Description of Collective Motion in Two-Dimensional Nuclei; Tomonaga’s Method Revisited

Seiya NISHIYAMA* and Joao da PROVIDêNCIA†

Centro de Física Computacional, Departamento de Física,
Universidade de Coimbra, P-3004-516 Coimbra, Portugal

Dedicated to the Memory of Toshio Marumori

July 15, 2014

Abstract

Four decades ago, Tomonaga proposed the elementary theory of quantum mechanical collective motion of two-dimensional nuclei of $N$ nucleons. The theory is based essentially on the neglect of $\frac{1}{\sqrt{N}}$ against unity. Very recently we have given exact canonically conjugate momenta to quadrupole-type collective coordinates under some subsidiary conditions and have derived nuclear quadrupole-type collective Hamiltonian. Even in the case of simple two-dimensional nuclei, we have a subsidiary condition to obtain exact canonical variables. Particularly the structure of the collective subspace satisfying the subsidiary condition is studied in detail. This subsidiary condition is important to investigate what is a structure of the collective subspace.

*Corresponding author. E-mail address: seikoceu@khe.biglobe.ne.jp, nisiyama@theor.fis.uc.pt
†E-mail address: providencia@theor.fis.uc.pt
1 Introduction

In studies of collective motions in nuclei, the very difficult problems of large-amplitude collective motions, which are strongly non-linear phenomena in quantum nuclear dynamics, still remain unsolved. How do we go beyond the usual mean field theories towards the construction of a theory for large-amplitude collective motions in nuclei? A proper treatment of collective variable was attempted. Four decades ago, Tomonaga proposed the elementary theory of quantum mechanical collective motion of two-dimensional nuclei of \( N \) nucleons. The theory is based essentially on the neglect of \( \frac{1}{\sqrt{N}} \) against unity \([1]\). Marumori et al. first gave a foundation of the unified model of collective motion and independent particle motion in nuclei and further investigated the collective motion from the standpoint of particle excitations \([2]\).

Applying Tomonaga’s basic idea in his collective motion theory to nuclei with the aid of the Sunakawa’s integral equation method \([3]\), one of the present authors (S.N.) developed a collective description of surface oscillations of nuclei \([4]\). This description is considered to give a possible microscopic foundation of nuclear collective motion derived from the famous Bohr-Mottelson model \([5]\) (see textbooks \([6, 7]\)). Introducing appropriate collective variables, this collective description was formulated by using the first quantized language, contrary to the second quantized manner in the Sunakawa method. Preceding the previous work \([4]\), extending the Tomonaga’s idea to three-dimensional case, Miyazima-Tamura \([8, 9]\) successfully proposed a collective description of the surface oscillations of nuclei. As they already pointed out, however, there exist two serious difficulties still remaining in traditional theoretical treatments of the nuclear collective motions: (i) Collective momenta defined according Tomonaga’s approach are not exact canonically conjugate to collective coordinates; (ii) The collective momenta are not independent from each other.

Very recently we have given exact canonically conjugate momenta to quadrupole-type collective coordinates and have derived a nuclear quadrupole-type collective Hamiltonian \([10]\) (referred to as I). The exact canonically conjugate momenta \( \Pi_{2\mu} \) to the quadrupole-type collective coordinates \( \phi_{2\mu} \) is derived by modifying the approximate momenta \( \pi_{2\mu} \) adopted by Miyazima-Tamura \([8]\) with the use of the discrete version of the Sunakawa’s integral equation \([3]\). We have shown that the exact canonical commutation relations between the collective variables \( \phi_{2\mu} \) and \( \Pi_{2\mu} \) and the commutativity of the momenta \( \Pi_{2\mu} \) and \( \Pi_{2\nu} \) under some subsidiary conditions. Using the exact canonical variables \( \phi_{2\mu} \) and \( \Pi_{2\mu} \), we found the collective Hamiltonian which includes the so-called surface phonon-phonon interaction corresponding to the Hamiltonian of Bohr-Mottelson model. Even in the case of simple two-dimensional nuclei, we have a subsidiary condition to obtain exact canonical variables. Particularly, we study structure of the collective subspace satisfying the subsidiary condition. This condition is important to investigate the structure of the collective subspace. This is an interesting problem which solved easily due to the simplicity of the two-dimensional nuclei.

In Section 2, we introduce collective variables \( \phi_{i} \) in two-dimensional nuclei and approximate conjugate momenta \( \eta_{i} \). In Section 3, we define the exact canonically conjugate momenta \( \pi_{i} \) and prove the commutativity of the exact collective momenta on the collective subspace. In Section 4, the exact \( \pi_{i} \)-dependent kinetic part \( T \) of Hamiltonian is derived. The \( \phi_{i} \)-dependent kinetic part \( T \) of the Hamiltonian including the constant term is also given. In the last section, we discuss on the subsidiary condition and give some concluding remarks. In Appendix A, we derive some approximate relations for the collective conjugate momenta which play crucial roles to determine the \( \phi_{i} \)-dependence of the \( T \).
2 Collective variables and associated relations

For the sake of simplicity, we focus on collective motions in the two-dimensional nuclei. The present illustration is oversimplified as far as we consider only the two-dimensional case.

Consider a two-dimensional nucleus consisting of $N$ nucleons interacting strongly with each other. Let us denote the coordinates of $n$-th nucleon by $(x_n, y_n)$ and by $(p_{xn}, p_{yn})$ their conjugate momenta, respectively.

The total Hamiltonian $H$ of our system is given by

$$H = T + V = \frac{\hbar^2}{2\mu} \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) + V(x_1, y_1; x_2, y_2; \cdots; x_N, y_N) ,$$  \hspace{1cm} (2.1)

where $T = 1/2m \sum_{n=1}^{N} (p_{xn}^2 + p_{yn}^2)$ is the total kinetic energy and $V$ is the interacting potential depending only on relative coordinates of the nucleons.

