THE ATTRACTOR AND THE QUANTUM STATES

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Received June 20, 2008

The dissipative dynamics anticipated in the proof of ’t Hooft’s existence theorem – “For any quantum system there exists at least one deterministic model that reproduces all its dynamics after prequantization” – is constructed here explicitly. We propose a generalization of Liouville’s classical phase space equation, incorporating dissipation and diffusion, and demonstrate that it describes the emergence of quantum states and their dynamics in the Schrödinger picture. Asymptotically, there is a stable ground state and two decoupled sets of degrees of freedom, which transform into each other under the energy-parity symmetry of Kaplan and Sundrum. They recover the familiar Hilbert space and its dual. Expectations of observables are shown to agree with the Born rule, which is not imposed a priori. This attractor mechanism is applicable in the presence of interactions, to few-body or field theories in particular.

Keywords: Emergent quantum states; determinism; foundations of quantum mechanics.

1. Introduction

There is a growing number of deterministic models of quantum mechanical objects which are based on conjectured fundamental information loss or dissipation mechanisms [123456].

These studies are largely motivated by the unresolved issues surrounding “quantum gravity”, i.e., by the conflict between quantum mechanics necessitating an external time and diffeomorphism invariance in general relativity, for example, which defies its existence. Furthermore, despite its great successes in describing the statistical aspects of experiments, quantum theory itself presents problems of interpretation, which arise from its indeterministic features and which are clearly seen, for example, in the unresolved measurement problem (also “wave function collapse” or “objective reduction”). Thus, concerning the foundations of quantum mechanics, there is an increasing impetus to try to reconstruct and to better understand the emergence of quantum mechanics from simpler structures beneath.

So far, the construction of models has proceeded case by case. It is guided by the idea that quantum states may actually represent large equivalence classes of deterministically evolving “classical” states – which become indistinguishable when affected by the conjectured information loss or dissipative process.
Furthermore, 't Hooft's existence theorem shows that generally the evolution of all quantum mechanical objects that are characterized by a finite dimensional Hilbert space can be captured by a dissipative process. This has recently been generalized for objects which are described by a set of mutually commuting Hermitian operators, i.e., for the case of a finite set of beables. For completeness, we review these results in the Appendix.

However, it has not been shown before how to complement the existence theorem by a dynamical theory. In order to describe the quantum mechanical world around us, such a theory has to deal with interacting as well as approximately isolated objects that exist throughout a large range of length and energy scales. We make a step in this direction, by showing that familiar aspects of quantum mechanics can be generated by an attractor mechanism, which is obtained by a generalization of the classical Liouville equation.

This will lead us to emerging quantum states which evolve according to the Schrödinger picture embodied in the von Neumann equation, to the Born rule, and to the Randall-Sundrum energy-parity symmetry, which may protect the cosmological constant against far too large corrections, which otherwise should be determined by particle physics scales.

2. A useful reformulation of Hamiltonian dynamics

For simplicity, we consider objects with a single continuous degree of freedom. However, it is straightforward to repeat the following derivations for interacting few-body systems and fields.

To begin with, we assume that there are only conservative forces and that Hamilton's equations are determined by the generic Hamiltonian function:

$$H(x, p) := \frac{1}{2} p^2 + V(x) ,$$

defined in terms of generalized coordinate $x$ and momentum $p$, and where $V(x)$ denotes the potential. An ensemble of such objects, for example, following trajectories with different initial conditions, is described by a distribution function $f$ in phase space, i.e., by the probability $f(x, p; t)dx dp$ to find a member of the ensemble in an infinitesimal volume at point $(x, p)$. This distribution evolves according to the Liouville equation:

$$- \partial_t f = \frac{\partial H}{\partial p} \cdot \frac{\partial f}{\partial x} - \frac{\partial H}{\partial x} \cdot \frac{\partial f}{\partial p} = \{ p \partial_x - V'(x) \partial_p \} f ,$$

with $V'(x) := dV(x)/dx$.

A Fourier transformation, $f(x, p; t) = \int dy e^{-ipy} f(x, y; t)$, replaces the Liouville equation by:

$$i \partial_t f = \{ - \partial_y \partial_x + y V'(x) \} f ,$$
without changing the symbol for the distribution function, whenever changing variables. Thus, momentum is eliminated in favour of doubling the number of coordinates. Finally, with the transformation:

\[ Q := x + y/2 \ , \quad q := x - y/2 \ , \]

we obtain the Liouville equation in the form:

\[ i\partial_t f = \{ \hat{H}_Q - \hat{H}_q + \Delta(Q, q) \} f \ , \]

\[ \hat{H}_\chi := -\frac{1}{2}\partial^2_\chi + V(\chi) \ , \text{ for } \chi = Q, q \ , \]

\[ \Delta(Q, q) := (Q - q)V'(\frac{Q + q}{2}) - V(Q) + V(q) = -\Delta(q, Q) . \]

Several comments are in order here:

• The presented reformulation of classical dynamics is rather independent of the number of degrees of freedom. It applies to matrix valued as well as to Grassmann valued variables, representing the “pseudoclassical” fermion fields introduced by Casalbuoni and by Berezin and Marinov. Field theories require the classical functional formalism employed previously in a related context \[2,11\]. Gauge theories or, generally, theories with constraints have to be examined carefully.

