A Faster Algorithm for Betweenness Centrality Based on Adjacency Matrices

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Abstract—Betweenness centrality is essential in complex network analysis; it characterizes the importance of nodes and edges in networks. It is a crucial problem that exactly computes the betweenness centrality in large networks faster, which urgently needs to be solved. We propose a novel algorithm for betweenness centrality based on the parallel computing of adjacency matrices, which is faster than the existing algorithms for large networks. The time complexity of the algorithm is only related to the number of nodes in the network, not the number of edges. Experimental evidence shows that the algorithm is effective and efficient. This novel algorithm is faster than Brandes’ algorithm on small and dense networks and offers excellent solutions for betweenness centrality index computing on large-scale complex networks.

Index Terms—Complex Network, Betweenness Centrality, Adjacency Matrix, Parallel Computing, GPU Acceleration.

1 INTRODUCTION

With the rapid developments of digital society, an increasing number of real networks appear in our lives, which are large scale and complex. Real networks can be regarded as complex networks in different fields, such as industrial networks [1], communication networks [2], transportation networks [3], social networks [4], and the Internet of Things [5]. Research on complex networks has attracted considerable attention from academic fields [6], [7], to deeply study the characteristics of a complex network, network science has emerged [8]. Network science focuses on network modeling, centrality measures and global characteristics of networks, link prediction and recommendation algorithms based on networks, the control and optimization of networks, and network dynamics.

In order to characterize the importance of nodes and edges in the networks, a large volume of research on networks has been devoted to the concept of centrality, such as degree centrality [9], closeness centrality [10], betweenness centrality [11], Katz centrality [12] and PageRank centrality [13]. Betweenness centrality is a core measure used to estimate the influence of vertices and edges in complex networks, and the calculation of betweenness centrality was incorporated into DARPA HPCS SSCA in 2008, which is a benchmark used extensively to evaluate the performance of emerging high-performance computing architectures for graph analysis [14], [15].

The scales of complex networks grow rapidly, which makes comparative centrality analysis with more than one million nodes prohibitive. Brandes’ algorithm is the fastest algorithm for computing betweenness centrality [16], [17], however, it cannot fully meet the demand for analysis of complex networks. There is a bottleneck in the existing algorithm, since Brandes’ algorithm is based on single source shortest-path traversal algorithms and the accumulation technique [4]. To a certain extent, this bottleneck restricts the acceleration of Brandes’ algorithm by parallel computing and thus makes it perform poorly on large-scale networks.

In previous studies on the calculation of betweenness centrality through parallel and distributed computation, there are four main technical schemes: solving multiple single-source shortest-path problems at the same time [14], [18], [19]; accelerating Brandes’ algorithm with GPUs [20], [21], [22], [23]; calculating the shortest paths and then searching for neighbors via parallelized processing after the source node is determined [24]; and dividing and conquering a large network, calculating the subgraphs individually, and synthesizing the calculation results to obtain the betweenness centrality [25], [26]. These studies have focused on sparse networks and have provided a variety of parallel optimizations of Brandes’ algorithm. However, these methods also severely degenerate in performance on dense networks. Exactly computing the betweenness centrality in large-scale net-works faster has become a critical issue in network science, and there is an urgent need for its application.

In this paper, we propose a more efficient algorithm based on adjacency matrix operations for computing betweenness centrality, named DAWN. The feasible network scale for the computation of betweenness centrality can be extended much further with the rapid development of hardware. DAWN has the following characteristics:

(1) DAWN is suitable for sparse networks, dense networks, unweighted networks, weighted networks, undirected networks and directed networks.

(2) The time complexity of DAWN depends only on the number of nodes and is insensitive to the density of the networks. Hence, its performance advantage is more prominent in large-scale networks.

(3) The operations of DAWN are mainly matrix operations which can be accelerated with GPUs, and improvements to hardware will bring greater computation efficiency.

