The cohomological support locus of pluri-canonical sheaves and the Iitaka fibration

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Abstract

Let \( \text{alb}_X : X \to \text{Alb}(X) \) be the Albanese map of a smooth projective variety and \( f : X \to Y \) the fibration arising from the Stein factorization of \( \text{alb}_X \). For a positive integer \( m \), if \( f \) and \( m \) satisfy two certain assumptions \( \text{AS}(1, 2) \), then the translates through the origin of all components of cohomological locus \( V^0(\omega_X, \text{alb}_X) \subset \text{Pic}^0(X) \) generate \( I^*\text{Pic}^0(S) \), where \( I : X \to S \) denotes the Iitaka fibration. As an application, we study pluri-canonical maps.

1. Introduction

Conventions. We work over complex numbers. For a smooth projective variety \( X \) with \( q(X) > 0 \), we usually denote by \( \text{alb}_X : X \to \text{Alb}(X) \) the Albanese map. A smooth variety \( X \) with Kodaira dimension \( \kappa(X) \geq 0 \) has Iitaka fibration \( I : X \to S \) which is defined by \( |mK_X| \) for sufficiently divisible \( m \). Blowing up \( X \) if necessary, throughout this paper, we always assume that \( I \) is a morphism. A good minimal model of a variety is a birational model with semi-ample canonical divisor and \( \mathbb{Q} \)-factorial canonical singularities. For a Cartier divisor \( D \) on a variety \( X \) with \( |D| \neq \emptyset \), we denote by \( \phi_{|D|} \) the map induced by the linear system \( |D| \). We use ‘\( \sim \)’ for the linear equivalence between two line bundles. Throughout this paper, a fibration \( f : X \to Y \) between two varieties means a projective surjective morphism with connected fibers.

Let \( a : X \to A \) be a map from a projective variety to an abelian variety, and \( F \) a sheaf on \( X \). The cohomological support locus of \( F \) with respect to \( a \) is defined as

\[
V^i(F, a) := \{ \alpha \in \text{Pic}^0(A) | h^i(F \otimes a^* \alpha) \neq 0 \}.
\]

The cohomological support locus of the canonical sheaf \( V^0(\omega_X, \text{alb}_X) \) plays an important role in studying irregular varieties (see, for example, \([3, 7, 8]\)), which is closely related to pluri-canonical maps and Iitaka fibrations. Recall the results of \([2, 14]\) on this topic.

Theorem 1.1. Let \( X \) be a smooth irregular variety with \( \kappa(X) \geq 0 \). Denote by \( I : X \to S \) the Iitaka fibration.

1. If \( X \) is of maximal Albanese dimension, then the subgroup of \( \text{Pic}^0(X) \) generated by the translates through the origin of the components of \( V^0(\omega_X, \text{alb}_X) \) is \( I^*\text{Pic}^0(S) \).

2. If \( X \) is of general type, and the Albanese fibers are of dimension 1, then \( V^0(\omega_X, \text{alb}_X) = \text{Pic}^0(X) \).

The result above is applied to study the birationality of pluri-canonical maps. Let \( X \) be a variety of general type with Albanese fibers being of dimension at most 1. It is proved that for
every \( n \geq 4 \) and \( \alpha \in \text{Pic}^0(X) \), \( |\omega_X^n \otimes \alpha| \) induces a birational map (cf. [14, 22]); moreover, if \( X \) is of maximal Albanese dimension, then \( |\omega_X^n| \) induces a birational map (see [13]).

If Albanese fibers have higher dimension, then the cohomological locus \( V^0(\omega_X, \text{alb}_X) \) may contain little information (for example, if a general fiber \( F \) has \( p_g(F) = 0 \)). We will study the cohomological support locus \( V^0(\omega_X^m, \text{alb}_X) \) of the pluri-canonical sheaf, which has been studied in [4, 18]. To state our main result, we introduce three assumptions for a fibration \( f : X \to Y \) between two smooth projective varieties and a positive integer \( m \).

**AS(1):** A general fiber \( F \) of \( f \) has a good minimal model, and \( |mK_F| \neq \emptyset \).

**AS(2):** For every smooth projective curve \( C \) on \( Y \), if \( X_C := X \times_Y C \) is smooth, then \( \deg(f_C^* \omega_{X_C/C}^m) > 0 \), where \( f_C = f|_{X_C} \), unless there exists an étale cover \( \tilde{C} \to C \) such that \( X_C \times_C \tilde{C} \) is birationally equivalent to \( F \times \tilde{C} \).

**WAS(2):** Let the notation be as in AS(2). Then \( \deg(f_C^* \omega_{X_C/C}^m) > 0 \), unless \( f_C : X_C \to C \) is birationally isotrivial, that is, there exists a flat base change \( \tilde{C} \to C \) such that \( X_C \times_C \tilde{C} \) is birationally equivalent to \( F \times \tilde{C} \).

Our main result is the following theorem.

**Theorem 1.2.** Let \( X \) be a smooth projective variety with \( \kappa(X) \geq 0 \), and denote by \( I : X \to S \) the Iitaka fibration. Let \( f : X \to Y \) be the fibration arising from the Stein factorization of the Albanese map \( \text{alb}_Y : X \to \text{Alb}(X) \). Suppose that assumptions AS(1, 2) are satisfied for the fibration \( f \) and the integer \( m > 0 \). Then the subgroup of \( \text{Pic}^0(X) \) generated by the translates through the origin of the components of \( V^0(\omega_X^m, \text{alb}_X) \) is \( I^\ast \text{Pic}^0(S) \); moreover, \( q(S) = q(X) - (\dim(X) - \kappa(X)) - (\dim(F) - \kappa(F)) \).

As an application, we prove the following theorem.

**Theorem 1.3.** Let \( X \) be a smooth projective variety, \( I : X \to S \) the Iitaka fibration and \( a = \text{alb}_S \circ I \). Let \( f : X \to Y \) be the fibration arising from the Stein factorization of \( a : X \to \text{Alb}(S) \) and \( F \) a general fiber. Suppose that the following conditions hold.

1. The assumptions AS(1, 2) are satisfied for \( f \) and \( m = 1 \).
2. For the integer \( n \geq 2 \), the pluri-canonical map \( \phi_{|nK_F|} \) is birational to the Iitaka fibration of \( F \).

Then for every \( \alpha \in \text{Pic}^0(S) \), the map \( \phi_{|(n+2)K_X \otimes I^\ast \alpha|} \) is birational to the Iitaka fibration.

**Remark 1.4.** In the theorem above, if assuming \( m > 1 \) in assumption (1), then we get an analogous result for \( \phi_{|(n+2m)K_X \otimes I^\ast \alpha|} \). However, this result is not better than that of [5, Theorem 2.8]. From this point of view, it is interesting to know whether AS(1, 2) are satisfied for certain fibrations and \( m = 1 \).

For assumptions AS(1, 2), we remark the following facts.

1. Assumption AS(1) is needed when proving that \( (\text{alb}_X)_\ast \omega_X^n \) is a GV-sheaf. The existence of good minimal models is still a conjecture; it has been proved for the varieties of general type or of maximal Albanese dimension. Please refer to [18] for the most recent results.

2. It is known that \( f_\ast \omega_{X/Y}^m \) is torsion free and weakly positive (see [24, Theorem III]), hence \( f_C^\ast \omega_{X_C/C}^m \) is a numerically effective vector bundle on \( C \). Combining Theorem 1.5 and [17,
Theorem 1.1], we get that if general fibers of \( f \) have good minimal models, then there always exists \( m > 1 \) such that assumptions AS(1, 2) hold.

(3) If \( f : X \to Y \) is birational or fibred by curves of genus at least 2, then AS(1,2) are satisfied for \( f \) and \( m = 1 \) (see [1, Chapter III, Theorem 18.2]). So Theorem 1.1 is a special case of Theorem 1.2. For a smooth variety \( X \) of general type, if a general Albanese fiber is composed of finitely many points or curves of genus at least 3, then applying Theorem 1.3, we get that \( \phi_{4K_X \otimes \alpha} \) is birational for every \( \alpha \in \text{Pic}^0(X) \), which coincides with the results of [22; 14, Theorem 5.3] (the cases \( g \geq 3 \)).

(4) Assumption AS(2) is stronger than WAS(2). By, Barth, Peters and Van de Ven [1, Chapter III, Theorem 18.2], for an elliptic fibration and \( m = 1 \), assumption WAS(2) is satisfied, but AS(2) may fail (see Section 6 for an example).

(5) Assumption AS(2) is necessary for Theorem 1.2: in Section 6, we construct a variety \( X \) such that \( \text{alb}_X \) is an elliptic fibration and the conclusion of Theorem 1.2 fails for \( \text{alb}_X \).

The following theorem tells us to what extent assumptions AS(2) and WAS(2) are equivalent.

**Theorem 1.5.** Let \( f : X \to Y \) be a fibration between two smooth varieties, \( F \) a general fiber and \( m \) a positive integer. Assume that AS(1) is satisfied for \( f \) and \( m \). If either \( m = 1 \) and the birational automorphism group Bir(\( F \)) acts faithfully on \( H^0(F, \omega_F) \), or \( m > 1 \), then WAS(2) is equivalent to AS(2).

