Spontaneous stripe order at certain even-denominator fractions in the lowest Landau level

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An understanding of the physics of half or quarter filled lowest Landau level has been achieved in terms of a Fermi sea of composite fermions, but the nature of the state at other even-denominator fractions has remained unclear. We investigate in this work Landau level fillings of the form \( \nu = (2n + 1)/(4n + 4) \), which correspond to composite fermion fillings \( \nu^* = n + 1/2 \). By considering various plausible candidate states, we rule out the composite-fermion Fermi sea and paired composite-fermion state at these filling factors, and predict that the system phase separates into stripes of \( n \) and \( n + 1 \) filled Landau levels of composite fermions.

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The Coulomb interaction between electrons in two dimensions confined to the lowest Landau level expresses itself most strongly through the binding of an even number of quantum mechanical vortices on each electron and thereby creating particles known as composite fermions [1,2,3,9–11]. In yet higher Landau levels, electrons do not capture vortices but instead exhibit a stripe phase [12,13].

Several interesting states have been discovered at electron filling factors \( \nu = n + 1/2 \), which will serve as the paradigms for the discussion below. In the lowest Landau level \( (\nu = 1/2) \) electrons transform into composite fermions which condense into a \(^{2}\text{CF}\) Fermi sea [4,8]. In the second LL \( (\nu = 2 + 1/2) \) electrons turn into composite fermions, which are believed to form Cooper pairs [3,4,11]. In yet higher Landau levels, electrons do not capture vortices but instead exhibit a stripe phase [12,13].

Which state actually occurs depends on the interaction matrix elements, and it is therefore important to have a good model for the inter-CF interaction, \( V^{CF}(r) \). We proceed as follows. In order to treat the state at \( \nu^* = n + 1/2 \), we start with the state with \( \nu^* = n \) filled Landau levels of composite fermions and add two additional composite fermions in the lowest empty CF Landau level. Following the standard procedure for writing the wave functions for composite fermions [3], the wave function for this state is given by \( P_{\text{LLL}} \Phi_1^2 \Phi_n^{m+1} \), where \( \Phi_1 \) is the wave function of one filled LL, \( \Phi_n^{m+1} \) is the wave function of the electron state in which \( n \) LLs are fully occupied and the \( (n+1) \)st LL contains two electrons in a relative angular momentum \( m \) state, and \( P_{\text{LLL}} \) is the lowest LL projection operator. The explicit form for the general wave functions of this kind is given in the literature [13], the calculation of energy requires evaluation of multi-dimensional integrals which is accomplished by the Monte Carlo method. (The spherical geometry [13] is used in our calculations.) This provides the pseudopotentials \( V^{CF}_m \), which completely specify the interaction between two composite fermions in the \((n+1)\)st CF-LL. Similar studies have been done previously [16], except

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that here $V_{m}^{CF}$ are evaluated for fairly large systems, believed to give a good approximation for the thermodynamic limit. We then construct a real space interaction between composite fermions; for convenience, we map the problem of composite fermions in any arbitrary CF-LL into a problem of fermions in the lowest LL. There is no unique prescription for this, because many real space interactions produce the same pseudopotentials, but we find it convenient to use a potential of the form [17]:

$$ V^{CF}(r) = \sum_{j} c_{j} r^{2j} e^{-r^{2}} + \left( \frac{2n+1}{r} \right)^{-\frac{5/2}{m}} \left[ \frac{r^{2}}{\epsilon_{l}} \right] $$

(1)

The last term gives the Coulomb interaction between two particles of appropriate fractional charge. The distance $r$ is measured in units of the effective magnetic length $\ell$, but the energies are measured in units of $e^{2}/\epsilon_{l}$ where $\epsilon_{l}$ is the magnetic length at the actual electron filling factor $\nu$ and $\epsilon$ is the dielectric constant of the background material. We fix the first few parameters $c_{j}$ by requiring that $V^{CF}(r)$ produce the first 5 to 6 odd pseudopotentials exactly. A comparison between $V_{m}^{CF}$ and the pseudopotentials of $V^{CF}(r)$ (Fig. 1) shows that $V^{CF}(r)$ is a good approximation for all distances. We have thus mapped the problem of $N$ composite fermions in the $(n+1)$st LL into that of $N$ fermions at an effective filling in the lowest LL interacting with an effective potential. Only the composite fermions in the topmost half filled CF-LLs will be considered explicitly; the completely occupied CF-LLs appear only through their role in determining the inter-CF interaction. We note that the interaction between composite fermions is remarkably different from that between electrons in the corresponding higher LLs. In the second CF-LL, it is most strongly repulsive in the $m = 3$ channel, and in higher CF-LLs, the interaction is actually attractive, with the lowest energy obtained in the $m = 1$ channel.

