Cellular automata that generate symmetrical patterns give singular functions

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Abstract

In this paper, we mainly study linear one-dimensional and two-dimensional elementary cellular automata that generate symmetrical spatio-temporal patterns. For spatio-temporal patterns of cellular automata from the single site seed, we normalize the number of nonzero states of the patterns, take the limits, and give one-variable functions for the limit sets. We can obtain a one-variable function for each limit set and show that the resulting functions are singular functions, which are non-constant, are continuous everywhere, and have a zero derivative almost everywhere. We show that for Rule 90, a one-dimensional elementary cellular automaton (CA), and a two-dimensional elementary CA, the resulting functions are Salem’s singular functions. We also discuss two nonlinear elementary CAs, Rule 22, and Rule 126. Although their spatio-temporal patterns are different from that of Rule 90, their resulting functions from the number of nonzero states equal the function of Rule 90.

Keywords: cellular automaton, fractal, singular function

1 Introduction

There are many studies about fractals generated by cellular automata, for example, Willson [1], Culik and Dube [2], Takahashi [3], and Haeseler et al. [4]. A cellular automaton (CA) is a discrete dynamical system, in some cases whose spatio-temporal pattern from a single site seed holds self-similarity and whose limit set is a fractal. In general, when we characterize the fractal, we calculate its fractal dimension, given by a specific numerical value. In this study, however, we assign a real one-variable function for the fractal to capture details of the fractal structure.
In this paper, we study linear elementary CAs that generate symmetrical spatio-temporal patterns. For linear automata, their spatio-temporal patterns from single site seeds hold self-similarity or partial self-similarity, and we can calculate the number of nonzero states of the patterns using the structure. We obtained results about the numbers for some elementary CAs in our previous paper [5]. From those results, we normalize the numbers of nonzero states, take the limits, and provide one-variable functions for limit sets. We show that if a CA is linear, it holds a counting equation in Lemma 1 which counts the number of nonzero states of the spatio-temporal pattern, and, using the equation, we obtain a one-variable function for each limit set of the automaton. This means one-variable functions characterize fractals by projecting fractals onto one-variable functions.

Next, we discuss the properties of the obtained one-variable functions and give sufficient conditions for the singularity of a function in Theorem 6. We also show that the obtained functions are singular functions, which are monotonically increasing (or decreasing), are continuous everywhere, and have a zero derivative almost everywhere. For the one-dimensional elementary CA Rule 90, the resulting function equals Salem’s singular function $L_{1/3}$, a self-affine function [6, 7, 8, 9], and for a two-dimensional elementary CA, the resulting function equals Salem’s singular function $L_{1/5}$ (numerical results were obtained in [10, 11] where we showed that the difference forms of the equations match Salem’s in [5]). For the one-dimensional CA Rule 150, we previously demonstrated that the resulting function is a singular function that strictly increases, is continuous, and is differentiable almost everywhere [12]. In addition, we discuss two nonlinear elementary CAs, Rule 22 and Rule 126. Their spatio-temporal patterns are similar to that of Rule 90 (see Figures 1, 9, and 10). We show that their resulting functions from the number of nonzero states equal the function of Rule 90.

The remainder of this paper is organized as follows. Section 2 describes the preliminaries concerning CAs and the previous results about the number of nonzero states of spatial and spatio-temporal patterns of CAs. For linear one-dimensional and two-dimensional elementary CAs, Section 3 reports our main results about real one-variable functions given using the self-similarities of the spatio-temporal patterns. We also provide sufficient conditions for the singularity of a function and show that the obtained functions are singular. Further, we discuss two nonlinear one-dimensional elementary CAs, Rule 22 and Rule 126, whose normalized functions equal that of Rule 90. Finally, Section 4 discusses the findings of this paper and highlights possible avenues for future studies.
2 Preliminaries

2.1 Definitions and notations

We provide some definitions and notations about CA. Let \( \{0, 1\} \) be a state set and \( \{0, 1\}^Z^d \) be a \( d \)-dimensional configuration space for \( d \in \mathbb{Z}_{>0} \). We define a configuration \( u_o \in \{0, 1\}^Z^d \) as

\[
(u_o)_i = \begin{cases} 
1 & \text{if } i = 0 \ (= (0, 0, \ldots, 0)), \\
0 & \text{if } i \in \mathbb{Z}^d \setminus \{0\}. 
\end{cases}
\] (1)

We call \( u_o \) the single site seed. Let \((\{0, 1\}^Z^d, S)\) be a discrete dynamical system for a transformation \( S \) on \( \{0, 1\}^Z^d \). The \( n \)-th iteration of \( S \) is denoted by \( S^n \).

We define one-dimensional and two-dimensional elementary CAs as follows.

**Definition 1.**

(i) A one-dimensional elementary cellular automaton \((1dECA) (\{0, 1\}^Z, S)\) is given by

\[
(Su)_i = s(u_{i-1}, u_i, u_{i+1})
\] (2)

for \( i \in \mathbb{Z} \) and \( u \in \{0, 1\}^Z \), where \( s: \{0, 1\}^3 \to \{0, 1\} \) is a local rule of \( S \).

(ii) A 1dECA \((\{0, 1\}^Z, S)\) is a symmetrical pattern generation 1dECA (SPG1dECA) if a local rule \( s \) satisfies \( s(u_{i-1}, u_i, u_{i+1}) = s(u_{i+1}, u_i, u_{i-1}), s(0, 0, 0) = 0, \) and \( s(0, 0, 1) = 1 \).

(iii) A 1dECA \((\{0, 1\}^Z, S)\) is linear if a local rule satisfies

\[
(Su)_i = c_0u_{i-1} + c_1u_i + c_2u_{i+1} \pmod{2},
\] (3)

where \( c_0, c_1, c_2 \in \{0, 1\} \).

** Remark 1.** There exist 256 1dECAs and 16 of them are SPG1dECAs. Only two SPG1dECAs, Rule 90 and Rule 150, are linear SPG1dECAs.

Table 1 shows the local rules of SPG1dECAs Rule 22, Rule 90, Rule 126, and Rule 150.

| \( u_{i-1}u_iu_{i+1} \) | \( 111 \) | \( 110 \) | \( 101 \) | \( 100 \) | \( 010 \) | \( 000 \) |
|-------------------------|--------|--------|--------|--------|--------|--------|
| \((S_{22}u)_i\)       | 0      | 0      | 0      | 1      | 1      | 0      |
| \((S_{90}u)_i\)       | 0      | 1      | 0      | 1      | 0      | 0      |
| \((S_{126}u)_i\)      | 0      | 1      | 1      | 1      | 0      | 0      |
| \((S_{150}u)_i\)      | 1      | 0      | 0      | 1      | 1      | 0      |

3
Definition 2. (i) A two-dimensional elementary cellular automaton (2dECA) 
\((\{0,1\}^2, T)\) is given by
\[
(Tu)_{i,j} = t \left( \begin{array}{ccc}
u_{i-1,j} & u_{i,j} & u_{i+1,j} \\
u_{i,j-1} & u_{i,j} & u_{i,j+1}
\end{array} \right)
\]
for \((i,j) \in \mathbb{Z}^2\) and \(u \in \{0,1\}^2\), where \(t : \{0,1\}^5 \to \{0,1\}\) is a local rule 
depending on the five states of the von Neumann neighborhood.

(ii) A 2dECA \((\{0,1\}^2, T)\) is a symmetrical pattern generation 2dECA (SPG2dECA) 
if a local rule \(t\) satisfies
\[
t \left( \begin{array}{c}
U \\
L
\end{array} \right) = t \left( \begin{array}{c}
D \\
R
\end{array} \right) = t \left( \begin{array}{c}
L \\
D
\end{array} \right),
\]
\[
t \left( \begin{array}{c}
U \\
L
\end{array} \right) = t \left( \begin{array}{c}
D \\
R
\end{array} \right) = t \left( \begin{array}{c}
L \\
D
\end{array} \right) = t \left( \begin{array}{c}
0 \\
0
\end{array} \right) = 0, ~ t \left( \begin{array}{c}
0 \\
1
\end{array} \right) = 1.
\]

(iii) A 2dECA \((\{0,1\}^2, T)\) is linear if a local rule satisfies
\[
(Tu)_{i,j} = c_0 u_{i,j} + c_1 u_{i+1,j} + c_2 u_{i,j-1} + c_3 u_{i-1,j} + c_4 u_{i,j+1} \pmod{2},
\]
where \(c_k \in \{0,1\}, k = 0,1,\ldots,4\).

