Topological Field Theory of Vortices over Closed Kähler Manifolds

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By dimensional reduction, Einstein-Hermitian equations of $n + 1$ dimensional closed Kähler manifolds lead to vortex equations of $n$ dimensional closed Kähler manifolds. A Yang-Mills-Higgs functional to unitary bundles over closed Kähler manifolds has topological invariance by adding the additional terms which have ghost fields. Henceforth we achieve the matter (Higgs field) coupled topological field theories in higher dimension.

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1. Introduction

At Yang-Mills gauge theory on four manifolds Witten\cite{1} set up the relativistic field theory with a global fermionic symmetry similar to BRST symmetry. This field theory has been called by topological quantum field theory (TQFT). TQFT could be constructed by BRST gauge fixing on the Yang-Mills instantons space of anti-self dual equations\cite{2}\cite{3}. That anti-self dual equations arise as minimizing conditions for gauge invariant functional.

In two and three manifolds as minimizing conditions we can choose the self dual and anti-self dual Yang-Mills equations, and vortex equations and monopole equations which have Higgs fields. Some authors showed that by choosing the solutions space of those equations as gauge fixing condition TQFT can be constructed\cite{4}\cite{5}\cite{6}.

If one considers a closed Kähler manifold and considers unitary connection on a complex bundle, the equations for the minima are the Einstein-Hermitian equations. The conditions for the existence of solutions of the Hermitian-Einstein equations can be related to the stability of the holomorphic bundle. On the complex surface one can set up TQFT\cite{7}\cite{8}.

Bradlow\cite{9}\cite{10} showed that existing the global minima of the Yang-Mills-Higgs is extended into arbitrary dimensional closed Kähler manifolds, and studied the necessary and sufficient conditions for the existence of solutions to the vortex equations.

We will show that the vortex equations is obtained by dimensional reduction of Einstein-Hermitian equations. In this scheme we will construct topological field theory over vortices, that is, a Yang-Mills-Higgs functional to unitary bundle over closed Kähler manifolds has fermionic symmetry.

In section 2, we review the moduli space on closed Kähler manifolds and intend to obtain the vortex equations from dimensional reduction $n+1$ into $n$ dimensional manifolds. In section 3, we show that the Yang-Mills-Higgs theory has fermionic symmetry (topological symmetry) over vortices space. In section 4, on 2-dimension Kähler manifolds we construct the topological invariant observables.
2. The moduli space of Yang-Mills Theory on closed Kähler Manifolds

2.1. The Einstein-Hermition Structure

Let $X$ be a closed Kähler manifold of complex dimension $n$ and let $\omega$ be the Kähler form. Let $E$ be a complex vector bundle of rank $n$ over $X$. We will consider the vector bundle $E$ to be endowed with a fixed hermitian metric $h$. The Hermitian metric $h$ in $E$ is a smooth field of Hermitian inner products in the fibers of $E$ such that

\[ h(\xi, \eta) \text{ is linear in } \xi, \quad \text{where } \xi, \eta \in E_x, \]
\[ h(\xi, \eta) = \overline{h(\eta, \xi)}, \]
\[ h(\xi, \xi) > 0 \quad \xi \neq 0, \]
\[ h(\xi, \eta) \text{ is a smooth function if } \xi \text{ and } \eta \text{ are smooth sections.} \]  

We call $(E, h)$ an Hermitian vector bundle. Let $A(h)$ denote connections on $E$ that are unitary with respect to $H$. We use the following;

\[ \Omega^{p,q}(X) = \text{the space of } (p,q)\text{-forms over } X, \]
\[ \Omega^{p,q}(X, E) = \text{the space of } (p,q)\text{-forms with values in } E, \]
\[ \Omega^{p,q}(X, \text{End } E) = \text{the space of } (p,q)\text{-forms with values in the endomorphism bundle of } E. \]

All the spaces carry hermitian metric induced by Kähler metric on $X$ and the metric on $E$. So we lead to identifications $E \approx E^*$ and $E \otimes E^* \approx \text{End } E$. We define the dual operator

\[ \bar{\ast} : \Omega^{p,q} \rightarrow \Omega^{n-p,n-q}. \]

