On Diffeomorphism Invariance and Black Hole Entropy

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Abstract. The Noether-charge realization and the Hamiltonian realization for the \textit{diff}(\mathcal{M}) algebra in diffeomorphism invariant gravitational theories are studied in a covariant formalism. For the Killing vector fields, the Noether-charge realization leads to the mass formula as an entire vanishing Noether charge for the vacuum black hole spacetimes in general relativity and the corresponding first law of the black hole mechanics. It is analyzed in which sense the Hamiltonian functionals form the \textit{diff}(\mathcal{M}) algebra under the Poisson bracket and shown how the Noether charges with respect to the diffeomorphism generated by vector fields and their variations in general relativity form this algebra. The asymptotic behaviors of vector fields generating diffeomorphism of the manifold with boundaries are discussed. In order to get more precise estimation for the "central extension" of the algebra, it is analyzed in the Newman-Penrose formalism and shown that the "central extension" for a large class of vector fields is always zero on the Killing horizon. It is also checked whether the Virasoro algebra may be picked up by choosing the vector fields near the horizon. The conclusion is unfortunately negative.

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I. INTRODUCTION

It is well known that the relation between the symmetry and related conservation law is one of the cornerstones in physics. For the spacetime manifolds in general relativity (GR), the diffeomorphisms form an infinite dimensional group under the composition. Therefore, it should also play certain important roles.

In the canonical formalism, Brown and Henneaux [1] showed that the asymptotic symmetry group for the three-dimensional gravity with negative cosmological constant is the pseudo-conformal group in 2-dimension. Its canonical generators form a pair of Virasoro algebras with nontrivial central charges. Strominger [2] suggested that with the help of the Cardy formula [3], the asymptotic symmetry and the central charge can be used to explain the statistical origin of the Bekenstein-Hawking entropy [4] of the 3-dimensional black hole. In 1999, Carlip attempted to generalize the Brown-Henneaux-Strominger construction to the black holes in any dimension [5] but met some conceptual problems [6]. Recently, the generalization is analyzed more carefully by Park in the canonical formalism [7].

Carlip [8,9] also tries to realize the Strominger’s idea in any dimension by using the covariant phase-space formalism developed by Wald et al [10]- [13]. Carlip’s approach has been generalized to the charged black hole [14], the dilaton black hole [15], the Kaluza-Klein black hole [16], the quantum correction [17], etc. In the meanwhile, Dreyer, Ghosh and Wiśniewski pointed out that this covariant phase-space formalism by Carlip contains technical flaws [18]. Koga related the flaws to the singular behavior of the asymptotic forms of the relevant quantities [19]. In the study of the asymptotic symmetries on the Killing horizon in spherically symmetric spacetimes, Koga also showed that the algebra of the Poisson brackets does not acquire nontrivial central charges [19]. An alternative version of this approach is also studied in the first-order gravity formalism [20].

In the covariant phase-space formalism [8,9], the Hamiltonian functional conjugate to a vector field is treated as the generator of diffeomorphism algebra just like the Hamiltonian in the canonical approach. The diffeomorphism algebra is assumed to be realized by the Poisson bracket or by Dirac bracket [21] on the constraint surface. Unfortunately, the Hamiltonian functional conjugate to a vector field $\xi^a$, as was pointed out by Wald [11,12], does not always exist for a given boundary condition. In addition, in the lack of the definition of the Poisson bracket and thus the Dirac bracket in the covariant phase-space formalism, Carlip borrowed the Poisson bracket and the Dirac bracket from the ADM formalism [21]- [23]. Although there are a number of studies on the definition and properties of the Poisson brackets in the covariant phase-space approach [24], such as the ones in the de Donder-Weyl field theory, it is still no explicit proof for their equivalence with the Poisson bracket in the ADM formalism. In fact, in the ADM formalism the Poisson bracket is defined on the basis of the 1+3 decomposition of a spacetime manifold, but in the covariant phase-space formalism there is no explicit splitting of the time and the space components.

On the other hand, however, in both classical and quantum field theories, if a Lagrangian possesses certain symmetries, such as gauge symmetry and Poincaré symmetry, the corresponding Lie algebras can be generated by the Noether charges of the conservation currents with respect to the symmetries. This general approach should also be available for the diffeomorphism invariance of the diffeomorphism-invariant theories of gravity such as GR.

The main purposes of the present paper are four folds. First, it is to obtain, in a co-
variant formalism, the Noether charge by the horizontal variation of the Lagrangian for
diffeomorphism invariant gravitational theories. It is shown that for the Killing vector fields
the covariant formalism leads to the vacuum black hole mass formula as a vanishing Noether
charge as well as the first law of the black hole mechanics in vacuum GR. Secondly, it is
analyzed in which sense the Hamiltonian functionals form the $\text{diff}(\mathcal{M})$ algebra in the covari-
ant formalism. It is also shown how the $\text{diff}(\mathcal{M})$ algebra with possible “central extension”
can be realized by virtue of the Noether charges in vacuum GR. The third purpose is to
show why the “central extension” of the algebra vanishes on shell near the horizon. When
the boundary of the (partial) Cauchy surface is fixed on the horizon, it is obvious that the
“central extension” vanishes because the realizations themselves are trivial. In the approach
[8,9], the “central extensions” of the $\text{diff}(\mathcal{M})$ and the Virasoro algebra are dependent on the
asymptotic behaviors of the vector fields specially chosen that correspond to the situation
that the boundary of the partial Cauchy surface is not fixed. In order to estimate more
precisely the asymptotic behaviors of such a kind of the vector fields, the Newman-Penrose
(N-P) formalism is employed in this paper. The advantage of the N-P formalism is obvious.
It not only can be set up on the horizon as well as the spacetime region with $r > r_h$, but
also may avoid the arbitrariness in the asymptotic behavior estimation. It is then shown
that the “central extension” of $\text{diff}(\mathcal{M})$ for the diffeomorphism generated by the vector field
$\xi^a = T^a + Rn^a$ with $R \sim O(\chi_K)$ is always zero on the horizon. The fourth purpose of
the present paper is to check whether the Virasoro algebra can be picked up by choosing
the vector fields near the horizon. It is shown in the N-P formalism that algebra of the
vector fields selected in [8] under the Lie bracket does not give the Virasoro algebra. Even
in the sense of [8] that the Lie brackets of the vector fields form a Virasoro algebra on the
horizon, the algebra of the Hamiltonian functionals conjugate to the vector fields is trivial,
too, and should not be regarded as a Virasoro algebra. This indicates that such a kind of
pure symmetry analysis is not enough to give the statistical or conformal field theory origin
of the black hole entropy.

The paper is organized as follows: In the next section, we review the covariant formalism
including the Noether currents with respect to the diffeomorphisms generated by vector
fields and their charges in a diffeomorphism-invariant theory. We also show how the vacuum
black hole mass formula as a whole is a total vanishing Noether charge with respect to the
combination of the Killing vector fields for the stationary axisymmetric black holes in GR
and re-derive the fist law of the black hole mechanics for this configuration [25]. In section
III, the Hamiltonian realization and the Noether-charge realization of the diffeomorphism
algebra in the covariant phase-space formalism are studied. In section IV, the diffeomorphism
algebra for a part of a manifold is discussed. Sec. V is devoted to the asymptotic behavior
of the vector fields generating the diffeomorphism. In section VI, the “central extension”
of $\text{diff}(\mathcal{M})$ is estimated in the N-P formalism and shown why it always vanishes near the
horizon. In section VII, we show in what sense the Virasoro algebra can be obtained as a
subalgebra of diffeomorphism algebra and why it is trivial. Finally, we summarize this paper
and make some discussion in section VIII. The correspondence, in the N-P formalism, of the
asymptotic conditions on the horizon proposed by Carlip [8,9] is given in Appendix A. In
Appendix B, some relations used in the calculation in the N-P formalism are listed.
II. NOETHER CHARGES IN DIFFEOMORPHISM INVARIANT GRAVITATIONAL THEORY

Let us consider a diffeomorphism-invariant gravitational theory on 4-dimensional space-time manifold $\mathcal{M}$ [26].

A. Noether currents and their charges with respect to diffeomorphism

The horizontal variation of the Lagrangian 4-form $L$ of such a kind of theories induced by a vector field $\xi^a$ can be written as [11,12,25]
\begin{equation}
\hat{\delta}_\xi L = E\hat{\delta}_\xi g + d\Theta(g, \hat{\delta}_\xi g), \tag{2.1}
\end{equation}
where $E = 0$ gives rise to the Euler-Lagrange equation for the theory and $\Theta(g, \hat{\delta}_\xi g)$ is the symplectic potential 3-form. On the other hand, using the Lie derivative $\mathcal{L}_\xi$, one has
\begin{equation}
\hat{\delta}_\xi L = \mathcal{L}_\xi L = d(\xi \cdot L). \tag{2.2}
\end{equation}
Equating Eqs.(2.1) and (2.2), one gets
\begin{equation}
d\ast \ast j(\xi) + E\hat{\delta}_\xi g = 0, \tag{2.3}
\end{equation}
where $\ast$ is the Hodge star and
\begin{equation}
j(\xi) = \ast(\Theta(g, \hat{\delta}_\xi g) - \xi \cdot L) \tag{2.4}
\end{equation}
is the Noether current 1-form with respect to the diffeomorphism generated by the given vector field $\xi^a$. Its (entire) Noether charge is given by the integral over a generic Cauchy surface $\Sigma \in \mathcal{M}$
\begin{equation}
Q(\xi) = \int_\Sigma \ast j(\xi). \tag{2.5}
\end{equation}

In order to obtain the differential formula for mass, i.e. the first law of the black hole mechanics, it is needed the variation relation of this Noether current. To this end, it is natural to calculate the following bi-variation with respect to vertical and horizontal variation $\delta$ and $\hat{\delta}$ of $L$,
\begin{equation}
\delta \hat{\delta}_\xi L = \delta(\mathcal{E}\hat{\delta}_\xi g + d\Theta(\hat{\delta}_\xi g)) = \delta d(\xi \cdot L). \tag{2.6}
\end{equation}
Therefore, from these equations, it follows the variation relation of this Noether current
\begin{equation}
0 = \delta(\mathcal{E}\hat{\delta}_\xi g) + d\delta \ast j, \tag{2.7}
\end{equation}
where we have used the commutative property between the vertical variation operator and the differential operator, i.e. $\delta d = d\delta$. Thus, for the conservation of variation of the Noether current (2.4), i.e.
\begin{equation}
d\delta \ast j = 0, \tag{2.8}
\end{equation}
the necessary and sufficient condition is
\begin{equation}
\delta(\mathcal{E}\hat{\delta}_\xi g) = 0. \tag{2.9}
\end{equation}
B. Noether charges for the Killing vectors and the vacuum mass formula in GR

In vacuum GR, the Lagrangian 4-form in units of $G = c = 1$ reads

$$L = \frac{1}{16\pi} R \epsilon,$$

(2.10)

where $R$ is the scalar curvature and $\epsilon$ the volume 4-form. The symplectic potential takes the form of

$$\Theta_{abc}(g, \delta \xi g) = \frac{1}{16\pi} \left[ \nabla^d (g_{ef} \delta \xi g^{ef}) - \nabla^e \delta \xi g^{de} \right] \epsilon_{dabc}.$$

(2.11)

Thus, the Noether current and its charge may be explicitly written as

$$j_a(\xi) = \frac{1}{8\pi} G_{ab} \xi^b - \frac{1}{16\pi} \nabla^b (\nabla_b \xi_a - \nabla_a \xi_b)$$

(2.12)

and

$$Q(\xi) = \frac{1}{8\pi} \int \ast (G_{ab} \xi^b) - \frac{1}{16\pi} \int_{\partial \Sigma} \ast d \xi,$$

(2.13)

respectively, where $G_{ab}$ is the Einstein tensor and $\partial \Sigma$ the boundary of the Cauchy surface.

