MAXIMAL BETTI NUMBER OF A FLAG SIMPLICIAL COMPLEX

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ABSTRACT. We prove that the homology of a flag simplicial complex with $n$ vertices has dimension at most $4^n/5$, which is approximately $1.32^n$. The same upper bound holds for the Euler characteristic and therefore, in more combinatorial language, for the alternating number of independent sets in any $n$-vertex graph. All bounds are asymptotically tight. We also study the same question for the independence complexes of bipartite graphs.

1. INTRODUCTION

Suppose $K$ is a simplicial complex with $n$ vertices. How big can the reduced homology groups $\tilde{H}_*(K; \mathbb{Q})$ be?

Of course there is a trivial upper bound: since $K$ has at most $2^n$ faces, it is homeomorphic to a CW-complex with at most $2^n$ cells, therefore $\dim \tilde{H}_*(K) \leq 2^n$. For arbitrary $K$ we can asymptotically almost achieve that value. The $k$-dimensional skeleton $\Delta_0^{(k)}$ of the $n$ simplex is known to be homotopy equivalent to the wedge of $\binom{n}{k+1}$ spheres, and for $k \approx n/2$ that is roughly $\binom{n}{5} \approx \frac{2^n}{\sqrt{n}}$ (by [2] this is in fact optimal).

A natural class of complexes for which the answer is less obvious are flag complexes. A simplicial complex is called flag if and only if all its minimal non-faces have dimension 1, but there is a much better definition, which we now state. An independent set in a simple, undirected graph $G$ is a collection of pairwise non-adjacent vertices. The independence complex of $G$ is a simplicial complex whose vertices are the vertices of $G$ and the simplices are the independent sets in $G$. Then a simplicial complex $K$ is flag if and only if it is the independence complex of some graph. The graph in question is the complement of the 1-skeleton of $K$.

In this language the question we ask is:

Given an $n$-vertex graph $G$, how big can the total reduced Betti number $\dim \tilde{H}_*(\mathbb{I}(G))$ be?

The same question can be asked about the reduced Euler characteristic $\tilde{\chi}(\mathbb{I}(G))$. This has a more combinatorial interpretation. If $f_i(G)$ is the number of independent sets of cardinality $i$ in $G$ then we define the independence polynomial $\mathbb{I}(G;x) = \sum_i f_i(G)x^i$ and then $\tilde{\chi}(\mathbb{I}(G)) = |\mathbb{I}(G)-1| = 4^n/5$. For this reason $|\tilde{\chi}(\mathbb{I}(G))|$ is sometimes called the alternating number of independent sets in $G$. For any space $K$ we have $|\tilde{\chi}(K)| \leq \dim \tilde{H}_*(K)$.

Example 1.1. Suppose $n$ is divisible by 5 and let $G = (n/5) \cdot K_5$ be the disjoint union of $n/5$ copies of the complete graph $K_5$. Then $\mathbb{I}(G;x) = (1+5x)^{n/5}$, so $|\tilde{\chi}(\mathbb{I}(G))| = |\mathbb{I}(G)-1| = 4^n/5$. In fact the complex $\mathbb{I}(G)$ is the join of $n/5$ copies of $\mathbb{I}(K_5)$. The latter is the discrete space with 5 points, or, in other words, the wedge sum $\bigvee^4 S^0$. That means we have a homotopy equivalence $\mathbb{I}(G) \simeq \bigvee^4 S^{n/5-1}$, and therefore

$$|\tilde{\chi}(\mathbb{I}(G))| = \dim \tilde{H}_*(\mathbb{I}(G)) = 4^n/5 \approx 1.32^n.$$

We could just as well take $G = (n/q) \cdot K_q$ for any $q \geq 2$ and get a wedge of $((q-1)/q)^n$ spheres. The number $(q-1)^{1/q}$ is maximized for $q = 5$.

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In this short note we prove that the graph of Example 1.1 is in fact extremal.

**Theorem 1.2.** For any $n$-vertex graph $G$ we have
\[
\dim \tilde{H}_n(I(G); \mathbb{Q}) \leq (4^{1/5})^n.
\]

The class of flag complexes includes clique complexes of graphs, Rips complexes of discrete metric spaces, order complexes of posets, matching and chessboard complexes and many others. There are several results that bound $|\chi(I(G))|$ or $\dim \tilde{H}_n(I(G))$ in terms of various parameters of $G$, see [3, 10, 11]. Recently graphs whose independence complexes have exponential Betti numbers have emerged in connection with supersymmetric lattice models in statistical physics, see [3, 4, 5, 6]. In this context it is interesting to have a benchmark against which one can compare the experimental estimates and upper bounds.