In Section 2, we introduce the theoretical foundation of
betweenness centrality and its typical algorithms on complex networks. In Section 3, we describe the novel algorithm based on the adjacency matrix. In Section 4, we demonstrate the experimental results and a comparative performance analysis with existing algorithms. Finally, we conclude the work of this paper and propose future problems in Section 5.

2 Theoretical Foundation

In this section, we describe the theoretical foundation of this paper, in three aspects: the concept and properties of betweenness centrality, typical algorithms for computing betweenness centrality and an introduction to GPUs.

2.1 Betweenness Centrality

Bavelas et al. proposed betweenness centrality for social network analysis and research [10]. Freeman created a rigorous mathematical definition of betweenness centrality and proposed a method of evaluating the influence of nodes based on betweenness centrality [9]. This method is based on the fact that a critical node with a high frequency on the shortest paths of a network has a greater impact in the network than a subordinate node with a low frequency on the shortest paths of the network, and it has complexities of $\Theta(n^3)$ in time and $\Theta(n^2)$ in space, where $n$ is the number of nodes in the network.

Brandes proposed a fast betweenness centrality algorithm that requires $\Theta(mn)$ space and runs in $\Theta(mn)$ and $\Theta(mn + n^2 \log n)$ time on unweighted and weighted networks, respectively, where $m$ is the number of links [1]. This algorithm has been integrated into a notable open-source project in network analysis “NetworkX” [27].

Construct a connected network $G = (V, E)$ according to the physical structure of the network, where $V$ represents the set of nodes in the network and $E$ represents the set of edges. We use $n$ and $m$ to denote the numbers of nodes and edges in the networks, respectively. Betweenness centrality is usually divided into node betweenness centrality and edge betweenness centrality. The definition of node betweenness is the ratio of the number of shortest paths passing through $v$ between $v_i$ and $v_j$ to the total number of shortest paths between two nodes in the network. The betweenness of node $v$ in the network is defined as:

$$Bet(v) = \sum_{v_i, v_j \in V} \frac{\sigma(v_i, v_j | v)}{\sigma(v_i, v_j)},$$

(1)

where $\sigma(v_i, v_j)$ represents the number of shortest paths passing between $v_i$ and $v_j$ and $\sigma(v_i, v_j | v)$ represents the number of shortest paths passing through $v$ between $v_i$ and $v_j$.

The definition of edge betweenness is similar to that of node betweenness, except that the object is changed from nodes to edges. The betweenness of edge $e$ is defined as:

$$Bet(e) = \sum_{v_i, v_j \in V} \frac{\sigma(v_i, v_j | e)}{\sigma(v_i, v_j)}.$$  

(2)

To facilitate comparative experiments, we adopt the normalized form of betweenness centrality in this paper, which is defined as follows:

$$bet(v) = \frac{2}{(n-1)(n-2)} \times Bet(v),$$

(3a)

$$bet(e) = \frac{2}{n(n-1)} \times Bet(e).$$

(3b)

Wang et al. proved that the sums of the normalized form of betweenness centrality satisfy the following identities in a connected network [28, 29]:

$$\sum_{v \in V} bet(v) = l - 1,$$

(4a)

$$\sum_{e \in E} bet(e) = l.$$

(4b)

where $l$ is the average length of the shortest paths in the network.

2.2 Typical Algorithms of Betweenness Centrality

At present, the most known algorithm that exactly computes the betweenness centrality is Brandes’ algorithm and Matthias et al. proposed its optimization [30]. It is computationally too expensive for us to calculate betweenness centrality based on Brandes’ algorithm in large-scale networks with tens of thousands of nodes. To solve the computation problem, Matteo Riondato et al. proposed an approximate Brandes algorithm (the randomized approximate Brandes algorithm, named the RA-Brandes algorithm) [31].

The motivation to complete operations faster makes approximation algorithms a viable choice in scenarios with low precision requirements. Therefore, approximate calculation methods have become an important research field. In this subsection, we introduce Brandes’ algorithm, the K-shell approximation algorithm and the RA-Brandes algorithm.

