QUANTITATIVE BOUNDS FOR CRITICALLY BOUNDED SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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ABSTRACT. We revisit the regularity theory of Escauriaza, Seregin, and Šverák for solutions to the three-dimensional Navier-Stokes equations which are uniformly bounded in the critical $L^3_x$ norm. By replacing all invocations of compactness methods in these arguments with quantitative substitutes, and similarly replacing unique continuation and backwards uniqueness estimates by their corresponding Carleman inequalities, we obtain quantitative bounds for higher regularity norms of these solutions in terms of the critical $L^3_x$ bound (with a dependence that is triple exponential in nature).

In particular, we show that as one approaches a finite blowup time $T_*$, the critical $L^3_x$ norm must blow up at a rate $(\log \log \log \frac{1}{T_* - t})^c$ or faster for an infinite sequence of times approaching $T_*$ and some absolute constant $c > 0$.

1. INTRODUCTION

This paper is concerned with quantitative bounds for solutions $u : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^3$, $p : [0, T) \times \mathbb{R}^3 \to \mathbb{R}$ to the Navier-Stokes equations

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= \Delta u - \nabla p \\
\nabla \cdot u &= 0.
\end{align*}
\]

(1.1)

To avoid technicalities, we shall restrict attention to classical solutions, by which we mean solutions that are smooth and such that all derivatives of $u, p$ lie in the space $L^\infty_t L^2_x([0, T) \times \mathbb{R}^3)$. As our bounds are quantitative and do not depend on any smooth norms of the solution, it is possible to extend the results here to weaker notions of solution, such as mild solutions of Kato [10], the weak Leray-Hopf solutions studied in [7], or the suitable weak solutions from [4], by using the regularity theory of such solutions; we leave the details to the interested reader. As is well known, such solutions have a maximal Cauchy development $u : [0, T_*) \times \mathbb{R}^3 \to \mathbb{R}^3$, $p : [0, T_*) \times \mathbb{R}^3 \to \mathbb{R}$ for some $0 < T_* \leq \infty$, with the restriction to $[0, T] \times \mathbb{R}^3$ a classical solution for all $T < T_*$, but for which no smooth extension to time $T_*$ is possible if $T_* < \infty$. We refer to $T_*$ as the maximal time of existence of such a classical solution.

The Navier-Stokes system enjoys the scaling symmetry $(u, p, T) \mapsto (\lambda^3 u, \lambda^4 p, \lambda^2 T)$ for any $\lambda > 0$, where

\[ u^\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x) \]

and

\[ p^\lambda(t, x) := \lambda^2 p(\lambda^2 t, \lambda x), \]

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Among other things, this means that the norm
\[ \| u \|_{L^\infty_t L^3_x([0,T] \times \mathbb{R}^3)} \]
is scale-invariant (or critical) for this equation. In [7] it was shown that as long as this norm stays bounded, solutions to Navier-Stokes remain regular. In particular, they showed an endpoint of the classical Prodi-Serrin-Ladyshenskaya blowup criterion [10], [19], [12] or the Leray blowup criterion [14]:

**Theorem 1.1 (Qualitative blowup criterion).** [7] Suppose \((u, p)\) is a classical solution to Navier-Stokes whose maximal time of existence \(T_\ast\) is finite. Then
\[
\limsup_{t \to T_\ast^-} \| u(t) \|_{L^3_x(\mathbb{R}^3)} = +\infty.
\]

There are now many proofs, variants and generalisations [7], [11], [8], [17], [15], [9], [5], [1], [2], [18], [21] of this theorem, including extensions to higher dimensions or other domains than Euclidean spaces, replacing \(L^3\) with another critical Besov or Lorenz space, or replacing the limit superior by a limit. However, in contrast to the more quantitative arguments of Leray, Prodi, Serrin and Ladyshenskaya, the proofs in the above references all rely at some point on a compactness argument to extract a limiting profile solution to which qualitative results such as unique continuation and backwards uniqueness for heat equations (as established in particular in [6]) can be applied. As such, the above proofs do not easily give any quantitative rate of blowup for the \(L^3\) norm.

On the other hand, the proofs of unique continuation and backwards uniqueness rely on explicit Carleman inequalities which are fully quantitative in nature. Thus, one would expect it to be possible, at least in principle, to remove the reliance on compactness methods and obtain a quantitative version of Theorem 1.1. This is the purpose of the current paper. More precisely, in Section 6 we will establish the following two results:

**Theorem 1.2 (Quantitative regularity for critically bounded solutions).** Let \(u : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3\), \(p : [0, T] \times \mathbb{R}^3 \to \mathbb{R}\) be a classical solution to the Navier-Stokes equations with
\[ \| u \|_{L^\infty_t L^3_x([0,T] \times \mathbb{R}^3)} \leq A \] (1.2)
for some \(A \geq 2\). Then we have the derivative bounds
\[ |\nabla_x^j u(t, x)| \leq \exp \exp (A^{O(1)}) t^{-\frac{j+1}{2}} \]
and
\[ |\nabla_x^j \omega(t, x)| \leq \exp \exp (A^{O(1)}) t^{-\frac{j+2}{2}} \]
whenever \(0 < t \leq T\), \(x \in \mathbb{R}^3\), and \(j = 0, 1\). (See Section 2 for the asymptotic notation used in this paper.)

**Remark 1.3.** It is not difficult to iterate using Schauder estimates in Hölder spaces and extend the above regularity bounds to higher values of \(j\) than \(j = 0, 1\) (allowing the implied constants in the \(O(1)\) notation to depend on \(j\)), and also control time derivatives (conceding a factor of \(t^{-1}\) for each time derivative); we leave this extension of Theorem 1.2 to the interested reader.
Theorem 1.4 (Quantitative blowup criterion). Let \( u : [0,T_*) \times \mathbb{R}^3 \to \mathbb{R}^3 \), \( p : [0,T_*) \times \mathbb{R}^3 \to \mathbb{R} \) be a classical solution to the Navier-Stokes equations which blows up at a finite time \( 0 < T_* < \infty \). Then

\[
\limsup_{t \to T_*^{-}} \| u(t) \|_{L^2(\mathbb{R}^3)} = +\infty
\]

for an absolute constant \( c > 0 \).

We now discuss the method of proof of these theorems, which uses many of the same key inputs as in previous arguments (most notably the Carleman estimates used to prove backwards uniqueness and unique continuation), but also introduces some other ingredients in order to avoid having to make some rather delicate results from the qualitative theory (such as profile decompositions) quantitative, as doing so would almost certainly lead to much poorer bounds than the ones given here.

The main estimate focuses on bounding the scale-invariant quantity

\[
N_0^{-1} |P_{N_0} u(t_0,x_0)|
\]

for various points \((t_0,x_0)\) in spacetime, and various frequencies \(N_0\), where \(P_{N_0}\) is a Littlewood-Paley projection operator to frequencies \(\sim N_0\) (see Section 2 for a precise definition). Using (1.2) and the Bernstein inequality, one can bound this quantity by \(O(A)\). It is well known that if one could improve this bound somewhat for sufficiently large \(N_0\), for instance to \(O(A_1^{-1})\) for a large constant \(C_0\), then (assuming \(A\) is large enough) the \(L^3\) norm becomes sufficiently “dispersed” in space and frequency that one could adapt the local well-posedness theory for the Navier-Stokes equation (or the local regularity theory from [4]) to obtain good bounds. Hence we will focus on establishing such a bound for (1.3) for \(N_0\) large enough (see Theorem 5.1 for a precise statement).

The first step in doing so is to observe (basically from the Duhamel formula and some standard Littlewood-Paley theory) that if the quantity (1.3) is large for some \(N_0,t_0,x_0\) with \(t_0\) not too close to the initial time 0, then the quantity

\[
N_1^{-1} |P_{N_1} u(t_1,x_1)|
\]

is also large (with exactly the same lower bound) for some \((t_1,x_1)\) a little bit to the past of \((t_0,x_0)\) (but more or less within the “parabolic domain of dependence”, in the sense that \(x_1 = x_0 + O((t_0 - t_1)^{1/2})\)) and with \(N_1\) comparable to \(N_0\); see Proposition 3.1(iv) for a precise statement. If one takes care to have exactly the same lower bounds for both (1.3) and (1.4), then this claim can be iterated, creating a chain of “bubbles of concentration” at various points \((t_n,x_n)\) and frequencies \(N_n\), propagating backwards in time, and for which

\[
N_n^{-1} |P_{N_n} u(t_n,x_n)|
\]

is bounded from below uniformly in \(n\). Furthermore, by using a “bounded total speed” property first observed in [20], one can ensure that \((t_n,x_n)\) stays in the “parabolic domain of dependence” in the sense that \(x_n = x_0 + O((t_0 - t_n)^{1/2})\). Due to the well

\[^1\text{Strictly speaking, it is the scale-invariant quantity } N_0^2 T \text{ that needs to be large, rather than } N_0 \text{ itself, where } T \text{ is the amount of time to the past of } x_0 \text{ for which the solution exists and obeys the bounds (1.2).}\]
known fact (dating back to the classical work of Leray [14]) that solutions to Navier-Stokes enjoy large “epochs of regularity” in which one has control of high regularity norms of the solution in large time intervals outside of a small dimensional singular set of times (see Proposition 3.1(ii) for a precise quantification of this statement), one can show that there are a large number of points \((t_n, x_n)\) for which the frequency \(N_n\) is basically as small as possible, in the sense that

\[
N_n \sim |t_0 - t_n|^{-1/2}.
\]

The (Littlewood-Paley component \(P_{N_n} u\) of) the solution \(u\) is large near \((t_n, x_n)\), and it is not difficult to then obtain analogous lower bounds on the vorticity

\[
\omega := \nabla \times u
\]

near \((t_n, x_n)\). The importance of working with the vorticity comes from the fact that it obeys the vorticity equation

\[
\partial_t \omega = \Delta \omega - (u \cdot \nabla) \omega + (\omega \cdot \nabla) u
\]

which can be viewed as a variable coefficient heat equation (in which the lower order coefficients \(u, \nabla u\) depend on the velocity field) for which the non-local effects of the pressure \(p\) do not explicitly appear. Using a quantitative version of unique continuation for backwards parabolic equations (see Proposition 4.3 for a precise statement) that can be established using Carleman inequalities, one can then obtain exponentially small, but still non-trivial, lower bounds\(^2\) for enstrophy-type quantities such as

\[
\int_{I_n} \int_{R_n \leq |x - x_n| \leq R_n'} |\omega(t, x)|^2 \, dx \, dt
\]

for various cylindrical annuli \(I_n \times \{x : R_n \leq |x - x_n| \leq R_n'\}\) surrounding \((t_n, x_n)\), with \(R_n'\) a large multiple of \(R_n\). Crucially, one can set \(R_n\) to be as large as one pleases (although the lower bound exhibits Gaussian decay in \(R_n\)). In order to apply the Carleman inequalities, it is important that the time interval \(I\) lies within one of the “epochs of regularity” in which one has good \(L^\infty\) estimates for \(u, \nabla u, \omega, \nabla \omega\), but this can be accomplished without much difficulty (mainly thanks to the energy dissipation term in the energy inequality).

For many choices of scale \(R_n\) (a bit larger than \(|t_0 - t_n|^{1/2}\)), one can use an “energy pigeonholing argument” (as used for instance by Bourgain [3]) to make the energy (or more precisely, a certain component of the enstrophy) small in an annular region \(\{x : R_n \leq |x - x_n| \leq R_n'\}\) at some time \(t'_n\) a little bit to the past of \(t_n\); by modifying the somewhat delicate analysis of local enstrophies from [20] that again takes advantage of the “bounded total speed” property, one can then propagate this smallness forward in time (at the cost of shrinking the annular region \(\{R_n \leq |x - x_n| \leq R_n'\}\) slightly), and in particular back up to time \(t_0\), and parabolic regularity theory can then be used to obtain good \(L^\infty\) estimates for \(u, \nabla u, \omega, \nabla \omega\) in these regions. This allows us to again use Carleman inequalities. Specifically, by using the Carleman inequalities used to prove the backwards uniqueness result in [7] (see Section 4 for precise statements), one can

\(^2\)One can think of this as applying (a quantitative version) of unique continuation “in the contrapositive”. Similarly for the invocation of backwards uniqueness below. Actually in practice the Carleman inequalities also require an additional term such as \(|\nabla \omega(t, x)|^2\) in the integrand, but we ignore this term for sake of discussion.
Figure 1. A schematic depiction of the main argument. Starting with a concentration of critical norm at a point \((t_0, x_0)\) in spacetime, one propagates this concentration backwards in time to generate concentrations at further points \((t_n, x_n)\) in spacetime. Restricting attention to an epoch of regularity \(I_n \times \mathbb{R}^3\) (depicted here in purple), Carleman estimates are then used to establish lower bounds on the vorticity at other locations in space, and in particular where the epoch intersects an “annulus of regularity” (depicted in green) arising from an energy (or enstrophy) pigeonholing argument. A further application of Carleman estimates are then used to establish a lower bound on the vorticity (or velocity) in the annular region at time \(t = t_0\), thus demonstrating a lack of compactness of the solution at this time which can be used to obtain a contradiction when \(N_0\) (or more precisely the scale-invariant quantity \(N_0^2 T\), where \(T\) is the lifespan of the solution) is large enough, by letting \(n\) vary.

then propagate the lower bounds on \(I_n \times \{x : R_n \leq |x - x_0| \leq R'_n\}\) forward in time until one returns to the original time \(t_0\) of interest, eventually obtaining a small but nontrivial lower bound for quantities such as

\[
\int_{R_n \leq |x - x_0| \leq R'_n} |\omega(t_0, x)|^2 \, dx
\]

(ignoring for this discussion some slight adjustments to the scales \(R_n, R'_n\) that occur during this argument), which after some routine manipulations (and using the fact that \((t_n, x_n)\) lies in the parabolic domain of dependence of \((t_0, x_0)\)) also gives a lower bound on quantities like

\[
\int_{R_n \leq |x - x_0| \leq R'_n} |u(t_0, x)|^3 \, dx.
\]

Crucially, this lower bound is uniform in \(n\). If one now lets \(n\) vary, the annuli \(\{R_n \leq |x - x_0| \leq R'_n\}\) end up becoming disjoint for widely separated \(n\), and one can eventually contradict (1.2) at time \(t = t_0\) if \(N_0\) is large enough.

Remark 1.5. The triply exponential nature of the bounds in Theorem 1.2 (which is of course closely tied to the triply logarithmic improvement to Theorem 1.1 in Theorem 1.4) can be explained as follows. One exponential factor comes from the Bourgain energy pigeonholing argument to locate a good spatial scale \(R\). A second exponential factor arises from the Carleman inequalities. The third exponential arises from locating enough disjoint spatial scales \(R_n\) to contradict (1.2). It seems that substantially new
ideas would be needed in order to improve significantly upon this triple exponential bound.

Remark 1.6. Of course, by Sobolev embedding, the $L^3_3(\mathbb{R}^3)$ norm in the above theorems can be replaced by the critical homogeneous Sobolev norm $\dot{H}^{1/2}_x(\mathbb{R}^3)$. It is likely that the arguments here can also be adapted to handle other critical Besov or Lorentz spaces (as long as the secondary exponent of such spaces is finite, so that the critical norm cannot simultaneously have a substantial presence at an unbounded number of scales), but we will not pursue this question here; based on Theorem 1.4 it is also reasonable to conjecture that the Orlicz norm $\parallel u(t) \parallel_{L^3_3(\log \log \log L)^{-c}(\mathbb{R}^3)}$ of $u$ also must blow up as $t \to T^*_\ast$ for some absolute constant $c > 0$. On the other hand, our argument relies heavily in many places on the fact that we are working in three dimensions. It may be possible to obtain a higher-dimensional analogue of our results by finding quantitative versions of the argument in [5], but we do not pursue this question here. Similarly, our arguments do not directly allow us to replace the limit superior in Theorem 1.1 with a limit, as is done in [17] (see also [1]); again, it may be possible to also find quantitative analogues of these results, but we do not pursue this matter here.

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2. Notation

We use the notation $X = O(Y)$, $X \lesssim Y$, or $Y \gtrsim X$ to denote the bound $|X| \leq CY$ for some absolute constant $C > 0$. If we need the implied constant $C$ to depend on parameters we shall indicate this by subscripts, for instance $X \lesssim_j Y$ denotes the bound $|X| \leq C_j Y$ where $C_j$ depends only on $j$.

Throughout this paper we will need a sufficiently large absolute constant $C_0$, which will remain fixed throughout the paper. For instance $C_0 = 10^5$ would suffice throughout our paper, if one worked out all the implied constants in the exponents carefully.

If $I \subset \mathbb{R}$ is a time interval, we use $|I|$ to denote its length. If $x_0 \in \mathbb{R}^3$ and $R > 0$, we use $B(x_0, R)$ to denote the ball $\{x \in \mathbb{R}^3 : |x - x_0| \leq R \}$, and if $B = B(x_0, R)$ is such a ball, we use $kB = B(x_0, kR)$ to denote its dilates for any $k > 0$.

