THE KOHN ALGORITHM ON DENJOY-CARLEMAN CLASSES

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Abstract. The equivalence of the Kohn finite ideal type and the D’Angelo finite type with the subellipticity of the \( \bar{\partial} \)-Neumann problem is extended to pseudoconvex domains in \( \mathbb{C}^n \) whose defining function is in a Denjoy-Carleman quasianalytic class closed under differentiation. The proof involves algebraic geometry over a ring intermediate between the ring of real-analytic functions \( C^\omega \) and the ring of smooth functions \( C^\infty \) whose Noetherianity or lack thereof is an open problem. It is also shown that such a ring satisfies the \( \sqrt{\text{acc}} \) property, one of the strongest properties a non-Noetherian ring could possess.

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1. Introduction

The systematic study of the subellipticity of the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains in \( \mathbb{C}^n \) was initiated by Joseph J. Kohn in his paper [16] published in Acta Mathematica in 1979. Kohn defined subelliptic multipliers for the \( \bar{\partial} \)-Neumann problem and showed they formed a multiplier ideal sheaf. Kohn was the first to construct a multiplier ideal sheaf, which has become since a standard object in algebraic geometry. Kohn defined his subelliptic multipliers to be germs of \( C^\infty \) functions and wrote down an algorithm that generates an increasing chain of ideals of multipliers, whose termination at the whole ring implies the subellipticity of the \( \bar{\partial} \)-Neumann problem. This termination condition is called Kohn finite ideal type. In the same paper [16], Joseph J. Kohn proved a three-way equivalence for pseudoconvex domains in \( \mathbb{C}^n \) with real-analytic \( C^\omega \) boundary, namely that the subellipticity of the \( \bar{\partial} \)-Neumann problem on \( (p, q) \) forms is equivalent to Kohn finite ideal type for \( (p, q) \) forms and is also equivalent to the condition that holomorphic varieties of complex dimension \( q \) have finite order of contact with the boundary of the domain. The latter condition is called finite D’Angelo type after John D’Angelo who investigated its properties in detail in [9] and proved the crucial fact that it is an open condition.
The next important development in the investigation of the subellipticity of the $\bar{\partial}$-Neumann problem came in the mid 1980’s with a series of three deep papers by David Catlin, [5], [6], and [7], in which he proved the equivalence of two out of the three properties that appear in Kohn’s theorem for real-analytic domains, namely that for a smooth, pseudoconvex domain in $\mathbb{C}^n$ the subellipticity of the $\bar{\partial}$-Neumann problem is equivalent to finite D’Angelo type. Catlin’s construction uses some of the subelliptic multipliers defined by Kohn but does not investigate whether the Kohn algorithm terminates, i.e. whether Kohn finite ideal type is also equivalent to the other two properties for smooth, pseudoconvex domains in $\mathbb{C}^n$. This three-way equivalence has been posited by Joseph J. Kohn and is called the Kohn conjecture.

The current investigation of the equivalence of types for Denjoy-Carleman quasianalytic classes is an intermediate case between the $C^\omega$ one settled by Kohn in 1979 and the Kohn conjecture. It involves classes of functions without necessarily convergent Taylor expansions that still satisfy the Lojasiewicz inequalities, properties crucial to the equivalence of types.

The Denjoy-Carleman quasianalytic classes are local subrings of the ring of smooth functions that satisfy certain bounds on their derivatives of various orders. The sequence of bounds behaves according to the Denjoy-Carleman Theorem, which means that the Taylor morphism is injective on any such Denjoy-Carleman quasianalytic class, i.e. there are no flat functions contained therein. Each Denjoy-Carleman quasianalytic class we will consider contains all real-analytic functions but is strictly larger, so it also contains functions with non-convergent Taylor expansions. The property that makes these Denjoy-Carleman quasianalytic classes both peculiar and highly interesting is that they fail to satisfy the Weierstrass Division Property as Childress proved in [8], which implies it is an open problem as to whether these are Noetherian rings or not. With one additional assumption on the sequence of bounds, each such Denjoy-Carleman quasianalytic class is closed under differentiation and therefore also closed under composition, division by a coordinate, and inverse. Edward Bierstone and Pierre Milman were thus able to extend in [2] the resolution of singularities algorithm to all Denjoy-Carleman quasianalytic classes satisfying this additional assumption. Their construction implies that all three Lojasiewicz inequalities hold. In addition, they proved topological Noetherianity, a weaker condition than Noetherianity but of great use for the equivalence of types considered here.

We shall show in this paper that for a pseudoconvex domain in $\mathbb{C}^n$ with boundary in any such Denjoy-Carleman quasianalytic class, all multipliers generated by the Kohn algorithm stay in the same Denjoy-Carleman class and the three-way equivalence proved by Kohn for pseudoconvex $C^\omega$ domains extends to this case:

**Main Theorem 1.1.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with boundary in a Denjoy-Carleman quasianalytic class $C_M(\overline{\Omega})$ defined in a neighborhood of $\overline{\Omega}$, where the sequence of bounds $M$ is such that $C_M(\overline{\Omega})$ is closed under differentiation. Let $x_0 \in \partial \Omega$ be any point on the boundary of the domain, and let $U_{x_0}$ be an appropriately small neighborhood around $x_0$. The following three properties are equivalent:

(i) The $\bar{\partial}$-Neumann problem for $(p,q)$ forms is subelliptic on $U_{x_0}$;
(ii) The Kohn algorithm on $(p,q)$ forms terminates at $x_0$ by generating the entire ring $C_M(U_{x_0})$;

(iii) The order of contact of holomorphic varieties of complex dimension $q$ with the boundary of the domain $\Omega$ in $U_{x_0}$ is finite.

As mentioned at the beginning of this introduction, the implication (ii) $\implies$ (i) was already done by Joseph J. Kohn for $C^\infty$ functions in [16], so his construction applies here as well. The implication (i) $\implies$ (iii) is the contrapositive of Catlin’s Theorem 1 in [5]. We should add that a complex variety passing through $x_0$ that has infinite order of contact with $b\Omega$ is already sitting inside $b\Omega$ since the Denjoy-Carleman quasianalytic functions considered here contain no flat functions. Thus, the aim of this paper is to prove the implication (iii) $\implies$ (ii). The proof that will be given is purely qualitative in nature. There is no attempt to compute an effective bound for the subelliptic gain in the $\bar{\partial}$-Neumann problem in terms of the D’Angelo type and the dimension. The focus is instead on the interaction of the algebraic properties of the Denjoy-Carleman quasianalytic classes with the algebra of the Kohn algorithm.

The construction given here is important for two reasons: First, it provides a glimpse into the algebra necessary for the direct proof of the Kohn conjecture. As it is not known whether these Denjoy-Carleman classes are Noetherian rings, certain finitely-generated subideals of the ideals of multipliers have to be constructed via which it can shown that the ascending chain in the Kohn algorithm stabilizes. This idea of constructing subideals of the ideals of multipliers that have special properties is prominent in the direct proof of the Kohn conjecture as the reader can easily see by comparing the set-up in [18] with this one. Second, it is the author’s hope that this work can serve as the first step of a two-step approximation proof of the Kohn conjecture, the second step being based on a theorem obtained by Szolem Mandelbrojt in 1940 in [17], which says that any smooth function can be decomposed as the sum of two Denjoy-Carleman quasianalytic functions belonging to potentially different classes.

