Differential operators and the symmetric groups

Ibrahim Nonkané
Departement d'économie et de mathématiques appliquées, IUFIC, Université Ouaga II,
Burkina Faso
E-mail: ononkon@univ-ouaga2.bf

Abstract. In this paper, we study the action of the rational quantum Calogero-Moser system on polynomials. In this vein, we study polynomials ring over the complex field \( \mathbb{C} \) as a module over a ring of differential operators by elaborating its irreducible submodules. We endowed the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) with a differential structure by using directly the action of the Weyl algebra associated with the ring of symmetric polynomial \( \mathbb{C}[x_1, \ldots, x_n] S_n \) after a localization. Then we study the polynomials representation of the ring of invariant differential operators under the symmetric group. We use the representation theory of symmetric groups to exhibit the generators of its simple components.

1. Introduction
The History of the Weyl algebra begins with the birth of quantum mechanics. The theory of groups has played a major role in the discovery of the general laws of quantum theory. It is not surprising that concepts arising in the theory of group find their applications in physics. As Hermann Weyl has said in [11], there exists a plainly discernible parallelism between the more recent developments of mathematics and physics. Then since symmetries are relevant in physics our attention has been drawn particularly to the symmetric group and its representation in connection to rational quantum Calogero-Moser system. In what follows we try to understand the actions of invariant differential operators on the polynomial ring through the representation theory of symmetric groups.

2. Preliminaries and motivations
2.1. The rational quantum Calogero-Moser system
Consider the differential operator

\[
H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - c(c + 1) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}.
\]

This is the quantum Hamiltonian for a system of \( n \) particles on the line of unit mass and the interaction potential (between particle 1 and 2) \( c(c + 1)/(x_1 - x_2)^2 \). This system is called the rational quantum Calogero-Moser system.
It turns out that the rational quantum Calogero-Moser system is completely integrable. Namely, we have the following theorem.
Theorem 2.1. [4] There exist differential operators $L_j$ with rational coefficients of the form

$$L_j = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^j + \text{lower order terms}, j = 1, \ldots, n,$$

which are invariant under the symmetric group $S_n$, homogeneous of degree $-j$, and such that $L_2 = H$ and $[L_j, L_k] = 0, \forall j, k = 1, \ldots, n$.

It is clear that the differential operators $L_j$ belong a localization of the ring of invariant differential operators under the symmetric group, so one way of understanding the actions of the $L_j$ on polynomials is to study the polynomials representation of the ring of differential operators that are invariant under the symmetric group localized at $\Delta = \prod_{i \neq j} (x_i - x_j)^2$.

2.2. Specht polynomials, Specht modules

In this subsection we recall some general facts about the actions of symmetric group on polynomials ring. The symmetric group $S_n$ is the group of permutations of the set of variables $\{x_1, \ldots, x_n\}$. Let $g \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, and $\sigma \in S_n$ we define

$$(\sigma g)(x_1, \ldots, x_n) = g(\sigma x_1, \ldots, \sigma x_n).$$

M.H. Peel gave the construction of irreducible submodules of $\mathbb{C}[x_1, \ldots, x_n]$ in the following way [9].

By a partition of $n$ we mean a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$ 

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition; we arrange the variables $x_1, \ldots, x_n$ in an array, with $r$ rows and $\lambda_1$ columns, containing a variable in the first $\lambda_i$ positions of the $i$th row; each variable occurs exactly once in the array. For example, one such array for the partition $(4, 2, 1)$ of 7 is

\[
\begin{array}{ccc}
 & x_1 & x_4 & x_6 \\
 x_2 & x_3 & & \\
 x_5 & & & \\
 x_7 & & & \\
\end{array}
\]

such an array is called a $\lambda$-tableau. there are $n!$ $\lambda$-tableaux for each partition $\lambda$ of $n$. We shall denote such tableaux by $t$. Suppose that the variable $a_1, \ldots, a_l$ occur in jth column of $\lambda$-tableau $t$, with $a_i$ in the $i$th row.

We form the difference product $\Delta(a_1, \ldots, a_l) = \prod_{i<k} (a_i - a_k)$, if $l > 1$, and if $l = 1$, $\Delta(a_1) = 1$. Multiplying these difference products for all the columns of $t$, we obtain a polynomial which we denote by $f(t)$. For $\sigma \in S_n$, let $\sigma t$ be the tableau obtained from $t$ by replacing $x_i$ in $t$ by $\sigma x_i$. Then $f(\sigma t) = f(\sigma t)$. It follows that the set of all linear combinations of the $n!$ polynomials $f(t)$, obtained from the $\lambda$-tableaux $t$, is a cyclic $\mathbb{C}[S_n]$-module generated by any $f(t)$. We denote this module by $S^\lambda$. A $\lambda$-tableau is said to be standard if the variables occur in increasing order ($x_i > x_j$ if $i > j$) along each row from left to right and down each column. M.H.Peel proved the following in [9].

Theorem 2.2. $B^\lambda = \{ f(t) : t \text{ is a standard } \lambda \text{-tableau} \}$ is a basis of $S^\lambda$.

