Spontaneous Scalarization in Scalar-Tensor Theories with Conformal Symmetry as an Attractor

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Motivated by constant-G theory, we introduce a one-parameter family of scalar-tensor theories as an extension of constant-G theory in which the conformal symmetry is a cosmological attractor. Since the model has the coupling function of negative curvature, we expect spontaneous scalarization occurs and the parameter is constrained by pulsar-timing measurements. Modeling neutron stars with realistic equation of states, we study the structure of neutron stars and calculate the effective scalar coupling with the neutron star in these theories. We find that within the parameter region where the observational constraints are satisfied, the effective scalar coupling almost coincides with that derived using the quadratic model with the same curvature. This indicates that the constraints obtained by the quadratic model will be used to limit the curvature of the coupling function universally in the future.

1. Introduction

The equivalence principle played the principal role in constructing general relativity (GR) by Einstein. The weak equivalence principle (WEP) states that the motion of a (uncharged) test body is independent of its internal structure and composition. WEP together with the local Lorentz invariance (independence of the results of nongravitational experiments from the velocity of the local Lorentz frame) and the local position invariance (the independence of the experimental results from the spacetime position) enables us to make the matter couple universally to gravity. Among various gravity theories, GR is the exception in that the equivalence principle is satisfied even for self-gravitating bodies: the strong equivalence principle (SEP).

In general scalar-tensor theories of gravity [1–3], the equation of motion of a self-gravitating massive body (at the first post-Newtonian approximation) depends on the inertial mass \( m_I \) and the (passive) gravitational mass \( m_G \) of the body. The ratio is given by

\[
\frac{m_G}{m_I} = 1 + \eta \frac{\Omega}{m_I}
\]

where \( \Omega \) is the gravitational self-energy of the body [4]. This implies that the motion of a massive body depends on its internal structure, being in violation of the SEP. The coefficient in front of the gravitational self-energy is \( \eta = 4\beta - \gamma - 3 \), where \( \beta \) and \( \gamma \) are the parametrized
post-Newtonian (PPN) parameters. For \( \eta \neq 0 \), the orbit of the Moon around the Earth is elongated toward the Sun (Nordtvedt effect [5]).

In GR, \( \beta = \gamma = 1 \) so that the equation of motion solely depends on the inertial mass and the velocity of its center of mass. However, even in scalar-tensor gravity, one may construct a theory with \( \eta = 0 \) so that SEP holds. The “constant-G” theory by Barker [7] is such an example. In fact, as shown by [8], it is the unique scalar-tensor theory which satisfies the SEP (at the first post-Newtonian approximation). Even more interestingly, the theory is cosmologically attracted toward the conformally symmetry [9] where the theory is scale-invariant. The conformal symmetry (and its spontaneous breaking) has been actively studied in constructing models of inflation [10, 11] and in constructing geodesically complete cosmologies [12, 13]. Here the conformal symmetry will be restored in the future. Motivated by this remarkable property of constant-G theory, we introduce a one-parameter family of scalar-tensor theories which exhibits the cosmological attraction toward conformal symmetry.

Since the curvature of the coupling function of these theories is negative, the scalar field experiences a tachyonic instability inside a compact star like a neutron star and the scalar field exhibits large deviation from its asymptotic value, a phenomenon so-called “spontaneous scalarization” [14], which is constrained by pulsar-timing observations.

In order to put constraints on the scalar-tensor theories from the binary pulsars, the dependence of the scalarization both on the equation of state (EOS) of a neutron star and on the coupling function (the function which determines the strength of the coupling to matter in the Einstein frame) must be taken into account. The dependence of the scalarization on the EOSs is studied in [15–20], and it is found that the threshold value of the scalarization is insensitive to EOS. Moreover, Ref.[19] find that there are several observational windows for the scalarization depending on the EOSs. However, the dependence on the coupling function has not been much studied and usually the quadratic model [14] is used. [21] studied binary pulsar constraints on two scalar-tensor theories (the quadratic model [14] and MO model [22]) and found that the constraints in the two theories are roughly the same.

We extend these previous studies by using our coupling function in light of new and updated pulsar data. We find that the window at \( \sim 1.7M_\odot \) is almost closed even if the dependence of the EOSs is taken into account. The coupling function used in this study deviates from a quadratic function with the same curvature for a large scalar field. However, we find that the deviation is small within the parameter region where the observational constraints are satisfied. Therefore, as far as the observational constraints on spontaneous scalarization are concerned, it is sufficient to employ the quadratic function for the coupling function.