In addition to the main result, Theorem 1.1 concerning the equivalence of types, we derive here a very simple consequence of Bierstone’s and Milman’s work in [2], namely that all the rings of Denjoy-Carleman functions closed under differentiation satisfy the $\sqrt{acc}$ condition for the real radical notion $\sqrt{}$. In commutative algebra, this condition is the nicest a non-Noetherian ring can satisfy. The reader may consult [19] and [12] for additional information on this type of rings.

**Theorem 1.2.** Let $U$ be an open set and let $C_M(U)$ be a Denjoy-Carleman quasianalytic class closed under differentiation. The ring $C_M(U)$ has the $\sqrt{acc}$ property on any open set $\bar{U} \subset U$. In other words, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals in $C_M(U)$, then the ascending chain of radical ideals $\sqrt{I_1} \subset \sqrt{I_2} \subset \cdots$ stabilizes, i.e. there exists a $k$ such that $\sqrt{I_j} = \sqrt{I_k}$ for all $j \geq k$.

The relevance of this theorem is showing that these Denjoy-Carleman quasianalytic classes are very natural examples of rings with the $\sqrt{acc}$ property, if indeed it turns out they are not Noetherian. As remarked above, the Noetherianity issue remains open.
The paper is organized as follows: Section 2 introduces the Denjoy-Carleman quasianalytic classes and outlines a number of their properties. At the end of this section, Theorem 1.2 is proven. Section 3 recalls the Kohn algorithm and other matters related to the subellipticity of the $\overline{\partial}$-Neumann problem. Finally, the equivalence of types, Theorem 1.1, is proven in Section 4.

I would like to thank Edward Bierstone and Pierre Milman for introducing me to their work on the Denjoy-Carleman quasianalytic classes and related literature as well as for very kindly taking the time to answer numerous questions on the properties of these classes.

2. The Denjoy-Carleman Quasianalytic Classes

We shall follow here the set-up of the Bierstone and Milman paper [2], but the reader is also directed to Vincent Thilliez’s expository paper [21] for the properties of more general quasianalytic local rings.

We start with the definition of a quasianalytic class. Let us make the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, where $z = (z_1, \ldots, z_n) = (x_1 + i x_{n+1}, \ldots, x_n + i x_{2n})$.

**Definition 2.1.** Let $U$ be a connected open set in $\mathbb{C}^n$, and let $M = \{M_0, M_1, M_2, \ldots\}$ be an increasing sequence of positive real numbers, where $M_0 = 1$. $C^\infty_M(U)$ consists of all $\mathbb{R}$-valued $f \in C^\infty(U)$ satisfying that for every compact set $K \subset U$, there exist constants $A, B > 0$ such that

$$\frac{1}{\alpha!} D^\alpha f(x) \leq A B^{||\alpha||} M_{||\alpha||}$$

for any $x \in K$, where $\alpha$ is a multi-index in $\mathbb{N}^{2n}$, $D^\alpha = \frac{\partial^{||\alpha||}}{\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}}$. $C^\infty_M(U)$ is called quasianalytic if the Taylor morphism assigning to each $f \in C^\infty_M(U)$ its Taylor expansion at $a \in U$ is injective for all $a \in U$.

Since the $\overline{\partial}$-Neumann problem is posed on $(p, q)$ forms in $\mathbb{C}^n$, we must also introduce another class $C_M(U)$ of $\mathbb{C}$-valued quasianalytic functions. We will examine the properties of both classes $C^\infty_M(U)$ and $C_M(U)$, going back and forth between them as necessary.

**Definition 2.2.** Let $U$ be a connected open set in $\mathbb{C}^n$, and let $M = \{M_0, M_1, M_2, \ldots\}$ be an increasing sequence of positive real numbers, where $M_0 = 1$. $C^\infty_M(U)$ consists of all $\mathbb{C}$-valued $f \in C^\infty(U)$ such that $f = g + ih$ for $g$ and $h$ real valued and $g, h \in C^\infty_M(U)$. $C_M(U)$ is called quasianalytic if the Taylor morphism assigning to each $f \in C_M(U)$ its Taylor expansion at $a \in U$ is injective for all $a \in U$.

**Remarks:**

1. Clearly, $f = g + ih \in C_M(U) \iff f, g \in C_M^\mathbb{R}(U)$

and

$C_M(U)$ quasianalytic $\iff C_M^\mathbb{R}(U)$ quasianalytic.
Consider any open set $\tilde{U} \subset U$. If $f \in C_M(U)$, then it is obvious from Definition 2.2 that the restriction $f|_{\tilde{U}} \in C_M(\tilde{U})$. The same holds for $C^R_M(U)$. We shall use this fact extensively.

In order to ensure that $C^R_M(U)$ is a local ring, which is essential for the algebraic considerations that will follow, we must make an additional assumption on the sequence $M$ of bounds. The interested reader should consult [21] for more information on this assumption and its implications.

**Definition 2.3.** The sequence $M = \{M_0, M_1, M_2, \ldots\}$ is called logarithmically convex if

$$\frac{M_{j+1}}{M_j} \leq \frac{M_{j+2}}{M_{j+1}}$$

for all $j \geq 0$, i.e. the sequence of subsequent quotients is increasing.

One of the consequences of Definition 2.3 along with the assumption that $M_0 = 1$ is that $\{(M_j)^{\frac{1}{j}}\}_{j \geq 1}$ is an increasing sequence of positive real numbers greater than or equal to 1. Since real-analytic functions satisfy estimate (2.1) with $M_{|\alpha|} = 1$ for all $\alpha$, it follows that $C^R_M(U)$ contains all $\mathbb{R}$-valued real-analytic functions and $C_M(U)$ contains all $\mathbb{C}$-valued real-analytic functions. In fact, Corollary 1 of Theorem 1 in [21] gives a condition on $M$ equivalent to having that $C^R_M(U)$ is exactly the ring of $\mathbb{R}$-valued real-analytic functions on $U$, $C^\omega(U)$:

**Proposition 2.4.** $C^R_M(U) = C^\omega(U)$ iff $\sup_{j \geq 1} (M_j)^{\frac{1}{j}} < \infty$.

Given that the sequence $\{(M_j)^{\frac{1}{j}}\}_{j \geq 1}$ is increasing, this proposition implies that we must assume

$$\lim_{j \to \infty} (M_j)^{\frac{1}{j}} = \infty \quad (2.2)$$

in order to guarantee that both $C^R_M(U)$ and $C_M(U)$ are strictly larger than the rings of $\mathbb{R}$-valued real-analytic functions and $\mathbb{C}$-valued real-analytic functions respectively.

Émile Borel was the first to give examples of quasianalytic functions that were not also real-analytic in two papers [3] and [4]. The reader should consult either these two papers of Borel or [21] for such examples. This work by Borel led Jacques Hadamard to pose the question in 1912 as to whether there exists a condition on the sequence $M$ that ensures the quasianalyticity of the corresponding class. The answer to Hadamard’s question is the Denjoy-Carleman Theorem:

**Denjoy-Carleman Theorem 2.5.** Let the sequence $M = \{M_0, M_1, M_2, \ldots\}$ be logarithmically convex, then both $C^R_M(U)$ and $C_M(U)$ are quasianalytic iff $\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty$.

