Legendre transformations on the triangular lattice

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1 Dual systems

Study of the nonlinear lattices

\[ \ddot{q}_n = r(\dot{q}_n)(g_{n+1} - g_n), \quad g_n = g(q_n - q_{n-1}) \quad (1) \]

have brought Toda to the notion of dual system, which is obtained from the original one by "replacement of particles by springs and springs by particles accordingly to the certain rules" \[3\]. These rules are the following. The lattice (1) is equivalent to the Euler-Lagrange equations for the functional

\[ \mathcal{L} = \int dt \sum_n (a(\dot{q}_n) - b_n), \quad a'' = 1/r, \quad b' = g, \]

where \( a \) and \( b \) define kinetic and potential energies respectively. The momentum conservation law (\( T_n \) denotes shift operator \( q_n \to q_{n+1} \))

\[ \frac{d}{dt} a'(\dot{q}_n) = (T_n - 1)g_n \]

allows to introduce the new variable \( Q_n \) accordingly to the formulae

\[ \dot{Q}_n = g_n, \quad Q_{n+1} - Q_n = a'(\dot{q}_n). \]

Following the terminology of the paper \[3\], we call this and analogous changes generalized Legendre transformations. The variable \( Q_n \) satisfies the dual lattice

\[ \ddot{Q}_n = R(\dot{Q}_n)(G_{n+1} - G_n) \]

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where the functions $R$ and $G$ are defined by equations $g' = R(g)$, $G(a'(q)) = q$.
For example, the Toda lattice $\ddot{q}_n = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1})$ is dual to the lattice $\ddot{Q}_n = \dot{Q}_n(Q_{n+1} - 2Q_n - Q_{n-1})$.

The following features of the considered transformation should be mentioned. Firstly, it is possible due to the invariance of the Lagrangian with respect to the shift $q \to q + \text{const}$. Secondly, it can be performed for arbitrary $r$ and $g$, and therefore it is not related with integrability. At last, it is involutory (up to the shift $T_n$) and therefore it cannot be used for reproducing of solutions of the lattice.

It was shown in [2] that the situation changes drastically in the case of the relativistic Toda type lattices

$$\ddot{q}_n = r(q_{n+1}h_{n+1} - \dot{q}_{n-1}h_n + g_{n+1} - g_n).$$

(2)

The Lagrangian for the lattice (2) is obtained from the one written above by adding of the term $-\dot{q}_nc_n, c' = h$ which is interpreted as the presence of magnetic field. This term is equivalent, up to the total divergence, to $-\dot{q}_nc_{n+1}$, and this leads to two equivalent forms of the momentum conservation law

$$\frac{d}{dt}(a'(q_n) - c_n) = (T_n - 1)(\dot{q}_nh_n + g_n) \iff \frac{d}{dt}(a'(q_n) - c_{n+1}) = (T_n - 1)(\dot{q}_{n-1}h_n + g_n)$$

and consequently to the pair of generalized Legendre transformations

$$T_- : \quad \dot{Q}_n = \dot{q}_nh_n + g_n, \quad Q_{n+1} - Q_n = a'(q_n) - c_n,$$

$$T_+ : \quad \dot{Q}_n = \dot{q}_{n-1}h_n + g_n, \quad \dot{Q}_{n+1} - \dot{Q}_n = a'(q_n) - c_{n+1}.$$ 

In contrast to the previous example, these transformations generally lead out of the class (2). The requirement that the transformed lattice must be of the same form turned out to be severe enough to isolate exactly the subclass of the integrable lattices (that is, possessing higher symmetries and conservation laws) [2]. It is not so surprisingly, because in this case the combination $T_-^{-1}T_+$ of the Legendre transformations defines the Bäcklund autotransformation of the lattice under consideration, and this property is indispensable feature of any integrable system. The situation is quite analogous to the case of the Bäcklund transformation for the KdV equation which is composition of two slightly different Miura transformations.

