FINITE ENERGY STANDING WAVES
FOR THE KLEIN-GORDON-MAXWELL SYSTEM:
THE LIMIT CASE

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ABSTRACT. In this paper we consider the Klein-Gordon-Maxwell system in the electrostatic case, assuming the fall-off large-distance requirement on the gauge potential. We are interested in proving the existence of finite energy (and finite charge) standing waves, having the phase corresponding to the mass coefficient in the Klein-Gordon Lagrangian.

1. INTRODUCTION

As it is well known, standing waves solving the nonlinear Klein-Gordon-Maxwell system in the electrostatic case can be obtained from the system

\[
\begin{cases}
-\Delta u + [(m^2 - \omega^2) + \varepsilon(2\omega - \varepsilon \phi)]u - u^{p-1} = 0 & \text{in } \mathbb{R}^3, \\
-\Delta \phi = \varepsilon(\omega - \varepsilon \phi)u^2 & \text{in } \mathbb{R}^3, \\
u > 0, \phi > 0 & \text{in } \mathbb{R}^3
\end{cases}
\]

for \(m, \varepsilon, \omega > 0\) and \(p > 1\) (we refer to [5] for the derivation of the system). The interest in these equations rests on the gauge theory from which they come. A couple \((u, \phi)\) solving (1) originates on one hand a matter field having the form of a standing wave

\[
\psi(x, t) = u(x)e^{-i\omega t}, \quad u > 0
\]
on the other the electromagnetic field

\[
(E(x, t), H(x, t)) = (-\nabla \phi(x), 0)
\]
interacting and influencing each other (see [6]). The physical relevance of this model is strengthened by the property of localization possessed by fields as in (2). Indeed, by the invariance of the original Lagrangian with respect to the Poincaré group of transformations, we are allowed to put a standing wave in motion by means of a Lorentz boost, obtaining a solitary wave behaving like a relativistic particle. In this sense, the system provides a relativistic consistent model for the description of the interaction between a particle embedded in the electromagnetic field generated by itself.

We point out that, as it is showed in [6], by the gauge invariance of the original Lagrangian with respect to transformations of the type

\[
u(x)e^{-i\omega t} \mapsto u(x)e^{-i(\omega t - \chi(t))}
\]

\[
\phi(x) \mapsto \phi(x) - \frac{\partial}{\partial t} \chi(t)
\]
where \(\chi \in C^\infty(\mathbb{R})\), starting from a solution \((u, \phi)\) of (1) and considering tranformations of the type \(\chi(t) = ct\) for \(c \in \mathbb{R}\), we can obtain standing waves \(\psi(x, t) = u(x)e^{-i\tilde{\omega} t}\) for an arbitray \(\tilde{\omega} \in \mathbb{R}\), remaining the electromagnetic field unvaried.

This gauge freedom is avoided by requiring, for example, a further condition on the behaviour

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of the gauge potential $\phi$ at infinity. In particular, in our project we are interested in potentials $\phi$ satisfying the so called large-distance fall-off requirement

\[(\mathcal{FO}) \quad \phi(x) \to 0 \text{ as } |x| \to +\infty,\]

in view to show that $\phi$ actually behaves like a Coulomb potential at infinity (see Remark 3.7).

The first existence and multiplicity results concerning problem (1) were obtained in [5]. These results were later improved in [8] and [2], while nonexistence theorems were proved in [9]. The Klein-Gordon-Maxwell coupling was also considered in a bounded domain with various boundary conditions for instance in [10, 11].

In all the quoted papers, the relation between the mass coefficient $m$ and the wave phase $\omega$ plays an important role to establish the existence of solutions.

In particular, putting together the results contained in [2, 5, 8], we have the following

**Theorem 1.1.** Let $p \in (2, 6)$ and assume that $0 < \omega < mg(p)$ where

$$g(p) = \begin{cases} \sqrt{(p-2)(4-p)} & \text{if } 2 < p < 3, \\ 1 & \text{if } 3 \leq p < 6. \end{cases}$$

Then (1) admits a nontrivial radial solution.

We remark that stationary solutions coming from Theorem 1.1 are definitely convincing for our theory, since both the energy

\[(4) \quad \mathcal{E}(u(x)e^{-i\omega t}, \phi, 0) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + |\nabla \phi|^2 + (m^2 + \omega^2)u^2 - 2e\omega \phi u^2 + e^2 \phi^2 u^2 - \frac{2}{p}|u|^p \right] dx\]

and the charge

\[(5) \quad Q(u(x)e^{-i\omega t}, \phi, 0) = e \int_{\mathbb{R}^3} (e\phi - \omega)u^2 dx\]

are finite (see [12, Section 2.3]). With an abuse of language, we will call finite energy solution a couple $(u, \phi)$ solving (1) and such that the energy related with (2) and (3) is finite.

The strategy to achieve the result in Theorem 1.1 consists in approaching (1) variationally and using usual tools of critical points theory to find solutions of the system as critical points of the functional

$$I_\omega(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 - |\nabla \phi|^2 + (m^2 - \omega^2)u^2 + 2e\omega \phi u^2 - e^2 \phi^2 u^2 \right] dx - \frac{1}{p}|u|^p dx$$

in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

It is well known (see for example [5]) that the reduction method permits to convert the problem of finding critical points of $I_\omega$ to the equivalent one of looking for critical points of the functional

$$J_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + (m^2 - \omega^2)u^2 + e\omega \phi u^2 \right] dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$$

defined in $H^1(\mathbb{R}^3)$, where $\phi_u$ represents the unique function in $D^{1,2}(\mathbb{R}^3)$ solving

\[(6) \quad - \Delta \phi = e(\omega - e\phi)u^2\]

in the dual of $D^{1,2}(\mathbb{R}^3)$. In order to promote the necessity of obtaining finite energy (and charge) solutions, the assumption $m > \omega$ seems to arise quite naturally to implement our variational strategy. On the other hand, carrying out a deeper analysis of the model, we realize that such an assumption turns out to be purposely technical, since there is no physical motivation preventing the existence of finite energy standing waves having the form $\psi(x, t) = u(x)e^{-i\omega t}$ (namely $m = \omega$), under the gauge choice $(\mathcal{FO})$. 
From the mathematical point of view, the idea of finding finite energy solutions to the system (1) in the limit case \(m = \omega\) appears immediately challenging.

