On a Problem of Jaak Peetre Concerning Pointwise Multipliers of Besov Spaces

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Abstract

We characterize the set of all pointwise multipliers of the Besov spaces $B^{s}_{p,q}(\mathbb{R}^d)$ under the restrictions $0 < p, q \leq \infty$ and $s > d/p$.

Key words: Pointwise multipliers; Besov spaces; characterization by differences; localization property of Besov spaces.

1 Introduction and main results

In his famous book \textit{New thoughts on Besov spaces}, page 151, Jaak Peetre posed the problem to determine the set of all pointwise multipliers $M(B^{s}_{p,q}(\mathbb{R}^d))$ of the Besov space $B^{s}_{p,q}(\mathbb{R}^d)$ in case $s > d/p$. Now, more than 40 years later, we are able to present the complete solution to this problem. To describe this we need to introduce a few related classes of functions. Here we are forced to distinguish between the cases $p \leq q$, $q < p < \infty$ and $p = \infty$.

Definition 1.1. Let $\psi \in C_{0}^\infty(\mathbb{R}^d)$ be a nonnegative nontrivial function. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then $B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}$ denotes the collection of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$\|f|B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}\| := \sup_{\lambda \in \mathbb{R}^d} \| f(\cdot) \psi(\cdot - \lambda) |B^{s}_{p,q}(\mathbb{R}^d)\| < \infty.$$ 

The spaces $B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}$ are quasi-Banach spaces independent of the choice of $\psi$ (in the sense of equivalent quasi-norms).

Theorem 1.2. Let $0 < p \leq q \leq \infty$, and $s > d/p$. Then

$$M(B^{s}_{p,q}(\mathbb{R}^d)) = B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}$$

in the sense of equivalent quasi-norms.
In proving this theorem we will make use of the characterization of Besov spaces by differences in a way similar to Strichartz in his paper [19], see also the monographs of Maz’ya and Shaposnikova [6], [7].

**Remark 1.3.** There is a large number of references dealing with pointwise multipliers for Besov or Lizorkin-Triebel spaces. Here we selected only those which include characterizations of $M(\dot{B}^s_{p,q}(\mathbb{R}^d))$.

- Strichartz [19] proved (1.1) for $p = q = 2$. In fact, he was dealing with the more general case of Bessel potential spaces $H^s_p(\mathbb{R}^d)$, $s > d/p$, but $B^s_{2,2}(\mathbb{R}^d) = H^s_2(\mathbb{R}^d)$ (in the sense of equivalent norms). His main tool were consisting in characterizations of $H^s_p(\mathbb{R}^d)$ by differences.

- Peetre [13], page 151, proved (1.1) for $1 \leq p = q < \infty$. He used a method nowadays called paramultiplication, which consists in a clever decomposition of the product in the Fourier image.

- Maz’ya and Shaposnikova, see [7, Theorems 4.1.1, 5.3.1, 5.3.2, 5.4.1], proved (1.1) for $1 \leq p = q < \infty$. Also these authors worked with characterizations by differences.

- Netrusov [9] proved characterizations of $M(\dot{B}^s_{p,q}(\mathbb{R}^d))$ in cases $0 < p = q \leq 1$ and $0 < p \leq 1$, $q = \infty$ in Fourier analytic terms. This has been the first contribution to the case $p \neq q$.

- Sickel [16] proved characterizations of $M(\dot{B}^s_{p,p}(\mathbb{R}^d))$ in terms of capacities for all $p$, $0 < p < \infty$, and all $s > d/p$. The used method here is again paramultiplication in connection with the Fourier analytic description of the spaces.

- Smirnov and S. [17] have shown the identity (1.1) in case $1 \leq p, q < \infty$ by using atomic characterizations of Besov spaces.

- Triebel [25, Proposition 2.22] proved a new characterization of $M(\dot{B}^s_{p,p}(\mathbb{R}^d))$, where either $0 < p \leq \infty$ and $s > d/p$ or $0 < p \leq 1$ and $s = d/p$.

Now we turn to the slightly more complicated case $q < p$. Here we have to introduce spaces with a different type of localization.

**Definition 1.4.** Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative function satisfying

$$\sum_{\mu \in \mathbb{Z}^d} \psi(x - \mu) = 1 \quad \text{for all } x \in \mathbb{R}^d. \quad (1.2)$$

Let $s > 0$, $0 < p, q \leq \infty$, $m \in \mathbb{N}$ and $s < m \leq s + 1$. By using $\psi\mu(\cdot) := \psi(\cdot - \mu)$, $\mu \in \mathbb{Z}^d$, the space $M^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\| f \|_{M^s_{p,q}(\mathbb{R}^d)} := \sup_{\| C_\mu \|_{L^p(\mathbb{Z}^d)}} \left\{ \| f(\cdot) \left( \sum_{\mu \in \mathbb{Z}^d} C_\mu \psi\mu(\cdot) \right) \|_{L^p(\mathbb{R}^d)} \right\}^q \quad (1.3)$$

$$+ \sum_{k=0}^\infty \left( 2^{ksp} \sup_{|h|<2^{-k}} \sum_{\mu \in \mathbb{Z}^d} |C_\mu|^p \| \Delta^m_h(\psi\mu f)(\cdot) \|_{L^p(\mathbb{R}^d)} \right)^{q/p} < \infty.$$
We fix $\nu$. By choosing $C_\mu := \delta_{\mu, \nu}$, $\mu \in \mathbb{Z}^d$, it is easily seen that the right-hand side in (1.3) reduces to the quasi-norm of $\psi_\nu f$ in $B_{p,q}^s(\mathbb{R}^d)$, see Proposition 2.1. This observation yields

$$M_{p,q}^s(\mathbb{R}^d) \hookrightarrow B_{p,q}^s(\mathbb{R}^d)_{\text{unif}}.$$ 

It is not difficult to see that this embedding is proper in case $q < p$ (all details will be given below). We have the following nice characterization of $M(B_{p,q}^s(\mathbb{R}^d))$.

**Theorem 1.5.** Let $0 < q < p < \infty$ and $s > d/p$. Let $\psi_\mu$, $\mu \in \mathbb{Z}^d$, be as in Definition 1.4. Then $f \in M(B_{p,q}^s(\mathbb{R}^d))$ if and only if $\sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f$ belongs to $B_{p,q}^s(\mathbb{R}^d)$ for all $\{C_\mu\}_\mu \in \ell_p(\mathbb{Z}^d)$ and

$$\sup_{\|\{C_\mu\}_\mu \|_{\ell_p(\mathbb{Z}^d)} \leq 1} \left\| \sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f \right\|_{B_{p,q}^s(\mathbb{R}^d)} < \infty.$$ 

The second main result of our paper can now formulated as follows.

**Theorem 1.6.** Let $0 < q < p < \infty$ and $s > d/p$. Then we have

$$M(B_{p,q}^s(\mathbb{R}^d)) = M_{p,q}^s(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms.

We supplement our findings with the more easy case of $p = \infty$.

**Theorem 1.7.** Let $s > 0$ and $0 < q \leq \infty$. Then it holds $M(B_{\infty,q}^s(\mathbb{R}^d)) = B_{\infty,q}^s(\mathbb{R}^d)$ in the sense of equivalent quasi-norms.

**Remark 1.8.** (i) Theorem 1.6 seems to be a novelty. We are not aware of any additional reference in this direction.

(ii) The simple characterization of $M(B_{\infty,q}^s(\mathbb{R}^d))$ has been known, see [17], but [17] was never published.

(iii) Of course, Theorem 1.7 and Theorem 1.2 overlap in case $p = q = \infty$. In this context it is of certain interest to notice that $M(B_{\infty,q}^s(\mathbb{R}^d)) = B_{\infty,q}^s(\mathbb{R}^d)_{\text{unif}}$ if and only if $q = \infty$.

Finally we collect results on the limiting situation $s = d/p > 0$.

**Theorem 1.9.** (i) Let $0 < p = q \leq 1$ and $s = d/p$. Then

$$M(B_{p,p}^s(\mathbb{R}^d)) = B_{p,p}^s(\mathbb{R}^d)_{\text{unif}}$$

holds in the sense of equivalent quasi-norms.

(ii) Let $0 < q < p \leq 1$ and $s = d/p$. Then we have

$$M(B_{p,q}^s(\mathbb{R}^d)) = M_{p,q}^s(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms.

(iii) Let $\psi_\mu$, $\mu \in \mathbb{Z}^d$, be as in Definition 1.4. Under the same restrictions as in (ii) $f \in M(B_{p,q}^s(\mathbb{R}^d))$ if and only if $\sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f$ belongs to $B_{p,q}^s(\mathbb{R}^d)$ for all $\{C_\mu\}_\mu \in \ell_p(\mathbb{Z}^d)$ and

$$\sup_{\|\{C_\mu\}_\mu \|_{\ell_p(\mathbb{Z}^d)} \leq 1} \left\| \sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f \right\|_{B_{p,q}^s(\mathbb{R}^d)} < \infty.$$ 

3
Remark 1.10. (i) In Corollary 3.18 below we shall prove that in the Banach space case Theorem 1.2 and Theorem 1.9 cover all cases where we have the coincidence $M(B_{p,q}^s(\mathbb{R}^d)) = B_{p,q}^s(\mathbb{R}^d)_{\text{unif}}$. (ii) Let $\Omega$ be an open, nontrivial and bounded subset of $\mathbb{R}^d$. Define $B_{p,q}^s(\Omega)$ as the collection of all distributions $g \in D'(\Omega)$ such that there exists some $f \in B_{p,q}^s(\mathbb{R}^d)$ satisfying $f(\varphi) = g(\varphi)$ for all $\varphi \in D(\Omega)$. Equipped with the quotient norm

$$
\| g | B_{p,q}^s(\Omega) \| := \inf \left\{ \| f | B_{p,q}^s(\mathbb{R}^d) \| : f|_{\Omega} = g \right\}
$$

$B_{p,q}^s(\Omega)$ becomes a quasi-Banach space. Intrinsic characterizations, e.g., in the spirit of Proposition 2.4, are known in case of a Lipschitz boundary, we refer to Dispa [2] and Triebel [25, 4.1.4]. Essentially as a consequence of Theorem 3.2 below it follows

$$
M(B_{p,q}^s(\Omega)) = B_{p,q}^s(\Omega)
$$

if $0 < p, q \leq \infty$ and either $s > d/p$ or $s = d/p > 0$ and $0 < q \leq 1$. Hence, the difficulties in determining $M(B_{p,q}^s(\mathbb{R}^d))$ are connected with the unboundedness of $\mathbb{R}^d$ and the difficult localization properties of Besov spaces, see Proposition 3.6.

