Nonlinear sigma models arise naturally in globally or locally supersymmetric field theories. The scalar fields which parametrize the sigma model manifold either arise from matter multiplets or they are part of a supergravity multiplet. In either case, once coupled to supergravity, scalar fields always seem to form a sigma model manifold, thereby lending themselves to a geometrical treatment. It is this geometrical aspect, together with the attendant symmetries of the system, which makes it possible to control what otherwise might be very complicated and nonlinear structure of the couplings of scalar fields to supergravity.

It is important to understand the structure of the supergravity theories in presence of scalar field couplings in all possible dimensions since the global and local symmetries which govern their structure turn out to be very powerful in capturing various important properties of a deeper underlying theory, such as M-theory. For example, the hidden symmetries of supergravity theory, which are intimately related to the structure of the sigma model sectors involved, shed light on the all important duality symmetries of M-theory.

It is impossible to list all possible sigma model manifolds that can arise in supergravity theories. However, it is useful to give a list of a large class of such manifolds,
at least to give a flavor of what is involved. In dimensions \( D > 4 \) the sigma model manifolds that are known to arise form almost a manageable list. To begin with they are primarily \( G/H \) coset spaces of one kind or another. Typically a subgroup \( K \) of the group \( G \) can be and has been gauged. A probably incomplete but rather extensive list of coset spaces and gauge groups involved in \( D > 4 \) supergravities is provided in Table 1, in the Appendix. The list gets more involved in \( D \leq 4 \) and we will not attempt to construct it here. Let us emphasize, however, that not in all \( D \leq 4 \) are the sigma model manifolds necessarily coset manifolds. For example, in \( N = 1 \) supergravity coupled to scalar multiplets in \( D = 2 \), it is well known that the scalar manifold can be an arbitrary riemannian manifold. What makes it difficult to review thoroughly all possible sigma models that can arise in various supersymmetric field theories is that supersymmetry imposes elaborate set of geometrical conditions on the geometry of the sigma model manifolds depending on the dimensions and number of supersymmetries involved. These conditions need to be analyzed case by case and they often require the existence of certain structures on the sigma model manifolds, the most typical of which are the complex structures. One pattern is clear though: as the number of supersymmetries increases, the constraints on the sigma model manifold become more and more stringent.

Most of the supergravity theories and their gaugings listed in Table 1 have already been constructed. For a detailed review, we refer the reader to [2]. Some are still to be constructed. For example, gauged \( D = 9 \) supergravities and (gauged) supergravities coupled to \( n \) Maxwell fields in \( N = (1,1), D = 6 \) and \( N = 2, D = 5 \) seem not to have been constructed, but the expected symmetries are nonetheless listed in Table 1. Gauging of the sigma models associated with the scalars of tensor multiplets do not seem to be possible because one does not know how to construct gauge covariant field strengths for tensor fields which are in non-singlet representations of the gauge group.

Sigma model manifolds consisting of a real line \( \mathbb{R} \) are not listed in Table 1. The pure \((1,1)\) supergravity in \( D = 6 \), for example has one scalar field. This theory also contains six vector fields of which four can be used to gauge an \( Sp(1) \times U(1) \) subgroup of the automorphism group. This example shows that gauging automorphism groups of a supergravity theory does not necessarily involve nonlinear sigma model sectors. Other examples of this phenomenon arise in \( N = 1,2 \) supergravities in \( D = 5 \). What is also not listed in Table 1 is the noncompact versions of \( K \) that can be gauged. The significance of the gauge group \( K' \) shown in Table 1 will be explained in section 9, when we consider the potential in gauged \( N = (1,0) \) supergravity in \( D = 6 \).

In contrast to other applications of sigma models in field theory, a characteristic property of gauged supergravity theories with scalar fields is that they have potentials in their Lagrangians. This is a consequence of the Noether procedure required to establish local supersymmetry. Although the perturbative treatment of string theories do not tend to produce gauged supergravity theories, they do arise as low
energy limits of M theory in certain backgrounds in a nonperturbative framework. In this context, it is natural to investigate the brane solutions of gauged supergravity theories in diverse dimensions. In doing so, the potentials mentioned above play an important role. Motivated by this fact, we will elucidate the structure of a potential which arises typically in gauged supergravity theories and we will derive a general and simple formula for it. As an application, we will apply this formula to the gauged (1,0) supergravity in six dimensions and derive an explicit formula for its potential. In doing so, we also exhibit the relation between various formulations of the gauged sigma models that exist in the supergravity literature.

For completeness and in view of their possible applications in $D < 4$, we have also included a section on the gauged sigma models with Wess-Zumino terms. This section is primarily based on [3].

This paper contains the following sections:

1. Introduction
2. Minimal formulation
3. Lifted formulation and coupling of fermions
4. Gauging $K \subseteq G$
5. Introducing a gauge invariant potential
6. Adapted coordinates and $H$-gauge condition
7. Introducing a gauged Wess-Zumino term
8. Gauged sigma model on a bundle of frames
9. Gauged sigma model on $G/H$
10. The potential in $(1,0)$ supergravity $D = 6$
11. Appendix: Table of gauged supergravity theories in $D > 5$

2 Minimal Formulation

In its minimal form, a nonlinear sigma model is a theory of scalar fields described by the Lagrangian

$$\mathcal{L}_\varphi = -\frac{1}{2f^2}\sqrt{-\gamma} \gamma^{\mu\nu}\partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta g_{\alpha\beta}(\varphi)$$  \(1\)
where $\gamma_{\mu\nu}$ is the spacetime metric, $g_{\alpha\beta}$ is a function of the fields (but not their derivatives), and $f$ is a coupling constant, which will be set equal to 1 in the rest of the paper. This model can be interpreted geometrically by saying that the fields $\varphi^\alpha$ are coordinate representatives of a map

$$\varphi : M \rightarrow N .$$

and that $g_{\alpha\beta}$ is the metric on $N$ in the chosen coordinate system. In field theory, and in particular in supergravity, $M$ is interpreted as spacetime and $N$ as an internal space; in the theory of extended objects, $M$ is the worldsheet and $N$ is interpreted as spacetime. In the rest of the paper, we shall assume that $M$ is flat Minkowskian spacetime, for simplicity.

It is usually desirable for physical reasons to assume that the theory has global invariance under a symmetry group $G$. Throughout this paper $G$ will denote a Lie group, not necessarily compact; the Lie algebra of $G$ will be denoted $\mathfrak{L}(G)$. We assume that in $\mathfrak{L}(G)$ there is given an inner product, not necessarily $Ad(G)$-invariant, and $\{T_I\}$, with $I = 1, \ldots, \dim G$ will be an orthogonal basis in $\mathfrak{L}(G)$. When the generators $T_I$ are represented by matrices, we will assume that they are normalized so that $\text{Tr}(T_I T_J) = -\frac{1}{2} \delta_{IJ}$. The structure constants $f_{IJ}^K$ are defined by

$$[T_I, T_J] = f_{IJ}^K T_K;$$

if the inner product in $\mathfrak{L}(G)$ is $Ad(G)$-invariant, then the structure constants are totally antisymmetric (note that since the metric in $\mathfrak{L}(G)$ is $\delta_{IJ}$ the distinction between upper and lower indices is immaterial).

