COHOMOLOGY OF PRESHEAVES WITH ORIENTED WEAK TRANSFERS

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Abstract. Over a field of characteristic zero, we establish the homotopy invariance of the Nisnevich cohomology of homotopy invariant presheaves with oriented weak transfers, and the agreement of Zariski and Nisnevich cohomology for such presheaves. This generalizes a foundational result in Voevodsky’s theory of motives. The main idea is to find explicit smooth representatives of the correspondences which provide the input for Voevodsky’s cohomological architecture.

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Section 1. Introduction

The goal of the theory of motives is the construction of a universal cohomology theory for algebraic varieties. While an abelian category of mixed motives has not yet been constructed, Voevodsky has defined a triangulated category of effective motivic complexes over a field \( k \), denoted \( \text{DM}^{\text{eff}}(k) \), which has the properties one would expect of the derived category of motives \([28]\). Moreover, this category has proved useful in its own right, for example in Voevodsky’s celebrated proof of the Milnor conjecture.

This approach succeeds in large part by transforming questions about the geometric category \( \text{DM}^{\text{eff}}(k) \) into questions about a category of a more homological nature. First, \( \text{DM}^{\text{eff}}(k) \) is a subcategory of \( D^-(\text{Sh}_{Nis}(\text{SmCor}(k))) \), a category of complexes of sheaves. Now the crucial ingredient is Voevodsky’s functor \( \mathbf{RC} : D^-(\text{Sh}_{Nis}(\text{SmCor}(k))) \to \text{DM}^{\text{eff}}(k) \) which is left adjoint to the inclusion and identifies \( \text{DM}^{\text{eff}}(k) \) with the localization of \( D^-(\text{Sh}_{Nis}(\text{SmCor}(k))) \) at a collection of morphisms expressing the invertibility of the affine line. The construction
of $RC$ itself depends critically on the cohomological theory of presheaves with transfers, in particular the result that (over a perfect field $k$), the Zariski cohomology and Nisnevich cohomology of a homotopy invariant presheaf with transfers coincide, and this cohomology is itself homotopy invariant [27]. The main result of this paper generalizes Voevodsky’s cohomological result to a class of presheaves with a weaker form of transfer structure.

**Theorem 1.1** ([2.1, 6.12, 6.13]). Let $k$ be a field of characteristic zero, and let $F$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then the Nisnevich cohomology presheaves $H^*_\text{Nis}(-, F_{\text{Nis}})$ are homotopy invariant, and Zariski cohomology coincides with Nisnevich cohomology for $F$.

Our transfers are very close to the weak transfers considered by Panin-Yagunov [21], Yagunov [30], and Hornbostel-Yagunov [8]. Motivation for our definition comes from oriented $T$-spectra, but our definition and results are independent of the theory of $T$-spectra. See Definition 2.1 for details.

The reader will see that we rely quite heavily on Voevodsky’s architecture. Our strategy is simply to make explicit the geometric input ($\mathbb{A}^1$-families of correspondences) of the cohomological theory, and to investigate “how nice” these correspondences can be made. In short, we find that all of the necessary constructions can be realized as finite morphisms between smooth affine schemes with trivial sheaf of Kähler differentials. See Subsection 4.1 for a more detailed description of the main idea. Along the way, we prove results about presheaves with (oriented) weak transfers which may be of independent interest: vanishing on semilocal schemes is detected at the generic point (Corollary 3.3), the Mayer-Vietoris sequence for open subschemes of the affine line (Theorem 4.8), Nisnevich excision (Theorem 5.1), and the Gersten resolution (Theorem 6.14). At some places we need the field $k$ to be infinite, but the more serious reason for the restriction on the characteristic of $k$ is our use of a Bertini theorem.

The basic example of a presheaf with transfers is a presheaf of cycles [26]; the basic non-example is algebraic $K$-theory. (Examples are discussed in further detail in Subsection 2.3.) Adaptations of Voevodsky’s argument to a wider class of theories, especially with the goal of comparing algebraic cycles and $K$-theory, have appeared before. Mark Walker extended much of Voevodsky’s theory to $K_0$-presheaves, a context which includes algebraic $K$-theory, and proved that two definitions of motivic cohomology (one cycle-theoretic, the other $K$-theoretic) are equivalent. In their construction of the motivic spectral sequence, Friedlander and Suslin used the notion of pseudo-pretheory (a generalization of Voevodsky’s notion of pretheory) to obtain fibrations of $K$-theory spectra with supports over semilocal schemes from such fibrations initially defined over fields [3]. Combining Voevodsky’s arguments with known results about Witt groups, Panin extended the cohomological results (homotopy invariance and the coincidence of Zariski and Nisnevich cohomology) to the Nisnevich sheafification of the Witt groups [20]. It is interesting to note that Witt theory is not orientable. More recently, Heller, Østvær, and Voineagu developed an equivariant version of presheaves with transfers [7]. Other recent generalizations of Voevodsky’s construction of motives, namely reciprocity sheaves and presheaves with traces, are discussed briefly in Subsection 2.4.
We felt there was some intrinsic interest in isolating and making explicit the geometric aspects of [27]. Additionally, we imagine that the geometric constructions presented here might prove useful to future adaptations of [27] (and hence [28]).

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Section 2. Presheaves with (oriented weak) transfers

2.1. Definition and basic properties. For a field \(k\), we denote by \(\text{Sm}/k\) the category of smooth separated \(k\)-schemes of finite type.

**Definition 2.1.** Let \(A\) be an additive category, and \(F : (\text{Sm}/k)^{\text{op}} \to A\) a functor. We assume \(F\) is additive in the sense that \(F(X_1 \coprod X_2) \cong F(X_1) \oplus F(X_2)\). We say the presheaf \(F\) is homotopy invariant if the canonical map \(\text{pr}_1^* : F(X) \to F(X_1 \times \mathbb{A}^1)\) is an isomorphism for all \(X \in \text{Sm}/k\). We say \(F\) has weak transfers if for any \(X, Y \in \text{Sm}/k\), and any closed embedding (of \(Y\)-schemes) \(X \hookrightarrow Y \times \mathbb{A}^n\) such that \(f : X \to Y\) is finite, flat, and generically étale, and so that the normal bundle \(N_X(Y \times \mathbb{A}^n)\) is trivial (and trivialized via \(\psi\)), we are given maps \(f^\psi_* : F(X) \to F(Y)\) satisfying the following properties.

1. The \(f_*\)’s are compatible with disjoint unions: if \(X = X_1 \coprod X_2\) and \(f_i : X_i \to Y\) denotes the induced morphism, then the diagram:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{f_*} & F(Y) \\
\downarrow & & \downarrow \\
F(X_1) \oplus F(X_2) & \xrightarrow{f_{1*}, f_{2*}} & F(Y)
\end{array}
\]

commutes. We have suppressed the \(\psi\)’s since there is a canonical isomorphism \(N_X(Y \times \mathbb{A}^n) = N_{X_1}(Y \times \mathbb{A}^n) \oplus N_{X_2}(Y \times \mathbb{A}^n)\).

2. The \(f_*\)’s are compatible with sections \(s : Y \to X\) which are isomorphisms onto connected components of \(X\). In the notation of the previous property, supposing \(s\) is an isomorphism onto \(X_1\), then we require \(f_1_* = s_*\) (for any embedding and trivialization).

3. The \(f^\psi_*\)’s are functorial for embeddings \(g : Y' \hookrightarrow Y\) of principal smooth divisors such that \(X' := X \times_Y Y' \in \text{Sm}/k\). That is, given such a \(g\), the diagram:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{g^*} & F(X') \\
\downarrow & & \downarrow f^\psi_* \\
F(Y) & \xrightarrow{f^\psi_*} & F(Y')
\end{array}
\]

commutes. Corollary A.4 implies this diagram makes sense and defines \(\psi\).

4. The \(f^\psi_*\)’s are functorial for smooth morphisms \(g : Y' \to Y\).

5. The \(f^\psi_*\)’s are compatible with the addition of irrelevant summands: suppose \(i = (f, \beta) : X \hookrightarrow Y \times \mathbb{A}^n\) is an \(N\)-trivial embedding, with the normal bundle
trivialized via $\psi$. Let $(i, 0) = (f, 0, \beta) : X \hookrightarrow Y \times \mathbb{A}^1 \times \mathbb{A}^n$. Then, using the notation of Lemma A.2, there is a canonical identification $N^Y_0 \oplus \mathcal{O} = \mathcal{N}^Y_{(i, 0)}$. Hence $\psi$ induces a trivialization $(\psi, 0)$ of $N_{(i, 0)}$. Then the requirement is that $f^0_* = f^0_{(\psi, 0)} : \mathcal{F}(X) \to \mathcal{F}(Y)$.

If $A$ is the category $\textbf{Ab}$ of abelian groups, then we require all of the weak transfers $f_*$ to be homomorphisms of abelian groups. We say $\mathcal{F}$ has oriented weak transfers if every map $f^0_*$ is independent of the trivialization $\psi$. We say $\mathcal{F}$ has weak transfers for affine varieties if we are given weak transfers as described above whenever $X$ and $Y$ are affine. A morphism of presheaves with some kind of transfer structure is a morphism of presheaves compatible with all transfer maps.

Remark 2.2. Our notion of weak transfers is very close to the notion of weak transfers studied by Panin-Yagunov [21] and Yagunov [30]. Our condition (1) is exactly the additivity condition, and our conditions (3) and (4) identify the particular types of transversal base change needed in the proof. Our condition (2) is a stronger form of the normalization condition of [21, Property 1.7], [30, Proposition 3.3]. Actually, we need condition (2) only for certain $X \in \text{Sm}/k$ (e.g., open subschemes of the affine line over a local scheme) which appear in various constructions. Our condition (5) is not used by these authors; it is motivated by the weak transfers in $\mathcal{T}$-spectra, and it is used in the proof of “independence of embedding” below. Rigidity theorems for presheaves with weak transfers are established in [21], [30], and [8].

Basic properties. Suppose $A$ is an abelian category. Then the category of homotopy invariant presheaves with ((oriented) weak) transfers (for affine varieties) with values in $A$ is abelian; and the inclusion of this category into the category of $A$-valued presheaves with ((oriented) weak) transfers (for affine varieties) is exact. Any subpresheaf with ((oriented) weak) transfers (for affine varieties) of a homotopy invariant presheaf with the same transfers is homotopy invariant. Any presheaf with transfer extension of homotopy invariant presheaves with the same transfer structure is homotopy invariant.

2.2. Consequences of orientation. Under the hypothesis of homotopy invariance, if the weak transfers are independent of the normal bundle trivialization, then they are independent of the $N$-trivial embedding. This justifies the omission of the map $X \to \mathbb{A}^n$ in the notation for the weak transfer. For clarity in the proof, we will decorate the weak transfer with further notation for the map $X \to \mathbb{A}^n$.

Lemma 2.3 (independence of embedding). Suppose $\mathcal{F}$ is a homotopy invariant presheaf on $\text{Sm}/k$ with oriented weak transfers, and let $f : X \to Y$ be a finite, generically étale morphism in $\text{Sm}/k$. Suppose $\Omega^1_{X/k}$ and $\Omega^1_{Y/k}$ are trivial. Then the map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is independent of the choice of $N$-trivial embedding.

Proof. Let $(f, \alpha) : X \hookrightarrow Y \times \mathbb{A}^n$ and $(f, \beta) : X \hookrightarrow Y \times \mathbb{A}^m$ be $N$-trivial embeddings. Consider the closed immersion $\iota : \mathbb{A}^1 \times X \to \mathbb{A}^1 \times Y \times \mathbb{A}^n \times \mathbb{A}^m$ defined by $i(t, x) = (t, f(x), t\alpha, (1 - t)\beta)$. Since both $\mathbb{A}^1 \times X$ and $\mathbb{A}^1 \times Y$ have trivial sheaf of Kähler differentials, Lemma A.1 implies that the product of $\iota$ with a constant morphism $\mathbb{A}^1 \times X \to \mathbb{A}^{\dim Y + 1}$ is an $N$-trivial embedding. We denote the product morphism by $(t, \xi)$.

Homotopy invariance implies the maps $i_0^* : \mathcal{F}(\mathbb{A}^1 \times X) \to \mathcal{F}(X)$ are both inverse to the isomorphism induced by the projection, hence are equal. Now the
compatibility of the weak transfers with the inclusions 0 × Y, 1 × Y → \AA^1 × Y implies (1 × f, \iota)_* = (f, \iota|_{0×X})_* = (f, \iota|_{1×X})_*.

We have \iota(0, x) = (0, f(x), 0, \beta, c). Since the weak transfers are compatible with the addition of irrelevant summands, we have (f, \iota|_{0×X})_* = (f, \beta)_*. Similarly, since \iota(1, x) = (1, f(x), \alpha, 0, c) we have (f, \iota|_{1×X})_* = (f, \alpha)_*.

Next we have the analogue of [27, Prop. 3.11].

**Lemma 2.4** (factorization through rational equivalence). Let X, S be smooth k-schemes, and suppose f : X → \AA^1 × S is a finite, generically étale morphism which admits an N-trivial embedding. Furthermore suppose the base change X_i := X ×_{\AA^1 × S} i × S is smooth for i = 0, 1 ∈ \AA^1(k). Let g_0 : X_0 → X and f_0 : X_0 → S denote the induced morphisms.

Let \mathcal{F} be a homotopy invariant presheaf on Sm/k with oriented weak transfers. Then we have f_0^* ∘ g_0^* = f_1^* ∘ g_1^* : \mathcal{F}(X) → \mathcal{F}(S).

**Proof.** The compatibility with the transverse squares determined by the embeddings 0 × S, 1 × S → \AA^1 × S implies that i_0^* ∘ f_* = f_0^* ∘ g_0^* : \mathcal{F}(X) → \mathcal{F}(S) (and similarly for the fiber at 1 × S). Since i_0^* = i_1^*, the result follows.

The analogue of [27, Prop. 3.12] is immediate because our transfers are defined on the groups \mathcal{F}(X) rather than as homomorphisms from a group of cycles. Thus the requirement that \mathcal{F} be a presheaf gives the following result.

**Lemma 2.5.** Let f : X → Y be a finite, generically étale morphism in Sm/k which admits an N-trivial embedding. Suppose i : X → C is a closed immersion into a smooth Y-curve C, and suppose j : C ⊂ C' is open immersion of smooth Y-curves.

Let \mathcal{F} be a homotopy invariant presheaf on Sm/k with oriented weak transfers. Then we have f_* ∘ (j ∪ i)^* = f_* ∘ i^* ∘ j^* : \mathcal{F}(C') → \mathcal{F}(Y).

### 2.3. Examples.

Presheaves with weak transfers in the sense of Voevodsky [27] are presheaves with oriented weak transfers. More generally, the K_0-presheaves considered by Mark Walker [29] have oriented weak transfers. (One can ignore the embeddings and normal bundles. Given a finite flat morphism f : Y → X, the transpose of the graph of f determines a class in K_0(X, Y), hence a morphism f_* : \mathcal{F}(Y) → \mathcal{F}(X).

See [29, Lemma 5.6].) To get a feeling for the differences among these, suppose Z = Z_1 ∪ Z_2 is a B-relative zero-cycle in a smooth curve C → B with B ∈ Sm/k. If \mathcal{F} is a presheaf with transfers (or a pretheory), then

\[\phi_Z = \phi_{Z_1} + \phi_{Z_2} : \mathcal{F}(C) → \mathcal{F}(B).\]

If \mathcal{F} is a K_0-presheaf (or a pseudo-pretheory), then (2.1) holds if the ideal sheaf of Z_1 is trivial upon restriction to Z_2 (or vice versa), but in general (2.1) may fail to hold. If \mathcal{F} is a presheaf with oriented weak transfers, then (2.1) holds provided Z_1 and Z_2 are themselves smooth (in particular, multiplicity-free), the morphisms Z_i → B admit N-trivial embeddings, and Z_1 ∩ Z_2 = ∅. Without these conditions, one or both sides of (2.1) may not be defined. In short, in our setting there are fewer transfer morphisms, and it is more difficult to verify relations among them.

Further examples come from T-spectra: the 0-space of a T-spectrum E (or its homotopy (pre)sheaves) has weak transfers. See [21] or [13, §9] for details on the construction. The condition (5) in Definition 2.1 roughly corresponds to passing to the 0-space. That the weak transfers in T-spectra satisfy condition (5) boils
down to a lemma of Spitzweck [25, Lemma 3.5] which asserts there is a canonical isomorphism $T \wedge X/T \wedge (X - Z) = Th(O_X)/Th(O_{X-Z}) \cong Th(N_ZX \oplus O_Z)$ in $H_{\bullet}(k)$. Here $H_{\bullet}(k)$ is the homotopy category of the Morel-Voevodsky category of pointed simplicial presheaves on $\textbf{Sm}/k$ with the $A_1$-Nisnevich model structure [17, Thm. 2.3.2, Defn. 3.2.1].

If the $T$-spectrum $E$ is oriented, then a vector bundle automorphism of a vector bundle $V$ over $X \in \textbf{Sm}/k$ induces the identity map on $E(Th(V))$ [19, Defn. 3.1.1]. In particular the choice of trivialization of the normal bundle does not influence the weak transfer map, so such a spectrum has oriented weak transfers. Following Yagunov’s observation [30, p. 30], we point out that we only use independence of trivialization for normal bundles arising in our constructions, which is a bit different from the full strength of the orientation.

Additionally, the weak transfers are inherited by various “support” constructions on $T$-spectra $E$ (possibly after passing to homotopy presheaves), in particular the spectra $E^Q$ and $E^{(q)}$ considered in [24], which are geometric models for the slice tower of $E$. Note that, in characteristic zero, Levine has shown the higher slices of a (not necessarily orientable) $S^1$-spectrum $E$ have filtrations whose subquotients are complexes of homotopy invariant presheaves with transfers. For the zero slice, one has a similar result after taking homotopy sheaves of the loop space; see [14]. In fact our original motivation for proving homotopy invariance of cohomology was to develop localization machinery applicable to the presheaves of spectra $E^Q$, $E^{(q)}$ (maybe assuming $E$ oriented) and thereby extend to all quasi-projective varieties the main result of [24]. We encountered difficulties in applying the Friedlander-Lawson moving lemma (as in [4]) to a general cohomology theory $E$.

2.4. Context. Nisnevich sheaves of abelian groups with homotopy invariant cohomology are called strictly $A_1$-invariant by Morel [16, Defn. 7]. A large supply of homotopy invariant cohomology is provided by the following result of Morel: the homotopy sheaves (in degree $\geq 2$) of a pointed motivic space (i.e., object of $H_{\bullet}(k)$) are strictly $A_1$-invariant, and in degree 1 are strongly $A_1$-invariant [16, Cor. 5.2]. Homotopy modules are the motivic analogues of stable homotopy groups [1, p. 520]. A theorem of Dégilde identifies the homotopy modules with transfers as those, among homotopy modules, which are orientable [1]. Roughly speaking, this says homotopy invariance of cohomology and orientability together imply the existence of Voevodsky transfers. Roughly speaking, our result says (assuming homotopy invariance for the presheaf itself) orientability and weak transfers together imply the homotopy invariance of cohomology. There is an important conceptual difference between these results and ours: in the more structural approaches of Morel and Dégilde, the Nisnevich topology is built in from the beginning, whereas we work with presheaves, so that proving Nisnevich excision is one of the main challenges.

