Eventually homological isomorphisms and Gorenstein projective modules

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Abstract

We prove that a certain eventually homological isomorphism between module categories induces a triangle equivalence between their singularity categories, Gorenstein defect categories and the stable categories of Gorenstein projective modules. Further, we show that Auslander-Reiten conjecture and Gorenstein symmetry conjecture can be reduced by eventually homological isomorphisms. Applying the results to arrow removal and vertex removal, we describe the Gorenstein projective modules over some non-monomial algebras, and we verify the Auslander-Reiten conjecture for certain algebras.

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1 Introduction

A functor $F : \mathcal{B} \rightarrow \mathcal{C}$ between abelian categories is called an eventually homological isomorphism if there is an integer $t$ such that for every $j > t$, there is an isomorphism $\text{Ext}^j_{\mathcal{B}}(X, Y) \cong \text{Ext}^j_{\mathcal{C}}(FX, FY)$ for all objects $X, Y \in \mathcal{B}$. This notion was introduced in [23], and it arises naturally in reducing homological properties of finite dimensional algebras. Indeed, Psaroudakis et al. [25] characterized when the functor $eA \otimes_A - : \text{mod}A \rightarrow \text{mod}eAe$ is an eventually homological isomorphism, and in that case, they transferred the
Gorensteinness, singularity categories and Fg condition of A to $eAe$, where A is an algebra and e is an idempotent of A. Recently, Erdmann et al. showed that the arrow removal operation, passing from a bound quiver algebra $A = kQ/I$ to $A/\langle \alpha \rangle$, yields an eventually homological isomorphism if $\alpha$ is an arrow which does not occur in a minimal generating set of $I$ [13]. We refer to [15, 26, 29] for more discussions on eventually homological isomorphisms.

Recall that the singularity category $D_{sg}(A)$ of an algebra A is the Verdier quotient of the bounded derived category of finitely generated modules over A by the full subcategory of perfect complexes [6]. According to [6], there is an embedding functor $F$ from the stable category $\text{Gproj} A$ of finitely generated Gorenstein projective modules to $D_{sg}(A)$, and the Gorenstein defect category of $A$ is defined to be Verdier quotient $D_{def}(A) := D_{sg}(A) / \text{Im} F$, see [5]. This category measures how far the algebra A is from being Gorenstein, because $A$ is Gorenstein if and only if $D_{def}(A)$ is trivial [5]. However, for non-Gorenstein algebras, not much is known about their Gorenstein defect categories. In recent years, many people described the Gorenstein projective modules over some special kinds of algebras, such as Nakayama algebras [28] and monomial algebras [10], and some experts compared the Gorenstein defect categories between two algebras related to one another [9, 18, 19, 20].

In [25], the authors proved that the Gorensteinness of algebras is invariant under certain eventually homological isomorphisms. Inspired by this, we consider the following natural question: is the Gorenstein defect categories preserved under eventually homological isomorphisms? and how about the homological conjectures related to the Gorensteinness? We answer these questions by the following theorem, which is listed as Theorem 3.3, Theorem 3.4, Theorem 3.8 and Theorem 3.9 in this paper.

**Theorem I.** Let $A$ and $B$ be two finite dimensional algebras, and $F : \text{mod} A \to \text{mod} B$ be an eventually homological isomorphism which is essentially surjective. Assume that F admits a left adjoint and a right adjoint. Then F induces the following triangle equivalences

$$D_{sg}(A) \cong D_{sg}(B), \quad \text{Gproj } A \cong \text{Gproj } B$$

and $A$ satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does $B$.

Theorem I can be applied to arrow removal and vertex removal to reduce the Gorenstein homological properties of algebras.

**Corollary I.** (Corollary 4.1) Let $A = kQ/I$ be a quotient of a path algebra $kQ$ over a field $k$. Choose an arrow $\alpha$ in $Q$ such that $\alpha$ does not occur in a
minimal generating set of I and define \( B = A/\langle \alpha \rangle \). Then \( \text{Gproj} A \cong \text{Gproj} B \) and \( D_{def}(A) \cong D_{def}(B) \). Moreover, \( A \) satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does \( B \).

We mention that the above arrow removal operation was used to reduce the Gorensteinness, singularity categories and the \( F_g \) condition in [13], and was investigated with respect to the Hochschild (co)homology and the finitistic dimension conjecture [12, 15].

Combining Theorem I with the result of Psaroudakis et al., we get the following corollary, which transfers the Gorenstein homological properties of \( A \) to \( eAe \). This transition is called vertex removal in [14, 15].

**Corollary II.** (Corollary 4.2) Let \( A \) be an algebra and \( e \) be an idempotent in \( A \). Assume that \( \text{pd}_{eAe} eA < \infty \) and \( \text{id}_A (\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) (or equivalently, \( \text{pd}_{(eA)e} eA < \infty \) and \( \text{pd}_A (\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \)). Then the functor \( eA \otimes_A - : \text{mod}A \to \text{mod}eAe \) induces triangle equivalences \( \text{Gproj} A \cong \text{Gproj}(eAe) \) and \( D_{def}(A) \cong D_{def}(eAe) \). Moreover, \( A \) satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does \( eAe \).

We mention that the functor \( eA \otimes_A - \) also induces a singular equivalence under the condition of Corollary II, see [7, 25]. Now Corollary II can be compared with a recent result of Li et al. [18]. Assume that \( \text{Tor}^i_{eAe} (Ae, G) = 0 \) for any \( G \in \text{Gproj}(eAe) \) and \( i \) sufficiently large. Then the functor \( eA \otimes_A - \) induces triangle equivalences \( D_{sp}(A) \cong D_{sp}(eAe) \), \( D_{def}(A) \cong D_{def}(eAe) \) and \( \text{Gproj} A \cong \text{Gproj}(eAe) \) if and only if \( \text{pd}_A (\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \), \( \text{pd}_{eAe} eA < \infty \) and both \( eA \otimes_A - \) and \( Ae \otimes_{eAe} - \) preserve modules of finite Gorenstein projective dimension, see [18, Corollary 1.2]. Here, we give a sufficient condition which seems more acceptable to achieve these equivalences. Indeed, according to our proof, the assumptions \( \text{pd}_{eAe} eA < \infty \) and \( \text{id}_A (\frac{A/AeA}{\text{rad}(A/AeA)}) < \infty \) infer that both \( eA \otimes_A - \) and \( Ae \otimes_{eAe} - \) preserve modules of finite Gorenstein projective dimension.

