NON-INVENTOR HOPF ALGEBRAS AND 3-MANIFOLD INVARIANTS

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Abstract. We present a definition of an invariant \( \#(M, H) \), defined for every finite-dimensional Hopf algebra (or Hopf superalgebra or Hopf object) \( H \) and for every closed, framed 3-manifold \( M \). When \( H \) is a quantized universal enveloping algebra, \( \#(M, H) \) is closely related to well-known quantum link invariants such as the HOMFLY polynomial, but it is not a topological quantum field theory.

This paper presents a definition of an invariant \( \#(M, H) \) which depends on a framed, closed 3-manifold \( M \) and a finite-dimensional Hopf algebra \( H \), and whose value lies in the ground field of \( H \). The Hopf algebra \( H \) need not be quasitriangular, triangular, ribbon, modular, a quantum deformation, involutory, or semisimple, nor does it need to have any other decoration or structural property. It can be any finite-dimensional example of the object defined by Sweedler [16] and Drinfel’d [5] or, more generally, a finite-dimensional Hopf super-algebra or a Hopf object in any category which sufficiently resembles the category of finite-dimensional vector spaces. In a previous paper [8], the author defined \( \#(M, H) \) for involutory Hopf algebras (Hopf algebras in which the square of the antipode is the identity) and for closed and unoriented but unframed 3-manifolds. An important intermediate class of finite-dimensional Hopf algebras is the class of balanced Hopf algebras, for which the 3-manifold \( M \) need only be oriented and combed rather than framed. Recall that a framing of a 3-manifold is a homotopy class of linearly independent triples of tangent vectors. A combing is defined as the homotopy class of a non-vanishing tangent vector field.

In a subsequent paper [7], we will define the related invariant \( \#(M, L, H) \), where \( M \) is a framed, closed 3-manifold, \( H \) is a Hopf algebra, and \( L \) is a framed link in \( M \). More generally, the invariant \( \#(M, G, H) \) can be defined, where \( G \) is a framed graph in \( M \). When \( M = S^3 \), these invariants coincide with the Reshetikhin-Turaev invariants of links and ribbons graphs derived from \( D(H) \), the quantum double of \( H \). In particular, if \( q \) is a root of unity and \( g \) is a simple Lie algebra, the Reshetikhin-Turaev invariants for the finite-dimensional quantum groups \( u_q(g) \) yield root-of-unity values of the familiar quantum link invariants, such as the Jones polynomial, the HOMFLY polynomial, the Kauffman polynomial, and the quantum \( G_2 \) link invariant. The Hopf algebra \( u_q(g) \) is almost the quantum double of \( u_q(g^+) \), a truncated quantum deformation of (the enveloping algebra) \( U(g^+) \), where \( g^+ \) is a Borel subalgebra of \( g \). Therefore \( H = u_q(g^+) \) is an important special case of the invariant \( \#(M, H) \) that we define here.

Some other important special cases of \( \#(M, H) \) are the following: \( \#(S^3, H) = 1 \) by normalization, while \( \#(S^2 \times S^1, H) = \text{Tr}(S^2) \) is \( \dim H \) when \( H \) is involutory and \( 0 \) when \( H \) is non-involutory, and \( \#(\mathbb{R} P^3, H) = \text{Tr}(S) \). (Here \( S \) is the antipode map of \( H \) and \( \text{Tr} \) is its trace as a linear endomorphism of \( H \). To distinguish the \( n \)th power of the antipode from the \( n \)-sphere, we write the former as \( S^n \) and the latter as \( S^n \).) Moreover, \( \#(M, H) \) is multiplicative under connected sums and under tensor products of Hopf algebras. If \( H = \mathbb{C}[G] \) is the Hopf algebra of a group, \( \#(M, H) \) is the number of homomorphisms from the fundamental group of \( M \) to \( G \), which can be written as \( |\text{Hom}(\pi_1(M), G)| = |[M : B_G]| = |H^1(M, G)| \). If \( H \) is an exterior algebra with one generator,
which is a two-dimensional graded Hopf algebra, then \( \#(M, H) = |H_1(M, \mathbb{Z})| \) when the right side is finite, and \( \#(M, H) = 0 \) otherwise.

Given the relation between \( \#(M, L, u_q(g^+)) \) and quantum link invariants, one might suspect, as the author once did, that \( \#(M, u_q(g^+)) \) includes or is equivalent to the Jones-Witten 3-manifold invariants, defined explicitly by Reshetikhin and Turaev [15] for the group SU(2) and by Turaev and Wenzl [17] for the group SU(\( n \)). However, they cannot be equal or equivalent up to normalization, because \( \#(S^2 \times S^1, H) = 0 \) when \( q \neq 1 \) (and in general when \( H \) fails to be semisimple and co-semisimple), which violates the axioms of a topological quantum field theory (TQFT), and the relevant Hopf algebras are non-involutory. Rather, \( \#(M, H) \) should agree with another invariant recently found by Kauffman and Radford [6] and considered by Reshetikhin [13] which has a Dehn surgery definition similar to the definition of the Reshetikhin-Turaev 3-manifold invariant.

If \( H \) is involutory and \( \dim H \neq 0 \), then \( \#(M, H) \) is a TQFT. In this case \( H \) must be a semisimple algebra [9], and \( \#(M, H) \) can equally well be defined in Turaev-Viro style using the representation category of \( H \), or in Reshetikhin-Turaev-Wenzl style using the representation theory of \( D(H) \). In the semisimple case, the Kauffman-Radford invariant also equals the corresponding Reshetikhin-Turaev-Wenzl-style invariant.

1. The basic idea; involutarity

Let \( G \) be a finite group, not necessarily commutative, and let \( C \) be a connected, finite simplicial complex with oriented edges and faces and with a distinguished vertex \( p \) chosen as a base point. Recall the definition of \( H^1(C, G) \), the non-commutative first cohomology of \( C \) with coefficients in \( G \): A 1-cochain is a function from the edges of \( C \) to \( G \). Given a cochain and a face, let \( g_1, g_2, \) and \( g_3 \) be the three group elements assigned to the edges of the face in the order given by the orientation of the face, and let \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) each be \( 1 \) or \( -1 \) if the orientation of the corresponding edge agrees or disagree with the orientation of the face:

\[
\begin{align*}
g_2 &\to g_3 \\
g_1 &\to g_2 \sigma_1 \sigma_2 ^{-1} \sigma_3 ^{-1}
\end{align*}
\]

The cochain is a co-cycle if

\[
g_1^{\sigma_1} g_2^{\sigma_2} g_3^{\sigma_3} = 1
\]

for every face; here \( \sigma_1 = \sigma_3 = -1 \) and \( \sigma_2 = 1 \). A 0-cochain is a function from the vertices of \( C \) to \( G \), and a 0-cochain is reduced if it is 1 on the base point \( p \). The 0-cochains form a group under pointwise multiplication with the reduced 0-cochains as a subgroup, and the coboundary operation can be understood as a group action of 0-cochains acting on 1-cochains: Given a 0-cochain \( c \) and a 1-cochain \( d \), if \( d \) has the value \( g \) at an edge \( e \) and \( c \) has the values \( h_1 \) and \( h_2 \) at the head and tail of \( e \), then the 1-cochain \( cd \) has the value \( h_1 g h_2 ^{-1} \) at \( e \). The cohomology set \( \tilde{H}^1(C, G) \) is defined as the set of orbits of the 0-cochains acting on 1-cochains, while the reduced cohomology set \( \tilde{H}^1(C, G) \) is defined as the set of orbits of the reduced 0-cochains. The reduced cohomology is the set of group homomorphisms from \( \pi_1(C) \) to \( G \), while the full cohomology is the set of conjugacy classes of these group homomorphisms. Since reduced 0-cochains act freely on 1-cochains, the number of 1-co-cycles can be computed as \( |G|^{v-1} |\text{Hom}(\pi_1(C), G)| \), where \( v \) is the number of vertices of \( C \).

The above analysis fully extends from simplicial complexes to polygonal complexes. A polygonal complex is a CW complex whose 2-skeleton consists of a collection of polygons whose edges are identified to each other by homeomorphisms and whose vertices may be identified through further equivalences. A co-cycle is still defined as a \( G \)-valued function on edges such that, up to orientations,
the product of the values of the edges around every face is the identity, and the conclusion that the total number of 1-co-cycles is $|G|^{v-1}|\text{Hom}(\pi_1(C), G)|$ still holds. Removing the powers of $|G|$ by normalization, define $\#(C, G)$ to be the remaining factor $|\text{Hom}(\pi_1(C), G)|$. Clearly, $\#(C, G)$ is a topological invariant.

The first goal is to redefine $\#(C, G)$ in terms of the Hopf algebra $\mathbb{F}[G]$, where $\mathbb{F}$ is a field of characteristic 0. Recall that the Hopf algebra structure on $\mathbb{F}[G]$ consists of an algebra structure which is the linear extension of group multiplication, and an algebra structure on the dual vector space $\mathbb{F}(G)$ which is pointwise multiplication of $\mathbb{F}$-valued functions on $G$. Recall also that the antipode map $S : \mathbb{F}[G] \rightarrow \mathbb{F}[G]$ is the linear extension of the group inverse. Let the elements of $G$ be both a standard basis and standard dual basis for $\mathbb{F}[G]$. Let $M_{a_1a_2...a_n}$ be the tensor corresponding to the multilinear form $\text{Tr}(A_1A_2...A_n)$, where the trace is the trace of left (or right) multiplication, and let $\Delta^{a_1a_2...a_n}$ be the same tensor defined by the dual algebra structure. Taking the indices as group elements, observe that $M_{a_1a_2...a_n}$ is $|G|$ if $a_1a_2...a_n = 1$ and is zero otherwise, while $\Delta^{a_1a_2...a_n} = 1$ if all $a_i$'s are equal and is zero otherwise. The fact that the matrix of an $M$ tensor expresses the co-cycle rule of non-commutative cohomology can be exploited as follows: Assign to each edge a $\Delta$ tensor with an index for each incident face, and assign to each face an $M$ tensor with an index for each edge. Assign to each pair consisting of a face and an edge it contains either the antipode map or the identity map from $G$, depending on whether the orientations disagree or agree. Take the tensor product of all tensors and contract according to incidence. The resulting element of $\mathbb{F}$ is evidently the sum of $|G|^f$ over all 1-co-cycles, and therefore equals $|G|^{v+f-1}\#(C, G)$ if $C$ has $v$ vertices and $f$ faces.

If a polygonal CW complex $C$ is a cell decomposition of an oriented 3-manifold $M$, an interesting symmetry appears. Consider the handle decomposition of $M$ corresponding to $C$, i.e., thicken each face and replace each edge by a prism with a long, thin side for each incident face prism:

Call the long thin sides of either kind of prism rectangles. The orientation of $M$ is a right-hand rule, which associates a co-orientation or a cyclic ordering of the rectangles of the prism of an oriented edge. Working from the handle decomposition, the tensorial definition of $\#(C, G)$ associates a $\Delta$ tensor to each edge prism, an $M$ tensor to each face prism, and either $I$ or $S$ to each rectangle.

If the indices of the $\Delta$ tensors are cyclically ordered according to the co-orientations, the tensorial expression is evidently invariant under an exchange of $\Delta$ and $M$ by Poincaré duality. As stated, it is a well-defined formula for any finite-dimensional Hopf algebra $H$. It is a familiar topological invariant for a large class of co-commutative Hopf algebras, namely group algebras, and it is the same invariant for a large class of commutative Hopf algebras, namely dual group algebras. One might conjecture, as the author once did, that it is always a topological invariant, but this is only true when the antipode $S$ is an involution, i.e., when $H$ is involutory. (See reference [3] for a proof of invariance. The involutory invariant was also independently discovered by Chung, Fukuma, and Shapere [8].)