In the following the components of all tensor fields on $N$ will be referred to the natural bases $\{\partial_\alpha\}$ and $\{dy^\alpha\}$. The left action of $G$ on $N$ is generated by vector fields $K_I = K_I^\alpha \partial_\alpha$ which under Lie brackets form an algebra anti-isomorphic to $\mathfrak{L}(G)$:

$$\mathfrak{L}(G) : \quad \mathcal{L}_{K_I} K_J^\alpha = -f_{IJ}^L K_L^\alpha .$$

The reason for this minus sign is that conventionally $\mathcal{L}(G)$ is defined as the algebra of left-invariant vector fields on $G$. These vector fields generate the right action of $G$ on itself. The left action of $G$ on itself is generated by the right-invariant vector fields, whose algebra is anti-isomorphic to $\mathcal{L}(G)$. Every left action of $G$ will be generated by vector fields satisfying such an algebra.

For the action (1) to be invariant, we assume that the vectors $K_I$ are Killing vectors for the metric $g$, that is, if $\mathcal{L}_v$ denotes the Lie derivative along $v$, for
If $\Lambda$ is an element of $\mathcal{L}(G)$, the infinitesimal variation of the fields under global $G$ transformation is

\[ \delta_\Lambda \varphi^\alpha = -\Lambda^I K_I^\alpha (\varphi) , \quad \partial_\mu \Lambda^I = 0 . \] (6)

In general, acting on any function of the fields, $\delta_\Lambda = \Lambda^I K_I^\alpha \bar{\partial}_\alpha$. Such variations satisfy an algebra isomorphic to the abstract algebra $\mathcal{L}(G)$: $[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{[\Lambda_1, \Lambda_2]}$. Invariance of the action based on the Lagrangian (1) follows directly from using (5).

3 Lifted Formulation and Coupling of Fermions

As we shall discuss later, in order to couple the system to fermions it is sometimes necessary to use a different formulation of the theory, where there are more fields than physical degrees of freedom. Some of the fields (or functions thereof) are then gauge degrees of freedom. This is completely analogous to what happens in gravity, where the coupling to fermions requires the use of the tetrad formalism.

The most general geometrical setup is to imagine a space $\tilde{N}$ with a map

\[ \pi : \tilde{N} \to N , \] (7)

which is surjective. In the new formulation the basic variables will be fields $\bar{\varphi}^\alpha$, describing a map from spacetime into $\tilde{N}$. Given this map $\bar{\varphi}$, one can construct in a unique way a map $\varphi$ from spacetime into $N$ by composing $\bar{\varphi}$ with the projection $\pi$, and the Lagrangian must be constructed in such a way that it has the same value for any two maps $\bar{\varphi}$ that project onto the same map $\varphi$.

This setup is unnecessarily general and in physical applications it is usually assumed that the projection $\pi$ amounts to factoring out the right action of some group $H$ acting on $\tilde{N}$. In the following we assume this to be the case.

Given a map $\varphi : M \to N$, we say that a map

\[ \bar{\varphi} : M \to \tilde{N} , \] (8)

is a lift of $\varphi$ if $\pi(\bar{\varphi}(x)) = \varphi(x)$. If $\bar{\varphi}$ is a lift of $\varphi$, then also $\bar{\varphi}'$, defined by $\bar{\varphi}'(x) = (\bar{\varphi}(x))h(x)$ for some map $h : M \to H$, is a lift of $\varphi$. Therefore, the lifted nonlinear sigma model has a nontrivial gauge group.
In general there are topological obstructions to the existence of lifts. Here we will deal only with local properties and we shall assume that lifts exist.

Let \( f_{abc} \) be the structure constants of \( H \). We have a right action of \( H \) on \( \bar{N} \), generated by vector fields \( F_a = F_\alpha^a \partial_\alpha \) whose algebra is isomorphic to \( \mathcal{L}(H) \):

\[
\mathcal{L}(H) : \quad \mathcal{L}_{F_a} F_b^\gamma = f_{abc} F_c^\gamma .
\]  

Given an element \( \eta = \eta^a T_a \) of \( \mathcal{L}(H) \), the infinitesimal variation of the fields \( \bar{\varphi} \) is

\[
\delta_\eta \bar{\varphi}^\alpha = \eta^a F_\alpha^a (\bar{\varphi})
\]  

and we have \([\delta_{\eta_1}, \delta_{\eta_2}] = \delta_{[\eta_1, \eta_2]}\).

If there is a global invariance under \( G \), it must be realized also on the lifted fields with Killing vectors \( \bar{K}_I^\alpha \) satisfying the same algebra as the fields \( K_\alpha^I \):

\[
\delta_{\Lambda} \bar{\varphi}^\alpha = -\Lambda^I \bar{K}_I^\alpha(\bar{\varphi}) , \quad \partial_\mu \Lambda^I = 0 .
\]  

We must have \( T\pi(\bar{K}_I) = K_I \) and this is possible if the Killing vectors \( \bar{K}_I \) are \( H \)-invariant, i.e.

\[
\mathcal{L}_{F_a} \bar{K}_I^\beta = 0 .
\]  

In order to rewrite the Lagrangian in terms of the lifted fields, we need a new geometrical ingredient: a connection in the bundle \( \pi : \bar{N} \to N \). By this we mean the following. The tangent space \( T_p \bar{N} \) at \( p \in \bar{N} \) contains a subspace \( V_p = ker T\pi \), called the vertical subspace, which is tangent to the orbit of \( H \). One can take the vectors \( F_a \) as a basis in \( V_p \). There is, however, no preferred choice of a complementary subspace in \( T_p \bar{N} \). A connection is precisely the assignment at each point \( p \) of a “horizontal” subspace \( H_p \) such that \( H_p \oplus V_p = T_p \bar{N} \) and such that the distribution of these spaces is \( H \)-invariant: for any \( h \in H \), \( H_{ph} = H_p h \).

These horizontal spaces can be defined as the kernels of a \( \mathcal{L}(H) \)-valued one-form \( \omega \) called the connection form, with the properties that

\[
\omega^a_\alpha F_b^\alpha = \delta^a_b .
\]  

and

\[
\mathcal{L}_{F_a} \omega^b_\alpha = -f_{ac}^b \omega^c_\alpha .
\]  

In addition, the connection is assumed to be \( G \)-invariant:
\[ L_{\bar{K}} \omega^b_\alpha = 0. \]  

(15)

It follows from (13) that the \( G \) invariant tensors

\[ V^\alpha_\beta = F_a \bar{\alpha} \omega^a_\beta, \]  

(16)

\[ H^\alpha_\beta = \delta^\alpha_\beta - V^\alpha_\beta, \]  

(17)

are the vertical and horizontal projectors.

