APPROXIMATIONS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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In this paper, we show that solutions of stochastic partial differential equations driven by Brownian motion can be approximated by stochastic partial differential equations forced by pure jump noise/random kicks. Applications to stochastic Burgers equations are discussed.

1. Introduction. Stochastic evolution equations and stochastic partial differential equations (SPDEs) are of great interest to many people. There exists a large amount of literature on the subject; see, for example, the monographs [5, 7, 8].

In this paper, we consider the following stochastic evolution equation:

\[ dY_t = -AY_t \, dt + \left[ b_1(Y_t) + b_2(Y_t) \right] \, dt + \sum_{i=1}^{m} \sigma_i(Y_t) \, dB_i^t, \]

\[ Y_0 = h \in H, \]

in the framework of a Gelfand triple:

\[ V \subset H \cong H^* \subset V^*, \]

where \( H, V \) are Hilbert spaces, \( H^*, V^* \) stand for the dual spaces of \( H \) and \( V \), \( A \) is the infinitesimal generator of a strongly continuous semigroup, \( b_1, \sigma_i, i = 1, \ldots, m \) are measurable mappings from \( H \) into \( H \), \( b_2 \) is a measurable mappings from \( V \) into \( V^* \), \( B_t = (B_1^t, \ldots, B_m^t), t \geq 0 \) is a \( m \)-dimensional Brownian motion. The solutions are considered to be weak solutions (in the PDE sense) in the space \( V \) and not as mild solutions in \( H \). The stochastic evolution equations of this type driven by Wiener processes were first studied in [20] and subsequently in [18]. For stochastic equations with general Hilbert space valued semimartingales replacing the Brownian motion, we refer the reader to [12–14, 22] and also [1].

The aim of this paper is to study the approximations of stochastic evolution equations of the above type by solutions of stochastic evolution equations driven by pure jump processes, namely forced by random kicks. One of the motivations is to shine some light on numerical simulations of SPDEs driven by pure jump noise. To include interesting applications, the drift of equation (1.1) will consist of
a “good” part $b_1$ and a “bad” part $b_2$. The crucial step of obtaining the approximation is to establish the tightness of the approximating equations in the space of Hilbert space-valued right continuous paths with left limits. This is tricky because of the nature of the infinite dimensions and the weak assumptions on the drift $b_2$. We first obtain the approximations assuming that the diffusion coefficients $\sigma_i$ take values in the smaller space $V$ and then remove this restriction by another layer of approximations. As far as we are aware of, this is the first paper to consider such approximations for SPDEs. The approximations of infinite activity Lévy processes were considered in [2]. Robustness of solutions of stochastic differential equations replacing the infinite activity of Lévy processes by Brownian motion was discussed in [3] and [6], and for the backward case in [9].

The rest of the paper is organized as follows. In Section 2, we lay down the precise framework. The main part is Section 3, where the approximations are established and the applications to stochastic Burgers equations are discussed.

2. Framework. Let $V$ and $H$ be two separable Hilbert spaces such that $V$ is continuously, densely imbedded in $H$. Identifying $H$ with its dual, we have

$$ (2.1) \quad V \subset H \cong H^* \subset V^*, $$

where $V^*$ stands for the topological dual of $V$. We assume that the imbedding $V \subset H$ is compact. Let $A$ be a self-adjoint operator on the Hilbert space $H$ satisfying the following coercivity hypothesis: There exist constants $\alpha_0 > 0$, $\alpha_1 > 0$ and $\lambda_0 \geq 0$ such that

$$ (2.2) \quad \alpha_0 \|u\|^2_V \leq 2\langle Au, u \rangle_H + \lambda_0 |u|^2_H \leq \alpha_1 \|u\|^2_V $$

for all $u \in V$.

$\langle Au, u \rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

We remark that $A$ is generally not bounded as an operator from $H$ into $H$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $\{B_t = (B_1^t, \ldots, B_m^t), t \geq 0\}$ be a $m$-dimensional $\mathcal{F}_t$-Brownian motion, $\nu(dx)$ a $\sigma$-finite measure on the measurable space $(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We assume $0 < \int_{|x| \leq 1} x^2 \nu(dx) < \infty$. Let $p_i = (p_i(t)), t \in D_{p_i}, i = 1, \ldots, m$ be mutually independent stationary $\mathcal{F}_t$-Poisson point processes on $\mathbb{R}_0$ with characteristic measure $\nu$. Here, $D_{p_i}$ represents a countable (random) subset of $(0, \infty)$. See [15] for the details on Poisson point processes. Denote by $N_i(dt, dx)$ the Poisson counting measure associated with $p_i$, that is, $N_i(t, A) = \sum_{s \in D_{p_i}, s \leq t} I_A(p_i(s))$. Let $\tilde{N}_i(dt, dx) := N_i(dt, dx) - dt \nu(dx)$ be the compensated Poisson random measure. Let $b_1, \sigma_i, i = 1, \ldots, m$ be measurable mappings from $H$ into $H$, and $b_2(\cdot)$ a measurable mapping from $V$ into $V^*$. Denote by $D([0, T], H)$ the space of all càdlàg paths from $[0, T]$ into $H$ equipped with the Skorohod topology. Consider the stochastic evolution equation:

$$ (2.3) \quad dX_t = -AX_t dt + [b_1(X_t) + b_2(X_t)] dt + \sum_{i=1}^m \sigma_i(X_t) dB_i^t, $$

$$ (2.4) \quad X_0 = h \in H. $$
Let us introduce the following conditions:

(H.1) $b_1(\cdot), \sigma_i(\cdot) : H \to H$ are globally Lipschitz maps, that is, there exists a constant $C < \infty$ such that

$$
|b_1(y_1) - b_1(y_2)|_H^2 + \sum_{i=1}^{m} |\sigma_i(y_1) - \sigma_i(y_2)|_H^2 \leq C |y_1 - y_2|_H^2 
$$

(2.5) for all $y_1, y_2 \in H$.

(H.2) $b_2(\cdot)$ is a mapping from $V$ into $V^*$ that satisfies:

(i) $\langle b_2(u), u \rangle = 0$ for $u \in V$,

(ii) there exist constants $C_1, \beta < \frac{1}{2}$ such that

$$
\langle b_2(y_1) - b_2(y_2), y_1 - y_2 \rangle \leq \beta \alpha_0 \|y_1 - y_2\|_V^2 + C_1 |y_1 - y_2|_H^2 (\|y_1\|_V^2 + \|y_2\|_V^2)
$$

(2.6) for all $y_1, y_2 \in V$,

where $\alpha_0$ is the constant appeared in (2.2).

(iii) There exists a constant $0 < \gamma < 1$ such that $\|b_2(u)\|_{V^*} \leq C_2 |u|_H^{2-\gamma} \|u\|_V^\gamma$ for $u \in V$.

DEFINITION 2.1. A continuous $H$-valued $(\mathcal{F}_t)$-adapted process $X = (X_t)_{t \geq 0}$ is said to be a solution to equation (2.3) if for any $T > 0$, $X \in L^2([0, T] \times \Omega; dt \times P, V)$ and $P$-a.s.

$$
X_t = h - \int_0^t AX_s \, ds + \int_0^t [b_1(X_s) + b_2(X_s)] \, ds + \sum_{i=1}^{m} \int_0^t \sigma_i(X_s) \, dB_i^s.
$$

Under the assumptions (H.1) and (H.2), it is known that equations (2.3) admits a unique solution (see, e.g., [19]).

We finish this section with two examples.

EXAMPLE 2.2. Let $D$ be a bounded domain in $\mathbb{R}^d$. Set $H = L^2(D)$. Let $V = H_0^{1,2}(D)$ denote the Sobolev space of order one with homogenous boundary conditions. Denote by $a(x) = (a_{ij}(x))$ a symmetric matrix-valued function on $D$ satisfying the uniform ellipticity condition:

$$
\frac{1}{c} I_{d \times d} \leq a(x) \leq c I_{d \times d} \quad \text{for some constant } c \in (0, \infty).
$$

Define

$$
Au = - \text{div}(a(x) \nabla u(x)).
$$

Then (2.2) is fulfilled for $(H, V, A)$. 

