Vertex 2-coloring without monochromatic cycles

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Abstract

In this paper we study a problem of vertex two-coloring of undirected graph such that there is no monochromatic cycle of given length. We show that this problem is hard to solve. We give a proof by presenting a reduction from variation of satisfiability (SAT) problem. We show nice properties of coloring cliques with two colors which plays pivotal role in the reduction construction.

I. Introduction

Vertex coloring problems (VCP) have been studied extensively since the inception of graph theory. In classical form, problem of $k$-coloring a graph is stated like this: can we color vertices of a graph using $k$ different colors, so that no neighbouring vertices have the same color? It is known that this problem is NP-complete [8]. VCPs have received much attention in the literature not only for its theoretical aspects and difficulty from the computational point of view, but also for its real world applications, for example in: scheduling [9], timetabling [6], register allocation [4], train platforming [2], frequency assignment [7], communication networks [12] and many other engineering fields.

In this paper we study a variation of the coloring problem. Using only two colors we want to color the vertices, so that there is no monochromatic cycle of given length. There have been some research in solving a slightly different problem: is there a 2-coloring such that there exists no monochromatic cycles (of any length). This problem can be viewed as partitioning a graph into two induces forests and it is known to be NP-complete [13] for directed graphs. Another result worth mentioning is by Nobinon et al. [10] where authors show that this problem is NP-complete even for oriented graphs. They also give implementation of three exact algorithms and some inapproximability results. The motivation to study this class of problems lies in economics – 2-coloring without monochromatic cycles can be used in the study of rationality of consumption behavior.

Many more papers have been written on subject of acyclic coloring (or partitioning). Papers relevant to ours include (among many others): [3], [11], [1].

II. Preliminaries

The purpose of this section is to introduce reader to notation used in later chapters as well as definitions of studied problems. Let $G = (V, E)$ be an undirected, unweighted graph. The cycle in $G$ is a vertex disjoint, closed, simple path in $G$. We denote $C_k$ to be a set of all cycles in $G$ of length $k$. Let $c : V → \{r, b\}$ be a mapping that for each vertex in $V$ assigns one of two colors (red or blue). We will call any such $c$: the coloring of graph $G$. Furthermore, we will say that given coloring $c$ is valid, if a certain predicate $P(c)$ is true. Let $K_n$ be a clique of size $n$, that is: a graph with $n$ vertices in which every vertex is connected by an edge to any other vertex.

Let $(2, k)$-COL be the decision problem of whether there exists a valid 2-coloring for given graph. We give the validity predicate $P_k(c)$ below. It is true only if the coloring $c$ does not contain any cycles of size $k$ with vertices of the same color.

$$P_k(c) \equiv \forall Q ∈ C_k \exists u, v ∈ Q \ c(u) ≠ c(v)$$

Formally, our problem can be expressed as:

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Theorem 2.1. For any integer \( k \geq 3 \), \((2,k)\)-\(\text{COL}\) is \(\text{NP}\)-complete.

In order to prove theorem 2.1 we will prove the following theorem:

Theorem 2.2. For any integer \( k \geq 3 \), there exists a computable function \( f \), such that for any boolean formula \( \phi \), \( \phi \in k\text{-NAE-SAT} \) if and only if \( f(\phi) \in (2,k)\text{-COL} \).

III. Two-coloring without monochromatic triangles

In this section we prove theorem 2.2 for \( k = 3 \). Let \( \phi \) be a boolean formula in 3CNF with \( n \) variables \( x_1, \ldots, x_n \) and \( m \) clauses \( C_1, \ldots, C_m \). We construct desired graph \( G_\phi \) in the following way. Let us begin by showing an abstract form of \( G_\phi \). The reduction consists of three gadgets: one for each variable, one for each clause, and one for each super-edge. The super-edge \( \{u, v\} \) is an edge with a property, that any valid coloring \( c \) implies that \( c(u) \neq c(v) \). For starters, assume that we already have such edges at our disposal. This is how we would construct \( G_\phi \): a gadget for variable \( x \) consists of two vertices labeled \( x \) and \( \neg x \) connected by a super-edge. Gadget for clause \( C = (u \lor v \lor w) \) consists of a triangle with vertices labeled \( u, v \) and \( w \). We connect each literal from variable gadget to its every occurrence in clause gadgets using super-edges. Example is given in figure 1 for formula \( \phi = (x_1 \lor \neg x_1 \lor x_2) \land (\neg x_2 \lor x_3 \lor \neg x_3) \). Dashed lines represent super-edges. We prove that this is indeed the correct reduction.