We call $f(t)$ the Specht polynomial corresponding to the $\lambda$-tableau $t$, we call $S^\lambda$ the Specht module corresponding to the partition $\lambda$, and $f(t)$ a standard Specht polynomial if $t$ is a standard tableau.

Theorem 2.3. The $S^\lambda$ for $\lambda \vdash n$ form a complete list of irreducible $S_n$-module over the complex field.
3. Decomposition Theorem
In this section we establish a decomposition theorem of the polynomial ring in \(n\) indeterminates.

3.1. Actions description
As we want to study the polynomials representation of the ring of invariant differential operators under the symmetric group localized at \(\Delta = \prod_{i \neq j} (x_i - x_j)^2\). It is convenient to precisely describe the action of that ring of invariant differential operators on the polynomials ring.

Let \(\mathcal{D}_X = \mathbb{C}\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle\) be the ring of differential operators associated with the polynomial ring \(\mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n]\), and \(\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[y_1, \ldots, y_n]\) be the ring of invariant under the symmetric group \(S_n\) where

\[
y_j = \sum_{i=1}^{n} \frac{x^j_i}{j} \quad \text{for} \quad j = 1, \ldots, n,
\]

We denote by \(\mathcal{D}_Y = \mathbb{C}\langle y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \rangle\) the ring of differential operators associated with \(\mathcal{O}_Y = \mathbb{C}[y_1, \ldots, y_n]\). By localization, \(\mathcal{O}_X\) is turned into a \(\mathcal{D}_Y\)-module, as the following lemma states.

**Notations** We adopt the following notations

\[
\tilde{\mathcal{O}}_X := \mathbb{C}[x_1, \ldots, x_n, \Delta^{-1}], \quad \tilde{\mathcal{O}}_Y := \mathbb{C}[y_1, \ldots, y_n, \Delta^{-2}], \quad \tilde{\mathcal{D}}_Y := \mathbb{C}[y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \Delta^{-2}]
\]

**Lemma 3.1.** \(\tilde{\mathcal{O}}_X\) is a \(\tilde{\mathcal{D}}_Y\)-module.

**Proof.** Let us make clear the action of \(\mathcal{D}_Y\) on \(\tilde{\mathcal{O}}_X\).

We have \(y_j = \sum_{i=1}^{n} \frac{x^j_i}{j}, j = 1, \ldots, n\), hence \(\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} x_i^{j-1} \frac{\partial}{\partial y_j}, i = 1, \ldots, n\). Let \(A = (x_i^{j-1})_{1 \leq i, j \leq n}\), we get the following equation

\[
\begin{pmatrix}
\frac{\partial}{\partial x_n} \\
\vdots \\
\frac{\partial}{\partial x_1}
\end{pmatrix} = A 
\begin{pmatrix}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{pmatrix}.
\]

Since \(\Delta \neq 0\), it follows that

\[
\begin{pmatrix}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{pmatrix} = A^{-1} 
\begin{pmatrix}
\frac{\partial}{\partial x_n} \\
\vdots \\
\frac{\partial}{\partial x_1}
\end{pmatrix}
\]

and it clear that \(\tilde{\mathcal{O}}_X\) is a \(\tilde{\mathcal{D}}_Y\)-module. \(\square\)

Is \(\tilde{\mathcal{O}}_X\) a \(\tilde{\mathcal{D}}_Y\)-semisimple module? If yes what are the simple components of \(\tilde{\mathcal{O}}_X\) as \(\tilde{\mathcal{D}}_Y\)-module and their multiplicities?
Example 3.2. For $n = 2$, $\mathcal{D}_Y = \mathbb{C}(x_1, x_2, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \Delta^{-2})$ and $\mathcal{O}_X = \mathbb{C}[x_1, x_2, \Delta^{-1}]$ where $\Delta = x_1 - x_2$ and

$$
\begin{align*}
\frac{\partial}{\partial y_1} &= \frac{x_2}{x_2 - x_1} \frac{\partial}{\partial x_1} - \frac{x_1}{x_2 - x_1} \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial y_2} &= \frac{1}{x_2 - x_1} \frac{\partial}{\partial x_1} + \frac{1}{x_2 - x_1} \frac{\partial}{\partial x_2}
\end{align*}
$$

We have that

$$\mathcal{O}_X = M_1 \oplus M_2$$

where $M_1 = \mathcal{O}_Y$ and $M_2 = \mathcal{D}_Y(x_1 - x_2)$ are $\mathcal{D}_Y$-simple modules.

3.2. Simple components and their multiplicities

In this paragraph, we state our first main result. We use the representation theory of symmetric groups to yield results on modules over the ring of differential operators. It is well-known that $\mathcal{O}_X = \mathbb{C}[S_n] \otimes \mathcal{O}_Y$ as $\mathcal{O}_Y$-modules.

