LOCAL AND GLOBAL WELL-POSEDNESS IN THE ENERGY SPACE FOR THE DISSIPATIVE ZAKHAROV-KUZNETSOV EQUATION IN 3D

Mohamad Darwich
Mathematics Department
Lebanese University, Hadat-Lebanon

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Abstract. In this paper, we consider the Zakharov-Kuznetsov equation in 3D, with a dissipative term of order $0 < \alpha \leq 2$ in the $x$ direction. We prove that the problem is locally well-posed in $H^s(\mathbb{R}^3)$, for $s > 1 - \frac{\alpha}{2}$, and by an a priori energy estimate, we prove that the problem is globally well-posed in $H^1(\mathbb{R}^3)$.

1. Introduction. The Zakharov-Kuznetsov (Z-K) equation is given by

$$\partial_t u + \partial_x \Delta u + uu_x = 0,$$

where $u$ is a real-valued function of $(x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R}^+$ and $\Delta$ is the laplacian operator in $\mathbb{R}^3$. This equation models the propagation of ionic-acoustic waves in magnetized plasma and was introduced by Zakharov and Kuznetsov (cf [16]).

The (Z-K) equation has the following conservation laws:

The $L^2$-norm:

$$M(u(t)) = \int_{\mathbb{R}^3} u^2(x, y, z, t) dx dy dz = \int_{\mathbb{R}^3} u^2(x, y, z, 0) dx dy dz = M(u(0)) \quad (2)$$

and the energy:

$$E(u(t)) = \frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2(x, y, z, t) dx dy dz - \frac{1}{3} \int_{\mathbb{R}^3} u^3(x, y, z, t) dx dy dz \right) = E(u(0)). \quad (3)$$

Therefore $L^2$ and $H^1$ are two natural spaces to study the well-posedness for the Z-K equation.

In 2D, Faminskii [6] proved the global well-posedness of the Cauchy problem for the (Z-K) equation in $H^s(\mathbb{R}^2)$, for $s \geq 1$, and the result was improved later by Linares and Pastor [17] who proved well-posedness in $H^s(\mathbb{R}^2)$, for $s > \frac{3}{4}$, where this result was improved also by Molinet and Pilot in [21] and Grunrock and Herr [10] to $s > \frac{1}{2}$.

In the 3D case, the well-posedness problem was treated by many authors: Ribaud and vento [24], Linares and Saut [18] and Molinet and Pilot [21], where the last authors was proved that the problem is globally well-posed in $H^s(\mathbb{R}^3)$, for $s > 1$. Note that the global well-posedness in the energy space $H^1(\mathbb{R}^3)$ is still an open problem.

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In this paper, we consider a dissipative version of the (Z-K) equation of the form
\[\partial_t u + \partial_x \Delta u + D_x^\alpha u + uu_x = 0,\] (4)
where \(\alpha \in (0, 2]\). The term \(D_x^\alpha\) represents the dissipation term, where \(D_x^\alpha\) is the Lévy operator defined through its Fourier transform by \(\mathcal{F}(D_x^\alpha u)(\xi) = |\xi|^\alpha \mathcal{F}u(\xi)\).

In a recent work, Hirayama [11] have studied the (Z-K) burgers equation (i.e equation (4) with \(\alpha = 2\)) in 2D, and he proved that the problem is locally and globally well-posed in \(H^{s,0}(\mathbb{R}^2)\) with \(s > -1/2\) and up to our knowledge there is no other results for this equation.