In [13] and [8] it is shown that the same upper bound holds for the Euler characteristic and total Betti number of the order complex of any $n$-element poset. Theorem 1.2 generalizes that result as every order complex is flag, but not conversely.

The paper is structured as follows. In §2 we prove Theorem 1.2. It implies that the same upper bound holds for the reduced Euler characteristic $|\chi(I(G))| = |I(G; -1)|$, and our argument also provides an independent, algebraic proof of that fact without using the comparison with homology. In §3 we discuss an analogous problem with the restriction that $G$ is bipartite.

2. Proof

For a graph $G$ we denote by $V(G)$ the set of vertices. If $v$ is a vertex of $G$ then we write $N_G[v]$ for the closed neighbourhood of $v$ which consists of $v$ and all its adjacent vertices in $G$. If $e = uv$ is an edge then we set $N_G[e] = N_G[u] \cup N_G[v]$. By $G - e$ we denote the graph obtained by removing an edge $e$ (but not its endpoints) from $G$. If $W \subseteq V(G)$ then $G \setminus W$ is the induced subgraph obtained by removing the vertices which belong to $W$. We also write $\deg_G(v)$ for the degree of $v$ in $G$ and $\mindeg(G)$ for the smallest degree of a vertex of $G$. For a simplicial complex $K$ we denote by $\Sigma K$ the suspension, that is the join $S^0 \ast K$. For basic notions of combinatorial topology see [9].

It is convenient to make the following general statement.

**Proposition 2.1.** Suppose $\beta(G)$ is any function that assigns to every graph $G$ a non-negative integer and which satisfies three conditions:
- $\beta(\emptyset) = 1$, where $\emptyset$ denotes the empty graph with no vertices,
- $\beta(G) = 0$ if $G$ has an isolated vertex,
- $\beta(G) \leq \beta(G - e) + \beta(G \setminus N_G[e])$ for any edge $e$ of $G$.

Given such a function define
\[
\beta_n = \max_{G: |V(G)| \leq n} \beta(G).
\]
Then we have
\[
\beta_n \leq 4^{n/5}.
\]

Our main theorem then follows from the next observation.

**Proposition 2.2.** The function $G \mapsto \dim \tilde{H}_n(I(G); \mathbb{Q})$ satisfies the assumptions of Prop. 2.1.

**Proof.** The first condition follows because $I(\emptyset)$ is the empty space and that has a single reduced homology group in the augmentation degree $-1$. The second condition holds because in this case $I(G)$ is a contractible space.

The third condition is a consequence of the observation of Meshulam [12], which can be summarized by saying that there is an inclusion map $\Sigma I(G \setminus N_G[e]) \hookrightarrow I(G)$ with homotopy cofibre $I(G - e)$. For completeness let us present a detailed argument.

An independent set in the graph $G - e$ can be of two kinds: it is either independent in $G$ or it contains both endpoints of $e$ plus some independent set in $G \setminus N_G[e]$. This means that we have a decomposition
\[
I(G - e) = K \cup L
\]
Proof. Let $\emptyset$ graph. The result holds for Proof of Proposition 2.1. The result follows from the previous lemma.

Then of the previous lemma we have an inclusion of vertex sets

$$\ldots \rightarrow \tilde{H}_{i-1}(I(G \setminus N_G[e])) \rightarrow \tilde{H}_i(I(G)) \rightarrow \tilde{H}_i(I(G - e)) \rightarrow \tilde{H}_{i-1}(I(G \setminus N_G[e])) \rightarrow \ldots .$$

Of course every time a sequence of vector spaces $A \rightarrow B \rightarrow C$ is exact at $B$ then $\dim B \leq \dim A + \dim C$. Summing over all $i$ we obtain the result. □

**Remark 2.3.** Another function which also satisfies Prop. 2.1 is $G \mapsto |\chi(I(G))| = |I(G; -1)|$. It follows from the above argument, but can also be seen purely algebraically. The first condition holds because $I(\emptyset; x) = 1$, the second follows because $I(G; x)$ is divisible by $1 + x$ and the third is a consequence of the equation $I(G; x) = I(G - e; x) - x^2 I(G \setminus N_G[e]; x)$.