2.2.1 Brandes’ Algorithm

Given network $G = (V, E)$, the betweenness centrality of node $v$ is defined as:

$$Bet(v) = \sum_{s \neq t \neq v \in V} \frac{\sigma_{st}(v)}{\sigma_{st}} = \sum_{s \neq t \neq v \in V} Bet_{st}(v).$$

(5)

Brandes proposed the following formula to calculate betweenness centrality:

$$Bet_{s,v}(v) = \sum_{w : v \in P_{sw}(w)} \frac{\sigma_{sw}}{\sigma_{sw}} [1 + Bet_{s,w}(w)],$$

(6)

$Bet_{s,v}(v)$ represents the betweenness centrality of node $v$, which has the shortest path to other nodes, with node $s$ as the source node in the network. As the network becomes increasingly dense, the performance of Brandes’ algorithm tends to degenerate, and it runs in $\Theta(n^2k)$ time on both unweighted and weighted networks, where $k$ is the length of the diameter.
2.2.2 K-Shell Approximation Algorithm

In 1983, Seidman introduced the concept of K-core decomposition [32]. In 2010, Maksim Kitsak et al. proposed the K-shell decomposition algorithm, which can be used for fast approximate calculation of betweenness centrality [33].

The K-shell decomposition algorithm layers the network according to degree, forms subnets with degree k, and extracts some nodes to form a new subnet within each subnet. The new subnets have some of the characteristics of the original networks, and betweenness centrality indices can be calculated quickly, however they are approximate. The open-source project NetworkX integrates this algorithm to perform the approximate computation of betweenness centrality.

2.2.3 RA-Brandes Algorithm

Matteo Riondato et al. proposed the RA-Brandes algorithm [31], which appended a sampling strategy to Brandes’ algorithm by selecting a certain subset of the entire network as a sampling set to calculate the betweenness centrality of node v.

In the open-source project Neo4j, the RA-Brandes algorithm uses two sampling strategies [34]. The first sampling strategy randomly selects nodes from the network to form a sample set. The sampling probability can be adjusted, and the default value is \( \frac{\log N}{\varepsilon^2} \). If the probability is 1, the algorithm runs in the same way as the exact betweenness centrality algorithm. The second sampling strategy is to select nodes by degree. The mean degree of the nodes is calculated, and only the nodes whose degrees are higher than the mean are selected to form the sample set.

The RA-Brandes algorithm pays more attention to nodes with higher degrees, and the K-shell algorithm can retain nodes with diverse degrees.

2.3 GPU Acceleration

A graphics processing unit (GPU) is a visual processor focused on graphics processing. Researchers have found that GPUs are more suitable for large-scale computing than central processing units (CPUs). This led to the birth of the general-purpose graphics processing unit (GPGPU).

NVIDIA has developed a dedicated GPU programming language, CUDA, and released high-performance computing libraries such as cuBLAS [35]. CPU, GPU, FPGA, etc. processors each have fields in which they perform best [36], and GPUs are more suitable for linear algebra. In experiments, the matrix multiplication function in cuBLAS takes only a tenth of the time of a function written based on Openmp. As the scale of the matrix grows, this gap continues to increase. In view of the powerful performance of GPUs in dealing with matrix operations, we utilize a GPU to accelerate the computation in this paper.

3 Designs of the Betweenness Centrality Algorithm

DAWN adopts a GPU for computational acceleration and uses Openmp and CUDA/C++ to compile. This requires \( \Theta(n^2) \) space and \( \Theta(n^{2.387}) \) time (approximate values in the limit state), where n is the number of nodes in the network. DAWN addresses the issue of calculating the betweenness centrality index in large-scale networks.

Classic complex network models mainly include the Erdős–Rényi random network model [37], Barabási–Albert scale-free network model [38], [39], [40], and the Watts–Strogatz small world network model [41], [42]. We choose the ER and BA models as the reference network models for algorithm testing. In this section, we introduce an algorithm for unweighted networks, an algorithm for weighted networks, the theoretical design of auxiliary functions and the parallel computing model of the algorithms.

