PARTITIONS AND DEFORMED CUMULANTS OF TYPE B WITH REMARKS ON THE BLITVIĆ MODEL

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ABSTRACT. Biane [B97] defined the number of restricted crossings of a partition as the intersections of corresponding arcs. A partition with restricted crossings provides the most important technical tool to investigate random variables on q-Fock space. This is due to the fact that this partition has a nice combinatorial interpretation. Later, this topic was deeply analyzed by Anshelevich [A01, A04a, A04b, A05]. Anshelevich gave the operator realization of a partition with restricted crossings, and using this tool he developed a beautiful theory of noncommutative q-Lévy processes. In this article we first define a gauge operator on (α, q)-Fock space that is analogous to Anshelevich, and then we introduce the partitions of type B, corresponding new cumulants and statistics. We also investigate the above property in the context of Blitvić model [B12], where some nice combinatorial structures appear, and give some remarks about t-free probability.

1. INTRODUCTION

Free probability was introduced by Voiculescu 30 years ago [V85, V86] in order to solve some problems in von Neumann algebras, and since then it has developed into a new field of mathematics with close connections to established fields like classical probability, combinatorics and analysis, in particular random matrices [V91], noncrossing partitions [NS06] and operator algebras. Other important examples of non-commutative probability spaces were constructed in [BS91, BS94, BGN03, B12, BS12, BEH15, V14, V16]. Biane, Goodman and Nica [BGN03] established connections between the lattices of noncrossing partitions of type B introduced by Reiner [R97], and the framework of free probability. They introduced the concept of noncrossing cumulant of type B. The inspiration for its definition is found by looking at an operation of restricted convolution of multiplicative functions, studied in parallel to functions on symmetric groups (of type A) and on hyperoctahedral groups (of type B). In this article, we describe a new partitions, which is also related to Coxeter groups of type B. Our construction is motivated by the idea of defining operators on the (α, q)-deformed Fock space introduced in [BEH15].

Now we recall a construction of [A01]. Bożejko and Speicher [BS91] defined a q-deformed Fock space as
\[ \mathcal{F}_q(H) = (\mathbb{C} \Omega) \oplus \bigoplus_{n=1}^{\infty} H^\otimes n, \]
where \( \Omega \) denotes the vacuum vector and \( H \) is the complexification of some real separable Hilbert space \( H_\mathbb{R} \). The inner product on \( \mathcal{F}_q(H) \), called the q-deformed inner product, is the sesquilinear extension of
\[ \langle x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_n \rangle_q = \delta_{m,n} \sum_{\sigma \in S(n)} q^{\vert \sigma \vert} \prod_{j=1}^{n} \langle x_j, y_{\sigma(j)} \rangle, \]
where \( q \in (-1, 1) \) and \( S(n) \) is the set of all the permutations of \( \{1, \ldots, n\} \) and \( \vert \sigma \vert := \# \{ (i, j) : i < j, \sigma(i) > \sigma(j) \} \) is the number of inversions of \( \sigma \in S_n \). On this space they defined creation and its adjoint, i.e. annihilation operators
\[ a_q^*(x) x_1 \otimes x_2 \otimes \ldots \otimes x_n = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes x, \]
\[ a_q(x) x_1 \otimes x_2 \otimes \ldots \otimes x_n = \sum_{i=1}^{n} q^{n-i} \langle x, x_i \rangle \otimes x_1 \otimes \ldots \otimes \hat{x_i} \otimes \ldots \otimes x_n, \]
where the superscript \( \hat{x_i} \) indicates that \( x_i \) has been deleted from the product. Note that in this paper we use right creation operators because we would like to be compatible with the article [BEH15]. Now let us recall the following construction by Anshelevich [A01] (in the sense of right creation). Let \( T_x \) be a
bounded self-adjoint operators on $H$ indexed by $x \in H_{\mathbb{R}}$. The corresponding gauge operator $p(T_x)$ is an operator on $\mathcal{F}_q(H)$ defined by

$$p(T_x)\Omega = 0,$$

$$p(T_x)x_1 \otimes x_2 \otimes \ldots \otimes x_n = \sum_{i=1}^{n} q^{n-i} x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_n \otimes (T_x x_i),$$

for $x_i \in H$. Then the joint moments of $A_q(x) = a_q(x) + a_q(x) + p(T_x)$, where $x \in H_{\mathbb{R}}$ (random variable on $q$-Fock space) are given by

$$\langle \Omega, A_q(x) \cdots A_q(x) \Omega \rangle_q = \sum_{\pi \in \mathcal{P}_{\geq 2}(n)} q^{rc(\pi)} \prod_{B \in \pi} \left( x_{\text{max}(B)} - 1 \prod_{i \in B, i \neq \text{min}(B, \text{max}(B))} T_x x_{\text{min}(B)} \right),$$

where by $\mathcal{P}_{\geq 2}(n)$ we denote the set partitions of $\{1, \ldots, n\}$ without singletons and $rc(\pi)$ denotes the number of restricted crossings of $\pi$ – introduced by Biane [B97]. At the beginning, we provide an informal definition of restricted crossings as the number of the intersections of arcs above the $x-$axis (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Examples of restricted crossing partition of 6 elements.}
\end{figure}

In this article we first define a gauge operator on $(\alpha, q)$-Fock space that is analogous to Anshelevich. Then using this operator, we calculate joint moments of corresponding operators, with new partitions and cumulants of type B. This new partitions of type B is a natural extension of two-colored pair partitions into two-colored block in a sense that we color by $\pm 1$ each arc in the block, similar to signed permutations of the numbers $\pm 1, \ldots, \pm n$ (Coxeter groups of type B). In the new situation we define restricted crossing of the partition in the same way as it was done by Biane [B97]. There appears here another statistic, which we called a restricted negative nesting; it means that we count how many negative arcs are covered by other arcs (see Figure 2). Note that in this article the arcs shown in figures with color $-1$ and $1$ are denoted by $\ldots_{-1}$ and $\ldots_1$, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Examples of restricted crossing partition of type B of 6 elements.}
\end{figure}

The paper is organized as follows. First we present definitions of $(\alpha, q)$-Fock space, creation and annihilation operators. Next, we introduce gauge operators and natural properties of these operators, including norm estimates and the self-adjoint relation, are presented in subsection 2.2. In subsections 2.3 and 2.4 we present partitions of type B and corresponding statistic. The generalized operators and cumulants of type B are studied in Section 3, where we give the main theorem, i.e. we show a direct
Wick formula for the mixed moments. Finally, in Section 4, we show that some similar combinatorial objects appear in the context of \((q, t)\)-probability spaces introduced by Blitvič [B12].

2. Preliminaries

2.1. \((α, q)\)-Fock space and creation, annihilation operators of type B. The Coxeter group of type \(B\) is generated by \(τ_0 = (1, -1), τ_i = (i, i + 1), i = 1, \ldots, n - 1\) and will be denoted by \(Σ(n)\). These generators satisfy the generalized braid relations \(τ_i^2 = e, 0 ≤ i ≤ n - 1, (τ_0τ_i)^4 = (τ_iτ_{i+1})^3 = e\) if \(|i - j| ≥ 2, 0 ≤ i, j ≤ n - 1\). Note that \(\{τ_i | i = 1, \ldots, n - 1\}\) generate a symmetric group \(S(n)\).

Let \(H_{\mathbb{R}}\) be a separable real Hilbert space and let \(H\) be its complexification with inner product \(⟨·, ·⟩\), linear on the right component and anti-linear on the left. When considering elements in \(H_{\mathbb{R}}\), it holds true that \(⟨x, y⟩ = ⟨y, x⟩\). We also assume that there exists a selfadjoint involution \(H ∋ x ↦ \bar{x} ∈ H\), is a selfadjoint linear bounded operator on \(H\) such that the double application of it becomes the identity operator. Let \(F_{\text{fin}}(H)\) be the (algebraic) full Fock space over \(H\)

\[
F_{\text{fin}}(H) := \bigoplus_{n=0}^{∞} H^⊗n,
\]

with convention that \(H^⊗0 = \mathbb{C}Ω\) is a one-dimensional normed unit vector. Note that elements of \(F_{\text{fin}}(H)\) are finite linear combinations of the elements from \(H^⊗n, n ∈ \mathbb{N} \cup \{0\}\) and we do not take the completion. For \(α, q ∈ (-1, 1)\) we define the type B symmetrization operator on \(H^⊗n\),

\[
P^{(n)}_{α,q} = \sum_{σ ∈ Σ(n)} α^{l_1(σ)}q^{l_2(σ)} σ, \quad n ≥ 1,
\]

\[
P^{(0)}_{α,q} = I_{H^⊗0},
\]

where \(σ ∈ Σ(n)\) is an irreducible form with minimal length and in this case let

\[
l_1(σ) \text{ be the number of } τ_0 \text{ appearing in } σ, \n
l_2(σ) \text{ be the number of } τ_i, 1 ≤ i ≤ n - 1, \text{ appearing in } σ, \n
τ_i(x_1 ⊗ \cdots ⊗ x_n) = x_1 ⊗ \cdots ⊗ x_{i-1} ⊗ x_{i+1} ⊗ x_i ⊗ x_{i+2} ⊗ \cdots ⊗ x_n, \quad n ≥ 2, \n
τ_0(x_1 ⊗ \cdots ⊗ x_n) = \bar{x}_1 ⊗ x_2 ⊗ \cdots ⊗ x_n, \quad n ≥ 1.
\]

Moreover let

\[
P_{α,q} := \bigoplus_{n=0}^{∞} P^{(n)}_{α,q},
\]

be the type-B symmetrization operator acting on the algebraic full Fock space. We deform the inner product by using the type-B symmetrization operator:

\[
⟨x_1 ⊗ \cdots ⊗ x_m, y_1 ⊗ \cdots ⊗ y_n⟩_{α,q} := ⟨x_1 ⊗ \cdots ⊗ x_m, P^{(n)}_{α,q} (y_1 ⊗ \cdots ⊗ y_n)⟩_{0,0},
\]

where \(⟨x_1 ⊗ \cdots ⊗ x_m, y_1 ⊗ \cdots ⊗ y_n⟩_{0,0} := δ_{m,n} \prod_{i=1}^{n} ⟨x_i, y_i⟩\). The \((α, q)\)-inner product is a semi-inner product for \(α, q ∈ [-1, 1]\). We restrict the parameters to the case \(α, q ∈ (-1, 1)\) so that the deformed semi-inner product is an inner product.

