Macrospin in external magnetic field: entropy production and fluctuation theorems

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Abstract. We consider stochastic rotational dynamics of a macrospin at a constant temperature, in the presence of an external magnetic field. Starting from the appropriate Langevin equation, which contains multiplicative noise, we calculate entropy production (EP) along stochastic trajectories, and obtain fluctuation theorems. The system remains inherently out of equilibrium due to a spin torque supporting an azimuthal current, leading to an excess EP in addition to the EP due to heat dissipation. The anomaly may be removed using a redefinition of dissipated heat and stochastic work done. Using numerical simulations, we obtain distribution functions for EP along stochastic trajectories to find good agreement with the detailed fluctuation theorem.

Keywords: driven diffusive systems (theory)
1. Introduction

With miniaturization of memory devices such as the magnetic read head and random access memory, thermal fluctuations are starting to play a non-trivial role in their performance, e.g. by activating magnetization reversal of ferromagnetic clusters [1]. The impact of thermal noise is stronger in smaller devices [2, 3], with the relative intensity being inversely proportional to system size. The thermally induced magnetization fluctuations will act as a fundamental limit to the performance of submicrometer magnetoresistive devices. Thus, even from the application perspective, it becomes crucial to understand the impact of thermal fluctuations, in order to reliably use small magnetic devices [4–8]. The simplest component of such magnetic devices is a single magnetic domain, a macrospin.

Thus it is interesting to understand stochastic thermodynamic properties of a macrospin under strong thermal fluctuations [9]. Stochastic counterparts of thermodynamic observables, e.g. energy, work, heat, and entropy, characterize stochastic trajectories in phase space. Statistical averages of these quantities lead to the corresponding macroscopic thermodynamic observables [10–12]. Several equalities involving the stochastic observables have been derived in the last two decades [13–20]. It is shown that negative entropy producing trajectories do occur, but their probability remains exponentially suppressed with respect to the positive entropy producing trajectories. The

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corresponding equality is known as the detailed fluctuation theorem (DFT) [21–28]. A related integral fluctuation theorem (IFT), and the Jarzynski equality, that expresses equilibrium free energy difference in terms of non-equilibrium work done, were also derived [13, 24]. Many of these theorems have been verified against experiments on colloids and granular matter [29–32], and successfully used to obtain free energy landscapes of bio-polymers such as RNA [33, 34].

Recently, stochastic thermodynamics has been extended to describe active Brownian particles that derive their motion using an internal energy source or ambient fuel [35–38]. Several active particle dynamics are describable in terms of non-linear velocity dependent forces having odd parity under time reversal. For these systems, it was shown that the entropy production (EP) in the environment has excess contributions, apart from the contribution from dissipated heat [37, 38]. This excess EP is shown to be related to the time-reversal symmetry breaking due to the presence of a non-linear velocity dependent force, and potential energy of interaction or trapping, together [38].

Using a single Ising spin undergoing Glauber dynamics, work distribution functions have been studied earlier, under various protocols of time-dependent variation of magnetic field [39]. If the external field is time independent, such a system reaches an equilibrium Boltzmann distribution when coupled to a heat bath. Thus to extract non-equilibrium EP, it is imperative to impose a time-dependent field on such a system. We instead focus on the three-dimensional stochastic motion of magnetization of a sufficiently small single magnetic domain, a macrospin, coupled to a Langevin heat bath, under external magnetic field [40, 41]. One fundamental difference in the dynamics of this system with respect to the Ising spin is the presence of a directed precessional motion around the external field, even if the field is time independent. Thus the system in the presence of a magnetic field is intrinsically out of equilibrium. Due to the small system size the dynamics of such a macrospin is strongly influenced by thermal noise.

We use the Langevin and corresponding Fokker–Planck equations describing stochastic motion of a macrospin under external magnetic field. The appropriate Langevin equation contains multiplicative noise, and the external magnetic field gives rise to a spin torque. The Fokker–Planck equation is used to calculate the rate of entropy change in the system, which has two terms: one is the EP due to non-equilibrium processes in the system, and the other term gives entropy flux to the reservoir. The derivation clearly shows that the total average EP in the system and reservoir is non-negative, as required by the second law of thermodynamics. Using probabilities of stochastic trajectories we derive fluctuation theorems for stochastic EP. The corresponding expression for EP in the reservoir, $\Delta s_r$, depends on the choice of time-reversed conjugate trajectories, only one of which is consistent with the Fokker–Planck equation. A direct derivation of the stochastic energy balance shows that the EP in the environment has an excess contribution, apart from the stochastic version of Clausius entropy associated with heat dissipation. However, this anomaly may be resolved by redefining both the dissipated heat $-\Delta Q \equiv T \Delta s_r$, with $T$ denoting the temperature of the reservoir, and the stochastic work done on the system, keeping the expression of change in internal energy unchanged. Within the redefined form, these two terms contain the contribution from rotational work done due to torque. We propose experiments to separately measure heat dissipation and reservoir EP in the presence of spin torque, to test the relation between the two. Such measurements will help to understand stochastic EP better.
2. Model

The deterministic dynamics of a macrospin having magnetization $\mathbf{m}$ under time-dependent magnetic field $\mathbf{H}(t)$ is described by $\dot{\mathbf{m}} = \gamma \mathbf{m} \times \mathbf{H}(t)$, where $\dot{\mathbf{m}} = d\mathbf{m}/dt$, and $\gamma$ denotes the gyromagnetic ratio. However, the macrospin is not isolated, and its dynamics is affected by the surrounding medium. In the simplest idealization, the effect of medium could be incorporated in terms of a stochastic force conjugate to magnetization, a magnetic field $\mathbf{h}(t)$ varying randomly with time $^4[42]$. Thus the effective dynamics becomes $\dot{\mathbf{m}} = \gamma \mathbf{m} \times (\mathbf{H}(t) + \mathbf{h}(t))$. If $\mathbf{h}(t)$ is modeled as a Gaussian noise, one obtains the well known Bloch equation $^4[41, 42]$. However, within such a description, the expectation value of magnetization relaxes to zero even in the presence of a constant external magnetic field. This situation corresponds to the case of infinite temperature. In a finite temperature Langevin dynamics, the stochastic force is necessarily coupled to a frictional dissipation, obeying the fluctuation-dissipation theorem. The corresponding Langevin dynamics has the Landau–Lifshitz–Gilbert (LLG) form $^3, 43]$

$$\dot{\mathbf{m}} = \gamma \mathbf{m} \times [\mathbf{H} + \mathbf{h}(t) - \eta \mathbf{m}],$$  

