FORMS ON VECTOR BUNDLES OVER COMPACT REAL HYPERBOLIC MANIFOLDS

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ABSTRACT. We study gauge theories based on abelian \( p \)-forms on real compact hyperbolic manifolds. The tensor kernel trace formula and the spectral functions associated with free generalized gauge fields are analyzed.

1. Introduction

Skew symmetric tensor fields play an important role in quantum field theory, supergravity, and string theory, where they naturally couple to two-form connections. The two-form in string theory (the so called \( B \)-field) is described at low energies by a Maxwell type gauge theory which can be extended to higher \( p \)-forms. The abelian two-forms are closely related to the theory of gerbes which play a role in string theory [1, 2, 3, 4, 5, 6, 7]. Such forms can be understood as a connection on an abelian gerbe.

In the abelian case (which will be considered in this paper) the self-dual two-form can be easily reduced to the abelian one-form gauge field. Generally, the covariant quantization of skew symmetric tensor fields has met difficulties with ghost counting and BRST-transformations. In the framework of functional integration the covariant quantization of free generalized gauge fields – \( p \)-forms and the BRST-transformations have been obtained in Ref. 8. We note the topological anomaly which can be computed using the Atiyah–Singer index theorem. Typically the anomaly is geometric, i.e. it is a smooth line bundle with hermitian metric and a compatible connection. The geometric anomaly (the pfaffian line bundle with metric and connection) may be computed using differential \( K \)-theory, a version of \( K \)-theory which includes differential forms as curvatures [9, 10].

In this paper we show that for \( p \)-forms on real hyperbolic manifolds the variables contribution reduces to a simple alternating sum of forms. We present a decomposition of the Hodge Laplacian and the tensor kernel trace formula associated to free generalized gauge fields – \( p \)-forms. The main ingredient required is a type of differential form structure on the physical, auxiliary, or ghost variables. We consider spectral functions on hyperbolic manifolds associated with physical degrees of freedom of the Hodge–de Rham operators on \( p \)-forms.


2. Exterior Forms of Real Hyperbolic Spaces

We shall work with an $n-$dimensional compact real hyperbolic space $X$ with universal covering $M$ and fundamental group $\Gamma$. We can represent $M$ as the symmetric space $G/K$, where $G = SO_1(n,1)$ and $K = SO(n)$ is a maximal compact subgroup of $G$. Then we regard $\Gamma$ as a discrete subgroup of $G$ acting isometrically on $M$, and we take $X$ to be the quotient space by that action: $X = \Gamma \backslash M = \Gamma \backslash G/K$. Let $\tau$ be an irreducible representation of $K$ on a complex vector space $V_\tau$, and form the induced homogeneous vector bundle $G \times_K V_\tau$ (the fiber product of $G$ with $V_\tau$ over $K$) $\rightarrow M$ over $M$. Restricting the $G$ action to $\Gamma$ we obtain the quotient bundle $E_\tau = \Gamma \backslash (G \times_K V_\tau) \rightarrow X = \Gamma \backslash M$ over $X$. The natural Riemannian structure on $M$ (therefore on $X$) induced by the Killing form $(\ ,\ )$ of $G$ gives rise to a connection Laplacian $L$ on $E_\tau$. If $\Omega_K$ denotes the Casimir operator of $K$—that is $\Omega_K = -\sum y_j^2$, for a basis $\{y_j\}$ of the Lie algebra $k_0$ of $K$, where $(y_j, y_\ell) = -\delta_{j\ell}$, then $\tau(\Omega_K) = \lambda_\tau 1$ for a suitable scalar $\lambda_\tau$. Moreover for the Casimir operator $\Omega$ of $G$, with $\Omega$ operating on smooth sections $\Gamma^\infty E_\tau$ of $E_\tau$ one has

$$\mathcal{L} = \Omega - \lambda_\tau 1;$$

see Lemma 3.1 of Ref. 11. For $\lambda \geq 0$ let

$$\Gamma^\infty (X, E_\tau)_\lambda = \{ s \in \Gamma^\infty E_\tau | -\mathcal{L}s = \lambda s \}$$