We define an inner product in the space of $(p,q)$-forms on $X$ by setting

\[ (\phi, \psi) = \int \phi \wedge \bar{\ast} \psi. \]

On Kähler manifold with Kähler form $\omega$, we can define a map

\[ L : \Omega^{p,q}(X, E) \rightarrow \Omega^{p+1,q+1}(X, E) \]
by
\[ L(\alpha) = \alpha \wedge \omega. \quad (2.5) \]

The adjoint of this map for the inner product (2.3) is given by \( \Lambda \) such that
\[ \Lambda : \Omega^{p,q}(X, E) \to \Omega^{p+1,q-1}(X, E). \quad (2.6) \]

Let \( d_A = d + A \) be a connection in \( E \). We can split of \( d_A \) into \( \partial_A + \bar{\partial}_A \) on complex vector bundle over \( X \) where
\[ \partial_A : \Omega^{p,q}(X, E) \to \Omega^{p+1,q}(X, E), \quad (2.7) \]
and
\[ \bar{\partial}_A : \Omega^{p,q}(X, E) \to \Omega^{p,q+1}(X, E), \quad (2.8) \]

We will denote \( F \) the curvature of \( d_A \), i.e. \( F = d_A \circ d_A \in \Omega^2(X, \text{End } E) \). Then we can decompose as
\[ F = F^{2,0} + F^{1,1} + F^{0,2}, \quad (2.9) \]
where
\[ F^{2,0} \equiv \partial_A \circ \partial_A \in \Omega^{2,0}(X, \text{End } E), \quad F^{0,2} \equiv \bar{\partial}_A \circ \bar{\partial}_A \in \Omega^{0,2}(X, \text{End } E), \quad (2.10) \]
\[ F^{1,1} \equiv (\partial_A \circ \bar{\partial}_A + \bar{\partial}_A \circ \partial_A) \in \Omega^{1,1}(X, \text{End } E). \]

\( F^{1,1} \) can be decompose into
\[ F^{1,1} = F^{1,1}_0 + F^0 \cdot \omega, \quad (2.11) \]
where \( F^{1,1}_0 \) consists of forms orthogonal to \( \omega \). A connection \( d_A \) is called integrable if \( (\bar{\partial}_A)^2 = 0 \). If \( d_A \) is a connection in \( E \) such that \( (\bar{\partial}_A)^2 = 0 \), \( \bar{\partial}_A \) defines a holomorphic structure on \( E \). A section \( \xi \in \Omega^0(X, E) \) is called holomorphic if and only if \( \bar{\partial}_A \xi = 0 \). The complex vector bundles which admit holomorphic structures is called vector bundle over a complex manifold \( M \). From now on we shall study holomorphic vector bundle.

Let \( \mathcal{H}(h) \) denote the subset of \( \mathcal{A}(h) \) consisting of \( d_A = \partial_A + \bar{\partial}_A \) such that \( \bar{\partial}_A \circ \bar{\partial}_A = 0 \) and let \( \mathcal{E}(h) \) denote the subset of \( \mathcal{H}(h) \) consisting of Einstein-Hermitian connections \( d_A \), i.e.
\[ \mathcal{E}(h) = \{ d_A \in \mathcal{H}(h) | i\Lambda F = cI_E \} \quad (2.12) \]
where \( c \) is a constant and \( I_E \in \Omega^0(x, \text{End } E) \) is the identity section.
2.2. Vortices

Let $\Omega^0(X, E)$ denote the smooth sections of $E$. We consider integrable unitary connections belonging to $A^{1,1}(h)$ such that

$$\mathcal{A}^{1,1}(h) = \{ dA \in A^{1,1} | F^{0,2} = 0 \}$$

(2.13)

and the pairs $(dA, \Phi)$ in $\varphi \subset A^{1,1} \times \Omega^0(X, E)$ where

$$\Theta = \{(\delta A, \Phi) \in A^{1,1} \times \Omega^0(X, E) | \bar{\partial}_A \Phi = 0 \}.$$  

(2.14)

The complex gauge group acts on both these spaces by

$$g(\bar{\partial}_A) = g \circ \bar{\partial}_A \circ g^{-1},$$

$$g(\Phi) = g\Phi.$$  

(2.15)

This vector valued function $\Phi$ is called a Higgs field.

