CONNECTIONS OVER TWISTED TENSOR PRODUCTS OF ALGEBRAS

JAVIER LÓPEZ PEÑA

ABSTRACT. Motivated by some results in classical differential geometry, we give a constructive procedure for building up a connection over a (twisted) tensor product of two algebras, starting from connections defined on the factors. The curvature for the product connection is explicitly calculated, and shown to be independent of the choice of the twisting map and the module twisting map used to define the product connection. As a consequence, we obtain that a product of two flat connections is again a flat connection. We show that our constructions also behaves well with respect to bimodule structures, namely being the product of two bimodule connections again a bimodule connection. As an application of our theory, all the product connections on the quantum plane are computed.

INTRODUCTION

One of the main tools in classical differential geometry is the use of the tangent bundle associated to a manifold. The role of the algebra of functions on the manifold is taken by the sections of the tangent bundle, namely, the vector fields. As a dual of the vector fields space, the algebra of differential forms (endowed with the exterior product) turns out to be an useful tool in the study of global properties of the manifold, giving rise to invariants such as the de Rham cohomology. A problem arises when trying to compare vector fields and differential forms at different points of the manifold, the solution to it being given by the concepts of (linear) connection and covariant derivative, that allow us to define the derivative of a curve on a point of orders higher than one, hence giving us a way to speak about accelerations on a path. The notion of connection also has another meanings in physics, like the existence of an electromagnetic potential, which is equivalent to the existence of a connection in a rank one trivial bundle with fixed trivialization.

Jean–Louis Koszul gave in [Kos60] a powerful algebraic generalization of differential geometry, in particular giving a completely algebraic description of the notion of connection. These notions were extended to a noncommutative framework by Alain Connes in [Con86], what meant the dawn of noncommutative differential geometry. Much research has been done about the theory of connections in this context. On the one hand, Joachim Cuntz and Daniel Quillen, in their seminal paper [CQ95] started the theory of quasi-free algebras (also named formally smooth by Maxim Kontsevich or quives

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by Lieven Le Bruyn), opening the way to an approach to noncommutative (algebraic) geometry (also dubbed \textit{nongeometry} to avoid confusions with Michael Artin and Michel Van den Bergh’s style of noncommutative algebraic geometry). These formally smooth algebras are characterized by the projectiveness (as a bimodule) of the first order universal differential calculus, or equivalently as those algebras that admit a universal linear connection. On the other hand, in Connes’ style of noncommutative geometry, the study of the general theory of connections leads to the definition of the Yang–Mills action, which turns out to be nothing but the usual gauge action when we specialize it to the commutative case (cf. \cite{Con86}, \cite{Lan97}, \cite{GBVF01} and references therein).

In this paper, we deal with the problem of building up products of those connection operators. Basically, there are two different notions of \textit{“product connection”} that one might want to build. Firstly, one might want to consider two different bundles over a manifold, each of them endowed with a connection, and then try to build a product connection on the (fibre) product bundle. A noncommutative version of this construction was given by Michel Dubois–Violette and John Madore in \cite{DV99}, \cite{Mad95}. Further steps on this direction, including its relations with the realization of vector fields as Cartan pairs as proposed by Andrzej Borowiec in \cite{Bor96}, have been given by Edwin Beggs in \cite{Beg}. The other possible notion of product connection, and the one with which we want to deal, refers to the consideration of the cartesian product of two given manifolds, and the building of a connection of the bundle associated to this product manifold.

Traditionally, when taking the passage from classical geometry to (noncommutative) algebra, the product space is associated with some kind of tensor product (the algebraic tensor product in the case of algebraic varieties, the topological tensor one when dealing with topological manifolds). In \cite{CSV95}, Andreas Cap, Herman Schichl and Jiří Vanžura pointed out the limitations of these approach and proposed a definition of \textit{“noncommutative cartesian product”} of spaces by means of the so–called \textit{twisted tensor product} of the algebras. A twisted tensor product is a particular case of the notion of \textit{distributive law} given by Jon Beck in \cite{Bec69}, and may be regarded as a sort of local version of a braiding in a braided monoidal category. Some further insights on the interpretation of this algebraic construction from a geometrical point of view has been given by Alfons Van Daele and S. Van Keer in \cite{VDVK94}, Stanislaw Woronowicz in \cite{Wor96}, and the author and some collaborators in \cite{JMLPPVO} (cf. also \cite{LPVO} for an interpretation of twisted tensor products from a deformation theory point of view). Following these ideas, we will show how to build a product connection on a twisted tensor product of two algebras.

In Section 1 we recall the notion of a (right) connection on an algebra, given as an operator $\nabla : E \rightarrow E \otimes_A \Omega^1 A$, where $E$ is a (right) $A$–module and $\Omega^1 A$ a first order differential calculus over $A$, motivating our choice of the differential calculus and the modules on the building of the product connection, for which an explicit formula is given in the simplest case of the usual tensor product. Then, we recall the main notions we need about twisted tensor products, defined by means of twisting maps $R : B \otimes A \rightarrow A \otimes B.$
In Section 2 we give the basic definition of our object of study, and prove that our definition actually yields a connection in the product space, and that this connection boils down nicely to the classical product connection in the commutative case.

To a connection we can always associate its curvature operator, obtained by squaring the extension of the connection to the whole differential calculus. The curvature operator leads to the definition of flat connections as those having 0 curvature. Flat connections have been used by Philippe Nuss in [Nus97] in relation with noncommutative descent theory, and also by Edwin Beggs and Tomasz Brzezinski in [BB05], where they are interpreted as the differential of a certain complex in order to build a noncommutative de Rham cohomology with coefficients. We deal with the problem of describing the curvature of our product connection in Section 3, stating our main theorem, that gives us an explicit formula to compute the curvature for the product connection in terms of the curvatures of the factors:

\[ \theta(e \otimes b, a \otimes f) = i_E(\theta^E(e)) \cdot b + a \cdot i_F(\theta^F(f)). \]

The most striking consequence of this theorem is the fact that the curvature does not depend neither on the twisting map \( R \) nor on the module twisting map that we use to get the module structure, suggesting that the curvature remains invariant under all the deformations obtained by means of a twisted tensor product. As an immediate corollary, we have that the product of two flat connections is again a flat connection.

In Section 4 we consider bimodule connections (in the sense introduced by Jihad Mourad in [Mon95]) instead of one sided connections, and we find necessary and sufficient conditions for the product of two bimodule connections to be a bimodule connection. We conclude, in Section 5, by illustrating our theory giving a complete description of all the product connections on the quantum plane \( k_q[x, y] \).

1. Preliminaries

1.1. Connections on algebras.

Let \( A \) be an associative, unital algebra over a field \( k \), and \( \Omega A = \bigoplus_{p \geq 0} \Omega^p A \) a differential calculus over \( A \), that is, a differential graded algebra generated, as a differential graded algebra, by \( \Omega^1 A \cong A \), with differential \( d = d_A \). Let \( E \) be a (right) \( A \)-module; a (right) connection on \( E \) is a linear mapping

\[ \nabla : E \longrightarrow E \otimes_A \Omega^1 A \]

satisfying the (right) Leibniz rule:

\[ \nabla(s \cdot a) = (\nabla s) \cdot a + s \otimes da \quad \forall \ s \in E, a \in A. \]

Under these conditions, the mapping \( \nabla \) can be extended in a unique way to an operator

\[ \nabla : E \otimes_A \Omega A \longrightarrow E \otimes_A \Omega A \]

of degree 1, by setting

\[ \nabla(s \otimes \omega) = \nabla s \otimes \omega + (-1)^p s \otimes d\omega \quad \forall \ s \in E, \omega \in \Omega^p A, \]
where we are using the identification \((E \otimes_A \Omega^1 A) \otimes_A \Omega^n A \cong E \otimes_A \Omega^{n+1} A\). Regarding \(E \otimes_A \Omega A\) as a right \(\Omega A\)-module, we find that the following graded Leibniz rule is satisfied:

\[
\nabla (\sigma \omega) = (\nabla \sigma) \omega + (-1)^p \sigma d\omega \quad \forall \sigma \in E \otimes_A \Omega^p A, \omega \in \Omega A.
\]

There are analogous concepts for left modules.