A short overview on further results in case $s \leq d/p$

As a service for the reader we finish this section with an overview about the knowledge on $M(B_{p,q}^s(\mathbb{R}^d))$ in case $s \leq d/p$.

- Let $1 \leq p < \infty$ and $0 < s \leq d/p$. Then $M(B_{p,p}^s(\mathbb{R}^d))$ has been characterized by Maz’ya and Shaposnikova, see [7, Theorems 4.1.1, 5.3.1, 5.3.2, 5.4.1].

- Let $0 < p = q \leq 1$ and $d\left(\frac{1}{p} - 1\right) < s \leq d/p$. Then $M(B_{p,p}^s(\mathbb{R}^d))$ has been characterized by Netrusov [9]. Here we wish to mention that the description of $M(B_{p,p}^s(\mathbb{R}^d))$ given by Netrusov looks different compared to Theorem 1.9.

- Let $0 < p \leq 1$ and $d\left(\frac{1}{p} - 1\right) < s \leq d/p$. Then $M(B_{p,\infty}^s(\mathbb{R}^d))$ has been characterized by Netrusov [9].

- Let $0 < p = q < \infty$ and $d \max(0, \frac{1}{p} - 1) < s \leq d/p$. Then $M(B_{p,p}^s(\mathbb{R}^d))$ has been characterized by Sickel [16].

- Let $p = \infty$ and $s = 0$. In [5] the spaces $M(B_{\infty,1}^s(\mathbb{R}^d))$ and $M(B_{\infty,\infty}^s(\mathbb{R}^d))$ have been characterized.

- Triebel [25, Theorem 2.25] has found a new characterization of $M(B_{p,p}^s(\mathbb{R}^d))$, $0 < p \leq 1$ and $s > d\left(\frac{1}{p} - 1\right)$, in terms of the quantity

$$
\sup_{\mu \in \mathbb{Z}^d} \sup_{j \in \mathbb{N}_0} \| \psi_\mu(\cdot) f(2^{-j} \cdot) | B_{p,p}^s(\mathbb{R}^d) \|
$$

where $\psi_\mu$ is defined as in Definition 1.4, see also Schneider and Vybiral [15].
The paper will be organized as follows. In Section 2 we collect all what we need about the function spaces under consideration. This will be followed by a short section including basic properties of pointwise multipliers. The next Section is devoted to the proof of Theorem 1.2 including some limiting cases with $s = d/p$. In Section 3.5 we deal with the proof of Theorem 1.6.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. For a real number $a$ we put $a_+ := \max(a, 0)$. The letter $d \in \mathbb{N}$, $d > 1$, is always reserved for the underlying dimension in $\mathbb{R}^d$ and $\mathbb{Z}^d$.

If $X$ and $Y$ are two (quasi-)normed spaces, the (quasi-)norm of an element $x$ in $X$ will be denoted by $\|x\|_X$. The symbol $X \hookrightarrow Y$ indicates that the identity operator is continuous. For two sequences $a_n$ and $b_n$ we will write $a_n \lesssim b_n$ if there exists a constant $c > 0$ such that $a_n \leq c b_n$ for all $n$. We use $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. The topological dual, the class of tempered distributions, is denoted by $S'(\mathbb{R}^d)$ (equipped with the weak topology). The Fourier transform on $S(\mathbb{R}^d)$ is given by

$$
\mathcal{F}\varphi(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} \varphi(x) \, dx, \quad \xi \in \mathbb{R}^d.
$$

The inverse transformation is denoted by $\mathcal{F}^{-1}$. We use both notations also for the transformations defined on $S'(\mathbb{R}^d)$.

2 Besov spaces

General references for Besov spaces are, e.g., the monographs of Nikol’skij [11], Peetre [13] and Triebel [22], [24], [25]. To introduce Besov spaces for the full range of parameters we make use of Fourier analysis, a way, originally introduced by Peetre [13] and later propagated also by Triebel [22], [24], [25].

2.1 Definition and basic properties

Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that $\varphi_0(x) = 1$ if $|x| \leq 1$ and $\varphi_0(x) = 0$ if $|x| \geq 3/2$. For $k \in \mathbb{N}$ we define

$$
\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^d.
$$

Because of

$$
\sum_{k=0}^{\infty} \varphi_k(x) = 1, \quad x \in \mathbb{R}^d,
$$

and

$$
\text{supp } \varphi_k \subset \{ x \in \mathbb{R}^d : 2^{k-1} \leq |x| \leq 3 \cdot 2^{k-1} \}, \quad k \in \mathbb{N},
$$
we call the system \((\varphi_k)_{k \in \mathbb{N}_0}\) a smooth dyadic decomposition of unity on \(\mathbb{R}^d\). Clearly, by the Paley-Wiener-Schwarz theorem,

\[
f_k(x) := \mathcal{F}^{-1}[\varphi_k \mathcal{F}f](x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N}_0.
\]

is a smooth function for all \(f \in \mathcal{S}'(\mathbb{R}^d)\).

**Definition 2.1.** Let \((\varphi_k)_{k \in \mathbb{N}_0}\) be the above system. Let \(0 < p, q \leq \infty\) and \(s \in \mathbb{R}\). Then \(B^s_{p,q}(\mathbb{R}^d)\) is the collection of all tempered distributions \(f \in \mathcal{S}'(\mathbb{R}^d)\) such that

\[
\|f|B^s_{p,q}(\mathbb{R}^d)\|_{\varphi_0} := \left(\sum_{k=0}^{\infty} 2^{ksq} \|\mathcal{F}^{-1}[\varphi_k \mathcal{F}f](\cdot)\|_{L^p(\mathbb{R}^d)}\right)^{1/q} < \infty
\]

(with usual modification if \(q = \infty\)).

Clearly, Besov spaces are quasi-Banach spaces independent of the generator \(\varphi_0\) (in the sense of equivalent quasi-norms). Therefore we will write \(\|f|B^s_{p,q}(\mathbb{R}^d)\|\) instead of \(\|f|B^s_{p,q}(\mathbb{R}^d)\|_{\varphi_0}\).

For us, embeddings into \(L^\infty_{loc}(\mathbb{R}^d)\) and \(L^\infty(\mathbb{R}^d)\) will be of some interest.

**Lemma 2.2.** Let \(0 < p, q \leq \infty\) and \(s \in \mathbb{R}\).

(i) Let \(s > d \max(0, \frac{1}{p} - 1)\). Then \(B^s_{p,q}(\mathbb{R}^d)\) is continuously embedded into \(L^\infty(\mathbb{R}^d)\).

(ii) The Besov space \(B^s_{p,q}(\mathbb{R}^d)\) is continuously embedded into \(L^\infty(\mathbb{R}^d)\) if and only if either \(s > d/p\) or \(s = d/p\) and \(0 < q \leq 1\).

**Remark 2.3.** (i) As it is well-known, \(B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)\) if and only if \(B^s_{p,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)\). Here \(C(\mathbb{R}^d)\) denotes the Banach space of all uniformly continuous functions on \(\mathbb{R}^d\) equipped with the supremum norm.

(ii) The embeddings described in Lemma 2.2 have a certain history. We would like to mention at least Grisvard, Peetre, Golovkin, Stein, Zygmund, Besov, Il’yin and Brudnij. For the Banach space case we refer to the supplement written by Lizorkin in the russian edition of Triebel’s monograph for more details. The general quasi-Banach case has been considered in [18].

**2.2 Tools - the characterization by differences and some inequalities**

For us the characterization of Besov spaces by differences will be more important. In case of a multivariate function \(f : \mathbb{R}^d \to \mathbb{C}, m \in \mathbb{N}, x, h \in \mathbb{R}^d\), we put

\[
\Delta^m_h f(x) := \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{m}{\ell} f(x + \ell h).
\]

The related modulus of smoothness is defined as

\[
\omega_m(f, t)_p := \sup_{|h| < t} \|\Delta^m_h f|L^p(\mathbb{R}^d)\|, \quad t > 0.
\]

**Proposition 2.4.** Let \(0 < p, q \leq \infty\), \(s > d \max(0, \frac{1}{p} - 1)\) and \(s < m\) for some natural number \(m\). Then the Besov space \(B^s_{p,q}(\mathbb{R}^d)\) is a collection of all \(f \in L^p(\mathbb{R}^d)\) such that

\[
\|f|B^s_{p,q}(\mathbb{R}^d)\| := \|f|L^p(\mathbb{R}^d)\| + \left(\sum_{k=0}^{\infty} (2^{ks}\omega_m(f, 2^{-k}))^q\right)^{1/q} < \infty.
\]
Furthermore, \( \| B_{p,q}^s(\mathbb{R}^d) \|_m \) and \( \| B_{p,q}^s(\mathbb{R}^d) \| \) are equivalent on \( L_{\max(1,p)}(\mathbb{R}^d) \) for any admissible \( m \).

**Remark 2.5.** The restriction \( s > d \) \( \max(0, \frac{1}{p} - 1) \) is natural in such a context. Since \( B_{p,q}^s(\mathbb{R}^d) \) contains singular distributions if \( s < d \) \( \max(0, \frac{1}{p} - 1) \) a characterization as in Proposition 2.4 becomes impossible. The version stated in Proposition 2.4 is a direct consequence of Theorem 2.5.12 in [22] using the monotonicity of \( \omega_m(f, t)_p \) with respect to \( t \).

The following construction of a maximal function is essentially due to Peetre, but based on earlier work of Fefferman and Stein. Let \( a > 0 \) and \( b > 0 \) be fixed. For \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) we define the Peetre maximal function \( P_{b,a}f \) by

\[
P_{b,a}f(x) := \sup_{z \in \mathbb{R}^d} \frac{|f(x - z)|}{1 + |bz|^a}, \quad x \in \mathbb{R}^d.
\]

**Proposition 2.6.** Let \( 0 < p \leq \infty \) and define \( \Omega := \{ x : |x| \leq b \} \) for some \( b > 0 \). Let further \( a > d/p \). Then there exists a positive constant \( C \), independent of \( b \), such that

\[
\| P_{b,a}f \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)}
\]

holds for all \( f \in L_{\max(1,p)}(\mathbb{R}^d) \) with \( \text{supp } (\mathcal{F}f) \subset \Omega \).

For a proof we refer to [22, Thm. 1.4.1]. A very useful relation between Peetre maximal function and differences is given by the following lemma, see [27] and also [22, page 102].