**Remark 1.6.** For a birational map \( \sigma : X' \to X \) between two smooth projective varieties, \( \sigma^* : H^0(X, \omega_X) \to H^0(X', \omega_{X'}) \) induces an 1–1 correspondence. So the action of Bir(\( F \)) on \( H^0(F, \omega_F) \) is well defined. It is known that the action is faithful if \( F \) is a curve with \( g(F) \geq 2 \).

For a fibration, it is interesting to find a minimal integer \( m > 0 \) satisfying (W)AS(2). We will consider whether (W)AS(2) holds for certain fibrations and \( m = 1 \). We can reduce this problem to a numerical criterion for the isotriviality of a fibration \( f : X \to C \) to a curve: whether \( \deg(f_* \omega_{X/C}) = 0 \) implies that \( f \) is birationally isotrivial. We propose the following problem.

**Problem 1.7.** Let \( f : X \to C \) be a fibration to a curve with general fibers having good minimal models and \( \deg(f_* \omega_{X/C}) = 0 \). Is the fibration \( f \) birationally isotrivial under one of the following conditions?

1. General fibers are of maximal Albanese dimension.
2. For a general fiber \( F \), the canonical ring \( \oplus_{m=0}^{+\infty} H^0(F, mK_F) \) is generated by \( H^0(F, K_F) \).

This numerical criterion works for the fibrations fibred by curves of genus \( g \geq 1 \) which are of type (1), and also works for those fibred by varieties with trivial canonical bundles by Theorem 1.9, which are of type (2). Precisely, we have the following results.

**Theorem 1.8.** Let \( X \) be a smooth projective variety and \( f : X \to C \) a fibration to a smooth projective curve. Suppose that \( f \) has a birational model \( \bar{f} : \bar{X} \to C \) such that the general fiber \( \bar{F} \) of \( \bar{f} \) is good, and the infinitesimal period map induced by the cup product

\[
\lambda : \Ext^1(\Omega^1_{\bar{F}}, \mathcal{O}_{\bar{F}}) \to \text{Hom}(H^0(\bar{F}, \omega_{\bar{F}}), \Ext^1(\Omega^1_{\bar{F}}, \omega_{\bar{F}}))
\]

is injective.

If \( \deg(f_* \omega_{X/C}) = 0 \), then \( f \) is birationally isotrivial.
As a corollary, we obtain the following theorem.

**Theorem 1.9.** Let $X$ be a smooth projective variety, $f : X \to C$ a fibration to a smooth projective curve and $m$ a positive integer. Suppose that a general fiber $F$ of $f$ has a good minimal model $\bar{F}$ and $\omega_{\bar{F}}^m = \mathcal{O}_{\bar{F}}$.

If $\deg(f_*\omega_{X/C}^m) = 0$, then $f$ is birationally isotrivial.

**Remark 1.10.** Theorem 1.8 and 1.9 slightly improve the analogous results appearing in [6; 16, Sections 1 and 4], where they considered the case when $\bar{F}$ is smooth. It is possible that the infinitesimal period map of a smooth variety is not injective, while that of the minimal model is injective (see the commutative diagram (5.1) for the relation between them). This is why we consider the infinitesimal period map of the minimal model of a general fiber.

2. Preliminaries

2.1. Fourier–Mukai transform

If $A$ denotes an abelian variety, then $\hat{A}$ denotes its dual $\text{Pic}^0(A)$, $\mathcal{P}$ denotes the Poincaré line bundle on $A \times \hat{A}$, and the Fourier–Mukai transform $R\Phi_\mathcal{P} : D^b(A) \to D^b(\hat{A})$ with respect to $\mathcal{P}$ is defined as

$$R\Phi_\mathcal{P}(\mathcal{F}) := R(p_2)_*(Lp_1^*\mathcal{F} \otimes \mathcal{P}),$$

where $p_1$ and $p_2$ are the projections from $A \times \hat{A}$ to $A$ and $\hat{A}$, respectively. Similarly, $R\Psi_\mathcal{P} : D^b(\hat{A}) \to D^b(A)$ is defined as

$$R\Psi_\mathcal{P}(\mathcal{F}) := R(p_1)_*(Lp_2^*\mathcal{F} \otimes \mathcal{P}).$$

**Theorem 2.1** ([19, Theorem 2.2]). Let $A$ be an abelian variety of dimension $d$. Then $R\Psi_\mathcal{P} \circ R\Phi_\mathcal{P} = (-1)^d[-d]$ and $R\Phi_\mathcal{P} \circ R\Psi_\mathcal{P} = (-1)^d[-d]$.

2.2. GV-sheaves, M-regular sheaves and ‘continuously globally generated’

**Definition 2.2** ([20, Definitions 2.1, 2.2 and 5.2]). Given a coherent sheaf $\mathcal{F}$ on an abelian variety $A$, its $i$th cohomological support locus is defined as

$$V^i(\mathcal{F}) := \{\alpha \in \hat{A} | h^i(\mathcal{F} \otimes \alpha) > 0\}.$$

The number $\text{gv}(\mathcal{F}) := \min_{i > 0} \{\text{codim}_{\hat{A}} V^i(\mathcal{F}) - i\}$ is called the generic vanishing index of $\mathcal{F}$. We say $\mathcal{F}$ is a GV-sheaf if $\text{gv}(\mathcal{F}) \geq 0$, and is an M-regular sheaf if $\text{gv}(\mathcal{F}) > 0$, and is an IT$^0$-sheaf if $V^i(\mathcal{F}) = 0$ for $i > 0$.

We say $\mathcal{F}$ is continuously globally generated if the sum of the evaluation maps

$$ev_U : \oplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^{-1} \to \mathcal{F}$$

is surjective for any non-empty open set $U \subset \hat{A}$.

**Theorem 2.3** ([9, Theorem 1.2, Corollary 3.2] or [20, Section 2]). Let $A$ be an abelian variety of dimension $d$ and $\mathcal{F}$ a GV-sheaf on $A$. Then we have the following results.

(i) The cohomological locus $V^0(\mathcal{F}) \supset V^1(\mathcal{F}) \supset \cdots \supset V^d(\mathcal{F})$, and $\mathcal{F} = 0$ if $V^0(\mathcal{F}) = \emptyset$. 
(ii) The complex \( R\Phi_P(R\Delta(F))[d] \cong R^d\Phi_P(R\Delta(F)) \) is a sheaf (where \( R\Delta(F) := RHom(F, O_A) \)), which we denote by \( R\Delta(F) \).

(iii) Applying Grothendieck duality gives \( Ext^i\hat{\Delta}(F), O_A) \cong (-1)^i\hat{\Delta}(F) \).

(iv) Any direct summand of \( F \) is a GV-sheaf.

**Proposition 2.4** ([20, Corollary 5.3]). An M-regular sheaf on an abelian variety is continuously globally generated.

### 2.3. Good minimal models and the cohomological support locus of \( \omega_X^m \)

On good minimal models recall the following proposition.

**Proposition 2.5** ([18, Theorem 4.5]). Let \( a : X \to A \) be a morphism from a smooth projective variety to an abelian variety. Assume that the general fibers of \( a \) have good minimal models. Then \( X \) has a good minimal model.

**Proposition 2.6.** Let \( a : X \to A \) be a morphism from a smooth projective variety to an abelian variety. Suppose that \( X \) has a good minimal model. Then for any torsion line bundle \( P \in \text{Pic}(X) / X \) (that is, \( nP \sim O_X \) for some integer \( n > 0 \)), the sheaf \( a_*(\omega_X^m \otimes P) \) is a GV-sheaf on \( A \), and every component of \( V^n(a_*\omega_X^m) \) is a translate of a sub-torus of \( \hat{A} \) via a torsion point.

**Proof.** Note that the argument of [18] applies for a general map to an abelian variety, not necessarily the Albanese map. By Lai [18, Theorem 3.5], it is known that every component of \( V^n(a_*\omega_X^m) \) is a translate of a sub-torus of \( \hat{A} \) via a torsion point. We still need to prove that \( a_*(\omega_X^m \otimes P) \) is a GV-sheaf on \( A \).

If \( P \neq O_X \), then consider the cyclic étale cover induced by \( nP \sim O_X \), where \( n \) is the order of \( P \). Note that \( X' \) also has a good minimal model, and the sheaf \( \omega_X^m \otimes P \) is a direct summand of \( \pi_*(\omega_X^m \otimes P) \), and so is \( a_*(\omega_X^m \otimes P) \) of \( a_*\pi_*(\omega_X^m \otimes P) \). To prove that \( a_*(\omega_X^m \otimes P) \) is a GV-sheaf, by considering the map \( a \circ \pi : X' \to A \) instead, we can assume \( P = O_X \).

