Prepared for submission to JHEP

Impure Aspects of Supersymmetric Wilson Loops

Valentina Cardinali\textsuperscript{a} Luca Griguolo\textsuperscript{b} and Domenico Seminara\textsuperscript{a}

\textsuperscript{a}Dipartimento di Fisica e Astronomia, Università di Firenze and INFN Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino, Italy
\textsuperscript{b}Dipartimento di Fisica, Università di Parma and INFN Gruppo Collegato di Parma, Viale G.P. Usberti 7/A, 43100 Parma, Italy
E-mail: cardinali@fi.infn.it, luca.griguolo@fis.unipr.it, seminara@fi.infn.it

ABSTRACT: We study a general class of supersymmetric Wilson loops operator in $\mathcal{N} = 4$ super Yang-Mills theory, obtained as orbits of conformal transformations. These loops are the natural generalization of the familiar circular Wilson-Maldacena operator and their supersymmetric properties are encoded into a Killing spinor that is not pure. We present a systematic analysis of their scalar couplings and of the preserved supercharges, modulo the action of the global symmetry group, both in the compact and in the non-compact case. The quantum behavior of their expectation value is also addressed, in the simplest case of the Lissajous contours: explicit computations at weak-coupling, through Feynman diagrams expansion, and at strong-coupling, by means of AdS/CFT correspondence, suggest the possibility of an exact evaluation.

KEYWORDS: Supersymmetric gauge theory, AdS-CFT Correspondence, Extended Supersymmetries
1 Introduction

Loop operators are probably the most basic observables of four dimensional gauge theories: they can be classified according to whether the particle running around the loop is electrically or magnetically charged, giving rise to Wilson or ’t Hooft operators respectively. They
play the role of order parameters for the phases that a gauge theory can exhibit, and serve as probes of the quantum gauge dynamics. In supersymmetric gauge theories loop operators become also ideal probes for checking some powerful, nonperturbative symmetry, as S-duality, that is conjectured to exchange weak and strong coupling behaviors. The possibility to compute exactly these observables allows for a quantitative study of S-duality and serves as a theoretical laboratory for gaining a deeper understanding of the inner workings of dualities among theories in different dimensions [1–4]. On the other hand, exact results in quantum field theory usually rely on powerful symmetry principles, such as supersymmetry: we could expect that particular loop operators, preserving some genuine fraction of the original supersymmetric invariance, are amenable of an exact quantum evaluation. A beautiful example is Pestun’s calculation of circular 1/2 BPS Wilson-Maldacena loops [5, 6] in a wide class of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories [7]. To be more precise, Pestun reduced the problem of computing this highly supersymmetric observable to a finite-dimensional matrix integral, proving and generalizing the statement of a conjecture, originally formulated in the \( \mathcal{N} = 4 \) case [8, 9]. It appears therefore important to find new BPS loop operators and to study these non-local gauge invariant observables at quantum level.

We consider Wilson loops in four-dimensional maximally supersymmetric Yang-Mills theory in Euclidean space-time: \( \mathcal{N} = 4 \) SYM is a superconformal theory, the fermionic subspace of its superconformal algebra being generated by Poincaré supercharges \( Q_\alpha \) and special conformal supercharges \( S^\alpha \). We call a Wilson loop supersymmetric if there exists at least one non-zero linear combination of \( Q_\alpha \) and \( S^\alpha \) leaving invariant the operator and we are interested in observables obtained from the ordinary electric loops by coupling them to the scalars of the \( \mathcal{N} = 4 \) supermultiplet. A certain number of such supersymmetric Wilson loops have been known for some time and analyzed previously [8–11]. They were captured by two classes: the loops of arbitrary shape found by Zarembo in [10] and the loops of arbitrary shape on a three-sphere \( S^3 \), embedded into space-time, found by Drukker-Giombi-Ricci-Trancanelli (DGRT) in [11]. Remarkably Zarembo’s observables are the same Wilson loops which appear in topological Langlands twist of \( \mathcal{N} = 4 \) SYM [12] and have trivial expectation value. The most familiar example of the loops in DGRT class is instead the 1/2 BPS circular loop coupled to one of the scalars: it can be computed exactly by Gaussian matrix model and the results perfectly agree with the string dual computation, suggested by AdS/CFT correspondence. The subset of DGRT loops restricted to \( S^2 \) was also recently studied in great details and an interesting connection with bosonic two-dimensional Yang-Mills on \( S^2 \) was established [11, 13–19].

An essential step in understanding the structure of supersymmetric Wilson loops in \( \mathcal{N} = 4 \) SYM was performed by Dymarsky and Pestun in [20]: they were able to list all possible Wilson operators \( \mathcal{W} \) that are invariant under at least one superconformal symmetry \( Q \) and to classify the interesting subclasses of \( \{\mathcal{W}, Q\} \) pairs modulo the action of the superconformal group. The main idea in their construction is to pack the data describing locally a supersymmetric Wilson loop, namely the tangent vector to the curve and the scalar couplings, into a ten-dimensional vector \( v^M(x) \). Requiring the invariance of the loop operator
with respect to a supersymmetry $Q$, generated by a given spinor $\epsilon(x)$ implies a certain system of linear equations on $v(x)$. The properties of this system depend crucially on whether the ten-dimensional spinor is pure or not. Actually the appearance of pure spinors is not completely surprising because the four-dimensional theory is a dimensional reduction [22] of the $\mathcal{N} = 1$ SYM in ten dimensions, where pure spinors show up naturally [23–25]. We remark that the space-time dependent spinor that parameterizes the superconformal transformations of $\mathcal{N} = 4$ SYM, can be viewed directly as a reduction of a chiral ten-dimensional spinor. If $\epsilon(x)$ is not a pure spinor, then the system for $v(x)$ has the unique solution, i.e. the tangent to the curve and the scalar couplings are completely fixed. The vector $v(x)$ is is determined by the ten-dimensional vector constructed in the canonical way as the bilinear in $\epsilon(x)$. The contours obtained in this way from a general supersymmetry parameter $\epsilon(x)$ are simply the orbits of the conformal transformation generated by $Q^2$ [20]. Interestingly, modulo conformal equivalence, the only resulting compact curves are the $(p,q)$ Lissajous figures where $p/q$ is the rational ratio of two eigenvalues of the $SO(4)$ matrix representing the action of $Q^2$. The situation changes if $\epsilon(x)$ is pure: in this case there are more solutions for the vector $v(x)$. Dymarsky and Pestun observed that a pure spinor defines ten-dimensional almost complex structure $J(x)$, and then the supersymmetry condition of the Wilson loop translates into the condition that $v(x)$ is anti-holomorphic vector with respect to $J(x)$. On the subspace of the space-time where $\epsilon(x)$ is pure there is richer space of solutions for supersymmetric Wilson operator: generically, for any curve sitting inside the subspace one can find scalar couplings to make the Wilson loop supersymmetric.

The analysis presented in [20] is mainly concentrated on the classification and the classical construction of pure spinor Wilson loops, that are, in a sense, more general and interesting than the impure ones, being supported on rather arbitrary curves and admitting a strong-coupling characterization in type IIB superstring theory, as calibrated surface on $AdS_5 \times S^5$. In this paper we are instead focussed on the less ambitious goal of studying the impure Wilson loop operators and their quantum aspects, both at weak and strong coupling. Our main concern is the compact case, therefore we study in details the $(p,q)$ Lissajous figures and their supersymmetry algebra: we find that, generically, five scalars couple to the supersymmetric Lissajous loop through a $6 \times 4$ constant rectangular matrix $M$ and a constant vector $B$. Both $M$ and $B$ generically possess complex entries and obey to some constraints that we solve explicitly. We recognize an apparent similarity with the scalar couplings introduced by Zarembo in [10] except for the additional coupling governed by $B$. This small deformation plays a crucial role since it prevents the loops from having a trivial VEV, as occurs for the ones considered in [10]. They also differ from the geometrically similar toroidal loops, introduced by [11], where only three scalars are coupled. The loops are generically 1/16 BPS but we observe enhancement of the supersymmetry for particular choices of the couplings. We also study the non-compact impure loops: we classify the orbits of the conformal group, writing down all the relevant contours modulo conformal equivalence and the corresponding couplings. We found convenient to rephrase this problem in six dimensional language, solving the orbit equations up to the action of an element of $SO(5,1)$: in so doing we construct some new families of supersymmetric loops,
as logarithmic spirals, helix and generalized straight lines. At quantum level we consider specifically the Lissajous Wilson loops: at weak coupling, the most striking property is that the combined vector-scalar propagator in Feynman gauge, stretching on the Wilson loop contour, is constant, exactly as for the circular case. Moreover an explicit two-loops evaluation shows that the contribution of interacting diagrams to the quantum expectation value sums to zero, suggesting that the exact answer could be obtained by summing only exchanged propagators on the loop contour: if this would be the case, a Gaussian matrix model underlies the computation and an exact localization procedure should be invoked. We test directly this possibility at strong coupling, by using the dual description of Lissajous Wilson loops by strings in $AdS_5 \times S^5$. More precisely the string duals propagate on a complexification of this space, as pointed out earlier in [26] where some cases of supersymmetric Wilson loops with complex scalar couplings were studied: we find indeed a perfect agreement with a Gaussian matrix model behavior.

The main question opened to future investigations is, of course, if localization could provide an exact computation of this class of supersymmetric Wilson loops, reproducing the weak and strong coupling results we have found in this paper: it should certainly rely on some generalization of Pestun’s procedure. The magnetic duals should also be constructed and studied at quantum level, providing new tests of S-duality. One could also wonder if some of the non-compact loops we found could be used in describing, at dual level, scattering processes or other observables in $\mathcal{N} = 4$ SYM, for example constructing generalized cusps [27]: helixes, similar to the ones appearing in this paper, have also been considered in [28].

The organization of the paper is the following: in Section 2 we briefly review the general strategy to classify supersymmetric Wilson loops in maximally supersymmetric four-dimensional Yang-Mills theory. In Section 3 we study Lissajous supersymmetric Wilson loops: we construct explicitly the scalar couplings and discuss their BPS properties and supersymmetry algebra. In Section 4 we perform the weak coupling computation at the second order in perturbation theory. In Section 5 we obtain the strong coupling solution by means of AdS/CFT correspondence. A number of Appendices is devoted to more technical aspects: Appendix A contains our conventions, Appendix B is dedicated to the complete classification of the conformal orbits, to the classification of scalar couplings and to the explicit construction of the relevant Killing spinors. In Appendix C the action of conformal transformations on the loops are discussed.

2 Impure Wilson loops

In $\mathcal{N} = 4$ super Yang-Mills (SYM)\(^1\) the most simple and common generalization of the familiar Wilson loop is obtained by considering extra couplings with the adjoint scalars $\Phi_a$ $(a = 1, \ldots, 6)$ [5, 6], namely by writing

$$W_R(\gamma) = \frac{1}{d(R)} \text{Tr}_R \left[ \text{Pexp} \int_\gamma \left( A_\mu(x(s)) \dot{x}^\mu(s) + \Phi_a(x(s)) v^a(s) \right) ds \right]$$  \hspace{1cm} (2.1)

\(^1\)See appendix A for our conventions on its action.
the suffix $R$ denoting the representation\footnote{The generator are taken anti-hermitian: $T^A = -T^A$.} of the gauge group $G$ where the trace is taken and $d(R)$ its dimensions. The six dimensional vector $v^a(s)$ identifies the new scalar couplings and, in general, its entries might be complex in the euclidean case. For these operators it is natural to use a ten dimensional notation (see app. A). In fact we can combine the gauge field $A_\mu$ and the scalar fields $\Phi_a$ into a vector $A_M \equiv (A_\mu, \Phi_a)\ (M = 1, 2, \ldots, 10)$ and we can merge the tangent vector $\dot{x}^\mu$ and $v^a(s)$ into a generalized vector of couplings $v^M \equiv (\dot{x}^\mu, v^a)$. Then the Wilson loop (2.1) can be rearranged in the compact form [20]

$$W_R(\gamma) = \frac{1}{d(R)} \mathrm{Tr}_R \left[ \exp \left( \oint_{\gamma} A_M v^M ds \right) \right].$$

(2.2)

An interesting subclass of these non-local operators is provided by the so-called supersymmetric Wilson loops, i.e. the operators (2.2) for which the combination $v^M A_M$ is invariant under, at least, one super-conformal transformation. This subset is determined by the vectors $v^M$ which obey the linear constraint

$$\delta_\epsilon (A_M v^M) = v^M(s) \psi_M \epsilon(x) = 0 \quad \Rightarrow \quad v^M(s) \gamma_M \epsilon(x) = 0,$$

(2.3)

where $\epsilon(x) = \epsilon_s + x^\mu \gamma_\mu \epsilon_c$ is the super-conformal Killing spinor associated to the transformation. Locally on the contour this implies that $v^M v_M = 0$ [21]. More generally, given $\epsilon(x)$, all possible solutions for $v^M$ of eq. (2.3) were obtained by Dymarsky and Pestun in [20]. They fall into two different classes depending on the value of the bilinear $u^M \equiv \epsilon^T C^{-1} \gamma^M \epsilon$.

**Case (A):** If $u^M$ vanishes identically on a submanifold $\Sigma_\epsilon \subseteq \mathbb{R}^4$, $\epsilon$ is a pure spinor on $\Sigma_\epsilon$ and consequently it induces an almost complex structure $J_\epsilon$ on this region [20]. The possible solutions $v^M$ of eq. (2.3) in a point $x \in \Sigma_\epsilon$ are then provided by all the anti-holomorphic vectors with respect to $J_\epsilon$ [20]. This result can be used to associate a supersymmetric Wilson loop to each closed contour $\gamma$ in $\Sigma_\epsilon$. An explicit construction of this class of operators, modulo equivalence under the action of the superconformal group, is given in [20]. All supersymmetric Wilson loops that have been studied previously are essentially captured by this case. In fact both the loops discussed by Zarembo in [10] and those found by Drukker-Giombi- Ricci-Trancanelli (DGRT) in [11] are of this type.

**Case (B):** When $u^M \neq 0$, the solution of eq. (2.3) is uniquely fixed up to a complex scale $\lambda$ and it is given by $v^M = \lambda u^M$ [20]. In other words, given the super-conformal spinor $\epsilon$, there is only one possible invariant Wilson loop:

$$W_R(\gamma) = \frac{1}{d(R)} \mathrm{Tr}_R \left[ \exp \left( \oint_{\gamma} (A_\mu u^\mu + \Phi_a u^a) \frac{ds}{(u^\mu u_\mu)^{1/2}} \right) \right],$$

(2.4)

where $s$ denotes the usual affine parameter which measures the length of the curve. In eq. (2.4), in order to identify the space-time couplings $u^\mu$ with the tangent vector to the
contour $\gamma$, we must require that $u^\mu$ is projectively equivalent to a real vector, \textit{i.e.} there is a $\lambda \in \mathbb{C}^*$ such that $\lambda u^\mu$ is real\(^3\). Then $\gamma$ is determined by the differential equation

$$\dot{x}^\mu = u^\mu,$$

(2.5)

where, for future convenience, we have chosen to fix the normalization of the spinor $\epsilon$ so that $\lambda = 1$. The path $\gamma$ defined by (2.5) has a natural and simple geometrical interpretation: it is just the orbit of the conformal transformation generated by $Q^2_\epsilon$, where $Q_\epsilon$ is the superconformal generator associated to the spinor $\epsilon$ (see [20]).

In the following we shall focus our attention on the supersymmetric Wilson loops of this second type for which we shall also use the term \textit{impure loops} to emphasize the difference with those of the CASE (A). Specifically, we shall determine the general form of the scalar couplings and provide an explicit construction of this family of loops. Next we shall discuss the complete sub-algebra of the superconformal group which leaves these loop invariant. Finally we shall analyze the properties of their VEVs both at weak and at strong coupling.

