Extending $\mathit{ALCQIO}$ with reachability
A description logic for shape analysis

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Abstract. We introduce an extension $\mathit{ALCQIO}_{b,Re}$ of the description logic $\mathit{ALCQIO}$, a sub-logic of the two-variable fragment of first order logic with counting quantifiers, with reachability assertions. $\mathit{ALCQIO}_{b,Re}$-formulae can define an unbounded number of trees. We show that finite implication of $\mathit{ALCQIO}_{b,Re}$-formulae is polynomial-time reducible to finite satisfiability of $\mathit{ALCQIO}$-formulae. As a consequence, we get that finite satisfiability and finite implication in $\mathit{ALCQIO}_{b,Re}$ are NEXPTIME-complete. Description logics with transitive closure constructors have been studied before, but $\mathit{ALCQIO}_{b,Re}$ is the first decidable description logic which allows at the same time nominals, inverse roles, counting quantifiers and transitive closures. $\mathit{ALCQIO}_{b,Re}$ is well-suited for applications in software verification and shape analysis. Shape analysis requires expressive logics with reachability which have good computational properties. We show that $\mathit{ALCQIO}_{b,Re}$ can describe complex data structures with a high degree of sharing and allows compositions such as list of trees.

1 Introduction

Description Logics (DLs) are a well established family of logics for Knowledge Representation and Reasoning [2]. They model the domain of interest in terms of concepts (classes of objects) and roles (binary relations between objects). These features make DLs very useful to formally describe and reason about graph-structured information. The usefulness of DLs is witnessed e.g. by the W3C choosing DLs to provide the logical foundations to the standard Web Ontology Language (OWL) [11]. Another application of DLs is formalization and static analysis of UML class diagrams and ER diagrams, which are basic modeling artifacts in object-oriented software development and database design, respectively [33]. In these settings, standard reasoning services provided by DLs can be used to verify e.g. the consistency of a diagram.

We list some DLs which allow to express some form of reachability from the literature. They extend the classical DL $\mathit{ALC}$, whose concept descriptions are a syntactic variant of the multimodal logic $\mathit{K}_m$ [16][14]. The important work of Schild [14] exposed a correspondence between variants of propositional dynamic logic (PDL), a logic for reasoning about program behavior, and variants of DLs extended with further role constructors, e.g. the transitive closure of a
role. Close correspondences between DLs extended with fixpoints and variants of the \( \mu \)-calculus have also been identified \cite{15,10}. Recently, extensions of DLs with regular expressions over roles have been proposed \cite{5}. ALCQIO is the extension of ALC with nominals, number restrictions and inverses, see e.g. \cite{3}. No decidable extensions of ALCQIO with reachability or transitive closure are known.

Our contribution: In this paper we introduce and develop decision procedures for the logic ALCQIO\(_{b,Re}\), which extends the closure of ALCQIO under Boolean operations (ALCQIO\(_b\)) with reachability assertions over finite structures. The reachability assertions guarantee that elements of the universe of a model are reachable in the graph-theoretic sense from initial sets of elements using prescribed sets of binary relation symbols. Alternatively, we can think of ALCQIO\(_{b,Re}\) as ALCQIO\(_b\) interpreted over structures containing an unbounded number of trees of bounded degree \( d \).

The main results of this paper are algorithms which decide the finite satisfiability and finite implication problems of ALCQIO\(_{b,Re}\). The algorithms are reductions to finite satisfiability in ALCQIO, which suggests relatively simple implementation using existing ALCQIO reasoners. The algorithms run in \( \text{NEXPTIME} \), which is optimal since ALCQIO is already \( \text{NEXPTIME}\)-hard.

We discuss below the use of ALCQIO\(_{b,Re}\) in shape analysis. ALCQIO\(_{b,Re}\) is a flexible and powerful formalism for describing complex data structures with sharing. This, together with the fact that we have a decision procedure for implication and not just satisfiability, makes ALCQIO\(_{b,Re}\) a promising candidate for software verification applications. The use of DLs in shape analysis has been previously suggested in \cite{9}, where a framework for verification is given based mainly on the description logics \( \mu \text{ALCQIO} \), which extends ALCQIO with fixed points, and its restriction \( \mu \text{ALCQO} \). However, unlike ALCQIO\(_{b,Re}\), \( \mu \text{ALCQIO} \) and \( \mu \text{ALCQO} \) are is unknown to be decidable on finite structures.

A related extension of the two-variable fragment: After we proved the results reported in this paper, we noticed a similarity in proof strategy with a recent deep result \cite{8} based on \cite{12}. There, the complexity of finite satisfiability of the two-variable fragment of first order logic extended with counting quantifiers (\( C^2 \)) and additionally with two forests (\( CT^2 \)) is studied.

The results in our paper and in \cite{8} are incomparable due to differences in several orthogonal aspects. (i) \( C^2 \) strictly contains ALCQIO\(_b\). (ii) \( CT^2 \) is restricted to at most two forests, whereas ALCQIO\(_{b,Re}\) allows an unbounded number of reachability conditions. The decidability of the extension of \( C^2 \) with three successor relations is not known, while extending ALCQIO\(_b\) with three successor relations is covered by the results of this paper. (iii) we have a decision procedure for implication in ALCQIO\(_{b,Re}\), while no such decision procedure is given for \( CT^2 \) in \cite{8}. (iv) to our knowledge, no reasoners for \( C^2 \) exist; the sophisticated construction in \cite{8} makes the worthy task of implementing a reasoner for \( CT^2 \) a considerable challenge. In contrast, our result reduces reasoning in ALCQIO\(_{b,Re}\) to satisfiability in ALCQIO, which is contained in the description logic SROIQ for which several reasoners have been implemented, e.g. \cite{18,20,17}.
Due to the intricate nature of the proof in [8], the exact relationship between our result and that of [8] is difficult to ascertain. It would be beneficial in future work to understand whether these results can be united within a natural logic containing both $CT^2$ and $ALCQIO_{b,Re}$.

Shape analysis attempts to analyze and verify correctness of programs with dynamically allocated data structures. This is a notoriously difficult task, because it necessitates efficient decision procedures for expressive logics on graphs and graph-like structures. In the last decade, model-theoretic approaches have been less prominent, and the leading approach is proof-theoretic [13]. Recent advances in finite model theory have created an opportunity for development of practical model-theoretic approaches in shape analysis.

To describe the memory of programs with dynamic data structures using a DL, a rather powerful DL must be chosen. The DL in question needs to allow a computationally problematic combination of constructors: (i) nominals are required to represent the program’s variables; (ii) number restrictions are required so that the program’s pointers are interpreted as functions; (iii) inverses are needed for defining data structures such as trees, where elements in the tree must have at most one parent, and for encoding program computation; and (iv) reachability is required since data structures should contain only elements which are reachable from program variables via program pointers.

The logic $ALCQIO_{b,Re}$ we introduce in this paper is especially suited to shape analysis, since $ALCQIO_{b,Re}$ contains nominals, number restrictions, inverses and reachability. We will see in Section 2.1 that $ALCQIO_{b,Re}$ is strong enough to describe e.g. lists, trees and lists of lists. $ALCQIO_{b,Re}$ supports programs with sharing, in which memory cells (which in model-theoretic terms are elements of the universe of the model) may participate in multiple data structures. The closure of the underlying logic $ALCQIO_b$ under Boolean operations allows to describe conditional statements in programs. The decision procedure for implication for $ALCQIO_{b,Re}$ is essential for verification applications, since it allows to show that specifications relating pre- and post-conditions are correct.

We show that $ALCQIO_{b,Re}$ supports modular reasoning expressed by so-called frame rules. The frame rules are discussed in Section 2.1. Frame Rule 1 is similar to the frame rule in [13] and deals with data structures with disjoint domains. Frame Rule 2 deals with data structures whose domain is not disjoint, but have disjoint pointers.

Since $ALCQIO_{b,Re}$ is a description logic, using $ALCQIO_{b,Re}$ for shape analysis brings an additional advantage. The verification community has focused mostly on a bottom-up approach to the analysis of programs with dynamic data structures, which examines pointers and the shapes induced by them. However, many real world programs manipulate complex data whose structure and content is most naturally described by formalisms from object oriented programming and databases such as UML and ER diagrams which are generalized by the framework of description logic. In a recent preprint [7] we discussed how to use a description logic to reason and verify correctness of entity-relations-type content of data structures on top of an existing shape analysis. Technically, [7]
is based on a reduction of a DL to satisfiability in $CT^2$. The DL in [7] cannot express reachability and the approach there depends on a combination of DL with an existing shape analysis. We believe the method of [7] can be modified to be based on $ALCQIO_{b,Re}$ alone. Exploration of this question is part of future work.

2 The Formalism and Examples

From the point of view of finite model theory, $ALCQIO_b$ and $ALCQIO_{b,Re}$ are syntactic variants of fragments of first or second order logic. In description logics terminology, binary relation symbols are called atomic roles, unary relation symbols are called atomic concepts, and constant symbols are called nominals. Let $N_R$, $N_C$ and $N_n$ denote the sets of atomic roles, atomic concepts and nominals. A vocabulary $\tau$ is then the union of $N_R$, $N_C$ and $N_n$. Let $N_F \subseteq N_R$ be a set of atomic roles. The roles in $N_F$ are called functional.

Formulae are built from the symbols in $\tau$. The various constructors available to build formulae determine the particular description logic, giving rise to a wide family of logics with varying expressivity, and decidability and complexity of reasoning. The semantics to formulae is given in terms of structures, where atomic concepts and atomic roles are interpreted as unary and binary relations in a structure, respectively, and constants are interpreted as elements in the structure’s universe.

We now define $ALCQIO_b$ and $ALCQIO_{b,Re}$ precisely. See Section 2.1 for examples.

Definition 1 (Syntax of $ALCQIO_b$). The set of roles, concepts and formulae of $ALCQIO_b$ are defined inductively:

– Atomic concepts and nominals are concepts; Atomic roles are roles;
– If $r$ is a role, $C, D$ are concepts and $n$ is a positive integer, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists r.C$ and $\exists \leq n r.C$ are concepts, and $r^-$ is a role;
– $C \sqsubseteq D$ (concept inclusion) where $C, D$ are concepts, are formulae;
– If $\varphi$ and $\psi$ are formulae, then $\varphi \land \psi$, $\varphi \lor \psi$, and $\neg \psi$ are formulae.

