On reducible partition of graphs

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Abstract

An undirected graph $H$ is called a minor of the graph $G$ if $H$ can be formed from $G$ by deleting edges and vertices and by contracting edges. If $G$ does not have a graph $H$ as a minor, then we say that $G$ is $H$-free. Hadwiger conjecture claim that the chromatic number of $G$ may be closely related to whether it contains $K_{n+1}$ minors. To study the color of a $K_{n+1}$-free $G$, we propose a new concept of reducible partition of vertex set $V_G$ of $G$. A reducible partition of a graph $G$ with $K_n$ minors and without $K_{n+1}$ minors is defined as a two-tuples $\{S_1 \subseteq V_G, S_2 \subseteq V_G\}$ which satisfy the following condisions:

(a) $S_2$ is dominated by $S_1$,
(b) the induced graph $G[S_1]$ is a forest,
(c) the induced graph $G[S_2]$ is $K_n$-free.

We will show that the reducible partition always exist and it can be used for study the color of $G$.

Keywords: Reducible partition, Graph color

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1. Exhaustive Reducible Partition(ERP) of graphs

Let $G = (V_G, E_G)$ be a graph with vertex set $V_G$ and edge set $E_G$. A subset $S \subset V_G$ is called a dominating set of $G$ if each vertex in $V - S$ is adjacent to at least one vertex of $S$. When the subgraph $G[S]$ induced by $S$ is a forest $F$, then
the forest $F$ is called a dominating forest (DF) of the graph $G$. If $G[S]$ is not a subgraph of any other dominating forests of $G$, then $G[S]$ is called a maximal DF. A DF $G[S]$ is maximum if the cardinality is the largest among all DFs.

If $F$ is one of maximal DFs of $G$, then we say $V - V_F$ is dominated by $V_F$ or $V - V_F$ is the dominated part of $V_G$, where $V_F$ is the vertex set of the graph $F$. The two-tuples $\{V_F, V - V_F\}$ is called a dominating forest partition of $G$. We will show that the dominating forest partition of $G$ can be obtained by using depth first search (DFS) of graphs.

First of all, we will obtain a the vertex sequence $L$ of graphs by using DFS. And then, we can pick vertices to form a DF from the sequence $L$ by checking whether $G[v \cup S_1]$ has cycles, where $S_1$ is the set of selected vertices and $v$ is the vertex to be checked. If $G[v \cup S_1]$ has no cycles, then $v$ will be put into $S_1$.

The pseudocode is listed as follow:

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**Algorithm for Finding a maximal dominating forest (AFMDF)**

**Require:** A given graph $G$ with $N$ vertices

**Ensure:** A dominating forest $S_1$

1: $L \leftarrow DFS(G)$.
2: $S_1 \leftarrow \{\}$
3: add $L(1)$ and $L(2)$ to $S_1$
4: for $i = 3 \rightarrow N$ do
5:   if $G[L(i) \cup S_1]$ has no cycles then
6:     add $L(i)$ to $S_1$.
7:   end if
8: end for
9: return $S_1$

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In the pseudocode of algorithm AFMDF, $\text{length}(S_1)$ is used to denote the number of elements in $S_1$, $DFS(G)$ is used to denote the depth first search (DFS) of graph $G$ and its output is a sequence of vertex of $G$. With the above algorithm, the following theorem can be obtain:
**Theorem 1.** Any graph $G$ possesses a DF.

*Proof.* The theorem can be proved by showing that the algorithm AFDF in Table 1 can always find a dominating forest. The result of AFDF is $S_1$, then $G[S_1]$ is a forest. Suppose that $G[S_1]$ is not a dominating forest, then there exists a vertex $v$ of $V_G - S_1$ such that $v$ has no neighbors in $S_1$, however from AFDF if such $v$ exists, then it will be added to $S_1$. And from the AFDF, if a vertex $u$ is not in $S_1$, then $u$ must have at least two neighbors in $S_1$, this means that $G[S_1]$ is a maximal DF. \qed 

**Lemma 1.** $G$ is a given graph and $S_1 \subseteq V_G$ is a subset of $V_G$. If the induced subgraph $G[S_1]$ is a maximal dominating tree, then for every vertex $v \in V_G - S$ there are at least 2 neighbors in one tree of $S$.