3 Lissajous figures: general properties and structure

In general, the integral curves of the conformal Killing vector $u^\mu$ do not define a closed loop: it can only occur when $u^\mu$, modulo conformal equivalence, specifies an orbit of the four dimensional rotation group. In this case $u^\mu$ can be always cast into the canonical form

$$u^\mu = \Omega^\mu_\nu x^\nu,$$

(3.1)

(see app. B.2.) The matrix $\Omega$ in (3.1) can be chosen to belong to the (Cartan) subalgebra $so(2) \oplus so(2)$ in $so(4)$ and its explicit form is

$$\Omega = \begin{pmatrix}
0 & \Omega_1 & 0 & 0 \\
-\Omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \Omega_2 \\
0 & 0 & -\Omega_2 & 0
\end{pmatrix}.$$ 

(3.2)

Then the orbits drawn by the tangent vector (3.1) are described by the parametric equations\(^4\)

$$x^\mu(s) = \left\{ \cos \frac{\theta}{2} \sin \Omega_1 s, \ \cos \frac{\theta}{2} \cos \Omega_1 s, \ \sin \frac{\theta}{2} \sin \Omega_2 s, \ \sin \frac{\theta}{2} \cos \Omega_2 s \right\},$$

(3.3)

where $\theta$ is a free parameter which runs from 0 to $\pi$. However, these paths define a closed circuit if and only if the ratio $\Omega_2/\Omega_1$ is a rational number $m/n$ with $m, n$ relatively prime. In this case the range of $s$ must be a multiple of $\frac{2\pi n}{\Omega_1}$.

\(^3\)The reality of $\lambda u^\mu$ is an implicit constraint on the possible spinors $\epsilon$.

\(^4\)We have dropped an irrelevant global scale in solving (2.5) and used the freedom to choose the initial point of the loop.
Geometrically, the curves (3.3) describe the superposition of two circular motions with different frequencies occurring in orthogonal planes and they can be considered a generalization of the familiar Lissajous figures. By construction they all lie on the sphere $S^3$ defined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and thus we can use the usual stereographic coordinates $(y_i \equiv \frac{x_i}{1-x_4}, i = 1, 2, 3)$ to picture them (see Fig. 1).

Figure 1. For fixed $\theta$ all these loops wrap a torus $T^2$ of equation $\left(\sqrt{y_1^2 + y_2^2} - \sec \frac{\theta}{2}\right)^2 + y_3^2 = \tan^2 \frac{\theta}{2}$. Fixing $\theta = \pi/2$, the loops for $\Omega_2/\Omega_1 = 1/3$ and $\Omega_2/\Omega_1 = 2/3$ are shown in (a) and (b) respectively.

When the ratio $\Omega_2/\Omega_1$ runs from 0 to 1, at fixed $\theta$, we are interpolating between a latitude ($\Omega_2/\Omega_1 = 0$, the red curve in Fig. 2) on the $S^2$ defined by $x_3 = 0$ and one of the equators of the sphere $S^3$ (the blue curve in Fig. 2). Instead, at fixed $\Omega_2/\Omega_1$, when the parameter $\theta$ goes from 0 to $\pi$, we have a family of circuits interpolating between two great circles in $S^3$: the former (at $\theta = 0$) with winding number $n$, the latter (at $\theta = \pi$) with winding number $m$.

The same contours were considered in [11] as an example of DGRT loops with more than 2 supersymmetries and they were named toroidal loops. However, as we shall see below, the scalar couplings are quite different in the two cases. For example, while DGRT loops in general couple to three scalars, ours will couple to five. Additional differences will become manifest when discussing the structure of the scalar couplings below.

For this family of operators the vector $u^a$ ($a = 1, \ldots, 6$), which couples the contour to the scalar fields $\Phi_a$, has a very simple structure. Up to terms vanishing along the loop, it is linear in $x^\mu$ or alternatively in $\dot{x}^\mu$ and it can be written as follows

$$u^a = M^a_\mu x^\mu + B^a = (M\Omega^{-1})^a_\mu \dot{x}^\mu + B^a,$$

where $M$ is a $6 \times 4$ constant rectangular matrix and $B$ is a constant vector. Both $M$ and $B$ generically possess complex entries. In addition the matrix $M$ must obey the following
algebraic constraints

\[(B \cdot M)_\mu = 0 \quad t^\dagger M + t^\dagger \Omega \Omega = -(B \cdot B) \mathbf{I}_{4 \times 4}, \]  

(3.5)

which encode the requirement of local supersymmetry \([i.e. \ u^M u_M = 0]\).

The impure loops will commonly couple to five independent scalars. In fact, because of eqs. (3.5), the columns of \(M\) and the vector \(B\) will provide in general a set of 5 orthogonal and thus independent vectors\(^5\). This is a crucial difference with the toroidal loops considered in [11] were the number of the coupled scalar was at most three.

In (3.4) we also recognize an apparent similarity with the scalar couplings introduced by Zarembo in [10] except for the additional coupling governed by \(B^A\). This small deformation plays a crucial role since it prevents the loops defined by (3.4) from having a trivial VEV, as occurs for the ones considered in [10]. Actually, from the point of view of perturbation theory, this family of loops can be considered the simplest generalization of the usual Wilson-Maldacena circle and in fact it enjoys very similar properties (see also Sect.4).

The details of the construction and the properties of these non-local operator starting from the impure Killing spinor \(\epsilon\) are presented in appendix B.

3.1 Supersymmetries

The Wilson loops introduced in Sect.3 are, by construction, invariant under the superconformal transformation defined by the spinor \(\epsilon = \epsilon_s + x^\mu \gamma_\mu \epsilon_c\), which generates the vector of the couplings. But this native invariance does not exhaust all possible supersymmetries. To classify all of them, we must solve the standard BPS-condition

\[\delta (u^M A_M) \propto \delta (x^\mu \gamma_\mu + u^a \gamma_a) \epsilon = (x^\nu \gamma_\nu + u^a \gamma_a)(\epsilon_s + x^\mu \gamma_\mu \epsilon_c) = 0,\]  

(3.6)

where we used the usual 32 \(\times\) 32 Dirac matrices \(\gamma_M\) to have a more efficient notation and we have also broken the range of their index \(M\) in two subsets \((\mu, a)\) with \(\mu = 1, \ldots, 4\) and \(a = 1, \ldots, 6\).

We can reorganize eq. (3.6) as a polynomial of second degree in the space-time coordinates \(x^\mu\). Then it takes the following form

\[x^\nu [\sigma_\nu \epsilon_s + (B_a \gamma^a) \gamma_\nu \epsilon_c] - (x \cdot \gamma) x^\nu [(B_a \gamma^a) \gamma_\nu \epsilon_s + \sigma_\nu \epsilon_c] = 0,\]  

(3.7)

where we have used that our closed contours must lie on the unit sphere, namely they obey the constraint \(x^2 = 1\). In eq. (3.7) we have also found convenient to introduce the auxiliary matrices

\[\sigma_\alpha = (\gamma_\mu \Omega^\mu_{\alpha} + \gamma_a M^a_{\alpha}).\]  

(3.8)

Because of eq. (3.5) they obey the four dimensional Clifford algebra

\[\sigma_\alpha \sigma_\beta + \sigma_\beta \sigma_\alpha = -2(B \cdot B) \delta_{\alpha\beta} \mathbf{1}\]  

(3.9)

\(^5\)Keep also in mind that the five functions \(\{1, x_\mu\}\) are linear independent for generic \(\Omega_1\) and \(\Omega_2\).
and a very simple set of anti-commutation relations with the standard Dirac matrices $\gamma^\nu$ and with $B_a \gamma^a$:

$$\gamma^\mu \sigma_\nu + \sigma_\nu \gamma^\mu = 2\Omega^\mu_\nu \mathbb{1} \quad \text{and} \quad \{\sigma_\mu, B_a \gamma^a\} = 0. \quad (3.10)$$

For a generic value of $\Omega_1$ and $\Omega_2$ [i.e. $\Omega_1^2 \neq \Omega_2^2 \neq 0$], the monomials in $x^\nu$, appearing in (3.7), provide an independent set of functions along the circuit, thus the two combinations between square brackets must vanish separately

$$\sigma_\nu \epsilon_s + (B^a \gamma_a) \gamma_\nu \epsilon_c = 0, \quad (3.11a)$$

$$\{\sigma_\mu, B_a \gamma^a\} = 0. \quad (3.11b)$$

For $(B \cdot B) \neq 0$, we can ignore the conditions (3.11b) since they can be shown to be equivalent to the set of equations (3.11a). Thus we are left with only four equations constraining the couple of constant spinors $(\epsilon_s, \epsilon_c)$. To solve this system, we first get rid of $\epsilon_c$ by solving (3.11a) for $\nu = 1$

$$\epsilon_c = \frac{1}{(B \cdot B)} (B^a \gamma_a) \gamma^1 \sigma_1 \epsilon_s. \quad (3.12)$$

Substituting (3.12) into the remaining three equations we learn that $\epsilon_s$ is annihilated by the following three linear operators

$$T_1 \epsilon_s \equiv (\gamma^2 \sigma_2 - \gamma^1 \sigma_1) \epsilon_s = [\gamma^2 \hat{M}_2 - \gamma^1 \hat{M}_1] \epsilon_s = 0, \quad (3.13a)$$

$$T_2 \epsilon_s \equiv (\gamma^3 \sigma_3 - \gamma^2 \sigma_2) \epsilon_s = [(\Omega_1 \gamma^{12} - \gamma^2 \hat{M}_2) - (\Omega_2 \gamma^{34} - \gamma^3 \hat{M}_3)] \epsilon_s = 0, \quad (3.13b)$$

$$T_3 \epsilon_s \equiv (\gamma^4 \sigma_4 - \gamma^3 \sigma_3) \epsilon_s = [\gamma^4 \hat{M}_4 - \gamma^3 \hat{M}_3] \epsilon_s = 0, \quad (3.13c)$$

where we have introduced the short-hand notation $\hat{M}_\mu \equiv \gamma_\alpha M_\alpha^\mu$. Since the six-component vectors $(M_1^a, M_2^a, M_3^a, M_4^a)$ generically define four orthogonal complex directions in $\mathbb{C}^6$ [see eq. (3.5)], the projectors on the kernels of the operators $T_i$ are easily constructed in terms of the matrices $\gamma^\mu$ and $\hat{M}_\mu$. One finds

$$P_1 = \frac{1}{2} \left( \mathbb{1} - \frac{(\gamma^2 \hat{M}_2)(\gamma^1 \hat{M}_1)}{(B \cdot B) + \Omega_1^2} \right),$$

$$P_2 = \frac{1}{2} \left( \mathbb{1} - \frac{(\Omega_1 \gamma^{12} - \gamma^2 \hat{M}_2)(\Omega_2 \gamma^{34} - \gamma^3 \hat{M}_3)}{(B \cdot B)} \right),$$

$$P_3 = \frac{1}{2} \left( \mathbb{1} - \frac{(\gamma^4 \hat{M}_4)(\gamma^3 \hat{M}_3)}{(B \cdot B) + \Omega_2^2} \right). \quad (3.14)$$

As they commute among themselves, the most general solution of the (3.13) can be always cast into the form

$$\epsilon_s = P_1 P_2 P_3 \eta_s, \quad (3.15)$$

where $\eta_s$ is a positive chiral spinor in ten dimensions. The number of linearly independent solutions can be also easily determined: in fact it is equal to the rank of the projector.
$P_1 P_2 P_3$ on the subspace of spinors of positive chirality. This last quantity is simply obtained by taking the trace of the combination $\frac{1}{2}(1 + \gamma^{11})P_1 P_2 P_3$:

$$\text{Tr} \left( \frac{1}{2}(1 + \gamma^{11})P_1 P_2 P_3 \right) = 2. \quad (3.16)$$

Our family of loops preserves generically two independent supercharges, being at least 1/16 BPS. The degree of supersymmetry can be of course enhanced for particular choices of the scalar couplings. In particular we see that the above analysis, done in terms of commuting projectors, appears to fail when $(B \cdot B)$ is equal to either $-\Omega_1^2$ or $-\Omega_2^2$ or to 0\(^6\). For those values the expression (3.14) for one of the three projectors is ill-defined.

We start by considering the case $(B \cdot B) = -\Omega_1^2$ (but $\Omega_1^2 \neq \Omega_2^2$)\(^7\). It is not difficult to show from the constraints (3.5) (see also Appendix B, above eq.(B.31)) that the first two columns of the matrix $M_\mu^a$ are given by two complex parallel light-like vectors: $M_1^a = m_1 V^a$, $M_2^a = m_2 V^a$ and $V^2 = 0$. As a consequence, eq.(3.13a) can be rearranged as follows

$$T_1 \epsilon_s = (m_1 \gamma^1 - m_2 \gamma^2) \hat{V} \epsilon_s = 0, \quad (3.17)$$

where we have defined $\hat{V} \equiv V^a \gamma_a$. If $m_1^2 \neq m_2^2$, the kernel of $T_1$ is simply equivalent to that of $V$ and the general solutions of our first equation can be written as $\hat{V} \eta$, being $\eta$ an arbitrary anti-chiral spinor. The solution of the full system (3.13) is then obtained by applying the projector $P_2$ and $P_3$ on $\hat{V} \eta$,

$$\epsilon_s = P_2 P_3 \hat{V} \eta, \quad (3.18)$$

and finding the independent components as $\eta$ is varied. The analysis can be performed in a pedestrian way and one ends up with only two independent spinor $\epsilon_{s_i}$, whose explicit form is

$$\epsilon_{s_i} = \sqrt{\Omega_1 - \Omega_2} \epsilon_{s_i}^+ + \sqrt{\Omega_1 + \Omega_2} \epsilon_{s_i}^-, \quad (3.19)$$

where $\epsilon_{s_i}^\pm$ are chiral spinors with respect to the matrix $\gamma^{1234}$ and they are given by

$$\begin{align*}
\epsilon_{s_1}^+ &= \{1, -i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
\epsilon_{s_2}^+ &= \{4p_1(B_1^1 - iB_1^2), 0, 2p_1(B_1^3 - iB_1^4), 2ip_1(B_1^3 - iB_1^4), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
\epsilon_{s_1}^- &= \{0, 0, 0, 0, ie^{i\alpha}, e^{i\alpha}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\
\epsilon_{s_2}^- &= \{0, 0, 0, 0, -4p_1(B_1^1 - iB_1^2) \sin \alpha, 4p_1(B_1^1 - iB_1^2) \cos \alpha, -2ie^{-i\alpha}p_1(B_1^3 + iB_1^4), -2ie^{-i\alpha}p_1(B_1^3 + iB_1^4), -i\epsilon^{i\alpha} \Omega_1^2, -e^{-i\alpha} \Omega_1^2, 0, 0, 0, 0, 0\}.
\end{align*} \quad (3.20)$$

The vector $B_1^a$ and the parameters $p_1$ and $\alpha$ are defined in appendices B.1 and B.3. To avoid a cumbersome notation we have also dropped the last sixteen entries, which obviously vanish for a spinor of positive chirality.

\(^6\)Since $(B \cdot B) = -2(B_0 \cdot B_1)$, these are exactly the same singular points encountered in the general discussion of the couplings in appendix B.3.

\(^7\)The case $(B \cdot B) = -\Omega_1^2$ (but $\Omega_1^2 \neq \Omega_2^2$) can be analyzed in a similar way. It can be obtained from this one by exchanging the role of $\Omega_1$ and $\Omega_2$. 

- 10 -
Since we have only the two independent solutions (3.20), the loops are still $1/16$ BPS. However, in this particular case, they are not anymore \textit{impure}: as one can easily check, any conformal Killing spinors $\epsilon$ associated to the solutions (3.19) solves $\epsilon^T C^{-1} \gamma^M \epsilon = 0$ on the unit sphere $S^3$. In other words they define a family of \textit{pure} loops coupled to four scalars.

