Bright and dark solitons in the systems with strong light-matter coupling: exact solutions and numerical simulations

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We theoretically study bright and dark solitons in an experimentally relevant hybrid system characterized by strong light-matter coupling. We find that the corresponding two-component model supports a variety of coexisting moving solitons including bright solitons on zero and nonzero background, dark-gray and gray-gray dark solitons. The solutions are found in the analytical form by reducing the two-component problem to a single stationary equation with cubic-quintic nonlinearity. All found solutions coexist under the same set of the model parameters, but, in a properly defined linear limit, approach different branches of the polariton dispersion relation for linear waves. Bright solitons with zero background feature an oscillatory-instability threshold which can be associated with a resonance between the edges of the continuous spectrum branches. ‘Half-topological’ dark-gray and nontopological gray-gray solitons are stable in wide parametric ranges below the modulational instability threshold, while bright solitons on the constant-amplitude pedestal are unstable.

I. INTRODUCTION

Optical solitons, as localized waves propagating in nonlinear fibers, were predicted [1] and experimentally observed [2] more than forty years ago. Since then optical solitary waves have been discovered and thoroughly studied in many optical systems. Apart from the fundamental importance, optical solitons can also be of interest from practical point of view, in particular, for information transmission and supercontinuum generation [3–5]. To describe formation of solitons, it is necessary to combine Maxwell equations with the equations accounting for the response of the material to the propagating electromagnetic field. This description is, however, in many cases so complicated that even numerical modelling of the dynamics of the light becomes impossible. Fortunately, the presence of small parameters often allows to simplify the problem. For example, slow varying amplitude approximation has proven to be a very powerful and precise model which allows to describe a large variety of optical phenomena using the nonlinear Schrödinger (NLS) equation and a family of its generalizations [6, 7]. In its basic form, the NLS equation is fully integrable and its soliton solutions are available in the analytical form. The knowledge of exact soliton solution for the NLS equation and other prototypical nonlinear models has two-fold importance. First, it helps to understand the properties of the localized waves in the considered system and, in particular, facilitates the stability study. Second, the analytical solutions can be used as a starting point for the development of a perturbation theory for more complex and general systems which take into account the originally neglected effects and do not always admit analytical solutions.

In this paper, we address bright and dark solitons propagating in optical waveguides with strong light-matter coupling which is a typical attribute of exciton-polariton systems. The system of such a kind consist of a dielectric waveguide with built-in quantum wells supporting excitons. If the losses are small, then, at the frequencies close to the exciton resonance, the photons and the excitons interact strongly so that at the crossing point the dispersions of the photons and the excitons hybridize and get split forming the lower and the upper polariton branches.

Without resonant material excitations, the nonlinear effects usually come to play at so high pulse energies that it reduces its practical applicability, especially in optical on-chip devices. The effective (material and waveguide) dispersion is also relatively low in these systems. For example, typical energy of 100 fs optical solitons in highly nonlinear optical waveguides is of the order of 100 pJ, and the soliton formation occurs at propagation distances of several centimeters. The advantage of the systems with strong light-matter coupling is that because of the material component of the eigenmodes the nonlinear effects are orders of magnitude stronger than in the systems with weak coupling. Another important fact is that the typical dispersion caused by the linear photon-exciton interaction is much stronger compared to the dispersion of pure photons. This allows to observe soliton formation at the propagation distances of order of hundreds microns and the energy of order of hundreds of fJ in 100 fs pulses. This makes the systems with strong light-matter interaction to be very promising for studying different nonlinear effects and explains why these systems have been attracting so much of attention over the recent years [8].

The simplest model describing the dynamics of pulses propagating in the waveguides with strong light-matter interaction neglects the dispersion and nonlinearity of pure photons, and hence the photonic component is described by a linear equation. This equation is coupled to the equation for excitons where the resonant frequency of the excitons is a function of their density. In the simplest case considered herein, the shift of the exciton fre-

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frequency is proportional to their density. In the context of meanfield approximation for polariton systems, the corresponding model was introduced in [9] and has been widely used. The adaptation of the model for the case when the frequency of excitation is much higher than the cut-off frequency of the waveguide is done in [10]. The latter model can be brought to the following form:

\[ i(\partial_t A + \partial_x A) = -\kappa \psi, \quad i\partial_t \psi = -\kappa A + g|\psi|^2 \psi. \quad (1) \]

Here \( t \) is the time normalized on some characteristic frequency \( \Omega_0 \), \( x \) is the coordinate normalized on the \( \Omega_0/\nu_g \), \( \nu_g \) is the group velocity of the pure photon mode at the resonant frequency of the material excitations \( \Omega_m \). In the equations (1) the coefficient \( \kappa \) accounts for the light-matter coupling strength and without loss of generality can be set to 1. This means that the normalization frequency \( \Omega \) is chosen to be equal to the light-matter coupling in the system.

The function \( A(x,t) \) in (1) is the slow varying amplitude of the photon field and \( \psi(x,t) \) is the order parameter describing the material excitations, for example \( \psi \) can be the order parameter of the coherent excitons in a semiconductor microcavity. To achieve analytical solutions, we disregard the dispersion of the pure guided photons assuming this is much smaller compared to the dispersion appearing due to the light-matter coupling (as it is typical for experimental conditions [10]). The effective mass of the coherent excitons is supposed to be much greater than the effective mass of guided photons (for semiconductor microcavities the typical values \( < 10^{-4} \)) and thus the frequency of linear material excitations does not depend on their wave vector. Let us remark that all frequencies used below in this paper are actually the detunings of the frequency from the linear frequency of the material excitations \( \Omega_m \). The wavevectors, in their turn, are the detunings of the wavevectors from the wavevector of the pure photonic mode of the frequency \( \Omega_0 \).

