Harnack inequalities for a class of semilinear stochastic partial differential equations

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Abstract: In this article, we study a class of semilinear stochastic partial differential equations driven by an additive noise. We establish Harnack inequalities for the semigroup associated with the solution by using coupling method, which implies the strong Feller property. The main results can be applied to SPDEs of various types such as stochastic heat equation and Fokker-Planck equation perturbed by space-time white noise.

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1 Introduction

In this paper, we are concerned with the following semilinear stochastic partial differential equations driven by an additive noise:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(t,x,u(t,x)) + \frac{\partial g(t,x,u(t,x))}{\partial x} + \dot{W}(t,x),$$

with the initial condition

$$u(x,0) = f(x) \in L^2(S^1),$$

where $Q = S^1 \times \mathbb{R}_+, S^1 := \{e^{i\theta}; \theta \in \mathbb{R}\}$ is a circular ring, and the random field $W(t,x) := W(e^{i\theta}; 0 \leq \theta \leq x) \times [0,t])$ is a regular Brownian sheet defined on a filtered probability space $(\Omega, P, \mathcal{F}; \mathcal{F}_t)$. The functions $b = b(t,x,r), g = g(t,x,r)$ are Borel functions of $(t,x,r) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}$.

There are several recent works about the semilinear SPDE (1.1). We mention some of them. For the bounded domain $[0, 1]$ and Dirichlet boundary condition, the existence and uniqueness of solutions to (1.1) driven by multiplicative noise was studied by Gyöngy in [2], where the author established global well-posedness of (1.1) in the space $C([0,T]; L^2([0,1]))$. Based on [2], Foondun and Setayeshgar [3] proved the large deviations principle uniformly on compact subsets of $C([0,T]; L^2([0,1]))$ for the law of the solutions to (1.1). Moreover, Zhang [8] established the Harnack inequalities and strong Feller property for (1.1) with reflection and $g \equiv 0$ by using coupling method introduced by Wang [6]. Under
the condition that \( g \equiv 0 \) in \((1.1)\) with circular ring \( S^1 \), Mueller proved its existence and uniqueness of invariant measures in [4].

The purpose of this paper is to prove Harnack inequalities for the semigroup associated with \((1.1)\). Motivated by [8], we utilize the coupling method proposed by Wang [6]. However, the proof of [8] cannot be directly applied to our model. This is due to the fact that it’s impossible to make \( L^2(S^1) \)–norm estimates to judge whether two processes are coupled successfully in the presence of an extra term \( \partial_r g \) appeared in \((1.1)\). To overcome this difficulty, we make the following appropriate modification. Firstly, based on the characteristics of our model, we construct a new coupling process of \((1.1)\). Secondly, instead of making \( L^2(S^1) \)–norm estimates, we use \( L^1(S^1) \)–norm estimates to verify the constructed new process can couple successfully with \((1.1)\). Based on these techniques and the comparison theorem (Theorem 5.1) proposed by Gyöngy in [2], we obtain the successful coupling under the assumption that the initial data \( f_1 \geq f_2 \). Further, using Girsanov theorem, we show Harnack inequalities hold. Finally, applying similar method as [4] or [7], we establish Harnack inequalities for the general data case (i.e., we do not assume \( f_1 \geq f_2 \)).

This paper is organized as follows. The mathematical formulation of semilinear stochastic partial differential equations is in Sect. 2. The main part of this paper is in Sect. 3, where Harnack inequalities are proved.

2 Framework

Let \( L^p(S^1), p \in (0, \infty) \) be the Lebesgue space, whose norm is denoted by \( | \cdot |_{L^p} \). In particular, denote \( H = L^2(S^1) \) with the corresponding norm \( | \cdot |_H \) and inner product \((\cdot, \cdot)_H\).

Define an operator \( A := \frac{\partial^2}{\partial x^2} \). Let \( G_t(x,y) = G(t, x, y), t \geq 0, x, y \in S^1 \) be the Green function for the operator \( \partial_t - A \) with circulate ring \( S^1 \). Denote by \( \{e_n(x)\}_{n \geq 1} \) the eigenvectors of \( A \) (equipped with circulate ring \( S^1 \)), which constitutes an orthonormal system of \( H \). Put

\[
\beta_n(t) = \int_0^t \int_{S^1} e_n(x)W(dsdx),
\]

then, \( \{\beta_n(t), n \geq 1\} \) is a sequence of independent Brownian motions. Define an \( H \)–cylindrical Brownian motion by \( W(t) = \sum_{n=1}^{\infty} \beta_n e_n \). Then, we have

\[
W(t, x)dt dx = dW(t),
\]

which is the stochastic Itô integral against the \( H \)–cylindrical Brownian motion.

