STAR OPERATIONS ON KUNZ DOMAINS

DARIO SPIRITO

Abstract. We study star operations on Kunz domains, a class of analytically irreducible, residually rational domains associated to pseudo-symmetric numerical semigroups, and we use them to refute a conjecture of Houston, Mimouni and Park. We also find an estimate for the number of star operations in a particular case, and a precise counting in a sub-case.

1. Introduction

Let \( D \) be an integral domain with quotient field \( K \), and let \( \mathcal{F}(D) \) be the set of fractional ideals of \( D \), i.e., the set of \( D \)-submodules \( I \) of \( K \) such that \( xI \subseteq D \) for some \( x \in K \setminus \{0\} \).

A star operation on \( D \) is a map \( \star : \mathcal{F}(D) \rightarrow \mathcal{F}(D) \), \( I \mapsto I^\star \), such that, for every \( I, J \in \mathcal{F}(D) \) and every \( x \in K \):

- \( I \subseteq I^\star \);
- if \( I \subseteq J \), then \( I^\star \subseteq J^\star \);
- \( (I^\star)^\star = I^\star \);
- \( x \cdot I^\star = (xI)^\star \);
- \( D = D^\star \).

A fractional ideal \( I \) is \( \star \)-closed if \( I = I^\star \).

The easiest example of a non-trivial star operation is the \( v \)-operation \( v : I \mapsto (D : (D : I)) \), where if \( I, J \in \mathcal{F}(D) \) we define \( (I : J) := \{ x \in K \mid xJ \subseteq I \} \). An ideal that is \( v \)-closed is said to be divisorial; if \( I \) is divisorial and \( \star \) is any other star operation then \( I = I^\star \). We denote by \( d \) the identity, which is obviously a star operation.

Recently, the cardinality of the set \( \text{Star}(D) \) of the star operations on \( D \) has been studied, especially in the case of Noetherian [6, 8] and Prüfer domains [5, 7]. In particular, Houston, Mimouni and Park started studying the relationship between the cardinality of \( \text{Star}(D) \) and the cardinality of \( \text{Star}(T) \), where \( T \) is an overring of \( D \) (an overring of \( D \) is a ring comprised between \( D \) and \( K \) ) [3, 4]: they called a domain star regular if \( |\text{Star}(D)| \geq |\text{Star}(T)| \) for every overring of \( T \). While even simple domains may fail to be star regular (for example, there are domains with just one star operations having an overring

Date: June 1, 2018.

2010 Mathematics Subject Classification. 13A15, 13E05, 13G05.

Key words and phrases. Star operations; pseudo-symmetric semigroups; Kunz domains; star regular domains.
with infinitely many star operations [3, Example 1.3]), they conjectured that every one-dimensional local Noetherian domain $D$ such that $1 < |\text{Star}(D)| < \infty$ is star regular, and proved it when the residue field of $D$ is infinite [3, Corollary 1.18].

In this context, a rich source of examples are semigroup rings, that is, subrings of the power series ring $K[[X]]$ (where $K$ is a field, usually finite) of the form $K[[S]] := K[[X^S]] := \{ \sum_{i} a_i X^i \mid a_i = 0 \text{ for all } i \notin S \}$, where $S$ is a numerical semigroup (i.e., a submonoid $S \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite). Star operations can also be defined on numerical semigroups [13], and there is a link between star operations on $S$ and star operations on $K[[S]]$: for example, every star operation on $S$ induces a star operation on $K[[S]]$, and $|\text{Star}(S)| = 1$ if and only if $|\text{Star}(K[[S]])| = 1$ [13, Theorem 5.3], with the latter result corresponding to the equivalence between $S$ being symmetric and $K[[S]]$ being Gorenstein [2, 10]. A detailed study of star operations on some numerical semigroup rings was carried out in [14].

In this paper, we study of star operations on Kunz domains, which are, roughly speaking, a generalization of rings in the form $K[[S]]$ where $S$ is a pseudo-symmetric semigroup (see the beginning of the next section for the definitions). We show that, if $R$ is a Kunz domain whose residue field is finite and the length of $R/R$ is at least 4 (where $R$ is the integral closure of $R$) then $R$ is a counterexample to Houston-Mimouni-Park’s conjecture; that is, $R$ satisfies $1 < |\text{Star}(R)| < \infty$ but there is an overring $T$ of $R$ with more star operations than $R$. In Section 3, we also study more deeply one specific class of domains, linking the cardinality of $\text{Star}(R)$ with the set of vector subspaces of a vector space over the residue field of $R$, and calculate the cardinality of $\text{Star}(R)$ when the value semigroup of $R$ is $\langle 4, 5, 7 \rangle$.

We refer to [12] for information about numerical semigroup, and to [1] for the passage from numerical semigroup to one-dimensional local domains.

2. Kunz Domains

Let $S$ be a numerical semigroup, and let $g := g(S) := \sup(\mathbb{Z} \setminus S)$. We say that $S$ is a pseudo-symmetric semigroup if $g$ is even and, for every $a \in \mathbb{N}$, $a \neq g/2$, either $a \in S$ or $g - a \in S$. If $a_1, \ldots, a_n$ are coprime integers, we denote by $\langle a_1, \ldots, a_n \rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$, i.e., $\langle a_1, \ldots, a_n \rangle = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N} \}$.