Such a class $C^R_M(U)$ or $C_M(U)$ satisfying the Denjoy-Carleman Theorem is called a Denjoy-Carleman quasianalytic class. We have just one more condition to impose on $M$, and we will have described completely the Denjoy-Carleman quasianalytic classes on which we will consider the equivalence of types. This condition guarantees that both $C^R_M(U)$ and $C_M(U)$ are closed under differentiation. The result is originally due to Szolem Mandelbrojt and can be found as Corollary 2 to Theorem 1 of [21]:
Proposition 2.6. Both $C^r_M(U)$ are $C_M(U)$ are closed under differentiation iff

$$\sup_{j \geq 1} \left( \frac{M_{j+1}}{M_j} \right)^{\frac{1}{j}} < \infty.$$ (2.3)

Remark: Clearly, $C^r_M(U)$ closed under differentiation implies $C_M(U)$ is also closed under differentiation.

We shall summarize now the results of Section 4 of [2] in a proposition that lists all the properties of a Denjoy-Carleman quasianalytic class of the type described above that enable the resolution of singularities. This result is only stated for $C^r_M(U)$ because some of the properties contained therein are awkward to give in complex coordinates.

Proposition 2.7. Let $C^r_M(U)$ be a Denjoy-Carleman quasianalytic class that also satisfies conditions (2.2) and (2.3) on its sequence of bounds $M$. $C^r_M(U)$ has the following properties:

(i) $C^r_M(U)$ contains all real-analytic functions on $U$ as well as the restrictions of all polynomials on $U$, $P(U)$;

(ii) $C^r_M(U)$ is closed under composition, namely if $U$ and $V$ are open subsets of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively, $f \in C^r_M(V)$, and $g = (g_1, \ldots, g_{2p}): U \rightarrow V$ is such that $g_j \in C^r_M(U)$ for $1 \leq j \leq 2p$, then $f \circ g \in C^r_M(U)$;

(iii) $C^r_M(U)$ is closed under differentiation;

(iv) $C^r_M(U)$ is closed under division by a coordinate, i.e. if $f \in C^r_M(U)$ and

$$f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_{2n}) \equiv 0,$$

then there exists $h \in C^r_M(U)$ such that $f(x) = (x_i - a_i) h(x)$;

(v) $C^r_M(U)$ is closed under inverse, namely let $U$ and $V$ be open subsets of $\mathbb{C}^n$ and let $\varphi = (\varphi_1, \ldots, \varphi_{2n}): U \rightarrow V$ be such that $\varphi_i \in C^r_M(U)$ for $1 \leq i \leq 2n$, $a \in U$, $\varphi(a) = b$, and the Jacobian matrix

$$\frac{\partial \varphi}{\partial x}(a) = \left( \frac{\partial (\varphi_1, \ldots, \varphi_{2n})}{\partial (x_1, \ldots, x_{2n})}(a) \right)$$

be invertible, then there exist neighborhoods $U'$ of $a$ and $V'$ of $b$ as well as a mapping $\psi = (\psi_1, \ldots, \psi_{2n}): V' \rightarrow U'$ such that $\psi_i \in C^r_M(V')$ for $1 \leq i \leq 2n$, $\psi(b) = a$ and $\varphi \circ \psi$ is the identity mapping on $V'$.

Remarks:

(1) Proposition 2.7 part (v) is equivalent to the Implicit Function Theorem, which can be stated in this context as follows: Let $U$ be an open subset of $\mathbb{C}^n \times \mathbb{C}^p$ with product coordinates $(x, y) = (x_1, \ldots, x_{2n}, y_1, \ldots, y_{2p})$. Let $f_1, \ldots, f_{2p} \in C^r_M(U)$, $(a, b) \in U$, $f(a, b) = 0 = (f_1(a, b), \ldots, f_{2p}(a, b))$, and $\frac{\partial f_i}{\partial y_j}(a, b)$ be invertible, then there exists a product neighborhood $V \times W$ of $(a, b)$ in $U$ and a mapping $g = (g_1, \ldots, g_{2p}): V \rightarrow W$ such that $g_i \in C^r_M(V)$ for all $1 \leq i \leq 2p$, $g(a) = b$, and $f(x, g(x)) = 0$ for all $x \in V$.

(2) Proposition 2.7 part (v) implies that $C^r_M(U)$ is closed under reciprocal, i.e. if $f \in C^r_M(U)$ is such that $f(x) \neq 0$ for all $x \in U$, then $\frac{1}{f} \in C^r_M(U)$. 
(3) Using the properties of $C^r_M(U)$ in Proposition 2.7, we can construct manifolds of class $C^r_M(U)$ as well as functions on such manifolds. Since real and imaginary parts of functions in $C_M(U)$ are in $C^r_M(U)$, we automatically also get well-defined manifolds of class $C_M(U)$ as well as functions on those manifolds.

As it will be seen in Section 3, the Kohn algorithm yields an increasing chain of ideals of functions defined on the boundary of the domain $\Omega \subset \mathbb{C}^n$. We thus have to explore here what it means to have germs of functions in the class $C_M(U)$ defined on a manifold of the same class as well as sheaves of ideals in this class of functions. Let $X$ be a manifold of class $C_M(U)$ such that $X \subset U \subset \mathbb{C}^n$. Since $X$ is embedded in $\mathbb{C}^n$, we equip it with the subset topology, where $\mathbb{C}^n$ has the standard Euclidean metric topology on it. Let $\mathfrak{G}_X^{C_M(U)}$ be the set of germs of functions in $C_M(U)$ defined at points of $X$. We consider a sheaf of ideals $\mathcal{I} \subset \mathfrak{G}_X^{C_M(U)}$, and we would like to define what it means for $\mathcal{I}$ to be of finite type.

**Definition 2.8.** A sheaf of ideals $\mathcal{I} \subset \mathfrak{G}_X^{C_M(U)}$ is said to be of finite type if for each $a \in X$, there exists a neighborhood $U_a$ of $a$ in $X$ and finitely many sections $f_1, \ldots, f_p \in C_M(U_a)$ such that for all $b \in U_a$ the stalk $\mathcal{I}_b$ is generated by the germs of $f_1, \ldots, f_p$ at $b$.

We will work with ideals of finite type $\mathcal{I} \subset \mathfrak{G}_X^{C_M(U)}$ and with the affine varieties $\mathcal{V}(\mathcal{I})$ corresponding to such ideals. In this language, an ideal of finite type $\mathcal{I} = (f_1, \ldots, f_p)$ is a subsheaf on some open set $\tilde{U} \subset U$ such that $\tilde{U} \cap X$ is open in the topology of $X$ and for every $b \in \tilde{U} \cap X$, the stalk at $b$ is generated by the germs of $f_1, \ldots, f_p$ at $b$ and $f_1, \ldots, f_p \in C_M(\tilde{U})$. A particularly simple case occurs when $\tilde{U} = X$, so $\mathcal{I} = (f_1, \ldots, f_p)$ is an ideal in the ring $C_M(\tilde{U})$. For a manifold $X$ of class $C^r_M(U)$ with germs $\mathfrak{G}_X^{C^r_M(U)}$ on it, the definition of an ideal of finite type is the same as in the case of $C_M(U)$.

We can now state two of the consequences of applying resolution of singularities to the Denjoy-Carleman quasianalytic class $C^r_M(U)$, which Edward Bierstone and Pierre Milman obtained in [2]. We shall state as corollaries in both instances the corresponding results for $C_M(U)$, which are the ones we are interested in here. Both of these are crucial for the equivalence of types on domains of class $C_M(U)$. The first of these important consequences is topological Noetherianity, which is Theorem 6.1 of [2].