The sub-logic $ALCQIO \subseteq ALCQIO_b$ contains all $ALCQIO_b$-formulae with no negations and no disjunctions.

A structure (or interpretation) is a tuple $M = (M, \tau, \cdot)$, where $M$ is a finite set (the universe), $\tau$ is a set of constants and unary and binary relation symbols (the vocabulary), and $\cdot$ is an interpretation function, which assigns to each constant $c \in \tau$ an element $c^M \in M$, and to each $n$-ary relation symbol $R \in \tau$ an $n$-ary relation $R^M$ over $M$. In this paper, each relation is either unary or binary (i.e. $n \in \{1, 2\}$). In this paper, all structures are finite. Satisfiability and implication always refer to finite structures only.

Definition 2 (Semantics of $ALCQIO_b$). The semantics to an $ALCQIO_b(\tau)$-formula $\varphi$ is given in terms of $\tau$-structures such that every $f \in N_F$, $f^M$ is a partial function. The function $\cdot^M$ is extended to the remaining concepts and roles.
inductively below. The satisfaction relation $\models$ is also given below. If $M \models \varphi$, then $M$ is a model of $\varphi$. We write $\psi \models \varphi$ and say that $\psi$ implies $\varphi$ if every model of $\psi$ is also a model of $\varphi$.

\[(C \cap D)^M = C^M \cap D^M\]
\[(C \cup D)^M = C^M \cup D^M\]
\[(r^-)^M = \{(e, e') \mid (e', e) \in r^M\}\]
\[(\exists r.C)^M = \{e \mid \exists e' : (e, e') \in r^M, e' \in C^M\}\]
\[\neg C^M = M \setminus C^M\]
\[(\exists r.n.C)^M = \{e \mid \exists e' : (e, e') \in r^M, e' \in C^M\}\]

$M \models C \subseteq D$ if $C^M \subseteq D^M$

$M \models \varphi \land \psi$ if $M \models \varphi$ and $M \models \psi$

$M \models \neg \varphi$ if $M \not\models \varphi$

$M \models \varphi \lor \psi$ if $M \models \varphi$ or $M \models \psi$

For $\mathcal{ALCQIO}_{b,Re}$, we define two new types of assertions.

**Reachability Assertion** $\text{Reach}(A, B, S)$ for every $A, B \in \text{N}_c$ and $S \subseteq \text{N}_r$.

**Disjointness Assertion** $\text{Disj}(A_1, A_2) = (A_1 \cap A_2 \equiv \bot)$ for every $A_1, A_2 \in \text{N}_c$.

Let $RE$ and $DI$ be sets of reachability respectively disjointness assertions.

**Compatibility** $RE$ and $DI$ compatible if for every $\text{Reach}(A_1, B_1, S_1)$ and $\text{Reach}(A_2, B_2, S_2)$ in $RE$ such that $S_1 \cap S_2 \neq \emptyset$, $\text{Disj}(A_1, A_2)$ is in $DI$.

**Definition 3 (Syntax of $\mathcal{ALCQIO}_{b,Re}$).** $\Psi = \psi \land RE \land DI$ is an $\mathcal{ALCQIO}_{b,Re}$-formula if

(A) $RE$ is a set of reachability assertions  
(B) $DI$ is a set of disjointness assertions  
(C) $\psi \in \mathcal{ALCQIO}_b$  
(D) $RE$ and $DI$ are compatible

For $RE$ we define $\text{CO}(RE) = \{B \subseteq A \mid \text{Reach}(A, B, S) \in RE\}$. For every $\beta = \text{Reach}(A, B, S) \in RE$ and $\tau$-structure $M$, let $D^M_\beta$ be the directed subgraph $\langle M, \cup_{s \in S}s^M \rangle$ induced by $A^M$.

**Definition 4 (Semantics of $\mathcal{ALCQIO}_{b,Re}$).** Let $\Psi = \psi \land RE \land DI \in \mathcal{ALCQIO}_{b,Re}$. For every $\tau$-structure $M$, $M \models \Psi$ if $M \models \psi \land DI \land \text{CO}(RE)$ (in $\mathcal{ALCQIO}_b$ semantics) and for every $\beta = \text{Reach}(A, B, S) \in RE$, every vertex of $D^M_\beta$ is reachable from a vertex of $B^M$.

In other words, denoting reflexive-transitive closure with $*$ and role composition with $\circ$, $M \models \Psi$ iff $M \models \psi \land DI \land \text{CO}(RE)$ and $B^M \times B^M \circ (\cup_{s \in S}s^M \cap A^M \times A^M)^* = B^M \times A^M$.

2.1 Examples

Lists and successor relations. Given a concept $L$, a nominal $\text{head}$ and a functional role $\text{next}$, $L^M$ is a singly-linked list from $\text{head}^M$ via $\text{next}^M$ if the directed subgraph of $\langle M, \text{ptr}^M \rangle$ induced by $L^M$ is a successor relation with minimal element $\text{head}^M$. This can be expressed by the $\mathcal{ALCQIO}_{b,Re}$ formula $\Phi_{\text{List}}$ obtained as the conjunction of $RE_1 = \{\text{Reach}(L, \text{head}, \text{next})\}$ and $DI_1 = \emptyset$. $RE_1$ expresses that $\text{head}^M \in L^M$ and all elements of $L^M$ are reachable from $\text{head}^M$ via $\text{next}^M$; $DI_1$ is empty since we have described no other data structure which could be disjoint from this list. $\Phi_{\text{List}}$ does not determine where the $\text{next}$ role of the last element of the list points. Acyclic and cyclic lists are defined as follows: $\Phi_{\text{aList}} = \Phi_{\text{List}} \land (L \subseteq \neg \exists \text{next}.(\neg L \cup \text{head}))$ and $\Phi_{\text{cList}} = \Phi_{\text{List}} \land (\exists \text{head} \subseteq \text{next}^- \cdot L)$. 

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\[ d \text{-ary trees. Given a concept } T, \text{ a nominal root and functional roles left and right, } T^M \text{ is a binary tree rooted at root}^M \text{ via left}^M \text{ and right}^M \text{ if the directed subgraph of } \langle M, \text{left}^M \cup \text{right}^M \rangle \text{ induced by } T^M \text{ is a directed tree rooted at root}^M \text{ in the graph-theoretic sense. This can be expressed by the ALCQIO formula } \Phi_T \text{ obtained as the conjunction of (a) root } \sqsubseteq \neg \exists \text{left}^-,T \text{ (b) root } \sqsubseteq \neg \exists \text{right}^-,T \text{ (c) } T \cap \neg \text{root } \subseteq \exists \leq \text{left}^-,T \lor \exists \leq \text{right}^-,T \text{ (d) } T \sqsubseteq \neg \exists \text{left}^-,T \lor \neg \exists \text{right}^-,T \text{ (e) } \text{RE}_i = \{ \text{Reach}(T, \text{root}, \{ \text{left}, \text{right} \}) \} \text{ and (f) } \text{DI}_i = \emptyset. \text{ (a-b) express that root}^M \text{ belongs to } T^M \text{ and is not pointed to from } T^M; \text{ (c-d) express that every element of } T^M \text{ besides the root has at most one incoming pointer from a } T^M \text{ element. (e) expresses that all elements of } T^M \text{ are reachable from root}^M \text{ via left}^M \text{ and right}^M. \text{ The case of } d \text{-ary trees, } d > 2, \text{ is similar, using } d \text{ functional roles child}_1, \ldots, \text{child}_d. \]

**Frame Rule 1.** ALCQIO\(_{b,Re}\) is closed under taking memory-disjoint union of data structures. E.g. ALCQIO\(_{b,Re}\)-formulae \( \Phi_i = \varphi_i \land \{ \text{Reach}(A_i, B_i, S_i) \}, \text{ } \) \( i = 1, 2 \), the following formula expresses that the domains of its models consist of two disjoint parts, corresponding to the \( \Phi_i: \Phi = \varphi_1 \land \varphi_2 \land \bigwedge \text{RE} \land \bigwedge \text{DI} \), where \( \text{RE} = \{ \text{Reach}(A_i, B_i, S_i) \} \mid i = 1, 2 \} \text{ and } \text{DI} = \{ \text{Disj}(A_1, A_2) \} \text{. There is no disjointness requirement on the roles is } S_1 \text{ and } S_2.\)

**Frame Rule 2.** ALCQIO\(_{b,Re}\) allows to define multiple data structures which may overlap in memory, as long as they do not share the same pointers. E.g. given ALCQIO\(_{b,Re}\)-formulae \( \Phi_i = \varphi_i \land \{ \text{Reach}(A_i, B_i, S_i) \}, \text{ } \) \( i = 1, 2 \), such that \( S_1 \) and \( S_2 \) are disjoint, the following formula expresses that the two data structures are defined simultaneously with possibly overlapping domain: \( \Phi = \varphi_1 \land \varphi_2 \land \bigwedge \text{RE} \), where \( \text{RE} = \{ \text{Reach}(A_i, B_i, S_i) \} \mid i = 1, 2 \} \).

**Composition of data structures.** Data structures such as list of lists, list of trees, tree of lists of lists, etc. can be expressed. Given a concept \( L \), a nominal head, and functional roles next\(_1\) and next\(_2\), \( L^M \) is an acyclic list of acyclic lists from head\(_1^M \) via next\(_1^M \) and next\(_2^M \) if there exists \( L_1^M \subseteq L^M \) such that \( L_1^M \) is an acyclic list from var\(_1^M \) via next\(_1^M \), and \( L^M \) is a disjoint union of acyclic lists via next\(_2^M \) whose heads belong to \( L_1^M \). This can be expressed similarly to \( \Phi_{\text{List}} \), using two reachability assertions \( \text{Reach}(L_1, \text{head}, \text{next}_1) \) and \( \text{Reach}(L, L_1, \text{next}_2) \). Since the two reachability assertions use disjoint roles, no disjointness assertion is required. Similarly, it is easy to express e.g. tree of acyclic lists of trees, etc.

### 3 Decision procedures for ALCQIO\(_{b,Re}\)

Let \( \Phi = \varphi \land \bigwedge \text{RE} \land \bigwedge \text{DI} \) be an ALCQIO\(_{b,Re}\)-formula. Let \( \tau = N_C \cup N_R \cup N_b \) be the vocabulary of \( \varphi \). We denote \( \text{RE} = \{ \text{Reach}(A_1, B_1, S_1), \ldots, \text{Reach}(A_h, B_h, S_h) \} \).

