Local Continuity and Asymptotic Behaviour of Degenerate Parabolic Systems

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Abstract

We study the local continuity and asymptotic behavior of solutions, \( u = (u_1, \cdots, u_k) \), of degenerate system

\[
  u'_i = \nabla \cdot \left( U^{m-1} \nabla u_i \right) \quad \text{for } m > 1 \text{ and } i = 1, \cdots, k
\]

describing the degenerate diffusion of the populations density vector, \( u \), of \( k \)-species whose diffusion is determined by their total population density \( U = u_1 + \cdots + u_k \). We adopt the intrinsic scaling and iteration arguments of DeGiorgi, Moser, and Dibenedetto for the local continuity of solutions, \( u' \). Under some regularity condition, we also prove that the population density function, \( u \), of \( i \)-th species with the population \( M_i \) converges to \( B M(x, t) \) in the space of differentiable functions of all order where \( B_M \) is the Barenblatt profile of the Porous Medium Equation with \( L^1 \) mass \( M = M_1 + \cdots + M_k \) while \( U \) converges to \( B_M \). As a consequence, each \( u' \) becomes a concave function after a finite time.

Keywords. Local Continuity, Asymptotic Behaviour, Degenerate Equation, Eventual Concavity

1 Introduction

Let us consider the evolution of population of different species in one system whose diffusion interacts each other. Under the closed system, we can consider the case when the evolution of population of each species are controlled by total population of all species in that system. For a given number of species \( k \in \mathbb{N} \), let \( u_i \geq 0 \), \( (i = 1, \cdots, k) \), represent the population density of \( i \)-th species and \( U \) be the total density of all species, i.e.,

\[
  U = u_1 + u_2 + \cdots + u_k = \sum_{i=1}^{k} u_i.
\]  

(1.1)

Now we consider a simple model case where each density function, \( u' \), diffuses following

\[
  u'_i = \nabla \cdot \left( U^{m-1} \nabla u_i \right) \quad \text{for } m > 1 \text{ and } i = 1, \cdots, k,
\]

(sPME)
where the diffusion coefficient is controlled by the total population density, $U$. Then we can observe $U$ satisfies the standard Porous Medium Equation (or PME):

$$U_t = \sum_{i=1}^{k} (u_i^t) = \sum_{i=1}^{k} \nabla \cdot (U^{m-1} \nabla u_i^t) = \nabla \cdot (U^{m-1} \nabla U) = \frac{1}{m} \Delta U^m \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty).$$

Moreover by (1.1),

$$u_i^t \leq U \quad \forall i = 1, \cdots, k. \quad (1.2)$$

Therefore, it is natural to consider that the density $u_i^t$, $(i = 1, \cdots, k)$ and total density $U$ both satisfy the same equation with the condition (1.2).

In this paper, first we are going to investigate the local continuity and asymptotic behavior of more general nonlinear parabolic partial differential equation

$$u_t = \nabla \cdot (U^{m-1} \nabla u) \quad (1.3)$$

in the range of exponents $m > 1$, with diffusion coefficients $U^{m-1}$ nonnegative and compactly supported.

Since the function $U(x, t)$ determines the diffusion coefficient of the equation (1.3), the evolutions described by equation (1.3) is strongly governed by the properties of $U$. If the function $U$ is equivalent to the solution $u$ of (1.3) in the sense that $U(x, t) = cu^\beta(x, t)$ $\forall (x, t) \in \mathbb{R}^n \times [0, \infty)$ for some constants $c > 0$ and $\beta \in \mathbb{R}$, the equation (1.3) appears in many physical phenomenons [Ar, DK, Va]. When $\beta(m-1)+1 > 1$, it is well known as the porous medium equation which arises in describing the flow of an ideal gas through a homogeneous porous medium [Ar]. Since $\beta(m-1) > 0$, the porous medium equation becomes degenerate when $u = 0$ and this degeneracy let the flow propagate slowly with finite speed. This implies that there exists an interface or free boundary which separates regions where $u > 0$ from regions where $u = 0$. [Va]. When $\beta(m-1)+1 = 1$ and $\beta(m-1)+1 < 1$, we call them the heat equation and the fast diffusion equation, respectively. Similar to the porous medium equation, the fast diffusion equation arises in many famous flows such as Yamabe flow and Ricci flow. We refer the readers to the papers [PS] for Yamabe flow and to the papers [Wu] for Ricci flow.

There are many studies on the regularity and asymptotic behaviour for the porous medium and fast diffusion equation. In [CF1, CF2, CF3], they showed that the free boundary of porous medium equation is locally Hölder continuous and as a consequence that the solution is also locally Hölder continuous for any initial data. In [CW], they proved that the interface is actually $C^{1,\alpha}$ with the initial data satisfying $u^{\beta(m-1)}_0 \in C^1$ in the support of $u_0$ and $\nabla u^{\beta(m-1)}_0 \neq 0$ along the free boundary. However, $C^{1,\alpha}$ continuity of the pressure of the solution was not guaranteed in their paper. In [DH], they showed that, under appropriate regularity assumptions on the initial data, the pressure of solution is smooth up to the interface and the free boundary is also a smooth surface for short period of time. The paper [LV] is devoted to investigating the geometric properties of the solutions of the porous medium equation posed in the whole space with the nonnegative, continuous and compactly supported initial data $u_0$. They showed that the pressure of solution becomes
a concave function with respect to the space variable after a finite time and there is a $C^\infty$ convergence between the pressure and the radially symmetric solution of porous medium equation called Barenblatt profile. For more information about the regularity and asymptotic behaviour on the porous medium and fast diffusion equation, we refer the readers to the papers [Di, HU, KL3] for regularity and to the papers [BBDGV, HK1, HK2, HKS, Va2] for asymptotic behaviour of solution of porous medium and fast diffusion equation.

Corresponding to the porous medium type equation, we can also consider the equation (1.3) as the $p$-Laplacian equation which is given by putting the diffusion coefficients $U_{m-1}$ to be equivalent to the gradients of the solution $u$ of (1.3) in the sense that

$$U_{m-1} = c |\nabla u|^{p-2}$$

in $\mathbb{R}^n \times (0, \infty)$ for some constant $c > 0$ and $p > 1$. Large number of literatures on the local continuity and asymptotic behaviour of solutions of $p$-laplacian equation can be also found. We refer the readers to the papers [CD, DF] for various estimates about local continuity and to the papers [KV] for the asymptotic behaviour of solution of $p$-laplacian equation.

As mentioned above, the behaviour of solution $u$ of (1.3) is strongly effected by the diffusion coefficients $U_{m-1}$. First, we are going to study the local continuity and asymptotic behavior of the solution of the problem

$$\begin{cases}
  u_t = \nabla \cdot (U_{m-1} \nabla u) \\
  u(x, 0) = u_0(x)
\end{cases} \quad \forall x \in \mathbb{R}^n$$

(PME$_u$)

in the range of exponents $m > 1$, with initial data $u_0$ nonnegative and integrable satisfying

$$0 \leq u(x, 0) \leq U(x, 0) \quad \forall x \in \mathbb{R}^n$$

(1.4)

where $U$ is the solution of

$$\begin{cases}
  U_t = \nabla \cdot (U^{m-1} \nabla U) = 1 \\
  U(x, 0) = U_0(x)
\end{cases} \quad \forall x \in \mathbb{R}^n$$

(PME)

with initial data $U_0$ nonnegative, integrable and compactly supported.

As the first result of this paper, we will prove the local continuity of solution $u$ of (PME$_u$) which satisfies (1.4). Among the methods for the local continuity, we will take the oscillation argument which will be used often for the Hölder regularity of solution. Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and consider the cylinder

$$(x_0, t_0) + Q(R, R^{2-\epsilon}) = (x_0, t_0) + B_R \times (-R^{2-\epsilon}, 0) \subset \mathbb{R}^n \times (0, \infty), \quad (0 < R \leq 1)$$

(1.5)

where $\epsilon > 0$ is a small number to be determined later and $B_R$ is the ball centered at $x = 0$ of radius $R > 0$. The main step of the oscillation argument is to show that the ratio between supremum and infimum on the set of $(x_0, t_0) + Q(R, R^{2-\epsilon})$ decreases as the radius $R$ shrinks to half (Oscillation Lemma). Thus it is very important to control the ratio on a given domain properly. To bound the ratio, we assume that the diffusion coefficients $U$ satisfies the following assumption:
**Assumption I:** There exist uniform constants $0 < \lambda \leq \Lambda < \infty$ and $\beta \geq 0$ such that for all $0 < R < R_0$

$$
\lambda u^\beta \leq U \leq \Lambda \quad \forall (x, t) \in (x_0, t_0) + Q(R, R^{2-\beta}) 
$$

(1.6)

holds for some constant $R_0 > 0$.

With this assumption, we now state the first result of our paper.

**Theorem 1.1.** Under the Assumption I, any weak solution of \( (\text{PME}_u) \) with initial data $u_0 \in L^1$ satisfying (1.4) is locally continuous in $\mathbb{R}^n \times (0, \infty)$.

As the second result of this paper, we will deal with the asymptotic behaviour of the solution $u$ of \( (\text{PME}_u) \). Denote by $B_M$ the self-similar Barenblatt solution of the porous medium equation with $L^1$ mass $M > 0$. If the function $U_0$ has the mass $M$ in $L^1(\mathbb{R}^n)$, then by \( [LV] \) it is well known that

$$
U(\cdot, t) \to B_M(\cdot, t) \quad \text{in } C^{\infty} \text{ as } t \to \infty
$$

under some degeneracy condition of $U$. Thus, it is natural to expect that if there is a limit of the solution $u$ of \( (\text{PME}_u) \) then the limit will satisfy

$$
\nu_t = \nabla \cdot \left( B_M^{n-1} \nabla v \right) \quad \text{and} \quad \nu \leq B_M \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).
$$

(1.7)

Since $cB_M$ is also a solution of (1.7) for any constant $c \in (0, 1)$, the constant $c$ could be $\frac{||u_0||_{L^1}}{M}$ if the solution $u$ maintains its $L^1$-mass. Under this expectation, we are going to state our second result of paper.

**Theorem 1.2.** Let $u$ be a nonnegative solution of \( (\text{PME}_u) \) with initial data $u_0 \in L^1$ satisfying (1.4). Let $||u_0||_{L^1(\mathbb{R}^n)} = M_0$. Then

$$
\lim_{t \to \infty} \left\| u(\cdot, t) - \frac{M_0}{M} B_M(\cdot, t) \right\|_{L^1} = 0
$$

(1.8)

and

$$
\lim_{t \to \infty} t^{\alpha_1} \left| u(x, t) - \frac{M_0}{M} B_M(x, t) \right| = 0
$$

(1.9)

uniformly in $\mathbb{R}^n$.

Denote by $\nu$ the pressure of $u$, i.e.,

$$
\nu(x, t) = u^{n-1}(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).
$$

For any $\lambda > 0$, let $v_\lambda$ be the rescaled function of $\nu$ by

$$
v_\lambda(x, t) = \lambda^{\frac{n-1}{n}} \nu \left( \lambda^{\frac{1}{n-1}} x, \lambda t \right), \quad \forall \lambda > 0, \ (x, t) \in \mathbb{R}^n \times (0, \infty).
$$

By Theorem 1.2 there is the uniform convergence such that

$$
v_\lambda(x, t) \to \left( \frac{M_0}{M} B_M(x, t) \right)^{n-1} \quad \text{in } L^p, \ (p \geq 1) \quad \text{as } \lambda \to \infty.
$$

By $C^{\infty}$ regularity in \( [Ko] \) and an argument similar to the proof of Theorem 3.2 of \( [LV] \), we can extend our convergence in $L^p$, $(p \geq 1)$, to the one in $C^{\infty}$ for some Euclidean metric $ds$ which will be mentioned later. The $C^{\infty}$ convergence of pressure $\nu$ is stated as follow.
Theorem 1.3 (cf. Theorem 3.2 of [LV]). For any \( k \in \mathbb{N} \),
\[
v_{\lambda}(x, 1) \to \left( \frac{M_0}{M} B_M(x, 1) \right)^{m-1} \quad \text{in } C^{k}_s \quad \text{as } \lambda \to \infty
\]
for each \( k \in \mathbb{N} \).

As a consequence of Theorem 1.3, we can also get the following geometric properties of pressure \( v \).

Corollary 1.4 (cf. Theorem 3.3 of [LV]). There exists a constant \( t_0 > 0 \) such that the pressure \( v(x, t) \) is strictly concave on \( \{ x \in \mathbb{R}^n : v(x, t) > 0 \} \) for all \( t > t_0 \). More precisely
\[
\lim_{t \to \infty} t \frac{\partial^2 v}{\partial x_i^2} = -\frac{1}{(m-1)n + 2} \quad \text{uniformly in } x \in \text{supp } v \quad (\forall i = 1, \cdots, n).
\]

Let \( u'(x, t), (1 \leq i \leq k) \), be nonnegative functions which are governed by evolutions of population of different species in one system whose diffusion interacts each other. Then, as a consequence of the Theorem 1.2 and Theorem 1.3, we can describe the large time asymptotic behaviour of \( u' \) as \( t \to \infty \).