**Lemma 2.7.** Let \( a > 0 \) and \( m \in \mathbb{N} \). Then there exists a constant \( C \) such that

\[
|\Delta^m_h f(x)| \leq C \max(1, |bh|^a) \min(1, |bh|^m) P_{b,a}f(x).
\]

holds for all \( b > 0 \), all \( h, x \in \mathbb{R}^d \) and all \( f \in S'(\mathbb{R}^d) \) satisfying \( \text{supp } (\mathcal{F}f) \subset \{ x : |x| \leq b \} \).

Later we shall need also the following modification.

**Lemma 2.8.** Let \( a > 0 \) and \( \psi \in C^k_0(\mathbb{R}^d) \) for some \( k \in \mathbb{N} \). Then, if \( m \in \mathbb{N}, m \leq k \), there exists a constant \( C \) such that

\[
|\Delta^m_h (\psi \cdot f)(x)| \leq C \max\{1, |bh|^a\} \min\{1, |bh|^m\} P_{b,a}f(x)
\]

holds for all \( x, h \in \mathbb{R}^d \) and all \( f \in S'(\mathbb{R}^d) \) such that \( \text{supp } (\mathcal{F}f) \subset \{ x : |x| \leq b \} \), \( b > 0 \). Here \( C \) can be chosen independent of \( b \).

**Proof.** The proof in one dimension can be found in [10]. The general case follows by the same type of argument. 

### 3 Pointwise multipliers

#### 3.1 Some generalities on pointwise multipliers

For a quasi-Banach space \( X \) of functions we shall call a function \( f \) a pointwise multiplier if \( f \cdot g \in X \) holds for all \( g \in X \) (this is includes, of course, that the operation \( g \mapsto f \cdot g \) must
be well defined for all $g \in X$). If $X \hookrightarrow L_p(\Omega)$ for some $p$ (here $\Omega$ is a domain in $\mathbb{R}^d$), as a consequence of the Closed Graph Theorem, we obtain that the linear operator $T_f : g \mapsto f \cdot g$, associated to such a pointwise multiplier, must be continuous in $X$, see [7, p. 33]. As usual we put
\[ \| T_f |\mathcal{L}(X)| := \sup_{\|g\| \leq 1} \| f \cdot g \| X . \]
The collection of all pointwise multipliers for a given space $X$ will be denoted by $M(X)$ and equipped with the quasi-norm
\[ \| f |M(X)| := \| T_f |\mathcal{L}(X)| . \]

Of course, beside the natural interpretation of $f \cdot g$ as a pointwise product of functions, one can define $f \cdot g$ on certain subsets of $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$, in particular on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$. Those extensions of the product will not play an important role within this paper. For that reason we will skip details and refer to Johnsen [3] and the monograph [14, 4.2].

Besov spaces are translation invariant, i.e., \[ \| g(\cdot - \mu) |B^{s}_{p,q}(\mathbb{R}^d)| = \| g |B^{s}_{p,q}(\mathbb{R}^d)| \text{ for all } \mu \in \mathbb{R}^d. \] This implies that the associated multiplier space $M(B^{s}_{p,q}(\mathbb{R}^d))$ is translation invariant as well. Let $f \in M(B^{s}_{p,q}(\mathbb{R}^d))$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative nontrivial function. Since $C_0^\infty(\mathbb{R}^d) \subset B^{s}_{p,q}(\mathbb{R}^d)$ we conclude that $\psi(\cdot - \mu) \cdot f(\cdot) \in B^{s}_{p,q}(\mathbb{R}^d)$ for all $\mu \in \mathbb{R}^d$. By assumption and translation invariance of $B^{s}_{p,q}(\mathbb{R}^d)$ we know that
\[ \| \psi(\cdot - \mu) \cdot f(\cdot) |B^{s}_{p,q}(\mathbb{R}^d)| \leq \| f |M(B^{s}_{p,q}(\mathbb{R}^d))| \| \psi |B^{s}_{p,q}(\mathbb{R}^d)| , \]
which proves $M(B^{s}_{p,q}(\mathbb{R}^d)) \hookrightarrow B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}$.

**Lemma 3.1.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then it follows $M(B^{s}_{p,q}(\mathbb{R}^d)) \hookrightarrow B^{s}_{p,q}(\mathbb{R}^d)_{\text{unif}}$.

### 3.2 On the algebra property of Besov spaces

We shall call a quasi-Banach space of functions $X$ an algebra with respect to pointwise multiplication (for short a multiplication algebra) if $f \cdot g \in X$ for all $f, g \in X$ and there exists a constant $c$ such that
\[ \| f \cdot g \| X \leq c \| f \| X \| g \| X \]
holds for all $f, g \in X$.

**Theorem 3.2.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then $B^{s}_{p,q}(\mathbb{R}^d)$ is a multiplication algebra if and only if one of the following conditions is satisfied

- $s > d/p$;
- $0 < p < \infty$, $s = d/p$ and $q \leq 1$.

**Remark 3.3.** The if-part with $p \geq 1$ has been proved for the first time 1970 in Peetre [12]. But he had called this assertion well-known in [12]. The extension to the quasi-Banach case has been obtained by Triebel [20, 21, 2.6.2]. There also necessity of the above conditions is shown. However, Triebel had overlooked that $B^{s}_{\infty,q}(\mathbb{R}^d)$, $0 < q \leq 1$, is not an algebra. For this correction we refer to [14, 4.6.4, 4.8.3].
3.3 Localized Besov spaces

This subsection has preparatory character.

Let $\psi$ be a non-negative $C^\infty_0(\mathbb{R}^d)$ function such that \( L_2 \) is satisfied. As above we put $\psi_\mu(x) := \psi(x - \mu)$, $\mu \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$.

**Definition 3.4.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

(i) $B^s_{p,q}(\mathbb{R}^d)$ denotes the collection of all $g \in \mathcal{S}'(\mathbb{R}^d)$ such that $\varphi \cdot g \in B^s_{p,q}(\mathbb{R}^d)$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

(ii) Let $0 < u \leq \infty$. Then $B^s_{p,q,v}(\mathbb{R}^d)$ is the collection of all $g \in B^s_{p,q}(\mathbb{R}^d)$ such that

$$\| g \|_{B^s_{p,q,v}(\mathbb{R}^d)} := \left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{B^s_{p,q}(\mathbb{R}^d)}^v \right)^{1/v} < \infty$$

with the usual modification in case $u = \infty$.

**Remark 3.5.** Obviously, the spaces $B^s_{p,q,\infty}(\mathbb{R}^d)$ and $B^s_{p,q}(\mathbb{R}^d)$ unif coincide. In addition we wish to mention that the classes $B^s_{p,q,v}$ are quasi-Banach spaces, independent of $\psi$ in the sense of equivalent quasi-norms.

**Proposition 3.6.** Let $0 < p, q, v \leq \infty$ and $s > d \max(0, \frac{1}{p} - 1)$.

(i) The embedding $B^s_{p,q,v}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)$ holds if and only if $v \leq \min(p, q)$.

(ii) The embedding $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q,v}(\mathbb{R}^d)$ holds if and only if $v \geq \max(p, q)$.

**Proof.** Step 1. Sufficiency.

**Substep 1.1.** Proof of (i). Let $v \leq \min(p, q)$ and let $m$ be a natural number with $m > s$. We suppose $g \in B^s_{p,q,v}(\mathbb{R}^d)$. In case $0 < v < \infty$ this implies $\psi_\mu g \in L_v(\mathbb{R}^d)$ with $r := \max(1, p)$, see Lemma 2.2, and at least formally

$$\left\| \sum_{\mu \in \mathbb{Z}^d} \psi_\mu g \right\|_{L^r(\mathbb{R}^d)} \lesssim \left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{L^r(\mathbb{R}^d)}^r \right)^{1/r} \lesssim \left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{L^r(\mathbb{R}^d)}^v \right)^{1/v} \lesssim \| g \|_{B^s_{p,q,v}(\mathbb{R}^d)}. \quad (3.1)$$

However, convergence of $\sum_{\mu \in \mathbb{Z}^d} \psi_\mu g$ in $L_r(\mathbb{R}^d)$ can be derived from \( (3.1) \) as well (because of $v < \infty$). This implies that $\sum_{\mu \in \mathbb{Z}^d} \psi_\mu g$ is a regular distribution. Hence, the following argument makes sense

