A model for stock returns and volatility

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HIGHLIGHTS

• We show that historic volatility is best described by the generalized inverse gamma distribution.
• We show that historic stock returns are best described by the generalized Student's distribution.
• We discuss stochastic stock and volatility models that produce these distributions.
• We obtain the mean and the variance of relaxation times on approach to steady state distributions.
• We examine 1/f noise in volatility and stock returns.

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ABSTRACT

We prove that Student's t-distribution provides one of the better fits to returns of S&P component stocks and the generalized inverse gamma distribution best fits VIX and VXO volatility data. We further prove that stock returns are best fit by the product distribution of the generalized inverse gamma and normal distributions. We find Brown noise in VIX and VXO time series and explain the mean and the variance of the relaxation times on approach to the steady-state distribution.

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1. Introduction

The generalized inverse gamma (GIGa) function (Appendix A) belongs to a family of distributions (Appendix B), which includes inverse gamma (IGa), lognormal (LN), gamma (Ga) and generalized gamma (GGa). The remarkable property of GIGa is its power-law tail; for a general three-parameter case, GIGa(α, β, γ) ∝ x^{1-αγ}, x → ∞. GIGa emerges as a steady state distribution in a number of problems, from a network model of economy [1,2], to ontogenetic mass growth [3], to response times in human cognition [4]. This common feature can be traced to a birth–death phenomenological model subject to stochastic perturbations (Appendix C). Here we argue that the GIGa distribution best describes the stock volatility distribution and the product distribution (Appendix D) of GIGa and normal (N) distributions, GIGa ∗ N, best describes the stock return distribution.

Numerically, we used the maximum likelihood method to determine the best parameters for each of the distributions in the above family of distributions 1 and found that GIGa provides the best fit for VIX and VXO volatility data [5–7]. We also found that among product distributions of the above family with normal distribution, GIGa’s product with N gives the best fit to the stock return distribution. Furthermore, among the better fits GIGa ∗ N fits are those with γ ≈ 2.

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In general, product distribution GIgα + N has $|x|^{1-\alpha \gamma}$ tails [left and right] and, for $\gamma = 2$ in particular, the product distribution GIgα(α, β, 2) * N for stock returns is the generalized Student's t-distribution, which has $|x|^{-1-2\alpha}$ tails [8–10]. Accordingly, our starting point is the geometric Brownian motion model of stock price [11,12], where the steady-state distribution of stock returns is given by the product distribution of volatility and normal distributions. Furthermore, the instantaneous variance of volatility (or square stochastic volatility—the terms used interchangeably) is described by the Nelson diffusion limit (NDL) of the GARCH(1, 1) model of stock volatility [13,14], whose stochastic term is uncorrelated from that in the equation for stock price; in the steady state, it is distributed as IGa, that is GIgα with $\gamma = 1$.

It should be noted that the product distributions of lognormal [16], gamma [22] and inverse gamma [8] distributions with normal distribution had been previously considered. Our key observation is that these distributions belong to a family of distributions and that it is this family of distributions that should be studied. Our work thus unifies and generalizes previous important contributions on stock returns.

This paper is organized as follows. In Section 2, we discuss stochastic stock and volatility models. In Section 3, we fit VIX and VXO, including direct evaluation of their power law tail exponents by the log–log plot. We also address Brown noise observed in the VIX/VXO time series. In Section 4, we discuss numerical results of fitting returns of S&P component stocks\(^2\) based on log-likelihood and discuss white noise in stock return series. In Section 5, we summarize our key findings.

2. Stochastic stock and volatility models

The widely accepted equation for stock price is given by
\[
\frac{dS}{S} = \mu dt + \sigma dW_1
\]
where $\mu$ is a constant and $\sigma$ volatility. The equation for the instantaneous volatility variance (square volatility) can be written in the following general form:
\[
dV = f(V)dt + \tilde{g}(V)dW_2.
\]
Here $dW_1$ and $dW_2$ are Wiener processes correlated by $(dW_1dW_2) = \rho dt$. Substituting $V = \sigma^2$ and using Ito calculus, we obtain the volatility equation
\[
d\sigma = f(\sigma)dt + g(\sigma)dW_2.
\]
The Fokker–Planck equation for the distribution function of $\sigma$, $P(\sigma, t)$, is given by
\[
\frac{\partial}{\partial t}P(\sigma, t) = \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} [g^2(\sigma)P(\sigma, t)] - \frac{\partial}{\partial \sigma} [f(\sigma)P(\sigma, t)].
\]
It has a stationary (steady-state) solution given by
\[
P(\sigma) = \frac{2}{g^2} \exp \left( \int \frac{2f}{g^2} d\sigma \right).
\]

In what follows, we shall assume that $dW_1$ and $dW_2$ are uncorrelated, that is $\rho = 0$. A number of possible forms of $f(\sigma)$ and $g(\sigma)$ are discussed in Appendix E; see also [16]. Here we concentrate on one particular form
\[
d\sigma = J(\theta \sigma^{1-\gamma} - \sigma)dt + \Sigma \sigma dW_2.
\]
The stationary (see Appendix F for discussion of relaxation times) solution of this equation is given by a three-parameter GIgα distribution
\[
\text{GIgα}(x; \alpha, \beta, \gamma) = \frac{\gamma}{\beta \Gamma(\alpha)} e^{-\left(\frac{x}{\beta}\right)^\alpha} \left(\frac{\beta}{x}\right)^{1+\alpha \gamma}
\]
as
\[
P(\sigma) = \text{GIgα} \left( \sigma; \left(1 + \frac{2J}{\Sigma^2}\right)^{-1/\gamma}, \left(\theta \frac{2J}{\Sigma^2 \gamma^{-1}}\right)^{-1/\gamma}, \gamma \right).
\]
where the parameter $\theta$ can be expressed using the mean $\bar{\sigma}$ as
\[
\theta = \gamma \frac{\Sigma^2}{2J} \left[ \frac{\bar{\sigma} \Gamma \left( \left(1 + \frac{2J}{\Sigma^2}\right)^{-1/\gamma} \right)^{-1/\gamma}}{\Gamma \left( \frac{2J}{\Sigma^2 \gamma^{-1}}\right)} \right]^{\gamma}.
\]
\(^2\)DJIA components are fitted in the same fashion leading to identical conclusions, which is described elsewhere.
In particular, when $\gamma = 1, \theta = \sigma$.

