Estimating Welfare Effects in a Nonparametric Choice Model: The Case of School Vouchers

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Abstract

We develop new robust discrete choice tools to learn about the average willingness to pay for a price subsidy and its effects on demand given exogenous, discrete variation in prices. Our starting point is a nonparametric, nonseparable model of choice. We exploit the insight that our welfare parameters in this model can be expressed as functions of demand for the different alternatives. However, while the variation in the data reveals the value of demand at the observed prices, the parameters generally depend on its values beyond these prices. We show how to sharply characterize what we can learn when demand is specified to be entirely nonparametric or to be parameterized in a flexible manner, both of which imply that the parameters are not necessarily point identified. We use our tools to analyze the welfare effects of price subsidies provided by school vouchers in the DC Opportunity Scholarship Program. We robustly find that the provision of the status quo voucher and a wide range of counterfactual vouchers of different amounts have positive benefits net of costs. This positive effect can be explained by the popularity of low-tuition schools in the program; removing them from the program can result in a negative net benefit. Relative to our bounds, we also find that comparable logit estimates potentially understate the benefits for certain voucher amounts, and provide a misleading sense of robustness for alternative amounts.

KEYWORDS: Discrete choice analysis, welfare analysis, demand analysis, nonparametrics, partial identification, price subsidy, school vouchers, Opportunity Scholarship Program.

JEL classification codes: C14, C25, D12, D61, I21.

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1 Introduction

Price subsidies are a common feature of many social programs that aim to encourage the use of certain alternatives or make them more affordable to disadvantaged populations. Important policy relevant examples include school vouchers that subsidize tuition for eligible private schools (Epple et al., 2017), subsidies on health insurance (Finkelstein et al., 2019), and price subsidies for various essential goods in developing countries (Dupas, 2014). Quantifying individuals’ willingness to pay for a price subsidy and its effects on demand are key inputs in performing cost benefit analyses of implemented subsidies and in their counterfactual design.

In this paper, our first contribution is to develop new discrete choice tools that show how to robustly learn about such welfare effects of a price subsidy given data with exogenous, discrete variation in prices. The starting point of our analysis is a nonparametric, nonseparable model of choice. In this model, we exploit the fact that our welfare parameters of interest can be expressed in terms of the demand for the various alternatives. The exogenous, discrete variation in prices—naturally arising in randomized evaluations of price changes—reveals the value of demand at the prices observed in the data. But, our parameters generally depend on values of demand beyond those observed in the data, which introduces an identification problem.

The traditional approach pursued in the literature to this problem is to consider parameterizations of demand through various models such as logit, probit, and mixed logit (e.g., Berry et al., 1995; McFadden, 1974; Train, 2009). Importantly, these parameterizations are carefully chosen such that they imply a unique demand function consistent with the data and hence such that the welfare parameters are point identified. However, a natural concern with this approach is that it may limit attention to only specific demand functions that can potentially drive the welfare estimates and resulting policy conclusions one draws from them.

To this end, our main methodological contribution is to show how to characterize what we can learn about our welfare parameters under more flexible specifications of demand. Our baseline specification leaves demand to be entirely nonparametric and only imposes a fundamental shape restriction that takes demand for each alternative to be increasing with the prices of other alternatives. In this case, there exists a space of infinite-dimensional demand functions consistent with the data, and hence that the parameters are generally only partially identified. The key complication is how to compute the sharp identified sets for the parameters generated by this space of functions. Our arguments show how to carefully exploit the geometric structure of the parameters as well as the information provided by the data and shape restrictions, such that the identified sets can be sharply computed using finite-dimensional optimization problems.

We also consider various extensions of this baseline result. In cases where the number of alternatives are large, the dimensions of these baseline optimization problems can be large and potentially impractical to compute. To ensure tractability in these cases, we show how to obtain
outer sets by considering specific sub-programs of the baseline ones as well as how to continue to
get sharp sets under additional separability assumptions on demand that reduce the dimensions of
the baseline programs. Moreover, we also show how to extend our baseline result to accommodate
additional parametric restrictions on demand. This is in the spirit of traditional methods, but
our analysis does not solely restrict attention to point identified demand functions and allows for
multiple parameterized demand functions to be consistent with the data.

The second contribution of the paper is to use the developed tools to perform a welfare analysis of
the price subsidy for eligible private schools provided by school vouchers, a program of active policy
debate. A large empirical literature has estimated the effects of vouchers on various outcomes using
data from programs that randomly allocate vouchers (e.g., Abdulkadiroğlu et al., 2018; Angrist
et al., 2002; Dynarski et al., 2018; Howell et al., 2000; Krueger and Zhu, 2004; Mayer et al., 2002;
Mills and Wolf, 2017; Muralidharan and Sundararaman, 2015; Wolf et al., 2010). However, as
surveyed in Epple et al. (2017), the evidence from these studies is mixed: some find positive effects,
while others find null or even negative effects. Our motivation arises from the fact that, despite this
mixed evidence on the effects on outcomes, the data across these studies indicate that a non-trivial
proportion of recipients choose to use the voucher. Revealed preference arguments then suggest
that recipients in general value vouchers and hence that vouchers may be welfare-enhancing. Yet,
little empirical work has attempted to quantify these welfare benefits and analyze whether they
can justify the costs of providing vouchers.

We apply our tools to data from the DC Opportunity Scholarship Program (OSP), a voucher
program in Washington, DC. Due to oversubscription, the program randomly allocated the voucher
to participants, which induces binary variation in prices, namely the prices of schools with and
without the application of the status quo voucher. Our estimated bounds reveal that provision of
the status quo amount of $7,500 as well as a range of counterfactual amounts can have a positive
welfare benefit net of the costs the government faces to provide them. In addition to being positive,
they reveal that potentially large net benefits are consistent with the data. We find that these
conclusions are robust to a range of alternative cost values as well as when we account for the fact
that individuals may be liquidity constrained and not able to afford all schools.

We find that these positive effects can be driven by the fact that there are a large number
of popular, low-tuition schools in the program. Counterfactuals measuring how the welfare effects
would change if these schools were removed from the program reveal that absent schools with tuition
at most $3,500, the program may have a negative net benefit. Intuitively, this suggests that these
schools induce a high welfare benefit for recipients relative to the net costs the government faces to
fund a voucher when redeemed at them. Indeed, a key rationale for school vouchers is that they may
subsidize private schools that provide services individuals value more efficiently than government-
funded schools (Friedman, 1962). A closer look at these schools reveals that the majority of them
are religious, and specifically Catholic, suggesting program participants particularly value this
component, in line with prior evidence that a school being Catholic is an important dimension affecting school choice (Altonji et al., 2005; Trivitt and Wolf, 2011). In fact, over half of the voucher recipients who choose to take up the voucher do so by redeeming it at a Catholic school.

When interpreting our findings on the benefits of vouchers, it is worth highlighting certain features of our analysis. Our analysis equates welfare with willingness to pay of individuals, and particularly parents who often make schooling decisions for their child, for the subsidy on school prices induced by the voucher. While a natural money metric for the welfare benefits, parents’ willingness to pay is of course only a proxy for the welfare benefits for students, as parents may not always know how to choose schools that are best for their child. Moreover, our analysis only captures the effects of vouchers through the decrease in school prices it induces, and not the potential equilibrium effects on the school system it may have. Consequently, it only informs the effect of marginal policies that provide a voucher to a student who applied to the program but was not admitted, and not of those that scale up the program. It is therefore important to emphasize that our results provide a partial picture on the overall welfare effects of vouchers, and one should be cautious when drawing broader policy conclusions based on them.

We also compare our empirical results to those one would obtain under various standard logit parameterizations. In general, these parameterizations imply demand estimates that match well the binary variation in enrollment shares induced by the receipt of the voucher. However, we find that they do not capture the range of demand functions credibly consistent with these shares, but limit attention to relatively price-inelastic functions—a feature of logit similarly documented in several other empirical settings (e.g., Compiani, 2022; Ho and Pakes, 2014; Tebaldi et al., 2021). Comparing to our bounds, we find this feature corroborates our concern that the parameterizations can drive the welfare effects towards certain values and provide a misleading picture of the true effects. Specifically, for voucher amounts where our estimates reveal net benefits, the logit estimates can understate their magnitude by systematically taking values close to our lower bounds. Alternatively, for the remaining amounts where our estimates highlight that the data is not sufficiently rich to robustly imply positive effects, the logit estimates provide a false sense of robustness on the benefits of vouchers by unambiguously predicting positive effects.

In the following subsection we describe the relation of our analysis to the literature, after which the remainder of the paper is organized as follows. Section 2 describes our setup and identification problem. Section 3 develops our procedures to compute the identified set. Section 4 applies our tools to analyze the welfare effects of school vouchers in the OSP. Section 5 compares our empirical results to those using traditional parametric methods. Section 6 concludes. Proofs of all results and additional details pertinent to the analysis are presented in the Supplementary Appendix. A Python package to implement our developed tools is available at https://github.com/vishalkamat/npdemand.
1.1 Related Literature

A growing literature studies nonparametric identification of various quantities in discrete choice settings. One approach pursued in this literature is to argue point identification, which is often based on requiring large amounts of exogenous variation in the data (e.g., Berry and Haile, 2009, 2014; Briesch et al., 2010; Chiappori and Komunjer, 2009; Matzkin, 1993). However, in many applications such as the ones we focus on, there exists only discrete variation, which generally gives rise to the case of partial identification. A number of recent papers have developed tools to evaluate various questions—such as estimating the effect of different prices and choice sets on demand, characterizing the underlying utility functions, and testing the premise of utility maximization—in setups that permit partial identification (e.g., Chesher et al., 2013; Kamat, 2021; Kitamura and Stoye, 2018; Manski, 2007; Tebaldi et al., 2021). As in our analysis, these papers carefully exploit the specific structure of their models and parameters to show how to construct the sharp identified set. But, as our setup and the parameters of interest are different from theirs, the developed arguments are distinct and complementary.

Our analysis is most closely related to the recent work in the literature on nonparametric welfare analysis. A building block of our analysis is the fact that we can express the average willingness to pay in terms of demand. To show this, we apply results from Bhattacharya (2015, 2018) who formally derived such expressions for the class of nonparametric choice models we consider.\footnote{In Appendix S.2, we also extend the arguments to show the validity of such expressions in cases with liquidity constraints, a result of potential independent interest.} If demand is point-identified, we can directly apply these results to identify the welfare effects of interest. Our novelty is to show how to exploit these results when demand might be only partially identified. Recently, Bhattacharya (2021) derives analytic nonparametric bounds for welfare effects in such cases for a binary choice problem with a single price dimension. As in our approach, the paper’s arguments are based on demand functions constant over a carefully constructed partition of the space of prices. However, as highlighted in Section 3.1.1, the arguments behind the construction of this partition rely on the unidimensionality of the space and require novel extensions to generalize to the case of multiple alternatives and prices we consider in our setup.

Our analysis is also conceptually related to the work of Mogstad et al. (2018) in an alternative setting of a binary treatment model. Their identification problem shares a similar structure where the parameters of interest can be expressed in terms of primitive functions—marginal treatment effects in their setup—that are only partially identified by the data. Indeed, our approach to incorporate parametric restrictions follows that in this paper. In contrast, as highlighted in Section 3.1.1, their arguments to compute nonparametric bounds rely again on the unidimensionality of their primitive functions, which arises due to the focus on a binary treatment. In this sense, our arguments which allow for multidimensional functions can provide insights to obtain nonparametric bounds in settings with multiple treatments (e.g., Kamat et al., 2022).
Our empirical analysis contributes to the literature on the evaluation of school voucher programs. Our choice based welfare analysis complements the large number of work cited above that primarily focuses on the effects of vouchers on outcomes. A smaller group of papers uses choice models to study various voucher-related school choice questions of interest (e.g., Allende, 2019; Arcidiacono et al., 2016; Carneiro et al., 2019; Gazmuri, 2019; Neilson, 2013). While these papers consider richer models that allow studying various effects of vouchers—such as equilibrium effects on the school setting—that generally go beyond the scope of our analysis, they do so using fully parameterized models. Our analysis complements these studies by evaluating a narrower, yet relevant question, but doing so using robust nonparametric tools.

2 Setup

2.1 Model

Let $J$ be a discrete set of choice alternatives such that $|J| \geq 2$. For each individual $i$, suppose that we observe $(D_i, P_i)$, where $D_i$ denotes the chosen alternative from $J$, and $P_i = (P_{ij} : j \in J)$ denotes a vector of prices for each alternative that the individual faces. Let $P_{obs}$ denote the support of the observed price vector, which we assume to be discrete. Certain alternatives potentially may not exhibit any price variation in which case we normalize their prices to 0. We assume the observed choice to be the product of a utility maximizing decision. Specifically, denoting by $Y_i$ the individual’s disposable income and by $U_{ij} : \mathbb{R} \to \mathbb{R}$ their (indirect) utility function for alternative $j \in J$, we take the observed choice to be given by

$$D_i = \arg \max_{j \in J} U_{ij}(Y_i - P_{ij})$$

i.e. the alternative maximizing the utility of the disposable income net of the price paid for that alternative.

Apart from the utility maximizing structure, we highlight that our choice model is nonparametric and nonseparable that does not impose any additional restrictions and allows for completely general unobserved heterogeneity. This is in contrast to traditional models employed in the literature that impose a combination of additional restrictions such as functional forms on the utility and parametric distributions on the unobserved heterogeneity—see Section 5 for details. A limitation of our setup, however, is that we do not model a supply side that generates prices as well as other factors beyond prices that may affect choice. This has potential implications on the interpretation and scope of our counterfactuals. We highlight this feature more concretely in the context of our application in Section 4.

Given the above structure, our analysis exploits the fact that our parameters of interest can be expressed in terms of demand functions. In turn, we frame our problem in terms of these
functions and consider various assumptions directly on them. The demand functions correspond to the distribution of choices across individuals at a given value of the price vector. More formally, let \( \mathcal{P} = \prod_{j \in \mathcal{J}} [p_j, \bar{p}_j] \) denote the domain of price vectors and let

\[
D_i(p) = \arg \max_{j \in \mathcal{J}} U_{ij}(Y_i - p_j)
\]

denote the individual’s choice had the price vector been counterfactually set to \( p \in \mathcal{P} \). Using this additional notation, we can respectively define the unconditional and conditional on \( P_i = p' \in \mathcal{P}_{\text{obs}} \) demand by

\[
q_j(p) = \text{Prob}\{D_i(p) = j\}, \quad (2)
\]

\[
q_j(p|p') = \text{Prob}\{D_i(p) = j|P_i = p'\} \quad (3)
\]

for each \( j \in \mathcal{J} \) and \( p \in \mathcal{P} \).

Our analysis is primarily based on the unconditional demand functions. But we also define conditional demand functions as they allow us to formally state the fact that our analysis throughout takes the observed variation in prices to be exogenous. In particular, we do so by assuming the following relation between the conditional and unconditional demand functions:

**Assumption E. (Exogeneity)** For each \( j \in \mathcal{J} \), \( q_j(p) = q_j(p|p') \) for all \( p \in \mathcal{P} \) and \( p' \in \mathcal{P}_{\text{obs}} \).

This assumption states that demand is invariant to values of the observed price vector, and in turn captures that the observed price is exogenous of the remaining underlying variables affecting choices. It follows from this assumption that conditional and unconditional demand are equal, and hence that the underlying demand functions can be uniquely captured by the vector \( q \equiv (q_j : j \in \mathcal{J}) \) of unconditional demand functions. As a result, in the remainder of our analysis, we focus solely on the unconditional demand; whenever we refer to demand, it is understood we are referring to the unconditional demand.

In our analysis, we also consider various additional assumptions on demand, which then restrict \( q \) to lie in some space of functions. Let \( \mathbf{F} \) generically denote this restricted space of functions. We postpone the description of these assumptions and their resulting \( \mathbf{F} \) to after we present our parameters of interest and state the objective of our analysis, as they will better motivate the purpose of the considered assumptions.

### 2.2 Parameters of Interest

We are interesting in evaluating the welfare effect of a price subsidy that decreases prices between two pre-specified price vectors.\(^2\) Let \( p^a, p^b \in \mathcal{P} \) respectively denote the larger and smaller pre-specified vectors in this price decrease in the sense that \( p^b_j \leq p^a_j \) for \( j \in \mathcal{J} \).

\(^2\)The welfare effects for general price changes cannot be simply expressed in terms of demand defined in (2), but rather require defining demand at counterfactual prices as well as disposable income—see Bhattacharya (2015, 2018).
We measure the welfare effect of the price subsidy by the willingness to pay for it. It provides a natural money metric for the gains and, equivalently, corresponds to the negative of the compensating variation for the price decrease induced by the subsidy. Formally, an individual’s willingness to pay for the subsidy can be defined by the variables $B_{a,b}^i$ that solves

$$\max_{j \in J} U_{ij}(Y_i - p_j^a) = \max_{j \in J} U_{ij}(Y_i - p_j^b - B_{a,b}^i) ,$$  

i.e. the amount of money to be subtracted from the individual’s income under the lower price so that they are indifferent and obtain the same utility as that under the higher price. Our analysis focuses on the average willingness to pay which is defined by

$$E[B_{a,b}^i] .$$

As mentioned, our analysis exploits the fact that our parameters can be expressed as functions of the demand functions. In order to show this for (5), we exploit results from Bhattacharya (2015, 2018) who precisely showed this in the context of a nonparametric, nonseparable model of choice as that in (1). In the following proposition, we reproduce this result in terms of our notation. To this end, it is useful to first introduce some additional notation. Let $\Delta_{a,b}^l \leq \ldots \leq \Delta_{a,b}^{|J|}$ denote the ordered values of $p_j^a - p_j^b$ across $j \in J$, i.e. the price decrements for the different alternatives, and let $J_{a,b}^l = \{j \in J : p_j^a - p_j^b \geq \Delta_{a,b}^l\}$ denote the alternatives whose price decrease is at least greater than the $l$th ordered price decrement. Moreover, with some abuse of notation, let $\min\{p_j^a, p_j^b + t\} = (\min\{p_j^a, p_j^b + t\} : j \in J)$ for $t \in \mathbb{R}$ denote the element wise minimum. Using this notation, we can then formally state the result as follows.