Consider indeed the system

\[
\begin{aligned}
-\Delta u + e(2\omega - e\phi)\phi u - u^{p-1} &= 0 & \text{in } \mathbb{R}^3, \\
-\Delta \phi = e(\omega - e\phi)u^2 &= 0 & \text{in } \mathbb{R}^3, \\
u > 0, \phi > 0 &= 0 & \text{in } \mathbb{R}^3,
\end{aligned}
\]

for \(e, \omega > 0\) and \(p > 1\).

The functional associated to the problem is

\[
I_m(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 - |\nabla \phi|^2 + 2e\omega \phi u^2 - e^2 \phi^2 u^2 \right) dx - \frac{1}{p} |u|^p dx
\]

whereas the formal reduced functional is

\[
J_m(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + e\omega \phi u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx.
\]

There is no difficulty in observing that, even if \(I_m\) is of course well defined in \(H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\), the lack of an explicit expression of the \(L^2\) norm of \(u\) in \(I_m\) makes it hard to apply standard critical points theory arguments in that space.

On the other hand, if we assume \(D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) as our setting, we have to face both the problem of inapplicability of the reduction method, and the difficulty in estimating the energy of any possible solution.

Finally, the introduction of an ad hoc functional setting as a sort of middle ground between those two spaces, does not immediately seem a feasible way.

A first attempt to solve (P) was made in [2], by means of a perturbation argument (see also [3,4]). In that paper, the problem was considered in presence of an inhomogeneous nonlinearity in the first equation, and a solution \((u, \phi)\) in the sense of distributions was obtained in \(D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) as the limit of a sequence of solutions of approximating problems like these

\[
\begin{aligned}
-\Delta u + \epsilon[e(2\omega - e\phi)\phi u - f(u)] &= 0 & \text{in } \mathbb{R}^3, \\
-\Delta \phi = e(\omega - e\phi)u^2 &= 0 & \text{in } \mathbb{R}^3, \\
u > 0, \phi > 0 &= 0 & \text{in } \mathbb{R}^3.
\end{aligned}
\]

Unfortunately, the lack of information about the \(L^2\) norm of \(u\) did not permit to estimate the energy in order to confirm the validity of the model.

In this paper, we bridge this gap following an idea in [13] where the application of a comparison principle leads to show the exponential decay property possessed by our solution \(u\).

Moreover, by similar arguments as those in [14] and a new upper bound estimate holding uniformly for the \(L^p\) norm of suitable solutions of

\[
\begin{aligned}
-\Delta u + \epsilon[e(2\omega - e\phi)\phi u - u^{p-1}] &= 0 & \text{in } \mathbb{R}^3, \\
-\Delta \phi = e(\omega - e\phi)u^2 &= 0 & \text{in } \mathbb{R}^3, \\
u > 0, \phi > 0 &= 0 & \text{in } \mathbb{R}^3,
\end{aligned}
\]

as \(\epsilon \to 0\), we are allowed to deal with a power-like nonlinearity.

The main result in this paper is the following.

**Theorem 1.2.** For any \(p \in (3, 6)\) there exists a finite energy solution \((u, \phi) \in C^2(\mathbb{R}^3) \times C^2(\mathbb{R}^3)\) to the problem (P).

The paper is organized in two sections.

In Section 2 we provide an inequality useful to get a uniform upper bound on certain \(L^2\) weighted norms. It will be used to control the \(L^p\) norms of approximating solutions.

In Section 3 we prove our main Theorem 1.2 by an argument based on the application of the comparison principle.

In what follows, the letter \(C\) denotes a positive constant which may change from line to line. We also point out that, everytime we will handle radial functions, with an abuse of
notation we will treat them as functions of one or three variables, denoting the argument by \( r \in (0, +\infty) \) or \( x \in \mathbb{R}^3 \).

2. PRELIMINARY RESULTS

As usual, for any \( u \in D^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \), we denote by \( \phi_u \) the unique function in \( D^{1,2}(\mathbb{R}^3) \) satisfying (6) in a weak sense.

It is well known that

1. \( \phi_u \geq 0 \);
2. \( e^\phi u \leq \omega \) in the set \( \{ x \in \mathbb{R}^3 \mid u(x) \neq 0 \} \);
3. if \( u \) is radial, then \( \phi_u \) is radial.

We denote by \( H^1_1(\mathbb{R}^3) \) and \( D^{1,2}(\mathbb{R}^3) \) the spaces of radial functions respectively in \( H^1(\mathbb{R}^3) \) and \( D^{1,2}(\mathbb{R}^3) \). For all \( M \geq 0 \), define

\[
B_M := \{ u \in D^{1,2}_r(\mathbb{R}^3) \cap L^{\frac{12}{5}}(\mathbb{R}^3) \mid \phi_u \in C^2(\mathbb{R}^3) \text{ and } \| \nabla \phi_u \|_2 \leq M \}.
\]

The main object of this section is to prove the following result.

