LIOUVILLE TYPE THEOREM FOR CRITICAL ORDER
HÉNON-LANE-EMDEN TYPE EQUATIONS ON A HALF SPACE AND
ITS APPLICATIONS

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Abstract. In this paper, we are concerned with the critical order Hénon-Lane-Emden type
equations with Navier boundary condition on a half space \( \mathbb{R}^n_+ \):

\[
\begin{cases}
(-\Delta)^{\frac{n}{2}} u(x) = f(x, u(x)), & x \in \mathbb{R}^n_+, \\
u(x) \geq 0, & x \in \mathbb{R}^n_+, \\
u = (-\Delta)u = \cdots = (-\Delta)^{\frac{n}{2}-1}u = 0, & x \in \partial \mathbb{R}^n_+,
\end{cases}
\]

where \( u \in C^n(\mathbb{R}^n_+) \cap C^{n-2}(\mathbb{R}^n_+) \) and \( n \geq 2 \) is even. We first consider the typical case \( f(x, u) = |x|^a u^p \) with \( 0 \leq a < +\infty \) and \( 1 < p < +\infty \). We prove the super poly-harmonic properties and establish the equivalence between (0.1) and the corresponding integral equations

\[
u(x) = \int_{\mathbb{R}^n_+} G(x, y)f(y, u(y))dy,
\]

where \( G(x, y) \) denotes the Green’s function for \((-\Delta)^{\frac{n}{2}}\) on \( \mathbb{R}^n_+ \) with Navier boundary conditions. Then, we establish Liouville theorem for (0.2) via “the method of scaling spheres” developed initially in [19] by Dai and Qin, and hence we obtain the Liouville theorem for (0.1) on \( \mathbb{R}^n_+ \). As an application of the Liouville theorem on \( \mathbb{R}^n_+ \) (Theorem 1.6) and Liouville theorems in \( \mathbb{R}^n \) established in Chen, Dai and Qin [4] for \( n \geq 4 \) and Bidaut-Véron and Giacomini [1] for \( n = 2 \), we derive a priori estimates and existence of positive solutions to critical order Lane-Emden equations in bounded domains for all \( n \geq 2 \) and \( 1 < p < +\infty \). Extensions to IEs and PDEs with general nonlinearities \( f(x, u) \) are also included.

Keywords: The method of scaling spheres; Critical order; Hénon-Lane-Emden type equations; Liouville theorems; a priori estimates; Navier problems.

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1. Introduction

1.1. Liouville theorems on a half space \( \mathbb{R}^n_+ \). In this paper, we first establish Liouville theorem for the following higher order Hénon-Lane-Emden equations with Navier boundary condition:

\[
\begin{cases}
(-\Delta)^{\frac{n}{2}} u(x) = |x|^a u^p(x), & x \in \mathbb{R}^n_+, \\
u(x) \geq 0, & x \in \mathbb{R}^n_+, \\
u = (-\Delta)u = \cdots = (-\Delta)^{\frac{n}{2}-1}u = 0 & x \in \partial \mathbb{R}^n_+,
\end{cases}
\]

where \( \mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0 \} \) is the upper half Euclidean space, \( u \in C^n(\mathbb{R}^n_+) \cap C^{n-2}(\mathbb{R}^n_+) \), \( n \geq 2 \) is even, \( 0 \leq a < +\infty \) and \( 1 < p < +\infty \).

For \( 0 < \alpha \leq n \), PDEs of the form

\[
(-\Delta)^{\frac{n}{2}} u(x) = |x|^a u^p(x)
\]

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are called the fractional order or higher order Hénon, Lane-Emden, Hardy equations for \( a > 0, a = 0, a < 0 \), respectively. These equations have numerous important applications in conformal geometry and Sobolev inequalities. In particular, in the case \( a = 0, (1.2) \) becomes the well-known Lane-Emden equation, which models many phenomena in mathematical physics and astrophysics. When \( a > 0 \), equations of type \( (1.2) \) was first proposed by Hénon in \([27]\) when he studied rotating stellar structures.

We say equations \( (1.2) \) have critical order if \( \alpha = n \) and non-critical order if \( 0 < \alpha < n \). Being essentially different from the non-critical order equations, the fundamental solution \( a \) of \((\Delta)^{\frac{\alpha}{2}} \) changes its signs in critical order case \( \alpha = n \). The nonlinear terms in \( (1.2) \) are called critical if \( p = p_{s}(a) := \frac{n+n+2a}{n-\alpha} (:= +\infty \text{ if } n = \alpha) \) and subcritical if \( 0 < p < p_{s}(a) \).

Liouville type theorems (i.e., nonexistence of nontrivial nonnegative solutions) and related properties for equations \( (1.2) \) in the whole space \( \mathbb{R}^{n} \) and the half space \( \mathbb{R}_{+}^{n} \) have been extensively studied (see \([1, 6, 7, 9, 13, 15, 18, 19, 20, 21, 22, 23, 25, 28, 29, 30, 32, 33, 37, 38, 39, 40, 41]\) and the references therein). These Liouville theorems, in conjunction with the blowing up and re-scaling arguments, are crucial in establishing a priori estimates and hence existence of positive solutions to non-variational boundary value problems for a class of elliptic equations on bounded domains or on Riemannian manifolds with boundaries (see \([4, 11, 19, 26, 36]\)).

In the critical order case \( \alpha = n \), Liouville theorems for \( (1.2) \) have been established in \([1]\) for \( n = 2 \) and in \([4]\) for \( n \geq 4 \) in whole space \( \mathbb{R}^{n} \). For the special case \( a = 0 \), their results can be concluded as the following theorem.

**Theorem 1.1.** \([1, 4]\) Suppose \( n \geq 2 \) is an even integer, \( 1 < p < +\infty \) and \( u \) is a nonnegative classical solution of

\[
(\Delta)^{\frac{\alpha}{2}} u(x) = u^{p}(x), \quad \forall x \in \mathbb{R}^{n}.
\]

Then, we have \( u \equiv 0 \) in \( \mathbb{R}^{n} \).

For \( a = 0 \), there are also many works on the Liouville type theorems for Lane-Emden equations on half space \( \mathbb{R}_{+}^{n} \), for instance, see \([6, 7, 9, 13, 15, 18, 19, 20, 21, 22, 31, 33, 37, 38, 39]\) and the references therein. Reichel and Weth \([38]\) proved Liouville theorem in the class of bounded nonnegative solutions for Dirichlet problem of higher order Lane-Emden equations \( (1.2) \) (i.e., \( a = 0 \) and \( \alpha = 2m \) with \( 1 \leq m < \frac{n}{2} \)) on \( \mathbb{R}_{+}^{n} \) in the cases \( 1 < p \leq \frac{n+2m}{n-2m} \), subsequently they also derived in \([39]\) Liouville theorem for general nonnegative solutions in the cases \( 1 < p < \frac{n+2m}{n-2m} \). In \([6]\), Chen, Fang and Li established Liouville theorem for Navier problem of Lane-Emden equation \( (1.2) \) on \( \mathbb{R}_{+}^{n} \) in the higher order cases \( \alpha = 2m \) with \( 1 \leq m < \frac{n}{2} \) and \( \frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m} \). In a recent work \([19]\), Dai and Qin developed the method of scaling spheres, which is essentially a frozen variant of the method of moving spheres initially used by Chen and Li \([10]\), Li and Zhu \([31]\) and Padilla \([35]\) and becomes a powerful tool in deriving asymptotic estimates for solutions. As one of many immediate applications, they established in \([19]\) the Liouville theorem for non-critical higher order Hénon-Lane-Emden type IEs and Lane-Emden type PDEs with Navier boundary conditions on \( \mathbb{R}_{+}^{n} \) for all \( 1 < p \leq \frac{n-2m}{n-2m} \).

For Liouville theorem on \( \mathbb{R}_{+}^{n} \), the cases \( a \neq 0 \) have not been fully understood. For \( a > 0 \), by using the method of scaling spheres developed initially in \([19]\), Dai, Qin and Zhang \([22]\) proved the Liouville theorem for non-critical higher order Hénon equations \( (1.2) \) (i.e., \( \alpha = 2m \) with \( 1 \leq m < \frac{n}{2} \)) with Navier boundary conditions on \( \mathbb{R}_{+}^{n} \) in the cases \( 1 < p < \frac{n+2m+2a}{n-2m} \).

In this paper, by applying the method of scaling spheres in integral forms, we will establish Liouville theorem for the Navier problem of critical order Hénon-Lane-Emden equation \( (1.1) \) on \( \mathbb{R}_{+}^{n} \) in all the cases that \( a \geq 0, n \geq 2 \) and \( 1 < p < +\infty \).
It's well known that the super poly-harmonic properties of solutions are crucial in establishing Liouville type theorems and the integral representation formulae for higher order or fractional order PDEs (see e.g. [4, 5, 6, 18, 22, 41]). In order to prove the equivalence between PDE (1.1) and corresponding integral equation, we will first prove the following generalized theorem on super poly-harmonic properties, namely, we allow $-n < a < 0$ and assume that $u \in C^n(\mathbb{R}^n_+) \cap C^{n-2}(\mathbb{R}^n_+)$ if $-n < a < 0$.

**Theorem 1.2.** (Super poly-harmonic properties) Assume $n \geq 4$ is even, $-n < a < +\infty$, $1 < p < +\infty$ and $u$ is a nonnegative solution of (1.1). If one of the following two assumptions \[ a \geq -2p - 2 \quad \text{or} \quad u(x) = o(|x|^2) \quad \text{as} \quad |x| \to +\infty \]
holds, then \[ (-\Delta)^i u(x) \geq 0 \]
for every $i = 1, 2, \ldots, \frac{n}{2} - 1$ and all $x \in \mathbb{R}^n_+$.

Based on the above super poly-harmonic properties, we can deduce the equivalence between PDE (1.1) and the following integral equation

\[ u(x) = \int_{\mathbb{R}^n_+} G^+(x, y)|y|^a u^p(y)dy, \]

where

\[ G^+(x, y) := C_n \left( \ln \frac{1}{|x - y|} - \ln \frac{1}{|\bar{x} - y|} \right) \]
denotes the Green’s function for $(-\Delta)^{\frac{n}{2}}$ on $\mathbb{R}^n_+$ with Navier boundary conditions, and $\bar{x} := (x_1, \ldots, -x_n)$ is the reflection of $x$ with respect to the boundary $\partial \mathbb{R}^n_+$. That is, we have the following theorem.

**Theorem 1.3.** If $u$ is a nonnegative classical solution of (1.1), then $u$ is also a nonnegative solution of integral equation (1.3), and vice versa.

Next, we consider the integral equations (1.3) instead of PDE (1.1). We will study the integral equation (1.3) via the method of scaling spheres in integral forms developed by Dai and Qin in [19]. Our Liouville type result for IE (1.3) is the following theorem.

**Theorem 1.4.** Assume $n \geq 1$, $1 \leq p < +\infty$ and $-n < a < +\infty$. If $u \in C(\mathbb{R}^n_+)$ is a nonnegative solution to (1.3), then $u \equiv 0$.

**Remark 1.5.** Note that we do not assume $n$ to be even in Theorem 1.4. It is also unexpected that the above Theorem 1.4 still holds for $n = 1$. One can see clearly from the proof that the assumption $u \in C(\mathbb{R}^n_+)$ in Theorem 1.4 can be weaken into $|x|^a u^{p-1} \in L^{1+\delta}_{loc}(\mathbb{R}^n_+)$ for some small $\delta > 0$.

As a consequence of Theorem 1.3 and 1.4, we obtain immediately the following Liouville type theorem on PDE (1.1).

**Theorem 1.6.** Assume $n \geq 2$ is even, $0 \leq a < +\infty$ and $1 < p < +\infty$. Suppose $u \in C^n(\mathbb{R}^n_+) \cap C^{n-2}(\mathbb{R}^n_+)$ is a nonnegative classical solution to (1.1), then $u \equiv 0$. 
1.2. A priori estimates and existence of positive solutions in bounded domains.