As an application of Corollary II, we reduce the Gorenstein homological properties of a triangular matrix algebra to its corner algebras under certain conditions, see Corollary 4.4 and Corollary 4.5. This reduction was also investigated in [18] with some different assumptions. We give some concrete examples illustrating that our results can be applied to study the Gorenstein projective modules and the Auslander-Reiten conjecture for some non-monomial algebras.
The paper is organized as follows. In section 2, we recall some relevant definitions and conventions. In section 3 we prove Theorem I. In section 4, Corollary I and Corollary II are proved, and the applications on triangular matrix algebras and concrete examples are given.

2 Definitions and conventions

In this section we will fix our notation and recall some basic definitions.

Let $S$ be a set of objects of a triangulated category $T$. We denote by $\text{thick} S$ the smallest triangulated subcategory of $T$ containing $S$ and closed under taking direct summands.

Throughout $k$ is a fixed field and $D := \text{Hom}_k(\mathbb{S}, k)$. All algebras are assumed to be finite dimensional associative $k$-algebras with identity unless stated otherwise. Let $A$ be such an algebra. We denote by $\text{mod} A$, $\text{proj} A$ and $\text{inj} A$ the full subcategories of $\text{Mod} A$ consisting of all finitely generated modules, finitely generated projective modules and finitely generated injective modules, respectively. Let $K_b(\text{proj} A)$ (resp. $K_b(\text{inj} A)$) be the bounded homotopy category of complexes over $\text{proj} A$ (resp. $\text{inj} A$). Let $\mathcal{D}(\text{Mod} A)$ (resp. $\mathcal{D}^b(\text{mod} A)$) be the derived category (resp. bounded derived category) of complexes over $\text{Mod} A$ (resp. $\text{mod} A$). Usually, we just write $\mathcal{D} A$ (resp. $\mathcal{D}^b(A)$) instead of $\mathcal{D}(\text{Mod} A)$ (resp. $\mathcal{D}^b(\text{mod} A)$).

Up to isomorphism, the objects in $K^b(\text{proj} A)$ are precisely all the compact objects in $\mathcal{D} A$. For convenience, we do not distinguish $K^b(\text{proj} A)$ from the perfect derived category $\mathcal{D}_{\text{per}}(A)$ of $A$, i.e., the full triangulated subcategory of $\mathcal{D} A$ consisting of all compact objects, which will not cause any confusion. Moreover, we also do not distinguish $K^b(\text{inj} A)$ (resp. $\mathcal{D}^b(A)$) from its essential image under the canonical embedding into $\mathcal{D} A$.

Recall that an algebra $A$ is said to be Gorenstein if $\text{id}_A A < \infty$ and $\text{id}_{A^{op}} A < \infty$. The Gorenstein symmetry conjecture states that $\text{id}_A A < \infty$ if and only if $\text{id}_{A^{op}} A < \infty$. This conjecture is listed in Auslander-Reiten-Smalø’s book [3, p.410, Conjecture (13)], and it closely connects with other homological conjectures. For example, it is known that the finitistic dimension conjecture implies the Gorenstein symmetry conjecture. But so far all these conjectures are still open.

Following [6] [23], the singularity category of $A$ is the Verdier quotient $D_{sg}(A) = \mathcal{D}^b(A)/K^b(\text{proj} A)$. Recall that an $A$-module $M$ is called Goren-
stein projective if there is an exact sequence

$$P^* = \ldots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \longrightarrow \ldots$$

of proj A with $M = \text{Ker} d^0$ such that $\text{Hom}_A(P^*, Q)$ is exact for every $Q \in$ proj A. Denote by Gproj A the subcategory of mod A consisting of Gorenstein projective modules. It is well known that Gproj A is a Frobenius category, and hence its stable category Gproj A is a triangulated category. Moreover, there is a canonical triangle functor $F : \text{Gproj} A \rightarrow D_{sg}(A)$ sending a Gorenstein projective object to the corresponding stalk complex concentrated in degree zero.

**Definition 2.1.** The Verdier quotient $D_{def}(A) := D_{sg}(A)/\text{Im} F$ is called the Gorenstein defect category of A.

For an algebra $A$, we define $\perp A := \{X \in \text{mod} A | \text{Ext}_A^i(X, A) = 0 \text{ for all } i > 0\}$, and denote $\perp A$ the stable category of $\perp A$ modulo finitely generated projective $A$-modules. According to [3] Theorem 2.12, $\perp A$ is a left triangulated category with the standard left triangulated structure, and it is clear that there is an embedding functor $\perp A \hookrightarrow D_{sg}(A)$ of left triangulated categories.

In an attempt to prove Nakayama conjecture, Auslander and Reiten [2] proposed the following conjecture: a finitely generated module $M$ is projective if $\text{Ext}_A^i(M, M \oplus A) = 0$, for any $i \geq 1$. This conjecture is called Auslander-Reiten conjecture, and it is true for several classes of algebras, such as algebras of finite representation type, syzygy-finite algebras, symmetric biserial algebras, algebras with radical square zero and local algebras with radical cube zero [2, 31, 32].

As a special case of Auslander-Reiten conjecture, Luo and Huang [21] proposed the Gorenstein projective conjecture: a finitely generated Gorenstein projective module $M$ is projective if $\text{Ext}_A^i(M, M) = 0$, for any $i \geq 1$. The Auslander-Reiten conjecture and the Gorenstein projective conjecture coincide when $A$ is a Gorenstein algebra, but it seems not true in general. Moreover, the Gorenstein projective conjecture is proved for CM-finite algebras [34]. For more development of this conjecture we refer to [22].

Let $A$ and $B$ be two algebras and $F : \mathcal{D}A \rightarrow \mathcal{D}B$ be a triangle functor. We say that $F$ restricts to $\mathcal{K}^b(\text{proj})$ (resp. $\mathcal{D}^b(\text{mod}), \mathcal{K}^b(\text{inj})$) if $F$ sends $\mathcal{K}^b(\text{proj} A)$ (resp. $\mathcal{D}^b(\text{mod} A), \mathcal{K}^b(\text{inj} A)$) to $\mathcal{K}^b(\text{proj} B)$ (resp. $\mathcal{D}^b(\text{mod} B), \mathcal{K}^b(\text{inj} B)$).
3 Eventually homological isomorphisms and Gorenstein projective modules

In this section, we will compare the singularity categories, Gorenstein defect categories and the stable categories of Gorenstein projective modules between two algebras linked by an eventually homological isomorphism. Moreover, we consider the Auslander-Reiten conjecture and the Gorenstein symmetry conjecture for these algebras.

Recall that a functor $F : \text{mod} A \to \text{mod} B$ is called an \textit{eventually homological isomorphism} if there is an integer $t$ such that for every $j > t$, there is an isomorphism $\text{Ext}^j_B(X, Y) \cong \text{Ext}^j_C(FX, FY)$ for all objects $X, Y \in B$. Given the smallest such $t$, we call the functor a $t$-\textit{eventually homological isomorphism}. The following two theorems from [25] will be used frequently.