*Proof.* We prove the lemma by contradiction. As $S$ is a dominating tree, then $v$ has at least one neighbor in $S$. Assume that $v \in V_G - S$ only has one neighbor in one tree of $G[S]$, then $v$ can be added into $S$ and $S + v$ is also a dominating tree. Which contradicts that $S$ is maximal. \qed 

**Theorem 2.** If $\{S_1, S_2\}$ is a dominating forest partition graph $G$ and $G[S_1]$ is a tree, then $\{S_1, S_2\}$ is a reducible partition of $G$.

*Proof.* The theorem can be proved by contradiction. Assume that a $K_{n+1}$-free $G$ contains $K_n$ minors and $\{S_1, S_2\}$ is not a reducible partition of $G$, then $G[S_2]$ has at least one $K_n$ minor, say $H$. From lemma 1, every vertex of $H$ has two neighbors in $S_1$, hence by contracting all edges of $G[S_1]$, one can obtain a $K_{n+1}$ minor, which contradicts that $G$ is $K_{n+1}$-free. \qed 

If the dominating forest $G[S_1]$ contains multiple disjoint tree, then $G[S_2]$ may contain a $K_n$, one example is shown in the following figure. For such partition $\{S_1, S_2\}$ of $G$ such that $G[S_1]$ contains $k$ disjoint trees $T_i, i = 1, \cdots, k$, we will obtain the reducible partition by introduce the concept of minimal $K_{n+1}$-free subgraph.
Figure 1: A $K_5$-free graph $G$. $T_1$ and $T_2$ are two disjoint trees in dominating forest $G[S_1]$, the red vertex set $S_2$ constitute a $K_4$. As both $G$ and $G[S_2]$ contain a $K_4$, then $\{S_1, S_2\}$ is not a reducible partition.

**Definition 1.** A $H$ is a $K_{n+1}$ minor of $G$, if $H'$ obtained from $H$ by deleting any edge is $K_{n+1}$-free, then $H$ is called a minimal $K_{n+1}$ subgraph or a minimal $K_{n+1}$ minor.

From the definition, $H'$ obtained by deleting any vertex is also $K_{n+1}$-free.

**Theorem 3.** If $G$ contains only one minimal $K_{n+1}$ minor $H$, then $G'$ obtained from $G$ by deleting any edge $e$ of $H$ is $K_{n+1}$-free.

*Proof.* The theorem can be proved by contradiction. Assume that the theorem is not true, then $G'$ has one $K_{n+1}$ minor $H'$. Because $H$ is the only one minimal $K_{n+1}$ minor, then $H$ is the intersection of all other subgraphs homeomorphic to $K_{n+1}$. Hence $H$ is a subgraph of $H'$, however $H'$ does not contain the deleted edge $e$, namely $H$ is not a subgraph of $H'$, which a contradiction. \(\square\)

**Theorem 4.** If $G$ contains $m$ minimal $K_{n+1}$ minors $H_i, i = 1, \cdots, m$ and the intersection of the $m$ $K_{n+1}$ minors $H$ is not null, then $G'$ obtained from $G$ by deleting any edge $e$ of $H$ is $K_{n+1}$-free.

*Proof.* As every $H_i, i = 1, \cdots, m$ is a minimal $K_{n+1}$ minor, then any $K_{n+1}$ minor $H_{n+1}$ contains at least one of $H_i$ as subgraph. Assume the theorem is not true,
then $G'$ has one $K_{n+1}$ minor $H'$ which does not contain any $H_i, i = 1, \ldots, m$ as subgraph which a contradiction.