Next we define the covariant derivative of \( \bar{\varphi} \) by

\[ D_\mu \bar{\varphi}^\alpha = H^\alpha_\beta \partial_\mu \bar{\varphi}^\beta = \partial_\mu \bar{\varphi}^\alpha - B^a_\mu F^a_\bar{\alpha} (\bar{\varphi}), \]  

(18)

where

\[ B^a_\mu = \partial_\mu \bar{\varphi}^\beta \omega^a_\beta (\varphi) \]  

(19)

is a composite gauge potential which is inert under the global left \( G \) transformations, and transforms as a gauge field under the composite local right \( H \) transformations:

\[ \delta_\Lambda B^a_\mu = 0, \]  

\[ \delta_\eta B^a_\mu = \partial_\mu B^a_{\eta} + f^{a}_{bc} B^b_{\mu} \eta^c. \]  

(20)

This result, together with (14) implies that the covariant derivative \( D_\mu \bar{\varphi}^\alpha \) transform as

\[ \delta_\Lambda D_\mu \bar{\varphi}^\alpha = -\Lambda^I \partial_\bar{\beta} \bar{K}^I_\bar{\beta} D_\mu \bar{\varphi}^\beta, \]  

\[ \delta_\eta D_\mu \bar{\varphi}^\alpha = \eta^a \partial_\bar{\beta} F^a_\bar{\alpha} D_\mu \bar{\varphi}^\beta. \]  

(21)

(22)

Let \( \bar{g} = \bar{g}_{\alpha\beta} d\bar{y}^\alpha \otimes d\bar{y}^\beta \) be a left \( G \) and right \( H \) invariant metric on \( \bar{N} \), such that \( V \perp H \) and that given any vectors \((v, w)\) on \( N \), and the unique vectors \((\bar{v}, \bar{w})\) on \( \bar{N} \) which are horizontal and project to \((v, w)\), then the inner product of \( \bar{v} \) and \( \bar{w} \) relative to \( \bar{g} \) must be equal to the inner product of \( v \) with \( w \) relative to \( g \). The Lagrangian of the lifted nonlinear sigma model can then be written as
\[ \mathcal{L}_{\phi} = -\frac{1}{2} \bar{g}_{\alpha\beta}(\bar{\phi}) D^\mu \bar{\phi}^\alpha D_\mu \bar{\phi}^\beta . \] (23)

Because of its gauge invariance this Lagrangian depends really only on \( \phi \) and it can be seen using (18) that it coincides with the Lagrangian (1).

We are now ready to couple fermions to the scalar fields. In the standard way of doing this, one assumes that the fermions carry a representation \( \rho \) of the group \( H \), so that when \( \bar{\phi} \) undergoes (10), the fermion undergoes

\[ \delta_\eta \psi = -\rho(\eta) \psi . \] (24)

(The minus sign is necessary to ensure that these transformations satisfy \([\delta_{\eta_1}, \delta_{\eta_2}] \psi = \delta_{[\eta_1, \eta_2]} \psi \), in accordance with (10).)

At each point \( x \) in \( M \) the fermion is given by an equivalence class of pairs \( (\bar{\phi}(x), \psi(x)) \) under \( H \). Therefore the fermion field can be thought of as a section of a vector bundle associated to the pullback by \( \phi \) of the principal \( H \) bundle \( \bar{N} \to N \). Note therefore that one cannot define what the fermions are before having given a scalar field configuration. This is in analogy with gravity where one cannot define what the fermions are prior to having given a metric. Thus the configurations space of scalars and fermions is not a product of the scalar and fermion configuration spaces, but rather a fiber bundle over the scalar configuration space.

In the action, a natural coupling between scalars and fermions arises through the gauge covariant derivative of \( \psi \), which is defined by

\[ D_\mu \psi = \partial_\mu \psi + B_\mu^a T_a \psi . \] (25)

with \( B_\mu \) defined as in (19). The fermionic kinetic term

\[ \mathcal{L}_\psi = \frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi , \] (26)

is manifestly \( H \)-gauge invariant.

To summarize, the total Lagrangian in the lifted formulation is given by

\[ \mathcal{L}_0 = -\frac{1}{2} \bar{g}_{\alpha\beta}(\bar{\phi}) D^\mu \bar{\phi}^\alpha D_\mu \bar{\phi}^\beta + \frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi , \] (27)

where the covariant derivatives are defined in (18) and (25).
4 Gauging $K \subseteq G$

We consider now the gauging of any subgroup of $G$, denoted by $K$, with generators $T_i (i = 1, ..., \dim K)$. In particular, the group $K$ can be chosen to be $G$ or $H$. In addition to the fields $\varphi^\alpha$ we have a dynamical $\mathcal{L}(G)$-valued field $A_i^\mu = A^i_\mu T_i$. Under an infinitesimal local $K$-transformation we have

\begin{align}
\delta \Lambda \bar{\varphi}^\alpha &= -\Lambda^i(x) \bar{K}^\alpha_i, \\
\delta \Lambda A^i_\mu &= \partial_\mu \Lambda^i + g f^i_{jk} A^j_\mu \Lambda^k,
\end{align}

where $g$ is the gauge coupling constant which will be set equal to 1 in the rest of the paper. This definition is such that the algebra (4) is satisfied. A relative sign between the two terms on the right hand side of (29) can be absorbed by a redefinition of $A$. Note also that $A_i^\mu$ is $\eta$-invariant:

\begin{equation}
\delta \eta A_i^\mu = 0.
\end{equation}

Next we define the $G$-covariant derivative of the lifted fields as

\begin{equation}
\nabla_\mu \bar{\varphi}^\alpha = \partial_\mu \bar{\varphi}^\alpha + A_i^\mu A^i_\mu \bar{\varphi}^\alpha.
\end{equation}

Upon using (10), (12) and (30) one verifies that

\begin{align}
\delta \Lambda \nabla_\mu \bar{\varphi}^\alpha &= -\Lambda^i(x) \partial_\mu \bar{K}^\alpha_i \nabla_\mu \bar{\varphi}^\alpha, \\
\delta \eta \nabla_\mu \bar{\varphi}^\alpha &= \left( \partial_\mu \eta^a \right) F^\alpha_a + \eta^a \partial_\mu F^\alpha_a \nabla_\mu \bar{\varphi}^\alpha.
\end{align}

Using these transformation properties, and (14), one can check that a composite $H$ gauge field defined by

\begin{equation}
B^a_\mu = \nabla_\mu \bar{\varphi}^\alpha \omega^a_\alpha,
\end{equation}

transform under the local left $G$ and local right $H$ transformation as

\begin{align}
\delta \Lambda B^a_\mu &= 0, \\
\delta \eta B^a_\mu &= \partial_\mu \eta^a + f^a_{bc} B^b_\mu \eta^c,
\end{align}
where (14) and (13) have been used. It follows that if we define the expression

\[ D_{\mu} \bar{\varphi}^\alpha = \nabla_{\mu} \bar{\varphi}^\alpha - B_{\mu}^a F_{\alpha}^a(\bar{\varphi}), \]  