**Theorem 3.1.** ([25, Theorem 4.3 (ii)]) Let $F : \text{mod} A \to \text{mod} B$ be a $t$-eventually homological isomorphism which is essentially surjective.

(i) For every $X \in \text{mod} A$, we have $\text{pd}_B F(X) \leq \sup \{t, \text{pd}_A X\}$ and $\text{id}_B F(X) \leq \sup \{t, \text{id}_A X\}$;

(ii) For any $Y \in \text{mod} B$, assume $F(Y') \cong Y$ for some $Y' \in \text{mod} A$. Then we have $\text{pd}_A Y' \leq \sup \{t, \text{pd}_B Y\}$ and $\text{id}_A Y' \leq \sup \{t, \text{id}_B Y\}$.

**Theorem 3.2.** ([25, Theorem 4.3 (v)]) Let $F : \text{mod} A \to \text{mod} B$ be an eventually homological isomorphism which is essentially surjective. Then $A$ is Gorenstein if and only if so is $B$.

Now we will reduce Gorenstein symmetry conjecture by eventually homological isomorphisms.

**Theorem 3.3.** Let $F : \text{mod} A \to \text{mod} B$ be an eventually homological isomorphism which is essentially surjective. Then $A$ satisfies the Gorenstein symmetry conjecture if and only if so does $B$.

**Proof.** Assume that $A$ satisfies the Gorenstein symmetry conjecture. If $\text{id}_B B < \infty$, then $K^b(\text{proj} B) \subseteq K^b(\text{inj} B)$ and by Theorem 3.1 we have $F(A) \in K^b(\text{proj} B) \subseteq K^b(\text{inj} B)$, that is, $\text{id}_B F(A) < \infty$. Using Theorem 3.1 again we have $\text{id}_A A < \infty$. Since $A$ satisfies the Gorenstein symmetry conjecture, we obtain that $A$ is Gorenstein. By Theorem 3.2, $B$ is Gorenstein and thus $\text{id}_{B^{op}} B < \infty$. Conversely, if $\text{id}_{B^{op}} B < \infty$, then $\text{pd}_B DB < \infty$. Therefore, $K^b(\text{inj} B) \subseteq K^b(\text{proj} B)$ and by Theorem 3.1 we have $F(DA) \in K^b(\text{inj} B) \subseteq K^b(\text{proj} B)$, that is, $\text{pd}_B F(DA) < \infty$. Using
Theorem 3.1 again we have \( \text{pd}_A DA < \infty \), that is, \( \text{id}_{A^{op}} A < \infty \). Since \( A \) satisfies the Gorenstein symmetry conjecture, we obtain that \( A \) is Gorenstein. By Theorem 3.2, \( B \) is Gorenstein and thus \( \text{id}_B B < \infty \).

Now assume that \( B \) satisfies the Gorenstein symmetry conjecture. If \( \text{id}_A A < \infty \), then \( K^b(\text{proj} A) \subseteq K^b(\text{inj} A) \). Let \( B' \in \text{mod} A \) such that \( F(B') \cong B \). Then it follows from Theorem 3.1 that \( B' \in K^b(\text{proj} A) \subseteq K^b(\text{inj} A) \), that is, \( \text{id}_A B' < \infty \). Using Theorem 3.1 again we have \( \text{id}_B B < \infty \). Since \( B \) satisfies the Gorenstein symmetry conjecture, we obtain that \( B \) is Gorenstein. By Theorem 3.2, \( A \) is Gorenstein and thus \( \text{id}_{A^{op}} A < \infty \). Conversely, if \( \text{id}_{A^{op}} A < \infty \), then the statement \( \text{id}_A A < \infty \) can be proved in a similar way. \( \square \)

Let \( A \) be an algebra and \( e \) be an idempotent of \( A \). Then \( A \) and \( eAe \) are singularly equivalent if \( eA \otimes_A - : \text{mod} A \to \text{mod} eAe \) is an eventually homological isomorphism, see [25, Main Theorem]. On the other hand, the arrow removal operation yields an eventually homological isomorphism which induces a singular equivalence, see [13, Main Theorem]. Now we will unify these two results by showing that two algebras linked by a certain eventually homological isomorphism are always singularly equivalent.

**Theorem 3.4.** Let \( F : \text{mod} A \to \text{mod} B \) be a \( t \)-eventually homological isomorphism which is essentially surjective. Assume that \( F \) admits a left adjoint \( H \) and a right adjoint \( G \). Then \( H \) and \( F \) induce a singular equivalence between \( A \) and \( B \).

**Proof.** Since \( (H, F, G) \) is an adjoint triple, it follows that \( H \) is right exact, \( G \) is left exact and \( F \) is exact. Therefore, these derived functors give rise to an adjoint triple \( (LH, F, RG) \) between \( \text{D} A \) and \( \text{D} B \). Moreover, the exactness of \( F \) implies that \( F \) restricts to \( \text{D}^b(\text{mod}) \), and then \( LH \) restricts to \( K^b(\text{proj}) \) by [1, Lemma 2.7]. It follows from Theorem 3.1 that \( F \) restricts to \( K^b(\text{proj}) \) and \( K^b(\text{inj}) \), and then \( LH \) restricts to \( \text{D}^b(\text{mod}) \) by [27, Lemma 1]. Therefore, the functors \( LH \) and \( F \) induce an adjoint pair between \( D_{sg}(A) \) and \( D_{sg}(B) \), see [23, Lemma 1.2]. Now we claim \( F : D_{sg}(A) \to D_{sg}(B) \) is fully faithful and dense, and then

\[
D_{sg}(B) \xrightarrow{LH} D_{sg}(A)
\]

is a mutually inverse equivalence.

For any \( X \in D_{sg}(B) \), there exists some \( n \in \mathbb{Z} \) and \( X \in \text{mod} B \) such that \( X \cong X[n] \) in \( D_{sg}(B) \), see [8, Lemma 2.1]. Since \( F : \text{mod} A \to \text{mod} B \),
is essentially surjective, we may assume that $X \cong F(X')$ for some $X' \in \text{mod}A$. Hence, we have $X^\bullet \cong F(X')[n] \cong F(X'[n])$ in $D_{sg}(B)$, and thus $F : D_{sg}(A) \to D_{sg}(B)$ is dense.

