The equations of CCC*

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Abstract

I review the equations of Conformal Cyclic Cosmology given by Penrose [10]. I suggest a slight modification to Penrose’s prescription and show how this works out for FRW cosmologies and for Class A Bianchi cosmologies.

1 Introduction

Penrose’s Conformal Cyclic Cosmology ([10], hereafter CCC) is a radical new cosmological model which assumes that the universe consists of a sequence of aeons, each a solution of the Einstein equations with sources and positive cosmological constant, each expanding from an initial Big Bang singularity to a space-like future null infinity $I^+$, the $I^+$ of one aeon being identified with the Big Bang of the next. There is therefore a conformal metric common to consecutive aeons, which is assumed to be smooth, and three-surfaces $\Sigma$ which are both $I^+$ of one aeon and what we’ll call the bang surface of the next. The surfaces $\Sigma$ are necessarily umbilic in a representative of the conformal metric.

Other assumptions are made and we shall spell these out below by concentrating on two consecutive aeons. Our concern in this talk is to set down the equations of CCC and by consideration of examples seek some understanding of them.

Thus we concentrate on two consecutive aeons, which we’ll call past and present, so we have three manifolds-with-metric:

- the previous aeon $(\hat{M}, \hat{g}_{ab})$;
- the present aeon $(\check{M}, \check{g}_{ab})$;
- the conformal extension $(\tilde{M}, g_{ab})$ of both, so that

$$\hat{g}_{ab} = \hat{\Omega}^2 g_{ab}, \quad \check{g}_{ab} = \check{\Omega}^2 g_{ab}, \quad g_{ab} = \tilde{\Omega}^2 g_{ab}. \quad (1)$$

We shall call $g_{ab}$ the bridging metric and suppose that $M = \hat{M} \cup \check{M} \cup \Sigma$ where $\Sigma$ is the common boundary:

$$\Sigma = \{\hat{\Omega}^{-1} = 0\} = \{\check{\Omega} = 0\}.$$  

Now necessarily $\Sigma$ is a singular surface for $\check{M}$ and we shall make enough assumptions below for $\Sigma$ to be $I^+$ for $\tilde{M}$. It also follows that the three Weyl tensors are equal:

$$\hat{\mathcal{C}}_{abc}^d = \check{\mathcal{C}}_{abc}^d = \mathcal{C}_{abc}^d,$$

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and then the Bianchi identity forces all three to vanish \( \Sigma \). This was one of the prime motivations for CCC, the desire to have Big Bang cosmological models with zero Weyl tensor at the bang.

Because of the conformal freedom, \( g \rightarrow \Theta^2 g \), in the choice of the bridging metric we can impose Penrose’s *reciprocal hypothesis*:

\[
\hat{\Omega} \hat{\Omega} = -1, \tag{2}
\]

and then necessarily, by (1),

\[
\hat{g}_{ab} = \hat{\Omega}^{-4} \hat{\gamma}_{ab} \tag{3}
\]

so that the metric of the present aeon is determined by the metric of the previous aeon, provided we can give an algorithm to determine a unique \( \hat{\Omega} \). This will be our main concern.

Penrose [10] assumes that the matter content in the previous aeon, at least near to \( I^+ \), is a radiation fluid together with a positive cosmological constant \( \Lambda \). There will also be Maxwell fields: these are assumed not to be dynamically important but they will propagate through \( I^+ \) into the present aeon. Now the matter content in \( M \) near \( \Sigma \), as determined through Einstein’s equations by the Einstein tensor \( \hat{\gamma}_{ab} \) of \( \hat{g}_{ab} \), will be fixed by \( \hat{\Omega} \) and \( \hat{\gamma} \). One no longer has freedom in choosing this matter content but it had better be roughly the same as in the previous aeon with, Penrose suggests, an extra field to represent dark matter.