(37)

it transforms as

\[ \delta_{\Lambda} D_{\mu} \bar{\varphi}^\alpha = -\Lambda^I(x) \partial_\beta \tilde{K}^I_{\mu} \bar{D}_{\mu} \bar{\varphi}^\beta, \]  

(38)

\[ \delta_{\eta} D_{\mu} \bar{\varphi}^\alpha = \eta^a \partial_\beta F_{\alpha}^a D_{\mu} \bar{\varphi}^\beta, \]  

(39)

so it deserves to be called the bi-covariant derivative of the lifted field. One can now write the kinetic term in terms of the lifted fields. In particular, the covariant derivative of the fermions takes the form

\[ D_{\mu} \psi = \partial_{\mu} \psi + B_{\mu}^a T_a \psi. \]  

(40)

Thus, a lifted gauged sigma model can be characterized by the Lagrangian

\[ \mathcal{L} = -\frac{1}{2} \bar{g}_{\alpha \beta}(\bar{\varphi}) D_{\mu} \bar{\varphi}^\alpha D_{\mu} \bar{\varphi}^\beta + \frac{1}{2} \bar{\psi} \gamma^\mu D_{\mu} \psi, \]  

(41)

where the fermions \( \psi \) carry a given representation of the group \( H \).

5 Introducing a Gauge Invariant Potential

There is no unique way to construct a gauge invariant potential in the context of bosonic sigma models. In the case of supersymmetric sigma models, however, the requirement of supersymmetry is often powerful enough to determine uniquely the form of the potential. In particular, when one gauges the automorphism group of supergravity theories which either contain scalar fields or are coupled to matter multiplets which contain scalar fields, the Noether procedure typically results in a potential. The important building block for the potential arises in the process of computing the supersymmetric variation of the gravitino kinetic term

\[ \mathcal{L}_{\psi} = \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^\rho D_{\mu} \psi \]  

(42)

where \( e \) is the determinant of the vielbein on \( M \), and the covariant derivative contains, in addition to the Lorentz connection, a composite gauge field \( B_{\mu}^a T_a \) with \( T_a \) in the fundamental representation of the automorphism group \( H_{\text{Aut}} \). In any
supergravity theory the supersymmetric variation of the gravitino must take the form

\[ \delta \psi_\mu = D_\mu \epsilon + \cdots, \]

(43)

where \( \epsilon(x) \) is the local supersymmetry parameter. Thus, in the process of varying the kinetic term (42) under (43), one encounters the commutator term

\[ [D_\mu, D_\nu] \epsilon = \mathcal{G}_{\mu \nu} \epsilon, \]

(44)

where \( \mathcal{G}_{\mu \nu} = \mathcal{G}^a_{\mu \nu} T_a \) is the \( \mathcal{L}(H) \) valued curvature tensor of the composite connection:

\[ \mathcal{G}^a_{\mu \nu} := \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f_{bc}^a B^b_\mu B^c_\nu, \]

(45)

and \( T^a \) is in the fundamental representation of \( H_{\text{Aut}} \). From the definition (34) it is straightforward to compute:

\[ \mathcal{G}^a_{\mu \nu} = F^i_{\mu \nu} C^a_i, \]

(46)

where \( F^i_{\mu \nu} = F^i_{\mu \nu} T_i \) is the \( \mathcal{L}(K) \)-valued curvature of \( A_\mu \) and

\[ C^a_i = \tilde{K}^a_i \omega^a. \]

(47)

Thus, from (42), (43) and (44) we see that a term of the form

\[ e \bar{\psi}_\mu \gamma^{\mu \rho \sigma} T_a \epsilon F^i_{\rho \sigma} C^a_i, \]

(48)

has to be cancelled by supersymmetric variation of other terms. The time tested Noether procedure to establish local supersymmetry then quickly leads to Yukawa couplings involving the \( C \)-function and a potential of the form (see, for example, [8] for details of how exactly this works)

\[ \mathcal{L}_C = -e^{-\varphi} \text{tr} C_i C^i, \]

(49)

where \( \varphi \) is the dilaton coming from the tensor multiplet and

\[ C_i := C^a_i T_a. \]

(50)

Using (4), (15), (12) and (14) it is easy to verify that
\[
\mathcal{L}_{K_i} C_j^a = f_{ij}^k C_k^a , \\
\mathcal{L}_{F_i} C_i^b = f_{ca}^b C_i^c .
\]  

(51)

Therefore the Lagrangian (49) is local left \(G\) and local right \(H\) invariant.

It should be emphasized that although the Noether procedure mentioned above primarily arises in the context of sigma model manifolds that are coset spaces, we can nonetheless introduce the potential (19) for general sigma model manifolds, as it has all the right properties. This will be done in what follows, until we come back to the application of the general formalism to specific examples. Without loss of generality, we will also continue to consider Minkowskian spacetimes.

6 Adapted Coordinates and \(H\)-Gauge Condition

As we said in section 2, the choice of coordinates on \(N\) is completely arbitrary: different choices amount to field redefinitions and thus lead to the same physical results. Similarly, in a lifted formulation the coordinates on \(\tilde{N}\) are also completely arbitrary. In practice it is sometimes convenient to use this freedom to choose a coordinate system adapted to the bundle structure. This means choosing locally a diffeomorphism of \(\tilde{N}\) to \(N \times H\), and using coordinates \(y^a\) on \(N\) and \(y^{\hat{a}}\) on \(H\) as coordinates on \(\tilde{N}\). We will then divide the fields as

\[
\tilde{\varphi}^{\hat{a}} = (\varphi^a, \varphi^{\hat{a}}) .
\]

(52)

where, obviously, \(\tilde{\varphi}^a = \varphi^a\) and \(\varphi^{\hat{a}}(x)\) are the coordinate components of an abstract \(H\)-valued field \(h(x)\). In this coordinate system the fundamental vector fields have components

\[
(F_a^\alpha, F_{\hat{a}}^{\hat{a}}) = (0, L_{\hat{a}}) .
\]

(53)

where \(L\) are the left-invariant vector fields on \(H\). The connection at a point with coordinates \((y^a, h)\) can be represented as

\[
\omega = h^{-1} B(y^a) h + h^{-1} dh ,
\]

(54)

\(B^b(y^a)\) being the local gauge potential on \(N\) and \(h^{-1} dh\) the left-invariant Maurer-Cartan form on \(H\). Thus \(\omega\) has components

\[
\omega = \sum_{\hat{a}} B^b(y^a) L_{\hat{a}}^b + h^{-1} dh .
\]

(55)
\[ \omega^a_\alpha = Ad(h^{-1})^a_b B^b_\alpha(y) , \]
\[ \omega^a_\hat{\alpha} = L^a_\hat{\alpha}(y) . \]  
(55)

Using these components and the fact that
\[ L^a_\hat{\beta} \partial_\hat{\beta} Ad(h^{-1})^b_c = -f_{ad}^b Ad(h^{-1})^d_c , \]  
(56)

one can verify separately the \( \alpha \) and \( \hat{\alpha} \) components of (14).

