Abstract

Koenigs constructed a family of two dimensional superintegrable (SI) models with one linear and two quadratic integrals in the momenta, shortly (1,2). More recently Matveev and Shevchishin have shown that this construction does generalize to models with one linear and two cubic integrals i.e. (1,3), up to the solution of a non-linear ordinary differential equation. Our explicit solution of this equation allowed for the construction of these SI systems and led to the proof that the systems globally defined on $\mathbb{S}^2$ are Zoll. We will generalize these results to the case $(1,n)$ for any $n \geq 2$. Our approach is again constructive and shows the existence, when $n$ is odd, of metrics globally defined on $\mathbb{S}^2$ which are indeed Zoll (under appropriate restrictions on the parameters), while if $n$ is even the metrics we found are never globally defined on $\mathbb{S}^2$, as it is already the case for the $(1,2)$ models constructed by Koenigs.
1 Introduction

As explained in the abstract, the starting point of our work is a set of SI models due to Koenigs [3], as popularized and generalized in [4],[5]. These models, defined on surfaces of revolution, exhibit an hamiltonian with one linear and two quadratic integrals in the momenta. Let us give an example, using the coordinates of [8], with hamiltonian

$$H = \frac{\cosh^2 x}{2(\rho + \sinh x)}(P_x^2 + P_\phi^2).$$ (1.1)

The symmetry of revolution shows that \((H, P_\phi)\) is already an integrable system. To reach SI we need a set of extra integrals

$$S_1 = \cos \phi (H - \sinh x P_\phi^2) + \sin \phi (\cosh x P_x P_\phi), \quad S_2 = \{P_\phi, S_1\}. \quad (1.2)$$

The extra integrals are not algebraically independent since we have

$$S_1^2 + S_2^2 = H^2 + 2\rho H P_x^2 - P_\phi^4.$$ (1.3)

However, the main problem, as pointed out in [8], is that the metric (1.1) is never globally defined on the manifold \(\mathbb{S}^2\). This unpleasant feature led Matveev and Shevchishin to take cubic extra integrals rather than quadratic ones. Still considering a surface of revolution

$$H = \Pi^2 + h_x^2 P_\phi^2 \quad \Pi = h_x P_x, \quad h_x = \frac{dh}{dx}, \quad (1.4)$$

they started from

$$S_1 = \cos \phi \Pi (H + \lambda_1(x) P_\phi^2) + \sin \phi P_\phi(\lambda_0(x) H + \lambda_2(x) P_\phi^2), \quad S_2 = \{P_\phi, S_1\}. \quad (1.5)$$

Here too one gets

$$S_1^2 + S_2^2 = H^3 + \sigma_1 H^2 P_\phi^2 + \sigma_2 H P_\phi^4 + \sigma_3 P_\phi^6.$$ (1.6)

with appropriate constants \(\sigma_i\).

However, Matveev and Shevchishin were led to a non-linear first order ODE which they could not solve. It was solved in [9] through appropriate coordinates changes and stemmed with the discovery of a metric (exhibiting two parameters) globally defined on \(\mathbb{S}^2\). In a subsequent work [10] we proved that this family of metrics is indeed Zoll.

We were led to consider the general case where the extra integrals \(S_1\) and \(S_2\) are of any integer degree \(n\) in the momenta, starting from Koenigs for \(n = 2\). In fact, to go through, the analysis needs to consider separately the odd and the even degrees.

The plan of this article is the following. In Section 2 we state our results in two Theorems, dealing successively with the case of extra integrals of degree \(2n + 1\) for \(n \geq 1\), and extra integrals of degree \(2n\) for \(n \geq 1\). Then Section 3 gives the proof of Theorem 1. In Section 4 the geodesics are constructed on the one hand using action-angle coordinates and on the other hand using the extra integrals. In Section 5 is given the proof of Theorem 2. Some concluding remarks are presented in the Section 6.
2 The results

When looking for a surface of revolution on $S^2$, as shown in [1][Proposition 4.10], one may start with the metric
\[ g = A^2(\theta) \, d\theta^2 + \sin^2 \theta \, d\phi^2 \quad \theta \in (0, \pi) \quad \phi \in S^1, \tag{2.1} \]
leading to the hamiltonian
\[ H = \Pi^2 + \frac{P_\phi^2}{\sin^2 \theta} \quad \Pi = \frac{P_\theta}{A(\theta)}. \tag{2.2} \]
The Killing vector $\partial_\phi$ implies, at the hamiltonian level, the conservation of $P_\phi$. In such a way the pair $(H, P_\phi)$ already defines an integrable system. To switch to a SI one, let us add two extra integrals
\[ S_1 = \cos \phi \, S + \sin \phi \, T, \quad S_2 = \{P_\phi, S_1\} = \cos \phi \, T - \sin \phi \, S, \tag{2.3} \]
where $S$ and $T$ are polynomials in $H$ and in $P_\phi^2$, of fixed degree in the momenta, denoted by $\sharp(S) = \sharp(T)$.

Our first result is:

**Theorem 1** *In the case where $\sharp(S_1) = \sharp(S_2) = 2n + 1$ with $n \geq 1$, the system defined by
\[ H, \quad P_\phi, \quad S_1, \quad S_2, \quad \tag{2.4} \]
where the extra integrals (2.3) are built with
\[ S = \Pi \sum_{k=0}^{n} \lambda_{2k-1}(\theta) \, H^{n-k} \, P_\phi^{2k}, \quad T = \sum_{k=0}^{n} \lambda_{2k}(\theta) \, H^{n-k} \, P_\phi^{2k+1}, \tag{2.5} \]
is superintegrable if one takes
\[ A(\theta) = 1 + \cos \theta \sum_{k=1}^{2n} \frac{e_k}{\sqrt{1 - m_k \sin^2 \theta}}, \quad \forall k : \quad e_k^2 = 1, \tag{2.6} \]
where all of the $2n$ real parameters $m_k$ are restricted to $m_k < 1$.

If, in addition, we have
\[ \sum_{k=1}^{2n} e_k = 0, \tag{2.7} \]
and
\[ A_0^{(n)} \equiv \sum_{k=1}^{n} \left| 1 - \sqrt{\frac{\mu_{2k-1}}{\mu_{2k}}} \right| < 1, \quad (\mu_k = 1 - m_k), \tag{2.8} \]
then the SI system is globally defined on $S^2$ and the metric is Zoll.

\(^1\)For $n = 1$ this restriction is not required.
The set of observables

\begin{align*}
H, \quad P_\phi, \quad S_+ = S_1 + iS_2, \quad S_- = S_1 - iS_2,
\end{align*}

generates a Poisson algebra, with the relations\footnote{The relation between the \( \sigma_l \) and the parameters \( m_k \) is given in Proposition 9.}

\begin{align}
S_+ S_- &= \sum_{l=0}^{2n+1} \sigma_l H^{2n+1-l} P_\phi^{2l} \\
\{S_+, S_-\} &= 2i \sum_{l=0}^{2n} (l + 1) \sigma_{l+1} H^{2n-l} P_\phi^{2l+1}.
\end{align}

Our second result is:

**Theorem 2** In the case where \( \sharp(S_1) = \sharp(S_2) = 2n \) with \( n \geq 1 \), the system defined by

\begin{align}
H, \quad P_\phi, \quad S_1, \quad S_2,
\end{align}

where the extra integrals (2.3) are built with

\begin{align}
S &= \sum_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} P_\phi^{2k}, \quad \mathcal{T} = \Pi \sum_{k=0}^{n-1} \lambda_{2k}(\theta) H^{n-k-1} P_\phi^{2k+1},
\end{align}

is SI if one takes

\begin{align}
A(\theta) &= 1 + \cos \theta \sum_{k=1}^{2n-1} \frac{e_k}{\sqrt{1 - m_k \sin^2 \theta}}, \quad e_k^2 = 1,
\end{align}

and all of the \( 2n - 1 \) real parameters \( m_k \) are restricted to \( m_k < 1 \).

This system is never globally defined on \( S^2 \).

Let us begin with the proof of Theorem 1.

## 3 Integrals of odd degree in the momenta

Here \( \sharp(S_1) = \sharp(S_2) = 2n + 1 \) for \( n \geq 1 \). Let us recall that the hamiltonian is given by

\begin{align}
H &= \Pi^2 + \sin^2 \theta P_\phi^2, \quad \Pi = \frac{P_\theta}{A(\theta)},
\end{align}

and the extra integrals by

\begin{align}
S_1 &= \cos \phi S + \sin \phi \mathcal{T}, \quad S_2 = \{P_\phi, S_1\} = \cos \phi \mathcal{T} - \sin \phi S,
\end{align}
where
\[ S = \prod_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} P_{\phi}^{2k}, \quad T = \sum_{k=0}^{n} \lambda_{2k}(\theta) H^{n-k} P_{\phi}^{2k+1}. \] (3.3)

These integrals are therefore defined by an array of functions of \( \theta \) of the form
\[ \begin{pmatrix} \lambda_{-1} = 1 & \lambda_1 & \lambda_3 & \ldots & \lambda_{2n-1} \\ \lambda_0 & \lambda_2 & \lambda_4 & \ldots & \lambda_{2n} \end{pmatrix}, \]
provided that they are determined by

**Proposition 1** \( S_1 \) and \( S_2 \) will be integrals if the \( \lambda \)'s solve the differential system\(^3\):
\[
0 \leq k \leq n : \begin{cases}
  s^2 \lambda'_{2k} = A \lambda_{2k-1}, & \text{(a)} \\
  s^2 \lambda'_{2k+1} = \lambda_{2k-1} - \frac{c}{s} \lambda_{2k-1} - A \lambda_{2k}, & \text{(b)}
\end{cases}
\] (3.4)

with the conventional value \( \lambda_{2n+1} = 0 \).

**Proof:** Both constraints \( \{H, S_1\} = 0 \) and \( \{H, S_2\} = 0 \) are seen to be equivalent to
\[
\{H, S\} = -2 \frac{P_{\phi}}{s^2} T \quad \{H, T\} = 2 \frac{P_{\phi}}{s^2} S.
\] (3.5)
Using the explicit form of \( S \) and \( T \) elementary computations give (3.4). \( \square \)

**Remark:** One can get rid of the derivatives in the right hand side of relation (3.4) by a simple recurrence which gives for \( 0 \leq k \leq n - 1 \):
\[
s^{2(k+1)} \lambda'_{2k+1} = -\frac{c}{s} (1 + s^2 \lambda_1 + s^4 \lambda_3 + \ldots + s^{2k} \lambda_{2k-1}) - (\lambda_0 + s^2 \lambda_2 + \ldots + s^{2k} \lambda_{2k}) A, \] (3.6)
while for \( k = n \) one gets the purely algebraic relation
\[
\frac{c}{s} (1 + s^2 \lambda_1 + s^4 \lambda_3 + \ldots + s^{2n} \lambda_{2n}) = -(\lambda_0 + s^2 \lambda_2 + \ldots + s^{2n} \lambda_{2n}) A. \] (3.7)

A simplifying approach to the differential system (3.4) makes use of generating functions, which encode all of the \( \lambda \)'s in a couple of objects.

**Definition 1** Let us define the generating functions
\[
\mathcal{L}(\theta, \xi) = \sum_{k=0}^{n} \lambda_{2k}(\theta) \xi^k, \quad \mathcal{M}(\theta, \xi) = \sum_{k=0}^{n} \lambda_{2k-1}(\theta) \xi^k, \quad \xi \in \mathbb{C}. \] (3.8)
These objects are mere polynomials in the variable \( \xi \). Their usefulness follows from
\(^3\)A prime is a \( \theta \) derivative while \( s = \sin \theta \) and \( c = \cos \theta \).
Proposition 2 \textit{The differential system (3.4) is equivalent, in terms of the generating functions, to}

\[ s^2 \partial_\theta \mathcal{L} = A \mathcal{M}, \quad s^2 (1 + \tau) \partial_\theta \mathcal{M} + \frac{c}{s} \mathcal{M} = -\xi A \mathcal{L}, \quad \tau = -\frac{\xi}{s^2}. \]  \hspace{1cm} (3.9)

\textbf{Proof:} Upon use of relations (a) in (3.4) we have

\[ s^2 \partial_\theta \mathcal{L} (\theta, \xi) = \sum_{k=0}^{n} \xi^k s^2 \lambda'_{2k}(\theta) = A \sum_{k=0}^{n} \xi^k \lambda_{2k-1}(\theta) = A \mathcal{M}(\theta, \xi). \]  \hspace{1cm} (3.10)

Conversely, expanding this relation in powers of \( \xi \) gives back the relations (a) in (3.4).

Similarly, using relations (b) in (3.4) we get

\[ s^2 \partial_\theta \mathcal{M} = \sum_{k=1}^{n} \xi^k s^2 \lambda'_{2k-1} = \sum_{k=1}^{n} \xi^k \left( \lambda'_{2k-3} - \frac{c}{s} \lambda_{2k-3} - A \lambda_{2k-2} \right), \]  \hspace{1cm} (3.11)

which becomes

\[ s^2 \partial_\theta \mathcal{M} = \partial_\theta \left( \xi \mathcal{M} - \xi^{n+1} \lambda_{2n-1} \right) - \frac{c}{s} \left( \xi \mathcal{M} - \xi^{n+1} \lambda_{2n-1} \right) - A \left( \xi \mathcal{L} - \xi^{n+1} \lambda_{2n} \right). \]  \hspace{1cm} (3.12)

We end up with

\[ s^2 (1 + \tau) \partial_\theta \mathcal{M} + \frac{c}{s} \mathcal{M} + \xi A \mathcal{L} = -\xi^{n+1} (\partial_\theta \lambda_{2n-1} - \lambda_{2n-1} - A \lambda_{2n}), \]  \hspace{1cm} (3.13)

and the right hand member does vanish thanks to relation (b) for \( k = n \) in (3.4). Conversely, expanding this relation in powers of \( \xi \) one recovers relations (b) in (3.4). \( \square \)

Let us describe the structure of the array of the \( \lambda_k \).

