Functions to Support Input and Output of Intervals

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Research Report DCS-311-IR
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Abstract

Interval arithmetic is hardly feasible without directed rounding as provided, for example, by the IEEE floating-point standard. Equally essential for interval methods is directed rounding for conversion between the external decimal and internal binary numerals. This is not provided by the standard I/O libraries. Conversion algorithms exist that guarantee identity upon conversion followed by its inverse. Although it may be possible to adapt these algorithms for use in decimal interval I/O, we argue that outward rounding in radix conversion is computationally a simpler problem than guaranteeing identity. Hence it is preferable to develop decimal interval I/O ab initio, which is what we do in this paper.

1 Introduction

Interval arithmetic endows every computation with the authority of proof — the theorem being that the real-valued solution \( x \) belongs to a set of reals \([a, b]\), where \( a \) and \( b \) are IEEE-standard floating-point numbers. Yet it can easily happen that we get as output something obviously wrong such as \([0.33333, 0.33333]\) when \( x = 1/3 \). It may well be that a correct interval has been computed. For example, if \([a, b]\) is the narrowest single-length IEEE-standard floating-point interval containing \( 1/3 \), and if we do not allow the output to be truncated, we get \([a, b]\) as

\[
\begin{align*}
[0.333333313465118408203125, \\
0.3333333432674407958984375],
\end{align*}
\]

which is best written as

\[
0.3333333313465118408203125, 0.33333343267407958984375,
\]

a notation proposed and analyzed in [9]. This is the unabridged version of

\[
[0.33333, 0.33333],
\]

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which is what we get with a typical default precision. The untruncated decimal numerals are exact representations of $[a, b]$ because every binary floating-point number can be represented as a decimal numeral. However, if one wants to ensure exact representation, then we get a decimal for every bit; not an economical representation. Then we might as well write the bits themselves, which gives

$$0.010101010101010101010101[0,1].$$

On output the only problem is that any reasonable choice of display precision causes the standard output routines to shorten the numerals for the bounds to the same result. The improvement needed here is output that is aware of whether an upper or a lower bound is to be displayed.

On input, however, we have a more serious problem. Suppose we want to initialize an interval variable at 0.1. Initializing the lower and upper bounds with

$$\text{lb} = \text{ub} = 0.1$$

is guaranteed to generate an interval that does not contain 0.1, for the simple reason that there is no floating-point number equal to 0.1. Thus the best the compiler can do is to produce one of the floating-point numbers closest to 0.1. We don’t know which one it is. The table in Figure 1 shows for each of $1/2, 1/3, \ldots, 1/10$ that the compiler picks the upper bound of the narrowest interval containing the fraction concerned. To remind us that we cannot count on this to happen, the other choice is made for $1/11$. Thus, for input we need an algorithm that produces, for every fraction or numeral as input, a narrow interval containing it; ideally the narrowest.

Floating-point numbers are normalized as a power of 2 multiplied by a mantissa that is between 1 and 2, like this:

$$2^{(-2)} \times 1.0101010101010101010101[0,1].$$

This shows the single-length format, which has 23 bits after the binary point.

Long strings of bits are usually shown in hexadecimal notation, which uses the 16 characters

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f$$

to represent four consecutive bits at a time. As there are only 23 bits to be displayed, we represent the first three bits in octal notation, with the characters

$$0, 1, 2, 3, 4, 5, 6, 7.$$  

Thus the 23 bits after the binary point are shown as an octal character followed by five hexadecimal characters. Thus, the narrowest interval containing $1/3$ is

$$2^{(-2)} \times 1.2aaaa[a,b].$$

In this way we get the table in Figure 1.

