Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Given a semistable vector bundle $E$ over $X$, we show that its direct image $F_*E$ under the Frobenius map $F$ of $X$ is again semistable. We deduce a numerical characterization of the stable rank-$p$ vector bundles $F_*L$, where $L$ is a line bundle over $X$.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$ and let $F : X \to X$ be the relative $k$-linear Frobenius map. It is by now a well-established fact that on any curve $X$ there exist semistable vector bundles $E$ such that their pull-back under the Frobenius map $F^*E$ is not semistable [LanP], [LasP]. In order to control the degree of instability of the bundle $F^*E$, one is naturally lead (through adjunction $\text{Hom}_{O_X}(F^*E, E') = \text{Hom}_{O_{X_1}}(E, F_*E')$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

1.1. Theorem. Assume that $g \geq 2$. If $E$ is a semistable vector bundle over $X$ (of any degree), then $F_*E$ is also semistable.

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that $F_*L$ is stable for a line bundle $L$ ([LanP] Proposition 1.2) and that in small characteristics the bundle $F_*E$ is stable for any stable bundle $E$ of small rank [JRXY]. The main ingredient of the proof is Faltings’ cohomological criterion of semistability. We also need the fact that the generalized Verschiebung $V$, defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank-$r$ vector bundles over $X_1$ with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map $F$,

$$V_r : \mathcal{M}_{X_1}(r) \to \mathcal{M}_X(r), \quad E \mapsto F_*E$$

is dominant for large $r$. We actually show a stronger statement for large $r$.

1.2. Proposition. If $l \geq g(p - 1) + 1$ and $l$ prime, then the generalized Verschiebung $V_l$ is generically étale for any curve $X$. In particular $V_l$ is separable and dominant.

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant $\nu$ of a vector bundle $E$, defined as

$$\nu(E) := \mu_{\text{max}}(F^*E) - \mu_{\text{min}}(F^*E),$$

where $\mu_{\text{max}}$ (resp. $\mu_{\text{min}}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of $F^*E$.

1.3. Proposition. For any semistable rank-$r$ vector bundle $E$

$$\nu(E) \leq \min((r - 1)(2g - 2), (p - 1)(2g - 2)).$$
We note that the inequality \( \nu(E) \leq (r-1)(2g-2) \) was proved in [SB] Corollary 2 and in [S] Theorem 3.1. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (=number of pieces) of the Harder-Narasimhan filtration of \( F^*E \) is at most \( p \) for semistable \( E \).

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant \( \nu \).

1.4. Proposition. Let \( E \) be a stable rank-\( p \) vector bundle over \( X \). Then the following statements are equivalent.

1. There exists a line bundle \( L \) such that \( E = F_*L \).
2. \( \nu(E) = (p-1)(2g-2) \).

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case \( \mu(E) = g-1 \).

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles \( E \) with slope \( \mu(E) = g-1 \).

Let \( E \) be a semistable vector bundle over \( X \) of rank \( r \) and let \( s \) be the integer defined by the equality
\[
\mu(E) = g - 1 + \frac{s}{r}.
\]
Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map \( F : X \to X_1 \), we obtain
\[
\mu(F_*E) = g - 1 + \frac{s}{pr}.
\]
Let \( \pi : \tilde{X} \to X \) be a connected étale covering of degree \( n \) and let \( \pi_1 : \tilde{X}_1 \to X_1 \) denote its twist by the Frobenius of \( k \) (see [R] section 4). The diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{F} & \tilde{X}_1 \\
\pi \downarrow & & \downarrow \pi_1 \\
X & \xrightarrow{F} & X_1
\end{array}
\]
is Cartesian and we have an isomorphism
\[
\pi_1^*(F_*E) \cong F_*(\pi_*E).
\]
Since semistability is preserved under pull-back by a separable morphism of curves, we see that \( \pi_*E \) is semistable. Moreover if \( F_*(\pi_*E) \) is semistable, then \( F_*E \) is also semistable.

Let \( L \) be a degree \( d \) line bundle over \( \tilde{X}_1 \). The projection formula
\[
F_*(\pi_*E \otimes F^*L) = F_*(\pi^*E) \otimes L
\]
shows that semistability of \( F_*(\pi_*E) \) is equivalent to semistability of \( F_*(\pi^*E \otimes F^*L) \).