Corollary 1.5. For \( 1 \leq i \leq k \), let \( u'(x, t) \) be nonnegative function with
\[
\|u'(t)\|_{L^1(\mathbb{R}^n)} = M_i > 0 \quad \forall t \geq 0
\]
and let \( v' \) be the pressure of \( u' \), i.e.,
\[
v'(x, t) = \left( u'(x, t) \right)^{m-1} \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty), \ 1 \leq i \leq k.
\]

Let the function \( U \) be constructed by (1.1). If the function \( u' \) is a solution of (SPME), then \( v' \) convergence to \( \left( \frac{M_i}{M} B_M \right)^{m-1} \) uniformly in \( L^p \), \( (p \geq 1) \) and \( C^\infty_s \) as \( t \to \infty \) where \( M = M_1 + \cdots + M_k \).

As a consequence of \( C^\infty_s \) convergence, the pressure of \( v' \) becomes strictly concave on \( \{ x \in \mathbb{R}^n : v > 0 \} \) after a finite time.

We end up this section by introducing the definition of solutions. We say that \( u \) is a weak solution of (PME) in \( \mathbb{R}^n \times (0, T) \) if \( u \) is a locally integrable function satisfying

1. \( u \) belongs to function space:
\[
U^{m-1} |\nabla u| \in L^2 \left( 0, T : L^2 \left( \mathbb{R}^n \right) \right)
\]

2. \( u \) satisfies the identity:
\[
\int_0^T \int_{\mathbb{R}^n} \left\{ U^{m-1} \nabla u \cdot \nabla \varphi - uu_{\varphi} \right\} \, dx \, dt = \int_{\mathbb{R}^n} u_0(x) \varphi(x, 0) \, dx \quad (1.10)
\]
holds for any test function \( \varphi \in C^1 \left( \mathbb{R}^n \times (0, T) \right) \) which has a compact support in \( \mathbb{R}^n \) and vanishes for \( t = T \).

This paper is divided into three parts: In Part 1 (Section 2) we study the properties of the solution of (PME), Part 2 (Section 3) is devoted to the proof of local continuity of solution, \( u \), (Theorem 1.1). As mentioned above, the main step is to show the Oscillation Lemma. In Part 3 (Section 4), we will investigate the \( C^\infty_s \) convergence between \( u \) and Barenblatt solution under some degenerate conditions.
2 Preliminary Results

In this section, we will study the existence and properties of solutions \( u \) and \( U \) of (PME) and (PME), respectively.

2.1 Properties of solution, \( U \), of Porous Medium Equations

As the first step of this section, we are going to deal with the existence and properties of function \( U \) of diffusion coefficients. The first one is existence of weak solution and the next one is mass conservation of (PME).

Lemma 2.1 (cf. Chapter 9 of [Va1]). Let \( m > 1 \). For every \( U_0 \in L^1(\mathbb{R}^n) \cap L^{m+1}(\mathbb{R}^n) \) there exists an unique weak solution \( U \) of (PME) with initial data \( U_0 \) such that \( U^m \in L^2(0, \infty : H^1(\mathbb{R}^n)) \). The solution \( U \) satisfies estimates

\[
\|U(\cdot, t)\|_{L^1} \leq \frac{2\|U_0\|_{L^1}}{(m-1)t}
\]

and

\[
|U(x, t)| \leq C\|U_0\|_{L^1}^{a_2}t^{-a_1}
\]

where \( a_1 = \frac{n}{n(m-1)+2} \), \( a_2 = \frac{1}{n(m-1)+2} \) and \( C > 0 \) depends only on \( m \) and \( n \). If \( U_0 \in L^p(\mathbb{R}^n) \) for \( 1 \leq p \leq \infty \), then \( U(\cdot, t) \in L^p(\mathbb{R}^n) \) and

\[
\|U(\cdot, t)\|_{L^p} \leq \|U_0\|_{L^p}.
\]

Lemma 2.2 (Mass conservation of PME in [Va1]). For every \( t > 0 \), we have

\[
\int_{\mathbb{R}^n} U(x, t)\, dx = \int_{\mathbb{R}^n} U_0(x)\, dx.
\]

2.2 Uniqueness and existence of solution \( u \)

With the properties of \( U \), we will consider the uniqueness and existence of weak solution of (PME).

Lemma 2.3 (Uniqueness of solutions). The Problem (PME) has at most one weak solution if \( u_0 \in L^2(\mathbb{R}^n) \).

Proof. Let \( u_1 \) and \( u_2 \) be two solutions of (PME) with initial data \( u_{0,1} \) and \( u_{0,2} \) respectively. Then \( v = u_1 - u_2 \) is also a solution of (PME) with initial data \( v_0 = u_{0,1} - u_{0,2} \). By an approximation argument similar to the proofs of Theorem 5.5 and Lemma 9.26 of [Va1], we have

\[
\int_0^T \int_{\mathbb{R}^n} U^{m-1} |\nabla v_+|^2 \, dxdt + \frac{1}{2} \int_{\mathbb{R}^n} v_+^2(x, T) \, dx = \frac{1}{2} \int_{\mathbb{R}^n} v_+^2(x, 0) \, dx.
\]

Thus if we have initial date \( u_{0,1} \) and \( u_{0,2} \) such that \( u_{0,1}(x) \leq u_{0,2}(x) \) for all \( x \in \mathbb{R}^n \), i.e., \( (v_0)_+ (x) = 0 \) for all \( x \in \mathbb{R}^n \), then by (2.2)

\[
v_+(x, t) = 0 \quad \text{a.e. in } \mathbb{R}^n \times (0, T)
\]

\[
\Rightarrow \quad u_1(x, t) \leq u_2(x, t) \quad \text{a.e. in } \mathbb{R}^n \times (0, T)
\]
Similarly, we can also have

$$u_1(x,t) \geq u_2(x,t) \quad \text{a.e. in } \mathbb{R}^n \times (0,T)$$  \hspace{1cm} (2.4)

if we have initial date $u_{0,1}$ and $u_{0,2}$ such that $u_{0,1}(x) \geq u_{0,2}(x)$ for all $x \in \mathbb{R}^n$. By (2.3) and (2.4) the lemma follows.

Let $u$ be a solution of (PME) with initial condition (1.4). Then by Lemma 2.3 we have

$$0 \leq u(x,t) \leq U(x,t) \quad \forall x \in \mathbb{R}^n, \ t \geq 0.$$  \hspace{1cm} (2.5)

As a consequence of (2.5), we can get the functional space to which the solutions of (PME) are belonging.

**Lemma 2.4.** Let $m > 1$ and let $U$ be the solution of (PME) which is given by Lemma 2.1. Then solution $u$ of (PME) and (1.4) satisfies

$$U^{m-1} |\nabla u| \in L^2(0,T : L^2(\mathbb{R}^n)).$$

**Proof.** Multiplying the first equation in (PME) by $U^{m-1}u$ and integrating over $\mathbb{R}^n \times (0,\infty)$, by (2.5) and Young’s inequality we have

$$\int_{\mathbb{R}^n} u^{m-1}u^2 \, dx(t) + \int_0^\infty \int_{\mathbb{R}^n} \left( U^{m-1} |\nabla u| \right)^2 \, dxdt \\
\leq \int_{\mathbb{R}^n} u^{m-1}u^2 \, dx(0) + \int_0^\infty \int_{\mathbb{R}^n} u^2 |\nabla U^{m-1}|^2 \, dxdt + (m-1) \int_0^\infty \int_{\mathbb{R}^n} U^{m-2}u^2 \, dxdt \\
\leq \int_{\mathbb{R}^n} u_0^{m+1} \, dx + \frac{(m-1)^2}{m^2} \int_0^\infty \int_{\mathbb{R}^n} |\nabla U|^2 \, dxdt + (m-1) \int_0^\infty \int_{\mathbb{R}^n} U^{m-2}u^2 \, dxdt.$$  \hspace{1cm} (2.6)

Since $U$ is the solution of (PME),

$$\int_0^\infty \int_{\mathbb{R}^n} U^{m-2}u^2 \, dxdt = \frac{1}{m} \int_0^\infty \int_{\mathbb{R}^n} \nabla \left( U^{m-2}u^2 \right) \cdot \nabla U \, dxdt \\
\leq \frac{|m-2|}{m^2} \int_0^\infty \int_{\mathbb{R}^n} |\nabla U|^2 \, dxdt + \frac{2}{m} \int_0^\infty \int_{\mathbb{R}^n} U^{m-1} |\nabla U| |\nabla U^m| \, dxdt \\
\leq \left( \frac{|m-2|}{m^2} + \frac{2}{m^2} \right) \int_0^\infty \int_{\mathbb{R}^n} |\nabla U|^2 \, dxdt + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^n} \left( U^{m-1} |\nabla u| \right)^2 \, dxdt$$  \hspace{1cm} (2.7)

By (2.6), (2.7) and Lemma 2.1

$$\int_0^\infty \int_{\mathbb{R}^n} \left( U^{m-1} |\nabla u| \right)^2 \, dxdt \leq C \left( ||u_0||_{L^1(\mathbb{R}^n)} , ||\nabla U^m||_{L^2(0,T; L^2(\mathbb{R}^n))} \right) < \infty$$

and the lemma follows.  \hspace{1cm} \Box

We now are ready for the existence of weak solution of (PME)w.

**Lemma 2.5.** Let $m > 1$ and let $U$ be the solution of (PME) which is given by Lemma 2.1. Let $u_0 \in L^1(\mathbb{R}^n)$ be a function with $0 \leq u_0 \leq U_0$. Then there exists a weak solution $u$ of (PME)w which satisfies (1.4).
Proof. For the functions $u_0, U$ and constants $M > 1, 0 < \epsilon < 1$, let
\[
\begin{cases}
  u_{0,M}(x,t) = \min(u_0(x), M) \\
  U_M(x,t) = \min(U(x,t), M) \\
  U_{e,M}(x,t) = \left(U_M^{-1}(x,t) + \epsilon\right)^{1/\epsilon}.
\end{cases}
\]

Since $\epsilon^{1/\epsilon} \leq U_{e,M} < M + 1$ in $\mathbb{R}^n \times (0, \infty)$, $U_{e,M}$ is uniformly parabolic in $\mathbb{R}^n \times (0, \infty)$. Thus, for any $0 < \epsilon < 1, M > 1$ there exists the solution $u_{e,M}$ of
\[
\begin{aligned}
  (u_{e,M})_t &= \nabla \left(U_M^{-1}\nabla u_{e,M}\right) & \text{in} \; & \mathbb{R}^n \times (0, \infty) \\
  u_{e,M}(x,0) &= u_{0,M}(x) & \forall x & \in \mathbb{R}^n.
\end{aligned}
\] (2.8)

Moreover,
\[
u_{0,M}(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus \text{supp} \; U_0 \quad \Rightarrow \quad u_{e,M}(x,t) \to 0 \quad \text{as} \; |x| \to \infty.
\]

Multiplying the first equation in (2.8) by $u_{e,M}$ and integrating over $\mathbb{R}^n \times (0, \infty)$, we have
\[
\sup_{0 \leq t < \infty} \left\|u_{e,M}\right\|_{L^2(\mathbb{R}^n)}^2 + \int_0^\infty \int_{\mathbb{R}^n} U_M^{-1} \left|\nabla u_{e,M}\right|^2 \, dx \, dt + \epsilon \int_0^\infty \int_{\mathbb{R}^n} \left|\nabla u_{e,M}\right|^2 \, dx \, dt \leq C \left\|u_0\right\|_{L^2(\mathbb{R}^n)}
\]
\[
\Rightarrow \quad \int_0^\infty \int_{\mathbb{R}^n} U_M^{-1} \left|\nabla u_{e,M}\right|^2 \, dx \, dt + \epsilon \int_0^\infty \int_{\mathbb{R}^n} \left|\nabla u_{e,M}\right|^2 \, dx \, dt \leq C \left(\|u_0\|_{L^2}\right).
\] (2.9)

Let $\{\epsilon_k\}_{k=1}^\infty$ be a sequence of real numbers such that $\epsilon_k \to 0$ as $k \to \infty$. Then by (2.9), the sequence $\{u_{\epsilon_k,M}\}_{k=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges in $H^1(\mathbb{R}^n \times (0, \infty))$ with weight $U_M^{-1}$ to a function $u_M$ in $\mathbb{R}^n \times (0, \infty)$ as $k \to \infty$. Since $u_{e,M}$ is the solution of (2.8), we have
\[
\int_0^\infty \int_{\mathbb{R}^n} \left\{U_M^{-1} \nabla u_{\epsilon,M} \cdot \nabla \varphi + \epsilon \nabla u_{\epsilon,M} \nabla \varphi - u_{\epsilon,M} \varphi\right\} \, dx \, dt = \int_{\mathbb{R}^n} u_{0,M}(x) \varphi(x,0) \, dx.
\] (2.10)

for any $\varphi \in C^2_0(\mathbb{R}^n \times [0, \infty))$. Letting $\epsilon \to 0$ in (2.10), by (2.9) we get
\[
\int_0^\infty \int_{\mathbb{R}^n} \left\{U_M^{-1} \nabla U_M \cdot \nabla \varphi - U_M \varphi\right\} \, dx \, dt = \int_{\mathbb{R}^n} u_{0,M}(x) \varphi(x,0) \, dx.
\] (2.11)

Hence $u_M$ is a weak solution of
\[
\begin{aligned}
  (u_M)_t &= \nabla \left(U_M^{-1}\nabla u_M\right) & \text{in} \; & \mathbb{R}^n \times (0, \infty) \\
  u_M(x,0) &= u_{0,M}(x) & \forall x & \in \mathbb{R}^n.
\end{aligned}
\] (2.12)

