REMARKS ON FACTORIALITY AND $q$-DEFORMATIONS

ADAM SKALSKI AND SIMENG WANG

Abstract. We prove that the mixed $q$-Gaussian algebra $\Gamma_q(H_R)$ associated to a real Hilbert space $H_R$ and a real symmetric matrix $Q = (q_{ij})$ with $\sup |q_{ij}| < 1$, is a factor as soon as $\dim H_R \geq 2$. We also discuss the factoriality of $q$-deformed Araki-Woods algebras, in particular showing that the $q$-deformed Araki-Woods algebra $\Gamma_q(H_R, U_t)$ given by a real Hilbert space $H_R$ and a strongly continuous group $U_t$ is a factor when $\dim H_R \geq 2$ and $U_t$ admits an invariant eigenvector.

1. Introduction

This paper studies the factoriality of some $q$-deformed von Neumann algebras. In the early 90’s, motivated by mathematical physics, Bożejko and Speicher introduced the von Neumann algebra $\Gamma_q(H_R)$ generated by $q$-Gaussian variables [BS91]. Since then, the von Neumann algebra $\Gamma_q(H_R)$ has been widely studied, and also its several generalizations have been introduced and fruitfully investigated. In particular, there are two interesting types of $q$-deformed algebras which generalize that of Bożejko and Speicher: the first one is the mixed $q$-Gaussian algebra introduced in [BS94], and the second one is the family of $q$-deformed Araki-Woods algebras constructed in [Hia03].

The question of factoriality of these $q$-deformed Neumann algebras remained a well-known problem in the field for many years. In 2005, Ricard [Ric05] proved that the von Neumann algebra $\Gamma_q(H_R)$ is a factor as soon as $\dim H_R \geq 2$, which solved the problem for $\Gamma_q(H_R)$ in full generality (for earlier partial results see also [Sm94], [Kr06], [BKS97]). However, the analogous problem for mixed $q$-Gaussian algebras and $q$-deformed Araki-Woods algebras has remained open. Among the known results, the factoriality of mixed $q$-Gaussian algebras was proved by Królak [Kr00] when the underlying Hilbert space is infinite-dimensional, and very recently by Nelson and Zeng [NZ16] when the size of the deformation parameters is sufficiently small; similarly, the factoriality of $q$-deformed Araki-Woods algebras was only established by Hiai in [Hia03] when the 'almost periodic part' (see Section 4 for an explanation of this term) of the underlying Hilbert space is infinite-dimensional, and by Nelson in [Nel15] when $q$ is small.

In this note we solve the problem of factoriality for mixed $q$-Gaussian algebras in full generality, following the ideas of [Ric05]. Our methods apply also to the $q$-deformed Araki-Woods algebras, and we show that the $q$-deformed Araki-Woods algebra $\Gamma_q(H_R, U_t)$ is a factor as soon as $\dim H_R \geq 2$ and the semigroup $U_t$ admits an invariant eigenvector. We remark that after the completion of this work, we learned that the last result mentioned above was also obtained independently by Bikram and Mukherjee in [BM16], as a part of a detailed study of maximal abelian subalgebras in $q$-deformed Araki-Woods algebras.

The scalar products below are always linear on the left. The plan of the paper is as follows: in Section 2 we present a Hilbert space lemma providing estimates for certain commutators to be used later, in Section 3 we establish the factoriality of mixed $q$-Gaussian algebras in full generality, and in Section 4 discuss several results concerning factoriality in the context of $q$-Araki-Woods von Neumann algebras.

2. A Convergence Lemma for $q$-Commutation Relations

The following purely Hilbert-space-theoretic lemma will play a key role in our discussions of factoriality in the following sections.

Lemma 1. Let $(H_n)_{n \geq 1}$ be a sequence of Hilbert spaces and write $H = \oplus_{n \geq 1} H_n$. Let $r, s \in \mathbb{N}$ and let $(a_i)_{1 \leq i \leq r}, (b_j)_{1 \leq j \leq s}$ be two families of operators on $H$ which send each $H_n$ into $H_{n+1}$ or
$H_{n-1}$, such that there exists $0 < q < 1$ with
\[
\| (a_i b_j - b_j a_i) |_{H_n} \| \leq q^n, \quad n \in \mathbb{N}.
\]
Assume that $K_n \subset H_n$ is a finite-dimensional Hilbert subspace for each $n \geq 1$ such that for $K = \oplus_n K_n$ we have
\[
a_i(K) \subset K, \quad 1 \leq i \leq r - 1, \quad \text{and } a_r |_K = 0.
\]
Then for any bounded nets $(\xi_\alpha), (\eta_\alpha) \subset K$ such that $\eta_\alpha \to 0$ weakly, we have
\[
(a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha) \to 0.
\]