In particular, for the stationary axisymmetric black holes in the vacuum GR and the Killing vector

$$\chi_a^K = t_a^K + \Omega_H \phi_a^K,$$

(2.14)

$Q(\chi^K)$ vanishes and leads to the black hole mass formula [25], where $t_a^K$ and $\phi_a^K$ are the timelike and spacelike Killing vectors of the spacetime, respectively, $\Omega_H$ the angular velocity on the horizon and hereafter, the subscript ‘$K$’ denotes the vector (or the corresponding one-form) being the Killing one. The reasons for the vanishing Noether charge leading to the black hole mass formula are as follows: For a Killing vector,

$$\frac{1}{2} (\nabla_b \nabla^a \chi^K - \nabla_b \nabla^b \chi^K) = R^a_b \chi^K,$$

(2.15)

which is zero on shell and leads to

$$j(\chi^K) = -\frac{1}{16\pi} R_{\xi a} + \frac{1}{4\pi} R_{ab} \xi^b = -\frac{1}{16\pi} R_{\xi a} - \frac{1}{8\pi} \nabla^b (\nabla_b \xi_a - \nabla_a \xi_b).$$

(2.16)

Consequently, $Q(\chi^K)$ vanishes on shell. By the definition of the mass and the angular momentum of the black hole,

$$Q_{\infty}(\chi^K) = -\frac{1}{8\pi} \int_{S_{\infty}} \ast d \chi^K = M - 2\Omega_H J.$$

(2.17)

On the other hand,

$$Q_{S_H}(\chi^K) = -\frac{1}{8\pi} \int_{S_H} \ast d \chi^K = \frac{\kappa}{4\pi} A,$$

(2.18)
where $\kappa$ is the surface gravity and $A$ the area of the cross section of the event horizon. $Q$ will be referred to as the partial Noether charge of a close surface. For the whole asymptotically flat region, the Cauchy surface emanates from the bifurcation surface and extends to the spatial infinity. Thus, the boundary of the Cauchy surface $\partial \Sigma$ should be $S_H \cup S_\infty$, and the Noether charges should take the form

$$Q(\chi) = Q(\chi^\infty_K) + Q(\chi^H_K) = Q_\infty(\chi_K) - Q_{SH}(\chi_K), \quad (2.19)$$

where $Q(\chi^\infty_K)$ and $Q(\chi^H_K)$ are the Noether charges for the Cauchy surface with compact interior and the one with compact infinity, respectively. This directly leads to the vacuum mass formula in GR as a vanishing (entire) Noether charge. Namely,

$$\frac{\kappa}{4\pi} A - M + 2\Omega_H J = 0. \quad (2.20)$$

It should be mentioned that there are somewhat differences between the present Noether charge approach and the one by Wald et al. First, in their approach the Noether charge for a spacetime manifold with the Cauchy surface possessing both interior and infinite asymptotically flat boundaries has not been considered rather what the Cauchy surface has been introduced is the one with compact infinity. Secondly, the orientation of the bifurcation surface $S_H$ is taken as positive rather than the one induced from the Cauchy surface with compact infinity. In fact, if the induced orientation could be taken, the entropy were no longer the Noether charge even in their approach but the negative one.

The Noether charge may be defined for a finite region of a spacetime. When $\Sigma$ is not chosen to be the whole of the Cauchy surface but its portion with two boundaries $B_1$ and $B_2$ such that $A_{B_2} > A_{B_1}$, where $A_B$ stands for the area of the surface $B$,

$$Q(\xi) = Q_{B_2}(\xi) - Q_{B_1}(\xi) \quad (2.21)$$

gives the Noether charge for the portion of the spacetime region $R \times \Sigma_p$.

**C. The first law of black hole mechanics in vacuum GR**

Let us now derive, from the variation relation of this vanishing Noether charge, the differential formula for mass, i.e. the first law of black hole mechanics, for the vacuum stationary axisymmetric black holes in GR by the variational approach.

For the vacuum gravitational fields in GR, the variation relation of this vanishing Noether current Eq. (2.9) becomes

$$0 = \frac{1}{16\pi} \delta(G_{ab} \delta \chi_K g^{ab}) + \delta[\nabla^a j_a(\chi_K)], \quad (2.22)$$

where $j_a(\chi_K)$ is given by Eq. (2.12). As was mentioned before, it is obvious that the conserved current $j$ vanishes for the Killing vector field $\chi^a_K$ (2.14) if the vacuum Einstein equation holds. Further, if the variation or the perturbation $\delta g$ is restricted in such a way that both $g$ and $g + \delta g$ are stationary axisymmetric black hole configurations and $\chi^a_K$ is the Killing vector (2.14), the vanishing Noether current and the variation of the dual of the conserved current should also vanish, i.e. $\delta * j = 0$ as well.
Thus it is straightforward to get

$$0 = \frac{1}{8\pi} \int_{\Sigma} \delta[R_{ab} \chi^{b}_{K} d\sigma^{a}] = \delta[M - \frac{\kappa}{4\pi} A - 2\Omega_{H} J],$$

(2.23)

and Eq. (2.9) is also satisfied. Here $\Sigma$ is a Cauchy surface.

It should be noticed that Eq. (2.23) is just the start point of Bardeen, Carter and Hawking’s calculation for the first law of the black hole mechanics [27]. As was required in [27], under the above perturbations, the positions of event horizon and the two Killing vector fields in Eq. (2.14) are unchanged. Consequently, as long as following what had been done in [27], Eq. (2.23) definitely leads to the differential formula for mass, i.e., the first law of black hole mechanics, among the stationary axisymmetric black hole configurations

$$\delta M - \frac{\kappa}{8\pi} \delta A - \Omega_{H} \delta J = 0.$$

(2.24)

Thus both the mass formula (2.20) and its differential formula (2.24), i.e., the first law of black hole mechanics for the stationary axisymmetric black hole configurations are all derived from the diffeomorphism invariance of the Lagrangian by the Noether charge via variational approach. Especially, they are in certain sense the vanishing Noether charge and its perturbation among the stationary axisymmetric black hole configurations.

III. ON REALIZATIONS OF $\text{diff}(\mathcal{M})$ ALGEBRA

In this section, we shall consider two different ways of realizations for the algebra $\text{diff}(\mathcal{M})$, i.e. the Hamiltonian realization and the Noether charge realization.

A. The Hamiltonian realization

It has been shown [28] that in gauge theories the flux of the symplectic current

$$\int_{\Sigma} *[\delta \mathbf{j}(\phi, \delta_{1}\phi) - \delta \mathbf{j}(\phi, \delta_{2}\phi)]$$

(3.1)

is equivalent to the Hamiltonian bracket in the reduced phase space. In Eq.(3.1) $\mathbf{j}$ is the Noether current, which is the dual of the symplectic potential $\Theta(\phi, \delta \phi) = \pi_{\phi}\delta \phi$ and $\delta \mathbf{j}(\phi, \delta_{1}\phi) - \delta \mathbf{j}(\phi, \delta_{2}\phi)$ is the (pre)symplectic current, where $\delta$ is a vertical variation.

The attempts to generalize the equivalence to the diffeomorphism invariant theories have been made [10]-[12] [29]. The (pre)symplectic current in a diffeomorphism invariant theory and its flux over a spacelike hypersurface are

$$\omega(\phi, \delta_{1}\phi, \delta_{2}\phi) = \delta_{1}\Theta(\phi, \delta_{2}\phi) - \delta_{2}\Theta(\phi, \delta_{1}\phi)$$

(3.2)

and

$$\Omega(\phi, \delta_{1}\phi, \delta_{2}\phi) = \int_{\Sigma} \omega(\phi, \delta_{1}\phi, \delta_{2}\phi),$$

(3.3)
respectively. In Eqs.(3.2) and (3.3), $\delta_1$ and $\delta_2$ have been generalized to arbitrary variations, which include both vertical and horizontal variations. The (pre-)symplectic structure, i.e. the flux of the (pre)symplectic current Eq.(3.3), has a remarkable property \[10,29\]

$$\Omega(\phi, \delta_1 \phi, \hat{\delta}_\xi \phi) = \Omega(\phi, \delta_1 \phi, L_{\xi} \phi) = 0.$$  

(3.4)

Namely, if one of the two variations is horizontal one, then the flux of the (pre)symplectic current must vanish. In \[11\] it has been shown explicitly that when the variation $\delta_2$ in Eq.(3.3) is restricted to the horizontal variation $\hat{\delta}_\xi$, $\delta H(\xi) = \Omega(\phi, \delta \phi, \hat{\delta}_\xi \phi)$,

where

$$H(\xi) = Q(\xi) - \int_{\partial \Sigma} \xi \cdot B,$$

(3.6)

$B$ is the 3-form satisfying

$$\delta \int_{\partial \Sigma} \xi \cdot B(\phi) = \int_{\partial \Sigma} \xi \cdot \Theta(\phi, \delta \phi).$$

(3.7)

It should be noted that in Eqs.(3.5)–(3.7)

$$\delta \xi = 0$$

(3.8)

is required \[11\]. In general, the vertical variations satisfy the condition (3.8), but the horizontal ones do not.

In \[8\] the variation $\delta$ in Eq.(3.5) is further restricted to the horizontal variation $\hat{\delta}_{\xi_2} = L_{\xi_2}$ so that the Cartan formula $L_{\xi} = d \circ i_{\xi} + i_{\xi} \circ d$ for any differential form can be used, and the identification of the flux of the presymplectic current and the Poisson bracket $\{ \cdot, \cdot \}$ is made. Then, Eq.(3.5) becomes

$$\hat{\delta}_{\xi_2} H(\xi_1) = \Omega(\phi, \hat{\delta}_{\xi_2} \phi, \hat{\delta}_{\xi_1} \phi) = \{H(\xi_1), H(\xi_2)\}.$$  

(3.9)

However, there are two difficulties in Eq.(3.9). First, in accordance with the requirement (3.8) for the first equality, we should have $\hat{\delta}_{\xi_2} \xi_1 = 0$, which is obvious in contradiction with the basic formula $L_{\xi_2} \xi_1 = [\xi_2, \xi_1]$. Second, it is the non-trivial flux of symplectic current that can be used to define the Poisson bracket. Unfortunately, the property (3.4) says that the flux of presymplectic current $\Omega(\phi, \hat{\delta}_{\xi_2} \phi, \hat{\delta}_{\xi_1} \phi)$ vanishes, so that the second equality is not valid. In order to reconcile the first contradiction, we have two choices:

A. The two vector fields are not arbitrary but fall into special classes. For example, the vector fields coincide up to a multiplication constant at the boundaries of Cauchy surfaces if $\hat{\delta}_\xi H$ is the algebraic summation of boundary terms.