**Proposition 1.** For each individual $i$, suppose $U_{ij}$ is continuous and strictly increasing for each $j \in J$. Then we have that $B_{a,b}^i$ defined in (4) exists and is unique, and that

$$E[B_{a,b}^i] = |J| - \sum_{l=1}^{\Delta_{a,b}^{|J|}} \int_{\Delta_{a,b}^{l-1}}^{\Delta_{a,b}^l} q_j \left( \min\{p_j^a, p_j^b + t\} \right) dt .$$

Proposition 1 requires utility to be increasing, which is captured by our restrictions on demand. Moreover, it requires them to be continuous. This rules out, for example, cases where a change in income may affect the availability of an alternative and hence discontinuously affect utility—see Section 4.2 for more concreteness on its empirical relevance and Appendix S.2 for an extension to such cases. To intuitively understand the expression in (6), observe for the $l$th price decrement that the price decrease for the alternatives in $J_{a,b}^l$ jointly goes from $\Delta_{a,b}^l$ to $\Delta_{a,b}^{l+1}$. As this price decrease can simply be viewed as a cash transfer conditional on choosing alternatives in $J_{a,b}^l$, the willingness

Identification in this case therefore not only requires variation and assumptions along the price dimension but also along that of disposable income, which we leave for future work. We note, however, that our analysis straightforwardly applies to evaluate the effects of general price changes solely on demand, i.e. (7) with $g_{a,b} = 0$. 

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to pay for it can potentially be only between the minimum and maximum value of the transfer, namely \( \Delta_{a,b}^l \) and \( \Delta_{a,b}^{l+1} \). The expression in (6) in turn states that the average willingness to pay for the \( l \)th decrement corresponds to the area under the demand curve for the alternatives in \( J_{a,b}^l \) as prices jointly vary between the minimum and maximum values in the presence of the transfer, and the total average willingness to pay is the sum across all the decrements—see Bhattacharya (2015, Section 2.1) for more discussion on the intuition for the above expression.

In addition to the above parameter, we are also interested in parameters that evaluate the effect of the price subsidy on demand. Moreover, we are interested in those that measure the difference in the welfare effect and a weighted change in demand, which can for example allow us to compare the benefits and costs of the subsidy as we do in our application. To this end, our analysis allows for a general class of parameters that can be expressed as functions of \( q \) as follows

\[
\theta(q) = g_{a,b} \sum_{l=1}^{\lvert J \rvert - 1} \sum_{j \in J_{a,b}^{l+1}} \int \Delta_{a,b}^{l+1} q_j \left( \min\{p^a, p^b + t\} \right) dt + \sum_{j \in J} g_j^a q_j(p^a) + g_j^b q_j(p^b) ,
\]

where \( a, b \), \( \{g_j^a : j \in J\} \) and \( \{g_j^b : j \in J\} \) are pre-specified values, i.e. linear combination of the expression in (6) and demand evaluated at \( p^a \) and \( p^b \). Indeed, by specifying \( g_{a,b} = 1 \) and \( g_j^a = g_j^b = 0 \) for \( j \in J \), we have the parameter in (6). Similarly, by specifying other values for \( g_{a,b} \), \( \{g_j^a : j \in J\} \) and \( \{g_j^b : j \in J\} \), we can analyze a range of additional parameters that capture the effect on demand as well as the potential cost incurred from the price subsidy—see our application in Section 4 for more concreteness.

2.3 Identified Set

The goal of the analysis is to learn about a pre-specified parameter of interest \( \theta(q) \) given by (7). Given the function \( \theta \) is known, what we can learn about the parameter translates to what we know about \( q \) through the data and imposed assumptions. From the data, we observe the distribution of \( (D_i, P_i) \), which for the purposes of the identification analysis is assumed to be perfectly known without uncertainty—we discuss estimation and inference in Section 4. Given the structure in (1), the definition of demand in (2)-(3) and Assumption E, it follows that demand must satisfy

\[
q_j(p) = \text{Prob}[D_i = j|P_i = p]
\]

for \( j \in J \) and \( p \in P_{\text{obs}} \), i.e. the random variation in prices reveals the value of demand at prices observed in the data. From the assumptions, we have that demand is restricted to lie in a space of functions \( F \). The admissible space of demand that satisfies the data and assumptions can be defined by

\[
Q = \{q \in F : q \text{ satisfies (8)}\} .
\]
What can be learned about the parameter of interest can then be formally captured by the identified set, which is defined by

$$ \theta(Q) = \{ \theta_0 \in \mathbb{R} : \theta(q) = \theta_0 \text{ for some } q \in Q \} \equiv \Theta, $$

(10)
i.e. the image of the space of admissible functions $Q$ under the function $\theta$. By construction, the identified set sharply captures all that we can learn about the parameter given the data and assumptions. It permits the parameter to be point identified in which case the identified set corresponds to a single point. Alternatively, if the parameter is partially identified, the identified set corresponds to the sharpest set of all possible parameter values consistent with the data and assumptions. Our objective is to compute the identified set under the assumptions we impose on demand, which we describe next.

### 2.4 Demand Specifications

Given the nonparametric nature of our model, the demand functions remain entirely unrestricted apart from the logical ones arising from the fact that they are distributions, namely

$$ q_j(p) \geq 0 \text{ for } j \in \mathcal{J}, $$

(11)

$$ \sum_{j \in \mathcal{J}} q_j(p) = 1, $$

(12)

for $p \in \mathcal{P}$, i.e. each demand function is positive and they sum together to one. On the other hand, observe that while the data restrictions in (8) reveal the value of demand at certain price points, the parameters of interest generally depend on values of demand beyond these prices. In order to reach informative conclusions, our analysis therefore considers additional assumptions that restrict how the demand functions vary with prices.

We consider two specifications of demand which define the restricted space of functions $\mathcal{F}$ and in turn the admissible set of functions $Q$ in (9). Our baseline specification imposes the following nonparametric shape restriction:

**Assumption B.** (Baseline) For each $j \in \mathcal{J}$, $q_j$ is weakly increasing in $p_m$ for each $m \in \mathcal{J} \setminus \{ j \}$.

This assumption, referred to as weak substitutes in Berry et al. (2013), is a fundamental shape restriction present in the majority of discrete choice models and is implied by taking $U_{ij}$ to be increasing for each $j \in \mathcal{J}$. It specifically imposes that for each $p, p' \in \mathcal{P}$ such that $p_j > p'_j$ for $j \in \mathcal{J}' \subseteq \mathcal{J}$ and $p_j = p'_j$ for $j \in \mathcal{J} \setminus \mathcal{J}'$, we have that

$$ q_j(p) \geq q_j(p') $$

(13)
for each $j \in \mathcal{J} \setminus \mathcal{J}'$. Under this specification, observe that the restricted space for demand is given by $F_B = \{ q \in \bar{F} : q \text{ satisfies } (11) - (13) \}$, and, in turn, the admissible space of functions in (9) by

$$Q_B = \{ q \in \bar{F} : q \text{ satisfies } (11) - (13) \text{ and } (8) \} ,$$

(14)

where $\bar{F}$ denotes the set of all functions from $\mathcal{P}$ to $\mathbb{R}^{|\mathcal{J}|}$.

Our second, auxiliary specification imposes an additional functional form restriction on demand. As elaborated in Section 5, this is in the spirit of traditional methods, whose analysis is based on imposing specific functional forms on demand. We consider the following general class of functional forms on demand:

**Assumption A. (Auxiliary)** For each $j \in \mathcal{J}$,

$$q_j(p) = \sum_{k=0}^{K_j} \alpha_{jk} \cdot b_{jk}(p)$$

(15)

for some unknown parameters $\{\alpha_{jk} : 0 \leq k \leq K_j\}$, where $\{b_{jk} : 0 \leq k \leq K_j\}$ denote some known functions.

This assumption imposes that demand is a linear function of some basis of prices, where the variable $\alpha \equiv (\alpha'_j : j \in \mathcal{J})'$ with $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jK_j})$ parameterizes the demand functions. As observed in Section 3.2, we focus on linear functions for their computational benefits. The assumption allows for a range of flexibility through the choice of $b_{jk}$ and $K_j$. It allows demand to be point identified in special cases, loosely when the number of unknown parameters in $\alpha_j$ for each $j \in \mathcal{J}$ is taken to be equal to the cardinality of the support of observed price variation. This is analogous to traditional methods that restrict the number of parameters in this manner to ensure point identification. However, the above assumption also allows for more general cases, where these functions may not be point-identified. Under this specification, observe that the restricted space for demand is given by $F_A = \{ q \in \bar{F} : q \text{ satisfies } (11) - (13) \text{ and } (15) \}$, and, in turn, the admissible space of functions in (9) by

$$Q_A = \{ q \in \bar{F} : q \text{ satisfies } (11) - (13), (15) \text{ and } (8) \} .$$

(16)

### 3 Identification Analysis

In this section, we develop procedures that show how to compute the identified set $\Theta$ in (10) under each of our specifications: first, in Section 3.1, under our baseline specification, i.e. when $Q = Q_B$ in (14); and then, in Section 3.2, under our auxiliary specification, i.e. when $Q = Q_A$ in (16).
3.1 Identified Set under Baseline Nonparametric Specification

In principle, the identified set in (10) can be computed by searching over $q$ in $Q$ and taking their image under the function $\theta$. However, under the baseline specification, this problem is infeasible as $Q_B$ is an infinite-dimensional space. To this end, the main idea behind our proposal is to show how to replace $Q_B$ by a finite-dimensional space $Q_{fd}^B$ such that there is no loss of information in the sense that $\theta(Q_B) = \theta(Q_{fd}^B)$. This allows us to compute the identified set by searching only through $q$ in $Q_{fd}^B$, which is a finite-dimensional problem and, hence, potentially feasible.

In particular, taking $W$ to be a finite partition of $P$, the finite dimensional space of functions we consider is given by

$$Q_{fd}^B = \left\{ q \in Q_B : q_j(p) = \sum_{w \in W} 1_w(p) \cdot \beta_j(w) \text{ for some } \{\beta_j(w)\}_{w \in W} \text{ for each } j \in J \right\},$$

(17)

where $1_w(p) \equiv 1\{p \in w\}$ and $\{\beta_j(w) : w \in W, j \in J\}$ are unknown parameters, i.e. a subset of $Q_B$ such that each $q$ is parameterized to be constant over the elements of $W$. The main challenge here is how to choose the partition $W$ such that we have $\theta(Q_B) = \theta(Q_{fd}^B)$. As we will observe below, we carefully do so such that the resulting $q$ is sufficiently rich to define the parameter of interest and data restrictions as well as preserve the information provided by the shape restrictions.

3.1.1 Partitioning the Space of Prices

Denoting by

$$P_{a,b}^l = \{ p \in \mathcal{P} : p_j = \min\{p_j^a, p_j^b + t\} \text{ for } t \in (\Delta_{l-1}^a, \Delta_{l+1}^b), \ j \in J \} \text{ for } 1 \leq l \leq |J| - 1,$$

(18)

$$\{p^a\} \text{ and } \{p^b\}$$

(19)

the various sets of prices that play a role in the definition of the parameter in (7), and by

$$\mathcal{P}_{obs}$$

(20)

the set of prices that play a role in the definition of the data restrictions in (8), let

$$\left\{ P_{a,b}^l : 1 \leq l \leq |J| - 1 \right\} \cup \left\{ p : p \in \{p^a, p^b\} \cup \mathcal{P}_{obs} \right\} \equiv \{\mathcal{P}_1^*, \ldots, \mathcal{P}_L^*\}$$

(21)

denote the collection of price sets that plays a role in the definition of the parameter and the data restrictions. Given these sets of prices, we define a collection of sets that is the key building block in our construction of $W$.

**Definition V.** (Partition $\mathcal{V}$) Let $\mathcal{V}$ denote a finite partition of $\mathcal{P}^* = \bigcup_{l=1}^L \mathcal{P}_l^*$ such that

(i) For each $l \in \{1, \ldots, L\}$, there exists $\mathcal{V}_l \subseteq \mathcal{V}$ such that $\mathcal{P}_l^* = \bigcup_{v \in \mathcal{V}_l} v$;
Figure 1: Various sets of prices for an example with $J = \{0, 1, 2\}$ and $P_{\text{obs}} = \{p'\}$, where $p_2^a - p_2^b > p_1^a - p_1^b$ and the price of alternative 0 is normalized to 0

\begin{align*}
\{p_a\} &\quad \{p_b\} &\quad \{p'\} \\
p_2^a &\quad p_2^b &\quad p_2' \\
p_1^a &\quad p_1^b &\quad p_1' \\
p_0^a &\quad p_0^b &\quad p_0'
\end{align*}

(a) Sets playing a role in the definition of $P^*$

(b) $V \equiv \{v_1, \ldots, v_7\}$ satisfying Definition V

- (ii) For each $v \in V$, we have that $v_{[j]} \equiv \{t \in \mathbf{R} : p_\cdot = t \text{ for some } p \in v\}$ is an interval for each $j \in J$; and

- (iii) For all $v, v' \in V$, we have either $v_{[j]} = v'_{[j]}$ or $v_{[j]} \cap v'_{[j]} = \emptyset$ for each $j \in J$.

Definition V states that $V$ is a finite partition of the union of the sets in (18)-(20) such that its elements satisfy certain properties: (i) requires that the elements can be used to build, by taking their unions, the sets in (18)-(20); (ii) requires each element to be connected in each coordinate; and (iii) requires any pair of elements to either completely overlap or be disjoint in each coordinate. Intuitively, we highlight that the first property, as the sets in (18)-(20) are based on the parameter of interest and data restrictions, is what ensures that our resulting finite-dimensional $q$ will be sufficiently rich to define the parameter and data restrictions. On the other hand, the latter two properties, which implies that the sets can be ordered and pairwise compared across each coordinate, is what ensures that our $q$ will preserve the information provided by the shape restrictions in (13), which we can observe are based on pairwise comparisons of prices.

To better understand the various sets of prices, Figure 1(a) first graphically illustrates those in (18)-(20) in the context of an example with three alternatives. Figure 1(b) then illustrates a partition of the union of the sets in Figure 1(a) to obtain a collection of sets satisfying Definition V. In particular, it breaks up any two sets in Figure 1(a) that partially overlap in a given coordinate such that Definition V(iii) is satisfied. In Appendix S.1.1, we formally describe how the partition can more generally be obtained in such a manner.
Using $V$, we can now construct our $W$. To this end, observe first that for each $j \in J$, the collection of sets determined by the prices in $v \in V$ in the $j$th coordinate, i.e. $\{v[j] : v \in V\}$, generates a partition of $[p^b_j, p^b_j] \cup \{p_j : P_{obs} \} \subseteq [\bar{p}_j, \bar{p}_j]$.\footnote{Note that stating $\{v[j] : v \in V\}$ is a partition of $[p^b_j, p^b_j]$ is not formally correct as the sets in (18) are open and hence their end points are not necessarily contained in the partition. To formally ensure it is partition, we need to carefully alter the boundaries of certain $v \in V$ to be either closed or open. However, for expositional ease, we abstract away from doing so as this distinction is not practically important for our analysis as our parameter of interest in (7) only takes Lebesgue integrals over these sets.} Moreover, observe that the collection of intervals between the observed price values outside $[p^b_j, p^b_j]$ generates a partition of $[\bar{p}_j, \bar{p}_j]$ with the ordered values of $p_j : P_{obs}, p_j \leq p^b_j \cup \{p^b_j\}$ and $p_j : P_{obs}, p_j \geq p^b_j \cup \{p^b_j\}$, respectively, generates a partition of $[\bar{p}_j, \bar{p}_j] \setminus ([p^b_j, p^b_j] \cup \{p_j : P_{obs}\})$. In turn, we together have that

$$V_j \equiv \left\{ \left( p^b_j, p^b_j \right) \right\} \cup \left\{ \left( \bar{p}_{m-1,j}^*, \bar{p}_{m,j}^* \right) \right\}_{m=2}^{M_j} \cup \left\{ v[j] : v \in V \right\} \cup \left\{ \left( \bar{p}_{m-1,j}^*, \bar{p}_{m,j}^* \right) \right\}_{m=2}^{M_j} \cup \{ p^*_j, \bar{p}_j \}$$

(22) generates a partition of $[\bar{p}_j, \bar{p}_j]$, i.e. the space of prices along the $j$th coordinate. We then take $W$ to be the Cartesian product of $V_j$ across the different coordinates, i.e.

$$W = \prod_{j \in J} V_j \equiv \{ w_1, \ldots, w_M \}.$$  

(23)

It is useful to highlight that our construction of $W$ simplifies in the case where demand is unidimensional—essentially arising when all alternatives except one have prices normalized to 0. As noted in Section 1.1, this special case is similar to the identification problem studied in Bhattacharya (2021) and Mogstad et al. (2018), who propose a finite dimensional space of functions comparable to that in (17) as a solution. In this case, we do not require to impose $V$ to additionally satisfy Definition $V(iii)$ as it is automatically implied by the fact that $V$ is a partition. In contrast, in the multidimensional case, this is not the case and, hence, we need to explicitly introduce it to ensure that the information provided by the shape restrictions in (13) is preserved. Moreover, given $V$, the construction of $W$ follows more straightforwardly in the unidimensional case as the sets in $V$ and those outside it, i.e. those in (22), directly generate a partition of the space of prices for a single coordinate. For the multidimensional case, an additional complication remains of how to combine these one dimensional partitions to partition the entire space of prices, which we propose to solve by taking their Cartesian product as in (23).

### 3.1.2 Equivalent Finite Dimensional Characterization

Given our constructed $W$, we next show that replacing the infinite-dimensional $Q_B$ with the finite-dimensional $Q_B^{fd}$ in (17) leads to no loss of information, i.e. $\theta(Q_B) = \theta(Q_B^{fd})$. Furthermore, we show that $\theta(Q_B^{fd})$ can be computed by two finite-dimensional optimization problems.
In order to state this result, it is useful to first rewrite $\theta \left( Q_{fd}^B \right)$ in terms of $\beta \equiv \left( \beta_j' : j \in J \right)'$, where $\beta_j = (\beta_j (w_1), \ldots, \beta_j (w_M))$, i.e. the variable parameterizing $q \in Q_{fd}^B$. To this end, observe that given $\theta$ is continuous in $q$ and that $q$ is continuous in $\beta$, there exists a continuous function $\theta_B$ of $\beta$ such that $\theta (q) = \theta_B (\beta)$. Similarly, observe that $Q_B$ can also be written in terms of $\beta$ by

$$B = \left\{ \beta \in \mathbb{R}^{d_{\beta}} : \sum_{w \in W} 1_w \cdot \beta_j (w) : j \in J \right\} \in Q_B ,$$

where $d_{\beta}$ denotes the dimension of $\beta$, i.e. the set of values of $\beta$ that ensure that the corresponding $q$ is in $Q_B$. Then, we can write $\theta \left( Q_{fd}^B \right)$ in terms of $\beta$ by

$$\{ \theta_0 \in \mathbb{R} : \theta_B (\beta) = \theta_0 \text{ for some } \beta \in B \} \equiv \Theta_B .$$

In the following proposition, we show that $\theta (Q_B)$ is equal to $\Theta_B$, and that $\Theta_B$ can be computed by two finite-dimensional optimization problems.