The dissipative term will help to establish the global existence in the energy space \(H^1(\mathbb{R}^3)\) and the quantities (2) and (3) are not longer conserved, more precisely we have:
\[\frac{d}{dt} \|u(t)\|^2_{L^2(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} (D_x^2 u)(x,y,z,t) dxdydz,\] (5)

and
\[\frac{d}{dt} E(u(t)) = -\frac{1}{2} \int_{\mathbb{R}^3} |D_x^2 \nabla u|^2 dxdydz + \frac{1}{2} \int_{\mathbb{R}^3} (D_x^\alpha u)^2 dxdydz.\] (6)

We will establish the well-posedness of the problem (4) in \(H^s(\mathbb{R}^3)\), \(s > 1^-\) and the paper is organized as follows:
- In Section 2, we give some notations and definitions, and we derive estimates in Bourgain spaces on the linear operators \(W\) and \(L\). The process is quite general and can be adapted to other dissipative dispersive semigroups.
- In Section 3, we prove a nonlinear estimate which enables us to obtain the Theorem 1, and section 4 is devoted to the proof of Theorem 2, by establishing an a priori energy estimate.

We state now our main results:

**Theorem 1.** Let \(\alpha \in [0, 2]\), then \(\forall s > 1 - \frac{\alpha}{2}\) and \(u_0 \in H^s(\mathbb{R}^3)\) there exist a positive \(T = T(\|\varphi\|_{H^s})\) and a unique solution \(u\) to (4) in
\[Y_T = C([0,T], H^s) \cap X_T^{s,\frac{1}{2}+1}.\] (7)
Moreover, the map \(\varphi \mapsto u\) is continuous from \(H^s\) to \(Y_T\).

**Theorem 2.** Let \(\alpha > 0\) and \(s = 1\), then the Cauchy problem of (4) is globally well-posed in \(H^1(\mathbb{R}^3)\).

2. **Notations, definitions and linear estimate.** For \(f \in \mathcal{S}'\) we denote by \(\hat{f}\) or \(\mathcal{F}(f)\) the Fourier transform of \(f\) i.e.
\[\hat{f}(\theta) = \int_{\mathbb{R}^n} e^{-i(z,\theta)} f(z) dz,\]

For a Banach space \(X\), we denote by \(\| \cdot \|_X\) the norm in \(X\). We will use the Sobolev spaces \(H^s(\mathbb{R}^3)\) and the homogeneous Sobolev spaces \(\dot{H}^s(\mathbb{R}^3)\) equipped with the norms
\[\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (\zeta)^{2s} |\hat{u}(\zeta)|^2 d\zeta, \quad \|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} (1 + |\zeta|^2)^{s} |\hat{u}(\zeta)|^2 d\zeta,\]
where \(\zeta = (\xi, \eta, \mu), |\zeta| = (|\xi|^2 + |\eta|^2 + |\mu|^2)^{\frac{1}{2}}\) and \(\langle \zeta \rangle = (1 + |\zeta|^2)^{\frac{1}{2}}\).

Next we consider the corresponding space-time Sobolev spaces \(H^{s,b}\) equipped with
Definition 1. The space \( \mathcal{N} = \{ \varphi \} \) as the weak closure of the test functions that are uniformly bounded by the norm next \( \varphi \) of the equation. We will rather work in its Besov version \( X \) the norm defined above, which take advantage of the mixed dispersive-dissipative part of the equation. In [22], the authors performed the iteration process in the space \( X \), which it endowed with the norm

\[
\| u \|_X = \| (i(\tau - P(\zeta)) + \xi^2 + \xi \mu) \hat{u}(\tau, \zeta) \|_{L^2(\mathbb{R}^4)},
\]

in [22], the authors performed the iteration process in the space \( X \) equipped with the norm defined above, which take advantage of the mixed dispersive-dissipative part of the equation. We will rather work in its Besov version \( X \) (with \( q = 1 \)) defined as the weak closure of the test functions that are uniformly bounded by the norm

\[
\| u \|_{X^{\frac{1}{2}, q}} = \left( \sum_{N,M} \left( \sum_{L} \langle L + M \rangle^\frac{3}{2} \langle N \rangle^{\frac{3}{2}} \| P_{N,M}Q_{L}u \|_{L^2_x(\mathbb{R}^4)}^q \right)^\frac{1}{q} \right)^\frac{1}{q}.
\]

and to do this we define the decomposition of Littlewood-Paley:

Let \( \eta \in C_0(\mathbb{R}) \) be such that \( \eta \geq 0 \), supp \( \eta \subset [-2, 2] \), \( \eta = 1 \) on \([-1, 1] \). We define next \( \varphi(\xi) = \eta(\xi) - \eta(2\xi) \).