A valid generalization to non-involutory Hopf algebras is the subject of the rest of the paper. The definition of $\#(M, H)$ for $H$ involutory mandates that a rectangle be replaced with $S^n$ (the $n$th power of the antipode), where $n \in \mathbb{Z}/2$ is 0 if the link of the edge prism and the rim of the face prism intersect positively and 1 if they intersect negatively on the surface formed by all rectangles, which is an example of a Heegaard surface. This rule is not natural (and does not lead to an invariant) when the exponent $n$ is an integer rather than an element of $\mathbb{Z}/2$; we must choose between all even
or all odd \( n \), depending on the sign of the orientation. The solution is to consider a logarithmic lifting of the unit tangent bundle of the Heegaard surface to an affine line bundle, which allows angles to take values in the real numbers rather than the circle and intersections to take values in the integers rather than in \( \mathbb{Z}/2 \). Unless the Heegaard surface is a torus, such a lifting must be singular, but its singularities can solve other problems that arise elsewhere in the definition. A singular combing of the Heegaard surface which extends to a combing of the 3-manifold can provide just such a logarithmic structure.

2. Topology

Throughout the paper, a manifold is a compact, oriented, triangulated, differentiable manifold. In addition, a manifold is assumed to have no boundary unless otherwise stated. PL and smooth constructions will be freely combined, a liberty which rarely leads to confusion in three and fewer dimensions.

2.1. Comblings and framings. Let \( M \) be a 3-manifold. A combing of \( M \) is a non-vanishing tangent vector field, and a framing of \( M \) consists of three linearly independent vector fields whose orientation agrees with that of the manifold. For convenience, view \( M \) as a Riemannian manifold, a combing as a section of the unit tangent bundle, and a framing as three orthogonal sections. Since the orientation of \( M \) induces a cross-product operation on tangent vectors, it suffices to describe a framing as a pair of orthogonal sections, the third section being given by the cross product.

The goal of this subsection is to classify the set of combings and the set of framings of a 3-manifold up to homotopy. It is important to describe these two sets explicitly, since the invariant \( \#(M, H) \) in general depends on a combing or framing of \( M \). In addition, the proofs of Lemmas 2.6 and 2.8 depend on the classification.

It is a classical result that the tangent bundle \( TM \) of a 3-manifold \( M \) is trivial. Using a trivialization of \( M \), which is itself a framing, a combing becomes a map from \( M \) to \( S^2 \) and a framing becomes a map from \( M \) to \( SO(3) \). Thus, the set of combings is bijective with the homotopy set \( [M, S^2] \) and the set of framings is bijective with \( [M, SO(3)] \). However, for many \( M \), since no one trivialization of \( M \) is completely natural, combings and framings are not canonically bijective with \( [M, S^2] \) and \( [M, SO(3)] \). Rather, it is important to understand the natural group action of \( [M, SO(3)] \) on \( [M, S^2] \) and \( [M, SO(3)] \).

Proposition 2.1. The homotopy class of a map from \( M \) to \( S^2 \) is described by a characteristic class \( c \in H^2(M, \mathbb{Z}) \), defined as the pullback of \( [S^2] \), and a Hopf degree \( d \) in an affine space of \( \mathbb{Z}/n \), where \( n \) is defined as follows: Let \( H^2_{\text{free}}(M, \mathbb{Z}) \) be the quotient of \( H^2(M, \mathbb{Z}) \) by its torsion, and let \( c_{\text{free}} \) be the image of \( c \) under this quotient. Then \( n \) is the maximal divisor of \( c_{\text{free}}/n \) exists in the lattice \( H^2_{\text{free}}(M, \mathbb{Z}) \).

Proof: (Sketch) In general, if \( X \) and \( Y \) are CW complexes, the homotopy set \( [X, Y] \) is approximated by the direct product of the cohomology groups of \( X \) with coefficients in the homotopy groups of \( Y \). More precisely, if \( X_n \) is the \( n \)-skeleton of \( X \), then restriction from \( X_n \) to \( X_{n-1} \) yields a map \( j_n : [X_n, Y] \to [X_{n-1}, Y] \), and of course \( [X, Y] \) is the inverse limit. Moreover, there is a group action of \( H^n(X, \pi_n(Y)) \) on \( \text{im} \ j_{n+1} \), and each orbit of this group action is \( j^{-1}(f) \cap \text{im} \ j_{n+1} \) for some \( f \in [X_{n-1}, Y] \). In simple cases, each \( j_n \) is a surjective group homomorphism with a complemented kernel and the action of \( H^n(X, \pi_n(Y)) \) is free, in which case \( [X, Y] \cong H^n(X, \pi_n(Y)) \). But the homotopy set \( [M, S^2] \) exhibits a complication that arises because \( S^2 \) has non-trivial homotopy groups of adjacent degree. Although \( \pi_2(S^2) \cong \pi_3(S^2) \cong \mathbb{Z} \), \( [M, S^2] \) resembles but does not necessarily equal \( H^2(M, \mathbb{Z}) \oplus H^3(M, \mathbb{Z}) \). To obtain the exact answer, consider the fibration sequence

\[
\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2 \to \mathbb{C}P^\infty \to \mathbb{H}P^\infty,
\]

where the first three terms are the Hopf fibration. The terms \( \mathbb{C}P^\infty \) and \( \mathbb{H}P^\infty \) are classifying spaces of \( \mathbb{S}^1 \) and \( \mathbb{S}^3 \) viewed as groups, and they extend the Hopf fibration because their loop spaces are
homotopy equivalent to $S^1$ and $S^3$. Sequence (1) would lead to the exact sequence

$$[M, S^1] \to [M, S^3] \to [M, S^2] \to [M, \mathbb{CP}^\infty] \to [M, \mathbb{HP}^\infty]$$

if all of the terms in the sequence were groups. Although three of the terms can be recognized as the cohomology groups of the 3-manifold $M$ and the last term is trivial, the homotopy set $[M, S^2]$ is not a group. Instead, corresponding exact sequence is

$$H^1(M, \mathbb{Z}) \to H^3(M, \mathbb{Z}) \to [M, S^2] \to H^2(M, \mathbb{Z}) \to 0,$$

where the second map is a group action and the first map varies according to a chosen orbit of the group action. Recall that the Hopf degree establishes the isomorphism $\pi_3(S^2) \cong \mathbb{Z}$; a calculation in obstruction theory using the Hopf degree establishes that the first arrow is:

$$H^1(M, \mathbb{Z}) \xrightarrow{\cup} H^3(M, \mathbb{Z}) \to [M, S^2] \to H^2(M, \mathbb{Z}) \ni c.$$

In other words, if an orbit of $[M, S^2]$ is given by a cohomology class $c \in H^2(M, \mathbb{Z})$, the stabilizer in $H^3(M, \mathbb{Z})$ of this orbit is the image of the cup product with $c$. Recall that $H^1(M, \mathbb{Z})$ is free, and that the cup product

$$\cup : H^1(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to H^3(M, \mathbb{Z})$$

annihilates the torsion of $H^2(M, \mathbb{Z})$ and is a unimodular bilinear form, or perfect pairing, on the free parts of the cohomology groups. It follows that the image of $d \mapsto d \cup c$ is generated by $n[M]$, where $n$ is the maximal divisor of $c_{\text{free}}$.

**Proposition 2.2.** The homotopy set $[M, SO(3)]$ is a central extension of $H^1(M, \mathbb{Z}/2)$ by $H^3(M, \mathbb{Z})$. The group action of $f \in [M, SO(3)]$ on $[M, S^2]$ is partly characterized as follows: The group $H^1(M, \mathbb{Z}/2) = [M, SO(3)]/H^3(M, \mathbb{Z})$ has a quotient $H^3(M, \mathbb{Z}) \ast \mathbb{Z}/2$ by the universal coefficient theorem; let $f'$ be the image of $f$ in $H^3(M, \mathbb{Z}) \ast \mathbb{Z}/2$. quotient. Then $f'$ acts on the characteristic class $c$ of an element of $[M, S^2]$ by $c \mapsto c + f'$, if $H^2(M, \mathbb{Z}) \ast \mathbb{Z}/2$ is understood as a subgroup of $H^2(M, \mathbb{Z})$. If $f \in H^3(M, \mathbb{Z})$, then $f$ acts on the degree $d$ of an element of $[M, S^2]$ by $d \mapsto d + f$.

**Proof:** (Sketch) The homotopy set $[M, SO(3)]$ is much easier to understand, since the non-trivial homotopy groups $\pi_1(SO(3)) \cong \mathbb{Z}/2$ and $\pi_3(SO(3)) \cong \mathbb{Z}$ are not adjacent. In general, $[M, SO(3)]$ is a central extension of $H^1(M, \mathbb{Z}/2)$ by $H^3(M, \mathbb{Z})$ as a group, where the group law is defined using the Lie group structure of $SO(3)$. In terms of the geometry on $M$, one framing $f_1$ converts a second framing $f_2$ into an element of $[M, SO(3)]$, and after this conversion the $H^1(M, \mathbb{Z}/2)$ class of $f_2$ describes its spin structure, while the $H^3(M, \mathbb{Z})$ class gives its degree. Since these classes are defined relative to $f_1$, the set of homotopy classes of framings of $M$ is only an affine space of the group $[M, SO(3)]$. However, the action of $[M, SO(3)]$ on $[M, S^2]$ is again more complicated. There is a fibration sequence analogous to sequence (1):

$$S^1 \to SO(3) \to S^2 \to \mathbb{CP}^\infty \to B_{SO(3)}.$$

Here is an explicit description of the third map: $SO(3)$ is a principal bundle over $S^2$ with fiber $S^1$, and since $\mathbb{CP}^\infty$ is the classifying space of the group $S^1$, the bundle over $S^2$ induces a map $S^2 \to \mathbb{CP}^\infty$ which is the map in the sequence. In sequence (1), the third map has degree 1, because the Hopf fibration has Chern class 1. By contrast, since $SO(3)$ has Chern class 2, the third map of sequence (2) has degree 2. Also, recall the familiar fact that the inclusion $S^1 \to SO(3)$ is a mod 2 reduction $\mathbb{Z} \to \mathbb{Z}/2$ of $\pi_1$.

Sequence (2) yields an exact sequence of homotopy sets of maps from $M$, which simplifies to another mixed exact sequence of sets and groups:

$$H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{Z}/2) \oplus H^3(M, \mathbb{Z}) \to [M, S^2] \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2).$$

The action of the $H^3(M, \mathbb{Z})$ subgroup of $[M, SO(3)]$ on $[M, S^2]$ can be understood by recalling that the covering map and group homomorphism $SU(2) \to SO(3)$ is an isomorphism of $\pi_3$, and, taking
Combed and framed Heegaard diagrams. Recall that a realization of a 3-manifold \( M \) as a polygonal CW complex yields a handle decomposition, where the handles are prisms that meet at rectangular sides. The union of all rectangles is a Heegaard surface, which is a surface that
divides $M$ into two handlebodies. In general, given a surface $R$, a handlebody with boundary $R$ can be described by a *handlebody diagram*, which is a collection of disjoint embedded circles in $R$ whose complement is planar. Given a handlebody diagram, the handlebody is constructed by attaching thickened disks to one side of a thickened copy of $R$ along the circles, and then attaching balls to the spherical boundary components that result. A *Heegaard diagram* is then a transverse pair of handlebody diagrams on $R$, and it defines a 3-manifold for which $R$ is a Heegaard surface.

By arbitrary convention, one of the handlebodies and its circles is called upper and the other is called lower. Given a polygonal cell decomposition, the perimeters of the faces are the upper circles and the links of the edges are the lower circles of a Heegaard diagram on the Heegaard surface of rectangles; the diagram can also be recognized as a dissection which is dual to the tiling by rectangles:

![Heegaard Diagram]

If a circle $c$ in a handlebody diagram separates two distinct components of the complement of the diagram, then $c$ is redundant in the sense that its removal yields another handlebody diagram which describes the same handlebody. It follows that the only *minimal* handlebody diagrams on a surface of genus $g$ are those with $g$ circles, since only those diagrams have connected complements. Unlike in reference [8], most handlebody diagrams in this paper will be assumed to be minimal. Note that $R$ admits only one minimal handlebody diagram up to homeomorphism. In this sense, only the relative position of the two handlebody diagrams determines the topology of a 3-manifold.