The paper is organized as follows. In Sec. 2, we review the properties of constant-G theory and introduce the conformal attractor model, a one-parameter family of scalar-tensor theories with the cosmological attraction toward the conformal symmetry. In Sec. 3, we study the structure of neutron stars in these theories and calculate the masses of neutron stars and the effective scalar coupling with the neutron star using three EOSs and compare them with the observational constraints. We also calculate the effective scalar coupling for the quadratic model. Sec. 4 is devoted to summary. We use the units of \( c = 1 \).

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\(^1\) From the lunar-laser-ranging experiment, \( \eta \) is constrained as \( \eta = (-0.2 \pm 1.1) \times 10^{-4} \) [6].
2. Cosmological Conformal Attractor

2.1. \( \eta \) and Constant-\( G \) Theory

We consider scalar-tensor theories of gravity whose action in the Jordan frame is given by

\[
S = \int d^4x \sqrt{-g} \frac{1}{16\pi} \left( \Phi R - \frac{\omega(\Phi)}{\Phi} (\nabla\Phi)^2 \right) + S_M(\psi, g_{\mu\nu}),
\]

where \( \Phi \) is the so-called Brans-Dicke scalar field, the inverse of which plays the role of the effective gravitational “constant”, \( \omega(\Phi) \) is the Brans-Dicke function which determines the strength of the coupling of the scalar field to gravity (and matter), \( S_M \) is the matter action and \( \psi \) denotes the matter field.

We note that for \( \omega = -3/2 \), the gravity part of the action is locally conformal invariant under the following Weyl scaling:

\[
g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu}, \quad \Phi \rightarrow e^{-2\sigma(x)} \Phi.
\]

From the post-Newtonian expansion of the theory, the effective gravitational constant \( G \) and the parametrized post-Newtonian (PPN) parameters \( \gamma \) and \( \beta \) of scalar-tensor theories are given by [4]

\[
G = \left. \frac{2\omega(\Phi)}{\Phi} \frac{d\omega}{d\Phi} \right|_{\Phi_0}
\]

\[
\gamma = \frac{\omega + 1}{\omega + 2},
\]

\[
\beta = 1 + \Phi \frac{d\omega}{d\Phi} \frac{1}{4(2\omega + 3)(\omega + 2)^2}.
\]

The equation of motion of a massive body is given in [4, 8]. It contains terms which depend on the gravitational self-energy, the coefficient of which is \( \eta = 4\beta - \gamma - 3 \). For \( \eta = 0 \), the motion of massive bodies does not depend on their internal structure and hence respects SEP.

Remarkably, \( \eta \) and \( G \) are related by (see [8] for a similar relation)

\[
\eta = (\gamma - 1) \frac{d\ln G}{d\ln \Phi}.
\]

Therefore, \( \eta = 0 \), for which the theory respects the SEP, implies either \( \gamma = 1 \) (GR) or \( G = \text{const} \). The latter is known as “constant-\( G \)” theory found by Barker [7] in which \( \omega(\Phi) \) is given by

\[
\omega(\Phi) = -\frac{3}{2} + \frac{1}{2G\Phi - 2}
\]

Note that \( |\omega| \rightarrow \infty \) (GR) for \( G\Phi \rightarrow 1 \) and that \( \omega \rightarrow -3/2 \) (conformal symmetry) for \( G|\Phi| \rightarrow \infty \). The constraint on \( \gamma \) from the measurement of the time delay of the Cassini spacecraft is \( \gamma - 1 = (2.1 \pm 2.3) \times 10^{-5} \) [23], which is satisfied for \( G\Phi - 1 = (-1.1 \pm 1.2) \times 10^{-6} \) at present.

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\(^2\)The action is still conformal invariant even if one includes a potential proportional to \( \Phi^2 \). However, the equation of the motion of the scalar field is not affected by such a potential as discussed in 2.2.

\(^3\)We do not use the units of the present-day gravitational constant \( G = 1 \) at this stage because we are interested in the cosmological evolution of \( \Phi \).
2.2. Cosmological Evolution

In order to study the dynamics of $\Phi$, it is useful to perform the following change of variables and moving to the so-called Einstein frame [24, 25] in which $\Phi$ is decoupled from the gravity sector:

$$g_{\mu\nu} = \frac{1}{G_* \Phi} \bar{g}_{\mu\nu} \equiv e^{2a(\varphi)} \bar{g}_{\mu\nu}$$

$$\frac{1}{2\omega + 3} = \frac{1}{4\pi G_*} \left( \frac{da(\varphi)}{d\varphi} \right)^2 \equiv \frac{1}{4\pi G_*} \alpha(\varphi)^2$$

where $G_*$ is the bare gravitational constant. The action (2) can be rewritten in terms of $\bar{g}_{\mu\nu}$ whose kinetic term is of Einstein-Hilbert form and a canonically normalized scalar field $\varphi$ as

$$S = \int d^4x \sqrt{-\bar{g}} \left( \frac{\bar{R}}{16\pi G_*} - \frac{1}{2} \left( \nabla \varphi \right)^2 \right) + S_M(\psi, e^{2a(\varphi)\bar{g}_{\mu\nu}})$$