be the space of eigensections of $\mathcal{L}$ corresponding to $\lambda$. Here we note that since $X$ is compact we can order the spectrum of $-\mathcal{L}$ by taking $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$; $\lim_{j \to \infty} \lambda_j = \infty$. We shall focus on the more difficult (and more interesting) case when $n = 2k$ is even, and we shall specialize $\tau$ to be the representation $\tau^{(p)}$ of $K = SO(2k)$ on $\Lambda^p \mathbb{C}^{2k}$, say $p \neq k$. The case when $n$ is odd will be dealt with later. It will be convenient moreover to work with the normalized Laplacian $\mathcal{L}_p = -c(n)\mathcal{L}$ where $c(n) = 2(n-1) = 2(2k-1)$. $\mathcal{L}_p$ has spectrum $\{c(n)\lambda_j , m_j\}_{j=0}^\infty$ where the multiplicity $m_j$ of the eigenvalue $c(n)\lambda_j$ is given by

$$m_j = \dim \Gamma^\infty (X, E_{\tau^{(p)}})_{\lambda_j}.$$  

2.1. Quantum dynamics of exterior forms. Let $T_{j_1j_2...j_k}$ be a skew-symmetric tensor of $(0,k)-$type, i.e. $T_{\sigma(j_1,...,j_k)} \overset{def}{=} \text{sgn} (\sigma) T_{j_1,j_2,...,j_k}$, where $\text{sgn}(\sigma) = \pm 1$ is the sign of a permutation $\sigma$. The exterior differential $p-$form is
Here $\wedge$ is the exterior product, $dx^j$ are the basis one–forms, and $j = 1, 2, ..., n$ $(\dim M = n)$. Let $\Lambda^*(M) \equiv \bigoplus_{p=0}^{\dim M} \Lambda^p$ be the graded Cartan exterior algebra of differential forms; $\Lambda^p$ is the space of all $p$–forms on $M$. Let $(\ast T)$ denote a skew symmetric tensor of $(0, n - k)$ type, i.e.

\[(\ast T)_{j_{k+1}...j_n} = (1/k!) \sqrt{|g|} \varepsilon_{j_1...j_n} T^{j_1...j_k}, \quad T^{j_1...j_k} = g^{j_1\ell_1} ... g^{j_k\ell_k} T_{\ell_1...\ell_k}, \]

where $\varepsilon_{j_1...j_n} = \pm 1$ is the Levi–Civita tensor density, and the metric $g_{j\ell}$ (an external gravitational field) has the signature $(+, +, ..., +)$. In local coordinates the exterior differential $d : \Lambda^p \rightarrow \Lambda^{p+1}$, and the co–differential $\delta : \Lambda^p \rightarrow \Lambda^{p-1}$ take respectively the form:

\[d\omega_p = (1/p!) \sum_{j_1,j_2,...,j_{p+1}} \frac{\partial T^{j_2...j_{p+1}}}{\partial x^{j_1}} dx^{j_1} \wedge ... \wedge dx^{j_{p+1}}, \]

\[\delta\omega_p = -(1/(p-1)!) \sum_{j_1,j_2,...,j_{p-1}} \frac{\partial T_{j_1...j_p}}{\partial x^{j_1}} dx^{j_1} \wedge ... \wedge dx^{j_p}. \]

From last equations it is easy to prove the following properties for operators and forms: $dd = \delta\delta = 0$, $\delta = (-1)^{np+n+1} \ast d\ast$, $\ast \ast \omega_p = (-1)^{p(n-p)} \omega_p$. Let $\alpha_p, \beta_p$ be $p$–forms; then the invariant inner product is defined by $(\alpha_p, \beta_p) \equiv \int_M \alpha_p \wedge \ast \beta_p$. The operators $d$ and $\delta$ are adjoint to each other with respect to this inner product for $p$–forms: $(\delta\alpha_p, \beta_p) = (\alpha_p, d\beta_p)$.