We use the mixed Lebesgue norms

$$\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^3)} := \left( \int_I \|u(t)\|_{L^r_x(\mathbb{R}^3)}^q \, dt \right)^{1/q}$$

where

$$\|u(t)\|_{L^r_x(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} |u(t,x)|^r \, dx \right)^{1/r}$$

with the usual modifications when $q = \infty$ or $r = \infty$. For any measurable subset $\Omega \subset I \times \mathbb{R}^3$, we write $\|u\|_{L^q_t L^r_x(\Omega)}$ for $\|u1_\Omega\|_{L^q_t L^r_x(I \times \mathbb{R}^3)}$, where $1_\Omega$ is the indicator function of $\Omega$. 
Given a Schwartz function $f : \mathbb{R}^3 \to \mathbb{R}$, we define the Fourier transform
\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-2\pi i \xi \cdot x} \, dx
\]
and then for any $N > 0$ we define the Littlewood-Paley projection $P_{\leq N}$ by the formula
\[
\overline{P_{\leq N}} f(\xi) := \varphi(\xi/N) \hat{f}(\xi)
\]
where $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is a fixed bump function supported on $B(0,1)$ that equals 1 on $B(0,1/2)$. We also define the companion Littlewood-Paley projections
\[
\begin{align*}
P_N & := P_N - P_{N/2} \\
P_{> N} & := 1 - P_{\leq N} \\
P_N & := P_{2N} - P_{N/4}
\end{align*}
\]
where 1 denotes the identity operator; thus for instance $P_{\leq N} f = \sum_{k=0}^{\infty} P_{2^{-k} N} f$ and $P_{> N} f = \sum_{k=1}^{\infty} P_{2^k N} f$ for Schwartz $f$ (with the convergence in a locally uniform sense). Also we have $P_N = P_N \hat{P}_N$. These operators can also be applied to vector-valued Schwartz functions by working component by component. These operators commute with other Fourier multipliers such as the Laplacian $\Delta$ and its inverse $\Delta^{-1}$, partial derivatives $\partial_t$, heat propagators $e^{t\Delta}$, and the Leray projection $\mathbb{P} := -\nabla \times \Delta^{-1} \nabla \times$ to divergence-free vector fields. To estimate such multipliers, we use the following general estimate:

**Lemma 2.1** (Multiplier theorem). Let $N > 0$, and let $m : \mathbb{R}^3 \to \mathbb{C}$ be a smooth function supported on $B(0,N)$ that obeys the bounds
\[
|\nabla^j m(\xi)| \leq M N^{-j}
\]
for all $0 \leq j \leq 100$ and some $M > 0$. Let $T_m$ denote the associated Fourier multiplier, thus
\[
\overline{T_m} f(\xi) := m(\xi) \hat{f}(\xi).
\]
Then one has
\[
\|T_m f\|_{L^p(\mathbb{R}^3)} \leq M N^{3/2 - \frac{3}{2p}} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.1}
\]
whenever $1 \leq p \leq q \leq \infty$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a Schwartz function. More generally, if $\Omega \subset \mathbb{R}^3$ is an open subset of $\mathbb{R}^3$, $A \geq 1$, and $\Omega_{A/N} := \{ x \in \mathbb{R}^3 : \text{dist}(x, \Omega) < A/N \}$ denotes the $A/N$-neighbourhood of $\Omega$, then we have a local version
\[
\|T_m f\|_{L^{p_1}(\Omega)} \leq M N^{\frac{3}{2p_1} - \frac{3}{2q}} \|f\|_{L^{p_1}(\Omega_{A/N})} + A^{-50} M |\Omega|^{\frac{1}{p_1} - \frac{1}{q}} N^{\frac{3}{2q} - \frac{3}{2p}} \|f\|_{L^{p_2}(\mathbb{R}^3)} \tag{2.2}
\]
of the above estimate, whenever $1 \leq p_1 \leq q_1 \leq \infty$ and $1 \leq p_2 \leq q_2 \leq \infty$ are such that $q_2 \geq q_1$, and $|\Omega|$ denotes the volume of $\Omega$.

By the usual limiting arguments, one can replace the hypothesis that $f$ is Schwartz with the requirement that $f$ lie in $L^p$. Also one can extend this theorem to vector-valued $f : \mathbb{R}^3 \to \mathbb{R}^3$ by working component by component. In practice, the $A^{-50}$ factor will ensure that the second term on the right-hand side of (2.2) is negligible compared to the first, and can be ignored on a first reading.
Proof. By homogeneity we can normalise $M = 1$; by scaling (or dimensional analysis) we may also normalise $N = 1$. We can write $T_m f$ as a convolution $T_m f = f * K$ of $f$ with the kernel

$$K(x) := \int_{\mathbb{R}^3} m(\xi) e^{2\pi i \xi \cdot x} \, d\xi.$$ 

By repeated integration by parts we obtain the bounds $K(x) \lesssim (1 + |x|)^{-90}$ (say), so in particular $\|K\|_{L^r(\mathbb{R}^3)} \lesssim 1$ for all $1 \leq r \leq \infty$. From Young’s convolution inequality we then conclude that

$$\|T_m f\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)},$$

giving \((2.1)\). To prove \((2.2)\), we see that the claim already follows from \((2.2)\) when $f$ is supported in $L^p(\Omega_A)$, so by the triangle inequality we may assume that $f$ is supported on $\mathbb{R}^3 \setminus \Omega_A$. In this case we may replace the convolution kernel $K$ by its restriction to the complement of $B(0, A)$, which allows us to improve the bound on the $L^p$ norm of the kernel to (say) $O(A^{-50})$. The claim follows from Young’s convolution inequality, after first using Hölder’s inequality to bound $\|T_m f\|_{L^q(\Omega)} \lesssim |\Omega|^{1/2} \|T_m f\|_{L^{2q}(\Omega)}$. \qed

Thus for instance, we have the Bernstein inequalities

$$\|\nabla^j f\|_{L^q(\mathbb{R}^3)} \lesssim_j N^j t^{-\frac{j}{2} - \frac{\beta}{q}} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.3}$$

whenever $1 \leq p \leq q \leq \infty$, $j \geq 0$, and $f$ is a Schwartz function whose Fourier transform is supported on $B(0, N)$, as can be seen by writing $f = P_{\leq 2N} f$ and applying Lemma \ref{lem:bernstein}. In a similar spirit, one has

$$\|P_N e^{t \Delta} \nabla^j f\|_{L^q(\mathbb{R}^3)} \lesssim_j \exp(-N^2 t/20) N^j t^{-\frac{j}{2} - \frac{2}{q}} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.4}$$

for any $t > 0$ and any Schwartz $f$. Summing this, we obtain the standard heat kernel bounds

$$\|e^{t \Delta} \nabla^j f\|_{L^q(\mathbb{R}^3)} \lesssim_j t^{-\frac{j}{2} - \frac{3}{p} + \frac{3}{q}} \|f\|_{L^p(\mathbb{R}^3)}. \tag{2.5}$$

3. Basic estimates

The purpose of this section is to establish the following initial bounds for $L^\infty_t L^3_x$-bounded solutions to the Navier-Stokes equations.

**Proposition 3.1** (Initial estimates). Let $u : [t_0 - T, t_0] \times \mathbb{R}^3 \to \mathbb{R}^3$, $p : [t_0 - T, t_0] \times \mathbb{R}^3 \to \mathbb{R}$ be a classical solution to Navier-Stokes that obeys the bound

$$\|u\|_{L^\infty_t L^3_x([t_0 - T, t_0] \times \mathbb{R}^3)} \leq A. \tag{3.1}$$

for some $A \geq C_0$. We adopt the notation

$$A_j := A C_0^j$$

for all $j$.

(i) (Pointwise derivative estimates) For any $(t, x) \in [t_0 - T/2, t_0] \times \mathbb{R}^3$ and $N > 0$, we have

$$P_N u(t, x) = O(AN); \quad \nabla P_N u(t, x) = O(AN^2); \quad \partial_t P_N u(t, x) = O(A^2 N^3). \tag{3.2}$$
similarly, the vorticity $\omega = \nabla \times u$ obeys the bounds
\[ P_N\omega(t, x) = O(AN^2); \quad \nabla P_N\omega(t, x) = O(AN^3); \quad \partial_t P_N\omega(t, x) = O(A^2N^4). \] (3.3)

(ii) (Bounded total speed) For any interval $I$ in $[t_0 - T/2, t_0]$, one has
\[ \|u\|_{L^1_t L^\infty_x(I \times \mathbb{R}^3)} \lesssim A^1|I|^{1/2}. \] (3.4)

(iii) (Epochs of regularity) For any interval $I$ in $[t_0 - T/2, t_0]$, there is a subinterval $I' \subset I$ with $|I'| \gtrsim A^{-2}|I|$ such that
\[ \|\nabla^j u\|_{L^\infty_t L^\infty_x(I' \times \mathbb{R}^3)} \lesssim A^{O(1)}|I|^{-(j+1)/2} \]
and
\[ \|\nabla^j \omega\|_{L^\infty_t L^\infty_x(I' \times \mathbb{R}^3)} \lesssim A^{O(1)}|I|^{-(j+2)/2} \]
for $j = 0, 1$.

(iv) (Back propagation) Let $(t_1, x_1) \in [t_0 - T/2, t_0] \times \mathbb{R}^3$ and $N_1 \geq A_3T^{-1/2}$ be such that
\[ |P_{N_1}u(t_1, x_1)| \geq A_1^{-1}N_1. \] (3.5)
Then there exists $(t_2, x_2) \in [t_0 - T, t_1] \times \mathbb{R}^3$ and $N_2 \in [A_2^1N_1, A_2N_1]$ such that
\[ A_3^{-1}N_1^{-2} \leq t_1 - t_2 \leq A_3N_1^{-2} \]
and
\[ |x_2 - x_1| \leq A_4N_1^{-1} \]
and
\[ |P_{N_2}u(t_2, x_2)| \geq A_1^{-1}N_2. \] (3.6)

(v) (Iterated back propagation) Let $x_0 \in \mathbb{R}^3$ and $N_0 > 0$ be such that
\[ |P_{N_0}u(t_0, x_0)| \geq A_1^{-1}N_0. \]
Then for every $A_4N_0^{-2} \leq T_1 \leq A_4^{-1}T$, there exists
\[ (t_1, x_1) \in [t_0 - T_1, t_0 - A_3^{-1}T_1] \times \mathbb{R}^3 \]
and
\[ N_1 = A_3^{O(1)}T_1^{-1/2} \]
such that
\[ x_1 = x_0 + O(A_4^{O(1)}T_1^{1/2}) \]
and
\[ |P_{N_1}u(t_1, x_1)| \geq A_1^{-1}N_1. \]

(vi) (Annuli of regularity) If $0 < T' < T/2$, $x_0 \in \mathbb{R}^3$, and $R_0 \geq (T')^{1/2}$, then there exists a scale
\[ R_0 \leq R \leq \exp(A_6^{O(1)})R_0 \]
such that on the region
\[ \Omega := \{(t, x) \in [t_0 - T', t_0] \times \mathbb{R}^3 : R \leq |x - x_0| \leq A_6R\} \]
we have
\[ \|\nabla^j u\|_{L^\infty_t L^\infty_x(\Omega)} \lesssim A_6^{-2}(T')^{-O(1)/2} \]
and
\[ \|\nabla^j \omega\|_{L^\infty_t L^\infty_x(\Omega)} \lesssim A_6^{-2}(T')^{-(j+2)/2} \]
for $j = 0, 1$. 


As $C_0$ is assumed large, any polynomial combination of $A = A_0, A_1, \ldots, A_{j-1}$ will be dominated by $A_j$ for any $j \geq 1$; we take advantage of this fact without comment in the sequel to simplify the estimates. The various numerical powers of $A$ (or $A_j$) that appear in the above proposition are not of much significance, except that it is important for iterative purposes that the negative power $A_1^{-1}$ appearing in (3.5) is exactly the same as the one appearing in (3.6).

In the remainder of this section $t_0, T, A, u, p$ are as in Proposition 3.1. Our objective is now to establish the claims (i)-(vi).

We begin with the proof of (i). It suffices to establish (3.2), as (3.3) then follows from (3.1) and (2.3). For the final claim, we first apply the Leray projection $\mathbb{P}$ to (1.1) to obtain the familiar equation

$$\partial_t u = \Delta u - \mathbb{P} \nabla \cdot (u \otimes u)$$

(3.7)

where the divergence $\nabla \cdot (u \otimes u)$ of the symmetric tensor $u \otimes u$ is expressed in coordinates as

$$\left(\nabla \cdot (u \otimes u)\right) = \partial_j(u_i u_j)$$

with the usual summation conventions. We apply $P_N$ to both sides of (3.7). From (3.1) and (2.3) we have

$$\|P_N \Delta u(t)\|_{L_\infty^3} \lesssim N^3 A.$$

From (3.1) and Hölder we have $\|u \otimes u(t)\|_{L_{L^2}^3} \lesssim A^2$, hence by Lemma 2.1 we have

$$\|P_N \mathbb{P} \nabla \cdot (u \otimes u)(t)\|_{L_\infty^3} \lesssim N^3 A,$$

and the final claim of (3.2) follows from the triangle inequality.

Now we prove (ii), (iii). It is not difficult to see that these estimates are invariant with respect to time translation (shifting $I, t_0, u$ accordingly) and also rescaling (adjusting $T, t_0, I, u$ accordingly). Hence we may assume without loss of generality that $I = [0, 1] \subset [t_0 - T/2, t_0]$, which implies that $[-1, 1] \subset [t_0 - T, t_0]$.

It will be convenient to remove a linear component from $u$, as it is not well controlled in $L_2^t$ type spaces. Namely, on $[-1, 1] \times \mathbb{R}^3$ we split $u = u^\text{lin} + u^\text{nlin}$, where $u^\text{lin}$ is the linear solution

$$u^\text{lin}(t) := e^{(t+1)\Delta} u(-1)$$

(3.8)

and $u^\text{nlin} := u - u^\text{lin}$ is the nonlinear component. From (3.1) we have

$$\|u^\text{lin}\|_{L_\infty^t \cdot L_2^3([-1,1] \times \mathbb{R}^3)} \lesssim A.$$

(3.9)

From (3.7) and Duhamel’s formula one has

$$u^\text{nlin}(t) = - \int_{-1}^t e^{(t-t')\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(t') \, dt'.$$

From (3.1), $u \otimes u$ has an $L_3^{3/2}((\mathbb{R}^3)$ norm of $O(A^2)$. From (2.5), the operator $e^{(t-t')\Delta} \mathbb{P} \nabla \cdot$ maps $L_2^t$ to $L_2^t$ with an operator norm of $(t - t')^{-3/4}$. From Minkowski’s inequality we conclude an energy bound for the nonlinear component:

$$\|u^\text{nlin}\|_{L_\infty^t \cdot L_2^3([-1,1] \times \mathbb{R}^3)} \lesssim A^2.$$

(3.10)
We now restrict attention to the slab $[-1/2, 1] \times \mathbb{R}^3$. Here $t + 1$ lies between 1/2 and 2, and we can use (3.1), (3.8), and (2.5) to obtain very good bounds on $u^\text{lin}$ (but only in spaces with an integrability exponent greater than or equal to 3). More precisely, we have

$$\|\nabla^j u^\text{lin}\|_{L^p_t L^q_x([-1/2, 1] \times \mathbb{R}^3)} \lesssim j A$$

(3.11)

for any $3 \leq p \leq \infty$ and $j \geq 0$.

To exploit the bound (3.10), we use the energy method. Since $u^\text{lin}$ solves the heat equation $\partial_t u^\text{lin} = \Delta u^\text{lin}$, we can subtract this from (1.1) to conclude that

$$\partial_t u^\text{lin} = \Delta u^\text{lin} - \nabla \cdot (u \otimes u) - \nabla p.$$  

(3.12)

Taking inner products with $u^\text{lin}$, which is divergence-free, and integrating by parts, we conclude that

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^3} |u^\text{lin}|^2 \, dx = - \int_{\mathbb{R}^3} |\nabla u^\text{lin}|^2 \, dx + \int_{\mathbb{R}^3} (\nabla u^\text{lin}) \cdot (u \otimes u) \, dx$$

where the quantity $(\nabla u^\text{lin}) \cdot (u \otimes u)$ is defined in coordinates as

$$(\nabla u^\text{lin}) \cdot (u \otimes u) = (\partial_i u^\text{lin}_j) u_i u_j.$$  

From the divergence-free nature of $u^\text{lin}$ and integration by parts we have

$$\int_{\mathbb{R}^3} (\nabla u^\text{lin}) \cdot (u^\text{lin} \otimes u^\text{lin}) \, dx = 0$$

and hence

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^3} |u^\text{lin}|^2 \, dx = - \int_{\mathbb{R}^3} |\nabla u^\text{lin}|^2 \, dx + \int_{\mathbb{R}^3} (\nabla u^\text{lin}) \cdot (u \otimes u - u^\text{lin} \otimes u^\text{lin}) \, dx.$$  

Integrating this on $[-1/2, 1]$ using (3.10) we conclude that

$$\int_{-1/2}^1 \int_{\mathbb{R}^3} |\nabla u^\text{lin}|^2 \, dx \, dt \leq A^2 + \int_{-1/2}^1 \int_{\mathbb{R}^3} |u^\text{lin} \otimes u - u^\text{lin} \otimes u^\text{lin}| \, dx \, dt,$$

and hence by Young’s inequality

$$\int_{-1/2}^1 \int_{\mathbb{R}^3} |\nabla u^\text{lin}|^2 \, dx \, dt \leq A^2 + \int_{-1/2}^1 \int_{\mathbb{R}^3} |u \otimes u - u^\text{lin} \otimes u^\text{lin}|^2 \, dx \, dt.$$  

Splitting $u \otimes u - u^\text{lin} \otimes u^\text{lin} = u_t \otimes u + u^\text{lin} \otimes u_t$ and using (3.1), (3.9), (3.11) (with $p = 6$, $j = 0$) and Hörder’s inequality, one has

$$\int_{-1/2}^1 \int_{\mathbb{R}^3} |u \otimes u - u^\text{lin} \otimes u^\text{lin}| \, dx \, dt \lesssim A^4$$

and thus

$$\int_{-1/2}^1 \int_{\mathbb{R}^3} |\nabla u^\text{lin}|^2 \, dx \, dt \lesssim A^4.$$  

(3.13)

By Plancherel’s theorem this implies in particular that

$$\sum_N N^2 \|P_N u^\text{lin}\|_{L^2_t L^2_x([-1/2, 1] \times \mathbb{R}^3)}^2 \lesssim A^4$$

(3.14)

where $N$ ranges over powers of two. Also, from Sobolev embedding one has

$$\|u^\text{lin}\|_{L^2_t L^2_x([-1/2, 1] \times \mathbb{R}^3)} \lesssim A^2.$$  

(3.15)
We are now ready to establish the bounded total speed property (ii), which is a variant of [20, Proposition 9.1]. If \( t \in [0,1] \) and \( N \geq 1 \) is a power of two, we see from (3.7) and Duhamel’s formula that

\[
P_N u^{\text{lin}}(t) = e^{(t+\frac{1}{2})\Delta} P_N u^{\text{lin}} \left( -\frac{1}{2} \right) - \int_{-1/2}^{t} e^{(t-t')\Delta} \nabla \cdot \tilde{P}_N (u \otimes u)(t') \, dt'.
\]

From (2.4), the operator \( P_N e^{(t-t')\Delta} \nabla \cdot \) has an operator norm of \( O(\sqrt{N} \exp(-N^2(t-t')/20)) \) on \( L^\infty_x \), while from (3.9), (2.4) we see that \( e^{(t+\frac{1}{2})\Delta} P_N v(-\frac{1}{2}) \) has an \( L^\infty_x \) norm of \( O(\sqrt{N} \exp(-N^2/20)) \). Thus by Young’s inequality

\[
\| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([0,1] \times \mathbb{R}^3)} \leq AN \exp(-N^2/20) + N^{-1} \| \tilde{P}_N (u \otimes u) \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}.
\]