It remains to prove Prop. 2.1 and we do so in a series of lemmas.

**Lemma 2.4.** Let $v$ be any vertex of a graph $G$ and let $e_1, \ldots, e_d$ be all the edges incident with $v$. Define $G_i = G$ and

$$G_i = G - \{e_1, \ldots, e_{i-1}\}, \quad i = 2, \ldots, d+1.$$

Then

$$\beta(G) \leq \sum_{i=1}^d \beta(G_i \setminus N_{G_i}[e_i]).$$

**Proof.** Since $G_{i+1} = G_i - e_i$, this follows immediately by induction from the third condition for $\beta$:

$$\beta(G) = \beta(G_1) \leq \beta(G_2) + \beta(G_1 \setminus N_{G_1}[e_1]) \leq \beta(G_3) + \beta(G_2 \setminus N_{G_2}[e_2]) + \beta(G_1 \setminus N_{G_1}[e_1]) \leq \ldots \leq \beta(G_{d+1}) + \sum_{i=1}^d \beta(G_i \setminus N_{G_i}[e_i]).$$

But $G_{d+1}$ has $v$ as an isolated vertex, so $\beta(G_{d+1}) = 0$. □

**Lemma 2.5.** If $|V(G)| = n$ and $\mindeg(G) = d$ then

$$\beta(G) \leq d \cdot \beta_{n-(d+1)}.$$

**Proof.** Let $v$ be any vertex of degree $d$ with incident edges $e_i = vu_i, i = 1, \ldots, d$. In the notation of the previous lemma we have an inclusion of vertex sets

$$V(G_i \setminus N_{G_i}[e_i]) \subseteq V(G_i \setminus N_{G_i}[v_i]) = V(G \setminus N_G[u_i]).$$

Since $\deg_G(u_i) \geq d$ the last set has at most $n - (d + 1)$ elements, therefore

$$\beta(G_i \setminus N_{G_i}[e_i]) \leq \beta_{n-(d+1)}.$$

The result follows from the previous lemma. □

**Proof of Proposition 2.1** The result holds for $n = 0$: the only graph with 0 vertices is the empty graph $\emptyset$ with $\beta(\emptyset) = 1 = 4^{0/5}$.

Now suppose $G$ is any graph with $n \geq 1$ vertices and let $d = \mindeg(G)$. Using the previous lemma and the induction hypothesis we get

$$\begin{align*}
\beta(G) & \leq d \cdot \beta_{n-(d+1)} \leq d \cdot 4^{(n-(d+1))/5} = \\
& = 4^{n/5} \cdot \frac{d}{4^{(d+1)/5}} \leq 4^{n/5}.
\end{align*}$$

The last inequality holds because the function $f(d) = \frac{d}{4^{(d+1)/5}}$ attains maximum (for integer values of $d$) when $d = 4$ and $f(4) = 1$. Since $G$ was arbitrary that completes the proof. □
Remark 2.6. A close inspection of the proof reveals that (when \( n \) is divisible by 5) the disjoint union of copies of \( K_5 \) is the only graph for which the total Betti number attains maximum. If \( n \) is not a multiple of 5 one must adjust the size of one or two copies of \( K_5 \). We omit the details which are analogous as in [3, 8].

3. Bipartite graphs

Our formulation of the problem makes it natural to ask the same question with various restrictions on \( G \). Consider the problem of maximizing the total Betti number \( \dim \tilde{H}_*(I(G)) \) for bipartite graphs \( G \).

Example 3.1. For \( q \geq 1 \) let \( K_{q,q} \) be the complete bipartite graph with parts of equal size \( q \) and let \( B_{2q} = K_{q,q} - M \) be the same graph with a perfect matching removed. Then the space \( I(B_{2q}) \) consists of two \( q \)-vertex simplices joined by \( q \) segments, hence it is homotopy equivalent to \( \bigvee^{q-1} S^1 \).

Now suppose \( n \) is divisible by \( 2q \) and let \( G = (n/2q) \cdot B_{2q} \). As before, we obtain that \( I(G) \) is homotopy equivalent to a wedge of \( (q-1)^n/2q = ((q-1)^{1/2q})^n \) spheres and that this expression is maximized for \( q = 5 \). We therefore have a bipartite graph \( G \) with \( n \) vertices and total Betti number \( (2^{1/5})^n \approx 1.15^n \).

Note that this graph is the so-called bipartite double cover of the graph of Example 1.1.

We conjecture that the graph from the previous example is extremal.