Among the triple $(\bar{\partial}_A, \Phi, h) \in \Theta$, if it satisfies the equations,

$$\bar{\partial}_A \Phi = 0,$$

$$i\Lambda F_{\bar{\partial}_A,h} + \frac{1}{2} \Phi \otimes \Phi^* h - \frac{1}{2} \tau I_E = 0,$$

(2.16)

we will call the metrics $h$ $\tau$-Hermitian-Yang-Mills-Higgs metrics. Here $F_{\bar{\partial}_A,h}$ is the curvature of the metric connection compatible with $\bar{\partial}_A$ and $h$ and $\Phi^* h$ denote that the adjoint is taken with respect to the metric $h$.

Bradlow[9] presented that the triple $(\bar{\partial}_A, \Phi, h)$ satisfied Eq.(2.16) are global minima of the Yang-Mills-Higgs action.

2.3. The Dimensional Reduction

We consider the $n+1$ dimensional closed Kähler complex manifold. We will show that with the dimensional reduction $n+1$ dimension to $n$ dimension manifold, the Einstein-Hermitian condition on the $n+1$ dimensional manifold leads to the Hermitian-Higgs condition on the $n$ dimensional manifold.
The connection is described by
\[ \tilde{A} = A'_1 dz_1 + \cdots + A'_{n+1} dz_{n+1} + A''_1 d\bar{z}_1 + \cdots + A''_{n+1} d\bar{z}_{n+1}. \] (2.17)

We now assume that the connections \( A_i \) are independent of the coordinates, \( z_{n+1} \) and \( \bar{z}_{n+1} \), and define functions of \((z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)\). Thus over \( n \) dimensional manifold we can define the connection
\[ A = A'_1 dz_1 + \cdots + A'_n dz_n + A''_1 d\bar{z}_1 + \cdots + A''_n d\bar{z}_n, \] (2.18)
and \( A'_{n+1} \) and \( A''_{n+1} \) relabel as \( \Phi^* \) and \( \Phi \) which are independent of \( z_{n+1} \) and \( \bar{z}_{n+1} \). We denote \( \tilde{F} \) the curvature form of vector bundle over \( n+1 \) dimensional manifold and \( F \) the curvature form of vector bundle over \( n \) dimensional manifold.

We assume that the curvature \( \tilde{F} \) of the vector bundle over the \((n+1)\) dimensional base manifold \( M \) can be rewritten by
\[ \tilde{F} = F^{2,0} + F^{0,2} + F^{1,1} + \partial_A \Phi^* \wedge dz_{n+1} + \bar{\partial}_A \Phi \wedge d\bar{z}_{n+1} + \Phi^* dz_{n+1} \wedge \Phi d\bar{z}_{n+1} \] (2.19)
where \( F \)'s are the curvature of vector bundle over the \( n \) dimensional base manifold \( X \) and \( \Phi \) independent of \( \{z_{n+1}, \bar{z}_{n+1}\} \) relabels the \( (n+1) \)th component of the connection.

The integrable condition \( \tilde{F}^{0,2} = 0 \) on the \( n+1 \) dimensional manifold leads to
\[ F^{0,2} = 0, \] (2.20)
on the \( n \) dimensional manifold. The first equation denotes the integrable condition on the \( n \) dimensional manifold and the second equation describes that \( \Phi \) has holomorphic condition. For the Einstein-Hermitian condition on the \( n+1 \) dimensional manifold, \( i\Lambda \tilde{F} = cI_E \) leads to vortex equation,
\[ i\Lambda F + \frac{1}{2} \Phi \otimes \Phi^* = \frac{1}{2} \tau I_E \] (2.21)
on the \( n \) dimensional manifold where we substitute \( \frac{\tau}{2} \) for \( c \). In fact,
\[ \Lambda(\Phi^* dz_{n+1} \wedge \Phi d\bar{z}_{n+1}) = \frac{1}{2} |\Phi|^2_h \] (2.22)
where \( \Lambda \) is adjoint operator of \( L \) which is defined locally by
\[ L \phi = \phi \wedge \left( \frac{i}{2} dz_{n+1} \wedge d\bar{z}_{n+1} \right) \] (2.23)