For any $X^\bullet, Y^\bullet \in D_{sg}(A)$, there exist $m, n \in \mathbb{Z}$ and $X, Y \in \text{mod}A$ such that $X^\bullet \cong X[m]$ and $Y^\bullet \cong Y[n]$ in $D_{sg}(A)$, see [8, Lemma 2.1]. Let $\eta$ be the counit of the adjoint pair $(LH, F)$ between $D^b(B)$ and $D^b(A)$. Then there is a canonical triangle

$$LHF X \to X \to \text{Cone}(\eta X) \to$$

in $D^b(A)$. Let $S$ be a simple $A$-module. Applying the functor $\text{Hom}_{D^A}(\cdot, S[i])$, we get an exact sequence

$$\text{Hom}_{D^A}(\text{Cone}(\eta X), S[i]) \to \text{Hom}_{D^A}(X, S[i]) \to \text{Hom}_{D^A}(LHF X, S[i]) \to.$$  

For any $i > t$, we have isomorphisms

$$\text{Hom}_{D^A}(LHF X, S[i]) \cong \text{Hom}_{D^B}(FX, FS[i])$$

$$\cong \text{Ext}^i_B(FX, FS)$$

$$\cong \text{Ext}^i_A(X, S)$$

$$\cong \text{Hom}_{D^A}(X, S[i]),$$

where the first isomorphism follows by adjunction, and the third one is the definition of $t$-eventually homological isomorphism. Therefore, we obtain $\text{Hom}_{D^A}(\text{Cone}(\eta X), S[i]) \cong 0$, for any $i > t+1$. Now we claim that $\text{Cone}(\eta X) \in K^b(\text{proj}A)$. Since $\text{Cone}(\eta X) \in D^b(A)$, $\text{Cone}(\eta X)$ is quasi-isomorphic to a minimal right bounded complex of finitely generated projective $A$-modules. If this complex is not bounded, then some indecomposable projective $A$-module with simple top $S$ occurs infinitely many times. It follows that there are nonzero morphisms from this complex to infinitely many positive shifts of $S$, that is, $\text{Hom}_{D^A}(\text{Cone}(\eta X), S[i]) \neq 0$ for infinite many $i \in \mathbb{Z}^+$. But this is a contradiction. Therefore, $\text{Cone}(\eta X) \in K^b(\text{proj}A)$ and thus $LHF X \cong X$ in $D_{sg}(A)$. As a result, we have isomorphisms

$$\text{Hom}_{D_{sg}(A)}(X^\bullet, Y^\bullet) \cong \text{Hom}_{D_{sg}(A)}(X[m], Y[n])$$

$$\cong \text{Hom}_{D_{sg}(A)}(LHF X[m], Y[n])$$

$$\cong \text{Hom}_{D_{sg}(B)}(FX[m], FY[n])$$

$$\cong \text{Hom}_{D_{sg}(B)}(FX^\bullet, FY^\bullet),$$

and then $F : D_{sg}(A) \to D_{sg}(B)$ is fully faithful.
Now we will investigate Gorenstein defect categories in the setting of eventually homological isomorphisms. We start with the following result.

**Lemma 3.5.** Let $F : \text{mod} A \to \text{mod} B$ be a $t$-eventually homological isomorphism which is essentially surjective. Then $\Omega^t(FX) \in \perp B$ for any $X \in \perp A$.

**Proof.** By definition, there exists some $B' \in \text{mod} A$ such that $F(B') \cong B$, and it follows from Theorem 3.1 that $\text{pd}_A B' \leq t$. For any $i > 0$, consider the isomorphisms

$$\text{Ext}_B^i(\Omega^t(FX), B) \cong \text{Ext}_B^{i+t}(FX, B) \cong \text{Ext}_B^{i+t}(FX, F(B')) \cong \text{Ext}_A^{i+t}(X, B'),$$

where the last equation holds because $F$ is a $t$-eventually homological isomorphism. Since $X \in \perp A$, we can use the projective resolution of $B'$ to do dimension shifting, that is, $\text{Ext}_A^{i+t}(X, B') \cong \text{Ext}_A^{i+2t}(X, \Omega^t B')$. Since $\text{pd}_A B' \leq t$, we have that $\Omega^t B' \in \text{proj} A$ and thus $\text{Ext}_A^{i+2t}(X, \Omega^t B') \cong 0$. Above all, we get $\text{Ext}_B^{i+t}(FX, B) \cong 0$ for any $i > 0$. \hfill $\Box$

Denote by $D^b(A)_{fGd}$ the full subcategory of $D^b(A)$ formed by those complexes quasi-isomorphic to bounded complex of Gorenstein projective objects. Here, the definition of $D^b(A)_{fGd}$ agrees with that in [17], where the objects in $D^b(A)_{fGd}$ are called complexes of finite Gorenstein projective dimension, see [17, Definition 2.7 and Proposition 2.10]. Moreover, $D^b(A)_{fGd}$ is a thick subcategory of $D^b(A)$ generated by all the Gorenstein projective modules, that is, $D^b(A)_{fGd} = \text{thick}(\text{Gproj} A)$, see [18, Theorem 2.7]. The following equivalence

$$\text{Gproj} A \cong D^b(A)_{fGd}/K^b(\text{proj} A)$$

is well known, see [6, Theorem 4.4.1], [9, Lemma 4.1] or [24, Theorem 4] for examples.

The following lemma is an alternative description of Gorenstein projective objects.

**Lemma 3.6.** ([16, Lemma 5.1]) An object $X \in \text{mod} A$ is Gorenstein projective if and only if there are short exact sequences $0 \to X^i \to P^{i+1} \to X^{i+1} \to 0$ in $\text{mod} A$ with $P^i$ projective and $X^i \in \perp A$ for all $i \in \mathbb{Z}$ such that $X^0 = X$.

Next we show that certain eventually homological isomorphisms preserve complexes of finite Gorenstein projective dimension.

**Lemma 3.7.** Let $F : \text{mod} A \to \text{mod} B$ be a $t$-eventually homological isomorphism which is exact and essentially surjective. Then $\Omega^t(FX) \in \text{Gproj} B$ for any $X \in \text{Gproj} A$, and $F$ induces a triangle functor from $D^b(A)_{fGd}$ to $D^b(B)_{fGd}$. \hfill $\Box$
Since \( F : \text{mod} A \to \text{mod} B \) is exact, we have an induced functor \( F : D^b(A) \to D^b(B) \). Let \( X \in \text{mod} A \) be a Gorenstein projective module. By Lemma 3.6 there are short exact sequences \( 0 \to X^i \to P^{i+1} \to X^{i+1} \to 0 \) in \( \text{mod} A \) with \( P^i \) projective and \( X^i \in \perp A \) for all \( i \in \mathbb{Z} \) such that \( X^0 = X \).

