Entanglement entropy in curved spacetimes with event horizons

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Abstract

We consider the computation of the entanglement entropy in curved backgrounds with event horizons. We use a Hamiltonian approach to the problem and perform numerical computations on a spherical lattice of spacing \(a\). We study the cosmological case and make explicit computations for the Friedmann-Robertson-Walker universe. Our results for a massless, minimally coupled scalar field can be summarized by \(S_{\text{ent}} = 0.30 r_H^2 / a^2\), which resembles the flat space formula, although here the horizon radius, \(r_H\), is time-dependent.

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I. INTRODUCTION

The formal proof of the validity of the first law of black hole thermodynamics \[1,2\] for arbitrary perturbations of a stationary black hole in any gravitational theory derivable from a diffeomorphism invariant Lagrangean gave an impulse and renewed interest on the computation of the black hole entropy. Recently, several authors have obtained expressions for the thermodynamic or coarse-grained black hole entropy in the form of an integral of the entropy density on any cross-section of the Killing horizon. More explicit expressions have been given for different higher derivative gravitational (interacting with matter) Lagrangeans \[3–5\]. The main techniques used to derive the black hole entropy have been the Noether charge construction \[6\], the Euclidean signature method \[7\], and the field redefinition approach \[8\].

All the above derivations make use of the first law of black hole thermodynamics where all its terms are known (mass, temperature, angular momentum, angular velocity of the horizon, electromagnetic charge and potential), but an integral on the horizon. One precisely identifies this unknown quantity on the horizon with the black hole entropy by analogy to any other ordinary thermodynamic system. This can be called the coarse–grained entropy because it is derived in the thermodynamic limit. It is, however, desirable to have an independent, statistical mechanical derivation of the entropy, which may be called the fine–grained entropy. Relatively early attempts \[9,10\] to compute this quantum-originated entropy considered the statistical entropy, \( S \approx k_B \sum_n p_n \ln p_n \approx k_B \ln N \), with \( N \) being the number of modes or configurations of field very close to the black hole horizon. 't Hooft \[9\] has computed the quantum corrections to the black hole entropy by counting the number of classical eigenmodes of a scalar field outside the event horizon. He immediately noted the appearance of an ultraviolet divergence in the entropy as one approaches the horizon in counting the modes, and discussed the need of imposing a short distance cut-off.

Bombelli et al. \[11\] studied instead the entanglement entropy \( S = - \text{Tr} [\rho \ln \rho] \) by considering in flat space a scalar quantum field in its ground state and computing its reduced density matrix, \( \rho = |\Phi \rangle \langle \Phi | \), where one has traced over the field degrees of freedom inside a region of space that is intended to represent the interior of a black hole. Thus, the entanglement entropy is associated to correlations across the boundary separating two regions. More recently, several \[12–17\] computations of the entanglement entropy agree in obtaining the proportionality to the area \( A \) of the surface dividing the known from the unknown regions and the \( 1/h^2 \) ultraviolet divergence, where \( h \) is the minimal distance to the horizon we reach in counting modes of the external field, typically given in Minkowskian or Schwarzschild radial coordinates. Their results can be summarized by

\[
S = g \frac{A}{\gamma h^2},
\]

where \( g \) accounts for the number of fields and its helicities and \( \gamma = 1/(360\pi) \) (see Refs. \[16,17\]).

In Refs. \[12,17,18\] the entropy is computed by identifying the dynamical degrees of freedom with the states of quantum fields propagating in the black hole’s interior. In Refs. \[14,16\] the entropy of fields outside the black hole is studied instead. The two kinds of computation are conciliated in Ref. \[15\] where it is shown that the density matrices of the two situations are identical.
The question of the ultraviolet divergence of the entropy is discussed in Ref. [19] where the Pauli-Villars regularization is used to avoid the introduction of a Planck scale cut-off in the computation of the entropy. In the paper [20], the entropy difference between two states is shown to be better behaved. Susskind [21] speculates about the possibility that the black hole entropy be finite in string theory.

