Introduction to Microwave Background Polarization

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1 Introduction

Through the hard work of many excellent experimenters, the detection and characterization of temperature fluctuations in the cosmic microwave background is now on sure footing. Following the initial detection by the DMR instrument aboard COBE (Smoot et al. 1990), numerous ground and balloon-based detections have been made, and the first reasonably large temperature maps at angular scales of a degree have been constructed (Devlin et al. 1998). The upcoming satellites MAP and Planck promise full sky temperature maps of unprecedented resolution and sensitivity, as detailed elsewhere in these lectures. Theoretically, much of this intensive effort has been motivated by the realization that the microwave background temperature fluctuations contain a wealth of fundamental cosmological information (Jungman et al. 1996, Zaldarriaga, Seljak, and Spergel 1997, Eisenstein, Hu and Tegmark 1999).

Polarization of the microwave background is a different story. Polarization is expected in every cosmological model, for the simple reason that the Thomson scattering which thermalizes the radiation has a polarization-dependent cross section. But the polarization signal is generically expected to be a factor of 10 to 50 smaller than the temperature fluctuations, presenting that much greater of an experimental challenge. Only upper limits on polarization of around a part in $10^5$ now exist, but a new generation of experiments optimized for polarization are currently being constructed, which potentially have both the raw sensitivity and the control over systematic errors necessary to make the

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first detection. In many ways, the experimental study of polarization today is at about the same stage that temperature was ten years ago.

This contribution aims to explain how polarization is physically characterized, how it is generated in the microwave background, the mathematical description of the associated power spectra, and the physical effects which might be probed via polarization measurements. Another elementary reference about microwave background polarization from a somewhat different perspective is White and Hu (1997).

2 Review of Stokes Parameters

Polarized light is conventionally described in terms of the Stokes parameters, which are presented in any optics text. Consider a nearly monochromatic plane electromagnetic wave propagating in the $z$-direction; nearly monochromatic here means that its frequency components are closely distributed around its mean frequency $\omega_0$. The components of the wave’s electric field vector at a given point in space can be written as

$$E_x = a_x(t) \cos[\omega_0 t - \theta_x(t)], \quad E_y = a_y(t) \cos[\omega_0 t - \theta_y(t)].$$

(1)

The requirement that the wave is nearly monochromatic guarantees that the amplitudes $a_x$ and $a_y$ and the phase angles $\theta_x$ and $\theta_y$ will vary slowly relative to the inverse frequency of the wave. If some correlation exists between the two components in Eq. (1), then the wave is polarized.

The Stokes parameters are defined as the following time averages:

$$I \equiv \langle a_x^2 \rangle + \langle a_y^2 \rangle;$$

(2)

$$Q \equiv \langle a_x^2 \rangle - \langle a_y^2 \rangle;$$

(3)

$$U \equiv \langle 2a_xa_y \cos(\theta_x - \theta_y) \rangle;$$

(4)

$$V \equiv \langle 2a_xa_y \sin(\theta_x - \theta_y) \rangle.$$  

(5)

The averages are over times long compared to the inverse frequency of the wave. The parameter $I$ gives the intensity of the radiation which is always positive and is equivalent to the temperature for blackbody radiation. The other three parameters define the polarization state of the wave and can have either sign. Unpolarized radiation, or “natural light,” is described by $Q = U = V = 0$.

The parameters $I$ and $V$ are physical observables independent of the coordinate system, but $Q$ and $U$ depend on the orientation of the $x$ and $y$ axes. If a
given wave is described by the parameters $Q$ and $U$ for a certain orientation of the coordinate system, then after a rotation of the $x-y$ plane through an angle $\phi$, it is straightforward to verify that the same wave is now described by the parameters

\[
Q' = Q \cos(2\phi) + U \sin(2\phi), \\
U' = -Q \sin(2\phi) + U \cos(2\phi).
\] (6)

From this transformation it is easy to see that the quantity $Q^2 + U^2$ is invariant under rotation of the axes, and the angle

\[
\alpha \equiv \frac{1}{2} \tan^{-1} \frac{U}{Q}
\] (7)
transforms to $\alpha - \phi$ under a rotation by $\phi$ and thus defines a constant orientation, which physically is parallel to the electric field of the wave. The Stokes parameters are a useful description of polarization because they are additive for incoherent superposition of radiation; note this is not true for the magnitude or orientation of polarization.

While polarization has a magnitude and an orientation, it is not a vector quantity because the orientation does not have a direction, describing only the plane in which the electric field of the wave oscillates. Mathematically, the Stokes parameters are identical for an axis rotation through an angle of $\pi$, whereas for a vector, such a rotation would lead to an inverted vector and a full rotation through $2\pi$ is required to return to the same situation. The transformation law in Eq. (6) is characteristic of the second-rank tensor

\[
\rho = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix},
\] (8)

which also corresponds to the quantum mechanical density matrix for an ensemble of photons (Kosowsky 1996) (the matrix is 2 by 2 because the photon has two helicity states).