By (2.7), for any $M > 0$ there exists a constant $t_M > 0$ such that
\[
t_M \to 0 \quad \text{as} \; M \to \infty \quad \text{and} \quad U_M(x,t) = U(x,t) \quad \forall x \in \mathbb{R}^n, \quad t \geq t_M.
\]

Thus by an argument similar to the identity (2.9), the sequence $\{u_M\}$ is bounded in $H^1(\mathbb{R}^n \times (\tau, \infty))$ with weight $U_M^{-1}$ for any $\tau > 0$. Then, for any $\tau > 0$ the sequence $\{u_M\}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges in $H^1(\mathbb{R}^n \times (\tau, \infty))$ with weight $U_M^{-1}$ to
a function $u$ in $\mathbb{R}^n \times (0, \infty)$ as $M \to \infty$. Choosing $\varphi \in C^{2,1}_0(\mathbb{R}^n \times (0, \infty))$ and letting $M \to \infty$ in (2.11), $u$ satisfies

$$
\int_0^\infty \int_{\mathbb{R}^n} \left\{ U^{m-1} \nabla u \cdot \nabla \varphi - u \varphi \right\} \, dx \, dt = 0 \quad \forall \varphi \in C^{2,1}_0(\mathbb{R}^n \times (0, \infty)).
$$

(2.13)

We now are going to show that

$$
u(\cdot, t) \to u_0 \quad \text{in } L^1 \text{ as } t \to 0^+.
$$

(2.14)

Let $\eta(x) \in C^2_0(\mathbb{R}^n)$. Then by an argument similar to the proof of (2.9),

$$
\left| \int_{\mathbb{R}^n} u_M(x, t) \eta(x) \, dx - \int_{\mathbb{R}^n} u_{0,M}(x) \eta(x) \, dx \right|
\leq \int_0^t \int_{\mathbb{R}^n} U^{m-1}_M(x, t) |\nabla u_M(x, t)|| \nabla \eta(x)| \, dx \, dt
\leq C \left( \|U_0\|_{L^1(\mathbb{R}^n)} \times \|U^{m-1}\|_{L^1(\mathbb{R}^n \times (0, 1))}, \|\nabla \varphi\|_{L^\infty} \right) t \quad \forall 0 < t < 1
$$

(2.15)

$$
\Rightarrow \int_{\text{supp } U(t)} u(t) \eta(t) \, dx - \int_{\text{supp } U(0)} u_0 \eta(0) \, dx
\leq C \left( \|U_0\|_{L^1(\mathbb{R}^n)} \times \|U^{m-1}\|_{L^1(\mathbb{R}^n \times (0, 1))}, \|\nabla \varphi\|_{L^\infty} \right) t \quad \forall 0 < t < 1.
$$

(2.16)

Letting $t \to 0$ in (2.16), the claim follows. Hence by (2.13), (2.14) and Lemma 2.4, $u$ is a weak solution of $\text{(PME)}$ which satisfies (1.4) and the lemma follows.

\section*{2.3 Equivalence properties on $u$ and $U$}

Since the equations satisfied by $u$ and $U$ have the same diffusion coefficients $U^{m-1}$, it is natural to expect that the solutions of $\text{(PME)}$ and $\text{(PML)}$ have many things in common. By an argument similar to the proof of 9.15 of [Va1], we also have an important conservation.

\textbf{Lemma 2.6.} For the solution $U$ of $\text{(PME)}$ which is given by Lemma 2.7 let $u$ be a weak solution of $\text{(PME)}$. Then, for every $t > 0$ we have

$$
\int_{\mathbb{R}^n} u(x, t) \, dx = \int_{\mathbb{R}^n} u_0 \, dx.
$$

\textit{Proof.} Let $\{\xi_l(x)\}_{l=1}^\infty \subset C^\infty(\mathbb{R}^n)$ be a sequence of functions such that $\xi_l(x) = 1$ for $|x| \leq l - 1$, $\xi_l(x) = 0$ for $|x| \geq l$ and $0 < \xi_l < 1$ for $l - 1 < |x| < l$. Multiplying the first equation in $\text{(PME)}$ by $\xi_l$ and integrating by parts,

$$
\int_{\mathbb{R}^n} u(x, t) \xi_l(x) \, dx - \int_{\mathbb{R}^n} u_0(x, t) \xi_l(x) \, dx = \int_0^t \int_{\mathbb{R}^n} (u)_t \xi_l \, dx \, d\tau
$$

$$
= \int_0^t \int_{\mathbb{R}^n} U^{m-1}(x, \tau) (\nabla u(x, \tau) \cdot \nabla \xi_l(x)) \, dx \, d\tau.
$$
Then by Lemma 2.4,
\[ \left| \int_{\mathbb{R}^n} u(x, t) \xi(t) \, dx - \int_{\mathbb{R}^n} u_0(x, t) \xi(t) \, dx \right| \leq ||\nabla \xi||_E \left( \int_0^t \int_{B(r \cdot t \cdot t)} |U^{m-1} \nabla u|^2 \, dx \, dt \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } l \rightarrow \infty \] (2.17)
and the lemma follows. \(\square\)

On any compact subset of the region where \(U > 0\), the solution \(u\) of \((\text{PME}_n)\) satisfies non-degenerate parabolic equation. By standard theory for non-degenerate parabolic equation \([\text{LSU}]\), we can get the following lemma.

**Lemma 2.7.** Let \(m > 1\) and \(t_0 \geq 0\). Let \(U\) be the solution of \((\text{PME})\) which is given by Lemma 2.7. Suppose that \(u \geq 0\) satisfies
\[ u_t = \nabla \left( U^{m-1} \nabla u \right) \quad \text{in the distribution sense in } \mathbb{R}^n \times (t_0, \infty) \] (2.18)
and (2.5). Then
\[ \text{supp } U(t) = \text{supp } u(t) \quad \forall t > t_0. \] (2.19)

**Proof.** By (2.5), we first have
\[ \text{supp } u(t) \subset \text{supp } U(t) \quad \forall t \geq t_0. \]
We now suppose that (2.19) fails for some \(t_1 > t_0\). Then, there exists a point \(x_0 \in \partial \text{supp } u(t_1)\) such that
\[ B_{2r}(x_0) \subset \text{supp } U(s) \quad \forall t_1 - 2\epsilon_1 \leq s \leq t_1 \] (2.20)
for sufficiently small \(r > 0\) and \(0 < \epsilon_1 < \frac{1}{2} (t_1 - t_0)\). By (2.20), the diffusion coefficients of (2.18) is uniformly parabolic in \(B_{2r}(x_0) \times [t_1 - 2\epsilon_1, t_1]\). Thus by standard theory for non-degenerate parabolic equation \([\text{LSU}]\), the solution \(u\) is continuous on \(B_r(x_0) \times [t_1 - \epsilon_1, t_1]\). This implies that
\[ u(\cdot, t) \neq 0 \quad \text{on } B_r(x_0) \quad \forall t \in [t_1 - \epsilon_1, t_1] \] (2.21)
for sufficiently small \(\epsilon_1 > 0\).

For \(0 < \tau < \epsilon_1\), let \(v_{0, \tau}(x) = u_t(x, t_1 - \tau) \chi_{B_r(x_0)}\). Then by (2.20), there exists an unique solution \(v^\tau\) of
\[ \begin{cases} 
  v_t(x, t) = \nabla \left( U^{m-1} (x, t + t_1 - \tau) \nabla v(x, t) \right) & \text{in } B_r(x_0) \times (0, \tau) \\
  v(x, 0) = v_{0, \tau}(x) & \text{in } B_r(x_0). 
\end{cases} \]
In addition, by (2.21) and standard theory for non-degenerate parabolic equation \([\text{LSU}]\), there exists a constant \(c_1 > 0\) such that
\[ v^\tau(x, \tau) \geq c_1 \quad \forall x \in B_{2r}(x_0). \] (2.22)
Since \(u(x, t + t_1 - \tau)\) is also a solution with initial data \(u(x, t_1 - \tau)\) which is bigger than \(v_{0, \tau}(x)\) in \(B_r(x_0)\), by (2.22) and the comparison principle we have
\[ u(x_0, t_1) \geq v^\tau(x_0, \tau) \geq c_1 > 0. \]
This contradicts the fact that \(u(x_0, t_1) = 0\). Hence (2.19) holds for all \(t \geq t_0\) and the lemma follows. \(\square\)
3 Local Continuity

Under the Assumption I, this section will be devoted to prove the local continuity of solution $u$ of \((PME_0)\) which satisfies (1.4). We start by stating well-known result, Sobolev-type inequality.

**Lemma 3.1** (cf. Lemma 3.1 of \([KL1]\)). Let $\eta(x,t)$ be a cut-off function compactly supported in $B_r$ and let $u$ be a function defined in $\mathbb{R}^n \times (t_1, t_2)$ for any $t_2 > t_1 > 0$. Then $u$ satisfies the following Sobolev inequalities:

$$
\|\eta u\|_{L^{2n/(2n-\alpha)}(\mathbb{R}^n)} \leq C \|\nabla(\eta u)\|_{L^2(\mathbb{R}^n)}
$$

(3.1)

and

$$
\|\eta u\|^2_{L^2(t_1, t_2; L^2(\mathbb{R}^n))} \leq C \left( \sup_{t_1 \leq \tau \leq t_2} \|\eta u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla(\eta u)\|^2_{L^2(t_1, t_2; L^2(\mathbb{R}^n))} \right) \|\eta u > 0\|^{\frac{2}{n-2}}
$$

(3.2)

for some $C > 0$.

From now on, we are going to focus on oscillation argument. To apply it to our case, we use a modification of the technique introduced in \([DI]\), \([KL1]\), \([HU]\).

Applying translation in (1.5), we may assume that $(x_0, t_0) = (0, 0)$. Set

$$
\mu^+ = \text{ess sup}_{Q(R, R^{2-\epsilon})} u, \quad \mu^- = \text{ess inf}_{Q(R, R^{2-\epsilon})} u, \quad \omega = \text{osc}_{Q(R, R^{2-\epsilon})} u = \mu^+ - \mu^-.
$$

By (1.6), the equation in \((PME_0)\) is non-degenerate on the region where $u > 0$. Thus if $\mu^- > 0$, then the equation $u_t = \nabla(U^{m-1}\nabla u)$ is uniformly parabolic in $Q(R, R^{2-\epsilon})$. By standard regularity theory for the parabolic equation \([LSU]\), Hölder estimates follows. Hence from now on, we assume that $\mu^- = 0$.

Construct the cylinder

$$
Q\left(R, \theta_0^{-\alpha_0} R^2\right) = B_R \times (-\theta_0^{-\alpha_0} R^2, 0) \quad \left(\theta_0 = \frac{\omega}{4}, \ \alpha_0 = \beta(m-1)\right)
$$

(3.3)

where $\beta$ is given by (1.6). If $U$ is uniformly parabolic, then the constant $\beta$ is zero. Thus the scaled parabolic cylinder $Q\left(R, \theta_0^{-\alpha_0} R^2\right)$ is equivalent to the standard $Q(R, R^2)$ with homogeneous of degree one. Therefore De Giorgi and Moser’s technique \([DG]\), \([Mo]\) on regularity theory for uniformly elliptic and parabolic PDE’s is enough to show the local continuity of solution $u$ of \((PME_0)\). Otherwise, $\theta_0^{\alpha_0}$ depends on the size of oscillation $\omega$. Thus the solution of \((PME_0)\) diffuses in a time scale determined by uniform constants $\lambda$, $\Lambda$ and the solution itself. Therefore we will use the intrinsic scaling technique to overcome the difficulties on local continuity stemmed from the relation between $u$ and $U$.

We will assume that the radius $0 < R < R_0$ is sufficiently small that

$$
\theta_0^{\alpha_0} > R^\epsilon.
$$

(3.4)

By (3.3) and (3.4),

$$
Q\left(R, \theta_0^{-\alpha_0} R^2\right) \subset Q\left(2R, R^{2-\epsilon}\right) \quad \text{and} \quad \text{osc}_{Q(R, \theta_0^{-\alpha_0} R^2)} u \leq \omega.
$$

To take care of the regularity problem in $u_t$, we introduce the Lebesgue-Steklov average $u_h$ of the weak solution $u$, for $h > 0$:

$$
u_h(\cdot, t) = \frac{1}{h} \int_{t-h}^{t+h} u(\cdot, \tau) d\tau.$$
$u_h$ is well-defined and it converges to $u$ as $h \to 0$ in $L^p$ for all $p \geq 1$. In addition, it is differentiable in time for all $h > 0$ and its derivative is 

$$u(t + h) - u(t)$$

with $h \to 0$.

Fix $t \in (0, T)$ and let $h$ be a small positive number such that $0 < t < t + h < T$. Then we can get the following formulation which is equivalent to (1.10)

$$
\int_{\mathcal{X}} \left[ (u_h)_t \varphi + \left( U^{m-1} \nabla u \right)_h \nabla \varphi \right] \, dx = 0, \quad \forall 0 < t < T - h. \tag{3.5}
$$

### 3.1 The First Alternative

We now start by stating the first alternative.