• The Eq. (5) appears as the von Neumann equation for a density operator \( \hat{f}(t) \), considering \( f(Q, q; t) \) as its matrix elements. However, a crucial difference is found in the interaction \( \Delta \) between the bra- and ket- states. The related Hilbert space and its dual, therefore are coupled, unlike the case of quantum mechanics.

• Alternatively, the Eq. (5) might be read as the Schrödinger equation for two identical (sets of) degrees of freedom. However, their respective Hamilton operators, \( \hat{H}_{Q, q} \), contribute with opposite sign. Since their interaction \( \Delta \) is antisymmetric under \( Q \leftrightarrow q \), the complete (Liouville) operator on the right-hand side of Eq. (5) has a symmetric spectrum with respect to zero and, generically, will not be bounded below. This Kaplan-Sundrum energy parity symmetry has been invoked before as a protection for a (near) zero cosmological constant which, otherwise, is threatened by many orders of magnitude too large zeropoint energies \[10,11\].

Only for free particle or harmonic oscillator \( \Delta \) vanishes. – Generally, with a coupling of the Hilbert space and its dual, or without a stable ground state, our reformulation of Hamiltonian dynamics does not qualify as a quantum theory.

The problem of a missing ground state, sometimes in disguise, has been the stumbling block in previous attempts at deterministic model building for quantum objects and has been overcome only in individual examples \[12,13,14,15,16\].
3. An attractor mechanism and emergent quantum features

Based on Hamilton’s equations, we have seen that deterministic ensemble theories imply the absence of a lowest energy state, when suitably rewritten in the form of a Schrödinger equation (or functional Schrödinger equation for field theories). On the other hand, a dissipative process was of paramount importance in the proof of existence of deterministic models for quantum mechanical objects, as reviewed in the Appendix. – In the following, we shall make use of these hints.

We propose an attractor mechanism that turns deterministic evolution, described by an ensemble theory as in Section 2., into the Schrödinger evolution of quantum states, based on a stable ground state. Our construction refers particularly to two assumptions:

(A) The emergence of quantum states originates from a microscopic process beneath which applies to all physical objects.

(B) The statistical interpretation of quantum states (Born rule) originates from the classical ensemble theory.

3.1. Expectations, operators and the Born rule

We begin with the normalization of the classical probability distribution:

$$1 \equiv \int \frac{dx dp}{2\pi} f(x, p; t) = \int dQ dq \delta(Q - q) f(Q, q; t) =: \text{Tr} \hat{f}(t) ,$$ (8)

incorporating the transformations of Section 2. Consider a complete set of orthonormal eigenfunctions of the operator $\hat{H}_\chi$ of Eq. (6), defined by $g_j(\chi) := \exp(-iE_j t)g_j(\chi)$ and $\hat{H}_\chi g_j(\chi) = E_j g_j(\chi)$, respectively, with a discrete spectrum, for simplicity. Then, we may expand $f$:

$$f(Q, q; t) = \sum_{j,k} f_{jk}(t) g_j(Q; t) g^*_k(q; t) .$$ (9)

Employing this, the normalization condition (8) can be stated as:

$$1 \equiv \sum_{j,k} f_{jk}(t) \exp(-i(E_j - E_k)t) \int dQ g_j(Q) g_k(Q) = \sum_j f_{jj}(t) .$$ (10)

Since the classical phase space distribution is real, the expansion coefficients form a Hermitean matrix, $f_{ij} = f^*_{ji}$, which we also denote by $\hat{f}$.

The classical expectation values are calculated as follows:

$$\langle x \rangle := \int \frac{dx dp}{2\pi} xf(x, p; t) = \int dQ dq \delta(Q - q) \frac{Q + q}{2} f(Q, q; t) ,$$ (11)

$$=: \text{Tr} \left( \hat{X} \hat{f}(t) \right) ,$$ (12)

$$\langle p \rangle := \int \frac{dx dp}{2\pi} pf(x, p; t) = \int dQ dq \delta(Q - q) ( -i ) \frac{\partial Q - \partial_q}{2} f(Q, q; t) ,$$ (13)

$$=: \text{Tr} \left( \hat{P} \hat{f}(t) \right) ,$$ (14)
introducing the operators ˆX and ˆP, with matrix elements

\[ X(q, Q) = \delta(Q - q)(Q + q)/2 \]

and

\[ P(q, Q) = -i(\delta(Q - q) \partial_Q - \partial_q \delta(Q - q)) \] (derivatives act left or right, as indicated). Eliminating one of the two integrations in the above equations with the help of the \( \delta \)-functions and suitable partial integrations, these operators are recognized as the coordinate and momentum operators of quantum theory.

We also find, for example:

\[
\text{Tr} \left( \left( \hat{X} \hat{P} + \hat{P} \hat{X} \right) \hat{f}(t) \right) = \frac{i}{2} + 2 \int \frac{dx dp}{2\pi} \, x p f(x, p; t) - \frac{i}{2},
\]

where the cancelling imaginary terms stem from the first and second term on the left-hand side, respectively, which both contribute equally to the integral.

