THIRD–ORDER AFFINE–INVARIANT (SYSTEMS OF) PDES IN TWO INDEPENDENT VARIABLES AS VANISHING OF THE FUBINI–PICK INVARIANT

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Abstract. In this paper we study 3rd order (system of) PDEs in two independent variables $x, y$ and one unknown function $u$ that are invariant with respect to the group of affine transformation $\text{Aff}(3)$ of $\mathbb{R}^3 = \{(x, y, u)\}$. After proving their relationship with the Fubini–Pick invariant, we derive the aforementioned PDEs by using a general method introduced in [3], which sheds light on some of their geometrical properties.

Keywords: Lie symmetries of PDEs; G-invariant PDEs; Jet Spaces; Fubini-Pick invariant

MSC 2020: 35A30; 35B06; 58A20; 58J70

Introduction

It is well known that the only $\text{Aff}(3)$–invariant second–order scalar PDE is the one given by $\det(u_{ij}) = 0$: a proof can be found, e.g., in [18] (see also the references therein, as well as [24] for a discussion of the general solution); remarkably, the contact geometry of this equation can be described in terms of its characteristics, see [2, 4, 5]. In this paper we show how to construct (non–trivial systems of) $\text{Aff}(3)$–invariant third–order PDEs in one unknown function and two independent variables by using a general method developed in [3, 1]. The geometric character of the proposed construction will be duly emphasised: first, by highlighting the relation of the so–obtained PDEs with the Fubini–Pick invariant, then by studying the compatibility conditions (if a system of $\text{Aff}(3)$–invariant of third–order PDEs is obtained) or the geometry of characteristics (if a scalar $\text{Aff}(3)$–invariant third–order PDE is obtained) and, finally, by clarifying the overall role played by the Blaschke metric.

A statistical manifold is a (pseudo)–Riemannian manifold $(M, h)$ equipped with a connection $\nabla$, such that $\nabla h$ is symmetric, see [14]; it is worth mentioning that the Blaschke metric, the Blaschke connection, and the Fubini–Pick $(0, 3)$–tensor, all satisfy the axioms of a statistical manifold. Therefore, our geometric departing point can be framed in this larger context.

The paper is organized as follows.

In Section 1 we focus on the basics of the affine geometry of surfaces of $\mathbb{R}^3$ and the affine structure of jet spaces. Contextually, we give the definition of a $G$–invariant (systems of) PDEs.

In Section 2, after recalling the main results of [3], we explain how to construct $G$–invariant (systems of) PDEs by using the aforementioned method.

In Section 3 we obtain the same $\text{Aff}(3)$–invariant (system of) PDE from two different viewpoints. The first one is based on the method used in [16]: $G$–invariant PDEs can be obtained as singular orbits of the prolonged action of $G$ on suitable jet spaces (Section 3.1). The second one shows that the same (system of) PDEs can be obtained by employing the affine second fundamental form $h^\xi$ associated to a transversal vector field $\xi$ and the corresponding affine $(0, 3)$–tensor $C^\xi$, rather than by using the Blaschke metric (Section 3.3).

In Section 4 we use the method outlined in Section 2 to construct $\text{Aff}(3)$–invariant (systems of) PDEs. The idea is to (equivalently) extend certain submanifolds defined in a fibre of the jet space of order 3 to the whole jet space by means of the affine action of $\text{Aff}(3)$, showing that only two cases are possible, accordingly to the signature of the Hessian matrix. Indeed, if the Hessian is non–degenerate (this is always the case if we assume that the affine second fundamental form is non–degenerate), we get an $\text{Aff}(3)$–invariant system of two PDEs of third order; on the other hand, if the determinant of the Hessian is negative, we obtain also a pair of $\text{Aff}(3)$–invariant scalar PDEs of third order: if these two scalar PDEs are considered as a system, the result is the same $\text{Aff}(3)$–invariant system of PDEs mentioned before.

In Section 5 we focus on the $\text{Aff}(3)$–invariant system of PDEs by computing its compatibility conditions. In particular, we show that a solution $u = u(x, y)$ to this system described a conic in both the planes $(x, u)$ and $(y, u)$.

In Section 6 we focus on the $\text{Aff}(3)$–invariant scalar PDEs: we show that their characteristic distribution degenerates into a 3–dimensional vector sub–distribution of the 5–dimensional Cartan distribution (i.e., the contact distribution of order 2) on $J^2$. We finally prove that such a 3–dimensional distribution completely characterizes the $\text{Aff}(3)$–invariant scalar PDEs, in the sense that their solutions are Legendrian submanifolds of $J^1$ whose prolongation on $J^2$ non–trivially intersect the aforementioned distribution.
**Notations and conventions.** We denote by $\oplus$ the symmetric product between tensors. In the case of differential forms, we will often omit the symbol $\oplus$, for instance:

$$dx^i dx^j := dx^i \oplus dx^j = \frac{1}{2} (dx^i \otimes dx^j + dx^j \otimes dx^i).$$

We denote by $S^k V$ the $k$th symmetric tensor power of $V$. For a subset $U \subseteq V$ of the vector space $V$, symbol $\langle U \rangle$ denotes the linear span of $U$ in $V$; we simply write $\mathbb{R}v$ for the linear span of $U = \{v\}$. If a group $G$ acts on a set $S$, then we let $G \cdot A := \{g \cdot a \mid g \in G, a \in A\}$ for any $A \subseteq S$. The module of vector fields on a manifold $M$ is denoted by $\mathfrak{X}(M)$. A system of coordinates on $\mathbb{R}^3$ will be denoted by $(x, y, u)$, unless otherwise specified. The Einstein convention on repeated indices will be used, unless otherwise specified.

**Acknowledgments.** D. Alekseevsky gratefully acknowledges support by the Grant Basis-Foundation Leader n. 22-7-1-34-1. G. Manno gratefully acknowledges support by the project “Connessioni proiettive, equazioni di Monge-Ampère e sistemi integrabili” (INdAM), “MIUR grant Dipartimenti di Eccellenza 2018-2022 (E11G18000350001)”, “Finanziamento alla Ricerca” 53_RBA17MANGIO and 53_RBA21MANGIO, and PRIN project 2017 “Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics” (code 2017JJ22SW5). G. Manno is a member of GNSAGA of INdAM. G. Moreno is supported by the Polish National Science Center project “Complex contact manifolds and geometry of secants”, 2017/26/E/ST1/00231.

## 1. Preliminaries

### 1.1. Basics on affine geometry of surfaces [6, 20].

Let $S$ be a surface of $\mathbb{R}^3$ and $\xi$ be a transversal field to it. We denote by $h^\xi$ the affine second fundamental form associated to $\xi$, i.e., the bilinear form defined by

$$D_X Y = \nabla_X Y + h^\xi(X, Y)\xi, \quad X, Y \in \mathfrak{X}(S),$$

where $D$ is the Levi-Civita connection of $\mathbb{R}^3$ and $\nabla_X Y$ is the tangential component. The rank of $h^\xi$ is independent of the choice of the transversal field $\xi$, so that a surface $S \subseteq \mathbb{R}^3$ is called non-degenerate if the rank of $h^\xi$ is equal to 2. From now on, unless otherwise specified, we shall work only with non-degenerate surfaces. Let $\omega \in \Omega^3(\mathbb{R}^3)$ be a $D$–parallel volume element: the induced volume form on $S$, associated with $\xi$, is the differential 2–form $\omega^\xi$ defined by

$$\omega^\xi(X, Y) := \omega(X, Y, \xi), \quad X, Y \in \mathfrak{X}(S),$$

and it is easy to see that

$$\nabla^\xi(\omega^\xi) = \tau^\xi \otimes \omega^\xi$$

for some 1–differential form $\tau^\xi$. The Codazzi equation for $h^\xi$,

$$(\nabla_X h^\xi)(Y, Z) + \tau^\xi(X) h^\xi(Y, Z) = (\nabla_X h^\xi)(X, Z) + \tau^\xi(Y) h^\xi(X, Z), \quad X, Y, Z \in \mathfrak{X}(S),$$

implies that the following $(0,3)$–tensor on $S$, that we call the affine $(0,3)$–tensor associated with $\xi$, is symmetric:

$$C^\xi(X, Y, Z) := (\nabla_X h^\xi)(Y, Z) + \tau^\xi(X) h^\xi(Y, Z), \quad X, Y, Z \in \mathfrak{X}(S).$$

Recalling that we are assuming the surface $S$ to be non-degenerate, there exists a unique, up to sign, transversal field $A$, called the affine normal, such that $\tau^A = 0$ and $\omega^A$ coincides with the volume element associated to $h^A$. The corresponding connection $\nabla := \nabla^A$, the $(0,2)$–tensor $h := h^A$ and the $(0,3)$–tensor $C := C^A$,

$$(4) \quad C(X, Y, Z) = (\nabla_X h)(Y, Z), \quad X, Y, Z \in \mathfrak{X}(S),$$

are called, respectively, the Blaschke connection, the Blaschke metric and the Fubini–Pick cubic form. The contraction of the Fubini-Pick form $C$ with the Blaschke metric, i.e.,

$$C^{ijk} C_{ijk} = h^{i1i2j1j2k1k2} C_{i1j1k1i2j2k2},$$

is a function called the Fubini-Pick invariant. Hence, the surface $S$ with the (pseudo)–Riemannian metric $h$ and the connection $\nabla$ is a statistical manifold as the Fubini–Pick cubic form $C = \nabla h$ is symmetric. In the context of statistical manifolds such tensor is called also the Amari–Chentsov tensor.

### 1.2. Jet spaces and PDEs [12, 23, 25].

From now on, $M$ will be a smooth manifold of dimension 3 and $S \subseteq M$ an embedded surface of $M$, unless specified otherwise. Locally, in an appropriate system of local coordinates $(x^1, x^2, u)$ of a neighborhood of $p \in S$, the surface $S$ is represented as $S = S_f = \{u = f(x^1, x^2)\}$ where $f$ is a smooth function of the variables $(x^1, x^2)$.