Since \( F \) is exact, the sequences

\[
0 \to FX^i \to FP^{i+1} \to FX^{i+1} \to 0
\]

are exact, and these lead to exact sequences

\[
0 \to \Omega^i(FX^i) \to \Omega^i(FP^{i+1}) \oplus Q^{i+1} \to \Omega^i(FX^{i+1}) \to 0,
\]

where \( Q^{i+1} \in \text{proj} B \). Since \( X^i \in \perp A \), it follows from Lemma 3.5 that \( \Omega^i(FX^i) \in \perp B \) for all \( i \in \mathbb{Z} \), and by Theorem 3.1 \( \Omega^i(FP^{i+1}) \in \text{proj} B \) for all \( i \in \mathbb{Z} \). Now Lemma 3.6 shows that \( \Omega^i(FX) \in \text{Gproj} B \), and then \( FX \in \text{thick}(\text{Gproj} B) \). Above all, we conclude that \( FX \in D^b(B)_{fGd} \) for any \( X \in \text{Gproj} A \).

Therefore,

\[
F(D^b(A)_{fGd}) = F(\text{thick}(\text{Gproj} A)) \subseteq \text{thick} F(\text{Gproj} A) \subseteq D^b(B)_{fGd}.
\]

Following [10], a triangle functor \( F : D^b(A) \to D^b(B) \) is said to be non-negative if \( F \) satisfies the following conditions: (1) \( F(X) \) is isomorphic to a complex with zero homology in all negative degrees, for all \( X \in \text{mod} A \); (2) \( F(A) \) is isomorphic to a complex in \( K^b(\text{proj} B) \) with zero terms in all negative degrees. Now we are ready to compare the Gorenstein defect categories and the stable categories of Gorenstein projective modules between two algebras linked by an eventually homological isomorphism.

**Theorem 3.8.** Let \( F : \text{mod} A \to \text{mod} B \) be a \( t \)-eventually homological isomorphism which is essentially surjective. Assume that \( F \) admits a left adjoint \( H \) and a right adjoint \( G \). Then \( F \) induces triangle equivalences \( \text{Gproj} A \cong \text{Gproj} B \) and \( D_{\text{def}}(A) \cong D_{\text{def}}(B) \).

**Proof.** Since \((H, F)\) is an adjoint pair, we infer that \( H \) preserves direct sums and \( H \) is right exact. By Watt’s theorem, \( H \) is isomorphic to \( H(B) \otimes_B - : \text{mod} B \to \text{mod} A \), where the right \( B \)-module structure of \( H(B) \) is given by \( B \cong \text{Hom}_B(B, B) \to \text{Hom}_A(H(B), H(B)) \). Now consider the derived functor \( LH \cong H(B) \otimes_B^L - : DB \to DA \). By the proof of Theorem 3.4 we have that \( LH \) and \( F \) restrict to both \( D^b(\text{mod}) \) and \( K^b(\text{proj}) \), and using [11] Lemma 2.8,
we get that $LH$ has a left adjoint which restricts to $K^b(\text{proj})$. It follows from [11, Lemma 3.4] that $LH$ restricts to a non-negative functor from $\mathcal{D}^b(B)$ to $\mathcal{D}^b(A)$, up to shifts. By [16, Proposition 5.2], the stable functor $\overline{LH}$ preserves Gorenstein projective modules. According to [16, Section 4.2], each $X \in \text{mod} B$ yields a triangle

$$P^*_X \to LH(X) \to \overline{LH}(X) \to$$

in $\mathcal{D}A$ with $P^*_X \in K^b(\text{proj}A)$. Therefore, $LHX \in D^b(A)_{fGd}$ for any $X \in \text{Gproj} B$, and then $LH$ sends the objects of $D^b(B)_{fGd}$ to $D^b(A)_{fGd}$. In view of Lemma [3.7], $LH$ and $F$ induce an adjoint pair between $D^b(B)_{fGd}/K^b(\text{proj}B)$ and $D^b(A)_{fGd}/K^b(\text{proj}A)$, see [16, Lemma 1.2]. Thanks to the equivalence $\text{Gproj} A \cong D^b(A)_{fGd}/K^b(\text{proj}A)$, we obtain an adjoint pair

$$
\begin{array}{c}
\text{Gproj} B \\
\overline{LH} \\
\downarrow \\
\text{Gproj} A \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{Gproj} A \\
\overline{LH} \\
\downarrow \\
\text{Gproj} B \\
\end{array}
,$$

and combining Theorem [3.4] we have the following exact commutative diagram

$$
\begin{array}{c}
0 \\
\overline{LH} \\
\downarrow \\
F \\
0 \\
\end{array}
\begin{array}{c}
\text{Gproj} A \\
\overline{LH} \\
\downarrow \\
\text{Gproj} B \\
\end{array}
\begin{array}{c}
D_{sg}(A) \\
\downarrow F \\
D_{sg}(B) \\
\end{array}
\begin{array}{c}
D_{def}(A) \\
\downarrow LH \\
D_{def}(B) \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
,$$

where the vertical functors between $D_{sg}(A)$ and $D_{sg}(B)$ are equivalences. Hence, $LH$ and $F$ induce an equivalence between $\text{Gproj} A$ and $\text{Gproj} B$, and also, there is an equivalence between $D_{def}(A)$ and $D_{def}(B)$.

Now let’s turn to the invariance of the Auslander-Reiten conjecture (resp. Gorenstein projective conjecture) under eventually homological isomorphisms.

**Theorem 3.9.** Assume that $F : \text{mod} A \to \text{mod} B$ satisfies all the conditions in Theorem [3.8]. Then $A$ satisfies the Auslander-Reiten conjecture (resp. Gorenstein projective conjecture) if and only if so does $B$.

**Proof.** By the proof of Theorem [3.8], the functor $LH$ restricts to a non-negative functor from $\mathcal{D}^b(B)$ to $\mathcal{D}^b(A)$. Moreover, $LH$ admits a right adjoint $F$ which preserves $K^b(\text{proj})$. Therefore, it follows from [16, Proposition 4.8 and Proposition 5.2] that there are two commutative diagrams

$$
\begin{array}{c}
\downarrow B \\
D_{sg}(B) \\
\overline{LH} \\
D_{sg}(A) \\
\downarrow A \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{Gproj} B \\
\overline{LH} \\
\downarrow \text{Gproj} A \\
\end{array}
,$$

and

$$
\begin{array}{c}
\downarrow B \\
D_{sg}(B) \\
\overline{LH} \\
D_{sg}(A) \\
\downarrow A \\
\end{array}
$$

and

$$
\begin{array}{c}
\text{Gproj} B \\
\overline{LH} \\
\downarrow \text{Gproj} A \\
\end{array}
.$$
It is clear that the embedding $\mathcal{A} \hookrightarrow D_{sg}(A)$ induces an isomorphism

$$\text{Ext}_A^i(X, Y) \cong \text{Hom}_{D_{sg}(A)}(X, Y[i])$$

for each $X, Y \in \mathcal{A}$ and for each $i > 0$. Moreover, it follows from Theorem 3.4 that $LH : D_{sg}(B) \rightarrow D_{sg}(A)$ is an equivalence. Hence, using the same judgment as [11, Lemma 3.3], we can prove that the Auslander-Reiten conjecture (resp. Gorenstein projective conjecture) holds for $B$ if it holds for $A$.