As an immediate application of the Liouville theorems (Theorem 1.1 for $\mathbb{R}^n$ and Theorem 1.6 for $\mathbb{R}^n_+$), we can derive a priori estimates and existence of positive solutions to critical order Lane-Emden equations in bounded domains $\Omega$ for all $1 < p < +\infty$.

In general, let the critical order uniformly elliptic operator $L$ be defined by

\begin{equation}
L := \left( \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^{\frac{p}{2}} + \sum_{|\beta| \leq n-1} b_{\beta}(x) D^\beta,
\end{equation}

where $n \geq 2$ is even and the coefficients $b_{\beta} \in L^\infty(\Omega)$ and $a_{ij} \in C^{n-2}(\Omega)$ such that there exists constant $\tau > 0$ with

\begin{equation}
\tau |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \tau^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \ x \in \Omega.
\end{equation}

Consider the Navier boundary value problem:

\begin{align}
Lu(x) &= f(x, u), \quad x \in \Omega, \\
u(x) &= Au(x) = \cdots = A^{\frac{p}{2}-1} u(x) = 0, \quad x \in \partial \Omega,
\end{align}

where $n \geq 2$ is even, $u \in C^n(\Omega) \cap C^{n-2}(\Omega)$ and $\Omega$ is a bounded domain with boundary $\partial \Omega \subset C^{n-2}$.

By virtue of the Liouville theorem in $\mathbb{R}^n$ established in [1, 4] (see also Theorem 1.1) and Liouville theorem in $\mathbb{R}^n_+$ (Theorem 1.6), using entirely similar blowing-up and re-scaling methods as in the proof of Theorem 6 in Chen, Fang and Li [6], we can derive the following a priori estimate for classical solutions (possibly sign-changing solutions) to the critical order Navier problem (1.7) in the full range $1 < p < +\infty$.

**Theorem 1.7.** Assume $n \geq 2$ is even, $1 < p < +\infty$ and there exist positive, continuous functions $h(x)$ and $k(x): \Omega \rightarrow (0, +\infty)$ such that

\begin{equation}
\lim_{s \to +\infty} \frac{f(x, s)}{s^p} = h(x), \quad \lim_{s \to -\infty} \frac{f(x, s)}{|s|^p} = k(x)
\end{equation}

uniformly with respect to $x \in \bar{\Omega}$. Then there exists a constant $C > 0$ depending only on $\Omega$, $n$, $p$, $h(x)$, $k(x)$, such that

\begin{equation}
\|u\|_{L^\infty(\Omega)} \leq C
\end{equation}

for every classical solution $u$ of Navier problem (1.7).

**Remark 1.8.** The proof of Theorem 1.7 is entirely similar to that of Theorem 6 in [6] (see also Theorem 1.13 in [19]). We only need to replace the Liouville theorems for non-critical order Lane-Emden equations in $\mathbb{R}^n$ (see Lin [28] for fourth order and Wei and Xu [41] for general even order) by Liouville theorems for critical order equations in $\mathbb{R}^n$ (see Bidaut-Véron and Giacomini [1] for $n = 2$ and Chen, Dai and Qin [4] for $n \geq 4$, see also Theorem 1.11), and replace the Liouville theorems for non-critical order Lane-Emden equations on $\mathbb{R}^n_+$ (Theorem 5 in [6], or further, Theorem 1.10 in [19]) by Theorem 1.6 in the proof. Thus we omit the details of the proof.
One can immediately apply Theorem 1.7 to the following critical order Navier problem:

\[
\begin{cases}
(-\Delta)^{\frac{n}{2}} u(x) = u^p(x) + t & \text{in } \Omega, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{\frac{n}{2}-1} u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( n \geq 2, \Omega \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial \Omega \in C^{n-2} \) and \( t \) is an arbitrary nonnegative real number.

We can deduce the following corollary from Theorem 1.7.

**Corollary 1.9.** Assume \( 1 < p < +\infty \). Then, for any nonnegative solution \( u \in C^n(\Omega) \cap C^{n-2}(\Omega) \) to the critical order Navier problem (1.10), we have

\[
\|u\|_{L^\infty(\Omega)} \leq C(n, p, \Omega).
\]

**Remark 1.10.** In [4], due to the lack of Liouville theorem in \( R^n_+ \) (Theorem 1.6), the authors first applied the method of moving planes in local way to derive a boundary layer estimates, then by using blowing-up arguments (see [2, 9]), they could only establish the a priori estimates for the critical order Navier problem (1.10) under the assumptions that either \( 1 < p < +\infty \) and \( \Omega \) is strictly convex, or \( 1 < p \leq \frac{n+2}{n-2} \) (see Theorem 1.3 in [4]). Now, as an immediate consequence of Theorem 1.7, we derive in Corollary 1.9 a priori estimates for the critical order Navier problem (1.10) for all the cases \( 1 < p < +\infty \) with no convexity assumptions on \( \Omega \), which extends Theorem 1.3 in [4] remarkably.

As a consequence of the a priori estimates (Corollary 1.9), by applying the Leray-Schauder fixed point theorem (see Theorem 4.1 in [4]), we can derive existence result for positive solution to the following Navier problem for critical order Lane-Emden equations in the full range \( 1 < p < +\infty \):

\[
\begin{cases}
(-\Delta)^{\frac{n}{2}} u(x) = u^p(x) & \text{in } \Omega, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{\frac{n}{2}-1} u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( n \geq 2 \) is even, \( 1 < p < +\infty \) and \( \Omega \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial \Omega \in C^{n-2} \).

By virtue of the a priori estimates (Theorem 1.3 in [4]), using the Leray-Schauder fixed point theorem, Chen, Dai and Qin [4] obtained existence of positive solution for the critical order Navier problem (1.12) under the assumptions that either \( p \in (1, +\infty) \) and \( \Omega \) is strictly convex, or \( p \in (1, \frac{n+2}{n-2}] \) (Theorem 1.4 in [4]). For existence results on non-critical higher order Hénon-Hardy equations on bounded domains, please see [16, 17, 18, 19, 24, 34] and the references therein. Since Corollary 1.9 extends Theorem 1.3 in [4] to the full range \( 1 < p < +\infty \) with no convexity assumptions on \( \Omega \), through entirely similar arguments, we can improve Theorem 1.4 in [4] remarkably and derive the following existence result for positive solution to the critical order Navier problem (1.12) in the full range \( 1 < p < +\infty \).

**Theorem 1.11.** Assume \( 1 < p < +\infty \). Then, the critical order Navier problem (1.12) possesses at least one positive solution \( u \in C^n(\Omega) \cap C^{n-2}(\Omega) \). Moreover, the positive solution \( u \) satisfies

\[
\|u\|_{L^\infty(\Omega)} \geq \left( \frac{\sqrt{2n}}{\text{diam } \Omega} \right)^{\frac{n}{p-1}}.
\]

**Remark 1.12.** The proof of Theorem 1.11 is entirely similar to that of Theorem 1.4 in [4]. We only need to replace Theorem 1.3 in [4] by Corollary 1.9 in the proof. Thus we omit the details of the proof.
Remark 1.13. The lower bounds (1.13) on the $L^\infty$ norm of positive solutions $u$ indicate that, if $\text{diam} \, \Omega < \sqrt{2n}$, then a uniform priori estimate does not exist and blow-up may occur when $p \to 1^+$. 

1.3. Extensions to general nonlinearities. Consider the following integral equations associated with Navier problems for general critical order elliptic equations on $\mathbb{R}^n_+$:

(1.14) \[ u(x) = \int_{\mathbb{R}^n_+} G^+(x, y)f(y, u(y))dy, \]

where $u \in C(\overline{\mathbb{R}^n_+})$, $n \geq 1$ and the nonlinear terms $f : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. 

Definition 1.14. We say that the nonlinear term $f$ has subcritical growth, provided that

(1.15) \[ \mu^n f(\mu x, u) \]

is strictly increasing with respect to $\mu \geq 1$ or $\mu \leq 1$ for all $(x, u) \in \mathbb{R}^n_+ \times \mathbb{R}_+$. 

Definition 1.15. A function $g(x, u)$ defined on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ is called locally Lipschitz on $u$, provided that for any $u_0 \in \mathbb{R}^n_+$ and $\omega \subseteq \mathbb{R}^n_+$ bounded, there exists a (relatively) open neighborhood $U(u_0) \subseteq \mathbb{R}^n_+$ such that $g$ is Lipschitz continuous on $u$ in $\omega \times U(u_0)$.

We need the following three assumptions on the nonlinear term $f(x, u)$.

(f1) The nonlinear term $f$ is non-decreasing about $u$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$, namely,

(1.16) \[ (x, u), (x, v) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \text{ with } u \leq v \implies f(x, u) \leq f(x, v). \]

(f2) There exists a $\sigma < n$ such that, $|x|^\sigma f(x, u)$ is locally Lipschitz on $u$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$.

(f3) There exist a cone $C \subseteq \mathbb{R}^n_+$ containing the positive $x_n$-axis with vertex at 0 (say, $C = \{x \in \mathbb{R}^n_+ \mid x_n > \frac{|x|}{\sqrt{n}}\}$), constants $C > 0$, $-n < a < +\infty$ and $0 < p < +\infty$ such that, the nonlinear term

(1.17) \[ f(x, u) \geq C|x|^a u^p \text{ in } C \times \mathbb{R}_+. \]

By applying the method of scaling spheres to the generalized integral equations (1.14), we can derive the following Liouville theorem.

Theorem 1.16. Assume $f$ is subcritical and satisfies the assumptions (f1), (f2) and (f3), then the Liouville type results in Theorem 1.4 are valid for integral equations (1.14).

Remark 1.17. By using the method of scaling spheres, Theorem 1.16 can be proved through a quite similar way as in the proof of Theorem 1.4, so we leave the details to readers. We would like to mention that, if the nonlinear term $f(x, u)$ satisfies subcritical conditions for $\mu \leq 1$ (see Definition 1.14), we only need to carry out calculations and estimates outside the upper half ball $B_\lambda^+(0)$ during the scaling spheres procedure.

Remark 1.18. In particular, $f(x, u) = |x|^a u^p$ with $a > -n$ satisfies all the assumptions in Theorem 1.16, thus Theorem 1.4 can also be regarded as a corollary of Theorem 1.16. In addition, $f(x, u) = |x|^a(x_n)^b u^p$ with $a + b > -n$ and $b \geq 0$ also satisfies all the assumptions in Theorem 1.16.

Next, we consider the following Navier problems for general critical order elliptic equations on $\mathbb{R}^n_+$:

(1.18) \[ \begin{cases} \ (-\Delta)^{\frac{n}{2}} u(x) = f(x, u(x)), & u(x) \geq 0, & x \in \mathbb{R}^n_+, \\ u = (-\Delta) u = \cdots = (-\Delta)^{\frac{n}{2}-1} u = 0 & \text{on } \partial \mathbb{R}^n_+, \end{cases} \]
where \( u \in C^n(\mathbb{R}^n_+) \cap C^{n-2}(\mathbb{R}^n_+) \), \( n \geq 2 \) is even and the nonlinear terms \( f : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \).

It is clear from the proof of Theorem 1.2 that (see Section 2), under the same assumptions, the super poly-harmonic properties in Theorem 1.2 also hold for nonnegative classical solutions to the generalized critical order elliptic equations (1.18) provided that

\[
(1.19) \quad f(x, u) \geq C|x|^a u^p \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+.
\]

Based on the super poly-harmonic properties, one can verify under some assumptions on \( f(x, u) \) that the proof of Theorem 1.3 can also be adopted to show the equivalence between the generalized critical order PDEs (1.18) and IE (1.14) (see Section 3). For these purpose, we need the following assumptions on the nonlinear term \( f(x, u) \).

\((f_2')\) The nonlinear term \( f(x, u) \) is locally Lipschitz on \( u \) in \( \mathbb{R}^n_+ \times \mathbb{R}_+ \).

\((f_3')\) There exist constants \( C > 0, 0 \leq a < +\infty \) and \( 0 < p < +\infty \) such that, the nonlinear term \( f(x, u) \) satisfies (1.19).