Up to some blowing-up maps, we can assume that there is a morphism \( \mu : X \to \hat{X} \) to one of its good minimal models. Since \( \hat{X} \) has at most canonical singularities which hence are rational, the map \( a \) factors through \( \mu \). Write \( a = \hat{a} \circ \mu \). Since \( \mu_*\omega_X^m = \omega_{\hat{X}}^m \), we have

\( \bigstar : a_*\omega_X^m = \hat{a}_*\omega_{\hat{X}}^m \).

Again up to some blowing-up maps, we can assume the following conditions.

1. We have \( K_X = \mu^*K_{\hat{X}} + \sum_i a_i E_i \), where \( E_i \) are the \( \mu \)-exceptional components and \( a_i \geq 0 \).

2. There exist a sufficiently divisible integer \( N > m \) and a smooth divisor \( D \in |\mu^*NK_X| \) by Bertini’s theorem, such that \( D + \sum_i E_i \) is a simple normal crossing divisor.

Then we have

\[
\mu^*mK_X = mK_X - \sum_i ma_i E_i \leq mK_X - \sum_i [(m - 1)a_i] E_i = mK_X - \sum_i [(m - 1)a_i] E_i - \left\lfloor \frac{m - 1}{N} \right\rfloor D.
\]
Note that $\mathcal{O}_X(-[((m-1)/N)D + \sum (m-1)a_iE_i])$ is contained in the multiplier ideal sheaf $\mathcal{I}(\!(m-1)K_X\!)$). So for an ample divisor $H$ on $A$ and $t > 0$, we have the natural inclusions

$$\mu^*\omega^n_X \subseteq \omega^n_X \otimes \mathcal{I}(\!(m-1)K_X\!) \subseteq \omega^n_X \otimes \mathcal{I}(\!(m-1)K_X + \frac{1}{t}a^*H\!)) \subseteq \omega^n_X.$$

Pushing forward via $a_+$, we obtain

$$a_+\omega^n_X \subseteq a_+(\omega^n_X \otimes \mathcal{I}(\!(m-1)K_X + \frac{1}{t}a^*H\!))) \subseteq a_+\omega^n_X.$$

Therefore, the two inclusions are equalities by $\clubsuit$. Then $a_+\omega^n_X$ is a GV-sheaf by Lai [18, Lemma 3.4].

2.4. Results on fibrations

Let $f: X \to S$ be a fibration between two quasi-projective varieties. The variation $\text{Var}(f)$, roughly speaking, is the number of moduli of fibers of $f$ in the sense of birational geometry. Please refer to [17, Section 1] for a precise definition. We say that $f$ is birationally isotrivial if $\text{Var}(f) = 0$, equivalently, there is a generically finite surjective base change $S' \to S$ such that $X \times_S S'$ is birationally equivalent to $F \times S'$, where $F$ is a general fiber of $f$; furthermore, if general fibers are isomorphic to each other, we say that $f$ is isotrivial.

Recall some results due to Kawamata.

**Proposition 2.7 ([17, Lemma 7.1, Corollary 7.3]).** Let $f: X \to S$ be a fibration between two normal varieties such that the generic fiber $\bar{X}$ is good over $K(S)$. Then there exist normal varieties $\bar{S}, S'$, a smooth non-empty open set $U \subset \bar{S}$ étale over $S$, a generically finite surjective morphism $\tau: \bar{S} \to S$, a surjective morphism $\phi: \bar{S} \to S'$ and two fibrations $\bar{f}: \bar{X} \to \bar{S}$ and $f^1: X^1 \to S^1$ of normal varieties fitting into the following commutative diagram:

$$
\begin{array}{ccc}
X & \xleftarrow{f} & \bar{X} \\
\downarrow^f & & \downarrow^\bar{f} \\
S & \xleftarrow{\tau} & S'
\end{array}
$$

such that the following conditions hold.

1. For every $t \in U$,
   $$\ker(d\phi_t \circ d\tau_s^{-1}) \cong \ker(\delta_s),$$
   where $s = \tau(t)$ and $\delta_s: T_{S,s} \to \text{Ext}^1(\Omega_{X_s}, \mathcal{O}_{X_s})$ denotes the Kodaira–Spencer map of $f$ at $s$.
2. The fibration $f^1: X^1 \to S^1$ has good generic fiber and $X \times_S U \cong X^1 \times_{S^1} U \cong \bar{X} \times_{\bar{S}} U$.
3. The dimension of $S^1 \dim(S^1) = \text{Var}(f)$.

In particular, $f$ is birationally isotrivial if and only if the Kodaira–Spencer map $\delta_s = 0$.

Here we remark the following results, which will be used in the sequel.

1. Let $X$ and $X'$ be two birational projective varieties with at most canonical singularities, and let $f: X \to Y$ and $f': X' \to Y$ be two birational fibrations. Then by considering a common resolution and comparing the push-forward of the pluri-canonical sheaves, we conclude that $f_*\omega^n_X \cong f'_*\omega^n_{X'}$.
2. Let $f: X \to Y$ be a birationally isotrivial fibration. Assume that a general fiber $F$ is of Kodaira dimension $\kappa(F) \geq 0$ and has a good minimal model. Then there exists a birational model $\bar{f}: \bar{X} \to \bar{Y}$ with good generic fiber, which is isotrivial by Proposition 2.7(2). If $X' \to X$
Here we give a simple explanation for (1) and (2). Up to some blowing-up maps we get a factorization of \( \hat{\iota} \), so we can assume that the Iitaka fibration of \( G \) coincides with its Iitaka fibration of \( G \) for a general fiber \( G \). Let \( F' \) be a general fiber of \( h \). Assume \( K_X \sim Q I^* H + V \), where \( H \) is an ample \( \mathbb{Q} \)-divisor on \( S \). By \( K_{F'} \sim K_X|_{F'} \), we have \( |m(I^* H + V)|_{F'} \subseteq |mK_{F'}| \) for any \( m > 0 \). Since \( \kappa(F') = 0 \), then \( I^* H|_{F'} \sim Q 0 \). This implies that \( F' \) is contained in some fiber of \( I : X \to S \); hence (1) follows. If \( g \) factors through \( I \), then (1) implies that the Iitaka fibration of \( G \) coincides with \( I|_G \); thus \( \kappa(G) = \dim I(G) \), so (2) follows.

2.5. Results on Iitaka fibration

Let \( X \) be a smooth variety with \( \kappa(X) \geq 0 \), and denote by \( I : X \to S \) the Iitaka fibration. Let \( g : X \to Z \) be a fibration and \( G \) a general fiber. Then we have the following results.

(1) Every Iitaka fiber of \( G \) is contained in some fiber of \( I : X \to S \).

(2) If \( g \) factors through \( I \), then \( \kappa(X) = \kappa(G) + \dim Z \).

(3) If \( \kappa(X) = 0 \), then its Albanese map is a fibration onto its Albanese variety (see [3, Theorem 1]).

Here we give a simple explanation for (1) and (2). Up to some blowing-up maps we get a fibration \( h : X \to W \), such that \( g \) factors through \( h \) and that the restriction map \( h|_G \) coincides with its Iitaka fibration of \( G \) for a general fiber \( G \). Let \( F' \) be a general fiber of \( h \). Assume \( K_X \sim Q I^* H + V \), where \( H \) is an ample \( \mathbb{Q} \)-divisor on \( S \). By \( K_{F'} \sim K_X|_{F'} \), we have \( |m(I^* H + V)|_{F'} \subseteq |mK_{F'}| \) for any \( m > 0 \). Since \( \kappa(F') = 0 \), then \( I^* H|_{F'} \sim Q 0 \). This implies that \( F' \) is contained in some fiber of \( I : X \to S \); hence (1) follows. If \( g \) factors through \( I \), then (1) implies that the Iitaka fibration of \( G \) coincides with \( I|_G \); thus \( \kappa(G) = \dim I(G) \), so (2) follows.

3. The main theorem

In this section, we will prove Theorems 1.2 and 1.3. Let the notation be as in Theorem 1.2.

3.1. The fibration induced by the cohomological support locus of \( \omega_X^m \)

Here we denote by \( \hat{T} \) the subgroup of \( \text{Pic}^0(X) \) generated by the translates through the origin of the components of \( V^0(\omega_X^m, \text{alb}_X) \), and denote by \( \iota : \text{Alb}(X) \to T \) the dual map of the inclusion \( \hat{T} \hookrightarrow \text{Pic}^0(X) \). So we can assume \( V^0(\omega_X^m, \text{alb}_X) \subset \tau + \hat{T} \), where \( \tau \) consists of finitely many torsion points on \( \text{Pic}^0(X) \), by Proposition 2.6. Let \( \text{alb}_Z \circ g : X \to Z \to T \) be the Stein factorization of \( \iota \circ \text{alb}_X \). Denote by \( G \) a general fiber of \( g \), by \( K \) a fiber of \( \iota \) and by \( a_G \) the restriction map of \( \text{alb}_X \) on \( G \). Then we get a commutative diagram.

\[
\begin{array}{ccc}
G & \longrightarrow & X \\
& \downarrow {a_G} & \downarrow \text{alb}_X \\
& \text{alb}_Z & \\
K & \longrightarrow & \text{Alb}(X) \downarrow \iota \\
& & \rightarrow T \\
\end{array}
\]

Claim 3.1. The cohomological locus \( V^0(a_*G\omega_G^m) \) has dimension 0.

