Quantum Stability of (2+1)-Spacetimes with Non-Trivial Topology

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Abstract

Quantum fields are investigated in the (2+1)-open-universes with non-trivial topologies by the method of images. The universes are locally de Sitter spacetime and anti-de Sitter spacetime. In the present article we study spacetimes whose spatial topologies are a torus with a cusp and a sphere with three cusps as a step toward the more general case. A quantum energy momentum tensor is obtained by the point stripping method. Though the cusps are no singularities, the latter cusps cause the divergence of the quantum field. This suggests that only the latter cusps are quantum mechanically unstable. Of course at the singularity of the background spacetime the quantum field diverges. Also the possibility of the divergence of topological effect by a negative spatial curvature is discussed. Since the volume of the negatively curved space is larger than that of the flat space, one see so many images of a single source by the non-trivial topology. It is confirmed that this divergence does not appear in our models of topologies. The results will be applicable to the case of three dimensional multi black hole [8].

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I. INTRODUCTION

When we consider matter fields in a spacetime with a non-trivial topology, the boundary effects of quantum fields appear. This is one of main targets in quantum field theory in curved spacetimes. Such effects have been studied well in spatially flat spacetimes, but not so well in spatially curved spacetimes. This is because of the complexity of the topology which is allowed in such curved spatial section. In other words, there will be new topological effects of quantum field in the curved spatial section with a little complex topology.

To construct a space with a non-trivial topology, we identify the points of covering space by the discrete subgroup of the isometry of the space. Then we want to consider the covering space with an appropriately simple isometry group such that the topology has some interesting characteristics. As the space with simple isometry, there are $S^n$, $R^n$ and $H^n$, so called closed-, flat- and open-universe. Furthermore, we decide to treat $H^n$ since this hyperbolic space allows various topologies possessing interesting characteristics.

To treat this open-universe as a background spacetime, we determine its time evolution. For simplicity we consider maximally symmetric spacetimes of de Sitter spacetime with a hyperbolic chart or anti de Sitter spacetime in a Robertson-Walker coordinate. Their spatial sections are $H^n$. The de Sitter spacetime with the hyperbolic chart may be important in a cosmological sense. It is believed that the global feature of our universe is homogeneous and isotropic. If our observation suggests that the spatial curvature of our universe is negative, the background spacetime is locally open-universe. In the inflation, the de Sitter spacetime with a hyperbolic chart is a good model for cosmology. If we prefer a universe with a finite volume, the de Sitter spacetime with a hyperbolic chart with non-trivial topology will become important.

The topology of the open-universe (in the present article, the ‘open-universe’ means not an open topology but only a negatively curved space) is well known in a two dimensional space. Then we construct the simple example of a two dimensional open-universe with interesting topology in (2+1)-de Sitter spacetimes and (2+1)-anti de Sitter spacetimes. Quantum
scalar field is studied in these spacetimes using point stripping manner. The divergences of the quantum fields will be discussed.

In the section 2, we prepare simple model of the universes with interesting topologies in the (2+1)-de Sitter spacetime and the (2+1)-anti de Sitter spacetime. The quantum field is investigated in the section 3. The last section is devoted to a summary and discussions.

II. (2+1) OPEN-UNIVERSE WITH NON-TRIVIAL TOPOLOGY

A. Two Dimensional Universe

First of all, we develop topologies of two-dimensional spatial sections. For the simplicity of topologies we treat (2+1)-spacetime in the present article. For a cosmological reason and a further simplicity, spatial two-dimensional sections are assumed to be $S^2$, $R^2$ or $H^2$ corresponding to closed-, flat- or open-universe, respectively. In a two-dimensional space, the topologies of complete manifolds are classified by Euler numbers. This is calculated from the number of handles and cusps (see Fig.4). The cusp is a point at infinity with a needle-like structure. Here it should be emphasized that the cusp is no unnatural or artificial. These points at infinity are no singularity. There is no reason to give them special treatment in a classical physics [3].
FIG. 1. A example of a two dimensional sphere with a finite volume is shown. There are six handles and five cusps. Its Euler number is $\chi = 2 - 2 \times 6 - 1 \times 5 = 15$.