Then we can find that the connections space \( \mathcal{E}(h) \) on the \( n+1 \) dimensional manifold leads to the stable pairs space, \( A^{1,1}(h) \times \Omega^0(X, E) \).
3. Topological Quantum Field Theory

We can obtain the following transformation laws from the curvature formula and Bianchi identity for the universal bundle over $M \times A/G$ for $n + 1$ dimensional manifold $M[11][12][2]$:

\[ s\tilde{A}' = \tilde{\psi}, \]
\[ s\tilde{A}'' = \tilde{\psi}, \]
\[ s\tilde{\psi} = -\partial_A \phi, \]
\[ s\tilde{\bar{\psi}} = -\bar{\partial}_A \phi, \]
\[ s\phi = 0, \]

(3.1)

Like Eq.(2.17), we assume that the ghost fields $\tilde{\psi}$ and $\tilde{\bar{\psi}}$ are independent of the coordinates $z_{n+1}$ and $\bar{z}_{n+1}$ and define functions of $(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$. The $(n + 1)$-th components $\tilde{\psi}_{n+1}$ and $\tilde{\bar{\psi}}_{n+1}$ relabel as $\alpha$ and $\bar{\alpha}$ which are independent of $z_{n+1}$ and $\bar{z}_{n+1}$. Then we can arrive at the following transformation laws;

\[ sA' = \psi, \quad sA'' = \tilde{\psi}, \]
\[ s\psi = -\partial_A \phi, \quad s\tilde{\psi} = -\bar{\partial}_A \phi, \]
\[ s\Phi^* = \alpha, \quad s\Phi = \bar{\alpha}, \]
\[ s\alpha = -[\Phi^*, \phi], \quad s\bar{\alpha} = -[\Phi, \phi], \]
\[ s\phi = 0. \]

(3.2)

We need additional multiplets to write Lagrangian. So we introduce two pairs $(\lambda, \eta)$, $(\xi, \rho)$ and $(\bar{\chi}, \bar{H})$ such that

\[ s\lambda = \eta, \quad s\eta = [\phi, \lambda], \]
\[ s\xi = \rho, \quad s\rho = [\phi, \xi], \]
\[ s\bar{\chi} = \bar{H}, \quad s\bar{H} = [\phi, \bar{\chi}]. \]

(3.3)

The multiplets $(\bar{\chi}, \bar{H})$ are self-dual 2-forms and have the ghost number $(-1, 0)$. So the self-dual 2-forms $\chi$ and $H$ can be written as

\[ \bar{\chi} = \chi^{2,0} + \chi^{0,2} + \chi^\omega, \]
\[ \bar{H} = H^{2,0} + H^{0,2} + H^\omega, \]

(3.4)
where \( \chi = \Lambda \bar{\chi} \) and \( H = \Lambda \bar{H} \). The multiplets \((\xi, \rho)\) and \((\lambda, \eta)\) are zero forms and have the ghost number \((-1, 0)\) and \((-2, 1)\), respectively.

In order to obtain the topological invariant action we are going to follow the procedure of standard BRST gauge fixing. We concern the stable pair sector with gauge coupled Higgs fields. So we choose the following gauge conditions

\[
F^{0,2} = 0,
\]

\[
\bar{\partial}_A \Phi = 0,
\]

\[
i\Lambda F + \frac{1}{2} \Phi \otimes \Phi^* = \frac{\tau}{2} I_E,
\]

and the slice condition,

\[
\Lambda(\bar{\partial}_A \psi - \partial_A \bar{\psi}) + \frac{1}{2} \Phi \otimes \alpha - \frac{1}{2} \Phi^* \otimes \bar{\alpha} = 0
\]

which are orthogonal to the variations in the pair \((A, \Phi)\) that can be obtained by a gauge transformations.