**Proof.** Put
\[
T_{ij}^{(n)} = (a_i b_j - b_j a_i) |_{H_n}, \quad 1 \leq i \leq r, 1 \leq j \leq s, n \geq 1.
\]
Then for each $i$ we may write
\[
a_i b_1 \cdots b_s \xi - b_1 \cdots b_s a_i \xi = \sum_{j=1}^s b_1 \cdots b_{j-1} T_{ij}^{(m(j,n))} b_{j+1} \cdots b_s \xi, \quad \xi \in H_n,
\]
where $m(j,n)$ is an integer greater than $n - s$. Iterating this formula we obtain
\[
a_r \cdots a_1 b_1 \cdots b_s \xi = b_1 \cdots b_s a_r \cdots a_1 \xi + \sum_{i=1}^r (a_i \cdots a_1 b_1 \cdots b_s a_r \cdots a_1 \xi - a_r \cdots a_1 b_1 \cdots b_s a_i \cdots a_1 \xi)
\]
\[
= b_1 \cdots b_s a_r \cdots a_1 \xi + \sum_{i=1}^r a_r \cdots a_{i+1} \sum_{j=1}^s b_1 \cdots b_{j-1} T_{ij}^{(m(i,j,n))} b_{j+1} \cdots b_s a_{i+1} \cdots a_1 \xi,
\]
where $\xi \in H_n$ and for each $i, j, n$ the integer $m(i,j,n)$ is greater than $n - s - r$. Now we consider two bounded nets $(\xi_\alpha), (\eta_\alpha) \subset K$ such that $\eta_\alpha \to 0$ weakly. Write
\[
\eta_\alpha = (\eta_\alpha^{(n)})_{n \geq 1}, \quad \eta_\alpha^{(n)} \in K_n.
\]
We have
\[
(a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha) = (\xi_\alpha, a_r \cdots a_1 b_1 \cdots b_s \eta_\alpha),
\]
and by the assumptions $a_r \cdots a_1 \eta_\alpha = 0$, so together with the previous computations for $a_r \cdots a_1 b_1 \cdots b_s \xi$, we obtain
\[
(2.1) \quad (a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha) = \sum_{n \geq 1} (\xi_\alpha, T_n \eta_\alpha^{(n)})
\]

where
\[
T_n = \sum_{i=1}^r a_r \cdots a_{i+1} \left( \sum_{j=1}^s b_1 \cdots b_{j-1} T_{ij}^{(m(i,j,n))} b_{j+1} \cdots b_s \right) a_{i+1} \cdots a_1.
\]
Recall that $\| T_{ij}^{(k)} \| \leq q^k$ for all $i, j, k$ by assumption. So for each $\alpha$ and $n$
\[
\| T_n \eta_\alpha^{(n)} \| \leq C(q, r, s) q^n \| \eta_\alpha^{(n)} \|,
\]
where $C(q, r, s)$ is a constant independent of $n$. Together with (2.1) we have
\[
(2.2) \quad | (a_1^* \cdots a_r^* \xi_\alpha, b_1 \cdots b_s \eta_\alpha) | \leq C(q, r, s) \sup_{\alpha} \| \xi_\alpha \| \sum_{n \geq 1} q^n \| \eta_\alpha^{(n)} \|.
\]
Since $\eta_\alpha \to 0$ weakly, we have for each $N \geq 1$,
\[
\sum_{n=1}^N q^n \| \eta_\alpha^{(n)} \| \to 0,
\]
and on the other hand,
\[
\sum_{n \geq N} q^n \| \eta_\alpha^{(n)} \| \leq \sup_n \| \eta_\alpha^{(n)} \| q^N / (1 - q).
\]
Therefore by (2.2) we get
\[ \forall N \geq 1, \quad \limsup_{\alpha} \left| \langle a_1^* \cdots a_N^* \xi, b_1 \cdots b_N \eta \rangle \right| \leq C'(r, s, q)q^N, \]
with a constant $C'(r, s, q)$ independent of $N$, which means that
\[ \langle a_1^* \cdots a_N^* \xi, b_1 \cdots b_N \eta \rangle \to 0, \]
as desired. \hfill \Box

3. Factoriality of mixed $q$-Gaussian algebras

Let $N \in \mathbb{N}$, let $Q = (q_{ij})_{i,j=1}^{N}$ be a symmetric matrix with $q_{ij} \in (-1,1)$, and let $H_{\mathbb{R}}$ be a finite-dimensional real Hilbert space with orthonormal basis $e_1, \ldots, e_N$. We recall briefly the construction of mixed Gaussian algebras, as introduced in [BS94]. Write $H = H_{\mathbb{R}} + iH_{\mathbb{R}}$ to be the complexification of $H_{\mathbb{R}}$. Let $\mathcal{F}_Q(H)$ be the Fock space associated to the Yang-Baxter operator
\[ T : H \otimes H \rightarrow H \otimes H, \quad e_i \otimes e_j \mapsto q_{ij} e_j \otimes e_i \]
constructed in [BS94]. Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{F}_Q(H)$ and let $\Omega$ be the vacuum vector. Denote by $\varphi(\cdot) = \langle \cdot, \Omega, \Omega \rangle$ the vacuum state. The left creation operators $l_i$ are defined by the formulas
\[ l_i \xi = e_i \otimes \xi, \quad \xi \in \mathcal{F}_Q(H), \]
and their adjoints, the left annihilation operators, can be characterised by equalities
\[ l_i^* \Omega = 0, \]
\[ l_i^*(e_{j_1} \otimes \cdots \otimes e_{j_n}) = \sum_{k=1}^{n} \delta_{i,j_k} q_{j_1} \cdots q_{j_{k-1}} e_{j_1} \otimes \cdots \otimes e_{j_{k-1}} \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_n}. \]
Similarly, we have the right creation/annihilation operators
\[ r_i \xi = \xi \otimes e_i, \quad \xi \in \mathcal{F}_Q(H), \]
\[ r_i^* \Omega = 0, \]
\[ r_i^*(e_{j_1} \otimes \cdots \otimes e_{j_n}) = \sum_{k=1}^{n} \delta_{i,j_k} q_{j_{k+1}} \cdots q_{j_n} e_{j_1} \otimes \cdots \otimes e_{j_{k-1}} \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_n}. \]
We consider the associated mixed $q$-Gaussian algebra $\Gamma_Q(H_{\mathbb{R}})$ generated by the self-adjoint variables $s_j = l_j^* + l_j$. Denote
\[ q = \max_{i,j} |q_{ij}| < 1. \]
By a word in $\mathcal{F}_Q(H)$ we mean a vector in $\mathcal{F}_Q(H)$ of the form $\zeta_1 \otimes \cdots \otimes \zeta_n$ with some $n \geq 1$ and $\zeta_1, \ldots, \zeta_n \in H$. Królik [Kr06] proved that any word $\xi \in \mathcal{F}_Q(H)$ corresponds to a Wick product $W(\xi) \in \Gamma_Q(H_{\mathbb{R}})$ with $W(\xi) \Omega = \xi$. Also, [BS94] remarked that $J\Gamma_Q(H_{\mathbb{R}})J$ is the commutant of $\Gamma_Q(H_{\mathbb{R}})$, where $J$ is the conjugation operator given by
\[ J(e_i \otimes \cdots \otimes e_i) = e_i \otimes \cdots \otimes e_i. \]
We write
\[ W_r(\xi) = JW(J\xi)J, \quad \xi \in \oplus_n H^\otimes n. \]
Then $W_r(\xi) \in \Gamma_Q(H_{\mathbb{R}})'$.