From the Gauss-Bonnet theorem for a complete 2-manifold, the Euler number $\chi$ is given by

$$\frac{1}{4\pi} \int dv \, R = \chi = 2 - 2N(\text{handle}) - N(\text{cusp}),$$

(1)

where $R$ is a two-dimensional scalar curvature and $N(\ast)$ is the number of $\ast$. The signature of $R$ restricts the variety of the topology. Since the Euler number is less than 2 ($\chi = 2$ is for a sphere), the case of negative curvature allows various topologies (various numbers of handles and cusps). In such negative curvature space, we expect new topological effects of a quantum field. Since the cusp is an infinitely small structure, it may cause the divergence of the quantum field. The negative curvature space has a crucial characteristic that the volume of the space is larger than that of the flat space at a distant region. Then one will see so many images of sources because of a non-trivial topology. The method of images may suffer the difficulty of a divergence. To discuss these speculations, we must treat general topologies of the negative curvature space $H^2$. In the present article, however, only two simple cases of them can be investigated since these cases possess the cusps and the above
mentioned characteristic.

To construct a non-trivial topology of the negative curvature space, we draw a polygon surrounded by geodesics on a hyperbolic space $H^2$ as a fundamental region and identify the geodesics. Poincaré model is one of the models of the hyperbolic space, which is conformally flat and a compact chart. The metric of the Poincaré model is

$$ds^2 = \frac{4(dr^2 + r^2d\phi^2)}{(1 - r^2)^2},$$

whose spatial curvature is $-1$. In this model, geodesics are circles crossing a circle with $r = 1$ at right angles. This circle with $r = 1$ is a sphere at infinity corresponding to the infinity of $H^2$.

Now we give simple topologies including cusps. The simplest polygon producing non-trivial topology is a tetragon. Drawing the tetragon $\Box ABCD$ of a fundamental region as in Fig.2, the tessellation of $H^2$ by this tetragon becomes simplest. In Fig.2, a half of the tetragon, $\Delta ABC$ and $\Delta ACD$, is a regular triangle and its vertices $A, B, C, D$ are on the sphere at infinity. $H^2$ are tessellated by parallel transformations of these triangles. The tessellation turns out to be a series of regular polygons whose centers are the origin of the Poincaré model. The triangles form a self-homothetic structure.
FIG. 2. Poincaré model and its tessellation are shown. A bold circle $r = 1$ is a sphere at infinity. A shaded region is a fundamental region of a torus with a cusp or a sphere with three cusps. $\triangle ABC$ is transformed to a $n$-th outward position by $(\Lambda_k^{(i)})^n$. Furthermore, $R(\theta)$ transform it to $\triangle A'B'C'$. 

Requiring orientability, there are two pairs of identifications for geodesics contained by the triangles, which provide complete manifold. One is

$$\triangle ABC \longrightarrow \triangle DCG \quad \text{by} \quad \Lambda_{DCG}^{(1)}$$

$$\triangle ABC \longrightarrow \triangle LAD \quad \text{by} \quad \Lambda_{LAD}^{(1)}, \quad \Lambda^{(1)} \in SO(2,1).$$

They generate a discrete subgroup $\Gamma^{(1)}$ of $SO(2,1)$. By these identifications, the topology of $H^2/\Gamma^{(1)}$ becomes a torus having a point at infinity, which is a cusp (see Fig. (3)).
FIG. 3. The upper shows that a torus with a cusp is constructed from a tetragon. The lower shows a tetragon becomes a sphere with three cusps.

Of course, the Gauss-Bonnet theorem gives its Euler number as

$$\chi = 2 - 2N(\text{handle}) - N(\text{cusp}) = 2 - 2 - 1 = -1 = \frac{1}{4\pi} \int_{H^2/\Gamma(1)} dv^{(2)} R. \quad (5)$$

The other is

$$\triangle ABC \longrightarrow \triangle GDC \quad \text{by } \Lambda_{GDC} \quad (6)$$

$$\triangle ABC \longrightarrow \triangle ADL \quad \text{by } \Lambda_{ADL}, \quad \Lambda^{(2)} \in SO(2,1). \quad (7)$$

They generate $\Gamma^{(2)} \subset SO(2,1)$. A resultant topology $H^2/\Gamma^{(2)}$ is a sphere with three cusps (Fig.3). The Euler number of it is

$$\chi = 2 - 2N(\text{handle}) - N(\text{cusp}) = 2 - 0 - 3 = -1 = \frac{1}{4\pi} \int_{H^2/\Gamma^{(2)}} dv^{(2)} R. \quad (8)$$

It is noted that these manifolds $H^2/\Gamma^{(1)}$ and $H^2/\Gamma^{(2)}$ are easier to handle than the double-torus which is a well known example of hyperbolic manifolds. For instance, though
the fundamental group of our manifolds is \((a, b)\), that of the double torus is \((a, b, c, d) : \; \; aba^{-1}b^{-1}cde^{-1}d^{-1} = 1\) [4].

