Reflection factors and exact $g$-functions
for purely elastic scattering theories

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Abstract

We discuss reflection factors for purely elastic scattering theories and relate them to perturbations of specific conformal boundary conditions, using recent results on exact off-critical $g$-functions. For the non-unitary cases, we support our conjectures using a relationship with quantum group reductions of the sine-Gordon model. Our results imply the existence of a variety of new flows between conformal boundary conditions, some of them driven by boundary-changing operators.
1 Introduction

There are two general approaches to the study of integrable boundary quantum field theories: infrared and ultraviolet. From the infrared point of view, solving the boundary Yang-Baxter and bootstrap constraints allows sets of reflection factors to be associated with various bulk scattering theories. However, there are many ambiguities left by these constraints, which cannot be lifted without the introduction of more selective criteria. This is to be expected, since a given bulk theory may admit many different integrable boundary conditions.

Alternatively, a theory might be defined via a classical Lagrangian, or as a perturbation of a boundary conformal field theory. In either case, it is natural to ask for a ‘UV/IR dictionary’, giving the relationship between the infrared and ultraviolet specifications of the model. In particular, this would reveal which among the infinite families of solutions to the boundary Yang-Baxter and bootstrap equations are physically realised, and to which boundary conditions they correspond.

Progress on this issue includes the relationship between the parameters in the boundary reflection factors and the couplings in the boundary sine-Gordon Lagrangian derived by Aliosha Zamolodchikov [1] (checked using the boundary TCSA [2] in [3]), and various perturbative results for cases where the theory admits a Lagrangian definition (see for example [4, 5]). However, these methods cannot be readily adapted to the study of general perturbed boundary conformal field theories. In particular, a boundary analogue of the thermodynamic Bethe ansatz (TBA) \( c \)-function, which allows bulk S-matrices to be identified with specific perturbed conformal field theories [6, 7], has been lacking. An obvious candidate is the \( g \)-function defined by Affleck and Ludwig in [8], but the initial proposal for a TBA-like equation for a fully off-critical version of \( g \), taking as input just the bulk S-matrix and the boundary reflection factors [9], turned out to be incorrect [10].

Recently it has been shown that the situation can be remedied [11]. The detailed work in [11] concentrated on the boundary scaling Lee-Yang model, for which the relationship between microscopic boundary conditions and reflection factors had already been established [2]. In this paper we extend the investigations of [11] to a collection of theories for which boundary UV/IR relations have yet to be found, namely the minimal purely elastic scattering theories associated with the ADET series of diagrams [12, 7, 13, 14, 15, 16, 17]. The bulk S-matrices of these models have long been known, but less progress has been made in associating solutions of the boundary bootstrap equations with specific perturbed boundary conditions. We present a collection of minimal reflection factors for the ADET theories, and test them by checking the \( g \)-function flows that they imply. We also show how these reflection factors can be modified to incorporate a free parameter, generalising a structure previously observed in the Lee-Yang model [2]. This enables us to predict a number of new flows between conformal boundary conditions.
In more detail, the rest of the paper is organised as follows. Bulk and boundary ADET scattering theories are discussed in sections 2 and 3, with section 3 containing a full set of reflection factors fulfilling certain minimality criteria. These solutions have no free parameters, and in section 4 the minimality requirement is relaxed and non-minimal one-parameter families of solutions are proposed to describe simultaneous bulk and boundary perturbations of boundary conformal field theories by relevant operators. In section 5 the one-parameter families of reflection amplitudes for the $T_r$ models are alternatively deduced by considering special reductions of the boundary sine-Gordon model.

After a discussion of the exact off-critical $g$-function in section 6, the physical consistency of the minimal solutions is checked in section 7, by comparing the predictions for the ultraviolet values of the $g$-function obtained from the reflection factors with conformal field theory results. An additional check in first order perturbation theory is performed for the 3-state Potts model in section 8. Section 9 summarises the $g$-function predictions for the one-parameter families of reflection factors and discusses their UV/IR relations. Our conclusions are given in section 10, and an appendix gathers together various results concerning the fields and $g$-function values in the diagonal $\hat{g}_1 \times \hat{g}_1/\hat{g}_2$ coset models which are needed in the main text.

2 The ADET family of purely elastic scattering theories

Purely elastic scattering theories are characterised by the property that their $S$-matrices are diagonal. The scattering of particles $a$ and $b$ with relative rapidity $\theta$ is then described by a single function $S_{ab}(\theta)$, which is a pure phase for physical rapidities. The unitarity and crossing symmetry conditions simplify to

\[
S_{ab}(\theta)S_{ab}(-\theta) = 1, \quad (2.1)
\]

\[
S_{ab}(\theta) = S_{\bar{b}a}(i\pi - \theta) \quad (2.2)
\]

respectively, where $\bar{b}$ is the antiparticle of $b$. The Yang-Baxter equation is trivially satisfied, but, as stressed by Zamolodchikov [18, 19], the bootstrap still provides a useful constraint. This states that whenever an on-shell three-point coupling $C^{abc}$ is nonzero, the $S$-matrix elements of particles $a$, $b$ and $\bar{c}$ with any other particle $d$ satisfy

\[
S_{dc}(\theta) = S_{da}(\theta - iU_{ac}^b)S_{db}(\theta + iU_{ac}^b) \quad (2.3)
\]

where the ‘fusing angles’ $U_{ab}^c$ are related to the masses of particles $a$, $b$ and $c$ by

\[
M_c^2 = M_a^2 + M_b^2 + 2M_aM_b\cos U_{ab}^c \quad (2.4)
\]

and $\bar{U} = \pi - U$. These angles also appear as the locations of certain odd-order poles in the $S$-matrix elements. More details of the workings of the bootstrap can be found in [18, 19] and, for example, [20].
The purely elastic scattering theories that we shall treat in this paper fall into two classes. The first class associates an S-matrix to each simply-laced Lie algebra $g$, of type A, D or E [12, 7, 13, 14]. These S-matrices are minimal, in that they have no zeros on the physical strip $0 \leq \Im \theta \leq \pi$, and one-particle unitary, in that all on-shell three-point couplings, as inferred from the residues of forward-channel S-matrix poles, are real. They describe particle scattering in the perturbations of the coset conformal field theories $\hat{g}_1 \times \hat{g}_1/\hat{g}_2$ by their $\{(1,1,\text{ad})\}$ operators, where $\hat{g}$ is the affine algebra associated with $g$. The unperturbed theories have central charge

$$c = \frac{2r_g}{(h+2)},$$

(2.5)

where $r_g$ is the rank of $g$, and $h$ is its Coxeter number. These UV central charges can be recovered directly from the S-matrices, using the thermodynamic Bethe ansatz [6, 7].

The ADE S-matrices describe the diagonal scattering of $r_g$ particle types, whose masses together form the Perron-Frobenius eigenvector of the Cartan matrix of $g$. This allows the particles to be attached to the nodes of the Dynkin diagram of $g$. Each S-matrix element can be conveniently written as a product of elementary blocks [12]

$$\{x\} = (x - 1)(x + 1), \quad (x)(\theta) = \frac{\sinh \left(\frac{\theta}{2} + \frac{i\pi x}{2h}\right)}{\sinh \left(\frac{\theta}{2} - \frac{i\pi x}{2h}\right)}$$

(2.6)

as

$$S_{ab} = \prod_{x \in A_{ab}} \{x\},$$

(2.7)

for some index set $A_{ab}$. Note that

$$(0) = 1, \quad (h) = -1, \quad (-x) = (x)^{-1}, \quad (x \pm 2h) = (x).$$

(2.8)

The notation (2.6) has been so arranged that the numbers $x$ are all integers. The sets $A_{ab}$ are tabulated in [12]; a universal formula expressing them in geometrical terms was found in [14], and is further discussed in [15, 16].

The S-matrices of the second class [21, 22, 23] are labelled by extending the set of ADE Dynkin diagrams to include the ‘tadpole’ $T_r$. They encode the diagonal scattering of $r$ particle types, and can be written in terms of the blocks (2.6) with $h = 2r+1$ [24]:

$$S_{ab} = \prod_{l=|a-b|+1}^{a+b-1} \{l\}\{h-l\}.$$  

(2.9)

These S-matrix elements are again minimal, but they are not one-particle unitary, reflecting the fact that they describe perturbations of the non-unitary minimal models $\mathcal{M}_{2,2r+3}$, with central charge $c = -2r(6r+5)/(2r+3)$. The perturbing operator this time is $\phi_{13}$. The $T_r$ S-matrices are quantum group reductions of the
sine-Gordon model at coupling $\beta^2 = 16\pi/(2r+3)$ [21], a fact that will be relevant later.

A self-contained classification of minimal purely elastic S-matrices is still lacking, but the results of [17] single out the ADET theories as the only examples having TBA systems for which all pseudoenergies remain finite in the ultraviolet limit.

The ADET diagrams are shown in figure 1, with nodes giving our conventions for labelling the particles in each theory. For the $D_r$ theories, particles $r-1$ and $r$ are sometimes labelled $s$ and $s'$, or $s$ and $\bar{s}$, for $r$ even or odd respectively.

![Diagram](image)

Fig. 1a: $A_r$  
Fig. 1b: $D_r$  
Fig. 1c: $T_r$

Fig. 1d: $E_6$  
Fig. 1e: $E_7$  
Fig. 1f: $E_8$

### 3 Minimal reflection factors for purely elastic scattering theories

The scattering of particles by a boundary in an integrable quantum field theory is described by a set of reflection factors. In this paper we shall restrict ourselves to situations where not only the bulk, but also the boundary scattering amplitudes are purely elastic. The reflection factors are then rapidity-dependent functions $R_a(\theta)$, one for each particle type $a$ in the theory. Unitarity [25, 26] and crossing-unitarity [26] constrain these functions as

$$R_a(\theta)R_a(-\theta) = 1,$$  \hspace{1cm} (3.1)

$$R_a(\theta)R_{\bar{a}}(\theta - i\pi) = S_{aa}(2\theta).$$  \hspace{1cm} (3.2)

In addition, whenever a bulk three-point coupling $C^{abc}$ is nonzero, a boundary bootstrap equation holds [25, 26]:

$$R_c(\theta) = R_a(\theta + iU^b_{ac})R_b(\theta - iU^a_{bc})S_{ab}(2\theta + iU^b_{ac} - iU^a_{bc}).$$  \hspace{1cm} (3.3)

For a given bulk S-matrix there are infinitely-many distinct sets of reflection factors consistent with these constraints, because any solution can be multiplied by a solution of the bulk bootstrap, unitarity and crossing equations to yield another solution [27]. To identify a set of reflection factors as physically relevant some more information is needed. A common basis for the conjecturing of bulk scattering amplitudes is a ‘minimality hypothesis’, that in the absence of other
requirements one should look for solutions of the constraints with the smallest possible number of poles and zeros. In this section we show how the same principle can be used to find sets of boundary amplitudes, one for each purely elastic S-matrix of type A, D or E. The amplitudes given below were at first conjectured [28] as a natural generalisation of the minimal versions of the \( A_r \) affine Toda field theory amplitudes found in [29, 30]. The reasoning behind these conjectures will be explained shortly. More recently, Fateev [31] proposed a set of reflection factors for the affine Toda field theories. These were in integral form, and not all matrix elements were given. However, modulo some overall signs and small typos, the coupling-independent parts of Fateev’s conjectures match ours. (This has also been found by Zambon [32], who obtained equivalent formulae to those recorded below taking Fateev’s integral formulae as a starting point.)