In quantum field theory the Lagrangian associated with $\omega_p$ takes the form: $L = d\omega_p \wedge \ast d\omega_p$ (gauge field); $L = \delta\omega_p \wedge \ast \delta\omega_p$ (co–gauge field). The Euler–Lagrange equations supplied with the gauge give $\Sigma_p \omega_p = 0$, $\delta\omega_p = 0$ (Lorentz gauge); $\Sigma_p \ast \omega_p = 0$, $d\omega_p = 0$ (co–Lorentz gauge). These Lagrangians give possible representation of tensor fields or generalized abelian gauge fields. The two representations of tensor fields are not completely independent. Indeed, there is a duality property in the exterior calculus which gives a connection between star–conjugated gauge tensor fields and co–gauge fields. The gauge $p$– forms map into the co–gauge $(n - p)$– forms under the action of the Hodge operator $(\ast)$. From the Hodge theory we have the orthogonal decomposition of $p$– forms

\[\omega_p = \delta\omega_{p+1} + d\omega_{p-1} + \text{Har}_p, \]

with $\text{Har}_p$ being a harmonic $p$– form. It is known that the $L^2$ harmonic $p$–
form $\text{Har}_p^{(2)}$ appear on even real hyperbolic manifolds only. The following result holds:

**Proposition 1:** (Ref. 12, p. 373). The manifold $\mathbb{H}^n$ admits $L^2$ harmonic $p-$forms if and only if $n = 2p$. For even dimensional real hyperbolic manifolds the space of $L^2$ harmonic $p-$forms is infinite dimensional.

One can consider the $L^2-$ de Rham complex:

$$0 \to \Lambda^0(M) \xrightarrow{d_0} \cdots \xrightarrow{d_p} \Lambda^p(M) \xrightarrow{d_{p+1}} \cdots \xrightarrow{d_{n+1}} \Lambda^n(M) \to 0,$$

and its associated $L^2-$ cohomology

$$H^p(M) = \ker (\Lambda^p(M) \xrightarrow{d_p} \Lambda^{p+1}(M)) / \text{range } (d_{p-1}\Lambda^{p-1}(M)).$$

The theorem of Kodaira Ref. 13, p. 165 gives the following injection:

$$\text{Har}_p^{(2)}(M) \xrightarrow{j} H^p(M).$$

The map (injection) $j$ is an isomorphism if and only if $d_{p-1}$ has closed range. If $j$ is not an isomorphism, then $j$ has infinite dimensional co–kernel. The associated Laplacian $\Sigma_p$ has closed range if and only if $d_p$ and $d_{p-1}$ have closed range Ref. 14, p. 446.

3. **The Trace Formula Applied to the Tensor Kernel**

Since $\Gamma$ is torsion free, each $\gamma \in \Gamma - \{1\}$ can be represented uniquely as some power of a primitive element $\rho : \gamma = \rho^{j(\gamma)}$ where $j(\gamma) \geq 1$ is an integer and $\delta$ cannot be written as $\gamma_1^j$ for $\gamma_1 \in \Gamma$, $j > 1$ an integer. Taking $\gamma \in \Gamma$, $\gamma \neq 1$, one can find $t_\gamma > 0$ and $m_\gamma \in \mathfrak{M} \stackrel{\text{def}}{=} \{m_\gamma \in \mathfrak{K} | m_\gamma a = am_\gamma, \forall a \in A\}$ such that $\gamma$ is $G$ conjugate to $m_\gamma \exp(t_\gamma H_0)$, namely for some $g \in G, g\gamma g^{-1} = m_\gamma \exp(t_\gamma H_0)$; that is, $\gamma$ is $G-$ conjugate to $m_\gamma \exp(t_\gamma H_0)$ and $m_\gamma \in SO(n-1)$. For $\text{Ad}$ denoting the adjoint representation of $G$ on its complexified Lie algebra, one can compute $t_\gamma$ as follows [16]:

$$e^{t_\gamma} = \max \{|c| | c = \text{an eigenvalue of } \text{Ad}(\gamma) : g \to g\},$$

