Optimal design on digital filter algorithm of Hankel transform

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Abstract. This study investigates the influence of the parameter c of Hankel transform pairs and sampling interval s on the digital filter of Hankel transform based on the idea of Hankel digital filtering algorithm proposed by Kong. The results show that the selection of the parameter c highly influence the precision of digital filter algorithm, i.e., an inappropriate c can induce algorithm errors. Upon the optimal selection of the parameter c and the constant length of the digital filter, it is found that an optimal sampling interval s also exists to achieve the best precision of digital filter algorithm. Finally, this study proposes six groups of 0-, 1-order Hankel transform digital filters with different length and promote its application.

1. Introduction
Electromagnetic method is an important branch of the geophysical method. In the forward calculation, it is often necessary to calculate the Hankel Transforms [1, 2]. The digital filter algorithm is an effective method to calculate such integrals. The length of the digital filter algorithm usually ranges from a few tens to several hundreds, thereby the digital filter is much faster than the common direct integrals, and has been referred to as the fast Hankel transform (FHT) [3, 4]. The digital filter algorithm of Hankel transform was first proposed by Ghosh [5], later improved by many scholars to handle problem-specific EM integrals [3, 4, 6-11], and new methods have been found for obtaining digital filter coefficients [7, 12]. When using the digital filter algorithm, the Hankel transforms are first transformed into convolution equations, which are then solved to obtain the filter coefficients. At present, there are three main methods to solve the convolution equations or to perform the deconvolution. Firstly, in the spectrum domain, the filter spectrum is obtained by using the division of the output spectrum and the input spectrum, then the filter response in the sample domain is obtained by performing an inverse Fourier transform on the filter spectrum, this method is most frequently used [4, 13]. Secondly, in the sample domain, the Wiener-Hopf minimization method uses input samples and output samples to obtain the filter response, it is a sample-domain method. This method was first introduced by Koefoed and Dirks [7], and later improved by Guptasarma [11] and Guptasarma and Singh [14]. Finally, this method also uses input samples and output samples in the sample domain to obtain the filter response. It was first introduced by Kong in 2007 [12]. This method uses 2N+1 groups of input and output samples to construct a matrix equation, and filter response is obtained by solving the matrix equation.

In the digital filter of Hankel transform, there are two factors of importance. One is the evaluation accuracy, which is the relative error of the results. The other is the length of the filter, which determines the computation speed of Hankel transform. Based on the new method proposed by Kong for the digital filter of the Hankel transform, this paper summarizes the relevant parameters that affect the digital filter algorithm of Hankel transform by comparing and analyzing many sets of digital filter,
and the short, medium, long digital filter coefficients for calculating the 0-, 1-order Hankel transform are given.

2. Digital filter algorithm for Hankel transform

2.1 Principle of digital filter algorithm for Hankel transform

In the electromagnetic method, the Hankel transform is defined as follows

\[ f(r) = \int_0^\infty k(\lambda) J_i(\lambda r) d\lambda \quad (1) \]

Where \( J_i(\lambda r) \) is i-order Bessel function of the first kind, \( i=0,1 \). \( k(\lambda) \) is a kernel function which depends on the subsurface physical properties.

Most electromagnetic modeling codes in use today evaluate Hankel transform by using the digital filter algorithm that was proposed by Ghosh (1971). Briefly, the digital filter algorithm can be found by substituting \( \lambda = e^{sm}, r = e^{jn} \) into Eq. 1:

\[ e^{jn} f(e^{jn}) = \int_{-\infty}^{\infty} k(e^{-jn}) \left[ s e^{(s-n)J_n(e^{(s-n)})} \right] dm \quad (2) \]

Where the parameter \( s \) is the sampling interval, Eq. 2 can be recast in the form of the convolution integral:

\[ F(n) = \int_{-\infty}^{\infty} K(m) H(n-m) dm = \int_{-\infty}^{\infty} K(n-m) H(m) dm \quad (3) \]

Where \( F(n)=e^{jn} f(e^{jn}) \) is the known output function, \( K(m)=k(e^{-jn}) \) is the known input function, and \( H(n-m)=se^{(s-n)J_n(e^{(s-n)})} \) is the filter response to be evaluated. Eq.3 can be rewritten in the discrete approximation form:

\[ F(n) = \sum_{m=-\infty}^{\infty} K(n-m) H(m) \quad (4) \]

