Nonparametric Range-Based Double Smoothing Spot Volatility Estimation for Diffusion Models

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Abstract
We consider nonparametric spot volatility estimation for diffusion models with discrete high frequency observations. Our estimator is carried out in two steps. First, using the local average of the range-based variance, we propose a crude estimator of the spot volatility. Second, we use usual nonparametric kernel smoothing to reconstruct the volatility function from the crude estimator. By inference, we find such a double smoothing operation can effectively reduce the estimation error.

1. Introduction

In recent years, stochastic differential equations have been widely studied by many researchers and some interesting results have appeared in the literature (see [1–5] and references therein). In particular, as a kind of stochastic differential equations, diffusion processes are usually involved in many popular models in financial mathematics, statistics, and econometrics. These models are widely used in option pricing, hedging, interest rate (exchange rate) modelling, and engineering management.

We consider a class of diffusion process as follows:

\[ X_t = X_0 + \int_0^t \alpha_u du + \int_0^t \beta_u dW_u, \quad t \in [0, T], \] (1)

with an initial condition \( X_0 \), where \( \{X_t\}_{t \geq 0} \) solves the equation and is often used to describe the logarithmic price of a security. \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion. The continuous process \( \{\alpha_t\}_{t \geq 0} \) and \( \{\beta_t\}_{t \geq 0} \) are the drift term and the diffusion term (spot volatility) of the process \( \{X_t\}_{t \geq 0} \).

Consider an arbitrary partition: \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = T \). For any nonnegative integer \( i \) \((0 \leq i \leq n)\), we define \( \delta_i = t_i - t_{i-1} \). For the sake of discussion, it is assumed that sample observations are selected at equal time interval, and the time interval between adjacent sample observations can be denoted as \( \delta = T/n \); hence, \( t_i = iT/n \). For the case of unequal time interval, it is enough to define \( \delta = \max_{0 \leq i \leq n} \{t_i - t_{i-1}\} \).

As it is known to all, volatility is very important in the derivative securities pricing, its correct estimation is the basis of reasonable pricing derivatives. In view of the robustness and accuracy of the nonparametric approach in modelling, many scholars use this approach to estimate volatility (see [6–8]). However, most of the nonparametric volatility estimation works were conducted around integral volatility (see [9, 10]).

Constraining a kernel function and choosing appropriate bandwidth parameters, Kristensen [11] proposed a filtered kernel-based spot volatility estimator for time-dependent diffusion models:

\[ \hat{\beta}^2_t = \frac{1}{h} \sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right) (X_{t_i} - X_{t_{i-1}})^2, \] (2)

where \( K(\cdot) \) and \( h \) were the kernel function and its bandwidth, respectively. Under certain conditions, they showed the following central limit theorem:

\[ \sqrt{h} (\hat{\beta}^2_t - \beta_t^2) \overset{d}{\longrightarrow} N\left(0, 2\beta_t^4 \int_R K^2(s)ds \right). \] (3)

Similar to Kristensen [11], Renault et al. [12] proposed a spot volatility estimator which can approach the

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We consider nonparametric spot volatility estimation for diffusion models with discrete high frequency observations. Our estimator is carried out in two steps. First, using the local average of the range-based variance, we propose a crude estimator of the spot volatility. Second, we use usual nonparametric kernel smoothing to reconstruct the volatility function from the crude estimator. By inference, we find such a double smoothing operation can effectively reduce the estimation error.
nonparametric bound arbitrarily with unevenly spaced data at hand. By using multiple observations, Liu et al. [13] provided an "ideal" preaveraging estimator, established its asymptotic distribution, and considered the asymptotic efficiency of the proposed estimator. Recently, more and more scholars began to consider estimating spot volatility (see [14–16]), and spot volatility inference has become a hot issue in the field of regression dynamic research. Under microstructure noise or Poisson jumps, based on delta sequences, Mancini et al. [17] prevented a nonparametric estimation method of spot volatility. Taking into account fixed time-horizon and infill asymptotics, they obtained the full limit theory under unequally sample observation. Under the coexistence of market microstructure noise and multiple transactions, with high frequency data, Liuet al. [18] gave a coexistence of market microstructure noise and multiple theory under unequally sample observation. Under the horizon and infill asymptotics, they obtained the full limit theory of the proposed estimator. Recently, more and more literatures included but nonparametric bound arbitrarily with unevenly spaced data. For a scaled Brownian motion $X_t$ based variance, and they found that the latter had higher efficiency of the proposed estimator. Recent literatures included but nonparametric bound arbitrarily with unevenly spaced data. Song [29] expanded this technique to the diffusion models with jumps and illustrated that this two-step smoothing estimator could narrow mean-squared error effectively. It is not difficult to find that the diffusion models in the above three literatures are both homogeneous. As far as we know, so far, few scholars have applied the double smoothing method to time-dependent diffusion models.