We can obtain the action as

\[
S = s \int_M Tr \left[ \chi^{0,2} \wedge \bar{\chi}(F^{0,2} + \frac{1}{16} H^{0,2}) + \xi \wedge \bar{\xi}(\bar{\partial}_A \Phi + \frac{1}{8} \rho) \right.
\]

\[
+ \chi(i\Lambda F + \frac{1}{2} \Phi \otimes \Phi^* - \frac{1}{2} \tau I_E + \frac{1}{4} H) \psi \wedge \bar{\psi}
\]

\[
+ \lambda(\Lambda(\bar{\partial}_A \psi - \partial_A \bar{\psi}) + \frac{1}{2} \Phi \otimes \alpha - \frac{1}{2} \Phi^* \otimes \bar{\alpha}) \psi \bar{\psi}
\]

Then by simple calculation one can find that

\[
S = \int_M Tr \left[ H^{0,2} \wedge \bar{\chi}(F^{0,2} + \frac{1}{16} H^{0,2}) + \rho \wedge \bar{\rho}(\bar{\partial}_A \Phi + \frac{1}{8} \rho) \right.
\]

\[
+ H(i\Lambda F + \frac{1}{2} \Phi \otimes \Phi^* - \frac{1}{2} \tau I_E + \frac{1}{4} H) \psi \wedge \bar{\psi}
\]

\[
+ \chi^0 \wedge (\bar{\bar{\partial}_A \bar{\psi}} + \frac{1}{16} [\phi, \bar{\chi}^{0,2}]) + \xi \wedge (\bar{\partial}_A \psi + \frac{1}{8} [\phi, \bar{\chi}^{2,0}])
\]

\[
+ \chi\{\Lambda(\bar{\partial}_A \bar{\psi} + \partial_A \psi) + \frac{1}{2} \alpha \otimes \Phi^* + \frac{1}{2} \Phi \otimes \alpha + \frac{1}{4} [\phi, H]\} \wedge \bar{\psi}
\]

\[
+ \eta\{\Lambda(\bar{\partial}_A \psi - \partial_A \bar{\psi}) + \frac{1}{2} \Phi \otimes \alpha - \frac{1}{2} \Phi^* \otimes \bar{\alpha}\} \wedge \bar{\psi}
\]

\[
+ \{\lambda((\bar{\partial}_A \partial A + \partial^* A \bar{\partial}_A) \phi - \frac{1}{2} \Phi \otimes [\Phi^*, \phi] + \frac{1}{2} \Phi^* \otimes [\Phi, \phi])
\]

\[
+ \Lambda(\psi \wedge [\lambda, \psi] - \psi \wedge [\lambda, \bar{\psi}])\} \wedge \bar{\psi}.
\]

\[
(3.5)
\]

\[
(3.6)
\]

\[
(3.7)
\]

\[
(3.8)
\]
$H$ is an auxiliary field important in closing the algebra but not propagating. By using the Euer-Lagrange equation we can eliminating $H$, one finds an action,

$$S = 4||F^{0,2}||^2 + 2||\bar{\partial}A\Phi||^2 + ||i\Lambda F + \frac{1}{2}\Phi \otimes \Phi^* - \frac{1}{2}\tau I||^2$$

$$+ \int_M Tr\big(\chi^{0,2} \wedge (\bar{\chi}\partial A\bar{\psi} + \frac{1}{16}[\phi, \bar{\chi}^{0,2}]) + \xi \wedge (\bar{\chi}\partial A\psi + \frac{1}{8}[\phi, \bar{\chi}^{2,0}])
$$

$$+ \chi\{\Lambda(\bar{\partial}A\bar{\psi} + \partial A\psi) + \frac{1}{2}\bar{\alpha} \otimes \Phi^* + \frac{1}{2}\Phi \otimes \alpha + \frac{1}{4}[\phi, H]\} \omega \wedge \bar{\omega}$$