Remark 2. There exist \(2^{32}\) 2dECAs and \(2^{10}\) of them are SPG2dECAs. Only 
two SPG2dECAs, \(T_0\) and \(T_{528}\), are linear SPG2dECAs.

Table 2 shows the local rules of SPG2dECAs \(T_0\) and \(T_{528}\).

| \(U\) | LCR | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
|-------|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(D\) |     | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |

\[
(T_0u)_{i,j} = 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0
\]

\[
(T_{528}u)_{i,j} = 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0
\]

Next, we introduce a singular function related to the spatio-temporal patterns of CAs.
Definition 3 \(\text{(7, 9)}\). Let \(\alpha\) be a parameter such that \(0 < \alpha < 1\) and \(\alpha \neq \frac{1}{2}\). The singular function \(L_\alpha : [0, 1] \to [0, 1]\) is defined as follows:

\[
L_\alpha(x) := \begin{cases} 
\alpha L_\alpha(2x) & (0 \leq x < 1/2), \\
(1 - \alpha)L_\alpha(2x - 1) + \alpha & (1/2 \leq x \leq 1).
\end{cases}
\]  

(9)

The functional equation has a unique continuous solution on the unit interval \([0, 1]\). The resulting function \(L_\alpha\) is strictly increasing, is continuous, and has the derivative zero almost everywhere. A difference form of the function \(L_\alpha\) is given by

\[
L_\alpha\left(\frac{2i + 1}{2k + 1}\right) = (1 - \alpha)L_\alpha\left(\frac{i}{2k}\right) + \alpha L_\alpha\left(\frac{i + 1}{2k}\right)
\]  

(10)

for \(0 \leq i \leq 2^k - 1, k \in \mathbb{Z}_{>0}\) \(\text{[9]}\). The end points are given by \(L_\alpha(0) = 0, L_\alpha(1) = 1\).

2.2 Previous results about the number of nonzero states of linear SPG1dECAs and a linear SPG2dECA

We introduce some previous results about the number of nonzero states in spatial and spatio-temporal patterns of linear SPG1dECAs and a linear SPG2dECA.

For a CA \(\{\{0, 1\}^d, T\}\), a subset of a \((d + 1)\)-dimensional Euclidean space \(V_T(n)\) is given by

\[
V_T(n) = \{(i, m) \in \mathbb{Z}^{d+1} \mid (T^m u_0)_i > 0, 0 \leq m \leq n\},
\]  

(11)

which consists of nonzero states from time step 0 to \(n\). Let \(V_T(n)/n\) be a contracted set of \(V_T(n)\) with a contraction rate of \(1/n\). A limit set of a CA is then defined by \(\lim_{n \to \infty} (V_T(n)/n)\) if it exists. For the limit sets of linear CAs, the following two theorems have been reported.

**Theorem 1** \(\text{[3]}\). Let \(p\) be a prime number and \(m \in \mathbb{Z}_{>0}\). For a \(p^m\)-state linear CA, if \(p^{m-1}\) divides time step \(n\), then \((T^m u_0)_i = (T^n u_0)_i\). If \(p^m\) divides \(n\) and at least one of the elements of \(i\) is indivisible by \(p\), then \((T^n u_0)_i\) equals 0.

**Theorem 2** \(\text{[3]}\). Let \(p\) be a prime number and \(m \in \mathbb{Z}_{>0}\). For a \(p^m\)-state linear CA, its limit set \(\lim_{k \to \infty} (V_T(p^k - 1)/p^k)\) exists.

From Theorems 1 and 2 we obtain the following results about linear CAs.

For a CA \(\{\{0, 1\}^d, T\}\), let \(\text{num}_T(n)\) be the number of nonzero states in a spatial pattern \(T^n u_0\) for time step \(n\), and let \(\text{cum}_T(n)\) be the cumulative sum of the number of nonzero states in a spatial pattern \(T^m u_0\) from time step \(m = 0\) to \(n\). Thus,

\[
\text{num}_T(n) = \sum_{i \in \mathbb{Z}^d} (T^n u_0)_i, \quad \text{cum}_T(n) = \sum_{m=0}^{n} \sum_{i \in \mathbb{Z}^d} (T^m u_0)_i,
\]  

(12)

where \(\text{cum}_T(-1) = \text{num}_T(-1) = 0\).

For SPG1dECA Rule 90 and SPG2dECA \(T_0\), we obtained the following results.
Proposition 1 (5). (i) For SPG1dECA Rule 90, let \( x = j/2^{k+1} \) for \( 0 \leq j \leq 2^{k+1} \), \( k \in \mathbb{Z}_{>0} \), and \( h_{S90}(x) = \frac{\text{cum}_{S90}(j-1)}{\text{cum}_{S90}(2^{k+1} - 1)} \). Thus,

\[
h_{S90}\left(\frac{2i+1}{2^{k+1}}\right) = \left(1 - \frac{1}{3}\right) h_{S90}\left(\frac{2i}{2^{k+1}}\right) + \frac{1}{3} h_{S90}\left(\frac{2i+2}{2^{k+1}}\right)
\]

for \( 0 \leq i \leq 2^k - 1 \). The boundary conditions are given by \( h_{S90}(0) = 0 \) and \( h_{S90}(1) = 1 \).

(ii) For SPG2dECA \( T_0 \), let \( x = j/2^{k+1} \) for \( 0 \leq j \leq 2^{k+1} \), \( k \in \mathbb{Z}_{>0} \), and \( h_{T0}(x) = \frac{\text{cum}_{T0}(j-1)}{\text{cum}_{T0}(2^{k+1} - 1)} \). Thus,

\[
h_{T0}\left(\frac{2i+1}{2^{k+1}}\right) = \left(1 - \frac{1}{5}\right) h_{T0}\left(\frac{2i}{2^{k+1}}\right) + \frac{1}{5} h_{T0}\left(\frac{2i+2}{2^{k+1}}\right)
\]

for \( 0 \leq i \leq 2^k - 1 \). The boundary conditions are given by \( h_{T0}(0) = 0 \) and \( h_{T0}(1) = 1 \).

Therefore, \( h_{S90} \) and \( h_{T0} \) equal the difference forms of the singular functions \( L_{1/3}(x) \) and \( L_{1/5}(x) \), respectively.

For SPG1dECA Rule 150, we obtained the following result.

Proposition 2 (12). For \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0,1] \), the function \( f_{S150} : [0,1] \rightarrow [0,1] \) is given by

\[
f_{S150}(x) = \lim_{k \to \infty} \frac{\text{cum}_{S150}(\sum_{i=1}^{k} x_i 2^{k-i}) - 1}{\text{cum}_{S150}(2^k - 1)} \]

\[
= \sum_{i=1}^{\infty} x_i \alpha \prod_{s=0}^{i-1} \left(\frac{(-1)^{s+1} + 2^{s+2}}{3}\right)^{p_{i,s}},
\]

where \( \alpha = (\sqrt{5}-1)/4 \) and \( p_{i,s} \) is the number of clusters consisting of \( s \) continuous 1s in the binary number \( 0.x_1x_2 \cdots x_{i-1} \).

3 Main results

We discuss the main results concerning SPG1dECAs and SPG2dECAs. Among the SPG1dECAs, Rule 90 and Rule 150 are linear and hold the equation in Lemma 1. Among SPG2dECAs, only two CAs, \( T_0 \) and \( T_{528} \), are both linear and hold with the equation in Lemma 1. In Section 3.1, we calculate the number of nonzero states of the spatial and spatio-temporal patterns of Rule 90, \( T_0 \), and \( T_{528} \). We normalized the dynamics of the number of nonzero states and obtained functions for them. (For Rule 150, we previously obtained the results in [12].) In Section 3.2, we provide a sufficient condition of singularity for a function and show that the resulting functions for the four CAs are singular functions, which are strictly increasing, continuous, and differentiable with the derivative.
zero almost everywhere. We also show that \( f_{S90} \) and \( f_{T0} \) are Salem’s singular function \( L_{1/\alpha} \), and the box-counting dimension of their limit sets are given by \(-\log \alpha / \log 2\). In Section 3.3, we discuss the nonlinear SPG1dECAs, Rule 22 and Rule 126. We discovered that the normalized functions for Rule 22 and Rule 126 equal the function for Rule 90.