Usually, we will be interested on working with the universal differential calculus over an algebra \(A\). Connections over the universal differential calculus will be called \textit{universal connections}. It is a well known fact (cf. \cite{CQ95}, Corollary 8.2) that a right \(A\)-module admits a universal connection if, and only if, it is projective over \(A\).

Whenever \(A\) is a commutative algebra, the tensor product \(E \otimes_A F\) of two \(A\)-modules \(E\) and \(F\) is again an \(A\)-module. If \(E\) and \(F\) carry respective connections \(\nabla^E\) and \(\nabla^F\), we may build the \textit{tensor product connection} on \(E \otimes_A F\) by defining

\[
\nabla^{E \otimes_A F} := \nabla^E \otimes F + E \otimes \nabla^F.
\]

A possible generalization of this construction was given by Dubois–Violette and Madore in \cite{DV99}, \cite{Mad95}. If \(E\) and \(F\) are \(A\)-bimodules equipped with right connections \(\nabla^E\) and \(\nabla^F\), and such that there exists a linear mapping

\[
\sigma : \Omega^1 A \otimes_A F \rightarrow F \otimes_A \Omega^1 A
\]

satisfying that

\[
\nabla^F(am) = a \nabla^F(m) + \sigma(da \otimes_A m) \quad \forall a \in A, m \in F,
\]

then we may define

\[
\nabla^{E \otimes_A F} : E \otimes_A F \rightarrow E \otimes_A F \otimes \Omega^1 A
\]

by setting

\[
\nabla^{E \otimes_A F} := (E \otimes \sigma) \circ (\nabla^E \otimes F) + E \otimes \nabla^F,
\]

and this \(\nabla^{E \otimes_A F}\) is a right connection on \(E \otimes_A F\).

Our aim is to define a different kind of “product connection” with a more geometrical flavour. Namely, consider that our algebras \(A = C^\infty(M)\) and \(B = C^\infty(N)\) represent the algebras of functions over certain manifolds \(M\) and \(N\), and that \(E = \mathfrak{X}(M)\) and \(F = \mathfrak{X}(N)\) are the modules of vector fields on the manifolds. The algebra associated to the cartesian product of the manifolds is \(C^\infty(M \times N) \cong C^\infty(M) \otimes C^\infty(N)\) (more precisely, a suitable completion of the latest). For the modules of vector fields and differential 1-forms, we have that

\[
\mathfrak{X}(M \times N) \cong \mathfrak{X}(M) \otimes C^\infty(N) \oplus C^\infty(M) \otimes \mathfrak{X}(N),
\]

\[
\Omega^1(C^\infty(M) \otimes C^\infty(N)) \cong \Omega^1(C^\infty(M)) \otimes C^\infty(N) \oplus C^\infty(M) \otimes \Omega^1(C^\infty(N)),
\]

hence, a “product connection” of two connections defined on \(E\) and \(F\) should be defined as a linear mapping

\[
\nabla : E \otimes B \oplus A \otimes F \rightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)
\]
Firstly, realize that if $E$ is a right (resp. left) $A$–module, and $F$ is a right (resp. left) $B$–module, then $E \otimes B \oplus A \otimes F$ is a right $(A \otimes B)$–module, with actions
\[
(e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta) := (e\alpha \otimes b\beta, a\alpha \otimes f\beta)
\]
(resp. $(\alpha \otimes \beta) \cdot (e \otimes b, a \otimes f) := (\alpha e \otimes \beta b, \alpha a \otimes \beta f)$)

For simplicity, we will only work with right connections. Left connections admit a similar treatment.

1.2. **Product Connection.**

Suppose then that $E$ is a right $A$–module endowed with a (right) connection $\nabla^E$, and that $F$ is a right $B$–module endowed with a (right) connection $\nabla^F$. Let us consider the mappings
\[
\nabla_1 : E \otimes B \rightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B), \\
\nabla_2 : A \otimes F \rightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)
\]
respectively given by
\[
\nabla_1 := (E \otimes \tau \otimes u_B) \circ (\nabla^E \otimes B) + (E \otimes u_A \otimes u_B \otimes \Omega^1 B) \circ (E \otimes d_B), \text{ and}
\]
\[
\nabla_2 := (A \otimes F \otimes u_A \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes \tau \otimes u_B) \circ (d_A \otimes F),
\]
where $\tau$ represent classical flips. If we use the shorthand notation $\nabla^E(e) = e_i \otimes d_A a_i$, where the summation symbol is omitted, the Leibniz rule for $\nabla^E$ is written as
\[
(1.7) \quad \nabla^E(e \alpha) = e_i \otimes (d_A a_i) \alpha + e \otimes d\alpha,
\]
and we have that
\[
\nabla_1((e \otimes b) \cdot (\alpha \otimes \beta)) = \nabla_1(e \alpha \otimes b\beta) =
\]
\[
= e_i \otimes b\beta \otimes_{A \otimes B} (d_A a_i) \alpha \otimes 1 + e \otimes b\beta \otimes_{A \otimes B} d\alpha \otimes 1 +
\]
\[
+ e \alpha \otimes 1 \otimes_{A \otimes B} 1 \otimes d_B(b\beta) =
\]
\[
= e_i \otimes b \otimes_{A \otimes B} (d_A a_i) \alpha \otimes \beta + e \otimes b \otimes_{A \otimes B} d\alpha \otimes \beta +
\]
\[
+ e \otimes 1 \otimes_{A \otimes B} \alpha \otimes d_B(b\beta) + e \otimes 1 \otimes_{A \otimes B} \alpha \otimes bd_B\beta =
\]
\[
= (e_i \otimes b \otimes_{A \otimes B} d_A a_i \otimes 1 + e \otimes 1 \otimes_{A \otimes B} 1 \otimes db) \cdot (\alpha \otimes \beta) +
\]
\[
+ e \otimes b \otimes_{A \otimes B} d_A \alpha \otimes \beta + e \otimes b \otimes_{A \otimes B} d\alpha \otimes d_B\beta =
\]
\[
= \nabla_1(e \otimes b) \cdot (\alpha \otimes \beta) + (e \otimes b) \otimes_{A \otimes B} d(\alpha \otimes \beta).
\]
A similar computation shows that
\[
\nabla_2((a \otimes f) \cdot (\alpha \otimes \beta)) = \nabla_2(a \otimes f) \cdot (\alpha \otimes \beta) + (a \otimes f) \otimes_{A \otimes B} d(\alpha \otimes \beta).
\]
Adding up these two equalities, we conclude that the map
\[
\nabla : E \otimes B \oplus A \otimes F \rightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)
\]
\[
(e \otimes b, a \otimes f) \mapsto \nabla_1(e \otimes b) + \nabla_2(a \otimes f)
\]
verifies that
\[ \nabla((e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta)) = \nabla(e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta) + (e \otimes b, a \otimes f) \otimes_{A \otimes B} d(\alpha \otimes \beta), \]
and henceforth, \( \nabla \) is a (right) connection on the module \( E \otimes B \oplus A \otimes F \). We shall call this map the \textbf{(classical) product connection} of \( \nabla^E \) and \( \nabla^F \).