**Lemma 3.2.** There exists a positive number $\rho_0$ depending on $\Lambda$ and $\omega$ such that if

$$\left| \left\{ (x, t) \in Q\left(R, \theta_0^{-\alpha_0} R^2 \right) : u(x, t) < \frac{\omega}{2} \right\} \right| \leq \rho_0 \left| Q\left(R, \theta_0^{-\alpha_0} R^2 \right) \right|$$

then,

$$u(x, t) > \frac{\omega}{4} \quad \text{for all } (x, t) \in Q\left(\frac{3}{2}, \theta_0^{-\alpha_0} \left(\frac{3}{4}\right)^2 \right).$$

**Proof.** We will use a modification of the proofs of proposition 3.1 of [HU] and Lemma 3.5 of [KL1] to prove the lemma. For $i \in \mathbb{N}$, we set

$$R_i = \frac{R}{2} + \frac{R}{2^i} \quad \text{and} \quad l_i = \mu_\omega + \left( \frac{\omega}{4} + \frac{\omega}{2^{i+1}} \right).$$

Consider a cut-off function $\eta_i(x, t) \in C^\infty (\mathbb{R}^n \times \mathbb{R})$ such that

$$
\begin{aligned}
0 \leq \eta_i & \leq 1 & \text{in } Q\left(R_i, \theta_0^{-\alpha_0} R_i^2 \right) \\
\eta_i = 1 & \text{in } Q\left(R_{i+1}, \theta_0^{-\alpha_0} R_{i+1}^2 \right) \\
\eta_i = 0 & \text{on the parabolic boundary of } Q\left(R_i, \theta_0^{-\alpha_0} R_i^2 \right) \\
|\nabla \eta_i| & \leq \frac{1}{R_i}, \quad |(\eta_i)| \leq \frac{2^{(i+1)\alpha_0}}{R_i^2} & \text{in } Q\left(R_i, \theta_0^{-\alpha_0} R_i^2 \right)
\end{aligned}
$$

In the weak formulation (3.5), we take $\varphi = (u_h - l_i)_- \eta_i^2$ and integrate over $(-\theta_0^{-\alpha_0} R_i^2, t)$ for $t \in (-\theta_0^{-\alpha_0} R_i^2, 0)$. Then

$$
\int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} (u_h)_t \left[ (u_h - l_i)_- \eta_i^2 \right] \, dx \, d\tau + \int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} \left( U^{m-1} \nabla u \right)_h \nabla \left[ (u_h - l_i)_- \eta_i^2 \right] \, dx \, d\tau = 0. \tag{3.7}
$$

On the first integral of (3.7), we have

$$
\int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} (u_h)_t \left[ (u_h - l_i)_- \eta_i^2 \right] \, dx \, d\tau
$$

$$
= \frac{1}{2} \int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} \left[ (u_h - l_i)_- \eta_i^2 \right] \, dx \, d\tau - \int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} (u_h - l_i)_-^2 \eta_i (\eta_i)_t \, dx \, d\tau
$$

$$
\geq \frac{1}{2} \int_{B_{l_i} \times [t]} (u_h - l_i)_-^2 \eta_i^2 \, dx - \frac{(\omega/2)^2 \, 2^{(i+1)\alpha_0}}{R_i^2} \int_{-\theta_0^{-\alpha_0} R_i^2}^t \int_{B_{R_i}} \chi_{[u_h \leq l_i]} \, dx \, d\tau. \tag{3.8}
$$
where \( u_\omega = \max \left( u, \frac{\omega}{4} \right) \). Next by Young’s inequality,

\[
\int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} (U^{m-1} \nabla u)_h \nabla \left[ (u_h - l) \eta_i^2 \right] \, dx \, d\tau \\
\to \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} U^{m-1} \nabla u \nabla \left[ (u - l) \eta_i^2 \right] \, dx \, d\tau \quad \text{as} \quad h \to 0
\]

\[
= \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} U^{m-1} \left| \nabla (u - l) \right|^2 \eta_i^2 \, dx \, d\tau \\
+ \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} U^{m-1} (u - l) \eta_i \left[ \nabla (u - l) \cdot \nabla \eta_i \right] \, dx \, d\tau
\]

\[
\geq \frac{1}{2} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \left| \nabla (u - l) \right|^2 \eta_i^2 \, dx \, d\tau
\]

\[
\geq \frac{\alpha \theta_0^{\alpha_0}}{2} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \left| \nabla (u - l) \right|^2 \eta_i^2 \, dx \, d\tau
\]

\[
\geq \frac{\alpha \theta_0^{\alpha_0}}{4} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \left| \nabla (u - l) \right|^2 \eta_i^2 \, dx \, d\tau
\]

Letting \( h \to 0 \) in (3.7), by (3.8) and (3.9) we get

\[
\sup_{-\theta_0^{-\alpha_0} R_i < t < 0} \int_{B_{R_i} \times [t]} (u_\omega - l) \eta_i^2 \, dx + \alpha \theta_0^{\alpha_0} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \left| \nabla (u_\omega - l) \eta_i \right|^2 \, dx \, d\tau \leq \frac{2^{2i+3} \left( 2 + \lambda \right) \theta_0^{\alpha_0} \left( \frac{\omega}{2} \right)^2}{R_i^2} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \chi_{\left[ u_\omega \leq l \right]} \, dx \, d\tau + \frac{2^{2i+4} \lambda^{m-1} \left( \frac{\omega}{2} \right)^2}{R_i^2} \int_{-\theta_0^{-\alpha_0} R_i}^0 \int_{B_{R_i}} \chi_{\left[ u_\omega \leq l \right]} \, dx \, d\tau.
\]

To control the quantity \( \theta_0^{\alpha_0} \), we consider the change of variables

\[
z = \theta_0^{\alpha_0} t
\]

and set the new functions

\[
\overline{u}_\omega (\cdot, z) = u_\omega (\cdot, \theta_0^{-\alpha_0} z) \quad \text{and} \quad \overline{\eta}_i (\cdot, z) = \eta_i (\cdot, \theta_0^{-\alpha_0} z).
\]

Then, by (3.10)

\[
\sup_{-R_i^2 < t < 0} \int_{B_{R_i} \times [t]} (\overline{u}_\omega - l) \overline{\eta}_i^2 \, dx + \int_{-R_i^2}^0 \int_{B_{R_i}} \left| \nabla [(\overline{u}_\omega - l) \overline{\eta}_i] \right|^2 \, dx \, dz \leq \frac{2^{2i+4} \left( 1 + \frac{1}{\lambda} \right) \left( \frac{\lambda^{m-1} \theta_0^{\alpha_0}}{2} \right) \left( \frac{2 + \lambda}{2} \right)^2}{R_i^2} A_i.
\]
where
\[ A_i = \int_{-R_i^2}^{0} \int_{B_{R_i}} X[I_{\omega \leq l}] \, dx \, dz. \]

By (3.11) and Lemma 3.1,
\[ \|\langle \mathbf{u}_{\omega} - l_i \rangle \eta_i \|_{L^2(Q(R_i, R_i^2))} \leq C \left( \Lambda, \theta_0^0 \right) \left( \omega \right)^2 2^{2(i+1)} R_i^{-2} A_i^{1+\frac{2}{n+2}}. \]  

(3.12)

Note that
\[ \int_{Q(R_i, R_i^2)} \|\langle \mathbf{u}_{\omega} - l_i \rangle \eta_i \|^2 \, dx \, dz \geq \left( l_i + 1 - l_i \right)^2 \int_{-R_i^2}^{0} \|x \in B_{R_i+1} : \mathbf{u}_{\omega}(x, z) < l_{i+1} \| \, dz \]
\[ = \left( \frac{\omega}{2^{n+2}} \right)^2 A_{i+1}. \]  

(3.13)

By (3.12) and (3.13),
\[ A_{i+1} \leq C \left( \Lambda, \theta_0^0 \right) 2^{2(i+1)} R_i^{-2} A_i^{1+\frac{2}{n+2}}. \]  

(3.14)

Let
\[ X_i = \frac{A_i}{Q(R_i, R_i^2)}. \]

Then by (3.14),
\[ X_{i+1} \leq C16^{i} X_i^{1+\frac{2}{n+2}}. \]

for some constant \( C = C \left( \Lambda, \theta_0^0 \right) > 0 \). If we take the constant \( \rho_0 > 0 \) in (3.6) sufficiently small that
\[ X_0 \leq C^{-\frac{n+2}{2}} 2^{-2(n+2)^2} \]
holds, then
\[ X_i \leq C^{-\frac{n+2}{2}} 2^{-\frac{(n+2)(n+2)}{2}} \]
and the lemma follows. \( \square \)

**Remark 3.3.** If \( U \) is equivalent to \( u^\beta \) in \( \mathbb{R}^n \times (0, \infty) \), i.e., there exists some constants \( 0 < c \leq C < \infty \) such that
\[ cu^\beta \leq U \leq Cu^\beta \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \]
then the constant \( \rho_0 \) in (3.6) is independent of \( U \) and \( \omega \).

### 3.2 The Second Alternative

Suppose that the assumption of Lemma 3.2 does not hold, i.e., for every sub-cylinder \( Q(R_i, \theta_0^{-\alpha}) \)
\[ \left| \left\{ (x, t) \in Q(R_i, \theta_0^{-\alpha})^2 : u(x, t) < \frac{\omega}{2} \right\} \right| > \rho_0 \left| Q(R_i, \theta_0^{-\alpha}) \right|. \]  

(3.15)

Then
\[ \frac{\omega}{2} \leq \mu^+ - \frac{\omega}{2}. \]  

(3.16)
Thus, by (3.15) and (3.16)
\[
\left\{ (x, t) \in Q \left( R, \theta_0^{-\alpha_0} R^2 \right) : u(x, t) > \mu^+ - \frac{\omega}{2} \right\} \leq (1 - \rho_0) \left| Q \left( R, \theta_0^{-\alpha_0} R^2 \right) \right|
\]
is valid for all cylinders
\[
Q \left( R, \theta_0^{-\alpha_0} R^2 \right) \subset Q \left( R, R^{2-\epsilon} \right).
\]
By an argument similar to the Lemma 4.2 of [KL2], we have the following lemma

**Lemma 3.4.** If (3.6) is violated, then there exists a time level
\[
t^* \in \left[ -\theta_0^{-\alpha_0} R^2, -\frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right]
\]
such that
\[
\left\{ x \in B_R : u(x, t^*) > \mu^+ - \frac{\omega}{2} \right\} < \left( 1 - \frac{\rho_0}{1 - \frac{\omega}{2}} \right) |B_R|.
\]

By lemma 3.4, there exists a time \( t^* < 0 \) such that the region in the ball \( B_R \) where \( u(\cdot, t^*) \) is close to its supremum is small. The next lemma shows that this continues for all \( t \geq t^* \).

**Lemma 3.5.** There exists a positive integer \( s_1 > 1 \) such that
\[
\left| \left\{ x \in B_R : u(x, t) > \mu^+ - \frac{\omega}{2} \right\} \right| < \left( 1 - \frac{\rho_0}{1 - \frac{\omega}{2}} \right) |B_R|, \quad \forall t \in [t^*, 0].
\]

**Proof.** We will use a modification of the proof of Lemma 3.7 of [KL1] to prove the lemma. Let
\[
H = \sup_{B_R \times [t^*, 0]} \left( u - \left( \mu^+ - \frac{\omega}{2} \right) \right)_+ \leq \frac{\omega}{2}
\]
and assume that there exists a constant \( 1 < s_2 \in \mathbb{N} \) such that
\[
0 < \frac{\omega}{2s_2+1} < H.
\]
If there’s no such integer \( s_2 \), (3.17) holds for any \( s_1 > 1 \) and the lemma follows.

