Classification of minimally unsatisfiable 2-CNFs

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Abstract

We consider minimally unsatisfiable 2-CNFs (short 2-MUs). Characterisations of 2-MUs in the literature have been restricted to the nonsingular case (where every variable occurs positively and negatively at least twice), and those with a unit-clause. We provide the full classification of 2-MUs $F$. The main tool is the implication digraph, and we show that the implication digraph of $F$ is a “weak double cycle” (WDC), a big cycle of small cycles (with possible overlaps). Combining logical and graph-theoretical methods, we prove that WDCs have at most one skew-symmetry, and thus we obtain that the isomorphisms between 2-MUs $F, F'$ are exactly the isomorphisms between their implication digraphs.

We obtain a variety of applications. For fixed deficiency $k$, the difference of the number of clauses of $F$ and the number $n$ of variables of $F$, the automorphism group of $F$ is a subgroup of the Dihedral group with $4k$ elements. The isomorphism problem restricted to 2-MUs $F$ is decidable in linear time for fixed $k$. The number of isomorphism types of 2-MUs for fixed $k$ is $\Theta(n^{4k-1})$. The smoothing (removal of linear vertices) of skew-symmetric WDCs corresponds exactly to the canonical normal form of $F$ obtained by 1-singular DP-reduction, a restricted form of DP-reduction (or “variable elimination”) only reducing variables of degree 2. The isomorphism types of these normal forms, i.e., the homeomorphism types of skew-symmetric WDCs, are in one-to-one correspondence with binary bracelets (or “turnover necklaces”) of length $k$.

1 Introduction

A CNF is minimally unsatisfiable (MU) iff it is unsatisfiable and removing any clause yields a satisfiable formula. We study the isomorphism problem for 2-CNF MUs (short 2-MUs), i.e., for given 2-MUs $F, F'$ decide whether $F$ is isomorphic to $F'$. The isomorphism problem for subclasses of MUs has been considered

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in [14, 10, 13, 1] and in the Handbook chapter [15]. It is shown in [16] that the isomorphism problem for MUs with fixed deficiency, the difference between the number of clauses and variables, is GI-complete (graph-isomorphism complete). Typical GI-complete problems are graph isomorphism, CNF isomorphism, and 2-CNF isomorphism. Furthermore even the class of Horn MUs (which is a subset of deficiency one) is still GI-complete. We give the first example of a class of restricted but interesting MUs, namely 2-MUs, with polytime isomorphism decision. Moreover we give a clear picture of 2-MUs, as big cycles of small cycles; the subclass of them not containing variables occurring just twice (the most trivial variables) corresponds to the class of binary strings called “bracelets” ([10]), which shows that there are exponentially many isomorphism types of 2-MUs (in dependency on the number of variables).

The starting point of the investigations of this paper are the most basic 2-MUs, the nonsingular 2-MUs. A nonsingular MU $F$ is characterised by the property that every variable occurs positively and negatively at least twice. It was first shown in the technical report [17] and then with a more general proof in [1], that a nonsingular 2-MU $F$ with $n(F) \geq 2$ (the number of variables occurring in $F$) is isomorphic to $B_{n(F)}$, a cycle of equivalences with a final negation, given as

$$B_n := \left( \begin{array}{c} v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow \ldots \leftrightarrow v_{n-1} \leftrightarrow v_n \leftrightarrow \neg v_1 = (\neg v_1 \lor v_2) \land (v_1 \lor \neg v_2) \land \ldots \land (v_{n-1} \lor \neg v_n) \land (v_1 \lor v_n) \land (\neg v_1 \lor \neg v_n) \end{array} \right)$$

for $n \geq 2$. Note that $B_n$ has $n$ variables and $2n$ clauses, and thus deficiency $n$.

In this paper we will unfold the structure of 2-MUs as digraphs. The concept of the implication digraph for a 2-CNF $F$ was introduced in [3]; for an overview see [7, Section 5.4.3]. The vertex-set of the implication digraph for $F$ consists of the $2n(F)$ literals of $F$, while a clause $(x \lor y)$ in $F$ yields the two arcs $x \rightarrow y$, $y \rightarrow x$; these arcs become one in case $x = y$. The fundamental property is that $F$ is unsatisfiable iff the implication digraph contains a contradictory cycle (a cycle containing a literal and its complement). The implication digraph of $F$, using the “labels” of vertices, allows exact reconstruction of $F$, since the vertices contain full information on literals and their complements. A more abstract image is obtained by only keeping the information about pairing of literals. From the abstract implication digraph (“unlabelled”) together with this information, which is a skew-symmetry of the digraph, we can reconstruct $F$ up to isomorphism. A main result of this paper is that from the isomorphism type of the implication digraph of 2-MUs alone, and indeed from the implication graph (the underlying graph), it is possible to reconstruct the original 2-MU up to isomorphism. This together with the understanding of the special digraphs obtained allows the complete classification of all 2-MUs. Before giving an overview on the paper, we mention some other related literature: [20, Lemma 19] states the basic information on unsatisfiable 2-CNFs, that they are refutable by resolving two chains of resolution steps. Random 2-CNFs are overviewed in [9]. MU-subsets of 2-CNFS are considered in [5]. Finally [13] considers related graphs, but is mainly interested in distinguishing satisfiability and unsatisfiability.

After discussing basic terminology in Section 2 we discuss implication digraphs of 2-CNFs in Section 3. Then 2-MUs of deficiency one are classified in Section 4. In Section 5 we discuss singular DP-reduction, and Section 6 provides the characterisation of the class of digraphs corresponding to the implication digraphs of 2-MUs. Finally in Section 7 we classify 2-MUs of higher deficiency.


2 Preliminaries

The concepts defined here are all standard, and need to be consulted only to look up details (e.g., what exactly is a “graph”) and notations.

The set of all variables is denoted by $\mathcal{V}A$, and we assume $\mathbb{N} = \{1, 2, \ldots\} \subseteq \mathcal{V}A$ (as in the DIMACS format). Literals are variables $v \in \mathcal{V}A$ and their complementations $\overline{v}$ ($v = \overline{v}$ for $v \in \mathbb{N}$). The underlying variable of a literal $x$ is $\text{var}(x) \in \mathcal{V}A$.

A clause is a finite and clash-free (i.e., non-tautological) set of literals. A clause-set is a finite set of clauses, and we use $\mathcal{CLS}$ for the set of all clause-sets. The empty clause-set is denoted by $\top \in \mathcal{CLS}$ and the empty clause by $\bot := \emptyset$. By $\mathcal{CLS}$ we denote the set of clause-sets $F \in \mathcal{CLS}$ such that for all clauses $C \in F$ holds $|C| \leq 2$. A clause-set $F$ is uniform resp. $k$-uniform, if all clauses of $F$ have the same length resp. length $k$. For a set $L$ of literals, we denote by $\exists L(x) := \{y : x \in L\}$ the elementwise complementation. The set of variables in a clause $C$ is denoted by $\text{var}(C) := \{\text{var}(x) : x \in C\} = (C \cup \overline{C}) \cap \mathcal{V}A$. The set of variables in $F$ is $\text{var}(F) := \bigcup_{C \in F} \text{var}(C) \subseteq \mathcal{V}A$, while $\text{lit}(F) := \text{var}(F) \cup \overline{\text{var}(F)}$ is the set of all literals whose variable is in $\text{var}(F)$. For $F \in \mathcal{CLS}$ we use $n(F) := |\text{var}(F)| \in \mathbb{N}_0$ for the number of variables and $c(F) := |F| \in \mathbb{N}_0$ for the number of clause. The deficiency of $F$ is $\delta(F) := c(F) - n(F) \in \mathbb{Z}$. For a literal $x$, the literal-degree $\text{ld}_F(x) := |\{C \in F : x \in C\}| \in \mathbb{N}_0$ is the number of clauses of $F$ containing $x$, while the degree of a variable $v$ is $\text{vd}_F(v) := \text{ld}_F(v) + \text{ld}_F(\overline{v}) \in \mathbb{N}_0$.

Two clause-sets $F, G \in \mathcal{CLS}$ are isomorphic, denoted by $F \cong G$, if there exists a complement-preserving bijection $f : \text{lit}(F) \to \text{lit}(G)$ such that $f(F) := \{f(C) : C \in F\} = G$. Two clauses $C, D$ are resolvable if they clash in exactly one variable $v \in \mathcal{V}A$, i.e., $C \cap \overline{D} = \{v\}$, in which case the resolvent on $v$ is $(C \setminus \{v\}) \cup (D \setminus \{v\})$. The DP-reduction for $F \in \mathcal{CLS}$ and a variable $v$, denoted by $\text{DP}_v(F) \in \mathcal{CLS}$, replaces all $C \in F$ with $v \in \text{var}(C)$ by all their resolvents on $v$. $\text{DP}_v(F)$ is satisfiability-equivalent to $F$. In fact, in this paper we do not need to handle assignments, and thus we define that $F$ is unsatisfiable, if repeated applications of DP-reduction yields $\bot$; otherwise $F$ is satisfiable (and we obtain $\top$ from DP-reduction). The set of minimally unsatisfiable clause-sets (unsatisfiable, while removal of any clause renders it satisfiable) is denoted by $\mathcal{MU}$. For some background on MUs, see [13] (though this article is self-contained). It is well-known that $\delta(F) \geq 1$ holds for $F \in \mathcal{MU}$ ([13]). We use $2\cdot\mathcal{MU} := \mathcal{MU} \cap 2\cdot\mathcal{CLS}$, and the subsets $2\cdot\mathcal{MU}_{\delta=k}$ resp. $2\cdot\mathcal{MU}_{\delta=k}$ given by $F \in 2\cdot\mathcal{MU}$ with $\delta(F) = k$ resp. $\delta(F) \geq k$. Since here often the empty clause $\bot$ is just in the way, by an upper-index “$*$” we exclude it: $2\cdot\mathcal{CLS}^* := \{F \in 2\cdot\mathcal{CLS} : \bot \notin F\}$, $2\cdot\mathcal{MU}^* := 2\cdot\mathcal{MU} \setminus \{\{\bot\}\}$, and $2\cdot\mathcal{MU}^*_{\delta=1} := 2\cdot\mathcal{MU}_{\delta=1} \setminus \{\{\bot\}\}$.

A graph resp. digraph $G$ is a pair $(V, E)$, where $V(G) := V$ is a finite set of vertices and $E(G) := E$ is the set of edges resp. arcs defined as two-element subsets $\{a, b\} \subseteq V$ resp. pairs $(a, b) \in V^2$ with $a \neq b$. Note that we do not allow (self-)loops, and that there are no parallel edges resp. arcs (though there might be antiparallel arcs). A (di)graph $G$ is a subgraph of another (di)graph $G'$ if $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$. For a digraph $G$, the transposed digraph, obtained by reversing the direction of all arcs, is denoted by $G^t$. For two digraphs $G_1, G_2$, an isomorphism from $G_1$ to $G_2$ is a bijection $f : V(G_1) \to V(G_2)$ such that $f(E(G_1)) = \{(f(a), f(b)) : (a, b) \in E(G_1)\} = E(G_2)$; if $G_1, G_2$ are graphs, then the condition is that $f(E(G_1)) = \{(f(a), f(b)) : (a, b) \in E(G_1)\} = E(G_2)$. If there is an isomorphism between
$G_1$ and $G_2$, then we write $G_1 \cong G_2$. A graph $G$ is promoted to a digraph by $\text{dg}(G) := (V(G), \{(a,b),(b,a) : \{a,b\} \in E(G)\})$, converting every edge $\{a,b\}$ into two arcs $(a,b),(b,a)$. The conversion of a digraph $G$ to its underlying graph (forgetting directions, and contracting antiparallel arcs into one edge) is denoted by $\text{ug}(G)$. A map $f$ is an isomorphism from graph $G$ to graph $G'$ iff $f$ is an isomorphism from $\text{dg}(G)$ to $\text{dg}(G')$. Every isomorphism $f$ from a digraph $G$ to a digraph $G'$ is also an isomorphism from $\text{ug}(G)$ to $\text{ug}(G')$ (but not vice versa).

