Low energy behavior of the astrophysical S-factor in radiative captures to loosely bound final states

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Abstract

The low-energy behavior of the astrophysical S-factor for $E1$ direct radiative captures $a(p,\gamma)b$ leading to loosely bound final states ($b=a+p$) is investigated. We derive a first-order integral representation for $S(E)$ and focus on the properties around zero energy. We show that it is the competition between various effects, namely the remnant Coulomb barrier, the initial and final centrifugal barriers and the binding energy, that defines the behavior of the $S(E \rightarrow 0)$. Contrary to previous findings, we prove that $S(E \rightarrow 0)$ is not determined by the pole corresponding to the bound state. The derivative $S'(0)$ increases with the increase of the centrifugal barrier, while it decreases with the charge of the target. For $l_i = l_f + 1$ the increase of the binding energy of the final nucleus increases the derivative $S'(0)$ while for $l_i = l_f - 1$ the opposite effect is found. We make use of our findings to explain the low energy behavior of the S-factors related to some notorious capture reactions: \textsuperscript{7}Be($p,\gamma$)\textsuperscript{8}B, \textsuperscript{14}N($p,\gamma$)\textsuperscript{15}O, \textsuperscript{16}O($p,\gamma$)\textsuperscript{17}F, \textsuperscript{20}Ne($p,\gamma$)\textsuperscript{21}Na and \textsuperscript{22}Mg($p,\gamma$)\textsuperscript{23}Al.

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I. INTRODUCTION

Direct radiative capture cross sections of charged particles $a(p, \gamma)b$ drop so sharply for the low energy region of stellar environment that it is often impossible to measure these rates at the appropriate energies in the laboratory [1]. Then, typically, the direct capture cross sections are measured at higher energies followed by an extrapolation down to zero energy. Therefore, the knowledge of the energy behavior of the cross section as $E \to 0$ is of crucial importance.

When an incident charged particle approaches a target it should penetrate through the Coulomb and centrifugal barriers in the initial state. The capture to a loosely bound state is the transition from the initial scattering state to the tail of the bound state wave function in the final state. This tail is affected by the Coulomb and centrifugal barriers in the final state. Thus, the radiative capture process is affected by four barriers, two Coulomb and two centrifugal barriers, implicitly present through the four parameters $\eta_i, \eta_f, l_i, l_f$. All barriers have a similar effect on the radiative cross section: they decrease the probability of the reaction at sub-barrier energies.

The cross section is usually factorized into the Gamow penetration factor, and the astrophysical S-factor:

$$\sigma(E) = e^{-2\pi \eta_i} \frac{E}{E} S(E).$$

(1)

The Gamow penetration factor is no more than the probability for an $s$-wave proton to penetrate through the pure Coulomb barrier (i.e. the Coulomb potential is extended down to the origin $r = 0$ assuming a pointlike target). Naturally, this factor defines only the gross energy behavior. In fact, as the capture to loosely bound states occurs under the Coulomb barrier, at a large distance from the target, the effective penetration factor is larger than the Gamow penetration factor in Eq. (1). We call remnant Coulomb barrier the remaining energy dependence of the astrophysical factor S(E) due to the initial Coulomb barrier, after removing the Gamow factor. This remnant effect behaves in an opposite way to the normal
It turns out that, depending on the system, S-factors can feature completely opposite behaviors as one approaches zero energy. Despite the long history of S-factor calculations and the numerous papers published on this subject, no satisfactory physical explanation for the different behaviors has been presented. In recent papers [3,4] some aspects of the low energy behavior of the direct capture S-factors were investigated. In [3], the low energy rise of the S-factor was attributed to the pole in the energy plane, located at $E = -\varepsilon$ (with $\varepsilon$ being the binding energy of the final nucleus $b = a + p$ relative to proton threshold). In [4], the integral form for the first three terms of the Taylor expansion were derived, using a potential model. In either work, the physical reasons for specific patterns in the S-factor energy behavior were not considered.

If the rise of $S(E \to 0)$ were in fact caused by the pole how could one understand the decreasing $S(E \to 0)$ for $^{16}\text{O}(p,\gamma)^{17}\text{F}(\text{gr. st.})$, a case where the pole is extremely close to the threshold? Why should the S-factor for $^{16}\text{O}(p,\gamma)^{17}\text{F}(2s_{1/2})$ have a steeper slope near the threshold than the S-factor for $^{7}\text{Be}(p,\gamma)^{8}\text{B}$? What are the crucial parameters defining the behavior of the S-factor for the direct captures to loosely bound states? Is it possible to define general thumb rules? It is clear that accurate numerical values for direct capture S-factors are trivially calculated with present day technology. For that reason analytical works have not been highly appreciated. Regardless, this work is a search for the understanding of the general physical features, based on analytical considerations.

The vast majority of the direct proton radiative capture reactions with interest to Astrophysics, are dominated by $E1$ transitions. We will thus concentrate on the $E1$ transition for the $a + p \to b + \gamma$ process. We also confine ourselves to loosely bound final states. The interest on the behavior of the S-factors for the direct captures reactions goes beyond $p$-shell nuclei. The heavier, $sd$-shell nuclei [3], when near the proton drip line and/or near shell closure, have very small $Q$-values for the radiative capture. Thus the S-factor is dominated by single resonant and direct captures. Consequently, the findings of this work have
implications for a great number of today’s interesting astrophysical cases.

In this paper we use an analytical/integral representation of the transition matrix element for the E1 process and impose the asymptotic approximation. Following, we identify the physical ingredients that affect the low-energy behavior of capture reactions to loosely bound states. In Section II the effect of the pole of $S(E)$ at negative energy is specifically addressed. The threshold behavior is studied in Section III. The competition between the remnant initial Coulomb barrier, the final Coulomb, the initial and final centrifugal barriers, as well as the trace of the singularity at $E = -\varepsilon$ are detailed and discussed. The S-factor dependence on the target charge is also studied. Section IV consists of an application to some important astrophysical capture processes. Finally, in section V, a summary of the results is presented.

II. THE S-FACTOR AROUND THE POLE

For E1 captures to loosely bound states, the reactions under consideration become purely peripheral and a simple solid sphere model is adequate. Such an approach is also used when calculating the non-resonant capture in the R-matrix method [6]. The S-factor for the capture from an initial state of $a + p$ with orbital angular momentum $l_i$, to a final state $J_f$ of $b = a + p$, with the angular orbital momentum $l_f$, can be written as [6]:

\[
S(E) = A \frac{e^{2\pi \eta_i}}{\mu_{ap} E} (k, r_0)^{2L+1} P_l_i \times \left| \frac{1}{r_0^{L+1}} \int_{r_0}^{\infty} dr r^L W_{-\eta_f, l_f+1/2} (2 \kappa r) [F_{l_i}(k, r) G_{l_i}(k, r_0) - F_{l_i}(k, r_0) G_{l_i}(k, r)] \right|^2,
\]

where $A$ is a factor depending on spin-angular characteristics, masses, charges and constants:

\[
A = 2\pi \cdot 10^4 \frac{(2 J_f + 1)}{(2 J_a + 1)(2 J_p + 1)} \frac{1}{2 l_f + 1} \left( \frac{m_L}{m_p} \right)^2 \left( \frac{Z_p}{m_L} + (-1)^L \frac{Z_a}{m_a} \right)^2 \times \frac{(L + 1)(2 L + 1)}{L} \frac{1}{((2 L + 1)!!)^2} C^2 (\mu_{ap} c)^2 m e^2 \frac{e^2}{\hbar c} \lambda^3.
\]

Here $C$ is the asymptotic normalization coefficient of the tail of the projection of the bound state wave function of the final nucleus $b$ onto the two-body channel $a+p$. The Sommerfeld
parameters for initial and final states, \( \eta_i = Z_a e^2 \mu_{ap} / k \) and \( \eta_f = Z_a e^2 \mu_{ap} / \kappa \), are defined in terms of the initial relative momentum of \( a + p \), \( k = \sqrt{2 \mu_{ap} E} \), and the bound state momentum, \( \kappa = \sqrt{2 \mu_{ap} \varepsilon} \), respectively. Note that the initial relative energy is \( E \) and the energy of the final state is \( -\varepsilon \), so the photon has energy \( E + \varepsilon \), momentum \( k_\gamma = (E + \varepsilon) / \hbar c \) and multipolarity \( L \). The penetration factor is defined in terms of the regular and irregular Coulomb functions at \( r = r_0 \):

\[
P_{li} = k r_0 / [F^2_{li}(k, r_0) + G^2_{li}(k, r_0)]
\]

and \( W_{-\eta_f, l_f + 1/2}(2 \kappa r) \) is the Whittaker function describing the radial behavior of the bound state wave function at \( r > r_0 \), where \( r_0 \) is the solid sphere radius. As usual, \( Z_a \) is the charge of \( a \) and \( \mu_{ap} \) is the reduced mass of the \( a + p \) system. The constants used are \( \lambda_p = 0.2118 \) fm for the proton Compton wave number and \( m c^2 = 931.5 \) MeV for the mass atomic unit. The factor \( 10^4 \) is introduced to provide the S-factor in keV b.