The next step is to explore the case $m_1^2 + m_2^2 = 0$, that provides an enlarged space of solutions. Taking $m_1 \neq 0$, besides the two conformal Killing spinors determined above, a third linearly independent solution surprisingly appears. For instance, for $m_2 = \text{i} m_1$, it is given by

$$
\epsilon s_3 = \sqrt{\Omega_1 - \Omega_2} \epsilon^+_{s_3} + \sqrt{\Omega_1 + \Omega_2} \epsilon^-_{s_3},
$$

with

$$
\epsilon_{s_3}^+ = \left\{ -e^{i\alpha} \sqrt{\Omega_1 - \Omega_2} - \frac{im_1 \sqrt{\Omega_1 + \Omega_2}}{\Omega_1} - \frac{e^{-i\alpha}}{8\Omega_1 \sqrt{\Omega_1 - \Omega_2}}, -ie^{i\alpha} \sqrt{\Omega_1 - \Omega_2} - \frac{m_1 + im_3}{8\Omega_1 \sqrt{\Omega_1 + \Omega_2}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}
$$

and

$$
\epsilon_{s_3}^- = \left\{ 0, 0, 0, 0, \frac{8i(\Omega_1 - \Omega_2)(\Omega_1 \sqrt{\Omega_1 + \Omega_2} + ie^{i\alpha} m_1 \sqrt{\Omega_1 - \Omega_2}) - i\sqrt{\Omega_1 + \Omega_2}}{8\Omega_1 \sqrt{\Omega_1 - \Omega_2} \sqrt{\Omega_1 + \Omega_2}}, -\sqrt{\Omega_1 + \Omega_2} (1 + 8\Omega_1 (\Omega_1 - \Omega_2) + 8ie^{i\alpha} m_1 (\Omega_1 - \Omega_2)^{3/2})}{8\Omega_1 \sqrt{\Omega_1 - \Omega_2} \sqrt{\Omega_1 + \Omega_2}}, e^{\alpha}(m_3 + im_4), e^{\alpha}(m_4 - im_3), 0, 0, 0, 0, 0, 0, 0 \right\}.
$$

The additional parameters $m_i$ in (3.22) and (3.23) are again defined in appendix B.3. The new spinor turns out to be \textit{impure} and in fact the vector $\epsilon s_3 C^{-1} \gamma^M \epsilon s_3$ is proportional to the vector of couplings.

A further and more clear enhancement in the solutions space is observed for $m_1 = m_2 = 0$, when the columns $M_1^a$ and $M_2^a$ do both vanish and the operator $T_1$ is identically zero. We loose eq.(3.13a) and the most general solution of the remaining two equations can be readily written as

$$
\epsilon s = P_2 P_3 \eta_8,
$$

with $\eta_8$ a spinor of positive chirality. We have an obvious augmentation of the supersymmetry and in fact the number of independent solutions of (3.13) is now given by

$$
\text{Tr} \left( \frac{1}{2} (1 + \gamma^{11}) P_2 P_3 \right) = 4.
$$

This particular subset of loops is therefore $1/8$ BPS and the number of scalars coupled to the contour is conversely reduced from five to three.

Finally, we consider the case $(B \cdot B) = 0$ (with $\Omega_1^2 \neq \Omega_2^2$ ), when the two sets of equations (3.11) are not linearly dependent. One can solve again both equations and verify that the loops are still $1/16$ BPS.
3.2 Supersymmetry algebra

For a generic value of $\Omega_1$ and $\Omega_2$ \( i.e. \Omega_1^2 \neq \Omega_2^2 \neq 0 \) and \((B \cdot B)\) different from either \(-\Omega_1^2\) or \(-\Omega_2^2\) or 0, the \textit{impure} loops are preserved by only two superconformal charges $Q_i$ \((i = 1, 2)\), which are generated by the two independent solutions, $\epsilon_i = \epsilon_s + x^\mu \gamma_\mu \epsilon_c$, \((i = 1, 2)\), of the linear system \((3.13)\). In the following we shall analyze the associated super-algebra.

To avoid a lengthy exercise in \textit{spinorology}, we focus our analysis on the little group preserving the origin of the coordinates \((x^\mu = 0)\) \[19\]. In this case for the anti-commutator of two super-charges can be cast in the following form \[19\]

\[
Q_i \{ Q_j \} = 2 \epsilon^T(C^{-1} \gamma_{ab} \epsilon_s) R_{ab} - 2 \epsilon^T(C^{-1} \gamma_{\mu\nu} \epsilon_s) R_{\mu\nu},
\]

(3.26)

where $R_{ab}$ denote the generators of the $R-$symmetry group $SO_R(6)$, while $R_{\mu\nu}$ are those of the euclidean Lorentz group $SO(4)$. Here, we have again split the ten-dimensional indices into two subsets: the greek ones range from 1 to 4 and the roman ones \((a, b, ...\)), which run from 1 to 6.

Exploiting the explicit form \((3.12)\) and \((3.15)\) of the solutions for $\epsilon_s$ and $\epsilon_c$, we find the following expression for the reduced super-algebra

\[
\{ \bar{Q}, Q \} = 4R_0 \{ Q, Q \} = \{ \bar{Q}, \bar{Q} \} = 0 \quad [R_0, Q] = [R_0, \bar{Q}] = 0, \quad (3.27)
\]

where

\[
\bar{Q} = \frac{1}{\sqrt{2}} (Q_1 + iQ_2) \quad Q = \frac{1}{\sqrt{2}} (Q_1 - iQ_2).
\]

The bosonic generator $R_0$ is a linear combination of the $R-$symmetry and rotation generators. In terms of the couplings appearing in the Wilson loops it is given by

\[
R_0 = \frac{1}{2} \Omega^{\mu\nu} \left( R_{\mu\nu} + M^a_\mu \left( M^T M \right)^{-1} b^b_\nu R_{ab} \right),
\]

(3.29)

The reduced algebra is $SU(1|1)$. The entire super-algebra is simply obtained by boosting up this one.

Next we consider the case when two columns of the matrix $M^a_\mu$ vanish and these Wilson loops are 1/8 BPS. Since only three \textit{effective} scalars couple to these second family of operators, there is an obvious invariance under the $SU(2)$ acting on the $R-$symmetry directions orthogonal to these scalars. Then the four complex supercharges can be organized in two doublets $\{ \tilde{Q}_\alpha \}$ \((2)\) and $\{ \tilde{Q}_{\bar{\alpha}} \}$ \((2)\) of this $SU(2)$ and the reduced super-algebra takes the form

\[
\left\{ \tilde{Q}_\alpha, \tilde{Q}_{\bar{\beta}} \right\} = 4 \left( C \sigma^I \right)_{\alpha\beta} R_I \quad \left\{ \tilde{Q}_\alpha, \tilde{Q}_{\bar{\beta}} \right\} = 4 \left( C \sigma^I \right)_{\alpha\beta} R_I \quad \left\{ \tilde{Q}_\alpha, \tilde{Q}_{\bar{\beta}} \right\} = 4 \delta_{\alpha\beta} R_0 \quad (3.30)
\]

where $C = i\sigma_2$ is the two-dimensional charge conjugation matrix. The bosonic part is instead given by

\[
[R_0, R_I] = 0 \quad [R_I, R_J] = i\epsilon_{IJK} R_K. \quad (3.31)
\]
The action of the bosonic generators on the supercharges contains the obvious transformation rule under the $SU(2)$
\[
[R_I, \tilde{Q}_\alpha] = -\frac{1}{2} i \sigma^I_{\alpha\beta} \tilde{Q}_\beta,
\]
\[
[R_I, \tilde{Q}_\bar{\alpha}] = \frac{1}{2} i \sigma^I_{\alpha\bar{\beta}} \tilde{Q}_{\bar{\beta}},
\]  
(3.32)
but also those under the $SO(2)$ generated by $R_0$
\[
[R_0, \tilde{Q}_\alpha] = -\frac{1}{2} C_{\alpha\beta} \tilde{Q}_{\beta},
\]
\[
[R_0, \tilde{Q}_{\bar{\alpha}}] = \frac{1}{2} C_{\bar{\alpha}\bar{\beta}} \tilde{Q}_{\bar{\beta}},
\]  
(3.33)
To provide an explicit form of the bosonic generators in terms of the Wilson loop couplings, we normalize the two non-vanishing columns of $M$ to obtain two orthonormal vectors
\[
m^c_\mu = M^c_\mu / (M^b_\mu M^b_\mu)^{\mu = 3, 4}.
\]
These vectors are orthogonal to $B$. In order to complete the orthonormal basis, we have to add three complex vectors, which will denote with $n^a_1, n^a_2$ and $w^a$. Then $SO(3)$ and $SO(2)$ symmetry associated to rotations of the scalar subspace spanned by $n_1, n_2, w$ are generated by
\[
R_0 = \frac{1}{2} \Omega^{\mu\nu} \left( R_{\mu\nu} + 2 \delta_{\mu3} \delta_{\nu4} m^3_3 m^4_4 R_{ab} \right)
\]
\[
R_1 = -n^a_1 n^b_2 R_{ab} \quad R_2 = n^a_1 w^b R_{ab} \quad R_3 = -n^a_2 w^b R_{ab}.
\]  
(3.34)
This is the usual $SU(1|2)$, which also appears in the case of DGRT loops living on $S^2$.

4 Perturbative aspects

In this Section we explore the quantum behavior at weak-coupling of the Lissajous Wilson loops: as we will see, the tight relation with the circular loop, that appears obvious at level of symmetries, will also become evident in the perturbative computation.

We start by considering the familiar perturbative expansion of the Wilson loop, directly derived from its definition as path-ordered exponential (in the following we will consider the Wilson loop in the fundamental representation):
\[
< W(\gamma) > = \frac{1}{N} \sum_{n=0}^{\infty} \int d\gamma \int_0^{s_1} ds_1 \int_0^{s_2} ds_2 \cdots \int_0^{s_{2n-1}} ds_{2n} \text{Tr} \langle A(s_1) \cdots A(s_{2n}) \rangle,
\]  
(4.1)
where we have expressed the expansion in terms of correlators of the effective connection
\[
A(x(s)) = A_\mu(x(s)) u^\mu(x(s)) + \Phi_a(x(s)) u^a(x(s)).
\]
In $\mathcal{N} = 4$ SYM theory Wilson loops with smooth contours that are locally supersymmetric exhibit an improved ultraviolet behavior, making them manifestly finite in perturbation theory [9]. This property nicely shows up at the first non-trivial order of the perturbative expansion, being encoded in the particular structure of the effective propagator appearing in the computation (in Feynman gauge):
\[ \langle A^A(s_1)A^B(s_2) \rangle_0 = \frac{g^2}{4\pi^2} \frac{u^a(s_1)u_a(s_2) + u^\mu(s_1)u_\mu(s_2)}{(x_1 - x_2)^2} \delta^{AB}. \] (4.2)

The finiteness of the first order contribution can be proved in full generality [14], but additional surprising properties are manifest for globally supersymmetric loops. In order to proceed we have, as first step, to compute in our case the combined vector-scalar propagator, effectively attaching on the loop contour. Taking into account the explicit form of the scalar couplings \( u^a \) in terms of \( M \) and \( B \) (see eq.(3.4)) and the relevant constraints (3.5), we obtain:

\[ \frac{u^a(s_1)u_a(s_2) + u^\mu(s_1)u_\mu(s_2)}{(x_1 - x_2)^2} = \frac{B^2}{2}. \] (4.3)

We recover therefore, for a general Lissajous loop, the very same result of the circular Wilson loops: in Feynman gauge, the relevant effective propagator appearing in the perturbative expansion, is constant when the initial and final points are attached on the loop. This peculiar property was taken originally as an indication that the path-integral computation of circular 1/2 BPS loops reduces to a matrix-model expectation value [8, 9], a fact that has been proved later by localization [7]. We find instructive to derive the above result also from a more general point of view, expressing the vector and the scalar couplings directly in terms of the Killing spinors associated to our loops. Let us consider, for a generic non-pure conformal Killing spinor \( \epsilon(x) \), the structure of the bilinear \( u^M(x) \) that defines the couplings:

\[ u^M(x) = \epsilon^T C^{-1} \gamma^M \epsilon = \epsilon_s \Gamma^M \epsilon_s + 2x_\mu \epsilon_c \Gamma^M \epsilon_s + 2\delta^M_\mu x_\mu \epsilon_c \epsilon_s + 2x_\mu x_\rho \delta^{\mu\rho}_c \epsilon_c \Gamma^\mu \epsilon_c - x^2 \epsilon_s \tilde{\Gamma}^M \epsilon_c. \]

[In the last equalities, we have shifted to the chiral notation introduced in App.A for Dirac matrices and spinors to have a more manageable notation.] Because \( u^M(x) u_M(x) = 0 \), we have that only quadratic terms in \( x_1, x_2 \) contribute, and the numerator of our effective propagator turns out to be:

\[ u^M(x_1) u_M(x_2) = 2(x_1 - x_2)_\nu \epsilon_c \Gamma^\mu \epsilon_c u_\nu(x_2) - (x_1 - x_2)^2 \epsilon_c \tilde{\Gamma}^M \epsilon_c u^M(x_2). \] (4.4)

The second term in this expression can be rewritten using

\[ \epsilon_c \tilde{\Gamma}^M \epsilon_c u^M(x_2) = \left( \epsilon_c \tilde{\Gamma}^M \epsilon_c \right) \left( \epsilon_s \Gamma^M \epsilon_s \right) + 4\epsilon_c \epsilon_s x_2 \rho \epsilon_c \Gamma^\rho \epsilon_c + 2x_2 \rho x_2 \sigma \epsilon_c \Gamma^\rho \epsilon_c \epsilon_c \Gamma^\sigma \epsilon_c, \]

and we see again that, in absence of the term related to special conformal transformations \( (\epsilon_c \Gamma^\rho \epsilon_c = 0) \), the effective propagator results constant and coincides, of course, with (4.3). The fact that special conformal transformations could change the value of the propagator was already noticed in [8] and it is at the very root of the difference between the expectation value of infinite lines and circular loops.

The first order contribution is

\[ W_1(\gamma) = \frac{g^2 N}{8\pi^2} \int_\gamma ds_1 ds_2 \frac{u^a(s_1)u_a(s_2) + u^\mu(s_1)u_\mu(s_2)}{(x_1 - x_2)^2} \equiv \frac{g^2 N}{8\pi^2} \Sigma_2[\gamma]. \] (4.5)
Because the periodicity of our loops is $2\pi n/\Omega_1$, as explained in Section 2, we obtain the same result of the circular loop up the replacement

$$g^2 \to -\frac{B^2 n^2}{\Omega_1^2} g^2, \quad (4.6)$$

where we have taken into account the difference in the constant effective propagator. The next step is to compute the second non-trivial order in the perturbative expansion: at the order $g^4$, the different contributions are not separately finite and we have to introduce the regularization procedure. On the other hand, in the circular case, it was shown in [9] that, using dimensional regularization, divergencies cancel and the remaining finite pieces can be easily evaluated. The same behavior was recognized for generic DGRT loops on $S^2$ [14], where a general and compact expression for the combined one-loop corrected propagators and internal vertices was provided. Here we follow the same strategy: firstly, we consider the effect of the one-loop correction to the effective propagator. The relevant diagrams are schematically displayed in fig. 3 and in the following we shall refer to them as the bubble diagrams.