In this scenario the previous aeon is straightforwardly defined as a solution of the Einstein equations with \( \Lambda > 0 \), and the Cauchy problem for this with data at \( I^+ \) is well-understood. Following Friedrich [6, 7] one has:

- for the vacuum Einstein equations with \( \Lambda > 0 \), [6], choose a Riemannian 3-metric \( a_{ij} \) and a symmetric tensor \( c_{ij} \) on \( \Sigma = I^+ \);
- if \( c_{ij} \) is trace-free and divergence-free w.r.t. the Levi-Civita connection of \( a_{ij} \), then there is a solution of the Einstein equations with this data;
- the data has gauge-freedom

\[
a_{ij} \rightarrow \hat{a}_{ij} = \theta^2 a_{ij}, \quad c_{ij} \rightarrow \hat{c}_{ij} = \theta^{-1} c_{ij}, \tag{4}
\]

in that data related like this give the same solution.

- The corresponding problem for any matter source described by a trace-free \( T_{ab} \) and \( \Lambda > 0 \) is formulated in [7], and worked out in detail for Yang-Mills sources;
- with a slightly different formalism, the corresponding result for pure radiation plus \( \Lambda \) is proved in [9].

Starting at the beginning of an aeon rather than at the end, there are some choices of matter model for which one can prove well-posed-ness of an initial value problem for the Einstein equations with data at a conformally-compactifiable initial singularity [3 4 2 14], and solutions obtained this way will have a big bang at which the Weyl tensor is finite. However, one cannot expect that solutions to the Cauchy problem with data at \( I^+ \), as described above, will join on to these solutions with finite-Weyl tensor big bang. One way to see this, for example for radiation-fluid, is to observe that there is much less freely-specifiable data at the bang (just the 3-metric of \( \Sigma \)) than at \( I^+ \) (both \( a_{ij} \) and \( c_{ij} \)). Furthermore, in this case, zero Weyl tensor at the bang forces the metric to be FRW [3]. Thus for CCC to work, the matter content after the bang cannot in general be as simple as a perfect fluid.

\[1\] where, for example, they might contribute to intergalactic magnetic fields.
Rather than try to guess what it should be we adopt a different strategy, still following Penrose \cite{10}: choose the desired matter model in $\hat{\mathcal{M}}$; devise a unique prescription for $\hat{\Omega}$; use the conformal rescaling to define the Einstein tensor $\hat{\mathcal{G}}_{ab}$ in $\hat{\mathcal{M}}$; and seek to interpret it.

There is an element of risk in this strategy in that the Einstein tensor after the bang may not be interpretable in terms of reasonable matter at all.

2 Starobinsky expansions

Starobinsky in a study of the asymptotics of metrics with positive $\Lambda$, \cite{13}, considered metrics written in the form

$$g = dt^2 - e^{2Ht}(a_{ij} + e^{-2Ht}b_{ij} + e^{-3Ht}c_{ij} + \ldots)dx^idx^j,$$

where $\Lambda = 3H^2$ and the spatial metrics $a_{ij}, b_{ij}, \ldots$ are time independent. This kind of expansion resembles the ambient metric construction of Riemannian geometers \cite{5}. To draw out the similarity to the present case, one can regard $e^{-Ht}$ as a defining function for $\mathcal{I}^+$ and (5) as an expansion in the defining function.

Rendall \cite{12} added a measure of rigour to the results of \cite{13} and showed in particular that Friedrich’s solutions had expansions like this. It is worth noting the existence of gauge freedom in an expansion like (5): the coordinate system is constructed by choosing a (late time) space-like surface to be $t = t_0 = \text{constant}$; then $t$ is proper time along the orthogonal congruence to this surface and the space-coordinates $x^i$ are co-moving. Thus one can make a new choice of the initial surface to produce a change

$$t \to \tilde{t} = t - \phi(x^i)$$

which must be accompanied by a redefinition of the co-moving space coordinates. This transformation entails

$$a_{ij} \to \tilde{a}_{ij} = e^{2H\phi}a_{ij}, \quad c_{ij} \to \tilde{c}_{ij} = e^{-H\phi}c_{ij},$$

which is (1) with $\theta = e^{H\phi}$.