Unitarity and crossing-unitarity together imply that the reflection factors must be \( 2\pi i \) periodic; unitarity then requires that they be products of the blocks \( \{ x \} \) introduced in the last section. The crossing-unitarity constraint is then key for the analysis of minimality. Each pole or zero of \( S_{aa}(2\theta) \) on the right hand side of (3.2) must be present in one or other of the factors on the left hand side of that equation. This sets a lower bound on the number of poles and zeros for the reflection factor \( R_a(\theta) \).

To exploit this observation, it is convenient to work at the level of the larger blocks \( \{ x \} \), the basic units for the bulk bootstrap equations [12, 14]. We define two complementary ‘square roots’ of these blocks as

\[
\langle x \rangle = \left( \frac{x-1}{2} \right) \left( \frac{2h-x-1}{2} \right)^{-1}, \quad \langle \tilde{x} \rangle = \left( \frac{x+1}{2} \right) \left( \frac{2h-x+1}{2} \right)^{-1}
\]

and record their basic properties

\[
\langle x \rangle(\theta) = \langle \tilde{x} \rangle(\theta + i\pi)
\]

and

\[
\langle x \rangle(\theta) \langle \tilde{x} \rangle(\theta) = \{ x \}(2\theta).
\]

The crossing-unitarity equation can then be solved, in a minimal fashion, by any product

\[
R_a = \prod_{x \in A_{aa}} f_x
\]

where each factor \( f_x \) can be freely chosen to be \( \langle x \rangle \) or \( \langle \tilde{x} \rangle \), modulo one subtlety: since \( (0) = 1 \), minimality requires that any factor \( f_1 \) be taken to be \( \langle 1 \rangle \) rather than \( \langle \tilde{1} \rangle \). In fact there is always exactly one such factor for the diagonal S-matrix elements relevant here [15].

\*We could also set \( \langle x \rangle = \left( \frac{x-1}{2} \right) \left( \frac{x+1}{2} \right), \langle \tilde{x} \rangle = \left( \frac{2h-x-1}{2} \right)^{-1} \left( \frac{2h-x+1}{2} \right)^{-1} \) throughout, but this would break the pattern previously seen for the \( A_r \) theories, and turns out not to fit with the \( g \)-function calculations later.
There remain the bootstrap equations (3.3). To treat these we need to know how the blocks $\langle x \rangle (\theta)$ and $\langle \tilde{x} \rangle (\theta)$ transform under general shifts in $\theta$. As in [14, 15], these shifts are best discussed by defining
\[
(x)_+ (\theta) = \sinh \left( \frac{\theta}{2} + \frac{i\pi x}{2h} \right) \tag{3.8}
\]
and then
\[
\langle x \rangle_+ = \left( \frac{x-1}{2} \right)_+ \left( \frac{2h-x-1}{2} \right)_+^{-1}, \quad \langle \tilde{x} \rangle_+ = \left( \frac{x+1}{2} \right)_+ \left( \frac{2h-x+1}{2} \right)_+^{-1} \tag{3.9}
\]
so that $\langle x \rangle = (x)_+ / (-x)_+$, $\langle x \rangle = \langle x \rangle_+ / \langle \tilde{x} \rangle_+$ and $\langle \tilde{x} \rangle = \langle \tilde{x} \rangle_+ / \langle -x \rangle_+$. Once a putative set of reflection factors (3.7) has been decomposed into these blocks, it is straightforward to implement the boundary bootstrap equations (3.3) using the properties
\[
\langle x \rangle_+ (\theta + i\pi y/h) = \langle x+y \rangle_+ (\theta), \quad \langle \tilde{x} \rangle_+ (\theta + i\pi y/h) = \langle \tilde{x}+y \rangle_+ (\theta). \tag{3.10}
\]
For the ADE theories it turns out that the boundary bootstrap equations can all be satisfied by minimal conjectures of the form (3.7), and that the choice of blocks is then fixed uniquely. (From the point of view of the bootstrap equations alone, an overall swap between $\langle x \rangle$ and $\langle \tilde{x} \rangle$ is possible, but the requirement that $f_1 = \langle 1 \rangle$ fixes even this ambiguity.) The final answers are recorded below.

$A_r$

The reflection factors for this case were given in [29]. In the current notation they are
\[
R_a = \prod_{x=1 \text{ odd}}^{2a-1} \langle x \rangle \tag{3.11}
\]

$D_r$

The reflection factors, $R_a$, for $a = 1, \ldots, r-2$ are
\[
R_a = \prod_{x=1 \text{ odd}}^{2a-1} \langle x \rangle \prod_{x=x+1 \text{ step 4}}^{2a-1} \langle h-x \rangle \prod_{x=1 \text{ step 4}}^{2a-5} \langle h-x-2 \rangle, \text{ for } a \text{ odd} \tag{3.12}
\]
\[
R_a = \prod_{x=1 \text{ odd}}^{2a-1} \langle x \rangle \prod_{x=1 \text{ step 4}}^{2a-3} \langle h-x \rangle \langle h-x-2 \rangle, \text{ for } a \text{ even}
\]
while for $a = r - 1$ and $r$ we have

$$R_{r-1} = R_r = \prod_{x=1}^{2r-5} \langle x \rangle, \text{ for } r \text{ odd}$$  
$$R_{r-1} = R_r = \prod_{x=1}^{2r-3} \langle x \rangle, \text{ for } r \text{ even}.$$  

(3.13)

$$E_6$$

$$R_1 = R_1 = \langle 1 \rangle \langle 7 \rangle$$

$$R_2 = \langle 1 \rangle \langle 5 \rangle \langle 7 \rangle \langle 11 \rangle$$

$$R_3 = R_3 = \langle 1 \rangle \langle 3 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle$$

$$R_4 = \langle 1 \rangle \langle 3 \rangle^2 \langle 5 \rangle^2 \langle 7 \rangle^2 \langle 9 \rangle \langle 9 \rangle \langle 11 \rangle$$  

(3.14)

$$E_7$$

$$R_1 = \langle 1 \rangle \langle 9 \rangle \langle 17 \rangle$$

$$R_2 = \langle 1 \rangle \langle 7 \rangle \langle 11 \rangle \langle 13 \rangle \langle 17 \rangle$$

$$R_3 = \langle 1 \rangle \langle 3 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 17 \rangle \langle 19 \rangle$$

$$R_4 = \langle 1 \rangle \langle 3 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 15 \rangle \langle 17 \rangle$$

$$R_5 = \langle 1 \rangle \langle 3 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle$$

$$R_6 = \langle 1 \rangle \langle 3 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 17 \rangle \langle 19 \rangle$$

$$R_7 = \langle 1 \rangle \langle 3 \rangle^2 \langle 5 \rangle^2 \langle 7 \rangle^3 \langle 9 \rangle^2 \langle 11 \rangle \langle 13 \rangle ^2 \langle 15 \rangle$$

(3.15)

$$E_8$$

$$R_1 = \langle 1 \rangle \langle 11 \rangle \langle 19 \rangle \langle 29 \rangle$$

$$R_2 = \langle 1 \rangle \langle 7 \rangle \langle 11 \rangle \langle 13 \rangle \langle 17 \rangle \langle 19 \rangle \langle 23 \rangle \langle 29 \rangle$$

$$R_3 = \langle 1 \rangle \langle 3 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 17 \rangle \langle 19 \rangle \langle 19 \rangle \langle 21 \rangle \langle 27 \rangle \langle 29 \rangle$$

$$R_4 = \langle 1 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle \langle 19 \rangle \langle 21 \rangle \langle 23 \rangle \langle 25 \rangle \langle 29 \rangle$$

$$R_5 = \langle 1 \rangle \langle 3 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle \langle 19 \rangle \langle 21 \rangle \langle 23 \rangle \langle 25 \rangle \langle 29 \rangle$$

(3.16)

$$R_6 = \langle 1 \rangle \langle 3 \rangle \langle 5 \rangle \langle 7 \rangle \langle 9 \rangle \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle \langle 19 \rangle \langle 21 \rangle \langle 23 \rangle \langle 25 \rangle \langle 27 \rangle \langle 29 \rangle$$

$$R_7 = \langle 1 \rangle \langle 3 \rangle^2 \langle 5 \rangle^2 \langle 7 \rangle^2 \langle 9 \rangle^2 \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle \langle 19 \rangle^2 \langle 21 \rangle^2 \langle 23 \rangle \langle 25 \rangle \langle 27 \rangle \langle 29 \rangle$$

$$R_8 = \langle 1 \rangle \langle 3 \rangle^2 \langle 5 \rangle^2 \langle 7 \rangle^3 \langle 9 \rangle^2 \langle 11 \rangle \langle 13 \rangle \langle 15 \rangle \langle 17 \rangle\langle 19 \rangle$$

(3.17)
For the T series the story is different: it is not possible to satisfy the boundary bootstrap equations with a conjecture of the form (3.7). This means that the minimal reflection factors for these models are forced by the bootstrap to have extra poles and zeros beyond those required by crossing-unitarity alone. Our general proposal will be given in eq. (5.4) below, but the situation can be understood using the boundary $T_1$, or Lee-Yang, model: the minimal reflection factor found in [2] for the single particle in this model bouncing off the \( |\text{BD}\rangle \) boundary is

\[
R_{|\text{BD}\rangle}(\theta) = \left( \frac{1}{2} \right) \left( \frac{3}{2} \right) \left( \frac{4}{2} \right)^{-1}.
\]

(3.17)

The simpler function \( \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{-1} \) would have been enough to satisfy crossing-unitarity, but then the boundary bootstrap would not have held, and so (3.17) really is a minimal solution. This observation fits nicely with the $g$-function calculations to be reported later: had the minimal reflection factors for the $T_r$ theories fallen into the pattern seen for other models, there would have been a mismatch between the predicted UV values of the $g$-functions and the known values from conformal field theory.