1.3. Twisted tensor products.

Let \( k \) be a field, used as a base field throughout. We denote \( \otimes_k \) by \( \otimes \), the identity \( id_V \) of an object \( V \) simply by \( V \), and by \( \tau : V \otimes W \to W \otimes V \), \( \tau(v \otimes w) = w \otimes v \), the usual flip. All algebras are assumed to be associative unital \( k \)-algebras; the multiplication and unit of an algebra \( D \) are denoted by \( \mu_D : D \otimes D \to D \) and respectively \( u_D : k \to D \) (or simply by \( \mu \) and \( u \) if there is no danger of confusion).

We recall the twisted tensor product of algebras from \cite{Tam90}, \cite{VDVK94}, \cite{CSV95}. If \( A \) and \( B \) are two algebras, a linear map \( R : B \otimes A \to A \otimes B \) is called a \textbf{twisting map} if it satisfies the conditions

\begin{align}
(1.8) \quad & R(b \otimes 1) = 1 \otimes b, \quad R(1 \otimes a) = a \otimes 1, \quad \forall a \in A, \, b \in B, \\
(1.9) \quad & R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A), \\
(1.10) \quad & R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R).
\end{align}

If we denote by \( R(b \otimes a) = a_R \otimes b_R \), for \( a \in A, \, b \in B \), then \((1.9)\) and \((1.10)\) may be written as:

\begin{align}
(1.11) \quad & (aa')_R \otimes b_R = a_R a'_R \otimes (b_R)_r, \\
(1.12) \quad & a_R \otimes (bb')_R = (a_R)_r \otimes b_r b'_R,
\end{align}

for all \( a, a' \in A \) and \( b, b' \in B \), where \( r \) is another copy of \( R \). If we define a multiplication on \( A \otimes B \), by \( \mu_R = (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B) \), that is

\begin{align}
(1.13) \quad & (a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b',
\end{align}

then this multiplication is associative and \( 1 \otimes 1 \) is the unit. This algebra structure is denoted by \( A \otimes_R B \) and is called the \textbf{twisted tensor product} of \( A \) and \( B \). This construction works also if \( A \) and \( B \) are algebras in an arbitrary monoidal category.

If \( A \otimes_{R_1} B, \, B \otimes_{R_2} C \) and \( A \otimes_{R_3} C \) are twisted tensor products of algebras, the twisting maps \( R_1, \, R_2, \, R_3 \) are called \textbf{compatible} if they satisfy

\[ (A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A), \]
see \cite{JMLPPVO}. If this is the case, the maps \( T_1 : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C \) and \( T_2 : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C) \) given by \( T_1 := (A \otimes R_2) \circ (R_3 \otimes B) \) and \( T_2 := (R_1 \otimes C) \circ (B \otimes R_3) \) are also twisting maps and \( A \otimes_{T_1} (B \otimes_{R_2} C) \equiv (A \otimes_{R_1} B) \otimes_{T_2} C; \) this algebra is denoted by \( A \otimes_{R_1} B \otimes_{R_2} C \). This construction may be iterated to an arbitrary number of factors, see \cite{JMLPPVO} for complete detail.
When we have a left $A$–module $M$, a left $B$–module $N$, a twisting map $R : B \otimes A \to A \otimes B$ and a linear map $\tau_{M,B} : B \otimes M \to M \otimes B$ such that
\begin{align}
\tau_{M,B} \circ (\mu_B \otimes M) &= (M \otimes \mu_B) \circ (\tau_{M,B} \otimes B) \circ (B \otimes \tau_{M,B}), \\
\tau_{M,B} \circ (B \otimes \lambda_M) &= (\lambda_M \otimes B) \circ (A \otimes \tau_{M,B}) \circ (R \otimes M),
\end{align}
then the map $\lambda_{\tau_{M,B}} : (A \otimes R) B \otimes (M \otimes N) \to M \otimes N$ defined by $\lambda_{\tau_{M,B}} := (\lambda_M \otimes \lambda_N) \circ (A \otimes \tau_{M,B} \otimes N)$ yields a left $(A \otimes R) B$–module structure on $M \otimes N$, which furthermore is compatible with the inclusion of $A$. In this case, we say that $\tau_{M,B}$ is a \textit{(left) module twisting map}. Unlike what happens for algebra twisting maps, usually is not enough to have a left $(A \otimes R) B$–module structure on $M \otimes N$ in order to recover a module twisting map. Some sufficient conditions for this to happen are, for instance, requiring that $M$ is projective and $N$ is faithful (cf. \cite[Theorem 3.8]{CSV95}).

Similarly, if we have a twisting map $R : B \otimes A \to A \otimes B$, a right $A$–module $M$ and a right $B$–module $N$, a linear map $\tau_{N,A} : N \otimes A \to A \otimes N$ such that
\begin{align}
\tau_{N,A} \circ (N \otimes \mu_A) &= (\mu_A \otimes N) \circ (A \otimes \tau_{N,A}) \circ (\tau_{N,A} \otimes A) \\
\tau_{N,A} \circ (\rho_B \otimes A) &= (A \otimes \rho_B) \circ (\tau_{N,A} \otimes B) \circ (N \otimes R),
\end{align}
then the map $\rho_{\tau_{N,A}} := (\rho_A \otimes \rho_B) \circ (M \otimes \tau_{N,A} \otimes B)$, yields a right $(A \otimes R) B$–module action on $M \otimes N$. In this case, we call $\tau_{N,A}$ a \textit{(right) module twisting map}.

Twisting maps also have a nice behaviour with respect to (universal) differential calculi. More concretely, we have the following result (cf. \cite{CSV95}):

\begin{theorem}
Let $A, B$ be two algebras. Then any twisting map $R : B \otimes A \to A \otimes B$ extends to a unique twisting map $\tilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ which satisfies the conditions
\begin{align}
\tilde{R} \circ (d_B \otimes \Omega A) &= (\varepsilon_A \otimes d_B) \circ \tilde{R}, \\
\tilde{R} \circ (\Omega B \otimes d_A) &= (d_A \otimes \varepsilon_B) \circ \tilde{R},
\end{align}
where $d_A$ and $d_B$ denote the differentials on the algebras of universal differential forms $\Omega A$ and $\Omega B$, and $\varepsilon_A, \varepsilon_B$ stand for the gradings on $\Omega A$ and $\Omega B$, respectively. Moreover, $\Omega A \otimes \tilde{R} \Omega B$ is a graded differential algebra with differential $d(\varphi \otimes \omega) := d_A \varphi \otimes \omega + (-1)^{|\varphi|} \varphi \otimes d_B \omega$.
\end{theorem}

\section{Twisted tensor product connection}

In the former section we introduced the definition of a connection within the formalism of differential calculus over algebras, and showed how to build the product connection for a tensor product of two algebras, extending the definition of the classical product connection in differential geometry. In \cite{JMLPPVO}, we advocated that in noncommutative geometry the cartesian product should not be replaced at the algebraic level by the usual tensor product of algebras, but by a deformation of it, known as the twisted tensor product. In this section, we will show how to extend the definition of the product connection to a twisted tensor product of two algebras under suitable conditions.