We now introduce the logarithmic function which appears in Section 2 of [Di] by
\[
\Psi(H, (u - k)_+, c) = \max \left\{ 0, \log \left( \frac{H}{H - (u - k)_+ + c} \right) \right\}
\]
for \( k = \mu^+ - \frac{\omega}{2} \) and \( c = \frac{\omega}{2s_2+1} \). Let \( \psi(u) = \Psi(H, (u - k)_+, c) \) for simplicity. Then \( \varphi \) satisfies
\[
\psi \leq s_2 \log 2, \quad 0 \leq \psi' \leq \frac{2s_2+1}{\omega u} \quad \text{and} \quad \psi'' = (\psi')^2 \geq 0.
\]
As the test function in (3.5), we take
\[
\varphi = \left( \psi^2 (u_h) \right)' \xi^2
\]
where \( u_h \) is the Lebesgue-Steklov average of \( u \) and \( \xi(x) \geq 0 \) is a smooth cut-off function such that
\[
\xi = 1 \quad \text{in} \ B_{(1-\gamma)R}, \quad \xi = 0 \quad \text{on} \ \partial B_R \quad \text{and} \quad |\nabla \xi| \leq \frac{C}{\gamma R}
\]
for some constants $0 < \nu < 1$ and $C > 0$. Then integrating (3.5) over $(t^*, t)$ for all $t \in (t^*, 0)$, we have
\[
\int_{t^*}^{t} \int_{B_R} \left( \psi^2 (u_h) \xi^2 \right) dxd\tau + \int_{t^*}^{t} \int_{B_R} \left( U^{m-1} \nabla u \right)_h \cdot \nabla \left( \left( \psi^2 (u_h) \right)' \xi^2 \right) dxd\tau = 0. \tag{3.20}
\]
On the first integral of (3.20), we have
\[
\int_{t^*}^{t} \int_{B_R} \left( \psi^2 (u_h) \xi^2 \right) dxd\tau = \int_{B_R \times [t^*, t]} \psi^2 (u_h) \xi^2 dx - \int_{B_R \times [t^*, t]} \psi^2 (u_h) \xi^2 dx \rightarrow \int_{B_R \times [t^*, t]} \psi^2 (u) \xi^2 dx - \int_{B_R \times [t^*, t]} \psi^2 (u) \xi^2 dx \quad \text{as } h \to 0. \tag{3.21}
\]
On the second integral of (3.20), by Young’s inequality
\[
\int_{t^*}^{t} \int_{B_R} \left( U^{m-1} \nabla u \right)_h \cdot \nabla \left( \left( \psi^2 (u_h) \right)' \xi^2 \right) dxd\tau \rightarrow \int_{t^*}^{t} \int_{B_R} U^{m-1} \nabla u \cdot \nabla \left( \left( \psi^2 (u) \right)' \xi^2 \right) dxd\tau \quad \text{as } h \to 0
\]
\[= 2 \int_{t^*}^{t} \int_{B_R} U^{m-1} (1 + \psi) (\psi')^2 \xi^2 \left| \nabla u \right|^2 dxd\tau + 2 \int_{t^*}^{t} \int_{B_R} U^{m-1} \psi \xi^2 \nabla \xi \cdot \nabla u dxd\tau \] \[\geq -2 \int_{t^*}^{t} \int_{B_R} U^{m-1} \psi \left| \nabla \xi \right|^2 dxd\tau. \tag{3.22}
\]
By (3.18), (3.19), (3.21), (3.22) and Lemma 3.4
\[
\int_{B_R \times [t^*, t]} \psi^2 (u) \xi^2 dx \leq s_2^2 (\log 2)^2 \left( \frac{1 - \rho_0}{1 - \frac{\rho_0}{2}} \right) + \frac{2C \Lambda^{m-1} s_2 \log 2}{\nu^2 R^2} \left( -t^* \right) |B_R| \] \[\leq s_2^2 (\log 2)^2 \left( \frac{1 - \rho_0}{1 - \frac{\rho_0}{2}} \right) + \frac{2C \Lambda^{m-1} s_2 \log 2}{\nu^2 \theta_0^{n_0}} |B_R|. \tag{3.23}
\]
holds for all $t \in (t^*, 0)$ with some constant $C > 0$. Let
\[S = \left\{ x \in B_{(1-\nu)R} : u(x, t) > \mu^+ - \frac{\omega_\mu}{2s_2+1} \right\}.
\]
Then the left hand side of (3.23) is bounded from below by
\[
\int_{B_R \times [t^*, t]} \psi^2 (u) \xi^2 dx \geq \int_{S} \psi^2 (u) \xi^2 dx \geq (s_2 - 1)^2 (\log 2)^2 |S| \quad \forall t \in (t^*, 0). \tag{3.24}
\]
Observe that
\[
\left| \left\{ x \in B_R : u(x, t) > \mu^+ - \frac{\omega_\mu}{2s_2+1} \right\} \right| \leq |S| + NV |B_R|. \tag{3.25}
\]
Thus by (3.23), (3.24) and (3.25)
\[
\left| \left\{ x \in B_R : u(x, t) > \mu^+ - \frac{\omega_\mu}{2s_2+1} \right\} \right| \leq \left( \frac{s_2}{s_2 - 1} \right)^2 \left( \frac{1 - \rho_0}{1 - \frac{\rho_0}{2}} \right) + \frac{2C \Lambda^{m-1} s_2}{\nu^2 \theta_0^{n_0} (s_2 - 1)^2 \log 2} |B_R|.
\]
To complete the proof, we choose $\nu$ so small that $\nu \leq \frac{3}{8p_0^2}$ and then $s_2$ so large that
\[
\left( \frac{s_2}{s_2 - 1} \right)^2 \leq \left( 1 - \frac{1}{2 \rho} \right) (1 + \rho) \quad \text{and} \quad \frac{2C \Lambda^{m-1} s_2}{\nu^2 \theta_0^{n_0} (s_2 - 1)^2 \log 2} \leq \frac{3}{8p_0^2}.
\]
Then (3.17) holds for $s_1 = s_2 + 1$ and the lemma follows.
Since \( t^* \in \left[ -\theta_0^{-\alpha_0} R^2, -\frac{\omega_0}{2} \theta_0^{-\alpha_0} R^2 \right] \), the previous lemma implies the following result.

**Corollary 3.6.** There exists a positive integer \( s_1 > s_0 \) such that for all \( t \in \left( -\frac{\omega_0}{2} \theta_0^{-\alpha_0} R^2, 0 \right) \)

\[
\left| \left\{ x \in B_R : u(x, t) > \frac{\mu^+ - \frac{\omega}{2 s_1}}{2} \right\} \right| < \left( 1 - \left( \frac{\rho_0}{2} \right)^2 \right) |B_R|.
\]  

(3.26)

To make the region where \( u \) is close to its supremum to be arbitrary small, we review the following lemma.

**Lemma 3.7 (De Giorgi [De]).** If \( f \in W^{1,1}(B_r) \) \((B_r \subset \mathbb{R}^n)\) and \( l, k \in \mathbb{R}, \ k < l \), then

\[
(l - k) \left| \left\{ x \in B_r : f(x) > l \right\} \right| \leq \frac{C_r^{\rho_1+1}}{\|x \in B_r : f(x) < k\|} \int_{k < f < l} |\nabla f| \, dx,
\]

where \( C \) depends only on \( n \).

By Corollary 3.5 and Lemma 3.7, we have the following lemma.

**Lemma 3.8.** If \((3.6)\) is isolated, for every \( \nu_* \in (0, 1) \) there exists a natural number \( s^* > s_1 > 1 \) depending on \( \Lambda \) and \( \omega \) such that

\[
\left| \left\{ (x, t) \in Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) : u(x, t) > \frac{\mu^+ - \frac{\omega}{2 s_1}}{2} \right\} \right| \leq \nu_* \left| Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) \right|.
\]

(3.27)

**Proof.** Since the proof of the lemma is the almost same as that of the proof of Lemma 4.5 of [HU], we will give sketch its proof here. Let \( k = \mu^+ - \frac{\omega}{2 s_1} \) for \( s \geq s_1 \). Take \( \varphi = (u_\tau - k)_+ \xi^2 \) in the weak formula \((3.5)\) where \( \eta(x, t) \in C^{\infty} \left( Q \left( 2R, \rho_0 \theta_0^{-\alpha_0} R^2 \right) \right) \) is a cut-off function such that

\[
\begin{align*}
0 &\leq \eta \leq 1 & \text{in } Q \left( 2R, \rho_0 \theta_0^{-\alpha_0} R^2 \right) \\
\eta &\equiv 1 & \text{in } Q \left( R, \rho_0 \theta_0^{-\alpha_0} R^2 \right) \\
\eta &\equiv 0 & \text{on the parabolic boundary of } Q \left( 2R, \rho_0 \theta_0^{-\alpha_0} R^2 \right) \\
|\nabla \eta| &\leq \frac{\theta_0 \rho_0}{2 \rho_0 R} & \text{in } Q \left( 2R, \rho_0 \theta_0^{-\alpha_0} R^2 \right)
\end{align*}
\]

Integrating over \( (-\rho_0 \theta_0^{-\alpha_0} R^2, t) \) for \( t \in \left( -\rho_0 \theta_0^{-\alpha_0} R^2, 0 \right) \) and taking the limit as \( h \to 0 \) in \((3.3)\), we have

\[
\frac{1}{2} \int_{-\rho_0 \theta_0^{-\alpha_0} R^2}^t \frac{d}{d\tau} \left( \int_{B_{2R}} \eta_u \, dx \right) d\tau - \int_{-\rho_0 \theta_0^{-\alpha_0} R^2}^t \int_{B_{2R}} \eta_u \, dx d\tau
\]

\[
+ \int_{-\rho_0 \theta_0^{-\alpha_0} R^2}^t \int_{B_{2R}} \eta_u \, dx d\tau
\]

\[
\Rightarrow \frac{\Lambda \theta_0}{2} \int_{-\rho_0 \theta_0^{-\alpha_0} R^2}^t \int_{B_{2R}} |\nabla (u - k)_+|^2 \eta^2 \, dx d\tau
\]

\[
\leq \left( \frac{\omega}{2 s^2} \right)^2 \frac{2 \theta_0}{R^2} \left( \frac{\theta_0}{\rho_0} + \Lambda^{m-1} \right)^2 \left| Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) \right|
\]

\[
\Rightarrow \int_{-\rho_0 \theta_0^{-\alpha_0} R^2}^t \int_{B_{2R}} |\nabla (u - k)_+|^2 \eta^2 \, dx d\tau
\]

\[
\leq C (\lambda, \Lambda, \omega) \left( \frac{\omega}{2 s^2} \right)^2 \frac{1}{R^2} \left| Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) \right|
\]

(3.28)
since \( \rho_0 \) depends on \( \lambda, \Lambda \) and \( \omega \) by Lemma 3.2.

For any \( t \in \left(-\frac{\omega}{2}\theta_0^{-\alpha_0}R^2, 0\right) \), let

\[
A_s(t) = \left\{ x \in B_R : u(x, t) > \mu^+ - \frac{\omega}{2s} \right\}
\]

and

\[
A_s = \int_{-\frac{\omega}{2}\theta_0^{-\alpha_0}R^2}^{0} |A_s(t)| \, dt.
\]

Applying Lemma 3.7 over the ball \( B_R \) and \( s \) thus if we choose \( \rho \) since Remark 3.9.

If \( U \) is equivalent to \( u \) then (3.27) holds and the lemma follows. □

By Corollary 3.6, Hölder inequality and (3.28) we have that

\[
\left( \frac{\omega}{2s+1} \right) |A_{s+1}(t)| \leq \frac{C}{\rho_0^2} R \int_{|k|<\omega} |\nabla u| \, dx
\]

\[
\Rightarrow \left( \frac{\omega}{2s+1} \right) A_{s+1} \leq \frac{C}{\rho_0^2} R \left( \int_{0}^{0} |\nabla u(k)|^2 \int_{B_R} |\nabla u - k|^2 \, dx \, dt \right)^{\frac{1}{2}} |A_s \setminus A_{s+1}|^{\frac{1}{2}}
\]

\[
\Rightarrow A_{s+1}^2 \leq C(\lambda, \Lambda, \omega) \left| Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) \right| |A_s \setminus A_{s+1}| \forall s = s_1, \cdots, s^*-1
\]

\[
\Rightarrow (s^*-s_1) A_{s_1}^2 \leq \sum_{s=s_1}^{s^*-1} A_{s+1}^2 \leq C(\lambda, \Lambda, \omega) \left| Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right) \right| |A_{s_1}| \setminus A_s|
\]

\[
\Rightarrow A_{s_1}^2 \leq \frac{C(\lambda, \Lambda, \omega)}{s^*-s_1} |Q \left( R, \frac{\rho_0}{2} \theta_0^{-\alpha_0} R^2 \right)|^2.
\]

Thus if we choose \( s^* \in \mathbb{N} \) sufficiently large that

\[
\frac{C(\lambda, \Lambda, \omega)}{s^*-s_1} \leq \nu^2,
\]

then (3.27) holds and the lemma follows. □

**Remark 3.9.** If \( U \) is equivalent to \( u^\beta \) in \( \mathbb{R}^n \times (0, \infty) \), i.e., there exists some constants \( 0 < c \leq C < \infty \) such that

\[
cu^\beta \leq U \leq Cu^\beta \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),
\]

then the constant \( s^* \) is independent of \( U \) and \( \omega \).

By Lemma 3.8 we have a similar assumption to the one in Lemma 3.2 for sufficiently small number \( \nu_1 > 0 \). Therefore, by an argument similar to the proof of Lemma 3.2 we can have the following result.

**Lemma 3.10.** The number \( \nu_1 \in (0,1) \) can be chosen such that

\[
u(x, t) \leq \mu^+ - \frac{\omega}{2s+1} \quad \text{a.e. on} \quad Q \left( \frac{\nu_1}{\nu}, \frac{\nu_0}{\nu} \theta_0^{-\alpha_0} \left( \frac{\nu_1}{\nu} \right)^2 \right).
\]

By Lemma 3.2 and Lemma 3.10 we have the following Oscillation Lemma.
Lemma 3.11 (Oscillation Lemma). There exist numbers $\rho_0$, $\sigma_0 \in (0, 1)$ depending on the $\lambda$, $\Lambda$ and $\omega$ such that if
\[
\text{osc}_{Q(R_0, R_0^2)} u = \omega
\]
then
\[
\text{osc}_{Q\left(\frac{R}{2}, \frac{R}{2}^2\right)} u = \sigma_0 \omega.
\]

Proof of Theorem 1.1. By Lemma 3.11, there exists a family of nest and shrinking cylinders $\{Q_n\}_{n=1}^{\infty}$ constructed recursively such that
\[
\text{ess sup}_{Q_n} u \leq \omega_n \quad \text{and} \quad \omega_n \to 0 \quad \text{as} \quad n \to 0.
\]
(3.30)
Thus, the continuity of $u$ follows. □

For the detail of the proof of (3.30), we recommend a reading of the survey paper [DUV].

Remark 3.12. 1. Under the Assumption I, the constant $\sigma_0$ in (3.29) may depend on the oscillation $\omega$.
Thus we can only get the local continuity of $u$ and can’t find the modulus of continuity at this stage.
See [Ur] for the details.