Following reference [8], three moves on Heegaard diagrams suffice to render any two Heegaard diagrams for a 3-manifold $M$ equivalent. The moves are the *circle slide*:

\[
\begin{array}{c}
\includegraphics[scale=0.5]{circle-slide.png}
\end{array}
\]

*stabilization*:

\[
\begin{array}{c}
\includegraphics[scale=0.5]{stabilization.png}
\end{array}
\]

and *isotopy*:

\[
\begin{array}{c}
\includegraphics[scale=0.5]{isotopy.png}
\end{array}
\]

We will need the following two well-known results concerning Heegaard diagrams and their moves:

**Theorem 2.3.** Any two minimal handlebody diagrams on a surface $R$ for a handlebody $B$ are equivalent by a sequence of circle slides.
PROOF: Let $d_1$ and $d_2$ be the two handlebody diagrams and let $D_1$ be a collection of disjoint disks in $B$ whose boundaries form $d_1$; similarly, let $D_2$ be another such collection whose boundaries form $d_2$. We first show by induction that we can eliminate the intersection of $D_1$ and $D_2$ by isotopy and handle slides. A disk $C_1 \in D_1$ with boundary $c_1$ intersects disks in $D_2$ in a pattern of arcs and circles:

If $a$ is an innermost arc of the intersection, as shown, then the endpoints of $a$ coincide with the endpoints of an arc $a'$ lying in $c_1$ whose interior does not meet any circles of $d_2$. Let $C_2 \in D_2$ be the other disk that contains $a$ and let $c_2$ be its boundary. We claim that $c_2$ can be repositioned to reduce intersection with $d_1$. To understand how isotopy and circle slides can be applied to $c_2$, we introduce the useful trick of cutting $R$ along every circle of $d_2$ other than $c_2$. The result $R'$ is a multiply-punctured torus for which $c_2$ is a meridian:

A circle slide applied to $c_2$ can be thought of as an isotopy of $c_2$ across a puncture. In this case, the arc $a'$ also lies in $R'$ as indicated. (There is only one topological position for it given that both endpoints lie on $c_2$.) Clearly, $c_2$ can be isotoped, perhaps across punctures of $R'$, so that an arc of it runs parallel to $a'$:

As a result, the intersection of $c_2$ with $d_1$ decreases by at least two points.

Thus, we can assume that $d_1$ and $d_2$ are disjoint. We show by induction that they can be made coincident, and in this case, the hypothesis that they describe the same handlebody is unnecessary. Let $c_1 \in d_1$ and let $R''$ be the result of cutting $R$ along all circles of $d_2$. The circle $c_1$ does not separate $R$, but it does separate $R''$. Therefore there exists a circle $c_2 \in d_2$ such that, if you glue the two cuffs of $R''$ corresponding to $c_2$ together to obtain $R'$, $c_1$ does not separate $R'$. The circles $c_1$ and $c_2$ are both meridians of the punctured torus $R'$, and therefore there is an isotopy of $c_1$ to $c_2$, possibly involving isotopy across punctures. Assuming that $c_1$ and $c_2$ are coincident, we can cut $R$ along $c_1$ to obtain $R'''$, and we can apply the argument inductively to the diagrams $d_1 - \{c_1\}$ and $d_2 - \{c_2\}$ to render $d_1$ and $d_2$ completely parallel.

Theorem 2.4 (Reidemeister,Singer). Any two Heegaard surfaces for a manifold $M$ are equivalent by a sequence of stabilization moves.

PROOF: (Sketch) As we have already suggested in the context of polygonal CW complexes, the 0- and 1-handles of a handle decomposition of $M$ are separated from the 2- and 3-handles by a
Heegaard surface. It is easy to realize every Heegaard surface this way. Following reference [8], it is also easy to realize every handle decomposition by means of a Morse function on $M$. Taking a generic path of Morse functions between any two given Morse functions, the only transitions in the corresponding Heegaard surfaces are stabilization moves.

Although neither Theorems 2.3 and 2.4 nor their proofs refer to the isotopy move, it is implicitly allowed given the possibility of isotopy of the lower diagram related to the upper diagram. The theorems show that the three given moves suffice.

If $R$ is a Heegaard surface in a 3-manifold $M$, the orientation of $M$ induces an orientation on $R$, by the convention that a positive tangent basis at a point for $R$ extends to a positive basis for $M$ by appending a normal vector that points from the lower side to the upper side. By convention, Heegaard circles are also oriented. Indeed, if a Heegaard diagram comes from a polygonal complex with oriented edges and faces, then an orientation of a face induces an orientation of its upper circle, while an orientation of an edge induces an orientation of its lower circle by the right-hand rule:

Reversing the orientation of a circle is an obvious move that renders any two orientations of a Heegaard diagram equivalent.

It will be important to sign-order the set of all upper and lower circles of a minimal Heegaard diagram, where a *sign-ordering* of a finite set is an orbit of the alternating group acting on the set of complete orderings of the set. Andrew Casson pointed out the following fact to the author:

**Proposition 2.5.** There is a canonical way to sign-order the Heegaard circles of a minimal Heegaard diagram, if the circles are oriented. Here canonical means that the sign-ordering is preserved by the handle slide move and reversed if the orientation of a Heegaard circle is reversed.

**Proof:** Given a Heegaard surface $R$ with a minimal Heegaard diagram, the vector space $H_1(R, \mathbb{R})$ has a symplectic structure given by the intersection form. The upper and lower handlebodies define Lagrangian subspaces $L_l$ and $L_u$ with respect to this form. Viewed as homological cycles, the circles for each handlebody form bases for $L_l$ and $L_u$, and a sign ordering of the circles yields an orientation of the (outer) direct sum $L_l \oplus L_u$. The vector space $H_1(R, \mathbb{R})$ also has an orientation induced by its symplectic structure. If $L_l$ and $L_u$ are transverse, which happens when the 3-manifold given by the diagram is a rational homology sphere, then $L_l \oplus L_u = H_1(R, \mathbb{R})$, and the canonical sign ordering of the circles is the one that agrees with the orientation of $H_1(R, \mathbb{R})$. But this is a special case. In the general case, the intersection form induces an isomorphism

$$(L_l \cap L_u)^* \cong H_1(R, \mathbb{R})/(L_l + L_u),$$

which means that there is an exact sequence

$$0 \to L_l \cap L_u \to L_l \oplus L_u \to H_1(R, \mathbb{R}) \to (L_l \cap L_u)^* \to 0.$$ 

Recall that an orientation for all but one term of a terminating exact sequence induces an orientation of the remaining term. In this case, taking either orientation of $L_l \cap L_u$ results in the same orientation of $L_l \oplus L_u$. This orientation yields the desired sign-ordering of the Heegaard circles. It is easy to check that a circle slide is a change of basis of $L_l$ or $L_u$ with determinant 1 and that an orientation reversal negates a basis vector, which means that the two operations respectively preserve and reverse the sign-ordering.
If we restrict the combing of a 3-manifold $M$ to a Heegaard surface $R$ and then orthogonally project to the tangent bundle $TR$, the result is in general a singular combing which can in principle have many types of singularities. However, in this paper we will only consider singular combings of $R$ with prescribed singularities in a standard position relative to a Heegaard diagram. We will show that such a combing canonically extends to a combing of $M$ (or rather, it is the projection of an $M$-tangent vector field that extends to $M$), and that these combings represent all homotopy classes of combings. Recall that the total index of all singularities of a combing on $R$ of genus $g$ is the Euler characteristic $2 - 2g$, that a singularity of index $-1$ has the geometry

\[
\begin{array}{c}
\downarrow \\
\bullet \\
\end{array}
\]

and that a singularity of index +2 has the geometry

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\end{array}
\]

Define a **combing** of a minimal Heegaard diagram on $R$ to be a combing of $R$ with $2g$ singularities of index $-1$, one on each circle, and one singularity of index +2 disjoint from all circles. The singularity of index $-1$ on a given circle, which is called the base point of the circle, should not lie on a crossing and the two outward-pointing vectors should be tangent to the circle.

The extension of a combing of a diagram for $M$ to a combing of $M$ will rely on a simple principle of interpolation to avoid discontinuities: Consider $X \times [0,1]$ embedded in $M$ for some 1- or 2-manifold $X$. Suppose that the $[0,1]$ fibers map to very short line segments, so that if $p \in X$, we may think of the tangent space to $M$ at $(p,t)$ as being the same vector space for different $t$. Let $v$ be a field of unit vectors tangent to $M$ defined on $X \times \{0,1\}$, and suppose further that $v(p,0)$ is never antiparallel to $v(p,1)$. Then $v$ may be continuously extended to $X \times [0,1]$ by setting $v(p,t)$ to be a geodesic path from $v(p,0)$ to $v(p,1)$ on the unit sphere of possible values of $v(p,\cdot)$.

The extension of $b$ to $M$ proceeds in five steps: Firstly, at a singularity of index $-1$, necessarily a base point of a circle, desingularize $b$ by making it point towards the disk attached to the circle. Secondly, on each disk which attaches to an upper or lower circle $c$, let $b$ be a fan emanating from the base point:

If this fan were extended all the way to $c$, $b$ would be discontinuous, because except in a neighborhood of the base point, $b$ is tangent to the Heegaard surface $R$, while the fan is normal. To maintain continuity, interpolate as above between the fan and $b$ as defined on $R$. At the point, the region on which $b$ is undefined consists of the sole upper ball $B_u$ which is attached to $R$ and the upper disks, the sole lower ball $B_l$ attached to $R$ and the lower disks, and a region which connects them around the singularity of index +2, which has not been desingularized. Geometrically, $B_u$
and $B_i$ are manifolds with corners, because they are handlebodies with sawed handles:

The third step consists of smoothing these round corners and extending $b$ correspondingly; note that the previous interpolation between the fan on each disk and $b$ on each circle does not produce a situation in which $b$ is an inward-pointing normal to the smoothed balls $B_u'$ and $B_l'$:

The fourth and fifth steps are a little more difficult to visualize (which is not to say that the first three steps are easy in this respect). Fourthly, identify a small topological ball $B_2$ around the singularity of index $+2$ which is geometrically a torus union a cylinder that plugs the hole of the torus, or the surface of revolution of a barbell shape in the plane:

Let $b$ be the outward normal on both the upper and lower flat disks of $B_2$, and extend $b$ as a pair of 3-dimensional fans on the balls $B_u'$ and $B_l'$ outside of $B_2$:

As in the second step, extending $b$ all the way to the boundary would result in discontinuities, but we may again interpolate as we have checked that $b$ is never an inward normal on $\partial B_u'$ or $\partial B_l'$, while the fans are outward normals near the boundaries.
Fifthly, we extend \( b \) to \( B_2 \). For this purpose, it is best to invert \( B_2 \) to recognize it as the complement of a cylinder \( B'_2 \) in the 3-sphere \( S^3 \):

If we coordinatize \( S^3 = \mathbb{R}^3 \cup \{ \infty \} \) by Cartesian coordinates \( x, y, \) and \( z \) and let \( B'_2 \) be the cylinder given by \( x^2 + y^2 \leq 1 \) and \( |z| \leq 1 \), then the restriction of \( b \) to \( \partial B_2 = \partial B'_2 \) can be given by a simple formula, for example \( (b_x, b_y, b_z) = (1, 0, -z) \). Indeed, \( b \) on \( \partial B'_2 \) extends to a vector field \( b' \) on \( B'_2 \) by the same formula. We extend \( b' \) to \( B_2 \subset S^3 \) so that \( b' \) is the ambidextrous combing of \( S^3 \); we then extend \( b \) to \( B_2 \subset M \) so that it agrees with \( b' \) on \( B_2 \). This completes the extension of \( b \) to \( M \).

