Two-sided bounds on free energy of directed polymers on strongly recurrent graphs

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Abstract

We study the directed polymers in random environment on an infinite graph \(G = (V, E)\) on which the underlying random walk satisfies sub-Gaussian heat kernel bounds with spectral dimension \(d_s\) strictly less than two. Our goal in this paper is to show (i) the existence and the coincidence of the quenched and the annealed free energy \(F_q(\beta), F_a(\beta)\) and (ii) that \(F_a(\beta) - F_q(\beta)\) is comparable to \(\beta^{\frac{4d_s}{4d_s-2}}\) for small inverse temperature \(\beta\).

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For a probability space \((\Omega, \mathcal{F}, P)\), we denote by \(P[X]\) the expectation of random variable \(X\) with respect to \(P\). Let \(\mathbb{N}_0 = \{0, 1, 2, \cdots\}\), \(\mathbb{N} = \{1, 2, 3, \cdots\}\), and \(\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}\). Let \(C_{x_1, \cdots, x_p}\) or \(C(x_1, \cdots, x_p)\) be a non-random constant which depends only on the parameters \(x_1, \cdots, x_p\).

1 Introduction

1.1 The model.

The directed polymers in random environment (DPRE) was introduced by Henley and Huse [23] in the physics literature to analyze an influence of random media to the shape of polymer chains. Later on, Imbrie-Spencer and Bolthausen succeeded to treat DPRE mathematically [24, 7]. Then, a lot of progress has been achieved by many authors [2, 10, 17, 16, 15]. In particular, it is known that the phase transition of delocalization-localization of polymer chain is characterized by the free energy[16]. The reader may refer the survey text [14].

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In the most studied model, polymer chain and random media are represented by a path of random walk on graph $\mathbb{Z}^d$ and time-space i.i.d. random variables $\omega$ respectively, and their interaction is described in terms of Gibbs measure with Hamiltonian $H(\omega, S)$. In this paper, we consider the model on a general infinite graph $G = (V, E)$ with bounded degree and replace the underlying walk by a reversible random walk $S$.

To formulate our model, we introduce a framework of a graph $G = (V, E)$ and a random walk $S = \{S_n\}_{n=0}^{\infty}$ with suitable conditions as in [5].

**Graph and random walk** Let $G = (V, E)$ be an infinite graph where $V$ is the set of vertices with countably infinite elements and $E$ is the set of undirected edges. Let $0 \in V$ be the distinguished vertex called the origin. We assume that graph does not have multiple edges. Then, each edge is identified as a pair of vertices $(x, y)$ and we denote by $N(x) = \{y \in V : \text{There exists } e \in E \text{ s.t. } e = (y, x)\}$.

We assume that graph $G$ is connected, (1.1) that is, for any $x, y \in V$, there exist an $n \in \mathbb{N}$ and a sequence of $x_0, x_1, \cdots, x_n \in V$ such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E$ for each $i = 1, \cdots, n$.

Also, we introduce the weight $\{\mu_{xy}\}_{x,y \in V}$ on graph $G$ such that

1. $\mu_{xy} = \mu_{yx}$.
2. $\mu_{xy} \geq 0$ for any $x, y \in V$.
3. $\mu_{xy} > 0$ if and only if $(x, y) \in E$.

We consider a random walk $S$ on weighted graph $G$ with transition probability given by

$$p_{x,y} = P_S(S_{n+1} = y | S_n = x) = \frac{\mu_{xy}}{\mu_x}, \quad y \in N(x),$$

where $\mu_x = \sum_{y \in N(x)} \mu_{xy}$ and we denote by $(\Omega_S, \mathcal{G}_S, P_S)$ the probability space on which $S$ is defined. In particular, $P^0_S$ denotes the law of $S$ starting from $x \in V$, and also by $P_S = P^0_S$ for simplicity.

We remark that a random walk $S$ is reversible since

$$\mu_x p_{x,y} = \mu_y p_{y,x}.$$ 

Throughout the paper, we assume the "bounded weights" condition: There exists a constant $C_\mu$ such that

$$C_\mu^{-1} \leq \mu_{xy} \leq C_\mu$$

(BW)

for any $x, y \in V$ with $(x, y) \in E$. 
Random media Let \( \{ \omega(n,x) : n \in \mathbb{N}, x \in V \} \) be i.i.d. \( \mathbb{R} \)-valued random variables which are independent of \( S \) and defined on the probability space \((\Omega, \mathcal{F}, Q)\). In this paper, we assume
\[
Q[\omega(n,x)] = 0, \quad \text{and} \quad Q[\exp(\beta \omega(n,x))] = e^{\lambda(\beta)} < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.2}
\]

For each \( n \in \mathbb{N}, z \in V \), let \( \theta_{n,z} : \Omega \to \Omega \) be a time-space shift of random media:
\[
\theta_{n,z} \circ \omega(m,y) = \omega(m+n,y+z)
\]
and we define \( \theta_{n,z} \circ f : f(\theta_{n,z} \circ \omega) \) for a measurable function \( f : \Omega \to \mathbb{R} \).

Then, Hamiltonian is defined by
\[
H_n(S, \omega) = \sum_{i=1}^n \omega(i,S_i)
\]
and the polymer measure is given by
\[
P_{n,x}^\beta(dS) = \frac{1}{Z_{n,\beta}} \exp(\beta H_n(S, \omega)) P_X^\beta(dS)
\]
where
\[
Z_{n,\beta}^X := Z_{n,\beta}^X = \left[ \exp(\beta H_n(S, \omega)) \right]
\]
is the normalizing constant called a partition function.

Then, the argument in [17, Theorem 3.2] assure the existence of a critical point \( \beta_1 \in [0, \infty] \) such that
\[
Q(W_x^{\infty, \beta} > 0) = \begin{cases} 
1 \quad \text{(weak disorder)} & \beta \in \{0\} \cup [0, \beta_1) \\
0 \quad \text{(strong disorder)} & \beta \in (\beta_1, \infty). 
\end{cases}
\]

One then defines the free energy of the systems.

**Proposition 1.1.** Assume that for each \( x \in V \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \tilde{V}(x,n) = 0, \quad \text{(subExp)}
\]
where \( \tilde{V}(x,n) \) is the number elements in the ball centered at \( x \) with radius \( n \).

Then, for any \( x \in V \), the following limit exists and is constant \( Q \)-a.s.:
\[
F_{q,x}(\beta) = \lim_{n \to \infty} \frac{1}{n} \log Z_{n,\beta}^X = \lim_{n \to \infty} \frac{1}{n} Q[\log Z_{n,\beta}^X(\beta)]
\]
\[
F_a(\beta) = \lim_{n \to \infty} \frac{1}{n} \log Q[Z_{n,\beta}^X] = \lambda(\beta).
\]

In particular, \( F_{q,x}(\beta) \) is constant in \( x \in V \) and we denote by \( F_q(\beta) = F_{q,x}(\beta) \) (\( x \in V \)).

