+r]]}{2r}
\]
strictly larger than 1 provides the norm equivalence
\[
\|f\|_{L^\infty(\mathbb{R})} \asymp \|f|\Lambda\|_{\ell^\infty(\Lambda)}, \quad f \in V_\infty(g).
\] (2)

(A similar property holds for all $L^p$ norms, $1 \leq p \leq \infty$; see [4, Theorems 5.2 and 3.1].)

In this article, we show the following uniqueness property for the absolute values of real-valued functions in $V_\infty(g)$.

**Theorem 1** (Sign retrieval) Let $g$ be a totally positive function of Gaussian type, as in (1), and $\Lambda \subseteq \mathbb{R}$ with lower Beurling density
\[
D^{-}(\Lambda) > 2.
\] (3)
Assume that $f_1, f_2 \in V_\infty(g)$ are real-valued and $|f_1| \equiv |f_2|$ on $\Lambda$. Then either $f_1 \equiv f_2$ or $f_1 \equiv -f_2$.

See [2] for motivation for sign retrieval in shift-invariant spaces. When $g$ is a Gaussian function—corresponding to $m = 0$ in (1)—Theorem 1 was recently obtained in [3, Theorem 1]. Here, that result is extended to all totally positive functions of Gaussian type.\footnote{In contrast to [3, Theorem 1], in Theorem 1 it is not assumed that $\Lambda$ is separated, or even relatively separated. See [3] for an example of a separated set with $D^-(\Lambda) < 2$ for which sign retrieval fails in a shift-invariant space with Gaussian generator.}

The intuition behind Theorem 1 is as follows. Suppose that $|f_1| \equiv |f_2|$ on $\Lambda$, and split $\Lambda$ into
\[
\begin{align*}
\Lambda_1 &= \{ \lambda \in \Lambda : f_1(\lambda) = f_2(\lambda) \}, \\
\Lambda_2 &= \{ \lambda \in \Lambda : f_1(\lambda) = -f_2(\lambda) \}.
\end{align*}
\] (4)
Then $f_1 - f_2$ vanishes on $\Lambda_1$ while $f_1 + f_2$ vanishes on $\Lambda_2$. Under (3), one may expect one of the two subsets $\Lambda_j$ to have lower Beurling density larger than 1. The sampling inequalities (2) would then imply that either $f_1 - f_2$ or $f_1 + f_2$ are identically zero. This argument breaks down, however, because Beurling’s lower density is not subadditive. For example, $\Lambda_1 := \Lambda \cap (-\infty, 0]$ and $\Lambda_2 := \Lambda \cap (0, \infty)$ have always zero lower Beurling density. The proof of the sign retrieval theorem for Gaussian generators in [3] resorts instead to a special property of the Gaussian function, namely that $V_\infty(g) \cdot V_\infty(g)$ is contained in a dilation of $V_\infty(\tilde{g})$ by a factor of 2, where $\tilde{g}$ is another Gaussian function. Thus, in the Gaussian case, the sampling theorem can be applied after rescaling to the set $\Lambda$ to conclude that $(f_1 - f_2) \cdot (f_1 + f_2) \equiv 0$, \hfill
and, by analyticity, that either $f_1 \equiv f_2$ or $f_1 \equiv -f_2$. A similar argument applies to Paley–Wiener spaces [1, Theorem 2.5]. We are unaware of an analogous dilation property for general totally positive generators.

To prove Theorem 1 for all totally positive generators of Gaussian type we take a different route. We define the upper average circular density of a set $\Lambda_1 \subseteq \mathbb{R}$ as

$$D^+_{\text{circ}}(\Lambda) = \limsup_{r \to \infty} \frac{4}{\pi r^2} \int_0^r \sum_{\lambda \in \Lambda \cap [-t,t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t},$$

with the convention that $D^+_{\text{circ}}(\Lambda) = \infty$ if $\Lambda$ is uncountably infinite. The density is named circular because, in (5), each point $\lambda \in [-t,t]$ is weighted with the measure of the largest vertical segment $\{\lambda\} \times (-a,a)$ contained in the two-dimensional open disk $B_t(0) \subseteq \mathbb{R}^2$.