From Eq. (11), the equation of motion of $\bar{g}_{\mu\nu}$ and $\varphi$ are given by

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi G_* \left( T_{\mu\nu} + \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \bar{g}_{\mu\nu} \left( \nabla \varphi \right)^2 \right)$$

$$\Box \varphi = -\alpha(\varphi)T$$

where $T_{\mu\nu} = -(2/\sqrt{-\bar{g}})\delta S_M/\delta \bar{g}^{\mu\nu}$ is the energy-momentum tensor in the Einstein frame. From Eq. (13), we find that $\varphi$ moves according to the effective potential $-a(\varphi)T$. The effective gravitational constant $G$ and the PPN parameter $\gamma$ are written as

$$G = G_* e^{2a(\varphi)} \left( 1 + 2\alpha(\varphi)^2/\kappa^2 \right) \bigg|_{\varphi_0}$$

$$\gamma - 1 = -\frac{4\alpha(\varphi)^2/\kappa^2}{2\alpha(\varphi)^2/\kappa^2 + 1} \bigg|_{\varphi_0}$$

where $\kappa = \sqrt{8\pi G_*}$ and $\varphi_0$ denotes the value of $\varphi$ at spatial infinity.

For constant-$G$ theory with $\omega(\Phi)$ in Eq. (8), from Eq. (9) and Eq. (10), $\varphi$ and $a(\varphi)$ are given by

$$\kappa \varphi = \sqrt{2} \tan^{-1} \sqrt{G\Phi - 1}$$

$$a(\varphi) = \ln \left( \cos \left( \kappa \varphi / \sqrt{2} \right) \right)$$

where $G\Phi \geq 1$ is assumed (cosh for $G\Phi < 1$) and $G_*$ coincides with $G$. We have fixed the integration constant so that $G\Phi = 1$ corresponds to $\kappa \varphi = 0$. The conformal symmetry ($G\Phi \to \infty$) now corresponds to $\kappa \varphi \to \pi/\sqrt{2}$. In Fig. 1, $a(\varphi)$ is shown. The constraint by the Cassini experiment is satisfied for $\kappa \varphi = (-2.6 \pm 4.0) \times 10^{-3}$ at present. From Eq. (13), one may see that $\varphi$ moves toward $\kappa \varphi \to \pi/\sqrt{2}$ according to the effective potential $-a(\varphi)T$ (see Fig. 1) during the matter-dominated epoch and during the dark energy-dominated epoch: The theory is thus cosmologically attracted toward the conformal symmetry [9].

\footnote{Note that our $\varphi$ is related to the scalar field in [8, 14] $\varphi_{\text{DEF}}$ via $\varphi_{\text{DEF}} = \kappa \varphi / \sqrt{2}$.}

\footnote{We note that the action Eq.(2) is still conformal invariant even if one includes a potential proportional to $\Phi^2$. In the Einstein frame action Eq.(11) such a potential corresponds to a constant and does not affect the evolution of the scalar field.}
2.3. Conformal Attractor Model

Although the scalar-tensor theory with $\eta = 0$ is unique, we may consider possible generalization of the models which exhibit cosmological attraction toward the conformal symmetry.

For example, we can consider the following generalization of the coupling function $a(\varphi)$ in Eq. (17):

$$a(\varphi) = \ln \left( \cos \left( \sqrt{p} \, \kappa \varphi \right) \right)$$

where $p$ is a non-negative parameter and $p = 1/2$ corresponds to constant-G theory. From Eq. (9) and Eq. (10), the corresponding $\Phi$ and $\omega(\Phi)$ are given by

$$G_\ast \Phi = \sec^2 \left( \sqrt{p} \, \kappa \varphi \right)$$

$$\omega(\Phi) = -\frac{3}{2} + \frac{1}{4p} \frac{1}{G_\ast \Phi - 1}.$$  

The conformal symmetry $\omega = -3/2$ now corresponds to $\kappa \varphi = \pi/(2\sqrt{p})$ and the shape of $a(\varphi)$ is similar to Eq. (17) ($p = 1/2$) and we expect similar cosmological attraction toward the conformal symmetry.  

