Abstract

We study the asymmetric one-dimensional telegraph process in the bounded domain. Lower boundary is absorbing and upper boundary is reflecting with delay. Point stays in the upper boundary until switch of regime occurs. We obtain the distribution of this process in terms of Laplace trasforms.

Mathematical subject classification: 60H30, 60K99, 44A10.
Keywords: telegraph process, Laplace transform.

1 Introduction

Telegraph process was introduced in [7] and [11]. It was shown in these works that telegraph process is closely related to the well-known PDE called telegrapher’s equation. Telegraph process has been applied to the number of problems in physics ([17], [10], [9]) and economics ([15], [8], [5], [21], [23], [22], [13]). A growing body of literature is dedicated to the properties of telegraph process: [18] and [6] derive the distribution of the symmetric telegraph process, its maximum and first passage time for unbounded domain, [16] solve telegrapher’s equation with reflecting and partially reflecting boundaries, [12] solves telegrapher’s equation by the Adomian decomposition method, [2] and [14] analyze asymmetric telegraph process on unbounded domain, [4] and [8] solve time-fractional telegrapher’s equation including the case of bounded domain with different types of boundary conditions, [19] solves telegrapher’s equation in the domain with variable borders.

We are interested in the dynamics of telegraph process in the bounded domain with absorbing lower boundary and reflecting upper boundary. Unlike [16], we assume that reflection is not instantaneous — point stays in the upper boundary until switch to the regime 0 occurs. That type of boundary conditions was analyzed in [1] and the dynamics for the case of two boundaries of this type was described in [20]. However, the case of mixed boundaries, to the best our knowledge, was not yet analyzed in the literature. We derive the system of two partial differential equations with boundary conditions and solve them by the method of Laplace transform. The solution is given in the terms of inverse Laplace transform of rather cumbersome functions.

2 Statement and solution of the problem

We consider telegraph process \( A(t) \) in the bounded domain \([0, B]\) for some \( B > 0 \). There are two regimes \( s(t) \in \{0, 1\} \) defined by velocities \( \mu_0 < 0 \) for the regime \( s = 0 \) and \( \mu_1 > 0 \) for the regime \( s = 1 \). The rates of occurrences of velocity switches are \( \Lambda_0 > 0 \) (for the switch from regime 0 to 1) and \( \Lambda_1 > 0 \) (for the opposite switch). The process starts at some point \( A(0) \in [0, B] \) in some regime \( s(0) \in \{0, 1\} \). When \( A(t) \) becomes negative for the first time, process stops.
hits \( B \), it stays in \( B \) until the switch to the regime 0 occurs. In order to describe the evolution of the process, we introduce functions

\[ F_s(t, A) = P(A(t) \geq A, s(t) = s). \]  

(1)

The definition of the process leads to the following description of dynamics

\[ F_0(t + \Delta t, A) = (1 - \Lambda_s \Delta t) F_s(t, A - \mu_s \Delta t) + \Lambda_1 - \Lambda_s \Delta t F_{1-s}(t, A), \quad s \in \{0, 1\} \]  

(2)

for small \( \Delta t \), which leads to the system of partial differential equations

\[
\begin{cases}
\frac{\partial}{\partial t} F_0(t, A) = -\mu_0 \frac{\partial}{\partial A} F_0(t, A) - \Lambda_0 F_0(t, A) + \Lambda_1 F_1(t, A), \\
\frac{\partial}{\partial t} F_1(t, A) = -\mu_1 \frac{\partial}{\partial A} F_1(t, A) - \Lambda_1 F_1(t, A) + \Lambda_0 F_0(t, A).
\end{cases}
\]  

(3)

In order to derive boundary conditions, substitute \( A = 0 \) and \( A = B \) to (2) and note that \( F_1(t, -\mu_1 \Delta t) = F_1(t, 0) \) and \( F_0(t, B - \mu_0 \Delta t) = 0 \). After some simple calculations we get

\[
\frac{\partial}{\partial A} F_1(t, 0) = 0, \quad F_0(t, B) = 0.
\]  