Incidentally, the result of Dégilde also suggests that the Nisnevich sheafification of a homotopy invariant presheaf with oriented weak transfers has Voevodsky transfers. Additional evidence is the following pair of results on $K_0$-presheaves: the Zariski separation of a homotopy invariant $K_0$-presheaf is again a homotopy invariant $K_0$-presheaf [29, Prop. 5.20], and a Zariski separated $K_0$-presheaf is a pretheory [29, Thm. 6.23].

Finally, we also wish to mention two recent expansions of Voevodsky’s theory in other directions. The theory of reciprocity sheaves aims to extend constructions in
the style of Voevodsky to capture non-homotopy invariant phenomena [10], [9], [11]. Kelly has defined presheaves with traces and the \( \ell \mathrm{dh} \) topology as a framework for motives in positive characteristic, using alterations of singular varieties as a replacement for resolution of singularities [12]. By contrast, our methods rely heavily on the hypothesis of homotopy invariance, and our goal is to work entirely with smooth correspondences.

SECTION 3. PRESHEAVES WITH WEAK TRANSFERS ON SEMILocal SCHEMES

3.1. Geometric Presentation Lemma. The following lemma enhances the presentation lemma of Panin-Zainoulline [22, 3.5] (which itself enhances the presentation lemma of Panin-Ojanguren [18, Sect. 10]) by controlling the singularities of the presentation at 0 and 1.

Lemma 3.1. Let \( k \) be a perfect infinite field. Let \( R \) be a semilocal essentially smooth \( k \)-algebra, and \( A \) an essentially smooth \( k \)-algebra which is finite over \( R[t] \). Suppose given an \( R \)-augmentation \( e : A \to R \) and that \( A \) is \( R \)-smooth at every prime containing \( I := \ker e \). Finally suppose given \( f \in A \) such that \( A/fA \) is \( R \)-finite. Then there exists \( s \in A \) such that:

(1) \( A \) is \( R[s] \)-finite;

(2) \( A/sA = A/I \times A/J \) for some ideal \( J \subset A \), with \( A/J \) essentially smooth and \( R \to A/J \) generically étale;

(3) \( J + fA = A \); and

(4) \( (s-1)A + fA = A \), with \( A/(s-1)A \) essentially smooth and \( R \cong R[s]/(s-1) \rightarrow A/(s-1)A \) generically étale.

Proof. One finds an element \( \alpha \in I \subset A \) with suitable vanishing properties. The main point is that for almost all \( r \in k^\times \), the element \( s := \alpha - rt^N \) (here \( N \) is large compared to the degrees of the coefficients \( p_i(t) \) of an equation expressing the integral dependence of \( \alpha \) on \( R[t] \); see the proof in [18] for further explanation of the notation) satisfies the conclusions. Then for almost all \( r \in k^\times \), the element \( s \) will have the required smoothness and étaleness properties.

The ideal \( I \) is locally principal since it is a section to smooth morphism of relative dimension 1. Locally on \( \mathrm{Spec} \ A \), we can write \( \alpha = aa', t = a't' \), with \( (a) = I \) and \( \alpha', t' \notin I \). By factoring out a suitable power of \( a \) we obtain a local defining equation for the residual part \( \mathrm{Spec} \ A/J \) of \( V(s) \). By Bertini’s theorem, \( V(s) \) is regular for most choices of \( r \in k^\times \). Since there is a finite cover of \( \mathrm{Spec} \ A \) on which \( I \) is principal, we can find an \( r \) which works uniformly. Similarly, most choices of \( r \) will produce a regular fiber at \( s = 1 \). The purity of the branch locus, together with the assumed \( R \)-smoothness along \( A/I \), implies the resulting finite extensions are generically étale.

□

3.2. Semilocal vanishing is detected generically. For a scheme \( X \) and a finite set \( S \subset X \), let \( \mathcal{O}_{X,S} \) denote the semilocalization of \( X \) at \( S \). For an irreducible variety \( X \), let \( \eta_X \in X \) denote its generic point. Since \( \eta_X \), for example, is not generally of finite type over \( k \), we define \( \mathcal{F}(\eta_X) \) as the limit of the values of \( \mathcal{F} \) on all (finite type) open subschemes of \( X \). For any open \( V \subset X \), let \( j_V : V \to X \) denote the open immersion. In this section we generalize results from [27, Sect. 4].

Theorem 3.2. Let \( k \) be a perfect infinite field. Suppose \( \mathcal{F} \) is a homotopy invariant presheaf of abelian groups on \( \mathrm{Sm}/k \) with weak transfers for affine varieties. Let \( X \)
be an irreducible smooth affine $k$-scheme, $\dim X = d$, such that $\Omega^1_{X/k} \cong O^d_X$. Let $S \subset X$ be a finite set, and $Z \to X$ a closed subset. Then there exists a neighborhood $U$ of $S$ and a map $a : \mathcal{F}(X \setminus Z) \to \mathcal{F}(U)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{j_*^X} & \mathcal{F}(X \setminus Z) \\
\downarrow j^* & & \downarrow a \\
\mathcal{F}(U) & & \mathcal{F}(U) 
\end{array}
$$

**Corollary 3.3.** For $k$ and $\mathcal{F}$ as in Theorem 3.2, and $U$ a nonempty open subscheme of a smooth semilocal $k$-scheme $S$, the map $\mathcal{F}(S) \to \mathcal{F}(U)$ is a split monomorphism. In particular the map $\mathcal{F}(\text{Spec}(O_{X,S})) \to \mathcal{F}(\eta_X)$ is a split monomorphism.

**Proof.** We can find a neighborhood of $S$ such that $\Omega^1_{X/k} \cong O^d_X$. □

**Corollary 3.4.** For $k$ and $\mathcal{F}$ as in Theorem 3.2, if $\mathcal{F}(\text{Spec} E) = 0$ for all fields $E \supset k$, then $\mathcal{F}_{\text{Zar}} = 0$.

**Corollary 3.5.** For $k$ and $\mathcal{F}$ as in Theorem 3.2, and $U$ a nonempty open subscheme of a smooth $k$-scheme $X$, the restriction map $\mathcal{F}_{\text{Zar}}(X) \to \mathcal{F}_{\text{Zar}}(U)$ is a monomorphism.

**Corollary 3.6.** Let $k$ be a perfect infinite field, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of homotopy invariant presheaves of abelian groups on $\text{Sm}/k$ with weak transfers for affine varieties. Suppose that for any field extension $E \supset k$ the morphism $\varphi_E : \mathcal{F}(\text{Spec} E) \to \mathcal{G}(\text{Spec} E)$ is an isomorphism. Then the morphism of associated sheaves $\varphi_{\text{Zar}} : \mathcal{F}_{\text{Zar}} \to \mathcal{G}_{\text{Zar}}$ is an isomorphism.

**Proof.** The presheaves $\ker \varphi$ and $\text{coker} \varphi$ inherit structures of homotopy invariant presheaves with weak transfers for affine varieties, so Theorem 3.2 applies to them. Since sheafification is exact it suffices to show the Zariski sheafification of the kernel and cokernel both vanish. By hypothesis we know they vanish on fields, hence their stalks vanish by Theorem 3.2. □

**Remark 3.7.** These results were obtained for $K_0$-presheaves by Mark Walker [29, 5.28, 5.30].

The proof of the following result is postponed because the methods are slightly more involved (see 5.9), in particular the orientation is involved.

**Corollary 3.8.** Let $k$ be a perfect infinite field. Suppose $\mathcal{F}$ is a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Let $S$ be a semilocal smooth $k$-variety and $S = U_0 \cup V$ a Zariski cover. Then there exists an open $U \subset U_0$ such that $S = U \cup V$ and the following sequence is exact:

$$
0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0.
$$

**Proof of Theorem 3.2.** By Quillen’s trick [23], we can find a finite surjective morphism $p : X \to \mathbb{A}^d$ such that composing with the projection away from the (say) first factor gives a morphism $X \to \mathbb{A}^{d-1}$ of relative dimension 1, smooth at $S$, and so that $Z$ is finite over $\mathbb{A}^{d-1}$. For any open $U \subset X$ (which we will assume contains
by applying \( \times_{\mathbb{A}^{d-1}} U \), we obtain the following diagram:

\[
\begin{array}{c}
X \times_{\mathbb{A}^{d-1}} U \xrightarrow{\text{cl.imm.}} X \xrightarrow{i} X \times_{\mathbb{A}^{d-1}} U \xrightarrow{\Delta} \mathbb{A}^1 \times U \cong \mathbb{A}^d \times_{\mathbb{A}^{d-1}} U
\end{array}
\]

We have an exact sequence of sheaves \( 0 \rightarrow p^\ast(\Omega^1_{X/k}) \rightarrow \Omega^1_X \rightarrow \Omega^1_{X/\mathbb{A}^{d-1}} \rightarrow 0 \), all of which are locally free near \( S \). Since \( \Omega^1_{\mathbb{A}^{d-1}/k} \cong \mathcal{O}_{\mathbb{A}^{d-1}}^{d-1} \), taking determinants shows \( \Omega^1_{X/\mathbb{A}^{d-1}} \cong \mathcal{O}_X \). As \( X \) is affine, the sequence also gives \( \Omega^1_{X/k} \cong \mathcal{O}_X \). (Hence also \( \Omega^1_{U/k} \cong \mathcal{O}_U \).)

By the geometric presentation lemma (Lemma 3.1), there is a morphism \( X \times_{\mathbb{A}^{d-1}} U \rightarrow \mathbb{A}^1 \) such that the induced morphism \( \pi : X \times_{\mathbb{A}^{d-1}} U \rightarrow U \times \mathbb{A}^1 \) of \( U \)-schemes has the following properties:

1. \( \pi \) is finite (hence flat, since source and target are regular of the same dimension);
2. \( \pi^{-1}(U \times 0) = \Delta(U) \bigcap R_0 \);
3. \( \delta : \Delta(U) \rightarrow U \) étale;
4. \( R_0 \) is regular (hence smooth), \( R_0 \subset (X \setminus Z) \times \mathbb{A}^{d-1} U \), and \( r_0 : R_0 \rightarrow U \) is generically étale; and
5. \( \pi^{-1}(U \times 1) =: F_1 \) is regular (hence smooth), \( F_1 \subset (X \setminus Z) \times \mathbb{A}^{d-1} U \), and \( f_1 : F_1 \rightarrow U \) generically étale.

We claim the morphism \( \pi \) admits an \( N \)-trivial embedding. By Lemma A.2 it suffices to show \( X \times_{\mathbb{A}^{d-1}} U \rightarrow U \) admits an \( N \)-trivial embedding. Since \( U \rightarrow \mathbb{A}^{d-1} \) is smooth, by Lemma A.3, it suffices to show \( X \rightarrow \mathbb{A}^{d-1} \) admits an \( N \)-trivial embedding. This is a consequence of Lemma A.1. We choose such an embedding. The geometric situation is summarized in the following diagram.

\[
\begin{array}{c}
X \times_{\mathbb{A}^{d-1}} U \xrightarrow{i} X \times_{\mathbb{A}^{d-1}} U \times \mathbb{A}^1 \xrightarrow{\pi} U \times \mathbb{A}^1
\end{array}
\]

Hence we have a commutative diagram in which the paired arrows are inverse isomorphisms, and the unlabeled arrows pointing down are induced by the base change \( 0 \rightarrow \mathbb{A}^1 \):

\[
\begin{array}{c}
\mathcal{F}(X \times \mathbb{A}^1) \xrightarrow{p_1^\ast} \mathcal{F}(X \times_{\mathbb{A}^{d-1}} U \times \mathbb{A}^1) \xrightarrow{i^\ast} \mathcal{F}(X \times_{\mathbb{A}^{d-1}} U) \xrightarrow{\pi^\ast} \mathcal{F}(U \times \mathbb{A}^1)
\end{array}
\]

We omit the choices of \( N \)-trivial embedding and trivializations in the notation, with the understanding that the embeddings are induced from an \( N \)-trivial embedding of \( \pi \), and whichever trivialization is used to define \( \pi^\ast \) is also used to define \( \delta_\ast, r_0\ast, f_1\ast \). Note that \( \delta_\ast \circ pr_{\mathcal{F}(\Delta)}(i_0^\ast) \circ p_1^\ast = j_U^\ast : \mathcal{F}(X) \rightarrow \mathcal{F}(U) \).
We can consider the similar diagram induced by $1 \mapsto \mathbb{A}^1$, replacing $\mathcal{F}(\Delta) \oplus \mathcal{F}(R_0)$ with $\mathcal{F}(F_1)$. Then we conclude $(\delta_+ + r_0) \circ i_0^* \circ p_1^* = f_1^* \circ i_1^* \circ p_1^* : \mathcal{F}(X) \to \mathcal{F}(U)$, as both are equal to $p_1^{\ast X} \circ \pi_+ \circ i^* \circ p_1^{\ast U} \circ \pi_1^{\ast X}$.

Since $R_0$ and $F_1$ avoid $Z$, we have a commutative diagram (for $C = R_0$ or $F_1$, $c = r_0, f_1$):

$$
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{p_1^*} & \mathcal{F}(X \times_{\mathbb{A}^{d-1}} U) \\
\downarrow{j_X/Z} & & \downarrow{(j_{X \times Z} \times 1)^*} \\
\mathcal{F}(X \setminus Z) & \xrightarrow{p_1^*} & \mathcal{F}(X \setminus Z \times_{\mathbb{A}^{d-1}} U)
\end{array}
$$

We conclude the map $(f_1^* \circ i_1^* \circ p_1^*) - (r_0 \circ p_{\mathcal{F}(R_0)}(i_0^*) \circ p_1^*) : \mathcal{F}(X) \to \mathcal{F}(U)$ factors through $j_{X \setminus Z}$. By our previous calculations, this map is exactly $j_{Z}^\ast$.

Observe we only used transfers along the finite generically étale morphism $\pi : X \times_{\mathbb{A}^{d-1}} U \to U \times \mathbb{A}^1$ and its base changes along $0, 1 \mapsto \mathbb{A}^1$, which are again generically étale by purity of the branch locus.

\section*{Section 4. Constructing smooth correspondences}

4.1. The main idea. In this section we analyze the geometric constructions used by Voevodsky [27]. A presheaf $\mathcal{F}$ has the structure of a pretheory if given any smooth relative curve $X \to S$ with $X, S \in \text{Sm}/k$ and any relative zero-cycle $Z$ on $X$, there is a transfer (wrong-way) homomorphism $\phi_Z : \mathcal{F}(X) \to \mathcal{F}(S)$. The collection of transfer homomorphisms is required to satisfy some natural conditions. This is a generalization of the more widely used notion of presheaf with transfers; presheaves with transfers are automatically pretheories of homological type.

Let $c_0(X/S)$ denote the group of relative zero-cycles on $X$ over $S$, and $h_0(X/S)$ the quotient of $c_0(X/S)$ by the relation of rational equivalence. If the pretheory $\mathcal{F}$ is in addition homotopy invariant, then the homomorphism $\phi_0 : c_0(X/S) \to \text{Hom}(\mathcal{F}(X), \mathcal{F}(S))$ factors through $h_0(X/S)$.

The main tool used to construct elements in $h_0(X/S)$ and to show relations among them is the following theorem of Suslin-Voevodsky [26, 3.1]: suppose $S$ is affine and $X \to S$ as above is quasi-affine and admits a compactification $\overline{X} \to S$ to a proper morphism of relative dimension 1, such that $\overline{X}$ is normal and $X_\infty := \overline{X} - X$ admits an $S$-affine neighborhood. Then $h_0(X/S)$ is isomorphic to the relative Picard group Pic($\overline{X}, X_\infty$), the group of isomorphism classes of pairs $(L, t)$, where $L$ is a line bundle on $\overline{X}$ and $t : L|_{X_\infty} \cong \mathcal{O}_{X_\infty}$ is a trivialization of the restriction. The map Pic($\overline{X}, X_\infty$) $\to h_0(X/S)$ is given by lifting the trivialization to a rational section, then taking the zero scheme. The ambiguity in the choice of lift corresponds exactly to $\mathbb{A}^1$-homotopy.

To ensure the zero schemes are effective relative divisors, we will need to lift elements of the relative Picard group to global regular sections. We will also need to ensure the zero schemes of these sections are smooth and multiplicity-free; this is accomplished by applying a Bertini theorem and we are thus restricted to characteristic zero. The explicit $\mathbb{A}^1$-homotopies constructed in Proposition 4.5 give the Mayer-Vietoris sequence for open subschemes of $\mathbb{A}^1$ (Theorem 4.8). In the next section similar constructions will be used to show that whether a local section of the presheaf $\mathcal{F}$ extends through a closed subset is invariant under Nisnevich covers (Theorem 5.1).
In the end the results of this section are restricted to the case where \( k \) is of characteristic zero, though several intermediate results do not require this assumption.

4.2. Mayer-Vietoris for open subschemes of the affine line. We establish the Mayer-Vietoris sequence for open subschemes of the affine line. As a consequence we compute the cohomology of open subschemes of \( \mathbb{A}^1 \) with coefficients in homotopy invariant presheaves with oriented weak transfers.

Proposition 4.1. Let \( k \) be a field. Let \( V \subset U \subset \mathbb{A}^1_k \) be nonempty open subschemes of the affine line over \( k \). On \( U \times \mathbb{P}^1 \) there exist a line bundle \( L \) and global sections \( s_0, s_1 \in H^0(U \times \mathbb{P}^1, L) \) with the following properties:

1. the zero scheme \( Z(s_0) \) is the disjoint union \( \Delta_U \sqcup R_0 \);
2. \( R_0 \) is smooth, \( U \)-finite, and contained in \( U \times V \);
3. the zero scheme \( Z(s_1) \) is smooth, \( U \)-finite, and contained in \( U \times V \);
4. \( s_0|_{U \times (\mathbb{P}^1 - U)} = s_1|_{U \times (\mathbb{P}^1 - U)} \); and
5. the intersection \( Z(s_0) \cap Z(s_1) \) is transverse.