Let $A$ and $B$ be algebras, $R : B \otimes A \to A \otimes B$ a twisting map, $E$ a right $A$–module endowed with a right connection $\nabla^E$, and $F$ a right $B$–module endowed with a right
connection $\nabla^F$. In [CSV95] it is shown that we can lift the twisting map $R$ to a twisting map $\tilde{R} : \Omega_B \otimes \Omega_A \to \Omega_A \otimes \Omega_B$ on the graded differential algebras of (universal) differential forms, and that the algebra

$$\Omega A \otimes_R \Omega B = \bigoplus_{n \in \mathbb{N}} \left( \bigoplus_{p+q=n} \Omega^p A \otimes \Omega^q B \right)$$

is a differential calculus over $A \otimes_R B$. For this differential calculus, the module of 1–forms can be identified as $\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B$, with the natural action induced by the twisting map. As the situation is pretty much the same as in the tensor product case, the natural way for defining a “twisted product” connection of $\nabla^E$ and $\nabla^F$ would be considering a linear map

$$\nabla : E \otimes B \oplus A \otimes F \to (E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B).$$

The first step on making this map becoming a connection is giving a right $(A \otimes_R B)$–module action on $E \otimes B \oplus A \otimes F$, which means finding a right $(A \otimes_R B)$–module structure on both $E \otimes B$ and $A \otimes F$. For the first one we may just use the twisting map and define:

$$(e \otimes b) \cdot (\alpha \otimes \beta) := e \alpha_R \otimes b_R \beta.$$ (2.1)

For the second one, a sufficient way of giving a module structure is finding a (right) module twisting map $\tau_{F,A} : F \otimes A \to A \otimes F$, and then taking

$$(a \otimes f) \cdot (\alpha \otimes \beta) := a \alpha_f \otimes f \beta.$$ (2.2)

The fact that the former definitions are indeed module actions follows directly from the fact that both $R$ and $\tau_{F,A}$ are right module twisting maps (cf. [CSV95], 3.12).

Following the lines given by the definition of the classical tensor product connection, in order to build $\nabla$ we have to find suitable maps $\nabla_1$ and $\nabla_2$. For the first one, it suffices to define

$$\nabla_1 : E \otimes B \to (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)$$

$$\nabla_1 := (E \otimes u_B \otimes \Omega^1 A \otimes B) \circ (\nabla^E \otimes B) + (E \otimes u_B \otimes u_A \otimes \Omega^1 B) \circ (E \otimes d_B).$$

With this definition, when $R$ is the classical flip is a definition trivially equivalent to the one given in the former section, and we have that

$$\nabla_1((e \otimes b) \cdot (\alpha \otimes \beta)) = \nabla_1(a \alpha_R \otimes b_R \beta) =$$

$$= (E \otimes u_B \otimes \Omega^1 A \otimes B)(\nabla^E(e \alpha_R) \otimes b_R \beta) +$$

$$+ (E \otimes u_B \otimes u_A \otimes \Omega^1 B)(e \alpha_R \otimes d(b_R \beta)) =$$

$$\stackrel{1}{=} e_1 \otimes 1 \otimes_{A \otimes_R B} (d_A a_i) \alpha_R \otimes b_R \beta +$$

$$+ e \otimes 1 \otimes_{A \otimes_R B} d \alpha_R \otimes b_R \beta +$$

$$+ e \alpha_R \otimes 1 \otimes_{A \otimes_R B} 1 \otimes (d_B b_R) \beta +$$

$$+ e \alpha_R \otimes 1 \otimes_{A \otimes_R B} 1 \otimes b_R d_B \beta =$$
\[ \begin{align*}
    &= e_i \otimes 1 \otimes_{A \otimes R B} (d_A a_i) \alpha_R \otimes b_R \beta + \\
    &\quad + e \otimes 1 \otimes_{A \otimes R B} d \alpha_R \otimes b_R \beta + \\
    &\quad + e \otimes 1 \otimes_{A \otimes R B} \alpha_R \otimes (d b_R) \beta + \\
    &\quad + e \otimes 1 \otimes_{A \otimes R B} \alpha_R \otimes b_R d_B \beta^2 \\
    \equiv &\ 2 (e_i \otimes 1 \otimes_{A \otimes R B} d_A a_i \otimes b + \\
    &\quad + e \otimes 1 \otimes_{A \otimes R B} 1 \otimes b) \cdot (\alpha \otimes \beta) + \\
    &\quad + e \otimes b \otimes_{A \otimes R B} d_A \alpha \otimes \beta + \\
    &\quad + e \otimes b \otimes_{A \otimes R B} \alpha \otimes d_B \beta = \\
    &= \nabla_1 (e \otimes b) \cdot (\alpha \otimes \beta) + e \otimes b \otimes_{A \otimes R B} d(\alpha \otimes \beta),
\end{align*} \]

where in 1 we are using Leibniz’s rules (for the connection \( \nabla^E \) and the differential \( d_B \)), in 2 the definition of the action (2.1) and the compatibility of the twisting map with the differential, as mentioned in equations (1.18) and (1.19).

The definition of \( \nabla_2 \) is more involved, and we are forced to assume some extra conditions on the maps \( R \) and \( \tau_{F,A} \). Namely, assume that \( R \) is invertible, with inverse \( S : A \otimes B \to B \otimes A \), that \( \tau_{F,A} \) is invertible with inverse \( \sigma_{A,F} : A \otimes F \to F \otimes A \), and such that the following relation, ensuring the compatibility of the module twisting map with the connection \( \nabla^F \), is satisfied:

\[ (A \otimes \nabla^F) \circ \tau_{F,A} = (\tau_{F,A} \otimes \Omega^1 B) \circ (F \otimes \tilde{R}) \circ (\nabla^F \otimes A). \]

From this condition, that in Sweedler’s like notation is written as

\[ (2.3) \quad a_{\tau} \otimes (f_{\tau})_j \otimes_B (d b_{\tau})_j = (a_{\tilde{R}})_\tau \otimes (f_j)_\tau \otimes_B ((d b_j)_{\tilde{R}})_\tau, \]

the module twisting conditions (1.16) and (1.17) for \( \tau_{F,A} \), and the twisting map conditions (1.9) and (1.10) for \( R \), we may easily deduce the following equalities:

\[ (2.5) \quad (\sigma_{A,F} \otimes \Omega^1 B) \circ (A \otimes \nabla^F) = (F \otimes \tilde{R}) \circ (\nabla^F \otimes A) \circ \sigma_{A,F}, \]
\[ (2.6) \quad (\mu_A \otimes F) \circ (A \otimes \tau_{F,A}) = \tau_{F,A} \circ (F \otimes \mu_A) \circ (\sigma_{A,F} \otimes A), \]
\[ (2.7) \quad \sigma_{A,F} \circ (A \otimes \lambda_F) \circ (\tau_{F,A} \otimes B) = (\lambda_F \otimes A) \circ (F \otimes S), \]
\[ (2.8) \quad \sigma_{A,F} \circ (\mu_A \otimes F) = (F \otimes \mu_A) \circ (\sigma_{A,F} \otimes A) \circ (A \otimes \sigma_{A,F}), \]
\[ (2.9) \quad \sigma_{A,F} \circ (B \otimes \lambda_F) \circ (R \otimes F) = (\lambda_F \otimes A) \circ (B \otimes \sigma_{A,F}) \]

and define the map \( \nabla_2 : A \otimes F \to (E \otimes B \oplus A \otimes F) \otimes_{A \oplus B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B) \)

\[ \nabla_2 := (A \otimes F \otimes u_B \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes F \otimes d_A \otimes u_B) \circ \sigma \]

then we have that

\[ \nabla_2 ((a \otimes f) \cdot (\alpha \otimes \beta)) = \nabla_2 (a a_{\tau} \otimes f_{\tau} \beta) = \]