In the present paper, we handle this problem, following Ref. [13], by making computations on a lattice of spacing \( a \), what renders the entropy finite. In Srednicki’s work the reduced ground state density matrix of a free, massless, scalar quantum field is obtained by tracing over the degrees of freedom located inside of an imaginary sphere of radius \( R \) and by making use of the analogy with the result [11] for a coupled system of harmonic oscillators. The final result is

\[
S = \frac{0.30}{4\pi} \left( \frac{4\pi R^2}{a^2} \right)
\]

which, as the author remarks, bears an striking similarity with the formula for the coarse–grained entropy of black holes

\[
S = \frac{A_H}{4\ell_P^2},
\]

where \( A_H \) is the area of the black hole’s event horizon. The similarity between these two equations is in fact surprising, since Srednicki’s computation has been carried out in flat spacetime while we know that the geometry generated by a black hole is curved. The argument here is that the modes which contribute mostly to the entropy are those of short wavelengths that do not “feel” whether the spacetime is curved [11]. However, the event horizon is a null surface which greatly differs from the spacelike surface \( R = \) constant. In addition, we expect the entanglement entropy of a quantum field computed in the curved metric of a black hole to give just a correction (due to quantum fluctuations) to the coarse-grained entropy produced by the classical curved black hole background, but not the whole value (given by Eq. (3)).

Comparatively much less attention has been paid to the cosmological case. Here one can also define a horizon temperature and the laws of thermodynamics [22,23], although their interpretation is not as transparent as in the black hole case. In our paper we will concentrate on the entanglement correction to the classical entropy of the universe. In Sec. II we generalize the approach of Ref. [13] to the case when the spacetime is curved and possesses a horizon acting as a barrier for our knowledge of the quantum state beyond it. In Sec. III we review some properties of the Friedmann-Robertson-Walker geometry while in Sec. IV we give our main results. The paper finishes with a discussion of the cosmological results and it possible extension to the black hole case.

II. COMPUTATION OF ENTANGLEMENT ENTROPY IN CURVED SPACETIME

In this paper, we want to extend the existing schemes for calculating the entanglement entropy to a curved spacetime with horizon. To compute the entanglement entropy, one
considers a quantum field in its ground state. The spacetime is divided into two parts which are separated by a boundary (in spacetimes with horizon, this subdivision appears quite naturally). A given observer has access to the information of only one of these two regions. In his/her subsystem, the field cannot be described by a pure state wave function. Because of the correlations that exists between the two regions, even in the ground state of the field, it must be described by a density matrix with an associated nonzero entropy. This is the origin of the entanglement entropy.

In Refs [11,13] a scheme was given for calculating the entanglement entropy in Minkowski space. They considered the quantum field as a set of coupled harmonic oscillators with Hamiltonian given by

$$H = \frac{1}{2} \sum_i \Pi_i^2 + \frac{1}{2} \sum_{i,j} \phi_i K_{ij} \phi_j.$$  \hspace{1cm} (4)

In the usual flat spacetime there is not horizon and the boundary can be chosen arbitrarily. From (4), it is then possible to compute the entanglement entropy numerically [13].

We consider a curved spacetime with an isotropic metric of the following form

$$ds^2 = C^2(r,t)dt^2 - B^2(r,t)(dx^2 + dy^2 + dz^2).$$  \hspace{1cm} (5)

For a massive, non minimally coupled scalar field $\phi$, the Lagrangean is given by

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} \left[ g^{\mu\nu} (\partial_{\mu} \phi)(\partial_{\nu} \phi) - (m^2 + \xi R) \phi^2 \right],$$  \hspace{1cm} (6)

which becomes for the metric (5)

$$\mathcal{L} = \frac{1}{2} C B^3 \left[ C^{-2} (\partial_t \phi)^2 - B^{-2} (\nabla \phi)^2 - (m^2 + \xi R) \phi^2 \right].$$  \hspace{1cm} (7)

The canonical momentum conjugate to $\phi$ is

$$\Pi = \frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = \sqrt{-g} g^{0\mu} \partial_{\mu} \phi = C^{-1} B^3 \partial_t \phi.$$  \hspace{1cm} (8)

We obtain the Hamiltonian of the scalar field from the definition

$$H = \int d^3 x \left[ \Pi \partial_t \phi - \mathcal{L} \right],$$  \hspace{1cm} (9)

which becomes in our case

$$H = \frac{1}{2} \int d^3 x \left[ CB^{-3} \Pi^2 + CB (\nabla \phi)^2 + CB^3 (m^2 + \xi R) \phi^2 \right].$$  \hspace{1cm} (10)