Operators appear here strictly by rewriting classical statistical formulae and not by a quantization rule, such as replacing \( x \) and \( p \) by operators \( \hat{X} \) and \( \hat{P} \), with \([\hat{X}, \hat{P}] = i\), acting on a Hilbert space (not necessarily related to phase space).

Furthermore, the Eqs. (8), (11)–(15) are in accordance with the interpretation of \( f(Q, q; t) \) as matrix elements of a density operator \( \hat{f}(t) \).

However, there is an important caveat: The eigenvalues of normalized quantum mechanical density operators are usually constrained to lie between zero and one, corresponding to the interpretation as standard probabilities. This is not necessarily the case with the operator \( \hat{f} \) obtained from a classical probability distribution. Similarly, the Wigner distribution – obtained from the matrix elements of a quantum mechanical density operator by applying the transformations leading from \( f(x, p) \) to \( f(Q, q) \) in reverse – generally, is not positive semi-definite on phase space, even though its marginal distributions are. Therefore, it does not necessarily qualify as a classical probability density.

The discussion of negative or larger-than-one probabilities is beyond the scope of the present article; we refer to the literature for arguments that make sense of them, see, for example, Refs. 12, 13, 14, 15. However, we anticipate that the proper quantum mechanical aspects of the states shall emerge dynamically, see Section 3.3. We will point out in the following, where the restriction to probabilities that lie in the interval \([0, 1]\) arises. Thus, the application of operators here, together with the density operator in particular, constitute examples for the Born rule, once they are complemented by a suitable dynamical framework.

3.2. Spacetime fluctuations and dissipative dynamics

We will propose a generalization of the conservative classical dynamics described by Eqs. (5)–(7) that incorporates dissipation as well as diffusion. For sufficiently long times, the evolving density operator should be attracted to solutions of the quantum mechanical von Neumann equation:

\[
i \partial_t \hat{f} = [\hat{H}_{\chi}, \hat{f}] .
\]

Equivalently, the expansion coefficients \( f_{ij} \) in Eq. (9) should become constants.
Employing this expansion, generally, the Eq. (5) can be rewritten as matrix equation for the coefficients:

$$i\partial_t f_{jk}(t) = \sum_{l,m} \Delta_{jklm} f_{lm}(t),$$

(17)

with the interaction double matrix $\Delta$ defined by:

$$\Delta_{jklm} := \int dQ dq g_j(Q)g_k(q)\Delta(Q,q)g_l(Q)g_m(q) = -\Delta_{kjml},$$

(18)

employing the antisymmetry of $\Delta$, Eq. (7). Consequently, $i\Delta$ maps a Hermitian matrix, such as $\hat{f}$, to a Hermitian matrix. This map produces zero when taking the trace, $\text{Tr} (\hat{\Delta} \hat{M}) = 0$, for any matrix $\hat{M}$, since:

$$\sum_j \Delta_{jjlm} = 0,$$

(19)

by definition (18), completeness, and Eq. (7). For example, the solution of Eq. (17), $\hat{f}(t) = \exp(-i\Delta t)\hat{f}(0)$, conserves the normalization of $\hat{f}$, Eq. (10).

The interaction $\Delta$ presents the unfamiliar coupling between Hilbert space and its dual, which prevented us from considering Eq. (5) as a truly quantum mechanical equation. We will now present some heuristic considerations which entail important modifications of the deterministic ensemble theory.

For very small intervals, spacetime itself may be thought to have atomistic structure and dynamics, as discussed, for example, in terms of classical causal sets.

Such a locally finite ordered set evolves by sequential growth, i.e., by the random (“sprinkling”) appearance of new set elements. Consider a “hypersurface” formed by the set elements that have no successors – i.e., which do not precede any other set elements, according to the causal order relation – at a certain stage of the evolution. Furthermore, consider the probability distribution $f$ of our coarse grained phase space picture and, in particular, the amount of probability that resides in the volume $V \propto N$ corresponding to $N$ arbitrarily chosen elements in this hypersurface. Following further growth, with a sufficiently large number of new elements added to the causal set, we pick those $N + \delta N$ elements from the evolved hypersurface that either have ancestors among the previously chosen $N$ elements or that are among the previously chosen elements and have not become ancestor to any new element. If $\delta N \neq 0$, this induces a change of probability density, if the amount of probability is conserved, as the volume evolves correspondingly, $V \rightarrow V + \delta V \propto N + \delta N$.

Comparing two different choices of the $N$ elements, say subsets $A$ and $B$ of the considered hypersurface, the corresponding values $\delta N_A$ and $\delta N_B$, generally, will differ, due to the random growth of the causal set. This induces random fluctuations in the probability density.

We represent the induced fluctuations by a homogeneous stochastic term $i\delta H f$ contributing to the right-hand side of Eq. (2). We shall see shortly that it leads to dissipation.
Relatively, such fluctuations become less important, i.e., typical fluctuations decrease, as the discrete spacetime associated with the growing causal set continues to evolve, supposedly towards a continuum limit. Accordingly, we treat $\delta H$ as a random variable, with distribution $\propto \exp(-t\delta H^2/4\epsilon)$ and $\epsilon$ an energy scale. Variability (“aging”) of $\epsilon$, incorporating an initial cut-off of the width, for example, is neglected.