Note that the functor $F : \text{mod}A \rightarrow \text{mod}B$ is exact, and $\text{pd}_B F(P) \leq t$ for any $P \in \text{proj}A$. Then the functor $F[-t] : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$ is non-negative and the stable functor $F[-t]$ is isomorphic to $\Omega^t F$. Combining [16, Proposition 4.8] with Lemma 3.5 and Lemma 3.7, we have two commutative diagrams

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F[-t]} & \mathcal{B} \\
\downarrow & & \downarrow \\
D_{sg}(A) & \xrightarrow{F[-t]} & D_{sg}(B)
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\text{Gproj}A & \xrightarrow{F[-t]} & \text{Gproj}B \\
\downarrow & & \downarrow \\
D_{sg}(A) & \xrightarrow{F[-t]} & D_{sg}(B)
\end{array}$$

By Theorem 3.4, the functor $F : D_{sg}(A) \rightarrow D_{sg}(B)$ is an equivalence, and so is $F[-t] : D_{sg}(A) \rightarrow D_{sg}(B)$. Therefore, the Auslander-Reiten conjecture (resp. Gorenstein projective conjecture) holds for $B$ implies that it holds for $A$.

4 Applications and examples

In this section, we will apply our main results to arrow removal and vertex removal. This will produce new reduction techniques for the study of Gorenstein defect categories, Gorenstein symmetry conjecture, Auslander-Reiten conjecture and Gorenstein projective conjecture.

Let $A$ be an admissible quotient $kQ/I$ of a path algebra $kQ$ over a field $k$. Choose an arrow $\alpha$ in $Q$ which does not occur in a minimal generating set of $I$ and define $B = A/\langle \alpha \rangle$. The arrow removal operation, transferring homological properties of $A$ to $B$, was investigated in [13, 15] with respect to finitistic dimension, Gorensteinness, singularity categories and the Fg condition. Now we will consider other homological invariants under this operation.
Corollary 4.1. Keep the above notations and assumptions. Then $\text{Gproj} A \cong \text{Gproj} B$ and $D_{\text{def}}(A) \cong D_{\text{def}}(B)$. Moreover, $A$ satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does $B$.

Proof. By [15, Proposition 4.6], there is a functor $e : \text{mod} A \to \text{mod} B$ which is essentially surjective and admits a left adjoint and a right adjoint. Further, it follows from [13, Corollary 3.3] that $e : \text{mod} A \to \text{mod} B$ is an eventually homological isomorphism. Now this corollary follows from Theorem 3.3, Theorem 3.8 and Theorem 3.9.

Let $A$ be an algebra and $e$ be an idempotent in $A$. In [25], the author proved that $eA \otimes_A - : \text{mod} A \to \text{mod} eA$ is an eventually homological isomorphism if and only if $\text{pd}_{eA} eA < \infty$ and $\text{id}_A(A/AeA)_{\text{rad}(A/AeA)} < \infty$ (or equivalently, $\text{pd}_{(eA)e} eA < \infty$ and $\text{pd}_A(A/AeA)_{\text{rad}(A/AeA)} < \infty$), and they compared the algebras $A$ and $eA$ with respect to Gorensteinness, singularity categories and the Fg condition under these conditions. Now we will investigate more homological invariants between $A$ and $eA$.

Corollary 4.2. Assume that $\text{pd}_{eA} eA < \infty$ and $\text{id}_A(A/AeA)_{\text{rad}(A/AeA)} < \infty$ (or equivalently, $\text{pd}_{(eA)e} eA < \infty$ and $\text{pd}_A(A/AeA)_{\text{rad}(A/AeA)} < \infty$). Then the functor $eA \otimes_A - : \text{mod} A \to \text{mod} eA$ induces triangle equivalences $\text{Gproj} A \cong \text{Gproj}(eA e)$ and $D_{\text{def}}(A) \cong D_{\text{def}}(eA e)$. Moreover, $A$ satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does $eA e$.

Proof. Clearly, any idempotent element $e$ induces a recollement between $\text{mod} A/AeA$, $\text{mod} A$ and $\text{mod} eA$. Therefore, the functor $eA \otimes_A - : \text{mod} A \to \text{mod} eA$ is essentially surjective, and it admits a left adjoint and a right adjoint. Further, it follows from [25, Main theorem] that $eA \otimes_A -$ is an eventually homological isomorphism. Now this corollary follows from Theorem 3.3, Theorem 3.8 and Theorem 3.9.

Using [25, Lemma 8.11, Lemma 8.9 and Proposition 8.7 ], we have the following special case of Corollary 4.2.

Corollary 4.3. Let $A = kQ/I$ be a quotient of a path algebra $kQ$ over a field $k$. Choose some vertices in $Q$ where no relations start and no relations end, and let $e$ be the sum of idempotents corresponding to all vertices except these. Then $\text{Gproj} A \cong \text{Gproj}(eA e)$ and $D_{\text{def}}(A) \cong D_{\text{def}}(eA e)$. Moreover, $A$ satisfies the Gorenstein symmetry conjecture (resp. Auslander-Reiten conjecture, Gorenstein projective conjecture) if and only if so does $eA e$.
Let $A$ and $B$ be algebras, $M$ an $A$-$B$-bimodule and $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Let $e_A = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$ and $e_B = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$. Denote by $S_{e_A} = e_A T \otimes_T - : \text{mod} T \to \text{mod} e_A T e_A$ and $S_{e_B} = e_B T \otimes_T - : \text{mod} T \to \text{mod} e_B T e_B$. From \cite{25}, $AM_B$ is compatible if $M \otimes_B -$ sends every acyclic complex of projective $B$-modules to acyclic complex, and $\text{Ext}_A^i(G, M) \cong 0$ for any $G \in \text{Gproj}A$ and $i > 0$. Assume that $AM_B$ is compatible, then $S_{e_A}$ induces triangle equivalences $D_{sg}(T) \cong D_{sg}(A)$, $D_{def}(T) \cong D_{def}(A)$ and $\text{Gproj}T \cong \text{Gproj}A$ if and only if $\text{gl.dim}B < \infty$, $\text{pd}_A M < \infty$ and $M \otimes_B -$ preserve modules of finite Gorenstein projective dimension, see \cite{18} Theorem 4.4 (2)]. Now we will use Corollary 4.2 to simplify these conditions.