2.1 Assumptions

In this paper, we adopt a special version of assumptions from [2] or [3]. The functions \( b = b(t, x, r) \), \( g = g(t, x, r) \) are Borel functions of \( (t, x, r) \in \mathbb{R}^+ \times S^1 \times \mathbb{R} \) satisfying the following assumptions:
(H1) There exists a constant $K > 0$ such that for all $(t, x, r) \in [0, T] \times S^1 \times \mathbb{R}$, we have
\[ \sup_{t \in [0, T]} \sup_{x \in S^1} |b(t, x, r)| \leq K(1 + |r|), \quad \sup_{t \in [0, T]} \sup_{x \in S^1} |g(t, x, r)| \leq K(1 + |r|). \]

(H2) $b$ and $g$ are Lipschitz, i.e., for all $(t, x, p, q) \in [0, T] \times S^1 \times \mathbb{R}^2$,
\[ |b(t, x, p) - b(t, x, q)| \leq L|p - q|, \quad |g(t, x, p) - g(t, x, q)| \leq L|p - q|. \]

**Definition 2.1.** A random field $u$ is a solution to (1.1) if $u = \{u(t, x), t \in \mathbb{R}^+, x \in S^1\}$ is an $L^2(S^1)$ valued continuous $\mathcal{T}_t$-adapted random field with initial value $f \in L^2(S^1)$ and satisfying for all $t \geq 0$, $\phi \in C^\infty(S^1)$, we have
\[ \int_{S^1} u(t, x)\phi(x)dx = \int_{S^1} f(x)\phi(x)dx + \int_0^t \int_{S^1} u(s, x)\frac{\partial^2 \phi(x)}{\partial x^2}dxdx + \int_0^t \int_{S^1} f(s, x, u(s, x))\phi(x)dxdx - \int_0^t \int_{S^1} g(s, x, u(s, x))\frac{\partial \phi(x)}{\partial x}dxdx + \int_0^t \int_{S^1} \phi(x)W(ds, dx), \quad P - a.s. \tag{2.2} \]

The existence and uniqueness of the solution of (1.1) is established in [2]. We recall here.

**Theorem 2.1.** Under assumptions (H1)-(H2), there exists a unique solution $u$ in the sense of Definition 2.1.

**Remark 1.** Referring to Proposition 3.5 in [2], under conditions in Theorem 2.1 (2.2) is equivalent to the following form: for all $t \geq 0$ and almost surely $\omega \in \Omega$,
\[ u(t, x) = \int_{S^1} G_t(x, y)f(y)dy + \int_0^t \int_{S^1} G_{t-s}(x, y)b(s, y, u(s, y))dyds - \int_0^t \int_{S^1} \partial_y G_{t-s}(x, y)g(s, y, u(s, y))dyds + \int_0^t \int_{S^1} G_{t-s}(x, y)W(dyds) \tag{2.3} \]
for almost every $x \in S^1$.

### 2.2 Notations

Denote by $\mathcal{B}(H)$ the $\sigma$–field of all Borel subsets of $H$ and by $\mathcal{M}(H)$ the set of all probability measures defined on $(H, \mathcal{B}(H))$. Let $u(t, x, f)$ be the solution of (1.1) and $P_t(f, \cdot)$ be the corresponding transition function
\[ P_t(f, \Gamma) = P(u(\cdot, t, f) \in \Gamma), \quad \Gamma \in \mathcal{B}(H), \quad t > 0, \]
where $f$ is the initial condition.

Denote by $B_b(H)$ the space of all bounded measurable functions on $H$. The semigroup $P_t$ associated with the solution $u(t, x, f)$ to (1.1) is defined by
\[ P_t\psi(f) = E[\psi(u(t, f))], \quad \psi \in B_b(H). \]

Referring to [1], we have

**Definition 2.2.** $P_t$ is strong Feller, if $P_t$ maps $B_b(H)$ into $C_b(H)$ for $t > 0$. 

2.3 A lemma

Define the linear operator \( J \) by

\[
J(v)(t, x) = \int_0^t \int_{S^1} H(r, t; x, y) v(r, y) dy dr, \quad t \in [0, T], \quad x \in S^1
\]  
(2.4)

for every \( v \in L^\infty([0, T]; L^1(S^1)) \).