\( F_{q,x}(\beta) \) and \( F_a(\beta) \) are called the quenched free energy and the annealed free energy of the systems, respectively.
Remark 1.2. Jensen’s inequality implies that
\[ \frac{1}{n} Q \log Z^x_{n, \beta} \leq \frac{1}{n} \log Q[Z^x_{n, \beta}] = \lambda(\beta) \]
and \( F_q(\beta) \leq \lambda(\beta) =: F_a(\beta) \). The same argument as in [17, Theorem 3.2] assures the existence of the critical point \( \beta_2 \in [0, \infty] \) such that
\[ F_q(\beta) - F_a(\beta) \begin{cases} 
0 & \beta \in [0, \beta_2] \\
< 0 \text{ (very strong disorder)} & \beta \in (\beta_2, \infty). 
\end{cases} \]

Our main result gives an asymptotic order of \( F_q(\beta) - F_a(\beta) \), where the detail of assumptions is discussed in subsection 1.2:

**Theorem 1.3.** (1) Suppose that the underlying random walk \( S \) satisfies upper heat kernel condition (UHK) with \( d_s < 2 \) and (VG). Then, there exist constants \( C_1 > 0 \) and \( \beta_0 > 0 \) such that for any \( \beta \in [0, \beta_0) \)
\[ F_q(\beta) - F_a(\beta) \leq -C_2 \beta^{\frac{4}{d_s}}. \]
In particular, \( \beta_1 = \beta_2 = 0. \)

(2) In addition, we suppose that the underlying random walk \( S \) satisfies lower heat kernel condition (LHK) with \( d_s < 2 \). Then, there exists a constant \( C_2 > 0 \) such that for any \( \beta \in [0, \beta_0) \)
\[ -C_2 \beta^{\frac{4}{d_s}} \leq F_q(\beta) - F_a(\beta). \]

For DPRE in 1+1 (\( d_f = 1 \) and \( d_w = 2 \)), the same upper bound (and lower bound for Gaussian environment) was obtained by H. Lacoin [25] and the same lower bound was obtained by K.S. Alexander and G. Yildirim [3].

**Remark 1.4.** Cosco, Seroussi, and Zeitouni study the DPRE on infinite graphs independently [18]. They discuss the critical points of phase transitions for large classes of graphs. They also obtain an upper bound of asymptotics of free energy under a similar set of conditions independently.

### 1.2 Heat kernel estimate

In this subsection, we review some selected facts on the random walk on graphs \( G = (V, E) \).

Let \( B(x, n) = \{ y \in V : d(x, y) \leq n \} \) be a ball centered at \( x \) with radius \( n \), where \( d(x, y) \) is a graph distance of \( x, y \in V \), that is \( d(x, y) = \inf \{ n \geq 0 : \exists \{ x_i \}_{i=0}^n \subset V, x = x_0, x_1, \ldots, x_n = y \text{ such that } (x_{i-1}, x_i) \in E \} \). For \( x \in V \) and a nonempty subset \( A, B \subset V \), we denote the distance between \( x \) and \( A \) by
\[ d(x, A) := \inf \{ d(x, y) : y \in A \} \]
and the distance between $A$ and $B$ by
\[ d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}. \]

Here, we list some properties of graphs and random walk. Let $d_f > 0$ and $d_w > 1$.

- (Upper heat kernel estimate) There exist constants $c_1, c_2 > 0$ such that for $n \geq 0$ and $x, y \in V$ with $d(x, y) \leq n$
\[
p_n(x, y) \leq \frac{c_1}{(n \lor 1)^{d_f/d_w}} \exp \left( -c_2 \left( \frac{d(x, y)^{d_w}}{n \lor 1} \right) \right). \tag{UHK}\]

- (Lower heat kernel estimate) There exist constants $c_3, c_4 > 0$ such that for $n \geq 0$ and $x, y \in V$ with $d(x, y) \leq n$
\[
\frac{c_3}{(n \lor 1)^{d_f/d_w}} \exp \left( -c_4 \left( \frac{d(x, y)^{d_w}}{n \lor 1} \right) \right) \leq p_n(x, y) + p_{n+1}(x, y) \tag{LHK}\]

- (Bounded geometry) The degree is bounded, that is
\[
\sup_{x \in V} |N(x)| < \infty \tag{BG}\]

- (Volume growth) There exists a non-random constant $C_V > 0$ s.t.
\[
C_V^{-1} n^{d_f} \leq V(x, n) \leq C_V n^{d_f} \quad \text{for } x \in V \text{ and } n \geq 1, \tag{VG}\]
where $V(x, n)$ be the volume growth function:
\[
V(x, n) = \# \Sigma_{x \in B(x, n)} = \sum_{y \in B(x, n)} \mu_y \quad \text{for } x \in V, n \geq 1.
\]

When (UHK) and (LHK) are satisfied, $d_w > 1$ is called the walk dimension of $S$ and $d_s := 2d_f/d_w$ is called the spectral dimension of $S$ [5, Definition 4.14].

We say the weighted graph is strongly recurrent if the random walk $S$ satisfies (UHK) and (LHK) with $d_f < d_w$, or equivalently, $d_s = 2d_f/d_w < 2$. [6, Section 1.1]

**Remark 1.5.**
- (UHK) and (LHK) imply (BW), (BG), (VG) [5, Lemma 4.17].
- (BW) and (VG) imply that there exists a non-random constant $C_V' > 0$ s.t.
\[
C_V'^{-1} n^{d_f} \leq \tilde{V}(x, n) \leq C_V' n^{d_f} \quad \text{for } x \in V, n \geq 1, \tag{VG'}
\]
where $\tilde{V}(x, n) = \#B(x, n)$ is the number of elements in the ball $B(x, n)$. 

Lemma 1.6. [5, Lemma 4.21] We assume (UHK). Then, there exist constants $c_5, c_6$ such that for $t \geq 1$,

$$P_x \left( S_n \notin B(x, tn^{\frac{d_f}{d_s}}) \right) \leq c_5 \exp \left( -c_6 t^{\frac{d_w}{d_s}+1} \right), \quad (1.3)$$

$$P_x \left( \tau(x, 2tn^{\frac{d_f}{d_s}}) < n \right) \leq 2c_5 \exp \left( -c_6 t^{\frac{d_w}{d_s}+1} \right), \quad (1.4)$$

where $\tau(x, r) = \inf \{ n \geq 0 : S_n \notin B(x, r) \}$ for $r \geq 1$.

1.2.1 Example of strongly recurrent graphs

Here, we will consider random walk on the Sierpinski gasket graph as an example of strongly recurrent graphs. See [5, Section 2.9].

Let $O = (0,0), a_0 = (1,0), b_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $G_0$ be the graph which consists of the vertices $O, a_0, b_0$ and the edges $\langle O, a_0 \rangle, \langle O, b_0 \rangle, \langle a_0, b_0 \rangle$. Then, we consider the sequence of graphs $\{G_n\}_{n=0}^\infty$ defined by

$$G_{n+1} = G_n \cup (G_n + 2^n a_0) \cup (G_n + 2^n b_0),$$

where $A + a = \{ x + a : x \in A \}$. Then, the graph $G = \bigcup_{n=0}^\infty G_n$ is the Sierpinski gasket graph.

![Sierpinski gasket graph](image)

Figure 1: Sierpinski gasket graph

When we consider a simple random walk $S_{SG}$ on $G$, it is known that $S_{SG}$ satisfies (UHK) and (LHK) with $d_f = \frac{\log 3}{\log 2}, d_w = \frac{\log 5}{\log 2}$ and hence $d_s = \frac{\log 9}{\log 5} < 1$ [5, Corollary 6.11].
1.3 Some remarks and Comments

For DPRE in $1 + 1$, one of authors obtained a sharper asymptotics as follows [28]. Suppose we assume (1.2) and $Q[\omega(n,x)^2] = 1$ with a certain technical condition on $\omega$, then

$$\lim_{\beta \to 0} \frac{F_q(\beta) - F_a(\beta)}{\beta^4} = \lim_{T \to \infty} \frac{1}{T} \mathbb{P}[\log \mathcal{Z}_T]$$

with $\mathcal{Z}_T = \int_{\mathbb{R}} \mathcal{Z}(T,x) dx$, where $\mathcal{Z}(t,x)$ is a solution to a stochastic heat equation

$$\partial_t \mathcal{Z} = \frac{1}{2} \Delta \mathcal{Z} + \sqrt{2} \mathcal{Z} \mathcal{W}, \quad \lim_{t \to 0} \mathcal{Z}(t,x) dx = \delta(dx),$$

where $\mathcal{W}$ is a time-space white noise on $[0, \infty) \times \mathbb{R}$.