Following Miyazima-Tamura [8], we introduce two collective coordinates

$$\phi_1 = \frac{1}{N r_0^2} \sum_{n=1}^{N} \frac{1}{2}(x_n^2 - y_n^2), \hspace{0.5cm} \phi_2 = \frac{1}{N r_0^2} \sum_{n=1}^{N} x_n y_n ,$$  \hspace{1cm} (2.2)

where $r_0$ is defined through $r_0^2 = 1/2 \cdot R_0^2$ and $R_0$ means the nuclear equilibrium radius.

Following Tomonaga [1], we introduce collective conjugate momenta to $\phi_1$ and $\phi_2$ in the sense of Tomonaga through

$$\eta_i = \mu N r_0^2 \dot{\phi}_i = \mu N r_0^2 \frac{l}{\hbar}[T, \phi_i] , \hspace{0.5cm} (i = 1, 2)$$  \hspace{1cm} (2.3)

whose explicit expressions are given as

$$\eta_1 = -i\hbar \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n} \right) , \hspace{1cm} (2.4)$$

$$\eta_2 = -i\hbar \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial y_n} + y_n \frac{\partial}{\partial x_n} \right) .$$

The commutation relations among these collective variables become as follows:

$$[\phi_i, \phi_j] = 0 , \hspace{0.5cm} [\eta_i, \phi_j] = -i\hbar \frac{r^2}{r_0^2} \delta_{ij} , \hspace{0.5cm} (i, j = 1, 2)$$  \hspace{1cm} (2.5)

$$[\eta_1, \eta_2] = -2i\hbar l ,$$

where $r^2$ and $l$ are defined, respectively as

$$r^2 \equiv \frac{1}{N} \sum_{n=1}^{N} (x_n^2 + y_n^2) , \hspace{0.5cm} l \equiv \sum_{n=1}^{N} (x_n p_{yn} - y_n p_{xn}) .$$  \hspace{1cm} (2.6)

The mean square of the nucleon distance from the center of the nucleus is $r^2$ and its equilibrium value is approximately $r_0^2$ appearing in Eq. (2.2), namely $\langle r^2 \rangle = r_0^2$. $l$ is the total angular momentum operator of the system in the original representation. From the second line of Eq. (2.5), an approximate commutation relation is derived as

$$[\eta_i, \phi_j] \cong -i\hbar \delta_{ij} . \hspace{0.5cm} (i, j = 1, 2)$$  \hspace{1cm} (2.7)

Thus, we can get approximate conjugate momenta $\eta_1$ and $\eta_2$ to the collective coordinates $\phi_1$ and $\phi_2$. However, as is shown from Eq. (2.5), the R.H.S. of the second line does not take the value $-i\hbar \delta_{ij}$ and the third one does not vanish. Then from these facts, it follows that the variables $\phi_i$ and $\eta_i$ are not canonically conjugate to each other.
3 Exact canonically conjugate momenta

We will now derive exact canonically conjugate momenta to \( \phi_i \). For this purpose, following [9] first we introduce operators \( \pi_i \) defined by

\[
\pi_i \equiv \frac{r_0^2}{2} \left( r^{-2} \eta_i + \eta_i r^{-2} \right). \quad (i = 1, 2)
\]  

(3. 1)

Then, the operators \( \phi_i \) and \( \pi_i \) satisfy the commutation relations

\[
[\pi_i, \phi_j] = -ih \delta_{ij}, \quad (i, j = 1, 2), \quad [\pi_1, \pi_2] = -2i\hbar \left( \frac{r_0^2}{r^2} \right)^2 (l - L_{\text{coll}}),
\]  

(3. 2)

where \( L_{\text{coll}} \) is defined as

\[ L_{\text{coll}} \equiv 2(\phi_1 \pi_2 - \phi_2 \pi_1), \]  

(3. 3)

which also appears in the Miyazima-Tamura’s collective description of a two-dimensional nucleus [8]. If the inhomogeneous term in the R.H.S. of the second equation of (3. 2) disappears, the operators \( \pi_i \) become independent from each other, i.e., \( [\pi_1, \pi_2] = 0 \) under the convection to conceive \( r^2 \) as constant c-number. We can give a proof on the commutativity of the operator \( \pi_i \) by vanishing of the inhomogeneous term. Then, it turns out that \( \pi_i \) are exact canonical conjugate to \( \phi_i \). Due to this fact and the form of (3. 3), the quantity \( L_{\text{coll}} \) can be regarded as a collective angular momentum operator. Therefore, it is concluded that the operators \( \{\phi_i, \pi_i\} \) is a set of exact canonical variables, if we restrict the Hilbert space to the collective subspace \(|\text{coll.subspace}\rangle\) which satisfies the subsidiary condition

\[
(l - L_{\text{coll}})|\text{coll.subspace}\rangle = 0, \quad [\pi_1, \pi_2]|\text{coll.subspace}\rangle = 0.
\]  

(3. 4)

This condition is very reasonable and also implies that our collective variables are the good ones in the collective subspace. This kind of subsidiary condition also appears in I.