Normally in the systems with strong light-matter coupling the dominating nonlinearity originates from the resonant frequency dependency of the material excitations on their density. In our model this frequency shift is taken to be proportional to the density of the material excitations and thus is equal to \( g|\psi|^2 \). Aiming to obtain analytical solutions, we consider a conservative problem. The conservative limit is a good approximation for polariton waves propagating over the distances sufficient to observe the formation of the solitons (hundreds of microns). The comprehensive studies of the effect of losses is definitely of interest but is out of the scope of the present paper and will be done elsewhere.

In the linear limit \( g = 0 \), for plane-wave solutions \( e^{-i\delta t + ikx} \), where \( \delta \) is the frequency and \( k \) is the wavenumber, we obtain two polariton branches (upper, with subscript 1, and lower, with subscript 2) of the dispersion relation (plotted in Fig. 1):

\[ \delta_{1,2}(k) = (k \pm \sqrt{k^2 + 4\kappa^2})/2. \quad (2) \]

System (1) can be reduced to the NLS equation written for the amplitude of the polariton mode belonging either to the lower or the upper branch provided the peak power of the pulse causes the exciton frequency shift much less compared to width of the gap between the upper and the lower polariton branches. However the soliton of higher intensities have to be considered taking into account the dispersion of the nonlinearity and the excitations belonging to both branches. That is why the analysis of the full two-component model is important. We also note that a generalized version of model (1) has been used to describe the formation of dark solitons in polariton fluids [11]. Another pertinent remark is that system (1) is mathematically similar to the coupled-mode equations describing gap solitons in optical fibers with grating [12]. The importance difference is that in (1) only the equation for \( \psi \) field (order parameter function describing coherent polaritons) is nonlinear whereas in the case of gap solitons both fields are nonlinear.

It has been found in [10] that system (1) admits an analytical bright soliton solution whose existence has been confirmed experimentally. In this paper, we perform a comprehensive study of different kinds of bright and dark solitons existing in the system and look into stability of the found analytical solitons. We find that apart from the bright solitons, the system admits 'semitopological' dark-gray solitons, nontopological gray-gray solitons, and bright solitons nesting in the background of nonzero constant amplitude. All these solutions, which are found in the analytical form, coexist in the system with the same set of model parameters. At the same time, in a properly defined linear limit, the frequency and wavevector of bright solitons approach the lower-frequency branch of the dispersion relation, while solitons of other types approach the upper polariton branch. We numerically observe that large-amplitude bright solitons are prone to oscillatory instabilities which, however, can have rather weak instability increment. Dark-gray and gray-gray solutions are stable in a vast range of parameters. The stability predictions are verified with direct numerical
modelling of soliton dynamics.

The rest of our paper is organized as follows. In Sec. III we present the analytical exact solutions for bright and dark solitons and discuss their spectral stability. In Sec. IV we present the result of direct numerical modelling of soliton dynamics. Section V concludes the paper.

II. EXACT MOVING SOLITON SOLUTIONS

A. Construction of solutions

We are looking for soliton solutions in the moving frame \( \zeta = x - v_s t \), where \( v_s \) is the velocity (hereafter, the subscript \('s\)' stays for 'soliton'). We therefore introduce the following substitutions \( A(x, t) = A_s(\zeta)e^{-i\delta_s t}, \) and \( \psi(x, t) = \psi_s(\zeta)e^{-i\delta_s t} \). The solitons are characterized by two parameters: velocity \( v_s \) and frequency in the moving frame \( \delta_s \) (which is generically different from the frequency \( \delta \) in the lab frame). Then system (1) reduces to

\[
i(1 - v_s)A_s' + \delta_s A_s = -\kappa \psi_s, \quad (3) \]
\[
-iv_s \psi'_s + \delta_s \psi_s = -\kappa A_s + g|\psi_s|^2\psi_s, \quad (4) \]

where prime means derivative with respect to the moving frame coordinate \( \zeta \). Differentiating the second equation of the latter system one more time, one can eliminate the wavefunction \( A_s(\zeta) \) and reduce the system to a single equation for \( \psi_s(\zeta) \). Using a substitution

\[
\psi_s(\zeta) = \phi_s(\zeta) \exp \left\{ -i \frac{\delta_s(1 - 2v_s)}{2v_s(1 - v_s)} \right\}, \quad (5) \]

where \( \phi_s(\zeta) \) is a new unknown, the problem transforms to

\[
-v_s^2 \phi_s'' + \left( \frac{\kappa^2 v_s}{1 - v_s} - \frac{\delta_s^2}{4(1 - v_s)^2} \right) \phi_s + \frac{g\delta_s}{2(1 - v_s)} |\phi_s|^2 \phi_s + igv_s|\phi_s|^2 \phi_s' = 0. \quad (6) \]

Next, we use the polar form \( \phi_s(\zeta) = \rho_s(\zeta)e^{i\Theta_s(\zeta)} \) and decompose equation (6) into real and imaginary parts. The latter results in the following relation:

\[
\Theta'_s \rho_s^2 = \frac{3g}{4v_s} \rho_s^4 + C, \quad (7) \]

where \( C \) is arbitrary constant of integration. Then the real part of the polar decomposition becomes

\[
-v_s^2 \rho_s'' + \left( \frac{\kappa^2 v_s}{1 - v_s} - \frac{\delta_s^2}{4(1 - v_s)^2} + \frac{gCv_s}{2} \right) \rho_s + \frac{g\delta_s}{2(1 - v_s)} \rho_s^3 - \frac{3g^2v_s^2}{16} \rho_s^5 + C^2v_s^2 \rho_s^2 = 0. \quad (8) \]

In the particular case \( C = 0 \) the latter equation can be considered as a stationary version of the cubic-quintic nonlinear Schrödinger equation, which is known to support a number of solutions in the form of bright and dark solitons, many of which can be found in the analytical form, see e.g. [8, 13]. This observation paves the way towards the systematic construction of analytical solitons for the original system (1). The quintic nonlinearity in (8) is focusing, while sign and effective strength of the cubic nonlinearity depend both on the soliton velocity \( v_s \) and frequency \( \delta_s \).