2. Let $\alpha_0$, $\theta_0$, $\sigma_0$ and $\rho_0$ be given by Lemma 3.11. If there exists constants $0 < c < C < \infty$ such that $cu^\beta \leq U \leq Cu^\beta$ in $\mathbb{R}^n \times (0, \infty)$, then the shrinking cylinders and oscillations in (3.30) can be represented by

\[
Q_n = Q\left(R_n, \theta_n^{-\alpha_0} R_n^2\right) \quad \text{and} \quad \omega_n = \sigma_0 \omega_0 \quad \left(R_i = \frac{R}{C^n}, \theta_n = \sigma_0 \theta_0\right)
\]
(3.31)
for some constant $C \geq \sqrt{\frac{\rho_0}{\sigma_0 \alpha_0}}$.

By (3.31) and an argument similar to the proof of Theorem 3.12 of [KLT], we can find the modulus of continuity of solution $u$ (Hölder regularity) when $u^\beta$ and $U$ are equivalent.

Theorem 3.13 (Hölder estimates). Suppose that $U$ is equivalent to $u^\beta$ in $\mathbb{R}^n \times (0, \infty)$, i.e., there exists some constants $0 < c \leq C < \infty$ such that
\[
cu^\beta \leq U \leq Cu^\beta \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),
\]
Then there exists constant $\sigma^* > 1$ and $\alpha \in (0, 1)$ that can be determined only in terms of data, such that
\[
\text{osc}_{Q(r, r^2 \theta_0^\alpha)} u \leq \sigma^* \left(\frac{r}{R}\right)^\alpha \quad (0 < r \leq R).
\]
Here, $\sigma^* = \frac{1}{\sigma_0} > 1$ and $\alpha = -\log_C \sigma_0 \in (0, 1)$.
4 Asymptotic Behaviour

In this section, we will investigate the uniform convergence between the solution of \( P_{ME_0} \) which satisfies (1.4) and Barenblatt profile of porous medium equation. The self-similar Barenblatt solution of the porous medium equation with \( L^1 \)-mass \( M \) is given explicitly by

\[
B_M(x, t) = t^{-a_1} \left( C_M - \frac{k|x|^2}{t^{2a_2}} \right)_{+}^{\frac{1}{m-1}}
\]  
(4.1)

where

\[
a_1 = \frac{n}{(m-1)n + 2}, \quad a_2 = \frac{a_1}{n}, \quad k = \frac{a_1(m-1)}{2mn}.
\]  
(4.2)

Here, the constant \( C_M > 0 \) is related to the \( L^1 \)-mass \( M \) of barenblatt solution. By [Va1], there exists a constant \( c_\star = c_\star(m, n) > 0 \) such that

\[
C_M = (c^* M^{a_3})^{m-1} \left( a_3 = \frac{2}{n} a_1 \right).
\]  
(4.3)

Denote by \( \rho_M(t) \) the radius of the support of Barenblatt solution \( B_M \) at time \( t \), i.e.,

\[
x \in \text{supp} B_M(\cdot, t) \iff |x| < \sqrt{(c^* M^{a_3})^{m-1}} t^{a_2} = \rho_M(t).
\]

Then by an argument similar to the proof of Lemma 3.5 of [KV], we have the following lemma.

**Lemma 4.1.** \( B_M(x, t) > B_M(x, t + \tau) \) in a region \( |x| \leq c(\tau, m, n) \rho_M(t) \) and \( B_M(x, t + \tau) > B_M(x, t) \) for \( c(\tau, m, n) \rho_M(t) < |x| < \rho_M(t + \tau) \). Moreover

\[
c(\tau, m, n) \to c_\# = \sqrt{(m-1) a_1} < 1 \quad \text{as } \tau \to 0.
\]

4.1 Properties of solutions with Barenblatt solution \( B_M \) as diffusion coefficients

For any \( M \geq M_0 > 0 \), let \( w \) be a solution of

\[
w_t = \nabla \cdot \left( B_M^{n-1} \nabla w \right) \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty)
\]  
(4.4)

with initial value \( w_0 \in L^1(\mathbb{R}^n) \) which satisfies

\[
w(x, t) \leq B_M(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).
\]  
(4.5)

and

\[
\int_{\mathbb{R}^n} w(x, t) \, dx = M_0 \quad \forall t \geq 0.
\]  
(4.6)

In the following lemma, we find \( L^\infty \) bounds of solution \( u \).
Lemma 4.2. Let \( w \) be a solution of (4.4) and (4.6). Suppose that
\[
w(x, t) \leq \frac{M_1}{M} \mathcal{B}_M(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty)
\] (4.7)
for any constant \( M_0 < M_1 \). Then there exists a constant \( M_2 \in (M_0, M_1) \) such that
\[
w(x, t) \leq \frac{M_2}{M} \mathcal{B}_M(0, t) = c^* M_2 M^{a_3 - 1} t^{-a_1}, \quad \forall t > 0
\] (4.8)
where constants \( a_1, a_3 \) and \( c^* \) are given by (4.2) and (4.3).

Proof. By an argument similar to the proof of Lemma 2.7
\[
\text{supp } w(t) = \text{supp } \mathcal{B}_M(t) \quad \forall t > 0.
\]
We first show that \( w(\cdot, 1) \) does not touch \( \frac{M_1}{M} \mathcal{B}_M(\cdot, 1) \) from below at any point in \( \text{supp } \mathcal{B}_M(1) \), i.e., for
\[
|\xi| < \sqrt{\frac{(c^* M^{a_3})^{m-1}}{k}} = \rho_M(1).
\]
Suppose that \( w(x, 1) \) touches \( \frac{M_1}{M} \mathcal{B}_M(x, 1) \) at a point \( x_0 \) with \( |x_0| < \rho_M(1) \). By radially symmetry and continuity of \( \mathcal{B}_M \), there exists a constant \( \epsilon_1 > 0 \) such that
\[
E_{\epsilon_1} = \{ x \in \mathbb{R}^n : |x| \leq |x_0| + \epsilon_1 \} \times [1 - \epsilon_1^2, 1] \subset \{ (x, t) \in \mathbb{R}^n \times [0, \infty) : \mathcal{B}_M(x, t) > 0 \}.
\]
On \( E_{\epsilon_1} \), there exists constant \( 0 < c < C < \infty \) such that
\[
c \leq \mathcal{B}_M(x, t) \leq C \quad \forall (x, t) \in E_{\epsilon_1}.
\]
Thus, the equation (4.4) is uniformly parabolic on \( E_{\epsilon_1} \). Therefore the function \( w - \frac{M_1}{M} \mathcal{B}_M \) is the classical solution of (4.4) on \( E_{\epsilon} \) which has its maximum at the point \( (x_0, 1) \) inside of \( E_{\epsilon_1} \) by (4.7). By Strong Maximum Principle,
\[
w(x, 1) \equiv \frac{M_1}{M} \mathcal{B}_M(x, 1) \quad \forall 0 \leq |x| \leq |x_0| + \epsilon_1.
\] (4.9)
By maximal interval argument, (4.9) can be extend to the support of \( \mathcal{B}_M(1) \). Since
\[
\int_{\mathbb{R}^n} \frac{M_1}{M} \mathcal{B}_M(x, 1) \, dx = M_1 \neq M_0 = \int_{\mathbb{R}^n} w(x, 1) \, dx,
\]
the contradiction arises and the claim follows.

By the claim, \( w(x, 1) < \frac{M_1}{M} \mathcal{B}_M(x, 1) \leq \frac{M_1}{M} \mathcal{B}_M(0, 1) = c^* M_1 M^{a_3 - 1} \) for all \( x \in \text{supp } \mathcal{B}_M(1) \). Hence there exists a constant \( M_2 \in (M_0, M_1) \) such that
\[
w(x, 1) \leq c^* M_2 M^{a_3 - 1} \quad \forall x \in \mathbb{R}^n.
\] (4.10)
To prove (4.8), we consider the rescaled function
\[
\tilde{w}(x, t) = T^{a_1} w(T^{a_2} x, T t), \quad (T > 0).
\]
Since
\[ B_M(x, t) = T^{a_1} B_M(T^{a_2} x, T t), \]
the function \( \hat{u} \) is a solution of (4.4) which satisfies (4.6) and (4.7). Then by an argument for (4.10), we have
\[ w(x, T) = \frac{1}{T^{a_1}} \hat{w} \left( \frac{x}{T^{a_1}} T t \right) \leq c^* M^2 M^{a_1} T^{-a_1} \quad \forall x \in \mathbb{R}^n \]
and the lemma follows. \( \square \)

By (4.5) and (4.6), there exists a constant \( M_0 \leq M' \leq M \) such that
\[ w(x, t) \leq \frac{M'}{M} B_M(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty) \tag{4.11} \]
We now consider the infimum of these bounds
\[ \overline{M} = \inf \left\{ M' : w(x, t) \leq \frac{M'}{M} B_M(x, t) \right\}. \tag{4.12} \]
We now are going to prove that \( \overline{M} = M_0 \).

**Theorem 4.3** (Uniqueness). Let \( 0 < M_0 \leq M \). Let \( w \) be non-negative solution of (4.4) which satisfies (4.5) and (4.6). Then
\[ w = \frac{M_0}{M} B_M \quad a.e. \text{ in } \mathbb{R}^n \times (0, \infty). \tag{4.13} \]

**Proof.** We will use a modification of the techniques of Lemma 3.5 of [KV] to prove theorem. By (4.11) and (4.12),
\[ \overline{M} \geq M_0 \quad \text{and} \quad w \leq \frac{M'}{M} B_M \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{4.14} \]
Suppose that \( \overline{M} > M_0 \). By Lemma 4.2 there exists a constant \( \overline{M} \in (M_0, \overline{M}) \) such that
\[ w(x, t) \leq c^* \overline{M} M^{a_1 - 1} T^{-a_1} \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty). \]
Let
\[ W(x, 1) = \min \left\{ c^* \overline{M} M^{a_1 - 1}, \frac{\overline{M}}{M} B_M(x, 1) \right\} \quad \forall x \in \mathbb{R}^n \]
and \( W \) be the solution of (4.4) in \( \mathbb{R}^n \times (1, \infty) \) with initial data \( W(x, 1) \) at time \( t = 1 \). By maximum principle,
\[ w(x, t) \leq W(x, t) \leq \frac{M'}{M} B_M(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [1, \infty). \]
Since \( W(0, 1) = c^* \overline{M} M^{a_1 - 1} \) is strictly less than \( \frac{\overline{M}}{M} B_M(0, 1) = c^* \overline{M} M^{a_1 - 1} \), by an argument similar to the proof of Lemma 4.2 there exists a constant \( t_1 > 1 \) such that
\[ W(x, t_1) < \frac{M'}{M} B_M(x, t_1) \quad \forall |x| < \rho_1(t_1). \tag{4.15} \]
By (4.15), $W(\cdot, t_1)$ and $\overline{\mathcal{B}}_M(\cdot, t_1)$ are strictly separated on the compact subset of $\text{supp} \, \mathcal{B}_M(t_1)$. Hence by Lemma 4.1 there exist constants $\delta > 0$ and $\tau > 0$ small enough that

$$W(x, t_1) < \frac{M}{M} \mathcal{B}_M(x, t_1 + \tau) \quad \forall |x| \leq c \rho_1(t_1) + \delta.$$  \hspace{1cm} (4.16)

On the other hand,

$$W(x, t_1) \leq \frac{M}{M} \mathcal{B}_M(x, t_1) < \overline{\mathcal{B}}_1(x, t_1 + \tau) \quad c \rho_1(t_1) + \delta \leq |x| \leq \rho_1(t_1 + \tau).$$  \hspace{1cm} (4.17)

By (4.16) and (4.16),

$$W(x, t_1) < \frac{M}{M} \mathcal{B}_M(x, t_1 + \tau) \quad \forall |x| \leq \rho_1(t_1 + \tau) \hspace{1cm} \Rightarrow \hspace{1cm} W(x, t_1) \leq \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, t_1 + \tau) \quad \forall x \in \mathbb{R}^n$$  \hspace{1cm} (4.18)

for sufficiently small constant $\epsilon > 0$. By (4.18) and maximum principle,

$$W(x, t) \leq \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, t + \tau) \quad \forall x \in \mathbb{R}^n, t \geq t_1.$$  \hspace{1cm} (4.19)

Since $w \leq W$ for $t \geq 1$, by (4.19)

$$w(x, t) \leq \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, t + \tau) \quad \forall x \in \mathbb{R}^n, t \geq t_1.$$  \hspace{1cm} (4.20)

We now consider the rescaled function

$$W_\theta(x, t) = \frac{1}{\theta^{\alpha_1}} W \left( \frac{x}{\theta^{\alpha_2}}, \frac{t}{\theta} \right).$$  \hspace{1cm} (4.21)

Then, $W_\theta$ is a solution of (4.4) in $\mathbb{R}^n \times (\theta, \infty)$ which satisfies on the initial data

$$W_\theta(x, \theta) = \min \left\{ c^\alpha \overline{M} M^{\alpha_1-1} M^{\alpha_1}, \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, \theta) \right\} \quad \forall x \in \mathbb{R}^n$$

since $\mathcal{B}_M$ is invariant under the rescaling (4.21). Since

$$w(x, t) \leq W_\theta(x, t) \quad \forall x \in \mathbb{R}^n, t \geq \theta t_1,$$

by an argument similar to the proof of (4.20),

$$w(x, t) \leq \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, t + \theta \tau) \quad \forall x \in \mathbb{R}^n, t \geq \theta t_1.$$  \hspace{1cm} (4.22)

Letting $\theta \to 0$ in (4.22),

$$w(x, t) \leq \frac{(M - \epsilon)}{M} \mathcal{B}_M(x, t) \quad \forall x \in \mathbb{R}^n, t > 0.$$  \hspace{1cm} (4.23)

Hence contradiction arises and $\overline{M} = M_0$. By (4.14),

$$0 \leq w(x, t) \leq \frac{M_0}{M} \mathcal{B}_M(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty).$$

Since $w$ has $L^1$ mass $M_0$, (4.13) holds and the theorem follows. \hspace{1cm} $\square$
4.2 Convergence of $U$

By [LV1], it is well known that there exists the uniform convergences between the solution $U$ of \textit{(PME)} which has $L^1$-mass $M$ and Barenblatt profile $B_M$.