4 One-parameter families of reflection factors

The minimal reflection factors introduced in the last section have no free parameters. However, combined perturbations of a boundary conformal field theory by relevant bulk and boundary operators involve a dimensionless quantity – the ratio of the induced bulk and boundary scales – on which the reflection factors would be expected to depend. To describe such situations, we must drop the minimality hypothesis, and extend our conjectures.

A first observation, rephrasing that of [27], is that given any two solutions $R_a(\theta)$ and $R'_a(\theta)$ of the reflection unitarity, crossing-unitarity and bootstrap relations (3.1), (3.2) and (3.3), their ratios $Z_a(\theta) = R_a(\theta)/R'_a(\theta)$ automatically solve one-index versions of the bulk unitarity, crossing and bootstrap equations (2.1), (2.2) and (2.3):

\[
Z_a(\theta)Z_a(-\theta) = 1, \quad Z_a(\theta) = Z_{\tilde{a}}(i\pi - \theta), \quad (4.1)
\]

\[
Z_c(\theta) = Z_a(\theta - iU^b_{ac})Z_b(\theta + iU^a_{bc}). \quad (4.2)
\]

The minimal reflection factors $R_a(\theta)$ can therefore be used as multiplicative ‘seeds’ for more general conjectures $R'_a(\theta) = R_a(\theta)/Z_a(\theta)$, with the parameter-dependent parts $Z_a(\theta)^{-1}$ constrained via (4.1) and (4.2). An immediate solution is $Z^{[d]}_a(\theta) = S_{da}(\theta)$ for any (fixed) particle type $d$ in the theory, where we have used the ket symbol to indicate that the label $|d\rangle$ might ultimately refer to one of the possible boundary states of the theory. However, this does not yet introduce a parameter. Noting that a symmetrical shift in $\theta$ preserves all the relevant equations, one possibility is to take

\[
Z^{[d,C]}_a(\theta) = S_{da}(\theta + C)S_{da}(\theta - C), \quad (4.3)
\]
with \( C \) at this stage arbitrary. This is indeed the solution adopted by the boundary scaling Lee-Yang example studied in [2]. This model is the \( r = 1 \) member of the \( T_r \) series described earlier, and corresponds to the perturbation of the non-unitary minimal model \( M_{25} \) by its only relevant bulk operator, \( \varphi \), of conformal dimensions \( \Delta_\varphi = \overline{\Delta}_\varphi = -\frac{1}{5} \). The minimal model has two conformally-invariant boundary conditions which were labelled \(|1\rangle \) and \(|\Phi\rangle \) in [2]. The \(|1\rangle \) boundary has no relevant boundary fields, and has a minimal reflection factor, given by (3.17) above. On the other hand, the \(|\Phi\rangle \) boundary has one relevant boundary field, denoted by \( \phi \), and gives rise to a one-parameter family of reflection factors

\[
R^{(b)}(\theta) = R^{(3)}(\theta)/Z^{(b)}(\theta)
\]

where the factor \( Z^{(b)}(\theta) \) has exactly the form mentioned above:

\[
Z^{(b)}(\theta) = S(\theta + \frac{\varphi}{m}(b+3))S(\theta - \frac{\varphi}{m}(b+3))
\]

where \( S(\theta) \) is the bulk S-matrix, and the parameter \( b \) can be related to the dimensionless ratio \( \mu^2/\lambda \) of the bulk and boundary couplings \( \lambda \) and \( \mu \) [2, 10]. (Since there is only one particle type in the Lee-Yang model, the indices \( a, d \) and so on are omitted. We also changed the notation slightly from that of [2] to avoid confusing the parameter \( b \) with a particle label.)

As an initial attempt to extend (4.4) to the remaining ADET theories one could therefore try

\[
R^{(d,C)}_a(\theta) = R_a(\theta)/Z^{(d,C)}_a(\theta),
\]

with \( Z^{(d,C)}_a(\theta) \) as in (4.3). This manoeuvre certainly generates mathematically-consistent sets of reflection amplitudes, but in the more general cases it is not the most economical choice. Consider instead the functions obtained by replacing the blocks \( \{x\} \) in (2.7) by the simpler blocks \( (x) \) [33]:

\[
S^{F}_{ab} = \prod_{x \in A_{ab}} (x).
\]

For the Lee-Yang model, \( S^{F}(\theta) \) coincides with \( S(\theta) \), but for other theories it has fewer poles and zeros. The bootstrap constraints are only satisfied up to signs, but if \( S^{F} \) is used to define a function \( Z^{(d,C)}_a(\theta) \) as

\[
Z^{(d,C)}_a(\theta) = S^{F}_{da}(\theta + \frac{\varphi}{m}C)S^{F}_{da}(\theta - \frac{\varphi}{m}C)
\]

then these signs cancel, and thus (4.8) provides a more “minimal” generalization of the family of Lee-Yang reflection factors which nevertheless preserves all of its desirable properties. Setting \( d \) equal to the lightest particle in the theory generally gives the family with the smallest number of additional poles and zeros, but we shall see evidence later that all cases have a role to play.

\[\text{\textsuperscript{†}}\text{There are also boundary-changing operators, but these were not considered in [2, 10].}\]
The normalisation of the shift was changed in passing from (4.3) to (4.8); this is convenient because, as a consequence of the property

$$(x - C)(\theta) \times (x + C)(\theta) = (x)(\theta + i\frac{\pi}{h}C) \times (x)(\theta - i\frac{\pi}{h}C),$$

we have

$$Z_a^{(d,C)} = \prod_{x \in A_u} (x - C)(x + C).$$

The factors $(Z_a^{(d,C)})^{-1}$ therefore coincide with the coupling-constant dependent parts of the affine Toda field theory S-matrices of [12, 7], with $C$ related to the parameter $B$ of [12] by $C = 1 - B$.

5 Boundary $T_r$ theories as reductions of boundary sine-Gordon

The reflection factors presented so far are only conjectures, and no evidence has been given linking them to any physically-realised boundary conditions. The best signal in this respect will come from the exact $g$-function calculations to be reported in later sections. However, for the $T_r$ theories, alternative support for our general scheme comes from an interesting relation with the reflection factors of the sine-Gordon model.

In the bulk, the $T_r$ theories can be found as particularly-simple quantum group reductions of the sine-Gordon model at certain values of the coupling, in which the soliton and antisoliton states are deleted leaving only the breathers [21]. Quantum group reduction in the presence of boundaries has yet to be fully understood, but the simplifications of the $T_r$ cases allow extra progress to be made. This generalises the analysis of [2] for $T_1$, the Lee-Yang model, but has some new features.

The boundary S-matrix for the sine-Gordon solitons was found by Ghoshal and Zamolodchikov in [26], and extended to the breathers by Ghoshal in [34]. To match the notation used above for the ADE theories, we trade the sine-Gordon bulk coupling constant $\beta$ for

$$h = \frac{16\pi}{\beta^2} - 2. \tag{5.1}$$

Ghoshal and Zamolodchikov expressed their matrix solution to the boundary Yang-Baxter equation for the sine-Gordon model\(^\dagger\) in terms of two parameters $\xi$ and $k$. However for the scalar part (which is the whole reflection factor for the breathers) they found it more convenient to use $\eta$ and $\vartheta$, related to $\xi$ and $k$ by

$$\cos \eta \cosh \vartheta = -\frac{1}{k} \cos \xi, \quad \cos^2 \eta + \cosh^2 \vartheta = 1 + \frac{1}{k^2}. \tag{5.2}$$

\(^\dagger\)also found by de Vega and Gonzalez Ruiz [35]
Ghoshal-Zamolodchikov’s reflection factor for the $a$\textsuperscript{th} breather on the $|\eta, \vartheta\rangle$ boundary can then be written as
\begin{equation}
R_{a}^{\eta, \vartheta}(\theta) = R_{0}^{(a)}(\theta)R_{1}^{(a)}(\theta) \tag{5.3}
\end{equation}
where
\begin{equation}
R_{0}^{(a)} = \left(\frac{a}{h}\right)\frac{(a + h)}{(a + \frac{3h}{2})} \prod_{l=1}^{a-1} \frac{(l(l + h))}{(l + \frac{3h}{2})^{2}} \tag{5.4}
\end{equation}
is the boundary-parameter-independent part and
\begin{equation}
R_{1}^{(a)}(\theta) = S^{(a)}(\eta, \theta)S^{(a)}(i\vartheta, \theta) \tag{5.5}
\end{equation}
with
\begin{equation}
S^{(a)}(\nu, \theta) = \prod_{\substack{l=1-a
\text{step 2}}^{a-1}} \frac{\left(\frac{2\nu}{\pi} - \frac{h}{2} + l\right)}{\left(\frac{2\nu}{\pi} + \frac{h}{2} + l\right)} \tag{5.6}
\end{equation}
contains the dependence on $\eta$ and $\vartheta$. In these formulae we used the same blocks ($x$) as defined in (2.6) for the minimal ADE theories, with $h$ now given by (5.1).

Now for the quantum group reduction. In the bulk, suppose that $\beta^{2}$ is such that
\begin{equation}
h = 2r + 1, \quad r \in \mathbb{N}. \tag{5.7}
\end{equation}
At these values of the coupling, Smirnov has shown [21] that a consistent scattering theory can be obtained by removing all the solitonic states, leaving $r$ breathers with masses $M_{a} = \frac{\sin(\pi a/h)}{\sin(\pi/h)}M_{1}$. The scattering of these breathers is then given by the $T_{r}$ S-matrix (2.9). In order to explain all poles without recourse to the solitons, extra three-point couplings must be introduced, some of which are necessarily imaginary, consistent with the $T_{r}$ models being perturbations of nonunitary conformal field theories. In the presence of a boundary, these extra couplings give rise to extra boundary bootstrap equations, which further constrain the boundary reflection factors and impose a relation between the parameters $\eta$ and $\vartheta$. This can be seen by a direct analysis of the boundary bootstrap equations but it is more interesting to take another route, as follows.