**Theorem 5.** Let $A_{ij}, i, j = 1, \ldots, m$ are intersections of every pair of $K_{n+1}$ minors $H_i$ and $H_j$ of $G$, if $A_{ij} = \emptyset$ is null then $A_{ij} = H_i$ and $A_{ji} = H_j$. $G'$ obtained from $G$ by deleting $|A|$ edge $e_k \in E_{A_k}, k = 1, \ldots, |A|$ is $K_{n+1}$-free.

**Proof.** As $A_{i}, i = 1, \ldots, |A|$ are intersections of every pair of $K_{n+1}$ minors $H_j$ and $H_k$ of $G$, then when $m$ edges $e_i \in E_{A_i}, i = 1, \ldots, m$ is deleted, there is no minimal $K_{n+1}$ minors, $G'$ is $K_{n+1}$-free.

**Definition 2.** For a $K_{n+1}$-free graph $G$ contains $m$ minimal $K_n$ minors $H_i, i = 1, \ldots, m$. A vertex set $\{v_i \in V_{H_i}, i = 1, \ldots, m\}$ is defined as a critical $K_n$ forest if $G\left[ \bigcup_{i=1}^{m} v_i \right]$ is a forest.

**Lemma 2.** For any $K_{n+1}$-free graph $G$ contains 3 minimal $K_n$ minors $H_i, i = 1, 2, 3$, critical $K_n$ forests always exist.

**Proof.** We will prove the theorem by contradiction. Assume that there is no critical $K_n$ forests in $G$, then for any vertex set $\{v_i \in V_{H_i}, i = 1, 2, 3\}$, $G\left[ \bigcup_{i=1}^{3} v_i \right]$ contains a cycle $C_3$ with 3 vertices, if this is true, then all vertices of $H_k$ are adjacency to $H_j$ for $j \neq k$. That leads to multiple $K_{n+2}$ minors, which is a contradiction as $G$ is $K_{n+1}$-free.

**Lemma 3.** For any $K_{n+1}$-free graph $G$ contains 4 minimal $K_n$ minors $H_i, i = 1, \ldots, 4$, critical $K_n$ forests always exist.

**Proof.** We will prove the theorem by contradiction. From lemma 2 one can pick a vertex set $F = \{v_i \in V_{H_i}, i = 1, 2, 3\}$ from any 3 of four minimal $K_n$ minors satisfies that $G[F]$ is a forest. If $F$ is an independent set, then lemma 3 is clear. Suppose $F$ is not an independent set, than (1):$G[F]$ will contains at a $K_2$ and one isolated vertex, or (2):$G[F]$ is a tree with three vertices. Assume that there is no critical $K_n$ forests in $G$. For case (1), the $K_2$ of $G[F]$ must be the neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains the subgraphs $H_4 + K_2$ which is a $K_{n+2}$ minor and is a contradiction because $G$ is $K_{n+1}$-free. For case (2), the
there are two vertices of $G[F]$ must be neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains a $K_{n+1}$ minor $G[F \cup V_{H_4}]$ which is a contradiction because $G$ is $K_{n+1}$-free.

Hence, the theorem is true and for $m = 4$. And there exists a set $F = \{v_i \in V_{H_i}, i = 1, \cdots, 4\}$ such that $G[F]$ is either a null subgraph, tree or a forest contains a tree with 3 vertices and an isolated vertex.

**Theorem 6.** For any $K_{n+1}$-free graph $G$ contains $m$ minimal $K_n$ minors $H_i, i = 1, \cdots, m$, there exists a set $F = \{v_i \in V_{H_i}, i = 1, \cdots, m\}$ such that $G[F]$ is critical $K_n$ forest.

**Proof.** We will prove the theorem by listing a set $F = \{a_i \in V_{H_i}, i = 1, \cdots, m\}$ such that $G[F]$ is critical $K_n$ forest from a procedure. In the procedure, we will legally add one vertex of $H_i$ to $F$, i.e., after a vertex is added, the result induced subgraph $G[F]$ does not contain cycle.