To explain the rough idea of (1.5), we need the following theorem obtained by T. Alberts, K. Khanin, and J. Quastel [1, Theorem 2.1]:

**Theorem A.** Suppose (1.2) and $Q[\omega(n,x)^2] = 1$. Then we have

$$\frac{Z_{0,T_n,\beta_n}}{Q[Z_{0,T_n,\beta_n}]} \Rightarrow \mathcal{Z}_T$$

where $\beta_n = n^{-\frac{1}{4}}$.

Then, (1.5) implies roughly that

$$(LHS) = \lim_{n \to \infty} \frac{1}{\beta_n^4} \lim_{T \to \infty} \frac{1}{T} Q \left[ \log \frac{Z_{0,T_n,\beta_n}}{Q[Z_{0,T_n,\beta_n}]} \right] = \lim_{T \to \infty} \lim_{n \to \infty} \frac{1}{T} Q \left[ \log \frac{Z_{0,T_n,\beta_n}}{Q[Z_{0,T_n,\beta_n}]} \right] = (RHS),$$

that is (1.5) means the interchangeability of two limits in $n$ and $T$. In several discrete disordered systems, it is known that such interchangeability of limits holds [9, 11, 12].

Thus, we have the following conjectures.

**Conjecture A.** Suppose the underlying random walk satisfies the local limit theorem in the sense discussed in [19]. Then, we have for each $T > 0$

$$\frac{Z_{0,T_n,\beta_n}}{Q[Z_{0,T_n,\beta_n}]} \Rightarrow \mathcal{Z}_T,$$

with $\beta_n = n^{-\frac{d_w - d_f}{2d_w}}$, $\mathcal{Z}_T = \int \mathcal{Z}_{T,x} \mu(dx)$, where $\mathcal{Z}_{t,x}$ satisfies the

$$\mathcal{Z}_{t,x} = p_t(x) + \hat{\beta} \int_0^t \int_{\mathcal{V}} \left( \int_{\mathcal{V}} p_{t-s}(x-z) \mathcal{Z}_{s,z-y} \mu(dy) \right) \mathcal{W}(ds,dy),$$

where $\hat{\beta}$ is a constant depends on underlying random walk and $p_t(x)$ is a heat kernel of Brownian motion in fractal graph and $\mathcal{W}$ is a white noise on $(\mathcal{V}, \mu)$. 
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**Remark 1.7.** Simple random walk on Sierpinski gasket graph satisfies local limit theorem \([19]\).

**Remark 1.8.** In \([22, 21]\), SPDE on measure space equipped with gradient and divergence are discussed and they prove the existence and uniqueness of solutions.

**Conjecture B.** Under the assumption in Conjecture A, we have that

\[
\lim_{\beta \to 0} \beta^{-\frac{1}{2\pi}} (F_q(\beta) - F_0(\beta)) = \lim_{T \to \infty} \frac{1}{T} \mathcal{P} [\log \mathcal{Z}_T].
\]

## 2 The existence of the free energy

In this section, we prove Proposition 1.1 and we assume (subExp) and (BW).

In \([15]\), the existence of the limit \(\lim_{n \to \infty} \frac{1}{n} Q[\log Z_n] \) follows from superadditivity of \(\{Q[\log Z_{n,\beta}^x]\}_{n=1}^{\infty}\), but we should remark that \(\{Q[\log Z_{n,\beta}^x]\}_{n=1}^{\infty}\) is not superadditive in our case due to loss of the shift invariance of the underlying random walk.

Letting \(Z_{n,\beta}^x\) be the point to point partition function

\[
Z_{n,\beta}^x = Z_{n,\beta}^x := P_x \left[ \exp \left( \beta \sum_{i=1}^{n} \omega(i,S_i) \right) : S_n = y \right],
\]

then \(\{Q[\log Z_{n}^x]\}_{n=1}^{\infty}\) is superadditive and hence the limit \(\lim_{n \to \infty} \frac{1}{n} Q[\log Z_n^x] = \sup_{n \geq 1} \frac{1}{n} Q[\log Z_n^x]\) exists.

**Proof of Proposition 1.1.** Theorem 6.1 \([27]\) combined with \([15, \text{Proposition 1.5}]\) gives the concentration inequality for \(\log Z_{n,\beta}^x\) and \(\log Z_{n}^x\): There exists a constant \(K > 0\) such that for any \(n \geq 1\), \(t \in (-1, 1)\), \(\delta > 0\), and \(x,y \in V\)

\[
Q[\exp(t(\log Z_n^x - Q[\log Z_n^x]))] \leq \exp \left( \frac{nKt^2}{1 - |t|} \right)
\]

(2.1)

and

\[
Q[|\log Z_n^x - Q[\log Z_n^x]| > n\delta] \leq \exp \left( -n \left( \sqrt{\delta + K} - \sqrt{K} \right)^2 \right),
\]

and the same inequality holds even if \(Z_n^x\) is replaced by \(Z_{n,\beta}^x\).

Thus, it is enough to show the convergence of \(\frac{1}{n} Q[\log Z_n^x]\) and independence of \(x \in V\). We will show that

\[
\lim_{n \to \infty} \frac{1}{n} Q[\log Z_n^x] = \lim_{n \to \infty} \frac{1}{n} Q[\log Z_{n,\beta}^x]
\]
by the same argument as in [13, Proposition 2.4]: $Z^x_n \geq Z^x_{n,x}$ is trivial so that
\[
\lim_{n \to \infty} \frac{1}{n} Q[\log Z^x_n] \leq \lim_{n \to \infty} \frac{1}{n} Q[\log Z^x_n].
\]
Also, $Q[\log Z^{x,x}_{2n}] \geq Q[\log Z^x_n] + Q[\log Z^{x,y}_{2n}]$ with (BW) implies that there exists a constant $c > 0$ such that for any $x, y \in V$
\[
Q[\log Z^{x,x}_{2n}] \geq 2Q[\log Z^x_{2n}] - c. \tag{2.2}
\]
On the other hand, we have that for any $0 < t < 1$
\[
\frac{1}{n} Q[\log Z^x_n] \leq \frac{1}{tn} \log \left[ \sum_{y \in V} (Z^x_{n,y})^t \right] \leq \frac{1}{tn} \log \left( \sum_{y \in V} Q[\exp(t(\log Z^{x,y}_{n} - Q[\log Z^x_n]))] \right) \exp(tQ[\log Z^{x,y}_{n}])
\]
\[
\leq \frac{1}{tn} \log \left[ \sum_{y \in V} \exp(t(\log Z^{x,y}_{n} - Q[\log Z^x_n])) \right] + \frac{nKt^2}{1-t} + \frac{c}{2nt}
\leq \frac{1}{tn} \left( \log \tilde{V}(x,n) + \frac{nKt^2}{1-t} + \frac{c}{2nt} \right), \tag{2.3}
\]
where we used Jensen’s inequality in the first line, the fact $(a + b)^\theta \leq a^\theta + b^\theta$ for $a, b \geq 0$ and $\theta \in (0, 1)$ in the second line, (2.2) in the fourth line, and (2.1) in the last line.

Taking $n \to \infty$ and then $t \to 0$, we obtain from (subExp) that
\[
\lim_{n \to \infty} \frac{1}{n} Q[\log Z^x_n] \leq \lim_{n \to \infty} \frac{1}{2n} Q[\log Z^{x,y}_{2n}]
\]
and thus \( \lim_{n \to \infty} \frac{1}{n} Q[\log Z^x_n] \) exists. The independence of $x$ for $F_{q,x}(\beta)$ follows by $Q[\log Z^{x}_{n+d(x,y)}] \geq Q[\log Z^x_n]$. \hfill \qed

### 3 Upper bound

In this section, we prove Theorem 1.3 (1). Throughout this section, we assume (BW), (VG) (which imply (VG')), (UHK).