Let $(V, M_V)$ be a discrete valuation ring with associated valuation $v$. Let $(R, M_R)$ be a local subring of $V$ with the following properties:

- $R$ and $V$ have the same quotient field;
- the integral closure of $R$ is $V$;
- $R$ is Noetherian;
- the conductor ideal $(R : V)$ is nonzero;
the inclusion $R \rightarrow V$ induces an isomorphism of residue fields $R/M_R \rightarrow V/M_V$. Equivalently, let $R$ be an analytically irreducible, residually rational one-dimensional Noetherian local domain having integral closure $V$. Note that for every such $R$ the set $\mathfrak{v}(R) := \{v(r) \mid r \in R\}$ is a numerical semigroup. We state explicitly a property which we will be using many times.

**Lemma 2.2.** Let $\mathfrak{v}(R)$ be an analytically irreducible, residually rational one-dimensional Noetherian local domain having integral closure $V$. Then,

$\mathfrak{v}(R) = \mathfrak{v}(R) \cap \mathfrak{v}(\mathfrak{m}) = \mathfrak{v}(R)$

where for every such $R$ the inclusion $R/M_R \rightarrow V/M_V$.

**Proposition 2.1** ([11, Corollary to Proposition 1]). Let $R$ be as above, and let $I \subseteq J$ be $R$-submodules of the quotient field of $R$. Then,

$$\ell_R(J/I) = |v(J) \setminus v(I)|,$$

where $\ell_R$ is the length of an $R$-module.

We say that $R$ is a Kunz domain if $\mathfrak{v}(R)$ is a pseudo-symmetric semigroup [11 Proposition II.1.12].

From now on, we suppose that $R$ is a Kunz domain, and we set $g := g(\mathfrak{v}(R))$ and $r := g/2$. The hypotheses on $R$ guarantee that, if $x \in V$ is such that $v(x) > g$, then $x \in R$ [10, Theorem, p.749].

**Lemma 2.2.** Let $y \in V$ be an element of valuation $g$, and let $T := R[y]$. Then:

(a) $T$ contains all elements of valuation $g$;
(b) $v(T) = \mathfrak{v}(R) \cup \{g\}$;
(c) $\ell_R(T/R) = 1$;
(d) $T = R + yR$.

**Proof.** Let $y' \in V$ be another element of valuation $g$. Then, $\mathfrak{v}(y/y') = 0$, and thus $y' := y/y'$ is a unit of $V$. Hence, there is a $c' \in R$ such that the images of $c$ and $c'$ in the residue field of $V$ coincide; in particular, $c = c' + m$ for some $m \in M_V$. Hence,

$$y' = cy = (c' + m)y = c'y + my.$$

Since $c' \in R$, we have $c'y \in R[y]$: furthermore, $\mathfrak{v}(my) = \mathfrak{v}(m) + \mathfrak{v}(y) > \mathfrak{v}(y) = g$, and thus $my \in R$. Hence, $y' \in R[y]$, and thus $R[y]$ contains all elements of valuation $g$.

The fact that $\mathfrak{v}(T) = \mathfrak{v}(R) \cup \{g\}$ is trivial; hence, $\ell_R(T/R) = |\mathfrak{v}(T) \setminus \mathfrak{v}(R)| = 1$. The last point follows from the fact that $R + yR$ is an $R$-module, from $R \subseteq R + yR \subseteq T$ and from $\ell_R(T/R) = 1$. □

In particular, the previous proposition shows that $T$ is independent from the element $y$ chosen. From now on, we will use $T$ to denote this ring.

We denote by $\mathcal{F}_0(R)$ the set of $R$-fractional ideals $I$ such that $R \subseteq I \subseteq V$. If $I$ is any fractional ideal over $R$, and $\alpha \in I$ is an element of minimal valuation, then $\alpha^{-1}I \in \mathcal{F}_0(R)$; hence, the action of any star operation is uniquely determined by its action on $\mathcal{F}_0(R)$. Furthermore, $V^* = V$ for all $\ast \in \text{Star}(R)$ (since $(R : (R : V)) = V$) and thus
$I^\ast \in \mathcal{F}_0(R)$ for all $I \in \mathcal{F}_0(R)$, i.e., $\ast$ restricts to a map from $\mathcal{F}_0(R)$ to itself.

To analyze star operations, we want to subdivide them according to whether they close $T$ or not. One case is very simple.

**Proposition 2.3.** If $\ast \in \text{Star}(R)$ is such that $T \neq T^\ast$, then $\ast = v$.

**Proof.** Suppose $\ast \neq v$; then, there is a fractional ideal $I \in \mathcal{F}_0(R)$ that is $\ast$-closed but not divisorial. By [1, Lemma II.1.22], $\nu(I)$ is not divisorial (in $\nu(R)$) and thus by [1, Proposition I.1.16] there is an integer $n \in \nu(I)$ such that $n + \tau \notin \nu(I)$.

Let $x \in I$ be an element of valuation $n$, and consider the ideal $J := x^{-1}I \cap V$: being the intersection of two $\ast$-closed ideals, it is itself $\ast$-closed. Since $\nu(x) > 0$, every element of valuation $g$ belongs to $J$; on the other hand, by the choice of $n$, no element of valuation $\tau$ can belong to $J$.

Consider now the ideal $L := (R : M_R)$: then, $L$ is divisorial (since $M_R$ is divisorial) and, using [1, Proposition II.1.16(1)],

$$\nu(L) = (\nu(R) - \nu(M_R)) = \nu(R) \cup \{\tau, g\}.$$  

We claim that $T = J \cap L$; indeed, clearly $J \cap L$ contains $R$, and if $y$ has valuation $g$ then $y \in J \cap L$ by construction; thus $T = R + yR \subseteq J \cap L$.