The old variables \( \eta_i \) and new variable \( \pi_i \) (3. 1) are related with \( \pi_i \) and \( \eta_i \), and \( \phi_i \) as

\[
\eta_i = r_0^{-2} r^{-2} \pi_i - ih 2r_0^2 r^{-2} \phi_i, \quad \pi_i = r_0^2 r^{-2} \eta_i + ih 2r_0^4 r^{-4} \phi_i, \quad (i = 1, 2)
\]  

(3. 5)

in which we have used the relations

\[
[\eta_i, r^{2n}] = -ih 4n r_0^2 r^{2(n-1)} \phi_i, \quad [\pi_i, r^{2n}] = -ih 4n r_0^4 r^{2(n-2)} \phi_i, \quad (i = 1, 2).
\]  

(3. 6)

The commutation relations between variables \( \eta_i \) and a kinetic operator \( T \) are calculated as

\[
[\eta_i, T] = -i\hbar \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial y_n^2} \right),
\]

\[
[\pi_i, T] = -i\hbar \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial y_n} + \frac{\partial}{\partial y_n} \frac{\partial}{\partial x_n} \right),
\]  

(3. 7)

and the commutator \([T, r^{-2}]\) is given as

\[
[T, r^{-2}] = \frac{2\hbar^2}{\mu N r^4} \sum_{n=1}^{N} x_n \left( \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) + \frac{2\hbar^2}{\mu r^4} \left( 1 - \frac{2}{N} \right),
\]  

(3. 8)

together with the commutators below

\[
[[\eta_i, T], r^{-2}] = -\frac{4\hbar^2}{\mu N r^4} \eta_i - i\hbar \frac{16h^2 r_0^2}{\mu N r^6} \phi_i,
\]

\[
[[T, r^{-2}], \eta_i] = -i\hbar \frac{16h^2 r_0^2}{\mu N r^6} \phi_i \sum_{n=1}^{N} x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} - i\hbar \frac{16h^2 r_0^2}{\mu r^6} \left( 1 - \frac{2}{N} \right) \phi_i.
\]  

(3. 9)

All of the above play important roles in the next Section to derive a collective Hamiltonian.
4 \( \pi_i \)- and \( \phi_i \)-dependence of kinetic part of Hamiltonian

4.1 \( \pi_i \)-dependence of kinetic part of Hamiltonian

In this subsection, a remaining task is to express the kinetic part \( T \) of the Hamiltonian in terms of \( \phi_i \) and \( \pi_i \). Along the same way as the one in 1, we first investigate \( \pi_i \)-dependence of \( T \). For this purpose, we expand it in a power series of the exact canonical conjugate momenta \( \pi_i \) as follows:

\[
T = T^{(0)}(\phi; r^2) + \sum_{i=1}^{2} T^{(1)}_i(\phi; r^2)\pi_i + \sum_{i,j=1}^{2} T^{(2)}_{ij}(\phi; r^2)\pi_i\pi_jr^2 + \cdots, \tag{4. 1}
\]

where \( T^{(n)}_{ij}(\phi; r^2) \) are unknown expansion coefficients. In order to get the explicit expression for \( T^n(n \neq 0) \), using the commutation relation \([\pi_i, \phi_j] = -i\hbar\delta_{ij}\), we take the commutators with \( \phi_i \) in the following way:

\[
\begin{align*}
[T, \phi_i] &= -i\hbar T^{(1)}_i(\phi; r^2), \quad (i = 1, 2) \\
[[T, \phi_i], \phi_j] &= (i\hbar)^2 T^{(2)}_{ij}(\phi; r^2), \quad (i, j = 1, 2) \\
[\ldots] &= (i\hbar)^3 T^{(2)}_{ijk}(\phi; r^2), \quad (i, j, k = 1, 2), \\
&\vdots
\end{align*}
\]

We can easily calculate the L.H.S. of (4. 2) by making explicit use of the definitions (2. 2) and (3. 1) and by taking commutators with \( \phi_i \) successively:

\[
\begin{align*}
[T, \phi_i] &= -i\hbar \frac{1}{\mu Nr_0^4} \eta_i = -i\hbar \frac{1}{2\mu Nr_0^4} (r^2\pi_i + \pi_i r^2), \quad (i = 1, 2) \\
[[T, \phi_i], \phi_j] &= (i\hbar)^2 \frac{r^2}{\mu Nr_0^4} \delta_{ij}, \quad (i, j = 1, 2) \\
[\ldots] &= (i\hbar)^3 6T^{(2)}_{ijk}(\phi; r^2), \quad (i, j, k = 1, 2) \\
&\vdots
\end{align*}
\]

Comparing (4. 3) with (4. 2), \( T^{(n)} \) are determined as

\[
\begin{align*}
T^{(1)}_i(\phi; r^2) &= 0, \\
T^{(2)}_{ij}(\phi; r^2) &= T^{(2)}_{ji}(\phi; r^2) = \frac{r^2}{2\mu Nr_0^4} \delta_{ij}, \\
T^{(n)}(\phi; r^2) &= 0. \quad (n \geq 3)
\end{align*}
\]

Substituting (4. 4) into (4. 1) and using the second relation of (3. 6), we can get the exact \( \pi_i \)-dependence of the kinetic part \( T \) of Hamiltonian in nuclei given by Miyazima-Tamura [8].

Finally, we should stress the fact that up to this stage, all the expressions are exact.

\[
T = T^{(0)}(\phi; r^2) + \frac{4\hbar^2}{\mu Nr^2} \sum_{i=1}^{2} \pi_i r^2 - \frac{r^2}{2\mu Nr_0^4} \sum_{i=1}^{2} \pi_i \pi_i + \frac{16\hbar^2 r_0^4}{\mu Nr_0^4} \sum_{i=1}^{2} \phi_i \phi_i - i\hbar \frac{2}{\mu Nr^2} \sum_{i=1}^{2} \phi_i \pi_i, \tag{4. 5}
\]

in which the third term is the kinetic energy part of the two-dimensional collective motion in nuclei given by Miyazima-Tamura [8].
4.2 $\phi_i$-dependence of kinetic part of Hamiltonian

Our next task is to determine the term $T^{(0)}(\phi; r^2)$. For this purpose, we expand it in a power series of the collective coordinates $\phi_j$ in the form

$$
T^{(0)}(\phi; r^2) = C_0(r^2) + \sum_{j=1}^2 C_{1j}(r^2)\phi_i + \sum_{i,j=1}^2 C_{2ij}(r^2)\phi_i\phi_j + \sum_{i,j,k=1}^2 C_{3ijk}(r^2)\phi_i\phi_j\phi_k + \cdots ,
$$

(4.6)

where $C_{2ij}(r^2) = C_{2ji}(r^2)$, etc. In the above, the expansion coefficients $C_n(r^2)$ are determined in a manner quite parallel to the manner used before.