When the binding energy of final state is small and the \( a - p \) relative energy is low, nuclear re-scattering \( a - p \) can be neglected. This corresponds to neglecting the term containing the irregular Coulomb function, \( G^2_{li}(k, r_0) \). One should keep in mind that whenever precise values for \( S(E > 0) \) are needed, phase shifts should be taken into account. However, here we are only concerned with the qualitative behavior of \( S(E) \) around zero. In section V comments on the accuracy of our results are presented. Eq. (2) then simplifies to:

\[
S(E) = A \frac{e^{2 \pi \eta_i}}{\mu_{ap} E} (k r_0) (k_\gamma r_0)^2 L + 1 \frac{1}{r_0} \int_{r_0}^\infty dr r^L W_{-\eta_f, l_f + 1/2}(2 \kappa r) F_{li}(k, r)^2.
\]

Both, Eq. (4) and Eq. (5), can be used for the sake of our following arguments.

Next, we realize that when \( k \to 0 \), extending the integral in Eq. (3) from \( r_0 \) to zero is a reasonable approximation, as then the dominant contribution comes from the asymptotic region. Replacing the bound state wave function by the Whittaker function down to zero radius is certainly not accurate for \( k > 0 \), but we have verified that the qualitative features for \( S(E) \) remain.

Using this approximation and the asymptotic expansion of the Whittaker function, we can obtain an analytical result for \( S(E) \), which is not intended to be quantitative but will
contain the main energy dependence as \( k \to 0 \). In the following pages we perform the
detailed derivation, but advise to follow onto Eq. (14) in case the reader is searching for a
final result only.

First, we use the asymptotic expansion of the Whittaker function (at \( r \to \infty \)) [7]:

\[
W_{-\eta_f,j_f+1/2}(2\kappa r) \to \frac{e^{-\kappa r}}{(2\kappa r)^{\eta_f}} \left\{ \sum_{j=0}^{\infty} \frac{(l_f - \eta_f)_j \ (l_f + \eta_f + j)_j}{j! \ (2\kappa r)^j} \right\}
\]

with \((l)_0 = 1\) and \((l)_j = 1 \cdot l \cdot (l - 1) \cdot \ldots (l - (j - 1))\). Substituting in Eq. (5) we arrive at:

\[
S(E) = A \frac{e^{2\pi \eta_i}}{\mu_{ap} E} (k r_0) (k \gamma r_0)^2 L + 1 \\
\times \left[ \frac{1}{r_0} \sum_{j=0}^{\infty} \frac{(l_f - \eta_f)_j \ (l_f + \eta_f + j)_j}{j!} \int_{r_0}^{\infty} dr \ r^L e^{-\kappa r} \frac{1}{(2\kappa r)^{\eta_f+j}} F_{l_i}(k, r) \right]^2.
\]

Before the integration over \( r \) is performed, by including the explicit behavior of the regular
Coulomb function, we can estimate the analytical behavior of the integral in each term of
the asymptotic expansion,

\[
T_j = \int_{r_0}^{\infty} dr \ r^L e^{-\kappa r} \frac{1}{(2\kappa r)^{\eta_f+j}} F_{l_i}(k, r).
\]

The regular Coulomb function is given by:

\[
F_{l_i}(k, r) = C_{l_i}(\eta_i) \ (k r)^{l_i+1} e^{-ikr} \ _1F_1[l_i + 1 - i\eta_i, 2l_i + 2; 2ikr],
\]

where \(_1F_1[l_i + 1 - i\eta_i, 2l_i + 2; 2ikr]\) is a confluent hypergeometric function (see Appendix, 
Eq. (A.3)), and

\[
C_{l_i}(\eta_i) = 2^{l_i} e^{-\pi \eta_i/2} |\Gamma(l_i + 1 + i\eta_i)| \Gamma(2l_i + 2).
\]

The asymptotic behavior of the Coulomb function as \( r \to \infty \) is well known:

\[
F_{l_i}(k, r) \sim \sin(kr - \frac{\pi}{2} l_i - \eta_i \ln 2kr + \sigma_{l_i}),
\]

with \( \sigma_{l_i} = arg \Gamma(l_i + 1 + i\eta_i) \). Although for the loosely bound states we need to know the
Coulomb wave function only for \( r > r_0 \), the physical meaning of the \( C_{l_i}(\eta_i) \) coefficient comes
from the behavior of \( F_{l_i}(k, r) \) at small \( r \):
\[ F_{ll}(k, r) \approx C_{ll}(r) \cdot (kr)^{l_l}. \]  

(12)

It is clear from Eq. (12) that \( C_{ll}(r) \) defines the probability of finding a charged particle in the vicinity of \( r = 0 \), under the influence of the Coulomb and centrifugal potentials. Note that

\[
C_{ll}^2(\eta) = 2^{2l+1} \pi \eta \frac{1}{e^{2\pi \eta} - 1} \frac{1}{k^{2l+1}} \frac{1}{\Gamma(2l + 2)} \prod_{j=1}^{l_l} (j^2 k^2 + \eta^2 \kappa^2). \tag{13}
\]

In brief, we have replaced the lower integration limit \( r_0 \) by zero, we have used the asymptotic expansion for the bound state Eq. (6) and the explicit representation of the Coulomb scattering wave in Eq. (9). After performing the integration, the S-factor simplifies to:

\[
S(E) = A \frac{e^{2\pi \eta}}{\mu_a p} k^{2l+3} C_{ll}^2(\eta) \frac{1}{(2\kappa)^2 \eta} \frac{1}{|(k - i \kappa)|^2 - \eta f + L + l_i - j, 2l_i + 2, \frac{2k}{k - i \kappa}|^2} \times \left( \sum_{j=0}^{\infty} \Upsilon_j \frac{(k - i \kappa)}{2\kappa} \right)^j \left( \sum_{j=0}^{\infty} \Upsilon_j \right) \left( \frac{\Gamma(2 - \eta f + L + l_i - j)}{j!} \right), \tag{14}
\]

where the expansion coefficients are given by:

\[
\Upsilon_j = (l_f - \eta_f)_j (l_f + \eta_f + j)_j \frac{\Gamma(2 - \eta_f + L + l_i - j)}{j!}. \tag{15}
\]

We note that the expression for \( S(E) \) in Eq. (14) is completely general. In particular, the expression for the neutron overlap integral can be obtained by calculating the limit \( \eta_f, \eta_i \to 0 \).