3 Polarization from Thomson Scattering

Polarization in the microwave background is generated through the polarization-dependent cross-section for Thomson scattering. Consider Thomson scattering of an incoming unpolarized beam of electromagnetic radiation by an electron; this discussion closely follows those in Kosowsky (1996) and Kosowsky (1998).
The total scattering cross-section, defined as the radiated intensity per unit solid angle divided by the incoming intensity per unit area, is given by

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi}|\hat{e}' \cdot \hat{e}|^2$$  \hspace{1cm} (9)

where $\sigma_T$ is the total Thomson cross section and the vectors $\hat{e}$ and $\hat{e}'$ are unit vectors in the planes perpendicular to the propagation directions which are aligned with the outgoing and incoming polarization, respectively. Consider first a nearly monochromatic, unpolarized incident plane wave of intensity $I'$ and cross-sectional area $\sigma_B$ which is scattered into the $z$-axis direction. Defining the $y$-axes of the incoming and outgoing coordinate systems to be in the scattering plane, the Stokes parameters of the outgoing beam, defined with respect to the $x$-axis, follow from Eq. (9) as

$$I = \frac{3\sigma_T}{8\pi\sigma_B}I'(1 + \cos^2 \theta),$$  \hspace{1cm} (10)

$$Q = \frac{3\sigma_T}{8\pi\sigma_B}I' \sin^2 \theta,$$  \hspace{1cm} (11)

$$U = 0,$$  \hspace{1cm} (12)

where $\theta$ is the angle between the incoming and outgoing beams. By symmetry, Thomson scattering can generate no circular polarization, so $V = 0$ always and will not be considered further. (Note that Eqs. (3) give the well-known result that sunlight from the horizon at midday is linearly polarized parallel to the horizon).

The net polarization produced by the scattering of an incoming, unpolarized radiation field of intensity $I'(\theta, \phi)$ is determined by integrating Eqs. (3) over all incoming directions. Note that the coordinate system for each incoming direction must be rotated about the $z$-axis so that the outgoing Stokes parameters are all defined with respect to a common coordinate system, using the transformation of $Q$ and $U$ under rotations. The result is

$$I(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B} \int d\Omega (1 + \cos^2 \theta) I'(\theta, \phi),$$  \hspace{1cm} (13)

$$Q(\hat{z}) - iU(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B} \int d\Omega \sin^2 \theta e^{2i\phi} I'(\theta, \phi).$$  \hspace{1cm} (14)

Expanding the incident radiation field in spherical harmonics,

$$I'(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi),$$  \hspace{1cm} (15)

leads to the following expressions for the outgoing Stokes parameters:
\[
I(\hat{z}) = \frac{3\sigma_T}{16\pi\sigma_B} \left[ \frac{8}{3} \sqrt{\pi} a_{00} + \frac{4}{3} \sqrt{\pi} a_{20} \right],
\]

\[
Q(\hat{z}) - iU(\hat{z}) = \frac{3\sigma_T}{4\pi\sigma_B} \sqrt{\frac{2\pi}{15}} a_{22}.
\]

Thus polarization is generated along the outgoing \( z \)-axis provided that the \( a_{22} \) quadrupole moment of the incoming radiation is non-zero. To determine the outgoing polarization in a direction making an angle \( \beta \) with the \( z \)-axis, the same physical incoming field must be multipole expanded in a coordinate system rotated through the Euler angle \( \beta \); the rotated multipole coefficients are

\[
\tilde{a}_{lm} = \int d\Omega Y_{lm}^*(R\Omega) I'(\Omega) = \sum_{m'=-m}^{m} D_{m'm}^l(R) \int d\Omega Y_{lm'}^*(\Omega) I'(\Omega),
\]

where \( R \) is the rotation operator and \( D_{m'm}^l \) is the Wigner D-symbol. (For a wonderfully complete reference on representations of the rotation group, see Varshalovich et al. (1988)). In the rotated coordinate system, the multipole coefficient generating polarization is \( \tilde{a}_{22} \) by Eqs. (3). The unrotated multipole components which contribute to polarization will all have \( l = 2 \) by the orthogonality of the spherical harmonics. If the incoming radiation field is independent of \( \phi \), as it will be for individual Fourier components of a density perturbation, then

\[
\tilde{a}_{22} = a_{20} d_{02}^2(\beta) = \frac{\sqrt{6}}{4} a_{20} \sin^2 \beta,
\]

which has used an explicit expression for the reduced D-symbol \( d_{m'm}^l \). The outgoing Stokes parameters are finally

\[
Q(\hat{n}) - iU(\hat{n}) = \frac{3\sigma_T}{8\pi\sigma_B} \sqrt{\frac{\pi}{5}} a_{20} \sin^2 \beta.
\]