As a consequence of Theorem 1.16, we derive the following Liouville theorem for the generalized critical order PDEs (1.18).

**Theorem 1.19.** Assume \( f \) is subcritical and satisfies the assumptions \((f_1), (f_2')\) and \((f_3')\), then the Liouville type results in Theorem 1.6 are valid for PDEs (1.18).

**Remark 1.20.** In particular, \( f(x, u) = |x|^a u^p \) with \( a \geq 0 \) satisfies all the assumptions in Theorem 1.19, thus Theorem 1.6 can also be regarded as a corollary of Theorem 1.19.

The rest of this paper is organized as follows. In section 2, we prove the super poly-harmonic properties for nonnegative solutions to (1.1) (i.e., Theorem 1.2) via a variant of the method used in [6]. In section 3, we show the equivalence between PDE (1.1) and IE (1.3), namely, Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.4, then Theorem 1.6 follows immediately as a consequence of Theorem 1.4.

In the following, we will use \( C \) to denote a general positive constant that may depend on \( n, a, p \) and \( u \), and whose value may differ from line to line.

**2. Super poly-harmonic properties**

In this section, we will prove Theorem 1.2. To this end, we make an odd extension of \( u \) to the whole space \( \mathbb{R}^n \). Define

\[
(2.1) \quad u(x', x_n) = -u(x', -x_n) \quad \text{for } x_n < 0,
\]

where \( x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \). Then \( u \) satisfies

\[
(2.2) \quad (-\Delta)^{\frac{n}{2}} u = |x|^a |u|^{p-1} u(x), \quad x \in \mathbb{R}^n.
\]

Let \( v_i := (-\Delta)^i u \). We aim to show that

\[
(2.3) \quad v_i(x) \geq 0
\]

for any \( x \in \mathbb{R}^n_+ \) and \( i = 1, 2, \cdots, \frac{n}{2} - 1 \). Our proof will be divided into two steps.

**Step 1.** We first show that

\[
(2.4) \quad v_{\frac{n}{2}-1} = (-\Delta)^{\frac{n}{2}-1} u \geq 0.
\]

If \((2.4)\) does not hold, then there exists \( x_0 \in \mathbb{R}^n_+ \), such that

\[
(2.5) \quad v_{\frac{n}{2}-1}(x_0) < 0.
\]
Now, let
\[
\bar{f}(r) = \bar{f}(|x - x_0|) := \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} f(x) d\sigma
\]
be the spherical average of \( f \) with respect to the center \( x_0 \). Then by the well-known property \( \Delta u = \Delta \bar{u} \), we have
\[
\begin{aligned}
-\Delta \bar{u}_{r-1}(r) &= |x|^a |u|^{p-1} u(r), \\
-\Delta \bar{u}_{r-2}(r) &= \bar{u}_{r-1}(r), \\
& \quad \cdots \\
-\Delta \bar{u}(r) &= \bar{u}(r).
\end{aligned}
\]
(2.7)

From the first equation in (2.7), integrating both sides from 0 to \( r \), we have
\[
-r^{n-1} \bar{u}_{r-1}(r) = \int_0^r s^{n-1} |x|^a |u|^{p-1} u(s) ds
\]
(2.8)
\[
= \frac{1}{\omega_n} \int_0^r \int_{\partial B_s(x_0)} |x|^a |u|^{p-1} u d\sigma ds
\]
\[
= \frac{1}{\omega_n} \int_{B_r(x_0)} |x|^a |u|^{p-1} u dx \geq 0,
\]
where \( \omega_n \) denotes the area of unit sphere in \( \mathbb{R}^n \). Here we have used the fact that, since \( x_0 \in \mathbb{R}^n_+ \), more than half of the ball \( B_r(x_0) \) is contained in \( \mathbb{R}^n_+ \), so (2.8) follows from the odd symmetry of \( u \) with respect to \( \partial \mathbb{R}^n_+ \). From (2.5) and (2.8), we have
\[
(2.9) \quad \bar{u}_{r-1}(r) \leq 0, \quad \bar{u}_{r-2}(r) \leq \bar{u}_{r-1}(0) = \bar{u}_{r-1}(x_0) < 0, \quad \forall r \geq 0.
\]
Then from the second equation in (2.7), we have
\[
(2.10) \quad -r^{n-1} \bar{u}_{r-2}(r)' = \bar{u}_{r-1}(r) \leq \bar{u}_{r-1}(0) := -c_0 < 0, \quad \forall r \geq 0,
\]
which means
\[
(2.11) \quad (r^{n-1} \bar{u}_{r-2}(r))' \geq c_0 r^{n-1}, \quad \forall r \geq 0.
\]
Integrating from 0 to \( r \) twice yields
\[
(2.12) \quad \bar{u}_{r-2}(r) \geq \frac{c_0}{2n} r^2 + \bar{u}_{r-2}(0) \geq \frac{c_0}{2n} r^2 + c_1, \quad \forall r \geq 0.
\]
Continuing this way, if \( \frac{n}{2} \) is odd, we can derive that
\[
(2.13) \quad \bar{u}(r) \leq -\bar{c}_0 r^{n-2} + \sum_{i=1}^{\frac{n}{2}-1} \bar{c}_i r^{2(\frac{n}{2}-1-i)}, \quad \forall r \geq 0,
\]
where \( \bar{c}_0 > 0 \) and \( \bar{c}_{\frac{n}{2}-1} = \bar{u}(0) = u(x_0) \geq 0 \). Then, similar to (2.8), by the definition of \( \bar{u} \) and (2.13), for \( r \) large, we obtain
\[
(2.14) \quad 0 \leq \frac{1}{\omega_n} \int_{B_r(x_0)} u dx = \int_0^r s^{n-1} \bar{u}(s) ds
\]
\[
\leq -\bar{c}_0 \frac{r^{2n-2}}{2n-2} + \sum_{i=1}^{\frac{n}{2}-1} \frac{\bar{c}_i}{2(n-1-i)} r^{2(\frac{n}{2}-1-i)+n} < 0,
\]
which is absurd. Hence in the following, we assume that \( \frac{n}{2} \) is even.

Set \( w_0(r) = \bar{u}(r) \) and \( w_k(r) = \Delta^k \bar{u}(r), \ k = 1, \cdots, \frac{n}{2} - 1 \). By (2.3), we have \( w_{\frac{n}{2} - 1}(0) = -\bar{v}_{\frac{n}{2} - 1}(0) > 0 \) and \( w_k \) satisfies

\[
\begin{aligned}
&\Delta w_{\frac{n}{2} - 1}(r) = |x|^a |u|^{p-1} u(r), \\
&\Delta w_k(r) = w_{k+1}(r), \ k = 0, \cdots, \frac{n}{2} - 2.
\end{aligned}
\]

We divide (2.15) into two parts. Let \( w_k := u_k + \phi_k, \ k = 0, \cdots, \frac{n}{2} - 1 \), where \( u_k \) satisfies

\[
\begin{aligned}
&\Delta u_{\frac{n}{2} - 1}(r) = |x|^a |u|^{p-1} u(r), \\
&\Delta u_k(r) = u_{k+1}(r), \ k = 0, \cdots, \frac{n}{2} - 2, \\
&u_k(0) = 0, \ k = 0, \cdots, \frac{n}{2} - 1,
\end{aligned}
\]

and \( \phi_k \) solves

\[
\begin{aligned}
&\Delta \phi_{\frac{n}{2} - 1}(r) = 0, \\
&\Delta \phi_k(r) = \phi_{k+1}(r), \ k = 0, \cdots, \frac{n}{2} - 2, \\
&\phi_k(0) = w_k(0), \ k = 0, \cdots, \frac{n}{2} - 1.
\end{aligned}
\]

From (2.17), by direct calculations, we have

\[
\phi_0(r) = \sum_{k=0}^{\frac{n}{2} - 1} c_k u_k(0) r^{2k},
\]

where \( c_k > 0, \ k = 0, \cdots, \frac{n}{2} - 1 \). It is easy to see that

\[
\phi_0(r) \geq c_0 \bar{u}(0) + c_{\frac{n}{2} - 1} w_{\frac{n}{2} - 1}(0) r^{n-2} - \sum_{k=1}^{\frac{n}{2} - 2} c_k |w_k(0)| r^{2k}.
\]

Let

\[
u_\lambda(x) = \lambda^{\frac{n+a}{2}} u(\lambda x)
\]

be the re-scaling of \( u \). Then one can verify that \( u_\lambda \) still satisfies the equation

\[
\Delta^{\frac{n}{2}} u_\lambda = |x|^a |u_\lambda|^{p-1} u_\lambda, \ x \in \mathbb{R}^n.
\]

Let \( w_{0,\lambda}(r) = \bar{w}(r) \) and \( w_{k,\lambda}(r) = \Delta^k \bar{w}(r), \ k = 1, \cdots, \frac{n}{2} - 1 \), where the spherical average is taken with respect to the center \( x_{0,\lambda} := \frac{r}{\lambda} \), we have

\[
w_{k,\lambda}(0) = \lambda^{\frac{n+a}{2} + 2k} w_k(0).
\]

Similar to \( w_k \), we decompose \( w_{k,\lambda} := u_{k,\lambda} + \phi_{k,\lambda}, \ k = 0, \cdots, \frac{n}{2} - 1 \), where \( u_{k,\lambda} \) and \( \phi_{k,\lambda} \) still satisfy (2.10) and (2.17) respectively if we substitute \( u_\lambda \) and \( w_{k,\lambda} \) for \( u \) and \( w_k \). Similar to (2.19), we can still conclude that

\[
\phi_{0,\lambda}(r) \geq \lambda^{\frac{n+a}{2}} \left( c_0 \bar{u}(0) + c_{\frac{n}{2} - 1} w_{\frac{n}{2} - 1}(0) \lambda^{n-2} r^{n-2} - \sum_{k=1}^{\frac{n}{2} - 2} c_k |w_k(0)| \lambda^{2k} r^{2k} \right).
\]

Next we use the iteration argument to derive a contradiction.

Set \( \Omega^+_r = B_r(x_{0,\lambda}) \cap \mathbb{R}^n_+, \Omega^-_r = B_r(x_{0,\lambda}) \cap (\mathbb{R}^n_+ \setminus \mathbb{R}^n_+) \). Let \( \tilde{\Omega^-_r} \) be the reflection of \( \Omega^-_r \) with respect to \( \partial \mathbb{R}^n_+ \) and \( \Omega^- = \Omega^+_r \setminus \tilde{\Omega^-_r} \).
We can choose \( \lambda \) large such that \( |x_{0,\lambda}| < \frac{1}{4} \), then it is easy to see that given \( 1 \leq \tau \leq 2 \), for all \( x \in \Omega_{\tau} \), we have \( |x|^a \geq (1 - |x_{0,\lambda}|)^a > \left( \frac{2}{3} \right)^a \) if \( a \geq 0 \), and \( |x|^a \geq (2 + |x_{0,\lambda}|)^a > \left( \frac{2}{3} \right)^a \) if \( a < 0 \). By the first equation of (2.16), Jensen’s inequality and odd symmetry of \( u_\lambda \) with respect to \( \partial \mathbb{R}^n_+ \), we have, for any \( 1 \leq r \leq 2 \),

\[
(2.24) \quad u_{\frac{1}{2} - 1, \lambda}(r) = \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} |x|^a |u_\lambda|^p u_\lambda ds d\tau \\
= \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} \frac{1}{\partial B_s(x_{0,\lambda})} \int_{\partial B_s(x_{0,\lambda})} |x|^a |u_\lambda|^p u_\lambda(x) d\sigma ds d\tau \\
= \frac{1}{\omega_n} \int_0^r \frac{1}{\tau^{n-1}} \int_{B_r(x_{0,\lambda})} |x|^a |u_\lambda|^p u_\lambda(x) dx d\tau \\
= \frac{1}{\omega_n} \int_0^r \frac{1}{\tau^{n-1}} \int_{\Omega_r} |x|^a u_\lambda^p(x) d\tau \geq C_0' \int_1^r \frac{1}{\tau^{n-1}} \int_{\Omega_r} u_\lambda^p(x) d\tau \\
\geq C_0' \int_1^r \frac{\Omega_r}{\tau^{n-1}} \left( \frac{1}{\Omega_r} \int_{\Omega_r} u_\lambda^p(x) dx \right)^{\frac{1}{p}} d\tau \\
= C_0'' \int_1^r \frac{1}{\tau^{n-1}} \left( \int_{B_r(x_{0,\lambda})} \frac{|B_r(x_{0,\lambda})|}{\Omega_r} \right)^{p-1} \left( \int_{\Omega_r} u_\lambda(x) dx \right)^p d\tau \\
\geq C_0'' \int_1^r \frac{1}{\tau^{np-p}} \left( \int_{B_r(x_{0,\lambda})} u_\lambda(x) dx \right)^p d\tau \\
= C_0'' \int_1^r \frac{1}{\tau^{np-p}} \left( \int_0^\tau \int_{\partial B_s(x_{0,\lambda})} u_\lambda(x) d\sigma ds \right)^p d\tau \\
=: C_0 \int_1^r \frac{1}{\tau^{np-p}} \left( \int_0^\tau \omega(s) s^{n-1} ds \right)^p d\tau,
\]

where \( C_0 \in (0, 1] \) is a positive constant depending only on \( n, p \) and \( a \) if \( \lambda \) is large enough.

