Abstract

I describe some recent work in which classical solutions of Dirac-Born-Infeld theory may be used to throw light on some properties of M-theory. The sources of Born-Infeld theory are the ends of strings ending on the world volume. Equivalently the fundamental string may be regarded as merely a thin and extended piece of the world volume.

1 Introduction

In the lecture I described some recent work I had been doing over a period of time assisted by my research student Dean Rasheed and with some initial assistance from Robert Bartnik. Some related ideas had been discussed in an unpublished note of Douglas,Shwarz and Lowe \[1\]. After the lecture I was informed by Curt Callan that he and Maldacena had also been thinking along the same lines. Their work is to be found in \[3\] and my own in \[2\]. What follows is a slightly extended and expanded version of what I said in Santiago. The bibliography below is mainly restricted to some relevant papers which appeared during the autumn after the lecture. For a full set of references to earlier work the reader is referred to \[2\] \[3\] \[1\].

As is well known, p-dimensional extended objects, ‘p-branes’ play a central role in establishing the various dualities between the the five superstring theories and eleven-dimensional supergravity theory and these in turn which have led to the conjecture that there exists a single over-arching structure, called ‘M-theory’, of which they may all be considered limiting cases. Whatever M-theory
ultimately turns out to be, it is already clear that it is a theory containing p-branes.

Until recently p-branes have been treated as

- Soliton-like BPS solutions of SUGRA theories.

or

- The ends of open superstrings satisfying $9 - p$ Dirichlet and $p + 1$ Neumann boundary conditions.

My intention is to consider the light brane approximation in which Newton’s constant $G \to 0$ but the string tension $\alpha'$ remains finite. In terms of actions we have:

$$S = \frac{1}{g_s^2} S_{\text{bulk}}^{\text{NS} \otimes \text{NS}} + S_{\text{bulk}}^{\text{R} \otimes \text{R}} + \frac{1}{g_s} S_{\text{brane}},$$

(1)

where $S_{\text{bulk}}$ is an integral over 10-dimensional spacetime and $S_{\text{brane}}$ is an integral over the $p + 1$-dimensional $p + 1$ dimensional world volume of the brane. Heavy branes correspond to the limit $g_s \to \infty$. Light branes correspond to the limit $g_s \to 0$. It is reasonable to ignore the fields generated by the motion of the brane and set the Ramond-Ramond fields to zero. We therefore consider a Dirichlet p-brane moving in flat $d + 1$ dimensional Minkowski spacetime $\mathbb{E}^{d,1}$ with constant dilaton and vanishing Kalb-Ramond 3-form.

## 2 The Dirac-Born-Infeld action

For purely bosonic fields $S_{\text{brane}}$ is then given by the Dirac-Born-Infeld action

$$-\int_{\Sigma_{p+1}} d^{p+1}x \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})},$$

(2)

where

$$G_{\mu\nu} = \partial_\mu Z^A \partial_\nu Z^B \eta_{AB}$$

(3)

is the pull back of the Minkowski metric $\eta_{AB}$ to the world volume $\Sigma_{p+1}$ using the embedding map $Z^A(x^\mu) : \Sigma_{p+1} \to \mathbb{E}^{d,1}$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(4)

is the curvature or field strength of an abelian connection $A_\mu(x^\nu)$ defined over the world volume.

The Dirac-Born-Infeld action is invariant under the semi-direct product of

- world volume diffeomorphisms and
- abelian gauge transformations.
To fix the former we adopt *Static Gauge*, called by mathematicians the *non-parametric representation*:

\[ Z^M = x^\mu, M = 0, 1 \ldots, p \]  \hspace{1cm} (5)

\[ Z^M = y^m, M = p + 1, \ldots, d - p. \]  \hspace{1cm} (6)

The transverse coordinates \( y^m \) behave like scalar fields on the world volume and the action becomes

\[ - \int_{\Sigma_{p+1}} d^{p+1}x \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu y^m \partial_\nu y^m + F_{\mu\nu})}. \]  \hspace{1cm} (7)

Note that original manifest global Poincaré symmetry \( E(d,1) \) has been reduced to a manifest \( E(p,1) \times SO(d-p) \).

