A study on determination of optimum gear ratios of a worm-helical gearbox

Vu Ngoc Pi¹, Tran Thi Hong², Le Xuan Hung¹, Luu Anh Tung¹, Nguyen Khac Tuan³

¹Mechanical Engineering Faculty, Thai Nguyen University of Technology, Thai Nguyen city, Vietnam
²Nguyen Tat Thanh University, Ho Chi Minh city, Vietnam
³Automotive and Power Machinery Faculty, Thai Nguyen University of Technology, Thai Nguyen city, Vietnam

vungocpi@tnut.edu.vn

Abstract. This paper introduces a new study on the optimum calculation of the gear ratios of a worm-helical gearbox. For finding the gear ratios, the cross section area of the gearbox was chosen as the objective function of the optimization problem. Moreover, the influence of the input parameters including the total gearbox ratio, the allowable contact stress and the wheel face width coefficient of the helical gear set, the output torque was investigated. For evaluation of the influence of these parameters on the optimum gear ratios, an “experiment” was designed and a computer program was built for performing the “experiment”. In addition, models for determining the optimum gear ratios of the gearbox were proposed. Based on those models, the gear ratios can achieve high accuracy with simplification.

1. Introduction

To date, there have been many study to determine the gear ratios of worm gearboxes. For a two-stage worm gearbox, for getting the reasonable housing structure of the gearbox, the gear ratios can be determined as follow ([1],[2]):

\[ u_1 \approx u_2 \approx \left( u_h \right)^{1/2} \] (1)

Also, the optimum gear ratio can be determined from tabulated form which was found based the practical data [3]. In addition, it is reported that the optimum gear ratio of the second stage of a two-stage worm gearbox is \( u_2 \approx 30.97 \) when the objective is the gearbox structure reasonable [4].

For worm–helical gearboxes, the gear ratio of the helical gear set is calculated by the following equation [1]:

\[ u_2 = \left( 0.03 \div 0.06 \right) u_h \] (2)

In fact, the above formula was found from an actual experience and it is quite simple. Nevertheless, the use of it may lead to the gearbox structure unreasonable. For ensuring the effectiveness of oil lubrication for both stages of the gearbox, in [5] presented a graph for determining the gear ratio of the worm gear set from the total ratio of the gearbox (Figure 1). From the figure it is
clear that, the determination of the gear ratio of the worm gear $u_1$ is very complicated. Moreover, for determining $u_1$, the user must select the coefficient $c$ (with $c = 1.3$ to $1.6$) so it is very complicated and we cannot reach to the optimum values. To exclude these points, the optimum gear ratios of the gearbox with the same objective as in [5] can be calculated by the following model [6]:

$$u_2 = 6.86 \cdot \psi_{bw2}^{1/2}$$

(3)

Where, $\psi_{bw2}$ is the coefficient of wheel face width ($\psi_{bw2} = 0.3 \ldots 0.4$).

Figure 1. Transmission ratio of worm gear unit versus the total transmission ratio [5].

Also for worm–helical gearboxes, for getting the minimum gearbox length it was reported that the optimum gear ratio of the helical gear unit $u_2$ was the allowable maximum gear ratio of it ($u_{2\text{max}} = 8 \ldots 10$) [7].

From the above analysis, although there have been several studies on for determining the optimum gear ratios of the worm gearboxes, there is still lack of a study in which the objective is the minimum acreage of the cross section. This paper introduces a study for calculation of the optimum gear ratios of the worm-helical gearboxes with this objective.

