Questions and remarks to the Langlands programme

A. N. Parshin

Abstract. A brief survey is given of the classical Langlands programme to construct a correspondence between \( n \)-dimensional representations of Galois groups of local and global fields of dimension 1 and irreducible representations of the groups \( \text{GL}(n) \) connected with these fields and their adelic rings. A generalization of the Langlands programme to fields of dimension 2 is considered and the corresponding version for 1-dimensional representations is described. A conjecture on the direct image of automorphic forms is stated which links the Langlands correspondences in dimensions 2 and 1. In the geometric case of surfaces over a finite field the conjecture is shown to follow from Lafforgue’s theorem on the existence of a global Langlands correspondence for curves. The direct image conjecture also implies the classical Hasse–Weil conjecture on the analytic behaviour of the zeta- and \( L \)-functions of curves defined over global fields of dimension 1.

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Contents

1. Introduction 510
2. Basic fields from the viewpoint of scheme theory 514
3. Two-dimensional generalization of the Langlands correspondence 516
4. Functorial properties of the Langlands correspondence 518
5. Relation to the geometric Drinfeld–Langlands correspondence 522
6. Direct image conjecture 526
7. A link with the Hasse–Weil conjecture 531
8. Appendix: zero-dimensional generalization of the Langlands correspondence 533
Bibliography 536

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1. Introduction

The goal of the Langlands programme is to establish a correspondence between representations of Galois groups (and their generalizations or versions) and representations of reductive algebraic groups. The starting point for the construction is a field. Six types of fields are considered: three types of local fields and three types of global fields [35], [14]. The former comprise the following:

1) finite extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers, the field $\mathbb{R}$ of real numbers, and the field $\mathbb{C}$ of complex numbers,
2) the fields of Laurent power series $\mathbb{F}_q((t))$, where $\mathbb{F}_q$ is the finite field of $q$ elements,
3) the field of Laurent power series $\mathbb{C}((t))$.

The global fields are:

4) fields of algebraic numbers (finite extensions of the field $\mathbb{Q}$ of rational numbers),
5) fields of algebraic functions in one variable with a finite field of constants (finite extensions of the field $\mathbb{F}_q(x)$),
6) fields of algebraic functions in one variable with $\mathbb{C}$ as the field of constants (finite extensions of the field $\mathbb{C}(x)$).

The local fields in the first list are completions of the global fields in the second list.

In the classical Langlands programme, only the first two types of local and global fields are considered. The programme, in its simplest (and most studied) variant, consists in the construction of a correspondence between finite-dimensional (of dimension $n$) representations of the Galois group $G_K = \text{Gal}(K^{\text{sep}}/K)$ of a separable closure of the field $K$ and irreducible infinite-dimensional (usually) representations of the group $GL(n, A_K)$, where $A_K = K$ for the local fields $K$ and $A_K = \mathbb{A}_K$ (ring of adeles) for the global fields $K$.

Recall that $\mathbb{A}_K = \{(f_v): f_v \in K_v \text{ for all } v, f_v \in \hat{O}_v \text{ for almost all } v\}$. Here $v$ runs through all valuation classes of the field $K$ with complete valuation rings $\hat{O}_v$ and local fields $K_v = \text{Frac}(\hat{O}_v) \supset K$ which are the local completions of $K$. In the case of number fields, one has to add finitely many Archimedean valuations of the field $K$ and the related embeddings of $K$ in the fields $\mathbb{R}$ or $\mathbb{C}$.

Globally, the representations we consider here must be automorphic, that is, realized in a space of functions on the quotient space $GL(n, \mathbb{A}_K)/GL(n, K)$. Since $n$-dimensional representations of the Galois group $G_K$ can be viewed as homomorphisms from $G_K$ into $GL(n, \mathbb{C})$, there is a natural generalization of this correspondence, when the group $GL(n)$ is replaced by an arbitrary reductive algebraic group $G$. In this more general situation, the hypothetical Langlands correspondence should appear (in a first approximation) as follows:

$$\text{Hom}(G_K, ^LG(\mathbb{C})) \leftrightarrow \text{irreducible (automorphic)}$$
$$\text{representations of the group } G(A_K).$$

Here $^LG$ is a reductive algebraic group, the Langlands dual group to $G$ (see the definition and basic properties in [51], [8]). It is an important and highly non-trivial fact that if the original group $G$ is regarded as a reductive group over the different
fields in the basic list, then the dual group $L^G$ is always a group defined over the complex numbers. The group $L^G$ contains the connected component of the identity $L^G_0$, which is a reductive algebraic group over $\mathbb{C}$. Its root system is constructed by application of the duality of tori, starting from the root system of the original group $G$. In addition, for the general linear group $G = GL(n)$ the root systems of $G$ and $L^G$ coincide. Hence, in this case the presence of duality is masked. Finally, the whole group $L^G$ is the semidirect product of the group $L^G_0$ and the Galois group $\text{Gal}(K^{\text{sep}}/K)$.

The representations of Galois groups included in the left-hand side of the Langlands correspondence require substantial generalization, in order that one can hope for the existence of a bijection with automorphic representations of reductive groups that are on the right-hand side. In the theory of Artin $L$-series one considers representations of finite Galois groups with values in the group $GL(n, \mathbb{C})$. If one passes to the Galois group $G_K$ of a separable closure of the ground field $K$, then all of its continuous representations in the group $GL(n, \mathbb{C})$ are obtained from representations of finite factors $\text{Gal}(L/K)$, where $L/K$ runs through all the normal finite extensions of $K$. Thus, there are not too many continuous complex representations of the Galois groups. On the other hand, there are continuous $l$-adic representations arising from the action of the Galois group on the étale cohomology of algebraic varieties defined over $K$. They usually do not factor through the finite factors of Galois groups. Deligne has devised a way of building an extension of $G_K$ whose complex representations ‘coincide’ with the $l$-adic representations of $G_K$. The Weil–Deligne group $WD_K$ is the semidirect product of the Weil group $W_K$ with $\mathbb{C}$ under the action $wz^{-1} = q^{-\nu(w)}z$. To be precise,

$$WD_K = \{w, z : w \in W_K, z \in \mathbb{C}\}$$

and

$$(w, z) \cdot (w', z') = (ww', z + q^{-\nu(w)}z'),$$

where $\nu : W_K \to \mathbb{Z}$ is the canonical map of the Weil group that takes the $n$th power of the Frobenius automorphism into $n$, and $q$ is the number of elements in the residue field.

In particular, the Langlands correspondence $LC_n$ for local fields and the group $GL(n)$ must have the form

$$\text{Hom}^{c,ss}(WD_K, GL(n, \mathbb{C})) \leftrightarrow \text{smooth irreducible representations of the group } GL(n, K),$$

1 Or over another field such as $\mathbb{Q}_l$, which has nothing in common with the ground field $K$.

2 We note that existence of the bijection was only assumed for the group $G = GL(n)$. Generally, the correspondence can fail to be one-to-one and have certain finite sets as pre-images, that is, so-called $L$-packets.

3 This is the case if all cycles on the variety are algebraic. By Tate’s conjectures, the converse should also be true.

4 The Weil group $W_K$ of a local field $K$ with finite residue field is the subgroup of $G_K$ consisting of those elements of it which act on the residue field by integer powers of the Frobenius automorphism.

5 See details in [51], § 4.1. Instead of the Weil–Deligne group one can consider representations of the Weil–Arthur group $WA_K = W_K \times SL(2, \mathbb{C})$. The transition from representations of one group to representations of the other is provided by the Jacobson–Morozov theorem (see [35]).
where this is the definitive exact form (the representations of the Weil–Deligne group are assumed to be continuous and completely reducible). Smoothness of a representation \( \pi : \text{GL}(n, K) \to \text{End}(V) \) means that for every \( v \in V \) there is an open compact subgroup \( \mathbb{K}' \) of \( \text{GL}(n, K) \) such that \( \pi(\mathbb{K}')(v) = v \). Irreducibility means the absence of any non-trivial invariant subspaces of \( V \). Let \( \rho, \rho' \) run through the \( n \)-dimensional representations of the Weil–Deligne group, and let \( \pi \) run through representations of \( \text{GL}(n, K) \). The Langlands correspondence has the following properties:

i) \( \text{LC}_1(\text{Det} \rho) = \text{central character of the representation LC}_n(\rho) \);

ii) \( \text{LC}_n(\rho \otimes \chi) = \text{LC}_n(\rho) \otimes (\text{LC}_1(\chi) \circ \text{Det}) \), where \( \chi \) is a 1-dimensional representation and \( \text{Det} : \text{GL}(n, K) \to K^* \) is the determinant;

iii) \( \text{LC}_{m+n}(\rho \times \rho') = \text{Ind}_{P}^{\text{GL}(m+n,K)} \circ \text{Res}_{M}^{P}[\text{LC}_m(\rho) \boxtimes \text{LC}_n(\rho')] \);

iv) if \( \rho \) is irreducible, then \( \text{LC}_n(\rho) \) is a cuspidal representation;

v) \( \text{LC}_n(\hat{\rho}) = \text{LC}_n(\rho) \);

vi) \( L(\rho, s) = L(\text{LC}_n(\rho), s) \);

vii) if \( \rho \) is unramified, then \( \text{LC}_n(\rho) \) is a spherical representation.

For the property iii) we have used the embedding of the product of the groups \( \text{GL}(m, K) \) and \( \text{GL}(n, K) \) in the group \( \text{GL}(m+n, K) \) as the Levi quotient \( M = \text{GL}(m, K) \times \text{GL}(n, K) \) of the standard parabolic subgroup \( P \). Concerning the operation \( \boxtimes \) see footnote 17. The subsequent operation is parabolic induction, which extends the representation of the Levi subgroup \( M \) to the subgroup \( P \) and then induces it to the whole group \( \text{GL}(m+n, K) \) (for details, and for the definition of cuspidal representations, see [4], [7], and [6], Chap. 4 for the case \( \text{GL}(2) \)).

Now let \( \hat{\rho} (\hat{\pi}) \) denote the dual (contragredient) representations arising from the representation \( \rho (\pi) \). Let \( L(\rho, s) \) be the Artin \( L \)-function associated with the representation \( \rho \) (see [51], §3 and [6], Chap. 1.8) and let \( L(\pi, s) \) be the \( L \)-function for the representation \( \pi \) (see §7 on the Hasse–Weil conjecture). A representation \( \pi \) of \( \text{GL}(n, K) \) on a space \( V \) is called spherical if there exists a vector \( v \in V \) such that \( \pi(\mathbb{K})(v) = v \), where \( \mathbb{K} = \text{GL}(n, \mathcal{O}_K) \) is a maximal compact subgroup. The list of properties can be further supplemented by an equality for local constants appearing in the functional equations for the \( L \)-functions (see [20]).

Similar properties should hold for the global Langlands correspondence.