Let \( \tilde{g} \) denote the genus of \( \tilde{X} \). By the Riemann-Hurwitz formula \( \tilde{g} - 1 = n(g - 1) \). We compute
\[
\mu(\pi^*E \otimes F^*L) = n(g - 1) + n\frac{s}{r} + pd = \tilde{g} - 1 + n\frac{s}{r} + pd,
\]
which gives
\[
\mu(F_*(\pi^*E \otimes F^*L)) = \tilde{g} - 1 + n\frac{s}{pr} + d.
\]

2.1. Lemma. For any integer \( m \) there exists a connected étale covering \( \pi : \tilde{X} \to X \) of degree \( n = p^km \) for some \( k \).
Proof. If the \( p \)-rank of \( X \) is nonzero, the statement is clear. If the \( p \)-rank is zero, we know by Corollaire 4.3.4 [R] that there exist connected étale coverings \( Y \to X \) of degree \( p^t \) for infinitely many integers \( t \) (more precisely for all \( t \) of the form \( (l - 1)(g - 1) \) where \( l \) is a large prime). Now we decompose \( m = p^s u \) with \( p \) not dividing \( u \). We then take a covering \( Y \to X \) of degree \( p^t \) with \( t \geq s \) and a covering \( \tilde{X} \to Y \) of degree \( u \).

Now the lemma applied to the integer \( m = pr \) shows existence of a connected étale covering \( \pi : \tilde{X} \to X \) of degree \( n = p^h m \). Hence \( n^{\mathbb{Z}/pr} \) is an integer and we can take \( d \) such that \( n^{\mathbb{Z}/pr} + d = 0 \).

To summarize, we have shown that for any semistable \( E \) over \( X \) there exists a covering \( \pi : \tilde{X} \to X \) and a line bundle \( L \) over \( \tilde{X}_1 \) such that the vector bundle \( \tilde{E} := \pi^*E \otimes F^*L \) is semistable with \( \mu(\tilde{E}) = \tilde{g} - 1 \) and such that semistability of \( F_*\tilde{E} \) implies semistability of \( F_*E \).

3. Proof of Theorem 1.1

In order to prove semistability of \( F_*E \) we shall use the cohomological criterion of semistability due to Faltings [F].

3.1. Proposition (Lemme 2.4). Let \( E \) be a rank-\( r \) vector bundle over \( X \) with \( \mu(E) = g - 1 \) and \( l \) an integer > \( \frac{2}{3}(g - 1) \). Then \( E \) is semistable if and only if there exists a rank-\( l \) vector bundle \( G \) with trivial determinant such that

\[
h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.
\]

Moreover if the previous condition holds for one bundle \( G \), it holds for a general bundle by upper semicontinuity of the function \( G \mapsto h^0(X, E \otimes G) \).

Remark. The proof of this proposition (see L section 2.4) works over any algebraically closed field \( k \).

By Proposition 1.2 (proved in section 4) we know that \( V_l \) is dominant when \( l \) is a large prime number. Hence a general vector bundle \( G \in \mathcal{M}_X(l) \) is of the form \( F^*G' \) for some \( G' \in \mathcal{M}_{X_1}(l) \). Consider a semistable \( E \) with \( \mu(E) = g - 1 \). Then by Proposition 3.1 \( h^0(X, E \otimes G) = 0 \) for general \( G \in \mathcal{M}_X(l) \). Assuming \( G \) general, we can write \( G = F^*G' \) and we obtain by adjunction

\[
h^0(X, E \otimes F^*G') = h^0(X_1, F_*E \otimes G') = 0.
\]

This shows that \( F_*E \) is semistable by Proposition 3.1.

4. Proof of Proposition 1.2

According to [MS] section 2 it will be enough to prove the existence of a stable vector bundle \( E \in \mathcal{M}_{X_1}(l) \) satisfying \( F^*E \) stable and

\[
h^0(X_1, B \otimes \text{End}_0(E)) = 0,
\]

because the vector space \( H^0(X_1, B \otimes \text{End}_0(E)) \) can be identified with the kernel of the differential of \( V_l \) at the point \( E \in \mathcal{M}_{X_1}(l) \). Here \( B \) denotes the sheaf of locally exact differentials over \( X_1 \) (see R section 4).