Types of existing solutions can be also anticipated from the phase portrait which can be obtained by multiplying Eq. (8) by \( \rho_s' \) and integrating. Representative phase portraits for different combinations of the parameters are presented in Fig. 2. For \( C = 0 \) and different values of the frequency \( \delta_s \) the system admits homoclinic orbits which join the equilibrium \( (\rho_s, \rho_s') = (0, 0) \) to itself and hence correspond to bright solitons [Fig. 2(a)] or homoclinic orbits corresponding to bright solitons situated on a nonzero background coexisting with and heteroclinic orbits corresponding to dark solitons [Fig. 2(b)]. For nonzero \( C \) [Fig. 2(c)], the system has homoclinic orbits of different types that correspond to bright solitons on a nonzero background (with maximal amplitude larger than that of the equilibrium) and to grey solitons (i.e., dips in the uniform background).

If the amplitude \( \rho_s(\zeta) \) is found from Eq. (8), one can recover the argument \( \Theta_s(\zeta) \) of the corresponding exci-
tonic field using \( \Theta_s(\zeta) \) and then find the photonic component \( A_s(\zeta) \) from Eq. (2). In the particular case \( C = 0 \) for the amplitude of the photonic field we compute

\[
|A_s(\zeta)|^2 = \frac{v_s}{1 - v_s} \left( \rho_s^2(\zeta) - \rho_\infty^2 \right) + \frac{\rho_s^2}{\kappa^2} \left( \frac{g}{4} \rho_s^2 - \frac{\delta_s}{2(1 - v_s)} \right)^2, \tag{9}
\]

where constant \( \rho_\infty \) is determined by the boundary conditions

\[
\lim_{\zeta \to \infty} \rho_s^2(\zeta) = \lim_{\zeta \to \infty} \rho_s'(\zeta) =: \rho_\infty^2, \tag{10}
\]

\[
\lim_{\zeta \to \infty} \rho_s'(\zeta) = \lim_{\zeta \to \infty} \rho_s'(\zeta) = 0. \tag{11}
\]

Therefore the squared amplitudes of \( \psi_s(\zeta) \) and \( A_s(\zeta) \) are proportional, except for an additive constant which is determined by the asymptotic behavior at the infinities.

### B. Bright solitons

As is evident from Fig. 2(a), for \( C = 0 \) in Eq. (3) the system supports bright solitons for which \( \rho_\infty = 0 \) in boundary conditions (10). As follows from (9) with \( \rho_\infty = 0 \), solutions of this type can only be meaningful for \( v_s \in (0, 1) \). Bright soliton solutions can be written down in the compact form if one introduces two auxiliary angles \( \alpha \in (0, \pi/2) \) and \( \theta \in (\pi/2, \pi/2) \) and adopts the following parametrization for the solution frequency and velocity:

\[
v_s = \sin^2 \alpha, \quad \delta_s = -\kappa \sin(2\alpha) \sin \theta. \tag{12}
\]

Then the following solution can be found (see also equation (10))

\[
\psi_s(\zeta) = \rho_s(\zeta)e^{2i\kappa \cot(2\alpha) \sin \theta + i\Theta_s(\zeta)},
\]

\[
A_s(\zeta) = \tan(\alpha)\rho_s(\zeta)e^{2\kappa \cot(2\alpha) \sin \theta + i\Theta_s(\zeta)/3}, \tag{13}
\]

where

\[
\rho_s^2(\zeta) = \frac{4\kappa g}{\sin \theta + \cosh(4\kappa \cos \theta \csc(2\alpha) \zeta)}
\]

\[
\Theta_s(\zeta) = 3 \arctan \left( \frac{1 - \sin \theta}{\cos \theta} \tanh(2\kappa \cos \theta \csc(2\alpha) \zeta) \right). \tag{14}
\]

Since bright solitons (13) are found in the frame moving with velocity \( v_s \), the frequency in the lab frame amounts to \( \delta = \delta_s + 2v_s \kappa \cot(2\alpha) \sin \theta = -\kappa \sin \theta \tan \alpha \), and the spatial wavenumber of soliton tails amounts to \( k_s = 2 \kappa \cot(2\alpha) \sin \theta \). In the limit \( \theta \to \pi/2 \) the soliton amplitude tends to zero, and the solution frequency \( \delta \) approaches from above the lower branch of the dispersion relation, i.e., \( \delta = \delta_1(k_s) \), where the polariton dispersion laws \( \delta_1(k) \) are defined in (2). This is shown schematically with the arrow ‘bs’ in Fig. 4. In the limit \( \theta \to -\pi/2 \), the solution frequency approaches (from below) the upper polariton branch \( \delta_2(k) \). In this limit, the shape of the solution becomes algebraic:

\[
\lim_{\theta \to -\pi/2} \frac{\rho_s^2(\zeta)}{\rho_\infty^2} = \frac{16\kappa}{g} \frac{\sin(2\alpha) \sin^2 \alpha}{\sin^2(2\alpha) + 16\kappa^2 \alpha^2}. \tag{15}
\]