The upper average circular density can be alternatively described as follows: as shown in Lemma 1 below, for any lattice $\alpha \mathbb{Z}$, $\alpha > 0$,

$$D^+_{\text{circ}}(\Lambda) = \limsup_{r \to \infty} \frac{2\alpha}{\pi r^2} \int_0^r \# \left( (\Lambda \setminus \{0\} \times \alpha \mathbb{Z}) \cap B_t(0) \right) \frac{dt}{t},$$

From here, it follows easily that $D^+_{\text{circ}}$ dominates Beurling’s lower density:

$$D^+_{\text{circ}}(\Lambda) \geq D^-(\Lambda);$$

see Lemma 1 below. We call $D^+_{\text{circ}}$ an upper density because, due to the sublinearity of lim sup,

$$D^+_{\text{circ}}(\Lambda_1 \cup \Lambda_2) \leq D^+_{\text{circ}}(\Lambda_1) + D^+_{\text{circ}}(\Lambda_2),$$

for any two sets $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}$.

Below we prove the following uniqueness result formulated in terms of the upper average circular density of the zero set $\{f = 0\}$ of a function $f$ (counted without multiplicities).

**Theorem 2** (Uniqueness theorem) Let $g$ be a totally positive function of Gaussian type, as in (1). Let $f \in V^\infty(g)$ be non-zero. Then $D^+_{\text{circ}}(\{f = 0\}) \leq 1$.

The uniqueness theorem (which applies also to non-real-valued functions) allows one to carry out the following natural proof of the sign retrieval theorem.

**Proof** (Proof of Theorem 1, assuming Theorem 2) Assume that $D^-(\Lambda) > 2$ and write $\Lambda = \Lambda_1 \cup \Lambda_2$ as in (4). Then, by (7) and (8), either $D^+_{\text{circ}}(\Lambda_1) > 1$ or $D^+_{\text{circ}}(\Lambda_2) > 1$ (possibly both). In the first case, Theorem 2 shows that $f_1 \equiv f_2$, while in the second, $f_1 \equiv -f_2$.

As the proof shows, Theorem 1 remains valid if (3) is relaxed to $D^+_{\text{circ}}(\Lambda) > 2$. Sign retrieval also holds under the same density condition for the shift-invariant spaces $V^p(g)$ defined with respect to $L^p$ norms, $1 \leq p \leq \infty$, as these are contained in $V^\infty(g)$.

Towards the proof of Theorem 2, we first prove (6) and (7).
Lemma 1 Let $\Lambda \subseteq \mathbb{R}$ and $\alpha > 0$. Then (6) and (7) hold true.

Proof We assume that $\Lambda$ has no accumulation points, since, otherwise, both sides of (6) are infinite. Denote provisionally the right hand side of (6) by $\tilde{D}_{\text{circ}}^+(\Lambda)$, and set $\Lambda' := \Lambda \setminus \{0\}$.

Step 1. Let $\epsilon \in (0, 1)$, and $C = C_{\alpha, \epsilon} > 0$ a constant to be specified. We claim:

$$D_{\text{circ}}^+(\Lambda) = \limsup_{r \to \infty} \frac{4}{\pi r^2} \int_0^r \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t}, \quad (9)$$

$$(1 + \delta)^2 \tilde{D}_{\text{circ}}^+(\Lambda) = \limsup_{r \to \infty} \frac{2\alpha}{\pi r^2} \int_0^r \#(\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1+\delta)t}(0) \frac{dt}{t}, \quad \delta = \pm \epsilon. \quad (10)$$

To prove the first claim, we first note that $D_{\text{circ}}^+(\Lambda) = D_{\text{circ}}^+(\Lambda')$, since, if $0 \in \Lambda$, the contribution of the point $\lambda = 0$ is

$$\frac{4}{\pi r^2} \int_0^r dt = \frac{4}{\pi r},$$

and does not affect the limit on $r$. Similarly, since $\sqrt{t^2 - \lambda^2} \leq t$,

$$\frac{4}{\pi r^2} \int_0^C \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t} \leq \#(\Lambda' \cap [-C, C]) \cdot \frac{4C}{\pi r^2}. \quad (11)$$

The last quantity is finite because $\Lambda$ has no accumulation points. This proves (9), because it shows that limiting the integral to $[C, r]$ does not affect the limit on $r$. Second, a change of variables shows that (10) holds with $C$ replaced by 0. In addition, since $\Lambda$ has no accumulation points, there exists $\eta > 0$ such that $\Lambda' \cap (-\eta, \eta) = \emptyset$. Therefore, we can estimate the part of the integral excluded in (10) as

$$\frac{2\alpha}{\pi r^2} \int_0^C \#((\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1+\delta)t}(0)) \frac{dt}{t} \leq \#((\Lambda' \times \alpha \mathbb{Z}) \cap B_{2C}(0)) \cdot \frac{2\alpha}{\pi r^2} \int_{\frac{\eta}{1+\epsilon}}^C \frac{dt}{t},$$

which is finite because $\eta > 0$. This proves (10).