2. Optimization problem

The cross section area of the gearbox is calculated by the following equation (Figure 2):

$$A = h \cdot L$$

(4)

In which,
\[ h = \max h_1 + \max h_2 \] (5)

Where, \( h_1 \) and \( h_2 \) are determined as (see Figure 2):

\[ h_1 = \max \left( \frac{d_{w11}}{2}; \frac{d_{w22}}{2} \right) \] (6)
\[ h_2 = \max \left( \frac{d_{w11}}{2} + a_{w1}; \frac{d_{w22}}{2} \right) \] (7)
\[ L = \frac{d_{w21}}{2} + a_{w2} + \frac{d_{w22}}{2} \] (8)

In the above equations, \( a_{w1} \) and \( a_{w2} \) are the center distances of the first and the second stages; \( d_{w11} \) and \( d_{w21} \) are the pitch diameters of the worm and the worm gear; \( d_{w22} \) is the pitch diameter of the driven gear of the helical gear unit. With the worm gear set, we have [8]:

\[ d_{w21} = m \cdot z_2 = 2 \cdot a_{w1} \cdot z_2 / (z_2 + q) \] (9)
\[ d_{w11} = m \cdot q = 2 \cdot a_{w1} \cdot q / (z_2 + q) \] (10)

In which, \( q \) is the worm pitch diameter coefficient and \( q \) can be determined as [8]:

\[ q = k_q \cdot z_2 \] (11)

In which, \( k_q \) is the coefficient for calculation of the worm pitch diameter coefficient: \( k_q = 0.25 \ldots 0.3 \) [8].

For the helical gear set, the driven diameter can be determined as [8]:

\[ d_{w22} = 2 \cdot a_{w2} \cdot u_2 / (u_2 + 1) \] (12)

From Equations (4) to (12), the following equation is given:

\[ A = f \left( a_{w1}, a_{w2}, k_q, u_2 \right) \] (13)

Thus, the optimization problem is defined as:

\[ \text{minimize } A = L \cdot h \] (14)

With the following constraints

\[ 8 \leq u_1 \leq 80 \] (15)
\[ 1 \leq u_2 \leq 9 \] (16)

From equations (13), (14) and (15), it is revealed that to solve the optimization problem it is required to determine the center distances of the worm gear set \( a_{w1} \) and the helical gear set \( a_{w2} \).

2.1. Determining the center distance of the worm gear set

The center distance of the worm gear set (nm) can be determined by [8]:

\[ a_{w1} = (z_2 + q) \left( \frac{170}{z_2 \cdot \left[ \sigma_H \right]} \right)^2 \cdot T_{21} \cdot K_H / q \] (16)

Substituting (11) into (16) we have:

\[ a_{w1} = \left( 1 + k_q \right) \left( \frac{170}{\left[ \sigma_H \right]} \right)^2 \cdot T_{21} \cdot K_H / k_q \] (17)
\[ a_{w1} = 60.168 \left( K_H \cdot \left[ T_{21} \right] / \left[ \sigma_H \right] \right) \] (18)

Where, \( T_{21} \) is the torque on the wheel (Nmm); With the worm-helical gearbox, \( T_{21} \) is calculated by:

\[ T_{21} = T_{m1} / \left( \eta_g \cdot \eta_b \cdot u_2 \right) \] (19)
In which, \( T_{out} \) is the output torque (Nmm); \( \eta_{tg} \) is the helical gear transmission efficiency (\( \eta_{tg} = 0.96 \ldots 0.98 \) [8]); \( \eta_b \) is the transmission efficiency of a pair of rolling bearing (\( \eta_b = 0.99 \ldots 0.995 \) [8]). Choosing \( \eta_{tg} = 0.97 \cdot \eta_t = 0.992 \) and substituting them into (19) gives:

\[
T_{21} = 1.056 \cdot \frac{T_{out}}{u_2}
\]

(20)

\( K_H \) is the load factor; \( K_H = 1.1 \ldots 1.3 \) [8]; Choosing \( K_H = 1.2 \) we have

\[
a_{w1} = 33.2084 \cdot \left(1 + k_q \right) \left( \frac{T_{out}}{k_q \cdot u_2 \cdot \left[ \sigma_H \right]^2} \right)^{1/3}
\]

(21)

Wherein, \( \left[ \sigma_H \right] \) is the allowable contact stress (N/mm\(^2\)); \( \left[ \sigma_H \right] \) depends on the wheel material. If the wheel material is tinless bronze or soft grey iron, from the tabulated data in [8], the following regression equation was found (with the determination coefficient \( R^2 = 0.9906 \) ) for determination of the allowable contact stress:

\[
\left[ \sigma_H \right] = 5.0515 \cdot v_{sl}^2 - 49.742 \cdot v_{sl} + 189.9
\]