An important message of this work is that static gauge cannot usually be globally well defined and it generates spurious singularities if the world volume is topologically non-trivial. Even if \( \Sigma_{p+1} \) is topologically trivial, static gauge may still break down if the brane bends back on itself. Geometrically we have projected \( \Sigma_{p+1} \) onto a \( p + 1 \) hyperplane in \( \mathbb{E}^{d,1} \) and the \( y^m \) are the height functions. However the projection need not be one-one.

Static gauge makes apparent that there are two useful consistent truncations

- \( y^m = 0 \) which is pure Born-Infeld theory in \( \mathbb{E}^{p,1} \)

- \( F_{\mu\nu} = 0 \) which corresponds to Dirac’s theory of minimal timelike submanifolds of \( \mathbb{E}^{d,1} \).

The basic time independent solutions in these two cases are the BlIon and the catenoid respectively. We shall see that there are two one-parameter families of solutions interpolating between them, rather analogous to the one parameter family of Reissner-Nordstrom black holes. An internal Harrison like \( SO(1,1) \) boost symmetry moves us along the two families, one of which is ‘sub-extreme’ and the other of which is ‘super-extreme’. The two families are separated by an extreme BPS type solution.

### 3 Blons

Let’s start with Born-Infeld theory and consider the case \( p = 3 \). Similar results hold if \( p \neq 3 \). The original aim of this theory was to construct classical finite energy pointlike solutions representing charged particles. It is these that I call ‘Blions’ and their study ‘Blonics’.

For time independent pure electric solutions the lagrangian reduces to

\[ L = -\sqrt{1 - \mathbf{E}^2} + 1. \]  \hspace{1cm} (8)
where \( \mathbf{E} = -\nabla \phi \) is the electric field. Thus the electric induction is

\[
\mathbf{D} = \frac{\partial L}{\partial \mathbf{E}} = \frac{\mathbf{E}}{\sqrt{1 - \mathbf{E}^2}}
\]

(9)

and thus

\[
\mathbf{E} = \frac{\mathbf{D}}{\sqrt{1 + \mathbf{D}^2}}
\]

(10)

Now if

\[
\nabla \cdot \mathbf{D} = 4\pi e \delta(x),
\]

(11)

\[
\mathbf{E} = \frac{e \hat{r}}{\sqrt{e^2 + r^4}}
\]

(12)

Clearly while the induction \( \mathbf{D} \) diverges at the origin the electric field remains bounded and attains unit magnitude at the origin. In other words the slope of the potential is 45 degrees at the origin.

The energy density is \( T_{00} = \mathbf{E} \cdot \mathbf{D} - L \) and it is easy to see that the total energy is finite.

It is important to realize the difference, not widely understood, between ‘BIons’ and conventional ‘solitons’. Originally Born-Infeld theory was intended as a ‘unitary’ theory of electromagnetism. In modern terms such a theory would be one in which the classical electron is represented by an everywhere non-singular finite energy of the source free non-linear equations of motion. In such theories the particle equations of motion follow from the equations of motion of the fields without having to be postulated separately. As such the theory was a failure because

- The BIon solutions have sources.
- The solutions are still singular at the location of the source.

and

- One must impose boundary conditions on the singularities in order to obtain the equations of motion. For a recent discussion of this point see [7, 19].

Nevertheless ‘BIonic’ solutions of field theories frequently have a sensible physical interpretation (cf. point defects in liquid crystals). To illustrate the point we consider briefly Born-Infeld electrostatics. Solutions of the equation of motion

\[
\nabla \cdot \frac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}}
\]

(13)

may be interpreted as spacelike maximal hypersurfaces \( t = \phi(x) \) in an auxiliary \( p + 1 \) Minkowski spacetime with coordinates \( (t, x) \). This allows one to use geometrical techniques from general relativity and p.d.e. theory to discuss the
existence and uniqueness of solutions. More significantly it allows us to construct
new solutions. For example boosting the trivial solution with velocity \( v = E \) in
the \( z \) direction gives rise to a uniform electric field \( \phi = -Ez \). The maximal field
strength in Born-Infeld theory corresponds to the maximum velocity in special
relativity.

One may also boost the BIon solution to give a static solution representing a
charged particle at rest in an asymptotically uniform electrostatic field \( E \). This
sounds paradoxical but it is not. The point is that the solution does not satisfy
the correct boundary conditions at the particle centre to be force free. It is
pinned by a force \( F = eE \) given by

\[
F_i = \int T_{ij} d\sigma_j
\tag{14}
\]

where the integral is taken over a sphere surrounding the Bion.