B. Since the first variations of fields are linear and homogeneous functions of $\xi$, we may define a new quantity by

$$\bar{\hat{\delta}}_{\xi_2} F(\xi_1) := \hat{\delta}_{\xi_2} F(\xi_1) - F(\hat{\delta}_{\xi_2} \xi_1) = \hat{\delta}_{\xi_2} F(\xi_1) - F([\xi_2, \xi_1]).$$

(3.10)

By use of Eqs.(3.10), (2.1), (2.4), one gets
\[ \bar{\delta}_{\xi_2} \star j(\xi_1) = \bar{\delta}_{\xi_2} \Theta(\xi_1) - \xi_1 \cdot \bar{\delta}_{\xi_2} L \]
\[ = \bar{\delta}_{\xi_2} \Theta(\xi_1) - \xi_1 \cdot [E \bar{\delta}_{\xi_2} g + d \Theta(\bar{\delta}_{\xi_2} g)] \]
\[ = \bar{\delta}_{\xi_2} \Theta(\xi_1) - L \xi_1 \Theta(\bar{\delta}_{\xi_2} g) + d[\xi_1 \cdot \Theta(\bar{\delta}_{\xi_2} g)]. \] (3.11)

where on-shell condition has been used. Therefore, if the variation equation
\[ \bar{\delta} \int_{\partial \Sigma} \xi \cdot B(\phi) = \int_{\partial \Sigma} \xi \cdot \Theta(\phi, \bar{\delta} \phi), \] (3.12)
then the first equality of Eq.(3.9) should be modified as
\[ \bar{\delta}_{\xi_2} H(\xi_1) = \Omega'(\phi, \hat{\delta}_{\xi_2} \phi, \hat{\delta}_{\xi_1} \phi), \] (3.13)
where
\[ \Omega'(\phi, \hat{\delta}_{\xi_1} \phi, \hat{\delta}_{\xi_2} \phi) = \int_{\Sigma} \omega'(\phi, \hat{\delta}_{\xi_1} \phi, \hat{\delta}_{\xi_2} \phi) \]
\[ = \int_{\Sigma} \bar{\delta}_{\xi_1} \Theta(\phi, \hat{\delta}_{\xi_2} \phi) - \bar{\delta}_{\xi_2} \Theta(\phi, \hat{\delta}_{\xi_1} \phi). \] (3.14)

In Eq.(3.14), \( L_{\xi_1} \Theta(\bar{\delta}_{\xi_2} g) \) has been written as \( \bar{\delta}_{\xi_1} \Theta(\phi, \bar{\delta}_{\xi_2} \phi) \) on account that the appearance of \( \bar{\delta}_{\xi} \) implies that \([\xi_1, \xi_2]\) is discarded in the calculation.

Even when the triviality problem of the flux of the presymplectic current is put by, there is ambiguity in the identification of the Poisson bracket with the flux of the presymplectic currents, taking the form of Eq.(3.9) or the form like
\[ \Omega'(\phi, \hat{\delta}_{\xi_1} \phi, \hat{\delta}_{\xi_2} \phi) = \{H(\xi_1), H(\xi_2)\}. \] (3.15)

According to the Brown-Henneaux analysis in the canonical approach [1], the diffeomorphism algebra in terms of the Dirac bracket for the boundary terms of the Hamiltonian functionals in the covariant phase space formalism is, in Ref. [8], represented in the form
\[ \{J(\xi_1), J(\xi_2)\}^* = J([\xi_1, \xi_2]) + K^H(\xi_1, \xi_2), \] (3.16)
where \( K^H(\xi_1, \xi_2) \) is the possible central extension in the Hamiltonian realization of \( \text{diff}(\mathcal{M}) \).
The upper index \( H \) stands for this realization.

**B. The Noether-charge realization**

In gauge theories the (pre)symplectic current (3.2) and the antisymmetric combination of the vertical variations of the Noether current
\[ \delta_1 \star j(\phi, \delta_2 \phi) - \delta_2 \star j(\phi, \delta_1 \phi) =: \tilde{\omega}(\phi, \delta_1 \phi, \delta_2 \phi). \] (3.17)

coincide. In general, however, they are different from each other in diffeomorphism invariant theories. With the help of the flux of the current (3.17), it is possible to realize the \( \text{diff} \) algebra by use of Noether charges [30].
In order to set up the Noether-charge realization, let us consider the two successive horizontal variations of the Lagrangian 4-form induced by two vector fields $\xi_1$ and $\xi_2$ denoted by $\delta_{\xi_1}$ and $\delta_{\xi_2}$, respectively,

$$\delta_{\xi_1}\delta_{\xi_2} L = \delta_{\xi_1}[E\delta_{\xi_2}g + d\Theta(g, \delta_{\xi_2}g)] = \delta_{\xi_1}d(\xi_2 \cdot L). \quad (3.18)$$

Exchanging the order of the variation, one obtains

$$\delta_{\xi_2}\delta_{\xi_1} L = \delta_{\xi_2}[E\delta_{\xi_1}g + d\Theta(g, \delta_{\xi_1}g)] = \delta_{\xi_2}d(\xi_1 \cdot L). \quad (3.19)$$

Subtraction of the two equations gives rise to

$$\delta_{\xi_1}\delta_{\xi_2} L - \delta_{\xi_2}\delta_{\xi_1} L = \delta_{\xi_1}[E\delta_{\xi_2}g + d\Theta(g, \delta_{\xi_2}g)] - \delta_{\xi_2}[E\delta_{\xi_1}g + d\Theta(g, \delta_{\xi_1}g)]$$

$$= \delta_{\xi_1}d(\xi_2 \cdot L) - \delta_{\xi_2}d(\xi_1 \cdot L). \quad (3.20)$$

Since the Lie bracket of two vector fields gives a new vector field, Eqs.(2.1) and (2.2) can also apply to the Lie bracket of the two vector fields, namely,

$$\delta_{[\xi_1, \xi_2]} L = E\delta_{[\xi_1, \xi_2]}g + d\Theta(g, \delta_{[\xi_1, \xi_2]}g) = d([\xi_1, \xi_2] \cdot L). \quad (3.21)$$

Subtracting Eq.(3.21) from Eq.(3.20) and using the identity

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1, \xi_2]}, \quad (3.22)$$

one has

$$0 = [\delta_{\xi_1}, \delta_{\xi_2}] L - \delta_{[\xi_1, \xi_2]} L$$

$$= \delta_{\xi_1}[E\delta_{\xi_2}g + d\Theta(g, \delta_{\xi_2}g)] - \delta_{\xi_1}[E\delta_{\xi_2}g + d\Theta(g, \delta_{\xi_2}g)] - E\delta_{[\xi_1, \xi_2]}g + d\Theta(g, \delta_{[\xi_1, \xi_2]}g)$$

$$= \delta_{\xi_1}d(\xi_2 \cdot L) - \delta_{\xi_2}d(\xi_1 \cdot L) - d([\xi_1, \xi_2] \cdot L). \quad (3.23)$$

It follows that

$$d\{\delta_{\xi_1}[*j(\xi_2)] - \delta_{\xi_2}[*j(\xi_1)] - [\xi_1, \xi_2]j\} = \delta_{\xi_2}(E\delta_{\xi_1}g) - \delta_{\xi_1}(E\delta_{\xi_2}g) + E\delta_{[\xi_1, \xi_2]}g. \quad (3.24)$$

Namely, the combination of the current 1-forms as a Noether-like current 1-form

$$k(\xi_1, \xi_2) = *\delta_{\xi_1}[*j(\xi_2)] - *\delta_{\xi_2}[*j(\xi_1)] - j([\xi_1, \xi_2]) \quad (3.25)$$

is conserved as long as

$$\delta_{\xi_2}(E\delta_{\xi_1}g) - \delta_{\xi_1}(E\delta_{\xi_2}g) + E\delta_{[\xi_1, \xi_2]}g = 0. \quad (3.26)$$

Obviously, Eq.(3.26) is valid on shell. It should be noted that the condition Eq.(3.26) is weaker than the on-shell condition. If the charge of $k(\xi_1, \xi_2)$ is denoted by $-K^Q(\xi_1, \xi_2)$ with the upper index $Q$ for the Noether-charge realization, then $K^Q(\xi_1, \xi_2)$ satisfies\footnote{In the present paper, we consider the case that the Cauchy surface keeps unchanged under the horizontal variation.}

\[ \]
$\hat{\delta}_{\xi_1}[Q(\xi_2)] - \hat{\delta}_{\xi_2}[Q(\xi_1)] = Q([\xi_1, \xi_2]) - K^Q(\xi_1, \xi_2). \quad (3.27)$

The left-hand side of Eq.(3.27) is nothing but the flux of the current (3.17) over a Cauchy surface with the substitution of the vertical variation by the horizontal one

$$\bar{\Omega}(\phi, \hat{\delta}_1 \phi, \hat{\delta}_2 \phi) = \int_\Sigma \bar{\omega}(\phi, \hat{\delta}_1 \phi, \hat{\delta}_2 \phi)$$

$$= \hat{\delta}_1 Q(\phi) - \hat{\delta}_2 Q(\phi) + \hat{\delta}_1 \phi). \quad (3.28)$$

Again, the requirement Eq.(3.8) should be satisfied in the second variation in Eq.(3.28) in accordance with Eq.(3.17), which results in either the vector fields falling into specific classes or $\hat{\delta}$ being replaced by $\bar{\delta}$ in the second variation. For the second choice, by use of Eq. (3.10) the left-hand side of Eq.(3.27) becomes

$$(\hat{\delta}_{\xi_1} Q(\xi_2)) + Q(\hat{\delta}_{\xi_1} \xi_2) - (\hat{\delta}_{\xi_2} Q)(\xi_1) - Q(\hat{\delta}_{\xi_2} \xi_1)$$

$$= (\hat{\delta}_{\xi_1} Q(\xi_2)) + Q([\xi_1, \xi_2]) - (\hat{\delta}_{\xi_2} Q)(\xi_1) - Q([\xi_2, \xi_1]) - Q([\xi_1, \xi_2])$$

$$= \hat{\delta}_{\xi_1} Q(\xi_2) - \hat{\delta}_{\xi_2} Q(\xi_1) + 2Q([\xi_1, \xi_2]). \quad (3.29)$$

Therefore, the Noether charges and their variations on shell form the algebraic relation of $\text{diff}(\mathcal{M})$

$$\hat{\delta}_{\xi_2} Q(\xi_1) - \hat{\delta}_{\xi_1} Q(\xi_2) = Q([\xi_1, \xi_2]) + K^Q(\xi_1, \xi_2), \quad (3.30)$$

where $K^Q(\xi_1, \xi_2)$ is the possible central extension.

It should be stressed again that in Eq.(3.30) the variation of $Q$ is carried out under the condition that the first vector $\xi$ keeps unchanged. In another word, the variation does not act on the vector field. The left-hand side of Eq.(3.30) is the flux of the exterior variation of the dual of Noether current Eq.(3.28) after the replacement of $\hat{\delta}$ by $\bar{\delta}$. It is remarkable that the ambiguity in the definition of the Poisson bracket in the Hamiltonian realization does not appear in the Noether-charge realization because the Poisson bracket does not come in the Noether-charge realization at all.

