Renormalisation group, trace anomaly and Feynman–Hellmann theorem

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We show that the logarithmic derivative of the gauge coupling on the hadronic mass and the cosmological constant term of a gauge theory are related to the gluon condensate of the hadron and the vacuum respectively. These relations are akin to Feynman–Hellmann relations whose derivation for the case at hand is complicated by the construction of the gauge theory Hamiltonian. We bypass this problem by using a renormalisation group equation for composite operators and the trace anomaly. The relations serve as possible definitions of the gluon condensates themselves which are plagued in direct approaches by power divergences. In turn these results might help to determine the contribution of the QCD phase transition to the cosmological constant and test speculative ideas.

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1. Introduction

The Feynman–Hellmann theorem relates the leading order variation of the energy to a local matrix element, providing a direct link between an observable and a theoretical quantity. Originally derived in quantum mechanics, its application to quantum field theory (QFT) is generally straightforward and widely used, see e.g. [1]. Exceptions are cases where the Hamiltonian is difficult to construct, which may arise as a QFT is usually defined from a Lagrangian where (most) symmetries are manifest. An example of such a case are gauge theories where the elimination of two degrees of freedom from the four vector $A_\mu$ is at the root of the problem. In this work we bypass the construction of the gauge field part of the Hamiltonian \(^1\) by using renormalisation group equations (RGE) for composite operators as well as the trace anomaly. We obtain a relation that relates the logarithmic derivative of the hadron mass with respect to the coupling constant, and the gluon condensate of the hadron state. Likewise we find a similar relation relating the derivative of the cosmological constant and the vacuum gluon condensate.

The corresponding relation for the quark mass was used in Ref. [3] to derive the leading scaling behaviour of the hadronic masses for a non-trivial infrared (IR) fixed point, deformed by the fermion mass parameter. The relation derived in this paper is used to compute the scaling corrections to the hadronic mass spectrum [4].

2. Preliminary results

We shall first rederive some results before assembling them to obtain the main relations of this paper.

2.1. RGE for matrix elements of local composite operators

In this section we outline the derivation of the standard RGE for local operator matrix elements on physical states which can be found in reference textbooks; e.g. [5,6]. We begin by defining the relation between the bare operator $O_i$ and the renormalised operator $\bar{O}_i$

$$O_i(g, m, \Lambda) = (\tilde{Z}_O)_{ij}^{-1}(\mu/\Lambda)\bar{O}_j(\bar{g}, \bar{m}, \mu).$$

(1)

where summation over indices is implied, $(g, m)$ are the bare gauge coupling and mass, $\Lambda$ is the UV cut-off of the theory, $(\bar{g}, \bar{m})$ are the set of renormalised couplings and $\mu$ is the renormalisation scale. As stated above the operators are understood to be evaluated between two physical states in order to avoid the issue of contact terms which arises upon insertion of additional operators. From the independence of the bare operator on the renormalisation scale,

$$\frac{d}{d\ln\mu} O_i(g, m, \Lambda) = 0,$$

(2)
one obtains an RGE of the form,
\[
\left( \frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial g} - \bar{m} \gamma_m \frac{\partial}{\partial m} + (\gamma_0)_{ij} \right) \bar{O}_j(\bar{g}, \bar{m}, \mu) = 0
\]
with
\[
\beta \equiv \frac{d \bar{g}}{d \ln \mu}, \quad \gamma_m = -\frac{d \ln \bar{m}}{d \ln \mu}
\]
\[
(\gamma_0)_{ij} = -\langle Z^0_0 \rangle_\mu \frac{d \langle Z_0 \rangle_{ij}}{d \ln \mu}.
\]
(4)