Substituting Eq. (13) in Eq. (14) one arrives at:

\[
S(E) = A 2^{2l+1} \pi \eta_f \kappa \frac{1}{\Gamma(2l_i + 2)} \prod_{j=1}^{l_l} (j^2 k^2 + \eta^2 \kappa^2) \times k^{2L+1} \left( \frac{1}{(2\kappa)^2 \eta} \frac{1}{|(k - i \kappa)|^2 - \eta f + L + l_i - j, 2l_i + 2, \frac{2k}{k - i \kappa}|^2} \sum_{j=0}^{\infty} \Upsilon_j \right) \left( \frac{(k - i \kappa)}{2\kappa} \right)^j \times 2F_1[l_i + 1 - i\eta_i, 2 - \eta_f + L + l_i - j, 2l_i + 2; \frac{2k}{k - i \kappa}|^2]. \tag{16}
\]

This equation bears all the necessary features to investigate the behavior of the S-factor approaching \( E \to -\varepsilon \).
Beforehand, we make some obvious remarks concerning the pole in $S(E)$. Taking into
count the asymptotic behavior of $F_l(k, r)$ given in Eq. ([1]), we conclude that the singu-
larity of $T_j$ on the physical $k$-half-plane ($Imk > 0$) is due to the divergence of the radial
integral on the upper limit ($r \rightarrow \infty$) at $k \rightarrow i \kappa$. More explicitly, $T_j$ behaves as:

$$T_j \propto \int_{r_0}^{\infty} dr r^L \frac{e^{-\kappa r}}{(r)^{\eta r+j}} e^{-ikr} r^{i\eta_l}.$$  \hspace{1cm} (17)

When $k \rightarrow i\kappa$ then $i\eta_f \rightarrow \eta_f$. Then, it is easy to verify that:

$$T_j \propto (k - i \kappa)^{j-L-1} \text{ for } j < L + 1$$  \hspace{1cm} (18)

$$\propto \ln(k - i \kappa) \text{ for } j = L + 1.$$

Note that $T_j$ is finite at $k = i \kappa$ for $j > L + 1$. Besides the square of the integral, $S(E)$
contains also a factor $k^{2L+1}$ (see Eq.[7]). Consequently it is the term $j = 0$ in the expansion
that generates the pole in $S(E)$, as first indicated in [3]. All higher order terms, $j > 0$, go to
zero as $k \rightarrow i \kappa$. Let $A_j(E)$ be the agglomerate of all energy dependent factors in Eq. ([7]):

$$A_j(E) = \frac{1}{E} e^{2\pi i \eta_l (k r_0)(k, r_{0})^{2L+1} |T_j|^2}.$$  \hspace{1cm} (20)

Then its behavior as $k \rightarrow i\kappa$ (or $E \rightarrow -\varepsilon$) is:

$$A_j(E) \propto \frac{1}{(E + \varepsilon)^{1-j}} \text{ for } j < L + 1$$  \hspace{1cm} (21)

$$\propto [\ln(E + \varepsilon)]^2 (E + \varepsilon)^{2L+1} \text{ for } j = L + 1.$$  \hspace{1cm} (22)

The term $j=0$ of Eq. (20) does not depend on $l_f$. Hence, if this pole really did govern
the behavior of the S-factor at zero energy, then $S(E)$ should increase rapidly as $E \rightarrow 0^+$,
independently of $l_f$. We know from experiment that this is not the case. In fact, we will
show that the rise of $S(E)$ at $E \rightarrow 0^+$ is dictated by higher order terms while the first order
term, the only one with a pole at $E = -\varepsilon$, actually decreases as $E \rightarrow 0^+$. To this end we
calculate the behavior of $S(E)$ around threshold for positive energies, $E \rightarrow 0^+$, but also for
negative energies, all the way down to $E \rightarrow -\varepsilon^+$.  

This equation bears all the necessary features to investigate the behavior of the S-factor
approaching $E \rightarrow -\varepsilon$. Let us begin with the term $j=0$: 

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For \( k \) necessarily be analytical at \( k \rightarrow 0 \) for the S-factor in the vicinity of the pole. Note that the continuation of \( S(E) \) will not be analyzed in Section III.

Using this result in Eq. (23) we conclude that the leading singular term for \( k \rightarrow i \kappa \):

\[
2F_1[l_i + 1 - i \eta, 2 - \eta_f + L + l_i, 2l_i + 2; \frac{2k}{k - i \kappa}] \sim c_1 (k - i \kappa)^{l_i + 1 - i \eta} + c_2 (k - i \kappa)^{2 - \eta_f + L + l_i}. \tag{24}
\]

Using this result in Eq. (23) we conclude that the leading singular term for \( k \rightarrow i \kappa \) is:

\[
S_{(0)}(E) \sim \frac{1}{(E + \varepsilon)}, \tag{25}
\]

whichever \( l_i, \eta_f \) or \( L \), consistent with the conclusions drawn from Eq. (21).

It is important to realize that the behavior of the hypergeometric function \( 2F_1[l_i + 1 - i \eta, 2 - \eta_f + L + l_i, 2l_i + 2; \frac{2k}{k - i \kappa}] \) is very different around the pole \( (z \rightarrow \infty) \) compared to its behavior around threshold \( (z \rightarrow 0) \). The analytical expression of \( S(E) \) at threshold will be derived in Section II.

\( 2F_1[l_i + 1 - i \eta, 2 - \eta_f + L + l_i, 2l_i + 2; \frac{2k}{k - i \kappa}] \) is an analytic function in the finite \( k \)-plane (except for \( k = \pm i \kappa \)). Hence, assuming that the cuts drawn from branching points \( k = \pm i \kappa \) go to infinity, we can make an analytic continuation of \( 2F_1[l_i + 1 - i \eta, 2 - \eta_f + L + l_i, 2l_i + 2; \frac{2k}{k - i \kappa}] \) into the complex \( k \)-plane. Using Eq. (23) we arrive at an expression for the S-factor in the vicinity of the pole. Note that the continuation of \( S(E) \) will not necessarily be analytical at \( k = 0 \), as the modulus operation may introduce a discontinuity.

For \( k = ip \) and \( E = -p^2 / 2 \mu_{ap} \), Eq. (23) takes the form

\[
S_{(0)}(-|E|) = A \frac{\gamma_0^2}{\Gamma(2 \eta_f +2)} \frac{1}{\Gamma(2 l_i +2)} \prod_{j=1}^{l_i} (j^2 k^2 + \eta_f^2 \kappa^2) \frac{k^{2 L+1}}{(2 \kappa)^{2 \eta_f}}
\times \frac{1}{(k - p)^{2 - 2 \eta_f + L + 2 l_i}} \left(2F_1[l_i + 1 - \eta_i, 2 - \eta_f + L + l_i, 2l_i + 2, \frac{2p}{p - \kappa}]\right)^2. \tag{26}
\]
where \( \eta' = Z_a e^2 \mu_{ap}/p \).

In Fig. 1 we present the energy behavior of \( S(0)(E) \) for \( a(p, \gamma)b \), for masses \( A_a = 7 \) and \( A_b = 8 \) and angular momentum \( l_i = 0, l_f = 1 \) and \( L = 1 \). It mimics \( ^7\text{Be}(p, \gamma)^8\text{B} \), but a different set of proton binding energies are used. The energy interval is \(-\varepsilon < E \leq \varepsilon\). All \( S(0)(E) \) are normalized to unity at zero energy. It is obvious that, even for the smallest binding energy, in which the pole is closest to threshold, \( S(0)(E) \) decreases as \( E \to 0^+ \). Whenever there is a rise of \( S(E) \) around zero, it can only be due to the sum effect of higher order terms in Eq. (16), and not the pole.

### III. THRESHOLD BEHAVIOR OF THE S-FACTOR

In this section, we look at the dependence of the derivative of \( S(E) \) on \( l_i \) and \( l_f \), and analyze the competition between the remnant initial Coulomb barrier and the trace of the singularity \( E = -\varepsilon \), discussed in the previous section. We also investigate the dependence of \( S(E) \) on charge and binding energy.

Using the integral representation of the initial scattering state, and the final bound state, in the asymptotic approximation, one can deduce an integral expression for the \( S \)-factor close to threshold. This derivation is presented in detail in Appendix A and the result we obtain is:

\[
S(E) = B k^2 \gamma \frac{\kappa^{2l+1}}{\eta_f} \prod_{j=0}^{l_i} (j^2 k^2 + \eta^2) \eta^2 \kappa^2
\times \left[ \int_0^\infty ds \mathcal{S}(s) \lim_{\epsilon \to 0} \frac{d^m}{d\epsilon^m} \left[ \frac{1}{((\kappa (1 + 2 s) + \epsilon)^2 + k^2)^{l_i+1}} e^{2\eta_j \arctan \frac{k}{(1 + 2 s) + \epsilon}} \right] \right]^2, \tag{27}
\]

where an energy independent constant \( B \) has been introduced (Eq. A.14), \( \mathcal{S}(s) \) is the spectral function defined in Eq. (A.8) and \( m = l_f + L + 1 - l_i \).