For a set $V$ and $m \in \mathbb{N}_0$, let $(V)_m := \{S \subseteq V : |S| = m\}$ be the set of $m$-element subsets of $V$. A **multigraph** is a pair $(V,E)$ where $V$ is a set and $E : (V)_2 \to \mathbb{N}_0$. A submultigraph $G'$ of a multigraph $G$ has $V(G') \subseteq V(G)$ and $\forall \{u,v\} \in (V)_2 : E(G')\{u,v\} \leq E(G)(\{u,v\})$. A graph $G$ is promoted to a multigraph $\text{mg}(G)$ by using the same vertex-set $V(G)$, and using the characteristic function of $E(G) \subseteq (V)_2$, while the underlying graph $\text{ug}(G)$ of a multigraph just forgets the multiplicities of edges and discards loops. A digraph $G$ is converted to a multigraph $\text{mg}(G)$ by forgetting the direction of arcs, while not contracting edges. An isomorphism from a multigraph $G$ to a multigraph $G'$ is a bijection $f : V(G) \to V(G')$ with $\forall v,w \in V(G) : E(G')(\{f(v),f(w)\}) = E(G)(\{v,w\})$. Every isomorphism $f : G \to G'$ between multigraphs is also an isomorphism $f : \text{ug}(G) \to \text{ug}(G')$ between the underlying graphs (but not vice versa). A map $f$ is an isomorphism from a graph $G$ to a graph $G'$ iff $f$ is an isomorphism from $\text{mg}(G)$ to $\text{mg}(G)$. Every isomorphism $f : G \to G'$ between digraphs is also an isomorphism $f : \text{mg}(G) \to \text{mg}(G')$ (but not vice versa).

The **in-degree** of a vertex $v \in V(G)$ of a digraph $G$ is the number of arcs going into $v$, the **out-degree** is the number of outgoing arcs, and the **degree** of $v$ is the sum of in- and out-degree. If $G$ is a graph, then the degree of $v$ is the number of vertices $w$ adjacent to $v$ (that is $|\{w \in V(G) : \{v,w\} \in E(G)\}| \in \mathbb{N}_0$). More generally, the degree of a vertex $v \in V(G)$ in a multigraph is $\text{deg}_G(v) := \sum_{w \in V(G)} E(G)\{v,w\} \in \mathbb{N}_0$, the number of adjacent edges. The set of neighbours of a vertex $v$ in a multigraph $G$ is $N_G(V) := \{w \in V(G) : E(G)(\{v,w\}) \neq 0\} \subseteq V(G)$ (the same as in the underlying graph). A linear vertex in a multigraph $G$ is a vertex $v \in G$ of degree two, while a linear vertex in a digraph is a vertex of in- and out-degree one.

A **cycle graph** is a connected graph, where every vertex is linear (so it has at least three vertices). The standardised cycle graph $\text{CG}_n$ for $n \geq 3$ has vertices $1,\ldots,n$ and edges $1 \to 2 \ldots \to n \to 1$. A **cycle multigraph** allows additionally for length 2 (two vertices and two parallel edges) and length 1 (one vertex with a loop). A cycle in a (multi)graph $G$ is a (sub)multigraph which is isomorphic to some cycle (multi)graph. A **cycle digraph** is a strongly connected digraph (from every vertex every other vertex is reachable by a (directed) path), where every vertex is linear (so it has at least two vertices). A cycle in a digraph is a subdigraph which is isomorphic to some cycle digraph.

### 3 Implication digraphs of 2-CNFSs

For $F \in 2\text{-CLS}^*$ the implication digraph $\text{idg}(F)$ is defined by $V(\text{idg}(F)) := \text{lit}(F)$ and $E(\text{idg}(F)) := \{(\overline{x},x) | \{x\} \in F\} \cup \{((\overline{x},y), (\overline{x},x)) | \{x,y\} \in F \land x \neq y\}$. The **implication graph** is $\text{ig}(F) := \text{ug}(\text{idg}(F))$. For a literal $x \in \text{lit}(F)$ its degree $\text{id}_F(x)$ is the in-degree of vertex $x$ in $\text{idg}(F)$, and the out-degree of vertex $\overline{x}$, while for a variable $v \in \text{var}(F)$ its degree $\text{vd}_F(v)$ is the degree of
vertex \( v \) as well as the degree of vertex \( \overline{v} \) in \( \text{idg}(F) \). The arc \( \overline{v} \to x \) is the **contraposition** of the arc \( v \to y \). In an implication digraph, a cycle with two clashing literals (i.e., a literal and its complement) is called **contradictory**. A clause-set \( F \in \mathcal{2-CLS}^* \) is unsatisfiable if there exists a contradictory cycle in \( \text{idg}(F) \) \((\text{[3]}\)). By forgetting complementation and translating clauses into arcs we get that for \( F_1, F_2 \in \mathcal{2-CLS}^* \) and an isomorphism \( f : F_1 \to F_2 \) also \( f : \text{idg}(F_1) \to \text{idg}(F_2) \) is an isomorphism. The reverse direction does not hold in general, and so the isomorphism type of implication digraphs is not a “complete isomorphism invariant” for 2-CNFs. When adding a notion of complementation to digraphs, then we obtain basically the same as 2-CNFs, as we now explain:

**Definition 3.1** (\(() 1\)) A **skew-symmetry** of a digraph \( G \) is a bijection \( \sigma : V(G) \to V(G) \), where \( \sigma \) is its own inverse (involution), i.e., \( \forall v \in V(G) : \sigma(\sigma(v)) = v \), for every vertex \( v \in V(G) \) we have \( \sigma(v) \neq v \) (i.e., \( \sigma \) has no fixed-point), and for every arc \( (a, b) \in E(G) \) holds \( \sigma(a, b) := (\sigma(b), \sigma(a)) \in E(G) \). A digraph \( G \) is called **skew-symmetric**, if there exists a skew-symmetry \( \sigma \) for \( G \), while a digraph **with skew-symmetry** is such a pair \((G, \sigma)\).

A skew-symmetry for \( G \) is an isomorphism \( f : G \to G^s \), where \( f \) as a map (from \( V(G) \) to itself) is an involution and fixed-point free. For \( F \in \mathcal{2-CLS}^* \) the digraph \( \text{idg}(F) \) has a natural skew-symmetry, namely the complementation of literals, and the corresponding digraph with skew-symmetry is denoted by \( \text{sidg}(F) := (\text{idg}(F), (\overline{v})_{v \in \text{idg}(F)}) \). An isomorphism from \((G_1, \sigma_1)\) to \((G_2, \sigma_2)\) is a digraph-isomorphism \( f : G_1 \to G_2 \) which is compatible with the skew-symmetries, i.e., for all \( v \in V(G_1) \) holds \( \sigma_2(f(v)) = f(\sigma_1(v)) \). For all \( F, F' \in \mathcal{2-CLS}^* \) holds \( F \cong F' \) iff \( \text{sidg}(F) \cong \text{sidg}(F') \). For any digraph with skew-symmetry \((G, \sigma)\) we can assume w.l.o.g. that \( V(G) \) is a set of literals closed under complementation, and that \( \sigma(x) = \overline{x} \) for all \( x \in V(G) \) holds, and then there is a unique \( F \in \mathcal{2-CLS}^* \) with \( \text{sidg}(F) = (G, \sigma) \). So in this sense digraphs with skew-symmetry are “the same” as 2-CNFs (however without variables, but just based on literals and their complements), and we can call skew-symmetries of digraphs just “complementations”. A digraph may have no complementation (e.g., digraphs with an odd number of vertices) or many. An arc \((x, y)\) is mapped by complementation to itself, i.e., \((\overline{x}, \overline{y}) = (x, y)\), iff \( x = \overline{y} \), iff the arc corresponds to the unit-clause \( \{y\} \).

## 4 2-MUs of deficiency one

The characterisation of the isomorphism types of \( F \in \mathcal{2-MU}_{k=1} \) is in principle not very difficult, but requires some attention to details. By \([3] \text{ Corollary 13}\) and \([3] \text{ 1-singular DP-reduction}\), i.e., DP-reduction for variables occurring exactly once positively and once negatively, applied to any MU \( F \) results in \( \{\bot\} \) iff \( \delta(F) = 1 \). So we can generate (exactly) all of \( \mathcal{2-MU}_{k=1} \) by starting from the empty clause, and repeatedly replacing a single clause \( C \) by two clauses \( C' \cup \{v\}, C'' \cup \{\overline{v}\} \) for \( C' \cup C'' = C \), \(|C'|, |C''| \leq 1 \), and a new variable \( v \). The clause-sets generated this way, starting with \( \{\{\bot\}\} \), together exactly yield \( \mathcal{2-MU}_{k=1} \). The complete details are in Section \([3] \text{ while we only state the results here. The four types of MUs obtained here are as follows. Let } M := \{\{-1, 2\}, \ldots, \{-(n-1), n\}\} \text{ for } n \in \mathbb{N} \text{ (using integers as literals)}:\)
1. \( U_n^2 := M \cup \{\{1\}, \{-n\}\} \) for \( n \geq 1 \).
2. \( U_{n,i}^1 := M \cup \{\{1\}, \{-n, -i\}\} \) for \( n \geq 2, 1 \leq i \leq n - 1 \).
3. \( U_{n,i}^0 := M \cup \{\{1, i\}, \{-n, -i\}\} \) for \( n \geq 3, 2 \leq i \leq \frac{n-1}{2} \) (occurs in \([5]\)).
4. \( U_{n,x,y}^0 := M \cup \{\{1, x\}, \{-n, -y\}\} \) for \( n \geq 4, 2 \leq x < y \leq n - 1, x + y \leq n + 1 \).