Given a combing \( b_1 \) of \( M \), an orthogonal combing \( b_2 \) is a section of the circle bundle orthogonal to \( b_1 \), and a section a circle bundle defined on a 2-skeleton of a cell complex always extends uniquely up to homotopy to the entire cell complex. Therefore, to describe a framing \((b_1, b_2)\) combinatorially, it suffices to describe \( b_1 \) as a diagram combing and then to describe \( b_2 \) on the Heegaard surface \( R \) and on all upper and lower disks. Unlike \( b_1, b_2 \) need not be in any special position in its combinatorial description; it need only be in general position relative to geometric objects such as crossings and upper and lower circles. Twist fronts indicate the position of \( b_2 \), where a twist front is an arc along which \( b_2 \) is normal to \( R \) and points from lower to upper handlebody. A twist front is transversely oriented in the direction that \( b_2 \) rotates by the right-hand rule relative to \( b_1 \), and transverse orientation is indicated by the symbol for cold fronts on weather maps:

Twist fronts terminate at base points. If the viewer’s eye is in the upper handlebody, a twist front points counterclockwise around an upper base point and clockwise around a lower base point:

By contrast, a twist front cannot end at the singularity of index \( +2 \) by conservation of twist front ends. In essence, a \( +2 \) singularity is a pair of \( +1 \) singularities of opposite type with a twist front connecting them, similar to the above figure.
Given a combing $b_1$ and given an orthogonal combing $b_2$ defined on $R$, $b_2$ may or may not extend orthogonally to upper and lower disks. Since the disks are 2-dimensional, if $b_2$ does extend, it extends uniquely up to homotopy. For each upper or lower circle $c$, define $\theta(c)$ to be the total counter-clockwise rotation, in units of $1 = 360^\circ$, of the tangent to $c$ relative to $b_1$ going around $c$ in the direction of its orientation. Let $\phi(c)$ be the total right-handed rotation of $b_2$ about $b_1$ going around $c$ in the direction of its rotation. In units of $1 = 360^\circ$, $\phi(c)$ is naturally a half-integer because of a fractional contribution at the base point. The number $\phi(c)$ can be computed as the the number of twist fronts that cross $c$ positively minus the number that cross negatively, with the base point counting half as much. An analysis of the extension of $b_1$ to the Heegaard disk attached at $c$ shows that $b_2$ extends if and only if $\phi(c) = -\theta(c)$ when $c$ is an upper circle and $\phi(c) = \theta(c)$ when $c$ is a lower circle.

To complete the combinatorial definition of combings and framings, we consider a set of elementary moves on them. All moves on uncombed minimal diagrams are allowed, except that a circle cannot be isotoped across the $+2$ singularity:

\[
\begin{array}{c}
+2 \\
\bigcirc \\
+2
\end{array}
\]

In particular, a $+2$ singularity inside an eyelet:

\[
\begin{array}{c}
+2 \\
u \\
\bigcirc \\
l
\end{array}
\]

prevents the usual isotopy move. This is restricted isotopy of Heegaard circles. The other moves, stabilization and a circle slides, are unrestricted. In addition, there are two new moves on the combings and framings that do not affect the underlying Heegaard diagram:

- A base point isotopy move:

\[
\begin{array}{c}
\begin{array}{c}
\bigtriangleup \\
-1
\end{array} \\
\bigcirc \\
\begin{array}{c}
\bigtriangleup \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bigtriangleup \\
+1
\end{array} \\
\bigcirc \\
\begin{array}{c}
\bigtriangleup \\
+1
\end{array}
\end{array}
\]

The diagram indicates an isotopy of a base point on a lower circle past an upper circle, but we also consider the isotopy of base points of upper circles. Also, in this diagram and later ones, a point labelled by either -1 or 2 indicates a singularity of the corresponding index.
- A base point spiral move:

Note also that homotopy of a diagram combing or framing is an allowed operation which must be respected when defining any combed or framed invariant, even though it is not a combinatorial move. Homotopy of the second combing of a framing can have a combinatorial effect on twist fronts, in the form of four possible moves:

- Two-point isotopy:

- Three-point isotopy:

- Circle birth:

- The exchange move:

Since only homotopy-invariant information will be used in the definition of invariants, these moves will automatically be respected. In particular, we will not explicitly use the four combinatorial moves on twist fronts, but it is useful for practical computations to know that they suffice to reproduce any homotopy of a second combing. Finally, a geometric analysis shows that a spiral move does not preserve a combing (or a framing), but rather it changes its degree by 1. More
precisely, we can take as a convention that a clockwise spiral move on an upper circle decrements the degree. In this case a clockwise spiral move on a lower circle increments the degree.

Before showing that these elementary moves suffice, we first show that all combings and framings of a 3-manifold can be realized combinatorially.

**Lemma 2.6.** Given a minimal Heegaard diagram $D$ of a 3-manifold $M$ of genus at least 1, every combing of $M$ is realized by some combing of $D$, and the same is true of framings.

**Proof:** It is easy to find at least one combing of $D$; let $b$ denote such a combing as well as its extension to $M$. Recall if $b'$ is another combing, $b$ and $b'$ have a relative characteristic class in $H^2(M, \mathbb{Z})$, and if that class is zero, they have a relative Hopf degree in $H^3(M, \mathbb{Z}) \cong \mathbb{Z}$. Geometrically, the relative characteristic class is the Poincaré dual of the antipodal intersection of $b'$ with $b$. I.e., let $-b$ be $b$ with its vectors negated, and consider $-b$ and $b'$ as sections of the unit tangent bundle of $M$ in general position. Then the intersection $-b \cap b'$, projected to $TM$, is a collection of oriented curves which represents a homology class in $H_1(M, \mathbb{Z})$ which is independent of homotopy of $b$ and $b'$. Its Poincaré dual is the relative characteristic class.

Suppose that $b'$ is a combing of $D$ whose singularities are coincident with those of $b$. Since the angle between $b'$ and $b$ extends continuously to the singularities, it defines a map $F : R \to S^1$ from the Heegaard surface to the circle. If $b$ and $b'$ are in general position, their anti-parallel intersection in $R$ yields an element $c$ in $H_1(R, \mathbb{Z})$ which is Poincaré dual to the cohomology class $F^*(\mathbb{S}^1)$. Since there are no restrictions on the map $F$, every element of $H_1(R, \mathbb{Z})$ is realized. Moreover, the extension of $b$ and $b'$ to $M$ excludes the possibility that they are anti-parallel anywhere other than on $R$. Therefore their anti-parallel intersection in $M$ is $i_*(c)$, where the map $i_* : H_1(R, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is induced by inclusion. Since this map is a surjection, every element in $H_1(M, \mathbb{Z}) \cong H^3(M, \mathbb{Z})$ is realized as a relative characteristic class as $b'$ varies.

The Hopf degree of $b'$ can also be set arbitrarily, because the spiral move changes it by 1. In conclusion, all combings of $M$ are realized by combings of $D$.

Since $b'$ can represent any combing of $M$, then in particular, it can represent the first combing $b_1$ of a framing $f$. At no point in the combinatorial definition of the second combing $b_2$ is there any restriction of its homotopy class, and therefore all framings are realized combinatorially also.

It is useful to explicitly identify the homotopy class of $b_2$ if it is defined combinatorially. Given two combings $b_2$ and $b_2'$ orthogonal to $b_1$, the homological difference of their twist fronts defines a homology class $c$ in $H_1(R, \mathbb{Z}) \cong H^1(R, \mathbb{Z})$. The combinatorial constraints on twist fronts imply that $c$ lies in the image of the injection $i^* : H_1(M, \mathbb{Z}) \to H^1(R, \mathbb{Z})$. (Alternatively, this can be seen directly by defining the angle between $b_2$ and $b_2'$ and noting that it extends to a function from $M$ to $\mathbb{S}^1$.) In turn, as we have already seen, the spin structure of $f = (b_1, b_2)$ differs from that of $(b_1, b_2')$ by

$$(i^*)^{-1}(c) \mod 2 \in H^1(M, \mathbb{Z}) \otimes \mathbb{Z}/2 \subseteq H^1(M, \mathbb{Z}/2),$$

while the Hopf degrees are equal.

Finally, we show that the given moves on combings and framings render two different realizations of the same combing or framing equivalent.

**Lemma 2.7.** Given a surface $R$ with a marked point $p$, any two minimal handlebody diagrams for a handlebody with boundary $R$ are equivalent by isotopy that avoids $p$ and handle slides.

**Proof:** The lemma is a corollary of Theorem 2.3; it does not replace it. Let $d$ be a minimal diagram on $R$ and let $C$ be a circle of $d$. We reproduce an isotopy of $C$ across the marked point $p$ by allowed moves. Let $R'$ be the result of cutting $R$ along all circles of $d$ other than $c$. Since $d$ is minimal, $R'$ is necessarily a multiply punctured torus with $c$ a non-separating circle in $R'$. Since the cuffs of the punctures of $R'$ are other circles of $d$, an isotopy of $C$ across such a puncture is effected by a circle slide. Therefore, if $C$ is on one side of $p$, it can be moved around $R'$ to the other side of $p$ by means of allowed operations. 


Lemma 2.8. Any two combings of a Heegaard diagram $D$ which extend to the same 3-manifold combing up to Hopf degree are equivalent by base point isotopy moves and base point spiral moves. Any two diagram framings which yield the same 3-manifold framing up to degree are equivalent by the same moves.

Proof: Suppose that $b_1$ and $b_2$ are two combings of $D$ with vanishing relative characteristic class in $M$. Using base point isotopies, move the base points of $b_2$ so that they coincide with those of $b_1$. Following the analysis in Lemma 2.6, the anti-parallel intersection of $b_1$ and $b_2$ is a class $c \in H_1(R, \mathbb{Z})$. The dual of the relative characteristic class is $i_*(c) = 0$, where $i : R \to M$ is inclusion. The kernel of $i_*$ is generated by the Heegaard circles of $D$, interpreted as homology cycles on $R$. At the same time, an isotopy of a base point of $b_2$ all the way around a circle $d$ in $D$ changes the class $c$ by the homology class represented by $d$. Therefore some collection of base point isotopies renders the intersection $c = 0$. Given $c = 0$, there is a homotopy between $b_1$ and $b_2$ that does not move the base points; however, this homotopy may involve base point spiral moves.

If $f_1$ and $f_2$ are two framings of $D$ which induce the same spin structure on $M$, then firstly their first combings have vanishing relative characteristic class. Therefore, by the previous argument, they may be arranged so that their first combings coincide. After this operation, the ratio of the second combings of $f_1$ and $f_2$ is a map $R \to S^1$ that represents a cohomology class $c' \in H^1(S, \mathbb{Z})$. The constraints on second combings mean that $c$ lies in the image of $i^* : H^1(M, \mathbb{Z}) \to H^1(R, \mathbb{Z})$. Furthermore, since $f_1$ and $f_2$ have the same spin structure, the class $(i^*)^{-1}(c') \mod 2 \in H^1(M, \mathbb{Z}) \otimes \mathbb{Z}/2 \subseteq H^1(M, \mathbb{Z}/2)$ vanishes. In other words, $c' \in 2i^*(H^1(M, \mathbb{Z}))$.

The problem is to apply base point isotopies to $f_2$ to make $c'$ vanish. The strategy for doing so is typified in the example of $M = S^1 \times S^2$ presented with its genus 1 Heegaard surface. In this case, we can say by convention that a base point isotopy all the way around the upper circle increases both $c$ (defined using the first combings) and $c'$ by 1. A base point isotopy around a lower circle in the opposite direction decreases $c$ but increases $c'$. The two operations together do not change the first combing but change $c'$ by 2, or any even number if the operations are repeated, as desired. In general, the upper and lower circles generate Lagrangian subgroups $L_u$ and $L_l$ of $H_1(R, \mathbb{Z})$. If we identify $H_1(R, \mathbb{Z}) \cong H^1(R, \mathbb{Z})$ by Poincaré duality, we can say that $L_u$ and $L_l$ are subgroups of $H^1(R, \mathbb{Z})$ also, with $L_u \cap L_l = i^*(H^1(M, \mathbb{Z}))$. A general collection of base point isotopies around circles is represented by a pair $(c_u, c_l) \in L_u \oplus L_l$. These isotopies change the homology class $c$ by $c_u + c_l$ but $c'$ by $c_u - c_l$. Therefore taking $c_u = -c_l \in L_u \cap L_l$, we can find base point isotopies to make $c'$ vanish without changing $c = 0$. 

3. Hopf algebras

This section is a review of a number of results about finite-dimensional Hopf algebras [1, 11, 12]. Although the results are not new, the arguments here separate axiomatic manipulation from concrete considerations about finite-dimensional associative algebra, thereby clarifying and in some cases slightly extending the results.