Proof. Consider the quotient group homomorphism \( \pi : \text{Pic}^0(X) \to \hat{K} \) whose kernel is \( \hat{T} \). We make the following observations.

(a) By definition, for \( \alpha \in \text{Pic}^0(X) \), we have \( V^0(\omega_X^m \otimes \alpha, \text{alb}_X) + \alpha = V^0(\omega_X^m, \text{alb}_X) \); thus \( V^0((\text{alb}_Z \circ g)_*(\omega_X^m \otimes \alpha)) = \hat{T} \cap V^0(\omega_X^m \otimes \alpha, \text{alb}_X) \cong (\hat{T} + \alpha) \cap V^0(\omega_X^m, \text{alb}_X) \).

(b) The locus \( V^0(\omega_X^m, \text{alb}_X) \) is projected to finite points \( \pi(\tau) \) on \( \hat{K} \) via \( \pi : \text{Pic}^0(X) \to \hat{K} \).
Let $\alpha_0 \in \hat{K}$ be a torsion point not contained in $\pi(\tau)$, and take a torsion point $\alpha \in \pi^{-1}\alpha_0$. By (a, b), we have $V^0((\text{alb}_Z \circ g)_*(\omega_X^m \otimes \alpha)) \cong (\tilde{T} + \alpha) \cap V^0(\omega_X^m, \text{alb}_X) = \emptyset$. Since $(\text{alb}_Z \circ g)_*(\omega_X^m \otimes \alpha)$ is a GV-sheaf by Proposition 2.6, we conclude that $(\text{alb}_Z \circ g)_*(\omega_X^m \otimes \alpha) = 0$ by Theorem 2.3(i); thus $V^0(a_{G^*} \omega_G^m \otimes \alpha_0) = \emptyset$. Applying Proposition 2.6 to $G$, we conclude that $V^0(a_{G^*} \omega_G^m) \subset \pi(\tau)$, and since $a_{G^*} \omega_G^m$ is a non-zero GV-sheaf, $V^0(a_{G^*} \omega_G^m)$ is non-empty and has dimension 0.

The structure of $G$ plays a key role in our proof, for which we have the following proposition with the proof postponed.

**Proposition 3.2.** There exists an étale cover $\tilde{K} \to K$ such that the fiber product $G \times_K \tilde{K}$ is birational to $F \times \tilde{K}$. In particular, the Iitaka fiber of $G$ is mapped onto $K$ via $\text{alb}_X$.

3.2. The dual of $T \overset{\tau}{\longrightarrow} I^* \text{Pic}^0(S)$

Let $I : X \to S$ be the Iitaka fibration and $G'$ a general fiber. We get a commutative diagram

$$
\begin{array}{ccc}
G' & \longrightarrow & X \\
\downarrow \scriptstyle a_{G'} & & \downarrow \scriptstyle \text{alb}_X \\
K' & \longrightarrow & \text{Alb}(X) \\
\downarrow \scriptstyle \text{alb}_S & & \downarrow \scriptstyle \tau' \\
T' & = & \text{Alb}(S)
\end{array}
$$

where $a_{G'} = \text{alb}_X|_{G'}$ and $K' = \text{alb}_X(G')$ which is a torus independent of $G'$ up to translates. Denote by $\pi' : \text{Pic}^0(X) \to K'$ the natural group homomorphism.

Since $G'$ has Kodaira dimension 0, the set $V^0(a_{G^*} \omega_G^m) \subset K'$ contains finitely many torsion points. Let $\tau'$ be the set of torsion points of $K'$. We conclude that $V^0(a_{G^*} \omega_G^m) \subset \tau'$. Take $\alpha_0 \in \hat{K}$ not contained in $\tau'$. We have $V^0(a_{G^*} \omega_G^m \otimes \alpha_0) = \emptyset$ and thus $h^0(G', \omega_G^m \otimes \alpha_0) = 0$. Therefore, for every $\alpha \in \pi'^{-1}\alpha_0$, $I_*(\omega_X^m \otimes \alpha)$ is zero at the generic point of $S$ and hence is zero on $S$ because it is torsion free. We conclude that $\text{alb}_S, I_*(\omega_X^m \otimes \alpha) = 0$ and thus $\pi'^{-1}\alpha_0 \cap V^0(\omega_X^m, \text{alb}_X) = \emptyset$.

This means that $V^0(\omega_X^m, \text{alb}_X)$ is projected into $\tau'$ on $K'$ via $\pi' : \text{Pic}^0(X) \to K'$. Since $\tau'$ consists of torsion points, the kernel of $\pi'$ contains $\tilde{T}$, that is, $\tilde{T} \leq I^* \tilde{T}' = I^* \text{Pic}^0(S)$.

3.3. Proof of Theorem 1.2

Consider the Iitaka fibration $I_G : G \to Z''$ and take a general fiber $F''$. Note that $F''$ is contained in some $G'$ (see Section 2.5). By Proposition 3.2, $F''$ is mapped onto $K$ via $\text{alb}_X$, so $K \subseteq K'$ up to a translate. Therefore, $\ker(\pi') \subseteq \ker(\pi)$, that is, $I^* \text{Pic}^0(S) \subseteq \tilde{T}$. Combining this and the result of Section 3.2 gives $\tilde{T} = I^* \text{Pic}^0(S)$.

We still need to calculate $\dim(\tilde{T})$. Observe that Iitaka fibration factors through $g : X \to Z$. By the results of Section 2.5, we have $\kappa(X) = \dim(Z) + \kappa(G)$. On the other hand, by Proposition 3.2, $\kappa(G) = \kappa(F)$. Then we are done by

$$
\begin{align*}
\dim(\tilde{T}) &= \dim(T) \\
&= \dim(Q) - \dim(K) \\
&= \dim(Q) - (\dim(G) - \dim(F)) \\
&= \dim(Q) - (\dim(X) - \dim(Z)) \\
&= \dim(Q) - (\kappa(X) - \kappa(G) - \dim(F)) \\
&= \dim(Q) - (\kappa(X)) - (\dim(F) - \kappa(F)).
\end{align*}
$$

(3.1)
3.4. Proof of Proposition 3.2

We will apply the argument of the proof of [14, Proposition 3.2]. Write \( a_G = \pi \circ f_G : G \rightarrow W \rightarrow K \), where \( W = f(G) \subset Y \), which coincides with the Stein factorization of \( a_G \). So assumptions AS(1, 2) hold for \( f_G \) and \( m \). Take a smooth curve \( i : B = H_1 \cap H_2 \cap \cdots \cap H_{d-1} \hookrightarrow K \), where \( d = \text{dim}(G) \) and \( H_i \) are some general very ample divisors on \( K \), such that both \( C = \pi^{-1}B \) and \( G_C = G \times_W C \) are smooth. We use \( a_C \) and \( f_C \) for the restriction maps of \( a_G \) and \( f \) on \( G_C \).

Arguing as in the proof of [14, Proposition 3.2], we can show both the following results.

(i) The degree of the push-forward \( \deg f_C \cdot \omega_G^m \rightarrow C = 0 \).

(ii) The map \( \pi : C \rightarrow B \) is unramified.

By (i) and assumption AS(2), we have that \( G_C \) is birationally isotrivial over \( C \). Take a birationally equivalent fibration \( \bar{f} : \bar{G} \rightarrow W \) of \( f|_G \) such that \( \bar{f} \) has a good generic fiber. Define \( \bar{G}_C = \bar{G} \times_W C \). The natural fibration \( \bar{G}_C \rightarrow C \) is birationally isotrivial and has a good generic fiber. By Proposition 2.7, for general \( t \in C \) the Kodaira–Spencer map \( \lambda_{C,t} : T_{C,t} \rightarrow \text{Ext}^1(\Omega_{\bar{G}_t}, \mathcal{O}_{\bar{G}_t}) \) is a zero map. We conclude that for general \( w \in W \), the Kodaira–Spencer map \( \lambda_w : \bar{T}_{W,w} \rightarrow \text{Ext}^1(\Omega_{\bar{G}_w}, \mathcal{O}_{\bar{G}_w}) \) is a zero map. Therefore, \( f|_G : G \rightarrow W \) is birationally isotrivial.

If \( D \subset K \) denotes the divisorial part of the branch locus of \( \pi \), then (ii) implies \( B \cdot D = 0 \); thus \( D = 0 \) since \( B \) is a complete intersection of ample divisors. We conclude that \( \pi : W \rightarrow K \) is unramified in codimension 1, thus \( \kappa(W) = 0 \), which means that \( W \) is birational to an abelian variety \( K' \) which is étale over \( K \) (see, for example, [15, Corollary 2]). Since \( \pi : W \rightarrow K \) is finite, it must be that \( W = K' \).