#### 3.1 Outline of proof

\( \varphi \land \bigwedge \text{CO}(\text{RE}) \land \bigwedge \text{DI} \) already belongs to ALCQIO\(_b\), but the reachability requirements are missing in order to capture \( \Phi \) completely. The models of \( \varphi \land
\(\wedge \text{CO(RE)} \wedge \wedge \text{DI}\) can be partitioned into two: standard and non-standard models, depending on whether they satisfy \(\wedge \text{RE}\). In general, we cannot augment \(\varphi \wedge \wedge \text{CO(RE)} \wedge \wedge \text{DI}\) to eliminate the non-standard models, since reachability is not expressible in \(\text{ALCQIO}\). However, we can augment it so that it is guaranteed that whenever a non-standard model exists, so does a standard model.

To do so, we define semi-connectedness, which is a weaker requirement than satisfying \(\wedge \text{RE}\). A model is semi-connected if every element of its universe which should be reachable according to some \(\text{Reach}(A_i, B_i, S_i)\) is not, is reachable from a cycle in \(A_i\). We show that semi-connectedness is expressible in \(\text{ALCQIO}\).

Under certain conditions, it is possible to apply an operation \(\triangleright\) which turns non-standard but semi-connected models into standard models, by eliminating the said cycles. The existence of a non-standard semi-connected model then implies the existence of a standard model. A sufficient condition under which semi-connected models can be turned to standard models using \(\triangleright\) is that there exist certain linear orderings of the types of their elements, which we call useful orderings. We show that having useful orderings is expressible in \(\text{ALCQIO}\).

As a consequence we get a decision procedure for satisfiability of \(\Phi\), which amounts to adding to \(\varphi \wedge \wedge \text{CO(RE)} \wedge \wedge \text{DI}\) the requirements that models are semi-connected and have useful orderings. The resulting \(\text{ALCQIO}\)-formula is satisfiable iff \(\Phi\) is. A decision procedure for implication is obtained as consequence. Decision procedures which are tight in terms of complexity are given in Section 3.5. In Section 3.4 we give simpler but complexity-wise suboptimal decision procedures. The decision procedures in Section 3.5 follow the same plan, and differ only in the construction and sizes of the formulae expressing the existence of useful orderings. Some proofs are omitted or only sketched here and are given in full in Appendix C.

### 3.2 Types and the operation \(\triangleright\)

We write \(C \in \varphi\) if there exists a concept \(D\) and an inclusion \(C \subseteq D\) or \(D \subseteq C\) which occurs in \(\varphi\).

**Definition 5 (\(\triangleright\)).** Let \(M\) be a \(\tau\)-structure. Let \(a_0, b_0, a_1, b_1 \in M\), \(r \in \text{NF}\) and \(t = (a_0, b_0, a_1, b_1, r)\) such that \(r^M(a_0) = b_0\) and \(r^M(a_1) = b_1\) and \(\{a_0, b_0\} \cap \{a_1, b_1\} = \emptyset\). Let \(M_{t^{\triangleright}}\) be the structure such that \(M\) and \(M_{t^{\triangleright}}\) have the same universe \(M\) and the same interpretations of every atomic concept, nominal and atomic role except for \(r\), and \(r^{M_{t^{\triangleright}}} = (r^M \setminus \{(a_0, b_0), (a_1, b_1)\}) \cup \{(a_0, b_1), (a_1, b_0)\}\).

For the main property of the operation \(\triangleright\) we need the notion of types.

**Definition 6 (Types).** \(M\) be a \(\tau\)-structure \(M\) and \(u \in M\). \(\text{tp}_M(u)\) is the set of \(C \in \varphi\) such that \(u \in C^M\). \(\text{tp}_M(u)\) extends \(\text{tp}_M(u)\) with a subset of \(\{\exists f.C \mid f \in \text{NF}, \varphi \in C\}\) such that \(\exists f.C \in \text{tp}_M(u)\) iff \(u \in (\exists f.C)^M\). We sometimes omit the subscript \(M\) when it is clear from the context.

**Lemma 1.** Let \(M_1\) and \(M_2\) be two \(\tau\)-structures with the same universe \(M\).
1. If for all \( u \in M \) we have \( \bar{tp}_{M_i}(u) = \bar{tp}_{M_2}(u) \), then \( M_1 \) and \( M_2 \) agree on \( \varphi \).

2. Let \( f \in F_\mathcal{F} \), and \( a_0, a_1, b_0, b_1 \in M \) such that \( \bar{tp}_{M_1}(a_0) = \bar{tp}_{M_1}(a_1) \) and \( (a_0, b_0) \in f^{M_1} \). There exist \( b_1 \in M \) such that \( \bar{tp}_{M_1}(b_0) = \bar{tp}_{M_1}(b_1) \) and \( (a_1, b_1) \in f^{M_1} \).

The crucial property of \( \triangleright \) is that \( M \) and \( M_{\triangleright} \) agree on \( \varphi \) if \( a_0 \) and \( a_1 \) have the same type.

**Lemma 2.** Let \( M \) be a \( \tau \)-structure, let \( r \in F_\mathcal{F} \), let \( a_0, b_0, a_1, b_1 \in M \) and let \( t = (a_0, b_0, a_1, b_1, r) \). If \( \bar{tp}_{M}(a_0) = \bar{tp}_{M}(a_1) \), \( r^M(a_0) = b_0 \), \( r^M(a_1) = b_1 \), and \( \{a_0,b_0\} \cap \{a_1,b_1\} = \emptyset \), then \( M \models \varphi \) if and only if \( M_{\triangleright} \models \varphi \).

The proof of Lemma 2 proceeds by induction on the construction of the concepts, showing that \( C^M = C^{M_{\triangleright}} \) for all \( C \in \varphi \). The only interesting cases are concepts of the form \( \exists \tau.C \) and \( \exists r^C \). Consider the more complicated case \( \exists r^{-}.C \). We define \( S_1(u) = \{v \mid (u,v) \in r^{-}^{M_1} \} \) and \( v \in C^{M_1} \) for \( M_1 = M \) and \( M_2 = M_{\triangleright} \). It is enough to show that \( |S_1(u)| = |S_2(u)| \) for all \( u \in M \), and this holds because \( S_2(u) \) can be obtained from \( S_1(u) \) by replacing \( a_0 \) for \( a_1 \) and \( a_1 \) for \( a_0 \).

If \( C^M = C^{M_{\triangleright}} \) for all \( C \in \varphi \), then every inclusion \( C \subseteq D \) in \( \varphi \) either holds in both \( M \) and \( M_{\triangleright} \) or does not hold in both structures. Moreover, we will use that for every \( u \in M \), \( \bar{tp}_{M}(u) = \bar{tp}_{M_{\triangleright}}(u) \) and \( \bar{tp}_{M}(u) = \bar{tp}_{M_{\triangleright}}(u) \).

### 3.3 Semi-connectedness and useful orderings

Here we define semi-connectedness and useful orderings exactly and prove that they capture reachability (Lemma 3).

**Definition 7 (semi-connected structure).** For every \( \beta_{h'} = \text{Reach}(A_{h'}, B_{h'}, S_{h'}) \), we write \( D_{h'}^M \) for the directed graph \( D_{h'}^\beta \). Let \( M \) be a \( \tau \)-structure. \( M \) is \( (RE,DI) \)-semi-connected if (I) \( M \models \bigwedge CO(RE) \land \bigwedge DI \) and (II) for every \( \text{Reach}(A_{h'}, B_{h'}, S_{h'}) \in RE \) and \( u \in A_{h'}^M \), either \( u \) is reachable in \( D_{h'}^M \) from \( B_{h'}^M \) or \( u \) is reachable from a cycle.

Observe that if \( M \) is \( (RE,DI) \)-semi-connected, then \( M \models \bigwedge RE \land \bigwedge DI \) if \( M \) satisfies the following strengthening of (II): for every \( \text{Reach}(A_{h'}, B_{h'}, S_{h'}) \in RE \) and \( u \in A_{h'}^M \), \( u \) is reachable from \( B_{h'}^M \).

**Definition 8 (Useful orderings).** Let \( M \) be a \( \tau \)-structure. Let \( \text{TYPES}(M) = \{\bar{tp}_{M}(u) \mid u \in M \} \) and \( 1 \leq h' \leq h \). A linear ordering \( R_{<} \) of \( \text{TYPES}(M) \) is \( h' \)-useful for \( M \) if for every element \( u \in A_{h'}^M \), either \( u \) is in \( B_{h'}^M \), or there exist elements \( v,w \in A_{h'}^M \) such that \( \bar{tp}_{M}(u) = \bar{tp}_{M}(v) \) and \( (\bar{tp}_{M}(w), \bar{tp}_{M}(v)) \in R_{<} \) and \( (w,v) \in \bigcup_{s \in S_{h'}} s^M \). Note that the size of \( \text{TYPES}(M) \) is at most \( 2^{|\varphi| \cdot |N_\mathcal{F}|} \).

**Lemma 3.** \( \Phi = \varphi \land \bigwedge RE \land \bigwedge DI \) is satisfiable if \( \varphi \) is satisfiable by a \( (RE,DI) \)-semi-connected structure with \( h' \)-useful orderings for every \( 1 \leq h' \leq h \).

**Proof (Sketch).** Consider \( M \) which satisfies \( \Phi \). \( M \) is trivially \( (RE,DI) \)-semi-connected. If \( \text{Reach}(A_{h'}, B_{h'}, S_{h'}) \) holds for \( M \), then the following ordering \( R_{<} \) of \( \text{TYPES}(M) \) obtained from a Depth-First Search run is \( h' \)-useful. The \( R_{<} \)
smallest elements in \(R_\prec\) are the types of the elements of \(B_{h'}\). A DFS is executed on \(D_{h'}^M\), always starting from elements of \(B_{h'}^M\) whenever all discovered vertices are fully explored. Whenever an element \(y\) whose type does not occur in \(R_{\prec}^{h'}\) is discovered in the search, \(\widetilde{tp}(y)\) is added as the largest element of \(R_{\prec}^{h'}\). All types which do not appear in \(R_{\prec}^{h'}\) after the DFS run ends are added as larger than the already ordered elements.