EXAMPLE 2.3. Let $A = -\Delta_\alpha$, where $\Delta_\alpha$ denotes the generator of a symmetric $\alpha$-stable process in $\mathbb{R}^d$, $0 < \alpha < 2$. It is known that $\Delta_\alpha = (-\Delta)^{\alpha/2}$, the fractional power of the Laplacian operator. The Dirichlet form associated with $\Delta_\alpha$ in $L^2(\mathbb{R}^d)$ is given by
\[
E(u, v) = K(d, \alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \, dx \, dy,
\]
\[
D(E) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty \right\},
\]
where $K(d, \alpha) = \alpha^2 \pi^{-(d+2)/2} \sin(\alpha\pi/2) \Gamma(d+\alpha/2) \Gamma(\alpha/2)$. The domain $D(E)$ can be identified as the fractional Sobolev space $H^{\alpha/2}$, see, for example, [11] for details about the fractional Laplace operator. To study equation (2.3), we choose $H = L^2(\mathbb{R}^d)$, and $V = D(E)$ with the inner product $\langle u, v \rangle = E(u, v) + (u, v)_{L^2(\mathbb{R}^d)}$. Then (2.2) is fulfilled for $(H, V, A)$.

3. Approximations of SPDEs by pure jump type SPDEs. Set, for $\varepsilon \in (0, 1)$,
\[
\alpha(\varepsilon) = \left( \int_{|x| \leq \varepsilon} x^2 \nu(dx) \right)^{1/2}.
\]
Consider the following SPDE driven by pure jump noise:
\[
X^\varepsilon_t = h - \int_0^t AX^\varepsilon_s \, ds + \int_0^t \left[ b_1(X^\varepsilon_s) + b_2(X^\varepsilon_s) \right] \, ds
\]
\[
+ \frac{1}{\alpha(\varepsilon)} \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} \sigma_i(X^\varepsilon_{s-}) x \tilde{N}_i(ds, dx).
\]

DEFINITION 3.1. A $H$-valued $(\mathcal{F}_t)$-adapted process $X^\varepsilon = (X^\varepsilon_t)_{t \geq 0}$ is said to be a solution to equation (3.1) if:

(i) for any $T > 0$, $X^\varepsilon \in D([0, T], H) \cap L^2([0, T] \times \Omega \times dt \times P, V)$,
(ii) for every $t \geq 0$, (3.1) holds $P$-almost surely.

The existence and uniqueness of the solution of equation (3.1) under the assumptions (H.1) and (H.2) can be found in [18, 19, 22]. Recall that $X$ is the solution to the SPDE (2.3):
\[
X_t = h - \int_0^t AX_s \, ds + \int_0^t \left[ b_1(X_s) + b_2(X_s) \right] \, ds
\]
\[
+ \sum_{i=1}^m \int_0^t \sigma_i(X_s) \, dB^i_s.
\]
Denote by $\mu^\varepsilon$, $\mu$, respectively, the laws of $X^\varepsilon$ and $X$ on the spaces $D([0, T], H)$ and $C([0, T], H)$ cf. Definitions 2.1 and 3.1. Consider the following conditions:
For every $i \leq m$, there exists a sequence of mappings $\sigma^i_n(\cdot): H \rightarrow V$ such that:

(i) $|\sigma^i_n(y_1) - \sigma^i_n(y_2)|_H \leq c|y_1 - y_2|_H$, where $c$ is a constant independent of $n$,

(ii) $|\sigma^i_n(y) - \sigma_i(y)|_H \rightarrow 0$ uniformly on bounded subsets of $H$ as $n \rightarrow \infty$.

**Remark 3.2.** In most of the cases, one simply chooses $\sigma^i_n$ to be the finite-dimensional projections of $\sigma_i$ into the subspaces of $V$.

(H.3)' The maps $\sigma_i(\cdot), i = 1, \ldots, m$ take the space $V$ into itself and satisfy $\|\sigma_i(y)\|_V \leq c(1 + \|y\|_V)$ for some constant $c$.

(H.4) There exists an orthonormal basis $\{e_k, k \geq 1\}$ of $H$ such that $Ae_k = \lambda_k e_k$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

We first prepare some preliminary results needed for the proofs of the main theorems.

The following estimate holds for $\{X^\varepsilon, \varepsilon > 0\}$.

**Lemma 3.3.** Let $X^\varepsilon$ be the solution of equation (3.1). If $\frac{\varepsilon}{\alpha(\varepsilon)} \leq C_0$ for some constant $C_0$, then we have for $p \geq 2$,

$$\sup_\varepsilon \left\{E\left[\sup_{0 \leq t \leq T} |X^\varepsilon_t|_H^p\right] + E\left[\left(\int_0^T \|X^\varepsilon_s\|_V^2 ds\right)^{p/2}\right]\right\} < \infty. \quad (3.3)$$

**Proof.** We prove the lemma for $p = 4$. Other cases are similar. In view of assumption (H.2), by Itô’s formula, we have

$$|X^\varepsilon_t|_H^2 = |h|^2 - 2 \int_0^t \langle X^\varepsilon_s, AX^\varepsilon_s \rangle ds + 2 \int_0^t \langle X^\varepsilon_s, b_1(X^\varepsilon_s) \rangle ds$$

$$+ \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} \left(\frac{1}{\alpha(\varepsilon)} \sigma_i(X^\varepsilon_{s-})x\right)^2_H ds$$

$$+ 2 \left(\int_{X^\varepsilon_{s-}}, \frac{1}{\alpha(\varepsilon)} \sigma_i(X^\varepsilon_{s-})x\right) \tilde{N}_i(ds, dx)$$

$$+ \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} \left(\frac{1}{\alpha(\varepsilon)} \sigma_i(X^\varepsilon_{s-})x\right)^2_H ds v(dx). \quad (3.4)$$

Let

$$M_t = \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} \left(\frac{1}{\alpha(\varepsilon)} \sigma(X^\varepsilon_{s-})x\right)^2_H$$

$$+ 2 \left(\int_{X^\varepsilon_{s-}}, \frac{1}{\alpha(\varepsilon)} \sigma(X^\varepsilon_{s-})x\right) \tilde{N}_i(ds, dx)$$

$$+ \sum_{i=1}^m M_i^t. \quad (3.5)$$
By Burkhölder’s inequality, for $t \leq T$, and a positive constant $C$, we have
\[
E\left[ \sup_{0 \leq u \leq t} |M_u|^2 \right] \leq C \sum_{i=1}^{m} E\left[ \sup_{0 \leq u \leq t} |M^i_u|^2 \right] \leq C \sum_{i=1}^{m} E[M^i, M^i],
\]
\[
= C \sum_{i=1}^{m} E\left[ \int_0^t \int_{|x| \leq \epsilon} \left( \left| \frac{1}{\alpha(\epsilon)} \sigma_i(X^\epsilon_s)x \right|^2 \right)_H \right.
\]
\[
+ 2 \left( X^\epsilon_{s-}, \frac{1}{\alpha(\epsilon)} \sigma_i(X^\epsilon_s)x \right)^2 N_i(ds, dx) \bigg] \bigg]
\[
= C \sum_{i=1}^{m} E\left[ \int_0^t \int_{|x| \leq \epsilon} \left( \left| \frac{1}{\alpha(\epsilon)} \sigma_i(X^\epsilon_s)x \right|^2 \right)_H \right.
\]
\[
+ 2 \left( X^\epsilon_{s-}, \frac{1}{\alpha(\epsilon)} \sigma_i(X^\epsilon_s)x \right)^2 ds v(dx) \bigg] \bigg]
\[
\leq C E\left[ \int_0^t (1 + |X^\epsilon_{s}|^4_H) ds \right],
\]
(3.6)

where the linear growth condition on $\sigma_i$ and the fact $\frac{\epsilon}{\alpha(\epsilon)} \leq C_0$ have been used.

Use first (2.2) and then square both sides of the resulting inequality to obtain from (3.4) that
\[
|X^\epsilon_t|^4_H + \left( \int_0^t \|X^\epsilon_s\|^2_V ds \right)^2 \leq C_T |h|^4_H + C_T \int_0^t (1 + |X^\epsilon_{s}|^4_H) ds + C_T M^2_i.
\]

Take supremum over the interval $[0, t]$ in (3.7), use (3.6) to get
\[
E\left[ \sup_{0 \leq s \leq t} |X^\epsilon_s|^4_H \right] + E\left[ \left( \int_0^t \|X^\epsilon_s\|^2_V ds \right)^2 \right]
\]
\[
\leq C |h|^4_H + C E\left[ \int_0^t (1 + |X^\epsilon_{s}|^4_H) ds \right].
\]
(3.8)

Applying Gronwall’s inequality proves the lemma. \qed

**Proposition 3.4.** Assume (H.1), (H.2), (H.3)', (H.4) and $\frac{\epsilon}{\alpha(\epsilon)} \leq C_0$ for some constant $C_0$. Then the family $\{X^\epsilon, \epsilon > 0\}$ is tight on the space $D([0, T], H)$.