We will prove the upper bound by the coarse graining scheme and change of measure method used in several polymer models [4, 8, 9, 25, 20, 30].
3.1 Coarse graining and change of measure

We define \( W_x^{n, \beta} = Z^{\beta, x} / Q[Z^{x, n}_n, \beta] = P_x^S[e_n] \) where
\[
e_n = e_n(\beta, \omega, S) := \exp(\beta H_n(S) - n\lambda(\beta)) \tag{3.1}
\]
and we denote by \( \{\mathcal{F}_n : n \geq 0\} \) the filtration generated by \( \omega(n, x) : \mathcal{F}_0 = \{\Omega, \emptyset\} \) and \( \mathcal{F}_n = \sigma[\omega(i, x) : 1 \leq i \leq n, x \in V] \) for \( n \geq 1 \).

We remark from Proposition 1.1 that
\[
F_q(\beta) - F_a(\beta) = \lim_{n \to \infty} \frac{1}{n} Q[\log W_{nN}^0(\beta)] = \lim_{N \to \infty} \frac{1}{nN} Q[\log W_{nN}^0(\beta)]. \tag{3.2}
\]

Therefore, we will estimate the RHS in (3.2) in the proof.

For small \( \beta > 0 \), let \( n \) be the smallest integer bigger than \( h_V(\beta) \), where \( h_V \) is a decreasing function on \((0, \beta_0)\) with \( \lim_{\beta \to +0} h_V(0) = \infty \) and given explicitly later. First, we consider a covering of graph by balls with radius \( n_w = n^\frac{1}{df} \). From Vitali’s covering lemma, we can find a set of vertices \( I^{(n)} = \{y_i^{(n)}\}_{i \in \mathbb{N}} \) such that
\[
\begin{align*}
(1) & \quad B(y_i^{(n)}, n_w) \cap B(y_j^{(n)}, n_w) = \emptyset \text{ for } i \neq j. \\
(2) & \quad \bigcup_i B(y_i^{(n)}, 5n_w) = V.
\end{align*}
\]

Denoting by \( \mathcal{N}_{y_i^{(n)}}(R) \) the elements in \( I^{(n)} \) which lie in the ball \( B(y_i^{(n)}, Rn_w) \) for \( R \geq 1 \), we have
\[
\bigcup_{j \in \mathcal{N}_{y_i^{(n)}}(R)} B(y_j^{(n)}, n_w) \subset B(y_i^{(n)}, (R + 1)n_w)
\]
and hence (VG’) implies that there exists a constant \( C_0 > 0 \) such that
\[
\# \mathcal{N}_{y_i^{(n)}}(R) \leq C_0 R^{df} \tag{3.3}
\]
for any \( n \geq 1 \) and \( y_i^{(n)} \in I^{(n)} \).

Jensen’s inequality implies that for each \( 0 < \theta < 1 \)
\[
\frac{1}{nN} Q[\log W_{nN}^0(\beta)] \leq \frac{1}{\theta nN} \log Q \left[ W_{nN}^0(\beta)^\theta \right].
\]
We will prove that there exists an \( m > 0 \) such that
\[
Q \left[ W_{nN}^0(\beta)^\theta \right] \leq e^{-mN} \tag{3.4}
\]
so that
\[
\lim_{n \to \infty} h_V(\beta)(F_q(\beta) - F_a(\beta)) \leq -\frac{m}{\theta}.
\]
To estimate $W^0_{nN}(\beta)^\theta$, we focus on the balls which the underlying random walk passes through in each $m = n, 2n, 3, \ldots, nN$:

$$W^0_{nN}(\beta) = P_0[\exp(\beta H_{nN}(S, \omega) - nN\lambda(\beta))] \leq \sum_{z_1, \cdots, z_N \in F^{(n)}} P_0[\exp(\beta H_{nN}(S, \omega) - nN\lambda(\beta)) : S_m \in B(z_i, 5n_w), i = 1, \cdots, N].$$

Therefore, we have that

$$Q \left[ W^0_{nN}(\beta)^\theta \right] \leq \sum_{z_1, \cdots, z_N \in F^{(n)}} Q \left[ P_0[\exp(\beta H_{nN}(S, \omega) - nN\lambda(\beta)) : S_m \in B(z_i, 5n_w), i = 1, \cdots, N]^\theta \right],$$

(3.5)

where we used the fact $(a + b)^\theta \leq a^\theta + b^\theta$ for $a, b \geq 0$ and $\theta \in (0, 1)$. For simplicity of notation, we denote by

$$W_{nN}(Z) = P_0[\exp(\beta H_{nN}(S, \omega) - nN\lambda(\beta)) : S_m \in B(z_i, 5n_w), i = 1, \cdots, N].$$

We will take $h_V(\beta) = C_1 \beta^{-\frac{1}{4\beta_1}}$, where $C_1 > 0$ will be chosen large later. Then, we have that

$$C_1 \beta^{-\frac{1}{4\beta_1}} \leq n < C_1 \beta^{-\frac{1}{4\beta_1}} + 1$$

(3.6)

For each $Z = (z_1, \cdots, z_N) \in (I^{(n)})^N$, we introduce a new probability measure which has a Radon-Nikodym derivative

$$\frac{dQ_Z}{dQ} = \exp \left( -\sum_{(l,x) \in J_Z} (\delta_n \omega(l,x) - \lambda(-\delta_n)) \right) = \prod_{i=0}^{N-1} \exp \left( -\sum_{(l,x) \in J_{i,z_i}} (\delta_n \omega(l,x) - \lambda(-\delta_n)) \right),$$

where

$$J_Z = \{(in + k, x) \in \mathbb{N} \times V : i = 0, \cdots, N - 1, k = 1, \cdots, n, x \in B(z_{i-1}, C_2n_w)\}$$

$$J_{i,z_i} = \{(in + k, x) \in \mathbb{N} \times V : k = 1, \cdots, n, x \in B(z_{i-1}, C_2n_w)\}$$

with $z_0 = 0$, $C_2$ will be chosen large later, and $\delta_n = (C_V n(C_2n_w)^{d_j})^{-\frac{1}{2}}$.

Then, Hölder’s inequality yields that

$$Q \left[ P_0[e_{nN} : S_m \in B(z_i, 5n_w), i = 1, \cdots, N]^\theta \right] = Q_Z \left[ \frac{dQ}{dQ_Z} P_0[e_{nN} : S_m \in B(z_i, 5n_w), i = 1, \cdots, N]^\theta \right]$$
where we have used $\lambda(\beta) + \lambda(-\beta) \leq C_3 \beta^2$ for small $\beta > 0$ with some constant $C_3 > 0$.

Markov property of $S$ implies that

$$Q \left[ P_0 \left[ e_{nN} : S_{in} \in B(z_i, \frac{5}{2}n_w), i = 1, \cdots, N \right] \right]$$

$$\leq P_0 \left[ Q \left[ \exp \left( -\sum_{i=1}^{N-1} \sum_{x \in B(z_i, \frac{5}{2}n_w)} \delta_n \omega(l, x) - \lambda(-\delta_n) \right) e_n : S_n \in B(z_{i+1}, \frac{5}{2}n_w) \right] \right]$$

$$\leq \prod_{i=0}^{N-1} \max_{x \in B(z_i, \frac{5}{2}n_w)} P_x \left[ \exp \left( (-C_4 \beta \delta_n)^\# \{ 1 \leq l \leq n : S_l \in B(z_i, C_2n_w) \} : S_n \in B(z_{i+1}, \frac{5}{2}n_w) \right) \right]$$

$$\leq \prod_{i=0}^{N-1} \max_{x \in B(z_i, \frac{5}{2}n_w)} \left[ \sup_{y \in \mathcal{F}_0, z \in \mathcal{F}_0} \max_{x \in B(y, \frac{5}{2}n_w)} P_x \left[ \exp \left( -C_4 \beta \delta_n^\# \{ 1 \leq l \leq n : S_l \in B(y, C_2n_w) \} : S_n \in B(z, \frac{5}{2}n_w) \right) \right] \right]$$

where we have used $\lambda(\beta - \delta_n) - \lambda(\beta) - \lambda(-\beta) \leq -C_4 \beta \delta_n$ for small $\beta > 0$ with some constant $C_4 > 0$.