3.1 The solution for integrals of odd degree

Let us define the functions

\[ \forall k \in \{1, 2, \ldots, 2n\} : \quad h_k(\theta) = e_k \sqrt{1 - m_k s^2}, \quad e_k^2 = 1, \quad m_k < 1, \]  \hspace{1cm} (3.14)

and

\[ \mathcal{H}(\theta, \xi) \equiv \prod_{k=1}^{2n} (1 + \xi h_k(\theta)) = \sum_{k=0}^{2n} (H)_k(\theta) \xi^k. \]  \hspace{1cm} (3.15)

The \( (H)_k(\theta) \) are nothing but the symmetric functions constructed in terms of the \( h_k(\theta) \). Their explicit form is

\[ (H)_0(\theta) = 1 \quad (H)_1(\theta) = \sum_{l=1}^{2n} h_l(\theta), \]  \hspace{1cm} (3.16)

and more generally

\[ (H)_k(\theta) = \sum_{1 \leq l_1 \leq l_2 \leq \ldots \leq l_k \leq 2n} h_{l_1}(\theta) h_{l_2}(\theta) \cdots h_{l_k}(\theta), \quad 2 \leq k \leq 2n. \]  \hspace{1cm} (3.17)

In terms of these objects, we can write down the solution for the \( \lambda \)'s.
Definition 2 Let us define, for \( k \in \{1, 2, \ldots, n\} \), the functions
\[
\lambda_{2k-1} = \frac{(-1)^k}{s^{2k}} \left\{ \sum_{l=0}^{k} \binom{n-l}{n-k} (H)_{2l} + c \sum_{l=0}^{k-1} \binom{n-l-1}{n-k} (H)_{2l+1} \right\}, \quad (3.18)
\]
and for \( k \in \{1, 2, \ldots, n-1\} \):
\[
\lambda_{2k} = \frac{(-1)^{k+1}}{s^{2k+1}} \left\{ \sum_{l=0}^{k} \binom{n-l}{n-k} (H)_{2l+1} + c \sum_{l=0}^{k} \binom{n-l}{n-k} (H)_{2l} \right\}, \quad (3.19)
\]
as well as
\[
\lambda_{2n} = \frac{(-1)^{n+1}}{s^{2n+1}} \left\{ \sum_{l=0}^{n-1} (H)_{2l+1} + c \sum_{l=0}^{n} (H)_{2l} \right\}. \quad (3.20)
\]
A direct proof that these formulae do solve the differential system (3.4) is rather cumber-
some. We will first compute their generating functions and then use Proposition 2.

Proposition 3 The generating functions \( \mathcal{L} \) and \( \mathcal{M} \) are given by
\[
-s \mathcal{L}(\theta, \xi) = \sum_{l=0}^{n-1} \psi_{l,n} (H)_{2l+1} + c \sum_{l=0}^{n} \psi_{l,n} (H)_{2l}, \quad (3.21)
\]
and by
\[
\mathcal{M}(\theta, \xi) = \sum_{l=0}^{n} \psi_{l,n} (H)_{2l} + c \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l+1}, \quad (3.22)
\]
where
\[
\tau = -\frac{\xi}{s^2}, \quad \psi_{l,n} = \tau^l (1+\tau)^{n-l}, \quad 0 \leq l \leq n. \quad (3.23)
\]

Proof: Starting from
\[
-s \mathcal{L}(\theta, \xi) = \sum_{k=0}^{n-1} \xi^k (-s \lambda_{2k}) + \xi^n (-s \lambda_{2n}), \quad (3.24)
\]
using (3.19), (3.20) and interchanging the order of the summations we get
\[
\sum_{l=0}^{n-1} (H)_{2l+1} \sum_{k=l}^{n-1} \binom{n-l}{n-k} \tau^k + c \sum_{l=0}^{n-1} (H)_{2l} \sum_{k=l}^{n-1} \binom{n-l}{n-k} \tau^k, \quad (3.25)
\]
to which we must add
\[
\xi^n (-s \lambda_{2n}) = \sum_{l=0}^{n-1} (H)_{2l+1} \tau^n + c \sum_{l=0}^{n} (H)_{2l} \tau^n, \quad (3.26)
\]
leading to

\[- s \mathcal{L}(\theta, \xi) = \sum_{l=0}^{n-1} \tau^l \left( (H)_{2l+1} + c (H)_{2l} \right) \sum_{k=l}^{n} \binom{n-l}{k-l} \tau^{k-l} + c \tau^n (H)_{2n}. \tag{3.27}\]

Using the binomial theorem we get

\[- s \mathcal{L}(\theta, \xi) = \sum_{l=0}^{n-1} \tau^l (1 + \tau)^{n-l} \left( (H)_{2l+1} + c (H)_{2l} \right) + c \tau^n (H)_{2n}, \tag{3.28}\]

leading to \((3.21)\).

For \(\mathcal{M}\), using \((3.18)\), we have

\[\mathcal{M}(\theta, \xi) = \sum_{k=0}^{n} \tau^k \sum_{l=0}^{k} \binom{n-l}{n-k} (H)_{2l} + c \sum_{k=1}^{n} \tau^k \sum_{l=0}^{k-1} \binom{n-l-1}{n-k} (H)_{2l+1}. \tag{3.29}\]

Interchanging the summation order and using the binomial theorem gives for the first sum

\[\sum_{l=0}^{n} \tau^l (H)_{2l} \sum_{k=l}^{n} \binom{n-l}{k-l} \tau^{k-l} = \sum_{l=1}^{n} \tau^l (1 + \tau)^{n-l} (H)_{2l} \tag{3.30}\]

and, by the same token, for the second one

\[c \sum_{k=1}^{n} \tau^k \sum_{l=0}^{k-1} \binom{n-l-1}{n-k} (H)_{2l+1} = c \sum_{l=0}^{n} \tau^l (H)_{2l+1} \sum_{k=l+1}^{n} \binom{n-l-1}{k-l-1} \tau^{k-l} = c \sum_{l=0}^{n-1} \tau^{l+1} (1 + \tau)^{n-l-1} (H)_{2l+1}. \tag{3.31}\]

This concludes the proof. \(\square\)

**Remark:** Using this Proposition one can check relation \((3.7)\) which becomes, in terms of the generating functions:

\[
\left( \frac{\xi}{s} \mathcal{M} + A \mathcal{L} \right) \bigg|_{\tau = -1} = 0. \tag{3.32}
\]

Up to now we have defined our \(\lambda_i\) and computed their generating functions. We reach the core of this first part: we will prove the PDE’s for the generating functions which determine the explicit form of the function \(A(\theta)\) and of the hamiltonian.

**Proposition 4** The generating functions given by \((3.21)\) and by \((3.22)\) are solutions of the following equations

\[s^2 \partial_\theta \mathcal{L} = A \mathcal{M} \quad s^2 (1 + \tau) \partial_\theta \mathcal{M} + \xi \frac{c}{s} \mathcal{M} = -\xi A \mathcal{L}, \tag{3.33}\]

where

\[A(\theta) = 1 + c \sum_{k=1}^{2n} \frac{c_k}{\sqrt{1 - m_k s^2}}. \tag{3.34}\]
It follows that the $\lambda$’s given by Definition 2 are indeed a solution of the differential system (3.4), and this implies that $S_1$ and $S_2$ are integrals for the hamiltonian

$$H = \Pi^2 + \frac{P_\theta^2}{s^2}, \quad \Pi = \frac{P_\theta}{A(\theta)},$$

(3.35)

**Proof:** Let us first define the following splitting of the generating functions:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

(3.36)

where

$$\mathcal{L}_1 = - \frac{1}{s} \sum_{l=0}^{n-1} \psi_{l,n}(H)_{2l+1}, \quad \mathcal{L}_2 = - \frac{c}{s} \sum_{l=0}^{n} \psi_{l,n}(H)_{2l},$$

(3.37)

and similarly

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

(3.38)

where

$$\mathcal{M}_1 = \sum_{l=0}^{n} \psi_{l,n}(H)_{2l}, \quad \mathcal{L}_2 = \sum_{l=0}^{n-1} \psi_{l+1,n}(H)_{2l+1}.$$  

(3.39)

Let us compute first $s^2 \partial_\theta \mathcal{L}_1$. It is made out of two pieces. The first one, which follows from:

$$- s \partial_\theta \psi_{l,n} = 2c \tau \partial_\tau \psi_{l,n} = c \left( 2l \psi_{l,n} + 2(n-l) \psi_{l+1,n} \right),$$

(3.40)

is given by

$$c \sum_{l=0}^{n-1} (2l+1) \psi_{l,n}(H)_{2l+1} + c \sum_{l=0}^{n-2} 2(n-l) \psi_{l+1,n}(H)_{2l+1} + 2c \psi_{n,n}(H)_{2n-1}.$$  

(3.41)

The second piece follows from relation (A.2) in Appendix A

$$- s \partial_\theta (H)_{2l+1} = -c \left( (2l+1)(H)_{2l+1} + (2n-2l+1)(H)_{2l-1} \right) + (A-1)(H)_{2l},$$

(3.42)

and is given by

$$- c \sum_{l=0}^{n-1} (2l+1) \psi_{l,n}(H)_{2l+1} - \sum_{l=0}^{n-2} (2n-2l+1) \psi_{l+1,n}(H)_{2l+1} + (A-1) \sum_{l=0}^{n-1} \psi_{l,n}(H)_{2l}. \quad (3.43)$$

Adding up we are left with

$$\sum_{l=0}^{n-1} \psi_{l+1,n}(H)_{2l+1} + (A-1) \sum_{l=0}^{n} \psi_{l,n}(H)_{2l} + \psi_{n,n} \left( c(H)_{2n-1} - (A-1)(H)_{2n} \right).$$

(3.44)

The last piece vanishes upon use of (A.8). The final result is

$$s^2 \partial_\theta \mathcal{L}_1 = \mathcal{M}_2 + (A-1)\mathcal{M}_1.$$  

(3.45)

\[4\text{Recall that here } \nu = 2n.\]
Similarly one can show
\[ s^2 \partial_\theta L_2 = \mathcal{M}_1 + (A - 1)\mathcal{M}_2. \] (3.46)

Adding up we get the first relation in (3.33).

Let us now compute \( s^2 (1 + \tau) \partial_\theta \mathcal{M}_1 \). It is made out of two pieces. The first one, which follows from \( s^2 (1 + \tau) \partial_\theta \psi_{l,n} = -\xi \psi_{l-1,n} \),
\[ s^2 (1 + \tau) \partial_\theta \psi_{l,n} = \xi^C \left( 2l \psi_{l-1,n} + 2(n - l) \psi_{l,n} \right), \quad s^2 (1 + \tau) \psi_{l,n} = -\xi \psi_{l-1,n}, \quad \] (3.47)
is given by
\[ \xi^C \sum_{l=1}^{n} 2l \psi_{l-1,n} (H)_{2l} + \xi^C \sum_{l=0}^{n-1} 2(n - l) \psi_{l,n} (H)_{2l}. \] (3.48)
The second piece, which follows from (A.2):
\[ \partial_\theta (H)_{2l} = \frac{c}{s} \left( 2l (H)_{2l} + 2(n - l + 1) (H)_{2(l-1)} \right) - \frac{A-1}{s} (H)_{2l-1}, \quad l \geq 1, \] (3.49)
is given by
\[ -\xi^C \sum_{l=1}^{n} 2l \psi_{l-1,n} (H)_{2l} - \xi^C \sum_{l=0}^{n-1} 2(n - l) \psi_{l,n} (H)_{2l} + \xi^C (A - 1) \sum_{l=1}^{n} \psi_{l-1,n} (H)_{2l-1}. \] (3.50)

Adding these two pieces one gets
\[ s^2 (1 + \tau) \partial_\theta \mathcal{M}_1 = -\xi (A - 1) \mathcal{L}_1. \] (3.51)

Adding to both members \( \xi^C \mathcal{M}_1 = -\xi \mathcal{L}_2 \) we conclude to
\[ s^2 (1 + \tau) \partial_\theta \mathcal{M}_1 + \xi^C \mathcal{M}_1 = \xi (\mathcal{L}_1 - \mathcal{L}_2) - \xi A \mathcal{L}_1. \] (3.52)

Similarly one can prove
\[ s^2 (1 + \tau) \partial_\theta \mathcal{M}_2 + \xi^C \mathcal{M}_2 = -\xi (\mathcal{L}_1 - \mathcal{L}_2) - \xi A \mathcal{L}_2. \] (3.53)

Adding them up we get the second relation in (3.33). Using Proposition 2 we can conclude that \( S_1 \) and \( S_2 \) are integrals of \( H \).

Having constructed a SI system with a linear integral and two extra integrals of degree \( 2n + 1 \) in the momenta, let us show that this solution, under appropriate restrictions on the parameters \( m_k \), is globally defined on \( M = S^2 \).