Static solutions which do satisfy the force free solution can be also found. Thus if \( p(x) \) is the Weirstrass elliptic function with \( g_3 = 0 \) and \( g_2 = 4 \) then

\[
p(x)p(y)p(z) = p(\phi)
\tag{15}
\]

we get a BIon crystal of NaCl type. In this case the forces on the BIons cancel by
symmetry. In general one may apply comparative statics and the virial theorem
to obtain some striking analogues of the results in black hole theory. If \( F^a \) is
the force on the \( a \)’th BIon which has position \( \mathbf{x}_a \), charge \( e_a \) and electrostatic
potential \( \Phi_a \) one has the ‘second law’

\[
dM = \Phi^a d\epsilon_a + F^a \cdot d\mathbf{x}_a
\tag{16}
\]

and the Smarr-Virial relation:

\[
M = \frac{1}{3} F^a \cdot \mathbf{x}_a + \frac{2}{3} e_a \Phi^a.
\tag{17}
\]

Here \( M \) is the total energy and there is a sum over the BIon index \( a \).

### 4 Catenoids

Consider one transverse coordinate \( y \). The lagrangian now becomes

\[
L = -\sqrt{1 + |\nabla y|^2} + 1.
\tag{18}
\]

If \( p = 3 \) We soon find that a spherically symmetric solution satisfies

\[
\partial_r y = \pm \frac{c}{\sqrt{r^2 - c^2}}
\tag{19}
\]
The solution breaks down at \( r = \sqrt{c} \) because of a breakdown of static gauge. In fact the spatial part of the world volume (i.e. the p-brane) consists of two copies of the solution for \( r > \sqrt{c} \) joined by a minimal throat. In other words, the solution has the geometry of the Einstein-Rosen throats familiar in Black Hole theory. (In fact the Einstein Rosen throats, i.e. the constant time surfaces of static black holes or of self-gravitating p-branes, are minimal submanifolds).

Near infinity the catenoid looks like two parallel \( p \)-planes situated a finite distance \( Y \) apart. Callan and Maldacena\[13\] have suggested that one should regard this as a D-brane-anti-D-brane configuration though it is a single connected surface. The catenoid is unstable in that it one can find a deformation which lowers the total volume. For that reason it was suggested by them that it should be thought of as some sort of sphaleron. It is interesting to note that subsequent to, and independent of, their discussion there appeared a paper \[8\] in which the it was shown, using the fact that that the Hessian (i.e second variation) of the Dirac energy is

\[
\int \sqrt{g} dx^{p} f \left( -\nabla_{\phi}^{2} - K_{ij} K^{ij} \right) f. \tag{20}
\]

where \( K_{ij} \) is the second fundamental form of the hypersurface, that quite generally any complete minimal hypersurface of \( \mathbb{E}^{p+1} \) with more than one end admits bounded harmonic functions and thus cannot be a true minimum of the energy.

### 5 Charged Catenoids

In general the relevant lagrangian is

\[
L = -\sqrt{1 - |\nabla \phi|^2 + |\nabla y|^2 + (\nabla y \cdot \nabla \phi)^2 - (\nabla \phi)^2 (\nabla y)^2} \tag{21}
\]

This is manifestly invariant under generalized Harrison transformations consisting of boosts in the \( \phi - y \) plane. Starting from the Bion or the catenoid we obtain the two one parameter families mentioned above. Note that the super-extreme solutions have a singular source on the world volume while the sub-extreme solutions are perfectly regular and have no source on the world volume. Starting with the catenoid and charging it up gives a narrower and narrower and longer and longer throat. Starting with the Bion and adding the scalar gives a bigger and bigger spike. The interesting question is what about the limiting case?