(22)

In which, \( v_{sl} \) is the slip velocity; \( v_{sl} \) can be calculated as [8]:

\[
v_{sl} = 0.0088 \cdot \left( P_i \cdot u \cdot n_t^2 \right)^{1/3}
\]

(23)

If the wheel material is tin bronze, \( \left[ \sigma_H \right] \) is determined by [8]:

\[
\left[ \sigma_H \right] = K_{HL} \cdot v_{sl} \cdot \left[ \sigma_{H0} \right]
\]

(24)

In which, \( \left[ \sigma_{H0} \right] \) is the allowable contact stress when the stress change cycle is \( 10^7 \) :

\[
\left[ \sigma_{H0} \right] = (0.7 \ldots 0.9) \cdot \sigma_t
\]

(25)

Where, \( \sigma_t \) is the tensile stress (N/mm\(^2\)); \( \sigma_t = 260 \) if \( v_{sl} = 5 \ldots 8 \); \( \sigma_t = 230 \) if \( v_{sl} = 8 \ldots 12 \) and \( \sigma_t = 285 \) if \( v_{sl} = 8 \ldots 25 \).

\( K_{HL} \) is the service life ratio; \( K_{HL} \) is determined by:

\[
K_{HL} = \left( 10^7 / N_{HE} \right)^{1/8}
\]

(26)

In which \( N_{HE} \) is the equivalent loading cycle number for the worm-wheel teeth for the whole lifetime of the gearing:

\[
N_{HE} = 60 \cdot n_2 \cdot t_s
\]

(27)

Where, \( t_s \) is the service lifetime of the gearing (h); \( n_2 \) is rotational frequencies of the wheel (rpm);

\[2.2. \text{Determining the center distance of the helical gear set}\]

The center distance of the second stage \( a_{w2} \) is calculated by [8]:

\[
a_{w2} = K_a \cdot \left( u_2 + 1 \right) \left( T_{12} \cdot k_{H,p} / \left[ \left( \left[ \sigma_H \right]^2 \cdot u_2 \cdot \psi_{bs2} \right) \right] \right)^{1/3}
\]

(28)

Similarly, for the second stage we have

\[
T_{uw} = T_{12} \cdot \eta_{tg} \cdot \eta_{be}^2 \cdot u_2
\]

(29)

Choosing \( \eta_{tg} = 0.97 \) and \( \eta_{be} = 0.992 \) as in section 2.1 gives

\[
T_{12} = 1.0476 \cdot \frac{T_{out}}{u_2}
\]

(30)

Substituting (30), \( k_a = 43 \) and \( k_{H,p} = 1.1 \) (as in section 2.1) into (28) gets:
\[ a_{u_2} = 45.0814 \cdot (u_2 + 1) \cdot \left( \frac{T_{\text{out}}}{\left[ \sigma_H \right]^2 \cdot u_2^2 \cdot \psi_{\text{ba}_2}} \right)^{1/3} \]  

2.3. Experimental work

To inspect the effect of the input parameters on the optimum gear ratios, an “experiment” was designed and performed. The experiment was designed with a 2-level full factorial design. Table 1 presents the input parameters which were selected for the investigation. Consequently, the design was organised with \(2^5 = 32\) number of experiments. To perform the experiment, a computer program was built based on equations (14) and (15). Table 2 shows different levels of the input parameters and the results of the output of the computer program (the optimum gear ratio of the worm gear set \(u_1\)).

| Factor                                      | Code | Unit | Low | High |
|---------------------------------------------|------|------|-----|------|
| Total gearbox ratio \(u_g\)                 |      |      | 50  | 300  |
| Coefficient for calculation \(q\)           | \(k_q\) |      | 0.25 | 0.3  |
| Coefficient of wheel face width of helical gear set \(\psi_{\text{ba}_2}\) | \(x_{\text{ba}_2}\) |      | 0.35 | 0.4  |
| Allowable contact stress of helical gear set \(\left[ \sigma_H \right]\) | AS  | MPa  | 350 | 410  |
| Output torque \(T_{\text{out}}\)            |      | N\text{mm} | 10^5 | 10^6 |