Our strategy will be to take $\hat{g}_{ab}$ in the Starobinski form and seek a unique prescription for $\hat{\Omega}$.

3 The example of FRW

The FRW metric sits very well with CCC as we shall see. Then guided by that calculation, we’ll extend the method to a wider class of cosmologies.

Consider the FRW metric

$$g = dt^2 - R(t)^2d\sigma_k^2$$

where $d\sigma_k^2$ with $k = -1, 0, 1$ is the usual constant curvature metric. If we assume a radiation fluid source, then the conservation equation can be integrated to give the density as $\rho = mR^{-4}$ for a constant of integration $m$. The Einstein equations, with cosmological constant $\Lambda$ now reduce to the Friedmann equation, which is

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{m}{3} - kR^2 + \frac{\Lambda}{3}R^4,$$

where we have written this in terms of conformal time defined by integrating $d\tau = dt/R$.

We’ll suppose that the metric in the past aeon took this form, and therefore put hats on everything: $\hat{g}, \hat{R}, \hat{t}, \hat{\rho}, \hat{\Lambda}, \ldots$
Now an obvious candidate for $\hat{\Omega}$ is a multiple of the scale factor: $\hat{\Omega} = c_1 \hat{R}$ for some constant $c_1$ to be fixed. With this choice

$$\hat{g} = \hat{\Omega}^{-4} \hat{g} = d\hat{t}^2 - \hat{R}(\hat{t})^2 d\sigma_{\hat{k}}^2,$$

(8)

with

$$\hat{R} = -c_1^{-2} \hat{R}^{-1}, \quad d\tau = d\hat{t} / \hat{R} = d\hat{t} / \hat{R}.$$  

(The minus sign in the first equation here has the same origin as the one in (2): $\hat{R}$ goes through zero at the bang and so is negative in $\hat{M}$.) With this choice, $\hat{g}$ is again in the FRW form, and with the choice $c_1 = (\hat{\Lambda} / \hat{m})^{1/4}$ the Friedmann equation transforms from a hatted to a checked version with

$$\check{\Omega} = \hat{\Omega} - \frac{4}{3} \hat{g} = d\check{t}^2 - \check{R}(\check{t})^2 d\sigma_{\check{k}}^2,$$

so the two aeons are diffeomorphic i.e they are the same solution of the EFEs.

It will be helpful below to note here that, introducing $\phi = \hat{\Omega}^{-1}$, the Friedmann equation implies

$$\hat{g} + 2H^2 \phi = k c_1^2 \phi^3,$$

(9)

from which it follows that the bridging metric has scalar curvature $6k c_1^2$, and the solution for $\phi$ takes the form

$$\phi = \phi_1 e^{-H\check{t}} + O(e^{-3H\check{t}})$$

(10)

with constant $\phi_1$.

The FRW metric is of course very special, being conformally flat. However we may take this case, or more precisely these three cases as $k$ varies, as the paradigm and seek to stay close to them even when there is some Weyl curvature.

### 4 Finding a unique $\hat{\Omega}$

Suppose now we have a more general metric $\hat{g}$ in the previous aeon which we take to be in Starobinski form (5). We may expand $\hat{\Omega}$ in the same way as

$$\phi := \hat{\Omega}^{-1} = e^{-H\hat{t}} \phi_1 + e^{-2H\hat{t}} \phi_2 + e^{-3H\hat{t}} \phi_3 + \ldots,$$

(11)

and then the metric of $\mathcal{I}^+$ will be $\phi_1^2 a_{ij}$.

We can find an equation for $\phi$ by noting, in line with (9), that the scalar curvature $s$ of the bridging metric $g$ satisfies

$$\hat{\square} \phi + 2H^2 \phi = \frac{1}{6} s \phi^3.$$  

(12)

Here we’ve used the fact that the scalar curvature of the previous aeon is $4\hat{\Lambda} = 12H^2$. With $s = 12H^2$, (12) is Penrose’s *phantom field equation* [10], but we can leave $s$ as a constant to be chosen later and proceed to solve this equation.