### 6 The BPS solution

It is a simple task to verify that taking

\[
\phi = \pm y = H \tag{22}
\]
where $H$ is an arbitrary harmonic function will solve the equations. If

$$H = \sum_a \frac{c_a}{|x - x_a|},$$

(23)

we get a superposition of arbitrarily many infinitely spiky solutions. One may verify that these solutions are supersymmetric and indeed they satisfy the effective equations of motion of the superstring to all orders [11]. We shall see why shortly. In the mean time we point out that the obvious natural interpretation of these solutions is that they represent infinitely long fundamental strings ending on a D-brane as first envisaged by Strominger and by Townsend. We can now see clearly where the source for the BIon comes from. It is carried by the string. Indeed one may check that the charge carried by the string equals that carried by the BIon using the fact that the coupling to the Neveu-Scharz field $B_{\mu\nu}$ in the Dirac-Born-Infeld action is obtained by the replacement

$$F_{\mu\nu} \rightarrow F_{\mu\nu} = F_{\mu\nu} - B_{\mu\nu}.$$  

(24)

Note that if $c = 1$ the D-brane spike has height $L$ at a distance $\frac{1}{L}$ from the source. The paper by Callan and Maldacena[3], see also [9] gives more detailed evidence for this viewpoint by showing that, by being careful about factors, the energy of a length $L$ of string agrees with the world volume energy of the fields outside a radius $\frac{1}{L}$.

7 Electric-magnetic duality and the inclusion of magnetic fields

We have $H = -\frac{\partial L}{\partial B} = -\nabla \chi$ where $\chi$ is the magnetostatic potential. Let $\Phi^A = (y, \phi, \chi)$ be coordinates in an auxiliary Minkowski spacetime $\mathbb{E}^{1,2}$ (with two negative signs).

By means of a suitable Legendre transformation one may obtain an effective action from which to deduce the equations of motion. It is

$$\sqrt{\text{det}(\nabla \Phi^A \cdot \nabla \Phi^B - \eta^{AB})}.$$  

(25)

This is manifestly invariant under $SO(2,1) \supset SO(2)$. The $SO(2)$ subgroup of rotations of $\phi$ into $\chi$ is of course just electric-magnetic duality rotations. It is well known that Born-Infeld theory has this symmetry. In fact its existence may be traced back to the basic S-duality of non-perturbative string theory. Acting on the solutions with it we can obtain magnetically charged BIons attached to D-strings. In fact classically there is an entire circle of dyonic BPS solutions but of course quantization breaks down $SL(2, \mathbb{R})$ to $SL(2, \mathbb{Z})$. 

7
8 Abelian Bogomol’nyi Monopoles

Setting $\phi = 0$ we find magnetic solutions with $B = H$. Thinking of $y$ as a Higgs field we recognize the equations

$$\nabla y = \pm B$$

(26)

as the abelian Bogomol’nyi equations of Yang-Mills theory, valid in the limit that the mass $m_W$ of the vector bosons goes to infinity. This is consistent with Witten’s ideas about nearby Dirichlet-branes. If two branes are well separated one has a gauge group $U(1) \times U(1)$, one of the factors corresponding to the centre of mass motion. If they are coincident one expects symmetry enhancement to $U(2)$. Taking out a $U(1)$ factor corresponding to the centre of mass the world volume gauge group is $SU(2)$. The distance $Y$ between the branes is supposed to be proportional to $m_W$. The abelian Born-Infeld theory does indeed seem able to capture the physics of the large vector-boson mass limit.

9 Dimensional Reduction and SUSY

It is rather convenient to obtain the Dirac-Born-Infeld lagrangian in static gauge by dimensionally reducing the pure Born-Infeld lagrangian

$$- \sqrt{-\det(\eta_{AB} + F_{AB})}$$

(27)

from ten dimensions to $p + 1$ dimensions. One sets

$$A_A = (A_m(x), A_\mu(x))$$

(28)

and identifies the transverse components $A_m$ of the gauge connection one-form $A_A$ with the transverse coordinates $y^m$ of the $p + 1$ brane. At lowest order one may use the supersymmetry transformations of 1-dimensional SUSY (abelian) Yang-Mills. Thus SUSY requires the existence of a sixteen component Majorana-Weyl Killing spinor $\epsilon$ such that

$$F_{AB}\gamma^A\gamma^B\epsilon = 0.$$  

(29)

In the electric case, our ansatz is

$$F_{\alpha i} = F_{0i},$$

(30)

so the BPS condition requires $(\gamma^0 + \gamma^5)\epsilon = 0$, which has eight real solutions. The self dual solutions are also easily seen to be BPS. One may check that both continue to admit killing spinors when the full non-linear supersymmetry transformations have been taken into account [18]. Moreover, in the electric case it has been argued that the solution gives an exact boundary conformal
field theory \[11\]. This is almost obvious because of the lightlike nature of the the electric ansatz. All contractions involving $F_{AB}$ must vanish.

