QUANTIZATION OF 2+1 GRAVITY ON THE TORUS
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ABSTRACT
We use the polygon representation of 2+1–dimensional gravity to explicitly carry out the canonical quantization of a universe with the topology of a torus. The mapping-class-invariant wave function for a quantum ”big bounce”, is reminiscent of the interference patterns of linear gratings. We consider the “problem of time” of quantum gravity: for one choice of internal time the universe recovers a semiclassical interpretation after the bounce, with a wave packet centered at a single geometry; for another choice of internal time, the quantum solutions involve interference between macroscopically distinct universes.
1. Introduction

As a consequence of Einstein’s equations, the spacetime $M$ of 2+1–dimensional gravity in the absence of matter sources is flat. If the universe is a Riemannian surface $\Sigma$ of genus $g$, there are topological degrees of freedom related to the rotation of gyroscopes which travel around non-contractible loops, and to the metric variables which represent the sizes of these loops. One of the interesting particularities of 2+1 gravity is that almost all classical solutions have either initial or final singularities [1], so the question arises whether the quantum generalizations of these solutions can satisfactorily handle the singularities.

For a universe $\Sigma$, the spacetime manifold $M$ is $\Sigma \times \mathbb{R}$; let $T$ be its tangent bundle and $T'$ a 3–dimensional vector bundle topologically equivalent to $T$ and containing the structure group $SO(2,1)$. The equivalence between $T$ and $T'$ implies the existence of at least one isomorphism $e_{i}^{a}$ which in this case corresponds to a dreibein, where $i$ and $a$ represent the tangent space and Lorentz indices, respectively. In general, the dreibein $e_{i}^{a}$ is assumed to be invertible in order for the metric tensor in $M$ to be non degenerate. Nevertheless, there exist regions in spacetime where $e_{i}^{a}$ is not everywhere invertible. Those regions are of special interest in general relativity as they correspond to “classical singularities” or regions of “zero volume”. In 2+1 quantum gravity Witten showed that one must include the singular solutions if the theory is to be renormalizable [2].

Among the various approaches to 2+1 quantum gravity, there are essentially three different formulations: (i) The “frozen time” formulation [2, 3, 4] is based on the first–order form of the Einstein–Hilbert action (Chern–Simons action); the holonomy invariants of flat connections lead to a complete set of Heisenberg observables; (ii) The Arnowitt–Deser–Misner (ADM) formulation [5, 6] with York’s extrinsic time; (iii) The polygon representation [7], which provides an explicit representation of the phase space, which is particularly convenient to consider dynamical issues. In this work, we will use this representation and develop the quantization in the Schrödinger picture.

We will consider only the simplest topological structure for $\Sigma$, namely that of a torus. In section 2, we review the polygon representation for a torus and present the generators of
the mapping class group. Section 3 is devoted to quantization. In Section 4, we examine in particular the quantum analogue of “big bounce” solutions, where the universe collapses to a singularity and then re-expands.

2. Polygon representation

A torus can be represented as a parallelogram with opposite edges identified. Let us denote by $E(1)$ and $E(2)$ the three–vectors which represent two of the edges in a Minkowski embedding, and $M(1)$ and $M(2)$ the Lorentz matrices used to identify the opposite sides $-M^{-1}(1)E(1)$ and $-M^{-1}(2)E(2)$, respectively. The reduced phase space can be parametrized by $E(\mu)$ and $M(\mu)$, $\mu = 1, 2$, with the following non–vanishing Poisson brackets ($a, b, c, \cdots \in \{0, 1, 2\}$, $\epsilon^{012} = -1$):

$$\{E^a(\mu), E^b(\mu)\} = \epsilon^{abc} E^c(\mu), \quad \{E^a(\mu), M^b_c(\mu)\} = \epsilon^{abd} M_{dc}(\mu). \quad (1)$$

The dynamics is generated by the constraints

$$J = E(1) + E(2) - M^{-1}(1)E(1) - M^{-1}(2)E(2) \approx 0, \quad (2)$$

which imply that the polygon closes, and the constraints

$$P^a = \frac{1}{2} \epsilon^{abc} W_{cb} \approx 0, \quad (3)$$

with $W = M(1)M^{-1}(2)M^{-1}(1)M(2)$, which represent the cycle condition for the identification matrices.