A case of particular importance is $\gamma = 2$, in which case the equation for the volatility variance is that of $\gamma = 1$ and reads as follows:

$$dV = \tilde{J}(\bar{V} - V)dt + \tilde{\Sigma} V dW_2.$$  \hfill (10)

Its stationary solution is given by an IGa distribution (the $\gamma = 1$ limit of (7)),

$$\text{IGa}(x; \alpha, \beta) = \frac{1}{\beta \Gamma(\alpha)} \exp\left[-\frac{\beta}{x}\right] \left(\frac{\beta}{x}\right)^{1+\alpha}$$  \hfill (11)

as

$$P(V) = \text{IGa}\left(V; 1 + \frac{2\tilde{J}}{\tilde{\Sigma}^2}, \frac{\tilde{\Sigma}^2}{2}\right).$$  \hfill (12)

Using $V = \sigma^2$ and Ito calculus, we obtain

$$d\sigma = \left(\frac{\tilde{J}^2}{2} \frac{\tilde{\Sigma}^2}{8} - \left(\frac{\tilde{J}^2}{2} + \frac{\tilde{\Sigma}^2}{8}\right) \sigma\right) dt + \frac{\tilde{\Sigma}}{2} \sigma dW_2.$$  \hfill (13)

On comparison with Eq. (6), we find the following parameter correspondence:

$$J = \frac{\tilde{J}^2}{2} + \frac{\tilde{\Sigma}^2}{8}$$
$$\theta = \tilde{V} \frac{\tilde{J}^2}{2} + \frac{\tilde{\Sigma}^2}{8}$$
$$\Sigma = \frac{\tilde{\Sigma}}{2}$$
$$\gamma = 2.$$  \hfill (14)

Substitution into Eq. (8), gives the distribution of $\sigma$ as,

$$\text{GlGa}\left(\sigma; 1 + \frac{2\tilde{J}}{\tilde{\Sigma}^2}, \sqrt{\tilde{V}} \frac{2\tilde{J}}{\tilde{\Sigma}^2}, 2\right).$$  \hfill (15)

It should be emphasized that a simple change of the variable to its square root produces the following transformation:

$$\text{GlGa}(\alpha, \beta, \gamma) \rightarrow \text{GlGa}(\alpha, \sqrt{\beta}, 2\gamma)$$ and in particular $\text{IGa}(\alpha, \beta, 1) \rightarrow \text{IGa}(\alpha, \sqrt{\beta}, 2)$, which is consistent with (10) and (15). From (9), the mean of $\sigma$ is given by

$$\sqrt{\tilde{V}} \sqrt{\frac{2\tilde{J}}{\tilde{\Sigma}^2} \Gamma\left(\frac{2\tilde{J}}{\tilde{\Sigma}^2} + \frac{1}{2}\right)}.$$  \hfill (16)

which is a monotonically increasing function which approaches $\sqrt{\tilde{V}}$ as $2\tilde{J}/\tilde{\Sigma}^2 \rightarrow \infty$.

Turning to Eq. (1), we observe that the stationary distribution of stock returns is a product distribution $P(\sigma) \ast N$. In Appendix D, we consider both formalism of the product distribution and various cases of $P(\sigma)$. Here we concentrate specifically on

$$\text{GlGa}(\alpha, \beta, 2) \ast N(0, 1) = \frac{\Gamma\left(\frac{1}{2} + \alpha\right)}{\sqrt{2\pi\beta \Gamma(\alpha)}} \left(\frac{2\beta^2}{\sigma^2 + 2\beta^2}\right)^{\frac{1}{2} + \alpha}.$$  \hfill (17)

which is the generalized Student’s $t$-distribution $T(0, \beta/\sqrt{\alpha}, 2\alpha)$ [15].

It should be mentioned that by Ito calculus and Eq. (1)

$$d\log S = \left(\mu - \frac{1}{2} \sigma^2\right) dt + \sigma dW_1.$$  \hfill (18)

In numerical calculations of stock returns, it is actually the $\Delta \log S$ that is being evaluated. However, it is clear that the premise of the stationary distribution being the product distribution of the volatility distribution and the normal distribution remains in force.
3. Market volatility

We analyze the Chicago Board Options Exchange (CBOE) volatility index [5–7]. On September 22, 2003, CBOE decided to change the manner in which it calculated the volatility index from VXO to VIX. However, both methods were applied to both the old and new data. Following CBOE convention [5], the VIX/VXO data from 1990 to 2004 are called vixarchive/vxoarchive and from 2004 to present vixcurrent/vxocurrent. In Fig. 1, we show the time series of the indices.
Table 1
Parameters of fitting distributions.

| Data          | IGa: $\alpha$, $\beta$ | GIGa: $\alpha$, $\beta$, $\gamma$ | LN: $\mu$, $\sigma$ |
|---------------|-------------------------|-----------------------------------|---------------------|
| vixarchive    | 10.7, 196               | 33.7, 1.03 \times 10^4, 0.557     | 2.96, 0.311         |
| vixcurrent    | 7.22, 127               | 0.721, 14.1, 3.96                  | 2.94, 0.398         |
| vxoarchive    | 9.63, 187               | 11.7, 295, 0.905                   | 3.02, 0.330         |
| vxocurrent    | 6.59, 113               | 0.678, 13.4, 3.94                  | 2.92, 0.419         |

Table 2
Power-law tail exponents of fitting distributions and the log–log plot.

| Data          | IGa: $-\alpha$ | GIGa: $-\alpha\gamma$ | Slope of log-log plot |
|---------------|----------------|-----------------------|-----------------------|
| vixarchive    | $-10.7$        | $-18.8$               | $-7.21$               |
| vixcurrent    | $-7.22$        | $-2.86$               | $-3.39$               |
| vxoarchive    | $-9.63$        | $-10.6$               | $-5.48$               |
| vxocurrent    | $-6.59$        | $-2.67$               | $-3.13$               |

We apply the maximum likelihood estimation method (Appendix G) to find the best fitting parameters of IGa, GIGa, Ga, and LN summarized in Table 1. Comparison of loglikelihood in Fig. 2 shows that the goodness of fit decreases in the following order: GIGa, IGa, LN, and Ga.

In Fig. 3, we plot histograms of vixarchive, vixcurrent, vxoarchive, and vxocurrent respectively, fitted with the best GIGa, IGa, LN, and Ga. We also measure the exponent of the power law tail of VIX and VXO directly (Appendix H), as shown in Fig. 4 and Table 2.

Finally, in Fig. 5 we clearly observe Brown noise in the volatility time series. This is entirely consistent with the Brown noise observed in the time series of the GIGa process of Eq. (6); in Fig. 6 we show Brown noise for an IGa process, $\gamma = 1$.

4. Stock returns

We analyze component stock prices (at close) of major indices. S&P 100 and S&P 500 lists can be found at the Standard & Poor’s website [17]. The historical daily prices of component stocks of S&P 100 and S&P 500 are downloaded from Yahoo! Finance [18]. The final date of S&P 100 and S&P 500 stocks used here is March 25, 2013. The daily S&P 500 and DJIA data is downloaded from the Research Division of the Federal Reserve Bank of St. Louis [19,20]. For S&P 500, we ignore five stocks that have less than 200 stock datapoints each.

We start with a simple test of the stochastic volatility model, Eqs. (1) and (6). In Fig. 7, we show stock returns of S&P 500 index and their distribution. Average daily return, $\log(S_{\text{tomorrow}}) - \log(S_{\text{today}}) \approx 0.00025$ corresponds to 6.4% annual return. For constant volatility, one would expect a normal distribution for stock returns. However, as is obvious from the figure, the normal distribution is not a good fit. On the other hand, the stochastic volatility model indicates that it is the ratio of stock return to volatility that should be normal. Visual inspection of Fig. 8 and the fit in Fig. 9 give initial validation to the model.

We proceed to rigorously analyze stock return data using maximum likelihood estimation. In our analysis, the stock returns are detrended and scaled into unit STDEV. The log-likelihood of the product distribution is evaluated by the numerical quadrature and maximized by the simplex algorithm. The numerical quadrature and the simplex algorithm are checked to be correct for the special case $\text{GIGa}(\alpha, \beta, 2) \ast N$, which is the generalized Student’s $t$-distribution, whose maximum likelihood estimation can be computed directly [21]. Figs. 10–12 convincingly show that the product distribution $\text{GIGa}(\alpha, \beta, \gamma) \ast N(0, 1)$ fits the stock return best. Distributions of best fit parameters are shown in Figs. 13 and 14.