Here we express all elements of \(\Gamma^{(i)}\) in an appropriate form for the rest of the present article. Using \(\{\Lambda^{(i)}_1, \Lambda^{(i)}_2, \Lambda^{(i)}_3\}\) acting as

\[
\begin{align*}
\triangle ABC & \rightarrow \triangle DCG \; \; \text{by} \; \Lambda^{(1)}_1, \quad (9) \\
& \rightarrow \triangle BIE \; \; \text{by} \; \Lambda^{(1)}_2, \quad (10) \\
& \rightarrow \triangle KFA \; \; \text{by} \; \Lambda^{(1)}_3, \quad (11) \\
& \rightarrow \triangle GDC \; \; \text{by} \; \Lambda^{(2)}_1, \quad (12) \\
& \rightarrow \triangle EBI \; \; \text{by} \; \Lambda^{(2)}_2, \quad (13) \\
& \rightarrow \triangle AKF \; \; \text{by} \; \Lambda^{(2)}_3. \quad (14)
\end{align*}
\]

\(\forall T^{(i)} \in \Gamma^{(i)}\) is given by

\[
T^{(i)} = R(\theta_j)(\Lambda^{(i)}_k)^n. \quad (15)
\]

For example, \(T^{(i)}\) transform \(\triangle ABC\) to \(\triangle A'B'C'\) in Fig.2. \(\triangle ABC\) is transformed to a \(2n\)-th outward position of triangles by \((\Lambda^{(i)}_k)^n\) and rotated around the origin by \(R(\theta_j)\) with an appropriate angle \(\theta_j\). \(k\) is selected from \(1 \sim 3\) so that the orders of the vertices match between \(\triangle ABC\) and \(\triangle A'B'C'\).

For the rest of the article, we give \(\Lambda^{(1)}_1\) in terms of the Poincaré model. Using \(z = re^{i\theta}\), this representation of the isometry of \(H^2\) becomes a subgroup of \(SL(2, C)\) in the coordinate of the Poincaré model and is given by

\[
z \rightarrow f(z) = \frac{az + b}{bz + a}, \quad \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \in SL(2, C). \quad (16)
\]

\(\Lambda^{(1)}_1\) is given by

\[
f(z) = \frac{a_0z + b_0}{b_0z + a_0}, \quad (17)
\]

\[
a_0 = \frac{1}{\sqrt{1 - r_0^2}} \sqrt{i \frac{r_0e^{i\pi/8} - 1}{r_0e^{-i\pi/8} - 1}}, \quad b_0 = \frac{1}{\sqrt{1 - r_0^2}} \frac{r_0e^{i\pi/8}}{a_0}. \quad (18)
\]
Here we should note that $\Lambda^{(1)}$'s and $\Lambda^{(2)}$'s are in the different category of the Lorentz group $SO(2, 1)$. It is known that all elements of $SO(2, 1)$ are $SO(2, 1)$-conjugate to an element of the following forms. We call them as standard forms. In the $SL(2, C)$ representation (16),

\begin{itemize}
  \item 1) An elliptic element is conjugate to

  \[ T_e : \; f(z) = e^{i\theta} z = \frac{e^{i\theta/2} z}{e^{-i\theta/2}} \]  

  with one fixed point on $H^2$.

  \item 2) A parabolic element is conjugate to

  \[ T_p : \; f(z) = \frac{(1 + i)z + i}{-iz + (1 - i)} \]  

  with one fixed point on the sphere at infinity.

  \item 3) A hyperbolic element is conjugate to

  \[ T_h : \; f(z) = \frac{z \cosh \beta + \sinh \beta}{z \sinh \beta + \cosh \beta}, \]  

  with two fixed point on the sphere at infinity.
\end{itemize}

These angle parameters $\theta, \beta$ are real numbers. Though $\Lambda^{(1)}$'s are conjugate the category 3), $\Lambda^{(2)}$'s are parabolic. We note that the fixed point of a parabolic element corresponds to a cusp produced by the parabolic element. It is revealed in the next section that these facts affect the quantum field.