Our conclusions below will be subject to two assumptions. (i) We assume that mixing with higher CF-LLs can be neglected, i.e., the inter-CF interaction is weak compared to the effective CF-cyclotron energy. There is evidence that this is an excellent approximation: the states containing several composite fermions are accurately described without considering mixing between CF-LLs [13]. (ii) We further assume that the interaction energy of many composite fermions in the $(n+1)$st CF-LL is well approximated by a sum of two-body terms. An examination of configurations containing three composite fermions in the second CF-LL indicates that this is a good approximation [13].

The first state that we consider is the Fermi sea, in which the $^{2}$CFs capture two additional vortices to become $^{4}$CFs, which then form a Fermi sea. (The composite fermions in the lower, fully occupied CF-LLs remain $^{2}$CFs; this state thus contains an admixture of two different flavors of composite fermions.) The wave function of the Fermi sea is

$$ \Psi_{FS} = P_{LLL} \Phi_{1}^{2} \Phi_{\infty} $$

(2)

The thermodynamic limit for the energy of the $^{4}$CF sea is obtained by an extrapolation of finite system results, as shown in Fig. (2). Since we are interested in comparing energies obtained by different methods, it is important to carefully define the total energy; we will consistently take the same form for the electron-electron, electron-background, and background-background interactions in all our calculations. All energies are quoted relative to the energy of an uncorrelated uniform state, explained below.

![Fig. 1. The pseudopotentials for the inter-composite fermion interaction in the second, third, and fourth CF Landau levels, calculated from the microscopic wave functions (circles). The error bars indicate the statistical error from the Monte Carlo sampling. The crosses are the pseudopotentials for the model interaction $V^{CF}(r)$ explained in the text.](image)

We do not expect the ground state to be the $^{4}$CF sea because the stability of the Fermi sea requires a strong short range repulsion, which is not the case with composite fermions in higher CF-LLs [20]. In fact, in the third and higher CF-LLs, the interaction between the composite fermions is attractive, which might suggest pairing of composite fermions. The paired state of composite fermions is represented by the Pfaffian wave function [11]

$$ \Psi_{Pf} = \Phi_{1}^{2} Pf[M] $$

(3)
where $\text{Pf}[M]$ is the Pfaffian of the $N \times N$ antisymmetric matrix $M$ with components $M_{ij} = (u_i v_j - v_i u_j)^{-1}$, where $u_j \equiv \cos(\theta_j/2) \exp(-i\phi_j/2)$ and $v_j \equiv \sin(\theta_j/2) \exp(i\phi_j/2)$. $\text{Pf}[M]$ is a real space BCS wave function, so $\Psi_{PF}$ describes a paired state of composite fermions. Again, since our base particle is a $^2$CF, $\Psi_{PF}$ contains pairing of $^4$CFs. The energy of this state, given in Fig. (3), beats the Fermi sea at $\nu = 3/8$ and $7/16$, raising the intriguing possibility of a FQHE, induced by pairing, at certain even denominator fractions in the lowest Landau level.

However, it is important to study the stability of any candidate FQHE state to quantum fluctuations. We consider the density-wave excitation of the Pfaffian wave function in the single mode approximation, described by the wave function $\mathcal{P}_{LLL, \rho k} \Psi_{PF}$, where $\rho_k$ is the density operator at wave vector $k$. We calculate the energy of this excitation following Ref. [21], with the help of the pair correlation function of $\Psi_{PF}$ [22]. The excitation energy is shown in Fig. (3) as a function of the wave vector, and indicates that the paired state is unstable.

These results rule out the $^4$CF Fermi sea as well as $^4$CF pairing at the filling factors considered here. We have also carried out exact diagonalization at the flux values corresponding to the Fermi sea and the Pfaffian wave function in the spherical geometry, and found that the ground state does not have orbital angular momentum $L = 0$, i.e., is not a uniform density state. The fact that the instability occurs at non-zero wave vectors in Fig. (3) also hints that the true ground state may not be a translationally invariant liquid.

Besides pairing, another possible consequence of an attractive interaction is phase separation. Due to the long range Coulomb interaction, the phase separation is likely to manifest itself here through the formation of stripes. We calculate the Hartree-Fock energy of the stripe state of $^2$CFs following the method of Koulakov et al. [12]. The interaction Hamiltonian is given by

$$\hat{V} = \frac{(2\pi)^3}{2L_x L_y} \sum_{\mathbf{q}} \hat{V}^{CF}(\mathbf{q}) \rho(-\mathbf{q}) \rho(\mathbf{q})$$

where $\rho(\mathbf{q})$ is the density operator and $\hat{V}^{CF}(\mathbf{q})$ is the Fourier transform of $V^{CF}(r)$. Subsequent to a Hartree-Fock decomposition, the expectation value of the interaction energy can be written as

$$<\hat{V}> = \frac{(2\pi)^3}{2L_x L_y} \sum_{\mathbf{q}} \hat{U}_{HF}(\mathbf{q}) \Delta(-\mathbf{q}) \Delta(\mathbf{q})$$

where $\Delta(\mathbf{q}) \equiv \frac{1}{2\pi} \sum_k e^{-ik\mathbf{q}\cdot\mathbf{r}} <a_{k+}^\dagger a_{k-}>$, $\hat{U}_{HF}(\mathbf{q}) = \hat{U}(\mathbf{q}) - e^2 U(q^2)$. The $\mathbf{q} = 0$ term, which corresponds to the uniform (uncorrelated) state, is treated separately: the direct part is canceled by the background, and the exchange part is taken as the reference energy, given by

$$E_0 = -N\tau \frac{U(0)}{2}$$

where $\tau = 1/2$. The contribution coming from nonzero values of $\mathbf{q}$ is called the coherence energy, $E_{coh}$.

