Color superconducting strange quark matter at finite temperature

Amruta Mishra\(^1\), Hiranmaya Mishra\(^2\)

\(^1\)Department of Physics, Indian Institute of Technology, New Delhi- 110016, India
\(^2\)Theory Division, Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

E-mail: \(^1\)amruta@physics.iitd.ac.in, \(^2\)hm@prl.ernet.in

Abstract. We investigate the phase diagram of strange quark matter in beta equilibrium where the lighter up and down quarks form the two flavor superconducting matter whereas the strange quark remains unpaired. This is studied within a Nambu-Jona-Lasinio model. The variational method as used here allows us to investigate simultaneous formation of condensates in quark–antiquark as well as in diquark channels. Color and electric charge neutrality conditions are imposed in the calculation of the thermodynamic potential. Medium dependence of strange quark mass plays a sensitive role in maintaining charge neutrality conditions. At zero temperature the system goes from the gapless phase to the usual BCS phase through an intermediate normal phase as density is increased. However, at higher temperature the gapless to BCS transition becomes a smooth transition. The gapless modes show a smooth behaviour with respect to temperature vanishing above a critical temperature which is larger than the BCS transition temperature.

1. Introduction
Color superconductivity has become a compelling topic in QCD during the last few years. At sufficiently high baryon densities, when nucleons get converted to quark matter, the resulting quark matter is in one kind or the other of the many different possible color superconducting phases at low enough temperatures [1]. The rich phase structure is essentially due to the fact that the quark quark interaction is not only strong and attractive in many channels but also many degrees of freedom are possible for quarks like color, flavor and spin so that various kinds of BCS pairing are possible. Although such phases seems to be almost unlikely to be created in present heavy ion collision experiments, perhaps, it is natural to expect some color superconducting phase to exist in the core of compact stars where the densities are above nuclear matter densities and temperatures are of the order of tens of keV. However, to consider quark matter for neutron stars, color and electrical charge neutrality conditions need to be imposed for the bulk quark matter. Recently it was observed that imposition of neutrality conditions lead to pairing of quarks with different fermi momenta giving rise to gapless modes [2, 3]. In the present work we generalise the variational approach of Ref. [3] to study the color superconductivity as well as the chiral symmetry breaking to include the effects of temperatures. This will be particularly relevant for the physics of proto neutron stars. Here, we compute the thermodynamic potential at finite temperatures for charge neutral ud-s matter where the two lighter quarks take part in diquark condensation. We also self consistently determine the quark antiquark condensates and the diquark condensates- the former being related to chiral symmetry breaking. We thus in the
present calculation keep all orders in the strange quark mass while computing the thermodynamic potential. The temperature dependence and the nature of the gapless modes as a function of the baryon density shall be particularly relevant for cooling of the neutron stars through neutrino emission [5].

2. An ansatz for the ground state and the thermodynamic potential

We here write down the ansatz for the variational state as a squeezed coherent state involving quark antiquark as well as diquark condensates as given by [3, 4]

$$|\Omega\rangle = U_d|\text{vac}\rangle = U_d U_Q|0\rangle, \quad (1)$$

Here, $U_Q$ and $U_d$ are unitary operators creating quark–antiquark and diquark pairs respectively. Explicitly,

$$U_Q = \exp \left( \int q^{ia}(k)^\dagger (\sigma \cdot k) h_i(k) q^{ia}(k) dk - h.c. \right). \quad (2)$$

In the above, $h_i(k)$ is a real function of $|k|$ which describes vacuum realignment for chiral symmetry breaking for quarks of a given flavor $i$. Similarly, the unitary operator $U_d$ describing diquark condensates is given as

$$U_d = \exp \left( B_d^\dagger - B_d \right) \int \left[ q^{ia}(k)^\dagger r f^{ia}(k) q^{ib}(k) + \bar{q}^{ia}(k) r f^{ia}_d(k) \bar{q}^{ib}_d(k) \right] dk - h.c. \right). \quad (3)$$