**B. de Sitter and anti de Sitter Spacetime with Non-trivial Topology**

Now we consider (2+1)-spacetime whose certain spatial sections have above mentioned topologies. For simplicity, Teichmüller deformation is not considered and every identifications are supposed to be done on the spatial 2-sections of the synchronous gauge. Then
the local geometry of the spacetime is that of the open FRW-universe. For further simplicity, we assume the spacetime is maximally symmetric. Such (2+1)-spacetimes allowing spatial $H^2$-sections are Minkowski spacetime, de Sitter spacetime $dS^3$ and anti de Sitter spacetime $AdS^3$. Though we treat only $dS^3$ and $AdS^3$ in the present article, the investigation of the present article is easily applicable to the case of the Minkowski spacetime.

The isometry groups of $dS^3$ and $AdS^3$ are $SO(3, 1)$ and $SO(2, 2)$, respectively. The open chart of $dS^3$ and the RW(Robertson-Walker)-coordinate of $AdS^3$ determine the natural extension of $SO(2, 1)$ which is the isometry of $H^2$ to $SO(3, 1)$ or $SO(2, 2)$ so that the extended action of $SO(2, 1)$ preserves their time-slicing, respectively. The identifications $\Gamma \subset SO(2, 1)$ on $H^2$ also extended to $\gamma \subset SO(3, 1)$, $SO(2, 2)$ on $dS^3$ and $AdS^3$, which preserves the time-slicing. $\gamma$ provides non-trivial topology $dS^3/\gamma$ or $AdS^3/\gamma$ to the spacetimes.

For the next section, we imbed $dS^3$ and $AdS^3$ as a covering spacetime of the concerning spacetime with non-trivial topology into four dimensional flat spacetimes with signatures (−+++ and (−+++), respectively. Using the coordinate of the Poincaré model for spatial sections,

$$ds^2 = dX^2 + dY^2 - dZ^2 \left\{ \begin{array}{l} + \\ - \end{array} \right\} dW^2 \quad (23)$$

$$X = \begin{cases} \sinh t \\ \cos t \end{cases} \frac{2r}{1 - r^2} \cos \theta \quad (24)$$

$$Y = \begin{cases} \sinh t \\ \cos t \end{cases} \frac{2r}{1 - r^2} \sin \theta \quad (25)$$

$$Z = \begin{cases} \sinh t \\ \cos t \end{cases} \frac{1 + r^2}{1 - r^2} \quad (26)$$

$$W = \begin{cases} \cosh t \\ \sin t \end{cases} \quad (27)$$

$$\Rightarrow ds^2 = -dt^2 + \left\{ \begin{array}{l} \sin^2 t \\ \cos^2 t \end{array} \right\} \frac{4(dr^2 + r^2 d\theta^2)}{(1 - r^2)^2}, \quad (28)$$

where the upper case is $dS^3$ and the lower case is $AdS^3$. We treat only the spacetime with a unit curvature radius for simplicity because the absolute value of the curvature is not essential for the following investigation.
III. QUANTUM FIELD

In this section Quantum field is investigated in the spacetime with non-trivial topology whose covering spacetime is $dS^3$ or $AdS^3$. We introduce conformally coupled massless scalar field $\phi$, with the action

$$S = - \int dx^3 \sqrt{g} \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{16} R \phi^2 \right),$$  \hspace{1cm} (29)

where $R$ is the scalar curvature. The field equation in $dS^3$ or $AdS^3$ with $R_{\mu\nu} = \pm 2g_{\mu\nu}$ is

$$\left( \nabla^\mu \nabla_\mu + \frac{3}{4} \right) \phi = 0$$ \hspace{1cm} (30)

with $R = \pm 6$ for our $dS^3$ or $AdS^3$ with a unit curvature radius.

Now we consider the Hadamard Green functions in the covering spacetime $dS^3$ or $AdS^3$. According to Steif \[6\], they are given by

$$\bar{G}(x, y) = \frac{1}{4\pi |x - y|},$$ \hspace{1cm} (31)

where $|x - y|$ is a chordal distance between $x$ and $y$ in the four dimensional imbedding spacetime \((24)\sim(27)\) and not a proper distance in $dS^3$ or $AdS^3$.