It will be convenient to use arrow notation for computations with tensors [8], which is just a graphical version of index notation. Given a finite-dimensional vector space $V$ and a tensor $T \in V_1 \otimes V_2 \otimes \ldots \otimes V_n$, where each $V_i$ is either $V$ or $V^*$, we denote $T$ by its letter together with an incoming arrow for each tensor factor of $V$ and an outgoing arrow for each tensor factor of $V^*$. Each arrow represents a specific tensor factor, and the permutation of tensor factors is denoted by permuting their free ends. For example, the equation

$$\rightarrow g = \leftarrow g$$
means that \( g \) is a symmetric bilinear form on \( V \). Just as with index notation, the tensor product of two tensors is denoted by juxtaposing them; for example,

\[
\begin{align*}
\mathbf{a} & \rightarrow \\
\mathbf{a} & \rightarrow
\end{align*}
\]

\( \mathbf{a} \otimes \mathbf{a} \in V \otimes V \), where \( \mathbf{a} \in V \). A contraction of two tensor factors \( V \) and \( V^\ast \) is denoted by joining arrows head to tail; for example,

\[
\begin{align*}
\mathbf{v} & \rightarrow L \rightarrow \\
\end{align*}
\]

denotes the vector \( L(v) \in V \) for a linear transformation \( L \) and a vector \( v \). Finally, the two diagrams

\[
\begin{align*}
\text{\[\text{identity linear transformation}\]} & \\
\text{\[\text{dim } V\]} & \\
\end{align*}
\]
denote the identity linear transformation and \( \text{dim } V \) (the trace of the identity), respectively.

Arrow notation is equally valid in any other pivotal, symmetric tensor category, by which we mean a category in which tensor products and duals have all of the usual properties that they do in the category of finite-dimensional vector spaces. More specifically, a tensor category is a category with an associative tensor product operation \( \otimes \) on objects and morphisms; the category is symmetric if there is a canonical isomorphism \( V \otimes W \cong W \otimes V \) that yields an action of the symmetric group on \( V \otimes^n \); and it is pivotal if there is a canonical isomorphism \( V^{**} \cong V \). For example, \( V \) might be a (finite-dimensional) \( \mathbb{Z}/2 \)-graded vector space or super-vector space. In this instructive case, contractions and permutations of tensor factors obey sign rules. For example, \( \text{dim } V \), which is the value of an oriented circle, must be taken as the graded dimension, the dimension of the even part minus the dimension of the odd part. Since tensors in arrow notation can also be interpreted as morphisms in the category, they must be even-graded for the notation to make sense. If the notation were extended to odd-graded tensors, the sign of the value of a diagram would depend on the permutation sign of an ordering of its odd-graded letters. This sign ambiguity results from the difference between the “internal Hom” between two graded vector spaces, which consists of all linear transformations, and the “external Hom”, which consists of even-graded linear transformations only.

Arrow notation partially extends to tensor categories which are not pivotal, e.g., the category of finite- and infinite-dimensional vector spaces. In such categories a diagram in arrow notation is still well-defined if it is acyclic, i.e., if there is no closed loop in the diagram in which all arrows point in the same direction along the loop. In the discussion below, derivations involving acyclic diagrams are valid in these categories also.

Indeed, the invariant \( \#(M, H) \) generalizes to the case in which \( H \) is a Hopf object in a tensor category in which addition is not defined. For example, reference \( \text{\[\ref{\text{[8]}}\]} \) shows that if \( H \) is a universal, involutory Hopf object in a universal tensor category, then \( \#(M, H) \) is a complete invariant of closed, oriented 3-manifolds. On the other hand, we do not consider braided tensor categories, which are categories in which the diagrams are embedded in three dimensions and the arrows may be knotted or linked. Braided categories appear in the definition of the Reshetikhin-Turaev link and 3-manifold invariants and they are implicit in \( \#(M, H) \) via the quantum double, but they are not used in any direct way in the definition of \( \#(M, H) \). Henceforth we will use the phrase “tensor category” to mean a pivotal, symmetric tensor category unless explicitly stated otherwise.

A Hopf object \( H \) (in particular a finite-dimensional Hopf algebra) consists of two tensors

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.5ex]
    \node (m) at (0,0) {M};
    \node (a) at (1,0) {\Delta};
    \draw [->] (m) -- (a);
\end{tikzpicture}
\end{align*}
\]

called multiplication and comultiplication, respectively, that satisfy several axioms. The axioms postulate the existence of three other tensors: the unit \( i \), the co-unit \( \epsilon \), and the antipode \( S \).
Multiplication is associative and unital:

\[
\begin{align*}
\Delta(M) \cdot \Delta(M) &= \Delta(M) \\
i \cdot \Delta(M) &= \Delta(M) &= \epsilon
\end{align*}
\]

as is comultiplication:

\[
\begin{align*}
- \Delta \Delta &= - \Delta \Delta \\
- \Delta \epsilon &= - \Delta \epsilon &= \epsilon
\end{align*}
\]

The two tensors are related by the bialgebra axiom:

\[
\begin{align*}
\Delta(M) \cdot \Delta &= - \Delta \Delta \\
- \Delta \cdot \Delta &= - \Delta \cdot \Delta
\end{align*}
\]

and the axiom of the antipode:

\[
\begin{align*}
- \Delta \cdot S &= - \Delta \cdot S \\
\end{align*}
\]

Group algebras were already mentioned as examples of Hopf algebras in Section 1. In a group algebra,

\[
\Delta(g) = g \otimes g
\]

and \(S(g) = g^{-1}\) for \(g\) a group element. An important example of a Hopf object in the graded category is an exterior algebra. The exterior algebra \(\Lambda^*(V)\) over a finite-dimensional vector space \(V\) is \(\mathbb{Z}/2\)-graded by degree and its multiplication structure is given by the wedge product. Comultiplication is generated by the relation

\[
\Delta(v) = v \otimes 1 + 1 \otimes v
\]

for all \(v \in V\).

The following lemma, proved in reference [8, Lemma 3.2], establishes some basic properties of Hopf objects:

**Lemma 3.1.** The following identities hold in any Hopf object:

1. The tensors

\[
\begin{align*}
- M &- M &- M &- M \\
1 &1 &1 &1 \\
- \Delta &- \Delta &- \Delta &- \Delta \\
1 &1 &1 &1 \\
- M &- M &- M &- M \\
1 &1 &1 &1 \\
S &S &S &S \\
1 &1 &1 &1 \\
- \Delta &- \Delta &- \Delta &- \Delta
\end{align*}
\]
called ladders, are inverses. E.g.,

\[
\begin{array}{c}
\xrightarrow{-M \rightarrow M} \\
\uparrow \\
\xrightarrow{S} \\
\downarrow \\
\xrightarrow{-\Delta \rightarrow \Delta}
\end{array}
\]

- The counit is a multiplication homomorphism and the unit is a co-multiplication homomorphism:

\[
\begin{array}{c}
\xrightarrow{-M \rightarrow \varepsilon} \\
\xrightarrow{-\varepsilon} \\
\xleftarrow{i \rightarrow \Delta} \\
\xrightarrow{i \rightarrow \varepsilon}
\end{array}
\]

- The antipode is a multiplication and co-multiplication anti-endomorphism:

\[
\begin{array}{c}
\xrightarrow{-M \rightarrow S} \\
\xleftarrow{-S} \\
\xrightarrow{-\Delta \rightarrow S} \\
\xleftarrow{-S \rightarrow \Delta^\text{op}}
\end{array}
\]

- The antipode fixes the unit and the co-unit:

\[
\begin{array}{c}
\xrightarrow{i \rightarrow S} \\
\xleftarrow{i \rightarrow \varepsilon} \\
\xleftarrow{-S \rightarrow \varepsilon} \\
\xrightarrow{-\varepsilon}
\end{array}
\]

Lemma 3.1 uses the following shorthand for reversed multiplication and co-multiplication:

\[
\begin{array}{c}
\xrightarrow{M^\text{op}} \\
\xleftarrow{\Delta^\text{op}}
\end{array}
\]

In addition, the following abbrevations for multiplication or comultiplication of more than two things will be useful:

\[
\begin{array}{c}
\xrightarrow{M} \\
\xleftarrow{i}
\end{array}
\]

Lemma 3.2. The contraction \(\varepsilon(i)\) is 1:

\[
i \rightarrow \varepsilon = 1
\]

Before proving the lemma, we explain its meaning. The proof actually shows that

\[
i \rightarrow \varepsilon
\]

For vector spaces over a field, or in any reasonable category, it is possible to contract the identity tensor with some vector and some dual vector to obtain 1, which would establish the lemma as stated. Indeed, taking the tensor product with \(\varepsilon(i)\) has no effect on any diagram which has an
arrow in it. But in the abstract setting of a tensor category, it is possible that there are scalars which cannot be expressed as contractions, and then $\epsilon(i)$ is not necessarily 1 in the commutative semigroup of scalars. However, we can pass to an equally useful subcategory consisting of the ideal generated by $\epsilon(i)$; since $\epsilon(i)$ is an idempotent, it will be 1 in the subcategory. This subcategory must contain all diagrams with at least one arrow; in particular, it contains $M, \Delta, \text{etc.}$

**Proof:** We compute:

\[
\begin{array}{c}
\rightarrow = \xrightarrow{i} M \rightarrow \Delta \xleftarrow{\epsilon} = \xrightarrow{i - \Delta} M \rightarrow \epsilon \\
= \xrightarrow{-\Delta} M \rightarrow \\
\leftarrow \Delta \xrightarrow{\epsilon} \xleftarrow{i} M \rightarrow \\
\end{array}
\]

Left integrals $\mu_L$, right integrals $\mu_R$, left co-integrals $\epsilon_L$, and right co-integrals $\epsilon_R$ are tensors that satisfy the equations

\[
\begin{array}{c}
\rightarrow \Delta \xleftarrow{\mu_R} = \rightarrow \mu_R \xrightarrow{i} \rightarrow \Delta \xrightarrow{\mu_L} = \rightarrow \mu_L \xrightarrow{i} \\
\rightarrow \Delta \xrightarrow{\epsilon_R} \xrightarrow{M} = \rightarrow \epsilon \xrightarrow{\epsilon_L} \xrightarrow{M} = \rightarrow \epsilon \xrightarrow{\epsilon_L} \\
\end{array}
\]

Traditionally, integrals are dual vectors and co-integrals are vectors, but it will be convenient to allow them to be tensors of arbitrary type. For example, a tensor $T$ that satisfies

\[
\rightarrow \Delta \xleftarrow{T} = \rightarrow i \xrightarrow{T}
\]

is also a left integral.

**Lemma 3.3.** (Existence of integrals) The tensor

\[
\rightarrow \xrightarrow{P_R} = \rightarrow M \xrightarrow{S} \xleftarrow{\Delta} \xrightarrow{S}
\]

is both a right integral and a right co-integral and has trace 1.

**Proof:** Applying a ladder to

\[
\begin{array}{c}
\rightarrow \Delta \xleftarrow{S} \xrightarrow{M} \\
\xrightarrow{\Delta}
\end{array}
\]
we obtain

\[
\begin{align*}
\Delta &\to \Delta &\to S &\to M \\
\downarrow & &\downarrow & &\downarrow \\
M &\to \Delta &\to M &\to
\end{align*}
\]

\[
\begin{align*}
\Delta &\to S &\to \Delta &\to M \\
\Delta &\to M &\to \Delta &\to M &\to
\end{align*}
\]

The fact that \( P_R \) is a right integral follows by applying the inverse ladder given by Lemma 3.1. The argument that it is a right co-integral is the same with \( M \) and \( \Delta \) switched and with arrows reversed. The final claim that \( \text{Tr}(P_R) = 1 \) is left as an exercise.

\[\square\]

**Lemma 3.4.** Given a right integral \( \mu_R \) and a right co-integral \( e_R \),

\[
e_R \to \Delta \to M \to \mu_R = \mu_R \to S
\]

**Proof:** Applying a ladder, we compute:

\[
e_R \to \Delta \to M \to \mu_R = e_R \to \mu_R \to S
\]

**Corollary 3.5.** (Uniqueness of integrals) Given a right integral \( \mu_R \) and a right co-integral \( e_R \),

\[
\sigma = \text{Tr}(P_R) = 1
\]

The corollary follows from Lemma 3.4 by replacing the ladder on the left side by its inverse on the right side.