Up to now, the Langlands correspondence has been constructed exactly in this form for local fields of type 1) (Harris and Taylor [18]; Henniart [20]) and type 2) (Laumon, Rapoport, and Stuhler [36]). For global fields of type 5) and the group \( G = \text{GL}(n) \), the correspondence was obtained for \( n = 2 \) by Drinfeld [9], [10] and for arbitrary \( n \) by Lafforgue [32]. In any case, the construction of the correspondence for number fields remains an open question, which is apparently very far from being resolved.

The previous constructions have a purely arithmetic origin. Based on his proof, which is concentrated around the moduli space of vector bundles on algebraic curves, Drinfeld has expanded the class of fields and introduced the geometric Langlands correspondence, which should hold for fields of types 3) and 6).

The initial object will then be a complete algebraic curve \( X \), defined over \( \mathbb{C} \), for which the field \( K \) is the field of rational functions on \( X \). The correspondence
appears as
\[ \text{local } L^G\text{-systems on } X \Leftrightarrow \text{Hecke sheaves on the scheme (stack) } \text{Bun}_{X,G} \]
and was proved for unramified local systems and the group \( G = \text{GL}(n) \) (Drinfeld for \( n = 2 \) and Frenkel, Gaitsgory, and Vilonen for arbitrary \( n \), based on a construction of Laumon). Since local (unramified) \( L^G \) systems on \( X \) are nothing but representations of the fundamental group of \( X \) in the group \( L^G \), this looks like a quite natural analogue of the arithmetical situation. On the other hand, there is \( \text{Bun}_{X,G} \), the stack of principal \( G \)-bundles on \( X \), and the ring of Hecke correspondences acts on that. For the stack \( \text{Bun}_{X,G} \) one needs not the usual local systems but rather perverse sheaves, which are considered instead of the automorphic forms in the classical Langlands correspondence. Such a sheaf is called a Hecke sheaf if it is semi-invariant with respect to Hecke correspondences. In this situation, the Hecke sheaves do not replace the representations of the adelic group that correspond via the arithmetic Langlands correspondence to representations of the Galois group, but correspond to the spherical vectors of these representations. Whether the representations themselves exist, I do not know, but if they do exist, then they should be unramified (spherical) in some reasonable sense. In the arithmetical situation the spherical vector is uniquely defined, up to a constant, and generates the whole representation space.

What are the problems in number theory that can be studied with the help of the Langlands programme?

1. **Non-Abelian class field theory.** The Langlands correspondence arose from consideration of the Abelian situation when \( G = \text{GL}(1) \) and the representations of the Galois group are homomorphisms of \( G_K \) into \( \text{GL}(1, \mathbb{C}) = \mathbb{C}^* \), that is, characters of \( G_K \). Previously, such maps were called quasi-characters, retaining the term ‘character’ for homomorphisms into the unitary group \( U(1) \subset \text{GL}(1, \mathbb{C}) \). By class field theory we have the reciprocity map,
\[ C_K \to G_K^{\text{ab}}, \]
where \( C_K = K^* \) for local fields of types 1) and 2) and \( C_K = \mathbb{A}_K/K^* \) for global fields of types 4) and 5). Characters of \( C_K \) (and they are automorphic forms on \( \text{GL}(1) \)) can be transferred to the group \( G_K \) as its Abelian characters, that is, 1-dimensional representations. We thus obtain an Abelian Langlands correspondence [34], which served as a sample and the starting point of the whole programme.

However, there is a fundamental difference between the general Langlands construction and its Abelian case. Actually, the difference is related to the problem of an explicit description of the Galois group. Class field theory permits explicit computation of the Galois group of the maximal Abelian extension of the original field (by specifying its generators and relations). Moreover, the cohomological formulation of class field theory makes it possible to compute the Galois group of certain non-Abelian extensions (\( l \)-extensions [49], [30]).

The Langlands programme enables us to describe the set of representations of the Galois group, providing it with a standard set of operations (primarily direct and tensor products). A further development of the programme was to construct the category of representations of the Galois group as a monoidal category
(more precisely, the Tannaka category, or rather its generalization; see the discussions in [35], [44]), which allows us to reconstruct the group uniquely (in view of a general theorem in the theory of Tannaka categories [5]). Note that this theorem is an existence theorem, and it is unclear how to obtain an explicit presentation through generators and relations.

2. **Conjectures on zeta- and L-functions** of arithmetic schemes (see §7 below on the Hasse–Weil conjecture). One great achievement in this direction is the proof of Artin’s conjecture on the holomorphy of L-series for a wide class of 2-dimensional representations of the Galois group over \( \mathbb{Q} \) (Langlands and Tunnell). Actually, the special case of octahedral representations was a starting point for Wiles’ proof of the Taniyama–Weil conjecture.

3. **Description** of the set of smooth irreducible representations of reductive groups over local and global fields.

The last direction is the most successful. We have seen how the viewpoint of the representation theory of reductive groups requires replacement of the Galois group of the field by the Weil–Deligne group or the Weil–Arthur group. This is most sharply shown by the example of the local field \( K = \mathbb{C} \). The group \( \text{GL}(n, K) \) has, for \( K = \mathbb{C} \), a non-trivial set of irreducible representations (they are all principal series representations, and their parameter space is a complex manifold, namely, the quotient space of the manifold of characters of the torus \((\mathbb{C}^*)^n\) by the permutation group \( S_n \)). At the same time, the Galois group \( G_K \) is trivial, but the Weil group \( W_{\mathbb{C}} \) is, according to the general theory ([2], Chap. XV and [51], §1), equal to \( \mathbb{C}^* \) and has exactly the same space of \( n \)-dimensional semisimple representations [29].

We now consider the basic fields of the Langlands programme in terms of the general principles of arithmetic algebraic geometry.

### 2. Basic fields from the viewpoint of scheme theory

The fields considered in Langlands theory are fields of functions on certain schemes. In scheme theory we have a classification by dimension. If the scheme is affine and corresponds to a ring \( A \), then we can consider its Krull dimension. We have the following table of rings arising in arithmetic, where \( \mathbb{F}_q \) is the finite field of \( q \) elements, \( \mathbb{Z} \) is the ring of integers, and \( \mathbb{F}_1 \) is the now popular field of one element.

| Dim(\( A \)) | geometric case | number case |
|--------------|----------------|-------------|
| > 2          | \( \ldots \)  | \( \ldots \) |
| 2            | \( \mathbb{F}_q[x, y] \)  | \( \mathbb{Z}[y] \) (surfaces) |
| 1            | \( \mathbb{F}_q[x] \) (curves) | \( \mathbb{Z} \) (arithmetical curves, \( \text{Spec}(\mathbb{Z}) \)) |
| 0            | \( \mathbb{F}_q \) | \( \mathbb{F}_1 \) |

\(^6\)Theorem of Zhelobenko and Naimark (1966; see the exposition in [56]). For the explicit form of the characters of the torus \( \mathbb{C}^* \), and consequently of the torus \((\mathbb{C}^*)^n\), see below and in more detail in [29].
These rings correspond to affine schemes and the fields of functions on them. In particular, here are the global fields of types 4) and 5) from the list of fields that are now being considered as the basic ones for the Langlands programme (see [35], [14]).

| Dim(A) | geometric case | number case |
|-------|---------------|-------------|
| 2     | \( \mathbb{F}_q(x,y) \) (rational functions) | \( \mathbb{Q}(y) \) (rational functions) |
| 1     | \( \mathbb{F}_q(x) \) (rational functions) | \( \mathbb{Q} \) (rational numbers) |
| 0     | \( \mathbb{F}_q \) | \( \mathbb{F}_1 \) |

Finally, here is a table of the local fields that arise in this situation:

| Dim(A) | geometric case | number case |
|-------|---------------|-------------|
| 2     | \( \mathbb{F}_q((x))((y)) \) (iterated Laurent power series) | \( \mathbb{Q}_p((y)), \mathbb{R}((y)), \mathbb{C}((y)) \) (Laurent power series) |
| 1     | \( \mathbb{F}_q((x)) \) (Laurent power series) | \( \mathbb{Q}_p, \mathbb{R}, \mathbb{C} \) (\( p \)-adic, real, and complex numbers) |

Here we see all the types 1)–3) of local fields, but also a number of new fields which have been introduced in higher adelic theory [42]. Such a classification is a development of the ideas of André Weil from his report to the Cambridge ICM in 1950 [53], where he stressed the importance of the concept of the Krull dimension for the classification of arithmetic problems. In his pre-war letter to his sister Simone [52] in which he also speaks of the analogy between the number field case and the geometric case, the situation is simplified, and so he adds the field \( \mathbb{C}((y)) \) and puts it ‘on the same level’ with such fields as \( \mathbb{F}_q((x)) \) and \( \mathbb{Q}_p \), which, as can be seen from the table, does not agree with the dimension.

Returning to the global fields, we may ask, where in these tables are schemes such as curves over \( \mathbb{C} \) associated with fields of type 6)? These curves and the fields of functions on them are the subject of the geometric Langlands correspondence generalizing the usual Langlands correspondence. Indeed, they are not here because they correspond to rings of the form \( \mathbb{C}[x] \). These objects have an intermediate nature; they are global and of dimension 1 over \( x \) (the coordinate on the curve), and local in the field of definition \( \mathbb{C} \). This field is included in the table as a 1-dimensional (!) local field. Thus, rings such as \( \mathbb{C}[x] \) correspond to geometric objects of dimension 2. That is, they are Archimedean fibres of arithmetic surfaces (on this topic see § 5 below).

This observation and the fundamental arithmetic analogies [41] suggest that one should consider, as a ‘partner’ of the geometric Langlands correspondence for curves over \( \mathbb{C} \), constructions of this kind for curves over the \( p \)-adic numbers \( \mathbb{Q}_p \). As far as

\[ \text{See their informal definition at the beginning of the next section.} \]
I know, this has not been considered by anyone, and later we will discuss how it might look.\footnote{As was pointed out to me by İlhan İkeda, a construction of this kind was recently found by A. G. M. Paulin for the Abelian case.}

Summarizing this discussion, we can say that the list of six basic fields could be completed as follows. Only the four types of fields 1), 2), 4), 5) belong to the 1-dimensional situation. Fields of types 3) and 6) are already in the 2-dimensional situation. In this case we have two types of local fields, namely,

1) finite extensions of the fields\footnote{If $K$ is a locally compact field with a non-Archimedean norm $|\cdot|$, then the field $K\{\{t\}\}$ consists of the series $\sum a_i t^i$, $a_i \in K$ which are infinite in both directions with $|a_i| \leq O(1)$ and $|a_i| \to 0$ as $i \to -\infty$. The fields $\mathbb{Q}_p\{\{t\}\}$ can arise on arithmetic surfaces.} $\mathbb{Q}_p((t))$, $\mathbb{R}((t))$, $\mathbb{C}((t))$, and $\mathbb{Q}_p\{\{t\}\}$,

2) the field of Laurent power series $\mathbb{F}_q((u))((t))$,

and two types of global fields, namely,

3) the fields of functions on arithmetic surfaces (finite extensions of the field $\mathbb{Q}(x)$ of rational functions),

4) the fields of functions on algebraic surfaces over a finite field of constants (finite extensions of the field $\mathbb{F}_q(x, y)$ of rational functions in two variables).