### 3.2 Global structure

We have seen that the metric and the hamiltonian
\[ g = A^2(\theta) d\theta^2 + s^2 d\phi^2, \quad H = \Pi^2 + \frac{P^2}{s^2}, \quad \Pi = \frac{P}{A(\theta)}, \] (3.54)
where

\[ A(\theta) = 1 + A(\theta), \quad A(\theta) = c \sum_{k=1}^{2n} \frac{e_k}{\sqrt{1 - m_k s^2}}, \quad \forall k : \left( e_k^2 = 1 \quad \& \quad m_k < 1 \right), \quad (3.55) \]

exhibits 3 integrals: \((P_\phi, S_1, S_2)\).

Let us first prove:

**Lemma 1** If \( \sum_{k=1}^{2n} e_k = 0 \) one has the uniform bound

\[ \forall \theta \in (0, \pi) : \quad |A(\theta)| \leq A_0^{(n)} = \sum_{k=1}^{n} \left| 1 - \sqrt{\frac{\mu_{2k-1}}{\mu_{2k}}} \right|, \quad \mu_k = 1 - m_k > 0. \quad (3.56) \]

**Proof:** Since we have \( \sum_{k=1}^{2n} e_k = 0 \), we can write

\[ A(\theta) = \sum_{k=1}^{n} \tilde{e}_k \left( \frac{c}{\sqrt{1 - m_{2k-1} s^2}} - \frac{c}{\sqrt{1 - m_{2k} s^2}} \right), \quad \forall k : \quad \tilde{e}_k^2 = 1. \quad (3.57) \]

Since \( A(\theta) \) is odd, it is sufficient to consider \( \theta \in [0, \pi/2) \). The substitution \( t = \tan \theta \) gives

\[ A(\theta(t)) = \sum_{k=1}^{n} \tilde{e}_k f_k(t) \quad f_k(t) = \left( \sqrt{1 + \mu_{2k-1} t^2} - \frac{1}{\sqrt{1 + \mu_{2k-1} t^2}} \right), \quad t \geq 0. \quad (3.58) \]

Writing

\[ f_k(t) = \frac{\sqrt{\mu_{2k} - \sqrt{\mu_{2k-1}}} \sqrt{\mu_{2k} t}}{\sqrt{1 + \mu_{2k-1} t^2}} \frac{1}{\sqrt{1 + \mu_{2k-1} t^2}} \frac{\sqrt{\mu_{2k} t + \sqrt{\mu_{2k-1} t^2}}}{\sqrt{1 + \mu_{2k} t^2 + \sqrt{1 + \mu_{2k-1} t^2}}}, \]

and observing that each term in the product is uniformly bounded, for \( t \geq 0 \) by 1, we get:

\[ \forall t \geq 0 : \quad |f_k(t)| \leq \left| 1 - \sqrt{\frac{\mu_{2k-1}}{\mu_{2k}}} \right|, \]

implying the lemma. \( \square \)

**Proposition 5** The SI system of observables

\[ H, \quad P_\phi, \quad S_1, \quad S_2 \]

constructed in Section 3.1 is globally defined on \( \mathbb{S}^2 \) and Zoll if

\[ \forall k : \quad m_k < 1, \quad \& \quad \sum_{k=1}^{2n} e_k = 0, \quad \& \quad A_0^{(n)} < 1. \quad (3.59) \]
**Proof:** Corollary (4.16) in [1] ensures that the metric is globally defined on $S^2$ and Zoll iff:

- $A(\theta)$ is odd in terms of $x = \cos \theta \in [-1, +1]$,

- $A(0) = A(\pi) = 0$.

- $A([0, \pi]) \subset (-1, +1)$.

The first property is obvious and the second one follows from

$$A(0) = -A(\pi) = \sum_{k=1}^{2n} e_k = 0. \quad (3.60)$$

The third property follows from $A_{0}^{(n)} < 1$ and Lemma 1.

The hamiltonian is therefore globally defined as well as $\Pi$ and $P_\phi/s$. Let us write the integrals

$$S_1 = \cos \phi S + \sin \phi T, \quad S_2 = \cos \phi T - \sin \phi S,$$

where

$$S = \Pi \sum_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} P_\phi^{2k}, \quad \lambda_{-1} = 1, \quad T = P_\phi \sum_{k=0}^{n} \lambda_{2k}(\theta) H^{n-k} P_\phi^{2k}. \quad (3.62)$$

A look at Definition 2 shows that we can write

$$\lambda_{2k-1} = \frac{(-1)^k}{s^{2k}} \mu_{2k-1}, \quad \lambda_{2k} = \frac{(-1)^{k+1}}{s^{2k+1}} \mu_{2k}, \quad (3.63)$$

where the $\mu$’s are $C^\infty$ for $\theta \in [0, \pi]$. It follows that

$$\begin{cases}
S = \sum_{k=0}^{n} (-1)^k \mu_{2k-1}(\theta) H^{n-k} \Pi \left( \frac{P_\phi}{s} \right)^{2k}, \\
T = \sum_{k=0}^{n} (-1)^{k+1} \mu_{2k}(\theta) H^{n-k} \left( \frac{P_\phi}{s} \right)^{2k+1},
\end{cases} \quad (3.64)$$

are globally defined as well. \hfill \square

**Remarks:**

1. Let us show that the set defined by the restriction $A_{0}^{(n)} < 1$ is not empty. Indeed the choice

$$\mu_{2k-1} = k^2 + k - 1, \quad \mu_{2k} = k^2(k + 1)^2 \quad \implies \quad A_{0}^{(n)} = \frac{n}{n + 1}.$$  

2. Let us give an example for which $A_{0}^{(n)} > 1$:

$$\mu_{2k-1} = 1, \quad \mu_{2k} = k^2(k + 1)^2 \quad \implies \quad A_{0}^{(n)} = \frac{n^2}{n + 1} > 1 \quad \text{for} \quad n \geq 2.$$
3. As stated in Theorem 1, the constraint $|A_0^{(n)}| < 1$ is needed only for $n \geq 2$. Indeed for $n = 1$ we can take

$$A(\theta) = c \left( \frac{1}{\sqrt{1-m_1 s^2}} - \frac{1}{\sqrt{1-m_2 s^2}} \right), \quad m_2 < m_1 < 1. \quad (3.65)$$

It is easy to prove that $A([0, \pi]) = [-f_0, +f_0]$ where

$$f_0 = \frac{1 - \rho}{\sqrt{(1-\rho)^2 + 3\rho}}, \quad \rho = \left( \frac{1-m_1}{1-m_2} \right)^{1/3}, \quad (3.66)$$

hence $f_0 \in (-1, 1)$.

4. In the proof of Lemma 1 one can write alternatively

$$f_k(t) = \frac{1 - \rho}{\sqrt{1 + \mu_{2k-1} t^2}} \frac{\sqrt{\mu_{2k-1} t}}{\sqrt{1 + \mu_{2k} t^2}} \frac{\sqrt{\mu_{2k} t + \sqrt{\mu_{2k-1} t^2}}}{\sqrt{1 + \mu_{2k-1} t^2}},$$

which gives

$$|f_k(t)| \leq \left| 1 - \sqrt{\frac{\mu_{2k}}{\mu_{2k-1}}} \right|.$$ 

So the bound

$$B_0^{(n)} \equiv \sum_{k=0}^{2n} \left| 1 - \sqrt{\frac{\mu_{2k}}{\mu_{2k-1}}} \right| < 1 \quad (3.67)$$

does also ensure that $|A(\theta)| < 1$ uniformly.

### 3.3 The Poisson algebra

Having defined

$$S_\pm = e^{-i\phi}(S + iT), \quad S_- = e^{i\phi}(S - iT), \quad (3.68)$$

let us begin with:

**Definition 3** The set of moments $\{\sigma_0, \sigma_1, \ldots, \sigma_{2n+1}\}$ and their generating function are defined by

$$S_+ S_- \equiv S^2 + T^2 = \sum_{l=0}^{2n+1} \sigma_l H^{2n-l+1} P_{\phi}^{2l}, \quad \Sigma(\xi) = \sum_{l=0}^{2n+1} \sigma_l \xi^l, \quad \xi \in \mathbb{C}. \quad (3.69)$$

The $\sigma_l$ are related to the $\lambda$’s by

**Proposition 6** The moments are given by:

$$0 \leq l \leq n : \quad \sigma_l \quad = \quad \sum_{k=0}^{l} S_{k,l-k} \quad (\Rightarrow \quad \sigma_0 = 1) \quad (3.70)$$

$$n + 1 \leq l \leq 2n + 1 : \quad \sigma_l \quad = \quad \sum_{k=l-n-1}^{n} S_{k,l-k}, \quad (3.70)$$
where
\[ S_{k,l} = \lambda_{2k-1} \lambda_{2l-1} + \lambda_{2k} \lambda_{2l-2} - \frac{1}{s^2} \lambda_{2k-1} \lambda_{2l-3}, \]  
(3.71)
and with the conventions that \( \lambda_{2n+1} = \lambda_{-2} = \lambda_{-3} = 0. \)

**Proof:** Using the formulae given for the \( S_1 \) and \( S_2 \) (and taking into account the conventional values) we have
\[ S_+ S_- = \sum_{k=0}^{n} \sum_{L=0}^{n+1} S_{k,L} H^{2n-k-L+1} P^2_{\phi}(k+L), \]  
(3.72)
where \( S_{k,L} \) is given by (3.71). The change of summation index \( L \rightarrow l = L + k \) gives
\[ S_+ S_- = \sum_{k=0}^{n} \sum_{l=k}^{n+k+1} S_{k,l-k} H^{2n-l+1} P^2_{\phi}(l). \]  
(3.73)
Interchanging the order of the summations we get
\[ S_+ S_- = \sum_{l=0}^{n} \left( \sum_{k=0}^{l} S_{k,l-k} \right) H^{2n+1-l} P^2_{\phi}(l) + \sum_{l=n+1}^{2n+1} \left( \sum_{k=l-n}^{n} S_{k,l-k} \right) H^{2n+1-l} P^2_{\phi}(l), \]  
(3.74)
from which the relations in (3.70) follow.

To relate the moments \( \sigma_l \), hence their generating function \( \Sigma(\xi) \), in terms of the parameters \( m_k \) appearing in \( A(\theta) \) several steps are needed. In the first one we need to relate \( \Sigma(\xi) \) to the generating functions:

**Proposition 7** The generating function of the moments is given by
\[ \Sigma(\xi) = \xi L^2(\theta, \xi) + (1 + \tau) M^2(\theta, \xi), \quad \tau = -\frac{\xi}{s^2}. \]  
(3.75)

**Proof:** Using (3.70) we have
\[ \Sigma(\xi) = \sum_{L=0}^{n} \xi^L \sum_{k=0}^{L} S_{k,L-k} + \sum_{L=n+1}^{2n+1} \xi^L \sum_{k=L-n-1}^{n} S_{k,L-k}. \]  
(3.76)
Interchanging the orders of the summations gives
\[ \Sigma(\xi) = \sum_{k=0}^{n} \sum_{L=k}^{n+k+1} \xi^L S_{k,L-k} = \sum_{k=0}^{n} \xi^k \sum_{l=0}^{n+l} \xi^l S_{k,l}. \]  
(3.77)
The terms in \( S_{k,l} \) give successively
\[ \sum_{k=0}^{n} \xi^k \lambda_{2k-1} \sum_{l=0}^{n+l} \xi^l \lambda_{2l-1} = M^2, \quad \sum_{k=0}^{n} \xi^k \lambda_{2k} \sum_{l=0}^{n+l} \xi^l \lambda_{2l-1} = \xi L^2, \]  
(3.78)
and
\[- \frac{1}{s^2} \sum_{k=0}^{n} \xi^{k} \lambda_{2k-1} \sum_{l=1}^{n+1} \xi^l \lambda_{2l-3} = \tau \mathcal{M}^2. \tag{3.79}\]

Adding all these pieces proves the Proposition. \(\square\)

In a second step we need a new writing of the generating functions

**Proposition 8** For \(\tau \geq 0\) (that is for \(\xi \leq 0\)) one has for the first generating function
\[
\mathcal{L}(\theta, \xi) = -\frac{(1 + \tau)^n}{2 \eta s} \left( (1 + \eta c) \mathcal{H}(\theta, \eta) - (1 - \eta c) \mathcal{H}(\theta, -\eta) \right), \tag{3.80}\]
where
\[
\tau = -\frac{\xi}{s^2}, \quad \eta = \sqrt{\frac{\tau}{\tau + 1}}, \quad \mathcal{H}(\theta, \xi) = \prod_{k=1}^{2n} (1 + \xi h_k(\theta)).
\]
The second generating function is given by
\[
\mathcal{M}(\theta, \xi) = \frac{(1 + \tau)^n}{2} \left( (1 + \eta c) \mathcal{H}(\theta, \eta) + (1 - \eta c) \mathcal{H}(\theta, -\eta) \right). \tag{3.81}\]

**Proof:** The relation (3.21), written out in detail gives
\[
(-s) \mathcal{L} = \sum_{l=0}^{n-1} \tau^l (1 + \tau)^{n-l} (H)_{2l+1} + c \sum_{l=0}^{n} \tau^l (1 + \tau)^{n-l} (H)_{2l}, \tag{3.82}\]
which becomes
\[
(-s) \mathcal{L} = (1 + \tau)^n \left( \frac{1}{\eta} \sum_{l=0}^{n-1} \eta^{2l+1} (H)_{2l+1} + c \sum_{l=0}^{n} \eta^{2l} (H)_{2l} \right). \tag{3.83}\]
These sums are given by relations (A.9) and (A.10) in Appendix A, and lead to (3.80). The proof of (3.81) is similar. \(\square\)