The problem can be reduced to an effectively one-dimensional one if we exploit its spherical symmetry. The metric (5) has an isotropic form then decomposition in terms of spherical harmonics is possible. Following Srednicki [13] in using the real spherical harmonics $Z_{lm}(\theta, \varphi)$ which are orthonormal and complete. We decompose the field and the canonical momentum into partial wave components.
\phi(\vec{x}) = \frac{1}{r} \sum_{lm} Z_{lm}(\theta, \varphi) \phi_{lm}(r), \quad (11)

\Pi(\vec{x}) = \frac{1}{r} \sum_{lm} Z_{lm}(\theta, \varphi) \Pi_{lm}(r). \quad (12)

When we substitute (11) and (12) into (10), we obtain after some algebra

\[ H = \frac{1}{2} \sum_{lm} \int_{0}^{\infty} dr \left[ \frac{1}{r^2} \phi_{lm}^2 + \frac{1}{r} \phi_{lm} \frac{\partial \phi_{lm}}{\partial r} + \frac{\partial^2 \phi_{lm}}{\partial r^2} \right] \quad (13) \]

We next perform a substitution in the canonical variables in such a way that the commutation relations are preserved:

\[ \Pi'_{lm} = F \Pi_{lm}, \quad \phi'_{lm} = \frac{1}{F} \phi_{lm}, \quad (14) \]

with \( F(r, t) = C^{1/2}B^{-3/2} \). In terms of the variables \( \phi'_{lm} \) and \( \Pi'_{lm} \), our Hamiltonian takes the form of Eq. (4)

\[ H = \frac{1}{2} \sum_{lm} \int_{0}^{\infty} dr \left[ \Pi'_{lm}^2 + \frac{\gamma_1 + \gamma_4}{a^2} \phi'_{lm}^2 + \gamma_2 \frac{\partial \phi'_{lm}}{\partial r} \phi'_{lm} + \gamma_5 \frac{l(l+1)}{r^2} \phi'_{lm}^2 + \gamma_6 \phi'_{lm}^2 \right] \quad (15) \]

We can write this as

\[ H = \frac{1}{2} \sum_{lm} \int_{0}^{\infty} dr \left[ \Pi'_{lm}^2 + \left( \frac{\gamma_1 + \gamma_4}{a^2} \phi'_{lm}^2 + \gamma_2 \frac{\partial \phi'_{lm}}{\partial r} \phi'_{lm} + \gamma_3 \frac{l(l+1)}{r^2} \phi'_{lm}^2 + \gamma_6 \phi'_{lm}^2 \right) \right] \quad (16) \]

where

\[ \gamma_1(r) = a^2 CB \left( \frac{\partial}{\partial r} C^{\frac{1}{2}} B^{-\frac{3}{2}} \phi'_{lm} \right)^2, \quad \gamma_2(r) = 2a C^{\frac{3}{2}} B^{-\frac{3}{2}} \frac{\partial}{\partial r} \left( C^{\frac{1}{2}} B^{-\frac{3}{2}} \right), \]

\[ \gamma_3(r) = \gamma_5(r) = C^2 B^{-2}, \quad \gamma_4(r) = \frac{a^2}{r} CB^{-3} \frac{\partial}{\partial r} (CB), \quad \gamma_6(r) = C^2 a^2 (m^2 + \xi R). \quad (17) \]

In Eqs (16) and (17), we have introduced an arbitrary length scale \( a \) to make the \( \gamma_i \) dimensionless. In the form (16), the Hamiltonian is in a suitable form for discretization.

In order to compute the entanglement entropy numerically, we put the system on a spherical lattice with \( N \) points in radial direction. The lattice spacing is denoted \( a \) so that \( r = ja \), where \( j \) is an integer. Discretization of (14) leads to

\[ H = \frac{1}{2a} \sum_{jlm} \left[ \Pi_{lmj}^2 + \left( \gamma_1 + \gamma_4 + \gamma_5 \frac{l(l+1)}{j^2} + \gamma_6 \right) \phi_{lmj}^2 \right] \]
To discretize the parameters $\gamma_i$, it is necessary to specify the form of the spacetime metric. This will be considered in the next sections. The Hamiltonian (18) is of the general form (4) with

$$K_{j,j} = \gamma_1(j) - \gamma_2(j) + \gamma_3(j + 1) + \gamma_4(j) + \gamma_5(j) \frac{l(l+1)}{j^2} + \gamma_6(j),$$