We mentioned that, by taking only the first term of the expansion of the Whittaker function defined in Eq. (6) we can arrive at a similar expression for \( S(0) \) where \( \mathcal{S}(s) \) should be replaced by \( \mathcal{S}_0(s) \) defined in Eq. (A.13). Although \( S(0)(E) \) has a pole at \( k = i \kappa \), as \( k \to +0 \) its first derivative \( S'(0)(E) > 0 \) for all binding energies, as can be verified in Fig. 1. The only
difference between \( S(E) \) and \( S_0(E) \) is in the spectral functions for \( W_{-\eta_f, l_f+1/2} (2 \kappa r) \), Eq. (A.1), and \( W^{(0)}_{-\eta_f, l_f+1/2} (2 \kappa r) \), Eq. (A.2). The spectral functions at \( s \to \infty \) are, from Eq. (A.8) and Eq. (A.15),

\[
S(s) = [s(1 + s)]^{l_f} \left[ \frac{s}{1 + s} \right]^{\eta_f} s^{2l_f} \quad \text{and} \quad S_0(s) = s^{l_f+\eta_f}.
\]

(28)

There are only two competing factors here: the centrifugal barrier \( (l_f) \) and the Coulomb barrier \( (\eta_f) \) of the final-state. Three situations may occur:

i) for \( l_f = \eta_f \), the S-factor is represented exactly by the first term: \( S(E) \equiv S_0(E) \);

ii) for \( l_f < \eta_f \), \( S_0(s)/S(s) \overset{s \to \infty}{\to} \infty \): the contribution of the large \( s \) values is more pronounced for \( S_0(E) \) than for \( S(E) \);

iii) for \( l_f > \eta_f \), \( S(s)/S_0(s) \overset{s \to \infty}{\to} \infty \): the contribution of the large \( s \) values is more pronounced for \( S(E) \).

In brief, if the final-state Coulomb barrier prevails, \( S_0'(0) > S'(0) \) otherwise, if the final-state centrifugal barrier prevails, \( S'(0) > S_0'(0) \).

Only two cases need to be considered for dipole transitions \( L = 1 \): \( m = 1 \) if \( l_i = l_f + 1 \) and \( m = 3 \) if \( l_i = l_f - 1 \). We consider them separately.

A. \( m = 1 \) when \( l_i = l_f + 1 \).

After taking the required derivative in Eq. (27), the expression for \( S(E) \) is

\[
S(E) = 4B \eta_f k^2 \frac{k^{L+1}}{\kappa} \prod_{j=1}^{l_i} (j^2 k^2 + \eta_f^2 \kappa^2) I^2(k^2),
\]

(29)

with

\[
I(k^2) = \int_0^\infty ds \left[ s(1 + s) \right]^{l_f} \left[ \frac{s}{1 + s} \right]^{\eta_f} \left[ \frac{(l_i + 1) (1 + 2 s) + \eta_f}{(1 + 2 s)^2 + E/\varepsilon} \right]^{l_i+2} \frac{\varepsilon^{2 \eta_i \arctan \frac{1}{\varepsilon} (1 + 2 s)}}{f_2(s,k)}.
\]

(30)

Let us first make some general qualitative remarks concerning the behavior of the integrand near threshold \( E \to 0^+ \). These will be useful later. In all cases considered, the
integrand peaks at $s_m > 0$. This peak occurs for a larger $s_m$, for $l+1 \to l$ than for $l \to l-1$, reflecting the effect of the centrifugal barriers in the initial and final state. In addition, the peak of the integrand also shifts to larger $s$-values when the energy increases.

Remarkably, the energy dependent part of the $S$-factor depends on $E$ rather than on $k$ and can be written as a positive definite function:

$$F(E) = \left[ k^2 \gamma \prod_{j=1}^{l_i} \left( j^2 k^2 + \eta_j^2 \kappa^2 \right) \right] I^2(k^2).$$

The part in square brackets decreases monotonically at $E \to 0$ (positive derivative) while the integral $I(k^2)$ increases as $E \to 0$ (negative derivative). The sign of the first derivative of $S(E \to 0^+)$, or in other words, the low-energy behavior of the $S$-factor, is defined by the sign of the derivative of $F(E)$.

Next, we focus our attention on $I(k^2)$. The integrand of $I(k^2)$ is a positive definite function of $s$ and $k$ and the two functions in the integrand of Eq. (31), $f_1(k^2, s)$ and $f_2(s, k)$, decrease monotonically with increasing energy. Now the interplay of the various ingredients becomes more transparent. Clearly, $f_2(s, k^2)$ is a remnant of the initial Coulomb barrier, as well as the second term of function $f_1(s, k^2)$, $\eta_f/[(1 + 2 s)^2 + E/\varepsilon]^l_{i+2}$.

Note that, as $l_i = l_f + 1$, the initial orbital cannot be an $s$-wave. As increasing $l_i, l_f$ shifts the peak of the integrand to higher values of $s$, the integrand, and, hence, $I(k^2)$, becomes less sensitive to energy variations. If $I'(E) < 0$ increases as $l_i = l_f + 1$ increases, the factor $k^2 \gamma^{l+1} \prod_{j=1}^{l_i} (j^2 k^2 + \eta_j^2 \kappa^2)$ eventually prevails leading to the positive derivative $S'(E)$.

In addition, increasing $\varepsilon$ shifts the pole away from threshold, such that $I(k^2)$ becomes less varying with energy around threshold. So, increasing the binding energy, decreases $|I'(k^2 = 0)|$.

We can derive an expression for the $S$-factor near threshold factorizing the energy dependence. First we rewrite $I(k^2)$ as

$$I(k^2) = \frac{1}{(E + \varepsilon)^2} \tilde{I}(k^2),$$

$$E \equiv 0^+ I(0) + I'(0) E + O([E/\varepsilon]^2),$$

$$E \equiv 0^+ I(0) + I'(0) E + O([E/\varepsilon]^2),$$

$$12$$
where \( \tilde{I}(0) \neq 0 \).

Let us estimate the impact of the pole singularity on \( I(k^2) \) in the physical region at \( E \to 0^+ \). In the case of a pure second order pole, \( \tilde{I}(k^2) \equiv \text{const} \), \( -\varepsilon I'(0)/I(0) = 2 \) (the order of the pole). Otherwise, the Taylor expansion holds: \( \tilde{I}(k^2) = \tilde{I}(0) + \tilde{I}'(0) E + O((E/\varepsilon)^2) \), where \( \tilde{I}'(0)/\tilde{I}(0) > 0 \). Then we get \( -\varepsilon I'(0)/I(0) = [2 - \varepsilon \tilde{I}'(0)/\tilde{I}(0)] \). If \( [\varepsilon \tilde{I}'(0)/\tilde{I}(0)] > 0 \), which corresponds to our case, the power of the singularity in \( I(k^2) \) as \( E \to 0^+ \) is weaker than 2, i.e. \( \varepsilon I'(0)/I(0) > -2 \).

The explicitly terms appearing in Eq. (33) can be derived:

\[
I(0) = \int_0^\infty ds (s + 1)^{2\ell_f + 3} \left[ \frac{1}{s} \right]^{\ell_f} \left[ \frac{(l_i + 1)(1 + 2s) + \eta_f (3(1 + 2s)^2 + 6)}{3(2s)^{2\ell_i + 6}} \right] e^{2\eta_f/(1+2s)} \quad \text{and}
\]

\[
I'(0) = -\frac{1}{\varepsilon} \int_0^\infty ds (s + 1)^{2\ell_f + 3} \left[ \frac{1}{s} \right]^{\ell_f} \left[ \frac{(l_i + 1)(1 + 2s) + \eta_f (3(1 + 2s)^2 + 6)}{3(1 + 2s)^{2\ell_i + 6}} \right] \left[ 3(l_i + 2)(1 + 2s) + 2\eta_f \right] e^{2\eta_f/(1+2s)}. \quad (34)
\]

Finally, we can also write down the threshold behavior of \( S(E) \), as \( S(E) \to 0^+ S(0) + S'(0) E + O([E/\varepsilon]^2) \) where:

\[
S(0) = 4\mathcal{B} \varepsilon^3 \kappa^{2\ell_f + 3} \eta_f^{2\ell_i + 1} \kappa^{2\ell_i} I^2(0), \quad (35)
\]

and

\[
S'(0) = 4\mathcal{B} \varepsilon^2 \kappa^{2\ell_f + 3} \eta_f^{2\ell_i + 1} \kappa^{2\ell_i} I^2(0) \left[ 3 + \frac{1 + 4 + \ldots + l_i^2}{\eta_f^2} + 2 \varepsilon \frac{I'(0)}{I(0)} \right]. \quad (36)
\]

From Eq. (36), it is clear that the sign of the derivative of the S-factor at \( E \to 0 \) is defined by the sign of \( 3 + (1 + 4 + \ldots + l_i^2)/\eta_f^2 - 2 \varepsilon |I'(0)/I(0)| \). Note that the larger is \( l_i \), the larger becomes the positive term in this factor.