Let us consider the multiplicative closed set $S = \{\Delta^k\}_{k \in \mathbb{N}} \subset \mathcal{O}_X$. It follows that:

$$S^{-1}\mathcal{O}_X = \mathbb{C}[S_n] \otimes S^{-1}\mathcal{O}_Y \text{ as } S^{-1}\mathcal{O}_Y\text{-modules.}$$

where $S^{-1}\mathcal{O}_X$ and $S^{-1}\mathcal{O}_Y$ are the localization of $\mathcal{O}_X$ and $\mathcal{O}_Y$ at $S$ respectively. But $S^{-1}\mathcal{O}_X = \mathcal{O}_X$ and $S^{-1}\mathcal{O}_Y = \mathcal{O}_Y$, whereby we get

$$\mathcal{O}_X = \mathbb{C}[S_n] \otimes \mathcal{O}_Y \text{ as } S_n\text{-modules.}$$

Lemma 3.3. There exists an injective map

$$\mathbb{C}[S_n] \hookrightarrow \text{Hom}_\mathbb{C}(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. the $S_n$-module $\mathbb{C}[S_n]$ acts on itself by multiplication, and this multiplication yields an injective map $\mathbb{C}[S_n] \hookrightarrow \text{Hom}_\mathbb{C}(\mathbb{C}[S_n], \mathbb{C}[S_n])$. Since $\mathcal{O}_Y$ is invariant under this action of $\mathbb{C}[S_n]$, we get the expected injective map. □

Proposition 3.4. There exists an injective map

$$\mathbb{C}[S_n] \hookrightarrow \text{Hom}_{\mathcal{D}_Y}(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. Since $\mathcal{D}_Y = \mathbb{C}(y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n, \Delta^{-2})$, we only need to show that every element of $\mathbb{C}[S_n]$ commute with $y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n$.

• It is clear that every element of $\mathbb{C}[S_n]$ commute with $y_i$, $i = 1, \ldots, n$.

• Let us show that every element of $\mathbb{C}[S_n]$ commute with $\partial y_i$, $i = 1, \ldots, n$. Let $D$ be a derivation on the field $\mathcal{O}_Y = \mathbb{C}(y_1, \ldots, y_n)$ the field of fractions of $\mathcal{O}_Y$, then $(\mathcal{O}_Y, D)$ is a differential field. Since $\mathcal{O}_X = \mathbb{C}(x_1, \ldots, x_n)$ (the field of fractions of $\mathcal{O}_X$) is an Galois extension of $\mathcal{O}_Y$, by [2, Théorème 6.2.6] there exists a unique derivation on $\mathcal{O}_X$ which extends $D$, then $(\mathcal{O}_X, D)$ is also a differential ring. In the same way, $\sigma^{-1}D\sigma = D$ for every $\sigma \in S_n$. Therefore $\sigma D = D\sigma$ and $\sigma$ commute with $D$. □


Corollary 3.5.
\[ \mathbb{C}[S_n] \cong \text{Hom}_{\mathcal{D}_Y}(\bar{O}_X, \bar{O}_X) \]

Proof. see [7, Corollary 26]

Before we state our first main result, let us recall some facts. By Maschke’s Theorem [6, Chap XVIII], we know that \( \mathbb{C}[S_n] \) is a semisimple ring, and

\[ \mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} R_\lambda, \]

where the sum is taken over all the partitions of \( n \) and \( R_\lambda \) are simple rings. We have the following corresponding decomposition of the identity element of \( \mathbb{C}[S_n] \):

\[ 1 = \sum_{\lambda \vdash n} r_\lambda, \]

where \( r_\lambda \) is the identity element of \( R_\lambda \), \( r_\lambda^2 = 1 \) and \( r_\lambda r_\mu = 0 \) if \( \lambda \neq \mu \), the set \( \{r_\lambda\}_{\lambda \vdash n} \) is the set of central idempotents of \( \mathbb{C}[S_n] \).

Let \( n \) be a positive integer, \( \lambda \) be a partition of \( n \), \( \text{Tab}(\lambda) \) the set of standard tableau of shape \( \lambda \) and \( \text{Tab}(n) = \bigcup_{\lambda \vdash n} \text{Tab}(\lambda) \). We have \( r_\lambda = \sum_{t \in \text{Tab}(\lambda)} e_t \), where \( e_t \) is the primitive idempotent associated to the standard tableau \( t \in \text{Tab}(\lambda) \) (see [5]).

Theorem 3.6. For every primitive idempotent \( e_i \in \mathbb{C}[S_n] \).

(i) \( e_i\bar{O}_X \) is a nontrivial \( \bar{D}_Y \)-submodule of \( \bar{O}_X \),

(ii) The \( \bar{D}_Y \)-module \( e_i\bar{O}_X \) is simple,

(iii) There exist a partition \( \lambda \vdash n \) and a Specht polynomial \( p_\lambda \) associated to a standard tableau of shape \( \lambda \) such that \( e_i\bar{O}_X = \bar{D}_Y p_\lambda \).

Proof.

(i) Let \( e_i \in \mathbb{C}[S_n] \) be a primitive idempotent, by [5, Theorem 4.3] there exists a Specht polynomial \( p_\lambda \) such that \( e_i p_\lambda \) is a scalar multiple of \( p_\lambda \), then \( 0 \neq p_\lambda \in e_i \bar{O}_X \) and \( e_i \bar{O}_X \neq \{0\} \). Since \( e_i \) commute with every element of \( \bar{D}_Y \) et \( \bar{O}_X \) is \( \bar{D}_Y \)-module, it follows that \( e_i \bar{O}_X \) is an \( \bar{D}_Y \)-module.