Thus, we have only the quarks of colors red and green $(a=1,2)$ and flavors $u,d$ $(i=1,2)$ taking part in diquark condensation. We have also introduced here (color, flavor dependent) functions $f^{ia}(k)$ and $f^{ia}_d(k)$ respectively for the diquark and diantiquark channels. As may be noted the state constructed in Eq.(3) is spin singlet and is antisymmetric in both color and flavor. To include the effects of temperature and density we write down the state at finite temperature and density $|\Omega(\beta, \mu)\rangle$ taking a thermal Bogoliubov transformation over the state $|\Omega\rangle$ using thermofield dynamics (TFD) as described in Refs. [6, 7]. We then have,

$$|\Omega(\beta, \mu)\rangle = U_{\beta, \mu}|\Omega\rangle = U_{\beta, \mu} U_d U_Q|0\rangle. \quad (4)$$

where $U_{\beta, \mu}$ is given as

$$U_{\beta, \mu} = e^{B(\beta, \mu) - B(\beta, \mu)}, \quad (5)$$

with

$$B(\beta, \mu) = \int \left[ q^{ia}_f(k)^\dagger \theta_-(k, \beta, \mu) \bar{q}^{ia}_d(k) + q^{ia}_d(k)^\dagger \theta_+(k, \beta, \mu) q^{ia}_f(k) \right] dk. \quad (6)$$

In Eq.(6) the ansatz functions $\theta_{\pm}(k, \beta, \mu)$ will be related to quark and antiquark distributions and the underlined operators are the operators in the extended Hilbert space associated with thermal doubling in TFD method. In Eq.(6) we have suppressed the color and flavor indices on the quarks as well as the functions $\theta(k, \beta, \mu)$. All the functions in the ansatz in Eq.(4) are to be obtained by minimising the thermodynamic potential.

Since we shall be dealing with non–asymptotic densities, we take here a simple model for quark interactioni, namely the Nambu Jona Lasino model, with four fermion point interaction explicitly given as

$$H = \sum_{i,a} \psi^{ia\dagger}(-i \alpha \cdot \nabla + \gamma^0 m_i) \psi^{ia} - G_s \sum_{A=0}^{8} \left[ (\bar{\psi} \lambda^A \psi)^2 - (\bar{\psi} \gamma^5 \lambda^A \psi)^2 \right] - G_D (\bar{\psi} \gamma^5 \epsilon \bar{\epsilon} \psi) (\bar{\psi} C \gamma^5 \epsilon \bar{\epsilon} \psi) \quad (7)$$
Finally, the contribution of the electron to the total thermodynamic potential is given as
\[
\bar{\Omega} = \Omega_{\text{ud}} + \Omega_s + \Omega_e,
\]
where, \(\Omega_{\text{ud}}\) is the contribution from the light quarks and is given as
\[
\Omega_{\text{ud}} = \frac{8}{(2\pi)^3} \int d\mathbf{k} \left[ \sqrt{k^2 + m^2} - \frac{1}{2} (\bar{\omega}_+ + \bar{\omega}_-) \right] - \frac{4}{\beta(2\pi)^3} \sum_{i=1,2} \int d\mathbf{k} \left[ \log(1 + \exp(-\beta \omega_{-i} - \epsilon_i)) + \log(1 + \exp(-\beta \omega_{+i} + \epsilon_i)) \right] + \frac{\Delta^2}{4G_D} + \sum_{i=1,2} \frac{(M^i - m)^2}{8G_s} + \frac{4}{(2\pi)^3} \int d\mathbf{k} \left[ \sqrt{k^2 + m^2} - \frac{1}{2} (\epsilon_1 + \epsilon_2) \right] - \frac{2}{\beta(2\pi)^3} \sum_{i=1,2} \int d\mathbf{k} \left[ \log(1 + \exp(-\beta (\epsilon_i - \mu^3_+))) + \log(1 + \exp(-\beta (\epsilon_i + \mu^3_+))) \right] + \frac{(M_s - m_s)^2}{8G_s}. \tag{9}
\]
Similarly, the contribution from the strange quarks to the thermodynamic potential, \(\Omega_s\) is given as
\[
\Omega_s = \frac{6}{(2\pi)^3} \int d\mathbf{k} \left[ \sqrt{k^2 + m_s^2} - \sqrt{k^2 + M_s^2} \right] - \frac{2}{\beta(2\pi)^3} \sum_{a=1,3} \int d\mathbf{k} \left[ \log(1 + \exp(-\beta (\epsilon_3 - \mu^{3a}_+))) + \log(1 + \exp(-\beta (\epsilon_3 + \mu^{3a}_+))) \right] + \frac{(M_s - m_s)^2}{8G_s}. \tag{10}
\]
Finally, the contribution of the electron to the total thermodynamic potential is given as
\[
\Omega_e = -\mu_e^L / 12\pi^2 (1 + 2\pi^2 T^2 / \mu_e^2), \tag{11}
\]
The first three lines in Eq.(9) correspond to the contribution from the quarks taking part in the condensation while the fourth and fifth lines correspond to the contribution from the two light quarks with the blue color. The quasi-particle energies are given by \(\omega_{-1} = \bar{\omega}_{-1} + \bar{\delta}_e + \bar{\delta}_u\) and \(\omega_{-2} = \bar{\omega}_{-2} - \bar{\delta}_e + \bar{\delta}_u\), for u and d quarks respectively. Similarly for the antiparticles the energies are given as \(\omega_{+1} = \bar{\omega}_{+1} + \bar{\delta}_e + \bar{\delta}_u\) and \(\omega_{+2} = \bar{\omega}_{+2} + \bar{\delta}_e - \bar{\delta}_u\), for u and d quarks. Here \(\bar{\omega}_{+} = \sqrt{\Delta^2 + (\bar{\epsilon} \pm \bar{\mu} m)^2}\), with \(\bar{\epsilon}\) and \(\bar{\mu}\) being the average energy and average chemical potential of the two quarks that condense. We do not write down here the
gap equations for the superconducting gap as well as the three mass gaps as they are not very illuminating for our discussions that follow. Interested reader however can find the same in Ref.[7]. Thus the thermodynamic potential is a function of three parameters: the two mass gaps and a superconducting gap which need to be minimised subjected to the conditions of electrical and color charge neutrality namely $Q_E = 2/3\rho^1 - 1/3\rho^2 - 1/3\rho^3 - \rho_e = 0$, and, $Q_8 = 1/\sqrt{3}\sum_i(\rho^1 + \rho^2 - 2\rho^3) = 0$. Here we have denoted $\rho^{ia} = \langle \psi^{ia\dagger}\psi^{ia}\rangle$ and $\rho^i = \sum_a \rho^{ia}$— the two indices in the superscript rering to flavor and color indices respectively. the results of such a constrained minimisation are discussed in the follwing section.