**Proof.** Write
\[
Y^\epsilon_t = \frac{1}{\alpha(\epsilon)} \sum_{i=1}^{m} \int_0^t \int_{|x| \leq \epsilon} \sigma_i(X^\epsilon_{s-})x \tilde{N}_i(ds, dx),
\]
(3.9)
and set $Z^\epsilon_t = X^\epsilon_t - Y^\epsilon_t$. It suffices to prove that both $\{Y^\epsilon, \epsilon > 0\}$ and $\{Z^\epsilon, \epsilon > 0\}$ are tight. This is done in two steps.
Step 1. Prove that \( \{Y^\varepsilon, \varepsilon > 0\} \) is tight.

In view of the assumption (H.3)' on \( \sigma_i \), we have \( Y^\varepsilon \in D([0, T], V) \). Since the imbedding \( V \subset H \) is compact, according to Theorem 3.1 in [16], it is sufficient to show that for every \( e \in H \), \( \{\langle Y^\varepsilon, e \rangle, \varepsilon > 0\} \) is tight in \( D([0, T], R) \). Note that

\[
\sup_{\varepsilon} E \left[ \sup_{0 \leq t \leq T} \langle Y^\varepsilon_t, e \rangle^2 \right] 
\leq \sup_{\varepsilon} E \left[ \sup_{0 \leq t \leq T} |Y^\varepsilon_t|^2_H \right] 
= C \sup_{\varepsilon} \frac{1}{\alpha(\varepsilon)^2} \sum_{i=1}^m E \left[ \int_0^T \int_{|x| \leq \varepsilon} |\sigma_i(X^\varepsilon_s)|^2_H x^2 \nu(dx) \, ds \right] 
= C \sum_{i=1}^m \sup_{\varepsilon} E \left[ \int_0^T |\sigma_i(X^\varepsilon_s)|^2_H ds \right] < \infty, 
\]

and for any stopping times \( \tau^\varepsilon \leq T \) and any positive constants \( \delta^\varepsilon \to 0 \) we have

\[
E \left[ |\langle Y^\varepsilon_{\tau^\varepsilon}, e \rangle - \langle Y^\varepsilon_{\tau^\varepsilon+\delta^\varepsilon}, e \rangle|^2 \right] 
\leq \frac{1}{\alpha(\varepsilon)^2} \sum_{i=1}^m E \left[ \int_{\tau^\varepsilon}^{\tau^\varepsilon+\delta^\varepsilon} \int_{|x| \leq \varepsilon} |\sigma_i(X^\varepsilon_s)|^2_H x^2 \nu(dx) \, ds \right] 
\leq C\delta^\varepsilon \sup_{\varepsilon} E \left[ 1 + \sup_{0 \leq t \leq T} |X^\varepsilon_t|^2_H \right] \to 0, 
\]

as \( \varepsilon \to 0 \). By Theorem 3.1 in [16] (see also [4]), (3.10) and (3.11) yield the tightness of \( \langle Y^\varepsilon, e \rangle, \varepsilon > 0 \).

Step 2. Prove that \( \{Z^\varepsilon, \varepsilon > 0\} \) is tight.

It is easy to see that \( Z^\varepsilon \) satisfies the equation:

\[
Z^\varepsilon_t = h - \int_0^t A Z^\varepsilon_s \, ds - \int_0^t A Y^\varepsilon_s \, ds 
+ \int_0^t b_1(Z^\varepsilon_s + Y^\varepsilon_s) \, ds + \int_0^t b_2(Z^\varepsilon_s + Y^\varepsilon_s) \, ds. 
\]

Recall \( \{e_k, k \geq 1\} \) is the orthonormal basis of \( H \) consisting of eigenvectors of \( A \) [see (H.4)]. We have

\[
\langle Z^\varepsilon_t, e_k \rangle = \langle h, e_k \rangle - \lambda_k \int_0^t \langle Z^\varepsilon_s, e_k \rangle \, ds - \lambda_k \int_0^t \langle Y^\varepsilon_s, e_k \rangle \, ds 
+ \int_0^t \langle b_1(Z^\varepsilon_s + Y^\varepsilon_s), e_k \rangle \, ds + \int_0^t \langle b_2(Z^\varepsilon_s + Y^\varepsilon_s), e_k \rangle \, ds. 
\]

By Corollary 5.2 in [16], to obtain the tightness of \( \{Z^\varepsilon, \varepsilon > 0\} \) we need to show:

(i) \( \{\langle Z^\varepsilon, e_k \rangle, \varepsilon > 0\} \) is tight in \( D([0, T], R) \) for every \( k \),
(ii) for any \( \delta > 0 \),

\[
\lim_{N \to \infty} \sup_{\varepsilon} P \left( \sup_{0 \leq t \leq T} R_N^\varepsilon(t) > \delta \right) = 0,
\]

where

\[
R_N^\varepsilon(t) = \sum_{k=N}^{\infty} \langle Z_t^\varepsilon, e_k \rangle^2.
\]

The proof of (i) is similar to that of the tightness of \( \langle Y^\varepsilon, e \rangle, \varepsilon > 0 \). It is omitted.

Let us prove (ii). By the chain rule, it follows that

\[
\langle Z_t^\varepsilon, e_k \rangle^2 = \langle h, e_k \rangle^2 - 2\lambda_k \int_0^t \langle Z_s^\varepsilon, e_k \rangle^2 ds - 2\lambda_k \int_0^t \langle Y_s^\varepsilon, e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t \langle b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t \langle b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds.
\]

By the variation of constants formula, we have

\[
\langle Z_t^\varepsilon, e_k \rangle^2 = e^{-2\lambda_k t} \langle h, e_k \rangle^2 - 2\lambda_k \int_0^t e^{-2\lambda_k (t-s)} \langle Y_s^\varepsilon, e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t e^{-2\lambda_k (t-s)} \langle b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t e^{-2\lambda_k (t-s)} \langle b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds.
\]

Hence,

\[
R_N^\varepsilon(t) = \sum_{k=N}^{\infty} \langle Z_t^\varepsilon, e_k \rangle^2 = \sum_{k=N}^{\infty} e^{-2\lambda_k t} \langle h, e_k \rangle^2 - 2 \int_0^t \sum_{k=N}^{\infty} \lambda_k e^{-2\lambda_k (t-s)} \langle Y_s^\varepsilon, e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k (t-s)} \langle b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds + 2 \int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k (t-s)} \langle b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle ds =: I_N^{(1)}(t) + I_N^{(2)}(t) + I_N^{(3)}(t) + I_N^{(4)}(t).
\]
Obviously,

\[ I_N^{(1)}(t) \leq \sum_{k=N}^{\infty} \langle h, e_k \rangle^2 \to 0, \]  

as \( N \to \infty \). For the third term on the right-hand side of (3.17), we have

\[ |I_N^{(3)}(t)| \leq 2 \int_0^t e^{-2\lambda_N(t-s)} \sum_{k=N}^{\infty} |b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k| |Z_s^\varepsilon, e_k| \, ds \]

\[ \leq 2 \int_0^t e^{-2\lambda_N(t-s)} ds \left( \sup_{0 \leq s \leq T} |Z_s^\varepsilon|_H \right) \left( \sup_{0 \leq s \leq T} |b_1(Z_s^\varepsilon + Y_s^\varepsilon)|_H \right) \]

\[ \leq C \frac{1}{\lambda_N} \left( 1 + \sup_{0 \leq s \leq T} |Z_s^\varepsilon|_H^2 + \sup_{0 \leq s \leq T} |Y_s^\varepsilon|_H^2 \right). \]

Hence,

\[ \sup_{\varepsilon} E \left[ \sup_{0 \leq t \leq T} |I_N^{(3)}(t)| \right] \]

\[ \leq C \frac{1}{\lambda_N} \left( 1 + \sup_{\varepsilon} E \left[ \sup_{0 \leq s \leq T} |Z_s^\varepsilon|_H^2 \right] + \sup_{\varepsilon} E \left[ \sup_{0 \leq s \leq T} |Y_s^\varepsilon|_H^2 \right] \right) \]

\[ \to 0 \quad \text{as} \quad N \to \infty. \]