**Theorem (Topological Noetherianity) 2.9.** Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be any sequence of ideals of finite type in $\mathfrak{G}_X^{C^r_M(U)}$ such that the corresponding varieties form a decreasing sequence $\mathcal{V}(\mathcal{I}_1) \supseteq \mathcal{V}(\mathcal{I}_2) \supseteq \cdots$. Given some compact set $K$ in the topology of $X$, the sequence $\mathcal{V}(\mathcal{I}_1) \supseteq \mathcal{V}(\mathcal{I}_2) \supseteq \cdots$ stabilizes in a neighborhood of $K$, i.e. there exists some $k$ such that in a neighborhood of $K$, $\mathcal{V}(\mathcal{I}_j) = \mathcal{V}(\mathcal{I}_k)$ for all $j \geq k$.

To any ideal $\mathcal{I} \in \mathfrak{G}_X^{C_M(U)}$ can be associated an ideal $\mathcal{I}^R \in \mathfrak{G}_X^{C^r_M(U)}$ defined by

$$\mathcal{I}^R = \{ f \in \mathfrak{G}_X^{C^r_M(U)} \mid \exists g \in \mathfrak{G}_X^{C_M(U)} \text{ s.t. } f = \text{Re} \, g \text{ or } f = \text{Im} \, g \}.$$

Clearly, $\mathcal{I}^R$ is of finite type if $\mathcal{I}$ is and $\mathcal{V}(\mathcal{I}^R) = \mathcal{V}(\mathcal{I})$. We thus obtain:
Corollary 2.10. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be any sequence of ideals of finite type in $\mathfrak{S}^{C_M(U)}_X$ such that the corresponding varieties form a decreasing sequence $\mathcal{V}(\mathcal{I}_1) \supseteq \mathcal{V}(\mathcal{I}_2) \supseteq \cdots$. Given some compact set $K$ in the topology of $X$, the sequence $\mathcal{V}(\mathcal{I}_1) \supseteq \mathcal{V}(\mathcal{I}_2) \supseteq \cdots$ stabilizes in a neighborhood of $K$, i.e. there exists some $\nu$ such that in a neighborhood of $K$, $\mathcal{V}(\mathcal{I}_j) = \mathcal{V}(\mathcal{I}_k)$ for all $j \geq k$.

The second of the important consequences of Edward Bierstone’s and Pierre Milman’s work is the full set of Lojasiewicz inequalities for the Denjoy-Carleman quasianalytic class $C^R_M(U)$, Theorem 6.3 of [2].

Theorem 2.11. The three Lojasiewicz inequalities hold on $C^R_M(U)$ as follows:

(I) Let $X$ be a manifold of class $C^R_M(U)$, and let $f, g \in C^R_M(X)$, i.e. these functions are defined in a neighborhood of $X$. If $\{x \in X \mid g(x) = 0\} \subseteq \{x \in X \mid f(x) = 0\}$ in a neighborhood of a set $K$ compact in the topology of $X$, then there exist $C, \alpha > 0$ such that

$$|g(x)| \geq C|f(x)|^\alpha$$

for all $x$ in a neighborhood of $K$. Furthermore, $\inf \alpha$ is a positive rational number.

(II) Let $f \in C^R_M(U)$, and set $Z = \{x \in U \mid f(x) = 0\}$. Suppose that $K \subseteq U$ is compact. Then there exist $C > 0$ and $\nu \geq 1$ such that

$$|f(x)| \geq C d(x, Z)^\nu$$

in a neighborhood of $K$, where $d(\cdot, Z)$ is the Euclidean distance to $Z$ and $\inf \nu \in \mathbb{Q}$.

(III) Let $f \in C^R_M(U)$, and let $K$ be a compact subset of $U$ such that $\nabla f(x) = 0$ only if $f(x) = 0$, where $\nabla f$ is the gradient in real coordinates $(x_1, \ldots, x_{2n})$. Then there exist $C > 0$ and $\mu$ satisfying $0 < \mu \leq 1$ such that

$$|\nabla f(x)| \geq C |f(x)|^{1-\mu}$$

in a neighborhood of $K$ and $\sup \mu \in \mathbb{Q}$.

Since the real and imaginary parts of elements in $C_M(U)$ are in $C^R_M(U)$, it is clear that parts (I) and (II) will also be true on $C_M(U)$. Part (III) is irrelevant for type equivalence, so we will not restate it for $C_M(U)$.

Corollary 2.12. The first two of the Lojasiewicz inequalities above hold on $C_M(U)$:

(I) Let $X$ be a manifold of class $C_M(U)$, and let $f, g \in C_M(X)$, i.e. these functions are defined in a neighborhood of $X$. If $\{x \in X \mid g(x) = 0\} \subseteq \{x \in X \mid f(x) = 0\}$ in a neighborhood of a set $K$ compact in the topology of $X$, then there exist $C, \alpha > 0$ such that

$$|g(x)| \geq C|f(x)|^\alpha$$

for all $x$ in a neighborhood of $K$. Furthermore, $\inf \alpha$ is a positive rational number.
(II) Let \( f \in C_M(U) \), and set \( Z = \{ x \in U \mid f(x) = 0 \} \). Suppose that \( K \subset U \) is compact. Then there exist \( C > 0 \) and \( \nu \geq 1 \) such that
\[
|f(x)| \geq C \, d(x, Z)^\nu
\]
in a neighborhood of \( K \), where \( d(\cdot, Z) \) is the Euclidean distance to \( Z \) and \( \inf \nu \in \mathbb{Q} \).

**Proof:** Part (II) follows from part (I) by setting \( g(x) = d(x, Z) \). To prove part (I), we look at \( g_1^2 + g_2^2 \) and compare it to \( f_1^2 + f_2^2 \), where \( f = f_1 + if_2 \) and \( g = g_1 + ig_2 \) for \( f_1, f_2, g_1, g_2 \in C_M^\infty(U) \). Clearly, \( \{ x \in U \mid f(x) = 0 \} = \{ x \in U \mid f_1^2 + f_2^2 = 0 \} \) and \( \{ x \in U \mid g(x) = 0 \} = \{ x \in U \mid g_1^2 + g_2^2 = 0 \} \) so \( \{ x \in U \mid g_1^2 + g_2^2 = 0 \} \subset \{ x \in U \mid f_1^2 + f_2^2 = 0 \} \). By part (I) of Theorem 2.11, we conclude that in a neighborhood of any set \( K \) compact in the topology of \( X \) there exist \( C, \alpha > 0 \) with \( \alpha \in \mathbb{Q} \) such that
\[
|(g_1^2 + g_2^2)(x)| \geq C \, |(f_1^2 + f_2^2)(x)|^\alpha
\]
for all \( x \) in a neighborhood of \( K \). This is equivalent to
\[
|g(x)|^2 \geq C \, |f(x)|^{2\alpha}.
\]
We conclude that
\[
|g(x)| \geq C' \, |f(x)|^{\alpha}
\]
must hold with \( C' = \sqrt{C} \). \( \square \)

We shall close this section with three important results for the equivalence of types on domains of class \( C_M(U) \) that follow from Corollaries 2.10 and 2.12 as well as the proof of Theorem 1.2. I am indebted to [20] for the observation contained in the first of these:

**Proposition 2.13.** Let \( X \) be a manifold, and let \( U \) be an open set such that \( X \subset U \subset \mathbb{C}^n \). Let \( \bar{U} \subset U \) be an open set such that \( X \cap \bar{U} \) is open and non-empty in the topology of \( X \). If \( \mathcal{I} \) is any ideal in \( \mathcal{C}_X^M(U) \) defined on \( \bar{U} \) and if \( K \) is any subset of \( \bar{U} \) compact in the topology of \( X \), then there exists a subideal of finite type \( \mathcal{J} \subset \mathcal{I} \) such that \( \mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{J}) \) in a neighborhood of \( K \), i.e. ideals \( \mathcal{I} \) and \( \mathcal{J} \) have the same corresponding affine varieties in a neighborhood of \( K \).