Also $\gamma = \delta^{j(r)}$ where $j(r) \geq 1$ is a whole number and $\delta \in \Gamma - \{1\}$ is a primitive element; ie. $\delta$ can not be expressed as $\gamma_1^j$ for some $\gamma_1 \in \Gamma$ and some whole number $j > 1$. The pair $(j(\gamma), \delta)$ is uniquely determined by $\gamma \in \Gamma - \{1\}$. These facts are known to follow since $\Gamma$ is torsion free.
Let \( a_0, n_0 \) denote the Lie algebras of \( A, N \) in an Iwasawa decomposition \( G = KAN \). The complexified Lie algebra \( g = g_0^C = so(2k + 1, \mathbb{C}) \) of \( G \) is of Cartan type \( B_k \) with the Dynkin diagram

\[
\begin{array}{c}
\circ - \circ - \circ \cdots \circ - \circ = \circ .
\end{array}
\]

Since the rank of \( G \) is one, \( \dim a_0 = 1 \) by definition, say \( a_0 = \mathbb{R}H_0 \) for a suitable basis vector \( H_0 \):

\[
H_0 = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & & & \\
. & . & & & \\
. & . & & & \\
0 & & & & \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix},
\]

(14)
is a \( (k + 1) \times (k + 1) \) matrix. By this choice we have the normalization \( \beta(H_0) = 1 \), where \( \beta : a_0 \to \mathbb{R} \) is the positive root which defines \( n_0 \); for more detail see Ref. 15. Define \( C(\gamma) \) on \( \Gamma - \{1\} \) by

\[
C(\gamma) \overset{\text{def}}{=} e^{-\rho_0 t_\gamma} |\det n_0 \left( \text{Ad}(m_\gamma e^{t_\gamma H_0})^{-1} - 1 \right) |^{-1}.
\]

(15)
Lastly, let \( C_\Gamma \subset \Gamma \) be a complete set of representations in \( \Gamma \) of its conjugacy classes. This means that any two elements in \( C_\Gamma \) are non-conjugate, and any \( \gamma \in \Gamma \) is \( \Gamma \)-conjugate to some element \( \gamma_1 \in C_\Gamma \) : \( x\gamma x^{-1} = \gamma_1 \) for some \( x \in \Gamma \). The reader may consult the appendix of [15] for further structural data concerning the Lie group \( SO_1(n, 1) \) (and other rank 1 groups). Note that the Killing form \( (, ) \) is given by \( (x, y) = (n - 1)\text{trace}(xy) \) for \( x, y \in g_0 \).

The standard systems of positive roots \( \Delta^+, \Delta_k^+ \) for \( g \) and \( \mathfrak{t} = \mathfrak{t}_0^C \) — the complexified Lie algebra of \( K \), with respect to a Cartan subgroup \( H \) of \( G \), \( H \subset K \), are given by

\[
\Delta^+ = \{ \epsilon_i | 1 \leq i \leq k \} \cup \Delta_k^+ ,
\]

(16)
where

\[
\Delta_k^+ = \{ \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq k \},
\]

(17)
and

\[
\Delta_k^+ \overset{\text{def}}{=} \{ \epsilon_i | 1 \leq i \leq k \}
\]

(18)
is the set of positive non-compact roots. Here

\[
(\varepsilon_i, \varepsilon_j) = \frac{\delta_{ij}}{(H_0, H_0)} = \frac{\delta_{ij}}{2(2k - 1)}, \tag{19}
\]

\[
(\varepsilon_i \pm \varepsilon_j, \varepsilon_i \pm \varepsilon_j) = \frac{1}{2k + 1}, \quad i < j; \quad \text{i.e.} \quad (\alpha, \alpha) = \frac{1}{2k - 1} \forall \alpha \in \Delta^+_k. \tag{20}
\]