Referring to [2], we have the following heat kernel estimate, which is very crucial to our proof.

**Lemma 2.1.** Let \( J \) be defined by \( H(s, t; x, y) = G_{t-s}(x, y) \) or by \( H(s, t; x, y) = \frac{\partial G_{t-s}(x, y)}{\partial y} \) in (2.7). Let \( \rho \in [1, \infty] \), \( q \in [1, \rho) \) and set \( \kappa = 1 + \frac{1}{\rho} - \frac{1}{q} \). Then \( J \) is a bounded linear operator from \( L^\gamma([0, T]; L^q(S^1)) \) into \( C([0, T]; L^\rho(S^1)) \) for \( \gamma > 2\kappa^{-1} \). Moreover, for any \( T \geq 0 \), there are constants \( C_1, C_2 \) such that

\[
|J(v)(t, \cdot)|_{L^\rho(S^1)} \leq C_1 \int_0^t (t-s)^{\frac{\kappa}{2}} |v(s, \cdot)|_{L^\rho(S^1)} ds \leq C_2 t^{\frac{\kappa}{2}-\frac{\gamma}{2}} \left( \int_0^t |v(s, \cdot)|^\gamma_{L^\rho(S^1)} ds \right)^{\frac{\gamma}{\gamma}}
\]  
(2.5)

3 Harnack Inequalities

We consider the following semilinear SPDE driven by an additive noise:

\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} & = \frac{\partial^2 u(t, x)}{\partial x^2} + b(u(t, x)) + \frac{\partial g(u(t, x))}{\partial x} + \dot{W}(t, x), \\
u(x, 0) & = f(x) \in L^2(S^1).
\end{aligned}
\]  
(3.6)

In order to prove Harnack inequalities for (3.6), we need an additional condition on \( b \):

**H3** There exists a constant \( C_0 \) such that \( b(\xi_1) - b(\xi_2) \leq C_0(\xi_1 - \xi_2) \), for any \( \xi_1, \xi_2 \in \mathbb{R} \).

**Theorem 3.1.** Under (H1)-(H3), for any \( t > 0, \psi \in B_0(H), \alpha > 1 \) and \( f_1, f_2 \in H \), we have

\[
(P_t \psi(f_1)\alpha \leq P_t(\psi\alpha)(f_2) \exp \left\{ \frac{\alpha}{2(\alpha - 1)} C(t)|f_1 - f_2|^2_H \right\},
\]  
(3.7)

where

\[
C(t) = \frac{2}{t} + 2C^2(1 + \frac{16}{9}t^{-\frac{1}{2}}) \exp \left\{ 4t + 4C_0^2t^2 \right\}.
\]

**Proof.** We will adopt the coupling method used by Wang in [6]. For \( r \in [0, t], f_1, f_2 \in H \), consider \( u(r, f_1) \) the solution of the following equations:

\[
\begin{aligned}
\frac{\partial u(r, f_1)}{\partial r} & = \frac{\partial^2 u(r, f_1)}{\partial x^2} + b(u(r, f_1)) + \frac{\partial g(u(r, f_1))}{\partial x} + \dot{W}(t, x), \\
u(x, 0) & = f_1(x) \in L^2(S^1).
\end{aligned}
\]  
(3.8)

Let \( v(r, f_2) \) be the solution of the following equations:

\[
\begin{aligned}
\frac{\partial v(r, f_2)}{\partial r} & = \frac{\partial^2 v(r, f_2)}{\partial x^2} + b(v(r, f_2)) + \frac{\partial g(v(r, f_2))}{\partial x} + \dot{W}(t, x) + C_0(u(r, f_1) - v(r, f_2)) + \frac{f_1(x) - f_2(x)}{t}, \\
v(x, 0) & = f_2(x) \in L^2(S^1).
\end{aligned}
\]  
(3.9)
Define a stopping time 

\[ \tau := \inf \{ s \geq 0 : u(s, f_1) = v(s, f_2) \}. \]

We firstly assume the initial data \( f_1 \geq f_2 \). Due to comparison theorem (Theorem 5.1) in [2], it gives that \( u(r, f_1, x) \geq v(r, f_2, x) \) for all \( r \in [0, T] \) and for \( dx \)-almost every \( x \in S^1 \).