In Fig. 15, we show histograms of S&P 500 index and IBM respectively and their fitting by product distributions. Clearly, the product distribution of GIGa and normal distributions is better able to capture the tail events.

In Figs. 16 and 17, we do direct fitting of tails of stock return of S&P 500 index and IBM respectively (Appendix H). Obviously the stock return is fat-tailed. However, the tail exponents obtained here deviate from those obtained by $\text{GIGa} \ast N$ fitting in Fig. 10. In Fig. 18, we show the Fourier transform of stock return series of S&P 500 index and IBM respectively. It exhibits white noise as opposed to the Brown noise of VIX and VXO in Fig. 5.

Notice that in the Heston model [12,22] (Appendix E) stock returns are given by $\text{GGa}(\alpha, \beta, 2) \ast N(0, 1)$, which cannot generate power-law tail. This contradicts both our and previous results [23–27] of $\approx 3$–5 for the tail exponent.

5. Summary

We demonstrated that $\text{GIGa}(\alpha, \beta, \gamma)$ provides the best fit to VIX/VXO volatility distributions and that the product distribution $\text{GIGa}(\alpha, \beta, \gamma) \ast N(0, 1)$ to the stock return distribution. Furthermore, for the latter we showed that $\gamma = 2$ is near the median/mode of the $\gamma$-distribution of best fits. For $\gamma = 2$, the stock return distribution is the generalized Student’s $t$-distribution $T(0, \beta/\sqrt{\alpha}, 2\alpha)$, which is among the better fits to empirical stock returns. Numerical evaluation of parameters of the fitting distributions was performed with the maximum likelihood estimation method.
Fig. 3. Histograms of VIX and VXO. Top down: vixarchive, vixcurrent, vxoarchive, and vxocurrent. In decreasing order of modal PDF (except for archives, where IGa is taller than GIGa), purple: GIGa; red: IGa; green: LN; and blue: Ga. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
Importance of $\gamma = 2$ puts our findings in excellent agreement with the stochastic volatility and stock return model defined by Eqs. (1) and (6). This model fully accounts for the power law tails observed in the volatility and stock return distributions. One could surmise that if VIX/VXO is thought of as implied volatility, volatility in the stock return model can be thought of as realized volatility. The fact that the steady-state distributions of both are described by GIGa points to a possibility of establishing an analytical relationship between the two.

Additional supporting evidence comes from the fact that Fourier transform of both empirical and simulated time series exhibits Brown noise for volatility and white noise for stock returns (notice that the change of volatility on the l.h.s. of (6) feeds back via its coupling to Wiener noise on the r.h.s., while there is no such feedback for stock return in (1)—this accounts for the difference in noise).

It should be underscored that our results are valid only for time lags which are shorter than the relaxation time in the stochastic equation for volatility. Otherwise, a problem of “subordination” may arise [31]. We tested our results on daily returns. Ideally, one needs to calculate relaxation times based on the model parameters, as prescribed in Appendix F, and find distributions over various time lags shorter than the relaxation time to determine how universal the product distribution predictions are. We hope to address this in a future work.

Lastly we point out that stock returns have been accumulated over very long time and that the definitions of VIX and VXO have changed over time. A question arises whether a different definition of implied volatility could be developed that would be a better match to realized volatility, as the comparison of historic implied and historic realized volatility here seems to suggest. Our approach may shed additional light on the “volatility smile” as well. We hope to address these issues in future work as well.

Appendix A. Properties of the GIGa distribution

We begin with the $\gamma = 1$ limit of GIGa, namely IGa distribution PDF

$$p_{\text{IGa}}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \exp \left[ -\frac{\beta}{x} \left( \frac{\beta}{x} \right)^{1+\alpha} \right].$$ (A.1)

Setting the mean to unity, the scaled distribution is

$$p_{\text{Scaled}}^{\text{IGa}}(x) = \frac{(\alpha - 1)^\alpha \exp \left( -\frac{\alpha-1}{x} \right)}{\Gamma(\alpha) x^{1+\alpha}}.$$ (A.2)
Fig. 5. Fourier transform of VIX and VXO time series. The top row is vixarchive and vixcurrent, and the bottom row is vxoarchive and vxocurrent. Jagged black line: discrete Fourier transform; red line: linear fit of the black line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 6. Discrete Fourier transform of a time series of the IGa process with $f = 0.1, \sigma = \sqrt{0.1}$ at times 1, 2, ..., 1000. Jagged black line: discrete Fourier transform; red line: linear fit of the black line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
Fig. 7. Top left: historical curve of daily S&P 500 index. Top right: historical curve of daily return of S&P 500 index. Bottom: histogram of return of S&P 500 index. The red line is a fit of the normal distribution. In the plots, return is computed from $\log S_{\text{tomorrow}} - \log S_{\text{today}}$ without other operations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 3
Parameters of GIGa distribution fitting. The first row is stock indices and stocks. The second and third rows correspond to GIGa $\alpha \gamma$ and $\gamma$ respectively. Similarly for other rows. From left to right and top down, the count of stock indices or stock corresponds to the $x$-coordinate in Fig. 10.

| Stock  | DJIA | SP 500 | AA | AXP | BA | BAC | CAT | CSCO |
|--------|------|--------|----|-----|----|-----|-----|------|
| $\alpha \gamma$ | 3.79  | 5.99   | 3.35| 3.61| 3.7 | 2.5 | 3.76| 2.92 |
| $\gamma$   | 1.17  | 0.69   | 2.32| 1.64| 2.11| 1.6 | 1.89| 2.6  |
| Stock  | CVX  | DD     | DIS| GE  | HD | HPQ | IBM | INTC |
| $\alpha \gamma$ | 3.58  | 3.78   | 3.4 | 3.02| 3.16| 3.28| 3.13| 3.01 |
| $\gamma$   | 2.89  | 1.89   | 2.23| 2.9 | 2.53| 2.66| 2.68| 4.3  |
| Stock  | JNJ  | JPM    | KO | MCD | MMM| MRK | MSFT| PFE  |
| $\alpha \gamma$ | 3.72  | 3.63   | 3.13| 2.91| 3.84| 3.2 | 2.95| 3.38 |
| $\gamma$   | 1.93  | 1.17   | 2.63| 4.27| 1.76| 3.59| 2.49| 3.02 |
| Stock  | PG   | T      | TRV| UNH | UTX| VZ  | WMT | XOM  |
| $\alpha \gamma$ | 3.37  | 3.95   | 3.23| 2.97| 3.93| 3.76| 3.17| 3.21 |
| $\gamma$   | 2.17  | 1.36   | 1.6 | 2.24| 1.86| 1.67| 2.08| 5.36 |

The mode of the above distribution is $x_{\text{mode}} = (\alpha - 1)/(\alpha + 1)$. The modal PDF is

$$P_{\text{Scaled}}(x_{\text{mode}}) = \frac{(1 + \alpha)^{1+\alpha} \exp(-1-\alpha)}{\Gamma(\alpha)(\alpha - 1)},$$

which has a minimum at $\alpha \approx 3.48$ as shown in Fig. 19. The change in PDF behavior on transition through this value is clearly observed in Fig. 20. Also plotted in Fig. 19 is the half-width of the distribution. Clearly, it highly correlates with the modal PDF and has a maximum in close proximity to its minimum.