**Proof:** This proposition is a consequence of topological Noetherianity. If \( \mathcal{I} \) is the zero ideal, there is nothing to prove, so we assume \( \mathcal{I} \neq (0) \). This means there exists some \( f_1 \in \mathcal{I} \) such that \( f_1 \neq 0 \) in a neighborhood of \( K \). Let \( \mathcal{I}_1 = (f_1) \) be the ideal generated by \( f_1 \). If \( \mathcal{I}_1 = \mathcal{J} \) in a neighborhood of \( K \), we are done; otherwise, there exists \( f_2 \in \mathcal{I}, f_2 \neq 0 \) in a neighborhood of \( K \) such that \( f_2 \notin \mathcal{I}_1 \). Set \( \mathcal{I}_2 = (f_1, f_2) \). \( \mathcal{I}_2 \) is clearly of finite type. Inductively, given \( \mathcal{I}_{n-1} = (f_1, \ldots, f_{n-1}) \) a subideal of finite type of \( \mathcal{I} \), either \( \mathcal{I}_{n-1} = \mathcal{I} \) in a neighborhood of \( K \) or there exists \( f_n \neq 0 \) in a neighborhood of \( K \) such that \( f_n \in \mathcal{I} \) but \( f_n \notin \mathcal{I}_{n-1} \). Set \( \mathcal{I}_n = (f_1, \ldots, f_{n-1}, f_n) \). We have thus constructed an increasing sequence of subideals of \( \mathcal{I} \) of finite type \( \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \). Consider now the decreasing sequence of the corresponding affine varieties \( \mathcal{V}(\mathcal{I}_1) \supseteq \mathcal{V}(\mathcal{I}_2) \supseteq \cdots \). By Corollary 2.10 there exists some \( k \) such that in a neighborhood of \( K \), \( \mathcal{V}(\mathcal{I}_j) = \mathcal{V}(\mathcal{I}_k) \) for all \( j \geq k \). Set \( \mathcal{J} = \mathcal{I}_k \), which is clearly a subideal of \( \mathcal{I} \) of finite type such that \( \mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{J}) \) in a neighborhood of \( K \). \( \square \)
The previous result and the Lojasiewicz inequalities, Corollary 2.12 imply a Lojasiewicz type Nullstellensatz. Before we can state this Nullstellensatz, we have to specify which notion of radical we will employ:

**Definition 2.14.** Let $C_M(U)$ be any Denjoy-Carleman quasianalytic class, and let $J \subset C_M(U)$ be an ideal, then the real radical of $J$ denoted by $\sqrt{J}$ is the set of $g \in C_M(U)$ such that there exists some $f \in J$ and some positive natural number $m \in \mathbb{N}^*$ such that $|g|^m \leq |f|$ on $U$.

The real radical is the correct generalization of the usual radical on the ring of holomorphic functions $\mathcal{O}$ for $\mathbb{C}$-valued $C^\omega$ functions, Denjoy-Carleman quasianalytic functions $C_M(U)$, and $C^\omega$ functions. Clearly, this definition can be made because $C_M(U)$ is constructed to be a ring via the assumption of logarithmic convexity on the sequence of bounds $M$.

**Theorem (Lojasiewicz Nullstellensatz) 2.15.** Let $X$ be a manifold, and let $U$ be an open set such that $X \subset U \subset \mathbb{C}^n$. Let $I = (f_1, \ldots, f_p)$ be any ideal of finite type in $\mathfrak{G}_X^{C_M(U)}$ defined on some open set $\tilde{U} \subset U$ such that $X \cap \tilde{U}$ is open and non-empty in the topology of $X$. Let $\mathcal{V}(I) \subset \tilde{U}$ be the affine variety corresponding to $I$, and let $I(\mathcal{V}(I))$ be the ideal of functions in $C_M(\tilde{U})$ vanishing on $\mathcal{V}(I)$. Given any subset $K$ of $\tilde{U}$ compact in the topology of $X$ and any neighborhood $U_K$ of $K$, $U_K$ open in $\mathbb{C}^n$ and such that $U_K \subset \tilde{U}$,

$$\sqrt{I}\big|_{U_K} = I(\mathcal{V}(I))\big|_{U_K},$$

where the real radical $\sqrt{I}|_{U_K}$ is taken in $C_M(U_K)$.

**Proof:** The inclusion $$\sqrt{I}\big|_{U_K} \subset I(\mathcal{V}(I))\big|_{U_K}$$ is clear. We only have to prove the reverse inclusion. Consider any $h \in I(\mathcal{V}(I))$, and set $g = |f_1|^2 + \cdots + |f_p|^2$, $g \in C_M(\tilde{U})$ since $f_1, \ldots, f_p \in C_M(\tilde{U})$. Since $\mathcal{V}(I) = \{x \in \tilde{U} \mid g(x) = 0\} \subseteq \{x \in \tilde{U} \mid h(x) = 0\}$, we apply part (I) of Corollary 2.12 on $U_K \subset \tilde{U}$ to conclude that there exist $C, \alpha > 0$ such that $|g(x)| \geq C|h(x)|^\alpha$ for all $x \in U_K$ with $\alpha$ rational. This precisely means that $h \in \sqrt{I}|_{U_K}$ as needed. □

We now have to use the Lojasiewicz inequalities to deduce that the affine variety corresponding to any ideal $I$ in a Denjoy-Carleman quasianalytic class, which is a ring, must have an open and dense set of smooth points. First, let us define a smooth point:
Definition 2.16. Let \( \mathcal{V} = \mathcal{V}(\mathcal{I}) \) be a variety corresponding to an ideal \( \mathcal{I} \subset C_M(U) \). \( x_0 \in \mathcal{V} \) is a smooth point of \( \mathcal{V} \) if there exist functions \( f_1, \ldots, f_s \in \mathcal{I}(\mathcal{V}(\mathcal{I})) \) and a neighborhood \( \tilde{U} \ni x_0 \) such that
\[
\mathcal{V} \cap \tilde{U} = \{ x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_s(x) = 0 \}
\]
is a \( C^\infty \) submanifold of \( \mathbb{C}^n \) of codimension \( s \), i.e. \( \partial f_1 \wedge \cdots \wedge \partial f_s(x) \neq 0 \) for all \( x \in \tilde{U} \).

Remarks:
1. The definition implies that the set of smooth points of a variety \( \mathcal{V} \) is open.
2. Each element in \( C_M(U) \) is a function of \( z = (z_1, \ldots, z_n) \) or its conjugate \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \), and if \( f \) vanishes on \( \mathcal{V} \) then so must \( \bar{f} \). This justifies giving the Jacobian restatement of what it means for \( \mathcal{V} \cap \tilde{U} \) to be a submanifold in terms of complex gradients only without any mention of their conjugates because \( f + \bar{f}, f - \bar{f} \in \mathcal{I}(\mathcal{V}(\mathcal{I})) \).