(1)

where $\eta$ is the Gilbert damping coefficient $^44]$. The stochastic magnetic field obeys Gaussian statistics with

$$\langle \mathbf{h}(t) \rangle = 0, \quad \langle \mathbf{h}(t) \otimes \mathbf{h}(t') \rangle = 2D_0 \mathbf{1} \delta(t-t')$$  

(2)

where $\mathbf{1}$ denotes the identity matrix and $D_0 = \eta k_B T / V$, with $T$ denoting the temperature, $k_B$ the Boltzmann constant, and $V$ the volume of the magnetic particle. To get an assessment of the strength of the stochastic field $\mathbf{h}(t)$, let us evaluate it for a magnetic particle of volume $V = (10 \text{ nm})^3$ at room temperature, over a time-span of 1 ps. The corresponding strength is about 10 mT. Compare this with the field strength near a magnetic tape, about 1 $\mu$T, or earth’s magnetic field, which varies between 25 and 65 $\mu$T. However, the very stochastic nature of $\mathbf{h}(t)$ ensures that its integrated impact over longer time-spans reduces in magnitude, leaving fluctuations in the orientations of $\mathbf{m}$. Note that the LLG equation conserves the amplitude of magnetization $m = |\mathbf{m}|$, $d(m^2)/dt = 0$. This is a valid approximation for macrospins made of ferromagnetic material such as Fe or Co with Curie temperature $T_c \sim 10^3$ K, three to four times higher than room temperature, leading to negligible fluctuations in $m$ $^45]$.

The above mentioned phenomenology can be derived from microscopic equations of motion (classical or quantum), by using the Zwanzig formalism of coupling the system dynamics with that of a heat bath composed of an infinitely large number of degrees of freedom, then integrating out the heat bath degrees of freedom from the system equation of motion, and finally using a Markovian approximation $^46]$. This was achieved in two earlier studies using two different kinds of heat bath. In $^41]$, the surrounding environment of the magnetization was assumed to be composed of spins. A bilinear coupling of the system magnetization with environmental spins led to the appropriate Langevin dynamics, which is equivalent to equation (1). In $^40]$, the dynamics of a single magnetic particle was coupled to a harmonic oscillator heat bath, using a bilinear coupling between the magnetization and displacements of the oscillators.
After integrating out the oscillator degrees of freedom one obtains the corresponding Langevin dynamics. This reduces to the above mentioned LLG form of equation (1) within the Markovian approximation. This approximation makes the memory kernel a constant $\eta$, the Gilbert damping coefficient, and gives rise to a delta-function correlated stochastic magnetic field $\mathbf{h}(t)$, having a mean value $\langle \mathbf{h}(t) \rangle = 0$, and obeying the fluctuation-dissipation relation $\langle \mathbf{h}(t) \otimes \mathbf{h}(t') \rangle = 2D_0 \delta(t - t')$ with $D_0 = \eta k_B T / V$ as expressed in equation (2) [40].

Thus, to describe the stochastic dynamics of a macrospin having magnetization $\mathbf{m}$ under an external field $\mathbf{H}(t)$, we use the LLG equation given in equation (1). The strength of stochastic noise in this equation depends on the magnetization $\mathbf{m}(t)$; i.e., the noise is multiplicative. In the above equation, $\mathbf{H}$ denotes the conservative field $\mathbf{H} = -\partial G / \partial \mathbf{m}$, where $G = -\mathbf{m} \cdot \mathbf{H}$ is the Gibbs free energy per unit volume. The macrospin undergoes a relaxation dynamics in the Langevin heat bath, settling into an average unidirectional precession around the field $\mathbf{H}$, conserving the amplitude $m$.

This angular dynamics of the instantaneous orientation of magnetization $\mathbf{m}$ may be represented in terms of polar and azimuthal angles $(\theta(t), \phi(t))$ on the surface of a sphere of a fixed radius $m$. Using this description, the LLG equation can be expressed as

$$
\dot{\theta} = h' m(H_\theta + h_\theta) - g'(\sin \theta)^{-1}(H'_\phi + h_\phi),
\sin \theta \dot{\phi} = g' m(H_\theta + h_\theta) + h' m(\sin \theta)^{-1}(H'_\phi + h_\phi),
$$

with $\dot{\theta} = \partial \theta / \partial t$, $\dot{\phi} = \partial \phi / \partial t$, and $H_\theta = -(1/m) \partial \theta G$, $H'_\phi = H_\phi \sin \theta = -(1/m) \partial \phi G$ so that one can express $\mathbf{H} = \hat{\theta} H_\theta + \hat{\phi} H_\phi$ in spherical polar coordinates. The components of the stochastic field are given by $h_\theta = h_x \cos \theta \cos \phi + h_y \cos \theta \sin \phi - h_z \sin \theta$, $h_\phi = -h_z \sin \theta \sin \phi + h_y \sin \theta \cos \phi$. Note that the radial component $h_r = h_x \sin \theta \cos \phi + h_y \sin \theta \sin \phi + h_z \cos \theta$ does not appear in the angular motion of magnetization. A typical steady state trajectory obtained from numerical simulations is shown in figure 1. The details of the simulations will be discussed in section 7.

In a recent study [47], it has been shown that the form of the Fokker–Planck equation derived from the LLG equation is independent of the choice of stochastic calculus—Itô, Stratonovich or a post-point discretization scheme [48, 49]. The Fokker–Planck equation was originally derived in [3] using the Stratonovich convention, which we use throughout this paper.

### 3. Fokker–Planck equation and EP

A statistical ensemble of magnetization orientations can be described by the surface probability density $P(\theta, \phi, t)$. The corresponding Fokker–Planck equation is expressed as
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\[ \partial_t P = -\nabla_{\Omega} \cdot J_{\Omega}, \quad J_{\Omega} = \hat{\theta} J_\theta + \hat{\phi} J_\phi \]

(4)

where the two-dimensional divergence on the surface of the unit sphere \( \nabla_{\Omega} \cdot J_{\Omega} = \frac{1}{\sin \theta} \partial_{\theta}(\sin \theta J_\theta) + \frac{1}{\sin \theta} \partial_{\phi} J_\phi \), \( \Omega \) denotes a solid angle. The two components of dissipative current are given by [3]

\[
J_\theta = m[h' H_\theta - g' H_\phi]P - k' \partial_\theta P
\]

\[
J_\phi = m[g' H_\theta + h' H_\phi]P - k'(\sin \theta)^{-1} \partial_\phi P.
\]

(5)

In the above relations, \( h' \) and \( g' \) play the role of mobility, and \( k' \) plays the role of diffusivity for angular dynamics. These mobility and diffusivity coefficients obey the Einstein-like relation \( k' = D_0 m^2 (h'^2 + g'^2) = k_B T \eta_\gamma^2 / V(1 + \gamma^2 \eta_\gamma^2 m^2) \).