\[ = (A \otimes F \otimes u_A \otimes \Omega^1 B)(a a_{\tau} \otimes \nabla^F (f_{\tau} \beta)) + \\
+ 1 \otimes (f_{\tau} \beta)_{\sigma} \otimes d_A ((a a_{\tau})_{\sigma}) \otimes 1 \overset{(2.9)}{=} \]
\[ a \alpha_\tau \otimes (f_\tau)_{ij} \otimes A \otimes_R B 1 \otimes d_B (b_\tau)_{ij} \beta + \\
+ a \alpha_\tau \otimes f_\tau \otimes A \otimes_R B 1 \otimes d_B \beta + \\
+ 1 \otimes (f_\tau)_{\sigma \tau} \otimes A \otimes_R B (d_A a_{\sigma \tau}) \alpha_{\tau \sigma} \otimes 1 + \\
+ 1 \otimes (f_\tau)_{\sigma \tau} \otimes A \otimes_R B a_{\sigma \tau} d_A (\alpha_{\tau \sigma}) \otimes 1 \]

\[ a(\alpha_R)_\tau \otimes (f_j)_\tau \otimes A \otimes_R B 1 \otimes (d_B b_j)_R \beta + \\
+ a \otimes f \otimes A \otimes_R B \alpha \otimes d_B \beta + \\
+ 1 \otimes (f_\beta s)_{\alpha \sigma} \otimes A \otimes_R B (d_A a_{\sigma}) \alpha_S \otimes 1 + \\
+ a \otimes f \beta_s \otimes A \otimes_R B d_A (\alpha_S) \otimes 1 \]

\[ (a \otimes f_j \otimes A \otimes_R B 1 \otimes (d_B b_j)) \cdot (\alpha \otimes \beta) + \\
+ a \otimes f \otimes A \otimes_R B \alpha \otimes d_B \beta + \\
+ 1 \otimes f \alpha \beta_S \otimes A \otimes_R B d_A (\alpha_S) \otimes 1 + \\
+ a \otimes f \otimes A \otimes_R B d_A \alpha \otimes \beta = \\
= (a \otimes f_j \otimes A \otimes_R B 1 \otimes (d_B b_j)) \cdot (\alpha \otimes \beta) + \\
+ 1 \otimes f \alpha \beta_S \otimes A \otimes_R B d_A (\alpha_S) \otimes 1 + \\
+ a \otimes f \otimes A \otimes_R B \alpha \otimes d_B \beta + \\
+ a \otimes f \otimes A \otimes_R B d_A \alpha \otimes \beta = \\
= \nabla_2 (a \otimes f) \cdot (\alpha \otimes \beta) + a \otimes f \otimes A \otimes_R B d (\alpha \otimes \beta). \]

Henceforth, the mapping
\[ \nabla : E \otimes B \oplus A \otimes F \to (E \otimes B \oplus A \otimes F) \otimes A \otimes_R B \left( \Omega^1 A \otimes B \oplus A \otimes \Omega^1 B \right) \]
defined as
\[ (2.10) \quad \nabla(e \otimes b, a \otimes f) := \nabla_1 (e \otimes b) + \nabla_2 (a \otimes f) \]
is a (right) connection on the module \( E \otimes B \oplus A \otimes F \). We will call this connection the \((twisted)\) product connection of \( \nabla^E \) and \( \nabla^F \).

3. Curvature on product connections

In this section our aim is to study the curvature for the formerly defined product connections. If we have a connection \( \nabla : E \to E \otimes A \Omega^1 A \), we will also denote by \( \nabla : E \otimes_A \Omega A \to E \otimes_A \Omega A \) the extension given by \([1,2]\), occasionally denoting by \( \nabla^{[n]} : E \otimes_A \Omega^n A \to E \otimes_A \Omega^{n+1} A \) its restriction to \( E \)-valued \( n \)-form. The curvature of the connection \( \nabla \) is defined to be the operator \( \theta := \nabla^{[1]} \circ \nabla^{[0]} : E \to E \otimes_A \Omega^2 A \). It is well
As curvature map may be extended to a (right) $\Omega$ of degree 2 given at degree $\nabla\theta$ using the definition of the product connection we have that known (cf. for instance [Lan97, Sect. 7.2]) that the map $\theta$ and $\nabla$ yield a right connection $e_f$. Symbols are omitted. In the same spirit, for $\nabla\in E_A$ let also $\nabla\in F_b$ $\nabla\in A\otimes_\sigma B$. With this notation, the respective curvatures are written as $\theta^E(e) = e_i \otimes_A d_Aa_i$, and $\theta^F(e_i) = e_{ij} \otimes_A d_Aa_{ij}$, where summation symbols are omitted. In the same spirit, for $f \in F$, we will denote $\nabla^F(f) = f_k \otimes B d_b b_k$, and $\nabla^F(f_k) := f_{kl} \otimes B d_B b_k$. With this notation, the respective curvatures are written as $\theta^E(e) = e_i \otimes_A d_Aa_i$, and $\nabla^F(f) = f_k \otimes B d_B b_k$. We will also denote by $i_E$ and $i_F$ the canonical inclusions (as vector spaces) of $E \otimes_A \Omega^2 A$ and $F \otimes B \Omega^2 B$ into $(E \otimes B \otimes A \otimes F) \otimes A \otimes_\sigma R A \otimes_\sigma B$. For a generic element $(e \otimes b, a \otimes f) \in (E \otimes B \otimes A \otimes F)$, using the definition of the product connection we have that

$$\nabla(e \otimes b, a \otimes f) = e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes b + e \otimes 1 \otimes A \otimes_\sigma R B 1 \otimes d_B b +$$

$$+ 1 \otimes f_\sigma \otimes A \otimes_\sigma R B d_A(a_\sigma) \otimes 1 + a \otimes f_k \otimes A \otimes_\sigma R b 1 \otimes d_B b_k$$

Applying $\nabla^{[1]}$ to each of these four terms we obtain:

$$\nabla^{[1]}(e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes b) = \nabla(e_i \otimes 1 \otimes (d_Aa_i \otimes b) +$$

$$+ (e_i \otimes 1) \otimes A \otimes_\sigma R B d(da_i \otimes b) \overset{1}{=}$$

$$= (e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes 1) \cdot (d_Aa_i \otimes b)$$

$$- e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes d_B b =$$

$$= e_{ij} \otimes 1 \otimes A \otimes_\sigma R B d_Aa_{ij} \otimes d_Aa_i \otimes b$$

$$- e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes d_B b =$$

$$= i_E(\theta^E(e)) \cdot b - e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes d_B b,$$

$$\nabla^{[1]}(e \otimes 1 \otimes A \otimes_\sigma R B 1 \otimes d_B b) = \nabla(e \otimes 1 \otimes 1 \otimes d_B b + (e \otimes 1) \otimes A \otimes_\sigma R B d(1 \otimes d_B b) =$$

$$= e_i \otimes 1 \otimes A \otimes_\sigma R B d_Aa_i \otimes d_B b,$$

$$\nabla^{[1]}(1 \otimes f_\sigma \otimes A \otimes_\sigma R B d_A(a_\sigma) \otimes 1) = \nabla(1 \otimes f_\sigma) \cdot (d_A(a_\sigma) \otimes 1) +$$

$$+ (1 \otimes f_\sigma) \otimes A \otimes_\sigma R B d(d_A(a_\sigma) \otimes 1) =$$

$$= (1 \otimes f_\sigma)_k \otimes A \otimes_\sigma R B 1 \otimes d_B(b_\sigma)_k \cdot (d_A(a_\sigma) \otimes 1) =$$