Any summations over capitalized variables such as \( N, L, M \) are presumed to be dyadic, i.e., these variables range over numbers of the form \( N = 2^j, j \in \mathbb{Z}, M = 2^k, k \in \mathbb{Z} \), and \( L = \{0\} \cup \{2^l\} \), \( l \in \mathbb{N} \) (see [5], [11] and [21]). We set \( \varphi(\xi) := \eta(\xi), \varphi_N(\xi) = \varphi(2^N \xi) \) and define the projections: \( F(P_{N,M}Q_{L}u)(\xi, \eta, \mu) = \varphi_N((\xi, \eta, \mu) F(u)(\xi, \eta, \mu), F(P_{N,M}Q_{L}u)(\xi, \eta, \mu) = \varphi_N((\xi, \eta, \mu) F(u)(\xi, \eta, \mu), F(Q_{L}u)(\tau, \xi, \eta, \mu) = \varphi_L(\tau - P(\xi, \eta, \mu)) F(u)(\tau, \xi, \eta, \mu).

Roughly speaking, the operators \( P_N, R_M \) and \( Q_L \) localize respectively in the annulus \( \{(|\xi, \eta, \mu|) \sim N \}, \{ |\xi| \sim M \} \) and \( \{ |\tau - P(\zeta)| \sim L \} \). We denote \( P_Nu \) by \( u_N \), \( Q_Lu \) by \( u_L \), \( R_Mu \) by \( u_M \) and \( P_{N,M}Q_{L}u \) by \( u_{N,M,L} \). Now we will define our space of resolution \( X^{\frac{1}{2}, 1} \) endowed with the Fourier restriction norm defined as follows:

**Definition 1.** 1. We define the function space \( X^{s,b,1} \) as the completion of the Schwartz class \( S(\mathbb{R}^4) \) equipped with norm

\[
\| u \|_{X^{s,b,1}} = \left( \sum_N \sum_M \sum_L \langle L + M \rangle^b \langle N \rangle^s \| P_{N,M}Q_{L}u \|_{L^2_x,y,z,t}^q \right)^\frac{1}{q}.
\]

2. For \( T \geq 0 \), we consider the localized spaces \( X^{s,b,1}_T \) endowed with the norm

\[
\| u \|_{X^{s,b,1}_T} = \inf_{w \in X^{s,b,1}} \{ \| w \|_{X^{s,b,1}} : w(t) = u(t) \text{ on } [0, T] \}.
\]

**Remark 1.** The space \( X^{s,b,1}_T \) is embedded in \( C([0,T]; H^s(\mathbb{R}^3)) \).
We will also use the space-time Lebesgue space $L^{q,r}$ endowed with the norm
\[ \|u\|_{L^{q,r}} = \left\| |x|^s u(x,t) \right\|_{L^q_t L^r_x}. \]

We denote by $W(\cdot)$ the semigroup associated with the free evolution of the equation (4) i.e.
\[ \forall t \geq 0, \ W(t)\varphi = \exp \left[ -|\xi|^\alpha - i\tau \right] \varphi, \ \varphi \in \mathcal{S}', \]
and we extend $W(\cdot)$ to a linear operator defined on the whole real axis by setting
\[ \forall t \in \mathbb{R}, \ W(t)\varphi = \exp \left[ -|\xi|^\alpha |t| + i\tau \right] \hat{\varphi}, \ \varphi \in \mathcal{S}'. \]