Let us now look into stability of the found bright gap solitons. Using the standard linear stability analysis, we consider perturbed stationary solutions in the form

\[
A(x, t) = e^{-i\delta_s t}[A_s(\zeta) + a_1(\zeta)e^{\lambda t} + a_2(\zeta)e^{\lambda^* t}],
\]

\[
\psi(x, t) = e^{-i\delta_s t}[\psi_s(\zeta) + p_1(\zeta)e^{\lambda t} + p_2(\zeta)e^{\lambda^* t}], \tag{16}
\]

where \( a_{1,2}(\zeta) \) and \( p_{1,2}(\zeta) \) describe the spatial shapes of the perturbations, and and complex \( \lambda \) characterizes the temporal behavior of the perturbations (positive real part of \( \lambda \) means that the perturbations grow and the soliton is therefore unstable). Substituting these expressions in Eq. (1) and keeping only linear (with respect to the small perturbations) terms, we arrive at the following system of linear stability equations which can be treated as an eigenvalue problem for the instability increment \( \lambda \):

\[
i\lambda a_1 = -i(1 - v_s)a_1' - \delta a_1 - \kappa \lambda a_1,
\]

\[
i\lambda a_2 = -i(1 - v_s)a_2' + \delta a_2 + \kappa \lambda a_2,
\]

\[
i\lambda p_1 = -\kappa a_1 + iv_s p_1' - (\delta - 2g|\psi_s|^2)p_1 + g\psi_s^2 p_2,
\]

\[
i\lambda p_2 = \kappa a_2 - g(\psi_s^*)^2p_2 + iv_s p_2' + (\delta - 2g|\psi_s|^2)p_2. \tag{18}
\]

It is known that dynamics of gap solitons in various setups can be affected by oscillatory instabilities (OIs) which correspond to unstable eigenvalues detaching from the edges of the continuous spectrum [14, 15, 16]. By definition [17], the instability of this type is associated with a quartet of complex eigenvalues \((\pm\lambda, \pm\lambda^*)\), see panel ‘OI’ in Fig. 4. Another common type of instabilities corresponds to a pair of purely real eigenvalues \((\lambda, -\lambda)\). The instability of this type can be referred to as the internal or exponential instability (EI), see schematic illustration ‘OI and EI’ in Fig. 4.

We solve the eigenvalue problem (18) by approximating the derivatives by finite differences and evaluating the spectrums of the resulting sparse matrix using the MatLab eig procedure. The fourth-order approximation has been used for the derivatives subject to the zero boundary conditions. Depending on the localization of eigenfunctions, we have used different computational windows \( \zeta \in [-L, L] \), with \( L \) ranging from 20 to 160 and number of grid nodes ranging from \( 10^4 \) to \( 10^5 \). In each case it has been checked that small variations of the grid parameters do not have any essential impact on the outcomes of the computation. Numerical solution of the linear stability eigenvalue problem (18) indicates that the oscillatory instabilities are indeed present in our system. More specifically, we observe that solitons with \( \delta_s < 0 \) are stable, whereas oscillatory instabilities can be found for \( \delta_s > 0 \). Precise detection of the instability threshold is
a numerically challenging problem, because for $\delta_s$ close to zero the instability increments are rather weak, and the decay of the tails of corresponding unstable eigenmodes is extremely slow as $\zeta$ approaches $\infty$ and $-\infty$. At the same time, the stability change at $\delta_s = 0$ (which, in terms of parametrization $[12]$, corresponds to $\theta = 0$) can be anticipated as one looks at the continuous spectrum associated with the linear stability system $[13]$. It has four branches of the continuous spectrum that occupy the following intervals of the imaginary axis:

\begin{align}
\lambda_{1,2} & \in i[\kappa \sin(2\alpha)(1 \pm \sin \theta), +\infty), \quad (19) \\
\lambda_{3,4} & \in (-\infty, -\kappa \sin(2\alpha)(1 \pm \sin \theta)]i. \quad (20)
\end{align}

Exactly at $\theta = 0$ the edges of the continuous spectrum coincide pairwise and become resonant, which, as we conjecture, results in the bifurcation of a quartet of oscillatory instability eigenvalues, with two eigenvalues emerging from $\lambda = i\kappa \sin(2\alpha)$ and two more eigenvalues emerging from $\lambda = -i\kappa \sin(2\alpha)$. Weak oscillatory instabilities (with the instability increment $\text{Re} \lambda \lesssim 10^{-2}$) emerging in the vicinity of the resonant spectrum edges have indeed been observed in our numerical simulations for small positive $\delta_s$ as shown in Fig. 3. As $\delta_s$ increases towards larger positive values, the increment of oscillatory instability grows.

As pointed out above in this section, apart from the oscillatory instabilities, the system can admit purely exponential instabilities associated with a pair of purely real eigenvalues $\pm \lambda \in \mathbb{R}$ in the linearization spectrum. The instability of this type emerges at the moment when two stable internal modes (i.e., purely imaginary and complex-conjugate eigenvalues) collide at the origin and then split into a pair of purely real eigenvalues of opposite sign. The moment of such an eigenvalue zero crossing can be obtained analytically using the multiple-scale analysis $[14, 16]$. To this end, we notice that the system $[11]$ with zero boundary conditions has two conserved quantities: $Q = \int_{-\infty}^{\infty} x(|A|^2 + |\psi|^2)$, and $P = \frac{1}{2} \int_{-\infty}^{\infty} dx (A \partial_x A^* - A^* \partial_x A + \psi \partial_x \psi^* - \psi^* \partial_x \psi)$. For bright solitons given by the exact solution $[13]$, these quantities become functions of $\alpha$ and $\theta$:

\begin{align}
Q_s &= \frac{2 \tan^2 \alpha}{g} (\pi - 2\theta), \quad (21) \\
P_s &= \frac{4\kappa \tan \alpha}{g \cos^2 \alpha} [(1 + 2 \cos^2 \alpha) \cos \theta - (\pi - 2\theta) \sin \theta]. \quad (22)
\end{align}