Since \( \tilde{K}_I \) projects onto \( K_I \), there is a unique generator \( v_I = v^a_I T_a \) of \( H \) such that
\[ \tilde{K}_I = K_I + v^a_I F_a . \]  
(57)

In adapted coordinates, the components of \( \tilde{K}_I \) are therefore given by \( \tilde{K}^\hat{\alpha}_I = (K^\alpha_I , v_I^a L^\hat{\alpha}_a) \). The vector \( v_I^a \) depends both on \( y^\alpha \) and \( y^{\hat{\alpha}} \), as one gets from (12)
\[ L^\hat{\alpha}_a \partial_\hat{\alpha} v_I^c = -f_{ab}^c v_I^b . \]  
(58)

Using this relation one finds that the \( \hat{\alpha} \) component of (15) is identically satisfied, and that the \( \alpha \) component gives
\[ \mathcal{L}_{K_I} \omega^b_\alpha = (\partial_\alpha v_I^b + f^b_{\alpha\gamma} \omega^\gamma_{\hat{\alpha}} v_I^\alpha) . \]  
(59)

For \( h = 1 \) this gives the familiar statement that a connection is invariant if the Lie derivative of the gauge potential is an infinitesimal gauge transformation. The advantage of using adapted coordinates is that they provide a clean separation of the gauge degrees of freedom (the \( \varphi^{\hat{\alpha}} \)) from the physical degrees of freedom (the \( \varphi^\alpha \)). Once the gauge degrees of freedom have been thus isolated, one can choose a gauge by simply fixing the functions \( \varphi^{\hat{\alpha}} \). For example, we may choose
\[ \varphi^{\hat{\alpha}} = \varphi^{\hat{\alpha}}_0 , \]  
(60)

where \( \varphi^{\hat{\alpha}}_0 \) is a constant. In this gauge, the sum of the Lagrangian (41) and (49), where a subgroup \( K \) of the full isometry group \( G \) has been gauged, takes the form
\[ \mathcal{L} = -\frac{1}{2} g_{\alpha\beta}(\varphi) D^\alpha_\mu \varphi^\alpha \varphi^\beta + \frac{1}{2} \bar{\psi_\gamma}^{\mu} D_{\mu} \psi - \text{tr} \ C_\gamma C^\gamma , \]  
(61)

where we have suppressed the tensor multiplet dilaton and
D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + A^\alpha_{\mu i} K^i_\alpha , \quad (62)
C^a_i = K^a_\alpha B^\alpha_i + v^a_i , \quad (63)

and $B^a_\mu$ occurring in the covariant derivative \((40)\) takes the form

$$B^a_\mu = D_\mu \varphi^\alpha B^\alpha_a T_a + A^a_{\mu i} v^a_i T_a . \quad (64)$$

Note that \((62)\) is the $G$-covariant derivative of the (unlifted) field $\varphi$ and the first term in \((61)\) is just the gauged version of \((1)\). We recall that $v^a_i(\varphi)$ is a function of the scalars which is to be determined from \((57)\). In section 9 we will derive a general and simple formula for the $C$-function in the case $N = G/H$, without having to compute the exact form of $v^a_i$.

7 Introducing a Gauged Wess-Zumino Term

In addition to the kinetic term and a potential, nonlinear sigma models may also contain higher derivative terms, or terms that are linear in the time derivative. A term of the latter kind is of particular interest and is known as the Wess-Zumino term. Nonlinear sigma models with Wess-Zumino terms are known as the Wess-Zumino-Witten (WZW) models. There exists a vast literature on this subject. Here we shall only review a general action formula valid in arbitrary dimensional spacetime $M$ and for scalar fields taking their values in an arbitrary riemannian manifold $N$.

Let $M$ be $(p+1)$-dimensional, and let us define the following forms on $N$:

$$b = \frac{1}{(p+1)!} d\varphi^{\alpha_1} \cdots d\varphi^{\alpha_{p+1}} b_{\alpha_1 \cdots \alpha_{p+1}} , \quad H = db ,$$
$$v^{(r)}_{i_1 \cdots i_k} = \frac{1}{r!} d\varphi^{\alpha_1} \cdots d\varphi^{\alpha_r} v_{i_1 \cdots i_r \cdots \alpha_1 \cdots \alpha_r} = v^{(r)}_{(i_1 \cdots i_k)} ,$$
$$r = p, p-2, \ldots, \varepsilon , \quad 2k + r = p + 2 ,$$
$$\varepsilon = 0 \text{ for even } p , \quad \varepsilon = 1 \text{ for odd } p . \quad (65)$$

As in the previous sections, we assume that $N$ has the isometries generating the group $G$, and we gauge the $K$ subgroup of $G$ generated by the Killing vectors $K^i_\alpha$. We shall work in the adapted coordinate system described in the previous section.

Let $M$ be the boundary of a $(p+2)$-dimensional manifold $Y$, and let us define the following covariant pull-backs to $Y$ ( For the purposes of this section only, we will
adhere to the convention of \[3\] for the covariant derivative according to which the replacement \( A \to -A \) is to be made in (62):

\[
\tilde{H}^{(p+2)} = \frac{1}{(p+2)!} dx^{\mu_1} \cdots dx^{\mu_{p+2}} D_{\mu_1} \varphi^{\alpha_{1}} \cdots D_{\mu_{p+2}} \varphi^{\alpha_{p+2}} H_{\alpha_{1} \cdots \alpha_{p+2}} ,
\]

\[
\tilde{v}^{(r)}_{i_1 \cdots i_k} = \frac{1}{r!} dx^{\mu_1} \cdots dx^{\mu_r} D_{\mu_1} \varphi^{\alpha_{1}} \cdots D_{\mu_r} \varphi^{\alpha_{r}} v_{\alpha_{1} \cdots \alpha_{r}, i_1 \cdots i_k} .
\]

Provided that \( \tilde{H} \) and \( \tilde{v} \) satisfy certain conditions (see below) the gauged WZW action can be written as an integral over \( Y \) as follows:

\[
S_{GWZ} = \int_{Y} \left( \tilde{H}^{(p+2)} + \tilde{v}^{(p)} \right) F^i + \tilde{v}^{(p-2)} F^i F^j + \cdots + \tilde{v}^{(\epsilon)} F^{i_1 \cdots i_N} \right) ,
\]

\[
\equiv \int_{Y} \mathcal{L}^{(p+2)} , \quad N = \frac{1}{2} (p + 2 - \epsilon) .
\]

Each term in this action is separately gauge invariant provided that

\[
\mathcal{L}_{\mathcal{K}, H} = 0 ,
\]

\[
\mathcal{L}_{\mathcal{K}, v^{(r)}_{i_1 \cdots i_k}} - k f^i_{j(i_1} \ell v^{(r)}_{i_2 \cdots i_k)\ell} = 0 .
\]