In this paper, the double smoothing technique is used to estimate the spot volatility of time-dependent diffusion models, and the consistency and asymptotic normality of the estimator are obtained. Compared with single smoothing, the double smoothing method has less error and less information loss. Our consideration is twofold. First, in the first step smoothing, we estimate the spot volatility at any time point by calculating local average of square of the range within a suitable time interval. This allows us to replicate as much as possible the actual features of the volatility function. Using the crude estimator obtained from the first smoothing, we then re-estimate the volatility function by means of usual kernel smoothing. Second, the inferiority on choosing moderate frequency data can be effectively overcome by using the range-based technique. Indeed, the return-based technique is a simple and convenient method to estimate volatility function; however, its disadvantages cannot be ignored either. When the sampling frequency is increased, the high frequency data is easily affected by microscopic noise. With high frequency data at hand, the range-based estimator is more precise than the return-based one.

2. Nonparametric Range-Based Double Smoothing Spot Volatility Estimation

Define

$$t_{i,1} = \inf \{t \geq 0: |t - t_i| \leq \varepsilon\},$$

$$t_{i,(j+1)} = \inf \{t \geq t_{i,j} + \delta: |t - t_i| \leq \varepsilon\},$$

where $\varepsilon$ satisfies $\varepsilon = o_p(1)$ and $\delta = o_p(1)$. For any $i (0 \leq i \leq n)$, define

$$m_i = \sum_{j=1}^{n} I_{|t_{i,j} - t_i| \leq \varepsilon} = \frac{2\varepsilon}{\delta} + 1,$$

where $I$ and $|.|$ denote indicative function and rounding function, respectively.

Remark 1. As a matter of fact, when $i < \lfloor \varepsilon/\delta \rfloor$ or $i > n - \lfloor \varepsilon/\delta \rfloor$, it has $m_i < \lfloor 2\varepsilon/\delta \rfloor + 1$. Fortunately, this is only a small number of cases. Therefore, without loss of generality, for any $i (0 \leq i \leq n)$, we suppose $m_i = \lfloor 2\varepsilon/\delta \rfloor + 1$ and $2\varepsilon/\delta$ is an integer, then $m_i = 2\varepsilon/\delta + 1 (m = 2\varepsilon/\delta + 1$ for short).

Some of the technical conditions necessary are given below, which are common in estimating volatility.

For the process $\{a_t\}_{t \geq 0}$ satisfies

$$\sup_{0 \leq t \leq T} |a_t| < \infty.$$  

[10]
T2 The process \( \{\beta_t\}_{t \geq 0} \) is differentiable and satisfies
\[
\sup \{|\beta_t - \beta_s|, s, t \in [0, T], |s - t| \leq \xi \} = O_p(\xi^{1/2}\log \xi^{1/2})
\]
\[
\sup_{0 \leq s \leq T} \beta_s^2 < \infty.
\]  
(11)

T3 The bandwidth \( h \) satisfies
\[
h = \frac{\delta^{1/2}}{\log(1/\delta)}.
\]  
(12)

T4 Kernel function \( K(\cdot) \) is differentiable with support \([-1, 1]\) and satisfies
\[
(i) \int_{-1}^{1} K(c)dc = 1,
\]
\[
(ii) \int_{-1}^{1} K'(c)dc < \infty,
\]  
(13)
\[
(iii) \int_{-1}^{1} K^2(c)dc < \infty,
\]
\[
(iv) \int_{-1}^{1} K^3(c)dc < \infty.
\]

Remark 2. As we know, the drift parameter is not important in volatility estimation, so we restrict it simply and indispensably in condition T1.

Remark 3. Condition T2 is a common assumption about the diffusion parameter. The same or similar assumptions are used in [16, 30].

Remark 4. Bandwidth selection is very important in kernel-based estimators. Condition T3 chooses the same bandwidth as [30].