Firstly, let $F = \{v_1 \in H_1\}$. Secondly, checking all vertices in $V_H = \bigcup_{i=2}^{m} V_{H_i}$, if there is no neighbors in $\bigcup_{i=2}^{m} H_i$, then let $F = \{v_1 \in V_{H_1}, v_2 \in V_{H_2}\}$ and $V_H = \bigcup_{i=3}^{m} V_{H_i}$; if there is a neighbor $v_j \in V_{H_i}$, then let $F = \{v_1 \in V_{H_1}, v_j \in H_j\}$ and $V_H = V_H - H_j$. Thirdly, check all vertices in $V_H$, if there are no neighbors in $V_H$, then let add $v_x \in V_{H_x}$ to $F$ and remove $V_{H_x}$ from $V_H$, if there exists some vertex $v_y \in H_y$ such that $G[F \cup v_y]$ has no cycles, then add $v_y$ to $F$ and remove $V_{H_y}$ from $V_H$. Fourthly, repeat the above procedures until $V_H$ becomes a null set. Note that, if there are no neighbors of $F$ in $V_H$, then it will produce multiple disjoint trees and cycles can arise only between the latest induced tree and $V_H$. And we will check whether $F$ has neighbors in $V_H$ or not at first.

The result induced graph $G[F]$ from above procedure is obvious a forest. If there are two edges between every vertex of $H_j$ and the latest tree obtained by the above procedures, then it will produce a $K_{n+1}$ minor, then there always exist a vertex $v_x \in V_{H_x}$ has no neighbors in the latest tree of $F$ or a vertex $v_y \in V_H$ only have one neighbor in $F$.

**Theorem 7.** For any connected graph $G$ has at most $K_{n+1}$ minors, there exists
at least one reducible partition \{S_1, S_2\}.

Proof. This theorem is clear by using the theorem\cite{6} one can let \( S_1 = F \) firstly, where \( F \) is defined in the proof of theorem\cite{6}. And then check each vertex \( v_j \) in \( V_G - S_1 \) in sequence, if \( G[S_1 \cup v_j] \) does not contain cycles, then update \( S_1 \) by adding \( v_j \) to \( S_1 \).

And then let \( S_2 = V_G - S_1 \), one can obtain a reducible partition \{\( S_1, S_2 \)\} of \( G \).

Now suppose \( \{S_1, S_2\} \) is a reducible partition of \( G \), if the induced subgraph \( G[S_2] \) is a forest then \( \{S_1, S_2\} \) is called an exhaustive reducible partition(ERP) of \( G \). If \( G[S_2] \) is not a forest, one can recursively perform the procedure showed in the proof of theorem\cite{6} to obtain the reducible partition of \( G[S_2] \).

**Definition 3.** For a \( K_{n+1} \)-free graph \( G \) contains \( K_n \) minors, a ERP \( \{S_1, S_2, \ldots, S_m\} \) is defined as a collection of subset of \( V_G \) such that:

(a) each \( G[S_i] \), \( i = 1, \ldots, m \) is a forest;

(b) \( S_j \) is dominated by \( S_i \), if \( i < j \);

(c) \( G \left[ \bigcup_{j=k}^{m} S_j \right] \) is \( K_{n-k+2} \)-free.

And \( m \) is defined as the depth of ERP.

**Proposition 1.** For a \( K_{n+1} \)-free graph \( G \) contains \( K_n \) minors, the depth of ERP of \( G \) is at most \( n - 1 \).

Proof. As \( G \left[ \bigcup_{j=k}^{m} S_j \right] \) is \( K_{n-k+2} \)-free, then take \( k = m \) and \( n - k + 2 = 3 \) to obtain

\[
m = n - 1
\]

at most.