The set of forms \( \tilde{v}^{(r)} \) are needed, however, so that the Lagrangian in (67) is closed. This property makes it possible to write the action on the boundary of \( Y \). Indeed, using the following identity, which is valid for any covariantly pulled-back form \( \tilde{\Omega} \),

\[
d\tilde{\Omega} = d\tilde{\Omega} - F^j \left( i_{K_j} \tilde{\Omega} \right) + A^j \left( \mathcal{L}_{\mathcal{K}, \tilde{\Omega}} \right) ,
\]

and using (68), one can show that

\[
d\mathcal{L}^{p+2} = 0 ,
\]

provided that the following additional conditions are satisfied:

\[
i_k H^{(p+2)} = dv^{(p)}_{k} ,
\]

\[
i_{(j} v^{(p)}_{i]} = dv^{(p-2)}_{ij} ,
\]
\[ i_{(k} v^{(p-2)}_{ij)} = dv^{(p-4)}_{ijk} , \]
\[ \vdots \]
\[ i_{(i_1 v^{(p+2)}_{i_2...i_N)}} = dv^{(e)}_{i_1...i_N} , \]

(71)

where we have used the notation \( i_{K_n} \equiv i_n \). Note that if \( H^{(p+2)} \) satisfies the property \( i_k H^{(p+2)} = 0 \), then the \( v \)-forms would not be necessary for gauging of the WZW model; one simply makes the replacement \( \partial_\mu \varphi \to D_\mu \varphi \) in that case.

In order to write the action as an integral over the \((p + 1)\) dimensional manifold \( M = \partial Y \), we need the variation of \( \mathcal{L}_{p+2} \) with respect to \( F^i \), which will be denoted by \( K_i \).

\[ K_i = \frac{\delta \mathcal{L}}{\delta F^i} . \]

(72)

Then, as shown in [3], the action on \( M \) can be written as

\[ S_{GWZ} = \int_M \int_0^1 dt A^i(tA) K_i(tA) , \]

(73)

where it is understood that the substitution \( A \to tA \) is to be made everywhere the gauge potential \( A \) occurs in the functional \( K \).

For \( p = 2 \), for example, the gauged Wess-Zumino action takes the form [3]

\[ S_{GWZ} = \int_M \left( b^{(3)} + A^i v^{(2)}_{i} + \frac{1}{2} A^i A^j v^{(1)}_{ij} - \frac{1}{6} A^i A^j A^k v^{(0)}_{ijk} + v^{(0)}_{ij} (A^i dA^j + \frac{1}{3} A^i f_{klj} A^k A^l) \right) , \]

(74)

where we have the earlier definitions

\[ i_{k} H^{(3)} = dv^{(2)}_{k} , \quad i_{(k} v^{(2)}_{j)} = v^{(0)}_{jk} , \]

(75)

as well as new ones defined in terms of these as

\[ dv^{(0)}_{ij} := v^{(1)}_{ij} , \quad i_{(k} v^{(1)}_{ij)} := v^{(0)}_{ijk} . \]

(76)

We conclude this section by noting that the gauged WZW model in arbitrary dimension with fundamental gauge fields on \( M \) is closely related to a general gauged
sigma model studied in the context of bosonic \( p \)-branes where the gauge fields are not fundamental vector fields on the worldvolume \( M \), but rather they are the target space background fields. The issue of gauge anomalies acquires different significance in these two cases.

8 Gauged Sigma Model on a Bundle of Frames

An example of a lifted sigma model encountered in supergravity is when \( N \) is any riemannian manifold and \( \tilde{N} = LN \) is the bundle of linear frames of \( N \). In this case \( H = GL(n) \), where \( n \) is the dimension of \( N \). The adapted coordinates in this case consist of coordinates on \( N \) and a matrix-valued field \( e^\alpha \beta \) representing a general basis on \( N \) (the index \( \hat{\alpha} \) in this case consists of the pair of indices \( (\alpha, \beta) \)). The gauge potential \( B^\alpha_\gamma \) is given by the components of the linear connection \( \Gamma^\gamma_\alpha \beta \), where \( \gamma \) is the form index and \( (\alpha, \beta) \) are the Lie algebra index. We take the connection to be the Levi-Civita connection of the metric \( g^\alpha \beta \). Since \( g \) is assumed to be invariant, also the corresponding linear connection is invariant. Under an infinitesimal isometry generated by \( K_\alpha \), the Levi-Civita connection transforms as

\[
\mathcal{L}_{K} \Gamma^\gamma_\alpha \beta = - (\partial_\gamma \partial_\beta K^\alpha_\gamma + \Gamma^\gamma_\alpha \delta \partial_\beta K^\delta_\gamma - \Gamma^\delta_\gamma \beta \partial_\delta K^\alpha_\gamma) .
\]

which is just (59) in the gauge \( h = 1 \) and with \( v^{\beta}_\gamma = \partial_\gamma K^\beta_\gamma \).

Consider the gauging of the \( K \) subgroup of the full isometry group \( G \). Under an infinitesimal isometry (28) a linear basis \( e^\alpha \beta \) transforms as

\[
\delta_L e^\alpha \beta = - \Lambda^i \partial_\gamma K^\alpha_i e^{\gamma} \beta .
\]

Therefore \( \tilde{K}^\alpha_\beta = \partial_\gamma K^\alpha_\gamma e^{\gamma} \beta \). From (63) we thus have

\[
\mathcal{D}_\mu e^\alpha \beta = \partial_\mu e^\alpha \beta + A^{\alpha}_\mu \partial_\gamma K^\alpha e^{\gamma} \beta .
\]

Under (28) we find

\[
\delta_L \mathcal{D}_\mu e^\alpha \beta = - \Lambda^i \partial_\gamma K^\alpha_i \mathcal{D}_\mu e^{\gamma} \beta + \Lambda^i \mathcal{D}_\mu \phi^\delta \partial_\delta \partial_\gamma K^\alpha e^{\gamma} \beta .
\]

which is in accordance with (32). For the composite gauge potential in adapted coordinates one finds

\[
B^\alpha_\beta = e^{-\alpha} \delta \mathcal{D}_\mu \phi^\gamma \Gamma^\gamma_\delta \phi e^{\delta} \beta + e^{-\alpha} \delta \mathcal{D}_\mu e^{\delta} \beta .
\]

The invariance of this potential under \( G \) follows by using (77), whereas under
\[ \delta \eta \varphi^\alpha = 0, \quad \delta \eta e^\alpha_\beta = \eta^\alpha \gamma e^\gamma_\beta, \quad (82) \]

one finds again (36).

