Finite Energy Electroweak Dyon

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ABSTRACT: We present finite energy analytic monopole and dyon solutions whose size is fixed by the electroweak scale. We discuss two types of solutions. The first type is obtained by regularizing the recent solutions of Cho and Maison by modifying the coupling strength of the quartic self-interaction of $W$-boson in Weinberg-Salam model. The second is obtained by enlarging the gauge group $SU(2) \times U(1)$ to $SU(2) \times SU(2)$. Our result demonstrates that one could actually construct genuine electroweak monopole and dyon whose mass scale is much smaller than the grand unification scale, with a minor modification of the electroweak interaction without compromising the underlying gauge invariance.
I. Introduction

Ever since Dirac [1] has introduced the concept of the magnetic monopole, the monopoles have remained a fascinating subject in theoretical physics. The Abelian monopole has been generalized to the non-Abelian monopoles by Wu and Yang [2, 3] who showed that the pure $SU(2)$ gauge theory allows a point-like monopole, and by ’t Hooft and Polyakov [4, 5] who have constructed a finite energy monopole solution in Georgi-Glashow model as a topological soliton. In the interesting case of the electroweak theory of Weinberg and Salam, however, it has generally been asserted that there exists no topological monopole of physical interest [6]. The basis for this “non-existence theorem” is, of course, that with the spontaneous symmetry breaking the quotient space $SU(2) \times U(1)/U(1)_{em}$ allows no non-trivial second homotopy. This has led many people to conclude that there is no topological structure in Weinberg-Salam model which can accommodate a spherically symmetric magnetic monopole.

This, however, has been shown to be not true. Indeed some time ago Cho and Maison [7] have established that Weinberg-Salam model and Georgi-Glashow model have exactly the same topological structure, and demonstrated the existence of a new type of monopole and dyon solutions in the standard Weinberg-Salam model. This was based on the observation that the Weinberg-Salam model, with the hypercharge $U(1)$, could be viewed as a gauged $CP^1$ model in which the (normalized) Higgs doublet plays the role of the $CP^1$ field. So the Weinberg-Salam model does have exactly the same nontrivial second homotopy as Georgi-Glashow model which allows topological monopoles. This would have been impossible without the hypercharge $U(1)$. Once this is understood, one could proceed to construct the desired monopole and dyon solutions in Weinberg-Salam model. Originally the solutions of Cho and Maison were obtained by a numerical integration. But a mathematically rigorous existence proof has since been established which endorses the numerical results, and the solutions are now referred to as Cho-Maison monopole and dyon [8, 9].

It should be emphasized that the Cho-Maison monopole is completely different from the “electroweak monopole” derived from the Nambu’s electroweak string. In his continued search for the string-like objects in physics, Nambu has demonstrated the existence of a rotating dumb bell made of the monopole anti-monopole pair connected by the neutral string of $Z$-field flux in Weinberg-Salam model [10]. Taking advantage of the Nambu’s pioneering work, others have claimed to discover an electroweak monopole, simply by making the neutral string infinitely long and removing the anti-monopole attached to the other end to infinity [11]. This type of “electroweak monopole”, however, must carry a fractional magnetic charge which can not be isolated, and obviously has no spherical symmetry which is manifest in the Cho-Maison monopole [7].

The Cho-Maison monopole may be viewed as a hybrid between the Abelian monopole and the ’t Hooft-Polyakov monopole, because it has a $U(1)$ point singular-
ity at the center even though the $SU(2)$ part is completely regular. Consequently it carries an infinite energy at the classical level, which means that physically the mass of the monopole remains arbitrary. A priori there is nothing wrong with this, but nevertheless one may wonder whether one can have an analytic electroweak monopole which has a finite energy. This has been shown to be possible [12, 13]. The purpose of this paper is to discuss the finite energy electroweak monopole and dyon solutions in detail.

The paper is organized as follows. In Section II we review the Cho-Maison monopole and dyon in Weinberg-Salam model, and discuss the difference between the Cho-Maison monopole and the “monopole” attached to Nambu’s electroweak string. In Section III we compare the Cho-Maison dyon with the Julia-Zee dyon to clarify the similarities between the two dyons. In doing so, we present the gauge-independent Abelian formalism of the Weinberg-Salam model. In Section IV we show how a minor modification of the electroweak interaction allows us, without compromising the gauge invariance, to construct the finite energy electroweak monopoles and dyons. Utilizing the gauge-independent Abelian formalism, we construct two types of solutions. The first is obtained by modifying the coupling strength of the quartic self-interaction of $W$-boson. The second is obtained by enlarging the gauge group to $SU(2) \times SU(2)$. In both cases the gauge-independent Abelian formalism plays a crucial role to guarantee the gauge invariance of the modified electroweak interactions. Finally in Section V we discuss the physical implications of the finite energy electroweak monopoles.

II. Cho-Maison Dyon in Weinberg-Salam Model

Before we construct the finite energy monopole and dyon solutions in electroweak theory we must understand how one could obtain the infinite energy solutions first. So we will briefly review the Cho-Maison solutions in Weinberg-Salam model. Let us start with the Lagrangian which describes (the bosonic sector of) the standard Weinberg-Salam model

\[
\mathcal{L} = -|D_\mu \phi|^2 - \frac{\lambda}{2}(\phi^\dagger \phi - \frac{\mu^2}{\lambda})^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}G_{\mu\nu}^2,
\]

\[
D_\mu \phi = (\partial_\mu - ig\vec{\tau} \cdot \vec{A}_\mu - ig' \vec{B}_\mu)\phi = (D_\mu - ig' \vec{B}_\mu)\phi,
\]

where $\phi$ is the Higgs doublet, $F_{\mu\nu}$ and $G_{\mu\nu}$ are the gauge field strengths of $SU(2)$ and $U(1)$ with the potentials $\vec{A}_\mu$ and $B_\mu$, and $g$ and $g'$ are the corresponding coupling constants. Notice that $D_\mu$ describes the covariant derivative of the $SU(2)$ subgroup only. From (2.1) one has the following equations of motion

\[
D^2 \phi = \lambda(\phi^\dagger \phi - \frac{\mu^2}{\lambda})\phi,
\]
\[ D_\mu \tilde{F}_{\mu\nu} = -j_\nu = i \frac{g}{2} \left[ \phi^\dagger \bar{\tau} (D_\nu \phi) - (D_\nu \phi)^\dagger \bar{\tau} \phi \right], \quad (2.2) \]

\[ \partial_\mu G_{\mu\nu} = -k_\nu = i \frac{g'}{2} \left[ \phi^\dagger (D_\nu \phi) - (D_\nu \phi)^\dagger \phi \right]. \]

Now we choose the following static spherically symmetric ansatz
\[ \phi = \frac{1}{\sqrt{2}} \rho(r) \xi(\theta, \varphi), \]
\[ \xi = i \begin{pmatrix} \sin(\theta/2) e^{-i\varphi} \\ -\cos(\theta/2) \end{pmatrix}, \quad \hat{n} = -\xi^\dagger \bar{\tau} \xi = \hat{r}, \]
\[ \vec{A}_\mu = \frac{1}{g} A(r) \partial_\mu \hat{t} \hat{n} + \frac{1}{g} (f(r) - 1) \hat{n} \times \partial_\mu \hat{n}, \quad (2.3) \]
\[ B_\mu = \frac{1}{g'} B(r) \partial_\mu \hat{t} + \frac{1}{g'} (1 - \cos \theta) \partial_\mu \varphi, \]

where \((t, r, \theta, \varphi)\) are the spherical coordinates. Notice that the apparent string singularity along the negative z-axis in \(\xi\) and \(B_\mu\) is a pure gauge artifact which can easily be removed with a hypercharge \(U(1)\) gauge transformation. Indeed one can easily exoclate the strings by making the hypercharge \(U(1)\) bundle non-trivial \([2]\). So the above ansatz describes a most general spherically symmetric ansatz of a \(SU(2) \times U(1)\) dyon. Here we emphasize the importance of the non-trivial \(U(1)\) degrees of freedom to make the ansatz spherically symmetric. Without the extra \(U(1)\) the Higgs doublet does not allow a spherically symmetric ansatz. This is because the spherical symmetry for the gauge field involves the embedding of the radial isotropy group \(SO(2)\) into the gauge group that requires the Higgs field to be invariant under the \(U(1)\) subgroup of \(SU(2)\). This is possible with a Higgs triplet, but not with a Higgs doublet \([14]\). In fact, in the absence of the hypercharge \(U(1)\) degrees of freedom, the above ansatz describes the \(SU(2)\) sphaleron which is not spherically symmetric \([15]\). To see this, one might try to remove the string in \(\xi\) with the \(U(1)\) subgroup of \(SU(2)\). But this \(U(1)\) will necessarily change \(\hat{n}\) and thus violate the spherical symmetry. This means that there is no \(SU(2)\) gauge transformation which can remove the string in \(\xi\) and at the same time keeps the spherical symmetry intact. The situation changes with the inclusion of the extra hypercharge \(U(1)\) in the standard model, which naturally makes \(\xi\) a \(CP^1\) field \([7]\). This allows the spherical symmetry for the Higgs doublet.