Putting together (3.5), (3.7), (3.8), and (3.9), we get

$$Q \left[ W^0_{nN}(\beta) \right]$$

$$\leq e^{C_5 N} \left( \sup_{y \in \mathcal{F}_0, z \in \mathcal{F}_0} \max_{x \in B(y, \frac{5}{2}n_w)} P_x \left[ \exp \left( -C_4 \beta \delta_n^\# \{ 1 \leq l \leq n : S_l \in B(y, C_2n_w) \} : S_n \in B(z, \frac{5}{2}n_w) \right) \right] \right)^N.$$
Thus, (3.4) follows when we show that for any $\varepsilon > 0$, there exists $C_1 > 0, C_2 > 0$ such that

$$\sup_{y \in I^{(n)}} \sum_{z \in I^{(n)}} \max_{x \in B(y, 5n_w)} P_x \left[ \exp \left( -C_4 \beta \delta_n \{ 1 \leq l \leq n : S_l \in B(y, C_2 n_w) \} \right) : S_n \in B(z, 5n_w) \right] \frac{1}{2} < \varepsilon.$$ 

For $y \in I^{(n)}$ and $R \in \mathbb{N}$, we consider

$$J_{y,R,1}^{(n)} = \{ z \in I^{(n)} : d(B(y, 5n_w), B(z, 5n_w)) \geq R n_w \},$$

$$J_{y,R,2}^{(n)} = \{ z \in I^{(n)} : d(B(y, 5n_w), B(z, 5n_w)) < R n_w \}.$$

Then, we obtain from (UHK) that for $x \in B(y, 5n_w)$

$$P_x (S_n \in B(z, 5n_w)) \leq \sum_{u \in B(y, 5n_w)} \frac{e_1}{n^d} \exp \left( -c_2 \left( \frac{d(x, u)^{d_y}}{n} \right) \right) \leq \tilde{V}(z, 5n_w) e_1 \frac{c_1}{n^d} \exp \left( -c_2 \left( \frac{d(B(y, 5n_w), B(z, 5n_w))^{d_y}}{n} \right) \right) \leq c_1 C'_Y 5^{d_f} \exp \left( -c_2 \left( \frac{d(B(y, 5n_w), B(z, 5n_w))^{d_y}}{n} \right) \right).$$

Thus, we obtain that

$$\sum_{z \in J_{y,R,1}^{(n)}} \max_{x \in B(y, 5n_w)} P_x (S_n \in B(z, 5n_w)) \frac{1}{2} \leq \sum_{k=R}^{\infty} \sum_{z \in I^{(n)}} \max_{x \in B(y, 5n_w)} P_x (S_n \in B(z, 5n_w)) \frac{1}{2} \leq \sum_{k=R}^{\infty} C_C(k + 10)^{d_f} \left( c_1 C'_Y 5^{d_f} \exp \left( -c_2 k^{\frac{1}{d_y-1}} \right) \right) \frac{1}{2} \leq \frac{\varepsilon}{2},$$

by taking $R > 0$ large enough since the summation in the second inequality converges.

For each $z \in J_{y,R,2}^{(n)}$, we have that

$$P_x \left[ \exp \left( -C_4 \beta \delta_n \{ 1 \leq l \leq n : S_l \in B(y, C_2 n_w) \} \right) : S_n \in B(z, 5n_w) \right] \leq P_x \left[ \exp \left( -C_4 \beta \delta_n \{ 1 \leq l \leq n : S_l \in B(y, C_2 n_w) \} \right) \right] \leq P_x \left( \tau(x, C_2 n_w) < n \right) + e^{-C_4 \beta \delta_n} \leq 2 c_5 \exp \left( -c_6 \left( \frac{C_2}{2} \right) \frac{d_y}{n^{d_y-1}} \right) + \exp \left( -C_4 \left( \frac{c_1}{n} \right) \frac{d_y-d_f}{2^{d_y}} \right) \left( C'_Y n (C_2 n_w)^{d_f} \right) \frac{1}{2} n).$$
\[
\leq 2c_5 \exp \left( -c_6 \left( \frac{C_2}{2} \right)^{\frac{d\mu}{d\nu-1}} \right) + \exp \left( -C_4 \frac{d\mu-d\nu}{\sqrt{C_1'C_2}} \right),
\]

and hence we have that for each \( y \in V \) and fixed \( R > 0 \),

\[
\sum_{z \in J(n)} \max_{x \in B(y,5n_\mu)} P_x(S_n \in B(z,5n_\mu))^\frac{1}{2} \\
\leq \sum_{z \in J(n)} \left( 2c_5 \exp \left( -c_6 \left( \frac{C_2}{2} \right)^{\frac{d\mu}{d\nu-1}} \right) + \exp \left( -C_4 \frac{d\mu-d\nu}{\sqrt{C_1'C_2}} \right) \right)^\frac{1}{2} \\
\leq C(C(10R)^{d\nu}) \left( 2c_5 \exp \left( -c_6 \left( \frac{C_2}{2} \right)^{\frac{d\mu}{d\nu-1}} \right) + \exp \left( -C_4 \frac{d\mu-d\nu}{\sqrt{C_1'C_2}} \right) \right)^\frac{1}{2} \\
< \frac{\varepsilon}{2},
\]

by taking \( C_2 > 0 \) and then \( C_1 > 0 \) large enough.

### 4 Lower bound

This section is devoted to the proof of Theorem 1.3 (2). Throughout this section, we assume (UHK) and (LHK), which imply (BW), (BG), (VG), and (VG').

Our proof of the lower bound is a modification of the one in [3]. Since \( G = (V,E) \) is infinite graph, there exists an infinite path \( \{x_n\}_{n=0}^\infty \) such that \( x_0 = 0, \langle x_i, x_{i+1} \rangle \in E \), and \( d(0,x_n) = 0 \) for \( n \geq 1 \).

For small \( \beta > 0 \), let \( n \) be the smallest integer bigger than \( h_V(\beta) = C_1 \beta^{-\frac{4}{2-\delta}} \), where \( C_1 > 0 \) will be taken small later.

We also define \( n_\mu = n^\frac{1}{\mu-\nu} \) for \( n \geq 1 \).

We consider two sets of balls

\[
\mathbb{B}_n = \{ B_i^{(n)} := B(x_{i\mu n}, n_\mu) : i \geq 0 \} \\
\tilde{\mathbb{B}}_n = \{ \tilde{B}_i^{(n)} := B(x_{i\mu n}, C_7 n_\mu) : i \geq 0 \},
\]

where \( C_7 \geq 5 \) will be chosen large enough later.

Then, it is clear that

\[
B_i^{(n)} \cap B_j^{(n)} \neq \emptyset \quad (i \neq j) \\
\tilde{B}_i^{(n)} \cap \tilde{B}_j^{(n)} = \emptyset \quad (|i-j| \geq 2C_7 + 2)
\]
$B_j^{(n)} \subset \tilde{B}_i^{(n)}$ \quad ($|i-j| \leq 1$).