Let us now express the moments in terms the parameters \(m_k\) which appear in \(A(\theta)\). To this end we will define, for the string \(M = (m_1, m_2, \ldots, m_{2n})\) the symmetric functions \((M)_l:\)
\[
\prod_{l=1}^{2n} (1 + \xi m_l) = \sum_{l=0}^{2n} \xi^l (M)_l. \tag{3.84}\]
We are now in position to prove:

**Proposition 9** The generating function of the moments is
\[
\Sigma(\xi) = (1 - \xi) \prod_{l=1}^{2n} (1 - \xi m_l), \quad \xi \in \mathbb{C}, \tag{3.85}\]
giving the explicit formulae

\[
\begin{align*}
\sigma_0 &= 1 \\
1 \leq l \leq 2n: \quad \sigma_l &= (-1)^l[(M)_l + (M)_{l-1}] \\ 
\sigma_{2n+1} &= -(M)_{2n} = -\prod_{k=1}^{2n} m_k.
\end{align*}
\] (3.86)

**Proof:** We will take \( \xi \leq 0 \) ensuring that \( \tau \geq 0 \). We have seen in (3.75) that \( \Sigma \) is given by

\[
\Sigma(\xi) = (1 + \tau)M^2 - \tau(sL)^2.
\] (3.87)

Upon use of relations (3.80) and (3.81) one obtains

\[
\Sigma(\xi) = (1 + \tau)^{2n+1}(1 - \eta^2c^2) \mathcal{H}(\theta, \eta) \mathcal{H}(\theta, -\eta).
\] (3.88)

The identities

\[
(1 + \tau)(1 - \eta^2c^2) = 1 - \xi, \quad (1 + \tau)^{2n} \mathcal{H}(\theta, \eta) \mathcal{H}(\theta, -\eta) = \prod_{k=1}^{2n} (1 - \xi m_k),
\] (3.89)

lead for \( \Sigma(\xi) \) to the relation (3.85). Analytic continuation extends it to \( \xi \in \mathbb{C} \). Expanding \( \Sigma(\xi) \) in powers of \( \xi \) gives (3.86). \( \square \)

**Remark:** It follows that

\[
\Sigma(1) = \sum_{l=0}^{2n+1} \sigma_l = 0.
\] (3.90)

Let us conclude with:

**Proposition 10** One has the relation

\[
\{S_+, S_-\} = 2i \sum_{l=0}^{2n} (l + 1)\sigma_{l+1} H^{2n-l} P^{2l+1}_\phi.
\] (3.91)

**Proof:** Extracting out from the bracket the \( \phi \) dependence gives

\[
\frac{\{S_+, S_-\}}{2i} = \frac{1}{2} \frac{\partial}{\partial P_\phi} (S^2 + T^2) - \{S, T\}.
\] (3.92)

The first term in the right hand side gives

\[
\sum_{l=0}^{2n} (l + 1)\sigma_{l+1} H^{2n-l} P^{2l+1}_\phi + \frac{1}{s^2} \sum_{l=0}^{2n} (2n + 1 - l) \sigma_l H^{2n-l} P^{2l+1}_\phi,
\] (3.93)
so that the relation (3.91) will hold true if we can prove the relation
\[ s^2 \{S, T\} = \sum_{l=0}^{2n} (2n + 1 - l) \sigma_l H^{2n-l} P_{\phi}^{2l+1}. \] (3.94)

Using the notation \( \Psi_{k+l}^{2n} = H^{2n-k-l} P_{\phi}^{2(k+l)+1} \), we have first
\[ \{S, T\} = \sum_{k,l} \left[ (n-k)\lambda_{2k-1} \Pi \{H, \lambda_{2l}\} - (n-l)\lambda_{2l} \{H, \lambda_{2k-1}\Pi \} + \right. \\
+ \left. H\{\lambda_{2k-1}\Pi, \lambda_{2l}\} \right] \frac{\Psi_{k+l}^{2n}}{H}. \] (3.95)

In the second sum let us change \( l \leftrightarrow k \), and let us notice that \( \{H, \lambda_{2l}\} = 2\Pi \frac{\lambda_{2l}}{A} \), thanks to relations (a) in (3.4). Computing the other brackets gives
\[ s^2 \{S, T\} = \sum_{k,l} \left[ 2(n-k)\Pi^2 \left( \lambda_{2k-1}\lambda_{2l-1} - \lambda_{2k} \frac{s^2\lambda_{2l-1}'}{A} \right) + \right. \\
+ \left. 2(n-k) \frac{c}{sA} \lambda_{2k}\lambda_{2l-1} P_{\phi}^2 + H\lambda_{2k-1}\lambda_{2l-1} \right] \Psi_{k+l}^{2n} \] (3.97)

Using \( \Pi^2 = H - \frac{P_{\phi}^2}{s^2} \) leads to
\[ s^2 \{S, T\} = \sum_{k,l} \left[ (2n-2k+1)\lambda_{2k-1}\lambda_{2l-1} - 2(n-k)s^2\lambda_{2k} \frac{s^2\lambda_{2l-1}'}{A} \right] \Psi_{k+l}^{2n} + \right. \\
+ \left. \sum_{k,l} \left[ 2(n-k) \frac{\lambda_{2k}}{A} \left( \lambda_{2l-1}' - \frac{c}{s} \lambda_{2l-1} \right) - 2(n-k) \frac{\lambda_{2k-1}\lambda_{2l-1}}{s^2} \right] \Psi_{k+l+1}^{2n} \] (3.98)

Noticing that \( \lambda'_{-1} = 0 \) we can write
\[ \sum_{k} \sum_{l=1}^{n} 2(n-k)\lambda_{2k} \frac{s^2\lambda_{2l-1}'}{A} \Psi_{k+l}^{2n} = \sum_{k,l} 2(n-k)\lambda_{2k} \frac{s^2\lambda_{2l+1}'}{A} \Psi_{k+l+1}^{2n} \] (3.99)

since \( \lambda_{2n+1} = 0 \). This allows to collect the three terms exhibiting a factor \( 1/A \)
\[ - \sum_{k,l} 2(n-k) \frac{\lambda_{2k}}{A} \left( s^2\lambda_{2l+1}' - \lambda_{2l-1}' + \frac{c}{s} \lambda_{2l-1} \right) \Psi_{k+l+1}^{2n} \] (3.100)

and upon use of relations (b) in (3.4) these terms reduce to
\[ \sum_{k,l} 2(n-k)\lambda_{2k}\lambda_{2l} \Psi_{k+l+1}^{2n}. \] (3.101)
So we end up with the left hand member of (3.94):
\[
s^2 \{S, T\} = \sum_{k,l} (2n - 2k + 1) \lambda_{2k-1} \lambda_{2l-1} \Psi_{k+l}^{2n} + \sum_{k,l} 2(n - k) \left( \lambda_{2k} \lambda_{2l} - \frac{\lambda_{2k-1} \lambda_{2l-1}}{s^2} \right) \Psi_{k+l+1}^{2n}.
\]

Let us consider now the right hand member of (3.94) with \(l \to L\). Exchanging the summation indices leads to
\[
\sum_{L=0}^{2n+1} (2n - L + 1) \sigma_L H^{2n-L} P_\phi^{2L+1} = \sum_k \sum_{L=k}^{k+n+1} (2n - L + 1) S_{k,L} H^{2n-L} P_\phi^{2L+1}.
\]
The change of summation index: \(l = L - k\) gives eventually
\[
\sum_k \sum_{l=0}^{n+1} (2n - k - l + 1) S_{k,l} \Psi_{k+l}^{2n}.
\]
Let us recall that
\[
S_{k,l} = \lambda_{2k-1} \lambda_{2l-1} + \lambda_{2k} \lambda_{2l-2} - \frac{1}{s^2} \lambda_{2k-1} \lambda_{2l-3}.
\]
The first term, due to the \(k \leftrightarrow l\) symmetry, gives
\[
\sum_{k,l} (2n - 2k + 1) \lambda_{2k-1} \lambda_{2l-1} \Psi_{k+l}^{2n}
\]
while the remaining two terms, with the change \(l \to l + 1\), give
\[
\sum_{k,l} 2(n - k) \left( \lambda_{2k} \lambda_{2l} - \frac{1}{s^2} \lambda_{2k-1} \lambda_{2l-1} \right) \Psi_{k+l+1}^{2n}.
\]
Adding these two terms and comparing with (3.102) establishes the relation (3.94), hence the Proposition.

Combining relations (3.69) and (3.91) we conclude to:

**Proposition 11** The set of observables
\[
H, \quad P_\phi, \quad S_+ = e^{-i\phi} (S + iT), \quad S_- = e^{i\phi} (S - iT),
\]
is indeed a Poisson algebra with
\[
S_+ S_- = \sum_{l=0}^{2n+1} \sigma_l H^{2n-l+1} P_\phi^{2l}, \quad \{S_+, S_-\} = 2i \sum_{l=0}^{2n} (l + 1) \sigma_{l+1} H^{2n-l} P_\phi^{2l+1}.
\]

This concludes the proof of Theorem 1. □
4 Geodesics

4.1 Geodesics from the action-angle coordinates

Since the hamiltonian defined by (3.54) and (3.55) is globally defined on $M = S^2$ it is interesting to study its geodesics.

Before computing the action and angle variables let us consider the torus $H = E$ and $P_\phi = L$. The Hamilton-Jacobi equation for the action:

$$\frac{\partial S}{\partial t} + \frac{1}{A^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{s^2} \left( \frac{\partial S}{\partial \phi} \right)^2 = 0,$$

allows for separation

$$S = -Et + L\phi + \int P_\theta d\theta$$

and leads to

$$P_\theta = \epsilon \sqrt{E} \sqrt{1 - \frac{s_0^2}{s^2}(1 + A(\theta))}, \quad s_0 = \sin i = \frac{L}{\sqrt{E}}, \quad i \in (0, \frac{\pi}{2}).$$

The sign $\epsilon$ is given by

$$\epsilon = \begin{cases} 
+1 & \text{if } \theta : i \to \pi - i \\
-1 & \text{if } \theta : \pi - i \to i.
\end{cases}$$

As may be seen in Figure 1, the plus sign corresponds to the first half of the geodesic where $\theta$ increases from $i$ to $\pi - i$, while the minus sign corresponds to the second half of the geodesic where $\theta$ decreases from $\pi - i$ to $i$.

![Figure 1: Geometry of the geodesics](image)
We will take the initial values:
\[ \theta = i \quad s = s_0 \quad c = c_0 \quad \phi = 0. \]
The first action is
\[ I_\phi = \frac{1}{2\pi} \oint L \, d\phi = L, \]
and the second one
\[ I_\theta = \frac{1}{2\pi} \oint P_\theta \, d\theta = \frac{\sqrt{E}}{\pi} \int_i^{\pi-i} \sqrt{1 - \frac{s_0^2}{s^2}} \, d\theta \]
because \( A(\theta) \) is odd. Coordinate change \( \sin \chi = \frac{c}{c_0} \) gives
\[ I_\theta = 2\sqrt{E} \frac{c_0^2}{\pi} \int_0^{\pi/2} \frac{\cos^2 \chi}{1 - c_0^2 \sin^2 \chi} \, d\chi = \sqrt{E} - L. \]
Hence we have obtained for the actions
\[ \begin{cases} 
I_\phi = P_\phi \\
I_\theta = \sqrt{H} - P_\phi 
\end{cases} \implies H = (I_\theta + I_\phi)^2. \tag{4.8} \]
Due to the superintegrability, the dynamical system is degenerate and we have a single frequency
\[ \nu = \frac{\partial H}{\partial I_\theta} = 2(I_\theta + I_\phi) = 2\sqrt{E} \implies \omega_\theta = \nu t + K, \quad \omega_\phi = \nu t + L, \tag{4.9} \]
which determines the time dependence of the angles.

Remark: Since \( A(\theta) \) is odd, it does not contribute to the action integrals. Hence the previous relations for the actions are in fact valid for any Zoll metric of revolution.

Let us now determine the angles:

**Proposition 12** For the first half of the geodesic one has
\[ 2\sqrt{E} \, t = \arccos \left( \frac{c}{c_0} \right) + \sum_{k=1}^{2n} e_k \Omega(\theta, m_k), \tag{4.10} \]
where
\[ \Omega(\theta, m_k) = \begin{cases} 
\frac{1}{\sqrt{|m_k|}} \arcsin \left( \sqrt{\frac{m_k}{1-m_k s_0^2} \sqrt{s^2 - s_0^2}} \right) & \text{if } m_k \in (0, 1) \\
\sqrt{s^2 - s_0^2} & \text{if } m_k = 0 \\
\frac{1}{\sqrt{|m_k|}} \arcsinh \left( \sqrt{\frac{|m_k|}{1+|m_k| s_0^2} \sqrt{s^2 - s_0^2}} \right) & \text{if } m_k < 0.
\end{cases} \tag{4.11} \]
Proof: We have, for the first half of the geodesic
\[ \omega_\theta = \frac{\partial S}{\partial I_\theta} = \int \frac{(1 + A(\theta))}{\sqrt{s^2 - s_0^2}} s \, d\theta, \] (4.12)
so that
\[ \omega_\theta = \int \frac{s}{\sqrt{s^2 - s_0^2}} d\theta + \sum_{k=1}^{2n} e_k \int \frac{sc}{\sqrt{s^2 - s_0^2} \sqrt{1 - m_k s^2}} d\theta. \] (4.13)
The second integral requires the change of variable \( u = \sqrt{s^2 - s_0^2} \).
\[ \square \]
From this we deduce

**Proposition 13 (Kepler’s law)** The period of the geodesic motion is given by \( T = \frac{\pi}{\sqrt{E}} \).

**Proof:** When \( \theta \) increases from \( i \) to \( \pi - i \) the time evolves from \( t = 0 \) to \( t = \frac{T}{2} \) while the right hand member in (4.10) evolves from 0 to \( \pi \).