Step 2. For $t > 0$, we note that

$$\#(\Lambda' \times \alpha \mathbb{Z}) \cap B_t(0) = \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \left\{ k \in \mathbb{Z} : \lambda^2 + \alpha^2 k^2 < t^2 \right\} = \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \left\{ k \in \mathbb{Z} : |k| < \alpha^{-1} \sqrt{t^2 - \lambda^2} \right\}.$$
Next, we choose the constant $C_{\alpha, \varepsilon}$ so that the following estimates hold for $t \geq C_{\alpha, \varepsilon}$:

$$
\# \left[ (\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1+\varepsilon)t}(0) \right] \geq \sum_{\lambda \in \Lambda' \cap [-t,t]} \left[ \frac{2}{\alpha} \sqrt{(1 + \varepsilon)^2 t^2 - \lambda^2 - 1} \right]
$$

$$
= \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1 + \varepsilon)^2 t^2 - \lambda^2 - \alpha \sqrt{(1 + \varepsilon)^2 t^2 - \lambda^2 + \alpha^2/4}}
$$

$$
\geq \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1 + \varepsilon)^2 t^2 - \lambda^2 - \alpha(1 + \varepsilon)t + \alpha^2/4}
$$

where the last estimate requires choosing $C_{\alpha, \varepsilon}$ large. Similarly,

$$
\# \left[ (\Lambda' \times \alpha \mathbb{Z}) \cap B_{(1-\varepsilon)t}(0) \right] \leq \sum_{\lambda \in \Lambda' \cap [-t,t]} \left[ \frac{2}{\alpha} \sqrt{(1 - \varepsilon)^2 t^2 - \lambda^2 + 1} \right]
$$

$$
= \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1 - \varepsilon)^2 t^2 - \lambda^2 + \alpha \sqrt{(1 - \varepsilon)^2 t^2 - \lambda^2 + \alpha^2/4}}
$$

$$
\leq \sum_{\lambda \in \Lambda' \cap [-t,t]} \frac{2}{\alpha} \sqrt{(1 - \varepsilon)^2 t^2 - \lambda^2 + \alpha(1 - \varepsilon)t + \alpha^2/4}
$$

where the last estimate requires choosing $C_{\alpha, \varepsilon}$ large. Combining the last two estimates with \( (9) \) and \( (10) \), and letting $\varepsilon \to 0$, we deduce \( (6) \).

**Step 3.** To prove \( (7) \), let $r > s > 0$ and use \( (6) \) with $\alpha = 1$ to estimate

$$
D_{\text{circ}}^+(\Lambda) \geq \liminf_{r \to \infty} \frac{2}{\pi r^2} \int_s^r \inf_{z \in \mathbb{R}^2} \# \left[ (\Lambda' \times \mathbb{Z}) \cap B_t(z) \right] \frac{dt}{t}
$$

$$
\geq \inf_{t \geq s} \left[ \frac{1}{|B_t(0)|} \inf_{z \in \mathbb{R}^2} \# \left[ (\Lambda' \times \mathbb{Z}) \cap B_t(z) \right] \right] \liminf_{r \to \infty} \frac{2}{\pi r^2} \int_s^r |B_t(0)| \frac{dt}{t}
$$

$$
= \inf_{t \geq s} \left[ \frac{1}{|B_t(z)|} \# \left[ (\Lambda' \times \mathbb{Z}) \cap B_t(z) \right] \right].
$$

Letting $s \to \infty$, we see that

$$
D_{\text{circ}}^+(\Lambda) \geq \liminf_{t \to \infty} \inf_{z \in \mathbb{R}^2} \frac{1}{|B_t(z)|} \# \left[ (\Lambda' \times \mathbb{Z}) \cap B_t(z) \right] = D^-(\Lambda') = D^-(\Lambda),
$$

where we used that the two-dimensional lower Beurling density of $\Lambda' \times \mathbb{Z}$ is $D^-(\Lambda)$. \( \square \)
We can now prove the uniqueness Theorem. The proof builds on [4, Lemma 5.1] and [5, Theorems 4.3 and 4.9].