3.1 Unweighted Networks

The unweighted network \( G = (V, E) \) is conveniently described as an adjacency matrix:

\[
A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix}.
\]

The entry \( a_{ij} \) in adjacency matrix A means that there is an edge with a distance of 1 between node i and node j. We strip node v and the edges connecting it to other nodes from the network, which means setting all the values in row \( v \) and column \( v \) to 0 in adjacency matrix A to obtain a matrix B with node v stripped out. We define the iterative formulas of matrices \( A(k), B(k) \) and \( C(k) \) as:

\[
\begin{align*}
A^{(k+1)} &= A^{(k)} \times A^{(1)}, \\
B^{(k+1)} &= B^{(k)} \times B^{(1)}, \quad 1 \leq k < k + 1 \leq n - 1 \\
C^{(k+1)} &= A^{(k+1)} - B^{(k+1)}.
\end{align*}
\]

(8)

Although the definitions of the three matrices are similar, the slight gaps between them are the focus of the DAWN algorithm. We define them as follows:

1. The element \( a_{ij}^{(k)} \) in matrix \( A^{(k)} \) represents the number of paths with length k between nodes i and j in the network.

2. The element \( b_{ij}^{(k)} \) in matrix \( B^{(k)} \) represents the number of paths with length k that do not pass through node v in going between nodes i and j in the network.

3. The element \( c_{ij}^{(k)} \) in matrix \( C^{(k)} \) represents the number of paths with length k that pass through node v in going between nodes i and j in the network.

Given the above definitions of the matrices, we propose a new betweenness centrality calculation formula:

\[
Bet_{AM}(v) = \sum_{1 \leq k \leq n-1} \sum_{1 \leq i,j \leq n, i \neq j} F_{ij}(k).
\]

(9)

We define the function \( F_{ij}(k) \) as:

\[
F_{ij}(k) = \begin{cases} 
\frac{c_{ij}^{(k)}}{a_{ij}^{(k)}}, & a_{ij}^{(k)} \neq 0 \land i \neq j \land \sum_{1 \leq p \leq k-1} a_{ij}^{(p)} = 0 \\
0, & \text{Otherwise}
\end{cases}
\]

(10)

To better analyze betweenness centrality, we suggest using the normalized formulas (11) and (12) for calculation.

We define the formula used for undirected networks as:

\[
Bet_{AM}(v) = \frac{2}{(n-1) \times (n-2)} \sum_{1 \leq k \leq n-1} F_{ij}(k).
\]

(11)
Similary, the formula used for directed networks is:
\[ \text{bet}_{AM}(v) = \frac{1}{(n-1) \times (n-2)} \sum_{1 \leq i < j \leq n} F_{ij}(k). \] (12)

We use pseudocode to describe the algorithm above.

**Algorithm 1 Unweighted Networks**

**Input:** \( A^{(n)}, B^{(n)} \)

**Output:** \( \text{bet}_{AM}(v) \)

1. function MAIN\( (A^{(n)}, B^{(n)}, \text{bet}_{AM}(v)) \)
2. \( \text{bet}_{AM}(v) \leftarrow 0 \)
3. for \( i = 1 \) to \( n-2 \) do
4. \( A^{(i+1)} \leftarrow A^{(i)} \times A \)
5. \( B^{(i+1)} \leftarrow B^{(i)} \times B \)
6. \( C^{(i+1)} \leftarrow A^{(i+1)} - B^{(i+1)} \)
7. end for
8. for \( k = 1 \) to \( n \) do
9. \( \text{sum} \leftarrow \text{sum} + F(k) \)
10. end for
11. \( \text{bet}_{AM}(v) \leftarrow \frac{2}{(n-1) \times (n-2)} \times \text{sum} \)
12. return \( \text{bet}_{AM}(v) \)
13. end function

### 3.2 Weighted Networks

The weighted network \( G = (V, E, W) \) is conveniently described as an adjacency matrix:

\[
A_w = \begin{bmatrix}
w_{11} & \cdots & w_{1n} \\
\vdots & \ddots & \vdots \\
w_{n1} & \cdots & w_{nn}
\end{bmatrix} .
\] (13)

Let \( w \) be the weight indices of the edges. We assume that \( w \in N \), where \( N \) is the set of natural numbers. In matrix \( A_w \), \( w_{ij} \) represents the weight of the edge between nodes \( i \) and \( j \). First, we define matrices \( A_{w}^{(k)}, B_{w}^{(k)}, C_{w}^{(k)} \) and \( H_{A_w}^{(k)} \).