**Proposition 2.1.** For all \( \alpha > \frac{1}{2} \) and for all \( M \geq 0 \) there exists \( C > 0 \) such that if \( u \in B_M \), then

\[
\int_{\mathbb{R}^3} \frac{u^2(x)}{\sqrt{|x|(1 + |\log |x||)^\alpha}} \, dx \leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + \int_{\mathbb{R}^3} \phi_u^2 \, dx.
\]

We start with some preliminary lemmas. The following one is essentially contained in [14]; we recall it, adapted to our need.

**Lemma 2.2.** For all \( \alpha > \frac{1}{2} \) there exists \( C_\alpha > 0 \) such that for any \( R > 1 \) and \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) measurable

\[
\left( \int_{3R}^{+\infty} h(r) \, dr \right)^2 \leq C_\alpha \int_{R}^{+\infty} \left( \int_{\frac{r}{2}}^{2r} (1 + \log r)^\alpha h(r)(1 + \log s)^\alpha h(s) \, ds \right) \, dr.
\]

**Proof.** Let \( \alpha > \frac{1}{2} \) and consider \( R > 1 \). Set \( k = \lfloor \log_2 R \rfloor + 1 \), where \( \lfloor \cdot \rfloor \) denotes the integer part. So \( R < 2^k < 3R \). Following the idea in [14], we use the inequality \( \sum_{n=k}^{\infty} a_n \leq \sum_{n=k}^{\infty} b_n \), for \( a_n = \int_{2^n}^{2^{n+1}} h(r) \, dr \) and \( b_n = (1 + n)^{2\alpha} \), and we have

\[
\left( \int_{3R}^{+\infty} h(r) \, dr \right)^2 \leq \sum_{n=k}^{\infty} \left( \int_{2^n}^{2^{n+1}} h(r) \, dr \right)^2.
\]

\[
\leq \sum_{n=0}^{\infty} \frac{1}{(1 + n)^{2\alpha}} \sum_{n=k}^{\infty} \left( \int_{2^n}^{2^{n+1}} h(r) \, dr \right)^2.
\]

\[
\leq C_\alpha \sum_{n=k}^{\infty} \left[ (1 + n)^{\alpha} \int_{2^n}^{2^{n+1}} h(r) \, dr \right]^2.
\]

Finally, again as in [14], we have

\[
\sum_{n=k}^{\infty} \left[ (1 + n)^{\alpha} \int_{2^n}^{2^{n+1}} h(r) \, dr \right]^2 \leq (\log_2 e)^{2\alpha} \int_{R}^{+\infty} \left( \int_{\frac{r}{2}}^{2r} (1 + \log r)^\alpha h(r)(1 + \log s)^\alpha h(s) \, ds \right) \, dr
\]

and we conclude. \( \square \)
Lemma 2.3. There exist two positive constants $C_1$ and $C_2$ such that for any $u \in D^{0,2}_r(\mathbb{R}^3) \cap L^{\frac{12}{5}}(\mathbb{R}^3)$ such that $\phi_u \in C^2(\mathbb{R}^3)$

$$
\int_{\max(1,C_1\|\nabla \phi_u\|^2)}^{+\infty} u^2(r)u^2(s) \min(r,s) \, ds \, dr \leq C_2 \int_{\mathbb{R}^3} \phi_u u^2 \, dx.
$$

Proof. Let $u$ be in $D^{0,2}_r(\mathbb{R}^3) \cap L^{\frac{12}{5}}(\mathbb{R}^3)$ and assume $\phi_u$ is in $C^2(\mathbb{R}^3)$.

By [7, Radial Lemma A.III] we know that there exists $C > 0$ which does not depend on $\phi_u$ such that for $|x| \geq 1$

$$
\phi_u(x) \leq \frac{C}{\sqrt{|x|}} \|\nabla \phi_u\|_2.
$$

Set $C_1 = \frac{4e^2C^2}{\omega^2}$. Since $\phi_u$ satisfies

$$
- (r^2 \phi_u')' = e\phi_u' (\omega - e\phi_u) u^2
$$

in $[0, +\infty)$, integrating (8) in $[0, t)$ for $t > \max(1, C_1 \|\nabla \phi_u\|^2_2)$, we get

$$
\phi_u'(t) = -\frac{e}{t^2} \int_0^t s^2 (\omega - e\phi_u(s)) u^2(s) \, ds \leq \frac{e\omega}{2t^2} \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{r^2} s^2 u^2(s) \, ds.
$$

Now, integrating in $(r, +\infty)$ for $r > \max(1, C_1 \|\nabla \phi_u\|^2_2)$, we obtain

$$
\phi_u(r) \geq \frac{e\omega}{2} \int_r^{+\infty} \frac{1}{t^2} \left( \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{t} s^2 u^2(s) \, ds \right) \, dt
$$

$$
= \frac{e\omega}{2} \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} s^2 u^2(s) \left( \int_{\max(r,s)}^{+\infty} \frac{1}{t^2} \, dt \right) \, ds
$$

$$
= \frac{e\omega}{2} \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} \frac{s^2}{\max(r,s)} u^2(s) \, ds.
$$

Finally, multiplying by $r^2 u^2(r)$ and integrating in $(\max(1, C_1 \|\nabla \phi_u\|^2_2), +\infty)$, we have

$$
\int_0^{+\infty} r^2 \phi_u(r) u^2(r) \, dr \geq \frac{e\omega}{2} \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} \left( \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} \frac{r^2 s^2}{\max(r,s)} u^2(r) u^2(s) \, ds \right) \, dr
$$

$$
= \frac{e\omega}{2} \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} \left( \int_{\max(1, C_1 \|\nabla \phi_u\|^2_2)}^{+\infty} rs \min(r,s) u^2(r) u^2(s) \, ds \right) \, dr
$$

from which we conclude. \qed

Lemma 2.4. For any $\alpha > \frac{1}{2}$ and $R_0 > 1$ there exists $C > 0$ such that for any measurable function $u : \mathbb{R}_+ \to \mathbb{R}$

$$
\int_0^{+\infty} \frac{u^2(r) r^{\frac{\alpha}{2}}}{(1 + |\log r|)^{\alpha}} \, dr \leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{R_0}^{+\infty} u^2(r) r u^2(s) \min(r,s) \, ds \right) \frac{1}{2}.
$$