**Lemma 3.3.** For any given \( \phi \), graph \( G_\phi \) has a property, that:

\[ \phi \in 3\text{-NAE-SAT} \iff G_\phi \in (2,3)\text{-COL} \]

**Proof.** First we assume that \( \phi \in 3\text{-NAE-SAT} \) and let \( \sigma(x) \) be the truth assignment that certify it. Each vertex with non-negated label \( x \) in vertex gadgets is colored red if \( \sigma(x) = T \) and blue otherwise. Coloring of every other vertex is forced by super-edges. Notice that the only place where there could be any monochromatic triangle is in some clause gadget. We cannot make that triangle using mixture of vertices from other clause gadgets or vertex gadgets because we always have to pass through a super-edge, hence we change a color of vertices on our path. Now if we assume on the contrary, that some clause \( C = (u \lor v \lor w) \) form a monochromatic triangle, then either \( \sigma(u) = \sigma(v) = \sigma(w) = T \) or \( \sigma(u) = \sigma(v) = \sigma(w) = F \), which gives a contradiction.

Now let \( c \) be the valid coloring of \( G_\phi \). Since \( G_\phi \) has no monochromatic triangles, and from the property of super-edge we simply assign value \( T \) for all variables from vertex gadgets that have color red, and \( F \) otherwise. This gives an assignment \( \sigma(x_1, \ldots, x_n) \) that proves that \( \phi \in 3\text{-NAE-SAT} \). To see that, observe that every clause corresponding to clause gadget will have at least one literal that is true, and at least one that is false, because this clause gadget does not form a monochromatic triangle, which was assumed. \qed

All we have to do now is construct a gadget for super-edge. Such gadget need to have a property, that some selected edge \( \{x, y\} \) in that gadget will always...
have \( c(x) \neq c(y) \), for any valid coloring \( c \) of that gadget (a valid coloring also has to exist). An example of the gadget is shown in figure 4. On the left picture edge \( \{x, y\} \) is pointed out. In the middle we have an example of valid coloring, and on the picture on the right we see how coloring \( \{x, y\} \) in one color gives a contradiction (vertex with a question mark cannot be colored neither red, nor blue). The existence of this gadget completes the proof of theorem 2.2 for \( k = 3 \) (and also theorem 2.1 with additional observation that our reduction is polynomial with respect to size of \( \phi \)).

We argue, that even if a super-edge in figure 4 is enough to verify the genuineness of theorem 2.2 (for \( k = 3 \)), it is not elaborate. We give a better construction of the gadget that uses a certain coloring property of \( K_4 \). Our method is also easier to generalize for \( k > 3 \).

The basic observation is that when we color any two vertices of \( K_4 \) in one selected color – lets say red – then the other two vertices will have to be colored blue (otherwise there would be a monochromatic triangle). Now if we were to loop another \( K_4 \) to those blue vertices (see figure 2) then the two non-colored vertices would have to be red, and so on, and so on. With this we can create strings of \( K_4 \)-s.

\[
\begin{align*}
|E(G_\phi)| &= |E_c| + |E_v| + |E_s| \\
&= 3m + 0 + 25(3m + n) \\
&= 78m + 25n
\end{align*}
\]

\[
\begin{align*}
|V(G_\phi)| &= |V_c| + |V_v| + |V_s| \\
&= 3m + 2n + (10 - 2)(3m + n) \\
&= 24m + 10n
\end{align*}
\]

This shows that reduction can be performed in polynomial time (with respect to \( n \) and \( m \)) and therefore completes (yet another) proof of theorem 2.2. But we can improve the reduction even further and push properties of our symmetric gadget to its limit.