By the results of Section 2.4, there exist a Galois cover \( Z' \rightarrow K' \) with Galois group \( H \) and a variety \( F' \) birational to \( F \) and with a faithful action of \( H \), such that \( F \) is birational to \( (F' \times Z')/H \), where \( H \) acts on \( F' \times Z' \) diagonally. For the fibration \( (F' \times Z')/H \rightarrow K' \), the fibers over the branch locus of \( Z' \rightarrow K' \) are multiple fibers. By assumption AS(2) again, \( G_C \rightarrow C \) has a birational model \( \bar{G}_C \rightarrow C \) such that every fiber is isomorphic to \( F' \), so \( C \) does not intersect the branch locus of \( Z' \rightarrow K' \). We conclude that \( Z' \rightarrow K' \) is unramified in codimension 1; hence \( Z' \) is birational to an abelian variety \( K \) which is étale over \( K' \). Then we finish the proof.

3.5. Proof of Theorem 1.3

Applying the argument above to the map \( a : X \rightarrow \text{Alb}(S) \), we can prove that the translates through the origin of the components of \( V^0(\omega_X, a) \) generate \( \text{Pic}^0(S) \). By assumption AS(1), \( X \) has a good minimal model (Proposition 2.5). Using the proof of Proposition 2.6, we have \( a_* (\mathcal{O}_X \otimes \mathcal{I}(\left( (k-1)K_X, || \right)) = a_* (\omega_X^k) \neq 0 \) for \( k > 0 \). Then we have the following results.

(I) For \( k \geq 2 \), by Jiang [12, Lemma 4.2] we have \( h^i(\text{Alb}(S), a_* (\omega_X^k \otimes \mathcal{I}(\left( (k-1)K_X, || \right)) \otimes Q) = 0 \) for any \( Q \in \text{Pic}^0(S) \) and \( i > 0 \), so \( a_* (\omega_X^k) \) is an \( \mathcal{I}^0 \) sheaf, and hence is continuously globally generated by Proposition 2.4.

(II) Writing \( I_x \otimes \omega_X^k = I_x \otimes \omega_X \otimes \omega_X^{k-1} \), then arguing similarly as in [22] or [14, Proposition 5.2], we can show that for general \( x \in X \) the sheaf \( a_* (I_x \otimes \omega_X^k) \) is \( M \)-regular for any \( k \geq 2 \), and hence is continuously globally generated.

Let \( \Gamma \) be the support of the cokernel of \( a^* a_* (\omega_X) \rightarrow \omega_X \). Take two distinct points \( x, y \in X \) not contained in \( \Gamma \) and separated by the Iitaka fibration. Suppose that \( x \) is general. We will prove that the linear system \( (n+2)K_X \otimes I^* \alpha \) separates \( x \) and \( y \). It suffices to find a section \( s \in H^0(X, I_x \otimes \omega_X^{n+2} \otimes I^* \alpha) \neq 0 \) not vanishing at \( y \).
We will apply the proof of [14, Theorem 5.3]. By condition (2) in this theorem, the following evaluation map is surjective:
\[ a^*a_*(I_x \otimes \omega_X) \to I_x \otimes \omega_X^c \to I_x \otimes \omega_X^c|_y \cong \mathbb{C}(y). \]
Then by (II), for general \( \beta \in \text{Pic}^0(S) \) there exists \( t_{-\beta} \in H^0(X, I_x \otimes \omega_X^c \otimes I^*\beta^{-1}) \) not vanishing at \( y \). Similarly, by (I), for general \( \beta \in \text{Pic}^0(S) \) there exists \( s_\beta \in H^0(X, \omega_X^c \otimes I^*\alpha \otimes I^*\beta) \) not vanishing at \( y \). Then the section \( t_{-\beta} \otimes s_\beta \in H^0(X, I_x \otimes \omega_X^{n+2} \otimes I^*\alpha) \) does not vanish at \( y \). We finish the proof.

4. Isotrivial fibration with zero-degree push-forward of relative pluri-canonical sheaves

In this section, we will study the isotrivial fibration \( f : X \to B \) to a curve with \( \deg(f_*\omega_X^{n}/B) = 0 \). As an application, we compare the two assumptions AS(2) and WAS(2). Throughout this section, for a projective morphism \( h : Y \to X \) between two varieties, we use \( \omega_Y/X \) for the relative dualizing sheaf. For duality theory and some basic properties of relative dualizing sheaves, we refer to [10, Chapter III, Section 8, 10 and Chapter V, Section 9], [11, Section 5] or [23, Section 6]. In particular, the relative dualizing sheaf \( \omega_{Y/X} \) commutes with flat base changes and coincides with the relative canonical sheaf if \( X \) and \( Y \) are smooth varieties. First, recall the following results.

**Lemma 4.1** ([24, Lemma 2.1]). Let \( X \) be a smooth quasi-projective variety, \( Y \) a normal variety and \( h : Y \to X \) a finite morphism. Assume that the discriminant \( \Delta(Y/X) \subset X \) is a normal crossing divisor. Take a desingularization \( d : Z \to Y \). Then we have the following results.

1. The map \( h \) is flat and \( Y \) has only rational singularities.
2. For an effective \( d \)-exceptional divisor \( E \), the dualizing sheaf \( \omega_Y \cong d_*\omega_Z(E) \).
3. The sheaf \( \omega_X \) is a direct factor of \( h_*d_*\omega_Z \).

Let \( \pi_0 : C \to B = C/G \) be a Galois cover between two smooth curves. Denote by \( R_{\pi_0} \) the ramification locus. Let \( F \) be a smooth projective variety with a faithful action of \( G, \bar{Y} = F \times C, \bar{X} = (F \times C)/G, \) where \( G \) acts on \( F \times C \) diagonally, \( \mu : X \to \bar{X} \) is a smooth resolution, \( \bar{\pi} : \bar{Y} \to X \) is the quotient map, \( \bar{g} : \bar{Y} \to C \) is the projection to \( C \) and \( f : X \to B \) is the natural fibration.

Consider the following commutative diagram:
\[
\begin{array}{ccc}
Y'' & \xrightarrow{\sigma} & Y' & \xrightarrow{\eta} & Y = X \times_B C & \xrightarrow{\pi} & X \\
\downarrow{\eta'} & \downarrow{g'} & \downarrow{g} & \downarrow{f} \\
C & \xrightarrow{id_C} & C & \xrightarrow{\pi_0} & B \\
\end{array}
\]
where \( \eta : Y' \to Y \) is the normalization and \( \sigma : Y'' \to Y' \) is a smooth resolution. Let \( \pi' = \pi \circ \eta : Y'' \to X \) and \( \pi'' = \pi' \circ \sigma : Y'' \to X \). We can assume that \( \pi' \) is branched along a normal crossing divisor on \( X \).

By construction, there is a birational morphism \( \nu : Y'' \to \bar{Y} \), and we can assume that the action of \( G \) lifts on \( Y'' \). Then \( \omega_{Y''}, \omega_C \) and \( \omega_F \) are all \( G \)-invariant sheaves, and \( G \) acts on their global sections via pulling back \( \zeta \mapsto \zeta^* \) for \( \zeta \in G \).

**Lemma 4.2.** We have \( \hat{\pi}^*G\omega_{\bar{Y}} \cong \mu_*\omega_X \), where \( \hat{\pi}^*G\omega_{\bar{Y}} \) denotes the \( G \)-invariant subsheaf of \( \hat{\pi}^*\omega_{\bar{Y}} \).
Proof. By Lemma 4.1(3), we know $\omega_X \cong \pi_*^{\nu} \omega_{Y''}$. Then the lemma follows by $\nu_* \omega_{Y''} = \omega_{\tilde{Y}}$. \hfill \Box

Lemma 4.3. Assume $r_1 = p_g(F) > 0$. Then $\deg(f_* \omega_{X/B}) = 0$ if and only if for every $P \in R_{\pi_0}$, the subgroup $G_P \leq G$ fixing $P$ acts trivially on $H^0(F, \omega_F)$.

Proof. Note that $G_P$ is cyclic. Assume that $G_P$ is of order $k_P$. Take a generator $\sigma$ of $G_P$ and a local parameter $t$ of $C$ at $P$. We can assume $\sigma^* t = \xi t$, where $\xi$ is a $k_P$th primitive root of 1. We can choose a basis $s_i$, $i = 1, 2, \ldots, r_1$ of $H^0(F, \omega_F)$ such that $\sigma^* s_i = \xi^{n_{P,i}} s_i$, where $0 \leq n_{P,i} \leq k_P - 1$. Since $\pi_0 : C \to B$ is a Galois cover, by Lemma 4.2 we have $f_* \omega_X \otimes O_{\pi_0(P),B} \cong (\tilde{g}_* \omega_{\tilde{Y}} \otimes O_{P,C})^{G_P}$, so $t^{k_P - n_{P,i} - 1} dt \otimes s_i$, $i = 1, 2, \ldots, r_1$ form a basis of $f_* \omega_X \otimes O_{\pi_0(P),B}$.