In this paper we develop algorithms to support software that performs correct input and output of intervals.
1/i floating-point number narrowest interval produced by standard containing 1/i I/O library via compiler

| 1/2     | 2^(-1) * 1.000000 | 2^(-1) * 1.000000[] | |
| 1/3     | 2^(-2) * 1.2aaaaab | 2^(-2) * 1.2aaaa[a,b] | |
| 1/4     | 2^(-2) * 1.000000 | 2^(-2) * 1.000000[] | |
| 1/5     | 2^(-3) * 1.4cccccd | 2^(-3) * 1.4cccc[c,d] | |
| 1/6     | 2^(-3) * 1.2aaaaab | 2^(-3) * 1.2aaaa[a,b] | |
| 1/7     | 2^(-3) * 1.124925 | 2^(-3) * 1.12492[4,5] | |
| 1/8     | 2^(-3) * 1.000000 | 2^(-3) * 1.000000[] | |
| 1/9     | 2^(-4) * 1.638e39 | 2^(-4) * 1.638e3[8,9] | |
| 1/10    | 2^(-4) * 1.4cccccd | 2^(-4) * 1.4cccc[c,d] | |
| 1/11    | 2^(-4) * 1.3a2e8c | 2^(-4) * 1.3a2e8[c,d] | |

Figure 1: Why standard input cannot be relied on to obtain an interval for some of the common fractions.

2 Previous work

The problems described in the introduction have motivated Rump [8] to develop a method for decimal input and output for intervals. He computes two 2-dimensional arrays low and upp of double-length floating-point numbers that satisfy

\[
\text{low}[d, e] \leq d \times 10^e \leq \text{upp}[d, e]
\]

for \( d \in \{1, 2, \ldots, 9\} \) and for \( e \in \{-340, -339, \ldots, 308\} \). These two arrays contain a total of 11682 double-length floating-point numbers.

With these precomputed numbers available, a decimal numeral \(0.d_1d_2\ldots d_k \times 10^e\) is converted on input to the interval

\[
\left[ \sum_{i=1}^{k} \text{low}[d_i, e - i], \sum_{i=1}^{k} \text{upp}[d_i, e - i] \right]
\]

of double-length floating-point numbers, where the inner brackets enclose array subscripts. The additions for the lower (upper) bound are performed in downward (upward) rounding mode. As a result of the possibly occurring roundings, the interval obtained is not in general as narrow as possible. To minimize the inevitable widening, Rump recommends performing the additions starting with the smallest array elements.

The work of Rump should be compared and contrasted with what we will call here general-purpose conversion algorithms. Though these are not specifically intended for interval conversions, they can perhaps be adapted to this purpose because of their guarantees on accuracy.

Let us briefly review these algorithms. Steele and White [5] formulated identity requirements for conversions between internal and external numerals. The internal identity requirement is that conversion of an internal numeral to external and back results in the
same. If the external numerals are unlimited in length, this requirement can always be met. Similarly, the external identity requirement is that conversion of an external numeral to internal and back results in the same. Usually, the internal numerals have a fixed length, so this requirement can only be met when the external numeral is not longer than is warranted by this fixed length.

Every binary numeral is representable as a decimal one. Hence, if the internal numerals are binary and the external ones are decimal, then it is trivial to satisfy the internal identity requirement by using the decimal equivalent of the internal binary numeral. The problem addressed by Steele and White is to do so with as few decimal digits as possible.

To transfer binary floating-point numerals from one computer to another by means of binary files, one needs to be assured that these files are written and read in a compatible way. Moreover, the layout of the floating-point numerals of both machines needs to be the same. Adherence to the IEEE floating-point format is not enough: in addition, both machines have to be big-endian or both need to be little-endian. Because of these difficulties it is attractive to convert internal binary numerals to an external text file containing decimal numerals and to have an algorithm that guarantees faithful translation back to internal format.

Such use of decimal I/O requires that the conversion is efficient. For this reason, Gay [3] and Burger and Dybvig [1] devised a faster decimal output. The identity requirement assumes sufficiently accurate decimal input. This was addressed by Clinger [2] and by Gay [3].

Compared to Rump’s approach, this has the advantage of not requiring a database of precomputed numbers.

The identity requirements satisfied by [5, 3, 1, 2] are convincing for the purposes envisaged by these authors. However, the requirements for decimal interval I/O are equally compelling and quite different:

1. On input, compute the narrowest floating-point interval containing the decimal numeral.

2. On output, compute the narrowest interval containing the floating-point number that is bounded by decimal numerals of specified length.

To satisfy these requirements it seemed simplest to develop our algorithms from first principles rather than to attempt to adapt the work referenced above.