Conversely, assume \(\mathcal{M} \models \varphi\), \(\mathcal{M}\) is \((RE,DI)\)-semi-connected and has \(h'\)-useful orderings \(R_{\prec}^{h'}\) for all \(h'\). We obtain from \(\mathcal{M}\) another structure \(\mathcal{M}'\) such that \(\mathcal{M}' \models \varphi\) by repeated applications of \(\triangleright\). If \(\mathcal{M} \not\models \varphi\), we choose \(h'\) whose reachability assertion does not hold and a tuple \(t = (a_0,b_0,a_1,b_1,s)\) as follows (see Figure 3.1). We choose \(a_1 \in A_{h'}^M\) to be a minimal element with respect to \(R_{\prec}^{h'}\), which is not reachable from \(B_{h'}^M\) and belongs to a \(S_h\)-cycle. The \(h'\)-usefulness of \(R_{\prec}^{h'}\) implies that there are \(f \in \overline{S}_h\), \(a_0, w \in A_{h'}^M\) such that \(\overline{tp}_\mathcal{M}(a_0) = \overline{tp}_\mathcal{M}(a_1), (w,a_0) \in f^\mathcal{M}\) and \((\overline{tp}_\mathcal{M}(w), \overline{tp}_\mathcal{M}(a_0)) \in R_{\prec}^{h'}\). By the minimality of \(a_1, w\) and hence \(a_0\) are reachable from \(B_{h'}^M\). There is \(s \in S_h\) and \(b_1 \in s^\mathcal{M}(a_1)\) such that \(b_1\) belongs to a \(S_h\)-cycle on which \(a_1\) lies in \(D_{h'}^M\). Let \(b_0 \in s^\mathcal{M}(a_0)\) with \(\overline{tp}_\mathcal{M}(b_0) = \overline{tp}_\mathcal{M}(b_1)\) as guaranteed by Lemma 4. We have that \(\{a_0,b_0\} \cap \{a_1,b_1\} = \emptyset\), because \(a_0\) and \(b_0\) are reachable from \(B_{h'}^M\), and \(a_1\) and \(b_1\) are not. Using Lemma 2, \(\mathcal{M}_{tp} \models \varphi \land \bigwedge CO(RE) \land \bigwedge DI\). The set of vertices reachable from \(B_{h'}^M\) in \(D_{h'}^M\) is strictly contained in the set of vertices reachable from \(B_{h'}^{M_{tp}}\) in \(D_{h'}^{M_{tp}}\). Repeated applications of \(\triangleright\) leads to \(\mathcal{M}'\) for which no such tuple \(t\) can be found, because all of the redundant cycles have been eliminated, and hence \(\mathcal{M}' \models \varphi \land \bigwedge RE \land \bigwedge DI\).

### 3.4 Reducing connected satisfiability to (plain finite) satisfiability

Here we show how to express semi-connectedness and existence of useful orderings in \(\mathcal{ACLQTO}_b\). For semi-connectedness, this is easy:

**Lemma 4.** There exists a formula \(\delta_{semi}^{RE,DI} \in \mathcal{ACLQTO}_b\) such that \(\mathcal{M} \models \delta_{semi}^{RE,DI}\) iff \(\mathcal{M}\) is \((RE,DI)\)-semi-connected.

\(\delta_{semi}^{RE,DI}\) is given by \(\delta_{semi}^{RE,DI} = \bigwedge DI \land \bigwedge CO(RE) \land \bigwedge_{1 \leq h' < h} \delta_{reach-cyc}^{h'}\), where \(\delta_{reach-cyc}^{h'} = A_{h'} \cap \neg B_{h'} \subseteq \bigcup_{s \in S_h} \exists s^- A_{h'}\), see Appendix C. Next we turn to expressing the existence of useful orderings.

**Definition 9.** Let \(\mathcal{M}\) be a structure, let \(k = |TYPES(\mathcal{M})|\). For every \(1 \leq h' \leq h\), \(ORD(\mathcal{M})\) is the set of structures \(\mathcal{N}\) with universe \(N\), \(M \subseteq N\), satisfying:

1. \(M\) is a new atomic concept; the substructure of \(\mathcal{N}\) with universe \(M\) is \(\mathcal{M}\);
2. \(\mathcal{N}\) has new distinct nominals \(\phi_1^N, \ldots, \phi_k^N\) with \(\{\phi_1^N, \ldots, \phi_k^N\} = N \setminus M\);
3. \(ord\) is a new role whose interpretation \(ord^N\) is a linear ordering of \(N \setminus M\);
Lemma 5. Let \( X \) defines the property \( D_1 \leq N \) to \( \hat{u}, v \) expresses that if two elements \( v \) of \( \hat{u} \) has smaller type \( \hat{u'} \). Theorem 1. \( \theta \) \( X \) \( M \)\( \models \neg \). Let \( R_\prec^h \subseteq \text{TYPES}(M)^2 \) be given as follows: \( (t_1,t_2) \in R_\prec^h \) iff exist \( u_1,u_2 \) with types \( \bar{t}_pM(u_1) = t_1 \) and \( \bar{t}_pM(u_2) = t_2 \) and \( (f_N^{ord,h'}(u_1),f_N^{ord,h'}(u_2)) \in \text{ord}^N \). For every \( 1 \leq h' \leq h \), \( R_\prec^h \) is an \( h' \)-useful ordering for \( M \).

By the definition of \( \text{ORD}(M) \) we have:

**Lemma 5.** Let \( M \) be a structure. \( \text{ORD}(M) \) is non-empty iff there exists an \( h' \)-useful ordering for \( M \) and every \( 1 \leq h' \leq h \).

**Lemma 6.** For every \( \xi \in \text{ALCQIO}_b \) there exists a formula \( \theta_\xi \) such that \( M \models \xi \) iff there exists \( N \in \text{ORD}(M) \) such that \( N \models \theta_\xi \).

**Proof.** The formula \( \theta = \theta^1 \land \theta^2 \land \theta^3 \land \theta^{4a} \land \theta^{4b} \land \theta^{4c} \) defines \( \text{ORD}(M) \), where \( \theta^X \) defines the property \( X \) in Definition 9. Let \( \bar{t}_pM(u) = \bar{t}_pM(v) \). For all \( u,v \in M \) and \( 1 \leq h' \leq h \), if \( \bar{t}_pM(u) = \bar{t}_pM(v) \), then \( \bar{t}_pM(u) = \bar{t}_pM(v) \).

**Theorem 1.** Let \( \Phi_i = \varphi_i \land \bigwedge \text{RE}_i \land \bigwedge \text{DI}_i \in \text{ALCQIO}_{b,\text{Re}}, \) for \( i = 1,2 \). There are \( \text{ALCQIO} \) formulas \( \mu_\varphi \) and \( \kappa_\varphi \) over an extended vocabulary such that

1. \( \Phi_1 \) is satisfiable iff \( \mu_\varphi \) is satisfiable.
2. \( \Phi_1 \) implies \( \Phi_2 \) iff \( \kappa_\varphi \) is not satisfiable.

**Proof.** (1) follows from Lemmas 3, 4, 5, and 6 with \( \mu_\varphi = \theta_\varphi \) and \( \psi = \delta^{\text{semi},\text{DI}} \land \varphi \).

Let \( X_{h'} \) be fresh atomic concepts, \( 1 \leq h' \leq h \). For every \( h' \), let \( \alpha_{h'} = (B_{h'} \subseteq X_{h'}) \land (A_{h'} \cap \neg X_{h'} \neq \bot) \land \bigwedge_{s \in S_{h'}} (\forall s, \neg X_{h'} \subseteq \neg X_{h'}) \). For every \( M \), \( M \models \neg \text{Reach}(A_{h'}, B_{h'}, S_{h'}) \) if and only if there exists \( X_{h'}^M \) such that \( \langle M, X_{h'}^M \rangle \models \alpha_{h'} \). Therefore, \( M \models \neg \Phi_2 \) iff there exist \( X_{h'}^M \subseteq M, 1 \leq h' \leq h \), such that \( \langle M, X_{h'}^M : 1 \leq h' \leq h \rangle \models \neg \Phi_2 \).
\( \neg \varphi_2 \lor \bigwedge_{1 \leq i' < h'} \alpha_{i'} \). Therefore, \( \kappa_\varphi = \Phi_1 \land \left( \neg \varphi_2 \lor \bigwedge_{1 \leq i' < h'} \alpha_{i'} \right) \) is satisfiable iff \( \Phi_1 \rightarrow \Phi_2 \) is not a tautology and we get (2). In both (1) and (2) we use that satisfiability in \( \mathcal{ALCQIO}_b \) is reducible to that in \( \mathcal{ALCQIO} \), see Appendix B.

### 3.5 NEXPTIME decision procedures

The algorithm in Theorem 1 produces, for a formula \( \varphi \), a formula \( \mu_\varphi \) whose size is exponential in the size of \( \varphi \). Most of the constructions along the proof introduce only a polynomial growth, except for the nominals in Definition 9 and the formulae that use them. We discuss here how to effectively compute an \( \mathcal{ALCQIO} \)-formula \( \eta_\varphi \) of polynomial size in \( \varphi \), which introduces the required linear ordering of exponential length without use of the nominals. Since satisfiability in \( \mathcal{ALCQIO} \) is NEXPTIME-complete [19], so is satisfiability and implication in \( \mathcal{ALCQIO}_{b,Re} \). We sketch the idea here. Appendix C.2 gives the details.