On the other hand, the effective gravitational constant $G$ defined by Eq. (4) is given by

$$G = 2pG_\ast + \frac{1-2p}{\Phi},$$

and the gravitational constant is no longer constant. PPN parameters $\gamma$ and $\beta$ and $\eta = 4\beta - \gamma - 3$ are given from Eq.(5) and Eq. (6) by

$$\gamma - 1 = -\frac{4p(G_\ast \Phi - 1)}{2pG_\ast \Phi + 1 - 2p}$$

$$\beta - 1 = -\frac{2p^2G_\ast \Phi(G_\ast \Phi - 1)}{(2pG_\ast \Phi + 1 - 2p)^2}$$

$$\eta = -\frac{4p(2p - 1)(G_\ast \Phi - 1)}{(2pG_\ast \Phi + 1 - 2p)^2}.$$  

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Fig. 1 $a(\varphi)$ in Eq. (17). $\kappa \varphi = 0$ corresponds to GR and $\kappa \varphi = \pi/\sqrt{2}$ corresponds to the conformal symmetry.

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We note that a theory with a quadratic function $a(\varphi) = -\frac{1}{2}p(\kappa \varphi)^2$ with $p > 0$ [14] is also cosmologically attracted toward the conformal symmetry.
In terms of $\gamma$, $\eta$ can be rewritten in a suggestive form:

$$\eta = (2p - 1) \left( \frac{\gamma + 1}{2} \right) (\gamma - 1).$$  \hspace{1cm} (25)

Therefore, although $\eta$ is no longer vanishing as long as $p \neq 1/2$, it is suppressed by $\gamma - 1$. From the bound by Cassini $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ \cite{23}, the constraint on $\eta$ from the lunar-laser-ranging experiment, $\eta = (-0.2 \pm 1.1) \times 10^{-4}$ \cite{6}, is satisfied for $p < 2.9$.

3. **Spontaneous Scalarization in Conformal Attractor Model**

The curvature of the coupling function Eq. (18) at $\varphi = 0$ is negative. For $a(\varphi)$ with negative curvature, the scalar field experiences a tachyonic instability inside a compact star like a neutron star and exhibits a large deviation from its asymptotic value, a phenomenon so-called “spontaneous scalarization” \cite{14}. From the observation of the pulsar-timings of several neutron star-white dwarf binaries, such a phenomenon has been constrained \cite{19, 26}.

Assuming a quadratic form of $a(\varphi) = -\frac{1}{2}p(\kappa\varphi)^2$, $p$ is constrained as $p \lesssim 2.2$.\footnote{Note that since $\varphi_{\text{DEF}}$ in \cite{14} corresponds to $\varphi_{\text{DEF}} = \kappa\varphi/\sqrt{2}$, $\beta_{\text{DEF}}$ in $a(\varphi_{\text{DEF}}) = \frac{1}{2}\beta_{\text{DEF}}\varphi_{\text{DEF}}^2$ corresponds to $\beta_{\text{DEF}} = -2p$.} Although the coupling function $a(\varphi)$ in Eq. (18) is well approximated as $a(\varphi) = -\frac{1}{2}p(\kappa\varphi)^2$ for $\kappa\varphi \ll 1$, the scalar field acquires a large value when the theory exhibits spontaneous scalarization and higher order terms in the Taylor expansion of $a(\varphi)$ may not be negligible and it is not clear whether the limit on $p$ from the quadratic model may apply here.

Hence, in this section, we study spontaneous scalarization for the conformal attractor model Eq. (18) and compare it with the quadratic model.

3.1. **TOV equation**

We study the spherically symmetric static solutions generated by perfect fluid neutron stars in scalar-tensor theories. The metric in the Einstein frame is assumed to be of the form

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -e^{\nu(r)} dt^2 + \frac{dr^2}{1 - 2\mu(r)/r} + r^2 d\Omega^2. \hspace{1cm} (26)$$

The perfect fluid energy-momentum in the Jordan frame is

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu\nu} \hspace{1cm} (27)$$

and is related to the energy-momentum tensor in the Einstein frame by $\bar{T}^\mu_{\nu} = e^{4\alpha} T^{\mu}_{\nu}$. The equation of motion Eq. (12) and Eq. (13) become

$$\mu' = 4\pi G_s r^2 e^{4\alpha(\varphi)} \epsilon + 2\pi G_s r(\rho - 2\mu) \varphi'^2 \hspace{1cm} (28)$$

$$\nu' = \frac{2\mu}{r(r - 2\mu)} + 8\pi G_s r^2 e^{4\alpha(\varphi)} \frac{\epsilon}{r - 2\mu} + 4\pi G_s r \varphi'^2 \hspace{1cm} (29)$$

$$\varphi'' = \frac{2(r - \mu)}{r(r - 2\mu)} \varphi' + 4\pi G_s r^2 e^{4\alpha(\varphi)} \frac{\epsilon - p}{r - 2\mu} + \frac{re^{4\alpha(\varphi)}}{r - 2\mu} (\epsilon - 3p) + \alpha(\varphi) \hspace{1cm} (30)$$