Substituting (11) into (12) and solving term by term we find that $\phi_1$ and $\phi_2$ are freely specifiable, with subsequent $\phi_n$ then determined. The example of FRW, (10), suggests choosing $\phi_2 = 0$ (which is Penrose’s *Delayed Rest Mass Hypothesis* [10]). To specify $\phi_1$, again following the example of FRW, we can seek to give the metric of $\mathcal{I}^+$, which is $\phi_1^2 a_{ij}$, constant scalar curvature. Doing this is essentially solving the Yamabe problem (see e.g. [8] or [11]) and what we can say is that, for a compact $\mathcal{I}^+$, $\phi_1$ should be chosen to minimise the Yamabe functional for $a_{ij}$. This may not work for a noncompact $\mathcal{I}^+$ but at least for a compact $\mathcal{I}^+$ we have a prescription for a unique $\phi$ up to scale and hence a unique $\hat{\Omega}$ up to scale.

It still remains to fix $s$ in (12) and again we’ll be guided by the example of FRW, where $s = 6k c_1^2 = s^\mathcal{I}$ and $s^\mathcal{I}$ is the scalar curvature of the metric of $\mathcal{I}^+$. These choices fix $\Omega$ or $\phi$ up
to a single constant which we may hope to fix by demanding \( \bar{\Lambda} = \hat{\Lambda} \). This remaining constant can be taken to be the scalar curvature of \( I^+ \) but we shall choose it to be

\[
a_1 := -\lim \left( \frac{d\phi_1}{d\tau} \right),
\]

where the limit is taken at \( I^+ \).

## 5 The matter content after the bang

The rescaling formula for the Ricci tensor with \( \bar{g} = \hat{\Omega}^{-4} \hat{g} \), and conventions as in \([11]\), is

\[
\bar{R}_{ab} = \hat{R}_{ab} + 2\nabla_a \Upsilon_b - 2\Upsilon_a \Upsilon_b + \bar{g}_{ab} \hat{g}^{ef}(\nabla_e \Upsilon_f + 2\Upsilon_e \Upsilon_f)
\]

with \( \Upsilon_a = 2\partial_a \log \hat{\Omega} \). The Einstein equations in \( \hat{M} \) are

\[
\bar{R}_{ab} = -\kappa \bar{T}_{ab} + \hat{\Lambda} \bar{g}_{ab},
\]

where

\[
\bar{T}_{ab} = \frac{1}{3} \bar{\rho}(4\bar{u}_a \bar{u}_b - \bar{g}_{ab}).
\]

To preserve the conservation equation we define

\[
\bar{T}_{ab} = \hat{\Omega}^4 \bar{T}_{ab} = \phi^{-4} \bar{T}_{ab}
\]

though this shouldn’t be regarded as all the matter content after the bang. Then solving (14) for \( \bar{G}_{ab} \) we find

\[
\bar{G}_{ab} = -\kappa \phi^4 \bar{T}_{ab} \left( \frac{4}{\phi} \hat{\nabla} a \hat{\nabla}_b \phi + \frac{4}{\phi^2} \phi_a \phi_b + \left( 8 \frac{|\nabla \phi|^2}{\phi^2} - 4 \frac{\Box \phi}{\phi} - \frac{\hat{\Lambda}}{\phi^4} \right) \bar{g}_{ab} \right).
\]

Since we have no freedom to change this we have to ask if it gives a sensible answer. We approach this question via an example.

## 6 An example: Class A Bianchi types

In this section we apply the procedure to spatially-homogeneous cosmological models with the symmetry of class A Bianchi types. These generically have nonzero Weyl curvature. Before restricting to CCC it is worth noting a few facts about the Bianchi metrics. Assume the metric to be

\[
g = dt^2 - R^2(e^{2\alpha} \sigma_1^2 + e^{2\beta} \sigma_2^2 + e^{2\gamma} \sigma_3^2),
\]

in terms of four functions of time, \((R(t), \alpha(t), \beta(t), \gamma(t))\), with \( \alpha + \beta + \gamma = 0 \), and the usual set of invariant one-forms \( \sigma_i \), which are taken to satisfy

\[
d\sigma_1 = n_1 \sigma_2 \wedge \sigma_3,
\]

and cyclic permutations, where each \( n_i \) is \( \pm 1 \) or \( 0 \).