Lemma 2. For each $n \in \mathbb{N}$ and $i, j = 1, \ldots, N$ the operators $T_i^{(n)}$ on $H^\otimes n$ characterised by the equalities
\[ l_i^* r_j - r_j l_i^* = \delta_{ij} \oplus_n T_i^{(n)}. \]
satisfy the norm estimate $\|T_i^{(n)}\| \leq q^n$. 
To see this, it suffices to consider the case $y = 1$. Observe that

$$l_i^* r_j (e_j \otimes \cdots \otimes e_j) = l_i^* (e_j \otimes \cdots \otimes e_j \otimes e_j)$$

$$\quad = \sum_{k=1}^{n} \delta_{i,k} q_i^j \cdots q_{ij-k} e_j \otimes \cdots \otimes e_{j-k-1} \otimes e_{j-k+1} \otimes \cdots \otimes e_j \otimes e_j$$

$$\quad \quad + \delta_{ij} q_{ij} \cdots q_{ij} e_j \otimes \cdots \otimes e_{j},$$

and

$$r_j l_i^* (e_j \otimes \cdots \otimes e_j) = \sum_{k=1}^{n} \delta_{i,j} q_{ij} \cdots q_{ij-k} e_j \otimes \cdots \otimes e_{j-k-1} \otimes e_{j-k+1} \otimes \cdots \otimes e_j \otimes e_j.$$  

Now take

$$T^{(n)}_i : H^\otimes n \to H^\otimes n, \quad e_j \otimes \cdots \otimes e_j \mapsto \delta_{ij} q_{ij} \cdots q_{ij} e_j \otimes \cdots \otimes e_j.$$  

The eigenspace of $T^{(n)}_i$ corresponding to $\delta_{ij} q_{ij} \cdots q_{ij}$ is spanned by the vectors of the type $E_{ij,\ldots,ij} = \{ e_{ij} \otimes \cdots \otimes e_{ij} : q_{ij} \cdots q_{ij} = q_{ij} \cdots q_{ij} \}$, which are orthogonal for distinct $j = \{j_1, \ldots, j_n\}$. So

$$\|T^{(n)}_i\| \leq \max\{q_{ij} \cdots q_{ij} : 1 \leq j_1, \ldots, j_n \leq N\} \leq q^n$$

and $T^{(n)}_i$ is the desired operator. \hfill $\square$

Now the following main result is in reach. The idea is partially inspired by the proof in [Ric05] in conjunction with Lemma 1.

**Theorem 3.** For each $1 \leq i \leq n$, the von Neumann subalgebra generated by $s_i$ is maximal abelian in $\Gamma_Q(H_\beta)$. In particular, $\Gamma_Q(H_\beta)$ is a factor if $n \geq 2$.

**Proof.** By [BKS97], we know that the spectral measure of $s_i$ is the $q$-semicircular law with $q = q_{ii}$. Therefore the von Neumann algebra $M$ generated by $s_i$ is diffuse and abelian, and hence isomorphic to the von Neumann algebra $L^\infty([0,1], dm)$ where $dm$ denotes the Lebesgue measure on $[0,1]$. As a result, we may find a sequence of unitaries $(u_\alpha)_{\alpha \in \mathbb{N}} \subset M$ which correspond to Rademacher functions via this isomorphism. In particular, we have

$$u_\alpha = u_\alpha^*, \quad u_\alpha^2 = 1, \quad u_\alpha \Omega \to 0 \text{ weakly in } F_Q(H).$$