We now illustrate these ideas with a few examples which relate to physical cases.

(i) \( l_i = 1, l_f = 0 \).

In Fig. 2 we present the behavior of \( S(E) \) for the \( l_i = 1 \to l_f = 0 \) capture \(^7\text{Be}(p, \gamma)^8\text{B}\)”, spanning a few values for the binding energy of \(^8\text{B}\). All \( S(E) \) are normalized to unity at \( E = 0 \). The smaller the binding energy, the larger the increase of \( S(E \to 0) \). The centrifugal
barrier in the initial state is not strong enough to overcome the negative derivative caused by the remnant Coulomb ($\eta_i$) and the singularity at $E = -\varepsilon$. In addition there is the dependence on the spectral function $[\frac{s}{1+s}]^{\eta_f}$, where $\eta_f$ depends on the inverse square root of the binding energy. For $\varepsilon = 0.06, 0.137, 0.35, 0.6$ MeV the Coulomb parameters are $\eta_f = 2.4, 1.6, 1.0, 0.76$, respectively. Replacing $l_f = 0$ in Eq. (34) we see that, for larger $\eta_f$, a significant contribution to the integral comes from smaller $s$-values, reducing $\varepsilon I'(0)/I(0) < 0$ and $1/\eta_f^2$, so that $\left[3 + 1/\eta_f^2 + 2\varepsilon I'(0)/I(0)\right]$ becomes negative. On the other hand, for smaller $\eta_f$, which corresponds to the pole at $E = -\varepsilon$ moving further away from threshold, the slope $\varepsilon I'(0)/I(0)$ and $1/\eta_f^2$ increase and, correspondingly, $\left[3 + 1/\eta_f^2 + 2\varepsilon I'(0)/I(0)\right]$ can become positive.

We also compare $S(E)$ with $S(0)(E)$, obtained retaining only the first term of the Whittaker expansion (Fig. 4). Contrary to $S'(E)$ we find that all calculated cases have positive $S'(0)(0+)$. This confirms that the negative derivative of $S(E \to 0)$ is not due to the nearby pole as concluded in Section 4. As $\eta_f > l_f = 0$, $S(0)'(0)$ is always larger than $S'(0)$.

(ii) $l_i = 2, l_f = 1$.  

In Fig. 5 are the S-factors for the $l_i = 2 \to l_f = 1$ capture $^7\text{Be}(p, \gamma)^{8}\text{B}$, assuming a set of binding energies for ”$^8\text{B}”$. Strikingly, all binding energies produce a positive derivative for $S(0)$. In this case the centrifugal barriers in the initial and final states overcome the remnant Coulomb barrier and the pole at $E = -\varepsilon$. Thus, even at the lowest binding energy, $\varepsilon = 0.06$ MeV, $S'(0)$ is positive. When the binding energy increases, the pole moves away from threshold (decreasing $\varepsilon I'(0)/I(0)$) and $\eta_f$ decreases, which means that $\left[3 + 5/\eta_f^2 + 2\varepsilon I'(0)/I(0)\right]$ becomes larger, as well as $S'(0)$. The general behavior for $l_i > 2 \to l_f > 1$ is identical.

Fig. 6 also shows the behavior of $S(0)(E)$, represented by the grey lines. This figure is a clear illustration of the points mentioned in the beginning of this section. For the cases $\varepsilon = 0.06$ and $\varepsilon = 0.137$ MeV, $\eta_f > l_f (= 1)$ and then $S(0)'(0) > S'(0)$. The binding $\varepsilon = 0.35$ MeV corresponds to $\eta_f = 1$ for which $S(0)(0) \equiv S(0)$. Finally, for $\varepsilon = 0.6$ MeV, $\eta_f < l_f$ and
consequently $S_0(0)'(0) < S'(0)$.

**B. $m = 3$ when $l_i = l_f - 1$.**

Directly from Eq. (27) we get

\[
S(E) = \mathcal{B} k^3 \frac{\gamma^{2l_f+1}}{\eta^2_f} \prod_{j=0}^{l_i} (j^2 k^2 + \eta_f \kappa^2) 
\times \left| \lim_{\varepsilon \to 0} \int_0^\infty dsS(s) \frac{d^3}{d\varepsilon^3} \left[ \frac{1}{(\kappa[1 + 2s] + \varepsilon)^2 + k^2]} \right] \right| \]

(37)

The explicit expression for the third derivative of the integrand is tedious, but the analysis is in all similar to the case with $m = 1$. The energy dependence of the S-factor has the same form as Eq. (31),

\[
F(E) = \left[ k^3 \prod_{j=0}^{l_i} (j^2 k^2 + \eta_f^2 \kappa^2) \right] I_3^2(k^2),
\]

(38)

only the integral changes due to the third derivative, which we here represent as $I_3(k^2)$. We keep in mind that the initial state now can be an s-wave ($l_i = 0$). $S(0)$ and $S'(0)$ are given by expressions identical to Eq. (36), replacing $I(0)$ and $I'(0)$ by $I_3(0)$ and $I_3'(0)$, which have rather complicated explicit forms. We apply these expressions for the two most common cases found in Astrophysics: transitions 0 → 1 and 1 → 2.

(i) $l_i = 0 \rightarrow l_f = 1$.

Similarly to what was done in the previous subsection, we consider $S(E)$ for the $l_i = 0 \rightarrow l_f = 1$ capture in the case of $^7\text{Be}(p, \gamma)^8\text{B}$ taking a set of binding energies for $^8\text{B}$. Results are presented in Fig. 4. In this case, there is no initial centrifugal barrier and the centrifugal barrier in the final state alone cannot overcome the rising of $I_3^2(E \sim 0)$ due to the pole and the remnant Coulomb. From Eq. (31) one can verify that the asymptotics of spectral function which goes as $s^2$ (see Eq. (28)), trends to push the peak of the integrand to high $s$-values. This would make $I_3(k^2)$ less sensitive to variation of $E$, yet it is the third derivative that enhances this sensitivity, creating the net effect.
By comparing Fig. 2 and Fig. 4, one realizes that $S'(0)$ increases with binding energy for both cases. While for $l_i = 1 \to l_f = 0$ the sign of $S'(0)$ becomes positive, all $l_i = 0 \to l_f = 1$ show $S'(0) < 0$, suggesting that the centrifugal barrier has a stronger impact in the initial state.

In Fig. 4 the comparison between $S(E)$ and $S(0)(E)$ (grey lines) is also done. Whenever $\eta_f > l_f$ ($\varepsilon = 0.06, 0.137$ MeV) $S(0)'(0) > S'(0)$; for $\eta_f = l_f$ ($\varepsilon = 0.35$ MeV) the two functions coincide and for $\eta_f < l_f$ ($\varepsilon = 0.6$ MeV) $S(0)'(0) < S'(0)$.

(ii) $l_i = 1 \to l_f = 2$.

In Fig. 5 are the S-factors for the $l_i = 1 \to l_f = 2$ capture for the reaction $^7\text{Be}(p, \gamma)^8\text{B}^+$ where "$^8\text{B}^+$ is bound by a similar set of binding energies (we chose instead of $\varepsilon = 0.35$ MeV, $\varepsilon = 0.09$ MeV for which $\eta_f = 2$ matching $l_f$). As in Fig. 4, $S'(0) > 0$ for all cases. Clearly here the centrifugal barriers are enough to overcome the influence of the remnant Coulomb barrier and the pole at $E = -\varepsilon$. Note that the factor preceding the integral is stronger in this case: $F(E, 1 \to 2) = k_3^3(k^2 + \eta_f^2\kappa^2)I_3^2(k^2)$, whereas $F(E, 0 \to 1) = k_1^3I_3^2(k^2)$. This factor is responsible for the decrease of $S(E)$ as $E \to 0$.