Taking \( \phi = 1 \) in (2.2), we have

\[
\int_{S^1} u(r, f_1, x)dx = \int_{S^1} f_1(x)dx + \int_0^r \int_{S^1} b(u(s, f_1, x))dxds + \int_0^r \int_{S^1} W(dxds),
\]

and

\[
\int_{S^1} v(r, f_2, x)dx = \int_{S^1} f_2(x)dx + \int_0^r \int_{S^1} b(v(s, x))dxds + \int_0^r \int_{S^1} W(dxds) + C_0 \int_0^r \int_{S^1} (u(s, f_1, x) - v(s, f_2, x))dxds + \frac{1}{t} \int_0^r \int_{S^1} (f_1(x) - f_2(x))dxds.
\]

Then, by (H3), we deduce that

\[
\int_{S^1} (u(r \wedge \tau, f_1, x) - v(r \wedge \tau, f_2, x))dx = \int_{S^1} (f_1(x) - f_2(x))dx + \int_0^{r \wedge \tau} \int_{S^1} b(u(s, f_1, x)) - b(v(s, f_2, x))dxds \\
- C_0 \int_0^{r \wedge \tau} \int_{S^1} (u(s, f_1, x) - v(s, f_2, x))dxds - \frac{1}{t} \int_0^{r \wedge \tau} \int_{S^1} (f_1(x) - f_2(x))dxds \\
\leq |f_1 - f_2|_{L^1(S^1)} - \frac{r \wedge \tau}{t} |f_1 - f_2|_{L^1(S^1)}.
\]  

This implies that \( \tau \leq t \) and \( u(t, f_1) = v(t, f_2) \). Otherwise, if \( t < \tau \), we deduce from (3.10) that \( u(t, f_1) = v(t, f_2) \), which contradicts the definition of the stopping time \( \tau \). Note that

\[
u(r, f_1, x) = \int_{S^1} G_r(x, y)f_1(y)dy + \int_0^r \int_{S^1} G_{r-s}(x, y)b(s, y, u(s, f_1, y))dyds \\
- \int_0^r \int_{S^1} \partial_y G_{r-s}(x, y)g(s, y, u(s, f_1, y))dyds + \int_0^r \int_{S^1} W(dyds),
\]

and

\[
v(r, f_2, x) = \int_{S^1} G_r(x, y)f_2(y)dy + \int_0^r \int_{S^1} G_{r-s}(x, y)b(s, y, v(s, f_2, y))dyds \\
- \int_0^r \int_{S^1} \partial_y G_{r-s}(x, y)g(s, y, v(s, f_2, y))dyds + \int_0^r \int_{S^1} W(dyds) \\
+ C_0 \int_0^r \int_{S^1} G_{r-s}(x, y)(u(s, f_1, y) - v(s, f_2, y))dydyds \\
+ \int_0^r \int_{S^1} G_{r-s}(x, y)\frac{f_1(y) - f_2(y)}{t}dydyds.
\]

Define

\[
\theta(r) = \int_{S^1} |u(r, f_1, x) - v(r, f_2, x)|^2dx = |u(r, f_1) - v(r, f_2)|^2_H.
\]
Using (H3), we deduce that
\[
\theta(r) \leq \int_{S^1} | \int_{S^1} G_r(x, y)(f_1(y) - f_2(y))dy |^2 dx + C_0^2 \int_{S^1} | \int_0^r \int_{S^1} G_{r-s}(x, y)(u(s, f_1, y) - v(s, f_2, y))dy ds |^2 dx
\]
\[
+ \int_{S^1} | \int_0^r \int_{S^1} \partial_s G_{r-s}(x, y)[g(u(s, f_1, y)) - g(v(s, f_2, y))]dy ds |^2 dx
\]
\[
+ \int_{S^1} | \int_0^r \int_{S^1} G_{r-s}(x, y) \frac{f_1(y) - f_2(y)}{t} dy ds |^2 dx
\]
\[:= J_1 + J_2 + J_3 + J_4.\]