(ii) Assume that \( 1 = \sum_{i=1}^s e_i \) where the \( \{e_i\}_{1 \leq i \leq s} \) is the set of primitive idempotents of \( \mathbb{C}[S_n] \), then \( \bar{O}_X = \sum_{i=1}^s e_i \bar{O}_X \). Let \( m \in e_i \bar{O}_X \cap e_j \bar{O}_X \) with \( i \neq j \) alors \( m = e_i m_i \) and \( m = e_j m_j \), but \( e_i e_j = 0 \) then \( e_i m = e_i e_j m = 0 \) hence \( m = 0 \). Therefore \( \bar{O}_X = \bigoplus_{i=1}^s e_i \bar{O}_X \) and we get:

\[ \text{Hom}_{\bar{D}_Y}(\bar{O}_X, \bar{O}_X) \cong \bigoplus_{i,j=1}^s \text{Hom}_{\bar{D}_Y}(e_i \bar{O}_X, e_j \bar{O}_X), \]

by Corollary 3.5 we get:

\[ \mathbb{C}[S_n] \cong \bigoplus_{i,j=1}^s \text{Hom}_{\bar{D}_Y}(e_i \bar{O}_X, e_j \bar{O}_X). \]

We also have, by [5, Proposition 3.29], that \( \mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} \text{End}_\mathbb{C}(S^\lambda) \) where \( S^\lambda \) is the Specht module associated with the partition \( \lambda \vdash n \). But

\[ \mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} r_\lambda \mathbb{C}[S_n] \text{ and } r_\lambda \mathbb{C}[S_n] \cong \text{End}_\mathbb{C}(C^f), \]

where \( C^f \) is the free abelian group generated by \( C \).
where $f^\lambda = \dim S^\lambda$. We recall that each standard tableau $t_i$ is associated with an idempotent $e_i$.

Let us show that $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} \left( \bigoplus_{t_i,t_j \in \text{Tab}(\lambda)} \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \right)$. Let $x$ be an element of $\mathbb{C}[S_n]$ and $r_\lambda$ a central idempotent with $\lambda \vdash n$. Then $x$ induces a $\mathcal{D}_Y$-homomorphism $\tilde{O}_X \to \Omega_X, m \mapsto x \cdot m$. Since $r_\lambda$ is in the centre of $\mathbb{C}[S_n]$, $x \cdot r_\lambda \tilde{O}_X = (x \cdot r_\lambda) \tilde{O}_X \subset r_\lambda \tilde{O}_X$, which means $x \in \bigoplus_{\lambda \vdash n} \text{Hom}_{\mathcal{D}_Y}(r_\lambda \tilde{O}_X, r_\lambda \tilde{O}_X)$. Then $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) = \{0\}$ if $t_i \in \text{Tab}(\lambda_i), t_j \in \text{Tab}(\lambda_j)$ and $\lambda_i \neq \lambda_j$. we get

$$\text{Hom}_{\mathcal{D}_Y}(\tilde{O}_X, \tilde{O}_X) \cong \bigoplus_{\lambda \vdash n} \left( \bigoplus_{t_i,t_j \in \text{Tab}(\lambda)} \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \right).$$

the number of direct factors in the sum $\bigoplus_{t_i,t_j \in \text{Tab}(\lambda)} \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X)$ is $(f^\lambda)^2$.

Let us show that $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \cong \mathbb{C}$ if $t_i,t_j \in \text{Tab}(\lambda)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}[S_n] & \xrightarrow{\phi} & \text{Hom}_{\mathcal{D}_Y}(\tilde{O}_X, \tilde{O}_X) \\
\alpha_\lambda \downarrow & & \beta_\lambda \\
r_\lambda \mathbb{C}[S_n] & \xrightarrow{\psi} & \text{Hom}_{\mathcal{D}_Y}(r_\lambda \tilde{O}_X, r_\lambda \tilde{O}_X)
\end{array}$$

where $\beta_\lambda : \bigoplus_{\mu \vdash n} \text{Hom}_{\mathcal{D}_Y}(r_\mu \tilde{O}_X, r_\mu \tilde{O}_X) \to \text{Hom}_{\mathcal{D}_Y}(r_\lambda \tilde{O}_X, r_\lambda \tilde{O}_X)$ and $\alpha_\lambda : \bigoplus_{\mu \vdash n} r_\mu \mathbb{C}[S_n] \to r_\lambda \mathbb{C}[S_n]$ are canonical projections et $\phi$ is the isomorphism of Corollary 3.5. It follows that $\psi$ is an isomorphism hence $r_\lambda \mathbb{C}[S_n] \cong \text{Hom}_{\mathcal{D}_Y}(r_\lambda \tilde{O}_X, r_\lambda \tilde{O}_X)$. Now we identify $r_\lambda \mathbb{C}[S_n]$ with the set $\text{Mat}_{f^\lambda}(\mathbb{C})$ of square matrices of order $f^\lambda$ with coefficients in $\mathbb{C}$ either with $\text{End}_\mathbb{C}(\mathcal{F}^\lambda)$.