The non-equilibrium Gibbs entropy is given by [24, 25]

\[
S = -k_B \int d\Omega \int P(\theta, \phi, t) \ln P(\theta, \phi, t) = \langle -k_B \ln P \rangle,
\]

where \( \int d\Omega = \int \sin \theta \, d\theta \, d\phi \) denotes integration over the phase space, which in this case is all possible solid angles, and \( \langle ... \rangle \) denotes the statistical average. Note that this definition of \( S \) is equivalent to the Shannon information entropy associated with any probability distribution [50, 51]. The generic paradigm of Maxwell’s demon paradox [52] helped building the connection between Shannon’s information entropy and thermodynamic entropy [53–55]. Recent experiments verified that it is possible to convert information to free energy [56]. The direct relation between information entropy and physical entropy was exemplified by Landauer’s principle linking the minimum heat dissipation associated with erasure of one bit of information as \( k_B T \ln 2 \) [57]. This has been recently verified experimentally [58]. Thus the above definition of entropy \( S \) has a much wider scope, going beyond equilibrium physics. This includes non-equilibrium systems as well. The stochastic entropy of a microscopic state of the

Figure 1. A typical trajectory of magnetization \( \mathbf{m} \) in the presence of a magnetic field \( \mathbf{H} = H \mathbf{\hat{z}} \) with \( H = 1 \) and a Langevin heat bath at temperature \( k_B T = 1 \), obtained from numerical simulations. The arrow heads denote the direction of motion.
system is \( s(\theta, \phi, t) = -k_B \ln P(\theta, \phi, t) \), with the entropy of the ensemble \( S = \langle s \rangle \). One can express the rate of change in stochastic entropy as

\[
\frac{\dot{s}}{k_B} = -\frac{\partial P}{P} \frac{\partial \theta}{\partial \theta} - \frac{\partial P}{P} \frac{\partial \phi}{\partial \phi} = -\frac{\partial P}{P} + \frac{J_\theta + J_\phi \sin \theta \dot{\phi}}{k'P} - \frac{\dot{s}_t}{k_B},
\]

where

\[
\frac{\dot{s}_t}{k_B} = \frac{m}{k'} \left[ h'(H_\theta + H_\phi \sin \theta \dot{\phi}) + g'(H_\theta \sin \theta \dot{\phi} - H_\phi \dot{\theta}) \right]
= V \frac{m}{k_B T} \left[ (H_\theta + H_\phi \sin \theta \dot{\phi}) + \frac{1}{m\eta\gamma} (H_\theta \sin \theta \dot{\phi} - H_\phi \dot{\theta}) \right].
\]

The first step in equation (6) identifies the explicit and implicit time dependences. The second step is obtained by using equation (5) to replace \( \partial \theta P \) and \( \partial \phi P \). In obtaining the second step in equation (7) we used \( k' = D_0 m^2 (h' + g') \), and the expressions for \( h' \) and \( g' \), along with the identity \( D_0 = \eta k_B T / V \).

At this point, we focus on the interpretation of the second term on the right-hand side of equation (6), \( \mathcal{T} = [J_\theta + J_\phi \sin \theta \dot{\phi}] / k'P \). Let us perform a two-step averaging on equation (6): (i) over trajectories and (ii) over the ensemble of all possible solid angles \( \Omega \) with probability \( P(\Omega, t) \). The trajectory average of the components of angular velocity leads to \( \langle \dot{\theta} | \theta, \phi, t \rangle = J_\theta / P \) and \( \langle \sin \theta \dot{\phi} | \theta, \phi, t \rangle = J_\phi / P \) [25]. Thus after averaging over trajectories, \( \mathcal{T} \) can be replaced by the expression \( \mathcal{T} = (J_\theta^2 + J_\phi^2) / k'P^2 \). Now, to perform averaging over the phase space probability \( P(\Omega, t) \), we multiply equation (6) throughout by \( P(\Omega, t) \) and integrate over \( \Omega \). The conservation of probability \( \int d\Omega \, P(\Omega, t) = 1 \) leads to \( \int d\Omega \, \partial_t P(\Omega, t) = 0 \). Thus one obtains the average EP in the system

\[
\dot{\mathcal{S}} \equiv \langle \dot{s} \rangle = k_B \int d\Omega \, \frac{J_\theta^2 + J_\phi^2}{k'P} - \langle \dot{s}_t \rangle \equiv \Pi - \langle \dot{s}_t \rangle,
\]

where

\[
\Pi \equiv k_B \langle \mathcal{T} \rangle = k_B \int d\Omega \, \frac{J_\theta^2 + J_\phi^2}{k'P}
\]

is the EP due to irreversible non-equilibrium processes occurring in the system quantified by \( J_\theta \) and \( J_\phi \), and \( \langle \dot{s}_t \rangle \) is the entropy flux from the system to the surrounding environment. This second term quantifies the EP in the environment. Thus the total EP in the combined system and environment,

\[
\dot{S}_t = \dot{\mathcal{S}} + \langle \dot{s}_t \rangle = \Pi \geq 0,
\]

obeys the second law of thermodynamics. At non-equilibrium steady states, \( \Pi = \langle \dot{s}_t \rangle \), i.e. whatever entropy is produced in the system flows out to the environment [59, 60].

The preceding analysis shows that \( \dot{s}_t \) is the stochastic EP in the environment. Note that the expression of \( \dot{s}_t \), as given by equation (7), consists of terms having dimensions of torque times angular velocity, similar to dissipated work that one obtains from usual Langevin dynamics of particles moving in a medium.
As we show in the following section, the average change in the total stochastic entropy $s_t = s + s_{\tau}$, calculated over a finite time interval $\tau_0$, can be interpreted as the Kullback–Leibler divergence between the time-forward and time-reversed distribution of trajectories. Thus, $\Delta s_t = \int_0^{\tau_0} dt \dot{s}_{\tau}$ quantifies the breakdown of time-reversal symmetry.