Lemma 3.5 suggests that \( P_R \) has a rank 1, \( i.e., \) that it factors as \( e_R \otimes \mu_R \) for some \( e_R \) and some \( \mu_R \). In the category of vector spaces, this is indeed the case, but in a general tensor category, there is a more subtle conclusion. Applying Corollary 3.5 twice, we obtain

\[
\begin{align*}
\sigma &\to P_R \to P_R \\
\sigma &\to P_R \to P_R
\end{align*}
\]

(The first equality is given by taking \( e_R = \mu_R = P_R \) in Corollary 3.5; the second is given by taking \( e_R \otimes \mu_R = P_R^2 \).) Let \( \sigma = \text{Tr}(P_R^2) \) be the scalar factor that appears. This is the transposition relation. Applying the transposition relation twice and using \( \text{Tr}(P_R) = 1 \), we obtain \( \sigma^2 = 1 \). If it
were the case that $\sigma = 1$, the transposition relation would say that the inward and outward arrows of copies of $P_R$ can be permuted separately, and it would therefore be valid to substitute

$$\rightarrow \mu_R \  e_R \leftarrow$$

for

$$\rightarrow P_R \leftarrow$$

in any diagram in which $P_R$ appears. (If necessary, we enlarge the tensor category in which we are working to factor $P_R$.) However, $\sigma = -1$ in many Hopf super-algebras, for example $\Lambda^*(V)$ when $V$ is odd-dimensional. When $\sigma = -1$, the transposition relation implies that $P_R = e_R \otimes \mu_R$ for some vectors $e_R$ and $\mu_R$, but it also implies that these vectors are odd-graded and are therefore not valid (external) morphisms in the category of odd-graded vector spaces. Nevertheless, it will be useful to extend arrow notation to factor $P_R$ in the general case. We define the notion of a formally odd-graded tensor: A diagram with a collection of odd-graded tensors is well-defined provided that the tensors are sign-ordered; if the sign ordering is switched, a factor of $\sigma$ arises. The tensor $P_R$ can be substituted by odd-graded tensors $e_R$ and $\mu_R$, and if there are the same number of each tensor, the sign-ordering can be indicated by dashed lines from each factor of $e_R$ to each factor of $\mu_R$, meaning that they are ordered in such a way that the factor of $\mu_R$ immediately follows the factor of $e_R$ that matches it, and that

$$\rightarrow \mu_R \  e_R \leftarrow = \sigma \rightarrow \mu_R \  e_R \leftarrow$$

As a rule, the sign-ordering can be inferred from context, and therefore it will sometimes be omitted below. By factoring $P_R$ and using the fact that its trace is 1, Lemma 3.3 can be rephrased as saying that there exists a right integral $\mu_R$ and a right co-integral $e_R$ such that

$$e_R \leftarrow \mu_R = 1$$

while Corollary 3.5 can be rephrased as

$$\rightarrow \mu'_R = \rightarrow \mu_R \ e_R \leftarrow \mu'_R$$

$$e'_R \leftarrow = e'_R \leftarrow \mu_R \ e_R \leftarrow$$

for any other right integral $\mu'_R$ and right co-integral $e'_R$.

Lemma 3.3 and Corollary 3.5 generalize to left integrals and left co-integrals by symmetry, taking

$$\rightarrow P_L \leftarrow = \rightarrow M \leftarrow \Delta \rightarrow$$
as a definition of $P_L$. *A priori*, this raises the possibility of two sign elements $\sigma_L$ and $\sigma_R$ and separate gradings for left and right integrals. Happily, $\text{Tr}(P_L^2) = \text{Tr}(P_R^2)$ by the following computation:

\[
\begin{align*}
\text{Tr}(P_L^2) &= \text{Tr}(P_R^2) \\
&= \text{Tr}(P_LP_R) = \text{Tr}(P_RP_L) \\
&= \text{Tr}(P_LP_R^\circ) = \text{Tr}(P_RP_L^\circ) \\
&= \text{Tr}(P_L^2) = \text{Tr}(P_R^2)
\end{align*}
\]

Therefore either trace can be called $\sigma$ and only one sign ordering is needed for all integrals and cointegrals. Unlike in the right integral case, we do not define $\mu_L$ and $e_L$ so that $P_L = \mu_L \otimes e_L$. Rather, the two quantities are proportional (as they must be by Corollary 3.5) in a way that will be convenient for normalization purposes.

Recall that a *group-like* element in a Hopf object is a vector $g$ such that

\[
g \to \Delta \xleftarrow{g} = g \to 1
\]

By Lemma 3.1b, $S(g)$ is also group-like:

\[
g \to S \to \Delta \xleftarrow{g} = g \to \Delta^\circ \xleftarrow{S} = g \to S \xleftarrow{S} = g \to S
\]

It follows from the axiom of the antipode that the antipode $S$ is an involution on group-like elements

\[
g \to S \xleftarrow{g} = g \to S^2 \xleftarrow{S} = g \to S \xleftarrow{\mu_R} = g \to S
\]

In a mixture of arrow notation with more standard algebraic notation, we will use $g^{-1}$ to mean $S(g)$ and $gh$ and $g^n$ to mean

\[
gh \xleftarrow{g} M \xrightarrow{h} = g \xrightarrow{g} h \xrightarrow{M} = g \xrightarrow{g} h \xrightarrow{M}
\]

for group-like elements $g$ and $h$ and $n$ an integer.

The tensor

\[
\text{Tr}(\Delta) \xleftarrow{\mu_R}
\]
is a right integral, because

\[ \Delta \mu_R = \Delta \mu_R \]

By Corollary 3.5,

\[ \Delta \mu_R = \mu_R e_R \Delta \mu_R \]

Moreover,

\[ e_R \Delta \mu_R = e_R \Delta \mu_R = e_R \Delta \mu_R e_R \Delta \mu_R \]

This means that

\[ a = e_R \Delta \mu_R \]

is a group-like element. The vector \( a \) is the phase element, and it generates the modular subgroup of the Hopf object \( H \). We similarly define the dual phase element \( \alpha \) as

\[ \alpha = e_R M \mu_R \]

It is group-like in \( H^* \). We define

\[ -\mu_L = a \sigma M \mu_R \quad e_L = e_R -\Delta \alpha \]

By Lemma 3.4, \( \mu_L \) is a left integral:

\[ \Delta \sigma M \mu_R = -\Delta M \mu_R = a -\Delta M \mu_R \]

and since \( a^{-1} \) exists, multiplication by \( a \) is invertible. The vector \( e_L \) is similarly a left co-integral. We define

\[ q = a -\alpha \]

In general, \( \alpha^n(a^k) = q^{nk} \). We verify that

\[ e_L -\mu_L = e_R -\Delta M \mu_R = a -\Delta M \alpha = q^{-1} \]

(Compare with the relation \( \mu_R(e_R) = 1 \).) Also,

\[ e_R -\mu_L = 1 = e_L -\mu_R \]
and
\[ e_r \mapsto \Delta \xrightarrow{\alpha} e_r \mapsto \Delta \xrightarrow{\alpha} e_t \mapsto q e_t \mapsto \]

\[ a \mapsto \mu_r = \xrightarrow{\alpha} \mu_l \quad a \mapsto \mu_r = \xrightarrow{\alpha} \mu_l \quad q \]

In conclusion,

**Lemma 3.6.** The vector \( a \) and the dual vector \( \alpha \) are group-like, the vector \( e_L \) and the dual vector \( \mu_L \) are a left integral and a left co-integral, and \( \mu_L(e_L) = q^{-1} \).

We turn to the subject of Hopf object dualities. The most important such duality, which we have already used, is the one that switches \( H = (M, \Delta) \) with its dual \( H^* = (\Delta, M) \). It is convenient to realize this duality by reversing arrows and reflecting diagrams about an axis, usually vertical:

\[ \xrightarrow{\mu_L} \quad \iff \quad \xrightarrow{\epsilon} \quad \iff \quad \xrightarrow{\mu_L} \]

\[ \xrightarrow{S} \quad \iff \quad \xrightarrow{S} \quad \xrightarrow{\mu} \quad \iff \quad \xrightarrow{\alpha} \]

In addition, it is easy to check that the pairs \((M^{op}, \Delta), (M, \Delta^{op})\), and \((M^{op}, \Delta^{op})\) form unital, counital bialgebras, and that the third pair is a Hopf algebra with antipode \( S \). Call these objects \( H^{op}, H^{cop}, \) and \( H^{op,cop} \), respectively. If either of \( H^{op} \) or \( H^{cop} \) were a Hopf algebra with antipode \( S' \), then \( S' = S^{-1} \), by the following derivation:

\[ \xrightarrow{\Delta} \quad \xrightarrow{S'} \quad \xrightarrow{S} \quad M \mapsto \xrightarrow{S'} \quad \xrightarrow{S} \quad M^{op} \mapsto \xrightarrow{S} \quad S \mapsto \]

\[ \xrightarrow{\epsilon} \quad i \mapsto \xrightarrow{S} \quad = \quad \xrightarrow{\Delta} \quad \xrightarrow{S} \quad M \mapsto \]

If we set

\[ \xrightarrow{\mu} \quad \xrightarrow{\mu} \quad \xrightarrow{\epsilon} \quad i \mapsto \xrightarrow{S} \quad = \quad \xrightarrow{\Delta} \quad \xrightarrow{S} \quad M \mapsto \]

then \( S' \) is the antipode for \( H^{op} \) and \( H^{cop} \) by the same argument as that of Lemma 3.4. Therefore,

**Lemma 3.7.** The antipode \( S \) is invertible.

Another formula for \( S' \) is

\[ \xrightarrow{\mu} \quad \xrightarrow{\mu} \quad \xrightarrow{\epsilon} \quad i \mapsto \xrightarrow{S} \quad = \quad \xrightarrow{\Delta} \quad \xrightarrow{S} \quad M \mapsto \mu_l \]

and \( S \) has a second formula as well:

\[ \xrightarrow{\mu} \quad \xrightarrow{\mu} \quad \xrightarrow{\epsilon} \quad i \mapsto \xrightarrow{S} \quad = \quad q e_l \mapsto \Delta \xrightarrow{\mu} M \mapsto \mu_l \]

and

\[ e_r \mapsto \Delta \xrightarrow{\alpha} e_r \mapsto \Delta \xrightarrow{\alpha} e_t \mapsto q e_t \mapsto \]

\[ a \mapsto \mu_r = \xrightarrow{\alpha} \mu_l \quad a \mapsto \mu_r = \xrightarrow{\alpha} \mu_l \quad q \]
Note that the existence of integrals and co-integrals is the first result which depends on the fact that the tensor category is pivotal. Contrariwise, suppose that \( H \) is defined in a non-pivotal category but that it has a left integral \( \mu_L \) and a right co-integral \( e_R \) such that \( \mu_L(e_R) = 1 \). Then since:

\[
\begin{align*}
 e_r &\to \Delta \to S \to \\
 &\to M \to \mu_l
\end{align*}
\]

any arrow in a diagram can be replaced by a tensor in which there is no directed path from the tail to the head. Therefore any diagram with cycles can be understood as an equivalent acyclic diagram, which establishes a pivotal structure for the tensor category containing \( H \) (more precisely, for the full subcategory whose objects are \( H^\otimes n \)).

**Lemma 3.8.** The antipode \( S \) has the following action on integrals and co-integrals:

\[
\begin{align*}
 - S \to \mu_l &= - \mu_r \sigma \\
 e_l \to S \to &= \sigma e_r \to \\
 e_r \to S \to &= q \sigma e_L \to \\
 e_L \to \Delta &\to S \to \mu_r \\
 e_r \to \mu_r &\to = q \sigma e_L \to
\end{align*}
\]

In particular, all integrals and co-integrals are eigenvectors of \( S^2 \) with eigenvalue \( q \).