The aim of the present paper is to apply these ideas in the totally discrete case: the differentiation $d/dt$ is replaced by the shift $T_m : q_{m,n} \to q_{m+1,n}$ on the second discrete variable. In order to abridge notation we will omit, as a rule, the subscripts $m,n : q = q_{m,n}, q_{1,0} = q_{m+1,n}$ and so on. Moreover, it is convenient to introduce notations for the differences (see figure below)

$$x = q - q_{-1,0}, \quad y = q - q_{0,-1}, \quad z = q - q_{-1,-1}.$$
Obviously the following identity is valid

\[(T_m - 1)y_{0,1} = (T_n - 1)x_{1,0}.\]  \hfill (3)

The analog of the lattice (1) is given by 5-point difference equation

\[(T_m - 1)f(x) = (T_n - 1)g(y)\]  \hfill (4)

of the discrete Toda lattice type \(3, 4, 5, 2\), and Legendre transformation is of the form

\[Y_{0,1} = Q_{0,1} - Q = f(x), \quad X_{1,0} = Q_{1,0} - Q = g(y).\]

It is not related to integrability as well as in continuous case. The analog of the relativistic lattices (2) is more symmetric than its continuous counterpart. Let us consider the functionals of the form

\[\mathcal{L} = \sum_{m,n} (a(x) + b(y) + c(z)).\]

Obviously, the rights of all terms (which play the roles of the kinetic and potential energies and magnetic field) are absolutely equal, and therefore it is more aesthetically to enumerate the variables \(q\) by the points of the rectilinear triangular lattice rather than square one. Nevertheless we use the standard grid which makes computations more easy. Euler equation is of the form

\[(T_m - 1)f(x) + (T_n - 1)g(y) + (T_m T_n - 1)h(z) = 0\]  \hfill (5)

where \(f = a', g = b', h = c'\) and we assume that \(f'g'h' \neq 0\) in order to exclude the case (4). Up to the author knowledge, the equations of this type appeared for the first time in the papers \(6, 7, 8\) by Suris, who also obtained them starting from the relativistic Toda lattices.
The third difference can be expressed through \( x, y \) in two different ways and this circumstance allows to rewrite (3) in two equivalent forms of the momentum conservation law:

\[(T_m - 1)(f(x) + h(z_{0,1})) + (T_n - 1)(g(y) + h(z)) = 0 \iff \]

\[(T_m - 1)(f(x) + h(z)) + (T_n - 1)(g(y) + h(z_{1,0})) = 0.\]

As a result we can define a pair of the Legendre transformations

\[T_+ : (X, Y_{-1,0}) = T(x, y_{0,1}), \quad T_- : (X_{0,-1}, Y) = T(x_{1,0}, y) \quad (6)\]

where the mapping \( T : (x, y) \to (X, Y) \) is given by formulae

\[X = g(y) + h(x + y), \quad Y = -f(x) - h(x + y). \quad (7)\]

Notice, that the lattices (3) are the continuous limit of the equations (5). Indeed, let us consider the family of the functionals

\[\mathcal{L}_\varepsilon = \sum_{m,n} (\varepsilon a(x_{m,n}/\varepsilon) + \varepsilon b(y_{m,n}) - c(y_{m,n}) + c(z_{m,n})),\]

and let \( q_{m,n} = q_n(t), t = m\varepsilon \). It is easy to see that passage to the limit \( \varepsilon \to 0 \) brings to the functional

\[\int dt \sum_n (a(\dot{q}_n) + b_n + \dot{q}_n c'_n)\]

which coincide with the one considered above, up to designations. Legendre transformations for the corresponding relativistic Toda lattice also are obtained from (3) by the mentioned limit.

**Definition 1** Equation (3) is called integrable if the generalized Legendre transformations (4), (5) are invertible (on the differences) and map it into the equation of the same type.