From [4] it seems that XSC does the conversions between binary and decimal numerals correctly, but the authors do not say how it was done. Although the source code of XSC is publicly available (GPL license), it contains more than 150 files. Moreover, the code is not documented in such a way as to facilitate finding the pieces of code that do the conversion, to gather them, and to understand how the conversion was done.

Our task, then, is to explain how to do the conversion, and to show how to implement it. As the utility of this kind of work requires executable programs, we needed to pick an implementation language. The primary language for numerical work is Fortran. However, this is more in the direction of systems programming, for which C/C++ is a reasonable choice of language.
3 Decimal input

The “scientific notation” for a number consisting of a decimal fraction and an exponent is almost universally used. Thus it would seem that one only has to cater to this format for I/O routines. Yet for input there is a strong case to be made for the pre-scientific notation of a fraction as a pair of integers. Therefore we consider these two in turn.

3.1 Rational fractions

In many situations the most convenient way to input a number is as a rational fraction $p/q$, where $p$ and $q$ are integers. Not only is it convenient, but there is also a convincing correctness criterion: as there exists, in a given format, a unique least floating-point interval that contains $p/q$, the input function should yield this interval.

An algorithm for this purpose is one that has been widely, but not universally, taught to children for at least two centuries. We will illustrate this algorithm with $p/q = 3/7$. As this number is less than 1, we know that the binary fraction has the form 0.0... How do we get the missing digits? Multiply by two to get 6/7. As the result is less than one, the result has the form 0.0... Multiply by two to get 12/7. As the result is not less than one, the result has the form 0.01... and subtract 1 from 12/7, so that we continue with 5/7. Double again, get 10/7, so that we continue with 3/7 after noting that the result has the form 0.011... We already know what comes after that, so we have determined that the binary equivalent of $3/7$ is 0. 011 011 011 ...

Let us check our computation. The first group of digits after the decimal point is worth $3/8$. Every next one is worth 1/8 times the previous group of three. So the value of the infinite string is $(3/8) \times (1 + (1/8) + (1/8)^2 + ...)$.

Let us now translate this procedure to a machine-executable algorithm. We can decide whether the next binary digit should be a 0 or a 1 by computing in a variable $pwr$ (from “power”) the successive powers $2^{-k}$ for $k = 0, 1, 2, ...$ and adding some of these powers in a variable named $frac$ (from “fraction”). We compute the next value of $pwr$ at every step, but only add it to $frac$ when deciding to write a 1 in the algorithm described above. This idea is embodied in the following code.

```java
float frac = 0.0;
// fraction to be built up out of powers of two
float pwr = 1.0; // power of 2 to add to fraction
// let r = p/q
while (frac + pwr > frac && p > 0) {
    // p > 0 and r - frac = (p/q)*pwr
    pwr = pwr/2.0; p = 2*p;
    if (p >= q) {
        p = p-q; frac = frac+pwr;
    }
    // if p = 0 then r = frac
    // if p > 0 then 0 < r - frac <= pwr
}
```
Typically, there are infinitely many binary digits. Only the first of a finite segment can be accommodated in a floating-point number. At every iteration \( pwr \) is halved. At some point this quantity becomes insignificant. The criterion for this is whether adding \( pwr \) to \( frac \) makes any difference to \( frac \). As soon as that is not the case, the iteration terminates.

The last assertion states that every time around the loop we have that \( frac \) is a lower bound for \( p/q \) and that the difference between the two is at most \( 2^{-i} \), where \( i \) is the number of times around the loop. This code builds in \( frac \) a lower bound to \( p/q \) that approaches \( p/q \) as closely as the precision allows. If \( p/q \) has can be represented in the floating-point number format, this shows by \( p \) becoming 0.

The above segment of code is the kernel of the function shown in Figure 2 for the IEEE standard 754 single-length format. It produces the least floating-point interval that contains the fraction \( p/q \).

```cpp
Interval* convert(int p, int q) {
    // Assumes p and q are positive and less than 2^30.
    // Returns the greatest float that is not greater than
    // the rational r = p/q.
    float sf = scaleFactor(p, q);
    float frac = 0.0;
    // fraction to be built up out of powers of two
    float pwr = 1.0; // power of 2 to add to fraction
    while (frac + pwr > frac && p > 0) {
        // p > 0 and r - frac = (p/q)*pwr
        pwr = pwr/2.0; p = 2*p;
        if (p >= q) {
            p = p-q; frac = frac+pwr;
        } // p > 0 and r - frac = (p/q)*pwr
    }
    frac = frac*sf;
    if (p==0) // r = frac
        return new Interval(frac, frac);
    else return new Interval(frac, next(frac));
}
```

Figure 2: A C++ function to convert a rational \( p/q \) to a floating-point number. The function `next()` produces the least floating-point number greater than its argument.

