Light Deflection Angle by Superentropic Black Holes

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Abstract

Motivated by recent works on light deflection and shadow behaviors on AdS geometries [1, 2], we investigate the deflection angle of light rays by superentropic black holes. Taking appropriate approximations, we first obtain the involved expression. For large values of the impact parameter, we get a specific value being zero for ordinary black holes without AdS backgrounds. Then, we examine and analyze such an optical quantity by providing graphical discussions in terms of a bounded region of the moduli space required by superentropic black hole conditions. Concretely, we study the deflection angle aspects by varying the black hole mass being fixed in the previous findings.

Keywords: Superentropic black holes, Deflection angle of lights, Gauss-Bonnet theorem.

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1 Introduction

Recently, many efforts have been devoted to study the black hole physics. Precisely, the analogy between the associated mechanics and the thermodynamic laws pushes one to consider the black hole as a thermodynamic object [3–6]. This has become more interesting by identifying the cosmological constant with its pressure. Concretely, the stability, the phase transitions, and the P-V criticality of various black holes have been investigated. For Anti de Sitter (AdS) black holes, many similarities with van der Waals fluids have been established [7–10].

More recently, the pulsar SGR J174-2900 near supermassive black holes SgrA* has been explored and investigated providing data on the horizon and the horizonless of events around compact objects [11, 12]. A special emphasis has been put on superentropic black holes as fascinating descriptions with non-compact horizon topologies having an entropy of maximum bounds [13–16]. Relaying on certain limits, the superentropic black holes have been approached by considering ultra-spinning limits of the Kerr-Newman-AdS solution [17]. Indeed, the thermodynamic of such black holes has been investigated by establishing a mapping with ordinary solutions based on ultra-spinning appropriate approximations [18–21].

After many years of investigations, the image of the supermassive black hole provided by the Event Horizon Telescope (ETH) and the detection of gravitational waves by LiGO have brought a greatest advancement in modern physics [22–26]. These have encouraged activities concerning the optical aspects of black hole objects. The obtained behaviors have unveiled more data which could be exploited to understand the corresponding interesting issue. Concretely, the shadow of various black holes has been engineered, in the celestial two dimensional space, using null geodesic equations based on the Hamilton-Jacobi approach. The corresponding physical information have been encoded in the shadow radius and the distortion parameter. These two geometrical parameters control the size and the shape of the involved shadows, respectively. For non-rotating black holes, for instance, it has been revealed that the shadows involve a circular configuration shape. The later has been distorted by introducing the rotating parameter to provide non-trivial geometries including either the D-shape, or cardioid forms [27–29]. Moreover, a particular interest has been put on the deflection angle behaviors of light rays using analytic and numerical methods and approaches. Precisely, Gibbons and Werner brought a direct way to compute such an optical quantity using the Gauss-Bonnet theorem applied to a spacial background described by the optic metric functions [30–40]. Moreover, it has been suggested an alternative approach based on an elliptic integral formalism using Weierstrass elliptic functions [41].

In this work, we compute and analyze the light deflection angle by superentropic black holes in four dimensions. Up to certain assumptions and appropriate approximations, we first find the associated expression. For large values of the impact parameter, we obtain a specific value being zero for ordinary black holes without AdS backgrounds. Then, we examine and discuss such an optical quantity by providing graphic representations in terms of the involved parameters. Motivated by the mass constraint required by superentropic
black hole conditions, we discuss the deflection angle aspect by varying the black hole mass being fixed in the previous investigations.

The organization of this work is as follows. Section 2 concerns the needed formalism to compute the involved optical quantity. Section 3 brings the obtained expression. Section 4 is developed to graphic representations. The last section provides concluding remarks.