To understand the physical content of the ansatz we now perform the following gauge transformation on \((2.3)\)
\[ \xi \rightarrow U \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad U = i \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \\ -\sin(\theta/2)e^{i\varphi} & \cos(\theta/2) \end{pmatrix}, \quad (2.4) \]
and find that in this unitary gauge we have

\[
\hat{n} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{A}_{\mu} \rightarrow \frac{1}{g} \begin{pmatrix} -f(r)(\sin \varphi \partial_\mu \theta + \sin \theta \cos \varphi \partial_\mu \varphi) \\ f(r)(\cos \varphi \partial_\mu \theta - \sin \theta \sin \varphi \partial_\mu \varphi) \\ A(r) \partial_\mu t + (1 - \cos \theta) \partial_\mu \varphi \end{pmatrix},
\]

(2.5)

So introducing the electromagnetic potential \( A_{\mu}^{(em)} \) and the neutral \( Z \)-boson potential \( Z_{\mu} \) with the Weinberg angle \( \theta_w \)

\[
\begin{pmatrix} A_{\mu}^{(em)} \\ Z_{\mu} \end{pmatrix} = \begin{pmatrix} \cos \theta_w & \sin \theta_w \\ -\sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} B_{\mu} \\ A_{\mu}^3 \end{pmatrix} = \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \begin{pmatrix} B_{\mu} \\ A_{\mu}^3 \end{pmatrix},
\]

(2.6)

we can express the ansatz (2.3) by

\[
\rho = \rho(r)
\]

\[
W_\mu = \frac{1}{\sqrt{2}}(A_\mu + i A_\mu^2) = \frac{i f(r)}{g \sqrt{2}} e^{i \varphi}(\partial_\mu \theta + i \sin \theta \partial_\mu \varphi),
\]

\[
A_{\mu}^{(em)} = e \left( \frac{1}{g^2} A(r) + \frac{1}{g'^2} B(r) \right) \partial_\mu t + \frac{1}{e} (1 - \cos \theta) \partial_\mu \varphi,
\]

\[
Z_{\mu} = \frac{e}{gg'}(A(r) - B(r)) \partial_\mu t,
\]

(2.7)

where \( \rho \) and \( W_\mu \) are the Higgs field and the \( W \)-boson, and \( e \) is the electric charge

\[
e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g \sin \theta_w = g' \cos \theta_w.
\]

This clearly shows that the ansatz is for the electromagnetic monopole and dyon.

The spherically symmetric ansatz (2.3) reduces the equations of motion to

\[
\ddot{f} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - A^2 \right) f,
\]

\[
\ddot{\rho} + \frac{2}{r} \dot{\rho} - \frac{f^2}{2r^2} \rho = -\frac{1}{4} (A - B)^2 \rho + \frac{\lambda}{2} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right) \rho,
\]

\[
\ddot{A} + \frac{2}{r} \dot{A} - \frac{2f^2}{r^2} A = \frac{g^2}{4} \rho^2 (A - B),
\]

(2.8)

\[
\ddot{B} + \frac{2}{r} \dot{B} = -\frac{g^2}{4} \rho^2 (A - B).
\]
Obviously this has a trivial solution

\[ f = 0, \quad \rho = \rho_0 = \sqrt{2\mu^2/\lambda}, \quad A = B = 0, \] (2.9)

which describes the point monopole in Weinberg-Salam model

\[ A^{(\text{em})}_\mu = \frac{1}{e} (1 - \cos \theta) \frac{\partial \mu}{\varphi}. \] (2.10)

This monopole has two remarkable features. First, this is not the Dirac’s monopole. It has the electric charge \(4\pi/e\), not \(2\pi/e\) [7]. Secondly, this monopole naturally admits a non-trivial dressing of weak bosons. Indeed, with the non-trivial dressing, the monopole becomes the Cho-Maison monopole and dyon.

To see this let us choose the following boundary condition

\[
\begin{align*}
    f(0) &= 1, \quad \rho(0) = 0, \quad A(0) = 0, \quad B(0) = b_0, \\
    f(\infty) &= 0, \quad \rho(\infty) = \rho_0, \quad A(\infty) = B(\infty) = A_0.
\end{align*}
\] (2.11)

Then we can show that the equation (2.8) admits a family of solutions labeled by the real parameter \(A_0\) lying in the range [7, 9]

\[ 0 \leq A_0 < \min \left( e\rho_0, \frac{g}{2}\rho_0 \right). \] (2.12)

In this case all four functions \(f(r), \rho(r), A(r),\) and \(B(r)\) must be positive for \(r > 0\), and \(A(r)/g^2 + B(r)/g^2\) and \(B(r)\) become increasing functions of \(r\). So we have \(0 \leq b_0 \leq A_0\). Furthermore, we have \(B(r) \geq A(r) \geq 0\) for all range, and \(B(r)\) must approach to \(A(r)\) with an exponential damping. Notice that, with the experimental fact \(\sin^2 \theta_w = 0.2325\), (2.12) can be written as \(0 \leq A_0 < e\rho_0\).

With the boundary condition (2.11) we can integrate (2.8). For example, with \(A_0 = 0\), we have the Cho-Maison monopole with \(A = B = 0\). In general, with \(A_0 \neq 0\), we find the Cho-Maison dyon solution shown in Fig.1 [7]. The solution looks very much like the well-known Prasad-Sommerfield solution of the Julia-Zee dyon. But there is a crucial difference. The Cho-Maison dyon now has a non-trivial \(A - B\), which represents the non-vanishing neutral \(Z\)-boson content of the dyon as shown by (2.7).

Near the origin the dyon solution has the following behavior,

\[
\begin{align*}
    f &\simeq 1 + \alpha_1 r^2, \\
    \rho &\simeq \beta_1 r^{\delta_-}, \\
    A &\simeq \alpha_1 r, \\
    B &\simeq b_0 + b_1 r^{2\delta_+},
\end{align*}
\] (2.13)

where \(\delta_\pm = (\sqrt{3} \pm 1)/2\). Asymptotically it has the following behavior,

\[ f \simeq f_1 \exp(-\omega r), \]
Figure 1: The Cho-Maison dyon solution. Here $Z(r) = A(r) - B(r)$ and we have chosen $\sin^2 \theta_w = 0.2325$, $\lambda/g^2 = M_H^2/4M_W^2 = 1/2$, and $A_0 = M_W/2$.

\[
\rho \simeq \rho_0 + \rho_1 \frac{\exp(-\sqrt{2}\mu r)}{r}, \\
A \simeq A_0 + \frac{A_1}{r}, \\
B \simeq A + B_1 \frac{\exp(-\nu r)}{r},
\]

where $\omega = \sqrt{(g\rho_0)^2/4 - A_0^2}$, and $\nu = \sqrt{(g^2 + g'^2)\rho_0}/2$. The physical meaning of the asymptotic behavior must be clear. Obviously $\rho$, $f$, and $A - B$ represent the Higgs boson, $W$-boson, and $Z$-boson whose masses are given by $M_H = \sqrt{2}\mu = \sqrt{(g\rho_0)$, $M_W = g\rho_0/2$, and $M_Z = \sqrt{g^2 + g'^2}\rho_0}/2$. So (2.14) tells that $M_H$, $\sqrt{1 - (A_0/M_W)^2} M_W$, and $M_Z$ determine the exponential damping of the Higgs boson, $W$-boson, and $Z$-boson to their vacuum expectation values asymptotically. Notice that it is $\sqrt{1 - (A_0/M_W)^2} M_W$, but not $M_W$, which determines the exponential damping of the $W$-boson. This tells that the electric potential of the dyon slows down the exponential damping of the $W$-boson, which is reasonable.

The dyon has the following electromagnetic charges

\[
q_e = -4\pi e \left[ r^2 \left( \frac{1}{g^2} A + \frac{1}{g'^2} B \right) \right] \bigg|_{r=\infty} = \frac{4\pi}{e} A_1 = -\frac{8\pi}{e} \sin^2 \theta_w \int_0^\infty f^2 A dr, \\
q_m = \frac{4\pi}{e}.
\]

Also, the asymptotic condition (2.14) assures that the dyon does not carry any neutral charge,

\[
Z_e = -\frac{4\pi e}{gg'} \left[ r^2 (\dot{A} - \dot{B}) \right] \bigg|_{r=\infty} = 0,
\]
\[ Z_m = 0. \] (2.16)

Furthermore, notice that the dyon equation (2.8) is invariant under the reflection

\[ A \rightarrow -A, \quad B \rightarrow -B. \] (2.17)

This means that, for a given magnetic charge, there are always two dyon solutions which carry opposite electric charges \( \pm q_e \). Clearly the signature of the electric charge of the dyon is determined by the signature of the boundary value \( A_0 \).