In Section 3.3 we used the extension of \( M \) to structures \( N \in \text{ORD}(M) \) to guarantee the existence of useful orderings. Here we guarantee this existence using structures \( J \) which extend \( M \) in a different though similar way. Let \( y = |\varphi| \cdot |N| \). We introduce new concepts \( P_1, \ldots, P_y \) and use them to require that \( J \setminus M \) is of size \( 2^y \) and that \( \text{succ} \) is interpreted as a successor relation in \( J \setminus M \). We think of the reflexive-transitive closure of \( \text{succ}^J \) as \( \text{ord}^N \) from Definition 9 but we will not compute \( \text{succ}^J \) explicitly. \( \text{succ}^J \) is defined so that for every binary word \( b_1 \ldots b_y \), there will be exactly one element of \( J \setminus M \) in \( \bigcap_{i:b_i = 1} P_i^J \setminus \bigcap_{i:b_i = 0} \neg P_i^J \). I.e., \( P_i^M \) represents elements whose corresponding binary word has \( b_i = 1 \). \( \text{succ}^J \) will be induced by the usual successor relation on binary words of length \( y \); an element \( u \in J \setminus M \) is smaller than \( v \in J \setminus M \) with respect to \( \text{succ}^J \) iff there is \( \ell \) such that \( u \) and \( v \) agree on \( P_i^J \), \( i > \ell \), and \( u \notin P_i^J \) and \( v \in P_i^J \). Similarly to the case \( N \), the functions \( f_{\text{ord},h'} \) together with \( (\text{succ}^J)^* \) induce orderings \( R_i^{h'} \) of types. Elements of \( u, v \in M \) have the same type in \( R_i^{h'} \) if they point via \( f_{\text{ord},h'} \) to the same element in \( J \setminus M \). \( u \) has a smaller type than \( v \) if \( u \) points via \( f_{\text{ord},h'} \) to an element which is smaller with respect to \( \text{succ}^J \) than \( v \). This allows us to express the analogous requirements to 4(b) and 4(c) in Definition 9 without using the nominals, but rather using the \( P_i \).

**Theorem 2.** Let \( \Phi_i = \varphi_i \land \bigwedge \text{RE}_i \land \bigwedge \text{DI}_i \in \mathcal{ALCQIO}_{b,Re} \) for \( i = 1, 2 \). There are polynomial-time computable \( \mathcal{ALCQIO} \) formulas \( \eta_\varphi \) and \( \rho_\varphi \) over an extended vocabulary such that

1. \( \Phi_1 \) is satisfiable iff \( \eta_\varphi \) is satisfiable.
2. \( \Phi_1 \) implies \( \Phi_2 \) iff \( \rho_\varphi \) is not satisfiable.
3. Satisfiability and implication in \( \mathcal{ALCQIO}_{b,Re} \) is NEXPTIME-complete.

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A  \( \text{ALCQIO}_b \) and \( C^2 \)

\( \text{ALCQIO}_{b,Re} \) has a fairly standard reduction to the two-variable fragment of first order logic with counting \( C^2 \) (see e.g. [5])

**Definition 10.** Let \( \text{tr} : \text{ALCQIO}_b \rightarrow C^2 \) be given as follows:

\[
\begin{align*}
\text{tr}_z(C) & = C(z) \quad \text{C is an atomic concept} \\
\text{tr}_{z,\bar{z}}(r) & = r(z, \bar{z}) \quad \text{r is an atomic role} \\
\text{tr}_z(C \cap D) & = \text{tr}_z(C) \land \text{tr}_z(D) \\
\text{tr}_z(C \cup D) & = \text{tr}_z(C) \lor \text{tr}_z(D) \\
\text{tr}_z(\neg C) & = \neg \text{tr}_z(C) \\
\text{tr}_{z,\bar{z}}(r^-) & = \text{tr}_{\bar{z},z}(r) \\
\text{tr}_z(\exists r.C) & = \exists y.\text{tr}_{z,\bar{z}}(r) \land \text{tr}_z(C) \\
\text{tr}_z(\exists^r \leq n C) & = \exists^y \leq n.\text{tr}_{z,\bar{z}}(r) \land \text{tr}_z(C) \\
\text{tr}(C \subseteq D) & = \forall x.\text{tr}_x(C) \rightarrow \text{tr}_x(D) \\
\text{tr}(\varphi \land \psi) & = \text{tr}(\varphi) \land \text{tr}(\psi) \\
\text{tr}(\neg \varphi) & = \neg \text{tr}(\varphi)
\end{align*}
\]

**Lemma 7.** For every \( \varphi \in \text{ALCQIO}_b \), \( \varphi \) and \( \text{tr}(\varphi) \) agree on the truth value of all \( \tau \)-structures.

B  From \( \text{ALCQIO}_b \) to \( \text{ALCQIO} \)

Here we show the reduction from satisfiability in \( \text{ALCQIO}_b \) to satisfiability in \( \text{ALCQIO} \).

**Lemma 8.** Let \( \tau \) be a vocabulary and \( \varphi \in \text{ALCQIO}_b(\tau) \). There exist a vocabulary \( \sigma \gneq \tau \) and \( \psi \in \text{ALCQIO}(\sigma) \) such that \( \varphi \) is satisfiable iff \( \psi \) is satisfiable, and the size of \( \psi \) is linear in the size of \( \varphi \). More precisely:

1. If \( \mathcal{M} \) is a \( \tau \)-structure satisfying \( \varphi \), then there exists an extension \( \mathcal{N} \) of \( \mathcal{M} \) such that \( \mathcal{N} \models \psi \). \( \mathcal{N} \) has the same universe as \( \mathcal{M} \) and agrees with \( \mathcal{M} \) on the interpretation of the symbols in \( \tau \).
2. If \( \mathcal{N} \) is a \( \sigma \)-structure satisfying \( \psi \), then the substructure of \( \mathcal{N} \) which corresponds to \( \tau \) satisfies \( \varphi \).

**Proof.** We prove the claim by induction on the construction of formulae in \( \text{ALCQIO}_{b,Re} \). The claim we prove is slightly augmented as follows:

- We assume without loss of generality that \( \varphi \) is given in negation normal form (NNF).
- \( \psi \) will not contain any negations.

We may assume without loss of generality that if \( \varphi \) is satisfiable, then it is satisfiable by a structure of size strictly larger than 1.

**Base** If \( \varphi = C \subseteq D \), then \( C \subseteq D \in \text{ALCQIO} \) and \( \sigma = \tau \).
Closure Let $\varphi_1, \varphi_2 \in \text{ALCQIO}_{h, Re}(\tau)$ in NNF, $\sigma_1, \sigma_2 \supseteq \tau$ be vocabularies, $\psi_1 \in \text{ALCQIO}(\sigma_1)$ and $\psi_2 \in \text{ALCQIO}(\sigma_2)$ as guaranteed. Without loss of generality, $(\sigma_1 \setminus \tau) \cap (\sigma_2 \setminus \tau) = \emptyset$.

1. $\varphi = \varphi_1 \land \varphi_2$: Let $\psi = \psi_1 \land \psi_2$ and $\sigma = \sigma_1 \cup \sigma_2$.

2. $\varphi = \neg \varphi_1$:

   By the assumption that that $\varphi$ is in NNF, $\varphi_1$ is of the form $(C \subseteq D)$. Let $\sigma$ be a fresh nominal which does not occur in $\sigma_1$. Let $\sigma = \sigma_1 \cup \{\sigma\}$. Let $\psi = (\sigma \subseteq C) \land (D \subseteq \neg \sigma)$.

3. $\varphi = \varphi_1 \lor \varphi_2$:

   Let $r$ be a fresh role and $o_1, o_2, o_X, o_Y$ be fresh nominals.

   $$\psi_{\text{prep}} = (o_X \subseteq \neg o_Y) \land (o_1 \subseteq \neg o_2) \land (o_X \cup o_Y \equiv o_1 \cup o_2) \land (\exists r. o_X \equiv T) \land (\exists r. o_Y \equiv \bot)$$

   For a structure $\mathcal{M}$ with universe $M$, $\mathcal{M} \models \psi_{\text{prep}}$ iff

   (a) $o_1^M \neq o_2^M$
   (b) $o_1^M \neq o_2^M$
   (c) $o_1^M = o_X^M$ and $o_2^M = o_Y^M$, or $o_1^M = o_X^M$ and $o_2^M = o_Y^M$.
   (d) $(\exists r. o_1)^M = M$ and $(\exists r. o_2)^M = \bot$, or
   $(\exists r. o_1)^M = \bot$ and $(\exists r. o_2)^M = M$.

   For $i \in \{1, 2\}$, let $\theta_i$ be the formula obtained from $\psi_1$ by replacing every atomic formula $C \subseteq D$ with $C \cap \exists r. o_i \subseteq D \cap \exists r. o_i$. Let $\sigma = \sigma_1 \cup \sigma_2 \cup \{r, o_1, o_2, o_X, o_Y\}$. Let $\psi = \psi_{\text{prep}} \land \theta_1 \land \theta_2$. The desired follows directly from the claim:

   **Claim.** Let $\mathcal{N}$ be a $\sigma$-structure such that $\mathcal{N} \models \psi_{\text{prep}}$, and let $\mathcal{M}_\mathcal{N}$ be the substructure of $\mathcal{N}$ which corresponds to $\tau$.

   (a) If $(\exists r. o_1)^\mathcal{N} = M$, then $\mathcal{N} \models \psi$ iff $\mathcal{M}_\mathcal{N} \models \varphi_1$.
   (b) If $(\exists r. o_1)^\mathcal{N} = \emptyset$, then $\mathcal{N} \models \psi$ iff $\mathcal{M}_\mathcal{N} \models \varphi_2$.

   **Proof.**

   (a) Let $\mathcal{N}$ be a $\sigma$-structure such that $(\exists r. o_1)^\mathcal{N} = M$. For every atomic formula $C \subseteq D$ in $\varphi_1$, $(C \cap \exists r. o_1)^\mathcal{N} = C^\mathcal{N} \cap M = C^\mathcal{N}$ and $(D \cap \exists r. o_1)^\mathcal{N} = D^\mathcal{N} \cap M = D^\mathcal{N}$. Hence, $\mathcal{M}_\mathcal{N} \models C \subseteq D$ iff $\mathcal{N} \models C \cap \exists r. o_1 \subseteq D \cap \exists r. o_1$. By construction of $\theta_1$, $\mathcal{M} \models \varphi_1$ iff $\mathcal{N} \models \theta_1$.

   For every atomic formula $C \subseteq D$ in $\varphi_2$, $(C \cap \exists r. o_2)^\mathcal{N} = C^\mathcal{N} \cap \emptyset = \emptyset$ and $(D \cap \exists r. o_2)^\mathcal{N} = D^\mathcal{N} \cap \emptyset = \emptyset$. Hence, $\mathcal{N} \models C \cap \exists r. o_2 \subseteq D \cap \exists r. o_2$. Since $\theta_2$ is a negation free Boolean combination of atomic formulae, $\mathcal{N} \models \theta_2$.