Let \( l \neq p \) be a prime number and let \( \alpha \in JX_1[l] \cong JX[l] \) be a nonzero \( l \)-torsion point. We denote by

\[\pi : \tilde{X} \to X \quad \text{and} \quad \pi_1 : \tilde{X}_1 \to X_1\]

the associated cyclic étale cover of \( X \) and \( X_1 \) and by \( \sigma \) a generator of the Galois group \( \text{Gal} (\tilde{X}/X) = \mathbb{Z}/l\mathbb{Z} \). We recall that the kernel of the Norm map

\[\text{Nm} : J\tilde{X} \longrightarrow JX\]
has $l$ connected components and we denote by

$$i: P := \text{Prym}(\tilde{X}/X) \subset J\tilde{X}$$

the associated Prym variety, i.e., the connected component containing the origin. Then we have an isogeny

$$\pi^* \times i: JX \times P \longrightarrow J\tilde{X}$$

and taking direct image under $\pi$ induces a morphism

$$P \longrightarrow \mathcal{M}_X(l), \quad L \mapsto \pi_* L.$$  

Similarly we define the Prym variety $P_1 \subset JX_1$ and the morphism $P_1 \to \mathcal{M}_X_1(l)$ (obtained by twisting with the Frobenius of $k$). Note that $\pi_1* L$ is semistable for any $L \in P_1$ and stable for general $L \in P_1$ (see e.g. [B]). Since $F^*(\pi_1* L) \cong \pi_1* (F^* L)$ — see diagram (2.4) — and since $F^*$ induces the Verschiebung $V_\pi: P_1 \to P$, which is surjective, we obtain that $\pi_1*, L$ and $F^*(\pi_1* L)$ are stable for general $L \in P_1$.

Therefore Proposition 1.2 will immediately follow from the next Proposition.

4.1. Proposition. If $l \geq g(p-1)+1$ then there exists a cyclic degree $l$ étale cover $\pi_1: \tilde{X}_1 \to X_1$ with the property that

$$h^0(X_1, B \otimes \text{End}_0(\pi_1* L)) = 0$$

for general $L \in P_1$.

**Proof.** By relative duality for the étale map $\pi_1$ we have $(\pi_1* L)^* \cong (\pi_1* L)^{-1}$. Therefore

$$\text{End}(\pi_1* L) \cong \pi_1* L \otimes \pi_1* L^{-1} \cong \pi_1* (L^{-1} \otimes \pi_1* L)$$

by the projection formula. Moreover since $\pi_1$ is Galois étale we have a direct sum decomposition

$$\pi_1* \pi_1* L \cong \bigoplus_{i=0}^{l-1} (\sigma^i)^* L.$$ 

Putting these isomorphisms together we find that

$$H^0(X_1, B \otimes \text{End}(\pi_1* L)) = H^0(X_1, B \otimes \pi_1* \left( \bigoplus_{i=0}^{l-1} (\sigma^i)^* L \right)$$

$$= 0 = \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \pi_1* (L^{-1} \otimes (\sigma^i)^* L))$$

$$= H^0(X_1, B \otimes \pi_1* O_{\tilde{X}_1}) \oplus \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \pi_1* (L^{-1} \otimes (\sigma^i)^* L)).$$

Moreover $\pi_* O_{\tilde{X}_1} = \bigoplus_{i=0}^{l-1} \alpha^i$, which implies that

$$(4.1) \quad H^0(X_1, B \otimes \text{End}_0(\pi_1* L)) = \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \alpha^i) \oplus \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \pi_1* (L^{-1} \otimes (\sigma^i)^* L)).$$

Let us denote for $i = 1, \ldots, l-1$ by $\phi_i$ the isogeny

$$\phi_i: P_1 \longrightarrow P_1, \quad L \mapsto L^{-1} \otimes (\sigma^i)^* L.$$  

Since the function $L \mapsto h^0(X_1, B \otimes \text{End}_0(\pi_1* L))$ is upper semicontinuous, it will be enough to show the existence of a cover $\pi_1: \tilde{X}_1 \rightarrow X_1$ satisfying

1. for $i = 1, \ldots, l-1$, $h^0(X_1, B \otimes \alpha^i) = 0$ (or equivalently, $P$ is an ordinary abelian variety).
2. for $M$ general in $P_1$, $h^0(X_1, B \otimes \pi_1* M) = 0$.