4. Fluctuation theorems

Now we proceed to derive EP along stochastic trajectories. Physically, EP characterizes the irreversibility of a trajectory. Consider the time evolution of a macrospin from $t = 0$ to $\tau_0$ through a path $X = [\theta(t), \phi(t), \mathbf{H}(t)]$, assuming for the moment a time-dependent protocol of controlling $\mathbf{H}(t)$. Let us divide the path into $i = 1, 2, \ldots, N$ segments of time-interval $\delta t$ with $N\delta t = \tau_0$. The transition probability $p_i^+(\theta', \phi', t + \delta t| \theta, \phi, t)$ on the $i$th infinitesimal segment is governed by the Gaussian random noise $(\delta \pi_j \delta \pi_j) =-\frac{P_h}{2} \delta \theta_j \delta \phi_j$, where $\pi_j^2$ is calculated at the $i$th instant. Denoting the equations (3) as $\dot{\theta} = \Theta(\theta, \phi, \mathbf{H})$ and $\dot{\phi} = \Phi(\theta, \phi, \mathbf{H})$, the transition probability on the $i$th segment $p_i^+ = J_i(\delta(\dot{\theta}_i - \Theta))\delta(\dot{\phi}_i - \Phi)) = J_i \int d\mathbf{h} P(\mathbf{h})\delta(\dot{\theta}_i - \Theta)\delta(\dot{\phi}_i - \Phi)$. The Jacobian of transformation $J_i = \det[\partial(h_{x_1}, h_{y_1}, h_{z_1})/\partial(m_{x_1}, \theta_1, \phi_1)]_{m_{x_1}=\text{constant}}$. The probability of a full trajectory is $P_t = \prod_{i=1}^{N} p_i^+$.

It is possible to choose conjugate dynamics and trajectories in several ways, each of which will give rise to a new entropy-like quantity obeying detailed and integral fluctuation theorems, as shown in appendix A. This fact has been discussed in the literature and questions have been posed as to which choice would be physically meaningful [12, 61, 62]. For example, consider EP in a soft matter system in external shear flow [61, 62]. While considering conjugate trajectories, one possibility is to assume that the external flow does not change direction with respect to the time-forward trajectories, treating the flow as a quantity similar to external force [61]. This gives rise to an expression of entropy which obeys fluctuation theorems [61]. On the other hand, one can also extend the operation of time reversal to the particles of fluid. Then for conjugate trajectories fluid flow changes sign. This leads to a different expression of entropy, again obeying fluctuation theorems. It was argued in [62] that this second choice is physically more appealing. It is clear that the conjugate dynamics has to be chosen carefully to produce physically meaningful expression of entropy [63].

For the present problem, the probability distribution of micro-states $P(\theta, \phi, t)$ evolves via the Fokker–Planck equation, from which we have already obtained EP. Calculation of EP using probabilities of time-forward and conjugate trajectories should agree with the outcome of the Fokker–Planck equation. Thus care has to be taken so that the choice of conjugate trajectories is consistent with the symmetries of the Fokker–Planck equation.

The time-forward trajectory $X$ considered above has $\mathbf{H}(t)$ as the control parameter. The above guiding principle gives us a unique choice of its conjugate form $\mathbf{H}(\tau_0 - t)$.
Note that this conjugation is different from a physical time-reversal operation, under which magnetic field changes sign. The microscopic variables \([\theta(t), \phi(t)]\) denote angular position, and are even functions under time reversal. Thus the time-reversed conjugate trajectory can be denoted as \(X^\dagger = [\theta(\tau_0 - t), \phi(\tau_0 - t), \mathbf{H}(\tau_0 - t)]\). The probability of conjugate trajectory \(\mathcal{P}_- = \prod_{i=1}^N \mathcal{P}_i^{-}\), where \(\mathcal{P}_i^{-} = \mathcal{J}_i^{-} \delta(\theta + \Theta(\tau_0 - t)) \delta(\phi + \Phi(\tau_0 - t))\), and \(\mathcal{J}_i^{-}\) denotes the relevant Jacobian. As \(\mathcal{J}_i^{-} = \mathcal{J}_i\), Jacobians drop out of the ratio \(\mathcal{P}_i^{\dagger} / \mathcal{P}_i^{-}\).

Let us outline the calculation of the Jacobian in the presence of multiplicative noise. Given a Langevin dynamics \(\dot{x} = \mathcal{F}(x) + g(x)\eta(t)\) with multiplicative noise \(g(x)\eta(t)\), one may rewrite it as \(\dot{x} / g(x) = h(x) + \eta(t)\) where \(h(x) = \mathcal{F}(x) / g(x)\). Using a transformation \(q(x)\) such that \(\dot{q} = \dot{x}(dq/dx)\) with \(dq/dx = 1/g(x)\), the Langevin equation may be expressed as \(\dot{q} = h(q) + \eta(t)\), transforming the multiplicative noise evolution of \(x\) in terms of \(q(x)\) evolving under additive noise. Then using Stratonovich discretization one can show

\[
\mathcal{J}_i = \text{det}[\partial_x \eta_i] = \frac{1}{\delta t} \left[ 1 - \frac{\delta t}{2} \frac{\partial}{\partial x_i} \left( \frac{\mathcal{F}(x_i)}{g(x_i)} \right) \right].
\]

Note that under time reversal \(x_i\) and as a result \(\mathcal{F}(x_i)\) and \(g(x_i)\) remain invariant, and so does \(\mathcal{J}_i\). This behavior is generic for over-damped Langevin equations with multiplicative noise, and remains valid for path probabilities of \((\theta, \phi)\) coordinates.

Let us assume that the trajectories considered above describe evolution from an initial steady state described by a distribution \(P_t(\theta_t, \phi_t, \mathbf{H}_t)\) to a final state \(P_f(\theta_f, \phi_f, \mathbf{H}_f)\). The total probabilities of time-forward and time-reversed conjugate trajectories are given by \(\mathcal{P}^f[X] = P_t^f \mathcal{P}_+\) and \(\mathcal{P}^b[X^\dagger] = P_t^b \mathcal{P}_-\) respectively, where \(X\) and \(X^\dagger\) denote the forward and conjugate processes. The Kullback–Leibler divergence of these probabilities

\[
D(\mathcal{P}^f || \mathcal{P}^b) = \sum_X \mathcal{P}^f[X] \ln \frac{\mathcal{P}^f[X]}{\mathcal{P}^b[X^\dagger]}.
\]