Now assume $x \in \Gamma_Q(H_\beta)$ with $xs_i = s_i x$, and hence

$$xy = yx, \quad y \in M.$$

Let $F_Q(C_{e_i}) \subset F_Q(H)$ be the Fock space associated to $e_i$. Observe that for any vector $\xi \in \bigcup_{m \in \mathbb{N}} H^\otimes m$ and all $\alpha \geq 1$ we have

$$\langle \xi, x\Omega \rangle = \varphi(x^* W(\xi)) = \varphi(x^* u_\alpha W(\xi)) = \varphi(u_\alpha x^* u_\alpha W(\xi)) = (W_\alpha(\xi)u_\alpha \Omega, xu_\alpha \Omega).$$

We remark that if further $\xi$ is orthogonal to $F_Q(C_{e_i})$ then

$$\forall y \in \Gamma_Q(H_\beta), \quad (W_\alpha(\xi)u_\alpha \Omega, yu_\alpha \Omega) \to 0.$$  

To see this, it suffices to consider the case $y \Omega \in H^\otimes n$ for an arbitrary $n \geq 0$ since it is easy to see that the functionals $y^* \Omega \mapsto (W_\alpha(\xi)u_\alpha \Omega, yu_\alpha \Omega)$ extend to uniformly bounded functionals on $F_Q(H)$ thanks to the traciality of $\varphi$ (BB94, Theorem 4.4). Now by the Wick formula in [Kr00, Theorem 1], it is enough to prove the convergence

$$\langle r_{i_1} \cdots r_{i_p} u_\alpha \Omega, l_{j_1} \cdots l_{j_q} u_\alpha \Omega \rangle \to 0$$

for any fixed indices $i_1, \ldots, i_p, j_1, \ldots, j_q$ with some $i_k \neq i$. Denote

$$s' = \min\{k : i_k \neq i\}.$$  

If $s' > s$, we have $r_{i_1}^* \cdots r_{i_p}^* u_\alpha \Omega = 0$ for all $\alpha \geq 1$ and the convergence (3.3) becomes trivial. So we assume in the following $s' \leq s$. Note that by definition

$$r_i l_j - l_j r_i = 0, \quad r_i^* l_i^* - l_i^* r_i^* = 0,$$
and by Lemma 2
\[ \|(l^*_i r_j - r_j l^*_i)\|_{H^\otimes n} \leq q^n, \quad n \geq 1. \]
Also, observe that by the choice of \( s' \),
\[ r^*_k \in \mathcal{F}_Q(\mathbb{R}e_i) = 0, \quad r^*_k(\mathcal{F}_Q(\mathbb{R}e_i)) \subset \mathcal{F}_Q(\mathbb{R}e_i), \quad 1 \leq k < s'. \]
So now applying Lemma 1 to the families of operators \( r^*_{i_1}, \ldots, r^*_{i_{s'}} \) and \( l_{j_1}, \ldots, l_{j_{t}}, l^*_{j_{t+1}}, \ldots, l^*_{j_{s'}} \), we obtain the convergence (3.3). As a consequence, the convergence (3.2) holds as well, which, together with (3.1), yields that
\[ \langle \xi, x\Omega \rangle = 0. \]
This means that \( x\Omega \in \mathcal{F}_Q(\mathbb{C}e_i) \) since \( \xi \) is arbitrarily chosen in a dense subset of \( \mathcal{F}_Q(\mathbb{C}e_i)^\perp \). We can then deduce that \( x \in M \) using the second quantization of the projection \( P : H_\mathbb{R} \rightarrow \mathbb{R}e_i \) (see [LP99, Lemma 3.1]). Thus we have shown that the von Neumann subalgebra \( M \) generated by \( s_i \) is maximal abelian in \( \Gamma_Q(H_\mathbb{R}) \).

Also, observe that by the choice of \( s \),
\[ \text{we can then deduce that } \Gamma_Q(H_\mathbb{R}) \cap \Gamma_Q(H_\mathbb{R}', \Re) \text{ is maximal abelian in } \Gamma_Q(H_\mathbb{R}). \]