\( \Box \)

2. Color of planar graphs

In the previous section, we have introduced the conception of ERP of a graph. In this section, we will apply ERP to the problem of colouring of planar
graphs. As the depth of ERG of any $K_{n+1}$-free graph is at most $n - 1$, then the depth of ERP is at most 3 for any planar graph. Hence for a planar graph $G$, there exists a ERP $\{S_1, S_2, S_3\}$ such that

1. each $G[S_i], i = 1, 2, 3$, is a forest;
2. $S_j$ is dominated by $S_i$, if $i < j$;
3. $G[S_2 \cup S_3]$ is $K_4$-free, and $G[S_3]$ is $K_3$-free, i.e., namely it contains disjoint trees or isolated vertices.

When the depth of a planar graph is 2, then $S_3$ is a null set. For this case the chromatic number $\chi(G) \leq 4$ is clear as $G[S_1 \cup S_2]$ is a subgraph of the sum of two complete bipartite graphs. Therefore, one only needs to consider the case where such planar graphs with depth 3. With the all the above theorem and lemma, we have the four-color theorem which is proved by Appel and Haken with computer[1].

**Theorem 8.** Any planar graph $G$ is 4-colorable.

**Proof.** From proposition[1] we know that the depth of ERP of any planar graph is 3. Assume the ERP of $G$ is $\{S_1, S_2, S_3\}$ that satisfies the condition (1), (2) and (3) listed at the beginning of this section. It is easy to say that $G[S_2 \cup S_3]$ is 4-colorable, assume $G[S_2 \cup S_3]$ is labelled by four integers “1”, “2”, “3” and “4”, if we can assign each vertex of $S_1$ by one of integers “1”, “2”, “3” or “4”, then the theorem is proved.

There are two case in which vertices of $S_1$ will be labelled by the fifth color. Case one:
The induced subgraph $G[N(v, S_2 \cup S_3)]$ has chromatic number 4, where $N(v, S_1 \cup S_2)$ denote the neighbors of vertex $v \in S_1$ in $S_2 \cup S_3$.

Case two:
The induced subgraph $G[N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)]$ has chromatic number 3, where $v_1, v_2 \in S_1$ are two adjacent vertices of $S_1$ and $N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)$ is used to denote $N(v_1, S_2 \cup S_3) \cap N(v_2, S_2 \cup S_3)$.

We will use contradiction to prove that these two situations are forbidden
as $G$ has no $K_5$ minors.

For case one, assume that the chromatic number of $G[N(v, S_2 \cup S_3)]$ is 4, then $v \in S_1$ must be labelled by the fifth color 5. As the chromatic number of $G[N(v, S_2 \cup S_3)]$ is 4, it has $K_4$ minors \[2\], hence $G[v \cup N(v, S_2 \cup S_3)]$ has $K_5$ minors which is a contradiction. This implies that the neighbors of vertex $v \in S_1$ in $S_2 \cup S_3$ can be labelled by 3 colors.

Now suppose that the induced subgraph $G[N(v, S_2 \cup S_3)]$ is already colored by some flawed algorithm and the number of the used color is 4 for some 3-colorable subgraphs. Next we will show that one can re-assign less than or equal 4 colors to $G[v \cup N(v, S_2 \cup S_3)]$. Let $SG_4$ denote the set of all such subgraphs with chromatic number 4 in $G[S_2 \cup S_3]$. Deleting such vertices of $SG_4$ with the color 4 in $G$, the result graph is denoted $H$, the set of these deleted vertices is denoted as $DC_4$. Clearly, $H$ is only 3-colorable, if this is not true then $H$ will be in $SG_4$ which contradicts that $H$ is not equal to $SG_4$. Hence, $H$ can be re-assigned 3 colors. Now we can re-add all deleted vertices $DC_4$ to $G$ and keep the adjacency same as before. Clearly, one can label them with the fourth color. This procedure shows that the neighbors of vertex $v \in S_1$ can always be assigned 3 or less colors and $v$ can be labelled by one of four colors.