By the Duhamel integral formulation, the equation (4) can be written as
\[ u(t) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t')\partial_x(u^2(t'))dt', \quad t \geq 0. \tag{10} \]
and to prove the local well posedness results, we shall apply a fixed point argument in $X^{s,\frac{1}{2}-1}$ to the extension of (10), which is defined on whole the real axis by:
\[ u(t) = \psi(t)[W(t)\phi - L(\partial_x(\psi^2u^2))(x,t)], \tag{11} \]
where $t \in \mathbb{R}$ and $L$ is the operator defined as
\[ L(f)(x,t) = W(t) \int e^{ix\xi t} \frac{e^{-t|\xi|^\alpha} - e^{-|t\xi|^\alpha}}{i\tau + \xi^\alpha} F(W(-t)f)(\xi,\tau)d\xi d\tau. \tag{12} \]
where $\psi$ is a time cut-off function satisfying
\[ \psi \in C_0^\infty(\mathbb{R}), \ \text{supp} \ \psi \subset [-1,1], \ \psi = 1 \text{ on } \left[ -\frac{1}{2}, \frac{1}{2} \right], \]
and $\psi_T(\cdot) = \psi(\cdot/T).$

Now, following Molinet-Ribaud [22], Baoxiang [27], Darwich [5] and exactly in the same way as the proof of the linear estimate established (2D case) in Section 2 in [11], we give the estimate of the linear term in the space $X^{s,\frac{1}{2}+\gamma},$ more precisely we have the following lemma:

**Lemma 1.** Let $s \in \mathbb{R}$, then:
\[ a) \text{ For all } \varphi \in H^s, \text{ we have } \]
\[ \|\psi(t)W(t)\varphi\|_{X^{s,\frac{1}{2}+\gamma}} \leq C\|\varphi\|_{H^s}. \tag{13} \]
\[ b) \text{ For all } f \in S(\mathbb{R}^3), \text{ we have } \]
\[ \|\psi(t)L(f)\|_{X^{s,\frac{1}{2}+\gamma}} \leq C\|f\|_{X^{s,-\frac{1}{2}+\gamma,1}}. \tag{14} \]
where $0 < \gamma < \frac{1}{2}.$

### 3. Bilinear estimate and the local existence result.

The aim of this section is to prove Theorem 1 and as it is standard for this type of problem, with the linear estimates in hand, we obtain the local existence result, once we estimate the nonlinear term $\partial_x(u^2)$ in $X^{s,\frac{1}{2}+\gamma,1}.$ More precisely we have the following proposition:

**Proposition 1.** For all $u, v \in X^{s,1/2,1}(\mathbb{R}^4), \ s > 1 - \frac{3}{2}$ with compact support in time included in the subset $\{(t, x, y, z) : t \in [-T, T]\},$ there exists $\beta > 0$ such that the following bilinear estimate holds
\[ \|\partial_x(uv)\|_{X^{s,-1/2+\gamma,1}} \leq CT^\beta \|u\|_{X^{s,1/2,1}}\|v\|_{X^{s,1/2,1}}. \tag{15} \]
Remark 2. We will mainly use the following version of (15), which is a direct consequence of Proposition 1, together with the triangle inequality
\[ \forall s > s' > 1 - \frac{\alpha}{2}, \quad \langle \xi \rangle^s \leq \langle \xi \rangle^{s'} \langle \xi_1 \rangle^{s-s'} + \langle \xi \rangle^{s'} \langle \xi - \xi_1 \rangle^{s-s'}, \]

\[ \|\partial_x (uv)\|_{X^{s,1/2,1}} \leq CT^\mu(s') \left( \|u\|_{X^{s',1/2,1}} \|v\|_{X^{s,1/2,1}} + \|u\|_{X^{s,1/2,1}} \|v\|_{X^{s',1/2,1}} \right). \]  
(16)

with \( \mu(\beta) > 0 \).