**Corollary 4.4.** (Compare \cite{18} Theorem 4.4 (2)]) Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular matrix algebra. Then $S_{e_A}$ induces triangle equivalences $D_{sg}(T) \cong D_{sg}(A)$, $D_{def}(T) \cong D_{def}(A)$ and $\text{Gproj}T \cong \text{Gproj}A$ if and only if $\text{gl.dim}B < \infty$ and $\text{pd}_A M < \infty$.

**Proof.** Assume that $\text{gl.dim}B < \infty$ and $\text{pd}_A M < \infty$. Then it follows from \cite{25} Lemma 8.15] that $\text{pd}_{e_A T e_A} e_A T < \infty$ and $\text{id}_A(\frac{T / T e_A T}{\text{rad}(T / T e_A T)}) < \infty$. According to \cite{25} Main theorem], $S_{e_A}$ induces a triangle equivalence $D_{sg}(T) \cong D_{sg}(A)$, and by Corollary 4.2, $S_{e_A}$ induces equivalences $\text{Gproj}T \cong \text{Gproj}A$ and $D_{def}(T) \cong D_{def}(A)$. Conversely, assume $S_{e_A}$ induces such triangle equivalences. Then it follows from \cite{18} Theorem 4.4 (2)] that $\text{gl.dim}B < \infty$ and $\text{pd}_A M < \infty$. Indeed, the compatibility of $M$ is not used in the only if part of \cite{18} Theorem 4.4 (2)].

Similarly, assume that $AM_B$ is compatible. Then $S_{e_B}$ induces triangle equivalences $D_{sg}(T) \cong D_{sg}(B)$, $D_{def}(T) \cong D_{def}(B)$ and $\text{Gproj}T \cong \text{Gproj}B$ if and only if $\text{gl.dim}A < \infty$, see \cite{18} Theorem 4.6 (2)]. Now we will give a sufficient conditions without the compatibility of $M$.

**Corollary 4.5.** (Compare \cite{18} Theorem 4.6 (2)]) Let $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular matrix algebra. Assume that $\text{gl.dim}A < \infty$ and $\text{pd}_{B^op}M < \infty$, then $S_{e_B}$ induces triangle equivalences $D_{sg}(T) \cong D_{sg}(B)$, $D_{def}(T) \cong D_{def}(B)$ and $\text{Gproj}T \cong \text{Gproj}B$.

**Proof.** Assume that $\text{gl.dim}A < \infty$ and $\text{pd}_{B^op}M < \infty$. Then it follows from \cite{25} Lemma 8.16] that $\text{pd}_{e_B T e_B} e_B T < \infty$ and $\text{id}_B(\frac{T / T e_B T}{\text{rad}(T / T e_B T)}) < \infty$. Therefore, the statement follows from \cite{25} Main theorem] and Corollary 4.2.
Now we will illustrate our results by three examples. In particular, the Gorenstein projective modules of some non-monomial algebras are described.

**Example 4.6.** Let $A$ be the $k$-algebra given by the following quiver

![Quiver Diagram](image)

with relations $\{\varepsilon^2, \beta\eta, \beta\alpha - \delta\gamma\}$. We write the concatenation of paths from right to left. Clearly, there is no relation starting and ending at the vertices 2 and 3, and then it follows from Corollary 4.3 that $\text{Gproj} A \cong \text{Gproj} B$ and $D_{def}(A) \cong D_{def}(B)$, where $B$ is the algebra

![Algebra Diagram](image)

with relations $\{\varepsilon^2\}$. Using Corollary 4.3 again, we have that $\text{Gproj} B \cong \text{Gproj} C$ and $D_{def}(B) \cong D_{def}(C)$, where $C$ is the algebra

![Algebra Diagram](image)

with relations $\{\varepsilon^2\}$. Since $C$ is selfinjective, we get $\text{Gproj} C \cong \text{mod} C \cong \text{mod} k$ and $D_{def}(C) \cong 0$. Above all, we conclude that $\text{Gproj} A \cong \text{mod} k$ and $D_{def}(A) \cong 0$. It is easy to check that the simple $A$-module corresponding to 4 is Gorenstein projective, and then all Gorenstein projective modules over $A$ are this simple module and projective modules.

**Example 4.7.** This is Example 1 from [30] and Example 6.5 from [15]. Let $A$ be the $k$-algebra given by the following quiver

![Quiver Diagram](image)

with relations $\{\alpha^3, \delta\alpha, \delta\beta, \xi\eta - \delta\gamma\}$. Clearly, there is no relation starting and ending at the vertices 3 and 5, and it follows from Corollary 4.3 that
Gproj\(A \cong \text{Gproj} B\) and \(D_{\text{def}}(A) \cong D_{\text{def}}(B)\), where \(B\) is the algebra

\[
\begin{array}{ccc}
1 & \overset{\alpha}{\longrightarrow} & 2 \\
\beta \swarrow & & \searrow \delta \\
& 4 & 
\end{array}
\]

with relations \(\{\alpha^3, \delta\alpha, \delta\beta\}\). Now consider the algebra \(B\). Let \(e = e_2 + e_4\) and let \(S_i\) denote the simple \(B\)-module associated to the vertex \(i\). Then \(\text{pd}_{(eBe)^{op}} Be < \infty\) and \(\text{pd}_B \left( \frac{B}{BeB} \right) = \text{pd}_B (S_1) < \infty\). Hence, according to Corollary 4.2 we have \(\text{Gproj} B \cong \text{Gproj} (C)\) and \(D_{\text{def}}(B) \cong D_{\text{def}}(C)\), where \(C = eBe\) is the algebra

\[
\begin{array}{ccc}
\gamma & \overset{\alpha}{\longrightarrow} & 2 \\
\beta \swarrow & & \searrow \delta \\
& 4 & 
\end{array}
\]

with relations \(\{\alpha^3, \delta\alpha\}\). Since \(C\) is a monomial algebra, its Gorenstein-projective modules were described in [10]. Indeed, \(C\) is CM-free and then \(\text{Gproj} A = \text{Gproj} B \cong \text{Gproj} C \cong 0\). Therefore, all Gorenstein projective modules over \(A\) are projective. Now consider the algebra \(C\). Since \(\text{pd} S_1 < \infty\) and \(\text{pd}_{e_2 A e_2} e_2 A < \infty\), it follows from [7, Theorem 2.1] that \(D_{\text{sg}}(C) \cong D_{\text{sg}} (k[x]/\langle x^3 \rangle)\). Then we get \(D_{\text{def}}(A) \cong D_{\text{def}}(C) \cong D_{\text{sg}}(C) \cong \text{mod}(k[x]/\langle x^3 \rangle)\).