C. The 2-cocycle condition for the central extension in Noether-charge realization

If an algebra realization satisfies the Jacobi identity, its central extension always obeys 2-cocycle condition from the Jacobi identity and vice versa. Now, let us consider

$$C_{\{1,2,3\}} \delta_{\xi_1} \{ \delta_{\xi_2} [Q(\xi_3)] - \delta_{\xi_1} [Q(\xi_2)] \} = C_{\{1,2,3\}} \delta_{\xi_1} [Q([\xi_2, \xi_3])] - C_{\{1,2,3\}} \delta_{\xi_1} [K^Q(\xi_2, \xi_3)], \quad (3.31)$$

where $C_{\{1,2,3\}}$ denotes the circular summation. By use of identity (3.22), the left-hand side of (3.31) can be written as

$$C_{\{1,2,3\}} \{ \delta_{\xi_1} \delta_{\xi_2} [Q(\xi_3)] - \delta_{\xi_2} \delta_{\xi_1} [Q(\xi_3)] \} = C_{\{1,2,3\}} \delta_{[\xi_1, \xi_2]} [Q(\xi_3)] = C_{\{1,2,3\}} \delta_{[\xi_2, \xi_3]} [Q(\xi_1)]. \quad (3.32)$$

It follows from (3.31) that

$$C_{\{1,2,3\}} \delta_{\xi_1} [K^Q(\xi_2, \xi_3)] = C_{\{1,2,3\}} \{ \delta_{\xi_1} [Q([\xi_2, \xi_3])] - \delta_{[\xi_2, \xi_3]} [Q(\xi_1)] \}. \quad (3.33)$$
On the other hand, from (3.27) one has
\[ \hat{\delta}_{[\xi_2, \xi_3]}[Q(\xi_1)] - \delta_{\xi_1}[Q([\xi_2, \xi_3])] = Q([\xi_2, \xi_3, \xi_1]) - K^Q([\xi_2, \xi_3], \xi_1). \] (3.34)

Eqs. (3.33) and (3.34) and the Jacobi identity
\[ C_{\{1,2,3\}}[\hat{\delta}_{\xi_1}, [\hat{\delta}_{\xi_2}, \hat{\delta}_{\xi_3}]] = 0 \] (3.35)
give rise to
\[ C_{\{1,2,3\}}[\hat{\delta}_{\xi_1}, K^Q(\xi_2, \xi_3)] = C_{\{1,2,3\}} K^Q([\xi_1, \xi_2], \xi_3). \] (3.36)

On account of identity (3.22), the Jacobi identity becomes
\[ C_{\{1,2,3\}}[\hat{\delta}_{\xi_1}, \hat{\delta}_{[\xi_2, \xi_3]}] = 0. \] (3.37)

Applying it on \( L \), one has
\[ C_{\{1,2,3\}} \{ \hat{\delta}_{\xi_1} [Q([\xi_2, \xi_3])] - \hat{\delta}_{[\xi_2, \xi_3]} [Q(\xi_1)] \} = 0, \] (3.38)
which leads to
\[ C_{\{1,2,3\}}[\hat{\delta}_{\xi_1}, K^Q(\xi_2, \xi_3)] = C_{\{1,2,3\}} K^Q([\xi_1, \xi_2], \xi_3) = 0. \] (3.39)

This is the 2-cocycle condition for the central extension.

IV. Diffeomorphism Algebra for a Portion of a Manifold

In this section, we focus on the case of stationary, asymptotically flat, axisymmetric spacetimes in vacuum GR. The Carter-Penrose diagram for the asymptotically flat region of a stationary axisymmetric spacetime is shown in the figure 1.

Now, we consider the Cauchy surface which is combined by two pieces \( \Sigma_1 \) and \( \Sigma_2 \) as shown in the figure 1. Namely,
\[ \Sigma = \Sigma_1 \cup \Sigma_2. \] (4.1)

\( \Sigma_1 \) emanates from the bifurcation surface \( S_H \), extends almost along the generator of the event horizon and ends at the certain place of the stretched Killing horizon [8] denoted by \( B_\epsilon \). At the end of calculation, \( \Sigma_1 \) tends to the event horizon by taking \( \epsilon \to 0 \). \( \Sigma_2 \) is a spacelike hypersurface matching \( \Sigma_1 \) at \( B_\epsilon \) and extending to the spatial infinity. The boundaries of \( \Sigma_1 \) and \( \Sigma_2 \) are \( \partial \Sigma_1 = S_H^{(-)} \cup B_\epsilon \) and \( \partial \Sigma_2 = B_\epsilon^{(-)} \cup S_\infty \), respectively. Such a choice of the Cauchy surface, may be called a combined Cauchy surface, will not affect the fact that the Noether charge corresponding to the Killing vector field (2.14) for the whole asymptotically flat region leads to the black hole mass formula [25]. In the following, we will discuss the problem in the Noether-charge realization and the Hamiltonian realization separately.
FIG. 1. The Carter-Penrose diagram of an asymptotically-flat region for stationary axisymmetric spacetimes. $\mathcal{I}^+$, $\mathcal{I}^-$, and $i^0$ are the future and past null infinity, and the spatial infinity, respectively. $\mathcal{H}^+$, $\mathcal{H}^-$, and $S_H$ are the future and past event horizon, and the bifurcation surface, respectively. $\Sigma = \Sigma_1 \cup \Sigma_2$ is the Cauchy surface for the whole of the asymptotically flat region. $\Sigma_1$ and $\Sigma_2$ match at $B_\epsilon$. $\epsilon$ tends to 0 so that $\Sigma_1$ tends to the event horizon.
A. Noether-charge realization

The Noether charge defined on the combined Cauchy surface (4.1) is combined in the following way:

\[ Q_\Sigma(\xi) = Q_{\Sigma_1}(\xi) + Q_{\Sigma_2}(\xi), \]  

(4.2)

where \( Q_{\Sigma_1}(\xi) \) and \( Q_{\Sigma_2}(\xi) \) are the Noether charges on \( \Sigma_1 \) and \( \Sigma_2 \), respectively. On shell, \( Q_{\Sigma_1} \) and \( Q_{\Sigma_2} \) may always be expressed, as mentioned before, in terms of the algebraic summation of the boundary terms

\[ Q_{\Sigma_1}(\xi) = Q_{B_\varepsilon}(\xi) - Q_H(\xi), \]  

(4.3)

and

\[ Q_{\Sigma_2}(\xi) = Q_\infty(\xi) - Q_{B_\varepsilon}(\xi), \]  

(4.4)

where \( Q_{B_\varepsilon}(\xi) \), \( Q_H(\xi) \), and \( Q_\infty(\xi) \) are evaluated at \( B_\varepsilon \), the bifurcation surface \( S_H \) and the spatial infinity \( S_\infty \), respectively. It is easy to show that for the Killing vector field (2.14),

\[ \lim_{\varepsilon \to 0} Q_{B_\varepsilon}(\chi_K) = Q_H(\chi_K) = \frac{\kappa}{8\pi} A. \]  

(4.5)

That is, \( \lim_{\varepsilon \to 0} Q_{B_\varepsilon} \) for the Killing vector (2.14) is proportional to the area \( A \) of the 2-dimensional horizon.

According to Sec. III.B, the Noether charges \( Q_{\Sigma_1}(\xi) \) and \( Q_{\Sigma_2}(\xi) \) over \( \Sigma_1 \) and \( \Sigma_2 \) and their (horizontal) variation form the algebraic relations for \( \text{diff}(R \times \Sigma_1) \) and \( \text{diff}(R \times \Sigma_2) \), respectively. For example, the algebraic relation for \( Q_{\Sigma_2}(\xi) \) takes the form

\[ \bar{\delta}_\xi Q_{\Sigma_2}(\xi_1) - \bar{\delta}_\xi Q_{\Sigma_2}(\xi_2) = Q_{\Sigma_2}([\xi_1, \xi_2]) + K^Q_{\Sigma_1}(\xi_2, \xi_2). \]  

(4.6)

By use of Eq.(4.4), Eq.(4.6) reduces to

\[ \bar{\delta}_\xi Q_{B_\varepsilon}(\xi_1) - \bar{\delta}_\xi Q_{B_\varepsilon}(\xi_2) - Q_{B_\varepsilon}([\xi_1, \xi_2]) - \tilde{K}^Q_{B_\varepsilon}(\xi_1, \xi_2) 
\]

\[ = \bar{\delta}_\xi Q_\infty(\xi_1) - \bar{\delta}_\xi Q_\infty(\xi_2) - Q_\infty([\xi_1, \xi_2]) - \tilde{K}^Q_\infty(\xi_1, \xi_2). \]  

(4.7)

The two sides of Eq.(4.7) are evaluated at two different places. Therefore, each of them should form an algebra independently. At most, they are equal to the same constant on \( \Sigma_2 \). (The value of the constant may be the vector-field dependent.) Absorbing the constant into the central extension \( K^Q(\xi_1, \xi_2) \), one has

\[ \bar{\delta}_\xi Q_{B_\varepsilon}(\xi_1) - \bar{\delta}_\xi Q_{B_\varepsilon}(\xi_2) = Q_{B_\varepsilon}([\xi_1, \xi_2]) + K^Q_{B_\varepsilon}(\xi_1, \xi_2), \]  

(4.8)

and

\[ \bar{\delta}_\xi Q_\infty(\xi_1) - \bar{\delta}_\xi Q_\infty(\xi_2) = Q_\infty([\xi_1, \xi_2]) + K^Q_\infty(\xi_1, \xi_2). \]  

(4.9)

Eqs.(4.8) and (4.9) are the algebraic relation for the partial Noether charges.
B. Hamiltonian realization

Since the Hamiltonian functional is also an additive quantity, it may be written as

$$H_{\Sigma}(\xi) = H_{\Sigma_1}(\xi) + H_{\Sigma_2}(\xi). \quad (4.10)$$

On $\Sigma_i$ ($i = 1, 2$), the Hamiltonian realization is

$$\{H_{\Sigma_i}(\xi_1), H_{\Sigma_i}(\xi_2)\} = H_{\Sigma_i}([\xi_1, \xi_2]) + K_{\Sigma_i}^{H}(\xi_1, \xi_2). \quad (4.11)$$

If the field $B$ exists on the intersection of $\Sigma_1$ and $\Sigma_2$, the Hamiltonian functionals, like the Noether charges, may be expressed in terms of their boundary terms

$$H_{\Sigma_1}(\xi) = J_B(\xi) - J_H(\xi), \quad (4.12)$$
$$H_{\Sigma_2}(\xi) = J_\infty(\xi) - J_B(\xi). \quad (4.13)$$

Similar to the Noether-charge realization,

$$\{J_H(\xi_1), J_H(\xi_2)\}^\ast = J_H([\xi_1, \xi_2]) + K_{H_i}^{H}(\xi_1, \xi_2), \quad (4.14)$$
$$\{J_B(\xi_1), J_B(\xi_2)\}^\ast = J_B([\xi_1, \xi_2]) + K_{B_i}^{H}(\xi_1, \xi_2), \quad (4.15)$$
$$\{J_\infty(\xi_1), J_\infty(\xi_2)\}^\ast = J_\infty([\xi_1, \xi_2]) + K_{\infty}^{H}(\xi_1, \xi_2). \quad (4.16)$$

Eq. (4.15) is used to calculate the central term in [8].

V. ASYMPTOTIC BEHAVIORS

A. Near the spatial infinity

In the present paper, we are only interested in the diffeomorphism of a manifold into itself. The horizontal variations induced by such kinds of diffeomorphism do not change the characters of a manifold. In particular, a Cauchy surface is mapped into another Cauchy surface under a horizontal variation in active point of view. For stationary asymptotically flat spacetimes, it requires that the infinite boundaries of Cauchy surfaces coincide at the same point $i^0$ on the Carter-Penrose diagram. If the discussion is confined in the deformation of ‘$t - r$’ plane, it requires that the vector fields generating diffeomorphisms must satisfy

$$\xi^a \sim C \left( \frac{\partial}{\partial t} \right)^a, \quad \text{as } r \to \infty, \quad (5.1)$$

where $C$ is a constant, $(\frac{\partial}{\partial t})^a$ is the timelike Killing vector. The Lie bracket of two vector fields of such a kind obviously vanishes at the spatial infinity.
B. Near the bifurcation surface $S_H$

For the stationary asymptotically flat spacetimes with a black hole, the inner boundary of the Cauchy surfaces coincide at the bifurcation surface $S_H$. Limited in the deformation of ‘$t - r$’ plane, it requires that the vector fields generating diffeomorphisms must satisfy

$$\xi^a \sim C' \chi^a_K,$$

where $C' \sim O(1)$ is another constant, $\chi^a_K$ the Killing vector Eq.(2.14). Since $\chi^a_K = 0$ on the bifurcation surface, the Lie bracket of two vector fields of such a kind is also zero at the bifurcation.