Applying the heat kernel estimates, we get
\[
J_1 \leq C \int_{S^1} dx \left( \int_{S^1} G_r(x, y)dy \right) \left( \int_{S^1} G_r(x, y) |f_1(y) - f_2(y)|^2 dy \right)
\]
\[\leq C \int_{S^1} dx \left( \int_{S^1} G_r(x, y) dy \right) |f_1(y) - f_2(y)|^2 dy \leq C|f_1 - f_2|_{H^1},\]

and
\[
J_2 \leq C_0^2 \int_{S^1} dx \left( \int_0^r \int_{S^1} |G_{r-s}(x, y)|^2 dy ds \right) \left( \int_0^r \int_{S^1} |u(s, f_1) - v(s, f_2)|^2 dy ds \right)
\]
\[\leq C_0^2 \left( \int_0^r \frac{1}{\sqrt{r-s}} ds \right) \left( \int_0^r \int_{S^1} |u(s, f_1) - v(s, f_2)|^2 dy ds \right) \leq 2C_0^2 r^{\frac{3}{2}} \int_0^r \theta(s) ds.
\]

Since g is Lipschitz continuous, it follows that
\[
J_3 \leq \int_{S^1} dx \left[ \int_0^r (r-s)^{-\frac{3}{4}} |g(u(s, f_1)) - g(v(s, f_2))| ds \right]^2
\]
\[\leq L^2 \left( \int_0^r (r-s)^{-\frac{3}{4}} ds \right) \left( \int_0^r (r-s)^{-\frac{3}{4}} |u(s, f_1) - v(s, f_2)|_{H^1}^2 ds \right) \leq 4L^2 r^{\frac{1}{2}} \int_0^r (r-s)^{-\frac{3}{4}} \theta(s) ds.
\]

Utilizing Hölder inequality, we obtain
\[
J_4 \leq \frac{1}{r^2} \int_{S^1} dx \left( \int_0^r \int_{S^1} |G_{r-s}(x, y)|^2 dy ds \right) \left( \int_0^r \int_{S^1} |f_1(y) - f_2(y)|^2 dy ds \right)
\]
\[\leq \frac{16}{9r^2} r^{\frac{3}{2}} |f_1 - f_2|_{H^1}^2.
\]

Based on the above, we conclude that
\[
\theta(r) \leq C|f_1 - f_2|_{H^1} + 2C_0^2 r^{\frac{3}{2}} \int_0^r \theta(s) ds
\]
\[+ 4L^2 r^{\frac{1}{2}} \int_0^r (r-s)^{-\frac{3}{4}} \theta(s) ds + \frac{16r^{\frac{3}{2}}}{9r^2} |f_1 - f_2|_{H^1}^2.
\]
According to (3.9), we deduce that
\[ \theta(r) \leq C(1 + \frac{16r^2}{9t^2}) \exp\{4C_0^2 r^2 + 4L^2 r \}. \] (3.13)

Set
\[ \xi(s) = (C_0(u(s, f_1) - v(s, f_2)) + \frac{f_1 - f_2}{t})1_{s < t}. \]

We deduce from (3.13) that
\[
\int_0^t |\xi(s)|_{H}^2 ds \leq \left[ \frac{2}{t} + 2C^2(1 + \frac{16}{9}t^{-\frac{1}{2}}) \exp\{4L^2 t + 4C_0^2 t^2\} \right] |f_1 - f_2|_{H}^2
\]
\[ := C(t)|f_1 - f_2|_{H}^2, \] (3.14)

where
\[ C(t) = \frac{2}{t} + 2C^2(1 + \frac{16}{9}t^{-\frac{1}{2}}) \exp\{4L^2 t + 4C_0^2 t^2\}. \]

Define
\[ \tilde{W}(r) = \int_0^r W(ds), \]

then \( \tilde{W}(r), r \geq 0 \) is a \( H \)-cylindrical Brownian motion under \( P \). Moreover, from (3.8), we have
\[
\begin{cases}
\frac{\partial u(r, f_1)}{\partial r} = \frac{\partial^2 u(r, f_1)}{\partial x^2} + b(u(r, f_1)) + \frac{\partial g(u(r, f_1))}{\partial x} + \tilde{W}(r), \\
u(0, f_1) = f_1.
\end{cases}
\] (3.15)