It follows from (2.23) and \( \frac{n+a}{p-1} + n - 2 > 0 \) that, we can choose \( \lambda \) sufficiently large to make \( \phi_{0,\lambda}(r) \) as large as we wish for \( 1 \leq r \leq 2 \). Apparently, \( u_{0,\lambda}(r) \geq 0 \), from \( \overline{u}_\lambda = u_{0,\lambda} + \phi_{0,\lambda} \) and (2.23), we can choose \( \lambda \) sufficiently large such that

\[
(2.25) \quad \overline{u}_\lambda(r) \geq a_0 (r - 1)^{\sigma_0}, \quad \forall 1 \leq r \leq 2,
\]

where \( a_0 \) and \( \sigma_0 \) are arbitrarily large, and will be determined later. By elementary calculation, it is easy to verify that

\[
(2.26) \quad \int_1^\tau (s - 1)^{\alpha} s^\beta ds \geq \frac{1}{\alpha + \beta + 1} (\tau - 1)^{\alpha + 1} \tau^\beta, \quad \forall \alpha, \beta > 0.
\]
By (2.24), (2.25) and (2.26), for any \(1 \leq r \leq 2\) and \(p > 1\), we obtain

\[
(2.27) \quad u_{r-1, \lambda}(r) \geq C_0 \int_1^r \frac{1}{\tau^{p(n-1)}} \left( \int_0^\tau u_{\lambda}(s)s^{n-1}ds \right)^p d\tau \\
\geq C_0 \int_1^r \frac{1}{\tau^{p(n-1)}} \left( \int_1^\tau a_0(s-1)^{\sigma_0}s^{n-1}ds \right)^p d\tau \\
\geq C_0 \int_1^r \frac{1}{\tau^{p(n-1)}} \left( \frac{a_0}{\sigma_0 + n} (\tau - 1)^{\sigma_0+1}\tau^{n-1} \right)^p d\tau \\
\geq \frac{C_0a_0^p}{(\sigma_0 + n)^p} \int_1^r (\tau - 1)^{(\sigma_0+1)p} d\tau = \frac{C_0a_0^p}{(\sigma_0 + n)^p[(\sigma_0 + 1)p + 1]}(r - 1)^{(\sigma_0+1)p+1}.
\]

Set \(\sigma_0\) large such that \(\sigma_0 \geq p + 2n\), then (2.27) implies that

\[
(2.28) \quad u_{r-1, \lambda}(r) \geq \frac{C_0a_0^p}{(2\sigma_0)^p(2\sigma_0p)}(r - 1)^{(\sigma_0+1)p+1}.
\]

From the second equation in (2.16) and (2.28), we have

\[
(2.29) \quad (r^{n-1}u_{r-2, \lambda})'(r) = r^{n-1}u_{r-1, \lambda}(r) \geq \frac{C_0a_0^p}{(2\sigma_0)^p(2\sigma_0p)}(r - 1)^{(\sigma_0+1)p+1}r^{n-1}, \quad \forall 1 \leq r \leq 2.
\]

Since \((r^{n-1}u_{r-2, \lambda})'(r) = r^{n-1}u_{r-1, \lambda}(r) \geq 0\) for any \(r \geq 0\), by (2.29), we derive

\[
(2.30) \quad r^{n-1}u_{r-2, \lambda}(r) = \int_0^r (\tau^{n-1}u_{\tau-2, \lambda}(\tau))'d\tau \\
\geq \frac{C_0a_0^p}{(2\sigma_0)^p(2\sigma_0p)} \int_1^r (\tau - 1)^{(\sigma_0+1)p+1}\tau^{n-1} d\tau \\
\geq \frac{C_0a_0^p}{(2\sigma_0)^p(2\sigma_0p)[(\sigma_0 + 1)p + n + 1]}(r - 1)^{(\sigma_0+1)p+2}r^{n-1}, \quad \forall 1 \leq r \leq 2,
\]

which means

\[
(2.31) \quad u_{r-2, \lambda}(r) \geq \frac{C_0a_0^p}{(2\sigma_0)^p(2\sigma_0p)[(\sigma_0 + 1)p + n + 1]}(r - 1)^{(\sigma_0+1)p+2}.
\]

By \((r^{n-1}u_{r-2, \lambda})'(r) = r^{n-1}u_{r-1, \lambda} \geq 0\) for any \(r \geq 0\), we have

\[
(2.32) \quad r^{n-1}u'_{r-2, \lambda}(r) = \int_0^r \tau^{n-1}u_{\tau-1, \lambda}(\tau)d\tau \geq 0, \quad \forall r \geq 0,
\]
thus $u'_{\frac{3}{2}, -\lambda}(r) \geq 0$, $\forall r \geq 0$. Similar to (2.30), by (2.31), we have

$$u_{\frac{3}{2}, -\lambda}(r) = \int_{0}^{r} u'_{\frac{3}{2}, -\lambda}(\tau) d\tau$$
$$\geq \int_{1}^{r} u'_{\frac{3}{2}, -\lambda}(\tau) d\tau$$

(2.33)

$$\geq \frac{C_{0}a_{0}^{p}}{(2\sigma_{0})^{p}(2\sigma_{0}p)[(\sigma_{0} + 1)p + n + 1]} \int_{1}^{r}(\tau - 1)^{(\sigma_{0} + 1)p + 2} d\tau$$
$$\geq \frac{C_{0}a_{0}^{p}}{(2\sigma_{0})^{p}(2\sigma_{0}p)[(\sigma_{0} + 1)p + n + 1][(\sigma_{0} + 1)p + 3]}(r - 1)^{(\sigma_{0} + 1)p + 3}$$

Continuing this way, we eventually obtain that

(2.34)

$$u_{0, \lambda}(r) \geq \frac{C_{0}a_{0}^{p}}{(2\sigma_{0})^{p}(2\sigma_{0}p)^{n-1}}(r - 1)^{(\sigma_{0} + 1)p + n - 1}, \quad \forall 1 \leq r \leq 2.$$  

Thus, we have

(2.35)

$$\frac{1}{u_{\lambda}}(r) \geq \frac{C_{0}a_{0}^{p}}{(2\sigma_{0})^{p}(2\sigma_{0}p)^{n-1}}(r - 1)^{(\sigma_{0} + 1)p + n - 1}, \quad \forall 1 \leq r \leq 2.$$  

We set $\sigma_{1} := 2p \sigma_{0}$ and $a_{1} = \frac{C_{0}a_{0}^{p}}{(2\sigma_{0})^{p}(2\sigma_{0}p)^{n-1}}$. Then, for any $1 \leq r \leq 2$, by (2.33), we have

$$\frac{1}{u_{\lambda}}(r) \geq a_{1}(r - 1)^{\sigma_{1}}, \quad \forall 1 \leq r \leq 2.$$  

Repeating the above arguments, we have

(2.36)

$$\frac{1}{u_{\lambda}}(r) \geq a_{k}(r - 1)^{\sigma_{k}}, \quad \forall 1 \leq r \leq 2,$$

where $\sigma_{k} = 2p \sigma_{k-1}$, $a_{k} = \frac{C_{0}a_{0}^{p}}{(2\sigma_{k-1})^{p}(2\sigma_{k-1}p)^{n-1}}$ and $k = 2, 3, \cdots$.  

We can prove that $a_{k} \to +\infty$ as $k \to +\infty$. In fact, by direct calculations, we have

(2.37)

$$a_{k} = \frac{C_{0}^{\frac{n-1}{p-1}} a_{0}^{\frac{1}{p-1}}}{(2p)^{(n-1+p)(k-1)p} \sigma_{0}^{\frac{n-1+p}{p-1}} \left[\left(\frac{C_{0}^{\frac{1}{p}}}{(2p)^{(n-1+p)p} \sigma_{0}^{\frac{n-1+p}{p-1}}}ight) \right]^{k-1}}$$

Take $a_{0} = 2C_{0}^{-\frac{1}{p-1}}(2p)^{(n-1+p)p} \sigma_{0}^{\frac{n-1+p}{p-1}}$, then by (2.36) and (2.37), we can see that

$$\frac{1}{u_{\lambda}}(2) \geq a_{k} \geq (2p)^{(n-1+p)p} \sigma_{0}^{\frac{n-1+p}{p-1}}k \to +\infty, \quad \text{as } k \to \infty,$$

which is absurd! Thus (2.4) must hold, that is, $(-\Delta)^{\frac{n}{2}-1}u \geq 0$ in $\mathbb{R}_{+}^{n}$.  

**Step 2.** Next, we will show that all the other $u_{k}(x) \geq 0$, $k = 1, 2, \cdots, \frac{n}{2} - 2$, $\forall x \in \mathbb{R}_{+}^{n}$.  

Suppose not, then there exists some $2 \leq i \leq \frac{n}{2} - 1$ and $x_{0} \in \mathbb{R}_{+}^{n}$ such that

(2.38) $u_{\frac{n}{2} - 1}(x) \geq 0, \quad v_{\frac{n}{2} - 2}(x) \geq 0, \quad \cdots, \quad v_{\frac{n}{2} - i + 1}(x) \geq 0, \quad \forall x \in \mathbb{R}_{+}^{n}$,

(2.39) $v_{\frac{n}{2} - i}(x_{0}) < 0$.  

Take spherical average with respect to the center \( x_0 \), we have \( \bar{v}_{\frac{n}{2} - i} \) satisfies

\[
\begin{cases}
-\Delta \bar{v}_{\frac{n}{2} - i}(r) = \bar{v}_{\frac{n}{2} - i + 1}, \\
-\Delta \bar{v}_{\frac{n}{2} - i - 1}(r) = \bar{v}_{\frac{n}{2} - i}(r), \\
\vdots \\
-\Delta \bar{u}(r) = \bar{u}(r).
\end{cases}
\]

By the first equation of (2.40), integrating both sides from 0 to \( r \), we arrive at

\[
-r^{n-1} \bar{v}_{\frac{n}{2} - i}(r) = \int_0^r s^{n-1} \bar{v}_{\frac{n}{2} - i + 1} ds = \frac{1}{\omega_n} \int_0^r \int_{\partial B_s(x_0)} v_{\frac{n}{2} - i + 1} d\sigma ds = \frac{1}{\omega_n} \int_{B_r(x_0)} v_{\frac{n}{2} - i + 1} dx \geq 0.
\]

Here we have used the fact that, since \( x_0 \in \mathbb{R}^n_+ \), more than half of the ball \( B_r(x_0) \) is contained in \( \mathbb{R}^n_+ \), so (2.41) follows from the odd symmetry of \( v_{\frac{n}{2} - i + 1} \) with respect to \( \partial \mathbb{R}^n_+ \).