**Proposition 2.** Suppose that $Q = Q_B$. Then, the identified set in (10) is equal to that in (25), i.e. $\Theta = \Theta_B$. In addition, if $B$ is empty then by definition $\Theta_B$ is empty; whereas, if $B$ is non-empty then $\Theta_B = [\theta_B, \bar{\theta}_B]$, where

$$\theta_B = \min_{\beta \in B} \theta_B (\beta) \text{ and } \bar{\theta}_B = \max_{\beta \in B} \theta_B (\beta) .$$

Proposition 2 shows that the identified set when not empty is given by a closed interval, where the endpoints can be obtained by solving two optimization problems. In the proof of the proposition, we explicitly derive $B$, the constraint set of these optimization problems, and observe that it is determined by constraints that are all linear in $\beta$. We also derive $\theta_B$, the objectives of these optimization problems, and observe that it is linear in $\beta$. These two observations then imply that the optimization problems are linear programs, a useful observation in their implementation. Lastly, observe that to compute the identified set using these linear programs, we specifically require that $B$ is non-empty or, equivalently, that the model is correctly misspecified. However, when this is not the case, the linear programs automatically terminate.

### 3.1.3 Dimension Reduction

While the problems in (26) are linear programs, they can nonetheless be computationally expensive when the dimension of the optimizing variable $\beta$ is large. Such a case arises especially when $|J|$ is large as in our application where it is equal to 70. To ensure tractability in such cases, we conclude this subsection by considering two lower-dimensional linear programs that are easier to compute and can continue to allow us to learn about our parameter.
Our first proposal considers sub-programs of (26) that obtain outer sets containing $\Theta_B$. In particular, given how $V$ was constructed, observe that the following subset of $W$

$$W'^r = \left\{ w \in W : w = \prod_{j \in \mathcal{J}} v_{[j]} \text{ for some } v \in V \right\} \equiv \{ w_1^r, \ldots, w_M^r \} ,$$

captures the sets of prices that play a role in the definition of the parameter, and in turn the subvector of $\beta$ defined over these sets given by

$$\beta'^r = \left( \beta'^r_j : j \in \mathcal{J} \right)' \equiv \phi(\beta),$$

is sufficient in determining $\theta_B$ in the sense that there exists a linear function $\theta'^r_B$ such that $\theta'^r_B(\beta'^r) = \theta_B(\beta)$. The lower-dimensional linear programs we consider are those in terms of the subvector $\beta'^r$ given by

$$\hat{\theta}^r_B = \min_{\beta'^r \in B'} \theta'^r_B(\beta'^r) \quad \text{and} \quad \overline{\theta}^r_B = \max_{\beta'^r \in B'} \theta'^r_B(\beta'^r) , \quad (27)$$

where $B'$ denotes a set of $\beta'^r$ determined by linear constraints. Indeed, if $B' = \phi(B)$, we have by construction that these programs are equivalent to those in (26). In turn, by taking $B'$ to be such that $\phi(B) \subseteq B'$, it follows that we have $\hat{\theta}^r_B \leq \theta_B$ and $\overline{\theta}^r_B \geq \theta_B$, and hence obtain an outer set for $\Theta_B$, i.e. $\Theta_B \in [\hat{\theta}^r_B, \overline{\theta}^r_B]$. In Appendix S.1.2, we provide a natural choice of such a $B'$ determined by restrictions on $\beta'^r$ implied by those in $B$, which we find in our empirical analysis can be tractably implemented and result in informative conclusions.

Our second proposal is to additionally impose separability on the demand functions given which we can sharply compute the identified set in tractable manner. In our empirical analysis, we specifically consider the following separability assumption that imposes that demand is a sum of lower-dimensional functions:

**Assumption S.** (Separability) For each $j \in \mathcal{J}$, $q_j(p) = \sum_{m \in \mathcal{J}} h_{jm}(p_m)$ for some unknown functions $\{ h_{jm} : m \in \mathcal{J} \}$.

Assumption S imposes the demand for each alternative to be additively separable in prices of all the alternatives. In Appendix S.1.3, we consider a more general separability assumption and show how Proposition 2 can be extended such that we can similarly use two linear programs as in (26) to compute the identified set under these additional assumptions. Importantly, we also highlight here that by requiring demand to be composed of lower dimension functions, the dimension of the optimizing variable in these programs can be substantially smaller than those in (26).

### 3.2 Identified Set under Auxiliary Parametric Specification

#### 3.2.1 Characterization

In contrast to the nonparametric specification, the problem in the auxiliary specification is finite-dimensional in nature due to the fact that $Q_A$ is a finite-dimensional parameterized space. In this
case, \( \theta(Q_A) \) can hence be directly characterized by searching over \( q \) in \( Q_A \) and then taking their image under the function \( \theta \).

In order to state the result that shows how to do this, it is useful as before to first rewrite \( \theta(Q_A) \) in terms of \( \alpha \), i.e. the variable parameterizing \( q \in Q_A \) through (15).\(^4\) Observe that given \( \theta \) is continuous in \( q \) and that \( q \) is continuous in \( \alpha \), there exists a continuous function \( \theta_A \) of \( \alpha \) such that \( \theta(q) = \theta_A(\alpha) \). Similarly, observe that \( Q_A \) can also be written in terms of \( \alpha \) by

\[
A = \left\{ \alpha \in \mathbb{R}^{d_\alpha} : \left( \sum_{k=0}^{K_j} \alpha_{jk} \cdot b_{jk} : j \in J \right) \in Q_B \right\}.
\] (28)

where \( d_\alpha \) denotes the dimension of \( \alpha \), i.e. the set of values of \( \alpha \) that ensure that the corresponding \( q \) is in \( Q_B \). Then, we can write \( \theta(Q_A) \) in terms of \( \alpha \) by

\[
\theta_A(A) = \{ \theta_0 \in \mathbb{R} : \theta_A(\alpha) = \theta_0 \text{ for some } \alpha \in A \} \equiv \Theta_A.
\] (29)

In the following proposition, we show that when \( A \) is connected and non-empty, the closure of \( \Theta_A \) is equal to an interval, where the endpoints can be characterized as solutions to two finite-dimensional optimization problems.

**Proposition 3.** If \( A \) is empty then by definition \( \Theta_A \) is empty; whereas, if \( A \) is connected and non-empty, then the closure of \( \Theta_A \) is given by \( [\theta_A, \bar{\theta}_A] \), where

\[
\theta_A = \inf_{\alpha \in A} \theta_A(\alpha) \quad \text{and} \quad \bar{\theta}_A = \sup_{\alpha \in A} \theta_A(\alpha).
\] (30)

### 3.2.2 Polynomial Specifications

Proposition 3 shows how to characterize the identified set under a general class of parametric restrictions. In our empirical analysis, for computational tractability, we consider parsimonious specifications that are parameterizations of the separable functions in Assumption S. In particular, we consider

\[
q_j(p) = \sum_{m \in J} \sum_{k=0}^{K} \alpha_{jm} \cdot p_m^k
\] (31)

for each \( j \in J \) and some unknown parameters \( \{ \alpha_{jm} : m \in J, \ 0 \leq k \leq K \} \), i.e. where the unknown functions in Assumption S are assumed to be polynomials of degree \( K \).

While the optimization problems in (30) are finite-dimensional, their computational tractability depends on the structure of the objective \( \theta_A \) and the constraint set \( A \). In Appendix S.1.4, we illustrate that under (31), \( \theta_A \) is a linear function of \( \alpha \) and that \( A \) is characterized by linear equality

\footnote{See also Mogstad et al. (2018) who previously showed how to characterize identified sets under similar parametric restrictions in the alternative context of a treatment effect model.}
and inequality restrictions on $\alpha$. However, we observe here that some of the linear restrictions are evaluated at every price in the continuous space $P$, which implies that the resulting optimization problems can be generally difficult to compute. To this end, we consider the following alternative optimization problems

$$\theta^A_r = \min_{\alpha \in A^r} \theta_A(\alpha) \quad \text{and} \quad \bar{\theta}^A_r = \max_{\alpha \in A^r} \theta_A(\alpha)$$

in our empirical analysis, where $A^r$ corresponds to a subset of $A$ that evaluates some of the restrictions on only a finite set of prices in $P$—the exact form of $A^r$ is provided in Appendix S.1.4. As the objective and the finite number of restrictions determining the constraint sets of these problems are linear in $\alpha$, they are linear programs and hence generally computationally tractable. But, since $A \subseteq A^r$, these problems only provide an outer set for $\Theta_A$, i.e. $\Theta_A \subseteq [\theta^A_r, \bar{\theta}^A_r]$, similar in spirit to those in (27) with respect to $\Theta_B$.

4 Evaluation of the DC Opportunity Scholarship Program

4.1 Background

The DC Opportunity Scholarship Program (OSP) was a federally-funded school voucher program established by Congress in January 2004, and which started accepting students for the 2004-2005 school year. The OSP was structured similarly to other voucher programs that existed at the time (Epple et al., 2017). It was open to students residing in Washington, DC, and whose family income was no higher than 185% of the federal poverty line ($18,850 for a family of four in 2004). It could be used only for K-12 education, and at the time of initial receipt was renewable for up to five years. It provided students a voucher worth $7,500 that could be used to offset tuition, fees, and transportation at any private school of their choice participating in the program.

The law that established the program also mandated its evaluation, which culminated with a final report to Congress (Wolf et al., 2010). The report exploited the fact that the OSP randomly allocated vouchers to participating students. In particular, Congress expected the program to be oversubscribed, i.e. the number of applicants would exceed the number of available vouchers. As a result, it required that vouchers be randomly allocated to applicants through a lottery if the program was oversubscribed—see Wolf et al. (2010) for details on the lottery. Wolf et al. (2010) exploited this random allocation by comparing various outcomes of voucher recipients to non-recipients to experimentally evaluate the effect of voucher receipt on these outcomes. The main findings, as listed in the executive summary, can be broadly summarized as follows. First, they find no conclusive evidence that the receipt of the voucher had any significant effects on various outcomes corresponding to student achievement. Second, they find that the receipt of the voucher significantly improved students’ chances of graduating from high school. Finally, they find that the receipt of the voucher raised parents’ ratings of school safety and satisfaction.
In what follows, we use the tools developed in the previous sections to complement these findings by analyzing the welfare effects of the price subsidy induced by the status quo voucher amount as well as counterfactual amounts. Our analysis is motivated by the fact that while the receipt of the voucher revealed mixed evidence on outcomes in the sense that there are zero as well as some positive effects, parents may nonetheless value the voucher, potentially across dimensions not easily captured by the outcomes. Indeed, as highlighted below, the data reveals that a non-trivial proportion of voucher recipients used the voucher, which, by revealed preference arguments, implies that they value receiving the voucher. Our analysis estimates these potential welfare benefits using data collected by the OSP.

4.2 Setting

In the context of our setup in Section 2, let $D_i$ correspond to the enrolled school and let it take values in $\mathcal{J} = \mathcal{J}_v \cup \{g, n\}$, where $\mathcal{J}_v$ denotes the set of private schools in the program, and $g$ and $n$ denote alternatives of enrolling in a government school (which includes charter schools) and a private school not in the program, respectively.\(^5\) In order to define the support of the price vector $P_i$, note that the voucher affected only the prices (tuition) of private schools in the program, and hence there is no variation in the prices of government and private schools not in the program. The prices of the alternatives $g$ and $n$ are therefore normalized to zero. For the private schools in the program, the variation in prices is determined by the receipt of the status quo voucher. For each $j \in \mathcal{J}_v$, let $p_j^* \in \mathbb{R}_+$ denote the original price of the school, and let $p_j(\tau) = \max\{p_j^* - \tau, 0\}$ denote its price under the application of a voucher of amount of $\tau \in \mathbb{R}_+$, as the voucher provided an amount of at most $\tau$ to cover tuition. Moreover, let $p(\tau) = (p_j(\tau) : j \in \mathcal{J})$ denote the vector of prices under a voucher amount of $\tau \in \mathbb{R}_+$, where note that $p_g(\tau) = p_n(\tau) = 0$ as their prices are normalized to 0. Denoting by $\tau_{sq}$ the status quo amount, the support of the prices is then given by $P_{obs} = \{p(0), p(\tau_{sq})\}$, i.e. the prices with and without the status quo voucher. Given that the voucher was randomly assigned to students, we have that Assumption E is satisfied.

The objective of our empirical analysis is to learn the welfare effects of the price decrease induced by the voucher, i.e. (5) when $p^a = p(0)$ and $p^b = p(\tau)$ given by

$$AB(\tau) \equiv E[B_{i}^{a,b}].$$  \hspace{1cm} (33)

Indeed, when $\tau = \tau_{sq}$, this corresponds to the effect of providing the status quo voucher amount, while when $\tau \neq \tau_{sq}$, it corresponds to the effect of providing a counterfactual voucher amount. To benchmark these benefits and perform a cost-benefit analysis, we also study additional parameters that measure the costs the government may face when individuals receive the voucher net of those

\(^5\)We separately consider the alternatives of enrolling in a government school and a private school not in the program, rather than combining them into a single alternative, as we take the costs associated with them in (34) to be different.
when they do not receive it, which can be straightforwardly written as (7). In particular, denoting by \( \{c_j(\tau) : j \in J\} \) the costs that the government associates with enrollment in the different alternatives under a voucher of amount \( \tau \), we take the average cost of the voucher to be

\[
AC(\tau) = \sum_{j \in J} c_j(\tau)q_j(p(\tau)) - c_j(0)q_j(p(0)),
\]

and the average surplus measuring benefit net of cost by

\[
AS(\tau) = AB(\tau) - AC(\tau).
\]

We take \( c_g(\tau) = c_g \), i.e. the cost associated with government schools is some known value \( c_g \); \( c_n(\tau) = 0 \), i.e. the cost associated with private schools not in the program is zero; and \( c_j(\tau) = \min\{p_j(0), \tau\} + \mu \cdot 1\{\tau > 0\} \), i.e. the cost associated with private schools in the program is the voucher amount spent to cover tuition plus some known administrative cost \( \mu \) of operating the program (i.e. charged only when the voucher amount is positive). For the known values, we take \( c_g = \$5,355 \), which corresponds to the educational expenditure reported by the US Census (2005). This is lower than total per-pupil expenditure from the Census (\$12,979, which includes some fixed costs), or educational expenditure as measured in other sources (\$8,105, Sable and Hill (2006)). However, as our surplus parameter is increasing in \( c_g \), we choose the smaller, more conservative value. On the other hand, we take \( \mu = \$200 \), which corresponds to cost of administration, adjudication and providing information to families for an alternative school voucher program reported in Levin and Driver (1997)—see Figure S.1 for robustness to a range of other values of \( c_g \) and \( \mu \).

6All cost values reported in this paragraph have been adjusted to 2004 dollars.

It is useful to highlight some limitations of our setup and how it affects the interpretation of our subsequent results. As noted in Section 2.1, our model does not capture channels through which the voucher may affect choice beyond a decrease in prices. Examples of such channels noted previously in the literature primarily correspond to various general equilibrium effects that affect utilities under different schools due to changes in school incentives to invest in quality (Allende, 2019; Neilson, 2013) or changes in peer composition (Allende, 2019; Gazmuri, 2019). Indeed, capturing such channels requires a richer model that explicitly introduces them in its structure. Our analysis can therefore more appropriately be viewed as a partial equilibrium one that takes these channels as fixed. More concretely, it can be viewed as analyzing the effects of a marginal policy that provides a voucher to an additional student who applied to the OSP but was not admitted, rather than those of scaling up the program that may induce general equilibrium effects.

Moreover, recall that our analysis relies on the expression for the average willingness to pay in Proposition 1, which requires the underlying utilities to be continuous in disposable income. In our empirical setting, this assumption, however, can potentially be suspect. As elaborated in Appendix S.2, this is due to the fact that individuals may be liquidity constrained and hence whether they

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\(6\)All cost values reported in this paragraph have been adjusted to 2004 dollars.
Table 1: Enrollment shares across school type by voucher receipt

| School Type                                | With Voucher | Without Voucher | Difference |
|--------------------------------------------|--------------|-----------------|------------|
| Government schools                        | 0.288        | 0.901           | -0.613     |
|                                           | [0.453]      | [0.299]         | (0.018)    |
| Private schools not in program            | 0.014        | 0.020           | -0.006     |
|                                           | [0.117]      | [0.140]         | (0.006)    |
| Private schools in program                | 0.698        | 0.079           | 0.619      |
|                                           | [0.459]      | [0.270]         | (0.018)    |
| Observations                              | 1,090        | 730             |            |

Observations rounded to the nearest 10. Standard deviations in square brackets and robust standard errors in parentheses.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

can afford to choose an alternative may discontinuously change with its price. In Proposition S.3, we extend Proposition 1 to show that in this case the expression in (6) continues to conservatively provide a valid lower bound for the average willingness to pay—the upper bound can intuitively go to infinity as the price decrease can make a new alternative affordable and hence is similar to an infinite price decrease that individuals may have an unbounded willingness to pay for. In this sense, our analysis can be conservatively viewed as providing a lower bound for the welfare effects. As we will observe below, our empirical conclusions are primarily based on the lower bounds of our estimates and therefore remain robust to this feature.

4.3 Data and Summary Statistics

The OSP collected detailed data for the first two years of the program, 2004 and 2005, and tracked students for at least four years. Across these years, the composition of applicants and private schools in the program changed. To keep prices and the set of eligible schools the same for all students, we focus on the second year of the program, 2005, which contains around 80% of the entire sample. In addition, to avoid complications from dynamics, we focus on the initial year of the data for students entering the program this year. In Appendices S.4.1-S.4.2, we provide details on how our analysis sample was constructed from the original evaluation data as well as various statistics on the schools and sample of individuals. Below, we present summary statistics for the main variables our analysis exploits, namely the enrollment shares and the prices of private schools in the program.