(4)

Our goal is to solve the system (3) with boundary conditions (4). To do so, we multiply both equations by \( e^{-\xi A} \) for some arbitrary \( \xi \) and integrate them with respect to \( A \):

\[
\begin{cases}
\frac{\partial}{\partial t} L_0(t, \xi) = -\mu_0 \int_0^B e^{-\xi A} \frac{\partial}{\partial A} F_0(t, A) \, dA - \Lambda_0 L_0(t, \xi) + \Lambda_1 L_1(t, \xi), \\
\frac{\partial}{\partial t} L_1(t, \xi) = -\mu_1 \int_0^B e^{-\xi A} \frac{\partial}{\partial A} F_1(t, A) \, dA - \Lambda_1 L_1(t, \xi) + \Lambda_0 L_0(t, \xi),
\end{cases}
\]  

(5)

where

\[ L_s(t, \xi) = \int_0^B e^{-\xi A} F_s(t, A) \, dA, \quad s \in \{0, 1\}. \]  

(6)

Integrating by parts and using the upper boundary condition, we get

\[
\begin{cases}
\frac{\partial}{\partial t} L_0(t, \xi) = \mu_0 \phi(t) - \mu_0 \xi L_0(t, \xi) - \Lambda_0 L_0(t, \xi) + \Lambda_1 L_1(t, \xi), \\
\frac{\partial}{\partial t} L_1(t, \xi) = -\mu_1 e^{-\xi B} \phi(t) + \mu_1 \psi(t) - \mu_1 \xi L_1(t, \xi) - \Lambda_1 L_1(t, \xi) + \Lambda_0 L_0(t, \xi),
\end{cases}
\]  

(7)

where

\[ F_0(t, 0) = \phi(t), \quad F_1(t, B) = \omega(t), \quad F_1(t, 0) = \psi(t). \]  

(8)

Now taking Laplace transform of (7) with respect to \( t \), we get

\[
\begin{align*}
\bar{L}_0(p, \xi) &= -\bar{\omega}(p) \frac{\Lambda_1 \mu_1 e^{-\xi B}}{(p - n)(p - m)} + \frac{\Lambda_1 \mu_1 \bar{\psi}(p)}{(p - n)(p - m)} + \frac{(\mu_1 \xi + p + \Lambda_1) \mu_0 \bar{\phi}(p)}{(p - n)(p - m)} + \\
&\quad \frac{\Lambda_1 L_1(0, \xi)}{(p - n)(p - m)} + \frac{(\mu_1 \xi + p + \Lambda_1) L_0(0, \xi)}{(p - n)(p - m)}, \\
\bar{L}_1(p, \xi) &= -\bar{\omega}(p) \frac{\mu_1 (\mu_0 \xi + p + \Lambda_0) e^{-\xi B}}{(p - n)(p - m)} + \frac{\mu_1 (\mu_0 \xi + p + \Lambda_0) \bar{\psi}(p)}{(p - n)(p - m)} + \frac{\Lambda_0 \mu_0 \bar{\phi}(p)}{(p - n)(p - m)} + \\
&\quad \frac{(\mu_0 \xi + p + \Lambda_0) L_1(0, \xi)}{(p - n)(p - m)} + \frac{\Lambda_0 L_0(0, \xi)}{(p - n)(p - m)}.
\end{align*}
\]  

(9)
where $p$ is the parameter of Laplace transform, tilde means Laplace transform of a function and

\[
\begin{align*}
q &= \xi^2 \mu_0^2 - 2 \xi^2 \mu_0 \mu_1 + \xi^2 \mu_1^2 + 2 \xi \Lambda_0 \mu_0 - 2 \xi \Lambda_0 \mu_1 - 2 \xi \Lambda_1 \mu_0 + 2 \xi \Lambda_1 \mu_1 + \\
&\quad \Lambda_0^2 + 2 \Lambda_0 \Lambda_1 + \Lambda_1^2,
\end{align*}
\]

\[\text{(10)}\]

We now prove the simple

**Proposition 1.** 1. $m$ is positive for sufficiently big absolute values of $\xi$ and any values of the parameters of the model.