Ghosh (1971) recognized that \( H(m) \) is essentially a vector of linear filter coefficients that could be predetermined, and subsequently applied to arbitrary kernel functions. The resulting 2N+1-point digital filter approximation to Eq.2 is:

\[ rf(r) = \sum_{j=-N}^{N} k(\lambda_j/r) \cdot h_j \quad (5) \]

Where \( \lambda_j \) is the logarithmically spaced filter abscissae, \( \lambda_j = e^{cj} \). \( h_j \) is digital filter coefficients of Hankel transform.

2.2 Direct method for obtaining digital filter coefficients

The convolution equation was constructed as a matrix equation by Kong (2007), then the matrix equation was solved to obtain digital filter coefficients in the sample domain. First, the following Hankel transform pairs are selected to evaluate the input function \( X \) and the output function \( Y \) of 0-, 1-order Hankel transforms [19].

\[ \int_{0}^{\infty} \lambda e^{-\lambda^2} J_0(\lambda r) d\lambda = \frac{1}{2c} e^{-\frac{r^2}{4c}} \quad (6) \]

\[ \int_{0}^{\infty} \lambda^2 e^{-\lambda^2} J_1(\lambda r) d\lambda = \frac{r}{4c^2} e^{-\frac{r^2}{4c^2}} \quad (7) \]

Second, it needs to fix the length 2N+1 of the digital filter, and select approximate sampling interval \( s \) and parameter \( c \). The input and output function of Eqs. 6-7 are sampled by Eq. 8. The sampled values are constructed as a matrix equation given by Eq. 5.

\[ r = e^{jn}, \lambda = e^{ms} \quad (8) \]

Where \( n = -N, -N+1, \ldots, N-1, N \); \( m = n - N, n - N + 1, \ldots, n + N - 1, n + N \)
Finally, the matrix equation is solved by iterative method or direct method. In this paper, the generalized minimal residual method (GMRES) \cite{16,17} is used to solve the matrix equation to obtain the digital filter coefficients.

3. The influence of relevant parameter and Error analysis

To verify the digital filter of the 0-,1-order Hankel transforms, we selected the frequency-domain expression of the non-grounding term of horizontal electric field excited by a horizontal electric dipole to verify the digital filter of 0-order Hankel transform, and selected the frequency-domain expression of vertical magnetic field to verify the digital filter of 1-order Hankel transform \cite{18}.

Under the condition of quasi-static, and time dependence $e^{i\omega t}$ is used, the non-grounding term of horizontal electric field is expressed as

$$P(r) = \frac{i\omega \mu_0}{2\pi} \int_0^\infty \frac{\lambda}{\lambda + u} J_0(\lambda r) d\lambda = \frac{1}{2\pi\sigma r} \left[ 1 - (ik \cdot r + 1)e^{-ikr} \right]$$  \hspace{1cm} (9)

The vertical magnetic field is expressed as

$$H_z = \frac{Idl}{2\pi} \int_0^\infty \frac{\lambda^2}{\lambda + u} J_1(\lambda r) d\lambda = \frac{Idl \cdot y}{2\pi r^3 (ik)^3} \left[ 3 - 3(3+3(ik)r+(ik)^2 r^2) e^{-ikr} \right]$$  \hspace{1cm} (10)

Where $I$ is the current intensity, $dl$ is the length of the electric dipole, $r$ is the offset, $r = \sqrt{x^2 + y^2}$. $ik = \sqrt{i\omega \mu_0 \sigma}$, $\omega$ is the angular frequency, $\omega = 2\pi f$. $\mu_0$ is magnetic permeability of air layer; $\sigma$ is the conductivity of the underground medium.

When verifying the accuracy of the digital filter, the parameters are set as follows: in the frequency domain $Idl = 1$, $\sigma = 0.01$ S/m, $x=50m$, $y=50m$; in the spatial domain $Idl = 1$, $\sigma = 0.01$ S/m, $y=50m$, $f = 1$Hz.