To prove the bilinear estimate, we will need the following lemma, which can be obtained in the same way as for the (Z-K) equation see [21] and [18]:

Lemma 2. Let \( \psi \in C^\infty_c(\mathbb{R}) \). Let \( u \in X^{0,\frac{1}{2},1} \) it holds
\[ \|\psi u\|_{L^4} \lesssim \|u\|_{X^{0,\frac{1}{2},1}} \quad (17) \]
and for any couple \((u, v)\) and any couple \((N_1, N_2)\) of dyadic numbers such that \( 4N_2 \) it holds:
\[ \|\psi P_{N_1} u P_{N_2} v\|_{L^2} \lesssim \frac{N_2}{N_1} \|P_{N_1} u\|_{X^{0,\frac{1}{2},1}} \|P_{N_2} v\|_{X^{0,\frac{1}{2},1}}. \]  
(18)

See [21] for the proof.

Remark 3. Following [8] it is easy to check that for any \( u \in L^2(\mathbb{R}^4) \) supported in \([-T,T]\) and any \( \delta > 0 \), there exists \( \beta = \beta(\delta) \) such that:
\[ \|\tilde{u}\|_{L^2(\mathbb{R}^4)} \lesssim CT^\beta \|u\|_{L^2}. \]  
(19)

Now we are ready to prove our crucial nonlinear estimate (Proposition 1):

Proof. We proceed by duality. Let \( w \in X^{0,0,\infty} \), we will estimate the following term
\[ J = \sum_{N, N_1, N_2} \sum_{M_1, M_2} \sum_{L_1, L_2} (L + M^\alpha)^{-\frac{1}{2} + \gamma} \langle N \rangle^s N | \int (\tilde{u}_{N_1, M_1, L_1} \ast \tilde{u}_{N_2, M_2, L_2}) \tilde{w}_{N, M, L} d\xi d\eta d\tau | \]

and let \( I = (L + M^\alpha)^{-\frac{1}{2} + \gamma} \langle N \rangle^s N | \int (\tilde{u}_{N_1, M_1, L_1} \ast \tilde{u}_{N_2, M_2, L_2}) \tilde{w}_{N, M, L} d\xi d\eta d\tau |. \)

We must separate the contributions \( N_1 \sim N_2 \) and \( N_1 \sim N_2. \)

Case 1: \( N \lesssim N_1 \sim N_2 \)

By Cauchy-Schwartz we have that:
\[ I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \|u_{N_1, M_1, L_1}\|_{L^4} \|v_{N_2, M_2, L_2}\|_{L^4} \|w_{N, M, L}\|_{X^{0,0,\infty}} \]

Now, using estimation (17), we obtain that:
\[ I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \|u_{N_1, M_1, L_1}\|_{X^{0,\frac{1}{2},1}} \|v_{N_2, M_2, L_2}\|_{X^{0,\frac{1}{2},1}} \|w_{N, M, L}\|_{X^{0,0,\infty}} \]

and then
\[ I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \]
\[ \times \langle N_1 \rangle^s \|u_{N_1, M_1, L_1}\|_{X^{0,\frac{1}{2},1}} \langle N_2 \rangle^s \|v_{N_2, M_2, L_2}\|_{X^{0,\frac{1}{2},1}} \|w_{N, M, L}\|_{X^{0,0,\infty}} \]  
(20)

Now we need to separate two subcases:
Case 1.1: $M \leq 1$
Remarks that $(L + M^\alpha)^{\frac{1}{2} - \gamma} \geq L^{\frac{1}{2} - \gamma}$, $(N_i)^{-s} \lesssim \langle N \rangle^{-s}$, $i = 1, 2$, this gives that:
\[
I \leq \langle N \rangle^{-s} M(L)^{-\frac{1}{2} + \gamma + \theta} \\
\quad \times (N_1)^s \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} (N_2)^s \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \langle L \rangle^{-\theta} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
where $\theta > 0$ small enough such that $-\frac{1}{2} + \gamma + \theta < 0$.
Now by summing in $L_1$, $N_1$, $M_1$, $L_2$, $N_2$, $M_2$, $L$, $M \leq 1$ and $N$ and using Remark 3 we get:
\[
J \lesssim T^\theta \|u\|_X \|v\|_X \|w_{N,M,L}\|_{X^{0,0,\infty}}.
\]