**Definition 1.** For \(α, q ∈ (-1, 1)\), the algebraic full Fock space \(F_{\text{fin}}(H)\) equipped with the inner product \(⟨·, ·⟩_{α,q}\) is called the Fock space of type B or the \((α, q)\)-Fock space. Let

\[
b_{α,q}(x) := r_q(x) + αℓ_q^N(x), \quad x ∈ H,
\]

where

\[
r_q(x)(x_1 ⊗ \cdots ⊗ x_n) = \sum_{k=1}^{n} q^{-k}⟨x_1, x_k⟩ x_1 ⊗ \cdots ⊗ \bar{x}_k ⊗ \cdots ⊗ x_n, \quad r_q(x)Ω = 0, \n
\]

\[
ℓ_q^N(x)(x_1 ⊗ \cdots ⊗ x_n) = q^{-n} \sum_{k=1}^{n} q^{-k-1}⟨x_1, \bar{x}_k⟩ x_1 ⊗ \cdots ⊗ \bar{x}_k ⊗ \cdots ⊗ x_n, \quad ℓ_q^N(x)Ω = 0.
\]
Operator \( b_{\alpha,q}^*(x) \) is its adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{\alpha,q} \). The operators \( b_{\alpha,q}^*(x) \) and \( b_{\alpha,q}(x) \) are called creation and annihilation operators of type B or \((\alpha, q)\)-creation and annihilation operators. Denote by \( \varphi \) the vacuum vector state \( \varphi(X) = \langle \Omega, X\Omega \rangle_{\alpha,q} \) and we denote \( \varphi_{\alpha,q}(X) = \varphi(X) \).

**Remark 1.** (1). For \( x \in H \) the creation operator of type B has the following form
\[
b_{\alpha,q}^*(x) |x_1 \otimes \cdots \otimes x_n \rangle := x_1 \otimes \cdots \otimes x_n \otimes x, \quad n \geq 1
\]
(2). Operators \( b_{\alpha,q}^*(x), b_{\alpha,q}(x) \) are bounded, for \( \alpha, q \in (-1, 1) \); see [BEH15].

**Additional facts.** In this part let us recall the properties of creation and annihilation operators from [BEH15]. We will use them in the next subsection.

**Proposition 1.** We have the decomposition
\[
p_{\alpha,q}^{(n)} = (p_{\alpha,q}^{(n-1)} \otimes I)R_{\alpha,q}^{(n-1)} \] on \( H^{\otimes n} \), \( n \geq 1 \),
where
\[
R_{\alpha,q}^{(n)} = 1 + \sum_{k=1}^{n-1} q^k \tau_{n-1} \cdots \tau_{n-k} + \alpha q^{n-1} \tau_{n-1} \tau_{n-2} \cdots \tau_0 \left( 1 + \sum_{k=1}^{n-1} q^k \tau_1 \cdots \tau_k \right),
\]
and the adjoint \( R_{\alpha,q}^{(n)}^* \) is taken with respect to \( \langle \cdot, \cdot \rangle_{0,0} \).

**Theorem 1.** For \( n \geq 1 \), we have
\[
b_{\alpha,q}(x) = r(x)R_{\alpha,q}^{(n)} \quad x \in H,
\]
where \( r(x) \) is the free right annihilation operator.

**Lemma 1.** For \( x \in H \), we get
\[
\|R_{\alpha,q}^{(n)}\|_{0,0} \leq (1 + |\alpha|q^{n-1})[n]_q, \quad n \geq 1.
\]

### 2.2. Gauge operators.

In this subsection we define a differential second quantization operator. First we introduce an operator which acts on \((\alpha, q)\)-Fock space, as
\[
p_0(T)\Omega = 0,
\]
\[
p_0(T)x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes x_n = x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes (Tx_n),
\]
where \( T \) is an operator on \( H \) with dense domain \( D \), where we assume that our involution on \( H \) and domain \( D \) are related by \( \overline{D} = D \). The adjoint of this operator satisfies \( \langle p_0(T)f|\zeta \rangle_{0,0} = \langle f|p_0(T)^*|\zeta \rangle_{0,0} \), and allows us to define a gauge operator (preservation or differential second quantization).

**Definition 2.** The gauge operator \( p(T) \) is an operator on \( \mathcal{F}_{\text{fin}}(H) \) with dense domain \( \mathcal{F}_{\text{fin}}(D) \) defined by
\[
p(T)\Omega = 0,
\]
\[
p(T) = p_0(T)R_{\alpha,q}^{(n)}.
\]

We also introduce the notation
\[
r_{q,T} := p_0(T) \left( 1 + \sum_{k=1}^{n-1} q^k \tau_{n-1} \cdots \tau_{n-k} \right),
\]
\[
\ell_{q,T}^{N} := p_0(T)q^{n-1} \left( \tau_{n-1} \tau_{n-2} \cdots \tau_1 \tau_0 \left( 1 + \sum_{k=1}^{n-1} q^k \tau_1 \cdots \tau_k \right) \right),
\]
i.e. \( p(T) = r_{q,T} + \alpha \ell_{q,T}^{N,T} \). Sometimes we will use the notation \( p_{\alpha,q}(T) = p(T) \).
Remark 2. (1) Let us observe that directly from the generalized braid relations for \( k \in [n-1] \), we have
\[
\pi_{n-1} \cdots \pi_{n-k} = \left( \begin{array}{cccc}
1 \cdots n-1 & n-1 & \cdots n-1 \\
k+1 \cdots n & n & \cdots n-1 & n-1 \\
1 \cdots k+1 \cdots n & n & \cdots n-1 & n-1 \\
\end{array} \right), \quad \pi_{n-1} \pi_{n-2} \cdots \pi_k = \left( \begin{array}{cccc}
1 \cdots k+1 \cdots n & n & \cdots n-1 & n-1 \\
1 \cdots k+2 \cdots n & n & \cdots n-1 & n-1 \\
\cdots & \cdots & \cdots & \cdots \\
1 \cdots n & n & \cdots n-1 & n-1 \\
\end{array} \right).
\]
This observation allows us to rewrite action of \( r_q^T \) and \( \ell_q^{N,T} \) on \( H^{\otimes n} \) as
\[
(2.12) \quad r_q^T(x_1 \otimes \cdots \otimes x_n) = \sum_{k=1}^{n} q^{n-k} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_n \otimes T(x_k),
\]
\[
(2.13) \quad \ell_q^{N,T}(x_1 \otimes \cdots \otimes x_n) = q^{n-1} \sum_{k=1}^{n} q^{k-1} x_1 \otimes \cdots \otimes x_k \otimes \cdots \otimes x_n \otimes T(x_k).
\]
(2) The operator \( p_{0,q}(T) \) is precisely the same as in [A01] (see introduction of this article). Visually, \( p_{0,q}(T) \) looks different than in [A01], but they are identical (in the sense that they are isomorphic) because we now use right creators. We would also like to emphasize that in the literature one can find other operators of this type but in general they are not symmetric; see [M93] [S00].

Proposition 2. If \( T \) is essentially self-adjoint on a dense domain \( \mathcal{D} \) and \( T(\mathcal{D}) \subset \mathcal{D} \), then \( p(T) \) is essentially self-adjoint on the dense domain \( \mathcal{F}_{\text{fin}}(\mathcal{D}) \).

Proof. We first observe that \( p_0(T^*)(P_{\alpha,q}^{(n-1)} \otimes I) = (P_{\alpha,q}^{(n-1)} \otimes I)p_0(T^*), \) indeed for \( x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathcal{F}_{\text{fin}}(\mathcal{D}) \), we have
\[
p_0(T^*)(P_{\alpha,q}^{(n-1)} \otimes I)(x_1 \otimes \cdots \otimes x_n) = p_0(T^*)(P_{\alpha,q}^{(n-1)} \otimes I)(x_1 \otimes \cdots \otimes x_n) - p_0(T^*)(x_1 \otimes \cdots \otimes x_n).
\]
Now we show that \( p(T) \) is symmetric on \( \mathcal{F}_{\text{fin}}(\mathcal{D}) \). Let us fix \( n, f, g \in \mathcal{D}^{\otimes n} \), then
\[
\langle p(T)f,g \rangle_{\alpha,q} = \langle p(T)f, P_{\alpha,q}^{(n-1)}g \rangle_{0,0} = \langle p_0(T)R_{\alpha,q}^{(n-1)}f, P_{\alpha,q}^{(n-1)}g \rangle_{0,0}
\]
\[
= \langle R_{\alpha,q}^{(n)}f, p_0(T^*)(P_{\alpha,q}^{(n-1)} \otimes I)P_{\alpha,q}^{(n-1)}g \rangle_{0,0} = \langle R_{\alpha,q}^{(n)}f, P_{\alpha,q}^{(n-1)} \otimes I)p_0(T^*)R_{\alpha,q}^{(n)}g \rangle_{0,0}
\]
by Proposition 1 we have
\[
= \langle f, R_{\alpha,q}^{(n)}(P_{\alpha,q}^{(n-1)} \otimes I)p_0(T^*)R_{\alpha,q}^{(n)}g \rangle_{0,0} = \langle f, p(T^*)g \rangle_{\alpha,q}.
\]
Now we show that \( T \) is essentially self-adjoint. Let \( E \) be the spectral measure of the closure of \( T \) and \( C \in \mathbb{R}_+ \). Let \( \{x_i\}_{i=1}^n \subset (E_{-C,C}H) \cap \mathcal{D}, \ x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \), then \( \|T x_i\| \leq C \|x_i\| \) and
\[
\langle p(T)x, p(T)x \rangle_{0,0} = \langle p_0(T)R_{\alpha,q}^{(n)}x, p_0(T)R_{\alpha,q}^{(n)}x \rangle_{0,0} \leq C^2 \|R_{\alpha,q}^{(n)}x\|^2_{0,0}
\]
by Lemma 1 we have
\[
\leq \left( C(1 + |\alpha||q|^{n-1})[n]_q \|x\|_{0,0} \right)^2 \leq (2nC \|x\|_{0,0})^2.
\]
Thus we get the following estimation for the norm of \( p(T)^k \)
\[
\|p(T)^k x\|^2_{\alpha,q} = \|p(T)^k x, P_{\alpha,q}^{(n)}p(T)^k x \rangle_{0,0} \leq \|P_{\alpha,q}^{(n)}\|^2_{0,0} \langle p(T)^k x, p(T)^k x \rangle_{0,0}
\]
\[
\leq \|P_{\alpha,q}^{(n)}\|^2_{0,0} (2k^2n^kC^k \|x\|_{0,0})^2.
\]
Now, we use the estimations from the proof of [BEH15] Theorem 2.9, equation (2.32) i.e.
\[
(2.14) \quad P_{\alpha,q}^{(n)} \leq (1 + |\alpha||q|^{n-1})[n]_q (P_{\alpha,q}^{(n-1)} \otimes I),
\]
with respect to the \((0,0)\)-inner product, so
\[
\left\|P_{\alpha,q}^{(n)}\right\|_{0,0} \leq \prod_{i=1}^{n} (1 + |\alpha||q|^{i-1})[i]_q \leq 2^n n!.
\]
However, it can be shown that
\[
\left\|p(T)^k x\right\|_{\alpha,q} \leq 2^n n! 2^k n^k C^k \|x\|_{0,0}, \text{ so series } \sum_{k=0}^{\infty} \frac{p(T)^k x}{k!} s^k \text{ has a positive radius of}
\]
By taking the square root of the operators from above inequality, we get
\[ \limsup_{k \to \infty} \sqrt[k]{\frac{\|p(T)^k\|_{\alpha,q}}{k!}} \leq \limsup_{k \to \infty} \sqrt[k]{\frac{2^n n! 2^k n^k C_k \|\bar{x}\|_0^0}{k!}} = 0. \]
Therefore \( \bar{x} \) is an analytic vector for \( p(T) \). The linear span of such vectors is invariant under \( p(T) \) and is a dense subset of \( \mathcal{D}^{\otimes n} \). Therefore by Nelson’s analytic vector theorem [N59] (see also [RS80]), \( p(T) \) is essentially self-adjoint on \( \mathcal{D}^{\otimes n} \).