$$+ \eta\{\Lambda(\bar{\partial}A\psi - \partial A\bar{\psi}) + \frac{1}{2}\Phi \otimes \alpha - \frac{1}{2}\Phi^* \otimes \bar{\alpha}\} \omega \bar{\omega}\big)$$

$$+ \{\Lambda(\bar{\partial}^* A\partial A\psi - \partial^* A\bar{\partial}A\bar{\psi}) - \frac{1}{2}\Phi \otimes [\Phi^*, \phi] + \frac{1}{2}\Phi^* \otimes [\Phi, \phi])$$

$$+ \Lambda(\bar{\psi} \wedge [\lambda, \psi] - \psi \wedge [\lambda, \bar{\psi}])\} \omega \wedge \bar{\omega}].$$

The first three terms of the action (3.9),

$$4||F^{0,2}||^2 + 2||\bar{\partial}A\Phi||^2 + ||i\Lambda F + \frac{1}{2}\Phi \otimes \Phi^* - \frac{1}{2}\tau I||^2$$

(3.10)

can be rewritten as

$$||F||^2 + ||D\Phi||^2 + \frac{1}{4}||\Phi \otimes \Phi^* - \tau I||^2 - \tau \int_X iTrF \wedge \omega^{n-1} - \int_X Tr(F \wedge F) \wedge \omega^{n-2}. \quad (3.11)$$

Here this follows from the identities,

$$|F|^2 \omega^n = |\Lambda F|^2 \omega^n + Tr(F \wedge F) \wedge \omega^{n-2} + 2(|F^{0,2}|^2 + |F^{2,0}|^2)\omega^n,$$

$$< i\Lambda F, \Phi \otimes \Phi^* > = -||\bar{\partial}A\Phi||^2 + ||\partial A\Phi||^2,$$

$$||i\Lambda F + \frac{1}{2}\Phi \otimes \Phi^* - \frac{1}{2}\tau I||^2 = ||i\Lambda F||^2 + < i\Lambda F, \Phi \otimes \Phi^* > - \tau < i\Lambda F, I >$$

$$+ \frac{1}{4}||\Phi \otimes \Phi^* - \tau I||^2,$$

$$< i\Lambda F, I > = \frac{i}{2\pi} \int_X Tr(F, \omega)\omega^n = i \int_X TrF \wedge \omega^{n-1}.$$

(3.12)

and the Kähler identies,

$$i[\Lambda, \bar{\partial}A] = \partial_A^*, \quad -i[\Lambda, \partial_A] = \bar{\partial}_A^*.$$

(3.13)
Finally we can obtain the invariant action

\[ S = \| F \|^2 + \| D\Phi \|^2 + \frac{1}{4} \| \Phi \otimes \Phi^* - \tau I \|^2 - \tau \int_X iTr F \wedge \omega^{n-1} - \int_X Tr (F \wedge F) \wedge \omega^{n-2} \]

\[ + \int_M \chi^{0,2} \wedge (\bar{\Phi} \partial A \psi + \frac{1}{16} [\phi, \bar{\chi}^{0,2}]) + \xi \wedge (\bar{\Phi} \partial A \psi + \frac{1}{8} [\phi, \bar{\chi}^{2,0}]) \]

\[ + \chi \{ \Lambda (\partial A \bar{\psi} + \bar{A} \psi) + \frac{1}{2} \check{\phi} \otimes \Phi^* + \frac{1}{2} \Phi \otimes \alpha + \frac{1}{4} [\phi, H] \} \omega \wedge \bar{\omega} \]

\[ + \eta \{ \Lambda (\bar{A} \psi - \partial A \bar{\phi}) + \frac{1}{2} \Phi \otimes \bar{\alpha} - \frac{1}{2} \Phi^* \otimes \alpha \} \omega \wedge \bar{\omega} \]

\[ + \{ \chi ((\partial A \bar{\phi} + \Phi^* \partial A A) \Phi - \frac{1}{2} \Phi \otimes [\Phi^*, \phi] + \frac{1}{2} \Phi^* \otimes [\Phi, \phi]) \]