Proof. Let $\alpha > \frac{1}{2}$ and $R_0 > 1$ and consider $C_\alpha > 0$ as in Lemma 2.2
Applying Lemma 2.2 for $h(r) = \frac{u^2(r)r^3}{(1 + |\log r|)^\alpha}$, by Hölder and Gagliardo-Nirenberg inequalities, we have that
\[
\left( \int_0^{+\infty} \frac{u^2(r)r^3}{(1 + |\log r|)^\alpha} \, dr \right)^2 \leq 2 \left( \int_0^{6R_0} \frac{u^2(r)r^3}{(1 + |\log r|)^\alpha} \, dr \right)^2 + 2 \left( \int_{6R_0}^{+\infty} \frac{u^2(r)r^3}{(1 + |\log r|)^\alpha} \, dr \right)^2
\leq 2 \left( \int_0^{6R_0} u^6(r) \, dr \right)^\frac{2}{3} \left( \int_0^{6R_0} r^\frac{4}{3} \, dr \right)^\frac{2}{3}
+ 2C_\alpha \int_{2R_0}^{+\infty} \left( \int_{\frac{3}{2}}^{2r} u^2(r)r^\frac{3}{2} u^2(s) \, ds \right) \, dr
\leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2
+ 2\sqrt{2}C_\alpha \int_{2R_0}^{+\infty} \left( \int_{\frac{3}{2}}^{+\infty} u^2(r)ru^2(s) \min(r,s) \, ds \right) \, dr
\leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2
+ 2\sqrt{2}C_\alpha \int_{R_0}^{+\infty} \left( \int_{R_0}^{+\infty} u^2(r)ru^2(s) \min(r,s) \, ds \right) \, dr.
\]
In the third inequality we have used that $\sqrt{r}s \leq \sqrt{2} \min(r,s)$ in the set \{(r, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \frac{3}{2} \leq s \leq 2r\}. \qed

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. Consider $\alpha > \frac{1}{2}$ and $M > 0$. Take $C_1$ and $C_2$ positive constants as in Lemma 2.3 and $R_0 > \max(1, C_1 M^2)$. The conclusion follows from Lemma 2.3 and Lemma 2.4. \qed

By Lemma 2.4 we also deduce the following estimate on the Lebesgue norms.

Proposition 2.5. Let $q \in (\frac{18}{5}, 6]$ and $R_0 > 1$. Then there exists $C > 0$ such that for every radial and measurable $u : \mathbb{R}^3 \to \mathbb{R}$ we have
\[
\left( \int_{\mathbb{R}^3} |u|^q \, dx \right)^\frac{1}{q} \leq C \left[ \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \left( \int_{R_0}^{+\infty} \left( \int_{R_0}^{+\infty} u^2(r)ru^2(s) \min(r,s) \, ds \right) \, dr \right)^{\frac{1}{2}} \right]^\frac{1}{q}.
\]

Proof. Fix $q \in (\frac{18}{5}, 6]$ and $R_0 > 1$. By continuous embedding theorems proved in [16, 17], there exists $\eta > \frac{1}{q}$ such that the following inequality holds for some positive constant $C$ and any $u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V(x) \, dx)$, where $V(x) = \frac{1}{1+|x|^{\eta}}$:
\[
\left( \int_{\mathbb{R}^3} |u|^q \, dx \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} \frac{u^2}{1+|x|^{\eta}} \, dx \right)^\frac{1}{q}.
\]
Now choose $\alpha > \frac{1}{2}$. Since
\[
\lim_{|x| \to 0} \frac{\sqrt{|x|(1 + |\log |x||)^\alpha}}{1 + |x|^{\eta}} = 0 = \lim_{|x| \to +\infty} \frac{\sqrt{|x|(1 + |\log |x||)^\alpha}}{1 + |x|^{\eta}},
\]
we deduce that
\[
\left( \int_{\mathbb{R}^3} |u|^q \, dx \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} \frac{u^2}{\sqrt{|x|(1 + |\log |x||)^\alpha}} \, dx \right)^\frac{1}{q}
\]
and then the conclusion follows by Lemma 2.4. \qed
3. Existence of a finite energy solution

In this section we assume $p \in (3, 6)$.

Looking at the proof of Theorem 1.1 (see in [2, 5, 8]), we have that for any $\varepsilon > 0$ there exists a solution $(u_\varepsilon, \phi_\varepsilon)$ to the problem $(P_\varepsilon)$ (positiveness can be deduced by standard arguments based on the maximum principle), where $u_\varepsilon$ is found as a critical point of the functional

$$J_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon u^2 + \varepsilon (2\omega - e\phi_u(x))\phi_u u^2\, dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p\, dx$$

and, called $c_\varepsilon$ the mountain pass level of $J_\varepsilon$, we have $J_\varepsilon(u_\varepsilon) \leq c_\varepsilon$.

Let $\varepsilon_n \to 0$ and consider the sequence $(u_n, \phi_n) \in H^1_0(\mathbb{R}^3) \times D^{1,2}_r(\mathbb{R}^3)$ of solutions of $(P_\varepsilon)$ built by Theorem 1.1 for $\varepsilon = \varepsilon_n$. We set $J_n = J_{\varepsilon_n}$ and call $c_n$ the corresponding mountain pass level.

By standard elliptic arguments, we can prove that both $u_n$ and $\phi_n$ are in $C^2(\mathbb{R}^3)$, for any $n \geq 1$. Moreover we have the following result about boundedness of the sequence.