Case 1.2: $M \geq 1$
Now using that $(L + M^\alpha)^{\frac{1}{2} - \gamma} \geq L^{\alpha(\frac{1}{2} - \gamma)} M^{\alpha(1 - \gamma)}$ for $\epsilon \in [0, 1]$ and that $M \leq N$, inequality (20) becomes:
\[
I \leq \langle N \rangle^{-s} M^{-s + \epsilon} M^{\alpha(\frac{1}{2} - \gamma)} M^{\alpha(-\frac{1}{2} + \gamma + \epsilon)} \\
\quad \times (N_1)^s \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} (N_2)^s \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \langle L \rangle^{-\frac{1}{2} - \gamma} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
Now if $s > 1 - (\frac{1}{2} - \gamma)\alpha + \epsilon(1 - \alpha\gamma + \frac{\alpha}{2})$ we can sum on $M > 1$ and we get that:
\[
J \lesssim T^\theta \|u\|_X \|v\|_X \|w_{N,M,L}\|_{X^{0,0,\infty}}.
\]

Case 2: $N_1 \approx N_2$
We assume without loss of generality that $N_1 \geq 4N_2$ to get
\[
I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} \|v_{N_2,M_2,L_2}\|_{L^2} \|w_{N,M,L}\|_{L^2}
\leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} N_2^{-1} \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
where we have used (18) in the last estimate.
Now as in case 1, $I$ is controled by:
\[
I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma + 1} N_2^{-1} \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \\
\quad \times (N_1)^s \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} (N_2)^s \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
\[
(24)
\]
We need to separate in two subcases:

Case 2.1: $N_1 \geq 1$
Then we have that:
\[
I \leq \langle N \rangle^s M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \langle N_1 \rangle^{-s-1} \langle N_2 \rangle^{1-s} \\
\quad \times (N_1)^s \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} (N_2)^s \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
\[
(25)
\]
If $s \geq 1$ then $\langle N_2 \rangle^{1-s} \leq 1$ and using that $N \lesssim N_1$ to get
\[
I \leq \langle N \rangle^{-1} M \langle L + M^\alpha \rangle^{-\frac{1}{2} + \gamma} \\
\quad \times (N_1)^s \|u_{N_1,M,L_1}\|_{X_0^{0,\frac{1}{2}}} (N_2)^s \|v_{N_2,M_2,L_2}\|_{X_0^{0,\frac{1}{2}}} \|w_{N,M,L}\|_{X^{0,0,\infty}}
\]
\[
(26)
\]
Now as in case 1 and by separating the two case $M \leq 1$ and $M \geq 1$, we obtain that:
\[
J \lesssim T^\theta \|u\|_X \|v\|_X \|w_{N,M,L}\|_{X^{0,0,\infty}}.
\]

If $s < 1$, then $\langle N_2 \rangle^{1-s} \leq \langle N_1 \rangle^{1-s}$ and this gives that:
\[
I \leq \langle N \rangle^{-s} M \langle L + M^\alpha \rangle^{-\frac{5}{2} + \gamma} \langle N_1 \rangle^{-s} \phi \langle N_2 \rangle^{s} \parallel u_{N_1,L_1} \parallel_{X^0 \frac{1}{2},1} \langle N_2 \rangle^{s} \parallel u_{N_2,L_2} \parallel_{X^0 \frac{1}{2},1} \parallel |w|_{N,M,L} \parallel_{X^{0,0,\infty}}
\]
and as above we get:
\[
J \lesssim T^\beta \parallel u \parallel_{X^0 \frac{1}{2},1} \parallel v \parallel_{X^0 \frac{1}{2},1} \parallel |w|_{N,M,L} \parallel_{X^{0,0,\infty}}.
\]