The stripe phase with period $\Lambda$ corresponds to the choice:

![FIG. 2. The energy per particle for the CF Fermi sea (squares) and the CF paired state (circles) as a function of $N$, the number of composite fermions in the $(n+1)$st CF-Landau level. The thermodynamic energy is also shown for the CF stripe and bubble phases (dash and cross, respectively). All energies are measured relative to the uncorrelated uniform density state, explained in the text.](image-url)

![FIG. 3. The excitation energy of the single mode approximation (SMA) excitation for the Pfaffian wave function.](image-url)
\[ \Delta(x, y) = \frac{1}{2\pi l^2} \sum_{q} \frac{2 \sin(\frac{\Delta q}{\Lambda})}{\Lambda q} e^{i q x} \]  
\[ E_{\text{coh}} = \frac{1}{2\pi l^2} \sum_{q} \tilde{U}_{HF}(q) \left( \frac{2 \sin(\frac{\Delta q}{\Lambda})}{\Lambda q} \right)^2 \]  

where \( q = \frac{2 \pi}{\Lambda} \), with \( j = \pm 1, \pm 2, \ldots \). This gives

\[ E_{\text{coh}} = \frac{1}{2\pi l^2} \sum_{q} \tilde{U}_{HF}(q) \left( \frac{2 \sin(\frac{\Delta q}{2\pi})}{\lambda q} \right)^2 \]

We compute it as a function of \( \Lambda \). The lowest energy, shown in Fig. (3), is obtained at \( \Lambda / l_0 = 10 \), 28, and 34 for \( \nu = 3/8 \), 5/12, and 7/16. The period is rather large compared to that for the electron stripes in higher LLs (for which \( \Lambda / l_0 \) is of order unity), which is not surprising because (i) the period of the stripes is controlled by the effective magnetic length, and (ii) the difference between the densities of the FQHE states on either side is quite small. The stripe phase has lower energy than both the Fermi-sea and paired states.

For completeness, we have also considered the Wigner crystal of “bubbles” [13], with each bubble containing in general several electrons. For a honeycomb lattice with lattice constant \( \Lambda_b \), the radius of a bubble is \( R = \frac{\sqrt{3} \pi \Lambda_b}{2} \) and

\[ \Delta(r) = \frac{2}{\sqrt{3} \Lambda_b} \sum_{q} \frac{R}{\lambda q} J_1(qR) e^{i q r} \]

where \( q = m \mathbf{e}_1 + m_2 \mathbf{e}_2 \) with \( \mathbf{e}_1 = \frac{4}{3 \Lambda_b} \hat{y}, \mathbf{e}_2 = \frac{2}{\sqrt{3} \Lambda_b} \hat{x} - \frac{2 \pi}{\sqrt{3} \Lambda_b} \hat{y} \), and \( m_1 \) and \( m_2 \) being integers. The coherence energy for the bubble phase is

\[ E_{\text{coh}} = \frac{4 \pi}{\sqrt{3} \Lambda_b} \sum_{q} \tilde{U}_{HF}(q) \left( \frac{R}{\lambda q} J_1(qR) \right)^2 \]

The lowest energy, shown in Fig. (3), is determined by considering bubbles with various occupancies, and has higher energy than the stripe phase at all of the filling factors considered here.

In the above, the filling factor \( \nu^* = n + 1/2 \) is viewed as half filling of CF particles on top of \( n \) filled CF-LLs. We have also considered [13] the complementary approach in which it is modeled as half filling of CF holes on the background of \( n + 1 \) filled CF-LLs. The stripe phase again has the lowest energy, which gives us further confidence in the robustness of our result.

The electron stripes in higher electronic Landau levels have revealed themselves through an anisotropic transport at temperatures below \( \sim 50 \) mK [13]. The conditions for the observation of CF stripes are more stringent. Estimates of the critical temperature from the Hartree Fock theory are not quantitatively reliable, but noting that the effective interaction between composite fermions at \( \nu^* = n + 1/2 \) is roughly an order of magnitude smaller than for electrons at \( \nu = n + 1/2 \), as measured by the pseudopotentials, we expect the critical temperature to also be similarly reduced. Also, the much larger period suggests the need for a high degree of density homogeneity.

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