Let $E_{ij}$ be the square matrix of order $f^\lambda$ with 1 at the position $(i,j)$ and 0 elsewhere and $E_i = E_{ii}$, then we identify the idempotent $e_i \in r_\lambda \mathbb{C}[S_n]$ to $E_i$ in $\text{Mat}_{f^\lambda}(\mathbb{C})$. Let $B = (a_{ij}) \in \text{Mat}_{f^\lambda}(\mathbb{C})$ we get $B = \sum_{i,j} a_{ij} E_{ij} = \sum_{i,j} E_i E_j$, in fact $E_i E_j$ is the matrix with $a_{ij}$ in the position $(i,j)$ and 0 elsewhere, if $R = \text{Mat}_{f^\lambda}$ we get that $E_i R E_j \cong \mathbb{C}$. This isomorphism $\psi$ implies that $\bigoplus_{t_i,t_j \in \text{Tab}(\lambda)} E_i R E_j \cong \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X)$; the restriction of $\psi$ to $E_i R E_j$ yields a map $E_i R E_j \to \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X)$ and this map is surjective, moreover we have $E_i R E_j \cong \text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X)$. Therefore $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \cong \mathbb{C}$.

Let us assume that $e_i\tilde{O}_X$ is not simple $\mathcal{D}_Y$-module, then $e_i\tilde{O}_X$ may be written as $e_i\tilde{O}_X = \bigoplus_{j \in J} N_j$ where the $N_j$ are simple $\mathcal{D}_Y$-modules and $|J| > 1$. It follows that $\dim_{\mathbb{C}}(\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X)) \geq |J|$ but $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_i\tilde{O}_X) \cong \mathbb{C}$ so we obtain that $J = 1$, which necessary implies that $e_i\tilde{O}_X$ is a simple $\mathcal{D}_Y$-module.

(iii) By the proof (i) there exist a Specht polynomial $p_\lambda \in e_i\tilde{O}_X, \lambda \vdash n$ such that $e_i\tilde{O}_X = \mathcal{D}_Y p_\lambda$.

\[\square\]

**Corollary 3.7.** With the above notations, $e_i\tilde{O}_X \cong_{\mathcal{D}_Y} e_j\tilde{O}_X$ if $t_i$ and $t_j$ have the same size (if there is a partition $\lambda \vdash n$ such that $t_i, t_j \in \text{Tab}(\lambda)$).

**Proof.** The $\mathcal{D}_Y$-modules $e_i\tilde{O}_X$ are simple and $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \cong \mathbb{C}$ whenever there exists a partition $\lambda \vdash n$ such that $t_i, t_j \in \text{Tab}(\lambda)$. Since $\text{Hom}_{\mathcal{D}_Y}(e_i\tilde{O}_X, e_j\tilde{O}_X) \neq \{0\}$, we conclude by using the Schur lemma.

\[\square\]
Proposition 3.8. For every young tableau $t$, let $f(t)$ be the associated Specht polynomial, then we have:

(i) $\tilde{O}_X = \bigoplus_{\text{Tab}(n)} \tilde{D}_Y f(t)$;

(ii) $\tilde{O}_X \cong_{\tilde{D}_Y} \bigoplus_{\lambda \vdash n} f^\lambda \tilde{D}_Y f(t_\lambda)$

where $t_\lambda \in \text{Tab}(\lambda)$.

Proof. We have by the proof of Theorem 3.6 that

$$\tilde{O}_X = \oplus e_i \tilde{O}_X$$

and the $e_i \tilde{O}_X$ are simple $\tilde{D}_Y$-modules. Since to each $e_i$ correspond a partition $\lambda \vdash n$ and a $\lambda$-tableau $t_i$ such that $e_i \tilde{O}_X = \tilde{D}_Y f(t)$ then $\tilde{O}_X = \bigoplus_{i \in \text{Tab}(n)} \tilde{D}_Y f(t)$. If $t$ and $t'$ are two $\lambda$-tableau by Corollary 3.7 $\tilde{D}_Y f(t) \cong_{\tilde{D}_Y} \tilde{D}_Y f(t')$. Therefore $\tilde{O}_X \cong_{\tilde{D}_Y} \bigoplus_{\lambda \vdash n} f^\lambda \tilde{D}_Y f(t_\lambda)$ with $t_\lambda \in \text{Tab}(\lambda)$.