Let us turn to \( I_N^{(2)}(t) \). By Hölder’s inequality,

\[ |I_N^{(2)}(t)| \leq 2 \int_0^t \left( \sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)} \lambda_k |Z_s^\varepsilon, e_k|^2 \right)^{1/2} \left( \sum_{k=N}^{\infty} \lambda_k |Y_s^\varepsilon, e_k|^2 \right)^{1/2} \, ds \]

\[ \leq 2 \int_0^t \left( \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)} \lambda_k |Z_s^\varepsilon, e_k|^2 \right)^{1/2} \left( |A Y_s^\varepsilon, Y_s^\varepsilon|_V^2 \right)^{1/2} \, ds \]

\[ \leq C \left( \sup_{0 \leq s \leq T} \| Y_s^\varepsilon \|_V \right) \left( \sup_{0 \leq s \leq T} |Z_s^\varepsilon|_H \right) \int_0^t e^{-\lambda_N(t-s)} \frac{1}{\sqrt{t-s}} \, ds \]

\[ \leq C \left( \frac{1}{\sqrt{\lambda_N}} \int_0^\infty e^{-u} \frac{1}{\sqrt{u}} \, du \right) \left( \sup_{0 \leq s \leq T} \| Y_s^\varepsilon \|_V \right) \left( \sup_{0 \leq s \leq T} |Z_s^\varepsilon|_H \right). \]
In view of the assumption (H.2), the last term on the right-hand side of (3.17) can be estimated as follows:

\[
|I_N^{(4)}(t)| = \left| \int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)} b_2(X_s^\varepsilon, e_k) |Z_s^\varepsilon, e_k| ds \right|
\]

\[
= \left| \int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)} \sqrt{\lambda_0 + \lambda_k} (A + \lambda_0 I)^{-1/2} b_2(X_s^\varepsilon, e_k)|Z_s^\varepsilon, e_k| ds \right|
\]

\[
\leq C \int_0^t \left( \sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)} (A + \lambda_0 I)^{-1/2} b_2(X_s^\varepsilon, e_k)^2 \right)^{1/2} \times \left( \sum_{k=N}^{\infty} (\lambda_0 + \lambda_k)|Z_s^\varepsilon, e_k|^2 \right)^{1/2} \hspace{1cm} ds
\]

\[
(3.22)
\]

\[
\leq C \int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)} \left( \sum_{k=N}^{\infty} (A + \lambda_0 I)^{-1/2} b_2(X_s^\varepsilon, e_k)^2 \right)^{1/2} \times \left( \sum_{k=N}^{\infty} (\lambda_0 + \lambda_k)|Z_s^\varepsilon, e_k|^2 \right)^{1/2} \hspace{1cm} ds
\]

\[
\leq C \int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)} \|b_2(X_s^\varepsilon)\|_V ds
\]

\[
\leq C \int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)} \|X_s^\varepsilon\|_H \|X_s^\varepsilon\|_V ds.
\]

This yields

\[
|I_N^{(4)}(t)| \leq C \sup_{0 \leq s \leq T} \|X_s^\varepsilon\|_H^{2-\gamma} \int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)} \|X_s^\varepsilon\|_V^{\gamma} ds
\]

\[
\leq C \sup_{0 \leq s \leq T} \|X_s^\varepsilon\|_H^{2-\gamma} \int_0^t e^{-2\lambda_N(t-s)} \left( \|X_s^\varepsilon\|_V^{1+\gamma} + \|X_s^\varepsilon\|_V^{\gamma} \|Y_s^\varepsilon\|_V \right) ds
\]

\[
\leq C \sup_{0 \leq s \leq T} \|X_s^\varepsilon\|_H^{2-\gamma} \left( \int_0^T e^{-\frac{4}{(1-\gamma)}\lambda_N(t-s)} ds \right)^{(1-\gamma)/2} \times \left( \int_0^T \left( \|X_s^\varepsilon\|_V^{1+\gamma} + \|X_s^\varepsilon\|_V^{\gamma} \|Y_s^\varepsilon\|_V \right)^{2/(1+\gamma)} ds \right)^{(1+\gamma)/2}
\]

\[
(3.23)
\]

\[
\leq C \left( \frac{1}{\lambda_N} \right)^{(1-\gamma)/2} \times \sup_{0 \leq s \leq T} \|X_s^\varepsilon\|_H^{2-\gamma} \left( \int_0^T \left( \|X_s^\varepsilon\|_V^{1+\gamma} + \|X_s^\varepsilon\|_V^{\gamma} \|Y_s^\varepsilon\|_V \right)^{2/(1+\gamma)} ds \right)^{(1+\gamma)/2}.
\]
Hence,
\[
\sup_{\varepsilon} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |I_N^{(4)}(t)| \right]
\leq C \left( \frac{1}{\lambda N} \right)^{(1 - \gamma)/2} \mathbb{E}\left[ \sup_{0 \leq s \leq T} \left| X_s^\varepsilon \right|_{H}^{2 - \gamma} \left( \int_0^T \left( \| X_s^\varepsilon \|_V^{1 + \gamma} + \| X_s^\varepsilon \|_V^{\gamma} + \| Y_s^\varepsilon \|_V \right) ds \right)^{1 + \gamma/2} \right]
\leq C \left( \frac{1}{\lambda N} \right)^{(1 - \gamma)/2} \times \sup_{\varepsilon} \mathbb{E}\left[ \sup_{0 \leq s \leq T} \left| X_s^\varepsilon \right|_{H}^{2 - \gamma} \left( \int_0^T \left( C \| X_s^\varepsilon \|_V^{2} + c \| Y_s^\varepsilon \|_V^{2} \right) ds \right)^{1 + \gamma/2} \right]
\to 0 \quad \text{as } N \to \infty,
\]
where we used the fact that
\[|ab| \leq C(|a|^p + |b|^q), \quad \frac{1}{p} + \frac{1}{q} = 1.\]

Putting together (3.17)–(3.24) and applying the Chebyshev inequality, we obtain (3.14). \square

Let \( \mathcal{D} \) denote the class of functions \( f \in C^3_b(H) \) that satisfy (i) \( \nabla f(z) \in D(A) \) and \( |A\nabla f(z)|_H \leq C(1 + |z|_H) \) for some constant \( C \), where \( \nabla f(z) \) stands for the gradient of \( f \) at \( z \), (ii) \( f''(z), f'''(z) \) are bounded, where \( f'', f''' \) denote the operators/multi-linear functionals associated with the second and the third derivatives of \( f \).

For \( f \in \mathcal{D} \), define
\[
L^\varepsilon f(z) = -\langle A\nabla f(z), z \rangle + \langle b_1(z), \nabla f(z) \rangle + \langle b_2(z), \nabla f(z) \rangle
+ \sum_{i=1}^m \int_{|x| \leq \varepsilon} \left[ f\left( z + \frac{1}{\alpha(\varepsilon)} \sigma_i(z)x \right) - f(z) \right. \\
\left. - \left( \nabla f(z), \frac{1}{\alpha(\varepsilon)} \sigma_i(z)x \right) \right] \nu(dx),
\]
(3.25)
and
\[
L f(z) = -\langle A\nabla f(z), z \rangle + \langle b_1(z), \nabla f(z) \rangle + \langle b_2(z), \nabla f(z) \rangle
+ \frac{1}{2} \sum_{i=1}^m (f''(z)\sigma_i(z), \sigma_i(z)).
\]
(3.26)
Lemma 3.5. Assume \( \lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha(\varepsilon)} = 0 \). Then for \( f \in \mathcal{D} \), it holds that
\[
L_\varepsilon f(z) \to Lf(z)
\] uniformly on bounded subsets of \( H \) as \( \varepsilon \to 0 \).