3.1 The influence of parameter $c$ on digital filter algorithm

The parameter $c$ is a variable in the Hankel transform pairs. In the literature \cite{12}, when the digital filter coefficients were calculated, the value of the parameter $c$ was taken as a constant by Kong, $c=3$; in the literature \cite{11}, the value of the parameter $c$ was taken as 1 for the design process, and different values of $c$ were tried for checking the error by Guptasarma and Singh. To investigate the effect of parameter $c$ on the filter, and to avoid the influence of filter length $2N+1$ and the sampling interval $s$ on the result, we use digital filter length and sampling interval used in literature \cite{15}, given as

Filter1: N= 025, s=0.212. Filter2: N= 050, s=0.150. Filter3: N=100, s=0.124.

The parameter $c$ takes 9 values in the interval $[10^{-3},10^1]$ by logarithmic distribution equidistant: 0.0010, 0.0032, 0.0100, 0.0316, 0.1000, 0.3162, 1.0000, 3.1623, 10.0000.

Relative errors of the filters with the different parameter $c$ are shown in Figs.1-2. The red line is the relative error of the filter for optimal $c$ value. The optimal value is selected from 101 values in the interval $[10^{-3},10^1]$ by logarithmic distribution equidistant when the accuracy of the filter is the highest in the frequency and spatial domains.

Fig.1-2 show that for the different-length filters of 0-,1-order Hankel transform, there is a big change of relative error in the frequency and spatial domains when the parameter $c$ takes different values, and an inappropriate $c$ value can induce digital filter algorithm error, i.e., the relative error reaches $10^{10}$ or greater when the parameter $c$ is equal to 10. The optimal parameter $c$ can cause the best performance of the filter, and the optimal parameter $c$ changes along with the filter length and sampling interval.
Fig1. Relative error comparison of 0-order Hankel Transform digital filter for different c values:
(a-b) Filter 1 (c-d) Filter 2 (e-f) Filter 3
3.2 The influence of the sampling interval $s$ on digital filter algorithm

To study the influence of the sampling interval $s$ on the digital filter, three different lengths of filters are selected, i.e., $N$ is set as 25, 50 and 100, respectively. In the literature\[11,12,15\] the sampling interval $s$ is generally from 0.1 to 0.3 on the natural log scale. In this paper, 10 values are taken equidistantly for the sampling interval $s$ in the interval $[0.05,0.5]$: 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50. The corresponding optimal $c$ values for different $N$ and $s$ groups are shown in Table 1, where $c_0$ is the optimal $c$ value of the 0-order Hankel transform, $c_1$ is the optimal $c$ value of the 1-order Hankel transform, and the $c_0$ and $c_1$ are calculated in the same way as discussed in Section 3.1.

Relative errors of the filters with different sampling interval are shown in Figs.3-4. The red line is the performance of the best filter among ten filters. Figures 3 and 4 show that the relative error of the filter with a constant $N$ is relatively large when the sampling interval $s$ is extremely small or large, i.e., the relative error is largest when the sampling interval $s$ is equal to 0.05, and the digital filter performs best only when sampling interval $s$ is taken reasonably. This indicates the existence of the optimal sampling interval in the interval $[0.05,0.50]$. Meanwhile, for a certain type of filter (0,1-order), the optimal sampling interval $s$ is inversely proportional to $N$, i.e., a large $N$ corresponds to a small optimal $s$ while a small $N$ corresponds to a large optimal $s$. Moreover, the optimal filter with a large $N$ generally gives a better performance than that with a small $N$. 

![Fig2. Relative error comparison of 1-order Hankel Transform digital filter for different c values: (a-b) Filter 1 (c-d) Filter 2 (e-f) Filter 3](image-url)
Fig 3. Relative error comparison of 0-order Hankel Transform digital filter for different s values:
(a-b) N=25 (c-d) N=50 (e-f) N=100

Fig 4. Relative error comparison of 1-order Hankel Transform digital filter for different s values:
(a-b) N=25 (c-d) N=50 (e-f) N=100