Let \( \tau = \tau^{(j)} \) be representation of \( K \) on \( \Lambda^j \mathbb{C}^{2k} \), \( \Lambda_{\tau^{(j)}} = \Lambda_j \) be highest weight of \( \tau \) is

\[
\begin{bmatrix}
\varepsilon_1 + \cdots + \varepsilon_j \\
\varepsilon_1 + \cdots + \varepsilon_{2k-j}
\end{bmatrix}
\begin{cases}
\text{if } j \leq k \\
\text{if } j > k
\end{cases}. \tag{21}
\]

Writing \( (\Lambda_j, \Lambda_j + 2\delta_k) = (\Lambda_j, \Lambda_j) + (\Lambda_j, 2\delta_k), \delta_k = \sum_{i=1}^{k} (k-i)\varepsilon_i \), for \( j \leq k \)
we have

\[
(\Lambda_j, \Lambda_j) = \left( \sum_{p=1}^{j} \varepsilon_p, \sum_{q=1}^{j} \varepsilon_q \right) = \sum_{p,q=1}^{j} (\varepsilon_p, \varepsilon_q) = \sum_{p,q=1}^{j} (\varepsilon_p, \varepsilon_p) = \frac{j}{(H_0, H_0)}, \tag{22}
\]

\[
(\Lambda_j, 2\delta_k) = \left( \sum_{p=1}^{j} \varepsilon_p, 2 \sum_{i=1}^{j} (k-i)\varepsilon_i + 2 \sum_{i=j+1}^{k} (k-i)\varepsilon_i \right) = 2 \sum_{p=1}^{j} (\varepsilon_p, (k-p)\varepsilon_p)
\]

\[
= \frac{2kj}{(H_0, H_0)} - 2 \sum_{p=1}^{j} p(\varepsilon_p, \varepsilon_p) = \frac{2kj}{(H_0, H_0)} - \frac{2j(j+1)}{(H_0, H_0)^2}. \tag{23}
\]

Therefore,

\[
(\Lambda_j, \Lambda_j + 2\delta_k) = \frac{j + 2kj - j(j+1)}{(H_0, H_0)} + \frac{2kj - j^2}{(H_0, H_0)}. \tag{24}
\]

In the case \( j > k \) we have

\[
(\Lambda_j, \Lambda_j) = \left( \sum_{p=1}^{2k-j} \varepsilon_p, \sum_{q=1}^{2k-j} \varepsilon_q \right) = \sum_{p=1}^{2k-j} (\varepsilon_p, \varepsilon_p) = \frac{2k-j}{(H_0, H_0)}, \tag{25}
\]

\[
(\Lambda_j, 2\delta_k) = 2 \left( \sum_{p=1}^{2k-j} \varepsilon_p, \sum_{i=1}^{k} (k-i)\varepsilon_i \right)
\]

\[
= 2 \left( \sum_{p=1}^{2k-j} \varepsilon_p, \sum_{i=1}^{2k-j} (k-i)\varepsilon_i + \sum_{i=2k-j+1}^{k} (k-i)\varepsilon_i \right)
\]
\[
\sum_{p=1}^{2k-j} \varepsilon_p \sum_{i=1}^{2k-j} k \varepsilon_i - \sum_{i=1}^{2k-j} i \varepsilon_i = 2k(2k-j) (H_0, H_0) - 2 \sum_{i=1}^{k} i(\varepsilon_i, \varepsilon_i) \]

\[
\sum_{p=1}^{2k-j} \varepsilon_p \sum_{i=1}^{2k-j} k \varepsilon_i - \sum_{i=1}^{2k-j} i \varepsilon_i = 2k(2k-j) - (2k-j)(2k-j+1) (H_0, H_0) = (2k-j)(j-1) (H_0, H_0) .
\] (26)

Thus for \( \Lambda_j = \Delta_j^+ - \) highest weight of 
\[
K = SO(2k) \text{ on } \Lambda^j \mathbb{C}^{2k}, \text{ we have } (\Lambda_j, \Lambda_j + 2\delta_k) = (2k-j^2)/(H_0, H_0) = (2k-j^2)/(2(2k-1)) \text{ for } 0 \leq j \leq 2k.
\]