2. $n$ is negative for any values $\xi$ and any values of the parameters of the model.

**Proof.** 1. Expanding $m$ in a Taylor series in the neighborhood of $\xi = +\infty$, we get

\[
m \approx -\mu_0 \xi - \Lambda_0 - \frac{\Lambda_0 \Lambda_1}{\xi (\mu_0 - \mu_1)},
\]

which is positive for sufficiently big positive values of $\xi$. Expanding $m$ in a Taylor series in the neighborhood of $\xi = -\infty$, we get

\[
m \approx -\xi \mu_1 - \Lambda_1 + \frac{\Lambda_0 \Lambda_1}{\xi (\mu_0 - \mu_1)},
\]

which is again positive for sufficiently big negative values of $\xi$.

2. Inequality $n < 0$ may be rewritten as

\[
- \mu_0 \xi - \xi \mu_1 - \Lambda_0 - \Lambda_1 < \sqrt{q}. \tag{11}
\]

If the expression on the left side is negative, inequality is proven. Assume it is positive:

\[
\mu_0 \xi + \xi \mu_1 + \Lambda_0 + \Lambda_1 < 0. \tag{12}
\]

Taking squares of both sides of \[(11),\] we get

\[
\xi (\xi \mu_0 \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0) < 0. \tag{13}
\]

Assume $\mu_0 + \mu_1 > 0$. Then \[(12)\] leads to

\[
\xi < \frac{-\Lambda_0 - \Lambda_1}{\mu_0 + \mu_1} < 0.
\]

Hence, first multiplier in \[(13)\] is negative. Consider the second one:

\[
\xi \mu_0 \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 > \mu_0 \mu_1 \frac{-\Lambda_0 - \Lambda_1}{\mu_0 + \mu_1} + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 = \frac{\Lambda_0 \mu_1^2 + \Lambda_1 \mu_0^2}{\mu_0 + \mu_1} > 0,
\]

hence \[(13)\] holds. Similarly, assume $\mu_0 + \mu_1 < 0$. Then \[(12)\] leads to

\[
\xi > \frac{-\Lambda_0 - \Lambda_1}{\mu_0 + \mu_1} > 0.
\]

Hence, first multiplier in \[(13)\] is positive. Consider the second one:

\[
\xi \mu_0 \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 < \mu_0 \mu_1 \frac{-\Lambda_0 - \Lambda_1}{\mu_0 + \mu_1} + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 = \frac{\Lambda_0 \mu_1^2 + \Lambda_1 \mu_0^2}{\mu_0 + \mu_1} < 0,
\]

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hence (13) holds. Obviously, if \( \mu_0 + \mu_1 = 0 \), (12) cannot hold. Proposition is proven.

Inverting Laplace transform in (9), we get

\[
L_0(t, \xi) = \frac{\Lambda_1 \mu_1 e^{-\xi B} e^{nt} \Omega_n(t)}{-n + m} - \frac{\Lambda_1 \mu_1 e^{-\xi B} e^{mt} \Omega_m(t)}{-n + m} + \left(\frac{\mu_1 \Psi_n(t) \Lambda_1}{-n + m} - \frac{\mu_0 (\xi \mu_1 + n + \Lambda_1) \Phi_n(t)}{-n + m}\right) e^{nt} + \left(\frac{\mu_1 \Psi_m(t) \Lambda_1}{-n + m} + \frac{\mu_0 (\xi \mu_1 + m + \Lambda_1) \Phi_m(t)}{-n + m}\right) e^{mt} + \left(-\frac{\Lambda_1 e^{nt}}{-n + m} + \frac{\Lambda_1 e^{mt}}{-n + m}\right) L_1(0, \xi) + \left(-\xi \mu_1 + n + \Lambda_1\right) e^{nt} + \left(\xi \mu_1 + m + \Lambda_1\right) e^{mt} \right) L_0(0, \xi),
\]