We denote by $J^\ell$ the space of $\ell$–jets of surfaces of $M$, i.e.,

$$J^\ell := \bigcup_{p \in M} \{[S]_p^\ell \mid S \text{ is a surface of } M \text{ passing through } p\},$$

where $[S]_p^\ell$ denotes the equivalence class of surfaces having with $S$ a contact of order $\ell$ at $p$. 


The natural map \( j^\ell : p \in S \mapsto [S]_p^\ell \in J^\ell \) is called the \( \ell \)-th jet extension of \( S \) and symbol
\[
S^{(\ell)} := j^\ell(S)
\]
denotes its image.

The coordinates \((x^1, x^2, u)\) defines local coordinates
\[
(x^1, x^2, f(x^1, x^2), \frac{\partial f}{\partial x^1}, \frac{\partial^2 f}{\partial x^1 \partial x^2}, \ldots, \frac{\partial^\ell f}{\partial x^1 \partial x^2 \cdots \partial x^\ell})
\]
of \( J^\ell \), such that the prolongation \( S^{(\ell)}_f \) of the surface \( S_f \) is locally described by
\[
\left( x, x^2, f(x^1, x^2), \frac{\partial f}{\partial x^1}, \frac{\partial^2 f}{\partial x^1 \partial x^2}, \ldots, \frac{\partial^\ell f}{\partial x^1 \partial x^2 \cdots \partial x^\ell} \right),
\]
which shows that \( S^{(\ell)}_f \) is a \( 2 \)-dimensional submanifold of \( J^\ell \).

The natural projections
\[
\pi_{\ell,m} : J^\ell \to J^m, \quad [S]_p^\ell \mapsto [S]_p^m, \quad \ell > m,
\]
define a tower of bundles. In what follows, a point \([S]_p^\ell \in J^\ell \) will be often denoted simply by \( a^\ell \): accordingly,
\[
J^\ell_{a^m} := \pi_{\ell,m}^{-1}(a^m), \quad a^m \in J^m,
\]
denotes the fiber of \( \pi_{\ell,m} \) over \( a^m \). The following definition will come in handy.

**Definition 1.1.** For \( \ell \geq 1 \), the tautological vector bundle \( T^\ell \subset \pi_{\ell-1}^* T(J^{\ell-1}) \) is the bundle over \( J^\ell \) given by
\[
T^\ell := \{(a^\ell, v) \in J^\ell \times T J^{\ell-1} \mid v \in T_{a^{\ell-1}} S^{(\ell-1)}\},
\]
i.e., the fiber \( T_{a^\ell}^\ell \) over the point \( a^\ell = [S]_p^\ell \) is \( T_{a^\ell}^\ell = T_{a^{\ell-1}} S^{(\ell-1)} \).

It is well known that \( \pi_{\ell-1} : J^{\ell-1} \to J^{\ell-1} \), for \( \ell \geq 2 \), are affine bundles modeled on the vector bundle
\[
\ker(d\pi_{\ell-1})
\]
on \( J^\ell \), where \( d\pi_{\ell-1} : J^{\ell} \to T J^{\ell-1} \): the contact distribution of order \( \ell \) on \( J^\ell \) can be then defined as:
\[
C^\ell := (d\pi_{\ell-1})^{-1}(T^\ell),
\]
see [13, 15, 26]; the next result is also classical, see [13, 23].

**Lemma 1.1.** There is an identification
\[
\ker(d\pi_{\ell-1}) = \pi_{\ell,1}^*(S^\ell T^* \otimes N),
\]
where \( T := T^1 \) and \( N \) is the vector bundle on \( J^1 \) whose fibres are defined by \( N_{[S]_p} := T_p M / T_p S \).

**Remark 1.1.** This affine bundle structure will be playing a key role in Section 2 below, since it allows to regard the difference between two points \([S]_p^\ell \) and \([\hat{S}]_p^\ell \) of the fibre \( J^\ell_{a^{\ell-1}} \) as an element of \( S^\ell T^* S \otimes \frac{T_p M}{T_p S} \); for instance, with \( \ell = 2 \), Lemma 1.1 locally reads
\[
[S]_p^2 - [\hat{S}]_p^2 = (x^i, u, u_i, u_{ij}) - (x^i, u, u_i, \hat{u}_{ij}) \approx (u_{ij} - \hat{u}_{ij}) \partial_{u_{ij}} \approx \frac{2}{1 + \delta_{ij}} (u_{ij} - \hat{u}_{ij}) dx^i dx^j \otimes \partial_u.
\]

A system of \( m \) PDEs of order \( \ell \) in \( 2 \) independent variables and one unknown (or dependent) variable on a manifold \( M = J^0 \) is a \( m \)-codimensional sub-bundle \( E \) of \( J^\ell \). Taking into account (8), we define the fiber of \( E \subset J^\ell \) over \( a^m \in J^m \), \( m < \ell \).

\[
E_{a^m} := E \cap J^\ell_{a^m}.
\]
A solution to \( E \) is a surface \( S \) of \( M \) such that its \( \ell \)-th jet extension \( S^{(\ell)} \) lies in \( E \), cf. (6).

A local diffeomorphism \( \phi \) of \( M \) naturally acts on all the jet spaces via its prolongation
\[
[S]_p^\ell \in J^\ell \mapsto [\phi(S)]_{\phi(p)}^\ell \in J^\ell
\]
to \( J^\ell \). This allows to speak of \( G \)-invariance for a system of PDEs \( E \subset J^\ell \), where \( G \) is a connected Lie group acting on \( M \): the system \( E \) is \( G \)-invariant if the subset \( E \) is preserved by \( G \). Let \( X \) be an infinitesimal generator of \( G \): its prolongation \( X^{(\ell)} \) to \( J^\ell \) is obtained by prolonging its local flow; therefore, if \( E \) is \( G \)-invariant, then \( X^{(\ell)} \) is tangent to \( E \). It is worth recalling the prolongation formula for \( X = X_0 \partial_u + X^i \partial_{x^i} \):
\[
X^{(\ell)} = X_0 \partial_u + X^i \partial_{x^i} + \sum_{|\sigma| = 1}^{\ell} X_\sigma \partial_{u_\sigma},
\]
where \( \sigma \) is a multi–index of length \(|\sigma|\) defined by \( \sigma = (\sigma_1, \ldots, \sigma_s) \), \( s \leq \ell \), \( \sigma_i \in \{1, \ldots, n\}, \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_s \), and the component \( X_{\sigma,j} \) are recursively defined by

\[
X_{\sigma,j} := D_x^{j}(X_\sigma) - u_{\sigma_j} D_x^{j}(X^i),
\]

where \( D_x^{j} \) is the operator of total derivation with respect to \( x^j \).

2. A GENERAL METHOD TO CONSTRUCT INVARIANT PDES ON HOMOGENEOUS MANIFOLDS

We review now one of the main result of [3]. Let us assume that \( J^0 = M \) is a homogeneous manifold, i.e., \( J^0 = G/H \) for a connected Lie group \( G \), and choose a point \( o \in J^0 \), as well as \( o^k \in J^k \), projecting onto \( o \): we can then consider, \( \forall \ell \geq 2 \), the fibre \( J^\ell_{o^k} \) as a vector space with origin \( o^\ell \). The natural lift of an element \( g \in G \) on each \( \ell \)–jet space \( J^\ell \) is given by (13), which in this case reads

\[
g: o^\ell = [S]_o^\ell \in J^\ell \mapsto g \cdot o^\ell := [g \cdot S]_{g(o)} \in J^\ell.
\]

We will denote by \( H^{(\ell)} \) the stabilizer of \( o^\ell \) in \( G \), that is

\[
H^{(\ell)} := G_{o^\ell},
\]

and by

\[
\tau : H^{(k-1)} \mapsto \text{Aff}(J^k_{o^k-1})
\]

the affine action of \( H^{(k-1)} \) on the fibre \( J^k_{o^k-1} \). Then

\[
W^k := \tau(H^{(k-1)}) \cdot o^k \subset J^k_{o^k-1}
\]

defines an affine subspace.

If we assume that there exists a \( o^k \in J^k \), with \( k \geq 2 \), such that

(A1) the orbit \( \tilde{J}^{k-1} := G \cdot o^{k-1} \subset J^{k-1} \) is open,

then

\[
\pi_{k,k-1} : J^k := \pi_{k,k-1}^{-1}(J^{k-1}) \mapsto \tilde{J}^{k-1} = G/H^{(k-1)}
\]

is a homogeneous affine fibre bundle defined by the affine representation \( \tau \) of the stability group \( H^{(k-1)} \) in the fiber \( J^k_{o^k-1} \).

Proposition 2.1 ([3]). Let \( M = G/H \) be a homogeneous manifold and let \( o^k \) be a point satisfying (A1). Any \( \tau(H^{(k-1)}) \)–invariant hypersurface \( \Sigma \subset J^k_{o^k-1} \) extends to a \( G \)–invariant hypersurface \( \mathcal{E} := G \cdot \Sigma \subset \tilde{J}^k \), which is a \( G \)–invariant PDE of order \( k \).

The assumption (A2) below reduces the classification of the aforementioned invariant PDEs to the description of hypersurfaces, that are invariant under a linear group. We denote by \( L_{H^{(k-1)}} = H^{(k-1)}_o \) the subgroup of the affine group \( \tau(H^{(k-1)}) \) stabilizing \( o^k \), which can be considered as the linear part of the affine group \( \tau(H^{(k-1)}) \).