Proof: We use the standard coordinates \( X_0, X_1 \) on \( \mathbb{P}^1 \), so that \( \infty \to \mathbb{P}^1 \) is defined by the vanishing of \( X_0 \), and the canonical coordinate on \( \mathbb{A}^1_{X_0 \neq 0} \subset \mathbb{P}^1 \). Let \( u \) denote the canonical coordinate on \( U \). Let \( F \) denote a homogeneous form generating the ideal of \( \mathbb{A}^1 - U \) with the reduced scheme structure, i.e., \( F \in \Gamma(\mathbb{P}^1, O_{\mathbb{P}^1}(\mathbb{A}^1 - U)) \). Let \( G \) denote a form generating the ideal of \( U - V \) with the reduced structure, and let \( s_\Delta = X_1 - uX_0 \) denote the canonical section of the diagonal bundle \( O(\Delta_U) \) on \( U \times \mathbb{P}^1 \). By rescaling we may assume \( F(X_0, X_1) = X_1^{\deg F} + \ldots \) and \( G(X_0, X_1) = X_1^{\deg G} + \ldots \). We let \( f, g \) denote the homogenizations of \( F, G \).

Let \( N = \deg(FG) + 1 \). The zero scheme of the section \( s_\Delta F(X)G(X) + X_0^N \in H^0(U \times \mathbb{P}^1, O(N)) \) has the following properties: it is disjoint from \( \Delta_U \); it is disjoint from \( U \times (\mathbb{P}^1 - V) \); it is the graph of a function \( V \to \mathbb{A}^1 \), hence it is smooth; and it is \( U \)-finite.

We set \( s_0 := s_\Delta (s_\Delta FG + X_0^N) \in H^0(U \times \mathbb{P}^1, L) \). Here \( L = O(\Delta_U) \otimes O(N) \cong O(N + 1) \). With this choice of \( s_0 \), properties (1) and (2) are satisfied. We write \( s_{R_0} := s_\Delta FG + X_0^N \).

Now we find sections \( s_1 \in H^0(U \times \mathbb{P}^1, L) \) satisfying (3)-(5). The method used here is explicit but does not readily generalize. There exist \( a(x), b(x) \in k[x] \) such that \( a(x)f(x) + b(x)g(x) = 1 \) and with \( \deg(b) < \deg(f) \) and \( \deg(a) < \deg(g) \).

Homogenizing we have (for some \( r > 0 \)):

\[
A(X)F(X)s_\Delta + B(X)G(X)s_\Delta = s_\Delta \cdot X_0^{N-r}.
\]

Now for any \( q(x) \) of degree \( \deg(g) + 2 \) such that \( f(x)q(x) = x^{N} + \ldots \) (i.e., the leading coefficient is 1), the section \( s_1(Q) := B(X)G(X)s_\Delta X_0^N + F(X)Q(X) \) has the following properties:

- along \( X_0 = 0 \), we have \( s_1(Q) = X_0^{N} (= s_0|_{U \times \infty}) \);
- along \( F = 0 \), we have \( s_1(Q) = s_\Delta X_0^{N} (= s_0|_{U \times (\mathbb{A}^1 - U)}) \); and
- along \( G = 0 \), we have \( s_1(Q) = F(X)Q(X) \).

The first two properties imply \( s_1(Q) = s_0 \) along \( U \times (\mathbb{P}^1 - U) \). The third implies \( s_1(Q) \) generates along \( U \times (U - V) \) for \( q \) generic. Now we show that having fixed \( B, G, F \), we can choose \( Q \) appropriately.
The first item shows $Z(s_1(Q)) \cap (U \times \infty) = \emptyset$ for any $Q$. Choosing $Q$ so that $Z(Q)$ is disjoint from $Z(GB)$, we see that $Z(s_1(Q)) \cap (U \times Z(GB)) \subset U \times (Z(F) \cap Z(GB))$. But the relation $af + bg = 1$ implies that $Z(F) \cap Z(GB) = \emptyset$, hence $Z(s_1(Q)) \cap (U \times Z(GB)) = \emptyset$ and $Z(s_1(Q))$ is the graph of the function $u = \frac{b(x)}{a(x)} + x$.

Hence for general $Q(X)$, the zero scheme $Z(s_1(Q))$ is smooth and disjoint from $U \times Z(G) = U \times (U - V)$.

We need to check $Q$ can be chosen so the intersections $Z(s_1(Q)) \cap \Delta_U$ and $Z(s_1(Q)) \cap R_0$, i.e.,

$$\{u = \frac{f(x)g(x)}{b(x)g(x)} + x\} \cap \{u = x\},$$

$$\{u = \frac{f(x)g(x)}{b(x)g(x)} + x\} \cap \left\{u = \frac{1}{f(x)g(x)} + x\right\},$$

are transverse. For general $q$ the polynomials $f_q$ and $f_q^2 - b$ have no multiple roots.

(The zero scheme of $F$ is reduced.)

\begin{proof} \textbf{Remark 4.2.} A more conceptual proof (valid only in characteristic zero) using methods which generalize goes as follows. Let $s_0$ be as in the proof. Consider the sheaf sequence:

$$0 \to I_{U \times (P^1 - V)} \otimes L \to L \to L|_{U \times (P^1 - V)} \to 0.$$ 

The bundle $I_{U \times (P^1 - V)} \otimes L$ induces the bundle $O(1)$ on every fiber, hence its $R^1\pi_{1*}$ vanishes. Since $U$ is affine, this implies the vanishing of $H^1$. Therefore the sheaf sequence remains exact after applying $H^0(U \times \mathbb{P}^1, -)$. (This also holds if we replace $I_{U \times (P^1 - V)}$ by $I_{U \times (P^1 - U)}$.) The Picard group of $U \times (U - V)$ is trivial. Then for any generator $g \in H^0(U \times (U - V), L|_{U \times (U - V)})$, the element $(s_0|_{U \times (P^1 - U)}, g) \in H^0(U \times (P^1 - V), L|_{U \times (P^1 - V)})$ lifts to a section $\tilde{s} \in H^0(U \times \mathbb{P}^1, L)$.

Choose such a generator $g$ and a lift $\tilde{s}$. Consider the pencil $\mathbb{P}_{s_0, \tilde{s}}$ determined by $s_0$ and $\tilde{s}$. First we observe this pencil does not contain $\Delta_U$ as a fixed component: $Z(\tilde{s}) \not\supset \Delta_U$ since $\tilde{s}$ does not vanish along $U \times (U - V)$, while $s_\Delta|_{U \times (U - V)}$ vanishes along $(U - V) \times (U - V) \neq \emptyset$. If the pencil $\mathbb{P}_{s_0, \tilde{s}}$ contains $R_0$ as a fixed component, then writing $\tilde{s} = s_1 \cdot s_0$, the pair $s_0 = s_\Delta, s_1$ for $L = \mathcal{O}(\Delta_U)$ satisfies the conclusion of the proposition. So we may assume the pencil contains neither $\Delta_U$ nor $R_0$ as a fixed component. Since we have the decomposition $Z(s_0) = \Delta_U \bigsqcup R_0$ into irreducible components, this implies the pencil does not contain any fixed component.

Every element $s_t \in \mathbb{P}_{s_0, \tilde{s}}$ of the pencil has the property that $s_t|_{U \times (P^1 - U)} = s_0|_{U \times (P^1 - U)}$. Since the element $\tilde{s} \in \mathbb{P}_{s_0, \tilde{s}}$ generates along $U \times (U - V)$, the same is true of the general element of the pencil.

Over a field of characteristic zero, the general element of a finite dimensional linear system on smooth quasi-projective variety is smooth away from the base locus of the linear system \cite[Cor. 10.9, Rmk. 10.9.2]{Ref}. However $Z(s_0)$ is smooth everywhere, hence the general element of $\mathbb{P}_{s_0, \tilde{s}}$ is smooth everywhere, i.e., even along the base locus. Note that it is enough to have a Bertini theorem over algebraically closed fields: if $k \xrightarrow{\cong} \overline{k}$ is an infinite subfield, the dense open locus of suitable sections in $\mathbb{P}^1$ contains many $k$-points essentially because $\mathbb{P}^1$ is a rational variety. For details see \cite[Prop. 2.8]{Ref}.

Finally we explain how $\tilde{s}$ can be modified to guarantee the intersection $Z(s_0) \cap Z(\tilde{s})$ is transverse. Let $e \in H^0(I_{U \times (P^1 - V)} \otimes L)$ be a section whose zero scheme is “horizontal,” i.e., independent of $u \in U$. Then $Z(s_0) \cap Z(e)$ is transverse since the
we define where the first map is induced by the morphism \( U \), scheme such that \( Z \) is the weak transfer along \( Z \). Corollary 4.4. \( Z \) generates along \( Z \). Let \( U \to X \times X' \) be a closed subscheme such that \( Z \to X \) finite flat and admits an \( N \)-trivial embedding. Then we define \( t_Z : F(X') \to F(X) \) to be the composition \( F(X') \to F(Z) \to F(X) \), where the first map is induced by the morphism \( Z \to X \times X' \to X' \) and the second is the weak transfer along \( Z \to X \).

**Corollary 4.4.** Let \( k \) be a field. Let \( F \) be a homotopy invariant presheaf on \( Sm/k \) with weak transfers for affine varieties. Let \( V \subseteq U \subseteq \mathbb{A}_k^n \) be nonempty open subschemes of \( \mathbb{A}_k^n \). Then the restriction map \( F(U) \to F(V) \) is injective.

**Proof.** The conditions guarantee the total space \( Z := Z(t \cdot s_0 + (1 - t) \cdot s_1) \subseteq \mathbb{A}_k^1 \times U \times \mathbb{P}^1 \) is \( k \)-smooth and \( (\mathbb{A}_k^1 \times U) \)-finite. The morphism \( Z \to \mathbb{A}_k^1 \times U \) admits an \( N \)-trivial embedding. Indeed the section \( d(t \cdot s_0 + (1 - t) \cdot s_1) \in H^0(Z, \Omega^1_{\mathbb{A}_k^1 \times U \times \mathbb{P}^1} |_Z) \) trivializes the conormal bundle of the embedding \( Z \to \mathbb{A}_k^1 \times U \times \mathbb{P}^1 \), which is canonically isomorphic to the conormal bundle of the embedding \( Z \to \mathbb{A}_k^1 \times U \times \mathbb{A}_k^1 \). Hence the homotopy invariance gives the relation:

\[
t_{\Delta} + t_{R_0} = t_{Z(s_0)} = t_{Z(s_1)} : F(U) \to F(U).
\]

Since the correspondences \( R_0 \) and \( Z(s_1) \) factor as \( U \to V \), we get:

\[
(t_{Z(s_1)} - t_{R_0}) \circ j^* = t_{\Delta} = id : F(U) \to F(U),
\]

and so \( j^* \) must be injective. \( \square \)

**Proposition 4.5.** Let \( k \) be a field of characteristic 0. Let \( U \subseteq \mathbb{A}_k^n \) be an open subscheme, and \( U = U_1 \cup U_2 \) a Zariski cover. Write \( U_{\infty} := \mathbb{P}^1 \setminus U \) and \( Z_i := U \setminus U_i \) for the pairwise disjoint reduced closed subschemes of \( \mathbb{P}^1 \). On \( U \times \mathbb{P}^1 \) there exist:

1. a line bundle \( M \); and
2. sections \( \gamma \in H^0(U \times \mathbb{P}^1, M) \) and \( s, s_1, s_2 \in H^0(U \times \mathbb{P}^1, \mathcal{O}(\Delta_U) \otimes M) \);

such that the following equalities hold:

1. \( s_1 = s_\Delta \cdot \gamma \) on \( U \times (U_{\infty} \coprod Z_1) \);
2. \( s_2 = s_\Delta \cdot \gamma \) on \( U \times (U_{\infty} \coprod Z_2) \); and
3. \( s_1 \cdot s_2 = s_\Delta \cdot \gamma \cdot s \) on \( U \times (U_{\infty} \coprod Z_1 \coprod Z_2) \).

The sections also satisfy:

1. \( \gamma \) generates along \( \Delta_U \) and along \( U \times (U_{\infty} \coprod Z_1 \coprod Z_2) \);
2. \( s_1 \) generates along \( U \times Z_2 \); and
3. \( s_2 \) generates along \( U \times Z_1 \);
4. \( Z(\gamma) \), \( Z(s_1) \), \( Z(s_2) \), \( Z(s) \) are \( k \)-smooth and \( U \)-finite; and
5. for \( i = 1, 2 \), the intersections \( Z(s_i) \cap (\Delta_{U_{12}} \coprod \mathbb{P}^1) \) and \( Z(s_i) \cap Z(s) \) are transverse in \( U_{12} \times \mathbb{P}^1 \).

Finally there is an open subscheme \( U_0 \subset U_{12} \) such that, letting \( (-)_0 \) denote the restriction, we have that \( Z(s_1)_0 \cap Z(s_2)_0 = \emptyset \) and \( Z(s)_0 \cap (\Delta_U \coprod \mathbb{P}^1 ) = \emptyset \).

**Remark 4.6.** The conditions imply that \( s_1 \) generates along \( U_1 \times (U_{\infty} \coprod Z_1) \), that \( s_2 \) generates along \( U_2 \times (U_{\infty} \coprod Z_2) \), and that \( s \) generates along \( U \times (U_{\infty} \coprod Z_1 \coprod Z_2) \); and moreover that \( s_2 = s \) on \( U \times Z_1 \) and \( s_1 = s \) on \( U \times Z_2 \).
Remark 4.7. For the application (Theorem 4.8), it would suffice to show the following weaker version of (5): for $i = 1, 2$, the intersections $Z(s_i) \cap Z(s)$ are transverse in $U_{12} \times \mathbb{P}^1$, possibly after ignoring irreducible components common to $Z(s_i)$ and $Z(s)$.

Proof. Let $F$ be a homogeneous form defining $U_\infty$, i.e., a global section of $O_{\mathbb{P}^1}(U_\infty)$; and let $G_1, G_2$ be forms defining $Z_1, Z_2$. Let $u$ be the coordinate on $U$ and $X_0, X_1$ the coordinates on $\mathbb{P}^1$. Then we set $\gamma := s_\Delta F(X) G_1(X) G_2(X) + X_0^n$, and $M = p_2^\ast (O(n))$. The section $\gamma$ generates along $\Delta_U$ and along $U \times (U_\infty \coprod Z_1 \coprod Z_2)$ as required.

Both $U \times Z_1$ and $U \times Z_2$ have trivial Picard group. We choose also sufficiently general generating sections $g_1$ of $M(\Delta_U) := M \otimes O(\Delta_U)$ along $U \times Z_2$ and $g_2$ along $U \times Z_1$. Now we explain how to find $s_1$ as desired; $s_2$ is gotten by switching the indices.

The bundle $\mathcal{I}_{U \times (U_\infty \coprod Z_1 \coprod Z_2)} \otimes M(\Delta_U)$ induces $O(2)$ on every fiber of the first projection, and $U$ is affine, so the sheaf sequence:

$$0 \to \mathcal{I}_{U \times (U_\infty \coprod Z_1 \coprod Z_2)} \otimes M(\Delta_U) \to M(\Delta_U) \to M(\Delta_U)|_{U \times (U_\infty \coprod Z_1 \coprod Z_2)} \to 0$$

remains exact after applying $H^0(U \times \mathbb{P}^1, -)$. So there are many global sections inducing $s_\Delta \cdot \gamma$ along $U \times (U_\infty \coprod Z_1)$ and $g_1$ along $U \times Z_2$. The general such lift is smooth since the zero scheme of $s_\Delta \cdot \gamma$ is smooth and the condition that the section generates along $U \times Z_2$ is more general than the behavior of $s_\Delta \cdot \gamma$ along $U \times Z_2$. More precisely one repeats the argument in Remark 4.2 (possibly changing the generating section $g_1$). This shows $Z(s_1)$ is $k$-smooth and intersects $\Delta \coprod Z(\gamma)$ transversely (in particular $Z(s_1)$ contains neither $\Delta$ nor $Z(\gamma)$ as an irreducible component).

To produce the section $s$, choose a general global section $s^a$ of $M(\Delta_U)$ such that $s^a$ agrees with $s_\Delta \cdot \gamma$ along $U \times U_\infty$, with $s_2$ on $U \times Z_1$, and with $s_1$ on $U \times Z_2$. We can find such a $s^a$ with smooth zero scheme since $Z(s_1)$ and $Z(s_2)$ are smooth. Then choose a general global section $e$ of $\mathcal{I}_{U \times (U_\infty \coprod Z_1 \coprod Z_2)} \otimes M(\Delta_U)$ with “horizontal” zero scheme.

The sections $s_1$ and $s_2$ cannot have horizontal zero schemes, so the pencil determined by $s^a$ and $e$ contains no component of $Z(s_1) \cup Z(s_2)$ as a fixed component. For $e$ sufficiently generic the pencil $\mathbb{P}(\lambda s^a + \mu e) \hookrightarrow U_1 \times \mathbb{P}^1 \times \mathbb{P}^1_{\lambda, \mu}$ will not have any base-points along $Z(s_1) \cup Z(s_2)$. Furthermore all elements of the pencil (written as $s^a + \frac{\mu}{\lambda} e$) have the correct behavior along $U \times (U_\infty \coprod Z_1 \coprod Z_2)$, so taking $s = s^a + \frac{\mu}{\lambda} e$, we can make the intersections transverse without disturbing the other properties.

To find the open subscheme $U_0 \subset U_{12}$, we simply discard the images of intersection points. More precisely we set $C := pr_1(Z(s_1) \cap Z(s_2)) \cup pr_1(Z(s) \cap (\Delta_{U_1} \coprod Z(\gamma)))) \hookrightarrow U_{12}$ and $U_0 := U_{12} - C$. The subscheme $U_0$ is nonempty since $Z(s_1)$, $Z(s_2)$ and $Z(s)$, $\Delta_{U_1} \coprod Z(\gamma)$ have no components in common.

\[ \square \]

Theorem 4.8. Let $k$ be a field of characteristic 0. Let $U \subset \mathbb{A}^1_k$ be an open subscheme, and $U = U_1 \cup U_2$ a Zariski cover. Write $U_{12} := U_1 \cap U_2$.

Let $F$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then the Mayer-Vietoris sequence:

$$0 \to F(U) \xrightarrow{j_{12}^1, j_{12}^2} F(U_1) \oplus F(U_2) \xrightarrow{j_{12,1}^1 - j_{12,2}^1} F(U_{12}) \to 0$$

is exact.
Proof. We have the open immersions (for $i = 1, 2$) : $j_i : U_i \to U$ and $j_{12/i} : U_{12} \to U_i$. We have also $j_{12} : U_{12} \to U$ and $j_0 : U_0 \to U_{12}$.

By Proposition 4.5 we obtain smooth correspondences $\gamma : U \to U_{12}, s_1 : U_1 \to U_{12}, s_2 : U_2 \to U_{12}$, and $s : U \to U_{12}$. These induce morphisms between various groups $F(-)$, e.g., we obtain a morphism $t_\gamma : F(U_{12}) \to F(\gamma) \to F(U)$ as described in Notation 4.3.