$$\left\| \sum_{\mu \in \mathbb{Z}^d} \psi_\mu g \right\|_{B^s_{p,q}(\mathbb{R}^d)}^v \lesssim \left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{L^p_v(\mathbb{R}^d)}^p \right)^{v/p} + \left( \sum_{k=0}^{\infty} \sup_{|h| < 2^{-k}} \left\| \sum_{\mu \in \mathbb{Z}^d} \left| \Delta^m_h (\psi_\mu g)(\cdot) \right| L_p(\mathbb{R}^d) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{v/q} \lesssim \| g \|_{B^s_{p,q,v}(\mathbb{R}^d)}^v + \left( \sum_{k=0}^{\infty} \sup_{|h| < 2^{-k}} \left\| \sum_{\mu \in \mathbb{Z}^d} \left| \Delta^m_h (\psi_\mu g)(\cdot) \right| L_{p/v}(\mathbb{R}^d) \right\|_{L^{q/v}(\mathbb{R}^d)}^{q/v} \right)^{v/q}.$$
Since $\Delta^m_h(\psi_\mu g)(x) \equiv 0$ if $|x - \mu| > c$ for some appropriate positive $c$ (independent of $\mu$) we get
\[
\left( \sum_{\mu \in \mathbb{Z}^d} |\Delta^m_h(\psi_\mu g)(x)|^v \right)^{1/v} \leq C \sum_{\mu \in \mathbb{Z}^d} |\Delta^m_h(\psi_\mu g)(x)|^v
\]
with a constant $C$ independent of $g$ and $x$. Inserting this into the previous inequality and applying triangle inequality with respect to $L_{p/v}(\mathbb{R}^d)$ first, afterwards with respect to $\ell_{q/v}$, we obtain
\[
\left\| \sum_{\mu \in \mathbb{Z}^d} \psi_\mu g \right\|_{B^s_{p,q}(\mathbb{R}^d)} \leq \sum_{\mu \in \mathbb{Z}^d} \left\| \psi_\mu g \right\|_{B^s_{p,q}(\mathbb{R}^d)}.
\]
Because of $g = \sum_{\mu \in \mathbb{Z}^d} \psi_\mu g$ in $L_v(\mathbb{R}^d)$, see [3.1], we have coincidence also almost everywhere and therefore also in $B^s_{p,q}(\mathbb{R}^d)$ by using Proposition 2.4. Hence we conclude
\[
\| g \|_{B^s_{p,q}(\mathbb{R}^d)} \leq \| g \|_{B^s_{p,q,v}(\mathbb{R}^d)}.
\]
This implies sufficiency in (i) in case $0 < v < \infty$. Now we consider the case $v = p = q = \infty$. For each $x \in \mathbb{R}^d$ we choose $\mu \in \mathbb{Z}^d$ such that $x \in \text{supp} \psi_\mu$. Denote
\[
\Omega_\mu := \{ \nu \in \mathbb{Z}^d : \text{dist} (\text{supp} \psi_\mu, \text{supp} \psi_\nu) \leq m \}
\]
and observe that the cardinality of the sets $\Omega_\mu$ is uniformly bounded in $\mu$. For $|h| < 1$, [12] and Proposition 2.4 in case $p = q = \infty$ yield
\[
|h|^{-s} |\Delta^m_h g(x)| \leq \sum_{\nu \in \Omega_\mu} |h|^{-s} |\Delta^m_h (g \psi_\nu)(x)| \leq \sum_{\nu \in \Omega_\mu} |h|^{-s} \left\| \Delta^m_h (g \psi_\nu)(\cdot) \right\|_{C(\mathbb{R}^d)} \leq C \left\| g \right\|_{B^s_{\infty,\infty,\infty}(\mathbb{R}^d)},
\]
which implies
\[
\| g \|_{B^s_{\infty,\infty}(\mathbb{R}^d)} = \| g \|_{C(\mathbb{R}^d)} + \sup_{|h| < 1} \sup_{x \in \mathbb{R}^d} |h|^{-s} |\Delta^m_h g(x)| \leq \| g \|_{B^s_{\infty,\infty,\infty}(\mathbb{R}^d)}.
\]
Substep 1.2. Proof of (ii). Let $g \in B^s_{p,q}(\mathbb{R}^d)$ and assume that $v \geq \max(p, q)$. By Lemma 2.2 we conclude that $g \in L_p(\mathbb{R}^d)$. It follows
\[
\left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{L_p(\mathbb{R}^d)} \right)^{1/v} \leq \left( \sum_{\mu \in \mathbb{Z}^d} \| \psi_\mu g \|_{L_p(\mathbb{R}^d)} \right)^{1/p} \leq \sup_{|h| < 2^{-k}} \| \Delta^m_h (\psi_\mu g)(\cdot) \|_{L_p(\mathbb{R}^d)} \| g \|_{B^s_{p,q}(\mathbb{R}^d)}. \quad (3.2)
\]
Next we consider the term
\[
A(g) := \left\{ \sum_{\mu \in \mathbb{Z}^d} \left( \sum_{k=0}^{\infty} 2^{skq} \sup_{|h| < 2^{-k}} \| \Delta^m_h (\psi_\mu g)(\cdot) \|_{L_p(\mathbb{R}^d)} \right)^{v/q} \right\}^{1/v}
\]
Since $v \geq q$ it follows
\[
A(g) \leq \left\{ \sum_{k=0}^{\infty} 2^{skq} \left( \sum_{\mu \in \mathbb{Z}^d} \sup_{|h| < 2^{-k}} \| \Delta^m_h (\psi_\mu g)(\cdot) \|_{L_p(\mathbb{R}^d)} \right)^{q/v} \right\}^{1/q}.
\]
For the next step of our estimate we shall use the convention that \( \varphi_{\ell} \equiv 0 \) if \( \ell < 0 \) and define
\[
g_{\ell} := \mathcal{F}^{-1}(\varphi_{\ell} \mathcal{F} g), \quad \ell \in \mathbb{Z},
\]
see \((2.1)\). It follows
\[
\psi_\mu g = \psi_\mu \sum_{\ell \in \mathbb{Z}} \mathcal{F}^{-1}(\varphi_{k+\ell} \mathcal{F} g) = \sum_{\ell \in \mathbb{Z}} \psi_\mu g_{k+\ell},
\]
which is valid in \( L_p(\mathbb{R}^d) \), see Lemma \((2.2)\). Hence, using the monotonicity of the \( \ell_r \)-norms, we find
\[
A(g) \leq \left\{ \sum_{k=0}^{\infty} 2^{sk\mu} \left( \sum_{\mu \in \mathbb{Z}^d} \sup_{|h| < 2^{-k}} \left\| \sum_{\ell \in \mathbb{Z}} | \Delta^m_h (\psi_\mu g_{k+\ell})(\cdot) | L_p(\mathbb{R}^d) \right\|^q \right)^{1/q} \right\}^{1/p}.
\]
Temporarily we assume \( p \leq 1 \). By means of \( |a + b|^p \leq |a|^p + |b|^p \), \( a, b \in \mathbb{R} \), this yields
\[
A(g) \leq \left\{ \sum_{k=0}^{\infty} 2^{sk\mu} \left( \sum_{\mu \in \mathbb{Z}^d} \sum_{|h| < 2^{-k}} \sup_{|h| < 2^{-k}} \left\| \Delta^m_h (\psi_\mu g_{k+\ell})(\cdot) | L_p(\mathbb{R}^d) \right\|^p \right)^{1/p} \right\}^{1/q}.
\]
Recall from Substep 1.1, that \( \Delta^m_h (\psi_\mu g_{k+\ell})(x) = 0 \) if \( |x - \mu| > c \). In case \( |x - \mu| < c \) we apply Lemma \((2.8)\) with \( \ell < 0 \) and obtain
\[
\sup_{|h| < 2^{-k}} | \Delta^m_h (\psi_\mu g_{k+\ell})(x) | \leq C 2^{m\ell} P_{2^{k+\ell}, a} g_{k+\ell}(x), \quad x \in \mathbb{R}^d,
\]
which leads to
\[
\sum_{\mu \in \mathbb{Z}^d} \sup_{|h| < 2^{-k}} \left\| \Delta^m_h (\psi_\mu f_{k+\ell})(\cdot) | L_p(\mathbb{R}^d) \right\|^p \leq 2^{m\ell p} \left\| P_{2^{k+\ell}, a} g_{k+\ell}(\cdot) \right\| L_p(\mathbb{R}^d) \right\| \leq 2^{m\ell p} \left\| g_{k+\ell}(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p
\]
as long as we choose \( a > d/p \), see Proposition \((2.6)\). In case \( \ell \geq 0 \) we use the obvious elementary inequality
\[
\sum_{\mu \in \mathbb{Z}^d} \sup_{|h| < 2^{-k}} \left\| \Delta^m_h (\psi_\mu g_{k+\ell})(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p \leq \left\| g_{k+\ell}(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p.
\]
Altogether we have found
\[
\sum_{\mu \in \mathbb{Z}^d} \sup_{|h| < 2^{-k}} \left\| \Delta^m_h (\psi_\mu g_{k+\ell})(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p \leq \min \{ 1, 2^{m\ell p} \} \left\| g_{k+\ell}(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p
\]
for all \( \ell \in \mathbb{Z} \). Inserting this into \((3.4)\) we obtain
\[
A(g) \leq \left\{ \sum_{k=0}^{\infty} 2^{sk\mu} \left( \sum_{\ell \in \mathbb{Z}} \min \{ 1, 2^{m\ell p} \} \left\| g_{k+\ell}(\cdot) \right\| L_p(\mathbb{R}^d) \right\|^p \right)^{1/q} \right\}^{1/q}.
\]
Now, if $q/p \leq 1$, we conclude

\[
A(g) \lesssim \left\{ \sum_{k=0}^{\infty} \left( \sum_{\ell \in \mathbb{Z}} 2^{s(k+\ell)} \| g_{k+\ell} \|_{L_p(\mathbb{R}^d)} \right)^q \right\}^{1/q} \\
\leq \left\{ \sum_{k=0}^{\infty} \left( \sum_{\ell \in \mathbb{Z}} 2^{s(k+\ell)} \| g_{k+\ell} \|_{L_p(\mathbb{R}^d)} \right)^q \right\}^{1/q} \\
\lesssim \| g \|_{B^s_{p,q}(\mathbb{R}^d)}
\]

(3.6)

since $s < m$. If $q/p > 1$ we use the triangle inequality in $\ell_{q/p}$ and find

\[
A(g) \lesssim \left\{ \sum_{k=0}^{\infty} \left( \sum_{\ell \in \mathbb{Z}} 2^{s(k+\ell)} \| g_{k+\ell} \|_{L_p(\mathbb{R}^d)} \right)^q \right\}^{1/p} \lesssim \| g \|_{B^s_{p,q}(\mathbb{R}^d)}.
\]

(3.7)

Summarizing, the embedding $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q,v}(\mathbb{R}^d)$ in case $0 < p \leq 1$ follows from (3.2), (3.6) and (3.7).

Now we turn to the case $p > 1$. Inequality (3.4) yields

\[
A(g) \lesssim \left\{ \sum_{k=0}^{\infty} 2^{sk} \left( \sum_{\ell \in \mathbb{Z}} \left( \sum_{\mu \in \mathbb{Z}^d \mid |h| < 2^{-k}} \sup \| \Delta_{\mu}^{m} (\psi_{\mu} g_{k+\ell}) (\cdot) \|_{L_p(\mathbb{R}^d)} \right)^p \right) \right\}^{1/q} \lesssim \| g \|_{B^s_{p,q}(\mathbb{R}^d)}.
\]

(3.5)

Next we divide into two cases: $q \geq 1$ and $q < 1$. Then we can continue as before and obtain $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q,v}(\mathbb{R}^d)$ also in case $1 \leq p \leq \infty$.

Step 2. Necessity.

Substep 2.1. The $p$-dependence. Without loss of generality we assume that $\psi \equiv 1$ on $[-\delta, \delta]^d$ for some $0 < \delta < 1$ and supp $\psi \subset [-1,1]^d$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a nontrivial function such that supp $\phi \subset [-\delta, \delta]^d$. Then we define a sequence $\{\mu_\ell\}_{\ell=0}^n$ with $\mu_\ell := (4m\ell, 0, \ldots, 0) \in \mathbb{Z}^d$ and a sequence of functions

\[
g_n := \sum_{\ell=0}^{n} C_\ell \phi(x - \mu_\ell), \quad x \in \mathbb{R}^d,
\]

where the real numbers $C_\ell > 0, \ell = 1, \ldots, n$, will be chosen later on. Elementary calculations, based on the use of $\| \cdot \|_{B^s_{p,q}(\mathbb{R}^d)}$, yield

\[
\| g_n \|_{B^s_{p,q,v}(\mathbb{R}^d)} \lesssim \left( \sum_{\ell=0}^{n} C_\ell \right)^{1/v} \quad \text{and} \quad \| g_n \|_{B^s_{p,q}(\mathbb{R}^d)} \lesssim \left( \sum_{\ell=0}^{n} C_\ell^p \right)^{1/p}.
\]

with hidden constants independent of $n$ and $(C_\ell)_\ell$. This implies the relations of $u$ to $p$.