### 3.1 Singular functions generated by linear SPG ECAs

For spatio-temporal patterns of an SPG1dECA Rule 90 and SPG2dECAs, \( T_0 \) and \( T_{528} \), we calculate the number of nonzero states, \( \text{cum}_T \) and \( \text{num}_T \), and provide normalized functions.

Let \( LS_1 \) be the set of linear SPG1dECAs, and let \( LS_2 \) be the set of linear SPG2dECAs. By Theorem 1, for a CA in \( LS_1 \) and \( LS_2 \), we can count the number of (partially) self-similar sets in each spatio-temporal pattern for time step \( n \). Then, we obtain the following lemma.

**Lemma 1.** Let \( n = \sum_{i=0}^{k-1} x_{k-i} 2^i \geq 0 \), where \( x_0 = 0 \). If a CA \( T \in LS_1 \cup LS_2 \), then

\[
\text{cum}_T(n - 1) = \sum_{i=1}^{k} x_i \text{num}_T \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) \text{cum}_T(2^{k-i} - 1). \quad (17)
\]

By Theorem 2 for CAs in \( LS_1 \cup LS_2 \), the following function \( f_T : [0, 1] \rightarrow [0, 1] \) exists.

**Definition 4.** For a CA \( (\{0, 1\}^d, T) \), a function \( f_T : [0, 1] \rightarrow [0, 1] \) is given by

\[
f_T(x) := \lim_{k \to \infty} \frac{\text{cum}_T \left( \sum_{i=1}^{k} x_i 2^{k-i} \right) - 1}{\text{cum}_T(2^k - 1)} \quad (18)
\]

for \( x = \sum_{i=1}^{\infty} (x_i / 2^i) \in [0, 1] \).

Next, we consider functions \( f_{S90}, f_{T0}, \) and \( f_{T528} \).

#### 3.1.1 Function \( f_{S90} \) generated by Rule 90

Figure 1 shows the spatio-temporal pattern of Rule 90 from the single site seed \( u_0 \), and Figure 2 shows the graph of the cumulative number of nonzero states in the spatio-temporal pattern of Rule 90. The values \( \text{cum}_{S90}(2^k - 1) \) and \( \text{num}_{S90}(n) \) were already obtained.

**Lemma 2 ([5]).** For time step \( n = \sum_{i=0}^{k-1} x_{k-i} 2^i \), we have \( \text{cum}_{S90}(2^k - 1) = 3^k \) and \( \text{num}_{S90}(n) = 2 \sum_{i=1}^{k} x_i \).

**Theorem 3.** For \( x = \sum_{i=1}^{\infty} (x_i / 2^i) \in [0, 1] \), the function \( f_{S90} : [0, 1] \rightarrow [0, 1] \) is given by

\[
f_{S90}(x) = \sum_{i=1}^{\infty} x_i 2 \sum_{j=0}^{i-1} x_j 3^{-i}. \quad (19)
\]
Proof. By Lemmas 1 and 2 we have

\[ f_{S90}(x) = \lim_{k \to \infty} \frac{cum_{S90} \left( \left( \sum_{i=1}^{k} x_i 2^{k-i} \right) - 1 \right)}{cum_{S90}(2^k - 1)} \]

\[ = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} x_i num_{S90} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) cum_{S90}(2^{k-i} - 1)}{cum_{S90}(2^k - 1)} \]

\[ = \lim_{k \to \infty} \sum_{i=1}^{k} x_i 2^{\sum_{j=0}^{i-1} x_j 3^{-i}}. \tag{22} \]

From Equation (22), we have \( x_i 2^{\sum_{j=0}^{i-1} x_j 3^{-i}} \leq \left( \frac{2}{3} \right)^i \).

Because \( \lim_{i \to \infty} \left| \left( \frac{2}{3} \right)^{i+1} / \left( \frac{2}{3} \right)^i \right| < 1 \), the infinite series \( \sum_{i=1}^{\infty} \left( \frac{2}{3} \right)^i \) absolutely converges. Thus, \( \sum_{i=1}^{\infty} x_i 2^{\sum_{j=0}^{i-1} x_j 3^{-i}} \) also absolutely converges.

We easily obtain \( f_{S90}(0) = 0 \) and \( f_{S90}(1) = 1 \). Therefore, Equation (19) is obtained.

Remark 3. When \( x \) is a dyadic rational, \( m/2^k \), we have two possible binary expansions. We will verify that the definition of \( f_{S90} \) is consistent for the values with two binary expansions. Let \( x = \sum_{i=1}^{k} (x_i/2^i) + 1/2^{k+1} \) and \( y = \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (1/2^i) \) for \( x_i \in \{0,1\} \) and \( k \in \mathbb{Z}_{>0} \). Hence, \( x = y \).

We have

\[ f_{S90}(y) - f_{S90}(x) = \left( \sum_{i=k+2}^{\infty} 2^{i-k-2} x_j 3^{-i} \right) - 2^{\sum_{j=1}^{k} x_j 3^{-k-1}} \]

\[ = 2^{\sum_{j=1}^{k} x_j 3^{-i}} \left( \sum_{i=1}^{\infty} 2^{i-1} 3^{-i} - 1 \right) = 0. \tag{24} \]
3.1.2 Function $f_{T_0}$ generated by $T_0$

Figure 3 shows the spatio-temporal pattern of an SPG2dECA $T_0$ from the single site seed $u_0$, and Figure 4 shows the graph of the cumulative number of nonzero states in the spatio-temporal pattern of $T_0$. The values $\text{cum}_{T_0}(2^k - 1)$ and $\text{num}_{T_0}(n)$ were already obtained.

**Lemma 3** ([5]). For time step $n = \sum_{i=0}^{k-1} x_k 2^i$, we have $\text{cum}_{T_0}(2^k - 1) = 5^k$ and $\text{num}_{T_0}(n) = 4 \sum_{i=1}^{k} x_i$.

![Figure 3: Spatio-temporal pattern of $T_0$](image)

**Theorem 4.** For $x = \sum_{i=1}^{\infty} (x_i / 2^i) \in [0, 1]$, the function $f_{T_0} : [0, 1] \to [0, 1]$ is given by

$$f_{T_0}(x) = \sum_{i=1}^{\infty} x_i 4 \sum_{j=0}^{i-1} x_j 5^{-i}.$$  (25)
Proof. By Lemmas 1 and 3, we have

$$f_{T_0}(x) = \lim_{k \to \infty} \frac{\text{cum}_{T_0} \left( \sum_{i=1}^{k} x_i 2^{k-i} \right) - 1}{\text{cum}_{T_0}(2^k - 1)}$$

$$= \lim_{k \to \infty} \frac{\sum_{i=1}^{k} x_i \text{num}_{T_0} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) \text{cum}_{T_0}(2^{k-i} - 1)}{\text{cum}_{T_0}(2^k - 1)}$$

$$= \lim_{k \to \infty} \frac{1}{5^k} \sum_{i=1}^{k} x_i \text{num}_{T_0} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) 5^{k-i}$$

$$= \lim_{k \to \infty} \sum_{i=1}^{k} x_i \text{num}_{T_0} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) 5^{i-2}$$

$$= \lim_{k \to \infty} \sum_{i=1}^{k} x_i \sum_{j=0}^{i-2} x_j 5^{-i-1}.$$  

From Equation (30), we have $x_i 4^{\sum_{j=0}^{i-1} x_j 5^{-i}} \leq (4/5)^i$.

Because $\lim_{i \to \infty} \left[ (4/5)^{i+1}/(4/5)^i \right] = 4/5 < 1$, the infinite series $\sum_{i=1}^{\infty} (4/5)^i$ absolutely converges. Thus, $\sum_{i=1}^{\infty} x_i 4^{\sum_{j=0}^{i-1} x_j 5^{-i}}$ also absolutely converges.

We easily obtain $f_{T_0}(0) = 0$ and $f_{T_0}(1) = 1$. Therefore, Equation (25) is obtained.