First, using (4.5), we have the commutation relation between the operator $\pi_i$ and $T^{(0)}(\phi; r^2)$ as

$$
[\pi_i, T^{(0)}(\phi; r^2)] = [\pi_i, T] + i\hbar \frac{48\hbar^2 r_0^4}{N r^6} \phi_i - i\hbar \frac{192\hbar^2 r_0^8}{N r^{10}} \phi_i \sum_{j=1}^2 \phi_j \phi_j
$$

$$
+ 2 \frac{\hbar^2}{N r^2} (N - 8) \phi_i \sum_{j=1}^2 \phi_j \pi_j + i\hbar \frac{2}{N r^2} \phi_i \sum_{j=1}^2 \pi_j \pi_j, \quad (i = 1, 2)
$$

(4.7)

in which we have used the first relation of (3.2) and the second relation of (3.6). The first commutator $[\pi_i, T]$ in (4.7) is computed as

$$
[\pi_i, T] = r_0^2 \frac{1}{2} [r^{-2}(\eta_n - \pi_n r^{-2}), T] = r_0^4 [T, r^{-2}]\eta_n + r_0^4 [T, \eta_n, T] - r_0^2 [T, r^{-2}] \pi_n + \frac{r_0^2}{2} [T, [T, r^{-2}], \pi_n]
$$

$$
= r_0^2 r^{-2}[\eta_n, T] - \frac{\hbar^2}{2\mu} \frac{4}{N} r^{-4} \sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_n + i\hbar \frac{16}{N} r_0^2 r^{-6} \phi_i \sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) (4.8)
$$

$$
+ 4 \left( 1 - \frac{1}{N} \right) r_0^2 r^{-6} \phi_i + i\hbar 16 \left( 1 - \frac{1}{N} \right) r_0^2 r^{-6} \phi_i, \quad (i = 1, 2)
$$

where we have used the commutators (3.8) and (3.9). Substituting (4.8) into (4.7) and the second equation of (3.6), we can get $[\pi_i, T^{(0)}(\phi; r^2)]$ as

$$
[\pi_i, T^{(0)}(\phi; r^2)] = r_0^2 r^{-2}[\eta_n, T] - \frac{\hbar^2}{2\mu} \frac{4}{N} r^{-4} \sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_n + i\hbar \frac{16}{N} r_0^2 r^{-6} \phi_i \sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right)
$$

$$
+ 4 \left( 1 - \frac{1}{N} \right) r_0^2 r^{-6} \phi_i + i\hbar 16 \left( 1 - \frac{1}{N} \right) r_0^2 r^{-6} \phi_i
$$

$$
+ \frac{\hbar^2}{2\mu} \left( i\hbar \frac{96}{N} r_0^2 r^{-6} \phi_i - i\hbar \frac{384}{N} r_0^2 r^{-10} \phi_i \sum_{j=1}^2 \phi_j \phi_j + \frac{4}{N} r^{-4} \pi_n + i\hbar \frac{8}{N} r_0^2 r^{-6} \phi_i
$$

$$
- \frac{16}{N} r_0^2 r^{-6} \phi_i \sum_{j=1}^2 \phi_j \pi_j \right) + i\hbar \frac{2}{N r^2} \phi_i \sum_{j=1}^2 \pi_j \pi_j. \quad (i = 1, 2)
$$

(4.9)

To carry further computation of (4.9), the following approximate relations play crucial roles:

$$
\sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_n \approx \eta_n + f(N) \eta_n + \frac{1}{h^2} N r^2 \left[ \eta_n, T \right] - i\hbar N r_0^2 \phi_i \sum_{n=1}^N \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right),
$$

(4.10)

$$
\sum_{j=1}^2 \phi_j \pi_j \approx -i\hbar \frac{1}{2} \sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) + i\hbar 2 r_0^4 r^{-4} \sum_{j=1}^2 \phi_j \phi_j,
$$

(4.11)

$$
\sum_{n=1}^N \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) - h^2 N r_0^2 r^{-2} \sum_{n=1}^N \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right),
$$

(4.12)
where we have used another approximate relation and an approximate mean-value
\[ \sum_{j=1}^{2} \phi_j \eta_j \approx -i \hbar \frac{1}{2} r_0^{-2} r^2 \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right), \quad \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) = f(N), \] (4. 13)

second equation of which, on the collective subspace |coll.subspace\rangle. The unknown function \( f(N) \) is determined later. A derivation of equation (4. 10) is given in detail in Appendix A. Substituting (4. 10), (4. 11) and (4. 12) into (4. 9), we reach the following final result:

\[
\begin{align*}
\left[ \pi_i, T(0)(\phi; r^2) \right] &= r_0^2 r^{-2} \eta_i, T - \frac{\hbar^2 r_0^2}{2 \mu} N r^{-4} \left\{ \eta_i + f(N) \eta_i + \frac{\mu}{\hbar^2} \frac{1}{2} N r^{2} \left[ \eta_i, T \right] - i \hbar N r_0^2 \phi_i \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) \right\} \\
&+ i \hbar \frac{16}{N} N^2 r_0^{-6} \phi_i \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) + i \hbar 2 r_0^4 r^{-4} \sum_{j=1}^{N} \phi_j \phi_j + 4 N r^{-4} \eta_i + i \hbar 8 N r_0^6 r^{-6} \phi_i \\
&+ \frac{16}{N} \frac{r_0^{-6}}{6} \phi_i \left\{ -i \hbar \frac{1}{2} \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) + i \hbar 2 r_0^4 r^{-4} \sum_{j=1}^{N} \phi_j \phi_j \right\}
\end{align*}
\] (4. 14)