**Proposition 3.1.** The sequence $(u_n)_n$ is bounded in $D^{1,2}_r(\mathbb{R}^3)$ and there exists $C > 0$ such that $\int_{\mathbb{R}^3} e\phi_n u_n^2\, dx \leq C$ for every $n \geq 1$. Moreover, there exists $M \geq 0$ such that $(u_n)_n$ is a sequence in $B_M$.

**Proof.** If $p \in (3, 4)$, then, proceeding as in [2, Theorem 1.1], by suitably combining the inequality $J_n(u_n) \leq c_n$ with Nehari and Pohozaev identities, we have that for any $\gamma \in \mathbb{R}$

$$D_{p,\gamma} \int_{\mathbb{R}^3} |\nabla u_n|^2\, dx + \int_{\mathbb{R}^3} [C_{p,\gamma} u_n + B_{p,\gamma} e\phi_n + A_{p,\gamma} e^2 \phi_n^2] u_n^2 \leq c_n$$

where

$$A_{p,\gamma} = \frac{1 + 2\gamma(p - 3)}{p},$$

$$B_{p,\gamma} = \frac{p - 10p\gamma - 4 + 24\gamma}{2p},$$

$$C_{p,\gamma} = \frac{(p - 2)(1 - 6\gamma)}{2p},$$

$$D_{p,\gamma} = \frac{p - 2p\gamma - 2 + 12\gamma}{2p}.$$

From the same computations as those in [2, Lemma A.1], we deduce that for $\gamma \in \left(\frac{2 - p}{2(6 - p)}, \frac{4 - p}{24 - 10p}\right)$ we have

$$A_{p,\gamma} > 0,$$

$$B_{p,\gamma} > 0,$$

$$C_{p,\gamma} > 0,$$

$$D_{p,\gamma} > 0,$$

and then, since it is a simple exercise to see that the sequence $(c_n)_n$ is bounded above, we conclude that both $(\|\nabla u_n\|_2)_n$ and $(\int_{\mathbb{R}^3} \phi_n u_n^2\, dx)_n$ are bounded.

If $p \in (4, 6)$, then we proceed as in [8] to obtain again boundedness of $(\|\nabla u_n\|_2)_n$ and $(\int_{\mathbb{R}^3} \phi_n u_n^2\, dx)_n$.

Since by the second equation in (7) we have

$$\int_{\mathbb{R}^3} |\nabla \phi_n|^2\, dx + e^2 \int_{\mathbb{R}^3} \phi_n^2 u_n^2\, dx = \int_{\mathbb{R}^3} e\omega \phi_n u_n^2\, dx,$$

we can establish the existence of some $M \geq 0$ such that every element in the sequence $(u_n)_n$ is in $B_M$.  \[\square\]

Now we proceed using the same notations and arguments as in [14]: by Proposition 2.1, we deduce that for any $\eta > \frac{1}{2}$ the sequence $(u_n)_n$ is bounded in $D^{1,2}_r(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V(x)dx)$.  \]
where \( V(x) = \frac{1}{|x|^2} \) and, by embedding theorems proved in [16, 17], \((u_n)_n\) is bounded also in \( L^q(\mathbb{R}^3)\) for all \( q \in \left( \frac{18}{5}, 6 \right)\). By standard compactness arguments based on the decay estimate holding for functions in \( D^{1,2}_r(\mathbb{R}^3)\) (see [7, Radial Lemma A.III]), this embedding is compact for \( q \in \left( \frac{18}{5}, 6 \right)\). Then, up to a subsequence, we have that there exists \( u_0 \in D^{1,2}_r(\mathbb{R}^3)\) such that

\[
\tag{10} u_n \rightharpoonup u_0 \text{ in } D^{1,2}_r(\mathbb{R}^3)
\]

and, for every \( q \in \left( \frac{48}{35}, 6 \right)\), \( u_0 \in L^q(\mathbb{R}^3)\) and

\[
\tag{11} u_n \rightarrow u_0 \text{ in } L^q(\mathbb{R}^3).
\]

In particular,

\[
\tag{12} \phi_n \rightarrow \phi_0 \text{ in } D^{1,2}_r(\mathbb{R}^3).
\]

Of course, \( u_0 \geq 0 \) and \( \phi_0 \geq 0 \). Moreover \( u_0 \neq 0 \) by (11) and the following result

**Proposition 3.2.** There exists \( C > 0 \) such that \( \| u_n \|_p \geq C \) uniformly.

**Proof.** Here we will follow an idea in [13]. Set \( R_0 > \max(1, C_1 M^2) \), where \( C_1 > 0 \) is the same as in Lemma 2.3, and define

\[
M[u] = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{R_0}^{+\infty} \left( \int_{R_0}^{+\infty} u^2(r)ru^2(s) \min(r, s) \, ds \right) \, dr
\]

and

\[
N[u] = \left[ \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \left( \int_{R_0}^{+\infty} \left( \int_{R_0}^{+\infty} u^2(r)ru^2(s) \min(r, s) \, ds \right) \, dr \right)^\frac{1}{2} \right)^\frac{1}{2} \right] ^{\frac{1}{2}}.
\]

Take \( C > 0 \) as in Proposition 2.5 in correspondence of \( p \) and \( R_0 \) and call \( u'_n(tx) = t^2 u_n(tx) \) for all \( t > 0 \). We have that

\[
\tag{13} \int_{\mathbb{R}^3} |u_n|^p \, dx = t^{3-2p} \int_{\mathbb{R}^3} |u'_n|^p \, dx \leq C t^{3-2p} N[u'_n].
\]