C. Near the stretched horizon $B_\epsilon$

Let $S = \lim_{\epsilon \to 0} B_\epsilon$ be on the Killing horizon. If $S$ is fixed on the Killing horizon, $\Sigma_1$ and $\Sigma_2$ are the Cauchy surfaces for the spacetime regions $\Sigma_1 \times R$ and $\Sigma_2 \times R$, respectively. In that case, if the discussion is again limited in ‘$t - r$’ plane,

$$\xi^a \to 0, \quad \text{as } \epsilon \to 0. \quad (5.3)$$

Thus, the Lie bracket of the two vector fields is still zero on $S$.

If $S$ is not fixed on the Killing horizon under the diffeomorphism, $\Sigma_1$ and $\Sigma_2$ are not the Cauchy surfaces of the spacetime regions $\Sigma_1 \times R$ and $\Sigma_2 \times R$, respectively, but $\Sigma_1 \cup \Sigma_2$ is still the Cauchy surface of the asymptotically flat region. In order $S$ to be mapped from a sphere to another sphere on the Killing horizon, the vector field should has the form

$$\xi^a = \tilde{T} \chi^a_K, \quad \text{on the Killing horizon.} \quad (5.4)$$

where $\tilde{T}$ is, at most, the function of $v$, $\theta$, and $\phi$, where $v$ is the parameter of the generator of the Killing horizon and $\theta$ and $\phi$ are coordinates on the cross section of the horizon. The Lie bracket of two vector fields of type (5.4) gives a new vector field of the same type. It should be noted that the vector field (5.4) does not satisfy the boundary condition (4.3) in [8] generally, which reads

$$\frac{\chi^a_K \chi^b_K \delta_{ab} g_{ab}}{\chi^2_K} \to 0, \quad \chi^2_K \to 0. \quad (5.5)$$

The boundary condition (4.3) in [8] is discussed in detail in Appendix A.

VI. THE NULL TETRAD AND CENTRAL EXTENSION

Now we are ready to calculate the central extension. It is obvious that both the Hamiltonian and Noether-charge realizations are trivial at the spatial infinity and the bifurcation surface $S_H$. Even on $S$, both realizations are also trivial when $S$ is fixed. In these cases, the central extensions are all zero.

In this section, we focus on the central extension on $S$ in the case that $S$ is unfixed under the diffeomorphism. As was emphasized in the introduction, in order to estimate the asymptotic behavior precisely we employ the N-P formalism.
A. Null tetrad

The (timelike) Killing vector $\chi^a_K$ defines the Killing horizon of a black hole via $\chi^2_K = 0$. Define a (spacelike) vector field $\rho^a$ orthogonal to the Killing vector $\chi^a_K$ by

$$\nabla_a \chi^2_K = -2\kappa \rho_a,$$

where $\kappa$ is the surface gravity on the horizon. Let $t^a_i$ ($i = 1, 2$) be the two independent tangent vector fields on the cross section of the horizon.

It is easy to check that $\rho^2 \to 0$, $\rho^a \to \chi^a_K$, and $\rho^2/\chi^2_K \to -1$ as the horizon is tended. These properties result in that the set of basis consisting of the four vector fields $\{\chi^a_K, \rho^a, t^a_i\}$ is ill-defined on the horizon.

In order to set up the basis that are well-defined on the horizon, we can first choose two real null vector fields $l^a$, $n^a$ as

$$l^a = \frac{1}{2}(\chi^a_K + \frac{|\chi^a_K|}{\rho} \rho^a)$$

$$n^a = -\frac{1}{\chi^2_K} \chi^a_K - \frac{|\chi^a_K|}{\rho} \rho^a),$$

or

$$\chi^a_K = l^a - \frac{\chi^2_K}{2} n^a$$

$$\rho^a = \frac{\rho}{|\chi^a_K|} (l^a + \frac{\chi^2_K}{2} n^a).$$

(6.2)

It is easy to see that $l^a n_a = -1$, $l^a l_a = n^a n_a = 0$ holds on the whole spacetime. Combined with other two complex null vector fields $m^a$ and $\bar{m}^a$, the set of the null vector fields $\{l$, $n$, $m$, $\bar{m}\}$ constitute the well-defined null tetrad fields on the whole spacetime.

Now, consider the vector fields

$$\xi^a = T l^a + R n^a = \tilde{T} \chi^a_K + \tilde{R} \rho^a.$$

(6.4)

Obviously,

$$T = \tilde{T} + \frac{\rho}{|\chi^a_K|} \tilde{R}$$

$$R = \frac{\chi^2_K}{2} (-\tilde{T} + \frac{\rho}{|\chi^a_K|} \tilde{R}).$$

(6.5)

If we require both $\tilde{T}$ and $\tilde{R}$ to be regular functions on the whole spacetime as required in $[8]$, we have $R \sim O(\chi^2_K)$.

B. The vector-field realization of $\text{diff}(\mathcal{M})$ under Lie bracket

Let us consider $[l, n]$ first. By definition,
\[ [l, n] = \frac{1}{2} \left\{ [\chi^a_K, -\frac{1}{\chi^2_K} \chi^a_K] - \left[ \frac{\chi_K}{\rho} \rho^a, \frac{1}{\chi^2_K} \chi^a_K \right] + [\chi^a_K, \frac{1}{\chi^2_K} \frac{\chi_K}{\rho} \rho^a] \\
+ \left[ \frac{\chi_K}{\rho} \rho^a, \frac{1}{\chi^2_K} \frac{\chi_K}{\rho} \rho^a \right] \right\} \tag{6.6} \]

The first term in the brace bracket in Eq.(6.6) is

\[ [\chi^a_K, \frac{1}{\chi^2_K} \chi^a_K] = \frac{1}{\chi^2_K} [\chi^a_K, \chi^a_K] + (D \frac{1}{\chi^2_K}) \chi^a_K = 0, \tag{6.7} \]

where \( D := \chi^a_K \nabla_a \). The second term is

\[ \left[ \frac{\chi_K}{\rho} \rho^a, \frac{1}{\chi^2_K} \chi^a_K \right] = \frac{\chi_K}{\rho} (\bar{D} \frac{1}{\chi^2_K}) \chi^a_K - \frac{1}{\chi^2_K} (D \frac{\chi_K}{\rho} \rho^a) + \frac{\chi_K}{\rho} \frac{1}{\chi^2_K} \{ \rho, \chi \} = -2\kappa \frac{1}{\chi^2_K} \frac{\rho}{\chi_K} \chi_K, \tag{6.8} \]

where \( \bar{D} := \rho^a \nabla_a \). The third term is

\[ \left[ \frac{\chi_K}{\rho} \rho^a, \frac{1}{\chi^2_K} \frac{\chi_K}{\rho} \rho^a \right] = \frac{D}{\chi^2_K} \frac{\chi_K}{\rho} \rho^a + \frac{1}{\chi^2_K} (D \frac{\chi_K}{\rho} \rho^a) + \frac{1}{\chi^2_K} \frac{\chi_K}{\rho} \{ \chi, \rho \} = 0. \tag{6.9} \]

The last term is

\[ \left[ \frac{\chi_K}{\rho} \rho^a, \frac{1}{\chi^2_K} \frac{\chi_K}{\rho} \rho^a \right] = -\frac{\chi^2_K}{\rho^2} (\bar{D} \frac{1}{\chi^2_K}) \rho^a + \frac{1}{\chi^2_K} \left[ \frac{\chi_K}{\rho} \rho^a, \frac{\chi_K}{\rho} \rho^a \right] = -2\kappa \frac{1}{\chi^2_K} \rho^a. \tag{6.10} \]

So, the Lie bracket of the null vector fields \( l \) and \( n \) is

\[ [l^a, n^a] = -\kappa \frac{\rho}{|\chi_K|} n^a. \tag{6.11} \]

For the sake of the later convenience, denote \( n^a \nabla_a \) as \( \Delta \) and \( l^a \nabla_a \) as \( \bar{D} \). Then, the above result is equivalent to

\[ [\bar{D}, \Delta] = -\kappa \frac{\rho}{|\chi_K|} \Delta \tag{6.12} \]

for any function \( f \). In comparison with the commutator of the N-P formalism [31], the following relation among the N-P coefficients are obtained:

\[ \tau + \bar{\pi} = 0, \]
\[ \gamma + \bar{\gamma} = 0, \]
\[ \epsilon + \bar{\epsilon} = \kappa \frac{\rho}{|\chi_K|}. \tag{6.13} \]

For the vector fields of Eq.(6.4),

\[ [\xi_1^a, \xi_2^a] = [T_1 l^a, T_2 l^a] + [R_1 n^a, T_2 n^a] + [T_1 l^a, R_2 n^a] + [R_1 n^a, R_2 n^a] \]
\[ = (T_1 \bar{D} T_2 - T_2 \bar{D} T_1) l^a + (R_1 \Delta T_2 - R_2 \Delta T_1) n^a + (T_1 \bar{D} R_2 - T_2 \bar{D} R_1) n^a \]
\[ + \kappa \frac{\rho}{|\chi_K|} (R_1 T_2 - T_1 R_2) n^a + (R_1 \Delta R_2 - R_2 \Delta R_1) n^a. \tag{6.14} \]
Because
\[
D\chi^2_K = \frac{1}{2}(\chi^a_K + \frac{|\chi_K|}{\rho}\rho^a)|\nabla_a\chi^2_K = \chi^2_K\frac{\rho}{|\chi_K|} = O(\chi^2_K),
\]
\[
\Delta\chi^2_K = -\frac{1}{\chi_K}(|\chi_K|\rho^a)|\nabla_a\chi^2_K = 2\chi^2_K\frac{\rho}{|\chi_K|} = O(1),
\]
(6.15)

it is easy to see that the Lie bracket of $\xi$’s is closed as long as $R = O(\chi^2_K)$.

### C. On the central extension

When $S$ is mapped from one sphere to another on the Killing horizon under the diffeomorphism, either $\Sigma_1$ or $\Sigma_2$ cannot be regarded as the Cauchy surface of subsystems $\Sigma_1 \times R$ and $\Sigma_2 \times R$ and the Cauchy surface of the whole asymptotically flat region as well. The legitimacy of applying Eq.(3.5) to the subsystem becomes questionable because the integral over the (partial) Cauchy surface is taken in Eq.(3.5). In this subsection, we do not plan to discuss the problems in the formulation. Instead, we will just estimate the central extension in the null tetrad according to Ref. [8].