A subtle feature appears that makes this example contrast with all previously considered cases: when the binding energy increases (the pole moves away from the threshold), $S'(E)$ decreases. To understand this property, one would have to develop the third derivative explicitly. The safe conclusion that can surely be drawn is that the strong centrifugal barrier washes out the dependence on binding energy.

The grey lines in Fig. 5 show the $S(0)(E)$ behavior. Identical conclusions to those obtained in (i) can be drawn from the results: when $\eta_f = l_f$ $S(E)$ coincides with $S(0)(E)$ (grey circles); when $l_f < \eta_f$ then $S(0)'(0) > S'(0)$ (solid lines), the opposite happening when $l_f > \eta_f$ (dashed and dot-dashed lines).
C. Charge dependence

The energy dependence of the S-factors is strongly influenced by the charge of the capturing nucleus. It is through $\eta_i$, the remnant Coulomb in the initial state, and $\eta_f$, the Coulomb parameter of the final state, that this dependence comes in. In order to illustrate the relative interplay between these parameters we chose again the typical case "$^7\text{Be}^\ (p, \gamma)\ ^8\text{B}^\ "$, $\varepsilon = 0.137$ MeV, where both $l_i = 0 \to l_f = 1$ and $l_i = 1 \to l_f = 2$ transitions are considered and the charge of the capturing nucleus is allowed to take the values $Z_a = 0, 4$ and 8 (naturally this last value is unphysical but intends to exaggerate the effect).

In order to isolate these effects we run three different calculations. First, the S-factors resulting from setting $Z_a = 0$, such that none of the Coulomb barrier is present. Secondly, the results of the calculations for which the remnant initial-state Coulomb barrier is set to zero ($\eta_i = 0$), but keeping $\eta_f$. It contains thus only the final-state Coulomb barrier. Finally, the full calculation for $S(E)$, including both initial and final Coulomb barriers. Results are plotted in Fig.(6) for a transition $1 \to 0$ and in Fig.(7) for a transition $2 \to 1$. From these figures we conclude:

1. the final Coulomb barrier increases the S-factor derivative, as a normal penetration barrier would;

2. the larger the charge, the stronger the effect of the final-state Coulomb barrier;

3. the effect of the final-state Coulomb barrier is significantly weaker than the initial remnant Coulomb barrier;

4. for all cases, the initial remnant Coulomb barrier significantly decreases the first derivative, acting as an attractive potential;

5. the larger the charge, the larger the decrease of $S'(0)$, such that in most cases it wins over all other real barriers making $S'(0)$ negative (note that this happens even for $l_i = 2$, where all previous cases had $S'(0) > 0$).
IV. COMPARATIVE BEHAVIOR OF S FACTORS FOR SOME IMPORTANT ASTROPHYSICAL PROCESSES

In this section we compare and explain the difference in the energy behavior of the direct capture S-factors for some important astrophysical processes. Since we are only interested in the comparative energy behavior, all S-factors are normalized to unity at \( E = 0 \). Four notorious reactions with either \( l_i = 1 \) or \( l_f = 1 \) are considered:

a) \( ^7\text{Be}(p, \gamma)^8\text{B}(2^+), \ \varepsilon = 0.137 \text{ MeV}, \ l_i = 0 \rightarrow l_f = 1; \)

b) \( ^{14}\text{N}(p, \gamma)^{15}\text{O}(3/2^+), \ \varepsilon = 0.504 \text{ MeV}, \ l_i = 1 \rightarrow l_f = 0; \)

c) \( ^{16}\text{O}(p, \gamma)^{17}\text{F}(1/2^+), \ \varepsilon = 0.105 \text{ MeV}, \ l_i = 1 \rightarrow l_f = 0; \)

d) \( ^{20}\text{Ne}(p, \gamma)^{21}\text{Na}(1/2^+), \ \varepsilon = 0.0064 \text{ MeV}, \ l_i = 1 \rightarrow l_f = 0. \)

Results are presented in Fig. 8. Since \( l_i, l_f \leq 1 \), the centrifugal barriers are not strong enough to win over the effect of the singularity at \( E = -\varepsilon \) and the initial remnant Coulomb barrier, leading to negative slopes for \( S(0) \). Within these examples, the largest charge system, also corresponds to the lowest binding energy, such that both effects result in a very negative derivative for \( S(0) \). A significant difference between \( S'(0) \) for the \( ^8\text{B} \) and \( ^{17}\text{F} \) cases, with similar binding energies, proves that the initial Coulomb remnant barrier which is stronger for \( ^{17}\text{F} \) than for \( ^8\text{B} \), wins over the initial centrifugal barrier \( l_i = 1 \), resulting in a more negative derivative \( S'(0) \) for \( ^{17}\text{F} \).

We also consider astrophysical examples for which either \( l_i = 2 \) or \( l_f = 2 \):

a) \( ^7\text{Be}(p, \gamma)^8\text{B}(2^+), \ \varepsilon = 0.137 \text{ MeV}, \ \text{with } l_i = 2 \rightarrow l_f = 1; \)

b) \( ^{16}\text{O}(p, \gamma)^{17}\text{F}(5/2^+), \ \varepsilon = 0.605 \text{ MeV}, \ \text{with } l_i = 1 \rightarrow l_f = 2; \)

c) \( ^{22}\text{Mg}(p, \gamma)^{23}\text{Al}(5/2^+), \ \varepsilon = 0.127 \text{ MeV}, \ \text{with } l_i = 2 \rightarrow l_f = 1. \)

The resulting S-factors are presented in Fig. 9. For the first two cases the slope is positive. For the \( ^8\text{B} \) case, the initial centrifugal barrier and smaller charge result in the highest \( S'(0) \). The fact that the remnant Coulomb barrier is stronger for \( ^{17}\text{F} \) than for \( ^8\text{B} \) and that the initial centrifugal barrier is lower, produces a smaller \( S'(0) \). For the capture on \( ^{22}\text{Na} \), despite the high initial orbital \( l_i = 2 \), the very large initial remnant Coulomb barrier (together with the
trace of the nearby singularity) is able to make the slope negative near zero.

V. SUMMARY

We investigated the low-energy behavior of the astrophysical $S(E)$ factors for direct radiative captures of protons by charged nuclei leading to loosely bound final states. For such processes a simple potential model, extending the asymptotic form of the final state to $r = 0$ and assuming a pure Coulomb scattering wave in the initial state are well justified, as most of the contribution to the low energy capture to loosely bound states comes from large distances.

We demonstrate that the behavior of the S-factors is governed by six essential ingredients. Two act in an attractive sense, creating a negative slope for $S(0^+)$: the remnant of the initial Coulomb barrier (left after extracting the Gamow penetration factor) and the singularity at $E = -\varepsilon$ (where $\varepsilon$ is the binding energy of the final state). Three act as \textit{real} penetration barriers, producing a positive slope for $S(0^+)$: both initial and final centrifugal barriers and the final-state Coulomb barrier. The effect of the final centrifugal and Coulomb barriers are minor compared with the initial centrifugal and Coulomb. Finally, there is still a photon factor $k_\gamma^{2L+1}$, which tends to increase the derivative of $S(E)$. We have derived analytical expressions for the S-factor in a few typical cases. We have tried to desmystified the idea that the energy behavior of the S-factor around threshold is dominated by the pole $E = -\varepsilon$. We have also shown that by taking only the first term of the Whittaker expansion (final state), one cannot reproduce the correct S-factor energy behavior at threshold, except when the Coulomb parameters of the final state is integer. Finally, we have not only illustrated our findings with a few sets of study cases but applied it to specific examples relevant in astrophysics.