**Case 2.1**: $\langle N \rangle < 1$ Here then necessary $M \leq 1$ and $N \leq 1$, and using that $M \leq N$, inequality (23) becomes:
\[
I \leq \langle N \rangle^{-s} M \langle L \rangle^{-\frac{5}{2} + \gamma + \theta} \times \langle N_1 \rangle^{s} \phi \langle N_2 \rangle^{s} \parallel u_{N_1,L_1} \parallel_{X^0 \frac{1}{2},1} \langle N_2 \rangle^{s} \parallel u_{N_2,L_2} \parallel_{X^0 \frac{1}{2},1} \parallel |w|_{N,M,L} \parallel_{X^{0,0,\infty}}
\]
where $\theta > 0$ small enough such that $-\frac{1}{2} + \gamma + \theta < 0$. Now by summing and by Remark 3, we obtain the estimate. 

3.1. Local existence result. Let $\Psi(u) := \psi_t W(t)(\varphi) + \int_0^t W(t-t') (\partial_x [\psi_2(t)]) u^2(t') dt'$, to obtain the well-posedness of (4), we prove that $\Psi$ is contraction map on closed subset of $X^{s,1/2,1}$. Lemma 1 and Proposition 1 yield
\[
\parallel \Psi(u) \parallel_{X^{s,1/2,1}} \leq C \parallel \varphi \parallel_{H^r} + C T^\beta \parallel u \parallel_{X^{s,1/2,1}^2}.
\] (27)

Next, since $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u - v)(u + v)]$, we get
\[
\parallel \Psi(u) - \Psi(v) \parallel_{X^{s,1/2,1}} \leq C T^\beta \parallel u - v \parallel_{X^{s,1/2,1}} \parallel u + v \parallel_{X^{s,1/2,1}}.
\] (28)

Now if we take $T < T(4C^2 \parallel \varphi \parallel_{H^r}^{-1/\beta}$, we deduce that $\Psi$ is strictly contractive in the ball of radius $2C \parallel \varphi \parallel_{H^r}$ in $X^{s,1/2,1}$. This proves the existence of a unique solution $u_1$ to (11) in $X^{s,1/2,1}$, with $T = T(\parallel \varphi \parallel_{H^r})$.

The above contraction argument gives the uniqueness of the solution to the truncated integral equation (11). The uniqueness of the solution to the integral equation (10) is obtained by the same argument as in section 4.2 of [22], then we omit it.

Now with remark (2) in hand, we obtain that if $s > s' > 1 - \frac{\alpha}{2}$ and $u_0 \in H^{s'}(\mathbb{R}^3)$, then there exists $T = T(\parallel u_0 \parallel_{H^{s'}})$ and $u \in X^{s,1/2,1}$ solution to (11).

4. Global existence result. We will prove in this section the global existence result in $H^1(\mathbb{R}^3)$ (Theorem 2). Now, for $\varphi \in H^1(\mathbb{R}^3)$, the local solution $u$ of (4), can be extended on a maximal existence interval $[0,T_*]$ such that
\[
\text{if } T_* < \infty, \text{ then } \limsup_{t \nearrow T_*} \parallel u(t) \parallel_{H^1} = +\infty.
\] (29)

We are going to see that the $H^1$-norm of the solution can’t blow up, which obviously ensures that $T_* = +\infty$.

Let us first, prove that the energy control the norm $\parallel \nabla u \parallel_{L^2}^2$, more precisely we have the following lemma:

**Lemma 3.** Let $u(t)$ be the solution of (4), then:
\[
\parallel \nabla u(t) \parallel_{L^2}^2 \lesssim \parallel u_0 \parallel_{L^2}^2 + E(u(t)).
\]
Proof. By interpolation we have:
\[ \|u\|_{L^3(\mathbb{R}^3)}^3 \leq \|u\|_{L^2(\mathbb{R}^3)}^{3/2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{3/2} \]
and by Young’s inequality, we obtain that:
\[ \|u\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{4} \|u\|_{L^2(\mathbb{R}^3)}^6 + \frac{3}{4} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \]
this inequality with (3) and (5), give that:
\[ E(u(t)) + \frac{1}{12} \|u_0\|_{L^2(\mathbb{R}^3)}^6 \geq \frac{1}{4} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2. \]

Proposition 2. Let \( u \) be the solution of (4), then \( \forall t \in [0, T] \), we have:
1. \( \|D_x^\alpha u\|_{L^2([0,t];L^2(\mathbb{R}^3))} \leq \|u(0)\|_{L^2(\mathbb{R}^3)} \).
2. \( E(u(t)) \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}, \|D_x^\alpha u\|_{L^2([0,t];L^2(\mathbb{R}^3))}, E(u_0)) \).