is a non-negative quantity and is a good candidate for the expression of average total EP \(\langle \Delta s_t \rangle\). The change in stochastic entropy of the system \(\Delta s = s_f - s_i = k_B \ln(P_f / P_t)\). This is a state function and depends on the exact initial and final micro-states. Similarly to \(\Delta s\), one can define the total entropy change \(\Delta s_t = k_B \ln \frac{P^f[X]}{P^b[X^\dagger]} = \Delta s + \Delta s_t\), where \(\Delta s_t = k_B \ln \frac{P_t}{P_f}\) is the change in entropy in the reservoir [64]. The above relation for \(\Delta s_t\) readily leads to the IFT [12], \(\langle e^{-\Delta s_t / k_B} \rangle = 1\). Note that in deriving IFT \(\sum_X = \sum_X^\dagger\) is used, as the Jacobian of transformation from the time-forward path \(X\) to the time-reversed path \(X^\dagger\) is unity [63]. As a result of the IFT and the Jensen inequality one obtains \(\langle \Delta s_t \rangle \geq 0\), the second law of thermodynamics. This is equivalent to the statement \(\dot{S}_t = d(s_t) / dt > 0\) derived above using the Fokker–Planck equation describing the irreversibility of the non-equilibrium dynamics.

After some algebra, it is possible to show that the ratio of the two probabilities of the forward and reverse paths \(\frac{P_t}{P_f} = \exp(\Delta s_t / k_B)\), where

\[
\Delta s_t = k_B \ln \frac{P_t}{P_f} = \frac{1}{2} \sum_X \mathcal{P}^f[X] \ln \frac{\mathcal{P}^f[X]}{\mathcal{P}^b[X^\dagger]}.
\]
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\[ \frac{\Delta s_t}{k_B} = V \frac{m}{k_B T} \int_0^\tau d\tau \left[ (H_0\dot{\theta} + H_0\sin \theta \dot{\phi}) + \frac{1}{m\eta\gamma} (H_0\sin \theta \dot{\phi} - H_0\dot{\theta}) \right]. \] (9)

In the last step we used \(D_0 = \eta k_B T / V\). The definition \(\Delta s_t\) in equation (10) directly leads to the expression of EP in the reservoir \(\dot{s}_t\) in equation (7) derived from the Fokker–Planck equation.

Further, in a steady state the total entropy change \(\Delta s_t\) along a time-forward path \(\Delta s_t^f(X)\) is equal and opposite to that along the time-reversed path, \(\Delta s_t^b(X) = -\Delta s_t^f(X)\). Using this, one obtains the following DFT [17, 24]:

\[ \rho(\Delta s_t) = e^{\Delta s_t / k_B} \rho(-\Delta s_t). \] (10)

As already mentioned, it is possible to consider conjugate trajectories in various other manners, e.g. considering \(X^\dagger = [-m(\tau_0 - t), -H(\tau_0 - t)]\), \(X^\dagger = [m(\tau_0 - t), -H(\tau_0 - t)]\), or \(X^\dagger = [-m(\tau_0 - t), H(\tau_0 - t)]\) (see appendix A). As can be shown easily, each one of these considerations will give the IFT and DFT, but with expressions of entropy in the reservoir different from that in equation (9).

5. Time-independent uniaxial field and detailed balance

In the presence of a uniaxial external field, the potential energy per unit volume \(G(\theta) = -H m \cos \theta\), i.e. \(H_\phi = 0\) as \(\partial_\phi G = 0\). Assuming the same uniaxial symmetry in the probability distribution \(P(\theta, t)\) independent of \(\phi\), \(\partial_\phi P = 0\), leads to \(\partial_\phi J_\phi = 0\) (see equation (5)). Thus the Fokker–Planck equation reduces to

\[ \partial_\theta P(\theta, t) = (\sin \theta)^{-1} \partial_\theta (\sin \theta J_\theta). \] (11)

The detailed balance condition requires vanishing of the dissipative current \(J_\theta = 0\) leading to the canonical Boltzmann distribution \(P = P_0 \exp[-G(\theta)/k_B T]\), which still allows for the presence of a divergence-less current (equation (5)) in the azimuthal direction \(J_\phi = -g'(\partial_\theta G)P(\theta)\) [3] (see figure 2(b)). A torque due to \(H\) acting on the magnetization \(m\) leads to precessional probability current \(J_\phi\) along \(\phi\), and to EP. The situation is similar to a particle in a harmonic trap under constant external torque, thereby producing entropy [65, 66]. The state under constant \(H\) is characterized by \(J_\phi = 0\), \(J_\phi(\theta) = 0\) controlled solely by the Boltzmann distribution \(P(\theta)\).

The change in system entropy between initial state \(P(\theta_i)\) and final state \(P(\theta_f)\) is

\[ \frac{\Delta s}{k_B} = [G(\theta_f) - G(\theta_i)] = -Hm[\cos \theta_f - \cos \theta_i]. \] (12)

Let us write down the expression of \(\Delta s_t\) for uniaxial time-independent magnetic field such that \(H_\phi = 0\). The expression simplifies to

\[ \frac{\Delta s_t}{k_B} = V \frac{m}{k_B T} \int_0^\tau d\tau \left[ H_0\dot{\theta} + \frac{1}{m\eta\gamma} H_0 \sin \theta \dot{\phi} \right]. \] (13)
6. Stochastic energy balance

The rate of stochastic energy gain per unit volume \( \dot{G} = -\mathbf{H} \cdot \dot{\mathbf{m}} - \mathbf{m} \cdot \dot{\mathbf{H}} \). In this expression, the rate of work done by the magnetic field is \( \dot{W} = -\mathbf{m} \cdot \dot{\mathbf{H}} \). The stochastic energy balance is given by \( \dot{G} = \dot{q} + \dot{W} \), where stochastic heat absorption by the system \( \dot{q} = -\mathbf{H} \cdot \dot{\mathbf{m}} \). In the spherical polar coordinates, the stochastic heat absorption per unit volume can be expressed as

\[
\dot{q} = -\mathbf{H} \cdot \dot{\mathbf{m}} = -[\dot{\phi} H_0 + \dot{\phi}_0 H_0] \cdot [\dot{\theta} m \phi + \dot{\phi} m \sin \theta \phi] \\
= -m [H_0 \theta + H_0 \phi \sin \theta \phi].
\]

(14)

The above result does not give \( \dot{Q} = V \dot{q} \) which will obey \( \dot{Q} = -T \dot{s}_r \). The expression of the discrepancy between \( -\dot{Q}/T \) and \( \dot{s}_r \) (see equation (7)) is the excess EP, and is given by the quantity \( (V/T) [(\dot{H}_0 \sin \theta \phi - H_0 \dot{\theta})/\eta \gamma] \). Note that, even for uniaxial time-independent external field with \( H_\phi = 0 \), the trajectory average \( J_\phi \), which depends on \( \sin \theta \phi \), remains non-zero, although the system equilibrates in the sub-space \( \theta \). The excess EP disappears if the motion of the macro-spin is restricted to two dimensions, as then the role of external field on the magnetization becomes equivalent to a tangential external force acting on a diffusing particle moving on the circumference of a circle (see appendix B).