**Proof:** We compute the effect of \( S \) on \( e_L \):

\[
\begin{align*}
 e_L \to \Delta \to M \to \mu_r \\
 e_L \to \mu_r &\to = q \sigma e_L \to
\end{align*}
\]

The argument is the same in the other three cases. \( \square \)

For involutory Hopf objects with invertible dimension, the trace of the regular representation is a right integral \( \mu_R \). In particular, \( \mu_R(AB) = \mu_R(BA) \) in this case. In the general case, \( \mu_R \) has a property similar to that of a trace:

**Lemma 3.9.** The tensors \( \mu_R \) and \( e_R \) satisfy:

\[
\begin{align*}
 e_r \to \Delta &\to M \to \mu_r \\
 e_r \to \mu_r &\to = \frac{\alpha}{q} e_r \to \Delta \\
 e_r \to \Delta &\to S \to \mu_r \\
 e_r \to \mu_r &\to = \frac{\alpha}{q} e_r \to \Delta \\
 e_r \to \Delta &\to S \to \\
 e_r \to \mu_r &\to = \frac{\alpha}{q} e_r \to \Delta \\
 e_r \to \Delta &\to M \to \mu_r
\end{align*}
\]

**Proof:** We give the proof for \( \mu_R \). Using expressions for \( S \) and \( S^{-1} \) and a formula for \( e_L \),

\[
\begin{align*}
 e_r \to \Delta &\to M \to \mu_r \\
 e_r \to \Delta &\to M \to \mu_r \\
 e_r \to \Delta &\to M \to \mu_r \\
 e_r \to \Delta &\to M \to \mu_r \\
 e_r \to \Delta &\to M \to \mu_r
\end{align*}
\]

Since \( S \) is invertible, the tensor

\[
 e_r \to \Delta
\]
is also and we can remove it from both sides. Applying $S^{-2}$ to both inward arrows on both sides then yields the desired result, using the fact that $\mu_R$ is an eigenvector with eigenvalue $q^{-1}$.

We define

$$\mu_{n-1/2} = a^n M \rightarrow \mu_R$$

for any integer $n$. Note that $\mu_{1/2} = \mu_L$, $\mu_{-1/2} = \mu_R$, $e_{1/2} = e_L$, and $e_{-1/2} = e_R$.

**Theorem 3.10** (Radford).

\[ a \quad \alpha \quad \rightarrow M \rightarrow \Delta \quad = \quad \rightarrow S^4 \rightarrow \\]

or, equivalently, $Ad^*_{\alpha} \circ Ad_a = S^4$.

**Proof:** First express $S^2$ as

\[ \rightarrow S^2 \rightarrow \quad = \quad q \quad e_\Delta \rightarrow \Delta \quad = \quad \mu L \quad \rightarrow \mu R \quad \rightarrow \Delta \rightarrow \Delta \quad M \rightarrow \mu R \]

using both formulas for $S$ and substitute for $S^{-1}$ in the middle. Applying $Ad^*_{\alpha-1}$, we obtain

\[ e_\Delta \rightarrow \Delta \rightarrow \Delta \rightarrow \Delta \rightarrow M \rightarrow \mu R \]

Applying $Ad_{\alpha-1}$, we obtain

\[ a \quad \alpha \quad \rightarrow M \rightarrow S^2 \rightarrow \Delta \quad = \quad e_\Delta \rightarrow \Delta \rightarrow \Delta \rightarrow \Delta \rightarrow M \rightarrow \mu R \quad = \quad \rightarrow S^2 \rightarrow \]

The result follows from the fact that the operators $Ad^*_{\alpha}$ and $Ad_a$ commute with $S^2$ (check).
and $\alpha$ have finite order. Since $\text{Ad}_a^*$ and $\text{Ad}_a$ commute with each other as well as with $S^2$, it follows that $S$ has finite order.

A Hopf object $H$ is balanced if $\text{Ad}_a = S^2$, which by Theorem 3.10 means that $H^*$ is also balanced. In the unbalanced case, define

$$\rightarrow T \rightarrow = - S \rightarrow \Delta \rightarrow$$

The tensor $T$ is the tilt of $H$, since it measures the extent to which $H$ fails to be balanced.

**Lemma 3.11.** The tilt map $T$ is an automorphism of $H$ that fixes integrals and co-integrals.

The proof is left as an exercise.

3.1. Quantum groups. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{g}^+$ be a Borel subalgebra. The universal enveloping algebras $U(\mathfrak{g})$ and $U(\mathfrak{g}^+)$ are Hopf algebras that admit deformations $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g}^+)$, where $q$ can be a non-zero complex number or an indeterminate $[5, 10]$. (It will appear as if $U_q(\mathfrak{g}^+)$ depends on a choice of $q^{1/2}$, but the dependence disappears with a slightly different choice of generators. Moreover, our convention is consistent with the definition $q = \alpha(a)$ given above.) Furthermore, if $q$ is a root of unity, these Hopf algebras admit finite-dimensional quotients $u_q(\mathfrak{g})$ and $u_q(\mathfrak{g}^+)$. Each of these four deformations has an established role in theory of quantum topological invariants $[14, 15]$. As mentioned in the introduction, the invariant $\#(M, u_q(\mathfrak{g}^+))$ is an important example of $\#(M, H)$. In light of this use, the goal of this section is the following lemma:

**Lemma 3.12.** The Hopf algebra $u_q(\mathfrak{g}^+)$ is balanced.

Before proving the lemma, we give a quick definition of $u_q(\mathfrak{g}^+)$ (which in any case varies in the literature): Let $\mathfrak{g}$ have rank $n$, let $\alpha_1, \ldots, \alpha_n$ be the simple roots of the root system of $\mathfrak{g}$, let $(\cdot, \cdot)$ be the dual Killing form, and let $\langle \alpha, \beta \rangle = (\alpha, \beta)/(\beta, \beta)$, so that $\langle \alpha_i, \alpha_j \rangle$ is the Cartan matrix. Let $r$ be the order of $q$, and choose a square root $q^{1/2}$. Then $u_q(\mathfrak{g}^+)$ is generated as a unital algebra by the generators $E_i$ and $K_i$ for $1 \leq i \leq n$ with the relations

$$K_i K_j = K_j K_i,$$
$$K_i E_j = q^{\langle \alpha_i, \alpha_j \rangle / 2} E_j K_i,$$
$$\text{Ad}_\Delta(E_i)^{1 - \langle \alpha_j, \alpha_i \rangle}(E_j) = 0,$$
$$K_i^{2r} = 1,$$
$$E_i^r = 0,$$

where $\text{Ad}_\Delta(X)$ is the linear operator defined as:

$$X \rightarrow \Delta \rightarrow \rightarrow M \rightarrow$$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$S(E_i) = - K_i^{-1} E_i,$$

and

$$S(K_i) = K_i^{-1}.$$
(The trick of expressing the quantum Serre relations in terms of the quantum adjoint was suggested to the author by Marc Rosso.) The rest of co-multiplication is generated as an algebra homomorphism from this relation and

\[ \Delta(K_i) = K_i \otimes K_i. \]

Similarly, the rest of the antipode map is generated as an algebra anti-automorphism. The unit is just written as 1. The counit \( \epsilon \) is generated as an algebra homomorphism by

\[ \epsilon(E_i) = 0, \]
\[ \epsilon(K_i) = 1. \]

**Proof:** (Sketch) The elements \( K_i \) are group-like and generate a subgroup \( C \) of \( U_q(\mathfrak{g}^+) \); the group algebra \( \mathbb{C}[C] \) is a Hopf subalgebra. There is a sequence \( i_1, \ldots, i_k \), where \( k \) is the number of positive roots of \( \mathfrak{g} \), such that

\[ E_{\text{max}} = E_{i_1} E_{i_2} \cdots E_{i_k} \]

is non-zero, but that, for any such sequence, \( E_{\text{max}} E_i = 0 \) for all \( i \). Moreover, each \( i \) appears \( m_i \) times as some \( i_j \), where \( 2\rho = \sum_i m_i \alpha_i \) is the sum of the positive roots of \( \mathfrak{g} \). Then

\[ e_L = \left( \sum_{K \in C} K \right) E_{\text{max}} \]

is clearly a left co-integral, and similarly

\[ e_R = E_{\text{max}} \left( \sum_{K \in C} K \right) \]

is a right co-integral. (We omit formulas for \( \mu_L \) and \( \mu_R \), since we will not use them here.) If we define \( \alpha \) as usual by

\[ X e_R = \alpha(X) e_R, \]

then

\[ \alpha(E_i) = 0 \]

and

\[ \alpha(K_i) = q^{(\rho, \alpha_i)} = q^{- \langle \alpha_i, \alpha_i \rangle}, \]

since \( \langle \rho, \alpha_i \rangle = 1 \) for all \( i \). We now compute the effect of \( \text{Ad}^*_\alpha \), which is an algebra automorphism, on \( E_i \) and \( K_i \):

\[ \text{Ad}^*_\alpha(K_i) = (\alpha \otimes I \otimes \alpha^{-1})(\Delta_3(K_i)) = \alpha(K_i) \otimes K_i \otimes \alpha^{-1}(K_i) = K_i, \]

while

\[ \text{Ad}^*_\alpha(E_i) = (\alpha \otimes I \otimes \alpha^{-1})(\Delta_3(E_i)) = \alpha(E_i) \otimes 1 \otimes 1 + \alpha(K_i) \otimes E_i \otimes 1 + \alpha(K_i) \otimes K_i \otimes \alpha^{-1}(E_i) = q^{- \langle \alpha_i, \alpha_i \rangle} E_i. \]

Here \( \Delta_3(X) \) is the triple co-product, meaning

\[ \Delta(X) = \Delta. \]

This value agrees with

\[ S^2(E_i) = K_i^{-1} E_i K_i = q^{- \langle \alpha_i, \alpha_i \rangle} E_i. \]

Since \( \text{Ad}^*_\alpha \) and \( S^2 \) agree on generators of \( U_q(\mathfrak{g}^+) \) and they are both algebra automorphisms, the two operators agree always and \( U_q(\mathfrak{g}^+) \) is balanced, as desired. \( \square \)
4. The invariant

Let $M$ be a closed 3-manifold with a minimal Heegaard diagram $D$. Orient all circles and give them the canonical sign-ordering relative to the orientation. Let $f = (b_1, b_2)$ be a framing of $D$. Recall the definition of $\theta(c)$ and $\phi(c)$ for Heegaard circles $c$. For each point $p$ on some circle $c$ of $D$ with base point $o$, we also define $\theta(p)$ be the counterclockwise rotation of the tangent to $c$ relative to $b_1$ from $o$ to $p$, as before in units of $1 = 360^\circ$. If $p$ is a crossing, then two rotation angles are defined; call them $\theta_l(p)$ and $\theta_u(p)$. Arrange the circles so that upper and lower circles are not only transverse, but also orthogonal when they cross, so that $\theta_l(p) - \theta_u(p)$ is always $\frac{n}{2} + \frac{1}{4}$ for some integer $n$. Let $\phi(p)$ be the total right-handed twist of $b_2$ around $b_1$ from $o$ to $p$, and similarly define $\phi_u(p)$ and $\phi_l(p)$. Using twist fronts, $\phi(p)$ can be computed as the total sign of all fronts crossed from $o$ to $p$, not counting the front that terminates at $o$ itself.

Let $H$ be a Hopf object in a pivotal tensor category. Let $\mu_R, \mu_L, e_R,$ and $e_L$ be integrals and co-integrals of $H$ such that $\mu_R(e_R) = \mu_R(e_L) = \mu_L(e_R) = 1$. Recall also the tensors, $a, \alpha,$ and $T$ associated to $H$ and the scalars $\sigma$ and $q$.

Define the quantity $\#(D, H)$ as follows: Replace each upper circle $c$ with an $M$ tensor with one inward arrow for each crossing and the outward arrow with an integral at the base point, with the arrows ordered as indicated:

Here $n = -\theta(c)$. Replace each lower circle $c$ with a $\Delta$ tensor with an outward arrow for each crossing and the inward arrow with a cointegral at the base point, with the arrows ordered as indicated:

Here $n = \theta(c)$. Since there is an integral or a co-integral for each circle, give them the same sign-ordering as that of the circles. Replace each crossing by the tensor:

$$\to S^\alpha T^k \to$$

where $n = 2(\theta_l(p) - \theta_u(p)) - \frac{1}{2}$, $k = \phi_l(p) - \phi_u(p)$, and $p$ is the crossing point. Finally, contract all tensors corresponding to circles and crossings according to incidence.

Theorem 4.1. The quantity $\#(D, H)$ depends only on $M$ and its framing.