**Proposition 3.** If \( T \) is a bounded operator on \( \mathbb{H} \), then \( p(T) \) is a bounded operator on the \((\alpha, q)\)-Fock space.

**Proof.** We begin by showing that \( p(T) \) is bounded in \( \mathcal{F}_{0,0}(\mathbb{H}) \). Next we show that \( p(T) \) is bounded. Using Proposition 2, it can be shown that \( p_{\alpha,q} p(T^*) = p(T)^* p_{\alpha,q} \) where \( p(T)^* \) is taken with respect to the \((0,0)\)-inner product. Indeed, for \( f, g \in \mathcal{H}^{\otimes n} \), we have
\[ \langle f, p(T^*)g \rangle_{\alpha,q} = \langle f, p_{\alpha,q} p(T)g \rangle_{0,0} \]
by Proposition 2, we get
\[ = \langle p(T) f, g \rangle_{\alpha,q} = \langle p(T) f, p_{\alpha,q} g \rangle_{0,0} = \langle f, p(T)^* p_{\alpha,q} g \rangle_{0,0}. \]
This gives us \( p_{\alpha,q} p(T^*) p(T) = p(T)^* p_{\alpha,q} p(T) \geq 0 \)
and
\[ p_{\alpha,q} p(T^*) p(T) [p(T)^* p(T)]^* p_{\alpha,q} \leq \|p(T)^* p(T) [p(T)^* p(T)]\|_{0,0} p_{\alpha,q}^2. \]
By taking the square root of the operators from above inequality, we get
\[ p_{\alpha,q} p(T^*) p(T) \leq \sqrt{\|p(T)^* p(T) [p(T)^* p(T)]\|_{0,0}} p_{\alpha,q} \leq \|p(T)^*\|_{0,0} p(T) \|_{0,0} p_{\alpha,q}. \]
If we take \( f \in \mathcal{H}^{\otimes n} \), then we get
\[ \langle p(T) f, p_{\alpha,q} p(T) f \rangle_{0,0} = \langle f | p(T^*) p(T) f \rangle_{\alpha,q} = \langle f | p_{\alpha,q} p(T) f \rangle_{0,0}. \]
It is clear by the definition of \( p_0 \) that \( \|p_0\|_{0,0} \leq \|T\| \), and thus
\[ \|p(T) f\|_{0,0} = \|p_0 (T) R_{\alpha,q}(n) f\|_{0,0} \leq \|p_0\|_{0,0} \|R_{\alpha,q}(n) f\|_{0,0}. \]
Now we use the estimation from Lemma 1 and we get
\[ \leq \|T\| (1 + |\alpha| |q|^{n-1}) |n| q \| f\|_{0,0} \leq \max\{1 + |\alpha|, (1 + |\alpha|)/(1 - q)\} \|T\| \| f\|_{0,0}. \]
Finally, since \( \|T^*\| = \|T\| \), we conclude that
\[ \|p(T)\|_{\alpha,q} \leq \sqrt{\|p(T^*)\|_{0,0} \|p(T)\|_{0,0}} \leq (1 + |\alpha|) \max\{1, 1/(1 - q)\} \|T\|. \]

**Remark 3.** In order to keep the essentially self-adjoint operators, we should make an additional assumption on the family of operators \( \{T_j\}_{j=1}^n \) see [A01] Subsection 2.5), but we do not, because the further computation can be done purely algebraically. **In the remainder of the article we assume that the operator \( T \) is bounded and self-adjoint.** We notice that in fact we have proved a more general Proposition 3, because in a forthcoming paper we are going to investigate Lévy processes of type B.

### 2.3. Partitions of type B

Let \( \Pi \) be the set \( \{1, \ldots, n\} \). For an ordered set \( S \), denote by \( \mathcal{P}(S) \) the lattice of set partitions of that set. We write \( B \in \Pi \) if \( B \) is a class of \( \Pi \) and we say that \( B \) is a **block** of \( \Pi \). A class of \( \Pi \) is called a **singleton** if it consists of one element, and let \( \Sigma(q) \) be the set of singletons of a set partition \( \Pi \). Given a partition \( \Pi \) of the set \( \Pi \), we write \( \text{Arc}(\Pi) \) for the set of pairs of integers \((i,j)\) which occur in the same block of \( \Pi \) such that \( j \) is the smallest element of the block greater than \( i \). The same notation \( \text{Arc}(B) \) is applied to a block \( B \in \Pi \). Thus, when we draw the points of block then we think that consecutive elements in every block (bigger than one) are connected by arcs above the x axis – see Figure 3.
In some sense, the above definition of type-B partitions is compatible with the groups \( \Sigma \) blocks of type B. Let \( P \)

The set of all partitions \( \pi \)

Remark 4.

(1) Our notation \( \pi_f \) should be understood in the sense that \( \pi_f = (\pi, f) \), where \( \pi \) is a set partition of \([n]\) and \( f \) is a coloring.

(2) Our definition of set partitions of type B is different from \([\text{BGN}03, \text{ChV}06, \text{R}97, \text{S}00, \text{RS}10]\).

(3) The set of all partitions \( \mathcal{P}(n) \) is a subset of \( \mathcal{P}^B(n) \). The relationship between them can be written as \( \pi \in \mathcal{P}(n) \cap \mathcal{P}^B(n) \iff \) all arcs have color 1.

(4) In some sense, the above definition of type-B partitions is compatible with the groups \( \Sigma(n) \). The Coxeter group of type B can be written as \( \Sigma(n) = \mathbb{Z}_2^n \rtimes S(n) \) and hence it can be defined as all signed permutations of the numbers \( \pm 1, \ldots, \pm n \). Thus \( \Sigma(n) \) consists of all signed permutations with signed entries in their window notation, i.e., we can assign arbitrary signs \( \pm 1 \) to the points \( 1, 2, \ldots, n \). Hence, the above construction of the colorings of type-B partitions is similar to that of signed permutations, because we assign color \( \pm 1 \) to every arc.

**Blocks of type B.** Let \( \pi_f \in \mathcal{P}^B(n) \), then we denote by \( B_c \in \pi_f \) each colored block of type B, where \( B = \{i_1, \ldots, i_m\} \) and \( c = (c_1, \ldots, c_{m-1}) \), \( c_j \in \{\pm 1\} \) is the color of arc \( \{i_j, i_{j+1}\} \) and

\[
\text{Arc}(B_c) = \{\{i_j, i_{j+1}\} | j \in [m-1]\}, \quad (m \geq 2),
\]

and

\[
\text{Arc}(\pi_f) = \cup_{B_c \in \pi_f} \text{Arc}(B_c).
\]

If \( B \) is a singleton, then sometimes we do not write its color, i.e., \( B := B_{[1]} \), because each singleton is necessarily colored by 1 and of course the same remark applies to \( \text{Sing}(\pi) := \text{Sing}(\pi_f) \). To be clear, we also denote by \( \max(B_c) := \max(B) \) its last element and by \( \min(B_c) := \min(B) \) its first element. We order the classes of \( \pi_f \) as \( \{B_{c_1}^1, \ldots, B_{c_l}^1\} \) according to the order of their last elements, i.e., \( \max(B_{c_1}^1) < \max(B_{c_2}^2) < \ldots < \max(B_{c_l}^l) \).

**2.4. Statistics.** Now, we introduce some partition statistics. Let \( B_c, \tilde{B}_c \in \pi_f \) with \#\( B_c, \#\tilde{B}_c \geq 2 \), where \( \pi_f \in \mathcal{P}^B(n) \).
**Restricted crossings.** Note that in the definition below the colorings of arcs are not important. Thus we use the same definition of *restricted crossings* as given in Biane [B97]. We say that the arc \{i, j\} is crossing the arc \{i’, j’\} if \( i < i’ < j < j’ \) or \( i’ < i < j’ < j \). Now we can define

\[
rc(B_c, \tilde{B}_c) = \# \{(V, W) \in \text{Arc}(B_c) \times \text{Arc}(\tilde{B}_c) \mid \text{V is crossing W}\}.
\]

The number of restricted crossings of \( \pi \)

\[
rc(\pi) = rc(\pi_\ell) = \sum_{i < j} \text{rc}(B^i_c, B^j_c),
\]

where \( \pi_\ell \setminus \text{Sing}(\pi) = \{B^1_{c_1}, \ldots, B^l_{c_l}\} \).

**Restricted negative nestings.** Now we define the number of *restricted negative nestings* of the partition \( \pi_\ell \in \mathcal{P}^B(n) \). We say that an arc \{i, j\} nests \{i’, j’\} if \( i < k < j \) for any \( k \in \{i’, j’\} \). The set of nestings of \( B_c, \tilde{B}_c \)

\[
\text{nest}(B_c, \tilde{B}_c) = \{(V, W) \in \text{Arc}(B_c) \times \text{Arc}(\tilde{B}_c) \mid V \text{ nests } W \text{ or } W \text{ nests } V\},
\]

and the set of nesting of \( \pi \)

\[
\text{nest}(\pi) = \text{nest}(\pi_\ell) = \bigcup_{i < j} \text{nest}(B^i_c, B^j_c),
\]

where \( \pi_\ell \setminus \text{Sing}(\pi) = \{B^1_{c_1}, \ldots, B^l_{c_l}\} \). The number of *restricted negative nestings* is defined by

\[
\text{rnarc}(\pi_\ell) = \# \{(V, W) \in \text{nest}(\pi) \mid V \text{ nests } W \text{ and } f(W) = -1 \text{ or } W \text{ nests } V \text{ and } f(V) = -1\}.
\]

**Negative arcs.** Let \( \text{Arc}(\pi_\ell, -1) := \{W \in \text{Arc}(\pi_\ell) \mid f(W) = -1\} \), then

\[
\text{Narc}(\pi_\ell) = \# \text{Arc}(\pi_\ell, -1),
\]

is the number of negative arcs.

**Remark 5.** In the above equation, we skip the index \( f \) in \( \pi \) if our statistic does not depend on coloring.