**Theorem 1** Equation (3) is integrable in the sense of the Definition 1 iff the inverse of (4) is of the form

\[x = G(Y) + H(X + Y), \quad y = -F(X) - H(X + Y). \quad (8)\]

In this case the dual equation is

\[(T_m - 1)F(X) + (T_n - 1)G(Y) + (T_mT_n - 1)H(Z) = 0. \quad (9)\]
Proof. Obviously, it is sufficient to consider only transformation $T_+$. Let $T^{-1}_+$ be of the form stated above, then one can easily check that identity (3) is equivalent to the equation (3). Conversely, assume that $T_+$ maps (1) into (2). Solving the formulae $(X, Y_{-1,0}) = T(x, y_{0,1})$ with respect to $x, y_{0,1}$ one obtains

$$x = \Phi(X, Y_{-1,0}), \quad y_{0,1} = \Psi(X, Y_{-1,0})$$

and identity (3) yields equation

$$\Phi(X_{1,1}, Y_{0,1}) - \Phi(X_{1,0}, Y) - \Psi(X_{1,0}, Y) + \Psi(X, Y_{-1,0}) = 0$$

which must be equivalent to (3), that is

$$F(X_{1,0}) - F(X) + G(Y_{0,1}) - G(Y) + H(X_{1,1} + Y_{0,1}) - H(X + Y_{-1,0}) = 0.$$  

Let us consider $X_{1,1}$ as the function on the rest variables involved in these equations, then

$$- \frac{\partial X_{1,1}}{\partial Y_{0,1}} = \frac{\partial \Phi}{\partial Y_{0,1}} / \frac{\partial \Phi}{\partial X_{1,1}} = \frac{G'(Y_{0,1}) + H'(X_{1,1} + Y_{0,1})}{H'(X_{1,1} + Y_{0,1})}$$

and hence $\Phi(X, Y) = \varphi(G(Y) + H(X + Y))$. Analogously the equality

$$- \frac{\partial Y_{-1,0}}{\partial X} = \frac{\partial \Psi}{\partial X} / \frac{\partial \Psi}{\partial Y_{-1,0}} = \frac{F'(X) + H'(X + Y_{-1,0})}{H'(X + Y_{-1,0})}$$

yields $\Psi(X, Y) = \psi(F(X) + H(X + Y))$. Finally, the relation

$$- \frac{\partial X_{1,0}}{\partial Y} = \frac{\partial (\Phi + \Psi)}{\partial Y} / \frac{\partial (\Phi + \Psi)}{\partial X_{1,0}} = \frac{G'(Y)}{F'(X_{1,0})}$$

proves $\Phi(X, Y) + \Psi(X, Y) = \chi(F(X) - G(Y))$. Using these relations one easily obtains $\varphi(z) = \alpha z + \beta, \psi(z) = -\alpha z + \gamma$ and finish the proof (notice that functions $F, G, H$ from (2) are defined up to the linear transformation).  

The immediate corollary is that equation which is dual to the dual equation coincides with the original one. This allows to use composition $T^{-1}_- T_+$ of the Legendre transformation for reproducing of the solutions.

2 Classification theorem

So, in order to classify the integrable equations (2) it is sufficient to find all functions $f, g, h$ such that inverse of the transformation (2) is given by (2). Hence the Jacobian $\Delta = f'g' + g'h' + h'f'$ of the map (2) must be nonzero and the following identities must hold