**Lemma 4.4** (cf. Theorem 2.8 of [LV1]). Let $U$ be the solution of \textit{(PME)} with initial data $U_0 \in L^1(\mathbb{R}^n)$ compactly supported. Let $M = \int_{\mathbb{R}^n} U_0(x) \, dx$. Then

$$
\lim_{t \to \infty} \| U(\cdot, t) - B_M(\cdot, t) \|_{L^1} = 0
$$

Convergence holds also in uniform norm in the proper scale:

$$
\lim_{t \to \infty} t^{\alpha_1} \| U(\cdot, t) - B_M(\cdot, t) \|_{L^\infty} = 0 \quad \text{uniformly } x \in \mathbb{R}^n.
$$ (4.24)

4.3 Scaling and Uniform estimates

Let $u$, $U$ be solutions of \textit{(PME)} with $L^1$-mass $M_0$, $M$, respectively. Construct the families of functions

$$
u_\lambda(x, t) = \lambda^{a_1} u(\lambda^{a_2} x, \lambda t) \quad \text{and} \quad U_\lambda(x, t) = \lambda^{a_1} U(\lambda^{a_2} x, \lambda t) \quad (\lambda > 0)
$$ (4.25)

where the exponents $a_1$ and $a_2$ are given by (4.2). Then by \textit{(PME)} and (2.5), $u_\lambda$ are solutions of

$$
\begin{cases}
(u_{\lambda})(t) = \nabla \cdot \left( U_{\lambda}^{m-1} \nabla u_{\lambda} \right) & \text{in } \mathbb{R}^n \times (0, \infty) \\
u_{\lambda}(x, 0) = u_{\lambda}(x, 0) & \forall x \in \mathbb{R}^n
\end{cases}
$$ (4.26)

which satisfies

$$
0 \leq u_{\lambda}(x, t) \leq U_{\lambda}(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).
$$ (4.27)

By Lemma 2.2 and Lemma 2.6,

$$
\int_{\mathbb{R}^n} U_\lambda(x, t) \, dx = \int_{\mathbb{R}^n} \lambda^{a_1} U(\lambda^{a_2} x, \lambda t) \, dx = \int_{\mathbb{R}^n} U(y, \lambda t) \, dy = M < \infty \quad \forall \lambda > 0, \ t \geq 0.
$$ (4.28)

and

$$
\int_{\mathbb{R}^n} \nu_\lambda(x, t) \, dx = \int_{\mathbb{R}^n} \lambda^{a_1} u(\lambda^{a_2} x, \lambda t) \, dx = \int_{\mathbb{R}^n} u(y, \lambda t) \, dy = M_0 < \infty \quad \forall \lambda > 0, \ t \geq 0.
$$ (4.29)

Hence the family $\{u_{\lambda}\}_{\lambda \geq 1}$ is uniformly bounded in $L^1(\mathbb{R}^n)$ for all $t > 0$. By (2.1) and (2.5),

$$
\|u_{\lambda}(\cdot, 1)\|_{L^\infty} \leq \|U_{\lambda}(\cdot, 1)\|_{L^\infty} = \lambda^{a_1} \|U(\cdot, \lambda)\|_{L^\infty} \leq \lambda^{a_1} \frac{C \|U_0\|_{L_{\lambda}^{\frac{2a_1}{a_1}}}}{\lambda^{a_1}} = CM^{\frac{2a_1}{a_1}}
$$

which is independent to $\lambda$. Similarly,

$$
\|u_{\lambda}(\cdot, t_0)\|_{L^\infty} \leq CM^{\frac{2a_1}{a_1}} t_0^{-a_1} \quad \forall t_0 > 0.
$$ (4.30)

By (4.29), (4.30) and Interpolation theory,

$$
\|u_{\lambda}(\cdot, t)\|_{L^p} \text{ is equibounded for all } p \in [1, \infty).
$$ (4.31)
By (4.28) and Lemma 4.4 there exists a constant \( \lambda_0 > 0 \) such that for any \( \lambda \geq \lambda_0 \) there exist constants \( 0 < c_{\lambda}, t_1 < 1 \) such that
\[
c_{\lambda} B_M (x, t_1) \leq U_\lambda (x, 0) \quad \forall x \in \mathbb{R}^n, \lambda \geq \lambda_0. \tag{4.32}
\]

Here,
\[
c_{\lambda} \to 1 \quad \text{and} \quad t_1 \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{4.33}
\]

By (4.32) and the maximum principle for porous medium equation, \cite{Va1}, we have
\[
c_{\lambda} B_M (x, t + t_1) \leq U_\lambda (x, t) \quad \forall x \in \mathbb{R}^n, t > 0, \lambda \geq \lambda_0
\Rightarrow c_{\lambda} B_M (x, t_0 + t_1) \leq U_\lambda (x, t_0) \quad \forall x \in \mathbb{R}^n, \lambda \geq \lambda_0 \tag{4.34}
\]
for any \( t_0 > 0 \). Observe that
\[
\text{supp} B_M (x, t_0) \subset \text{supp} B_M (x, t_0 + t_1) \quad \forall \lambda \geq \lambda_0. \tag{4.35}
\]

Since \( B_M \) is continuous in \( \mathbb{R}^n \times (0, \infty) \), by (4.33) and (4.35) there exists a constant \( \lambda_1 (t_0) > \lambda_0 \) such that
\[
c_{\lambda} \geq \frac{3}{4} \quad \text{and} \quad \frac{2}{3} B_M (x, t_0) \leq B_M (x, t_0 + t_1) \quad \forall \lambda \geq \lambda_1. \tag{4.36}
\]

By (4.34) and (4.36),
\[
\frac{1}{2} B_M (x, t_0) \leq U_\lambda (x, t_0) \quad \forall \lambda \geq \lambda_1. \tag{4.37}
\]

By (4.37) and the maximum principle for porous medium equation, \cite{Va1}, we have
\[
\frac{1}{2} B_M (x, t) \leq U_\lambda (x, t) \quad \forall t \geq t_0, \lambda \geq \lambda_1. \tag{4.38}
\]

Multiplying the first equation in (4.26) by \( u_\lambda \) and integrating over \( \mathbb{R}^n \times (t_0, t) \) for all \( t > t_0 \), the we have
\[
\int_{\mathbb{R}^n} u_\lambda^2 (x, t) \, dx + \int_{t_0}^t \int_{\mathbb{R}^n} U_\lambda^{m-1} |\nabla u_\lambda|^2 \, dxd\tau = \int_{\mathbb{R}^n} u_\lambda^2 (x, t_0) \, dx
\Rightarrow \int_{t_0}^t \int_{\mathbb{R}^n} U_\lambda^{m-1} |\nabla u_\lambda|^2 \, dxd\tau \leq C (||u_\lambda (t_0)||_{L^2}) \quad \forall t \geq t_0 > 0
\Rightarrow \int_{t_0}^t \int_{\mathbb{R}^n} B_M^{m-1} |\nabla u_\lambda|^2 \, dxd\tau \leq C (||u_\lambda (t_0)||_{L^2}) \quad \forall t \geq t_0 > 0, \lambda \geq \lambda_1. \tag{4.39}
\]

4.4 Limit function of solution \( u \)

As the first result of the convergence, we prove that there exists an uniform convergence in \( L^p \) between \( u \) and \( \frac{M_0}{M} B_M \).

**Proof of Theorem 4.2** We will use a modification of the proof of Theorem 18.1 of \cite{Va1}. For any \( \lambda > 0 \), let \( u_\lambda, U_\lambda \) be given by (4.25). By (4.30), the family \( \{u_\lambda\} \) is uniformly bounded in \( \mathbb{R}^n \times (t_0, \infty) \) for any \( t_0 > 0 \). Thus \( \{u_\lambda\} \) is relatively compact in \( L^1_{loc} (\mathbb{R}^n \times (0, \infty)) \). Therefore for sequence \( \lambda_n \to 0 \) as \( n \to \infty \), the sequence \( \{u_{\lambda_n}\} \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges
in $L^1_{loc}(\mathbb{R}^n \times (0,\infty))$ to some function $u_\infty$ in $\mathbb{R}^n \times (0,\infty)$ as $n \to \infty$.

Let $0 < t_0 < t_1$ and let $\varphi \in C^\infty_0(\mathbb{R}^n \times (0,\infty))$ be a test function such that

$$\varphi(\cdot,t) = 0 \quad \forall 0 < t < t_0, \ t > t_1.$$ 

Multiplying the first equation in (4.26) by $\varphi \in C^\infty_0(\mathbb{R}^n \times (0,\infty))$, we have

$$\int_0^\infty \int_{\mathbb{R}^n} U^{-m-1}_\lambda \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^n} u_\lambda \varphi_t \, dx \, dt = 0$$ 

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}^n} \delta_{M}^{m-1} \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt$$ 

$$+ \int_0^\infty \int_{\mathbb{R}^n} \left(U^{-m-1}_{\lambda} - \delta_{M}^{m-1}\right) \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^n} u_\lambda \varphi_t \, dx \, dt = 0 \quad (4.40)$$

Let $\lambda_1 > 0$ be given by (4.36) and let $\epsilon > 0$. Then by (4.38),

$$\text{supp} \, \delta_{M}(\cdot, t) \subset \text{supp} \, U_{\lambda}(\cdot, t) \quad \forall t \geq t_0, \ \lambda \geq \lambda_1. \quad (4.41)$$

By Lemma 4.4 there exists a constant $\lambda_2 \geq \lambda_1$ such that

$$|\text{supp} \, U_{\lambda}(t) \setminus \text{supp} \, \delta_{M}(t)| < \epsilon \quad \forall t \in [t_0, t_1] \ \lambda \geq \lambda_2 \quad (4.42)$$

and

$$\left|U^{-m-1}_{\lambda}(x, t) - \delta_{M}^{m-1}(x, t)\right| < \epsilon \quad \forall x \in \mathbb{R}^n, \ t \in [t_0, t_1] \ \lambda \geq \lambda_2. \quad (4.43)$$

Let $E_{\delta_{M}, t_0, t_1} = \{(x, t) \in \mathbb{R}^n \times [t_0, t_1] : \delta_{M}(x, t) > 0\}$

and let $\mathcal{K}$ be a compact subset of $E_{\delta_{M}, t_0, t_1}$ such that

$$|E_{\delta_{M}, t_0, t_1} \setminus \mathcal{K}| < \epsilon. \quad (4.44)$$

By Lemma 4.4, there exists a constant $\lambda_3 > \lambda_2$ such that $\{U_{\lambda}\}_{\lambda \geq \lambda_3}$ is uniformly parabolic in $\mathcal{K}$. Then by parabolic Schauder estimates [LSU] there exists a constant $C_\mathcal{K} < \infty$ such that

$$|\nabla u_\lambda| \leq C_\mathcal{K} \quad \forall \lambda \geq \lambda_3, \ (x, t) \in \mathcal{K}. \quad (4.45)$$