(A2) The affine group \( \tau(H^{(k-1)}) \) is the semidirect product \( \tau(H^{(k-1)}) = L_{H^{(k-1)}} \rtimes T_{W^k} \), where \( T_{W^k} \) is the group of translations or, equivalently, the orbit \( W^k \) defined by (15) is a vector space.

A homogeneous manifold \( M = G/H \), such that both assumptions (A1) and (A2) are satisfied has been referred to as a \( k \)–admissible manifold in [1]: to simplify the exposition of the present paper, we will use the stronger assumption of the existence of a fiducial surface in \( M \).

Definition 2.1. A surface \( S \subset M \) is called a fiducial surface (of order \( k \) for the group \( G \) at the origin \( o \)) if

1. \( o \in S \),
2. \( S \) is homogeneous with respect to a subgroup of \( G \),
3. \( o^{k-1} = [S]_o^{k-1} \) satisfies (A1),
4. \( o^k = [S]_o^k \) satisfies (A2), that is the affine subspace \( W^k = \tau(H^{(k-1)}) \cdot o^k \) given by (15) is a linear subspace.

We can now state one of the main result of [3], duly recast in the context of systems of PDEs. Recall that \( L_{H^{(k-1)}} \) denotes the linear part of \( \tau(H^{(k-1)}) \).

Theorem 2.1. There is a natural 1–1 correspondence between \( L_{H^{(k-1)}} \)–invariant \( m \)-codimensional submanifolds \( \Sigma \) of \( J^k_{o^k-1}/W^k \) and \( G \)–invariant systems of \( m \) PDEs \( \mathcal{E}_\Sigma := G \cdot \Sigma \) of \( J^k \), where \( \Sigma \subset J^k_{o^k-1} \) is the pre-image of \( \Sigma \) (see Figure 1).

3. AFFINE INVARIANT THIRD–ORDER PDEs

Let \((x, y, u)\) be local coordinates on \( \mathbb{R}^3 \). Then, according to Section 1.2, they define local coordinates on any \( k \)-jet space:

\[
J^k := \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{yyy}, u_{xxx}, \ldots)\}.
\]
3.1. Lie symmetry method for obtaining the Aff(3)–invariant PDE. The Lie algebra \( \mathfrak{aff}(3) \) of the vector fields on \( J^3 \) associated with the action of the affine group Aff(3) is spanned by the prolongations \( X_i^{(3)} \) to \( J^3 \) of \( X_i \), where \( X_i \) is one of the twelve generators
\[
\begin{align*}
\partial_x, & \quad \partial_y, \quad \partial_u, \quad x\partial_x, \quad x\partial_y, \quad x\partial_u, \quad y\partial_x, \quad y\partial_y, \quad u\partial_x, \quad u\partial_y, \quad u\partial_u,
\end{align*}
\]
with \( i = 1, \ldots, 12 \), cf. (14). The assignment
\[
J^3 \ni \alpha \mapsto D_\alpha := \text{Span}\{X_1^{(3)}|_\alpha, X_2^{(3)}|_\alpha, \ldots, X_{12}^{(3)}|_\alpha\} \subset T_\alpha J^3
\]
defines a distribution on \( J^3 \), which is of dimension \( 12 = \dim J^3 \) everywhere, except for a closed subset \( \mathcal{E} \subset J^3 \). Since \( \mathcal{E} \) is Aff(3)–invariant, it leads to a Aff(3)–invariant (system of) PDEs, as clarified in the next Proposition (see [16] for more details).

Proposition 3.1. The subset \( \mathcal{E} \subset J^3 \) described by
\[
\mathcal{E} : (u_{xx}u_{yy}^2 - u_{xy}^2) \cdot F = 0,
\]
where \( F \) is
\[
F = 6u_{xx}u_{xx}u_{xy}u_{yy}^2 - 6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 18u_{xx}u_{xy}^2u_{xy}u_{yy} + 12u_{xx}u_{xxy}u_{xyy}^2 + 6u_{xx}u_{xyy}^2u_{yy}^2 - 9u_{xx}u_{xyy}u_{yy}^2 - 3u_{xx}u_{yy}^3 - 6u_{xx}u_{xyy}u_{xyy}^2 + 9u_{xx}u_{xyy}u_{yy}^2 + u_{xx}u_{yy}^3 - 6u_{xx}u_{xyy}u_{xyy}^2 - 2u_{xy}u_{xyy} = 0.
\]
is Aff(3)–invariant.

As we will see later, the aforementioned subset \( \mathcal{E} \) encompasses all possible Aff(3)–invariant (systems of) PDEs.

Remark 3.1. A solution of \( u_{xx}u_{yy} - u_{xy}^2 = 0 \) is also a solution of \( F = 0 \), where \( F \) is given by (19). This can be seen by considering the differential consequences of \( u_{xx}u_{yy} - u_{xy}^2 = 0 \), i.e., \( u_{xxx}u_{yy} + u_{xx}u_{xyy} - 2u_{xy}u_{xyy} = 0 \) and \( u_{xx}u_{xyy} + u_{xxx}u_{yy} = 0 \).

Proposition 3.2. Let \( u_{xx}u_{yy} - u_{xy}^2 = 0 \) and let \( F \) be given by (19). Then \( F = 0 \) if and only if the Fubini–Pick invariant (4) vanishes.

Proof. It is a straightforward computation. □

3.2. Codimension and smoothness of (systems of) Aff(3)–invariant PDEs. An immediate consequence of (18) is that \( \mathcal{E} \) is the union of the Monge–Ampère equation \( u_{xx}u_{yy} - u_{xy}^2 = 0 \) and of the third–order PDE described by \( F = 0 \), where \( F \) is given by (19). It is now convenient to define
\[
J^k_+ := \{ \text{points of (16) such that } \text{det}(\text{Hess}(u)) > 0 \}, \quad J^k_- := \{ \text{points of (16) such that } \text{det}(\text{Hess}(u)) < 0 \},
\]
with \( k \geq 2 \): indeed, in order to study the Aff(3)–invariant PDE \( \mathcal{E} \) given by (18), we can focus on
\[
\mathcal{E}_+ := \mathcal{E} \cap J^k_+, \quad \mathcal{E}_- := \mathcal{E} \cap J^k_-. \quad \mathcal{E}_+ := \mathcal{E} \cap J^k_+.
\]
Proposition 3.3. \( \mathcal{E}_+ \) is a smooth submanifold of \( J^3 \) of codimension 2, whereas \( \mathcal{E}_- \) is the union
\[
\mathcal{E}_- = \mathcal{E}_1 \cup \mathcal{E}_2^2
\]
of two smooth hypersurfaces of \( J^3 \).
Proof. It can be proved by direct computations: by assuming $u_{yy} > 0$, the function $F$ defined by (19) can be brought to the form

\begin{equation}
(3u_{xxx}u_{xyy}u_{yyy} - 3u_{xx}u_{xyy}u_{yy}^2 - 3u_{xxx}u_{yy}^2u_{yy} - 3u_{xx}u_{xyy}u_{yy}^2 - 4u_{xy}^3u_{yy} + 6u_{xy}^2u_{xyy}u_{yy})
\end{equation}

that is the sum (resp., difference) of two squares in the case when det$(\text{Hess}(u))$ is positive (resp., negative), i.e.,

\begin{equation}
\mathcal{E}_a^1 \cup \mathcal{E}_a^2 : \left( \sqrt{-\det(\text{Hess}(u))} \right) (u_{xxx}u_{xyy}u_{yyy} - 3u_{xx}u_{xyy}u_{yy}^2 - 4u_{xy}^2u_{xyy} + 6u_{xy}u_{xyy}u_{yy})
\end{equation}

Remark 3.2. The intersections $\mathcal{E}_a^\alpha = \mathcal{E} \cap J_a^\alpha$ are cut out by the functions $F|_{J_a^\alpha}$ respectively, where $F$ is given by (19). Immediate computations shows that

\begin{equation}
F|_{J_a^2} = (3u_{xy} - u_{yy})^2 + (3u_{yy} - u_{xx})^2,
\end{equation}

\begin{equation}
F|_{J_a^3} = (u_{xxx} - 3u_{xxy} + 3u_{yy} - u_{yyy})(-u_{xxx} - 3u_{xxy} - 3u_{yy} - u_{yyy}).
\end{equation}

Directly from (25) it follows that $\mathcal{E}_a^2$ is the intersection of the two hypersurfaces

\begin{equation}
3u_{xx} - u_{yy} = 0, \quad 3u_{xy} - u_{xx} = 0,
\end{equation}

of $J_a^3$, whereas it follows from (26) that $\mathcal{E}_a^3$ is the union of the two hypersurfaces

\begin{equation}
u_{xxx} - 3u_{xxy} + 3u_{yy} - u_{yyy} = 0, \quad -u_{xxx} - 3u_{xxy} - 3u_{yy} - u_{yyy} = 0.
\end{equation}

Proposition 3.3 can be also proved by extending, respectively, submanifolds (27) and (28) to the whole of $J_a^2$ by means of $\text{Aff}(3)$.

3.3. A tensorial derivation of the (system of) $\text{Aff}(3)$–invariant PDEs.

Proposition 3.4. Let $S = \{ u = f(x, y) \}$ be a surface of $\mathbb{R}^3$, $h^\xi$ as in (2) and $C^\xi$ the cubic form (3).

- If $\det(\text{Hess}(u)) > 0$, then the condition of existence of a 1-form $\alpha \in \Omega^1(S)$ such that

\begin{equation}
C^\xi = \alpha \circ h^\xi,
\end{equation}

is independent of the choice of the transversal field $\xi$.

- If $\det(\text{Hess}(u)) < 0$, then the condition of existence of a 1-form $\alpha \in \Omega^1(S)$ and a symmetric 2-form $\beta$ on $S$ such that

\begin{equation}
C^\xi = \alpha \circ h^\xi + \beta \circ \theta,
\end{equation}

where $\theta \in \Omega^1(S)$ is a non-zero 1–form such that $(h^\xi)^{-1}(\theta, \theta) = 0$, is independent of the choice of the transversal field $\xi$.