\[
L_1(t, \xi) = -\frac{\Omega_m(t) \mu_1 (\mu_0 \xi + m + \Lambda_0) e^{-\xi B + nt} \Omega_n(t)}{-n + m} + \frac{\mu_1 (\mu_0 \xi + n + \Lambda_0) e^{-\xi B + nt} \Omega_n(t)}{-n + m} + \left(-\frac{\Lambda_0 \mu_0 \Phi_n(t)}{-n + m} - \frac{\mu_1 (\mu_0 \xi + n + \Lambda_0) \Psi_n(t)}{-n + m}\right) e^{nt} + \left(\frac{\Lambda_0 \mu_0 \Phi_m(t)}{-n + m} + \frac{\mu_1 (\mu_0 \xi + m + \Lambda_0) \Psi_m(t)}{-n + m}\right) e^{mt} + \left(-\frac{\Lambda_0 e^{nt}}{-n + m} - \frac{\Lambda_0 e^{mt}}{-n + m}\right) L_0(0, \xi) + \left(-\xi \mu_0 + n + \Lambda_0\right) e^{nt} + \left(\xi \mu_0 + m + \Lambda_0\right) e^{mt} \right) L_1(0, \xi),
\]

where \( \Pi_k(t) = \int_0^t \pi(\tau) e^{-kr} d\tau \) for \( \pi \in \{\phi, \psi, \omega\} \) and \( k \in \{m, n\} \). Since the process stops with probability one, \( L_0 \) and \( L_1 \) tend to zero as \( t \) tends to infinity for any \( \xi \). In view of Proposition 1, the necessary condition for this is that the coefficients of \( e^{nt} \) tend to zero as \( t \) tends to infinity. This leads to the system of two equations, one of them turns out to be identity and the second one is

\[
L_1(0, \xi) = -\frac{\xi \mu_1 + n + \Lambda_1}{\Lambda_1} L_0(0, \xi) - \frac{\mu_0 (\xi \mu_1 + n + \Lambda_1) \Phi_n}{\Lambda_1} - \frac{\mu_1 \Psi_m + \mu_1 e^{-\xi B} \Omega_m}{-n + m},
\]

where \( \Pi_k = \int_0^{\infty} \pi(\tau) e^{-kr} d\tau \). We now consider the lower boundary condition. Twice integrating (6) for \( s = 1 \) by parts, we get

\[
0 = \frac{\partial}{\partial A} F_1(t, 0) = \xi^2 L_1(t, \xi) + \xi F_1(t, B) e^{-\xi B} - \xi F_1(t, 0) + e^{-\xi B} \frac{\partial}{\partial A} F_1(t, B) - \int_0^B \left(\frac{\partial^2}{\partial A^2} F_1(t, A)\right) e^{-\xi A} dA.
\]

Applying mean theorem for the integral, we get

\[
0 = \frac{\partial}{\partial A} F_1(t, 0) = \xi^2 L_1(t, \xi) + \xi F_1(t, B) e^{-\xi B} - \xi F_1(t, 0) + e^{-\xi B} \frac{\partial}{\partial A} F_1(t, B) - \frac{1}{\xi} \frac{\partial^2}{\partial A^2} F_1(t, \hat{A}(t, \xi)) + \frac{e^{-\xi B}}{\xi} \frac{\partial^2}{\partial A^2} F_1(t, \hat{A}(t, \xi))
\]

for some point \( \hat{A}(t, \xi) \). Tending \( \xi \) to \( +\infty \), we get
\[ 0 = \lim_{\xi \to \infty} (\xi^2 L_1 (t, \xi) - \xi \psi (t)). \]  