There is an additional condition that must be taken into account. Not all parallelograms generate different tori, but some of them are equivalent under transformations of the mapping class group, i.e., the group of isomorphisms that cannot be smoothly deformed to the identity. For the torus, there are two different generators of mapping class group transformations. The first one

$$E(1) \to E(1) + E(2), \quad E(2) \to M^{-1}(1)E(2),$$

$$M(1) \to M(1), \quad M(2) \to M^{-1}(1)M(2) \quad (4)$$

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corresponds to a displacement along the base vector $E(2)$ such that the new vector $E(1)$ coincides with the diagonal of the initial parallelogram. For the second generator

$$E(1) \rightarrow E(1), \quad E(2) \rightarrow M(2)M^{-1}(1)E(1) + E(2),$$

$$M(1) \rightarrow M(1)M^{-1}(2), \quad M(2) \rightarrow M(2)$$

the displacement is along the vector opposite to $E(1)$.

The Poisson brackets (1), the polygon closure relations (2) and the cycle conditions (3), are all invariant under mapping class transformations.

3. Reduced phase space quantization

A. Canonical variables

The reduced phase space is defined by the variables $E^a(\mu)$ and $M^a(\mu)$. To carry out the quantization programme, it is convenient to introduce the canonical variables $X^a(\mu)$ and their corresponding “momenta” $P^a(\mu)$ by means of the regular transformation

$$X(\mu) = \frac{1}{P^2(\mu)} \left[ P(\mu) \wedge J(\mu) + 2 \frac{[E(\mu), P(\mu)] P(\mu)}{\text{tr}M(\mu) - 1} \right],$$

and

$$P^a(\mu) = \frac{1}{2} \epsilon^{abc} M_{cb}(\mu),$$

where $J(\mu) = [I - M^{-1}(\mu)]E(\mu)$ and the exterior product is defined by $(A \wedge B)^a = \epsilon^{abc} A_c B_b$ (some of the sign conventions differ from those of ref. [7]). The Poisson brackets of these new variables are canonical $\{P^a(\mu), X_b(\mu)\} = \delta^a_b$. The closure condition may be written as

$$J(\mu) = \sum_\mu X(\mu) \wedge P(\mu) \approx 0,$$

while the translation constrains $P \approx 0$ are given implicitly in terms of $P(\mu)$. The inverse transformation can be calculated explicitly from Eqs. (5) and (6) [7]. To first order in the momenta $P(\mu)$,

$$M^a_b(\mu) = \delta^a_b + \epsilon^{ca} P_c(\mu) + O(P^2), \quad E(\mu) = X(\mu) - \frac{1}{2} X(\mu) \wedge P(\mu) + O(P^2).$$
B. Mapping class transformations

The canonical variables $X(\mu)$ and $P(\mu)$ defined above are related to a particular choice of the base vectors $E(1)$ and $E(2)$ for the torus. Another choice of base vectors would lead to a different set of canonical variables for the same spacetime. The different choices are related by mapping class transformations. A convenient representation of the mapping class group is achieved by considering a parallelogram on $\mathbb{R}^2$ with two basis vectors $\hat{e}_1$ and $\hat{e}_2$. The mapping class images of this parallelogram are $\hat{e}_1' = a_{11}\hat{e}_1 + a_{12}\hat{e}_2$ and $\hat{e}_2' = a_{21}\hat{e}_1 + a_{22}\hat{e}_2$, where $a_{11}, \ldots \in \mathbb{N}$ and $a_{11}a_{22} - a_{12}a_{21} = 1$. The corresponding transformation for the polygon vectors embedded in Minkowski space is:

\[
E'(1) = \left[1 + M(2) + \cdots + M^{a_{11}-1}(2)\right] E(1) \\
+ \left[1 + M^{-1}(1) + \cdots + M^{-a_{12}+1}(1)\right] M^{a_{11}-1}(2)E(2),
\]

\[
E'(2) = \left[1 + M^{-1}(1) + \cdots + M^{-a_{22}+1}(1)\right] E(2) \\
+ \left[1 + M(2) + \cdots + M^{a_{21}-1}(2)\right] M^{-a_{22}}(1)M(2)E(1),
\]

\[
M'(1) = M^{-a_{21}}(2)M^{a_{22}}(1),
\]

\[
M'(2) = M^{-a_{12}}(1)M^{a_{11}}(2).
\]