Proof. By direct computations. □

Theorem 3.1. Let $S = \{ u = f(x, y) \}$ be a surface of $\mathbb{R}^3$, $h^\xi$ as in (2) and $C^\xi$ the cubic form (3). Then the function $f(x, y)$ is a solution to the equation $\mathcal{E}_+$ (resp., $\mathcal{E}_-$) (cf. (21)) if and only if condition (29) (resp., (30)) of Proposition 3.4 is satisfied.

Proof. To begin with, let us consider the case when condition (29) holds true. Taking into account the local expression of $C^\xi$ and that $h^\xi$ is proportional to the Hessian matrix of $u = f(x, y)$, it is easy to see that condition (29) reads

\begin{equation}
(u_{xxx}dx^3 + 3u_{xxy}dx^2dy + 3u_{xyy}dx^2y^2 + u_{yyy}dy^3) = (\alpha_1dx + \alpha_2dy)(u_{xx}dx^2 + 2u_{xy}dx^2y + u_{yy}dy^2),
\end{equation}
for some functions $\alpha_1, \alpha_2$. A direct computation shows that, after eliminating $\alpha_1$ and $\alpha_2$, condition (31) becomes a system that locally describes $\mathcal{E}_+$:

\begin{align}
\mathcal{E}_+ : \begin{cases}
  u_{xx}^2 u_{yyy} - 3 u_{xx} u_{yy} u_{xy} + 2 u_{xy} u_{yy} u_{xx} = 0, \\
  u_{yy}^2 u_{xxx} - 3 u_{xx} u_{yy} u_{xy} + 2 u_{xy} u_{xx} u_{yy} = 0.
\end{cases}
\end{align}

System (32) coincides with the system coming from (23) in the case when $\det(\text{Hess}(u)) > 0$.

The case when condition (30) holds true can be treated analogously: indeed, such condition reads

\begin{equation}
(u_{xxx} dx^3 + 3 u_{xxy} dx^2 dy + 3 u_{xyy} dx dy^3 + u_{yyy} dy^3) = (\alpha_1 dx + \alpha_2 dy)(u_{xx} dx^2 + 2 u_{xy} dx dy + u_{yy} dy^2) + \\
(\beta_1 dx^2 + 2 \beta_{12} dx dy + \beta_{22} dy^2) \left( u_{xy} \pm \sqrt{-u_{xx} u_{yy} + u_{xy}^2} \right) dx + u_{yy} dy,
\end{equation}

for some functions $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{22}$. A direct computation shows that after eliminating $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{22}$, one obtains two equations (see (24)) locally describing $\mathcal{E}_- = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ corresponds to the PDE obtained by equating to zero the expression given by the first two lines of (24), whereas $\mathcal{E}_2$ corresponds to the PDE obtained by equating to zero the expression given by the last two lines of (24).

Theorem 3.1 gives a geometric interpretation of solutions of equations (29) and (30). Equation (29) means that the cubic form $C^5$ is divided by the metric $h^5$, whereas equation (30) means that the remainder of the division of $C^5$ by $h^5$ is a decomposable cubic form, more precisely a product of an isotropic 1-form and a quadratic form.

Remark 3.3. Theorem 3.1 might also be proved by working in a specific fibre of $\pi_{3,2}$ and then acting by $\text{Aff}(3)$, see Remark 3.2. We will check this only in the case when $\det(\text{Hess}(u)) > 0$, since the case of a negative $\det(\text{Hess}(u))$ is virtually the same: if we fix the point

$$ a^2 = (0, 0, 0, 0, 0, 1, 0, 1) \in J^2_+, $$

then condition (31) becomes

$$ (u_{xxx} dx^3 + 3 u_{xxy} dx^2 dy + 3 u_{xyy} dx dy^3 + u_{yyy} dy^3) = (\alpha_1 dx + \alpha_2 dy)(dx^2 + dy^2), $$

i.e.,

$$ u_{xxx} = \alpha_1, \quad 3 u_{xxy} = \alpha_2, \quad 3 u_{xyy} = \alpha_1, \quad u_{yyy} = \alpha_2. $$

By eliminating $\alpha_1$ and $\alpha_2$, we obtain the system (27).

4. Affine-invariant PDEs as extensions of invariant subsets of the fiber of $J^3$

4.1. Preliminary results needed to apply the main theorem. We focus now on the three-dimensional affine space $\mathbb{R}^3 = \text{Aff}(3) / \text{GL}(3)$, regarded as a homogeneous manifold $J^0 = G/H$, with the origin $a^0 = (0, 0, 0)$. In order to apply the algorithm outlined in Theorem 2.1, we need fiducial hypersurfaces for the affine group $\text{Aff}(3)$, and we need to compute the stabilizer subgroups $H^{(k)}$.

Lemma 4.1. The paraboloid $S_+ \text{ (resp. the hyperboloid } S_-)$

\begin{equation}
S_{\pm} := \{ u = Q_{\pm}(x, y) \},
\end{equation}

where

\begin{equation}
Q_{\pm}(x, y) = \frac{1}{2}(x^2 \pm y^2),
\end{equation}

is a fiducial hypersurface of order 2 for the affine group $\text{Aff}(3)$ at origin $a^0$.

Proof. A straightforward computation shows that both $S_{\pm}$ fulfill all conditions of Definition 2.1, see also [1].

We will denote by

\begin{equation}
a^{(k)} := [S_{\pm}]_{(0, 0)},
\end{equation}

where $S_{\pm}$ is given by (34), the origins associated to the fiducial hypersurfaces $S_{\pm}$.

Lemma 4.2 ([1]). For $k = 0, 1, 2, 3$, the subgroups $H^{(k)}$ of $G = \text{Aff}(3)$ that stabilize the origins $a^{(k)}$, are:

\begin{align}
H^{(0)} &= \text{GL}(3), \\
H^{(1)} &= (\mathbb{R}^2 \times \text{GL}(2)) \times \mathbb{R}^*, \\
H^{(2)} &= (\mathbb{R}^2 \times \text{O}(1, 1)) \times \mathbb{R}^*, \quad a^2_{\pm} \in J^2_+, \\
H^{(3)} &= \text{O}(1, 1) \times \mathbb{R}^*, \quad a^3_{\pm} \in J^3_+.
\end{align}
Remark 4.1. Since \( \text{O}(2) \times \mathbb{R}^* \) is a \( \mathbb{Z}_2 \)-covering of the conformal group \( \text{CO}(2) \), we will employ the notation \( \tilde{\text{CO}}(2) := \text{O}(2) \times \mathbb{R}^* \); an analogous symbol \( \tilde{\text{CO}}(1,1) \) will be used for the Lorentzian signature case.

Now we describe the linear subspace \( W_3^3 \) that is associated with the fiducial hypersurface \( S_3 \) introduced in Lemma 4.1 above, cf. Definition 2.1.

If we start from the fiducial surface \( S_3 \), then the (open) orbit of \( a^3 \) turns out to be \( J^2 \) and the linear space \( W_3^3 \) is the (two-dimensional) linear subspace of the (four-dimensional) space \( S^3 \mathbb{R}^2^* \) made of cubic forms on \( \mathbb{R}^2 \) that are proportional to \( Q_3(x,y) \) via a linear factor:

\[
W_3^3 = \mathbb{R}^2^* \odot \mathbb{R}Q_3.
\]

Taking into account Remark 1.1, Theorem 2.1 claims that the only \( \text{Aff}(3) \)-invariant third-order conditions we can impose on a surface \( S \subset \mathbb{R}^3 \) at a point \( p \in S \) are:

1. (\( + \)) if \( p \) is such that \( [S]^p \) is in the same \( \text{Aff}(3) \)-orbit as the paraboloid \( S_+ \): the condition is that, by taking the difference \( [S]^p - [S_+]^p \) modulo \( W_3^3 \), the result lands in a \( \tilde{\text{CO}}(2) \)-invariant proper subset \( \Sigma \) in \( S^3 \mathbb{R}^2^* / W_3^3 \);
2. (\( - \)) if \( p \) is such that \( [S]^p \) is in the same \( \text{Aff}(3) \)-orbit as the hyperboloid \( S_- \): the condition is that, by taking the difference \( [S]^p - [S_-]^p \) modulo \( W_3^3 \), the result lands in a \( \tilde{\text{CO}}(1,1) \)-invariant proper subset \( \Sigma \) in \( S^3 \mathbb{R}^2^* / W_3^3 \);
3. (\( 0 \)) if \( p \) is not any of the previous ones: the condition coincides with the prolongation to \( J^3 \) of the second-order PDE

\[
\det u_{ij} = 0.
\]

Now we are ready to apply Theorem 2.1 to the case of \( G = \text{Aff}(3) \) acting on \( J^0 = \mathbb{R}^3 \).

Theorem 4.1. Let \( Q_3 \) be the quadratic forms and \( S_3 \) be the fiducial hypersurfaces introduced in Lemma 4.1 above. Let

\[
S_0^3 \mathbb{R}^2^* := \left\{ \frac{S^3 \mathbb{R}^2^*}{\mathbb{R}^2^* \odot \mathbb{R}Q_3} \right\}.
\]

Then, for any \( \tilde{\text{CO}}(2) \)-invariant (resp., \( \tilde{\text{CO}}(1,1) \)-invariant) subset

\[
\Sigma \subset S_0^3 \mathbb{R}^2^*,
\]

the condition that the difference \( [S]^3 \equiv [S_+]^3 \) modulo \( \mathbb{R}^2^* \odot \mathbb{R}Q_3 \) be \( \Sigma \)-valued defines the \( \text{Aff}(3) \)-invariant (system of) third-order PDEs \( \mathcal{E}_\Sigma \subset J^3_3 \).