Both minimum and maximum above clearly separate the regime of small $\alpha$: $\alpha \to 1$, where the approximate form of the scaled PDF is

$$P_{\text{Scaled}}(x) \approx \frac{(\alpha - 1) \exp[-(\alpha - 1)/x]}{x^2},$$

(A.4)
whose mode is \((\alpha - 1)/2\) and the magnitude of the maximum is \(4 \exp[-(\alpha - 1)^2/2]/(\alpha - 1) \approx 4/(\alpha - 1)\), from the regime of large \(\alpha\), \(\alpha \to \infty\), where
\[
P_{\text{IGa}}^{\text{Scaled}}(x) \to \delta(x - 1).
\]  

We now turn to the GI\(\alpha\) distribution and the effect of parameter \(\gamma\). In Fig. 21 we give the contour plots of modal PDF and total half-widths in the \((\eta, \gamma)\) plane, where \(\eta = \alpha \gamma\) and \(-1 - \eta\) is the exponent of the power law tail. We observe an interesting scaling property of GI\(\alpha\): for \(\gamma \approx 2.1/\eta\), the dependence of the PDF on \(\eta\) is very weak, as demonstrated in Fig. 22, where it is plotted for integer \(\eta\) from 2 to 7. An alternative way to illustrate this is to plot PDF for a fixed \(\eta\) and variable \(\gamma\), as shown in Fig. 23. Following the thick line we notice that, for \(\eta > 3\), mode and half-width change very little with \(\eta\). The key implication of the scaling property is that IG\(\alpha\) contains all essential features pertinent to GI\(\alpha\).
Fig. 10. Log-likelihood of DJIA, S&P 500 index, and DJIA component stocks. From 1 to 32, the x-coordinates correspond to stock indices and stocks; the symbols and their fitting parameters are given in Table 3 below. All the values of log-likelihood are relative to the LN distribution. Red down-pointing triangles: GIGa∗N, magenta up-pointing triangles: GIGa(α, β, 2)∗N, orange diamonds: IGa∗N, black squares: LN∗N, blue dots: GGa∗N computed from the simplex method with iteration numbers as 1000, cyan stars: Ga∗N, and green circles: GGa(α, β, 2)∗N. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 11. Log-likelihood of S&P 500 component stocks. All the values of log-likelihood are relative to the LN distribution. Red down-pointing triangles: GIGa∗N, magenta up-pointing triangles: GIGa(α, β, 2)∗N, orange diamonds: IGa∗N, black squares: LN∗N, blue dots: GGa∗N computed from the simplex method with iteration numbers as 1000, cyan stars: Ga∗N, and green circles: GGa(α, β, 2)∗N. The top plot for all the ranges of log-likelihood and the bottom plot for a shorter range. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Appendix B. Parameterization of the GIGa family of distributions

This Appendix is a self-contained re-derivation of a LN limit of GIGa [28]. The three-parameter GIGa distribution is given by

\[
GIGa(x; \alpha, \beta, \gamma) = \frac{\gamma}{\beta \Gamma(\alpha)} e^{-\left(\frac{\beta}{x}\right)\gamma} \left(\frac{\beta}{x}\right)^{1+\alpha\gamma}
\]  

(B.1)

for \( x > 0 \) and 0 otherwise. We require that \( \alpha, \beta, \gamma > 0 \). IGa is the \( \gamma = 1 \) case of GIGa, given by Eq. (A.1): note that GIGa and IGa have power-law tails \( x^{-1-\alpha\gamma} \) and \( x^{-1-\alpha} \) respectively for \( x \gg \beta \).
Fig. 12. Log-likelihood of S&P 100 component stocks. All the values of log-likelihood are relative to the LN distribution. Red down-pointing triangles: GIGa = N, magenta up-pointing triangles: GIGa(α, β, 2) = N, orange diamonds: IGa = N, black squares: LN = N, blue dots: GGa = N computed from the simplex method with iteration numbers as 1000, cyan stars: Ga = N, and green circles: GGa(α, β, 2) = N. The top plot for all ranges of log-likelihood and the bottom plot for a shorter range. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 13. Parameters of GIGa used in fitting component stocks of S&P 500 (top) and S&P 100 (bottom). αγ is the theoretical exponent of the power-law tails in stock return, γ is the control parameter. If γ = 2, the volatility variance is described by IGa, which can be generated by a mean-field theory of [10] [1,2]. The mean of αγ is 4.3 and the median is 3.6. The mean of γ is 1.9 and the median is 1.5 for S&P 500 and 2.1 and 2.0 for S&P 100.
Fig. 14. Parameter $\alpha$ of $\text{GIG}(\alpha, \beta, 2)$ used in fitting component stocks of S&P 500 (left) and S&P 100 (right). $2\alpha$ is the theoretical exponent of the power-law tails in stock return. The mean of $2\alpha$ is 3.2 and the median is 3.1 for S&P 500 and 3.3 and 3.4 for S&P 100.

We proceed to rewrite $\text{GIG}$ in the following form:

$$\text{GIG}(x; \alpha, \beta, \gamma) = \frac{\gamma}{\Gamma(\alpha)} \exp \left[ \alpha \ln \left( \frac{x}{\beta} \right)^{-\gamma} - \left( \frac{x}{\beta} \right)^{-\gamma} \right].$$

(B.2)

A re-parameterization

$$\mu = \ln \beta - \frac{1}{\gamma} \ln \frac{1}{\lambda^2}$$

(B.3)

$$\sigma = \frac{1}{\gamma \sqrt{\alpha}}$$

(B.4)

$$\lambda = \frac{1}{\sqrt{\alpha}},$$

(B.5)

with $\sigma > 0$ and $\lambda > 0$, allows to express the old parameters in terms of the new:

$$\alpha = \frac{1}{\lambda^2}$$

(B.6)

$$\beta = e^{\mu} \lambda^{-2\alpha}$$

(B.7)

$$\gamma = \frac{\lambda}{\sigma},$$

(B.8)

leading, in turn, to

$$\left( \frac{x}{\beta} \right)^{-\gamma} = e^{-\frac{\lambda}{\sigma} (\ln x - \mu)} \lambda^{-2}$$

(B.9)

$$\ln \left( \frac{x}{\beta} \right)^{-\gamma} = -\frac{\lambda}{\sigma} (\ln x - \mu) + \ln(\lambda^{-2})$$

(B.10)

and

$$\alpha \ln \left( \frac{x}{\beta} \right)^{-\gamma} - \left( \frac{x}{\beta} \right)^{-\gamma} \approx \frac{\ln(\lambda^{-2}) - 1}{\lambda^2} - \frac{(\ln x - \mu)^2}{2\sigma^2},$$

(B.11)

where we have used the Taylor expansion of the exp term in Eq. (B.9), which depends on $\lambda/\sigma = \gamma \rightarrow 0^+$. We can also prove that

$$\frac{\gamma}{\Gamma(\alpha)} \exp \left[ \frac{\ln(\lambda^{-2}) - 1}{\lambda^2} \right] = \frac{1}{\sqrt{2\pi \sigma}}.$$

(B.12)

based on Stirling’s approximation when we let $\lambda^{-2} = \alpha \rightarrow +\infty$.