Denoting by \(d_{O_i} \equiv d_{\bar{O}_i}\) the engineering dimension of \(O_i\) one gets by dimensional analysis
\[
\left( \frac{\partial}{\partial \ln \mu} + \bar{m} \frac{\partial}{\partial m} - d_{O_i} \right) \bar{O}_j(\bar{g}, \bar{m}, \mu) = 0,
\]
(5) an equation which can be combined with (3) into
\[
\left( \frac{\partial}{\partial \ln \mu} - (1 + \bar{m}) \frac{\partial}{\partial m} + (\Delta_O)_{ij} \right) \bar{O}_j(\bar{g}, \bar{m}, \mu) = 0.
\]
(6)

Eq. (6) is an RGE equation for the composite operator, sometimes referred to as the ’t Hooft–Weinberg or Callan–Symanzik equation. The symbol \(\Delta_O \equiv d_O + \gamma_0\) is, as usual, the scaling dimension of the operator \(\bar{O}\). Eq. (6) can be solved by the method of characteristics by introducing a parameter which has the interpretation of a blocking variable. This is for instance used in Ref. [4] to identify the scaling corrections to correlators at a non-trivial IR fixed point.

To this end we note that in this paper the \(O_i\) considered are physical quantities (no anomalous scaling) which in addition do not mix with other operators and therefore \((\Delta_O)_{ij} = d_{O_i} d_{O_j}\).

### 2.3. Feynman–Hellmann theorem in QFT

Let us start by recalling the main steps in the derivation of the Feynman–Hellmann theorem in quantum mechanics, which is a simple but powerful relation which has been obtained by a number of authors [8]. Let us consider a quantum-mechanical system, whose dynamics is determined by a Hamiltonian \(H(\lambda)\), which depends on some parameter \(\lambda\). The Feynman–Hellmann theorem states that the \(\lambda\)-derivative of the energy equals the derivative of the Hamiltonian when evaluated on the corresponding eigenstates:

\[
\frac{\partial}{\partial \lambda} E(\lambda) = \langle \Psi_{E(\lambda)} | \frac{\partial}{\partial \lambda} H(\lambda) | \Psi_{E(\lambda)} \rangle.
\]
(14)

It relies on the observation that
\[
\langle \Psi_{E(\lambda)} | \frac{\partial}{\partial \lambda} \Psi_{E(\lambda)} \rangle = 1 \Rightarrow \frac{\partial}{\partial \lambda} \langle \Psi_{E(\lambda)} | \Psi_{E(\lambda)} \rangle = 0.
\]
(15)

The adaption to QFT, in the simplest cases, necessitates solely to take into account the relativistic state normalisation (7). E.g. with \(\mathcal{H}_m = Q = \bar{Q}\) \((11)\), where \(\mathcal{H}_\text{tot} = \mathcal{H}_m + \ldots\), the Feynman–Hellmann theorem for the mass reads:

\[
\bar{m} \frac{\partial}{\partial m} E_H^2 = \langle \bar{Q} \rangle_{E_H},
\]
(16)

\[
\bar{m} \frac{\partial}{\partial m} (\Delta \mathcal{A}_G) = \langle \bar{Q} \rangle_0.
\]
(17)

In (16) we have identified \((2\pi)^{D-1} \delta(D-1) (\bar{g} - \bar{p}) \rightarrow -\bar{p} \Rightarrow \int d^{D-1}x\) in the sense of distributions. Note, the \(E_H^2\) instead of \(E_H\) in (14) on the LHS originates from the additional factor of \(2E_H\) in the normalisation (7). In (17) the normalisation \(\langle 0|0 \rangle = 1\) was assumed. Furthermore we note that in a mass independent scheme \((\bar{m} \equiv \bar{Z}_m = 0)\)

\[
\bar{m} \frac{\partial}{\partial m} = m \frac{\partial}{\partial m}
\]
and since \(Q = \bar{Q}\), therefore the relation (16) also holds for bare quantities. Eq. (16) is widely known [1] and used in lattice simulation to extract the corresponding contribution to the nucleon mass for example [11]. Noting that \(m \frac{\partial}{\partial m} E_H^2 = \bar{m} \frac{\partial}{\partial m} E_H^2\), it follows that \(\langle Q \rangle_{E_H} = \langle Q \rangle_{M_H}\) with normalisation (7) is a static quantity.