\( \text{idg}(U_n^2) \) is a cycle digraph with \( 2n \) vertices and \( 2n \) edges (where all vertices have degree 2). The labelled digraph, actually a graph with skew-symmetry, is shown as follows. Here arcs from unit-clauses are drawn as double-arcs (if multidigraphs would be used, then unit-clauses indeed would yield two parallel arcs):

\[
\text{idg}(U_n^2) = \begin{array}{cccccccc}
1 & 2 & \cdots & n-1 & n \\
\downarrow & \downarrow & \cdots & \cdots & \cdots & \cdots & \downarrow & \downarrow \\
-1 & -2 & \cdots & -(n-1) & -(n) \\
\end{array}
\]

We note that \( U_n^2 = U_{n,n}^1 \) (allowing this degeneration for the moment). \( \text{idg}(U_{n,i}^1) \) has \( 2n \) vertices and \( 2n + 1 \) edges, and consists of two cycle digraphs of length \( n + i \), which overlap in a path of length \( 2i - 1 \geq 1 \) (we note \( 2(n + i) - (2i - 1) = 2n + 1 \)); two vertices have degree 3, all other vertices have degree 2:

\[
\text{idg}(U_{n,i}^1) = \begin{array}{cccccccc}
1 & \cdots & i & \cdots & n \\
\downarrow & \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\
-1 & \cdots & -i & \cdots & -n \\
\end{array}
\]

The digraph \( \text{idg}(U_{n,i}^0) \) has \( 2n \) vertices and \( 2n + 2 \) edges, and two vertices have degree 4, while all other vertices have degree 2:

\[
\text{idg}(U_{n,i}^0) = \begin{array}{cccccccc}
1 & \cdots & i & \cdots & n \\
\downarrow & \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\
-1 & \cdots & -i & \cdots & -n \\
\end{array}
\]

The implication digraph of \( U_{n,x,y}^0 \) has \( 2n \) vertices and \( 2n + 2 \) edges, and four vertices have degree 3, while all other vertices have degree 2:

\[
\text{idg}(U_{n,x,y}^0) = \begin{array}{cccccccc}
1 & \cdots & x & \cdots & y & \cdots & n \\
\downarrow & \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\
-1 & \cdots & -x & \cdots & -y & \cdots & -n \\
\end{array}
\]

The classification of \( 2 \cdot \mathcal{M}_{\delta=1}^* \) is stated as follows:

**Theorem 4.1** For input \( F \in 2 \cdot \mathcal{M}_{\delta=1}^* \) exactly one of the four cases applies. Let \( u(F) \in \{0, 1, 2\} \) be the number of unit-clauses in \( F \). Then in polynomial time the unique parameter-list \( L(F) \) of length 0, 1, 2, according to the applicable case can be computed, such that \( F \cong U_{u(F), L(F)}^u \) holds. This map canon : \( 2 \cdot \mathcal{M}_{\delta=1}^* \to 2 \cdot \mathcal{M}_{\delta=1}^* \) given by \( \text{canon}(F) := U_{u(F), L(F)}^u \) is a polytime computable clause-set-canonicalisation, that is, for \( F, F' \in 2 \cdot \mathcal{M}_{\delta=1}^* \) holds \( F \cong F' \) iff \( \text{canon}(F) = \text{canon}(F') \). Furthermore the map \( F \in 2 \cdot \mathcal{M}_{\delta=1}^* \mapsto \text{canon}'(F) := \text{idg}(\text{canon}(F)) \) to the class of graphs is a polytime computable graph-canonicalisation, that is, for \( F, F' \in 2 \cdot \mathcal{M}_{\delta=1}^* \) holds \( F \cong F' \) iff \( \text{canon}'(F) = \text{canon}'(F') \). From \( \text{canon}'(F) \) in polytime \( F \) can be reconstructed up to isomorphism.
5 Singular DP-reduction and smoothing

The fundamental tool for the analysis of MUs is “singular DP-reduction”, i.e., the reduction \( F \in \mathcal{MU} \rightarrow \text{DP}_v(F) \): in general \( \text{DP}_v(F) \notin \mathcal{MU} \), however if \( v \) is a singular variable, i.e., \( \text{ld}_F(v) = 1 \) or \( \text{ld}_F(\overline{v}) = 1 \) holds, then \( \text{DP}_v(F) \in \mathcal{MU} \) is guaranteed. For \( F \in 2\text{-\mathcal{MU}}_{\delta=1} \), by definition a variable \( v \) is singular iff vertex \( v \) in \( \text{idg}(F) \) has in- or out-degree 1. The application of DP-reduction for singular variables in \( F \in \mathcal{MU} \) is called singular DP-reduction. These reductions do not yield tautological resolvents, and neither between the resolvents nor between resolvents and old clauses a contraction (two previously different clauses become equal) happens, since otherwise \( F \) wouldn’t be MU. So the class of MUs with fixed deficiency \( k \geq 1 \) is stable under singular DP-reduction ([19, Lemma 9]). Since 2-CLS is stable under resolution, also the classes 2\-\mathcal{MU}_{\delta=k} are stable under singular DP-reduction. An \( F \in \mathcal{MU} \) is called nonsingular, if \( F \) does not contain a singular variable; the set of all nonsingular MUs is denoted by \( \mathcal{MU}' \subset \mathcal{MU} \). For \( F \in \mathcal{MU} \), the set of all nonsingular MUs reachable from \( F \) by singular DP-reduction is denoted by \( \text{sDP}(F) \subset \mathcal{MU}' \). So for any \( F' \in \text{sDP}(F) \) we have \( \delta(F') = \delta(F) \). A fundamental lemma in [19] is that the elements of \( \text{sDP}(F) \) all have the same number of variables (but in general they are non-isomorphic). The basic result, established in [17, 1], for this work is that for all \( F \in 2\text{-\mathcal{MU}} \) holds 1\-\text{sDP}(F) = \{\bot\} iff \( \delta(F) = 1 \). As mentioned, once we have removed all 1-singular variables, singular-DP-reduction never reintroduces them:

[5.1 1-singular DP-reduction]

For an MU \( F \), the nicest case of singular DP-reduction is the confluent case, that is, \( |\text{sDP}(F)| = 1 \). By [19, Section 5] we have confluence, when performing only “1-singular DP-reduction”, that is, applying sDP-reduction only in case of 1-singular variables (i.e., of degree 2). And furthermore it follows by the general results there, that once all 1-singular variables are eliminated, none are being reintroduced. The simple reason for confluence is that 1-singular DP-reduction does not remove 1-singular variables other than the eliminated variable (but we note that new 1-singular variables in general are created). We denote by \( \mathcal{MU}^+ \subset \mathcal{MU} \) the set of non-1-singular \( F \in \mathcal{MU} \), i.e., where every variable of \( F \) has degree at least 3 (while for nonsingular \( F \in \mathcal{MU}' \) every variable has degree at least 4). For \( C \subseteq \mathcal{MU} \) we use \( C^+ := C \cap \mathcal{MU}^+ \). We use 1\text{sDP}(F) \in \mathcal{MU}^+ for \( F \in \mathcal{MU} \) to denote the (unique) non-1-singular MU obtained by 1-singular DP-reduction from \( F \). The basis for Section 4 is that for all \( F \in \mathcal{MU} \) holds 1\text{sDP}(F) = \{\bot\} iff \( \delta(F) = 1 \). As mentioned, once we have removed all 1-singular variables, singular-DP-reduction never reintroduces them:

\textbf{Lemma 5.1} \( \mathcal{MU}^+ \) \text{ is stable under singular-DP-reduction.}

\textbf{Proof}: The only possibility of a singular DP-reduction on \( v \) with main clause \( v \in C \subseteq F \in \mathcal{MU} \) and side-clauses \( \overline{v} \in D_1, \ldots, D_m \) \( F \) decreasing the degree of a literal \( x \in \text{lit}(F) \) is that \( x \in C \cap D_1 \cap \cdots \cap D_m \) — but since \( F \in \mathcal{MU}^+ \), we have \( m \geq 2 \), and thus the literal-degree of \( x \) in \( \text{DP}_v(F) \) is at least two. \( \square \)
The analysis of singular DP-reduction for a class $\mathcal{C} \subseteq \mathcal{MU}$, where always stability of $\mathcal{C}$ under singular DP-reduction is assumed, now can proceed by first considering the simple confluent reduction $F \in \mathcal{C} \sim 1sDP(F) \in \mathcal{C}^+$ and characterising the elements of $\mathcal{C}^+$. The second stage then can start with $\mathcal{C}^+ \subseteq \mathcal{C}$, and need only to consider non-1-singular DP-reductions to arrive at $\mathcal{C}' = \mathcal{C} \cap \mathcal{MU}'$.

5.2 Smoothing of (multi-)graphs

We will now see that a general reduction operation for graphs, strongly related to the concept of “homeomorphism” of graphs, covers most cases of 1-singular DP-reduction. Indeed it is essential to consider multigraphs here, which allow loops and parallel edges. Following [12, Section 7.2.4, D37], a smoothing step for a multigraph $G$ chooses a linear vertex $v \in V(G)$ of degree 2 and with $v \notin N_G(v)$, removes the vertex $v$ and the two edges from $G$ incident with $v$, and for the vertices $u, w$ with $N_G(v) = \{u, w\}$ (note that possibly $u = w$) adds an edge connecting $u$ and $w$: thus the obtained multigraph $G'$ has $V(G') = V(G) \setminus \{v\}$ and $E(G')(\{u, w\}) = E(G)(\{u, w\}) + 1$. We note that the degree of the remaining vertices is not changed except for the case $u = w$, in which case the degree of $u$ decreases by one. Especially linear vertices in $G$ different from $v$ stay linear vertices in $G'$, except for the case when we have two linear vertices $v \neq v'$ forming a 2-cycle (i.e., $E(G)(\{v, v'\}) = 2$), in which case the degree of $v'$ in $G'$ is 1 (namely $E(G')(\{v\}) = 1$). So performing smoothing steps on a multigraph $G$ as long as possible results in a multigraph $G'\ (with \ V(G') \subseteq V(G))$, where $G'$ is uniquely determined except for isolated cycles $C \subseteq V(G)$ (all vertices of $C$ are linear in $G$) of length at least two, where exactly one $v \in C$ is chosen, and the whole cycle $C$ is replaced by a loop at $v$. For 1-singular DP-reduction this choice does not happen, since the result of this situation is the (unique) empty clause. For example consider $U_2^2 = \{\{1\}, \{-1, 2\}, \{-2\}\}$: we can perform 1-singular DP-reduction on variables 1 or 2, obtaining $\{\{2\}, \{-2\}\}$ or $\{\{1\}, \{-1\}\}$, which corresponds to isolated cycles of length 2 — but now resolution in both cases yields $\bot$, while for the corresponding smoothing-operation one of the two literals is selected to label the remaining loop. We have shown:

**Lemma 5.2** We assume some linear order on the universe of vertices is given. For a (finite) digraph $G$ denote by $\text{sm}(G)$ the multigraph obtained from $G$ by performing smoothing steps as long as possible, where in case of a choice the first element in the given linear order is chosen. For every nonlinear vertex $v \in V(G)$ we have $v \in V(\text{sm}(G))$ with $\deg_{\text{sm}(G)}(v) = \deg_G(v)$.

The only influence of the linear order on $\text{sm}(G)$ is for isolated cycles $C \subseteq V(G)$ of length $|C| \geq 2$, where for the last element $v \in C$ according to the linear order we have $V(\text{sm}(G)) \cap C = \{v\}$, with $\deg_{\text{sm}(G)}(v) = 1$ and $N_{\text{sm}(G)}(v) = \{v\}$. The vertices of $\text{sm}(G)$ are the nonlinear vertices of $G$ plus these selected vertices for each isolated cycle of $G$.

So the results obtained for different linear orders are isomorphic, and for isomorphic multigraphs $G, G'$ also $\text{sm}(G) \cong \text{sm}(G')$.

Following [12, Section 7.2.4, D38], two multigraphs $G, G'$ are homeomorphic, if $\text{sm}(G) \cong \text{sm}(G')$. So two isomorphic multigraphs are homeomorphic, but not vice versa. In order to present the connection between 1-singular DP-reduction for 2-MUs and smoothing of the implication graphs, we need indeed to introduce...
the implication multigraph \( \text{img}(F) \) for \( F \in 2\text{-CLS} \), which for \( \perp \notin F \) is \( \text{img}(F) := \text{mg}(\text{idg}(F)) \), while for \( \perp \in F \) we add a new vertex \( v_\perp \) to \( \text{img}(F \setminus \{\perp\}) \), and add a loop at \( v_\perp \) (of multiplicity 1). So for a variable \( v \) its variable-degree \( \text{vd}_F(v) \) equals the degree of vertex \( v \) as well as the degree of vertex \( \tau \) in \( \text{img}(F) \).

Thus a vertex \( x \in V(\text{img}(F)) \) is linear iff \( \tau \) is linear, while a variable \( v \) is 1-singular in \( F \) iff vertices \( \tau, v \) are linear in \( \text{img}(F) \). If \( f : F \to F' \) is an isomorphism between \( F, F' \in 2\text{-CLS} \), then \( f' : \text{img}(F) \to \text{img}(F') \) is also an isomorphism, where \( f' \) just possibly extends the map \( f \) by mapping \( f'(v_\perp) = v_\perp \).