### 3. A DEFORMED PROBABILITY OF TYPE B

#### 3.1. Operators and cumulants of type B

In this non-commutative setting, random variables are understood to be the elements of the *-algebra generated by \( \{b_{\alpha,q}(x), b_{\alpha,q}^*(x), p(T_x) \mid x \in \mathbb{H}_\mathbb{R}\} \). Particularly interesting are their joint mixed moments, i.e. expressions

\[
\varphi \left( [b_{\alpha,q}(x_n) + b_{\alpha,q}^*(x_n) + p(T_x)] \ldots [b_{\alpha,q}(x_1) + b_{\alpha,q}^*(x_1) + p(T_x)] \right).
\]

In order to work effectively on this object we need to define corresponding cumulants. This topic in the case of q-deformed Fock space was deeply analyzed in the literature; see [Nic95, Nic96, A01, L05].

One of the first definitions of cumulants appropriate for q-deformed probability theory has already been given in [Nic95], based on an analog of the canonical form introduced by Voiculescu in the context of free probability. The advantage of the approach of that paper is that Nica’s cumulants are defined for any probability distribution whose all moments are finite, however, the canonical form in that paper is not self-adjoint. Also Lehner [L05] developed some general formulas, computed cumulants and partitioned cumulants for generalized Toeplitz operators. Our approach is close to [HP84, Sch91, A01].

We provide an explicit formula for the combinatorial cumulants, involving the number of restricted crossings and negative nestings of a partition. First, we need to define the operators and cumulants, where we assume that \( T_x \) are fixed bounded self-adjoint operators on \( \mathbb{H} \) indexed by \( x \in \mathbb{H}_\mathbb{R} \).

**Definition 3.** The operator

\[
B_{\alpha,q}^\lambda(x) := b_{\alpha,q}(x) + b_{\alpha,q}^*(x) + p(T_x) + \lambda I, \quad x \in \mathbb{H}_\mathbb{R}, \quad \lambda \in \mathbb{R},
\]

on \( \mathcal{F}_{\text{fin}}(H) \) is called *operator of type B* and \( B_{\alpha,q} := B_{\alpha,q}^0 \).
Definition 4. Let $\tau_i \in P^B(n)$, $B_c = \{i_1, \ldots, i_m\}(c_1, \ldots, c_m \in \tau_i)$, $x = (x_1, \ldots, x_n) \in \mathbb{H}^n_\mathbb{R}$ and $\lambda_i \in \mathbb{R}$. The deformed cumulant of type $B$ is defined by

$$
K^x(B_c) := \begin{cases} 
\lambda_{\min}(B_c) & \text{if } #B_c = 1, \\
(x_\max(B_c), f_{m-1}T_{x_{i_{m-1}}} \cdots f_3T_{x_{i_3}} f_2T_{x_{i_2}} f_1x_{\min}(B_c)) & \text{if } #B_c \geq 2,
\end{cases}
$$

where $f_i$ is a color operator $f_i : H \mapsto H$ defined as follows:

$$
f_i(x) = \begin{cases} 
x & \text{if } c_i = 1, \\
\bar{x} & \text{if } c_i = -1.
\end{cases}
$$

The following theorem is the main result of the paper. Its proof is given in Section 3.3.

Theorem 2. Suppose that $x = (x_1, \ldots, x_n) \in \mathbb{H}^n_\mathbb{R}$, then

$$
\varphi\left(\prod_{B \in \tau}^\lambda_{\alpha_q}(x_n) \cdots \prod_{B \in \tau}^\lambda_{\alpha_1}(x_1)\right) = \sum_{\tau_i \in P^B(n)} \alpha^{N_{\operatorname{Narc}}(\tau_i)} q^{\text{rc}(\tau)+2\text{max}(\tau_i)} K^x_{\tau_i},
$$

Corollary 1. Assume that $x_i \in \mathbb{H}_\mathbb{R}$ for $i \in [n]$.

1. For $\alpha = \lambda = 0$ and $\bar{x}_i = x_i$, we obtain the $q$-deformed formula for moments of random variable on $q$-Fock space (see [A01] or [A04a] Proposition 6)

$$
\varphi_{0,q}(\prod_{B \in \tau}^\lambda_{\alpha_q}(x_n) \cdots \prod_{B \in \tau}^\lambda_{\alpha_1}(x_1)) = \sum_{\tau_i \in P^B(n)} q^{\text{rc}(\tau)} \prod_{B \in \tau} x_{\max}(B) \prod_{i \in B, i \neq \min(B), \max(B)} T_{x_i} x_{\min}(B).
$$

2. For $T = \mathbf{0}$ and $\lambda = 0$ we get the formula for Gaussian operator of type $B$ [BEH15, Corollary 3.9]

$$
\varphi_{\alpha,q}(\prod_{B \in \tau}^\lambda_{\alpha_q}(x_n) \cdots \prod_{B \in \tau}^\lambda_{\alpha_1}(x_1)) = \sum_{\tau_i \in P^B(n)} \alpha^{N_{\operatorname{Narc}}(\tau_i)} q^{\text{rc}(\tau)+2\text{max}(\tau_i)} \prod_{\{i,j\} \in \tau, f(\{i,j\}) = 1} \langle x_i, x_j \rangle \prod_{\{i,j\} \in \tau, f(\{i,j\}) = -1} \langle x_i, x_j \rangle.
$$

3. For $q = \lambda = 0$ and $\bar{x}_i = x_i$, we get

$$
\varphi_{\alpha,0}(\prod_{B \in \tau}^\lambda_{\alpha_q}(x_n) \cdots \prod_{B \in \tau}^\lambda_{\alpha_1}(x_1)) = \sum_{\tau_i \in P^B(n)} \left(1 + \alpha \right)^{\#\text{OutArc}(\tau)} x_{\max}(B) \prod_{i \in B, i \neq \min(B), \max(B)} T_{x_i} x_{\min}(B).
$$

where

- $\#\text{OutArc}(\tau)$ is the number of outer arcs in $\tau$ in a sense that these arcs are not nested by others;
- $N_{\operatorname{Narc}}(n)$ is the set of noncrossing partitions of $[n]$ without singletons.

Proof. (1) If we put $\alpha = \lambda = 0$ in (3.2), then all arcs have color 1 and

$$
K^x(B_c) = \langle x_{\max}(B), \prod_{i \in B, i \neq \min(B), \max(B)} T_{x_i} x_{\min}(B) \rangle \text{ if } c = (1, \ldots, 1).
$$

(2) Under the assumption that $T = \mathbf{0}$ and $\lambda = 0$, we see

$$
K^x(B_c) = \langle x_{\max}(B), f_{m-1}T_{x_{i_{m-1}}} \cdots f_3T_{x_{i_3}} f_2T_{x_{i_2}} f_1x_{\min}(B_c) \rangle \neq 0
$$

$\iff$ $#B_c = 2$, which means that $\tau_i \in P^B_2(n)$.

(3) When $q = \lambda = 0$, then the nonzero terms in part (2) of Theorem 2 occur only when $\pi$ is noncrossing and inner arc are colored by 1. Thus, for $\pi \in N_{\operatorname{Narc}}(n)$ we count how many times we can assign color $-1$ to outer arcs and so we get

$$
\sum_{i=0} \alpha^i \binom{\#\text{OutArc}(\tau)}{i} = (1 + \alpha)^{\#\text{OutArc}(\tau)}.
$$
Example 1. The deformed cumulants and partitions of type B for $\varphi(B_{\alpha,q}(x_4)B_{\alpha,q}(x_3)B_{\alpha,q}(x_2)B_{\alpha,q}(x_1))$ can be graphically represented in Figure 5.

![Figure 5](image)

**FIGURE 5.** Deformed cumulants, statistics and partitions of type B for $\varphi(B_{\alpha,q}(x_4)B_{\alpha,q}(x_3)B_{\alpha,q}(x_2)B_{\alpha,q}(x_1))$.

### 3.2. The Poisson distribution of type B.

For a probability measure $\mu$ with finite moments of all orders, let us orthogonalize the sequence $(1, y, y^2, y^3, \ldots)$ in the Hilbert space $L^2(\mathbb{R}, \mu)$, following the Gram-Schmidt method. This procedure yields orthogonal polynomials $(P_0(y), P_1(y), P_2(y), \ldots)$ with $\deg P_n(y) = n$. Multiplying by constants, we take $P_n(y)$ to be monic, i.e. the coefficient of $y^n$ is 1.

It is known that they satisfy a recurrence relation

$$yP_n(y) = P_{n+1}(y) + \beta_n P_n(y) + \gamma_{n-1} P_{n-1}(y), \quad n = 0, 1, 2, \ldots$$

with the convention that $P_{-1}(y) = 0$. The coefficients $\beta_n$ and $\gamma_n$ are called *Jacobi parameters* and they satisfy $\beta_n \in \mathbb{R}$ and $\gamma_n \geq 0$. The following representation of type-B Poisson distribution is in the spirit of [A01] and rather different from that of [SY00a, SY00b].

**Definition 5.** $(\alpha, q)$-Poisson of type B polynomials are defined by the recursion relations

$$yP_n^{(\alpha,q)}(y) = P_{n+1}^{(\alpha,q)}(y) + [n]_q(1 + \alpha q^{n-1})P_n^{(\alpha,q)}(y) + [n]_q(1 + \alpha q^{n-1})P_{n-1}^{(\alpha,q)}(y), \quad n \geq 1$$

with initial conditions $P_{-1}^{(\alpha,q)}(y) = 0$, $P_0^{(\alpha,q)}(y) = 1$ and $P_1^{(\alpha,q)}(y) = y$. There exists a probability measure $\mu_{\alpha,q}$ which associates the orthogonal polynomials $P_n^{(\alpha,q)}$.

**Remark 6.** Professor M. Ismail informed us that the measure of orthogonality of the above polynomial sequence is not known. The $(\alpha, q)$-Poisson of type B polynomials are $q$-analogues of the polynomials studied in [IK12] (equation (5.11) with $\nu = 1$). In special cases, we can identify this measure:

1. the measure $\mu_{\alpha,1}$ is the classical Poisson law;
2. the measure $\mu_{0,0}$ is the Marchenko-Pastur distribution;
3. the measure $\mu_{0,q}$ is the $q$-Poisson law and the orthogonal polynomials $P_n^{(0,q)}(y)$ are called $q$-Poisson-Charlier polynomials (see [A01]);
4. the measure $\mu_{\alpha, -1}$ is a non-symmetric Bernoulli distribution.
(5) the measure $\mu_{\alpha,0}$, $\alpha \neq 0$ is a two-state free Meixner distribution because its Jacobi parameters are independent of $n$ for $n \geq 2$ (see [AM12, DGIX09]). The corresponding measure belongs to the Bernstein-Szegő class i.e. has at most 3 atoms and absolutely continuous part of $\mu_{\alpha,0}$ is

$$\frac{\sqrt{4 - (x - 1)^2}}{p(x)} \, dx,$$

where $p(x)$ is a cubic polynomial.