$$xxy + yxy = 0, \quad xxy = xxx, \quad yxy = yy.$$
These three conditions are equivalent. Indeed, the Jacobi matrix is
\[
\begin{pmatrix}
x_X & y_X \\
x_Y & y_Y \\
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
-h' & f' + h' \\
-g' - h' & h' \\
\end{pmatrix}
\]
that is \(x_X = -y_Y\). Straightforward computation proves that functions \(f(x), g(y), h(x + y)\) must satisfy the equation
\[
(g' + h')\frac{f''}{f} + (f' + h')\frac{g''}{g'} = (f' + g')\frac{h''}{h'}.
\]
Designations \(f' = 1/u, g' = 1/v, h' = 1/w\) rewrite it in more convenient form
\[
[v(y) + w(x + y)]u'(x) + [u(x) + w(x + y)]v'(y) = [u(x) + v(y)]w'(x + y). \tag{10}
\]
The classification problem is reduced to solving of this functional equation. Of course, the functions \(u, v, w\) can be multiplied on the arbitrary constant and linear transformation \(\tilde{x} = c(x - x_0), \quad \tilde{y} = c(y - y_0) \Leftrightarrow \tilde{q}_{m,n} = c(q_{m,n} - mx_0 - ny_0 - \text{const})\)
can be applied, as well as permutation of the coordinate axes.

At first let us consider some degenerate cases. Assume that two of the functions \(u, v, w\) are constant, say \(u\) and \(v\). Then (11) yields that either \(w\) is constant as well or \(u = -v\) and \(w\) is arbitrary. In the first case the equation (5) is linear and in the second it is of the form
\[
\alpha(q_{1,0} + q_{-1,0} - q_{0,1} - q_{0,-1}) + h(q_{1,1} - q) - h(q - q_{-1,-1}) = 0
\]
and admits one integration
\[
\alpha(q_{m+1,n} - q_{m,n+1} - q_{m,n}) + h(q_{m+1,n+1} - q_{m,n}) = c_{m-n}.
\]
Other degenerate case corresponds to the case of zero Jacobian, that is \(u + v + w = 0\) (this implies that all three functions are linear). In this case one can prove that equation (5) can be reduced to the equation on the variable \(p = y_0, x / x\).

Further on we will not consider these degenerate cases.

**Lemma 1** Functions \(u, v, w\) satisfy equations
\[
(u')^2 = \delta u^2 + 2\alpha u + \varepsilon, \quad (v')^2 = \delta v^2 + 2\beta v + \varepsilon, \quad (w')^2 = \delta w^2 + 2\gamma w + \varepsilon. \tag{11}
\]
**Proof.** At first prove that functions \(u(x)\) and \(v(y)\) satisfy equation
\[
(u'' - v'')(u + v) - (u')^2 + (v')^2 = k(u - v), \quad k = \text{const}. \tag{12}
\]
Let us eliminate \( w \) from (10). Applying the operator \( \partial_x - \partial_y \) one obtains the linear system on \( w, w' \):

\[
\begin{pmatrix}
  u' + v' & -u - v \\
  u'' - v'' & v' - u'
\end{pmatrix}
\begin{pmatrix}
w + u \\
w'
\end{pmatrix}
= (u - v)
\begin{pmatrix}
u' \\
u''
\end{pmatrix}.
\]

Its determinant \( \Delta \) is exactly the left hand side of the equation (12). If it is identically zero then (12) is proved, otherwise one finds

\[
w + u = \frac{u - v}{\Delta} (uu'' - (u')^2 + u''v + u'v'), \quad w' = \frac{u - v}{\Delta} (u''v' + u'v'')
\]

and consequently \( ((u - v)/\Delta)_y (uu'' - (u')^2 + u''v + u'v') = 0 \). Assume that the expression in the second bracket vanishes. If \( u' \neq 0 \) then \( v' = -u''v/u' + u' - uu''/u' \), \( v'' = -u''v'/u' \), but then, as one easily checks, \( \Delta = 0 \). If \( u' = 0 \) then \( w + u = 0 \) that is we come to the degenerate solution excluded above. Therefore \( ((u - v)/\Delta)_y = 0 \). Due to the symmetry between \( u \) and \( v \) one can prove analogously \( ((u - v)/\Delta)_x = 0 \) and obtain (13).