We split \( u \otimes u = u^{\text{lin}} \otimes u^{\text{lin}} + u^{\text{lin}} \otimes u^{\text{lin}} + u^{\text{lin}} \otimes u^{\text{lin}} + u^{\text{lin}} \otimes u^{\text{lin}} \). From (3.11) one has

\[
\| \tilde{P}_N (u^{\text{lin}} \otimes u^{\text{lin}}) \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)} \lesssim A^2.
\]

From (2.3), Hölder’s inequality, and (3.11), (3.15) one has

\[
\| \tilde{P}_N (u^{\text{lin}} \otimes u^{\text{lin}}) \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)} \lesssim N^{1/2} \| u^{\text{lin}} \otimes u^{\text{lin}} \|_{L^1_t L^2_x ([1/2,1] \times \mathbb{R}^3)} \lesssim A^3 N^{1/2}.
\]

Similarly with \( u^{\text{lin}} \otimes u^{\text{lin}} \) replaced by \( u^{\text{lin}} \otimes u^{\text{lin}} \). We then split \( u^{\text{lin}} \otimes u^{\text{lin}} = P_N u^{\text{lin}} \otimes P_N u^{\text{lin}} + P_N u^{\text{lin}} \otimes P_N u^{\text{lin}} + P_N u^{\text{lin}} \otimes P_N u^{\text{lin}} + P_N u^{\text{lin}} \otimes P_N u^{\text{lin}} \). We have from Hölder that

\[
\| P_N u^{\text{lin}} \otimes P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)} \lesssim \| P_N u^{\text{lin}} \|_{L^2_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2 \lesssim \| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)} \lesssim A^2 N^{-1/2} + N^{-1} \| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2.
\]

Putting this all together, we conclude that

\[
\| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([0,1] \times \mathbb{R}^3)} \lesssim A^3 N^{-1/2} + N^{-1} \| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2 + N^2 \| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2.
\]

By (2.3) and Cauchy-Schwarz we have

\[
\| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2 \lesssim \sum_{N \leq N'} (N')^{3/2} \| P_N u^{\text{lin}} \|_{L^2_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2 \lesssim N^{3/2} \sum_{N \leq N'} (N')^{3/2} \| P_N u^{\text{lin}} \|_{L^2_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2
\]

where \( N' \) ranges over powers of two, while from Plancherel’s theorem one has

\[
\| P_N u^{\text{lin}} \|_{L^2_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2 \lesssim \sum_{N \geq N'} \| P_{N'} u^{\text{lin}} \|_{L^2_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)}^2.
\]

Summing in \( N \), and using the triangle inequality followed by (3.14), we conclude that

\[
\| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([0,1] \times \mathbb{R}^3)} \lesssim A^3 + \sum_{N'} (N')^2 \| P_N u^{\text{lin}} \|_{L^1_t L^\infty_x ([1/2,1] \times \mathbb{R}^3)} \lesssim A^4.
\]
From (3.9) and (2.3) we also have
\[ \|u^{\text{lin}}\|_{L_t^1 L_x^\infty([0,1] \times \mathbb{R}^3)}, \|P_{\text{c1}} u^{\text{lin}}\|_{L_t^1 L_x^\infty([0,1] \times \mathbb{R}^3)} \leq A \]
and we conclude
\[ \|u\|_{L_t^1 L_x^\infty([0,1] \times \mathbb{R}^3)} \leq A^4 \]
which gives (ii).

Now we establish (iii). For \( t \in [0,1] \) we define the enstrophy-type quantity
\[ E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u^{\text{lin}}(t,x)|^2 \, dx, \]
Taking the gradient of (3.12) and then taking the inner product with \( \nabla u^{\text{lin}} \), we see upon integration by parts that
\[ \partial_t E(t) = -\int_{\mathbb{R}^3} |\nabla^2 u^{\text{lin}}|^2 \, dx + \int_{\mathbb{R}^3} \Delta u^{\text{lin}} \cdot (\nabla \cdot (u \otimes u)) \, dx \]
and hence by Young’s inequality
\[ \partial_t E(t) \leq -\frac{1}{2} \|\nabla^2 u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)}^2 + O \left( \|\nabla \cdot (u \otimes u)\|_{L_x^2(\mathbb{R}^3)}^2 \right). \]
By the Leibniz rule and Hölder’s inequality, one has
\[ \|\nabla \cdot (u \otimes u)\|_{L_x^3(\mathbb{R}^3)} \leq \|u\|_{L_x^2(\mathbb{R}^3)} \|\nabla u\|_{L_x^3(\mathbb{R}^3)}. \]
From (3.11) and the triangle inequality one has
\[ \|u\|_{L_x^3(\mathbb{R}^3)} \leq A + \|u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)} \]
and
\[ \|\nabla u\|_{L_x^3(\mathbb{R}^3)} \leq A + \|\nabla u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)} \]
while from Sobolev embedding and Hölder one has
\[ \|u^{\text{lin}}\|_{L_x^3(\mathbb{R}^3)} \leq \|\nabla u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)} \leq E(t)^{1/2} \]
and
\[ \|\nabla u^{\text{lin}}\|_{L_x^3(\mathbb{R}^3)} \leq \|\nabla u^{\text{lin}}\|_{L_x^{3/2}(\mathbb{R}^3)}^{1/2} \|\nabla^2 u^{\text{lin}}\|_{L_x^{3/2}(\mathbb{R}^3)}^{1/2} \leq E(t)^{1/4} \|\nabla^2 u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)}^{1/2}. \]
We conclude that
\[ \partial_t E(t) \leq -\frac{1}{2} \|\nabla^2 u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)}^2 + O \left( (A^2 + E(t))(A^2 + E(t)^{1/2} \|\nabla^2 u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)}) \right) \]
and hence by Young’s inequality
\[ \partial_t E(t) \leq -\frac{1}{4} \|\nabla^2 u^{\text{lin}}\|_{L_x^2(\mathbb{R}^3)}^2 + O((A^2 + E(t))A^2 + (A^2 + E(t))^2 E(t)). \] (3.16)
In particular we have
\[ \partial_t E(t) \leq O(A^4 + A^4 E(t) + E(t)^3). \] (3.17)
From (3.13) we have
\[ \int_0^1 E(t) \, dt \leq A^4, \]
and hence by the pigeonhole principle, we can find a time \( t_1 \in [0, 1/2] \) such that
\[ E(t_1) \leq A^4. \]
A standard continuity argument using (3.17) then gives $E(t) \lesssim A^4$ for $t \in [t_1, t_1 + c A^{-8}] = [\tau(0), \tau(1)]$, where $\tau(s) := t_1 + s c A^{-8}$ and $c > 0$ is a small absolute constant. Inserting this back into (3.16) one has

$$\partial_t E(t) \leq -\frac{1}{4} \|\nabla u^{\text{min}}\|_{L^2_x(R^3)}^2 + O(A^{12})$$

and hence by the fundamental theorem of calculus

$$\int_{\tau(0)}^{\tau(1)} \int_{R^3} |\nabla u^{\text{min}}|^2 \, dx \, dt \lesssim A^4.$$ (3.18)

Thus we have

$$\|\nabla u^{\text{min}}\|_{L^p_t L^2_x([\tau(0), \tau(1)] \times R^3)} + \|\nabla^2 u^{\text{min}}\|_{L^1_t L^2_x([\tau(0), \tau(1)] \times R^3)} \lesssim A^2.$$ (3.19)

From the Gagliardo-Nirenberg inequality

$$\|u^{\text{min}}\|_{L^p_t L^2_x} \lesssim \|\nabla u^{\text{min}}\|_{L^1_t L^2_x}^{1/2} \|\nabla^2 u^{\text{min}}\|_{L^1_t L^2_x}^{1/2}$$

and Hölder’s inequality, one concludes in particular that

$$\|u^{\text{min}}\|_{L^1_t L^2_x([\tau(0), \tau(1)] \times R^3)} \lesssim A^2$$ (3.20)

and hence by (3.11)

$$\|u\|_{L^1_t L^6_x([\tau(0), \tau(1)] \times R^3)} \lesssim A^2;$$ (3.21)

also from Sobolev embedding and (3.19) one has

$$\|\nabla u^{\text{min}}\|_{L^1_t L^6_x([\tau(0), \tau(1)] \times R^3)} \lesssim A^2$$

and hence by (3.11)

$$\|\nabla u\|_{L^1_t L^6_x([\tau(0), \tau(1)] \times R^3)} \lesssim A^2.$$ (3.22)

These are subcritical regularity estimates and can now be iterated to obtain even higher regularity. For $t \in [\tau(0.1), \tau(1)]$, we see from (3.7) that

$$u(t) = e^{(t-\tau(0))\Delta} u(\tau(0)) - \int_{\tau(0)}^{t} e^{(t-t')\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(t') \, dt'.$$ (3.23)

From (2.5) the operator $e^{(t-t')\Delta} \mathbb{P} \nabla$ has norm $O((t-t')^{-1/2})$ on $L^\infty_x$, while $e^{(t-\tau(0))\Delta}$ maps $L^2_x$ to $L^\infty_x$ with norm $O((t-\tau(0))^{-1/2}) = O(A^{O(1)})$. We conclude from (3.1) that

$$\|u(t)\|_{L^\infty_t L^2_x(\mathbb{R}^3)} \lesssim A^{O(1)} + \int_{\tau(0)}^{t} (t-t')^{-1/2} \|u(t')\|_{L^2_t L^2_x(\mathbb{R}^3)} \, dt'.$$

From (3.21) and Young’s convolution inequality, we conclude that

$$\|u\|_{L^1_t L^6_x([\tau(0.1), \tau(1)] \times R^3)} \lesssim A^{O(1)}.$$ (3.24)

Repeating the above argument, we now also see for $t \in [\tau(0.2), \tau(1)]$ that

$$\|u(t)\|_{L^\infty_t L^2_x(\mathbb{R}^3)} \lesssim A^{O(1)} + \int_{\tau(0.1)}^{t} (t-t')^{-1/2} \|u(t')\|_{L^2_t L^2_x(\mathbb{R}^3)} \, dt'$$

so from Hölder’s inequality we conclude that

$$\|u\|_{L^\infty_t L^6_x([\tau(0.2), \tau(1)] \times R^3)} \lesssim A^{O(1)}.$$ (3.24)

Now we differentiate (3.7) to conclude that

$$\nabla u(t) = \nabla e^{(t-\tau(0.2))\Delta} u(\tau(0.2)) - \int_{\tau(0.2)}^{t} \nabla e^{(t-t')\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(t') \, dt'.$$
for $t \in [\tau(0.3), \tau(1)]$. From (3.1), the first term $\nabla e^{(t-\tau(0.2))\Delta P} u(\tau(0.2))$ has an $L^\infty_x$ norm of $O(A^{O(1)})$. From (2.5), the operator $\nabla e^{(t-t')\Delta P}$ maps $L^6_x$ to $L^\infty_x$ with norm $O((t-t')^{-3/4})$, thus

$$\|\nabla (u(t))\|_{L^\infty_x} \lesssim A^{O(1)} + \int_0^t (t-t')^{-3/4} \|\nabla \cdot (u \otimes u)(t')\|_{L^6_x} dt'.$$

From (3.22), (3.24), Leibniz and Hölder one has

$$\|\nabla \cdot (u \otimes u)\|_{L^4_x([\tau(0.2), \tau(1)] \times \mathbb{R}^3)} \lesssim A^{O(1)}$$

and hence by fractional integration

$$\|\nabla u\|_{L^4_x([\tau(0.3), \tau(1)] \times \mathbb{R}^3)} \lesssim A^{O(1)}.$$  

From this, (3.24), Leibniz, and Hölder one has

$$\|\nabla \cdot (u \otimes u)\|_{L^4_x([\tau(0.3), \tau(1)] \times \mathbb{R}^3)} \lesssim A^{O(1)}.$$  

By (2.5), $\nabla e^{(t-t')\Delta P}$ has an operator norm of $O((t-t')^{-1/2})$ on $L^\infty_x$, thus

$$\|\nabla u(t)\|_{L^\infty_x} \lesssim A^{O(1)} + \int_0^t (t-t')^{-1/2} \|\nabla \cdot (u \otimes u)(t')\|_{L^\infty_x} dt'$$

for $t \in [\tau(0.4), \tau(1)]$, and hence by Hölder’s inequality

$$\|\nabla u\|_{L^\infty_x([\tau(0.4), \tau(1)] \times \mathbb{R}^3)} \lesssim A^{O(1)}.$$  

From the vorticity equation (1.5), we now have

$$\partial_t \omega = \Delta \omega + O(A^{O(1)}(|\omega| + |\nabla \omega|))$$

on $[\tau(0.4), \tau(1)] \times \mathbb{R}^3$, and also $\omega = O(A^{O(1)})$ on this slab. Standard parabolic regularity estimates (see e.g., [14]) then give

$$\|\nabla \omega\|_{L^\infty_x([\tau(0.5), \tau(1)] \times \mathbb{R}^3)} \lesssim A^{O(1)}.$$  

Setting $I' := [\tau(0.5), \tau(1)]$, we obtain the claim (iii). We remark that it is also possible to control higher derivatives $\nabla^j u, \nabla^j \omega$ with $j > 1$, for instance by using parabolic Schauder estimates in Hölder spaces, but we will not need to do so here.

Now we establish (iv). Let $t_1, x_1, N_1$ be as in that part of the proposition. By rescaling we may normalise $N_1 = 1$, and by translation invariance we may normalise $(t_1, x_1) = (0, 0)$, so that $t_0 - T \leq \mathcal{T} \leq -\frac{A_3^2}{2}$, so in particular $[-2A_3, 0] \subset [t_0 - T, t_0]$. From (3.5) we have

$$|P_1 u(0, 0)| \geq A_1^{-1}.$$  

Assume for contradiction that the claim fails, then we have

$$\|P_N u\|_{L^\infty_x L^\infty([\tau(0.3), \tau(1)] \times \mathbb{R}^3 \times B(0, A_4))} \lesssim A_1^{-1} N$$

for all $A_2^{-1} \leq N \leq A_2$. From (3.2) and the fundamental theorem of calculus in time, we can enlarge the time interval to reach $t = 0$, so that

$$\|P_N u\|_{L^\infty_x L^\infty([-A_3, 0] \times B(0, A_4))} \lesssim A_1^{-1} N.$$
Suppose now that $N \geq A_2^{-1}$. For $t \in [-A_3, 0]$, we can use Duhamel’s formula, (3.7), and the triangle inequality to write

$$\|P_N u(t)\|_{L^3_x(B(0,A_3))} \leq \|e^{(t+2A_3)} \|_{L^2_x(B(0,A_4))} \\|P_N u(-2A_3)\|_{L^3_x(B(0,A_4))} + \int_{-2A_3}^t \|e^{(t-t')} \|_{L^2_x(R^3)} \|P_N \nabla \cdot (u(t') \otimes u(t'))\|_{L^3_x} dt'.$$

From (2.4), $e^{(t+2A_3)} \|_{L^2_x} P_N$ has an operator norm of $O(\exp(-N^2 A_3/20))$ on $L^3_x$, and $e^{(t-t')} \|_{L^2_x} P_N \nabla \cdot$ similarly has an operator norm of $O(N \exp(-N^2 (t - t')/20))$ on $L^3_x$. Applying (3.1) and Hölder’s inequality, we conclude that

$$\|P_N u(t)\|_{L^3_x(B(0,A_4))} \leq A_4 \exp(-N^2 A_3/20) + A^2 N^{-1}$$

and hence in the range $N \geq A_2^{-1}$ we have

$$\|P_N u\|_{L^T_x L^3_x([-A_3,0] \times B(0,A_4))} \leq A^2 N^{-1}. \quad (3.26)$$

Now suppose that $N \geq A_2^{-1/2}$. For $t \in [-A_3/2, 0]$, we again use Duhamel’s formula, (3.7) and the triangle inequality to write

$$\|P_N u(t)\|_{L^1_x(B(0,A_4/2))} \leq \|e^{(t+2A_3)} \|_{L^2_x(B(0,A_4/2))} \\|P_N u(-A_3)\|_{L^1_x(B(0,A_4/2))} + \int_{-A_3}^t \|e^{(t-t')} \|_{L^2_x(R^3)} \|P_N \nabla \cdot \tilde{P}_N(u(t') \otimes u(t'))\|_{L^1_x} dt'.$$

From (2.4), (3.1), and Hölder as before we have

$$\|e^{(t+2A_3)} \|_{L^2_x} P_N \nabla \cdot \tilde{P}_N(u(t') \otimes u(t')) \leq A_4 \exp(-N^2 A_3/40).$$

From (2.2) one has

$$\|e^{(t-t')} \|_{L^2_x(R^3)} \|P_N \nabla \cdot \tilde{P}_N(u(t') \otimes u(t'))\|_{L^1_x(B(0,A_4/4))} \leq N \exp(-N^2 (t - t')/20) \\|\tilde{P}_N(u(t') \otimes u(t'))\|_{L^3_x} + A_4^{-1} A_4^{1/2} \|\tilde{P}_N(u(t') \otimes u(t'))\|_{L^3_x}$$

and hence by (3.1)

$$\|P_N u\|_{L^T_x L^1_x([-A_3/2,0] \times B(0,A_4/2))} \leq A_4 A_4^{1/2} N^{-1} \|\tilde{P}_N(u(t') \otimes u(t'))\|_{L^1_x([-A_3,0] \times B(0,A_4/4))}.$$ 

Since $\tilde{P}_N (P_{\leq N/100} u(t') \otimes P_{\leq N/100} u(t'))$ vanishes, we can write

$$\tilde{P}_N(u(t') \otimes u(t')) = \tilde{P}_N (P_{> N/100} u(t') \otimes u(t')) + \tilde{P}_N (P_{> N/100} u(t') \otimes P_{> N/100} u(t')) \quad (3.27)$$

From (2.2), (3.1) we have

$$\|\tilde{P}_N (P_{> N/100} u(t') \otimes u(t'))\|_{L^T_x L^1_x([-A_3,0] \times B(0,A_4/4))} \leq \|P_{> N/100} u(t') \otimes u(t')\|_{L^T_x L^1_x([-A_3,0] \times B(0,A_4))} + A_4^{-40}.$$ 

From (3.26) (and the triangle inequality) as well as (3.1) and Hölder’s inequality, we thus have

$$\|\tilde{P}_N (P_{> N/100} u(t') \otimes u(t'))\|_{L^T_x L^1_x([-A_3,0] \times B(0,A_4/4))} \leq A^4 N^{-1}.$$ 