With the ansatz (2.3) we have the following energy of the dyon

\[ E = E_0 + E_1, \]

\[ E_0 = \frac{2\pi}{g^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{g^2}{r^2} + \left( f^2 - 1 \right)^2 \right\}, \]

\[ E_1 = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ \frac{g^2}{2} \left( r \dot{\rho} \right)^2 + \frac{g^2}{4} f^2 \rho^2 + \frac{g^2 r^2}{8} (B - A)^2 \rho^2 + \frac{\lambda g^2 r^2}{8} \left( \rho^2 - \frac{2\mu^2}{\lambda} \right)^2 \right\} + \left( \dot{f} \right)^2 + \frac{1}{2} (r \dot{A})^2 + \frac{g^2}{2g^2} (r \dot{B})^2 + f^2 A^2 \right\}. \] (2.18)

The boundary condition (2.11) guarantees that \( E_1 \) is finite. As for \( E_0 \) we can minimize it with the boundary condition \( f(0) = 1 \), but even with this \( E_0 \) becomes infinite. Of course the origin of this infinite energy is obvious, which is precisely due to the magnetic singularity of \( B_\mu \) at the origin. This means that one can not predict the mass of dyon. Physically it remains arbitrary.

The numerical solutions assures the existence of the electroweak monopole and dyon in Weinberg-Salam model. In spite of this one may still like to have a mathematically rigorous existence proof of the Cho-Maison solutions. The mathematical existence proof is non-trivial, because the equation of motion (2.8) is not the Euler-Lagrange equation of the positive definite energy (2.18), but that of the indefinite action (2.1). Fortunately the existence proof has been given by Yang [8, 9].

At this point it should be mentioned that the existence of a different type of “electroweak monopole” has been asserted in the literature which has a fractional magnetic charge [11],

\[ \tilde{q}_m = \frac{4\pi}{e} \sin \theta_w. \] (2.19)

This assertion has made a wrong impression that the fractionally charged monopole is the only monopole which could exist in Weinberg-Salam model [6], which has led many people to question the correctness of the Cho-Maison monopole. So it is worth
to clarify the situation before we close this section. As we have pointed out in the introduction, long time ago Nambu has shown the existence of electroweak string in Weinberg-Salam model which has monopole anti-monopole pair with the fractional charge $\pm q_m$ at the ends [10]. Obviously one could try to isolate the monopole at one end by extending the string to infinity. Simply by doing this some people have claimed to discover the fractionally charged “electroweak monopole” [11]. But clearly the Nambu’s monopole can not be identified as an electroweak monopole, because one can not really isolate it. To do so one has to pump in an infinite energy. This means that the fractionally charged monopoles, just like the quarks in QCD, can only be paired with the anti-monopoles to form confined objects which can not be isolated with finite energy [10]. Indeed this confinement of the fractionally charged monopoles in the electroweak theory was precisely the motivation of the Nambu’s pioneering work.

Even if one neglects the confinement and simply considers the monopole configuration with infinite string, one can not regard it as an electroweak monopole. The reason is because along the string the Higgs field must vanish, so that asymptotically the Higgs field does not approach its vacuum value in the vicinity of the string. This forbids us to identify the monopole as an electroweak object.

Nevertheless, as far as the Higgs doublet is concerned, the ansatz (2.3) is identical to Nambu’s ansatz. If so, one may wonder how Nambu did not discover the Cho-Maison monopole. The reason is that he concentrated on the trivial sector of the hypercharge $U(1)$ bundle because he was interested in the string configuration. As long as one stays in the trivial sector, of course, one can not remove the string and must treat it as physical. This is how Nambu constructed the electroweak string which confines the monopole anti-monopole pair. Notice, however, one can always make the string disappear by making the $U(1)$ bundle non-trivial. In this case one can easily remove the string by a gauge transformation, and have a genuine isolated monopole which has the integer magnetic charge $4\pi/e$ [7]. This is how the spherically symmetric Cho-Maison monopole has been constructed. This clarifies the difference between the Nambu’s monopole and the Cho-Maison monopole, which emphasizes again the fact that the non-trivialty of the hypercharge $U(1)$ bundle is crucial for the Cho-Maison monopole.

**III. Comparison with Julia-Zee Dyon**

At this stage one may ask whether there is any way to make the energy of the Cho-Maison solutions finite. A simple way to make the energy finite is to introduce the gravitational interaction [16]. But the gravitational interaction is not likely remove the singularity at the origin, and one may still wonder if there is any way to regularize the Cho-Maison solutions. To answer this question it is important to understand that the finite energy non-Abelian monopoles are really nothing but
the Abelian monopoles whose singularity at the origin is regularized by the charged vector fields. This can best be demonstrated by the t’Hooft-Polyakov monopole in Georgi-Glashow model [12, 17]. So in this section we discuss the gauge invariant Abelian formalism of Georgi-Glashow model and review how the charged vector field regularizes the Abelian monopole singularity at the origin.

Consider Georgi-Glashow model

\[
\mathcal{L}_{GG} = -\frac{1}{2} (D_\mu \vec{\Phi})^2 - \frac{\lambda}{4} \left( \vec{\Phi}^2 - \frac{\mu^2}{\lambda} \right)^2 - \frac{1}{4} F_{\mu\nu}^2, \tag{3.1}
\]

where \( \vec{\Phi} \) is the Higgs triplet. A best way to Abelianize it is to start from the gauge-independent decomposition of the \( SU(2) \) gauge potential into the restricted binding potential \( \hat{A}_\mu \) and the gauge covariant valence potential \( \vec{W}_\mu \) [18, 19], which has recently been referred to as Cho decomposition or Cho-Faddeev-Niemi-Shabanov decomposition [20, 21]. Let

\[
\vec{\Phi} = \rho \hat{n}, \tag{3.2}
\]

and identify \( \hat{n} \) to be the unit isovector which selects the charge direction in \( SU(2) \) space. Then the Cho decomposition of an arbitrary \( SU(2) \) gauge potential is given by [18, 19]

\[
\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \vec{W}_\mu = \hat{A}_\mu + \vec{W}_\mu, \quad (A_\mu = \hat{n} \cdot \vec{A}_\mu, \quad \hat{n} \cdot \vec{W}_\mu = 0), \tag{3.3}
\]

where \( A_\mu \) is the “electric” potential. Notice that the restricted potential \( \hat{A}_\mu \) is precisely the connection which leaves \( \hat{n} \) invariant under parallel transport,

\[
\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n} = 0. \tag{3.4}
\]

Under the infinitesimal gauge transformation

\[
\delta \hat{n} = -\hat{\alpha} \times \hat{n}, \quad \delta \hat{A}_\mu = \frac{1}{g} \partial_\mu \hat{\alpha}, \tag{3.5}
\]

one has

\[
\delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \hat{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \hat{\alpha}, \quad \delta \vec{W}_\mu = -\hat{\alpha} \times \vec{W}_\mu. \tag{3.6}
\]

This tells that \( \hat{A}_\mu \) by itself describes an \( SU(2) \) connection which enjoys the full \( SU(2) \) gauge degrees of freedom. Furthermore the valence potential \( \vec{W}_\mu \) forms a gauge covariant vector field under the gauge transformation. But what is really remarkable is that the decomposition is gauge independent. Once \( \hat{n} \) is chosen, the decomposition follows automatically, regardless of the choice of gauge [18, 19].
Remarkably \( \hat{A}_\mu \) retains all the essential topological characteristics of the original non-Abelian potential. First, \( \hat{n} \) defines \( \pi_2(S^2) \) which describes the non-Abelian monopoles [2, 3]. Secondly, it characterizes the Hopf invariant \( \pi_3(S^2) \cong \pi_3(S^3) \) which describes the topologically distinct vacua [22, 23]. Furthermore \( \hat{A}_\mu \) has a dual structure,

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g \hat{A}_\mu \times \hat{A}_\nu = (F_{\mu\nu} + H_{\mu\nu})\hat{n},
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu, \quad (3.7)
\]

where \( \tilde{C}_\mu \) is the “magnetic” potential. Notice that one can always introduce the magnetic potential, since \( H_{\mu\nu} \) forms a closed two-form locally sectionwise [18, 19]. In fact, replacing \( \hat{n} \) with a \( CP^1 \) field \( \xi \) by

\[
\hat{n} = -\xi^\dagger \tilde{\tau} \xi, \quad (3.8)
\]

we have

\[
\tilde{C}_\mu = \frac{2i}{g} \xi^\dagger \partial \mu \xi, \quad H_{\mu\nu} = \frac{2i}{g} (\partial_\mu \xi^\dagger \partial \nu \xi - \partial_\nu \xi^\dagger \partial \mu \xi). \quad (3.9)
\]

To see that \( \tilde{C}_\mu \) does describe the monopole, notice that with the ansatz (2.3) we have

\[
\tilde{C}_\mu = \frac{1}{g} (1 - \cos \theta) \partial_\mu \varphi. \quad (3.10)
\]

This is nothing but the Abelian monopole potential, which justifies \( \tilde{C}_\mu \) as the magnetic potential. The corresponding non-Abelian monopole potential is given by

\[
\tilde{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad (3.11)
\]

in terms of which the magnetic field is expressed by

\[
\tilde{H}_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu + g \tilde{C}_\mu \times \tilde{C}_\nu = H_{\mu\nu} \hat{n}. \quad (3.12)
\]

This provides the gauge independent separation of the monopole field \( \tilde{H}_{\mu\nu} \) from the generic non-Abelian gauge field \( \tilde{F}_{\mu\nu} \). The monopole potential (3.11) is now referred to as the Cho connection by Faddeev [20, 21].