On the other hand, $\nu(J \cap L) \subseteq \nu(J) \cap \nu(L) = \nu(R) \cup \{g\}$, and thus $J \cap L \subseteq T$.

Hence, $T = J \cap L$; since $J$ and $L$ are both $\ast$-closed, so is $T$. Therefore, if $T \neq T^\ast$ then $\ast$ must be the divisorial closure, as claimed. \hfill $\square$

Suppose now that $T = T^\ast$. Then, $\ast$ restricts to a star operation $\ast_1 := \ast |_{\mathcal{F}(T)}$: the amount of information we lose in the passage from $\ast$ to $\ast_1$ depends on the $R$-fractional ideals that are not ideals over $T$. We can determine them explicitly.

**Lemma 2.4.** Let $I \in \mathcal{F}_0(R)$, $I \neq R$. Then, the following are equivalent.

(i) $\nu(I) = \nu(R) \cup \{\tau\}$;
(ii) $I$ does not contain any element of valuation $g$;
(iii) $TI \neq I$;
(iv) $I$ is the canonical ideal of $R$.

Furthermore, in this case, $I^\ast = (R : M_R)$.

**Proof.** (i) $\Rightarrow$ (ii): since $R \subseteq I$, there is an element $x$ of $I$ of valuation $0$; hence, $IT$ contains an element of valuation $g$, and thus $IT \neq I$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i): suppose there is an $x \in I$ such that $\nu(x) \notin \nu(R) \cup \{\tau\}$. Since $\nu(R)$ is pseudo-symmetric, there is an $y \in R$ such that $\nu(y) = g - \nu(x)$; hence, $I$ contains an element (explicitly, $xy$) of valuation $g$ and, by the proof of Lemma 2.2, it follows that it contains every element of valuation $g$. 


Fix now an element \( y \in V \) of valuation \( g \). Since \( IT \neq I \), there are \( i \in I, t \in T \) such that \( it \notin I \). By Lemma 2.2 there are \( r, r' \in R \) such that \( t = r + yr' \); hence, \( it = i(r + yr') = ir + iyr' \). Both \( ir \) and \( iyr' \) are in \( I \), the former since it belongs to \( IR = I \) and the latter because its valuation is at least \( g \). However, this contradicts \( it \notin I \); therefore, \( v(I) \leq v(R) \cup \{ \tau \} \).

If \( v(I) = v(R) \), then we must have \( I = R \), against our hypothesis; therefore, \( v(I) = v(R) \cup \{ \tau \} \).

\[ \text{(i)} \iff \text{(iv)} \] by \([9\text{, Satz 5}]\), \( I \) is the canonical ideal of \( R \) if and only if \( v(I) \) is the canonical ideal of \( v(R) \). The claim follows since \( v(R) \) is pseudo-symmetric and since the canonical ideal of a numerical semigroup \( S \) is \( S \cup \{ x \in \mathbb{N} \mid g(S) - x \notin S \} \), which in this case is \( S \cup \{ \tau \} \).

For the last claim, we first note that \((R : M_R)\) is divisorial (since \( M_R \) is divisorial) and that contains \( I \): indeed, if \( x \in I \) has valuation \( \tau \), and \( m \in M_R \), then \( xm \in M_R \), for otherwise \( m \notin R \) and thus \( R + mR \) would be an ideal properly between \( R \) and \( I \), against \( \ell_R(I/R) = 1 \).

Hence, \( I^v \) can only be \( I \) or \((R : M_R)\). However, \((R : I) \subseteq M_R \), and thus \( I^v = (R : (R : I)) \supseteq (R : M_R) \). Hence, \( I^v = (R : M_R) \).

**Proposition 2.5.** The map

\[
\Psi : \text{Star}(R) \setminus \{d, v\} \longrightarrow \text{Star}(T)
\]

\[ \star \longmapsto \star|_{F(T)} \]

is well-defined and injective.

**Proof.** By Proposition 2.3 if \( \star \neq v \) then \( T = T^\star \), and thus \( \star|_{F(T)} \) is a star operation on \( T \); hence, \( \Psi \) is well-defined. We claim that it is injective: suppose \( \star_1 \neq \star_2 \). Then, there is an \( I \in F_3(R) \) such that \( I^\star_1 \neq I^\star_2 \). If \( I \) is a \( T \)-module then \( \Psi(\star_1) \neq \Psi(\star_2) \); suppose \( I \) is not a \( T \)-module.

By Lemma 2.4 \( I \) can only be \( R \) or a canonical ideal of \( R \). In the former case, \( R^\star_1 = R = R^\star_2 \), a contradiction. In the latter case, \( I^\star \) can only if \( I \) or \((R : M_R)\) (since \( \ell((R : M_R)/I) = 1 \)); suppose now that \( I^\star = I \) for some \( \star \in \text{Star}(R) \). By definition of the canonical ideal, \( J = (I : (I : J)) \) for every ideal \( J \); since \( (I : L) \) is always \( \star \)-closed if \( I \) is \( \star \)-closed, it follows that \( \star \) must be the identity. Since \( \star_1, \star_2 \neq d \), we must have \( I^\star_1 = (R : M_R) = I^\star_2 \), against the assumptions. Thus, \( \Psi \) is injective. \( \square \)