In Feynman gauge they can be easily computed with the help of [9], where the one-loop correction to the gauge and scalar propagator has been calculated. The final result is (here $D = 2\omega$)

$$S_2 = -g^4(N^2 - 1)\frac{\Gamma^2(\omega - 1)}{2\pi^2(2 - \omega)(2\omega - 3)}\Sigma_{4\omega-6}[\gamma], \quad (4.7)$$

that clearly exhibits a pole at $\omega = 2$. The next step, at this order, is to investigate the so-called spider diagrams, namely the perturbative contributions coming from the gauge vertex $A^3$ and the scalar-gauge vertex $\phi^2 A$ (see fig. 4). We have to compute

$$S_3 = g^3\frac{3}{N} \int_{\gamma} ds_1 ds_2 ds_3 \eta(s_1, s_2, s_3) \langle \text{Tr}[A(s_1)A(s_2)A(s_3)] \rangle_0, \quad (4.8)$$

where

$$\eta(s_1, s_2, s_3) = \theta(s_1 - s_2)\theta(s_2 - s_3) + \text{cyclic permutations}. \quad (4.9)$$

After a simple computation $S_3$ takes the form

$$S_3 = g^4(N^2 - 1)\frac{B^2}{8} \int_{\gamma} ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3)(x_1 - x_3)^2 x_2^\mu \frac{\partial I_1(x_3 - x_1, x_2 - x_1)}{\partial x_3^\mu}, \quad (4.10)$$
where we have introduced the symbol

$$\epsilon(s_1, s_2, s_3) = \eta(s_1, s_2, s_3) - \eta(s_2, s_1, s_3),$$

that is a totally antisymmetric object in the permutations of \((s_1, s_2, s_3)\) and its value is 1 when \(s_1 > s_2 > s_3\). The quantity \(I_1(x, y)\) is defined as the following integral in momentum space

$$I_1(x, y) \equiv \int \frac{d^2\omega}{(2\pi)^4} \frac{e^{ip_1 x + ip_2 y}}{p_1^2 p_2^2 (p_1 + p_2)^2}.$$  \hfill (4.11)

Following closely the same steps in reference [14], we can factor out from (4.10) a contribution that completely cancels the divergent and finite part of the bubble, leaving us with a regular expression proportional to

$$S_2 + S_3 \simeq \oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) \frac{(x_3 - x_2) \cdot \hat{x}_2}{(x_3 - x_2)^2} \log \left[ \frac{(x_2 - x_1)^2}{(x_3 - x_1)^2} \right].$$

that in the parametrization (3.3) turns out to be

$$\oint ds_1 ds_2 ds_3 \epsilon(s_1, s_2, s_3) \frac{\Omega_1 \cos^2 \frac{\theta}{2} \sin \Omega_1 (s_3 - s_2) + \Omega_2 \sin^2 \frac{\theta}{2} \sin \Omega_2 (s_3 - s_2)}{2 \left( 1 - \cos^2 \frac{\theta}{2} \cos \Omega_1 (s_3 - s_2) - \sin^2 \frac{\theta}{2} \cos \Omega_2 (s_3 - s_2) \right)} \times$$

$$\times \log \left[ \frac{(1 - \cos^2 \frac{\theta}{2} \cos \Omega_1 (s_2 - s_1) - \sin^2 \frac{\theta}{2} \cos \Omega_2 (s_2 - s_1))}{(1 - \cos^2 \frac{\theta}{2} \cos \Omega_1 (s_3 - s_1) - \sin^2 \frac{\theta}{2} \cos \Omega_2 (s_3 - s_1))} \right].$$

The integral is potentially complicated, but it actually vanishes because the integrand is antisymmetric in the exchange \(s_2 \leftrightarrow s_3\) while the measure and the integration domains are symmetric. This parallels exactly the circular case.

To get the complete two-loop answer we have still to consider the double-exchange diagrams to the perturbative expansion of the Wilson loop, namely we have to analyze the contribution

$$\frac{g^4}{N} \oint ds_1 ds_2 ds_3 ds_4 \theta(s_1 - s_2) \theta(s_2 - s_3) \theta(s_3 - s_4) \langle \text{Tr}[A(s_1)A(s_2)A(s_3)A(s_4)] \rangle_0.$$  \hfill (4.12)

It is quite clear that, due to the constant character of the effective propagator, we simply recover again the circular result up the rescaling (4.6): we are led therefore to conjecture

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4.png}
\caption{Spider-diagrams: gauge and scalar contribution}
\end{figure}
that the exact quantum expectation value of the Lissajous Wilson loop is simply obtained from the circular Wilson loop, once (4.6) is taken into account

\[ < W(\gamma) > = \frac{1}{N} L^2_{N-1} \left( g^2 \frac{B^2 n^2}{\Omega_1^2} \right) \exp \left[ -\frac{g^2}{2} \frac{B^2 n^2}{\Omega_1^2} \right]. \] (4.13)

The original derivation of the above formula [8, 9] relied mainly on the assumption that interacting contributions vanish to all order in perturbation theory: the resummation of all the exchange diagrams can be easily performed, a job effectively done by a Gaussian matrix-model. The ultimate reason of the correctness of such derivation stands, of course, on the proof, given by Pestun [7], that the full $N = 4$ path-integral computing the 1/2 BPS circular loop reduces to the matrix-model, exploiting a localization procedure. In our case we have not been able to prove a similar result and we can only provide further evidences for the conjecture. Taking the large $N$ limit and defining $\lambda = g^2 N$ we get

\[ \langle W \rangle = \frac{2}{\sqrt{\lambda \delta}} I_1 \left( \sqrt{\lambda \delta} \right) \] (4.14)

where $\delta = -B^2 n^2 / \Omega_1^2$ and $I_1$ is the Bessel function. Due to AdS/CFT correspondence, we expect that as $\lambda$ becomes large this result, if correct, should match a classical string theory solution, i.e we should recover on the string side the behavior

\[ \langle W \rangle \sim \exp \left( \sqrt{\lambda \delta} \right). \] (4.15)

We will see indeed the matching in next section.

**5 Strong coupling: classical solution**

In the following we shall discuss the string theory duals of this family of loops. There is an obvious issue that we have to address before proceeding: the scalar couplings $u^a$ are in general complex, hence the usual interpretation of $u^a$ as a 6-component vector drawing a contour in $S^5$ is apparently lost. A similar situation was considered in [26], where the strong coupling regime of loops lying on a hyperbolic sub-manifold of the space-time was investigated. There, it was suggested that the dual open string does not move in the usual $S^5$ sphere, but in its complexification. In other words, if $Y^a$, with $Y^a Y^a = 1$, are the six flat coordinates spanning the sphere, they must be allowed to assume complex values: $y^i \in \mathbb{C}^6$.

This prescription is not uncommon in the AdS/CFT correspondence: a typical situation arises when considering charged local operators in the Euclidean theory. In this case, studying for example operators like $\text{Tr}[Z^2]$ (the BMN ground state [29]), we have a semiclassical description of these objects, in the Lorentzian theory, in terms of particle trajectories or giant gravitons. Of course in the Euclidean theory there is no time and real propagations
from the boundary of $AdS_5$ into the bulk are not available. It was suggested (see the
discussion in [30]) that considering a tunneling picture or, equivalently, a complexification
of the space, the problem can be resolved and it basically corresponds to Wick-rotate one
of the directions on $S^5$.

We come now to examine our specific problem: determining the classical string solution
dual to the 1/16 BPS Lissajous Wilson loops. To construct it, we have to minimize the
Polyakov action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{h} h^{\alpha\beta} G_{MN}(X) \partial_\alpha X^N \partial_\beta X^M,$$

(5.1)

with the boundary conditions fixed by the Wilson loop. Here $G_{MN}$ is the $AdS_5 \times S^5$
metric, while $h^{\alpha\beta}$ is the world-sheet metric. In what follows we shall use the conformal
gauge and we shall set $\sqrt{h} h^{\alpha\beta} = \delta^{\alpha\beta}$. Since we are dealing with a single loop operator,
the $AdS_5$ and the $S^5$ parts of the sigma model are completely decoupled and they can be
solved separately. In particular the Virasoro constraints of the two sectors must be satisfied
independently.

To begin with, we shall discuss the euclidean $AdS_5$ sector of the $\sigma$–model, following closely
[11, 31] where a general techniques for investigate toroidal loops is presented. Firstly, they
parametrize the $AdS_5$ metric as follows

$$ds^2 = -dr_0^2 + r_0^2 dv^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2$$

(5.2)

where the radial coordinates $r_i$ obey the constraint

$$-r_0^2 + r_1^2 + r_2^2 = -1,$$

(5.3)

to ensure that we are describing euclidean $AdS_5$. Since the boundary conditions are set by
a Lissajous figure of the form

$$\gamma(\tau) : x^\mu(\tau) = \left\{ \cos \frac{\theta}{2} \cos \Omega_1 \tau, \ \cos \frac{\theta}{2} \sin \Omega_1 \tau, \ \sin \frac{\theta}{2} \sin \Omega_2 \tau, \ \sin \frac{\theta}{2} \cos \Omega_2 \tau \right\},$$

(5.4)

a natural ansatz for describing the world-sheet is provided by

$$r_i = r_i(\sigma), \ \ v = v_0 \ (\text{const.}), \ \ \phi_1 = \Omega_2 \tau \ \text{and} \ \ \phi_2 = \Omega_1 \tau.$$ 

(5.5)

Then the reduced action for the remaining dynamical variable can be written as

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma (-r_0^2 + r_1^2 + r_2^2 + \Omega_2^2 r_1^2 + \Omega_1^2 r_2^2 + \Lambda (-r_0^2 + r_1^2 + r_2^2 + 1))$$

(5.6)

where $\Lambda$ is a Lagrange multiplier and the prime denotes the derivative with respect to $\sigma$.
Since we are working in the conformal gauge, this action must be supplemented with the
Virasoro constraint

$$-r_0^2 + r_1^2 + r_2^2 - \Omega_2^2 r_1^2 - \Omega_1^2 r_2^2 = 0.$$ 

(5.7)

\[8\text{We have redefined the solution for mere convenience.}\]
In [11, 31] it was pointed out that the dynamics described by the action (5.6) is integrable and hence one can find a complete set of integrals of motion. For example one can consider

\[ I_0 = r_0^2 - \frac{1}{\Omega_1^2} (r_0 r_1' - r_1 r_0')^2 - \frac{1}{\Omega_2^2} (r_0 r_2' - r_2 r_0')^2 = 1, \]

\[ I_1 = r_1^2 - \frac{1}{\Omega_1^2} (r_0 r_1' - r_1 r_0')^2 + \frac{1}{(\Omega_1^2 - \Omega_2^2)} (r_1 r_2' - r_2 r_1')^2 = 0, \]

(5.8)

together with the Virasoro constraint. We can now change variables from \( r \) to \((\zeta, \sigma)\)

\[ r_0 = \frac{\zeta_1 \zeta_2}{\Omega_1 \Omega_2}, \quad r_1 = \sqrt{\frac{(\zeta_1^2 - \Omega_1^2) (\zeta_2^2 - \Omega_2^2)}{\Omega_1^2 (\Omega_1^2 - \Omega_2^2)}}, \quad r_2 = \sqrt{\frac{(\zeta_1^2 - \Omega_1^2) (\zeta_2^2 - \Omega_2^2)}{\Omega_2^2 (\Omega_1^2 - \Omega_2^2)}} \]

(5.9)

with \( \Omega_2 \leq \zeta_1 \leq \Omega_1 \leq \zeta_2 \). The equation of motion for these new unknowns can be then determined from the integral of motions and they are given by

\[ \zeta_1' = \pm \frac{\zeta_1^2 - \Omega_2^2}{\zeta_1^2 - \Omega_1^2}, \quad \zeta_2' = \pm \frac{\zeta_2^2 - \Omega_2^2}{\zeta_2^2 - \Omega_1^2}. \]

(5.10)

Actually we do not need to solve explicitly the equations of motion to determine the value of the classical action on the solutions. Exploiting again the integral of motions, the action turns out to be

\[
S_{AdS_5} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left( \Omega_2^2 r_1^2 + \Omega_1^2 r_2^2 \right) = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left( \zeta_1' + \zeta_2' \right) = \\
= - \frac{\sqrt{\lambda}}{2\pi} \left( \int_{\Omega_2}^{\Omega_1} \frac{d\zeta_1}{\sqrt{\Omega_2^2 \sin^2 \frac{\zeta_1}{2} + \Omega_1^2 \cos^2 \frac{\zeta_1}{2}}} + \int_{\infty}^{\Omega_1} d\zeta_2 \right) \int d\tau = \\
\simeq - \frac{\sqrt{\lambda}}{2\pi} \left( \Omega_2 + \Omega_1 - \frac{\Omega_2 \Omega_1}{\sqrt{\Omega_2^2 \sin^2 \frac{\theta}{2} + \Omega_1^2 \cos^2 \frac{\theta}{2}}} \right) \int d\tau,
\]

(5.11)

where in the last expression the divergence was removed by hand. The integration domains are determined by (5.9); for \( r_1 \) and \( r_2 \) the integration is from the boundary to the interior of \( AdS \): at the boundary we have \( r_1, r_2 \to \infty \) with \( \frac{r_1}{r_2} \to \tan \frac{\theta}{2} \), while in the interior \( r_1, r_2 \to 0 \). The integration over \( \tau \) is trivial and it simply produces the range of this variable: \( \tau = (0, 2\pi n/\Omega_1) \). Summarizing we have obtained

\[ S_{AdS_5} = - \frac{n}{\Omega_1} \sqrt{\lambda} \left( \Omega_1 + \Omega_2 - \frac{\Omega_1 \Omega_2}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}} \right). \]

(5.12)

Next we consider the \( S^5 \) sector of the \( \sigma \)–model. We again prefer to use euclidean flat coordinates and to write the action as follows

\[
S_{S^5} = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \left[ \partial_a Y^a \partial^a Y^a - \Lambda (Y^a Y^a - 1) \right],
\]

(5.13)
where \( a \) run from 1 to 6 and \( \Lambda \) is a Lagrange multiplier which ensures that the target space is a sphere. In order to simplify the explicit form of the boundary conditions to be imposed, we introduce a new set of coordinates defined by

\[
y_0 = b^a Y_a, \quad y_5 = b^a_0 Y_a \quad \text{and} \quad y_\mu = m^a_\mu Y^a, \tag{5.14}
\]

where \( b^a \) is the normalized vector \( B^a / \sqrt{(B\cdot B)} \), \( m^a_\mu \) are the four orthonormal vectors given by \( m^a_\mu = M^a_\mu / \sqrt{(M\cdot M)} \) and finally \( b^a_0 \) is a sixth vector of unit norm and orthogonal to the previous ones. At the level of the gauge theory, this corresponds to a (complex) redefinition of the scalars such that the matrix \( M^a_\mu \) of couplings is diagonal.