After applying $F$, these morphisms are subject to the following relations, where we have omitted $F(-)$ to simplify the notation. (The first equation, e.g., means $t_s \circ j_{12} = t_\gamma \circ j_{12} + \text{id} : F(U) \to F(U)$.)

\begin{align*}
(4.1) \quad j_{12} \circ s &= j_{12} \circ \gamma + \text{id} : U \to U \\
(4.2) \quad j_{12/1} \circ s_1 &= j_{12/1} \circ \gamma \circ j_1 + \text{id}_1 : U_1 \to U_1 \\
(4.3) \quad j_{12/2} \circ s_2 &= j_{12/2} \circ \gamma \circ j_2 + \text{id}_2 : U_2 \to U_2 \\
(4.4) \quad s_1 \circ j_{12/1} + s_2 \circ j_{12/2} \circ j_0 &= (s \circ j_{12} + \gamma \circ j_{12} + \text{id}_{12}) \circ j_0 : U_0 \to U_{12} \\
(4.5) \quad j_{12/1} \circ s_2 &= j_{12/1} \circ s \circ j_2 : U_2 \to U_1 \\
(4.6) \quad j_{12/2} \circ s_1 &= j_{12/2} \circ s \circ j_1 : U_1 \to U_2 \\
(4.7)
\end{align*}

Now we define the contracting homotopy, where we have written $j_\gamma$ for $j_\gamma^*$ in the top row to be consistent with the rest of the notation:

\[
\begin{array}{cccc}
0 & \xrightarrow{j_{12} \circ s = j_{12} \circ \gamma + \text{id}} & F(U) & \xrightarrow{j_{12/1} \circ s_1 = j_{12/1} \circ \gamma \circ j_1 + \text{id}_1} & F(U_1) \\
0 & \xrightarrow{0 + B_2} & F(U) & \xrightarrow{12/2 \circ s_2 = j_{12/2} \circ s \circ j_2} & F(U_2) \\
& & \xrightarrow{A_1, C - A_2} & \xrightarrow{1/2 \circ s \circ j_1 - s_1} & 0 \\
& & & & \xrightarrow{s_1 \circ j_{12/1} + s_2 \circ j_{12/2}} & F(U_{12}) \\
& & & & \xrightarrow{\gamma \circ j_{12} + \gamma \circ j_{12} + \text{id}_{12}} & F(U_{12}) \\
& & & & \xrightarrow{s \circ j_{12} + \gamma \circ j_{12} + s_2 \circ j_{12/2} + \gamma \circ j_{12} + \gamma \circ j_{12} - s_1} & F(U_{12}) \\
& & & & \xrightarrow{\text{id}_{12}} & 0
\end{array}
\]

where $B_2 = j_{12/2} \circ (s - \gamma), A_i = \gamma \circ j_i - s_i$, and $C = (\gamma - s) \circ j_2$.

We verify the homotopy relations (using the convention $dh - hd$).

- $j_2 \circ B_2 + j_1 \circ 0 \overset{\text{defn}}{=} j_2 \circ (j_{12/2} \circ (s - \gamma)) = j_{12} \circ (s - \gamma) \overset{4.1}{=} \text{id}$
- $-(j_{12/1} \circ A_1) \overset{\text{defn}}{=} -(j_{12/1} \circ (\gamma \circ j_1 - s_1)) \overset{4.2}{=} \text{id}_1$
- $B_2 \circ j_1 + j_{12/2} \circ A_1 \overset{\text{defn}}{=} j_{12/2} \circ (s - \gamma) \circ j_1 + j_{12/2} \circ (\gamma \circ j_1 - s_1) \overset{4.3}{=} j_{12/2} \circ s_1 - j_{12/2} \circ \gamma \circ j_1 + j_{12/2} \circ s_2 \circ j_1 = 0$
- $B_2 \circ j_2 + j_{12/2} \circ (C - A_2) = j_{12/2} \circ (s - \gamma) \circ j_2 + j_{12/2} \circ (\gamma - s) \circ j_2 + j_{12/2} \circ (s_2 - \gamma \circ j_2) \overset{4.4}{=} \text{id}_2$
- $0 \circ j_2 + j_{12/1} \circ (C - A_2) = j_{12/1} \circ (\gamma - s) \circ j_2 + j_{12/1} \circ (s_2 - \gamma \circ j_2) = -j_{12/1} \circ s \circ j_2 + j_{12/1} \circ s_2 \overset{4.5}{=} 0$
- $A_1 \circ j_{12/1} \circ (C - A_2) \circ j_{12/2} = \gamma \circ j_{12} - s_1 \circ j_{12/1} + (s - \gamma) \circ j_{12} + \gamma \circ j_{12} - s_2 \circ j_{12/2} = \gamma \circ j_{12} + s \circ j_{12} - s_1 \circ j_{12/1} - s_2 \circ j_{12/2} = \text{id}_{12}$

The final equality holds upon precomposition with $j_0$, and on the presheaf this corresponds to composing with the injection $F(U_{12}) \to F(U_0)$. Hence the equation holds by Corollary 4.4. \qed
Remark 4.9. Given a reduced but reducible divisor $D$ in $\mathbb{A}^n$ with smooth irreducible components $D_1, \ldots, D_r$, our strategy is to define $t_D$ as the sum $\sum_i t_{D_i}$, then verify relations among various $t_D$’s after removing the points of incidence. This requires the weak transfers to be oriented, or at least “oriented over all open subschemes of $A^1$."

More precisely, the definition of each $t_{D_i}$ requires a trivialization of the normal bundle $N_{D_i}(\mathbb{A}^n)$. If there are nontrivial incidences $D_i := D_i \cap D_j$, these trivializations will not glue to a trivialization of $N_{D_i \cup D_j}(\mathbb{A}^n - \cup_{i,j} D_{ij})$.

Here is a simple example which captures our situation étale locally. Consider $V(xy) = V(x) \cup V(y) \hookrightarrow \mathbb{A}^2$. Then $dy$ trivializes the conormal bundle $N^*_y(\mathbb{A}^2)$ and $dx$ trivializes the conormal bundle $N^*_x(\mathbb{A}^2)$. These do not glue to a trivialization of the conormal bundle $N^*_{(xy)}(\mathbb{A}^2) - (0,0)$. Indeed $d(xy) = xdy + ydx$ trivializes $N^*_{(xy)}(\mathbb{A}^2) - (0,0)$, but restricts to $xdy$ on $V(y)$ and to $ydx$ on $V(x) - 0$.

In this example, we require the weak transfers to be compatible with the open immersion $V(x) - 0 \cong \mathbb{A}^1 - 0 \subset V(x) \cong \mathbb{A}^1$, and also that the trivialization $dx|_{V(x) - 0}$ induces the same weak transfer as the trivialization $ydx$. Thus the proof of Theorem 4.8 requires the weak transfers to be oriented. In general, discarding some incidence signals the orientation is necessary.

Lemma 4.10. Let $\mathcal{F}$ be a presheaf of abelian groups on $\text{Sm}/k$, and suppose for any subscheme $U \subset \mathbb{A}^1_k$ and any Zariski covering $U = U_1 \cup U_2$ we have that the sequence:

$$0 \to \mathcal{F}(U) \to \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \to \mathcal{F}(U_1 \cap U_2) \to 0$$

is exact. Then for all such $U$ we have $H^0_{\text{Zar}}(U, \mathcal{F}_{\text{Zar}}) = \mathcal{F}(U)$ and $H^i_{\text{Zar}}(U, \mathcal{F}_{\text{Zar}}) = 0$ for $i \neq 0$.

Proof. This isolates the implication shown in [27, Thm. 4.15].

Corollary 4.11. Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then for any open subscheme $U \subset \mathbb{A}^1_k$, we have $H^0_{\text{Zar}}(U, \mathcal{F}_{\text{Zar}}) = \mathcal{F}(U)$ and $H^i_{\text{Zar}}(U, \mathcal{F}_{\text{Zar}}) = 0$ for $i \neq 0$.

Proof. Combine Theorem 4.8 and Lemma 4.10.

Corollary 4.12. Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then the morphism $\mathbb{A}^1_k \to \text{Spec } k$ induces an isomorphism $\mathcal{F}_{\text{Zar}}(\text{Spec } k) \cong \mathcal{F}_{\text{Zar}}(\mathbb{A}^1_k)$.

4.3. The affine line over a smooth local scheme. We will also need the injectivity of the restriction map for open subschemes of the affine line over a local base.

Proposition 4.13. Let $k$ be a field of characteristic 0, and let $S$ be a smooth local $k$-scheme. Let $V \subset U \subset \mathbb{A}^1_S$ be nonempty open subschemes with $0_S \hookrightarrow V$, and suppose $U - 0_S$ is affine. On $U \times_S \mathbb{P}^1_S$ there are a line bundle $L$ and global sections $s_0, s_1 \in H^0(U \times_S \mathbb{P}^1_S, L)$ with the following properties:

1. the zero scheme $Z(s_0|_{U - 0_S} \times_S \mathbb{P}^1_S)$ is the disjoint union $U - 0_S \coprod R_0$;
2. $R_0$ is $k$-smooth, $(U - 0_S)$-finite, and contained in $(U - 0_S) \times_S (V - 0_S)$;
(3) the zero scheme $Z(s_1)$ is $k$-smooth, $(U - 0_S)$-finite, and contained in $(U - 0_S) \times_S (V - 0_S)$;

(4) $s_0|(U-0_S)\times_S (0_S \amalg (\mathbb{P}^1_S-U)) = s_1|(U - 0_S)\times_S (0_S \amalg (\mathbb{P}^1_S-U))$; and

(5) the intersection $Z(s_0) \cap Z(s_1)$ is transverse.

Proof. We follow the proof of Proposition 4.4 in Remark 4.2: since $U - 0_S$ is affine, if a coherent sheaf $M$ on $(U - 0_S) \times_S \mathbb{P}^1_S$ satisfies $R^1pr_1_*M = 0$, then $H^1((U - 0_S) \times_S \mathbb{P}^1_S, M) = 0$. Less significantly, $0_S$ plays the role of $\infty$.

We use the standard coordinates $X_0, X_1$ on $\mathbb{P}^1_S$, so that $0_S \hookrightarrow \mathbb{P}^1_S$ is defined by the vanishing of $X_1$, and $\Delta_{\mathbb{P}^1_S}$ is the canonical coordinate on $(\mathbb{A}^1_S)_{X_0 \neq 0} \subset \mathbb{P}^1_S$. Let $u$ denote the canonical coordinate on $U$. We have Pic$(U \times_S \mathbb{P}^1_S) \cong \mathbb{Z}$ via the degree in the second factor. Let $F$ denote a homogeneous form (i.e., section of some $\mathcal{O}(a)$ on $\mathbb{P}^2_S$) such that $Z(F) \hookrightarrow \mathbb{P}^1_S$ satisfies $Z(F) \supset \mathbb{P}^1_S - U$, and let $G$ be such that $Z(G) \supset U - V$. Note we can achieve that $Z(FG) \cap 0_S = \emptyset$. Let also $s_\Delta = X_1 - uX_0$ denote the canonical section of the diagonal bundle $\mathcal{O}(\Delta_U)$ on $U \times_S \mathbb{P}^1_S$.

Now we let $\gamma := s_\Delta FG + X_N^1 \in H^0(U \times_S \mathbb{P}^1_S, pr_2^*\mathcal{O}(N))$ and $s_0 := s_\Delta \cdot \gamma \in H^0(U \times_S \mathbb{P}^1_S, \mathcal{O}(\Delta_U) \otimes pr_2^*\mathcal{O}(N))$. Of course $R_0 = Z(\gamma)$ and $L = \mathcal{O}(\Delta_U) \otimes pr_2^*\mathcal{O}(N) \cong pr_2^*\mathcal{O}(N + 1)$. To verify the properties of $Z(s_0|(U-0_S)\times_S \mathbb{P}^1_S)$ we observe the following:

- $\gamma|_{U \times_S Z(FG)} = X_N^1|_{U \times_S Z(FG)} \neq 0$;
- $\gamma|_{\Delta_U}$ has a zero only along $0_S \times S 0_S$, hence $\gamma|_{\Delta_U-0_S}$ is nowhere zero; and
- the restriction $\gamma|_{U_{\kappa(S)} \times_S \mathbb{P}^1_S}$ is the graph of a rational map $\mathbb{P}^1_{\kappa(S)} \to U_{\kappa(S)}$, hence the closed fiber $(R_0)_{\kappa(S)}$ is $k$-smooth, hence $R_0$ is $k$-smooth.

Consider the exact sheaf sequence:

$$0 \to \mathcal{I}_{(U-0_S)\times_S (0_S \amalg Z(FG))} \otimes L \to L \to L|_{(U-0_S)\times_S (0_S \amalg Z(FG))} \to 0.$$

The bundle $\mathcal{I}_{(U-0_S)\times_S (0_S \amalg Z(FG))} \otimes L$ induces $\mathcal{O}(1)$ on every fiber, hence has no $R^1pr_1_*$. Since $U - 0_S$ is affine, this bundle has no $H^1$ and the sequence remains exact after applying $H^0(-)$. The Picard group of $(U-0_S) \times_S Z(G)$ is trivial. Therefore any element of $H^0(L|_{(U-0_S)\times_S (0_S \amalg Z(FG))})$ given by a pair of the form:

$$(s_0 \text{ along } (U-0_S) \times_S (0_S \amalg Z(F)), \text{ some generator } g \text{ along } (U-0_S) \times_S Z(G)),$$

can be lifted to a global section of $L$. Since $Z(s_0)$ is $k$-smooth, for a general lift $s_1$ of $(s_0, g)$ for a general $g$, we will have $Z(s_1)$ is $k$-smooth and intersects $Z(s_0)$ transversely.

To see $R_0 = Z(\gamma)$ and $Z(s_1)$ are contained in $(U - 0_S) \times_S (V - 0_S)$, we notice both $\gamma$ and $s_1$ were chosen to generate $L$ along $(U - 0_S) \times_S (0_S \amalg Z(FG)) \supset (U - 0_S) \times_S (0_S \amalg (\mathbb{P}^1_S - V))$. □

Corollary 4.14. Let $k$ be a field of characteristic 0. Let $F$ be a homotopy invariant presheaf on $\text{Sm}_k$ with weak transfers for affine varieties. Let $S$ be a smooth local $k$-scheme, and let $0_S \to V \subset U \subset \mathbb{A}^1_k$ be open subschemes of $\mathbb{A}^1_k$. Then the restriction map $F(U - 0_S) \to F(V - 0_S)$ is injective.

Proof. The conditions guarantee the total space $Z(t \cdot s_0 + (1 - t) \cdot s_1) \hookrightarrow \mathbb{A}^1_k \times_k (U - 0_S) \times_S \mathbb{P}^1_S$ is $k$-smooth and finite over $\mathbb{A}^1_k \times_k (U - 0_S)$, with trivial normal bundle. Hence the homotopy invariant gives the relation:

$$t_{\Delta_{U-0_S}} + t_{R_0} = t_{Z(s_0)} = t_{Z(s_1)} : F(U - 0_S) \to F(U - 0_S).$$
Since the correspondences $R_0, Z(s_1)$ factor as $U - 0_S \to V - 0_S \overset{\partial}{\to} U - 0_S$, we get:

\[(t_{Z(s_1)} - t_{R_0}) \circ j^* = t_{\Delta_{U - 0_S}} = \text{id} : \mathcal{F}(U - 0_S) \to \mathcal{F}(U - 0_S),\]

and so $j^*$ must be injective.

The constructions made here are similar to the constructions we will make in Proposition 5.3. We employ the following notation. On $\mathbb{A}^1_S \times S \mathbb{P}^1_S$, $u$ is the canonical coordinate on the first factor and $X_0, X_1$ are homogeneous coordinates on the second factor. In the second factor we have $0_S = Z(X_1), \infty_S = Z(X_0), 1_S = Z(X_1 - X_0)$; also $s_\Delta = X_1 - uX_0$, and $0_S = V(u)$ in the first factor. Finally we set $U_\infty := \mathbb{P}^1_S - U$.

**Proposition 4.15.** Let $k$ be an infinite field, and let $S$ be a smooth local $k$-scheme. Let $U \subset \mathbb{A}^1_S$ be an affine open neighborhood of $0_S \hookrightarrow \mathbb{A}^1_S$. On $\mathbb{A}^1_S \times S \mathbb{P}^1_S$ there are a line bundle $L$ and global sections $s_0, s_1 \in H^0(\mathbb{A}^1_S \times S \mathbb{P}^1_S, L), s_2 \in H^0(U \times S \mathbb{P}^1_S, L)$ with the following properties:

1. $s_0$ and $s_1$ agree along $\mathbb{A}^1_S \times S U_\infty$ and generate there;
2. along $\mathbb{A}^1_S \times S 0_S$ we have $\frac{s_1}{s_0} = u$, the canonical coordinate on the first factor;
3. $s_1$ generates along $\mathbb{A}^1_S \times S 0_S$;
4. $s_\Delta \cdot s_1 = (X_1 - X_0) \cdot s_0$ along $\mathbb{A}^1_S \times S (\infty_S \coprod 0_S)$; both are sections of $L \otimes \text{pr}^*_2 \mathcal{O}(1)$;
5. $s_\Delta \cdot s_2 = (X_1 - X_0) \cdot s_0$ along $U \times S (U_\infty \coprod 0_S)$;
6. $s_2$ generates along $U \times S (U_\infty \coprod 0_S)$;
7. the incidence $Z(s_1) \cap \Delta$ is disjoint from $0_S \times S \mathbb{P}^1_S$;
8. the incidence $Z(s_0) \cap (\mathbb{A}^1_S \times 1_S)$ is disjoint from $0_S \times S \mathbb{P}^1_S$;
9. there is a closed subset $Z_1 \hookrightarrow \mathbb{A}^1_S$ disjoint from $0_S$ such that:
   a) $Z(t \cdot s_\Delta \cdot s_1 + (1 - t) \cdot (X_1 - X_0) \cdot s_0) \hookrightarrow \mathbb{A}^1_S \times S (\mathbb{A}^1_S - Z_1 - 0_S) \times S (\mathbb{A}^1_S - 0_S)$ is smooth,
   b) the fiber at $t = 0$ is the disjoint union $((\mathbb{A}^1_S - Z_1 - 0_S) \times S 1_S) \coprod Z(s_0)$, and
   c) the fiber at $t = 1$ is the disjoint union $\Delta_{\mathbb{A}^1_S - Z_1 - 0_S} \coprod Z(s_1)$;
10. the incidence $Z(s_2) \cap \Delta$ is disjoint from $0_S \times S \mathbb{P}^1_S$;
11. there is a closed subset $Z_2 \hookrightarrow U$ disjoint from $0_S$ such that:
   a) $Z(t \cdot s_\Delta \cdot s_2 + (1 - t) \cdot (X_1 - X_0) \cdot s_0) \hookrightarrow \mathbb{A}^1_S \times S (U - Z_2 - 0_S) \times S (U - 0_S)$ is smooth,
   b) the fiber at $t = 0$ is the disjoint union $((U - Z_2 - 0_S) \times S 1_S) \coprod Z(s_0)$, and
   c) the fiber at $t = 1$ is the disjoint union $\Delta_{U - Z_2 - 0_S} \coprod Z(s_2)$; and
12. via the first projections, the zero schemes are finite and admit $N$-trivial embeddings.