As in Step 1, we can also derive a contradiction if \( \frac{n}{2} - i \) is even, hence we assume that \( \frac{n}{2} - i \) is odd hereafter. By the same arguments as in deriving (2.13) in Step 1, we can obtain that

\[
\bar{u}(r) \geq c_0 r^{2\left(\frac{n}{2} - i\right)} + \sum_{j=1}^{\frac{n}{2} - i} c_j r^{2\left(\frac{n}{2} - i - j\right)}, \quad \forall r \geq 0,
\]

where \( c_0 > 0 \) and \( c_{\frac{n}{2} - i} = \bar{u}(0) = u(x_0) \geq 0 \). Therefore, if we assume that \( u(x) = o(|x|^2) \) as \( |x| \to +\infty \), we will get a contradiction from (2.42) immediately. We only need to discuss under the assumptions \(-2p - 2 < a < +\infty \) hereafter.

Notice that there exists \( r_0 \) large enough such that if \( r \geq r_0 \), then \( \forall x \in \Omega_r, |x|^a \geq (r - |x_0|)^a \geq Cr^a \) if \( a \geq 0 \), and \( |x|^a \geq (r + |x_0|)^a \geq Cr^a \) if \( a < 0 \), furthermore, by (2.42), we can also get, if \( r \geq r_0 \),

\[
\bar{u}(r) \geq \frac{c_0}{2} r^{2\left(\frac{n}{2} - i\right)}.
\]
Similar to (2.24), by (2.8), Jensen’s inequality and odd symmetry of \( u \) with respect to \( \partial \mathbb{R}^n_+ \), we have for any \( r \geq r_0 \),

\[
- r^{n-1} \varphi_{\frac{n}{2}-1}(r) = \frac{1}{\omega_n} \int_{B_r(x_0)} |x|^a |u|^{p-1} u \, dx \\
= \frac{1}{\omega_n} \int_{\partial B_r(x_0)} |x|^a u^p \, d\sigma \geq C r^a \int_{\Omega_r} u^p \, dx \\
\geq \frac{C|\Omega_r|r^a}{\omega_n} \left( \frac{1}{|\Omega_r|} \int_{\Omega_r} u^p \, dx \right) \\
\geq \frac{C|\Omega_r|r^a}{\omega_n} \left( \frac{1}{|\Omega_r|} \int_{\Omega_r} u \, dx \right)^p \\
= C r^a \left( \frac{|B_r(x_0)|}{|\Omega_r|} \right)^{p-1} \frac{1}{\omega_n |B_r(x_0)|^{p-1}} \left( \int_{B_r(x_0)} u \, dx \right)^p \\
\geq \frac{C r^a}{\omega_n |B_r(x_0)|^{p-1}} \left( \int_{B_r(x_0)} u \, dx \right)^p \\
= \frac{C r^a}{\omega_n |B_r(x_0)|^{p-1}} \left( \int_{\partial B_r(x_0)} u \, d\sigma \right)^p \\
= \frac{C r^a}{\omega_n |B_r(x_0)|^{p-1}} \left( \int_0^r \bar{u}(s) s^{n-1} \, ds \right)^p \\
= \frac{C}{r^{np-n-a}} \left( \int_0^r \bar{u}(s) s^{n-1} \, ds \right)^p.
\]

Combining (2.44) with (2.43), we have, for all \( r \geq 2r_0 \),

\[
- r^{n-1} \varphi_{\frac{n}{2}-1}(r) \geq C \frac{1}{r^{np-n-a}} \left( \int_0^r \bar{u}(s) s^{n-1} \, ds \right)^p \\
\geq C \frac{1}{r^{np-n-a}} \left( \int_0^r s^{2(n-2)} s^{n-1} \, ds \right)^p \\
\geq C \frac{1}{r^{np-n-a}} r^{np+2(n-2)} \\
\geq C r^{n+2(n-2) - p + a}.
\]

That is,

\[
- \varphi_{\frac{n}{2}-1}(r) \leq - C r^{2(n-2) - p + a + 1}.
\]

Integrating (2.46) in both sides from fixed \( r_1 \geq 2r_0 \) to \( r \), we obtain, if \( a > -2 - 2p \), then

\[
- \varphi_{\frac{n}{2}-1}(r) \leq - C \left( r^{2p+a+2} - r_1^{2p+a+2} \right) + \varphi_{\frac{n}{2}-1}(r_1) \to -\infty, \quad \text{as } r \to +\infty;
\]

if \( a = -2 - 2p \), then

\[
- \varphi_{\frac{n}{2}-1}(r) \leq - C \left( \ln r - \ln r_1 \right) + \varphi_{\frac{n}{2}-1}(r_1) \to -\infty, \quad \text{as } r \to +\infty.
\]
However, from the proven fact that \( v_{n-1}^+ \geq 0, x \in \mathbb{R}^n_+ \), the odd symmetry of \( v_{n-1}^+ \) with respect to \( \partial \mathbb{R}^n_+ \) and (2.47), (2.48), we get that

\[
(2.49) \quad 0 \leq \frac{1}{\omega_n} \int_{B_r(x_0)} v_{n-1}^+ (x) dx = \frac{1}{\omega_n} \int_{B_r(x_0)} v_{n-1}^+ (x) dx + \frac{1}{\omega_n} \int_{B_r(x_0) \setminus B_{r_1}(x_0)} v_{n-1}^+ (x) dx
\]

\[
= \frac{1}{\omega_n} \int_{B_{r_1}(x_0)} v_{n-1}^+ (x) dx + \int_{r_1} r^{n-1} \frac{v_{n-1}^+}{n-1} (s) ds \to -\infty, \quad \text{as} \ r \to +\infty.
\]

This is a contradiction! Therefore, we arrive at \( v_k (x) \geq 0, k = 1, 2, \ldots, \frac{n}{2} - 2, \forall x \in \mathbb{R}^n_+ \).

This finishes our proof of Theorem 1.2.

3. Equivalence between PDE and IE

In this section, we prove the equivalence between PDE (1.1) and IE (1.3), namely, Theorem 1.3. We only need to prove that any nonnegative solution of PDE (1.1) also satisfies IE (1.3).

(i) We first consider the cases that \( n = 4 \) is even.

In section 2, we have proved that \( v_i = (-\Delta)^i u \geq 0 \) for \( i = 1, 2, \ldots, \frac{n}{2} - 1 \), then (1.1) is equivalent to the following system

\[
(3.1) \quad \begin{cases} 
-\Delta v_{n-1}^\pm (x) = |x|^a |u|^p (x), \\
-\Delta v_{n-2}^\pm (x) = v_{n-1}^\pm (x), \\
\ldots \\
-\Delta u (x) = v_1 (x).
\end{cases}
\]

In the following, similar as in [6], we define

\[
(3.2) \quad G(x, y, i) := c_n \left( \frac{1}{|x - y|^{n-i}} - \frac{1}{|\bar{x} - y|^{n-i}} \right), \quad x, y \in \mathbb{R}^n, \ i = 2, 4, \ldots, n - 2,
\]

and \( G^+ (x, y) := G(x, y, n) = C_n \left( \ln \frac{1}{|x - y|} - \ln \frac{1}{|\bar{x} - y|} \right) \).

Let \( B_R^\pm = B_R(0) \cap R^2_+ \). In [3], Cao and Chen derived the Green’s function \( G_R (x, y, 2) \) associated with \(-\Delta\) for the half ball \( B_R^\pm \), that is,

\[
G_R(x, y, 2) := \frac{c_n}{|x - y|^{n-2}} - \frac{c_n}{\left( |x - y|^2 + (R - |x|^2 R) \right) \left( |\bar{x} - y|^2 + (R - |\bar{x}|^2 R) \right)^{n-2}}
\]

\[
- \left( \frac{c_n}{|\bar{x} - y|^{n-2}} - \frac{c_n}{\left( |\bar{x} - y|^2 + (R - |\bar{x}|^2 R) \right) \left( |\bar{x} - y|^2 + (R - |\bar{x}|^2 R) \right)^{n-2}} \right).
\]

Now we list some essential properties of the above Green’s functions, which have been proved in [3] and [6]. First,

\[
(3.3) \quad G_R(x, y, 2) \to G(x, y, 2) = \frac{c_n}{|x - y|^{n-2}} - \frac{c_n}{|\bar{x} - y|^{n-2}}, \quad \text{as} \ R \to +\infty.
\]

Second, let \( \Gamma_R \) be the hemisphere part of \( \partial B_R^\pm \), then for each fixed \( x \in B_R^\pm \) and any \( y \in \Gamma_R \),

\[
(3.4) \quad \frac{\partial G_R (x, y, 2)}{\partial v_y} = (2 - n) R \left( 1 - \frac{|x|^2}{R^2} \right) \left( \frac{1}{|x - y|^{n-2}} - \frac{1}{|\bar{x} - y|^{n-2}} \right) < 0,
\]
moreover, for each fixed \( x \in B_R^+ \), any \( y \in \Gamma_R \) and \( R \) sufficiently large,
\[
G(x, y, 2) = \frac{c_n}{|x - y|^{n-2}} - \frac{c_n}{|\bar{x} - y|^{n-2}} \sim \frac{y_n}{R^n}, \quad \text{as } R \to +\infty,
\]
\[
|\partial G_R(x, y, 2)| = (n - 2)R \left(1 - \frac{|x|^2}{R^2}\right) \left(\frac{1}{|x - y|^n} - \frac{1}{|\bar{x} - y|^n}\right) \sim \frac{y_n}{R^{n+1}}.
\]

In [6], the authors prove the following property of Green’s function with different orders on a half space:
\[
G(x, y, 2k) = \int_{\mathbb{R}^n_+} G(x, z, 2)G(z, y, 2(k - 1))dz, \quad \forall k = 2, \cdots, \frac{n}{2} - 1.
\]

To continue, we need to prove that the above property is still valid for the critical order \( k = \frac{n}{2} \), this is the following crucial lemma.

**Lemma 3.1.** Assume that \( n \geq 4 \) is even, then
\[
G^+(x, y) = \int_{\mathbb{R}^n_+} G(x, z, 2)G(z, y, n - 2)dz.
\]

**Proof.** By elementary calculations, one can easily verify that
\[
-\Delta_y G^+(x, y) = G(x, y, n - 2),
\]
\[
G^+(x, y) = 0, \quad \text{if } x \text{ or } y \in \partial \mathbb{R}^n_+.
\]

For fixed \( x, z \in B_R^+ \), \( z \neq x \) and \( \epsilon \) sufficiently small, multiplying both side of (3.9) by \( G_R(y, z, 2) \) and integrating on \( B_R^+ \setminus B_\epsilon(z) \) by parts, we have
\[
\text{LHS} = \int_{B_R^+ \setminus B_\epsilon(z)} (-\Delta_y G^+(x, y))G_R(y, z, 2)dy
\]
\[
= \int_{B_R^+ \setminus B_\epsilon(z)} G^+(x, y) (-\Delta_y G_R(y, z, 2))dy
\]
\[
+ \int_{\partial B_\epsilon(z)} \left\{ -\frac{\partial G^+(x, y)}{\partial\nu_y}G_R(y, z, 2) + \frac{\partial G_R(y, z, 2)G^+(x, y)}{\partial\nu_y} \right\} d\sigma_y
\]
\[
+ \int_{\partial B_R^+} \left\{ -\frac{\partial G^+(x, y)}{\partial\nu_y}G_R(y, z, 2) + \frac{\partial G_R(y, z, 2)G^+(x, y)}{\partial\nu_y} \right\} d\sigma_y
\]
\[
= \int_{\partial B_\epsilon(z)} \left\{ -\frac{\partial G_R(y, z, 2)G^+(x, y)}{\partial\nu_y} + \frac{\partial G^+(x, y)}{\partial\nu_y} \right\} d\sigma_y
\]
\[
+ \int_{\Gamma_R} \left\{ \frac{\partial G_R(y, z, 2)G^+(x, y)}{\partial\nu_y} \right\} d\sigma_y
\]