Table 1 presents enrollment shares across the three types of schools, i.e. government schools and private schools in and not in the program, by voucher receipt. A relatively large proportion (69.8%) choose to take up the voucher as revealed by those enrolled in private schools in the
Figure 2: Tuition prices across voucher private schools, and enrollment shares across them as well as government and non-voucher private schools.

(a) Tuition of private schools in the program

(b) Enrollment shares

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

program. By revealed preference, this implies that recipients value the voucher. In addition, the voucher increases the proportion enrolling in voucher private schools by 61.9 percentage points, suggesting that prices play an important role in inducing private school enrollment. The voucher also produces a nearly symmetric decline in the proportion enrolled in government schools (-61.3 percentage points) implying that nearly all students induced into voucher schools would be in government ones absent the voucher.

In 2005, there were approximately 70 private schools in the program (out of a total of about 110 in Washington, DC). Figure 2 summarizes the variation in prices across these schools as well as the enrollment shares across various ranges of these prices. Figure 2(a) reveals that a large number of voucher schools had low prices—around 80% had prices below the status quo voucher amount. Figure 2(b) reveals that the voucher induced a significant proportion to enroll in these low-price schools—out of the 61.9 percentage point increase in the proportion attending a voucher private school, a full 59 percentage points (95%) was into schools with prices less than the status quo voucher amount. Similarly, a large proportion of recipients (81%) redeem the voucher at schools with prices below the cost of a government school. Given that the majority of these recipients would have enrolled in government schools absent the voucher as observed from Table 1, this suggests that the government may face only small net costs or even savings from the provision of a voucher. Our

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7Figures are rounded to the nearest ten for privacy purposes.
Table 2: Estimated bounds and 90% confidence intervals on welfare effects for status quo voucher amount

|                  | Nonparametric (NP) | Parametric (P) Separable, $K$ |
|------------------|--------------------|-----------------------------|
|                  | Baseline Separable | (1)  | (2)  | (3)  | (4)  | (5)  | (6)  | (7)  |
| $AB(\tau_{sq})$ | 156                | 1,583 | 1,011 | 723  | 541  | 403  |      |      |
|                  | 362                | 1,752 | 1,168 | 891  | 710  | 585  |      |      |
|                  | 5,239              | 1,853 | 2,426 | 2,752| 2,952| 3,114|      |      |
|                  | 5,570              | 1,996 | 2,607 | 2,958| 3,171| 3,333|      |      |
|                  | -168               | -168  | -168  | -168 | -168 | -168 |      |      |
| $AC(\tau_{sq})$ | 113                | 113   | 113   | 113  | 113  |      |      |      |
|                  | 332                | 332   | 332   | 332  | 332  |      |      |      |
| $AS(\tau_{sq})$ | 80                 | 1,433 | 873   | 610  | 428  | 303  |      |      |
|                  | 249                | 1,639 | 1,055 | 778  | 597  | 472  |      |      |
|                  | 5,126              | 1,740 | 2,313 | 2,639| 2,839| 3,001|      |      |
|                  | 5,557              | 1,921 | 2,519 | 2,870| 3,083| 3,245|      |      |

For each parameter, the inner panel reports the estimated bounds and the outer panel reports confidence intervals, respectively. Lower and upper bounds are not repeated if they coincide.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

estimates below make this point more precisely.

4.4 Welfare Estimates

Table 2 presents estimated bounds for the welfare effects of providing the status quo voucher amount. Each row corresponds to the parameters in (33)-(35) taking $\tau = \tau_{sq} \equiv 7,500$. Each column corresponds to a specification of demand, which is either the baseline nonparametric specification, or when separability in Assumption S or their parameterized version in (31) for some value of $K$ is imposed. We consider $K$ from 1 to 5. Here and in the remainder of the empirical analysis, bounds under the nonparametric baseline specification are estimated using (27) with the choice of $B^r$ described in Appendix S.1.2, those under nonparametric separability assumptions are estimated using (S.11) described in Appendix S.1.3, and those under the parametric specifications are estimated using (32) with choices of $A^r$ described in Appendix S.1.4, where in all cases the enrollment shares in the restriction in (8) are replaced by their empirical counterparts.\(^8\) We also

\(^8\)We find that all the specifications exactly match the data and hence that the estimated bounds are non-empty. If this was not the case, one could straightforwardly apply the estimation procedure from Mogstad et al. (2018) that allows the specification to not exactly match the data due to sampling uncertainty and provides non-empty estimates of the bounds.
report 90% confidence intervals, which are constructed using a bootstrap procedure from Bugni et al. (2017) described in Appendix S.3.

The estimates for $AB(\tau_{sq})$ under the baseline nonparametric specification in Column (1) reveal that the average benefit from the status quo voucher can range between $362 and $5,239. While these bounds are potentially not sharp as noted in Section 3.1.3, comparing them to the sharp ones under the separability assumption in Column (2) reveal that the lower bounds are equal and hence that at least the lower bound is sharp. The estimates for $AC(\tau_{sq})$ reveal that it is point identified. This is because it is a function of demand at values of prices observed in the data, namely the prices with and without the status quo voucher. The average cost, equal to $113, is low compared to the amount of $7,500 that the voucher provides. As highlighted above, this is due to the fact that a large proportion of recipients redeem the voucher at low-cost private schools relative to government schools they would have enrolled absent the voucher.

Taking the difference between benefits and costs, the estimates for $AS(\tau_{sq})$ reveal that the average benefit net of costs of the status-quo voucher is positive. Importantly, this finding robustly holds even under the nonparametric baseline specification in which case the net benefit is at least $249. Moreover, from the upper bounds, we can observe that a potentially high net benefit is consistent with the data, where they can range from $1,740 to $5,126 depending on the strength of assumptions one is willing to impose. Overall, this suggests that the voucher recipients can have a large welfare benefit from the private schools at which they redeem the voucher relative to the costs the government faces to fund the voucher at these schools. The confidence intervals reveal that these findings are also statistically significant.

Figure 3 presents estimated bounds for the parameters in (33)-(35) for a range of counterfactual voucher amounts beyond the status quo, for the baseline and separable nonparametric specification as well as the parametric specification from Column (5) in Table 2. As in the case of the status quo, for values below the status quo amount, Figure 3(c) reveals that under the baseline specification we continue to robustly find positive, potentially large net benefits—Figure S.2 reveals that this finding is also generally statistically significant. However, for those above the status quo amount, the conclusion on the presence of positive effects is dependent on the strength of the assumptions. In particular, we have positive net benefits only if we are willing to impose parametric restrictions in which case the net benefits are positive for a range of values above the status quo amount.

This can be explained from the estimates of the underlying average benefits and cost in Figures 3(a) and (b). In both figures, the bounds for the nonparametric specification appear to be significantly tighter for values of the voucher below the status quo rather than those above it. Intuitively, this is because, in contrast to the parametric specification, the nonparametric specifications allow for much more flexibility in the substitution patterns between schools and also, in contrast to values of the voucher below the status quo, there is no additional data at higher voucher amounts to provide information on the potential substitution patterns. In turn, the bounds are wide, highlighting
that a range of patterns are nonparametrically consistent with the data.

4.5 Role of Low-Tuition Schools

In summary, our welfare estimates reveal that voucher provision has positive, potentially large net benefits under the status quo as well as a range of counterfactual voucher amounts. While discussing our results above, we highlighted that the positive effects arose in part due to the presence of low-tuition schools in the program that many recipients attend, but that have a small net cost to the government. We conclude our analysis by further exploring the importance of these schools in the program under the status quo voucher amount.

We analyze how our estimates change when we remove schools having prices less than a certain amount from the program. For a given $\kappa \in \mathbb{R}_+$, let $\mathcal{J}_\kappa = \{ j \in \mathcal{J}_v : p_j(0) \leq \kappa \}$ denote the set of private schools in the program with prices no more than $\kappa$, and let $p^\kappa_j(\tau)$ be equal to $p_j(0)$ if $j \in \mathcal{J}_\kappa$ and $p_j(\tau)$ otherwise, i.e. the voucher amount is applied to only schools with prices above $\kappa$. Then we are interested in studying the parameter in (5) when $p^a = p(0)$ and $p^b = p^\kappa(\tau)$,

$$ AB^\kappa(\tau_{sq}) = E[B^a_{1},b] , $$

as well as analogous versions of those in (34) and (35) given by

$$ AC^\kappa(\tau_{sq}) = \sum_{j \in \mathcal{J}} c^\kappa_j(\tau_{sq}) \cdot q_j(p^\kappa(\tau_{sq})) - \sum_{j \in \mathcal{J}} c_j(0) \cdot q_j(p(0)) , $$

Figure 3: Estimated bounds on welfare effects for a range of voucher amounts
Figure 4: Estimated bounds on welfare effects when schools with tuition at most $\kappa$ are removed from the program

\begin{align*}
    \text{Average surplus} &= AB^c (\tau_{sq}) - AC^c (\tau_{sq}), \\
    \text{Average cost} &= AC^c (\tau_{sq}), \\
    \text{Average benefit} &= AB^c (\tau_{sq}),
\end{align*}

where $c_j^c (\tau_{sq}) = c_j (\tau_{sq})$ for $j \in J \setminus \kappa$ and $c_j^c (\tau_{sq}) = 0$ for $j \in \kappa$, i.e. we take the same costs as that in (34) except with the difference that we take the schools that are removed from the program to have zero costs.

Figure 4 presents estimated bounds for a range of values of $\kappa$. Intuitively, as the baseline nonparametric specification imposes no restriction on substitution patterns and as the data provides no information on variation where the voucher is applied to only certain voucher schools, the bounds under the baseline nonparametric specification can be wide. The bounds under the parametric specification in contrast can be substantially smaller. Across all specifications, Figure 4(c) suggests that the removal of low-tuition schools from the program generally results in the reduction of average surplus. Importantly, it reveals that removing schools with tuition of $3,500 and lower from the program could potentially cause it to have a negative surplus. A closer look at Figure 2(a) reveals that nearly 30% of schools in the program have tuition of at most this value. The estimates reveal that the presence of these low-tuition schools in the program plays an essential role in explaining the positive net benefits we find for the provision of the status quo voucher amount.

To provide some suggestive evidence on what are the features of the low-tuition schools that are so compelling to voucher recipients, Table 3 compares various average characteristics between schools charging above and below $3,500 in tuition. Consistent with spending less money on instruction, the low-tuition schools have larger student-teacher ratios, and are somewhat less likely
Table 3: Average school characteristics by tuition level for voucher private schools

|                      | Tuition ≤ $3,500 | Tuition > $3,500 | Difference |
|----------------------|------------------|------------------|------------|
| School size          | 196.103          | 281.843          | -85.740    |
| Student/teacher ratio| 13.000           | 9.860            | 3.140      |
| Catholic (=1)        | 0.829            | 0.159            | 0.671      |
| Other religious (=1) | 0.003            | 0.256            | -0.253     |
| Gifted program (=1)  | 0.243            | 0.397            | -0.155     |
| Learning difficulties program (=1) | 0.446 | 0.547 | -0.101 |
| Individual tutors available (=1) | 0.609 | 0.774 | -0.165 |
| Students tracked by ability (=1) | 0.763 | 0.729 | 0.034 |
| Remedial classes available (=1) | 0.646 | 0.619 | 0.027 |
| Number of schools    | 20               | 50               |            |

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations. Observations rounded to the nearest 10.

to have individual tutors or programs for students with learning difficulties. The most striking difference, however, is that they are much more likely to be Catholic—83% versus 16%. This suggests that voucher recipients particularly value this feature of low-tuition schools and is what drives their high welfare benefit for the voucher relative to the low cost of funding it at such schools.

5 Comparison to Traditional Parametric Methods

In this section, we compare our empirical results and conclusions from the previous section to those we would obtain when applying traditional methods.

5.1 Logit Specifications

Recall from Section 2.3, our identification problem requires imposing restrictions on how the demand functions vary with price. Traditional methods do so by imposing a parametric functional form on demand such that it is point identified by the variation in the data, which in turn point-identifies our parameters of interest. These parametrizations are commonly implied by imposing functional forms on the utilities and parametric distributions on the unobserved heterogeneity.

In our comparison, we consider various versions of a standard logit parameterization of our model that begins by assuming

$$U_{ij}(Y_i - p_j) = \xi_j - \gamma_ip_j + \epsilon_{ij}$$

(39)
for \( j \in \mathcal{J} \), i.e. utility is linear in prices, with alternative-specific intercepts, individual-specific price coefficients, and individual and alternative-specific shocks. The difference between the various versions arises from the distributions imposed on the unobserved heterogeneity \( \gamma_i \) and \( \epsilon_{ij} \) as follows:

**Logit I:** \( \gamma_i = \bar{\gamma}_0 \); and \( \epsilon_{ij} \) is distributed independently across \( j \) as Type I extreme value.

**Logit II:** \( \gamma_i = \bar{\gamma}_0 + \bar{\gamma}'_1 X_i \); and \( \epsilon_{ij} \) is distributed the same as in Logit I.

**Mixed Logit:** \( \gamma_i = \bar{\gamma}_0 + \bar{\gamma}'_1 X_i + v_i \), where \( v_i \) is normally distributed with mean 0 and variance \( \sigma^2 \); and \( \epsilon_{ij} \) is distributed the same as in Logit I.

**Nested Logit:** \( \gamma_i = \bar{\gamma}_0 + \bar{\gamma}'_1 X_i \); and \( \epsilon_i = (\epsilon_{ij} : j \in \mathcal{J}) \) has a CDF evaluated at \( \epsilon \) equal to

\[
\exp \left( - \sum_{k \in \{1,2\}} \left( \sum_{j \in \mathcal{N}_k} e^{-\epsilon_j / \lambda_k} \right) \lambda_k \right)
\]

for some \( \lambda_1, \lambda_2 \in \mathbb{R} \), where \( \mathcal{N}_1 = \mathcal{J}_v \cup \{n\} \) and \( \mathcal{N}_2 = \{g\} \).

The first specification is a basic logit one that takes the price coefficient to be constant across individuals. The second and third respectively introduce observed and then unobserved heterogeneity in the price coefficient. The final specification introduces observed heterogeneity in the price coefficient, and dependence in the shocks across alternatives in the same nest, where there are two nests with one consisting of private schools and the other of government schools. Importantly, the flexibility of all these parameterizations is carefully chosen such that the underlying parameters are point identified given the binary variation in prices induced by the voucher, after imposing the usual location and scale normalizations—see, for example, Train (2009, Chapter 2.5). Table S.4 reports the parameter estimates for the various specifications, estimated using maximum likelihood. Here we take \( X_i \) to be a vector of indicators for which bin the family income lies in, where there are four bins determined by quartiles of its empirical distribution.

As the underlying parameters are point identified, it follows that the corresponding demand functions are also point identified by plugging in the point identified parameter values in the expressions for demand implied by the various specifications—the expressions are provided in Table S.5. Similarly, as we do in what follows, we can estimate demand by plugging in the underlying parameter estimates from Table S.4 in the expressions in Table S.5, and then our parameters of interest by plugging in estimated demand in (7). In our implementation, the integrals with respect to \( \phi \) in the expressions in Table S.5 are numerically solved using simulation with 200 draws, while those with respect to \( f_X \) are computed using the empirical distribution of \( X_i \), and the integrals in the expression in (7) are numerically solved using a fine grid of points.

### 5.2 Observed and Counterfactual Demand Estimates

Before proceeding to the welfare estimates under the different logit specifications, we first present various estimates of the underlying demand functions implied by these specifications. In Figure
Figure 5: Demand estimates for various specification under observed and counterfactual voucher amounts

(a) Enrollment shares  
(b) Probability of voucher takeup  
(c) Probability of takeup at schools with tuition at least voucher amount

SOURCE: *Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018)*, U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

5(a), we analyze how well the implied demand functions match the observed shares by plotting them over the empirical enrollment shares in Figure 2(b). In contrast to our specifications that exactly match the observed shares, we can observe that this is not the case with the logit specifications. Nonetheless, the discrepancies are small. Heuristically, this is due to the fact that there is only binary variation in the data, which is not too demanding to match relative to the flexibility of the logit models. We also statistically test the null hypothesis of no discrepancies by bootstrapping (using 200 draws) the test statistic based on the sum of squared difference between the estimated implied and observed shares, i.e.

\[ TS_{\text{fit}} = \sum_{j \in J} \sum_{p \in P_{\text{obs}}} \left( \hat{q}_j(p) - \hat{\text{Prob}}[D_j = j | P_t = p] \right)^2, \]

where, for \( j \in J \) and \( p \in P_{\text{obs}} \), \( \hat{q}_j(p) \) denotes estimates of the implied demand function and \( \hat{\text{Prob}}[D_j = j | P_t = p] \) denotes the empirical enrollment share. The \( p \)-values reported in Figure 5(a) reveal that the discrepancies are not statistically significant.

Next, to analyze the implied demand functions at prices beyond those observed in the data, Figure 5(b)-(c) presents estimates capturing demand for the voucher in various dimensions at various counterfactual voucher amounts. In particular, Figure 5(b) presents demand for the voucher, i.e. probability of choosing \( j \in J_v \), while Figure 5(c) presents the demand for a school with tuition at least the voucher amount, i.e. probability of choosing \( j \in J_v \) such that \( p_j^v \geq \tau \). In this case, comparing to the estimates under our baseline nonparametric and the reported parametric specifi-
Figure 6: Welfare estimates for status quo and counterfactual amounts under various specifications

| Voucher amount ($) | Status quo amount | NP Baseline | P Separable, K = 3 | Logit I | Logit II | Mixed Logit | Nested Logit |
|-------------------|------------------|-------------|--------------------|--------|---------|-------------|--------------|
| -6,000            | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| -4,000            | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| -2,000            | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 0                 | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 2,000             | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 4,000             | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 6,000             | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 8,000             | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 10,000            | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |
| 12,000            | 0                | 0           | 0                  | 0      | 0       | 0           | 0            |

In Figure 6(a): for the nonparametric baseline and parametric separable specifications, the intervals denote the estimated lower and upper bounds; and, for the logit specifications, the markers denote the point estimates and the dashed intervals denote 90% confidence intervals computed using the percentile bootstrap with 200 draws.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

5.3 Welfare Estimates

In Figure 6(a), we present the estimates for our various welfare parameters for the status quo voucher amount under the logit specifications as well as our baseline nonparametric and one of our parametric specifications. Moreover, in Figure 6(b), we additionally present estimates for the average surplus parameter under various counterfactual voucher amounts.