The multiple-scale analysis relies on the assumption that if the increment of the newly emerged exponential instability is small, then the initial stage of the dynamical instability development results in the adiabatic change of the solitons frequency and velocity, i.e., one can introduce functions $v_s = v_s(T)$ and $\delta_s = \delta_s(T)$, where $T = \epsilon t$ is a 'slow time', and $\epsilon$ is a formal small parameter. Respectively, the auxiliary parameters are also to be considered as function of the slow time: $\alpha = \alpha(T)$ and $\theta = \theta(T)$. Carrying out the corresponding calculations, we observe that a new small unstable eigenvalue appears in (or disappears from) the linearization spectrum at the instance when the following condition is satisfied

\begin{equation}
\frac{\partial P_s}{\partial \alpha} \frac{\partial Q_s}{\partial \theta} - \frac{\partial P_s}{\partial \theta} \frac{\partial Q_s}{\partial \alpha} = 0. \quad (23)
\end{equation}

Direct computation reduces $[23]$ to the following simple equation for $\theta$:

\begin{equation}
(\pi^2 - 4\pi \theta + A \theta^2 - 3) \cos \theta + 2(\pi - 2\theta) \sin \theta = 0. \quad (24)
\end{equation}

Within the interval $\theta \in (-\pi/2, \pi/2)$, the latter equation has a single root $\theta_s \approx -1.189$, with $\sin \theta_s \approx -0.928$. In terms of the soliton frequency and velocity, the found threshold corresponds to the following dependence

\begin{equation}
\delta_{s*, s} = -2\kappa \sqrt{v_s (1 - v_s)} \sin \theta_s = -\kappa \sin(2\alpha) \sin \theta_s. \quad (25)
\end{equation}
The emergence of a new pair of real eigenvalues as the soliton frequency $\delta_s$ increases above the found threshold value $\delta_{s,1}$ has been verified by means of direct evaluation of the linear stability eigenvalues. At the same time, since $\delta_{s,1}$ is well above zero, these new eigenvalues emerge in the parametric region where the bright solitons are already unstable due to the oscillatory instabilities described above. Therefore, for a generic initial perturbation, this additional exponential instability has no significant impact on the overall behavior of the system.

Results of the stability analysis for bright solitons are summarized in the diagram shown in Fig. 2. It shows the domain of existence of bright solitons on the plane $(\delta_s, v_s)$ and demarcates the stability and instability regions.

C. Dark-gray solitons and bright solitons on a nonzero pedestal

Phase portrait shown in Fig. 2(b) indicates that for $C = 0$ and sufficiently large frequencies $\delta_s$ the system supports solitons of two more types. First, the heteroclinic orbit in Fig. 2(b) correspond to dark solitons for which the profile $\rho_s(\zeta)$ increases monotonically from $-\rho_\infty$ to $+\rho_\infty$ and becomes zero at some $\zeta$, where the excitonic wavefunction $\psi_s(\zeta)$ has a topological $\pi$ phase jump. Regarding the corresponding photonic field $A_s(\zeta)$, from (30) we observe that for $v_s \in (0, 1)$ the amplitude $|A_s|^2$ corresponds to a dip in the uniform background. Moreover, the amplitude of the wavefunction nowhere vanishes, i.e., $|A_s(\zeta)| > 0$ for all $\zeta$, i.e., corresponds to a nontopological (gray) soliton without the phase jump.

The second type of soliton solutions in the phase portrait Fig. 2(b) corresponds to bright solitons on nonzero pedestal. For these solutions both fields $\psi_s(\zeta)$ and $A_s(\zeta)$ correspond to bumps on the constant-amplitude background.

To present these solutions, for $v_s \in (0, 1)$ it is convenient to introduce the following parametrization

$$v_s = \sin^2 \alpha, \quad \delta_s = \frac{\kappa}{2} \sin(2\alpha)(3e^\theta - e^{-\theta}),$$

(26)

where $\alpha \in (0, \pi/2)$ and $\theta > 0$, and the following constants

$$\rho_\infty^2 = \frac{4\kappa}{3} \tan \alpha \sinh \theta, \quad b = \frac{1}{4}(1 - e^{-2\theta}),$$

$$p = \kappa \csc(2\alpha) \sqrt{3e^{2\theta} - e^{-2\theta} - 2}.$$  

(27)

Then the dark soliton profile reads

$$\psi_s(\zeta) = \rho_s(\zeta)e^{-i\kappa(3e^\theta - e^{-\theta})\cot(2\alpha)\zeta + i\Theta_s(\zeta)},$$

(28)

where

$$\rho_s(\zeta) = \frac{\rho_\infty \sinh(\rho_s \zeta)}{\sqrt{\cosh^2(p_s \zeta) - b}},$$

$$\Theta_s(\zeta) = 6\kappa \sinh \theta \csc(2\alpha) \zeta \left[-3 \arctan \left(\frac{1 - e^{-2\theta}}{3 + e^{-2\theta}} \tanh(p_s \zeta)\right)\right].$$

(30)

For bright solitons on the background, it is sufficient to redefine

$$b = (3 + e^{-2\theta})/4,$$

(31)

and the resulting solution is obtained from (28) with

$$\rho_s(\zeta) = \frac{\rho_\infty \cosh(\rho_s \zeta)}{\sqrt{\cosh^2(p_s \zeta) - b}},$$

$$\Theta_s(\zeta) = 6\kappa \sinh \theta \csc(2\alpha) \zeta \left[-3 \arctan \left(\frac{1 - e^{-2\theta}}{3 + e^{-2\theta}} \coth(p_s \zeta)\right) + 3\pi/2\right],$$

(33)

where (33) is only valid for $\zeta \geq 0$, and for $\zeta < 0$ the argument must be redefined as an odd function, i.e., $\Theta_s(\zeta) := -\Theta_s(-\zeta)$.