Furthermore, the evolution of $f$ must be modified by diffusion, caused by the randomly growing structure beneath $19$. Somehow reminding of Galton’s board, additional causal set elements keep appearing on which a moving phase space distribution “scatters” and, thus, spreads into different microscopic evolutionary paths.

Asymptotically, diffusion and dissipation may balance each other in such a way that probability is conserved, resulting in a Hermitian matrix $\hat{g}$, with unit trace, to which $\hat{f}$ must be attracted.

These effects are incorporated in our minimalist model:

$$i\partial_t \hat{f}(t) = (\hat{\Delta} + \delta H)(\hat{f}(t) - \hat{g}(t)),$$

which generalizes Eq. (17). The dissipative term stems from $+\delta H f(Q,q)$ entering the right-hand side of Eq. (5), which means adding $+\delta H \sum_{l,m} \delta_{jl}\delta_{km} \hat{f}_{lm}(t) = \delta H f_{jk}(t)$ in Eq. (17). Together with its prefactors, the matrix $\hat{g}$ enters as a source term here.

The solution of the linear first order Eq. (20) is:

$$\hat{f}(t) = e^{-i(\hat{\Delta} + \delta H)t} \left( \hat{f}(0) + i(\hat{\Delta} + \delta H) \int_0^t ds \ e^{i(\hat{\Delta} + \delta H)s} \hat{g}(s) \right).$$

Averaging over the Gaussian fluctuations $\delta H$ gives:

$$\hat{f}(t) = e^{-i(\hat{\Delta} - i\epsilon)t} \left( \hat{f}(0) + i(\hat{\Delta} - i\epsilon) \int_0^t ds \ e^{i(\hat{\Delta} - i\epsilon)s} \hat{g}(s) \right)$$

$$= \hat{g}(t) + e^{-i(\hat{\Delta} - i\epsilon)t} (\hat{f}(0) - \hat{g}(0)) - \int_0^t ds \ e^{-i(\hat{\Delta} - i\epsilon)(t-s)} \partial_s \hat{g}(s),$$

which shows the nonunitary dissipative decay caused by the fluctuations. Taking the trace of Eq. (21) or (22), with the help of Eq. (19) and with $\text{Tr} \ \hat{f}(0) = 1$, however, we find that probability is conserved, i.e., the normalization $\text{Tr} \ \hat{f}(t) = 1$, provided that $\text{Tr} \ \hat{g}(t) = 1$. Thus, the source term compensates the dissipative loss.

Furthermore, if $\hat{g}(t)$ becomes constant sufficiently fast, for large $t \gg 1/\epsilon$, then $\hat{f}(t) \approx \hat{g}(t) \to \hat{g}(\infty)$, asymptotically. In this limit, dissipation effectively eliminates the coupling $\Delta$ and our simplistic account of dissipation/diffusion leads to constant matrix elements $f_{ij}(t) \to g_{ij}(\infty)$. – Via Eq. (9), this implies that the von Neumann Eq. (16) becomes a valid approximation for sufficiently large $t$; i.e., in this limit quantum theory will be recovered.

However, we still need a relation between the matrices $\hat{g}(t)$ and $\hat{f}(t)$, in order to complete the dynamical model and to take into account that the properties of common quantum mechanical objects are highly reproducible.
3.3. Attraction towards quantum states

The minimalist model of Eq. (20) is completed by a nonlinear equation for the source matrix $\hat{g}$:

$$\partial_t \hat{g}(t) = \tau^{-1} (\hat{f}(0) - \langle \hat{f}(0) \rangle \hat{g}(t)) \hat{g}(t),$$  \hspace{1cm} (23)

with $\tau$ a time scale and $\langle \hat{f} \rangle \hat{g} := \text{Tr} (\hat{f} \hat{g}) / \text{Tr} \hat{g}$.

This equation is structurally similar to nonlinear wave equations that have been considered as extensions of the Schrödinger equation, incorporating dissipation or measurement processes into quantum mechanics \textsuperscript{20,21,22}. However, presently we consider it as a phenomenological description of how – following a measurement or state preparation – the source evolves. The diffusive effects, alluded to before, are assumed to result in a source term which reflects the initial condition of the classical theory, represented by $\hat{f}(0)$.

The solution of Eq. (23) is given simply by:

$$\hat{g}(t) = e^{\hat{f}(0)t/\tau} / \text{Tr} e^{\hat{f}(0)t/\tau}. \hspace{1cm} (24)$$

This implies $\text{Tr} \hat{g}(t) = 1$, as needed for the conservation of probability.

For all times, the matrix $\hat{g}(t)$ can be diagonalized by the unitary transformation which diagonalizes the Hermitian matrix $\hat{g}(0)$, $\hat{U} \hat{g}(0) \hat{U}^\dagger = \text{diag}(f'_{11}, f'_{22}, \ldots)$. Furthermore, for all finite initial $\hat{f}(0)$, the eigenvalues of the matrix $\hat{g}(t)$ are positive and, by its unit trace, constrained to lie in $[0, 1]$. In particular, note that initially $\hat{g}(0) = 1 / \text{Tr} \hat{1}$, i.e., the matrix starts out with a homogeneous distribution of eigenvalues (with a regularization to be implemented). This gives no preference to any energy eigenstate, recalling that we are working with the basis of eigenstates of the operator $\hat{H}_\chi$, cf. Section 3.1.