Note that since we have specialized \( \tau \) to be \( \tau^{(p)} \), the space of smooth sections \( \Gamma \mathfrak{g} E_{\tau} \) of \( E_{\tau} \) is just the space of smooth \( p \)-forms on \( X \). We can therefore apply the version of the trace formula developed by Fried in Ref. 17. First we set up some more notation. For \( \sigma_j \) the natural representation of \( SO(2k-1) \) on \( \Lambda^j \mathbb{C}^{2k-1} \) one has the corresponding Harish–Chandra–Plancherel density given, for a suitable normalization of Haar measure \( dx \) on \( G \), by

\[
\mu_{\sigma_p(r)} = \frac{\pi}{24k-4|\Gamma(k)|^2} \left( \frac{2k-1}{p} \right) r P_{\sigma_p(r)} \tanh(\pi r) ,
\] (27)

for \( 0 \leq p \leq k-1 \), where

\[
P_{\sigma_p(r)} = \prod_{\ell=2}^{p+1} \left[ r^2 + (k - \ell + \frac{3}{2})^2 \right] \prod_{\ell=p+2}^{k} \left[ r^2 + (k - \ell + \frac{1}{2})^2 \right]
\] (28)

is an even polynomial of degree \( 2k-2 \). One has that \( P_{\sigma_p(r)} = P_{\sigma_{2k-1-p}(r)} \) and \( \mu_{\sigma_p(r)} = \mu_{\sigma_{2k-1-p}(r)} \) for \( k \leq p \leq 2k-1 \). Define the Miatello coefficients [18] \( a_{2\ell}^{(p)} \) for \( G = SO(2k+1,1) \) by

\[
P_{\sigma_p(r)} = \sum_{\ell=0}^{k-1} a_{2\ell}^{(p)} r^{2\ell} , \quad 0 \leq p \leq 2k-1 .
\] (29)

Let \( \text{Vol}(\Gamma \mathfrak{g} G) \) will denote the integral of the constant function \( 1 \) on \( \Gamma \mathfrak{g} G \) with respect to the \( G \)-invariant measure on \( \Gamma \mathfrak{g} G \) induced by \( dx \). For \( 0 \leq p \leq n-1 \) the Fried trace formula applied to kernel \( K_t \) holds[17]:

\[
\text{Tr} (e^{-t\mathcal{C}_p}) = I^{(p)}_\Gamma (K_t) + I^{(p-1)}_\Gamma (K_t) + H^{(p)}_\Gamma (K_t) + H^{(p-1)}_\Gamma (K_t) ,
\] (30)
where \( I^{(p)}_t(K_t), H^{(p)}_t(K_t) \) are identity and hyperbolic orbital integral respectively:

\[
I^{(p)}_t(K_t) \overset{\text{def}}{=} \frac{\chi(1) \text{Vol}(\Gamma \setminus G)}{4\pi} \int_{\mathbb{R}} \mu_{\sigma_p}(r) e^{-t(r^2+\rho_0^2)} dr, \tag{31}
\]

\[
H^{(p)}_t(K_t) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \mathcal{C}_t \setminus \{1\}} \frac{\chi(\gamma)}{j(\gamma)} t \zeta(\gamma) \chi_{\sigma_p}(m_{\gamma}) \exp \left\{ -t(\rho_0^2 + p) - t^2/(4t) \right\}, \tag{32}
\]

with \( \rho_0 = (n - 1)/2 \), and \( \chi_{\sigma}(m) = \text{Tr}(m) \) for \( m \in SO(2n - 1) \).