Further on, rewriting (12) in the form

\[
\left\frac{u'}{u + v}\right_x = \frac{v'' + k}{u + v} \frac{(v')^2 + 2kv}{(u + v)^2}
\]

multiplying by \( u'/u + v \) and integrating with respect to \( x \) yield

\[
(u')^2 = \delta(y)(u + v)^2 - 2(u'' + k)(u + v) + (v')^2 + 2kv.
\]

Replacing \( u \) and \( v \) one obtains

\[
(v')^2 = \bar{\delta}(x)(u + v)^2 - 2(u'' + k)(u + v) + (u')^2 + 2ku.
\]

Subtracting one equation from the other one and using (12) one obtains \( \bar{\delta} = \delta = \text{const} \). Summing and dividing by \( u + v \) give \( u'' + v'' = \delta(u + v) - k \). Separation of variables yields \( (u')^2 = \delta u^2 + 2\alpha u + \varepsilon \), \( (v')^2 = \delta v^2 + 2\beta v + \bar{\varepsilon} \), where \( \alpha + \beta = -k \), and substitution into (12) proves \( \varepsilon = \bar{\varepsilon} \). The last of the equations (11) is obtained in virtue of symmetry of the coordinate axes.

It is clear that solutions of the equations (11) must satisfy also some additional relations. However their analysis is not in principle difficult, and direct examination of all solutions brings to the following list.

**Theorem 2** The equations (5) integrable in the sense of Definition 1, are exhausted, up to changes \( \tilde{q}_{m,n} = c(q_{m,n} - mx_0 - ny_0) \) and permutations of coordinate axes, by the following sets of the functions \( f, g, h \). In formulae (A), (B), (C) parameters are constrained by relation \( \lambda + \mu + \nu = 0 \), and in (I) by relation

\[
\lambda + \mu = \nu = 0.
\]
\[ \lambda \mu \nu = -1. \]

\[(A) \quad f = \frac{\mu}{x}, \quad g = \frac{\nu}{y}, \quad h = \frac{\lambda}{z}, \]
\[(B) \quad f = \mu \coth x, \quad g = \nu \coth y, \quad h = \lambda \coth z, \]
\[(C) \quad f = \frac{1}{2} \log \frac{x + \mu}{x - \mu}, \quad g = \frac{1}{2} \log \frac{y + \nu}{y - \nu}, \quad h = \frac{1}{2} \log \frac{z + \lambda}{z - \lambda}, \]
\[(D) \quad f = \log x, \quad g = \log y, \quad h = \log(1 - 1/z), \]
\[(E) \quad f = -e^{x} - 1, \quad g = -e^{y}, \quad h = \frac{1}{1 + e^{x}}, \]
\[(F) \quad f = \log(e^{x} - 1), \quad g = \log(e^{y} - 1), \quad h = -\log(e^{x} - 1), \]
\[(G) \quad f = -\log(e^{-x} - 1), \quad g = \log(e^{-y} - 1), \quad h = -z, \]
\[(H) \quad f = \log(\lambda^{-1}(e^{x} + 1)), \quad g = \log(\lambda^{-1}e^{y} - 1), \quad h = \log \frac{e^{x} + \lambda}{e^{y} - \lambda}, \]
\[(I) \quad f = \log \frac{\mu e^{x} + 1}{\mu e^{x} + \mu}, \quad g = \log \frac{\nu e^{y} + 1}{\nu e^{y} + \nu}, \quad h = \log \frac{\lambda e^{z} + 1}{\lambda e^{z} + \lambda}. \]

Legendre transformations \((6), (7)\) link together the equations corresponding to solutions \((B)\) and \((C)\), \((D)\) and \((E)\), \((F)\) and \((G)\), while equations corresponding to solutions \((A)\), \((H)\) and \((I)\) are self-dual.

Notice, that cases \((A)\) and \((B)\) are connected by point transformation \(q = \exp(2\tilde{q})\). It is explained by fact that the Lagrangian \(\sum (\mu \log x + \nu \log y + \lambda \log z)\) of the equation \((8)\), \((A)\) is invariant under the dilations \(q \rightarrow Cq\) as well as under the shifts \(q \rightarrow q + C\). From the other hand, inversions \(q \rightarrow q/(1 - Cq)\) preserves Lagrangian as well but the change \(q = 1/\tilde{q}\) which maps this group into the shift group does not bring to new equation.