With the decomposition (3.3), one has

\[
\tilde{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \tilde{W}_\nu - \hat{D}_\nu \tilde{W}_\mu + g \tilde{W}_\mu \times \tilde{W}_\nu, \quad (3.13)
\]

so that the Yang-Mills Lagrangian is expressed as

\[
\mathcal{L}_{YM} = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{4} (\hat{D}_\mu \tilde{W}_\nu - \hat{D}_\nu \tilde{W}_\mu)^2 - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\tilde{W}_\mu \times \tilde{W}_\nu) - \frac{g^2}{4} (\tilde{W}_\mu \times \tilde{W}_\nu)^2. \quad (3.14)
\]
This shows that the Yang-Mills theory can be viewed as a restricted gauge theory made of the restricted potential, which has the valence gluons as its source [18, 19]. For a long time it has generally been asserted that the non-Abelian gauge symmetry uniquely determines its dynamics. This has led many people to believe that the Yang-Mills theory is the only theory which has the full non-Abelian gauge symmetry. But the above analysis clearly demonstrates that this is not true. Evidently the non-Abelian gauge symmetry allows a simpler gauge theory, the restricted gauge theory, which enjoys the full non-Abelian gauge degrees of freedom yet contains much less physical degrees of freedom. In this view the Yang-Mills theory is nothing but the restricted gauge theory which has an extra gauge covariant vector field as the colored source. This observation plays a central role in our understanding of QCD, in particular the monopole condensation in QCD [18, 19]. Only recently this important fact has become to be appreciated [20, 21].

An important advantage of the decomposition (3.3) is that it can actually Abelianize (or more precisely “dualize”) the non-Abelian gauge theory [18, 19]. To see this let \( \hat{n}_1, \hat{n}_2, \hat{n} \) be a right-handed orthonormal basis of SU(2) space and let

\[
\hat{W}_\mu = W_1^\mu \hat{n}_1 + W_2^\mu \hat{n}_2, \quad (W_1^\mu = \hat{n}_1 \cdot \hat{W}_\mu, \ W_2^\mu = \hat{n}_2 \cdot \hat{W}_\mu).
\]

With this one has

\[
\hat{D}_\mu \hat{W}_\nu = \left[ \partial_\mu W_1^\nu - g(A_\mu + \tilde{C}_\mu)W_2^\nu \right] \hat{n}_1 + \left[ \partial_\mu W_2^\nu + g(A_\mu + \tilde{C}_\mu)W_1^\nu \right] \hat{n}_2, \quad (3.15)
\]

so that with

\[
A_\mu = A_\mu + \tilde{C}_\mu, \quad W_\mu = \frac{1}{\sqrt{2}}(W_1^\mu + iW_2^\mu),
\]

one could express the Lagrangian explicitly in terms of the dual potential \( A_\mu \) and the complex vector field \( W_\mu \),

\[
\mathcal{L}_{YM} = -\frac{1}{4}(\mathcal{F}_{\mu\nu} + W_{\mu\nu})^2 - \frac{1}{2}|\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu|^2; \quad (3.16)
\]

where now \( \hat{D}_\mu = \partial_\mu + igA_\mu \) is an Abelian covariant derivative, and

\[
\mathcal{F}_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu}, \quad W_{\mu\nu} = -ig(W_\mu^*W_\nu - W_\nu^*W_\mu).
\]

This describes an Abelian gauge theory coupled to the charged vector field \( W_\mu \). In this form the equations of motion of Yang-Mills theory is expressed by

\[
\partial_\mu(\mathcal{F}_{\mu\nu} + W_{\mu\nu}) = igW_\nu^*(\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu) - igW_\mu^*(\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu)*,
\]

\[
\hat{D}_\mu(\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu) = igW_\mu(\mathcal{F}_{\mu\nu} + W_{\mu\nu}). \quad (3.17)
\]

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. An important point of the Abelian formalism is that, in addition to the local
dynamical degrees $W_\mu^1$, $W_\mu^2$, and $A_\mu$ of Yang-Mills theory, it has an extra magnetic potential $\tilde{C}_\mu$. Furthermore the the Abelian potential $A_\mu$ which couples to $W_\mu$ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. Clearly $\tilde{C}_\mu$ represents the topological degrees of the non-Abelian symmetry which does not show up in the naive Abelianization that one obtains by fixing the gauge [18, 19].

An important feature of this Abelianization is that it is gauge independent, because here we have never fixed the gauge to obtain this Abelian formalism. So one might ask how the non-Abelian gauge symmetry is realized in this Abelian formalism. To discuss this let

\[
\vec{\alpha} = \alpha_1 \hat{n}_1 + \alpha_2 \hat{n}_2 + \theta \hat{n}, \quad \alpha = \frac{1}{\sqrt{2}}(\alpha_1 + i \alpha_2),
\]

\[
\tilde{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n} = -C_\mu^1 \hat{n}_1 - C_\mu^2 \hat{n}_2, \quad C_\mu = \frac{1}{\sqrt{2}}(C_\mu^1 + i C_\mu^2). \tag{3.18}
\]

Then the Lagrangian (3.16) is invariant not only under the active gauge transformation (3.6) described by

\[
\delta A_\mu = \frac{1}{g} \partial_\mu \theta - i(C_\mu^* \alpha - C_\mu \beta^*), \quad \delta \tilde{C}_\mu = -\delta A_\mu, \quad \delta W_\mu = 0, \tag{3.19}
\]

but also under the following passive gauge transformation described by

\[
\delta A_\mu = \frac{1}{g} \partial_\mu \theta - i(W_\mu^* \alpha - W_\mu \beta^*), \quad \delta \tilde{C}_\mu = 0, \quad \delta W_\mu = \frac{1}{g} \hat{D}_\mu \alpha - i \theta W_\mu. \tag{3.20}
\]

Clearly this passive gauge transformation assures the desired non-Abelian gauge symmetry for the Abelian formalism. This tells that the Abelian theory not only retains the original gauge symmetry, but actually has an enlarged (both the active and passive) gauge symmetries. But we emphasize that this is not the “naive” Abelianization of Yang-Mills theory which one obtains by fixing the gauge. Our Abelianization is a gauge-independent Abelianization. Besides, here the Abelian gauge group is $U(1)_e \otimes U(1)_m$, so that the theory becomes a dual gauge theory [18, 19]. This is evident from (3.19) and (3.20).

With this we can now obtain the gauge invariant Abelianization of Georgi-Glashow model. From (3.2) and (3.3) we have

\[
\mathcal{L}_{GG} = -\frac{1}{4} \left( \hat{D}_\mu \tilde{\Phi} \right)^2 - \frac{g^2}{2} (\hat{W}_\mu \times \tilde{\Phi})^2 - \frac{\lambda}{4} \left( \tilde{\Phi}^2 - \frac{\mu^2}{\lambda} \right)^2 - \frac{1}{4} (\hat{D}_\mu \hat{W}_\nu - \hat{D}_\nu \hat{W}_\mu)^2 - \frac{1}{4} (\hat{F}_{\mu\nu} + g \hat{W}_\mu \times \tilde{W}_\nu)^2
\]

\[
= -\frac{1}{2} (\partial_\mu \rho)^2 - g^2 \rho^2 W_\mu^* W_\mu - \frac{\lambda}{4} \left( \rho^2 - \frac{\mu^2}{\lambda} \right)^2 - \frac{1}{2} (\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu)^2
\]

\[
- \frac{1}{4} \mathcal{F}_{\mu\nu}^2 + ig \mathcal{F}_{\mu\nu} W_\mu^* W_\nu + \frac{g^2}{4} (W_\mu^* W_\nu - W_\nu^* W_\mu)^2. \tag{3.21}
\]
Now, the spherically symmetric ansatz of the Julia-Zee dyon
\[ \Phi = \rho(r) \hat{n}, \quad \hat{n} = \hat{r}, \]
\[ \vec{A}_\mu = \frac{1}{g} A(r) \partial_\mu t \hat{n} + \frac{1}{g} (f(r) - 1) \hat{n} \times \partial_\mu \hat{n}, \quad (3.22) \]
can be written in this Abelian formalism as
\[ \rho = \rho(r) \]
\[ W_\mu = \frac{i f(r)}{g} \frac{e^{i \phi}}{\sqrt{2}} \left( \partial_\mu \theta + i \sin \theta \partial_\mu \phi \right), \]
\[ \mathcal{A}_\mu = \frac{1}{g} A(r) \partial_\mu t + \frac{1}{g} (1 - \cos \theta) \partial_\mu \phi. \quad (3.23) \]

With the ansatz one has the following equation of motion
\[ \ddot{f} - \frac{f^2 - 1}{r^2} f = \left( g^2 \rho^2 - A^2 \right) f, \]
\[ \ddot{\rho} + \frac{2}{r} \dot{\rho} - 2 \frac{f^2}{r^2} \rho = \lambda \left( \rho^2 - \frac{\mu^2}{\lambda} \right) \rho, \quad (3.24) \]
\[ \ddot{A} + \frac{2}{r} \dot{A} - 2 \frac{f^2}{r^2} A = 0. \]