To get the commutativity \( \left[ [\pi_i, T(0)(\phi; r^2)], \phi_j \right] = 0, \ (i, j = 1, 2) \), namely, \( \left[ \pi_i, T(0)(\phi; r^2) \right] \) depends only on the variables \( \phi \), we strongly demand that the term \( \eta_i \) in the first term of the second line from the bottom in (4.14) must vanish on the collective subspace |coll.subspace\rangle. Then, the unknown function \( f(N) \) should satisfy a simple relation

\[ f(N) = -N \left( 1 - \frac{1}{N} \right). \] (4. 15)

Using (4. 14), (4. 15) and the commutation relations, the first equation of (3. 2) and the second equation of (3. 4) and for the sake of simplicity discarding the contributions from the effects by the terms \( [\pi_i, r^{-6}] \) and \( [\pi_i, r^{-6}] \), we get the following commutation relations:

\[ \left[ \pi_i, \pi_j, T(0)(\phi; r^2) \right] = -\frac{\hbar^2 r_0^2}{2 \mu} r^2 \left\{ 16 \left( 1 - \frac{1}{N} \right) r_0^2 r^{-6} \delta_{ij} + \frac{4}{N} r_0^2 r^{-6} \delta_{ij} + \frac{496}{N} r_0^6 r^{-10} \phi_i \phi_j \right\}, \] (4. 16)

\[ \left[ \pi_i, \pi_j, \pi_k, T(0)(\phi; r^2) \right] = \frac{\hbar^2 r_0^2}{2 \mu} i \hbar^2 2 \frac{496}{N} r_0^6 r^{-10} (\delta_{ij} \phi_k + \phi_j \delta_{ik}), \] (4. 17)

\[ \left[ \pi_i, \pi_j, \pi_k, \pi_l, T(0)(\phi; r^2) \right] = \frac{\hbar^2 r_0^2}{2 \mu} i \hbar^2 2 \frac{496}{N} r_0^6 r^{-10} (\delta_{jk} \delta_{il} + \delta_{ik} \delta_{jl}), \] (4. 18)

\[ \left[ \pi_i, \pi_j, \pi_k, \pi_l, \pi_m, T(0)(\phi; r^2) \right] = 0, \ (i, j, k, l, m = 1, 2). \] (4. 19)

In the derivation of (4.16), we have used the approximate relation \( \sum_{i=1}^{2} \phi_i \phi_i \approx \frac{1}{4} r_0^4 r^4 \).
According to the way quite similar to the previous one, we take the commutation relations between \( T^{(0)}(\phi; r^2) \) expanded as (4.6) with \( \pi_i \):

\[
\begin{align*}
[\pi_i, T^{(0)}(\phi; r^2)] &= -i\hbar C_i(r^2) - 2i\hbar \sum_{j=1}^{2} C_{2ij}(r^2) \phi_j - 3i\hbar \sum_{j,k=1}^{2} C_{3ijk}(r^2) \phi_j \phi_k + \cdots , \\
[\pi_i, [\pi_j, T^{(0)}(\phi; r^2)]] &= -2\hbar^2 \left\{ C_{2ij}(r^2) + 3 \sum_{k=1}^{2} C_{3ijk}(r^2) \phi_k + 6 \sum_{k,l=1}^{2} C_{4ijkl}(r^2) \phi_k \phi_l \right\} + \cdots , \\
[\pi_i, [\pi_j, [\pi_k, T^{(0)}(\phi; r^2)]]] &= 6i\hbar^2 \left\{ C_{3ijk}(r^2) + 4 \sum_{l=1}^{2} C_{4ijkl}(r^2) \phi_l \right\} + \cdots , \\
[\pi_i, [\pi_j, [\pi_k, [\pi_l, T^{(0)}(\phi; r^2)]]]] &= 24\hbar^4 C_{4ijkl}(r^2) + \cdots , \\
[\pi_i, [\pi_j, [\pi_k, [\pi_m, T^{(0)}(\phi; r^2)]]]] &= 0, (i, j, k, l, m = 1, 2).
\end{align*}
\]

Comparing (4.20) with the last line of (4.14) and with equations from (4.16) to (4.19) and using the approximate relation \( \sum_{i=1}^{2} \phi_i \phi_i \approx \frac{1}{4} r_0^{-4} r^4 \), \( C_n \) are determined as

\[
\begin{align*}
C_1(r^2) &= 0 , \\
C_{2ij}(r^2) &= \frac{\hbar^2 r_0^2}{2\mu} \left\{ 8 \left(1 - \frac{1}{N}\right) r_0^2 r^{-6} \delta_{ij} + \frac{2}{N} r_0^2 r^{-6} \delta_{ij} + \frac{496}{N} r_0^6 r^{-10} \phi_i \phi_j \right\} , \\
C_{3ijk}(r^2) &= \frac{\hbar^2 r_0^2}{2\mu} \frac{1}{3 N} r_0^6 r^{-10} (\delta_{ij} \phi_k + \phi_j \delta_{ik}) , \\
C_{4ijkl}(r^2) &= \frac{\hbar^2 r_0^2}{2\mu} \frac{1}{3 N} r_0^6 r^{-10} (\delta_{jk} \delta_{il} + \delta_{ik} \delta_{jl}) , \\
C_n(r^2) &= 0, \quad (n \geq 5).
\end{align*}
\]

Substituting (4.21) into (4.6), we have

\[
T^{(0)}(\phi; r^2) = C_0(r^2) \\
+ \sum_{i,j,k,l=1}^{2} \frac{\hbar^2 r_0^2}{2\mu} \left\{ 8 \left(1 - \frac{1}{N}\right) r_0^2 r^{-6} \sum_{i=1}^{2} \phi_i \phi_i + \frac{2}{N} r_0^2 r^{-6} \sum_{i=1}^{2} \phi_i \phi_i + \frac{496}{N} r_0^6 r^{-10} \sum_{i=1}^{2} \phi_i \phi_i \sum_{j=1}^{2} \phi_j \phi_j \right\} \phi_i \phi_j + \cdots , \\
= C_0(r^2) \\
+ \frac{\hbar^2 r_0^2}{2\mu} \left\{ 8 \left(1 - \frac{1}{N}\right) r_0^2 r^{-6} \sum_{i=1}^{2} \phi_i \phi_i + \frac{2}{N} r_0^2 r^{-6} \sum_{i=1}^{2} \phi_i \phi_i + \frac{496}{N} r_0^6 r^{-10} \sum_{i=1}^{2} \phi_i \phi_i \sum_{j=1}^{2} \phi_j \phi_j \right\} \phi_i \phi_j + \cdots .
\]