**Proposition 4.** Suppose that $\alpha, q \in (-1, 1)$ and $x \in H$, $\|x\| = 1$, $x = \pm x$ and $T = \text{Id}$. Then the probability distribution of $B_{\alpha,q}$ with respect to the vacuum state is given by $\mu_{\alpha, q}$ if $x = x$ and $\mu_{-\alpha,q}$ if $x = -x$.

**Proof.** First assume that $x = x$. Note that for $n = 1$ $P_{\alpha,q} \{ B_{\alpha,q}(x) \} \Omega = B_{\alpha,q}(x) \Omega = x$ and by induction

$$P_{n+1} \{ B_{\alpha,q}(x) \} \Omega = B_{\alpha,q}(x) \Omega - [n]_q (1 + \alpha q^{-1}) P_{n} \{ B_{\alpha,q}(x) \} \Omega - [n]_q (1 + \alpha q^{-1}) P_{n} \{ B_{\alpha,q}(x) \} \Omega = B_{\alpha,q}(x) x^{-n} - [n]_q (1 + \alpha q^{-1}) x^{-n} - [n]_q (1 + \alpha q^{-1}) x^{-n+1} = x^{-n+1}.$$

Hence, from above it follows that

$$\| x^{-n} \|_{\alpha,q} = \| P_{n} \{ \alpha q \} \|_{L^2}, \quad n \in \mathbb{N} \cup \{0\}.$$

Therefore the map $\Phi: (\text{span}[x^{-n} \ | \ n \geq 0], \| \cdot \|_{\alpha,q}) \to L^2(\mathbb{R}, \mu_{\alpha,q})$ defined by $\Phi(x^{-n}) = P_n \{ \alpha q \}(y)$ is an isometry. Since $\Phi$ is an isometry, we get $\langle \Omega, B_{\alpha,q}(x) \Omega \rangle_{\alpha,q} = m_n(\mu_{\alpha,q})$. By Proposition 3 we conclude that $B_{\alpha,q}(x)$ is bounded, so its vacuum distribution is compactly supported. Hence, the moments uniquely determine the measure and we conclude that $B_{\alpha,q}(x)$ has the distribution $\mu_{\alpha,q}$. We proceed analogously if $x = -x$. \hfill \Box

### 3.3. Proof of the main theorem.

#### Extended partition.

In order to prove the main theorem we need the set $P_{\mathcal{E}}(n)$ of so-called extended partitions, which contains the set $P_{\mathcal{B}}(n)$. We use these partitions in the proof of Theorem 3 only. Here each block of size at least two can be additionally marked by $\mathcal{E}$. More precisely, for the fixed $\pi_\tau \in P_{\mathcal{B}}(n)$, and $B_c \in \pi_\tau$, where $\#B_c \geq 2$, we consider additional numbers $\bar{1}, \bar{2}, \ldots, \bar{n}$ and define $B_c' = \{ 1, \ldots, m \} = \bar{1}, \ldots, \bar{m} \}$. $\hspace{1cm}$

**Definition 6.** A set partitions $P_{\mathcal{E}}(n)$ is defined by $P_{\mathcal{B}}(n)$, as follows:

$$P_{\mathcal{E}}(n) = \{ S' \cup (\pi_\tau \setminus S) \ | \ S \subset \pi_\tau \setminus \text{Sing}(\pi), \pi_\tau \in P_{\mathcal{B}}(n) \},$$

where $S' := \{ A' \ | \ A \in S \}$ and $\emptyset' = \emptyset$.

**Remark 7.** We can think about operation $\tau$ in this way, that we pick some blocks of size at least two and marked them by $\mathcal{E}$.

**Example 2.** For example,

$$P_{\mathcal{E}}(3) = \{ \{1, 2, \{3\}\}, \{1, 2\} \times \{\pm1\}, \{2\} \times \{\pm1\}, \{1, 3\} \times \{\pm1\}, \{2, 3\} \times \{\pm1\}, \{1, 2, 3\} \times \{\pm1\} \}. \hspace{1cm}$$

For $\pi_\tau \in P_{\mathcal{E}}(n)$ we denote by $\text{Block}'(\pi_\tau)$ the blocks of $\pi_\tau$ which are marked by $\mathcal{E}$ and $\text{Block}(\pi_\tau) = \pi_\tau \setminus (\text{Block}'(\pi_\tau) \cup \text{Sing}(\pi))$. Thus we can decompose an extended partition of type B as a disjoint subset

$$\pi_\tau = \text{Block}'(\pi_\tau) \cup \text{Block}(\pi_\tau) \cup \text{Sing}(\pi).$$

**Cover and left of max.** We also need some other statistics: MaxC$\pi_\tau$ and MaxL$\pi_\tau$ for $\pi_\tau \in P_{\mathcal{E}}(n)$ which we will use in Theorem 3 only. Namely, we define

MaxC$\pi_\tau = \# \{ (V, W) \in [\text{Block}'(\pi_\tau) \cup \text{Sing}(\pi)] \times \text{Arc}(\pi_\tau) \ | \ i < \max(V) < j \text{ for } i, j \in W \}$,

MaxL$\pi_\tau = \# \{ (V, W) \in [\text{Block}'(\pi_\tau) \cup \text{Sing}(\pi)] \times \text{Arc}(\pi_\tau, -1) \ | \ \max(V) < j \text{ for all } j \in W \}.$
**Remark 8.** Note that $\text{MaxC}(\pi)$ represents the number of covered singletons and $\text{MaxL}(\pi_f)$ the number of singletons to the left of negative arcs, whenever we have $\text{Block}'(\pi_f) = \emptyset$.

**Example 3.** For the partition in Figure 6 we see that

- if $\pi = \{1, 4, 6, 7\}_{(1,-1,-1)},\{2\},\{3, 5, 10\}_{(1,-1)},\{8, 12\}_{(-1)},\{9, 11\}_{(1)}$, then $\text{rc}(\pi) = 5$, $\text{rrarc}(\pi_f) = 1$, $\text{Narc}(\pi_f) = 4$, $\text{MaxC}(\pi) = 3$ and $\text{MaxL}(\pi_f) = 3$;
- if $\pi = \{1, 4, 6, 7\}_{(1,-1,-1)},\{2\},\{3, 5, 10\}_{(1,-1)},\{8, 12\}_{(-1)},\{9, 11\}_{(1)}$, then $\text{rc}(\pi) = 5$, $\text{Narc}(\pi_f) = 4$, $\text{rrarc}(\pi_f) = 1$, $\text{MaxC}(\pi) = 1$ and $\text{MaxL}(\pi_f) = 3$;
- if $\pi = \{1, 4, 6, 7\}_{(1,-1,-1)},\{2\},\{3, 5, 10\}_{(1,-1)},\{8, 12\}_{(-1)},\{9, 11\}_{(1)}$, then $\text{rc}(\pi) = 5$, $\text{Narc}(\pi_f) = 4$, $\text{rrarc}(\pi_f) = 1$, $\text{MaxC}(\pi) = 2$ and $\text{MaxL}(\pi_f) = 4$.

![Figure 6. A partition of 12 elements with five blocks](image)

In order to simplify notation, we define the following operators, which map $H$ into $H$ and which are indexed by the block $B_c = \{i_1, \ldots, i_m\}_{(c_1, \ldots, c_m=1)} \in \pi$, i.e.

$$
\hat{T}_{B_c}^x = \begin{cases}
1 & \text{if } B_c \in \text{Sing}(\pi) \ (c = 1) \\
T_{x_{i_1}} f_{m-1} \ldots f_3 T_{x_{i_3}} f_2 T_{x_{i_2}} f_1 & \text{if } B_c \in \text{Block}'(\pi_f),
\end{cases}
$$

$$
T_{B_c}^x = \begin{cases}
f_1 & \text{if } \#B_c = 2 \text{ and } B_c \in \text{Block}(\pi_f), \\
f_{m-1} T_{x_{i_m-1}} \ldots f_3 T_{x_{i_3}} f_2 T_{x_{i_2}} f_1 & \text{if } \#B_c > 2 \text{ and } B_c \in \text{Block}(\pi_f),
\end{cases}
$$

where $x = (x_1, \ldots, x_n) \in H^n$, $\pi_f \in \mathcal{P}_B^n(n)$ and $f_i$ is a color operator introduced in the Definition 4 of cumulants. With the above notations we also introduce

$$
\hat{R}_{\pi_f}^x = \prod_{B_c \in \text{Block}(\pi_f)} \langle x_{\text{max}(B_c)}, T_{B_c}^x x_{\text{min}(B_c)} \rangle.
$$

Notice that in the above formula we use the following bracket notation $\{x\}_{\text{max}(B_c)}$, which should be understood that the position of $\ast$ (in the tensor product) is ordered with respect to the max$(B_c)$. For example, if $\text{Block}'(\pi_f) \cup \text{Sing}(\pi) = \{\{4\}, \{6\}, \{2, 5, 7\}_{(1,-1)}, \{1, 3\}_{(-1)}\}$, then

$$
\hat{R}_{\pi_f}^x = T_{x_7} x_1 \otimes x_4 \otimes x_6 \otimes T_{x_5} \bar{x}_{x_1 x_2}.
$$

We also use the following conventions for $\epsilon \in \{1, *, t\}$

$$
b_{\alpha, q}^\epsilon (x) = \begin{cases}
b_{\alpha, q}^* (x) & \text{if } \epsilon = *, \\
r_q(x) + \alpha \ell_q^N(x) & \text{if } \epsilon = 1, \\
r_q^t(x) + \alpha \ell_q^N T_n(x) & \text{if } \epsilon = t.
\end{cases}
$$

Now we prove the following theorem, which shows the relationship between partitions of type B (corresponding statistic) and a joint action of operators on a vacuum vector introduced in Section 2.
Theorem 3. For any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and any \( \epsilon = (\epsilon(1), \ldots, \epsilon(n)) \in \{1, *, l\}^n \), we have
\[
b_{\alpha,q}^{(n)}(x_1) \cdots b_{\alpha,q}^{(1)}(x_1) = \sum_{\pi \in \mathcal{P}_{E,\epsilon}(n)} \alpha^{\text{Narc}(\pi_l)} q^{\text{rc}(\pi)} + \text{MaxC}(\pi) + 2\text{rnarc}(\pi_l) + 2\text{MaxL}(\pi_l) \mathcal{R}_{\pi_l}^{B_\pi} \mathcal{R}_{\pi_l}^{S_\pi},
\]
(3.5)

Remark 9. (1) If \( \text{Block}'(\pi_l) \cup \text{Sing}(\pi) = \emptyset \), then \( \mathcal{R}_{\pi_l}^{S_\pi} = \emptyset \).