With these relations, one can compute the action of the mapping class group on the canonical variables. To first order in $P$,

\[
X'(1) = a_{11}X(1) + a_{12}X(2) - \frac{a_{11}}{2}[(a_{22} - 1)P(1) - (a_{11} + a_{21} - 1)P(2)] \wedge X(1) \\
- \frac{a_{12}}{2}[(a_{12} + a_{22} - 1)P(1) - (2a_{11} + a_{21} - 1)P(2)] \wedge X(2) + O(P^2),
\]

\[
X'(2) = a_{21}X(1) + a_{22}X(2) - \frac{a_{21}}{2}[(2a_{22} - a_{12} - 1)P(1) + (a_{11} - a_{21} - 1)P(2)] \wedge X(1) \\
- \frac{a_{22}}{2}[(a_{22} - a_{12} - 1)P(1) + (a_{11} - 1)P(2)] \wedge X(2) + O(P^2).
\]

These expressions will be needed to calculate the mapping class invariant wave function, below.

C. The internal time
Our goal is the calculation of wave functions for the torus “big bounce”. First we have to choose one of the variables as “internal time” and calculate the corresponding Hamiltonian. Since we are not considering punctures on the torus, the Lorentz matrices $M(\mu)$ are pure boosts. Without loss of generality we can choose a frame in which the momentum $P(1)$ is parallel to a spatial axis, say $x$. Then

$$P^t(1) = P^y(1) = 0$$  \hspace{1cm} (16)

and

$$M^a_{\ b}(1) = \begin{pmatrix}
\cosh \beta & 0 & \sinh \beta \\
0 & 1 & 0 \\
\sinh \beta & 0 & \cosh \beta
\end{pmatrix},$$  \hspace{1cm} (16)

where $\beta$ is the boost parameter. These conditions fix the “gauge symmetries” generated by the constraints $J^t = J^y = 0$, which must be solved for the variables conjugate to the gauge conditions:

$$X^t(1) = \frac{X^x(2)P^t(2) - X^t(2)P^x(2)}{P^x(1)}, \quad X^y(1) = \frac{X^x(2)P^y(2) - X^y(2)P^x(2)}{P^x(1)}.$$  \hspace{1cm} (17)

The further constraints $P \approx 0$ become

$$P^y(2) \approx 0, \quad P^t(2) \approx 0.$$  \hspace{1cm} (18)

It follows from the constraint $J^x = 0$ that $X^t(2) = 0$. With the remaining components of the canonical variable $X(\mu)$, one can construct the internal time which must be a variable that does not commute with the constraints (18). The choice

$$t = \cos \alpha X^y(2) - \sin \alpha X^x(2)$$  \hspace{1cm} (19)

satisfies this criterion, where $\alpha$ is a nonvanishing parameter. Defining new variables $x_\mu$ and $p_\nu$ ($\mu, \nu \in \{1, 2\}$) with canonical Poisson brackets $\{x_\mu, p_\nu\} = \delta_{\mu\nu}$ in the following way

$$x_1 = X^x(1), \quad p_1 = P_x(1), \quad x_2 = \cos \alpha X^x(2) + \sin \alpha X^y(2), \quad p_2 = P_x(2)/\cos \alpha,$$  \hspace{1cm} (20)

the Hamiltonian corresponding to the internal time (19) can be written as

$$H = -\tan \alpha p_2.$$  \hspace{1cm} (21)
D. The wave function

The calculation of wave functions implies two different aspects: (1) the dynamical problem, i.e. determination of the Hamiltonian, and (2) the mapping class problem, i.e. finding expressions which are invariant with respect to transformations of the mapping class group. The first problem has been solved for the torus in the last subsection. Here we will attack the second problem.

Consider a wave packet described by the function $\tilde{a}(\vec{x}_T)$ at some initial time $T$. The propagation of this packet from $T$ to an arbitrary time $t$ can be described by a wave function involving a path integral of the form

$$\tilde{\psi}(\vec{x}_t, t) = \int d\vec{x}_T \tilde{a}(\vec{x}_T) \int D\vec{x}(\tau) D\vec{p}(\tau) e^{i \int_{T}^{t} (\vec{p} \cdot \dot{\vec{x}} - H) d\tau} \delta(G) \delta(K) |\text{Det}\{G, K\}|,$$

where $G \approx 0$ are the gauge constraints, $K \approx 0$ represent the gauge conditions, and $|\text{Det}\{G, K\}|$ is the determinant of the corresponding Jacobian. Splitting up the interval $t-T$ into $N$ infinitesimal time intervals, the expression $\int D\vec{x} D\vec{p}$ includes $N-1$ integrations over $d<x_i>$, where $<x_i>$ is the main value of $x(\tau_i)$ in each infinitesimal interval, and $N$ integrations $d\vec{p}_T \ldots d\vec{p}_t$. Since $H$ is a function of $p$ only, the integration over each $<x_i>$ yields a term of the form $\delta(\vec{p}_{i+1} - \vec{p}_i)$. The further integration over $d\vec{p}_T \ldots d\vec{p}_t$ leads to