Similarly for the other component of (3.27). We conclude that

$$\|P_N u\|_{L^T_x L^1_x([-A_3/2,0] \times B(0,A_4/2))} \leq A^3 N^{-2} \quad (3.28)$$

for all $N \geq A_2^{-1/2}$. 
Now suppose that $A_2^{-1/3} \leq N \leq A_2^{1/3}$. For $t \in [-A_3/3,0]$, we again use Duhamel’s formula, (3.7), and the triangle inequality as before to write

$$\|P_N u(t)\|_{L^2(B(0,A_4/4))} \leq \|e^{(t+A_3/2)A} P_N u(-A_3/2)\|_{L^2(B(0,A_4/4))} + \int_{-A_3/2}^t \|e^{(t-t')A} P_N \nabla \cdot \tilde{P}_N (u(t') \otimes u(t'))\|_{L^2_x(B(0,A_4/4))} \, dt'.$$

Arguing as before we have

$$\|e^{(t+A_3/2)A} P_N u(-A_3/2)\|_{L^2_x(B(0,A_4/4))} \leq AA_4^{1/2} \exp(-N^2 A_3/120)$$

and

$$\|e^{(t-t')A} P_N \nabla \cdot \tilde{P}_N (u(t') \otimes u(t'))\|_{L^2_x(B(0,A_4/4))} \leq N^{5/2} \exp(-N^2 (t-t')/20)$$

(3.29)

We can split $\tilde{P}_N (u(t') \otimes u(t'))$ into $O(1)$ paraproduct terms of the form $\tilde{P}_N (P_N' u(t') \otimes P_{N/100} u(t'))$ where $N' \sim N$, $O(1)$ terms of the form $\tilde{P}_N (P_{N/100} u(t') \otimes P_N u(t'))$, and a sum of the form $\sum_{N_1-N_2 \geq N} \tilde{P}_N (P_{N_1} u(t') \otimes P_{N_2} u(t'))$. For the “high-low” term $\tilde{P}_N (P_N' u(t') \otimes P_{N/100} u(t'))$, we observe from (3.26), (3.2) and the triangle inequality that

$$\|P_{N/100} u\|_{L^\infty_t L^2_x([-A_3/4,0] \times B(0,A_4))} \leq A_4^{40} + N^{1/2} \|\tilde{P}_N (u(t') \otimes u(t'))\|_{L^\infty_t L^2_x([-A_3/2,0] \times B(0,A_4))}.$$

Using this, (2.2), (3.28) (for the high frequency factor $P_N u(t')$), and Hölder’s inequality, we conclude that the contribution of this term to (3.29) is $O(A^3 A_1^{-1} N^{-1/2})$. Similarly for the “low-high” term $\tilde{P}_N (P_{N/100} u(t') \otimes P_N u(t'))$. Finally, to control the “high-high” term $\sum_{N_1-N_2 \geq N} \tilde{P}_N (P_{N_1} u(t') \otimes P_{N_2} u(t'))$, we use (2.2), the triangle inequality, Hölder, and (3.28) to control this contribution by

$$\|\tilde{P}_N (u(t') \otimes u(t'))\|_{L^\infty_t L^2_x([-A_3/4,0] \times B(0,A_4))} \leq A_4^{40} + N^{1/2} \sum_{N_1-N_2 \geq N} A^3 N_1^{-2} \|P_{N_2} u\|_{L^\infty_t L^2_x([-A_3/4,0] \times B(0,A_4))}.$$

Using (3.26) when $N_2 \leq A_2$ and (3.2) otherwise, we see that this term also contributes $O(A^3 A_1^{-1} N^{-1/2})$. We have thus shown that

$$\|P_N u\|_{L^\infty_t L^2_x([-A_3/4,0] \times B(0,A_4))} \leq A^3 A_1^{-1} N^{-1/2}$$

(3.30)

for $A_2^{-1/3} \leq N \leq A_2^{1/3}$.

We now return once again to Duhamel’s formula to estimate

$$|P_1 u(0,0)| \leq |e^{A_3 \Delta/4} P_1 u(-A_3/4)(0) + \int_{-A_3/4}^0 |e^{(t-t')A} P_1 \nabla \cdot \tilde{P}_1 (u(t') \otimes u(t'))(0) \, dt'|.$$

From (2.4), (3.1), the first term is $O(A_3 \exp(-A_3^2/320))$, thus from (3.25) we have

$$\int_{-A_3/4}^0 |e^{(t-t')A} P_1 \nabla \cdot \tilde{P}_1 (u(t') \otimes u(t'))(0) \, dt' \lesssim A_1^{-1}.$$

From (2.2), (3.1) one has

$$|e^{(t-t')A} P_1 \nabla \cdot \tilde{P}_1 (u(t') \otimes u(t'))(0) \leq \exp(-t-t')/20)(\|\tilde{P}_1 (u(t') \otimes u(t'))\|_{L^1(B(0,A_1))} + A_1^{-50})$$

(3.31)
and hence by the pigeonhole principle we have
\[ \| \tilde{P}_1(u(t') \otimes u(t')) \|_{L^2_t(B(0,A_1))} \geq A_1^{-1}. \]
for some \(-A_3/4 \leq t' \leq 0\).

Fix this \(t'\). As before, we can split \(\tilde{P}_1(u(t') \otimes u(t'))\) into the sum of \(O(1)\) “low-high” terms \(\tilde{P}_1(P_{N_i}u(t') \otimes P_{z_1/100}u(t'))\) and “high-low” terms \(\tilde{P}_1(P_{z_1/100}u(t') \otimes P_{N_i}u(t'))\) with \(N' \sim 1\), plus a “high-high” term \(\sum_{N_1,N_2 \geq 1} \tilde{P}_1(P_{N_1}u(t') \otimes P_{N_2}u(t'))\). For the first two types of terms, we use (2.2) (for frequencies larger than \(A_2^{-1/3}\)), (3.1), and Hölder to conclude that
\[ \| P_{z_1/100}u(t') \|_{L^2_t(B(0,2A_1))} \leq A^3A_1^{-1} \]
and then from (3.30), (2.2) (and (3.1)) to control the global contribution of (2.2) we see that the contribution of those two types of terms is \(O(A^6A_1^{-2})\). For the high-high terms with \(N_1,N_2 \leq A_2^{1/3}\), we again use (3.30), (2.2), (3.1) to again obtain a bound of \(O(A^6A_1^{-2})\). For the cases when \(N_1,N_2 \geq A_2^{1/3}\), we use (3.26), (3.1) to obtain a much better bound \(O(A^3A_2^{-1/3})\). Putting all this together we obtain
\[ A_1^{-1} \leq A^6A_1^{-2} \]
giving the required contradiction. This establishes (iv).

Now we prove (v). We may assume that \(A_4^4N_0^{-2} \leq A_4^{-1}T\), since the claim is trivial otherwise. Thus we have \(N_0 \geq A_4T^{-1/2}\).

By iteratively applying (iv), we may find a sequence \((t_0, x_0), (t_1, x_1), \ldots, (t_n, x_n) \in [t_0 - T, t_0]\) and \(N_0, N_1, \ldots, N_n > 0\) for some \(n \geq 1\), with the properties
\[ |P_{N_i}u(t_i, x_i)| \geq A_1^{-1}N_i \]
\[ A_2^{-1}N_{i-1} \leq N_i \leq A_2N_{i-1} \]
\[ A_3^{-1}N_{i-1}^{-2} \leq t_{i-1} - t_i \leq A_3N_{i-1}^{-2} \]
\[ |x_i - x_{i-1}| \leq A_4N_{i-1}^{-1} \]
for all \(i = 1, \ldots, n\), with \(t_i \in [t_0 - T/2, t_0]\) and \(N_i \geq A_3T^{-1/2}\) for \(i = 0, \ldots, n - 1\) and either \(t_n \in [t_0 - T, t_0 - T/2]\) or \(N_n < A_3T^{-1/2}\). To see that this process terminates at a finite \(n\), observe from the classical nature of \(u\) that the \(P_{N_i}u(t_i, x_i)\) are uniformly bounded in \(i\), which by (3.31) implies that the \(N_i\) are uniformly bounded above, and hence by (3.33) \(t_{i-1} - t_i\) are uniformly bounded below; since \(t_i\) must stay above \(t_0 - T\), we obtain the required finite time termination. By (3.33), the first time \(t_1\) after \(t_0\) lies in the interval
\[ t_1 \in [t_0 - A_2N_0^{-2}, t_0 - A_2^{-1}N_0^{-2}]. \]
If \(N_n < A_3T^{-1/2}\), then by (3.33), (3.32)
\[ t_{n-1} - t_n \geq A_3^{-1}N_n^{-2} \geq A_3^{-2}N_n^{-2} \geq A_3^{-4}(t_{n-1} - t_n) \leq A_3^{-4}T \]
so in particular \(t_n \leq t_0 - A_3^{-4}T\). Of course this inequality also holds if \(t_n \in [t_0 - T, t_0 - T/2]\). In either case, we see from the hypothesis \(A_1N_0^{-2} \leq T_1 \leq A_4^{-1}T\) that
\[ t_n < t - T_1 \leq t_1. \]
Let \( m \) be the largest index for which \( t_m \geq t - T_1 \), thus \( 1 \leq m \leq n - 1 \) and \( t_{m+1} > t - T_1 \). By telescoping (3.33), we conclude that

\[
\sum_{i=0}^{m} A_3 N_i^{-2} = \sum_{i=1}^{m+1} A_3 N_{i-1}^{-2} \geq t - t_{m+1} \geq T_1. \tag{3.35}
\]

On the other hand, from (3.31) and (3.2) we have

\[
|P_N u(t, x_i)| \gtrsim A_1^1 N_i
\]

for \( t \in [t_i - A_1^2 N_i^{-2}, t_i] \); as \( P_N \) is bounded on \( L^\infty \) by (2.3), this implies that

\[
\|P_N u(t)\|_{L_T^T(R^3)} \gtrsim A_1^1 N_i
\]

for such \( t \). From (3.33) we see that the time intervals \([t_i - A_1^2 N_i^{-2}, t_i]\) are disjoint and lie in \([t - T_1, t]\) for \( i = 0, \ldots, m - 1 \). Applying (3.4), we conclude that

\[
\sum_{i=0}^{m-1} A_1^1 N_i \times A_1^2 N_i^{-2} \lesssim A_4^2 T_1^{1/2}
\]

and thus

\[
\sum_{i=0}^{m-1} N_i^{-1} \lesssim A_4^2 T_1^{1/2}.
\]

Using (3.32) to extend this sum to the final index \( m \), we conclude that

\[
\sum_{i=0}^{m} N_i^{-1} \lesssim A_4^2 T_1^{1/2}. \tag{3.36}
\]

Comparing this with (3.35), we conclude that there exists \( i = 0, \ldots, m \) such that

\[
N_i^{-1} \gtrsim A_3^{-2} T_1^{1/2}.
\]

Since \( A_4 N_0^{-2} \lesssim A_4^1 T_1 \), \( i \) cannot be zero, thus \( 1 \leq i \leq m \). From (3.33), (3.32) we have

\[
t_0 - t_i \geq t_{i-1} - t_i \\
\geq A_3^{-1} N_i^{-2} \\
\geq A_3^{-2} N_i^{-2} \\
\geq A_3^{-6} T_1.
\]

Since \( t_0 - t_i \) is also bounded by \( T_1 \), we also have from (3.33) that \( A_3^{-1} N_i^{-2} \leq T_1 \), thus \( N_i \gtrsim A_3^{-1/2} T_1^{-1/2} \). Finally, from telescoping (3.34) and using (3.36), we conclude that

\[
|x_i - x_0| \lesssim A_4^2 T_1^{1/2},
\]

and the claim follows.

Finally, we prove (vi), which is the most difficult estimate. The claim is invariant with respect to time translation and rescaling, so we may assume that \([t_0 - T', t_0] = [0, 1]\). In particular \([-1, 1] \subset [t_0 - T, t_0]\), so we may decompose \( u = u_{\text{lin}} + u_{\text{nlin}} \) as before with the estimates (3.10), (3.11), (3.13).

From (3.13) we can find a time \( t_1 \in [-1/2, 0] \) such that

\[
\int_{R^3} |\nabla u_{\text{nlin}}(t_1, x)|^2 \, dx \lesssim A^4.
\]
Fix this time $t_1$. From (3.9) we thus have

$$\int_{\mathbb{R}^3} |\nabla u^\text{lin}(t_1, x)|^2 + \sum_{j=0}^{4} |\nabla^j u^\text{lin}(t_1, x)|^3 \, dx \leq A^4.$$  

By the pigeonhole principle, we can thus find a scale

$$A^{100}_6 R_0 \leq R \leq \exp(A^{O(1)}_6) R_0$$  

(3.37)

such that

$$\int_{A^{10}_6 R \leq \xi \leq A^6 R} |\nabla u^\text{lin}(t_1, x)|^2 + \sum_{j=0}^{4} |\nabla^j u^\text{lin}(t_1, x)|^3 \, dx \leq A^{-10}_6.$$  

(3.38)

Fix this $R$. We now propagate this estimate forward in time to $[t_1, 1]$. We first achieve this for the linear component $u^\text{lin}$, which is straightforward. From Sobolev embedding we have

$$\sup_{A^8 R \leq \xi \leq A^6 R} |\nabla^j u^\text{lin}(t_1, x)| \leq A^{-3}_6$$

for $j = 0, 1, 2$. Since $\nabla^j u^\text{lin}$ solves the linear heat equation, we conclude from this, (2.2), and (3.11) that

$$\sup_{t \leq 1 \leq A^8 R \leq \xi \leq A^6 R} \sup_{A^8 R \leq \xi \leq A^6 R} |\nabla^j u^\text{lin}(t, x)| \leq A^{-3}_6$$

(3.39)

for $j = 0, 1, 2$. This estimate (when combined with (3.11)) will suffice to control all the terms involving the linear component $u^\text{lin}$ of the velocity (or the analogous component $\omega^\text{lin} := \nabla \times u^\text{lin}$ of the vorticity).

The vorticity $\omega := \nabla \times u$ obeys the vorticity equation (1.5). On $[t_1, 1] \times \mathbb{R}^3$, we decompose $\omega = \omega^\text{lin} + \omega^\text{nlin}$, where $\omega^\text{lin} := \nabla \times u^\text{lin}$ is the linear component of the vorticity and $\omega^\text{nlin} := \nabla \times u^\text{nlin}$ is the nonlinear component. As $\omega^\text{lin}$ solves the heat equation, we have

$$\partial_t \omega^\text{lin} = \Delta \omega^\text{lin} - (u \cdot \nabla) \omega + (\omega \cdot \nabla) u.$$  

(3.40)

As in [20 §10], we apply the energy method to this equation with a carefully chosen time-dependent cutoff function. Namely, let

$$R_- \in [A^8_6 R, 2A^8_6 R]; \quad R_+ \in [A^8_6 R/2, A^8_6 R]$$  

(3.41)

be scales to be chosen later, and define the time-dependent radii

$$R_-(t) := R_- + C_0 \int_{t_1}^{t} (A^8_6 + \|u(t)\|_{L^\infty(\mathbb{R}^3)}) \, dt$$

$$R_+(t) := R_+ - C_0 \int_{t_1}^{t} (A^8_6 + \|u(t)\|_{L^\infty(\mathbb{R}^3)}) \, dt$$

that start at $R_-, R_+$ respectively, and contract inwards at a rate faster than the velocity field $u$. From the bounded total speed property (3.4), (3.37), and the hypothesis $R_0 \geq 1$, we conclude that

$$R_-(t) \in [A^8_6 R, 3A^8_6 R]; \quad R_+(t) \in [A^8_6 R/3, A^8_6 R]$$

for all $t \in [t_1, 1]$.