It has been written in [8] that for vacuum GR
\[
\{J_B(\xi_1), J_B(\xi_2)\} = \frac{1}{16\pi} \int_{B_\epsilon} \epsilon_{abcd}[\xi^c_2 \nabla_e(\nabla^d_1 \xi^e_1) - \xi^c_1 \nabla_e(\nabla^d_2 \xi^e_2)].
\]
(6.16)

For a class of vector fields considered in [8],
\[
[\xi_1, \xi_2] \cdot B = 0
\]
(6.17)
on the boundary. Thus, the boundary term of the Hamiltonian functional reads
\[
J_B([\xi_1, \xi_2]) = Q_B([\xi_1, \xi_2]) = -\frac{1}{16\pi} \int_{B_\epsilon} \epsilon_{abcd}\nabla^c(\xi^e_1 \nabla_\epsilon \xi^d_2 - \xi^e_2 \nabla_\epsilon \xi^d_1),
\]
(6.18)

and then the possible central term is
\[
K^H_S(\xi_1, \xi_2) = \lim_{\epsilon \to 0} (\{J_B(\xi_1), J_B(\xi_2)\})^* - J_B([\xi_1, \xi_2])
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{8\pi} \int_{B_\epsilon} \epsilon_{abcd}\left[\nabla_\epsilon(\xi^e_1 \nabla^d_2 \xi^c_2) - 2\xi^e_1 \nabla_\epsilon \xi^c_2 \nabla_\epsilon \xi^d_2 \right].
\]
(6.19)

For the vector field (6.4), the right-hand side of Eq.(6.16) becomes
\[
\frac{1}{16\pi} \int_{B_\epsilon} \epsilon_{abcd}\{(T_2 l^e + R_2 n^e)\nabla_\epsilon[\nabla^e(T_1 l^d + R_1 n^d) - \nabla^d(T_1 l^e + R_1 n^e)] - [1 \leftrightarrow 2]\}.
\]
(6.20)

The first term of the integrand in Eq.(6.20) is
\[ \epsilon_{abcd}\{ (T_2^c + R_2^c) \nabla_e \nabla^e (T_1^d + R_1^d) - [1 \leftrightarrow 2] \} \]

\[ = \epsilon_{abcd}\{ 2T_2^c (\nabla_{e}T_1^d) \nabla^e [1 \leftrightarrow 2] \} \]

\[ = \epsilon_{abcd}\{ 2T_2^c (\nabla_{e}T_1^d) \nabla^e [1 \leftrightarrow 2] \} \]

Similarly, the second term is

\[ \epsilon_{abcd}\{ (T_2^c + R_2^c) \nabla_e \nabla^d (T_1^e + R_1^e) - [1 \leftrightarrow 2] \} \]

\[ = \epsilon_{abcd}\{ 2T_2^c (\nabla_{e}T_1^d) \nabla^d [1 \leftrightarrow 2] \} \]

Now, Eq.(6.20) becomes

\[ \text{Eq.(6.20)} = \frac{1}{16\pi} \int_{B_e} \epsilon_{abcd}\{ 2T_2^c (\nabla_{e}T_1^d) \nabla^d [1 \leftrightarrow 2] \} \]

In the N-P formalism, we have \( g_{ab} = -l_a n_b - n_a l_b + m_a \bar{m}_b + \bar{m}_a m_b, \delta^a_\beta = -l^a n_b - n^a l_b + m^a \bar{m}_b + \bar{m}^a m_b, \) and

\[ \nabla_a = -l^a \Delta - n^a \bar{D} + \bar{m}^a \delta + m^a \bar{\delta}, \]  

where \( \delta = m^a \nabla_a \) and \( \bar{\delta} = \bar{m}^a \nabla_a. \) These lead to

\[ \nabla_a \nabla^a = -\bar{D} \Delta - \Delta \bar{D} + \bar{\delta} \delta + \delta \bar{\delta} - [(\epsilon + \bar{\epsilon}) - (\rho + \bar{\rho})] \Delta \]

\[ + [(\gamma + \bar{\gamma}) - (\mu + \bar{\mu})] \bar{D} + (\pi - \bar{\pi} - \alpha + \beta) \delta + (\bar{\pi} - \pi - \bar{\alpha} + \beta) \bar{\delta} \]

\[ = -\bar{D} \Delta - \Delta \bar{D} + \bar{\delta} \delta + \delta \bar{\delta} - \frac{\rho}{|x_K|} - (\rho + \bar{\rho})] \Delta - (\mu + \bar{\mu}) \bar{D} \]

\[ + (\pi - \bar{\pi} - \alpha + \beta) \delta + (\bar{\pi} - \pi - \bar{\alpha} + \beta) \bar{\delta}, \]

where Eq.(6.13) has been used in the second equality. In addition, \( \epsilon_{abcd} = i_{a} \wedge n_{b} \wedge m_{c} \wedge \bar{m}_{d}. \)

Now, we may much more precisely analyze Eq.(6.24) term by term under the assumptions that

\[ R, \delta R, \bar{\delta} R \sim O(\chi^2_K) \]

and

\[ \rho + \bar{\rho} = -\bar{m}^a m^b \nabla_a l_b - m^a \bar{m}^b \nabla_a l_b \sim O(\chi^2_K). \]

Since the volume element on \( B_e \) is \( \epsilon_{ab} = i_{a} m_{b} \wedge \bar{m}_{b}, \) only those terms in the brace bracket in Eq.(6.24), which contains \( i \wedge n, \) will contribute to the integral. So, we will only write out
those terms in following calculation and denote those equalities as ‘≡’. In the calculation, the relations for the null tetrad in Appendix B will be used. The first term is

\[
2T_2 l^e (\nabla_e T_1) \nabla^e l^d = 2T_2 l^e (-l^d n_f - n^d l_f + m^d \bar{m}_f) (\nabla_e T_1) \nabla^e l^f \\
≡ -2T_2 l^e n_f (\nabla_e T_1) \nabla^e l^f \\
= -T^e n^d T_2 (\nabla_e T_1) \nabla^e l^2 = 0. 
\] (6.29)

The second term is

\[
T_2 l^e n^d \nabla_e \nabla^e R_1 = l^e n^d T_2 \{ -\bar{D} \Delta - \Delta \bar{D} - \kappa \frac{\rho}{|\chi_K|} \Delta + (\bar{\rho} + \bar{\rho}) \Delta \} R_1. 
\] (6.30)

The third term is

\[
2T_2 l^e (\nabla_e R_1) \nabla^e n^d \equiv -2T_2 l^e n^d (\nabla_e R_1) l_f \nabla^e l^f \\
= l^e n^d (2\kappa \frac{\rho}{|\chi_K|} T_2 \Delta R_1) + O(\chi_K^2). 
\] (6.31)

The forth term is [32]

\[
-2T_2 l^e (\nabla_e \nabla^d T_1) l^e \equiv T_2 l^e n^d l_f \nabla_e \nabla^f T_1 \\
= l^e n^d [T_2 \bar{D}^2 T_1 + T_2 (l^e n^f \nabla_e l_f) \bar{D} T_1 - T_2 (l^e \bar{m}^f \nabla_e l_f) \delta T_1 - T_2 (l^e m^f \nabla_e l_f) \bar{\delta} T_1] \\
= l^e n^d [T_2 \bar{D}^2 T_1 - \kappa \frac{\rho}{|\chi_K|} T_2 \bar{D} T_1 + i T_2 \delta T_1 + i T_2 \bar{\delta} T_1]. 
\] (6.32)

The fifth term is

\[
-2T_2 l^e (\nabla^d T_1) \nabla_e l^e \equiv l^e n^d T_2 (\bar{D} T_1) \nabla_e l^e \\
= l^e n^d (\kappa \frac{\rho}{|\chi_K|} T_2 \bar{D} T_1) + O(\chi_K^2), 
\] (6.33)

The sixth term is

\[
-2T_2 l^e (\nabla_e T_1) \nabla^d l^e \equiv l^e n^d T_2 (\nabla_e T_1) l_f \nabla^f l^e \\
= l^e n^d T_2 (\kappa \frac{\rho}{|\chi_K|} T_2 \bar{D} T_1 - i \delta T_1 - i \bar{\delta} T_1). 
\] (6.34)

The seventh term is

\[
-2T_2 l^e (\nabla_e \nabla^d R_1) n^e \equiv l^e n^d T_2 \nabla_e \nabla^f R_1 \\
= l^e n^d (T_2 \Delta \bar{D} R_1) + O(\chi_K^2). 
\] (6.35)

The eighth term is

\[
-2T_2 l^e (\nabla^d R_1) \nabla_e n^e \equiv l^e n^d T_2 (\bar{D} R_1) \nabla_e n^e = O(\chi_K^2). 
\] (6.36)

The last term is
\[-T_2 \varepsilon^c (\nabla_e R_1) \nabla^d n^e \overset{\circ}{=} \varepsilon^c n^d T_2 (\nabla_e R_1) l_f \nabla^f n^e \]
\[= -\varepsilon^c n^d (\kappa \rho \chi K) T_2 \Delta R_1 + O(\chi^2 K). \quad (6.37)\]

Thus,

\[
\text{Eq.}(6.20) = \frac{1}{16\pi} \int B_c \varepsilon_{abcd} \varepsilon^c n^d \{[T_2 (-\bar{D} \Delta - \Delta \bar{D} - \kappa \rho \chi K) \Delta] R_1 \\
+ 2\kappa \rho \chi K T_2 \Delta R_1 + T_2 \bar{D}^2 T_1 - \kappa \rho \chi K T_2 \bar{D} T_1 \\
+ \kappa \rho \chi K T_2 \bar{D} T_1 + \kappa \rho \chi K T_2 \bar{D} T_1 + T_2 \Delta \bar{D} R_1 - \kappa \rho \chi K T_2 \Delta R_1] \\
+ O(\chi^2 K) - [1 \leftrightarrow 2]\}
= \frac{1}{16\pi} \int B_c \varepsilon_{abcd} \varepsilon^c n^d \{[-T_2 \bar{D} \Delta R_1 + T_2 \bar{D}^2 T_1 + \kappa \rho \chi K T_2 \bar{D} T_1] \\
+ O(\chi^2 K) - [1 \leftrightarrow 2]. \quad (6.38)\]

On the other hand, the boundary term of the Hamiltonian functional is equal to the partial Noether charge for the given vector fields, which may be written on shell as

\[
J_B(\xi) = Q_B(\xi) = -\frac{1}{16\pi} \int_{B_c} *d\xi \\
= -\frac{1}{16\pi} \int_{B_c} *(dT \wedge l + T d l + dR \wedge n + R d n) \\
= -\frac{1}{16\pi} \int_{B_c} *(dT \wedge l + T d l + dR \wedge n) + O(\chi^2 K). \quad (6.39)\]

Since

\[
dT \wedge l \overset{\circ}{=} (\bar{D} T) l \wedge n \quad (6.40)\]
\[
(dl)_{ab} \overset{\circ}{=} -2n_a \bar{D} l_b - 2l_a \Delta l_b \\
\quad \overset{\circ}{=}- (n^c \bar{D} l_c) l \wedge n \\
= \kappa \rho \chi K l \wedge n + O(\chi^2 K) \quad (6.41)\]
\[
dR \wedge n \overset{\circ}{=} -(\Delta R) l \wedge n, \quad (6.42)\]

the boundary term of the Hamiltonian functional with respect to the vector field takes the form of

\[
J_B(\xi) = -\frac{1}{16\pi} \int_{B_c} \varepsilon_{abcd} \varepsilon^c n^d (\bar{D} T + \kappa \rho \chi K T - \Delta R) + O(\chi^2 K). \quad (6.43)\]

Thus,

\[
J_B([\xi_1, \xi_2]) = -\frac{1}{16\pi} \int_{B_c} \varepsilon_{abcd} \varepsilon^c n^d [\bar{D} (T_1 \bar{D} T_2 + R_1 \Delta T_2) + \kappa \rho \chi K (T_1 \bar{D} T_2 + R_1 \Delta T_2) \\
- \Delta (T_1 \bar{D} R_2 + \kappa \rho \chi K R_1 T_2 + R_1 \Delta R_2)] - (1 \leftrightarrow 2) \\
= -\frac{1}{16\pi} \int_{B_c} \varepsilon_{abcd} \varepsilon^c n^d (T_1 \bar{D}^2 T_2 + \kappa \rho \chi K T_1 \bar{D} T_2 - T_1 \bar{D} \Delta R_2) + O(\chi^2 K) \\
- (1 \leftrightarrow 2). \quad (6.44)\]
Therefore,
\[
\{ J_S(\xi_1), J_S(\xi_2) \}^* - J_S([\xi_1, \xi_2]) = 0. \tag{6.45}
\]
This means that even when \( S \) is mapped from one point to another on the Killing horizon as considered in [8], \textit{the central extension is also zero} as long as the conditions (6.27) and (6.28) are satisfied!