Inversing the process above, we can take a basis $\alpha_i$, $i = 1, 2, \ldots, r_1$ of $f_* \omega_X \otimes O_{\pi_0(P),B}$ such that $t^{-(k_P - n_{P,i} - 1)} \pi_0^* \omega_{K_P} \otimes O_{P,C}$. We conclude that

$$\deg(\tilde{g}_* \omega_{\tilde{Y}}) = \deg(\pi_0^* \det(f_* \omega_X)) + \sum_{P \in R_{\pi_0}} \sum_{i=1}^{r_1} (k_P - n_{P,i} - 1);$$

thus

$$\deg(\tilde{g}_* \omega_{\tilde{Y}}/C) = \deg(\pi_0^* \det(f_* \omega_X)) + \sum_{P \in R_{\pi_0}} \sum_{i=1}^{r_1} (k_P - n_{P,i} - 1) - r_1 \deg(\omega_C)$$

$$= \deg(\pi_0^* \det(f_* \omega_{X/B})) + r_1 \deg(\pi_0^* \omega_B) + \sum_{P \in R_{\pi_0}} \sum_{i=1}^{r_1} (k_P - n_{P,i} - 1) - r_1 \deg(\omega_C)$$

$$= \deg(\pi_0) \deg(f_* \omega_{X/B}) + \sum_{P \in R_{\pi_0}} \sum_{i=1}^{r_1} (k_P - n_{P,i} - 1) - r_1 \deg(\omega_{C/B})$$

$$= \deg(\pi_0) \deg(f_* \omega_{X/B}) + \sum_{P \in R_{\pi_0}} \sum_{i=1}^{r_1} (-n_{P,i}).$$

We can see $\deg(\tilde{g}_* \omega_{\tilde{Y}}/C) = \deg(f_* \omega_{X/B}) = 0$ if and only if $n_{P,i} = 0$ for every $P$ and $i$, which is equivalent to that for every $P \in R_{\pi_0}$, $G_P$ acts trivially on $H^0(F, \omega_F)$. \hfill \Box

Lemma 4.4. Assume $m > 1$ and $r_m = P_m(F) > 0$. Then $\deg(f_* \omega_{X/B}^m) = 0$ if and only if $\pi_0 : C \to B$ is an étale cover.

Proof. The direction ‘if’ is easy. We consider the other direction, and assume $\deg(f_* \omega_{X/B}^m) = 0$.

We argue by contradiction. Let $P$ be a ramification point of $\pi_0$, $F_P$ the fiber of the fibration $\tilde{Y} \to C$ over $P$, $F_P''$ the fiber of the fibration $Y'' \to C$ over $P$ and $\tilde{F}_P''$ the strict transform of $F_P$ via $\nu$. Denote by $E_i$ the $\nu$-exceptional components. Write $K_{Y''} \sim \nu^* K_{\tilde{Y}} + \sum_i a_i E_i$ and $\nu^* F_P = \tilde{F}_P'' + \sum_i b_i E_i$, where $a_i > 0$, $b_i \geq 0$ and $b_i = 0$ if $E_j$ does not intersect $\tilde{F}_P''$. Here we claim

$$\blacklozenge : \sum_i a_i E_i \geq \tilde{F}_P'' \geq \nu^* F_P.$$
Indeed, the adjunction formula gives
\[ K_{F''_{\bar{p}}}|_{\bar{p}^\prime} \sim (K_{Y''} + F''_{\bar{p}})|_{\bar{p}^\prime} \sim \left( \nu^*(K_{\tilde{Y}} + F_{\bar{p}}) + \sum (a_i - b_i)E_i \right)|_{\bar{p}^\prime} \sim \nu^*K_{F_{\bar{p}}} + \sum (a_i - b_i)E_i|_{\bar{p}^\prime}. \]

Since \( F_{\bar{p}} \) is smooth, we have \( (a_i - b_i) \geq 0; \) thus \( \sum (a_i - b_i)E_i + F_{\bar{p}} \geq \nu^*F_{\bar{p}}. \)

The following argument refines the proof of [24, Lemmas 3.2 and 3.3]. Since \( \pi_0 : C \to B \) is flat, we have \( \omega_{Y/C} \cong \pi^*\omega_{Y/B} \). From the finite morphism \( \eta : Y' \to Y \), we have the trace map
\[ \alpha_0 : \eta_*\omega_{Y'/C} \to \omega_{Y/C} \]
and the pull-back homomorphism
\[ \alpha_1 : \eta^*\omega_{Y'/C} \to \eta^*\omega_{Y/C}. \]
The pull-back homomorphism \( \eta^*\eta_*\omega_{Y'/C} \to \omega_{Y'/C} \) is surjective since \( \eta \) is affine, and the kernel is torsion. Since \( \omega_{Y/C} \) is invertible, we have \( \text{Hom}(\omega_{Y'/C}, \eta^*\omega_{Y/C}) \cong \text{Hom}(\eta_*\omega_{Y'/C}, \eta^*\omega_{Y/C}); \)
thus the homomorphism \( \alpha_1 \) factors through a homomorphism
\[ \alpha_2 : \omega_{Y'/C} \to \eta^*\omega_{Y/C}. \]
By the isomorphism \( \sigma_*\omega_{Y''/C} \cong \omega_{Y'/C} \) (Lemma 4.1(2)), we get
\[ \sigma^*\omega_{Y''/C} \cong \sigma^*\sigma_*\omega_{Y'/C} \to \omega_{Y''/C}, \]
which is isomorphic outside the exceptional locus. Both \( \omega_{Y''/C} \) and \( \sigma^*\eta^*\omega_{Y/C} \) are invertible sheaves. So for some effective \( \sigma \)-exceptional divisor \( E \), the pull-back \( \sigma^*\alpha_2 \) induces an injection
\[ \alpha_3 : \omega_{Y''/C} \to \sigma^*\eta^*\omega_{Y/C}(E) \cong \pi''^*\omega_{Y/X/B}(E). \]
Note that \( \alpha_3 \) is not necessarily surjective. Since \( \pi'' : Y'' \to X \) is unramified along \( \tilde{F}_{\bar{p}}'' \) while \( \pi_0 : C \to B \) is ramified along \( P \), the image of \( \alpha_3 \) is contained in \( \mathcal{O}_{Y''}(\tilde{F}_{\bar{p}}'') \otimes \pi''^*\omega_{Y/X/B}(E). \) So we get a homomorphism
\[ \alpha_4 : \omega_{Y''/C}(\tilde{F}_{\bar{p}}'') \to \sigma^*\eta^*\omega_{Y/C}(E) \cong \pi''^*\omega_{X/B}(E). \]
By \( \bullet \), we have an injection \( \nu^*\omega_{Y/C}(\tilde{F}_{\bar{p}}') \to \omega_{Y''/C}(\tilde{F}_{\bar{p}}'') \), and get the following homomorphism by composing this injection with \( \alpha_4' \):
\[ \alpha_5 : \nu^*\omega_{Y/C}(\tilde{F}_{\bar{p}}') \to \sigma^*\eta^*\omega_{Y/C}(E). \]
Tensoring \( \alpha_4^{m-1} \) with \( \text{id}_{\omega_{Y''/C}} \) gives
\[ \alpha_5 : \nu^*\omega_{Y/C}^{m-1}((m-1)\tilde{F}_{\bar{p}}') \otimes \omega_{Y''/C} \to \sigma^*\eta^*\omega_{Y/C}^{m-1} \otimes \omega_{Y''/C}((m-1)E). \]
Applying \( \sigma_* \) to \( \alpha_5 \), by the projection formula and Lemma 4.1(2) we get
\[ \alpha_6 : \sigma_*(\nu^*\omega_{Y/C}^{m-1}((m-1)\tilde{F}_{\bar{p}}') \otimes \omega_{Y''/C}) \to \eta^*\omega_{Y/C}^{m-1} \otimes \sigma_*\omega_{Y''/C} \cong \eta^*\omega_{Y/C}^{m-1} \otimes \omega_{Y'/C}. \]
Similarly, applying \( \eta_* \) to \( \alpha_6 \), then composing with \( \text{id}_{\omega_{Y'/C}} \otimes \alpha_0 \) gives
\[ \alpha_7 : \eta_*\sigma_*(\nu^*\omega_{Y/C}^{m-1}((m-1)\tilde{F}_{\bar{p}}') \otimes \omega_{Y''/C}) \to \omega_{Y/C}^{m-1} \otimes \eta_*\omega_{Y'/C} \to \omega_{Y/C}^{m-1} \otimes \omega_{Y/C} \cong \pi^*\omega_{X/B}^{m}. \]
Applying \( g_* \) to \( \alpha_7 \), we obtain the injection
\[ g_*\eta_*\sigma_*(\nu^*\omega_{Y/C}^{m-1}((m-1)\tilde{F}_{\bar{p}}') \otimes \omega_{Y''/C}) \]
\[ \cong g_*\nu_*(\nu^*(\omega_{Y/C}^{m-1} \otimes \tilde{g}^*\mathcal{O}_C((m-1)P)) \otimes \omega_{Y'/C}) \]
\[ \cong \oplus^m \mathcal{O}_C((m-1)P) \to g_*\pi^*\omega_{X/B}^{m} \cong \pi_0^*f_*\omega_{X/B}^{m}, \quad (4.1) \]
where the last \( \cong \) is due to the property of flat base changes. A contradiction follows by \( \deg(f_*\omega_{X/B}^{m}) = 0 \). \( \Box \)
Proof of Theorem 1.5. Let $C, X_C, f_C$ be as in AS(2), and assume WAS(2). By the results of Section 2.4, there exists a finite cover $\pi_0 : \tilde{C} \to C$ such that $X_C \times_C \tilde{C}$ is birational to $F \times \tilde{C}$. We can assume that $\pi_0$ is a Galois cover with Galois group $G$, and a variety $F'$ birational to $F$ and with a faithful action of $G$, such that $X_C$ is birational to $(F' \times \tilde{C})/G$, where $G$ acts on $F' \times \tilde{C}$ diagonally.