The frequency in the lab frame can be computed from the soliton frequency as $\delta = -\kappa(3e^\theta - e^{-\theta})\cot(2\alpha)v_s + \delta_s = (\kappa/2)(3e^\theta - e^{-\theta})\tan \alpha$. For dark solitons, it is natural to define the small-amplitude limit as $\rho_\infty \rightarrow 0$, which in the case at hand corresponds to $\theta \rightarrow +0$. In this limit the frequency $\delta$ approaches the upper polariton branch of the dispersion relation from above as shown schematically with arrow ‘ds’ in Fig. 1.

Before we proceed to stability of the found solutions, it is important to examine the potential modulational instability [7] of the corresponding constant-amplitude background. Far from the soliton core, the solutions asymptotically transform to constant-amplitude waves $\psi_0(\zeta) = \rho_\infty e^{i\Omega \zeta}$, where

$$\Omega = \kappa \csc(2\alpha) \left[6 \sinh \theta - (3e^\theta - e^{-\theta}) \cos(2\alpha)\right].$$

(34)

For these constant-amplitude solutions, we perform the standard modulational stability analysis using a substitution similar to (16) with perturbations $a_{1,2}(\zeta) = \tilde{a}_{1,2} e^{\pm i\Omega \zeta + ik\zeta}$, $p_{1,2}(\zeta) = \tilde{p}_{1,2} e^{\pm i\Omega \zeta + ik\zeta}$, where $\tilde{a}_{1,2}$ and $\tilde{p}_{1,2}$ are constants, and real $k$ characterizes the wavenumber of the perturbation. Then the modulational instability eigenvalues $\lambda(k)$ can be found as roots of a quartic characteristic equation. The exhaustive classification of all roots is a tedious task, but it is possible to describe their behavior the limit $k \rightarrow \infty$. Using computer algebra, one can show, that in this limit the roots of the characteristic equation have asymptotic behavior
as follows
\[ \lambda_{1,2}(k) = ik \cos^2 \alpha - ic_{1,2} + o(1), \]
\[ \lambda_{3,4}(k) = -ik \sin^2 \alpha - ic_{3,4} + o(1), \]
where
\[ c_{1,2} = \pm \kappa e^{-\theta} \cot \alpha, \quad c_{3,4} = \pm \kappa \tan \alpha \sqrt{4 - 3e^{2\theta}}. \]

Therefore the coefficients \( c_{1,2} \) are real, while \( c_{3,4} \) are real if and only if \( e^\theta < 2/\sqrt{3} \), i.e., \( \theta \lesssim 0.144 \). If the latter condition is violated, then \( \lambda_{3,4}(k) \) acquire a nonzero real part, and constant-amplitude solutions (and, respectively, the soliton solutions on the constant-amplitude background) become unstable with respect to small-wavelength (\( |k| \gg 1 \)) perturbations. Systematic numerical evaluation of roots of the characteristic equation indicates that below the found instability threshold the constant-amplitude waves are stable.

In the region where the modulational instability is absent, we perform an additional search for possible unstable modes by numerical solutions of the linear stability equations (18). It indicates that below the modulational instability threshold the dark-gray solitons are stable, except for a narrow parametric interval of weak instabilities centered around \( v_s = 1/2 \). Bright solitons on the pedestal are unstable even below the modulational instability threshold due to the presence of internal unstable modes associated with localized eigenfunctions of the linear-instability operator.

### D. Gray-gray solitons

Phase portrait in Fig. 2(c) indicates that when the constant of integration \( C \) in Eq. (8) is nonzero, the bright solitons on the pedestal coexist with gray solitons, i.e., non-topological dips in the constant-amplitude background, with the amplitude at the dip being nonzero. While the shape of the bright solitons is similar to those presented in the previous subsection, the gray-gray solitons constitute a significant generalization of the dark-gray solitons presented above. The respective solutions can also be found in analytical form, although the resulting expressions are rather bulky. To simplify the presentation, in this subsection we set \( \kappa = g = 1 \). Then the amplitude and phase of the excitonic field is given as

\[ \rho_s(\zeta) = \rho_\infty \sqrt{\frac{\sin^2(p\zeta) + c}{\cosh^2(p\zeta) - b}}, \]

and

\[ \Theta_s(\zeta) = \int_0^\zeta \left( \frac{3}{4v_s} \rho_s^2 + C_\rho_s^{-2} \right) d\zeta' = \frac{\pi}{2p} \left( \frac{3\rho_\infty^2 \cos^2 s_2 \tan s_1 + C \rho_\infty^2 \cos^2 s_1 \tan s_2}{4v_s} \right) + \frac{3\rho_\infty^2}{4v_s} \cosh^2(p\zeta) \arctan(\tan(s_1) \tanh(p\zeta)) - \cos^2 s_2 \tan s_1 \arctan(\cot s_1 \coth(p\zeta)) \right) \]

\[ + \frac{C}{\rho_\infty^2} \left( \frac{\sin^2 s_1 \cot s_2}{p} \arctan(\tan(s_2) \tanh(p\zeta)) - \cos^2 s_1 \tan s_2 \arctan(\cot s_2 \coth(p\zeta)) \right), \]

where \( s_1 = \arcsin \sqrt{b} \) and \( s_2 = \arccos \sqrt{c} \). For negative \( \zeta, \Theta_s(\zeta) \) must be redefined as an odd function: \( \Theta_s(-\zeta) := -\Theta_s(\zeta) \). Then the solution can be found as

\[ \psi_s(\zeta) = \rho_s(\zeta) \exp \left\{ i\Theta_s(\zeta) - ic \frac{\delta s(1 - 2v_s)}{2v_s(1 - v_s)} \right\}. \]