Conjecture 3.2. For any \( n \)-vertex bipartite graph \( G \) we have

\[
\dim \tilde{H}_*(I(G); \mathbb{Q}) \leq (2^{1/5})^n.
\]

There is also another, equivalent formulation of this conjecture.

Conjecture 3.3. If \( K \) is any simplicial complex (not necessarily flag) with \( a \) vertices and \( b \) maximal faces then

\[
\dim \tilde{H}_*(K; \mathbb{Q}) \leq (2^{1/5})^{a+b}.
\]

Proof of the equivalence of Conjectures 3.2 and 3.3. Consider a bipartite \( n \)-vertex graph \( G \) with parts \( U, W \) of sizes \( |U| = a, |W| = b, a + b = n \). Let \( K_G \) be the simplicial complex with vertex set \( U \) and with maximal faces of the form

\[
U \setminus N_G(w) \quad \text{for all} \quad w \in W.
\]

It is a known fact (see [1, Thm.3.7], [2, Sect. 3]) that \( I(G) \simeq \Sigma K_G \), so the spaces \( I(G) \) and \( K_G \) have the same total reduced Betti numbers. Since \( K_G \) has \( a \) vertices and at most \( b \) maximal faces, Conjecture 3.3 implies

\[
\dim \tilde{H}_*(I(G)) = \dim \tilde{H}_*(K_G) \leq (2^{1/5})^{a+b} = (2^{1/5})^n.
\]

Conversely, if \( K \) is any complex with \( a \) vertices and \( b \) maximal faces, we construct a bipartite graph \( G_K \). It has one vertex for each vertex and for each maximal face of \( K \) and a vertex \( u \) is adjacent to a maximal face \( f \) if \( u \notin f \). Now the previous homotopy equivalence translates into \( I(G_K) \simeq \Sigma K \) and \( G_K \) has \( a + b \) vertices so Conjecture 3.2 implies

\[
\dim \tilde{H}_*(K) = \dim \tilde{H}_*(I(G_K)) \leq (2^{1/5})^{a+b}.
\]

\( \square \)

The methods of the previous section suffice to prove a partial result, weaker than the conjectured optimum. First consider, for every \( d \geq 1 \), the function

\[
f_d(x) = x^{-(d+1)} + x^{-(d+2)} + \cdots + x^{-2d}
\]

and let \( 1 \leq \alpha_d < 2 \) be the unique solution to \( f_d(\alpha_d) = 1 \). Then one checks easily that \( \alpha = \alpha_3 \approx 1.25 \) is the largest of all the \( \alpha_d \) and we have the next result.
Proposition 3.4. For every triangle-free \( n \)-vertex graph \( G \) we have
\[
\dim \tilde{H}_*(\mathcal{I}(G); \mathbb{Q}) \leq \alpha^n.
\]

Proof. Once again the proof works for any function \( \beta \) which satisfies the conditions of [2,1]. Define
\[
\beta'_n = \max_{G \text{ triangle-free} \atop |V(G)| \leq n} \beta(G).
\]
We follow the inductive argument of the previous section, but triangle-freeness gives a better estimate of the size of the removed neighbourhoods.

The result holds for \( n = 0 \). Now suppose \( G \) is a triangle-free graph with \( n \geq 1 \) vertices and \( \mindeg(G) = d \). Let \( v \) be a vertex of degree \( d \). In the graph \( G_i \) of Lemma [2,4] the endpoints of \( e_i \) have degrees at least \( d - i + 1 \) and \( d \) and their neighbourhoods are disjoint, so
\[
|V(G_i \setminus N_{G_i}[e_i])| \leq n - (d - i + 1) - d = n - 2d + i - 1.
\]
All graphs \( G_i \setminus N_{G_i}[e_i] \) are triangle-free, so by Lemma [2,4] we get
\[
\beta(G) \leq \beta'_{n-2d} + \beta'_{n-(2d-1)} + \cdots + \beta'_{n-(d+1)} \leq \alpha^{n-2d} + \alpha^{n-(2d-1)} + \cdots + \alpha^{n-(d+1)} = \alpha^n f_d(\alpha) \leq \alpha^n
\]
since \( f_d(\alpha) \leq f_d(\alpha_d) = 1 \). Since \( G \) was arbitrary that completes the proof. \( \square \)

As a consequence Conjectures 3.2 and 3.3 hold with the constant \( 2^{1/5} \) replaced by \( \alpha \).

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