The fermions carry a representation of the group \( GL(n) \) and under (82) transform as

\[ \delta \eta \psi^\alpha = -\eta^\alpha \beta \psi^\beta. \quad (83) \]

Under an infinitesimal isometry (28) with the attendant transformation (78) of the linear frames, one finds

\[ \delta \Lambda \psi^\alpha = -\Lambda^i \partial_\beta K^\alpha_i \psi^\beta. \quad (84) \]

Eq. (40) becomes

\[ D_\mu \psi^\alpha = \partial_\mu \psi^\alpha + D_\mu \varphi^\gamma e^{-1\alpha}\delta \Gamma^\gamma_\phi e^{\phi \lambda} \psi^\lambda + e^{-1\alpha}_\gamma \partial_\mu e^\gamma_\delta \psi^\delta. \quad (85) \]

Upon using the \( H \) gauge freedom one can choose \( e^\alpha_\beta = \delta^\alpha_\beta \), in which case

\[ D_\mu \psi^\alpha = \partial_\mu \psi^\alpha + D_\mu \varphi^\gamma \Gamma^\alpha_\beta \psi^\beta - A^i_\mu \partial_\beta K^\alpha_i \psi^\beta. \quad (86) \]

Taking into account obvious notational differences (a redefinition of \( \Lambda \) by a sign), this corresponds to the formula given in [7].

9 Gauged Sigma Models on \( G/H \)

The most frequently encountered sigma models are based on coset spaces. Let us assume therefore that \( N = G/H \), where the coset space \( G/H \) is reductive, i.e. there exists an \( Ad(H) \)-invariant subspace \( \mathcal{P} \) of \( \mathcal{L}(G) \) such that

\[ \mathcal{L}(G) = \mathcal{L}(H) \oplus \mathcal{P}. \quad (87) \]

The space \( \mathcal{P} \) can be identified with the tangent space to \( G/H \) at the coset \( eH \). Note that if the basis is chosen in such a way that \( \{T_a\} \) with \( a = 1, \ldots, \text{dim } H \) is a basis in \( \mathcal{L}(H) \) and \( \{T_r\} \) with \( r = 1, \ldots, \text{dim } G/H \) is a basis in \( \mathcal{P} \), then

\[ f_{ab}^r = 0 \quad ; \quad f_{ar}^b = 0. \quad (88) \]

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The group $G$ acts on $G/H$ from the left by $g(g'H) = (gg')H$. On the group we have left-invariant and right-invariant vector fields $L_I$ and $R_I$. They are chosen to agree at the identity: $L_I(e) = R_I(e)$, and they commute: $[L_I, R_I] = 0$. The vector fields $K_I$ generating the left action of $G$ are the projections of the right-invariant vector fields $R_I$ under the map $g \to gH$. We assume that the restriction to $P$ of the inner product in $L(G)$ is $\text{Ad}(H)$-invariant; via standard theorems, this gives rise to a $G$-invariant metric $g = g_{\alpha\beta} dy^\alpha \otimes dy^\beta$ on $G/H$.

In the lifted formulation, we have $\tilde{N} = G$, $\tilde{K}_I = R_I$ (the right-invariant vector fields on $G$) and $F_a = L_a$ (the left-invariant vector fields on $H$). For the invariant connection we take the $\mathcal{L}(H)$-component of the left-invariant Maurer-Cartan form $g^{-1} \partial_h g$ on $G$. This example illustrates the reason why the groups are chosen to act as they were. Traditionally one chooses to work with right cosets $gH$. This fixes the action of $H$ on $G$ to be from the right. The remaining action of $G$ on the coset space is from the left. It arises from the action of $G$ on itself from the left.

We shall now review a well-known way of writing sigma models in terms of matrices, and recast the earlier results in this formalism. Traditionally one works in a gauge-fixed version of the lifted formalism, the gauge fixing being given by a locally defined section $L : G/H \to G$. This section is just a choice of a coset representative for each coset. In addition, as usual when working with groups, it is very convenient to use matrix representations, so we also write $L(y)$ for the matrix representing the abstract group element $L(y)$. Under the action of a group element $g$, $y \to y'$ and

$$L(y') = gL(y)h^{-1},$$

where $h = h(g, y)$ is a compensating gauge transformation that restores the chosen gauge. Infinitesimally, if $g = 1 + \Lambda$, we can write $h = 1 + v$, where $v = v(y, \Lambda)$ is the matrix representing the Lie algebra element $v$ that was defined in (57). Inserting in (89) one gets the formula

$$K_i^\alpha \partial_\alpha L = T_iL - L\upsilon_i T_a,$$

which is just a matrix way of rewriting (57) (the right invariant vector field $R_I$ at $L(y)$ is represented by the matrix $T_iL(y)$ and so on).

The pull-back the Maurer-Cartan form by the section $L$ can be decomposed as

$$L^{-1} \partial_\alpha L = V^\alpha_T + B^\alpha_a T_a,$$

where $V^\alpha_T$ is the vielbein and $B^\alpha_a$ is a gauge potential on $G/H$. It is also convenient to define

$$L^{-1} \partial_\mu L = F^\mu_T + B^\mu_a T_a,$$

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where
\[ P^r_\mu = \partial_\mu \phi^\alpha V^r_\alpha, \quad B^a_\mu = \partial_\mu \phi^\alpha B^a_\alpha. \] (93)

It is easy to show that \( P^r_\mu \) transforms covariantly and \( B^a_\mu \) as a gauge field under the composite local \( H \)-transformations. Indeed, \( B^a_\mu \) coincides with (19) in adapted coordinates (see (53)).

The ungauged sigma model Lagrangian (27) can be written as
\[ L_0 = \frac{1}{2} \bar{\psi} \gamma^{\mu} \left( \partial_\mu + B^a_\mu T_a \right) \psi. \] (94)

Upon the gauging of a subgroup \( K \) of \( G \), the decomposition (92) has to be modified as follows
\[ L^{-1} \left( \partial_\mu + A^i_\mu T_i \right) L = P^r_\mu T_r + B^a_\mu T_a. \] (95)

We will now show that the \( H \)-connection \( B \) takes the form given earlier in (64) and that the quantity \( P^r_\mu \) can also be represented in terms of the covariant derivative of the scalars. To this end, we multiply (90) with \( L^{-1} \) from the left and use (91) to obtain
\[ K^\alpha_i V^r_\alpha = \left( L^{-1} T_i L \right)^r, \] (96)
\[ K^\alpha_i \omega^a_\alpha = \left( L^{-1} T_i L \right)^a - \psi^a_i. \] (97)

Using these relations in (95), we find
\[ P^r_\mu = D_\mu \phi^\alpha V^r_\alpha, \]
\[ B^a_\mu = D_\mu \phi^\alpha B^a_\alpha + A^i_\mu \psi^a_i. \] (98)

As a by product, we find that the expression for the \( C \)-function given in (63) can now be written as
\[ C^a_i = \left( L^{-1} T_i L \right)^a. \] (99)

In summary, the gauge invariant sigma model Lagrangian (61) can be written as
\[ \mathcal{L} = \frac{1}{2} P_{\mu r} P^{\mu r} + \bar{\psi} \gamma^{\mu} \left( \partial_\mu + B^a_\mu T_a \right) \psi - \text{tr} C^i C^r. \] (100)
As an application of the formula (99), in the next section we will compute the potential that arises in the gauged \((1,0)\) supergravity in six dimensions. The formula (99) can readily be applied also to a class of supergravity theories where \(N = G/H\) and the dimension of the gauge group \(K\) equals the dimension of the defining representation of \(G\). In these cases, the generator \(T_i\) occurring in (99) becomes a structure constant of the group \(K\).