Set \( Z(t) = \exp(-\int_0^t \xi(s) d\tilde{W}_s) - \frac{1}{2} \int_0^t |\xi(s)|_{H}^2 ds \), and define a new probability measure \( Q \) as
\[ \frac{dQ}{dP} \bigg|_{\tilde{\mathcal{F}_t}} = Z(t). \]

By the Girsanov Theorem, \( \tilde{W}(r) = \tilde{W}(r) + \int_0^t \xi(s) ds, r \geq 0 \) is a \( H \)-cylindrical Brownian motion under \( Q \). According to (3.9), we deduce that
\[
\begin{cases}
\frac{\partial v(r, f_2)}{\partial r} = \frac{\partial^2 v(r, f_2)}{\partial x^2} + b(v(r, f_2)) + \frac{\partial g(v(r, f_2))}{\partial x} + \tilde{W}(r), \\
v(0, f_2) = f_2.
\end{cases}
\] (3.16)

Hence, we deduce from (3.15) and (3.16) that the law of \( v(r, f_2) \) under \( Q \) is the same as that of \( u(r, f_2) \) under \( P \). To avoid confusion, we denote by \( E_P \) the expectation under \( P \) and \( E_Q \) the expectation under \( Q \).

For any \( \alpha > 1 \), by Hölder inequality and the fact that \( u(t, f_1) = v(t, f_2) \), it follows that
\[
P_t \psi(f_1) = E_P[\psi(u(t, f_1))] = E_P[\psi(v(t, f_2))]
\]
\[
= E_P[\psi(v(t, f_2))Z_t^\frac{1}{\alpha}Z_t^{-\frac{1}{\alpha}}]
\]
\[
\leq (E_P(\psi^\alpha(v(t, f_2))Z_t))^\frac{1}{\alpha} (E_P(Z_t^{-\frac{1}{\alpha}}))^\frac{\alpha}{\alpha}
\]
\[
= (E_Q(\psi^\alpha(v(t, f_2))Z_t))^\frac{1}{\alpha} (E_P(Z_t^{-\frac{1}{\alpha}}))^\frac{\alpha}{\alpha}
\]
\[
= (E_P(\psi^\alpha(u(t, f_2))))^\frac{1}{\alpha} (E_P(Z_t^{-\frac{1}{\alpha}}))^\frac{\alpha}{\alpha}
\]
\[
= (P_t(\psi^\alpha)(f_2))^\frac{1}{\alpha} (E_P(Z_t^{-\frac{1}{\alpha}}))^\frac{\alpha}{\alpha}.
\]
Taking into account (3.14), we deduce that

\[
\left( E_p(Z_t^{-\frac{1}{\alpha-1}}) \right)^{\frac{\alpha-1}{\alpha}} \leq \left( E_p[\exp\left\{ \frac{1}{\alpha-1} \int_0^t (\xi_s, d\tilde{W}(s))_H - \frac{1}{2(\alpha-1)^2} \int_0^t |\xi_s|_H^2 ds + \frac{1}{2(\alpha-1)^2} \int_0^t |\xi_s|^2 ds \right\}] \right)^{\frac{\alpha-1}{\alpha}} 
\]

\[
\leq \exp \left\{ \frac{1}{2(\alpha-1)} \int_0^t |\xi_s|_H^2 ds \right\} 
\leq \exp \left\{ \frac{1}{2(\alpha-1)} C(t)|f_1 - f_2|_H^2 \right\},
\]

where we have used the fact that \(\exp\left\{ \frac{1}{\alpha-1} \int_0^t (\xi_s, d\tilde{W}(s))_H - \frac{1}{2(\alpha-1)^2} \int_0^t |\xi_s|_H^2 ds \right\}\) is a martingale.