Next, we will estimate the above integrals on the boundary. For arbitrarily fixed \( x \in B_R^+ \), by mean value theorem, we have
\[
G^+(x, y) = C_n \left(\ln \frac{1}{|x - y|} - \ln \frac{1}{|\bar{x} - y|}\right)
\]
\[
= C_n \left\{ \ln \left(|\bar{x} - y|\right)^2 - \ln \left(|x - y|\right)^2 \right\}
\]
\[
= C \frac{x_n y_n}{\xi^2} \sim \frac{y_n}{R^2}, \quad \forall y \in \partial B_R^+, \quad \text{as } R \to +\infty,
\]
where $\xi$ is valued between $|x - y|^2$ and $|x - y|^2$. By (3.6) and (3.12), we obtain

\begin{equation}
(3.13) \quad \left| \int_{\Gamma_R} \left\{ \frac{\partial G_R}{\partial \nu_y} (y, z, 2) G^+(x, y) \right\} d\sigma_y \right| \sim \frac{y^2_R}{R^3} \leq \frac{C}{R^2} \to 0, \quad \text{as } R \to +\infty. \tag{3.13}
\end{equation}

Since $\frac{\partial G_R}{\partial \nu_y}(y, z, 2) \sim \frac{1}{\epsilon^2}$, $G_R(y, z, 2) \sim \frac{1}{\epsilon}$ on $\partial B_\epsilon(z)$ and $G^+(x, y)$ is smooth in $B_\epsilon(z)$ for $\epsilon$ sufficiently small, we have

\begin{equation}
(3.14) \quad \int_{\partial B_\epsilon(z)} \left\{ -\frac{\partial G^+(x, y)}{\partial \nu_y} G_R(y, z, 2) + \frac{\partial G_R}{\partial \nu_y} (y, z, 2) G^+(x, y) \right\} d\sigma_y \to G^+(x, z), \tag{3.14}
\end{equation}

as $\epsilon \to 0+$. First let $\epsilon \to 0+$ and then let $R \to +\infty$, combining (3.11), (3.13) and (3.14), we derive

\begin{equation}
(3.15) \quad \text{LHS} \to G^+(x, z). \tag{3.15}
\end{equation}

As to the right-hand side, we have

\begin{equation}
(3.16) \quad \text{RHS} = \int_{B_R^n \setminus B_\epsilon(z)} G(x, y, n - 2) G_R(y, z, 2) dy. \tag{3.16}
\end{equation}

Since $G_R(y, z, 2)$ is smooth in $B_R^n \setminus B_\epsilon(z)$, $G(x, y, n - 2) \sim \frac{1}{|x-y|^2}$ near $x = y$ and $n \geq 4$, we derive from (3.15) that

\begin{equation}
(3.17) \quad \int_{B_R^n \setminus B_\epsilon(z)} G(x, y, n - 2) G_R(y, z, 2) dy \leq C < +\infty. \tag{3.17}
\end{equation}

First let $\epsilon \to 0$ and then let $R \to \infty$, by (3.3) and dominated convergence theorem, we deduce

\begin{equation}
(3.18) \quad \text{RHS} = \int_{B_R^n \setminus B_\epsilon(z)} G(x, y, n - 2) G_R(y, z, 2) dy \to \int_{R_n^+} G(x, y, n - 2) G(y, z, 2) dy \tag{3.18}
\end{equation}

Combining (3.15) and (3.18), we arrive at

\begin{equation}
(3.19) \quad G^+(x, z) = \int_{R_n^+} G(x, y, n - 2) G(y, z, 2) dy \tag{3.19}
\end{equation}

This completes the proof of Lemma 3.1

Now we can prove $u$ is also a solution of the integral equation (1.3) using Lemma 3.1. Multiplying both sides of (3.11) by $G_R(x, y, 2)$ and integrating on $B_R^+$ by parts, we have

\begin{equation}
(3.20) \quad \begin{cases}
\int_{B_R^n} G_R(x, y, 2) |y|^a u^p(y) dy = v_{n-1}^{\frac{n}{2}}(x) + \int_{\Gamma_R} v_{n-1}^{\frac{n}{2}}(y) \frac{\partial G_R(x, y, 2)}{\partial \nu_y} d\sigma_y, \\
\int_{B_R^n} G_R(x, y, 2) v_{n-2}^{\frac{n}{2}}(y) dy = v_{n-2}^{\frac{n}{2}}(x) + \int_{\Gamma_R} v_{n-2}^{\frac{n}{2}}(y) \frac{\partial G_R(x, y, 2)}{\partial \nu_y} d\sigma_y, \\
\quad \cdot \\
\int_{B_R^n} G_R(x, y, 2) v_1(y) dy = u(x) + \int_{\Gamma_R} u(y) \frac{\partial G_R(x, y, 2)}{\partial \nu_y} d\sigma_y.
\end{cases} \tag{3.20}
\end{equation}
By Theorem 1.2, (3.4) and (3.20), we derive

\[
\begin{align*}
\int_{B^+} G_R(x,y,2) y^\alpha u^p(y) dy & \leq v^\frac{1}{4} - 1(x), \\
\int_{B^+} G_R(x,y,2) v^\frac{1}{4} - 2(y) dy & \leq v^\frac{1}{4} - 2(x), \\
\cdot \cdot \cdot \\
\int_{B^+} G_R(x,y,2) v_1(y) dy & \leq u(x).
\end{align*}
\]

(3.21)

Letting \( R \to \infty \), and by (3.3), we deduce

\[
\begin{align*}
\int_{R^n} G(x,y,2) y^\alpha u^p(y) dy & < \infty, \\
\int_{R^n} G(x,y,2) v^\frac{1}{4} - 1(y) dy & < \infty, \\
\cdot \cdot \cdot \\
\int_{R^n} G(x,y,2) v_1(y) dy & < \infty.
\end{align*}
\]

(3.22)

By (3.22), we conclude that there exists a sequence \( R_j \to \infty \) such that

\[
\begin{align*}
R_j \int_{\Gamma_{R_j}} G(x,y,2) v_i(y) d\sigma_y & \to 0, \quad \text{as } R_j \to \infty, \quad i = 1, \ldots, \frac{n}{2} - 1, \\
R_j \int_{\Gamma_{R_j}} G(x,y,2) |y|^\alpha u(y) d\sigma_y & \to 0, \quad \text{as } R_j \to \infty.
\end{align*}
\]

(3.23)

Then from (3.23), it follows easily that

\[
\begin{align*}
\frac{1}{R_j^{n-1}} \int_{\Gamma_{R_j}} v_i(y) y_n d\sigma_y & \to 0, \quad \text{as } R_j \to \infty, \quad i = 1, \ldots, \frac{n}{2} - 1, \\
\frac{1}{R_j^{n-1-a}} \int_{\Gamma_{R_j}} u^p(y) y_n d\sigma_y & \to 0, \quad \text{as } R_j \to \infty.
\end{align*}
\]

(3.24)

As an immediate consequence of (3.25), we have

\[
\begin{align*}
\frac{1}{R_j^{n+1}} \int_{\Gamma_{R_j}} v_i(y) y_n d\sigma_y & \to 0, \quad \text{as } R_j \to \infty, \quad i = 1, \ldots, \frac{n}{2} - 1.
\end{align*}
\]

(3.26)

By (3.26), Hölder inequality and the fact that \( 1 + a + p > 0 \), we derive

\[
\begin{align*}
\frac{1}{R_j^{n+1}} \int_{\Gamma_{R_j}} u(y) y_n d\sigma_y & \leq \frac{R_j^{-\frac{1}{p} - \frac{a}{p}}}{R_j^{\frac{n}{p} + 1}} \left( \frac{1}{R_j^{n-1-a}} \int_{\Gamma_{R_j}} u^p(y) y_n d\sigma_y \right)^{\frac{1}{p}} \left( \int_{\Gamma_{R_j}} y_n d\sigma_y \right)^{1 - \frac{1}{p}} \\
& \leq \frac{R_j^{-\frac{1}{p} - \frac{a}{p}}}{R_j^{\frac{n}{p} + 1}} \left( \frac{1}{R_j^{n-1-a}} \int_{\Gamma_{R_j}} u^p(y) y_n d\sigma_y \right)^{\frac{1}{p}} R_j^{n(1 - \frac{1}{p})} \\
& \leq R_j^{\frac{1 + a + p}{p}} o(1) \to 0, \quad \text{as } R_j \to \infty.
\end{align*}
\]

(3.27)
Substituting (3.6) into (3.20), and by (3.3), (3.27), (3.28), we arrive at

\[
\begin{align*}
    v_{n-1}^{+}(x) &= \int_{\mathbb{R}^n_+} G(x, y, 2) |y|^a u^p(y) dy, \\
    v_{n-2}^{+}(x) &= \int_{\mathbb{R}^n_+} G(x, y, 2) v_{n-1}^{+}(y) dy, \\
    &\quad \ldots, \\
    u(x) &= \int_{\mathbb{R}^n_+} G(x, y, 2) v_1(y) dy.
\end{align*}
\]

Then by (3.7) and Lemma 3.1, it is easy to see that

\[
u(x) = \int_{\mathbb{R}^n_+} G(x, y, 2) v_1(y) dy
\]

(ii) Next, we consider the case \(n = 2\).

For \(n = 2\), the Green’s function \(G_R^{+}(x, y)\) associated with \(-\Delta\) for the half disk \(B_R^{+}\) is given by

\[
G_R^{+}(x, y) := \frac{1}{2\pi} \left( \ln \left| \frac{1}{x-y} - \ln \frac{1}{R^2 |x|^2 - y} \right) - \frac{1}{2\pi} \left( \ln \frac{1}{|x-y|} - \ln \frac{1}{R^2 |x|^2 - y} \right) \right).
\]

Now we give some essential properties of the above Green’s functions. First,

\[
G_R^{+}(x, y) \rightarrow G^{+}(x, y) = \frac{1}{2\pi} \left( \ln \left| \frac{1}{x-y} - \ln \frac{1}{|x-y|} \right) \right), \quad \text{as } R \rightarrow +\infty.
\]

Second, let \(\Gamma_R\) be the semi-circle part of \(\partial B_R^{+}\), then for each fixed \(x \in B_R^{+}\) and any \(y \in \Gamma_R\),

\[
\frac{\partial G_R^{+}(x, y)}{\partial y^i} = -\frac{1}{2\pi} R \left( 1 - \frac{|x|^2}{R^2} \right) \left( \frac{1}{|x-y|^2} - \frac{1}{|\bar{x}-y|^2} \right) < 0,
\]

moreover, for each fixed \(x \in B_R^{+}\), any \(y \in \Gamma_R\) and \(R\) sufficiently large,

\[
G^{+}(x, y) = \frac{1}{2\pi} \left( \ln \frac{1}{|x-y|} - \ln \frac{1}{|\bar{x}-y|} \right) \sim \frac{y_2}{R^2}, \quad \text{as } R \rightarrow +\infty,
\]

\[
\left| \frac{\partial G_R^{+}(x, y)}{\partial y^i} \right| = \frac{1}{2\pi} R \left( 1 - \frac{|x|^2}{R^2} \right) \left( \frac{1}{|x-y|^2} - \frac{1}{|\bar{x}-y|^2} \right) \sim \frac{y_2}{R^3}.
\]

Multiplying both sides of PDE (1.11) by \(G_R^{+}(x, y)\) and integrating on \(B_R^{+}\) by parts, we have

\[
\int_{B_R^{+}} G_R^{+}(x, y) |y|^a u^p(y) dy = u(x) + \int_{\Gamma_R} u(y) \frac{\partial G_R^{+}(x, y)}{\partial y^i} d\sigma_y.
\]
By (3.33) and (3.36), we derive

\[(3.37) \quad \int_{B_R^+} G_R^+(x, y)|y|^{a} u^p(y)dy \leq u(x).\]

By letting \(R \to \infty\) and (3.32), we deduce

\[(3.38) \quad \int_{\mathbb{R}^n_+} G^+(x, y)|y|^{a} u^p(y)dy < \infty.\]

By (3.38), we conclude that there exists a sequence \(R_j \to \infty\) such that

\[(3.39) \quad R_j \int_{\Gamma_{R_j}} G^+(x, y)|y|^{a} u(y)d\sigma_y \to 0, \quad \text{as } R_j \to \infty,\]

and hence, from (3.34), it follows easily that

\[(3.40) \quad \frac{1}{R_j^{a-p}} \int_{\Gamma_{R_j}} u^p(y) y_2 d\sigma_y \to 0, \quad \text{as } R_j \to \infty.\]

By (3.40), Hölder inequality and the fact that \(1 + a + p > 0\), we derive

\[(3.41) \quad \frac{1}{R_j^{a-p}} \int_{\Gamma_{R_j}} u(y) y_2 d\sigma_y \leq \frac{R_j^{1-a}}{R_j^{a-p}} \left( \frac{1}{R_j^{a}} \int_{\Gamma_{R_j}} u^p(y) y_2 d\sigma_y \right)^{\frac{1}{p}} \left( \int_{\Gamma_{R_j}} y_2 d\sigma_y \right)^{1-\frac{1}{p}} \leq R_j^{\frac{1-a}{p}} o(1) \to 0, \quad \text{as } R_j \to \infty.\]

Substituting (3.35) into (3.36), and by (3.32), (3.41), we finally arrive at

\[(3.42) \quad u(x) = \int_{\mathbb{R}^n_+} G^+(x, y)|y|^{a} u^p(y)dy.\]

This completes our proof of Theorem 1.3.