3.3. Example

We consider the case $n = 3, 4$

- For $n = 3$ the Specht polynomials corresponding to standard tableaux are

$$f(t_1) = 1, f(t_2) = (x_1 - x_2), f(t_3) = (x_1 - x_2)$$

and

$$f(t_4) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

correspondingly we have that

$$\tilde{O}_X = \tilde{O}_Y \oplus \tilde{D}_Y (x_1 - x_2) \oplus \tilde{D}_Y (x_1 - x_3) \oplus \tilde{D}_Y (x_1 - x_2)(x_1 - x_3)(x_2 - x_3);$$

- For $n = 4$ the Specht polynomials corresponding to standard tableaux are

$$f(t_1) = 1, f(t_2) = (x_1 - x_2), f(t_3) = (x_1 - x_3), f(t_4) = (x_1 - x_4),$$

$$f(t_5) = (x_1 - x_2)(x_3 - x_4), f(t_6) = (x_1 - x_3)(x_2 - x_4),$$

$$f(t_7) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3), f(t_8) = (x_1 - x_2)(x_1 - x_4)(x_2 - x_4),$$

$$f(t_9) = (x_1 - x_3)(x_3 - x_4)(x_3 - x_4)$$

and

$$f(t_{10}) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

correspondingly we have that

$$\tilde{O}_X = \tilde{O}_Y \oplus \tilde{D}_Y (x_1 - x_2) \oplus \tilde{D}_Y (x_1 - x_3) \oplus \tilde{D}_Y (x_1 - x_4) \oplus \tilde{D}_Y (x_1 - x_2)(x_1 - x_3)(x_2 - x_4)$$

$$\oplus \tilde{D}_Y (x_1 - x_3)(x_2 - x_4) \oplus \tilde{D}_Y (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

$$\oplus \tilde{D}_Y (x_1 - x_2)(x_1 - x_4)(x_2 - x_4) \oplus \tilde{D}_Y (x_1 - x_3)(x_3 - x_4)(x_3 - x_4)$$

$$\oplus \tilde{D}_Y (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$
4. Change of differential operators
In this section we replace some of the generators of the differential ring in order to describe the actions of the differential operators on the $S_n$-modules. Let us consider the following differential operators:

$$T_i = \sum_{j=1}^{n} x^{i+1} \frac{\partial}{\partial x_j} \text{ for } i \in \mathbb{N}.$$  

For every $i \in \mathbb{N}$, $T_i$ is a homogeneous differential operator of degree $i$ and we have that

$$[T_i, T_j] = (i - j)T_{i+j}, \text{ for } i, j \in \mathbb{N},$$

where the bracket $[,]$ the commutator operator. The operators $T_i$ preserve the ideal $\langle \Delta \rangle$ generated by $\Delta$ (see [8]).

Lemma 4.1. $\tilde{\mathcal{D}}_Y = \mathbb{C}(y_1, \ldots, y_n, T_{n-1}, \ldots, T_{n-2}, \Delta^{-2})$.

Proof. By definition of $T_i$, we have that

$$\begin{pmatrix} T_{n-1} \\ T_0 \\ \vdots \\ T_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$  

By the proof of Lemma 3.1 we know that

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$  

it follows that

$$\begin{pmatrix} T_{n-1} \\ T_0 \\ \vdots \\ T_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$  

Hence

$$\begin{pmatrix} T_{n-1} \\ T_0 \\ \vdots \\ T_{n-2} \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & & \vdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$  

where $p_k = \sum_{j=1}^{n} x_j^k \in \tilde{\mathcal{D}}_Y$, $k \in \mathbb{N}$. Therefore

$$T_{j-1} = p_j \frac{\partial}{\partial y_1} + p_{j+1} \frac{\partial}{\partial y_2} + \cdots + p_{j+n-1} \frac{\partial}{\partial y_n}, \text{ for } j = 0, \ldots, n-1.$$  

(4.1)
Conversely we have
\[
\begin{vmatrix}
\frac{\partial}{\partial y_1} & \cdots & \frac{\partial}{\partial y_n} \\
\frac{\partial}{\partial y_2} & \cdots & \frac{\partial}{\partial y_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial y_n} & \cdots & \frac{\partial}{\partial y_n}
\end{vmatrix} = \begin{pmatrix} T_1 & \cdots & T_{n-1} \\ T_2 & \cdots & T_{n-2} \\ \vdots & \ddots & \vdots \\ T_n & \cdots & T_{n-2} \end{pmatrix}
\]
that implies that the coefficients of $\Gamma^{-1}$ can be written as polynomials in $p_0, p_1, \ldots, p_{2n-2}$ and $\Delta^{-2}$. It follows that
\[
\frac{\partial}{\partial y_i} \in \mathbb{C}(y_1, \ldots, y_n, T_{-1}, \ldots, T_{n-2}, \Delta^{-2}).
\]
We conclude that $\mathcal{D}_Y = \mathbb{C}(y_1, \ldots, y_n, T_{-1}, \ldots, T_{n-2}, \Delta^{-2}).$ \hfill \Box

**Example 4.2.** Let us denote $\mathcal{D}_Y(\Delta)$ the $\mathcal{D}_Y$-module generated by $\Delta$. We claim that $\mathcal{D}_Y(\Delta) = \mathcal{O}_Y \cdot \Delta$. For $i \in \mathbb{N}$, there exists a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $T_i(\Delta) = f \cdot \Delta$. Now let $\pi \in S_n$ be a permutation, we have
\[
\pi(T_i(\Delta)) = T_i(\pi \Delta)
= T_i(\text{sgn}(\pi) \Delta)
= \text{sgn}(\pi) T_i(\Delta)
= \text{sgn}(\pi) \Delta \cdot f
= \pi(\Delta) \cdot f,
\]
It follows that $\pi(f) = f$ so $f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$, then $\mathcal{D}_Y(\Delta)$ is the sign representation.