Now, for any \( n \geq 1 \), set \( t_n = (M[u_n])^{-\frac{1}{2}} \) so that \( M[u'_n] = t_n^3 M[u_n] = 1 \). Since for every \( v : \mathbb{R}^3 \rightarrow \mathbb{R} \) measurable we have that \( M[v] \leq 1 \) implies \( \frac{1}{2} (N[v])^2 \leq M[v] \), from (13) we deduce

\[
\tag{14} \int_{\mathbb{R}^3} |u_n|^p \, dx \leq C t_n^{3-2p} N[u'_n] \leq C \sqrt{2} (M[u_n])^{2p-3}. \]

Now, by our choice of \( R_0 \) and the fact that \( u_n \in B_M \) for every \( n \geq 1 \), by Lemma 2.3 and (14) we infer that for every \( n \geq 1 \)

\[
\int_{\mathbb{R}^3} |u_n|^p \, dx \leq C \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^3} \phi_n u_n^2 \, dx \right)^{\frac{2p-3}{3}}.
\]

Since the functions \( u_n \) satisfy the Nehari identity, we have

\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 + \varepsilon \omega \phi_n u_n^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon_n u_n^2 + \varepsilon (2\omega - e \phi_n(x)) \phi_n u_n^2 \, dx
\]

\[
= \int_{\mathbb{R}^3} |u_n|^p \, dx \leq C \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^3} \phi_n u_n^2 \, dx \right)^{\frac{2p-3}{3}}.
\]

The conclusion follows recalling the fact that \( p > 3 \). \( \square \)
Proposition 3.3. The couple \((u_0, \phi_0)\) solves \((P)\) in the sense of distributions, namely
\[
\int_{\mathbb{R}^3} (\nabla u_0 \nabla \psi + c(2\omega - c\phi_0)\phi_0 u_0 \psi - u_0^{p-1} \psi) \, dx = 0,
\]
\[
\int_{\mathbb{R}^3} \nabla \phi_0 \nabla \psi \, dx = \int_{\mathbb{R}^3} c(\omega - c\phi_0)u_0^2 \psi \, dx,
\]
for all \(\psi \in C_0^\infty(\mathbb{R}^3)\).

Taking into account (10), (11) and (12), the proof is definitely similar to that of [2, Theorem 1.2], so we omit it. Moreover, since for all \(q \in (\frac{18}{19}, 6)\) we have \(u_0 \in D^{1,2}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)\), a direct application of Holder inequality and standard density arguments show the following

Corollary 3.4. For all \(v \in D^{1,2}(\mathbb{R}^3) \cap L^h(\mathbb{R}^3)\) and \(w \in D^{1,2}(\mathbb{R}^3) \cap L^k(\mathbb{R}^3)\) with \((h, k) \in \left[1, \frac{3}{2}\right) \times \left[1, \frac{6}{7}\right)\)
\[
\int_{\mathbb{R}^3} (\nabla u_0 \nabla v + c(2\omega - c\phi_0)\phi_0 u_0 v - u_0^{p-1} v) \, dx = 0,
\]
\[
\int_{\mathbb{R}^3} \nabla \phi_0 \nabla w \, dx = \int_{\mathbb{R}^3} c(\omega - c\phi_0)u_0^2 w \, dx.
\]

We emphasize the fact that, until this step, we cannot establish a stronger form of relationship between the couple \((u_0, \phi_0)\) and the equations in \((P)\).

In particular, the possibility that our particle possesses infinite energy should compromise our theory making it inconsistent with any physical purpose. To this end, first we prove the following generalization of the Strauss’ radial Lemma [15]

Lemma 3.5. Let \(N \geq 3\) and \(q \in [2, 2N)\). Then there exists \(C > 0\) such that for any \(u \in D^{1,2}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)\) and \(|x| > 1\) we have
\[
|u(x)| \leq C \frac{\|\nabla u\|_2 + \|u\|_q}{|x|^\frac{2N-2}{2q}}.
\]

Moreover \(u\) is almost everywhere equal to a continuous function in \(\mathbb{R}^N \setminus \{0\}\).

Proof. Set \(u \in C_0^\infty(\mathbb{R}^3)\) and consider \(N \geq 3\). As in [15], we obtain the inequality
\[
r^{N-1}u^2(r) \leq \int_0^r [u'(s)]^2 + u^2(s)]s^{N-1} \, ds + mr^{N-2}u^2(r)
\]
where \(m = \frac{N-1}{2}\). Now we proceed with the following estimate
\[
\int_0^r u^2(s)s^{N-1} \, ds = \int_0^r u^2(s)s^{\frac{2(N-1)}{\eta}}s^{\frac{(N-1)(\eta-2)}{\eta}} \, ds
\]
\[
\leq \left( \int_0^r |u(s)|^\eta s^{N-1} \, ds \right)^\frac{2}{\eta} \left( \int_0^r s^{N-1} \, ds \right)^\frac{\eta-2}{\eta},
\]
and then, comparing the two inequalities, we arrive to
\[
r^{N-1}u^2(r) \leq C \left( \|\nabla u\|_2^2 + r \frac{N(u-2)}{\eta} \|u\|_q^2 \right) + mr^{N-2}u^2(r),
\]
that is
\[
\left( 1 - \frac{m}{r} \right) u^2(r) \leq C \frac{r}{\eta} \left( \|\nabla u\|_2^2 + \|u\|_q^2 \right)^2,
\]
corresponding to our estimate. We conclude by density arguments. \(\square\)

Now we can prove the following integrability result.

Proposition 3.6. The function \(u_0\) is in \(L^2(\mathbb{R}^3)\).
Proof. In this proof we combine ideas in [13] and [3] adapting them to our not trivial situation.