4. THE PROOF OF THEOREM 1.4

In this section, we will carry out the proof of Theorem 1.4 by applying the method of scaling spheres in integral forms developed by Dai and Qin in [19].

Suppose \(u\) is a nonnegative continuous solution of IE (1.3) but \(u \not\equiv 0\), we will derive a contradiction via the method of scaling spheres in integral forms.

In order to apply the method of scaling spheres, we first give some definitions. One can easily see that \(u \not\equiv 0\) implies \(u > 0\) in \(\mathbb{R}^n_+\). Let \(\lambda > 0\) be an arbitrary positive real number and let the scaling half sphere be

\[(4.1) \quad S^+_\lambda = \{ x \in \mathbb{R}^n_+ : |x| = \lambda \}.\]

We denote the reflection of \(x\) about the sphere \(\{ x \in \mathbb{R}^n_+ : |x| = \lambda \}\) by \(x^\lambda := \lambda^2 x^2 |x|^2\) and let

\[(4.2) \quad B^+_\lambda(0) := B_\lambda(0) \cap \mathbb{R}^n_+, \quad \widetilde{B}^+_\lambda(0) := \{ x \in \mathbb{R}^n_+ : x^\lambda \in B^+_\lambda(0) \}.\]
Define the Kelvin transform of \( u \) centered at 0 by
\[
(4.3) \quad u_\lambda(x) = u \left( \frac{\lambda^2 x}{|x|^2} \right)
\]
for arbitrary \( x \in \mathbb{R}^n_+ \setminus \{0\} \). It’s obvious that the Kelvin transform \( u_\lambda \) may have singularity at 0 and \( \lim_{|x| \to \infty} u_\lambda(x) = u(0) = 0 \). By (4.3), one can infer from the regularity assumptions on \( u \) that \( u_\lambda \in C(\mathbb{R}^n_+ \setminus \{0\}) \).

By direct calculations, one can verify that
\[
(4.4) \quad G^+(x, y) > G^+(x, y^\lambda), \quad \forall \, x, y \in B_\lambda^+(0),
\]
\[
(4.5) \quad |x^\lambda - y| = \frac{|y|}{|x|} |x - y^\lambda|, \quad |x| = |\bar{x}|, \quad \bar{x}^\lambda = \bar{x}^\lambda.
\]

We can deduce from (1.3), (4.3) and (4.5) that (for the invariance properties of fractional or higher order Laplacians under the Kelvin type transforms, please refer to \[8, 9, 12, 28, 41\])
\[
(4.6) \quad u_\lambda(x) = u \left( \frac{\lambda^2 x}{|x|^2} \right) = \int_{\mathbb{R}^n_+} G^+(x^\lambda, y) |y|^a u^p(y) dy
\]
\[
= C_n \int_{\mathbb{R}^n_+} \left( \ln \frac{1}{|x^\lambda - y|} - \ln \frac{1}{|x - y^\lambda|} \right) |y|^a u^p(y) dy
\]
\[
= C_n \int_{\mathbb{R}^n_+} \left( \ln \frac{1}{|x^\lambda - y|} - \ln \frac{1}{|\bar{x} - y^\lambda|} \right) |y|^a u^p(y) dy
\]
\[
= C_n \int_{\mathbb{R}^n_+} \left( \ln \frac{|x|}{|y||x - y^\lambda|} - \ln \frac{|\bar{x}|}{|y||\bar{x} - y^\lambda|} \right) |y|^a u^p(y) dy
\]
\[
= C_n \int_{\mathbb{R}^n_+} \left( \ln \frac{1}{|x - y^\lambda|} - \ln \frac{1}{|\bar{x} - y^\lambda|} \right) |y|^a u^p(y) dy
\]
\[
= \int_{\mathbb{R}^n_+} G^+(x, y^\lambda) |y|^a u^p(y) dy.
\]

Let \( \omega^\lambda(x) := u_\lambda(x) - u(x) \) for any \( x \in \overline{B_\lambda^+(0)} \setminus \{0\} \). Since \( u \) satisfies (1.3), by changing variables, we have
\[
(4.7) \quad u(x) = \int_{\mathbb{R}^n_+} G^+(x, y) |y|^a u^p(y) dy
\]
\[
= \int_{B_\lambda^+(0)} G^+(x, y) |y|^a u^p(y) dy + \int_{\mathbb{R}^n_+ \setminus B_\lambda^+(0)} G^+(x, y) |y|^a u^p(y) dy
\]
\[
= \int_{B_\lambda^+(0)} G^+(x, y) |y|^a u^p(y) dy + \int_{B_\lambda^+(0)} \left( \frac{\lambda}{|y|} \right)^{2(n+a)} G^+(x, y^\lambda) |y|^a u_\lambda^p(y) dy.
\]

Similarly, by (4.6), we obtain
\[
(4.8) \quad u_\lambda(x) = \int_{\mathbb{R}^n_+} G^+(x, y^\lambda) |y|^a u^p(y) dy
\]
\[
= \int_{B_\lambda^+(0)} G^+(x, y^\lambda) |y|^a u^p(y) dy + \int_{B_\lambda^+(0)} \left( \frac{\lambda}{|y|} \right)^{2(n+a)} G^+(x, y) |y|^a u_\lambda^p(y) dy.
\]
Then, by (4.7) and (4.8), we arrive at
\begin{equation}
\omega^{\lambda}(x) = u^{\lambda}(x) - u(x)
\end{equation}
\begin{equation}
= \int_{B_{\lambda}(0)} \left( G^+(x,y) - G^+(x,\lambda^{\lambda}) \right) |y|^a \left( \left( \frac{\lambda}{|y|} \right)^{2(n+a)} u^{\lambda}_{\lambda}(y) - u^{\lambda}(y) \right) dy
\end{equation}
for every $x \in B_{\lambda}^+(0)$.

Now we can carry out the process of scaling spheres in two steps.

**Step 1.** Start dilating the sphere from near $\lambda = 0$. We will first show that, for $\lambda > 0$ sufficiently small,
\begin{equation}
\omega^{\lambda}(x) \geq 0, \quad \forall \ x \in B_{\lambda}^+(0).
\end{equation}

Define
\begin{equation}
(B_{\lambda}^+)^- := \{ x \in B_{\lambda}^+(0) | \omega_{\lambda}(x) < 0 \}.
\end{equation}

Through elementary calculations, one can obtain that for any $x, y \in B_{\lambda}^+(0)$, $x \neq y$,
\begin{equation}
G^+(x, y) = C_n \left( \ln \frac{1}{|x-y|} - \ln \frac{1}{|x-y|} \right)
\end{equation}
\begin{equation}
= C_n \frac{1}{2} \ln \frac{|\vec{x} \cdot \vec{y}|^2}{|x-y|^2}
\end{equation}
\begin{equation}
= C \ln \left( 1 + \frac{4x_n y_n}{|x-y|^2} \right)
\end{equation}
\begin{equation}
\leq C \ln \left( 1 + \frac{4\lambda^2}{|x-y|^2} \right).
\end{equation}

It is well known that
\begin{equation}
\ln (1 + t) = o(t^\varepsilon), \quad \text{as } t \to +\infty,
\end{equation}
where $\varepsilon$ is an arbitrary positive real number. This implies, for any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that
\begin{equation}
\ln (1 + t) \leq t^\varepsilon, \quad \forall \ t > \frac{4}{\delta(\varepsilon)^2}.
\end{equation}

Therefore, by (4.12), (4.14) and straightforward calculations, we have the following lemma that states some basic estimates for Green’s function $G^+(x, y)$.

**Lemma 4.1.** Assume $G^+(x, y)$ be the Green’s functions in integral equation (1.3). Then we have
\begin{equation}
G^+(x, y) \leq C \lambda^{2\varepsilon} \frac{1}{|x-y|^{2\varepsilon}}, \quad \forall \ x, y \in B_{\lambda}^+(0), \ |x-y| < \lambda \delta(\varepsilon);
\end{equation}
\begin{equation}
G^+(x, y) \leq C \ln \left( 1 + \frac{4}{\delta(\varepsilon)^2} \right), \quad \forall \ x, y \in B_{\lambda}^+(0), \ |x-y| \geq \lambda \delta(\varepsilon);
\end{equation}
\begin{equation}
G^+(x, y) \leq C' \frac{4x_n y_n}{|x-y|^2}, \quad \forall \ x, y \in \mathbb{R}^n_+, \ x \neq y;
\end{equation}
\begin{equation}
G^+(x, y) \geq C'' \frac{4x_n y_n}{|x-y|^2}, \quad \forall \ x, y \in \mathbb{R}^n_+, \ \frac{|x|}{|y|} \leq \frac{1}{100} \text{ or } \frac{|y|}{|x|} \leq \frac{1}{100}.
\end{equation}
By the assumption \( a > -n \), (4.11), (4.12), (4.15) and (4.16), we have, for any \( x \in (B^+_\lambda)^- \),

\[
\omega^\lambda(x) = \int_{B^+_\lambda(0)} (G^+(x, y) - G^+(x, y^\lambda)) \frac{(\lambda)^{2(n+\alpha)}}{|y|} |u_\lambda^p(y) - u^p(y)| dy
\]

\[
> \int_{B^+_\lambda(0)} (G^+(x, y) - G^+(x, y^\lambda)) |y| \frac{(\lambda)^{2(n+\alpha)}}{|y|} |u_\lambda^p(y) - u^p(y)| dy
\]

(4.19)

\[
\geq \int_{(B^+_\lambda)^-} (G^+(x, y) - G^+(x, y^\lambda)) |y| a \frac{(\lambda)^{2(n+\alpha)}}{|y|} |u_\lambda^p(y) - u^p(y)| dy
\]

\[
\geq \int_{(B^+_\lambda)^-} C\lambda^{2\epsilon} \int_{(B^+_\lambda)^- \cap B_{\delta(x)}(x)} \frac{1}{|x - y|^{2\epsilon}} |y| a u^{p-1}(y) \omega^\lambda(y) dy
\]

\[
+ C(\delta(x)) \int_{(B^+_\lambda)^- \setminus B_{\delta(x)}(x)} |y|^a u^{p-1}(y) \omega^\lambda(y) dy.
\]