An immediate corollary of the previous proposition is that \( |\text{Star}(R)| \leq |\text{Star}(T)| + 2 \). Our counterexample thus involves finding star operations of \( T \) that do not belong to the image of \( \Psi \); to do so, we restrict to the case \( \ell_R(V/R) \geq 4 \) or, equivalently, \( |\mathbb{N} \setminus v(R)| \geq 4 \). This excludes exactly two pseudo-symmetric numerical semigroups, namely \( \langle 3, 4, 5 \rangle \) and \( \langle 3, 5, 7 \rangle \).
Lemma 2.6. Let $S$ be a pseudo-symmetric numerical semigroup, let $g := \max(\mathbb{N} \setminus S)$ and let $S' := S \cup \{g(S)\}$. If $|\mathbb{N} \setminus S| \geq 4$, then there are $a, b \in (S' - M_{S'}) \setminus S'$, $a \neq b$, such that $2a, 2b, a + b \in S'$.

Proof. We claim that $a := \tau$ and $b := g - \mu$ are the two elements we are looking for.

Since $a + M_S \subseteq S$ and $a + g > g$ (and so $a + g \in M_S$) we have $a \in (S' - M_{S'})$. Furthermore, since $|\mathbb{N} \setminus S| \geq 4$, we have $g > \mu$, and thus $b + m \geq g$ for all $m \in M_{S'}$.

By the previous point, $a + m, b + m \in S' \cup \{a, b\}$ for every $m \in M_{S'}$. We always have $2a \geq g$, and thus $2a \in S'$.

If $g > 2\mu$ then $a > \mu$, and so $a + b \geq g$, which implies $a + b \in S'$; moreover, also $b > \mu$, and thus $2b \in S'$.

If $g < 2\mu$, then $g$ must be equal to $2\mu - 2$ or to $\mu - 1$; the latter case is impossible since $|\mathbb{N} \setminus S| \geq 4$. Hence, $b = \mu - 2$ and $a = \mu - 1$. Then, $2b = 2\mu - 4$ and $a + b = 2\mu - 3$; again since $|\mathbb{N} \setminus S| \geq 4$, we must have $\mu > 3$, and thus $2b > a + b \geq \mu$. Furthermore, in this case $S' = \{0, \mu, \ldots\}$, and so $a + b, 2b \in S'$, as claimed. \hfill \Box

Proposition 2.7. Let $K$ be the residue field of $R$, and suppose that $\ell_{\mathfrak{p}}(V/R) \geq 4$. There are at least $|K| + 1$ star operations on $T$ that do not close $(R : M_R)$.

Proof. We first note that $(R : M_R)$ is a $T$-module. Indeed, let $x \in (R : M_R)$ and $t \in T$; then, $t = r + ax$, with $r \in R$ and $v(y) = g$, and so $xt = xr + axy$. Both $xr$ and $axy$ belong to $(R : M_R)$, the former because $(R : M_R)$ is a $R$-module and the latter since its valuation is at least $g$: hence, $xt \in (R : M_R)$. Thus, it makes sense to ask if a star operation on $T$ closes $(R : M_R)$. We also note that $T \subseteq (R : M_R) \subseteq (T : M_T)$, and thus $(R : M_R)^{\star_T} = (T : M_T)$ (where $\star_T$ is the $v$-operation on $T$).

Let $S' := v(T)$: by Lemma 2.2 we can find $a, b \in (S' - M_{S'}) \setminus S'$ such that $2a, 2b, a + b \in S'$. Choose $x, y \in (T : M_T)$ such that $v(x) = a$ and $v(y) = b$ (and, without loss of generality, suppose $y \notin (R : M_R)$): they exist since $v((T : M_T)) = (S' - M_{S'})$ [H Proposition II.1.16].

Let $\{\alpha_1, \ldots, \alpha_q\}$ be a complete set of representatives of $R/M_R$ (or, equivalently, of $T/M_T$), and let $T_i := T[x + \alpha_i y]$; then, by the choice of $v(x)$ and $v(y)$, we have $T_i = T + (x + \alpha_i y)T$, and in particular $T_i \subseteq (T : M_T)$. Define $\ast_i$ as the star operation

$$I \mapsto I^{\ast_T} \cap IT_i.$$ 

We claim that $\ast_i$ closes $T_i$ but not $T_j$ for $j \neq i$.

Indeed, clearly $T_i^{\ast_i} = T_i$. If $j \neq i$, then $T_i T_j$ contains both $x + \alpha_i y$ and $x + \alpha_j y$, and thus it contains their difference $(\alpha_i - \alpha_j)y$. Since $\alpha_i$ and $\alpha_j$ are units corresponding to different residues, it follows that $\alpha_i - \alpha_j$ is a unit of $R$, and thus of $T$; hence, $y \in T_i T_j$. By construction, $y \in (T : M_T)$; thus, $y \in T_i^{\ast_i}$. On the other hand, $y \notin T_i$, and thus $T_i^{\ast_i} \neq T_i.$
Thus, \( \{ \ast_1, \ldots, \ast_q \} \) are \( q = |K| \) different star operations. Furthermore, none of them closes \( (R : M_R) \), since
\[
(R : M_R)^* = (T : M_T) \cap (R : M_R)T[x + a_i y]
\]
contains \( y \), while \( y \notin (R : M_R) \).