Apparently, the cases \((E)\), \((F)\), \((G)\), \((H)\) and \((A)\), \((B)\) for \(\lambda = 0\) appeared in the papers [5, 6, 7, 8] by Suris for the first time. The rest cases are new, up to the author knowledge.

### 3 Higher symmetries

Study of the properties of the presented equations goes out of the scope of this paper. Of course, one has to prove that the Definition 1 brings to equations which are integrable in more habitual sense, that is satisfying the Painlevé test or possessing the higher symmetries and conservation laws. Now we restrict ourself by presenting of the zero curvature representation and some higher symmetries for the most simple equation \((8)\), \((A)\), that is

\[
\frac{\mu}{q_{1,0} - q} + \frac{\mu}{q_{-1,0} - q} + \frac{\nu}{q_{0,1} - q} + \frac{\nu}{q_{0,-1} - q} + \frac{\lambda}{q_{1,1} - q} + \frac{\lambda}{q_{1,-1} - q} = 0, \tag{13}
\]

where \(\lambda + \mu + \nu = 0\). Notice, that all formulae in this section remain valid also for vanishing \(\lambda\) when the equation \((13)\) degenerates into equation of the form \((11)\).

(Continuous) symmetry of the discrete equation \(E = 0\) is a vector field \(q_{t} = \Phi\) on the lattice which preserves this equation: \(\partial_{t}(E) = 0|_{E=0}\). At first let us consider, for completeness, the classic symmetries \(q_{t} = 1, q_{t} = q, q_{t} = q^{2}\)
corresponding to the group of linear-fractional transformations. Since these symmetries are variational, hence in virtue of the Noether theorem multiplying of (13) by their right hand sides yields some conservation laws. Indeed, the shift symmetry generates the momentum conservation law (5), and dilations and inversions give respectively the conservation laws

\[
(T_m - 1) \frac{\mu q}{q - q_{-1,0}} + (T_n - 1) \frac{\nu q}{q - q_{0,-1}} + (T_m T_n - 1) \frac{\lambda q}{q - q_{-1,-1}} = 0,
\]

\[
(T_m - 1) \frac{\mu q q_{-1,0}}{q - q_{-1,0}} + (T_n - 1) \frac{\nu q q_{0,-1}}{q - q_{0,-1}} + (T_m T_n - 1) \frac{\lambda q q_{0,-1}}{q - q_{0,-1}} = 0.
\]

The structure of the higher symmetries of the equation (13) is rather original, as one can infer out of the simplest representatives, and it would be very interesting to obtain the complete description of the symmetry algebra.

**Theorem 3** The following formulae define the symmetries of the equation (13):

\[
\frac{1}{q \xi} = \frac{\mu}{q_{1,0} - q} + \frac{\nu}{q_{0,1} - q} + \frac{\lambda}{q_{-1,1} - q}, \quad \frac{1}{q \eta} = \frac{\mu}{q_{1,0} - q} + \frac{\nu}{q_{0,1} - q} + \frac{\lambda}{q_{-1,1} - q},
\]

\[
\frac{1}{q \zeta} = \frac{\mu}{q_{1,0} - q} + \frac{\nu}{q_{0,1} - q} + \frac{\lambda}{q_{1,1} - q}.
\]

Vector fields \(\partial_\xi, \partial_\eta, \partial_\zeta\) commute in virtue of the equation (13).

The proof can be obtained by straightforward, although tedious computa-
tion. It is sufficient to check the determining equation \(\partial_t(E) = 0|_{E=0}\) only for one symmetry, because two others are obtained by cyclic permutation of the axes \(x, y, z\).