$$p' = -(\epsilon + p) \left( \frac{\mu}{r(r - 2\mu)} + 4\pi G_s r^2 e^{4\alpha(\varphi)} \frac{\rho - 2\mu}{r - 2\mu} + 2\pi G_s r \varphi'^2 + \alpha(\varphi) \varphi' \right) \hspace{1cm} (31)$$

where the prime denotes a derivative with respect to $r$.\footnote{Note that since $\varphi_{\text{DEF}}$ in \cite{14} corresponds to $\varphi_{\text{DEF}} = \kappa\varphi/\sqrt{2}$, $\beta_{\text{DEF}}$ in $a(\varphi_{\text{DEF}}) = \frac{1}{2}\beta_{\text{DEF}}\varphi_{\text{DEF}}^2$ corresponds to $\beta_{\text{DEF}} = -2p$.}
With the coupling function $a(\varphi)$ Eq. (18), given the initial conditions at $r = 0$, we numerically integrate these equations outward using a 4th-order Runge-Kutta method. The pressure goes to zero at the surface of star, and beyond that only the metric and scalar equations with vanishing $p$ and $\epsilon$ are necessary. Specifically, we set $p$ and $\epsilon$ to zero if $p$ is less than $10^{-7} m_B n_0$ (with $m_B = 1.66 \times 10^{-24}$g and $n_0 = 0.1$fm$^{-3}$) well below neutron drip and integrate Eq. (28) $\sim$ Eq. (30) further outward. The regularity at $r = 0$ requires $\mu(0) = \nu(0) = \varphi'(0) = 0$. We can freely specify $\varphi(0)$ and $p(0)$. But in order to satisfy the bound by the Cassini satellite experiment $|\gamma - 1| < 2.3 \times 10^{-5}$, from Eq. (15) the asymptotic value $\varphi_0$ should satisfy $\alpha(\varphi_0)/\kappa < 2.4 \times 10^{-3}$. Considering the limit on $\gamma$ and in order to put conservative constraints on the theory, we require $\alpha(\varphi_0)/\kappa = 1.0 \times 10^{-4}$ at large $r$. Hence, for a given $p(0)$, a particular value of $\varphi(0)$ can satisfy this condition. We employ the shooting method to find $\varphi(0)$.

3.2. Equation of State

As an equation of state (EOS), we adopt piecewise-polytropic parametrizations for the nuclear EOS by Read et al. [27] for APR4 [28] and H4 [29, 30] and MS1 [31] EOSs. APR4 (variational method) and MS1 (relativistic mean-field theory) are EOSs for nuclear matter composed of neutrons, protons, electrons, and muons, whereas H4 (relativistic mean-field theory) includes the effect of hyperons in addition.

A piecewise polytropic EOS consists of several polytropic EOSs

$$p(\rho) = K_i \rho^{\Gamma_i}, \quad \rho_{i-1} \leq \rho \leq \rho_i,$$

(32)

where $\rho$ is the rest-mass density. From the continuity of the pressure at $\rho_i$ determines $K_{i+1}$ at the next interval as $K_{i+1} = p(\rho_i)/\rho_i^{\Gamma_{i+1}}$. The energy density $\epsilon$ is determined by the first law of thermodynamics, $d(\epsilon/\rho) = -pd(1/\rho)$ as

$$\epsilon = (1 + a_i)\rho + \frac{K_i}{\Gamma_i - 1} \rho^{\Gamma_i}, \quad \rho_{i-1} \leq \rho \leq \rho_i,$$

(33)

where $a_i$ is an integration constant given by

$$a_i = \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}} - 1 - \frac{K_{i-1}}{\Gamma_i - 1} \rho_{i-1}^{\Gamma_i - 1},$$

(34)

from the continuity of the energy density at $\rho_{i-1}$.

In the four-parameter model of [27], the EOS at low densities (crust EOS) is fixed to the EOS of Douchin and Haensel [32] and is matched to a polytrope with adiabatic exponent $\Gamma_1$. At a fixed rest-mass density$^8$ $\rho_1 = 10^{14.7}$g/cm$^3$ and pressure $p_1 = p(\rho_1)$, the EOS is joined continuously to a second polytrope with $\Gamma_2$. Finally, at $\rho_2 = 10^{15}$g/cm$^3$, the EOS is joined to a third polytrope with $\Gamma_3$. Further details are described in [27].

In Fig. 2, we show the mass-radius relation for three EOSs for the conformal attractor model Eq. (18) with $p = 2.3$ (solid) and GR (dotted). The physical radius of a neutron star $R$ is defined in terms of the surface of the star $r_*$ by $R = e^{\alpha(\varphi(r_*))} r_*$. Among three EOSs, MS1 is the stiffest (large $p_1$) EOS and the radius of a $1.4 M_\odot$ neutron star is large ($\sim 14.6$km in GR), while APR4 is a soft EOS and the radius of a $1.4 M_\odot$ neutron star is small ($\sim 11.2$km).