Although in this work, we were mainly concerned with the energy behavior of the astrophysical factors, our approximate equations can reproduce, the absolute values of the capture rates within a few percent. The smaller the binding energy, the better accuracy of
our equations. The application of Eq. (5) can be extended to energies of a few hundred keV, \textit{i.e.} to the explosive nucleosynthesis energy range, when binding energies are not larger than 1 MeV. Specifically, for charges $Z_a = 4 - 10$ and binding energies $\varepsilon \leq 1.0$ MeV, S-factors obtained from Eq. (5) do not deviate more than a few \% from the exact potential model, for relative energies up to 1 MeV. However for incident energies $\geq 400$ keV and/or higher binding energies, a simple potential model becomes less accurate: nuclear interactions in the initial state and microscopic effects, such as antisymmetrization, should to be included.

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APPENDIX A: INTEGRAL REPRESENTATION OF S(E)

In the Appendix we derive the expression for the S-factor for captures to loosely bound states. Our starting point is the definition in Eq. (5). We simplify this expression, reducing it to a first-order integral with an integrand expressed in terms of elementary functions. To this end, we use the integral representation for both bound and scattering states, under the assumption of the asymptotic approximation for the bound state wave function discussed in the main text.

The Whittaker function has the following integral representation [7]:

\[
W_{-\eta_f, l_f+1/2}^{(2\kappa r)} = \frac{(2\kappa r)^{l_f+1} e^{-\kappa r}}{\Gamma(l_f + \eta_f + 1)} \int_0^\infty ds \, e^{-2\kappa rs}s(1+s)^{l_f} \left[ \frac{1}{1+s} \right]^{\eta_f}.
\]

(A.1)

Furthermore, the integral representation of the first term of its asymptotic expansion in powers $1/\kappa r$ ($j = 0$ in Eq.6), is given by:

\[
W_{-\eta_f, l_f+1/2}^{(0)} = \frac{e^{-\kappa r}}{(2\kappa r)^{\eta_f}} = \frac{(2\kappa r)^{l_f+1} e^{-\kappa r}}{\Gamma(l_f + \eta_f + 1)} \int_0^\infty ds \, e^{-2\kappa rs}s^{l_f+\eta_f}.
\]

(A.2)

The difference between $W_{-\eta_f, l_f+1/2}^{(2\kappa r)}$ and the first term $W_{-\eta_f, l_f+1/2}^{(0)}$ integral representations is only in the spectral function.

The Coulomb regular solution, $F_{l_i}(k, r)$, is given by Eq. (9) in terms of the confluent hypergeometric function. This can be written in integral form as [7]:

\[
_{1}F_{1}(l_i + 1 - i\eta_i, 2 l_i + 2 ; 2 i kr) = \frac{1}{B(l_i + 1 - i\eta_i, l_i + 1 + i\eta_i)}
\]

\[
\times \int_0^1 dt \, e^{2ikrt}[t(1-t)]^{l_i} \left[ \frac{1-t}{t} \right]^{i\eta_i}.
\]

(A.3)

Here, $B(l_i + 1 - i\eta_i, l_i + 1 + i\eta_i)$ is a $\beta$-function expressed in terms of $\Gamma$-functions:

\[
B(l_i + 1 - i\eta_i, l_i + 1 + i\eta_i) = \frac{\Gamma(2 l_i + 2)}{\Gamma(l_i + 1 - i\eta_i) \Gamma(l_i + 1 + i\eta_i)}.
\]

(A.4)

Substituting Eq. (A.1) and Eq. (A.3) into Eq. (9), allowing for Eq. (13) and changing the order of the integrations, we arrive at the following expression for the S-factor:
\[
S(E) = A \frac{1}{\Gamma(l_f + \eta_f + 1)^2} \frac{1}{\Gamma(2 \ell_i + 2)^2} |B(l_i + 1 - i \eta_i, l_i + 1 + i \eta_i)|^2 \\
\times k_\gamma^{2L+1} k_{\eta_f}^{2l_f+1} \prod_{j=0}^{l_i} (j^2 k^2 + \eta_f^2 \kappa^2) |J|^2,
\]

(A.5)

We have merged the three integrations into \( J \):

\[
J = \int_0^\infty ds \left[ s(1 + s) \right]^{l_f} \left[ \frac{s}{1 + s} \right]^{\eta_f} \int_0^1 dt \left[ t(1 - t) \right]^{l_i} \left[ \frac{1 - t}{t} \right]^{i \eta_i} \\
\times \int_{r_0}^\infty dr r^{L+l_i+l_f+2} e^{-r(\kappa(1+2s)+i k(1-2t))}.
\]

(A.6)

We now replace the lower limit of the integration by \( r_0 = 0 \), keeping in mind the application to loosely bound states (where the contribution from the interior is not significant). The result, after integrating over \( r \), holds:

\[
J = V \int_0^\infty ds \left[ s(1 + s) \right]^{l_f} \left[ \frac{s}{1 + s} \right]^{\eta_f} \int_0^1 dt \left[ t(1 - t) \right]^{l_i} \left[ \frac{1 - t}{t} \right]^{i \eta_i} \\
\times \left[ \kappa(1+2s)+i k(1-2t) \right]^{l_i+l_f+L+3},
\]

(A.7)

with \( V = \int_0^\infty dv v^{l_i+l_f+L+3} e^{-v} \). Let us define the spectral function of \( s \), as indicated above, \( S(s) \) and the remaining integral over \( t \) as \( J_t \). The expression for \( S(s) \) is:

\[
S(s) = \left[ s(1 + s) \right]^{l_f} \left[ \frac{s}{1 + s} \right]^{\eta_f}.
\]

(A.8)

The integral over \( t \) can be reduced to:

\[
J_t \equiv \int_0^1 dt \left[ t(1 - t) \right]^{l_i} \left[ \frac{1 - t}{t} \right]^{i \eta_i} \left[ \kappa(1+2s)+i k(1-2t) \right]^{l_i+l_f+L+3} \\
= (-1)^{\eta_i} \frac{(2 l_i + 1)!}{(l_f + L + l_i + 2)!} \\
\times \lim_{\epsilon \to 0} \frac{d^m}{d \epsilon^m} \int_0^1 dt \left[ t(1 - t) \right]^{l_i} \left[ \frac{1 - t}{t} \right]^{i \eta_i} \left[ \kappa(1+2s)+\epsilon + i k(1-2t) \right]^{2l_i+2},
\]

(A.9)

where we introduce \( m = l_f + L + 1 - l_i \). One can further simplify the integrand by factorizing the \( t \)-independent terms, such that the remaining integral consists only on the integral representation of the hypergeometrical function \( _2F_1 \):
\[ J_t = (-1)^m \frac{(2 l_i + 1)!}{(l_f + L + l_i + 2)!} B(l_i + 1 - i \eta_i, l_i + 1 + i \eta_i) \]
\times \lim_{\epsilon \to 0} \frac{d^m}{d \epsilon^m} \frac{1}{[\kappa (1 + 2 s) + \epsilon + i k]^{2l_i+2}} 2F_1(l_i + 1 - i \eta_i, 2l_i + 2, 2l_i + 2; y). \] \quad (A.10)

The argument of the hypergeometric Gauss function is \( y = 1 - \frac{2i k}{\kappa (1 + 2 s) + \epsilon + i k} \). Taking into account the property: \( 2F_1(\alpha, \beta; \gamma; y) = (1 - y)^{\gamma - \alpha - \beta} 2F_1(\gamma - \alpha, \gamma - \beta, \gamma; y) \) \( (\text{Eq. 9.131-1 of [I]}), \) and the fact that \( 2F_1(\alpha, 0; \gamma; y) = 1 \), we can arrive at a simplified expression:

\[ J_t = (-1)^m \frac{(2 l_i + 1)!}{(l_f + L + l_i + 2)!} B(l_i + 1 - i \eta_i, l_i + 1 + i \eta_i) \]
\times \lim_{\epsilon \to 0} \frac{d^m}{d \epsilon^m} \frac{1}{[\kappa (1 + 2 s) + \epsilon + k]^{2l_i+2}} e^{2i \eta_i \arctan \frac{k}{\kappa (1 + 2 s) + \epsilon + k}}. \] \quad (A.11)