For each $I,J \in \mathbb{N}$ with $0 \leq J \leq I$, we set a rectangle in $\mathbb{N}_0 \times V$

$$R^{(n)}(I,J) = [In, (I+1)n] \times \tilde{B}_j^{(n)}.$$  

We introduce a coarse grained time-space lattice embedded into $\mathbb{N}_0 \times \mathbb{Z}$:

$$\mathbb{L}_{CG} = \{(I,J) \in \mathbb{N}_0 \times \mathbb{N}_0 : 0 \leq J \leq I, I-J \in 2\mathbb{N}_0\}.$$  

A path $\Gamma = \Gamma_{(I,J)}$ from $(0,0)$ to $(I,J)$ in $\mathbb{L}_{CG}$ is a sequence of sites $\{(i, \gamma_i) : i = 0, \ldots, I\}$ with $\gamma_0 = 0, \gamma_{i+1} - \gamma_i = 1 (i = 0, \ldots, I-1)$, and $\gamma = J$. Also, an infinite path $\Gamma = \Gamma_\infty$ is an infinite sequence of sites $\{(i, \gamma_i) : i \geq 0\}$ with $\gamma_0 = 0, \gamma_{i+1} - \gamma_i = 1 (i \geq 0)$. A length of a path $\Gamma_{(I,J)}$ denoted by $|\Gamma|$ is $I$ and we define $|\Gamma| = \infty$ for an infinite path $\Gamma$. We denote by $\Gamma(i) = \gamma_i$ the spatial site of a path $\Gamma$ at time $i$. For $\Gamma_{(I,J)}$ and $\Gamma_{(J,I)}$, we say that $\Gamma_{(I,J)}$ is closer to $0$ than $\Gamma_{(J,I)}$ if $\Gamma(i) \leq \Gamma(i)$ for $0 \leq i \leq I$.  

Given a finite path $\Gamma = \Gamma_{(I,J)}$, we denote by $\Gamma_+ = \Gamma_{(I,J),+}$ and $\Gamma_- = \Gamma_{(I,J),-}$ the path to $(I+1, J+1)$ and $(I+1, J-1)$ whose path up to time $I$ coincides with $\Gamma$, respectively.  

For an $I \geq 0$ and a path $\Gamma$ with $|\Gamma| \geq I$, we consider a set $\Omega_I^{(n)}(\Gamma)$ of trajectories of random walk $S$ up to time $In$ by

$$\Omega_I^{(n)}(\Gamma) = \left\{ s = \{(i,s_i)_{i=0}^{In} : s_0, s_{In} \in B_n^{(n)} \text{ for } L \leq I, s \subseteq \bigcup_{L \leq I} R^{(n)}(L, \gamma_L) \} \right\},$$  

where we remark that random walk doesn’t have to start at $0$ in the definition.

We obtain a lower bound of the free energy by the following lemma:

**Lemma 4.1.** Taking $C_1 > 0$ small enough in $hv(\beta)$ and $C_7 > 0$ large enough, then there exists a $\bar{c} \in (0,1)$ and $p > 0$ such that $\beta \in (0, \beta_0]$

$$Q\left(\text{There exists a random infinite path } \Gamma \text{ such that } W_{In}\left(\Omega_I^{(n)}(\Gamma)\right) \geq \bar{c}^I \text{ for all } I \geq 0\right) > p,$$

where we define

$$W_{In}(A) = P^x_S[e_n : A] \quad \text{for } A \in \mathcal{G}_S, x \in V.$$  

Indeed, it is trivial that for any infinite path $\Gamma$

$$W_{In} \geq W_{In}\left(\Omega_I^{(n)}(\Gamma)\right)$$  

and hence Lemma 4.1 implies that

$$Q\left(\lim_{I \to \infty} \frac{1}{In} \log W_{In} \geq \frac{\bar{c}}{n}\right) > p.$$  

and Proposition 1.1 tells us that $F_q(\beta) \geq \frac{\bar{c}}{n} Q$-a.s.
4.1 Proof of Lemma 4.1

Suppose 1-0 is assigned to each site \((I, J) \in L_{CG}\) in a certain manner. Then, we say that \((I, J)\) is open (closed) if 1 (0) is assigned to \((I, J)\).

We say a path \(\Gamma_{(I,J)}\) is

- **open** if all the site \((i, \gamma_i)\) is open,
- **maximal** if it has the maximum number of open sites in \(0 \leq i \leq I - 1\) among all paths to \((I, J)\),
- **optimal** if it is the maximal path which is closer to 0 than any other paths \(\Gamma'_{(I,J)}\).

For each \((I, J) \in L_{CG}\), we denote by \(\Gamma_{opt}^{(I,J)}\) the optimal path to \((I, J)\), which is uniquely determined by the configuration in the time-space site up to time \(I - 1\).

Now, we will assign 1-0 to each site by induction in \(I\). Let \(\tilde{c} > 0\) be a constant which will be given explicitly later.

We assign 1 to \((0, 0)\) if
\[
W_n(\Omega_1^{(n)}(\Gamma_{(0,0),+})) \geq \tilde{c}
\]
and 0 otherwise. Given the 1-0 state to sites \((i, j) \in L_{CG}\) for \(i \leq I - 1\), then we assign 1 to the site \((I, J)\) for \(0 < J \leq I\) if
\[
W_n(\Omega_1^{(n)}(\Gamma_{(I,J),+})) \geq \tilde{c} \quad \text{and} \quad W_n(\Omega_1^{(n)}(\Gamma_{(I,J),-})) \geq \tilde{c},
\]
and to the site \((I, 0)\) if
\[
W_n(\Omega_1^{(n)}(\Gamma_{(I,0),+})) \geq \tilde{c} ,
\]
otherwise 0.

The construction implies that if the optimal path to the site \((I, J), \Gamma_{opt}^{(I,J)}\), is open, then
\[
W_{(I+1)n}(\Omega_1^{(n)}(\Gamma_{(I,J),*})) \geq \tilde{c}^{I+1}, \quad * \in \{+,-\}.
\]

Also, if there exists an infinite open path \(\Gamma\), then
\[
W_{In}(\Omega_1^{(n)}(\Gamma)) \geq \tilde{c}^I \quad \text{for any} \ I \geq 0.
\]

Thus, it is enough to show that there exists \(p_1 > 0\) such that
\[
Q(\text{There exists an infinite open path } \Gamma) > p_1,
\]
where $\Gamma|_J$ is the path of $\Gamma$ up to time $I$.

We introduce random probability measures on $B_J^{(n)}$ by

$$V_{(I,J)}^{(n)}(x) = \frac{1}{W_{In}(\Omega_I^{(n)}(\Gamma_{opt}^{(I,J)}))} W_{In} \left( (\Omega_I^{(n)}(\Gamma_{opt}^{(I,J)})) \cap \{S_{In} = x\} \right), \quad x \in B_J^{(n)}.$$  

Then, we find that

$$\tilde{W}_{(I,J),*} = \sum_{x \in B_J^{(n)}} V_{(I,J)}^{(n)}(x) \theta_{In} \circ W_n x \left( \{S_i \in \hat{B}_J^{(n)}, i = 0, 1, \ldots, n\} \cap \{S_n \in B_J^{(n)}\} \right).$$

It is easy to see that

$$Q \left[ \tilde{W}_{(I,J),*} \mid \mathcal{F}_{In} \right] = \sum_{x \in B_J^{(n)}} V_{(I,J)}^{(n)}(x) P_S^{(n)} \left( \{S_i \in \hat{B}_J^{(n)}, i = 1, \ldots, n\} \cap \{S_n \in B_J^{(n)}\} \right)$$

for $* \in \{+,-\}$. Let $\tilde{\Omega}_{J*,n}(S) = \{S_i \in \hat{B}_J^{(n)}, i = 1, \ldots, n\} \cap \{S_n \in B_J^{(n)}\}$.