Proof: We demonstrate invariance or covariance under each type of diagram move. For most of the moves, we use the duality between $H$ and $H^*$ to cut the calculation in half.

- Orientation reversal. If the orientation of a lower circle $c$ is reversed, $n = \theta(c)$ is subtracted from $\theta_l(p)$ for each crossing $p$ on $c$ and then $\theta(c)$ is negated. The crossings on $c$ reverse order,
and the sign-ordering of all circles also reverses. Therefore a tensor for a lower circle such as:

\[ e_n \Delta \rightarrow \]

is replaced with

\[ e_n \Delta^\text{op} \rightarrow S^{2n} \]

By Lemma 3.8, these two are equal.

- **Rotation front spiral.** This move has the effect of changing \( \phi_l(p) \) or \( \phi_u(p) \) by one for all crossings on an upper or lower circle. A tensor for a lower circle such as

\[ e_n \Delta \rightarrow \]

might change to

\[ T \rightarrow \]

These are equal because \( T \) is an automorphism of all of the intrinsic structure of \( H \).

- **Two-point isotopy.** After appropriate orientation reversal, a digon which is the starting point of a two-point isotopy move can look like:

The corresponding tensor is

\[ \rightarrow \Delta \rightarrow S^{2n+1} \rightarrow M \rightarrow = S^{2n} \rightarrow \varepsilon \rightarrow i \rightarrow S^{2n} \rightarrow = \varepsilon \rightarrow i \rightarrow \]

for some \( n \), which is equivalent to the tensor after the move by the computation shown.

- **Base point isotopy.** For simplicity, we assume a rotation of a lower circle \( c_l \) that moves its base point past a crossing \( p \) with an upper circle \( c_u \), such that the identity tensor \( I = S^0T^0 \)
is assigned to $p$. Following Figure 18, the relevant piece of $\#(M, H)$ might be:

\[
e_R \Delta \xrightarrow{\alpha^n} M \xrightarrow{\mu_R} = e_R \Delta \xrightarrow{\alpha^n} a \xrightarrow{S^2} M \xrightarrow{\mu_R}
\]

\[
e_R \Delta \xrightarrow{T^{n-1}S^{2n}} a \xrightarrow{S^2T^{-1}} M \xrightarrow{\mu_R}
\]

where $n = \theta(c_1) + \frac{1}{2}$ and $k = \frac{1}{2} - \theta(c_a)$. The computation shown uses Lemma 3.9. The final expression matches the result of the rotation, because $\phi_l(p)$ changes from 0 to $n - 1$, $\theta_l(p)$ changes from 0 to $n - \frac{1}{2}$, $\theta_u(p)$ decreases by $\frac{1}{2}$, and $\theta(c_a)$ and $\theta(q)$ decrease by one for every point $q$ after $p$ on $c_a$ (and similarly $\phi(c_a)$ and $\phi(q)$ change in the opposite direction).

- Circle slide. A circle slide might typically look like this:

\[
\text{Circle slide:}
\]

\[
\text{Diagram of circle slide:}
\]
We again assume for simplicity that all crossings before the move have tensor \( I = S^0T^0 \). Then the relevant piece of \( \#(M, H) \) before the move is:

\[
\begin{array}{c}
\vdots \\
\rightarrow M \xrightarrow{\mu_R} M \\
\text{a} & \rightarrow M \xrightarrow{\Delta} M \\
\rightarrow \Delta \\
\rightarrow \Delta \\
\text{a} & \rightarrow M \\
\end{array}
\]

where \( c_1 \) slides past \( c_2 \) and \( n = \frac{1}{2} - \theta(c_2) \). The derivation given arrives at the result of the handle slide move, except for one omitted step. Just as with the circle rotation move, a factor of \( a^n \) appears which is in the wrong position. Conjugating by \( a^n \) yields \( S^{2n}T^{-n} \) tensors. These tensors arise in the move because \( \theta(p) \) decreases and \( \phi(p) \) increases by \( n \) for all points \( p \) along \( c_2 \) after the position of the circle slide. In the given example, \( n = 0 \).

- The stabilization move yields a scalar factor which equals 1:

\[
e_R = \mu_k = 1
\]

- Base point spiral: A clockwise spiral on a lower circle \( c \) increases \( \theta_1(p) \) by one at every crossing of \( p \in c \). This brings in a factor of \( S^2 \), and by Lemma 3.8, the invariant gains a factor of \( q \). In general, changing the Hopf degree of the framing of \( M \) by \( n \) changes \( \#(M, H) \) by \( q^n \).

**4.1. The combed invariant.** The definition of \( \#(M, H) \) for \( H \) balanced and \( M \) combed is in fact exactly the same as for \( H \) arbitrary and \( M \) framed. Since the tilt map \( T \) is the identity, the second combing \( b_2 \) of a framing \( f = (b_1, b_2) \) is irrelevant. In this case, all constructions of Section 4 are valid if \( b_1 \) is a combing that does not extend to a framing, with the conclusion that \( \#(M, H) \) is defined even when the combing of \( M \) has non-zero Euler class.

**5. Properties and examples**

The manifold \( \mathbb{R}P^3 \), if interpreted as the group \( \text{SO}(3) \), has two natural combings: The left-invariant combing and the right-invariant combing. These two combings differ in their characteristic class, but not in their Euler class; since \( H^2(\mathbb{R}P^3, \mathbb{Z}) \) is entirely 2-torsion, all combings extend to framings. (One consequence is the well-known fact that the left-invariant and right-invariant framings of \( \mathbb{R}P^3 \) differ in spin structure.) Indeed, we could just as well say that \( \mathbb{R}P^3 \) is framed rather than combed. A simple computation using the genus 1 Heegaard diagram of \( \mathbb{R}P^3 \) demonstrates that

\[
\#(\mathbb{R}P^3, H) = \text{Tr}(S)
\]

with one combing for any \( H \), and that

\[
\#(\mathbb{R}P^3, H) = \text{Tr}(S^{-1})
\]
for the other combing. If \( H = u_q(\text{sl}(2)^+) \), the basis
\[
\{ K^j(\text{K}^{-1}E^2)^k, K^j(\text{K}^{-1}E^2)^k E \}
\]
is convenient, because \( S \) permutes the basis elements up to scalar factors. The only basis elements that are (projectively) fixed are those of the form \((\text{K}^{-1}E^2)^k\) and \(K^r(\text{K}^{-1}E^2)^k\) with \(0 < k < r/2\).

The result is that
\[
\text{Tr}(S) = 2 \frac{1 - q^{-\lfloor \frac{k}{n} \rfloor}}{1 - q^{-1}}
\]
and
\[
\text{Tr}(S^{-1}) = 2 \frac{1 - q^{\lfloor \frac{k}{n} \rfloor}}{1 - q}.
\]
These values are similar to values for the Reshetikhin-Turaev 3-manifold invariants, but not quite the same.

The manifold \( S^2 \times S^1 \) has many different combings, and in general the value of the invariant is
\[
\#(S^2 \times S^1, H) = \mu_R(a^n)e_R(a^k)
\]
for some \( n \) and \( k \). If \( q \neq 1 \), this expression must be zero. When \( n = k = 0 \), \( \#(S^2 \times S^1, H) \) reduces to \( \text{Tr}(S^2) \), which is non-zero for Hopf algebras precisely when \( H \) is semisimple and co-semisimple.

Furthermore, in characteristic 0, it is known that \( H \) is semisimple if and only if it is involutory [9].

A connected sum \( M_1 \# M_2 \) of two 3-manifolds has a Heegaard diagram which is also a connected sum; the Heegaard circles of the two pieces are disjoint. With suitable combings or framings,
\[
\#(M_1 \# M_2, H) = \#(M_1, H)\#(M_2, H).
\]

The symmetries in the definition of \( \#(M, H) \) immediately yield the identities:
\[
\#(M, H) = \#(-M, H^{op}) = \#(-M, H^{cop})
\]
and
\[
\#(M, H^*) = \#(M, H),
\]
where \(-M \) is \( M \) with reversed orientation.

Suppose that a combing of \( M \) has divisor \( d \) but \( q^d \neq 1 \), where as before \( q = \alpha(a) \) for some Hopf algebra \( H \). Then \( \#(M, H) = 0 \), because on the one hand, changing the degree of the combing by \( d \) changes \( \#(M, H) \) by \( q^d \), but on the other hand, it does not change the combing at all.

Since \( \Lambda^*(\mathbb{C}) \) is commutative and co-commutative in the graded sense, \( \#(M, \Lambda^*(\mathbb{C})) \) does not depend on the order of the crossings on each upper and lower Heegaard circle. Therefore \( \#(M, \Lambda^*(\mathbb{C})) \) can only depend on the intersection matrix between the upper and lower crossings. Indeed, it is the determinant of this matrix up to sign; and the sign-ordering normalization ensures that \( \#(M, \Lambda^*(\mathbb{C})) \) is non-negative. Therefore
\[
\#(M, \Lambda^*(\mathbb{C})) = |H_1(M, \mathbb{Z})|
\]
when the right side is finite, and
\[
\#(M, \Lambda^*(\mathbb{C})) = 0
\]
otherwise.

If a finite group \( G \) acts by automorphisms on a Hopf algebra or super-algebra \( H \), then \( G \rtimes H \) is well defined as a Hopf algebra semi-direct product. In particular, if \( V \) is a linear representation of \( G \), then \( G \rtimes \Lambda^*(V) \) is an interesting Hopf super-algebra. Unfortunately, \( \#(M, G \rtimes \Lambda^*(V)) \) is not interesting. It expands as a sum over homomorphisms from \( \pi_1(M) \) to \( G \). If a given homomorphism \( \pi_1(M) \rightarrow G \) is composed with the representation \( G \rightarrow \text{End}(V) \), the result is a flat vector bundle \( E \) over \( M \) with fiber \( V \) at the base point. The corresponding term in \( \#(M, G \rtimes \Lambda^*(V)) \) is the determinant of the middle term of the chain complex
\[
0 \rightarrow C_3(M, E) \rightarrow C_2(M, E) \rightarrow C_1(M, E) \rightarrow C_0(M, E) \rightarrow 0.
\]
Unless the homomorphism $\pi_1(M) \to G$ is trivial, this determinant is 0. The determinant of the whole complex is the Reidemeister torsion of $E$ and is an interesting invariant, but it does not appear in $\#(M, G \ltimes \Lambda^*(V))$. Thus, $\#(M, G \ltimes \Lambda^*(V))$ almost sums the Reidemeister torsion of $M$ over linear representations of $\pi_1(M)$ into $V$ that factor through $G$, but instead evaluates to $\#(M, \Lambda^*(\mathbb{C}))$.

There are several reasons to believe that there is a generalization of $\#(M, H)$ that involves the 0-cells and the 3-cells of a cell decomposition of $M$ rather than just the edges and faces:

- There might be an analogue of the non-reduced cohomology set $H^1(M, G)$, as well as the weighted enumeration of $H^1(M, G)$ considered by Dijkgraaf and Witten [4].
- The invariant $\#(M, G \ltimes \Lambda^*(V))$ comes close to an interesting, classically considered invariant.
- The invariant $\#(M, H)$ is not always a TQFT. In particular, $\#(M, u_q(\mathfrak{g}^+))$ is not a Jones-Witten TQFT.

The treatment of $\#(M, H)$ in this paper is from the point of view of deriving information about 3-manifolds using Hopf algebras and Hopf objects. An equally useful point of view, which we did not address, is the converse: What can we learn about Hopf algebras using 3-manifolds? Reference [8] establishes a noteworthy result in this direction: Two expressions in an abstract involutory Hopf object (or in the universal involutory Hopf object) are axiomatically equal if and only if two related 3-manifolds with boundary are homeomorphic. It would be interesting to extend this result to arbitrary Hopf objects. Such an extension would involve interpreting the axioms of a Hopf object as moves on some kind of topological object, probably a suitably decorated 3-manifold.

One of the merits of $\#(M, H)$ in the involutory case is that the definition is particularly simple. Unfortunately, the general definition of $\#(M, H)$ has been a disappointment by comparison. Perhaps if the axioms of a Hopf object were properly understood topologically, it would lead to a simpler definition of $\#(M, H)$.

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