For the ring of holomorphic functions \( \mathcal{O} \) and for the ring of \( \mathbb{C} \)-valued real-analytic functions \( C^\omega \), it is known that any affine variety has an open and dense set of smooth points. For the ring of smooth functions \( C^\infty \), this same statement is not always true but was proven by René Thom in [22] for \( \mathcal{V} = \mathcal{V}(\mathcal{I}) \) under the hypothesis that the ideal \( \mathcal{I} \subset C^\infty \) is Lojasiewicz. An ideal \( \mathcal{I} \subset C^\infty \) is called Lojasiewicz if it is of finite type and its generators satisfy the Lojasiewicz inequality with respect to distance, which is the content of part (II) of Corollary 2.12 for all elements of the ring \( C_M(U) \). Given any ideal \( \mathcal{I} \subset C_M(U) \), Proposition 2.13 guarantees that \( \mathcal{V}(\mathcal{I}) \) can be presented in a neighborhood of a compact set as the variety \( \mathcal{V}(\mathcal{J}) \) corresponding to a subideal \( \mathcal{J} \) of \( \mathcal{I} \) of finite type. This result combined with part (II) of Corollary 2.12 leads us to expect that indeed any affine variety corresponding to an ideal in \( C_M(U) \) must have an open and dense set of smooth points:

Proposition 2.17. Let \( X \) be a manifold, and let \( U \) be an open set such that \( X \subset U \subset \mathbb{C}^n \). Let \( \mathcal{I} \) be any ideal in \( \mathfrak{S}^{C_M(U)}_X \) defined on some open set \( \tilde{U} \subset U \) such that \( X \cap \tilde{U} \) is open and non-empty in the topology of \( X \). Let \( \mathcal{V}(\mathcal{I}) \subset \tilde{U} \) be the affine variety corresponding to \( \mathcal{I} \), then \( \mathcal{V}(\mathcal{I}) \) has an open and dense set of smooth points.

Proof: We know the set of smooth points of \( \mathcal{V}(\mathcal{I}) \) is open, so we only have to show that it is dense. For any \( x_0 \in \mathcal{V}(\mathcal{I}) \), the set \( \{ x_0 \} \) is compact in the topology chosen for \( X \), so we set \( K = \{ x_0 \} \) and apply Proposition 2.13 to conclude that there exists a neighborhood \( U' \) of \( x_0 \), \( U' \subset \tilde{U} \), as well as a subideal of finite type \( \mathcal{J} = (f_1, \ldots, f_p) \subset \mathcal{I} \) such that \( \mathcal{V}(\mathcal{J}) = \mathcal{V}(\mathcal{I}) \) on \( U' \). Set \( f = |f_1|^2 + \cdots + |f_p|^2 \). \( f \in C_M(\tilde{U}) \) since \( f_1, \ldots, f_p \in C_M(\tilde{U}) \). It is obvious that \( \mathcal{V} \cap U' = \{ x \in U' \mid f(x) = 0 \} \). By part (II) of Corollary 2.12, the Lojasiewicz inequality with respect to distance holds for \( f \) on \( U' \), so the proof of Rene Thom in [22] or the even simpler proof of the same result given by Jean-Claude Tougeron in [23] (proof of Proposition 4.6 in subsection V.4) apply verbatim.

Remark: Thom’s theorem does not apply directly to this case because we are considering here an ideal \( \mathcal{I} \) of elements in the ring \( C_M(U') \) and not in the ring \( C^\infty(U) \).

Finally, we can prove Theorem 1.2.
Proof of Theorem 1.2: Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be an increasing chain of ideals such that $I_j = \sqrt[n]{I_j}$ for all $j \geq 1$. For any open $\tilde{U} \subset U$ and any $j \geq 1$, we apply Proposition 2.13 with $X = U$ to conclude that there exists a subideal of finite type $J_j \subset I_j$ such that $\mathcal{V}(J_j) = \mathcal{V}(I_j)$. We apply next the Lojasiewicz Nullstellensatz, Proposition 2.15, to conclude that $\mathcal{I}(\mathcal{V}(I_j)) = \sqrt[n]{\mathcal{I}(\mathcal{V}(J_j))}$ in a neighborhood of $\tilde{U}$ for all $j \geq 1$. By Topological Noetherianity, Corollary 2.10, the decreasing chain of varieties $\mathcal{V}(I_1) \supset \mathcal{I}(I_2) \supset \ldots$ stabilizes, namely there exists some $k \in \mathbb{N}$ such that for all $j \geq k$, $\mathcal{V}(I_j) = \mathcal{V}(I_k)$. The largest ideal of functions, however, that vanishes on $\mathcal{V}(I_j)$ is

$$I_j = \sqrt[n]{I_j} = \sqrt[n]{J_j} = \sqrt[n]{J_k} = \sqrt[n]{I_k}$$

for all $j \geq k$. \hfill \Box

3. Subellipticity of the $\bar{\partial}$-Neumann Problem and the Kohn Algorithm

We shall give here only a brief outline of the properties of subelliptic multipliers and the Kohn algorithm. The interested reader should consult Joseph J. Kohn’s paper [16] for the set-up of the $\bar{\partial}$-Neumann problem as well as the proofs of the results cited in this section. We first give Kohn’s characterization of subellipticity of the $\bar{\partial}$-Neumann problem on $(p, q)$ forms:

Definition 3.1. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $x_0 \in \overline{\Omega}$. The $\bar{\partial}$-Neumann problem on $\Omega$ for $(p, q)$ forms is said to be subelliptic at $x_0$ if there exist a neighborhood $U$ of $x_0$ and constants $C, \epsilon > 0$ such that

$$||\varphi||^2_\epsilon \leq C (||\bar{\partial}\varphi||^2_0 + ||\bar{\partial}^*\varphi||^2_0 + ||\varphi||^2_0)$$

for all $(p, q)$ forms $\varphi \in C_0^\infty(U) \cap \text{Dom}(\bar{\partial}^*)$, where $|| \cdot ||_\epsilon$ is the Sobolev norm of order $\epsilon$ and $|| \cdot ||_0$ is the $L^2$ norm.

The non-ellipticity of the $\bar{\partial}$-Neumann problem is coming precisely from the boundary condition given by $\varphi \in \text{Dom}(\bar{\partial}^*)$. If the point $x_0$ is inside the domain $\Omega$ then automatically estimate 3.1 holds at $x_0$ with the largest possible $\epsilon$ allowed by the $\bar{\partial}$-Neumann problem, namely $\epsilon = 1$. The problem is thus elliptic rather than subelliptic inside as it was proven by Kohn in [13] and [14] for strongly pseudoconvex domains as well as by Hörmander in [11] and Kohn in [15] via weighted estimates for pseudoconvex domains. This implies subellipticity only needs to be studied on the boundary of the domain $b\Omega$.

We shall now give Kohn’s definition of a subelliptic multiplier:

Definition 3.2. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $x_0 \in \overline{\Omega}$. A $C^\infty$ function $f$ is called a subelliptic multiplier at $x_0$ for the $\bar{\partial}$-Neumann problem on $\Omega$ if there exist a neighborhood $U$ of $x_0$ and constants $C, \epsilon > 0$ such that

$$||f\varphi||^2_\epsilon \leq C (||\bar{\partial}\varphi||^2_0 + ||\bar{\partial}^*\varphi||^2_0 + ||\varphi||^2_0)$$

for all $(p, q)$ forms $\varphi \in C_0^\infty(U) \cap \text{Dom}(\bar{\partial}^*)$. We will denote by $\mathcal{I}(x_0)$ the set of all subelliptic multipliers at $x_0$. 
The notation $I^q(x_0)$ for subelliptic multipliers at $x_0$ for $(p,q)$ forms drops reference to $p$, the holomorphic part of such forms, which is irrelevant in the $\bar{\partial}$-Neumann problem.