Remark 4. For $x = \sum_{i=1}^{\infty} (x_i/2^i) + 1/2^{k+1}$ and $y = \sum_{i=1}^{k} x_i/2^i + \sum_{i=k+1}^{\infty} (1/2^i)$, for $x_i \in \{0, 1\}$ and $k \in \mathbb{Z}_{>0}$, we verify that $f_{T_0}(x) = f_{T_0}(y)$ because $x = y$. We have

$$f_{T_0}(y) - f_{T_0}(x) = \left( \sum_{i=k+2}^{\infty} 4^{i-k-2} + \sum_{j=1}^{k} x_j \sum_{i=1}^{\infty} 5^{-i} \right) - 4^{\sum_{j=1}^{k} x_j} \sum_{i=1}^{\infty} 5^{-i} - 1 = 0.$$  

3.1.3 Function $f_{T_{528}}$ generated by $T_{528}$

We study the spatio-temporal pattern of $((0, 1)^{\mathbb{Z}^2}, T_{528})$ from the initial configuration $u_0$ (Figure 6). Figure 5 shows the cumulative number of nonzero states of $T_{528}$. First, we obtain $\text{cum}_{T_{528}}(2^k - 1)$ and $\text{num}_{T_{528}}(n)$.

Lemma 4. For time step $n = \sum_{i=0}^{k-1} x_{k-i} 2^i$, we have

$$\text{cum}_{T_{528}}(2^k - 1) = 2^{k-1} \left( 1 + \sqrt{2}^{k+1} + (1 - \sqrt{2})^{k+1} \right),$$

$$\text{num}_{T_{528}}(n) = \prod_{s=0}^{i+1} \left( \frac{17 + 7\sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^s + \frac{17 - 7\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^s \right)^{p_s},$$

where $p_s$ is the number of times the $s$-th iteration of $T_{528}$ is applied.
where $p_s$ is the number of clusters of $s$ consecutive 1-states in the binary number of $n$, $x_1 x_2 \ldots x_k$. 

**Figure 4:** Dynamics of the cumulative number of nonzero states of $T_0$

**Figure 5:** Dynamics of the cumulative number of nonzero states of $T_{528}$

**Proof.** For some $k \in \mathbb{Z}_{\geq 0}$ we study a set of nonzero states of the spatio-temporal pattern $\{T_{528}^{u_0}\}_{n=0}^{2^k-1}$, $V_{T_{528}}(2^k - 1)$. We provide three types of partially self-similar sets, $A_k$, $B_k$, and $C_k$, based on the four-sided pyramid $V_{T_{528}}(2^k - 1)$ (see Figure 7). Let $A_k$ be $V_{T_{528}}(2^k - 1)$ itself. We remove a quadrangular prism whose size is $1 \times 1 \times (2^k - 1)$ from $A_k$ and crop it vertically to quarter of its size through the top of the pyramid $V_{T_{528}}(2^k - 1)$. We combine a piece of the pyramid and the quadrangular prism of size $1 \times 1 \times (2^k - 1)$, and call it $B_k$. Let $C_k$ be the remaining three pieces of the quartered pyramid without the quadrangular prism of size of $1 \times 1 \times (2^k - 1)$.

**Figure 6:** Spatio-temporal pattern of $T_{528}$
Let \( z_k \) be the number of nonzero states in \( A_k \). We can easily find that the number of nonzero states in \( B_k \) is \( (z_k - 2^k) / 4 + 2^k \) and that the number of nonzero states in \( C_k \) is \( 3(z_k - 2^k) / 4 \). We set the initial value \( z_0 = 1 \), and \( z_{-1} = 1/2 \) because of technical reason. We construct \( A_{k+1} \) from one \( A_k \), two \( A_{k-1} \)s, eight \( B_{k-1} \)s, and four \( C_k \)s (see Figure 8). Then, we obtain the following recurrence formula:

\[
z_{k+1} = z_k + 2z_{k-1} + 8 \left( \frac{z_{k-1} - 2^{k-1}}{4} + 2^{k-1} \right) + 4 \left( \frac{3(z_k - 2^k)}{4} \right) \tag{35}
\]

\[
= 4(z_k + z_{k-1}). \tag{36}
\]

Set \( y_{k+1} = 2z_k \), and we have

\[
\begin{cases}
z_0 = 1, & y_0 = 1, \\
z_{k+1} = 4z_k + 2y_k, & \\
y_{k+1} = 2z_k. \tag{37}
\end{cases}
\]

For a vector \( a \), a matrix \( M \), and a vector \( v_0 \) given by

\[
a = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \quad u_0 = \begin{pmatrix} z_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{38}
\]

we have

\[
cum_{T528}(2^k - 1) = aM^k u_0 = 2^{k-1} \left( (1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1} \right). \tag{39}
\]

Let

\[
M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \tag{40}
\]

and for the binary number of time step \( n = \sum_{i=0}^{k-1} x_{k-i} 2^i \), we have

\[
um_{T528}(n) = aM_{x_0} M_{x_1} \cdots M_{x_{k-1}} u_0, \tag{41}\]

where \( x_0 = 0 \). For \( k \geq 0 \),

\[
aM_0^k u_0 = 1, \tag{42}
\]

\[
aM_1^k u_0 = \frac{17 + 7\sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^k + \frac{17 - 7\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^k. \tag{43}
\]

The matrices \( M_0 \) and \( M_1 \) hold the following:

\[
aM_1^k M_0^k u_0 = aM_1^k u_0 = aM_0^k M_1^k u_0, \tag{44}
\]

\[
aM_1^{k_0} M_0^{k_1} M_1^{k_1} \cdots M_1^{k_i} M_0^{k_i} u_0 = (aM_1^{k_0} u_0) (aM_1^{k_1} u_0) \cdots (aM_1^{k_i} u_0). \tag{45}
\]
Let $p_s$ be the number of clusters of $s$ continuous 1-states in the binary number of $n = \sum_{i=0}^{k-1} x_{k-i}2^i$. Thus,

$$num_{T528}(n) = \prod_{s=0}^{l+1} (aM_1^s u_0)^{p_s}$$

(44)

$$= \prod_{s=0}^{l+1} \left( \frac{17 + 7\sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^s + \frac{17 - 7\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^s \right)^{p_s}.$$

(45)

\[
\]

Figure 7: Partially self-similar sets of $\{T_{528}^n u_0\}_{n=0}^{2^k-1}$

Hence, we have the following results for $T_{528}$.

**Theorem 5.** For $x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1]$, the function $f_{T528} : [0, 1] \rightarrow [0, 1]$ is given by

$$f_{T528}(x) = \sum_{i=1}^{\infty} x_i \alpha^i \prod_{s=0}^{l+1} \left( \frac{17 + 7\sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^s + \frac{17 - 7\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^s \right)^{p_{i,s}},$$

(46)

where $\alpha = (\sqrt{2} - 1)/2$, and $p_{i,s}$ is the number of clusters consisting of $s$ continuous 1s in the binary number $0.x_1x_2\cdots x_{i-1}$. 

13
Proof. By Lemmas 1 and 4, for \( k \in \mathbb{Z}_{>0} \), we have

\[
f_{T528} (x) = \frac{\text{cum}_{T528} \left( \sum_{i=1}^{k} x_i 2^{k-i} \right) - 1}{\text{cum}_{T528} (2^k - 1)}
\]

\[
= \sum_{i=1}^{k} x_i \text{num}_{T528} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) \text{cum}_{T528} (2^{k-i} - 1)
\]

\[
= \sum_{i=1}^{k} x_i r(x)_i \frac{2^{-i} \left( (1 + \sqrt{2})^{k-i+1} + (1 - \sqrt{2})^{k-i+1} \right)}{(1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1}}
\]

\[
= \frac{1}{1 + (2\sqrt{2} - 3)^{k+1}} \sum_{i=1}^{k} x_i r(x)_i \alpha^i
\]

\[
+ \frac{2^{k+1}}{(1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1}} \sum_{i=1}^{k} x_i r(x)_i \frac{(1 - \sqrt{2})^{k-i+1}}{2^{k+i+1}},
\]

where \( r(x)_i = a_{x_0} M_{x_1} \cdots M_{x_{i-1}} u_0 \) with \( x_0 = 0 \).