Between them lie the fields of intermediate type (partially local, partially global). In the situation of a surface over a finite field the intermediate fields are

2a) finite extensions of the fields $\mathbb{F}_q(u)((t))$,

2b) the field of quotients of the ring $\mathbb{F}_q[[u, t]]$ of formal power series in two variables.

In the case of arithmetic surfaces, the field $\mathbb{C}(x)$ (the field 6) from the Langlands list) is also an intermediate field, but it is not included in the standard set of fields appearing in 2-dimensional adelic theory (see $\S$ 3 below and \cite{42}). This is due to an additional structure, namely, the existence of a morphism from the 2-dimensional scheme onto a 1-dimensional scheme $B$ and its fibres, defined over 1-dimensional local fields of the scheme $B$.

Remark 1. Adelic techniques allow us to consider the local and global fields of the original Langlands programme in a unified way. As we have seen, in both cases these fields are the fields of functions on a scheme $X$. In the local (non-Archimedean!) case we have $X = \text{Spec}(\mathcal{O})$, where $\mathcal{O}$ is a local discrete valuation ring with finite residue field. In the global situation, $X$ is either a curve over a finite field or the spectrum of the ring of integers of an algebraic number field. In both cases, $A_K = \mathbb{A}_X$ if we apply the general definition of the ring of adeles \cite{11} \cite{42}, which may include Archimedean components as well.

3. Two-dimensional generalization of the Langlands correspondence

We may wonder about the existence of a 2-dimensional generalization of the Langlands correspondence. Let $X$ be a 2-dimensional scheme and $K$ the field of functions on it. Then the $n$-dimensional representations of the Galois group $G_K$ of a separable closure of $K$ could correspond to irreducible automorphic representations of the adelic group $G(\mathbb{A}_X)$. Here, the ring of adeles $\mathbb{A}_X$ was introduced in higher adelic theory in the 1970s (see the surveys \cite{11} \cite{42}).
If \( x \in C \subset X \) is a flag on \( X \) consisting of a point \( x \) and an irreducible curve \( C \) that is smooth at \( x \), then one can introduce a 2-dimensional local field \( K_{x,C} \). In local coordinates \( u, t \) of a formal neighbourhood of \( x \), the field \( K_{x,C} \) is equal to \( k(x)((u))((t)) \) (if \( X \) is a smooth surface), where \( C = (t = 0) \) and \( k(x) \) is the residue field of \( x \). In general, one can associate a direct sum of finitely many 2-dimensional local fields with the flag \( x, C \). The adelic ring \( \mathbb{A}_X \) is defined as the adelic (part of the total) product of the fields \( K_{x,C} \) for all flags \( x, C \).

The field \( K_{x,C} \) contains a discrete valuation ring \( \mathcal{O}_{x,C} \) (which is \( k(x)((u))[[t]] \) if \( X \) is a surface). For an irreducible curve \( C \) we take \( K_C = \text{Frac}(\mathcal{O}_C) \), and for a point \( x \) we set \( K_x = K\mathcal{O}_x \subset \text{Frac}(\mathcal{O}_x) \). Here \( \mathcal{O}_C = k(C)((t)) \) and \( \mathcal{O}_x = k(x)[[u, t]] \). The adelic product of these rings gives rise to three subrings \( \mathbb{A}_{12}, \mathbb{A}_{01}, \mathbb{A}_{02} \) of \( \mathbb{A}_X \). These rings are embedded in \( \mathbb{A}_X \) in a diagonal way (exactly like the principal adeles in the case of 1-dimensional schemes). We can compare the structure of the local adelic components in dimensions one and two:

\[
\begin{array}{ccc}
K_x & \rightarrow & K_{x,C} \\
K & \rightarrow & K \\
K_x & \rightarrow & K_{C} \\
\end{array}
\]

A generalization of the Langlands programme that uses this theory of adeles was outlined by Kapranov [26] (and detailed in [22]). It does not use the ordinary notion of representation but rather its generalization associated with 2-categories [15].

Here the number 2 stands for the dimension of the scheme \( X \). We have the following table of categories, where \( k \) is an arbitrary field:

| \( n = \text{Dim} \) | \( n \)-categories | objects | distinguished object |
|----------------|-----------------|--------|---------------------|
| 2              | 2-Vect /k       | Vect-mod/k | Vect /k |
| 1              | Vect /k         | V/k    | k                   |
| 0              | k               | A \subset k | \{1\}            |

where \( k \) is a ground field (or a ring) with a subset \( A \), \( V \) is a vector space over \( k \), and \( \text{Vect} /k \) is the category of all vector spaces over \( k \). The latter is a tensor category, and can be considered as a categorical generalization of the notion of ring. Then we can define the categories-modules \( \text{Vect-mod}/k \) over such a ‘ring’, and they form a 2-category whose objects are such categories.

We see an obvious inductive structure that can easily be compared with the inductive structure of the \( n \)-dimensional local fields. In addition, given a group \( G \), we can introduce a concept of an \( n \)-representation of it. A 1-representation is an ordinary homomorphism into the group of automorphisms of a vector space. Two-representations are obtained by replacing spaces by categories and homomorphisms by functors. More precisely, instead of the space \( V \) we choose the category-module \( \text{Vect-mod}/k \), and the 2-representation \( \pi \) associates with an
element $g \in G$ a functor $\pi(g) : \text{Vect-mod}/k \to \text{Vect-mod}/k$ satisfying a multiplicativity condition. This generalization is a special example of the general categorification process for concepts and structures.$^{10}$

Then according to Kapranov, in a certain approximation the Langlands correspondence for 2-dimensional local fields $K$ appears as

\[
\text{n-dimensional complex (or } l\text{-adic) Galois representations of } G_K \quad \Leftrightarrow \quad \text{irreducible 2-representations of the group } \text{GL}(2n, K),
\]

and for a global field $K$ on the scheme $X$ the field $K$ on the right-hand side of the correspondence is replaced by the ring $\mathbb{A}_X$ of adeles. The 2-representation must satisfy a certain automorphy condition. These definitions and related concepts need to be clarified, and one must be aware of certain restrictions they involve. In particular, the characteristic of the basic field is assumed to be 0 and nothing is said about local fields of Archimedean type. Surely, the construction should be valid for these fields and for fields of finite characteristic. We shall return to this issue.$^{11}$

4. Functorial properties of the Langlands correspondence

The usual Langlands correspondence has a large number of functorial properties. The theory includes two kinds of objects: a scheme or a field and a reductive algebraic group (or, more properly speaking, root data). The Langlands principle of functoriality $^{35}$ says what happens when one changes the group. We will restrict ourselves here to properties related to a change of the schemes.$^{12}$ First, these are properties associated with finite extensions of fields (base change and automorphic induction or lifting). Second, they are properties associated with the transition from local to global fields. All these properties are special cases of the general functorial construction of inverse and direct images.

The first properties are related to finite surjective morphisms of 1-dimensional schemes $f : X \to Y$, where both $X$ and $Y$ are either curves (or rings of integers in algebraic number fields) or the spectra of local rings. The second properties arise for morphisms $f : X \to Y$, where $X$ is the spectrum of a local ring, and $Y$ is a curve. Let $B$ be a curve over $\mathbb{F}_q$ and let $b \in B$. Then one has the commutative diagram

\[
\begin{array}{ccc}
\text{sheaves } \mathcal{F} \text{ on } B & \longrightarrow & \text{automorphic forms on } \text{GL}(\mathbb{A}_B) \\
\downarrow i^* & & \downarrow i^* \\
\text{sheaves } i^*(\mathcal{F}) \text{ on } \text{Spec}(\hat{\mathcal{O}}_b) & \longrightarrow & \text{smooth functions on } \text{GL}(K_b)
\end{array}
\]

$^{10}$representations will be discussed later in the Appendix.

$^{11}$Recently, Osipov, in developing the ideas of Kapranov, proposed a definition of an unramified local Langlands correspondence for 2-dimensional local fields and the group $\text{GL}(n)$ for any $n$.

$^{12}$These two are not independent. In fact, the functoriality principle implies automorphic induction and base change for finite extensions.
where \( i : \text{Spec}(\mathcal{O}_b) \to B \) is the canonical embedding, the left-hand map \( i^* \) is the inverse image of sheaves, and the right-hand map \( i^* \) is the taking of the local \( b \)-component of a function on the whole adelic group (for non-trivial examples of this diagram see [18]).

If the Langlands correspondence exists in one form or another for higher dimensions, then it should have very many more functorialities. Indeed, let us consider a 2-dimensional scheme \( X \) together with a structure morphism \( f : X \to B \), where \( B \) is a 1-dimensional scheme.

Let us assume for simplicity that our schemes are varieties defined over a finite field. For \( l \)-adic sheaves (representations of Galois groups), there are direct (push-forward) and inverse (pullback) images under the map \( f \). Then by the Langlands correspondence the same kind of operations should exist for representations of adelic groups.

Without discussing this question in full generality, we consider a special and already extremely interesting case. Let us start with a 1-dimensional representation of the Galois group on \( X \). It determines a certain sheaf \( \mathcal{F} \) of rank 1 on \( X \) and the direct images \( R^i f_*(\mathcal{F}) \) on \( B \), \( i = 0, 1, 2 \).

In this Abelian situation on the surface \( X \), we already have the Langlands correspondence between characters of the Galois group \( G_K \) and characters of the group \( K_2(\mathbb{A}_X) \). By 2-dimensional class field theory (see surveys [11], [23], [45]) there is a canonical map

\[
\varphi_X : K_2(\mathbb{A}_X) \to G_K^{ab}
\]

which gives a map of the character groups in the opposite direction. The reciprocity laws on \( X \) are

\[
\varphi_X(K_2(\mathbb{A}_{01})) = (1), \quad \varphi_X(K_2(\mathbb{A}_{02})) = (1).
\]

We can assume that on the ‘automorphic side’ of the Langlands correspondence we have direct image and inverse image operations between characters of \( K_2(\mathbb{A}_X) \) and automorphic forms on \( \mathbb{A}_B \).