To conclude the proof, it is enough to note that none of the \( \ast_i \) are the divisorial closure (since they close one of the \( T_i \), none of which are divisorial), and thus we have another star operation that does not close \( (R : M_R) \).

\[ \square \]

We are now ready to show that \( R \) is the desired counterexample.

**Theorem 2.8.** Let \( R \) be a Kunz domain with finite residue field, and suppose that \( \ell_R(V/R) \geq 4 \). Then, \( 1 < |\text{Star}(R)| < \infty \), but \( R \) is not star regular.

**Proof.** Since \( K \) is a finite field and \( R \) is not Gorenstein, by [6, Theorem 2.5] \( 1 < |\text{Star}(R)| < \infty \), and the same for \( T \).

By Proposition 2.5 we have \( |\text{Star}(R)| \geq 2 + |\Psi(\text{Star}(R))| \); by Proposition 2.7 we have \( |\Psi(\text{Star}(R))| \leq |\text{Star}(T)| - |K| - 1 \). Hence,
\[
|\text{Star}(R)| \leq 2 + |\text{Star}(T)| - |K| - 1 = |\text{Star}(T)| - |K| + 1 < |\text{Star}(T)|
\]
since \( |K| \geq 2 \). The claim is proved. \[ \square \]

3. The case \( v(R) = \langle n, n+1, \ldots, 2n-3, 2n-1 \rangle \)

In this section, we specialize to the case of Kunz domains \( R \) such that \( v(R) = \langle n, n+1, \ldots, 2n-1, 2n-3 \rangle = \{0, n, n+1, \ldots, 2n-1, 2n-3, \ldots \} \), where \( n \geq 4 \) is an integer. It is not hard to see that this semigroup is pseudo-symmetric, with \( g = 2n - 2 \) and \( \tau = n - 1 \).

We note that this semigroup is pseudo-symmetric also if \( n = 3 \), for which the number of star operations has been calculated in [8, Proposition 2.10]: we have \( |\text{Star}(R)| = 4 \).

By Lemma 2.4 the only \( I \in \mathcal{F}_0(R) \) such that \( IT \neq I \) are \( R \) and the canonical ideals. From now on, we denote by \( \mathcal{G} \) the set \( \{ I \in \mathcal{F}_0(R) \mid IT = I \} \); we want to parametrize \( \mathcal{G} \) by subspaces of a vector space.

**Lemma 3.1.** Let \( K \) be the residue field of \( R \). Then, there is an order-preserving bijection between \( \mathcal{G} \) and the set of vector subspaces of \( K^{n-1} \).

**Proof.** Every \( I \in \mathcal{G} \) contains \( T \). The quotient of \( R \)-modules \( \pi : V \mapsto V/T \) induces a map
\[
\tilde{\pi} : \mathcal{G} \longrightarrow \mathcal{P}(V/T)
I \mapsto \pi(I),
\]
where \( \mathcal{P}(V/T) \) denotes the power set of \( V/T \). It is obvious that \( \tilde{\pi} \) is injective.
The map \( \pi \) induces on \( V/T \) a structure of \( K \)-vector space of dimension \( n - 1 \). If \( I \in \mathcal{G} \), then its image along \( \tilde{\pi} \) will be a vector subspace; conversely, if \( W \) is a vector subspace of \( V/T \) then \( \pi^{-1}(W) \) will be an ideal in \( \mathcal{G} \). The claim is proved. \( \square \)

For an arbitrary domain \( D \) and a fractional ideal \( I \) of \( D \), the star operation \emph{generated} by \( I \) is the map \[ \star_I : J \mapsto (I : (I : J)) \cap J^v = J^v \cap \bigcap_{\gamma \in (I : J) \setminus \{0\}} \gamma^{-1}I; \]

this star operation has the property that, if \( I \) is \( \star \)-closed for some \( \star \in \text{Star}(R) \) and \( J \) is \( \star_f \)-closed, then \( J \) is also \( \star \)-closed. If \( \Delta \subseteq \mathcal{F}(S) \), we define \( \star_\Delta \) as the map \[ \star_\Delta : J \mapsto \bigcap_{I \in \Delta} J^{\star_I}. \]

In the present case, we can characterize when an ideal is \( \star_\Delta \)-closed.

**Proposition 3.2.** Let \( I, J \in \mathcal{G} \) and let \( \Delta \subseteq \mathcal{G} \) be a set of nondivisorial ideals.

(a) \( I \) is divisorial if and only if \( n - 1 \in \mathfrak{v}(I) \);

(b) \( I^v = I \cup \{x \mid \mathfrak{v}(x) \geq n - 1\} \);

(c) if \( I, J \) are nondivisorial, then \( I = I^{\star_J} \) if and only if \( I \subseteq \gamma^{-1}J \) for some \( \gamma \) of valuation 0;

(d) if \( I \) is nondivisorial, then \( I \) is \( \star_\Delta \)-closed if and only if \( I \subseteq \gamma^{-1}J \) for some \( J \in \Delta \) and some \( \gamma \) of valuation 0.

**Proof.** (a) If \( I \) is divisorial, then (since \( I \neq R \)) we must have \( (R : M_R) \subseteq I \); in particular, \( n - 1 \in \mathfrak{v}(I) \).