The function assumes that \( 0.5 \leq p/q < 1 \), which is of course not in general the case. It therefore needs the function in Figure 3.

### 3.2 Input of decimal floating-point numerals

Many programming languages and data files use a similar format for fractional numbers. Though details may vary, it is easy to extract from such files an integer \( e \) containing the
float scaleFactor(int& p, int& q) {
    // Let r = p/q.
    // Returns a scale factor sf such that p/q is in [0.5,1)
    // and r = (p/q)*sf
    float sf = 1.0; // the scale factor
    r = (p/q)*sf; assume q < 2^(30) to avoid overflow
    while (q > p) { p = 2*p; sf /= 2.0; }
    // r = (p/q)*sf and q <= 2*p;
    // assume p < 2^(30) to avoid overflow
    while (p >= q) { q = 2*q; sf *= 2.0; }
    // r = (p/q)*sf and p < q <= 2*p;
    // therefore 0.5 <= (p/q) < 1
    return sf;
}

Figure 3: A C++ function to scale the fraction p/q.

exponent and a string containing the decimal digits d₁, . . . , dₙ of the fraction, where d₁ ≠ 0.
We assume that these source code or data file elements are intended to denote the rational
number r = 10^e ∑_{i=1}^{n} dᵢ10⁻ⁱ. The function we aim at here takes as input e and d₁, . . . , dₙ
and returns the narrowest floating-point interval containing r.

Such numerals can be treated analogously to the rational fractions discussed in Sec-
tion 3.1. A difference is that instead of subjecting a rational fraction p/q to repeated dou-
bling possibly combined with subtracting 1, we do this with a string of decimal digits rep-
resenting the mantissa of the number to be input.

A more significant difference compared to the rational-fraction case is the presence of
a power of 10. We need to convert to binary not only the mantissa, but also the input power
of 10. This happens in a preliminary stage we call binarization.

For example, suppose we desire to convert 0.0123 to binary. According to our assumed
convention, we have e = −1 and have d₁, . . . , dₙ in the form of the string "123".

In this example the power of 10 is exchanged with power of 2 and a corresponding
change in mantissa as follows: 0.123 * 10⁻¹ = 0.246 * 10⁻¹ * 2⁻¹ = 0.492 * 10⁻¹ * 2⁻² =
0.984 * 10⁻¹ * 2⁻³ = 0.1968 * 2⁻⁴.

To implement binarization, we need a function to double a decimal mantissa given
as a string of decimal digits. Each digit is doubled by integer arithmetic operating on
the numerical equivalent of the digit. In this operation neither rounding nor overflow can
occur. See the function mul2 in Figure 4. This function is not a general-purpose doubling
routine: it is specific to its argument being the decimal digits d₁, . . . , dₙ of a mantissa of
the form ∑_{i=1}^{n} dᵢ10⁻ⁱ. Accordingly, when the result is 1 or greater, this additional digit is
not inserted into the resulting string. Instead the last carry is returned. By inspecting it, the
calling code can determine whether the mantissa has overflowed.

With the doubling function available, the binarization function is straightforward. See
Figure 5.