In other words, an azimuthally-symmetric radiation field will generate a polarized scattered field if it has a non-zero \( a_{20} \) multipole component, and the magnitude of the scattered polarization will be proportional to \( \sin^2 \beta \). Since the incoming field is real, \( a_{20} \) will be real, \( U = 0 \), and the polarization orientation will be in the plane of the \( z \)-axis and the scattering direction. Similar relationships can be derived for radiation fields which are not azimuthally symmetric, which occur in the cases of vector and tensor metric perturbations.

In short, what this section shows is that unpolarized quadrupolar radiation fields get Thomson scattered into polarized radiation fields. This is the key
fact which must be appreciated to understand why the microwave background should be polarized and what the magnitude of the polarization is expected to be.

4 Generation of Microwave Background Polarization

At times significantly before decoupling, the universe is hot enough that protons and electrons exist freely in a plasma. During this epoch, the rate for photons to Thomson scatter off of free electrons is large compared to the expansion rate of the universe. Thus, the photons and electrons stay in thermal equilibrium at a common temperature and are said to be tightly coupled. As the universe drops below a temperature of around 0.1 eV at a redshift of around 1300, the electrons and protons begin to “recombine” into neutral hydrogen. Within a short time, almost all the free electrons are converted to neutral hydrogen, the rapid Thomson scattering ceases for lack of scatterers, and the radiation is said to decouple. At this point, the radiation will propagate freely until the universe reionizes at some redshift greater than 5.

During the tight coupling epoch, the photons must have a distribution which mirrors that of the electrons. An immediate consequence is that the angular dependence of the radiation field at a given point can only possess a monopole (corresponding to the temperature) and a dipole (corresponding to a Doppler shift from a peculiar velocity) component, and that the radiation field is unpolarized. Any higher multipole moment will rapidly damp away as the electrons scatter off the free electrons, and no net polarization can be produced through scattering.

A quadrupole is subsequently produced at decoupling as free streaming of the photons begins. A single Fourier mode of the radiation field can be described by the temperature distribution function \( \Theta(k, \mu, \eta) \) where \( k \) is the wavenumber, \( \mu = \hat{k} \cdot \hat{n} \) is the angle between the vector \( \mathbf{k} \) and the propagation direction \( \hat{n} \), and \( \eta \) is conformal time. (For mathematical simplicity only a flat universe is considered here, although the non-flat cases are no more complicated conceptually.) Ignoring gravitational potential contributions, free streaming of the photons is described by the Liouville equation \( \dot{\Theta} + ik\mu \Theta = 0 \). If the free streaming begins at time \( \eta_s \), then the solution at a later time is simply \( \Theta(k, \mu, \eta) = \Theta(k, \mu, \eta_s) \exp(-ik\mu(\eta-\eta_s)) \). We can reexpress the \( \mu \) dependence as a multipole expansion

\[
\Theta(k, \mu, \eta) = \sum_{l=0}^{\infty} (-i)^l \Theta_l(k, \eta) P_l(\mu); \quad (21)
\]
using the identity
\[ e^{iz \cos \phi} = \sum_{n=0}^{\infty} (2n+1)i^n j_n(z) P_n(\cos \phi) \] (22)

the free streaming becomes
\[ \Theta_l(k, \eta) = (2l+1)[\Theta_0(k, \eta) j_l(k\eta - k\eta_*) + \Theta_1(k, \eta) j'_l(k\eta - k\eta_*)] \] (23)

where \( j_l \) is the usual spherical Bessel function.

We are interested in the behavior of the free streaming at times near decoupling; at later times, the number density of free electrons which can Thomson scatter has dropped to negligible levels and no further polarization can be produced. The physical length scales of interest for microwave background fluctuations will be larger than the thickness of the last scattering surface, so \( k(\eta - \eta_*) \) will be small compared to unity. For small arguments \( x \ll 1 \), \( j_l(x)/j'_l(x) \sim x/l \), which implies that if the monopole and dipole radiation components are initially of comparable size, free streaming through the region of polarization generation with thickness \( \Delta \) will generate a quadrupole component from the dipole which is a factor of \( 2/(k\Delta) \) larger than the quadrupole component from the monopole. In other words, on length scales large compared to the thickness of the surface of last scattering, the quadrupole moment and thus the polarization couples much more strongly to the velocity of the baryon-photon fluid than to the density. Note that on smaller scales with \( k\Delta \gtrsim 1 \), the polarization can couple more strongly to either the velocity or the density, depending on the scale, but for standard recombination these scales are always small enough that the microwave background fluctuations are strongly diffusion damped.