We first recall the observation of [2], that the reflection factors for the boundary Lee-Yang model match solutions of boundary Yang-Baxter equation for the sine-Gordon model at
\begin{equation}
\xi \rightarrow i\infty. \tag{5.8}
\end{equation}
The Lee-Yang case corresponds to $r = 1, \ h = 3$, but we shall suppose that the same constraint should hold more generally (see also [9]). The condition must be translated into the $(\eta, \vartheta)$ parametrisation. One degree of freedom can be retained by allowing $k$ to tend to infinity as the limit (5.8) is taken. The relations (5.2) then become
\begin{equation}
\cos \eta \cosh \vartheta = A, \quad \cos^{2} \eta + \cosh^{2} \vartheta = 1. \tag{5.9}
\end{equation}
The constant $A$ can be tuned to any value by taking $\xi$ and $k$ to infinity suitably, and so the first equation in (5.9) is not a constraint; solving the second for $i\vartheta$,

$$i\vartheta = \eta + \frac{\pi}{2} + (r-d)\pi, \quad d \in \mathbb{Z}. \quad (5.10)$$

(The shift by $r$ is included for later convenience.) This appears to give a countable infinity of one-parameter families of reflection factors, but this is not so: the blocks $(x)$ in (5.5) and (5.6) depend on the boundary parameters only through the combination

$$\frac{2i\vartheta}{\pi} = \frac{2\eta}{\pi} + 2(r-d) + 1. \quad (5.11)$$

Since $(x + 2h) = (x)$, the reflection factors for $d$ are therefore the same as those for $d + h$. In addition, the freedom to redefine the parameter $\eta$ gives an extra invariance of the one-parameter families under $d \to 2r - d$. Thus the full set of options is realised by

$$d = 0, 1, \ldots r. \quad (5.12)$$

Note also that $R_1^{(a)}(\theta)$, the coupling-dependent part of the reflection factor, is trivial if $d = 0$. Thus the limit (5.8) corresponds to $r$ one-parameter families of breather reflection factors, and one ‘isolated’ case. This matches the counting of conformal boundary conditions (the set of bulk Virasoro primary fields [36]) for the $\mathcal{M}_{2,2r+3}$ minimal models, and the fact that of these boundary conditions, all but one have the relevant $\phi_{13}$ boundary operator in their spectra.

To see how this fits in with our earlier ideas, we swap $\eta$ for $C$, defined by

$$\frac{2\eta}{\pi} = d - \frac{h}{2} + C. \quad (5.13)$$

Then, starting from (5.5) and relabelling,

$$R_1^{(a)} = \prod_{l=1-a}^{a-1} \frac{(-h + d + l + C)}{(d + l + C)} \frac{(-d + l + C)}{(h - d + l + C)}$$

$$= \prod_{l=1-a}^{a-1} \frac{(-d - l - C)}{(-d - l + C)} \frac{(-h + d + l + C)}{(-h + d + l + C)}$$

$$= \prod_{l=1-a+d}^{a+d-1} \frac{(-l - C)}{(-l + C)} \frac{(-h + l + C)}{(-h + l + C)} \frac{(-h + l - C)}{(-h + l - C)}. \quad (5.14)$$

Since the first $a - d$ terms in the last product cancel if $a > d$, it is easily seen that $R_1^{(a)}$ coincides with $(Z_a^{(d,C)})^{-1}$, as expressed by (4.10), for the $T_r$ S-matrix (2.9).
6 Some aspects of the off-critical \(g\)-functions

Our main tool in linking reflection factors to specific boundary conditions will be the formula for an exact off-critical \(g\)-function introduced in \([11]\). In this section we begin by reviewing the main formula, and then discuss some further properties and modifications which will be relevant later.

6.1 The exact \(g\)-function for diagonal scattering theories

The ground-state degeneracy, or \(g\)-function, was introduced by Affleck and Ludwig as a useful characterisation of the boundary conditions in critical quantum field theories \([8]\). In a massive model, the definition can proceed along similar lines \([9, 2, 11]\). One starts with the partition function \(Z_{\langle \alpha|\alpha \rangle[L,R]}\) for the theory on a finite cylinder of circumference \(L\), length \(R\) and boundary conditions of type \(\alpha\) at both ends. In the \(R\)-channel description, time runs along the length of the cylinder, and the partition function is represented as a sum over an eigenbasis \(\{|\psi_k\rangle\}\) of \(H_{\text{circ}}^{\alpha}(M,L)\), the Hamiltonian which propagates states along the cylinder:

\[
Z_{\langle \alpha|\alpha \rangle[L,R]} = \sum_{k=0}^{\infty} (G_{\alpha}^{(k)}(l))^2 e^{-RE_{\text{circ}}^{\alpha}(M,L)}. \tag{6.1}
\]

Here \(l = ML\), \(M\) is the mass of the lightest particle in the theory, and

\[
G_{\alpha}^{(k)}(l) = \frac{\langle \alpha|\psi_k \rangle}{\langle \psi_k|\psi_k \rangle^{1/2}}, \tag{6.2}
\]

where \(|\alpha\rangle\) is the (massive) boundary state \([26]\) for the \(\alpha\) boundary condition. At finite values of \(l\) any possible infinite-volume vacuum degeneracy is lifted by tunneling effects, making the ground state \(|\psi_0\rangle\) unique. This gives \(Z_{\langle \alpha|\alpha \rangle}\) the following leading and next-to-leading behaviour in the large-\(R\) limit:

\[
\ln Z_{\langle \alpha|\alpha \rangle[L,R]} \sim -RE_{\text{circ}}^{\alpha}(M,L) + 2 \ln G_{\alpha}^{(0)}(l). \tag{6.3}
\]

It is the second, sub-leading term which characterises the massive \(g\)-function. Subtracting a linearly-growing piece \(-f_{\alpha}L\), the \(g\)-function for the boundary condition \(\alpha\) at system size \(l\) is defined to be

\[
\ln g_{\alpha}(l) = \ln G_{\alpha}^{(0)}(l) + f_{\alpha}L. \tag{6.4}
\]

The constant \(f_{\alpha}\) is equal to the constant (boundary) contribution to the ground-state energy \(E_{\text{strip}}^{\alpha}(R)\) of the \(L\)-channel Hamiltonian \(H_{\text{strip}}^{\alpha}(R) \equiv H_{\text{strip}}^{\alpha}(R)\), which propagates states living on a segment of length \(R\) and boundary conditions \(\alpha\) at both ends.

Although this definition is in principle unambiguous, the difficulty of characterising the finite-size states \(|\alpha\rangle\) and \(|\psi_0\rangle\) limits its utility in practice. An alternative expression for \(g\) can be obtained by comparing (6.3) with the \(L\)-channel
representation, a sum over the full set \( \{ E_{\text{strip}}(R) \} \) of eigenvalues of \( H_{\text{strip}}(R) \):  
\[
Z_{\{\alpha|\alpha\}}[L, R] = \sum_{k=0}^{\infty} e^{-LE_{\text{strip}}(M,R)}. \tag{6.5}
\]

As \( R \) is sent to infinity, we have  
\[
\ln g_{(\alpha)}(l) = \frac{1}{2} \lim_{R \to \infty} \left[ \ln \left( \sum_{k=0}^{\infty} e^{-LE_{\text{strip}}(M,R)} \right) + 2f_{(\alpha)}L + RE_{0}^{\text{circ}}(M,L) \right]. \tag{6.6}
\]

In theories with only massive excitations in the bulk and no infinite-volume vacuum degeneracy, \( \ln g_{(\alpha)}(l) \) tends exponentially to zero at large \( l \), while in the UV limit \( l \to 0 \) it reproduces the value of the \( g \)-function in the unperturbed boundary conformal field theory.

The subleading nature of the term being extracted makes it hard to evaluate (6.6) directly. In [11], considerations of the infrared (large-\( l \)) asymptotics and a conjectured resummation led to the proposal of a general formula for the exact \( g \)-function of a purely elastic scattering theory. However in section 6.2 below we shall argue that the result of [11] needs to be modified by the introduction of a simple symmetry factor whenever there is coexistence of vacua at infinite volume. This happens in the low-temperature phases of the \( E_6, E_7, A \) and \( D \) models. The more-general result is  
\[
\ln g_{(\alpha)}(l) = \ln C_{(\alpha)} + \frac{1}{4} \sum_{a=1}^{r_{g}} \int_{\mathbb{R}} d\theta \left( \phi_{a}^{(\alpha)}(\theta) - \delta(\theta) - 2\phi_{aa}(2\theta) \right) \ln \left( 1 + e^{-\varepsilon_{a}(\theta)} \right) + \Sigma(l) \tag{6.7}
\]

where \( C_{(\alpha)} \) is the symmetry factor to be discussed shortly, and the functions \( \phi_{a}^{(\alpha)}(\theta) \) and \( \phi_{ab}(\theta) \) are related to the bulk and boundary scattering amplitudes \( S_{ab}(\theta) \) and \( R_{a}^{(\alpha)}(\theta) \) by  
\[
\phi_{a}^{(\alpha)}(\theta) = -\frac{i}{\pi} \frac{d}{d\theta} \ln R_{a}^{(\alpha)}(\theta), \tag{6.8}
\]
\[
\phi_{ab}(\theta) = -\frac{i}{2\pi} \frac{d}{d\theta} \ln S_{ab}(\theta). \tag{6.9}
\]

The boundary-condition-independent piece \( \Sigma(l) \) can be expressed in terms of solutions to the bulk thermodynamic Bethe ansatz (TBA) equations as  
\[
\Sigma(l) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{a_{1} \ldots a_{n}=1}^{r_{g}} \frac{1}{n} \int_{\mathbb{R}^{n}} \frac{d\theta_{1}}{1 + e^{\varepsilon_{a_{1}}(\theta_{1})}} \cdots \frac{d\theta_{n}}{1 + e^{\varepsilon_{a_{n}}(\theta_{n})}} \times 
(\phi_{a_{1},a_{2}}(\theta_{1} + \theta_{2})\phi_{a_{2},a_{3}}(\theta_{2} - \theta_{3}) \cdots \phi_{a_{n},a_{n+1}}(\theta_{n} - \theta_{n+1})) \tag{6.10}
\]
with \( \theta_{n+1} = \theta_{1}, a_{n+1} = a_{1} \). The functions \( \varepsilon_{a}(\theta) \), called pseudoenergies, solve the bulk TBA equations  
\[
\varepsilon_{a}(\theta) = M_{a} L \cosh \theta - \sum_{b=1}^{r_{g}} \int_{\mathbb{R}} d\theta' \phi_{ab}(\theta - \theta') L_{b}(\theta'), \quad a = 1, \ldots, r_{g}. \tag{6.11}
\]
where \( L_a(\theta) = \ln(1 + e^{-e_a(\theta)}) \). We also recall that the ground-state energy of the theory on a circle of circumference \( L \) is given in terms of these pseudoenergies by

\[
E^{\text{circ}}_0(M,L) = -\sum_{a=1}^{r_g} \int_{\mathbb{R}} \frac{d\theta}{2\pi} M_a \cosh \theta L_a(\theta) + \mathcal{E}LM^2
\]  

(6.12)

where \( \mathcal{E}LM^2 \) is the bulk contribution to the energy.