Substep 2.2. The $q$-dependence. What concerns the $q$-dependence it is not longer convenient to work with differences. We will switch to wavelets. Wavelet bases in Besov spaces are a well-developed concept. We refer to the monographs of Meyer [8], Wojtasczyk [28] and Triebel
for the general $d$-dimensional case (for the one-dimensional case see also the book of Kahane and Lemarie-Rieusset [4]). Let $\phi$ be a compactly supported sufficiently smooth scaling function and let $\tilde{\psi}_1, \ldots, \tilde{\psi}_{2d-1}$ be associated wavelets, all defined on $\mathbb{R}^d$. In addition we assume that supp $\tilde{\psi}_1 \subset [-T, T]$ for some $T > 0$. For $j \in \mathbb{N}_0$, $k \in \mathbb{Z}^d$, and $i = 1, \ldots, 2d - 1$, we shall make use of the standard abbreviations in this context:

$$\phi_{0,k}(x) := \phi(x - k) \quad \text{and} \quad \tilde{\psi}_{i,j,k}(x) := 2^{jd/2} \tilde{\psi}_i(2^j x - k), \quad x \in \mathbb{R}^d.$$  

Then any $g \in B_{p,q}^s(\mathbb{R}^d)$ admits an unique representation

$$g = \sum_{k \in \mathbb{Z}^d} a_k \phi_{0,k} + \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} a_{i,j,k} \tilde{\psi}_{i,j,k}$$  

in $\mathcal{S}'(\mathbb{R}^d)$, where

$$a_k := \langle g, \phi_{0,k} \rangle \quad \text{and} \quad a_{i,j,k} := \langle g, \tilde{\psi}_{i,j,k} \rangle.$$  

Moreover,

$$\| g | B_{p,q}^s(\mathbb{R}^d) \| \asymp \left( \sum_{k \in \mathbb{Z}^d} |a_k|^p \right)^{1/p} + \left[ \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} 2^{j(s+d/2)q} \left( \sum_{k \in \mathbb{Z}^d} 2^{-jd} |a_{i,j,k}|^p \right)^{q/p} \right]^{1/q},$$  

(3.9)

in the sense of equivalent quasi-norms; see, e.g., [26, Theorem 1.20]. Now we define

$$g_{\alpha,\mu} := \sum_{j=1}^{\infty} \alpha_j \tilde{\psi}_{1,j,\mu_j}$$  

for some sequence $\alpha := (\alpha_j)_j$ of positive numbers and some sequence $\mu := (\mu_j)_j \subset \mathbb{Z}^d$ to be chosen later on. It follows from (3.9)

$$\| g_{\alpha,\mu} | B_{p,q}^s(\mathbb{R}^d) \| \asymp \left( \sum_{j=1}^{\infty} 2^j \left( s + d \left( \frac{1}{2} - \frac{1}{p} \right) \right) q |\alpha_j|^q \right)^{1/q}$$  

(3.10)

with hidden constants independent of $\alpha$ and $\mu$. Without loss of generality we may assume that our function $\psi$ used in the definition of $B_{p,q,\psi}^s(\mathbb{R}^d)$ satisfies $\psi \equiv 1$ on $[-\delta, \delta]^d$ for some $0 < \delta < 1$ and supp $\psi \subset [-1, 1]^d$. Next we choose a natural number $M$ such that

$$\text{supp} \tilde{\psi}_{1,j,0} \subset [-\delta, \delta]^d \quad \text{for all} \quad j \geq M.$$  

Then it is easily checked that we get

$$\| g_{\alpha,\mu} | B_{p,q,v}^s(\mathbb{R}^d) \| \asymp \left( \sum_{j=1}^{\infty} 2^j \left( s + d \left( \frac{1}{2} - \frac{1}{p} \right) \right) v |\alpha_j|^v \right)^{1/v}$$  

(3.11)

for all sequences $\alpha$ satisfying $\alpha_1 = \ldots = \alpha_M = 0$ and $\mu$ chosen as $\mu_j := (j 2^j, 0, \ldots, 0)$. Based on (3.10) and (3.11), the relations of $q$ to $v$ follow.

**Remark 3.7.** Probably Bourdaud [1] was the first who had considered the classes $B_{p,q,v}^s(\mathbb{R}^d)$ with $p \neq q$ and $v \neq \infty$. He already investigated the embeddings in Proposition 3.6 in case $v = p$.  

13
For convenience of the reader we state one obvious but important consequence of Proposition 3.6.

Corollary 3.8. Let \( 0 < p \leq q \leq \infty \) and \( s > d \max(0, \frac{1}{p} - 1) \). Then \( B^s_{p,q}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d) \) in the sense of equivalent quasi-norms.

Remark 3.9. Corollary 3.8 is well-known, we refer to Peetre [13, page 150] \((p \geq 1)\), Maz’ya and Shaposhnikova [6, Lem. 3.1.1.9], [7, Prop. 4.2.6] \((p \geq 1)\), and Triebel [24, 2.4.7] (general case).

3.4 Proof of Theorem 1.2
The heart of the matter consists in the following proposition.

Proposition 3.10. Let \( 0 < p \leq q \leq \infty \) and \( s > d/p \). Then there exists a constant \( C > 0 \) such that

\[
\| fg | B^s_{p,q}(\mathbb{R}^d) \| \leq C \| g | B^s_{p,q}(\mathbb{R}^d) \| \| f | B^s_{p,q}(\mathbb{R}^d) \| \]  \hspace{1cm} (3.12)

holds for all \( g \in B^s_{p,q}(\mathbb{R}^d) \) and all \( f \in B^s_{p,q}(\mathbb{R}^d) \).

Proof. Let \( m - 1 \leq s < m \). For technical reasons we shall estimate \( \| fg | B^s_{p,q}(\mathbb{R}^d) \|_{2m} \). Let \( \psi, \phi \in C_0^\infty(\mathbb{R}^d) \) are chosen such that the following holds: \( \psi \) is nontrivial and satisfies (1.2) and \( \phi \equiv 1 \) on \( \text{supp} \psi \). As explained in the proof of Proposition 3.6 we have

\[
\| fg | B^s_{p,q}(\mathbb{R}^d) \|_{2m} = \left\| \sum_{\mu \in \mathbb{Z}^d} \phi_\mu g \psi_\mu f | B^s_{p,q}(\mathbb{R}^d) \| \right\|_{2m}.
\]

Clearly,

\[
\left\| \sum_{\mu \in \mathbb{Z}^d} \phi_\mu g \psi_\mu f | L_p(\mathbb{R}^d) \right\| \leq \left\| \sum_{\mu \in \mathbb{Z}^d} |\phi_\mu g| \cdot \|\psi_\mu f | L_\infty(\mathbb{R}^d) \| \right\| L_p(\mathbb{R}^d) \right\| \leq \| g | L_p(\mathbb{R}^d) \| \cdot \sup_{\mu \in \mathbb{Z}^d} \|\psi_\mu f | L_\infty(\mathbb{R}^d) \| \right\| \leq \| g | B^s_{p,q}(\mathbb{R}^d) \| \| f | B^s_{p,q}(\mathbb{R}^d) \| \] \hspace{1cm} (3.13)

where we used in the last step the embedding \( B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \), see Lemma 2.2. Next we need some identities for differences. Note that if \( F, G : \mathbb{R}^d \rightarrow \mathbb{C} \) are two functions and \( n \in \mathbb{N} \) we have

\[
\Delta_h^n(F \cdot G)(x) = \sum_{j=0}^n \binom{n}{j} \Delta_h^{n-j}F(x + jh) \Delta_h^jG(x), \quad x, h \in \mathbb{R}^d.
\]

This can be proved by induction on \( n \). Making use of this formula we obtain

\[
|\Delta_h^{2m} (fg)(x)| \leq \sum_{u=0}^{2m} \binom{2m}{u} \sum_{\mu \in \mathbb{Z}^d} |\Delta_h^{2m-u}(\phi_\mu g)(x + uh) \Delta_h^u(\psi_\mu f)(x)|, \quad h, x \in \mathbb{R}^d.
\] \hspace{1cm} (3.14)

Let \( |h| \leq 1 \). Since \( \Delta_h^n(\phi_\mu g)(x + uh) \equiv \Delta_h^n(\psi_\mu f)(x) \equiv 0 \) if \( |x - \mu| > c \) for some appropriate positive \( c \) (independent of \( \mu \)) this yields

\[
S_u : = \left\{ \sum_{k=0}^{\infty} 2^{ks} \sup_{|h|<2^{-k}} \left\| \sum_{\mu \in \mathbb{Z}^d} \Delta_h^{2m-u}(\phi_\mu g)(\cdot + uh) \Delta_h^u(\psi_\mu f)(\cdot) | L_p(\mathbb{R}^d) \right\|^q \right\}^{1/q} \leq \left\{ \sum_{k=0}^{\infty} 2^{ks} \sup_{|h|<2^{-k}} \left\{ \| \sum_{\mu \in \mathbb{Z}^d} \| \Delta_h^{2m-u}(\phi_\mu g)(\cdot + uh) \Delta_h^u(\psi_\mu f)(\cdot) | L_p(\mathbb{R}^d) \|^p \right\}^{q/p} \right\}^{1/q}
\]

14
for any \( u, 0 \leq u \leq 2m \). To estimate \( S_u \) we have to distinguish into different cases.