In general, we can observe that the logit specifications all generate estimates that lie within our bounds. Intuitively, this is because the implied demand functions approximately fall in the estimated version of $Q_B$ due to the fact they match well the data, as observed in Figure 5(a),

(caption removed)
as well as satisfy the shape restrictions in (13) as \( U_{ij} \) is generally increasing given that the price coefficients are negative, as observed in Table S.4. However, given that they limit attention to only certain demand functions as observed from Figures 5(b)-(c), they generate welfare effects that all lie within specific areas of our bounds. Relative to our estimates, this can potentially provide a misleading sense of the true effects that are consistent with the data—this also holds true when accounting for statistical uncertainty in the logit estimates as observed from the relative tight confidence intervals for the parameters in Figure 6(a) and Figure S.3.

Importantly, this can affect the interpretation of the empirical conclusions relative to those based on our estimates, previously discussed in Section 4. In particular, for voucher amounts equal to and below the status quo, our estimates can be observed to reveal positive and potentially large net benefits. In contrast, as the logit estimates all systematically fall close to our lower bounds, they can understate the potential benefits by implying only lower valuations can be consistent with the data for such voucher amounts. Alternatively, for amounts above the status quo, our estimates reveal that the data is not sufficiently rich to robustly reveal positive effects of the voucher, and caution that positive effects can be driven by only stronger assumptions. In this case, the logit estimates instead unambiguously predict positive effects and provide a false sense of robustness on the potential benefits, that are in fact driven by the parameterizations.

6 Conclusion

In this paper, we develop new discrete choice tools to robustly learn about the average willing to pay for a price subsidy and its effects on demand given exogenous, discrete variation in prices. Specifically, our tools show how to characterize what we can learn when demand is allowed to be nonparametric as well as flexibly parameterized, both of which imply that our parameters are generally partially identified. We use our tools to perform a welfare analysis of the price subsidy provided by school vouchers in the DC Opportunity Scholarship Program. We also compare our empirical results to those one would obtain under standard logit parameterizations of demand and highlight how they can provide a potentially misleading picture of the true effects.
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Supplementary Appendix to “Estimating Welfare Effects in a Nonparametric Choice Model: The Case of School Vouchers”

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Abstract

This document presents proofs and additional details pertinent to the main analysis for the authors’ paper titled “Estimating Welfare Effects in a Nonparametric Choice Model: The Case of School Vouchers.” Section S.1 presents additional details pertinent to the identification analysis. Section S.2 presents the validity of the average willingness to pay demand expressions under liquidity constraints. Section S.3 presents the procedures used to perform statistical inference in the empirical analysis. Section S.4 presents additional details pertinent to the empirical analysis. Section S.5 presents proofs of all results. Section S.6 presents additional tables and figures.
S.1 Additional Details Relevant for Identification Analysis

S.1.1 Constructing \( \mathcal{V} \)

In this section, we show how to obtain a collection of sets \( \mathcal{V} \) that partitions \( \mathcal{P}^* = \bigcup_{l=1}^{L} \mathcal{P}_l^* \) satisfying Definition \( \mathcal{V} \).

To this end, observe first that the union of the sets in \( \mathcal{V}^{a,b} \) and \( \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \} \) equals \( \mathcal{P}^* \), where

\[ \mathcal{V}^{a,b} = \left\{ \mathcal{P}_{l}^{a,b}, \ldots, \mathcal{P}_{|\mathcal{J}|-1}^{a,b}, \{ p^a \}, \{ p^b \} \right\}, \]

and further that \( \mathcal{V}^{a,b} \) and \( \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \} \) each correspond to a partition of the union of the sets in it that satisfies Definition \( \mathcal{V}(i)-(iii) \) with respect to its union. However, for a given \( v \in \mathcal{V}^{a,b} \) and \( v' \in \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \} \), it may be that Definition \( \mathcal{V}(iii) \) is not satisfied. In particular, it may be the case that there exists \( p \in \mathcal{P}_{\text{obs}} \) that intersects some \( v \in \mathcal{V}^{a,b} \) in the \( j \)th dimension, i.e. \( \{ p_j \} \subset v_{[j]} \) for some \( j \in \mathcal{J} \), and hence we have \( \{ p_j \} \not\subset v_{[j]} \) and \( \{ p_j \} \cap v_{[j]} \neq \emptyset \).

To obtain our partition, we therefore further partition the elements in \( \mathcal{V}^{a,b} \) to obtain \( \hat{\mathcal{V}}^{a,b} \) such that Definition \( \mathcal{V}(iii) \) is satisfied between the elements of \( \hat{\mathcal{V}}^{a,b} \) and \( \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \} \) by accounting for whether \( p \in \mathcal{P}_{\text{obs}} \) and \( v \in \mathcal{V}^{a,b} \) intersect in each of the \( j \in \mathcal{J} \) dimensions. In particular, let \( \mathcal{T} = \{ \Delta_l^{a,b} : 1 \leq l \leq |\mathcal{J}|-1 \} \cup \{ \max \{ \min \{ p_j^a, p_j^b \}, p_j^b \} - p_j^b : j \in \mathcal{J}, p \in \mathcal{P}_{\text{obs}} \} \) denote the set of points such that \( \{ \min \{ p_j^a, p_j^b + t \} : t \in \mathcal{T} \} \) correspond to the end-points of the sets in \( \mathcal{P}^{a,b}_l \), \( 1 \leq l \leq |\mathcal{J}|-1 \), for each of the \( j \in \mathcal{J} \) dimensions as well as where \( p \in \mathcal{P}_{\text{obs}} \) may intersect the sets \( \mathcal{V}^{a,b} \) in each of the \( j \in \mathcal{J} \) dimensions—note here that the minimums and maximums ensure that we only include the intersection points if they are in \([p_j^b, p_j^a]\). Denoting by \( t_1 < \ldots < t_{|\mathcal{T}|} \) the ordered values of \( \mathcal{T} \), then take \( \hat{\mathcal{V}}^{a,b} = \left\{ \hat{\mathcal{P}}^{a,b}_l, \ldots, \hat{\mathcal{P}}^{a,b}_{|\mathcal{T}|-1}, \{ p^a \}, \{ p^b \} \right\} \), where

\[ \hat{\mathcal{P}}^{a,b}_l = \left\{ p \in \mathcal{P} : p_j = \min \{ p_j^a, p_j^b + t_l \}, t_l \in (t_i, t_{i+1}) \text{ for } j \in \mathcal{J} \right\} \text{ for } 1 \leq l \leq |\mathcal{T}|-1. \]

As before, observe that \( \hat{\mathcal{V}}^{a,b} \) and \( \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \} \) each correspond to a partition of the union of the sets in it that satisfy Definition \( \mathcal{V}(i)-(iii) \) with respect to its union. However, now by construction, we also have that Definition \( \mathcal{V}(iii) \) is satisfied for each \( v \in \hat{\mathcal{V}}^{a,b} \) and \( v' \in \mathcal{P}_{\text{obs}} \). To this end, we take

\[ \mathcal{V} = \hat{\mathcal{V}}^{a,b} \cup \{ \{ p \} : p \in \mathcal{P}_{\text{obs}} \}, \]

which corresponds to a partition of \( \mathcal{P}^* \) satisfying Definition \( \mathcal{V} \).

S.1.2 Example of a Set \( \mathcal{B}^r \)

In this section, we describe a set \( \mathcal{B}^r \) such that \( \phi(\mathcal{B}) \subseteq \mathcal{B}^r \) that we use in our empirical analysis when implementing the linear programs in (27). To this end, it is first useful to consider an equivalent representation of the restrictions in (S.29) written in terms of pairs \( w, w' \in \mathcal{W} \).
Proposition S.1. \( \beta \) satisfies (S.29) if and only if
\[
\sum_{j \in J^+} \beta_j(w) \geq \sum_{j \in J_{w,w'}^+ \cup J^+} \beta_j(w^{'})
\]
for each \( J^+ \subseteq J_{w,w'}^+ \) and \( w, w' \in W \), where
\[
J_{w,w'}^+ = \left\{ j \in J : t > t' \text{ for all } t \in w[\{j\}], \ t' \in w'[\{j\}] \right\}
\]
\[
J_{w,w'}^- = J \setminus \left( J_{w,w'}^+ \cup J_{w',w}^- \right)
\]
and \( w[\{j\}] = \{ t \in R : p_j = t \text{ for some } p \in w \} \) for each \( w \in W \) and \( j \in J \).

Given the equivalence between the restrictions in (S.29) and (S.1), we can alternatively write \( B \) in (S.31) as
\[
B = \left\{ \beta \in R^{d_\beta} : \beta \text{ satisfies } (S.27) - (S.28), \ (S.1), \text{ and } (S.30) \right\}.
\]
In this set, observe that each restriction on \( \beta \) is for a given \( w \in W \) or for a pair of \( w, w' \in W \).

Our choice of \( B' \) corresponds to the subset of these restrictions on \( \beta \) for \( w \in W^r \) or for pairs of \( w, w' \in W^r \), i.e. the subset of restrictions that directly correspond those that are in terms of \( \beta^r \).

More specifically, these restrictions correspond to the following
\[
\beta^r_j(w) \geq 0 \text{ for each } j \in J \text{ and } w \in W^r, \quad (S.2)
\]
\[
\sum_{j \in J} \beta^r_j(w) = 1 \text{ for each } w \in W^r, \quad (S.3)
\]
\[
\sum_{j \in J_{w,w'}^+ \cup J^+} \beta^r_j(w) \geq \sum_{j \in J_{w,w'}^+ \cup J^+} \beta^r_j(w^{'}) \text{ for each } J^+ \subseteq J_{w,w'}^+ \text{ and } w, w' \in W^r, \quad (S.4)
\]
\[
\beta^r_j(\{p\}) = \text{Prob}[D_i = j | P_i = p] \text{ for each } j \in J, \ p \in P_{\text{obs}}. \quad (S.5)
\]
Then, denoting by \( d_\beta^r \) the dimension of \( \beta^r \), the set we consider is given by
\[
B' = \left\{ \beta^r \in R^{d_\beta^r} : \beta^r \text{ satisfies } (S.2) - (S.5) \right\}.
\]

### S.1.3 Extension to Separability Assumptions

We noted in Section 3.1.3 that we can additionally impose separability assumptions on demand to reduce the dimension of the linear programs to compute the identified set under the baseline nonparametric specification. In this section, we show how to extend the arguments in Section 3.1 to compute the identified set under a general class of such separability assumptions.

For each \( p \in P \) and \( J' \subseteq J \), let
\[
p_{\{J'\}} = (p_j : j \in J') \in \prod_{j \in J'} [p_j, \bar{p}_j]
\]
denote the sub-vector of \( p \) with indices in \( J' \). Using this notation, the general separability assumption we consider can be stated as follows:

**Assumption GS.** (General Separability) For each \( j \in J \),

\[
q_j(p) = \sum_{l \in L_j} h_{jl}(p|_{J_{jl}}) ,
\]

(S.6)

where \( L_j \) and \( J_{jl} \) are pre-specified subsets of \( J \), and \( h_{jl} \) are some unknown functions.

Assumption GS imposes demand to be a sum of functions that are of lower dimension than demand. Observe that Assumption S is a special case of this assumption. It corresponds to the case with \( L_j = J \) and \( J_{jl} = \{l\} \). We highlight that it also allows for the case where no dimension reduction is imposed and demand is as that in baseline non-separable case, by taking \( L_j \) to be a singleton set such as \( \{j\} \) and taking \( J_{jl} = J \)—see proof of Proposition 2 for more details. Under this assumption, our admissible set of demand functions is given by

\[
Q_S = \{ q \in F : q \text{ satisfies (11) – (12), (13) and (S.6)} \} ,
\]

(S.7)

i.e. \( Q_B \) in (14), but with the additional separability restriction in (S.6).

In order to describe how to compute the identified set \( \theta(Q_S) \), we show as in Proposition 2 that there is no loss of information in first replacing \( Q_S \) with a certain finite-dimensional space, given which the identified set can then be computed using linear programs. To define this finite dimensional space, let \( W \) denote the constructed partition of \( P \) from (23), and, for each \( w \in W \) and \( J' \subseteq J \), let

\[
w_{[J']} = \{ p_{[J']} : p \in w \} \in \prod_{j \in J'} V_j \equiv W_{[J']}
\]

denote the set that includes the sub-vector of prices in \( w \) with indices in \( J' \). The finite dimensional space we consider is then given by

\[
Q_{fd}^S = \left\{ q \in Q_B : q_j(p) = \sum_{w \in W_j} \sum_{l \in L_j} \psi_{jl}(w_{[J_{jl}]}) \text{ for some } \psi_{jl}(w_{[J_{jl}]}) w_{[J_{jl}]} \in W_{[J_{jl}]}, j \in J \right\} ,
\]

(S.8)

i.e. the same as (17) but with the additional restriction that the constant valued functions satisfy (S.6). Let \( \psi \) capture finite-dimensional variables \( \{ \psi_{jl}(w_{[J_{jl}]}) : w \in W_{[J_{jl}]}, l \in L_j, j \in J \} \) in vector form. Observe that the dimension of this variable is given by \( \sum_{j \in J} \sum_{l \in L_j} \prod_{m \in J_{jl}} |V_m| \), while that of \( \beta \) in (17) is \( |J| \prod_{m \in J} |V_m| \). In our empirical application, the former can be substantially smaller than the latter under Assumption S—see Table S.3.
As in Section 3.1.2, to show that \( \theta(Q_S) = \theta(Q_{fd}S) \) and how to compute \( \theta(Q_{fd}S) \), it is next useful to rewrite \( \theta(Q_{fd}S) \) in term of \( \psi \). Let \( \theta_S \) denote the continuous function that rewrites the parameter of interest in terms of \( \psi \) in the sense that \( \theta(q) = \theta_S(\psi) \). Similarly, let

\[
S = \left\{ \psi \in \mathbb{R}^{d\psi} : \left( \sum_{w \in W} 1_w \sum_{l \in L_j} \psi_j(l) : j \in J \right) \in Q_B \right\} \tag{S.9}
\]

where \( d\psi \) denotes the dimension of \( \psi \), capture the restrictions on demand in terms of \( \psi \). We can then write \( \theta(Q_{fd}S) \) in terms of \( \psi \) by

\[
\Theta_S \equiv \left\{ \theta_0 \in \mathbb{R} : \theta_S(\psi) = \theta_0 \text{ for some } \psi \in S \right\}. \tag{S.10}
\]

As in Proposition 2, we show in the following proposition that \( \theta(Q_S) \) is equal to \( \Theta_S \), and that it can be compute by solving two optimization problem. We again highlight in its proof that \( \theta_S \) is linear and that \( S \) is linear, both of which again imply that the optimization problems are linear programs.

**Proposition S.2.** Suppose that \( Q = Q_S \). Then, the identified set in (10) is equal to that in (S.10), i.e. \( \Theta = \Theta_S \). In addition, if \( S \) is empty then by definition \( \Theta_S \) is empty; whereas, if \( S \) is non-empty then the closure of \( \Theta_S \) is given by \( \bar{\Theta}_S \), where

\[
\bar{\theta}_S = \inf_{\psi \in S} \theta_S(\psi) \quad \text{and} \quad \bar{\theta}_S = \sup_{\psi \in S} \theta_S(\psi). \tag{S.11}
\]

### S.1.4 Implementation Details for Auxiliary Parametric Assumptions

In this section, we characterize \( \theta_A \) and \( A \) for the optimization problems in (30) under the specification in (31). As highlighted in Section 3.2, we compute bounds under this specification using the linear programs in (32), which are based on an alternative constraint set \( A^r \). As we will observe more precisely below, this is because the corresponding sets \( A \) are based on restrictions evaluated on all possible prices in \( P \), which makes the problems in (30) difficult to compute. The set \( A^r \) we consider is based on taking these same restrictions but only evaluated on a given, finite set of prices in \( P \). The set of prices we consider is given by \( P^r = \prod_{j=1}^J P^r_j \), where

\[
P^r_j = \left\{ p_j + \frac{l}{\bar{L}} (\bar{p}_j - p_j) : 0 \leq l \leq \bar{L} \right\} \cup \left\{ p_j : p \in \{ p^a, p^b \} \cup P_{\text{obs}} \right\} \text{ for some pre-specified value of } \bar{L},
\]

i.e. a set of \( (\bar{L} + 1) \) equidistant points in \( [p_j, \bar{p}_j] \) along with the values in the price decrease and observed data. In our empirical results, we take \( \bar{L} = 6 \). In unreported results, we find that increasing the value of \( \bar{L} \) generally tightens the bounds in a gradual manner, but at the cost of increased computational time, specifically for the confidence intervals.

Below, for completeness, we also present the set of restrictions that determine \( A^r \) for each of the specifications. For these purposes, let \( \{ p_{0,j}, \ldots, p_{L,j} \} \) denote the set of ordered values of \( P^r_j \) for
each \( j \in J \) and let \( \Delta p_{l,j}^k = p_{l,j}^k - p_{l-1,j}^k \) denote the difference in two consecutive prices in this set, where each of these two prices is raised to the power of \( k \).

To see the form of the parameter \( \theta_A \) under (31), observe that the parameter of interest in (7) can be written in terms of \( \alpha \) as follows

\[
\theta_A(\alpha) \equiv g^{a,b}(\alpha) = \sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^a} \Delta_{l+1}^{a,b} \left( \sum_{m \in J_{l+1}^a} \alpha_{jmk} \cdot (p_m^b)^k + \sum_{m \in J_{l+1}^a} \alpha_{jmk} \cdot (p_m^b + t)^k \right) dt
\]

\[+ \sum_{j \in J} \sum_{m \in J} \sum_{k=0}^K \alpha_{jmk} \cdot \left( g_a^j \cdot (p_m^a)^k + g_b^j \cdot (p_m^b)^k \right) \]

\[= g^{a,b}(\alpha) = \sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^a} \left( \sum_{m \in J_{l+1}^a} \alpha_{jmk} \cdot (p_m^a)^k \cdot (\Delta_{l+1}^{a,b} - \Delta_{l}^{a,b}) + \sum_{m \in J \setminus J_{l+1}^a} \sum_{k=0}^K \alpha_{jmk} \cdot \left( \frac{(p_m^b + t)^k}{k+1} \right) \Delta_{l}^{a,b} \right) \]

\[+ \sum_{j \in J} \sum_{m \in J} \sum_{k=0}^K \alpha_{jmk} \cdot \left( g_a^j \cdot (p_m^a)^k + g_b^j \cdot (p_m^b)^k \right) \quad (S.12)\]

where the first line follows from directly substituting the relation between \( q \) and \( \alpha \) from (31) in (7), and the second line from evaluating the integrals in the first part of the expression.