10 The Potential in \((1,0)\) Supergravity in \(D = 6\)

The \(n\) copies of the hypermultiplets in this theory parametrize a noncompact quaternionic coset manifold. A generic example of such a manifold is

\[
\frac{G}{H} = \frac{Sp(n,1)}{Sp(n) \times Sp(1)},
\]

where \(Sp(1)\) is the automorphism group of the supersymmetry algebra. Thus the supersymmetry parameter \(\epsilon^A(x)\) carries the \(Sp(1)\) doublet index \(A = 1,2\). The group \(K = Sp(1) \times Sp(1) \subset Sp(n,1)\) has been gauged in [8]. The group \(K'\) in Table 1 refers to \(Sp(n)\) in this example. Let the index \(a' = 1,...,2n\) label the fundamental representation of this group. The index \(i = 1,...,\dim K\) splits into the symmetric pairs \((AB)\) and \((a'b')\), and we have the \(C\)-function matrices

\[
C^a_i \rightarrow (C^a_{AB}, C^a_{a'b'})
\]

\[
a = 1,2,3, \quad A = 1,2, \quad a' = 1,...,2n.
\]

From (101) we have

\[
C^a_{AB} = (L^{-1}T_{AB}L)^a, \quad C^a_{a'b'} = (L^{-1}T_{a'b'}L)^a,
\]

where \(T_{AB}\) and \(T_{a'b'}\) are the generators of \(Sp(1)\) and \(Sp(n)\), respectively. To compute the explicit form of these functions, let us choose the standard representation of the coset (101) as follows

\[
L = \exp \left( \begin{array}{cc} 0 & \varphi^{a'} A \\ (\varphi^{B'})^T & 0 \end{array} \right),
\]

where the scalar fields satisfy the following conditions

\[
\varphi^{a'B} = (\varphi^{b'B})^* = \Omega^{a'b'} \varphi^{a'} \epsilon_{AB} \quad (105)
\]

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The Ω and ε-tensors are constant antisymmetric invariant tensors of $Sp(n)$ and $Sp(1)$, respectively. Using matrix notation, we have

$$L = \begin{pmatrix} \cosh \sqrt{\varphi^\dagger \varphi} & \varphi \frac{\sinh \sqrt{\varphi^\dagger \varphi}}{\sqrt{\varphi^\dagger \varphi}} \\ \varphi^\dagger \frac{\sinh \sqrt{\varphi^\dagger \varphi}}{\sqrt{\varphi^\dagger \varphi}} & \cosh \sqrt{\varphi^\dagger \varphi} \end{pmatrix}, \quad (106)$$

where $\varphi$ represents an $n \times 2$ matrix. Note that $(\varphi^\dagger \varphi)^{AB}_{\phantom{AB}CD} = \varphi^A_{\phantom{a}c} \varphi^B_{\phantom{b}d} \delta_{c}^{\phantom{c}D} \delta_{d}^{\phantom{d}C}$ with $\varphi^2 := \text{tr} \varphi^\dagger \varphi$. We can map a pair of symmetric $Sp(1)$ indices to an $Sp(1)$ vector index through the relation $V^a = \frac{1}{2} (\sigma^a)^{AB} V^A_B$. Let us also recall the explicit form of the defining representations of $Sp(1)$ and $Sp(n)$:

$$(T_{AB})^{CD} = \varepsilon_{CB} \delta_{DA}^D + \varepsilon_{CA} \delta_{DB}^D , \quad (T_{a'b'})^{cd'} = \Omega^{cd'}_{cb} \delta_{a'a'}^a + \Omega^{cd'}_{ca} \delta_{b'b'}^b. \quad (107)$$

From (103), (106) and (107), we find

$$C_{AB} = \cosh^2 \sqrt{\varphi^2} \sigma_{AB}^2 \quad C_{a'b'} = \frac{2}{\varphi^2} \sinh^2 \sqrt{\varphi^2} (\varphi \sigma^a_{ab'})_{a'b'} \quad (108)$$

The total potential in the gauged $(1, 0)$ supergravity in $D = 6$ constructed in is the sum of the squares of these $C$-functions multiplied by an exponent of the dilaton field which comes from the tensor multiplet. The $C$-functions were also computed in but for a different choice of the coset representative $L$. This corresponds to a redefinition of the scalar fields and therefore the physical content is the same.

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| $D$ | $N$ | Scalar Manifold $G/H$ | Gauge Group $K \subseteq G$ | Matter Sector |
|-----|-----|----------------------|------------------------|--------------|
| 10  | (2,0) | $SU(1,1)/U(1)$ | — | — |
| 9   | 2   | $GL(2,R)/SO(2)$ | $SO(2)$ | — |
| 1   |     | $SO(n,1)/SO(n)$ | $\dim K \subseteq n + 1$ | $n$ Maxwell |
| 8   | 2   | $SL(3,R)/SO(3) \times SL(2,R)/SO(2)$ | $SO(3)$ | — |
| 1   |     | $SO(n,2)/SO(n) \times SO(2)$ | $\dim K \subseteq n + 2$ | $n$ Maxwell |
| 7   | 2   | $SL(5,R)/SO(5)$ | $SO(5)$ | — |
| 1   |     | $SO(n,3)/SO(n) \times SO(3)$ | $\dim K \subseteq n + 3$ | $n$ Maxwell |
| 6   | (2,2) | $SO(5,5)/SO(5) \times SO(5)$ | $SO(5)$ | — |
|     | (2,0) | $SO(n,5)/SO(n) \times SO(5)$ | — | $n$ Tensor |
|     | (1,1) | $SO(n,4)/SO(n) \times SO(4)$ | $\dim K \subseteq n + 4$ | $n$ Maxwell |
|     | (1,0) | Quaternionic Kahler | $Sp(1) \times K'$ | $n$ Hyper |
|     |     | $SO(n,1)/SO(n)$ | — | $n$ Tensor |
| 5   | 4   | $E_6/USp(8)$ | $SO(6)$ | — |
| 3   |     | $SU^*(6)/USp(6)$ | $SU(3) \times U(1)$ | — |
| 2   |     | $SO(n,5)/SO(n) \times SO(5)$ | $\dim K \subseteq n + 5$ | $n$ Maxwell |
|     | Quaternionic Kahler | $Sp(1) \times K'$ | $n$ Hyper |
| 1   |     | $SO(n - 1,1) \times SO(1,1)/SO(n - 1)$ | $\dim K \subseteq n$ | $n$ Maxwell |
|     | $E_{6(-26)}/F_4$ | $SU(3)$ | 25 Maxwell |
|     | $SU^*(6)/Sp(3)$ | $SU(3)$ | 13 Maxwell |
|     | $SL(3,C)/SU(3)$ | $SU(3)$ | 7 Maxwell |
|     | $SL(3,R)/SO(3)$ | $SO(2)$ | 4 Maxwell |

Table 1: Supergravities in $D \geq 4$ dimensions with $N$ supersymmetry and nontrivial sigma model sectors.
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