Substituting (15) to (14), we obtain

\[
L_1 (t, \xi) = -\frac{\Omega_m (t)}{-n + m} \mu_1 (\mu_0 \xi + m + \Lambda_0) e^{-\xi B + mt} + \frac{\mu_1 (\mu_0 \xi + n + \Lambda_0) e^{-\xi B + nt} \Omega_n (t)}{-n + m} + \\
\left( -\frac{\Lambda_0 \mu_0 \Phi_n (t)}{-n + m} - \frac{\mu_1 (\mu_0 \xi + n + \Lambda_0) \Psi_n (t)}{-n + m} \right) e^{nt} + \\
\left( -\frac{\Lambda_0 \mu_0 \Phi_m (t)}{-n + m} + \frac{\mu_1 (\mu_0 \xi + m + \Lambda_0) \Psi_m (t)}{-n + m} \right) e^{mt} + \left( \frac{e^{nt} \Lambda_0}{-n + m} - \frac{e^{nt} \Lambda_0}{-n + m} \right) L_0 (0, \xi) + \\
\left( -\frac{(\mu_0 \xi + n + \Lambda_0) e^{nt}}{-n + m} + \frac{(\mu_0 \xi + m + \Lambda_0) e^{mt}}{-n + m} \right) + \\
\left( -\frac{(\xi \mu_1 + m + \Lambda_1) L_0 (0, \xi)}{\Lambda_1} - \frac{\mu_0 (\xi \mu_1 + m + \Lambda_1) \Phi_m}{\Lambda_1} - \frac{\mu_1 \Psi_m + \mu_1 e^{-\xi B} \Omega_m}{\Lambda_1} \right). 
\]

Now we apply the following representation of every integral:

\[
\int_0^T \pi (\tau) e^{-k \tau} d\tau = -\frac{\pi (T) e^{-k T}}{k} + \frac{\pi (0)}{k} - \frac{\pi' (T) e^{-k T}}{k^2} + \frac{\pi' (0)}{k^2} + \frac{\pi'' (\theta)}{k^3} - \frac{\pi'' (\theta) e^{-k T}}{k^3}, T \in \{ t, \infty \}. 
\]

Substituting it to (17) and applying Taylor series for big \( \xi \), we get

\[
L_1 (t, \xi) \approx \psi (t) \xi \mu_1 - \psi (t) \Lambda_1 + \phi (t) \Lambda_0 - \psi' (t) \frac{\mu_1 \xi^2}{\mu_1}. 
\]

Substituting (18) to (16) we get

\[
\psi (t) \Lambda_1 - \phi (t) \Lambda_0 + \psi' (t) = 0. 
\]

Substituting (19) to (15) and integrating by parts, we get

\[
L_1 (0, \xi) = \\
\frac{\mu_0 \mu_1 \xi \psi (0) - L_0 (0, \xi) \xi \Lambda_0 \mu_1 + \mu_0 m \psi (0) + \mu_0 \psi (0) \Lambda_1 - L_0 (0, \xi) m \Lambda_0 - L_0 (0, \xi) \Lambda_1 \Lambda_0}{\Lambda_0 \Lambda_1} \\
\frac{(m \xi \mu_0 \mu_1 + \xi \Lambda_1 \mu_0 \mu_1 + m^2 \mu_0 + 2 m \Lambda_1 \mu_0 + \Lambda_0 \Lambda_1 \mu_1 + \Lambda_1^2 \mu_0) \int_0^\infty \psi (\tau) e^{-m \tau} d\tau}{\Lambda_0 \Lambda_1} + \\
\mu_1 e^{-\xi B} \int_0^\infty \omega (\tau) e^{-m \tau} d\tau. 
\]