However, for sufficiently large times, the diagonalized matrix $\hat{g}_d(t) := \hat{U} \hat{g}(t) \hat{U}^\dagger$ approaches exponentially fast an onedimensional projector $\hat{P}$:

$$\hat{g}_d(t) \longrightarrow \frac{1}{\sum_j e^{(f'_{jj}-\lambda)t/\tau}} \text{diag} \left( e^{(f'_{11}-\lambda)t/\tau}, \ldots, e^{(f'_{kk}-\lambda)t/\tau}, \ldots \right) \hspace{1cm} (25)$$

$$\longrightarrow \text{diag} (0, \ldots, 1, 0, \ldots) =: \hat{P}, \hspace{1cm} (26)$$

where the only nonvanishing entry appears in the position of the largest eigenvalue of $\hat{g}_d$, $\lambda := \max_k f'_{kk}$, assuming that it is not degenerate, for simplicity. This particular position is indicated by $\bar{l}$, for example, in $P_{jk} = \delta_{\bar{j}\bar{l}} \delta_{\bar{k}\bar{l}}$.

We recall from the previous Section 3.2. that the time dependent matrix $\hat{f}$ of expansion coefficients of Eq. (9) is attracted to the asymptotic value of the source matrix $\hat{g}$, for sufficiently large $t$. Undoing the diagonalizing transformation, which depends on $\hat{f}(0)$, we obtain:

$$\hat{f}(t) \longrightarrow \hat{g}(\infty) = \hat{U}^\dagger \hat{P} \hat{U} \hspace{1cm} (27)$$
Thus, if $\hat{f}(0)$ is diagonal, corresponding to $\hat{U} = \hat{U}^\dagger = 1$, then the distribution $f$ of Eq. (28) is attracted to become a \textit{density matrix representing a stationary state}:

$$f(Q, q; t) \rightarrow \sum_{j,k} P_{jk} g_j(Q; t) g_k^*(q; t) = g_i(Q) g_i(q) ,$$

(28)

which presents a solution of the von Neumann equation [10] with $\partial_t \hat{f} = 0$. All diagonal $\hat{f}(0)$ that have their largest eigenvalue in the same position (referring to the chosen basis), and which are related to classical phase space distributions via Eq. (9), lead to the same density matrix. In this precise sense, a stationary quantum state represents a large equivalence class of classical distributions.

More generally, if $\hat{f}(0)$ is not diagonal, corresponding to $\hat{U} \neq 1$, then the distribution $f$ is attracted to become a \textit{density matrix representing a pure state}:

$$f(Q, q; t) \rightarrow \sum_{j,k} (\hat{U}^\dagger \hat{U})_{jk} g_j(Q; t) g_k^*(q; t) .$$

(29)

Also a solution of Eq. (10), this includes superpositions of stationary states, which are again seen as representing large equivalence classes of classical distributions. Finally, mixed states are formed by ensembles of pure states.

Thus, the quantum states emerge asymptotically, exponentially fast. They are represented by properly normalized density matrices with eigenvalues in $[0, 1]$, which agrees with the standard probability interpretation.

However, this does not exclude initial phase space distributions $f$ which assume nonstandard values according to the underlying ensemble theory, cf. Section 3.1. Here we share a pragmatic point of view: Due to the normalization of a phase space distribution $f$, any negative or larger-than-one probability must be compensated so that the sum over all alternatives of events, to which probabilities are assigned, equals one. Therefore, it must be possible to perform a coarse graining, i.e., a re-partitioning of the set of alternatives, such that nonstandard probabilities are avoided altogether. This amounts to a partitioning of the space of events, e.g., phase space, corresponding to independent alternatives which, in principle, are the ones that can be explored experimentally.

In order to address such issues in detail, a microscopic theory is needed of dissipation and diffusion, due to spacetime fluctuations, and an understanding of how a measurement or state preparation fixes an initial $\hat{f}(0)$ or an equivalence class of such initial state matrices. This may lead to new insights concerning the measurement problem. Its solution must transcend quantum theory, in its usual formulation, as is well known. In particular, the stochastic nature of measurement results might be related to the fact that quantum states represent large equivalence classes of classical phase space distributions, as we have seen, rather than ontological “elements of reality”.

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$\therefore$
4. Conclusions

We have generalized Liouville’s phase space equation for Hamiltonian systems by incorporating an attractor mechanism, comprising a homogeneous stochastic term and a source term. The latter compensates dynamically the dissipative loss of information induced by the former. – For suitably chosen variables, this classical model can be cast into the form of Eqs. (20) and (23).

Its solutions are attracted to density matrices which solve the von Neumann equation. With respect to the classical ensemble theory, these quantum states are recognized as large equivalence classes of states with varying initial conditions.

Motivated by the assumption of an atomistic structure of spacetime – such as represented by a causal set – our model reflects the question: Does quantum mechanics originate, as a phenomenon of coarse graining, from fluctuations which are induced by the growth of discrete spacetime beneath?