Now, we denote the nonparametric range-based double smoothing spot volatility estimator of \( \beta_t^2 \) as
\[
\tilde{\beta}_t^2 = \frac{\delta}{h} \sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right) \cdot \tilde{\beta}_i^2,
\]  
(14)
where
\[
\tilde{\beta}_i^2 = \frac{1}{dmh^2} \sum_{j=1}^{m} \tau_{X_{i,j}, \delta}^2.
\]  
(15)

Remark 5. In equation (15), we take a certain point as the midpoint and set a reasonable period, during which the volatility is only a function of state variables and has nothing to do with time. By calculating a local average of the range square in the time period and supposing it as the initial value of the spot volatility estimator at the time point, the real characteristics of volatility can be reproduced as much as possible. At the same time, the complete sample data are used in ranges to ensure the integrity of information.

3. The Consistency and the Asymptotic Normality of the Estimator

**Theorem 1.** Assume that the process \( \{X_t\}_{t \geq 0} \) satisfies the model in equation (1) and conditions T1–T4 hold; then, as \( \delta \to 0 \),
\[
\tilde{\beta}_t^2 \xrightarrow{P} \beta_t^2.
\]  
(16)

**Proof.** \( \tilde{\beta}_t^2 \) can be decomposed into \( A + B \), where
\[
A = \frac{\delta}{h} \sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right) \cdot \left(\tilde{\beta}_i^2 - \beta_i^2\right),
\]
\[
B = \frac{\delta}{h} \sum_{i=1}^{n} K\left(\frac{t_i - t}{h}\right) \cdot \beta_i^2.
\]  
(17)

We discuss the term \( B \) first. It can be further decomposed into \( B_1 + B_2 + B_3 \), where
\[
B_1 = \frac{1}{h} \sum_{i=1}^{n} \beta_i^2 \int_{t_i}^{t} \left(K\left(\frac{u - t}{h}\right) - K\left(\frac{u - t}{h}\right)\right)du,
\]
\[
B_2 = \frac{1}{h} \sum_{i=1}^{n} \int_{t_i}^{t} K\left(\frac{u - t}{h}\right) \cdot \left(\beta_i^2 - \beta_i^2\right)du,
\]  
(18)
\[
B_3 = \frac{1}{h} \sum_{i=1}^{n} \int_{t_i}^{t} K\left(\frac{u - t}{h}\right) \cdot \beta_i^2 du.
\]

For \( B_1 \), using Taylor’s formula, we can obtain
\[
B_1 = \frac{1}{h} \sum_{i=1}^{n} \beta_i^2 \int_{t_i}^{t} \left(K'\left(\frac{u - t}{h}\right) \cdot \frac{t_i - u}{h} + o\left(\frac{t_i - u}{h}\right)\right)du
\]
\[
= O_p\left(\frac{\delta}{h} \int_{0}^{T} K'\left(\frac{u - t}{h}\right)du\right)
\]
\[
= O_p\left(\frac{\delta}{h} \int_{-1}^{1} K'(s)ds\right).
\]  
(19)

By the conditions of T3 and T4 (ii), we can get \( B_1 = O_p(\delta/h) = O_p(1) \). From the conditions of T2 and T4 (i), it is obvious that \( B_2 = O_p(\delta^{1/2}\log \delta^{1/2}) = o_p(1) \). Consider \( B_3 \) term below. Let
\[
\frac{u - t}{h} = s.
\]  
(20)

Then,
\[ B_3 = \int_{-1}^{1} K(s)\beta_t^3 \, ds \]
\[ = \int_{-1}^{1} K(s)(\beta_t^2 + O_p(h^{1/2}|\log h|^{1/2})) \, ds \]
\[ = \beta_t^2 + O_p(h^{1/2}|\log h|^{1/2}). \]

Combining \( B_1, B_2 \) with \( B_3 \), we have \( B \longrightarrow^p \beta_t^2. \) In order to prove Theorem 1, we just have to prove \( A \longrightarrow^p 0 \), namely, \( \beta_t^2 \longrightarrow^p \beta_t^2. \) Let

\[ \omega_{i,j} = \frac{1}{\delta m\lambda_2} \beta_t^2 r_{W_{i,j}}. \]

It can be known from equation (5) that

\[ \sum_{j=1}^{m} E[\omega_{i,j}] = \beta_t^2. \]

Now, suppose

\[ \theta_{i,j} = \omega_{i,j} - E[\omega_{i,j}], \]

we can obtain

\[ E[\theta_{i,j}^2] = \frac{\Lambda}{m^2} \beta_t^4, \]

where \( \Lambda = (\lambda_4 - \lambda_2^2)/\lambda_2^2. \) Furthermore,

\[ \sum_{j=1}^{m} E[\theta_{i,j}^2] \to^p 0. \]