5 Polarization and Sound Waves

Inflation produces acoustic oscillations in the early universe which are coherent: all Fourier modes of a given wavelength have the same phase. Such acoustic oscillations have a very specific relationship between velocity and density perturbations, which shows up in the relative angular scales of features in the temperature and polarization power spectra. As emphasized in Hu and Sugiyama (1996), the photon-baryon density perturbation in the tight-coupling regime obeys the differential equation for a forced, damped harmonic oscillator with the damping coming from the expansion of the universe and the forcing from gravitational potential perturbations. The solution is of the
form

\[ \Theta_0(k, \eta) = A_1(\eta) \cos(kr_s) + A_2(\eta) \sin(kr_s) \]  \hspace{1cm} (24)

where the amplitudes vary slowly in time and \( r_s \simeq \eta/\sqrt{3} \) is the sound horizon. The velocity perturbation follows from the photon continuity equation \( \dot{\Theta}_0 = -k\Theta_1/3 \), again neglecting gravitational potential perturbations. A detailed consideration of boundary conditions reveals that initial isentropic density perturbations couple to the cosine harmonic in the small-scale limit, and this approximation is good even for the largest-wavelength acoustic oscillations (Hu and White 1996). Thus in an inflationary model, at the surface of last scattering, the photon monopole has a \( k \)-dependence of approximately \( \cos(k\eta_\star/\sqrt{3}) \), while the dipole, which is the main contributor to the polarization, has a \( k \)-dependence of approximately \( \sin(k\eta_\star/\sqrt{3}) \). For initial isocurvature perturbations, the density perturbations couple instead to the sine harmonic, but the photon monopole and dipole are still \( \pi/2 \) out of phase.

Squaring these amplitudes gives the rough behavior of the CMB power spectra. Acoustic peaks in the temperature power spectrum occur at scales where \( \cos^2(k\eta_\star/\sqrt{3}) \) has its maxima. The amplitude of the velocity perturbations are suppressed by a factor of \( c_s \) with respect to the density perturbations, so the temperature peaks reflect only the density perturbations. The polarization couples to the temperature dipole on scales larger than the thickness of the last scattering surface, and acoustic peaks in the polarization power spectrum will be present at scales where \( \sin^2(k\eta_\star/\sqrt{3}) \) has its maxima. In other words, the temperature peaks represent density extrema, the polarization peaks represent velocity extrema, and for coherent oscillations these two sets of maxima are at \textit{interleaved angular scales} (see Fig. 1). This is a generic signature of coherent acoustic oscillations and is likely the most easily measurable physics signal in microwave background polarization. If two peaks are detected in the temperature power spectrum, the angular scale between the two makes a tempting target for polarization measurements.

The cross-correlation between the temperature and polarization will have extrema as \( -\cos(k\eta_\star/\sqrt{3}) \sin(k\eta_\star/\sqrt{3}) \) which fall between the temperature and polarization peaks. (The correlation between the polarization and the velocity contribution to the temperature averages to zero because of their different angular dependences.) The sign of the cross-correlation peaks can be used to deduce whether a temperature peak represents a compression or a rarefaction, which can be checked against the alternating peak-height signature if the universe has a large enough baryon fraction (Hu and Sugiyama 1996).

A combination of isentropic and isocurvature fluctuations shifts all acoustic phases by the same amount if the ratio of their amplitudes is independent of scale, thus leaving the acoustic signature intact. If the amplitude ratio depends
Fig. 1. The power spectra for temperature fluctuations (top), polarization (center) and temperature-polarization cross-correlation (bottom) for a typical inflationary model. The oscillations remain in phase up to $l = 3000$.

on scale, the coherent acoustic oscillations could be modified, but fine tuning would be required to wash them out completely. Multi-field inflation models generically produce both isocurvature and isentropic perturbations (Kofman and Linde 1987, Mukhanov and Steinhardt 1998) but the resulting microwave background power spectra are just beginning to be studied in detail (Kanazawa et al. 1998).

6 The Tensor Harmonic Expansion

The last two sections have pulled a fast one. We began by discussing polarization as a two component tensor quantity, but then started discussing the production of polarization as if only its amplitude were relevant. A more complete formalism for describing the polarization field has been worked out and will be presented in this section (see Kamionkowski, Kosowsky, and Stebbins
(1997) for a more extensive discussion). An equivalent formalism employing
spin-weighted spherical harmonics has been used extensively by Zaldarriaga
and Seljak (1997). Note that the normalizations employed by Seljak and Zal-
darriaga are slightly different than those adopted here and by Kamionkowski,
Kosowsky, and Stebbins (1997).