Since the above transformation is an element of \( SO(6, \mathbb{C}) \), the action (5.13) is substantially unchanged. The original normalized vector \( u^a \) of scalar couplings takes the following simpler form

\[
\Theta_0 = b^a u_a = \frac{\sqrt{-B\cdot B}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}, \quad \Theta_5 = b^a_0 u_a = 0,
\]

\[
\Theta_4 + i\Theta_3 = m^a_4 u_a + i m^a_3 u_a = -i \sin \frac{\theta}{2} \sqrt{-B\cdot B - \Omega_2^2} e^{i \Omega_2 \tau} \frac{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}},
\]

\[
\Theta_1 + i\Theta_2 = m^a_1 u_a + i m^a_2 u_a = -i \cos \frac{\theta}{2} \sqrt{-B\cdot B - \Omega_1^2} e^{i \Omega_1 \tau} \frac{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}},
\]

in this new basis. Next we perform a further change of variables and instead of the four \( y_\mu \) we introduce the toroidal coordinates

\[
r_1 e^{i \phi_1} = y_4 + iy_3 \quad \text{and} \quad r_2 e^{i \phi_2} = y_1 + iy_2. \tag{5.16}
\]

In term of these coordinates, the \( \sigma \)-model action can be cast into the form

\[
S_{S^5} = \frac{\sqrt{\Lambda}}{4\pi} \int d^2 \sigma \left[ \partial_\sigma y_0 \partial^\sigma y_0 + \partial_\sigma y_1 \partial^\sigma y_1 + \partial_\sigma r_1 \partial^\sigma r_1 + r_1^2 \partial_\sigma \phi_1 \partial^\sigma \phi_1 + \partial_\sigma r_2 \partial^\sigma r_2 + r_2^2 \partial_\sigma \phi_2 \partial^\sigma \phi_2 - \Lambda ((y_0)^2 + (y_5)^2 + r_1^2 + r_2^2 - 1) \right]. \tag{5.17}
\]

The boundary conditions for the \( S^5 \) sector are then simply given by\(^9\)

\[
y_0 = \frac{\sqrt{-B\cdot B}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}, \quad y_5 = 0,
\]

\[
r_1 e^{i \phi_1} = -i \frac{\sin \frac{\theta}{2} \sqrt{-B\cdot B - \Omega_2^2} e^{i \Omega_2 \tau}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}, \quad r_2 e^{i \phi_2} = -i \frac{\cos \frac{\theta}{2} \sqrt{-B\cdot B - \Omega_1^2} e^{i \Omega_1 \tau}}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}}. \tag{5.18}
\]

The above equality are meant to hold when \( \sigma \) reaches its boundary value. Since we prefer to have boundary conditions, which are (formally) real, we perform another Wick-rotation:

\(^9\)We choose \( \sqrt{-B\cdot B} \) and \( \sqrt{-B\cdot B - \Omega_{1,2}^2} \) to be real. This choice will simplify our treatment.
$y_k \mapsto iy_k$ for $k = 1, \ldots, 5$. After these procedure $S^5$ becomes euclidean $AdS_5$ and we can choose the following ansatz for the solutions

$$y_0 = 0, \quad y_5 = 0 \quad r_i = r_i(\sigma) \quad i = 1, 2 \quad \phi_1 = \Omega_2 \tau \quad \phi_2 = \Omega_1 \tau.$$ (5.19)

With this choice, apart from an overall sign in the action and for the boundary conditions, we are left with the same problem encountered in the $AdS_5$ sector. Thus we can follow the same procedure and introduce the new coordinates $(\hat{\zeta}_1, \hat{\zeta}_2)$

$$y_0 = \frac{\hat{\zeta}_1 \hat{\zeta}_2}{\Omega_1 \Omega_2}, \quad r_1 = \sqrt{\frac{(\hat{\zeta}_1^2 - \Omega_2^2) (\hat{\zeta}_2^2 - \Omega_1^2)}{\Omega_2^2 (\Omega_1^2 - \Omega_2^2)}} \quad r_2 = \sqrt{\frac{(\Omega_1^2 - \hat{\zeta}_1^2) (\hat{\zeta}_2^2 - \Omega_1^2)}{\Omega_1^2 (\Omega_1^2 - \Omega_2^2)}}$$ (5.20)

with $\Omega_2 \leq \hat{\zeta}_1 \leq \Omega_1 \leq \hat{\zeta}_2$. The action has again the simple form

$$S = \frac{n}{\Omega_1} \sqrt{\lambda} \int d\sigma \left( \hat{\zeta}_1' + \hat{\zeta}_2' \right).$$ (5.21)

The integration is from the boundary values to $r_1, r_2 = 0$. For the latter we have $\hat{\zeta}_1 = \Omega_2$ and $\hat{\zeta}_2 = \Omega_1$. The boundary values at infinity are trickier to obtain, but after some algebraic manipulations, the boundary condition on $r_i$ translates into

$$\hat{\zeta}_1^2 = -B \cdot B \quad \hat{\zeta}_1 = \frac{\Omega_1^2 \Omega_2^2}{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}} \quad \hat{\zeta}_2^2 = \frac{\Omega_1^2 \Omega_2^2}{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}$$ (5.22)

or

$$\hat{\zeta}_2^2 = \frac{\Omega_1^2 \Omega_2^2}{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}} \quad \hat{\zeta}_2 = -B \cdot B.$$ (5.23)

The choice between the two possibilities depends on the values of the parameters $B, \Omega_1, \Omega_2, \theta$, but this does not affect the result for the action:

$$S_{S^5} = \frac{n}{\Omega_1} \sqrt{\lambda} \left( \Omega_1 + \Omega_2 - \sqrt{-B \cdot B} - \frac{\Omega_1 \Omega_2}{\sqrt{\Omega_1^2 \cos^2 \frac{\theta}{2} + \Omega_2^2 \sin^2 \frac{\theta}{2}}} \right).$$ (5.24)

The total result is obtained by adding the two contributions: $S_{AdS_5} + S_{S^5}$. We have

$$S = -\frac{n}{\Omega_1} \sqrt{(-B \cdot B) \lambda}.$$ (5.25)

As anticipated in the previous Section, we have been able to reproduce by AdS/CFT correspondence the strong-coupling result conjectured from the perturbative computation: we notice that the expression suggested from the weak coupling expansion originates from a precise cancellation between the $AdS_5$ and the $S^5$ sectors of the $\sigma$-model. Another remark concerns the subtractions we have done by hands to get a finite action: the very same result would be obtained by applying the Legendre transformation procedure, proposed in [8], that is generally considered the correct one. We feel therefore quite confident that an exact localization underlies our results.
Acknowledgements

This work was supported in part by the MIUR-PRIN contract 2009-KHZKRX. We are pleased to thank Marco Bertolini and Fabrizio Pucci for participating at the early stage of the project. We also thank Antonio Bassetto and Nadav Drukker for useful discussions.
Appendices

A General framework and Conventions

Dirac Algebra and spinors in D=10: The Euclidean Dirac algebra in ten dimensions is defined by the anti-commutation rules

\[ \gamma^M \gamma^N + \gamma^N \gamma^M = 2 \delta^{MN} \mathbb{1}, \quad (A.1) \]

where the \( \gamma^M \) are 32 \( \times \) 32 matrices. We shall use the Weyl representation, where the chiral operator, \( \gamma^{11} = -i \gamma^1 \gamma^2 \cdots \gamma^{10} \), is diagonal and it is given by

\[ \gamma^{11} = \begin{pmatrix} \mathbb{1}_{16 \times 16} & 0 \\ 0 & -\mathbb{1}_{16 \times 16} \end{pmatrix}. \quad (A.2) \]

With this choice we can write the Dirac matrices \( \gamma^M \) in block form

\[ \gamma^M = \begin{pmatrix} 0 \ \tilde{\Gamma}^M \\ \Gamma^M \ 0 \end{pmatrix}. \quad (A.3) \]

The non-vanishing blocks are related by hermitian conjugation \( \tilde{\Gamma}^M = (\Gamma^M)^\dagger \) and they obey the chiral algebra

\[ \Gamma^M \tilde{\Gamma}^N + \tilde{\Gamma}^N \Gamma^M = 2 \eta^{MN} \mathbb{1} \quad \text{and} \quad \tilde{\Gamma}^M \Gamma^N + \Gamma^N \tilde{\Gamma}^M = 2 \eta^{MN} \mathbb{1}. \quad (A.4) \]

In addition they can be taken symmetric, i.e. \( (\Gamma^M)^t = \Gamma^M \) and \( (\tilde{\Gamma}^M)^t = \tilde{\Gamma}^M \). Differently from what occurs in the case of Lorentz signature we cannot choose the above blocks to be also real, we can only impose

\[ \begin{cases} (\Gamma^M)^* = (\Gamma^M)^t & \text{for } M \neq 10 \\ (\tilde{\Gamma}^{10})^* = -\tilde{\Gamma}^{10} \\ (\tilde{\Gamma}^{10})^* = -\tilde{\Gamma}^{10} \end{cases}. \quad (A.5) \]

The reality condition (A.5) combined with the previous requirements will also imply that \( \tilde{\Gamma}^M = \Gamma^M \) for \( M \neq 10 \) and \( \tilde{\Gamma}^{10} = -\Gamma^{10} \).

An explicit representation for these blocks can be constructed by identifying the \( \Gamma^M \) with the real representation of the Euclidean Clifford algebra in 9 dimensions for \( M \neq 10 \) and by posing \( \Gamma^{10} = i \mathbb{1} \).

Finally the matrices \( \Gamma^M \) and \( \tilde{\Gamma}^M \) obey two important Fierz identities which play a key role in any computation involving ten dimensional supersymmetry

\[ (\Gamma^M)_{\alpha_1 \alpha_2} (\Gamma_M)_{\alpha_3 \alpha_4} = 0 \quad \text{and} \quad (\tilde{\Gamma}^M)_{\alpha_1 \alpha_2} (\tilde{\Gamma}_M)_{\alpha_3 \alpha_4} = 0 \quad (A.6) \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, \ldots, 16 \) are the matrix indices of \( \Gamma^M \) and \( \tilde{\Gamma}^M \).
In this representation the 32-component Dirac spinor $\psi$ naturally splits into two Weyl spinors of opposite chirality with 16 components each

$$
\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
$$

(A.7)

The blocks $\Gamma^M$ and $\tilde{\Gamma}^M$ will act on $\psi_+$ ($\Gamma^M \psi_+$) and $\psi_-$ ($\tilde{\Gamma}^M \psi_-$) respectively by reversing their chirality. Since $\Gamma^M$ and $\tilde{\Gamma}^M$ coincide for $M \neq 10$, we shall not distinguish between them when it is not necessary and for instance we shall sometimes write $\Gamma^\mu \psi_-$ (with $\mu = 1, \ldots, 4$) instead of $\tilde{\Gamma}^\mu \psi_-$. It is also convenient to introduce the matrices

$$
\Gamma^{MN} = \frac{1}{2}(\tilde{\Gamma}^M \Gamma^N - \tilde{\Gamma}^N \Gamma^M) \quad \text{and} \quad \tilde{\Gamma}^{MN} = \frac{1}{2}(\Gamma^M \tilde{\Gamma}^N - \Gamma^N \tilde{\Gamma}^M),
$$

(A.8)

which are proportional to the generators of the two irreducible chiral representations of the rotation group and preserve chirality.

In euclidean space there is no need for complex conjugation to write down a fermionic invariant bilinear. In the 32-component notation we can in fact introduce the complex contraction $\psi C^{-1} \psi$ where $C = -i\gamma^{10}$ is the charge conjugation matrix $[C^{-1} \gamma^M C = -\gamma^M]^\dagger$.

In the Weyl representation this combination can be equivalently written in terms of the 16-component Weyl spinors as follows

$$
\tilde{\psi} \psi = 2 \psi_+ \psi_-.
$$

(A.9)

This unusual choice for fermionic bilinears is important when considering supersymmetric theories in euclidean space since it avoids the introduction of $\psi^\dagger$ in the construction of an invariant action.

**Euclidean $\mathcal{N}=4$ SYM:** Let us discuss the euclidean version of $\mathcal{N} = 4$ SYM on a flat $d = 4$ space-time, with gauge group $G$. As we have extensively done in this paper, the theory can be viewed as the dimensional reduction of $\mathcal{N} = 1$ SYM in $d = 10$ and its action takes the following compact form,

$$
S = -\frac{1}{2g_{YM}^2} \int d^4x \text{Tr} \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi \right) \quad N, M = 1...10,
$$

(A.10)

when using the ten-dimensional notation \(^\text{10}\). The matrices $\Gamma^M$ have been introduced in the previous subsection. In eq. (A.10) all fields take value in the Lie algebra of $G$ while the covariant derivative and the field strength are given by

$$
D_M \equiv \partial_M + A_M \quad F_{MN} \equiv [D_M, D_N].
$$

Denoting the space-time directions with greek indices $\mu, \nu = 1...4$ and the remaining ones with $A, B = 5...10$, the bosonic part of the familiar $\mathcal{N} = 4$ Lagrangian emerges by expanding (A.10) in terms of $(A_\mu, \Phi_A) \equiv A_M$.

\(^\text{10}\)In the following we shall closely follow the convention of [20].
The superconformal transformations, which leave invariant the action of $\mathcal{N} = 4$ SYM, in ten dimensional notation are given by

$$
\delta A_M = \epsilon (x) \Gamma_M \Psi, \quad \delta \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon (x) - 2 \Phi_A \tilde{\Gamma}^A \epsilon_c.
$$

(A.11)

The parameter $\epsilon (x)$ is a conformal Killing spinor in flat space, i.e.

$$
\epsilon (x) = \epsilon_s + x^\mu \Gamma^\mu \epsilon_c,
$$

(A.12)

where $\epsilon_s$ and $\epsilon_c$ are two constant spinors. In particular $\epsilon_s$ has positive (ten-dimensional) chirality and it generates the usual 16 Poincaré supersymmetries, while $\epsilon_c$ is anti-chiral and yields the remaining 16 fermionic superconformal symmetries.

In the subsequent appendices shall use the above chiral notation ($\Gamma^M, \tilde{\Gamma}^M$), while in the main text we have adopted for simplicity the standard $32 \times 32$ Dirac matrices $\gamma^M$.

B The classification of the impure Wilson loops

In this appendix we give a general classification of the possible impure Wilson loops, constructing explicitly both the contours and the scalar couplings. In doing so, we heavily rely on the properties of the Killing spinors associated to the loops and on the well-known six-dimensional embedding of the conformal group. Our strategy is to reduce the classification to independent representatives up to conformal transformations and to solve properly the basic constraints in the different cases. The general situation is reached by ”conformally boosting” the relevant contours and couplings, using the appropriate conformal transformations described in appendix C.

B.1 General Properties

We start the construction of an impure Wilson loop $\mathcal{W}(\gamma)$ with assigning a conformal Killing spinor,

$$
\epsilon = \epsilon_s + x^\mu \Gamma^\mu \epsilon_c,
$$

(B.1)

for which the vector $\epsilon \Gamma^M \epsilon \neq 0$ does not vanish and in particular its space-time components, $\epsilon \Gamma^\mu \epsilon$, define a real four-vector with unit norm\(^{11}\). Then the contour $\gamma$ is obtained by solving the differential equation

$$
\dot{x}^\mu = \epsilon \Gamma^\mu \epsilon = \epsilon_s \Gamma^\mu \epsilon_s + 2 \epsilon_s \epsilon_c x^\mu + 2 \epsilon_s \tilde{\Gamma}^\mu \epsilon_c x^\nu + 2 x^\mu x^\nu \epsilon_c \tilde{\Gamma}_\nu \epsilon_c - x^2 \epsilon_c \tilde{\Gamma}^\mu \epsilon_c =
$$

$$
= a^\mu + \lambda x^\mu + \Omega^\mu x^\nu + 2 x^\mu (b \cdot x) - x^2 b^\mu.
$$

(B.2)

On the r.h.s of (B.2) we recognize an infinitesimal generator of the conformal algebra, which is a combination of a translation ($a^\mu \equiv \epsilon_s \Gamma^\mu \epsilon_s$), a dilation ($\lambda \equiv 2 \epsilon_s \epsilon_c$), a rotation

\(^{11}\)We can weaken this condition and require that $\epsilon \Gamma^\mu \epsilon$ is proportional to a real vector, since $\epsilon$ is defined up to the scaling $\epsilon \mapsto \lambda \epsilon$ with $\lambda \in \mathbb{C}^*$
\((\Omega^\mu_\nu \equiv 2\epsilon_a \tilde{\Gamma}^\mu_\nu \epsilon_c)\) and a special conformal transformation \((b^\mu \equiv \epsilon_c \tilde{\Gamma}^\mu \epsilon_c)\). In other words, as we have anticipated in the main text, \(x^\mu(s)\) is an orbit of the conformal group.