Proof of Theorem 2 We proceed by induction on $m$ in (1).

In the case $m = 0$, the function $g$ is a Gaussian $g(x) = C_0 e^{-ax^2}$, with $a = \frac{\pi^2}{\gamma} > 0$. Let $f = \sum_k c_k g(\cdot - k) \in V^\infty(g)$ with $c$ bounded be non-zero, and denote by $\Lambda \subseteq \mathbb{R}$ its zero set.

As shown in [5, Lemma 4.1], $f$ possesses an extension to an entire function on $\mathbb{C}$, satisfying the growth estimate $|f(x + iy)| \lesssim e^{a(y^2)}$ for $x, y \in \mathbb{R}$, where the implied constant depends on $f$. Let $n \geq 0$ be the order of $f$ at $z = 0$, and consider the analytic function

$$F(z) := C_1 z^{-n} f(z) e^{g z^2},$$

where $C_1 \in \mathbb{C}$ is chosen so that $F(0) = 1$. Then $F$ satisfies

$$|F(x + iy)| \leq C e^{a(y^2)} e^{\frac{g}{2}(x^2 - y^2)} = C e^{\frac{g}{2}(x^2 + y^2)}, \quad x, y \in \mathbb{R},$$

for some constant $C > 0$. Moreover, the zero set of $f$ is invariant under addition of $i \frac{\pi}{a} \mathbb{Z}$ [5, Lemma 4.2]. Therefore, the set of complex zeros of $F$ contains $(\Lambda \setminus \{0\}) + i \frac{\pi}{a} \mathbb{Z}$:

$$F(\lambda + i \frac{\pi}{a} k) = 0, \quad \lambda \in \Lambda \setminus \{0\}, k \in \mathbb{Z}.$$ 

By (6) with $\alpha = \frac{\pi}{a}$, the zero-counting function of $F$,

$$n_F(t) := \#\{z \in \mathbb{C} : F(z) = 0, |z| \leq t\},$$

satisfies

$$\limsup_{r \to \infty} \frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt \geq \frac{a}{2} D^+_{\text{circ}}(\Lambda).$$

(12)

On the other hand, by Jensen’s formula combined with (11), for all $r > 0$,

$$\frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt = \frac{1}{2\pi r^2} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta \leq \frac{\log(C)}{r^2} + \frac{a}{2}.$$ 

(13)

By combining (12) and (13) we conclude that $D^+_{\text{circ}}(\Lambda) \leq 1$, as claimed. Note, in particular, that $\Lambda$ has no accumulation points.

Inductive step. Let $g$ be as in (1) and $f = \sum_k c_k g(\cdot - k) \in V^\infty(g)$ non-zero. Without loss of generality we assume that, in (1), $\delta_1, \ldots, \delta_m$ are non-zero. Since $g$ is real-valued, by replacing $c_k$ with $\Re(c_k)$ or $\Im(c_k)$ if necessary, we may assume that $f$ is real-valued. Let $g_1$ be given by (1), except that the product runs only up to $m - 1$. Then

$$f_1 := f + \delta_m f' = \sum_{k \in \mathbb{Z}} c_k g_1(\cdot - k)$$
Fig. 1 Each positive zero $\lambda_k$ of $f$ is assigned to a zero $\gamma_k$ of $f_1$ in such a way that the corresponding vertical segments contained in the disk $B_t(0)$ become longer.

belongs to $V^\infty(g_1)$ and is real-valued and non-zero, because the coefficients $c_k$ are real and at least one of them is non-zero. We assume by inductive hypothesis that $D^+_\text{circ}((f_1 = 0)) \leq 1$.

Let $\Lambda = \{ f = 0 \}$ and $\Gamma = \{ f_1 = 0 \}$. Suppose that $\Lambda$ has no accumulation points. Assume for the moment that, in addition, $f$ has a non-negative zero, and order the set of non-negative zeros of $f$ increasingly:

$$\Lambda \cap [0, \infty) = \{ \lambda_k : k = 0, \ldots, N \},$$

where $N \in \mathbb{N} \cup \{0, \infty\}$, and we do not count multiplicities. Hence, $\lambda_k > 0$, if $k > 0$.