Remarks:
(1) If there exists a subelliptic multiplier $f \in I^q(x_0)$ such that $f(x_0) \neq 0$, then a subelliptic estimate holds at $x_0$ for the $\bar{\partial}$-Neumann problem.
(2) If $x_0 \in b\Omega$ but $f = 0$ on $U \cap b\Omega$, then estimate (3.2) holds for $\epsilon = 1$, which is the largest possible value. This is the case if we set $f = r$, where $r$ is the defining function of the domain $\Omega$.
(3) If $x_0 \in b\Omega$, the highest possibly gain in regularity in the $\bar{\partial}$-Neumann problem given by $\epsilon$ in estimate (3.1) is $\frac{1}{2}$ under the strongest convexity assumption, namely strong pseudoconvexity of $\Omega$, as proved by Kohn in [13] and [14].

Kohn’s Theorem 1.21 in [16] encapsulates the properties of subelliptic multipliers he proved in his paper. These motivate the way he sets up his algorithm, which determines whether or not the $\bar{\partial}$-Neumann problem is subelliptic:

**Theorem 3.3.** If $\Omega$ is pseudoconvex with a $C^\infty$ boundary and if $x_0 \in \overline{\Omega}$, then we have:
(a) $I^q(x_0)$ is an ideal.
(b) $I^q(x_0) = \sqrt{I^q(x_0)}$.
(c) If $r = 0$ on $b\Omega$, then $r \in I^q(x_0)$ and the coefficients of $\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}$ are in $I^q(x_0)$.
(d) If $f_1, \ldots, f_{n-q} \in I^q(x_0)$, then the coefficients of $\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q-j}$ are in $I^q(x_0)$, for $j \leq n - q$.

The Kohn Algorithm:
Step 1
$$I^q_1(x_0) = \sqrt{r, \text{coeff} \{r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\}}$$

Step (k+1)
$$I^q_{k+1}(x_0) = \sqrt{(I^q_k(x_0), A^q_k(x_0))},$$
where
$$A^q_k(x_0) = \text{coeff} \{\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q-j}\}$$
for $f_1, \ldots, f_j \in I^q(x_0)$ and $j \leq n - q$. Note that $(\cdot)$ stands for the ideal generated by the functions inside the parentheses and $\text{coeff} \{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\}$ is the determinant of the Levi form for $q = 1$, namely in the $\bar{\partial}$-Neumann problem for $(0,1)$ forms. Evidently, $I^q_k(x_0) \subset I^q(x_0)$ at each step $k$, and furthermore the algorithm generates an increasing chain of ideals
$$I^q_1(x_0) \subset I^q_2(x_0) \subset \cdots.$$

We will employ the following notation pertaining to the varieties corresponding to ideals of multipliers just as Kohn does in [16]:
$$V^q_k(x_0) = \mathcal{V}(I^q_k(x_0))$$
We will now recall from [16] the definition of the Zariski tangent space to an ideal and to a variety, which are crucial in testing the progress of the Kohn algorithm. We will state everything in terms of ideals in the ring $C_M(U)$, which is the underlying ring in our construction.

**Definition 3.4.** Let $\mathcal{I}$ be an ideal in $C_M(U)$ and let $\mathcal{V}(\mathcal{I})$ be the variety corresponding to $\mathcal{I}$. If $x \in \mathcal{V}(\mathcal{I})$, then we define $Z_x^{1,0}(\mathcal{I})$ the Zariski tangent space of $\mathcal{I}$ at $x$ to be

$$Z_x^{1,0}(\mathcal{I}) = \{ L \in T_x^{1,0}(U) \mid L(f) = 0 \quad \forall f \in \mathcal{I} \},$$

where $T_x^{1,0}(U)$ is the $(1,0)$ tangent space to $U \subset \mathbb{C}^n$ at $x$. If $\mathcal{V}$ is a variety, then we define

$$Z_x^{1,0}(\mathcal{V}) = Z_x^{1,0}(\mathcal{I}(\mathcal{V})).$$

where $\mathcal{I}(\mathcal{V})$ is the ideal of all functions in $C_M(U)$ vanishing on $\mathcal{V}$.

The next lemma is Lemma 6.10 of [16] that relates $Z_x^{1,0}(\mathcal{I})$ with $Z_x^{1,0}(\mathcal{V}(\mathcal{I}))$:

**Lemma 3.5.** Let $\mathcal{I}$ be an ideal in $C_M(U)$. If $x \in \mathcal{V}(\mathcal{I})$, then

$$Z_x^{1,0}(\mathcal{V}(\mathcal{I})) \subset Z_x^{1,0}(\mathcal{I}). \quad (3.3)$$

Equality holds in (3.3) if the ideal $\mathcal{I}$ satisfies the Nullstellensatz, namely $\mathcal{I} = \mathcal{I}(\mathcal{V}(\mathcal{I}))$.

We define

$$\mathcal{N}_x = \{ L \in T_x^{1,0}(b\Omega) \mid \langle (\partial \bar{\partial} r)_x, L \wedge \bar{L} \rangle = 0 \},$$

which is the subspace of $T_x^{1,0}(b\Omega)$ consisting of the directions in which the Levi form vanishes. With this notation, we can define the holomorphic dimension of a variety lying in the boundary of the domain $\Omega$.

**Definition 3.6.** Let $\mathcal{V}$ be a variety in $U$ corresponding to an ideal $\mathcal{I}$ in $C_M(U)$ such that $\mathcal{V} \subset b\Omega$. We define the holomorphic dimension of $\mathcal{V}$ by

$$\text{hol. dim } (\mathcal{V}) = \min_{x \in \mathcal{V}} \dim Z_x^{1,0}(\mathcal{V}) \cap \mathcal{N}_x.$$
4. Equivalence of Types

We start by showing that the Kohn algorithm on a domain with boundary in $C_M(U)$ only produces elements of $C_M(U)$.

**Theorem 4.1.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with boundary in a Denjoy-Carleman quasianalytic class $C_M(\Omega)$ defined in a neighborhood of $\overline{\Omega}$, where the sequence of bounds $M$ is such that $C_M(\Omega)$ is closed under differentiation. Let $I^q_1(x_0) \subset I^q_2(x_0) \subset \cdots$ be the increasing chain of ideals generated by the Kohn algorithm, then $I^q_k(x_0) \subset C_M(\Omega)$ at every step $k$ of the Kohn algorithm.

**Proof:** Let $r$ be the defining function of $\Omega$. The hypothesis says that $r \in C_M(\Omega)$. As seen in Section 3, the Kohn algorithm involves only differentiations, taking of real radicals, addition, and multiplication starting with $r$. Addition and multiplication appear not only in the generation of various ideals but also in the computation of all Jacobians involved. $C_M(\Omega)$ is defined to be closed under differentiation and such that it is a ring, hence closed under addition and multiplication. Finally, Definition 2.14 was given such that the real radical of an ideal $I$ stays in the same ring to which $I$ belongs.

Next, we will show that the decreasing chain of varieties $V^q_1(x_0) \supseteq V^q_2(x_0) \supseteq \cdots$ given by the Kohn algorithm stabilizes in a neighborhood of any point $x_0 \in b\Omega$:

**Proposition 4.2.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with boundary in a Denjoy-Carleman quasianalytic class $C_M(\Omega)$ defined in a neighborhood of $\overline{\Omega}$, where the sequence of bounds $M$ is such that $C_M(\Omega)$ is closed under differentiation. Let $V^q_1(x_0) \supseteq V^q_2(x_0) \supseteq \cdots$ be the decreasing chain of varieties generated by the Kohn algorithm, and let any point $x_0 \in b\Omega$ be given. The sequence $V^q_1(x_0) \supseteq V^q_2(x_0) \supseteq \cdots$ stabilizes in a neighborhood of $x_0$, i.e. there exists some $k$ and a neighborhood $U_{x_0}$ of $x_0$ such that $V^q_j(x_0) = V^q_k(x_0)$ for all $j \geq k$ and all $x \in U_{x_0}$.