   (b) This case is symmetric to the previous case. Let $\mathcal{N}$ be a $\sigma$-structure such that $(\exists r. o_2)^\mathcal{N} = \emptyset$. Then $(\exists r. o_2)^\mathcal{N} = M$. For every atomic formula $C \subseteq D$ in $\varphi_2$, $(C \cap \exists r. o_2)^\mathcal{N} = C^\mathcal{N} \cap M = C^\mathcal{N}$ and $(D \cap \exists r. o_2)^\mathcal{N} = D^\mathcal{N} \cap M = D^\mathcal{N}$. Hence, $\mathcal{M} \models C \subseteq D$ iff $\mathcal{N} \models C \cap \exists r. o_2 \subseteq D \cap \exists r. o_2$. By construction of $\theta_2$, $\mathcal{M}_\mathcal{N} \models \varphi_2$ iff $\mathcal{N} \models \theta_2$.

   For every atomic formula $C \subseteq D$ in $\varphi_1$, $(C \cap \exists r. o_1)^\mathcal{N} = C^\mathcal{N} \cap \emptyset = \emptyset$ and $(D \cap \exists r. o_1)^\mathcal{N} = D^\mathcal{N} \cap \emptyset = \emptyset$. Hence, $\mathcal{N} \models C \cap \exists r. o_1 \subseteq D \cap \exists r. o_1$. Since $\theta_1$ is a negation free Boolean combination of atomic formulae, $\mathcal{N} \models \theta_1$. 

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C Full Proofs

Lemma 1. Let $M_1$ and $M_2$ be two $\tau$-structures with the same universe $M$.

1. If for all $u \in M$ we have $\overline{t}_p M_1(u) = \overline{t}_p M_2(u)$, then $M_1$ and $M_2$ agree on $\varphi$.
2. Let $f \in N_F$, and $a_0, a_1, b_0, b_1 \in M$ such that $tp_{M_1}(a_0) = tp_{M_1}(a_1)$, and such that $(a_0, b_0) \in f^{M_1}$. There exist $b_1 \in M$ such that $tp_{M_1}(b_0) = tp_{M_1}(b_1)$ and $(a_1, b_1) \in f^{M_1}$.

Proof.

1. For every $C \in \varphi$ and $u \in M$, $u \in C^{M_1}$ iff $u \in C^{M_2} M_1$. Hence, $C^{M_1} = C^{M_2}$.
   Therefore, every inclusion assertion $C \subseteq D$ holds in $M_1$ iff it holds in $M_2$, and consequently the same is true for every Boolean combination of inclusion assertions.
2. Let $C \in \varphi$, $b_0 \in C^{M_1}$, $(a_0, b_0) \in f^{M_1}$, $a_0 \in (\exists f.C)^{M_1}$, $\overline{t}_p M_1(a_0) = \overline{t}_p M_1(a_1)$, $a_1 \in (\exists f.C)^{M_1}$, there exists $b_1 \in C^{M_1}$ such that $(a_1, b_1) \in f^{M_1}$.
   From the functionality of $f$, there exists for every $C$ a unique $b_1$ such that $b_1, C, f = b_1$.

Lemma 2. Let $M$ be a $\tau$-structure, $r \in N_F$, $a_0, b_0, a_1, b_1 \in M$ and $t = (a_0, b_0, a_1, b_1, r)$. If $tp_{M}(a_0) = tp_{M}(a_1)$, $r^M(a_0) = b_0$, $r^M(a_1) = b_1$, and $\{a_0, b_0\} \cap \{a_1, b_1\} = \emptyset$, then $C^{M} = C^{M_{1,0}}$ for all $C \in \varphi$, and consequently, for every $u \in M$, $\overline{t}_p M(u) = \overline{t}_p M_{1,0}(u)$ and $\overline{t}_p M(u) = \overline{t}_p M_{1,0}(u)$.

Proof. We prove this by construction of the concepts:

1. If $A \in N_C$, then $A^{M_1} = A^{M_2}$ since none of the atomic concepts change between $M_1$ and $M_2$.
2. If $\sigma \in N_\sigma$, then similarly, there is no change.
3. If $C_1$ and $C_2$ are concepts satisfying the property, then $C_1 \cap C_2$, $C_1 \cup C_2$ and $\neg C_1$ also satisfy the property.
4. For a role $s$, a concept $C$ and a non-negative integer $n$, we consider the concepts $\exists s.C$, $\geq n.s.C$, $\exists \neg s.C$, $\geq n \neg s.C$.
   (a) If $s \neq r$ is a role, $C$ is a concept and $n$ is a non-negative integer, then $\exists s.C^{M} = (\exists s.C)^{M_{1,0}}$ and $(\geq n.s.C)^{M} = (\geq n.s.C)^{M_{1,0}}$ since $s^{M} = s^{M_{1,0}}$ and by induction $C^{M} = C^{M_{1,0}}$.
   (b) If $s = r$:
      - $\exists r.C$: $\overline{t}_p M_1(a_0) = \overline{t}_p M_1(a_1)$ implies $\overline{t}_p M_1(b_0) = \overline{t}_p M_1(b_1)$, i.e. $b_0 \in C^{M_1}$ iff $b_1 \in C^{M_1}$. By induction, $b_0 \in C^{M_1}$ iff $b_0 \in C^{M_2}$ and $b_1 \in C^{M_2}$ iff $b_1 \in C^{M_2}$. So, $a_0 \in (\exists r.C)^{M_1}$ iff $b_0 \in C^{M_1}$ iff $b_0 \in C^{M_2}$ iff $a_0 \in (\exists r.C)^{M_2}$ and similarly, $a_1 \in (\exists r.C)^{M_1}$ iff $a_1 \in (\exists r.C)^{M_2}$. Since the only difference between $M_1$ and $M_2$ is the values of $r^{M_1}$ on $a_0$ and $a_1$, we have $(\exists r.C)^{M_1} = (\exists r.C)^{M_2}$ and $(\geq n r.C)^{M_1} = (\geq n r.C)^{M_2}$. 

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have a predecessor, so $M$ does not have a predecessor in $A$ for some $h$. It remains to show that for each $u \in M$, let 
\[ S_i(u) = \{ v \mid (u, v) \in (r^-)^{M_i} \text{ and } v \in C^{M_i} \} \]
for $i = 1, 2$.

We divide into cases depending on $u$:

- If $u \notin \{b_0, b_1\}$, then $S_0(u) = S_1(u)$, using that $(r^-)^{M_1}$ and $(r^-)^{M_2}$ agree on $u$ and that by induction $C^{M_1} = C^{M_2}$. Therefore, $u \in (\geq n r^- . C)^{M_1}$ iff $u \in (\geq n r^- . C)^{M_2}$.

- If $u = b_i$, $i \in \{0, 1\}$: for every $v \notin \{a_0, a_1\}$, $v \in S_0(u)$ iff $v \in S_1(u)$, using that $(u, v) \in (r^-)^{M_1}$ iff $(u, v) \in (r^-)^{M_2}$ and that by induction $C^{M_1} = C^{M_2}$. For every $j \in \{0, 1\}$, $a_j \in S_1(b_i)$ iff $(a_j, b_i) \in g^{M_1}$ and $b_j \in C^{M_1}$ iff $(a_j, b_i) \in r^{M_2}$ and $b_i \in C^{M_1}$ iff $(a_j, b_i) \in r^{M_2}$ and $b_i \in C^{M_1}$ iff $a_j \in S_2(b_i)$. So, $|S_1(b_i)| = |S_2(b_i)|$, $i = 1, 2$. Hence, $b_i \in (\geq n r^- . C)^{M_1}$ iff $b_i \in (\geq n r^- . C)^{M_2}$. Since $\mathcal{TP}_{M_1}(b_0) = \mathcal{TP}_{M_2}(b_1)$, $b_i \in (\geq n r^- . C)^{M_1}$ iff $b_i \in (\geq n r^- . C)^{M_2}$ iff $b_i \in (\geq n r^- . C)^{M_2}$.

We got that for all $u \in M$, $u \in (\geq n r^- . C)^{M_1}$ iff $u \in (\geq n r^- . C)^{M_2}$.

Lemma 4. There exists a formula $\delta^{RE, DI}_{semi} \in ALCQIO_b$ such that $M \models \delta^{RE, DI}_{semi}$ iff $M$ is $(RE, DI)$-semi-connected.

Proof. Let $\delta^{RE, DI}_{semi} = \bigwedge DI \land \bigwedge CO(RE) \land \bigwedge_{1 \leq h' \leq h} \delta^{h' \text{ cyc}} - \text{reach} \land \exists s. A_h'$. Assume $M$ is $(RE, DI)$-semi-connected. Every $u \in A_h'$ is reachable from $B_{h'}^M$ or is reachable from a cycle, and therefore $u$ has a predecessor in $A_h^M$ with respect to $\bigcup_{s \in S_h'} s^M$, unless $u \in B_h'$, so $M \models \delta^{RE, DI}_{\text{reach cyc}}$.

Conversely assume that $M$ is not $(RE, DI)$-semi-connected. There exists a vertex $u$ which is not reachable from $B_{h'}^M$ nor from a cycle. There must exist a vertex $v$ in $A_h^M$ which is a predecessor of $u$ (possibly $u$ itself) and which does not have a predecessor in $A_h^M$, otherwise $u$ lies on a cycle (using the finiteness of the universe). Since $u$ is reachable from $v$, we must have that, like $u$, $v$ not reachable from $B_{h'}^M$. Therefore, $v$ is not in $B_{h'}^M$ but belongs to $A_h^M$ and does not have a predecessor, so $M \models \delta^{RE, DI}_{\text{reach cyc}}$.

Lemma 5. Let $M$ be a structure, $ORD(M)$ is non-empty iff there exists an $h'$-useful ordering for $M$ and every $1 \leq h' \leq h$.

Proof. If $ORD(M)$ is non-empty, then let $N \in ORD(M)$. By definition, $ord^N$ is an $h'$-useful order for $M$.

Conversely, let $R^N_h$, $1 \leq h' \leq h$, be $h'$-useful orderings for $M$. Clearly, $M$ can be extended to a structure $N$ satisfying properties 1, 2, and 3 from Definition 9. It remains to show that for each $h'$, $f_{ord,h'}$ can be given an interpretation in $N$ such that property 4 holds without changing the interpretation of any other
symbol (thereby retaining properties 1, 2, and 3). Since $k = |TYPES(M)|$, for every $h'$ there exists a subset $T_{h'} \subseteq N \setminus M$ of size $k' \leq k$ and an isomorphism $iso_{h'} : TYPES(M) \rightarrow T_{h'}$ between $R^N_{h'}$ and the restriction of $ord^N$ to $T_{h'}$. We define $f^N_{ord,h'}(u) = iso_{h'}(\bar{tp}(u))$, for all $u \in M$. For this $f^N_{ord,h'}$, property 4 holds.