**Lemma 4.3.** If $P$ is polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ that divides $\Delta$, then $P$ divides $D(P)$ for all derivation $D \in \mathcal{D}_Y$.

**Proof.** For $j = -1, \ldots, n-1$, $T_j(\Delta)$ belongs to the ideal $\langle \Delta \rangle$ of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $\Delta$ [2]. It follows that $D(\Delta) \in \langle \Delta \rangle$ for all derivation $D \in \mathcal{D}_Y$. Assume that $P$ divides $\Delta$ then $P$ divides $D(\Delta)$ and there exists a polynomial $Q$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that $\Delta = PQ$ where $P$ and $Q$ are coprime. But $D(\Delta) = PD(Q) + QD(P)$, since $P$ divides $D(\Delta)$, $P$ must divide $D(P)$. \hfill \Box

Let $\lambda \vdash n$ and $t$ a $\lambda$-tableau, it clear that the Specht polynomial $f(t)$ associated with $t$ divides $\Delta$. Then for every derivation $D \in \mathcal{D}_Y$, we have that $D(f(t)) = f(t)g$ with $g \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. In others the effect of a derivation of $\mathcal{D}_Y$ on a Specht polynomial yields a multiple of that Specht polynomial, the following lemma gives precise information about the factor $g$.

Let $C(t)$ be the set of all permutations $\sigma \in S_n$ such that for each $i$, $x_i, \sigma(x_i)$ occur in the same column of $t$; such that a a permutation is called a column stabilizer of $t$, and we have for all $\sigma \in C(t), \sigma(f(t)) = \text{sgn}(\sigma) f(t)$.

**Lemma 4.4.** Let $\lambda \vdash n$, $t$ a $\lambda$-tableau and $D$ a derivation in $\mathcal{D}_Y$ such that $D(f(t)) \neq 0$ then there exists a polynomial in $g \in \mathbb{C}[x_1, x_2, \ldots, x_n]^{C(t)}$ such that $D(f(t)) = f(t)g$.

**Proof.** Let $\sigma \in C(t)$ and $D \in \mathcal{D}_Y$, by the proof of Proposition 3.4 the elements of $\mathbb{C}[S_n]$ commute with those $\mathcal{D}_Y$, hence
\[
\sigma(D(f(t))) = D(\sigma f(t)) = D(\text{sgn}(\sigma)f(t)) = \text{sgn}(\sigma)D(f(t)).
\]
Since $D(f(t)) = f(t)g$ avec $g \in \mathbb{C}[x_1, \ldots, x_n]$, it follows that $\sigma(g) = g$, hence $g \in \mathbb{C}[x_1, \ldots, x_n]^{C(t)}$. \hfill \Box
Theorem 4.5. Let \( \lambda \vdash n \) and \( D \in \tilde{D}_n \) such that \( D(f(t)) \neq 0 \) for \( t \in \text{Tab}(\lambda) \). Then the image of the Specht module \( S^\lambda \) by \( D \) is an \( S_n \)-module an isomorphic copy of \( S^\lambda \).

Proof. Let \( \lambda \vdash n, D \in \tilde{D}_n \) such that \( D(f(t)) \neq 0 \) for \( t \in \text{Tab}(\lambda) \) and set \( \lambda^D = D(S^\lambda) \) the image of the Specht module \( S^\lambda \) by \( D \). \( \lambda^D \) is the vector space spanned by the set \( \{D(f(t))| t \in \text{Tab}(\lambda)\} \) is a \( \lambda \)-tableau. By Proposition 3.8 The elements of \( \{f(t)| z \in \text{Tab}(\lambda)\} \) are linearly independent over \( \tilde{D}_n \). It follows that the elements \( \{D(f(t))| t \in \text{Tab}(\lambda)\} \) are linear independent over \( \mathbb{C} \). Let \( t' \) a \( \lambda \)-tableau which is not standard, by [9] \( f(t') \) can be written as linear combination of elements \( \{f(t)| t \in \text{Tab}(\lambda)\} \). Then \( D(f(t')) \) can also be written as a linear combination of elements \( \{D(f(t))| t \in \text{Tab}(\lambda)\} \). Hence \( \{D(f(t))| t \in \text{Tab}(\lambda)\} \) is a base of \( \lambda^D \) over \( \mathbb{C} \). Since \( D \) commute with elements of \( \mathbb{C}[S_n] \), \( \lambda^D \) is an \( S_n \)-module isomorphic to \( S^\lambda \). \( \square 

Remark 4.6.