By (4.39), (4.41), (4.42), (4.43), (4.44) and (4.45),

$$\left|\int_0^\infty \int_{\mathbb{R}^n} \left(U^{-m-1}_{\lambda} - \delta_{M}^{m-1}\right) \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt\right|$$ 

$$\leq \int_0^\infty \int_{\mathbb{R}^n} \left|U^{-m-1}_{\lambda} - \delta_{M}^{m-1}\right| \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt$$ 

$$\leq \int_{\mathcal{K}} \left|U^{-m-1}_{\lambda} - \delta_{M}^{m-1}\right| \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt + \int_{\mathcal{K}} \left(U^{-m-1}_{\lambda} + \delta_{M}^{m-1}\right) \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt$$ 

$$+ \int_{t_0}^{t_1} \int_{\text{supp} \, U_{\lambda}(t) \setminus \text{supp} \, \delta_{M}^{m-1}(t)} U^{-m-1}_{\lambda} \nabla u_\lambda \cdot \nabla \varphi \, dx \, dt$$ 

$$\leq C_\mathcal{K} \|\nabla \varphi\|_{L^\infty} \epsilon + C \left(||u_\lambda(t_0)||_{L^2}\right) \left(1 + \sqrt{\left(1 - t_0\right)} \||u_\lambda\|_{L^2}^{2m} + ||\delta_{M}\|_{L^2}^{2m}\right) \|\nabla \varphi\|_{L^\infty} \epsilon^{\frac{1}{2}} \quad \forall \lambda \geq \lambda_3.$$
Since $\epsilon$ is arbitrary, 
\[
\int_0^\infty \int_{\mathbb{R}^n} \left( U_{m}^{m-1} - \mathcal{B}_{M}^{m-1} \right) \nabla u_{\lambda} \cdot \nabla \varphi \, dx \, dt \to 0 \quad \text{as } \lambda \to 0. \tag{4.46}
\]
By (4.39) and Lemma 4.4,
\[
\begin{cases}
  u_{\lambda} \to u_{\infty} & \text{locally in } L^1 \\
  \nabla u_{\lambda} \to \nabla u_{\infty} & \text{locally in } L^2 \text{ with weight } \mathcal{B}_{M}^{m-1}
\end{cases}
\tag{4.47}
\]
Letting $\lambda \to \infty$ in (4.40), by (4.46) and (4.47) we have
\[
\int_0^\infty \int_{\mathbb{R}^n} \mathcal{B}_{M}^{m-1} \nabla u_{\infty} \cdot \nabla \varphi \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^n} u_{\infty} \varphi_t \, dx \, dt = 0.
\]
Thus
\[
u_{\infty} \text{ is a weak solution of } u_t - \nabla \left( \mathcal{B}_{M}^{m-1} \nabla u \right) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{4.48}
\]
By an argument similar to the proofs of Lemma 18.4 and Lemma 18.6 of [Va1], we can also have
\[
u_{0,\lambda}(x) \to M_0 \delta(x) \quad \text{as } \lambda \to \infty \quad \text{and} \quad \nu_{\infty}(x,t) \to M_0 \delta(x) \quad \text{as } t \to 0 \tag{4.49}
\]
By (4.27), (4.29), (4.48) and (4.49), $u_{\infty}$ is a solution of (4.4) which satisfies (4.5) and (4.6). Thus by Theorem 4.3,
\[
u_{\infty}(x,t) = \frac{M_0}{M} \mathcal{B}_{M}(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times [0, \infty). \tag{4.50}
\]
By (4.50) an argument similar to the proof of Theorem 2.8 of [LV], we have (1.8), (1.9) and the theorem follows. \hfill \Box

### 4.5 $C^\infty_s$-convergence

We finish this section by improving Theorem 1.2 (the uniform convergence in $L^p$, $p \geq 1$) up to $C^\infty_s$-convergence.

Denote by $\Omega_0$ the set of all points in $\mathbb{R}^n$ where $U_0 > 0$, i.e.,
\[
\Omega_0 = \{ x \in \mathbb{R}^n : U_0(x) > 0 \}.
\]
Since $U_0$ is compactly supported, there exists a constant $R > 0$ such that $\overline{\Omega_0}$ is contained in a ball of radius $R > 0$. For the existence of non-degenerate Lipschitz solution, some conditions are needed to be imposed on the initial data $u_0$, see [CVW] for the detail.

**Conditions for $C^\infty_s$-convergence**

- **Support**: $\text{supp } u_0 = \text{supp } U_0$.
- **Regularity**: $u_0^{m-1}, U_0^{m-1} \in C^1(\overline{\Omega_0})$.
- **Non-degeneracy**: there exists a constant $K > 0$ such that
  \[
  0 < \frac{1}{K} < u_0^{m-1} + |\nabla u_0^{m-1}| < K \quad \text{and} \quad 0 < \frac{1}{K} < U_0^{m-1} + |\nabla U_0^{m-1}| < K \quad \text{in } \overline{\Omega_0}. \tag{4.51}
  \]
Lemma 4.5. Under the conditions for $C^\infty$-convergence, there exists a constant $\epsilon_1 > 0$ such that

$$\epsilon_1 U_0(x) \leq u_0 \quad \forall x \in \mathbb{R}^n.$$ 

Then by (2.5) and the maximum principle for porous medium equation, we can get the following lemma.

\textbf{Lemma 4.5.} \textit{Under the conditions for $C^\infty$-convergence, there exists a constant $\epsilon_1 > 0$ such that}

$$0 \leq u(x, t) \leq U(x, t) \leq \frac{1}{\epsilon_1} u(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Denote by $v$ and $V$ the pressures of $u$ and $U$ respectively, i.e.,

$$v(x, t) = u^{m-1}(x, t) \quad \text{and} \quad V(x, t) = U^{m-1}(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Then

$$u_t = \frac{1}{m} \nabla \cdot (A \nabla u^m) \quad \text{in} \ \mathbb{R}^n \times (0, \infty)$$

$$\Rightarrow \quad v_t = A \left( v \Delta v + \frac{1}{m-1} |\nabla v|^2 \right) + v \nabla A \cdot \nabla v \quad \text{in} \ \mathbb{R}^n \times (0, \infty) \quad (4.52)$$

where $A = \left( \frac{U}{u} \right)^{m-1}$.

To explain the concept of $C^\infty$-convergence, we first consider the change of coordinates by which the free boundary $v = 0$ has been transformed into the fixed boundary. By Implicit Function Theorem, we can solve the equation $z = v(x_1, \cdots, x_{n-1}, x_n, t)$ with respect to $x_n$ locally around the points $(x^0_1, \cdots, x^0_{n-1}, x^0_n, t^0)$ on free boundary, i.e., for sufficiently small $\eta > 0$ there exists a function $x_n = h(x_1, \cdots, x_{n-1}, z, t)$ defined on a small box

$$\mathcal{B}_\eta = \{ 0 \leq z \leq \eta, |x_i - x^0_i| \leq \eta, -\eta \leq t - t^0 \leq 0 \} \quad \forall i = 1, \cdots, n-1.$$ 

On the set $\mathcal{B}_\eta$,

$$z = v(x', h(x', z, t), t) \quad (x' = (x_1, \cdots, x_{n-1})). \quad (4.53)$$

Thus by simple computation, we have

$$v_{x_n} = \frac{1}{h_z}, \quad v_{x_i} = -\frac{h_{x_i}}{h_z}, \quad v_t = \frac{h_t}{h_z}$$

$$v_{x_n x_n} = -\frac{h_{zz}}{h_z^2}, \quad v_{x_i x_i} = -\frac{1}{h_z} \left( \frac{h_{x_i}^2 h_{zz}}{h_z^3} - \frac{2 h_{x_i} h_{x_i z}}{h_z^2} + h_{x_i x_i} \right) \quad \forall i = 1, \cdots, n-1. \quad (4.54)$$

Then by (4.52) and (4.54), $h$ satisfies

$$h_t = A z \Delta x' h + A z^{-\sigma} \left( z^{1+\sigma} F(\nabla h) \right)_z + z \nabla x' A \cdot \nabla x' h + z A_z F(\nabla h)$$

$$= z^{-\sigma} \nabla x' \left( A_z z^{1+\sigma} \nabla x' h \right) + z^{-\sigma} \left( A z^{1+\sigma} F(\nabla h) \right)_z \quad (4.55)$$

where

$$\sigma = \frac{1}{m-1} - 1 \quad \text{and} \quad F(\nabla h) = -\frac{1 + |\nabla x' h|^2}{h_z}.$$
Observe that $A$ is uniformly parabolic in $\mathbb{R}^n \times (0, \infty)$ by (2.5) and Lemma 4.5. Therefore by an argument similar to the paper [DH], it can be easily checked that the equation (4.55) is governed by the Riemannian metric $ds$ where

$$ds^2 = \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dz^2}{2z}.$$ 

The distance between two points $P_1 = (x_1^1, \ldots, x_{n-1}^1, z_1^1, t_1)$ and $P_2 = (x_1^2, \ldots, x_{n-1}^2, z_2^2, t_2)$ in this metric is equivalent to the function

$$\bar{s}(P_1, P_2) = \frac{\sum_{i=1}^{n-1} |x_i^1 - x_i^2|^2 + |z_1^1 - z_2^1|^2}{\sum_{i=1}^{n-1} \sqrt{x_i^1 + \sqrt{|z_1^1 - z_2^1|}} + \sqrt{|t_1 - t_2|^2}}.
$$

Under this distance, Hölder semi-norm, $C^s$ norm and $C^{2, \alpha}_s$ norm of a function $f$ defined on a compact subset $\mathcal{A}$ of the half space $\{(x_1, \ldots, x_{n-1}, z, t) : z \geq 0\}$ are given as follow.

$$\|f\|_{H^s(\mathcal{A})} = \sup \left\{ \frac{|f(P_1) - f(P_2)|}{s[P_1, P_2]^s} : \forall P_1, P_2 \in \mathcal{A} \right\},$$

$$\|f\|_{C^s(\mathcal{A})} = \|f\|_{L^\infty(\mathcal{A})} + \|f\|_{H^s(\mathcal{A})},$$

$$\|f\|_{C^{2+\alpha}_s(\mathcal{A})} = \|f\|_{C^s(\mathcal{A})} + \sum_{i=1}^{n-1} \|f_{x_i}\|_{C^s(\mathcal{A})} + \|f_{z}\|_{C^s(\mathcal{A})} + \|f_{t}\|_{C^s(\mathcal{A})} + \sum_{i,j=1}^{n-1} \|f_{x_i x_j}\|_{C^s(\mathcal{A})} + \sum_{i,j=1}^{n-1} \|f_{x_i z_j}\|_{C^s(\mathcal{A})} + \|f_{z t}\|_{C^s(\mathcal{A})}.$$

The concept of $C^{\infty}_s$ space can be obtained by extending these definitions to spaces of higher order derivatives. For any $k \in \mathbb{N}$, we denote by $C^{k, \epsilon_1+\alpha}_s(\mathcal{A})$, $(\epsilon_1 = 0, 2)$, the space of all functions $f$ whose $k$-th order derivatives $D^{i_1}_{x_1} \cdots D^{i_{n-1}}_{x_{n-1}} D_j^l f$, $(i_1 + \cdots + i_{n-1} + j + l = k)$, exists and belong to the space of $C^{\epsilon_1+\alpha}_s(\mathcal{A})$. Then we say that a function $f$ belongs to the space $C^{\infty}_s(\mathcal{A})$ by

$$f \in C^{\infty}_s(\mathcal{A}) \iff f \in C^{k,2+\alpha}_s(\mathcal{A}) \forall k \in \mathbb{N}.$$ 

From now on, we are going to focus on $C^{\infty}_s$-convergence. For any $\lambda > 0$, let $v_\lambda$ be the rescaled function of $v$ by

$$v_\lambda(x, t) = \lambda^{(m-1)\alpha_1} v(\lambda^{a_2} x, \lambda t), \quad \forall \lambda > 0, \quad (x, t) \in \mathbb{R}^n \times (0, \infty)$$

where the exponents $a_1$ and $a_2$ are given by (4.2) and let $h_\lambda$ be the function from (4.55) with $v$ being replaced by $v_\lambda$. By Theorem 4.3 of [LV], there exists a constant $\lambda_0 > 0$ such that

The free boundary $\{ (x, t) : v_\lambda(x, t) > 0 \}$ is $C^{1, \alpha}$ surface for all $\lambda > \lambda_0$. \hspace{1cm} (4.56)

By (2.5) and Lemma 4.5 the coefficients $A(x, t)$ is uniformly parabolic in $\mathbb{R}^n \times (0, \infty)$. Moreover by Theorem 5.13

$$A \in C^\alpha_s. \hspace{1cm} (4.57)$$

Thus the equation (4.52) belongs to the same class of equations studied in [KO]. Hence by (4.57) and an argument similar to the proof of Theorem 5.6.1 in [KO], $h_\lambda$ have the $C^{1, \alpha}_s$-estimates up to the boundary.
Applying the standard bootstrap argument, we can even get \( C_{s,k}^{k,\alpha} \)-estimates of \( h_{\lambda} \) for any \( k \in \mathbb{N} \). Therefore, converting coordinates back to the original \((x, t)\), we can get the uniform estimate of derivatives of \( v_{\lambda} \).

**Theorem 4.6** (cf. Theorem 3.1 of [LV]). *For every \( k \in \mathbb{N} \), there exist constants \( \lambda_k > 0 \) and \( C_k > 0 \) such that*

\[
\|v_{\lambda}\|_{C^k(\Omega_0(u_{\lambda}))} < C_k \quad \forall \lambda > \lambda_k
\]

*where*

\[
\Omega_0(u_{\lambda}) = \{(x, t) : v_{\lambda}(x, t) > 0, \ 1 < t < 2\}.
\]

We finish this work by proving the Theorem 1.3.

**Proof of Theorem 1.3** By Theorem 1.2, there is the uniform convergence such that

\[
v_{\lambda}(x, t) \to \left( \frac{M_0}{M} B_M(x, t) \right)^{m-1} = r^{-a_1(m-1)G\left(\frac{x}{p_{Mz}}\right)} \quad \text{as} \ \lambda \to \infty
\]

where \( a_1 \) and \( a_2 \) are given by (4.2). By an argument similar to the explanation between Theorem 3.1 and Theorem 3.2 of [LV], there exists a function \( g_{\lambda}(x, t) \) such that

\[
(x, v_{\lambda}(x, t)) = \left( x, r^{-a_1(m-1)G\left(\frac{x}{p_{Mz}}\right)} \right) + g_{\lambda}(x, t)N\left(\frac{x}{p_{Mz}}\right)
\]

where \( N(x) \) is a smooth unit vector field, transverse to the surface \((x, G(x))\) and parallel to the \( x \)-plane in a neighborhood of the boundary \( \partial \{ x : G(x) > 0 \} \). By Theorem 4.6 and Arzelà-Ascoli Theorem, there exists the \( C^\infty_s \)-convergence between \( v_{\lambda} \) and \( \left( \frac{M_0}{M} B_M(x, t) \right)^{m-1} \) and the theorem follows. \( \square \)

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