**Step 1.** The case \( 0 \leq u < m \). Recall, \( B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \) under the given restrictions, see Lemma 2.2. It follows

\[
\left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot + uh) \Delta^u_h(\psi \mu f)(\cdot) \right\|_{L^p_\mu(\mathbb{R}^d)} \leq \left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot + uh) \right\|_{L^p_\mu(\mathbb{R}^d)} \cdot \left\| \Delta^u_h(\psi \mu f)(\cdot) \right\|_{L_\infty(\mathbb{R}^d)}
\]

\[
\lesssim \left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot + uh) \right\|_{L^p_\mu(\mathbb{R}^d)} \cdot \left\| \psi \mu f \right\|_{L_\infty(\mathbb{R}^d)}
\]

\[
\lesssim \left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot) \right\|_{L^p_\mu(\mathbb{R}^d)} \cdot \left\| f \right\|_{B^s_{p,q}(\mathbb{R}^d)}.
\]

Now we deal with the term \( \{ \ldots \}^{1/q} \) on the right-hand side. As in proof of Proposition 3.6, see formula (3.3), we use the decomposition

\[
\phi \mu g = \sum_{\ell \in \mathbb{Z}} (\phi \mu g_{k+\ell}).
\]

This yields

\[
\left\{ \sum_{k=0}^{\infty} \left( \sum_{|h| < 2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot) \right\|_{L^p_\mu(\mathbb{R}^d)}^{q/p} \right)^{1/q} \right\}
\]

\[
\leq \left\{ \sum_{k=0}^{\infty} \left( \sum_{|h| < 2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \left\| \sum_{\ell \in \mathbb{Z}} \Delta^{2m-u}_h(\phi \mu g_{k+\ell}) (\cdot) \right\|_{L^p_\mu(\mathbb{R}^d)}^{q/p} \right)^{1/q} \right\}.
\]

Since \( 2m - u \geq m \) we can proceed as in Substep 1.2, proof of Proposition 3.6 starting at formula (3.4). As a result we find

\[
S_u \lesssim \| g \|_{B^s_{p,q}(\mathbb{R}^d)} \cdot \| f \|_{B^s_{p,q}(\mathbb{R}^d)} \leq (3.15)
\]

for all \( u, 0 \leq u \leq m \).

**Step 2.** The case \( m < u \leq 2m \). We have

\[
\left\| \Delta^{2m-u}_h(\phi \mu g) (\cdot + uh) \Delta^u_h(\psi \mu f)(\cdot) \right\|_{L^p_\mu(\mathbb{R}^d)} \leq \left\| \Delta^{2m-u}_h(\phi \mu g) \right\|_{L^\infty(\mathbb{R}^d)} \cdot \left\| \Delta^u_h(\psi \mu f) \right\|_{L^p_\mu(\mathbb{R}^d)}
\]

\[
\lesssim \left\| \phi \mu g \right\|_{L^\infty(\mathbb{R}^d)} \cdot \left\| \Delta^u_h(\psi \mu f) \right\|_{L^p_\mu(\mathbb{R}^d)}.
\]

Inserting this into \( S_u \) we find

\[
S_u \leq \left\{ \sum_{k=0}^{\infty} \left( \sum_{|h| < 2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \left\| \phi \mu g \right\|_{L^\infty(\mathbb{R}^d)} \cdot \left\| \Delta^u_h(\psi \mu f) \right\|_{L^p_\mu(\mathbb{R}^d)}^{q/p} \right)^{1/q} \right\}.
\]

(3.16)
Using the triangle inequality in $L_{q/p}$ we arrive at

$$S_u = \left\{ \sum_{\mu \in \Z^d} \left( \| \phi_\mu g \|_{L_\infty(\R^d)}^q \sum_{k=0}^{\infty} 2^{k sq} \sup_{|h| < 2^{-k}} \| \Delta^k_\mu f \|_{L_\mu(\R^d)}^q \right)^{p/q} \right\}^{1/p}$$

$$\leq \left\{ \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{L_\infty(\R^d)} |p| \| \psi_\mu f \|_{B^{s}_{p,q}(\R^d)} |p| \right\}^{1/p}$$

$$\leq \left\{ \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{L_\infty(\R^d)} |p| \right\}^{1/p} \| f \|_{L_{p,q}(\R^d)} \text{unif.}$$

Since $s > d/p$, there exists $\varepsilon > 0$ such that $s - \varepsilon > d/p$. This implies $B^{s-\varepsilon}_{p,p}(\R^d) \hookrightarrow L_{\infty}(\R^d)$, see Lemma 2.2. Hence we obtain the estimate

$$\left( \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{L_\infty(\R^d)} |p| \right)^{1/p} \lesssim \left( \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{B^{s-\varepsilon}_{p,p}(\R^d)} |p| \right)^{1/p} \lesssim \| f \|_{B^{s-\varepsilon}_{p,p}(\R^d)}$$

where we used Corollary 3.8 with respect to $B^{s-\varepsilon}_{p,p}(\R^d)$. The elementary embedding $B^{s}_{p,q}(\R^d) \hookrightarrow B^{s-\varepsilon}_{p,p}(\R^d)$ implies

$$S_u \lesssim \| g \|_{B^{s}_{p,q}(\R^d)} \| f \|_{B^{s}_{p,q}(\R^d)} \text{unif.}$$

(3.18) also in this situation. Summarizing (3.13), (3.14)-(3.18), prove the claim (3.12). \[ \square \]

**Remark 3.11.** Let us mention that the inequality (3.17) is not true in the limiting case $s = d/p$, $p < q \leq 1$. From the Sobolev-type embedding $B^{d/p}_{p,q}(\R^d) \hookrightarrow B^{d/q}_{q,q}(\R^d)$, see [22, 2.7.1], we derive

$$\left( \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{L_\infty(\R^d)} |q| \right)^{1/q} \lesssim \left( \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{B^{d/q}_{q,q}(\R^d)} |q| \right)^{1/q} \lesssim \| g \|_{B^{d/q}_{p,q}(\R^d)}$$

The exponent $q$ is best possible, i.e., if

$$\left( \sum_{\mu \in \Z^d} \| \phi_\mu g \|_{L_\infty(\R^d)} |v| \right)^{1/v} \lesssim \| g \|_{B^{d/p}_{p,q}(\R^d)}$$

holds for all $g \in B^{d/p}_{p,q}(\R^d)$, then $v \geq q$ follows.

**Proposition 3.12.** Let $0 < p \leq 1$ and $s = d/p$. Then there exists a constant $C > 0$ such that

$$\| fg \|_{B^{d/p}_{p,p}(\R^d)} \leq C \| g \|_{B^{d/p}_{p,p}(\R^d)} \| f \|_{B^{d/p}_{p,p}(\R^d)} \text{unif.}$$

holds for all $g \in B^{s}_{p,p}(\R^d)$ and all $f \in B^{s}_{p,p}(\R^d)$.

**Proof.** Let $\phi$ and $\psi$ be as in proof of Proposition 3.10. Employing Corollary 3.8 and the algebra property of $B^{d/p}_{p,p}(\R^d)$, see Theorem 3.2, we get

$$\| fg \|_{B^{s}_{p,p}(\R^d)} \approx \left( \sum_{\mu \in \Z^d} \| (\psi_\mu f)(\phi_\mu g) \|_{B^{s}_{p,p}(\R^d)} |p| \right)^{1/p} \leq \left( \sum_{\mu \in \Z^d} \| \psi_\mu f \|_{B^{s}_{p,p}(\R^d)} |p| \| \phi_\mu g \|_{B^{s}_{p,p}(\R^d)} |p| \right)^{1/p} \lesssim \| f \|_{B^{s}_{p,p}(\R^d)} \| g \|_{B^{s}_{p,p}(\R^d)} \text{unif.}$$

16
which proves the claim. ■

**Remark 3.13.** As already mentioned above this result can be found in Netrusov [9] and Triebel [25, Proposition 2.22].

**Proof of Theorem 1.2.** From Proposition 3.10 we derive $B^s_{p,q}(\mathbb{R}^d) \hookrightarrow M(B^s_{p,q}(\mathbb{R}^d))$, whereas from Lemma 3.1 $M(B^s_{p,q}(\mathbb{R}^d)) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)_{\text{unif}}$ follows.

### 3.5 Proofs of Theorems 1.6 and 1.5

**Lemma 3.14.** Let $0 < p, q \leq \infty$ and $s > d(1/p - 1)_+$. Then we have

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow M^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)_{\text{unif}}.$$

**Proof.** The right-hand embedding has been explained just before Theorem 1.6. Also the left-hand embedding is easily seen. From the trivial inequality

$$\|g| M^s_{p,q}(\mathbb{R}^d)\| \leq \|g| L_p(\mathbb{R}^d)\| + \left\{ \sum_{k=0}^{\infty} \left( \frac{2^{ksp}}{|h|<2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \| \Delta^m_h(\psi_\mu g)(\cdot)| L_p(\mathbb{R}^d)|^p \right)^{q/p} \right\}^{1/q}$$

and the same argument as used in Substep 1.2 of the proof of Proposition 3.6, see formula (3.41), the left-hand embedding follows. ■

**Proof of Theorem 1.6.** Step 1. Let $m$ be a natural number such that $m - 1 < s < m$. We claim that

$$\|fg| B^s_{p,q}(\mathbb{R}^d)\|_{2m} \leq C \|f| M^s_{p,q}(\mathbb{R}^d)\| \|g| B^s_{p,q}(\mathbb{R}^d)\|$$

holds for all $f \in M^s_{p,q}(\mathbb{R}^d)$ and $g \in B^s_{p,q}(\mathbb{R}^d)$. As a first step of the proof we observe that in (3.15) we did not use the condition $p \leq q$. Hence we can apply (3.15) for all terms $S_u$ with $0 \leq u \leq m$. Since $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)_{\text{unif}}$ this is sufficient for us. It remains to deal with the case $m < u \leq 2m$. Starting point for us is formula (3.16)

$$S_u \leq \left\{ \sum_{k=0}^{\infty} \left( \frac{2^{ksp}}{|h|<2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \| \phi_\mu g| L_\infty(\mathbb{R}^d)|^p \cdot \| \Delta^m_h(\psi_\mu f)(\cdot)| L_p(\mathbb{R}^d)|^p \right)^{q/p} \right\}^{1/q}.$$

Now we choose

$$C_\mu := c \frac{\| \phi_\mu g| L_\infty(\mathbb{R}^d)|}{\| g| B^s_{p,q}(\mathbb{R}^d)|}, \quad \mu \in \mathbb{Z}^d,$$

where $c$ will be fixed later on. Of course, here we assume that $\| g| B^s_{p,q}(\mathbb{R}^d)| > 0$, otherwise there is nothing to prove. With a proper choice of $c$, the sequence $\{C_\mu\}_{\mu \in \mathbb{Z}^d}$ belongs to the set $\ell_p^+$ because of

$$\sum_{\mu \in \mathbb{Z}^d} C_\mu^p = c^p \sum_{\mu \in \mathbb{Z}^d} \frac{\| \phi_\mu g| L_\infty(\mathbb{R}^d)|^p}{\| g| B^s_{p,q}(\mathbb{R}^d)|^p} \leq 1,$$

see (3.17). Notice that $c$ can be chosen independent of $g$. Hence, we conclude that

$$S_u \lesssim \| g| B^s_{p,q}(\mathbb{R}^d)| \cdot \left\{ \sum_{k=0}^{\infty} \left( \frac{2^{ksp}}{|h|<2^{-k}} \sum_{\mu \in \mathbb{Z}^d} C_\mu^p \| \Delta^m_h(\psi_\mu f)(\cdot)| L_p(\mathbb{R}^d)|^p \right)^{q/p} \right\}^{1/q}.$$

17
which proves $M_{p,q}^s(\mathbb{R}^d) \hookrightarrow M(B_{p,q}^s(\mathbb{R}^d))$.