\[ + \Lambda (\bar{\psi} \wedge [\lambda, \psi] - \psi \wedge [\lambda, \bar{\psi}]) \} \omega \wedge \bar{\omega} \]

\[ (3.14) \]

under the following fermionic transformations;

\[ sA' = \psi, \quad sA'' = \bar{\psi}, \]

\[ s\psi = -\partial A \phi, \quad s\bar{\psi} = -\bar{\partial} A \bar{\phi}, \]

\[ s\Phi^* = \alpha, \quad s\Phi = \bar{\alpha}, \]

\[ s\alpha = -[\Phi^*, \phi], \quad s\bar{\alpha} = -[\Phi, \phi], \]

\[ s\phi = 0, \quad s\xi = \bar{\partial} A \Phi, \]

\[ s\chi^{0,2} = 4F^{0,2}, \quad \chi = iA F + \frac{1}{2} \Phi \otimes \Phi^* - \frac{\tau}{2} I_E, \]

\[ s\lambda = \eta, \quad s\eta = [\phi, \lambda]. \]

\[ (3.15) \]

4. Observables

In this section, we hope to find the topological invariant observables. The observables must be BRST invariant, not depend explicitly on the metric, and not be written as \( s \)-exact, \( s \rho \). These operators can be constructed as certain operators

\[ I(\mathcal{O}) = \int_M \mathcal{W}_{p,q} \wedge \mathcal{O}, \]

\[ (4.1) \]

where \( \mathcal{W}_{2-p,2-q}, (p, q = 0, 1, 2) \), is a \( (2-p, 2-q) \)-form on constructed out of the fields and
\( O \) is \((p, q)\)-form on \( M \). These are

\[
\begin{align*}
W_{0,0}^4 &= Tr\phi^2, \\
W_{0,1}^3 &= Tr(2\bar{\psi}\phi), \\
W_{0,2}^2 &= Tr(\bar{\psi} \wedge \bar{\psi} + 2F^{0,2}\phi), \\
W_{1,1}^2 &= Tr(2\bar{\psi} \wedge \psi + 2F^{1,1}\phi), \\
W_{1,0}^3 &= Tr(2\psi\phi), \\
W_{2,0}^2 &= Tr(\psi \wedge \psi + 2F^{2,0}\phi), \\
W_{2,1}^1 &= Tr(2F^{2,0} \wedge \bar{\psi} + 2F^{1,1} \wedge \psi), \\
W_{1,2}^1 &= Tr(2F^{0,2} \wedge \psi + 2F^{1,1} \wedge \bar{\psi}), \\
W_{2,2}^0 &= Tr(2F^{2,0} \wedge F^{0,2} + F^{1,1} \wedge F^{1,1}).
\end{align*}
\]

These arise from the components \( c^{(2,2)}_{-i} \) of the second Chern class, \( c_2 = Tr(\mathcal{F}\mathcal{F}) \), in the universal bundle.

There are the other observables obtained by Higgs fields,

\[
\begin{align*}
W_{2,1}^0 &= \partial_A \Phi^* \wedge F^{1,1} + \bar{\partial}_A \Phi \wedge F^{2,0}, \\
W_{2,0}^1 &= \partial_A \Phi^* \wedge \psi + \alpha F^{2,0} + \bar{\alpha} F^{2,0}, \\
W_{1,1}^1 &= \partial_A \Phi^* \wedge \bar{\psi} + \bar{\partial}_A \Phi \wedge \psi + \alpha F^{1,1} + \bar{\alpha} F^{1,1}, \\
W_{0,2}^1 &= \bar{\partial}_A \Phi \wedge \bar{\psi} + \alpha F^{0,2} + \bar{\alpha} F^{0,2}, \\
W_{1,2}^0 &= \bar{\partial}_A \Phi \wedge F^{1,1} + \partial_A \Phi^* \wedge F^{0,2}, \\
W_{1,0}^2 &= \partial_A \Phi^* \phi + \alpha\psi + \bar{\alpha}\bar{\psi}, \\
W_{2,0}^1 &= \bar{\partial}_A \Phi \phi + \alpha\bar{\psi} + \bar{\alpha}\psi, \\
W_{0,1}^3 &= \alpha\phi + \bar{\alpha}\phi.
\end{align*}
\]

These are obtained by \( c' = Tr(\mathcal{K}\mathcal{F}) \) arised from the dimensional reduction where \( \mathcal{K} = \partial_A \Phi^* + \bar{\partial}_A \Phi + \alpha + \bar{\alpha} \).