3. Results and discussions

For numerical calculations we have taken the values of the parameters of NJL model as follows. The cutoff $\Lambda$ and the scalar coupling $G_s$ are chosen by fitting the pion decay constant $f_\pi = 93$ MeV and the chiral condensate $\langle \bar{u}u \rangle^{1/3} = -250$ MeV=$\langle \bar{d}d \rangle^{1/3}$. We also take the current quark masses of u and d quarks as zero. This leads to the coupling constant $G_s$ and the cut off $\Lambda$ as $G_s = 5.0163$ GeV$^{-2}$ and $\Lambda = 0.6533$ GeV. Similar to Ref.[9] we take the current quark mass of strange quarks as $m_s = 120$ MeV as a typical value giving “reasonable” vacuum properties. With this choice of parameters, the constituent quark masses at zero temperature and density are given as $M_1 = 0.313$ GeV=$M_2$, and for strange quark $M_3 = 0.541$ GeV. These values are similar to those obtained in Ref.[10], where the parameters have been fixed by fitting vacuum masses and decay constants of pesudoscalar mesons. In Fig.1 we show the gap parameters as a function of density without imposition of charge neutrality condition. At $\mu_B = 960$ MeV, the system goes from vacuum solution to a color superconducting phase while the masses of the light quarks drop to zero from their vacuum values.

![Figure 1](image1.png)  
**Figure 1.** Gap parameters when charge neutrality conditions are not imposed. Solid curve refers to masses of u and d quarks, dotted curve refers to mass of strange quarks and the dashed curve refers to the superconducting gap.

![Figure 2](image2.png)  
**Figure 2.** Phase diagram in the $(\mu, T)$ plane. Middle curve is the critical line and the outer lines are the lower and upper spinodals.