Now smoothing of \( \text{img}(F) \) for \( F \in 2\text{-MU} \) corresponds exactly to 1-singular DP-reduction for \( F \), except that unit-clauses \( \{x\} \) obtained by contraction, i.e., from \( \{v, x\}, \{\tau, x\} \), with \( v \) being 1-singular, never participate in the reduction process, since the multiplicity of the edge between \( \tau \) and \( x \) here is increased by two. And except that the clause-set-process does not record multiplicities of edges, of course:

**Lemma 5.3** Consider \( F \in 2\text{-MU} \). Then the underlying graph of \( \text{sm}(F) := \text{sm}(\text{img}(F)) \) is \( \text{ig}(F') \), where \( F' \) is obtained from \( F \) by any series of 1-singular DP-reductions without using new unit-clauses obtained by contraction, where the series is maximal, and where in case \( F' = \{\perp\} \) we let the single vertex \( v_\perp \) of \( \text{img}(F') \) be the final vertex in the smoothing-sequence of \( \text{img}(F) \).

We call \( \text{sm}(F) \) the **homeomorphism type** of \( F \). By Theorem 4.1 we obtain that \( \text{sm}(F) \) for \( F \in 2\text{-MU}_\delta=1 \) is equal to exactly one of the homeomorphism types of the four cases \( U_{n}^{2}, U_{n,i}^{1}, U_{n,i}^{0}, U_{n,x,y}^{0} \):

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\]

6 **Weak double cycles for 2-MUs of higher deficiency**

The operation of **splitting a vertex** \( x \) in a digraph \( G \) consists of replacing \( x \) with two new vertices \( u, v \) and an arc \( (u, v) \), such that all arcs coming into \( x \) come into \( u \), and all arcs going out of \( x \) go out of \( v \). **Splitting an arc** in \( G \) replaces an arc \( (x, y) \in E(G) \) with \( (x, v), (v, y) \) for a new (linear) vertex \( v \):

\[
\begin{array}{c}
x \quad y
\end{array}
\]

\[
\begin{array}{c}
x \quad y
\end{array}
\]

The main class of digraphs studied here is obtained from “double cycles” by the above two operations, where a **double \( m \)-cycle** for \( m \geq 3 \) is the digraph obtained from some cycle graph, that is, a digraph isomorphic to \( \text{dg}(\text{CG}_m) \). Double cycles are strongly connected, every vertex has degree 4, and for every arc also the reverse arc exists (this characterises the class of double cycles).

**Definition 6.1** ([22]) An **\( m \)-weak-double-cycle (WDC)** is a digraph obtained from some double \( m \)-cycle (\( m \geq 3 \)) by splitting some vertices or arcs.
A double $m$-cycle $G$ has $m$ (small) cycles of length two, and two (big) cycles of length $m$. These $m + 2$ cycles are precisely all the cycles in $G$. It is easy to see that splitting arcs or vertices in any digraph $G$ maintains the number the cycles of $G$ (just enlarges some of them). So an $m$-WDC $G$ has precisely $m + 2$ cycles. The small cycles in $G$ are characterised by having at most 4 nonlinear vertices (in $G$), with at most two of them of degree 4 (in $G$). The big cycles (“clockwise” and “anticlockwise”) contain all the overlapping vertices between small cycles, and alternately choose the “outer section” and “the inner section” of a small cycle, using the natural planar drawing of $G$.

To get a better grasp on the cycles in WDCs, we introduce the general concept of the cycle-multigraph $\text{cmg}(G)$ for a graph/multigraph/digraph $G$, which is a multigraph with vertex-set the cycles of $G$ (recall these are subgraph/multigraph/digraphs), and for any two vertices $g, g'$ the number of edges between them is $|V(g) \cap V(g')|$, i.e., the number of common vertices. For an isomorphism $f : G \to G'$, we obtain an induced isomorphism $f' : \text{cmg}(G) \to \text{cmg}(G')$ in the obvious way (just mapping via $f$ the vertices of $G$ inside the structure $\text{cmg}(G)$). For an $m$-WDC $G$ the cycle-multigraph $\text{cmg}(G)$ is as follows:

1. There are $m + 2$ vertices, $m$ of them in an $m$-cycle (the “small cycles”), and two central vertices connected to every other vertex (the “big cycles”).
2. Every vertex $g$ of $\text{cmg}(G)$ has a loop, multiplicity the size of the subgraph $(|V(g)|)$.
3. Every small-cycle-vertex connects with its neighbouring small cycles, multiplicity of the connecting edges being the number of vertices in the overlap.
4. The multiplicity of the edge between the two central vertices is the sum of these overlap-sizes.
5. The multiplicity of the edge connecting one central vertex $g$ with a small-cycle $g'$ is the sum of the overlaps of $g'$ with its small-cycle-neighbours plus the number of vertices in the “outer-/inner-section” of $g'$ as chosen by $g$.

An isomorphism $f : G \to G'$ of WDCs $G, G'$ maps small cycles of $G$ to small cycles of $G'$ (by the above invariant characterisation of small cycles in WDCs), and the appropriate restrictions of $f$ yield isomorphisms of these small cycles (as subdigraphs). Let $S, S'$ be the induced submultigraphs of $\text{cmg}(G), \text{cmg}(G')$ given by the small-cycle-vertices. As stated above, we have the induced multigraph-isomorphism $f' : \text{cmg}(G) \to \text{cmg}(G')$, which induces a multigraph-isomorphism $f'' : S \to S'$, since small cycles are mapped by $f'$ to small cycles. Furthermore, from $f''$ one can reconstruct $f$ in polynomial time: $f$ must respect the overlaps of the cycles, and then the map is fixed also on the interior vertices of the cycles. The underlying graph $\text{ug}(S)$ is an $m$-cycle graph. The automorphism group (the self-isomorphisms together with the composition of maps) of $CG_m$ is the Dihedral group with $2m$ elements ($m$ rotations and $m$ reflections). We have arrived at an efficient process for computing the isomorphisms between WDCs:

**Lemma 6.2** Consider WDCs $G, G'$. The isomorphisms $f : G \to G'$ can be determined in polynomial time as follows, where we assume that both $G, G'$ are $m$-WDCs for some $m \geq 3$ (otherwise $G \not\cong G'$):
1. Choose any isomorphisms $\alpha, \beta$ between the cycles $S, S'$ of small cycles in $\text{cmg}(G), \text{cmg}(G')$, as graphs, with $CG_m$.

2. Run through the $2m$ automorphisms of the Dihedral group, as permutations of $\{1, \ldots, m\}$, considered via $\alpha, \beta$ as an isomorphism $f'' : S \to S'$.

3. Keep those $f : V(G) \to V(G')$, where the extension process from $f''$ succeeds.

We see that the automorphism groups of $m$-WDCs are subgroups of the Dihedral group with $2^m$ elements (obtained in Lemma 6.2 by a natural filtering process). We also obtain a reasonably direct procedure for deciding isomorphism of WDCs (which indeed follows immediately from [21] by the fact that the maximum degrees of WDCs is 4 and using a general procedure):

**Corollary 6.3** The class of WDCs has polytime isomorphism decision.

In general from the (unlabelled) $\text{cmg}(G)$ one can not reconstruct $G$ (up to isomorphism), but for a WDC $G$ this is possible, and this even from $\text{cmg}(\text{ug}(G))$. This implies that WDCs can be reconstructed up to isomorphism from their underlying graphs. We prove this however in a more direct way, avoiding to unfold here the “full cycle-picture” (we note that $\text{cmg}(\text{ug}(G))$ has more elements than $\text{cmg}(G)$, which corresponds to the wlog’s in the direct proofs).

Between the base level of double cycles and the general level of WDCs there is the middle level of **nonlinear WDCs**, i.e., WDCs without linear vertices. Once one linear vertex has been produced via splitting of arcs, we will always keep one, and thus nonlinear WDCs are exactly generated from double cycles by (only) splitting vertices. Nonlinear $m$-WDCs arise from 0 to $m$ splittings of vertices, where splitting a degree-4-vertex yields two degree-3-vertices, and the new arc is an overlap between the two neighbouring cycles involved.

**Lemma 6.4** For any nonlinear WDC $G$, from the unlabelled $\text{mg}(G)$ we can reconstruct $G$ up to isomorphism (in polynomial time).

**Proof:** We need to give directions to the arcs of $\text{mg}(G)$, which is a big cycle of small cycles (each of length 2, 3, 4). We just choose one of the small cycles, and choose one direction for it (doesn’t matter which). Now those neighbouring cycles, which have a nontrivial overlap with that cycle, obtain their direction from the one arc in them, and so on. If we come to a one-point-connection between cycles, then we are free to choose a direction for the new cycle, and we force again the neighbouring cycles with an overlap. In this way we necessarily can give all edges a direction, and we obtain a digraph isomorphic to $G$. □

It is easy to see that general WDCs $G$ are produced by first producing some nonlinear WDC $G'$, and then splitting arcs in $G'$, obtaining $G$. In other words, using the smoothing operation (Section 5.2), an arbitrary digraph $G$ is a WDC iff $\text{sm}(G)$ is a nonlinear WDC. Adding linear vertices still allows to apply the proof of Lemma 6.4, and so we obtain

**Corollary 6.5** For any WDC $G$, from the unlabelled $\text{mg}(G)$ we can reconstruct $G$ up to isomorphism (in polynomial time).
So transpositions of WDCs are isomorphic WDCs (which allows Lemma 6.2 to be applied in the determination of skew-symmetries for WDCs). We can even forget the multiplicity of edges:

**Corollary 6.6** For any WDC $G$, from the unlabelled underlying graph of $G$ we can reconstruct $G$ up to isomorphism (in polynomial time).

**Proof:** If in the big cycle there are “single edges”, not part of a small cycle, then these edges are replaced by a pair of parallel edges. To the obtained multigraph, Corollary 6.5 is applied. \qed

So for any WDCs $G, G'$ we have $G \cong G'$ iff $\text{ug}(G) \cong \text{ug}(G')$.

Now that we know that $\text{mg}(G)$ for WDCs $G$ contains the essential information of $G$, we can consider the homeomorphism type of $G$, i.e., the homeomorphism type of $\text{mg}(G)$. So we reduce $\text{mg}(G)$ to $\text{sm}(\text{mg}(G))$ according to Lemma 5.2. We have $\text{sm}(\text{mg}(G)) = \text{mg}(G')$, where $G'$ is the nonlinear WDC which is obtained in the first phase of generating $G$ (only splitting vertices in double cycles). So the homeomorphism type of $G$ (all digraphs homeomorphic to $G$) is the set of all WDCs $H$ with $\text{sm}(\text{mg}(H)) \cong \text{sm}(\text{mg}(G))$, which is equivalent to $G' \cong H'$, where $G', H'$ are the nonlinear WDCs obtained of the first phase of generation. We will conclude this section on WDCs by giving a concrete description of the isomorphism type of nonlinear WDCs, in terms of “binary bracelets” (instead of “bracelet” also “turnover necklace” is used). Since formulas for counting bracelets are known ([10]), this yields an explicit formula for the number of isomorphism types of nonlinear $m$-WDCs.