**Proposition 4.2.** There exists $c > 0$ such that

$$\inf_{n \geq 1} \inf_{J \in \Gamma_0} \inf_{x \in B_J^{(n)}} P_S^{(n)} \left( \{S_i \in \hat{B}_J^{(n)}, i = 1, \ldots, n\} \cap \{S_n \in B_J^{(n)}\} \right) > c.$$  

**Proof.** For fixed $n \geq 1, J \geq 1, x \in B_J^{(n)}$,

$$P_S^{(n)} \left( \{S_i \in \hat{B}_J^{(n)}, i = 1, \ldots, n\} \cap \{S_n \in B_J^{(n)}\} \right) \geq P_S^{(n)} \left( S_n \in B_J^{(n)} \right) - P_S^{(n)} \left( S_i \notin \hat{B}_J^{(n)}, i = 1, \ldots, n \right).$$

It is easy to see that for any $J \geq 1, x \in B_J^{(n)}$

$$\sum_{y \in B(x,J_{(u),\mu}, \frac{1}{2} n_u)} (P_S^{(n)}(S_n = y) + P_S^{(n)}(S_{n+1} = y)) \leq \sum_{y \in B(x,J_{(u),\mu}, \frac{1}{2} n_u)} P_S^{(n)}(S_n = y) + \sum_{z \in \mathcal{V}} \sum_{y \in B(x,J_{(u),\mu}, \frac{1}{2} n_u + 1)} P_S^{(n)}(S_n = z) P_S^{(n)}(S_1 = y) \leq C_8 P_S^{(n)} \left( S_n \in B_J^{(n)} \right),$$

where $C_8 > 0$ is a constant depending only on $\mu$. Thus, (LHK) implies that there exists a constant $C_9 > 0$ such that for any $J \geq 1, x \in B_J^{(n)}$

$$P_S^{(n)} \left( S_n \in B_J^{(n)} \right) \geq C_9.$$
From (1.4), we can take
\[ P_x \left( S_i \notin \tilde{B}^{(n)}_i, i = 1, \ldots, n \right) \leq 2c_5 \exp \left( -c_6 \left( \frac{C_7 - 1}{2} \right)^{\frac{\delta n}{2(N^2 - 1)}} \right) \]
small arbitrarily by letting \( C_7 > 0 \) large enough.

Throughout this section, we will fix a constant \( \tilde{c} = \frac{1}{2}c \), where \( c > 0 \) is a constant appeared in Proposition 4.2.

Let \( X^{(n)}_{(I,J)} = 1 \{ (I,J) \) is open} for \( (I,J) \in \mathbb{L}_{CG} \). In the proof of Lemma 4.1, we will show that the \( \{ X^{(n)}_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \) stochastically dominates a super-critical oriented percolation.

**Lemma 4.3.** For any \( \epsilon > 0 \), there exists \( C_1 > 0 \) such that for any \( (I,J) \in \mathbb{L}_{CG} \) and for any small \( \beta > 0 \)
\[ Q \left( X^{(n)}_{(I,J)} = 1 | \mathcal{F}_n \right) > 1 - \epsilon. \]

The following lemma which is a modification of [26, Theorem 1.3] tells us that 1-0-states \( \{ X^{(n)}_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \) stochastically dominate non-trivial Bernoulli random variables \( \{ Y_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \).

**Lemma 4.4.** Suppose that \( Q \left( X^{(n)}_{(I,J)} = 1 \right) > p. \) There exists a \( p_0 \in (0,1) \) such that if \( p > p_0 \), then there exist i.i.d. Bernoulli random variables \( \{ Y_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \) with density \( 0 < \rho(p) < 1 \) which are stochastically dominated from above by \( \{ X^{(n)}_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \).

Furthermore, \( \rho(p) \to 1 \) as \( p \to 1. \)

**Proof of Lemma 4.1.** Lemma 4.3 and Lemma 4.4 yield that \( \{ X^{(n)}_{(I,J)} : (I,J) \in \mathbb{L}_{CG} \} \) dominates oriented site percolation on \( \mathbb{N}_0 \times \mathbb{N}_0 \) with density \( 0 < \rho(p) < 1 \) from above.

In particular, we can find by the standard contour argument that the critical probability \( \tilde{p} \) of oriented site percolation is non-trivial. Thus, taking \( C_1 > 0 \) small such that \( Q \left( X^{(n)}_{(I,J)} = 1 \right) > p \) with \( \rho(p) > \tilde{p} \), we have that
\[ Q \left( \text{There exists an infinite nearest neighbor path } \Gamma \right) > 0. \]

**4.2 Proof of Lemma 4.3**

When we prove that
\[ Q \left( \left( \frac{\tilde{W}^{(n)}_{(I,J),*}}{\mathcal{F}_n} \right)^2 \bigg| \mathcal{F}_n \right) - Q \left( \frac{\tilde{W}^{(n)}_{(I,J),*}}{\mathcal{F}_n} \bigg| \mathcal{F}_n \right)^2 \leq \frac{\varepsilon}{8} \quad \text{for } * \in \{ +, - \} \]
\(Q\)-a.s., we have from Chebyshev’s inequality that

\[
Q\left(\left|\bar{W}^{(n)}_{(I,J),*} - Q\left[\bar{W}^{(n)}_{(I,J),*} \mid \mathcal{F}_n\right]\right| \geq \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2}.
\]

In particular, we have that

\[
1 - \frac{\varepsilon}{2} < Q\left(\bar{W}^{(n)}_{(I,J),*} \geq \frac{1}{2} Q\left[\bar{W}^{(n)}_{(I,J),*} \mid \mathcal{F}_n\right] \mid \mathcal{F}_n\right) \leq Q\left(\bar{W}^{(n)}_{(I,J),*} > c \mid \mathcal{F}_n\right).
\]

Combining Proposition 4.2, it is enough to show that when we take \(C_1 > 0\) small,

\[
Q\left(\left(\bar{W}^{(n)}_{(I,J),*}\right)^2 \mid \mathcal{F}_n\right) - Q\left(\bar{W}^{(n)}_{(I,J),*} \mid \mathcal{F}_n\right)^2
\]

could be small.

It is easy to see from the second moment argument in DPRE that

\[
Q\left(\left(\bar{W}^{(n)}_{(I,J),*}\right)^2 \mid \mathcal{F}_n\right) - Q\left(\bar{W}^{(n)}_{(I,J),*} \mid \mathcal{F}_n\right)^2
\]

\[
= \sum_{x,x' \in B_{\delta'}^{n}} v_{(I,J)}^{(n)}(x) v_{(I,J)}^{(n)}(x') P_{S,S'}^{x,x'} \left[\exp(\gamma(\beta)L_n(S,S')) - 1 : \tilde{\Omega}^{(n)}(S) \cap \tilde{\Omega}^{(n)}(S')\right]
\]

\[
\leq \sum_{x,x' \in B_{\delta'}^{n}} v_{(I,J)}^{(n)}(x) v_{(I,J)}^{(n)}(x') P_{S,S'}^{x,x'} \left[\exp(\gamma(\beta)L_n(S,S')) - 1\right]
\]

where \(\gamma(\beta) = \lambda(2\beta) - 2\lambda(\beta)\), \(P_{S,S'}^{x,x'}\) is the product of probability measures \(P_S^x\) and \(P_{S'}^{x'}\), and \(L_n(S,S') = \sum_{i=1}^{n} 1\{S_i = S'_i\}\) is the collision local time of two simple random walks \(S\) and \(S'\) up to time \(n\).