Note that these two conditions imply that the vector space (4.1) equals $\{0\}$ for general $L \in P_1$, because the $\phi_i$’s are surjective.

We recall that $\ker (\pi_1^*: JX_1 \rightarrow J\tilde{X}_1) = \langle \alpha \rangle \cong \mathbb{Z}/l\mathbb{Z}$ and that

$$P_1[l] = P_1 \cap \pi_1^*(JX_1) \cong \alpha^\perp/\langle \alpha \rangle$$

where $\alpha^\perp = \{ \beta \in JX_1[l] \mid \omega(\alpha, \beta) = 1 \}$ and $\omega: JX_1[l] \times JX_1[l] \rightarrow \mu_l$ denotes the symplectic Weil form. Consider a $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. Then $\pi_1^* \beta \in P_1[l]$ and

$$\pi_1* \pi_1* \beta = \bigoplus_{i=0}^{l-1} \beta \otimes \alpha^i.$$
Again by upper semicontinuity of the function $M \mapsto h^0(X_1, B \otimes \pi_1^* M)$ one observes that the conditions (1) and (2) are satisfied because of the following lemma (take $M = \pi_1^* \beta$).

4.2. Lemma. If $l \geq g(p - 1) + 1$ then there exists a pair $(\alpha, \beta) \in JX_1[l] \times JX_1[l]$ satisfying

1. $\alpha \neq 0$ and $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$,
2. for $i = 1, \ldots, l - 1$ $h^0(X_1, B \otimes \alpha^i) = 0$,
3. for $i = 0, \ldots, l - 1$ $h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0$.

Proof. We adapt the proof of [R] Lemme 4.3.5. We denote by $\mathbb{F}_l$ the finite field $\mathbb{Z}/l\mathbb{Z}$. Then there exists a symplectic isomorphism $JX_1[l] \cong \mathbb{F}_l^g \times \mathbb{F}_l^g$, where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in $\mathbb{F}_l^g$ and additively in $\mathbb{F}_l^g$. A quick computation shows that the number of isotropic 2-planes in $\mathbb{F}_l^g \times \mathbb{F}_l^g$ equals

$$N(l) = \frac{(l^2g - 1)(l^2g - 2 - 1)}{(l^2 - 1)(l - 1)}.$$ 

Let $\Theta_B \subset JX_1$ denote the theta divisor associated to $B$. Then by [R] Lemma 4.3.5 the cardinality $A(l)$ of the finite set $\Sigma(l) := JX_1[l] \cap \Theta_B$ satisfies

$$A(l) \leq l^2g - 2g(p - 1).$$

Suppose that there exists an isotropic 2-plane $\Pi \subset \mathbb{F}_l^g \times \mathbb{F}_l^g$ which contains $\leq l - 2$ points of $\Sigma(l)$. Then we can find a pair $(\alpha, \beta)$ satisfying the 3 properties of the Lemma as follows: any nonzero point $x \in \Pi$ determines a line (= $\mathbb{F}_l$-vector space of dimension 1). Since a line contains $l - 1$ nonzero points, we obtain at most $(l - 1)(l - 2)$ nonzero points lying on lines generated by $\Sigma(l) \cap \Pi$. Since $(l - 1)(l - 2) < l^2 - 1$ there exists a nonzero $\alpha$ in the complement of these lines. Now we note that there are $l - 1$ affine lines parallel to the line generated by $\alpha$ and the $l$ points on any of these affine lines are of the form $\beta \alpha^i$ for $i = 0, \ldots, l - 1$ for some $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. The points $\Sigma(l) \cap \Pi$ lie on at most $l - 2$ such affine lines, hence there exists at least one affine line parallel to $\langle \alpha \rangle$ avoiding $\Sigma(l)$. This gives $\beta$.