$$= 1 \otimes (f_\sigma)_k \otimes A \otimes_\sigma R B d_A(a_\sigma) \otimes d_B(b_\sigma)_k \overset{2}{=}$$

$$= -1 \otimes (f_\sigma)_k \otimes A \otimes_\sigma R B d_A(a_\sigma) \otimes d_B(b_\sigma)_k \overset{\text{sym}}{=}$$

$$= -1 \otimes (f_\sigma)_k \otimes A \otimes_\sigma R B d_A(a_\sigma) \otimes d_B b_k,$$
\[ \nabla^{[1]}(a \otimes f_k \otimes_{A \otimes R} B 1 \otimes d_B b_k) = \nabla(a \otimes f_k) \cdot (1 \otimes d_B b_k) + \]

\[ + a \otimes f_k \otimes_{A \otimes R} B d(1 \otimes d_B b_k) = \]

\[ = (a \otimes f_{kl} \otimes_{A \otimes R} B 1 \otimes d_B b_{kl}) \cdot (1 \otimes d_B b_k) + \]

\[ + (1 \otimes (f_k)_{\sigma} \otimes_{A \otimes R} B d_A a_{\sigma} \otimes 1) \cdot (1 \otimes d_B b_k) = \]

\[ = a \otimes f_{kl} \otimes_{A \otimes R} B 1 \otimes d_B b_{kl} d_B b_k + \]

\[ + 1 \otimes (f_k)_{\sigma} \otimes_{A \otimes R} B d_A a_{\sigma} \otimes d_B b_k = \]

\[ = a \cdot i_F(\theta^F(f)) + 1 \otimes (f_k)_{\sigma} \otimes_{A \otimes R} B d_A a_{\sigma} \otimes d_B b_k. \]

where in 1 we are using the definitions of \( \nabla \) and the differential \( d \), in 2 the compatibility of \( \tilde{R} \) with \( d_A \). Adding up these four equalities we obtain the following result:

**Theorem 3.1.** The curvature of the product connection is given by

\[ \theta(e \otimes b, a \otimes f) = i_E(\theta^E(e)) \cdot b + a \cdot i_F(\theta^F(f)). \]

An interesting remark at the sight of the former result is that the product curvature does not depend neither on the twisting map \( R \) nor on the module twisting map \( \tau_{F,A} \), but only on the curvatures of the factors. As an immediate consequence of Equation (3.1) we obtain the following result:

**Corollary 3.2.** The product connection of two flat connections is a flat connection.

Henceforth, one might ask the question of describing the de Rham cohomology with coefficients in the sense of Beggs and Brzezinski (ref. [BB05]) for the (twisted) product connection of two flat connections. We will leave this problem for future works. It is also worth noticing that formula (3.1) drops down in the commutative case to the classical formula for the curvature on a product manifold.

4. **Bimodule connections**

For many purposes, only considering right (or left) modules is not enough. On the one hand, if we want to apply our theory to \( * \)-algebras, then sooner or later we will be bond to deal with \( * \)-modules and hermitian modules, but since the involution reverses the order of the products, these notions only make sense when we consider bimodules. On the other hand, there is a special kind of connections, known as **linear connections**, obtained when we take \( E = \Omega^1 A \). Since \( \Omega^1 A \) is a bimodule in a natural way, there is no reason to neglect one of its structures restraining ourselves to look at it just as a one-sided module. Reasons for extending the notion of connection to bimodules have been largely discussed at [Mon95], [DV99] and references therein.

Different approaches for dealing with this problem have been tried. The first one, described by Cuntz and Quillen in [CQ95], consists on considering a couple \( (\nabla^l, \nabla^r) \) where \( \nabla^l \) is a left connection which is also a right \( A \)-module morphism, and \( \nabla^r \) a right connection which is also a left \( A \)-module morphism. As it was pointed out in [DHLP96], this approach, though rising a very interesting algebraic theory, is not well suited for our geometrical point of view, since it doesn’t behave as expected when restricted to the commutative case. A different approach was introduced by Mourad in [Mon95] for
the particular case of linear connections and later generalized to arbitrary bimodules by Dubois-Violette and Masson in [DVM96] (see also [DV99, Chapter 10]). Their approach goes as follow: let $E$ be an $A$–bimodule; a \textit{(right) bimodule connection} on $E$ is a right connection $\nabla : E \rightarrow E \otimes_A \Omega^1 A$ together with a bimodule homomorphism $\sigma : \Omega^1 A \otimes_A E \rightarrow E \otimes_A \Omega^1 A$ such that

$$\nabla(ma) = a\nabla(m) + \sigma(d_A(a) \otimes_A m) \quad \text{for any } a \in A, m \in E. \quad (4.1)$$

Giving a right bimodule connection in the above sense is equivalent to give a pair $(\nabla^L, \nabla^R)$ consisting in a left connection $\nabla^L$ and a right connection $\nabla^R$ that are $\sigma$–compatible, meaning that

$$\nabla^R = \sigma \circ \nabla^L. \quad (4.2)$$

\textbf{Remark.} A weaker definition of $\sigma$–compatibility, namely requiring that equation (1.2) holds only in the center $Z(E) := \{ m \in E : \text{am = ma } \forall a \in A \}$ of $E$ rather than in the whole bimodule, has also been studied in [DHL96].

So, assume that we have $E$ bimodule over $A$, $\nabla^E$ a bimodule connection on $E$ with respect to the morphism $\varphi : \Omega^1 A \otimes_A E \rightarrow E \otimes_A \Omega^1 A$, and $F$ a bimodule over $B$ endowed with $\nabla^F$ a bimodule connection with respect to the bimodule morphism $\psi : \Omega^1 B \otimes_B F \rightarrow F \otimes_B \Omega^1 B$. As before, let $R : B \otimes A \rightarrow A \otimes B$ an invertible twisting map with inverse $S$, and assume also that we have a right module twisting maps $\tau_{F,A} : F \otimes A \rightarrow A \otimes F$ satisfying condition (2.3) and a left module twisting map $\tau_{B,E} : B \otimes E \rightarrow E \otimes B$ satisfying condition

$$\nabla^E \otimes B) \circ \tau_{B,E} = (E \otimes \tilde{R}) \circ (\tau_{B,E} \otimes \Omega^1 A) \circ (B \otimes \nabla^E), \quad (4.3)$$

which is the analogous of condition (2.4), and such that $(E \otimes B) \oplus (A \otimes F)$ becomes an $A \otimes_R B$ bimodule with left action

$$(\alpha \otimes \beta) \cdot (e \otimes b, a \otimes f) := (\alpha e_\tau \otimes \beta_\tau b, \alpha a_R \otimes \beta_R f),$$

then we have that

$$\nabla((\alpha \otimes \beta)(e \otimes b)) = \nabla_1(\alpha e_\tau \otimes \beta_\tau b) =$$

$$= (\alpha e_\tau) \otimes 1 \otimes_{A \otimes_R B} d_A(d_\tau') \otimes \beta_\tau b +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau b) =$$

$$= \alpha(e_\tau) \otimes 1 \otimes_{A \otimes_R B} d_A(a_\tau) \otimes \beta_\tau b +$$

$$+ (e_\tau) \varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha) \varphi \otimes \beta_\tau b +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau b) +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes \beta_\tau d_B b \quad \text{[E3]}$$

$$= \alpha(e_\tau) \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha)_{\tilde{R}} \otimes (\beta_\tau)_{\tilde{R}} b +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes \beta_\tau d_B b +$$

$$+ (e_\tau) \varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha) \varphi \otimes \beta_\tau b +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau b) =$$

$$= (\alpha e_\tau) \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha) \otimes \beta_\tau b +$$

$$+ (e_\tau) \varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha) \varphi \otimes \beta_\tau b +$$

$$+ \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau b) =$$
\[ \nabla((\alpha \otimes \beta)(e \otimes b)) = (\alpha \otimes \beta)\nabla(e \otimes b) + \xi(\varphi(e) \otimes A \otimes R B (d_A \alpha) \otimes \beta b) + \alpha \varepsilon(e) \otimes 1 \otimes (A \otimes R B 1 \otimes d_B(\beta) b). \]