It should be mentioned that in [8] the asymptotic conditions had been proposed to specify a subset of the class of vector fields for getting the non-vanishing center extension. However, those vector fields do satisfy conditions (6.27) and (6.28). This will be shown in details in Appendix A.

Since the boundary terms of the Hamiltonian functionals coincide with the partial Noether charges, all the calculations in this subsection are available to the Noether-charge realization. Namely, the central extension in the Noether-charge realization also vanishes [30].

\section*{VII. COMMENT ON VIRASORO ALGEBRA}

Now, we turn to discuss the realization of the subalgebra of the \textit{diff}(\mathcal{M}) algebra on the boundary \( S \) in the basis of \( \{ \chi^a_K, \rho^a, t^a_1, t^a_2 \} \).

\subsection*{A. The vector fields and the boundary condition}

Following Ref. [8], consider the vector fields
\[
\xi^a = \bar{T} \chi^a_K + \tilde{R} \rho^a \tag{7.1}
\]
with
\[
\tilde{R} = \frac{1}{\kappa} \frac{\chi^2}{\rho^2} \chi^a_K \nabla_a \bar{T} \tag{7.2}
\]
and
\[
\rho^a \nabla_a \bar{T}|_H = 0. \tag{7.3}
\]

Since the tangent vector fields on a spacetime manifold with either Lorentzian signature or Euclidean signature are all \textit{real} vector fields, \( \bar{T} \) must be a real function. In order to obtain the subalgebra on the boundary \( S \), \( \bar{T} \) may be further chosen as
\[
A_n = \frac{1}{\kappa} \cos[n(\kappa v + \varphi)] \tag{7.4}
\]
and
\[
B_n = \frac{1}{\kappa} \sin[n(\kappa v + \varphi)], \tag{7.5}
\]
where \( v \) is the parameters of the integral curves of \( \chi^\alpha_K \) such that \( \chi^\alpha_K \nabla_a v = 1 \), the integers \( n \geq 0 \), and \( \varphi \) is the coordinate on \( B_\varepsilon \) with \( 2\pi \) period. Obviously, the complex combinations of \( A_n \) and \( B_n \),

\[
\tilde{T}_n = A_n \pm iB_n = \frac{1}{\kappa} e^{in(\kappa v + \varphi)},
\]

has the form of

\[
\tilde{T}_n = \frac{1}{\kappa} e^{ink \varphi} f_n, \quad \text{with} \ n \in \mathbb{Z},
\]

where \( f_n, \forall n \in \mathbb{Z}, \) satisfy

\[
\int_{S^1} \tilde{\varepsilon}_{ab} f_m f_n = \delta_{m+n,0}.
\]

Eq.(7.7) with Eq.(7.8) has been used to obtain the Virasoro algebra in [8].

**B. Realization of Virasoro algebra by vector fields**

In [8], the Lie bracket of two vector fields of type (7.1) was

\[
[\xi_1, \xi_2]^a = (\tilde{T}_1 D \tilde{T}_2 - \tilde{T}_2 D \tilde{T}_1) \chi^\alpha_K + \frac{1}{\kappa} \frac{\chi^2_K}{\rho^2} D(\tilde{T}_1 D \tilde{T}_2 - \tilde{T}_2 D \tilde{T}_1) \rho^a.
\]

(7.9)

When \( \tilde{T} = A_m \) and \( B_m \), Eq.(7.9) is given by equivalent to

\[
[A_m, A_n] = \kappa (mB_m A_n - nA_m B_n),
\]

(7.10)

\[
[B_m, B_n] = -\kappa (mA_m B_n - nB_m A_n),
\]

(7.11)

\[
[A_m, B_n] = \kappa (nA_m A_n + mB_m B_n),
\]

(7.12)

which lead to

\[
[A_m, A_n] - [B_m, B_n] = (m - n) B_{m+n},
\]

(7.13)

\[
[A_m, B_n] - [A_n, B_m] = (m - n) A_{m+n}.
\]

(7.14)

They are equivalent to

\[
i[\tilde{T}_m, \tilde{T}_n] = (m - n) \tilde{T}_{m+n}.
\]

(7.15)

Eq.(7.15) is nothing else but the complex expression of the classical Virasoro algebra [34], (more precisely, the Witt algebra \( \text{diff}(S^1) \) [35]), while (7.13) and (7.14) are the real expressions for the same algebra. Therefore, \( \tilde{T}_n \) (or equivalently, the set of \( A_n \) and \( B_n \)) chosen in such a way would form a representation of the classical Virasoro algebra if the condition (7.3) was reasonable and if Eq.(7.9) was valid.

It has been argued in [18] that the condition (7.3) implies \( \tilde{D} \tilde{T} = 0 \) on the horizon which is conflict to the basic requirement \( \tilde{D} \tilde{T} \neq 0 \). According to the discussion in the Sec. VI and Appendix A, the Lie bracket of the two vector fields of type (7.1) should be
\[ [\xi_1^a, \xi_2^a] = (T_1 \bar{D} T_2 - T_2 \bar{D} T_1) l^a + (R_1 \Delta T_2 - R_2 \Delta T_1) l^a + O(\chi_K^2), \quad (7.16) \]

which leads to

\[
[\xi_1^a, \xi_2^a] = \frac{1}{2} [(\bar{T}_1 + \frac{\rho}{|\chi|} \bar{R}_1)(D + \frac{|\chi|}{\rho} \bar{D})(\tilde{T}_2 + \frac{\rho}{|\chi|} \bar{R}_2)] l^a + \frac{1}{2} \chi^2 [(-\bar{T}_1 + \frac{\rho}{|\chi|} \bar{R}_1) \frac{1}{\chi^2}(-D + \frac{|\chi|}{\rho} \bar{D})(\tilde{T}_2 + \frac{\rho}{|\chi|} \bar{R}_2)] l^a - (1 \leftrightarrow 2) + O(\chi_K^2) \quad (7.17)
\]

under the condition (7.3). It is obviously different from Eq.(7.9). Hence, even for the choice (7.7), Eq.(7.17) does not give the Virasoro algebra.

C. The Hamiltonian realization of Virasoro algebra

1. The non-zero Hamiltonian functionals

By definition,

\[
Q_S(\tilde{T}_n) = \frac{1}{16\pi} \int_S \tilde{e}_{ab} \left(2\kappa \tilde{T}_n - \frac{1}{\kappa} D^2 \tilde{T}_n \right) = \frac{2 + n^2}{16\pi} A_H \delta_{n,0} = \frac{A_H}{8\pi} \delta_{n,0}. \quad (7.18)
\]

Namely, the partial Noether charges and thus the boundary terms of the Hamiltonian functionals with respect to the vector fields chosen in the above way all vanish except \( Q_S(\tilde{T}_0) \) and \( J_S(\tilde{T}_0) \), which are equal to \( A_H/8\pi \).

In the Virasoro algebra, \( \{L_n\} \) are the generators of the conformal symmetry, which should not be equal to zero. In the present case, the boundary terms of the Hamiltonian functionals act as the generators of the conformal symmetry. The vanishing boundary terms of the Hamiltonian functionals imply the Virasoro algebra is trivial. Namely, when \( n, m \neq 0 \) and \( n \neq -m \),

\[
\{ J_S(T_m), J_S(T_n) \}^* = (m - n) J_S(\tilde{T}_{m+n}) \quad (7.19)
\]

reads \( 0 = 0 \) actually.

2. The surface densities the classical Virasoro algebra

The problem may not be so serious because the integrands of the boundary terms of the Hamiltonian functionals may form the required algebraic relation. Unfortunately, the following calculation shows that the surface densities with respect to the vector fields chosen above do not agree with the classical Virasoro algebra.

For the vector fields (7.1) with (7.2) and (7.3), Eq.(6.16) in the (ill-defined) basis \( \{\chi_K^a, \rho^a, t_1^a, t_2^a\} \) gives rise to [33]
\[
\{ J_S(\tilde{T}_m), J_S(\tilde{T}_n) \}^* = \frac{1}{16\pi} \int_S \tilde{\epsilon}_{ab} \left[ \frac{1}{\kappa} (D\tilde{T}_n D^2\tilde{T}_m - D\tilde{T}_m D^2\tilde{T}_n) \\
- 2\kappa (\tilde{T}_n D\tilde{T}_m - \tilde{T}_m D\tilde{T}_n) \right] \\
= -i(m-n) J_S(\tilde{T}_{m+n}) + \frac{i(m^3 - n^3)\kappa}{16\pi} \int_S \tilde{\epsilon}_{ab}\tilde{T}_{m+n}
\]

(7.20)

It looks like the classical Virasoro algebra because\( \int_S \tilde{\epsilon}_{ab}\tilde{T}_{m+n} = 0\). Unfortunately, \( J_S(\tilde{T}_{m+n}) = 0 \) (as well as \( J_S(\tilde{T}_m) = 0, \forall m \neq 0 \)) on the same footing as \( \int_S \tilde{\epsilon}_{ab}\tilde{T}_{m+n} = 0 \). Thus, Eq.(7.20) cannot be regarded as the Virasoro algebra. It is obvious from Eq.(7.20) that the surface densities of the boundary terms of the Hamiltonian functionals do not fulfill the classical Virasoro algebra. Therefore, it is difficult to convince that Eq.(7.20) is the realization of a Virasoro algebra because all terms deviating from the Virasoro algebra as well as all boundary terms of Hamiltonian functionals except one vanish on the same footing.

**VIII. CONCLUSION AND DISCUSSION**

In a diffeomorphism invariant theory of gravity, the Noether charges themselves, as in other classical and quantum field theories, may be used to realize the symmetry (in the present case, the diffeomorphism invariance) of the theory. This is a complete covariant approach without using the Poisson bracket, which is well defined in the canonical formalism but still contains some ambiguities in the covariant formalism, at least, for the horizontal variations. It is emphasized that the Noether charge on shell are well defined for any given boundary condition and any given vector field and may always be expressed on shell in terms of the algebraic summation of the boundary terms on a Cauchy surface [12,25]. The latter property results in the Noether-charge realization being separated into the realization of the partial Noether charge of the two-dimensional closed surface that is the boundary of the Cauchy surface.