We will now show what we call The Necklace Reduction. If we look at a loop of size \( l \), we will spot as many as \( l \) candidates for choosing the edge \( \{x, y\} \). This is easily seen in figure 5. The symmetry of our gadget guarantees, that any edge on the juncture of \( K_4 \)-s can be considered \( x, y \). But that leaves \( l - 1 \) candidates unused. In necklace reduction we get rid of wasting so many useful edges (to some extent). We simply weave all vertex gadgets on a single loop of length \( 2n + 1 \). Vertex gadget for variable

\[
\begin{align*}
\phi
\end{align*}
\]
$x_i$ (for $i = 1..n$) now becomes edge on the juncture of $(2i)$-th and $(2i + 1)$-th $K_4$-s (numeration can start at any arbitrary $K_4$). We leave the rest of reduction the same as before. We have now created a beautiful necklace of which example can be seen in figure 3 (it uses formula from previous example; some labels were omitted).

Number of edges and vertices drops down to:

$|E(G_\phi)| = 78n + 10n + 5, \quad |V(G_\phi)| = 27n + 4n + 2$

We can further improve the necklace by weaving all other super-edges, but the construction is rather complicated. Details will be available in extended version of this paper.

**IV. Two-coloring without monochromatic squares**

In this section we extend our reduction to cycles of length 4. The abstract form of $G_\phi$ remains almost the same and the only difference is that we have squares in place of triangles for clause gadgets. In fact we use the similar graph for higher values of $k$. Proof of correctness is the same as before, so we leave the details to the reader.

The most important part is to construct a gadget for super-edges. Now, we want to create a graph with a selected edge $\{x,y\}$ that there exists a valid coloring (without monochromatic squares) and that in every valid coloring $c: c(x) \neq c(y)$. We use $K_6$ as a building block for the gadget and exploit its coloring property.

In figure 6 on the left we see $K_6$. On the right it is the same $K_6$, but with rearranged edges. Three arbitrarily chosen, disjoint edges have been pointed out and stretched in three different directions. Rest of the edges are less significant so we placed dotted lines in their place. Notice, that when we color vertices of top edge in a single color – let’s say red – then by using easy pigeon hole argument we can conclude, that exactly one of two bottom edges will have both of its vertices colored blue.

To further simplify the $K_6$, imagine that the selected edges become nodes and that there are lines between top node and two bottom nodes. This creates a reverse v-shaped component. The node which has two different colors associated to it, we label as $X$ (see figure 7).

Now we present the trick to our gadget. We build a full binary tree of height 4, consisting of reverse v-shaped components. It follows from coloring property of $K_6$ discussed before, that if we color root node in red, then there exist a path from root to leaf with alternating colors (see figure 8). Notice the analogy to the construction of strings in previous section.
Figure 4: The super-edge gadget

Figure 5: The symmetric super-edge gadget

Figure 8: Super-edge gadget. All leafs are connected to root.

To achieve a contradiction we connect all leafs to the root using two edges for each leaf in a way that they form a square. This completes the construction. We choose root node as \( \{x, y\} \).

Chosing the height 4 for \( T \) is not a coincidence, as using any tree of smaller size would either not lead to contradiction (heights 1 or 3) or would not be colorable – for height 2 we can find a monochromatic square in any coloring. We again leave verification to the reader.

It remains to show that our gadget has a valid coloring. We simply label all nodes by \( X \). We now prove that this will not create any monochromatic square. There are two places in our gadget that require special attention:

- \( P1 \). Connections between inner nodes of the tree, and
- \( P2 \). Connections between leafs and root.

Both of them can be handled in a straightforward way. For the former look at figure \( \square \) where we reverse the process of \( K_6 \)-simplification for some subtree of \( T \). We quickly verify, that there are no monochromatic squares. This is the smallest, nontrivial subtree in which there could lurk some hidden monochromatic squares. Thanks to regular structure and symmetry of full binary trees, any other combinations of nodes need not be checked. One could use induction for formal proof, but we will leave it like this.
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$P_2$ causes some minor troubles. Take a look at figure 10. Notice that we found a monochromatic square. This leads to conclusion that not every coloring that labels each node by $X$ is valid. We can fix that by coloring both leafs so that they form alternating squares with the root (the color is alternating). Now any path that passes from leaf to root have to change the color, so there are no more threat to find a monochromatic square.