Acknowledgments

It is a pleasure to thank A. Khrennikov and V. Manko for discussions and M. Genovese for the invitation to present this work at the 4th Workshop ad memoriam of Carlo Novero “Advances in Foundations of Quantum Mechanics and Quantum Information with Atoms and Photons” (Torino, May 2008).

Appendix: Deterministic models of quantum objects do exist

Having mentioned individual cases of deterministic dynamical models of quantum objects in Section 1, we review here statements about their existence in general.

The existence theorem of ’t Hooft concerns the Schrödinger equation for a quantum system with a $d$-dimensional Hilbert space

$$\frac{d\psi}{dt} = -i\hat{H}\psi,$$

where $\hat{H}$ denotes the Hamiltonian, a $d \times d$ matrix here.

As it turns out, the dynamics of Eq. (30) is reproduced in a deterministic system with two degrees of freedom, one periodic variable, $\varphi \in [0, 2\pi]$, and another real variable, $\omega$, which evolve according to the classical equations of motion:

$$\frac{d\varphi(t)}{dt} = \omega,$$

$$\frac{d\omega(t)}{dt} = -\kappa f(\omega)f'(\omega), \quad f(\omega) := \det (\hat{H} - \omega),$$

where $\kappa > 0$ is a parameter. – It is easy to see that $\omega$ moves exponentially fast towards one of the eigenvalues of $\hat{H}$, since multiplying $f$ by minus one times its derivative $f'$ makes all corresponding zeros attractive (see Figure 1 of Ref. [4] for an illustration). The initial condition for Eq. (32) determines which eigenvalue $E_i$ is approached, resulting in a limit cycle for $\varphi$ with period $T_i = 2\pi\omega_i^{-1} = 2\pi E_i^{-1}$. 
It is useful to introduce two auxiliary operators:
\[
\hat{p}_\varphi := -i \frac{\partial}{\partial \varphi}, \quad \hat{p}_\omega := -i \frac{\partial}{\partial \omega},
\]
which are not related to classical observables. We also define:
\[
\hat{h} := \omega \hat{p}_\varphi - \frac{\kappa}{2} \left(f(\omega)f'(\omega), \hat{p}_\omega\right),
\]
with \(\{x, y\} := xy + yx\). This operator generates the evolution described by the classical equations of motion (31)–(32). Indeed, we can rewrite them as:
\[
\frac{d\varphi(t)}{dt} = -i [\varphi(t), \hat{h}],
\]
\[
\frac{d\omega(t)}{dt} = -i [\omega(t), \hat{h}],
\]
with \([x, y] := xy - yx\). Thus, the operator formalism, which is familiar in quantum theory, turns out to be useful in this classical context as well. The generator \(\hat{h}\) is Hermitian, despite the dissipative character of the equations motion.

The Hilbert space on which these operators act is composed of elements which we call prequantum states. They can be employed as usual, in order to calculate the observable properties of the classical system, which are functions \(O(\varphi, \omega)\).

Let us consider the evolution of those prequantum states \(\psi\) which describe the trajectory of the classical system for an arbitrary but fixed initial condition:
\[
\psi(\varphi, \omega; t) = \sum_n e^{in\varphi} \psi_n(\omega; t) (37)
\]
\[
\underset{t \to \infty}{\longrightarrow} \sum_n e^{in(\varphi - \omega_i t)} \psi_n(\omega_i; 0),
\]
where \(\omega_i\) is the particular fixed point to which \(\omega(t)\) is attracted, depending on its initial condition; the Fourier transformation takes periodicity in the angular variable into account. Then, in a superselection sector where the absolutely conserved “quantum number” \(n\) is fixed to a particular value \(n'\), the prequantum states are directly related to the energy eigenstates of the quantum system described by Eq. (30):
\[
e^{-iEi't} \psi(E_i) = e^{-in'\omega_i t} \psi_n'(\omega_i; 0),
\]
evolving in the usual way, with \(t' := n't\). Probabilistic superpositions of prequantum states with different \(\omega_i\) can be formed and result in pure quantum states showing interference.

In conclusion, characteristics of quantum systems described by Eq. (30) can emerge from the dissipative evolution of deterministic systems beneath.

Next, we consider quantum mechanical objects that require, for a complete characterization of their state, a set of simultaneous eigenvalues of a number of linearly independent and mutually commuting Hermitian operators, \(A_n, n = 1, \ldots, N\), collectively denoted by \(\vec{A} := (A_1, \ldots, A_N)^t\), which are the beables. These operators,
with eigenvalues denoted by $A_{jn}$, $j = 1, \ldots, d$, act linearly on a finite dimensional Hilbert space, corresponding to a finite number of degrees of freedom.

While a particular GL($N$, $\mathbb{R}$) symmetry of beables has been useful in generalizing the previous existence theorem for the case at hand, we assume here for simplicity that the set of beables is fixed.

Then, the proof of existence of a deterministic model accounting for a finite number of beables is analogous to the previous one, Eqs. (31)–(39). The model here comprises $N$ real degrees of freedom which are periodic, $\vec{\varphi} := (\varphi_1, \ldots, \varphi_N)^t$, $\varphi_n \in [0, 2\pi]$, and evolve according to the classical equation of motion:

$$\frac{d\vec{\varphi}(t)}{dt} = \vec{\omega},$$

involving a second set of $N$ real degrees of freedom, $\vec{\omega} := (\omega_1, \ldots, \omega_N)^t$.