With the boundary condition
\[ f(0) = 1, \quad A(0) = 0, \quad \rho(0) = 0, \]
\[ f(\infty) = 0, \quad A(\infty) = A_0, \quad \rho(\infty) = \rho_0, \quad (3.25) \]
one can integrate (3.24) and obtain the Julia-Zee dyon. Again it must be clear from (3.24) that, for a given magnetic charge, there are always two dyons with opposite electric charges. Moreover, for the monopole solution with \( A = 0 \), the equation reduces to the following Bogomol’nyi-Prasad-Sommerfield equation in the limit \( \lambda = 0 \)
\[ \dot{f} \pm \epsilon \rho f = 0, \]
\[ \dot{\rho} \pm \frac{1}{er^2} (f^2 - 1) = 0, \quad (3.26) \]
which has the analytic solution [5]
\[ f = \frac{e \rho_0 r}{\sinh(e \rho_0 r)}, \quad \rho = \rho_0 \coth(e \rho_0 r) - \frac{1}{er}, \quad (3.27) \]
Notice that the boundary condition \( A(0) = 0 \) and \( f(0) = 1 \) is crucial to make the \( SU(2) \) potential \( \vec{A}_\mu \) regular at the origin.
IV. Finite Energy Electroweak Dyon

The above analysis tells us two things. First, the Julia-Zee dyon is nothing but an Abelian dyon whose singularity at the origin is regularized by the charged vector field $W_\mu$ and scalar field $\rho$. Secondly, the Cho-Maison dyon in the unitary gauge can also be viewed as an Abelian dyon whose singularity at the origin is only partly regularized by the weak bosons. Obviously this makes the two dyons very similar to each other. This suggests that one could also try to make the energy of the Cho-Maison solutions finite by introducing additional interactions and/or charged vector fields. In this section we will present two ways which allow us to achieve this goal along this line, and construct analytic electroweak monopole and dyon solutions with finite energy.

A. Electromagnetic Regularization

We first regularize the magnetic singularity with a judicious choice of an extra electromagnetic interaction of the charged vector field with the Abelian monopole [12, 13]. This regularization provides a most economic way to make the energy of the Cho-Maison solution finite, because here we could use the already existing $W$-boson without introducing a new source.

For this we need to Abelianize Weinberg-Salam model first. With $\phi = \frac{1}{\sqrt{2}} \rho \xi$, $\hat{n} = -\xi^\dagger \vec{\tau} \xi$, ($\xi^\dagger \xi = 1$), we have

$$L = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\rho^2}{2} |D_\mu \xi|^2 - \frac{\lambda}{8} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right)^2 - \frac{1}{4} \tilde{F}_{\mu\nu}^2 - \frac{1}{4} G_{\mu\nu}^2$$

$$= \frac{1}{2} (\partial_\mu \rho)^2 - \frac{\rho^2}{2} \left( |D_\mu \xi|^2 - |\xi^\dagger D_\mu \xi|^2 \right) + \frac{\rho^2}{2} \left( \xi^\dagger D_\mu \xi - ig' B_\mu \right)^2 - \frac{\lambda}{8} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right)^2 - \frac{1}{4} (F_{\mu\nu} + W_{\mu\nu})^2 - \frac{1}{4} G_{\mu\nu}^2 - \frac{1}{2} |\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu|^2$$

$$= -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{g^2}{4} \rho^2 W_\mu W_\mu - \frac{\rho^2}{8} (gA_\mu - g'B_\mu)^2 - \frac{\lambda}{8} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right)^2 - \frac{1}{4} (F_{\mu\nu} + W_{\mu\nu})^2 - \frac{1}{4} G_{\mu\nu}^2 - \frac{1}{2} |\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu|^2.$$  \hspace{1cm} (4.2)

There are three points to be emphasized here. First, the Lagrangian is explicitly invariant under the $U(1)$ gauge transformation of the $\xi$ field. This means that Weinberg-Salam model can also be viewed as a gauged $CP^1$ model [7]. This of course is why Weinberg-Salam model allows the Cho-Maison solutions. Secondly, the charged vector field $W_\mu$ is nothing but the $W$-boson. Indeed, with the identification of $W_\mu$ as the physical $W$-boson, the Lagrangian becomes formally identical to
what we have in the unitary gauge. But there is an important difference. Here we
did not obtain the above Lagrangian by fixing the gauge. As a result our Abelian
gauge potential which couples to $W$-boson is given by $A_\mu + \tilde{C}_\mu$, and has a dual struc-
ture. Thirdly, in this gauge invariant Abelianization the electromagnetic potential
and $Z$-boson are given by

\begin{align}
A^{(em)}_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}(g' A_\mu + gB_\mu), \\
Z_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}(gA_\mu - g'B_\mu),
\end{align}

(4.3)

so that, in terms of the physical fields, the Lagrangian (4.2) is expressed by

\begin{align}
\mathcal{L} &= -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{g^2}{4} \rho^2 W^*_\mu W_\mu - \frac{g^2 + g'^2}{8} \rho^2 Z^2_\mu - \frac{\lambda}{8} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right)^2 \\
&- \frac{1}{4} F^{(em)}_{\mu\nu}^2 - \frac{1}{4} Z^2_{\mu\nu} - \frac{1}{2} |(D^{(em)}_\mu W_\nu - D^{(em)}_\nu W_\mu)|^2 \\
&+ i e F^{(em)}_{\mu\nu} W^*_\mu W_\nu + i g' \frac{g}{g'} Z_{\mu\nu} W^*_\mu W_\nu + \frac{g^2}{4} (W^*_\mu W_\nu - W^*_\nu W_\mu)^2,
\end{align}

(4.4)

where $D^{(em)}_\mu = \partial_\mu + ie A^{(em)}_\mu$.

Already at this level the above Lagrangian provides us an important piece of
information. In the absence of the electromagnetic interaction (i.e., with $A^{(em)}_\mu = W_\mu = 0$) the Lagrangian describes a spontaneously broken $U(1)_Z$ gauge theory,

\begin{align}
\mathcal{L} &= -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{g^2}{4} \rho^2 W^*_\mu W_\mu - \frac{g^2 + g'^2}{8} \rho^2 Z^2_\mu - \frac{\lambda}{8} \left( \rho^2 - \frac{2 \mu^2}{\lambda} \right)^2 - \frac{1}{4} Z^2_{\mu\nu},
\end{align}

(4.5)

which is nothing but the Ginsburg-Landau theory of superconductivity. Furthermore,
here $M_H$ and $M_Z$ corresponds to the coherence length (of the Higgs field) and the
penetration length (of the magnetic field made of $Z$-field). So, when $M_H > M_Z$ (or $M_H < M_Z$), the theory describes a type II (or type I) superconductivity, which is
well known to admit the Abrikosov-Nielsen-Olesen vortex solution. This confirms the
existence of Nambu’s string in Weinberg-Salam model. What Nambu showed was
that he could make the string finite by attaching the fractionally charged monopole
anti-monopole pair to this string [10].

To regularize the Cho-Maison dyon we now introduce an extra interaction $\mathcal{L}_1$ to
(4.4),

\begin{align}
\mathcal{L}_1 = i e F^{(em)}_{\mu\nu} W^*_\mu W_\nu + \frac{\beta}{4} g^2 (W^*_\mu W_\nu - W^*_\nu W_\mu)^2,
\end{align}

(4.6)

where $\alpha$ and $\beta$ are arbitrary constants. Notice that the extra interaction still respects
the gauge invariance of the theory, because with the decomposition (3.3) the extra
interaction can be expressed in a gauge invariant form with the help of the gauge
covariant multiplet $\tilde{W}_\mu$. With this additional interaction the Lagrangian (4.4) is
modified to

\[
\hat{\mathcal{L}} = \mathcal{L} + \mathcal{L}_1
\]

\[
= -\frac{1}{2}(\partial_{\mu}\rho)^2 - \frac{g^2}{4}\rho^2 W_{\mu}^* W_{\mu} - \frac{\lambda}{8}\left(\rho^2 - \frac{2\mu^2}{\lambda}\right)^2 - \frac{1}{4}F_{\mu\nu}^{(em)}^2
\]

\[
- \frac{1}{2}(D_{\mu}^{(em)} W_{\nu} - D_{\nu}^{(em)} W_{\mu}) + i\frac{g}{g'}(Z_{\mu} W_{\nu} - Z_{\nu} W_{\mu})^2 - \frac{1}{4}Z_{\mu\nu} - \frac{g^2 + g'^2}{8}\rho^2 Z_{\mu}
\]

\[
+ i\frac{g}{g'}Z_{\mu\nu} W_{\mu}^* W_{\nu} + i\frac{f^2}{g^2} F_{\mu\nu}^{(em)} W_{\mu}^* W_{\nu} + (1 + \beta)\frac{g^2}{4}(W_{\mu}^* W_{\nu} - W_{\nu}^* W_{\mu})^2, 
\]