4. **Factoriality of \( q \)-Araki-Woods algebras**

Now we discuss the factoriality of \( q \)-Araki-Woods algebras. We refer to [Hia03] for the detailed description of the construction of these algebras and only sketch the outline below. Following the notation of [Hia03], given a real Hilbert space \( H_\mathbb{R} \) with a strongly continuous group \( U_t \) of orthogonal transformations on \( H_\mathbb{R} \), we may introduce a deformed inner product \( \langle \cdot, \cdot \rangle_U \) on \( H_\mathbb{C} := H_\mathbb{R} + iH_\mathbb{R} \). Denote by \( H \) the completion of \( H_\mathbb{C} \) with respect to \( \langle \cdot, \cdot \rangle_U \) and denote by \( \mathcal{F}_Q(H) \) the \( q \)-Fock space associated to \( H \). We define the left and right creation operators
\[ l(\xi)\eta = \xi \otimes \eta, \quad r(\xi)\eta = \eta \otimes \xi, \quad \xi, \eta \in \mathcal{F}_Q(H) \]
and the left and right annihilation operators
\[ l^*(\xi) = l(\xi)^*, \quad r^*(\xi) = r(\xi)^*, \quad \xi \in H. \]
We denote by \( \Gamma_q(H_\mathbb{R}, U_t) \) (resp. \( C_\gamma^*(H_\mathbb{R}, U_t) \) ) the von Neumann algebra (resp. C*-algebra) generated by \( \{ l(e) + l^*(e) : e \in H_\mathbb{R} \} \) in \( B(\mathcal{F}_Q(H)) \), to be called the \( q \)-Araki-Woods von Neumann algebra. Properties of the vacuum state guarantee the existence of the Wick product map \( W : \Gamma_q(H_\mathbb{R}, U_t) \Omega \rightarrow \Gamma_q(H_\mathbb{R}, U_t) \) such that \( W(\xi)\Omega = \xi \). On the other hand, denote
\[ H_\mathbb{R}' = \{ \xi \in H : \forall \eta \in H_\mathbb{R}, \langle \xi, \eta \rangle \in \mathbb{R} \}. \]
Then the von Neumann algebra \( \Gamma_q, r(H_\mathbb{R}, U_t) \) generated by \( \{ r(e) + r^*(e) : e \in H_\mathbb{R}' \} \) in \( B(\mathcal{F}_Q(H)) \) is the commutant of \( \Gamma_q(H_\mathbb{R}, U_t) \), and again there exist a right Wick product \( W_r : \Gamma_q, r(H_\mathbb{R}, U_t) \Omega \rightarrow \Gamma_q, r(H_\mathbb{R}, U_t) \) such that \( W_r(\xi)\Omega = \xi \). We denote by \( i \) the standard complex conjugation on \( H_\mathbb{R} + iH_\mathbb{R} \), and by \( I_r \) the complex conjugation on \( H_\mathbb{R}' + iH_\mathbb{R}' \). The following observations are well-known and we state them here for later use.

**Lemma 4.** (1) Suppose that \( e_1, \ldots, e_n \in H_\mathbb{C} \). Then we have the following Wick formula
\[ W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^n \sum_{i_1, \ldots, i_k, j_{k+1}, \ldots, j_n} l(e_{i_1}) \cdots l(e_{i_k}) l^*(Ie_{j_{k+1}}) \cdots l^*(Ie_{j_n}) q^{i_1, j_2}, \]
where \( I_1 = \{ i_1, \ldots, i_k \} \) and \( I_2 = \{ j_{k+1}, \ldots, j_n \} \) form a partition of the set \( \{ 1, \ldots, n \} \) and \( i(I_1, I_2) \) is the number of crossings. A similar formula holds for \( W_r(e_1 \otimes \cdots \otimes e_n) \) as well.

(2) Let \( f \in H_\mathbb{R}, e \in H_\mathbb{R}' + iH_\mathbb{R}'. \) If \( \langle e, f \rangle = 0 \), then \( \langle I_r e, f \rangle = 0. \)

**Proof.** (1) See [BKS97, Proposition 2.7], [Was16, Lemma 3.1].

(2) Write \( e = e_1 + ie_2 \) with \( e_1, e_2 \in H_\mathbb{R}' \). Since \( \langle e_1, f \rangle \in \mathbb{R}, \langle e_2, f \rangle \in \mathbb{R} \), we see that the identity \( \langle e, f \rangle = 0 \) yields
\[ \langle e_1, f \rangle = \langle e_2, f \rangle = 0. \]
Therefore
\[ \langle I_r e, f \rangle = \langle e_1 - ie_2, f \rangle = 0. \]
\( \square \)
According to Shlyakhtenko [Shl97], we have the decomposition
\[(H_R, U_t) = (K_R, U'_t) \oplus (L_R, U''_t),\]
where \(U'_t\) is almost periodic and \(U''_t\) is ergodic. Then \(K_R \subset H_R\) is the real closed subspace spanned by eigenvectors of \(U_t = \Lambda^t\). Let \(K_C = K_R + iK_R\) be the complexification and \(K\) be the completion of \(K_C\) with respect to the deformed norm as above, and similarly for \(L\). Note that the orthogonal projection \(P : H_R \to K_R\) commutes with \(U_t\). So by the second quantization, \(Γ_q(K_R, U_t|K)\) embeds as a von Neumann subalgebra of \(Γ_q(H_R, U_t)\). For an operator \(T\) we denote by \(F_q(T)\) its second quantization.

The following observation shows that in looking at the centre of the q-Araki-Woods algebra it suffices to consider the ‘\(K\)-part’ of the algebra (we do not really use this fact in the sequel).

**Lemma 5.** (1) The semigroup \(F_q(U_t)\) admits no eigenvectors in \(F_q(K)^\perp \subset F_q(H)\); (2) Assume \(x \in Γ_q(H_R, U_t) \cap Γ_q(H_R, U_t)'\). Then \(xΩ ∈ F_q(K)\) and \(x \in Γ_q(K_R, U_t|K_R)\).