For $t \in [t_1, 1]$, we define the local enstrophy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega^\text{lin}(t, x)|^2 \eta(t, x) \, dx$$
where $\eta$ is the time-varying cutoff
\[
\eta(t, x) = \max(\min(A_6, |x| - R_-(t), R_+(t) - |x|), 0),
\]
thus $\eta$ is supported in the annulus \( \{R_-(t) \leq |x| \leq R_+(t)\} \), is Lipschitz with norm 1, and equals $A_6$ in the smaller annulus \( \{R_-(t) + A_5 \leq |x| \leq R_+(t) - A_5\} \). From (3.38) we have the initial bound
\[
E(t_1) \leq A_6^{-9}. \tag{3.42}
\]
Now we control the time derivative $\partial_t E(t)$ for $t \in [t_1, 1]$. From (3.40) and integration by parts we have
\[
\partial_t E(t) = -Y_1(t) - Y_2(t) + Y_3(t) + Y_4(t) + Y_5(t) + Y_6(t) + Y_7(t) + Y_8(t) + Y_9(t)
\]
where $Y_1$ is the dissipation term
\[
Y_1(t) := \int_{\mathbb{R}^3} |
abla \omega^{\text{lin}}(t, x)|^2 \, dx,
\]
$Y_2(t)$ is the recession term
\[
Y_2(t) := -\frac{1}{2} \int_{\mathbb{R}^3} |\omega^{\text{lin}}(t, x)|^2 \partial_t \eta(t, x) \, dx,
\]
$Y_3(t)$ is the heat flux term
\[
Y_3(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega^{\text{lin}}(t, x)|^2 \Delta \eta(t, x) \, dx,
\]
$Y_4(t)$ is the transport term
\[
Y_4(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega^{\text{lin}}(t, x)|^2 u(t, x) \cdot \nabla \eta(t, x) \, dx,
\]
$Y_5(t)$ is a correction to the transport term arising from $\omega^{\text{lin}}$
\[
Y_5(t) := -\int_{\mathbb{R}^3} \omega^{\text{lin}}(t, x) \cdot (u(t, x) \cdot \nabla) \omega^{\text{lin}}(t, x) \, \eta(t, x) \, dx,
\]
$Y_6(t)$ is the main nonlinear term
\[
Y_6(t) := \int_{\mathbb{R}^3} \omega^{\text{lin}}(t, x) \cdot (\omega^{\text{lin}}(t, x) \cdot \nabla) u^{\text{lin}}(t, x) \, \eta(t, x) \, dx
\]
and $Y_7(t), Y_8(t), Y_9(t)$ are corrections to the transport term arising from the $u^{\text{lin}}$ and $\omega^{\text{lin}}$
\[
Y_7(t) := \int_{\mathbb{R}^3} \omega^{\text{lin}}(t, x) \cdot (\omega^{\text{lin}}(t, x) \cdot \nabla) u^{\text{lin}}(t, x) \, \eta(t, x) \, dx,
\]
\[
Y_8(t) := \int_{\mathbb{R}^3} \omega^{\text{lin}}(t, x) \cdot (\omega^{\text{lin}}(t, x) \cdot \nabla) u^{\text{lin}}(t, x) \, \eta(t, x) \, dx,
\]
\[
Y_9(t) := \int_{\mathbb{R}^3} \omega^{\text{lin}}(t, x) \cdot (\omega^{\text{lin}}(t, x) \cdot \nabla) u^{\text{lin}}(t, x) \, \eta(t, x) \, dx.
\]
Here all derivatives of the Lipschitz function $\eta$ are interpreted in a distributional sense. We now aim to control $Y_3(t), \ldots, Y_9(t)$ in terms of $Y_1(t), Y_2(t), E(t)$, and some other quantities that are well controlled. From definition of $\eta$ we see that
\[
-\partial_t \eta(t, x) = C_0(A_6 + \|u(t)\|_{L^\infty(\mathbb{R}^3)}) |\nabla \eta(t, x)|
\]
so in particular we have that $Y_3(t)$ is non-negative and
\[
Y_4(t) \leq C_0^{-1} Y_2(t).
\]
A direct computation of $\Delta \eta$ in polar coordinates yields the bound

$$Y_3(t) \lesssim \int_{|x| \in [R_-(t), R_-(t) + A_6]} \frac{|\omega^{\text{nlin}}(t, x)|^2}{|x|} \, dx$$

$$+ \sum_{r = R_-(t), R_-(t), R_-(t) + A_6} r^2 \int_{S^2} |\omega^{\text{nlin}}(t, r \theta)| \, d\theta$$

where $d\theta$ is surface measure on the sphere (in fact the $r = R_-(t) + A_6, R_+(t) - A_6$ terms are non-positive and could be discarded if desired). This expression is difficult to estimate for fixed choices of $R_-, R_+$. However, if selects $R_- , R_+$ uniformly at random from the range (3.41), we see from Fubini’s theorem that the expected value $\mathbb{E}[Y_3]$ of $|Y_3|$ can be estimated by

$$\mathbb{E}[Y_3(t)] \lesssim A_6 \int_{|x| \in [A_6^8, 3A_6^8, \ldots, 3A_6^8]} \frac{|\omega^{\text{nlin}}(t, x)|^2}{|x|^2} \, dx$$

and hence by (3.13), (3.37)

$$\mathbb{E} \int_{t_1}^1 |Y_3(t)| \, dt \lesssim A_6^{-10}$$

(say). Thus we can select $R_-, R_+$ so that

$$\int_{t_1}^1 |Y_3(t)| \, dt \lesssim A_6^{-10} \quad (3.43)$$

and we shall now do so.

To treat $Y_5(t)$, we use Young’s inequality to bound

$$Y_5(t) \lesssim E(t) + \int_{\mathbb{R}^3} |(u \cdot \nabla) \omega^{\text{nlin}}| \, dx.$$ 

Using (3.1), (3.11), (3.39), Hölder’s inequality, we then have

$$Y_5(t) \lesssim E(t) + A_6^{-2}$$

(say).

In a similar vein, from (3.39) and Hölder’s inequality one has

$$Y_7(t) \lesssim E(t)$$

(with plenty of room to spare) and from Young’s inequality one has

$$Y_9(t) \lesssim E(t) + \int_{\mathbb{R}^3} |(\omega^{\text{nlin}} \cdot \nabla) u^{\text{nlin}}| \, dx$$

and hence by (3.1), (3.11), (3.39), and Hölder

$$Y_9(t) \lesssim E(t) + A_6^{-2}.$$ 

For $Y_8$, we again use Young’s inequality to bound

$$Y_8(t) \lesssim E(t) + \int_{\mathbb{R}^3} |(\omega^{\text{nlin}} \cdot \nabla) u^{\text{nlin}}| \, dx$$

and hence by (3.39)

$$Y_8(t) \lesssim E(t) + Y_{10}(t)$$

where

$$Y_{10}(t) = A_6^{-3} \int_{\mathbb{R}^3} |\nabla u^{\text{nlin}}(t, x)|^2 \, dx.$$
Observe from (3.13) that
\[
\int_{t_1}^{1} |Y_{10}(t)| \, dt \lesssim A_6^{-2}.
\] (3.44)

We are left with estimation of the most difficult term \(Y_6(t)\). Following [20], we cover the annulus \(\{R_- \leq |x| \leq R_+(t)\}\) by a boundedly overlapping Whitney decomposition of balls \(B = B(x_B, r_B)\), where the radius \(r_B\) of the ball is given as \(r_B = \frac{1}{100} \eta(t, r_B)\). In particular, we have \(\eta(t, x) \sim r_B\) on the dilate \(10B = B(x_B, r_B)\) of the ball. We can then write
\[
Y_6(t) \sim \sum_B r_B \int_B |\omega^{\text{lin}}|^2 |\nabla u^{\text{lin}}| \, dx
\]
where we suppress the explicit dependence on \(t, x\) for brevity. Similarly one has
\[
E(t) \sim \sum_B r_B \int_{10B} |\omega^{\text{lin}}|^2 \, dx
\] (3.45)
and
\[
Y_1(t) \sim \sum_B r_B \int_{10B} |\nabla \omega^{\text{lin}}|^2 \, dx
\] (3.46)
To control \(Y_6(t)\), we need to control \(\nabla u^{\text{lin}}\). The Biot-Savart law suggests that this function has comparable size to \(\omega^{\text{lin}}\), but we need to localise this intuition to the ball \(B\) and thus must address the slightly non-local nature of the Biot-Savart law. Fortunately this can be handled using standard cutoff functions. Namely, we have \(\Delta u^{\text{lin}} = -\nabla \times \omega^{\text{lin}}\), hence if we let \(\psi_B\) be a smooth cutoff adapted to \(3B\) that equals 1 on \(2B\), then
\[
u^{\text{lin}} = -\Delta^{-1}(\nabla \times (\omega^{\text{lin}} \psi_B)) + v
\]
where \(v\) is harmonic on \(2B\). From Sobolev embedding and Hölder one has
\[
\|v\|_{L^3_x(2B)} \lesssim \|\omega^{\text{lin}} \psi_B\|_{L^6_{t,x}(R^3)} + \|u^{\text{lin}}\|_{L^3_x(2B)} \lesssim r_B^{3/2} \|\omega^{\text{lin}}\|_{L^3_x(3B)} + \|u^{\text{lin}}\|_{L^3_x(2B)}
\]
and hence by elliptic regularity for harmonic functions
\[
\|\nabla v\|_{L^\infty_x(B)} \lesssim r_B^{-5/2} \|\nu\|_{L^3_x(2B)} \lesssim r_B^{-1} \|\omega^{\text{lin}}\|_{L^3_x(3B)} + r_B^{-5/2} \|u^{\text{lin}}\|_{L^2_x(2B)}
\]
We conclude the pointwise estimate
\[
\nabla u^{\text{lin}} = -\nabla \Delta^{-1}(\nabla \times (\omega^{\text{lin}} \psi_B)) + O(r_B^{-1} \|\omega^{\text{lin}}\|_{L^3_x(3B)}) + O(r_B^{-5/2} \|u^{\text{lin}}\|_{L^2_x(2B)})
\] (3.47)
on \(B\). By elliptic regularity, \(\nabla \Delta^{-1}(\nabla \times (\omega^{\text{lin}} \psi_B))\) has an \(L^3_x(B)\) norm of \(O(\|\omega^{\text{lin}}\|_{L^3_x(3B)})\).
From Hölder’s inequality we thus have
\[
\int_B |\omega^{\text{lin}}|^2 |\nabla u^{\text{lin}}| \, dx \lesssim \|\omega^{\text{lin}}\|_{L^3_x(3B)}^3 + r_B^{-5/2} \|\omega^{\text{lin}}\|_{L^2_x(3B)}^2 \|u^{\text{lin}}\|_{L^2_x(3B)}
\]
and hence \(Y_6(t) \lesssim Y_{6,1}(t) + Y_{6,2}(t)\), where
\[
Y_{6,1}(t) := \sum_B r_B \|\omega^{\text{lin}}\|_{L^3_x(3B)}^3
\]
and
\[
Y_{6,2}(t) := \sum_B r_B^{-3/2} \|\omega^{\text{lin}}\|_{L^2_x(3B)}^2 \|u^{\text{lin}}\|_{L^2_x(3B)}
\]
For \(Y_{6,2}(t)\), we first consider the contribution of the large balls in which \(r_B \gtrsim A_6\). Here we simply use (3.10) to bound \(\|u^{\text{lin}}\|_{L^2_x(3B)} \lesssim A^2\). Since \(r_B^{-3/2} A^2 \lesssim \lesssim \) for large balls \(B\),
the contribution of this case is $O(E(t))$ thanks to (3.45). Now we look at the small balls in which $r_B < A^{10}$. Here we use Hölder to bound

$$
\|u^{\text{lin}}\|_{L^2(3B)} \lesssim r_B^{3/2} \|u^{\text{lin}}\|_{L^6_t(\mathbb{R}^3)} \lesssim r_B^{3/2} \left( A^2 + \|u\|_{L^6_t(\mathbb{R}^3)} \right)
$$

so the contribution of this case is bounded by

$$
\sum_{B: r_B < A^{10}} \|\omega^{\text{lin}}\|_{L^2(3B)}^2 \left( A^2 + \|u\|_{L^6_t(\mathbb{R}^3)} \right).
$$

For small balls $B$, $3B$ is completely contained inside the region in which $\partial_t \eta \gtrsim C_0(A_6 + \|u\|_{L^6_t(\mathbb{R}^3)})$, so the contribution of this case can be bounded by $O(C_0^{-1}Y_2(t))$. Thus

$$
Y_{6,2}(t) \lesssim E(t) + C_0^{-1}Y_2(t).
$$

Now we control $Y_{6,1}(t)$. For each ball $B$, define the mean vorticity $\omega_B$ by

$$
\omega_B := \frac{\int_{\mathbb{R}^3} \omega^{\text{lin}} \psi_B \, dx}{\int_{\mathbb{R}^3} \psi_B \, dx}.
$$

From the Poincaré inequality, Sobolev embedding, and the triangle inequality we have

$$
\|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)} \lesssim \|\nabla \omega^{\text{lin}}\|_{L^3(10B)}
$$

and similarly

$$
|\omega_B - \omega_{B'}| \lesssim r_B^{-1/2} \|\nabla \omega^{\text{lin}}\|_{L^3(10B)} \tag{3.48}
$$

when $B, B'$ are overlapping Whitney balls. We can now use Hölder’s inequality to bound

$$
Y_{6,1}(t) \lesssim \sum_B r_B^4A_6^3 + \sum_B r_B \|\omega^{\text{lin}} - \omega_B\|^3_{L^2(3B)}
\lesssim \sum_B r_B^4A_6^3 + \sum_B r_B \|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)}^{3/2} \|\nabla \omega^{\text{lin}}\|_{L^3(10B)}^{3/2}.
$$

By Young’s inequality and (3.46), we then have

$$
Y_{6,1}(t) \lesssim \frac{1}{2} Y_1(t) + O(\sum_B r_B^4A_6^3 + \sum_B r_B \|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)}^6).
$$

From the triangle inequality and Cauchy-Schwarz, and (3.45), one has

$$
\|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)} \lesssim \|\omega^{\text{lin}}\|_{L^2(3B)} \lesssim r_B^{-1/2} E(t)^{1/2}
$$

and hence from Hölder, (3.46)

$$
\sum_B r_B \|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)}^6 \lesssim \sum_B r_B \|\omega^{\text{lin}} - \omega_B\|_{L^2(3B)}^2 E(t)^2 \lesssim E(t)^2 Y_1(t).
$$

Now we estimate $\sum_B r_B^4A_6^3$. We can arrange the Whitney decomposition so that all the radii $r_B$ are powers of 1.001, and that every ball $B$ of radius less than (say) $A_6/100$ has a “parent” ball $p(B)$ that overlaps $B$ and has radius 1.001$r_B$. From the triangle inequality we have

$$
|\omega_B| \leq |\omega_{p^k(B)}| + \sum_{i=0}^{k-1} |\omega_{p^{i+1}(B)} - \omega_{p^i(B)}|
$$

for any Whitney ball $B$, where $k = k_B$ is the first natural number for which the iterated parent $p^k(B)$ has radius larger than $A_6^{1/2}$. By Hölder we then have

$$
|\omega_B|^3 \leq |\omega_{p^k(B)}|^3 + \sum_{i=0}^{k-1} |\omega_{p^{i+1}(B)} - \omega_{p^i(B)}|^3.
$$
From a volume packing argument we see that for a given \( i \), a Whitney ball \( B' \) is of the form \( p'(B) \) for at most \( O((1.001)^{2i}) \) choices of \( B \). One can then sum the geometric series (exactly as in [20, §10]) and conclude that

\[
\sum_{B} r_B^4 \omega_B^3 \lesssim \sum_{B : r_B < A_6/100} r_B^4 \omega_B^3 + \sum_{B : r_B \leq A_6^{1/2}} r_B^4 |\omega_B - \omega_{p(B)}|^3.
\]

For the small balls in which \( r_B < A_6/100 \), we observe from (3.46) and Cauchy-Schwarz that

\[
\omega_B, \omega_{p(B)} \lesssim r_B^{-2} E(t)^{1/2}
\]

and thus from (3.48), (3.46)

\[
\sum_{B : r_B < A_6/100} r_B^4 |\omega_B - \omega_{p(B)}|^3 \lesssim E(t)^{1/2} Y_1(t).
\]

For the large balls in which \( r_B \geq A_6/100 \), we write \( \omega^{\text{lin}} = \nabla \times u^{\text{lin}} \) and integrate by parts using Cauchy-Schwarz to find that

\[
\omega_B \lesssim r_B^{-5/2} \|u^{\text{lin}}\|_{L^2(B)}
\]

and hence using (3.10) and the bounded overlap of the Whitney balls

\[
\sum_{B : r_B \geq A_6/100} r_B^4 \omega_B^3 \lesssim \sum_{B : r_B \geq A_6/100} A^2 r_B^{-7/2} \|u^{\text{lin}}\|_{L^2(B)}^2 \lesssim A^4 A_6^{-7/2} \lesssim A_6^{-2}.
\]

Thus we have

\[
Y_{6,1} \leq \frac{1}{2} Y_1(t) + O(E(t)^{1/2} Y_1(t) + A_6^{-2} + E(t)^2 Y_1(t)).
\]

Putting all this together, we see that

\[
\partial_t E(t) \leq -\frac{1}{2} Y_1(t) + O \left( E(t) + |Y_3(t)| + |Y_{10}(t)| + A_6^{-2} + E(t)^{1/2} Y_1(t) + E(t)^2 Y_1(t) \right).
\]

A standard continuity argument using (3.42), (3.43), (3.44) then gives

\[
E(t) \lesssim A_6^{-2}
\]

for all \( t_1 \leq t \leq 1 \), and also

\[
\int_{t_1}^1 Y_1(t) \, dt \lesssim A_6^{-2}.
\]

These are subcritical regularity estimates and can now be iterated\(^3\) as in the proof of (iii) to obtain higher regularity. First we move from control of the vorticity back to control of the velocity. From (3.47) and elliptic regularity one has

\[
\|\nabla u^{\text{lin}}\|_{L^2(B)}^2 \lesssim \|\omega^{\text{lin}}\|_{L^2(B)}^2 + r_B^{-2} \|u^{\text{lin}}\|_{L^2(2B)}^2
\]

for any ball \( B \); summing this on balls of radius \( A_6^{10} \) (say) using (3.49), (3.10), we conclude that

\[
\int_{A_6^{-7} R \leq |x| \leq A_6^7 R} |\nabla u^{\text{lin}}(t, x)|^2 \, dx \lesssim A_6^{-2}
\]

for all \( t_1 \leq t \leq 1 \). Similarly we have

\[
\|\nabla^2 u^{\text{lin}}\|_{L^2(B)}^2 \lesssim \|\nabla \omega^{\text{lin}}\|_{L^2(B)}^2 + r_B^{-2} \|\nabla u^{\text{lin}}\|_{L^2(2B)}^2
\]

\(^3\)It is likely that one can also proceed at this point using the local regularity theory from [4].
and using (3.51), (3.50) in place of (3.10), (3.49) we conclude that
\[
\int_{t_1}^1 \int_{A_6^{-6} R \leq |x| \leq A_6^{-2} R} |\nabla^2 u_{\text{lin}}(t, x)|^2 \, dx \, dt \leq A_6^{-2}.
\]
Using the Gagliardo-Nirenberg inequality (3.20) as before we see that
\[
\|u_{\text{lin}}\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (A_6^{-3} R \leq |x| \leq A_6^{-2} R))} \leq A_6^{-2},
\]
which when combined with (3.39) gives
\[
\|u\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (A_6^{-5} R \leq |x| \leq A_6^{-4} R))} \leq A_6^{-2}.
\]
By repeating the arguments in (iii) (using (2.2) in place of (2.1) to handle the long-range components of the heat kernel, which can be controlled with extremely good bounds using (3.1)), one can then show iteratively that
\[
\|u\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (A_6^{-4} R \leq |x| \leq A_6^{-3} R))} \leq A_6^{-2},
\]
then
\[
\|\nabla u\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (A_6^{-3} R \leq |x| \leq A_6^{-2} R))} \leq A_6^{-2},
\]
then finally
\[
\|\nabla \omega\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (R \leq |x| \leq 2 A_6 R))} \leq A_6^{-2},
\]
and
\[
\|\nabla \omega\|_{L^4_1 L^\infty_2 ([t_1, 1] \times (R \leq |x| \leq A_6 R))} \leq A_6^{-2},
\]
giving (vi).

4. Carleman inequalities for backwards heat equations

We will need some Carleman inequalities for backwards heat equations which are essentially contained in previous literature (most notably [7], [6]), but made slightly more quantitative for our application (also it will be convenient to not demand that the functions involved vanish at the starting and final time). Following [7], we shall reverse the direction of time and work here with backwards heat equations rather than forward ones.