In the next stage, normalization, we ensure that the mantissa is in the interval [0.5,1).
In our current example this happens by means of the steps 0.1968 * 2⁻⁴ = 0.3936 * 2⁻⁵ =
int mul2(string& mnts) {
    // Multiplies decimal mantissa in argument by 2.
    // Returns last carry.
    int carry = 0;
    int dd;  // Result of doubling a decimal digit.
    for(int i = mnts.length()-1; i >= 0; i--){
        dd = 2*(mnts[i] - '0') + carry;
        mnts[i] = dd%10 + '0'; carry = dd/10;
    }
    if (mnts[mnts.length()-1] == '0')
        mnts.erase(mnts.length()-1,1);
    // mnts is a mantissa, hence no trailing zeros
    return carry;
}

void binarizeExp(string& mnts, int& exp) {
    // When called, exp is a decimal exponent.
    // On exit, exp is a binary exponent and mnts is adjusted,
    // so that the same number is denoted.
    int binExp = 0; int carry = 0;
    while (exp < 0 ) {
        carry = mul2(mnts); binExp--;
        if (carry != 0) { exp++; mnts.insert(0,"1"); }
    }
    while (exp > 0 ) {
        div2(mnts); binExp++;
        if (mnts[0] == '0') { exp--; mnts.erase(0,1); }
    }
    // exp = 0
    exp = binExp;
}

Figure 4: A C++ function to double a number of the form $\sum_{i=1}^{n} d_i 10^{-i}$ where the $d_1, \ldots, d_n$ are given in the string argument mnts.

Figure 5: A C++ function to binarize a number of the form $10^e \sum_{i=1}^{n} d_i 10^{-i}$ where the digits $d_1, \ldots, d_n$ are given in the string argument mnts. If we denote the values on exit by adding primes, we have $10^e \sum_{i=1}^{n} d_i 10^{-i} = 2^e' \sum_{i=1}^{n'} d_i' 10^{-i'}$.
void normalize(string& mnts, int& exp) {
    // Maintaining the value of the denoted number,
    // adjusts exp so that 0.5 <= 0.mnts < 1.
    while ((mnts[0] - '0') < 5) {
        exp--; mul2(mnts); // discard zero carry
    }
}

0.7872 * 2^{-6} The function for normalizing is in Figure 6.

Figure 6: A C++ function to normalize a number of the form $\sum_{i=1}^{n} d_i 10^{-i}$ where the mantissa $d_1, \ldots, d_n$ is given in the string argument mnts.

After binarization and normalization we are ready to start the conversion of the decimal mantissa to binary. A good starting point is one of the radix conversion methods given by Knuth [7], section 4.4, where he converts from radix $b$ to radix $B$. In our case we have $b = 10$ and $B = 2$. The options are to divide by 10 using radix-2 arithmetic and to multiply by $B = 2$ using radix-10 arithmetic. We select the latter.

Thus we have a fractional number $u$ given as a string of decimal digits. Knuth obtains the digits $U_1, U_2, \ldots$ of the binary representation as follows:

\[
\begin{align*}
U_1 &= \lfloor uB \rfloor \\
U_2 &= \lfloor \{uB\}B \rfloor \\
U_3 &= \lfloor \{\{u\}B\}B \rfloor \\
\vdots
\end{align*}
\]

where $\{x\}$ denotes $x \mod 1$, which is $x - \lfloor x \rfloor$.

We must not only obtain the successive binary digits $U_1, U_2, \text{ and } U_3$, but we must also pack them as a floating-point number. Hence we modify Knuth’s iteration to the following, of which we show the first few steps, continuing the previous example.

\[
\begin{align*}
0.7872 &= 0.7872 \times 1 \\
       &= 1.5744 \times (1/2) \\
       &= 1/2 + 0.5744 \times (1/2) \\
       &= 1/2 + 1.1488 \times (1/4) \\
       &= 1/2 + 1/4 + 0.1488 \times (1/4) \\
       &= 1/2 + 1/4 + 0.2976 \times (1/8) \\
       &= 1/2 + 1/4 + 0.5952 \times (1/16) \\
       &= 1/2 + 1/4 + 1.1904 \times (1/32) \\
       &= 1/2 + 1/4 + 1/32 + 0.1904 \times (1/32), \\
\end{align*}
\]
Giving $0.11001\ldots$ as the binary representation of 0.7872. We see that in this way a sum of powers of 2 is built up while the remaining decimal mantissa is multiplied by an ever smaller factor. The iteration is terminated when this product is less than the machine precision. At that point the sum of the powers of 2 is the left bound of the narrowest interval containing 0.7872. The multiplications and additions, though floating-point operations, are, by their special nature, performed without rounding error.