Its explicit form can be easily constructed in terms of the solutions of the six-dimensional linear system

\[
\dot{Y}^m(s) = W^m_n Y^n(s) \quad (m, n = 1, \ldots, 6),
\]

where constant matrix \(W\) realizes the canonical embedding of the (euclidean) conformal transformation in the r.h.s of (B.2) into the algebra of \(SO(5,1)\), i.e.

\[
W \equiv \begin{pmatrix}
0 & \Omega^1_2 & \Omega^1_3 & \Omega^1_4 & a^1 + b^1 & a^1 - b^1 \\
-\Omega^1_2 & 0 & \Omega^2_3 & \Omega^2_4 & a^2 + b^2 & a^2 - b^2 \\
-\Omega^1_3 & -\Omega^2_3 & 0 & \Omega^3_4 & a^3 + b^3 & a^3 - b^3 \\
-\Omega^1_4 & -\Omega^2_4 & -\Omega^3_4 & 0 & a^4 + b^4 & a^4 - b^4 \\
-a^1 - b^1 & -a^2 - b^2 & -a^3 - b^3 & -a^4 - b^4 & 0 & -\lambda \\
a^1 - b^1 & a^2 - b^2 & a^3 - b^3 & a^4 - b^4 & -\lambda & 0
\end{pmatrix}.
\]

In fact we can write that

\[
x^\mu(s) = \frac{Y^\mu(s)}{Y^5(s) + Y^6(s)} \quad (\mu = 1, \ldots, 4),
\]

where \(Y^m(s)\) is subject to the initial condition \(Y^m(0) = y_0^m\) with \((y_0 \cdot y_0) = 0\).

The scalar couplings of the \(impure\) Wilson loops are instead parametrized, in general, by two constant six-dimensional vectors \(B^a_0\) and \(B^a_1\) and a \(6 \times 4\) rectangular matrix \(M^a_\mu\), the vector \(u^a = \epsilon \Gamma^a \epsilon\) (with \(a = 1, \ldots, 6\)) in (2.4) being naturally arranged as follows

\[
u^a = \epsilon \Gamma^a \epsilon = (\epsilon_s \Gamma^a \epsilon_s - x^2 \epsilon_s \tilde{\Gamma}^a \epsilon_c) + 2 \epsilon_s \Gamma^a \tilde{\Gamma}_\nu \epsilon_c x^\nu \equiv B^a_0 - x^2 B^a_1 + M^a_\mu x^\mu.
\]

Remarkably, the coefficients \(B^a_0\), \(B^a_1\) and \(M^a_\mu\) in (B.6) are not free quantities but they are subject to some constraints which keep track of their \textit{spinorial} origin. For example the Fierz identity

\[
2(\epsilon_s \Gamma_M \epsilon_s)(\epsilon_s \Gamma^M \tilde{\Gamma}_\nu \epsilon_c) = 0
\]

translates into

\[
a_\nu \Omega^\nu_\mu + \lambda a_\mu + B^a_0 M^a_\mu = 0
\]

and similarly we can also show

\[
b_\nu \Omega^\nu_\mu + B^a_1 M^a_\mu - \lambda b_\mu = 0,
\]

\[
(a \cdot a) + (B_0 \cdot B_0) = (b \cdot b) + (B_1 \cdot B_1) = 0,
\]

\[
M^a_\mu M^a_\nu + (\lambda \delta^a_\mu + \Omega^a_\mu)(\lambda \delta^a_\nu + \Omega^a_\nu) + 2(a_\mu b_\nu + a_\nu b_\mu) = 2 \delta_{\mu\nu} [(a \cdot b) + (B_0 \cdot B_1)].
\]

The converse is also true: \textit{i.e.} if the couplings \(M^a_\mu\), \(B^a_0\) and \(B^a_1\) obey the constraints (B.8) and (B.9), they define a supersymmetric Wilson loop.

In Subsec. B.3 we shall discuss in detail the possible solutions of the above set of constraints.
B.2 Classification of the possible orbits

The circuits for the impure Wilson loops are provided by orbits of the conformal group: we shall write down all the relevant contours modulo conformal equivalence, i.e. we shall identify loops which differ by a conformal transformation. It is convenient to rephrase this problem in six dimensional language and to list all the possible forms of the matrix (B.4) up to the adjoint action of an element of SO(5,1). This classification is achieved by separating the matrices $W$ into two classes according to the value of their determinant.

$\det(W) \neq 0$:

In this case the kernel of $W$ is trivial and the matrix possesses $6$ eigenvalues different from zero which can be generically paired in three couples:

$$ (i\Omega_1, -i\Omega_1), \quad (i\Omega_2, -i\Omega_2) \quad \text{and} \quad (\rho, -\rho), $$

with $\Omega_1, 2$ and $\rho$ real numbers. The eigenvectors associated to $\rho$ and $-\rho$ are two real independent light-like vectors with respect to SO(5,1) invariant metric $\eta = \text{diag}(1,1,1,1,1,-1)$.

The linear space orthogonal to these eigenvectors is an invariant subspace and on such subspace $W$ is an anti-hermitian matrix defining the generator of an SO(4) rotation.

Thus the matrix (B.4) can be always cast in the following canonical form

$$ W_C \equiv \begin{pmatrix} 0 & \Omega_1 & 0 & 0 & 0 & 0 \\ -\Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_2 & 0 & 0 & 0 \\ 0 & 0 & -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho \end{pmatrix}, $$

(B.11)

up to an SO(5,1) transformation. The corresponding contour is then given by

$$ x^1 = e^{\rho s} \cos \theta_0 \cos(\Omega_1 s) \quad x^3 = e^{\rho s} \sin \theta_0 \cos(\Omega_2 s) $$

$$ x^2 = e^{\rho s} \cos \theta_0 \sin(\Omega_1 s) \quad x^4 = e^{\rho s} \sin \theta_0 \sin(\Omega_2 s) $$

(B.12)

For $\rho > 0$ it is an infinite open path that starts from the origin $(x^\mu = 0)$ at $s = -\infty$ and it reaches infinity when $s = \infty$. It is obtained by composing two planar motions on orthogonal planes. The motion on each plane is the well-known logarithmic spiral.

$\det(W) = 0$:

In this case the kernel of $W$ is not empty and we have three different possibilities.

(A) The kernel of $W$ includes a time-like vector: $W$ defines a rotation and up to an
$SO(5,1)$ transformation the matrix (B.4) takes the form

$$W_C \equiv \begin{pmatrix}
0 & \Omega_1 & 0 & 0 & 0 & 0 \\
-\Omega_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Omega_2 & 0 & 0 \\
0 & 0 & -\Omega_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{B.13}$$

The contour is simply given by

$$\begin{align*}
x^1 &= \cos \theta_0 \cos(\Omega_1 s) \\
x^2 &= \cos \theta_0 \sin(\Omega_1 s) \\
x^3 &= \sin \theta_0 \cos(\Omega_2 s) \\
x^4 &= \sin \theta_0 \sin(\Omega_2 s),
\end{align*} \tag{B.14}$$

and it lies on a sphere $S^3$. It is generically an open orbit but it closes if the ratio $\Omega_1 / \Omega_2 \in \mathbb{Q}$ (Lissajous figures). This class of contours are discussed in details in the main text.

(B) The kernel of $W$ contains a light-like vector $v_\ell$: up to an $SO(5,1)$ we can always choose $v_\ell = (0,0,0,0,-1,1)$ and the matrix $W$ can be rearranged as follows

$$W_C \equiv \begin{pmatrix}
0 & \Omega_1 & 0 & 0 & a_1 & a_1 \\
-\Omega_1 & 0 & 0 & 0 & a_2 & a_2 \\
0 & 0 & 0 & \Omega_2 & a_3 & a_3 \\
0 & 0 & -\Omega_2 & 0 & a_4 & a_4 \\
-a_1 & -a_2 & -a_3 & -a_4 & 0 & 0 \\
a_1 & a_2 & a_3 & a_4 & 0 & 0
\end{pmatrix}, \tag{B.15}$$

where we have used the residual $SO(4)$ invariance of $v_\ell$ to put $\Omega^\mu_{\,\nu}$ in its canonical form. If $\Omega_1$ and $\Omega_2$ are different from zero, we can further set $a_i = 0$ by means of the adjoint action of a translation, i.e. we have again obtained the canonical form (B.13).

If only one of the $\Omega_i$ vanishes, e.g. $\Omega_2 = 0$, we can always reduce $W$ to

$$W_C \equiv \begin{pmatrix}
0 & \Omega_1 & 0 & 0 & 0 & 0 \\
-\Omega_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a \ a \\
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0
\end{pmatrix}, \tag{B.16}$$

by means of a translation in the plane $(1,2)$ and of a rotation in the plane $(3,4)$. Thus the corresponding contour is an helix whose parametric equation is

$$\begin{align*}
x^1 &= \cos \theta_0 \cos(\Omega_1 s) \\
x^2 &= \cos \theta_0 \sin(\Omega_1 s) \\
x^3 &= x_0^3 \\
x^4 &= at.
\end{align*} \tag{B.17}$$

If both the $\Omega_i$ vanish, $W$ generates a translation and its canonical form is given by (B.16) for $\Omega_1 = 0$. The path is then given by a straight-line parallel to the coordinate axis $x^4$. 

– 28 –
The kernel consists only of space-like vectors: we can easily show that its dimensions is at least 2 and thus \( W \) reduces to a generator of \( SO(n, 1) \) (with \( n = 1, 3 \)) on the sub-space orthogonal to the kernel.

For \( n = 1 \) we have a pure dilation, namely we obtain (B.11) with \( \Omega_1 = \Omega_2 = 0 \). The contour is again a straight-line given by

\[
x^\mu = x_0^\mu e^{\lambda t}.
\]

(B.18)

Instead for \( n = 3 \) we have a dilation and a planar rotation, i.e. (B.11) with \( \Omega_1 = 0 \). The contour is obtained from (B.12) by posing \( \Omega_1 = 0 \).

### B.3 Classification of the couplings

We investigate here the structure of the scalar couplings for the orbits classified in Subsec.B.2. More precisely, we shall show how to solve the constraints (B.8) and (B.9) and to parameterize the different solutions.

\( \text{det}(W) = 0: \)

**A** We consider first the case when \( W_C \) is a pure rotation. Since \( a^\mu \) and \( b^\mu \) vanish, \( B_0^a \) and \( B_1^a \) are 6-component light-like complex vectors because of (B.9b). Let us assume that \( (B_0 \cdot B_1) \neq 0 \), then, for instance, \( B_0^a \) can be put in the following canonical form\(^\text{12}\)

\[
B_0^a = p_1(0, 0, 0, 0, 1, i),
\]

(B.19)

up to an \( SO(6) \) rotation. The light-like vector \( B_1^a \) can be instead parameterized as follows

\[
B_1^a = \left( \hat{B}_1^1, \hat{B}_1^2, \hat{B}_1^3, \hat{B}_1^4, \frac{1}{2} \left[ m - \frac{\hat{B}_1^1 \hat{B}_1^1}{m} \right], \frac{i}{2} \left[ m + \frac{\hat{B}_1^1 \hat{B}_1^1}{m} \right] \right).
\]

(B.20)

The special form of \( B_0 \) reduces the condition (B.8) to

\[
M_5^\mu + i M_6^\mu = 0,
\]

(B.21)

which is solved by setting

\[
M_5^\mu = p_1 q_\mu \quad \text{and} \quad M_6^\mu = i p_1 q_\mu,
\]

(B.22)

where \( q \) is a four-vector fixed by the condition (B.9a). We get

\[
q_\mu = -\frac{1}{(B_0 \cdot B_1)} \sum_{i=1}^4 \hat{B}_1^i \hat{M}_i^\mu,
\]

(B.23)

\(^{12}\text{A non-vanishing light-like vector with respect to the euclidean metric is complex. Its real and imaginary part defines two real orthogonal vectors of equal norm. The form (B.19) is a trivial consequence of these properties.}\)
where \( \hat{M}_i^\mu \) is a \( 4 \times 4 \) matrix obtained from \( M^a_\mu \) by erasing the last two rows. We remark that \( \rho_\mu(B_i^1 \hat{B}_i^1) = -(B_0^0 B_1^1) \). Finally, we have to consider the quadratic condition (B.9c)

\[
\sum_{a=1}^{4} \hat{M}_i^\mu \hat{M}_i^\nu + \Omega^\mu_\alpha \Omega^\nu_\alpha = 2\delta_{\mu\nu}(B_0 \cdot B_1),
\]

which in turn implies

\[
\sum_{a=1}^{4} \hat{M}_i^\mu \hat{M}_i^\nu = \begin{pmatrix}
2(B_0 \cdot B_1) - \Omega_1^2 & 0 & 0 & 0 \\
0 & 2(B_0 \cdot B_1) - \Omega_2^2 & 0 & 0 \\
0 & 0 & 2(B_0 \cdot B_1) - \Omega_2^2 & 0 \\
0 & 0 & 0 & 2(B_0 \cdot B_1) - \Omega_2^2
\end{pmatrix}.
\]

(B.25)

The general solution of (B.25) is provided by

\[
\hat{M}_i^\mu = S \begin{pmatrix}
a_1 & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 \\
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & a_2
\end{pmatrix},
\]

(B.26)

where \( S \) is a matrix of \( SO(4,C) \) and \( a_2^2 = 2(B_0 \cdot B_1) - \Omega_1^2 \). Equivalently we can say that the columns of \( \hat{M}_i^\mu \) define four complex orthogonal vectors whose norms are given by the diagonal element in the r.h.s. of eq. (B.25).

An apparent singular point in our analysis occurs when either \( \Omega_1^2 \) or \( \Omega_2^2 \) are equal to \( 2(B_0 \cdot B_1) \). Let us consider, for instance, the case \( \Omega_1^2 = 2(B_0 \cdot B_1) \): the first two columns of the reduced matrix \( \hat{M}_i^\mu \) are two light-like orthogonal complex vectors and up to an \( SO(4) \) rotation we can set

\[
\hat{M}_1^i = (im_1, m_1, 0, 0).
\]

(B.27)

In turn the vector \( \hat{M}_2^i \) can be taken of the form

\[
\hat{M}_1^i = (im_2, m_2, im_2, n_2).
\]

(B.28)

and the remaining two columns can be parameterized as follows

\[
\hat{M}_3^i = (im_3, m_3, \sqrt{\Omega_1^2 - \Omega_2^2} \cos \alpha, \sqrt{\Omega_1^2 - \Omega_2^2} \sin \alpha)
\]

\[
\hat{M}_4^i = (im_4, m_4, \sqrt{\Omega_1^2 - \Omega_2^2} \cos \beta, \sqrt{\Omega_1^2 - \Omega_2^2} \sin \beta).
\]

(B.29)

If \( \Omega_1^2 \neq \Omega_2^2 \), the orthogonality between \( \hat{M}_2^i \) and \( \hat{M}_{3,4}^i \) implies \( n_2 = 0 \), while \( (\hat{M}_3 \cdot \hat{M}_4) = 0 \) is equivalent to \( \beta - \alpha = \frac{\pi}{2} \). We end up with the following matrix

\[
\hat{M}_i^\mu = \begin{pmatrix}
im_1 & im_2 & im_3 & im_4 \\
m_1 & m_2 & m_3 & m_4 \\
0 & 0 & \sqrt{\Omega_1^2 - \Omega_2^2} \cos \alpha & -\sqrt{\Omega_1^2 - \Omega_2^2} \sin \alpha \\
0 & 0 & \sqrt{\Omega_1^2 - \Omega_2^2} \sin \alpha & \sqrt{\Omega_1^2 - \Omega_2^2} \cos \alpha
\end{pmatrix},
\]

(B.30)
up to an \( SO(4) \) rotation, and the four-component vectors \( \hat{M}_1^i \) and \( \hat{M}_2^i \) are not only light-like but also parallel. The same property is actually shared by the complete first two columns of \( M^a_\mu \) : by means of (B.23), we can easily check that \( M_1^a = m_1 V^a \) and \( M_2^a = m_2 V^a \), where

\[
V^a = \left( i, 1, 0, 0, -\frac{p_1}{(B_0 \cdot B_1)}(i\hat{B}_1^1 + \hat{B}_1^3), -\frac{ip_1}{(B_0 \cdot B_1)}(i\hat{B}_1^1 + \hat{B}_1^3) \right). \tag{B.31}
\]

The remaining two columns of the matrix \( M^a_\mu \) can be instead reorganized as follows

\[
M_3^a = m_3 V^a + \sqrt{\Omega_1^2 - \Omega_2^2} \left( 0, 0, \cos \alpha, \sin \alpha, -\frac{p_1(\hat{B}_1^1 \cos \alpha + \hat{B}_1^3 \sin \alpha)}{(B_0 \cdot B_1)}, -\frac{ip_1(\hat{B}_1^1 \cos \alpha + \hat{B}_1^3 \sin \alpha)}{(B_0 \cdot B_1)} \right) = m_3 V^a + S_3^a, \tag{B.32a}
\]

\[
M_4^a = m_4 V^a + \sqrt{\Omega_1^2 - \Omega_2^2} \left( 0, 0, -\sin \alpha, \cos \alpha, -\frac{p_1(\hat{B}_1^1 \cos \alpha - \hat{B}_1^3 \sin \alpha)}{(B_0 \cdot B_1)}, -\frac{ip_1(\hat{B}_1^1 \cos \alpha - \hat{B}_1^3 \sin \alpha)}{(B_0 \cdot B_1)} \right) = m_4 V^a + S_4^a, \tag{B.32b}
\]

where \( S_3 \) and \( S_4 \) are orthogonal to \( V \).