The easily verified fact that the abelian anti-self-duality equations are sufficient conditions for solutions the Born-Infeld equations leads to an interesting and useful relation to minimal 2-surfaces in $\mathbb{E}^4$. One assumes that $A_\mu$ depends only on $z = x^1 + ix^2$. Setting $A_3 + iA_4 = w = x_3 + ix_4$ one then easily calculates that $F_{\mu\nu} = -\ast F_{\mu\nu}$ reduces to the Cauchy-Riemann equations, i.e. to the condition that $w$ is a locally holomorphic function of $z$. In this way one may obtain a variety of interesting minimal surfaces which can in fact be regarded as exact solutions of M-theory. Thus if

$$wz = c, \quad (31)$$

and $c \neq 0$, we obtain a smooth connected 2-brane with topology $\mathbb{C}^* \equiv \mathbb{C} \setminus 0$ looking like two 2-planes connected by a throat. If $c = 0$ this degenerates to two 2-planes, $z = 0$ and $w = 0$ intersecting at a point. Constructions of minimal surfaces in $\mathbb{E}^4$ using holomorphic embeddings were pioneered by Kommerell in 1911 \[12\] so it seems reasonable to refer to them as Kommerell solutions. The reader is referred \[13\] for an account of recent applications to gauge theory.

10 Calibrated geometries

The holomorphic solutions of the last section are in fact a special case of a more general class of solutions in which the familiar Wirtinger’s inequality for the area, or more generally the $p$-volume, of holomorphically embedded $p$-cycles is replaced by a more general inequality. The basic idea is to replace a suitable power of the Kähler form by some other closed $p$-form. The form is called by Harvey and Lawson \[14\] a calibrating $p$-form. Using their work one see that the Dirac-Born-Infeld equations have a very rich set of solutions. Here is not the place to discuss them and their applications to gauge theory in detail. I will simply remark that one encounters various kinds of topological defects on the world volume, such as vortices and global monopoles. These latter may be relevant to discussions of the non-abelian monopoles in the limit that the gauge field decouples. Of special interest are the BPS solutions and their Bogomoľnyi bounds. It turns out that there is a close connection between this, kappa symmetry, and the calibration condition of Harvey and Lawson \[14\]. In fact he calibration condition turns out to be the condition for supersymmetry \[15\].

11 Non-Abelian Born-Infeld

It is widely believed that when a number of $D$-branes coincide there is symmetry enhancement. The current most popular suggestion for the relevant generaliza-
tion of the Born-Infeld action is that of Tseytlin
\begin{equation}
- \text{Str} \sqrt{-\det(\eta_{AB} + F_{AB})}
\end{equation}
where $F_{AB}$ is in the adjoint representation of the gauge group $G$ and Str denotes
the symmetrized trace of any product of matrices in the adjoint representation
that it precedes. The Tseytlin action has the property, that solutions
of the non-abelian Bogomol'nyi equations are also solutions. In the case of
$SU(2)$ one may even establish a generalized Bogomol'nyi bound. Very
recently BPS bounds have also been established in a rather different way
using an apparently different energy functional.

12 Conclusion

A striking aspect of the work reported above is the way in which Born-Infeld
theory has finally found a home on the world-volume and its mysterious sources
have been shown to be just the ends of strings extending into higher dimensions.
Even more striking is the way in which the fundamental string solution emerges
as a limiting case of M-theory solutions. It seems to reinforce the widely held
viewpoint that in the ultimate formulation of the theory, strings as such may
have no fundamental role to play and may indeed appear only as effective
excitations. However, as always, it is worth exercising some caution. After all,
who would have thought a few years ago that Born-Infeld theory and Dirac’s
doomed attempt to construct an extended model of the electron, long since rel-
egated to the dustbin of history and condemned as a last nostalgic gasp at the
fag-end of the classical world-picture should re-emerge at the cutting edge of
post modernist physics?

There will no doubt be many more fag-ends and even a few cigars before the
final story is told. In the meantime it is my pleasant duty to thank Claudio and
Jorge, so ably assisted by the Chilean Air Force, for organizing such a wonderful
conference and making our stay in Chile and Antartica so memorable.

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