The microwave background temperature pattern on the sky \( T(\hat{n}) \) is conven-
tionally expanded in a complete set of orthonormal basis functions, the spher-
ical harmonics:

\[
\frac{T(\hat{n})}{T_0} = 1 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{(lm)}^T Y_{(lm)}(\hat{n})
\]

(25)

where

\[
a_{(lm)}^T = \frac{1}{T_0} \int d\hat{n} T(\hat{n}) Y_{(lm)}^*(\hat{n})
\]

(26)

are the temperature multipole coefficients and \( T_0 \) is the mean CMB tempera-
ture. Similarly, we can expand the polarization tensor for linear polariza-
tion,

\[
P_{ab}(\hat{n}) = \frac{1}{2} \begin{pmatrix} Q(\hat{n}) & -U(\hat{n}) \sin \theta \\ -U(\hat{n}) \sin \theta & -Q(\hat{n}) \sin^2 \theta \end{pmatrix}
\]

(27)

(compare with Eq. 8; the extra factors are convenient because the usual spher-
ical coordinate basis is orthogonal but not orthonormal) in terms of tensor
spherical harmonics, a complete set of orthonormal basis functions for sym-
metric trace-free \( 2 \times 2 \) tensors on the sky,

\[
\frac{P_{ab}(\hat{n})}{T_0} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[ a_{(lm)}^G Y_{(lm)ab}(\hat{n}) + a_{(lm)}^C Y_{(lm)ab}(\hat{n}) \right],
\]

(28)

where the expansion coefficients are given by

\[
a_{(lm)}^G = \frac{1}{T_0} \int d\hat{n} P_{ab}(\hat{n}) Y_{(lm)ab}^G(\hat{n}),
\]

(29)

\[
a_{(lm)}^C = \frac{1}{T_0} \int d\hat{n} P_{ab}(\hat{n}) Y_{(lm)ab}^C(\hat{n}),
\]

(30)

which follow from the orthonormality properties

\[
\int d\hat{n} Y_{(lm)ab}^G(\hat{n}) Y_{(l'm')}^{G*}(\hat{n}) = \int d\hat{n} Y_{(lm)ab}^C(\hat{n}) Y_{(l'm')}^{C*}(\hat{n}) = \delta_{ll'} \delta_{mm'},
\]

(31)
These tensor spherical harmonics have been used primarily in the literature of gravitational radiation, where the metric perturbation can be expanded in these tensors. Explicit forms can be derived via various algebraic and group theoretic methods; see Thorne (1980) for a complete discussion. A particularly elegant and useful derivation of the tensor spherical harmonics (along with the vector spherical harmonics as well) is provided by differential geometry (Stebbins 1996). Given a scalar function on a manifold, the only related vector quantity at a given point of the manifold is the covariant derivative of the scalar function. The tensor basis functions can be derived by taking the scalar basis functions $Y_{lm}$ and applying to them two covariant derivative operators on the manifold of the two-sphere (the sky):

$$Y_{(lm)ab}^G = N_l \left( Y_{(lm):ab} - \frac{1}{2} g_{ab} Y_{(lm):c}^c \right),$$  \hspace{1cm} (33)

and

$$Y_{(lm)ab}^C = \frac{N_l}{2} \left( Y_{(lm):ac}^c \epsilon_b + Y_{(lm):bc}^c \epsilon_a \right),$$  \hspace{1cm} (34)

where $\epsilon_{ab}$ is the completely antisymmetric tensor, the ";" denotes covariant differentiation on the 2-sphere, and

$$N_l \equiv \sqrt{\frac{2(l - 2)!}{(l + 2)!}}$$  \hspace{1cm} (35)

is a normalization factor. Note that the somewhat more familiar vector spherical harmonics used to describe electromagnetic multipole radiation can likewise be derived as a single covariant derivative of the scalar spherical harmonics.

While the formalism of differential geometry may look imposing at first glance, the expansion of the polarization field has been cast into exactly the same form as for the familiar temperature case, with only the extra complication of evaluating covariant derivatives. Explicit forms for the tensor harmonics are given in Kamionkowski, Kosowsky, and Stebbins (1997). Note that the underlying manifold, the two-sphere, is the simplest non-trivial manifold, with a constant Ricci curvature $R = 2$, so the differential geometry is easy. One particularly useful property for doing calculations is that the covariant derivatives are
subject to integration by parts:

$$\int d\hat{n}AB_{,a} = -\int d\hat{n}A_{,a}B$$  \hspace{1cm} (36)$$

with no surface term if the integral is over the entire sky. Also, the scalar spherical harmonics are eigenvalues of the Laplacian operator:

$$Y_{(lm):a} \equiv \nabla^2 Y_{(lm)} = -l(l+1)Y_{(lm)}. \hspace{1cm} (37)$$

The existence of two sets of basis functions, labeled here by “G” and “C”, is due to the fact that the symmetric traceless $2 \times 2$ tensor describing linear polarization is specified by two independent parameters. In two dimensions, any symmetric traceless tensor can be uniquely decomposed into a part of the form $A_{,ab} - (1/2)g_{ab}A_{,cc}$ and another part of the form $B_{,ac}\epsilon^c_b + B_{,bc}\epsilon^c_a$ where $A$ and $B$ are two scalar functions. This decomposition is quite similar to the decomposition of a vector field into a part which is the gradient of a scalar field and a part which is the curl of a vector field; hence we use the notation G for “gradient” and C for “curl”. In fact, this correspondence is more than just cosmetic: if a linear polarization field is visualized in the usual way with headless “vectors” representing the amplitude and orientation of the polarization, then the G harmonics describe the portion of the polarization field which has no handedness associated with it, while the C harmonics describe the other portion of the field which does have a handedness (just as with the gradient and curl of a vector field).

This geometric interpretation leads to an important physical conclusion. Consider a universe containing only scalar perturbations, and imagine a single Fourier mode of the perturbations. The mode has only one direction associated with it, defined by the Fourier vector $k$; since the perturbation is scalar, it must be rotationally symmetric around this axis. (If it were not, the gradient of the perturbation would define an independent physical direction, which would violate the assumption of a scalar perturbation.) Such a mode can have no physical handedness associated with it, and as a result, the polarization pattern it induces in the microwave background couples only to the G harmonics. Another way of stating this conclusion is that primordial density perturbations produce no C-type polarization as long as the perturbations evolve linearly. This property is very useful for constraining or measuring other physical effects, several of which are considered below.

Finally, just as temperature fluctuations are commonly characterized by their power spectrum $C_l$, polarization fluctuations possess analogous power spectra. We now have three sets of multipole moments, $a_{T(lm)}$, $a_{G(lm)}$, and $a_{C(lm)}$, which fully describe the temperature/polarization map of the sky. Statistical isotropy implies that
\[ \langle a_{(lm)}^{\Gamma^*} a_{(l' m')}^\Gamma \rangle = C_l^\Gamma \delta_{ll'} \delta_{mm'}, \quad \langle a_{(lm)}^{G^*} a_{(l' m')}^G \rangle = C_l^G \delta_{ll'} \delta_{mm'}, \]
\[ \langle a_{(lm)}^{C^*} a_{(l' m')}^C \rangle = C_l^{TC} \delta_{ll'} \delta_{mm'}, \quad \langle a_{(lm)}^{G^*} a_{(l' m')}^C \rangle = C_l^{GC} \delta_{ll'} \delta_{mm'}, \]
\[ \langle a_{(lm)}^{T^*} a_{(l' m')}^C \rangle = C_l^{TC} \delta_{ll'} \delta_{mm'}, \quad \langle a_{(lm)}^{G^*} a_{(l' m')}^C \rangle = C_l^{GC} \delta_{ll'} \delta_{mm'}, \]  

where the angle brackets are an average over all realizations of the probability distribution for the cosmological initial conditions. Simple statistical estimators of the various \( C_l \)'s can be constructed from maps of the microwave background temperature and polarization.

For Gaussian theories, the statistical properties of a temperature/polarization map are specified fully by these six sets of multipole moments. In addition, the scalar spherical harmonics \( Y_{(lm)} \) and the G tensor harmonics \( Y_{(lm)ab}^{G} \) have parity \((-1)^l\), but the C harmonics \( Y_{(lm)ab}^{C} \) have parity \((-1)^{l+1}\). If the large-scale perturbations in the early universe were invariant under parity inversion, then \( C_l^{TC} = C_l^{GC} = 0 \). The arguments in the previous paragraph about handedness further imply that for scalar perturbations, \( C_l^{C} = 0 \). A question of substantial theoretical and experimental interest is what kinds of physics produce measurable nonzero \( C_l^{C} \), \( C_l^{TC} \), and \( C_l^{GC} \). This question is addressed in the following section.

The power spectra can be computed for a given cosmological model through well-known numerical techniques. A set of power spectra for scalar and tensor perturbations in a typical inflation-like cosmological model, generated with the CMBFAST code (Seljak and Zaldarriaga 1996) are displayed in Fig. 2.

### 7 Polarization and Physical Effects

What is microwave background polarization good for? One basic and model-independent answer to this question was outlined above: polarization can provide a clean demonstration of the existence of acoustic oscillations in the early universe. The fact that three of the six polarization-temperature power spectra are zero for linear scalar perturbations gives several other interesting and model-independent probes of physics.