Next, in (a) and (b), we show that Equation (50) converges to \( \sum_{i=1}^{\infty} x_i r(x)_i \alpha^i \) as \( k \) tends to infinity. In (a), we consider the first term of Equation (50), and in (b), we consider the second term of Equation (50).

(a) We show that the first term of Equation (50) converges \( \sum_{i=1}^{\infty} x_i r(x)_i \alpha^i \) as \( k \to \infty \).
First, we have \(0 \leq x_i r(x) \alpha^i \leq (4^i/2 + \alpha) \alpha^i\) for any \(i > 0\) because

\[
r(x)_i \leq aM_0 M_i^{-1} u_0 \tag{51}
\]

\[
= 17 + 7\sqrt{17} \left( \frac{3 + \sqrt{17}}{2} \right)^{i-1} + 17 - 7\sqrt{17} \left( \frac{3 - \sqrt{17}}{2} \right)^{i-1} \tag{52}
\]

\[
= 17 - \sqrt{17} \left( \frac{3 + \sqrt{17}}{2} \right)^i + 17 + \sqrt{17} \left( \frac{3 - \sqrt{17}}{2} \right)^i \tag{53}
\]

\[
= \frac{1}{2} \left( \left( \frac{3 + \sqrt{17}}{2} \right)^i + \left( \frac{3 - \sqrt{17}}{2} \right)^i \right) \nonumber
\]

\[
+ \frac{\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^i - \left( \frac{3 + \sqrt{17}}{2} \right)^i \tag{54}
\]

\[
< \frac{4^i}{2} + \alpha. \tag{55}
\]

Because \(\lim_{i \to \infty} \left| ((4^i+1/2 + \alpha) \alpha^{i+1}) / ((4^i/2 + \alpha) \alpha^i) \right| < 1\), the infinite series \(\sum_{i=1}^{\infty} (4^i/2 + \alpha) \alpha^i\) absolutely converges. Thus, \(\sum_{i=1}^{\infty} x_i r(x) \alpha^i\) also absolutely converges.

(b) We show that the second term of Equation (50) converges to 0 as \(k\) tends to infinity.

For the coefficient of the second term of Equation (50), we easily calculate
\[
2^{k+1}/((1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1}) \to 0 \ (k \to \infty). \tag{56}
\]

Next, we calculate \(\sum_{i=1}^{k} x_i r(x) \frac{(1 - \sqrt{2})}{2}^{k-i+1}\).

When \(k\) is even, i.e., \(k = 2m\) for \(m \in \mathbb{Z}_{>0}\), we have

\[
\sum_{i=1}^{2m} x_i r(x) \frac{(1 - \sqrt{2})^{2m-i+1}}{2^{2m+i+1}} \tag{56}
\]

\[
= \sum_{i=1}^{m} x_{2i-1} r(x) \frac{(1 - \sqrt{2})^{2m-(2i-1)+1}}{2^{2m+(2i-1)+1}} + \sum_{i=1}^{m} x_{2i} r(x) \frac{(1 - \sqrt{2})^{2m-2i+1}}{2^{2m+2i+1}} \tag{57}
\]

\[
= 4\alpha^{2m+2} \sum_{i=1}^{m} x_{2i-1} r(x) \frac{(\sqrt{2}+1)}{2}^{2i} - \alpha^{2m+1} \sum_{i=1}^{m} x_{2i} r(x) \frac{(\sqrt{2}+1)}{2}^{2i}. \tag{58}
\]
Here, we evaluate the first term of Equation (58). By Equation (55),

\[ 4\alpha^{2m+2} \sum_{i=1}^{m} x_{2i-1} r(x)_{2i-1} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]  

(59)

\[ \leq 4\alpha^{2m+2} \sum_{i=1}^{m} \left( \frac{4^{2i-1}}{2} + \alpha \right) \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]  

(60)

\[ = \frac{\alpha^{2m+2}}{2} \sum_{i=1}^{m} (2\sqrt{2} + 2)^{2i} + 4\alpha^{2m+3} \sum_{i=1}^{m} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]  

(61)

\[ = \frac{1 - \alpha^{2m}}{2(11 + 8\sqrt{2})} + \frac{4^{-2m} - \alpha^{2m}}{2\sqrt{2} - 1} \]  

(62)

\[ \to \frac{8\sqrt{2} - 11}{14} \quad (m \to \infty). \]  

(63)

Because Equation (60) increases with \( m \), we have

\[ 0 \leq 4\alpha^{2m+2} \sum_{i=1}^{m} x_{2i-1} r(x)_{2i-1} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq (8\sqrt{2} - 11)/14. \]  

Next, we evaluate the second term of Equation (58). By Equation (55),

\[ \alpha^{2m+1} \sum_{i=1}^{m} x_{2i} r(x)_{2i} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]  

(64)

\[ \leq \alpha^{2m+1} \sum_{i=1}^{m} \left( \frac{4^{2i}}{2} + \alpha \right) \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]  

(65)

\[ = \frac{(\sqrt{2} + 1)(1 - \alpha^{2m})}{11 + 8\sqrt{2}} + \frac{4^{-2m} - \alpha^{2m}}{4(2\sqrt{2} - 1)} \]  

(66)

\[ \to \frac{5 - 3\sqrt{2}}{7} \quad (m \to \infty). \]  

Because Equation (65) increases with \( m \), we have

\[ 0 \leq \alpha^{2m+1} \sum_{i=1}^{m} x_{2i} r(x)_{2i} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq (5 - 3\sqrt{2})/7. \]  

Hence, when \( k \) is even, \(-(5 - 3\sqrt{2})/7 \leq \sum_{i=1}^{2m} x_i r(x)_i (1 - \sqrt{2})^{2m-i+1}/2^{2m+i+1} \leq (8\sqrt{2} - 11)/14.\]

Next, when \( k \) is odd, i.e., \( k = 2m - 1 \) for \( m \in \mathbb{Z}_{>0} \), we calculate the summation of the second term in Equation (50). We have
We evaluate the first term of Equation (69). Then,

\[ \alpha^2 \sum_{i=1}^{m-1} x_{2i} r(x)_{2i} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq \alpha^2 \sum_{i=1}^{m-1} \left( \frac{4^{2i-1} + \alpha}{2} \right) \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]

\[ \rightarrow \frac{8\sqrt{2} - 11}{14}, \quad (m \rightarrow \infty). \]

Because Equation (70) is increasing, \( 0 \leq \alpha^2 \sum_{i=1}^{m-1} x_{2i} r(x)_{2i} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq \frac{8\sqrt{2} - 11}{14} \). We also evaluate the second term of Equation (69). Thus, we have

\[ 4\alpha^{2m+1} \sum_{i=1}^{m} x_{2i-1} r(x)_{2i-1} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq 4\alpha^{2m+1} \sum_{i=1}^{m} \left( \frac{4^{2i-1} - \alpha}{2} \right) \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \]

\[ \rightarrow \frac{5 - 3\sqrt{2}}{7}, \quad (m \rightarrow \infty). \]

Because Equation (73) is increasing, \( 0 \leq 4\alpha^{2m+1} \sum_{i=1}^{m} x_{2i-1} r(x)_{2i-1} \left( \frac{\sqrt{2} + 1}{2} \right)^{2i} \leq \frac{5 - 3\sqrt{2}}{7} \). Thus, when \( k \) is odd, we have \(-\frac{5 - 3\sqrt{2}}{7} \leq \sum_{i=1}^{2m-1} x_i r(x) \leq \frac{5 - 3\sqrt{2}}{7} \). Therefore, for the second term of Equation (60),

\[ \frac{2^{k+1}}{(1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1}} \sum_{i=1}^{k} x_i r(x_i) \frac{(1-\sqrt{2})^k - i+1}{2^k + i+1} \rightarrow 0, \quad (k \rightarrow \infty), \]

because \( |(5 - 3\sqrt{2})/7| < 1 \) and \( |(8\sqrt{2} - 11)/14| < 1 \).
By the definition of $f_{T528}$, we verify $f_{T528}(0) = 0$ and $f_{T528}(1) = 1$. Then, we have $f_{T528} : [0, 1] \rightarrow [0, 1]$.