A binary bracelet of length $m \in \mathbb{N}$ is a binary string of length $m$, where bracelets are equivalent, if one can be obtained from the other by rotation or reflection. Numerical data on the number of equivalence-classes of binary bracelets of length $m$ is given in the OEIS ([23, Sequence A000029]); for example for $m = 3$ one has 4 classes 000, 100, 110, 111, for $m = 4$ there are 6 classes 0000, 1000, 1100, 1010, 1110, 1111. A nonlinear $m$-WDC $G$ is big cycle of $m$ small cycles, where the overlap of two neighbouring cycles is either a vertex or a single edge. Such multigraphs are equivalent to a binary bracelet of length $m$, where the one- resp. the two-vertex overlap is represented by 0 resp. 1. This can be seen by considering $\text{cmg}(G)$ of a nonlinear $m$-WDC $G$, and the multi-cycle $S$ of small cycles in $\text{cmg}(G)$ (recall the discussion before Lemma 6.2). Since due to nonlinearity the small cycles have no internal structure other than given in the overlaps, $G$ can be reconstructed up to isomorphism from $S$. The information we are seeking is contained in the multiplicity of connecting edges in the multi-cycle, and we can drop the loops at the vertices. So the isomorphism type of $G$ is represented by the cycle-multigraph $S'$ of length $m$ obtained from $S$ by removing all loops: neighbouring vertices in $S'$ are connected by one/two edges if they intersect in one/two vertices. Translating “one edge” into 0 and “two edges” into 1, we obtain the derived bracelet (up to equivalence of bracelets — we pick an arbitrary starting point in the cycle, and pick one of the two directions). We have shown that two nonlinear WDCs $G, G'$ are isomorphic iff the derived binary bracelets are equivalent. Thus the number of isomorphism types of nonlinear $m$-WDCs is the number of equivalence classes of binary bracelets.
7 Classifying 2-MUs of higher deficiency

We fix now \( k \geq 2 \) and consider \( F \in 2\cdot\mathcal{MU}_{6=k} \). Two reminders: First, \( F \) does not have unit-clauses (but is 2-uniform; \( \geq 2 \) [20, Lemma 8], \( \geq 3 \) [4, Lemma 5.1]) — if it had unit-clauses, then unit-clause propagation would yield necessarily the empty clause (otherwise \( F \) had a non-trivial autarky, which contradicts MU), thus \( F' \) would be renamable Horn, and so \( \delta(F) = 1 \). Second, every literal occurs in \( F \) at most twice — otherwise we would set this literal to false, obtain from the result an MU by removing some other clauses, and obtain a 2-MU with at least three unit-clauses, but in any 2-MU there are at most two unit-clauses (also remarked in \( \geq 3 \)).

Lemma 7.1 From \( 2\cdot\mathcal{MU}_{6=k}^+ \) we obtain \( 2\cdot\mathcal{MU}_{6=k} \) by repeated applications of 1-singular extension, which means that for \( F \in 2\cdot\mathcal{MU}_{6=k} \) one chooses \( \{x, y\} \in F \) (\( x \neq y \) holds) and a new variable \( v \), and replaces \( \{x, y\} \) by \( \{v, x\} \). The related reduction by 1-singular DP-reduction reduces \( F \) to \( F' := 1sDP(F) \in 2\cdot\mathcal{MU}_{6=k}^+ \). For the implication digraphs \( G := idg(F) \) and \( G' := idg(F') \) we have now \( mg(G') = sm(mg(G)) \) by Lemma 5.3 (contractions are impossible).

Our new starting point is now \( F' \), and we perform singular DP-reductions, which by Lemma 5.4 are necessarily non-1-singular, that is, eliminate variables of degree 3 (also called “2-singular variables”, since there are two side-clauses). So consider a variable \( v \) of \( F \) of degree 3, with occurrences \( \{v, x\}, \{\bar{v}, y\}, \{\bar{v}, z\} \in F \). Again we do not have contraction here, that is \( x \neq y \) and \( x \neq z \) (otherwise a unit-clause would be created), thus DP-reduction for \( v \) increases the literal-degree of \( x \) by one, and thus not only literal \( v \) occurs only once in \( F \), but also literal \( x \) (and \( \bar{v} \) is also 2-singular). We have shown that a singular DP-reduction for any \( F' \in 2\cdot\mathcal{MU}_{6=k}^+ \) removes one 2-singular variable (degree-3-variable), transforms one 2-singular variable (degree-3-variable) into a non-singular variable (degree-4-variable), and leaves other literal-degrees unchanged. Since this reduction process for \( F' \) ends with a clause-set isomorphic to \( B_k \), which has \( k \) variables, all of degree 4 (non-singular), there can be at most \( k \) singular DP-reductions for \( F' \).

Lemma 7.2 We obtain \( 2\cdot\mathcal{MU}_{6=k}^+ \) by starting with any clause-set isomorphic to \( B_k \), and then repeatedly applying up to \( k \) times the following process for \( F \in 2\cdot\mathcal{MU}_{6=k}^+ \): choose some literal \( x \) which occurs positively and negatively twice in \( F \), and for the occurrences \( \{x, a\}, \{x, b\}, \{\bar{x}, c\}, \{\bar{x}, d\} \in F \) and a literal \( y \) with underlying new variable \( \var(y) \), replace the two \( x \)-clauses by \( \{y, x\}, \{\bar{y}, a\}, \{\bar{y}, b\} \).

So altogether we obtain all \( F \in 2\cdot\mathcal{MU}_{6=k} \) by first applying Lemma 7.2 obtaining \( F' \in 2\cdot\mathcal{MU}_{6=k}^+ \), which is taken as starting point for applying Lemma 7.1. Such a generation sequence can be computed in polynomial time for \( F \), by first computing 1sDP(\( F \)), which in turn is reduced by singular DP-reduction, and then reversing the whole reduction sequence.

We now show that the implication digraphs of \( F \in 2\cdot\mathcal{MU}_{6=k} \) are 2k-WDCs. To start, we have \( idg(B_k) \cong \delta(G_{2k}) \):

\[
\text{idg}(B_k) = \begin{pmatrix}
1 & 2 & \cdots & k - 1 & k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-(k - 1) & -(k - 2) & \cdots & -2 & -1 \\
\hline
\end{pmatrix}
\]
Now consider Lemma\textsuperscript{7.2}. We replace two clauses $\{x, a\}, \{x, b\}$ by three clauses
$\{y, x\}, \{\overline{y}, a\}, \{\overline{y}, b\}$. For the implication digraph this means the transition:

We see that this can be obtained up to isomorphism of digraphs by first, say, splitting vertex $x$, and then splitting vertex $\overline{x}$ (recall that the vertices in implication digraphs are just placeholders, and also do not know about complementation). We have shown:

**Lemma 7.3** The implication digraph of $F \in 2 \cdot \mathcal{M}_U^{+} = k$ is a nonlinear $2k$-WDC.

In the same way, obviously one step of 1-singular extension in Lemma\textsuperscript{7.1} is captured by two applications of arc-splitting. Altogether we have shown

**Theorem 7.4** The implication digraph of $F \in 2 \cdot \mathcal{M}_U^{+} = k$ is a $2k$-WDC.

We are now ready to prove the main result, showing that the implication digraph of $F$ has a unique skew-symmetry, which yields the complementation of literals, and thus one can reconstruct $F$ from the (unlabelled) idg($F$). Since $F$ does not have unit-clauses, we have to exclude skew-symmetries which yield them (otherwise uniqueness wouldn’t hold). So we define, that a skew-symmetry $\sigma$ of a digraph $G$ is **unit-free** if $\forall v \in V(G) : (v, \sigma(v)) \notin E(G)$. We start with a lemma on the skew-symmetries of the directed cycle:

**Lemma 7.5** Consider a cycle digraph $G$ with $n \geq 2$ vertices. If $n$ is odd then there is no complementation. For even $n$ there are exactly $n/2$ complementations $\sigma$. All clause-sets given by $(G, \sigma)$ are isomorphic to $U^2_n$ (and thus $\sigma$ has a unit).

**Proof:** W.l.o.g. we assume $G = 1 \rightarrow \ldots \rightarrow n \rightarrow 1$. Recall, the skew-symmetries are the digraph-isomorphisms $f : G \rightarrow G^t$, which as permutations of $V(G)$ are involutions and don’t have fixed-points. The isomorphisms from $G$ to $G^t$ are given by the $n$ rotations, the $n$ symmetries of $G$, composed with one fixed isomorphism from $G$ to $G^t$, where one can use the rotation “anticlockwise”, i.e., $1 \mapsto 1, 2 \mapsto n, \ldots, n \mapsto 2$. This yields that precisely the $n$ reflections of the (undirected) cycle $CG_n$ are the sought isomorphisms. They all are involutions, and exactly half of them are fixed-point free. Recalling the implication digraph of $U^2_n$ (Section\textsuperscript{3}), they all yield clause-sets isomorphic to $U^2_n$. □

We also need a variation:

**Lemma 7.6** Consider a digraph $G$ which is the union of two directed cycles $G', G''$, i.e., $V(G) = V(G') \cup V(G'')$ and $E(G) = E(G') \cup E(G'')$, such that the overlap $V(G') \cap V(G'')$ is not empty, and the induced subdigraph on it is a path of length $|V(G) \cap V(G')| - 1$. Then every skew-symmetry of $G$ has a unit.
Proof: Assume a unit-free complementation $\sigma$ of $G$, and let $F \in 2\cdot\mathcal{CLS}$ be the corresponding clause-set. $F$ is unsatisfiable, since $G$ is strongly connected. Indeed $F$ is minimally unsatisfiable, since otherwise there would be a subdigraph $G'$ stable under $\sigma$ with at least two arcs less, corresponding to an MU inside $F$, but $G'$ cannot have a contradictory cycle. The homeomorphism type of $G$ is that of two cycles, either with a one-point connection or with a nontrivial overlap. If $\delta(F) \geq 2$, then by Lemma 7.3 there would be $2k$ cycles in it, which is not possible. So $\delta(F) = 1$. But also this requires at least three cycles, since $F$ does not have a unit-clause (see the homeomorphism types shown after Lemma 5.3).

We are ready to show that WDCs can yield at most one 2-MU (in the precise sense, not just up to isomorphism):

**Theorem 7.7** Every WDC has at most one unit-free complementation.

**Proof:** Consider a WDC $G$ and a unit-free skew-symmetry $\sigma$ for $G$. We show that $\sigma$ is unique. As above, $\sigma : G \to G^t$ is an isomorphism, where $G^t$ is also a WDC. We obtain the induced isomorphism $\sigma' : \text{cmg}(G) \to \text{cmg}(G^t)$ (recall the discussion before Lemma 6.2). Furthermore, there is the induced isomorphism $\sigma'' : S \to S'$, where $S, S'$ are the induced subgraphs given by the small cycles in $G, G^t$ (here indeed just as the subgraphs, not as submultigraphs). $\sigma'$ is just $\sigma$ on the vertices, transported to the small cycles as subdigraphs of $G$. Now the small cycles of $G^t$ are essentially the same as the small cycles of $G$, except of the reversed direction of the arcs. Thus w.l.o.g. we can consider $\sigma''$ as an automorphism (symmetry) of the undirected $m$-cycle $S$ (where $G$ is an $m$-WDC), that is, $\sigma''$ is one of the $m$ rotations and $m$ reflections.

If $\sigma''$ had a fixed-point (would map one small cycle of $G$ to itself), then by Lemma 7.3, $\sigma$ would not be unit-free. If $m$ would be odd, then the only symmetries without fixed-points are the nontrivial rotations, but for odd $n$ none of them is an involution. So $m$ is even. This leaves for $\sigma''$ the $m$ reflections and the point-symmetry, the rotation by 180 degrees. We now exclude the reflections, which proves the theorem (since from $\sigma''$ one can reconstruct $\sigma$). And this is indeed easy now: Assume $\sigma$ is a reflection. As already used in Lemma 7.3, there are two neighbouring vertices of $S$ which are mapped by $\sigma''$ to each other. Now by Lemma 7.6, $\sigma$ again would not be unit-free. □

We finally have shown the main result of the paper:

**Theorem 7.8** Consider $F, F' \in 2\cdot\mathcal{ML}_k$. Then the set of isomorphisms $f : F \to F'$, as maps $f : \text{lit}(F) \to \text{lit}(F')$, is equal to the map of isomorphisms $f : \text{idg}(F) \to \text{idg}(F')$ (as maps $f : V(\text{idg}(F)) \to V(\text{idg}(F'))$).