Let us compute now the angle \( \omega_\phi \) which is more interesting since it will give a first description of the geodesics:

**Proposition 14** The analytic structure of the geodesics, when \( \theta \) increases from \( i \) to \( \pi - i \) and \( \phi \) from 0 to \( \pi \), is given by:
\[ \phi = \arccos \left( \frac{s_0 \, c}{s \, c_0} \right) + \sum_{k=1}^{2n} e_k \arcsin \left( \frac{1}{\sqrt{1 - m_k s_0^2}} \sqrt{1 - \frac{s_0^2}{s^2}} \right), \] (4.14)
and when \( \theta \) decreases from \( \pi - i \) to \( i \) while \( \phi \) increases from 0 to \( 2\pi \), is given by:
\[ \phi = 2\pi - \arccos \left( \frac{s_0 \, c}{s \, c_0} \right) - \sum_{k=1}^{2n} e_k \arcsin \left( \frac{1}{\sqrt{1 - m_k s_0^2}} \sqrt{1 - \frac{s_0^2}{s^2}} \right). \] (4.15)

It follows that all the geodesics are closed.

**Proof:** We have
\[ \omega_\phi = \frac{\partial S}{\partial I_\phi} = \phi + \omega_\theta - s_0 \epsilon \int \frac{(1 + A(\theta))}{s \sqrt{s^2 - s_0^2}} d\theta. \] (4.16)
The change of variable \( u = \frac{\sqrt{s^2 - s_0^2}}{s} \) allows to evaluate the integral and gives
\[ \phi = L - K + \epsilon \arccos \left( \frac{s_0 \, c}{s \, c_0} \right) + \epsilon \sum_{k=1}^{2n} e_k \arcsin \left( \frac{1}{\sqrt{1 - m_k s_0^2}} \sqrt{1 - \frac{s_0^2}{s^2}} \right). \] (4.17)
When \( \theta \) increases from \( i \) (starting with \( \phi = 0 \)) to \( \pi - i \) we will have \( \epsilon = +1 \), hence (4.14). When \( \theta \) decreases from \( \pi - i \) (starting with \( \phi = \pi \)) to \( i \) we have \( \epsilon = -1 \), hence (4.15). At the end of the turn \( \phi \) has increased from 0 to \( 2\pi \) and the geodesic does close, as it should, since the metric is Zoll.

For future use let us prove
Proposition 15 When $\theta$ increases from $i$ to $\pi - i$ one has

$$K e^{i\phi} = \left( \frac{c}{s_0} + i \sqrt{1 - \frac{s_0^2}{s^2}} \right)^{2n} \prod_{k=1}^{2n} \left( \frac{s_0}{s} \sqrt{1 - m_k s^2} + i e_k \sqrt{1 - \frac{s_0^2}{s^2}} \right),$$

where

$$K = \frac{c_0}{s_0} \prod_{k=1}^{2n} \sqrt{1 - m_k s_0^2}.$$

Proof: This exponential produces as a first term

$$\exp \left( i \arccos \left( \frac{s_0 c}{c_0 s} \right) \right) = \frac{1}{c_0} \left( \frac{s_0}{s} + i \sqrt{1 - \frac{s_0^2}{s^2}} \right),$$

multiplied by the product involving the factors

$$\exp \left( i \arcsin \left( \frac{1}{\sqrt{1 - m_k s_0^2}} \right) \right) = \frac{1}{\sqrt{1 - m_k s_0^2}} \left( \frac{s_0}{s} \sqrt{1 - m_k s^2} + i e_k \sqrt{1 - \frac{s_0^2}{s^2}} \right),$$

leading to (4.18).

□

Remark: this gives another description of the first half of the geodesics. For the second half it is sufficient to change $\phi \to 2\pi - \phi$ in (4.18).

4.2 Geodesics from the integrals

As pointed out in [10] for the cubic case, one can recover rather conveniently the geodesics from the very extra integrals. It is therefore interesting to check that, by this rather different approach, we do get the relation (4.18) for the first half of the geodesic.

On the torus $H = E$ and $P_\phi = L$ we have

$$\frac{S_1 + i S_2}{E^{n+1/2}} = e^{-i\phi} \frac{(S + iT)}{E^{n+1/2}}.$$

These quantities are easily extracted out from the generating functions

$$\frac{T}{E^{n+1/2}} = \frac{L}{\sqrt{E}} \sum_{k=0}^{n} \lambda_{2k}(\theta) \left( \frac{L^2}{E} \right)^k = s_0 L(\theta, s_0^2),$$

where $L$ is given by (3.21), and

$$\frac{S}{E^{n+1/2}} = \sqrt{1 - \frac{s_0^2}{s^2}} \sum_{k=0}^{n} \lambda_{2k-1}(\theta) \left( \frac{L^2}{E} \right)^k = \sqrt{1 - \frac{s_0^2}{s^2}} M(\theta, s_0^2),$$

where $M$ is given by (3.22). One can write

$$(-s) L(\theta, s_0^2) = \left( \sqrt{1 - \frac{s_0^2}{s^2}} \right)^{2n} \left( \sum_{k=0}^{n-1} (-1)^k \mu^{2k} \mu_{2k}(\theta) + \frac{c}{\mu} \sum_{k=0}^{n} (-1)^k \mu^{2k+1} \mu_{2k+1}(\theta) \right).$$
with \( \mu = \frac{s_0}{\sqrt{s^2 - s_0^2}} \). Using the relations (A.9) and (A.10) with \( \xi \to i\mu \) we obtain

\[
is_0 \mathcal{L}(\theta, s_0^2) = -\sqrt{1 - \frac{s_0^2}{s^2}} \left( \mathcal{P} - \overline{\mathcal{P}} \right) + s_0 \frac{c}{s} \frac{1}{2i} \left( \mathcal{P} + \overline{\mathcal{P}} \right),
\]

where

\[
\mathcal{P} = \prod_{k=1}^{2n} \left( \sqrt{1 - \frac{s_0^2}{s^2}} + ie_k \frac{s_0}{s} \sqrt{1 - m_k s^2} \right).
\]

It remains to compute

\[
\mathcal{M}(\theta, s_0^2) = \left( \sqrt{1 - \frac{s_0^2}{s^2}} \right)^{2n} \left( \sum_{k=0}^{n-1} (-1)^k \mu^{2k} (H)_{2k}(\theta) \right.
\]

\[
- c \mu \sum_{k=0}^{n-1} (-1)^k \mu^{2k+1} (H)_{2k+1}(\theta),
\]

which leads to

\[
\sqrt{1 - \frac{s_0^2}{s^2}} \mathcal{M}(\theta, s_0^2) = \sqrt{1 - \frac{s_0^2}{s^2}} \left( \mathcal{P} - \overline{\mathcal{P}} \right) - s_0 \frac{c}{s} \frac{1}{2i} \left( \mathcal{P} + \overline{\mathcal{P}} \right).
\]

Hence we conclude to

\[
\frac{S_1 + iS_2}{E^{n+1/2}} = e^{-i\phi} \left[ \sqrt{1 - \frac{s_0^2}{s^2}} - is_0 \frac{c}{s} \right] \overline{\mathcal{P}},
\]

which can be written

\[
(-i)(-1)^n e^{-i\phi} \left[ s_0 \frac{c}{s} + i \sqrt{1 - \frac{s_0^2}{s^2}} \right]^{2n} \prod_{k=1}^{2n} \left( s_0 \frac{1}{s} \sqrt{1 - m_k s^2} + ie_k \sqrt{1 - \frac{s_0^2}{s^2}} \right).
\]

This conserved quantity, evaluated for \( t = 0 \) and \( s = s_0 \), has for value

\[
(-i)(-1)^n c_0 \prod_{k=1}^{2n} \sqrt{1 - m_k s_0^2} = (-i)(-1)^n K.
\]

And we do recover the relation (4.28).

Let us proceed to the second part of this article, devoted to the proof of Theorem 2.

5 Integrals of even degree in the momenta

Before dealing with the general case, let us first consider integrals which are quadratic in the momenta i.e., one of the Koenigs SI models [3]. Using the coordinates of [8] one has

\[
H^{(K)} = \frac{\cosh^2 x}{\rho + \sinh x} (P_x^2 + P_y^2) \quad \rho > 0,
\]

where

\[
\rho = \sqrt{1 - \frac{s_0^2}{s^2}}.
\]
and we will consider only the first extra integral
\[ S_1^{(K)} = \cos y(H^{(K)} - 2 \sinh x P_y^2) + 2 \sin y \cosh x P_x P_y, \] (5.2)
In the coordinates used throughout this work, and anticipating on the results of the next sections, we have
\[ H = \Pi^2 + \frac{P^2}{\sin^2 \theta}, \quad \Pi = \frac{P}{A(\theta)}, \quad A(\theta) = 1 + A(\theta), \quad A = \frac{c}{\sqrt{1 - m s^2}}, \] (5.3)
and for the first extra integral
\[ S_1 = \cos \phi \left( H - \frac{(1 + c \sqrt{1 - m s^2})}{s^2} P^2 \right) - \sin \phi \left( c + \sqrt{1 - m s^2} \right) \Pi P_\phi, \quad m < 1. \] (5.4)
That this metric is not globally defined on \( S^2 \) stems from the fact that \( A([0, \pi]) = [-1, +1] \) instead of \( A([0, \pi]) \subset (-1, +1) \). Nevertheless, these two metrics are related by the following local diffeomorphism:

**Proposition 16** Provided that \( m < 0 \) one has
\[ H^{(K)} = H/\lambda^2, \quad 2\lambda^2 = \sqrt{|m|} = \rho + \sqrt{\rho^2 + 1}, \]
and
\[ e^x = \frac{1}{\sqrt{|m|}} \frac{1 + \sqrt{1 + |m| s^2}}{1 - c}, \quad \lambda^2 s^2 = \frac{\rho + \sinh x}{\cosh^2 x}. \] (5.5)

**Proof:** Elementary computational check. \(\Box\)

**Remarks:**

1. The coordinates \((\theta, \phi)\) appear rather weird when compared to the coordinates \((x, y)\) which lead to a simple structure for the integrals given in (5.2).

2. The fact that for trigonometric integrals the metric is not globally defined on \( S^2 \) was first observed in \([3]\). However, in this same reference, it was shown that there could be, for special choices of the parameters of Koenigs models, SI systems globally defined either on \( \mathbb{R}^2 \) or on \( \mathbb{H}^2 \) which cannot be obtained in our approach since, as shown in \([1]\), the metric structure
\[ g = A^2 d\theta^2 + \sin^2 \theta d\phi^2 \quad \theta \in (0, \pi) \quad \phi \in S^1, \] (5.6)
is locally adapted only to \( S^2 \).

Let us turn ourselves to the general case where \( \sharp(S_1) = \sharp(S_2) = 2n \) for \( n \geq 1 \). The hamiltonian remains
\[ H = \frac{1}{2} \left( \Pi^2 + \frac{P^2}{\sin^2 \theta} \right) \quad \Pi = \frac{P}{A(\theta)}. \] (5.7)
The extra integrals will be again
\[ S_1 = \cos \phi S + \sin \phi T, \quad S_2 = \cos \phi T - \sin \phi S, \quad (5.8) \]
but this time we have
\[ S = \sum_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} p_{\phi}^{2k}, \quad T = \prod_{k=0}^{n-1} \lambda_{2k}(\theta) H^{n-k-1} p_{\phi}^{2k+1}, \quad (5.9) \]
defining an array of functions of \( \theta \) of the form
\[
\left( \begin{array}{cccc}
\lambda_{-1} = 1 & \lambda_1 & \lambda_3 & \ldots & \lambda_{2n-1} \\
\lambda_0 & \lambda_2 & \ldots & \lambda_{2(n-1)}
\end{array} \right).
\]

**Remark:** Since most of the proofs are similar to those of Section 3, we will proceed speedily.

**Proposition 17** \( S_1 \) and \( S_2 \) will be integrals iff the \( \lambda \)'s solve the differential system:

\[
0 \leq k \leq n : \quad \begin{cases} 
 s^2 \lambda_{2k-1}'(\theta) = - A \lambda_{2(k-1)} & (a) \\
 s^2 \lambda_{2k}'(\theta) = \lambda_{2(k-1)}' - \frac{c}{s} \lambda_{2(k-1)} + A \lambda_{2k-1} & (b)
\end{cases}
\quad (5.10)
\]

with the conventional values \( \lambda_{2n} = \lambda_{-2} = 0 \).