We can easily show that

\[
\partial W_{0,0}^4 = sW_{1,0}^3, \quad \bar{\partial} W_{0,0}^4 = sW_{0,1}^3,
\]

(4.4)
and one finds recursively
\[ \partial W_{p,q}^{4-p-q} = s W_{p+1,q}^{4-p-1-q}, \quad \bar{\partial} W_{p,q}^{4-p-q} = s W_{p,q+1}^{4-p-q} \]
\[ \partial W_{2,2}^0 = 0, \quad \bar{\partial} W_{2,2}^0 = 0. \] (4.5)

Then the integral \( I(\mathcal{O}) \) is BRST invariant, since
\[ s I(\mathcal{O}) = \int_M s W_{2-p,2-q} \wedge \mathcal{O} = \int_\gamma \partial W_{2-(p-1),2-q} \wedge \mathcal{O} = 0 \] (4.6)
or
\[ s I(\mathcal{O}') = \int_M s W_{2-p,2-q} \wedge \mathcal{O}' = \int_M \bar{\partial} W_{2-p,2-(q-1)} \wedge \mathcal{O}' = 0. \] (4.7)

Assume that \( \mathcal{O} = \partial \sigma \) or \( \mathcal{O} = \bar{\partial} \sigma \),
\[ I(\mathcal{O}) = \int_M W_{2-p,2-q} \wedge \partial \sigma = (-1)^{p+q} \int_M \partial W_{2-p,2-q} \wedge \sigma \]
\[ = (-1)^{p+q} \int_M s W_{2-(p+1),2-q} \wedge \sigma = s((-1)^{p+q} \int_M W_{2-(p+1),q} \wedge \sigma) \] (4.8)
or
\[ I(\mathcal{O}) = \int_M W_{2-p,2-q} \wedge \bar{\partial} \sigma = (-1)^{p+q} \int_M \bar{\partial} W_{2-p,2-q} \wedge \sigma \]
\[ = (-1)^{p+q} \int_M s W_{2-p,2-(q+1)} \wedge \sigma = s((-1)^{p+q} \int_M W_{2-p,2-(q+1)} \wedge \sigma). \] (4.9)

The expectation values of such observables vanish Therefore we can obtain the observables of topological BRST symmetry from the de Rham cohomology classes of \( M \).

Let us define the map
\[ \Psi_{\mathcal{O}} : \sum_{p+q=r} H^{p,q}(M) \to H^r(A^{1,1} \times \Omega^0), \] (4.10)
which takes the form
\[ \Psi_{\mathcal{O}} = \int_M W_{2-p,2-q} \wedge \mathcal{O} \] (4.11)
where \( \mathcal{O} \) is a \( \bar{\partial} \)-closed or \( \partial \)-closed \( (p, q) \)-form on \( M \). We can see that such a map corresponds to Donaldson map defining over cohomology.
5. Conclusion

We have shown that the vortex equations of \( n \)-dimensional Kähler manifold are given by dimensional reduction of Einstein-Hermitian equations of \((n + 1)\)-dimensional Kähler manifold. The topological invariant action over the vortex sector led to Yang-Mills-Higgs field theory with fermionic symmetry on closed Kähler manifold. Two kinds of observables were obtained. One are observables from the pure gauge fields and the others are observables coupled the gauge fields with the Higgs fields. These facts follow also dimensional reduction. However we need further to study the relation vortex moduli space over \( n \) dimension with Einstein-Hermitian moduli space over \( n+1 \) dimension in dimensional reduction.

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