On the other hand,
\[
\nabla((\alpha \otimes \beta)(a \otimes f)) = \nabla_2(\alpha a_R \otimes \beta_R f) = 1 \otimes (\beta_R f) \otimes A \otimes R B d_A((\alpha a_R) \otimes 1 + 1 \otimes (\beta_R f) \otimes A \otimes R B 1 \otimes d_B(b_k) = 1 \otimes (\beta f) \otimes A \otimes R B d_A(\alpha a) \otimes 1 + 1 \otimes (\beta f) \otimes A \otimes R B (\alpha a) \otimes 1 + 1 \otimes (\beta f) \otimes A \otimes R B 1 \otimes d_B(\psi) = (\alpha \otimes \beta)\nabla_2(a \otimes f) + 1 \otimes (\beta f) \otimes A \otimes R B d_A(\alpha a) \otimes 1 + 1 \otimes (\beta f) \otimes A \otimes R B 1 \otimes (d_B(\beta)) \psi. \]

Adding up these two equalities we obtain
\[
\nabla((\alpha \otimes \beta)(e \otimes b, a \otimes f)) = (\alpha \otimes \beta)\nabla(e \otimes b, a \otimes f) + \xi(\varphi(e) \otimes A \otimes R B (e \otimes b, a \otimes f)), \]

where the map \( \xi : (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B) \otimes A \otimes R B (E \otimes B \oplus A \otimes F) \otimes A \otimes R B (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B) \) is defined by \( \xi := \xi_{11} + \xi_{12} + \xi_{21} + \xi_{22} \), being
\[
\xi_{11}(d_A \alpha \otimes \beta) \otimes A \otimes R B e \otimes b) := (e \cdot \varphi(1 \otimes A \otimes R B d_A(\alpha) \otimes \beta b), \\
\xi_{12}(\alpha \otimes d_B \beta) \otimes A \otimes R B e \otimes b) := \alpha \varepsilon(e) \otimes 1 \otimes A \otimes R B 1 \otimes d_B(\beta) b, \\
\xi_{21}(d_A \alpha \otimes \beta \otimes A \otimes R B a \otimes f) := 1 \otimes (\beta f) \otimes A \otimes R B d_A(\alpha a) \otimes 1, \\
\xi_{22}(\alpha \otimes d_B \beta) \otimes A \otimes R B a \otimes f) := \alpha a_R \otimes f \otimes A \otimes R B 1 \otimes (d_B(\beta) \psi). \]

Hence, in order to show that the product connection \( \nabla \) is a bimodule connection we only have to show that \( \xi \) is a bimodule morphism, which is equivalent to prove that all the \( \xi_{ij} \) are bimodule morphisms.

**Lemma 4.1.** The map \( \xi_{11} \) is a left \((A \otimes R B)\)-module morphism, if, and only if, the equality
\[
(\varphi \otimes B) \circ (\Omega^1 A \otimes \tau_{B,E}) \circ (\bar{R} \otimes E) = (E \otimes \bar{R}) \circ (\tau_{B,E} \otimes \Omega^1 A) \circ (B \otimes \varphi) \]
is satisfied in \( B \otimes \Omega^1 A \otimes E \).
Proof In order to check that the compatibility condition is necessary, just apply the compatibility with the module action to an element of the form $1 \otimes b \otimes \omega \otimes 1 \otimes e \otimes 1$.

Conversely, assuming condition (4.4), we have that

$$\begin{align*}
\xi_{11}((x \otimes y) \cdot (d\alpha \otimes \beta \otimes \tau_{A \otimes R} e \otimes b)) &= \\
&= \xi_{11}(x(d\alpha) \bar{R} \otimes y\bar{R}\beta \otimes \tau_{A \otimes R} e \otimes b) \\
&= (e_{\tau})_{\varphi} \otimes 1 \otimes \tau_{A \otimes R} ((d\alpha)_{\bar{R}} \varphi \otimes (y\bar{R}\beta)_{\tau} b) \tag{1} \\
&= x(e_{\tau})_{\varphi} \otimes 1 \otimes \tau_{A \otimes R} ((d\alpha)_{\bar{R}} \varphi \otimes (y\bar{R}\beta)_{\tau} b) \tag{2} \\
&= x((e_{\tau})_{\varphi} \otimes 1 \otimes \tau_{A \otimes R} ((d\alpha)_{\bar{R}} \varphi \otimes (y\bar{R}\beta)_{\tau} b) = \\
&= x((e_{\tau})_{\varphi} \otimes 1 \otimes y_{\tau} \otimes \tau_{A \otimes R} ((d\alpha)_{\varphi} \otimes \beta_{\tau} b) = \\
&= (x \otimes y)\xi_{11}(d\alpha \otimes \beta \otimes \tau_{A \otimes R} e \otimes b),
\end{align*}$$

where in [1] we are using that $\varphi$ is a left module map, in [2] that $\tau$ is a module twisting map. □

It is straightforward checking that $\xi_{11}$ is a right module map, and thus left to the reader. In a completely analogous way, it is straightforward to check that $\xi_{22}$ is a left module map, whilst for the right module condition we need a compatibility relation similar to (4.4). More concretely, we have the following result, whose proof is analogous to the one of Lemma 4.1.

Lemma 4.2. The map $\xi_{22}$ is a right $A \otimes_R B$–module morphism if, and only if, the equality

$$(A \otimes \psi) \circ (\bar{R} \otimes F) \circ (\Omega^1 B \otimes \tau_{F,A}) = (\tau_{F,A} \otimes \Omega^1 B) \circ (F \otimes \bar{R}) \circ (\psi \otimes A)$$

is satisfied in $\Omega^1 B \otimes F \otimes A$.

For $\xi_{12}$ and $\xi_{21}$, the right (resp. left) module map conditions are also straightforward. We will show now that $\xi_{12}$ is a left module map, the proof that $\xi_{21}$ is a right module map being analogous.