(2) If \#(\{i \in [j] | \epsilon(i) = 1\}) > \#(\{i \in [j] | \epsilon(i) = *\}) \) for some \( j \in [n] \), then we have
\[
b_{\alpha,q}^{(n)}(x_1) \cdots b_{\alpha,q}^{(1)}(x_1) = 0.
\]
This case is also covered by (3.5) if we understand that the sum over the empty set is 0 since \( \mathcal{P}_{E,\epsilon}(n) = \emptyset \) in this case.

Proof. The proof is given by induction. When \( n = 1 \), \( b_{\alpha,q}(x_1) = p(T_{x_1}) = 0 \) and \( b_{\alpha,q}^{*}(x_1) = x_1 \) and hence the formula is true. Suppose that the formula is true for \( n = k \). Then for any \( \epsilon \in \{1, *, l\}^k \), we get
\[
b_{\alpha,q}^{(k)}(x_k) \cdots b_{\alpha,q}^{(1)}(x_1) = \sum_{\pi_l \in \mathcal{P}_{E,\epsilon}(k)} \alpha^{\text{Narc}(\pi_l)} q^{\text{rc}(\pi_l)} + \text{MaxC}(\pi_l) + 2\text{rnarc}(\pi_l) + 2\text{MaxL}(\pi_l) \mathcal{R}_{\pi_l}^{B_\pi} \mathcal{R}_{\pi_l}^{S_\pi}.
\]

We will show that the action of \( b_{\alpha,q}^{(k+1)}(x_{k+1}) \) corresponds to the inductive graphic description of set partitions of type B. From now on, we fix a partition \( \pi_l \in \mathcal{P}_{E,\epsilon}(k) \) and one block \( B_c \in \text{Block}'(\pi_l) \cup \text{Sing}(\pi) \) in the \( i^\text{th} \) position i.e. \( i = \text{max}(B_c) \). In this situation, the block \( B_c \) contributes to the element \( \mathcal{R}_{B_c,x_{\min}(B_c)}^{B_\pi} \) (in a sense that \( \mathcal{R}_{\pi_l}^{S_\pi} = \cdots \otimes \{\mathcal{R}_{B_c,x_{\min}(B_c)}^{B_\pi}\} \otimes \cdots \)). Suppose that \( \pi_l \) has
- \( p \) blocks \( B_c \in \text{Block}'(\pi_l) \cup \text{Sing}(\pi) \), such that \( \text{max}(B_c) < i \) – we call them Left(\( B_c \)),
- \( r \) blocks \( B_c \in \text{Block}'(\pi_l) \cup \text{Sing}(\pi) \), such that \( \text{max}(B_c) > i \) – we call them Right(\( B_c \)).

We understand that \( p = 0 \) (\( r = 0 \), respectively) when there are no blocks \( B_c \in \text{Block}'(\pi_l) \cup \text{Sing}(\pi) \) such that \( \text{max}(B_c) \) is to the left (right, respectively) of \( i \). Here we assume more, that \( \pi_l \) has
- arcs \( U_1, \ldots, U_u \) with color 1 to the right of \( i^\text{th} \) in the strict sense,
- arcs \( V_1, \ldots, V_t \) with color \(-1\) to the right of \( i^\text{th} \) in the strict sense,
- arcs \( W_1, \ldots, W_u \) which cover \( i^\text{th} \) in a sense that \( b_* < i < b_* \) for \( b_*, b_* \in W_* \).

\[ \text{Figure 7. The main structure of partition } \pi_l \in \mathcal{P}_{E,\epsilon}(k) \text{ in the induction step.} \]

There may be arcs to the left of \( i^\text{th} \), but they do not matter – see Figure 7. Note here that \( p + r + 1 = \#\text{Sing}(\pi) + \#\text{Block}'(\pi_l) \). In the proof we use the notation \( \tilde{B}_{(\epsilon, \pm 1)} \), which denotes the block created from \( B_c \) by adding color \( \pm 1 \) to the last coordinate of \( c \), i.e. \( (c, \pm 1) \) and \( \tilde{B} = B \cup \{k + 1\} \). In the cases below, each subsequent operator \( b_{\alpha,q}^{(n)}(x_{k+1}) \) contributes to
- a new singleton in \( \text{Sing}(\pi) \), if \( \epsilon = * \) – Case 1;
- a new block in \( \text{Block}(\pi_l) \), if \( \epsilon = 1 \) – Case 2;
- a new block in \( \text{Block}'(\pi_l) \), if \( \epsilon = l \) – Case 3.

Case 1. If \( \epsilon(k + 1) = * \), then the operator \( b_{\alpha,q}^{*}(x_{k+1}) \) acts on the tensor product, putting \( x_{k+1} \) on the right. This operation graphically corresponds to adding the singleton \( \{k + 1\} \) (with color 1) to \( \pi_l \in \mathcal{P}_{E,\epsilon}(k) \), to yield a new type-B partition \( \pi_l \in \mathcal{P}_{E,\epsilon}(k + 1) \). This map \( \pi_l \mapsto \tilde{\pi}_l \) does not change the numbers \( \text{Narc, rc, rnarc, MaxL or MaxC} \), because a new singleton is the right element of \( \tilde{\pi}_l \).
This is compatible with the fact that the action of $b^*_{x,q}(x_{k+1})$ does not change the coefficient. Hence, the formula (3.5) holds when $n = k + 1$ and $e(k + 1) = *$.

Now we move to Cases 2 and 3, where we assume $\text{Block}'(\pi_t) \cup \text{Sing}(\pi) \neq \emptyset$.

Case 2. If $e(k + 1) = 1$, then we have two cases.

Case 2a. If $r_q(x_{k+1})$ acts on the tensor product, then new $p + r + 1$ terms appear by using (2.5). In the $i$th term the inner product $\langle x_{k+1}, \hat{T}_{B_c}^{x \text{min}(B_c)} \rangle$ appears with coefficient $q^r$. Graphically this corresponds to getting a set partition $\tilde{\pi}_t \in \mathcal{P}^p_{\leq e}(k+1)$ by adding $k+1$ to $\pi_t$ and creating the block $\tilde{B}_{(C,1)} \in \text{Block}(\tilde{\pi}_t)$, i.e. now the last arc has color $1$ -- see Figure 8. This new arc $\{i, k + 1\}$

- crosses the arcs $W_1, \ldots, W_u$ and so increases the number of crossings by $u$;
- decreases by $u$ the number of blocks $B_c \in \text{Block}'(\tilde{\pi}_t) \cup \text{Sing}(\tilde{\pi})$ such that $\max(B_c)$ is between the arcs $W_1, \ldots, W_u$ (originally $i$ was between the arcs $W_1, \ldots, W_u$);
- increases by $r$ the number of block $B_c \in \text{Block}'(\tilde{\pi}_t) \cup \text{Sing}(\tilde{\pi})$ such that the $\max(B_c)$ is covered by the new arc;
- covers the negative arcs $V_1, \ldots, V_t$;
- causes that in the new situation $B_c$ is not a block in $\tilde{\pi}_t$, so the number $\text{MaxL}(\pi_t)$ decreases by $t$.

Altogether we have: $rc(\tilde{\pi}) = rc(\pi) + u$, $\text{MaxC}(\tilde{\pi}) = \text{MaxC}(\pi) - u + r$, $r\text{narc}(\tilde{\pi}_t) = r\text{narc}(\pi_t) + t$, $\text{MaxL}(\tilde{\pi}_t) = \text{MaxL}(\pi_t) - t$ and $\text{Narc}(\tilde{\pi}_t) = \text{Narc}(\pi_t)$. So the exponent of $q$ increases by $r$. Summarizing this, we get the factor $q^r$ and the inner product

$$\langle x_{k+1}, \hat{T}_{B_c}^{x \text{min}(B_c)} \rangle = \langle x_{\text{max}(B_{(C,1)}),}, T_{B_{(C,1)}^\pi}^{x \text{min}(B_{(C,1)})} \rangle,$$

which is exactly the expression when $r_q(x_{k+1})$ acts on $\hat{T}_{B_c}^{x \text{min}(B_c)}$.

**Remark 10.** Note that in the above algorithm, we create a new pair in $\text{Block}(\tilde{\pi}_t)$ with color $1$, whenever we have $B_c \in \text{Sing}(\pi)$. An analogous remark applies to the remainder of the proof.

Case 2b. If $\alpha \ell_N(x_{k+1})$ acts on the tensor product, then new $p + r + 1$ terms appear by using (2.6). In the $i$th term the inner product $\langle x_{k+1}, \hat{T}_{B_c}^{x \text{min}(B_c)} \rangle$ appears with coefficient $\alpha q^p(p+r+1)$. This means that we create a new block $B_{(C,1)} \in \text{Block}(\tilde{\pi}_t)$, i.e. the last arc $\{i, k + 1\}$ has color $-1$ -- see Figure 9. Similarly to Case 2a, we calculate the changes of statistics and get: $rc(\tilde{\pi}) = rc(\pi) + u$, $\text{MaxC}(\tilde{\pi}) = \text{MaxC}(\pi) - u + r$, $r\text{narc}(\tilde{\pi}_t) = r\text{narc}(\pi_t) + t$, $\text{MaxL}(\tilde{\pi}_t) = \text{MaxL}(\pi_t) - t + p$, and $\text{Narc}(\tilde{\pi}_t) = \text{Narc}(\pi_t) + 1$. Altogether, when moving from $\pi_t$ to $\tilde{\pi}_t$, the exponent of $\alpha$ increases by $1$ and the exponent of $q$ increases by $2p + r$, which coincides with the coefficient appearing in the action of $\alpha \ell_N(x_{k+1})$. In the end, we get the factor $q^{2p+r}$ and the inner product

$$\langle x_{k+1}, \hat{T}_{B_c}^{x \text{min}(B_c)} \rangle = \langle x_{\text{max}(B_{(C,1)}),}, T_{B_{(C,1)}^\pi}^{x \text{min}(B_{(C,1)})} \rangle.$$
$B_c \in \text{Block}'(\pi_f)$

$B_{(C),l} \in \text{Block}(\pi_f)$

**Figure 9.** The visualization of the action $\alpha l_q^N(x_{k+1})$ on $\hat{T}^{x}_{B_{c}}x^{\min}(B_c)$.

\[ k + 1): \]

$B_c \in \text{Block}'(\pi_f)_{B_c \neq B_c}$

with coefficient $q^r$. Then we get a new partition $\tilde{\pi}_f \in \mathcal{P}_{E; e_c}(k + 1)$ by adding $k + 1$ to $\pi_f$ and creating the block marked by $i$ with the last arc colored by $-1$, i.e. $\hat{T}^{x}_{B_{c},l}x^{\min}(B_{c},l)$ (marked by $i$) with the last arc colored by $-1$, see Figure 10. Now $\max(\hat{T}^{x}_{B_{c},l}) = k + 1$, so we can calculate the change in the statistic generated by the new arc, analogously to Case 2a, because $\max(\hat{T}^{x}_{B_{c},l})$ cannot be covered or be to the left of some negative arc, i.e. the exponent of $q$ will increase by $r$. This situation is also compatible with changes inside the tensor product, i.e. $\hat{T}^{x}_{B_{c},l}x^{\min}(B_{c},l) = \hat{T}^{x}_{x_{k+1}}(\hat{T}^{x}_{B_{c}}x^{\min}(B_c))$.