The case of the inverse image is not hypothetical, but follows from the known results of higher adelic theory. There is a direct image operation for Abelian groups [27], [39]:\(^{13}\)

\[
f_* : K_2(\mathbb{A}_X) \to K_1(\mathbb{A}_B).
\]

Since \( K_1(\mathbb{A}_B) = \text{GL}(1, \mathbb{A}_B) \), we obtain a map of automorphic forms on \( \text{GL}(1, \mathbb{A}_B) \) to automorphic forms (that is, characters) on \( K_2(\mathbb{A}_X) \). Their ‘automorphy’ is ensured

\(^{13}\)In this paper the map \( f \) is smooth. The analogous homomorphism for differential forms was constructed in [38] for non-smooth maps. Quite recently [37], a direct image morphism for \( K \)-groups was constructed for non-smooth maps of algebraic and also arithmetic surfaces.
by higher reciprocity laws on the scheme $X$. We have the commutative diagram

\[
\begin{array}{ccc}
\text{sheaves } f^*(\mathcal{F}) & \rightarrow & \text{characters of the group } K_2(\mathbb{A}_X) \\
\downarrow f^* & \downarrow f^* \\
\text{sheaves } \mathcal{F} & \rightarrow & \text{automorphic forms on } \text{GL}(1, \mathbb{A}_B)
\end{array}
\]

(4)

The direct image must\(^\text{14}\) consist in constructing automorphic forms $f^*_i(\chi)$ on $\text{GL}(r_i, \mathbb{A}_B)$ corresponding to characters of $K_2(\mathbb{A}_X)$ on $X$. The following commutative diagram must be valid:

\[
\begin{array}{ccc}
\text{sheaves } \mathcal{F} & \rightarrow & \text{characters of the group } K_2(\mathbb{A}_X) \\
\downarrow R^if_* & \downarrow f^*_i \\
\text{sheaves } R^if_*(\mathcal{F}) & \rightarrow & \text{automorphic forms on } \text{GL}(r_i, \mathbb{A}_B)
\end{array}
\]

(5)

Here $r_i$ is the rank of the sheaf $R^if_*(\mathcal{F})$. It is quite possible that instead of the groups $\text{GL}(r_i, \mathbb{A}_B)$ one must consider the ‘even’ groups $\text{GL}(2, \mathbb{A}_B) \sim \text{GL}(r_0, \mathbb{A}_B) \times \text{GL}(r_2, \mathbb{A}_B)$ and the ‘odd’ groups $\text{GL}(r_1, \mathbb{A}_B)$ separately. Also, in the case of the ‘middle’ group corresponding to $i = 1$ the monodromy which preserves the skew-symmetric cup product could be taken into account, and the orthogonal group corresponding to the symplectic group (of the Galois action) according to the $L$-duality could be considered.\(^\text{15}\)

Remark 2. We can compare this picture and the number field case in which a 2-dimensional regular scheme $X$ is mapped onto a 1-dimensional scheme $B$ (the ring of integers in a number field). We expect the same kind of construction in this case also. If the general fibre of the map $f$ is an elliptic curve over $\mathbb{Q}$, $B = \text{Spec}(\mathbb{Z})$, and $\chi$ is a trivial character, then the automorphic form $f^1(\chi)$ is just an adelic version

\(^{14}\)We deliberately describe this hypothetical construction in a preliminary vague form. Generally, the Langlands correspondence includes irreducible automorphic representations on the right-hand side. Nevertheless, in many cases it is possible to find a generating vector in the representation spaces. These vectors lead to automorphic forms on the adelic groups. Thus, in the unramified case there are spherical vectors and unramified automorphic forms; for generic representations one can use the Whittaker functions. At the end of this note we give an exact formulation in a simple but still non-trivial case.

\(^{15}\)More precisely, the group $\text{GSpin}(2g+1, \mathbb{C})$ which is $L$-dual to the group $\text{GSp}(2g)$ of simplectic similarities.
of the parabolic form corresponding to the curve according to the Taniyama–Weil conjecture (now a theorem!). In the Langlands programme this form generates a cuspidal representation of the group $GL(2, \mathbb{A}_B)$ [33], [16]. At the same time, the less complicated representation associated (in our notation) with the forms $f^0_*(\chi)$ and $f^2_*(\chi)$ has somehow remained in the shadows. These forms are defined on the group $GL(1, \mathbb{A}_B)$, that is, they are characters of $\mathbb{A}_B^*$ and hence give a character $\eta$ of the standard maximal torus $T$ in $G = GL(2, \mathbb{A}_B)$. Parabolic induction applied to $\eta$ (if it is not equal to 0) gives an induced representation from the principal series of $G$ which is generated by an Eisenstein series. Thus, the original character $\chi$ can be associated with two infinite-dimensional representations of $GL(2, \mathbb{A}_B)$, where the ‘even’ one belongs to the continuous spectrum, and the other ‘odd’ one belongs to the discrete spectrum.

Let us return to our diagram. Since we are considering the case of curves over a finite field, the correspondence on the bottom line has already been constructed (Drinfeld–Lafforgue), and one ‘only’ needs to close the entire diagram. It seems natural to construct first the local and semilocal direct images that occur in the following diagram:

$$
\begin{array}{ccc}
\text{characters} & \rightarrow & \text{characters} \\
\text{of the group} & & \text{of the group} \\
K_2(\mathbb{A}_X) & \rightarrow & K_2(\mathbb{A}_{X_b}) \\
\downarrow f^* & & \downarrow f^* \\
\text{automorphic} & \rightarrow & \text{automorphic} \\
\text{forms} & & \text{forms} \\
on GL(r_i, \mathbb{A}_B) & \rightarrow & on GL(r_i, K_b) \\
f^* & & f^* \\
\end{array}
$$

(6)

Here $b = f(x), C$ is a curve on $X$ passing through $x$ (in particular, the fibre $F_b$ of the map $f), X_b$ is the 2-dimensional scheme, and $X_b = X \times_B \text{Spec}(\mathcal{O}_b)$ is an infinitesimal neighbourhood of the fibre $f^{-1}(b)$.

The next step should be to prove the global automorphic property for the image-form on $B$, starting from the reciprocity laws for the character on $K_2(\mathbb{A}_X)$. Note that these issues can be considered completely independently from how one might arrange the Langlands correspondence for sheaves $\mathcal{F}$ of rank $> 1$ on $X$.

Also, it is important to consider functorial properties of the Langlands correspondence for 1-dimensional schemes (conventional theory) and schemes (finite fields) of dimension 0. This question was raised by me in the late 1970s and discussed then with A. Beilinson. Although it was clear that the correspondence in dimension 0 must be something simple, we could not think it out. When Kapranov sent me the first version of his paper [26], I suggested that he should find the right definition and add it to his 2-dimensional construction, which he did in the final version. However, there he did not consider functorial properties associated with surjective maps of schemes of different dimensions.
In the case of schemes of dimension 1 and 0 we have a diagram

\[
\begin{array}{ccc}
\text{sheaves } \mathcal{F} \text{ of rank } r \text{ on } X & \longrightarrow & \text{automorphic forms on } \text{GL}(r, \mathbb{A}_X) \\
R^i f_* & & f^*_i \\
\text{sheaves } R^i f_*(\mathcal{F}) \text{ on } \text{Spec}(\mathbb{F}_q) & \longrightarrow & L\text{-functions}
\end{array}
\]

where \( X \) is a curve over the finite field \( \mathbb{F}_q \), the sheaves on \( \text{Spec}(\mathbb{F}_q) \) are vector spaces with the Frobenius action, and the \( L \)-functions are characteristic polynomials of the Frobenius automorphism. The lower line refers to the 0-dimensional situation and represents the Langlands correspondence for dimension zero according to Kapranov’s proposal.\(^{16}\)

This diagram should be consistent with the hypothetical diagram which we proposed above for a morphism of the scheme \( X_b \) on \( \text{Spec}(\mathcal{O}_b) \) if we take a closed fibre \( F_b = X \times \text{Spec}(k(b)) \) as the curve \( X \). The starting point of this concordance is the Cartesian commutative diagram

\[
\begin{array}{ccc}
F_b & \longrightarrow & X_b \\
\downarrow & & \downarrow \\
\text{Spec}(k(b)) & \longrightarrow & \text{Spec}(\mathcal{O}_b)
\end{array}
\]

5. Relation to the geometric Drinfeld–Langlands correspondence

Let us now consider the same picture for the number field case when the scheme \( X \) is mapped onto a 1-dimensional scheme \( B \) (the ring of integers) and completed in accordance with the Arakelov theory by Archimedean fibres \( X \times_B \mathbb{C} \) or \( X \times_B \mathbb{R} \)\(^{[41]}\).

Semilocal schemes \( X_b \) in the new Archimedean situation no longer exist, but we must find an analogue of the above structures in this case as well (according to the general principles of the ‘numbers-functions’ analogy in arithmetic\(^{[41]}\)). It would be natural to expect that such a 2-dimensional (!) Kapranov analogue of the Langlands correspondence for the scheme \( X \times_B \mathbb{C} \) (so far only very hypothetical) might be given by the (already formulated and partially proved) geometric Drinfeld–Langlands correspondence on the curve \( X \) which is defined over the complex field \( \mathbb{C} \), and which appears here as an Archimedean fibre. We will discuss this possibility later.

Let us assume now that in this situation we also have direct images and the commutative diagram

\[
\begin{array}{ccc}
\text{sheaves } \mathcal{F} \text{ of rank 1 on } X & \longrightarrow & \text{characters } C_X \\
R^i f_* & & f^*_i \\
\text{sheaves } R^i f_*(\mathcal{F}) \text{ on } \text{Spec}(\mathbb{C}) & \longrightarrow & \text{automorphic forms on } \text{GL}(r_i, \mathbb{C})
\end{array}
\]

where \( f : X \to \text{Spec}(\mathbb{C}) \), and \( C_X \) is a suitable class formation (see\(^{[2]}\)) in class field theory for the field \( \mathbb{C}(X) \) of functions on \( X \). Such formations were constructed in class field theory for fields of type \( \mathbb{C}(X) \) in the 1950s by Serre and

\(^{16}\)See a discussion of this proposal below in the Appendix.
Hasewinkel [47], [19] and by Kawada and Tate [28]. They have the form

\[ C_X = \pi_1(\mathbb{A}_X^{(0)}/K^*) = \pi_1(\lim_{\leftarrow} J_m(X)), \]

where \( (0) \) denotes the degree-0 ideles, the \( J_m(X) \) are generalized Jacobian varieties with ‘module’ \( m \) [46], and the limit is taken over all \( m \) and is a pro-algebraic group in the sense of Serre [47].

The bottom line of the diagram is well known in the classical theory of the local Langlands correspondence for the field \( \mathbb{C} \). The ‘sheaves’ on \( \text{Spec}(\mathbb{C}) \) are vector spaces of finite dimension over \( \mathbb{C} \) equipped with a Hodge structure. Note that the cohomology of the local system (sheaf) on \( X \) can be equipped with a canonical Hodge structure. Indeed, it is this structure that defines a representation of the Weil group \( W_{\mathbb{C}} \) for \( \mathbb{C} \), which plays the role of the Galois group in this situation. We have \( W_{\mathbb{C}} = \mathbb{R}_{\mathbb{C}}/\mathbb{R}_{\mathbb{G}_m} = \mathbb{C}^* \).

Thus, the following question arises.

**Question 1.** How can one construct the direct image (that is, the arrow on the right-hand side of the diagram (7)) explicitly?

But one first needs to figure out the correct definition of the Langlands correspondence in this situation. As above, we can and must begin with the local Langlands correspondence, local on \( X \). The existing Drinfeld theory (see the survey [13]) considers a local correspondence associated with local fields of the form \( \mathbb{C}((t_P)) \) which correspond to points \( P \) of \( X \) (here \( t_P \) is a formal local parameter at the point \( P \)). The local correspondence must describe the \( n \)-dimensional representations of the group, which is a local version of the Galois group (or of the fundamental group in the unramified case). It is already interesting to study the case of an Abelian group and \( n = 1 \). Thus, there arises the question of the local analogue of class field theory for such fields. Since they are Archimedean variants of 2-dimensional local fields, it is natural to ask the following question.