The conditions (2) and (3) imply that $s_0$ generates along $(\mathbb{A}^1_S - 0_S) \times S 0_S$. The conditions (4) and (5) imply $s_2 = s_1$ along $U \times S (\infty_S \coprod 0_S)$. With (2) this implies that $\frac{s_1}{s_2} = u$ along $U \times S 0_S$.

**Proof.** The complement of $U \subset \mathbb{P}^1_S$ is an effective Cartier divisor, so we write $Z(F) = U_\infty$ with $F \in \Gamma(\mathbb{P}^1_S, \mathcal{O}(a))$ for some $a \in \mathbb{Z}_{\geq 0}$. Note that $U_\infty \supset \infty_S$, so $X_0$ divides $F$. Now we take $L = \text{pr}^*_2(\mathcal{O}(N))$ with $N = a + 1$. Additionally we choose a global section $A$ of $L|_{(U \times S)U_\infty}$ such that $A|_{U \times S U_\infty} = \frac{X^N_0(X_1 - X_0)}{s_\Delta}|_{U \times S U_\infty}$, and such that $A$ vanishes along $U \times S 0_S$. (This is possible since $\mathcal{T}_{\mathbb{A}^1_S \times S(\infty_S \coprod 0_S)} \otimes L$ is trivial
on every fiber of the first projection, hence has no \( H^1 \).) In particular \( A \) generates along \( U \times_S U_\infty \).

We make the following observations. The form \( F \) is homogeneous in the variables \( X_0, X_1 \).

- The zero schemes of the sections \( X_1^N \) and \( (X_1 - X_0) \cdot F \) are disjoint on \( \mathbb{A}^1_S \times_S \mathbb{P}_S^1 \) since \( Z(F) \cap \emptyset = \emptyset \), so the total space of the pencil they determine is smooth. Indeed even over the residue field, the general element of the pencil \( \mathbb{P}(\lambda X_1^N + \mu(X_1 - X_0) \cdot F) \hookrightarrow \mathbb{P}^1_{S(u)} \times \mathbb{P}^1_{A,\mu} \) is reduced.
- The zero schemes of the sections \( X_1^N \) and \( s_\Delta \cdot F \) are disjoint on \( (\mathbb{A}^1_S - 0_S) \times_S \mathbb{P}_S^1 \) (If \( X_1 = 0 \) then \( F \neq 0 \), so a common zero occurs at \( X_1 = X_1 - uX_0 = 0 \), but \( u \) is invertible so the equations imply \( X_0 = X_1 = 0 \).) Hence the total space of this pencil is also smooth. (In fact a general element of this pencil is the graph of a morphism even over the residue field, since \( Z(X_1^N + s_\Delta \cdot F) \cap Z(X_0 \cdot F) = \emptyset \).
- Since \( A \) generates along \( U \times_S Z(F) \), the pencil determined by \( A \) and \( F \cdot X_0 \) on \( U \times_S \mathbb{P}_S^1 \) is base-point free. So the total space \( \mathbb{P}(\lambda A + \mu F \cdot X_0) \hookrightarrow U \times_S \mathbb{P}^1_S \times \mathbb{P}^1_{A,\mu} \) is smooth over \( k \).

Therefore possibly after rescaling \( F \) by an element from the ground field, we have that \( s_1 := X_1^N + (X_1 - X_0) \cdot F \) and \( s_0 := X_1^N + s_\Delta \cdot F \in H^0(\mathbb{A}^1_S \times_S \mathbb{P}_S^1, L) \), and also \( s_2 := A + F \cdot X_0 \in H^0(U \times_S \mathbb{P}^1_S, L) \), have smooth zero schemes.

Additionally we can achieve that the section \( X_1^N + s_\Delta \cdot F \) is nonzero along \( 0_S \times 1_S \): if we write \( F = \sum f_i X_0^i X_1^{n-i} \) with \( f_i \in \Gamma(S, \mathcal{O}_S) \), then we have \( X_1^N + s_\Delta \cdot F |_{0_S \times 1_S} = X_1^N(1 + \sum f_i) \). So we require that \( 1 + \sum f_i \in \Gamma(S, \mathcal{O}_S)^{\times} \).

There is one further open requirement on \( s_2 \): having chosen the sections \( s_0, X_1 - X_0, s_\Delta \), their zero schemes are finite over \( \mathbb{A}^1_S \), in particular finite over the local scheme \( S \cong 0_S = V(u) \hookrightarrow \mathbb{A}^1_S \). Therefore the scheme \( Z(s_0) \cup Z(X_1 - X_0) \cup Z(s_\Delta) \) has finitely many closed points lying over \( V(u) \times_S \mathbb{P}_S^1 \). Now we require the section \( s_2 \) is nonzero at all of these points. This can be achieved since \( s_2 \) was chosen from a base-point free pencil. Thus (10) is satisfied.

It is now straightforward to check the equalities and generating behavior asserted in (1)-(6).

To see (7), observe that \( s_1 = s_\Delta = u = 0 \) implies \( X_1 = 0 \). But then \( s_1|_{X_1=0} = -X_0 \) has zero scheme disjoint from \( X_1 = 0 \).

To see (8), observe that \( s_0 = X_1 - X_0 = u = 0 \) implies that \( s_0 = X_0^N(1 + \sum f_i) \) if \( u \). But one of our conditions is that \( 1 + \sum f_i \) is a unit, hence \( s_0 = 0 \) implies \( X_0 = 0 \), whence \( X_1 = 0 \).

To see (9), first we remove the image in \( \mathbb{A}^1_S \) of the incidences of (7) and (8); as it is the image of the closed incidence set via the proper morphism \( \mathbb{A}^1_S \times_S \mathbb{P}_S^1, \) this set is closed. By the properties (7) and (8), the image of the incidence is disjoint from \( 0_S \hookrightarrow \mathbb{A}^1_S \). This handles the singularities of the fibers at \( t = 0 \) and \( t = 1 \). The singularities of the total space are supported on the base locus, so it suffices to show the base locus has non-dense image disjoint from \( 0_S \). There are four pairs of components to check:

- \( s_1 = s_0 = 0 \). This implies \( uX_0F = X_0F \). Now \( s_1|_{F=0} = X_0^N \), which has zero scheme disjoint from \( \mathbb{A}^1_S \times_S Z(F) \). Therefore \( F \neq 0 \). Since \( X_0 \) divides \( F \), we conclude \( X_0 \neq 0 \) as well. Therefore the base locus arising from \( Z(s_1) \cap Z(s_0) \) is supported over \( u = 1 \).
• $s_1 = X_1 - X_0 = 0$. Since $s_1|_{X_1 - X_0 = 0} = X_1^{N}$, this implies $X_1 = 0$, whence $X_0 = 0$.

• $s_{\Delta_{\Delta} - 0} = s_0 = 0$. Since $s_0|_{\Delta} = X_1^{N}$, this implies $X_1 = 0$, whence $uX_0 = 0$.

Since $u$ is invertible, this means $X_0 = 0$.

• $s_{\Delta_{\Delta} - 0} = X_1 - X_0 = 0$. If $X_1 = X_0$ then neither can be zero. Then $s_{\Delta_{\Delta} - 0}|_{X_1 - X_0 = 0} = (1 - u)X_0$, so the base locus arising from $\Delta \cap (A_1 \times S \times 1_{S})$ is supported over $u = 1$.

Hence the total space is smooth if we remove $V(u - 1) \rightarrow A_1 - 0$. Then if we take $Z$ to be $V(u - 1)$ together with the image of the incidences discussed above, the total space also decomposes at $t = 0, 1$ as required.

The set $Z_2$ in (11) is obtained similarly: we remove the image in $U$ of the incidences in (7) and (10). By our choice of $s_2$ we know the incidences $Z(s_2) \cap Z(s_0)$ and $Z(s_2) \cap U \times S 1_S$ are disjoint from $0_S \times S P_1^2$. Then we take $Z_2$ to be the union of the images of these incidences in $U$. By construction the total space of $Z(t \cdot s_0 \cdot (X_1 - X_0) + (1 - t) \cdot s_{\Delta} \cdot s_0) \mapsto \mathbb{A}_1 \times (U - Z_2 - 0_S) \times S (U - 0_S)$ is $k$-smooth and decomposes at $t = 0, 1$ as desired.

All of the zero schemes are contained in $\mathbb{A}_1 \times S \mathbb{A}_1$, or the product of one of these schemes with the affine line over $k$. These are smooth affine $k$-schemes with trivial sheaf of Kähler differentials, hence the conormal bundle of a smooth Cartier divisor is trivial. Therefore the zero schemes admit $N$-trivial embeddings via the first projection; they are finite since the first projection factors as a closed immersion followed by the projection away from $P^1$. This shows (12).

\[ \square \]

**Proposition 4.16.** Let $k$ be a field of characteristic $0$. Let $F$ be a homotopy invariant presheaf on $Sm/k$ with oriented weak transfers. Let $S$ be a smooth local $k$-scheme. Let $U$ be an open affine neighborhood of $0_S \hookrightarrow \mathbb{A}_1$. Then the canonical map $F(\mathbb{A}_1 - 0_S)/F(\mathbb{A}_1) \rightarrow F(U - 0_S)/F(U)$ is an isomorphism.

**Proof.** We will define a map $\psi : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)$ with the properties of the map $\psi$ constructed in Theorem 5.1. By the first property $\psi$ descends to a map $F(U - 0_S)/F(U) \rightarrow F(\mathbb{A}_1 - 0_S)$. Then the second and third properties show $\psi$ induces the inverse to the natural map.

We use the line bundles and sections constructed in Proposition 4.15 to construct $\psi$. The sections $s_0$ and $s_1$ determine smooth correspondences $(\mathbb{A}_1 - 0_S) \rightarrow (U - 0_S)$, hence morphisms $t_{Z(s_0)}, t_{Z(s_1)} : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)$ on the presheaf $F$. Now we define

$\psi := t_{Z(s_0)} - t_{Z(s_1)} : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)$.

We have $Z(X_1 - X_0) \mapsto \mathbb{A}_1 \times S 1_S \mapsto \mathbb{A}_1 \times S (U - 0_S)$. Therefore $t_{Z(X_1 - X_0)} : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)/F(\mathbb{A}_1)$ is zero. Similarly, considering where $s_1$ was required to generate, we have $Z(s_1) \mapsto \mathbb{A}_1 \times S (U - 0_S)$, hence $t_{Z(s_1)} : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)/F(\mathbb{A}_1)$ is zero. Therefore $\psi = t_{Z(s_0)} : F(U - 0_S) \rightarrow F(\mathbb{A}_1 - 0_S)/F(\mathbb{A}_1)$.

**Property (1).** We claim the composition

$F(U) \xrightarrow{\psi} F(U - 0_S) \xrightarrow{\psi} F(\mathbb{A}_1 - 0_S)$

is zero. For this it suffices to construct an $\mathbb{A}_1$-homotopy between $s_0$ and $s_1$ on $(\mathbb{A}_1 - 0_S) \times S U$. Since $s_0 = s_1$ along $\mathbb{A}_1 \times S U_{\infty}$ and both sections generate there, we may use the zero scheme $Z(t \cdot s_0 + (1 - t) \cdot s_1) \mapsto \mathbb{A}_1 \times (\mathbb{A}_1 - V(u - 1) - 0_S) \times S U$. 

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We removed the base locus so the total space is guaranteed to be smooth. Since the transfers are compatible with open immersions in the base, this homotopy shows that $\mathcal{F}(U) \rightarrow \mathcal{F}(U - 0_S) \xrightarrow{\psi} \mathcal{F}(\mathbb{A}_S^1 - 0_S) \rightarrow \mathcal{F}(\mathbb{A}_S^1 - V(u - 1) - 0_S)$ is zero. But $\mathcal{F}(\mathbb{A}_S^1 - 0_S) \rightarrow \mathcal{F}(\mathbb{A}_S^1 - V(u - 1) - 0_S)$ is injective by Proposition 4.13, so the claim follows.

**Property (2).** We claim the composition

$$
\mathcal{F}(\mathbb{A}_S^1 - 0_S)/\mathcal{F}(\mathbb{A}_S^1) \xrightarrow{j^*_{U - 0_S \subset \mathbb{A}_S^1 - 0_S}} \mathcal{F}(U - 0_S)/\mathcal{F}(U) \xrightarrow{\psi} \mathcal{F}(\mathbb{A}_S^1 - 0_S)/\mathcal{F}(\mathbb{A}_S^1)
$$

is the identity. We use the total space $Z(t \cdot s_\Delta \cdot s_1 + (1 - t) \cdot (X_1 - X_0) \cdot s_0)$, regarded as a closed subscheme in $\mathbb{A}_1^1 \times (\mathbb{A}_1^1 - Z_1 - 0_S) \times S (\mathbb{A}_1^1 - 0_S)$. This shows that, as maps $\mathcal{F}(\mathbb{A}_S^1 - 0_S) \rightarrow \mathcal{F}(\mathbb{A}_S^1 - Z_1 - 0_S)$,

$$
0 = j^*_{\mathbb{A}_S^1 - Z_1 - 0_S} \circ (t\Delta_{\mathbb{A}_S^1 - 0_S} + (tZ(s_1) - tZ(X_1 - X_0) - tZ(s_0))) \circ j^*_U \circ 0_S \subset \mathbb{A}_S^1 - 0_S.
$$

By Proposition 4.13 we conclude that, as maps $\mathcal{F}(\mathbb{A}_S^1 - 0_S) \rightarrow \mathcal{F}(\mathbb{A}_S^1 - 0_S)$,

$$
0 = t\Delta_{\mathbb{A}_S^1 - 0_S} + (tZ(s_1) - tZ(X_1 - X_0) - tZ(s_0)) \circ j^*_U \circ 0_S \subset \mathbb{A}_S^1 - 0_S.
$$

Since $tZ(s_1) = tZ(X_1 - X_0) = 0$ and $\psi = tZ(s_0)$, we get

$$
0 = t\Delta_{\mathbb{A}_S^1 - 0_S} - (\psi \circ j^*_U \circ 0_S) : \mathcal{F}(\mathbb{A}_S^1 - 0_S) \rightarrow \mathcal{F}(\mathbb{A}_S^1 - 0_S)/\mathcal{F}(\mathbb{A}_S^1)
$$

and the claim follows.

**Property (3).** We claim the composition

$$
\mathcal{F}(U - 0_S)/\mathcal{F}(U) \xrightarrow{\psi} \mathcal{F}(\mathbb{A}_S^1 - 0_S)/\mathcal{F}(\mathbb{A}_S^1) \xrightarrow{j^*_{U - 0_S \subset \mathbb{A}_S^1 - 0_S}} \mathcal{F}(U - 0_S)/\mathcal{F}(U)
$$

is the identity. The ideas are similar, though here we require a homotopy respecting all of $U_\infty$. We use the total space of $Z(t \cdot s_\Delta \cdot s_2 + (1 - t) \cdot (X_1 - X_0) \cdot s_0) \rightarrow \mathbb{A}_1^1 \times (U - Z_2 - 0_S) \times S (U - 0_S)$. By looking at how the fibers at $t = 0, 1$ decompose, we conclude that, as maps $\mathcal{F}(U - 0_S) \rightarrow \mathcal{F}(U - Z_2 - 0_S)$,

$$
0 = j^*_U \circ Z_2 - 0_S \subset U - 0_S \circ (t\Delta_{U - 0_S} + tZ(s_2) + (j^*_U \circ 0_S \subset \mathbb{A}_S^1 - 0_S) \circ (tZ(s_0) - tZ(X_1 - X_0))).
$$

By Proposition 4.13 we conclude that

$$
t\Delta_{U - 0_S} + tZ(s_2) = j^*_U \circ 0_S \subset \mathbb{A}_S^1 - 0_S \circ (tZ(s_0) + tZ(X_1 - X_0)) : \mathcal{F}(U - 0_S) \rightarrow \mathcal{F}(U - 0_S).
$$

We have $tZ(s_2) = tZ(X_1 - X_0) = 0 : \mathcal{F}(U - 0_S) \rightarrow \mathcal{F}(U - 0_S)/\mathcal{F}(U)$ since the sections $s_2, X_1 - X_0$ extend to nonvanishing sections on $U \times S (U_\infty \coprod 0_S)$, i.e., through $0_S$ in the base. Combining this with $\psi = tZ(s_0)$ and the fact that the transfers are compatible with the étale base change $U - 0_S \rightarrow \mathbb{A}_S^1 - 0_S$, we find

$$
t\Delta_{U - 0_S} = j^*_U \circ 0_S \subset \mathbb{A}_S^1 - 0_S \circ \psi : \mathcal{F}(U - 0_S) \rightarrow \mathcal{F}(U - 0_S)/\mathcal{F}(U)
$$

as desired.

**Proposition 4.17.** Let $\mathcal{F}$ be a presheaf of abelian groups on $\text{Sm} / k$ such that for any smooth local $k$-scheme $S$ and any open affine neighborhood $U$ of $0_S \rightarrow \mathbb{A}_S^1$, the natural map $\mathcal{F}(\mathbb{A}_S^1 - 0_S)/\mathcal{F}(\mathbb{A}_S^1) \rightarrow \mathcal{F}(U - 0_S)/\mathcal{F}(U)$ is an isomorphism. Then for any smooth $k$-scheme $Y$ there is a canonical isomorphism of sheaves

$$
(\mathcal{F}_1)_{\text{Zar}} \cong \mathcal{F}(Y \times \mathbb{A}_S^1, Y \times 0)
$$

on $Y_{\text{Zar}}$.
Proof. The limit on the right hand side may be computed over all affine open
eighborhoods of 0. Then this simply isolates the implication shown in [27, Prop. 4.11]. □

Corollary 4.18. Let k be a field of characteristic 0. Let F be a homotopy invariant
presheaf on Sm/k with oriented weak transfers. Then for any smooth k-scheme Y
there is a canonical isomorphism of sheaves

\[(F^{-1})_{Zar} \cong F(Y \times \mathbb{A}^1, Y \times 0)\]
on Y_{Zar}.

Proof. Combine Proposition 4.16 and Proposition 4.17. □

Section 5. Nisnevich excision

We will need the following excision statement for open subschemes of the affine
line. Let \( U \subset \mathbb{A}^1_k \) be an open subscheme. Then for any a \( \in F_{Nis}(U) \), and any closed
point \( x \in U \), there exists an open neighborhood \( V \ni x \) such that the restriction of
a belongs to \( F(V) \subset F_{Nis}(V) \). More generally we will need an excision statement
for a local smooth pair. Both of these follow from the analogue of [27, 4.12]. We
also discuss applications to “contractions” of presheaves.