Introducing Eq. (A.11) and Eq. (A.8) into Eq. (A.7), \( J \) reduces to

\[ J = V (-1)^m \frac{(2 l_i + 1)!}{(l_f + L + l_i + 2)!} B(l_i + 1 - i \eta_i, l_i + 1 + i \eta_i) \]
\times \int_0^\infty ds S(s) \lim_{\epsilon \to 0} \frac{d^m}{d \epsilon^m} \frac{1}{[\kappa (1 + 2 s) + \epsilon + k]^{2l_i+2}} e^{2i \eta_i \arctan \frac{k}{\kappa (1 + 2 s) + \epsilon + k}}. \] \quad (A.12)

Consequently, replacing Eq. (A.12) in Eq. (A.5), \( S(E) \) takes the form:

\[ S(E) = B k^{\gamma L+1} \eta_f \prod_{j=0}^{l_i} (j^2 k^2 + \eta_f^2 k^2) \]
\times \left| \lim_{\epsilon \to 0} \int_0^\infty ds S(s) \frac{d^m}{d \epsilon^m} \frac{1}{[\kappa (1 + 2 s) + \epsilon + k]^{2l_i+2}} e^{2i \eta_i \arctan \frac{k}{\kappa (1 + 2 s) + \epsilon + k}} \right|^2, \] \quad (A.13)

where a new constant, putting in evidence energy independent factors, is introduced:

\[ B = A 2^{2l_i+2l_f+4} \frac{1}{\Gamma(l_f + \eta_f + 1)} \frac{[(2 l_i + 1)!]^2}{[(l_f + L + l_i + 2)!]^2} \frac{1}{\Gamma(2 l_i + 2)^2}. \] \quad (A.14)

The constant \( A \) was introduced in the beginning of the text by Eq. (3).

Similarly, the function \( S_{(0)}(E) \), resulting from the first term of the Whittaker expansion, can be derived. The result holds an identical expression to \( S(E) \), the only difference being the spectral function over \( s \). Thus, Eq. (A.13) is applicable to \( S_{(0)}(E) \) when replacing \( S(s) \) by \( S_0(s) \):

\[ S_0(s) = s^{l_f+\eta_f}. \] \quad (A.15)
It is important to note that the replacement of the lower integration limit in the radial integral of Eq. (A.5) imposes conditions for the convergence of the integrals over $s$ at $\infty$:

1) for $S(E)$, $l_f < 2 + L + l_i$, which is always satisfied;

2) for $S_{(0)}(E)$, $\eta_f < 2 + L + l_i$, which limits the charge of the capturing nucleus.

Finally, we should mention that, as we focus only on dipole transitions, then $|l_i - l_f| = 1$ and $L = 1$. Hence, there are two possibilities for $m$: $l_i - l_f = 1 \rightarrow m = 1$ and $l_f - l_i = 1 \rightarrow m = 3$. In each case an evaluation of Eq. (A.13) can be explicitly performed. This is presented and discussed in section III.
REFERENCES

[1] E. G. Adelberger et al., Rev. Mod. Phys. 70, 1265 (1998).

[2] C.E. Rolfs and W.S. Rodney, Cauldrons in Cosmos, The University of Chicago Press, 1988.

[3] B. K. Jennings, S. Karataglidis and T. D. Shoppa, Phys. Rev. C 58, 3711 (1998); Phys. Rev. C 58, 579 (1998).

[4] D. Baye and E. Brainis, Phys. Rev. C 61, 025801 (2000).

[5] H. Herndl, J. Göres, M. Wiescher, B. A. Brown and L. Van Wormer, Phys. Rev. C 52, 1078 (1995).

[6] F. C. Barker and T. Kajino, Austr. J. Phys. 44, 369, 1991.

[7] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, San Diego, 1980.
FIG. 1. $S_0(E)$ for the $l_i = 0, \rightarrow l_f = 1$ proton capture, for a set of binding energies of the final state. Masses and charges are the same as for $^7\text{Be}(p,\gamma)^8\text{B}$. $S_0(E)$ is normalized to unity at zero energy.

FIG. 2. Proton capture $l_i = 1, \rightarrow l_f = 0$ for a set of binding energies: $S(E)$ (dark lines) versus $S_0(E)$ (grey lines). Masses and charges are the same as for $^7\text{Be}(p,\gamma)^8\text{B}$. All S-factors are normalized to unity at zero energy.
FIG. 3. Proton capture $l_i = 2, \rightarrow l_f = 1$ for a set of binding energies: $S(E)$ (dark lines) versus $S_{(0)}(E)$ (grey lines). For $\varepsilon = 0.09$ MeV, $S_{(0)}(E)$ is plotted in circles to distinguish from $S(E)$. Masses and charges are the same as for $^7$Be(p,γ)$^8$B. All S-factors are normalized to unity at zero energy.

FIG. 4. Proton capture $l_i = 0, \rightarrow l_f = 1$ for a set of binding energies: $S(E)$ (dark lines) versus $S_{(0)}(E)$ (grey lines). For $\varepsilon = 0.35$ MeV, $S_{(0)}(E)$ is plotted in circles to distinguish from $S(E)$. Masses and charges are the same as for $^7$Be(p,γ)$^8$B. All S-factors are normalized to unity at zero energy.
FIG. 5. Proton capture $l_i = 1, \rightarrow l_f = 2$ for a set of binding energies: $S(E)$ (dark lines) versus $S(0)(E)$ (grey lines). For $\varepsilon = 0.09$ MeV, $S(0)(E)$ is plotted in circles to distinguish from $S(E)$. Masses and charges are the same as for $^7\text{Be}(p,\gamma)^8\text{B}$. All S-factors are normalized to unity at zero energy.

FIG. 6. S-factors for the E1-capture $^7\text{Be}(p,\gamma)^8\text{B}$: The solid line is the result for $S(E)$ assuming $Z_a = 0$; the dashed line and dot-dashed correspond to $S(E)$ for $Z_a = 4$ calculated with and without the initial Coulomb barrier, correspondingly; the squares and circles correspond to $S(E)$ for $Z_a = 8$ calculated with and without the initial Coulomb barrier, correspondingly. All S-factors are normalized to unity at zero energy.
FIG. 7. S-factors for the same reaction as Fig. 6, but $l_i = 2 \to l_f = 1$: The solid line is the result for $S(E)$ assuming $Z_a = 0$; the dashed line and dot-dashed correspond to $S(E)$ for $Z_a = 4$ calculated with and without the initial Coulomb barrier, correspondingly; the squares and circles correspond to $S(E)$ for $Z_a = 8$ calculated with and without the initial Coulomb barrier, correspondingly. All S-factors are normalized to unity at zero energy.

FIG. 8. S-factors for some astrophysical reactions: $^7\text{Be}(p, \gamma)^8\text{B}(2^+)$, corresponding to a transition $l_i = 0 \to l_f = 1$ with binding energy $\varepsilon = 0.137$ MeV (solid); $^{14}\text{N}(p, \gamma)^{15}\text{O}(3/2^+)$, corresponding to a transition $l_i = 1 \to l_f = 0$ with binding energy $\varepsilon = 0.504$ MeV (dotted); $^{16}\text{O}(p, \gamma)^{17}\text{F}(1/2^+)$, corresponding to a transition $l_i = 1 \to l_f = 0$ with binding energy $\varepsilon = 0.105$ MeV (dashed); $^{20}\text{Ne}(p, \gamma)^{21}\text{Na}(1/2^+)$, corresponding to a transition $l_i = 1 \to l_f = 0$ with binding energy $\varepsilon = 0.0064$ MeV (dot-dashed). All the astrophysical factors are normalized to unity at zero energy.
FIG. 9. S-factors for some astrophysical reactions: $^7\text{Be}(p, \gamma)^8\text{B}(2^+)$, corresponding to a transition $l_i = 2 \rightarrow l_f = 1$ and binding energy of the final state $\varepsilon = 0.137$ MeV (solid); $^{16}\text{O}(p, \gamma)^{17}\text{F}(5/2^+)$, corresponding to a transition $l_i = 1 \rightarrow l_f = 2$, and binding energy of the final state $\varepsilon = 0.605$ MeV (dashed); $^{22}\text{Mg}(p, \gamma)^{23}\text{Al}(5/2^+)$, corresponding to a transition $l_i = 1 \rightarrow l_f = 2$ and binding energy of the final state $\varepsilon = 0.127$ MeV (dot-dashed). All the astrophysical factors are normalized to unity at zero energy.