In the decomposition of the group algebra [10, Proposition], every irreducible appears with a certain multiplicity. For each partition, Peel describe a basis of one irreducible component by the Specht polynomials [9]. Later Ariki, Terasoma and Yanada describe the bases of other irreducible components by higher Specht Polynomials [1]. Their methods was purely combinatorial. Since Theorem 4.5 states that the actions of derivations of \( \tilde{D}_n \) on a Specht module yields an isomorphic copy of that Specht module. Now is it possible recover the other irreducible components in [1] by the actions of differential operators on the ones in [9]?

Proposition 4.7. Let \( \lambda \vdash n, t \) a \( \lambda \)-tableau, then \( T_{-1}(f(t)) = 0 \) and \( T_i(f(t)) \neq 0 \) for \( i \neq -1 \).

Due to the restriction of Theorem 4.5, to get get a non trivial isomorphic copy of an Specht modules, it better to deal with monomial of \( \tilde{D}_n \). For each partition, Peel describe a basis of one irreducible component by the Specht polynomials [9]. Later Ariki, Terasoma and Yanada describe the bases of other irreducible components by higher Specht Polynomials [1]. Their methods was purely combinatorial. Since Theorem 4.5 states that the actions of derivations of \( \tilde{D}_n \) on a Specht module yields an isomorphic copy of that Specht module. Now is it possible recover the other irreducible components in [1] by the actions of differential operators on the ones in [9]?

Let us consider the bilinear form on \( O_X = \mathbb{C}[x_1, \ldots, x_n] \) defined as follows:

\[
\langle P, Q \rangle = \frac{\sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(PQ)}{\Delta} \bigg|_{x_1=\cdots=x_n=0}, \text{ for } P, Q \in O_X = \mathbb{C}[x_1, \ldots, x_n].
\]

The kernel of this bilinear is the ideal \( IA_n \) of \( O_X \) generated by homogeneous symmetric polynomials of strictly positive degree. In fact The bilinear form \( \langle P, Q \rangle \) defines a scalar product on the ring of harmonic polynomials \( H_n = O_X/IA_n \).

Lemma 4.8. For \( i \geq 1 \) we have

\[
\langle P, T_i(Q) \rangle = -\langle T_i(P), Q \rangle \text{ for } P, Q \in \mathbb{C}[x_1, \ldots, x_n].
\]

Set \( \Omega(P, Q) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(PQ) \Delta \), For \( i \geq 1 \) we have

\[
T_i(\Delta \Omega(P, Q)) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(T_i(PQ))
\]

\[
T_i(\Delta) \Omega(P, Q) + \Delta T_i(\Omega(P, Q)) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(T_i(P)Q) + \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(PT_i(Q))
\]

Since \( T_i(\Delta) = R \Delta \) where \( R \) is a homogeneous polynomial of degree \( i \) (see [8]),

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(T_i(P)Q) + \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma(PT_i(Q)) \bigg|_{x_1=\cdots=x_n=0} = 0
\]

and \( \langle P, T_i(Q) \rangle = -\langle T_i(P), Q \rangle \).
Proposition 4.9. Let $M \subset \mathcal{H}_n$ be a $\mathcal{D}_Y$-module stable under the action of $T_i, i \geq 1$ then the orthogonal $M^\perp$ of $M$ is also stable par $T_i$

Proof. We have that $M^\perp = \{ P; \langle P, Q \rangle = 0, \text{ for all } Q \in M \}$. Let $P \in M^\perp$, and since for all $Q \in M, T_i(Q) \in M$ since $M$ is stable under $T_i, i \geq 1$ by assumption, $\langle P, T_i(Q) \rangle = 0$. I follows from the preceding lemma that $\langle T_i(P), Q \rangle = 0$, hence $T_i(P) \in M^\perp$.

Acknowledgments
I am deeply thankful to Professor Rikard Bogvad of Stockholm University for instructive comments during the writing of this paper. This Research is financially supported by the International Science Program (ISP)

References
[1] Ariki, S. Terasoma, T. and Yamada, H., Higher Specht polynomials, Hiroshima Math, 27 (1997), no. 1, 177-188
[2] Chambert-Loir, A. Algèbre corporelle, Les éditions de l’Ecole polytechnique (2005).
[3] Coutinho, S. C., A primer of algebraic $D$-modules, 33, Cambridge University Press, Cambridge, 1995.
[4] Etingof, Pavel., Calogero-Moser Systems and Representation Theory, Zurich Lectures in Advanced Mathematics, European Mathematical Society Publishing House, (2007).
[5] Fulton, W. and Harris, J. Representation Theory. A First Course, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, (1991).
[6] Lang, S, Algebra. Revisited third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.
[7] Nonkané, I., Specht polynomials and modules over the Weyl algebra, Afr. Mat, 30, Issue 11 pp 279-290 (2019).
[8] Orlik, P and Terao, H., Arrangement of hyperplanes, 300, Springer-Verlag, 1991
[9] Peel, M.H, Specht modules and symmetric groups , J. Algebra, 36, 1975. no. 1, 88-97
[10] Sagan, Bruce. E., The symmetric group. Representations, combinatorial algorithms, and symmetric functions, Second edition. Graduate Texts in Mathematics, 203. Springer-Verlag, New York, (2001).
[11] Weyl, Hermann, The theory of groups and quantum mechanics, Dover, New York.