**Question 2.** What is the relationship with 2-dimensional adelic class field theory?

If \( K \) is a 2-dimensional local field with finite last residue field, then the class field theory involves constructing the canonical reciprocity map

\[ K_2(K) \to G^\text{ab}_K, \]

which is a local component of the global map considered above. The kernel of this map coincides with the subgroup of divisible elements in \( K_2(K) \) (Fesenko’s theorem; see his survey [23]). Examples of 2-dimensional local fields include the fields \( \mathbb{F}_q((u))((t)) \) and \( \mathbb{Q}_p((t)) \), for which the class formations can be determined (in a very non-trivial way) and the Weil group is defined as the extension of the Galois group by the group \( K_2(K) \) [31], [1].

In the case of the local field \( \mathbb{C}((t)) \) with Archimedean residue field, we have the following structure of \( K \)-groups:

\[ K_2\mathbb{C}((t)) \to K_1\mathbb{C} = \mathbb{C}^*, \]

where the group \( \mathbb{C}^* \) is not the Galois group of \( \mathbb{C} \), but its Weil group (!), and one can assume that the Weil group of the field \( \mathbb{C}((t)) \) is defined by the same construction as
above, with use of the group $K_2 \mathbb{C}((t))$. This gives us the left-hand side of the local Abelian Langlands correspondence for $\mathbb{C}((t))$. The question of the right-hand side of the correspondence can be connected (according to Kapranov) with the study of 2-representations.

In any case, the resulting structure is basically different from the local geometric Drinfeld correspondence, which includes connections on line bundles, $D$-modules, and the Fourier–Mukai transformation (see the survey [13]). The fundamental point of this difference is the fact that the proposed construction takes into account the arithmetic nature of the ground field $\mathbb{C}$, which manifests itself in that it may be the completion of a global number field at some infinite point. When a curve over $\mathbb{C}$ is an Archimedean fibre of an arithmetic surface $X$, the field $\mathbb{C}$ can be identified with the local completions of the number fields corresponding to all horizontal curves on $X$.

The following picture, borrowed from [41], §4, is a simplest example of such a situation.

On the arithmetic surface $X = \mathbb{P}^1 / \text{Spec}(\mathbb{Z})$ we see three horizontal curves defined by the equations $t = 0$, $t = 1/5$, $t = 2$, where $t$ is a coordinate on the general fibre $\mathbb{P}^1 / \mathbb{R}$, which is the curve lying over the infinite place ($\infty$) of the ground ‘curve’ (‘compactification of $\text{Spec}(\mathbb{Z})$).

At the same time, the Drinfeld construction makes sense over any ground field (of characteristic 0?) and does not depend on its nature.

Remark 3. There should also be a similar picture for curves over $\mathbb{R}$. Here one has local fields of the form $\mathbb{R}((t))$, for which

$$K_2 \mathbb{R}[[t]] \hookrightarrow K_2 \mathbb{R}((t)) \rightarrow K_1 \mathbb{R} = \mathbb{R}^*,$$

$$K_2 \mathbb{R}[[t]] \rightarrow K_2 \mathbb{R} \rightarrow \{\pm 1\}.$$  

These expansions correspond to the unramified extension $\mathbb{C}((t)) \supset \mathbb{R}((t))$ and to the tamely ramified extension $\mathbb{R}((t^{1/2})) \supset \mathbb{R}((t))$. There are no other Abelian extensions of the field $\mathbb{R}((t))$. The class field theory for the field $K = \mathbb{R}((t))$ is
a canonical isomorphism

\[ K_2(K)/\{\text{divisible elements}\} \to \text{Gal}(K^{ab}/K). \]

As above, we assume that the Weil group of the field \( \mathbb{R}(\!(t)\!) \) can be constructed from the group \( K_2\mathbb{R}(\!(t)\!) \).

Consequently, the basic fields of the Langlands programme should be supplemented by finite extensions of \( \mathbb{R}(\!(t)\!) \) and \( \mathbb{Q}_p(\!(t)\!) \) associated with algebraic curves defined over \( \mathbb{R} \) and \( \mathbb{Q}_p \). To the list of local fields we should add the fields \( \mathbb{R}(\!(t)\!)/K((t)) \) with \( K \supset \mathbb{Q}_p \).

Remark 4. In the Langlands correspondence, a fundamental role is played by \( L \)-functions. One can associate with representations of Galois groups (and more generally of Weil groups) certain \( L \)-functions (in particular, these are the Dedekind and Hecke \( L \)-functions for Abelian representations and Artin’s generalization of them to the non-Abelian case). On the other hand, Langlands (and then Godement and Jacquet) introduced \( L \)-functions for automorphic representations of adelic groups of reductive groups. Like arithmetic \( L \)-functions, they are defined in terms of an Euler product of local \( L \)-factors connected with local components of a global automorphic representation (see §7 on the Hasse–Weil conjecture).

An important property of the Langlands correspondence is the equality of the \( L \)-functions which appear in both parts of the correspondence (arithmetic \( L \)-functions and automorphic \( L \)-functions). Global equality is preceded by equality of local \( L \)-functions for all the local correspondences.

In the case of the geometric Langlands correspondence introduced by Drinfeld, there is apparently no proper definition of \( L \)-functions, and hence there is no equality of this kind. Our interpretation of the geometric correspondence suggests that there must exist \( L \)-functions associated with the local fields \( \mathbb{C}(\!(t_P)\!) \) of points \( P \) on the curve \( X \) defined over \( \mathbb{C} \). They should be included in the local geometric Langlands correspondence. Since there is a structure map \( X \to \text{Spec}(\mathbb{C}) \), one needs to take into account that the field \( \mathbb{C} \) has its own \( L \)-functions which are involved in the local Langlands correspondence for this field. They are the gamma functions connected with the Hodge structures according to Serre [48].

Since we expect that there are certain direct and inverse images of automorphic forms associated with the map \( f: X \to \text{Spec}(\mathbb{C}) \), this suggests that the hypothetical \( L \)-functions of the local fields \( \mathbb{C}(\!(t_P)\!) \) of the points on the curve \( X \) must be connected with these gamma functions. Let \( z \in \mathbb{C}^* \) and let \( [z] = z/|z|, |z|_\mathbb{C} = |z|^2 \). Then the characters of the group \( \mathbb{C}^* \) have the form

\[ \chi: z \mapsto [z]^l|z|_\mathbb{C}^t, \quad l \in \mathbb{Z}, \quad t \in \mathbb{C}. \]

As a preliminary definition of the local \( L \)-function corresponding to unramified extensions at the point \( P \) of \( X \), one can propose the same gamma-factors for the character \( \chi \) as in the 1-dimensional situation of the local field \( \mathbb{C} \) [29]:

\[ L_P(s, \chi) = 2(2\pi)^{-(s+t+|l|/2)}\Gamma(s+t+|l|/2). \]

Here we assume that the unramified ‘extensions’ of the field \( K_P = \mathbb{C}(\!(t_P)\!) \) can be described by the Weil group of its residue field \( \mathbb{C} \). As we noted above, although the
field \( \mathbb{C} \) has no non-trivial extensions, the role of the Galois group when we construct the local Langlands correspondence is played by its Weil group.

Next, we can assume that the global \( L \)-function of a sheaf \( \mathcal{F} \) on \( X \) will decompose into a product, over all points of the curve, of these local \( L \)-functions.

### 6. Direct image conjecture

Let \( f : X \to B \) be a proper morphism of a surface \( X \) onto a regular curve \( B \) with a smooth general fibre. Let us return to our previously stated hypothesis about the existence of the direct image from the set of Abelian characters of \( K \), a smooth general fibre. Let us return to our previously stated hypothesis about the number field case, only for Galois cyclic extensions \([3]\). In the geometric case the unramified automorphic forms on \( \text{GL}(n, \mathbb{A}_F) \) to the set of automorphic representations of \( \text{GL}(n, \mathbb{A}_F) \) on \( B \). We formulate a number of properties which must be satisfied. For our starting point we take well-known properties of the direct image (automorphic induction) in the 1-dimensional case and unramified (spherical) representations. Here they are (see, for example, \([3]\) and \([21]\), Proposition 4.5).

Let \( f : B' \to B \) be a finite unramified covering of degree \( m \) of the curves \( B' \) and \( B \). Then the direct image \( f_* \) (automorphic induction AI) is a map \( \mathcal{A} \mathcal{F}^{nr}(n, B') \to \mathcal{A} \mathcal{F}^{nr}(nm, B) \), and the inverse image \( f^* \) (base change BC) is a map \( \mathcal{A} \mathcal{F}^{nr}(n, B) \to \mathcal{A} \mathcal{F}^{nr}(n, B') \). Here \( \mathcal{A} \mathcal{F}^{nr}(n, B) \) is the space of all unramified automorphic forms on \( \text{GL}(n, \mathbb{A}_B) \):

\[
\mathcal{A} \mathcal{F}^{nr}(n, B) = \{ \text{smooth functions on } \text{GL}(n, \mathcal{O})/\text{GL}(n, \mathbb{A}_B)/\text{GL}(n, K) \},
\]

where \( \mathcal{O} = \prod_x \mathcal{O}_x \), \( x \in B \), and \( K = \mathbb{F}_q(B) \). The following properties hold:

i) if \( \phi \in \mathcal{A} \mathcal{F}^{nr}(n, B') \) and \( \psi \in \mathcal{A} \mathcal{F}^{nr}(n', B) \), then we have the projection formula\(^{17}\)

\[
f_*(\phi \boxtimes f^*(\psi)) = f_*(\phi) \boxtimes \psi;
\]

ii) \( f_*(\tilde{\phi}) = \tilde{f}_*(\phi) \) and \( f^*(\tilde{\phi}) = \tilde{f}^*(\phi) \), where \( \tilde{\phi}(g) = \phi(g^{-1}) \) (duality formula);

iii) if \( n = 1 \), then \( f_*(\chi)(gz) = \omega(z)f_*(\chi)(g) \) and \( f_*(1)(gz) = \omega(\eta)z f_*(1)(g) \), where \( z \in \mathbb{A}_B^* \) (which is the centre of the group \( \text{GL}(\mathbb{A}_B) \)), and \( \omega \eta^{-1} = \beta^*(\chi) \) for the natural embedding \( \beta : \mathbb{A}_B^* \to \mathbb{A}_{B'}^* \);

iv) for unramified coverings \( f' : B'' \to B' \) and \( f : B' \to B \),

\[
(f' \circ f)^* = f'^* \circ f^* \quad \text{and} \quad (f' \circ f)_* = f_* \circ f'_*;
\]

v) if \( k : B' \to B' \) and \( j : B \to B \) are automorphisms of curves for which \( f \circ k = j \circ f \), then the base change\(^{18}\) formulae \( k^* \circ f_* = j^* \circ f_* \) and \( k_* \circ f^* = j_* \circ f^* \) hold;

vi) if \( \phi \in \mathcal{A} \mathcal{F}^{nr}(n, B') \), then \( L_{\text{GL}(n, \mathcal{A}_B)}(\phi) = L_{\text{GL}(nm, \mathcal{A}_B)}(f_*(\phi)) \).