**Proof:** In general every isomorphism from $F$ to $F'$ is an isomorphism from $\text{idg}(F)$ to $\text{idg}(F')$; so assume that $f$ is an isomorphism from $\text{idg}(F)$ to $\text{idg}(F')$, and we have to show that $f$ is an isomorphism from $F$ to $F'$. This follows by observing that $f$ transports any skew-symmetry $\sigma$ for $\text{idg}(F)$ to a skew-symmetry $\sigma_f$ for $\text{idg}(F')$, and $f$ then becomes an isomorphism from $(\text{idg}(F), \sigma)$ to $(\text{idg}(F'), \sigma_f)$. By Theorem 7.7, $\sigma$ is the natural skew-symmetry of $\text{idg}(F)$ as given by the complementation of $F$, and $\sigma_f$ is the natural skew-symmetry as
given by $F'$. Since digraphs with given skew-symmetry are the same as 2-CNFs, the statement follows. □

We obtain a number of applications:

**Corollary 7.9** For $F, F' \in 2\mathcal{MU}_{\delta=k}$ holds $F \cong F' \iff \text{idg}(F) \cong \text{idg}(F')$, where the implication digraphs are 2k-WDCs.

**Corollary 7.10** For $F, F' \in 2\mathcal{MU}_{\delta=k}$ the number of isomorphisms between $F$ and $F'$ is at most $4k$. The automorphism group of $F$ is a subgroup of the Dihedral group with $4k$ elements, and construction of the group table can be done in time $O(k \cdot \|F\|)$, using $\|F\|$ for the length of $F$.

**Corollary 7.11** The isomorphism problem for inputs $F, F' \in 2\mathcal{MU}$ can be decided in time $O(\delta(F) \cdot \|F+F'\|)$, assuming $\delta(F) = \delta(F')$ (otherwise $F \not\cong F'$).

**Corollary 7.12** The number of isomorphism types of $F \in 2\mathcal{MU}_{\delta=k}$ with (exactly) $n(F) = n \in \mathbb{N}_0$ variables is $\Theta(n^{3k-1})$ (for fixed $k$).

**Proof:** There are $2k$ cycles in $\text{idg}(F)$, with half of them duplicated by skew-symmetry, so that we have $k$ essential cycles. These cycles are arranged in a big cycle, and so have three non-overlapping parts, say the upper, right, and lower parts, which makes $3k$ numbers adding up to $n$, and so the number of isomorphism types is $O(n^{3k-1})$. By Corollary 7.10 the equivalence-classes are of constant size. □

**Corollary 7.13** The homeomorphism types of $2\mathcal{MU}_{\delta=k}$ are in one-to-one correspondence with the equivalence classes of binary bracelets of length $k$.

**Proof:** The homeomorphism types are the isomorphism types of $2\mathcal{MU}_{\delta=k}^+$, which correspond to the isomorphism types of nonlinear 2k-WDCs with skew-symmetry. Isomorphism types of nonlinear 2k-WDCs correspond to equivalence classes of binary bracelets of length 2k, and due to skew-symmetry, half of them are discarded. □

**References**

[1] Hoda Abbasizanjani and Oliver Kullmann. Minimal unsatisﬁability and minimal strongly connected digraphs. In Olaf Beyersdorff and Christoph M. Wintersteiger, editors, Theory and Applications of Satisﬁability Testing - SAT 2018, volume 10929 of Lecture Notes in Computer Science, pages 329–345. Springer, 2018. doi:10.1007/978-3-319-94144-8_20.

[2] Ron Aharoni and Nathan Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. Journal of Combinatorial Theory, Series A, 43(2):196–204, November 1986. doi:10.1016/0097-3165(86)90060-9.
[3] Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A linear-
time algorithm for testing the truth of certain quantified boolean for-
mulas. Information Processing Letters, 8(3):121–123, March 1979. doi:10.1016/0020-0190(79)90002-4

[4] Hans Kleine Büning, Piotr Wojciechowski, and K. Subramani. On the com-
putational complexity of read once resolution decidability in 2CNF for-
ulas. In T. Gopal, G. Jger, and S. Steila, editors, International Conference
on Theory and Applications of Models of Computation (TAMC 2017), vol-
ume 10185 of Lecture Notes in Computer Science (LNCS), pages 362–372.
Springer, 2017. doi:10.1007/978-3-319-55911-7_26.

[5] Joshua Buresh-Oppenheim and David Mitchell. Minimum witnesses for
unsatisfiable 2CNFs. In Armin Biere and Carla P. Gomes, editors, The-
ory and Applications of Satisfiability Testing - SAT 2006, volume 4121 of
Lecture Notes in Computer Science, pages 42–47. Springer, 2006. doi:
10.1007/11814948_6.

[6] V. Chvátal and Bruce Reed. Mick gets some (the odds are on his side).
In Proc. 32th Annual Symposium on Foundations of Computer Science
(Pittsburgh, PA), pages 620–627. IEEE Comput. Soc. Press, 1992. doi:
10.1109/SFCS.1992.267789.

[7] Yves Crama and Peter L. Hammer. Boolean Functions: Theory, Algo-
rithms, and Applications, volume 142 of Encyclopedia of Mathematics and
Its Applications. Cambridge University Press, 2011. ISBN 978-0-521-84751-
3.

[8] Gennady Davydov, Inna Davydova, and Hans Kleine Büning. An efficient
algorithm for the minimal unsatisfiability problem for a subclass of CNF.
Annals of Mathematics and Artificial Intelligence, 23(3-4):229–245, 1998.
doi:10.1023/A:1018924526592.

[9] W. Fernandez de La Vega. Random 2-SAT: results and problems. Theoret-
ical Computer Science, 265:131–146, 2001. doi:10.1016/S0304-3975(01)
00156-6.

[10] E.N. Gilbert and John Riordan. Symmetry types of periodic seque-
nces. Illinois Journal of Mathematics, 5(4):657–665, 1961. doi:10.1215/ijm/
1255631587.

[11] Andrew V. Goldberg and Alexander V. Karzanov. Path problems in
skew-symmetric graphs. Combinatorica, 16:353–382, 1996. doi:10.1007
BF01261321.

[12] Jonathan L. Gross and Jay Yellen, editors. Handbook of Graph Theory.
Discrete Mathematics and Its Applications. CRC Press, 2003. ISBN 1-
58488-090-2; QA166.H36.

[13] Vaibhav Karve and Anil N. Hirani. The complete set of minimal simple
graphs that support unsatisfiable 2-CNFs. Discrete Applied Mathematics,
2020. In press. doi:10.1016/j.dam.2019.12.017. 

17
[14] Hans Kleine Büning. On subclasses of minimal unsatisfiable formulas. *Discrete Applied Mathematics*, 107(1-3):83–98, 2000. doi:10.1016/S0166-218X(00)00245-6

[15] Hans Kleine Büning and Oliver Kullmann. Minimal unsatisfiability and autarkies. In Armin Biere, Marijn J.H. Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*, chapter 11, pages 339–401. IOS Press, February 2009. doi:10.3233/978-1-58603-929-5-335

[16] Hans Kleine Büning and Daoyun Xu. The complexity of homomorphisms and renamings for minimal unsatisfiable formulas. *Annals of Mathematics and Artificial Intelligence*, 43(1-4):113–127, 2005. doi:10.1007/s10472-005-0422-8

[17] Hans Kleine Büning and Xishun Zhao. Minimal unsatisfiability: Results and open questions. Technical Report tr-ri-02-230, Series Computer Science, University of Paderborn, University of Paderborn, Department of Mathematics and Computer Science, 2002. http://wwcs.uni-paderborn.de/cs/ag-klbue/de/research/MinUnsat/index.html

[18] Oliver Kullmann. An application of matroid theory to the SAT problem. In *Proceedings of the 15th Annual IEEE Conference on Computational Complexity*, pages 116–124, July 2000. See also TR00-018, Electronic Colloquium on Computational Complexity (ECCC), March 2000. doi:10.1109/CCC.2000.856741

[19] Oliver Kullmann and Xishun Zhao. On Davis-Putnam reductions for minimally unsatisfiable clause-sets. *Theoretical Computer Science*, 492:70–87, June 2013. doi:10.1016/j.tcs.2013.04.020

[20] Paolo Liberatore. Redundancy in logic II: 2CNF and Horn propositional formulae. *Artificial Intelligence*, 172(2-3):265–299, February 2008. doi:10.1016/j.artint.2007.06.003

[21] Eugene M. Luks. Isomorphism of graphs of bounded valence can be tested in polynomial time. *Journal of Computer and System Sciences*, 25(1):42–65, August 1982. doi:10.1016/0022-0000(82)90009-5

[22] Paul Seymour and Carsten Thomassen. Characterization of even directed graphs. *Journal of Combinatorial Theory Series B*, 42(1):36–45, 1987. doi:10.1016/0095-8956(87)90061-X

[23] Neil J.A. Sloane. The On-Line Encyclopedia of Integer Sequences (OEIS), 2008. Available from: http://oeis.org/
A The running example

The running example of the paper is based on $B_3$, which has the following implication digraph (3 variables, thus 6 vertices, and 6 clauses, thus $2 \cdot 6 = 12$ arcs):

$$
\text{idg}(B_3) = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{b3_idg.png}
\end{array}
$$

$idg(B_3)$ is a double 6-cycle and so has six cycles of length two, the cycles $K_1, K_2, K_3$, and their contrapositions $-K_1, -K_2, -K_3$, where the contraposition of an arc $(x, y)$ is the arc $(y, x)$ (we don’t use the notation $\overline{K_i}$ here for typographical reasons). We note here, that the contraposition of each small cycle is its “antipodal” cycle, on the “opposite side” of the digraph. $idg(B_3)$ has also two big cycles, namely $K_4 : v_1 \to v_2 \to v_3 \to v_4 \to v_5 \to v_6 \to v_1$ and its contraposition $-K_4$, and these two cycles are exactly the contradictory cycles. In general, as mentioned in the paper, the implication digraph of $B_n$ is a double $2n$-cycle with $2n$ small cycles (non-contradictory), and two big cycles (contradictory), so that together $idg(B_n)$ has exactly $2n + 2$ cycles.

The pairs of complementary literals in $idg(B_3)$ are shown in the following image as $\bullet \leftrightarrow \bullet$, $\circ \leftrightarrow \circ$, $\times \leftrightarrow \times$ (note their antipodal positions); we also show the abstract implication digraph alone, which has lost the information on the complementation, that is, the “unlabelled” $idg(B_3)$:

$$
\text{idg}(B_3) = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{b3_idg_unlabelled.png}
\end{array}
$$

The final abstraction for a 2-MU $F$ is the homeomorphism type of the implication graph of $F$. The homeomorphism type of $B_3$, shown below, is a cycle of 6 (small) cycles where the connection of these small cycles is precisely one vertex.

$$
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{b3_homeomorphism_type.png}
\end{array}
$$

As we used natural numbers for variables (like in the DIMACS file format), the clause $\{-1, 2\}$ stands for the usual clause $\{\overline{v_1}, v_2\}$. The following implication digraph, which is a 6-WDC, is our running example which we have obtained from
idg($\mathcal{B}_3$) by splitting some vertices and arcs.