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2 We consider the states as used in (8) as momentum eigenstates and not boosted states and therefore \(\bar{p}\) has no dependence on \(M_H\). More precisely in a lattice simulation the states originate from interpolating operators of the form: \(\Phi(\bar{p}) = \int d^{D-1}x e^{i\bar{p} \cdot x} \Phi(x, \bar{Q})\).

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3. Gluon condensates through RGE and Feynman–Hellmann theorem

The adaption of the analogous relations (16), (17) with regard to the gauge coupling $g$ is complicated by the fact that in gauge theories the construction of the Hamiltonian itself is rather involved, see e.g. Ref. [9]. As stated in the introduction, we bypass the construction of the Hamiltonian, and its primary and secondary constraints, by using the RGE, the trace anomaly, and the relations for the mass.

The RGE (6) for the $M_H^2$ and $\Lambda_{\text{GT}}$,

$$
\left( \beta \frac{\partial}{\partial g} - (1 + \gamma_m)\bar{m} \frac{\partial}{\partial \bar{m}} + 2 \right) M_H^2 = 0, \tag{18}
$$

$$
\left( \beta \frac{\partial}{\partial g} - (1 + \gamma_m)\bar{m} \frac{\partial}{\partial \bar{m}} + D \right) \Lambda_{\text{GT}} = 0, \tag{19}
$$

where $\Delta M_H^2 = 2$ and $\Delta \Lambda_{\text{GT}} = D$ are simply the engineering dimensions as $M_H^2$ and $\Lambda_{\text{GT}}$ are observables which are free from anomalous scaling. Using Eqs. (18), (12), (16) and Eqs. (19), (13), (17) we obtain:

$$
\left( \beta \frac{\partial}{\partial g} M_H^2 + \frac{1}{2g} (\bar{G})_{EH} \right) = \beta \left( \frac{\partial}{\partial g} \Lambda_{\text{GT}} + \frac{1}{2g} (\bar{G})_0 \right) = 0 \tag{20}
$$

and from there we read off our main results,

$$
\bar{g} \frac{\partial}{\partial \bar{g}} E_H^2 = -\frac{1}{2} (\bar{G})_{EH}, \tag{21}
$$

$$
\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{GT}} = -\frac{1}{2} (\bar{G})_0. \tag{22}
$$

For the first relation we have used $\frac{\partial}{\partial \bar{g}} M_H^2 = \frac{\partial}{\partial \bar{g}} E_H^2$, where the same remark applies as for the derivative with respect to $m$ given earlier on. In particular this implies that $(\bar{G})_{EH} = (\bar{G})_{M_H}$ is a static quantity. The relation remains valid in the case where the quark masses are degenerate as one can replace $m_\Psi \rightarrow m_\Psi q_1 + m_\Psi q_2$ in all relations as well as for the corresponding anomalous mass dimension. By taking the coupling to be dimensionless we have implicitly assumed the space-time dimension to be $D = 4$. For the second relation it is possible to derive a relation in $D$-dimensions to replace $\beta / 2g \rightarrow \beta / 2g + (D - 4)/4$. Since $\beta$ disappears from the final results (21), (22), the latter are valid for any integer $D \geq 2$. It seems worthwhile to point out that the relations (21), (22) have been checked explicitly [2]; Eq. (21) for the Schwinger model and the Seiberg–Witten theory as well as Eq. (22) for the massive multi flavour Schwinger model. The relations (21), (22) (c.f. also (12), (13)) are akin to Gell-Mann Oakes Renner relations [12] in that they relate an operator expectation value to physical quantities. The scheme dependence of the gluon condensates, inherent in the earlier statement $G \neq \bar{G}$, is made manifest through $\frac{\partial}{\partial \bar{g}} m_\Psi \neq \bar{G}$ and the fact that $E_H^2$ and $\Lambda_{\text{GT}}$, being physical quantities, do not renormalise. Thus we wish to stress that it is vital to distinguish bare and renormalised quantities when discussing the relations (21), (22). The condensates may be computed through (21), (22) with lattice Monte Carlo simulation in some fixed scheme. The conversion to other schemes, say, the $\overline{\text{MS}}$-scheme can be done through a perturbative computation at some large matching scale. For example defining two schemes $a$ and $b$ through $g = a g_a = b g_b$, one gets:

$$
\bar{g}_a \frac{\partial}{\partial \bar{g}_a} = 2 Z_a \frac{\partial}{\partial g} b g_b \frac{\partial}{\partial \bar{g}_b}, \quad Z_a^{gb} = \left( 1 + \frac{\bar{g}_a}{g_a} \frac{\partial}{\partial \bar{g}_a} \ln \frac{Z_\lambda}{Z_b} \right). \tag{23}
$$

The transformation between schemes $a$ and $b$ is therefore given by $(G_a) = Z_a^{gb} (G_b)$ according to Eqs. (21), (22) for both the vacuum and particle gluon matrix element. The derivation of the relations (21), (22) with bare couplings would surely be possible, but we do not consider it a necessity.

It is worthwhile to illustrate the importance of using eigenstates of the Hamiltonian for the matrix elements considered in the Feynman–Hellmann theorem by an example at hand. One might be tempted to obtain the relation (16) directly from the trace anomaly (10) assuming a mass independent scheme (which entails that $\beta$ and $\gamma_m$ are independent of $m$) via

$$
\frac{\partial}{\partial m} M_H^2 = \frac{\partial}{\partial m} \left( \frac{1}{2} (T_{\mu}^\mu)_{M_H} \right) = \frac{1}{2} (1 + \gamma_m) (\bar{Q})_{M_H} + \text{corrections}, \tag{24}
$$

which without corrections and $\gamma_m \neq 1$ contradicts (16). The necessary corrections originate from the fact that $T_{\mu}^\mu$ does not commute with the Hamiltonian in general and therefore is not an eigenoperator of the physical states (7). Thus differentiation of the states with respect to $\frac{\partial}{\partial m}$ is required for consistency and exemplifies the importance of the energy eigenstates in the Feynman–Hellmann theorem.

4. Conclusions and discussion

In this work we have derived relations between the logarithmic derivative of the mass of a state (and the vacuum energy) with respect to the gauge coupling in terms of the corresponding gluon condensates as given in Eqs. (21), (22). For the readers convenience we restate the relations within slightly more standard notation:

$$
\bar{g} \frac{\partial}{\partial \bar{g}} M_H^2 = -\frac{1}{2} (H(E_H)) \frac{1}{2} \bar{G}^2 |H(E_H)|, \tag{25}
$$

$$
\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{GT}} = -\frac{1}{2} (0) \frac{1}{2} \bar{G}^2 |0|, \tag{26}
$$

where $\frac{\partial}{\partial \bar{g}} M_H^2 = \frac{\partial}{\partial \bar{g}} E_H^2$ as argued earlier on and barred quantities correspond to renormalised quantities. In particular $\Lambda_{\text{GT}}$ and $M_H^2$ originate from the trace of the energy momentum tensor which is known to be finite after renormalisation of the basic parameters of the theory (7). Hence the relation above relates finite quantities with each other. We shall comment on the interest of these equations for various aspects in the paragraphs below.