$$K_{j,j+1} = K_{j+1,j} = \frac{1}{2} \gamma_2(j) - \gamma_3(j).$$

Knowing the explicit form of the $\gamma_i$, it is thus immediately possible to calculate numerically the entanglement entropy using Srednicki’s algorithm [13]: The lattice is divided into two parts. We want to trace over the first $n$ lattice sites (the region inside the horizon). To this end, the matrix $K$ is written via singular value decomposition as $K = U^T K_D U$ where $K_D$ is diagonal and $U$ orthogonal. One defines the square root of $K$ as $\omega = U^T K_D^{\frac{1}{2}} U$ and writes it in the form

$$\omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where the submatrix $A$ is $n \times n$ and $C = (N-n) \times (N-n)$. Next we introduce $\tilde{B} = \frac{1}{2} B^T A^{-1} B$ and $\gamma = C - \tilde{B}$. We perform a singular value decomposition of $\gamma$: $\gamma = V^T \gamma_D V$. We finally define the matrix $\tilde{B}' = \gamma_D^{-\frac{1}{2}} V \tilde{B} V^T \gamma_D^{-\frac{1}{2}}$ which has $N-n$ eigenvalues $\tilde{B}'_i$. The entanglement entropy associated with the trace over the $n$ sites inside the horizon is

$$S = \sum_{i=0}^{\infty} (2l + 1) S_l,$$

where

$$S_l = \sum_{i=1}^{N-n} \left[ \log(1 - \xi_i) - \frac{\xi_i}{1-\xi_i} \log \xi_i \right],$$

and $\xi_i = \tilde{B}'_i/(1 + \sqrt{1 - \tilde{B}'_i^2})$. A derivation of this algorithm as well as its physical discussion can be found in [13].

### III. GEOMETRY OF ROBERTSON-WALKER UNIVERSES

It is our aim in this paper to calculate the entanglement entropy of a quantum field in a curved spacetime with horizon. Specifically, we are interested in the behavior of the entropy for different classes of cosmological models. We consider therefore the Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = dt^2 - \beta^2 C^2(t) \left[ d\chi^2 + \frac{\sin^2(\sqrt{k} \chi)}{k}(d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where

$$+ \gamma_2 \phi_{lmj} (\phi_{lmj+1} - \phi_{lmj}) + \gamma_3 (\phi_{lmj+1} - \phi_{lmj})^2 \right].$$

$$\left( \phi_{lmj} + 1 - \phi_{lmj} \right)^2 + \gamma_3 (\phi_{lmj} + 1 - \phi_{lmj}) \frac{l(l+1)}{j^2} + \gamma_6 (\phi_{lmj} + 1 - \phi_{lmj})^2 \right].$$

(18)
and treat the three different cases \( k = 1, -1, 0 \) corresponding to a closed, open and spatially flat universe. The constant \( \beta \) has the dimension of a length and determines the spacetime curvature.

In the form (23), the metric is not very useful for calculating the entanglement entropy because it is not in the isotropic form (3). It can obtained by means of the coordinate transformation 

\[
d\eta = C^{-1}(t)dt, \quad x = \frac{2\beta}{\sqrt{k}} \tan \left( \sqrt{k} \frac{\chi}{2} \right) \sin \theta \cos \varphi, \\
y = \frac{2\beta}{\sqrt{k}} \tan \left( \sqrt{k} \frac{\chi}{2} \right) \sin \theta \sin \varphi, \quad z = \frac{2\beta}{\sqrt{k}} \tan \left( \sqrt{k} \frac{\chi}{2} \right) \cos \theta.
\]

Note that it does not mix space and time variables. For the sake of simplicity, we restrict ourselves in the following to times \( \eta > 0 \).

With (24), the line element becomes

\[
ds^2 = C^2(\eta) \left[ d\eta^2 - \frac{1}{\left( 1 + k \frac{r^2}{4\beta^2} \right)^2} (dx^2 + dy^2 + dz^2) \right].
\]

It is of the form (3) and it will be the starting point for our subsequent calculation of the entanglement entropy.