Upon substitution of Eqs. (B.11) and (B.12) into Eq. (B.1), we obtain the LN distribution

$$\text{LN}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma x}} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right].$$

(B.13)
In conclusion, GIGa has the limit of LN when $\lambda$ tends to 0 in such a way that $\alpha$ tends to $+\infty$ quadratically and $\gamma$ tends to 0 linearly.

GIGa (IGa) are also transparently related to GGa (Ga) distributions: $\text{GGa}(x; \alpha, \beta, \gamma) \rightarrow -\text{GGa}(x; \alpha, \beta, -\gamma) = -\text{GIGa}(x; \alpha, \beta, \gamma)$ and $\text{GGa}(x; \alpha, \beta, \gamma) \leftrightarrow \text{GIGa}(1/x; \alpha, 1/\beta, \gamma)$. Note, finally, that Lawless [29] derived the LN limit of GGa in a manner similar to ours, which solidifies the concept of the “family” that unites these distributions.

Appendix C. Stochastic “birth–death” model

Many natural and social phenomena fall into a stochastic “birth–death” model, described by the equation

$$dx = c_1x^{1-\gamma}dt - c_2xdt + \sigma xdW.$$  \[(C.1)\]
Fig. 16. Log–log plot of the stock return rate of S&P 500. Left: the left tail of negative stock return. The x-axis is the logarithm of absolute values of stock return and the y-axis is CDF. Right: the right tail of (positive) stock return.

Fig. 17. Log–log plot of the stock return rate of IBM. Left: the left tail of negative stock return. The x-axis is the logarithm of absolute values of stock return and the y-axis is CDF. Right: the right tail of (positive) stock return.

Fig. 18. Fourier transform of stock return of S&P 500 (left) and IBM (right).

where $x$ can alternatively stand for additive quantities such as wealth [1], body mass of a species [3], human response time [4], etc., and volatility variance in this work.
Thesecondtermontherhsdescribesanexponentiallyfastdecay,suchasthelossofwealthandmassduetothedecayofone’sownresources,orthereductionofvolatilityintheabsenceofcompetinginputsandofadditionaltimesduetolearning.
Thefirstrhstermmayalternativelydescribemetabolicconsumption,acquisitionofwealthineconomicexchange,plethora
ofmarketsignals,andalsovolatility.
Thethird,stochastictermistheonethatchangesthedeterministicdynamicequation,characterizedbythesaturation
toafinalvalueofthequantity,withtheadditionaltermsthatactasacolourfuldistributionofvalues—asthereadingcouldebewritteninthesolidstate.
Furthermore,justasthewealthmodelhasmicroscopicunderpinningsinanetworkmodelofeconomicexchange[1],itis
likelythatstochasticontogeneticgrowth[3]couldbedescribedbyananalogousnetworkmodelbasedon
capillaryexchange.

**Appendix D. Product distribution of GIGa and GGa with normal distribution**

Given two distributions of $x$ and $y$ with PDF $f(x)$ and $g(y)$ respectively, the product distribution is defined as

$$z = xy,$$

whitespace PDF is given by

$$\int_{-\infty}^{\infty} \frac{1}{|x|} f(x) g \left( \frac{z}{x} \right) dx.$$  \hfill (D.2)

We are interested in the circumstance when the distribution of $x$ is generated from a stochastic volatility model, such as (3),
and $y$ is normally distributed with 0 mean, such as assumed in (1). Since the standard deviation of $y$ can always be absorbed
in $x$, it can be set to 1 without loss of generality so that $y$ has the standard normal distribution $N(0, 1)$.

The closed form of $GGa(\alpha, \beta, \gamma) \ast N(0, 1)$ or $GIGa(\alpha, \beta, \gamma) \ast N(0, 1)$ cannot be obtained in the general case. Below we
giveexpressionsforsomeimportantlimits:

$$GGa(\alpha, \beta, 2) \ast N(0, 1) = \sqrt{\frac{2}{\pi}} \left( \frac{|z|}{\sqrt{2\beta}} \right)^{\alpha - \frac{1}{2}} K_{\alpha - \frac{1}{2}} \left( \frac{\sqrt{2|z|}}{\beta} \right),$$  \hfill (D.3)
Fig. 21. Top: contours of modal PDF of GIGa distributions with mean 1. Thin lines: contours of modal PDF at mode. Thick line: $\gamma = 2.1/\eta$. Bottom: contours of total half-widths of GIGa distributions with mean 1. Thick line: $\gamma = 2.5/\eta$.

Fig. 22. Scaled PDF of GIGa distributions with mean 1. In the plots, $\gamma = 2.1/\eta$. Six lines correspond to $\eta = 2, 3, \ldots, 7$.

where $K$ is the modified Bessel function of the second kind.

\[
\text{GIGa}(\alpha, \beta, 2) * N(0, 1) = \frac{I^* \left( \frac{1}{2} + \alpha \right) \left( \frac{2\beta^2}{z^2 + 2\beta^2} \right)^{\frac{1+z}{2}}}{\sqrt{2\pi} \beta \Gamma(\alpha)} ,
\]  

which is the generalized Student’s $t$-distribution $T(0, \frac{\beta}{\sqrt{\alpha}}, 2\alpha)$.  

\[
Fig. 23. Scaled PDF of GIGa distributions with mean 1. In each subplot with constant \( \eta \), from left to right, \( \gamma = 0.5/\eta, 1/\eta, 1.5/\eta, 2/\eta, 2.5/\eta, 3/\eta, \) and \( 3.5/\eta \), corresponding to red, magenta, orange, green, cyan, blue, and purple lines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

\[
\text{IGa}(\alpha, \beta) \ast N(0, 1) = 2^{-\frac{\gamma}{2} - 1} \alpha \left( \frac{\beta}{2} \right)^\alpha U \left( \frac{\alpha+1}{2}, 1, \frac{\beta^2}{2z^2} \right) \frac{\sqrt{\pi z}}{\beta^2}. \tag{D.5}
\]

where \( U \) is Tricomi’s confluent hypergeometric function.

Conversely, the mean and the variance of variable \( z \) can be analytically evaluated in the general case:

\[
\text{mean } z = 0 \quad \text{var } z = \frac{\beta^2 \Gamma \left( \alpha \pm \frac{1}{z} \right)}{\Gamma \left( \alpha \right)}. \tag{D.6}
\]

where the plus and minus correspond to distributions \( \text{GGa}(\alpha, \beta, \gamma) \ast N(0, 1) \) and \( \text{GIGa}(\alpha, \beta, \gamma) \ast N(0, 1) \) respectively. Notice also that the variance of the GGa/GIGa itself is given by

\[
\frac{\beta^2 \left[ \Gamma(\alpha) \Gamma \left( \alpha \pm \frac{1}{z} \right) - \Gamma \left( \alpha \pm \frac{1}{\gamma} \right)^2 \right]}{\Gamma(\alpha)^2}. \tag{D.7}
\]

Finally, it can be shown that Ref. [21]

\[
\text{GGa}(\alpha, \beta, \gamma) \ast N(0, 1) \rightarrow |z|^{-1-\alpha \gamma}, \quad |z| \rightarrow \infty. \tag{D.8}
\]

that is the tail of the GIGa carries through into its product distribution with the normal.
Appendix E. Stochastic differential equations of volatility

Per Eq. (5), any transformation \( f \rightarrow f \sigma^{2a}, \ g \rightarrow f \sigma^a \) does not change the functional form of the integral. In turn, this means that there exists a family of stochastic differential equations (SDE) for each type of the distribution, such as GIGa, GGa, etc. However, as already mentioned, we are interested in a number of specific SDE rooted in modeling of various phenomena. With this in mind, we discuss several SDE for volatility and the corresponding steady state distributions.