In [11] the formula (6.7) was checked in detail, but only for the \( r_g = 1 \), \( C|_{\alpha} = 1 \) case corresponding to the Lee-Yang model. Our results below will confirm that it holds in more general cases, provided \( C|_{\alpha} \) is appropriately chosen.

### 6.2 Models with internal symmetries

The scaling Lee-Yang model, on which the analysis of [11] was mostly concentrated, is a massive integrable quantum field theory with fully diagonal scattering and a single vacuum. However many models lack one or both these properties. In this section we shall extend our analysis to models possessing, in infinite volume, \( N \) equivalent vacua but still described by purely elastic scattering theories. For the ADE-related theories the \( N \) equivalent vacua are related by a global symmetry \( Z \). We shall see that when these systems are in their low-temperature phases, the formula for \( g(l) \) as originally proposed in [11] should be slightly corrected, by including a 'symmetry factor' \( C|_{\alpha} \).

Low-temperature phases are common features of two dimensional magnetic spin systems below their critical temperatures. A discrete symmetry \( Z \) of the Hamiltonian \( H \) is spontaneously broken, and a unique ground state is singled out from a multiplet of equivalent degenerate vacua. This contrasts with the high-temperature phase, where the ground state is \( Z \)-invariant.

The prototype of two-dimensional spin systems is the Ising model, and we shall treat this case first. The Ising Hamiltonian for zero external magnetic field is invariant under a global spin reversal transformation \( (Z = Z_2) \), and at low temperatures \( T < T_c \) this symmetry is spontaneously broken and there is a doublet \( \{|+\rangle, |-\rangle\} \) of vacuum states transforming into each other under a global spin-flip:

\[
|\pm\rangle \rightarrow |\mp\rangle.
\]  

(6.13)

The \( Z_2 \) symmetry of \( H \) is preserved under renormalisation and it also characterizes the continuum field theory version of the model: in the low-temperature phase the bulk field theory has two degenerate vacua \( \{|+\rangle, |-\rangle\} \) with excitations, the massive kinks \( K_{[+\mp]}(\theta), K_{[-\mp]}(\theta) \), corresponding to field configurations interpolating between these vacua.

In infinite volume the ground state is either \(|+\rangle \) or \(|-\rangle \), and the transition from \(|+\rangle \) to \(|-\rangle \) (or vice-versa) can only happen in an infinite interval of time. On the other hand in a finite volume \( V \), tunneling is allowed: a kink with finite speed

\[^{§}\text{Integrable models with non-equivalent vacua are typically associated to non-diagonal scattering like, for example, the } \phi_{13} \text{ perturbations of the minimal } \mathcal{M}_{p,q} \text{ models [41].}\]
can span the whole volume segment in a finite time interval. We are interested in the scaling limit of the Ising model on a finite cylinder and we interpret the open-segment direction of length $R$ as the space coordinate. Time is then periodic with period $L$. If the boundary conditions are taken to be fixed, of type $+$, at both ends of the segment, then the only multi-particle states which can propagate have an even number of kinks, of the form

$$K_{[-+]}(\theta_1)K_{[-+]}(\theta_2) \ldots K_{[-+]}(\theta_n), \quad (n \text{ even}).$$

For $R$ large the rapidities $\{\theta_j\}$ are quantised according to the Bethe ansatz equations

$$r \sinh \theta_j - i \ln R^{(+)}(\theta_j) = \pi j, \quad (r = MR, j = 1, 2, \ldots),$$

where $R^{(+)}(\theta)$ is the amplitude describing the scattering of particles off a wall with fixed boundary conditions of type $+$. At low temperatures $Z_{(++)}[L, R]$ therefore receives contributions only from states with an even number of particles. It is conveniently written in the form (see for example [42]):

$$Z_{(++)}[L, R] = \frac{1}{2}Z_{(++)}^{(0)}[L, R] + \frac{1}{2}Z_{(++)}^{(1)}[L, R]$$

where

$$Z_{(++)}^{(b)}[L, R] = e^{-LE_0^{\mathrm{strip}}(M, R)} \prod_{j > 0} \left(1 + (-1)^b e^{-l \cosh \theta_j}\right), \quad (l = ML),$$

$E_0^{\mathrm{strip}}(M, R)$ is the ground-state energy, and the set $\{\theta_j\}$ is quantised by (6.15). In order to extract the subleading contributions to $Z_{(++)}$, as in [11] we first set

$$Z_{(++)}[L, R] = \frac{1}{2}e^{\ln Z_{(++)}^{(0)}[L, R]} + \frac{1}{2}e^{\ln Z_{(++)}^{(1)}[L, R]},$$

and in the limit $R \to \infty$ use Newton’s approximation to transform sums into integrals. The result is

$$\ln Z_{(++)}^{(b)}[L, R] \sim \frac{1}{2} \int_R d\theta \left(\frac{1}{\pi} \cosh(\theta) + \phi^{(+)}(\theta) - \delta(\theta)\right) \ln(1 + (-1)^b e^{-l \cosh \theta})$$

$$= 2 \ln g_{(+)}^{(b)}(l) - RE_0^{\mathrm{circle}, b}(M, L), \quad (b = 0, 1)$$

(6.19)

From (6.19) we see that at finite values of $l$ as $R \to \infty$

$$\ln Z_{(++)}^{(0)}[L, R] - \ln Z_{(++)}^{(1)}[L, R] \to +\infty$$

(6.20)

and therefore

$$Z_{(++)}[L, R] \sim (g_{(+)}(l))^2 e^{-RE_0^{\mathrm{circle}}(M, L)} \sim \frac{1}{2} Z_{(++)}^{(0)}[L, R]$$

$$= \frac{1}{2}(g_{(+)}^{(0)}(l))^2 e^{-RE_0^{\mathrm{circle}}(M, L)},$$

(6.21)
where \( E^\text{circ}_0(M, L) \equiv E^\text{circ,0}_0(M, L) \) is the ground state energy for the system with periodic boundary conditions.

Notice that \( Z^{(0)}_{(+|+)}[L, R] \) takes contributions from states with both even and odd number of particles. In fact, following [11], we have \( g_{|+\rangle}^{(0)} = g_{\text{ref}}^{\text{fixed}} \) and

\[
g_{|+\rangle}(l) = g_{|\rightarrow\rangle}(l) = \frac{1}{\sqrt{2}} g_{\text{ref}}^{\text{fixed}}.
\]

In conclusion, the appearance of the extra symmetry factor \( C_{|+\rangle} = \frac{1}{\sqrt{N}} \) is related to the kink selection rule restricting the possible multi-particle states. In the Ising model this was seen in the exact formula for the low-temperature partition function \( Z^{(+|+)}_\langle[L, R] \), by its being written as an averaged sum of \( Z^{(0)}_{(+|+)}[L, R] \) and \( Z^{(1)}_{(+|+)}[L, R] \).

In more general theories with discrete symmetries, similar considerations apply. Consider a set of \( i = 1, 2, \ldots, N \) ‘fixed’ boundary conditions, matching the \( i = 1, 2, \ldots, N \) vacua. A derivation of \( Z_{(i|i)} \) as in [11], including all multiparticle states satisfying the Bethe momentum quantisation conditions, will give an incorrect answer, since some states will be forbidden by the kink structure. Instead, let us construct a new boundary state

\[
|U\rangle = \sum_{j=1}^{N} |j\rangle,
\]

and consider the partition function \( Z_{(U|i)} \). Since boundary scattering does not mix vacua it is clear that the only effect of replacing \( |i\rangle \) by \( |U\rangle \) is to eliminate the kink condition on allowed multiparticle states. Thus the counting of states on an interval with \( U \) at one end and \( i \) at the other is exactly the same as it would be in a high-temperature phase with the reflection factor at both ends being \( R_{|i\rangle}(\theta) \), and the derivation in [11] goes through to find at large \( R \)

\[
\ln Z_{(U|i)}[L, R] \sim -R E^\text{circ}_0(M, L) + 2 \ln G^{(0)}(l)
\]

where \( G^{(0)} \) is – up to a linear term – given by (6.7) with \( \phi_{\alpha}^{(i)}(\theta) = \phi_{\alpha}^{(j)}(\theta) \) and \( C_{|\alpha\rangle} = 1 \). On the other hand, since all vacua are related by the discrete symmetry \( Z \) and the finite-volume vacuum state \( |\psi_0\rangle \) must be symmetrical under this symmetry, \( \langle \psi_0|i\rangle = \langle \psi_0|j\rangle \forall i, j \), and so

\[
\left(G^{(0)}(l)\right)^2 = \frac{\langle U|\psi_0\rangle\langle \psi_0|i\rangle}{\langle \psi_0|\psi_0\rangle} = N \frac{\langle \psi_0|i\rangle^2}{\langle \psi_0|\psi_0\rangle} = N \left(G^{(0)}_{|i\rangle}(l)\right)^2.
\]