Using the Fokker–Planck equation we have shown that the specific form of \( s_r \) derived there leads to total EP \( \dot{S}_r \geq 0 \), consistent with the second law of thermodynamics. One may redefine the rate of heat dissipation as \( -\dot{Q} \equiv T \dot{s}_r \), independent of how it is to be split into the rate of work done and the rate of change in internal energy [12]. Thus it
is possible to rewrite \( \dot{q} \) and \( \dot{W} \) keeping \( \dot{G} = \dot{q} + \dot{W} \) intact, such that \( \dot{Q} \equiv -T \dot{s}_t \) is obeyed. At this point, note that equation (7) gives the form

\[
T \dot{s}_t = V \left[ \mathbf{H} \cdot \mathbf{m} + \frac{1}{\eta \gamma} (H_\theta \sin \theta \phi - H_\phi \dot{\theta}) \right].
\]

Thus the redefined quantities will have the following forms:

\[
\dot{q} \equiv -\mathbf{H} \cdot \mathbf{m} - \frac{1}{\eta \gamma} (H_\theta \sin \theta \phi - H_\phi \dot{\theta}),
\]

\[
\dot{W} \equiv -\mathbf{m} \cdot \mathbf{H} + \frac{1}{\eta \gamma} (H_\theta \sin \theta \phi - H_\phi \dot{\theta}).
\]

The redefined rate of work done has an excess contribution from angular motion due to spin torque. In the context of changing one equilibrium state to another, such a redefinition is disadvantageous, as the change in the corresponding free energy can not be related to quasi-static work done \([12]\). However, the current system is intrinsically out of equilibrium, and such a guiding principle is not strictly applicable.

Note that in the presence of a time-independent uniaxial magnetic field the system remains out of equilibrium, and the work done on the system due to spin torque should be dissipated as heat with \( \langle \dot{Q} \rangle = \frac{HV}{\eta \gamma} \langle \sin^2 \theta \phi \rangle \). Positivity of total EP requires this spin torque contribution in entropy.

Recently we performed a separate study of EP \([67]\) using a generalized Langevin dynamics of spins, where, unlike the case in the current study, the spin amplitude is allowed to fluctuate \([68]\). This Langevin equation does not involve multiplicative noise, unlike equation (1). Our calculation showed that the excess EP is related to the behavior of phase space variables under time reversal, and does not depend on the constraint of constant \( m \) imposed by the LLG equation.

\section{Distribution of EP}

We numerically evaluate the distribution of total EP \( \Delta s_t = \Delta s + \Delta s_t \) over trajectories of various durations \( \tau_0 \), in the presence of a time-independent magnetic field \( H \) along the \( z \)-direction. This is done by integrating equation (3), expressing the magnetization in units of \( m \), energy in units of \( k_B T \), and using \( H = k_B T / m \). In numerical integration, we used a stochastic generalization of the Heun scheme \([69]\). This method is known to converge to the solutions of stochastic differential equations interpreted in the Stratonovitch sense. In simulations, we used time step \( \delta t = 0.001 \tau \), where \( \tau = m / \gamma k_B T \) sets the unit of time.

To test the validity of our numerical integration, we first obtain the equilibrium distribution \( P(\theta) \) that agrees with analytical form \( P_0 \exp(-G(\theta)/k_B T) \) with \( G(\theta) = -mH \cos \theta \) (figure 2(a)). The steady state supports a probability current \( J_\phi(\theta) \) in the azimuthal direction. A typical steady state trajectory is shown in figure 1. Figure 2(b) shows the
polar dependence of azimuthal current obtained from simulations, and its comparison with the theoretical expression

\[ J_\theta = g' m H_0 P(\theta) \]

In figure 3 we show the probability distributions of EP \( \rho(\Delta s_t) \) calculated numerically using \( \Delta s_t = \Delta s + \Delta s_e \), and expressions of \( \Delta s \) and \( \Delta s_t \) from equations (12) and (13) respectively. The distributions are calculated after collecting data over \( 10^7 \) realizations for various durations of \( \tau_0 \) as indicated in figure 3. An appreciable probability of negative EP is clearly visible. With increase in \( \tau_0 \), the distribution broadens and the peak position shifts towards higher values of entropy. From each \( \rho(\Delta s_t) \) curve, we obtain the ratio of probabilities of positive and negative EPs, \( \rho(\Delta s_t) / \rho(-\Delta s_t) \). As is shown in figure 4, this ratio shows good agreement with the DFT, \( \rho(\Delta s_t) / \rho(-\Delta s_t) = \exp(\Delta s_t / k_B) \). The deviation of data from the analytic function at \( \Delta s_t / k_B \gtrsim 10 \) is due to poor statistics at large negative entropies.

Note that the probability distribution \( \rho(\Delta s_0) \), that obeys DFT, was obtained from time evolution in the forward direction, without any need to define a conjugate trajectory under time reversal. This again shows that the property of the distribution function \( \rho(\Delta s_t) \) is encoded in the dynamics. The choice of conjugate trajectories used for deriving fluctuation theorems has to be consistent with the expression of EP obtained from the Fokker–Planck equation, governing the dynamics of the probability distribution of micro-states. This requirement restricts the choice of conjugate trajectories to obey specific symmetry, which we presented in the main text. We discuss other possible choices of conjugate trajectories in appendix A. This is to demonstrate that, although, the corresponding entropy-like quantities obey DFT and IFT, all of them fail to capture the physical EP. For time independent \( H \) that generates a directed spin torque, the only contribution to EP in the environment comes from the azimuthal dynamics. The fact that the simulated probability distribution \( \rho(\Delta s_t) \) obeys DFT means that \( \langle \Delta s_t \rangle \gtrsim 10 \) with \( \Delta s_t \) containing the excess EP due to gyroscopic motion.
8. Outlook

We have derived fluctuation theorems involving EP in a macrospin undergoing stochastic rotational dynamics in the presence of external magnetic field. The fluctuation theorems were derived using the ratio of probabilities of time-forward and reversed trajectories. While it is possible to choose the reversed trajectories in various manners, we argued that the choice needs to be consistent with the Fokker–Planck equation. The constraint of constant amplitude of magnetization renders a dynamics that naturally involves a multiplicative noise. The magnetic field generates a spin torque, even when the field is time independent, driving the system out of equilibrium. This led to an excess EP in the environment, which appears to be inconsistent with the expression of heat dissipated derived from energy conservation. However, this anomaly can be lifted by redefining the expressions of dissipated heat and stochastic work done. Using numerical simulations, we obtained the distributions of EP over various time intervals, and showed that they agree with the detailed fluctuation theorem.