$^8$In terms of the nuclear saturation density $\rho_{\text{nuc}} \simeq 2.7 \times 10^{14}$g/cm$^3$, $\rho_1 \simeq 1.9 \rho_{\text{nuc}}$.  

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Fig. 2  Mass-radius relation for APR (blue, left), H4(magenta, middle), MS1(green, right) EOSs. The solid curves are for the conformal attractor model Eq. (18) with $p = 2.3$ and the dotted curves are for GR.

In fact, a strong correlation between the pressure at around the nuclear saturation density $\rho_{\text{nuc}}$ and the radius of $1.4M_\odot$ neutron stars has been found [33]. H4 is in between the two EOSs. On the other hand, the maximum mass of a neutron star (in GR) is the largest for MS1 ($\sim 2.75M_\odot$), while the smallest for H4 ($\sim 2.01M_\odot$) but is still compatible with the most massive pulsar ($2.08 \pm 0.07M_\odot$) measured by [34, 35].

We also show the radial profile of the energy density $\epsilon$ in unit of $m_Bn_0 = 1.66 \times 10^{14}$g/cm$^3$ for APR4 EOS in the left of Fig. 3 and the radial profile of the scalar field $\phi$ in the right.

Fig. 3  Left: The energy density $\epsilon$ (in unit of $m_Bn_0 = 1.66 \times 10^{14}$g/cm$^3$) as a function of the physical radius $e^{a(\phi)r}$ for APR4 EOS. A solid curve is for the conformal attractor model with $p = 2.3$ and a dotted curve is for GR. Right: the profile of the scalar field $\phi$. $p = 2.3, 2.2, 2.1$ from top to bottom. All cases have a fixed gravitational mass of $1.9M_\odot$.

For each EOS, we compute the total gravitational (ADM) mass $M_A$ of the neutron star which is easily read off from the asymptotic behavior of $\mu(r)$ at infinity as $G_sM_A = \lim_{r \to \infty} \mu(r)$. Also, from the asymptotic behavior of $\phi$, $\kappa\phi \to \kappa\phi_0 + G_sM_S/r$, we compute
the scalar charge $M_S$ of the neutron star. The effective scalar coupling $\alpha_A$ introduced in [8, 14] is written in terms of $M_A$ and $M_S$ as (note again that our $\varphi$ is related to the scalar field $\varphi_{\text{DEF}}$ in [8, 14] via $\varphi_{\text{DEF}} = \kappa \varphi / \sqrt{2}$)

$$\alpha_A = \frac{\sqrt{2}}{\kappa} \frac{\partial \ln M_A}{\partial \varphi} = - \frac{M_S}{\sqrt{2} M_A}. \quad (35)$$

3.3. Pulsar Constraints

| Pulsar       | Orbital period $P_b$ (d) | $P_b$ (10$^{-12}$ s)$^1$ | Companion mass | Pulsar mass | $\alpha_A$ |
|--------------|--------------------------|---------------------------|----------------|-------------|------------|
| J0348+0432   | 0.102424062722(7)        | -0.273(45)                | 0.172(3)       | 2.01(4)     | $< 5.0 \times 10^{-3}$ |
| J1738+0333   | 0.3547907398724(13)      | -0.0170(31)               | 0.181(+0.008,-0.007) | 1.46(+0.06,-0.05) | $< 2.7 \times 10^{-3}$ |
| J1012+5307   | 0.60467271355(3)         | -0.061(4)                 | 0.165(15)      | 1.72(16)    | $< 4.7 \times 10^{-3}$ |
| J1713+0747   | 67.8251299228(5)         | -0.34(15)                 | 0.290(11)      | 1.33(10)    | $< 3.4 \times 10^{-3}$ |
| J2222-0137   | 2.44576437(2)            | -0.2509(76)               | 1.319(4)       | 1.831(10)   | $< 5.0 \times 10^{-3}$ |
| J1909-3744   | 1.533449474305(5)        | -0.51087(13)              | 0.209(1)       | 1.492(14)   | $< 4.0 \times 10^{-3}$ |

Table 1 Parameters of NS-WD binaries, PSRs J0348+0432[26], J1738+0333[36], J1012+5307[37–39], J1713+0747[40], and J2222-0137[41, 42], J1909-3744[43] and the 2$\sigma$ limits on $\alpha_A$.