We recall that the graphs of the functions \( Q_3 \) given by (35) are fiducial hypersurfaces in the sense of Definition 2.1. In particular, they describe two open \( \text{Aff}(3) \)-orbits in \( J^3 \), that we have denoted by \( J^3_+ \) and \( J^3_- \), respectively.

In order to use Theorem 4.1 to produce examples of \( \text{Aff}(3) \)-invariant (systems of) PDEs, we need to study \( \tilde{\text{CO}}(2) \)- or \( \tilde{\text{CO}}(1,1) \)-invariant subset \( \Sigma \subset S_0^3 \mathbb{R}^2^* \): it turns out that such an invariance implies that \( \Sigma \) can only take two forms, as it will be proved below: the aforementioned two possibilities will correspond exactly to the \( \text{Aff}(3) \)-invariant PDEs \( \mathcal{E}_\Sigma \) and \( \mathcal{E}_- \), introduced earlier, see (21).

We warn the reader that, while keep using coordinates \( \{x, y, u\} \) in \( \mathbb{R}^3 \), the same symbols \( \{x, y\} \) will also denote a basis of \( \mathbb{R}^2^* \): accordingly,

\[
S^3 \mathbb{R}^2^* = \left\{ x^3, 3x^2y, 3xy^2, y^3 \right\},
\]

and this will be the standard basis of \( S^3 \mathbb{R}^2^* \). Moreover, since the point \( a^3_\pm \) (see (36)) allows to identify \( J^3_\pm \) with \( S^3 \mathbb{R}^2^* \), the standard coordinates \( \{u_{xxx}, u_{xyy}, u_{yy}, u_{yyy}\} \) of \( J^3_\pm \) turn out to be the dual coordinates to (40).

4.2. The case when the fiducial hypersurface is the paraboloid \( S_+ \). In this section we set \( Q = Q_+ \). It will be convenient to identify the space

\[
S_0^3 \mathbb{R}^2^* = \left\{ \frac{S^3 \mathbb{R}^2^*}{\mathbb{R}^2^* \odot \mathbb{R}Q} \right\} = \left\{ \frac{x^3, 3x^2y, 3xy^2, y^3}{xQ, yQ} \right\} = \left\{ \frac{x^3, 3x^2y, 3xy^2, y^3}{x^3 + xy^2, yx^2 + y^3} \right\}
\]

with its subspace

\[
\left\{ x^3, y^3 \right\}.
\]

Keeping this identification in mind, the same symbol \( S_0^3 \mathbb{R}^2^* \) will be used for both (41) and (42); it is also necessary to introduce a basis

\[
\left\{ \xi, \eta \right\}
\]

of \( (S_0^3 \mathbb{R}^2^*) \) that is dual to \( \{x^3, y^3\} \).

To study the \( \tilde{\text{CO}}(2) \)-invariant subsets \( \Sigma \) of the two-dimensional space \( S_0^3 \mathbb{R}^2^* \), we have to clarify first the \( \tilde{\text{CO}}(2) \)-action on it.
4.2.1. The action of $\mathbb{C}O(2)$ on $(x^3, y^3)$. Since
\[ (44) \quad \mathbb{C}O(2) = O(2) \times \mathbb{R}^2, \]
we can consider separately the action of a rotation, a reflection and a dilation. To begin with, if
\[ R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in SO(2) \]
is a rotation, then
\[ R^*_t \left( x^3 \right) = (R^*_t(x))^3 \]
\[ = (x \cos(t) - y \sin(t))^3 \]
\[ = x^3 \cos^3(t) - 3x^2y \cos^2(t) \sin(t) + 3xy^2 \cos(t) \sin^2(t) - y^3 \sin^3(t) \]
\[ = x^3 \cos^3(t) + 3y^3 \cos^2(t) \sin(t) - 3x^2 \cos(t) \sin^2(t) + y^3 \sin^3(t) \]
\[ = \cos(3t)x^3 + \sin(3t)y^3, \]
\[ R^*_t \left( y^3 \right) = -\sin(3t)x^3 + \cos(3t)y^3. \]
In other words, if we let a $2 \times 2$ matrix act on $S_0^3\mathbb{R}^2$ by identifying the latter with $\mathbb{R}^2$ via the basis $\{x^3, y^3\}$, then $R_t \in SO(2)$ is the rotation $R_{-3t}$; similarly, the reflection $x \rightarrow -x$ corresponds to the reflection $x^3 \rightarrow -x^3$ and the scaling $(x, y) \rightarrow \lambda(x, y)$ will correspond to the scaling $(x^3, y^3) \rightarrow \lambda^3(x^3, y^3)$. In view of (44), we can conclude that the $\mathbb{C}O(2)$–action on $S_0^3\mathbb{R}^2$ can be identified with the standard $\mathbb{C}O(2)$–action on $\mathbb{R}^2$, that possesses only two invariant subsets: $\{0\}$ and $\mathbb{R}^2$ itself.

Since we are not interested in trivial PDEs, the only choice for the subset $\Sigma$ in Theorem 4.1 is $\Sigma = \{0\}$.

4.2.2. The system of PDEs associated with $\Sigma = \{0\}$. According to Theorem 4.1, the (system of) PDEs associated with $\Sigma = \{0\}$ describes surfaces $S$ of $\mathbb{R}^3$ such that $[S]_p^3 - [S]_p^3$ lands in $\Sigma = W^* = \mathbb{R}^2 \otimes \mathbb{R}Q$, that is, a codimension–two linear subspace of $S^3\mathbb{R}^2$. By employing the aforementioned dual coordinates $\{u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}\}$, we see that the subspace $\mathbb{R}^2 \otimes \mathbb{R}Q$ is cut out precisely by the two linear equations
\[ u_{xxx} - 3u_{xyy} = 0, \quad u_{yyy} - 3u_{xx} = 0. \]
Since the system (45) can be recast as a unique equation $(u_{xxx} - 3u_{xyy})^2 + (u_{yyy} - 3u_{xx})^2 = 0$, the fiber $(\mathcal{E}_\Sigma)_{a^3_3} \subset J^3_{a^3_1}$ of the Aff(3)–invariant PDE $\mathcal{E}_\Sigma$ constructed by Theorem 4.1 can be given by
\[ (\mathcal{E}_\Sigma)_{a^3_3} = \{ f = 0 \} \subset J^3_{a^3_1}, \]
where
\[ f(u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}) = (u_{xxx} - 3u_{xyy})^2 + (u_{yyy} - 3u_{xx})^2 \]
is a function on $J^3_{a^3_3}$. The $H(2)$–invariant subset $(\mathcal{E}_\Sigma)_{a^3_1}$ of $J^3_{a^3_1}$ can be extended to an $H(1)$–invariant subset
\[ (\mathcal{E}_\Sigma)_{a^3_1} = H(1) \cdot (\mathcal{E}_\Sigma)_{a^3_3} \]
of $J^3_{a^3_1}$. In fact, the group $H(1)$ in formula (46) can be replaced by its factor $GL(2)$, that acts naturally on $J^2_{a^1_1} = GL(2) \cdot Q$. The latter is the orbit in $S^2\mathbb{R}^2$ of $Q$, i.e., (one half) the Euclidean squared norm, whose associated matrix is $\frac{1}{2} I_2$:
\[ J^2_{a^1_1} = \{ A \cdot \left( \frac{1}{2} I_2 \right) \cdot A^t \mid A \in GL(2) \} = \{ A \cdot A^t \mid A \in GL(2) \}. \]
Therefore, any point $(u_{xxx}, u_{xxy}, u_{yyy}) \in J^3_{a^3_3}$ can be brought to the form $A \cdot A^t$ for some $A \in GL(2) \subset H(2)$. It follows that
\[ (\mathcal{E}_\Sigma)_{(u_{xxx}, u_{xxy}, u_{yyy})} = A \cdot (\mathcal{E}_\Sigma)_{a^3_3} \subset J^3_{(u_{xxx}, u_{xxy}, u_{yyy})} \]
must be cut out by the equation
\[ F(u_{xxx}, u_{xxy}, u_{yyy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}) = A^{-1} \cdot (f) = 0, \]
where $A^{-1}$ is regarded as a diffeomorphism from the fiber $J^3_{(u_{xxx}, u_{xxy}, u_{yyy})}$ to the fiber $J^3_{a^3_1}$, by applying the prolongation formula (13) to the diffeomorphism $A^{-1}$ of $J^0$. Then, the same function $F$, that has no dependency upon $x, y, u, u_x, u_y$, cuts out the whole equation $\mathcal{E}_\Sigma$ in $J^3_{a^3_1}$.