Proof. Note that for any \( f \in \mathcal{D} \) we have
\[
f(y + w) - f(y) - \langle \nabla f(y), w \rangle = \int_0^1 \int_0^\alpha \langle f''(y + \beta w)w, w \rangle d\beta.
\]
Thus, for every \( z \in H \),
\[
L_\varepsilon f(z) - Lf(z)
\]
\[
= \sum_{i=1}^m \left\{ \int_{|x| \leq \varepsilon} \int_0^1 d\alpha \int_0^\alpha d\beta \left[ f'' \left( z + \beta \frac{1}{\alpha(\varepsilon)} \sigma_i(z)x \right) \right] \right\}
\]
\[
= \frac{1}{\alpha(\varepsilon)^2} \int_{|x| \leq \varepsilon} x^2 v(dx) \int_0^1 d\alpha \int_0^\alpha d\beta \sum_{i=1}^m \left[ f'' \left( z + \beta \frac{1}{\alpha(\varepsilon)} \sigma_i(z)x \right) \right] \sigma_i(z), \sigma_i(z) \right] - \left\{ f''(z) \sigma_i(z), \sigma_i(z) \right\}
\]
Hence, for \( z \in B_N = \{ z \in H ; |z|_H \leq N \} \) we have
\[
|L_\varepsilon f(z) - Lf(z)|
\]
\[
\leq C \frac{1}{\alpha(\varepsilon)^2} \int_{|x| \leq \varepsilon} x^2 v(dx) \int_0^1 d\alpha \int_0^\alpha d\beta \beta |x| \frac{1}{\alpha(\varepsilon)}
\]
\[
\times \sum_{i=1}^m \left( |\sigma_i(z)|_H |\sigma_i(z)|_H^2 \right)
\]
\[
\leq CN \frac{\varepsilon}{\alpha(\varepsilon)} \to 0,
\]
uniformly on \( B_N \) as \( \varepsilon \to 0 \), where we have used the local Lipschitz continuity of \( f''(z) \). \( \square \)

Theorem 3.6. Suppose (H.1), (H.2), (H.3)', (H.4) hold and \( \lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha(\varepsilon)} = 0 \). Then, for any \( T > 0 \), \( \mu_\varepsilon \) converges weakly to \( \mu \), for \( \varepsilon \to 0 \), on the space \( D([0, T], H) \) equipped with the Skorohod topology.

Proof. Since the mappings \( \sigma_i \) take values in the space \( V \), by Proposition 3.4, the family \( \{ \mu_\varepsilon, \varepsilon > 0 \} \) is tight. Let \( \mu_0 \) be the weak limit of any convergent sequence
{\mu_{\varepsilon_n}} on the canonical space \((\Omega = D([0, T], H), \mathcal{F})\) as \(\varepsilon_n \to 0\). We will show that \(\mu_0 = \mu\). Denote by \(X_t(\omega) = w(t), \omega \in \Omega\) the coordinate process. Set \(J(X) = \sup_{0 \leq s \leq T} |X_s - X_{s-}|_H\). Since

\[ E^{\mu_\varepsilon}[J(X)] = E[J(X^\varepsilon)] \]

(3.30)

\[ \leq \frac{\varepsilon}{\alpha(\varepsilon)} \sum_{i=1}^{m} E\left[ \sup_{0 \leq s \leq T} |\sigma_i(X^\varepsilon_s)|_H \right] \]

\[ \leq C \frac{\varepsilon}{\alpha(\varepsilon)} \left( 1 + E\left[ \sup_{0 \leq s \leq T} |X^\varepsilon_s|_H \right] \right) \to 0, \]

as \(\varepsilon \to 0\), it follows from Theorem 13.4 in [4] that \(\mu_0\) is supported on the \(C([0, T], H)\), the space of \(H\)-valued continuous functions on \([0, T]\). As a consequence, the finite-dimensional distributions of \(\mu_{\varepsilon_n}\) converge to that of \(\mu_0\).

Let us fix \(f \in D\). Then by Itô’s formula,

\[ f(X^\varepsilon_t) - f(h) - \int_0^t L^\varepsilon f(X^\varepsilon_s) \, ds \]

(3.31)

\[ = \sum_{i=1}^{m} \int_0^t \int_{|x| \leq \varepsilon} \left\{ f \left( X^\varepsilon_{s-} + \frac{1}{\alpha(\varepsilon)} \sigma_i(X^\varepsilon_{s-}) x \right) - f(X^\varepsilon_{s-}) \right\} \tilde{N}_i(ds, dx) \]

is a martingale. Such an Itô formula for stochastic evolution equations/SPDEs driven by continuous martingales can be found in [20] (Theorem 3.2, page 147). In our pure jump case, due to the strong assumptions on the function \(f\) this Itô formula can be obtained by an approximation argument as follows. For \(n \geq 1\), consider the finite-dimensional projection:

\[ X^n_{t,\varepsilon} = \sum_{i=1}^{n} \langle X^\varepsilon_t, e_i \rangle e_i, \]

where \(e_i, i \geq 1\) are the eigenvectors of \(A\) [see (H.4)]. We first apply Itô’s formula to the finite-dimensional process \(X^n_{t,\varepsilon}\) and the function \(F_n(x_1, \ldots, x_n) = f(\sum_{i=1}^{n} x_i e_i)\), and then take the limit as \(n \to \infty\) to get (3.31).

By the martingale property, for any \(s_0 < s_1 < \cdots < s_n \leq s < t\) and \(f_0, f_1, \ldots, f_n \in C_b(H)\) it holds that

\[ E^{\mu_\varepsilon}\left[ \left( f(X_t) - f(X_s) - \int_s^t L^\varepsilon f(X_u) \, du \right) f(X_{s_0}) \cdots f(X_{s_n}) \right] = 0. \]

(3.32)

For any positive constant \(M > 0\), by Lemma 3.5 we have

\[ \lim_{n \to \infty} E^{\mu_{\varepsilon_n}} \left[ \int_s^t |L^{\varepsilon_n} f(X_u) - L f(X_u)| \, du, \sup_{0 \leq u \leq T} |X_u|_H \leq M \right] = 0. \]

(3.33)
On the other hand, in view of the assumptions on $f$ we have

$$\sup_n E^{\mu_{\varepsilon n}} \left[ \int_s^t |L^{\varepsilon_n} f(X_u) - Lf(X_u)| \, du \right] \leq C \frac{1}{M} \sup_n E^{\mu_{\varepsilon n}} \left[ \sup_{0 \leq u \leq T} |X_u|^3_H \right] \leq C' \frac{1}{M}.$$  \hspace{1cm} (3.34)

Combining (3.33) with (3.34), we arrive at

$$\lim_{n \to \infty} E^{\mu_{\varepsilon n}} \left[ \int_s^t |L^{\varepsilon_n} f(X_u) - Lf(X_u)| \, du \right] = 0. \hspace{1cm} (3.35)$$

By the weak convergence of $\mu_{\varepsilon n}$ and the convergence of finite distributions, it follows from (3.32) and (3.35) that

$$E^{\mu_0} \left[ \left( f(X_t) - f(X_s) - \int_s^t Lf(X_u) \, du \right) f(X_{s_0}) \cdots f(X_{s_n}) \right] = \lim_{n \to \infty} E^{\mu_{\varepsilon n}} \left[ \left( f(X_t) - f(X_s) - \int_s^t Lf(X_u) \, du \right) f(X_{s_0}) \cdots f(X_{s_n}) \right]$$

$$= \lim_{n \to \infty} E^{\mu_{\varepsilon n}} \left[ \left( f(X_t) - f(X_s) - \int_s^t L^{\varepsilon_n} f(X_u) \, du \right) f(X_{s_0}) \cdots f(X_{s_n}) \right]$$

$$= \lim_{n \to \infty} E^{\mu_{\varepsilon n}} \left[ \left( f(X_t) - f(X_s) - \int_s^t L^{\varepsilon_n} f(X_u) \, du \right) f(X_{s_0}) \cdots f(X_{s_n}) \right] = 0. \hspace{1cm} (3.36)$$