Hence

\[
g_{|i\rangle}(l) = \frac{1}{\sqrt{N}} g(l)
\]

and \( g_{|i\rangle}(l) \) is given by the formula (6.7) with \( C_{|i\rangle} = \frac{1}{\sqrt{N}} \).
An immediate consequence is that in the infrared, \( g_{(i)}(l) \) does not tend to one. Instead,

\[
\lim_{l \to \infty} g_{(i)}(l) = C_{(i)} = \frac{1}{\sqrt{N}}.
\]

This perhaps-surprising claim can be justified independently. In infinite volume, boundary states can be described using a basis of scattering states [26]:

\[
|B\rangle = \left(1 + \frac{1}{2} \int_{-\infty}^{\infty} K^{ab}(\theta) A_a(-\theta) A_b(\theta) + \ldots\right)|0\rangle
\]

(6.28)

where \(|0\rangle\) is the bulk vacuum for an infinite line. In finite but large volumes, the same expression provides a good approximation to the boundary state, the only modification normally being that the momenta of the multiparticle states must be quantised by the relevant Bethe ansatz equations. However a subtlety arises when there is a degeneracy among the bulk vacua, in situations where the boundary condition distinguishes between these vacua. (Similar issues are encountered in comparing exact bulk VEVs of fields with those obtained from finite-volume approximations [43, 44].) Take the case of a fixed boundary condition which picks out one of the degenerate vacua, say \( i \); then the state \(|0\rangle\) on the RHS of (6.28) is \( |i\rangle \). However when calculating the finite-volume \( g \)-function, we must consider \( \langle \psi_0 | B \rangle \), where \( \langle \psi_0 | \) is the finite-volume vacuum. (Note that the bulk energy is normalised to zero when considering (6.28), so this inner product will give us \( g \) directly, rather than \( G \).) In large but finite volumes, the tunneling amplitude between the infinite-volume bulk vacua is non-zero and so the appropriately-normalised finite volume ground state is the symmetric combination

\[
\langle \psi_0 | = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \langle j |.
\]

The limiting value of \( g \) in the infrared is therefore not 1, but \( 1/\sqrt{N} \), as found in the exact calculation earlier.

Table 1 lists the central charge, the Coxeter number, the symmetry group \( \mathcal{Z} \), and the number of degenerate vacua for the ADET models. For the \( A_r, D_r, E_6 \) and \( E_7 \) theories at low temperatures there is a coexistence of \( N \) vacuum states \( |i\rangle, i = 1, 2, \ldots, N, \) with \( N = r + 1, 4, 3 \) and 2 respectively. The corresponding symmetry factor \( C_{(i)} \), for fixed-type boundary conditions which single out a single bulk vacuum, is always \( 1/\sqrt{N} \). (For more general boundary conditions one can anticipate other symmetry factors, but we shall leave a detailed discussion of this point for another time.)

### 6.3 Further properties of the exact \( g \)-function

An important property of the sets of TBA equations that we are considering concerns the so-called Y-functions [37],

\[
Y_a(\theta) = e^{\varepsilon_a(\theta)}.
\]

(6.30)
These satisfy a set of functional relations called a Y-system:

\[ Y_a(\theta + \frac{i\pi}{h})Y_a(\theta - \frac{i\pi}{h}) = \prod_{b=1}^{r_g} (1 + Y_b(\theta))^{A_{ba}^G}, \quad (6.31) \]

where \( A_{ba}^G \) is the incidence matrix of the Dynkin diagram \( G \) labelling the TBA system. Defining an associated set of T-functions through the relations

\[ Y_a(\theta) = \prod_{c=1}^{r_g} (T_c(\theta))^{A_{ca}^G}, \quad 1 + Y_a(\theta) = T_a(\theta + \frac{i\pi}{h})T_a(\theta - \frac{i\pi}{h}) \quad (6.32) \]

we have

\[ T_a(\theta + \frac{i\pi}{h})T_a(\theta - \frac{i\pi}{h}) \prod_{c=1}^{r_g} (T_c(\theta))^{-A_{ca}^G} = 1 + Y_a^{-1}(\theta) \quad . \quad (6.33) \]

In addition the T-functions satisfy

\[ T_a(\theta + \frac{i\pi}{h})T_a(\theta - \frac{i\pi}{h}) = 1 + \prod_{b=1}^{r_g} (T_b(\theta))^{A_{ba}^G} \quad . \quad (6.34) \]
Fourier transforming the logarithm of equation (6.33), solving taking the large $\theta$ asymptotic into account, and transforming back recovers the (standard) formula

$$\ln T_a(\theta) = \frac{LM_a}{2\cos \frac{\theta}{\pi}} \cosh \theta - \sum_{b=1}^{r_g} \int_{\mathbb{R}} d\theta' \chi_{ab}(\theta - \theta') L_b(\theta'), \quad a = 1, \ldots, r_g,$$  

(6.35)

where [37, 17]

$$\chi_{ab}(\theta) = - \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik\theta} \left( 2 \cosh (k\pi/h) \mathbb{1} - A^{(G)} \right)^{-1}_{ab} = - \frac{i}{2\pi} \frac{d}{d\theta} S^F_{ab}(\theta).$$  

(6.36)

The result (6.35) allows us to make a connection between the exact $g$-functions for our parameter-dependent reflection factors and the $T$ functions. Taking the reflection factors defined by (4.8) and (4.6)

$$R^{[b,C]}_a = R^{[a]}(\theta)/\left( S^F_{ab}(\theta - i\frac{\pi}{h}C)S^F_{ab}(\theta + i\frac{\pi}{h}C) \right)$$  

(6.37)

and using (6.7), we find

$$\ln g_{[b,C]}(l) = \ln g(l) - \sum_{a=1}^{r_g} \int_{\mathbb{R}} d\theta' \chi_{ab}(\theta' - i\frac{\pi}{h}C)L_b(\theta')$$

$$- \frac{1}{2} \sum_{a=1}^{r_g} \int_{\mathbb{R}} d\theta' \chi_{ab}(\theta' + i\frac{\pi}{h}C)L_b(\theta').$$  

(6.38)

Comparing this result with (6.35) and using the property $T_b(\theta) = T_b(-\theta)$,

$$\ln g_{[b,C]}(l) = \ln g(l) + \ln T_b(i\frac{\pi}{h}C) - \frac{LM_b}{2\cos \frac{\pi}{h}} \cos(\frac{\pi}{h}C)$$  

(6.39)

or, subtracting the linear contribution:

$$G^{(0)}_{[b,C]}(l) = G^{(0)}(l)T_b(i\frac{\pi}{h}C).$$  

(6.40)

Exact relations between $g$- and $T$- functions in various situations where the bulk remains critical were observed in [38, 39, 40]. These were extended off-criticality to a relation between the $G$- and $T$- function of the Lee-Yang model in [10]. The generalisation of [10] proposed here relies on the specific forms of our one-parameter families of reflection factors, and thus provides some further motivation for their introduction.

It is sometimes helpful to have an alternative representation for the infinite sum $\Sigma(l)$ in (6.7), the piece of the $g$-function which does not depend on the specific boundary condition. It is readily checked that, for values of $l$ such that the sum converges,

$$\Sigma(l) = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{Tr} K^n - \sum_{n=1}^{\infty} \frac{1}{2n} \text{Tr} H^n$$  

(6.41)
where
\[ K_{ab}(\theta, \theta') = \frac{1}{2} \frac{1}{\sqrt{1 + e^{\varepsilon_a(\theta)}}} \frac{1}{\sqrt{1 + e^{\varepsilon_b(\theta')}}} \left( \phi_{ab}(\theta + \theta') + \phi_{ab}(\theta - \theta') \right) \]  
(6.42)
and
\[ H_{ab}(\theta, \theta') = \frac{1}{\sqrt{1 + e^{\varepsilon_a(\theta)}}} \phi_{ab}(\theta - \theta') \frac{1}{\sqrt{1 + e^{\varepsilon_b(\theta')}}}. \]  
(6.43)
The identity
\[ \ln \det(I - M) = \text{Tr} \ln(I - M) = -\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} M^n \]  
(6.44)
then allows \( \Sigma(l) \) to be rewritten as
\[ \Sigma(l) = \frac{1}{2} \ln \det \left( \frac{I - H}{(I - K)^2} \right). \]  
(6.45)
This formula makes sense even when the original sum diverges, a fact that will be used in the next section.

7 UV values of the \( g \)-function

Taking into account the effect of vacuum degeneracies described above, a first test of the reflection factors found earlier is to calculate the \( l \to 0 \) limit of (6.7). The value of \( g_{|\alpha\rangle}(0) \) should match the value of a conformal field theory \( g \)-function, either of a Cardy state or of a superposition of such states.

As \( l \to 0 \), the pseudoenergies \( \varepsilon_a(\theta) \) tend to constants, which we shall denote as \( \epsilon_a \). Their values were tabulated by Klassen and Melzer in [7], who also observed an elegant formula the integrals of the logarithmic derivatives of the bulk S-matrix elements, later proved in [15]:
\[ \int_{\mathbb{R}} d\theta \phi_{ab}(\theta) = -N_{ab}, \]  
(7.1)
where \( N_{ab} \) is related to the Cartan matrix \( C \) of the associated Lie algebra by \( N = 2C^{-1} - 1 \). *We shall also need the integrals of the logarithmic derivatives of the reflection factors. Writing
\[ R_a = \prod_{x \in A} (x) \]  
(7.2)
*The Cartan matrix for \( T_r \) is taken to be
\[
\begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& & 2 & -1 \\
& & & 2 & -1 \\
& & & & 1
\end{pmatrix}
\]
for some set $A$, the required integrals are

$$\int_{\mathbb{R}} d\theta \phi_a^{(\alpha)}(\theta) = -2 \sum_{x \in A} \left( 1 - \frac{x}{h} \right) \text{sign}[x/h]$$

(7.3)

with $\text{sign}[0] = 0$. For the minimal ADE reflection factors found in section 3 these evaluate to

$$\int_{\mathbb{R}} d\theta \phi_a^{(\alpha)}(\theta) = 1 - N_{aa},$$

(7.4)

while for the $T_r$ reflection factors (5.4),

$$\int_{\mathbb{R}} d\theta \phi_a^{(\alpha)}(\theta) = -2N_{aa}.$$ (7.5)

To calculate the UV limit of $\Sigma(l)$, we define a matrix $\mathcal{M}$ whose elements are

$$M_{ab} = -\frac{N_{ab}}{1 + e^{\kappa}}.$$ (7.6)

in terms of which

$$\Sigma(0) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} (e_1^n + \ldots + e_r^n) = \frac{1}{4} \ln \left((1 - e_1) \ldots (1 - e_r)\right)$$

(7.7)

where $e_1, \ldots, e_r$ are the eigenvalues of $\mathcal{M}$. When one of these eigenvalues is larger than 1, the infinite sum does not converge (this is the case for $A_r, r \geq 5$, $D_r$, $r \geq 4$ and $E_r$) but the RHS of (7.7) gives the correct analytic continuation, since it follows from (6.45).