2 Deflection angle computations

To compute the deflection angle of light rays around four dimensional black holes, several methods have been explored. They have been extensively investigated to approach a large class of solutions. One of them has been relied on the geodesic equations of the massless particle motion. This road, however, provides solutions using non-trivial elliptic functions. An alternative method, which will be used in the present work, exploits the Gauss-Bonnet formalism results based on optic metric calculations [34, 42–44]. Placing the observer and the source at finite distance in the equatorial plane, the deflection angle can be derived using the following equation

\[ \Theta = \Psi_R - \Psi_S + \phi_{SR}. \] (2.1)

In this equation, \( \Psi_R \) and \( \Psi_S \) indicate angles between the light ray and the radial direction at the observer and the source position, respectively. It is denoted that \( \phi_{SR} \) represents the longitude separation angle specified later on [43]. These angles are linked to the components of the unit tangential vector along the light \( e^i \) defined by

\[ (e^r, e^\theta, e^\phi) = \epsilon \left( \frac{dr}{d\phi}, 0, 1 \right). \] (2.2)

where \( \epsilon \) can be obtained from the metric. Using the following metric relation

\[ ds^2 = -A(r, \theta)dt^2 + B(r, \theta)dr^2 + C(r, \theta)d\theta^2 + D(r, \theta)d\phi^2 - 2H(r, \theta)dt d\phi, \] (2.3)

\( \epsilon \) can be determined as as a function of the metric parameters. Indeed, it is given by

\[ \epsilon = \frac{A(r)D(r) + H^2(r)}{A(r)(H(r) + A(r)b)} \] (2.4)

where \( b \) is the impact parameter specified later on. In the equatorial plane \( (\theta = \pi/2) \) at a constant \( t \) of the space-time metric, it is possible to consider a 2-dimensional curved space defined by the following line element

\[ dl^2 \equiv \gamma_{ij}dx^i dx^j \] (2.5)

where \( \gamma_{ij} \) represents a spatial metric [33, 34, 42]. To obtain the angle measured from the radial direction, one should use the expression of the unit basis vector and the spatial metric as following

\[ \cos \Psi \equiv \gamma_{ij}e^i R^j, \] (2.6)
where the components of the radial vector $R^j$ are $(\frac{1}{\sqrt{\gamma}}, 0, 0)$ taken at the particular angle $\theta = \pi/2$. Using Eq.(2.2) and Eq.(2.6), the sin $\Psi$ expression is found to be

$$\sin \Psi = \frac{H(r) + A(r)b}{\sqrt{A(r)D(r) + H^2(r)}}.$$ (2.7)

Having presented shortly the formalism needed to compute the deflection angle of lights, we move to superentropic black hole applications.

## 3 Deflection angle of superentropic black holes

Motivated by a recent work on black hole shadows, we would like to investigate the deflection angle of lights by special black hole solutions called superentropic black holes. These solutions have been proposed as the Kerr-Neuman-AdS black hole limits. Before calculating such a deflection angle of light rays, we first present relevant physical and mathematical elements corresponding to these solutions. Following [2, 13, 17], the line element of the associated metric reads as

$$ds^2 = -\frac{\Delta_r}{\Sigma} (dt - \ell \sin^2 \theta d\phi)^2 + \Sigma \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\sin^2 \theta}\right) + \frac{\sin^4 \theta}{\Sigma} \left(\ell dt - \frac{r^2}{\ell^2} d\phi\right)^2,$$ (3.1)

where $\ell$ is a cosmological parameter. The involved metric terms

$$\Sigma = r^2 + \ell^2 \cos^2 \theta, \quad \Delta_r = (\ell + \frac{r^2}{\ell})^2 - 2mr + q^2.$$ (3.2)

It is noted that $m$ and $q$ are the mass and the charge parameters, respectively. In this solution, the usual local coordinate $\phi$, being a noncompact direction, should be compactified via the relation $\phi \sim \phi + \alpha$, where $\alpha$ is a dimensionless parameter controlling the associated size. The later has been identified with a new chemical potential as proposed in [17]. The solution of the equation $\Delta_r = 0$ provides a large root $r_+$ which reveals the existence of the black hole horizon. A close examination shows that this solution requires a constraint on the mass parameter. The calculation has given

$$m \geq 2r_c \left(\frac{r_c^2}{\ell^2} + 1\right),$$ (3.3)

where one has a critical radius via the relations $r_c^2 = \frac{\ell^2}{3} \left[(4 + \frac{3}{2}q^2)^{1/2} - 1\right]$. This constraint provides a relationship between $m$, $\ell$ and $q$ black hole parameters. It turns out that the thermodynamics of such black hole solutions has been investigated in many places including in [18, 19]. In particular, certain quantities have been approached in terms of usual charged and rotating black holes. In the present investigation, we attempt to complete the associated optical behaviors. This goes beyond the previous works where the mass parameter has been fixed [28, 40]. Here, however, this black hole parameter will be varied up to the previous
constraint. To inspect the deflection angle behaviors of such four dimensional black holes, we present the associated backgrounds from the orbit equation on the equatorial plane. More precisely, it is expressed as follows

\[
\left( \frac{dr}{d\phi} \right)^2 = \frac{(A(r)D(r) + H^2(r)) (D(r) - 2H(r)b - A(r)b^2)}{B(r) (H(r) + A(r)b)^2},
\]