**Step 2.** Let $f \in M(B_{p,q}^s(\mathbb{R}^d))$ and $\{C_\mu\}_\mu \in \ell_p(\mathbb{Z}^d)$. We subdivide $\mathbb{Z}^d$ into a finite number of disjoint sets $\Omega_\ell$, $\ell = 1, \ldots, n$, such that

$$\text{dist} \left( \text{supp} \psi_\mu, \text{supp} \psi_{\mu'} \right) \geq 2m,$$

for all $\mu, \mu' \in \Omega_\ell$, $\mu \neq \mu'$, and all $\ell = 1, \ldots, n$. With

$$P_\mu := \{ x \in \mathbb{R}^d : \text{dist} (\text{supp} \psi_\mu, x) \leq m \}, \quad \mu \in \mathbb{Z}^d,$$

we find

$$\sum_{\mu \in \Omega_\ell} |C_\mu|^p \int_{P_\mu} |\Delta_h^m(\psi_\mu f)(x)|^p dx = \int_{\mathbb{R}^d} |\Delta_h^m \left( \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu f \right)(x)|^p dx, \quad |h| < 1.$$

This implies

$$\left\{ \sum_{k=0}^{\infty} \left( 2^{ksp} \sup_{|h| < 2^{-k}} \sum_{\mu \in \mathbb{Z}^d} |C_\mu|^p \| \Delta_h^m(\psi_\mu f)(\cdot) \|_{L_p(\mathbb{R}^d)} \|^{p/q} \right) \right\}^{1/q} \leq \sum_{\ell=1}^{n} \left\{ \sum_{k=0}^{\infty} \left( 2^{ksp} \sup_{|h| < 2^{-k}} \sum_{\mu \in \Omega_\ell} |C_\mu|^p \| \Delta_h^m(\psi_\mu f)(\cdot) \|_{L_p(\mathbb{R}^d)} \|^{p/q} \right) \right\}^{1/q} \leq \sum_{\ell=1}^{n} \left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\|.$$  

(3.19)

Next, we make use of the definition of $M(B_{p,q}^s(\mathbb{R}^d))$

$$\left\| f \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\| \leq \left\| f \right\|_{M(B_{p,q}^s(\mathbb{R}^d))} \cdot \left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\|.  \quad (3.20)$$

A simple calculation leads to

$$\left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\| \leq \left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{L_p(\mathbb{R}^d)} \right\| + \left\{ \sum_{k=0}^{\infty} 2^{ksp} \sup_{|h| < 2^{-k}} \left( \sum_{\mu \in \Omega_\ell} |C_\mu|^p \| \Delta_h^m(\psi_\mu f)(\cdot) \|_{L_p(\mathbb{R}^d)} \|^{p} \right) \right\}^{q/p} \left\| f \right\|_{M(B_{p,q}^s(\mathbb{R}^d))} \cdot \left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\|.$$

Inserting this into (3.20) we find

$$\sup_{\|C_\mu\|_{\ell_p(\mathbb{Z}^d)} \leq 1} \left\| f \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\| \leq \left\| f \right\|_{M(B_{p,q}^s(\mathbb{R}^d))} \cdot \left\| \sum_{\mu \in \Omega_\ell} C_\mu \psi_\mu \|_{B_{p,q}^s(\mathbb{R}^d)} \right\|.  \quad (3.21)$$

for all $\ell = 1, \ldots, n$. Consequently, we obtain

$$\left\| f \right\|_{M(B_{p,q}^s(\mathbb{R}^d))} \leq \left\| f \right\|_{M(B_{p,q}^s(\mathbb{R}^d))}.$$  

The proof is complete.

In Step 2 of the preceding proof we did not use the restrictions in $p$, $q$ and $s$. We only used the possibility to describe the quasi-norm of $B_{p,q}^s(\mathbb{R}^d)$ by differences as explained in Proposition 24. This yields the following.
Lemma 3.15. Let $0 < p, q \leq \infty$ and $s > d(\frac{1}{p} - 1)_+$. Then we have the continuous embedding

$$M(B^s_{p,q} (\mathbb{R}^d)) \hookrightarrow M^s_{p,q} (\mathbb{R}^d).$$

Proof of Theorem 1.5 Step 1. Let $f \in M(B^s_{p,q} (\mathbb{R}^d))$ and $(C_\mu)\mu \in \ell_p (\mathbb{Z}^d)$. Then (3.21) yields what we need.

Step 2. Let $f$ be a function such that $\sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f$ belongs to $B^s_{p,q} (\mathbb{R}^d)$ for all $\{C_\mu\}_\mu \in \ell_p (\mathbb{Z}^d)$ and

$$\sup_{\|\{C_\mu\}_\mu \| \ell_p (\mathbb{Z}^d) \leq 1} \left\| \sum_{\mu \in \mathbb{Z}^d} C_\mu \psi_\mu f \right\|_{B^s_{p,q} (\mathbb{R}^d)} < \infty.$$

By choosing $\{C_\mu\}_\mu$ appropriate it is immediate that $f \in B^s_{p,q} (\mathbb{R}^d)$.

Proof of Theorem 1.7 and Theorem 1.9

3.6 Proof of Theorem 1.7 and Theorem 1.9

Proof of Theorem 1.7. Because of $s > 0$ the space $B^s_{\infty,q} (\mathbb{R}^d)$ forms an algebra with respect to pointwise multiplication, see Theorem 3.2, i.e., $B^s_{\infty,q} (\mathbb{R}^d) \hookrightarrow M(B^s_{\infty,q} (\mathbb{R}^d))$. On the other hand, the function $g \equiv 1$ belongs to $B^s_{\infty,q} (\mathbb{R}^d)$. This implies that a pointwise multiplier $f$ has to belong to $B^s_{\infty,q} (\mathbb{R}^d)$ as well.

Proof of Theorem 1.9. Part (i) follows from Proposition 3.12 and Lemma 3.1. Now we turn to (ii). Let $s = d/p$ and $q \leq \min(1, p)$. The needed modifications of Step 1 of the proof of Theorem 1.6 are based on the estimate

$$\sum_{\mu \in \mathbb{Z}^d} \left\| \phi_\mu f \right\|_{L^\infty (\mathbb{R}^d)}^p \lesssim \sum_{\mu \in \mathbb{Z}^d} \left\| \phi_\mu f \right\|_{B^s_{p,q} (\mathbb{R}^d)}^p \lesssim \left\| f \right\|_{B^s_{p,q} (\mathbb{R}^d)}^p,$$

where we first employed Lemma 2.2 and afterwards Proposition 3.6. This yields sufficiency. Necessity is a consequence of Lemma 3.15.

Finally, the arguments from the proof of Theorem 1.5 carry over to this limiting situation described in (iii).
3.7 Localized Besov spaces as subspaces of $M(\dot{B}^s_{p,q}(\mathbb{R}^d))$ and consequences

Proposition 3.16. Let $0 < q < p \leq \infty$ and $0 < v \leq \infty$. Let either $s > \frac{d}{p}$ or $s = \frac{d}{p}$ and $q \leq 1$. Then we have

$$B^s_{p,q,v}(\mathbb{R}^d) \hookrightarrow M(\dot{B}^s_{p,q}(\mathbb{R}^d))$$

if and only if $\frac{1}{v} \geq \frac{1}{q} - 1/p$.

**Proof.** Step 1. Sufficiency. Proposition [3.6](#equation-3.6) yields that it is enough to consider the case $1/v = 1/q - 1/p$. Let $s < m \leq s + 1$. We shall prove that

$$\| fg \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} \leq C \| g \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} \| f \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)}$$

holds for all $g \in \dot{B}^s_{p,q}(\mathbb{R}^d)$ and $f \in \dot{B}^s_{p,q,v}(\mathbb{R}^d)$. From now on we shall follow the proof of Proposition [3.10](#equation-3.10). Observe $\dot{B}^s_{p,q,v}(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{p,q,\infty}(\mathbb{R}^d) = \dot{B}^s_{p,q}(\mathbb{R}^d)_{\text{unif}}$. Since the condition $p \leq q$ is not needed in Step 1 and Substep 2.1 of the proof of Proposition [3.10](#equation-3.10) it is enough to deal with the case $m < u \leq 2m$. From [3.16](#equation-3.16), $q < p$ and Proposition [2.4](#equation-2.4) we derive that

$$S_u \leq \left\{ \sum_{k=0}^{\infty} \sup_{\| \phi \|_2 < 2^{-k}} \sum_{\mu \in \mathbb{Z}^d} \| \phi \|_{L^p(\mathbb{R}^d)}^{q/p} \cdot \| \Delta^u_k(\phi) \|^{q/p} \right\}^{1/q} \leq \left\{ \sum_{\mu \in \mathbb{Z}^d} \| \phi \|_{L^p(\mathbb{R}^d)}^{q/p} \cdot \sum_{k=0}^{\infty} \left( \sup_{\| \phi \|_2 < 2^{-k}} \| \Delta^u_k(\phi) \|^{q/p} \right) \right\}^{1/q} \leq \left\{ \sum_{\mu \in \mathbb{Z}^d} \| \phi \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)}^{q/p} \cdot \psi_k f \right\}^{1/q}.$$

Now Hölder’s inequality with $\frac{1}{\nu} + \frac{1}{\nu} = \frac{1}{q}$, Lemma [2.2](#equation-2.2) and Corollary [3.8](#equation-3.8) yield

$$S_u \leq \left\{ \sum_{\mu \in \mathbb{Z}^d} \| \phi \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)}^{q/p} \right\}^{1/v} \leq \left\{ \sum_{\mu \in \mathbb{Z}^d} \| \psi_k f \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)}\right\}^{1/v} \leq \| g \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)} \| f \|_{\dot{B}^s_{p,q,v}(\mathbb{R}^d)}.$$