Let $X'_C$ be a smooth resolution of $(F' \times \tilde{C})/G$ and $f'_C : X'_C \to C$ the natural fibration. Then $f_C^* \omega^n_{X_C/C} \cong f'_C \omega^n_{X'_C/C}$. Applying Lemmas 4.3 and 4.4 to the fibration $f'_C$, we can conclude Theorem 1.5.

5. Numerical criterion for the isotriviality of a fibration

We will prove Theorems 1.8 and 1.9. As preparation, we introduce two lemmas.

Lemma 5.1. Let $C' \to C$ be a finite morphism between two smooth curves, let $X'$ be a resolution of the fiber product $X \times_C C'$, and denote by $f' : X' \to C'$ the natural fibration. If $\text{deg}(f_* \omega_{X/C}) = 0$, then $\text{deg}(f'_* \omega_{X'/C'}) = 0$.

Proof. This follows from [17, Corollary 5.4].

Using [16, Theorem 3] and the notation there, we have the following lemma.

Lemma 5.2. Let $U \subset C$ be an open set, $H_0$ a variation of Hodge structure with unipotent local monodromies and $H$ the extension of $H_0$ on $C$. If for a general point $t \in C$ the natural homomorphism $T_{C,t} \to \text{Hom}(F^n, F_n)$ is injective, then $\text{deg}(F^n) > 0$.

Proof of Theorem 1.8. Let $f : X \to C$ and $\tilde{f} : \tilde{X} \to C$ be as in Theorem 1.8. Replacing $X$ by a resolution of $\tilde{X}$ if necessary, then we can assume that there is a morphism $\mu : X \to \tilde{X}$. Take a general point $t \in C$, and denote by $F, \tilde{F}$ the fibers of $f, \tilde{f}$ over $t$ respectively.

We assume deg($f_* \omega_{X/C}$) = 0. By Proposition 2.7, we only need to prove that the Kodaira–Spencer map is zero:

$$\delta : T_{t,C} \to \text{Ext}^1(\Omega_F^1, \mathcal{O}_F).$$

We break the proof into three steps.

Step 1. With the help of Lemma 5.1, up to a base change we can assume that over an open set $U \subset C$, the natural variation of the Hodge structure on $R^nf_*\mathcal{C}$ has unipotent local monodromies. Using Lemma 5.2, by the assumption that deg($f_* \omega_{X/C}$) = 0, the following composite map must be zero:

$$\lambda \circ \delta : T_{t,C} \to H^2(F, T_F) \to \text{Hom}(H^0(F, \Omega_F^n), H^1(F, \Omega_F^{n-1})) \cong \text{Hom}(H^0(F, \omega_F), \text{Ext}^1(\Omega_F^1, \omega_F)),$$

where $\lambda : H^1(F, T_F) \to \text{Hom}(H^0(F, \Omega_F^n), H^1(F, \Omega_F^{n-1}))$ is the infinitesimal period map, and $\delta : T_{t,C} \to H^2(F, T_F)$ is the Kodaira–Spencer map.

Step 2. Denote by $K_F \in D^{[-n,0]}(\tilde{F})$ the dualizing complex of $\tilde{F}$ (see [11, Proposition 5.19]). We have

$$\text{Ext}^1(\Omega_F^1, \omega_F) \xrightarrow{\beta_1} \text{Ext}^1(\mu^* \Omega_F^1, \omega_F) \xrightarrow{\beta_2} \text{Ext}^1(L\mu^* \Omega_F^1, \omega_F) \xrightarrow{\beta_3} \text{Ext}^1(\Omega_F^1, K_F[-n]) \xrightarrow{\beta_4} \text{Ext}^1(\Omega_F^1, \omega_F),$$

where

(a) The map $\beta_1$ is induced by $\mu^* \Omega_F^1 \to \Omega_F^1$;
(b) The map $\beta_2$ is induced by $L \mu^* \Omega_F \to \mu^* \Omega_F$;
(c) The map $\beta_3$ is induced from applying Grothendieck duality:
\[
R\text{Hom}(L \mu^* \Omega_F^1, \omega_F) \cong R\text{Hom}(L \mu^* \Omega_F^1, \mu^* \mathcal{K}_F[-n]) \cong R\text{Hom}(R\mu_* L \mu^* \Omega_F^1, \mathcal{K}_F[-n])
\cong R\text{Hom}(\Omega_F^1 \otimes R\mu_* \mathcal{O}_F, \mathcal{K}_F[-n]) \cong R\text{Hom}(\Omega_F^1, \mathcal{K}_F[-n]);
\]
(d) The map $\beta_4$ is induced by the natural homomorphism $\omega_F \cong \mathcal{H}^0(\mathcal{K}_F[-n]) \to \mathcal{K}_F[-n] \in D^{[0,n]}(\mathcal{F})$, and is injective by spectral sequence.

Step 3. By $H^0(\mathcal{F}, \omega_F) \cong H^0(\bar{\mathcal{F}}, \omega_{\bar{F}})$, we have the following commutative diagram:
\[
\begin{array}{ccc}
T_{t, C} & \xrightarrow{\delta} & H^1(F, T_F) \xrightarrow{\lambda} \text{Hom}(H^0(F, \omega_F), \text{Ext}^1(\Omega_F^1, \omega_F)) \\
\downarrow & & \downarrow \beta' \\
T_{t, C} & \xrightarrow{\delta} & \text{Ext}^1(\Omega_F^1, \mathcal{O}_F) \xrightarrow{\lambda} \text{Hom}(H^0(\bar{\mathcal{F}}, \omega_{\bar{F}}), \text{Ext}^1(\Omega_{\bar{F}}^1, \omega_{\bar{F}}))
\end{array}
\tag{5.1}
\]
where $\alpha : H^1(F, T_F) \cong \text{Ext}^1(\Omega_F^1, \mathcal{O}_F) \to \text{Ext}^1(\Omega_F^1, \mathcal{O}_F)$ is the tangent map of the natural map between the deformation functors $D_F \to D_{\bar{F}}$, and $\beta', \beta'_4$ are induced from $\beta_3 \circ \beta_2 \circ \beta_1, \beta_4$, respectively.

Recall that $\beta' \circ \lambda \circ \delta = 0$ by Step 1, and $\lambda$ and $\beta'_4$ are injective by assumption and Step 2. Then we conclude that $\delta = 0$ by the commutative diagram above. The proof is completed. □

Remark 5.3. The proof above heavily relies on the infinitesimal Torelli theorem, that is, the injectivity of the infinitesimal period map
\[
\bar{\lambda} : \text{Ext}^1(\Omega_{\bar{F}}^1, \mathcal{O}_{\bar{F}}) \to \text{Hom}(H^0(\bar{\mathcal{F}}, \omega_{\bar{F}}), \text{Ext}^1(\Omega_{\bar{F}}^1, \omega_{\bar{F}})),
\]
so it does not work when the fibers have non-injective period maps (for example, curves of genus 2). Reider proved that the infinitesimal Torelli theorem holds for certain irregular surfaces (see [21]).

Proof of Theorem 1.9. The case $m = 1$ follows immediately by applying Theorem 1.8, because then $\omega_F = \mathcal{O}_F$, which implies that the infinitesimal period map $\bar{\lambda} : \text{Ext}^1(\Omega_F^1, \mathcal{O}_F) \to \text{Hom}(H^0(\bar{\mathcal{F}}, \omega_{\bar{F}}), \text{Ext}^1(\Omega_{\bar{F}}^1, \omega_{\bar{F}}))$ is injective.

Let $m_0$ be the minimal positive integer such that $\omega_{\bar{F}}^{m_0} = \mathcal{O}_{\bar{F}}$. Then $m_0 \mid m$, and there exists a natural injection of two line bundles
\[
\otimes \omega_{X/C}^{m_0} f_* \omega_{X/C}^m \hookrightarrow f_* \omega_{X/C}^m.
\]
So $(m/m_0) \deg(f_* \omega_{X/C}^{m_0}) \leq \deg(f_* \omega_{X/C}^m)$.

In the following, we assume $m_0 > 1$ and $\deg(f_* \omega_{X/C}^{m_0}) = 0$. We aim to prove that $f$ is birationally isotrivial.