The Hadamard functions for spacetimes with non-trivial topologies $dS^3/\gamma^{(i)}$, $AdS^3/\gamma^{(i)}$ can be obtained from the Hadamard function for their covering spacetime \((31)\) by the method of images. Since the images of $y$ are generated by elements of $\gamma^{(i)}$, the Green function is

$$G^{(i)}(x, y) = \sum_{T^{(i)} \in \gamma^{(i)}, T^{(i)} \neq id} \bar{G}(x, T^{(i)} \circ y).$$ \hspace{1cm} (32)

where the summation is over all elements of $\gamma^{(i)}$ except for identity. The identity is excluded to subtract all local contributions of the quantum field. This procedure ought to regularize the energy-momentum tensor of the quantum field.

When $\gamma$ is Abelian group, the summation can be easily evaluated like the three dimensional black hole case \[4\] (for example, $AdS^3/\gamma_B H : \gamma_B H = (A_1^{(1)})^n$ is equivalent to the three dimensional black hole). On the other hand, our non-Abelian $\gamma$ makes rigorous evaluations impossible. The simple universe shown in the previous section, however, allows us to evaluate some divergences. The abstract summation of \((32)\) is decomposed into
\[ G^{(i)}(x, y) = \sum_{n=1}^{\infty} \sum_{j=1}^{3} \sum_{k} \bar{G}(x, R(\theta_j)(\Lambda_k^{(i)})^n \circ y) \] (33)

by eq.(15) A quantum energy-momentum tensor is given by

\[ < T_{\mu\nu} > = \lim_{y \to x} \mathcal{D}_{\mu\nu} G(x, y), \] (34)

in the point stripping method, where \( \mathcal{D}_{\mu\nu} \) is a certain differential operator (see [3], for example). Hence, investigating the zero of the distance \(|x - T(x)|\) and the summations of (33), we can discuss the divergences of \( < T_{\mu\nu} > \) since the divergences of \( < T_{\mu\nu} > \) come out from the divergences of the Green function.

First of all we discuss the characteristics of \( \gamma^{(1)} \). As stated in the previous section, \( T^{(1)} \in \gamma^{(1)} \) can be written as

\[ T^{(1)} = A^{-1} T_h A \]

by an appropriate \( A \in SO(2,1) \), where \( A \) corresponds to an isometric coordinate transformation of \( \theta \to \theta' \) and \( r \to r' \) in eq.(28). \( T_h \) in eq.(28) corresponds to a Lorentz boost about \( X-Z \) direction in the imbedding spacetime. From imbedding equation (24)\( \sim \) (27), a chordal distance \(|x - T^{(1)}(x)|\) is given by

\[ |x - A^{-1} T_h A \circ x| = |x' - T_h \circ x'| \] (35)

\[ = \sqrt{(2 \cosh 2\chi - 1)(X'^2 - Z'^2)} \] (36)

\[ = \left\{ \frac{\sinh t}{\cos t} \right\} \sqrt{(2 \cosh 2\chi - 1)\frac{4r'^2 \cos^2 \theta' - (1 + r'^2)^2}{(1 - r'^2)^2}}, \] (37)

where \{\( X', Y', Z', W' \)\} is an imbedding coordinate for \( x' = A(x) \). \( T_h \) in the imbedding spacetime \{\( X, Y, Z, W \)\} is given by the matrix,

\[
\begin{pmatrix}
\cosh \chi & 0 & \sinh \chi & 0 \\
0 & 1 & 0 & 0 \\
\sinh \chi & 0 & \cosh \chi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (38)

From (24)\( \sim \) (27), \( \chi \) is given by the parameter \( \beta \) of (22) as
\[ \chi = 2\beta. \]  

(39)

In the \(SL(2, C)\) representation (16), for a hyperbolic element \(T^{(1)}\)

\[ \chi = 2 \cosh^{-1} \frac{Tr_{SL(2, C)}(T^{(1)})}{2} = 2 \cosh^{-1} \frac{a + \bar{a}}{2}, \]  

(40)

where \(Tr_{SL(2, C)}\) is a trace about \(SL(2, C)\) matrix. Especially, by the rotational symmetry of \(\Lambda^{(1)}_{1, 3}\)’s, all \(\chi\)’s of \(\Lambda^{(1)}_{1, 3}\)’s are the same. We write it as \(\chi_0 = 2 \cosh^{-1}((a_0 + \bar{a}_0)/2)\).