(3.4)

where \( b \) is the impact parameter given by

\[
b = \frac{|L|}{E} = \frac{D(r) \frac{d\phi}{dt} - H(r)}{H(r) \frac{d\phi}{dt} + A(r)}.
\]

(3.5)

The involved radial functions read as

\[
A(r) = \frac{\Delta_r - \ell^2}{\Sigma}, \quad D(r) = \frac{(r^2 + \ell^2) \ell - \Delta_r \ell}{\Sigma},
\]

\[
B(r) = \frac{\Sigma}{\Delta_r}, \quad H(r) = \frac{(r^2 + \ell^2)^2 - \Delta_r \ell^2}{\Sigma}.
\]

(3.6) \hspace{1cm} (3.7)

Using the expressions given in Eq.(3.6) and Eq.(3.7) and taking \( u = \frac{1}{r} \), Eq.(3.4) becomes

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1 + \ell^2 u^4 + \alpha}{\ell^2 \left( \ell(1 - \ell^2 u^2 + \alpha) - b(1 + \alpha) \right)^2},
\]

(3.8)

where one has

\[
\alpha = \ell^2 u^2 (2 - 2mu + q^2 u^2).
\]

(3.9)

To simplify the computations, certain approximations should be exploited. Considering the following order \( O(m^2, q^3, \ell^2) \), the orbit equation can take the following form

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{b^2} + \frac{2\ell}{b^3} - 2u^2 + 2mu^3 - q^2 u^4.
\]

(3.10)

The calculation of the deflection angle of light rays needs the explicit expression of \( \phi_{RS} \). Indeed, it is given by

\[
\phi_{RS} = \int_S^R d\phi = \int_{u_S}^{u_0} \frac{1}{\sqrt{G(u)}} du + \int_{u_R}^{u_0} \frac{1}{\sqrt{G(u)}} du,
\]

(3.11)

where one has used \( G(u) \equiv \left( \frac{du}{d\phi} \right)^2 \). It is recalled that \( u_S \) and \( u_R \) represent the inverse of the source and the observer distance from the black hole. Moreover, \( u_0 \) denotes the inverse of the closest approach \( r_0 \).

To write down the desired expression, we take weak field and slow rotation approximations. For simplicity reasons, we consider the following order \( O(m^2, \ell^2, q^2) \). Eq.(3.8) provides a relation between the impact parameter \( b \) and \( u_0 \) which reads as

\[
b = \ell + \frac{1}{\sqrt{2}} \left( \frac{m}{2} + \frac{1}{u_0} \right)
\]

(3.12)
where \( u_0 \) solves the constraint \( G(u) = 0 \). Combining the above equations, we obtain

\[
\phi_{RS} = \int_{u_g}^{u_0} \left( \frac{1}{\sqrt{2\sqrt{u_0^2 - u^2}}} + m \frac{(u^2 + uu_0 + u_0^2)}{2\sqrt{2}(u + u_0)\sqrt{u_0^2 - u^2}} - q^2 \frac{u_0(u^2 + uu_0 + u_0^2)}{4\sqrt{2}(u + u_0)\sqrt{u_0^2 - u^2}} \right) du \\
+ \int_{u_R}^{u_0} \left( \frac{1}{\sqrt{2\sqrt{u_0^2 - u^2}}} + m \frac{(u^2 + uu_0 + u_0^2)}{2\sqrt{2}(u + u_0)\sqrt{u_0^2 - u^2}} - q^2 \frac{u_0(u^2 + uu_0 + u_0^2)}{4\sqrt{2}(u + u_0)\sqrt{u_0^2 - u^2}} \right) du \\
+ \mathcal{O}(m^2, \ell^2, q^3).
\]