Analytical expressions for constant \( p, h_i \) and \( c \) are available in a computer algebra program, but are too bulky to be presented herein. Instead, in Table I we present three numerical sets for gray and bright solitons with different velocities.

### III. DIRECT NUMERICAL MODELLING OF SOLITON DYNAMICS

Now let us consider the dynamical development of the instabilities of the soliton solutions discussed above. The numerical simulations of the field evolution are done by well known split-step method. At the first step, we solve a linear part of the equation in Fourier space and, at the second step, we solve the nonlinear part. The typical step of the spatial mesh was about \( \Delta x = 10^{-3} \) and the time step \( \Delta t = 10^{-4} \). The boundary conditions are periodic, the width of the simulation window \( L \) is much larger than the soliton width (\( L \approx 30 \)). In the case of the solitons on a background, the width of the window was adjusted to provide the continuity of both fields at all points of the simulation interval. It was specially checked that neither spatial no temporal discretizations affected the results of
noise. Soliton parameters are taken in the form of a soliton solution perturbed by a weak noise. Stronger instabilities, such as those with \( \theta < 0 \), correspond to a poorly localized in space unstable eigenfunction. The initial conditions for the simulations are taken in the form of a soliton solution perturbed by a weak noise. Then we perform numerical simulation and for each moment of time evaluate the perturbation on the soliton background. The perturbation is defined as \( \vec{U}(x,t) = (A(x,t) - A_s(x-\xi) \exp(i\Phi), \psi(x,t) - \psi_s(x-\xi) \exp(i\Phi))^T \) with the phase \( \Phi \) and the displacement \( \xi \) giving the minimum of the norm \( N = \int |\vec{U}|^2 dx \) (here \( A_s, \psi_s \) are the analytical solution, \( A \) and \( \psi \) are the fields distributions found by numerical simulations). A typical evolution of the perturbation \( \vec{U} \) is shown in Fig. 6(a) showing the spatiotemporal evolution of \( |\vec{U}|^2 \) in the reference frame moving with the soliton. The oscillatory growth of the perturbation is clearly seen in this figure. Panel (b) of the figure shows the dynamics of the norm \( N \) in logarithmic scale. It is seen that after some time, when the growing mode start dominating over other components of the perturbation, the norm \( N \) grows exponentially in time. This allows us to extract the growth rate from the results of numerical simulations. We can also calculate the mutual phase between the soliton and the perturbation defined as \( \varphi = \arg U_1(x_m,t) \) where \( U_1 \) is the first component of \( \vec{U} \) and \( x_m \) is the coordinate of the maximum of the field intensity distribution. The temporal behaviour of \( \cos \varphi \) is shown in Fig. 6(c). At larger \( t \) the dependency becomes quasisinusoidal and its inverse period gives an estimate for the imaginary part of the eigenvalue (the frequency) of the growing perturbation.

This way we can compare the growth rates (real parts of \( \lambda \)) and the frequencies (imaginary parts of \( \lambda \)) of the growing perturbations obtained from direct numerical modeling and from numerical solution of the spectral problem. These values are shown in Fig. 6(d),(e) for different values of \( \theta \). One can see a good quantitative agreement between the eigenvalues. Therefore the results of numerical simulations support the conclusion on the presence of oscillatory instabilities in linear-instability spectra of bright solitons.

To compare the results of the direct numerical simulations with the predictions of the linear stability analysis obtained from the numerical solution of the spectral problem, we have extracted the growth rate and the oscillation frequency of the perturbation destroying the solitons. The results of the numerical simulations are summarized in Fig. 6(a–c) for \( \theta = -0.6 \). We take the initial conditions in the form of analytical soliton solution perturbed by a weak noise. Then we perform numerical simulation and for each moment of time evaluate the perturbation on the soliton background. The perturbation is defined as \( \vec{U}(x,t) = (A(x,t) - A_s(x-\xi) \exp(i\Phi), \psi(x,t) - \psi_s(x-\xi) \exp(i\Phi))^T \) with the phase \( \Phi \) and the displacement \( \xi \) giving the minimum of the norm \( N = \int |\vec{U}|^2 dx \) (here \( A_s, \psi_s \) are the analytical solution, \( A \) and \( \psi \) are the fields distributions found by numerical simulations). A typical evolution of the perturbation \( \vec{U} \) is shown in Fig. 6(a) showing the spatiotemporal evolution of \( |\vec{U}|^2 \) in the reference frame moving with the soliton. The oscillatory growth of the perturbation is clearly seen in this figure. Panel (b) of the figure shows the dynamics of the norm \( N \) in logarithmic scale. It is seen that after some time, when the growing mode start dominating over other components of the perturbation, the norm \( N \) grows exponentially in time. This allows us to extract the growth rate from the results of numerical simulations. We can also calculate the mutual phase between the soliton and the perturbation defined as \( \varphi = \arg U_1(x_m,t) \) where \( U_1 \) is the first component of \( \vec{U} \) and \( x_m \) is the coordinate of the maximum of the field intensity distribution. The temporal behaviour of \( \cos \varphi \) is shown in Fig. 6(c). At larger \( t \) the dependency becomes quasisinusoidal and its inverse period gives an estimate for the imaginary part of the eigenvalue (the frequency) of the growing perturbation.