On the contrary, \(T^{(2)}\) has a different standard form (21). With appropriate \(B \in SO(2, 1)\) and \(T_p\) in (21), \(T^{(2)}\) is written as \(B^{-1}T_pB\). We dare express the distance in the coordinate of the Poincaré model (24) ∼ (27) to evaluate it at infinity. \(|x - T^{(2)}(x)|\) becomes

\[ |x - B^{-1}T_pB \circ x| = |x'' - T_p \circ x''| \]  

(41)

\[ = (Z'' + Y'') \]  

(42)

\[ = \left\{ \begin{array}{l} \sinh t \frac{2r'' \sin \theta'' + 1 + r''^2}{1 - r''^2}, \\ \cos t \end{array} \right. \]  

(43)

where \(\{X'', Y'', Z'', W''\}\) and \(r'', \theta''\) are coordinates for \(x'' = B(x)\). When \(\theta''\) is \(3\pi/2\), the distance vanishes at infinity \((r'' \to 1)\). This point \((r'' = 1, \theta'' = 3\pi/2)\) is a fixed point of \(T_p\) corresponding to a cusp. \(T^{(2)}\) forms a cusp at \(x_c\) with \(B(x_c) = (r'' = 1, \theta'' = 3\pi/2)\). Then \(|x - T^{(2)}(x)|\) should vanish at the cusp on each timeslice, while \(|x - T^{(1)}(x)|\) never approaches to zero except for the initial or final singularity.

There are three possibilities of divergences for the quantum field. The quantum field will diverge at the singularity of the background spacetime as in the case of three dimensional black hole [3]. Also the infinitely small structure of cusps will cause the divergence of the quantum field though the cusp is not a singularity. Furthermore, the summation of the images in the method of images is expected to diverge since the volume of the hyperbolic space is larger than that of the flat space at a distant region. One will see so many images of a source.

First two possibilities are investigated in the following. For \(T^{(1)} \in \gamma^{(1)}\), eq.(37) tells that \(|x - T^{(1)}(x)|\) vanishes at the planes \(X'^2 - Z'^2 = 0\). Since \(A \in SO(2, 1)\) is a proper-Lorentz
group in a part of the imbedding spacetime \( \{X, Y, Z, W = \pm 1\} \), the planes \( X'^2 - Z'^2 = 0 \) cannot invade into the inside of a light cone at \( \{0, 0, 0, \pm 1\} \) which is initial or final singularity of \( dS^3 \) and \( AdS^3 \). Moreover, one or two pairs of null geodesic, \( \{X'^2 - Z'^2 = 0, Y' = 0, W' = \pm 1\} \), lie on the singularities. Since the quantum field diverges on these planes \( < T_{\mu\nu} > \) diverge when one approaches to the singularity of the background spacetimes by the topological effect.

On the other hand, \( T^{(2)} \in \gamma^{(2)} \) has a different characteristic about cusps. From eq.(43), \( \|x - T^{(2)}(x)\| \) vanishes at the cusps on each time-slice, while \( \|x - T^{(1)}(x)\| \) never vanishes on each time-slice even at their cusps. Therefore \( < T_{\mu\nu} > \) is singular at the cusps of \( dS^3/\gamma^{(2)} \) and \( AdS^3/\gamma^{(2)} \) and regular at the cusps of \( dS^3/\gamma^{(1)} \) and \( AdS^3/\gamma^{(1)} \).

Finally we discuss the third possibility of the divergences estimating the summation over all transformations in (33). From a rotational symmetry around the origin of each time-slice \( (X = Y = 0, Z = \{\sinh t, \cos t\}, W = \{\cosh t, \sin t\}) \),

\[
|x - R(\theta)(\Lambda_k^{(1)})^n \circ x|_{x=\text{origin}} = |x - (\Lambda_k^{(1)})^n \circ x|_{x=\text{origin}} = \begin{cases} \sinh t \\ \cos t \end{cases} \sqrt{2 \cosh(2n\chi_0) - 1},
\]

where we use \( A^{-1}\Lambda^A = (A^{-1}\Lambda A)^n \). If we consider the point \( x = x_0 \),

\[
\begin{align*}
\begin{cases} \sinh t \\ \cos t \end{cases} \sqrt{2 \cosh(2n\chi_0) - 1} - 2|x_0| & \leq |x - R(\theta)(\Lambda_k^{(1)})^n|_{x=x_0} \\
& \leq \begin{cases} \sinh t \\ \cos t \end{cases} \sqrt{2 \cosh(2n\chi_0) - 1} + 2|x_0|
\end{align*}
\]