The steady state of a macrospin dynamics under a time-independent magnetic field behaves differently in two subspaces—while the axial current is zero, leading to a Boltzmann distribution, azimuthal current remains non-zero. The corresponding heat dissipation may be measured by calorimetry. On the other hand, whole trajectories $[\theta(t),\phi(t)]$ may be followed using Kerr microscopy [70, 71]. These two independent measurements may be used to check experimentally whether equation (14) or equation (16) describe the actual heat dissipation. Such measurements would improve our understanding of the relationship between stochastic heat dissipation and EP due to torque. A similar situation arises, e.g., for a particle in harmonic trap under constant external torque, producing entropy [65, 66]. We hope that our work will elicit further discussions on this technologically relevant and fundamentally important topic.
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Appendix A. Alternative choices of conjugate trajectories

We present three more choices of conjugate trajectories and their consequences. Denoting the path probability of the reverse trajectory \( X^\dagger = [-m(\tau_0 - t), -H(\tau_0 - t)] \) by \( \mathcal{P}^{(1)} \), one can show that the ratio \( \mathcal{P}_r/\mathcal{P}^{(1)} = \exp(\Delta s^{(1)}/k_B) \), where
\[
\frac{\Delta s^{(1)}}{k_B} = V \frac{m}{k_B T} \int_0^{\tau_0} dt [H_\phi \dot{\phi} + H_\theta \sin \theta \dot{\phi}] = -\frac{\Delta Q}{T}.
\] (A.1)

This implies that \( \Delta s^{(1)} = \Delta s + \Delta s^{(1)} \) obeys the IFT, and as a result \( \langle \Delta s^{(1)} \rangle \geq 0 \). The EP under the time-reversal symmetry considered here is associated with dissipative components of probability currents under the same symmetry [67]. As the external driving \( H \) in the present case is not symmetric under time reversal, the DFT will have the form \( \rho(\Delta s^{(1)}) = e^{\Delta s^{(1)}/k_B} \rho^*(\Delta s^{(1)}) \), where \( \rho^* \) denotes the probability calculated along the conjugate trajectory.

For the choice of conjugate trajectory in which \( H \) alone changes sign, such that the probability of conjugate trajectory \( X^\dagger = [m(\tau_0 - t), -H(\tau_0 - t)] \) is denoted by \( \mathcal{P}^{(2)} \), one obtains the ratio \( \mathcal{P}_r/\mathcal{P}^{(2)} = \exp(\Delta s^{(2)}/k_B) \), where
\[
\frac{\Delta s^{(2)}}{k_B} = V \frac{m}{k_B T} \int_0^{\tau_0} dt \frac{1}{\gamma} [H_\theta \sin \theta \dot{\phi} - H_\phi \dot{\phi}] = \frac{\Delta s^{(3)}}{k_B}.
\] (A.2)

Again, \( \Delta s^{(2)} = \Delta s + \Delta s^{(2)} \) obeys the IFT and DFT. It is interesting to note that EP in the environment as shown in the main text \( \Delta s_r = \Delta s^{(1)} + \Delta s^{(2)} \) (see equation (9)).

The third alternative is to consider conjugate trajectories in which \( m \) alone changes sign, i.e. \( X^\dagger = [-m(\tau_0 - t), H(\tau_0 - t)] \). Denoting the probability of conjugate trajectory \( \mathcal{P}^{(3)} \), one obtains \( \mathcal{P}_r/\mathcal{P}^{(3)} = \exp(\Delta s^{(3)}/k_B) \), with \( \Delta s^{(3)} = 0 \). By construction, \( \Delta s^{(3)} = \Delta s \) also obeys the IFT and DFT.

Appendix B. Stochastic thermodynamics: spin confined to 2D

Here we consider that the stochastic spin-rotation is confined to two dimensions (2D). The Langevin dynamics is described by
\[
\frac{dm}{dt} = \gamma \left[ H_\phi \dot{\phi} + h(t) - \eta \frac{dm}{dt} \right] \times m
\] (B.1)
where we assumed an external field perpendicular to this 2D plane. If the spin is restricted to rotate on the \((m, \phi)\) plane, \( dm/dt = m \dot{\phi} \phi \) with \( \phi = d\phi/dt \). In a strictly 2D
dynamics, neglecting the out of plane motion due to Gilbert damping, the equation of motion simplifies to
\[
\dot{\phi} = \gamma [H_0 + h_2(t)],
\]
with \(\langle h_2(t) \rangle = 0\), \(\langle h_2(t) h_2(t') \rangle = 2D_0 \delta(t - t')\). This is equivalent to the over-damped Langevin motion in 1D \(\dot{x} = \mu [\xi(t) + f]\) with mobility \(\mu\), Gaussian white noise \(\xi(t)\), and external force \(f\). Thus one obtains a stochastic version of the first law of thermodynamics \(\Delta q + \Delta W = 0\) with heat absorbed by the system \(\Delta q = \int dt \dot{\phi} [-\dot{\phi}/\gamma + h_2(t)]\) and work done \(\Delta W = \int dt \dot{\phi} H_0\).

Let us assume that the initial and final micro-states are described by \(\phi_i\) and \(\phi_f\) respectively. We consider time evolution starting from a single micro-state picked up from the initial distribution \(P_i(\phi_i)\), which evolves to one of the final micro-states obeying a distribution \(P_{\tau}(\phi_{\tau})\). The probability of a time-forward path evolved from \(t = 0\) to \(\tau\) is described by the stochastic field \(h_2(t) = \dot{\phi}/\gamma - H_0\) with \(P_+ \propto \exp \left[-(1/4D_0) \int_0^\tau dt h_2^2(t)\right]\), where \(D_0 = k_B T/\gamma\). Under time reversal the stochastic noise is described by \([-\dot{\phi}/\gamma - H_0]\).