For case two, assume that the chromatic number of $G[N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)]$ is 3, then the chromatic number of $G[v_1 \cup v_2 \cup N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)]$ is 5. In this configuration, $G[N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)]$ has $K_3$ minors and it is fully connected to a $K_2$ in induced by $v_1 \cup v_2$. Therefore, $G[v_1 \cup v_2 \cup N(v_1, S_2 \cup S_3; v_2, S_2 \cup S_3)]$ has $K_5$ minors, which is a contradiction. That is to say, the common neighbors of two connected vertices can be labelled by 2 or less colors. \[\square\]

3. Color of graph without $K_{n+1}$

In the previous section, we have demonstrated that planar graph is 4-colorable by using the method of ERP. Scholars believe that four color theorem is a special case of Hadwiger conjecture which states that if $G$ is loopless and has no $K_1$ minor then its chromatic number satisfies $\chi(G)$. It is known to be true for
$1 \leq t \leq 6^3 [4]$. 

**Conjecture 1.** [3] Every connected n-chromatic graph $G$ contracts to $K_n$, or to a copy of $K_n$ with some multiple adjacencies.

Before studying the color of a graph without $K_{n+1}$, we will first discuss the depth of ERP of a graph without $K_{n+1}$ minor.

Proposition 1 tell us that the vertex set of a graph without $K_{n+1}$ minor can always be divided into at most $n-1$ subsets, and the subgraph induced by vertices in each of the $n-1$ subsets is a collection of disjoint trees, see Fig. 2.

![Figure 2: Schematic diagram of ERP of a graph $G$ has no $K_{n+1}$ minor. The subgraph $G[S_k]$ induced by $S_k$ only contains disjoint cliques or trees and $G[S_j]$ is dominated by $G[S_k]$ for $j > k$.](image)

For a graph $G$ has no $K_6$ minor, Bela Bollobas and P. Catlin show that $G$ is 5-colorable[5]. Now we show that this conclusion can be obtained by using the ERP. As the depth of ERP of $G$ without $K_6$ minor is at most 4, hence the vertex set of $G$ can be divided into four subsets, say $S_1, S_2, S_3, S_4$.

**Theorem 9.** A graph $G$ has no $K_6$ minor is 5-colorable.

**Proof.** From Proposition 1 the depth of $G$ is at most 4, that is to say the vertex set $V_G$ can be divided into 4 subsets $S_1, S_2, S_3, S_4$ and $G[S_j]$ only contains disjoint trees. $G[S_j]$ is dominated by $G[S_k]$ for $j > k$. Clearly, subgraph $G[S_2 \cup S_3 \cup S_4]$ has no $K_5$ minor, i.e. it is a planar graph. Based on four color theorem, $G[S_2 \cup S_3 \cup S_4]$ is at most 4-colorable. Assume the chromatic number
of \( G[S_2 \cup S_3 \cup S_4] \) is 4, then we only need to assign colors to \( S_1 \). For \( G[S_1] \), the chromatic number is 2. If the chromatic number of \( G \) is 6, then there exist two adjacency vertices \( v_1 \) and \( v_2 \) of \( G[S_1] \), and the subgraph \( G[N(v_1) \cap N(v_2)] \) has chromatic number 4. As \( G[N(v_1) \cap N(v_2)] \) has at most \( K_4 \) minor and 4-colorable by the ERP, this implies that \( G[v_1 \cup v_2 \cup (N(v_1) \cap N(v_2))] \) has \( K_6 \) minor, which contradicts that \( G \) has no \( K_6 \). Hence, there are no such two vertices \( v_1 \) and \( v_2 \). If \( \chi(G[S_2 \cup S_3 \cup S_4]) \) is 3, then \( G \) is 5-colorable as \( G[S_1] \) is 2-colorable. If ERP contains only 3(or lese) \( S_1, S_2, S_3 \) subsets, then \( G[S_2 \cup S_3] \) is a planar graph. The above procedure of coloring still work. Therefore, \( G \) is 5-colorable.