While Eq. (40) replaces Eq. (31), presently the vector $\vec{\omega}$ evolves according to the classical equation:

$$\frac{d\vec{\omega}(t)}{dt} = -\kappa \frac{\partial}{\partial \vec{\omega}} F^2(t),$$

with $\kappa > 0$. This equation determines $\vec{\omega}$, once its initial value $\vec{\omega}(t_0)$ is supplied and the function $F$ is given by:

$$F(t) := \sum_{n=1}^N \det^2 (\vec{A} - \vec{\omega}(t))_n,$$

where the sum is over the components of the vector inside (…) and the determinant refers to the Hilbert space on which the operators act that are collected in $\vec{A}$.

We remark that operators and Hilbert space have only been introduced for convenient book keeping. Essentially needed, so far, are the real numbers $A_{jn}$ which parametrize $F$, in analogy to the energy eigenvalues entering Eqs. (32). An important property of beables is that related eigenvalues are invariant under unitary transformations in Hilbert space. Therefore, the function $F$ had to be a scalar under such transformations. Other symmetry aspects are discussed in Ref. 8.

Furthermore, the sum of squares of determinants in Eq. (42) is zero, if and only if the $N$-dimensional vector $\vec{\omega}$ corresponds to one of the points of the $N$-dimensional finite lattice defined by $d \times N$ numbers $A_{jn}$, i.e., by the $d$ eigenvalues of each one of the $N$ operators $\vec{A}$. In this way, the Eqs. (41)–(42) generalize the Eqs. (32). As in the previous case, the zeros of $F$ are attractive. Thus, the vector $\vec{\omega}$ is attracted to a fixed vector with components determined by eigenvalues of the operators $\hat{A}_n$:

$$\omega_n(t) \xrightarrow{t \to \infty} A^{(n)}_n =: \omega^*_n.$$

Which particular eigenvalues contribute, indexed by $j(n)$, $j = 1, \ldots, d$, depends on the arbitrary initial condition for $\vec{\omega}$.

Finally, the considerations of Eqs. (37)–(39) are easily generalized. We consider prequantum states $\psi$ which describe the trajectory of the deterministic system:

$$\psi(\vec{\varphi}, \vec{\omega}^*; t) = \sum_{n} e^{i\vec{n} \cdot \vec{\varphi}} \psi_n(\vec{\omega}^*; 0),$$

(44)
where the fixed point $\mathcal{J}^*$ for a given initial condition and periodicity in $\mathcal{E}$ are taken into account. All components of the vector $\mathcal{J}^*$ contribute to the phase of the evolving state, i.e., all $N$ Hermitian operators $\hat{A}_n$ contribute, each with one of its set of $d$ eigenvalues $A^j_n$. The states fall into superselection sectors that can be classified by the absolutely conserved vector $\vec{n}$.

Three qualitatively different situations may arise. – First, the model universe may find itself in a state where all components of $\vec{n}$ are equal, denoted by $\vec{n}' \equiv (n'_1, \ldots, n'_N)$. Here, the emergent Hamiltonian must be identified as:

$$\hat{H} = \sum_{n=1}^{N} \hat{A}_n ,$$

with an eigenvalue $E_* = \sum_n A^j_n = \sum_n \omega^*_n$ corresponding to a particular initial condition for the deterministic trajectory. The quantum states are related to the prequantum states by:

$$e^{-iE_*t'} \psi(E_*) = e^{-in'_t} \sum_n \omega^*_n \psi(\vec{\omega}^*; 0) ,$$

with $t' := n't$, cf. Eq. (39). One of the beables, corresponding to $\omega^*_1$, for example, could be eliminated in favour of the Hamiltonian and $E_*$, respectively, such that the above relation becomes $\psi(E_*) \propto \psi(\vec{\omega}^*; 0)$. Thus, we find degenerate energy eigenstates, which are further resolved by the eigenvalues of the $N - 1$ remaining beables, i.e., by the values of $\omega^*_2, \ldots, \omega^*_N$. – Second, assuming that all $\omega^*_n$ are of the same order of magnitude, one of the components of the superselection vector $\vec{n}$, say $n_1$, may be very much larger than all others. In this case, we consider the Hamiltonian:

$$\hat{H}_1 = \hat{A}_1 ,$$

with eigenvalues $E_* = \omega^*_1$, which presents a valid approximation, as long as only sufficiently small eigenvalues $\omega^*_{n>1}$ have to be taken into account. In this case:

$$e^{-iE_*t'} \psi(E_*) = e^{-in_1t} \sum_{n} \omega^*_n \psi(\vec{\omega}^*; 0) ,$$

with $t' := n_1t$. That is, one contribution to the phase is dominant; this leads to degenerate energy eigenstates, to be resolved as before. – There will be only accidental degeneracies, if any, in the third case, when all beables possibly contribute:

$$\hat{H}_{alt} = \vec{n} \cdot \vec{A} ,$$

with eigenvalues of the form $E_* = \vec{n} \cdot \vec{\omega}^*$. Here, we obtain:

$$e^{-iE_*t} \psi(E_*) = e^{-in \vec{n} \cdot \vec{\omega}^*} \psi(\vec{\omega}^*; 0) .$$

One of the eigenvalues $\omega^*_k$ could be replaced by $E_* - \sum_{m \neq k} n_m \omega^*_m / n_k$, provided $n_k \neq 0$. Thus, in this most general case, there still exist a unique Hamiltonian and a related energy variable, which govern the evolution of the emergent states.