**Proof:** Similar to the proof of Proposition [1] in Section 3. \( \square \)

**Remark:** Here too one can get rid of the derivatives in the right hand side of relation (5.10) by a simple recurrence which gives for \( 0 \leq k \leq n - 1 \):

\[
s^{2(k+1)} \lambda_{2k}' = - sc(\lambda_0 + s^2 \lambda_2 + \ldots + s^{2(k-1)} \lambda_{2(k-1)}) + (1 + s^2 \lambda_1 + \ldots + s^{2k} \lambda_{2k-1})A, \quad (5.11)
\]
while for \( k = n \) one gets the purely algebraic relation

\[
s c(\lambda_0 + s^2 \lambda_2 + \ldots + s^{2(n-1)} \lambda_{2(n-1)}) = (1 + s^2 \lambda_1 + \ldots + s^{2n} \lambda_{2n-1})A. \quad (5.12)
\]

Let us consider again the generating functions, defined this time by:

**Definition 4** The generating functions are now
\[
\mathcal{L}(\theta, \xi) = \sum_{k=0}^{n} \lambda_{2k}(\theta) \xi^k \quad \mathcal{M}(\theta, \xi) = \sum_{k=0}^{n} \lambda_{2k-1}(\theta) \xi^k. \quad (5.13)
\]

**Proposition 18** The differential system (5.10) is equivalent, in terms of generating functions, to

\[
(1 + \tau) \partial_{\theta} \mathcal{L} - \tau \frac{c}{s} \mathcal{L} = \frac{A}{s^2} \mathcal{M}, \quad \partial_{\theta} \mathcal{M} = \tau A \mathcal{L}, \quad \tau = - \frac{\xi}{s^2}. \quad (5.14)
\]

**Proof:** Similar to the proof of Proposition [2] \( \square \)
5.1 The solution for integrals of even degree

For \( n \geq 1 \), let us first define the functions

\[
\forall k \in \{1, 2, \ldots, 2n - 1\} : \quad h_k(\theta) = e_k \sqrt{1 - m_k s^2}, \quad m_k < 1, \quad (5.15)
\]

and

\[
\mathcal{H}(\theta, \xi) \equiv \prod_{k=1}^{2n-1} (1 + \xi h_k(\theta)) = \sum_{k=0}^{2n-1} (H)_k(\theta) \xi^k. \quad (5.16)
\]

The \((H)_k\) are nothing but the symmetric functions constructed in terms of the \( h_k \). In terms of these objects, we can write down the solution for the \( \lambda \)'s.

**Definition 5** Let us define, for \( k \in \{0, 1, \ldots, n - 1\} \)

\[
\lambda_{2k} = \frac{(-1)^{k+1}}{s^{2k+1}} \left\{ \sum_{l=0}^{k} \left( \frac{n - l - 1}{n - k - 1} \right) (H)_{2l+1} + c \sum_{l=0}^{k} \left( \frac{n - l - 1}{n - k - 1} \right) (H)_{2l} \right\}, \quad (5.17)
\]

for \( k \in \{0, 1, \ldots, n - 2\} \)

\[
\lambda_{2k+1} = \frac{(-1)^{k+1}}{s^{2(k+1)}} \left\{ \sum_{l=0}^{k+1} \left( \frac{n - l}{n - k - 1} \right) (H)_{2l+1} + c \sum_{l=0}^{k} \left( \frac{n - l - 1}{n - k - 1} \right) (H)_{2l+1} \right\}, \quad (5.18)
\]

and for \( k = n - 1 \)

\[
\lambda_{2n-1} = \frac{(-1)^{n}}{s^{2n}} \left\{ \sum_{l=0}^{n-1} (H)_{2l+1} + c \sum_{l=0}^{n-1} (H)_{2l+1} \right\}. \quad (5.19)
\]

**Remark:** For \( n = 1 \) we have a single parameter \( m_1 = m \) and the previous formulae give

\[
\lambda_0 = -\frac{1}{s} (H)_1 + c \quad \lambda_1 = -\frac{1}{s^2} (1 + c(H)_1) \quad (H)_1 = \sqrt{1 - m \, s^2}. \quad (5.20)
\]

Since the integrals are given by

\[
S = H^2 + \lambda_1 P_{\phi}^2 \quad T = \lambda_0 \Pi P_{\phi}, \quad (5.21)
\]

we have proved the relation anticipated in (5.4).

Let us compute the generating functions.

**Proposition 19** The generating functions \( \mathcal{L} \) and \( \mathcal{M} \) are given by

\[
(-s)\mathcal{L}(\theta, \xi) = \sum_{l=0}^{n-1} \psi_{l,n-1} (H)_{2l+1} + c \sum_{l=0}^{n-1} \psi_{l,n-1} (H)_{2l} \quad (5.22)
\]

and by

\[
\mathcal{M}(\theta, \xi) = \sum_{l=0}^{n-1} \psi_{l,n} (H)_{2l+1} + c \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l+1} \quad (5.23)
\]

where

\[
\tau = -\frac{\xi}{s^2}, \quad \psi_{l,n} = \tau^l (1 + \tau)^{n-l}, \quad 0 \leq l \leq n. \quad (5.24)
\]

\[\text{This set is empty for } n = 1.\]
Proof: From the definition of $L$ and upon use of the formulae in (5.17) we have, after an exchange of the summation orders

$$(-s)L = \sum_{l=0}^{n-1} (H)_{2l+1} \sum_{k=l}^{n-1} \left( \frac{n-l-1}{n-k-1} \right) \tau^k + c \sum_{l=0}^{n-1} (H)_{2l} \sum_{k=l}^{n-1} \left( \frac{n-l-1}{n-k-1} \right) \tau^k.$$  

(5.25)

Noticing that

$$\sum_{k=l}^{n-1} \left( \frac{n-l-1}{n-k-1} \right) \tau^k = \tau^l \sum_{s=0}^{n-l-1} \left( \frac{n-l-1}{n-l-1-s} \right) \tau^s = \tau^l (1 + \tau)^{n-l-1} = \psi_{l,n-1},$$  

(5.26)

implies relation (5.22).

In $M$ the piece having $c$ factored out is

$$\sum_{k=1}^{n} \tau^k \sum_{l=0}^{k-1} \left( \frac{n-l-1}{n-k} \right) (H)_{2l+1} = \sum_{l=0}^{n-1} \sum_{k=l+1}^{n} \left( \frac{n-l-1}{n-k} \right) \tau^k = \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l+1}.$$  

(5.27)

The remaining piece is

$$\sum_{k=0}^{n-1} \tau^k \sum_{l=0}^{k} \left( \frac{n-l}{n-k} \right) (H)_{2l} + \tau^n \sum_{l=0}^{n-1} (H)_{2l} = \sum_{l=0}^{n-1} (H)_{2l} \sum_{k=l}^{n} \left( \frac{n-l}{n-k} \right) \tau^k.$$  

(5.28)

The last term, using again the binomial theorem, becomes

$$\sum_{l=0}^{n} \psi_{l,n} (H)_{2l},$$  

(5.29)

and this ends up the proof. □

Remark: Here too, using these generating functions, one can check easily the algebraic relation (5.12) which is

$$(scL - AM) \bigg|_{\tau=-1} = 0.$$  

(5.30)

Now let us use Proposition 18 to prove

Proposition 20  The generating functions $L$ and $M$, given by (5.22) and (5.23), are solutions of the equations

$$(1 + \tau) \partial_\theta L - \frac{c}{s} L = \frac{A}{s^2} M, \quad \partial_\theta M = \tau A L,$$  

(5.31)

where

$$A(\theta) = 1 + c \sum_{k=1}^{2n-1} \frac{e_k}{\sqrt{1 - m_k s^2}}.$$  

(5.32)

It follows that the $\lambda$'s given by Definition 5 are indeed a solution of the differential system (18), and this implies that $S_1$ and $S_2$ are indeed integrals for the hamiltonian (5.7).
Proof: Similar to the proof of Proposition 4 in Section 3. One has to define the splitting
\begin{align}
L_1 &= -\frac{1}{s} \sum_{l=0}^{n-1} \psi_{l,n-1} (H)_{2l+1} \\
L_2 &= -\frac{c}{s} \sum_{l=0}^{n-1} \psi_{l,n-1} (H)_{2l}
\end{align}
(5.33)
and similarly
\begin{align}
M_1 &= \sum_{l=0}^{n-1} \psi_{l,n} (H)_{2l} \\
M_2 &= c \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l+1}.
\end{align}
(5.34)
Let us begin with the derivative of $L_1$. We have
\begin{equation}
-(1 + \tau) \partial_\theta (s^{-1}) \Psi_{l,n-1} = \frac{c}{s} \Psi_{l,n},
\end{equation}
(5.35)
and
\begin{align}
(1 + \tau) \partial_\theta \Psi_{l,n-1} &= -\frac{c}{s} 2\tau(1 + \tau) \partial_l \Psi_{l,n-1} = -\frac{c}{s} \left[2l\Psi_{l,n} + 2(n - l - 1)\Psi_{l+1,n} \right],
\end{align}
(5.36)
while the derivatives of $(H)_{2l+1}$ are given in Appendix A with $\nu = 2n - 1$. Adding all the terms we get
\begin{equation}
(1 + \tau) \partial_\theta L_1 = \frac{(A - 1)}{s^2}.
\end{equation}
(5.37)
Observing that
\begin{equation}
\tau \psi_{l,n-1} = \psi_{l+1,n} \quad \Rightarrow \quad -\tau \frac{c}{s} L_1 = \frac{M_2}{s^2},
\end{equation}
(5.38)
we obtain
\begin{equation}
(1 + \tau) \partial_\theta L_1 - \tau \frac{c}{s} L_1 = \frac{M_2}{s^2} + \frac{(A - 1)}{s^2} M_1.
\end{equation}
(5.39)
Similarly one can prove
\begin{equation}
(1 + \tau) \partial_\theta M_1 - \tau \frac{c}{s} M_1 = \frac{M_1}{s^2} + \frac{(A - 1)}{s^2} M_2.
\end{equation}
(5.40)
The sum of these two relations proves (5.31).
Let us proceed with the derivative of $M_1$. We have
\begin{equation}
\partial_\theta \Psi_{l,n} = -\frac{c}{s} \left[2l\Psi_{l,n} + 2(n - l)\Psi_{l+1,n} \right]
\end{equation}
(5.41)
and using Appendix A we get, after easy algebra and use of (A.8):
\begin{equation}
\partial_\theta M_1 = -\frac{c}{s} \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l} - \frac{(A - 1)}{s} \sum_{l=0}^{n-1} \psi_{l+1,n} (H)_{2l+1} = \tau L_2 + \tau (A - 1)L_1.
\end{equation}
(5.42)
Similarly one can prove
\begin{equation}
\partial_\theta M_2 = \tau L_1 + \tau (A - 1)L_2.
\end{equation}
(5.43)
From which the second relation in (5.31) follows.
5.2 The Poisson algebra

Let us begin with:

**Proposition 21** One defines the moments \(\sigma_l\) and their generating functions as follows

\[
S_+ S_- = S^2 + T^2 = \sum_{l=0}^{2n} \sigma_l H^{2n-l} P_{\phi}^{2l}, \quad \Sigma(\xi) = \sum_{l=0}^{2n} \sigma_l \xi^l. \tag{5.44}
\]

These moments are related with the \(\lambda's\) according to

\[
l = 0 : \quad \sigma_0 = 1
\]

\[
1 \leq l \leq n : \quad \sigma_l = \sum_{k=0}^{l} S_{k,l-k}
\]

\[
n + 1 \leq l \leq 2n : \quad \sigma_l = \sum_{k=l-n}^{n} S_{k,l-k}, \tag{5.45}
\]

where

\[
S_{k,l} = \lambda_{2k-1} \lambda_{2l-1} + \lambda_{2(k-1)} \lambda_{2l} - \frac{1}{s^2} \lambda_{2(k-1)} \lambda_{2(l-1)}, \tag{5.46}
\]

and with the convention that \(\lambda_{-2} = 0\).

**Proof:** Since \(\lambda_{-2} = 0\) it is convenient to write

\[
T = \prod \sum_{k=0}^{n-1} \lambda_{2k} H^{n-1} P_{\phi}^{2k+1} = \prod \sum_{k=0}^{n} \lambda_{2(k-1)} H^{n-k} P_{\phi}^{2k-1}. \tag{5.47}
\]

It follows that

\[
S_+ S_- = \sum_{k,L=0}^{n} S_{k,L} H^{2n-k-L} P_{\phi}^{2(k+L)}, \tag{5.48}
\]

where \(S_{k,l}\) is given by \([5.46]\). Defining \(l = L + k\) we get

\[
S_+ S_- = \sum_{k=0}^{n} \sum_{l=k}^{n+k} S_{k,l-k} H^{2n-l} P_{\phi}^{2l}, \tag{5.49}
\]

and upon exchange of the summations we end up with

\[
S_+ S_- = \sum_{l=0}^{n} \left( \sum_{k=0}^{l} S_{k,l-k} \right) H^{2n-l} P_{\phi}^{2l} + \sum_{l=n+1}^{2n} \left( \sum_{k=l-n}^{n} S_{k,l-k} \right) H^{2n-l} P_{\phi}^{2l}, \tag{5.50}
\]

which concludes the proof. \(\square\)

Let us compute \(\Sigma(\xi)\) in terms of the generating functions.