E.1. GIGa

The equation for volatility, which was obtained from NDL of GARCH(1, 1), was already discussed in Section 2 and is added here for completeness. The steady-state, normalized solution of

\[
d\sigma = J(\theta \sigma^{1-\gamma} - \sigma) \, dt + \Sigma \sigma \, dW,
\]

is given by

\[
P(\sigma) = \text{GIGa} \left( \sigma; \left( 1 + \frac{2f}{\Sigma^2} \right) \gamma^{-1}, \left( \frac{\theta}{\gamma} \frac{2f}{\Sigma^2} \right)^{1/\gamma} \right),
\]

where

\[
\theta = \gamma \Sigma^2 \left[ \frac{\overline{\sigma} \Gamma^\prime \left( \left( 1 + \frac{2f}{\Sigma^2} \right) \left( \gamma^{-1} \right) \right)}{\Gamma \left( \frac{2\gamma}{\Sigma^2} \gamma^{-1} \right)} \right] \gamma
\]

and \( \overline{\sigma} \) is the mean value of \( \sigma \). Above, \( \overline{\sigma}, J \) and \( \Sigma \) are positive constants. Similar assumptions are made throughout this section. Since \( \theta(1) = \sigma \), for \( \gamma = 1 \) the distribution reduces to

\[
P(\sigma) = \text{IGa} \left( 1 + \frac{2f}{\Sigma^2} \cdot \frac{2/\overline{\sigma}}{\Sigma^2} \right).
\]

E.2. GGa

A natural generator of GGa is Eq. (6) with reversed signs of \( J \) and \( \gamma \),

\[
d\sigma = J(\sigma - \theta \sigma^{1+\gamma}) \, dt + \Sigma \sigma \, dW.
\]

Its steady-state solution is given by

\[
P(\sigma) = \text{GGa} \left( \sigma; \left( -1 + \frac{2f}{\Sigma^2} \right) \gamma^{-1}, \left( \frac{\gamma \Sigma^2}{\theta} \frac{2f}{\Sigma^2} \right)^{1/\gamma} \right).
\]

Taking into account the scaling property discussed at the top of the Appendix, another two possible SDEs are

\[
d\sigma = J(1 - \theta \sigma^{\gamma}) \, dt + \Sigma \sqrt{\sigma} \, dW,
\]

and

\[
d\sigma = J(\sigma^{-1} - \theta \sigma^{-\gamma}) \, dt + \Sigma \, dW.
\]

The latter equation is particularly significant since it is a direct consequence of the Heston model for \( \gamma = 2 \). Indeed, the Heston model [12,22] for volatility variance \( V \) reads

\[
dV = J \left( V - \overline{V} \right) \, dt + \phi \sqrt{V} \, dW.
\]

Absorbing \( \overline{V} \) in \( J \), we rewrite the equation as

\[
dV = J \left( 1 - \frac{V}{\overline{V}} \right) \, dt + \phi \sqrt{V} \, dW.
\]

The latter yields a steady-state distribution given by

\[
P(V) = \text{Ga} \left( V; \frac{2f}{\phi^2}, \left( \frac{\phi^2 V}{2f} \right) \right).
\]
whose mean is $\bar{V}$. Changing variable to volatility $\sigma = \sqrt{V}$, Ito calculus yields
\[ d\sigma = \left[ \frac{1}{2} J \left( \sigma^{-1} - \bar{V} / \sigma^{-1} \right) - \frac{\phi^2}{8} \sigma^{-1} \right] dt + \frac{1}{2} \phi dW, \quad (E.12) \]
whose steady-state distribution is given by
\[ P(\sigma) = G\Gammaa \left( \sigma; \frac{2J}{\phi^2}, \phi \sqrt{\frac{\bar{V}}{2J}} \right). \quad (E.13) \]

E.3. LN

An Ornstein–Uhlenbeck process
\[ dx = \theta(\mu - x) dt + \sigma dW \quad (E.14) \]
yields a normal steady state distribution
\[ P(x) = \frac{1}{\sqrt{\pi \sigma^2}} \exp \left[ -\frac{\theta(x - \mu)^2}{\sigma^2} \right], \quad (E.15) \]
whose mean is $\mu$. A change of variable $x = \log X$ leads, with Ito calculus, to Refs. [16,30]
\[ dX = \theta X (\mu - \log X) dt + \frac{1}{2} \sigma^2 X dt + X \sigma dW \quad (E.16) \]
and the steady state distribution
\[ P(X) = \frac{1}{\sqrt{\pi \sigma X}} \exp \left[ -\frac{\theta(\log X - \mu)^2}{\sigma^2} \right], \quad (E.17) \]
whose mean is $\exp \left( \mu + \sigma^2 / 2\theta \right)$.

Just as before, when we showed that LN distribution can be obtained as a limit of the GiGa distribution, we can show that LN SDE can be obtained as a limit of GiGa SDE. Changing notations in (6), we rewrite it as
\[ dY = J(\Theta Y^{1-\gamma} - Y) dt + Y \Sigma. \quad (E.18) \]
It can be shown that (E.16) is a limiting case of (E.18) if we set
\[ J = \theta, \quad Y = \exp \left( \mu + \frac{2\sigma^2}{2\theta} \right), \quad \Sigma = \sigma \quad (E.19) \]
and let $\gamma \to 0^+$ linearly and $(2J / \Sigma^2) \gamma^{-1}$ ($\alpha$ in GiGa($\alpha, \beta, \gamma$)) tend to $+\infty$ quadratically. Details of the derivation can be found in Ref. [21].

Appendix F. Relaxation time

Consider an IGa process defined as
\[ dX = J(1 - X) dt + \Sigma X dW, \quad (F.1) \]
where $J$ and $\Sigma$ are constants and $dW$ is the Wiener process. This is the process described by Eq. (6) (Eq. (E.1)) for GiGa with $\gamma = 1$ and $\bar{X} = 1$. As previously pointed out in Appendix A, a GiGa process can be understood from that of IGa. The stationary distribution of $X$ is an IGa distribution [1],
\[ P_s(X) = G\Gammaa \left( X; 1 + \frac{2J}{\Sigma^2}, \frac{2J}{\Sigma^2} \right). \quad (F.2) \]
The purpose of this Appendix is to estimate the mean and the standard deviation of the relaxation time on approach to the steady-state distribution and to test these results numerically.