**Remark 5.** For $x = \sum_{i=1}^{k} (x_i/2^i) + 1/2^{k+1}$ and $y = \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (1/2^i)$, for $x_i \in \{0, 1\}$ and $k \in \mathbb{Z}_{>0}$, we verify that $f_{T528}(x) = f_{T528}(y)$ because $x = y$. Then,

$$f_{T528}(y) - f_{T528}(x) = \sum_{i=k+2}^{\infty} r(y)_i \alpha^i - r(x)_{k+1} \alpha^{k+1} = r(x)_{k+1} \alpha^{k+1}$$

$$= \left( \sum_{i=1}^{\infty} \left( \frac{17 + 7\sqrt{17}}{34} \left( \frac{3 + \sqrt{17}}{2} \right)^{i-1} + \frac{17 - 7\sqrt{17}}{34} \left( \frac{3 - \sqrt{17}}{2} \right)^{i-1} \right) \alpha^i - 1 \right)$$

$$= 0.$$  

### 3.2 $f_{S90}$, $f_{T0}$, and $f_{T528}$ are singular functions

Based on the results in Section 3.1, we provide a sufficient condition for singularity and show that the resulting functions, $f_{S90}$, $f_{T0}$, and $f_{T528}$, are singular. We also show that $f_{S90}$ and $f_{T0}$ are Salem’s singular function.

**Theorem 6.** Let $\alpha$ be a parameter such that $0 < \alpha < 1/2$. For $x = \sum_{i=1}^{\infty} x_i/2^i \in [0, 1]$, a function $r(x)_i$ is given by $\prod_{s=0}^{\infty} (q(s))^{p_s}$, where $q(s) \in \mathbb{Z}_{>0}$ is a function for $s \in \mathbb{Z}_{>0}$, and $p_s$, is the number of clusters consisting of $s$ continuous $1$s in the binary number $0.x_1x_2\ldots x_{i-1}$. If $q(s)$ satisfies the following four conditions:

(a) $q(0) = 1$,

(b) $q(s_1) q(s_2 - 1) < q(s_1 + s_2) \leq q(s_1) q(s_2)$ for $s_1, \ s_2 > 0$,

(c) $q(s_1) q(s_1 + s_2) < q(s_1 + s_2) q(i-1) \alpha^i$ for $s > 0$, and

(d) $\sum_{i=1}^{\infty} q(i-1) \alpha^i = 1$,

then a function $f(x) = \sum_{i=1}^{\infty} x_i r(x)_i \alpha^i$ for $x = \sum_{i=1}^{\infty} x_i/2^i \in [0, 1]$ satisfies the following three properties:

(i) $f$ is strictly increasing,

(ii) $f$ is continuous, and

(iii) $f$ is differentiable with derivative zero almost everywhere.

**Proof of Theorem 6.** (i). Suppose $0 \leq x < y \leq 1$. We can choose some $k \in \mathbb{Z}_{>0}$ such that $x = \sum_{i=1}^{k} (x_i/2^i) + \sum_{i=k+2}^{\infty} (x_i/2^i)$ and $y = (\sum_{i=1}^{k} (x_i/2^i)) + 1/2^{k+1} + \ldots + 1/2^{n+1}$ with $n > k$. Then, $y = x - \sum_{i=k+1}^{\infty} x_i/2^i$. By the definition of $r(x)_i$, we have

$$r(x)_i = \begin{cases} 1 & \text{if } i = k+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$r(y)_i = \begin{cases} 1 & \text{if } i = k+2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $|r(x)_i - r(y)_i| = 1$ for $i = k+1$ and $|r(x)_i - r(y)_i| = 0$ otherwise. Hence, $|f(x) - f(y)| = |r(x)_i - r(y)_i| = 1$ for some $i_k$, where $i_k$ is the smallest index such that $|r(x)_i - r(y)_i| = 1$. Therefore, $f(x) = f(y) + 1$ and $f(x) - f(y) = 1$, which implies that $f$ is strictly increasing.
strictly increasing by Theorem 6 (\(x_i = 1\), where \(P_{i=k+2}^\infty x_i = 0\). When \(k = 0\), we have \(x = \sum_{i=1}^\infty (x_i/2^i)\) and \(y = 1/2 + \sum_{i=1}^\infty (y_i/2^i)\), where \(P_{i=2}^\infty x_i = 0\). Let \(\{x\}\) be the fractional part of \(x \in \mathbb{R}_{\geq 0}\), i.e., \(x - [x]\), where \([x]\) is the greatest integer less than or equal to \(x\). Thus, we have

\[
f(y) - f(x) = r(x)_{k+1}a^{k+1} + \sum_{i=k+2}^\infty y_i r(y)_i a^i - \left(\sum_{i=k+2}^\infty x_i r(x)_i a^i\right)
\]

(79)

\[
f(y) - f(x) = r(x)_{k+1}a^{k+1} \left(1 - \sum_{i=1}^\infty x_{i+k+1} r(\{2^{k+1}x\}_i) a^i\right) + \sum_{i=k+2}^\infty y_i r(y)_i a^i.
\]

(80)

Based on the definition \(r(x)_{k+1}a^{k+1} > 0\), because \(P_{i=k+2}^\infty x_i = 0\) and condition (d), we have \(1 - \sum_{i=1}^\infty x_{i+k+1} r(\{2^{k+1}x\}_i) a^i > 0\) and \(P_{i=k+2}^\infty y_i r(y)_i a^i \geq 0\). Hence, if \(y > x\), then \(f(y) > f(x)\).

Proof of Theorem 6 (ii). Suppose \(x = \sum_{i=1}^k (x_i/2^i) + \sum_{i=k+2}^\infty (x_i/2^i) \in [0,1)\) and \(y = (\sum_{i=1}^k (x_i/2^i)) + 1/2^{k+1} + (\sum_{i=k+2}^\infty (y_i/2^i)) \in (0,1]\), where \(P_{i=k+2}^\infty x_i = 0\) for some \(k \in \mathbb{Z}_{\geq 0}\). Then \(0 < y - x \leq 1/2^k\). We set \(\epsilon := 2q(1)q(k)a^{k+1}\) and \(\delta := \epsilon/(q(k)(2a)^{k+1})\). If \(y - x \leq 1/2^k < \delta\), then

\[
f(y) - f(x) = r(x)_{k+1}a^{k+1} \left(1 - \sum_{i=1}^\infty x_{i+k+1} r(\{2^{k+1}x\}_i) a^i\right) + \sum_{i=k+2}^\infty y_i r(y)_i a^i
\]

(81)

\[
\leq r(x)_{k+1}a^{k+1} + \sum_{i=k+2}^\infty y_i r(y)_i a^i
\]

(82)

\[
\leq r(x)_{k+1}a^{k+1} + r(x)_{k+1}q(1)a^{k+1} \sum_{i=1}^\infty y_{i+k+1} r(\{2^{k+1}y\}_i) a^i
\]

(83)

\[
\leq (1 + q(1))q(k)a^{k+1} < \epsilon.
\]

(84)

By (b), we have \(q(s_1)q(0) < q(s_1)q(1)\) if \(s_2 = 1\), and then \(q(1) > 1\). By (c), the sequence \(q(k)a^{k+1}\) is strictly decreasing. Thus, Equation (84) is obtained.

Therefore, as \(f\) is a function on a finite bounded interval \([0,1]\), \(f\) is continuous.

Proof of Theorem 6 (iii). The function \(f\) has bounded variation because \(f\) is strictly increasing by Theorem 6 (i). Hence, \(f\) is differentiable almost everywhere on \([0,1]\) (e.g., [13 Theorem 6.3.3]).