In Fig.2 we show the resulting phase diagram for the chiral transition in the temperature and
quark chemical potential ($\mu_B/3$) plane in the present model when charge neutrality conditions are not imposed. The middle line is the critical line in this plane. We also show here the upper ($T_1$) and the lower ($T_2$) spinodal lines constraining the region of spinodal instability region. In this region there are solutions of the mass gap equations which are metastable and fluctuations in this region is important. The tricritical point ($T_c, \mu_c$) turns out to be ($74.9, 285$ MeV) beyond which the transition is second order.

We next extend our discussions to the case where the charge neutrality conditions ($\mu_E=0=\mu_8$) are imposed. At lower baryon densities charge neutrality condition forces the d-quark density to be larger than u- quarks. This makes the d-quark masses vanishing much earlier than that of the u- quarks as density is increased. At higher chemical potential the strange quarks help in maintaining the charge neutrality condition. The superconducting gap is shown in Fig.3 and Fig.4 for different temperatures. As may be seen for smaller values of the baryon chemical potential ($\mu_B < 1470$ MeV) we have smaller values of the gap which reaches a maximum about 80 MeV at $\mu_B = 1390$ MeV at zero temperature. Then it decreases and vanishes at $\mu_B = 1470$ MeV=$\mu_2$. The gap remains zero till $\mu_B = \mu_3 = 1530$ MeV. At $\mu_3$ the system jumps from the normal phase to the BCS phase with a gap about 132 MeV. The baryon number density jumps from $\rho_B = 0.96$ fm$^{-3}$ to 1.54 fm$^{-3}$ i.e. from 6 to 9.5 times the nuclear matter density. The interval in the baryon chemical potential between the gapless and BCS phase within which it is normal quark matter decreases with increase in temperature and then disappears as may be noted in Fig.s4. The sharp transition between gapless phase to the BCS phase as a function of baryon chemical potential at smaller temperature becomes a smooth transition as temperature increases as may be seen from Fig.4.

Let us next discuss now some of the characteristics of the gapless modes which occur for smaller values of the baryon chemical potential ($\mu_B < \mu_2$). At zero temperature, gapless modes occur when the gap is less than half the difference of the chemical potential $\delta\mu$ of the
two condensing quarks. It is easy to show also that the excitation energy $\omega_2$ of the d-quark vanishes at momenta $\mu_- = \bar{\mu} - \sqrt{\delta \mu^2 - \Delta^2}$ and $\mu_+ = \bar{\mu} + \sqrt{\delta \mu^2 - \Delta^2}$ at zero temperature. In Fig.6 we plot the dispersion relations - the excitation energies of the quasi particles as functions of momentum for quark chemical potential $\mu_q = (1/3)\mu_B \simeq 379\text{MeV}$ and at zero temperature. The gap in this case turns out to be $\Delta = 59 \text{ MeV}$ and the difference of the chemical potential turns out to be $\delta \mu \simeq -66 \text{ MeV}$ at zero temperature. Far from the pairing region $\bar{\mu} \simeq 354 \text{ MeV}$, the spectrum looks like that of usual BCS type dispersion relations. Of the two excitation energies, $\omega_1$ shows a minimum at the average fermi momentum $\bar{\mu}$ with a value $\omega_1(|k| = \bar{\mu}) = \Delta + \delta \mu \simeq 125\text{MeV}$. On the other hand, $\omega_2$ becomes gapless ($\omega_2 = 0$) at momenta $\mu_-$ and $\mu_+$. In this ‘breached’ pairing region [11] we have only unpaired down quarks and no up quarks. In this phase the number densities of the condensing u and d quarks are not the same. The occupation numbers in the momentum space for the u and d quarks are plotted in Fig.5. In the region between $\mu_-$ and $\mu_+$, the occupation numbers resemble like that of normal matter as $\rho_u$ becomes almost zero but for the vanishingly small antiparticle contribution. On the other hand, in the same region, the occupation number for d- quarks becomes unity in the same limit for the antiparticle contributions. This gives rise to difference in number densities of the u and d quarks that take part in diquark condensation and in fact is given as This is in fact given by $\delta \rho_{sc}(T = 0) = 2/(3\pi^2)(\mu_+^3 - \mu_-^3)$. We also note here that the ratio of $T_C$ to the gap $\Delta(T = 0)$ for the gapless phase is not the BCS value of 0.567 but can be as large as 0.81 for the chemical potential considered here.