For the unramified case, the definition of \( L \)-functions is given in \( \S 7 \) on the Hasse–Weil conjecture. These properties are assumed to hold for the much more general case of ramified coverings. As far as I know, they have been established, in the number field case, only for Galois cyclic extensions \([3]\). In the geometric case the

\(^{17}\)The operation \( \boxtimes \) associates with forms on \( \text{GL}(n, \mathbb{A}_B) \) and \( \text{GL}(n', \mathbb{A}_B) \) a form on \( \text{GL}(n, \mathbb{A}_B) \times \text{GL}(n', \mathbb{A}_B) \) corresponding to the external tensor product of the representations.

\(^{18}\)We make a remark about terminology. Here the expression ‘base change’ takes its customary meaning in algebraic geometry. The same expression was used earlier in the sense in which it is understood in the theory of automorphic representations.
existence of automorphic induction with these properties follows from Lafforgue’s theorem (see footnote 20) and the corresponding construction for Galois groups.

We show how to obtain the property iii) from the corresponding property for direct images of \(l\)-adic sheaves. We have the following equality, which is well known in representation theory:

\[
\det \rho(g) \det r(g)^{-1} = \chi(\text{Ver}(g))
\]  

for representations \(\rho\) and \(r\) of a group \(G\). Here \(H \subset G\) is a subgroup of finite index, \(\chi: H \to \mathbb{C}^*\) is a character, \(\rho = \text{ind}_{H}^{G}(\chi), r = \text{ind}_{H}^{G}(1), g \in G\), and \(\text{Ver}: G^{ab} \to H^{ab}\) is the transfer (Verlagerung) (for the definition of the latter see [2], Chap. XIII.2).

For a finite unramified covering \(f: B' \to B\), we have an embedding of the fundamental group \(\pi_1(B) \to \pi_1(B')\) onto a subgroup of finite index. Applying (8) to their representations, we obtain the following formula for the determinant of the image of a locally constant \(l\)-adic sheaf \(F\) of rank 1 on the curve \(B'\):

\[
\det f_*(F) \otimes (\det f_* \mathbb{Q}_l)^{-1} = \text{Nm}_{B'/B} F,
\]

where \(\det(F)_b = \det(g_b), b \in B\), and the norm of the sheaf is defined as

\[
\text{Nm}_{B'/B} F_b = \bigotimes_{b' \mapsto b} \text{Nm}_{k(b')/k(b)} F_{b'}.
\]

To obtain from this the property iii) one needs to apply the commutative diagram

\[
\begin{array}{ccc}
\text{sheaves } F & \longrightarrow & \text{characters } \chi \text{ of } K_1(\mathbb{A}_{B'}) \\
\downarrow & & \downarrow \\
\text{sheaves } \text{Nm}_{B'/B} F & \longrightarrow & \text{characters } \chi|_{K_1(\mathbb{A}_B)} \text{ of } K_1(\mathbb{A}_B)
\end{array}
\]

arising from class field theory, where the horizontal arrows are Abelian Langlands correspondences, and in order to construct the right-hand vertical arrow we made use of the natural embedding \(K_1(\mathbb{A}_B) \to K_1(\mathbb{A}_{B'})\). Then one applies the formula for the central character of an automorphic representation of the adele group which corresponds to the \(l\)-adic sheaf according to the Langlands correspondence (Lafforgue’s theorem). That is, the central character is the image of the sheaf \(\det f_* F\) of rank 1 on \(B\) with respect to the Abelian Langlands correspondence.

When we turn to the 2-dimensional case, the number of functorial properties will be increased and the properties themselves are more complicated. Recall that we consider maps \(f_i, i = 0, 1, 2\), from the group \(\mathcal{A} \mathcal{F}^{nr}(K_2(\mathbb{A}_X))\) of unramified characters of \(K_2(\mathbb{A}_X)\) satisfying the reciprocity law (2) (and trivial on the subgroup \(K_2(\mathbb{A}_{12,X})\)) to the space \(\mathcal{A} \mathcal{F}^{nr}(\text{GL}(r_i, \mathbb{A}_B))\) of unramified automorphic forms. Note that the ranks \(r_i\) depend in general on \(\chi\). We denote by \(g\) the genus of a general fibre of the map \(f\).

**Direct image conjecture.** For smooth proper morphisms \(f: X \to B\) of a smooth surface \(X\) onto a smooth proper curve \(B\) that are defined over the finite field \(\mathbb{F}_q\), there exist homomorphisms \(f_i^*: \mathcal{A} \mathcal{F}^{nr}(K_2(\mathbb{A}_X)) \to \mathcal{A} \mathcal{F}^{nr}(\text{GL}(r_i, \mathbb{A}_B)), i = 0, 1, 2\), such that:
but it involves going beyond the Abelian group $K$

Theorem.

The existence of maps $f^i_*$ satisfying the properties o)–vi) for smooth proper morphisms $f: X \to B$ follows from the existence of the global Langlands correspondence for algebraic curves $B$ over finite fields.

We prefer to regard this statement as a hypothesis, since it is a highly non-trivial task to obtain an explicit construction of the direct image without using the Langlands correspondence and the general theory of étale cohomology. We now give an outline of how to obtain this construction, using the Langlands correspondence and its properties. Referring to the diagram (5), let us define the maps $f^i_*$ using the two horizontal maps in this diagram and the left-hand vertical map. The upper arrow, the 2-dimensional Abelian Langlands correspondence, is a bijection, which allows us to invert it and thus to begin the construction of the direct image. The

---

A composition rule for finite unramified coverings of the scheme $X$ must also be in place, but it involves going beyond the Abelian group $K_2(\mathbb{A}_X)$.  

---
left arrow follows from the theory of étale $l$-adic sheaves. Finally, the least trivial part of the construction is the bottom arrow, the global Langlands correspondence for $B$ (Lafforgue’s theorem\textsuperscript{20} [32]).

We sketch proofs of the least trivial properties of the direct image.

In the property $o$), we say that the character $\chi$ is trivial on a fibre if its restriction to the fibre $F_b$ is trivial on the group $\text{Pic}^0(F_b)$ for some $b \in B$. The property follows easily from well-known facts about the cohomology of $l$-adic sheaves.

Since $X$ is a surface over a finite field, the duality formula $\text{ii)}$ follows from the relative duality theorem for the $l$-adic sheaf $\mathcal{F}$ on $X$ corresponding to the character $\chi$,

$$R^i f_*(\mathcal{F}) \otimes R^{2-i} f_*(\text{Hom}(\mathcal{F}, \mathbb{Q}_l)(1)) \to R^2 f_*(\mathcal{F} \otimes \text{Hom}(\mathcal{F}, \mathbb{Q}_l)(1))$$

$$\to R^2 f_*(\mathbb{Q}_l)(1) = \mathbb{Q}_l,$$

and subsequent application of the Langlands correspondence.

To obtain the property $\text{iii)}$, it is necessary to formulate an analogue of (9) in the new situation. We set

$$(\text{Nm}_{X/B}\mathcal{F})_b = \bigotimes_{x \text{ inf}^{-1}(b)} \mathcal{F}^{\nu_x}_x(\omega_{X/B}),$$

where $\mathcal{F}$ is a sheaf of rank $1$ on $X$, $b \in B$, $\omega_{X/B}$ is a section of the relative cotangent bundle, and $\nu_x$ is the valuation at a point $x$ on a fibre of the map $f$. We then have

$$\bigotimes_i (\text{Det} R^i f_* \mathcal{F})^{(-1)^i} \bigotimes_i (\text{Det} R^i f_* \mathbb{Q}_l)^{-1)(i+1) = \text{Nm}_{X/B}\mathcal{F}.$$ 

This relation follows when we apply to the fibres of the morphism $f$ Deligne’s formula for a locally constant sheaf $\mathcal{F}$ of rank $1$ on a smooth projective geometrically irreducible curve $C$ of genus $g$ defined over a field $k$. It says that

$$\bigotimes_i (\text{Det} H^i(C, \mathcal{F}))^{(-1)^i} = \bigotimes_{x \in C} \mathcal{F}^{\nu_x}_x(\omega)(1 - g), \quad (11)$$

where $\omega$ is a non-zero differential form on $C$ of degree $1$, and $\mathcal{F}(n)$ is the Tate twist of the sheaf $\mathcal{F}$ (see, for instance, [40], §3.1).\textsuperscript{21} The morphism $f$ gives the inverse map

$$i: K_1(\mathbb{A}_B) \to K_2(\mathbb{A}_X), \quad i(a) = (a, (\omega_{X/B})_{02}), \quad a \in K_1(\mathbb{A}_B),$$

\textsuperscript{20}In Lafforgue’s paper the correspondence was constructed for irreducible representations of the Galois group and cuspidal representations of the adele group of the field of functions on the curve. Evidently, the parabolic induction technique enables us to reduce the general case of completely irreducible representations to this one. The Galois representations related to the sheaves $R^i f_* \mathcal{F}$ are completely irreducible. This follows from the semisimplicity of the Tate module of Abelian varieties, proved by Yu. G. Zarhin.

\textsuperscript{21}If $k = \mathbb{F}_q$ and the sheaf $\mathcal{F}$ corresponds to a character $\chi$ of the idele group $K_1(\mathbb{A}_C)$, then (11) is a reformulation of the expression for the elementary factor of the classical functional equation

$$L_C(s, \chi) = \chi((\omega))q^{s(2-2g)}q^{-1}L_C(1 - s, \chi^{-1})$$

for $L$-functions $L_C(s, \chi)$ of the curve $C$. The elementary factor $\chi((\omega))q^{s(2-2g)}q^{-1}$ corresponds to an action of the Frobenius automorphism of the ground field on the 1-dimensional $l$-adic space dual to the space from the left-hand side in (11). The factor $\chi((\omega))q^{s(2-2g)}$ is connected with the product $\bigotimes_{x \in C} \mathcal{F}^{\nu_x}_x(\omega)$, and the twisting $(1 - g)$ corresponds to $q^{-1}$.\textsuperscript{21}
where \((\omega_{X/B})_{02}\) is the idele associated with the section \(\omega_{X/B}\) (see [40], §2.2). We can now formulate the analogue of the diagram (10):

\[
\begin{array}{ccc}
\text{sheaves } \mathcal{F} \text{ of rank 1 on } X & \longrightarrow & \text{characters } \chi \text{ of } K_2(\mathbb{A}_X) \\
\downarrow & & \downarrow \\
\text{sheaves } \text{Nm}_{X/B}\mathcal{F} \text{ on } B & \longrightarrow & \text{characters } i^*\chi \text{ of } K_1(\mathbb{A}_B)
\end{array}
\]

Then the same arguments as in the case of a finite covering give the property iii).