Theorem 5.1. Let k be a perfect infinite field, and let F be a homotopy invariant
presheaf on Sm/k with oriented weak transfers for affine varieties. Let \( f : X_1 \to X_2 \)
be an étale morphism in Sm/k with \( \dim(X_1) = \dim(X_2) = n \), and let \( Z_2 \hookrightarrow X_2 \)
be a smooth divisor such that \( Z_1 := f^{-1}(Z_2) \to Z_2 \) is an isomorphism. Then for any
closed point \( z \in Z_2 \) there exists an open subscheme \( z \in V \subset X_2 \) and a morphism
\( \psi : F(X_1 - Z_1) \to F(V - V \cap Z_2) \) with the following properties.

(1) The composition \( F(X_1) \to F(X_1 - Z_1) \xrightarrow{\psi} F(V - V \cap Z_2) \) is zero.

(2) The following diagram commutes:

\[
\begin{array}{ccc}
F(X_2 - Z_2) & \xrightarrow{f^*} & F(X_1 - Z_1) \\
\downarrow & & \downarrow \\
F(V - V \cap Z_2) & \xrightarrow{\overline{\psi}} & F(V) \\
\end{array}
\]

(3) The following diagram commutes:

\[
\begin{array}{ccc}
F(X_1 - Z_1) & \xrightarrow{\overline{\psi}} & F(V - V \cap Z_2) \\
\downarrow & & \downarrow \\
F(f^{-1}(V) - f^{-1}(V) \cap Z_1) & \xrightarrow{\overline{f^*}} & F(f^{-1}(V)) \\
\end{array}
\]

(We use a bar to indicate an induced quotient map; unlabeled maps are restriction
maps or maps induced by restriction maps.)

We begin with a lemma which realizes the situation in Theorem 5.1 as a morphism
of relative curves.

Lemma 5.2. Let k be an infinite field. Let \( f : X_1 \to X_3 \) be an étale morphism
in Sm/k with \( \dim(X_1) = \dim(X_2) = n \), and let \( Z_2 \hookrightarrow X_2 \) be a smooth divisor
such that \( Z_1 := f^{-1}(Z_2) \to Z_2 \) is an isomorphism. Let \( z \in Z_2 \). Then, possibly after
shrinking \( X_2 \) about \( z \), there exist:
The morphism $p$ with its normalization. The morphism is quasi-finite. Hence it is finite.

Then for any $A$ satisfying the following properties:

1. The morphism $p$ is étale along $\pi^{-1}(\pi(z))$ (hence $p \circ \pi$ is smooth at $z$).
2. The morphism $p \circ \pi|_{z} : Z_2 \to A^{n-1}$ is finite and (by the previous property) étale at $z$.
3. In the cartesian diagram

\[
\begin{array}{ccc}
(X_2)_0 & \to & X_2 \\
\downarrow & & \downarrow \\
A^{n-1} & \to & A^n
\end{array}
\]

we have $(X_2)_0 = Z_2 \cup R_2$ with $z \notin R_2$. (The étaleness guarantees $Z_2$ is the unique branch through $z$.) In particular $pr_1(Z_2) = 0 \in A^1$.

The morphism $X_2 \to A^n \cong A^1 \times A^{n-1} \subset \mathbb{P}^1 \times A^{n-1}$ is quasi-finite, hence by Zariski’s Main Theorem we can find an open immersion $X_2 \subset X_2$ and a finite morphism $\overline{\pi} : X_2 \to \mathbb{P}^1 \times A^{n-1}$ which extends $\pi$. We may replace an arbitrary $X_2$ with its normalization. The morphism $pr_2 \circ \overline{\pi} : X_2 \to A^{n-1}$ is our candidate for $\overline{\pi} : X_2 \to S$. The morphism $\overline{\pi}$ is finite, hence projective; and $pr_2 : \mathbb{P}^1 \times A^{n-1} \to A^{n-1}$ is projective.

We have a closed immersion $X_{\infty,2} \hookrightarrow \overline{X_2}$, hence the induced morphism $X_{\infty,2} \to A^{n-1}$ is proper. Since also $X_{\infty,2} \subset \pi^{-1}(\infty \times A^{n-1})$, the morphism $X_{\infty,2} \to A^{n-1}$ is quasi-finite. Hence it is finite.

We claim $Z_2 \cup X_{\infty,2}$ admits an open neighborhood in $\overline{X_2}$ which is affine over $A^{n-1}$. We have $pr_1(Z_2) = 0 \in A^1$ and $Z_2$ is irreducible, hence $pr_1(Z_2) = 0 \in A^1$. Then for any $t \in \mathbb{P}^1$ different from 0 and $\infty$, the variety $\overline{X_2} - \overline{\pi}(t \times A^{n-1})$ is an affine open neighborhood of $Z_2 \cup X_{\infty,2}$. Also $Z_2 \cap X_{\infty,2} = \emptyset$; this even holds after applying $pr_1$.

We claim $X_1$ can be compatibly compactified. By applying Zariski’s Main Theorem to the morphism $X_1 \to X_2 \subset \overline{X_2}$, we obtain an open immersion $X_1 \subset \overline{X_1}$ (with $\overline{X_1}$ normal) and a finite morphism $\overline{f} : \overline{X_1} \to \overline{X_2}$. Therefore $\overline{X_1}$ is a projective curve over $S = A^{n-1}$. By the same arguments as above we conclude the subvariety $X_{\infty,1}$ is finite over $A^{n-1}$, and $Z_1 \cup X_{\infty,1}$ has an open neighborhood in $X_1$ which is
affine over \(\mathbb{A}^{n-1}\). We have also \(\overline{f}^{-1}(X_{\infty,2}) \subset X_{\infty,1}\), hence \(\overline{f}\) restricts to a morphism \(X_1 \to X_2\) which is nothing but our original \(f\).

A finite surjective morphism \(f : X \to Y\) of normal varieties is flat away from a set of codimension at least 2. Hence in this situation for any line bundle \(L\) on \(X\) we have the line bundle \(\det f_*L\) on \(Y\). To a section \(s\) of \(L\) there is associated a section \(N(s)\) of \(\det f_*L\); if \(s\) cuts out the Cartier divisor \(D\) on \(X\), then \(N(s)\) cuts out \(f_*D\) on \(Y\). If \(M\) is a line bundle on \(Y\), then \(\det f_*M \cong M^\otimes \deg(f)\).

We apply these constructions to \(V \times_S \overline{X}_1 \to V \times_S \overline{X}_2\), which is the base change of \(\overline{X}_1 \to \overline{X}_2\) via the smooth morphism \(V \times_S \overline{X}_2 \to \overline{X}_2\). This is permissible since the nonflat locus of \(V \times_S \overline{X}_1 \to V \times_S \overline{X}_2\) still has codimension at least 2.

To simplify the notations in the next proposition we use \(\overline{f}_2 := 1 \times \overline{f}\) and \(f_1 := f \times 1\). We use the same general structure as in Proposition 4.15.

**Proposition 5.3.** Let \(k\) be a perfect infinite field. Let the hypotheses and notation be as in Theorem 5.1, and suppose a compactification as in Lemma 5.2 has been chosen. Let \(d := \deg(\overline{f})\). Then there exist:

1. an open neighborhood \(V\) of \(z_2\);
2. a line bundle \(L\) on \(V \times_S \overline{X}_1\);
3. sections \(s_0, s_1 \in H^0(V \times_S \overline{X}_1, \overline{f}_2^* \mathcal{O}(\Delta_V) \otimes L)\);
4. a section \(t_1 \in H^0(V \times_S \overline{X}_2, \mathcal{O}(d - 1) \mathcal{O}(\Delta_V) \otimes \det f_2^*L)\); and
5. a section \(u_1 \in H^0(f^{-1}(V) \times_S \overline{X}_1, \mathcal{O}(\Delta_{\overline{f}^{-1}(V)} f_1^*(\overline{f}_2^* \mathcal{O}(\Delta_V) \otimes L))\);

satisfying:

1. \(s_0\) and \(s_1\) agree on \(V \times_S X_{\infty,1}\);
2. \(s_1\) generates along \(V \times_S (X_{\infty,1} \coprod Z_1)\);
3. \(s_0\) and \(s_1\) are given by \(\overline{f}_2(s_\Delta) \cdot \ell\), where \(\ell\) is a generating section of \(L|_{V \times_S \overline{f}^{-1}(X_{\infty,2})}\);
4. \(g\) is a generating section of \(L|_{V \times_S Z_2}\);
5. \(f_1^*(s_0)\) and \(s_{\Delta_{\overline{f}^{-1}(V)}} u_1\) agree on \(f^{-1}(V) \times (X_{\infty,1} \coprod Z_1)\), and \(u_1\) generates along \(f^{-1}(V) \times_S (X_{\infty,1} \coprod Z_1)\);
6. \(N(s_0)\) and \(s_{\Delta_{\overline{f}^{-1}(V)}} t_1\) agree on \(V \times_S (X_{\infty,2} \coprod Z_2)\), and \(t_1\) generates along \(V \times_S (X_{\infty,2} \coprod Z_2)\);
7. \(f_1^*(s_0)\) and \(s_{\Delta_{\overline{f}^{-1}(V)}} u_1\) agree on \(f^{-1}(V) \times (X_{\infty,1} \coprod Z_1)\), and \(u_1\) generates along \(f^{-1}(V) \times_S (X_{\infty,1} \coprod Z_1)\);
8. the zero schemes \(Z(s_0), Z(s_1), Z(t_1), Z(u_1)\) are \(k\)-smooth;
9. the zero schemes:
   a. \(Z(t \cdot s_0 + (1 - t) \cdot s_1) \hookrightarrow A_1 \times (V - V \cap Z_2) \times_S (X_1 - Z_1)\);
   b. \(Z(t \cdot N(s_0) + (1 - t) \cdot s_{\Delta_{V-\cap Z_2}} t_1) \hookrightarrow A_1 \times (V - V \cap Z_2) \times_S (X_2 - Z_2)\), and
   c. \(Z(t \cdot f_1^*s_0 + (1 - t) \cdot s_{\Delta_{\overline{f}^{-1}(V)-f^{-1}(V\cap Z_2)}} u_1) \hookrightarrow A_1 \times (f^{-1}(V) - f^{-1}(V \cap Z_2)) \times_S (X_1 - Z_1)\)
are \(k\)-smooth;
10. the zero schemes \(Z(u_1), Z(t_1)\) are disjoint from the relevant diagonals:
    a. \(\Delta_{V-\cap Z_2} \cap Z(t_1) = \emptyset\) in \((V - V \cap Z_2) \times_S (X_2 - Z_2)\), and
    b. \(\Delta_{f^{-1}(V)-f^{-1}(V\cap Z_2)} \cap Z(u_1) = \emptyset\) in \((f^{-1}(V) - f^{-1}(V \cap Z_2)) \times_S (X_1 - Z_1)\);
(11) via the first projections, the zero schemes are finite and admit \( N \)-trivial embeddings.

**Remark 5.4.** The sections \( N(s_0) \) and \( s_{\Delta V} \cdot t_1 \) are compared via the isomorphism:

\[
\det f_{2*}(\overline{f_2}^* \mathcal{O}(\Delta_V) \otimes L) \cong \mathcal{O}(\Delta_V) \otimes \mathcal{O}((d-1)\Delta_V) \otimes \det \overline{f_2}_* L.
\]

The sections \( f_1^*(s_0) \) and \( s_{\Delta f^{-1}_1(V)} \cdot u_1 \) are compared via the isomorphism:

\[
f_1^*(\overline{f_2}^* \mathcal{O}(\Delta_V) \otimes L) \cong \mathcal{O}(\Delta_{f^{-1}_1(V)}) \otimes \mathcal{O}(R) \otimes f_1^* L
\]

where \( R \leftarrow f^{-1}(V) \times X_1 \) is a smooth Cartier divisor disjoint from \( \Delta_{f^{-1}_1(V)} \).

**Proof.** The zero schemes are automatically \( V \)-finite: we have \( Z(s) \hookrightarrow V \times_S X \) closed, and \( V \times_S \overline{X} \to V \) is projective. Hence the zero scheme is \( V \)-proper. Note that since we can choose \( V \) smooth over \( S \), we have \( V \times_S X_2 \) is smooth over \( X_2 \), hence also \( V \times_S X_2 \) is \( k \)-smooth.

The choice of \( L \) in (2) is dictated by the following criteria. The first is that for any of the line bundles \( M \) appearing in (3)-(5) (all of these involve \( L \) and some twists) and any of the closed subschemes \( Z \) among \( V \times_S (X_{\infty,1} \coprod Z_1) \), \( V \times_S (X_{\infty,2} \coprod Z_2) \), \( f^{-1}(V) \times_S (X_{\infty,1} \coprod Z_1) \), it holds that \( R^1 f_2^* (\mathcal{O}_Z \otimes M) = 0 \). Replacing \( S \) by an affine neighborhood of \( p_2(z) \), the vanishing of \( R^1 f_* \) is equivalent to the standard restriction sequence remaining exact after applying \( H^0(-) \). The second criterion is that for all \( Z, M \) as above, the linear subsystem \( H^0(Z_2 \otimes M) \) of \( H^0(M) \) is base-point free outside of \( Z \). Both can be achieved by choosing \( L \) sufficiently ample relative to the morphism \( V \times_S \overline{X}_1 \to V \).

Since we can shrink \( V \) about \( z \), we can ensure the zero schemes are smooth if they intersect the curves \( z \times_S \overline{X}_1 \) transversely. We will deal carefully with the sections \( s_0 \) and \( s_1 \) and the curve \( C_z := z \times_S \overline{X}_1 \); the others are similar. Note \( C_z \) contains the smooth curve \( z \times_S \overline{X}_1 \) as a dense open subscheme.

Let \( \ell \) be a global section of \( L \) which generates \( L \) along \( V \times_S (\mathcal{O}_V)^{-1}(X_{\infty,2}) \) and is not a \( p \)-th power of a section (of a line bundle which is a \( p \)-th root of \( L \)). Let \( g \) be a global section which generates along \( V \times_S Z_1 \). Since \( Z_1 \) finite over \( S, Z_1 \) is equal to its closure in \( \overline{X}_1 \). Since \( f \) is \( \acute{e}tale \) along \( Z_1 \), we have \( \overline{f}^{-1}(Z_2) = Z_1 \coprod T \), where \( T \subset X_{\infty,1} \).

We specify an element of \( H^0(\overline{f_2}^* \mathcal{O}(\Delta_V) \otimes L) \mid_{V \times_S (X_{\infty,1} \coprod Z_1)} \), as in the statement, as follows:

- on \( V \times_S (\overline{f}_2)^{-1}(X_{\infty,2}) \) the section restricts to \( s_{\mathcal{F}_2}^* \Delta \cdot \ell \);
- on \( V \times_S ((\overline{f}_2)^{-1}(Z_2) - Z_1) \) it generates; and
- on \( V \times_S Z_1 \) it restricts to \( s_{\mathcal{F}_2}^* \Delta \cdot g \).

By our assumption on \( L \), we can find a global section \( s_{\mathcal{F}_2}^{init} \) of \( \overline{f_2}^* \mathcal{O}(\Delta_V) \otimes L \) with this behavior on \( V \times_S (X_{\infty,1} \coprod Z_1) \). Now let \( \epsilon_1 \in H^0(\overline{f}_2 \mid_{V \times_S (\overline{f}_2^{-1}(X_{\infty,2}) \coprod Z_1)} \otimes \overline{f_2}^* \mathcal{O}(\Delta_V) \otimes L) \) be a section which does not vanish along \( V \times_S ((\overline{f}_2)^{-1}(Z_2) - Z_1) \) and satisfies \( Z(\epsilon_1) \cap Z(s_{\mathcal{F}_2}^{init}) = V \times_S Z_1 \).

Now we restrict the pencil determined by \( s_{\mathcal{F}_2}^{init} \) and \( \epsilon_1 \) to the curve \( C_z \), i.e., we consider the zero scheme \( Z(\lambda s_{\mathcal{F}_2}^{init} + \mu \epsilon_1) \hookrightarrow C_z \times \mathbb{P}_\lambda^1 \). This pencil has the unique base point \( z \times_S z \). Hence \( Z(\lambda s_{\mathcal{F}_2}^{init} + \mu \epsilon_1) \) consists of two irreducible components: the graph of a morphism \( C_z \to \mathbb{P}_1 \); and the component \( (z \times_S z) \times \mathbb{P}_1 \). Since \( k \) is perfect the extension \( k \to \kappa(z) \) is separable and hence the morphism \((z \times_S z) \times \mathbb{P}_1 \to \mathbb{P}_1 \) is
separable. By our choice of \( \ell \) we know the morphism \( \overline{C}_z \to \P^1 \) induces a separable field extension over the point \( \mu = 0 \), hence \( \overline{C}_z \to \P^1 \) is separable. Therefore \( Z(\lambda_0^{\text{init}} + \mu_1) \to \P^1 \) has smooth generic fiber, in other words the general element of the pencil \( Z(\lambda_0^{\text{init}} + \mu_1) \to (V \times S X_1^*) \times \P^1 \) intersects \( C_z \) transversely. (In particular it avoids the singular locus of \( \overline{C}_z \).) We choose a general element \( s_0(\lambda_0, \mu_0) \) of this pencil.

Now we can find \( e_2 \in H^0(\mathcal{I}_V \times S X_{x,1}) \otimes \mathcal{J}_S \mathcal{O}(\Delta V) \otimes L \) such that \( e_2 \) does not vanish along \( V \times S Z_1 \). Since the subsystem to which \( e_2 \) belongs has no unassigned base locus, we can choose \( e_2 \) so that \( Z(s_0(\lambda_0, \mu_0)) \cap Z(e_2) \cap \overline{C}_z = \emptyset \). Now we let \( s_0 := s_0(\lambda_0, \mu_0) \), and we let \( s_1 \) be a general element of the pencil determined by \( s_0 \) and \( e_2 \). Since \( s_1 \) is more general than \( s_0 \), the zero scheme of \( Z(s_1) \) also intersects \( \overline{C}_z \) transversely. Since \( s_0 = e_2 = 0 \) has no solution along \( \overline{C}_z \), the total space \( Z(t \cdot s_0 + (1 - t) \cdot s_1) \leftrightarrow \mathbb{A}^1 \times \overline{C}_z \) is \( k \)-smooth, hence (possibly after shrinking \( V \)) the total space \( Z(t s_0 + (1 - t) s_1) \leftrightarrow \mathbb{A}^1 \times (V \times S X_1^*) \) is \( k \)-smooth. (We just need: \( Z(t \cdot s_0 + (1 - t) \cdot s_1) \leftrightarrow \mathbb{A}^1 \times (V - V \cap Z_2) \times S (X_1 - Z_1) \) is \( k \)-smooth.)