As mentioned before, the implication digraph together with complementation of vertices is essentially the same as the original clause-set. The underlying clause-set of the above implication digraph is

$$F = \{\{1, -5\}, \{3, 5\}, \{-1, 4\}, \{-4, 6\}, \{-6, 2\},$$

$$\{-2, 3\}, \{-1, -3\}, \{1, -2\}, \{-3, 4\}\} \in 2\cdot\mathcal{M}U.$$

idg($F$) has six small cycles $K_1, K_2, K_3$ and their contraposition $-K_1, -K_2, -K_3$, as in idg($\mathcal{B}_3$). We see that idg($F$) has four linear vertices, namely 5, -5, 6, -6. In order to understand better the structure of $F$ we do some abstraction, namely we remove all the linear vertices (obtaining a nonlinear WDC) as follows:

The underlying clause-set for this implication digraph is

$$F' = \{\{1, 3\}, \{-1, 4\}, \{2, -4\}, \{-2, 3\}, \{-1, -3\}, \{1, -2\}, \{-3, 4\}\} \in 2\cdot\mathcal{M}U.$$

We have shown in the paper, that the homeomorphism types of the implication multigraph for any 2-MU of deficiency greater than one is a cycle of cycles, where the overlap of every small cycle with its neighbouring small cycles is either a vertex or an edge. The homeomorphism type of idg($F$) is as follows:
As a nonlinear 6-WDC, the derived bracelet (starting top-left) is 100100.

B Examples for digraphs and skew-symmetries

Non-isomorphic 2-CNFs can have isomorphic digraphs, and so the isomorphism type of implication digraphs is not a “complete isomorphism invariant” for 2-CNFs, as the following example shows:

Example B.1 We consider any digraph \( G \) being the disjoint union of two (directed) cycles. If the cycles have different lengths, then they cannot be the contraposition of each other, and so each cycle must be a contradictory cycle (as for every literal its complement must be in the same cycle). That is, every \( F \in 2-\text{CLS}^* \) with \( \text{idg}(F) \cong G \) is unsatisfiable. We assume now that the cycles have equal length. So we have two possibilities, namely that the cycles are the contraposition of each other, or they both are contradictory. The first case corresponds to a satisfiable 2-CNF, while the second case yields an unsatisfiable 2-CNF as before, e.g., consider \( F = \{\{−1, 2\}, \{−2, 3\}, \{−3, 4\}, \{1, −4\}\}, F′ = \{\{−1\}, \{1, 2\}, \{−2\}, \{−3\}, \{3, 4\}, \{−4\}\}. The implication digraphs are

\[
\text{idg}(F) = \begin{array}{cccc}
1 & 2 & −1 & −4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
4 & 3 & −2 & −3
\end{array}
\quad \begin{array}{cccc}
1 & −1 & 3 & −3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
−2 & 2 & −4 & 4
\end{array}
\]

\( \text{idg}(F′) \) has two contradictory cycles, and so \( F′ \) is unsatisfiable, while \( F \) is satisfiable. Therefore \( F \not\cong F′ \), while \( \text{idg}(F) \cong \text{idg}(F′) \).

Example B.2 Continuing with the \( F, F′ \) from Example B.1, we consider the digraph \( G := \{(v_1, \ldots, v_4, w_1, \ldots, w_4), \{(v_1, v_2), \ldots, (v_4, v_1), (w_1, w_2), \ldots, (w_4, w_1)\}\} \).

There are four complementations yielding a digraph with skew-symmetry isomorphic to \( \text{sidg}(F) \), namely one can choose \( \overline{v_i} = w_i \) for any \( i \), and then the other complementations are determined. And there are \( 2 \cdot 2 = 4 \) complementations yielding \( \text{sidg}(F′) \), namely one can say \( \overline{v_1} = v_2 \) or \( \overline{v_2} = v_3 \) for the first cycle (which determines the complementations in this cycle), and the same for the second cycle. Altogether \( G \) has exactly \( 4 + 4 = 8 \) complementations, which yield exactly two isomorphism-types of digraphs with skew-symmetry.

C 2-MUs of deficiency one

In Section 4 of the paper, we characterised all the isomorphism types of \( F \in 2-\text{MU}_{δ=1} \); here now we provide the proof of these results (we also repeat the statements, so that this section is self-contained).

By [8, Corollary 13] and [8] 1-singular DP-reduction, i.e., DP-reduction for variables occurring exactly once positively and once negatively, applied to any MU \( F \) results in \( \{\bot\} \) if \( δ(F) = 1 \). So we can generate (exactly) all of \( 2-\text{MU}_{δ=1} \) by starting from the empty clause, and repeatedly replacing a single clause \( C \) by two clauses \( C′ \cup \{v\}, C″ \cup \{\overline{v}\} \) for \( C′ \cup C″ = C, |C′|, |C″| \leq 1 \), and a new variable \( v \). The clause-sets generated this way, starting with \( \{\{\bot\}\}, \) together exactly yield \( 2-\text{MU}_{δ=1} \).
Here we consider generating the elements of $2\cdot\mathcal{MU}_{δ=1}^r$, and so the starting point are the 2-MUs with precisely one variable, namely $\{v, \{\overline{v}\}\}$. We need indeed not to create all of $2\cdot\mathcal{MU}_{δ=1}^r$, but only up to isomorphism. We have w.l.o.g. the following cases, for $|C| = 1, 2$:

(i) $C = \{x\}$:

A

$C' = \{x, v\}, C'' = \{\overline{v}\}$.

B

$C'' = \{x, v\}, C'' = \{x, \overline{v}\}$.

(ii) $C = \{x, y\}, x \neq y$:

$C' = \{x, v\}, C'' = \{y, \overline{v}\}$.

We note that there are always at most two unit-clauses (this holds in general for $2\cdot\mathcal{MU}$, but can be observed easily here). We now standardise the process, to minimise the number of case distinctions needed. It is possible to start only with variable $v = 1$, that is, with $\{1\}, \{-1\}$, and for each new variable to choose the next natural number. In our examples we will sometimes proceed in this way, but in general it is more convenient to have a free choice of variables (restricting this doesn’t save anything from a proof perspective). To really simplify the generation, the application of rules need to be restricted.

Rule (ii) cannot be used at the start, and Rule B can be applied at most twice. Indeed the generation process can be restricted w.l.o.g. to have two phases, where Rules A, B are only used in the first phase, and Rule (ii) only in the second phase. For each rule, the clause we choose ($C$ above) is the **main clause**, while the **side clauses** are the replacement clauses ($C', C''$ above). If Rule (ii) is followed by Rule A or B, then we can swap the applications, as the side clauses for Rule (ii) are binary and thus disjoint with the main clause for Rule A or B. Also if Rule B is followed by Rule A, then because of disjointness of the side clauses of B and the main clause of A we can swap the rules. So we can assume that a generation process has first applications of Rule A, then at most two applications of B, and then applications of (ii).

Furthermore, two consecutive applications of Rule A can be replaced by one application of Rule A, followed by one application of Rule (ii); this is shown by considering the very first applications, w.l.o.g. first applied to $\{1\}$, then to $\{-2\}$, yielding

\[
\{\{1\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2, 3\}, \{-3\}, \{-1\}\}.
\]

The simulation is

\[
\{\{1\}, \{-1\}\} \leadsto \{\{1, 3\}, \{-3\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2, 3\}, \{-3\}, \{-1\}\}.
\]

The last point in this standardisation process is to consider exactly one application of A, followed by at least one application of B. Here it doesn’t matter, whether the first application of B uses as main clause the original clause or the new unit-clause produced by A, while we note that the unit-clause for a second application of B is unique. The reason is that both clause-sets are isomorphic: the first case yields $\{\{1\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2, 3\}, \{-1, -3\}\}$, the second case yields $\{\{1, 2\}, \{-2\}, \{-1\}\} \leadsto \{\{1, 2\}, \{-2, 3\}, \{-2, -3\}, \{-1\}\}$, and the isomorphism swaps variables 1 and 2. We summarise:
Lemma C.1 We can generate up to isomorphism the elements of $2\mathcal{MU}_{k=1}^*$ by a sequence of applications of Rules A, B, (ii), with the following restrictions: First at most one application of A, then at most two applications of B, and finally arbitrarily many application of (ii) (if at least one application of Rules (ia) or (ib) took place). If we have one A and at least one B, then as main clause of the first B the new unit-clause is used.

Concerning Rule (ii), it is easy to see that it produces just a chain as follows:

Lemma C.2 Applying Rule (ii) $n \geq 1$ times to $\{x, y\}$ yields a clause-set isomorphic to $\{x, 1\}, \{-1, 2\}, \ldots, \{-n, n\}, \{-n, y\}.$

Important to note here that when there are several binary clauses to start with, the applications of Rule (ii) don’t interfere, and so for each of the starting binary clauses we can apply Lemma C.2, with the new variables made disjoint.

We are now ready to derive five basic types of $F \in 2\mathcal{MU}_{k=1}^*$, according to the number of applications of Rules A, B (while Rule (ii) is applied arbitrarily often). The five starting points for the applications of Lemma C.2 (and Rule (ii)) are as follows, showing the sequence of applications of Rules A, B, (and after the colon the number of unit-clauses:

(A) $\{\{1, 2\}, \{-2\}, \{-1\} : 2.$

(B) $\{\{1, 2\}, \{1, -2\}, \{-1\} : 1.$

(AB) $\{\{1, 2\}, \{-2, 3\}, \{-2, -3\}, \{-1\} : 1.$

(BB) $\{\{1, 2\}, \{1, -2\}, \{-1, 3\}, \{-1, -3\} : 0.$

(ABB) $\{\{1, 2\}, \{-2, 3\}, \{-2, -3\}, \{-1, 4\}, \{-1, -4\} : 0.$

First consider case (A). Replacing $\{1, 2\}$ according to Lemma C.2, using new variables $3, \ldots, n$ yields the clauses $\{1, 2\}, \{-2, 3\}, \ldots, \{-n, n\}, \{-n, 2\}, \{-2\}, \{-1\}.$ For better formatting we swap variables $n$ and 2, and flip variable 1. This yields “$U_n^2$” listed below (with “U” for “unit”), which makes sense for $n \geq 1$, and thus covers all cases with exactly two unit-clauses. To give an overview, we list also the other three cases now (discussed below). They all generalise just the two unit-clauses. Let $F := \{-1, 2\}, \ldots, \{-n, 1\}$ for $n \in \mathbb{N}$ (and with $n-1$ clauses) be the invariant “middle part” (compare Lemma C.2):

1. $U_n^2 := M \cup \{\{1\}, \{-n\}\}$ for $n \geq 1.$

2. $U_{n,i}^1 := M \cup \{\{1\}, \{-n, -i\}\}$ for $n \geq 2, 1 \leq i \leq n - 1.$

3. $U_{n,i}^0 := M \cup \{\{1, i\}, \{-n, -i\}\}$ for $n \geq 3, 2 \leq i \leq \frac{n - 1}{2}.$

4. $U_{n,x,y}^0 := M \cup \{\{1, x\}, \{-n, -y\}\}$ for $n \geq 4, 2 \leq x < y \leq n - 1, x + y \leq n + 1.$

For the first case, with two units, we have shown:

Lemma C.3 For $F \in 2\mathcal{MU}_{k=1}^*$ holds $F \cong U_n^2(F)$ iff $F$ has two unit-clauses.
The implication digraph of $U_n^2$ is a cycle digraph with $2n$ vertices and $2n$ edges (where all vertices have degree 2). The labelled digraph, actually a graph with skew-symmetry, is shown as follows. Here arcs from unit-clauses are drawn as double-arcs (if multigraphs would be used, then unit-clauses indeed would yield two parallel arcs):

$$\text{idg}(U_n^2) = \begin{array}{c}
1 & \cdots & n-1 & \cdots & 2 & \cdots & n \\
\parallel & \cdots & \parallel & \cdots & \parallel & \cdots & \parallel \\
-1 & \cdots & -(n-1) & \cdots & -2 & \cdots & -n
\end{array}$$

We now come to cases (B), (AB), i.e., exactly one unit-clause. First we note that we can merge chains based on two binary clauses $\{x, z\}, \{y, \overline{z}\}$, using $z$ to connect the chains:

**Lemma C.4** Applying Rule (ii) $n \geq 1$ times to $\{\{x, z\}, \{y, \overline{z}\}\}$ yields a clause-set isomorphic to $\{x, 1\}, \{-1, 2\}, \ldots, \{-n-1, n\}, \{-n, y\}$. Thus from Case (B) we obtain $\{\{-1\}, \{1, 2\}, \{-2, 3\}, \ldots, \{-n-1, n\}, \{-n, 1\}\}$ for $n \geq 2$. We note that after flipping literal 1, this is $U_n^2$, when adding to the last clause the literal $-1$, i.e., we get $U_{n,1}^1$ (recall the list above).