Our main tool is the following general inequality (cf. [6] Lemma 2):

**Lemma 4.1** (General Carleman inequality). Let \([t_1, t_2]\) be a time interval, and let \(u \in C^\infty_c([t_1, t_2] \times \mathbb{R}^d \to \mathbb{R}^m)\) be a (vector-valued) test function solving the backwards heat equation
\[
Lu = f
\]
with \(L\) the backwards heat operator
\[
L := \partial_t + \Delta,
\]
and let \(g : [t_1, t_2] \times \mathbb{R}^n \to \mathbb{R}\) be smooth. Let \(F : [t_1, t_2] \times \mathbb{R}^d \to \mathbb{R}\) denote the function
\[
F := \partial_t g - \Delta g - |\nabla g|^2.
\]
Then we have the inequality
\[ \partial_t \int_{\mathbb{R}^d} \left( |\nabla u|^2 + \frac{1}{2} F|u|^2 \right) e^\varphi dx \geq \int_{\mathbb{R}^d} \left( \frac{1}{2} (LF)|u|^2 + 2D^2g(\nabla u, \nabla u) - \frac{1}{2} |Lu|^2 \right) e^\varphi dx \]
for all \( t \in I \), where \( D^2g \) is the quadratic form expressed in coordinates as
\[ D^2g(v, w) = \left( \partial_i \partial_j g \right) v_i w_j \]
with the usual summation conventions. In particular, from the fundamental theorem of calculus one has
\[ \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left( \frac{1}{2} (LF)|u|^2 + 2D^2g(\nabla u, \nabla u) \right) e^\varphi dx dt \leq \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |Lu|^2 e^\varphi dx dt + \int_{\mathbb{R}^d} \left( |\nabla u|^2 + \frac{1}{2} F|u|^2 \right) e^\varphi dx |_{t=t_1}^{t=t_2}. \]

The above inequality is valid in all dimensions, but in this paper we will only need this lemma in the case \( d = m = 3 \).

**Proof.** By breaking \( u \) into components, we may assume without loss of generality that we are in the scalar case \( m = 1 \).

We use the usual commutator method. Introducing the weighted (and time-dependent) inner product
\[ \langle u, v \rangle := \int_{\mathbb{R}^n} uv e^\varphi dx \]
for test functions \( u, v : I \times \mathbb{R}^n \to \mathbb{R} \), we compute after differentiating under the integral sign and integrating by parts
\[ \langle Lu, v \rangle + \langle u, Lv \rangle = \int_{\mathbb{R}^n} \left( \partial_i(uv) + \Delta(uv) - 2\nabla u \cdot \nabla v \right) e^\varphi dx \]
\[ = \partial_i\langle u, v \rangle + \int_{\mathbb{R}^n} \left( -(\partial_i g)uv + (\Delta g + |\nabla g|^2)uv - 2\nabla u \cdot \nabla v \right) e^\varphi dx \]
\[ = \partial_i\langle u, v \rangle - \langle Fu, v \rangle - 2\langle \partial_i u, \partial_i v \rangle \]
with the usual summation conventions. We can write
\[ -\langle \partial_i u, \partial_i v \rangle - \frac{1}{2} \langle Fu, v \rangle = \langle Su, v \rangle \]  \hspace{1cm} (4.2)
where \( S \) is the differential operator
\[ Su := \Delta u + \nabla g \cdot \nabla u - \frac{1}{2} Fu \]
which is then formally self-adjoint with respect to the inner product \( \langle , \rangle \); one can view \( S \) as the self-adjoint component of \( L \). We can then rewrite the above identity as
\[ \partial_i\langle u, v \rangle = \langle Lu, v \rangle + \langle u, Lv \rangle - 2\langle Su, v \rangle. \]
In particular, by the self-adjointness of $S$ we have for any test functions $u, v$ that
\[ \partial_t \langle Su, v \rangle = \langle LSu, v \rangle + \langle Su, Lv \rangle - 2 \langle Su, Sv \rangle \]
\[ = \langle [L, S]u, v \rangle + \langle SLu, v \rangle + \langle Su, Lv \rangle - 2 \langle Su, Sv \rangle \]
\[ = \langle [L, S]u, v \rangle + \frac{1}{2} \langle Lu, Lv \rangle - \frac{1}{2} \langle (L - 2S)u, (L - 2S)v \rangle. \]

Among other things, this shows that the differential operator $[L, S]$ (which does not involve any time derivatives) is formally self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. Specialising to the case $u = v$, we conclude in particular the inequality
\[ \partial_t \langle Su, u \rangle \leq \langle [L, S]u, u \rangle + \frac{1}{2} \langle Lu, Lu \rangle. \quad (4.3) \]

Now we compute $[L, S]$. As previously noted, $[L, S]$ is a formally self-adjoint differential operator that does not involve any time derivatives. Since the second order operator $L$ commutes with the second order component $\Delta$ of $S$, we see that $[L, S]$ is a second-order operator. The highest order terms can be easily computed in coordinates as
\[ [L, S]u = 2(\partial_t \partial_{jy}) \partial_j \partial_t u + \text{l.o.t.} \]
and hence after integrating by parts the symmetric quadratic form $\langle [L, S]u, v \rangle$ must take the form
\[ \langle [L, S]u, v \rangle = 2 \int_{\mathbb{R}^d} (D^2 g(\nabla u, \nabla v) + Huv) \ e^g \ dx \]
for some function $H$; setting $u = 1$, we see that $H$ must equal
\[ H = [L, S]1 = LS1 = LF. \]

We conclude that
\[ \langle [L, S]u, u \rangle = 2 \int_{\mathbb{R}^d} (D^2 g(\nabla u, \nabla u) + (LF)|u|^2) \ e^g \ dx. \]

Inserting this identity back into (4.3) and using (4.2), we obtain the claim. \qed

The inequality below is a quantitative variant of [6, Lemma 4].

**Proposition 4.2** (First Carleman inequality). Let $T > 0$, $0 < r_- < r_+$, and let $A$ denote the cylindrical annulus
\[ A := \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : t \in [0, T]; r_- \leq |x| \leq r_+ \}. \]
Let $u : A \to \mathbb{R}^3$ be a smooth function obeying the differential inequality
\[ |Lu| \leq C_0^{-1}T^{-1}|u| + C_0^{-1/2}T^{-1/2} |\nabla u| \]
\[ (4.4) \]
on $A$. Assume the inequality
\[ r_-^2 \geq 4C_0T. \]
\[ (4.5) \]
Then one has
\[ \int_0^{T/4} \int_{10r_- \leq |x| \leq 12r_+} (T^{-1}|u|^2 + |\nabla u|^2) \ dx dt \leq C_0^2 e^{-\frac{r_-^2}{4C_0T}}(X + e^{2r_2/Y}C_0T) \]
where
\[ X := \int_A e^{2|x|^2/C_0T} (T^{-1}|u(t, x)|^2 + |\nabla u(t, x)|^2) \ dx dt \]
and

\[ Y := \int_{r_- \leq |x| \leq r_+} |u(0, x)|^2 \, dx. \]

The key feature here is the gain of \( e^{-\frac{4C_0 r_+}{T^2}} \), which can be compared against the trivial bound of \( e^{-2r^2/C_0 T} X \) that follows by lower bounding the factor \( e^{2|x|^2/C_0 T} \) appearing in \( X \) by \( e^{2r^2/C_0 T} \). Thus, this lemma becomes powerful when the ratio \( r_+/r_- \) is large. Informally, Proposition 4.2 asserts that if \( u \) solves \( (4.4) \) on \( \mathcal{A} \), has some mild Gaussian decay as \( |x| \to \infty \), and is extremely small at \( t = 0 \), then it is also very small in the interior of \( \mathcal{A} \) near \( t = 0 \). The various numerical constants such as 1/4 or 10 appearing in the above proposition can be modified (and optimised) if desired, but we fix a specific choice of constants for sake of concreteness. The weight \( e^{2|x|^2/C_0 T} \) in \( X \) is inconvenient, but it is negligible when compared against the “natural” decay rate of \( e^{-|x|^2/4t} \) arising from the fundamental solution of the heat equation, and it can be managed in our application by using the second Carleman inequality given below. Specialising Proposition 4.2 to the case \( u(0, x) = 0 \) (so that \( Y = 0 \)) and sending \( r_+ \) to infinity, one recovers a variant of the backwards uniqueness result in [6, Lemma 4].

**Proof.** We may assume that

\[ r_+ \geq 20r_- \tag{4.6} \]

since the claim is vacuous otherwise. By the pigeonhole principle, one can find a time \( T_0 \in [T/2, T] \) such that

\[ \int_{r_- \leq |x| \leq r_+} e^{2|x|^2/C_0 T} (T^{-1} \|u(T_0, x)\|^2 + \|\nabla u(T_0, x)\|^2) \, dx \lesssim T^{-1} X. \tag{4.7} \]

Fix this time \( T_0 \). In the discussion below we implicitly restrict \((t, x)\) to the region \( \mathcal{A} \cap ([0, T_0] \times \mathbb{R}^3) \). We set

\[ \alpha := \frac{r_+}{2C_0 T^2} \tag{4.8} \]

and observe from (4.5), (4.6) that

\[ \alpha \geq \frac{40}{r_- T}. \tag{4.9} \]

Following [4], we apply Lemma 4.1 on the interval \([0, T_0]\) with the weight

\[ g := \alpha(T_0 - t) |x| + \frac{1}{C_0 T} |x|^2 \]

and \( u \) replaced by \( \psi u \), where \( \psi(x) \) is a smooth cutoff supported on the region \( r_1 \leq |x| \leq r_2 \) that equals 1 on \( 2r_1 \leq |x| \leq r_2/2 \) and obeys the estimates \( |\nabla^j \psi(x)| = O(1/|x|^j) \) for \( j = 0, 1, 2 \). Since \( \alpha(T_0 - t) |x| \) is convex in \( x \), we have

\[ D^2 g(\nabla(\psi u), \nabla(\psi u)) \geq 2 - \frac{1}{C_0 T} |\nabla(\psi u)|^2. \]
The function $F$ defined in Lemma 4.1 can be computed on $A$ as

$$F = -\alpha|x| - \frac{2\alpha(T_0 - t)}{|x|} + \frac{6}{C_0 T} - \left(\alpha(T_0 - t) + \frac{2|x|}{C_0 T}\right)^2$$

$$= -\alpha|x| - \frac{2\alpha(T_0 - t)}{|x|} + \frac{6}{C_0 T} - \alpha^2(T_0 - t)^2 - 4\alpha \frac{T_0 - t}{C_0 T} |x| - \frac{4|x|^2}{C_0 T^2}.$$  

In particular by (4.5) we have $\frac{4|x|^2}{C_0 T^2} \geq \frac{6}{C_0 T}$, and hence $F$ is negative. We also calculate

$$LF = 2\alpha^2(T_0 - t) + \frac{4\alpha|x|}{C_0 T} - \frac{8\alpha(T_0 - t)}{C_0 T|x|} - \frac{24}{C_0^2 T^2}.$$  

We see from (4.5) that $\frac{T_0 - t}{|x|} \leq \frac{1}{4}|x|$, so that

$$LF \geq \frac{2\alpha|x|}{C_0 T} - \frac{24}{C_0^2 T^2}.$$  

By (4.9) we thus have $LF \geq \frac{56}{C_0^2 T^2}$. Applying Lemma 4.1 and discarding some terms, we conclude that

$$\int_0^{T_0} \int_{x \in [r_-, r_+]/2} \left(28C_0^{-2}T^{-2}|u|^2 + 4C_0^{-1}T^{-1}|
abla u|^2\right) e^g \, dx \, dt$$

$$\leq \frac{1}{2} \int_0^{T_0} \int_{R^3} |L(\psi u)|^2 e^g \, dx \, dt$$

$$+ \int_{R^3} |
\nabla (\psi u)(T, x)|^2 e^{g(T_0, x)} \, dx + \int_{R^3} |(\psi u)(0, x)|^2 |F(0, x)| e^{g(0, x)} \, dx.$$  

In the region $2r_- \leq |x| \leq r_+/2$ we have from (4.4) that

$$|L(\psi u)|^2 = |Lu|^2 \leq 2C_0^{-2}T^{-2}|u|^2 + 2C_0^{-1}T^{-1}|
abla u|^2.$$  

In the regions $r_- \leq |x| \leq r_-$ or $r_+/2 \leq |x| \leq r_+$, we have

$$|L(\psi u)|^2 \leq |Lu|^2 + |x|^{-2}|
abla u|^2 + |x|^{-4}|u|^2 \leq C_0^{-2}T^{-2}|u|^2 + C_0^{-1}T^{-1}|
abla u|^2.$$  

thanks to (4.4), (4.5). For all other $x$, $L(\psi u)$ vanishes. A similar calculation gives

$$|
\nabla (\psi u)|^2 \leq |
\nabla u|^2 + C_0^{-1}T^{-1}|u|^2.$$  

We therefore have

$$\int_0^{T_0} \int_{x \in [r_-, r_+]/2} \left(C_0^{-2}T^{-2}|u|^2 + C_0^{-1}T^{-1}|
abla u|^2\right) e^g \, dx \, dt$$

$$\leq \int_0^{T_0} \int_{x \in [r_-, r_+]/2} \left(C_0^{-2}T^{-2}|u|^2 + C_0^{-1}T^{-1}|
abla u|^2\right) e^g \, dx \, dt$$

$$+ \int_{r_- \leq |x| \leq r_+} (C_0^{-1}T^{-1}|u|^2 + |
\nabla u|^2) e^{g(T_0, x)} \, dx + \int_{r_- \leq |x| \leq r_+} |u(0, x)|^2 |F(0, x)| e^{g(0, x)} \, dx.$$  

From (4.7) one has

$$\int_{r_- \leq |x| \leq r_+} (C_0^{-1}T^{-1}|u|^2 + |
\nabla u|^2) e^{g(T_0, x)} \, dx \leq T^{-1} X.$$  

When $t \in [0, T_0]$ and $|x| \in [r_+/2, r_+]$, one has

$$e^g \leq e^{\alpha|T| |x|^{-2}/C_0 T} e^{2|x|^2/C_0 T} \leq e^{2|x|^2/C_0 T}$$
We conclude that by (4.8). When instead \( t \in [0, T_0] \) and \( |x| \in [r_-, 2r_-] \), one has
\[
e^g \leq e^{\alpha T|x|^{\frac{1}{2}} |x|^2/\epsilon_0^2} e^{2|x|^2/C_0 T} \leq e^{2\alpha T r_-} e^{2|x|^2/C_0 T}.
\]
We conclude that
\[
\int_0^{T_0} \int_{2r_- \leq |x| \leq r_0/2} \left( C_0^{-2} T^{-2} |u|^2 + C_0^{-1} T^{-1} |\nabla u|^2 \right) e^g \, dx \, dt \\
\leq e^{2\alpha T r_-} T^{-1} X + \int_{r_- \leq |x| \leq r_+} |u(0, x)|^2 |F(0, x)| \, e^{g(0, x)} \, dx.
\]
In the region \( t \in [0, T/4] \), \( 10r_- \leq |x| \leq r_0/2 \), one has
\[
e^g \geq e^{\frac{\alpha T |x|}{4} - \frac{|x|^2}{\epsilon_0 T}} \geq e^\frac{1}{2} u^{\alpha T r_-}
\]
and hence
\[
\int_0^{T/4} \int_{10r_- \leq |x| \leq r_0/2} \left( C_0^{-2} T^{-2} |u|^2 + C_0^{-1} T^{-1} |\nabla u|^2 \right) \, dx \, dt \\
\leq e^{-\alpha T r_-^2} \left( T^{-1} X + \int_{r_- \leq |x| \leq r_+} |u(0, x)|^2 |F(0, x)| \, e^{g(0, x)} \, dx \right).
\]
From (4.8) we have
\[
e^{-\alpha T r_-^2/2} = e^{-\frac{r_- r_+}{4\epsilon_0 T}}.
\]
Finally, for \( t = 0 \) and \( r_- \leq |x| \leq r_+ \) one has
\[
e^g \leq e^{\alpha T|x|^{\frac{1}{2}} |x|^2/\epsilon_0^2} \leq e^{\frac{\alpha T r_+}{2}}
\]
and
\[
|F| \leq \alpha r_+ + \alpha T r_- + C_0^{-1} T^{-1} + \alpha^2 T^2 + C_0^{-1} \alpha r_+ + C_0^{-2} T^{-2} r_+^2 \\
\leq \frac{r_+^2}{T^2} + \frac{r_-}{T} + \frac{1}{T} + \frac{r_+^2}{T^2} + \frac{r_+^2}{T^2} + \frac{r_+^2}{T^2} \\
\leq \frac{r_+^2}{T} \left( T^{-1} + \frac{1}{r_+ r_-} \right) \\
\leq \frac{r_+^2}{T} \left( T^{-1} \right)
\]
since \( \frac{1}{r_+ r_-} \leq \frac{1}{r_-^2} \leq \frac{1}{T} \) by (4.6), (4.5). Bounding \( \frac{r_+^2}{T} e^{\frac{3\alpha T}{2}} \leq \frac{2\alpha}{2} \) and multiplying by \( T \), we conclude that
\[
\int_0^{T/4} \int_{10r_- \leq |x| \leq r_0/2} \left( C_0^{-2} T^{-1} |u|^2 + C_0^{-1} |\nabla u|^2 \right) \, dx \, dt \\
\leq e^{-\frac{T r_-^2}{4\epsilon_0^2}} \left( X + e^{2r_+^2/ T} \int_{r_- \leq |x| \leq r_+} |u(0, x)|^2 \, dx \right)
\]
giving the claim. \[\square\]

Our second application of Lemma 4.1 is the following quantitative version of standard parabolic unique continuation results.
Proposition 4.3 (Second Carleman inequality). Let $T, r > 0$, and let $C$ denote the cylindrical region

$$C = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 : t \in [0, T]; |x| \leq r \}.$$  

Let $u : C \to \mathbb{R}^3$ be a smooth function obeying the differential inequality (4.4) on $C$. Assume the inequality

$$r^2 \geq 4000 T. \tag{4.10}$$

Then for any

$$0 < t_1 \leq t_0 < \frac{T}{1000} \tag{4.11}$$

one has

$$\int_{t_0}^{2t_0} \int_{|x| \leq r/2} \left( T^{-1} |u|^2 + |\nabla u|^2 \right) e^{-\alpha x^2/4t} \, dx \, dt \leq e^{-\frac{r^2}{1600} t_0} X + t_0^{3/2} (et_0/t_1)^{O(e^{r/2})} Y$$

where

$$X := \int_0^T \int_{|x| \leq r} \left( T^{-1} |u|^2 + |\nabla u|^2 \right) \, dx \, dt$$

and

$$Y := \int_{|x| \leq r} |u(0, x)|^2 (e^{-3/2} e^{-\alpha x^2/4t_1}) \, dx.$$

As with the previous inequality, the numerical constants here such as 1000, 500 can be optimised if desired, but this explicit choice of constants suffices for our application. The key feature here is the gain of $e^{-\frac{r^2}{1600} t_0}$. Specialising to the case where $u$ vanishes to infinite order at $(0, 0)$, sending $t_1 \to 0$ (which sends $(et_0/t_1)^{O(e^{r/2})} Y$ to zero thanks to the infinite order vanishing), and then sending $t_0 \to 0$, we obtain a variant of a standard unique continuation theorem for backwards parabolic equations (see e.g., [7, Theorem 4.1]).