Note that monomial algebras satisfy the Gorenstein symmetry conjecture, Auslander-Reiten conjecture and Gorenstein projective conjecture. Hence, we conclude that \(A\) satisfies these conjectures by Corollary 4.1 and Corollary 4.3.

**Example 4.8.** This is Example 6.3 from [15]. Let \(A\) be the \(k\)-algebra given by the following quiver

\[
\begin{array}{ccc}
6 & \overset{g}{\longrightarrow} & 1 \\
\downarrow f_1 & & \downarrow f_2 \\
5 & \overset{c}{\longrightarrow} & 4 \\
\downarrow e \swarrow & & \searrow d \\
2 & & 3 \\
\end{array}
\]

with relations \(\{ca - db, f_1 ec, ed, gf_1 e, bgf_1, ag\}\). Then \(A\) is a representation infinite non-monomial algebra and \(\text{gl.dim} A = \infty\). However, \(A\) can be reduced by Corollary 4.1 since \(f_2\) is not occurring in any relations. Note that the algebra \(A/(f_2)\) is of finite representation type, and then it satisfies the Gorenstein symmetry conjecture, Auslander-Reiten conjecture and Gorenstein projective conjecture. Hence, we conclude that \(A\) satisfies these conjectures by Corollary 4.1.
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References

[1] L. Angeleri Hügel, S. König, Q. Liu and D. Yang, Ladders and simplicity of derived module categories, J. Algebra 472 (2017), 15–66.

[2] M. Auslander and I. Reiten, On a generalized version of the Nakayama conjecture, Proc. Amer. Math. Soc. 52 (1975), 69–74.

[3] M. Auslander, I. Reiten and S.O. Smalø, Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995.

[4] A. Beligiannis and N. Marmaridis, Left triangulated categories arising from contravariantly finite subcategories, Comm. Algebra 22 (1994), no. 12, 5021–5036.

[5] P. A. Bergh, D. A. Jørgensen and S. Oppermann, The Gorenstein defect category, Q. J. Math 66 (2015), 459–471.

[6] R.-O. Buchweitz, Maximal cohen-macaulay modules and Tate-cohomology over gorenstein rings, unpublished (1987), available at http://hdl.handle.net/1807/16682 (1987).

[7] X. W. Chen, Singularity categories, Schur functors and triangular matrix rings, Algebr. Represent. Theor. 12 (2009), 181–191.

[8] X. W. Chen, The singularity category of an algebra with radical square zero, Doc. Math. 16 (2011), 921–936.

[9] X. W. Chen and W. Ren, Frobenius functors and Gorenstein homological properties, arXiv:2008.11467v2.

[10] X.W. Chen, D. W. Shen, and G. D. Zhou, The Gorenstein-projective modules over a monomial algebra, Proc. Royal Soc. Edin. 148 A (2018), 1115–1134.

[11] Y. P. Chen, W. Hu, Y. Y. Qin and R. Wang, Singular equivalences and Auslander-Reiten conjecture, arXiv:2011.02729v1.
[12] C. Cibils, M. Lanzilotta, E. N. Marcos and A. Solotar, Deleting or adding arrows of a bound quiver algebra and Hochschild (co)homology, Proc. Amer. Math. Soc. 148 (2020), no. 6, 2421–2432.

[13] K. Erdmann, C. Psaroudakis and Ø. Solberg, Homological invariants of the arrow removal operation, arXiv:2108.04891.

[14] K. Fuller and M. Saorin, On the finitistic dimension conjecture for Artinian rings, Manuscripta Math. 74 (1992), no. 2, 117–132.

[15] E. L. Green, C. Psaroudakis and Ø. Solberg, Reduction techniques for the finitistic dimension, Trans. Am. Math. Soc 374 (2021), 6839–6879.

[16] W. Hu and S. Pan, Stable functors of derived equivalences and Gorenstein projective modules, Math. Nachr. 290 (2017), no. 10, 1512–1530.

[17] Y. Kato, On derived equivalent coherent rings, Comm. Algebra 30 (2002), 4437–4454.

[18] H. H. Li, J. S. Hu and Y. F. Zheng, When the Schur functor induces a triangle-equivalence between Gorenstein defect categories, Sci China Math 65 (2022), https://doi.org/10.1007/s11425-021-1899-3

[19] M. Lu, Gorenstein defect categories of triangular matrix algebras, J. Algebra 480 (2017), 346–367.

[20] M. Lu, Gorenstein Properties of Simple Gluing Algebras, Algebr. Represent. Theor. 22 (2019), 517–543.

[21] R. Luo and Z. Y. Huang, When are torsionless modules projective? J Algebra 320 (2008), 2156–2164

[22] R. Luo and D. M. Jian, On the Gorenstein projective conjecture: IG-projective modules, J. Algebra Appl. 15 (2016), no. 6, 1650117, 11pp.

[23] D. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, Trudy Steklov Math. Institute 204 (2004), 240–262.

[24] S. Y. Pan and X. J. Zhang, Derived equivalences and Cohen-Macaulay Auslander algebras, Front. Math. China 10 (2015), no. 2, 323–338.
[25] C. Psaroudakis, Ø. Skartsæterhagen and Ø. Solberg, Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements, Trans. Am. Math. Soc. Ser. B 1 (2014), 45–95.

[26] Y. Y. Qin, Eventually homological isomorphisms in recollements of derived categories, J. Algebra 563 (2020), 53–73.

[27] Y. Y. Qin and Y. Han, Reducing homological conjectures by n-recollements, Algebr. Represent. Theor. 19 (2016), no. 2, 377–395.

[28] C. M. Ringel, The Gorenstein projective modules for the Nakayama algebras. I, J. Algebra 385 (2013), 241–261.

[29] K. L. Wu and J. Q. Wei, Syzygy properties under recollements of derived categories, J. Algebra 589 (2022), 215–237.

[30] C. C. Xi, On the finitistic dimension conjecture I: Related to representation-finite algebras, J. Pure Appl. Algebra 193 (2004), no. 1-3, 287–305.

[31] D. M. Xu, A note on the Auslander-Reiten conjecture, Acta. Math. Sin. (Engl. Ser.) 29 (2013), no.10, 1993–1996.

[32] D. M. Xu, Auslander-Reiten conjecture and special biserial algebras, Arch. Math. (Basel) 105 (2015), no. 1, 13–22.

[33] P. Zhang, Gorenstein-projective modules and symmetric recollements. J Algebra 388 (2013), 65–80.

[34] X. J. Zhang, A note on Gorenstein projective conjecture II, Nanjing Daxue Xuebao Shuxue Bannian Kan 29 (2012), no. 2, 155–162.