Next, to see the linear restrictions that determine \( A \), observe that by substituting the relation between \( q \) and \( \alpha \) into the various restrictions in (11)-(12), (13) and (8) we obtain

\[
\sum_{m \in J} \sum_{k=0}^K \alpha_{jmk} \cdot p_m^k \geq 0 \quad \text{for each } j \in J, \quad (S.13)
\]

\[
\sum_{j \in J} \sum_{m \in J} \alpha_{jmk} \cdot p_m^k = 1 \quad (S.14)
\]

for all \( p \in P \),

\[
\sum_{m \in J} \sum_{k=0}^K \alpha_{jmk} \cdot \left( p_m^k - p_m^k \right) \geq 0 \quad (S.15)
\]

for each \( j \in J \setminus J' \) and \( p,p' \in P \) such that \( p_j > p'_j \) for \( j \in J' \) \( \subseteq J \) and \( p_j = p'_j \) for \( j \in J \setminus J' \), and

\[
\sum_{m=1}^J \sum_{k=0}^K \alpha_{jmk} \cdot (p_m^k) = \text{Prob}[D = j|P_i = p], \quad (S.16)
\]

for each \( j \in J \) and \( p \in P_{\text{obs}} \). In turn, we have that \( A = \{ \alpha \in \mathbb{R}^{da} : \alpha \text{ satisfies } (S.13) - (S.16) \} \).
As mentioned above, considering the above restrictions at all prices in $P$ can be generally difficult. We therefore only consider a subset of these restrictions that are evaluated at prices in $P^r$. Simplifying and removing some jointly redundant restrictions using some algebra, these restrictions can be given by

$$\sum_{k=0}^{K} \alpha_{jjk} \cdot p_{l,j}^k + \sum_{m \in \mathcal{J} \setminus \{j\}} \sum_{k=0}^{K} \alpha_{jm} \cdot p_{0,m}^k \geq 0 \quad \text{for each } j \in \mathcal{J}, \ 0 \leq l \leq L, \quad (S.17)$$

$$\sum_{j \in \mathcal{J}} \sum_{m \in \mathcal{J}} \sum_{k=0}^{K} \alpha_{jm} \cdot p_{0,m}^k = 1, \quad (S.18)$$

$$\sum_{j \in \mathcal{J}} \sum_{k=0}^{K} \alpha_{jm} \cdot \Delta p_{l+1,m}^k = 0 \quad \text{for } m \in \mathcal{J}, \ 0 \leq l \leq L - 1, \quad (S.19)$$

$$\sum_{k=0}^{K} \alpha_{jm} \cdot \Delta p_{l+1,m}^k \geq 0 \quad \text{for each } j \in \mathcal{J}, \ m \neq j \in \mathcal{J}, \ 0 \leq l \leq L - 1, \quad (S.20)$$

and (S.16), and, in turn, we have that $A^r = \{ \alpha \in \mathbb{R}^{d_\alpha} : \alpha \text{ satisfies (S.17) - (S.20) and (S.16)} \}$.

### S.2 Robustness to Liquidity Constraints

As highlighted in Section 4.2, the indirect utility may be discontinuous due to the fact that individuals may be liquidity constrained. To see this, suppose that an individual can borrow only up to an amount of $I_i \geq 0$ and, in turn, their set of affordable alternatives when prices are set to $p \in P$ is given by

$$C_i(p) = \{ j \in \mathcal{J} : p_j \leq Y_i + I_i \}.$$

Moreover, let $\tilde{U}_{ij} : \mathbb{R} \to \mathbb{R}$ be the utility function for alternative $j \in \mathcal{J}$ if it is affordable. In this case, the counterfactual choice at price $p \in P$ is given by

$$D_i(p) = \arg \max_{j \in C_i(p)} \tilde{U}_{ij}(Y_i - p_j),$$

i.e. the utility maximizing choice from the set of affordable alternatives. This falls in our setup by taking the indirect utility $U_{ij}$ to be related to $C_i(p)$ and $\tilde{U}_{ij}$ as follows

$$U_{ij}(Y_i - p_j) = \begin{cases} \tilde{U}_{ij}(Y_i - p_j) & \text{if } j \in C_i(p), \\ \max_{m \in C_i(p)} \tilde{U}_{im}(Y_i - p_m) - \epsilon & \text{otherwise} \end{cases},$$

for some $\epsilon > 0$, i.e. it is equal to $\tilde{U}_{ij}$ if the alternative is affordable and sufficiently small if not so that it is never chosen. In turn, we can observe that even if $\tilde{U}_{ij}$ is continuous, $U_{ij}$ can be discontinuous at $-I_i$, i.e. when the liquidity constraint just binds.
In the following proposition, we show that for the willingness to pay for a price decrease from $p_a \in P$ to $p_b \in P$ in this case defined by the variable $\tilde{B}_{a,b}^i$ that solves

$$\max_{j \in C_i(p)} \tilde{U}_{ij}(Y_i - p_j) = \max_{j \in C_i(p)} \tilde{U}_{ij}(Y_i - p_j - \tilde{B}_{a,b}^i),$$

(S.21)

the expression in (6) continues to be valid for its average value or, more conservatively, provides a valid lower bound.

**Proposition S.3.** For each individual $i$, suppose $\tilde{U}_{ij}$ is continuous and strictly increasing for each $j \in J$.

(i) If there exists $j_0 \in J$ with prices normalized to 0 such that $j_0 \in C_i(p)$ for all $p \in P$ and that $\tilde{U}_{ij_0}(Y_i) \geq \tilde{U}_{ij}(-I_i)$ for each $j \in J \setminus \{j_0\}$, then we have that $\tilde{B}_{a,b}^i$ defined in (S.21) exists and is unique, and that

$$E[\tilde{B}_{a,b}^i] = \sum_{l=1}^{\lvert J \rvert-1} \sum_{j \in J_{l+1}^{a,b}} \Delta_{a,b}^i \int_{\Delta_{a,b}^i} q_j \left( \min\{p^a, p^b + t\} \right) dt,$$

(S.22)

where $\Delta_{a,b}^i$ and $J_{l+1}^{a,b}$ for $1 \leq l \leq \lvert J \rvert$ are defined as in Proposition 1.

(ii) Alternatively, if there exists $j_0 \in J$ with prices normalized to 0 such that $j_0 \in C_i(p)$ for all $p \in P$ and that $\tilde{U}_{ij_0}(Y_i) \geq \tilde{U}_{ij}(Y_i - p_j)$ for each $j \in J \setminus \{j_0\}$ as $p_j \to \infty$, then we have that $\tilde{B}_{a,b}^i$ defined in (S.21) exists and is unique, and that

$$E[\tilde{B}_{a,b}^i] \geq \sum_{l=1}^{\lvert J \rvert-1} \sum_{j \in J_{l+1}^{a,b}} \Delta_{a,b}^i \int_{\Delta_{a,b}^i} q_j \left( \min\{p^a, p^b + t\} \right) dt.$$

(S.23)

Proposition S.3(i) provides a scenario when the expression in (6) continues to be valid for the average willingness to pay in the case of liquidity constraints. In particular, it supposes that there exists an alternative with price normalized to 0 that is always affordable, such that individuals prefer to choose this alternative over borrowing to choose an alternative that may not be affordable. Intuitively, in this case, the liquidity constraint is never binding and hence the willingness to pay and its resulting average value continue to remain the same with or without liquidity constraints. In our empirical analysis, such an alternative corresponds to government schools, where it is reasonable to assume that students prefer to enroll in government schools rather than borrow to enroll in private schools. This assumption is nonetheless not straightforwardly verifiable as it is based on unobserved quantities, which motivates Proposition S.3(ii).

Proposition S.3(ii) considers a weaker scenario that supposes that there exists an alternative that is always affordable that individuals prefer over another alternative when the price of that
alternative goes to infinity. In our empirical analysis, this translates to the even more reasonable case that students will prefer to enroll in government schools over a private school if its price is extremely large. In this case, however, Proposition S.3(ii) shows that the expression in (6) only provides a valid lower bound for the average willingness to pay. Intuitively, this is due to the fact that the price decrease here corresponds to an infinite one for the alternatives for which it relaxes the liquidity constraint, and hence individuals can have an unbounded willingness to pay for such a price decrease.

S.3 Statistical Inference

In this section, we describe how we construct confidence intervals for our parameters in our empirical analysis in Section 4 using the bootstrap procedure from Bugni et al. (2017). To this end, let

\[ \{(D_i, P_i) : 1 \leq i \leq N\} \]  

(S.24)

denote our sample of \( N \) observations, assumed to be independently and identically distributed, on which our statistical tests are based.

Recall that our parameters of interest are generally bounded across our various specifications, where the lower and upper bounds are given by minimization and maximization problems, respectively. We construct confidence intervals such that each point in these bounds lies in the interval with probability at least \((1 - \alpha)\) for some pre-specified value of \( \alpha \in (0, 1) \). In order to describe the common procedure that we use across all the parameters and specifications, it is useful to first define the common structure present in all these cases. To this end, note that each point in the bounds across these cases can be written as \( c'x \) for some vector \( x \in \mathbb{R}^{d_x} \) of dimension \( d_x \) that satisfies

\[
A_1x = b_1 , \\
A_2x \leq b_2 ,
\]

(S.25)  

(S.26)

where \( x \) corresponds to the optimizing variable in the minimization and maximization problems, \( c \) corresponds to the vector defining the objective in these problems that depends on the choice of parameter, (S.25) capture the restrictions imposed on the optimizing variable by the observed shares through (8), and (S.26) capture the restrictions imposed by the shape restrictions in (11)-(13). Note that across these cases the values \( c, A_1, A_2 \) and \( b_2 \) are known and deterministic, and only \( b_1 \) needs to be estimated as it corresponds to the observed enrollment shares.

We construct confidence intervals for the various parameters across the various specifications by test inversion. In particular, we test the null hypothesis at level \( \alpha \) that there exists a \( x \) satisfying the restrictions in (S.25) and (S.26) such that \( c'x = \theta_0 \) for some given value of \( \theta_0 \in \mathbb{R} \). Confidence intervals are then constructed by collecting the values of \( \theta_0 \) that are not rejected.
We test this null hypothesis using a bootstrap procedure from Bugni et al. (2017), who show it can have several desirable theoretical properties that account for the fact that the parameter of interest is generally partially identified. Our above setup can be mapped into their general framework, given which the test procedure can be described in the following steps:

1. Compute the test statistic

\[ TS_N(\theta_0) = N \cdot \min_x (A_1 x - \hat{b}_1)'\hat{\Sigma}^{-1}(A_1 x - \hat{b}_1) \]

subject to \( x \) satisfying \( c'x = \theta_0 \) and the restrictions in (S.26), where \( \hat{b}_1 \) corresponds to the empirical counterpart of \( b_1 \) using the data in (S.24) and \( \hat{\Sigma} \) corresponds to a diagonal matrix consisting of the sample variances of the entries of \( \hat{b}_1 \).

2. For \( l = 1, \ldots, B \), compute the so-called minimum resampling bootstrap test statistics of the form

\[ TS_{l,N}^{MR}(\theta_0) = \min \{ TS_{l,N}^{DR}(\theta_0), TS_{l,N}^{PR}(\theta_0) \} \]

where

\[ TS_{l,N}^{DR}(\theta_0) = N \cdot (\hat{b}_1 - \hat{b}_{l,1})'\hat{\Sigma}^{-1}(\hat{b}_1 - \hat{b}_{l,1}) \]

and

\[ TS_{l,N}^{PR}(\theta_0) = N \cdot \min_x \left( \frac{1}{\kappa_N} (A_1 x - \hat{b}_1) \right)' \hat{\Sigma}^{-1} \left( \frac{1}{\kappa_N} (A_1 x - \hat{b}_1) \right) \]

subject to \( x \) satisfying \( c'x = \theta_0 \) and the restrictions in (S.26), and \( \hat{b}_{l,1} \) corresponds to the analogue of \( \hat{b}_1 \) using the \( l \)th bootstrap sample drawn with replacement from the data in (S.24). Here note that \( \kappa_N \) is a tuning parameter that satisfies \( \kappa_N \to \infty \) and \( \kappa_N/\sqrt{N} \to 0 \) as \( N \to \infty \). For our empirical results, following Bugni et al. (2017), we take \( \kappa_N = \sqrt{\ln N} \). Moreover, we take \( B = 200 \).

3. Compute the critical value of the test \( \hat{c}(1 - \alpha, \theta_0) \) by taking the \((1 - \alpha)\)-quantile of the distribution of computed bootstrap test statistics. The test procedure is given by \( \phi_N(\theta_0) = 1\{TS_N(\theta_0) > \hat{c}(1 - \alpha, \theta_0)\} \), i.e we reject if the test statistic is greater than the critical value, and do not reject if it is equal to it or below.

Given the above test procedure for a given value of \( \theta_0 \), we can then use it construct confidence intervals by collecting the set all points we don’t reject, namely \( \{\theta_0 \in \mathbb{R} : \phi_N(\theta_0) = 0\} \).
S.4 Additional Details on Empirical Analysis

S.4.1 Data Construction

In this section, we describe how we construct the data used in our empirical analysis in Section 4. The original data sample comes from the replication files for the evaluation of OSP, which are available from the US Department of Education (Wolf et al., 2010). Recall that our analysis focuses on the initial school choice for students who entered the experiment in 2005. Beginning with this subsample, we make the following data-cleaning choices to reach our final analysis data.

Our analysis requires only the prices (as measured by the tuition) of the participating private schools and the school choices of the students (to compute their enrollment shares). For all participating private schools, we observe tuition in either the first or second year of the study, but not necessarily both. If we observe tuition only in the first year, we assume that it was unchanged between the first and second year (recall we use only the second year of data). For school choices, we have that they are missing for around 36% of the students in our data. By a fortunate quirk of the research design, however, participating private schools reported all voucher students to the researchers. Unobserved school choices must therefore be either in non-participating private schools, or government-funded schools. For these students, we assume they enroll in these two groups at the same relative rate that students with observed choices enroll in non-participating private and government-funded schools. Once we obtain these school choices, we weight these observed choices using the baseline weights of the original evaluation—see Wolf et al. (2010, Appendix A.7) for details on how these weights were constructed.

S.4.2 Summary Statistics on School Setting

In this section, we describe some additional information on the students and schools in our analysis data. Table S.1 reports mean characteristics of students and their families. Only families making less than 185% of the federal poverty line were eligible for the program, and so unsurprisingly the students are relatively disadvantaged. Approximately 50% of the children’s mothers were married, and fewer than 50% were employed at baseline. Family income was slightly less than $17,000. Baseline achievement reflects both positive and negative selection: families chose to participate in the experiment, but they also had to be relatively poor to qualify. The table also reveals that voucher recipients and non-recipients are balanced in terms of the various predetermined characteristics. This suggests that the receipt of the voucher was random, in line with Assumption E.

Table S.2 reports characteristics of the private and government-funded schools in the sample, both unweighted and weighted by attendance. Panel A reveals that the private schools are substantially whiter, have smaller student/teacher ratios, and are more likely to track students by ability.
Table S.1: Student and family characteristics by voucher receipt

|                                      | With voucher | Without voucher | Difference |
|--------------------------------------|--------------|-----------------|------------|
| Mother married (=1)                  | 0.52         | 0.55            | -0.034     |
| Mother years education               | 12.20        | 12.25           | -0.057     |
| Mother works full time (=1)          | 0.35         | 0.38            | -0.028     |
| Mother works part time (=1)          | 0.11         | 0.11            | 0.007      |
| Family income (dollars)              | 16,725       | 17,372          | -647       |
| HH receives govt transfers (=1)      | 0.03         | 0.01            | 0.016      |
| Household size                       | 4.11         | 4.14            | -0.030     |
| Black (=1)                           | 1.00         | 1.00            | -0.001     |
| Male (=1)                            | 0.50         | 0.47            | 0.030      |
| Grade ≤ 5 (=1)                       | 0.65         | 0.65            | 0.000      |
| Grade 6-8 (=1)                       | 0.21         | 0.21            | -0.000     |
| Grade ≥ 9 (=1)                       | 0.14         | 0.14            | 0.000      |
| Child learning disabilities (=1)     | 0.09         | 0.09            | -0.004     |
| Observations                         | 1,090        | 730             |            |

Table shows mean student and family characteristics by treatment group, weighted using the baseline weights. Observations rounded to the nearest 10.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

Most strikingly, many of the private schools are religious—35% of them are Catholic, and an additional 20% another religion. In addition, private schools tend to have lower share of minorities, lower share of student/teacher ratio, lower school sizes, have more students tracked by ability and have lower learning difficulties program. Comparing Panel A and Panel B reveals that among the schools that students actually attended (as reported in the attendance-weighted results in Panel B), there are smaller differences between private and government-funded schools. For example, while the average private school is only 73% minority (relative to 96% for the government schools), the average private school attended by a voucher student is 96% minority.

S.5 Proofs

S.5.1 Proof of Proposition 1

The proof of this proposition follows from Bhattacharya (2018, Proposition 1) and Bhattacharya (2018, Theorem 1). We reproduce these proofs here in the context of our setup and notation for completeness. For convenience, we drop the $i$ sub-index here.