Based on the above estimates, we conclude that

\[
(P_t \psi(f_1))^\alpha \leq P_t(\psi^\alpha)(f_2) \exp \left\{ \frac{\alpha}{2(\alpha-1)} C(t)|f_1 - f_2|_H^2 \right\}.
\]

For the general case, i.e., we do not assume \(f_1 \geq f_2\). Similarly to the proof of Theorem 1.1 in [4], we consider a solution \(u, v^1, v^2\) of the following SPDEs:

\[
\begin{align*}
\frac{\partial u(x, r)}{\partial r} & = \frac{\partial^2 u(x, r)}{\partial x^2} + b(u(x, r)) + \partial_x g(u(x, r)) + W(x, r), \\
\frac{\partial v^i(x, r)}{\partial r} & = \frac{\partial^2 v^i(x, r)}{\partial x^2} + b(v^i(x, r)) + \partial_x g(v^i(x, r)) + W(x, r) \\
& \quad + C_0(u(x, r) - v^i(x, r)) + \frac{u(x, 0) - v^i(x, 0)}{t}, \\
u(x, 0) & = \max_{i=1,2} v^i(x, 0),
\end{align*}
\]

Define \(\tau_i = \inf\{s \geq 0 : u(s, u(x, 0)) = v^i(s, v^i(x, 0))\}\). From the above, we have obtained

\[
|u(r \wedge \tau_i, u(x, 0)) - v^i(r \wedge \tau_i, v^i(x, 0))|_{L^1(S^1)} 
\leq |u(x, 0) - v^i(x, 0)|_{L^1(S^1)} - \frac{r \wedge \tau_i}{t} u(x, 0) - v^i(x, 0)|_{L^1(S^1)},
\]

which implies that \(\tau_i \leq t\) and \(u(t, u(x, 0)) = v^i(t, v^i(x, 0))\) for \(i = 1, 2\). Hence, \(v^1(t, v^1(x, 0)) = v^2(t, v^2(x, 0))\). We complete the proof.

\[\square\]

**Corollary 3.2.** The semigroup \(P_t, t \geq 0\) is strong Feller.

**Proof.** Fix \(t > 0\) and \(\psi \in B_b(H)\). We aim to \(P_t \psi(f)\) is continuous in \(f \in H\). Let \(f_n\) be any sequence functions in \(H\) satisfying \(f_n \to f\). From Theorem [3.1], we have

\[
(P_t \psi(f_n))^\alpha \leq P_t(\psi^\alpha)(f) \exp \left\{ \frac{\alpha}{2(\alpha-1)} C(t)|f_n - f|_H^2 \right\}.
\]
Let $n \to \infty$ and sending $\alpha \downarrow 1$, we get
\[
\limsup_{n \to \infty} P_t \psi(f_n) \leq P_t(\psi)(f).
\]
Again, by Theorem 3.1, we also have
\[
P_t(\psi^\alpha)(f) \geq (P_t \psi(f_n))^\alpha \exp \left\{ \frac{-\alpha}{2(\alpha - 1)} C(t)|f_n - f|^2_H \right\}.
\]
Let $n \to \infty$ and sending $\alpha \downarrow 1$, we get
\[
\liminf_{n \to \infty} P_t \psi(f_n) \geq P_t(\psi)(f).
\]
Hence, we conclude that
\[
\lim_{n \to \infty} P_t \psi(f_n) = P_t(\psi)(f).
\]
□

References

[1] G. Da Prato, J. Zabczyk: *Ergodicity for infinite dimensional systems*. Cambridge University Press. Cambridge (1996).

[2] I. Gyöngy: *Existence and uniqueness results for semilinear stochastic partial differential equations*. Stochastic Process. Appl. 73 (1998), no. 2, 271-299.

[3] M. Foondun, L. Setayeshgar: *Large deviations for a class of semilinear stochastic partial differential equations*. Statist. Probab. Lett. 121 (2017), 143-151.

[4] C. Mueller: *Coupling and invariant measures for the heat equation with noise*. Ann. Probab. 21 (1993), no. 4, 2189-2199.

[5] S. Peszat and J. Zabczyk: *Strong Feller property and irreducibility for diffusions on Hilbert spaces*. Annals of Probability. Vol. 23, No. 1 (Jan., 1995), pp. 157-172.

[6] F. Wang: *Harnack inequality and applications for stochastic generalized porous media equations*. Ann. Probab. 35 (2007), no. 4, 1333-1350.

[7] J. Yang, T. Zhang: *Existence and uniqueness of invariant measures for SPDEs with two reflecting walls*. J. Theoret. Probab. 27 (2014), no. 3, 863-877.

[8] T. Zhang: *White noise driven SPDEs with reflection: strong Feller properties and Harnack inequalities*. Potential Anal. 33 (2010), no. 2, 137-151.