Proof. By the pigeonhole principle, we can select a time

$$\frac{T}{200} \leq T_0 \leq \frac{T}{100} \tag{4.12}$$

such that

$$\int_{|x| \leq r} \left( T^{-1} |u|^2 + |\nabla u|^2 \right) \, dx \leq T^{-1} X. \tag{4.13}$$

We define

$$\alpha = \frac{r^2}{400t_0} \tag{4.14}$$

so from (4.10), (4.11) we have

$$\alpha \geq 10. \tag{4.15}$$

We apply Lemma 4.1 on $[0, T_0] \times \mathbb{R}^3$ with the weight

$$g := -\frac{|x|^2}{4(t + t_1)} - \frac{3}{2} \log(t + t_1) - \alpha \log \frac{t + t_1}{T_0 + t_1} + \alpha \frac{t + t_1}{T_0 + t_1}.$$
(which is a modification of the logarithm of the fundamental solution \( \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t} \) of the heat equation) and \( u \) replaced by \( \psi u \), where \( \psi(x) \) is a smooth cutoff supported on the region \( |x| \leq r \) that equals 1 on \( |x| \leq r/2 \) and obeys the estimates

\[
|\nabla^j \psi(x)| = O(r^{-j})
\]

for \( r/2 \leq |x| \leq r \) and \( j = 0, 1, 2 \). Clearly

\[
D^2 g(\nabla(\psi u), \nabla(\psi u)) = -\frac{1}{2(t + t_1)} |\nabla(\psi u)|^2.
\]

We can calculate

\[
F = \frac{|x|^2}{4(t + t_1)^2} - \frac{3}{2(t + t_1)} - \frac{\alpha}{t + t_1} + \frac{\alpha}{T_0 + t_1} + \frac{3}{2(t + t_1)} + \left| \frac{x}{2(t + t_1)} \right|^2
\]

and hence

\[
LF = \frac{\alpha}{(t + t_1)^2}.
\]

From Lemma 4.1 we thus have

\[
\partial_t \int_{\mathbb{R}^3} \left( |\nabla(\psi u)|^2 - \frac{\alpha}{2(t + t_1)} |\psi u|^2 + \frac{\alpha}{2(T_0 + t_1)} |\psi u|^2 \right) e^g dx
\]

\[
\geq \int_{\mathbb{R}^3} \left( \frac{\alpha}{2(t + t_1)^2} |\psi u|^2 - \frac{1}{t + t_1} |\nabla(\psi u)|^2 - \frac{1}{2} |L(u)|^2 \right) e^g dx.
\]

To exploit this differential inequality we use the method of integrating factors. If we introduce the energy

\[
E(t) := \int_{\mathbb{R}^3} \left( |\nabla(\psi u)|^2 - \frac{\alpha}{2(t + t_1)} |\psi u|^2 + \frac{\alpha}{2(T_0 + t_1)} |\psi u|^2 \right) e^g dx
\]

then we conclude from the product rule that

\[
\partial_t \left( \left( t + t_1 + \frac{(t + t_1)^2}{10T_0} \right) E(t) \right)
\]

\[
\geq \left( 1 + \frac{t + t_1}{5(T_0 + t_1)} \right) E(t) + \left( t + t_1 + \frac{(t + t_1)^2}{10T_0} \right) \int_{\mathbb{R}^3} \left( \frac{\alpha}{2(t + t_1)^2} |\psi u|^2 + \frac{1}{t + t_1} |\nabla(\psi u)|^2 - \frac{1}{2} |L(\psi u)|^2 \right) e^g dx
\]

\[
= \int_{\mathbb{R}^3} \left( \frac{t + t_1}{10(T_0 + t_1)} |\nabla(\psi u)|^2 + \frac{5(T_0 + t_1) - (t + t_1)}{10(T_0 + t_1)^2} |\psi u|^2 - \frac{1}{2} \left( t + t_1 + \frac{(t + t_1)^2}{10(T_0 + t_1)} \right) |L(\psi u)|^2 \right) e^g dx
\]

\[
\geq \int_{\mathbb{R}^3} \left( \frac{t + t_1}{10(T_0 + t_1)} |\nabla(\psi u)|^2 + \frac{\alpha}{10(T_0 + t_1)} |\psi u|^2 - (t + t_1) |L(\psi u)|^2 \right) e^g dx
\]

and hence by the fundamental theorem of calculus

\[
\int_0^{T_0} \int_{\mathbb{R}^3} \left( \frac{t + t_1}{10(T_0 + t_1)} |\nabla(\psi u)|^2 + \frac{\alpha}{10(T_0 + t_1)} |\psi u|^2 \right) e^g dx dt
\]

\[
\leq \int_0^{T_0} \int_{\mathbb{R}^3} |L(\psi u)|^2 e^g dx dt + \left( t + t_1 + \frac{(t + t_1)^2}{10(T_0 + t_1)} \right) E(t)|_{t=0}^{t=T_0}.
\]
Discarding some terms, we conclude that

$$
\int_0^{T_0} \int_{|x| \leq r/2} \left( \frac{t + t_1}{10(T_0 + t_1)} |\nabla u|^2 + \frac{\alpha}{10(T_0 + t_1)} |u|^2 \right) e^s dx dt
\leq \int_0^{T_0} \int_{|x| \leq r} (t + t_1) |L(\psi u)|^2 e^s dx dt
$$

(4.17)

When $|x| \leq r/2$, one has

$$
|L(\psi u)|^2 = |Lu|^2 \leq 2T^{-2} |u|^2 + 2T^{-1} |\nabla u|^2
$$

thanks to (4.4). By (4.12), (4.15) the contribution of this case is less than half of the left-hand side of (4.17). When $r/2 \leq |x| \leq r$, we have from (4.16), (4.10) that

$$
|L(\psi u)|^2 \leq T^{-2} |u|^2 + T^{-1} |\nabla u|^2 + r^{-4} |u|^2 + r^{-2} |\nabla|^2
\leq T^{-2} |u|^2 + T^{-1} |\nabla u|^2.
$$

Finally, $L(\psi u)$ vanishes for $|x| > r$. Putting all this together, we conclude that

$$
\int_0^{T_0} \int_{|x| \leq r/2} \left( \frac{t + t_1}{T_0 + t_1} |\nabla u|^2 + \frac{\alpha}{T_0 + t_1} |u|^2 \right) e^s dx dt
\leq \int_0^{T_0} \int_{r/2 \leq |x| \leq r} (t + t_1)(T^{-2} |u|^2 + T^{-1} |\nabla u|^2) e^s dx dt
$$

$$
+ T_0 \int_{x \in R^3} |\nabla (\psi u)(T_0, x)|^2 e^s dx + \alpha \int_{|x| \leq r} |u(0, x)|^2 e^s dx.
$$

Restricting the left-hand integral to the region $t_0 \leq t \leq 2t_0$ and also bounding $t_1 \leq t_0 \leq T_0 \leq T$ in several places, we conclude that

$$
\int_{t_0}^{2t_0} \int_{|x| \leq r/2} \left( \frac{t_0}{T_0} |\nabla u|^2 + \frac{\alpha}{T_0} |u|^2 \right) e^s dx dt
\leq \int_0^{T_0} \int_{r/2 \leq |x| \leq r} (T^{-1} |u|^2 + |\nabla u|^2) e^s dx dt
$$

$$
+ T \int_{R^3} |\nabla (\psi u)(T_0, x)|^2 e^s dx + \alpha \int_{|x| \leq r} |u(0, x)|^2 e^s dx.
$$

From elementary calculus we have the inequality

$$
\frac{a}{t} - b \log t \leq b \log \frac{b}{ae}
$$
for any $a, b, t > 0$ (the left-hand side attains its maximum when $t = a/b$). When $r/2 \leq |x| \leq r$ and $0 \leq t \leq T_0$, we then have

\[
g \leq -\frac{|x|^2}{4(t + t_1)} - \left(\alpha + \frac{3}{2}\right) \log(t + t_1) + \alpha \log(T_0 + t_1) + \alpha
\]

\[
\leq \left(\alpha + \frac{3}{2}\right) \log \frac{|x|^2}{e|x|^2} + \alpha \log(T_0 + t_1) + \alpha
\]

\[
\leq \left(\alpha + \frac{3}{2}\right) \log \frac{32\alpha}{er^2} + \alpha \log(T_0 + t_1) + \alpha
\]

\[
\leq \alpha \log \frac{32\alpha(T_0 + t_1)}{r^2} + \frac{3}{2} \log \frac{32\alpha}{er^2}
\]

and thus by (4.14)

\[
\int_0^{T_0} \int_{r/2<|x|<r} \left(T^{-1}|u|^2 + |\nabla u|^2\right) e^g dx dt \lesssim t_0^{-3/2} \exp(\alpha \log \frac{32\alpha(T_0 + t_1)}{r^2}) X.
\]

When $|x| \leq r$ and $t = T_0$, then

\[
g \leq -\frac{3}{2} \log t_0 + \alpha
\]

and $\nabla(\psi u)$ is supported on the ball $\{|x| \leq r\}$ and obeys the estimate

\[
|\nabla(\psi u)| \lesssim |\nabla u| + r^{-1}|u| \lesssim T^{-1}|u| + |\nabla u|
\]

thanks to (4.10), and hence by (4.13)

\[
T \int_{\mathbb{R}^3} |\nabla(\psi u)(T_0, x)|^2 e^g dx \lesssim t_0^{-3/2} \exp(\alpha) X.
\]

From (4.10), (4.15) we have $\log \frac{32\alpha(T_0 + t_0)}{r^2} \geq 1$. Thus

\[
\int_0^{t_0} \int_{|x| \leq r/2} \left(\frac{t_0}{T_0} |\nabla u|^2 + \frac{\alpha}{T_0} |u|^2\right) e^g dx dt
\]

\[
\lesssim t_0^{-3/2} \exp(\alpha \log \frac{32\alpha(T_0 + t_1)}{r^2}) X + \alpha \int_{|x| \leq r} \frac{1}{|x|^2} |u(0, x)|^2 e^g dx.
\]

In the region $t_0 \leq t \leq 2t_0$, $|x| \leq r/2$, we have

\[
g \gtrsim -\frac{|x|^2}{4t} - \frac{3}{2} \log(3t) - \alpha \log \frac{3t_0}{T_0 + t_1}
\]

so that

\[
e^g \gtrsim t_0^{-3/2} e^{-|x|^2/4t} \exp \left(-\alpha \log \frac{3t_0}{T_0 + t_1}\right).
\]

Finally, when $t = 0$ and $|x| \leq r$, we have

\[
g \leq -\frac{|x|^2}{4t_1} - \frac{3}{2} \log t_1 - \alpha \log \frac{t_1}{T_0 + t_1} + \alpha
\]

so that

\[
e^g \leq t_1^{-3/2} e^{-|x|^2/4t_1} \exp \left(\alpha \log \frac{e(T_0 + t_1)}{t_1}\right),
\]
We conclude that
\[
\int_{t_0}^{2t_0} \int_{|x| \leq r/2} \left( \frac{t_0}{T_0} |\nabla u|^2 + \frac{\alpha}{T_0} |u|^2 \right) \, dx \, dt \\
\lesssim \exp \left( \alpha \log \frac{96\alpha t_0}{r^2} \right) X + \alpha \exp \left( \alpha \log \left( \frac{3e t_0}{t_1} \right) \right) t_0^{\frac{3}{2}} \int_{|x| \leq r} |u(0, x)|^2 t_1^{-3/2} e^{-\frac{|x|^2}{4t_1}} \, dx.
\]
From (4.14) we have \( \log \frac{96\alpha t_0}{r^2} \leq -1 \), while from (4.15), (4.12), (4.10), (4.14) we have \( \frac{\alpha}{T_0} \geq T^{-1} \) and \( \frac{t_0}{T_0} \geq \frac{t_0}{r^2} \geq \alpha^{-1} \). We conclude that
\[
\int_{t_0}^{2t_0} \int_{|x| \leq r/2} \left( |\nabla u|^2 + T^{-1} |u|^2 \right) \, dx \, dt \\
\lesssim \alpha^2 e^{-\alpha} X + \exp \left( \alpha \log \left( \frac{3e t_0}{t_1} \right) \right) t_0^{\frac{3}{2}} \int_{|x| \leq r} |u(0, x)|^2 t_1^{-3/2} e^{-\frac{|x|^2}{4t_1}} \, dx.
\]
From (4.14) we have \( \alpha = O(r^2/t_0) \) and \( \alpha^2 e^{-\alpha} \lesssim e^{-\frac{1}{9600}} \), and the claim follows. \( \square \)

5. Main estimate

In this section we combine the estimates in Proposition 3.1 with the Carleman inequalities from the previous section to obtain

**Theorem 5.1** (Main estimate). Let \( t_0, T, u, p, A \) obey the hypotheses of Proposition 3.1, and suppose that there exists \( x_0 \in \mathbb{R}^3 \) and \( N_0 > 0 \) such that
\[
|P_{N_0} u(t_0, x_0)| \geq A_1^{-1} N_0
\]
where as before we set \( A_j = A_{4j}^0 \). Then
\[
T N_1^2 \lesssim \exp(\exp(\exp(A_6^{O(1)}))).
\]

**Proof.** After translating in time and space we may normalise \( (t_0, x_0) = (0, 0) \). Let \( T_1 \) be a time scale in the interval
\[
A_1^{-1} N_0^{-2} \leq T_1 \leq A_4^{-1} T.
\]
By Proposition 3.1(vi), there exists
\[
(t_1, x_1) \in [-T_1, -A_3^{O(1)} T_1] \times B(0, A_4^{O(1)} T_1^{1/2})
\]
and
\[
N_1 = A_3^{O(1)} T_1^{-1/2}
\]
such that
\[
P_{N_1} u(t_1, x_1) \geq A_1^{-1} N_1.
\]
From the Biot-Savart law we have
\[
P_{N_1} u(t_1, x_1) = -\Delta^{-1} P_{N_1} \nabla \times \tilde{P}_{N_1} \omega(t_1, x_1),
\]
and hence by (2.2)
\[
P_{N_1} u(t_1, x_1) \lesssim N_1^{-1} \| \tilde{P}_{N_1} \omega(t_1) \|_{L^\infty(B(x_1, A_1/N_1))} + A_1^{-50} N_1^{-1} \| \tilde{P}_{N_1} \omega(t_1) \|_{L^\infty(\mathbb{R}^3)}.
\]
From (3.1), (2.3) one has
\[
\| \tilde{P}_{N_1} \omega(t_1) \|_{L^\infty(\mathbb{R}^3)} \lesssim AN^2
\]
and thus we have
\[ \tilde{P}_{N_1}(t_1, x'_1) \gtrsim A_1^{-1} N_1^2 \]
for some \( x'_1 = x_1 + O(A_1/N_1) = O(A_4^{O(1)} T_1^{1/2}) \). By Proposition 3.1(i), one has
\[ \nabla \tilde{P}_{N_1} \omega = O(A N_1^3); \quad \partial_t \tilde{P}_{N_1} \omega = O(A N_1^3) \]
and thus
\[ \tilde{P}_{N_1} \omega(t, x) \gtrsim A_1^{-1} N_1^2 \]  
for all \( (t, x) \in [t_1, t_1 + A_1^{-2} N_1^{-2}] \times B(x'_1, A_1^{-2} N_1^{-1}) \). By Proposition 3.1(iii), there is an interval
\[ I' \subset [t_1, t_1 + A_1^{-2} N_1^{-2}] \cap [-T_1, -A_3^{O(1)} T_1] \]
with \( |I'| = A_3^{O(1)} T_1 \) such that
\[ u(t, x) = O(A_3^{O(1)} T_1^{-1/2}), \nabla u(t, x) = O(A_3^{O(1)} T_1^{-1}) \]
and
\[ \omega(t, x) = O(A_3^{O(1)} T_1^{-1}), \nabla \omega(t, x) = O(A_3^{O(1)} T_1^{-3/2}) \]  
for all \( t \in I' \). From (5.3), (5.1), (5.2) one has
\[ \int_{B(0, A_4^{O(1)} T_1^{1/2})} |\tilde{P}_{N_1} \omega(t, x)|^2 \, dx \, dt \gtrsim A_3^{O(1)} T_1^{-1/2} \]
for all \( t \in I' \).

Write \( I' = [t', t' - T'] \), and let \( x_* \in \mathbb{R}^3 \) be any point with \( |x_*| \gtrsim A_5 T_1^{-1/2} \). We apply Proposition 4.3 on the slab \([0, T'] \times \mathbb{R}^3 \) with \( r := A_5 |x_*|, t_0 = T'/2, \) and \( t_1 = A_5^4 T' \), and \( u \) replaced by the function
\[ (t, x) \mapsto \omega(t' - t, x_* + x) \]
(so that the hypothesis (4.4) follows from the vorticity equation and (5.5)) to conclude that
\[ Z \leq \exp(-A_5 |x_*|^2 / T') X + (T')^{3/2} \exp(O(A_3^3 |x_*|^2 / T')) Y \]
where
\[ X = \int_{I'} \int_{B(x_*, A_5 |x_*|)} ((T')^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \]
and
\[ Y := (T')^{-3/2} \int_{B(x_*, A_5 |x_*|)} |\omega(t', x)|^2 e^{-A_5^3 |x-x'|^2 / 4 T'} \, dx \]
and
\[ Z := \int_{t'-T'}^{t'-T'/2} \int_{B(x_*, A_5 |x_*|/2)} (T')^{-1} |\omega|^2 e^{-|x-x'|^2 / 4 (t'-t)} \, dx \, dt. \]
From (5.6) we have
\[ Z \gtrsim A_3^{-O(1)} \exp(-|x_*|^2 / 100 T') (T')^{-1/2}. \]
From (5.4) we have
\[ X \lesssim (T')^{-2} A_3^3 |x_*|^3 \lesssim A_3^3 \exp(|x_*|^2/T')(T')^{-1/2} \]
and hence the expression \( \exp(-A_3 |x_*|^2/T')X \) is negligible compared to \( Z \). We conclude that
\[ Y \gtrsim \exp(-O(A_3^3 |x_*|^2/T'))(T')^{-2}. \]
Using (5.4), the contribution to \( Y \) outside of the ball \( B(x_*, |x_*|/2) \) is negligible, thus
\[ \int_{B(x_*, |x_*|/2)} |\omega(t', x)|^2 \, dx \gtrsim \exp(-O(A_3^3 |x_*|^2/T'))(T')^{-1/2} \]
and therefore
\[ \int_{B(0,2R) \backslash B(0,R/2)} |\omega(t', x)|^2 \, dx \gtrsim \exp(-O(A_3^3 R^2/T'))(T')^{-1/2} \]
whenever \( R \geq A_5 T_1^{1/2} \). A similar argument holds with \( t' \) replaced by any time in \( [t' - T'/4, t'] \). We conclude in particular that we have the Gaussian lower bound
\[ \int_{-A_1^1 T_1}^{A_1^1 T_1} \int_{B(0,2R) \backslash B(0,R/2)} |\omega(t, x)|^2 \, dx \, dt \gtrsim \exp(-A_5^1 R^2/T_1)T_1^{1/2} \] (5.7)
whenever \( A_1 N_0^{-2} \leq T_1 \leq A_4^{-1} T_1 \) and \( R \geq A_5 T_1^{1/2} \).