### 7.1 $T_r$

We start with the $T_r$ theories, whose UV limits are the non-unitary minimal models $\mathcal{M}_{2,2r+3}$. There are $r + 1$ ‘pure’ conformal boundary conditions, and the corresponding values of the conformal $g$-functions can be found using the formula

$$g_{(1,1+d)} = \frac{S_{(1,1+r),(1,1+d)}}{|S_{(1,1+r),(1,1)}|^{1/2}}, \quad d = 0, \ldots, r$$

(7.8)

where $(1,1+r)$ is the ground state of the bulk theory, with lowest conformal weight, $(1,1)$ is the ‘conformal vacuum’ – with conformal weight 0 – and $S$ is the modular S-matrix (see for example [45]). The components of $S$ for a general minimal model $\mathcal{M}_{p,p'}$ are

$$S_{(n,m);(\rho,\sigma)} = 2\sqrt{\frac{2}{pp'}} (-1)^{1+m\rho+n\sigma} \sin(\pi \frac{p}{p'} n \rho) \sin(\pi \frac{p'}{p} m \sigma),$$

(7.9)

where $1 \leq n, \rho \leq p' - 1, 1 \leq m, \sigma \leq p - 1$ and, to avoid double-counting, $m < \frac{p}{p'} n$ and $\rho < \frac{p'}{p} \sigma$. For the $T_r$ theories, $p' = 2$ and $p = 2r + 3$, and so $n$ and $\rho$ are both equal to 1 while $m$ and $\sigma$ range from 1 to $r + 1$, as in (7.8).
The values predicted by (7.8) should be compared with the values of $g_{(\alpha)}(0)$ calculated from (6.7). For the boundary-parameter-independent part of the reflection factor from (5.4), which is the $d = 0$ case of the ‘quantum group reduced’ options (5.12), the second term of (6.7) evaluates to

$$
\frac{1}{4} \sum_{a=1}^{r} \int_{\mathbb{R}} d\theta \left( \phi_{a}(\theta) - \delta(\theta) - 2\phi_{aa}(2\theta) \right) \ln \left( 1 + e^{-\epsilon_{a}} \right)
$$

$$
= -\frac{1}{2} \sum_{a=1}^{r} a \ln \left( 1 + \frac{\sin^{2} \left( \frac{\pi}{2r+3} \right)}{\sin \left( \frac{a\pi}{2r+3} \right) \sin \left( \frac{(a+2)\pi}{2r+3} \right)} \right)
$$

$$
= \ln \left( \frac{\sin \left( \frac{\pi}{2r+3} \right) \sin^{r} \left( \frac{(r+2)\pi}{2r+3} \right)}{\sin^{r+1} \left( \frac{(r+1)\pi}{2r+3} \right)} \right),
$$

(7.10)

while $\Sigma(0)$, calculated using (7.7), is

$$
-\frac{1}{4} \ln \left( (1 - e_{1}) \ldots (1 - e_{r}) \right) = \frac{1}{2} \ln \left( \frac{2}{\sqrt{2r+3}} \sin \left( \frac{(r+1)\pi}{2r+3} \right) \right). \quad (7.11)
$$

Adding these terms together and using some simple trigonometric identities reveals a dramatic simplification:

$$
g(l)|_{l=0} = \left( \frac{2}{\sqrt{2r+3}} \sin \left( \frac{\pi}{2r+3} \right) \right)^{1/2} = g(1,1) \quad (7.12)
$$

for all $r$. This suggests that the minimal $T_{r}$ reflection factors (5.5) describe bulk perturbations of the boundary conformal field theory with $(1,1)$ boundary conditions. Notice that among all of the possible conformal boundary conditions in the unperturbed theory, this is the only one with no relevant boundary operators, matching the fact that the minimal reflection factors (5.5) have no free parameters.

### 7.2 $A_{r}$, $D_{r}$ and $E_{r}$

The ADE cases exhibit an interestingly uniform structure: substituting (7.1) and (7.4) into (6.7) shows that the UV limit of the second, reflection-factor-dependent, term of (6.7) is always zero when the minimal reflection factors from section 3 are used. This result can be confirmed by examining the contributions of the blocks $(x)$ of $S(2\theta)$, and $(x)$ (or $(\overline{x})$) of $R^{(\alpha)}(\theta)$, as follows. Since the bulk S-matrix can be written as

$$
S_{ab}(\theta) = \prod_{x \in A_{ab}} \{x\}(\theta) \quad (7.13)
$$

where $\{x\} = (x + 1)(x - 1)$, $S_{ab}(2\theta)$ can be written similarly:

$$
S_{ab}(2\theta) = \prod_{x \in A_{ab}} [x](\theta) \quad (7.14)
$$
where
\[
[x] = \left(\frac{x+1}{2}\right) \left(\frac{x-1}{2}\right) \left(\frac{2h-x+1}{2}\right)^{-1} \left(\frac{2h-x-1}{2}\right)^{-1}.
\] (7.15)

Summing the contributions of each block \([x]\), it is easily seen that
\[
2 \int_{\mathbb{R}} d\theta \phi_{\alpha}^a(2\theta) = \sum_{x \in A_{\alpha a}} \left(\frac{2x}{h} + \delta_{x,1} - 2\right).
\] (7.16)

The minimal reflection factors discussed in section 3 can be written in a similar way:
\[
R_a = \prod_{x \in A_{\alpha a}} f_x
\] (7.17)

where each \(f_x\) is either \(\langle x \rangle\) or \(\langle \tilde{x} \rangle\). The contribution to \(\int d\theta \phi_a^{(\alpha)}(\theta)\) from each \(\langle x \rangle\) is \(-2 + 2x/h + 2\delta_{x,1}\) and from each \(\langle \tilde{x} \rangle\) is \(-2 + 2x/h\). Noting that every minimal reflection factor contains the block \((1)\) exactly once, we find
\[
\int_{\mathbb{R}} d\theta \phi_a^{(\alpha)}(\theta) = \sum_{x \in A_{\alpha a}} \left(\frac{2x}{h} + 2\delta_{x,1} - 2\right),
\] (7.18)

and so
\[
\int_{\mathbb{R}} d\theta \left(\phi_a^{(\alpha)}(\theta) - \delta(\theta) - 2\phi_{\alpha a}(2\theta)\right) = 0
\] (7.19)

for every A, D and E theory, as claimed. Working backwards, this gives a general proof of the formula (7.4), given (7.1).

With the reflection-factor-dependent term giving zero, the sum of \(\Sigma(0)\) and the symmetry factor should correspond to a conformal \(g\)-function value. To find these values in a uniform way, we shall use the diagonal coset description \(\widehat{g}_1 \times \widehat{g}_1/\widehat{g}_2\) where \(\widehat{g}_l\) is the affine Lie algebra at level \(l\) associated to one of the A, D or E Lie algebras [46]. Coset fields are specified by triples \(\{\mu, \nu, \rho\}\) of \(\widehat{g}\) weights at levels 1, 1 and 2, and the \(g\)-function for the corresponding conformal boundary condition can be written in terms of the modular S-matrix of the coset model \(S\{0,0,0\}\{\mu,\nu,\rho\}\) as
\[
g_{\{\mu,\nu,\rho\}} = \frac{S\{0,0,0\}\{\mu,\nu,\rho\}}{\sqrt{S\{0,0,0\}\{0,0,0\}}}.
\] (7.20)

The integrable highest weights \(\mu\) of \(\widehat{g}\) can be expressed in terms of the fundamental weights \(\Lambda_i\) of \(\widehat{g}\) as:
\[
\mu = \sum_{i=0}^{r_0} n_i \Lambda_i
\] (7.21)

where the non-negative integers \(n_i, i = 0, \ldots, r_0\), are the Dynkin labels. The fundamental weights can be written as
\[
\Lambda_0 = (0, 1, 0)
\]
\[
\Lambda_i = (\lambda_i, \tilde{a}_i, 0)
\]
where $\lambda_i$ are the fundamental weights of $g$ and the colabels, $\tilde{a}_i$, are given for each case below:

- $A_r : \tilde{a}_i = 1$ for all $i$
- $D_r : \tilde{a}_1 = \tilde{a}_{r-1} = \tilde{a}_r = 1$ all others are 2
- $E_6 : \tilde{a}_i = (1, 1, 2, 2, 2, 3)$
- $E_7 : \tilde{a}_i = (1, 2, 2, 2, 3, 3, 4)$
- $E_8 : \tilde{a}_i = (2, 2, 3, 3, 4, 4, 5, 6)$

The highest weights $\mu_0$ of the corresponding finite-dimensional Lie algebra, $g$, can be written in a similar way, with Dynkin labels, $n_1, \ldots, n_{rg}$, taken from the same set:

$$\mu_0 = \sum_{i=1}^{rg} n_i \lambda_i.$$  \hfill (7.24)

Once the level $l$ of $\hat{g}$ has been specified, the Dynkin labels must satisfy

$$l = n_0 + \sum_{i=1}^{rg} n_i \tilde{a}_i$$  \hfill (7.25)

so the representation of $\hat{g}_l$ is completely determined by the Dynkin labels of the corresponding representation of $g$. For example, the label 0 is given to the vacuum representation, where $n_i = 0, i = 1, \ldots, rg$ for both $\hat{g}_1$ and $\hat{g}_2$.

In appendix A we show among other things that the coset conformal $g$-function (7.20) depends only on the level 2 weight, via the level 2 modular $S$-matrix\footnote{We’d like to thank Daniel Roggenkamp for his help in explaining this observation.}:

$$g_{\mu, \nu, \rho} = g_\rho = \frac{S_{0\rho}^{(2)}}{\sqrt{S_{00}^{(2)}}}$$  \hfill (7.26)

Clearly, this will not fix the coset field in general. For example, the number of boundary conditions with $g$-function equal to $g_0$ is $r + 1, 4, 3, 2, 2$ for the $A_r, D_r, E_6, E_7$ and $E_8$ models respectively. More details on this point are given in appendix A.