An important concept for our discussion is the notion of the cosmological event horizon. It represents the maximal coordinate distance a photon emitted at time \( t \) can travel. For a metric of the form (23), it is given by

\[
r_H = \frac{2\beta}{\sqrt{k}} \tan \left( \sqrt{k} \frac{\Delta \eta}{2\beta} \right),
\]

where

\[
\Delta \eta = \eta(t_f) - \eta(t) = \int_t^{t_f} \frac{dt'}{C(t')}.
\]

Here, \( t_f \) equals infinity for \( k = 0, -1 \), while for \( k = +1 \) it corresponds to the time when the universe collapses to a final singularity.

We will be interested in the area spanned by the cosmological event horizon. Generally, in a curved spacetime with diagonal metric, the proper area of a spherical surface \( r = r_0 = \) constant is defined by

\[
A_0 = \int_{r=r_0} \sqrt{g_{\theta\theta} g_{\varphi\varphi}} d\theta d\varphi.
\]

For the metric (23), this becomes

\[
A_0 = 4\pi C^2(t) \frac{\sin^2 \left( \sqrt{k} \chi_0 \right)}{k} = 4\pi C^2(\eta) \frac{r_0^2}{\left( 1 + k \frac{r_0^2}{4\beta^2} \right)^2}.
\]
As a concrete example, one can consider the evolution of a universe with negligible matter contribution but endowed with a cosmological constant $\Lambda$. In that case, we have for $k = \pm 1, 0$ the following evolution

$$C(t) = \frac{1}{2} \left( e^{t/\beta} + ke^{-t/\beta} \right),$$

(30)

where $\beta = \sqrt{3/\Lambda}$. In terms of the conformal time

$$\Delta \eta = \frac{2\beta}{\sqrt{k}} \arctan \left[ \sqrt{k} e^{-t/\beta} \right],$$

(31)

we have

$$C(\eta) = \frac{\sqrt{k}}{2} \left[ \tan \left( -\frac{\sqrt{k} \Delta \eta}{2\beta} \right) + \tan^{-1} \left( -\frac{\sqrt{k} \Delta \eta}{2\beta} \right) \right].$$

(32)

The “radius” of the cosmological horizon can then be explicitly computed from (29):

$$r_H = \frac{2\beta}{k} \left( C - \sqrt{C^2 - k} \right).$$

(33)

The area of the horizon can be calculated by inserting (33) into (29). We obtain

$$A_H = 4\pi \beta^2.$$  

(34)

It is noteworthy that it does not depend on $C(\eta)$, i.e., it is time independent.

In the $k = 0$ case, the evolution law is $C(t) = \exp(t/\beta)/2$, corresponding to $C(\eta) = \beta/\Delta \eta$. The horizon radius is

$$r_H = \Delta \eta = \frac{\beta}{C},$$

(35)

and the area is again given by (34).

**IV. ENTANGLEMENT ENTROPY IN FRIEDMANN-ROBERTSON-WALKER UNIVERSES**

With the formalism described in Sec. II and the geometry reviewed in Sec. III, it is now possible to calculate numerically the entanglement entropy of the quantum field which is associated with the FRW cosmological horizon. Let us first note that the metric (25) has the form (3) with

$$C = C(\eta), \quad B = \frac{C(\eta)}{1 + k\frac{r^2}{4\beta^2}}.$$  

(36)

Evaluating Eqs. (17), we find that the coefficients $\gamma_i$ are

$$\gamma_1 = \frac{9k^2 j^2}{16 b^4}, \quad \gamma_2 = \frac{3k j(j^2 + 4b^2)}{8 b^4},$$

where $b^4 = \frac{4}{3} \sqrt{3/\Lambda}$. In the $k = 0$ case, the entanglement entropy is

$$S = a \beta^2 \int \frac{d\eta}{C(\eta)},$$

(37)

where $a$ is a constant. The coefficient $a$ is a function of the cosmological constant $\Lambda$ and is given by

$$a = \frac{\sqrt{3}}{16\pi \beta^2}.$$  

(38)
Here, the dimensionless quantity $b$ is defined by $b = \beta/a$. Note that in the case $m = 0 = \xi$ the $\gamma_i$ are independent of $\eta$, as will also be the Hamiltonian. We shall now consider this simpler case.