**Proof.** (1) Let \((e_i)\) be an orthonormal basis in \(H_R\). Since \(F_q(P)\) is the orthogonal projection onto \(F_q(K)\), we have
\[F_q(P)(F_q(K)^\perp) = 0.\]

Hence
\[F_q(K)^\perp = \text{span}\{e_{i_1} \otimes \cdots \otimes e_{i_n} : n \geq 1, \exists 1 \leq m \leq n, e_{i_m} \in L_R\}.\]

Denote
\[K_n = \text{span}\{e_{i_1} \otimes \cdots \otimes e_{i_n} ∈ F_q(K)^\perp\} = \text{span}\{H_{i_1} \otimes \cdots \otimes H_{i_n}, H_i = K \text{ or } L, \exists H_i = L\}.\]

Note that \(U_t\) is unitarily equivalent to a multipliers map on some \(L^2(μ)\). So by the definition of \(K\) and \(L\) and the fact that at least one of \(H_{i_k}\) is equal to \(L\), it is easy to see that \(F_q(U_t)|_{H_{i_1} \otimes \cdots \otimes H_{i_n}} = U_t|_{H_{i_1} \otimes \cdots \otimes U_t|_{H_{i_n}}}\) admits no eigenvectors. Since each \(H_{i_n}\) is invariant under \(U_t\), \(F_q(U_t)\) admits no eigenvectors in \(K_n\) either. Then the lemma follows immediately. Indeed, let
\[ξ = \sum_n ξ_n ∈ F_q(K)^\perp, \quad ξ_n ∈ K_n,\]
be an eigenvector. Then we get
\[\sum_n (U_t ξ_n - λξ_n) = 0,\]
for some \(λ\) and hence \(U_t ξ_n - λξ_n = 0\) for all \(n\), which yields a contradiction.

(2) Assume \(x ∈ Γ_q(H_R, U_t) \cap Γ_q(H_R, U_t)'\). Note that \(x\) is in the centralizer of the vacuum state \(φ\). So we have for all \(t ∈ \mathbb{R}\),
\[σ_t(x)Ω = Δ^t x Ω = xΩ.\]
Recall the Tomita-Takesaki theory for \(Γ_q(H_R, U_t)\) and the vacuum state. We see that \(xΩ\) is a fixed point of \(F_q(U_t)\), and hence \((F_q(P)^\perp)(xΩ)\) is an eigenvector by orthogonal decomposition. So by the above lemma \((F_q(P)^\perp)(xΩ) = 0\). That is, \(xΩ ∈ F_q(K)\) and \(x ∈ Γ_q(K_R, U_t|K_R)\). \(\square\)

**Proposition 6.** Let \(D_R \subset H_R\) be a real finite-dimensional Hilbert subspace and let \(M\) be a diffuse abelian von Neumann subalgebra of \(Γ_q(H_R, U_t)\) such that \(MΩ < F_q(D)\), where \(D = D_R + iD_R\). Assume \(x ∈ Γ_q(H_R, U_t) \cap M'\). (1) If \(x ∈ C^*_q(H_R, U_t)\), then \(xΩ ∈ F_q(D)\); (2) If \(M\) is contained in the centralizer of \(Γ_q(H_R, U_t)\), then \(xΩ ∈ F_q(D)\).

**Proof.** The proof is similar to that of Theorem 3, so we only present a sketch. Since \(M\) is diffuse and \(MΩ < F_q(D)\), we may find a sequence of unitaries \((u_α)_{α ∈ N} < M\) such that
\[u_α = u_α^*, \quad u_α^2 = 1, \quad u_α Ω → 0 \text{ weakly in } F_q(D).\]
We may show that for any vector \(ξ ∈ H^{⊗n}\) with \(n ≥ 1\) which is orthogonal to \(F_q(D)\), and for \(w ∈ Γ_q(H_R, U_t)\), if one of the following conditions is satisfied:
(a) \(w ∈ C^*_q(H_R, U_t)\);
(b) the operator $z\Omega \mapsto zu_{n}\Omega$ is uniformly bounded on $F_{q}(H)$; then
\begin{equation}
\varphi(u_{n}w^{*}u_{n}W(\xi)) = (W_{r}(\xi)u_{n}\Omega, wu_{n}\Omega) \to 0.
\end{equation}
Indeed, we note that the anti-linear functional $z \mapsto \varphi(u_{n}z^{*}u_{n}W(\xi))$ is uniformly bounded on $C_{*}^{r}(H_{R}, U_{r})$ with respect to $\alpha$, and if (b) is satisfied, the anti-linear functional $z \mapsto \varphi(u_{n}z^{*}u_{n}W(\xi))$ is uniformly bounded on $F_{q}(H)$ with respect to $\alpha$. So if any one of (a) and (b) is satisfied, we may find a sequence of vectors $(\eta_{k})_{k=1}^{\infty}$ in the algebraic span of $\{H^{\otimes n} : n \geq 1\}$ such that we have the convergence
\begin{equation}
\varphi(u_{n}W(\eta_{k})^{*}u_{n}W(\xi)) \to \varphi(u_{n}w^{*}u_{n}W(\xi)), \quad k \to \infty
\end{equation}
which is uniform with respect to $\alpha$. This means that in order to see (4.2) under the condition (a) or (b), it suffices to assume that $w$ belongs to the the algebraic span of $\{H^{\otimes n} : n \geq 1\}$. On the other hand, recall that $\xi \perp F_{q}(D)$, which means that $\xi$ is the combination of words of the form
\[ e_{m_{1}} \otimes \cdots \otimes e_{m_{n}}, \quad e_{m_{1}}, \ldots, e_{m_{n}} \in H \cup D^{\perp}, \exists 1 \leq k \leq n, \; e_{m_{k}} \in D^{\perp}. \]
Thus by the Wick formula in Lemma 4.1, it suffices to prove the convergence
\begin{align*}
\langle r(e_{1}) \cdots r(e_{m})r^{*}(I_{e_{1}}e_{m+1}) \cdots r^{*}(I_{e_{k}}e_{m+n})u_{n}\Omega, l(e_{j_{1}}) \cdots l(e_{j_{p}})l^{*}(I_{e_{j_{p+1}}}) \cdots l^{*}(I_{e_{j_{p+n}}})u_{n}\Omega \rangle \to 0,
\end{align*}
where there is $1 \leq k \leq n$ such that $e_{i_{k}} \in D^{\perp}, e_{i_{j}} \in H$ for $1 \leq k < k'$. By Lemma 4, $I_{e_{i_{k}}} \in D^{\perp}$ holds as well. Consequently, if $k \geq m+1$, then $r^{*}(I_{e_{i_{k}}}) \cdots r^{*}(I_{e_{i_{k+1}}})u_{n}\Omega = 0$ and the above convergence is trivial. Hence we assume $k \leq m$. Recall that $l^{*}(r^{*}(g) - r^{*}(g))l^{*}(f) = \langle f, g \rangle q^{k} (\oplus k \geq \text{id}_{H^{\otimes k}}), \quad f, g \in H$.