For a sufficiently large \( n \), \( \sqrt{2 \cosh(2n\chi_0) - 1} \) and \( |x - R(\theta)(\Lambda_k^{(2)})^n(x)| \) behaves as \( e^{n|\chi_0|} \). Furthermore, \( |x - R(\theta)(\Lambda_k^{(2)})^n| \) also behaves as \( e^{n|\chi_0|} \) for a large \( n \), since the tessellation of \( \gamma^{(2)} \) is the same tessellation as \( \gamma^{(1)} \). From (17)(18)(19)(40) the summation about \( j \) and \( k \) in (33) is estimated as

\[
\lim_{y \to x} \sum_{j=1}^{3} \sum_{k} \bar{G}(x, R(\theta_j)(T_k)^i \circ y) \sim \frac{4^n}{e^{n|\chi_0|}} = \left( \frac{4}{4.74} \right)^n = 0.8438^n.
\]

Though a rigorous estimation may be possible, it is too complicated and will give us no essential information. \( < T_{\mu\nu} > \sim O(\sum_n 0.843^n) \) barely converge because of the exact value.
of $\chi_0$. If the investigation could be done in other topology with negative curvature, a different value of $\chi$ might cause the divergences of the summation.

IV. SUMMARY AND DISCUSSION

In the present article, we have investigated new topological effects of a quantum field in a torus universe with a cusp and a sphere universe with three cusps. Their covering spacetime is de Sitter spacetime or anti de Sitter spacetime. The cusp is a point at infinity with regular local structure and needle-like global structure. Three possibilities of divergences of the energy-momentum tensor have been studied.

First, a divergence appears on the coordinate singularity of the classical background spacetime, which is initial or final singularity in cosmological sense. This is similar to the case of the three dimensional black hole [6].

Next possibility is a divergence at the cusps. In the present article, we show there are two types of the cusps. One is a cusp made by hyperbolic transformations of $SO(2,1)$ and the other is made by a parabolic transformation. We observed that $< T_{\mu\nu} >$ diverge at the latter cusps, which are included in the sphere with three cusps. This aspect means that the latter cusp is quantum mechanically unstable. Only the latter cusps will require a treatment in quantum gravity.

The last possibility is the divergence of the summation of images. This corresponds to the effect that we see more images of a source in a negatively curved universe than in a flat universe at a distant region. The summation, however, converges in the spacetimes given in the present article. The convergence of the image summation strongly depends on the values of a boost angle $|\chi|$ and the shape of a tessellation in the covering space. Though $e^{|\chi_0|}$ is 4.74, if $e^{|\chi_0|}$ were less than 4 with the same tessellation, $< T_{\mu\nu} >$ would diverge everywhere and the divergence is hard to remove. In the case of other topology, other $e^{|\chi|}$ and other tessellation may make the summation diverge. If so, it will turn out that there are topologies accepting quantum field and not accepting. When we consider a compact (without a cusp)
topology with a negative curvature, such situation may occur, though the compact topology is very difficult to treat.

Recently Brill [8] shows that a three dimensional multi-black hole solution can be constructed in the three dimensional anti-de Sitter spacetime. It is easily found that $AdS^3/\gamma^{(2)}$ with a larger boost angle $|\chi'|$ than $|\chi_0|$ is regarded as a two-black hole solution. Of course, the summation of images converges in this solution.

By a regularization performed in the present article, we perfectly subtract local divergences. We, however, have observed the topological divergences. They cannot be regularized and will have physical meanings.

Can we carry out a similar investigation in other topology with a negative curvature. A compact topology (without a cusp) seems impossible since the tessellation is so complicated. On the other hand, it may be possible to treat other topologies with cusps. At least, we can decide whether each cusp causes a divergence of quantum field or not knowing whether the identification providing the cusp is parabolic or hyperbolic. The divergence of summation of images sensitively depends on the shape of the tessellation in the covering space and will be difficult to treat without a sufficient symmetry. In (3+1)-dimension the similar investigation may be possible. There will be convenient models of non-trivial topology.

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