Finally let us suppose that any isotropic 2-plane contains $\geq l - 1$ points of $\Sigma(l)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \{(x, \Pi) \mid x \in \Pi \cap \Sigma(l) \text{ and } \Pi \text{ isotropic 2-plane}\}.$$ 

with cardinality $|S|$. Then by our assumption we have

$$(4.2) \quad |S| \geq (l - 1)N(l).$$

On the other hand, since any nonzero $x \in \mathbb{F}_l^g \times \mathbb{F}_l^g$ is contained in $\frac{l^2g - 2 - 1}{l - 1}$ isotropic 2-planes, we obtain

$$(4.3) \quad |S| \leq \frac{l^2g - 2 - 1}{l - 1}A(l).$$

Putting (4.2) and (4.3) together, we obtain

$$A(l) \geq \frac{l^2g - 1}{l + 1}.$$ 

But this contradicts the inequality $A(l) \leq l^2g - 2g(p - 1) - 1$ if $l \geq g(p - 1) + 1$. □

This completes the proof of Proposition 4.1. □

Remark. It has been shown [O] Theorem A.6 that $V_r$ is dominant for any rank $r$ and any curve $X$, by using a versal deformation of a direct sum a $r$ line bundles.

Remark. We note that $V_r$ is not separable when $p$ divides the rank $r$ and $X$ is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in \mathcal{M}_{X_1}(r)$ identifies with the quotient...
satisfying the inequalities

\[ \mu \geq \frac{1}{p} (\mu_{\text{min}}(F^*E) + (p-1)(g-1)) \]

hence

\[ \mu(F^*E) \leq \mu_{\text{min}}(F^*E) + (p-1)(g-1). \]

Similarly we consider the subbundle \( S \hookrightarrow F^*E \) with maximal slope, i.e., \( \mu(S) = \mu_{\text{max}}(F^*E) \) and \( S \) semistable. Taking the dual and proceeding as above, we obtain that

\[ \mu(F^*E) \geq \mu_{\text{max}}(F^*E) - (p-1)(g-1). \]

Now we combine both inequalities and we are done.

**Remark.** We note that the inequality of Proposition 1.3 is sharp. The maximum \((p-1)(2g-2)\) is obtained for the bundles \( E = F_*E' \) (see \[JRXY\] Theorem 5.3).

6. Characterization of direct images

Consider a line bundle \( L \) over \( X \). Then the direct image \( F_*L \) is stable (\[LanP\] Proposition 1.2) and the Harder-Narasimhan filtration of \( F^*F_*L \) is of the form (see \[JRXY\])

\[ 0 = V_0 \subset V_1 \subset \ldots \subset V_{p-1} \subset V_p = F^*F_*L, \quad \text{with} \quad V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}. \]

In particular \( \nu(F_*L) = (p-1)(2g-2) \). In this section we will show a converse statement.

More generally let \( E \) be a stable rank-\( rp \) vector bundle with \( \mu(E) = g - 1 + \frac{d}{rp} \) for some integer \( d \) and satisfying

1. the Harder-Narasimhan filtration of \( F^*E \) has \( l \) terms.
2. \( \nu(E) = (p-1)(2g-2) \).

**Questions.** Do we have \( l \leq p \)? Is \( E \) of the form \( E = F_*G \) for some rank-\( r \) vector bundle \( G \)? We will give a positive answer in the case \( r = 1 \) (Proposition 6.1).

Let us denote the Harder-Narasimhan filtration by

\[ 0 = V_0 \subset V_1 \subset \ldots \subset V_{p-1} \subset V_p = F^*E, \quad V_i/V_{i-1} = M_i. \]

satisfying the inequalities

\[ \mu_{\text{max}}(F^*E) = \mu(M_1) > \mu(M_2) > \ldots > \mu(M_l) = \mu_{\text{min}}(F^*E). \]