**Proposition 22** One has

\[
\Sigma(\xi) = \xi (1 + \tau) L^2 + M^2, \quad \tau = -\frac{\xi}{s^2}. \tag{5.51}
\]
Proof: Using the relations in (5.45) we get
\[ \Sigma(\xi) = \sum_{l=0}^{n} \left( \sum_{k=0}^{l} S_{k,l-k} \right) \xi^l + \sum_{l=n+1}^{2n} \left( \sum_{k=l-n}^{n} S_{k,l-k} \right) \xi^l = \sum_{k,l=0}^{n} S_{k,l} \xi^{k+l}. \] (5.52)

Expressing \( S_{k,l} \) in terms of the \( \lambda \)'s we have to compute the following terms:
\[ \sum_{k=0}^{n} \lambda_{2k-1} \xi^k \sum_{l=0}^{n} \lambda_{2l-1} \xi^l = M^2, \quad \sum_{k=1}^{n} \lambda_{2(k-1)} \xi^k \sum_{l=0}^{n-1} \lambda_{2l} \xi^l = \xi L^2, \] (5.53)
and
\[ -\frac{1}{s^2} \sum_{k=1}^{n} \lambda_{2(k-1)} \xi^k \sum_{l=1}^{n} \lambda_{2(l-1)} \xi^l = \xi \tau L^2. \] (5.54)

Adding up ends up the proof. \( \square \)

Let us compute the explicit form of the moments in terms of the parameters \( m_k \) appearing in \( A(\theta) \). To this end we will define, for the string \( M = (m_1, m_2, \ldots, m_{2n-1}) \) the symmetric functions \( (M)_k \):
\[ \prod_{k=1}^{2n-1} (1 + \xi m_k) = \sum_{k=0}^{2n-1} \xi^k (M)_k, \] (5.55)
and we will prove

**Proposition 23** The generating function of the moments is
\[ \Sigma(\xi) = (1 - \xi) \prod_{k=1}^{2n-1} (1 - \xi m_k), \quad \xi \in \mathbb{C}, \] (5.56)
giving the explicit formulae
\[ k = 0 : \quad \sigma_0 = 1 \]
\[ 1 \leq k \leq 2n - 1 : \quad \sigma_k = (-1)^k [(M)_k + (M)_{k-1}] \] (5.57)
\[ k = 2n : \quad \sigma_{2n} = (M)_{2n} = \prod_{k=1}^{2n-1} m_k. \]

Proof: Let us restrict ourselves to \( \tau > 0 \). Relation (5.23) gives
\[ \mathcal{M} = (1 + \tau)^n \left\{ \sum_{l=0}^{n-1} \left( \frac{\tau}{1+\tau} \right)^l (H)_{2l} + c \sum_{l=0}^{n-1} \left( \frac{\tau}{1+\tau} \right)^{l+1} (H)_{2l+1} \right\}. \] (5.58)

The coordinate change
\[ \frac{\tau}{1+\tau} = \eta^2, \quad 1 + \tau = \frac{1}{1+\eta^2}, \] (5.59)
allows to write
\[ \mathcal{M} = (1 + \tau)^n \left\{ \sum_{l=0}^{n-1} \eta^{2l} (H)_{2l} + \eta c \sum_{l=0}^{n-1} \eta^{2l+1} (H)_{2l+1} \right\}. \] (5.60)

Using the relations (A.9) and (A.10) given in Appendix A leads to
\[ \mathcal{M} = (1 + \tau)^n \left\{ \frac{1}{2} (1 + \eta c) \mathcal{H}(\theta, +\eta) + \frac{1}{2} (1 - \eta c) \mathcal{H}(\theta, -\eta) \right\} \] (5.61)

Using relation (5.22), and after similar steps, one gets
\[ -s(1 + \tau) \mathcal{L} = (1 + \tau)^n \left\{ \frac{1}{2\eta} (1 + \eta c) \mathcal{H}(\theta, +\eta) - \frac{1}{2\eta} (1 - \eta c) \mathcal{H}(\theta, -\eta) \right\}. \] (5.62)

The last step uses (5.51):
\[ \Sigma(\xi) = \mathcal{M}^2 - \eta^2 \left( s(1 + \tau) \mathcal{L} \right)^2 = (1 + \tau)^{2n} (1 - \eta^2 c^2) \mathcal{H}(\theta, +\eta) \mathcal{H}(\theta, -\eta). \] (5.63)

The proof of (5.56) follows from the relations
\[ 1 - \eta^2 c^2 = \frac{1 - \xi}{1 + \tau} \quad (1 + \tau)^{2n-1} \mathcal{H}(\theta, +\eta) \mathcal{H}(\theta, -\eta) = \prod_{l=1}^{2n-1} (1 - \xi m_l). \] (5.64)

Analytic continuation extends this result to \( \xi \in \mathbb{C} \). Expanding \( \Sigma(\xi) \) in powers of \( \xi \) gives the relations in (5.57). \( \square \)

**Remark:** It follows again that
\[ \Sigma(1) = \sum_{l=0}^{2n} \sigma_l = 0. \] (5.65)

Let us conclude this section with

**Proposition 24** One has the relation
\[ \{S_+, S_-\} = 2i \sum_{l=0}^{2n-1} (l + 1) \sigma_{l+1} H^{2n-l-1} P_{\phi}^{2l+1}. \] (5.66)

**Proof:** Extracting out from the bracket the \( \phi \) dependence gives
\[ \frac{\{S_+, S_-\}}{2i} = \frac{1}{2i} \frac{\partial}{\partial P_{\phi}} (S^2 + T^2) - \{S, T\}. \] (5.67)

The first term in the right hand side gives
\[ \sum_{l=0}^{2n-1} (l + 1) \sigma_{l+1} H^{2n-l-1} P_{\phi}^{2l+1} + \frac{1}{s^2} \sum_{l=0}^{2n} (2n - l) \sigma_l H^{2n-l} P_{\phi}^{2l+1}, \] (5.68)
so that the relation \((5.66)\) will hold true if we can prove the relation

\[
s^2 \{S, T\} = \sum_{l=0}^{2n} (2n - l) \sigma_l H^{2n-l} P_{\phi}^{2l+1}. \tag{5.69}
\]

Defining \(\Psi_{k+l}^2 = H^{2n-k-l} P_{\phi}^{2(k+l)-1}\) let us first compute the left hand member:

\[
\{S, T\} = \sum_{k,l=0}^{n} \frac{\Psi_{k+l}^2}{H} \left[ (n - k) \lambda_{2(k-1)} \{H, \lambda_{2(l-1)}\Pi\} - (n - l) \lambda_{2(l-1)} \{H, \lambda_{2k-1}\} + H \{\lambda_{2k-1}, \lambda_{2(l-1)}\Pi\} \right]. \tag{5.70}
\]

Let us change, in the second sum, \(l \rightarrow k\). Since we have

\[
\{H, \lambda_{2l-1}\} = 2\Pi \frac{\lambda_{2l-1}'}{A} = -\frac{2\Pi}{s^2} \lambda_{2(l-1)}, \tag{5.71}
\]

using relation (a) in \((5.10)\), one gets

\[
s^2 \{S, T\} = \sum_{k,l} \frac{\Psi_{k+l}^2}{H} \left[ 2(n - k) \lambda_{2k-1} \frac{s^2 \lambda_{2(l-1)}'}{A} \Pi^2 + 2(n - k) \lambda_{2(k-1)} \lambda_{2(l-1)} \Pi^2 + 2(n - k) \frac{c}{sA} \lambda_{2k-1} \lambda_{2(l-1)} P_{\phi}^2 + H \lambda_{2(k-1)} \lambda_{2(l-1)} \right]. \tag{5.72}
\]

Using \(\Pi^2 = H - \frac{P_{\phi}^2}{s^2}\) leads to

\[
s^2 \{S, T\} = \sum_{k,l} \left[ (2n - 2k + 1) \lambda_{2(k-1)} \lambda_{2(l-1)} + 2(n - k) \lambda_{2k-1} \frac{s^2 \lambda_{2(l-1)}'}{A} \right] \Psi_{k+l}^2
\]

\[
+ \sum_{k,l} 2(n - k) \left[ - \lambda_{2k-1} \frac{\lambda_{2(l-1)}'}{A} - \frac{1}{s^2} \lambda_{2(k-1)} \lambda_{2(l-1)} + \frac{c}{sA} \lambda_{2k-1} \lambda_{2(l-1)} \right] \Psi_{k+l+1}^2. \tag{5.73}
\]

Changing the summation index \(l - 1 \rightarrow l\) (recall that \(\lambda_{2n} = \lambda_{-2} = 0\)), we have

\[
\sum_k \sum_{l=1}^{n} 2(n - k) \lambda_{2k-1} \frac{s^2 \lambda_{2(l-1)}'}{A} \Psi_{k+l}^2 = \sum_{k,l} 2(n - k) \lambda_{2k-1} \frac{s^2 \lambda_{2l}'}{A}. \tag{5.74}
\]

Collecting the terms which display a factor \(A^{-1}\) we obtain

\[
\sum_{k,l} 2(n - k) \lambda_{2k-1} \frac{\lambda_{2k-1}}{A} \left( s^2 \lambda_{2l}' - \lambda_{2(l-1)}' + \frac{c}{s} \lambda_{2(l-1)} \right) \Psi_{k+l+1}^2 = \sum_{k,l} 2(n - k) \lambda_{2k-1} \lambda_{2l-1} \Psi_{k+l+1}^2, \tag{5.75}
\]

using the relations (b) in \((5.10)\). So we conclude to

\[
s^2 \{S, T\} = \sum_{k,l} (2n - 2k + 1) \lambda_{2(k-1)} \lambda_{2(l-1)} \Psi_{k+l}^2 + \sum_{k,l} 2(n - k) \left( \lambda_{2k-1} \lambda_{2l-1} - \frac{1}{s^2} \lambda_{2(k-1)} \lambda_{2(l-1)} \right) \Psi_{k+l+1}^2. \tag{5.76}
\]
Let us now consider the right hand member of (5.69):

$$\sum_{L=0}^{2n} (2n - L) \sigma_L H^{2n-L-1} P^{2L+1}_\phi. \quad (5.77)$$

Expressing the $\sigma_L$ as in (5.45) and exchanging the summations we get

$$\sum_{k} \sum_{L=k}^{k+n} (2n - L) S_{k,L-k} H^{2n-L-1} P^{2L+1}_\phi = \sum_{k,l} (2n - k - l) S_{k,l} \Psi_{k+l+1}^{2n}. \quad (5.78)$$

Let us recall that

$$S_{k,l} = \lambda_{2(k-1)} \lambda_{2l} + \lambda_{2k-1} \lambda_{2l-1} - \frac{1}{s^2} \lambda_{2(k-1)} \lambda_{2(l-1)}. \quad (5.81)$$

So we have a first piece

$$\sum_{k,l} 2(n - k) \left( \lambda_{2k-1} \lambda_{2l-1} - \frac{1}{s^2} \lambda_{2(k-1)} \lambda_{2(l-1)} \right) \Psi_{k+l+1}^{2n}, \quad (5.79)$$

while in the second the change $l \rightarrow l - 1$ leads to

$$\sum_{k,l} (2n - k - l) \lambda_{2(k-1)} \lambda_{2l} \Psi_{k+l+1}^{2n} = \sum_{k,l} (2n - 2k + 1) \lambda_{2(k-1)} \lambda_{2(l-1)} \Psi_{k+l}^{2n}. \quad (5.80)$$

These two pieces prove (5.69), hence the Proposition.

We can therefore conclude to:

**Proposition 25** The set of observables

$$H, \quad P_\phi, \quad S_+ = e^{-i\phi} (S + iT), \quad S_- = e^{i\phi} (S - iT), \quad (5.81)$$

is indeed a Poisson algebra with

$$S_+ S_- = \sum_{l=0}^{2n} \sigma_l H^{2n-l} P^{2l}_\phi, \quad \{S_+, S_-\} = 2i \sum_{l=0}^{2n-1} (l + 1) \sigma_{l+1} H^{2n-l-1} P^{2l+1}_\phi. \quad (5.82)$$

### 5.3 Global aspects

We have considered the metric and the SI hamiltonian

$$g = A^2(\theta) \, d\theta^2 + s^2 \, d\phi^2, \quad H = \Pi^2 + \frac{P^2_{\phi}}{s^2}, \quad \Pi = \frac{P_{\theta}}{A(\theta)}. \quad (5.83)$$

where

$$A(\theta) = 1 + A(\theta), \quad A(\theta) = c \sum_{k=1}^{2n} \frac{e_k}{\sqrt{1 - m_k s^2}}. \quad (5.84)$$

Let us prove
Proposition 26 The metric constructed above is \textbf{never} globally defined on $M = \mathbb{S}^2$.

\textbf{Proof:} The metric will be globally defined on $\mathbb{S}^2$ provided that $\mathcal{A}([0, \pi]) \subset (-1, +1)$. This is not the case since

$$\mathcal{A}(\theta = 0) = -\mathcal{A}(\theta = \pi) = S + e_{2n-1} \quad S = \sum_{k=1}^{2n-2} e_k. \quad (5.85)$$

If $S$ is strictly positive, then $\mathcal{A}(\theta = 0) \geq 1$. If $S = 0$ and $e_{2n-1}$ is positive we have $\mathcal{A}(\theta = 0) = 1$. If $S = 0$ and $e_{2n-1}$ is negative, then $\mathcal{A}(\theta = 0) = -1$. If $S$ is strictly negative just reverse $\theta = 0$ and $\theta = \pi$.

It cannot be Zoll since $\mathcal{A}(0) = -\mathcal{A}(\pi)$ cannot vanish. \hfill \Box

\textbf{This concludes the proof of Theorem 2.} \hfill \Box

\textbf{Remarks:}

1. The difference between the case of extra integrals with even degrees and odd degrees is quite surprising. However one could already observe this phenomenon for the $(1, 2)$ Koenigs case.

2. Since these metrics are not globally defined, they are of little interest. Nevertheless the formulae we gave for the geodesics in Section 3, for the case of extra integrals of odd degrees are still valid: one needs just to change everywhere the summations over $\{1, 2, \ldots, 2n\}$ into summations over $\{1, 2, \ldots, 2n-1\}$.

\section{An example: the quartic case}

Since it was studied by Novichkov in \cite{7}, let us examine this case more closely.