3. Optimization results and discussions

For imagining the influence of the input parameters on the response and for evaluating the relative strength of the effect, a graph of the main effect of each parameter is plotted in Figure 3. From the Figure, it is found that the optimum gear ratio of the worm gear set \(u_1\) increases significantly with the increase of the total gearbox ratio \(u_g\). Also, it is effected by coefficient \(k_q\), the allowable contact stress of the helical gear set AS, the coefficient of wheel face width of the helical gear set \(\psi_{\text{ba}_2}\) and the output torque \(T_{\text{out}}\).

| StdOrder | RunOrder | CenterPt | Blocks | \(u_g\) | Xba1 | Xba2 | AS (MPa) | Tout (N\text{m}) | \(u_1\) |
|----------|----------|----------|--------|--------|------|------|----------|-----------------|-------|
| 28       | 1        | 1        | 1      | 30     | 0.35 | 0.35 | 410      | 1000            | 9.53  |
| 14       | 2        | 1        | 1      | 30     | 0.3  | 0.4  | 410      | 100             | 8.66  |
| 13       | 3        | 1        | 1      | 5      | 0.3  | 0.4  | 410      | 100             | 2.62  |
| 32       | 4        | 1        | 1      | 30     | 0.35 | 0.4  | 410      | 1000            | 9.11  |
| 30       | 5        | 1        | 1      | 30     | 0.3  | 0.4  | 410      | 1000            | 8.66  |
| 6        | 6        | 1        | 1      | 30     | 0.3  | 0.4  | 350      | 100             | 8.66  |
| 3        | 7        | 1        | 1      | 5      | 0.35 | 0.35 | 350      | 100             | 2.88  |
| 27       | 8        | 1        | 1      | 5      | 0.35 | 0.35 | 410      | 1000            | 2.88  |
| ...      | 20       | 31       | 1      | 1      | 30   | 0.35 | 0.35 | 350     | 1000 | 9.53  |
| 8        | 32       | 1        | 1      | 30     | 0.35 | 0.4  | 350      | 100             | 9.11  |
Figure 3. Main effects plot for optimum gear ratio of the worm gear set.

Figure 4. Pareto Chart of the Standardized Effects.

Figure 4 presents the Pareto chart of the standardized effects from the largest effect to the smallest effect. It can be seen from this graph that the bars that represent parameters including the total gearbox ratio (factor A), the coefficient $k_q$ (factor B), the coefficient of wheel face width of the helical gear set (factors C), the allowable contact stress of the helical gear set (factor D), the output torque (factor E), and the interactions between them (AD, AE, AC, AB…) cross the reference line. Consequently, these factors are statistically significant at the 0.05 level with the response model.

Figure 5. Normal Plot for $u_1$. 
Figure 6. Estimated Effects and Coefficients for $u_1$.

As the Pareto chart cannot show which effects increase or decrease the response, the Normal Plot of the standardized effects (Figure 5) is used for that. From Figure 5, it was found that the total gearbox ratio is the most significant factor for the optimum gear ratio of the worm gear set. Furthermore, the total gearbox ratio (factor A), the coefficient $k_q$ (factor B) and the interaction AB have a positive standardized effect. If they increase, the optimum gear ratio of the worm gear set increases. Also, the coefficient of wheel face width of the helical gear set, the allowable contact stress of the helical gear set, the output torque and the interactions AC, AD and AE have a negative standardized effect. When they increase the optimum gear ratio of the worm gear set decreases.