The quotient \( F^*E \to M_i \) gives via adjunction a nonzero map \( E \to F_*M_i \). Since \( F_*M_i \) is semistable, we obtain that \( \mu(E) \leq \mu(F_*M_i) \). This implies that \( \mu(M_i) \geq g - 1 + \frac{d}{p} \). Similarly taking the dual of the inclusion \( M_i \subset F^*E \) gives a map \( F^*(E^*) \to M_i^* \) and by adjunction \( E^* \to F_*(M_i^*) \). Let us denote \( \mu(M_i^*) = g - 1 + \delta \), so that \( \mu(F_*(M_i^*)) = g - 1 + \frac{d}{p} \). Because of semistability of \( F_*(M_i^*) \), we obtain \( -(g - 1 + \frac{d}{rp}) = \mu(E^*) \leq \mu(F^*(M_i^*)), \) hence \( \delta \geq -2p(g - 1) - \frac{d}{r} \). This implies that
\[ \mu(M_1) \leq (2p - 1)(g - 1) + \frac{d}{r}. \]
Combining this inequality with \( \mu(M_i) \geq g - 1 + \frac{d}{r} \) and the assumption \( \mu(M_1) - \mu(M_l) = (p - 1)(2g - 2) \), we obtain that
\[ \mu(M_1) = (2p - 1)(g - 1) + \frac{d}{r}, \quad \mu(M_l) = g - 1 + \frac{d}{r}. \]

Let us denote by \( r_i \) the rank of the semistable bundle \( M_i \). We have the equality
\[ \sum_{i=1}^{l} r_i = rp. \]

Since \( E \) is stable and \( F_*(M_i) \) is semistable and since these bundles have the same slope, we deduce that \( r_i \geq r \). Similarly we obtain that \( r_1 \geq r \).

Note that it is enough to show that \( r_l = r \). Since \( E \) is stable and \( F_* M_i \) semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce the integers for \( i = 1, \ldots, l - 1 \)
\[ \delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g - 1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i). \]
Then we have the equality
\[ \sum_{i=1}^{l-1} \delta_i = \mu(M_l) - \mu(M_1) + 2(l - 1)(g - 1) = 2(l - p)(g - 1). \]

We note that if \( \delta_i < 0 \), then \( \text{Hom}(M_i, M_{i+1} \otimes \omega) = 0 \).

6.1. Proposition. Let \( E \) be stable rank-\( p \) vector bundle with \( \mu(E) = g - 1 + \frac{d}{p} \) and \( \nu(E) = (p - 1)(2g - 2) \). Then \( E = F_* L \) for some line bundle \( L \) of degree \( g - 1 + d \).

Proof. Let us first show that \( l = p \). We suppose that \( l < p \). Then \( \sum_{i=1}^{l-1} \delta_i = 2(l - p)(g - 1) < 0 \) so that there exists a \( k \leq l - 1 \) such that \( \delta_k < 0 \). We may choose \( k \) minimal, i.e., \( \delta_i \geq 0 \) for \( i < k \).

Then we have
\[ \mu(M_k) > \mu(M_i) + 2(g - 1) \quad \text{for } i > k. \]

We recall that \( \mu(M_i) \leq \mu(M_{k+1}) \) for \( i > k \). The Harder-Narasimhan filtration of \( V_k \) is given by the first \( k \) terms of the Harder-Narasimhan filtration of \( F_* E \). Hence \( \mu_{\text{min}}(V_k) = \mu(M_k) \).

Consider now the canonical connection \( \nabla \) on \( F_* E \) and its first fundamental form
\[ \phi_k : V_k \leftrightarrow F_* E \xrightarrow{\nabla} F_* E \otimes \omega_X \xrightarrow{\nabla} (F_* E/V_k) \otimes \omega_X. \]

Since \( \mu_{\text{min}}(V_k) > \mu(M_i \otimes \omega) \) for \( i > k \) we obtain \( \phi_k = 0 \). Hence \( \nabla \) preserves \( V_k \) and since \( \nabla \) has zero \( p \)-curvature, there exists a subbundle \( E_k \subset E \) such that \( F_* E_k = V_k \).