**Proof:** If we can represent $V^q_1(x_0) \supseteq V^q_2(x_0) \supseteq \cdots$ as corresponding to an increasing sequence of subideals of finite type $J_1 \subseteq J_2 \cdots$, where $J_k \subset I^q_k(x_0)$ for all $k \geq 1$, then Corollary 2.10 implies the result. By Proposition 2.13 there exists a subideal of finite type $J_1 \subset I^q_1(x_0)$ such that $\mathcal{V}(J_1) = \mathcal{V}(I^q_1(x_0))$ on $U_{x_0}$. Given a subideal of finite type $J_{s-1} \subset I^q_{s-1}(x_0)$ such that $\mathcal{V}(J_{s-1}) = \mathcal{V}(I^q_{s-1}(x_0))$ on $U_{x_0}$, we would like to produce the next subideal $J_s$. We start with $J_{s-1}$ in the proof of Proposition 2.13 instead of the zero ideal and keep adding elements of $I^q_{s-1}(x_0)$ until $\mathcal{V}(J_s) = \mathcal{V}(I^q_s(x_0))$ on $U_{x_0}$. Corollary 2.10 guarantees only finitely many elements of $I^q_s(x_0)$ need to be added just as in the proof of Proposition 2.13.

We now recall Theorem 3 proven by Eric Bedford and John Erik Fornæss in 1981 in [1]. This theorem gives a generalization to $C^\infty$ boundaries of the Diederich-Fornæss theorem in [10] that provides the crucial geometrical step connecting the failure of the Kohn algorithm to generate the whole ring with the existence of a complex variety in the boundary of the domain:
Theorem 4.3. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and let $M \subset b\Omega$ be a smooth submanifold. If $M$ has holomorphic dimension $q$ with respect to $b\Omega$ at some point $p \in M$, then there exists a germ of a complex $q$-dimensional manifold with $V \subset b\Omega$. Further, if $V$ cannot be chosen so that $V \cap M \neq \emptyset$, then there is a manifold $V' \subset b\Omega$ with complex dimension $q + 1$.

We are finally ready to tackle the proof of the equivalence of types.

Proof of Theorem 4.3 As explained in the introduction, the implication (iii) $\implies$ (ii) is the only one that needs to be proved. We will prove the contrapositive statement that the failure of the Kohn algorithm to terminate at the whole ring, negation of (ii), implies the existence of a holomorphic variety in the boundary of the domain, negation of finite D’Angelo type (iii). Let $x_0 \in b\Omega$ be any point on the boundary of the domain. By Proposition 4.2, there exists a neighborhood $U_{x_0}$ of $x_0$ and some natural number $k$ such that $V^j(x_0) = V^k(x_0)$ for all $j \geq k$ and all $x \in U_{x_0}$. By assumption, $T^j_k(x_0) \neq C(U_{x_0})$, so $V^k(x_0) \neq \emptyset$. By Proposition 2.13 there exists a subideal of finite type $J_k \subset T^k_k(x_0)$ such that $V(J_k) = V^k_k(x_0)$ on $U_{x_0}$. We apply the Lojasiewicz Nullstellensatz, Theorem 2.13, to $J_k$ to conclude that

$$\sqrt{\mathcal{J}_k}\bigg|_{U_{x_0}} = \mathcal{I}(V(J_k))\bigg|_{U_{x_0}}.$$  

Since $J_k \subset T_k^k(x_0) = \sqrt{T_k^k(x_0)}$ and $V(J_k) = V_k^k(x_0) = V(T_k^k(x_0))$ on $U_{x_0}$, then

$$\sqrt{T_k^k(x_0)}\bigg|_{U_{x_0}} = \mathcal{I}(V(T_k^k(x_0)))\bigg|_{U_{x_0}} = \mathcal{I}(V(J_k))\bigg|_{U_{x_0}}.$$  

$V^j(x_0) = V^k(x_0)$ for all $j \geq k$. Proposition 3.7 and Lemma 3.5 together imply that $V^k_k(x_0) \cap U_{x_0}$ has holomorphic dimension at least $q$. By Proposition 2.17, $V^k_k(x_0) \cap U_{x_0}$ has an open and dense set of smooth points, so Theorem 4.3 implies that there exists a complex manifold $\mathcal{W} \subset U_{x_0} \cap b\Omega$ such that $\dim C \mathcal{W} \geq q$. This contradicts condition (iii), finite D’Angelo type.  

References

[1] Eric Bedford and J. E. Fornaess. Complex manifolds in pseudoconvex boundaries. *Duke Math. J.*, 48(1):279–288, 1981.
[2] Edward Bierstone and Pierre D. Milman. Resolution of singularities in Denjoy-Carleman classes. *Selecta Math. (N.S.)*, 10(1):1–28, 2004.
[3] Émile Borel. Sur la généralisation du prolongement analytique. *C. R. Acad. Sci. Paris*, 130:1115–1118, 1900.
[4] Émile Borel. Sur les séries de polynômes et de fractions rationnelles. *Acta Math.*, 24:309–387, 1901.
[5] David Catlin. Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem. *Ann. of Math. (2)*, 117(1):147–171, 1983.
[6] David Catlin. Boundary invariants of pseudoconvex domains. *Ann. of Math. (2)*, 120(3):529–586, 1984.
[7] David Catlin. Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains. *Ann. of Math. (2)*, 126(1):131–191, 1987.
[8] C. L. Childress. Weierstrass division in quasianalytic local rings. *Canad. J. Math.*, 28(5):938–953, 1976.
[9] John P. D’Angelo. Real hypersurfaces, orders of contact, and applications. *Ann. of Math. (2)*, 115(3):615–637, 1982.
[10] Klas Diederich and John E. Fornaess. Pseudoconvex domains with real-analytic boundary. *Ann. Math. (2)*, 107(2):371–384, 1978.

[11] Lars Hörmander. $L^2$ estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.

[12] Irving Kaplansky. *Commutative rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[13] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math. (2)*, 78:112–148, 1963.

[14] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. II. *Ann. of Math. (2)*, 79:450–472, 1964.

[15] J. J. Kohn. Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds. *Trans. Amer. Math. Soc.*, 181:273–292, 1973.

[16] J. J. Kohn. Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions. *Acta Math.*, 142(1-2):79–122, 1979.

[17] S. Mandelbrojt. Sur les fonctions indéfiniment dérivables. *Acta Math.*, 72:15–29, 1940.

[18] Andreea Nicoara. Equivalence of types and Catlin boundary systems. *Preprint. arXiv:0711.0429v1 [math.CV]*.

[19] Jack Ohm and R. L. Pendleton. Rings with noetherian spectrum. *Duke Math. J.*, 35:631–639, 1968.

[20] Federica Pieroni. On the real algebra of Denjoy-Carleman classes. *Selecta Math. (N.S.)*, 13(2):321–351, 2007.

[21] Vincent Thilliez. On quasianalytic local rings. *Expo. Math.*, 26(1):1–23, 2008.

[22] René Thom. On some ideals of differentiable functions. *J. Math. Soc. Japan*, 19:255–259, 1967.

[23] Jean-Claude Tougeron. *Idéaux de fonctions différentiables*. Springer-Verlag, Berlin, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71.

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