Obtaining $F$ out of $f$ is not complicated: it turns out that
\[ A^{-1} \cdot (u_{xxx}) = \text{det}(A)^{-3}(a_{13}^2 u_{xxx} - 3a_{12} a_{12} u_{xxy} + 3a_{12} a_{11} u_{yyy} + a_{11}^2 (-u_{yyy})), \]
\[ A^{-1} \cdot (u_{xxy}) = \text{det}(A)^{-3}(-a_{13}^2 a_{12} u_{xxx} + (2a_{12} a_{12} + a_{11} a_{22}) u_{xxy} + (a_{13}^2 - 3a_{12} a_{12} a_{11} u_{xyy} + a_{11}^2 a_{12} u_{yyy})), \]
\[ A^{-1} \cdot (u_{xyy}) = \text{det}(A)^{-3}(a_{13}^2 a_{22} u_{xxx} + (-a_{11} - 3a_{12} a_{12} a_{11}) u_{xxy} + (a_{11} a_{22} + 2a_{11} a_{11}) u_{xyy} - a_{12} a_{11} u_{yyy})), \]
\[ A^{-1} \cdot (u_{yyy}) = \text{det}(A)^{-3}(-a_{13} u_{xxx} + 3a_{12} a_{11} u_{xxy} - 3a_{12} a_{11} a_{11} u_{xyy} + a_{11}^2 u_{yyy}), \]
whence
\[ F = u_{xxx}(-6a_{12}(a_{11} + a_{22})(a_{12}^2 + a_{22})^2u_{xyy} + 6(-a_{12}^2 + a_{22}^2a_{11}^2 + 4a_{11}a_{22}a_{12} + a_{11}(a_{12}^2 + a_{22}))(a_{12}^2 + a_{22})u_{xyy} + \\
+ 2a_{12}(a_{11} + a_{22})(3a_{12}^4 - 2a_{22}a_{12}^2 - 8a_{11}a_{22}a_{12}^2 - a_{11}(a_{12}^2 - 3a_{22}))u_{yy} + (a_{12}^2 + a_{22})^2u_{xxx} + \\
+ u_{xxx}(6(a_{11}^2 + a_{12}^2)(-a_{12}^2 + a_{22}^2a_{12} + 4a_{11}a_{22}a_{12} + a_{11}(a_{12}^2 - a_{22}))(a_{12}^2 + a_{22}))u_{yy} - 18a_{12}(a_{11} + a_{22})(a_{11} + a_{22})u_{xyy} + \\
+ 9(a_{12}^2 + a_{22})^2(a_{11}^2 + a_{12}^2)u_{xx} - 6a_{12}(a_{11} + a_{22})(a_{11} + a_{12})u_{xyy}u_{yy} + 9(a_{12}^2 + a_{22})(a_{11} + a_{12})^2u_{xyy} + (a_{11}^2 + a_{12})^3u_{yy}, \]
up to a nonzero factor \( \det(A)^{-6} \). Without loss of generality, the matrix \( A \) has been assumed to be symmetric, meaning that
\[ A \cdot A^t = A^2 = \begin{pmatrix}
a_{11} + a_{12}^2 & a_{11}a_{12} + a_{22}a_{12} \\
a_{11}a_{12} + a_{22}a_{12} & a_{12}^2 + a_{22}^2
\end{pmatrix} = \begin{pmatrix}
u_{xx} & u_{xy} \\
u_{xy} & u_{yy}
\end{pmatrix}. \]

Taking the last relation into account, one finally obtains \( F = 0 \), where \( F \) is given by (19).

4.3. The case when the fiducial hypersurface is the hyperboloid \( S_- \). If we set now \( Q = Q_- \), then the Aff(3)–invariant PDE \( \mathcal{E}_\Sigma \) constructed by Theorem 4.1 will be a subset of \( J^3_s \). In analogy with (41) we identify
\[ S^3_0 \mathbb{R}^2 = \left\{ x^3, 3x^2y, 3xy^2, y^3 \right\}_{x^3 - xy^2, yx^2 - y^3} \]
with the subspace \( \left\{ x^3, y^3 \right\} \), that we keep denoting by the same symbol \( S^3_0 \mathbb{R}^2 \).

4.3.1. The action of \( \mathbb{C}O(1, 1) \) on \( \left\{ x^3, y^3 \right\} \). By employing hyperbolic rotations, the same technique used in Section 4.2.1 allows to show that the action of \( \mathbb{C}O(1, 1) \) on \( S^3_0 \mathbb{R}^2 \) can be identified with the standard one on \( \mathbb{R}^2 \); however, differently from the case treated in Section 4.2, in addition to \( \Sigma = \{0\} \), here we have a codimension–one \( O(1, 1) \)–invariant submanifold, namely the (degenerate) quadric
\[ \Sigma := \{ \xi^2 - \eta^2 = 0 \}, \]
where \( \xi \) and \( \eta \) are defined by (43). As a manifold, \( \Sigma \) is singular, since the quadric \( \xi^2 - \eta^2 = 0 \) is the union of the lines \( \xi = \eta = 0 \) and \( \xi = -\eta = 0 \). Nevertheless, it will be easier to work with the whole quadric \( \xi^2 - \eta^2 = 0 \), rather than with each factor separately.

4.3.2. The PDE associated with \( \Sigma := \{ \xi^2 - \eta^2 = 0 \} \). It suffices to observe that, by the very definition of \( \xi \) and \( \eta \),
\[ \xi = u_{xxx} + 3u_{xxy} \quad \eta = u_{yy} + 3u_{xyy}, \]
whence the \( H^{(2)} \)–invariant hypersurface \( (\mathcal{E}_\Sigma)_\xi \) of \( J^3_s \) associated with \( \Sigma \) reads
\[ f(u_{xxx}, u_{xxy}, u_{xyy}, u_{yy}) = (u_{xxx} + 3u_{xxy})^2 - (u_{yy} + 3u_{xyy})^2 = 0. \]
As before, we use \( \text{GL}(2) \), that acts naturally on \( J^2_s \) \( \text{GL}(2) \cdot Q \), to bring any point \( (u_{xx}, u_{xy}, u_{yy}) \) to the form
\[ A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A^t, \]
where \( A \in \text{GL}(2) \subset H^{(2)} \). The formulas for obtaining \( F \) out of \( f \) are analogous to the ones employed before and we omit them. Surprisingly enough, the resulting PDE \( F = 0 \) will be given exactly by the same function \( F \) as above, that is, (19).

5. Compatibility conditions and solutions in the case of the Aff(3)–invariant system of PDEs

A system of PDEs whose number of equations is strictly greater than the number of unknown functions could be overdetermined, i.e., it could possess a certain number of non–trivial compatibility conditions [21].

For instance, in the case when \( \det(\text{Hess}(u)) > 0 \) we have obtained\(^1\) the system (32) of two PDEs in one unknown function \( u = u(x, y) \); we will see that it is indeed an overdetermined system. Before computing the compatibility conditions of the system (32), let us observe that functions \( u = u(x, y) \), with either \( u_{xx} = 0 \) or \( u_{yy} = 0 \), are solutions to the system, so that it makes sense to assume, from now on, that both \( u_{xx} \) and \( u_{yy} \) be nonzero. Thus, we write system (32) as follows:
\[ \mathcal{E}_+: \begin{cases} u_{xxx} = -u_{xxy}(2u_{xy}u_{yy} - 3u_{xyy}u_{yy}), \\
u_{xxy} = u_{xxx}u_{yy} - 4u_{xy}u_{xx} + 6u_{xyy}u_{xx}u_{yy}. \end{cases} \]

\(^1\)In fact, as we stressed in Section 4.3, the same system has been obtained in the case \( \det(\text{Hess}(u)) < 0 \) as well.
5.1. Compatibility conditions and solutions of the system of PDEs (48). The technique for obtaining the compatibility conditions is standard. First, one takes the total derivatives of both sides of both the equations the system (48) consists of: this allows to express \( u_{xxxx}, u_{xxxy} \) and \( u_{xxyy} \) in terms of \( u_{xyyy}, u_{yyyy}, u_{xyy}, u_{yy} \) and second order derivatives; then one observes that the cross total differentiation \( D_y(u_{xxxx}) - D_x(u_{xxyy}) \), that is
\[
\frac{8}{3} (u_{xy} u_{yy} u_{yyyy} - 2 u_{xy} u_{yy}^2 + 2 u_{xyy} u_{yy} u_{yy} - u_{xyyy} u_{yy}^2 ) (u_{xx} u_{yy} - u_{xyy}^2 )
\]
has to vanish. Since we are assuming \( \det(\text{Hess}(u)) \neq 0 \) and \( u_{yy} \neq 0 \), the vanishing of the above expression yields a fourth–order compatibility condition:
\[
(49) \quad u_{xy} u_{yy} u_{yyyy} - 2 u_{xy} u_{yy}^2 + 2 u_{xyy} u_{yy} u_{yy} - u_{xyyy} u_{yy}^2 = 0.
\]
Since (49) allows to express also \( u_{xyyy} \) in terms of \( u_{yyyy}, u_{xyy}, u_{yy} \) and derivatives of second order, we find out that, in view of what already seen above, all but one fourth–order derivatives, i.e., \( u_{xxxx}, u_{xxxy}, u_{xxyy} \) and \( u_{xyyy} \), can be expressed in terms of the remaining fourth–order derivative, i.e., \( u_{yyyy} \), and of the third–order derivatives \( u_{xyy}, u_{yy} \), and of the derivatives of second order as well. This scheme repeats verbatim for fifth–order compatibility conditions. Indeed, taking into account the relations that we have obtained so far, the fifth–order derivatives \( u_{xxxx}, u_{xxxy}, u_{xxyy}, u_{xyyy}, u_{xyyy} \) can be expressed in terms of \( u_{yyyy}, u_{xyy}, u_{xy}, u_{yy} \), and of derivatives of second order: the vanishing of the cross total differentiation \( D_y(u_{xxxx}) - D_x(u_{xxyy}) \), that is
\[
\frac{1}{9} u_{yy} (u_{xx} u_{yy} - u_{xy}^2) (9 u_{yy}^2 u_{yyyy} - 45 u_{yy} u_{yy} u_{yyyy} + 40 u_{yyyy}^3 ) = 0,
\]
gives then
\[
(50) \quad 9 u_{yy}^2 u_{yyyy} - 45 u_{yy} u_{yy} u_{yyyy} + 40 u_{yyyy}^3 = 0,
\]
since both \( u_{xx} \) and \( u_{yy} \) are nonzero, and we are assuming \( \det(\text{Hess}(u)) \neq 0 \). These results have been obtained also in [7] by using different techniques. Let us stress that (50) is actually a well-known ODE: its solution space is made of the conics in the \((y,u)-\)plane, see, e.g., [19, 21]. In fact the following result holds.