If \( \Omega_1^2 = \Omega_2^2 = 2(B_0 \cdot B_1) \) we find, instead, the following matrix

\[
\hat{M}_i^\mu = \begin{pmatrix}
im_1 & im_2 & im_3 & im_4 \\
m_1 & m_2 & m_3 & m_4 \\
0 & in_2 & in_3 & in_4 \\
0 & n_2 & n_3 & n_4
\end{pmatrix}. \tag{B.33}
\]

When \( (B_0 \cdot B_1) = 0 \), the above analysis is not substantially altered if either \( B_0^a \) or \( B_1^a \) do not vanish. For example, if we consider again the case \( B_0^a \neq 0 \) (with \( \Omega_1 \neq 0 \) and \( \Omega_2 \neq 0 \)), the four vectors \( M_i^\mu \) are still orthogonal and the condition (B.9a) implies

\[
\sum_{i=1}^{4} \hat{B}_i^i \hat{M}_i^\mu = 0 \quad \Rightarrow \quad \hat{B}_1^i = 0 : \tag{B.34}
\]

in other words, the vector \( B_1^a \) is parallel to \( B_0^a \). In this case, the vector \( q_\mu \) is undetermined.

If \( B_0^a = B_1^a = 0 \) the only surviving constraint is

\[
M_\nu^a M_\nu^a = -\begin{pmatrix}
\Omega_1^2 & 0 & 0 & 0 \\
0 & \Omega_1^2 & 0 & 0 \\
0 & 0 & \Omega_2^2 & 0 \\
0 & 0 & 0 & \Omega_2^2
\end{pmatrix}, \tag{B.35}
\]

which simply states that the columns of \( M^a_\nu \) are six-component complex orthogonal vectors. This exhausts all the possibilities contained in the case (A).

\( \text{(B)} \) We have to analyze the canonical form (B.16) for the matrix \( W \). The vector \( B_1^i \) is still light-like and complex and, if different from the null vector, it can be chosen to be

\[
B_1^a = p_1(0, 0, 0, 1, i), \tag{B.36}
\]
up to an $SO(6)$ rotation. The condition (B.9a) again implies

$$M_\mu^5 + iM_\mu^6 = 0,$$  \hspace{1cm} (B.37)

which is solved by setting

$$M_\mu^5 = p_1 q_\mu \quad \text{and} \quad M_\mu^6 = i p_1 q_\mu.$$  \hspace{1cm} (B.38)

The vector $q_\mu$ is now determined by the orthogonality conditions with respect to $B_0$ and one obtains

$$q_\mu = -\frac{1}{(B_0 \cdot B_1)} \sum_{i=1}^4 \hat{B}_i \hat{M}_\mu^i.$$  \hspace{1cm} (B.39)

The quadratic condition (B.9c) is substantially unaltered with respect to the case (A) and in fact it can be arranged in the following way

$$\sum_{a=1}^4 \hat{M}_\mu^i \hat{M}_\nu^i = \begin{pmatrix} 2(B_0 \cdot B_1) - \Omega_1^2 & 0 & 0 & 0 \\ 0 & 2(B_0 \cdot B_1) - \Omega_1^2 & 0 & 0 \\ 0 & 0 & 2(B_0 \cdot B_1) & 0 \\ 0 & 0 & 0 & 2(B_0 \cdot B_1) \end{pmatrix}. \hspace{1cm} (B.40)$$

The discussion of its solution is very similar to the previous case.

If $B_1^a$ is identically zero, the vector $B_0^a$ cannot vanish since it is time-like and it can be chosen to be

$$B_0^a = \left(0, 0, 0, 1, \frac{1}{2} \left[m - \frac{a^2}{m}\right], \frac{i}{2} \left[m + \frac{a^2}{m}\right]\right).$$  \hspace{1cm} (B.41)

up to an $SO(6)$ rotation. The orthogonality condition (B.8) now implies

$$M_\mu^5 = \frac{1}{2} \left(m + \frac{a^2}{m}\right) q_\mu \quad \text{and} \quad M_\mu^6 = \frac{i}{2} \left(m - \frac{a^2}{m}\right) q_\mu.$$  \hspace{1cm} (B.42)

The quadratic condition now takes the form

$$\sum_{a=1}^4 \hat{M}_\mu^i \hat{M}_\nu^i + \Omega_\mu^\rho \Omega_\nu^\nu + a^2 q_\mu q_\nu = 0,$$  \hspace{1cm} (B.43)

whose general solution is provided by

$$\hat{M}_\nu^i = i S^j \left(\Omega_\nu^r + \frac{a}{P^2} P_r^r q_\nu\right),$$  \hspace{1cm} (B.44)

where $P^r$ is a vector of the kernel of $\Omega$ and $S$ is a matrix of $SO(4, C)$.

$\text{det}(W) \neq 0$ :

This case is not really different from the case (A) for $\text{det}(W) = 0$. We have only a redefinition of the $a_i$’s in (B.26):

$$a_i^2 = 2(B_0 \cdot B_1) - \Omega_i^2 - \lambda^2.$$
B.4 Construction of the Killing spinors generating the Wilson loops

We shall focus our attention here on the general case $\epsilon_c \tilde{\Gamma}^M \epsilon_c \neq 0$\(^\text{13}\). The other possibilities can be discussed in a similar way.

Since the four vector $b^\mu$ vanishes for all the “fundamental” orbits considered in the subsect. B.2, the ten dimensional vector $\epsilon_c \tilde{\Gamma}^M \epsilon_c$ (with $M = 1, 2, \ldots, 9, 10$) can be always put in the canonical form

$$\epsilon_c \tilde{\Gamma}^M \epsilon_c = (0, 0, 0, 0, 0, 0, 0, 0, 1, -i)_{10}.$$ \tag{B.45}

modulo an $R$–symmetry rotation and a dilation. Then the Fierz identities (A.6) allow us to translate eq. (B.45) into an equivalent and simpler statement for the spinor $\epsilon_c$, namely $\epsilon_c$ is an eigen-spinor of positive chirality of the matrix $\tilde{\Gamma}^9$. For future convenience, we shall decompose $\epsilon_c$ in eigenstates of $\tilde{\Gamma}^5 \equiv \tilde{\Gamma}^1 \ldots \tilde{\Gamma}^4$ and we shall write

$$\epsilon_c = \cos t \epsilon_c^+ + \sin t \epsilon_c^- \quad \text{where} \quad \epsilon_c^+ \epsilon_c^- = \epsilon_c^- \epsilon_c^+ = 1.$$ \tag{B.46}

We then proceed to construct the spinor $\epsilon_s$.

$\text{det}(W) \neq 0$ : 

By imposing that $W$ has the canonical form (B.11), we find that $\epsilon_s$ admits the following expansion

$$\epsilon_s = \frac{1}{2} \left( 1 + \sum_{i=1}^{4} \frac{\hat{B}^i_0 \Gamma_{i+4}}{(B_0 \cdot \hat{B}^i_0)} \left[ \sec(t) \left( \lambda_+ 1 + \Omega_- \hat{\Gamma}^{12} \right) \epsilon_c^+ - \csc(t) \left( \lambda_- 1 + \Omega_+ \hat{\Gamma}^{12} \right) \epsilon_c^- \right] \right), \tag{B.47}$$

where $\lambda_{\pm} = \frac{\psi \pm \sigma}{2}$, $\Omega_{\pm} = \frac{\Omega_{1} \pm \Omega_{2}}{2}$, $\sigma$ is an arbitrary real number and $\hat{B}^i_0$ are four complex arbitrary entries.

We can now evaluate the remaining parameters in the scalar couplings. We first compute the constant vector $B_0$ and we obtain

$$B_0 = \left( \hat{B}^1_0, \hat{B}^2_0, \hat{B}^3_0, \hat{B}^4_0, \frac{1}{2} \left( (B_1 \cdot B_0) - \frac{|B_0|^2}{(B_1 \cdot B_0)} \right) \right) \cdot \frac{i}{2} \left( (B_1 \cdot B_0) + \frac{|B_0|^2}{(B_1 \cdot B_0)} \right) \tag{B.48}$$

with $|B_0|^2 = (\hat{B}^1_0)^2 + (\hat{B}^2_0)^2 + (\hat{B}^3_0)^2 + (\hat{B}^4_0)^2$. In eq. (B.48) the symbol $(B_1 \cdot B_0)$ is actually a short-hand notation for the following combination of the parameters

$$\frac{1}{2} \left( \csc^2(t) (\lambda_+^2 + \Omega_+^2) + \sec^2(t) (\lambda_-^2 + \Omega_-^2) \right), \tag{B.49}$$

but it also denotes its meaning in terms of the Wilson-loop couplings. Next we shall determine the matrix $M^i_\mu$. The first four rows $(i = 1, \ldots, 4)$ can be summarized in the following expression

$$M^i_\mu = (\lambda_+ \tan(t) - \lambda_- \cot(t)) \epsilon_c^+ \Gamma^{i+4} \epsilon_c^- - \Omega_+ \cot(t) \epsilon_c^+ \Gamma^{i+4} \Gamma_\mu \Gamma^{12} \epsilon_c^- - \Omega_- \tan(t) \epsilon_c^- \Gamma^{i+4} \Gamma_\mu \Gamma^{12} \epsilon_c^+. \tag{B.50}$$

\(^\text{13}\)Equivalently we can say that the vector $B^i_0$ does not vanish.
The expression for the remaining two rows is not particularly elegant, but at the end we simply find that they are given by eqs. (B.38) and (B.39).

In this framework the $SO(4, \mathbb{C})$ freedom in constructing the matrix $\hat{M}_\mu^a$, emphasized in the previous subsection, is translated into the freedom to choose the spinor $\epsilon_c$ without altering the other couplings. This arbitrariness obviously corresponds to the complex rotation in the directions $(5, 6, 7, 8)$.

$$\text{det}(W) = 0$$

(A) This case is obtained from the previous analysis by setting $\rho = 0$.

(B) Also this case requires small changes. Apart from setting to zero $\rho$ and $\Omega_2$ in (B.47) we have to add a term proportional to $a$:

$$a \frac{\sin(2t)}{\sigma^2 + \Omega_1^4} \left[ \sin(t) \left( \sigma \Gamma^4 \epsilon_c^+ + \Omega_1 \Gamma^3 \epsilon_c^+ \right) - \cos(t) \left( \sigma \Gamma^4 \epsilon_c^- + \Omega_1 \Gamma^3 \epsilon_c^- \right) \right].$$

(B.51)

On the contrary the matrix $\hat{M}_\mu^i$ is unaffected by the new parameter $a$ and it is simply obtained from (B.50) by posing $\Omega_2$ and $\rho$ to zero.

(C) It does not require new ingredient with respect to the cases (A) and (B) and it can be obtained from them by choosing some of the free parameters to be zero.

### C Conformal transformations

In the previous appendix we have briefly discussed the possible *impure* Wilson loops modulo conformal equivalence. In this appendix, for completeness, we shall discuss how a conformal transformation would act on the couplings and the contour of our loops. We find useful first to investigate conformal transformations on the relevant Killing spinors and then to extend the analysis on the scalar couplings.

#### C.1 Conformal transformations on Killing spinors

The simplest way to construct these transformations is to view the couple $(\epsilon_s, \epsilon_c)$ as a positive chiral spinor in the spinor representation of the ten dimensional conformal group $SO(11, 1)$. This representation is realized in terms of the $64 \times 64$ Dirac matrices

$$\sigma_A \sigma_B + \sigma_B \sigma_A = 2 \eta_{AB} \mathbb{I},$$

(C.1)

where $\eta_{AB} = \text{diag}(1, \ldots, 1, -1)$. The chiral representation for the $\sigma_A$ can be given in terms of the following block-antidiagonal matrices

$$\sigma^a = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix}, \quad (a = 1, \ldots, 10) \quad \sigma^{11} = \begin{pmatrix} 0 & \gamma^{11} \\ \gamma^{11} & 0 \end{pmatrix} \quad \sigma^{12} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

(C.2)
where $\gamma^\alpha$ are the ten dimensional Euclidean Dirac matrices. Consider now a spinor of positive chirality

$$\sigma^{13} \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix}.$$  \hfill (C.3)

The action of the generators of $SO(11,1)$ on this spinor can be rewritten in ten dimensional language as follows

\begin{equation}
M^{ab} \epsilon = \frac{1}{2} \gamma^{ab} \epsilon \quad (a,b = 1, \ldots, 11) \quad M^{12,a} \epsilon = \frac{1}{2} \gamma^a \epsilon \quad (a = 1, \ldots, 11),
\end{equation}

where $\epsilon$ is now viewed as a 32 component Dirac spinor in ten dimensions. There is an obvious embedding of the conformal subgroup $SO(5,1)$ in the representation (C.4). It is obtained by setting

\begin{equation}
P_{\mu} \epsilon = \frac{1}{2} \gamma^\mu \epsilon \Delta \epsilon = M^{12,11} \epsilon = \frac{1}{2} \gamma^{11} \epsilon, \quad
K_{\mu} \epsilon = (M^{12,\mu} - M^{11,\mu}) \epsilon = \frac{1}{2} \gamma^\mu \epsilon - \frac{1}{2} \gamma^{11,\mu} \epsilon = \frac{1}{2} \gamma^\mu (1 + \gamma^{11}) \epsilon,
\end{equation}

with $\mu = 1, \ldots, 4$. We can also embed the R-symmetry group $SO(6)$ of $\mathcal{N} = 4$ SYM by choosing

\begin{equation}
R^{i-4,j-4} \epsilon = M^{ij} \epsilon = \frac{1}{2} \gamma^{ij} \epsilon, \quad (i,j = 5, \ldots, 10).
\end{equation}

The original couple of ten dimensional chiral spinors $(\epsilon_s, \epsilon_c)$ is recovered by decomposing the spinor $\epsilon$ in eigenstates of $\gamma^{11}$. We shall write

$$\epsilon = \epsilon_s + \epsilon_c \quad \text{with} \quad \gamma^{11} \epsilon_s = \epsilon_s \quad \text{and} \quad \gamma^{11} \epsilon_c = -\epsilon_c.$$  \hfill (C.7)