In the final stage, we scale the lower bound by the factor $2^{-6}$ resulting from binarization and normalization. We perform this scaling by successive divisions or multiplications by 2, again assuring the absence of rounding errors. Figure 7 displays a function along these lines that finds the narrowest floating-point interval. This function that the fraction $x$ to be converted is non-negative. In that case the result is $[a, a']$, where $a'$ is the least floating-point number greater than floating-point number $a$. This function can also be used for a negative fraction $y$. If our function gives $[a, a']$ with input $-y$, then we change the output

```c++
interval convertFrac(string& mnts, int& exp) {
  interval result; // interval to be returned
  // $r = 10^{\exp} \times 0.mnts$
  binarizeExp(mnts, exp);
  // $r = 2^{\exp} \times 0.mnts$
  normalize(mnts, exp);
  // $r = 2^{\exp} \times 0.mnts$ and $0.5 <= 0.mnts < 1$
  // $r' = 0.mnts$ and $0.5 <= 0.mnts < 1$
  float frac = 0.0; float pwr = 1.0;
  int carry = 0;
  while (frac+pwr > frac && mnts.length() > 0) {
    // $0 <= r' - frac = 0.mnts \times pwr$
    pwr /= 2.0; // $pwr = 2^{-i}$
    carry = mul2(mnts);
    if (carry != 0) { frac += pwr;
      // subtract 1 by discarding nonzero carry
    }
  } // $0 <= r' - frac = 0.mnts \times pwr$
  // and pwr is negligible compared to frac
  // hence frac is the greatest flpt less than r'
  // scale frac with exp
  while (exp > 0) {frac *= 2.0; exp--;
  } while (exp < 0) {frac /= 2.0; exp++;
  result.lb = frac;
  result.ub = (mnts.length() == 0) ? frac : next(frac);
  return result;
}
```

Figure 7: A C++ function to find the narrowest floating-point interval that contains a given number of the form $10^e \sum_{i=1}^{n} d_i 10^{-i}$ where the $d_1, \ldots, d_n$ are given in the string argument mnts and $e$ is given in the argument exp.
to $[-a',-a]$.

This function uses some auxiliary declarations: one that defines interval and one that defines a function to determine the next floating-point number after a given one. The auxiliary definitions are displayed in Figure 8.

typedef struct interval { float lb, ub; }

float next(float x) {
    // returns the least flpt number greater than nonnegative x
    // if x is normalized and if it is less than the greatest
    // flpt number
    // x0 = x
    float sf = 1.0; // becomes the scale factor
    // x0 = x*sf
    while (x < 1.0) { x *= 2.0; sf /= 2.0; }
    // x0 = x*sf and 1 <= x
    while (x >= 2.0) { x /= 2.0; sf *= 2.0; }
    // x0 = x*sf and 1 <= x < 2
    float eps = 1.0; // becomes the machine epsilon
    while (((float)1.0 + eps) > (float)1.0) eps *= 0.5;
    // eps is the first one that did not make a difference
    eps *= 2.0;
    // eps is the last one that did make a difference
    // by definition the machine epsilon
    return (x + eps) * sf;
}

Figure 8: Some definitions auxiliary to Program 7

4 Decimal output

As in decimal input, the main concern is to avoid rounding errors. This is of course taken care of by representing all digits, binary or decimal, separately as small integers. Operations on these are free from rounding errors because the operands are integer; they are immune to overflow because of their smallness. However, we would also like to speed up conversion as much as possible by using operations on floating-point numbers when we can be sure that no rounding errors occur or by using operations of full-size integers when we can be sure that no overflow can occur.