The existence of the stationary distribution is possible due to the first term in Eq. (F.1). For $J = 0$, on the other hand, it reduces to a lognormal process described by the time dependent distribution (obtained with Ito calculus) given by
\[ P_t(X, t) = \frac{1}{\sqrt{2\pi t \Sigma X}} \exp \left[ -\frac{(\ln X + \Sigma^2 t/2)^2}{2\Sigma^2 t} \right]. \quad (F.3) \]
Clearly, (F.3) describes a normalized distribution which tends to zero for every $X$ as time tends to infinity.
**Fig. 24.** Relaxation times of an inverse gamma process. Top: $\Sigma = \sqrt{0.1}$ and bottom: $\Sigma = 1$. Mean: black squares: numerical results; red line: theoretical estimate (F.4) with $c_1 = 1$; blue line: theoretical estimate (F.5). STDEV: black squares: numerical results; green line: theoretical estimate (F.6) with $c_2 = 1/4$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Fig. 25.** Top: plots of PDF of LN($x; \mu, \sigma$) with mean 1. The left red, middle green, and right blue curves correspond to parameters $\sigma = 1, 0.5,$ and 0.2 respectively. Bottom: log–log plots of simulated data sampled from the LN distributions. Below $-1$ of the y-axis, the left blue, middle green, and right red curves correspond to $\sigma = 0.2, 0.5,$ and 1 respectively. The dashed lines are fitting of $\log_{10}(1 - \text{CDF}(x))$ vs. $\log_{10}x$ in a range of CDF from 0.9 to 0.99. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The mean relaxation time can be defined as the time scale $t$ such that $\ln X_t \approx \ln X_1$, where the mean is evaluated with distributions (F.2) and (F.3) respectively. Simple calculation yields

$$\text{mean} = -\frac{2c_1}{\Sigma^2} \left[ -\psi^{(0)} \left( \frac{2}{\Sigma^2} \right) + \ln \left( \frac{2}{\Sigma^2} \right) \right],$$

(F.4)
where $\psi(0)$ is the digamma function and $c_1$ is a constant to account for the approximate nature of the estimate.\(^3\) Note that when $2J/\Sigma^2 \gg 1$, (F.4) becomes

$$\text{mean} \approx \frac{c_1}{2J}. \quad (F.5)$$

Similarly, the rms of the relaxation time can be estimated from

$$\text{STDEV} = \frac{c_2}{\Sigma^2} \psi^{(1)} \left( 1 + \frac{2J}{\Sigma^2} \right), \quad (F.6)$$

where $\psi^{(1)}$ is the polygamma function of order 1 and $c_2$ is a constant.

Numerically, we consider an ensemble of paths described by (F.1). The relaxation time is then defined as such when the $p$-value of the ensemble of $X$ conforming to the IgA distribution is larger than 0.1. In our computation, 5000 paths are considered for each relaxation time. Our results are shown in Fig. 24. Clearly, our estimates (F.4)–(F.6) with $c_1 = 1$ and $c_2 = 1/4$ fit the data quite well.

**Appendix G. Maximum likelihood estimation of GGa and GIGa**

For PDF $f(x|\theta)$, with parameter(s) $\theta$ and a dataset $\{x_i\}$ of size $n$, the likelihood function is

$$\prod_{i=1}^{n} f(x_i|\theta) \quad (G.1)$$

and the log-likelihood function is

$$\frac{1}{n} \sum_{i=1}^{n} \log f(x_i|\theta). \quad (G.2)$$

---

\(^3\) The same result can be obtained by equating the modes of the two distributions.
Fig. 27. Log–log plots of simulated data sampled from GIGa distributions GIGa \((x; \alpha, \beta, 0.5)\) (top) and GIGa \((x; \alpha, \beta, 2)\) (bottom) with mean 1. Below \(-1\) of the \(y\)-axis, the left blue, middle green, and right red curves correspond to \(\alpha = 2.5, 2,\) and \(1.5\) respectively. The dashed lines with slopes \(-2.8, -2.4,\) and \(-2.0\) respectively (top) and \(-4.3, -3.6,\) and \(-2.8\) (bottom) are fitting of \(\log_{10}(1 - \text{CDF}(x))\) vs. \(\log_{10} x\) in a range of CDF from 0.9 to 0.99. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The maximum likelihood estimation of \(\theta\) should maximize the log-likelihood function. Here we consider the case of GGa and GIGa. The generalized Student’s \(t\)-distribution is discussed in Ref. [21].

Since, as already mentioned, the PDFs of GGa and GIGa formally follow \(\text{GGa}(x; \alpha, \beta, \gamma) \xrightarrow{\gamma \rightarrow -\gamma} -\text{GIGa}(x; \alpha, \beta, -\gamma) = -\text{GIGa}(x; \alpha, \beta, \gamma),\) it is sufficient to consider GGa. Setting to zero partial derivatives over \(\alpha, \beta,\) and \(\gamma\) of the log-likelihood function for \(\text{GGa}(x; \alpha, \beta, \gamma)\) gives

\[
\left( \frac{1}{n} \sum_{i=1}^{n} \log x_i^\gamma \right) - \log \beta^\gamma - \psi(\alpha) = 0 \quad \text{(G.3)}
\]

\[
\frac{1}{n} \sum_{i=1}^{n} x_i^\gamma \beta^\gamma - \alpha = 0 \quad \text{(G.4)}
\]

\[
1 - \frac{1}{n} \sum_{i=1}^{n} \left( \frac{x_i^\gamma}{\beta^\gamma} - \alpha \right) \log \frac{x_i^\gamma}{\beta^\gamma} = 0 \quad \text{(G.5)}
\]

With the definition

\[
\frac{1}{n} \sum_{i=1}^{n} f(x_i) = \bar{f}(x),
\]

we obtain

\[
\log x^\gamma - \log \beta^\gamma - \psi(\alpha) = 0 \quad \text{(G.7)}
\]

\[
\frac{x^\gamma}{\beta^\gamma} - \alpha = 0 \quad \text{(G.8)}
\]
Fig. 28. Local slope of the log–log plot of IGa distribution $IGa(x; \alpha, \beta)$ with mean $1(\beta = \Gamma(\alpha)/\Gamma(\alpha - 1)).$ The left column is the log–log plot and the right one is the local slope of the log–log plot from Eq. (H.4). $\alpha$ is 2, 3, 5, and 7 for the first, second, third, and fourth rows respectively. The red lines are $-\alpha$: the limit of the local slope when $x \to \infty.$ (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

1. $1 - \left(\frac{x'\log x'}{\beta' / \beta} - \alpha\right) \log \frac{x'}{\beta'} = 0.$ \hspace{1cm} (G.9)

Substitution of Eq. (G.8) into (G.7) and Eq. (G.9) yields

$$\log \frac{x'}{\beta'} - \log \frac{x}{\beta'} + \log \alpha - \psi(\alpha) = 0$$ \hspace{1cm} (G.10)

and

$$\alpha = \left[\left(\frac{x' \log x'}{\beta'} - \log \frac{x'}{\beta'}\right)^{-1}\right]. \hspace{1cm} (G.11)$$
Fig. 29. Local slope of the log-log plot of GIGa \((x; \alpha, \beta, \gamma)\) with mean \(1 (\beta = \Gamma(\alpha)/\Gamma(\alpha - 1/\gamma))\). The left column is the log-log plot and the right one is the local slope of the log-log plot from Eq. (H.4). \((\alpha, \gamma)\) is \(\{6, 0.5\}, \{10, 0.5\}, \{1.5, 2\}, \text{and} \{2.5, 2\}\) for the first, second, third, and fourth rows respectively. The red lines are \(-\alpha\gamma\): the limit of the local slope when \(x \to \infty\). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Eqs. (G.10) and (G.11) form the basis of a maximum likelihood estimation program. Given a \(\gamma\), from (G.11), we calculate \(\alpha\) and then insert \(\alpha\) into the lhs of (G.10). A bisection method can be realized over \(\gamma\). For more details, see Ref. [21]. We note that Eqs. (G.10) and (G.11) result in either GGa \((\gamma > 0)\), or GIGa \((\gamma < 0)\) [21].