The action of $P^\mu$, $K^\mu$ and $\Delta$ in terms of $\epsilon_s$ and $\epsilon_c$ can be also rewrite as follows

\begin{equation}
\Delta \epsilon_s = \frac{1}{2} \gamma^{11} \epsilon_s = \frac{\epsilon_s}{2} \quad \Delta \epsilon_c = \frac{1}{2} \gamma^{11} \epsilon_s = -\epsilon_c/2,
\end{equation}

\begin{equation}
P_{\mu} \epsilon = \frac{1}{2} \gamma^\mu (1 - \gamma^{11}) \epsilon = \gamma_\mu \epsilon_c, \quad
K_{\mu} \epsilon = \frac{1}{2} \gamma^\mu (1 + \gamma^{11}) \epsilon = \gamma_\mu \epsilon_s.
\end{equation}

Exploiting (C.8) we can easily compute the action of a finite translation or of a finite special conformal transformation on the spinor $\epsilon$. For a translation, we obtain

\begin{equation}
\exp (v^\mu P_\mu) \epsilon = \epsilon + v^\mu \gamma_\mu \epsilon_c + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} (v^\mu P_\mu)^n \epsilon = \epsilon + v^\mu \gamma_\mu \epsilon_c,
\end{equation}

since

\begin{equation}(v^\alpha P_\alpha)^2 \epsilon = \frac{1}{4} v^\alpha v^\beta \gamma_\alpha (1 + \gamma^{11}) \gamma_\beta (1 + \gamma^{11}) = 0.
\end{equation}

In other words under a translation the spinors $\epsilon_s$ and $\epsilon_c$ transform as follows

$$\epsilon_c \mapsto \epsilon_c \quad \epsilon_s \mapsto \epsilon_s + v^\mu \gamma_\mu \epsilon_c.$$  \hfill (C.11)
For a special conformal transformation, we find instead
\[
\exp (v^\mu K_\mu) \epsilon = \epsilon + v^\mu \gamma_\mu \epsilon_s + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} (v^\mu K_\mu)^n \epsilon = \epsilon + v^\mu \gamma_\mu \epsilon_s
\]  
(C.12)

since \((v^\alpha K_\alpha)^n\) also vanishes for \(n \geq 2\). In terms of the spinors \(\epsilon_s\) and \(\epsilon_c\), we have the following transformation
\[
\epsilon_s \mapsto \epsilon_s \quad \epsilon_c \mapsto \epsilon_c + v^\mu \gamma_\mu \epsilon_s.
\]  
(C.13)

The dilation instead yields
\[
\epsilon_s \mapsto e^{\frac{\hat{b}}{2}} \epsilon_s \quad \epsilon_c \mapsto e^{-\frac{\hat{b}}{2}} \epsilon_c.
\]  
(C.14)

The action of rotations and \(SO(6)\) \(R\)-symmetry is instead the obvious one.

### C.2 Conformal transformations and contours

In the following we shall illustrate how conformal transformations reflect on the set of parameters which defines the Wilson loop.

**Special conformal transformations:** The special conformal transformation defined by the vector \(v^\mu\) maps \((\epsilon_s, \epsilon_c)\) to \((\epsilon'_s, \epsilon'_c) = (\epsilon_s, \epsilon_c + v^\mu \Gamma_\mu \epsilon_s)\). The new parameters for the circuit are then given
\[
\begin{align*}
\alpha'^\mu &= \epsilon'_s \Gamma^\mu \epsilon'_c = \epsilon_s \Gamma^\mu \epsilon_s = \alpha^\mu \\
\lambda' &= 2\epsilon'_s \epsilon'_c = 2\epsilon_s (\epsilon_c + v^\alpha \Gamma_\alpha \epsilon_s) = \lambda + 2(v \cdot a) \\
\hat{\Omega}^\mu &\equiv \epsilon'_s \Gamma_\nu \epsilon'_c = 2\epsilon_s \Gamma_\nu (\epsilon_c + v^\alpha \Gamma_\alpha \epsilon_s) = \Omega^\mu + 2\epsilon_s \Gamma_\nu \Gamma_\alpha \epsilon_s v^\alpha = \Omega^\mu + 2a^\mu v_\nu - 2v^\mu a_\nu,
\end{align*}
\]  
(C.15)

since \(\epsilon_s \Gamma_\alpha \epsilon_s = 0\). The parameter \(b'^\mu\) is instead given by
\[
\hat{b}' = \epsilon'_s \Gamma^\mu \epsilon'_c = (\epsilon_c + v^\beta \epsilon_c \Gamma_\beta)(\epsilon_c + v^\alpha \Gamma_\alpha \epsilon_s) = \epsilon_c \Gamma^\mu \epsilon_c + 2\epsilon_s \epsilon_c v^\mu - 2\epsilon_s \Gamma_\alpha \epsilon_s v^\alpha + v^\alpha v^\beta \epsilon_c \Gamma_\beta \Gamma_\alpha \epsilon_s = \epsilon_c \Gamma^\mu \epsilon_c + 2\epsilon_s \epsilon_c v^\mu - 2\epsilon_s \Gamma_\alpha \epsilon_s v^\alpha + v^\alpha v^\beta \epsilon_c \Gamma_\beta \Gamma_\alpha \epsilon_s = \epsilon_c \Gamma^\mu \epsilon_c + 2\epsilon_s \epsilon_c v^\mu - 2\epsilon_s \Gamma_\alpha \epsilon_s v^\alpha + v^\alpha v^\beta \Gamma_\beta \Gamma_\alpha \epsilon_s = (\epsilon_c + v^\beta \epsilon_c \Gamma_\beta)(\epsilon_c + v^\alpha \Gamma_\alpha \epsilon_s) = b^\mu + \lambda v^\mu - \Omega^\mu \epsilon_c + 2v^\mu (a \cdot v) - v^2 a^\nu.
\]  
(C.16)

The new circuit \(y'^\mu(s)\) is obviously obtained from the original one through the conformal transformation generated by the vector \(v^\mu\)
\[
y'^\mu = \frac{x^\mu - v^\mu x^2}{1 - 2(v \cdot x) + v^2 x^2}.
\]  
(C.17)

The transformed couplings are instead given by
\[
\begin{align*}
M'^a &= \epsilon'_c \Gamma^a \epsilon'_c = \epsilon_c \Gamma^a \epsilon_c + v^\alpha \epsilon_c \Gamma^a \Gamma_\alpha \epsilon_s = M^a + B^a_0 v_\mu \\
B'^a_0 &= \epsilon'_c \Gamma^a \epsilon'_s = \epsilon_s \Gamma^a \epsilon_s = B^a_0 \\
B'^a_1 &= \epsilon'_c \Gamma^a \epsilon'_c = (\epsilon_c + v^\alpha \epsilon_c \Gamma_\alpha)(\epsilon_c + v^\beta \Gamma_\beta \epsilon_s) = \epsilon_c \Gamma^a \epsilon_c + 2\epsilon_s \epsilon_c \Gamma^a \Gamma_\alpha \epsilon_s = B^a_0 - 2v^\beta M^a_\beta - v^2 B^a_0.
\end{align*}
\]  
(C.18)
TRANSLATIONS: If we perform the translation defined by the vector $v^\mu$, the couple $(\epsilon_s, \epsilon_c)$ is mapped to $(\epsilon'_s, \epsilon'_c) = (\epsilon_s + v^\mu \Gamma_\mu \epsilon_c, \epsilon_c)$, while the parameters $(a', \lambda', \Omega', b')$ become

$$\begin{align*}
a'^\mu &= \epsilon'_s \Gamma^\mu \epsilon'_s = (\epsilon_s + v^\beta \epsilon_c \Gamma_\beta) \Gamma^\mu (\epsilon_s + v^\alpha \Gamma_\alpha \epsilon_c) = \\
&= a^\mu + \lambda v^\mu + \Omega^\mu \epsilon^\nu v_\nu + 2v^\mu (b \cdot v) - v^2 b^\nu.
\end{align*}$$

$$\begin{align*}
\lambda' &= 2(\epsilon_s + u^\alpha \epsilon_c \Gamma_\alpha) \epsilon_c = \lambda + 2(b \cdot v). \\
\Omega'^\mu \epsilon'_\nu &= 2\epsilon'_s \epsilon'_c = 2(\epsilon_s + v^\alpha \epsilon_c \Gamma_\alpha) \Gamma^\mu \epsilon_c = \Omega^\mu + 2v^\mu b_\nu - 2b^\mu v_\nu \\
b'^\mu &= \epsilon'_s \Gamma^\mu \epsilon'_c = \epsilon_c \Gamma^\mu \epsilon_c = b^\mu.
\end{align*}$$

(C.19)

The new contour is obviously $y^\mu = x^\mu + v^\mu$, while the scalar couplings are

$$\begin{align*}
M'^a_\mu &= \epsilon'_s \Gamma^a \tilde{\Gamma}_\mu \epsilon'_c = \epsilon_s \Gamma^a \tilde{\Gamma}_\mu \epsilon_c + v^\alpha \epsilon_c \tilde{\Gamma}_\alpha \Gamma^a \tilde{\Gamma}_\mu \epsilon_c = M^a_\mu - B^a_1 v_\mu \\
B'^a_1 &= \epsilon'_c \Gamma^a \epsilon'_c = \epsilon_c \Gamma^a \epsilon_c = B^a_1 \\
B'^0_0 &= \epsilon'_s \Gamma^a \epsilon'_s = (\epsilon_s + v^\beta \epsilon_c \Gamma_\beta) \Gamma^a (\epsilon_s + v^\alpha \tilde{\Gamma}_\alpha \epsilon_c) = \\
&= \epsilon_s \tilde{\Gamma}^a \epsilon_c + 2v^\beta \epsilon_s \Gamma^a \tilde{\Gamma}_\beta \epsilon_c + v^\beta \epsilon_c \tilde{\Gamma}_\beta \Gamma^a \tilde{\Gamma}_\alpha \epsilon_c = B^a_0 + 2v^\beta M^a_\beta - v^2 B^a_1.
\end{align*}$$

(C.20)

DILATIONS: When considering a dilation the couple $(\epsilon_s, \epsilon_c) \mapsto (\epsilon'_s, \epsilon'_c) = (e^{\rho/2} \epsilon_s, e^{-\rho/2} \epsilon_c)$ and $(a, \lambda, \Omega, b) \mapsto (e^{\rho} a, \lambda, \Omega, e^{-\rho} b)$. The new circuit is a constant rescaling of the original one: $y = e^\rho x$. Finally the couplings are almost unchanged

$$\begin{align*}
B'^0_0 &= e^\rho B^0_0, \\
B'^a_1 &= e^{-\rho} B^a_1 \\
M'^a_\mu &= M^a_\mu.
\end{align*}$$

(C.21)

We shall not discuss in details Lorentz rotations and $R-$symmetry transformations since they are realized on the circuit and on the couplings in the obvious way.
References

[1] D. Gaiotto, “N=2 dualities,” arXiv:0904.2715 [hep-th].

[2] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91, 167 (2010) [arXiv:0906.3219 [hep-th]].

[3] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, “Loop and surface operators in N=2 gauge theory and Liouville modular geometry,” JHEP 1001, 113 (2010) [arXiv:0909.0945 [hep-th]].

[4] N. Drukker, J. Gomis, T. Okuda and J. Teschner, “Gauge Theory Loop Operators and Liouville Theory,” JHEP 1002, 057 (2010) [arXiv:0909.1105 [hep-th]].

[5] W. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Witter supergravity,” Eur. Phys. J. C 22, 379 (2001) [arXiv:hep-th/9803001].

[6] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859 (1998) [arXiv:hep-th/9803002].

[7] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824 [hep-th].

[8] N. Drukker and D. J. Gross, “An Exact prediction of N=4 SUSYM theory for string theory,” J. Math. Phys. 42, 2896 (2001) [arXiv:hep-th/0010274].

[9] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859 (1998) [arXiv:hep-th/9803002].

[10] J.K. Erickson, G.W. Semenoff, K. Zarembo, Wilson loops in N=4 supersymmetric Yang-Mills theory, hep-th/0003055.

[11] K. Zarembo, “Supersymmetric Wilson loops,” Nucl. Phys. B 643, 157 (2002) [arXiv:hep-th/0205160].

[12] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, “Supersymmetric Wilson loops on S^3,” JHEP 0805, 017 (2008) [arXiv:0711.3226 [hep-th]].

[13] A. Kapustin and E. Witten, “Electric-magnetic duality and the geometric Langlands program,” arXiv:hep-th/0604151.

[14] A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, “Supersymmetric Wilson loops at two loops,” JHEP 0810, 088 (2008) [arXiv:0805.0665 [hep-th]].

[15] S. Giombi, V. Pestun and R. Ricci, “Notes on supersymmetric Wilson loops on a two-sphere,” JHEP 0807 (2010) 088 [arXiv:0905.0665 [hep-th]].

[16] A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, S. Thambyahpillai and D. Young, “Correlators of supersymmetric Wilson-loops, protected operators and matrix models in N=4 SYM,” JHEP 0908, 061 (2009) [arXiv:0905.1943 [hep-th]].

[17] A. Bassetto, L. Griguolo, F. Pucci, D. Seminara, S. Thambyahpillai and D. Young, “Correlators of supersymmetric Wilson loops at weak and strong coupling,” JHEP 1003, 038 (2010) [arXiv:0912.5440 [hep-th]].

[18] S. Giombi and V. Pestun, “Correlators of local operators and 1/8 BPS Wilson loops on S**2 from 2d YM and matrix models,” JHEP 1010 (2010) 033 [arXiv:0906.1572 [hep-th]].
[19] V. Pestun, “Localization of the four-dimensional N=4 SYM to a two-sphere and 1/8 BPS Wilson loops”, arXiv:0906.0638 [hep-th].

[20] A. Dymarsky and V. Pestun, “Supersymmetric Wilson loops in N=4 WYM and pure spinors,” JHEP 1004, 115 (2010) [arXiv:0911.1841 [hep-th]].

[21] N. Drukker, D. J. Gross and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D 60, 125006 (1999) [hep-th/9904191].

[22] L. Brink, J. H. Schwarz and J. Scherk, “Supersymmetric Yang-Mills Theories,” Nucl. Phys. B 121, 77 (1977).

[23] B. E. W. Nilsson, “Pure Spinors As Auxiliary Fields In The Ten-Dimensional Supersymmetric Yang-Mills Theory,” Class. Quant. Grav. 3, L41 (1986).

[24] P. S. Howe, “Pure Spinors Lines In Superspace And Ten-Dimensional Supersymmetric Theories,” Phys. Lett. B 258, 141 (1991) [Addendum-ibid. B 259, 511 (1991)].

[25] P. S. Howe, “Pure Spinors, Function Superspaces And Supergravity Theories In Ten-Dimensions And Eleven-Dimensions,” Phys. Lett. B 273, 90 (1991).

[26] V. Branding and N. Drukker, “BPS Wilson loops in N=4 SYM: Examples on hyperbolic submanifolds of space-time,” Phys. Rev. D 79, 106006 (2009) [arXiv:0902.4586 [hep-th]].

[27] N. Drukker and V. Forini, “Generalized quark-antiquark potential at weak and strong coupling,” JHEP 1106, 131 (2011) [arXiv:1105.5144 [hep-th]].

[28] A. Irrgang and M. Kruczenski, “Double-helix Wilson loops: Case of two angular momenta,” JHEP 0912, 014 (2009) [arXiv:0908.3020 [hep-th]].

[29] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, “Strings in flat space and pp waves from N = 4 Super Yang Mills,” AIP Conf. Proc. 646 (2003) 314.

[30] T. Yoneya, “What is holography in the plane wave limit of AdS$_5$/SYM$_4$ correspondence ?,” Prog. Theor. Phys. Suppl. 152, 108 (2004) [hep-th/0304183].

[31] N. Drukker and B. Fiol, “On the integrability of Wilson loops in AdS$_5 \times S^5$: Some periodic ansatze,” JHEP 0601, 056 (2006) [hep-th/0506058].