Table 1 The optimal parameter c of different groups of N and s

| N  | Sampling interval s | Parameter c₀ | Parameter c₁ |
|----|---------------------|--------------|--------------|
| 25 | 0.05                | 1.778e+00    | 1.413e-03    |
| 25 | 0.10                | 3.162e-01    | 3.981e-03    |
| 25 | 0.15                | 3.162e+00    | 6.310e+00    |
| 25 | 0.20                | 1.585e+00    | 3.981e+00    |
| 25 | 0.25                | 1.995e+00    | 3.162e+00    |
| 25 | 0.30                | 1.778e+00    | 2.512e+00    |
| 25 | 0.35                | 1.585e+00    | 1.995e+00    |
| 25 | 0.40                | 1.585e+00    | 1.778e+00    |
| 25 | 0.45                | 5.012e-01    | 1.259e+00    |
| 25 | 0.50                | 1.413e+00    | 1.259e+00    |
| 50 | 0.05                | 3.548e+00    | 3.981e-03    |
3.3 Error analysis of several digital filters

In order to get a filter of higher precision, the sampling interval is taken by a step size 0.005 in the interval [0.05,0.5], and N is taken as 25, 50, 100. The optimal sampling interval and the optimal parameter c for different N are shown Table 2.

To facilitate comparison with several common digital filters, the following definitions are made:

Table 2 The optimal sampling interval s and optimal parameter c corresponding to different N

| N   | Sampling interval s | Parameter c | Sampling interval s | Parameter c |
|-----|---------------------|-------------|---------------------|-------------|
| 25  | 0.365               | 1.585e+00   | 0.240               | 2.239e+00   |
| 50  | 0.200               | 1.778e+00   | 0.150               | 1.259e+00   |
| 100 | 0.115               | 7.943e-01   | 0.125               | 5.623e-02   |

(1) New-051, New-101, New-201: the 51-point (N=25), 101-point (N=50) and 201-point (N=100) digital filters of 0-,1-order Hankel transform calculated in the paper.

(2) K.K-051, K.K-101, K.K-201: the 51-point, 101-point and 201-point digital filters of 0-,1-order Hankel transform calculated by Kerry Key [15].

(3) Anders-801: the 801-point digital filters of 0-,1-order Hankel transforms calculated by Anderson [8].

(4) G&S-061, G&S-120: the 61-point and 120-point digital filters of 0-order Hankel transform by Guptasarma and Singh [11].

(5) G&S-047, G&S-140: the 47-point and 140-point digital filters of 1-order Hankel transform by Guptasarma and Singh [11].

The result of comparison is shown in Fig.5. Big difference is found between the three sets of filters in this paper and the filters of Kerry Key, who also use the direct method to obtain digital filter coefficients. i.e., there is almost no difference between the relative error of the 51-point (N=25) 1-order Hankel transform digital filter in this paper, and that of Kerry Key’s 101-point (N=50) 1-order Hankel transform digital filter; and for a fixed length of a filter, the filter in this paper perform better
than Kerry Key’s filter, it further shows that it is very important to select the optimal sampling interval $s$ and the optimal parameter $c$ in the process of obtaining the digital filter coefficients.

Big difference is also found between 51-point digital filter calculated in this paper, and Guptasarma and Singh’s 61-point, 47-point digital filters. i.e., the 51-point digital filter is superior to Guptasarma and Singh’s 61-point digital filter with a relative error of at least two orders of magnitude, and 47-point digital filter with a relative error with about one orders of magnitude.

From the perspective of computing accuracy, there is no big difference among the 101-point digital filter calculated in this paper, Guptasarma and Singh’s 120-point, 140-point digital filter and Anderson’s 801-point digital filter. However, from the perspective of the computing speed, the length of the 101-point digital filter is the smallest, it means that the calculation speed of the length of the 101-point digital is fastest. In addition, the 201-point digital filter calculated in this paper performs best and has the highest computing accuracy.

Fig.5. Relative error comparison of several digital filters
(a-b) the digital filters of 0-order Hankel transform, (c-d) the digital filters of 1-order Hankel transform

4. Conclusion
This paper investigates the influence of the sampling interval $s$ and the parameter $c$ on the digital filter algorithm based on the direct method of digital filter algorithm of Hankel transform by Kong in the sample domain, the main conclusions are drawn as follows:

1) In the process of obtaining the digital filter coefficients of Hankel transform, it is very important to select the appropriate parameter $c$ for the filter. There is a great difference between among the filters of the different $c$ values, and an inappropriate parameter $c$ can induce a completely wrong filter.