Our starting point is the assumption that a floating-point number can be represented by an integer for the exponent part and by an integer that is represented by the same sequence of bits as there are in the mantissa of the floating-point number. In the case of the IEEE standard single-length format, this integer is 24 bits with the most significant bit equal to 1. That is, for given floating-point $f$, we find integers $e$ and $m$ such that $f = m \cdot 2^e$. As before, we use the case of the IEEE standard 754 single-length floating-point format as example.
Consider the assignment \( m = f \), which is legal in C/C++. It implicitly converts the floating-point number \( f \) to an integer if \( m \) is an integer. If \( f \) is not an integer, then this assignment does not result in \( m \) containing the bits of the mantissa of \( f \). If \( f \) is an integer, then this assignment may also not result in \( m \) containing the bits of the mantissa of \( f \): \( f \) may be larger than the largest integer. However, if we ensure that \( f \) is in the interval \([2^{23}, 2^{24})\), it is both assured to be an integer, while at the same time not causing overflow when assigned to a 32-bit integer. Thus we may extract the bits of \( f \) and place them in \( m \) by simply performing the assignment \( m = f \) provided that we first scale \( f \) to be in this range. The scaling is performed by doubling or halving of \( f \) and therefore does not cause rounding errors. The required scaling operations are accounted for in the binary exponent \( e \). See Figure 9 for a function to perform the conversion from \( f \) to the corresponding \( e \) and \( m \).

```c
void fltoem(const float& f0, int& e, int& m) {
    // returns m and e such that f0 = m * 2^e
    // with 2^{23} <= m < 2^{24} that is
    // m has the same bits as 1.significand of f0
    float f = f0;
    float e23 = 1.0;
    for (int i = 0; i < 23; i++) e23 *= 2.0; // e23 = 2^{23}
    float e24 = 2.0 * e23;
    e = 0; // f0 == f*2^e
    while (f < e23) {f *= 2.0; e--;} // f0 == f * 2^e
    while (f >= e24) {f /= 2.0; e++;} // f0 == f * 2^e
    // f is an integer and 2^{23} <= f < 2^{24}
    m = (int)f; // f0 == m*2^e and 2^{23} <= m < 2^{24}
}
```

Figure 9: A function to realize the relation \( f_0 = m * 2^e \), with \( f_0 \) as input and \( m \) and \( e \) as output. Example for single-length floating-point format.

It remains to determine the string of decimal digits that is equal to \( m * 2^e \) with \( m \) and \( e \) given as variables of type integer. If we place no limit on the length of the string, this is always possible: decimal strings are closed under doubling and halving. We first convert the integer variable \( m \) to a decimal string.

This is done by one of the best known algorithms in programming. One of the forms in which it appears is the function `itoa` in [6]. A streamlined version is the following, though it prints the digits in reverse order:

```c
void print(int n) {
    while (n>0) { cout << n%10; n = n/10; }
    cout << endl;
}
```

What we need is basically the same algorithm, but elaborated by the need to output to a string rather than to print. We also need to ensure that the digits are not reversed. Thus we arrive at the function in Figure 10.
void decimalizeMnts(int& n, string& s) {
    s.erase(0, s.length()); // ensures s is empty
    while (n > 0) {
        s.insert(s.begin(),1,(char)('0' + (n%10)));
        // inserts digit n%10 at the beginning of s
        n = n/10;
    }
}

Figure 10: Decimalizes the integer mantissa.

The result of `decimalizeMnts` is the representation of the integer `m` such that \( f = m \times 2^e \). What we want is a `mantissa`. That is, we need to move the decimal point from the right to the left. This is done by multiplying by a suitable power of 10, which is the length of the decimal numeral. Hence the function in Figure 11.

```cpp
int normalize(const string& mnts) {
    // Returns the decimal exponent necessary to turn mnts
    // from a whole number into a normalized mantissa.
    // Assumes mnts has no leading zeros.
    return mnts.length();
}
```

Figure 11: Transforms the integer mantissa to a fractional one.

In the final stage we convert the binary exponent `e` to increments or decrements of the decimal exponent, with a concomitant change in the mantissa. This happens in the function shown in Figure 12.

To summarize, we transform a floating-point number `f` to an integer `exp` and a mantissa `mnts` such that `f = 0.mnts \times 10^{exp}` by first transforming it to an integer mantissa with a binary exponent (function `fltoem`), then decimalizing the mantissa (function `decimalizeMnts`), then transforming the integer mantissa to a fractional one (function `normalize`), and finally folding the binary exponent into the decimal exponent produced by the normalization stage. This completes the decimal output of a floating-point number. See the function `decOut` in Figure 13.