From (4.22), we obtain the final result in the following form:

\[
T^{(0)}(\phi; r^2) = C_0(r^2) + \frac{\hbar^2 r_0^2}{2\mu} \frac{3}{N} \left(4 - \frac{3}{N}\right) \left(1 + 3 \sum_{i=1}^{2} \phi_i \phi_i + \frac{2728}{3N} \right) \phi_i \phi_i + \cdots .
\]
4.3 Determination of constant term \( C_0(r^2) \) and final expression for kinetic part \( T \) of Hamiltonian

In this Subsection, we determine the constant term \( C_0(r^2) \). Using (4.23) and (4.25), \( C_0(r^2) \) is given as

\[
C_0(r^2) = T^{(0)}(\phi; r^2) = \frac{\hbar^2 r_0^2}{2\mu} \left( 2 \left( \frac{1}{N} \right) + \frac{1}{2N} + \frac{31}{6N} \right) r_0^{-2} r^{-2} - \frac{\hbar^2 r_0^2}{2\mu} \left( 2 + \frac{166}{3N} \right) r_0^{-2} r^{-2}.
\]

Substituting (4.11) and (4.12) into (4.24), \( C_0(r^2) \) is expressed as

\[
C_0(r^2) = T - \frac{r^2}{2\mu N r_0^2} \left\{ \hbar^2 r_0^4 r^{-4} - \hbar^2 20 r_0^8 r^{-8} \sum_{j=1}^{2} \phi_j \phi_j + \hbar^2 2 r_0^4 r^{-4} \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) \right\} + \frac{i\hbar}{\mu N r_0^2} \left\{ -i\hbar \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) + i\hbar 2 r_0^4 r^{-4} \sum_{j=1}^{2} \phi_j \phi_j \right\} + \frac{\hbar^2 r_0^2 32}{2\mu N r_0^2 r^{-2}} \sum_{k=1}^{2} \phi_k \phi_k - \frac{\hbar^2 r_0^2}{2\mu} \left( 2 + \frac{166}{3N} \right) r_0^{-2} r^{-2},
\]

which is rearranged as

\[
C_0(r^2) = T + \frac{h^2 r_0^2}{2\mu} \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) - \frac{h^2 r_0^2 2}{2\mu N r_0^2 r^{-2}} \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) - \frac{h^2 r_0^2 2}{2\mu N r_0^2 r^{-2}} \left\{ 2 - 10 r_0^4 r^{-4} \sum_{j=1}^{2} \phi_j \phi_j \right\} - \frac{h^2 r_0^2 8}{2\mu N r_0^2 r^{-2} r^{-6} \sum_{j=1}^{2} \phi_j \phi_j} + \frac{h^2 r_0^2}{2\mu} \sum_{k=1}^{2} \phi_k \phi_k - \frac{h^2 r_0^2}{2\mu} \left( 2 + \frac{166}{3N} \right) r_0^{-2} r^{-2}.
\]

in the R.H.S. of which the terms in the first and second lines cancel out each other. Using again the approximate relation \( \sum_{k=1}^{2} \phi_k \phi_k \approx \frac{1}{4} r_0^{-4} r^4 \), the constant term \( C_0(r^2) \) is determined as

\[
C_0(r^2) = -\frac{\hbar^2 r_0^2}{2\mu} \left( 2 + \frac{166}{3N} \right) r_0^{-2} r^{-2}.
\]

Thus, the final expression for the kinetic part \( T \) of the Hamiltonian is given as follows:

\[
T = T^{(0)}(\phi; r^2) - \frac{4\hbar^2}{\mu N r^2} + \left( \frac{r^2}{2\mu N r_0} \right) \sum_{i=1}^{2} \pi_i \pi_i + 16 \frac{\hbar^2 r_0^4}{\mu N r_6} \sum_{i=1}^{2} \phi_i \phi_i - \frac{2}{\mu N r^2} \sum_{k=1}^{2} \phi_k \phi_k \left\{ \frac{1}{N} - \frac{3}{N} \right\} + \frac{\hbar^2 r_0^2 2728}{2\mu 3N^9} \sum_{k=1}^{2} \phi_k \phi_k + \frac{\hbar^2 r_0^4 2728}{2\mu 3N^9} \sum_{k=1}^{2} \phi_k \phi_k \right\} \sum_{k=1}^{2} \phi_k \phi_k \right\}
\]

\[
C_0(r^2) = -\frac{\hbar^2 r_0^2}{2\mu} \left( 2 + \frac{169}{3N} \right) r_0^{-2} r^{-2}.
\]
5 Discussion and concluding remarks

As for the present two-dimensional nuclei, particularly, we study the structure of the collective subspace satisfying the subsidiary condition (3.2). This subsidiary condition is important to investigate what is the structure of the collective subspace. We denote a wave function of the collective subspace as \( \langle \text{coll.subspace} | \Psi \rangle \). On this wave function, we assume a useful anzatz: \( \Psi[\cdot] = \Psi[r] \). Then on the \( \Psi[r] \) we have

\[
\frac{\partial}{\partial x_n} \Psi[r] = \frac{1}{N} \frac{x_n}{r} \frac{\partial}{\partial r} \Psi[r], \quad \frac{\partial}{\partial y_n} \Psi[r] = \frac{1}{N} \frac{y_n}{r} \frac{\partial}{\partial r} \Psi[r],
\]