**Proposition 5.1.** The function \( u = u(x,y) \) is a solution to (50) if and only if
\[
a(x)y^2 + b(x)yu + c(x)u^2 + d(x)y + e(x)u + f(x) = 0.
\]

Now we wonder about higher–order compatibility conditions: since equation (50) allows to obtain the fifth–order derivative \( u_{yyyy} \) in terms of those of lower order, in view of all we have obtained so far, it turns out that all fifth–order derivatives can be expressed in terms of the lower–order ones, forming a system of six equations. Such system is not overdetermined because a direct computation shows that it gives no other compatibility conditions.

Since the system (32) is symmetric with respect to the interchanging of \( x \) and \( y \), the consequences (49) and (50) have to still hold true after switching \( x \) with \( y \), i.e., we get the system:
\[
\begin{align*}
& u_{xy} u_{yy} u_{yyyy} - 2 u_{xy} u_{yy}^2 + 2 u_{xyy} u_{yy} u_{yy} - u_{xyyy} u_{yy}^2 = 0, \\
& u_{xy} u_{xx} u_{xxxx} - 2 u_{xy} u_{xx} u_{xyy} + 2 u_{xyy} u_{xx} u_{xyy} - u_{xyyy} u_{xyy}^2 = 0, \\
& 9 u_{yy}^2 u_{yyyy} - 45 u_{yy} u_{yy} u_{yyyy} + 40 u_{yyyy}^3 = 0, \\
& 9 u_{xx}^2 u_{xxxx} - 45 u_{xx} u_{xx} u_{xxxx} + 40 u_{xxxx}^3 = 0.
\end{align*}
\]

Proposition 5.1 says precisely that the solutions to the sub–system of (51) made of the last two equations are those conics in the \((y,u)-\)plane (with coefficients depending upon \( x \)) that are simultaneously conics in the \((x,u)-\)plane (with coefficients depending upon \( y \)): in other words, \( u = u(x,y) \) is a solution to the aforementioned sub–system of (51) if and only if
\[
(52) \quad a u^2 + (h_0 + h_1 x + h_2 y + h_3 xy) u + k_0 + k_1 x + k_2 y + k_3 x^2 + k_4 xy + k_5 y^2 + k_6 x^2 y + k_7 xy^2 + k_8 x^2 y^2 = 0, \quad a, h_i, k_i \in \mathbb{R}.
\]

5.2. A tensorial (re)formulation of the compatibility conditions. In this section we show that the compatibility conditions can be given a tensorial interpretation, much as we did in Section 3.3 above for the system (48) itself: this can be achieved by considering the prolongation to higher–order jet spaces of a system of PDEs, equipped with its compatibility conditions, and the higher–order Hessian. To begin with, we interpret the system (48) as a 10–dimensional submanifold of \( J^3 \): local coordinates on \( \mathcal{S} \) are
\[
x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxy}, u_{yy}.
\]
The fourth–order compatibility conditions (49), together with the original system (48) and the outcomes of its total differentiation, describes an 11–dimensional submanifold of \( J^3 \), which we denote by \( \mathcal{E}^{(I)}_+ \): indeed, similarly as we have showed above, the eleven functions

\[
(53) \quad x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxy}, u_{xyy}, u_{yyy}, u_{yxy}, u_{yxx}, u_{yyxy}, u_{yxyx}, u_{xyyy}, u_{xyxy}, u_{yyyy}, u_{yxyx}, u_{xyyy}, u_{xyxy}
\]

can be taken as local coordinates on \( \mathcal{E}^{(I)}_+ \), since \( u_{xxx}, u_{xxy}, u_{xxyx}, u_{xyy}, u_{xyyy}, u_{yxyx}, u_{xyxy}, u_{yyxy}, u_{yxyxy}, u_{xyxyx}, u_{yxyyy}, u_{xyxyy} \) and \( u_{xyxyx}, u_{yxyy}, u_{xyxyx}, u_{yxyxy} \) can be expressed in terms of (53). This very reasoning can be repeated at the fifth order: we consider the system made of the equation (50), the system describing the submanifold \( \mathcal{E}^{(I)}_+ \), together with the outcomes of its total differentiation, and interpret this new system as a submanifold of \( J^5 \), which will be accordingly denoted by \( \mathcal{E}^{(II)}_+ \). Also the submanifold \( \mathcal{E}^{(II)}_+ \) turns out to be 11–dimensional since, as we have seen above, all fifth–order derivatives can be expressed in terms of lower–order ones and then we can take again (53) as local coordinates on \( \mathcal{E}^{(II)}_+ \). The following result, that can be obtained by direct computations, generalizes the fact that system (48) can be obtained by requiring that \( u_{xxx}dx^3 + 3u_{xxy}dx^2dy + 3u_{xyy}dxdy^2 + u_{yyyy}dy^3 \) be proportional to \( u_{xxx}dx^2 + 2u_{xxy}dxy + u_{yyyy}dy^2 \), see (31).

**Proposition 5.2.** The following tensorial relations

\[
\begin{align*}
&u_{xxx}dx^3 + 3u_{xxy}dx^2dy + 3u_{xyy}dxdy^2 + u_{yyyy}dy^3 \propto u_{xxx}dx^2 + 2u_{xxy}dxy + u_{yyyy}dy^2, \\
&u_{xxxx}dx^4 + 4u_{xxyy}dx^3dy + 6u_{xxyy}dxdy^2 + u_{xx}dxdy + u_{yy}dy^2, \\
&u_{xxxxx}dx^5 + 5u_{xxyyy}dx^4dy + 10u_{xxyxy}dx^3dy^2 + \cdots + u_{yyyy}dy^5 \propto u_{xxx}dx^2 + 2u_{xxy}dxy + u_{yyyy}dy^2,
\end{align*}
\]

hold true, once restricted, respectively, to \( \mathcal{E}_+ \), \( \mathcal{E}^{(I)}_+ \) and \( \mathcal{E}^{(II)}_+ \).

6. **The Geometry of Characteristics Lines in the Case of Aff(3)–Invariant Scalar PDEs**

A particularly simple and geometrically well–behaved class of scalar PDEs in two independent variables is made of those whose characteristic conic distribution degenerates to a vectorial one, see [2, 5, 8, 9, 17]: such PDEs are called also of Goursat type. In this section we will show that the 3rd–order scalar PDEs given by (24) are precisely of this kind: in particular, we shall see that their symbol has rank one and that they are completely determined by a 3–dimensional vector sub–distribution of the 2nd–order contact distribution \( \mathcal{C}^2 \) on \( J^2 \) (cf. (10)), called the characteristic distribution.

For the sake of simplicity, here we will deal only with one of the two factors of (24), for instance, with

\[
(54) \quad \mathcal{E}^1 : \sqrt{\det(\Hess(u))}\left(u_{xx}u_{yy}u_{yyy} - 3u_{xxy}u_{yy}^2 - 4u_{xxy}u_{yy} + 6u_{xxy}u_{yy}ight)
\]

\[
+ \left( -3u_{xx}u_{xy}u_{yy}u_{yy} + 3u_{xx}u_{xx}u_{xy}^2 - u_{xxx}u_{yy}^2 + 3u_{xxy}u_{xy}u_{yy}^2 + 4u_{xxy}u_{xy}u_{yy} - 6u_{xy}u_{xy}u_{yy} \right) = 0,
\]

since the whole machinery applies as well to the other factor; we stress also that we work all the time over \( J^2_+ \). The (3–dimensional) characteristic distribution \( \mathcal{V} \) of (54) will be studied below.

6.1. **Characteristic lines of a 3rd–order scalar PDE.** The construction of characteristic lines of a 3rd order Monge–Ampère equation in two independent variables of Goursat type is explained in [17, Section 5], and it goes as follows.

A scalar 3rd order PDE in one unknown function \( u = u(x^1, x^2) = u(x, y) \) and 2 independent variables \( (x^1, x^2) = (x, y) \) is locally described by

\[
\mathcal{E} = \{ F(x^1, u, u_i, u_{ij}, u_{ijk}) = 0 \}.
\]

We can assume that (55) be a hypersurface of \( J^3 \). The departing point of our construction is the symmetric 3–form

\[
(56) \quad \sum_{i,j,k} \frac{\partial F}{\partial u_{ijk}} \eta_i \eta_j \eta_k,
\]

that can be associated with the function \( F \), called the symbol of \( F \) (see, e.g., [10, 11, 22]), and the symmetric 3–tensor

\[
(57) \quad \sum_{i,j,k} \frac{\partial F}{\partial u_{ijk}} D^{(3)}_{x^i} D^{(3)}_{x^j} D^{(3)}_{x^k},
\]

belonging to \( \mathcal{T}^3 \otimes \mathcal{T}^3 \otimes \mathcal{T}^3 \), where \( \mathcal{T}^3 \) is the tautological bundle over \( J^3 \) (see Definition 1.1) and \( D^{(3)}_{x^i} \) denotes the total derivative operator with respect to \( x^i \), truncated to the third order.

The symmetric 3–form (56) (as well as (57)), once restricted to the PDE \( \mathcal{E} \) given by (55), gives the so–called symbol of \( \mathcal{E} \). Even if the symbol of \( \mathcal{E} \) is defined up to a non–vanishing factor, we will be using only its properties that do not depend on this factor.

Let us assume that the symbol of \( \mathcal{E} \) be a perfect cube, i.e., the cubic power of a linear factor: then there must exist functions \( f^1 \) on \( \mathcal{E} \), such that

\[
(58) \quad \left( \sum_{i,j,k} \frac{\partial F}{\partial u_{ijk}} \eta_i \eta_j \eta_k \right)_{\mathcal{E}} = (f^1 \eta_i)^3.
\]
Since the symbol of $\mathcal{E}$ satisfies (58), it is called a rank-one symbol: from a tensorial viewpoint, the restriction of (57) to $\mathcal{E}$ breaks into the 3rd power of a single 1-tensor:

\[
\left( \sum_{i,j,k} \frac{\partial F}{\partial u_{ij}} D^{(3)}_x D^{(3)}_y D^{(3)}_z \right)_{\mathcal{E}} = \left( f^i D^{(3)}_x \right)^3_{\mathcal{E}}.
\]

This allows to associate, with each point $a^3 \in \mathcal{E}_{a^2} = \mathcal{E} \cap J^3_{a^2}$ (cf. (12)), the tangent line at $a^2$

\[
l_{a^3} = \left( f'(a^3) D^{(3)}_x \right)_{a^3},
\]

called a characteristic line of $\mathcal{E}$ at $a^3$.