Therefore,

\[ \sum_{j=1}^{m} \omega_{i,j} \to^p \beta_t^2. \]

Next, we prove \( \beta_t^2 \to^p \sum_{j=1}^{m} \omega_{i,j}. \) Notice that

\[ \beta_t^2 - \sum_{j=1}^{m} \omega_{i,j} = \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} (r_{X_{i,j}} + \beta_t r_{W_{i,j}})(r_{X_{i,j}} - \beta_t r_{W_{i,j}}) \]
\[ = D_1 + D_2, \]

where

\[ D_1 = \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} \left( r_{X_{i,j}} - \beta_t r_{W_{i,j}} \right)^2, \]
\[ D_2 = \frac{2}{\delta m\lambda_2} \sum_{j=1}^{m} (r_{X_{i,j}} - \beta_t r_{W_{i,j}}) \beta_t r_{W_{i,j}}. \]

For \( D_1 \), we further decompose it as follows:

\[ D_1 \leq \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} \alpha_u \, du + \int_{\tau}^{t} (\beta_u - \beta_t) \, dW_u \right)^2 \]
\[ \leq \frac{2}{\delta m\lambda_2} \sum_{j=1}^{m} \left( \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} \alpha_u \, du \right)^2 \right) \]
\[ + \frac{2}{\delta m\lambda_2} \sum_{j=1}^{m} \left( \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} (\beta_u - \beta_t) \, dW_u \right)^2 \right). \]

From condition T1, it is obvious that \( D_{1,1} = O_p(\epsilon). \) For \( D_{1,2}, \) by the Burkholder–Davis–Gundy inequality (BDG inequality for short), there exists a constant \( C(>0) \) to make

\[ E[D_{1,2}] \leq \frac{2C}{\delta m\lambda_2} \sum_{j=1}^{m} \left( \int_{t_{i,j-1}}^{t_{i,j}} (\beta_u - \beta_t)^2 \, du \right). \]

By the condition of T2 and the definition of \( t_{i,j} \), we can obtain

\[ E[D_{1,2}] = O_p(\epsilon \log |\epsilon|). \]

Therefore, \( D_1 = o_p(1). \) For \( D_2 \), using the decomposition similar to \( D_1 \), it holds that

\[ D_2 \leq \frac{2}{\delta m\lambda_2} \sum_{j=1}^{m} \beta_t r_{W_{i,j}} \left( \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} \alpha_u \, du + \int_{\tau}^{t} (\beta_u - \beta_t) \, dW_u \right)^2 \right). \]

Using Hölder’s inequality, we obtain

\[ D_2 \leq 2 \left( \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} (\beta_t r_{W_{i,j}})^2 \right)^{1/2} \]
\[ \cdot \left( \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} \alpha_u \, du + \int_{\tau}^{t} (\beta_u - \beta_t) \, dW_u \right)^2 \right)^{1/2}. \]

Using Hölder’s inequality again, we may further obtain

\[ E[D_2] \leq 2 \left( E \left[ \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} (\beta_t r_{W_{i,j}})^2 \right] \right)^{1/2} \]
\[ \cdot \left( E \left[ \frac{1}{\delta m\lambda_2} \sum_{j=1}^{m} \sup_{t_{i,j}\leq t \leq t_{i,j+1}} \left( \int_{\tau}^{t} \alpha_u \, du + \int_{\tau}^{t} (\beta_u - \beta_t) \, dW_u \right)^2 \right] \right)^{1/2}. \]

From the discussion of \( D_1 \), it is true that \( E[D_1] = O_p(\epsilon^{1/2}|\log |\epsilon||^{1/2}). \) So, \( \beta_t - \sum_{j=1}^{m} \omega_{i,j} \to^p 0, \) and these complete the proof. □
Theorem 2. Assume that the process \( \{X_t\}_{t \geq 0} \) satisfies the model in equation (1) and the condition T1–T4 hold, given \( \delta \rightarrow 0 \), such that

\[
\frac{e \delta^2 |\log h|}{\delta^2} = o_p(1),
\]

\[
\frac{eh^2 |\log h|}{\delta^2} = o_p(1).
\]

Then,

\[
\frac{\sqrt{eh}}{\delta} \left( \beta_t^2 - \beta_t^2 \right) \xrightarrow{p} N \left( 0, \frac{\Lambda_2 \beta_t^4}{2} \int_{-1}^{1} K^2(s) ds \right),
\]

where \( \Lambda_2 = (\lambda_4 - \lambda_2^2) / \lambda_2^2 \).