1. Matrix \( A_{w}^{(k)} \) represents the number of paths with weight \( k \) between nodes \( i \) and \( j \) in the network. Matrix \( A_{w}^{(1)} \) represents the adjacency matrix corresponding to the subnetwork formed by the edges with weight 1 in the adjacency matrix \( A_w \).

2. Matrix \( B_{w}^{(k)} \) represents the number of paths with weight \( k \) that do not pass through node \( v \) between nodes \( i \) and \( j \) in the network. Matrix \( B_{w}^{(1)} \) represents the adjacency matrix corresponding to the subnetwork formed by the edges with weight 1 that do not pass through node \( v \) in the adjacency matrix \( A_w \).

3. Matrix \( C_{w}^{(k)} \) represents the number of paths with weight \( k \) that pass through node \( v \) between nodes \( i \) and \( j \) in the network. Matrix \( C_{w}^{(1)} \) represents the adjacency matrix corresponding to the subnetwork formed by the edges with weight 1 that pass through node \( v \) in the adjacency matrix \( A_w \).

4. Matrix \( H_{A_w}^{(k)} \) represents the adjacency matrix corresponding to the subnetwork formed by the edges with weight \( k \) in the adjacency matrix \( A_w \). If there is no edge with weight \( k \) in the network, then \( H_{A_w}^{(k)} \) is a null matrix, and we also define \( H_{A_w}^{(1)} \) as a null matrix. Second, we define iterative formulas for matrix \( A_{w}^{(k)} \) as follows:

\[
\begin{align*}
A_{w}^{(k+1)} &= A_{w}^{(k)} \times A^{(1)} + H_{A_w}^{(k+1)}, & 1 \leq k < k + 1 \leq w_0 \\
A_{w}^{(k)} &= A_{w}^{(r)} \times A_{w}^{(w_0)}, & w_0 < k < n
\end{align*}
\] (14)

Third, similar to matrix \( A_{w}^{(k)} \), we define iterative formulas for matrix \( B_{w}^{(k)} \) as follows:

\[
\begin{align*}
B_{w}^{(k+1)} &= B_{w}^{(k)} \times B^{(1)} + H_{B_w}^{(k+1)}, & 1 \leq k < k + 1 \leq w_v \\
B_{w}^{(k)} &= B_{w}^{(r)} \times B_{w}^{(w_v)}, & w_v < k < n
\end{align*}
\] (15)

\( H_{B_w}^{(k)} \) represents the adjacency matrix corresponding to the subnetwork formed by the edges with weight \( k \) in the adjacency matrix \( B_{w}^{(k)} \), and \( H_{A_w}^{(1)} \) is a null matrix. \( w_0 \) represents the upper limit of the weight of the weighted network represented by matrix \( A_{w}^{(k)} \), and \( w_v \) represents the upper limit of the weight of the weighted network represented by matrix \( B_{w}^{(k)} \). Finally, according to [8], [14] and [15], we define the matrix \( C_{w}^{(k)} \) as:

\[
C_{w}^{(k)} = A_{w}^{(k)} - B_{w}^{(k)}. \] (16)

We define the variable parameters \( r \) and \( \lambda \) in the above formulas as:

\[
\begin{align*}
\lambda &= k \mod w_0, & \lambda &= \lfloor \frac{k-r}{w_0} \rfloor, \quad \text{In formulas} \ [14] \\
\lambda &= k \mod w_v, & \lambda &= \lfloor \frac{k-r}{w_v} \rfloor, \quad \text{In formulas} \ [15]
\end{align*}
\] (17)

The calculation processes of the unweighted and weighted networks are the same, except for the different matrix definitions and iterative formulas. Finally, we calculate the betweenness centrality of unweighted and weighted networks with [9] ~ [12].