Thus the ratio of these path probabilities is given by \(P_+ / P_- = \exp \left[(1/D_0\gamma) \int_0^\tau dt H_0 \dot{\phi}\right]\).

Now using the definition of work done and the first law derived above, we may rewrite the relation as \(P_+/P_- = \exp[-(1/D_0\gamma)\Delta q]\). This quantity accounts for the entropy change in the reservoir \(P_+/P_- = \exp(\Delta s_t/k_B)\) [25], with \(\Delta s_t = -k_B \Delta q/(D_0\gamma) = -\Delta q/T\). Thus the 2D counterpart of the full 3D dynamics does not have any discrepancy in terms of the definition of dissipated heat with EP in the reservoir, unlike the case in 3D, as described in the main text. The anomaly in EP and its subsequent resolution via redefinition of heat and work is required in 3D, as the system maintains non-equilibrium rotational current in azimuthal direction even when the external magnetic field is time independent.

References

[1] Koch R H, Grinstein G, Keefe G A, Lu Y, Trouilloud P L, Gallagher W J and Parkin S S P 2000 Phys. Rev. Lett. 84 5419
[2] Blanter Y M and Büttiker M 2000 Phys. Rep. 336 1
[3] Brown W F 1963 Phys. Rev. 130 1677
[4] Tserkovnyak Y and Brataas A 2001 Phys. Rev. B 64 214402
[5] Foros J, Brataas A, Tserkovnyak Y and Bauer G E W 2005 Phys. Rev. Lett. 95 016601
[6] Foros J, Brataas A, Bauer G E W and Tserkovnyak Y 2007 Phys. Rev. B 75 092405
[7] Bandopadhyay S, Brataas A and Bauer G E W 2011 Appl. Phys. Lett. 98 083110
[8] Covington M 2006 US Patent No 7,042,685
[9] Utsumi Y and Taniguchi T 2015 Phys. Rev. Lett. 114 186601
[10] Sekimoto K 1998 Prog. Theor. Phys. Suppl. 130 17
[11] Jarzynski C 2011 Annu. Rev. Condens. Matter Phys. 2 329
[12] Seifert U 2012 Rep. Prog. Phys. 75 126001
[13] Jarzynski C 1997 Phys. Rev. Lett. 78 2690
[14] Baiesi M, Maes C and Wynants B 2009 J. Stat. Phys. 137 1094
[15] Baiesi M, Boksenbojm E, Maes C and Wynants B 2010 J. Stat. Phys. 139 492
[16] Hummer G and Szabo A 2010 Proc. Natl Acad. Sci. USA 107 21441
[17] Kurchan J 2007 J. Stat. Mech. P07005
[18] Narayan O and Dhar A 2004 J. Phys. A: Math. Gen. 37 63
[19] Jayannavar A M and Sahoo M 2007 Phys. Rev. E 75 032102

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[20] Lahiri S and Jayannavar A M 2014 arXiv:1402.5588
[21] Evans D J, Cohen E G D and Morriss G P 1993 Phys. Rev. Lett. 71 2401
[22] Gallavotti G and Cohen E G D 1995 Phys. Rev. Lett. 74 2694
[23] Lebowitz J L and Spohn H 1999 J. Stat. Phys. 95 333
[24] Crooks G E 1999 Phys. Rev. E 60 2721
[25] Seifert U 2005 Phys. Rev. Lett. 95 040602
[26] Saha A, Lahiri S and Jayannavar A M 2009 Phys. Rev. E 80 011117
[27] Lahiri S and Jayannavar A M 2009 Eur. Phys. J. B 69 87
[28] Sahoo M, Lahiri S and Jayannavar A M 2011 J. Phys. A: Math. Theor. 44 205001
[29] Wang G, Sevick E, Mittag E, Searles D and Evans D 2002 Phys. Rev. Lett. 89 050601
[30] Blickle V, Speck T, Heldt L, Seifert U and Bechinger C 2006 Phys. Rev. Lett. 96 24
[31] Speck T, V Blickle, Bechinger C and Seifert U 2007 Europhys. Lett. 79 30002
[32] Joubaud S, Lohse D and van der Meer D 2012 Phys. Rev. Lett. 108 210604
[33] Lipparini J, Dumont S, Smith S B, Tinoco I and Bustamante C 2002 Science 296 1832
[34] Collin D, Ritort F, Jarzynski C, Smith S B, Tinoco I and Bustamante C 2005 Nature 437 231
[35] Hayashi K, Ueno H, Imo R and Noji H 2010 Phys. Rev. Lett. 104 218103
[36] Seifert U 2011 Eur. Phys. J. E 34 26
[37] Ganguly C and Chaudhuri D 2013 Phys. Rev. E 88 032102
[38] Chaudhuri D 2014 Phys. Rev. E 90 022131
[39] Marathe R and Dhar A 2005 Phys. Rev. E 72 066112
[40] Jayannavar A M 1991 Z. Phys. B 82 153
[41] Seshadri V and Lindenberg K 1982 Physica A 115 501
[42] Kubo R 1962 Fluctuation, Relaxation and Resonance in Magnetic Systems ed D ter Haar (Edinburgh: Oliver and Boyd)
[43] Kubo R and Hashitsume N 1970 Prog. Theor. Phys. Suppl. 46 210
[44] Gilbert T L 1955 Phys. Rev. 100 1243
[45] Blundell S 2001 Magnetsim in Condensed Matter (Oxford: Oxford University Press)
[46] Zwanzig R 2001 Nonequilibrium Statistical Mechanics (Oxford: Oxford University Press)
[47] Aron C, Barci D G, Cugliandolo L F, Arenas Z G and Lozano G S 2012 J. Stat. Mech.: P09008
[48] L A W C and Lubensky T C 2007 J. Phys. Rev. E 011123
[49] Jayannavar A M 1991 Z. Phys. B 82 153
[50] Chaudhuri D 2011 Phys. Rev. Lett. 106 140601
[51] Chaudhuri D, Bandopadhyay S, Chaudhuri D and Jayannavar A M 2015 Phys. Rev. E 92 032143
[52] Ma P-W and Dudarev S L 2012 Phys. Rev. B 86 054416
[53] Garcia-Palacios J L and Lázaro F J 1998 Phys. Rev. B 58 14937
[54] Hiebert W K, Lagae L, Dus J, Bekaert J, Wirix-Speetjens R and De Boeck J 2003 J. Appl. Phys. 93 6906
[55] Kruglyak V V, Barman A, Hicken R J, Childress J R and Katine J A 2005 J. Appl. Phys. 97 10A706

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