\[ \hat{T}^{x}_{B_{c},l}x^{\min}(B_{c},l) \]

$B_c \in \text{Block}'(\pi_f)$

$B_{(C),l} \in \text{Block}'(\pi_f)$

**Figure 10.** The visualization of the action $T_q^{x_{k+1}}$ on $\hat{T}^{x}_{B_{c}}x^{\min}(B_c)$.

Case 3b. We use equation (2.11), then we get in the $i^{th}$ term of the operator $\alpha l_q^{N,T_{k+1}}$

$$\bigotimes_{B_c \in \text{Block}'(\pi_f)_{B_c \neq B_c}} \left\{ \hat{T}^{x}_{B_{c}}x^{\min}(B_c) \right\} \bigotimes_{\max(\hat{T}^{x}_{B_{c}})} \left\{ \hat{T}^{x}_{x_{k+1}}(\hat{T}^{x}_{B_{c}}x^{\min}(B_c)) \right\}$$

with coefficient $\alpha q^{p+1-(p+r+1)-1}$. Thus we obtain $\tilde{\pi}_f \in \mathcal{P}_{E; e_c}(k + 1)$ by adding $k + 1$ to $\pi_f$ and create the block marked by $i$ with the last arc colored by $-1$, i.e. $\hat{T}^{x}_{B_{c},l}x^{\min}(B_{c},l)$ – see Figure 11. Similarly to the Case 3a $\max(\hat{T}^{x}_{B_{c},l}) = k + 1$ holds, so we can use a change in statistic from Case 2b to get that the exponent of $\alpha$ increases by 1, the exponent of $q$ increases by $2p + r$ and we have

$$\hat{T}^{x}_{B_{c},l}x^{\min}(B_{c},l) = \hat{T}^{x}_{x_{k+1}}(\hat{T}^{x}_{B_{c}}x^{\min}(B_c)).$$

Case 4. We have Block$(\pi_f) \cup $ Sing$(\pi) = \emptyset$, then $\hat{R}_{\pi_f} = \Omega$ and the action of $b_{\alpha,q}(x)$ and $p(T_x)$ on the vacuum vector gives zero, which is compatible with the fact that under this action we cannot create an arc from the $\Omega$.

Note that as $\pi_f$ runs over $\mathcal{P}_{E; e(1),...e(k)}(k)$, every set partition

$$\tilde{\pi}_f \in \mathcal{P}_{E; e(1),...e(k),e(k+1)}(k + 1)$$

appears exactly once in one of Cases 1, 2, 3 or 4, which shows by induction that the formula (3.5) holds for all $n \in \mathbb{N}$. \qed
We now present the proof of the main result.

Proof. First, let us notice that for \( \epsilon \in \{1, *, \prime\} \) we have

\[
\varphi \left( b^{e(n)} \alpha q (x_n) \cdots b^{e(1)} \alpha q (x_1) \right) = \sum_{\pi \in P_{\geq 2}^{\epsilon}(n)} \alpha_{\text{Narc}}(\pi) \q^{\text{rc}(\pi) + 2\text{marc}(\pi)} R_{\pi}^{*}
\]

Indeed, from equation (3.5) we see that the following condition must hold: \( \hat{R}_x \pi = \Omega \). This will happen if and only if

\[
\text{Block} \prime(\pi) \cup \text{Sing}(\pi) = \emptyset,
\]

which implies (3.6). From our definition it follows that

\[
K_{x} \pi = R_{x} \pi \text{ if } \text{Sing}(\pi) = \emptyset,
\]

so by taking the sum over all \( \epsilon \) from equation (3.6), we see that

\[
\varphi \left( (b_{\alpha q}^\lambda (x_n) - \lambda_n I) \cdots (b_{\alpha q}^\lambda (x_1) - \lambda_1 I) \right) = \sum_{\pi \in P_{\geq 2}^{\epsilon}(n)} \alpha_{\text{Narc}}(\pi) q^{\text{rc}(\pi) + 2\text{marc}(\pi)} K_{x} \pi,
\]

We also see that

\[
\varphi \left( (b_{\alpha q}^\lambda (x_n) - \lambda_n I + \lambda_n I) \cdots (b_{\alpha q}^\lambda (x_1) - \lambda_1 I + \lambda_1 I) \right)
\]

by equation (3.6), we get

\[
= \sum_{\nu \subset \langle n \rangle} \left[ \prod_{i \in \nu} \lambda_i \sum_{\pi \in P_{\geq 2}^{\epsilon}(n) \backslash \nu} \alpha_{\text{Narc}}(\pi) \q^{\text{rc}(\pi) + 2\text{marc}(\pi)} K_{x} \pi \right],
\]

which by induction implies (3.2).

\( \square \)

4. Remarks about \((q, t)\)-probability space

We would like to stress that this section is loosely related to the previous part of the article.

4.1. Blitvić model. In this section we will show that very similar and interesting partitions appear in the context of \((q, t)\)-probability spaces introduced by Blitvić [B12, B14]. First, we remind Blitvić construction in the context of right creators. Let \( \mathcal{T}_B(H) \) be the (algebraic) full Fock space over \( H \) given in (2.1), with property \( \hat{x} = x \). For \( q \in (-1, 1) \) and \( |q| < t < 1 \), we define the type \((q, t)\)-symmetrization operator on \( H \otimes^n \) as

\[
\tilde{P}^{(n)}_{q, t} := t^{(n)}_{*, q} p_{0, q/t}, \quad \tilde{P}_{q, t} := \bigoplus_{n=0}^{\infty} \tilde{P}^{(n)}_{q, t},
\]

and a corresponding inner product as

\[
\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n \rangle_{q, t} := \langle x_1 \otimes \cdots \otimes x_m, \tilde{P}^{(n)}_{q, t}(y_1 \otimes \cdots \otimes y_n) \rangle_{0, q}.
\]
Definition 7. For $q \in (-1, 1)$ and $|q| < t < 1$, the algebraic full Fock space $\mathcal{F}_{\text{fin}}(H)$ equipped with the inner product $\left< \cdot , \cdot \right>_{q,t}$ is called the $(q,t)$—*Fock space*. Blitvić introduced the following annihilation operator:

$$a_{q,t}(x)(x_1 \otimes \cdots \otimes x_n) = \sum_{k=1}^{n} t^{k-1} q^{n-k} \langle x, x_k \rangle x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_n, \quad n \geq 1$$

$$a_{q,t}(x)\Omega = 0,$$

and the creation operator:

$$a^*_{q,t}(x)(x_1 \otimes \cdots \otimes x_n) := x_1 \otimes \cdots \otimes x_n \otimes x, \quad n \geq 1$$

$$a^*_{q,t}(x)\Omega = x,$$

i.e. $a^*_{q,t}(x)$ is adjoint to $a_{q,t}(x)$ with respect to the inner product $\left< \cdot , \cdot \right>_{q,t}$. The operators $a^*_{q,t}(x)$ and $a_{q,t}(x)$ are called $(q,t)$-*creation and annihilation operators*. Denote by $\Phi(X)$ the vacuum vector state $\Phi(X) := \Phi_{q,t}(X) = (\Omega, X\Omega)_{q,t}$.

Remark 11. We restrict our parameters to $q \in (-1, 1)$ and $|q| < t < 1$, because then $\tilde{P}_{q,t}$ is strictly positive; see [B12, Lemma 4]. In the article [B12] the allowed range of parameters is $|q| = t < 1$, but then $\tilde{P}_{q,t}$ is positive. From the combinatorial point of view, the main result of this section is that Theorem 4 is also true when $|q| = t < 1$.

$(t,q)$-*gauge operator*. Let $T$ be a bounded and self-adjoint operator on $H$. The corresponding *gauge operator* $\tilde{p}(T)$ is an operator on $\mathcal{F}_{\text{fin}}(H)$, defined by

$$\tilde{p}(T) := t^{n-1} p_0(T) R_{0,q/t}^{(n)} = t^{n-1} p_{0,q/t}(T).$$

An explicit form of the operator $\tilde{p}(T)$ on $H^*^n$ is:

$$\tilde{p}(T)(x_1 \otimes \cdots \otimes x_n) = \sum_{k=1}^{n} t^{k-1} q^{n-k} x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_n \otimes T(x_k).$$

Remark 12. (1). The above definition of a gauge operator is motivated by simply noticing that an annihilator operator is of the form $a_{q,t}(x) = t^{n-1} r(x) R_{0,q/t}^{(n)}$.

(2). It is not difficult to see that when the parameters are restricted to $|q| < t < 1$, the operator $\tilde{p}(T)$ has properties desired in Propositions 2 and 3, i.e. if $T$ is self-adjoint, then $\tilde{p}(T)$ is self-adjoint, and if $T$ bounded on $H$, then $\tilde{p}(T)$ is bounded on the $(q,t)$-Fock space. Indeed, for $f, g \in H^*^n$ and by Proposition 2 and 3 we have

$$\left< \tilde{p}(T)f, g \right>_{q,t} = t^{n-1} t\left(\frac{q}{t}\right) \langle p_{0,q/t}(T)f, g \rangle_{0,q/t} = t^{n-1} t\left(\frac{q}{t}\right) \langle f, p_{0,q/t}(T^*)g \rangle_{0,q/t} = \langle f, \tilde{p}(T^*)g \rangle_{q,t}$$

and

$$\left< \tilde{p}(T)f, \tilde{p}(T)f \right>_{q,t} = t^{2n-2} t\left(\frac{q}{t}\right) \langle p_{0,q/t}(T)f, p_{0,q/t}(T)f \rangle_{0,q/t} \leq \left( \max\{1, t/(t-q)\} \right)^2 \|T\|^2 \|f\|^2_{0,q/t}.$$

Here we also see that under the assumption $|q| = t < 1$ we cannot get the above-mentioned property. This property might also be true but then we need a different argument in order to prove it.

Now define the following operators

$$Y_{q,t}(x) := a_{q,t}(x) + a^*_{q,t}(x) + \tilde{p}(T_x), \quad x \in H_R,$$

where $T_x$ is a bounded self-adjoint operator on $H$, indexed by $x \in H_R$. The main theorem of this section is a nice Wick formula, which expresses the joint distribution in the collection of their joint cumulants.

Theorem 4. Suppose that $(x_1, \ldots, x_n) \in H^n_R$, then

$$\Phi\left(Y_{q,t}(x_1) \cdots Y_{q,t}(x_n)\right) = \sum_{\pi \in \mathcal{P}_{\geq 2}(n)} q^{\text{area}(\pi)} t^{\text{arc}(\pi)} \prod_{B \in \pi} \langle X_{\max(B)}, \prod_{i \in B, i \neq \min(B)} \max(B) T_{x_i} X_{\min(B)} \rangle.$$
Remark 13. (1). The proof is similar in spirit to the proof of Theorem 3. This is not entirely obvious but we leave the formal proof to the reader (we just sketch a heuristic proof), because it can be obtained by modifications of Theorem 3.