In terms of the orthonormal tetrad \((\theta^0, \theta^1, \theta^2, \theta^3) = (dt, R e^\alpha \sigma_1, R e^\beta \sigma_2, R e^\gamma \sigma_3)\) the components of the Einstein tensor (still with conventions as in \([11]\)) are

\[
G_{00} = - \left( 3 \frac{\ddot{R}}{R} + \dot{\alpha} \dot{\beta} + \dot{\beta} \dot{\gamma} + \dot{\gamma} \dot{\alpha} + AB + BC + CA \right)
\]

\[
G_{11} = - \left( -2 \frac{\ddot{R}}{R} - \frac{\dot{R}}{R}^2 + 3 \frac{\dot{R}}{R} + \dot{\alpha} - \dot{\alpha}^2 + \dot{\beta} \dot{\gamma} - AB + BC - CA \right)
\]
with the overdot being $d/dt$, cyclic permutations giving $G_{22}, G_{33}$, and all other components zero. Also here

$$A = \frac{1}{2R}(-n_1 e^{2\alpha} + n_2 e^{2\beta} + n_3 e^{2\gamma}),$$

and cyclic permutations for $B, C$.

We can also note the components of the Weyl tensor for the metric (16) in the orthonormal basis. These are

$$E_{11} = -\frac{\dot{R}\dot{\alpha}}{R} + \frac{1}{3}(2\beta\dot{\gamma} - \dot{\gamma}\dot{\alpha} - \dot{\alpha}\beta - 4BC + 2CA + 2AB), \quad (19)$$

$$B_{11} = \dot{\alpha}(B + C) - \dot{\beta}C - \dot{\gamma}B, \quad (20)$$

with the other components obtained by symmetry.

With a metric of this form in the previous aeon, $\mathcal{I}^+$ will be at $\tau = \infty$. The leading term $a_{ij}$ in the Starobinsky expansion (5) will have constant scalar curvature so the leading term $\phi_1$ in $\phi$ in (11) will be constant. This forces $\phi$ and hence $\Omega$ to be functions only of $\tau$ and therefore $\dot{g}$ has the same symmetry as $\dot{g}$. If $\dot{g}, \ddot{g}$ are both of the form of (16) with respectively hats and checks on all quantities then

$$\ddot{R} = -\theta^2 \dot{R}, \quad d\bar{t} = -\theta^2 d\bar{t}, \quad \ddot{\alpha} = \ddot{\alpha} \text{ etc.}$$

From the expression for $A, B, C$ it also follows that $\dot{A} = \phi^{-2} \dot{A}$, and permutations of this.

We shall suppose that $\dot{M}$ contains radiation fluid plus $\dot{\Lambda}$ so that the Einstein equations in the previous aeon become

$$\dot{G}_{00} = -\dot{\rho} + \dot{\Lambda}, \quad \dot{G}_{11} = \dot{G}_{22} = \dot{G}_{33} = -\frac{1}{3}(\dot{\rho} - \dot{\Lambda}).$$

From the symmetry, it is inevitable that the Einstein tensor of $\ddot{g}$ is also diagonal but it won’t have the perfect-fluid form. We set

$$\dot{G}_{00} = -\dot{\rho} + \dot{\Lambda}, \quad \dot{G}_{11} = -\dot{\rho}_1 - \dot{\Lambda}, \quad \dot{G}_{22} = -\dot{\rho}_2 - \dot{\Lambda}, \quad \dot{G}_{33} = -\dot{\rho}_3 - \dot{\Lambda}, \quad (21)$$

in terms of principal pressures $\dot{\rho}_i$, and then we can find these diagonal entries as power series in $\tau$, the proper-time coordinate in $\dot{M}$. This is most easily accomplished by calculating first in conformal time $\tau$ where