Suppose \( n - 1 \in \mathfrak{v}(I) \); since \( I \) contains every element of valuation at least \( n \), it contains also all elements of valuation \( n - 1 \). Let \( x \) be such that \( \mathfrak{v}(x) = n - 1 \); then, \( \mathfrak{v}(x + r) \geq n - 1 \) for every \( r \in V \), and thus \( x + I \subseteq I \). Hence, \( I \) is divisorial by \[ \text{[II Proposition II.1.23]}. \]

(b) Let \( L := I \cup \{x \mid \mathfrak{v}(x) \geq n - 1\} \). If \( n - 1 \in \mathfrak{v}(I) \), then \( L = I \) and \( I^v = L \) by the previous point. If \( n - 1 \notin \mathfrak{v}(I) \), then (since \( I \) contains any element of valuation at least \( n \)), \( L \) is a fractional ideal of \( R \) such that \( \mathfrak{v}(L) = \mathfrak{v}(I) \cup \{n - 1\} \); hence, it is divisorial and \( \ell(L/I) = 1 \). It follows that \( L = I^v \), as claimed.

(c) Suppose \( I \subseteq \gamma^{-1}J \), where \( \mathfrak{v}(\gamma) = 0 \). Since \( J \) is not divisorial, \( n - 1 \notin \mathfrak{v}(J) = \mathfrak{v}(\gamma^{-1}J) \); hence, using the previous point, \( I = I^v \cap \gamma^{-1}J \) is closed by \( \star_J \).

Conversely, suppose \( I = I^{\star_J} \). Since \( I \) is nondivisorial, there must be \( \gamma \in (I : J), \gamma \neq 0 \) such that \( I \subseteq \gamma^{-1}J \) and \( I^v \nsubseteq \gamma^{-1}J \). If \( \mathfrak{v}(\gamma) > 0 \), then \( \gamma^{-1}J \) contains the elements of valuation \( n - 1 \); it follows that \( I^v \subseteq \gamma^{-1}J \) and thus that \( I^v \subseteq I^{\star_J} \), against \( I = I^{\star_J} \). Hence, \( \mathfrak{v}(\gamma) = 0 \), as claimed.

(d) If \( I \subseteq \gamma^{-1}J \) for some \( J \in \Delta \) and some \( \gamma \) such that \( \mathfrak{v}(\gamma) = 0 \), then \( I^{\star_\Delta} \subseteq I^{\star_J} = I \), and thus \( I \) is \( \star_\Delta \)-closed.
Conversely, suppose \( I = I^*\Delta \). For every \( J \in \Delta \), the ideal \( I^*J \) is contained in \( I^* = I \cup \{ x \mid v(x) \geq n - 1 \} \); since \( \ell(I^n/I) = 1 \), it follows that \( I^*J \) is either \( I \) or \( I^n \). Since \( I = I^*\Delta \), it must be \( I^*J = I \) for some \( J \); by the previous point, \( I \subseteq \gamma^{-1}J \) for some \( \gamma \), as claimed. \( \square \)

An important consequence of the previous proposition is the following: suppose that \( \Delta \) is a set of nondivisorial ideals in \( \mathcal{F}_0(R) \) such that, when \( I \neq J \) are in \( \Delta \), then \( I \not\subseteq \gamma^{-1}J \) for all \( \gamma \) having valuation 0. Then, for every subset \( \Lambda \subseteq \Delta \), the set of ideals of \( \Delta \) that are \( \ast_{\Lambda} \)-closed is exactly \( \Lambda \); in particular, each nonempty subset of \( \Delta \) generates a different star operation.

We will use this observation to estimate the cardinality of \( \text{Star}(R) \) when the residue field is finite.

**Proposition 3.3.** Let \( R \) be a Kunz domain such that \( v(R) = \langle n, n + 1, \ldots, 2n - 3, 2n - 1 \rangle \), and suppose that the residue field of \( R \) has cardinality \( q < \infty \). Then,

\[
|\text{Star}(R)| \geq 2^\frac{q^{n-2} - 1}{q - 1} \geq 2^q^{n-3}.
\]

**Proof.** Let \( L := \{ x \in V \mid v(x) \geq n \} \); then, \( A := V/L \) is a \( K \)-algebra. Let \( e_1 \) be an element of valuation 1, and let \( e_i := e_1^i \); then, \( \{ 1 = e_0, e_1, \ldots, e_{n-1} \} \) projects to a \( K \)-basis of \( A \), which for simplicity we still denote by \( \{ e_0, \ldots, e_{n-1} \} \). The vector subspace spanned by \( e_0 \) is exactly the field \( K \).

Since \( V \) and \( L \) are stable by multiplication by every element of valuation 0, asking if \( \gamma I \subseteq J \) for some \( I, J \in \mathcal{F}_0(R) \) and some \( \gamma \) is equivalent to asking if there is a \( \gamma \in A \) of “valuation” 0 such that \( \gamma \mathcal{T} \subseteq \mathcal{J} \), where \( \mathcal{T} \) and \( \mathcal{J} \) are the images of \( I \) and \( J \), respectively, in \( A \). Hence, instead of working with ideals in \( \mathcal{F}_0(R) \) we can work with vector subspaces of \( A \) containing \( e_0 \).

Furthermore, if \( V \) is a vector subspace of \( A \) and \( \gamma \) has valuation 0, then \( \gamma V \) has the same dimension of \( V \); thus, if \( V \) and \( W \) have the same dimension, \( \gamma V \subseteq W \) if and only if \( \gamma V = W \). Let \( \sim \) denote the equivalence relation such that \( V \sim W \) if and only if \( \gamma V = W \) for some \( \gamma \) of valuation 0.