Step 2. Necessity. We shall employ the same type of arguments as in Substep 2.2 of the proof Proposition [3.6](#equation-3.6) see in particular [3.8](#equation-3.8) and [3.9](#equation-3.9). Then we choose

$$f_{M,N,\alpha}(x) := \sum_{j=M}^{N} \alpha_j \overline{\psi}_{1,\mu_j}(x), \quad \mu_j := (4^j,0,\ldots,0)$$

for some sequence $\alpha := (\alpha_j)_{j \geq M}$, $N$ and $M$ (to be fixed later). Define $\gamma_j := 2^j (s + d(\frac{1}{2} - \frac{1}{p})) |\alpha_j|$, $j \geq M$. Then, by making use of the same conventions as in [3.11](#equation-3.11), we obtain

$$\| f_{M,N,\alpha} \|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} \approx \left( \sum_{j=M}^{N} 2^j (s + d(\frac{1}{2} - \frac{1}{p})) |\alpha_j|^q \right)^{1/q} = \left( \sum_{j=M}^{N} |\gamma_j|^q \right)^{1/q}.$$
and
\[ \| f_{M,N,\alpha} |B_{p,q,v}^{s}(\mathbb{R}^d)\| \leq \left( \sum_{j=M}^{N} 2^j \left( \frac{d}{p} \right)^{\frac{1}{p}} |\alpha_j|^{\frac{1}{v}} \right)^{1/v} = \left( \sum_{j=M}^{N} |\gamma_j|^{\frac{1}{v}} \right)^{1/v}. \]

Defining
\[ g_{M,N}(x) := \sum_{j=M}^{N} \psi(x - 2^{-j} \mu_j), \quad x \in \mathbb{R}^d, \]
we conclude
\[ \| g_{M,N} |B_{p,q,v}^{s}(\mathbb{R}^d)\| \leq (N - M)^{1/p}. \]

The lower bound is trivial (it even holds for \( \| g_{M,N} |L_p(\mathbb{R}^d)\| \)). For the proof of the upper bound one uses the information on the supports of the functions \( \psi(\cdot - 2^{-j} \mu_j) \) and Proposition 2.4. Observe, all hidden constants are independent of \( M, N \) and \( \alpha \). By construction we have the identity \( f_{M,N,\alpha} \cdot g_{M,N} = f_{M,N,\alpha} \). Hence, \( B_{p,q,v}^{s}(\mathbb{R}^d) \) holds for some \( \gamma_j \). Applying Proposition 3.16 once again, we find
\[ B_{p,q,v}^{s}(\mathbb{R}^d) \rightarrow M(B_{p,q,v}^{s}(\mathbb{R}^d)) \]
and therefore
\[ \| f_{M,N,\alpha} |B_{p,q,v}^{s}(\mathbb{R}^d)\| = \| f_{M,N,\alpha} \cdot g_{M,N} |B_{p,q,v}^{s}(\mathbb{R}^d)\| \leq c \| f_{M,N,\alpha} |B_{p,q,v}(\mathbb{R}^d)\| \| g_{M,N} |B_{p,q,v}(\mathbb{R}^d)\| \]
for some \( c > 0 \). Consequently
\[ \left( \sum_{j=M}^{N} |\gamma_j|^{\frac{1}{q}} \right)^{1/q} \leq c' \left( \sum_{j=M}^{N} |\gamma_j|^{\frac{1}{v}} \right)^{1/v} (N - M)^{1/p} \]
holds for some \( c' \) independent of \( N, M \) and \( \alpha \). Choosing \( \gamma_j = 1 \) for all \( j \) the necessity of \( \frac{1}{v} \geq \frac{1}{q} - \frac{1}{p} \) follows.

**Remark 3.17.** Proposition 3.16 in case \( 1 \leq q < p \leq \infty \) can be found in [17].

Proposition 3.16 has an interesting consequence. From Lemma 3.1 we know that always \( M(B_{p,q,v}^{s}(\mathbb{R}^d)) \rightarrow B_{p,q,v}^{s}(\mathbb{R}^d) \) holds. Now we ask for coincidence.

**Corollary 3.18.** Let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \). Then \( M(B_{p,q,v}^{s}(\mathbb{R}^d)) \) coincides with \( B_{p,q,v}^{s}(\mathbb{R}^d) \) if and only if either \( 1 \leq p \leq q \leq \infty \) and \( s > d/p \) or \( p = q = 1 \) and \( s = d \).

**Proof.** Step 1. Necessity. From \( M(B_{p,q,v}^{s}(\mathbb{R}^d)) = B_{p,q,v}^{s}(\mathbb{R}^d) \) we conclude \( B_{p,q,v}^{s}(\mathbb{R}^d) \rightarrow M(B_{p,q,v}^{s}(\mathbb{R}^d)) \). Now Theorem 3.2 yields either \( s > d/p \) or \( 0 < p < \infty \), \( 0 < q \leq 1 \) and \( s = d/p \). Next we employ Proposition 3.16. Hence, if \( s > d/p \) then \( B_{p,q,v}^{s}(\mathbb{R}^d) \) \( \subset \) \( M(B_{p,q,v}^{s}(\mathbb{R}^d)) \) if \( 0 < q < p \leq \infty \). Now we turn to \( s = d/p \). Applying Proposition 3.16 once again, we find \( B_{p,1,v}^{d/p}(\mathbb{R}^d) \) \( \subset \) \( M(B_{p,1,v}^{d/p}(\mathbb{R}^d)) \) if \( 1 < p \leq \infty \).

Step 2. Sufficiency. This follows immediately from Corollary 3.8 and Theorem 1.2.

**Proof of Remark 1.10** Theorem 3.2 yields
\[ \| f \cdot g |B_{p,q,v}(\Omega)\| \leq c \| f |B_{p,q,v}(\Omega)\| \| g |B_{p,q,v}(\Omega)\| \]
whenever this inequality is true on \( \mathbb{R}^d \). Hence, \( B_{p,q,v}^{s}(\mathbb{R}^d) \rightarrow M(B_{p,q,v}^{s}(\Omega)) \) under the restrictions of Theorem 3.2. On the other hand, the function \( f \equiv 1 \) belongs to all spaces \( B_{p,q,v}^{s}(\Omega) \), since \( \Omega \) is bounded. Hence, \( f \in B_{p,q,v}^{s}(\Omega) \) is also a necessary condition for \( f \) to belong to \( M(B_{p,q,v}^{s}(\Omega)) \).
References

[1] G. Bourdaud, Localisations des espaces de Besov, *Studia Math.* 90 (1988), 153–163.

[2] S. Dispa, Intrinsic characterizations of Besov spaces on Lipschitz domains, *Math. Nachr.* 260 (2003), 21–33.

[3] J. Johnsen, Pointwise multiplication of Besov and Triebel-Lizorkin spaces, *Math. Nachr.* 175 (1995), 85–133.

[4] J.-P. Kahane and P.-G. Lemarie-Rieuser, *Fourier series and wavelets*, Gordon and Breach Publ., 1995.

[5] H. Koch and W. Sickel, Pointwise multipliers of Besov spaces of smoothness zero and spaces of continuous functions, *Rev. Mat. Iberoamericana* 18 (2002), 587–626.

[6] V.G. Maz’ya and T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman, Boston, 1985.

[7] V.G. Maz’ya and T.O. Shaposhnikova, *Theory of Sobolev multipliers with applications to differential and integral operators*, Springer, Berlin, 2009.

[8] Y. Meyer, *Wavelets and operators*, Cambridge Univ. Press, Cambridge, 1992.

[9] Yu. V. Netrusov, Theorems on traces and multipliers for functions in Lizorkin-Triebel spaces, *Zap. Nauchn. Sem. St-Petersburg, Otdel. Mat. Inst. Steklov (POMI)* 200 (1992), 132–138. English translation in: *J. Math. Sci.* 77 (1995), 3221–3224.

[10] V.K. Nguyen, M. Ullrich and T. Ullrich, Change of variable in spaces of mixed smoothness and numerical integration of multivariate functions on the unit cube, *Constr. Approx.* (2017), DOI: 10.1007/s00365-017-9371-9.

[11] S.M. Nikol’skij, *Approximation of functions of several variables and imbedding theorems*, Springer, Berlin, 1975.

[12] J. Peetre, Interpolation of Lipschitz operators and metric spaces, *Mathematica (Cluj)* 12 (1970), 1–20.

[13] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Press, Durham, 1976.

[14] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications 3, Walter de Gruyter & Co., Berlin, 1996.

[15] C. Schneider and J. Vybiral, Non-smooth atomic decompositions, traces on Lipschitz domains, and pointwise multipliers in function spaces, *J. Funct. Anal.* 264 (2013), 1197–1237.

[16] W. Sickel, On pointwise multipliers for $F_{p,q}^s (\mathbb{R}^n)$, the case $\sigma_{p,q} < s < n/p$, *Ann. Mat. Pura Appl. (4)* 176 (1999), 209–250.
[17] W. Sickel and I. Smirnow, Localization properties of Besov spaces and its associated multiplier spaces, Jenaer Schriften Math/Inf 21/99, Jena, 1999.

[18] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type, Z. Anal. Anwendungen 14 (1995), 105–140.

[19] R. S. Strichartz, Multiplier on fractional Sobolev spaces, J. Math. Mech. 16 (1967) 1031–1060.

[20] H. Triebel, Multiplication properties of the spaces $B^s_{p,q}$ and $F^s_{p,q}$. Quasi-Banach algebras of functions, Ann. Mat. Pura Appl. (4) 113 (1977), 33–42.

[21] H. Triebel, Besov-Sobolev-Hardy spaces, Teubner-Texte Math. 9, Teubner, Leipzig, 1978.

[22] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[23] H. Triebel, Theory of function spaces, (Russian), Moscow, Mir, 1986.

[24] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

[25] H. Triebel, Theory of function spaces III, Birkhäuser, Basel, 2006.

[26] H. Triebel, Function spaces and wavelets on domains, EMS Publishing House, Zürich, 2008.

[27] T. Ullrich, Function spaces with dominating mixed smoothness. Characterizations by differences, Jenaer Schriften zur Mathematik und Informatik Math/Inf/05/06, Jena, 2006.

[28] P. Wojtaszczyk, A mathematical introduction to wavelets, Cambridge Univ. Press, Cambridge, 1997.