To see why the variable $B^{a,b}$ exists and is unique, note first that the right hand side of (4) is
Table S.2: Characteristics of sample schools

|                      | Private | Government-funded | Difference |
|----------------------|---------|-------------------|------------|
| **Panel A: Unweighted characteristics** |         |                   |            |
| Share minority       | 0.73    | 0.96              | -0.227     |
| School size          | 222.97  | 325.90            | -102.924   |
| Student/teacher ratio| 8.92    | 12.82             | -3.897     |
| Catholic (=1)        | 0.35    | 0.00              | 0.354      |
| Other religious (=1) | 0.20    | 0.00              | 0.200      |
| Secular (=1)         | 0.45    | 1.00              | -0.554     |
| Gifted program (=1)  | 0.35    | 0.39              | -0.040     |
| Learning difficulties program (=1) | 0.48 | 0.93 | -0.447 |
| Individual tutors available (=1) | 0.64 | 0.69 | -0.052 |
| Students tracked by ability (=1) | 0.79 | 0.60 | 0.192 |
| Remedial classes available (=1) | 0.61 | 0.68 | -0.070 |
| **Panel B: Attendance-weighted characteristics** |         |                   |            |
| Share minority       | 0.96    | 0.98              | -0.017     |
| School size          | 205.86  | 419.46            | -213.605   |
| Student/teacher ratio| 13.17   | 13.72             | -0.551     |
| Catholic (=1)        | 0.53    | 0.00              | 0.534      |
| Other religious (=1) | 0.25    | 0.00              | 0.252      |
| Secular (=1)         | 0.21    | 1.00              | -0.786     |
| Gifted program (=1)  | 0.34    | 0.34              | -0.000     |
| Learning difficulties program (=1) | 0.45 | 0.96 | -0.518 |
| Individual tutors available (=1) | 0.80 | 0.77 | 0.026 |
| Students tracked by ability (=1) | 0.70 | 0.55 | 0.157 |
| Remedial classes available (=1) | 0.68 | 0.73 | -0.048 |
| Observations         | 60      | 160               |            |

Displays school characteristics for private and government-run schools. We do not break out the private schools by participation status because the non-participating schools almost never responded. Observations rounded to the nearest 10.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

continuous in $B^{a,b}$ as $U_j$ is a continuous function for each $j \in \mathcal{J}$. In addition, since $Y - p^a_j > Y - p^b_j - B^{a,b}$ and $Y - p^a_j < Y - p^b_j - B^{a,b}$ for $j \in \mathcal{J}$ when $B^{a,b} > \Delta^{a,b}_{\mathcal{J}}$ and $B^{a,b} < \Delta^{a,b}_{\mathcal{J}}$, respectively, note that it follows from the fact that $U_j$ is strictly increasing for each $j \in \mathcal{J}$ that if $B^{a,b} > \Delta^{a,b}_{\mathcal{J}}$ then the right hand side of (4) is strictly smaller than its left hand side, whereas if $B^{a,b} < \Delta^{a,b}_{\mathcal{J}}$ then the right hand side will be strictly greater than the left hand side. Form these two points, it then follows by the intermediate value theorem that there exists a $B^{a,b} \in \left[\Delta^{a,b}_{\mathcal{J}}, \Delta^{a,b}_{\mathcal{J}}\right]$ such that the right hand side equals the left hand side, i.e. a solution to (4) exists. Furthermore, given that $U_j$ is strictly increasing for each $j \in \mathcal{J}$, it follows that the solution must be unique.
To see why the average value of $B^{a,b}$ is given by (6), note from above that $B^{a,b} \in [\Delta_1^{a,b}, \Delta_{|J|}^{a,b}]$ and hence that $\text{Prob}[B^{a,b} \leq t] = 0$ for $t < \Delta_1^{a,b}$ and $\text{Prob}[B^{a,b} \leq t] = 1$ for $t \geq \Delta_{|J|}^{a,b}$. To calculate this probability for $t \in [\Delta_1^{a,b}, \Delta_{|J|}^{a,b})$, note that since $U_j$ is strictly increasing for each $j \in J$, we have that $B^{a,b} \leq t$ is equivalent to

$$\max_{j \in J} U_j (Y - p_j^a) \geq \max_{j \in J} U_j (Y - p_j^b - t) ,$$

It follows that for $t \in [\Delta_1^{a,b}, \Delta_{l+1}^{a,b})$ for $l = 1, \ldots, |J| - 1$, we have

$$\text{Prob}[B^{a,b} \leq t] = \sum_{j \in J} \text{Prob} \left[ U_j (Y - p_j^a) \geq \max \left\{ \max_{m \in J \setminus \{j\}} U_m (Y - p_m^a), \max_{m \in J} U_m (Y - p_m^b - t) \right\} \right]$$

$$= \sum_{j \in J \setminus J_{l+1}^{a,b}} \text{Prob} \left[ U_j (Y - p_j^a) \geq \max \left\{ \max_{m \in J \setminus J_{l+1}^{a,b} \cup \{j\}} U_m (Y - p_m^a), \max_{m \in J_{l+1}^{a,b}} U_m (Y - p_m^b - t) \right\} \right]$$

$$= \sum_{j \in J \setminus J_{l+1}^{a,b}} q_j \left( \min \{p_j^a, p_j^b + t\} \right) ,$$

where the second equality follows from the fact that $U_j$ is strictly increasing for each $j \in J$ along with $Y - p_j^b - t \geq Y - p_j^a$ for $j \in J_{l+1}^{a,b}$ and $Y - p_j^b - t \leq Y - p_j^a$ for $j \in J \setminus J_{l+1}^{a,b}$, and the final equality follows from the definition of the average demand functions along with noting that $\min \{p_j^a, p_j^b + t\} = p_j^b + t$ for $j \in J_{l+1}^{a,b}$ and $\min \{p_j^a, p_j^b + t\} = p_j^a$ for $j \in J \setminus J_{l+1}^{a,b}$. Finally, given that for a positive random variable $X$ we have that its expectation is given by

$$E[X] = \int_0^\infty [1 - \text{Prob}[X \leq x]] \, dx ,$$

it follows from the above characterization of $\text{Prob}[B^{a,b} \leq t]$ along with noting that

$$\sum_{j \in J_{l+1}^{a,b}} q_j \left( \min \{p_j^a, p_j^b + t\} \right) = 1 - \sum_{j \in J \setminus J_{l+1}^{a,b}} q_j \left( \min \{p_j^a, p_j^b + t\} \right) ,$$

that the average value of $B^{a,b}$ is given by (6). This concludes the proof.

### S.5.2 Proof of Proposition 2

In order to prove the proposition, we need to show that $\Theta \subseteq \Theta_B$ and $\Theta_B \subseteq \Theta$, and that $\Theta_B = [\bar{\Theta}_B, \bar{\Theta}_B]$ if $B$ is non-empty. The proof of the first two parts, i.e. $\Theta \subseteq \Theta_B$ and $\Theta_B \subseteq \Theta$, respectively follow directly from the first two parts of Proposition S.2 because $Q_B$ corresponds to a special case of $Q_S$ in (S.7) when taking $L_j$ to be a singleton set such as $\{j\}$ and $J_{jl} = J$ in (S.6) for each $j \in J$. It remains to prove the third part, i.e. $\Theta_B = [\bar{\Theta}_B, \bar{\Theta}_B]$ if $B$ is non-empty, as the third part of Proposition S.2 only shows that the closure of the set is a closed interval. Here we additionally
show that $B$ is not only convex but also compact, which implies that image of it through $\theta_B$ is a closed interval with end points given by (26).

In order to show $B$ is convex and compact, it is useful to rewrite (S.34)-(S.37) in terms of $\beta$ when taking $L_j$ to be a singleton set such as $\{j\}$ and $J_{jl} = J$. In this case, we have that (S.34)-(S.35) corresponds to

$$\beta_j(w) \geq 0 \text{ for each } j \in J, \quad (S.27)$$

$$\sum_{j \in J} \beta_j(w) = 1 \quad (S.28)$$

for each $w \in W$; (S.36) corresponds to

$$\beta_j(w) \geq \beta_j(w') \quad (S.29)$$

for each $w, w' \in W$ and $j \in J \setminus J'$ such that $t > t'$ for all $t \in w_{\{m\}}$, $t' \in w'_{\{m\}}$ for $m \in J' \subseteq J$ and $w_{\{m\}} = w'_{\{m\}}$ for $m \in J \setminus J'$; and (S.37) corresponds to

$$\beta_j(\{p\}) = \text{Prob}[D_i = j | P_i = p], \quad (S.30)$$

for each $j \in J$ and $p \in P_{\text{obs}}$. Hence, it follows that

$$B = \left\{ \beta \in \mathbb{R}^d : \beta \text{ satisfies } (S.27) - (S.30) \right\}. \quad (S.31)$$

Given that $B$ is determined by linear inequality restrictions and further that each variable is bounded by the restrictions in (S.27)-(S.28), it follows that it is bounded and compact, which concludes the proof.

S.5.3 Proof of Proposition 3

Note $A$ is a connected and non-empty set. Then, since $\theta_A$ is a continuous real-valued scalar function, it follows that the image of this function over $A$ given by $\Theta_A$ is a connected and non-empty set on the real line, i.e. an interval whose closure has endpoints given by (30).

S.5.4 Proof of Proposition S.1

To see why the restrictions in (S.1) imply those in (S.29), consider $w, w' \in W$ such that $t > t'$ for all $t \in w_{\{j\}}$, $t' \in w'_{\{j\}}$ for $j \in J' \subseteq J$ and $w_{\{j\}} = w'_{\{j\}}$ for $j \in J \setminus J'$. In this case, note that $J_{w,w'} = J'$, $J_{w',w} = \emptyset$, and $J_{w,w'} = J \setminus J'$. Then, taking $J^\dagger = \{j\}$ for each $j \in J_{w,w'}$ in (S.1) implies that (S.29) holds for each $j \in J'$. To see why the restrictions in (S.29) imply those in (S.1), consider $w, w' \in W$ as well as a $w'' \in W$ such that we have $w_{\{j\}} = w'_{\{j\}} = w_{\{j\}}$ for $j \in J \setminus \left( J_{w,w'}^\dagger \cup J_{w',w}^\dagger \right)$, $w_{\{j\}} = w_{\{j\}}$ for $j \in J_{w,w'}^\dagger$, and $w''_{\{j\}} = w'_{\{j\}}$ for $j \in J_{w',w}^\dagger$. Since this
implies that \( t'' > t \) for all \( t \in w_{\{j\}} \), \( t'' \in w_{\{j\}}' \) for \( j \in J_{w,w}^\succ \) and \( w_{\{j\}} = w_{\{j\}}'' \) for \( j \in J \setminus J_{w,w}^\succ \), it follows from (S.29) that
\[
\beta_j(w'') \geq \beta_j(w)
\] (S.32)
for each \( j \in J_{w,w}^\succ \cup J_{w,w}^\succ ^\prime \). Similarly, since it also implies that \( t'' > t' \) for all \( t \in w_{\{j\}}' \), \( t'' \in w_{\{j\}}'' \) for \( j \in J_{w,w}^\succ \) and \( w_{\{j\}}' = w_{\{j\}}'' \) for \( j \in J \setminus J_{w,w}^\succ \), it also follows from (S.29) that
\[
\beta_j(w'') \geq \beta_j(w')
\] (S.33)
for each \( j \in J_{w,w}^\succ \cup J_{w,w}^\succ ^\prime \). Then, for each \( J^\dagger \subseteq J_{w,w}^\prec \) \( \cup \) \( J_{w,w}^\prec ^\prime \), this implies that (S.1) holds as
\[
\sum_{j \in J_{w,w}^\prec \cup J^\dagger} \beta_j(w') \leq \sum_{j \in J_{w,w}^\prec \cup J^\dagger} \beta_j(w'') = 1 - \sum_{j \in J \setminus (J_{w,w}^\prec \cup J^\dagger)} \beta_j(w') = \sum_{j \in J \setminus (J_{w,w}^\prec \cup J^\dagger)} \beta_j(w)
\]
where the first inequality follows from (S.33), the second equality follows from (S.28), the third inequality from (S.32), and the final equality from (S.28).

S.5.5 Proof of Proposition S.2

In order to prove the proposition, we need to show that \( \Theta \subseteq \Theta_S \) and \( \Theta_S \subseteq \Theta \), and that closure(\( \Theta_S \)) = \([\theta_S, \bar{\theta}_S] \) if \( S \) is non-empty. Below, we divide the proof into three parts respectively showing each of these statements. First, we show \( \Theta \subseteq \Theta_S \), i.e. for every \( \theta_0 \in \Theta \) there exists a \( \psi \in S \) such that \( \theta_S(\psi) = \theta_0 \). Second, we show \( \Theta_S \subseteq \Theta \), i.e. for every \( \theta_0 \in \Theta_S \) there exists a \( q \in Q_S \) such that \( \theta(q) = \theta_0 \). Third, we show that if \( S \) is non-empty then it its closure is given by \( \Theta_S = [\theta_S, \bar{\theta}_S] \).

Before proceeding, it is useful to first explicitly state the restrictions on \( \psi \) that characterize \( S \) as well as the expression for \( \theta_S \). To this end, note that \( S \) corresponds to all \( \psi \) such that the corresponding \( q \) determined by (S.8) satisfy (11)-(12), (13) and (8). Given that (S.8) requires \( q \) to be a constant valued function over \( w \in W \), it follows that (11)-(12), (13) and (8) need to be satisfied across only values in \( W \). In particular, observe that (11) and (12) equivalently correspond to
\[
\sum_{l \in L_j} \psi_{jl}(w_{\{j\}l}) \geq 0 \quad \text{for each } j \in J, \quad (S.34)
\]
\[
\sum_{j \in J} \sum_{l \in L_j} \psi_{jl}(w_{\{j\}l}) = 1 \quad (S.35)
\]
for each $w \in \mathcal{W}$, and, given the way $\mathcal{W}$ was constructed, (13) corresponds to
\[
\sum_{l \in L_j} \psi_{jl}(w_{[Jjl]}) \geq \sum_{l \in L_j} \psi_{jl}(w'_{[Jjl]}) \tag{S.36}
\]
for each $w, w' \in \mathcal{W}$ and $j \in \mathcal{J} \setminus \mathcal{J}'$ such that $t \neq t'$ for all $t \in w_{[\{m\}]}, t' \in w'_{[\{m\}]}$ for $m \in \mathcal{J}' \subseteq \mathcal{J}$ and $w_{[\{m\}]} = w'_{[\{m\}]}$ for $m \in \mathcal{J} \setminus \mathcal{J}'$. Similarly, since $\{p\} \in \mathcal{W}$ for each $p \in \mathcal{P}_{\text{obs}}$ given the way $\mathcal{W}$ was constructed, observe that (8) corresponds to
\[
\sum_{l \in L_j} \psi_{jl}(\{p_{[Jjl]}\}) = \text{Prob} [D_i = j | P_i = p] , \tag{S.37}
\]
for each $j \in \mathcal{J}$ and $p \in \mathcal{P}_{\text{obs}}$. Then, it follows $S$ can be written as
\[
S = \left\{ \psi \in \mathbb{R}^{d\psi} : \psi \text{ satisfies (S.34) – (S.37)} \right\} . \tag{S.38}
\]
For the parameter of interest, observe that (6) can be written in terms of $\psi$ as
\[
E[B_{t_i}^{a,b}] = \sum_{l=1}^{|\mathcal{J}|-1} \sum_{j \in \Delta_{l+1}^{a,b}} \int q_j \left( \min\{p^a, p^b + t\} \right) dt , \tag{S.39}
\]
\[
= \sum_{l=1}^{|\mathcal{J}|-1} \sum_{j \in \Delta_{l+1}^{a,b}} \sum_{v \in \mathcal{V}_{t_i+1}^{a,b}} \int q_j \left( \min\{p^a, p^b + t\} \right) dt , \tag{S.40}
\]
\[
= \sum_{l=1}^{|\mathcal{J}|-1} \sum_{j \in \Delta_{l+1}^{a,b}} \sum_{v \in \mathcal{V}_{t_i+1}^{a,b}} \left( \bar{t}_v - t_v \right) \sum_{l \in L_j} \psi_{jl}(w(v)_{[Jjl]}), \tag{S.41}
\]
where $\mathcal{V}_{t_i+1}^{a,b} \subseteq \mathcal{V}$ is such that $\bigcup_{v \in \mathcal{V}_{t_i+1}^{a,b}} v = \mathcal{P}_{t_i+1}^{a,b}$, which exists given Definition $V(i)$, and $w(v) = \prod_{j \in \mathcal{J}} v_{[j]} \in \mathcal{W}$ given $v \in \mathcal{V}$. In particular, the first line simply recalls (6), the second line follows from rewriting it in terms of the collection of sets $\mathcal{V}$ and exploiting the fact that given the form of $\mathcal{P}_{t_i+1}^{a,b}$, we have that, for $v \in \mathcal{V}_{t_i+1}^{a,b}$, $v = \{p \in \mathcal{P} : p = \min\{p^a, p^b + t\}$ for $t \in (t_v, \bar{t}_v)$, for some values of $t_v$ and $\bar{t}_v$, and the third line then directly follows from substituting in the equation from (S.8). In turn, along with the fact that $\{p^a\}, \{p^b\} \in \mathcal{W}$ given how $\mathcal{W}$ was constructed, observe that we have that (7) can be written in terms of $\psi$ as follows
\[
\theta_S(\psi) = g_{a}^{a,b} \sum_{l=1}^{|\mathcal{J}|-1} \sum_{j \in \Delta_{l+1}^{a,b}} \sum_{v \in \mathcal{V}_{t_i+1}^{a,b}} (\bar{t}_v - t_v) \sum_{l \in L_j} \psi_{jl}(w(v)_{[Jjl]})
\]
\[
+ \sum_{j \in \mathcal{J}} \left( g_{j}^{a} \sum_{l \in L_j} \psi_{jl}(\{p^a_{[Jjl]}\}) + g_{j}^{b} \sum_{l \in L_j} \psi_{jl}(\{p^b_{[Jjl]}\}) \right) . \tag{S.42}
\]
Given the explicit characterizations of $S$ and $\theta_S$, we now proceed to presenting the proofs of each of the three parts.
Part 1: Since $\theta_0 \in \Theta$, there exists by definition a $q \in Q$ such that $\theta(q) = \theta_0$. Using this $q$, we construct a $\psi^\dagger$ such that $\psi^\dagger \in S$ and $\theta_S(\psi^\dagger) = \theta_0$. In particular, given that $q$ satisfies (S.6), we take $\psi^\dagger$ to be such that
\[
\psi^\dagger_{jl}(w_{[J,j]}) = \int_0^1 p_{[J,j]}(t,w) \, dt
\]  
for each $w \in W$, $j \in J$ and $l \in L_j$, where, for $t \in (0,1)$, $p(t,w) = (p_j(t,w) : j \in J)$ with $p_j(t,w) = w_{[j]} + (\bar{w}_{[j]} - w_{[j]}) t$, and $\bar{w}_{[j]} = \sup\{t : t \in w_{[j]}\}$ and $w_{[j]} = \inf\{t : t \in w_{[j]}\}$.