Let \( X \) be the set of 2-dimensional subspaces of \( A \) that contain \( e_0 \) but not \( e_{n-1} \). Then, the preimage of every element of \( X \) is a nondivisorial ideal.

An element of \( X \) is in the form \( \langle e_0, \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} \rangle \), where at least one among \( \lambda_1, \ldots, \lambda_{n-2} \) is not 0; since \( \langle e_0, f \rangle = \langle e_0, \lambda f \rangle \) for all \( \lambda \in K, \lambda \neq 0 \), there are exactly \( (q^{n-1} - q)/(q - 1) \) such subspaces.

Let \( V \in X \), say \( V = \langle e_0, f \rangle \), and consider the equivalence class \( \Delta \) of \( V \) with respect to \( \sim \). Then, \( W \in \Delta \) if and only if \( \gamma W = V \) for some \( \gamma \); since \( 1 \in W \), it follows that such a \( \gamma \) must belong to \( V \). Since \( \gamma \) has valuation 0, it must be in the form \( \lambda_0 e_0 + \lambda_1 f \) with \( \lambda_0 \neq 0 \); furthermore,
if $\gamma' = \lambda \gamma$ then $\gamma^{-1}V = \gamma'^{-1}W$. Hence, the cardinality of $\Delta$ is at most 
\[
\frac{q^{n-1} - q}{q - 1} = \frac{q^{n-2} - 1}{q - 1} \geq q^{n-3}
\]
equivalence classes; let $X'$ be a set of representatives of such classes, and let $Y$ be the preimage of $X'$ in the power set of $\mathcal{F}_0(R)$. Then, every subset of $Y$ generates a different star operation (with the empty set corresponding to the $v$-operation); it follows that 
\[
|\text{Star}(R)| \geq 2^{\frac{q^{n-2} - 1}{q - 1}} \geq 2^{q^{n-3}},
\]
as claimed. \hfill \Box

For $n = 4$, we can even calculate $|\text{Star}(R)|$.

**Proposition 3.4.** Let $R$ be a Kunz domain such that $v(R) = \langle 4, 5, 7 \rangle$, and suppose that the residue field of $R$ has cardinality $q < \infty$. Then, 
\[
|\text{Star}(R)| = 2^{q^2} + 3.
\]

**Proof.** Consider the same setup of the previous proof. We start by claiming that two vector subspaces $W_1, W_2$ of $A$ of dimension 3 that contain $e_0$ but not $e_3$ are equivalent under $\sim$.

Indeed, any such subspace must have a basis of the form $\{e_0, e_1 + \theta_1 e_3, e_2 + \theta_2 e_3\}$, and different pairs $(\theta_1, \theta_2)$ induce different subspaces; let $W(\theta_1, \theta_2) := \langle e_0, e_1 + \theta_1 e_3, e_2 + \theta_2 e_3 \rangle$. To show that two such subspaces are equivalent, we prove that they are all equivalent to $W(0, 0)$. Let $\gamma := e_0 - \theta_2 e_1 - \theta_1 e_2$: we claim that $\gamma W(\theta_1, \theta_2) = W(0, 0)$. Indeed, 
\[
\gamma (e_1 + \theta_1 e_3) = e_1 + \theta_1 e_3 - \theta_2 e_2 - \theta_1 e_3 = e_1 - \theta_2 e_2 \in W(0, 0),
\]
and likewise 
\[
\gamma (e_2 + \theta_2 e_3) = e_2 + \theta_2 e_3 - \theta_2 e_3 = e_2 \in W(0, 0).
\]
Hence, $W(\theta_1, \theta_2) \sim W(0, 0)$.

Consider now the set $\Delta$ of nondivisorial ideals in $\mathcal{F}_0(R)$. By Lemma 2.2 and Proposition 3.2, $\Delta$ is equal to the union of the set of the canonical ideals and the set $\mathcal{G}$ of the $I \in \mathcal{F}_0(R)$ such that $IT = T$. By Lemma 3.1 and Proposition 3.2, the elements of the latter correspond to the subspaces of $V/T$ containing $e_0$ but not $e_3$: hence, we can write $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where $\mathcal{G}_i$ contains the ideals of $\mathcal{G}$ corresponding to subspaces of dimension $i$.

Given $\star \in \text{Star}(R)$, let $\Delta(\star) := \{I \in \Delta \mid I = I^\star\}$. We claim that $\Delta(\star)$ is one of the following:

- $\mathcal{G}$;
- $\Delta \setminus \{J\}$;
- $\Lambda \cup \{T\}$ for some $\Lambda \subseteq \mathcal{G}_2$;
By Proposition 2.3 if $T \neq T^*$ (i.e., if $T \notin \Delta(\star)$) then $\star = v$, and
$\Delta(\star) = \emptyset$. 

If $\Delta(\star)$ contains a canonical ideal then $\star$ is the identity, and thus $\Delta(\star) = \Delta$.

If $I$ is $\star$-closed for some $I \in G_3$, then every element of $G_3$ must be closed, since any other $I' \in G_3$ is in the form $\gamma I$ for some $\gamma$ of valuation 0 (by the first part of the proof); furthermore, every element of $G_2$ is the intersection of the elements of $G_3$ containing it, and thus it is $\star$-closed. It follows that $\Delta(\star) = \Delta \setminus \{J\}$; in particular, there is only one such star operation.