(4.7)

so that with the ansatz (2.7) the energy of dyon is given by

\[
\hat{E} = \hat{E}_0 + \hat{E}_1,
\]

\[
\hat{E}_0 = \frac{2\pi}{g^2} \int_0^{\infty} \frac{dr}{r^2} \left\{ \frac{g^2}{g'^2} + 1 - 2(1 + \alpha)f^2 + (1 + \beta)f^4 \right\},
\]

\[
\hat{E}_1 = E_1. 
\]

(4.8)

Notice that with \(\alpha = \beta = 0\), \(\hat{E}_0\) reduces to \(E_0\) and becomes infinite. For the energy (4.8) to be finite, the integrand of \(\hat{E}_0\) must be free from both \(O(1/r^2)\) and \(O(1/r)\) singularities at the origin. This requires us to have

\[
1 + \frac{g^2}{g'^2} - 2(1 + \alpha)f^2(0) + (1 + \beta)f^4(0) = 0,
\]

\[
(1 + \alpha)f(0) - (1 + \beta)f^3(0) = 0. 
\]

(4.9)

Thus we arrive at the following condition for a finite energy solution

\[
1 + \beta = (1 + \alpha)^2 \sin^2 \theta_w = (1 + \alpha)^2 \frac{e^2}{g^2}, \quad f(0) = \frac{1}{\sqrt{(1 + \alpha) \sin^2 \theta_w}}. 
\]

(4.10)

But notice that, although obviously sufficient for a finite energy solution, this condition does not guarantee the smoothness of the gauge potentials at the origin. One might try to impose the condition \(f(0) = 1\), because in this case the \(SU(2)\) potential \(\vec{A}_\mu\) of the ansatz (2.3) becomes regular everywhere including the origin. Unfortunately this condition does not remove the point singularity of \(B_\mu\) at the origin.

The condition for an analytic solution is given by [13]

\[
\alpha = 0, \quad 1 + \beta = \frac{e^2}{g^2}, \quad f(0) = \frac{1}{\sin \theta_w} = \frac{g}{e}. 
\]

(4.11)

Notice that this amounts to changing the coupling strength of the \(W\)-boson quartic self-interaction from \(g^2/4\) to \(e^2/4\). To derive this analyticity condition it is important to remember that Weinberg-Salam model, just like Georgi-Glashow model, can be
viewed as a gauged $CP^1$ model [7]. This means that it can also be expressed by a Higgs triplet. To see this we introduce a Higgs triplet $\Phi$ and an “electromagnetic” $SU(2)$ gauge potential $A_\mu$ by

$$\Phi = -\rho \xi \hat{\tau} \xi = \rho \hat{n}, \quad (\phi = \frac{1}{\sqrt{2}} \rho \xi),$$

$$A_\mu = (A_\mu^{(em)\rho}) - \frac{2i}{g} \xi^i \partial_\mu \xi \hat{n} - \frac{1}{e} \hat{n} \times \partial_\mu \hat{n} + \hat{W}_\mu = \hat{A}_\mu + \hat{W}_\mu. \quad (4.12)$$

With this the electroweak Lagrangian (4.7) with (4.10) can be expressed by

$$\hat{L} = -\frac{1}{2} \left( \hat{D}_\mu \Phi \right)^2 - \frac{g^2}{8} (\hat{W}_\mu \times \Phi)^2 - \frac{\lambda}{8} \left( \Phi^2 - \frac{2\mu^2}{\lambda} \right)^2$$

$$- \frac{1}{4} \left( \hat{F}_{\mu \nu} + (1 + \alpha) e \hat{W}_\mu \times \hat{W}_\nu \right)^2 - \frac{1}{4} \left( \hat{D}_\mu \hat{W}_\nu - \hat{D}_\nu \hat{W}_\mu + e \frac{g}{g'} \hat{n} \times (Z_\mu \hat{W}_\nu - Z_\nu \hat{W}_\mu) \right)^2$$

$$- \frac{1}{4} \left( \hat{Z}_{\mu \nu} \right)^2 - \frac{g^2 + g'^2}{8} \Phi^2 \hat{Z}_\mu^2 - \frac{e^2 g}{2 g'} \hat{n} \cdot (\hat{W}_\mu \times \hat{W}_\nu), \quad (4.13)$$

where $\hat{D}_\mu = \partial_\mu + e \hat{A}_\mu \times$. Notice that the Lagrangian is explicitly gauge invariant, due to the fact that $\hat{W}_\mu$ is gauge covariant. This reassures that the extra interaction (4.6) is indeed gauge invariant. Although the Lagrangian (because of the appearance of $\hat{n} = \Phi/|\Phi|$) looks to contain a non-polynomial interaction, the problematic non-polynomial interaction disappears when it is expressed in terms of the physical fields.

In this form the Lagrangian describes a “generalized” Georgi-Glashow model, which has extra interaction with the $Z$-boson. In particular, in the absence of the $Z$-boson, the theory reduces to an $SU(2)_{em}$ gauge theory

$$\hat{L} \rightarrow -\frac{1}{2} \left( \hat{D}_\mu \Phi \right)^2 + \frac{1}{2} (e^2 - \frac{g^2}{4}) (\hat{W}_\mu \times \Phi)^2 - \frac{\lambda}{8} \left( \Phi^2 - \frac{2\mu^2}{\lambda} \right)^2 - \frac{1}{4} \hat{F}_{\mu \nu}^2$$

$$- \alpha \frac{e}{2} \hat{F}_{\mu \nu} \cdot (\hat{W}_\mu \times \hat{W}_\nu) - \alpha^2 \frac{e^2}{4} (\hat{W}_\mu \times \hat{W}_\nu)^2. \quad (4.14)$$

Evidently, with (4.11), the Lagrangian becomes almost identical to (3.1). The only difference is that here we have the extra interaction $(e^2 - g^2/4)(\hat{W}_\mu \times \Phi)^2/2$. Furthermore, in this form the ansatz (2.7) is written as

$$\Phi = \rho(r) \hat{n}, \quad \hat{n} = \hat{r},$$

$$A_\mu = e \left( \frac{1}{g} A(r) + \frac{1}{g'} B(r) \right) \partial_\mu t \hat{n} + \frac{1}{e} \left( \frac{g}{g'} f(r) - 1 \right) \hat{n} \times \partial_\mu \hat{n},$$

$$Z_\mu = \frac{e}{gg'} (A(r) - B(r)) \partial_\mu t. \quad (4.15)$$

Comparing this with the Julia-Zee ansatz (3.22) we conclude that the ansatz (2.7) becomes smooth everywhere when $A(0)/g^2 + B(0)/g'^2 = 0$ and $f(0) = g/e$. In particular the monopole singularity disappears when $f(0) = g/e$. This gives us the analyticity condition (4.11).
But we emphasize that, to have a finite energy solution, the condition (4.10) is enough. Indeed, viewing the electroweak theory as an Abelian gauge theory described by (4.4), there seems no apparent reason why the ansatz (2.7) should satisfy the analyticity condition (4.11). For this reason we will leave $\alpha$ (and $f(0)$) arbitrary in the following, unless specified otherwise.

With (4.10) the equation of motion is given by

$$\ddot{f} - \frac{(1 + \alpha)}{r^2} \left( \frac{f^2}{f^2(0)} - 1 \right) f = \left( \frac{g^2}{4} \rho^2 - A^2 \right) f,$$

$$\ddot{\rho} + \frac{2}{r} \dot{\rho} - \frac{f^2}{2r^2} \rho = -\frac{1}{4} (A - B)^2 \rho + \frac{\lambda}{2} \left( \rho^2 - \frac{2\mu^2}{\lambda} \right) \rho,$$

$$\ddot{A} + \frac{2}{r} \dot{A} - \frac{2f^2}{r^2} A = \frac{g^2}{4} (A - B) \rho^2,$$

(4.16)

$$\ddot{B} + \frac{2}{r} \dot{B} = -\frac{g^2}{4} (A - B) \rho^2.$$  

One could integrate this with the boundary conditions near the origin,

$$f/f(0) \approx 1 + \alpha_1 r^{\delta_1},$$

$$\rho \approx \beta_1 r^{\delta_2},$$

$$A \approx a_1 r^{\delta_3},$$

$$B \approx b_0 + b_1 r^{\delta_4},$$

(4.17)

and the finite energy condition (2.14) near the infinity. Inserting (4.17) to the equation we have

$$\delta_1 = \frac{1}{2} (1 + \sqrt{8\alpha + 9}),$$

$$\delta_2 = \frac{1}{2} (\sqrt{1 + 2f^2(0)} - 1),$$

$$\delta_3 = \frac{1}{2} (\sqrt{1 + 8f^2(0)} - 1),$$

$$\delta_4 = \sqrt{1 + 2f^2(0)} + 1.$$  

(4.18)

Notice that all four deltas are positive (as far as $\alpha > -1$), so that the four functions are well behaved at the origin. Furthermore, when $\alpha = 0$ and $f(0) = 1$, this reduces to (2.13). Clearly the solution describes a finite energy electroweak dyon, even though the gauge potential of the ansatz (4.15) has a (harmless) mathematical singularity at the origin when $\alpha \neq 0$.