For \( p \geq 1 \) there is a measure \( \mu_{\sigma}(r) \) corresponding to a general irreducible representation \( \sigma \) of \( \mathfrak{M} \). Let \( \sigma_p \) be the standard representation of \( \mathfrak{M} = SO(n - 1) \) on \( \Lambda^p \mathbb{C}^{(n-1)} \). If \( n = 2k \) is even then \( \sigma_p (0 \leq p \leq n - 1) \) is always irreducible; if \( n = 2k + 1 \) then every \( \sigma_p \) is irreducible except for \( p = (n - 1)/2 = k \), in which case \( \sigma_k \) is the direct sum of two spin \(-1/2\) representations \( \sigma^\pm : \sigma_k = \sigma^+ \oplus \sigma^- \). For \( p = k \) the representation \( \tau_k \) of \( K = SO(2k) \) on \( \Lambda^k \mathbb{C}^{2k} \) is not irreducible, \( \tau_k = \tau^+_k \oplus \tau^-_k \) is the direct sum of spin \(-1/2\) representations.

**3.1. Case of the trivial representation.** For \( p = 0 \) (i.e. for smooth functions or smooth vector bundle sections) the measure \( \mu(r) \equiv \mu_0(r) \) corresponds to the trivial representation of \( \mathfrak{M} \). Therefore we take

\[
I^{(-1)}_t(K_t) = H^{(-1)}_t(K_t) = 0. \tag{33}
\]

Let \( \chi_{\sigma}(m) = \text{trace}(\sigma(m)) \) be the character of \( \sigma \), for \( \sigma \) a finite-dimensional representation of \( \mathfrak{M} \). Since \( \sigma_0 \) is the trivial representation one has \( \chi_{\sigma_0}(m_{\gamma}) = 1 \). In this case formula (30) reduces exactly to the trace formula for \( p = 0 \) [11, 19, 20, 15, 21, 22],

\[
I^{(0)}_t(K_t) = \frac{\chi(1) \text{Vol}(\Gamma \setminus G)}{4\pi} \int_{\mathbb{R}} \mu_{\sigma_0}(r) e^{-t(r^2+\rho_0^2)} dr + H^{(0)}_t(K_t) \tag{33}
\]

The function \( H^{(0)}_t(K_t) \) has the form

\[
H^{(0)}_t(K_t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \mathcal{C}_t \setminus \{1\}} \chi(\gamma) t \zeta(\gamma)^{-1} C(\gamma) \exp \left\{ -t\rho_0^2 - t^2/(4t) \right\}. \tag{34}
\]

**3.2. Odd dimensional manifolds with cusps.** Taking into account the fixed Iwasawa decomposition \( G = KAN \), consider a \( \Gamma \)-cuspimal minimal parabolic subgroup \( P \) of \( G \) with the Langlands decomposition \( P = BAN \), \( B \) being the centralizer of \( A \) in \( K \). Let us define the Dirac operator \( \mathfrak{D} \),
assuming a spin structure for $\Gamma \backslash \text{Spin}(2k+1,1)/\text{Spin}(2k+1)$. The spin bundle $E_{\tau_s}$ is the locally homogeneous vector bundle defined by the spin representation $\tau_s$ of the maximal compact group $\text{Spin}(2k+1)$. One can decompose the space of sections of $E_{\tau_s}$ into two subspaces, which are given by the half spin representations $\sigma_{\pm}$ of $\text{Spin}(2k) \subset \text{Spin}(2k+1)$. Let us consider a family of functions $K_t$ over $G = \text{Spin}(2k+1,1)$, which is given by taking the local trace for the integral kernel $\exp(-tD^2)$ (or $D\exp(-tD^2)$).

The Selberg trace formula has the form [23]:

$$
\sum_{\sigma=\sigma_{\pm}} \sum_{\lambda_k \in \sigma_p} \hat{K}_t(\sigma, i\lambda) - \frac{d(\sigma_{\pm})}{4\pi} \int_{\mathbb{R}} \text{Tr}(s\Gamma(\sigma_{\pm},-i\lambda)) \times (d/ds)\Gamma(\sigma_{\pm}, s)|_{s=i\lambda}\pi\Gamma(\sigma_{\pm}, i\lambda)(K_t)) \, d\lambda = I_{\Gamma}(\mathcal{K}_t) + H_{\Gamma}(\mathcal{K}_t) + U_{\Gamma}(\mathcal{K}_t),
$$