As the chemical potential is increased beyond 1530 MeV at zero temperature the solution for the gap jumps to a higher value of about 130 MeV almost similar to the case when charge neutrality condition is not imposed. This corresponds to the usual BCS solution. In this case, the numbers of u quarks and d quarks participating in condensation are the same. The charge
neutrality condition however is maintained by the third color, the electron as well as the strange quarks. One essential effect of including strange quarks is that the electron density starts decreasing for chemical potentials greater than the strange quark mass as the strange quark can carry the negative charge to maintain electric charge neutrality condition. This has the effect that the lowest excitation energy $\omega_2 = \bar{\omega} + \delta_\mu$ in the BCS pairing case becomes large due to both the large value of the gap as well as the smaller magnitude of the electron chemical potential.

4. Summary

We have analysed here in NJL model the structure of vacuum in terms of quark–antiquark as well as diquark pairing at finite temperature. The methodology uses an explicit variational construct of the trial state. The gap function as well as the thermal distribution functions are also determined variationally. To consider neutron star matter, we have imposed the color and electric charge neutrality conditions through the introduction of appropriate chemical potentials.

We observe that the mass of the strange quark plays a sensitive role in maintaining charge neutrality condition. Further, we observed that solutions for the gap quations which may not be free energetically preferable solutions, can be the preferred solutions when constraints of charge neutrality conditions are imposed. We also find that with increase in chemical potential, the quark matter has a transition from a gapless phase to the BCS phase through an intermediate normal quark matter phase and this interval decreases and finally vanishes at higher temperature. The sharp transition at zero temperature becomes a smooth transition at higher temperatures as density is increased.

We have focussed our attention here to the two flavor superconducting phase with strange quarks. The variational method adopted can be directly generalised to include color flavor locked phase and one can then make a free energy comparison regarding the possibility of which phase would be thermodynamically favourable at what density. This will be particularly interesting for cooling of neutron stars with a CFL core through neutrino emission.

[1] For reviews see Rajagopal, K. and Wilczek, F. arXiv:hep-ph/0011333; Hong, D.K. Acta Phys. Polon. B32,1253 (2001); Alford, M.G., Ann. Rev. Nucl. Part. Sci 51, 131 (2001); Nardulli,G., Riv. Nuovo Cim. 25N3, 1 (2002); Reddy, S., Acta Phys Polon. 33, 4101(2002); Schaefer, T., arXiv:hep-ph/0304281; Rischke, D.K., Prog. Part. Nucl. Phys. 52, 197 (2004); Ren, H.C., arXiv:hep-ph/0404074; Huang, M. arXiv: hep-ph/0409167; Shovkovy, L., arXiv:nucl-th/0410191.

[2] Shovkovy Igor, Huang Mei, Phys. Lett. B 564, 205 (2003), Huang Mei and Shovkovy Igor, Nucl. Phys. A729, 835 (2003).

[3] Amruta Mishra and Hiranmaya Mishra, Phys. Rev. D 69, 014014 (2004).

[4] H. Mishra and J.C. Parikh, Nucl. Phys. A679, 597 (2001).

[5] Jaikumar P. and Prakash, M. Phys. Lett. B 516, 345 (2001).

[6] H. Umezawa, H. Matsumoto and M. Tachiki Thermofield dynamics and condensed states (North Holland, Amsterdam, 1982) ; P.A. Henning, Phys. Rep. 253, 235 (1995).

[7] Amruta Mishra and Hiranmaya Mishra, J. Phys. G: Nucl. Phys. 231431997.

[8] Mishra, A. and Mishra, H., arXiv: hep-ph/0412213.

[9] Buballa M. and Oertel M., Nucl. Phys. A703, 770 (2002).

[10] P. Rehberg, S.P. Klevansky and J. Haeffner, Phys. Rev. C 53, 410 (1996).

[11] W.V. Liu and F. Wilczek, Phys. Rev. Lett. 900470022003, E. Gubankova, W.V. Liu and F. Wilczek, Phys. Rev. Lett. 910320012003.