Now with the boundary condition

$$f(0) = g/e, \quad \rho(0) = 0, \quad A(0) = 0, \quad B(0) = b_0,$$

$$f(\infty) = 0, \quad \rho(\infty) = \rho_0, \quad A(\infty) = B(\infty) = A_0,$$

(4.19)
we can integrate (4.16) numerically. Notice that, strictly speaking, the Coulomb potential of the dyon retains a mathematical singularity at the origin when $b_0 \neq 0$. The results of the numerical integration for the monopole and dyon solution are shown in Fig.2 and Fig.3. It is really remarkable that the finite energy solutions look almost identical to the Cho-Maison solutions, even though they no longer have the magnetic singularity at the origin. The reason for this similarity must be clear. All that we need to have the analytic monopole and dyon in the electroweak theory is a simple modification of the coupling strength of $W$-boson quartic self-interaction from $g^2/4$ to $e^2/4$.

Of course, with an arbitrary $\alpha$, we can still integrate (4.16) and have a finite energy solution. In this case the gauge potential $A_\mu$ in general has a (harmless) mathematical singularity at the origin. Even in this case, however, the generic feature of the solutions remain the same.

Clearly the energy of the dyon must be of the order of $M_W$. Indeed for the monopole the energy can be expressed as

$$E = \frac{4\pi}{e^2} C(\alpha, \sin^2 \theta_w, \lambda/g^2) M_W$$  \hspace{1cm} (4.20)

where $C$ the dimensionless function of $\alpha$, $\sin^2 \theta_w$, and $\lambda/g^2$. With $\alpha = 0$ and experimental value $\sin^2 \theta_w$, $C$ becomes slowly varying function of $\lambda/g^2$ with $C = 1.407$ for $\lambda/g^2 = 1/2$. This demonstrates that the finite energy solutions are really nothing but the regularized Cho-Maison solutions which have a mass of electroweak scale.

Notice that we can even have an explicitly analytic monopole solution, if we add
**Figure 3:** The electroweak dyon solution. The solid line represents the finite energy dyon and dotted line represents the Cho-Maison dyon, where we have chosen $\lambda/g^2 = 1/2$ and $A_0 = M_W/2$.

An extra term $\mathcal{L}_2$ to the Lagrangian (4.13)

$$\mathcal{L}_2 = -\frac{1}{2}(e^2 - \frac{g^2}{4})(\vec{W}_\mu \times \vec{\Phi})^2 = -(e^2 - \frac{g^2}{4})\rho^2 W_\mu W_\mu. \quad (4.21)$$

This amounts to changing the mass of the $W$-boson from $g\rho_0/2$ to $e\rho_0$. More precisely, with (4.11) the extra term effectively reduces the Lagrangian (4.13) to that of Georgi-Glashow model in the absence of the $Z$-boson,

$$\hat{\mathcal{L}} \rightarrow -\frac{1}{2}(D_\mu \vec{\Phi})^2 - \frac{\lambda}{8} \left(\vec{\Phi}^2 - \frac{2\mu^2}{\lambda}\right)^2 - \frac{1}{4} F_{\mu\nu}^2. \quad (4.22)$$

Obviously this (with $A = B = 0$) allows the well-known Bogomol’nyi-Prasad-Sommerfield equation in the limit $\lambda = 0$ [5]

$$\dot{f} \pm e\rho f = 0,$$

$$\dot{\rho} \pm \frac{1}{er^2} \left(\frac{f^2}{f^2(0)} - 1\right) = 0. \quad (4.23)$$

This has the analytic monopole solution

$$f = f(0)\frac{e\rho_0 r}{\sinh(e\rho_0 r)} = \frac{g\rho_0 r}{\sinh(e\rho_0 r)}, \quad \rho = \rho_0 \coth(e\rho_0 r) - \frac{1}{er}. \quad (4.24)$$

whose energy is given by the Bogomol’nyi bound

$$E = \frac{4\pi}{e^2} M'_W = \frac{8\pi}{e^2} \sin \theta_w M_W, \quad (M'_W = e\rho_0). \quad (4.25)$$
But we emphasize that, even with this extra term, the electroweak dyon becomes different from the Prasad-Sommerfield dyon because it has a non-trivial dressing of the $Z$-boson.

**B. Embedding $SU(2) \times U(1)$ to $SU(2) \times SU(2)$**

As we have noticed the origin of the infinite energy of the Cho-Maison solutions was the magnetic singularity of $U(1)_{em}$. On the other hand the ansatz (2.3) also suggests that this singularity really originates from the magnetic part of the hypercharge $U(1)$ field $B_\mu$. So one could try to obtain a finite energy monopole solution by regularizing this hypercharge $U(1)$ singularity. This could be done by introducing a hypercharged vector field to the theory [12]. A simplest way to do this is, of course, to enlarge the hypercharge $U(1)$ and embed it to another $SU(2)$.

To construct the desired solutions we generalize the Lagrangian (4.2) by adding the following Lagrangian

$$\mathcal{L}' = -\frac{1}{2} |\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 + ig' G_{\mu\nu} X^*_\mu X_\nu + \frac{1}{4} g'^2 (X^*_\mu X_\nu - X^*_\nu X_\mu)^2$$

$$- \frac{1}{2} (\partial_\mu \sigma)^2 - g'^2 \sigma^2 X^*_\mu - \frac{\kappa}{4} \left( \sigma^2 - \frac{m^2}{\kappa} \right)^2,$$

where $X_\mu$ is a hypercharged vector field, $\sigma$ is a Higgs field, and $\hat{D}_\mu = \partial_\mu + ig'B_\mu$. Notice that, if we introduce a hypercharge $SU(2)$ gauge field $\tilde{B}_\mu$ and a scalar triplet $\tilde{\Phi}$ and identify

$$X_\mu = \frac{1}{\sqrt{2}} (B_1^\mu + iB_2^\mu), \quad B_\mu = B_3^\mu, \quad \tilde{\Phi} = (0, 0, \sigma),$$

the above Lagrangian becomes identical to

$$\mathcal{L}' = -\frac{1}{2} (\hat{D}_\mu \tilde{\Phi})^2 - \frac{\kappa}{4} (\tilde{\Phi}^2 - \frac{m^2}{\kappa})^2 - \frac{1}{4} G_{\mu\nu}^2,$$

as far as we interpret $B_\mu$ and $G_{\mu\nu}$ as the dual gauge field of the hypercharge $U(1)$. This clearly shows that Lagrangian (4.26) is nothing but the embedding of the hypercharge $U(1)$ to an $SU(2)$ Georgi-Glashow model.

From (4.2) and (4.26) one has the following equations of motion

$$\partial_\mu (\partial_\mu \rho) = \frac{g^2}{2} W^*_\mu W_\mu \rho + \frac{1}{4} (gA_\mu - g'B_\mu)^2 \rho + \frac{\lambda}{2} \left( 2 \rho^2 - \frac{2 \rho^2}{\lambda} \right) \rho,$$

$$\hat{D}_\mu (\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu) = igF_{\mu\nu} W_\mu - g^2 W_\mu (W_\nu W^*_\mu - W^*_\nu W_\mu) + \frac{g^2}{4} \rho^2 W_\nu,$$

$$\partial_\mu F_{\mu\nu} = \frac{g}{4} \rho^2 (gA_\nu - g'B_\nu) + ig \left( W^*_\mu (\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu) - (\hat{D}_\mu W_\nu - \hat{D}_\nu W_\mu)^* W_\mu \right)$$

$$+ ig \partial_\mu (W^*_\mu W_\nu - W^*_\nu W_\mu),$$
\[ \partial_{\mu} G_{\mu\nu} = \frac{g'}{4} \rho^2 (g'B_{\nu} - gA_{\nu}) + ig' \left( X_{\mu}^* (\tilde{\nabla}_{\mu} X_{\nu} - \tilde{\nabla}_{\nu} X_{\mu}) - (\tilde{\nabla}_{\mu} X_{\nu} - \tilde{\nabla}_{\nu} X_{\mu})^* X_{\mu} \right) + ig' \partial_{\mu} (X_{\mu}^* X_{\nu} - X_{\nu}^* X_{\mu}), \]

\[ \partial_{\mu} (\partial_{\mu} \sigma) = 2g'^2 X_{\mu}^* X_{\mu} \sigma + \kappa \left( \sigma^2 - \frac{m^2}{\kappa} \right) \sigma, \]

\[ \tilde{\nabla}_{\mu} (\tilde{\nabla}_{\mu} X_{\nu} - \tilde{\nabla}_{\nu} X_{\mu}) = ig' G_{\mu\nu} X_{\mu} - g^2 X_{\mu} (X_{\mu}^* X_{\nu} - X_{\nu}^* X_{\mu}) + g^2 \sigma^2 X_{\nu}, \quad (4.28) \]

Now for a static spherically symmetric ansatz we choose (2.3) and assume

\[ \sigma = \sigma(r), \]

\[ X_{\mu} = \frac{i}{g'} \frac{h(r)}{\sqrt{2}} e^{i\phi} (\partial_{\mu} \theta + i \sin \theta \partial_{\mu} \varphi). \quad (4.29) \]