In order to show $\psi^\dagger \in S$, we need to show it satisfies the restrictions in (S.34)-(S.37). The restriction in (S.34) is satisfied for each $w \in W$ and $j \in J$ as
\[
\sum_{l \in L_j} \psi^\dagger_{jl}(w_{[J,j]}) = \sum_{l \in L_j} \int_0^1 p_{[J,j]}(t,w) \, dt = \int_0^1 q_j(p(t,w)) \, dt \geq 0 ,
\]
where the equalities follow from (S.43) and (S.6), respectively, and the inequality from (11). Similarly, the restriction in (S.35) is satisfied for each $w \in W$ and $j \in J$ as
\[
\sum_{j \in J} \sum_{l \in L_j} \psi^\dagger_{jl}(w_{[J,j]}) = \int_0^1 \sum_{l \in L_j} h_{jl} \left( p_{[J,j]}(t,w) \right) \, dt = 1 ,
\]
where the equalities follows from (S.43) and (12), respectively. To see why (S.36) is satisfied, take $w, w' \in W$ such that $t > t'$ for all $t \in w_{[j]}$, $t' \in w'_{[j]}$ for $j \in J' \subseteq J$ and $w_{[j]} = w'_{[j]}$ for $j \in J \setminus J'$. For $t \in (0,1)$, observe that $p_j(t,w) > p_j(t,w')$ for $j \in J'$ and $p_j(t,w) = p_j(t,w')$ for $j \in J \setminus J'$. In turn, it follows from (13) that $q_j(t,w) \geq q_j(t,w')$ for each $t \in (0,1)$ and $j \in J \setminus J'$. Taking the integral over $t \in (0,1)$ and then rewriting using (S.6) and (S.43), it directly follows that (S.36) is satisfied. Finally, for the restriction in (S.37), observe that it is satisfied as
\[
\sum_{l \in L_j} \psi^\dagger_{jl} \left( \{p_{[J,j]}\} \right) = \sum_{l \in L_j} h_{jl} \left( p_{[J,j]} \right) = q_j(p) = P_{jl0} ,
\]
for each $j \in J$ and $p \in P_{\text{obs}}$, where the equalities follow from (S.43), (S.6), and (8), respectively.

In order to show $\psi^\dagger$ satisfies $\theta_S(\psi^\dagger) = \theta_0$, note that it is sufficient to show $\theta_S(\psi^\dagger) = \theta(q)$ as $\theta(q) = \theta_0$. To this end, observe that the various components in (S.40) can be written in terms of $\psi^\dagger$ as
\[
\int_{\bar{t}_v}^{\bar{t}_v} q_j \left( \min\{p^a, p^b + t\} \right) \, dt = (\bar{t}_v - \bar{t}_v) \int_0^1 q_j(p(t,w(v))) \, dt ,
\]
for each $j \in J$ and $t \in (0,v)$.
for each \( v \in \mathcal{V}_l^{a,b} \), \( 1 \leq l \leq |\mathcal{J}| - 1 \), where the first equality follows from the change of variables \( t = t_v + (t_v - t_w) \cdot t' \) for \( t' \in [0, 1] \) along with the above definition of \( p(t, w(v)) \), and the second and third equalities follow from (S.43) and (S.6), respectively; and that

\[
\sum_{j \in J} g_j^q(p^q) + g_j^p(p^p) = \sum_{j \in J} \left( g_j^q \sum_{l \in L_j} h_{jl} \left( p_{\mathcal{J}\backslash l}^q(t, w(v)) \right) + g_j^p \sum_{l \in L_j} h_{jl} \left( p_{\mathcal{J}\backslash l}^p(t, w(v)) \right) \right) \\
= \sum_{j \in J} \left( g_j^q \sum_{l \in L_j} \psi_{jl}^q \left( \{ p_{\mathcal{J}\backslash l}^q \} \right) + g_j^p \sum_{l \in L_j} \psi_{jl}^p \left( \{ p_{\mathcal{J}\backslash l}^p \} \right) \right)
\]

where the equalities follow from (S.6) and (S.43), respectively. Substituting these terms in (7), we obtain the expression in (S.42) from which it follows that \( \theta(q) = \theta_S(\psi^\dagger) \). This completes the first part of the proof.

**Part 2:** Since \( \theta_0 \in \Theta_S \), there exist by definition a \( \psi \in \mathbf{S} \) such that \( \theta_S(\psi) = \theta_0 \) and, in turn, by how \( \Theta_S \) and \( \theta_S \) were constructed, a \( q^\dagger \in \mathbb{Q}_{\mathcal{S}}^\dagger \) that is related to \( \psi \) by the equation in (S.8) such that \( \theta(q^\dagger) = \theta_S(\psi) = \theta_0 \). Since it holds that \( \mathbb{Q}_{\mathcal{S}}^\dagger \subseteq \mathbb{Q}_S \), it follows that \( q \in \mathbb{Q} \). This completes the second part of the proof.

**Part 3:** Given the various linear restrictions that define \( \mathbf{S} \) in (S.38), observe that \( \mathbf{S} \) is a convex set. In addition, it is also a non-empty set by assumption. Then, since \( \theta_S \) is a continuous real-valued scalar function, it follows that the image of this function over \( \mathbf{S} \) given by \( \Theta_S \) is a convex and non-empty set on the real line, i.e. an interval whose closure has endpoints given by (26). This completes the final part of the proof.

**S.5.6 Proof of Proposition S.3**

In order to show (i), observe that as there is \( j_0 \in C_i(p) \) such that \( \tilde{U}_{ij_0}(Y_i - p_{j_0}) \geq \tilde{U}_{ij}(-I_i) \geq \tilde{U}_{ij}(Y_i - p_j) \) for each \( j \in \mathcal{J} \setminus C_i(p) \) for all \( p \in \mathcal{P} \), where the first inequality follows from the additional condition on the utilities in (i) and the second from the fact that \( \tilde{U}_{ij} \) is increasing, we have that

\[
\arg \max_{j \in C_i(p)} \tilde{U}_{ij}(Y_i - p_j) = \arg \max_{j \in \mathcal{J}} \tilde{U}_{ij}(Y_i - p_j)
\]

for all \( p \in \mathcal{P} \). The definition of \( \tilde{B}_{ij}^{a,b} \) in (S.21) can therefore be equivalently replaced by

\[
\max_{j \in \mathcal{J}} \tilde{U}_{ij}(Y_i - p_j^a) = \max_{j \in \mathcal{J}} \tilde{U}_{ij}(Y_i - p_j^b - \tilde{B}_{ij}^{a,b})
\]
As this is the same as that in (4) with $U_{ij}$ replaced by $\tilde{U}_{ij}$, Proposition 1 directly applies to give (i), which completes the first part of the proof.

In order to show (ii), let $C$ denote the set of all possible values of $C_i(p)$ for $p \in \mathcal{P}$. Fixing $(C_i(p^a), C_i(p^b)) = (c^a, c^b) \in C \times C$, consider the value of $B_i^*$ that solves

$$
\max_{j \in c^b} \tilde{U}_{ij}(Y_i - \tilde{p}_j^a) = \max_{j \in c^b} \tilde{U}_{ij}(Y_i - p_j^b - B_i^*) ,
$$

where $\tilde{p}_j^a = p_j^a$ if $j \in c^a$ and $\tilde{p}_j^a = p_j^b + r$ if $j \notin c^a$ for some $r \geq 0$. Analogously applying the arguments from Proposition 1 to that in (S.44), we can show that $B_i^*$ exists and is unique, and that

$$
E \left[ \tilde{B}_i^* \right](C_i(p^a), C_i(p^b)) = (c^a, c^b) = \sum_{k=1}^{|c^h|-1} \sum_{j \in C_{k+1}} \int_{\Delta_k}^{\Delta_{k+1}} q_{j|c^a,c^b} \left( \min\{\tilde{p}_j^a, p_j^b + t\} \right) dt \quad (S.45)
$$

where $\Delta_1^* \leq \cdots \leq \Delta_{|c^b|}^*$ denote the ordered values of $\tilde{p}_j^a - p_j^b$ across $j \in c^b$, $C_k = \{ j \in c^b : \tilde{p}_j^a - p_j^b \geq \Delta_k^* \}$ for $1 \leq k \leq |c^b|$, and

$$
q_{j|c^a,c^b}(p) = \operatorname{Prob} \left\{ \arg \max_{m \in c^b} \tilde{U}_{im}(Y_i - p_m) = j \mid (C_i(p^a), C_i(p^b)) = (c^a, c^b) \right\}
$$

for each $j \in J$. For each $(c^a, c^b) \in C \times C$, as the problem in (S.44) reduces to that in (S.21) as $r \to \infty$ due to the additional condition on the utilities in (ii), it follows that $\tilde{B}_i^{a,b}$ also exists and is unique, and that

$$
E \left[ \tilde{B}_i^{a,b} \right](C_i(p^a), C_i(p^b)) = (c^a, c^b) = \lim_{r \to \infty} E \left[ \tilde{B}_i^* \right](C_i(p^a), C_i(p^b)) = (c^a, c^b)
$$

$$
= \sum_{k=1}^{|c^a|-1} \sum_{j \in C_{k+1}} \int_{\Delta_k}^{\Delta_{k+1}} q_{j|c^a,c^b} \left( \min\{\tilde{p}_j^a, p_j^b + t\} \right) dt + \sum_{j \in c^b \setminus c^a \Delta_{|c^a|}} \int_{\Delta_{|c^a|}}^{\infty} q_{j|c^a,c^b} \left( \min\{\tilde{p}_j^a, p_j^b + t\} \right) dt ,
$$

where the second equality follows from expanding the expression in (S.45) using the fact that for large enough $r$ we have that $\Delta_i^* = r$ and $C_i = c^b \setminus c^a$ for $|c^a| < l \leq |c^b|$. It remains to show that (S.23) holds. To this end, we argue that

$$
E \left[ \tilde{B}_i^{a,b} \right](C_i(p^a), C_i(p^b)) = (c^a, c^b) \geq \sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^{a,b} \Delta_{l}^{a,b}} \int_{\Delta_{l}^{a,b}}^{\Delta_{l+1}^{a,b}} q_{j|c^a,c^b} \left( \min\{p_j^a, p_j^b + t\} \right) dt ,
$$

which by taking expectations over the values of $(C_i(p^a), C_i(p^b))$ will then imply (S.23). In order to show this and complete the proof, it is useful to first introduce some additional notation. Let
\(\tilde{\Delta}_1 \leq \cdots \leq \tilde{\Delta}_{|c_a|+1}\) be such that \(\tilde{\Delta}_l = \Delta^*_l\) for \(1 \leq l \leq |c^a|\) and \(\tilde{\Delta}_{|c_a|+1} = \Delta_{a,b}^{a,b}\). Moreover, for each \(l \in \{1, \ldots, |J|\}\), let \(k_l\) be the smallest value in \(\{1, \ldots, |c^a|+1\}\) such that \(J_{l+1}^{a,b} \cap c^a = \mathcal{C}_{k_l} \cap c^a\); and, for each \(k \in \{1, \ldots, |J|\}\), let \(l_k\) be the smallest value in \(\{1, \ldots, |J|\}\) such that \(\Delta_{l_k}^{a,b} = \tilde{\Delta}_k\). Using this notation, observe for \(r\) large enough such that \(r > \Delta_{|J|}^{a,b}\), we have that

\[
\sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{p^a, p^b + t\}\right) dt = \sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^{a,b} \cap c^b} \int q_{j[c^a,c^b]} \left(\min\{p^a, p^b + t\}\right) dt
\]

\[
\leq \sum_{l=1}^{|J|-1} \sum_{j \in J_{l+1}^{a,b} \cap c^b} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
\leq \sum_{l=1}^{|J|-1} \sum_{j \in \mathcal{C}_{k_{l+1}}^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
= \sum_{k=1}^{|c^a|+1} \sum_{l=1}^{|J|-1} \sum_{j \in \mathcal{C}_k^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
= \sum_{k=2}^{|c^a|+1} \sum_{l=1}^{k-1} \sum_{j \in \mathcal{C}_k^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
= \sum_{k=1}^{|c^a|} \sum_{j \in \mathcal{C}_{k+1}^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
= \sum_{k=1}^{|c^a|} \sum_{j \in \mathcal{C}_{k+1}^{a,b}} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt
\]

\[
+ \sum_{j \in c^b \setminus c^a} \int q_{j[c^a,c^b]} \left(\min\{\tilde{p}^a, p^b + t\}\right) dt,
\]

where the first line follows from the fact that \(q_{j[c^a,c^b]}(p) = 0\) for \(j \in J_{l+1}^{a,b} \setminus c^b\); the second line from the fact that for each \(j \in J_{l+1}^{a,b}\), we have that \(\min\{\tilde{p}^a_j, p^b_j + t\} = \min\{p^a_j, p^b_j + t\} = p^a_j + t\) for \(t \in (\Delta_1^{a,b}, \Delta_{l+1}^{a,b})\) as \(p^a_j + t \leq p^a_j \leq \tilde{p}^a_j\) and that \(\min\{\tilde{p}^m_j, p^m_j + t\} \geq \min\{p^a_j, p^b_j + t\}\) for \(m \neq j \in \mathcal{J}\), which then implies that \(q_j(\min\{p^a_j, p^b_j + t\}) \geq q_j(\min\{p^a_j, p^b_j + t\})\) given the assumption that the utilities are increasing; the third line from the fact that \(J_{l+1}^{a,b} \cap c^b = \mathcal{C}_{k_{l+1}} \cup \left(J_{l+1}^{a,b} \setminus (c^b \setminus c^a)\right) \subseteq \mathcal{C}_{k_{l+1}}^{a,b}\).
using the definition of $C_{k+1}$; the fourth line from rearranging the summations; the fifth line from
the definition of $l_k$ and the fact that $k = 1$ can be dropped as $l_0 = l_1 = 1$ due to the assumption that
there is an alternative with prices normalized to 0; the sixth line from summing up the integrals
using the fact that for each $j \in C_k$ and $l_{k-1} \leq l \leq l_k - 1$ we have that $\min\{\tilde{p}^a_j, p^b_j + t\}$ equals $p^b_j + t$
for $j \in C_k$ and $\tilde{p}^a_j$ for $j \in c^b \setminus C_k$ and that $q_{j|c^a,c^b}$ is invariant to prices of alternatives in $j \notin c^b$;
the seventh line from shifting index $k$ by a value of one and using the introduced notation that
$\Delta_{a,b}^{a,b} = \tilde{\Delta}_k$; and final equality from the relation between $\tilde{\Delta}_k$ and $\Delta^*_k$. Taking $r \to \infty$ then completes
the proof.

S.6 Additional Table and Figures

Table S.3: Dimension of optimizing variable under different specifications for the programs estimating welfare effects under the status quo voucher amount

|                 | Nonparametric | Parametric Separable, $K$ |
|----------------|---------------|---------------------------|
| True Baseline  | 1.91×10^67   | 5,618                     |
| Outer Baseline | 1,855         | 8,427                     |
| Separable      | 65,773        | 11,236                    |
|                | 5,618         | 14,045                    |
|                | 8,427         | 16,854                    |
|                | 11,236        |                           |
|                | 14,045        |                           |
|                | 16,854        |                           |

True Baseline true denotes the programs under baseline specification in (26), and Outer Baseline outer denotes the programs under the baseline specification in (27).

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.
Table S.4: Estimates of the underlying parameters for the various logit specifications

| Specification | Logit I | Logit II | Mixed Logit | Nested Logit |
|---------------|---------|----------|-------------|--------------|
| $\bar{\gamma}_0$ | 7.22    | 8.34     | 8.66        | 8.58         |
|                | [0.36]  | [0.67]   | [0.67]      | [0.68]       |
| $\bar{\gamma}_{11}$ | -0.63   | -0.72    | -0.41       |               |
|                | [0.81]  | [0.82]   | [0.88]      |              |
| $\bar{\gamma}_{12}$ | -0.63   | -0.64    | -0.11       |              |
|                | [0.80]  | [0.84]   | [0.91]      |              |
| $\bar{\gamma}_{13}$ | -2.38   | -2.25    | -2.43       |              |
|                | [0.69]  | [0.69]   | [0.76]      |              |
| $\sigma$      |         |          | 1.22        |              |
|                |         |          | [0.51]      |              |
| $\lambda_1$   |         |          | 1.93        |              |
|                |         |          | [0.32]      |              |

Standard errors reported in parentheses. $\bar{\gamma}_l$ corresponds to the coefficient on the $l$th income bin, i.e. incomes between the $l$ and $l+1$ quartiles of the empirical distribution. The coefficient on the fourth income bin is normalized to 0 due to avoid perfect multicollinearity with the respect to the constant term. $\lambda_2$ is normalized to 1 as there is a single alternative in $N_2$.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

Table S.5: Expressions for demand under the various logit specifications

| Specification | Demand function for each $j \in J$ |
|---------------|-----------------------------------|
| Logit I       | $q_j(p) = \frac{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l}}{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l}} f_X(x)$ |
| Logit II      | $q_j(p) = \int \frac{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j}}{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j}} f_X(x) dx$ |
| Mixed Logit   | $q_j(p) = \int \frac{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j - \gamma_2 p_j}}{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j - \gamma_2 p_j}} \phi(v) f_X(x) dx dv$ |
| Nested Logit  | $q_j(p) = \int \frac{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j - \gamma_2 p_j}}{\sum_{l \in J} e^{\xi_l - \gamma_0 p_l - \gamma_1 p_j - \gamma_2 p_j}} f_X(x) dx$ |

$f_X$ is the density of $X_i$, $\phi$ is the density of a normal distribution with mean 0 and variance $\sigma^2$, and $N' = N_1$ and $N' = N_2$ and $\lambda' = \lambda_2$ otherwise.
Figure S.1: Estimated bounds on average surplus under nonparametric baseline specification for alternative values of government school costs \((c_g)\) and administrative costs \((\mu)\)

Observe that as long as we assume that \(c_g\) is at least around $5,000, and that \(\gamma\) is at most around $500, we continue to robustly find positive net average benefits.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.

Figure S.2: 90% confidence intervals on average surplus for a range of voucher amounts

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.
Figure S.3: Estimates along with 90% confidence intervals for the logit specifications for various demand and welfare parameters at various counterfactual voucher amounts

For the nonparametric baseline and parametric separable specifications, the intervals denote the estimated lower and upper bounds. For the various logit specifications, the markers denote the point estimates and the dashed intervals denote 90% confidence intervals computed using the percentile bootstrap with 200 draws.

SOURCE: Evaluation of the DC Opportunity Scholarship Program: Final Report (NCEE 2010-4018), U.S. Department of Education, National Center for Education Statistics previously unpublished tabulations.
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