$$\tilde{\psi}(\vec{x}_t, t) = C \int d\vec{x}_T \tilde{a}(\vec{x}_T) \int d\vec{p}_T e^{i [\vec{p}_T (\vec{x}_t - \vec{x}_T) - H (t-T)]} \delta(G) \delta(K) |\text{Det}\{G, K\}|,$$

where $C$ is a constant, and the Hamiltonian $H$ is evaluated at $\vec{p}_T$.

The expression (23), however, is not invariant with respect to mapping class transformations. To reach this invariance, we apply the method of images, i.e. we consider the sum over all mapping class images of $x$ (including $x_0 = t$). Then

$$\psi_{\text{inv}}(\vec{x}_t, t) \sim \sum_{\gamma} \hat{O}_\gamma \tilde{\psi}(\vec{x}_t, t),$$

where the sum is over all transformations ($\gamma$) of the mapping class group, and $\hat{O}_\gamma$ is an operator which represents the action of each transformation on the non-invariant wave function.
function. To determine the action of $\hat{O}_\gamma$ on $\tilde{\psi}$, it is convenient to transform the scalar products entering Eq.(23) into covariant expressions. This can be done by multiplying the integrand in (23) by $\delta(p_0 - H)\delta(x_0 - t)$ and changing $d\vec{x}_T d\vec{p}_T$ by $dx_T dp_T$. Clearly, this change does not alter the result of the integration, but allows us to write the argument of the exponential in Eq.(23) in a covariant way: $p_T(x_T - x_T)$. On the resulting wave function we apply the operator $\hat{O}_\gamma$ that modifies $x$ into $x'_\gamma$, and then integrate over $x'_\gamma$, with the factor $\delta(x_0' - t)$; thus, the parameter $t$ is invariant with respect to mapping class transformations, and the invariant wave function (24) can be written as

$$\psi_{inv}(\vec{x}_t, t) \sim \sum_{\gamma} \int d\vec{x}_T d\vec{p}_T \tilde{a}(\vec{x}_T) e^{i[p_T(x_T' - x_T) - H(t - T)]} \delta(G_\gamma) \delta(K_\gamma) |\text{Det}\{G_\gamma, K_\gamma\}|,$$  

(25)

where the index $\gamma$ was omitted in $x'_\gamma$ for the sake of clarity.

To proceed with the integration of the wave function (25), we specify the initial wave packet as a Gaussian distribution, at $T = 0$:

$$\tilde{a} \sim e^{-\frac{(x_1 - a)^2 + (x_2 - b)^2}{2\sigma^2}} e^{i\vec{k} \cdot \vec{x}},$$

where $a, b, \vec{k}$ and $\sigma$ are constants. The explicit values of the mapping class images $\vec{x}'$ can be obtained by using the defining equations (20) and the relations (14) and (15), which represent the action of mapping class transformations on the original canonical variables $X(\mu)$. The resulting wave function is given by

$$\psi_{inv}(\vec{x}_t, t) \sim \sum_{\gamma} \int dp_1 dp_2 e^{-\frac{p_1^2}{2}((p_1 - k_1)^2 + (p_2 - k_2)^2)}$$

$$\times e^{i[(p_1 - k_1)(x_1 - a + \Delta x_1) + (p_2 - k_2)(x_2 - b + \Delta x_2 + \tan \alpha t)]} \delta(G_\gamma) \delta(K_\gamma) |\text{Det}\{G_\gamma, K_\gamma\}|,$$  

(26)

where

$$\Delta x_1 = (a_{11} - 1)x_1 + a_{12}x_2 \cos \alpha - a_{12}t \sin \alpha + O(p^2),$$

$$\Delta x_2 = (a_{22} - 1)x_2 + a_{21}x_1 \cos \alpha - a_{21} \sin \alpha \cos \alpha (x_2 \sin \alpha + t \cos \alpha) \frac{p_2}{p_1} + O(p^2).$$

The sum in Eq.(26) is over all integer values of $a_{ij}$ such that $a_{11}a_{22} - a_{12}a_{21} = 1$. This wave function (26) describes the propagation of an initial packet $\tilde{a}(\vec{x}_T)$; we will consider in
particular the case where this packet is centered about a collapsing universe, and examine
the quantum “big bounce”.