By contradiction, assume that \( \|u_0\|_2 = +\infty \). By Proposition 3.1 and (9), we know that there exists \( R_1 > 0 \) such that for any \( n \geq 1 \) and \( r > R_1 \),

\[
\phi_n(r) \geq \frac{e\omega}{2} \int_{R_1}^{+\infty} \frac{s^2}{\max(r, s)} u_n^2(s) ds.
\]

Now, for every \( r, s \) with \( 0 < r < s \), we set \( A^r_s = B_s \setminus B_r \), where \( B_r \) and \( B_s \) are the balls centered in \( 0 \) and with radius respectively \( r \) and \( s \). Since \( u_n \to u_0 \) in \( L^2(A^r_s) \) for every \( r < s \), we have that for any \( K > 0 \) there exists \( R_K > 0 \) for which

\[
\lim_n \|u_n\|_{L^2(A^r_{R_K})} > K.
\]

Then by (15) we have that there exist three positive numbers \( C, R_1 \) and \( R_2 \) and \( n_0 \in \mathbb{N} \) such that \( R_1 < R_2 \) and

\[
\phi_n(r) \geq \frac{e\omega}{2} \int_{R_1}^{+\infty} \frac{s^2}{\max(r, s)} u_n^2(s) ds \geq \frac{e\omega}{2r} \int_{R_1}^{R_2} s^2 u_n^2(s) ds > \frac{C}{r}
\]

for every \( r > R_2 \) and \( n \geq n_0 \). Since, up to a subsequence, \( \phi_n \to \phi_0 \) pointwise, we deduce that

\[
\phi_0(r) \geq \frac{C}{r}, 	ext{ for } r > R_2
\]

and, of course, \( \omega - e\phi_0 \geq 0 \). Since, exactly as in [13, Theorem 6.1], we have that for every \( R > 0 \) there exists \( \bar{R} > R \) such that \( u_0(\bar{R}) \leq \phi_0(\bar{R}) \), we can consider \( \bar{R} > R_2 \) such that the function \( \varphi \) defined as follows

\[
\varphi(x) = \begin{cases} 
0 & \text{if } |x| < \bar{R}, \\
(u_0 - \phi_0)_+ & \text{if } |x| \geq \bar{R}
\end{cases}
\]

is in \( D^{1,2}_r(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \) for any \( q \in \left( \frac{18}{7}, 6 \right] \), and by Lemma 3.5 and [7, Radial Lemma A.III] we have

\[
e(\omega - e\phi_0) u_0^2 - u_0^{q-1} > 0 \text{ in } [\bar{R}, +\infty[.
\]

Since \( \frac{9}{7} < \frac{18}{7} < \frac{9}{5} \), by Corollary 3.4 the function \( \varphi \) is a test function for the second equation, but it is not for the first. Then we approximate it by means of a family of cut off functions in the following way.

Define \( k : \mathbb{R}^3 \to [0, 1] \) as a smooth radial function, radially decreasing and such that \( k \equiv 1 \) in \( |x| \leq 1 \) and \( k \equiv 0 \) in \( |x| \geq 2 \). For any \( M > 0 \), define \( v_M = k_M \varphi \), where \( k_M(x) = k(x/M) \). Of course \( v_M \geq 0 \) and since \( \text{supp} v_M \) is compact and \( \nabla \varphi \in L^2(\mathbb{R}^3) \), we have that \( \varphi_M \) is a test function for both the equations in the system. Moreover

\[
\varphi_M \to \varphi \text{ in } L^q \text{ for all } q \in \left( \frac{18}{7}, 6 \right],
\]

and, taken an arbitrary \( h \in (\frac{18}{7}, 6) \),

\[
\|\nabla \varphi - \nabla \varphi_M\|_2^2 \leq C \int_{|x|\geq M} |\nabla \varphi|^2 dx + \frac{C}{M^2} \int_{A^M_2} \varphi^2 dx \\
\leq o_M(1) + \frac{C}{M^2} \|\varphi\|_h^2 |A^2_2|^\frac{a-2}{a}
\leq o_M(1) + \frac{C}{M^{a-2}} \|\varphi\|_h^2,
\]

and so

\[
\varphi_M \to \varphi \text{ in } D^{1,2}_r(\mathbb{R}^3).
\]
By Corollary 3.4,
\[
\int_{\mathbb{R}^3} (\nabla u_0 \nabla \varphi_M + e(2\omega - e\phi_0)\phi_0 u_0 \varphi_M - u_0^{p-1} \varphi_M) \, dx = 0,
\]
\[
\int_{\mathbb{R}^3} \nabla \phi_0 \nabla \varphi_M \, dx = \int_{\mathbb{R}^3} e(\omega - e\phi_0)u_0^2 \varphi_M \, dx
\]
so, comparing and using the fact that \((2\omega - e\phi_0)\phi_0 u_0 \varphi_M \geq 0,
\]
\[
\int_{\mathbb{R}^3} \nabla (u_0 - \phi_0) \nabla \varphi_M \, dx = \int_{\mathbb{R}^3} [-e(2\omega - e\phi_0)\phi_0 u_0 - e(\omega - e\phi_0)u_0^2 + u_0^{p-1}] \varphi_M \, dx
\]
\[
\leq \int_{\mathbb{R}^3} [-e(\omega - e\phi_0)u_0^2 + u_0^{p-1}] \varphi_M \, dx.
\]
Letting \(M\) go to \(+\infty\), by continuity we have
\[
\int_{\mathbb{R}^3} \nabla (u_0 - \phi_0) \nabla \varphi \, dx \leq \int_{\mathbb{R}^3} [-e(\omega - e\phi_0)u_0^2 + u_0^{p-1}] \varphi \, dx.
\]
By definition of \(\varphi\) and (17), we deduce
\[
\int_{|x| \geq R} |\nabla (u_0 - \phi_0)+|^2 \, dx = \int_{\mathbb{R}^3} \nabla (u_0 - \phi_0) \nabla \varphi \, dx
\]
\[
\leq \int_{\mathbb{R}^3} [-e(\omega - e\phi_0)u_0^2 + u_0^{p-1}] \varphi \, dx
\]
\[
= \int_{|x| \geq R} [-e(\omega - e\phi_0)u_0^2 + u_0^{p-1}] (u_0 - \phi_0)_+ \, dx \leq 0
\]
and then \(u_0 \leq \phi_0\) in \((\bar{R}, +\infty)\).