With the spherically symmetric ansatz (4.28) is reduced to

\[ \ddot{f} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - A^2 \right) f, \]

\[ \ddot{\rho} + \frac{2}{r} \dot{\rho} - \frac{f^2}{2r^2} \rho = -\frac{1}{4} (A - B)^2 \rho + \frac{\lambda}{2} \left( \rho^2 - \frac{2\mu^2}{\lambda} \right) \rho, \]

\[ \ddot{\lambda} + \frac{2}{r} \dot{\rho} - \frac{2f^2}{r^2} \lambda = \frac{g^2}{4} \rho^2 (A - B), \quad (4.30) \]

\[ \ddot{h} - \frac{h^2 - 1}{r^2} h = (g^2 \sigma^2 - B^2) h, \quad (4.31) \]

\[ \ddot{\sigma} + \frac{2}{r} \dot{\sigma} - \frac{2h^2}{r^2} \sigma = \kappa \left( \sigma^2 - \frac{m^2}{\kappa} \right) \sigma, \]

\[ \ddot{B} + \frac{2}{r} \dot{B} - \frac{2h^2}{r^2} B = \frac{g^2}{4} \rho^2 (B - A). \]

Furthermore, the energy of the above configuration is given by

\[ E = E_W + E_X, \quad (4.32) \]

\[ E_W = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ (\dot{f})^2 + (f^2 - 1)^2 + \frac{1}{2} (r \dot{A})^2 + f^2 A^2 + \frac{g^2}{2} (r \dot{\rho})^2 + \frac{g^2}{4} f^2 \rho^2 + \frac{g^2 r^2}{8} (A - B)^2 \rho^2 + \frac{\lambda g^2 r^2}{8} \left( \rho^2 - \frac{2\mu^2}{\lambda} \right)^2 \right\} \]

\[ = \frac{4\pi}{g^2} C_1 (\lambda/g^2) M_W, \]
Figure 4: The $SU(2) \times SU(2)$ monopole solution, where the dashed line represents hypercharge part which describes Bogomol’nyi-Prasad-Sommerfield solution.

\[
E_X = \frac{4\pi}{g'^2} \int_0^\infty dr \left\{ \left( \dot{h} \right)^2 + \frac{(h^2 - 1)^2}{2r^2} + \frac{1}{2}(rB)^2 + h^2B^2 + \frac{g'^2}{2}(r\dot{\sigma})^2 + g'^2h^2\sigma^2 + \frac{\kappa g'^2r^2}{4}(\sigma^2 - \sigma_0^2)^2 \right\} = \frac{4\pi}{g'^2}C_2(\kappa/g'^2)M_X,
\]

where $M_X = g'\sigma_0 = g'\sqrt{m^2/\kappa}$. The boundary conditions for a regular field configuration can be chosen as

\[
\begin{align*}
  f(0) &= h(0) = 1, \quad A(0) = B(0) = \rho(0) = \sigma(0) = 0, \\
  f(\infty) &= h(\infty) = 0, \quad A(\infty) = A_0, \quad B(\infty) = B_0, \quad \rho(\infty) = \rho_0, \quad \sigma(\infty) = \sigma_0. \quad (4.33)
\end{align*}
\]

Notice that this guarantees the analyticity of the solution everywhere, including the origin.

With the boundary condition (4.33) one may try to find the desired solution. From the physical point of view one could assume $M_X \gg M_W$, where $M_X$ is an intermediate scale which lies somewhere between the grand unification scale and the electroweak scale. Now, let $A = B = 0$ for simplicity. Then (4.32) decouples to describes two independent systems so that the monopole solution has two cores, the one with the size $O(1/M_W)$ and the other with the size $O(1/M_X)$. With $M_X = 10M_W$ we obtain the solution shown in Fig.4 in the limit $\lambda = \kappa = 0$. In this limit we find $C_1 = 1.946$ and $C_2 = 1$ so that the energy of the solution is given by

\[
E = \frac{4\pi}{e^2} \left( \cos^2 \theta_w + 0.195\sin^2 \theta_w \right)M_X.
\]
Clearly the solution describes the Cho-Maison monopole whose singularity is regularized by a Prasad-Sommerfield monopole of the size $O(1/M_X)$.

It must be emphasized that, even though the energy of the monopole is fixed by the intermediate scale, the size of the monopole is fixed by the electroweak scale. Furthermore from the outside the monopole looks exactly the same as the Cho-Maison monopole. Only the inner core is regularized by the hypercharged vector field. This tells that the monopole should be interpreted as an electroweak monopole.

V. Conclusions

In this paper we have discussed two ways to regularize the Cho-Maison monopole and dyon solutions of the Weinberg-Salam model, and explicitly constructed genuine finite energy electroweak monopole and dyon solutions which are analytic everywhere including the origin. The finite energy solutions are obtained with a simple modification of the interaction of the $W$-boson or with the embedding of the hypercharge $U(1)$ to a compact $SU(2)$. It has generally been believed that the finite energy monopole must exist only at the grand unification scale [24]. But our result tells that this belief is unfounded, and endorses the existence of a totally new class of electroweak monopole whose mass is much smaller than the monopoles of the grand unification. Obviously the electroweak monopoles are topological solitons which must be stable.

Strictly speaking the finite energy solutions are not the solutions of the Weinberg-Salam model, because their existence requires a modification or generalization of the model. But from the physical point of view there is no doubt that they should be interpreted as the electroweak monopole and dyon, because they are really nothing but the regularized Cho-Maison solutions whose size is fixed at the electroweak scale. In spite of the fact that the Cho-Maison solutions are obviously the solutions of the Weinberg-Salam model one could try to object them as the electroweak dyons under the presumption that the Cho-Maison solutions could be regularized only at the grand unification scale. Our work shows that this objection is groundless, and assures that it is not necessary for us to go to the grand unification scale to make the energy of the Cho-Maison solutions finite. This really reinforces the Cho-Maison dyons as the electroweak dyons which must be taken seriously. Certainly the existence of the finite energy electroweak monopoles should have important physical implications [25].

We close with the following remarks:

1) A most important aspect of our result is that, unlike the original Dirac monopole, the magnetic charge of the electroweak monopoles must satisfy the Schwinger quantization condition $q_m = 4\pi n/e$. Since the Weinberg-Salam model has an unbroken $U(1)_{em}$, one might try to embed the original Dirac monopole with the charge $q_m = 2\pi/e$ to it and obtain the monopole as a classical solution of the Weinberg-Salam model. However, we emphasize that this is possible strictly within the electrodynamics. The electroweak unification simply forbids such an embedding. So within
the framework of the electroweak unification the unit of the magnetic charge must be \(4\pi/e\), not \(2\pi/e\). The existence of a monopole with \(q_m = 2\pi/e\) is simply not compatible with the Weinberg-Salam model. This point has never been well-appreciated before.

2) It has generally been believed that the topological aspects of Weinberg-Salam model and Georgi-Glashow model are quite different, because the Weinberg-Salam model is based on a Higgs doublet but the Georgi-Glashow model is based on a Higgs triplet. Our analysis shows that this is not true. Both of them can be viewed as a gauged \(CP^1\) model which have exactly the same topology \(\pi_2(S^2)\). Furthermore, in the absence of the \(Z\)-boson, even the dynamics becomes very similar. In fact we have shown that, with a simple change of the mass and quartic self-interaction of the \(W\)-boson, the two theories in this case become identical in the limit \(\lambda = 0\). Only the presence of the \(Z\)-boson in Weinberg-Salam model makes them qualitatively different. This point also has not been well-appreciated.

3) The electromagnetic regularization of the Abelian point monopole with the charged vector fields by the interaction (4.6) is nothing new. In fact it is this regularization which makes the energy of the 't Hooft-Polyakov monopole finite. Furthermore it is well-known that the 't Hooft-Polyakov monopole is the only analytic solution (with \(\alpha = \beta = 0\)) which one could obtain with this technique [17]. What we have shown in this paper is that the same technique also works to regularize the Cho-Maison solutions, but with \(\alpha = 0\) and \(\beta = -g^2/(g^2 + g'^2)\).

4) The introduction of the additional interactions (4.6) and (4.21) to the Lagrangian (2.1) could spoil the renormalizability of the Weinberg-Salam model (although this issue has to be examined in more detail). How serious would this offense be, however, is not clear at this moment. The existence of the monopole makes the renormalizability difficult to enforce in the electroweak theory. Here we simply notice that the introduction of a non-renormalizable interaction has been an acceptable practice to study finite energy classical solutions.

5) The embedding of the electroweak \(SU(2) \times U(1)\) to a larger \(SU(2) \times SU(2)\) could naturally arise in the left-right symmetric grand unification models, in particular in the \(SO(10)\) grand unification, although the embedding of the hypercharge \(U(1)\) to a compact \(SU(2)\) may turn out to be too simple to be realistic. Independent of the details, however, our discussion suggests that the electroweak monopoles at an intermediate scale \(M_X\) could be possible in a realistic grand unification.

For a long time it has been asserted that the standard electroweak theory of Weinberg and Salam has no topological properties of interest. Obviously this assertion is not based on the facts. We hope that our paper will correct this misunderstanding once and for all.
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