First the $\ln \bar{g}$-derivative of $M_H^2$ and $\Lambda_{\text{GT}}$ may be taken as a definition of the gluon condensates. This means that the LHS, computable in lattice Monte Carlo simulations, serves as a definition of the condensates on the RHS. An important point is that by computing the condensates indirectly via derivatives from physical quantities, problems with power divergences, which plague direct approaches, are absent. In this respect our approach constitutes a paradigm shift. For $(\bar{G})_{EH}$ this should be straightforward since $M_H^2$ is easily computable whereas for the gluon vacuum condensate $(\bar{G})_0$ this comes with the caveat that $\Lambda_{\text{GT}}$ is not easy to compute.
by itself. We note that for the former the disconnected part is automatically absent since it does not contribute to the mass $M^2_\phi$. The scheme dependence of the condensate is determined by the scheme dependence of the LHS and has been discussed in the text. The transition from one scheme to another can be achieved by a perturbative computation provided the matching scale is high enough for perturbation theory to be valid.

We shall add a few remarks on the gluon condensates. In QCD the matter condensates $\bar{\beta}/(2g)(\bar{G})_{EH}$ are known indirectly through the mass (Eq. (12)) for light mesons, other than the pseudo Goldstone bosons $\pi, K, \eta, \ldots$, as for the latter $\bar{Q}$ is negligible since it is $O(m_{\text{Higgs}})$. For the nucleon this was first discussed in [13]. For the $B$-meson $\bar{\beta}/(2g)(\bar{G})_{EH}$ is related to a non-perturbative definition of the heavy quark scale $\Lambda_{\text{HQ}}$ [14]. The determination of the gluon vacuum condensate is of importance for QCD sum rules [15] as well as for the cosmological constant problem to be discussed further below. The value of the gluon condensate cannot be regarded as settled. This is, in part, due to the fact that there is no direct first principle determination of the gluon condensate.

Let us comment on aspects of the cosmological constant, which is a topic of more speculative nature. Without gravity only energy differences matter. Thus the cosmological constant is only determined up to a constant in flat space. Yet the difference of the cosmological constant due to the QCD phase transition itself is generally seen to be a tractable quantity, given by Eq. (13) provided the condensates are well defined. The quark condensate is known through the Gell-Mann Oakes Renner relation [12]:

$$\langle \bar{Q} \rangle_0 = -f_\pi^2 m_\pi^2 + O(m),$$

with $m_\pi$ and $f_\pi$ being the pion mass and decay constants. It would seem that any undetermined constant of the gluon condensate should drop out in Eq. (22). Therefore $\langle \bar{G} \rangle_0$ determined from this equation could be reinserted into Eq. (13), where scheme dependence cancels provided the appropriate $\bar{\beta}$ and $\gamma_\text{m}$ are used. Scheme independence in turn might be used as a consistency check of the ideas brought forward in this paragraph.

Let us add that if the gluon condensate can be determined, then it could be checked to what degree the lowest $J^{PS} = 0^{+}$ state in a confining gauge theory saturates the partial dilaton conserved current hypothesis, see e.g. Ref. [16]. This could serve as a quantitative measure to identify what is commonly referred to as a dilaton in the literature. The possibility that the Higgs boson candidate discovered at the LHC might be a dilaton of a gauge theory with slow running coupling (walking technicolor) is a possibility that is still considered within the particle physics community e.g. [17].

It might be interesting to make use of the relation (25) in approaches where hadron masses can be computed. We are thinking not only of lattice QCD but also of AdS/QCD or Dyson-Schwinger approaches. The gluon condensate could be reinserted, along with the quark condensate, into the trace anomaly (12) and this allows for the extraction of information on the beta function and the anomalous dimension of the mass. In the case where there are either no fermions or fermions with zero mass, the relation in footnote 5 serves as a definition of the beta function of the theory. Moreover, since the relation applies to any state one can check for the robustness of the results by applying it to many states.