\( r = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (x_n^2 + y_n^2)} \), \hspace{1cm} (5.1)

The subsidiary condition is expressed as

\( (l-L_{\text{coll}}) \Psi[r] = 0, \quad L_{\text{coll}} \equiv 2(\phi_1 \pi_2 - \phi_2 \pi_1). \) \hspace{1cm} (5.2)

Substituting the explicit expression for the total angular momentum operator \( l \) and the second relation of (3.5) into (5.2), the subsidiary condition is calculated as follows:

\[
(l-L_{\text{coll}}) \Psi[r] = \left[ \frac{\hbar}{i} \sum_n \left( x_n \frac{\partial}{\partial y_n} - y_n \frac{\partial}{\partial x_n} \right) - 2 \{ \phi_1 (r_0^2 r^{-2} \eta_2 + i \hbar 2 r_0^4 r^{-4} \phi_2) - \phi_2 (r_0^2 r^{-2} \eta_1 + i \hbar 2 r_0^4 r^{-4} \phi_1) \} \right] \Psi[r]
\]

\[
= \frac{\hbar}{i} \left[ \frac{1}{N} \sum_n (x_n y_n - y_n x_n) - 2 r_0^2 r^{-2} \{ \phi_1 \sum_n \left( x_n \frac{\partial}{\partial y_n} + y_n \frac{\partial}{\partial x_n} \right) - \phi_2 \sum_n \left( x_n \frac{\partial}{\partial y_n} - y_n \frac{\partial}{\partial x_n} \right) \} \right] \Psi[r]
\]

\[
= -\frac{\hbar}{i} 2 r_0^2 r^{-2} \left[ \phi_1 \sum_n \left( x_n y_n + y_n x_n \right) \frac{\partial}{\partial r} - \phi_2 \sum_n \left( x_n^2 - y_n^2 \right) \frac{\partial}{\partial r} \right] \Psi[r]
\]

\[
= -\frac{\hbar}{i} 4 r_0^4 r^{-3} \left[ \phi_1 \frac{1}{N r_0^2} \sum_n x_n y_n \frac{\partial}{\partial r} - \phi_2 \frac{1}{N r_0^2} \sum_n \frac{1}{2} (x_n^2 - y_n^2) \frac{\partial}{\partial r} \right] \Psi[r]
\]

\[
= 4 i \hbar r_0^4 r^{-3} [\phi_1 \phi_2 - 2 \phi_2 \phi_1] \frac{\partial}{\partial r} \Psi[r] = 0,
\]

where we have used the relations (5.1) and the collective coordinates (2.2) and conjugate momenta (2.4). Thus we could prove the subsidiary condition (3.4). The anzatz \( \Psi[\cdot] = \Psi[r] \) is one of the possible solutions for the wave function of the collective subspace.

In this paper, the exact canonically conjugate momenta \( \pi_i \) to \( \phi_i \) is derived by modifying the approximate momenta \( \eta_i \). We showed the commutativity of momenta \( \pi_i \) and \( \pi_j \) under the subsidiary condition. Using the exact canonical variables \( \phi_i \) and \( \pi_i \), we found the collective Hamiltonian including the so-called phonon-phonon interaction under the replacement of operator \( r^2 \) which is not a collective variable, by an average value \( \bar{r}^2 = r_0^2 \). In the simple two-dimensional nuclei, discussions of the couplings between the individual particle motion and the collective motion will be possible if we investigate in detail the collective subspace relating to the individual particle motion through the variable \( r^2 \). This work will be presented elsewhere.

Acknowledgements

S. N. would like to express his sincere thanks to Professor Manuel Fiolhais for kind and warm hospitality extended to him at the Centro de Física Computacional, Universidade de Coimbra, Portugal. This work was supported by FCT (Portugal) under the project CERN/FP/83505/2008.
Appendix

A Derivation of (4.10)

We here derive the approximate relation (4.10) for the case of \( \eta \).

\[
\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta = -i\hbar \sum_{n=1}^{N} \left( \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \left( x_n' \frac{\partial}{\partial x_n'} - y_n' \frac{\partial}{\partial y_n'} \right) \right) \]

\[
\approx -i\hbar \sum_{n,n'=1}^{N} \left\{ \left( x_n \frac{\partial}{\partial x_n} \right) x_n' \frac{\partial}{\partial x_n'} - \left( x_n \frac{\partial}{\partial x_n} \right) y_n' \frac{\partial}{\partial y_n'} \right\} \]

\[
- i\hbar \left( \sum_{n=1}^{N} x_n^2 \sum_{n'=1}^{N} \frac{\partial}{\partial x_n'} - \sum_{n=1}^{N} x_n' \sum_{n'=1}^{N} \frac{\partial}{\partial y_n'} \right) + \frac{\hbar}{i} \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n} \right) \]

\[
\approx -i\hbar \left\{ \frac{1}{2} \sum_{n=1}^{N} \left( x_n^2 - y_n^2 \right) \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_n'} + \frac{\partial}{\partial y_n'} \right) + \frac{1}{2} \sum_{n=1}^{N} \left( x_n^2 + y_n^2 \right) \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_n'} - \frac{\partial}{\partial y_n'} \right) \right\} \]

\[
+ \eta_1 + \sum_{n,n'=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \frac{\hbar}{i} \sum_{n'=1}^{N} \left( x_n' \frac{\partial}{\partial x_n'} - y_n' \frac{\partial}{\partial y_n'} \right), \]

in which \( \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \) stands for the mean-value of the operator \( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \) on the collective subspace \( |\text{coll.subspace}\rangle \).