Suppose that \(x = \sum_{i=1}^\infty (x_i/2^i)\) is a differentiable point on \([0,1]\). For any \(k \in \mathbb{Z}_{>0}\), we can choose \(y = \sum_{i=1}^k (x_i/2^i)\) such that \(x \leq y \leq y + 1/2^k\). Let \(l_k\) be
max \{i \mid x_i = 0, 1 \leq i \leq k\} if \( \prod_{i=1}^{k} x_i = 0\), and 0 if \( \prod_{i=1}^{k} x_i = 1\). Then,

\[
\frac{f(y + 1/2^k) - f(y)}{1/2^k} = \frac{f \left( y + \left( \sum_{i=k+1}^{\infty} 1/2^i \right) \right) - f(y)}{1/2^k} = 2^k \sum_{i=k+1}^{\infty} r \left( y + \frac{1}{2^k} \right) \alpha^i \\
= 2^k r(x) \alpha^k \sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i. 
\]

(85)

Assuming the derivative at \( x \) is not zero, the derivative is finite and positive because \( f \) is strictly increasing. Let \( \hat{y} = y + x_{k+1}/2^{k+1} \). When \( x_{k+1} = 1 \),

\[
\frac{2^{k+1} (f(\hat{y} + 1/2^{k+1}) - f(\hat{y}))}{2^k (f(y + 1/2^k) - f(y))} = \frac{2 \sum_{i=k-l_k+2}^{\infty} q(i-1) \alpha^i}{\sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i} = \frac{2 q(k-l_k) \alpha^{k-l_k+1}}{\sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i}. 
\]

(88)

When \( x_{k+1} = 0 \),

\[
\frac{2^{k+1} (f(\hat{y} + 1/2^{k+1}) - f(\hat{y}))}{2^k (f(y + 1/2^k) - f(y))} = \frac{2^{k+1} r(x) \alpha^k \sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i}{\sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i}. 
\]

(90)

By contrast, because \( f \) is differentiable at \( x \), we have

\[
\lim_{k \to \infty} \frac{2^{k+1} (f(\hat{y} + 1/2^{k+1}) - f(\hat{y}))}{2^k (f(y + 1/2^k) - f(y))} = 1. 
\]

(92)

Based on Equations (89) and (91):

\[
\lim_{k \to \infty} \frac{2 q(k-l_k) \alpha^{k-l_k+1}}{\sum_{i=k-l_k+1}^{\infty} q(i-1) \alpha^i} = 1, 
\]

(93)

\[
\lim_{k \to \infty} \frac{q(k-l_k) \alpha^{k-l_k+1} + \sum_{i=k-l_k+2}^{\infty} q(i-1) \alpha^i}{2 q(k-l_k) \alpha^{k-l_k+1}} = 1, 
\]

(94)

\[
\lim_{k \to \infty} \frac{\sum_{i=k-l_k+2}^{\infty} q(i-1) \alpha^i}{q(k-l_k) \alpha^{k-l_k+1}} = 1. 
\]

(95)

By (c), for any \( s \in \mathbb{Z}_{>0} \), we have \( q(s) \alpha^{s+1} < \sum_{i=s+2}^{\infty} q(i-1) \alpha^i \). This contradicts the assumption that the derivative at \( x \) is not zero. Hence, the derivative at \( x \) is zero when \( f \) is differentiable at \( x \). \( \square \)
Corollary 1. By Theorem 6, $f_{S90}$, $f_{S150}$, $f_{T0}$, and $f_{T528}$ are singular functions.

Remark 6. For function $q$ in Theorem 6(b), the equal sign is used only when $q$ is an exponential function. For example, for $f_{S90}$ and $f_{T0}$, the functions are $2^i$ and $4^i$, respectively.

Corollary 2. We can show that the function $f_{S90}$ is Salem’s singular function $L_{1/3}$, i.e.,

$$f_{S90}(x) = \begin{cases} 
\frac{1}{3} f_{S90}(2x) & (0 \leq x < \frac{1}{2}) \\
\frac{2}{3} f_{S90}(2x-1) + \frac{1}{3} & (\frac{1}{2} \leq x \leq 1)
\end{cases}, \quad (96)$$

We can show that the function $f_{T0}$ is Salem’s singular function $L_{1/5}$, i.e.,

$$f_{T0}(x) = \begin{cases} 
\frac{1}{5} f_{T0}(2x) & (0 \leq x < \frac{1}{2}) \\
\frac{4}{5} f_{T0}(2x-1) + \frac{1}{5} & (\frac{1}{2} \leq x \leq 1)
\end{cases}, \quad (97)$$

These results match the difference equations in Proposition 7.

Corollary 3. For a CA $((0,1)^2, T) \in LS_1 \cup LS_2$, if a function $f_T$ is given by Salem’s singular function $L_{1/\alpha}$, the box-counting dimension of the limit set $\lim_{k \to \infty} V_T(2^k - 1)/2^k$ is $-\log \alpha/\log 2$.

3.3 Function $f_{S90}$ obtained by nonlinear SPG1dECAs

In this section, we focus on two nonlinear SPG1dECAs, Rule 22 and Rule 126. Sixteen SPG1dECAs exist. Two of them, Rule 90 and Rule 150, are linear, and we already discussed them in the previous sections. The others are nonlinear, and the spatio-temporal patterns of Rule 18, Rule 146, and Rule 218 are the same as that of Rule 90. For nonlinear SPG1dECAs Rule 50, Rule 54, Rule 94, Rule 122, Rule 178, Rule 182, Rule 222, Rule 250, and Rule 254, the limit sets are not fractals because their box-counting dimensions are 2. Thus, this section discusses the other nonlinear SPG1dECAs, Rule 22 and Rule 126. Although their spatio-temporal patterns are different from that of Rule 90 (see Figures 9 and 10), their resulting functions, $f_{S22}$ and $f_{S126}$, are equal to $f_{S90}$.

3.3.1 Function $f_{S90}$ by a nonlinear SPG1dECA Rule 22

Although an SPG1dECA Rule 22 is nonlinear, it holds Lemma 1 and the following results are obtained.

Lemma 5. Rule 22 holds the equation in Lemma 7.
Proof. The local rule of Rule 22 is given by \((S_{22}u)_i = u_{i-1} + u_i + u_{i+1} + u_{i-1}u_i + u_{i+1}(mod 2)\) for \(u \in \{0, 1\}^\mathbb{Z}\). Thus, we have

\[
(S_{22}^2 u)_i = (S_{22}(S_{22}u))_i = u_{i-2} + u_{i-2}u_i + u_{i-2}u_iu_{i+2} + u_iu_{i+2} + u_{i+2} + u_{i-2}u_{i+1}u_{i+1} + u_{i-2}u_{i-1}u_{i+2} + u_{i-2}u_{i+1} + u_{i-1}u_{i+2} + u_{i-1}u_{i+1}u_{i+2} + u_{i-1}u_i + u_{i+1}(mod 2).
\]

If \(u_{2m-1} = 0\) for any \(m \in \mathbb{Z}\), then \((S_{22}^2 u)_{2m-1} = 0\) and \((S_{22}^2 u)_{2m} = u_{2m-2} + u_{2m-2}u_{2m} + u_{2m-2}u_{2m}u_{2m+2} + u_{2m}u_{2m+2} + u_{2m+2}(mod 2)\). Inductively, for the odd-numbered columns, we have \((S_{22}^k u)_{2m-1} = 0\) for any \(k \in \mathbb{Z}^\geq 0\).

Next, if \(u_{2m-1} = 0\) and \(u_{4m} = 0\) for any \(m \in \mathbb{Z}\), then \((S_{22}^2 u)_{4m-2} = 0\) and \((S_{22}^2 u)_{4m} = u_{4m-2} + u_{4m+2}(mod 2)\). Thus, for the even-numbered columns, we have \((S_{22}^2 u)_{2m} = u_{2m-2} + u_{2m+2}(mod 2)\), which equals the local rule of Rule 90. For odd time steps, we easily know \((S_{22}^{2k+1} u)_{2m-1} = (S_{22}^{2k+1} u)_{2m} = (S_{22}^{2k+1} u)_{2m+1} = 1\) if and only if \((S_{22}^2 u)_{2m} = 1\), and \((S_{22}^{2k+1} u)_i = 0\) otherwise.

Therefore, the number of nonzero states of Rule 22 from the initial configuration \(u_0\) is given by the equation in Lemma 6.

**Lemma 6.** For time step \(n = \sum_{i=0}^{k-1} x_k \cdot 2^i\), we have \(\text{cum}_{s_{22}}(2^k - 1) = 4 \cdot 3^{k-1}\) and 
\(\text{num}_{s_{22}}(n) = 2 \sum_{i=1}^{k} x_i \cdot 3^{2i}\).