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Now we proceed to various solitons that nestle on a background of constant nonzero amplitude. According to the results of Sec. IV event instability of these

| No. | \( v_s \) | \( \rho_{\infty} \) | \( \delta_s \) | \( C \) | \( p \) | \( b_{\text{gray}} \) | \( c_{\text{gray}} \) | \( b_{\text{bright}} \) | \( c_{\text{bright}} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 0.25 | 0.5 | 1.07 | 0.0506 | 1.0946 | 0.0466 | 0.0301 | 0.9534 | 0.9699 |
| 2   | 0.5  | 0.5 | 1.14 | 0.0557 | 0.6881 | 0.0264 | 0.0899 | 0.9735 | 0.9101 |
| 3   | 0.75 | 1   | 1.2  | 0.7080 | 1.1396 | 0.0440 | 0.2547 | 0.9560 | 0.7453 |

TABLE I: Three numerical sets for gray-gray and bright solitons on constant-amplitude background presented in Sec. II D. Values \( v_s, \rho_{\infty}, \delta_s \) are exact and values in other columns are approximate; subscripts gray and bright correspond to parameters for the coexisting gray-gray and bright solitons.
solutions can originate either in the modulational instability of the constant-amplitude background or in internal unstable modes of the soliton itself. In order to illustrate the instability of the former type, we consider a dark-gray soliton nestling on an unstable background. In this case the unstable modes can be characterized by a wavevector because far away of the soliton core the asymptotic behavior of the eigenfunctions corresponds to plane waves. The spectral analysis shows that all modes with relatively high $k$ are unstable. So to demonstrate the instability and be sure that the numerical method is valid, we take the noise with localized spatial spectrum and check that at large simulation times we do not see the growth of the modes with very high $k$ and thus our discretization does not affect the results of the simulations. To this end, we prepare the initial random perturbation as follows: we take random field distribution, calculate its Fourier transform, multiplied it with a Gaussian function centered at some $k$, and then calculate the inverse Fourier transform. The spectra of $A$ and $\psi$ fields at different propagation times are shown in Fig. 7. The spectra of the initial distribution are shown by black lines. The spectra of pure soliton solutions are shown as a reference by a dashed black lines. We observe that the perturbation grows and the satellite spectral lines appear. At large times the instability destroys the background and, correspondingly, destroys the solitons, see Fig. 8 illustrating this process.

We have also examined the stability of bright solitons
FIG. 9: The decay of a bright soliton on the background. Panel (a) shows the initial distribution of $A$ field (lower part), the evolution of the field (in the middle), and the field distribution at $t = 46$ (the upper part). Panel (b) shows the same for $\psi$ field. The initial conditions for the simulations are taken in the form of a soliton solution perturbed by a weak noise. Soliton parameters are $\alpha = \pi/6$ and $\theta = 0.05$ [see Eqs. (26) for the adopted parameterization].

FIG. 10: The evaluations of $A$ (a) and $\psi$ (b) fields are shown for a generalized grey soliton. The horizontal axis $\xi = x - v_s t$ where $v_s$ is the velocity of the analytically found soliton. The numerical soliton parameters correspond to No. 1 in Table I.

FIG. 11: The evaluations of $A$ (a) and $\psi$ (b) fields are shown for a generalized bright soliton. The numerical soliton parameters correspond to No. 1 in Table I.

A field causing complete destruction of the soliton.

IV. CONCLUSION

In this work, we have thoroughly investigated, analytically and numerically, different families of the solitons that exist in the experimentally relevant model describing the evolution of optical pulses in the conservative systems with strong light-matter coupling. In particular, we found analytically the solutions for bright solitons on zero and non-zero backgrounds (solitons on pedestal), Ising-like dark solitons having a point where the excitation field intensity is exactly zero and the phase of the field shifts by $\pi$ and grey (Bloch-like) solitons where the intensity has a deep and the phase of the field is continuous and rotates in the soliton core. The corresponding two-component solutions can be termed to as dark-gray (‘half-topological’) and gray-gray (nontopological) solitons. All found solutions coexist in the system but, in a properly defined linear limit, detach from different polariton branches of the dispersion law: bright solitons bifurcate from the lower branch towards the gap, and all other solutions detach from the upper branch.

We have found that stability of bright solitons on zero background can be affected by oscillatory (radiative) instabilities which emerge when the soliton frequency becomes positive. The oscillatory instability increment is initially weak, but grows distinctively as the soliton
frequency increases. Instability of the solitons on the nonzero background can develop either from the modulational instability of the constant-amplitude waves or from the internal unstable modes. All examined bright-bright solitons are unstable, while dark-gray and gray-gray solitons are stable in vast parametric regions below the modulational instability threshold. The stability predictions, including the prominent role of the radiative instability, have been verified in direct numerical modelling of soliton dynamics.

The results of the paper shed light on possible localized solutions that may exist in the conservative system with strong light-matter coupling when only material excitations are nonlinear. The variety of found analytical solutions can be used as a starting point for developing a perturbation theory for the dissipative and driven-dissipative systems where the dissipative terms can be considered as perturbations. We believe that this will facilitate the theoretical studies of the hybrid systems with light-matter interactions.

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