The most important is that the “curl” polarization power spectrum directly reflects the existence of any vector (vorticity) or tensor (gravitational wave) metric perturbations. Inflation models generically predict a nearly-scale-invariant spectrum of tensor perturbations, while defects or other active sources produce significant amounts of both vector and tensor perturbations. If the measured temperature power spectrum of the microwave background turns out to look different than what is expected in the broad class of inflation-like cosmological models, polarization will tell what part of the temperature
Fig. 2. Theoretical predictions for the four nonzero CMB temperature-polarization spectra as a function of multipole moment $l$. The solid curves are the predictions for a COBE-normalized scalar perturbations, while the dotted curves are COBE-normalized tensor perturbations. Note that the panel for $C^C_l$ contains no dotted curve since scalar perturbations produce no “C” polarization component; instead, the dashed line in the lower right panel shows a reionized model with optical depth $\tau = 0.1$ to the surface of last scatter.
anisotropies arise from vector and tensor perturbations. More intriguingly, in inflation models, the amplitude of the tensor perturbations is directly proportional to the energy scale at which inflation occurred, so characterizing the gravitational wave background becomes a probe of GUT-scale physics at $10^{16}$ GeV! Inflation also predicts potentially measurable relationships between the amplitudes and power law indices of the primordial density and gravitational wave perturbations (see (Lidsey et al. 1997) a comprehensive overview), and measuring a $C_l^T$ power spectrum appears to be the only way to obtain precise enough measurements of the tensor perturbations to test these predictions. A microwave background map with foreseeable sensitivity could measure gravitational wave perturbations with amplitudes smaller than $10^{-3}$ times the amplitude of density perturbations (Kamionkowski and Kosowsky 1998), thanks to the fact that the density perturbations don’t contribute to $C_l^T$. The tensor perturbations generally contribute significantly to the temperature perturbations at angular scales larger than two degrees ($l \lesssim 100$) in a flat universe but have a much broader range of scales in polarization ($50 \lesssim l \lesssim 500$). For tensor and vector perturbations, the amplitude of the C-polarization is generally about the same as that of the G-polarization; if the perturbations inducing the COBE temperature anisotropies are 10% tensors, then we expect the peak of $l^2C_l^C \simeq 10^{-15}$ at angular scales around $l = 80$. An experimental challenge not for the faint of heart!

A second source of C-type polarization is gravitational lensing. The mass distribution in the universe between us and the surface of last scatter will bend the geodesics of the microwave background photons. This lensing can be described by an effective displacement field, in which the temperature and polarization at each point of the sky in an unlensed universe is mapped to a nearby but different point on the sky when lensing is accounted for. The displacement alters the shape of temperature contours in the microwave background, and likewise distorts the polarization pattern, inducing some curl component to the polarization field. Detailed calculations of this effect and the induced $C_l^C$ have been made by Zaldarriaga and Seljak (1998). The amplitude of this effect is expected to be around $l^2C_l^C \simeq 10^{-14}$ on a broad range of subdegree angular scales ($200 \lesssim l \lesssim 3000$) with the power spectrum peaking around $l = 1000$ in a flat universe. This lensing polarization signal is just at the limit of detectability for the upcoming Planck satellite; future polarization satellites with better sensitivity could make detailed lensing maps based on the curl component of microwave background polarization. It is interesting to note that tensor perturbations and gravitational lensing are substantially distinguishable by their different angular scales. Note that the most recent version of the publicly available CMBFAST code by Seljak and Zaldarriaga (Seljak and Zaldarriaga 1996) computes polarization from both tensor modes and from gravitational lensing.

A third source of C-type polarization is a primordial magnetic field. If a mag-
netic field was present at recombination, the linear polarization of electromagnetic radiation would undergo a Faraday rotation as it propagated through the surface of last scatter while significant numbers of free electrons were still present. (Such rotation could also occur after reionization, but both the electron density and the field strength would be much smaller and the resulting rotation is small compared to the primordial signal). This effect rotates an initial G-type polarization field into a C-type polarization field. A detailed estimate of the magnitude of this effect (Kosowsky and Loeb 1997) shows that a primordial field with present strength $10^{-9}$ gauss induces a measurable one-degree rotation in the polarization at a frequency of 30 GHz. Faraday rotation depends quadratically on wavelength of the radiation, so down at 3 GHz, the rotation would be a huge 100 degrees (although the polarized emission from synchrotron radiation would also be correspondingly larger). Such a rotation will induce $P_{TC}$ at a level of between $10^{-15}$ for one degree of rotation and $10^{-11}$ for large rotations. Additionally, it has been pointed out that Faraday rotation will contribute also to the $C_{l}^{TC}$ cross-correlation at corresponding levels (Scannapieco and Ferreira 1997) as well as to $C_{l}^{GC}$. Investigation of the angular dependence and detectability of such a signal is ongoing (Mack and Kosowsky 1999). The best current constraints on a homogeneous component of a primordial magnetic field come from COBE constraints on anisotropic Bianchi spacetimes (Barrow, Ferreira and Silk 1997), because a universe which contains a homogeneous magnetic field cannot be statistically isotropic. Detection of a significant primordial magnetic field would both provide the seed field needed to generate current galactic and subgalactic-scale magnetic fields via the dynamo mechanism, and also provide a very interesting constraint on fundamental particle physics, particularly if a field on large scales is detected (see, e.g., Turner and Widrow (1988) or Gasperini et al. (1995)).