Let $\star$ be any star operation different from the three above. Then, $\Delta(\star)$ must contain $T$ and cannot contain any canonical ideal nor any element of $G_3$. Hence, $\Delta(\star)$ must be equal to $\Lambda \cup \{T\}$ for some $\Lambda \subseteq G_2$. Moreover, $\Lambda \cup \{T\}$ is equal to $\Delta(\star)$ for some $\star$ if and only if $\Lambda$ is the (possibly empty) union of equivalence classes under $\sim$. It follows that $|\text{Star}(R)| = 2^x + 3$, where $x$ is the number of such equivalence classes.

By the proof of Proposition 3.3 the image of an element of $G_2$ is in the form $(e_0, f)$, where $f = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ with at least one between $\lambda_1$ and $\lambda_2$ nonzero. Let $V(\lambda_1, \lambda_2, \lambda_3)$ denote the subspace $(e_0, f)$; clearly, $V(\lambda_1, \lambda_2, \lambda_3) = V(c\lambda_1, c\lambda_2, c\lambda_3)$ for every $c \in K \setminus \{0\}$. The subspaces equivalent to $V$ must have the form $(e_0 + \theta f)^{-1}V$ for some $\theta \in K$, and, by using the basis $\{e_0, e_0 + \theta f\}$ of $V$, we see that $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = (e_0, (e_0 + \theta f)^{-1})$. If $\theta = 0$, then $e_0 + \theta f = e_0$, and thus $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = V(\lambda_1, \lambda_2, \lambda_3)$; suppose, from now on, that $\theta \neq 0$.

To calculate $(e_0 + \theta f)^{-1} = e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, we can simply expand the product $(e_0 + \theta f)(e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)$, using $e_i = 0$ for $i > 3$, and then impose that the coefficients of $e_1, e_2$ and $e_3$ are zero; we obtain
\[
\begin{align*}
\alpha_1 &= -\theta \lambda_1 \\
\alpha_2 &= -\theta (\lambda_1 \alpha_1 + \lambda_2) \\
\alpha_3 &= -\theta (\lambda_1 \alpha_2 + \lambda_2 \alpha_1 + \lambda_3).
\end{align*}
\]

Since $\theta \neq 0$, the set $\{e_0, (e_0 + \theta f)^{-1} - e_0\}$ is a basis of $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3)$; hence, $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = V(\alpha_1, \alpha_2, \alpha_3)$. We distinguish two cases.

If $\lambda_1 = 0$, then $\lambda_2 \neq 0$, and so we can suppose $\lambda_2 = 1$. Then, we have
\[
\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 &= -\theta \\
\alpha_3 &= -\theta \lambda_3.
\end{align*}
\]
and so $(e_0 + \theta f)^{-1}V(0, 1, \lambda_3) = V(0, -\theta, -\theta \lambda_3) = V(0, 1, \lambda_3)$ since $\theta \neq 0$. It follows that the only subspace equivalent to $V(0, 1, \lambda_3)$ is
$V(0, 1, \lambda_3)$ itself; since we have $q$ choices for $\lambda_3$, this case gives $q$ different equivalence classes.

If $\lambda_1 \neq 0$, we can suppose $\lambda_1 = 1$. Then, we get

$$
\begin{align*}
\alpha_1 &= -\theta \\
\alpha_2 &= -\theta(\alpha_1 + \lambda_2) = -\theta(-\theta + \lambda_2) \\
\alpha_3 &= -\theta(-\theta(-\theta + \lambda_2) - \theta\lambda_2 + \lambda_3).
\end{align*}
$$

Since $\theta \neq 0$, we can divide by $-\theta$, obtaining

$$(e_0 + \theta f)^{-1}V(1, \lambda_2, \lambda_3) = V(1, -\theta + \lambda_2, \theta^2 - 2\theta\lambda_2 + \lambda_3).$$

Since $-\theta + \lambda_2 \neq -\theta' + \lambda_2$ if $\theta \neq \theta'$, we have $(e_0 + \theta f)^{-1}V(1, \lambda_2, \lambda_3) \neq (e_0 + \theta' f)^{-1}V(1, \lambda_2, \lambda_3)$ for all $\theta \neq \theta'$; thus, every equivalence class is composed by $q$ subspaces. Since there are $q^2$ such subspaces, we get other $q$ equivalence classes.

Therefore, $G_2$ is partitioned into $2q$ equivalence classes, and so $|\text{Star}(R)| = 2^{2q} + 3$, as claimed. \hfill \Box

Remark 3.5.

(1) The estimate obtained in Proposition 3.3 grows very quickly; for example, if $q$ is fixed, it follows that the double logarithm of $|\text{Star}(R)|$ grows (at least) linearly in $n = \ell(V/R) + 1$. This should be compared with [8, Theorem 3.21], where the authors analyzed a case where the growth of $|\text{Star}(R)|$ was linear in $\ell(V/R)$.

(2) Thanks to Theorem 2.8, Proposition 3.3 also gives lower bounds for the cardinality of the set of star operations of $T := R \cup \{x \in V \mid v(x) = 2n - 2\}$. If $V = K[[X]]$ is the ring of power series, then $T$ will be equal to $K + X^nK[[X]]$. In particular, for $n = 4$, we have $|\text{Star}(T)| \geq 2^{2q} + 3$, which is an estimate pretty close to the precise cardinality of $\text{Star}(T)$, namely $2^{2q+1} + 2^{q+1} + 2$ [14, Corollary 4.1.2].

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Dipartimento di Matematica e Fisica, Università degli Studi “Roma Tre”, Roma, Italy
E-mail address: spirito@mat.uniroma3.it