We now exclude \( \xi \) from (20). To do so, we express \( \xi \) through \( m \) from (10). It can be done in two ways:

\[
\xi_1 = -\frac{\mu_0 m - m \mu_1 - \Lambda_0 \mu_1 - \Lambda_1 \mu_0 + \sqrt{r}}{2 \mu_0 \mu_1}, \xi_2 = -\frac{\mu_0 m + m \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 + \sqrt{r}}{2 \mu_0 \mu_1}, \\
r = \frac{m^2 \mu_0^2 - 2 m^2 \mu_0 \mu_1 + m^2 \mu_1^2 - 2 m \Lambda_0 \mu_0 \mu_1 + 2 m \Lambda_0 \mu_0^2 + 2 m \Lambda_1 \mu_0^2 + 2 m \Lambda_1 \mu_0 \mu_1 + \Lambda_0^2 \mu_1^2 + 2 \Lambda_0 \Lambda_1 \mu_0 \mu_1 + \Lambda_1^2 \mu_0^2. 
\]
We also introduce
\[ U = m \mu_0 + m \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 - \sqrt{r}, \quad W = m \mu_0 + m \mu_1 + \Lambda_0 \mu_1 + \Lambda_1 \mu_0 + \sqrt{r}. \] (22)
Substituting (21) into (20), we get:

\[
L_1 \left( 0, -\frac{U}{2 \mu_0 \mu_1} \right) = -\frac{2 m \mu_0 \psi (0) - 2 \mu_0 \psi (0) \Lambda_1 + U \psi (0)}{2 \Lambda_0 \Lambda_1} + \]
\[
\mu_1 e^{\frac{w_B}{2 \mu_0 \mu_1}} \int_0^\infty \omega (\tau) e^{-m \tau} d\tau + \frac{2 m \mu_0 - 2 \Lambda_1 \mu_0 + U}{2 \Lambda_1 \mu_0} L_0 \left( 0, -\frac{U}{2 \mu_0 \mu_1} \right) \]
\[
\frac{(-2 m^2 \mu_0 - 4 m \Lambda_1 \mu_0 - 2 \Lambda_0 \Lambda_1 \mu_1 - 2 \Lambda_1^2 \mu_0 + U m + U \Lambda_1) \int_0^\infty \psi (\tau) e^{-m \tau} d\tau}{2 \Lambda_0 \Lambda_1},
\]
\[
L_1 \left( 0, -\frac{W}{2 \mu_0 \mu_1} \right) = -\frac{2 m \mu_0 \psi (0) - 2 \mu_0 \psi (0) \Lambda_1 + W \psi (0)}{2 \Lambda_0 \Lambda_1} + \]
\[
\mu_1 e^{\frac{w_B}{2 \mu_0 \mu_1}} \int_0^\infty \omega (\tau) e^{-m \tau} d\tau + \frac{2 m \mu_0 - 2 \Lambda_1 \mu_0 + W}{2 \Lambda_1 \mu_0} L_0 \left( 0, -\frac{W}{2 \mu_0 \mu_1} \right) \]
\[
\frac{(-2 m^2 \mu_0 - 4 m \Lambda_1 \mu_0 - 2 \Lambda_0 \Lambda_1 \mu_1 - 2 \Lambda_1^2 \mu_0 + W m + W \Lambda_1) \int_0^\infty \psi (\tau) e^{-m \tau} d\tau}{2 \Lambda_0 \Lambda_1}.
\]

This is the linear system of equations on two unknowns. We derive its solution as the combination of fundamental solutions. Indeed, let \( \psi_A^0 (t) \) and \( \omega_A^0 (t) \) be solutions of (23) for \( F_0 (0, A) = 0 \), \( F_1 (0, A) = \delta_A (x) \) and \( \psi (0) = 0 \) and \( \psi_A^1 (t) \) and \( \omega_A^1 (t) \) be solutions of (23) for \( F_0 (0, A) = \delta_A (x) \), \( F_1 (0, A) = 0 \) and \( \psi (0) = 0 \) and \( \psi_A^2 (t) \) and \( \omega_A^2 (t) \) be solutions of (23) for \( F_0 (0, A) = 0 \), \( F_1 (0, A) = 0 \). Then the general solution of (23) can be found as

\[
\psi (t) = \int_0^B \psi_A^0 (\tau) F_0 (0, A) dA + \int_0^B \psi_A^1 (\tau) F_1 (0, A) dA + \psi^2 (\tau),
\]
\[
\omega (t) = \int_0^B \omega_A^0 (\tau) F_0 (0, A) dA + \int_0^B \omega_A^1 (\tau) F_1 (0, A) dA + \omega^2 (\tau).
\] (24)