Now, possibly replacing \(\bar{R}\) with a larger value, by Lemma 3.5 we can assume \(u_0^{p-3}(r) < \frac{e\omega}{2}\) in \((\bar{R}, +\infty)\), so that, by (16),
\[
e(2\omega - e\phi_0(r))\phi_0(r) - u_0^{p-2}(r) \geq e\omega \phi_0(r) - \frac{e\omega}{2} u_0(r) \geq \frac{e\omega}{2} \phi_0(r) \geq \frac{e\omega C}{2} \frac{1}{r}
\]
Take \(\gamma \in (0, \frac{e\omega}{2\omega})\) consider the problem
\[
\begin{cases}
-\Delta w + \frac{\gamma}{|x|} w = 0 & \text{if } |x| > \bar{R}, \\
w = u_0 & \text{if } |x| = \bar{R}, \\
w \to 0 & \text{as } |x| \to +\infty
\end{cases}
\]
and let \(v\) be a radial solution. Now we again use the comparison principle by approximation. Consider the function \(\psi : \mathbb{R}^3 \to \mathbb{R}\) such that
\[
\psi(x) = \begin{cases}
0 & \text{if } |x| < \bar{R}, \\
(u_0 - v)_+ & \text{if } |x| \geq \bar{R}
\end{cases}
\]
As before, define \(\psi_M = k_M \psi\) and multiply the first equation of the system and equation
\[
-\Delta v + \frac{\gamma}{|x|} v = 0
\]
by \(\psi_M\) (which is a test function for both the equations) and integrate. Comparing, we obtain
\[
\int_{\mathbb{R}^3} \nabla (u_0 - v) \nabla \psi_M \, dx + \int_{\mathbb{R}^3} \frac{\gamma}{|x|} (u_0 - v) \psi_M \, dx
\]
\[
= \int_{\mathbb{R}^3} \left( u_0^{p-2} - e(2\omega - e\phi_0)\phi_0 + \frac{\gamma}{|x|} \right) u_0 \psi_M \, dx.
\]
Observe that for any $M \geq \tilde{R}$ it is
\[
\int_{\mathbb{R}^3} \frac{\gamma}{|x|} (u_0 - v) \psi_M \, dx = \int_{A_M^R} k_M \frac{\gamma}{|x|} |(u_0 - v)_+|^2 \, dx \geq 0
\]
and
\[
\int_{\mathbb{R}^3} \left( u_0^{p-2} - e(2\omega - e\phi_0)\phi_0 + \frac{\gamma}{|x|} \right) u_0 \psi_M \, dx = \int_{A_M^R} k_M \left( u_0^{p-2} - e(2\omega - e\phi_0)\phi_0 + \frac{\gamma}{|x|} \right) u_0 (u_0 - v)_+ \, dx \leq 0.
\]
Then
\[
\int_{\mathbb{R}^3} \nabla (u_0 - v) \nabla \psi_M \, dx \leq 0
\]
and, passing to the limit as $M$ goes to infinity, we have
\[
\int_{|x| \geq R} |\nabla (u_0 - v)_+|^2 \, dx \leq 0,
\]
and then $u_0 \leq v$ almost everywhere in $(\tilde{R}, +\infty)$. The contradiction arises since $v$ exponentially decays at infinity (see [1]).

Finally we conclude with the following

**Proof of Theorem 1.2.** Since $u_0 \in L^2(\mathbb{R}^3)$, by Proposition 3.3 and a density argument, we have that for all $(v, w) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$
\[
\int_{\mathbb{R}^3} (\nabla u_0 \nabla v + e(2\omega - e\phi_0)\phi_0 u_0 v - u_0^{p-1} v) \, dx = 0,
\]
\[
\int_{\mathbb{R}^3} \nabla \phi_0 \nabla w \, dx = \int_{\mathbb{R}^3} e(\omega - e\phi_0) u_0^2 w \, dx.
\]
We deduce that

1. by uniqueness $\phi_0 = \phi_{u_0}$,
2. by ellipticity $u_0 \in C^2(\mathbb{R}^3)$ and $\phi_{u_0} \in C^2(\mathbb{R}^3)$, and equations are satisfied pointwise,
3. by the strong maximum principle $u_0 > 0$ and $\phi_{u_0} > 0$,
4. by Berestycki - Lions’ radial lemma $\phi_{u_0}$ satisfies (FO),
5. since $(u_0, \phi_{u_0}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, the energy (4) and the charge (5) are finite.

**Remark 3.7.** Observe that, since $u_0 \in L^2(\mathbb{R}^3)$ and $\phi_{u_0} \in C^2(\mathbb{R}^3)$, the function $\phi_{u_0}$ satisfies (8) for $u = u_0$ and then, by direct computations, we deduce that there exist two positive constants $K_1$ and $K_2$ such that for any $r \geq 1$,
\[
\frac{K_1}{r} \leq \phi_{u_0}(r) \leq \frac{K_2}{r}.
\]
By this fact and using the same arguments as those in the proof of Proposition 3.6 (actually we do not need anymore truncations to apply the comparison principle), we show that $u_0$ decays exponentially at infinity.

We conclude that the majority of standing wave’s charge is localized inside a bounded region, in line with the particle-like interpretation.
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