$$\begin{align*}
\xi_{12}((x \otimes y) \cdot (\alpha \otimes d\beta \otimes \tau_{A \otimes R} e \otimes b)) &= \\
&= \xi_{12}(x\alpha R \otimes yd\beta \otimes \tau_{A \otimes R} e \otimes b) = \\
&= \xi_{12}(x\alpha R \otimes (yd\beta) \otimes \tau_{A \otimes R} e \otimes b) - \\
&= -\xi_{12}(x\alpha R \otimes (yd\beta) \otimes \tau_{A \otimes R} e \otimes \beta_{\tau} b) = \\
&= x\alpha R e_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes (d(yd\beta)_{\tau}) b = \\
&= -x\alpha R(e_{\tau})_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes d((yd\beta)_{\tau}) b \tag{4} \\
&= -x\alpha R(e_{\tau})_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes d((yd\beta)_{\tau}) b = \\
&= -x\alpha R(e_{\tau})_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes d((yd\beta)_{\tau}) b = \\
&= -x\alpha R(e_{\tau})_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes d((yd\beta)_{\tau}) b = \\
&= -x\alpha R(e_{\tau})_{\tau} \otimes 1 \otimes \tau_{A \otimes R} 1 \otimes d((yd\beta)_{\tau}) b =
\end{align*}$$
\[ x\alpha R(e_\tau) \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d((y_R)_\tau)\beta_\tau b + \\
+ x\alpha R(e_\tau) \otimes 1 \otimes_{A \otimes_R B} 1 \otimes (y_R)_\tau(d(\beta_\tau))b = \\
- x\alpha R(e_\tau) \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d((y_R)_\tau)\beta_\tau b = \\
= x\alpha R(e_\tau) \otimes 1 \otimes_{A \otimes_R B} 1 \otimes (y_R)_\tau(d(\beta_\tau))b \overset{[2]}{=} \\
\overset{[2]}{=} x(\alpha e_\tau) \otimes 1 \otimes_{A \otimes_R B} 1 \otimes y_\tau(d\beta_\tau)b = \\
= x(\alpha e_\tau) \otimes y_\tau \otimes_{A \otimes_R B} 1 \otimes (d\beta_\tau)b = \\
= (x \otimes y) \cdot \xi_{12}(\alpha \otimes d\beta_{A \otimes_R B})e \otimes b, \]

where in [1] and [2] we use that \( \tau_{F, A} \) is a module twisting map.

Summarizing, we have proved the following result:

**Theorem 4.3.** Let \( E \) be a bimodule over \( A \), \((\nabla^E, \varphi)\) a bimodule connection on \( E \), \( F \) a bimodule over \( B \), \((\nabla^F, \psi)\), \( R : B \otimes A \to A \otimes B \) an invertible twisting map; \( \tau_{F, A} : F \otimes A \to A \otimes F \) a right module twisting map satisfying condition (4.3) and \( \tau_{B, E} : B \otimes E \to E \otimes B \) a left module twisting map satisfying condition (4.3). Assume also that conditions (4.4) and (4.5) are satisfied, then the product connection of \( \nabla^E \) and \( \nabla^F \) is a bimodule connection with respect to the morphism \( \xi \).

5. **Examples**

Let us start by recalling some facts from [CQ95]. For any projective (right) module \( E \) over an algebra \( A \), there exists a module \( E' \) such that \( E \oplus E' = A^n \), and we have two canonical mappings

\[ p : A^n = E \oplus E' \longrightarrow E \quad \text{and} \quad \lambda : E \hookrightarrow E \oplus E', \]

we can then define the map \( \nabla_0 := (p \otimes \text{Id}) \circ (A^n \otimes d) \circ (\lambda \otimes \text{Id}) \) as the composition given by

\[ E \otimes_A \Omega^p A \xrightarrow{\lambda \otimes \text{Id}} A^n \otimes_A \Omega^p A \xrightarrow{A^n \otimes d} \Omega^{p+1} A \xrightarrow{p \otimes \text{Id}} E \otimes_A \Omega^{p+1} A \]

The operator \( \nabla_0 \) is a (flat) connection on \( E \), called the **Grassmann connection** on \( E \).

**Remark.** Physicists sometimes use the shorthand notation \( \nabla_0 = pd \) to denote the Grassmann connection.

It is also well known (cf. for instance [CQ95]) that the space of all linear connections over a projective module \( E \) is an affine space modeled on the space of \( A \)-module morphisms \( \text{End}_A(E) \otimes_A \Omega^1 A \), and henceforth we can write any linear connection \( \nabla \) on \( E \) as \( \nabla = \nabla_0 + \alpha \), being \( \alpha \in \text{End}_A(E) \otimes_A \Omega^1 A \), where the “matrix” \( \alpha \) is called the **gauge potential** of the connection \( \nabla \).
5.1. **Product connections on the quantum plane** $k_q[x, y]$.

Consider now $A := k[x]$ the polynomial algebra in one variable. Since any projective module over $A$ is free (actually, by Quillen-Suslin Theorem, any projective module over any polynomial ring is free) it is enough to consider connections for modules of the form $E = A^n$. If we denote by $\{e_i\}_{i=1,...,m}$ the canonical generator set for $E$, we may write the Grassmann connection on $E$ as

$$\nabla^E_0(e_1, \ldots, e_m) = e_1 \otimes_A da_1 + \cdots + e_m \otimes_A da_m \in E \otimes_A \Omega^1 A.$$  

(5.1)

Analogously, let $B := k[y]$, $F := B^n$ with canonical generating system $\{f_j\}_{j=1,...,n}$ and Grassmann connection

$$\nabla^F_0(b_1, \ldots, b_n) = f_1 \otimes_B db_1 + \cdots + f_n \otimes_B db_n.$$  

(5.2)

Recall that the quantum plane $k_q[x, y]$ may be seen as the twisted tensor product $k[x] \otimes_R k[y]$ with respect to the twisting map obtained by extension of $R(y \otimes x) := qx \otimes y$. This is an invertible twisting map which extends to an invertible module twisting map $\tau_{F, A} : F \otimes A \rightarrow A \otimes F$ in a natural way. For elements $e \otimes b \in E \otimes B$, where $e = (a_1, \ldots, a_m)$, and a generator $x \otimes f$ with $f = (y^i, \ldots, y^n)$ of $A \otimes F$, using the definition of our product connection given by Equation (2.10), we have that the product of the Grassmann connections is

$$\nabla^{gr}(e \otimes b, x \otimes f) = \left( \sum e_i \otimes 1 \otimes da_i \right) \otimes b + e \otimes 1 \otimes 1 \otimes db +$$

$$+ x \otimes \left( \sum f_k \otimes 1 \otimes dy^k \right) + 1 \otimes (q^{-i_1}y^{i_1}, \ldots, q^{-i_n}y^{i_n}) \otimes dx \otimes 1$$

**Remark.** If we introduce the notation $\lambda_q(p(y)) := p(qy)$, we can give the former expression for an element $a \otimes f$ of the form $a = x^j$, $f = (b_1, \ldots, b_n) \in F$ as

$$\nabla^{gr}(e \otimes b, a \otimes f) = \sum e_i \otimes 1 \otimes da_i \otimes b + e \otimes 1 \otimes 1 \otimes db +$$

$$+ \sum a \otimes f_k \otimes 1 \otimes db_k + \sum 1 \otimes \lambda_{q^{-j}}(b_k) \otimes d(x^j) \otimes 1$$

Now, for a generic connection $\nabla^E$ over the module $E$, there must exist a potential $\alpha^E = \varphi \otimes \omega \in \text{End } E \otimes_A \Omega^1 A$ given by $\alpha^E(a_1, \ldots, a_m) = \sum_{j,i} \varphi_j(a_j) \otimes \omega_i$ such that $\nabla^E = \nabla^E_0 + \alpha^E$. In the same way, for a generic connection $\nabla^F$ on $F$ there must exist a potential $\alpha^F = \sum k \psi_k \otimes \eta_k$, given by $\alpha^F(b_1, \ldots, b_n) = \sum_{k,j} \psi_k b_j \otimes \eta_k$, and such that $\nabla^F = \nabla^F_0 + \alpha^F$. Applying the formula for the product connection to $\nabla^E$ and $\nabla^F$ we easily observe that

$$\nabla(e \otimes b, a \otimes f) = \nabla^{gr}(e \otimes b, a \otimes f) + \sum_{i,j} \varphi_i(a_j) \otimes 1 \otimes \omega_i \otimes b + \sum_{k,i} a \otimes \psi_k(b_i) \otimes 1 \otimes \eta_k,$$

expression that tells us the formula for all possible product connections on the quantum plane.


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