(3.13)

The computing integrals give

\[
\phi_{RS} = \left( \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} - \arcsin \left( \frac{u_S}{u_0} \right) \right) + \frac{m}{2\sqrt{2}} \frac{(2u_0 + u_S)\sqrt{u_0^2 - u_S^2}}{u_0 + u_R} - q^2 \frac{u_0(u_s + 3u_0)\sqrt{u_0^2 - u_s^2}}{4\sqrt{2}(u_s + u_0)} \right) \\
+ \left( \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} - \arcsin \left( \frac{u_R}{u_0} \right) \right) + \frac{m}{2\sqrt{2}} \frac{(2u_0 + u_R)\sqrt{u_0^2 - u_R^2}}{u_0 + u_R} - q^2 \frac{u_0(u_r + 3u_0)\sqrt{u_0^2 - u_r^2}}{4\sqrt{2}(u_r + u_0)} \right) \\
+ \mathcal{O}(m^2, \ell^2, q^3).
\]

(3.14)

It is remarked that Eq.(3.14) is expressed as a function of \( u_0 \). To get \( \phi_{RS} \) in term of the impact parameter \( b \), however, we should exploit Eq.(3.12). Indeed, the calculations lead to

\[
\phi_{RS} = \frac{1}{\sqrt{2}} \left( \pi - \arcsin(\sqrt{2}bu_r) - \arcsin(\sqrt{2}bu_s) \right) \\
+ m \left( \frac{1 - b^2u_r^2}{b\sqrt{1 - 2b^2u_r^2}} + \frac{1 - b^2u_s^2}{2b\sqrt{1 - 2b^2u_s^2}} \right) + \ell \left( \frac{u_r}{\sqrt{1 - 2b^2u_r^2}} + \frac{u_s}{\sqrt{1 - 2b^2u_s^2}} \right) \\
+ m \ell \left( \frac{2\sqrt{2}bu_r + 1 + b^2u_r^2}{b^2 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} + \frac{2\sqrt{2}bu_s + 1 + b^2u_s^2}{b^2 (2bu_s + \sqrt{2}) \sqrt{1 - 2b^2u_s^2}} \right) \\
- q^2 \left( \frac{\sqrt{2}bu_r + 3}{8b^2 (2bu_r + \sqrt{2})} \right) + \left( \frac{\sqrt{2}bu_s + 3}{8b^2 (2bu_s + \sqrt{2})} \right) \\
- mq^2 \left( \frac{5\sqrt{2}bu_r + 3}{8b^3 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} + \frac{5\sqrt{2}bu_r + 3}{8b^3 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} \right) \\
- q^2 \left( \frac{10bu_r + 3\sqrt{2}}{4b^3 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} + \frac{10bu_r + 3\sqrt{2}}{4b^3 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} \right) \\
- mlq^2 \left( \frac{50bu_r + 9\sqrt{2}}{8b^4 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} + \frac{50bu_r + 9\sqrt{2}}{8b^4 (2bu_r + \sqrt{2}) \sqrt{1 - 2b^2u_r^2}} \right) + \mathcal{O}(m^2, \ell^2, q^3).
\]

(3.15)

To obtain the expression that we are after, Eq.(2.7) should be used. Up to certain approximations, we obtain the following relation

\[
\sin \Psi = b\ell u^2 - b\ell mu^3 + \frac{1}{2}b\ell q^2 u^4 - 1 + \mathcal{O}(m^2, \ell^2, q^3).
\]

(3.16)
This gives

$$
\Psi_R - \Psi_S = \sqrt{2b}u_r + \sqrt{2b}u_s - \frac{m\sqrt{b}u_r^2}{\sqrt{2}} - \frac{m\sqrt{b}u_s^2}{\sqrt{2}} - 2\pi + O(m^2, \ell^2, q^3, q^3u_r^3, q^3u_s^3).
$$

(3.17)