Now applying Lemma 1 as in Theorem 3, we obtain the desired convergence (4.2).

Now the conclusion of the theorem is immediate. Take $x \in \Gamma_{q}(H_{R}, U_{r}) \subset M'$. We have for all $\alpha \geq 1$ and every $\xi \in H^{\otimes n}$ with $n \geq 1$ which is orthogonal to $F_{q}(D)$,
\[ \langle \xi, x\Omega \rangle = \varphi(x^{*}W(\xi)) = \varphi(x^{*}u_{n}^{*}W(\xi)) = \varphi(u_{n}x^{*}u_{n}W(\xi)) = \langle W_{r}(\xi)u_{n}\Omega, xu_{n}\Omega \rangle. \]
If now the assumption of (1) holds, then by (a) and (4.2) we see that
\[ \langle \xi, x\Omega \rangle = \langle W_{r}(\xi)u_{n}\Omega, xu_{n}\Omega \rangle \to 0. \]
Similarly if the assumption of (2) holds, then the $u_{n}$'s belong to the centralizer of $\Gamma_{q}(H_{R}, U_{r})$, and hence
\[ \|zu_{n}\Omega\|^2 = \varphi(u_{n}z^{*}zu_{n}) = \varphi(z^{*}zu_{n}^{2}) = \varphi(z^{*}z) = \|z\Omega\|^2, \]
so (b) is satisfied. By (4.2) this yields that
\[ \langle \xi, x\Omega \rangle = \langle W_{r}(\xi)u_{n}\Omega, xu_{n}\Omega \rangle \to 0. \]
So $\langle \xi, x\Omega \rangle = 0$ for all words $\xi \in F_{q}(D)^{\perp}$ and hence $x\Omega \in F_{q}(D)$. \hfill \Box

We are ready to state the second main result of this article.

**Theorem 7.** Assume $\dim H_{R} \geq 2$.

1. If there exists $\xi_{0} \in H_{R}$ such that $U_{r}\xi_{0} = \xi_{0}$, then $\Gamma_{q}(H_{R}, U_{r})$ is a factor;
2. Let $H_{R}^{(1)}$, $H_{R}^{(2)}$ be two finite-dimensional Hilbert subspaces of $H_{R}$ which are invariant under $U_{r}$ and are orthogonal with respect to the real inner product of $H_{R}$. Assume that for $k = 1, 2$ the centralizer of $\Gamma_{q}(H_{R}^{(k)}, U_{r}^{(k)})$ contains a diffuse element. Then $\Gamma_{q}(H_{R}, U_{r})$ is a factor;
3. $\Gamma_{q}(H_{R}, U_{r}) \cap C_{*}^{r}(H_{R}, U_{r}) = C_{1}$. \hfill 

**Proof.** (1) Since $\dim H_{R} \geq 2$ and $U_{r}\xi_{0} = \xi_{0}$, the subspace $(C\xi_{0})^{\perp} \subset H$ is invariant under $U_{r}$, and we may find a vector $\eta \in (C\xi_{0})^{\perp}$ such that $\eta \in H_{R}^{0}$, $\eta \perp \xi_{0}$. Note that in this case $W_{r}(\eta) = W_{r}(\eta)^{*}$ and $I_{\eta} \perp \xi_{0}$. Take $x \in \Gamma_{q}(H_{R}, U_{r}) \cap C_{*}^{r}(H_{R}, U_{r})$ and denote $\xi = x\Omega$. Note that $W(\xi_{0})$ belongs to the centralizer of $\Gamma_{q}(H_{R}, U_{r})$ by the assumption $U_{r}\xi_{0} = \xi_{0}$, and that the spectral measure of $W(\xi_{0})$ is $q$-semicircular ([Not06, Remarks p.298-299]) and hence $W(\xi_{0})$ generates a diffuse abelian von Neumann subalgebra. So by Proposition 6(2), we have
\[ \xi \in F_{q}(C\xi_{0}), \quad \eta \perp \xi, I_{\eta} \perp \xi. \]
Then we see that
\[ W(\xi)\eta = xW(\eta)\Omega = W(\eta)x\Omega = W(\eta)\xi \]
\[ = I(\eta)\xi + I'(\eta)\xi = \eta \otimes \xi. \]