Substituting \( F_0 (0, A) = 0 \), \( F_1 (0, A) = \delta_A (x) \) and \( \psi (0) = 0 \) into (23) and solving the system, we get

\[
\int_0^\infty \omega_A^0 (\tau) e^{-m \tau} d\tau = \frac{2 \Lambda_1 \Lambda_0 \mu_1 \int_0^\infty \omega_A^0 (\tau) e^{-m \tau} d\tau}{S_2} + \frac{\mu_B}{S_2^0 \mu_0 \mu_1},
\]
\[
\int_0^\infty \psi_A^0 (\tau) e^{-m \tau} d\tau = \frac{2 \Lambda_1 \Lambda_0 \mu_1 \int_0^\infty \omega_A^0 (\tau) e^{-m \tau} d\tau}{S_2} + \frac{\mu_A}{S_2 \mu_0} + \frac{\Lambda_0 (m \mu_0 - m \mu_1 - \Lambda_0 \mu_1 + \Lambda_1 \mu_0 + s)}{S_2 \mu_0} e^{\frac{w_B}{2 \mu_0 \mu_1}}.
\] (25)

Substituting \( F_0 (0, A) = \delta_A (x) \), \( F_1 (0, A) = 0 \) and \( \psi (0) = 0 \) into (23) and solving the system, we get
\[
\int_0^\infty \omega_A^1 (\tau) e^{-m \tau} d\tau = \frac{\left( e^{\frac{W_A}{\mu_1}} S_2 + e^{\frac{U_A}{\mu_1}} S_1 \right)}{\mu_1 \left( e^{\frac{W_B}{\mu_1}} S_2 + S_1 e^{\frac{U_B}{\mu_1}} \right)},
\]
\[
\int_0^\infty \psi_A^1 (\tau) e^{-m \tau} d\tau = \frac{2 \Lambda_1 \Lambda_0 \mu_1 \int_0^\infty \omega_A^1 (\tau) e^{-m \tau} d\tau e^{\frac{U_B}{\mu_1}}}{S_2} - \frac{2 \Lambda_1 \Lambda_0}{S_2} e^{\frac{U_A}{\mu_1}}.
\]

Substituting \(F_0(0, A) = 0\) and \(F_1(0, A) = 0\) into (23) and solving the system, we get

\[
\int_0^\infty \omega^2 (\tau) e^{-m \tau} d\tau = \frac{2 s \psi(0)}{\left( e^{\frac{W_B}{\mu_1}} S_2 + S_1 e^{\frac{U_B}{\mu_1}} \right)},
\]
\[
\int_0^\infty \psi^2 (\tau) e^{-m \tau} d\tau = \frac{-\psi(0) (-2 m \mu_0 - 2 \Lambda_1 \mu_0 + U) e^{\frac{W_B}{\mu_1}} + \psi(0) (-2 m \mu_0 - 2 \Lambda_1 \mu_0 + W) e^{\frac{U_B}{\mu_1}}}{\left( e^{\frac{W_B}{\mu_1}} S_2 + S_1 e^{\frac{U_B}{\mu_1}} \right)}.
\]

Hence, in order to derive explicit formulae for \(\omega^0_A, \psi^0_A, \omega^1_A, \psi^1_A, \omega^2, \psi^2\), we need to find corresponding inverse Laplace transforms in (25), (26) and (27). After that \(\omega\) and \(\psi\) can be found from (24) and \(\phi\) can be found from (19). After that \(L_0\) and \(L_1\) can be found from (14), which gives the full description of the dynamics of the process.

**References**

[1] V. Balakrishnan, C. V. der Broeck, and P. Hanggi. First-passage times of non-markovian processes: The case of a reflecting boundary. *Physical Review A*, 38(8):4213–4222, 1988.