This completes our review of the existence theorems \cite{7,8}. They do not present constructive theorems, since spectral information is needed as input. In the main part of this article, we aim for a constructive theory instead.
References

1. G. ’t Hooft, *A mathematical theory for deterministic quantum mechanics*, J. Phys.: Conf. Ser. 67 (2007) 012015; arXiv:quant-ph/0604008; *dito*, Int. J. Theor. Phys. 42 (2003) 355-361; *dito*, Class. Quant. Grav. 16 (1999) 3263-3279.

2. H.-T. Elze, *Deterministic models of quantum fields*, J. Phys.: Conf. Ser. 33 (2006) 399-404; arXiv:gr-qc/0512016; *dito*, Braz. J. Phys. 35 (2005) 343-350; *dito*, Phys. Lett. A310 (2003) 110-118.

3. M. Blasone, P. Jizba and H. Kleinert, *Quantum behavior of deterministic systems with information loss. Path integral approach*, Ann. Phys. 320 (2005) 468-486; arXiv:quant-ph/0504200; *dito*, Braz. J. Phys. 35 (2005) 497-502; *dito*, Phys. Rev. A71 (2005) 052507.

4. S. L. Adler, *“Quantum Mechanics as an Emergent Phenomenon”* (Cambridge Univ. Press, Cambridge, 2005).

5. F. Markopoulou and L. Smolin, *Quantum theory from quantum gravity*, Phys. Rev. D70 (2004) 124029; arXiv:gr-qc/0311059.

6. M. Blasone, P. Jizba and G. Vitiello, *Dissipation and quantization*, Phys. Lett. A287 (2001) 205-210; arXiv:hep-th/0007138.

7. G. ’t Hooft, *Emergent quantum mechanics and emergent symmetries*, presented at PASCOS 13, Imperial College, London, July 6, 2007; arXiv:0707.4568.

8. H.-T. Elze, *Note on the existence theorem in “Emergent quantum mechanics and emergent symmetries”*, J. Phys. A: Math. Theor. (2008), to appear; arXiv:0710.2765.

9. J.S. Bell, *“Speakable and Unspeakable in Quantum Mechanics”*, 2nd edition (Cambridge Univ. Press, Cambridge, 2004).

10. D. E. Kaplan and R. Sundrum, *A symmetry for the cosmological constant*, JHEP 0607 (2006) 042; arXiv:hep-th/0505265.

11. H.-T. Elze, *Quantum fields, cosmological constant and symmetry doubling*, Int. J. Theor. Phys. 46, No 8 (2007) 2063-2081; arXiv:hep-th/0510267.

12. P.A.M. Dirac, *The physical interpretation of quantum mechanics*, Proc. Roy. Soc. London A180 (1942) 1-40.

13. R.P. Feynman, *Negative probability*, in: “Quantum Implications: Essays in Honour of David Bohm, ed. B.J. Hiley and F.D. Peat (Routledge and Kegan Paul, London and New York, 1987) 235-248.

14. A. Khrennikov, *Interpretations of Probability* (VSP, Utrecht and Boston, 2003).

15. J.B. Hartle, *Quantum mechanics with extended probabilities*, arXiv:0801.0688.

16. R.D. Sorkin, *Causal Sets: Discrete Gravity* in: “Proceedings of the Valdivia Summer School”, Jan. 2002, A. Gomberoff and D. Marolf (eds.), arXiv:gr-qc/0309009.

17. F. Dowker, *Causal sets and the deep structure of spacetime* in: “100 Years of Relativity - Space-Time Structure: Einstein and Beyond”, A. Ashtekar (ed.) (World Scientific, Singapore 2005), arXiv:gr-qc/0508109.

18. D.P. Rideout and R.D. Sorkin, *A classical sequential growth dynamics for causal sets*, Phys. Rev. D61 (2000) 024002, arXiv:gr-qc/9904062.

19. F. Dowker, J. Henson and R.D. Sorkin, *Quantum gravity phenomenology, Lorentz invariance and discreteness*, Mod. Phys. Lett. A19 (2004) 1829-1840, arXiv:arXiv:gr-qc/0311055v3.

20. N. Gisin, *A simple nonlinear dissipative quantum evolution equation*, J. Phys. A: Math. Gen. 14 (1981) 2259-2267.

21. A.N. Grigorenko, *Measurement description by means of a nonlinear Schrödinger equation*, J. Phys. A: Math. Gen. 28 (1995) 1459-1466.

22. N. Gisin and M. Rigo, *Relevant and irrelevant nonlinear Schrödinger equations*, J. Phys. A: Math. Gen. 28 (1995) 7375-7390.