Figure 6 describes the estimated effects and coefficients for $u_1$. It is found from the figure that parameters which have a significant effect on a response have P-values lower than 0.05 are the total gearbox ratio $u_g$, the coefficient $k_q$, the allowable contact stress of the helical gear set $\sigma_H$, the coefficient of wheel face width of the helical gear set $X_{ba}$, the output torque $T_{out}$ and their interactions (except $k_q \cdot X_{ba}$, $k_q \cdot AS$ and $k_q \cdot T_{out}$). Therefore, the relation between the optimum gear ratio of the worm gear set and the significant effect factors can be described as follows:

$$u_1 = 13.58 + 0.78021 \cdot u_g - 17.7 \cdot k_q - 26.87 \cdot \psi_{ba} - 0.02639 \cdot [\sigma_H] - 0.003985 \cdot T_{out} +$$
$$+ 0.1446 \cdot u_g \cdot k_q - 0.348 \cdot u_g \cdot \sigma_H - 0.000674 \cdot u_g \cdot [\sigma_H] - 0.000041 \cdot u_g \cdot T_{out} +$$
$$+ 26.3 \cdot k_q \cdot \psi_{ba} + 0.0208 \cdot k_q \cdot [\sigma_H] - 0.000011 \cdot k_q \cdot T_{out} + 0.457 \cdot \psi_{ba} \cdot [\sigma_H] +$$
$$+ 0.000418 \cdot \psi_{ba} \cdot T_{out} + 0.000006 \cdot [\sigma_H] \cdot T_{out}$$

(32)

From Figure 6, the high values of adj-R2 and pred-R2 indicate that the estimated model (Equation (32)) fit the experimental data very well. Equation (32) is used to calculate the gear ratio of the first stage $u_1$. After having $u_1$, the gear ratio of the helical gear set is determined by $u_2 = u_g / u_1$. 

| Term          | Effect  | Coef  | SE Coef | T-Value | P-Value | VIF |
|---------------|---------|-------|---------|---------|---------|-----|
| Constant      | 41.0717 | 0.0147|         | 2792.00 | 0.000  | 1.0 |
| $u_g$         | 41.0717 | 0.0147|         | 1396.00 | 0.000  | 1.0 |
| $k_q$         | 0.7229  | 0.03615|        | 24.57   | 0.000  | 1.0 |
| $X_{ba}$      | -1.7402 | -0.8701|        | -59.15  | 0.000  | 1.0 |
| $A_S$         | -4.0468 | -2.0234|        | -137.55 | 0.000  | 1.0 |
| $T_{out}$     | -3.6712 | -1.8356|        | -124.78 | 0.000  | 1.0 |
| $u_g \cdot k_q$ | 0.3615  | 0.1807|         | 12.29   | 0.000  | 1.0 |
| $u_g \cdot X_{ba}$ | -0.8701 | -0.4351|        | -29.57  | 0.000  | 1.0 |
| $u_g \cdot A_S$ | -2.0234 | -1.0117|        | -68.77  | 0.000  | 1.0 |
| $u_g \cdot T_{out}$ | -1.8356 | -0.9178|        | -62.39  | 0.000  | 1.0 |
| $k_q \cdot X_{ba}$ | 0.0328  | 0.0164|         | 1.12    | 0.281  | 1.0 |
| $k_q \cdot A_S$ | 0.0312  | 0.0156|         | 1.06    | 0.305  | 1.0 |
| $k_q \cdot T_{out}$ | -0.0025 | -0.0013|        | -0.09   | 0.933  | 1.0 |
| $X_{ba} \cdot A_S$ | 0.0685  | 0.0343|         | 2.33    | 0.033  | 1.0 |
| $X_{ba} \cdot T_{out}$ | 0.0940  | 0.0470|         | 3.19    | 0.006  | 1.0 |
| $A_S \cdot T_{out}$ | 0.1741  | 0.0870|         | 5.92    | 0.000  | 1.0 |

Model Summary

| S | R-sq | R-sq(adj) | R-sq(pred) |
|---|------|-----------|------------|
| 0.0832151 | 100.00%  | 100.00%  | 100.00%   |
4. Conclusions
A study on determination of the optimum gear ratios of a worm helical gearbox in order to get the minimum cross-sectional acreage of the gearbox was carried out. In the study, the effect of the input factors including the total gearbox ratio, the allowable contact stress and the wheel face width coefficient of the helical gear set, the output torque was analysed. Also, a model for calculation of the optimum gear ratio of the worm gear set for getting the minimum cross-sectional acreage of the gearbox was proposed. The gear ratios can be calculated simply by using the explicit model.

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