(35)

where $\sigma_p \in \text{Spec} \mathcal{D}$, $d(\sigma_{\pm})$ is the degree of the half spin representation of $\text{Spin}(2k)$, $\Gamma(\sigma_{\pm}, i\lambda)$ is the scattering matrix and $U_{\Gamma}(\mathcal{K}_t)$ is an unipotent orbital integral. The analysis of the unipotent orbital integral $U_{\Gamma}(\mathcal{K}_t)$ gives the following result [24, 23]: all of the unipotent terms vanish in the Selberg trace formula applied to the integral kernel function. This means that using the Fried result [17] one obtains a result in the case of cusps similar to that in the case of compact odd dimensional manifolds.

4. The Spectral Functions on $p-$ Forms

If $\{\lambda^{(p)}\}_{\ell=0}^\infty$ are eigenvalues of the operator $\delta d$ restricted on $p-$ forms and $\{\nu^{(p)}\}_{\ell=0}^\infty$ are the eigenvalues of $d\delta$, then $\lambda^{(p)} = \nu^{(p+1)}$ with equal multiplicity. There are two equivalent eigenvalues problems

$$
\mathcal{L}_p \omega_p = \lambda \omega_p \iff \mathcal{L}_{p+1} d\omega_p = \lambda d\omega_p
$$

$$
\mathcal{L}_{p-1} \delta\omega_p = \lambda \delta\omega_p.
$$

(36)

This means that the spectra of the Hodge Laplacian acting on exact $p-$ forms and on co–exact $(p-1)-$ forms are the same. The transverse part of the skew symmetric tensor is represented by the co–exact $p-$ form $\omega_p^{(CE)} = \delta\omega_{p+1}$, which trivially satisfies $\delta\omega_p^{(CE)} = 0$, and we denote by $\mathcal{L}_p^{(CE)} = \delta d$ the restriction of the Laplacian on the co–exact $p-$ form.

The goal now is to extract the co–exact $p-$ form on manifold which describes the physical degrees of freedom of the system. Choosing a basis $\{\omega^\ell_p\}$ of $p-$ forms (eigenfunctions of the Laplacian) we get for an arbitrary suitable function $F(\mathcal{L}_p)$ [25]:

$$
\sum_{\ell} \langle \omega^\ell_p, F(\mathcal{L}_p) \omega^\ell_p \rangle = \sum_{\ell} \langle \omega_p^{(CE)}^{(CE)}, F(\mathcal{L}_p) \omega_p^{(CE)} \rangle
$$
where \( b_p \) are the Betti numbers, \( b_p \equiv b_p(M) = \text{rank}_\mathbb{Z} H_p(M; \mathbb{Z}) \). Using the properties of the Hodge Laplacian one can obtain
\[
\sum \ell \langle d\omega^{\ell}_{p-1}, F(\mathcal{L}^{\ell}_p) d\omega^{\ell}_{p-1} \rangle = \sum \ell \langle d\omega^{\ell}_{p-1}, F(\mathcal{L}^{\ell-1}_p) d\omega^{\ell}_{p-1} \rangle
\]
(38)

Finally we get
\[
\text{Tr} F(\mathcal{L}^{(CE)}_p) = \sum_{j=0}^{p} (-1)^j (\text{Tr} F(\mathcal{L}^{p-j}) - F(0) \sum_{j=0}^{p} (-1)^j b_{p-j}).
\]
(39)

Making the choice \( F(x) = \exp(-tx) \), one can rewrite Eq. (39) using the Fried’s result (30) in the form
\[
\text{Tr} \left( e^{-t \mathcal{L}^{(CE)}_p} \right) = \sum_{j=0}^{p} (-1)^j \left( I^{(p-j)}_t(\mathcal{K}_t) + I^{(p-j-1)}_t(\mathcal{K}_t) \right)
\]
\[
+ H^{(p-j)}_t(\mathcal{K}_t) + H^{(p-j-1)}_t(\mathcal{K}_t) - b_{p-j}).
\]
(40)

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