The zero curvature representation for the equation (13) is defined as compatibility condition for the auxiliary linear problems \(\Psi_{1,0} = A \Psi\) and \(\Psi_{1,1} = B \Psi_{1,0}\). In order to retain the equal rights of the directions on the lattice, let us consider also the equation \(\Psi = C \Psi_{1,1}\) and obtain

\[
BA = A_{0,1} B_{-1,0}, \quad AC = C_{1,0} A_{1,1}, \quad CB = B_{-1,-1} C_{0,-1}.
\]

Consistency with the problem of the form \(\Psi_t = U \Psi\) corresponding to the continuous symmetry, brings to the equations

\[
A_t = U_{1,0} A - A U, \quad B_t = U_{1,1} B - B U_{1,0}, \quad C_t = U C - C U_{1,1}.
\]

Matrices \(A, B, C\) are of the form

\[
A = (k - \mu) I - \nu P(q, q_{0,-1}) - \lambda P(q, q_{-1,-1}),
\]

\[
B = k I - \mu P(q, q_{1,0}) - \lambda P(q, q_{1,1}),
\]

\[
C = (k + \nu) I - \mu P(q, q_{-1,0}) - \nu P(q, q_{0,1}),
\]

\[
9
\]
where $k$ is spectral parameter, $I$ is unit matrix and projector $P$ is given by formula

$$P(u, v) = \frac{1}{u - v} \begin{pmatrix} -v & -uv \\ 1 & u \end{pmatrix}$$

(the coefficients on $I$ can be chosen more symmetric by shift $k \to k + (\mu - \nu)/3$.) On the picture above each of the matrices $A, B, C$ depends on the vertices of the triangle in which it is placed (the center of the hexagon corresponds to the variable $q$). Notice also that successive application of the operators $A, B, C$ maps the wave function into itself up to the scalar factor, since $CBA = k(k - \mu)(k + \nu)I$. The check of the equivalence of (13) and equations (14) is not difficult when using the properties

$$P(u, v)P(p, q) = \frac{(u - q)(p - v)}{(u - v)(p - q)} P(p, v), \quad P(u, v) + P(v, u) = I.$$

The representations (15) for the symmetry $\partial_t = \partial_z$ are given by the matrix $U = k^{-1}(\tilde{U} - \frac{1}{2}I)$, where $\tilde{U}$ is the projector

$$\tilde{U} = \frac{1}{\alpha + q_{-1,0}^\beta} \begin{pmatrix} \alpha & q_{-1,0}^\alpha \\ \beta & q_{-1,0}^\beta \end{pmatrix},$$

$\alpha = \lambda q_{-1,-1} + \mu q_{-1,-1}q_{0,-1} + \nu q_{0,-1}q, \quad \beta = \lambda q_{0,-1} + \mu q + \nu q_{-1,-1}$.

It should be stressed that the symmetries presented in Theorem 3 are not integrable equations by themselves and it makes sense to consider them only together with the constraint (13). In other words, they are 1+1 rather than 1+2-dimensional equations. Indeed, denoting $u_m = q_{m,n}, \quad v_m = q_{m,n-1}$ and using equation (13) one can rewrite the symmetries $\partial_t, \partial_z$ as the one-dimensional lattices

$$\frac{1}{u_n} = \frac{\mu}{u_1 - u} + \frac{\nu}{v - u} + \frac{\lambda}{v - u}, \quad -\frac{1}{v_n} = \frac{\mu}{v_1 - v} + \frac{\nu}{u - v} + \frac{\lambda}{u - v},$$

(16)
Remarkable fact is that these lattices are equivalent to one of the relativistic lattices (2), namely to the lattice, corresponding to the isotropic Heisenberg model.

\textbf{Theorem 4} Elimination of the variables \( v_m \) from the equations (16) brings to the lattice

\[
- \frac{1}{u_\zeta} = \frac{\mu}{u_{-1} - u} + \frac{\nu}{v - u} + \frac{\lambda}{v_{-1} - u}, \quad \frac{1}{v_\eta} = \frac{\mu}{v_1 - v} + \frac{\nu}{u - v} + \frac{\lambda}{u_1 - v}
\]

(17)

(in order to write down \( \partial_\xi \) in analogous form one should choose dynamical variables along the axis \( y \) or \( z \)).