It has been observed that the divergence terms linked to \(u_r \to 0\) and \(u_s \to 0\) have been removed in the Eq.(3.15) and Eq.(3.17). With these limits, we can get the corresponding deflection angle of light rays associated with the present black hole solutions. Performing certain computations and using periodic conditions, we obtain the following expression

$$
\Theta = \frac{m}{b} + \frac{\ell m}{b^2} - \frac{3q^2}{4\sqrt{2b^3}} - \frac{3\ell q^2}{2\sqrt{2b^3}} - \frac{3mq}{8b^3}
\quad - \frac{9\ell mq^2}{8b^4} + \frac{\pi}{\sqrt{2}} + O(m^2, \ell^2, q^3, \ell u_r^3, \ell u_s^3, q^3u_r^3, q^3u_s^3).
$$

(3.18)

This light angle deflection of superentropic black hole solutions is given in terms of \(m, \ell\) and \(q\) parameters. Such parameters will be used to discuss and illustrate the associated behaviors. A particular focus will be on the mass parameter variation being fixed in the previous investigations.

### 4 Graphical discussions

In this section, we analyze the effect of the above mentioned parameters on the deflection angle of light rays by superentropic black holes. In particular, we discuss two situations relying on the mass parameter variations. In the case of the first situation associated with a fixed mass parameter, the variation behaviors are illustrated in Fig.(1) for different values of the charge and the AdS length. The left plot of this figure reveals the aspects of the deflection angle by varying the impact parameter \(b\) for different values of \(\ell\) corresponding to a neutral solution. The deflection angle increases by increasing \(\ell\). For large values of \(b\), the angle deflection goes to a particular value \(\frac{\pi}{\sqrt{2}}\). Fixing the AdS length size, the right plot

![Figure 1: The deflection angle behaviors by varying the impact parameter b with different values of the relevant parameters: the mass m, the AdS radius \(\ell\) and the charge q.](image-url)
shows the deflection angle behaviors by varying the impact parameter $b$ for different charge values of $q$. For small values of $b$ and small values of $q$, the deflection angle does not bring consistent acceptable behaviors. For large values of $q$, however, the angle deflection keeps the same previous variations. For large values of $b$, this optical angle goes to $\frac{\pi}{\sqrt{2}}$. It has been remarked that the charge can play a relevant optical role for such black hole solutions.

The involved constraint Eq.(3.3) associated with the building of such black holes requires the implementation of the mass parameter in the investigation of the deflection angle variations. This way could be considered as an extended vision of such an optical quantity. In particular, we investigate the effect of the mass parameter. For neutral solutions providing relevant aspects, the variations are presented in Fig.(2). Considering the mass as a parameter, the left plot shows the variation in terms of the impact parameter. It has been observed that the deflection angle increases with the mass parameter. For large values of $b$, the angle $\Theta$ goes the previous specific value. Taking the mass as a variable, the right plot provides the deflection variations for different values of $b$ by fixing $\ell$ and $q$. It has been remarked a linear variation. For generic values of $b$, the deflection angle increases with $m$. In particular, it increases significantly for small values of $b$. Fixing the mass parameter, we recover the same result obtained in the previous illustrations.

5 Conclusions and final remarks

In this paper, we have investigated the weak gravitational lensing in the framework of superentropic black hole solutions. In particular, we have considered the photon rays into the equatorial plane. Then, we have computed the optical metric in order to get the corresponding orbit equations. Using Gauss-Bonnet theorem, we have derived the expression of the total deflection angle of lights by superentropic black holes. Theses computations have been based on certain appropriate approximations. To inspect the behaviors of the angle deflection, many parameters have been used to elaborate graphical representations. These parameters have provided a bounded region in the associated moduli space required by the
superenentropic black hole building. It has been observed that the relevant parameter is the mass pushing one to consider two situations. In the first situation, we have considered the mass as a parameter. For large values of the impact parameter, it has been remarked that the deflection angle does not go to zero contrary to the ordinary black holes without AdS backgrounds. We have found a specific value given by $\frac{\pi}{\sqrt{2}}$. This deflection angle distinction could be linked to the presence of AdS radius and the black hole geometry effects even for large values of $b$. This result matches perfectly with the previous investigations [2]. In the second situation, the mass has been considered as a variable. In particular, it has been found that the deflection angle involves linear behaviors. For a generic mass value, this angle increases by decreasing the impact parameter.

This work could open certain new investigation gates. It should be also interesting to implement the dark sector effect. We could anticipate that one can extend the present work to others related topics using Gauss-Bonnet theorem results.

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