Proof. Decompose \( \beta_t^2 - \beta_t^2 \) as follows:

\[
\beta_t^2 - \beta_t^2 = \left( \beta_t^2 - \frac{\delta}{h} \sum_{n} K \left( \frac{t_i - t}{h} \right) \cdot \sum_{j} \omega_{i,j} \right)
\]

\[
+ \frac{\delta}{h} \sum_{n} K \left( \frac{t_i - t}{h} \right) \cdot \sum_{j} \theta_{i,j}
\]

\[
\left( \frac{\delta}{h} \sum_{n} K \left( \frac{t_i - t}{h} \right) \cdot \sum_{j} E \left[ \omega_{i,j} \right] - \beta_t^2 \right)
\]

\[= F_1 + F_2 + F_3.\]

By the proof of Theorem 1, we know

\[F_1 = o_p \left( \epsilon^{1/2} |\log h|^{1/2} \right),\]

\[F_3 = o_p \left( h^{1/2} |\log h|^{1/2} \right).\]

It can be seen from equations (36) and (37) that

\[
\frac{\sqrt{eh}}{\delta} F_1 = o_p(1),
\]

\[
\frac{\sqrt{eh}}{\delta} F_3 = o_p(1).
\]

Now, we discuss

\[
\frac{\sqrt{eh}}{\delta} F_2 = \sqrt{\frac{eh}{h}} \sum_{n} \frac{K \left( \frac{t_i - t}{h} \right) \cdot \sum_{j} \theta_{i,j}}{h}.
\]

Let

\[
\beta_t = \left( \frac{\epsilon}{h} K \left( \frac{t_i - t}{h} \right) \cdot \sum_{j} \theta_{i,j} \right).
\]

For \( i = 1, 2, \ldots, n \), \( j, k = 1, 2, \ldots, m \), as \( j \neq k, \theta_{i,j} \) and \( \theta_{i,k} \) are independent of each other, so

\[
E [\beta_t^2] = \frac{\epsilon^2}{h^2} \left( t_i - t \right) \sum_{j=1}^{m} E [\theta_{i,j}^2]
\]

\[
= \frac{\epsilon^2}{m h^2} \Lambda_2 \beta_t^2 \int_{-1}^{1} K^2(s) ds.
\]

Furthermore,

\[
\sum_{i=1}^{n} E [\beta_t^2] = \frac{\epsilon^2}{m h^2} \sum_{i=1}^{n} \frac{K^2 \left( \frac{t_i - t}{h} \right) \cdot \theta_{i,j}^2}{\beta_t^2} + \beta_t^4
\]

\[
= \frac{\epsilon^2}{m h^2} \sum_{i=1}^{n} \frac{K^2 \left( \frac{t_i - t}{h} \right) \cdot \theta_{i,j}^2}{\beta_t^2}
\]

\[
= \frac{\epsilon^2}{m h^2} \Lambda_2 \beta_t^2 \int_{-1}^{1} K^2(s) ds.
\]

Similarly,

\[
\sum_{i=1}^{n} E [\beta_t^2] = \frac{\epsilon^{3/2}}{m h^{1/2} \delta} \beta_t^6 \int_{-1}^{1} K^2(s) ds + O_p \left( h |\log h| \right).
\]

From Remark 1 and condition T4 (iv), it holds that

\[
\sum_{i=1}^{n} E [\beta_t^2] = O \left( \frac{\delta}{\epsilon^{3/2} h^{1/2}} \right) = o_p(1).
\]

It can be obtained from Lyapunov's central limit theorem that

\[
\frac{\sqrt{eh}}{\delta} F_2 = \frac{d}{N} \left( 0, \frac{\Lambda_2 \beta_t^2}{2} \right) \int_{-1}^{1} K^2(s) ds.
\]
better than the return-based single smoothing estimator. By comparing equations (7) and (38), we find that the range-based double smoothing estimator is more accurate than the single one because $\delta \sqrt{\varepsilon h} = \mathcal{O}(\sqrt{\delta h})$.

4. Conclusions

Using the entire sample data, range-based estimator not only guarantees the integrity of information but also reduces influence of microstructure noise. Compared with single smoothing, the double smoothing method can replicate the actual features of the volatility function more. In this paper, combining the range technique with the double smoothing method, we propose a range-based double smoothing spot volatility estimator and prove its consistency and asymptotic normality. We also find that the range-based single smoothing estimator is better than the return-based single smoothing one, while the range-based double smoothing one is the best.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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