2) The effect of the sampling interval $s$ on the filter is discussed with a fixed length of a digital filter and must be established in the case of selecting the optimal $c$ value. For digital filters of different length, the optimal sampling interval is inversely proportional to $N$, i.e., the larger the $N$, the smaller the optimal sampling interval. Meanwhile, the larger the filter length, the higher the evaluation accuracy.

3) For 0- and 1-order Hankel transforms, three filters of different length are calculated in this paper: long filter of 201 points, medium filter of 101 points, short filter of 51 points. In practical applications, the filter of the smaller length should be selected to improve the calculation speed in the case of meeting the accuracy requirements.
References

[1] LIU Ying, LI Yu-guo. Marine controlled-source electromagnetic fields of an arbitrary electric dipole over a layered anisotropic medium[J]. Oil Geophysical Prospecting, 2015, 50(4): 755–765 +7–8.

[2] LI Jian-hui, LIU Shu-cai, ZHU Zi-qiang, et al. Relationship between electromagnetic field and magnetic field’s symmetric excited by rectangular loop[J]. Journal of Central South University, 2010, 41(02): 638–642.

[3] ANDERSON W L. Numerical integration of related Hankel transforms of orders 0 and 1 by adaptive digital filtering[J]. Geophysics, 1979, 44(7):1287–1305.

[4] JOHANSEN H K, SØRENSEN K. Fast Hankel transforms[J]. Geophysical Prospecting, 1979, 27(4): 876–901.

[5] GHOSH D P. The application of linear filter theory to the direct interpretation of geoelectrical resistivity sounding measurements [J]. Geophysical Prospecting, 1971, 19(2): 192–217.

[6] KOEFOED O, GHOSH D P, POLMAN G J. Computation of type curves for electromagnetic depth sounding with a horizontal transmitting coil by means of a digital linear filter[J]. Geophysical Prospecting, 1972, 20(2): 406–420.

[7] KOEFOED O, DIRKS F J H. Determination of resistivity sounding filters by the Wiener-Hopf least-squares method[J]. Geophysical Prospecting, 1979, 27(1): 245–250.

[8] ANDERSON W L. Fast Hankel-transforms using related and lagged convolutions[J]. ACM Transactions on Mathematical Software, 1982, 8(4): 369–370.

[9] ANDERSON W L. A hybrid fast Hankel transform algorithm for electromagnetic modeling[J]. Geophysics, 1989, 54(2): 263–266.

[10] SØRENSEN K, CHRISTENSEN N. The fields from a finite electrical dipole—A new computational approach[J]. Geophysics, 1994, 59(6): 864–880.

[11] GUPTASARMA D, SINGH B. New digital linear filters for Hankel J0 and J1 transforms[J]. Geophysical Prospecting, 1997, 45(5): 745–762.

[12] KONG F N. Hankel transform filters for dipole antenna radiation in a conductive medium[J]. Geophysical Prospecting, 2007, 55(1): 83–89.

[13] WANG Hua-jun. Digital Filter algorithm of the sine and cosine transform[J]. Chinese Journal of Engineering Geophysics, 2004, 1(04): 329–335.

[14] GUPTASARMA D. Optimization of short digital linear filters for increased accuracy[J]. Geophysical Prospecting, 1982, 30(4): 501–514.

[15] KEY K. Is the fast Hankel transform faster than quadrature? [J] Geophysics, 2012, 77(3): F21–F30.

[16] SAAD Y, SCHULTZ M. GMRES: a generalized minimal residual algorithm for solving nonsymmetric Linear Systems[J]. SIAM Journal on Scientific and Statistical Computing, 1986, 7(3): 856–869.

[17] SAAD Y. Iterative methods for sparse linear systems[M]. Society for Industrial and Applied Mathematics, 2003:275-292.

[18] NABIGHIAN M N. Electromagnetic methods in applied geophysics [M]. ZHAO Jing-xiang, WANG Yan-jun, trans. Beijing: Geology Press, 1992: 217–224.

[19] GRAF U. Introduction to hyperfunctions and their integral transforms: An Applied and Computational Approach[M]. Birkhäuser Basel, 2010:405.