**Theorem 7.** For \(x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1]\), the function \(f_{s_{22}} : [0, 1] \rightarrow [0, 1]\) is given by
\[
f_{s_{22}}(x) = f_{s_{96}}(x).
\]

Lemma 6
Proof. Based on Lemma 5 for \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1] \), we have

\[
f_{S_{22}}(x) = \lim_{k \to \infty} \frac{\text{cum}_{S_{22}} \left( \left( \sum_{i=1}^{k} x_i 2^{k-i} \right) - 1 \right)}{\text{cum}_{S_{22}}(2^k - 1)}
\]

(101)

\[
= \lim_{k \to \infty} \frac{\sum_{i=1}^{k} x_i \text{num}_{S_{22}} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right)}{\text{cum}_{S_{22}}(2^k - 1)}
\]

(102)

\[
= \lim_{k \to \infty} \sum_{i=1}^{k} x_i \left( 2 \sum_{j=1}^{i-1} x_j \right) 3^{-i}
\]

(103)

\[
= f_{S_{20}}(x).
\]

(104)

\[\square\]

3.3.2 Function \( f_{S_{90}} \) by a nonlinear SPG1dECA Rule 126

Rule 126 is also a nonlinear SPG1dECA, and it does not hold Lemma 1. However, if we consider the spatio-temporal pattern from another initial configuration \( \hat{u}_0 \), we can consider the function \( f_{S_{126}} \).

We give a configuration \( \hat{u}_0 \in \{0, 1\}^Z \) by \( (\hat{u}_0)_i = 1 \) if \( i \in \{0, 1\} \), and \( (\hat{u}_0)_i = 0 \) if \( i \in Z \setminus \{0, 1\} \). Let \( \text{num}_{S_{126}}(n) \) be the number of nonzero states in a spatial pattern \( S_{126} \hat{u}_0 \) and \( \text{cum}_{S_{126}}(n) \) be the cumulative sum of the number of nonzero states in a spatial pattern \( S_{126} \hat{u}_0 \) from time step \( m = 0 \) to \( n \). We can now obtain the following relationship similar to Lemma 1.

Lemma 7. Let \( n = \sum_{i=0}^{k-1} x_k 2^i \geq 0 \), where \( x_0 = 0 \). For SPG1dECA \( S_{126} \), we have

\[
\text{cum}_{S_{126}}(n-1) = \frac{1}{2} \sum_{i=1}^{k} x_i \text{num}_{S_{126}} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) \text{cum}_{S_{126}}(2^{k-i} - 1).
\]

(105)

Proof. The local rule of Rule 126 is given by \( (S_{126}u)_i = u_{i-1} + u_i + u_{i+1} + u_{i-1}u_i + u_iu_{i+1} + u_{i-1}u_{i+1} \mod 2 \) for \( i \in Z \). The differences between the local rule of Rule 126 and the local rule of Rule 90 are the transitions for \( (u_{i-1}, u_i, u_{i+1}) = (1, 0, 1) \) and \( (0, 1, 0) \) (see Table 1).

We set the initial configuration \( u \in \{0, 1\}^Z \) such that \( u_{2m+1} = u_{2m} \) for any \( m \in Z \). Then, we have

\[
(S_{126}u)_{2m-1} = u_{2m-2} + u_{2m} \mod 2 = (S_{90}u)_{2m-1},
\]

(106)

\[
(S_{126}u)_{2m} = u_{2m-1} + u_{2m+1} \mod 2 = (S_{90}u)_{2m},
\]

(107)

\[
(S_{126}u)_{2m+1} = u_{2m} + u_{2m+2} \mod 2 = (S_{90}u)_{2m+1},
\]

(108)

\[
(S_{126}u)_{2m+2} = u_{2m+1} + u_{2m+3} \mod 2 = (S_{90}u)_{2m+2}.
\]

(109)

Because of the assumption of \( u \), we have the relationships \( (S_{126}u)_{2m-1} = (S_{126}u)_{2m} \) and \( (S_{126}u)_{2m+1} = (S_{126}u)_{2m+2} \) for any \( m \in Z \). Hence, we show
that for the initial configuration \( \hat{u}_o \), the spatio-temporal patterns of Rule 126 and Rule 90 are the same, and the number of nonzero states of \( \{ S_{126}^{n} u_o \}_{n=0}^{2^k-1} \) is double the number of nonzero states of \( \{ S_{90}^{n} u_o \}_{n=0}^{2^k-1} \).

From the result of Lemma 7, we obtain the following result.

**Lemma 8.** \( \text{cum}_{S_{126}}(2^k-1) = 2 \cdot 3^k \) and \( \text{num}_{S_{126}}(n) = 2^{1+\sum_{i=1}^{k} x_i} \).

**Remark 7.** If we remove the nonzero states in the center column of the spatio-temporal pattern \( \{ S_{n}^{126} \hat{u}_o \}_{n=0}^{2^k-1} \), we can calculate \( \text{cum}_{S_{126}} \) and \( \text{num}_{S_{126}} \). For time step \( n = 2 \sum_{i=0}^{k-1} x_{i-2^i} \), we have \( \text{cum}_{S_{126}}(2^k-1) = 2 \cdot 3^k - k - 1 \) and \( \text{num}_{S_{126}}(n) = 4 + \prod_{i=1}^{k} (1-x_i) \cdot 2^{2\sum_{i=1}^{k-1} x_i - \prod_{i=1}^{k} x_i} \).

For Rule 126, we have the following results.

**Theorem 8.** For \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1] \), the function \( f_{S_{126}} : [0, 1] \rightarrow [0, 2] \) is given by
\[
 f_{S_{126}}(x) = 2 f_{S_{90}}(x). \tag{110}
\]

**Proof.** For \( x = \sum_{i=1}^{\infty} (x_i/2^i) \in [0, 1] \), we have
\[
 f_{S_{126}}(x) = \lim_{k \to \infty} \frac{\text{cum}_{S_{126}} \left( (\sum_{i=1}^{k} x_i 2^{k-i}) - 1 \right)}{\text{cum}_{S_{126}}(2^k-1)} \tag{111}
 = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} x_i \text{num}_{S_{126}} \left( \sum_{j=0}^{i-1} x_j 2^{k-j} \right) \text{cum}_{S_{126}}(2^{k-i} - 1)}{\text{cum}_{S_{126}}(2^k-1)} \tag{112}
 = \lim_{k \to \infty} 2 \sum_{i=1}^{k} x_i \left( 2^{\sum_{j=1}^{i-1} x_j} \right) 3^{-i} \tag{113}
 = 2 f_{S_{90}}(x). \tag{114}
\]

### 4 Concluding remarks

In this paper, we shared our results concerning SPG1dECAs and SPG2dECAs. In Section 3.1 we discussed linear SPG1dECAs and linear SPG2dECAs. Because the CAs hold the equation in Lemma 7, we can calculate the numbers of nonzero states of their spatial and spatio-temporal patterns, \( \text{num}_T \) and \( \text{cum}_T \), for each CA. We normalized the numbers and obtained the functions \( f_{S_{90}}, f_{T_0}, \) and \( f_{T_{528}} \). In Section 3.2 we showed that the functions for linear SPG1dECAs and linear SPG2dECAS are singular functions that strictly increase, are continuous, and are differentiable with derivative zero almost everywhere. We provided a sufficient condition of singularity for the function \( f \) in Theorem 6. From this theorem we showed that \( f_{S_{90}}, f_{S_{150}}, f_{T_0}, \) and \( f_{T_{528}} \) are singular functions. We
also discussed the relationship with Salem’s singular function $L_{1/\alpha}$. We have $f_{S90} = L_{1/3}$ and $f_{T0} = L_{1/5}$, and the box-counting dimension of their limit sets are $- \log 3 / \log 2$ for Rule 90 and $- \log 5 / \log 2$ for $T_0$. In Section 3.3, we discussed nonlinear 1dECAs, specifically Rule 22 and Rule 126. From their spatio-temporal patterns, we obtained the functions $f_{S22}$ and $f_{S126}$, which equals $f_{S90}$.

In future work, we plan to study the other SPG2dECAs. We will study their number of nonzero states and their normalized functions. In this paper, we showed that the resulting functions are singular, and in [14], we had shown that the functions are discontinuous and Riemann integrable. We will study other pathological functions, not only singular functions emerging from 2dECAs, and we will provide generalized conditions for Theorem 6.

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Data Availability Statement

The data that supports the findings of this work are available within this paper.

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