$$d\tau = d\bar{t}/\dot{R} = d\bar{t}/\dot{R},$$

and this was done in [14], from which we quote. The (00) Einstein equation in $\dot{M}$ forces $\dot{R}$ to blow up at finite $\tau$. Suppose this is at $\tau = \tau_F$ and set $\bar{\tau} = \tau_F - \tau$, then

$$\dot{R} = \frac{1}{H\bar{\tau}}(1 + a\bar{\tau}^2 + O(\bar{\tau}^3)) \quad (22)$$

$$\alpha = \alpha_0 + \alpha_1 \bar{\tau}^2 + O(\bar{\tau}^3)$$

$$\beta = \beta_0 + \beta_1 \bar{\tau}^2 + O(\bar{\tau}^3)$$

$$\gamma = \gamma_0 + \gamma_1 \bar{\tau}^2 + O(\bar{\tau}^3)$$

$$\phi = \alpha_1 \bar{\tau} + a_3 \bar{\tau}^3 + O(\bar{\tau}^4)$$

where $a = \operatorname{Lim} \frac{\dot{R}^2(AB + BC + CA)}{18}$, and the limit is onto $\mathcal{I}^+$; $a_1 \neq 0$ is a free constant in $\phi$; $\alpha_0, \beta_0, \gamma_0$ are the values of $\alpha, \beta, \gamma$ at $\mathcal{I}^+$ and, with $a_1$, determine the metric of $\mathcal{I}^+$ which is free data; $\alpha_1 = \operatorname{Lim} \frac{\dot{R}^2(2BC - AB - AC)}{18}$ with $\beta_1, \gamma_1$ obtained by cyclic permutation; and $a_3$ is obtained by solving (12) as

$$a_3 = -4aa_1 + \frac{a_1^3 s}{12H^2}. $$

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We can simplify $a_3$ with the aid of the metric of $I^+$, which is

$$\frac{a_1^2}{H^2}(e^{2\alpha_0} \sigma_1^2 + e^{2\beta_0} \sigma_2^2 + e^{2\gamma_0} \sigma_3^2).$$  (23)

The scalar curvature of (23) is $s^I = \frac{36H^2 a_1^2}{a_1^4}$ and our prescription above was to take $s = s^I$, in which case $a_3 = -aa_1$.

For the Weyl tensor components we find from (20) that $B_{ij} = O(\hat{\tau}^3)$ while from (19) $E_{ij} = O(\hat{\tau}^2)$ (recall that the Weyl tensor is expected to vanish at $I^+$, and these components are conformally invariant).

For the matter content after the bang we start with (21) and (18):

$$\dot{\rho} + \dot{\Lambda} = -\dot{G}_{00} = 3\ddot{R}^2 \dddot{R} \left( \frac{d(\phi^2 \dot{R})}{d\tau} \right)^2 + \frac{1}{\phi^4 \dot{R}^2} (\alpha_\tau \beta_\tau + \beta_\tau \gamma_\tau + \gamma_\tau \alpha_\tau) + \dot{A} \dot{B} + \dot{B} \dot{C} + \dot{C} \dot{A}$$

$$= \frac{3}{\phi^8 \dot{R}^4} \left( \frac{d(\phi^2 \dot{R})}{d\tau} \right)^2 + \frac{1}{\phi^4 \dot{R}^2} (\alpha_\tau \beta_\tau + \beta_\tau \gamma_\tau + \gamma_\tau \alpha_\tau) + \frac{1}{\phi^4} (\dot{A} \dot{B} + \dot{B} \dot{C} + \dot{C} \dot{A})$$

$$= \frac{1}{\phi^4} (\dot{\rho} + \dot{\Lambda}) + \frac{12}{\phi^6 \dot{R}^3} \frac{d\phi}{d\tau} \frac{d(\phi \dot{R})}{d\tau}$$

$$= \frac{3H^2}{a_1^4 \dot{\tau}^4} + \frac{12H^2 a_1^2}{a_1^4 \dot{\tau}^2} + O(1),$$

using the expansions of (22) in the last line.