### 3.3 Auxiliary Functions

For a network with \( n \) nodes, DAWN needs at most \( k_0 \) loops to complete the calculation, where \( k_0 \) is the maximum length of the shortest paths. We build auxiliary functions to reduce the number of loops, which are the search function based on the matrix and the threshold function.

#### 3.3.1 Search Function for the Matrix

We define the matrix \( \Gamma^{(n)} \) to register the value of the function \( F(k) \). According to the definition of \( F(k) \), the new operations do not affect the betweenness centrality indices after the matrix is filled, so we suggest ending the loop at this time and calculating \( \text{bet}_{AM}(v) \).

For example, we define the matrix \( \Gamma^{(8)} \) as:

\[
\Gamma^{(8)} = \begin{bmatrix}
# & # & # & # & # & # & # \\
# & # & * & * & * & * & * \\
* & # & * & * & * & * & * \\
* & * & # & * & * & * & * \\
* & * & * & * & # & * & * \\
* & * & * & * & * & # & * \\
* & * & * & * & * & * & #
\end{bmatrix} .
\] (18)

“#” represents that the value of this position is not included in the calculation of the betweenness centrality indices. “*”
represents that the value is to be calculated, and if the value does not appear, it will finally be recorded as 0. $\Gamma^{(8)}$ above corresponds to the operations of betweenness centrality for node 0 in the networks. The values in row $i_0$ and column $i_0$ are not included in the calculations. We do not consider the situation in which the path starts and ends overlap in calculating the shortest path, so we exclude the case of $i = j$.

### 3.3.2 Threshold Function

The threshold function is used to estimate the number of loops based on the average path length of the network. We define the threshold of the Erdős–Rényi network as:

$$K_0 = \lceil \mu \times \frac{\ln(n)}{\ln(deg)} \rceil,$$

(19)

$\lceil \star \rceil$ is the ceiling function, $\mu$ is defined as the tolerance coefficient ($\mu \geq 1$), $\text{deg}$ represents the average degree of the network, and $\frac{\ln(n)}{\ln(deg)}$ represents the average shortest path length in the network. $\langle \text{deg} \rangle$ can be expressed as:

$$\langle \text{deg} \rangle = p \times (n - 1),$$

(20)

where $p$ represents the connection probability of any pair of nodes in the network.

The degree distribution of the BA network obeys $p(k) \sim k^{-\gamma}$, where $\gamma$ is the power exponent. We define the threshold of the Barabasi-Albert network as:

$$K_0 = \lceil \mu \times \ln(\ln(n)) \rceil, \quad 2 < \gamma < 3$$

(21)

$$K_0 = \lceil \mu \times \ln(\ln(n)) \rceil, \quad 2 < \gamma < 3$$

After adding the auxiliary functions, the maximum number of loops is reduced from $n - 1$ to $K_0$, and the time complexity of the algorithm is also reduced. When the number of loops far exceeds $K_0$, special circumstances such as isolated nodes or disconnected subnets in the networks need to be considered. In these circumstances, there is no connected path between some nodes. Even if the operation is performed $n - 1$ times, the matrix $\Gamma^{(n)}$ will not be filled, and the calculation can be ended earlier.

### 3.4 Parallel Computing Model of the Algorithm

We propose a parallel computing model named EVENT parallelism for betweenness centrality calculation, which aggregates the advantages of data parallelism and task parallelism. Event parallelism means dividing and conquering the target event and generating subevents, assigning subevents to each thread, selecting the running processor according to the characteristics of each program in the subevents, supporting data sharing and for load balancing across threads.