To understand the property vi), we must represent the group \(K_2(\mathbb{A}_X)\) as an adelic product over all flags \(x \in C\) and then compute it in the following way:

\[
K_2(\mathbb{A}_X) \to K_2(\mathbb{A}_X)/K_2(\mathbb{A}_{12}) = \bigoplus_{x \in C} \mathbb{Z} = \bigoplus_{x \in X} \bigoplus_{C \ni x} \mathbb{Z} \to \bigoplus_{x \in X} \mathbb{Z} \cdot 1_x.
\]

Here, the last arrow is the sum of all \((x, C)\)-components around the given point \(x\). Finally, since the character \(\chi\) is unramified and automorphic, the reciprocity law implies that \(\chi\) is defined on the last group.\(^{22}\)

Then

\[
L_X(\chi) := \prod_{x \in |X|} (1 - \chi(1_x))^{-1},
\]

where \(1_x\) is a generator of the local group at the point \(x \in X\). The \(L\)-functions \(L_{GL(r_i, \mathbb{A}_B)}(f_i^*(\chi))\) are the standard \(L\)-functions associated with the representations generated by the forms \(f_i^*(\chi)\) (see [17] and the next section).

The property vi) follows from the analogous property

\[
L_X(\mathcal{F}) = \prod_i L_B(R^i f_* (\mathcal{F})) (-1)^i
\]

for the corresponding \(l\)-adic sheaves \(\mathcal{F}\) and \(R^i f_* (\mathcal{F})\) and the Langlands correspondences on \(X\) and \(B\), which imply that \(L_X(\mathcal{F}) = L_X(\chi)\) and \(L_B(R^i f_* (\mathcal{F})) = L_{GL(r_i, \mathbb{A}_B)}(f_i^*(\chi))\) for all \(i\).

**Remark 5.** It is not difficult to state local and semilocal versions of this conjecture (as we indicated above) and to formulate their interrelations. We can even give an explicit expression for these direct images in the case of a (smooth) map \(f: X_b \to \text{Spec} \mathcal{O}_b\) with general fibre of genus \(g\), where the character \(\chi\) is trivial (or is the \(s\)th power of the norm \(|·|\)). Then \(f_b^*(\chi) = |·|^s\), \(f_b^*(\chi) = |·|^s-1\), and \(f_b^*(\chi)\) is an Eisenstein series generating an unramified principal series representation of the group \(GL(2g, K_b)\). The Satake parameters of this representation are eigenvalues of the Frobenius automorphism of the field \(k(b)\), acting on the 1-dimensional cohomology of the curve \(F_b = X_b \times_{\mathbb{A}_b} k(b)\). However, we can define them without using \(L\)-adic cohomology, by making use of the Tate–Iwasawa method on the curve \(F_b\). The appropriate version of this method was given in [43].

\(^{22}\)It is actually also defined on the quotient \(K_2(\mathbb{A}_X)/K_2(\mathbb{A}_{12})K_2(\mathbb{A}_{01})K_2(\mathbb{A}_{02})\), which is isomorphic to the second Chow group. This group is an extension of \(\mathbb{Z}\) by a finite group according to the finiteness theorems of Bloch and Kato.
Remark 6. Certainly, the choice of smooth morphisms in our conjecture is too restrictive (since we have a complete base \( B \)). It would be more reasonable to consider the semistable families \( X \) over a proper curve \( B \). In this case, we can expect that the target of the direct image (still for unramified \( \chi \) on \( X \)) will be tamely ramified automorphic representations \((\pi, V_\pi) = \bigotimes_{b \in B} V_{\pi,b}\) of the groups \( GL(\mathbb{A}_B) \). If \( \dim_{\mathbb{C}} V_{\pi,b}^I = 1 \) for the Iwahori subgroups \( I \subset GL(\mathcal{O}_b) \) and all the points \( b \) of bad reduction, then we associate with \( \pi \) a tamely ramified automorphic function on \( GL(\mathbb{A}_B) \). Since \( \dim_{\mathbb{C}} V_{\pi}^I > 1 \) for general tamely ramified representations \( V_{\pi} \) of a group \( GL(K_b) \), the construction of such a function requires additional considerations. This is possible for semistable elliptic families \cite{33}, \cite{16} when the special representations are the local components for points of bad reduction.

Remark 7. In the geometric case, the direct image conjecture is a consequence of 2-dimensional class field theory and the Langlands correspondence on the curve \( B \) (Lafforgue’s theorem). In general, if it were possible to determine not only the Abelian automorphic representations but also the automorphic representations of non-commutative adelic groups on schemes \( X \) of any dimension (for example, by developing Kapranov’s idea), then one could try to define the direct image operation \( f_\ast \) by expanding the map \( f \) into the composition of the projection of a projective bundle and an embedding. This is suggested by Grothendieck’s proof of the Riemann–Roch theorem.

7. A link with the Hasse–Weil conjecture

We assume that the direct image conjecture can be stated and proved in a much more general situation. Namely, one can remove the smoothness condition for schemes and morphisms (see Remark 6) and also consider the ramified characters. Finally, there is an extension of the conjecture to the number field case. Then an (explicit) construction of direct images of this type would provide an opportunity to prove the Hasse–Weil conjecture \cite{53} on meromorphic continuation and the functional equation for \( L \)-functions of algebraic curves defined over algebraic number fields (and to re-prove the corresponding theorem of Grothendieck on \( L \)-functions of algebraic curves defined over fields of algebraic functions with a finite field of constants, without any use of \( l \)-adic cohomology).

Curves with this field of definition have models, namely, 2-dimensional schemes \( X \) with a structure morphism onto a 1-dimensional scheme \( B \), which is either a curve over a finite field or the spectrum of the ring of integers in an algebraic number field. The existence of the direct image from \( X \) to \( B \) makes it possible to construct an automorphic form on \( B \), starting from a character of the group \( K_2(\mathbb{A}_X) \). The Mellin transformation of this form will be the \( L \)-function of interest to us. Automorphy of the form constructed in this way will, according to the general theory, imply the meromorphic continuation and the functional equation for this \( L \)-function \cite{54}, \cite{25}, \cite{17}.

We give a precise formulation of the corresponding theorem for the case of algebraic curves defined over a finite field. Let \( \mathcal{A} \mathcal{F}(n, B) \) be the space of automorphic forms on \( GL(n, \mathbb{A}_B) \), that is, the space of functions \( F \) on \( GL(n, \mathbb{A}_B) \) such that:

i) there exists a character \( \omega \) of the centre \( Z = \mathbb{A}_B^* \) of \( GL(n) \) such that \( F(zg) = \omega(z)F(g) \), \( z \in Z \), \( g \in GL(n, \mathbb{A}_B) \);
ii) if $\gamma \in \text{GL}(n, K)$, where $K = \mathbb{F}_q(B)$, then $F(\gamma g) = F(g)$;

iii) there exists an open compact subgroup $K' \subset \text{GL}(n, \mathcal{O})$, where $\mathcal{O} = \prod_x \mathcal{O}_x$, $x \in B$, such that $F(kg) = F(g)$ for $k \in K'$, $g \in \text{GL}(n, \mathcal{A}_B)$.

The group $\text{GL}(n, \mathcal{A}_B)$ acts on the space of automorphic forms by left translation.

A smooth representation of $\text{GL}(n, \mathcal{A}_B)$ is said to be automorphic if it is embedded as a subrepresentation in the space $\mathcal{A}F(n, B)$ of automorphic forms.

Each irreducible automorphic representation $\pi$ is a tensor product $\otimes_{x \in B} \pi_x$ of smooth irreducible representations of the local groups $\text{GL}(n, K_x)$. For almost all $x$ the representation $\pi_x$ is a spherical one ([12], [6], Chap. 3.4). According to the general theory, such a representation belongs to the principal series, that is, it can be obtained by parabolic induction from an unramified character $\chi: T \rightarrow \mathbb{C}^*$ of a maximal torus $T$ of the group $\text{GL}(n, K_x)$ (see [6], Theorem 4.6.4 for the case $n = 2$, the proof of which can be extended to arbitrary $n$). This character is given by a set of $n$ complex numbers $z_1, \ldots, z_n$ (defined up to a permutation). Then we can define the local $L$-function for spherical representations $\pi_x$ as

$$L_B(s, \pi_x) = \prod_{1 \leq i \leq n} (1 - z_i q^{-s})^{-1}$$

and the global $L$-function as the Euler product

$$L_B(s, \pi) = \prod_{x \in B} L_B(s, \pi_x).$$

(12)

The missing factors for a finite set of points $x \in B$ are defined in a more complicated way (see [25], Chap. 1, Theorem 2.18, [16], Chap. 2, Theorem 8.2, and Table A for $n = 2$, and [17], Chap. 1, Theorem 3.3 for arbitrary $n$).

**Theorem.** Let $B$ be a smooth projective curve over the finite field $\mathbb{F}_q$ and let $\pi$ be an automorphic representation of $\text{GL}(n, \mathcal{A}_B)$. Then:

i) the Euler product (12) for the $L$-function $L_B(s, \pi)$ converges in the right half-plane $\text{Re } s > n$;

ii) $L_B(s, \pi)$ extends meromorphically to the whole $s$-plane as a rational function of $q^{-s}$;

iii) $L_B(s, \pi) = \varepsilon(s, \pi)L_B(n - s, \pi)$, where $\varepsilon(s, \pi)$ is an elementary factor of the form $aq^{bs}$, $a, b \in \mathbb{R}$.

This theorem was proved for cuspidal representations in [17], Theorem 8.13. The general case can be reduced to this one using parabolic induction [24], Theorem 6.2.23 For the group $\text{GL}(2)$, the result was obtained by Jacquet and Langlands in 1970 ([25], Chap. II, Theorem 11.1). We note that in these works an analogue of the theorem was also proved for number fields.

If all the local components of the representation $\pi$ are spherical, then it is generated by an unramified automorphic form (a tensor product of the local spherical vectors) and its Mellin transform is equal to the $L$-function $L_B(s, \pi)$. This fact allows us to apply this theorem to the automorphic forms $f_i^*(x)$ arising as direct images of an unramified automorphic character $\chi$ of the group $K_2(\mathcal{A}_X)$ of the surface $X$ which is mapped onto the curve $B$.

23We have changed $1 - s$ in the formulation from [24] to $n - s$, which is achieved by a simple renormalization of the argument of the $L$-function.
Such a construction can be seen as a way of solving the general problem discussed in [42], §5.2. It is a matter of generalizing to higher-dimensional (in this case 2-dimensional) schemes the Tate–Iwasawa approach [50], [57] to zeta- and L-functions of 1-dimensional schemes. It seems that harmonic analysis on 2-dimensional adelic spaces ([42], §2) and the theory of representations of discrete Heisenberg groups ([42], §§3, 4) are the tools needed to construct this (hypothetical) direct image.