Vector fields defined by equations (16) and (17) commute and variables \( u = u_m, \ v = v_m \) satisfy the system (now we assume that \( \mu \neq 0 \))

\begin{align*}
    u_\eta \zeta &= u_\eta^2 \left( \frac{\mu u_{1,\eta}}{(u_1 - u)^2} - \frac{\mu u_{-1,\eta}}{(u - u_{-1})^2} - \frac{1}{u_1 - u} + \frac{1}{u - u_{-1}} \right), \\
    v_\eta \zeta &= v_\eta^2 \left( \frac{\mu v_{1,\eta}}{(v_1 - v)^2} - \frac{\mu v_{-1,\eta}}{(v - v_{-1})^2} - \frac{1}{v_1 - v} + \frac{1}{v - v_{-1}} \right)
\end{align*}

in virtue of these equations.

So, the relation between the continuous equations (3) and their discrete analogues (4) is more profound than it seemed at first sight. Remind, that the relation between the lattices (2) and equations of the form (4) has been already established in paper [2]. More precisely, it was demonstrated there that the composition of the Legendre transformations \( T_{-1} T_+ \) for relativistic Toda lattices is equivalent to the shift in Toda lattices (1) (however, the analogy with discrete equations is slightly failed at this point, since the composition \( T_{-1} T_+ \) of the transformations (6) cannot be rewritten as equation of the form (4)) and the nonlinear superposition principle for the lattices of these two types brings to the equations (4). The example considered above demonstrates that the equations (5) can be interpreted as nonlinear superposition principle as well, but for the pair of relativistic lattices.

In conclusion of this section notice that, although the equation (13) was not subjected to the Painlevé test, but the simplest examples of 2-periodic reductions (several variants are possible) demonstrate that it is Painlevé integrable even at \( \lambda + \mu + \nu \neq 0 \). Of course in this case the above zero curvature representations and symmetries are lacking. Possibly, this situation is analogous to the case of difference Hirota-Miwa and KdV equations which were verified by the Painlevé test in the paper [3].
4 Concluding remarks

The author’s main purpose was to demonstrate that condition of invariance with respect to the Legendre transformations allows effectively isolate the class of integrable equations. The obtained examples of difference equations at the triangular lattice provide the discrete analogues of relativistic Toda lattices and prompt expectation that the method of Legendre transformations suggested in [2] is not just a trick and its scope is rather wide. The open problem is further generalizations on the new classes of equations, in particular multidimensional ones.

As in continuous case the weakest point of the method is the conjecture about the shift invariance of the Lagrangian. The study of the relativistic Toda lattices [2] implies that the difference analog of the Landau-Lifshitz equation must exist which is characterized by absence of the classic symmetries.

¿From the other hand, it is not difficult to find some multifield generalizations, although their classification is far from completeness, as well as in continuous case. Now we present only the vector analog of the discrete Heisenberg equation (13):

$$\mu(T_m - 1) \frac{x}{|x|^2} + \nu(T_n - 1) \frac{y}{|y|^2} = (\mu + \nu)(T_m T_n - 1) \frac{z}{|z|^2}$$

where $q \in \mathbb{R}^d$ and $|q|$ denotes Euclidean norm. Legendre transformation $T_+$ is of the form

$$X = \nu \frac{y_{0,1}}{|y_{0,1}|^2} - (\mu + \nu) \frac{x + y_{0,1}}{|x + y_{0,1}|^2}, \quad Y_{1,0} = -\mu \frac{x}{|x|^2} + (\mu + \nu) \frac{x + y_{0,1}}{|x + y_{0,1}|^2}$$

and, as in the scalar case, maps equation into itself. Apparently this example admits generalizations for arbitrary Jordan triple systems with invertible elements.

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