Note that $f_* \omega_{X/C}^{m_0} \in \text{Pic}^0(C)$, and there is a line bundle $L \in \text{Pic}^0(C)$ such that $L^{m_0} \sim f_* \omega_{X/C}^{m_0}$. We have
\[
h^0(X, (\omega_{X/C} \otimes f^* L^{-1})^{m_0}) = h^0(C, f_* \omega_{X/C}^{m_0} \otimes L^{-m_0}) = 1.
\]
Let
\[
\mathcal{L} = \omega_{X/C} \otimes f^* L^{-1}.
\]
Then there exists an effective Cartier divisor $D \subset X$ such that $L^{m_0} \sim D$. Up to a resolution, we can assume that $D$ is a simple normal crossing divisor.

Let $Z' := \text{Spec} \oplus_{i=0}^{m_0-1} L^{-i}$. Denote by $Z$ the normalization of $Z'$ and by $\pi : Z \to X$ the natural covering map. Then by Kawamata [17, Proposition 2.1(i)], we have
\[
\pi_* \omega_Z \cong \oplus_{i=0}^{m_0-1} \omega_X(D'_i) \otimes L^{-i},
\]
where $D'_i = [(i/m_0)D] + [(i/m_0)D]_{\text{red}}$ and $[(i/m_0)D]_{\text{red}}$ is the reduced divisor of the fractional part of $(i/m_0)D$. Note that $D'_i \leq D$ and $D'_i | F \sim \omega_{F_0}^{m_0}$, thus for $n \in \mathbb{Z}$,
\[
h^0(F, \omega_F^n(D'_i | F)) \leq h^0(F, \omega_F^n(D | F)) = h^0(F, \omega_F^{n+m_0}).
\]
So if $m_0 \neq n$, then $h^0(F, \omega_F^n(D'_i | F)) = 0$.

Let $\sigma : W \to Z$ be a desingularization. Then $\sigma_* \omega_W \cong \omega_Z$ by Lemma 4.1. Denote by $g, h$ the natural fibrations from $W, Z$ to $C$. We have
\[
g_* \omega_{W/C} \cong h_* \omega_Z/C \cong f_*(\oplus_{i=0}^{m_0-1} \omega_X(C) (D'_i) \otimes L^{-i})
\]
\[
\cong \oplus_{i=0}^{m_0-1} f_*\omega_X(C) (D'_i) \otimes L^i \quad \text{by the projection formula}
\]
\[
\cong f_* \mathcal{O}_X(D'_1) \otimes L \quad \text{since } h^0(F, \omega_{F_0}^{-i}(D'_i | F)) = h^0(F, \omega_{F_0}^{-i}) = 0 \text{ if } i \neq 1
\]
\[
\cong L \quad \text{since } D'_1 \leq D \text{ and } f_* \mathcal{O}_X(D) \cong f_* \mathcal{L}^{m_0} \cong \mathcal{O}_C
\]
and thus $\deg(g_* \omega_{W/C}) = 0$.

By construction, general fibers of $g : W \to C$ have good minimal models with trivial canonical bundles. Therefore, $g$ is birationally isotrivial, and so is $f$.

6. Example

In this section, to illustrate the necessity of $\text{AS}(2)$, we construct a variety $X$, such that $\kappa(X) = 1$ and $\text{alb}_X$ is fibred by elliptic curves, but $\dim V^0(\omega_X, \text{alb}_X) = 0$.

(i) Let $E$ be an abelian curve, $a \in E$ a torsion point of order 2, $t_a$ the translate by $a$ and $i$ the involution induced by $(-1)_E$.

(ii) Let $C$ be a curve of genus at least 2 with an involution $\eta$ such that the quotient $C/\eta$ is an elliptic curve.

(iii) Let $Z = E \times C \times E$, with $\sigma = t_a \times \eta \times \text{id}_E$ and $\tau = i \times \text{id}_C \times t_a$ two involutions on $Z$.

Since $\sigma$ and $\tau$ commute, the group $G = < \sigma, \tau > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $G$ acts on $Z$ freely, so the quotient map $\pi : Z \to X := Z/G$ is étale. It follows that $\kappa(X) = 1$, $q(X) = 2$, and the natural map $X \to C/\eta \times E/t_a$ is connected and hence coincides with the Albanese map $\text{alb}_X$. Then we have the following commutative diagram:
\[
\begin{array}{ccc}
E \times C \times E & \xrightarrow{\pi} & X \\
p_2 \times p_3 \downarrow & & \downarrow \text{alb}_X \\
C \times E & \xrightarrow{\pi_2 \times \pi_3} & C/\eta \times E/t_a
\end{array}
\]
where $p_2$ and $p_3$ are the projections from $E \times C \times E$ to the second and the third factors, and $\pi_2, \pi_3$ are the quotient maps from $C, E$ to $C/\eta, E/t_a$, respectively.

We can give $\omega_Z$ a $G$-invariant structure such that $\pi_2^* \omega_Z \cong \omega_X$, and similarly give $\omega_C$ an $\eta$-invariant structure and $\omega_E$ a $t_a$-invariant and $i$-invariant structure; for $\alpha \in \text{Pic}^0(C/\eta)$ and $\beta \in \text{Pic}^0(E/t_a)$, we give $\pi_2^* \alpha$ an $\eta$-invariant structure such that $(\pi_2)_0^*(\pi_2^* \alpha) \cong \alpha$ and give $\pi_3^* \beta$ a $t_a$-invariant structure such that $(\pi_3)_0^*(\pi_3^* \beta) \cong \beta$. Then we have $\alpha \boxtimes \beta \in \text{Pic}^0(X) \cong \text{Pic}^0(C/\eta) \times \text{Pic}^0(E/t_a)$, and naturally $\omega_Z \otimes \pi_3^*(\alpha \boxtimes \beta)$ is a $G$-invariant sheaf such that
\[
\pi_2^*(\omega_Z \otimes \pi_3^*(\alpha \boxtimes \beta)) \cong \omega_X \otimes \alpha \boxtimes \beta.
\]
For $\alpha \boxtimes \beta \in \text{Pic}^0(X) \cong \text{Pic}^0(C/\eta) \times \text{Pic}^0(E/t_a)$, we have

$$H^0(X, \omega_X \otimes \alpha \boxtimes \beta)$$

$$\cong H^0(Z, \omega_Z \otimes \pi^*(\alpha \boxtimes \beta))^G$$

$$\cong (H^0(E, \omega_E) \otimes H^0(C, \omega_C \otimes \pi_2^*(\alpha) \otimes H^0(E, \omega_E \otimes \pi_3^*(\beta))^G$$

$$\cong (H^0(E, \omega_E) \otimes H^0(C, \omega_C \otimes \pi_2^*(\alpha) \otimes H^0(E, \omega_E \otimes \pi_3^*(\beta)))^\tau$$

$$\cong (H^0(E, \omega_E) \otimes H^0(C, \omega_C \otimes \pi_2^*(\alpha)) \otimes H^0(E, \omega_E \otimes \pi_3^*(\beta))\beta$$

$$\cong H^0(E, \omega_E) \otimes H^0(C, \omega_C \otimes \pi_2^*(\alpha)) \otimes H^0(E, \omega_E \otimes \pi_3^*(\beta))^{t_\alpha}, \quad (6.1)$$

where $H^0(E, \omega_E \otimes \pi^*(\beta))^{t_\alpha}$ is the $t_\alpha$-anti-invariant subspace, and the fourth $\cong$ is due to the fact that $H^0(E, \omega_E)$ is $t_\alpha$-invariant and $i$-anti-invariant.

Note that $H^0(C, \omega_C \otimes \pi_2^*(\alpha)) \cong H^0(C/\eta, \omega_{C/\eta} \otimes \alpha)$ and since $(\pi_3)_*(\omega_E) \cong \omega_{E/t_a} \otimes L$, where $L$ is a torsion line bundle on $E/t_a$ of order 2, so $(\pi_3)_*(\omega_E \otimes \pi_3^*(\beta)) \cong \omega_{E/t_a} \otimes \beta \otimes L \otimes \beta$, where $\omega_{E/t_a} \otimes \beta$ is $t_\alpha$-invariant and $L \otimes \beta$ is $t_\alpha$-anti-invariant. So we conclude that $H^0(X, \omega_X \otimes \alpha \boxtimes \beta) \cong H^0(E, \omega_E) \otimes H^0(C, \omega_C \otimes \alpha) \otimes H^0(E/t_a, L \otimes \beta) \neq 0$ if and only if $\alpha = O_{C/\eta}$ and $\beta = L$. Therefore, $V^0(\omega_X) = \{O_{C/\eta} \boxtimes L\}$.

Considering the fibration $g : S := (E \times C)/<t_a \times \eta \to B := C/\eta$, we have $\deg(g_*\omega_{S/B}) = 0$ by the proof of Lemma 4.3, and $AS(2)$ fails for $f$ and $m = 1$ since $f$ has multiple fibers. Since $S$ is isomorphic to a fiber of $X \to C/\eta \times E/t_a \to E/t_a$, we can see that $AS(2)$ fails for $\text{alb}_X$ and $m = 1$.

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