The betweenness centrality calculations of the nodes in the networks are independent of each other. We define the betweenness centrality calculation of the entire network as the target event and the betweenness centrality calculation of each node as a subevent. The calculation time is inversely proportional to the number of threads enabled, and thread conflicts are effectively avoided.

### 3.5 Optimized Matrix Algorithm

We achieve acceleration of the matrix algorithms through data precomputation and matrix block multiplication, thereby significantly reducing the time complexity of the algorithm.

#### 3.5.1 Precomputation

The matrix $A^{(k)}$ is constant for nodes in the network, and the betweenness centrality calculation of each node requires the data of matrices $A^{(1)} \sim A^{(k)}$. We calculate this part of the data before dividing and conquering the event and then set it as shared data. These data do not need to be kept in memory; instead, they can be read from the disk when needed.

This reduces the number of matrix multiplication operations from $2k_0n$ to $k_0(n + 1)$, when calculating the betweenness centrality of all nodes; here, $k_0$ represents the number of times the algorithm runs until the matrix $\Gamma^{(k)}$ is filled. This optimization does not fundamentally change the spatial complexity of the algorithm, but it significantly reduces its temporal complexity. For large-scale networks, we recommend storing this part of the data on disk to save the relatively limited available memory.

#### 3.5.2 Matrix Block Multiplication

We adopt the Coppersmith-Winograd matrix multiplication algorithm to propose a novel implementation by GPUs [43], [44], [45]. This novel implementation uses 7 multiplications and 3 nested loops ($O(n^2)$) which are calculated by GPUs and CPUs, respectively, and it can be executed iteratively. The following is the pseudocode for the proposed novel implementation.

**Algorithm 2** Matrix Block Multiplication

**Input:** $A^{(n)}$, $B^{(n)}$

**Output:** $C^{(n)}$

1: function MAIN($A^{(n)}$, $B^{(n)}$, $C^{(n)}$)
2: parallel section 1:
3: $S1 \leftarrow A21 + A22$, $T1 \leftarrow B12 - B11$
4: $S3 \leftarrow A11 - A21$, $T3 \leftarrow B22 - B12$
5: $S2 \leftarrow S1 - A11$, $T2 \leftarrow B22 - T1$
6: parallel section 2:
7: $M5 \leftarrow S1 \times T1$
8: $M7 \leftarrow S3 \times T3$
9: $M6 \leftarrow S2 \times T2$
10: $M1 \leftarrow A11 \times B11$
11: $M2 \leftarrow A12 \times B21$
12: parallel section 3:
13: $C11 \leftarrow M1 + M2$, $U2 \leftarrow M1 + M6$
14: $U3 \leftarrow U2 + M7$, $C22 \leftarrow U3 + M5$
15: $U4 \leftarrow U2 + M5$, $S4 \leftarrow A12 - S2$
16: $T4 \leftarrow T2 - B21$
17: parallel section 4:
18: $M4 \leftarrow S4 \times B22$
19: $M3 \leftarrow A22 \times T4$
20: parallel section 5:
21: $C21 \leftarrow U3 - M4$, $C12 \leftarrow U4 + M3$
22: return $C^{(n)}$
23: end function
We believe that this order of calculations achieves the best balance between temporal and spatial complexity. It requires $\Theta(3n^2 + \frac{7}{2} \omega^2)$ space and $\Theta(n^\omega)$ time, where $2.387 < \omega < 2.81$. Changing the order of the calculations may lead to calculation errors. The details of the implementation code are given in the appendix.

When the order of the matrix is odd, we add a new row and column to the matrix in which all values are set to 0. This operation, which requires at most $\log_2 n$ time, does not affect the result of matrix block multiplication and extends the applicability of the algorithm. The temporal complexity of the algorithm slightly increases to $\Theta((n + \log_2 n)^\omega)$.

## 4 Experimental Results

We analyze the experimental results in this section, which is divided into two subsections: the introduction of the experimental environment and parameters, the experimental results and analysis. The calculation of betweenness centrality for a network with millions of nodes and connection probabilities at the level of tens of percent can be achieved on a single-GPU system. The experimental results show that a multi-GPU system achieves nearly linear speedup.