Proof. If we integrate in time the identity (5), we obtain \( \forall t \geq 0: \)
\[ \|u(t)\|_{L^2(\mathbb{R}^3)}^2 - \|u(0)\|_{L^2(\mathbb{R}^3)}^2 = -\int_0^t \int_{\mathbb{R}^3} (D_x^\alpha u)^2 dx dy dz ds = -\|D_x^\alpha u\|_{L^2([0,t];L^2(\mathbb{R}^3))}^2, \]
this give immediately the first inequality of the proposition.

Now let \( u_0 \in H^4(\mathbb{R}^3) \), then there exists \( T = T(\|u_0\|_{H^1}) \) such that the solution exists in \( H^4(\mathbb{R}^3) \) and since \( C_{\infty}^\infty(\mathbb{R}^3) \) dense in \( H^4(\mathbb{R}^3) \), then all calculations will be justified.

Now by (6), integration by parts and Cauchy-Schwarz, we can write:
\[ \frac{d}{dt} E(u(t)) + \frac{1}{2} \int_{\mathbb{R}^3} |D_x^\alpha \nabla u|^2 \leq \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)} \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)}^2, \]
notice that the fractional Leibniz rule (see [15]) leads to:
\[ \|D_x^\alpha u^2\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{L^3(\mathbb{R}^3)} \|D_x^\alpha u\|_{L^3(\mathbb{R}^3)} \]
and by interpolation we obtain:
\[ \|u\|_{L^3(\mathbb{R}^3)} \leq \|u\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{1/2}, \]
and
\[ \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)} \leq \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla (D_x^\alpha u)\|_{L^2(\mathbb{R}^3)}^{1/2}. \]

Inequalities (30), (32) and (33) give that:
\[ \frac{d}{dt} E(u(t)) + \frac{1}{2} \int_{\mathbb{R}^3} |D_x^\alpha \nabla u|^2 \leq \frac{\epsilon}{2} \|D_x^\alpha \nabla u\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2\epsilon} \|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)}^2, \]
and then for \( \epsilon \) small enough and using Lemma 3, we obtain that:
\[ \frac{d}{dt} E(u(t)) + \|u_0\|_{L^2} \leq \frac{1}{2\epsilon} \|u\|_{L^2(\mathbb{R}^3)} \|D_x^\alpha u\|_{L^2(\mathbb{R}^3)}^2 (E(u(t)) + \|u_0\|_{L^2}^{3/2} \]
and by Gronwall inequality we obtain the desired estimate.
Now we are ready to prove Theorem 2: Let \( u_0 \) in \( H^1(\mathbb{R}^3) \) and \( T^* \) be the maximal time of the existence of the solution emanating from \( u_0 \).

By Proposition 2 we have that:
\[
\|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 \lesssim E(u(t)) + \|u_0\|_{L^2}^2
\]
\[
\leq C(||u_0||_{L^2(\mathbb{R}^3)}), ||D_x^2 u||_{L^2(0,T;L^2(\mathbb{R}^3))}, E(u_0)), \forall t \in [0,T^*]
\]
but \( ||D_x^2 u||_{L^2(0,T;L^2(\mathbb{R}^3))} \leq ||u(0)||_{L^2(\mathbb{R}^3)}, \forall t \in [0,T^*] \), this gives directly that
\[
\lim_{t \to T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^3)} < +\infty
\]
and thus the solution \( u \) is global in time.

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E-mail address: Mohamad.Darwich@lmpt.univ-tours.fr