C.1 Lemma 3

We first prove two auxiliary lemmas.

Lemma 9. Let $M$ be a $\tau$-structure. If $M \models \bigwedge RE \land \bigwedge DI$, then there exist $h'$-useful orderings for $M$ for every $1 \leq h' \leq h$.

Proof. Let $1 \leq h' \leq h$. Let $R_<$ be an ordering of $TYPES(M)$ built as follows.

1. $R_<$ is initialized as any linear orderings of the types of $B_{h'}^M$. (They will remain the smallest elements as we continue to the next stages.)
2. Let $R_{DFS}$ be a linear ordering of the elements of $M$ obtained by running a depth first search of $D_{h'}^M$ starting from elements of $B_{h'}^M$ whenever all discovered vertices are fully explored.
3. Considering the vertices $y$ of $D_{h'}^M$ from smallest to largest in $R_{DFS}$, $\bar{tp}(y)$ is added as the largest element of $R_<$, unless it already exists in $R_<$. The elements of $TYPES(M)$ which do not occur in $R_<$ are added to $R_<$ as the maximal elements. Any ordering within elements added in this stage is allowed, as long as they are larger than all elements added at previous stages.

Now consider an element $u \in M$ which does not belong to $B^M$. Let $v$ be the first element with the same type as $\bar{u}$ which is reached in the DFS. When $v$ is considered in (3), $\bar{tp}(v)$ does not belong to $R_<$ and is added to $R_<$.

Lemma 10. Let $M$ be a structure, $1 \leq h' \leq h$, $r \in S_{h'}$, and $t = (a_0, b_0, a_1, b_1, r)$ such that $a_0, b_0, a_1, b_1 \in A_{h'}^M$, $\bar{tp}_{A_{h'}^M}(a_0) = \bar{tp}_{A_{h'}^M}(a_1)$, $\bar{r}_{A_{h'}^M}(a_0) = b_0$, $\bar{r}_{A_{h'}^M}(a_1) = b_1$. $a_1$ is not reachable in $D_{h'}^M$ from $B_{h'}^M$ and belongs to a cycle. $a_0$ is reachable from $B_{h'}^M$. Let $M^1 = M$ and $M^2 = M_{A_{h'}}$. If the following hold for $i = 1$, then they hold for $i = 2$:

(a) $M^1 \models \varphi \land \bigwedge RE_{h' - 1} \land \bigwedge DI$; (b) $M^i$ is $(RE, DI)$-semi-connected; (c) $M^i$ has $h^i$-useful orderings for all $1 \leq h^i \leq h$.

Moreover, the set of vertices reachable from $B_{h'}^M$ in $D_{h'}^M$ is strictly contained in the set of vertices reachable from $D_{h'}^{M_{A_{h'}}}$ in $D_{h'}^{M_{A_{h'}}}$.

Proof. For all $1 \leq \ell \leq h$, $D^{\ell}_{h'} = D^{M_{A_{h'}}}_{\ell}$, $A^{\ell}_{h'} = A^{M_{A_{h'}}}_{\ell}$, $B^{\ell}_{h'} = B^{M_{A_{h'}}}_{\ell}$.

For every $\alpha = A_\ell \cap A_j$ or $\alpha = B_{h'} \subseteq A_{h'}$, $M_{A_{h'}} \models \alpha$, since the interpretations of the $A_\ell$ and $B_\ell$ are the same in $M$ and $M_{A_{h'}}$. It also remains true that for every $1 \leq \ell \leq h$, $h' \neq \ell$, every vertex of $D^\ell_{h'}$ is either reachable from
\(B^\triangledown_h\) or from a cycle by the compatibility of \(RE\) and \(DI\). Moreover, for every \(1 \leq \ell < h',\) every vertex of \(D^\triangledown_{h'}\) is reachable from \(B^\triangledown_h\). So, \(\mathcal{M}_\triangledown \models \bigwedge RE_{h'} \land \bigwedge DI\) and \((RE\{\text{Reach}(A',B',S')\},DI)-\text{semi-connected}.

Let \(R\) be the set of vertices reachable from \(b_1\) in \(D^\triangledown_M\). Every vertex \(z\) in \(R\) in reachable in \(D^\triangledown_{h'}\) from \(B^\triangledown_h\) by the concatenation of the path witnessing that \(z\) is reachable from \(b_1\), the edge \((a_0,b_1)\), and a path \(\pi_z\) witnessing that \(z\) is reachable from \(b_1\) in \(D^\triangledown_M\). Note that in particular, \(b_1\) and \(a_1\) belong to \(R\).

On the other hand, any vertex which was reachable from \(B'y = B^\triangledown_{h'}\) in \(D^\triangledown_M\) remains reachable in \(D^\triangledown_{h'}\), by replacing any witnessing path which includes the edge \((a_0,b_0)\) in \(D^\triangledown_M\) with the concatenation of \((a_0,b_1)\) with \(\pi_{a_1}\) and then \((a_1,b_0)\).

As a result, the set of vertices reachable from \(B^\triangledown_h\) in \(D^\triangledown_{h'}\) strictly contains the analogous set of \(D^\triangledown_M\), and \(a_1\) is reachable in \(D^\triangledown_{h'}\) but not in \(D^\triangledown_M\). Consequently \(\mathcal{M}_{\triangledown}\) is \((RE,DI)-\text{semi-connected}.

By Lemma 3, \(\mathcal{M}_{\triangledown} \models \varphi\).

Let \(h' \leq h'' \leq h\) and \(R'_<\) an \(h''\)-useful ordering for \(\mathcal{M}\). Let \(u \in A^M_{h''} = A_{h''}^{\triangledown}\), either \(u\) is in \(B^M_{h''}\), or there exist elements \(v,w \in A^M_{h''}\) such that \(\bar{t}_p_M(u) = \bar{t}_p_M(v) = \bar{t}_p_M(w)\) and \((u,v) \in \bigcup_{s \in S}\{s\}\). After applying \(\triangledown\), there exists \(w'\) such that \((w',v) \in \bigcup_{s \in S} s^M_\triangledown\) and \(\bar{t}_p_{M\triangledown}(w) = \bar{t}_p_{M\triangledown}(v)\): if \((w,v) = (a_1,b_1), i \in \{0,1\}\), then \(w' = a_{i-1}\), and otherwise \(w = w'\). By Lemma 2, \(\bar{t}_p_M(x) = \bar{t}_p_{M\triangledown}(x)\) for every \(x \in M\). We get that \(R''_<\) is an \(h''\)-useful ordering for \(\mathcal{M}_{\triangledown}\).

**Lemma 3.** \(\varphi \land \bigwedge RE \land \bigwedge DI\) is satisfiable iff \(\varphi\) is satisfiable by a \((RE,DI)-\text{semi-connected}\) structure with \(h'\)-useful orderings for every \(1 \leq h' \leq h\).

**Proof.** By Lemma 3 if \(\varphi \land \bigwedge RE \land \bigwedge DI\) is satisfied, then that structure has \(h'\)-useful orderings for every \(h'\).

For the other direction we need to prove that if \(\mathcal{M}\) is a \((RE,DI)-\text{semi-connected}\) structure such that \(\mathcal{M} \models \varphi\) and \(\mathcal{M}\) has \(h'\)-useful ordering for every \(h'\), then \(\varphi \land \bigwedge RE \land \bigwedge DI\) is satisfiable.

The lemma from the following claim with \(\mathcal{M}_h\) as the desired structure:

**Claim.** For every \(0 \leq h' \leq h\), there exists \(\mathcal{M}_{h'}\)

1. \(\mathcal{M}_{h'} \models \varphi \land \bigwedge RE_{h'} \land \bigwedge DI\),
2. \(\mathcal{M}_{h'}\) is \((RE,DI)-\text{semi-connected})
3. \(\mathcal{M}_{h'}\) has with \(h''\)-useful orderings for all \(1 \leq h'' \leq h\).

**Proof.** We prove the claim by induction. The base case \(h' = 0\) holds trivially for \(\mathcal{M}_0 = \mathcal{M}\). Now assume the induction hypothesis holds for \(h' - 1\). Let \(R_<\) be an \(h'\)-useful ordering of \(\mathcal{M}_{h'-1}\). We denote \(\mathcal{M}' = \mathcal{M}_{h'-1}\).

Let \(a_1 \in A^{M'}_{h'}\) be a minimal element with respect to \(R_<\), which is not reachable from \(B^M_{h'}\) and belongs to a cycle. There exists such \(a_1\) unless all elements of \(A^{M'}_{h'}\) are reachable from \(B^M_{h'}\) by the semi-connectedness. In particular \(a_1 \notin B^M_{h'}\). By the \(h'\)-usefulness of \(R_<\), there are \(f \in S_{h'}, a_0, w \in A_{h'}^{M'}\) such that \(\bar{t}_p_{M'}(a_0) = \bar{t}_p_{M'}(a_1), (w,a_0) \in f^{M'}\) and \((\bar{t}_p_{M'}(w),\bar{t}_p_{M'}(a_0)) \in R_<\). By
the minimality of \( a_1 \), \( w \) is reachable from \( B_{h'}^{M'} \) and therefore the same is true of \( a_0 \). By Lemma 1, \( tp_M(a_0) = \tilde{tp}_{M'}(a_1) \). Let \( s \in S_{h'} \) and \( b_1 \in s^{M'}(a_1) \) such that \( b_1 \) belongs to the same cycle as \( a_1 \) in \( D_{h'}^{M'} \) as guaranteed. Let \( b_0 \in s^{M'}(a_0) \) with such that \( \tilde{tp}_{M'}(b_0) = \tilde{tp}_{M'}(b_1) \); there exists such \( b_0 \) by Lemma 1. We have that \( \{a_0, b_0\} \cap \{a_1, b_1\} = \emptyset \), because \( a_0 \) and \( b_0 \) are reachable from \( B_{h'}^{M'} \), and \( a_1 \) and \( b_1 \) are not.