For the Killing vectors, the Noether charge approach may give rise to certain relations among the Noether charges with respect to the symmetries generated by the Killing vectors. For the stationary axisymmetric spacetimes with a black hole, the vacuum black hole mass formula as a whole can be viewed as a (total) Noether charge for the combination of the Noether charges for the Killing vectors in GR. The first law of black hole mechanics can also be re-derived within the Noether charge formalism under certain conditions. In addition, only the horizontal variations are needed in the Noether charge approach. This is also in agreement with the spirit of Noether’s theorem.

In vacuum GR and when \( \xi \cdot \mathbf{B} = 0 \) on the boundary, the Noether-charge realization should coincide with the Hamiltonian realization because in this case the boundary terms of the Hamiltonian functional are equal to the partial Noether charges of boundary surface. Unfortunately, Hamiltonian functionals do not always exist because they are related to Noether charges by Eq. (3.6) with the definition of \( \mathbf{B} \) given in variational equation (3.7) and because the variational equation does not always have a solution for a given boundary condition and a vector field. This problem, in fact, has been pointed out by Ward and his collaborator [11,12]. Obviously, if the 3-form \( \mathbf{B} \) does not exist, one cannot define the Hamiltonian functional and thus the Hamiltonian realization of \( \text{diff}(\mathcal{M}) \) algebra.
As mentioned before, the Hamiltonian realization has not well set up because the problem in the definition of the Poisson bracket in the covariant phase-space formalism has not yet been solved without debate. Discarding this problem, we may conclude from Eqs. (4.15), (6.16) and (6.44) that the central extension on the Killing horizon for the large class of vector fields including those studied in [8] should vanish, even though in that case the boundary of \( \Sigma_1 \) or \( \Sigma_2 \) is not fixed in the variation so that they cannot be treated as the Cauchy surfaces of portions of the manifold. The main reason that our conclusion is different from [8,9], [14]-[17] is that the null tetrad is used here, which is well-defined on the horizon, while in the previous works the vector fields \( \{ \chi_K^a, \rho^a, t_1^a, t_2^a \} \) are treated as a set of basis, which are ill-defined on the horizon. The vanishing central term implies that the black hole entropy cannot be explained by such a kind of the classical symmetry analysis on the horizon.

It should be mentioned, however, that in the abstract and the section I all the phrases "central term" are always with the quotation mark. This is due to the fact that it is not really the central term in usual sense. It is in fact the one proportional to the partial Noether-like charge of the Noether-like current in (3.25). In order to justify whether the one corresponding to the entire Noether-like charge is vanishing, it is needed to consider the whole Cauchy surface \( \Sigma \) from the bifurcation surface \( S_H \) to the spacial infinity \( \vartheta^0 \) rather than the partial Cauchy surface. But, even for the null-frame has been used in this paper there are still some flaws to be solved. In addition, the event horizon is not a real boundary of the spacetime with a black hole. Therefore, it is also questionable to merely consider the Hamiltonian functionals with respect to the partial Cauchy surface \( \Sigma_2 \). How to solve these problems are still under investigation.

Although the algebra of the vector fields satisfying (7.1), (7.2) and (7.3) under the Lie bracket seems to be equivalent to the classical Virasoro algebra in the ill-defined basis \( \{ \chi_K^a, \rho^a, t_1^a, t_2^a \} \), the analysis in the N-P formalism shows that it does not give a Virasoro algebra. Even in the sense of [8] that the Lie bracket algebra of the vector fields is equivalent to the Virasoro algebra on the horizon, the algebra of the corresponding boundary terms of Hamiltonian functionals is trivial because all the boundary terms except one vanish. The surface densities of the boundary terms do not form the algebraic relation of the Virasoro algebra. It again challenges the proposal that the origin of the black hole entropy can be explained by the classical symmetry analysis on the horizon.

Finally, as was mentioned at beginning, though the present discussion is confined in 4 dimension, it is easy to generalize the discussion to any dimension in principle.

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APPENDIX A: ASYMPTOTIC CONDITIONS ON HORIZON

In [8], two asymptotic conditions had been given
\[ \frac{\chi^a K}{\chi^2 K} \hat{\delta} g_{ab} \to 0 \] \hspace{1cm} (A1)

and

\[ \chi^a K t^b \hat{\delta} g_{ab} \to 0 \] \hspace{1cm} (A2)
as \chi^2_K \to 0. Because

\[ \frac{\chi^a K}{\chi^2 K} \hat{\delta} g_{ab} = \frac{\chi^a K}{\chi^2 K} \mathcal{L}_\xi g_{ab} \]
\[ = \frac{2}{\chi^2_K} (l^a - \frac{\chi^2_K}{2} n^a) (l^b - \frac{\chi^2_K}{2} n^b) [T \nabla_a l_b + T \nabla_b l_a + (\nabla_a R) n_b + R \nabla_a n_b] \]
\[ = \frac{2}{\chi^2_K} \left[ -(\bar{\mathcal{D}} R) + R l^a l^b \nabla_a n_b + \frac{\chi^2_K}{2} (\Delta R) - \frac{\chi^2_K}{2} R n^a l^b \nabla_a n_b \right. \]
\[ \left. + \frac{\chi^2_K}{2} (\bar{\mathcal{D}} T) - \frac{\chi^2_K}{2} R l^a n^b \nabla_a l_b - \frac{\chi^4}{4} (\Delta T) + \frac{\chi^4}{4} T n^a n^b \nabla_a l_b \right] \]
\[ = \frac{2}{\chi^2_K} \left[ -(\bar{\mathcal{D}} R + \kappa \frac{\rho}{|\chi^2_K|} R) + \Delta R + \bar{\mathcal{D}} T + \kappa \frac{\rho}{|\chi^2_K|} T + O(\chi^2_K), \right. \] \hspace{1cm} (A3)

the condition (A1) becomes

\[ \left[ \frac{1}{\chi^2_K} \left( -(\bar{\mathcal{D}} R + \kappa \frac{\rho}{|\chi^2_K|} R) + \Delta R + \bar{\mathcal{D}} T + \kappa \frac{\rho}{|\chi^2_K|} T \right) \right]_H = 0. \] \hspace{1cm} (A4)

The condition (A2) is equivalent to

\[ \chi^a_K m^b \hat{\delta} g_{ab} \to 0 \quad \text{as} \quad \chi^2_K \to 0. \] \hspace{1cm} (A5)

Due to

\[ \chi^a_K m^b \hat{\delta} g_{ab} = (l^a - \frac{\chi^2_K}{2} n^a) m^b \mathcal{L}_\xi g_{ab} \]
\[ = (l^a - \frac{\chi^2_K}{2} n^a) m^b (\nabla_a T) l_b + T \nabla_a l_b + (\nabla_a R) n_b + R \nabla_a n_b \]
\[ + (\nabla_b T) l_a + T \nabla_b l_a + (\nabla_b R) n_a + R \nabla_b n_a \]
\[ = T l^a m^b \nabla_a l_b + R l^a m^b \nabla_b l_a - m^a \nabla_a R + R l^a m^b \nabla_b n_a + O(\chi^2_K) \]
\[ = -t T - \delta R + (\bar{\pi} + \bar{\alpha} + \beta) R + O(\chi^2_K), \] \hspace{1cm} (A6)

Eq.(A5) requires

\[ t T \sim -\delta R \sim O(\chi^2_K). \] \hspace{1cm} (A7)

Namely, \( t \sim O(\chi^2_K) \). Therefore, the asymptotic conditions on the horizon specify a special class of the vector fields which has the form of \( \xi^a = T l^a + R n^a \) with \( R \sim O(\chi^2_K) \).
APPENDIX B: SOME USEFUL RELATIONS

In this appendix, some useful relations for the null tetrad are listed. In the proof of the relations, the identities in Appendix A of [8] are used.

First,

\[
l^a l^b \nabla_{a b} = \frac{1}{2} \tilde{D} l^2 = 0, \quad (B1)
\]

\[
n^a l^b \nabla_{a b} = \frac{1}{2} \Delta l^2 = 0, \quad (B2)
\]

\[
l^a n^b \nabla_{a b} = \frac{1}{2} \tilde{D} n^2 = 0, \quad (B3)
\]

\[
n^a n^b \nabla_{a b} = \frac{1}{2} \Delta n^2 = 0, \quad (B4)
\]

\[
m^a l^b \nabla_{a b} = \frac{1}{2} \delta l^2 = 0, \quad (B5)
\]

\[
m^a n^b \nabla_{a b} = \frac{1}{2} \delta n^2 = 0. \quad (B6)
\]

Because of Eq.(6.11),

\[
l^a l^b \nabla_{a b} = l^b n^a \nabla_{a b} - \kappa \frac{\rho}{|\chi_K|} n_b l^b = \kappa \frac{\rho}{|\chi_K|}, \quad (B7)
\]

\[
n^a l^b \nabla_{a b} = -n^a n^b \nabla_{a b} = n^b n^a \nabla_{a b} - \kappa \frac{\rho}{|\chi_K|} n_b n^b = 0, \quad (B8)
\]

\[
l^a n^b \nabla_{a b} = -l^a l^b \nabla_{a b} = -l^b n^a \nabla_{a b} + \kappa \frac{\rho}{|\chi_K|} n_b n^b = -\kappa \frac{\rho}{|\chi_K|}, \quad (B9)
\]

\[
n^a n^b \nabla_{a b} = n^b l^a \nabla_{a b} + \kappa \frac{\rho}{|\chi_K|} n_b n^b = 0. \quad (B10)
\]

In addition,

\[
\nabla_a l^a = \frac{1}{2} (\nabla_a \chi_K + \tilde{D} \frac{|\chi_K|}{\rho} + \chi_K \nabla_a \rho^2)
\]

\[
= \frac{1}{2} \left\{ -\frac{1}{2} \frac{\rho}{|\chi_K|} \tilde{D} \frac{\chi_K^2}{\rho^2} + \frac{|\chi_K|}{\rho} \left[ -2 \kappa \frac{\rho^2}{\chi_K^2} + O(\chi_K^2) \right] \right\}
\]

\[
= \kappa \frac{\rho}{|\chi_K|} + O(\chi_K^2). \quad (B11)
\]

\[
\nabla_a n^a = -\frac{1}{\chi_K^2} \nabla_a \chi_K - \chi_K \nabla_a \frac{1}{\chi_K} + \frac{|\chi_K|}{\chi_K} \tilde{D} \frac{1}{\rho} \nabla_a n^2 + \frac{1}{\chi_K^2} \tilde{D} \frac{|\chi_K|}{\rho} + \frac{1}{\chi_K} \frac{|\chi_K|}{\rho} \nabla_a \rho^2
\]

\[
= \frac{|\chi_K|}{\rho} \tilde{D} \frac{1}{\chi_K} + \frac{1}{\chi_K} \tilde{D} |\chi_K| + \frac{1}{\chi_K^2} |\chi_K| \nabla_a \rho^2
\]

\[
= \frac{|\chi_K|}{\rho} \frac{1}{\chi_K} \frac{(2 \kappa) \rho^2}{2 \chi_K^2} - \frac{1}{2 \chi_K^2} |\chi_K| \frac{\tilde{D} \chi_K^2}{\rho^2} + \frac{1}{\chi_K^2} |\chi_K| \nabla_a \rho^2
\]

\[
= -2 \kappa \frac{\rho}{\chi_K} \frac{1}{\chi_K} \frac{\rho}{2 \chi_K^2} |\chi_K| O(\chi_K^2) + \frac{1}{\chi_K^2} \frac{|\chi_K|}{\rho} \left[ -2 \kappa \frac{\rho^2}{\chi_K^2} + O(\chi_K^2) \right]
\]

\[
= O(1). \quad (B12)
\]
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