Remark 8. Let $X$ be an $n$-dimensional regular scheme and $f : X \to B$ a flat proper morphism onto a 1-dimensional regular scheme $B$ with smooth general fibre. In the number field case, the schemes $X$ and $B$ must be considered in the Arakelov geometry, completed by Archimedean fibres. The $n$-dimensional class field theory suggests that there exists a reciprocity map

$$\varphi_X : K^n_M(\mathbb{A}_X) \to G^\text{ab}_K,$$

where $K^n_M(\mathbb{A}_X)$ is the Milnor part of the $n$th Quillen K-functor, and $G^\text{ab}_K$ is the Galois group of the maximal Abelian covering of the scheme $X$. We may assume that the direct image conjecture is also valid (with the appropriate changes) for these morphisms and the automorphic characters of the group $K^n_M(\mathbb{A}_X)$. Automorphy of the characters means that they are trivial on the $n$ subgroups of the form

$$K^n_M(\mathbb{A}_{0, \hat{i}, 2, \ldots, n}), \quad K^n_M(\mathbb{A}_{0, 1, \hat{i}, \ldots, n}), \quad \ldots, \quad K^n_M(\mathbb{A}_{0, 1, \ldots, \hat{n}}),$$

where $\hat{i}$ means that the $i$th index must be omitted. These relations are a generalization of the formulae (2). Just as above, this conjecture implies the Hasse–Weil conjecture for the scheme $X$.

The first step towards the conjecture is to construct direct images $f_* : K_n(\mathbb{A}_X^\bullet) \to K_{n-k}(\mathbb{A}_Y^\bullet)[k]$ for proper morphisms $f : X \to Y$ of relative dimension $k$, full adelic complexes, and $n - k \geq 0$. We do not consider the natural conditions for the existence of these direct images. We only mention that for closed regular embeddings the map $f_*$ depends on the choice of the local equations for $Y$ in $X$. Nevertheless, the resulting homomorphism $f_*$ is defined uniquely up to homotopy.

8. Appendix: zero-dimensional generalization of the Langlands correspondence

Let us consider in more detail an analogue of the Langlands correspondence in the 0-dimensional situation, that is, for finite fields and their extensions. As in the case of fields of higher dimension, class field theory gives us a canonical reciprocity map

$$K_0(K) \to G^\text{ab}_K = G_K,$$

which appears explicitly as

$$\mathbb{Z} \to \hat{\mathbb{Z}}$$

and maps 1 into the Frobenius automorphism $\text{Fr} \ (\text{Fr}(x) = x^q)$ for the field $K = \mathbb{F}_q$. Here $K_0(\mathbb{F}_q)$ is the Grothendieck $K$-group of the category of finite-dimensional vector spaces, and $1 \in K_0(\mathbb{F}_q)$ corresponds to the trivial 1-dimensional space.
If $K'/K$ is a finite extension of degree $m$, then we have the commutative diagrams

$$
\begin{array}{ccc}
K_0(K') & \longrightarrow & G_{K'} \\
\downarrow \text{Nm} & & \downarrow \text{Ver} \\
K_0(K) & \longrightarrow & G_K \\
\end{array}
$$

where the embeddings $i$ and $j$ are induced by the inclusion of $K$ in $K'$, $\text{Nm}$ is the norm map, and $\text{Ver}$ is the transfer from $G_{K'}^{\text{ab}} = G_K^{\text{ab}}$. In this case, $\text{Nm}$ is multiplication by $m$, and $j$ is an isomorphism (of groups equal to $\mathbb{Z}$).

As above, the map (13) leads to a homomorphism of the groups of characters (1-dimensional representations) in the opposite direction. We can define functors of direct and inverse images for the characters, and the diagrams given lead to properties of them analogous to those in the previous section.

We recall that Kapranov’s proposal for the 0-dimensional correspondence was to take the $L$-function $L(s, \rho)$ of the Galois representation $\rho$. If $\lambda_1, \ldots, \lambda_n$ is the spectrum of $\rho$, then $L(s, \rho) = \det(I - \rho(Fr)q^{-s}) = \prod_i(1 - \lambda_i q^{-s})$. On the other hand, there is a structure of $d$-representations ($d = 1$ or 2) for the correspondences in dimension $d$. We can expect that something similar to 0-representations might appear on the right-hand side of the 0-dimensional correspondence. To understand the situation, we give the necessary definitions of the $d$-representation theory ($d = 0, 1, 2$).

This is a purely algebraic theory that has nothing to do with arithmetic. We fix a group $G$, a ground field $k$, and a cardinal number $n$. The inductive structure of the theory then appears as follows.

| $d$ | representation spaces | $d$-representations | $n = \text{dimension of representation}$ | sets of 1-dimensional representations |
|-----|-----------------------|---------------------|----------------------------------------|--------------------------------------|
| 2   | $V$-mod/$k$           | functor $\pi(g)$: $V$-mod $\rightarrow V$-mod | $\text{rk} V$-mod/$k$ | $H^2(G, k^*)$ |
| 1   | $V/k$                 | homomorphism $\pi(g): V \rightarrow V$ | $\dim_k V$ | $H^1(G, k^*) = \text{Hom}(G, k^*)$ |
| 0   | $A \subset \text{Sym}^n(k^*)$ | just $A$, no $\pi(g)$ | $\deg A = n$ | $H^0(G, k^*) = k^*$ |

| $d$ | distinguished object | Hom’s | sets of characters |
|-----|---------------------|-------|-------------------|
| 2   | $\text{Vect}/k$    | $\text{Func}(\mathcal{C}, \mathcal{D})$ | $\{g \mapsto V_g\}$ |
| 1   | $k$                 | $\text{Hom}_k(V, U)$ | $\text{Map}(G, k)$ |
| 0   | 1                   | $\left(\sum_i n_i a_i^{-1}\right) \left(\sum_j m_j b_j\right)$ | $k$ |

Here $V$-mod/$k$ is a category-module (see §3 above), $\text{rk} V$-mod/$k$ is its rank (see [15], [26]), $V_g$ is a vector space over $k$ depending on $g \in G$, $V/k$ or $V$ is a vector
space over \( k \), \( \text{Sym}^n(k^*) \) is the set of cycles of degree \( n \) with non-negative multiplicities, and \( \text{Func}(\mathcal{C}, \mathcal{D}) \) is the set of functors from the category \( \mathcal{C} \) to the category \( \mathcal{D} \) which commute with the action of the category \( \text{Vect}/k \) (in the categorical sense).

We see that for \( d = 0 \) the group \( G \) itself is not involved in all the constructions. For \( d = 0 \) the direct sum of the representations \( A \) and \( B \) is the sum of the cycles \( A \) and \( B \), their tensor product is their Pontryagin product in the group \( k^* \), and the dual to \( A = \sum_i n_i a_i \) is \( \hat{A} = \sum_i n_i a_i^{-1} \). The trace \( \text{Tr}(A) \) is \( \sum_i n_i a_i \), where the sum is taken in the group \( k \). For \( d = 2 \) the dual ‘object’ to a category \( \mathcal{C} \) is the category of functors \( \text{Func}(\mathcal{C}, \text{Vect}/k) \).

We now try to apply this little theory to the local Langlands correspondence, first for the field \( k = \mathbb{C} \). The group on the left-hand side of the correspondence is the same Galois group for all \( d \). But on the right-hand side, we see 2-representations of the group \( \text{GL}(2n, K) \) for \( d = 2 \) and 1-representations of the group \( \text{GL}(n, K) \) for \( d = 1 \). We may guess that for \( d = 0 \) one needs to consider 0-representations of the group \( \text{GL}(0, K) \), a group that is simply trivial. Then the Langlands correspondence could appear as follows:

\[
\text{Rep}^{(n)}(G_K) = \{ \text{n-dimensional semisimple representations of } G_K \text{ over } \mathbb{C} \} \\
\Leftrightarrow \{ \text{irreducible 0-representations of the group } \text{GL}(0, K) = (1) \},
\]

where \( \rho \in \text{Rep}^{(n)}(G_K) \) goes to the determinant \( \lambda_1 \cdots \lambda_n \) of the spectrum \( \lambda_1, \ldots, \lambda_n \) of \( \rho \). We see that this construction does not coincide completely with Kapranov’s proposal. The \( L \)-function of his choice does not belong to the field \( \mathbb{C} \), as we might expect according to the table above. We choose a determinant which is an element of \( \mathbb{C} \). It also satisfies the same multiplicative property for exact sequences of representations as the \( L \)-function. But the correspondence will not then be a bijection, in contrast to the behaviour of the \( L \)-functions which are actually characteristic polynomials of the Frobenius automorphism. This problem is left as a question.

We have to say that we also encounter another difficulty. The Langlands correspondence is in perfect accordance with the reciprocity law for \( d = 1 \) or 2. However, we do not see here the group \( K_0(K) \) on the right-hand side of the correspondence for \( d = 0 \). This unexpected puzzle can be solved along the following lines.

In his paper [26] Kapranov justifies the appearance of the non-commutative group \( \text{GL}(2, K) \) on the right-hand side of the Abelian (!) 2-dimensional correspondence by the relation

\[
K_2(K) = H_2(\text{GL}(2, K), \mathbb{Z})/H_2(\text{GL}(1, K), \mathbb{Z}),
\]

which is a particular case of the general Suslin theorem on the Milnor group \( K_n^M(K) \) for a field \( K \). In the case \( n = 2 \) this should follow from Matsumoto’s theorem that \( K_2^M(K) = K_2(K) = H_2(\text{SL}(K), \mathbb{Z}) \). Applying this, Kapranov has shown that 1-dimensional 1-representations of the group \( K_2(K) \) are in one-to-one correspondence with 1-dimensional 2-representations of the group \( \text{GL}(2, K) \) (with a trivial restriction to the subgroup \( \text{GL}(1, K) \)). Since \( K_1(K) = K^* = H_1(\text{GL}(1, K), \mathbb{Z}) \), we have an analogous relation for 1-representations when \( d = 1 \).

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For \( d = 0 \) we can add the relation

\[
K_0(K) = H_0(\text{GL}(0, K), \mathbb{Z}) = \mathbb{Z},
\]

and then the 0-dimensional Langlands correspondence appears as follows:

\[
\text{Rep}^{(n)}(G_K) = \{ n\text{-dimensional semisimple }1\text{-representations of } G_K \text{ over } \mathbb{C} \}
\]

\[
\Leftrightarrow \{ n\text{-dimensional semisimple }1\text{-representations of } K_0(K) \text{ over } \mathbb{C} \}
\]

\[
\Rightarrow \{ 1\text{-dimensional }0\text{-representations of the group } \text{GL}(0, K) = (1) \}. 
\]

The same versions hold for \( d = 1 \) or 2, but it appears that the bijection can fail to be valid for \( d = 2 \). Moreover, it is unusual that we have to consider all semisimple representations on the right-hand side of the correspondence and not just the irreducible ones as in the case of dimension 1.

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A. N. Parshin
Steklov Mathematical Institute
of the Russian Academy of Sciences
E-mail: parshin@mi.ras.ru