We consider the following 6 NS-WD binaries: PSRs J0348+0432[26], J1738+0333[36], J1012+5307[37–39], J1713+0747[40], and J2222-0137[41, 42], J1909-3744[43]. For PSR J1012+5307, the limit on the scalar coupling $\alpha_A$ from the absence of the dipole radiation is recently improved in [39] by the precise measurement of the distance by VLBI. Moreover, the mass of the neutron star is determined recently due to an estimate of the mass of the white dwarf companion using binary evolution models [38]. For PSR J2222-0137, the limit on the scalar coupling is recently improved in [42] compared with [41] by the improved analysis of VLBI data together with the extended timing data. For PSR 1713+0747, PSR J2222-0137 and PSR J1909-3744, the pulsar mass is determined from the Shapiro time-delays, while for others the pulsar mass is determined from the combination of the white dwarf mass determined from the optical spectrum and the mass ratio determined from the orbital velocity.

The measurements of the orbital decay of the binary systems are consistent with the orbital decay due to the emission of gravitational waves predicted by GR [44]:

$$\dot{P}_b^{GR} = -\frac{192\pi}{5} \frac{(1 + 73e^2/24 + 37e^4/96)}{(1 - e^2)^{7/2}} \left( \frac{2\pi G_s M_c}{P_b c^3} \right)^{5/3} \frac{q}{q + 1} \left( q + \frac{1}{3} \right), \quad (36)$$

where $e$ is the orbital eccentricity, $P_b$ is the orbital period, $M_c$ is the companion (white dwarf) mass and $q = M_A/M_c$ is the ratio of the pulsar mass to the companion mass. In scalar-tensor theory, scalar waves are also emitted and contribute the orbital decay. For the binary systems, the dominant contribution comes from dipolar waves [8]:

$$\dot{P}_b^D = -2\pi \frac{(1 + e^2/2)}{(1 - e^2)^{5/2}} \left( \frac{2\pi G_s M_c}{P_b c^3} \right) \frac{q}{q + 1} \frac{(\alpha_A - \alpha_c)^2}{}, \quad (37)$$
Fig. 4  Effective scalar coupling $\alpha_A$ as a function of the gravitational mass of the neutron star for APR4(top), H4(middle) and MS1(bottom). The curves are for $p = 2.0, 2.1, 2.2, 2.3$ from bottom to top. Red points are the $2\sigma$ limits on $\alpha_A$. 
where $\alpha_c$ are the effective scalar coupling to white dwarf. Since the self-gravity of white dwarf is small, $\alpha_c$ is no different from its asymptotic value: $\alpha_c \simeq \sqrt{2}\alpha(\phi_0)/\kappa \ll 1$. The measurements of $\dot{P}_b$ constrain the dipole contribution, hence $\alpha_A - \alpha_c \simeq \alpha_A$. The parameters of six NS-WD binaries and the 2σ limits on $\alpha_A$ are shown in Table 1.

In Fig. 4, $\alpha_A$ as a function of the gravitational mass of the neutron stars for three EOSs together with the 2σ limits on $\alpha_A$ from the pulsar-timings are shown. We find that spontaneous scalarization occurs if $p \gtrsim 2.2$ and $p$ is constrained as $p < 2.3$ irrespective of EOSs. However, more detailed constraints on $p$ depend on the EOS: $p$ is constrained to be $p < 2.2$ for APR4 and H4, while $p = 2.2$ is allowed for MS1. We may place a conservative limit on $p$ to be $p < 2.3$. Although there exist small windows for the scalarization at $\sim 1.9M_\odot$ for H4 and at $\sim 2.3M_\odot$ for MS1 [19], a “window at $\sim 1.7M_\odot$” [19] is now closed with the inclusion of PSR J1012+5307 and PSR J2222-0137.

3.4. Choice of Coupling Function

![Fig. 5](image)

**Fig. 5** $a(\phi)$ of the conformal attractor model Eq. (18) (solid line) and of the quadratic model $-\frac{1}{2}p(\kappa\phi)^2$ (dotted line) for $p = 2.3$.

So far, the effective scalar coupling is computed for the coupling function Eq. (18). However, most frequently studied gravity theory is the so-called DEF theory [14] based on the quadratic coupling function:

$$a(\phi) = \frac{1}{2}\beta(\kappa\phi)^2. \quad (38)$$

Eq. (18) can be expanded as $a(\phi) = -\frac{1}{2}p(\kappa\phi)^2 + O((\kappa\phi)^4)$ for $\kappa\phi \ll 1$, hence $p$ corresponds to $-\beta$ in the quadratic model. Note that since $\phi_{\text{DEF}}$ in [14] corresponds to $\phi_{\text{DEF}} = \kappa\phi/\sqrt{2}$, $\beta_{\text{DEF}}$ in $a(\phi_{\text{DEF}}) = \frac{1}{2}\beta_{\text{DEF}}\phi_{\text{DEF}}^2$ corresponds to $\beta_{\text{DEF}} = 2\beta$.