Since \(\exp(\gamma(\beta)L_n(S,S')) = \prod_{i=1}^{n} \left(e^{\gamma(\beta)1\{S_i = S'_i\}} - 1 + 1\right) = \prod_{i=1}^{n} \left(e^{\gamma(\beta)} - 1\right) 1\{S_i = S'_i\} + 1\), we have

\[
P_{S,S'}^{x,x'} \left[\exp(\gamma(\beta)L_n(S,S')) - 1\right]
\]

\[
= \sum_{k=1}^{n} \sum_{1 \leq j_1 < \ldots < j_k \leq n} \prod_{i=1}^{k} \left(e^{\gamma(\beta)} - 1\right) 1\{S_{j_i} = S'_{j_i}\}
\]

\[
= \sum_{k=1}^{n} (e^{\gamma(\beta)} - 1)^k \sum_{1 \leq j_1 < \ldots < j_k \leq n} \prod_{i=1}^{k} P_{S,S'}^{x,x'}(S_{j_i} = S'_{j_i} = x_i) \prod_{i=1}^{k-1} P_S^{x_i}(S_{j_{i+1}} - j_i = x_{i+1})^2
\]

\[
\leq \sum_{k=1}^{n} (e^{\gamma(\beta)} - 1)^k \left(Cn^{1 - \frac{d}{4\nu}}\right)^k,
\]
where we have used the fact that
\[
\sum_{i=1}^{n} \sum_{y \in V} P^x(S_i = y)^2 \leq \sum_{i=1}^{n} \sup_{y \in V} P^x(S_i = y) \sum_{z \in V} P^x(S_i = z)^2 \leq \sum_{i=1}^{n} \frac{c_1}{s^d} \leq C n^{1 - \frac{d}{m}},
\]
in the last line. Since we have that
\[
\lim_{\beta \to 0} e^{\gamma(\beta)} - 1 = \gamma''(0),
\]
it follows that for small $\beta > 0$
\[
Q \left( \left( \widetilde{W}_{(I,J)^*}^{(n)} \right)^2 \middle| \mathcal{F}_{in} \right) - Q \left( \left( \widetilde{W}_{(I,J)^*}^{(n)} \right)^2 \middle| \mathcal{F}_{in} \right) \leq \sum_{k=1}^{n} \left( 2(2\gamma''(0)\beta)^2 \left( C n^{1 - \frac{d}{m}} \right) \right)^k \leq \sum_{k=1}^{n} \left( 2(2\gamma''(0)\beta)^2 CC_1^{2\gamma''(0)\beta - 2} \right)^k \leq \frac{2\gamma''(0)CC_1^{2\gamma''(0)\beta - 2}}{1 - 2\gamma''(0)CC_1^{2\gamma''(0)\beta - 2}} \leq 2\gamma''(0)CC_1^{2\gamma''(0)\beta - 2} \leq 1 - 2\gamma''(0)CC_1^{2\gamma''(0)\beta - 2},
\]
(4.1)
when $C_1 > 0$ is taken small enough such that $2\gamma''(0)CC_1^{2\gamma''(0)\beta - 2} < 1$. Furthermore, letting $C_1 > 0$ small, the RHS in (4.1) could be smaller than $\frac{\delta}{8}$ and hence Lemma 4.3 follows.

A Proof of Lemma 4.4

First, we will introduce some notations. Let $S$ be a countable set and $\Omega(S) = \{0, 1\}^S$ with product topology and corresponding Borel $\sigma$-algebra, $\mathcal{B}$. We define a partial order to $\Omega(S)$ by saying that for $\omega_1, \omega_2 \in \Omega(S)$, $\omega_1 \leq \omega_2$ if
\[
\omega_1(s) \leq \omega_2(s), \quad \text{for all } s \in S.
\]
We say a measurable function $f : \Omega(S) \to \mathbb{R}$ is increasing if $f(\omega_1) \leq f(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega(S)$ with $\omega_1 \leq \omega_2$.

Given two probability measures $\mu, \nu$ on $(\Omega(S), \mathcal{B})$, we say that $\mu$ stochastically dominates $\nu$ ($\mu \succeq \nu$) if for any continuous increasing function $f$,
\[
\int_{\Omega(S)} f(\omega) \mu(d\omega) \geq \int_{\Omega(S)} f(\omega) \nu(d\omega).
\]

For $\rho \in [0, 1]$, we denote by $\pi_\rho = \prod_{s \in S} \mu_\rho(\omega(s))$ a probability measure on $(\Omega(S), \mathcal{B})$ with marginal distribution $\pi_\rho(\{\omega : \omega(s) = 1\}) = \mu_\rho^*(\omega = 1) = \rho$ for $s \in S$.

We shall come back to the proof of Lemma 4.4.

The followings is the first key lemma.
Lemma A.1. [29, Lemma 1], [26, Lemma 1.1] Suppose that \( \{X_s : s \in S\} \) is a family of \( \{0,1\} \)-valued random variables, indexed by \( S \), with joint law \( \mu \). Suppose \( S \) can be totally ordered in such a way that, given any finite subset of \( S \), \( s_1 < s_2 < \cdots < s_{j+1} \), and any choice of \( \varepsilon_1, \ldots, \varepsilon_j \in \{0,1\} \), then, whenever \( \mathbb{P}(X_{s_1} = \varepsilon_1, \cdots, X_{s_j} = \varepsilon_j) > 0 \),

\[
\mathbb{P}\left( X_{s_{j+1}} = 1 \mid X_{s_1} = \varepsilon_1, \cdots, X_{s_j} = \varepsilon_j \right) \geq \rho.
\]

Then, \( \mu \succ \pi_\rho \).

We define a total order to \( \mathbb{L}_{CG} \) by saying that

\[
(I_1, J_1) < (I_2, J_2) \quad \text{if} \quad \begin{cases} I_1 < I_2, \\
I_1 = I_2 \text{ and } J_1 < J_2.
\end{cases}
\]

We remark that \( X^{(n)}_{(I,J)} \) and \( X^{(n)}_{(I',J')} \) are independent conditioned on \( \mathbb{P}_n \) if \( |I_1 - J_1| \geq 4 \). For each \( (I,J) \in \mathbb{L}_{CG} \), we say that \( (I', J') \) is adjacent to \( (I,J) \) if \( I' = I \) and \( |J - J'| < 4 \).

The following lemma is a modification of [26, Lemma 1.2].

Lemma A.2. We denote by \( \mu^{(n)} \) the law of \( \{X_{i,j}^{(n)} : (I,J) \in \mathbb{L}_{CG}\} \). Let \( \varepsilon > 0 \) small enough such that there exist \( \alpha, r \in (0,1) \) with

\[
(1 - \alpha)(1 - r)^5 \geq \varepsilon \\
(1 - \alpha)\alpha^5 \geq \varepsilon
\]

Let \( \{Y_{i,j} : (I,J) \in \mathbb{L}_{CG}\} \) be a family independent of \( \{X_{i,j}^{(n)} : (I,J) \in \mathbb{L}_{CG}\} \) with joint law \( \pi_r \), and let \( Z_{i,j}^{(n)} = X_{i,j}^{(n)} Y_{i,j} \). Then, for each \( (I,J) \), and for any choice of \( \varepsilon_{i,j} \in \{0,1\} \) with \( (i,j) < (I,J) \), we have

\[
\mathbb{P}\left( Z_{(I,J)}^{(n)} = 1 \mid Z_{(i,j)}^{(n)} = \varepsilon_{i,j}, \text{ for all } (i,j) < (I,J) \right) \geq \alpha r.
\]

Proof. We omit the proof since it is the same as the one in [26].

Remark A.3. We don’t know whether \( \mathbb{Q}\left( X_{(I,J)}^{(n)} = 1 \mid \sigma[X_{(I',J')}^{(n)} : I' > I] \right) > 1 - \varepsilon \). So, we can’t apply [26, Theorem 1.3] directly to prove Proposition 4.2.

Proof of Lemma 4.4. We denote by \( \nu_{\alpha, r}^{(n)} \) the law of \( \{Z_{(I,J)}^{(n)} : (I,J) \in \mathbb{L}_{CG}\} \). Then, we find by coupling of \( X \) and \( Z \) that \( \mu^{(n)} \succ \nu_{\alpha, r}^{(n)} \). Combining with Lemma A.1 and Lemma A.2 yields \( \nu_{\alpha, r}^{(n)} \succ \pi_{\alpha r} \). Letting \( \varepsilon \to 0 \), we can take \( \alpha r \) close to 1 arbitrarily.
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