Now we claim \( Z(s_0) \) maps isomorphically onto its image \( Z(N(s_0)) \). We can calculate the degree of the morphism \( Z(s_0) \to Z(N(s_0)) \) along the base change \( V \times S Z_2 \to V \times S X_2 \). We have \( f^{-1}(V \times S Z_2) \subset (V \times S (Z_1 \cup X_{x,1})) \) scheme-theoretically along \( Z_1 \); along \( X_{x,1} \) there might be multiplicities. Also \( Z(s_0) \cap (V \times S (Z_1 \cup X_{x,1})) = Z_2 \times S Z_1 \) scheme-theoretically, and \( Z_2 \times S Z_1 \) is degree 1 (indeed, an isomorphism) onto its image \( Z_2 \times S Z_2 \). Therefore \( Z(s_0) \) itself maps with degree 1, i.e., isomorphically, onto \( Z(N(s_0)) \).

Very similar arguments give the \( k \)-smoothness of \( Z(t \cdot N(s_0) + (1 - t) \cdot s_{\Delta V - V \cap Z_2} ; t_1) \).

After shrinking \( V \) about \( z \) if necessary we have \( Z(N(s_0)) \cap \Delta V - V \cap Z_2 = \emptyset \). Then an arbitrary \( t_1 \) with the prescribed behavior along \( V \times (X_{x,2} \sqcup Z_2) \) may not work, but we are free to add to any given \( t_1 \) a global section of \( \mathcal{I}_V \times S (X_{x,2} \sqcup Z_2) \otimes \mathcal{O}((d - 1) \Delta V) \odot \det \mathcal{J}_S L \). Since this subsystem has no unassigned base locus, we can make \( Z(t_1) \) disjoint from \( Z(N(s_0)) \) and \( \Delta V \) in the fiber \( z \times S \overline{X}_2 \), hence this holds in a neighborhood of \( z \) as well.

The situation with \( f_1^*(s_0) \) and \( u_1 \) is slightly easier since we pull-back along an étale morphism rather than push-forward along a finite generically étale morphism. The ideas are the same.

In all cases, we conclude the existence of \( N \)-trivial embeddings by shrinking \( V \) so that \( V \) and the smooth schemes finite over it have trivial sheaf of Kähler differentials. \( \square \)

**Proof of Theorem 5.1.** Because the desired properties of \( \psi \) are stable under precomposition with the restriction \( \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(f^{-1}(U) - f^{-1}(U) \cap Z_1) \), we may replace \( X_2 \) by an open neighborhood \( U \) of the point \( z \), and \( X_1 \) by the preimage \( f^{-1}U \). We put the excisive morphism \( f \) into the convenient relative form guaranteed by Lemma 5.2. Now we define \( \psi : \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(V - V \cap Z_2) \) as the difference \( \psi := t_{Z(s_0)} - t_{Z(s_1)} \), where \( Z(s_0) \) and \( Z(s_1) \) are the smooth correspondences constructed in Proposition 5.3, and \( L \) is defined in Notation 4.3.

**Property (1).** The correspondences \( Z(s_0), Z(s_1) : (V - V \cap Z_2) \to (X_1 - Z_1) \subset X_1 \) are \( \mathbb{A}^1 \)-homotopic, i.e., they become homotopic when allowed to pass through \( Z_1 \). To see this, we use the zero scheme \( Z(t \cdot s_0 + (1 - t) \cdot s_1) \leftrightarrow \mathbb{A}^1 \times (V - V \cap Z_2) \times S X_1 \). This is \( k \)-smooth by the properties in 5.3, and \( Z(t \cdot s_0 + (1 - t) \cdot s_1) \leftrightarrow \mathbb{A}^1 \times (V - V \cap Z_2) \) admits an \( N \)-trivial embedding if \( V \) is sufficiently small.
Property (2). Considering where \( s_1 \) was required to generate, we see \( Z(s_1) \hookrightarrow (V - V \cap Z_2) \times_S (X_1 - Z_1) \) extends through \( V \cap Z_2 \), i.e., to a smooth \( V \)–finite correspondence in \( V \times_S (X_1 - Z_1) \). In geometric terms we have a factorization

\[
\begin{array}{ccc}
V - V \cap Z_2 & \xrightarrow{Z(s_1)} & X_1 - Z_1 \\
\downarrow & & \downarrow \\
V & \xrightarrow{Z(s_1)^{-}} & \\
\end{array}
\]

where the arrow \( V \to X_1 - Z_1 \) is the (smooth) finite correspondence obtained by taking the closure of \( Z(s_1) \). Since the weak transfer along \( Z(s_1)^{-} \) is compatible with the open immersion \( V - V \cap Z_2 \subset V \), we conclude \( t_{Z(s_1)} : \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(V - V \cap Z_2)/\mathcal{F}(V) \) is zero. Hence we have

\[
(5.1) \quad \overline{\psi} = t_{Z(s_0)} : \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(V - V \cap Z_2)/\mathcal{F}(V).
\]

Now the commutative diagram:

\[
\begin{array}{ccc}
Z(s_0)|_{V - V \cap Z_2} & \xrightarrow{pr_2} & (V - V \cap Z_2) \times_S (X_1 - Z_1) \\
\downarrow \cong & & \downarrow 1 \times f \\
Z(N(s_0))|_{V - V \cap Z_2} & \xrightarrow{pr_2} & (V - V \cap Z_2) \times_S (X_2 - Z_2) \\
\end{array}
\]

shows that

\[
(5.2) \quad t_{Z(s_0)} \circ f^* = t_{Z(N(s_0))} : \mathcal{F}(X_2 - Z_2) \to \mathcal{F}(V - V \cap Z_2).
\]

The homotopy \( Z(t \cdot N(s_0) + (1 - t) \cdot s_{\Delta_{V - V \cap Z_2}} : t_1) \to \mathbb{A}^1 \times (V - V \cap Z_2) \times_S (X_2 - Z_2) \) shows that \( t_{Z(N(s_0))} = t_{\Delta_{V - V \cap Z_2}} + t_{Z(t_1)} : \mathcal{F}(X_2 - Z_2) \to \mathcal{F}(V - V \cap Z_2) \). Furthermore \( Z(t_1) \) extends to a correspondence \( V \to X_2 - Z_2 \), so the map \( t_{Z(t_1)} : \mathcal{F}(X_2 - Z_2) \to \mathcal{F}(V - V \cap Z_2)/\mathcal{F}(V) \) is zero.

Therefore we have:

\[
(5.3) \quad t_{Z(N(s_0))} = t_{\Delta_{V - V \cap Z_2}} : \mathcal{F}(X_2 - Z_2) \to \mathcal{F}(V - V \cap Z_2)/\mathcal{F}(V).
\]

Putting everything together, we conclude:

\[
\overline{\psi} \circ f^* = t_{Z(s_0)} \circ f^* = t_{Z(N(s_0))} = t_{\Delta_{V - V \cap Z_2}} : \mathcal{F}(X_2 - Z_2) \to \mathcal{F}(V - V \cap Z_2)/\mathcal{F}(V)
\]

as desired.

Property (3). Our goal is to show \( \overline{\psi} = t_{Z(s_0)} : \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(f^{-1}(V) - f^{-1}(V \cap Z_2))/\mathcal{F}(f^{-1}(V)) \) is the restriction map. Since \( \overline{\psi} = t_{Z(s_0)} \) and the transfer maps are compatible with the étale base change \( f : f^{-1}(V) - f^{-1}(V \cap Z_2) \to V - V \cap Z_2 \) (in particular \( N \)-triviality is preserved by étale base change), it suffices to show

\[
(5.4) \quad t_{Z(f_1^*(s_0))} = t_{\Delta_{f^{-1}(V) - f^{-1}(V \cap Z_2)}}.
\]

The argument is the same: the zero scheme of

\[
t \cdot f_1^*(s_0) + (1 - t) \cdot s_{\Delta_{f^{-1}(V) - f^{-1}(V \cap Z_2)}} : \mathbb{A}^1 \times (f^{-1}(V) - f^{-1}(V \cap Z_2)) \times_S X_1
\]

in \( \mathbb{A}^1 \times (f^{-1}(V) - f^{-1}(V \cap Z_2)) \times_S X_1 \) gives the relation

\[
t_{Z(f_1^*(s_0))} = t_{\Delta_{f^{-1}(V) - f^{-1}(V \cap Z_2)}} + t_{Z(u_1)} : \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(f^{-1}(V) - f^{-1}(V \cap Z_2)).
\]
Furthermore \( Z(u_1) \) is closed in \( f^{-1}(V) \times_S (X_1 - Z_1) \), i.e., extends through \( f^{-1}(V \cap Z_2) \). Therefore \( t_{Z(u_1)}: \mathcal{F}(X_1 - Z_1) \to \mathcal{F}(f^{-1}(V) - f^{-1}(V \cap Z_2))/\mathcal{F}(f^{-1}(V)) \) is zero and \( t_{Z(f(s_0))} = t_{\Delta_{f^{-1}(V) - f^{-1}(V \cap Z_2)}} \), as desired.

\[ \square \]

As a result of Theorem 5.1 we have the statement of [27, Cor. 4.13] for \( \mathcal{F} \) as in the theorem.

**Corollary 5.5.** Let \( k \) be a perfect infinite field, and let \( \mathcal{F} \) be a homotopy invariant presheaf on \( \text{Sm}/k \) with oriented weak transfers for affine varieties. Let \( f: X_1 \to X_2 \) be an étale morphism of smooth \( k \)-schemes and \( Z \to X_2 \) a smooth divisor such that \( f^{-1}(Z) \to Z \) is an isomorphism. Then the canonical morphism of sheaves

\[ \mathcal{F}(X_2, Z) \to \mathcal{F}(X_1, f^{-1}(Z)) \]

on \( Z_{\text{Zar}} \) is an isomorphism.

Then we have the analogue of [27, Thm. 4.14], [15, Thm. 23.12].

**Theorem 5.6.** Let \( k \) be a perfect infinite field, and let \( \mathcal{F} \) be a homotopy invariant presheaf on \( \text{Sm}/k \) with oriented weak transfers for affine varieties. Let \( X \) be a smooth \( k \)-scheme and \( Z \to X \) a smooth divisor. Then about any \( x \in X \) there is an open neighborhood \( U \) and isomorphisms

\[ \mathcal{F}(U \times_Y (U \cap Z) \times Y) \cong \mathcal{F}(A^1 \times (U \cap Z) \times Y \times Y) \]

of sheaves on \((U \cap Z) \times Y)_{\text{Zar}} \) for all \( Y \in \text{Sm}/k \), natural in \( Y \).

**Proof.** We use the constructions of [27, Thm. 4.14], then apply Corollary 5.5 to the arrows in the diagram appearing at the end of the proof of [27, Thm. 4.14]. \( \square \)

**Corollary 5.7.** In the situation of Theorem 5.6 there are isomorphisms

\[ \mathcal{F}(U \times_Y (U \cap Z) \times Y) \cong (\mathcal{F}_{-1})_{\text{Zar}}. \]

**Proof.** Use Proposition 4.17. \( \square \)

**Definition 5.8.** A presheaf \( \mathcal{F} \) has contractions if it satisfies the conclusion of 5.7. By \( \mathcal{F}_{-1} \) we denote the presheaf whose value on a scheme \( X \) is \( \text{coker}(\mathcal{F}(X \times A^1) \to \mathcal{F}(X \times (A^1 - 0))) \).

**Corollary 5.9** (see [15] Lemma 22.10). Let \( k \) be a perfect infinite field, and let \( \mathcal{F} \) be a homotopy invariant presheaf of abelian groups on \( \text{Sm}/k \) with oriented weak transfers for affine varieties. Let \( S \) be the semilocal scheme of a finite set of points on a smooth \( k \)-scheme, and let \( S = U_0 \cup V \) be a Zariski open cover. Then there exists an open \( U \subset U_0 \) such that \( S = U \cup V \) and the Mayer-Vietoris sequence:

\[ 0 \to \mathcal{F}(S) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0 \]

is exact.

**Proof.** If \( k \) is infinite, all of the constructions entering into Proposition 5.3 and Theorem 5.1 work for \( z \) replaced by a finite set of points, and the neighborhood \( V \) of \( z \) can be made to contain any given finite set of points (not necessarily on the divisor \( Z_2 \)). If the sections are chosen sufficiently generic, the incidences of their zero schemes will avoid a finite set.

Let \( Z \) be a smooth divisor passing through the (finitely many) closed points not contained in \( U \); this is a closed subscheme of \( V \) which is also closed in \( S \). Then
Let $U := S - Z \subset U_0$ be an open subscheme of $U_0$ such that $U \cup V = S$. Furthermore we have $V - Z = V \cap (X - Z) = V \cap U$.

We apply Theorem 5.1 with $f : X_1 \to X_2$ the open immersion $j : V \subset S$ and $Z_2 = Z$. Since $Z$ is a closed subscheme of $V$ which remains closed in $S$, the morphism $j^{-1}Z \to Z$ is an isomorphism. Hence we get a map $\psi : \mathcal{F}(V - Z) \to \mathcal{F}(S - Z)$ inducing an isomorphism $\mathcal{F}(V - Z)/\mathcal{F}(V) \cong \mathcal{F}(S - Z)/\mathcal{F}(S)$ inverse to the natural map, hence the natural restriction map $\mathcal{F}(U)/\mathcal{F}(S) \to \mathcal{F}(U \cap V)/\mathcal{F}(V)$ is also an isomorphism. Thus we may apply the 9-lemma to the diagram (whose border of outer zeroes has been omitted):

\[
\begin{array}{ccc}
0 & \to & \mathcal{F}(V) \\
\downarrow & & \downarrow \psi \\
\mathcal{F}(S) & \to & \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \\
\downarrow \cong & & \downarrow p_1 \\
\mathcal{F}(S) & \to & \mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{F}(S)
\end{array}
\]

to conclude the middle row is exact. \hfill \Box

We can compute the Nisnevich cohomology of $U \subset \mathbb{A}^1_k$ with coefficients in a presheaf with oriented weak transfers.

**Proposition 5.10.** Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then for any open subscheme $U \subset \mathbb{A}^1_k$, we have $H^0_{\text{Nis}}(U, \mathcal{F}_{\text{Nis}}) = \mathcal{F}(U)$ and $H^i_{\text{Nis}}(U, \mathcal{F}_{\text{Nis}}) = 0$ for $i \neq 0$.

**Proof.** By Theorem 4.8 we have $\mathcal{F} \cong \mathcal{F}_{\text{Zar}}$ on $U_{\text{Zar}}$. Since any Nisnevich cover contains a dense open immersion (to cover the generic point), Corollary 3.3 implies the map $\mathcal{F}(U) \to \mathcal{F}_{\text{Nis}}(U)$ is injective. Hence we may assume $\mathcal{F}$ is separated as a presheaf on $U_{\text{Nis}}$.

Now Theorem 5.1 implies that given any $a \in \mathcal{F}_{\text{Nis}}(U)$ and any closed point $x \in U$, there exists an open $x \in V \subset U$ such that $a|_V \in \mathcal{F}(V) \subset \mathcal{F}_{\text{Nis}}(V)$. Thus Zariski locally, $a$ is induced by sections in $\mathcal{F}$, and since $\mathcal{F}$ is a Zariski sheaf, we get an element of $\mathcal{F}(U)$ inducing $a$. Thus the map $\mathcal{F}(U) \to \mathcal{F}_{\text{Nis}}(U)$ is also surjective.

Since $U$ is a curve only the case $i = 1$ remains, and Theorem 5.1 produces the Mayer-Vietoris sequence for a Nisnevich cover of $U$ (via the method of Corollary 5.9). \hfill \Box

### Section 6. Homotopy invariance of cohomology

We carry out the rest of Voevodsky’s argument using certain simplifications due to Mazza-Voevodsky-Weibel [15]. Observe that we could have worked exclusively with affine varieties: if we know the (functorial) agreement of Zariski and Nisnevich cohomology on affine varieties, then we obtain an isomorphism of Čech to derived functor spectral sequences. Similarly, homotopy invariance of cohomology on the category of affine varieties implies homotopy invariance of cohomology on all of $\text{Sm}/k$. 29
Proposition 6.1. Let $k$ be a field of characteristic 0, let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties, and let $X$ be a smooth $k$-scheme. Then $\mathcal{F}_{\text{Zar}}(X \times k^1) \cong \mathcal{F}_{\text{Zar}}(X)$.

Proof. We may assume $X$ is integral with generic point $\eta$. The zero section $0_X : X \to \mathbb{A}^1_X$ induces a splitting $\mathcal{F}_{\text{Zar}}(\mathbb{A}^1_X) = \mathcal{F}_{\text{Zar}}(X) \oplus \ker(0^*_X)$ compatible with the splitting $\mathcal{F}_{\text{Zar}}(\eta) = \mathcal{F}_{\text{Zar}}(\eta) \oplus \ker(0^*_\eta)$. By Corollary 3.5 the map $\mathcal{F}_{\text{Zar}}(\mathbb{A}^1_X) \to \mathcal{F}_{\text{Zar}}(\eta)$ is injective, hence the maps $\mathcal{F}_{\text{Zar}}(X) \to \mathcal{F}_{\text{Zar}}(\eta)$ and $\ker(0^*_X) \to \ker(0^*_\eta)$ are both injective. By Corollary 4.12 we have $\mathcal{F}(\eta) \cong \mathcal{F}_{\text{Zar}}(\eta^1)$, hence ker$(0^*_\eta)$ is trivial. But then ker$(0^*_X)$ must be trivial as well, and the result follows. \hfill \Box

Notation 6.2. For a presheaf $\mathcal{F}$ of abelian groups on $\text{Sch}/k$ we denote by $s_{\text{Zar}}(\mathcal{F})$ its Zariski separation. On a $k$-scheme $X$ the value of $s_{\text{Zar}}(\mathcal{F})$ is $\mathcal{F}(X)/\cup(\ker(\mathcal{F}(X) \to \oplus_i(\mathcal{F}(U_i))))$, where the union is taken over the partially ordered set of Zariski covers $\{U_i\}$ of $X$.

Proposition 6.3. Let $k$ be a perfect infinite field, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with (oriented) weak transfers for affine varieties. Then $s_{\text{Zar}}(\mathcal{F})$ has a unique structure of homotopy invariant presheaf with (oriented) weak transfers for affine varieties such that the canonical morphism $\mathcal{F} \to s_{\text{Zar}}(\mathcal{F})$ is a morphism of presheaves with weak transfers for affine varieties.

Proof. The presheaf $s_{\text{Zar}}(\mathcal{F})$, being a quotient of a homotopy invariant presheaf, is homotopy invariant. So suppose $f : X \to Y$ is a morphism for which a weak transfer $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ exists and for which a weak transfer $f_* : s_{\text{Zar}}(\mathcal{F})(X) \to s_{\text{Zar}}(\mathcal{F})(Y)$ must be constructed.