(2). Similarly as in Corollary 1 we can state that for $t \to 1$, we obtain the $q$-deformed formula for moments of $q$-random variable from the article [A01, A04b], and for $T = \theta$ we get the formula for moments of $(q, t)$-Gaussian operator from [B12].

(3). The number of restricted nestings is defined as $\text{rarc}(\pi) := |\{(V, W) \in \text{nest}(\pi)\}|$, i.e. it is a number of covered arcs. For a partition in Figure 12, we see that $\text{rc}(\pi) = 7$ and $\text{rarc}(\pi) = 5$. Partitions with restricted crossings and nestings appear in this context in many combinatorial articles; see [KZ06, ChDRS07, RS10].

The following is the sketch of a proof: Let us first observe that it is sufficient to focus on Cases 1, 2a and 3a of Theorem 3. Here, we just emphasize how to modify these cases in order to obtain Theorem 4. We keep the notation from this proof, where we assume that all arcs have color 1, and the corresponding analog of the statistic $\text{MaxL}$ is redefined as

$$\text{MLLeft}(\pi) := |\{(V, W) \in [\text{Block}'(\pi) \cup \text{Sing}(\pi)] \times \text{Arc}(\pi) : \max(V) < i \text{ for } i \in W\}|.$$

In the induction step we assume that

$$a^{(k)}_{q, t}(x_k) \cdots a^{(1)}_{q, t}(x_1) \Omega = \sum_{\pi \in \mathcal{P}_{E, x}(k)} q^{\text{rc}(\pi) + \text{MaxC}(\pi)} t^{\text{rarc}(\pi) + \text{MLLeft}(\pi)} R_{\pi} R_{\pi}^*,$$

and analogical map $\pi \to \tilde{\pi}$ from the proof of Theorem 3 is obtained in the following way:

(a) in Case 1 the operator $a^{(k)}_{q, t}(x_k)$ contributes to adding the singleton $\{k + 1\}$ to $\text{Sing}(\pi)$;
(b) in Case 2a the annihilator $a^{(k)}_{q, t}(x_k)$ contributes to a new block in $\text{Block}(\tilde{\pi})$, with the last arc $\{i, k + 1\}$ and $\text{rarc}(\tilde{\pi}) = \text{rc}(\pi) + u$, $\text{MaxC}(\tilde{\pi}) = \text{MaxC}(\pi) - u + r$, $\text{rarc}(\tilde{\pi}) = \text{rarc}(\pi) + s + t$ and $\text{MLLeft}(\tilde{\pi}) = \text{MLLeft}(\pi) + p - (t + s)$. So the exponent of $q$ increases by $r$ and the exponent of $t$ increases by $p$;
(c) in Case 3a the gauge operator $\tilde{p}(T_{x_k})$ contributes to a new block in $\text{Block}'(\tilde{\pi})$ and the change in the statistic is the same as in case (b).

Finally, when we put $\text{Block}'(\pi) \cup \text{Sing}(\pi) = \emptyset$, we obtain the formula (4.3).

\[\square\]

The orthogonal polynomial.

**Definition 8.** $(q, t)$-Poisson polynomials are defined by the recursion relations:

$$y^{\tilde{P}^n_{n+1}(q, t)}(y) = \tilde{P}^n_{n+1}(q, t)(y) + [\mathcal{N}]_{q, t}^{\tilde{P}^n_{n-1}(q, t)}(y) + [\mathcal{N}]_{q, t}^{\tilde{P}^n_{n}(q, t)}(y), \quad n \geq 1$$

where $[\mathcal{N}]_{q, t} = \sum_{i=1}^{n} q^{-i} t^{n-i}$, and initial conditions $\tilde{P}^n_{-1}(q, t) = 0, \tilde{P}^n_{0}(q, t) = 1$ and $\tilde{P}^n_{1}(q, t) = y$.

Let $\tilde{\mu}_{q, t}$ be a probability measure, which associates the orthogonal polynomials $\tilde{P}^n_{q, t}$.

**Proposition 5.** Suppose that $x \in H, \|x\| = 1$ and $T = \text{Id}$. Then the probability distribution of $Y_{q, t}$ with respect to the vacuum state is given by $\tilde{\mu}_{q, t}$. 

\[\square\]
Proof. Note that for \( n = 1 \) \( \tilde{P}_1^{(q,t)}(Y_{q,t}(x))\Omega = Y(x)\Omega = x \) and by induction
\[
\tilde{P}_{n+1}^{(q,t)}(Y_{q,t}(x))\Omega = Y_{q,t}(x)\tilde{P}_n^{(q,t)}(Y_{q,t}(x))\Omega - [n]_{q,t}\tilde{P}_n^{(q,t)}(Y_{q,t}(x))\Omega = Y_{q,t}(x)\Omega - [n]_{q,t}x^{\otimes n} - [n]_{q,t}x^{\otimes (n-1)} = x^{\otimes n+1}.
\]
The rest of argument is similar to that of the Proposition 4. \( \square \)

Corollary 2. By [F80] Propositions 7A and 7B or [KZ06] Proposition 4.1] we conclude that the moment generating function of the measure \( \tilde{\mu}_{q,t} \) has the following elegant continued fraction expansion:
\[
\sum_{n \geq 0} m_n(\tilde{\mu}_{q,t})z^n = \frac{1}{1 - z - \frac{[2]_{q,t}z^2}{1 - [2]_{q,t}z - \frac{[3]_{q,t}z^2}{1 - [3]_{q,t}z - \frac{[4]_{q,t}z^2}{\ddots}}}}.
\] (4.4)

4.2. t-deformed free probability. When \( q = 0 \) and \( t \in (0,1) \), the case is reduced to a new t-deformed free probability (the term t-free probability also appeared in [BW01], but in a completely different context of t-transformation of measures). Blitvić [B12] showed that the statistics of the t-deformed semicircular element can be described in an elegant form drawing from the deformed Catalan numbers, the generalized Rogers-Ramanujan continued fraction, and the t-Airy function of Ismail [I03]. In light of its present interpretation, a t-deformed free probability resembles the combinatorial theory of free probability of Voiculescu [V85]. In a t-deformed free probability, when we illustrate noncrossing partitions graphically, we connect all consecutive points in a block by a semicircle, and count how many arcs are covered by other arcs; see Figure 13. In particular, we obtain Voiculescu probability when we pass with \( t \) to 1. This situation was described for pair partitions in [B12], where it was shown that moments of t-semicircular element can be represented by noncrossing nesting pair partitions.

![Figure 13: Noncrossing cumulants and partitions for \( \tilde{\phi}_{0,t}(Y_{0,t}^3(x)) = t^2 + 2t + 3 \), with \( T = 1d \) and \( \|x\| = 1 \).](image)

**t-free Poisson distribution.** Al-Salam and Ismail in [AI83] Equation 5.6] introduced the three-term recurrence
\[
yU_n(y, a, b) = U_{n+1}(y, a, b) - at^nU_n(y, a, b) + bt^{n-1}U_{n-1}(y, a, b), \quad n \geq 1
\] (4.5)
with \( U_0(y) = 1 \) and \( U_1(y) = cy \). They also showed [AI83] Theorem 5.1], that when \( c > 0, b > 0, 1 + at > 0 \) and \( t \in (0, 1) \), then there exists a unique purely discrete positive measure with bounded support orthogonalizing polynomial (4.5).

**Definition 9.** For \( t \in (0, 1) \), we call the measure \( \tilde{\mu}_{0,t} \) the t-free Poisson or t-Marchenko-Pastur distribution, i.e. they are determined by the following three-term recurrence:
\[
y\tilde{P}_n^{(0,1)}(y) = \tilde{P}_n^{(0,1)}(y) + t^{n-1}\tilde{P}_n^{(0,1)}(y) + t^{n-1}\tilde{P}_{n-1}^{(0,1)}(y), \quad n \geq 1
\] (4.6)
with initial conditions \( \tilde{P}_0^{(0,t)}(y) = 1, \tilde{P}_1^{(0,t)}(y) = y. \)

**Proposition 6.** The measure \( \tilde{\mu}_{0,t} \) is a purely discrete positive measure with bounded support.

**Proof.** Recall that random variables \( Y_{0,t}(x) \), with \( \|x\| = 1 \) have the distribution \( \tilde{\mu}_{0,t} \), i.e. they orthogonalize (4.6). If we put \( a = -1, b = t^2 \) and \( c = 1 \) in the recurrence (4.5) then \( 1 + at = 1 - t > 0 \) for \( t \in (0, 1) \) and

\[
(4.7) \quad yU_n(y, -1, t^2) = U_{n+1}(y, -1, t^2) + t^nU_n(y, -1, t^2) + t^{n+1}U_{n-1}(y, -1, t^2), \quad n \geq 1.
\]

Now, let us substitute \( L_n(y) = U_n(y, -1, t^2)/t^n \), multiply (4.7) by \( t^{-n-1} \) and replace \( y \) by \( ty \), then we get the recursion

\[
yL_n(y) = L_{n+1}(y) + t^{-n}L_n(y) + t^{-n-1}L_{n-1}(y), \quad n \geq 1
\]

with \( L_0(y) = 1 \) and \( L_1(y) = y \), so we see that \( L_n(y) = \tilde{P}_0^{(0,t)}(y) \). This observation means that the recurrence (4.7) corresponds to monic orthogonal polynomials which orthogonalize the distribution of \( tY_{0,t}(x) \). So we deduce from [AI83, Theorem 5.1] that the law of \( tY_{0,t}(x) \) is a purely discrete positive measure with bounded support, hence the distribution of \( Y_{0,t}(x) = \tilde{\mu}_{0,t} \) also has these properties. \( \square \)

**Remark 14.** (1). In the contexts of combinatorics and of number theory, we can conclude that the \( t \)-free Poisson distribution is an object of significant interest. It turns out that certain familiar objects, for example the generating function of this distribution, can be represented by the continued fraction expansion (4.4) with \( q = 0 \), which is called the generalized Rogers-Ramanujan continued fraction; see [AI83].

(2). In this paper we have shown that the \( t \)-free Poisson is a discrete probability measure. A similar result was obtained by Blitvić [B12] in the case of the \( t \)-semicircular distribution. This observation suggests, that \( t \)-free probability is related to paper [CHS15].

Acknowledgments

The author would like to thank Marek Bożejko and Franz Lehner for suggesting topics, several discussions and helpful comments. The author also thanks Mourad E. H. Ismail, who suggested the proof of Proposition 6. The work was supported by the Austrian Science Fund (FWF) Project No P 25510-N26 and the Narodowe Centrum Nauki grant 2014/15/B/ST1/00064.

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