Now let \( T_2 \) be a scale for which
\[ A_4^2 N_0^{-2} \leq T_2 \leq A_4^{-1} T. \] (5.8)
By Proposition 3.1(vi), there exists a scale
\[ A_6 T_2^{1/2} \leq R \leq \exp(A_6^{O(1)}) T_2^{1/2} \] (5.9)
such that on the cylindrical annulus
\[ \Omega := \{(t, x) \in [-T_2, 0] \times \mathbb{R}^3 : R \leq |x| \leq A_6 R\} \]
one has the estimates
\[ \nabla^j u(t, x) = O(A_6^{-2} T_2^{-\frac{j+1}{2}}), \quad \nabla^j \omega(t, x) = O(A_6^{-2} T_2^{-\frac{j+2}{2}}) \] (5.10)
for \( j = 0, 1 \). We apply Proposition 4.2 on the slab \([0, T_2/C_0] \times \mathbb{R}^3\) with \( r_- := 10R \), \( r_+ := A_6 R/10 \), and \( u \) replaced by the function
\[ (t, x) \mapsto \omega(-t, x) \]
(so that the hypothesis (4.4) follows from the vorticity equation and (5.10)) to conclude that
\[ Z' \leq \exp(-A_6^{1/2} R^2/T_2) X' + \exp(\exp(A_6^{O(1)})) Y' \]
where
\[ X' := \int_{-T_2/C_0}^{0} \int_{10 R \leq |x| \leq A_6 R / 10} e^{2\frac{t^2}{T_2}} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \]
and
\[ Y' := \int_{10 R \leq |x| \leq A_6 R / 10} |\omega(0, x)|^2 \, dx \]
and
\[ Z' := \int_{-T_2/4 C_0}^{0} \int_{100 R \leq |x| \leq A_1 R / 20} T_2^{-1} |\omega|^2 \, dx \, dt. \]
From (5.7) we have

$$Z' \geq \exp(-A_5^2 R^2 / T_2) T_2^{-1/2}.$$  

Thus we either have

$$X' \geq \exp(A_1^{1/3} R^2 / T_2) T_2^{-1/2}$$ \hspace{1cm} (5.11)

or

$$Y' \geq \exp(- \exp(A_0^{(1)}) T_2^{-1/2}).$$ \hspace{1cm} (5.12)

Suppose for the moment that (5.11) holds. From the pigeonhole principle, we can then find a scale

$$10R \leq R' \leq A_0 R / 10$$ \hspace{1cm} (5.13)

such that

$$\int_{-T_2/C_0}^0 \int_{R' \leq |x| \leq 2 R'} e^{\frac{20 t^2}{2 \pi}} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \geq \exp(A_1^{1/4} R^2 / T_2) T_2^{-1/2}$$

and thus

$$\int_{-T_2/C_0}^0 \int_{R' \leq |x| \leq 2 R'} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \geq \exp(-10(R')^2 / T_2) T_2^{-1/2}.$$

From (5.10) we see that the contribution to the left-hand side arising from those times $t$ in the interval $[-\exp(-20(R')^2 / T_2), T_2, 0]$ is negligible, thus

$$\int_{-T_2/C_0}^{-\exp(-20(R')^2 / T_2) T_2} \int_{R' \leq |x| \leq 2 R'} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \geq \exp(-10(R')^2 / T_2) T_2^{-1/2}.$$  

Thus by a further application of the pigeonhole principle, one can locate a time scale

$$\exp(-20(R')^2 / T_2) T_2 \leq t_0 \leq T_2 / C_0$$ \hspace{1cm} (5.14)

such that

$$\int_{-2t_0}^{-t_0} \int_{R' \leq |x| \leq 2 R'} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \geq \exp(-10(R')^2 / T_2) T_2^{-1/2}.$$

Covering the annulus $R' \leq |x| \leq 2 R'$ by $O(\exp(O((R')^2 / T_2))$ balls of radius $t_0^{1/2}$, one can then find $x_*$ with $R' \leq |x_*| \leq 2 R'$ such that

$$\int_{-2t_0}^{-t_0} \int_{B(x_*, t_0^{1/2})} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt \geq \exp(-O((R')^2 / T_2)) T_2^{-1/2}.$$ \hspace{1cm} (5.15)

Now we apply Proposition 4.3 on the slab $[0, T_2] \times \mathbb{R}^3$ with $r := |x| / 2$, $t_1 := t_0$, and $u$ replaced by the function

$$(t, x) \mapsto \omega(-t, x_* + x)$$

(so that the hypothesis (4.4) follows from the vorticity equation and (5.5)) to conclude that

$$Z'' \leq \exp(-\frac{|x_*|^2}{1000 t_0^2}) X'' + T_2^{3/2} \exp(O(|x_*|^2 / t_0)) Y''$$ \hspace{1cm} (5.16)

where

$$X'' := \int_{-T_2}^0 \int_{B(x_*, |x_*|/2)} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) \, dx \, dt$$

and

$$Y'' := t_0^{3/2} \int_{B(x_*, |x_*|/2)} |\omega(0, x)|^2 e^{-|x-x_*|^2 / 4 t_0} \, dx.$$
and
\[ Z'' := \int_{-2t_0}^{t_0} \int_{B(x_*, |x_*|/4)} (T^{-1}_2 |\omega|^2 + |\nabla \omega|^2) e^{-|x-x_*|^2/4|t|} \, dx \, dt. \]

From (5.15) one has
\[ Z'' \gtrsim \exp(-O((R')^2/T_2))T_2^{-1/2}. \]

From (5.10) one has
\[ X'' \lesssim T_2^{-2}(R')^3 \lesssim \exp((R')^2/T_2)T_2^{-1/2}. \]

Since \([x_*]^2 \geq \frac{C_0}{1000} \frac{(R')^2}{T_2}\), the first term on the right-hand side of (5.16) can be absorbed by the left-hand side, so we conclude that
\[ Y'' \gtrsim \exp(-O((R')^2/\ell_0))T_2^{-2} \]

and hence
\[ \int_{R'/2 \leq |x| \leq 2R'} |\omega(0, x)|^2 \, dx \gtrsim \exp(-O((R')^2/\ell_0))T_2^{-2}t_0^{-3/2}. \]

Using the bounds (5.13), (5.9), (5.14), we conclude in particular that
\[ \int_{2R \leq |x| \leq A_R R/2} |\omega(0, x)|^2 \, dx \gtrsim \exp(-\exp(A_0^{(1)})T_2^{-1/2}). \]  

(5.17)

Note that this bound is also implied by (5.12). Thus we have unconditionally established (5.17) for any scale \(T_2\) obeying (5.8), and for a suitable scale \(R\) obeying (5.9) and the bounds (5.10).

We now convert this vorticity lower bound (5.17) to a lower bound on the velocity. The annulus \(2R \leq |x| \leq A_R R/2\) has volume \(O(\exp(\exp(A_0^{(1)})T_2^{3/2}))\) by (5.9), hence by the pigeonhole principle there exists a point \(x_*\) in this annulus for which
\[ |\omega(0, x_*)| \gtrsim \exp(-\exp(A_0^{(1)}))T_2^{-1}. \]

Comparing this with (5.10), we see that
\[ |\int_{\mathbb{R}^3} \omega(0, x_* - r y) \varphi(y) \, dy| \gtrsim \exp(-\exp(A_0^{(1)}))T_2^{-1} \]

for some bump function \(\varphi\) supported on \(B(0,1)\), where \(r\) is a radius of the form \(r = \exp(-\exp(A_0^{(1)}))T_2^{-1/2}\). Writing \(\omega = \nabla \times u\) and integrating by parts, we conclude that
\[ |\int_{\mathbb{R}^3} u(0, x_* - r y) \nabla \times \varphi(y) \, dy| \gtrsim \exp(-\exp(A_0^{(1)}))T_2^{-1/2} \]

and hence by Hölder’s inequality
\[ \int_{\mathbb{R}^3} |u(0, x_* - r y)|^3 \, dy \gtrsim \exp(-\exp(A_0^{(1)}))T_2^{-3/2} \]

or equivalently
\[ \int_{B(x_*, r)} |u(0, x)|^3 \, dx \gtrsim \exp(-\exp(A_0^{(1)})). \]

We conclude that for any scale \(T_2\) obeying (5.8), we have
\[ \int_{T_2^{1/2} \leq |x| \leq \exp(\exp(A_7))T_2^{1/2}} |u(0, x)|^3 \, dx \gtrsim \exp(-\exp(A_0^{(1)})). \]
Summing over a set of such scales $T_2$ increasing geometrically at ratio $\exp(\exp(A_{\ell}))$, we conclude that if $T \geq A_{\ell}^2 N_0^{-2}$, then
\[
\int_{\mathbb{R}^3} |u(0, x)|^3 \, dx \geq \exp(- \exp(A_{\ell}^{O(1)})) \log(T N_0^2).
\]
Comparing this with (3.1), one obtains the claim. \hfill \Box

6. Applications

Using the main estimate, we now prove the theorems claimed in the introduction.

We begin with Theorem 1.2. By increasing $A$ as necessary we may assume that $A \geq C_0$, so that Theorem 5.1 applies. By rescaling it suffices to establish the claim when $t = 1$, so that $T \geq 1$. Applying Theorem 5.1 in the contrapositive, we see that
\[
\|P_N u\|_{L^2_t L^p_x([1/2,1] \times \mathbb{R}^3)} \leq A_1^{-1} N \tag{6.1}
\]
whenever $N \geq N_*$, where
\[
N_* := \exp(\exp(\exp(A C_0^5))).
\]
We now insert this bound into the energy method. As before, we split $u = u^{\text{lin}} + u^{\text{nlin}}$ on $[1/2,1] \times \mathbb{R}^3$, where
\[
\begin{aligned}
u^{\text{lin}}(t) &:= e^{t \Delta} u(0) \\
u^{\text{nlin}} &:= u - u^{\text{lin}}
\end{aligned}
\]
and $u^{\text{nlin}} := u - u^{\text{lin}}$, and similarly split $\omega = \omega^{\text{lin}} + \omega^{\text{nlin}}$. From (2.5), (3.1) we have
\[
\|\nabla^j u^{\text{lin}}\|_{L^p_t L^q_x([1/2,1] \times \mathbb{R}^3)} \lesssim_j A \tag{6.2}
\]
for all $j \geq 0$ and $3 \leq p \leq \infty$. We introduce the nonlinear enstrophy
\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega^{\text{nlin}}(t, x)|^2 \, dx
\]
for $t \in [1/2,1]$, and compute the time derivative $\partial_t E(t)$. From the vorticity equation (3.40) and integration by parts we have
\[
\partial_t E(t) = -Y_1(t) + Y_2(t) + Y_3(t) + Y_4(t) + Y_5(t) + Y_6(t) \tag{6.3}
\]
where
\[
\begin{aligned}
Y_1(t) &= \int_{\mathbb{R}^3} |\nabla \omega^{\text{nlin}}(t, x)|^2 \, dx \\
Y_2(t) &= -\int_{\mathbb{R}^3} \omega^{\text{nlin}} \cdot (u \cdot \nabla) \omega^{\text{lin}} \, dx \\
Y_3(t) &= \int_{\mathbb{R}^3} \omega^{\text{nlin}} \cdot (\omega^{\text{nlin}} \cdot \nabla) u^{\text{nlin}} \, dx \\
Y_4(t) &= \int_{\mathbb{R}^3} \omega^{\text{nlin}} \cdot (\omega^{\text{nlin}} \cdot \nabla) u^{\text{lin}} \, dx \\
Y_5(t) &= \int_{\mathbb{R}^3} \omega^{\text{nlin}} \cdot (\omega^{\text{lin}} \cdot \nabla) u^{\text{nlin}} \, dx \\
Y_6(t) &= \int_{\mathbb{R}^3} \omega^{\text{nlin}} \cdot (\omega^{\text{lin}} \cdot \nabla) u^{\text{lin}} \, dx.
\end{aligned}
\]
From Hölder, (6.2), (3.1) we have
\[
Y_2(t), Y_6(t) \lesssim A^2 E(t)^{1/2} \lesssim A^4 + E(t)
\]
and similarly \( Y_4(t), Y_3(t) \lesssim AE(t) \),

using Plancherel’s theorem to control
\[
\| \nabla u^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)} \lesssim \| \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)}.
\] (6.4)

For \( Y_3(t) \) we apply a Littlewood-Paley decomposition to all three factors to bound
\[
Y_3(t) \lesssim \sum_{N_1,N_2,N_3} \int_{\mathbb{R}^3} P_{N_1} \omega^{\text{lin}} \cdot (P_{N_2} \omega^{\text{lin}} \cdot \nabla) P_{N_3} u^{\text{lin}} \, dx
\]
where \( N_1, N_2, N_3 \) range over powers of two. The integral vanishes unless two of the \( N_1, N_2, N_3 \) are comparable to each other, and the third is less than or comparable to the other two. Controlling the two highest frequency terms in \( L^2_x \) and the lower one in \( L^\infty_x \), and using the Littlewood-Paley localised version of (6.4), we conclude that
\[
Y_3(t) \lesssim \sum_{N_1,N_2,N_3} \| P_{N_1} \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)} \| P_{N_2} \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)} \| P_{N_3} \omega^{\text{lin}} \|_{L^\infty_x(\mathbb{R}^3)}.
\]
From (3.1), (2.3), the quantity \( \| P_{N_1} \omega^{\text{lin}} \|_{L^\infty_x(\mathbb{R}^3)} \) is bounded by \( O(A N_3^2) \); for \( N_3 \geq N_* \), we have the superior bound \( O(A_1 N_*^2) \). We thus see that
\[
\sum_{N_3 \leq N_2} \| P_{N_3} \omega^{\text{lin}} \|_{L^\infty_x(\mathbb{R}^3)} \lesssim A_1^{-1} N_*^2 + AN_*^2
\]
and thus by Cauchy-Schwarz
\[
Y_3(t) \lesssim \sum_{N_1} \| P_{N_1} \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)}^2 (A_1^{-1} N_1^2 + AN_*^2).
\]
On the other hand, from Plancherel’s theorem we have
\[
Y_1(t) \sim \sum_{N_1} \| P_{N_1} \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)}^2 N_1^2
\]
and
\[
E(t) \sim \sum_{N_1} \| P_{N_1} \omega^{\text{lin}} \|_{L^2_x(\mathbb{R}^3)}^2
\]
and hence
\[
Y_3(t) \lesssim A_1^{-1} Y_1(t) + AN_*^2 E(t).
\]
Putting all this together, we conclude that
\[
\partial_t E(t) + Y_1(t) \lesssim AN_*^2 E(t) + A^4.
\]
In particular, from Gronwall’s inequality we have
\[
E(t_2) \lesssim E(t_1) + A^4
\]
whenever \( 1/2 \leq t_1 \leq t_2 \leq 1 \) is such that \( |t_2 - t_1| \leq A^{-1} N_*^{-2} \). On the other hand, from a (slightly rescaled) version of (3.13) we have
\[
\int_{1/2}^1 E(t) \, dt \lesssim A^4
\]
and hence on any time interval in \([1/2,1]\) of length \( A^{-1} N_*^{-2} \) there is at least one time \( t \) with \( E(t) \lesssim A^5 N_*^2 \). We conclude that
\[
E(t) \lesssim A^5 N_*^2 \lesssim N_*^{O(1)},
\]
for all $t \in [3/4, 1]$, which then also implies

$$
\int_{3/4}^1 Y_1(t) \leq N_*^{O(1)}.
$$

Iterating this as in the proof of Proposition 3.1(iii) (or Proposition 3.1(vi)), we now have the estimates

$$
|u(t, x)|, |
abla u(t, x)|, |\omega(t, x)|, |\nabla \omega(t, x)| \leq N_*^{O(1)}
$$
on $[7/8, 1] \times \mathbb{R}^3$. This gives Theorem 1.2.

Now we prove Theorem 1.1. We may rescale $T_* = 1$. Let $c > 0$ be a sufficiently small constant, and suppose for contradiction that

$$
\limsup_{t \to 1^-} \frac{\|u(t)\|_{L^3_x(\mathbb{R}^3)}}{(\log \log \log \frac{1}{1-t})^c} < +\infty,
$$

thus we have

$$
\|u(t)\|_{L^3_x(\mathbb{R}^3)} \leq M(\log \log \log (1000 + \frac{1}{1-t}))^c
$$

(6.5)

for all $0 \leq t < 1$ and some constant $M$. Applying Theorem 1.2, we obtain (for $c$ small enough) the bounds

$$
\|u(t)\|_{L^\infty_t(L^3_x(\mathbb{R}^3)), \nabla u(t)\|_{L^\infty_t(L^3_x(\mathbb{R}^3)), \|\omega(t)\|_{L^\infty_t(L^3_x(\mathbb{R}^3))}, \nabla \omega(t)\|_{L^\infty_t(L^3_x(\mathbb{R}^3))} \leq M (1-t)^{-1/10}
$$

(6.6)
say for all $1/2 \leq t < 1$. In particular, $u$ is bounded in $L^2_t L^\infty_x$, contradicting the classical Prodi-Serrin-Ladyshenskaya blowup criterion [16], [19], [12]. The claim follows.

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