For a principal pressure we have

$$\dot{p}_1 - \dot{\Lambda} = -\dot{G}_{11} = -2\ddot{R}^2 \dddot{R} + \frac{1}{\dot{R}^2} \left( \frac{1}{\dot{R} \ddot{R}} \right) - 3\frac{\dot{R} \ddot{\alpha}}{\dot{R}^2} + 3 \left( \frac{\xi}{\dot{R}^3} \right)^2 + \frac{3}{\dot{R}^2} \frac{d\dot{R}}{d\tau} \frac{da_1}{d\tau}$$

$$+ \frac{1}{\dot{R}^2} \left( \frac{1}{\dot{R} \ddot{\alpha}} \right) + \frac{1}{\dot{R}^2} \left( \frac{d\beta}{d\tau} \frac{d\gamma}{d\tau} - \frac{d\alpha_t}{d\tau} \right)^2 + \frac{1}{\phi^4} (\dot{A} \dot{B} + \dot{B} \dot{C} - \dot{C} \dot{A}).$$

Put $\dot{R} = -\phi^2 \dot{R}$ and simplify to find

$$= \frac{1}{\phi^4} \left( \frac{1}{3} \dot{\rho} - \dot{\Lambda} \right) + \frac{6}{\phi^5 \dot{R}^2} \frac{da_1}{d\tau} \frac{d\phi}{d\tau} - \frac{4}{\phi^6 \dot{R}^3} \frac{d\phi}{d\tau} \frac{d(\phi \dot{R})}{d\tau} - \frac{4}{\phi^4 \dot{R}^2} \frac{d\dot{\phi}}{d\tau} \frac{1}{\phi}$$

$$= \frac{H^2}{a_1^4 \dot{\tau}^4} + \frac{4H^2 (a + 3a_1)}{a_1^4 \dot{\tau}^2} + O(1).$$

It is natural to write quantities after the bang in terms of proper time rather than conformal time so that we need an expression for $\dot{t}$ in terms of $\tau$ which we obtain by solving

$$d\dot{t} = \ddot{R} d\tau = -\phi^2 \dot{R} d\tau.$$  

We find

$$\dot{t} = \frac{a_1^2}{2H} \dot{\tau}^2 - \frac{aa_1^2}{4H} \dot{\tau}^4 + h.o.$$  

choosing a common origin for $\dot{t}$ and $\tau$, which inverts to

$$\ddot{\tau}^2 = \frac{2H}{a_1^2} \dot{\tau} + \frac{2H^2 a}{a_1^4} \dot{t}^2 + h.o.
In terms of $\hat{t}$ we find
\[ E_{11} = -\frac{\Lambda H \alpha_1}{a_1^2} \hat{t} + \text{h.o.} \quad (24) \]
\[ \dot{\rho} = 3 \frac{\hat{t}^2}{4\hat{t}^2} + \frac{9aH}{2a_1^2} + O(1) \]
\[ \dot{p}_1 = \frac{1}{4\hat{t}^2} + \frac{3H}{2a_1^2}(a + 4\alpha_1) + O(1). \]

This is our final result: to leading order, which is $O(\hat{t}^{-2})$ the matter after the bang is radiation perfect fluid with isotropic pressures equal to one-third of density; to next order the pressure $\dot{p}_1$ differs from one-third of density by a multiple of $\alpha_1$ which in turn is a multiple of $dE_{11}/d\hat{t}$ where $E_{11}$ is the (11)-component of the electric Weyl tensor, so that $\alpha_1$ is proportional to a component of the gravitational radiation field at $I^+$; the sum of the pressures is still equal to the density at this order, so that the energy-momentum tensor is still trace-free; at the next order, the sum of the pressures will cease to equal the density.

These are satisfactory answers, to these orders. The next step, which we leave to later work, is to see what happens without the assumption of spatial homogeneity.

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