6.2. The 3-dimensional distribution associated with the Aff(3)–invariant PDE with $\det(\text{Hess}(u)) < 0$. In this section, to keep the notation light, we will denote a point of $J^k$ simply by $a^k$.

Since the symbol of the PDE (54) turns out to be of rank one, being proportional to

\[
\left( \sqrt{-\det(\text{Hess}(u))} u_{yy} \eta_1 \right)^3,
\]

it makes sense to consider the line (59) for each point $a^3 = (x, y, u, u_i, u_{ij}, u_{ijk})$ that satisfies (54):

\[
l_{a^3} = \left( \sqrt{-\det(\text{Hess}(u))} u_{yy} D^{(3)}_x - \left( \sqrt{-\det(\text{Hess}(u))} u_{xy} + u_{xx} u_{yy} - u_{yy}^2 \right) D^{(3)}_y \right),
\]

where

\[
D^{(3)}_x = \partial_x + u_{xx} \partial_u + u_{xx} \partial_{u_x} + u_{x} \partial_{u_y} + u_{xx} \partial_{u_{xy}} + u_{xx} \partial_{u_{yy}} + u_{yy} \partial_{u_{yy}},
\]

\[
D^{(3)}_y = \partial_y + u_{yy} \partial_u + u_{yy} \partial_{u_x} + u_{y} \partial_{u_y} + u_{yy} \partial_{u_{xy}} + u_{yy} \partial_{u_{yy}},
\]

and $u_{xx}$ is obtained from (54).

Let us recall that $C^2$ denotes the 2nd order contact distribution on $J^2$, cf. (10). We are then in position of defining a conic subset $\mathcal{V}_{a^2}$ of $C^2_{a^2}$ at $a^2 = (x, y, u, u_i, u_{ij}) \in J^2$: for each point $a^3 = (x, y, u, u_i, u_{ij}, u_{ijk})$ of the fiber $(\mathcal{E}^1)_{a^2}$ of the PDE (54) over $a^2 = (x, y, u, u_i, u_{ij})$, we consider the line $l_{a^3}$ given by (60) and we let

\[
\mathcal{V}_{a^2} = \bigcup_{a^3 \in (\mathcal{E}^1)_{a^2}} l_{a^3},
\]

i.e., as the point $a^3$ describes $(\mathcal{E}^1)_{a^2}$, the corresponding line $l_{a^3}$ sweeps the conic subset $\mathcal{V}_{a^2}$.

Proposition 6.1. The distribution of conic subsets of $C^2$

\[
J^2 \ni a^2 \rightarrow \mathcal{V}_{a^2} \subset C^2_{a^2}
\]

defined by (61) turns out to be a vector distribution locally given by

\[
\mathcal{V} = \left\{ -u_{yy} D^{(3)}_x + b D^{(2)}_y, 2b \partial_{u_{xx}}, y^2 u_{yy} \right\},
\]

where

\[
b = u_{xy} - \sqrt{-\det(\text{Hess}(u))}.
\]

Proof. A direct (and tedious) computation shows that $\mathcal{V}_{a^2}$ is locally described by

\[
\mathcal{V}_{a^2} : \left\{ \begin{array}{l}
y^1 u_{xy} \sqrt{-\det(\text{Hess}(u))} + y^2 u_{yy} \sqrt{-\det(\text{Hess}(u))} + y^1 u_{xx} u_{yy} - y^1 u_{xy}^2 = 0, \\
2p_{11} u_{yy} \sqrt{-\det(\text{Hess}(u))} - 2\sqrt{-\det(\text{Hess}(u))} p_{22} u_{xy} - 2p_{12} u_{xx} u_{yy} - 2u_{xx} u_{yy}^2 + 2p_{22} u_{xy}^2 = 0,
\end{array} \right.
\]

where $y^1, y^2, p_{11}, p_{12}, p_{22}$ are local coordinates on $C^2_{a^2}$ with respect to the basis $D^{(2)}_{a^2}, D^{(2)}_{a^2}, \partial_{u_{xx}} | a^2, \partial_{u_{xy}} | a^2, \partial_{u_{yy}} | a^2$ of $C^2_{a^2}$. In other words, for each point $a^2 = (x, y, u, u_i, u_{ij}) \in J^2$, the subset $\mathcal{V}_{a^2}$ of $C^2_{a^2}$ is described by the system (64) of two (independent) linear equations in $y^1, y^2, p_{11}, p_{12}, p_{22}$: this means that the correspondence (62) defines a 3-dimensional linear distribution on $J^2$ inscribed in the 2nd order contact distribution $C^2$. The space of solutions of the system (64) is precisely (63).

For the sake of paper’s self-consistency, we explain how PDE (54) can be actually recovered out of $\mathcal{V}$. Recalling that the PDE (54) can be interpreted as a hypersurface of $J^3$, one can use $\mathcal{V}$ to define the subset

\[
\mathcal{E}_\mathcal{V} := \left\{ a^3 \in J^3 | \mathcal{V}_{a^2} \cap \mathcal{V}_{a^3} \neq 0, a^2 \in J^2, \pi_{3,2}(a^3) = a^2 \right\}
\]
of $J^3$, where $T^3$ is the tautological bundle on $J^3$, see Definition 1.1. Since $T^3_a$ and $V_a$ are subspaces of dimension 2 and 3, respectively, of the 5-dimensional vector space $C^2_{a^2}$, their intersection is generically 0-dimensional; the condition that such intersection be non-trivial reads
\[
\det \begin{pmatrix}
1 & 0 & u_{xxx} & u_{xxy} & u_{xyy} \\
0 & 1 & u_{xxx} & u_{xxy} & u_{yy} \\
-u_{yy} & b & 0 & 0 & 0 \\
0 & 0 & 2b & u_{yy} & 0 \\
0 & 0 & -b^2 & 0 & u_{yy}^2
\end{pmatrix} = 0,
\]
which gives exactly the equation (54). In other words, we have proved that $E^1 = E_V$, i.e., $E^1$ can be recovered out of $V$.

Another approach is based on the Aff(3)-invariance of the equation (54). If we fix the point
\[
a^2 = (0, 0, 0, 0, -1, 0, 1) \in J^2,
\]
then the equation (54) becomes (cf. (28) and (47))
\[
(66) \quad u_{xxx} + 3u_{xxy} + 3u_{yy} = 0,
\]
whose symbol is simply $(\eta_1 + \eta_2)^3$, whence the characteristic line is $D_x^{(3)} + D_y^{(3)}$, with $u_{xxx} = -3u_{xxy} + 3u_{yy}$. In the expression of $D_x^{(3)}$, cf. (28) and (47). This immediately leads to the 3-dimensional linear subspace
\[
(67) \quad V := \{ D_x^{(2)}|_{a^2} + D_y^{(2)}|_{a^2}, -2\partial u_{xxx}|_{a^2}, -\partial u_{xxy}|_{a^2}, -\partial u_{xxy}|_{a^2}, -\partial u_{yy}|_{a^2} \}
\]
of $C^2_{a^2}$, and it is easy to see that (68) gives precisely the evaluation at the point $a^2$ of (63).

If we now compute the fiber
\[
(\mathcal{E}_V)_{a^2} = \{ a^3 \in J^2 | T^3_a \cap V \neq \emptyset \}
\]
of the equation (65) at the point (66), where $V$ is given by (68), we find out that such a fiber is given by (67). In other words, the equation (54) and the Goursat–Monge–Ampère equation (65), that are both Aff(3)-invariant, have the same fiber over the point $a^2 \in J^2$: this means that they must coincide.

6.3. The symbol of (19) as a feature of the Blaschke metric. Now we go back to the whole equation $E$, that is the zero locus in $J^3$ of the function $F$ given by (19): by direct computations, that we will omit, it is possible to establish the equation of $F$ directly from the Blaschke metric and the Fubini–Pick cubic form (see Section 1.1).

**Proposition 6.2.** Let $h = h_{ij}$ be the Blaschke metric and $C = C_{abc}$ the Fubini–Pick cubic form: then
\[
(69) \quad 8(|\det(\text{Hess}(u))|^2 h_{ai} h_{bj} h_{ck} C_{abc} \eta_i \eta_j \eta_k = \text{sgn}(\det(\text{Hess}(u))) \sum_{t \leq j < k} \frac{\partial F}{\partial u_{ijk}} \eta_i \eta_j \eta_k,
\]
where $F$ is given by (19).

It should be stressed that the cubic form at the right-hand side of (69) is the symbol of the function $F$ and not of the equation $E = \{ F = 0 \}$ since, as we have seen above, for $\det(\text{Hess}(u)) > 0$, the equation is actually a system of two equations. Nevertheless, for $\det(\text{Hess}(u)) < 0$, we have that $F = f^1 f^2$, where $f^1$ are the two functions given by the left-hand side of (24) and it turns out that
\[
\sum_{i \leq j < k} \frac{\partial F}{\partial u_{ijk}} \eta_i \eta_j \eta_k = \sum_{i \leq j < k} (f^1 f^2 \frac{\partial f^1}{\partial u_{ijk}} \eta_i \eta_j \eta_k + f^1 \frac{\partial f^2}{\partial u_{ijk}} \eta_i \eta_j \eta_k).
\]

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The summation at the left–hand side of (69) is over all indices.