Appendix H. Log-log plot of distribution tails

The exponent of a power law tail can be easily calculated once we notice that

\[
1 - \text{CDF}(x) = \int_x^{\infty} \text{PDF}(x)dx.
\]
Fig. 30. Local slope of the log–log plot of the lognormal distribution. The mean of the distribution is set as 1 through $\mu = -\sigma^2/2$. The left column is the log–log plot and the right one is the local slope of the log–log plot in Eq. (H.5). $\sigma$ is 0.2, 0.5, 1, and 2 for the first, second, third, and fourth rows respectively. The jagged part of the top right plot is due to computational precision.

If $\text{PDF}(x) \propto x^{-1-k}$ with $x \gg 1$, then \(^4\)

$$\log(1 - \text{CDF}(x)) \propto \text{const} - k \log x.$$  \hspace{1cm} (H.2)

In Figs. 25 and 26, we show the log–log plot of the tail of LN and IGa distributions respectively. Clearly, a straight line fit is considerably better for the latter, even though the fitted slope does not agree with the theoretical value. Towards this end, in Fig. 27, we show log–log plots of the tail of GIGa distributions for $\gamma = 0.5$ and $\gamma = 2$. The empirical trend emerging form the IGa and GIGa plots is that the straight line fits of log–log plots become progressively better as $\gamma$ gets larger.

---

\(^4\) When calculating the empirical CDF of a sorted sequence $\{x_1, x_2, \ldots, x_n\}$, we set the empirical CDF as $\{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\}$ for two reasons. First, empirical CDF $\{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\}$ renders log$(1 - \text{CDF})$ meaningless for $x_n$. Second, it is symmetrical as needed for symmetrical distributions such as (generalized) Student’s $t$-distribution.
To understand this $\gamma$-dependence the difference between the theoretical and fitted slopes, we consider the local slope of the log–log plot.

$$\frac{d \log(1 - \text{CDF}(x))}{d \log x}.$$  \hfill (H.3)

For GIGa (and IGa, $\gamma = 1$), the local slope is given by

$$\gamma e^{-\left(\frac{\mu}{\alpha}\right)^\gamma} \frac{\Gamma\left(\alpha\right)}{\left(\text{Q}\left(\alpha, \left(\frac{\mu}{\alpha}\right)^\gamma\right) - 1\right)}$$  \hfill (H.4)

with the regularized gamma function $Q(s, x) = \frac{\Gamma(s, x)}{\Gamma(s)}$, where $\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the incomplete gamma function. The local slopes are shown, as a function of $x$ in Figs. 28 and 29 respectively. It is clear that the local slope can differ substantially from its limiting (saturation) value. As $\gamma$ becomes larger, the local slope tends closer to its limiting value.

For the LN distribution, the local slope is given by

$$\sqrt{\frac{2}{\pi}} e^{-\left(\frac{\log(x-\mu)^2}{2\sigma^2}\right)} \left(1 + \text{erf}\left(\frac{\log(x-\mu)}{\sqrt{2\sigma}}\right)\right),$$  \hfill (H.5)

which slowly decreases with $x$. But as is clear from (H.5) and Fig. 30, the local slope does not saturate when $x \to \infty$.

References

[1] T. Ma, J.G. Holden, R.A. Serota, Physica A 392 (2013) 2434.
[2] J.-P. Bouchaud, M. Mézard, Physica A 282 (2000) 536.
[3] D. West, B. West, Internat. J. Modern Phys. B 26 (2012) 1230010.
[4] T. Ma, J.G. Holden, R.A. Serota, arXiv:1305.6320, 2013.
[5] VIX Historical Price Data, http://www.cboe.com/micro/vix/historical.aspx.
[6] Volatility Indexes, http://www.cboe.com/micro/volatility/introduction.aspx.
[7] VXO, http://www.cboe.com/micro/vxo/.
[8] P.D. Praetz, J. Bus. 45 (1972) 49.
[9] E. Platen, R. Rendek, J. Stat. Theory Pract. 2 (2008) 233.
[10] A. Gerig, J. Vicente, M.A. Fuentes, Phys. Rev. E 80 (2009) 065102.
[11] J. Hull, Options, Futures, and Other Derivatives, third ed., Prentice Hall, Upper Saddle River, NJ, 1997.
[12] J. Gatheral, The Volatility Surface: A Practitioner’s Guide, Wiley & Sons, Hoboken, New Jersey, 2006.
[13] D.B. Nelson, J. Econometrics 45 (1990) 7.
[14] J.-C. Duan, Math. Finance 5 (1995) 13. GARCH(1, 1) model is used in the theory of option pricing.
[15] S. Jackman, Bayesian Analysis for the Social Sciences, Wiley & Sons, 2005.
[16] J.B. Wiggins, J. Financ. Econ. 19 (1987) 351.
[17] Standard & Poor’s, http://www.standardandpoors.com/.
[18] Yahoo! Finance, http://finance.yahoo.com/.
[19] Economic Research Federal Reserve Bank of St. Louis: S&P 500 Stock Price Index (SP500), http://research.stlouisfed.org/fred2/series/SP500/ downloaddata?cid=32255.
[20] Economic Research Federal Reserve Bank of St. Louis: Dow Jones Industrial Average (DJIA), https://research.stlouisfed.org/fred2/series/DJIA/ downloaddata?cid=32255/.
[21] T. Ma, Ph.D. Thesis, University of Cincinnati, 2013.
[22] S.L. Heston, Rev. Financ. Stud. 6 (1993) 327.
[23] M.H. Stanley, L.A. Amaral, S.V. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, M.A. Salinger, H.E. Stanley, Nature 379 (1996) 804.
[24] P. Cizeau, Y. Liu, M. Meyer, C.-K. Peng, H.E. Stanley, Physica A 245 (441) (1997).
[25] Y. Liu, P. Gopikrishnan, Cizeau, Meyer, Peng, H.E. Stanley, Phys. Rev. E 60 (1999) 1390.
[26] P. Gopikrishnan, V. Plerou, L.A.N. Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E 60 (1999) 5305.
[27] V. Plerou, P. Gopikrishnan, L.A.N. Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E 60 (1999) 6519.
[28] The Generalized Gamma Distribution and Reliability Analysis, http://www.weibull.com/hotwire/issue15/hottopics15.htm.
[29] J. Lawless, Statistical Models and Methods for Lifetime Data, Wiley & Sons, New York, 1982.
[30] L.O. Scott, J. Financ. Quant. Anal. 22 (1987) 419.
[31] A.C. Silva, V.M. Yakovenko, Physica A 382 (2007) 278; See also A.A. Dragulescu, V.M. Yakovenko, Quant. Finance 2 (2002) 443; A.C. Silva, R.E. Prange, V.M. Yakovenko, Physica A 344 (2004) 227.