The function `decOut` in Figure 13 gives all of the decimals of a numeral that is equal to the floating-point number in question. To obtain the lower bound of the required narrowest interval bounded by decimal numerals of specified length, the output of `decOut` needs to be truncated to this length. The upper bound is then obtained by adding to it one unit of the last decimal place.
void decimalizeExp(string& mnts, int& binExp, int& decExp) {
// Reduces the binary exponent binExp to zero.
// Assumes that mnts has no leading zeros.
while (binExp > 0) {
    binExp--;
    if (mul2Mnts(mnts) == 1) {
        decExp++; mnts.insert(0,"1");
    }
    if (mnts[mnts.length()-1] == '0')
        mnts.erase(mnts.length()-1,1);
}
while (binExp < 0) {
    div2(mnts); binExp++;
    if (mnts[0] == '0') {
        mnts.erase(0,1); decExp--;
    }
}
}

Figure 12: Decimalizes the binary exponent.

void decOut(float f, int& exp, string& mnts) {
// returns the decimal string equal to f
    int e, m;
    fltoem(f, e, m); // f = m * 2^e
decimalizeMnts(m, mnts);
    // mnts is decimal equivalent of m
    exp = normalize(mnts);
decimalizeExp(mnts, e, exp);
}

Figure 13: Finds the decimal equivalent of floating-point number.
5 Conclusions

To input an interval given as a pair of decimal numerals, one needs to assure that the resulting pair of floating-point numbers contains the input pair. The fact that a decimal numeral is typically not representable as a binary floating-point number makes it necessary that the internal result is wider than the input. We want it to be wider by as little as possible. The standard library functions are not adequate for this purpose. In this paper we have developed functions that support such interval input.

For output, there is in principle no problem: for every finite binary floating-point numeral there is a decimal numeral that is equal to it. Therefore, if one is willing to accept lengthy numerals, it is possible to obtain as output an interval that is equal to the internally stored floating-point interval. But to obtain an output interval that is both correct and not too long, it is in general necessary to round the decimal bounds in the correct directions.

In this paper we have developed functions that can be used as basic building blocks for the required input and output for intervals. These include

1. For every decimal numeral, computing the unique narrowest floating-point interval containing it.

2. For every finite floating-point number, the decimal numeral that is equal to it.

In designing these building blocks, we have been guided by the criteria of efficiency, language-independence, and machine-independence.

Efficiency To avoid rounding errors and overflow one can perform all arithmetic on numbers represented by strings of digits. These digits are small integers and are therefore operated upon without rounding error and without overflow. The disadvantage is the resulting slowness of the algorithms. We have therefore taken advantage of several situations in which floating-point operations do not give rounding errors and are much faster.

Language-independence The most extreme form of language-independence is to write the algorithms in pseudo-code. To ensure that our algorithms can be verified, we have chosen a simple subset of C++ that almost fits inside the C language. The few excursions beyond the bounds of C are amply rewarded, for example, by the availability of the string class. Of course, the most important part of verification is the understanding of the code. In this respect C is not better than pseudo-code, but not worse either. The reason for presenting our algorithms in executable form is that execution is a useful check on verification by reading the code.

We have facilitated reading the code by including as assertions comments. Of each of these we claim that it holds whenever execution passes the assertion.

But the fact that we want executable algorithms, and hence rely on a specific programming language, does not imply that we feel free to use all the facilities of that language. For example, to determine the next floating-point number after a given one, we can use the facilities of C to access the bits of a floating-point number. But such facilities vary widely among programming languages. We therefore prefer to determine the next floating point number by means of arithmetic operations, so that the resulting algorithm is more readily transferable to another language.
**Machine-independence** For input it is desired to find the smallest floating-point interval containing the number denoted by the input numeral. Usually such a requirement is met by modifying the default rounding mode of the processor. We have refrained from using this option, as the specification of the programming language does not include these operations. As a result, the different implementations of C come with implementation-specific functions to perform these operations.

The motivation of this paper was the usual situation where the semantics of an interval is that it contains one or a finite number of solutions. In this situation correctness of the conversion process means obtaining an interval that contains the original one. It also happens that the semantics of the original interval is that all points in it are solutions, as can happen with inequalities. In such a situation the correctness criterion is to transform the original interval to one that is contained in it, again differing by as little as possible. The functions presented here can serve as building blocks in this situation also.

**Acknowledgements**

This research was supported by the University of Victoria and by the Natural Science and Engineering Research Council of Canada.

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