Faraday rotation from magnetic fields is a special case of cosmological birefringence: rotation of polarization by differing amounts depending on direction of observation. Such rotation could arise from interactions between photons and other unknown fields. Constraints on the C-polarization of the microwave background could strongly constrain new pseudoscalar particles (see, e.g., Carroll and Field (1997)). More generally, non-zero cosmological contributions to the $C_{l}^{TC}$ and $C_{l}^{GC}$ cross correlations, which must be zero if parity is a valid symmetry of the cosmological perturbations, would indicate some intrinsic parity to either the primordial perturbations (Lue, Wang, and Kamionkowski 1998) or to some interaction of the microwave background photons (Carroll 1998). These types of effects are generally independent of photon frequency, so they can be distinguished from Faraday rotation through microwave background frequency dependence.

The above signals are all model-independent probes of new physics using microwave background polarization. An additional less daring but initially more useful and important use of polarization is in determining and constraining the
basic background cosmology of the universe. It has been appreciated for several years now that the microwave background offers the cleanest and most powerful constraint on the gross features of the universe (Jungman et al. 1996). If the universe is described by an inflation-type model, with nearly scale-invariant initial adiabatic perturbations which evolve via gravitational instability, then the power spectrum of microwave background temperature fluctuations can strongly constrain nearly all cosmological parameters describing the universe: densities of various matter and energy components, amplitudes and power laws of initial density and gravitational wave perturbations, the Hubble parameter, and the redshift of reionization. More recent work (Zaldarriaga, Seljak, and Spergel 1997, Eisenstein, Hu and Tegmark 1999) has shown that the addition of polarization information can help tighten these constraints considerably, mainly because the new information now gives four theoretical power spectra to match instead of just one. Polarization particularly helps constrain the reionization redshift and the baryon density (Zaldarriaga and Harari 1995). Polarization will also be important for deciphering the universe if measurements of the temperature anisotropies reveal that the universe is not described by the simple class of inflation-like cosmological models: it is a strong discriminator between vector and tensor perturbations and scalar perturbations (Kamionkowski, Kosowsky, and Stebbins 1997).

Finally, no discussion of this sort would be completely honest without mentioning the thorny issue of foreground emission. We are gradually concluding that foregrounds have some non-negligible effect on temperature anisotropies, but that the amplitudes of various foregrounds are small enough that they will not substantially hinder our ability to draw cosmological conclusions from microwave background temperature maps (see, e.g., Tegmark (1998) for a recent estimate). Whether the same will prove true for polarization is unknown at present. Free-free emission is likely to have only negligible polarization, but synchrotron emission will be strongly polarized, and the polarization of dust emission is difficult to estimate reliably (Draine and Lazarian 1998). Polarized emission from radio point sources is another potential problem. No present measurements have had sufficient sensitivity to detect polarized emission from any of these foreground sources, so it is difficult to predict the foreground impact. My own guess is that the G-polarization component, from which acoustic oscillations can be confirmed and from which parameter estimation can be significantly improved, will face foreground contamination comparable to the temperature anisotropies. If so, and if the polarization foregrounds are divided evenly between C and G polarization components, then control of foregrounds will become crucial for the very interesting physics probed by the cosmological C polarization. But I fully expect that through a combination of techniques, including carefully tailored sky cuts, measurements at many frequencies, improved theoretical understanding, foreground non-Gaussianity, and foreground template matching, we will separate out the small cosmological polarization signals from whatever polarized foregrounds are out there.
The next five years will bring us microwave background temperature maps of vastly improved sensitivity and resolution, and almost certainly the first detection of microwave background polarization. These observations will provide us with very tight constraints on our cosmological model, or else will reveal some new and unexpected aspect of our universe. Either way, the microwave background will be the cornerstone of a mature cosmology. What is left to do after Planck? One good answer to this question, I believe, is very high sensitivity measurements of microwave background polarization. Such observations hold the promise of probing the potential driving inflation, detecting primordial magnetic fields, mapping the matter distribution in the universe, and likely a variety of other interesting physics yet to be explored.

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