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References

[1] J. Gasser, A. Zepeda, Approaching the chiral limit in QCD, Nucl. Phys. B 174 (1980) 445.
[2] V. Prochazka, R. Zwicky, Gluon condensates from the Hamiltonian formalism, arXiv:1312.5495 [hep-ph].
[3] L. Del Debbio, R. Zwicky, Scaling relations for the entire spectrum in mass-deformed conformal gauge theories, Phys. Lett. B 708 (2011) 217–220, arXiv:1009.2894 [hep-ph].
[4] L. Del Debbio, R. Zwicky, Conformal scaling and the size of $m$-hadrons, Phys. Rev. D 89 (2014) 014503, arXiv:1306.4038 [hep-ph].
[5] J. Zinn-Justin, Quantum field theory and critical phenomena, Int. Ser. Monogr. Phys. 113 (2002) 1.
[6] J.C. Collins, Renormalization. An Introduction to Renormalization, the Renormalization Group, and the Operator Product Expansion, Univ. Pr., Cambridge, UK, 1984, p. 380.
[7] S.L. Adler, J.C. Collins, A. Duncan, Energy-momentum tensor trace anomaly in spin 1/2 QED, Phys. Rev. D 15 (1977) 1712.
[8] J.C. Collins, A. Duncan, S.D. Joglekar, Trace and dilatation anomalies in gauge theories, Phys. Rev. D 16 (1977) 438.
[9] N.K. Nielsen, The energy momentum tensor in a nonabelian quark gluon theory, Nucl. Phys. B 120 (1977) 212.
[10] P. Minkowski, On the anomalous divergence of the dilatation current in gauge theories, Print-76-0813-BERN.
[11] P. Güttinger, Das verhalten von atomen im magnetischen dreifeld, Z. Phys. 73 (3–4) (1952) 169.
[12] W. Pauli, Principles of Wave Mechanics, Handbuch der Physik, vol. 24, Springer, Berlin, 1933, p. 162.
[13] H. Hellmann, Einführung in die Quantenchemie, Franz Deuticke, Leipzig, 1937, p. 285.
[14] R.P. Feynman, Forces in molecules, Phys. Rev. 56 (1939) 340.
[15] F. Ramond, Field theory. A modern primer, Front. Phys. 51 (1981) 1.
[16] L. Del Debbio, R. Zwicky, Hyperscaling relations in mass-deformed conformal gauge theories, Phys. Rev. D 82 (2010) 014502, arXiv:1005.2371 [hep-ph].
[17] M. Foster, et al., UKQCD Collaboration, Phys. Rev. D 59 (1999) 074503, arXiv:hep-lat/9810021.
[18] M. Gell-Mann, R.J. Oakes, B. Renner, Behavior of current divergences under SU(3) × SU(3), Phys. Rev. 175 (1968) 2195.
[19] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Remarks on Higgs boson interactions with nucleons, Phys. Lett. B 78 (1978) 443.
[20] L.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev, A.I. Vainshtein, Sum rules for heavy flavor transitions in the $S$ limit, Phys. Rev. D 52 (1995) 196, arXiv:hep-ph/9405410.
[21] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, QCD and resonance physics. Sum rules, Nucl. Phys. B 147 (1979) 385.
[22] S. Coleman, Dilatations (1971), in: Aspects of Symmetry. Selected Erice Lectures, CUP, 1985.
[23] S. Matsuzaki, K. Yamawaki, Is 125 GeV techni-dilaton found at LHC?, Phys. Lett. B 719 (2013) 378, arXiv:1207.5911 [hep-ph].
[24] Z. Fodor, K. Holland, J. Kuti, D. Nagyadi, C. Schroeder, C.H. Wong, Can the nearly conformal sextet gauge model hide the Higgs impostor?, Phys. Lett. B 718 (2012) 657, arXiv:1209.0391 [hep-lat].

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5 A more complete discussion would include the mass degeneracy of the flavours. The effect of heavy flavours ($m_{\text{quark}} > M_{\text{QCD}}$) can in principle be absorbed into the beta function. The precise discussion of which goes beyond the scope of this Letter.