As a result, writing
\[ \lambda = \langle \xi, \Omega \rangle, \quad \zeta = \xi - \lambda \Omega, \]
we have
\[
\| \eta \otimes \xi \|^2 = \langle \eta \otimes \xi, W(\xi)\eta \rangle = \langle \eta \otimes \xi, W_r(\eta)\xi \rangle = \langle W_r(\eta)(\eta \otimes \xi), \xi \rangle \\
= \lambda(W_r(\eta)\eta, \xi) + \langle W_r(\eta)(\eta \otimes \xi), \xi \rangle \\
= \lambda\|\eta\|^2\|\xi\| + \lambda\langle \eta \otimes \eta, \xi \rangle + \langle \eta \otimes \zeta \otimes \eta, \xi \rangle \\
= |\lambda|^2\|\eta\|^2
\]
where we have used the relation \( \eta \perp \zeta_0 \) in the last inequality. However
\[
\| \eta \otimes \xi \|^2 = \| \eta \otimes (\lambda \Omega + \zeta) \|^2 = |\lambda|^2\|\eta\|^2 + \| \eta \otimes \zeta \|^2.
\]
Thus the above two equalities yield that \( \eta \otimes \zeta = 0 \). Therefore \( \zeta = 0 \) and \( x\Omega = \xi = \lambda \Omega \). This proves that
\[
\Gamma_q(H_R, U_t) \cap \Gamma_q(H_R, U_t) = \mathbb{C}1.
\]

(2) This assertion follows directly from Proposition 6(2) since according to that result any \( x \in \Gamma_q(H_R, U_t)' \cap \Gamma_q(H_R, U_t) \) should satisfy
\[
x\Omega \in F_q(H^{(1)}) \cap F_q(H^{(2)}) (= \mathbb{C}\Omega).
\]

(3) Since \( \dim H_R \geq 2 \), we may find two vectors \( e_1, e_2 \in H_R \) which are orthogonal with respect to the real inner product of \( H_R \). Then \( W(e_1) \) and \( W(e_2) \) are self-adjoint diffuse elements as discussed before, and \( F_q(C_{e_1}) \cap F_q(C_{e_2}) = \mathbb{C}\Omega \). Then according to Proposition 6(1), any \( x \in \Gamma_q(H_R, U_t)' \cap C_q(H_R, U_t) \) should satisfy
\[
x\Omega \in F_q(C_{e_1}) \cap F_q(C_{e_2}) (= \mathbb{C}\Omega).
\]
Therefore the assertion is proved. \( \square \)

Acknowledgment. The authors would like to thank Éric Ricard and Mateusz Wasilewski for helpful discussions, and the anonymous referee for careful reading of our manuscript. The authors were partially supported by the NCN (National Centre of Science) grant 2014/14/E/ST1/00525.

References

[BKS97] M. Bożejko, B. Kümmerer, and R. Speicher. \( q \)-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.*, 185(1):129–154, 1997.

[BM16] P. Bikram and K. Mukherjee. Generator masas in \( q \)-deformed Araki-Woods von Neumann algebras and factoriality (preprint). Available at arXiv:1606.04752.

[BS91] M. Bożejko and R. Speicher. An example of a generalized Brownian motion. *Comm. Math. Phys.*, 137(4):519–531, 1991.

[BS94] M. Bożejko and R. Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Math. Ann.*, 300(1):97–120, 1994.

[Hia03] F. Hiai. \( q \)-deformed Araki-Woods algebras. In *Operator algebras and mathematical physics (Constanța, 2001)*, pages 169–202. Theta, Bucharest, 2003.

[Kr60] I. Królak. Wick product for commutation relations connected with Yang-Baxter operators and new constructions of factors. *Comm. Math. Phys.*, 210(3):685–701, 2000.

[Kr60] I. Królak. Factoriality of von Neumann algebras connected with general commutation relations – finite dimensional case. Banach Center Publ., 73:277–284, 2006.

[LP99] F. Lust-Piquard. Riesz transforms on deformed Fock spaces. *Comm. Math. Phys.*, 205(3):519–549, 1999.

[Nel15] B. Nelson. Free monotonic transport without a trace. *Comm. Math. Phys.*, 334(3):1245–1298, 2015.

[Nou06] A. Nou. Asymptotic matricial models and QWEP property for \( q \)-Araki–Woods algebras. *J. Funct. Anal.*, 232(2):295–327, 2006.

[NZ16] B. Nelson and Q. Zeng. An application of free transport to mixed \( q \)-Gaussian algebras. *Proc. Amer. Math. Soc.*, online, 2016.

[Ric05] É. Ricard. Factoriality of \( q \)-Gaussian von Neumann algebras. *Comm. Math. Phys.*, 257(3):659–665, 2005.

[Shi97] D. Shlyakhtenko. Free quasi-free states. *Pacific J. Math.*, 177(2):329–368, 1997.
[Śni04] P. Śniady. Factoriality of Bożejko-Speicher von Neumann algebras. Comm. Math. Phys., 246(3):561–567, 2004.

[Was16] M. Wasilewski. $q$-Araki-Woods algebras: extension of second quantisation and Haagerup approximation property. Proc. AMS, to appear. Available at arXiv:1605.06034.

Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00–656 Warszawa, Poland

E-mail address: a.skalski@impan.pl

Laboratoire de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France and Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00–956 Warszawa, Poland

E-mail address: simeng.wang@univ-fcomte.fr