In Minkowski space, the radius up to which the field has to be traced over has been chosen arbitrarily [13]. The entropy was then calculated for different values of this radius. In a spacetime with horizon, however, this procedure is not appropriate. The horizon sets a natural radius for the region to be traced over since it represents a boundary for the information an observer inside (in the cosmological case, outside for a black hole) can have about the outside region (inside for a black hole). We therefore proceed in the following way: At a fixed value of $C(\eta)$, we consider a set of universes with different values of $\beta$. According to (33), they possess different horizon radii $r_H$. We calculate the entanglement entropy numerically for each of these universes by tracing up to the horizon radius (33) (this is equivalent to tracing for radius bigger than the horizon due to the validity of the identity of the density matrices in both cases [15]). For a special value of the parameter $C$, the result is plotted in Fig. 1 as a function of the proper area of the horizon which depends on $\beta$ via (34). We find straight lines in all of the cases $k = \pm 1, 0$. This shows that the entanglement entropy is proportional to the horizon area for the three spacetime classes:

$$S = f_k(C) \frac{A_H}{4\pi a^2}. \quad (38)$$

The slope of these curves is given by the a priori unknown quantity $f_k(C)/(4\pi a^2)$. It depends on $k$ as well as on the expansion parameter $C(\eta)$. We can try to find the functional form of $f_k(C)$ by repeating the above procedure for different values of $C$ and plotting $f_k$ versus $C$. Doing so, one finds the results shown in Fig. 2 where the dots represent our numerical computation of $f(C)$ for $k = \pm 1, 0$.

It would be desirable to know the analytical form of the functions $f_k(C)$ and therefore of the entanglement entropy. Within our numerical approach, we can only try to find a function that matches our data points sufficiently well. In Fig. 2, the solid lines are plots of the functions

$$f_k(C) \approx 0.30 \frac{r_H^2}{\beta^2} = 0.30 \cdot \frac{4}{k^2}(C - \sqrt{C^2 - k^2}). \quad (39)$$

We see that the numerical values are perfectly fitted within the numerical precision. Note that there are no free fitting parameters apart from the numerical factor 0.30. For $k = 0$ we recover the flat space Hamiltonian and $f(C) \approx 0.30C^{-2}$. The first equation in (39) holds for general FRW spaces, while the second equation, with the particular dependence on $C(\eta)$, corresponds to the simplified case where we can neglect the matter contribution to the evolution of the universe (de Sitter–like geometry).

Taking into account (33) and (34), this can be written

$$S_{\text{ent}} = 0.30 \frac{r_H^2}{a^2}. \quad (40)$$
This simple formula is our final result for the open, asymptotically open and closed FRW universe. Formally, it resembles very much the Minkowski space formula (2), although one should keep in mind the conceptual differences due to the curvature of the spacetime and the existence of an event horizon, which are also reflected in its specific calculation, as explained above.

V. DISCUSSION

As we have seen, in the conformal time gauge for the metric, Eq (25), the Hamiltonian of a massless, minimally coupled scalar field is time independent. However, since the boundary \( r_H \) up to which we trace over depends on time, the entanglement entropy varies as the universe evolves. In particular, it will increase as the universe grows and decrease as it shrinks (for \( k = 1 \)) in the same way the boundary of the observer’s knowledge (given by the cosmological event horizon) does. The dependence on the particular way we take the slices of time has its origin in the Hamiltonian approach we have used.

It is also interesting to recall here that in Ref [3] it was found, for black holes in quadratic theories of gravitation, also a proportionality between the entropy and the square of the event horizon radius rather than to the horizon area. Thus, in some sense, Eq (40) seems to be a more fundamental relation than Eq (1). Note however, that the above proportionality for black holes in quadratic theories holds in the conformal approximation to the solution made in Ref [4], but it may well not survive in an exact solution (see Refs [4,5]). The other possibility to further investigate this point is to directly apply the procedure developed in this paper to the black hole case. This is presently under study by the authors.

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FIGURES

FIG. 1. This figure shows the proportionality of our computed entanglement entropy to the cosmological horizon area in a given slice of time, i.e. for fixed conformal factor $C(\eta)$. Here $k = 1, -1, 0$ correspond respectively to a FRW closed, open and spatially flat universe.

FIG. 2. Here we plot the function $f_k(C)$ appearing in Eq (38) and defined by the proportionality entropy–horizon area. The dots correspond to our numerical computation while the continuous line to the fit by $0.30 r_H^2 / \beta^2$. The $k$ parameter reflects the effect of different spatial curvatures. The dependence on $C(\eta)$ reflects time dependent character of the de Sitter metric.