We now evaluate \( \mu(E_k) \). By assumption \( \delta_i \geq 0 \) for \( i < k \). Hence
\[ \mu(M_i) \geq \mu(M_1) - 2(i - 1)(g - 1) \quad \text{for } i \leq k, \]
which implies that
\[ \deg(V_k) = \sum_{i=1}^{k} r_i \mu(M_i) \geq \text{rk}(V_k) \mu(M_1) - 2(g - 1) \sum_{i=1}^{k} r_i (i - 1). \]

Hence we obtain
\[ p \mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g - 1)C, \]
where \( C \) denotes the fraction \( \frac{\sum_{i=1}^{k} r_i (i - 1)}{\text{rk}(V_k)} \). We will prove in a moment that \( C \leq \frac{\nu - 1}{2} \), so that we obtain by substitution
\[ p \mu(E_k) \geq (2p - 1)(g - 1) + d - (g - 1)(p - 1) = p(g - 1) + d = p \mu(E), \]
contradicting stability of $E$. Now let us show that $C \leq \frac{p-1}{2}$ or equivalently
\[
\sum_{i=1}^{k} i r_i \leq \frac{p+1}{2} \sum_{i=1}^{k} r_i.
\]
But that is obvious if $k \leq \frac{p-1}{2}$. Now if $k > \frac{p-1}{2}$ we note that passing from $E$ to $E^*$ reverses the order of the $\delta_i$’s, so that the index $k^*$ for $E^*$ satisfies $k^* \leq \frac{p-1}{2}$. This proves that $l = p$.

Because of (6.1) we obtain $r_i = 1$ for all $i$ and therefore $E = F_* M_p$. □

7. Stability of $F_* E$?

Is stability also preserved by $F_*$?

We show the following result in that direction.

7.1. Proposition. Let $E$ be a stable vector bundle over $X$. Then $F_* E$ is simple.

Proof. Using relative duality $(F_* E)^* \cong F_*(E^* \otimes \omega^{1-p}_X)$ we obtain
\[
H^0(X_1, \text{End}(F_* E)) = H^0(X, F_* F_* E \otimes E^* \otimes \omega^{1-p}_X).
\]
Moreover the Harder-Narasimhan filtration of $F_* F_* E$ is of the form (see [JRXY])
\[
0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F_* F_* E, \quad \text{with} \quad V_i/V_{i-1} \cong E \otimes \omega^{p-i}_X.
\]
We deduce that
\[
H^0(X, F_* F_* E \otimes E^* \otimes \omega^{1-p}_X) = H^0(X, V_1 \otimes E^* \otimes \omega^{1-p}_X) = H^0(X, \text{End}(E)),
\]
and we are done. □

References

[B] A. Beauville: On the stability of the direct image of a generic vector bundle, preprint available at
\[http://math.unice.fr/~beauvill/pubs/imdir.pdf\]
[F] G. Faltings: Projective connections and G-bundles, J. Alg. Geometry 2, No. 3 (1993), 507-568
[JRX Y] K. Joshi, S. Ramanan, E.Z. Xia, J.K. Yu: On vector bundles destabilized by Frobenius pull-back, Compositio Math. 142 (2006), 616-630
[L] J. Le Potier: Module des fibrés semi-stables et fonctions thêta, Moduli of vector bundles (Sanda 1994, Kyoto 1994) 83-101, Lecture Notes in Pure and Appl. Math. 179, Dekker, New York, 1996
[LauP] H. Lange, C. Pauly: On Frobenius-destabilized rank-2 vector bundles over curves, [math.AG/0309450]
[LasP] Y. Laszlo, C. Pauly: The Frobenius map, rank 2 vector bundles and Kummer’s quartic surface in characteristic 2 and 3, Adv. Math. 185, No. 2 (2004), 246-269
[MS] V.B. Mehta, S. Subramanian: Nef line bundles which are not ample, Math. Zeit. 219 (1995), 235-244
[O] B. Osserman: The generalized Verschiebung map for curves of genus 2, Math. Ann., to appear
[R] M. Raynaud: Sections des fibrés vectoriels sur une courbe, Bull. Soc. Math. France, Vol. 110 (1982), 103-125
[SB] N.I. Shepherd-Barron: Semistability and reduction mod p, Topology, Vol. 37, No. 3 (1998), 659-664
[S] X. Sun: Remarks on semistability of $G$-bundles in positive characteristic, Compositio Math. 119 (1999), No. 1, 41-52

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