|\begin{align*}
|E(G_\phi)| &= |E_c| + |E_v| + |E_s| \\
&= 4m + 0 + 243(4m + n) \\
&= 976m + 243n
\end{align*}|

|\begin{align*}
|V(G_\phi)| &= |V_c| + |V_v| + |V_s| \\
&= 4m + 2n + (62 - 2)(4m + n) \\
&= 244m + 62n
\end{align*}|

This completes the proof of theorem 2.2 for $k = 4$. We see that this is a polynomial reduction. For sake of completeness let’s count number of edges and vertices in a single super-edge gadget, and then in entire graph $G_\phi$:

#edges-in-gadget = 243,  #vertices-in-gadget = 62

**V. The general case**

In this section we finally prove theorem 2.2 for $k > 4$. We do this by expanding the binary tree gadget from last section. The tree will grow exponentially with respect to $k$, but remember that $k$ is a constant associated with the problem $(2, k)$-$COL$, so our reduction will still be polynomial in size of $\phi$ (but very, very big). Our goal now is to construct a graph with a selected edge $\{x, y\}$, that there exists a valid coloring (without monochromatic cycles of length $k$) and that in every valid coloring $c$: $c(x) \neq c(y)$. For now assume that $k$ is even. This will simplify our reasoning.

First we construct a binary tree $T$ consisting of reverse v-shaped components introduced in previous section. Let height of $T$ be $h = 4\lceil \frac{k-1}{2} \rceil$. For $i = 1..\lceil \frac{k-1}{2} \rceil$, we will call all nodes of depth $4i$: cycle-inducing (notice that root and leafs are also cycle-inducing). Let $CI$ be the set of all cycle-inducing nodes in $T$. If we color root node in a single color – let’s say red – then there exists a path $P$ from root to some leaf, with alternating colors. Notice that all nodes in $P \cap CI$ are now colored red. Those nodes will create a monochromatic cycle of length $k$. To achieve this, we add edges between cycle-induced nodes in the following way.

First, we connect root and leafs just like in previous section. Next, for each cycle-induced node of depth $4i$ ($(i = 1..\lceil \frac{k-1}{2} \rceil - 1)$) we connect it to all its descendants on depth $4(i + 1)$ (they also belong to $CI$). We add edges between them the same way we did with root and leafs. The example of how this produces monochromatic cycle is shown on in figure 11. If we take the graph induced by $P \cap CI$, it forms a donut shown in the right picture. We can easily identify a monochromatic cycle of length $k$. 
Figure 11: Super-edge gadget for general case and how to achieve contradiction.

Last thing to do is to prove that there exist a valid coloring of our super-edge gadget. Again we begin with labeling all nodes in tree by $X$. We know from previous section how to handle connections between root and leafs – we have to do the same with all cycle-inducing nodes and their first cycle-inducing descendants. This way we will not be able to form a monochromatic cycle that passes through two different nodes that are in CI.

Note that at this point the gadget is correct only when value $\lfloor \frac{k-1}{2} \rfloor$ is odd. This is true because of the way we color nodes in CI: the coloring of nodes on level $4i$ force the coloring on nodes on level $4(i+1)$. This problem can be easily fixed by expanding tree another 4 levels and treating nodes at level $h-4$ as dummy nodes.

We are left with the case when $k$ is an odd number. Note that the construction above is not working in this case, as we will not achieve a contradiction. The fix is as follows: we change connections between leafs and root. Choose one vertex of root node and connect all vertices in leafs to it. This creates triangles rather than squares and the donut now looks like someone has taken a bite, but we can now find a monochromatic cycle of length $k$ for all odd numbers (if we color root node in red). The valid coloring does not change.

For sake of completeness we count the number of edges and vertices in entire reduction:

$$|V(G_\varphi)| = |V_c| + |V_v| + |V_s| = km + 2n + (2(2^4\lfloor \frac{k-1}{2} \rfloor + 1) - 2)(km + n)$$

Thus, we have proved theorem 2.1.

VI. Conclusions

We have shown that using symmetry, one can conceive many interesting combinatorial structures and in graph theory there is nothing more symmetric and regular than a clique. The obvious question is: can we make the reduction smaller? We have proved that string gadget from section 3 can be used as a tool to greatly decrease the number of edges and vertices used, but we do not know if the same can be said about tree gadget from sections 4 and 5.

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