We need to show that if $a \in \mathcal{F}(X)$ is a locally trivial element, then so is $f_*(a) \in \mathcal{F}(Y)$. Since the weak transfers are compatible with open immersions, we may assume $Y$ is local. Then $X$ is semilocal and all of the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U_i)$ are injective by Corollary 3.3. But then already $a$ vanishes on the semilocal scheme $X$, hence $f_*(a)$ is zero as well. \hfill \Box

Proposition 6.4. Let $k$ be a perfect infinite field, let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties, and let $S$ be a smooth semilocal $k$-scheme. Then $\mathcal{F}_{\text{Zar}}(S) = \mathcal{F}(S)$.

Proof. The canonical map $\mathcal{F}(S) \to s_{\text{Zar}}(\mathcal{F})(S)$ is injective by Corollary 3.3, hence an isomorphism, so by Proposition 6.3 we may assume $\mathcal{F}$ is a separated presheaf. As in [27, 4.24], we use induction on the number of open sets in a cover together with Corollary 5.9 to construct a candidate lift of an element $a \in \mathcal{F}_{\text{Zar}}(S)$. Then the assumption of separatedness and Corollary 3.3 show the candidate lift works. \hfill \Box

Proposition 6.5. Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then $\mathcal{F}_{\text{Zar}}$ regarded as a sheaf on $\text{Sm}/k$ has a unique structure of presheaf with oriented weak transfers for affine varieties such that $\mathcal{F} \to \mathcal{F}_{\text{Zar}}$ is a morphism of presheaves with transfer structure. Furthermore the presheaf $\mathcal{F}_{\text{Zar}}$ is homotopy invariant and has contractions.

Proof. We may assume $\mathcal{F}$ is separated by Proposition 6.3. Consider a morphism $f : X \to Y$ in $\text{Sm}/k$ along which we must construct a transfer, and an element $a \in \mathcal{F}_{\text{Zar}}(X)$. Since $\mathcal{F}$ is separated and a finite cover of a local scheme is semilocal,
by Proposition 6.4 we can find a cover \( \{U_i\} \) of \( Y \) such that the restriction \( a_t := a|_{f^{-1}U_t} \in \mathcal{F}_{\text{Zar}}(f^{-1}U_t) \) belongs to the subgroup \( \mathcal{F}(f^{-1}U_t) \). Let \( b_t \) denote the element \( f|_{f^{-1}U_t}(a_t) \in \mathcal{F}(U_t) \). Since the weak transfers are compatible with the open immersions \( U_{ij} \to U_i \), the elements \( b_t \) and \( b_j \) coincide on \( U_i \cap U_j \), hence they glue to a global element \( b \in \mathcal{F}_{\text{Zar}}(Y) \).

Homotopy invariance is the statement of Proposition 6.1. Hence \( \mathcal{F}_{\text{Zar}} \) has contractions by Theorem 5.6.

**Lemma 6.6.** Let \( k \) be a field, and let \( \mathcal{F} \) be a presheaf of abelian groups on \( \text{Sm}_k \) with oriented weak transfers for affine varieties. Then the Nisnevich sheafification \( \mathcal{F}_{\text{Nis}} \) has a unique structure of a presheaf with oriented weak transfers making the canonical morphism \( \mathcal{F} \to \mathcal{F}_{\text{Nis}} \) a morphism of presheaves with oriented weak transfers for affine varieties.

**Proof.** First we note \( \mathcal{F}_{\text{Nis}} \) inherits the transfer structure from \( \mathcal{F} \). This follows more or less immediately from the fact that a finite cover of a Henselian local scheme is a disjoint union of Henselian local schemes. The requirement that \( \mathcal{F} \) be compatible with disjoint unions then determines the induced weak transfers \( f_* : \mathcal{F}_{\text{Nis}}(X) \to \mathcal{F}_{\text{Nis}}(Y) \) along finite morphisms to suitably shrunken targets \( Y \). For details see the \( i = 0 \) case of Lemma 6.10. \( \square \)

We have the first comparison results.

**Theorem 6.7.** Let \( k \) be a field of characteristic 0, and let \( \mathcal{F} \) be a homotopy invariant presheaf of abelian groups on \( \text{Sm}_k \) with oriented weak transfers for affine varieties. Then \( \mathcal{F}_{\text{Zar}} = \mathcal{F}_{\text{Nis}} \).

**Proof.** Since we have Lemma 6.6, we can follow the argument of [15, 22.2]: the kernel and cokernel of the presheaf map \( \mathcal{F} \to \mathcal{F}_{\text{Nis}} \) have the same transfer structure and have trivial Nisnevich sheafification. It suffices to show they have trivial Zariski sheafification, so we may assume \( \mathcal{F}_{\text{Nis}} = 0 \). By Proposition 6.5 we have that \( \mathcal{F}_{\text{Zar}} \) is homotopy invariant with oriented weak transfers for affine varieties, and for any presheaf we have \( \mathcal{F}_{\text{Nis}} = (\mathcal{F}_{\text{Zar}})_{\text{Nis}} \), hence we may assume \( \mathcal{F} \) is a Zariski sheaf.

So we need that \( \mathcal{F}(S) = 0 \) for \( S \) local. But Theorem 5.1 implies that a section which is trivialized by a Nisnevich cover must already be Zariski locally trivial. \( \square \)

**Corollary 6.8.** Let \( k \) be a field of characteristic 0. Then Theorem 5.6 and Corollary 5.7 hold with the Zariski topology replaced by the Nisnevich topology.

**Proposition 6.9** (application to contractions). Let \( k \) be a field of characteristic 0, and let \( \mathcal{F} \) be a homotopy invariant presheaf of abelian groups on \( \text{Sm}_k \) with oriented weak transfers for affine varieties. Then \( (\mathcal{F}_{\text{Nis}})_{-1} \cong (\mathcal{F}_{-1})_{\text{Nis}} \).

**Proof.** Proposition 6.5 and Theorem 6.7 together imply that \( \mathcal{F}_{\text{Nis}} \) is homotopy invariant with oriented weak transfers. The presheaf \( \mathcal{F}_{-1} \) inherits homotopy invariance and the transfer structure, since if \( f : X \to Y \) admits an \( N \)-trivial embedding then so does \( f \times \text{id}_T : X \times T \to Y \times T \). (We use this for \( T = \mathbb{A}^1, \mathbb{A}^1 - 0 \).) Thus we have a canonical map of homotopy invariant presheaves with oriented weak transfers \( (\mathcal{F}_{-1})_{\text{Nis}} \to (\mathcal{F}_{\text{Nis}})_{-1} \). To apply Corollary 3.4 to the kernel and cokernel of this map, we need to show that \( \mathcal{F}_{-1}(\text{Spec } E) = (\mathcal{F}_{\text{Nis}})_{-1}(\text{Spec } E) \) for all fields \( E \supset k \). In other words, we need \( \mathcal{F}(\mathbb{A}^1_E - 0)/\mathcal{F}(\mathbb{A}^1_E) = \mathcal{F}_{\text{Nis}}(\mathbb{A}^1_E - 0)/\mathcal{F}_{\text{Nis}}(\mathbb{A}^1_E) \), which follows from Theorem 6.7 and Theorem 4.8. \( \square \)
Lemma 6.10. Let $k$ be a field, and let $\mathcal{F}$ be a Nisnevich sheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then the cohomology presheaves $H^n_{Nis}(-, \mathcal{F})$ have oriented weak transfers.

Proof. Following [15, Ex. 6.20], we observe the canonical flasque resolution of $\mathcal{F}$ consists of terms which may be endowed with oriented weak transfers. More precisely, $\mathcal{F}$ admits a canonical injection into the sheaf $E_{\mathcal{F}}$ whose value on $X \in \text{Sm}/k$ is $\prod_{x \in X} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x})$, i.e., the product of the values of $\mathcal{F}$ on the Hensel local rings of the closed points in $X$. (Then we take the cokernel and repeat.) Given a finite flat morphism $f : X \to Y$ along which $E_{\mathcal{F}}$ should have an oriented weak transfer, the fiber $f^{-1}(\text{Spec } \mathcal{O}_{Y,y})$ splits as $\prod_{x \in f^{-1}(y)} \text{Spec } \mathcal{O}_{X,x}$, Therefore we may identify $E_{\mathcal{F}}(X) = \prod_{x \in f^{-1}(y)} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x}) = \prod_{y \in Y} \prod_{x \in f^{-1}(y)} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x})$. To define the morphism $f_* : E_{\mathcal{F}}(X) \to E_{\mathcal{F}}(Y)$, it suffices to define for every $y \in Y$ a morphism $\prod_{x \in f^{-1}(y)} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x}) \to \mathcal{F}(\text{Spec } \mathcal{O}_{Y,y})$, and for this we take the sum of the weak transfers $f_* : \mathcal{F}(\text{Spec } \mathcal{O}_{X,x}) \to \mathcal{F}(\text{Spec } \mathcal{O}_{Y,y})$. □

Remark 6.11. Since the canonical flasque resolution can be constructed using only affine varieties, the Nisnevich cohomology of $\mathcal{F}$ has oriented weak transfers even if $\mathcal{F}$ itself has such transfers only for affine varieties. Also note Lemma 6.10 fails for the Zariski topology: we have a restriction morphism $\mathcal{F}(f^{-1}(\text{Spec } \mathcal{O}_{Y,y})) \to \prod_{x \in f^{-1}(y)} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x})$ and a weak transfer $f_* : \mathcal{F}(f^{-1}(\text{Spec } \mathcal{O}_{Y,y})) \to \mathcal{F}(\text{Spec } \mathcal{O}_{X,x})$, but there is no reason the morphism $f_*$ should extend to $\prod_{x \in f^{-1}(y)} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x})$. The lemma holds for the Zariski topology with the homotopy invariance hypothesis (in characteristic zero) by Proposition 6.5.

Corollary 6.12. Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then $H^n_{Nis}(-, \mathcal{F}_{Nis})$ is a homotopy invariant presheaf with oriented weak transfers.

Proof. The proof of [15, 24.1-4] goes through: the arguments are sheaf-theoretic and use properties of presheaves with transfers that we have verified extend to presheaves with oriented weak transfers. The case $n = 0$ follows from Proposition 6.5 and Theorem 6.7. Hence we are reduced to [15, 24.2] by the same Leray spectral sequence argument.

The proof of [15, 24.2] goes through in our setting since we have the computation of the Nisnevich cohomology for open subschemes of $\mathbb{A}^1_E$ by Proposition 5.10. Thus we are reduced to proving [15, 24.3] in our setting; copying notation, we need to show the composition

$$H^n_{Nis}(S \times \mathbb{A}^1, \mathcal{F}) \xrightarrow{\tau} H^n_{Nis}(S \times \mathbb{A}^1, j_* j^* \mathcal{F}) \xrightarrow{\eta} H^n_{Nis}(U \times \mathbb{A}^1, j^* \mathcal{F})$$

is injective. The proof that $\eta$ is injective applies since we have Proposition 6.9 and Corollary 6.8. The proof that $\tau$ is injective also works in our setting. Corollary 3.5 gives the injectivity $\mathcal{F} \to j_* j^* \mathcal{F}$, and Corollary 6.8 allows us to identify the cokernel as a contraction. Finally, the argument for [15, 24.4] is sheaf-theoretic. □

Corollary 6.13. Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Then $H^n_{Nis}(-, \mathcal{F}_{Nis}) = H^n_{Zar}(-, \mathcal{F}_{Zar})$ for all $n$. 

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The Gersten resolution. We have developed enough machinery that we can also imitate the construction of the Gersten resolution. Corollaries 6.12, 3.3, 4.11, and 6.13 provide us with the analogues of [15, 24.1, 11.1, 22.7]. Theorem 5.6 is the analogue of [15, 23.12], and [15, 2.10, 23.4, 23.7] are sheaf-theoretic.

**Theorem 6.14.** Let $k$ be a field of characteristic 0, and let $\mathcal{F}$ be a homotopy invariant presheaf of abelian groups on $\text{Sm}/k$ with oriented weak transfers for affine varieties. Let $X \in \text{Sm}/k$ be a smooth $k$-scheme of dimension $n$, and let $X^{(i)}$ denote the set of codimension $i$ points of $X$. Then there is a canonical exact sequence on $X_{\text{Zar}}$:

$$0 \to \mathcal{F} \to \bigoplus_{x \in X^{(0)}} i_x(\mathcal{F}) \to \bigoplus_{x \in X^{(1)}} i_x(\mathcal{F}_{-1}) \to \cdots \to \bigoplus_{x \in X^{(n)}} i_x(\mathcal{F}_{-n}) \to 0.$$  

### Appendix A. Trivializing normal bundles

We collect some basic lemmas about normal bundles. Let $k$ be a field. We say a morphism of smooth $k$-schemes $f : X \to Y$ admits an $N$-trivial embedding if there is a closed embedding of $Y$-schemes $X \hookrightarrow Y \times \mathbb{A}^n$ such that the normal bundle $N_X(Y \times \mathbb{A}^n)$ to $X$ in $Y \times \mathbb{A}^n$ is trivial. The existence of any embedding $X \hookrightarrow Y \times \mathbb{A}^n$ implies $f$ is affine. If $f$ is not itself an embedding, the existence of an embedding $X \hookrightarrow Y \times \mathbb{A}^n$ implies $X$ admits a nonconstant regular function. If $f$ is étale, then any embedding $X \hookrightarrow Y \times \mathbb{A}^n$ is $N$-trivial.

**Lemma A.1.** Let $f : X \to Y$ be a morphism in $\text{Sm}/k$ such that $\Omega^1_{X/k}$ and $\Omega^1_{Y/k}$ are trivial. Then $f$ admits an $N$-trivial embedding.

**Proof.** For any embedding $i : X \to Y \times \mathbb{A}^n$ with ideal sheaf $I$, in the exact sequence

$$0 \to I/I^2 \to i^*\Omega^1_{Y \times \mathbb{A}^n/k} \to \Omega^1_{X/k} \to 0$$

there is a section $\Omega^1_{X/k} \to i^*\Omega^1_{Y \times \mathbb{A}^n/k}$ since both of these bundles are trivial. Hence $N_X(Y \times \mathbb{A}^n) + \mathcal{O}^{\dim X} \cong \mathcal{O}^{\dim Y}$. Therefore in any embedding $X \hookrightarrow Y \times \mathbb{A}^n \hookrightarrow Y \times \mathbb{A}^{\dim X}$ induced by a constant map $X \to \mathbb{A}^{\dim X}$, the normal bundle is trivial. □

**Lemma A.2.** Let $f : X \to Y$ be a morphism in $\text{Sm}/k$ that admits an $N$-trivial embedding. Then any morphism of $Y$-schemes $X \to \mathbb{A}^1 \times Y$ admits an $N$-trivial embedding.

**Proof.** Suppose $i = (f, \beta) : X \hookrightarrow Y \times \mathbb{A}^n$ is an $N$-trivial embedding, and $(f, \alpha) : X \to Y \times \mathbb{A}^1$ is the given morphism. Then we claim $(f, \alpha, \beta) : X \hookrightarrow Y \times \mathbb{A}^1 \times \mathbb{A}^n$ is $N$-trivial. We denote by $N^{(-)}$ the conormal bundle of the embedding $(-)$. We have an exact square:
Let \( \text{Corollary A.4.} \)

Therefore this last sequence is split exact and \( \Omega^1_{X/k} \) the normal bundle to \( Y \) embedding. Let \( Y \) hypothesis, and \( \Omega^1_{1/k} \) Lemma A.3.

The choice of \( \Omega^1_{\alpha/\beta} \) transfers along a finite flat morphism \( \Omega^1_{\beta/\alpha} \) vanishes in \( \Omega^1_{X/k} \), hence factors through \( \Omega^1_{\alpha/\beta} \), so the left hand column is split exact. Now \( N^1_{\alpha/\beta} \) is trivial by hypothesis, and \( \Omega^1_{\alpha/\beta} \) is trivial.

\( \Box \)

**Lemma A.3.** Let \( f : X \to Y \) be a morphism in \( \text{Sm}/k \) that admits an \( N \)-trivial embedding. Let \( Y' \to Y \) be a flat morphism. Then \( f' : X' \to Y' \) admits an \( N \)-trivial embedding.

**Proof.** By [5, B.7.4], the normal bundle to \( X' \) in \( Y' \times \mathbb{A}^n \) is simply the pull-back of the normal bundle to \( X \) in \( Y \times \mathbb{A}^n \).

**Corollary A.4.** Let \( f : X \to Y \) be a flat morphism in \( \text{Sm}/k \), \( X \to Y \times \mathbb{A}^n \) an \( N \)-trivial closed embedding (of \( Y \)-schemes), and \( Y' \to Y \) a closed immersion in \( \text{Sm}/k \). Suppose that \( X' = X \cap (Y' \times \mathbb{A}^n) \) is smooth. Then \( X' \to Y' \times \mathbb{A}^n \) is an \( N \)-trivial embedding (of \( Y' \)-schemes).

**Proof.** Note the flatness of \( f \) implies \( X \) and \( Y' \times \mathbb{A}^n \) intersect properly in \( Y \times \mathbb{A}^n \). The embedding \( X' \to Y' \times \mathbb{A}^n \) is regular, hence so is the embedding \( X' \to Y \times \mathbb{A}^n \). Then [5, B.7.4] asserts that we have an exact sequence:

\[
N_X(Y \times \mathbb{A}^n) = N_X(Y \times \mathbb{A}^n)|_{X'} \oplus N_{Y' \times \mathbb{A}^n}(Y \times \mathbb{A}^n)|_{X'}.
\]

We have also the exact sequence:

\[
0 \to N_{X'}(Y' \times \mathbb{A}^n) \to N_{X'}(Y \times \mathbb{A}^n) \to N_{Y' \times \mathbb{A}^n}(Y \times \mathbb{A}^n)|_{X'} \to 0.
\]

Therefore this last sequence is split exact and \( N_{X'}(Y' \times \mathbb{A}^n) = N_X(Y \times \mathbb{A}^n)|_{X'} \). \( \Box \)

If a morphism of \( k \)-schemes \( f : X \to Y \) admits an \( N \)-trivial embedding \( X \to Y \times \mathbb{A}^n \), then we say a morphism \( g : Y' \to Y \) is transversal to \( f \) if the canonical morphism \( N_X(Y' \times \mathbb{A}^n) \to g^*(N_X(Y \times \mathbb{A}^n)) \) is an isomorphism. This is independent of the choice of \( N \)-trivial embedding since it is equivalent to the canonical morphism \( \Omega^1_{Y'/Y}|_{X'} \to \Omega^1_{X'/X} \) being an isomorphism. In the application, we require the weak transfers along a finite flat morphism \( f : X \to Y \) in \( \text{Sm}/k \), whose definition involves the choice of an \( N \)-trivial embedding, to be compatible with certain types of base change. By Corollary A.4, any base change \( g : Y' \to Y \) is transversal to such an \( f \). We only needed compatibility with base change by smooth morphisms, and by inclusions of a smooth divisor.

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