Although the coupling function $a(\phi)$ in Eq. (18) is well approximated as $a(\phi) = -\frac{1}{2}p(\kappa\phi)^2$ as long as $\kappa\phi \ll 1$ (see Fig. 5), the scalar field may acquire a large value when the theory exhibits spontaneous scalarization and the scalar field can experience a wider portion of the coupling function.

However, as shown in Fig. 6, even for $p = 2.3$, the difference of $\alpha_A$ between the two coupling functions is very small. For $p = 2.2$, two $\alpha_A$s almost coincide. In fact, as shown in Fig. 7,

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9 The weak dependence of the onset of the scalarization on the EOS was found in [15, 16].

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the maximum value of the scalar field ($\varphi(0)$) is at most $\kappa\varphi(0) \lesssim 0.2$ even for $p = 2.3$ and higher order terms in the Taylor expansion of $a(\varphi)$ is negligible. Similar results are obtained by [21] from the comparison of the quadratic model and MO model [22].

This behavior can be understood from the stability analysis of neutron stars in scalar-tensor theories [15, 45]. Spontaneous scalarization is triggered by a tachyonic instability of the scalar field of a general relativistic star. From the linear stability analysis of neutron stars, the threshold value of the curvature of the coupling function is found to be insensitive to EOSs [15]. In the linear analysis, only the quadratic term in $a(\varphi)$ is relevant and possible higher order terms are not important in the analysis. Moreover, using the catastrophe theory,
it is shown that the onset of spontaneous scalarization in the quadratic coupling function corresponds to cusp catastrophe [45] which is structurally stable, i.e. the theory is stable against adding higher order terms in the coupling function. Therefore, we expect that the structure of the theory near the onset of the scalarization is described by the quadratic coupling function universally.

Since pulsar data already disfavor \( p \geq 2.3 \), we thus conclude that as far as the observational constraints on spontaneous scalarization are concerned, it is sufficient to employ the quadratic function for the coupling function.

3.5. Comment on the Cosmological Evolution

Finally we comment on the cosmological evolution of \( \varphi \) which defines its asymptotic value \( \varphi_0 \).

As we have seen in 2.2, the conformal attractor model is cosmologically attracted toward the conformal symmetry \( \omega \to -3/2 \), in vast disagreement with the solar system experiments. Therefore, similar to the quadratic function with negative curvature, a severe fine-tuning of the initial conditions is required to satisfy the solar system constraints today\(^\text{[46]}\).

One obvious and the simplest possibility to avoid this problem is to introduce a mass term for \( \varphi \) (in the Einstein frame)\(^\text{[47–49]}\). Massive scalar-tensor theories with mass \( m_\varphi \) of \( 10^{-28}\text{eV} \lesssim m_\varphi \lesssim 10^{-10}\text{eV} \) may both exhibit spontaneous scalarization (upper bound) and cosmological attraction toward GR in the matter-dominated era (lower bound). According to \([48]\), the effective scalar coupling of a neutron star is not different from that of a massless theory if \( m_\varphi \lesssim 10^{-13}\text{eV} \) but is much suppressed for \( m_\varphi \gtrsim 10^{-12}\text{eV} \).

4. Summary

Motivated by constant-G theory which respects the SEP and is cosmologically attracted toward the conformal symmetry, we introduce a one-parameter family of scalar-tensor theories which exhibit cosmological attraction toward the conformal symmetry. From the constraint on the violation of SEP by the lunar-laser-ranging experiment, the parameter \( p \) is constrained to \( p < 2.9 \).

We have studied the structure of neutron stars in these theories for three realistic EOSs (APR4, H4, MS1). From the constraints on the effective scalar coupling from six neutron star-white dwarf binaries, the parameter \( p \) is constrained to be \( p < 2.2 \) for APR4 and H4, while \( p < 2.3 \) for MS1. With new pulsar data a window for the scalarization at \( \sim 1.7M_\odot \) is closed, but small windows are still open at \( \sim 1.9M_\odot \) for H4 and at \( \sim 2.3M_\odot \) for MS1.

We have also compared in detail our coupling function and the quadratic model and have found that the difference of the effective scalar coupling between the two theories is very small for \( p \leq 2.3 \). Combining these results with the structural stability of the quadratic model at the onset of scalarization \([45]\), we conclude that as far as the observational constraints on spontaneous scalarization are concerned, the dependence on the coupling function is small and hence we can safely employ the quadratic function as the coupling function and the constraints obtained by the quadratic model will be used to put constraints on the curvature of the coupling function universally in the future.

\(^{10} m_\varphi \lesssim 10^{-17}\text{eV} \) may be required for the success of the big-bang nucleosynthesis \([49]\). If this is the case, the structure of neutron stars would be almost the same as that of a massless theory.
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