\subsection{Our solution}

Our solution of the problem was obtained using the coordinates $(\theta, \phi)$. We have for hamiltonian

$$H = \Pi^2 + \frac{P^2_{\phi}}{s^2}, \quad \Pi = \frac{P_{\theta}}{A}, \quad A = 1 + c \sum_{k=1}^{3} \frac{e_k}{\sqrt{1 - m_k s^2}}. \quad (6.1)$$

The extra integrals are given by

$$S_1 = \cos \phi S + \sin \phi \mathcal{T}, \quad S_2 = \cos \phi \mathcal{T} - \sin \phi S, \quad (6.2)$$

with

$$S = H^2 + \lambda_1 H P^2_{\phi} + \lambda_3 P^4_{\phi}, \quad \mathcal{T} = \Pi P_{\phi} \left( \lambda_0 H + \lambda_2 P^2_{\phi} \right). \quad (6.3)$$
where

\[
\begin{align*}
\lambda_0 &= -\frac{1}{s} ((H)_1 + c), \\
\lambda_1 &= -\frac{1}{s^2} ((H)_2 + c (H)_1 + 2), \\
\lambda_2 &= \frac{1}{s^3} ((H)_3 + c (H)_2 + (H)_1 + c), \\
\lambda_3 &= \frac{1}{s^4} (c (H)_3 + (H)_2 + c (H)_1 + 1).
\end{align*}
\]

(6.4)

Later on we will need also the quadratic relations which follow from the conservation of \( S_1^2 + S_2^2 \):

\[
\begin{align*}
(a) & \quad \lambda_0^2 + 2\lambda_1 = \sigma_1, \\
(b) & \quad \lambda_1^2 - \frac{\lambda_0^2}{s^2} + 2\lambda_0 \lambda_2 + 2\lambda_3 = \sigma_2, \\
(c) & \quad \lambda_2^2 - 2\frac{\lambda_0 \lambda_2}{s^2} + 2\lambda_1 \lambda_3 = \sigma_3, \\
(d) & \quad \lambda_3^2 - \frac{\lambda_2^2}{s^2} = \sigma_4.
\end{align*}
\]

(6.5)

Now let us state Novichkov’s results.

### 6.2 Novichkov results

They are expressed in the coordinates \((x, \phi)\) with the hamiltonian

\[
H = \Pi^2 + h_x^2 P_{\phi}^2, \quad \Pi = h_x P_x, \quad h_x = D_x h(x).
\]

(6.6)

Up to slight changes, his extra integrals \((S_1, S_2)\) are constructed from

\[
S = H^2 + l_1(x) H P_{\phi}^2 + l_3(x) P_{\phi}^4, \quad T = P_{x} P_{\phi} (l_0(x) H + l_2(x) P_{\phi}^2).
\]

(6.7)

Let us first state his main result and present a short check:

**Theorem 3 (Novichkov)** The previously defined dynamical system is SI provided that \( h \) be a solution of the first order non-linear ODE:

\[
\mathcal{N}(h) \equiv 2 P h h_x^3 + \left[ - (h^2 + 2A_2) P' + A_5 \right] h_x^2 - P^2 = 0,
\]

(6.8)

where

\[
P = A_3 \cosh x + A_4 \sinh x.
\]

(6.9)

**Check:** The conservation of \( S_1 \) gives

\[
l_1' = -l_0, \quad l_2' = -l_3, \quad l_3' = -l_2,
\]

(6.10)

as well as

\[
l_0' - \frac{h_{xx} l_0}{h_x} = h_x^2, \quad l_2' - \frac{h_{xx} l_2}{h_x} = h_x^2 \left( h_x^2 + l_1 + \frac{h_{xx} l_0}{h_x} \right).
\]

(6.11)
An obvious consequence is \( l''_2 = l_2 \) which gives
\[
l_2(x) \equiv P(x) = A_3 \cosh x + A_4 \sinh x, \quad l_3 = -P'.
\]
(6.12)

Integrating for \( l_0 \) one gets
\[
l_0 = (h + A_1)h_x \quad \implies \quad l_1 = -\frac{1}{2}(h + A_1)^2 - A_2.
\]
(6.13)

It remains to use the relation involving \( l_2 \), which produces a second order ODE
\[
(h + A_1)^3 h_x^3 + P h_{xx} + h_x^5 + \left( -\frac{1}{2}(h + A_1)^2 - A_2 \right) h_x^3 - P' h_x = 0,
\]
(6.14)
and this is nothing but the second order ODE (2.3.3) obtained in [7]. Let us observe that we can take \( A_1 = 0 \) by a translation of \( h \), since only \( h_x \) appears in the hamiltonian.

Then, by the construction of an integrating factor, Novichkov reduces (6.14) to the first order ODE (6.8).

\[\square\]

6.3 Connection relations

They follow from a comparison of the hamiltonians (6.1) and (6.6) and read
\[
h_x(\theta) = \frac{1}{s} \quad D_\theta x = \frac{A}{s} \quad D_\theta h(\theta) = \frac{A}{s^2}.
\]
(6.15)

Integrating for \( x(\theta) \) and \( h(\theta) \) one obtains
\[
x(\theta) - x_0 = \ln \left( \frac{s}{1 + c} \right) + \sum_{k=1}^{3} \ln \frac{s}{1 + h_k(\theta)}, \quad h(\theta) = -\frac{1}{s}((H)_1 + c),
\]
(6.16)
where \( h_k(\theta) = e_k \sqrt{1 - m_k s^2} \).

A comparison of the extra integrals gives two new relations
\[
P = \frac{1}{s^4} \left( (H)_3 + c(H)_2 + (H)_1 + c \right) \quad P' = -\frac{1}{s^4} \left( c(H)_3 + (H)_2 + c(H)_1 + 1 \right).
\]
(6.17)

Now we are in position to connect both formulations. We have first
\[
e^x = \frac{e^{x_0}}{M} \frac{1}{s^4} E_-, \quad E_- = (1 - c) \prod_{k=1}^{3} (1 - h_k), \quad M = m_1 m_2 m_3,
\]
(6.18)
and
\[
e^{-x} = e^{-x_0} \frac{1}{s^4} E_+, \quad E_+ = (1 + c) \prod_{k=1}^{3} (1 + h_k).
\]
(6.19)

Let us notice the useful relations
\[
\frac{1}{2}(E_+ + E_-) = c(H)_3 + (H)_2 + c(H)_1 + 1,
\]
(6.20)
\[
\frac{1}{2}(E_+ - E_-) = (H)_3 + c(H)_2 + (H)_1 + c.
\]
So, starting from

\[ P = \alpha e^x + \beta e^{-x}, \quad \alpha = \frac{A_3 + A_4}{2}, \quad \beta = \frac{A_3 - A_4}{2}, \]  

(6.21)

using the previous formulae for the exponentials, and with the help of the relations

\[ \alpha e^{x_0} - \beta e^{-x_0} = -1 \]  

(6.22)

we conclude that \( P \) and \( P' \) are indeed given by the relations already obtained in (6.17).

So the various objects appearing in Novichkov’s ODE (6.8) are, in our notations:

\[ h = \lambda_0, \quad h_x = \frac{1}{s}, \quad P = \frac{\lambda_2}{s}, \quad P' = -\lambda_3. \]  

(6.23)

As a side remark, let us point out that our solution gives a \textit{parametric} solution of (6.8) in terms of the coordinate \( \theta \).

Now we will check this ODE. We have

\[ \frac{N(h)}{h_x^2} = 2P h h_x - (h^2 + 2A_2)P' + A_5 - \frac{P^2}{h_x^2}, \]  

(6.24)

which, translated in our notations, becomes:

\[ \frac{N(h)}{h_x^2} = 2\frac{\lambda_0 \lambda_2}{s^2} + \lambda_2^2 + 2A_2 \lambda_3 + A_5 - \lambda_2^2. \]  

(6.25)

Using (6.5)(c) leads to

\[ \frac{N(h)}{h_x^2} = \lambda_3(\lambda_0^2 + 2\lambda_1) + 2A_2 \lambda_3 + A_5 - \sigma_3, \]  

(6.26)

and using (6.5)(a), we end up with

\[ \frac{N(h)}{h_x^2} = (2A_2 + \sigma_1)\lambda_3 + A_5 - \sigma_3, \]  

(6.27)

which does vanish by the identification of parameters

\[ A_2 = -\frac{\sigma_1}{2}, \quad A_5 = \sigma_3. \]  

(6.28)

Let us explain now why there is no way to give an \textit{explicit} solution to (6.8) keeping the coordinate \( x \). The solution we obtained is \textit{explicit} provided that one is using for coordinate \( \theta \). Now, looking at the formula (6.16) for \( x(\theta) \) it is clear that its reciprocal function cannot be explicit.
7 Conclusion

Let us conclude with the following remarks:

- We have seen the importance of a “good” choice of the coordinates in order to be able to solve explicitly the differential systems of SI systems. Unfortunately the choice of “good” coordinates is not algorithmic.

- We have proved the existence of a solution for the differential systems (3.4) and (5.10). However the problem of uniqueness is left open.

- The main surprise of this article is probably that SI systems are not necessarily Zoll, even for metrics of revolution! This is particularly striking for the case of integrals of even degrees. Our conjecture that the converse is true, i.e. that any Zoll metric of revolution generates a SI system, remains an open problem.

- In the approach of Matveev and Shevchishin [6], one considers extra integrals having three different dependences with respect to the coordinate $\phi$:

  1. A trigonometric dependence, considered in this work. For extra integrals of odd degree in the momenta we have obtained SI systems globally defined on $S^2$.

  2. A hyperbolic dependence. In the cubic case this choice led to no globally defined metric, so it does not seem very attractive to generalize it to higher degrees.

  3. A quadratic dependence. This case was solved in [11] for any degree of the extra integrals: it leads to metrics globally defined either on $\mathbb{R}^2$ or on $\mathbb{H}^2$ but never on $S^2$.

- If one starts looking for a SI system with one Killing vector $\partial_\phi$ and a quadratic integral of the form

  $$S = A(\theta, \phi) P_\theta^2 + B(\theta, \phi) P_\theta P_\phi + C(\theta, \phi) P_\phi^2,$$  

  (7.1)

  one can prove that the only possible $\phi$-dependence of the various functions is, as considered in [6], either trigonometric or hyperbolic or quadratic and there is no other possibility. This is Koenigs theorem [3]. However, it is an open problem to ascertain whether this remains true for SI systems with cubic and higher degree integrals.

- The study of the quantization of all of these models could be interesting albeit difficult. The conformally invariant quantization constructed in [2] may play a prominent role.
Appendices

A Appendix A

The functions $h_k(\theta)$ such that
\[ \forall k \in \{1, 2, \ldots, \nu\} : \quad h_k(\theta) = e_k \sqrt{1 - m_k s^2} \quad e_k^2 = 1, \]
allow to define the functions $(H)_k(\theta)$ by the generating function
\[ H(\theta, \xi) \equiv \nu \prod_{k=1}^{\nu} (1 + \xi h_k(\theta)) = \sum_{k=0}^{\nu} (H)_k(\theta) \xi^k. \quad \text{(A.1)} \]

**Proposition 27** The derivatives with respect to $\theta$ of the functions $(H)_k$ are given by
\[ \forall k \in \{1, 2, \ldots, \nu\} : \quad (H)_k' \equiv \frac{c}{s} \left( h_k - \frac{1}{h_k} \right), \quad (A.2) \]
provided that
\[ (H)_{-1}(\theta) \equiv 0, \quad A(\theta) = 1 + c \sum_{k=1}^{\nu} \frac{1}{h_k(\theta)}. \quad \text{(A.3)} \]

**Proof:** Using the relations
\[ h_k' = \frac{c}{s} \left( h_k - \frac{1}{h_k} \right), \quad \text{(A.4)} \]
we deduce
\[ \frac{\partial_\theta H}{H} = \frac{c}{s} \left( \sum_{k=1}^{\nu} \frac{\xi h_k}{1 + \xi h_k} - \sum_{k=1}^{\nu} \frac{\xi}{h_k(1 + \xi h_k)} \right) = \frac{c \xi \partial_\xi H}{s H} - \frac{c}{s} \sum_{k=1}^{\nu} \frac{\xi}{h_k(1 + \xi h_k)}. \quad \text{(A.5)} \]
The last sum is transformed according to
\[ \sum_{k=1}^{\nu} \frac{\xi(1 - \xi^2 h_k^2 + \xi^2 h_k^2)}{h_k(1 + \xi h_k)} = \sum_{k=1}^{\nu} \frac{\xi(1 - \xi h_k)}{h_k} + \xi^2 \sum_{k=1}^{\nu} \frac{\xi h_k}{1 + \xi h_k} \]
\[ = \xi \frac{A - 1}{c} - \nu \xi^2 + \xi^2 \frac{\xi \partial_\xi H}{H}. \quad \text{(A.6)} \]
Hence we have obtained
\[ \partial_\theta H = \frac{c}{s} \left( \xi \partial_\xi H + \xi^2 (\nu - \xi \partial_\xi) H \right) - \xi \frac{A - 1}{s} H. \quad \text{(A.7)} \]
Expanding in powers of $\xi$ gives \(\text{(A.2)}\). \hfill \square

Let us mention also the useful relation
\[ c (H)_{\nu-1} = (A - 1) (H)_\nu. \quad \text{(A.8)} \]
Splitting in (A.1) the even and the odd powers of $\xi$ gives

$$\frac{1}{2}(H(\theta, \xi) + H(\theta, -\xi)) = \sum_{k=0}^{\nu} \xi^{2k} (H)_{2k}(\theta),$$

(A.9)

and

$$\frac{1}{2}(H(\theta, \xi) - H(\theta, -\xi)) = \sum_{k=0}^{\nu-1} \xi^{2k+1} (H)_{2k+1}(\theta).$$

(A.10)

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