By (4.19), Hardy-Littlewood-Sobolev inequality and Hölder inequality, for arbitrary \( \frac{n}{2\epsilon} < q < \infty \), we obtain

\[
\|\omega^\lambda\|_{L^q((B^+_\lambda)^-)} \leq C\lambda^{2\epsilon} \left\| \int_{(B^+_\lambda)^- \cap B_{\delta(x)}(x)} \frac{1}{|x - y|^{2\epsilon}} |y|^a u^{p-1}(y) \omega^\lambda(y) dy \right\|_{L^q((B^+_\lambda)^-)}
\]

\[
+ C(\delta(x)) (B^+_\lambda)^{-\frac{1}{2}} \int_{(B^+_\lambda)^-} |y|^a u^{p-1}(y) \omega^\lambda(y) dy
\]

\[
\leq C\lambda^{2\epsilon} \left\| \int_{(B^+_\lambda)^- \cap B_{\delta(x)}(x)} \frac{1}{|x - y|^{2\epsilon}} |y|^a u^{p-1}(y) \omega^\lambda(y) dy \right\|_{L^q((B^+_\lambda)^-)}
\]

(4.20)

\[
+ C(\delta(x)) (B^+_\lambda)^{-\frac{1}{2}} \int_{(B^+_\lambda)^-} |y|^a u^{p-1}(y) \omega^\lambda(y) dy
\]

\[
\leq C\lambda^{2\epsilon} \left\| |x|^a u^{p-1} \omega^\lambda \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)}
\]

\[
+ C(\delta(x)) (B^+_\lambda)^{-\frac{1}{2}} \left\| |x|^a u^{p-1} \omega^\lambda \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)}
\]

\[
\leq C\lambda^{2\epsilon} \left\| |x|^a u^{p-1} \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)} \|\omega^\lambda\|_{L^q((B^+_\lambda)^-)}
\]

\[
+ C(\delta(x)) (B^+_\lambda)^{-\frac{1}{2}} \left\| |x|^a u^{p-1} \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)} \|\omega^\lambda\|_{L^q((B^+_\lambda)^-)}.
\]

Since \( a > -n \), we first choose \( \epsilon > 0 \) sufficiently small such that \( \frac{n}{n - 2\epsilon} > -n \), then choose \( q > \frac{n}{2\epsilon} \) sufficiently large such that \( \frac{2q}{q - 1} > -n \). Then from the assumption that \( u \) is continuous in \( \mathbb{R}_+^n \), it is obvious that \( |x|^a u^{p-1} \in L^{\frac{n}{2\epsilon}}(\mathbb{R}_+^n) \cap L^{\frac{n}{q - 1}}(\mathbb{R}_+^n) \). Therefore, there exists \( \delta_0 > 0 \) sufficiently small, such that

(4.21)

\[
C\lambda^{2\epsilon} \left\| |x|^a u^{p-1} \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)} + C(\delta(x)) (B^+_\lambda)^{-\frac{1}{2}} \left\| |x|^a u^{p-1} \right\|_{L^{\frac{n}{2\epsilon}}((B^+_\lambda)^-)} < \frac{1}{2}
\]
for any $0 < \lambda < \delta_0$. Thus, by (4.20), we must have
\begin{equation}
\|\omega^\lambda\|_{L^q((B_\lambda^+)^-)} = 0
\end{equation}
for any $0 < \lambda < \delta_0$. From the continuity of $\omega^\lambda$ in $\mathbb{R}^n_+ \setminus \{0\}$ and the definition of $(B_\lambda^+)^-$, we immediately get that $(B_\lambda^+)^- = \emptyset$ and hence (4.10) holds true for any $0 < \lambda < \delta_0$. This completes Step 1.

Step 2. Dilate the half sphere $S_\lambda^+$ outward until $\lambda = +\infty$ to derive lower bound estimates on $u$ in a cone. Step 1 provides us a starting point to dilate the half sphere $S_\lambda^+$ from near $\lambda = 0$. Now we dilate the half sphere $S_\lambda^+$ outward as long as (4.10) holds. Let
\begin{equation}
\lambda_0 := \sup\{\lambda > 0 \mid |\omega^\mu| \geq 0 \text{ in } B_\mu^+(0), \ \forall 0 < \mu \leq \lambda\} \in (0, +\infty],
\end{equation}
and hence, one has
\begin{equation}
\omega^\lambda(x) \geq 0, \quad \forall x \in B_\lambda^+(0).
\end{equation}

In what follows, we will prove $\lambda_0 = +\infty$ by driving a contradiction under the assumption that $\lambda_0 < +\infty$.

In fact, suppose $\lambda_0 < +\infty$, we must have
\begin{equation}
\omega^\lambda_0(x) \equiv 0, \quad \forall x \in B_\lambda^+(0).
\end{equation}

Suppose on the contrary that (4.25) does not hold, that is, there exists a $x_0 \in B_\lambda^+(0)$ such that $\omega^\lambda_0(x_0) > 0$. Then, by (4.19) and (4.24), we have
\begin{equation}
\omega^\lambda_0(x) > 0, \quad \forall x \in B_\lambda^+(0).
\end{equation}

Choose $\delta_1 > 0$ sufficiently small, which will be determined later. Define the narrow region
\begin{equation}
A_{\delta_1} := \{x \in B_{\lambda_0}^+(0) \mid \text{dist}(x, \partial B_{\lambda_0}^+(0)) < \delta_1\}.
\end{equation}
Note that $A_{\delta_1} = \{x \in B_{\lambda_0}^+(0) \mid x_n < \delta_1 \text{ or } \lambda_0 - \delta_1 < |x| < \lambda_0\}$.

Since that $\omega^\lambda_0$ is continuous in $\mathbb{R}^n_+ \setminus \{0\}$ and $A_{\delta_1} := B_{\lambda_0}^+(0) \setminus A_{\delta_1}$ is a compact subset, there exists a positive constant $C_0$ such that
\begin{equation}
\omega^\lambda_0(x) > C_0, \quad \forall x \in A_{\delta_1}^c.
\end{equation}
By continuity, we can choose $\delta_2 > 0$ sufficiently small, such that, for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2],$
\begin{equation}
\omega^\lambda(x) > \frac{C_0}{2}, \quad \forall x \in A_{\delta_1}^c.
\end{equation}

Hence we must have
\begin{equation}
(B_\lambda^-)^- \subset B_\lambda^+(0) \setminus A_{\delta_1} := (B_\lambda^+(0) \setminus B_{\lambda_0}^+(0)) \cup A_{\delta_1}
\end{equation}
for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2]$. By (4.20) and local integrability of $|x|^a u^{p-1}$ in $\mathbb{R}^n_+$, we can choose $\delta_1$ and $\delta_2$ sufficiently small such that, for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2],$
\begin{equation}
C_\lambda 2^{2r} \|x|^a u^{p-1}\|_{L^{\frac{p}{2}}(\mathbb{R}^n_+ \setminus (B_\lambda^+)^-) + C(\delta(\varepsilon))(B_\lambda^-)^- \|x|^a u^{p-1}\|_{L^{\frac{p}{2}}(\mathbb{R}^n_+ \setminus (B_\lambda^+)^-)} < \frac{1}{2}.
\end{equation}
Then, by the same argument as in step 1, we obtain that for any $\lambda \in [\lambda_0, \lambda_0 + \delta_2],$
\begin{equation}
\omega^\lambda(x) \geq 0, \quad \forall x \in B_\lambda^+(0).
\end{equation}
This contradicts the definition of $\lambda_0$. Hence, (4.25) must hold true.
However, by (4.24), (4.33), (4.25) and the fact that \( n + a > 0 \), we obtain
\[
0 = \omega^{\lambda_0}(x) = u_{\lambda_0}(x) - u(x)
\]
\[
= \int_{B^+_\lambda_0(0)} \left( G^+(x, y) - G^+(x, y^{\lambda_0}) \right) |y|^a \left( \left( \frac{\lambda_0}{|y|} \right)^{2(n+a)} - 1 \right) u^p(y) dy > 0
\]
for any \( x \in B^+_\lambda_0(0) \), which is absurd. Thus we must have \( \lambda_0 = +\infty \), that is,
\[
\lambda_0 \geq \lambda_0 \geq \lambda , \quad \forall \ 0 < \lambda < +\infty.
\]
Therefore, we obtain that \( u \) is radially nondecreasing. For arbitrary \( |x| \geq 1 \), \( x \in \mathbb{R}^n_+ \), let \( \lambda := \sqrt{|x|} \), then (4.34) yields that
\[
\lambda_0 \geq \lambda_0 \geq \lambda , \quad \forall \ 0 < \lambda < +\infty.
\]
and hence, we arrive at the following lower bound estimate:
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

The lower bound estimate (4.36) can be improved remarkably using the “Bootstrap” iteration technique and the integral equation (1.3).

In fact, let \( \mu_0 := 0 \), we infer from the integral equation (1.3), (4.18) and (4.36) that, for any \( |x| \geq 1 \) and \( x_n \geq \frac{|x|}{\sqrt{n}} \),
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

Therefore, we obtain that \( u \) is radially nondecreasing. For arbitrary \( |x| \geq 1 \), \( x \in \mathbb{R}^n_+ \), let \( \lambda := \sqrt{|x|} \), then (4.34) yields that
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

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In fact, let \( \mu_0 := 0 \), we infer from the integral equation (1.3), (4.18) and (4.36) that, for any \( |x| \geq 1 \) and \( x_n \geq \frac{|x|}{\sqrt{n}} \),
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

where we have used the fact \( (n + a) + p\mu_0 > 0 \) since \( a > 0 \) and \( 1 < p < +\infty \). Let \( \mu_1 := p\mu_0 + (n + a) \). Due to \( 1 < p < +\infty \) and \( a > 0 \), our important observation is
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

Thus we have obtained a better lower bound estimate than (4.36) after one iteration, that is,
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

For \( k = 0, 1, 2, \cdots \), define
\[
u_n \geq u \left( \frac{x}{|x|} \right), \quad \forall \ |x| \geq 1 \quad \forall \ n \geq \frac{|x|}{\sqrt{n}}.
\]

Since \( a > 0 \) and \( 1 < p < +\infty \), it is easy to see that the sequence \( \{\mu_k\} \) is monotone increasing with respect to \( k \) and \( (n + a) + p\mu_k > 0 \) for any \( k = 0, 1, 2, \cdots \). Continuing the above iteration
process involving the integral equation (1.3), we have the following lower bound estimates for every $k = 0, 1, 2, \cdots$,

\[(4.41) \quad u(x) \geq C_k |x|^{\mu_k}, \quad \forall \ |x| \geq 1, \ x_n \geq \frac{|x|}{\sqrt{n}}. \]

From (4.41) and the obvious property that

\[(4.42) \quad \mu_k \to +\infty, \quad \text{as} \quad k \to +\infty, \]

we can conclude the following lower bound estimates for positive solution $u$.

**Theorem 4.2.** Assume $n \geq 1$, $a > -n$ and $1 \leq p < +\infty$. Suppose $u \in C(\mathbb{R}_+^n)$ is a positive solution to IE (1.3), then it satisfies the following lower bound estimates: for all $x \in \mathbb{R}_+^n$ satisfying $|x| \geq 1$ and $x_n \geq \frac{|x|}{\sqrt{n}}$,

\[(4.43) \quad u(x) \geq C_\kappa |x|^{\kappa}, \quad \forall \ \kappa < +\infty. \]

The lower bound estimates in Theorem 4.2 obviously contradicts the integral equation (1.3). In fact, since $a > -n$, by (4.18) and (4.36), we have

\[(4.44) \quad +\infty > u \left( \frac{e_n}{100} \right) = \int_{\mathbb{R}_+^n} G \left( \frac{e_n}{100}, y \right) |y|^a u^p(y) dy \]

\[\geq \int_{|y| \geq 1, y_n \geq \frac{|y|}{\sqrt{n}}} G \left( \frac{e_n}{100}, y \right) |y|^a u^p(y) dy \]

\[\geq C \int_{|y| \geq 1, y_n \geq \frac{|y|}{\sqrt{n}}} \frac{y_n}{|y|^2} |y|^{a+1} dy \]

\[\geq C \int_{|y| \geq 1, y_n \geq \frac{|y|}{\sqrt{n}}} |y|^a dy \]

\[= +\infty, \]

where the unit vector $e_n := (0, \cdots, 0, 1) \in \mathbb{R}_+^n$. This is a contradiction! Thus we must have $u \equiv 0$ in $\mathbb{R}_+^n$.

This concludes our proof of Theorem 1.4.

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