4. Conclusions: The “Big Bounce”, and the Problem of Time

Let us consider the initial packet centered at \( \{a,b\} \), with initial momentum distributed
about \( \vec{k} = \{k_1,k_2\} \). The classical evolution is given by the dynamical equations \( \dot{x}_1 = 0, \)
\( \dot{x}_2 = -\tan \alpha \), so the classical trajectory goes through a singular universe \( \{a,0\} \) at time
\( t = \frac{b}{\tan \alpha} \) and then unfolds into the universe \( \{a,-b\} \) at \( t = \frac{2b}{\tan \alpha} \). There can be other
classical trajectories from \( \{a,b\} \) to \( \{a,-b\} \); indeed, the final value of the variables, \( \{a,-b\} \),
might well correspond to a different set of basis loops; correcting for the mapping class
transformation between the initial and final states, we would then look at the classical
evolution from \( \{a,b\} \) to
\[
\{a + \Delta x_1, -b + \Delta x_2\},
\]
where \( \Delta x_i \) are evaluated at \( \vec{x} = \{a,-b\} \). Since \( \dot{x}_1 = 0 \), this can be a classical trajectory
only if \( a_{12} = 0 \) and \( a_{11} = 1 \), which implies that \( a_{22} = 1 \). There is then one classical
trajectory for each \( a_{21} \), which reaches the “detector” (of universes...) at \( \{a,-b\} \) in a time
given implicitly by \( t \tan \alpha = 2b - \Delta x_2 \). Conversely, at time \( t = \frac{2b}{\tan \alpha} \) this classical solution
reaches
\[
x_2 = -b - \Delta x_2 = -b - a_{21}(a \cos \alpha - \frac{\sin 2\alpha}{2}(-b \sin \alpha + t \cos \alpha) \frac{k_2}{k_1}).
\]

In the quantum theory one has a sum of amplitudes, and the question arises of whether
the amplitudes interfere. Since \( \Delta x_2 \sim a_{21}a \) and \( a \) is a macroscopic constant of the motion,
one does not expect interference of images with different values of \( a_{21} \); so only one of the
classical trajectories described above will contribute. On the other hand, with \( \Delta x_1 \sim a_{12}x_2 \), one finds that the images for various values of \( a_{12} \) are distributed as in a linear
grating. Therefore, one expects interference to occur when the variance of the wave packet
is at least of the same order of magnitude as the spacing \( \eta \) between slits of the “grating”,
\[
\eta = x_2 \cos \alpha.
\]
This indicates that quantum interference is surely going to be significant near the singularity, where \( x_2 \simeq 0 \).

One might worry about the summability of the expression for the invariant wave function in the limit \( x_2 \to 0 \), as the number of terms in the sum grows without bound – this is related to the delicate issue of how the quantum theory handles the singularity. However, the wave function is nothing but the unitary evolution of the initial packet with a smooth Hamiltonian, so by construction it is normed at all times; the challenge would be to find the appropriate regularization of the sum near \( x_2 = 0 \), or to analytically continue an expression which is computable for all \( x_2 \neq 0 \).

Another question, which we will consider in greater detail, is whether there can be interference between macroscopically distinct universes after the “big bounce”.

Since we have chosen an internal time which leads to a Hamiltonian linear in \( p_2 \), there is no dispersion and therefore the variance is equal to the initial variance \( \sigma^2 \) at all times. Assuming that this initial variance is microscopic, one finds interference effects only near the singularity.

What if one had chosen a different internal time? Consider for instance

\[
    t' = \frac{\cos \alpha \ Y(2) - \sin \alpha \ X(2)}{-2 \sin \alpha \ P(2)}. \tag{30}
\]

The corresponding Hamiltonian is

\[
    H = \tan^2 \alpha \ p_2^2, \tag{31}
\]

which leads to the dispersion \( \sigma^2(t) = \sigma^2 + \tan^2 \alpha \ t' \). Thus, after a sufficiently large amount of time (such as the time to go through the singularity and re-expand to a large universe), the variance becomes large and one has interference between macroscopically distinct universes.

This is a particularly striking manifestation of one of the “problems of time” of quantum gravity, known as the multiple choice problem: different choices of internal time lead to
different physical predictions. In this case, the interference lines with the second choice of time will lead to sizeable amplitudes for non-classical histories. In contrast, with the first choice of internal time one finds a significant amplitude only near the predictions of the classical theory.

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