[2] L. Beghin, L. Nieddu, and E. Orsingher. Probabilistic analysis of the telegrapher’s process with drift by means of relativistic transformations. *Journal of Applied Mathematics and Stochastic Analysis*, 14(1):11–25, 2001.

[3] Y. Bondarenko. Probabilistic model for description of evolution of financial indices. *Cybernetics and Systems Analysis*, 36(5):738–742, 2000.

[4] J. Chen, F. Liu, and V. Anh. Analytical solution for the time-fractional telegraph equation by the method of separating variables. *Journal of Mathematical Analysis and Applications*, 338(1):1364–1377, 2008.

[5] A. D. Crescenzo and F. Pellerey. On prices’ evolutions based on geometric telegrapher’s process. *Applied Stochastic Models in Business and Industry*, 18(2):171–184, 2002.

[6] S. K. Foong and S. Kanno. Properties of the telegrapher’s random without a trap. *Stochastic Processes and their Applications*, 53(1):147–173, 1994.

[7] S. Goldstein. On diffusion by discontinuous move-ments and on the telegraph equation. *The Quarterly Journal of Mechanics and Applied Mathematics*, 4(2):129–156, 1951.
[8] F. Huang. Analytical solution for the time-fractional telegraph equation. *Journal of Applied Mathematics*, 2009(1):1–9, 2009.

[9] A. Ishimaru. Diffusion of light in turbid material. *Applied Optics*, 28(12):2210–2215, 1989.

[10] D. D. Joseph and L. Preziosi. Heat waves. *Reviews of Modern Physics*, 61(3):41–73, 1989.

[11] M. Kac. A stochastic model related to the telegraphers equation. *Rocky Mountain Journal of Mathematics*, 4(3):497–509, 1974.

[12] D. Kaya. A new approach to the telegraph equation: an application of decomposition method. *Bulletin of the Institute of Mathematics Academia Sinica*, 28(1):51–57, 2000.

[13] O. Lopez and N. Ratanov. Option pricing driven by a telegraph process with random jumps. *Journal of Applied Probability*, 49(3):838–849, 2012.

[14] O. Lopez and N. Ratanov. On the asymmetric telegraph processes. *Journal of Applied Probability*, 51(1):569–589, 2014.

[15] G. B. D. Masi, Y. Kabanov, and W. Runggaldier. Mean-variance hedging of options on stocks with markov volatilities. *Theory of Probability and Its Applications*, 39(1):172–182, 1995.

[16] J. Masoliver, J. M. Porra, and G. H. Weiss. Solution to the telegrapher’s equation in the presence of reflecting and partly reflecting boundaries. *Physical Review E*, 48(2):939–944, 1993.

[17] D. Mugnai, A. Ranfagni, R. Ruggeri, and A. Agresti. Path-integral solution of the telegrapher equation: An application to the tunneling time determination. *Physical Review Letters*, 68(259):259–262, 1992.

[18] E. Orsingher. Probability law, flow function, maximum distribution of wave-governed random motions and their connections with kirchoff’s laws. *Stochastic Processes and their Applications*, 34(1):49–66, 1990.

[19] V. A. Ostapenko. Mixed initial-boundary value problem for telegraph equation in domain with variable borders. *Advances in Mathematical Physics*, 2012(2012):24–28, 2012.

[20] A. A. Pogorni and R. M. Rodriguez-Dagnino. Stationary distribution of random motion with delay in reflecting boundaries. *Applied Mathematics*, 1(1):24–28, 2010.

[21] N. Ratanov. A jump telegraph model for option pricing. *Quantitative Finance*, 7(5):575–583, 2007.

[22] N. Ratanov. Option pricing model based on a markov-modulated diffusion with jumps. *Brazilian Journal of Probability and Statistics*, 24(2):413–431, 2010.

[23] N. Ratanov and A. Melnikov. On financial markets based on telegraph processes. *Stochastics*, 80(2-3):247–268, 2008.