Let \( t = (a_0, b_0, a_1, b_1, s) \). By Lemma 10, \( M_{\omega} \models (\bigwedge RE_{h'-1} \land \bigwedge DI, M_{\omega} \text{ is } (RE, DI)\text{-semi-connected}, M_{\omega} \text{ has with } h'' \text{ useful orderings for all } 1 \leq h'' \leq h\), \( M_{\omega} \models \varphi \), and the set of vertices reachable from \( B_{h'}^{M} \) in \( D_{h'}^{M} \) is strictly contained in the set of vertices reachable from \( B_{h'}^{M_{\omega}} \) in \( D_{h'}^{M_{\omega}} \).

We repeatedly find \( a_0, a_1, b_0, b_1, s \) as above and apply \( \triangleright \) until no such \( a_1 \) can be found, obtaining a structure \( M_{h'} \). At this point, all elements of \( A_{h'}^{M_{h'}} \) are reachable in \( D_{h'}^{M_{h'}} \) from \( B_{h'}^{M_{h'}} \). We get that \( M_{h'} \models (\bigwedge RE_{h'} \land \bigwedge DI, \text{ and therefore } M_{h'} \) is the desired structure.

### C.2 NEXPTIME decision procedures

Here we give the detailed proof of the discussion in Section 3.5.

**Lemma 11.** There exists a formula \( \zeta_y \) of size polynomial in \( y \) with an atomic concept \( M \) and an atomic role \( succ \), such that the models \( J \) of \( \zeta_{ord} \) are exactly the structures satisfying:

1. \( \text{succ}^J \) is a successor relation on \( J \setminus M \) and \( |J \setminus M| = 2^y \);
2. for all \( 1 \leq h' \leq h \), \( f_{ord}^J : M \to J \setminus M \);\n3. for all \( u, v \in M \) and \( 1 \leq h' \leq h \), if \( f_{ord}^J (u) = f_{ord}^J (v) \), then \( \tilde{tp}_M(u) = \tilde{tp}_M(v) \);
4. For every \( h' \), the ordering induced by \( (\text{succ}^J)^* \) and \( f_{ord}^J \) on \( \text{TYPES}(M) \) is an \( h' \)-useful ordering for \( M \).

\( (\text{succ}^J)^* \) denotes here the reflexive-transitive closure of \( \text{succ}^J \), and, importantly, \( (\text{succ}^J)^* \) itself is not a relation in \( J \).

**Proof.** For every \( 1 \leq i \leq y \), \( 1 \leq h' \leq h \) and \( s \in S_{h'} \), let \( P_i, E_{h',s} \) and \( E_{h',s,i} \), be fresh atomic concepts. We require that our structures satisfy \( \zeta_{P,E} \), which is the conjunction of \( P_i \subseteq \neg M, E_{h',s} \subseteq M \) and \( E_{h',s,i} \subseteq M \) for all \( i, h' \). The successor relation \( \text{succ} \) is defined so that for every binary word \( b_1 \ldots b_y \), there will be exactly one element of \( J \setminus M \) in

\[
( \bigwedge_{i: b_i = 1} P_i \bigwedge_{i: b_i = 0} \neg P_i )^J .
\]

\( \text{succ} \) will be induced by the usual successor relation on binary words of length \( y \). The formulae

\[
\zeta_{\text{first}} = \exists \text{succ}^\neg \equiv \neg M \sqcap \neg (\neg P_1 \sqcap \cdots \sqcap \neg P_y) \\
\zeta_{\text{last}} = \exists \text{succ} \equiv \neg M \sqcap \neg (P_1 \sqcap \cdots \sqcap P_y)
\]

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specify the first respectively last elements of $succ$.

$$\zeta_{fune} = (\exists succ \equiv \leq 1 succ) \land (\exists succ^- \equiv \leq 1 succ^-)$$

expresses that $s$ is a successor relation. $\zeta_{consec}$ below expresses that the successor relation mimics the binary words: two words $b_y \ldots b_1$ and $d_y \ldots d_1$ are consecutive in $succ$ if there exists an index $i$ such that $b_i \ldots b_1 = 01^{i-1}$, $d_i \ldots d_1 = 10^{i-1}$, and $b_y \ldots b_{i+1} = d_y \ldots d_{i+1}$. $\zeta_1$ applies the aforementioned formulae to $succ$. The concepts $P_j$ are used to represent the bits $b_j$ and $d_j$. For every $1 \leq i \leq y$, let

$$C_i = P_j \land \exists s, P_j$$

$$C_{\leq i} = \bigcap_{j<i} (P_j \land \exists succ, \neg P_j)$$

$$C_{>i} = \bigcap_{i<j\leq y} (P_j \land \exists succ, P_j \lor \neg P_j \land \exists succ, \neg P_j)$$

$$\zeta_{consec} = (\neg M \sqsubseteq \bigcup_{1\leq i\leq y} C_{<i} \cap C_i \cap C_{>i})$$

We have

$$\zeta_1 = \zeta_{fune} \land \zeta_{first} \land \zeta_{last} \land \zeta_{consec}.$$

Notice that $\zeta_1$ requires that $|J \setminus M| = 2^y$.

The formulae $\zeta_2 = \theta^{4a}$ and $\zeta_3 = \theta^{4b}$ from Lemma 6 state that $f_{ord,h'}^{\mathcal{J}}$ is a function from $M$ to $J \setminus M$ for all $1 \leq h' \leq h$ and, for all $1 \leq k' \leq k$, all elements in $(f_{ord,h'}^{\mathcal{J}}(k'))$ have the same type $\tilde{f}_{\mathcal{M}}$.

We do not define the transitive closure $(succ^\mathcal{J})^*$ explicitly. Instead we define directly the $E_{h',s;i}^\mathcal{J}$, $E_{h',s;i}^\mathcal{J}$ and $E_{h',s;i}^\mathcal{J}$ will contain all of the elements $u \in M$ such that the types of $u$ and $s^{\mathcal{J}}(u)$ agree on membership in $P_1^{\mathcal{J}}, \ldots, P_i^{\mathcal{J}}$, $u \notin P_i^{\mathcal{J}}$, and $s^{\mathcal{J}}(u) \in P_{i+1}^{\mathcal{J}}$. $\zeta_{h',s;i, pos}$ and $\zeta_{h',s;i,neg}$ below require that $u$ and $s^{\mathcal{J}}(u)$ agree on $P_1^{\mathcal{J}}, \ldots, P_i^{\mathcal{J}}$, and $\zeta_{h',s;i}$ makes the requirements on $P_{i+1}^{\mathcal{J}}$. $\zeta_{h',s;i, pos}$ is the conjunction for all $1 \leq j \leq i$ of

$$E_{h',s;i}^{\mathcal{J}} \sqsubseteq \exists f_{ord,h'}^{\mathcal{J}}, P_j \sqsubseteq \exists s, \exists f_{ord,h'}^{\mathcal{J}}, P_j.$$
such that \( v \) is again in \( A_{h'}^J \) and \( v \) has a previous element which is smaller than \( v \). We get that \( \zeta_4 \) expresses that \((\text{succ}^J)^*\) is induced an \( h'\)-useful ordering on the types \( \tilde{t}_pM \) for all \( h' \). \( \zeta_4 \) is the conjunction of

\[
A_{h'} \cap \neg B_{h'} \subseteq \exists f_{\text{ord},h'} \exists f_{\text{ord},h'}^-. A_{h'} \cap \bigcup_{s \in S_{h'}} \exists s^- . E_{h',s} \wedge \bigwedge_{s \in S_{h'}} \zeta_{E_{h',s}}
\]

for every \( 1 \leq h' \leq h \). The desired formula is

\[
\zeta_y = \zeta_{P,E} \land \zeta_1 \land \zeta_2 \land \zeta_3 \land \zeta_4.
\]

**Lemma 12.** Let \( M \) be a structure. There exists \( J \) such that the substructure of \( J \) with universe \( M \) is \( M \) and \( J \models \zeta_y \) iff there exists an \( h'\)-useful ordering for \( M \) for every \( 1 \leq h' \leq h \).

**Proof.** If there exists such \( J \), then by Lemma 11 the desired useful orderings are the orderings induced by \((\text{succ}^J)^*\) and \( f_{\text{ord},h'}^J \) on the types realized by \( M \).

Conversely, let \( R_{h'}^< \) be \( h'\)-useful orderings for \( M \) for every \( 1 \leq h' \leq h \). By Lemma \( 5 \) there exists \( N \in \text{ORD}(M) \). Let \( J \) be obtained by extending \( M \) such that

1. \( J = N \),
2. \( \text{succ}^J = \{(o_i^N, o_{i+1}^N) \mid 1 \leq i \leq r - 1\} \), and
3. \( f_{\text{ord},h'}^J = f_{\text{ord},h'}^N \) for every \( h' \).

It is not hard to verify that \( J \) satisfies the properties of Lemma 11 and hence \( J \models \zeta_y \).

**Theorem 2.** Let \( \Phi_i = \varphi_i \land \bigwedge RE_i \land \bigwedge DI_i \in \text{ALCQIO}_{b,\text{Re}} \) for \( i = 1, 2 \). There are polynomial-time computable \( \text{ALCQIO} \) formulas \( \eta_{\varphi} \) and \( \rho_{\varphi} \) over an extended vocabulary such that

1. \( \Phi_1 \) is satisfiable iff \( \eta_{\varphi} \) is satisfiable.
2. \( \Phi_1 \) implies \( \Phi_2 \) iff \( \rho_{\varphi} \) is not satisfiable.
3. Satisfiability and implication in \( \text{ALCQIO}_{b,\text{Re}} \) is \( \text{NEXPTIME}-\text{complete} \).

(1) follows from Lemmas 3, 4 and 12 with \( \eta_{\varphi} = \theta_{\varphi} \land \zeta_y \) and where \( \psi = \delta_{\text{semi}}^{RE, DI} \land \varphi \). (2) follows from (1) similarly to Theorem 1. We also here the reduction from \( \text{ALCQIO}_{b} \) to \( \text{ALCQIO} \) in Appendix B. Satisfiability in \( \text{ALCQIO} \) is \( \text{NEXPTIME}-\text{complete} \) [19]. Since \( \text{ALCQIO}_{b,\text{Re}} \) contains \( \text{ALCQIO} \), and at the same time, satisfiability and implication of \( \text{ALCQIO}_{b,\text{Re}} \) formulae are polynomial-time reducible to \( \text{ALCQIO} \) satisfiability, (3) holds.