For Case (AB), we rename variable 2 to some $x$ not used as new variable. First from $\{1, x\}$ we obtain either $\{1, x\}$ or $\{1, 2\}, \ldots, \{-p, x\}$ for some $p \geq 2$. And from $\{-x, 3\}, \{-x, -3\}$, renamed to $\{-x, p+1\}, \{-x, -(p+1)\}$, we obtain $\{-x, p+1\}, \ldots, \{-q, -x\}$ for some $q \geq p+1$. Appending these chains yields with the original $\{\{-1\}\}$ a clause-set isomorphic to $\{\{-1\}\}, \{1, 2\}, \ldots, \{-n-1, n\}, \{n, -i\}$ for some $2 \leq i < n$ and $n \geq 2$. After flipping literal 1, this is $U_n^2$, when adding to the last clause the literal $-i$, i.e., we get $U_{n,1}^i$. We have shown:

**Lemma C.5** For $F \in 2\mathcal{M}d_{n+1}$ holds $F \cong U_{n(F),i}^1$ for some $1 \leq i < n(F)$ iff $F$ has exactly one unit-clause.

We note that $U_n^2 = U_{n,n}^1$ (allowing this degeneration for the moment). The implication digraph of $U_{n,1}^1$ has $2n$ vertices and $2n+1$ edges, and consists of two cycle digraphs of length $n+i$, which overlap in a path of length $2i-1 \geq 1$ (we note $2(n+i) - (2i-1) = 2n+1$); two vertices have degree 3, all other vertices have degree 2:

$$\text{idg}(U_{n,1}^1) = \begin{array}{c}
1 & \cdots & i & \cdots & n \\
\parallel & \cdots & \parallel & \cdots & \parallel \\
-1 & \cdots & -i & \cdots & -n
\end{array}$$

$$\begin{array}{c}
n & \cdots & n-1 & \cdots & i & \cdots & i+2 & \cdots & i+1 \\
\leftrightarrow & \cdots & \leftrightarrow & \cdots & \leftrightarrow & \cdots & \leftrightarrow & \cdots & \leftrightarrow \\
-i & \cdots & -(i-1) & \cdots & -1 & \cdots & i-1 & \cdots & i
\end{array}$$

$$\begin{array}{c}
-(i+1) & \cdots & -(i+2) & \cdots & -(n-1) & \cdots & -n
\end{array}$$

We note here that the digraphs $\text{idg}(U_{n,1}^1)$ have a unique skew-symmetry (we don’t prove that here, but the basic fact used is that a skew-symmetry is an “anti-automorphism”, and has to pair vertices of identical degree).

Finally we come to cases (BB), (ABB) (without unit-clauses, and thus all clause-sets considered are 2-uniform). For case (BB), we apply Lemma C.4.
twice, and similar to above, we obtain $U_{n,i}^0$ (defined above), for the moment allowing all $2 \leq i \leq n - 1$. The implication digraph of $U_{n,i}^0$ has $2n$ vertices and $2n + 2$ edges, and two vertices have degree 4, while all other vertices have degree 2:

$$\text{idg}(U_{n,i}^0) = 1 \rightarrow \cdots \rightarrow i \rightarrow \cdots \rightarrow n = -1 \rightarrow \cdots \rightarrow -i \rightarrow \cdots \rightarrow -n$$

The two paths from $-i$ to $i$ have length $i$, while the two paths from $i$ to $-i$ have length $n - i + 1$. If $i > n - i + 1$, then we reverse the direction of all arcs in this digraph, which corresponds to flipping all literals in $U_{n,i}^0$. So then we obtained an isomorphic clause-set, where the upper two paths are swapped with the lower two paths, and thus w.l.o.g. one can assume $i \leq n - i + 1$, that is, $i \leq \frac{n+1}{2}$. We note here that the digraphs $\text{idg}(U_{n,i}^0)$ again have a unique skew-symmetry.

Similarly, for case (ABB), in a sense the most general case, we obtain $U_{n,x,y}^0$ (defined above), allowing for the moment all $2 \leq x, y \leq n - 1$, where $x < y$ (this comes from the chaining-order). Allowing degenerations, we have $U_n^2 = U_{n,1,n}^0$, $U_{n,i}^1 = U_{n,1,i}^0$ and $U_{n,i}^0 = U_{n,i,i}^0$. The implication digraph of $U_{n,x,y}^0$ has $2n$ vertices and $2n + 2$ edges, and four vertices have degree 3, while all other vertices have degree 2:

$$\text{idg}(U_{n,x,y}^0) = 1 \rightarrow \cdots \rightarrow x \rightarrow \cdots \rightarrow y \rightarrow \cdots \rightarrow n = -1 \rightarrow \cdots \rightarrow -x \rightarrow \cdots \rightarrow -y \rightarrow \cdots \rightarrow -n$$

The two paths from $-x$ to $x$ have length $x$, the two paths from $y$ to $-y$ have length $n - y + 1$. As above, w.l.o.g. we can assume $x \leq n - y + 1$, i.e., $x + y \leq n + 1$.

The classification of 2-uniform elements of $2\mathcal{ML}_{\delta=1}$ is summarised in the following lemma:

**Lemma C.6** For 2-uniform $F \in 2\mathcal{ML}_{\delta=1}$ with $n := n(F)$ holds:
• If \( F \) has a variable of degree 4 (occurring twice positively and twice negatively), then \( n \geq 3 \) and \( F \cong U^0_{n,i} \) for some \( i \in \{2, \ldots, n-1\} \).

• Otherwise \( n \geq 4 \) and \( F \cong U^0_{n,x,y} \) for some \( x, y \in \{2, \ldots, n-1\} \) with \( x < y \).

Altogether we achieved the classification of \( 2\cdot\mathcal{MU}^*_{6=1} \):

**Theorem C.7** For input \( F \in 2\cdot\mathcal{MU}^*_{6=1} \) exactly one of the four cases in Lemmas C.3, C.5, and C.6 applies. Let \( u(F) \in \{0,1,2\} \) be the number of unit-clauses in \( F \). Then in polynomial time the unique parameter-list \( L(F) \) of length 0, 1, 2, according to the applicable case can be computed, such that \( F \cong U^u(F)_{n(F),L(F)} \) holds. This map \( \text{canon} : 2\cdot\mathcal{MU}^*_{6=1} \to 2\cdot\mathcal{MU}^*_{6=1} \) given by \( \text{canon}(F) := U^u(F)_{n(F),L(F)} \), is a polytime computable clause-set-canonisation, that is, for \( F, F' \in 2\cdot\mathcal{MU}^*_{6=1} \) holds \( F \cong F' \iff \text{canon}(F) = \text{canon}(F') \).

Furthermore the map \( F \in 2\cdot\mathcal{MU}^*_{6=1} \mapsto \text{ig}(\text{canon}(F)) \) to the class of graphs is a polytime computable graph-canonisation, that is, for \( F, F' \in 2\cdot\mathcal{MU}^*_{6=1} \) holds \( F \cong F' \iff \text{ig}(\text{canon}(F)) = \text{ig}(\text{canon}(F')) \). From \( \text{canon}(F) \) in polytime \( F \) can be reconstructed up to isomorphism.

**Proof:** A contraction of two arcs into one edge, when transitioning from the implication digraph \( \text{idg}(F) \) to the implication graph \( \text{ig}(F) \), happens exactly for the two following cases:

1. For complementary unit-clauses in \( F \), \( \text{idg}(F) \) is the cycle digraph of length 2, while \( \text{ig}(F) \) is the complete graph with two vertices.

2. Equivalence-clauses \( \{x, y\}, \{\overline{x}, \overline{y}\} \in F \) yield a cycle digraph of length 2 in \( \text{idg}(F) \), and a single edge in \( \text{ig}(F) \).

We see from the implication digraphs, that here only the first case happens, i.e., when \( u(F) = 1 \). This is the case \( U^1_2 \), which doesn’t pose any problems. For the sequel of the proof we assume \( u(F) \geq 2 \), and thus no contractions happen when transitioning from \( \text{idg}(F) \) to \( \text{ig}(F) \).

The four cases \( U^2_n, U^1_{n,i}, U^0_{n,i}, U^0_{n,x,y} \) are separated by vertex-degrees in the implication graph, since their degree-spectra as triples in \( (\mathbb{N}_0 \cup \{+\infty\})^3 \) for the numbers of degree-2/3/4-vertices, with “inf” meaning “unbounded”, are resp. \( (\inf, 0, 0) \), \( (\inf, 2, 0) \), \( (\inf, 0, 4) \) and \( (\inf, 4, 0) \). Some parameter-list \( L(F) \) can be computed in polytime by performing the standardisation of the chain of 1-singular-DP-reductions leading to 0, as in the proofs of Lemmas C.3, C.5, and C.6, or they are determined from the implication graph, as in the following uniqueness argument. Namely that the parameters are uniquely determined, is read off the unlabelled implications graphs (i.e., vertices are “anonymised”) as follows:

- The parameter \( i \) in \( \text{ig}(U^1_{n,i}) \) can be computed from the length of the shared path \( 2i - 1 \) of the two cycles.

- For \( \text{ig}(U^0_{n,i}) \) there are exactly two vertices of degree 4, and there are two paths of length \( i \) and two paths of length \( n - i + 1 \) between them (no more). Since \( i \leq n - i + 1 \), we can compute \( i \).
• For \( \text{ig}(U_{n,x,y}^0) \) there are exactly four vertices \( a, b, c, d \) of degree 3, which we can identify in such a way that \( \text{ig}(U_{n,x,y}^0) \) consists of a cycle running through these vertices in the given order, and where between \( a, b \) and \( c, d \) there are parallel paths to the path between \( a, b \) resp. \( c, d \) on that cycle, such that this is all of the graph. These parallel paths have the same length \( p \) resp. \( q \). W.l.o.g. \( p \leq q \), and now \( x := p \) and \( y := n - q + 1 \).

This shows all the statements in the theorem.

By adding up the contributions we obtain that the exact number of isomorphism types of \( F \in 2\cdot\mathcal{MU}_{U_{n,x,y}^0} = 1 \) with \( n(F) = n \in \mathbb{N}_0 \) is

\[
\begin{cases} 
1 & \text{if } n = 0 \\
\frac{1}{2}n(n + 2) & \text{if } n \text{ is even} \\
\frac{1}{2}(n + 1)^2 & \text{if } n \text{ is odd}
\end{cases}
\]

This is \( A076921 \) (where that sequence starts with index 1).