Since $s_0 < s_1 < \cdots < s_n \leq s < t$ are arbitrary, (3.36) implies that for any $f \in D$,

$$M^f_t := f(X_t) - f(h) - \int_0^t Lf(X_s) \, ds, \hspace{1cm} t \geq 0,$$

is a martingale under $\mu_0$. In particular, let $f(z) = \langle e_k, z \rangle$ and $f(z) = \langle e_k, z \rangle \langle e_j, z \rangle$, respectively, to obtain that under $\mu_0$

$$M^k_t := \langle e_k, X_t \rangle - \langle e_k, h \rangle + \int_0^t \langle A e_k, X_s \rangle \, ds - \int_0^t \langle b_1(X_s), e_k \rangle \, ds$$

$$- \int_0^t \langle b_2(X_s), e_k \rangle \, ds \hspace{1cm} (3.37)$$

and

$$M^{k;j}_t := \langle e_k, X_t \rangle \langle e_j, X_t \rangle - \langle e_k, h \rangle \langle e_j, h \rangle + \int_0^t \left\{ \langle A e_k, X_s \rangle \langle e_j, X_s \rangle + \langle A e_j, X_s \rangle \langle e_k, X_s \rangle \right\} \, ds$$

$$- \int_0^t \langle b_1(X_s), e_k \rangle \langle e_j, X_s \rangle + e_j \langle e_k, X_s \rangle \rangle \, ds. \hspace{1cm} (3.38)$$
are martingales. This together with Itô’s formula yields that

\begin{equation}
\langle M^k, M^j \rangle_t = \sum_{i=1}^{m} \int_0^t \langle \sigma_i(X_s), e_k \rangle \langle \sigma_i(X_s), e_j \rangle ds,
\end{equation}

where \( \langle M^k, M^j \rangle \) stands for the sharp bracket of the two martingales. Now by Theorem 18.12 in [17] (or Theorem 7.1’ in [15]), there exists a probability space \((\Omega', \mathcal{F}', P')\) with a filtration \(\mathcal{F}'_t\) such that on the standard extension

\((\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathcal{F}_t \times \mathcal{F}'_t, \mu_0 \times P')\)

of \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) there exists a \(m\)-dimensional Brownian motion \(B_t = (B_1^1, \ldots, B_m^m), t \geq 0\) such that

\begin{equation}
M^k_t = \sum_{i=1}^{m} \int_0^t \langle \sigma_i(X_s), e_k \rangle dB_i^i,
\end{equation}

namely,

\begin{equation}
\langle e_k, X_t \rangle - \langle e_k, h \rangle = -\int_0^t \langle A e_k, X_s \rangle ds + \int_0^t \langle b_1(X_s), e_k \rangle ds + \int_0^t \langle b_2(X_s), e_k \rangle ds + \sum_{i=1}^{m} \int_0^t \langle \sigma_i(X_s), e_k \rangle dB_i^i
\end{equation}

for any \(k \geq 1\). Thus, under \(\mu_0\), \(X_t, t \geq 0\) is a weak solution (both in the probabilistic and in PDE sense) of the SPDE

\[ X_t = h - \int_0^t A X_s ds + \int_0^t b_1(X_s) ds + \int_0^t b_2(X_s) ds + \sum_{i=1}^{m} \int_0^t \sigma_i(X_s) dB_i^i. \]

By the uniqueness of the above equation, we conclude that \(\mu_0 = \mu\) completing the proof of the theorem. □

**Theorem 3.7.** Suppose (H.1), (H.2), (H.3) and (H.4) hold and \(\lim_{\epsilon \to 0} \frac{\epsilon}{\alpha(\epsilon)} = 0\). Then, for any \(T > 0\), \(\mu_\epsilon\) converges weakly to \(\mu\), for \(\epsilon \to 0\), on the space \(D([0, T], H)\) equipped with the Skorohod topology.
PROOF. Let \( \sigma_i^j(\cdot) \) be the mappings specified in (H.3). Let \( X^{n,\varepsilon}, X^n \) be the solutions of the SPDEs

\[
X^{n,\varepsilon}_t = h - \int_0^t A X^{n,\varepsilon}_s \, ds + \int_0^t b_1(X^{n,\varepsilon}_s) \, ds + \int_0^t b_2(X^{n,\varepsilon}_s) \, ds
\]
\[
+ \frac{1}{\alpha(\varepsilon)} \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} \sigma_i^j(X^{n,\varepsilon}_{s-}) x \tilde{N}_i(ds, dx),
\]
\[
X^n_t = h - \int_0^t A X^n_s \, ds + \int_0^t b_1(X^n_s) \, ds + \int_0^t b_2(X^n_s) \, ds
\]
\[
+ \sum_{i=1}^m \int_0^t \sigma_i^j(X^n_s) \, dB^i_s.
\]

We claim that for any \( \delta > 0 \),

\[
limit_{n \to \infty} \sup_{\varepsilon} P \left( \sup_{0 \leq t \leq T} \|X^{n,\varepsilon}_t - X^\varepsilon_t\|_H > \delta \right) = 0,
\]
\[
limit_{n \to \infty} P \left( \sup_{0 \leq t \leq T} \|X^n_t - X^t\|_H^2 > \delta \right) = 0.
\]

Let us only prove (3.44). The proof of (3.45) is simpler. As the proof of (3.3), we can show that

\[
\sup_n \sup_{\varepsilon} \left\{ E \left[ \sup_{0 \leq t \leq T} \|X^{n,\varepsilon}_t\|_H^2 \right] + E \left[ \int_0^T \|X^{n,\varepsilon}_s\|_V^2 \, ds \right] \right\} < \infty,
\]
\[
\sup_n \left\{ E \left[ \sup_{0 \leq t \leq T} \|X^n_t\|_H^2 \right] + E \left[ \int_0^T \|X^n_s\|_V^2 \, ds \right] \right\} < \infty.
\]

Applying Itô’s formula first to \( \|X^{n,\varepsilon}_t - X^\varepsilon_t\|_H^2 \) and then applying the integration by parts formula, we obtain

\[
e^{-\gamma \int_0^t (\|X^{n,\varepsilon}_s\|_V^2 + \|X^\varepsilon_s\|_V^2) \, ds} \|X^{n,\varepsilon}_t - X^\varepsilon_t\|_H^2
\]
\[
= -\gamma \int_0^t e^{-\gamma \int_0^s (\|X^{n,\varepsilon}_u\|_V^2 + \|X^\varepsilon_u\|_V^2) \, du} \|X^{n,\varepsilon}_s - X^\varepsilon_s\|_H^2 \left( \|X^{n,\varepsilon}_s\|_V^2 + \|X^\varepsilon_s\|_V^2 \right) \, ds
\]
\[
- 2 \int_0^t e^{-\gamma \int_0^s (\|X^{n,\varepsilon}_u\|_V^2 + \|X^\varepsilon_u\|_V^2) \, du} \langle X^{n,\varepsilon}_s - X^\varepsilon_s, A(X^{n,\varepsilon}_s - X^\varepsilon_s) \rangle \, ds
\]
\[
+ 2 \int_0^t e^{-\gamma \int_0^s (\|X^{n,\varepsilon}_u\|_V^2 + \|X^\varepsilon_u\|_V^2) \, du} \langle X^{n,\varepsilon}_s - X^\varepsilon_s, b_1(X^{n,\varepsilon}_s) - b_1(X^\varepsilon_s) \rangle \, ds
\]
\[
+ 2 \int_0^t e^{-\gamma \int_0^s (\|X^{n,\varepsilon}_u\|_V^2 + \|X^\varepsilon_u\|_V^2) \, du} \langle X^{n,\varepsilon}_s - X^\varepsilon_s, b_2(X^{n,\varepsilon}_s) - b_2(X^\varepsilon_s) \rangle \, ds
\]
\[
+ \sum_{i=1}^m \int_0^t \int_{|x| \leq \varepsilon} e^{-\gamma \int_0^s (\|X^{n,\varepsilon}_u\|_V^2 + \|X^\varepsilon_u\|_V^2) \, du}
\]

(3.48)
\[ \times \left( \frac{1}{\alpha(\epsilon)} \left( \sigma_n^i(X^n_{s-}) x - \sigma_i(X^n_{s-}) x \right)^2 H \right) \\
+ 2 \left( (X^n_{s-} - X^\epsilon_{s}), \frac{1}{\alpha(\epsilon)} (\sigma_n^i(X^n_{s-}) x - \sigma_i(X^n_{s-}) x) \right) \tilde{N}_i(ds, dx) \]
\[ + \sum_{i=1}^m \int_0^t \int_{|x| \leq \epsilon} e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} du \]
\[ \times \left| \frac{1}{\alpha(\epsilon)} (\sigma_n^i(X^n_{s-}) x - \sigma_i(X^n_{s-}) x) \right|^2 H ds \nu(dx) \]
:= \sum_{k=1}^6 I_{k,n}(t). 

