Comment on covariant Stora–Zumino chain terms

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Abstract

In a recent paper, Ekstrand proposed a simple expression from which covariant anomaly, covariant Schwinger term and higher covariant chain terms may be computed. We comment on the relation of his result to the earlier work of Tsutsui.

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1 Introduction

There are several methods for the computation of anomalies, anomalous commutators (Schwinger terms) and higher terms of the Stora–Zumino chain, both for the consistent and covariant case. Among these are perturbative, functional and algebraic methods [1]–[17] (see [18] for a review).

One particularly well-known and useful method for the computation of the consistent anomalous chain terms is the method of descent equations a la Stora and Zumino [11, 12]. In this framework, all consistent chain terms are computed from the Chern–Simons form \( \Omega_{2n-1}(A, F) \) with the help of some algebraic cohomology methods (here \( A, F \) are Lie-algebra-valued forms, \( A = A_{\mu}^a T^a dx^\mu, \ F = dA + A^2 \)). It turns out that the anomalous chain terms (which are determined up to exact terms) are just the expansion coefficients \( \Omega_{2n-1-k}^k(v, A, F) \) of the shifted Chern–Simons form

\[
\Omega_{2n-1}(A, F) = \sum_{k=0}^{2n-1} \Omega_{2n-1-k}^k(v, A, F)
\]  

in powers of \( v \), where \( v \) is the ghost field with ghost number 1.

For the covariant case, an analogous cohomological analysis was performed by Tsutsui, [13]. There, a procedure for computing all the covariant anomalous chain terms was derived, and an expression in terms of a shifted Chern–Simons form, analogous to (1), was given (see (11) below).

Recently, a simple expression for a shifted Chern–Simons form for the covariant chain terms was proposed by Ekstrand [19] (see (6) below). Further, he proved that his expression (i.e., the linear and quadratic expansion terms in \( v \)) correctly reproduce the covariant anomaly and Schwinger term. In addition, he noticed that his expression deviates from the one given by Tsutsui, [13], already for the Schwinger term (i.e., at order \( v^2 \)).

It is the purpose of this paper to comment on that difference. We will show that the algebraic construction of Tsutsui, [13], is completely equivalent to the proposed shift formula of Ekstrand, [19], and leads to the same covariant chain terms in all orders. The mentioned difference is due to the fact that the explicit shift formula for the covariant chain terms that was given in [13] is incorrect and does not reproduce the algebraic results of that paper.

2 Covariant chain terms a la Ekstrand and Tsutsui

As already mentioned, the (non-integrated) consistent anomaly, Schwinger term and higher chain terms are given by the expansion in powers of the ghost of the shifted Chern–Simons density

\[
\Omega_{2n-1}(A + v, F) = \Omega_{2n-1}(A + v, dA + A^2)
= \Omega_{2n-1}(A + v, (d + \delta)(A + v) + (A + v)^2)
\]  

(2)
where $\delta$ is the BRS operator

$$
\delta A = -Dv, \quad \delta F = [F, v], \quad \delta v = -v^2
$$

(3)

(all commutators are graded w.r.t. form and ghost degree), and the Russian formula $\hat{\delta} \equiv A + v$ has been used. It was observed in [13] that it is precisely the occurrence of the BRS operator $\delta$ in (2) that makes the resulting expressions for the anomaly and Schwinger term non-covariant. Therefore, it was proposed in [13] that a similar expression for covariant chain terms may be found by simply dropping $\delta$ in (2). Further, it was proven that this proposition is correct, i.e., that the expression

$$
\Omega_{2n-1}(A, F) \equiv \bar{\Omega}_{2n-1}(A, dA)
$$

(5)

has linear and quadratic (in $v$) contributions that agree with the covariant anomaly and Schwinger term as computed by other methods.

As we want to relate this result (6) to the cohomological computations of [13], we should briefly review the latter. In [13] the following two even (w.r.t. their total grading) operators $m, l$ are introduced

$$
mA = v, \quad mF = 0
$$

$$
mv = 0, \quad mdv = -v^2
$$

$$
lA = 0, \quad lF = -Dv
$$

$$
lv = 0, \quad ldv = -v^2.
$$

(7)

Both $m$ and $l$ act algebraically on formal polynomials of $A, F, v$ and $dv$. The consistent chain terms may be recovered by the action of $m$ alone. Indeed,

$$
\Omega_{2n-1-k}^k(v, A, F) = \frac{1}{k!} m^k \Omega_{2n-1}(A, F).
$$

(9)

On the other hand, both $m$ and $l$ are needed for the computation of the covariant chain terms. It was proved in [13] that the covariant chain terms will indeed be covariant provided they are computed as

$$
\bar{\Omega}_{2n-1-k}^k(v, dv, A, F) = \frac{1}{k!} (m - l)^k \Omega_{2n-1}(A, F).
$$

(10)
As a shift formula that should incorporate all these covariant chain terms, the following (incorrect) expression was given in [13]

$$\Omega_{2n-1}(A + v, F + Dv).$$

(11)

This expression does not reproduce (10) because $m$ and $l$ do not commute. More precisely, the problem is that the algebraic restrictions of $m$ and $l$ to derivations on $(A, F)$ (these restrictions commute with $v$ and $dv$) do not commute which each other, because only these restrictions are relevant for the difference $m - l$. Even if $m$ and $l$ are changed w.r.t. their action on $(v, dv)$ so that they commute which each other as derivations on $(A, F, v, dv)$, which may be achieved by choosing $mdv = ldv = -2v^2$ instead of $-v^2$ like in (7), (8), the shift formula (11) would be wrong and the formula (12) below would be correct.

The correct shift formula that takes this non-commutativity into account and reproduces (10), is

$$\exp(v \frac{\delta}{\delta A} + Dv \frac{\delta}{\delta F})\Omega_{2n-1}(A, F)$$

(12)

as may be checked easily (see also [17], journal version). Here the exponential is defined as its power series, and the derivative is understood in an algebraic sense, e.g., $v(\delta/\delta A)P(A) = P(A + v)|_{v^1}$ (i.e., the component of $P(A + v)$ linear in $v$). This formula is equal to expression (6) that was proposed in [19], as may be proved easily. Indeed,

$$e^{v \frac{\delta}{\delta A} + Dv \frac{\delta}{\delta F}} \Omega_{2n-1}(A, F) =$$

$$e^{(Dv + v^2) \frac{\delta}{\delta F}} e^{v \frac{\delta}{\delta A}} \Omega_{2n-1}(A, F) =$$

$$\Omega_{2n-1}(A + v, F + Dv + v^2) =$$

$$\Omega_{2n-1}(A + v, d(A + v) + (A + v)^2)$$

(13)

where we used the Baker–Campbell–Hausdorff formula in the first step.

3 Summary

We have shown that the “shift formula” (6) that was proposed in [19] for the generation of covariant chain terms and the cohomological analysis of [13] lead to identical results for all chain terms. The discrepancy that was mentioned in [19] is simply due to an incorrect expression for the shift formula in [13], which is not supported by the (correct) cohomological computation of the same paper.

Our finding further points towards the correctness of the shift formula proposed in [19] for all covariant chain terms. It should be mentioned at this point, however, that a physical interpretation of the higher covariant chain terms has not yet been found.
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