\[
\begin{array}{c}
S_1 \quad \cdots \quad \cdots \\
S_2 \quad \cdots \quad \cdots \\
S_3 \quad \cdots \quad \cdots \\
S_4 \quad \cdots \quad \cdots \\
\end{array}
\]

Figure 3: Schematic diagram of ERP of a graph \( G \) has no \( K_6 \) minor. The subgraph \( G[S_k] \) induced by \( S_k \) only contains disjoint cliques or trees and \( G[S_j] \) is dominated by \( G[S_k] \) for \( j > k \)

With the similar procedure, one can prove Hadwiger conjecture

**Theorem 10.** A graph \( G \) has no \( K_{n+1} \) minor is \( n \)-colorable.

**Proof.** We will prove the theorem by induction. Firstly, we need to show this theorem is correct for \( n = 5 \) which is four color theorem. Secondly we assume the theorem is correct for \( n = t \), namely if \( G \) has no \( K_{t+1} \) then \( G \) is \( t \)-colorable. And we will check whether the theorem is correct for \( n = t + 1 \), namely if \( G \) has no \( K_{t+2} \), then \( G \) is \( (t+1) \)-colorable. For \( n = t + 1 \), the depth of ERP of \( G \) is \( t + 1 \), the vertex set \( V_G \) of \( G \) can be divided into \( t \) subsets \( S_1, S_2, \ldots, S_{t+1} \) and \( G[S_j] \)
is dominated by $G[S_k]$ for $j > k$. Based on the assumption that the theorem is correct for $n = t$, namely $G \left[ \bigcup_{j=2}^{t+1} S_j \right]$ is $t$-colorable. Without loss of generality, we take $\chi(G \left[ \bigcup_{j=2}^{t+1} S_j \right]) = t$. Then if $\chi(G) \geq t + 2$, there exist two adjacent vertices $v_1, v_2 \in S_1$ in $G[S_1]$ and the chromatic number $\chi(G \left[ N(v_1) \cap N(v_2) \right])$ of $G \left[ N(v_1) \cap N(v_2) \right]$ is $t$. As $G \left[ N(v_1) \cap N(v_2) \right]$ has $K_t$ minor by using the assumption of the induction, then $G[v_1 \cup v_2 \cup (N(v_1) \cap N(v_2))]$ has $K_{t+2}$ minor, which contracts the assumption that $G$ has no $K_{t+2}$. Hence, there are no such pairs of vertices. If the neighbors of some vertex $v \in S_1$ is $t$-colorable, then $G$ is at most $t + 1$-colorable.

4. Conclusions and Discussions

In this theme, we propose a concept of reducible partition and exhaustive reducible partition of graphs. We showed that any graph $G$ possess exhaustive reducible partitions and the depth of ERP of $G$ is at most $n - 1$ if $G$ does not contain $K_{n+1}$ minor. The theme illustrates that the four color theorem can be proved by using the ERP. If the depth of ERP of a planar graph is 3, then one can obtain that any subgraphs induced by the union of two of these 3 subsets is 4-colorable, to keep the graph is planar, the fifth color is forbidden. If the depth of ERP of a planar graph is 2, the chromatic number is at most 4. For general graphs, we have shown that the ERP can also work on how to prove Hadwiger conjecture, i.e. it also implies Hadwiger conjecture. The method is similar to the procedure of proving four color theorem of planar graphs. For a $K_{n+1}$-free graph, any two vertices of $S_1$ of the ERP have common neighbors whose induced subgraph is $K_{n-1}$-free, hence it is $n$-colorable.

Some structural information of a graph can be acquired by list the ERP to some degree. With these information, can we design algorithms to improve the performance for coloring a graph? That will be a work we want to consider in the future.
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