Itô’s formula for \(|X^n_{t} - X^\epsilon_{t}|^2_H|\ in the continuous setting can be found in [18], Theorem 3.1, or in [21], Theorem 4.2.5. In the current pure jump case, Itô’s formula can be seen through finite-dimensional projections of \(X^n_{t} - X^\epsilon_{t}\) and a limiting procedure. The expression in (3.48) is a result of further use of the integration by parts formula for the real-valued semimartingales \(e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du}, t \geq 0\) and \(|X^n_{t} - X^\epsilon_{t}|^2_H, t \geq 0\).

In view of assumption (2.6), we see that
\[ I_{1,n}^{n,e}(t) + I_{2,n}^{n,e}(t) + I_{4,n}^{n,e}(t) \]
(3.49)
\[ \leq -(1 - 2\beta)\alpha_0 \int_0^t e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} \|X^n_{s} - X^\epsilon_s\|_V^2 ds, \]
if \(\gamma \geq 2C_1\), where \(C_1\) is the constant appeared in (2.6).

Similar to the proofs of (3.6), (3.8), using Burkholder’s inequality, we obtain from (3.48), (3.49) that for \(t \leq T\),
\[ E\left[ \sup_{0 \leq s \leq t} e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} |X^n_{s} - X^\epsilon_s|^2_H \right] \]
\[ + E\left[ \int_0^t e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} \|X^n_{s} - X^\epsilon_s\|_V^2 ds \right] \]
\[ \leq \frac{1}{4} E\left[ \sup_{0 \leq s \leq t} e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} |X^n_{s} - X^\epsilon_s|^2_H \right] \]
\[ + C E\left[ \int_0^t e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} |X^n_{s} - X^\epsilon_s|^2_H ds \right] \]
\[ + C \sum_{i=1}^m E\left[ \int_0^t \int_{|x| \leq \epsilon} e^{-\gamma \int_0^u (\|X^n_{u-}\|_V^2 + \|X^\epsilon_u\|_V^2) du} \right] \]
\[ \times \left| \frac{1}{\alpha(\epsilon)} (\sigma_n^i(X^n_{s-}) x - \sigma_i(X^n_{s-}) x) \right|^2 H ds \nu(dx) \]
(3.50)
+ C \sum_{i=1}^{m} E \left[ \int_0^t \int_{|x| \leq \epsilon} e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} ds \right.
\times \left. \left| \frac{1}{\alpha(\epsilon)} \left( \sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s) \right) \right|^2_H \, \nu(dx) \right]
\leq \frac{1}{4} E \left[ \sup_{0 \leq s \leq t} e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} \left( X^{n,\epsilon}_s - X^\epsilon_s \right)^2_H \right]
+ C E \left[ \int_0^t e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} \left( X^{n,\epsilon}_s - X^\epsilon_s \right)^2_H ds \right]
+ C \sum_{i=1}^{m} E \left[ \int_0^t \int_{|x| \leq \epsilon} e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} ds \right.
\times \left. \left| \frac{1}{\alpha(\epsilon)} \left( \sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s) \right) \right|^2_H \, \nu(dx) \right],

where the uniform Lipschitz constant of \( \sigma^i_n \) in (H.3)(i) has been used. Applying the Gronwall’s inequality, we obtain

\begin{equation}
E \left[ \sup_{0 \leq s \leq t} e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} \left( X^{n,\epsilon}_s - X^\epsilon_s \right)^2_H \right]
+ E \left[ \int_0^t e^{-\gamma \int_0^s \left( \|X^{n,\epsilon}_u\|_V^2 + \|X^{\epsilon}_u\|_V^2 \right) du} \left( X^{n,\epsilon}_s - X^\epsilon_s \right)^2_H ds \right]
\leq C \sum_{i=1}^{m} E \left[ \int_0^T |\sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s)|^2_H ds \right].
\end{equation}

(3.51)

For any \( M > 0 \), we have

\begin{equation}
\sum_{i=1}^{m} E \left[ \int_0^T \left| \sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s) \right|^2_H ds \right]
= \sum_{i=1}^{m} E \left[ \int_0^T \left| \sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s) \right|^2_H ds \right.
\times \left. \sup_{0 \leq s \leq T} |X^\epsilon_s|_H \leq M \right]
+ \sum_{i=1}^{m} E \left[ \int_0^T \left| \sigma^i_n(X^\epsilon_s) - \sigma_i(X^\epsilon_s) \right|^2_H ds \right.
\times \left. \sup_{0 \leq s \leq T} |X^\epsilon_s|_H > M \right]
\leq T \sum_{i=1}^{m} \sup_{|z| \leq M} \left| \sigma^i_n(z) - \sigma_i(z) \right|^2_H + C T \frac{1}{M} \left( 1 + E \left[ \sup_{0 \leq s \leq T} |X^\epsilon_s|^3_H \right] \right)
\leq T \sum_{i=1}^{m} \sup_{|z| \leq M} \left| \sigma^i_n(z) - \sigma_i(z) \right|^2_H + C T \frac{1}{M},
\end{equation}

(3.52)
where (3.3) has been used. Since $M$ can be chosen as large as we wish, together with (3.51) and (H.3)(ii) we deduce that
\[
\lim_{n \to \infty} \sup_{\varepsilon} E \left[ \sup_{0 \leq s \leq T} e^{-\gamma \int_0^s (\|X^n_{s,\varepsilon}\|^2_V + \|X^\varepsilon_s\|^2_V) du} \left| X^n_{s,\varepsilon} - X^\varepsilon_s \right|^2_H \right] = 0.
\]

(3.53)

For any given $\delta_1 > 0$, in view of (3.46), (3.47), we can choose a positive constant $M_1$ such that
\[
\sup_{n,\varepsilon} P \left( \sup_{0 \leq t \leq T} \left| X^n_{t,\varepsilon} - X_t^\varepsilon \right|_H > \delta, \int_0^T (\|X^n_{s,\varepsilon}\|^2_V + \|X^\varepsilon_s\|^2_V) ds > M_1 \right) \leq \frac{\delta_1}{2}.
\]

(3.54)

On the other hand,
\[
\sup_{\varepsilon} P \left( \sup_{0 \leq t \leq T} \left| X^n_{t,\varepsilon} - X_t^\varepsilon \right|_H > \delta, \int_0^T (\|X^n_{s,\varepsilon}\|^2_V + \|X^\varepsilon_s\|^2_V) ds \leq M_1 \right) \leq e^{\gamma M_1 / \delta^2} \sup_{\varepsilon} E \left[ \sup_{0 \leq s \leq T} e^{-\gamma \int_0^s (\|X^n_{s,\varepsilon}\|^2_V + \|X^\varepsilon_s\|^2_V) du} \left| X^n_{s,\varepsilon} - X^\varepsilon_s \right|^2_H \right].
\]

(3.55)

It follows from (3.53) and (3.55) that there exists $N > 0$ such that for $n \geq N$,
\[
\sup_{n,\varepsilon} P \left( \sup_{0 \leq t \leq T} \left| X^n_{t,\varepsilon} - X_t^\varepsilon \right|_H > \delta, \int_0^T (\|X^n_{s,\varepsilon}\|^2_V + \|X^\varepsilon_s\|^2_V) ds \leq M_1 \right) \leq \frac{\delta_1}{2}.
\]

(3.56)

Combining (3.54) and (3.56) together yields (3.44).

Finally, we prove that $\mu^\varepsilon$ converges to $\mu$. Let $\mu_n^\varepsilon$, $\mu_n$ denote, respectively, the laws of $X^n_{t,\varepsilon}$ and $X^n$. Let $G$ be a bounded, uniformly continuous function on $E := D([0, T], H)$. For any $n \geq 1$, we write
\[
\int_E G(w) \mu^\varepsilon(dw) - \int_E G(w) \mu(dw) = \int_E G(w) \mu^\varepsilon(dw) - \int_E G(w) \mu_n^\varepsilon(dw) + \int_E G(w) \mu_n^\varepsilon(dw)
\]
\[
- \int_E G(w) \mu_n(dw) + \int_E G(w) \mu_n(dw) - \int_E G(w) \mu(dw)
\]
\[
= E[G(X^\varepsilon) - G(X^n_{t,\varepsilon})] + \left( \int_E G(w) \mu_n^\varepsilon(dw) - \int_E G(w) \mu_n(dw) \right)
\]
\[
+ E[G(X^n) - G(X)].
\]
Give any $\delta > 0$. Since $G$ is uniformly continuous, there exists $\delta_1 > 0$ such that

$$
\left| E \left[ \left( G(X^\varepsilon) - G(X^{n,\varepsilon}) \right) \right], \sup_{0\leq s \leq T} \left| X^{n,\varepsilon}_s - X^\varepsilon_s \right|_H \leq \delta_1 \right| \leq \frac{\delta}{4}
$$

for all $n \geq 1$, $\varepsilon > 0$. In view of (3.44) and (3.45), there exists $N_1$,

$$
\sup_{\varepsilon} \left| E \left[ \left( G(X^\varepsilon) - G(X^{N_1,\varepsilon}) \right) \right], \sup_{0\leq s \leq T} \left| X^{N_1,\varepsilon}_s - X^\varepsilon_s \right|_H > \delta_1 \right| \leq \frac{\delta}{4},
$$

and

$$
\left| E \left[ \left( G(X^{N_1}) - G(X) \right) \right] \right| \leq \frac{\delta}{4}.
$$