By Rolle’s Theorem applied to the differential operator $I + \delta_m \partial_x$, there is a sequence of zeros of $f_1$ that interlaces $\Lambda \cap [0, \infty)$. (Indeed, we note that $\frac{d}{dx} e^{x/\delta_m} f_1(x) = \frac{1}{\delta_m} e^{x/\delta_m} f_1(x)$ and apply the standard version of Rolle’s theorem; see [4, Lemma 5.1] or [5, Lemma 4.8] for details.) We parameterize that sequence as $\Gamma^+ = \{ \gamma_k : k = 1, \ldots, N \}$, with

$$\lambda_{k-1} < \gamma_k < \lambda_k, \quad k > 0.$$

The set $\Gamma^+$ is empty if $N = 0$. For $t > 0$, this ordering associates with each $\lambda_k \in (0, t]$, $k \neq 0$, a distinct zero of $f_1, \gamma_k \in (0, t]$, such that the vertical segment through $\gamma_k$ contained in the disk $B_t(0)$ is longer than the corresponding segment through $\lambda_k$:

$$\sum_{k>0, \lambda_k \in (0, t]} \sqrt{t^2 - \lambda_k^2} \leq \sum_{k>0, \gamma_k \in (0, t]} \sqrt{t^2 - \gamma_k^2} \leq \sum_{\gamma \in \Gamma \cap (0, t]} \sqrt{t^2 - \gamma^2};$$

see Fig. 1.

We now add the term corresponding to $\lambda_0$, and bound $\sqrt{t^2 - \lambda_0^2} \leq t$ to obtain

$$\sum_{\lambda \in \Lambda \cap [0, t]} \sqrt{t^2 - \lambda^2} \leq t + \sum_{\gamma \in \Gamma \cap (0, t]} \sqrt{t^2 - \gamma^2}.$$
The previous estimate is also trivially true if \( f \) has no non-negative zeros. Arguing similarly with the non-positive zeros of \( f \) we conclude that

\[
\sum_{\lambda \in \Lambda \cap [-t,t]} \sqrt{t^2 - \lambda^2} \leq 2t + \sum_{\gamma \in \Gamma \cap [-t,t]} \sqrt{t^2 - \gamma^2}.
\]

This shows that \( D_{\text{circ}}^+(\{f = 0\}) \leq D_{\text{circ}}^+(\{f_1 = 0\}) \). The latter quantity is bounded by 1 by inductive hypothesis. Finally, if \( \Lambda = \{f = 0\} \) has an accumulation point, Rolle’s theorem applied as before shows that so does \( \Gamma = \{f_1 = 0\} \), and, therefore, \( D_{\text{circ}}^+(\{f_1 = 0\}) = \infty \) contradicting the inductive hypothesis. \( \square \)

**Acknowledgements** J. L. R. gratefully acknowledges support from the Austrian Science Fund (FWF): Y 1199 and P 29462.

**Funding** Open Access funding provided by University of Vienna

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Alaifari, R., Grohs, P.: Phase retrieval in the general setting of continuous frames for Banach spaces. SIAM J. Math. Anal. 49(3), 1895–1911 (2017)
2. Cheng, C., Jiang, J., Sun, Q.: Phaseless sampling and reconstruction of real-valued signals in shift-invariant spaces. J. Fourier Anal. Appl. 25(4), 1361–1394 (2019)
3. Gröchenig, K.: Phase-retrieval in shift-invariant spaces with Gaussian generator. J. Fourier Anal. Appl. 26(3), 15 (2020). (Paper No. 52)
4. Gröchenig, K., Romero, J.L., Stöckler, J.: Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions. Invent. Math. 211(3), 1119–1148 (2018)
5. Gröchenig, K., Romero, J.L., Stöckler, J.: Sharp results on sampling with derivatives in shift-invariant spaces and multi-window Gabor frames. Constr. Approx. 51(1), 1–25 (2020)
6. Jia, R.Q., Micchelli, C.A.: Using the refinement equations for the construction of pre-wavelets. II. Powers of two. In: Curves and Surfaces (Chamonix-Mont-Blanc, 1990), pp 209–246. Academic, Boston (1991)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.