\[
\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta = \eta_1 + \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_1 \]

\[
- i\hbar N r_0^2 \frac{1}{2} \phi_1 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right) - i\hbar \frac{1}{2} N r_0^2 \phi_1 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} - \frac{\partial}{\partial y_n^2} \right) \]  \hspace{1cm} (A.2)

\[
= \eta_1 + f(N) \eta_1 + \frac{\mu}{\hbar^2} \frac{1}{2} N r_0^2 [\eta_1, T] - i\hbar N r_0^2 \phi_1 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right), \]

where we have used the first equation of (2.2), (2.3), the first equation of (2.4) and (2.6). Then we have the relation

\[
\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta = \eta_1 + f(N) \eta_1 + \frac{\mu}{\hbar^2} \frac{1}{2} N r_0^2 [\eta_1, T] \]

\[
- i\hbar N r_0^2 \phi_1 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right). \]  \hspace{1cm} (A.3)

In the above equations (A.2) and (A.3), we have used the second relation of (4.13) for the unknown function \( f(N) \).
Next we derive the approximate relation (4.10) for the case of $\eta_2$.

$$\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_2 = -i\hbar \sum_{n,n'=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \left( x_{n'} \frac{\partial}{\partial x_{n'}} + y_{n'} \frac{\partial}{\partial y_{n'}} \right)$$

$$= -i\hbar \sum_{n,n'=1}^{N} \left\{ x_n \left( \frac{\partial}{\partial x_n}, x_{n'} \right) + x_{n'} \left( \frac{\partial}{\partial x_{n'}}, x_n \right) \right\} + x_n y_{n'} \frac{\partial}{\partial y_n \partial x_{n'}} + x_{n'} y_n \frac{\partial}{\partial y_{n'} \partial x_n}$$

$$+ y_n x_{n'} \frac{\partial}{\partial y_n \partial x_{n'}} + y_{n'} x_n \frac{\partial}{\partial y_{n'} \partial x_n}$$

$$\approx -i\hbar \sum_{n,n'=1}^{N} \left\{ \langle x_n \frac{\partial}{\partial x_n} \rangle x_{n'} \frac{\partial}{\partial x_{n'}} + \langle x_n \frac{\partial}{\partial x_n} \rangle y_{n'} \frac{\partial}{\partial y_{n'}} \right\}$$

$$+ \left\{ \langle y_n \frac{\partial}{\partial y_n} \rangle x_{n'} \frac{\partial}{\partial x_{n'}} + \langle y_n \frac{\partial}{\partial y_n} \rangle y_{n'} \frac{\partial}{\partial y_{n'}} \right\}$$

$$= -i\hbar \left\{ \sum_{n=1}^{N} x_n y_n \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_{n'}} + \frac{\partial}{\partial y_{n'}} \right) + \frac{1}{2} \sum_{n=1}^{N} \left( x_n^2 + y_n^2 \right) \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_{n'}} + \frac{\partial}{\partial y_{n'}} \right) \right\}$$

$$+ \frac{\hbar}{i} \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right)$$

$$= -i\hbar \left\{ \sum_{n=1}^{N} x_n y_n \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_{n'}} + \frac{\partial}{\partial y_{n'}} \right) + \frac{1}{2} \sum_{n=1}^{N} \left( x_n^2 + y_n^2 \right) \sum_{n'=1}^{N} \left( \frac{\partial}{\partial x_{n'}} + \frac{\partial}{\partial y_{n'}} \right) \right\}$$

$$+ \eta_2 + \frac{\hbar}{i} \sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right)$$

in which $\langle x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \rangle$ also stands for the mean-value of the operator $x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}$ on the collective subspace (coll.subspace). Then we reach the final goal

$$\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_2 = \eta_2 + \sum_{n=1}^{N} \langle x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \rangle \eta_2$$

$$- i\hbar N r_0^2 \phi_2 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right) - i\hbar \frac{1}{2} N r^2 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial y_n} + \frac{\partial}{\partial y_n} \frac{\partial}{\partial x_n} \right)$$

$$= \eta_2 + f(N) \eta_2 + \frac{\mu}{\hbar^2} \frac{1}{2} N r^2 \left[ \eta_2, T \right] - i\hbar N r_0^2 \phi_2 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right),$$

where we have used the second equation of (2.2), (2.3), the second equation of (2.4) and (2.6). Finally, we obtain the relation

$$\sum_{n=1}^{N} \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right) \eta_2 = \eta_2 + f(N) \eta_2 + \frac{\mu}{\hbar^2} \frac{1}{2} N r^2 \left[ \eta_2, T \right]$$

$$- i\hbar N r_0^2 \phi_2 \sum_{n=1}^{N} \left( \frac{\partial}{\partial x_n^2} + \frac{\partial}{\partial y_n^2} \right).$$

In the above equations (A.5) and (A.6), we also have used the second relation of (4.13) for the $f(N)$. Both the equations (A.3) and (A.6) are combined into a single equation. Thus we can derive the approximate relation (4.10).
References

[1] S. Tomonaga, Prog. Theor. Phys. 5 (1950) 544; 13 (1955) 467, 482.

[2] T. Marumori, Prog. Theor. Phys. 24 (1960) 331.

[3] S. Sunakawa, Y. Yoko-o and H. Nakatani, Prog. Theor. Phys. 27 (1962) 589, 600.

[4] S. Nishiyama, Prog. Theor. Phys. 58 (1977) 1316; in Collective Motion of Nuclei, Report of the Pre-Symposium of Tokyo Conference (1977) pp.98-108.

[5] A. Bohr and B. Mottelson, Nuclear Structure, Volume II, W. A. Benjamin, 1974.

[6] J.M. Eisenberg and W. Greiner, Nuclear Models, North-Holland Physics Publishing, Elsevier Science Publisher Company, Inc. 1987.

[7] D. J. Rowe and J. L. Wood, Fundamentals of Nuclear Models, Foundational Models, World Scientific Publishing Co. Pte. Ltd., 2010.

[8] T. Miyazima and T. Tamura, Prog. Theor. Phys. 15 (1956) 255.

[9] T. Tamura, Nuovo Cimento 4 (1956) 713.

[10] S. Nishiyama and João da Providencia, Nucl. Phys. A923 (2014) 51.