On the other hand, by Theorem 3.6, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \leq \varepsilon_1$,

$$
\left| \int_E G(w)\mu^\varepsilon_{N_1}(dw) - \int_E G(w)\mu_{N_1}(dw) \right| \leq \frac{\delta}{4}.
$$

Putting (3.57)–(3.60) together we obtain that for $\varepsilon \leq \varepsilon_1$,

$$
\left| \int_E G(w)\mu^\varepsilon(dw) - \int_E G(w)\mu(dw) \right| \leq \delta.
$$

Since $\delta > 0$ is arbitrarily small, we deduce that

$$
\lim_{\varepsilon \to 0} \int_E G(w)\mu^\varepsilon(dw) = \int_E G(w)\mu(dw)
$$

completing the proof of the theorem. \qed

**EXAMPLE 3.8.** Approximations of stochastic Burgers equations.

Consider the stochastic Burgers equations on $[0, 1]$:

$$
du(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) \, dt + \frac{1}{2} \frac{\partial}{\partial \xi} [u^2(t, \xi)] \, dt + \sum_{i=1}^{m} \sigma_i(u(t, \xi)) \, dB_i^i,
$$

$$
u(t, 0) = u(t, 1) = 0, \quad t > 0,
$$

$$
\begin{aligned}
du^\varepsilon(t, \xi) &= \frac{\partial^2}{\partial \xi^2} u^\varepsilon(t, \xi) \, dt + \frac{1}{2} \frac{\partial}{\partial \xi} [(u^\varepsilon)^2(t, \xi)] \, dt \\
&+ \frac{1}{\alpha(\varepsilon)} \sum_{i=1}^{m} \int_{|x| \leq \varepsilon} \sigma_i(u^\varepsilon(t-, \xi)) x \tilde{N}_i(dt, dx),
\end{aligned}
$$

$$
u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, \quad t > 0,
$$

where $\sigma_i(\cdot), i = 1, \ldots, m$ is a Lipschitz continuous functions with $\sigma_i(0) = 0$. 


Let $V = H_0^1(0, 1)$ with the norm
\[
\|v\|_V := \left( \int_0^1 \left( \frac{\partial u(\xi)}{\partial \xi} \right)^2 d\xi \right)^{1/2} = \|v\|.
\]
Let $H := L^2(0, 1)$ be the $L^2$-space with inner product $(\cdot, \cdot)$.

Set
\[
Au = -\frac{\partial^2}{\partial \xi^2}u(\xi) \quad \forall u \in D(A) = H^2(0, 1) \cap V.
\]

Define for $k \geq 1$,
\[
e_k(\xi) = \sqrt{2}\sin(k\pi \xi), \quad \xi \in [0, 1].
\]
Then $e_k, k \geq 1$ are eigenvectors of the operator $A$ with eigenvalues $\lambda_k = \pi^2 k^2$, which forms an orthonormal basis of the Hilbert space $H$. For $u \in V$, define
\[
B(u) := u(\xi) \frac{\partial}{\partial \xi} u(\xi), \quad \sigma(u) := \sigma(u(\xi)).
\]
Here, $B(u) \in H$ because $\|u\|_{L^\infty} < \infty$ for $u \in V$. By the Lipschitz continuity of $\sigma_i$, it is easily seen that
\[
\|\sigma_i(u)\|_V \leq C(1 + \|u\|_V), \quad i = 1, \ldots, m,
\]
hence $(H.3)'$ holds. Now let us show that $B(u)$ satisfies condition $(H.2)$. First, $(H.2)(i), (B(u), u) = 0$, is well known (see, e.g., [8]). Note that $\bar{e}_k = \frac{1}{\sqrt{\lambda_k}}e_k, k \geq 1$ forms an orthonormal basis of $V$. Recall that any element $l \in V^*$ can be identified through the Riesz representation theorem as an element $\bar{l}$ in $V$, and moreover,
\[
\|l\|_{V^*}^2 = \|\bar{l}\|_V^2 = \sum_{k=1}^\infty \langle \bar{l}, \bar{e}_k \rangle^2 = \sum_{k=1}^\infty (l(\bar{e}_k))^2.
\]
Thus, for $u \in V$, we have
\[
\|B(u)\|_{V^*}^2 = \sum_{k=1}^\infty (B(u)(\bar{e}_k))^2
\]
\[
= \sum_{k=1}^\infty \left( \frac{1}{2} \frac{1}{\sqrt{\lambda_k}} \int_0^1 \frac{\partial}{\partial \xi} \left[ u^2(\xi) \right] e_k(\xi) d\xi \right)^2
\]
\[
= \sum_{k=1}^\infty \left( \frac{1}{2} \frac{1}{\sqrt{\lambda_k}} \int_0^1 u^2(\xi) \frac{\partial}{\partial \xi} e_k(\xi) d\xi \right)^2
\]
\[
= \sum_{k=1}^\infty \left( \frac{1}{2} \int_0^1 u(\xi)^2 \sqrt{2}\cos(k\pi \xi) d\xi \right)^2
\]
\[
\leq C \int_0^1 u^4(s, \xi)^4 d\xi = C \|u\|^4_{L^4},
\]
where we have used the fact that \( \{ \sqrt{2} \cos(k\pi \xi); k \geq 1 \} \) also forms an orthonormal system of \( L^2(0, 1) \). By the Sobolev embedding theorem (see, e.g., Theorem 6, Chapter 5 in [10]), we have

\[
L^4(0, 1) \subset H^{1/4}(0, 1),
\]

where \( H^{1/4}(0, 1) \) is the usual Sobolev space of order \( \frac{1}{4} \). Combing the above embedding theorem with the following well-known interpolation inequality (see, e.g., Section 4.3 in [23])

\[
\|u\|_{H^{1/4}} \leq C|u|_H^{3/4}\|u\|_V^{1/4},
\]

we obtain from (3.68) that

\[
\|B(u)\|_V^* \leq C|u|_H^{3/2}\|u\|_V^{1/2}
\]

proving (H.2)(iii) with \( \gamma = \frac{1}{2} \). Finally, we will check (H.2)(ii). Let \( u, v \in V \). We have

\[
\langle B(u) - B(v), u - v \rangle = \frac{1}{2} \int_0^1 \frac{\partial}{\partial \xi} \left[ u^2(\xi) - v^2(\xi) \right] (u(\xi) - v(\xi)) d\xi
\]

\[
= -\frac{1}{2} \int_0^1 (u^2(\xi) - v^2(\xi)) \frac{\partial}{\partial \xi} (u(\xi) - v(\xi)) d\xi
\]

\[
\leq \frac{1}{2} \int_0^1 \left( \frac{\partial}{\partial \xi} (u(\xi) - v(\xi)) \right)^2 d\xi
\]

\[
+ C \int_0^1 (u(\xi) - v(\xi))^2 (u(\xi) + v(\xi))^2 d\xi
\]

\[
\leq \frac{1}{2}\|u - v\|_V^2 + C\|u - v\|_H^2 (\|u\|_\infty^2 + \|v\|_\infty^2)
\]

\[
\leq \frac{1}{2}\|u - v\|_V^2 + C\|u - v\|_H^2 (\|u\|_V^2 + \|V\|_V^2),
\]

which is (H.2)(ii).

Now we can apply Theorem 3.7 to obtain the following convergence of the solutions of stochastic Burgers equations.

**Theorem 3.9.** Let \( u^\varepsilon, u \) be solutions to the stochastic Burgers equations (3.64) and (3.62). Then \( u^\varepsilon \) converges weakly to \( u \) in the space \( D([0, T]; H) \).

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