### 4.1 Experimental Environment Introduction

We compiled the algorithmic solution and compared it with the code based on NetworkX. NetworkX is a Python package for complex networks analysis; it has excellent performance and a wide range of application scenarios [27]. We noticed that the maximum of matrices order in multiplications that TASLE V100 can support is 45000 when DAWN is running, and this upper limit is slightly floating. When the scale of the network is above this upper limit, matrix block multiplication becomes an inevitable choice. Table 1 shows some of the parameters of the test machine.

| Hardware | Parameters |
|----------|------------|
| CPU      | Intel Xeon Gold 6151(2) |
| RAM      | 1TB |
| GPU      | NVIDIA TESLA V100(4) |
| IDE      | Visual Studio 2019 |
| OS       | Windows Server 2019 |
| Toolkit  | CUDA 11.4 |

### 4.2 Results and Analysis

Unweighted undirected networks (UWD) and weighted directed networks (WD) are the models with the highest and lowest operating efficiency for Brandes’ algorithm, respectively. We show the running time of NetworkX and DAWN on Erdős–Rényi networks and Barabasi-Albert networks with 10000 nodes. The unit for the running time is seconds, with 6 significant digits reserved. The time shown in the table is the average time of 5 tests. The scale, density and complexity of the networks will have a significant impact on the efficiency of Brandes’ algorithm. As the connection probability increases, the calculation time of DAWN on the networks decreases. Katzav et al. proposed the phenomenon of network diameter shrinkage [46]. As the density increases, the diameter of the network shrinks and the calculation time decreases.

A. McLaughlin et al. studied how to accelerate Brands’ algorithm with GPUs and used the edge traversal speed to evaluate the algorithm performance [20]. Most of the data used in the research of A. McLaughlin et al. were represented by relatively sparse UWD networks. The public dataset released by Stanford University includes dense networks, such as: Social Circles( Google+) [47]; Email-Eu-core Temporal Network [48]; Dynamic Face-to-Face Interaction Networks [49]; Social Network(MOOC User Action Dataset) [49]; Disease-gene Association Network [50]; Enhanced Tissue-specific Gene Interaction Networks [51].

These dense networks have appeared in datasets in recent years, which, in some sense, indicates that dense networks have gradually become a research focus in the field of complex network analysis. Therefore, we believe that it is also necessary to evaluate the performance of our algorithm on dense networks. The results include experiments running on the UWD-ER model with a connection probability of 0.1. We show that the betweenness centrality calculation time DAWN operations can be completed on a single-GPU system, and multi-GPU acceleration has no significant advantage when the number of nodes in the network is fewer than $2^{15}$. We present separate calculation times for a network with a number of nodes in the range of $2^{10}$ to $2^{15}$.

Figure 1(b) shows that DAWN achieves approximately linear speedup on the multi-GPU systems. A. McLaughlin et al. gave the calculation time of sampling and edge-parallel methods running on delaunay-n20 via 192 GPUs [20], [52], which is a random triangulation network with over a million nodes and an average degree of about 3. DAWN has performance advantages within the scope of the experiment and running on an Erdős–Rényi network with connection probability of 10%.

The distribution of the nodes and edges in a real network is uneven. We performed validation experiments using the real datasets mentioned in this section, and the results show that the computation times for some nodes exceed the values shown in Figure 1, while the times for other nodes are lower than these values. If the node to be calculated is a marginal node or an isolated node in the network, the
addresses the insufficient calculation efficiency of Brandes’ algorithm on dense networks with thousands of nodes and solves the urgent problem in the field of complex network analysis: how to exactly compute the betweenness centrality index on large networks faster. DAWN can be widely used in large-scale network application fields such as social network analysis, information analysis and recommendation on the internet, biological network structure analysis, and communication network vulnerability analysis.

**APPENDIX A**

Please refer to the GITHUB library for the verification code of this paper. https://github.com/lxrzlyr/Betweenness-centrality.git

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