Algorithms for computing the modular $S$-matrices are given by Gannon in [47]; Schellekens has also produced a useful program for their calculation [48]. The level 2 modular $S$-matrix elements for $A, D$ and $E$ theories needed to calculate the UV values of the $g$-functions using eq.(7.26) are given in table 2. The representations are labelled by the Dynkin labels, $n_i, i = 0, \ldots, r$. The number of coset fields corresponding to each label is equal to the order of the orbit of that level 2 weight, under the outer automorphism group $O(\hat{g})$. Note for $D_r$ even, the weights $\rho_{r-1}$ and $\rho_r$ are in different orbits, each with order 2, whereas for $r$ odd they are in the same orbit with order 4, so in both cases there are 4 coset fields with the same $g$-function value.
Table 2: Level 2 modular S-matrix elements for A, D and E models

| Model | S-matrix element | Labels |
|-------|------------------|--------|
| A<sub>r</sub> | $s_{00}^{(2)} = \frac{2}{\sqrt{r+2}} \sin \left( \frac{\pi}{h+2} \right) \prod_{k=1}^{r} \sin \left( \frac{k\pi}{h+2} \right)$ | $0 : n_0 = 2, n_i = 0$ for $i = 1, \ldots, r$ |
| | $s_{0\rho_{\ell}}^{(2)} = \frac{2^{\ell+1}}{(h+2)\sqrt{2}} \sin \left( \frac{(\ell+1)\pi}{h+2} \right) \prod_{k=1}^{\ell} \sin \left( \frac{k\pi}{h+2} \right)$ | $\rho_j : n_0 = 1, n_i = \left\{ \begin{array}{ll} 0 & i \neq j \\ 1 & i = j \end{array} \right.$ for $j = 1, \ldots, [h/2]$. |
| D<sub>r</sub> | $s_{00}^{(2)} = \frac{1}{\sqrt{2r}}$ | $0 : n_0 = 2, n_i = 0$ for $i = 1, \ldots, r$ |
| | $s_{0\rho_{\ell}}^{(2)} = s_{0\rho_{\ell}}^{(2)} = 2^{\ell+1}$ | $\rho_1 : n_0 = n_1 = 1$, all other $n_i = 0$ |
| | $s_{0\rho_{\ell-1}}^{(2)} = s_{0\rho_{\ell}}^{(2)} = \frac{1}{2\sqrt{2}}$ | $\rho_j : n_0 = 0, n_i = \left\{ \begin{array}{ll} 0 & i \neq j \\ 1 & i = j \end{array} \right.$ for $j = 2, \ldots, r/2$ |
| | $s_{0\rho_{r-1}}^{(2)} = s_{0\rho_{r-1}}^{(2)} = 1$ | $\rho_{r-1} : n_0 = n_{r-1} = 1$, all other $n_i = 0$ |
| | $s_{0\rho_{r}}^{(2)} = s_{0\rho_{r}}^{(2)} = 1$ | $\rho_{r} : n_0 = n_r = 1$, all other $n_i = 0$ |
| E<sub>6</sub> | $s_{00}^{(2)} = \frac{1}{2\sqrt{2}} \sin \left( \frac{2\pi}{h+2} \right)$ | $0 : n_0 = 2, n_i = 0$ for $i = 1, 1, 2, 3, 3, 4$ |
| | $s_{0\rho_{j}}^{(2)} = \frac{1}{2\sqrt{2}} \sin \left( \frac{2(4-j+1)\pi}{h+2} \right)$ for $j = 1, 2$ | $\rho_1 : n_0 = 1, n_1 = 1$ all other $n_i = 0$ |
| | | $\rho_2 : n_0 = 0, n_2 = 1$ all other $n_i = 0$ |
| E<sub>7</sub> | $s_{00}^{(2)} = \frac{2}{\sqrt{2(h+2)}} \sin \left( \frac{4\pi}{h+2} \right)$ | $0 : n_0 = 2, n_i = 0$ for $i = 1, \ldots, 7$ |
| | $s_{0\rho_{j}}^{(2)} = \frac{2}{\sqrt{2(h+2)}} \sin \left( \frac{8\pi}{h+2} \right)$ for $j = 1, \ldots, 7$ | $\rho_1 : n_0 = 1, n_1 = 1$ all other $n_i = 0$ |
| | | $\rho_2 : n_0 = 0, n_2 = 1$ all other $n_i = 0$ |
| | | $\rho_3 : n_0 = 0, n_3 = 1$ all other $n_i = 0$ |
| E<sub>8</sub> | $s_{00}^{(2)} = s_{0\rho_{2}}^{(2)} = \frac{1}{2}$ | $0 : n_0 = 2, n_i = 0$, for all $i = 1, \ldots, 8$ |
| | $s_{0\rho_{j}}^{(2)} = s_{0\rho_{j}}^{(2)} = \frac{1}{\sqrt{2}}$ | $\rho_2 : n_0 = 0, n_2 = 1$, all other $n_i = 0$ |
| | | $\rho_3 : n_0 = 0, n_3 = 1$ all other $n_i = 0$ |

where $[x]$ is the integer part of $x$ and $h$ is the Coxeter number.
For each coset, the possible CFT values of the \( g \)-function can now be calculated using eq.(7.26) and the modular S-matrix elements given in table 2. Working case-by-case, we checked these numbers against the sums \( \ln C_\alpha + \Sigma(0) \), the UV limits of the off-critical \( g \)-functions \( g(l) \) for the minimal reflection factors described in section 3. In every case, we found that

\[
\ln g(l)|_{l=0} = \ln C_\alpha + \Sigma(0) = \ln g_0
\]

(7.27)

provided that the symmetry factors \( C_\alpha \) were assigned as in table 1. The explicit \( g \)-function values are given in table 3.

Table 3: UV \( g \)-function values calculated from the minimal reflection factors for the A, D and E models

| \( C_\alpha \) | \( g_0 \) | Number of fields |
|---|---|---|
| \( A_r \) | \( \frac{1}{\sqrt{r+1}} (\frac{2^{h+1}}{(h+2)\sqrt{h}} \sin \left( \frac{\pi}{h+2} \right) \prod_{k=1}^{[h+1]} \sin \left( \frac{k\pi}{h+2} \right) )^{1/2} \) | \( r + 1 \) |
| \( D_c \) | \( \frac{7}{2} \) \( \left( \frac{2}{7} \right)^{1/4} \) | 4 |
| \( E_6 \) | \( \frac{1}{\sqrt{3}} \) \( \frac{2}{\sqrt{21}} \sin \left( \frac{2\pi}{h+2} \right) \) \( ^{1/2} \) | 3 |
| \( E_7 \) | \( \frac{1}{\sqrt{2}} \) \( \frac{2}{\sqrt{h+2}} \sin \left( \frac{4\pi}{h+2} \right) \) \( ^{1/2} \) | 2 |
| \( E_8 \) | \( 1 \) \( \frac{1}{\sqrt{2}} \) | 2 |

For some minimal models, we can compare \( g_0 \) to the values of the \( g \)-function corresponding to known cases, thereby matching our reflection factors to physical boundary conditions. For the three-state Potts model \( (A_2) \), the tricritical Ising model \( (E_7) \) and the Ising model \( (E_8) \) the corresponding boundary condition is the ‘fixed’ condition in each case [49, 50, 8]. Note the number of coset fields with \( g \)-function equal to \( g_0 \) corresponds to the number of degenerate vacua, and hence to the number of possible ‘fixed’ boundary conditions for all A, D and E models.

8 Checks in conformal perturbation theory

The off-critical \( g \)-functions only match boundary conformal field theory values in the far ultraviolet. Moving away from this point one expects a variety of corrections, some of which were analysed using conformal perturbation theory in [10]. The expansion provided by our exact \( g \)-function result is instead about the infrared, but convergence is sufficiently fast that the first few terms of the UV expansion can be extracted numerically, allowing a comparison with conformal perturbation theory to be made. In [11] this was done for the boundary Lee-Yang model; in this section we test our more general proposals for the case of the three-state Potts model. Since the treatment of conformal perturbation theory
in [10] concentrated on the non-unitary Lee-Yang case, we begin with a general discussion of the leading bulk-induced correction to the \( g \)-function in the (simpler to treat) unitary cases.

Consider a unitary conformal field theory on a circle of circumference \( L \), perturbed by a bulk spinless primary field \( \varphi \) with scaling dimension \( x_{\varphi} = \Delta_{\varphi} + \overline{\Delta}_{\varphi} \). The perturbed Hamiltonian is then

\[
\hat{H} = \hat{H}_0 + \lambda \hat{H}_1 \tag{8.1}
\]

where

\[
\hat{H}_0 = \frac{2\pi}{L} \left( L_0 + \mathcal{L}_0 - \frac{c}{12} \right) \tag{8.2}
\]

and

\[
\hat{H}_1 = \left( \frac{L}{2\pi} \right)^{1-x_{\varphi}} \oint \varphi(e^{i\theta}) d\theta. \tag{8.3}
\]

For \( \lambda \) real in the ADE models we also expect a \( \lambda - M \) relation of the form

\[
M(\lambda) = \kappa |\lambda|^{1/(2-x_{\varphi})}; \quad |\lambda(M)| = \left( \frac{M}{\kappa} \right)^{2-x_{\varphi}} \tag{8.4}
\]

with \( \kappa \) a model-dependent constant.

Leaving the boundary unperturbed, we have a conformal boundary condition \( \alpha \), with boundary state \( |\alpha\rangle \). Set \( g_{(\alpha)} = g_{(\alpha)}^{0} = \langle \alpha | 0 \rangle \) and \( g_{(\alpha)}^{\varphi} = \langle \alpha | \varphi \rangle \), where \( |0\rangle \) and \( |\varphi\rangle \) are the states corresponding to the fields 1 and \( \varphi \). Since the theory is unitary, \( |0\rangle \) is also the unperturbed ground state; and since \( \varphi \) is primary, we have \( \langle 0| \varphi |0\rangle = 0 \).

The aim is to calculate \( \ln G_{(\alpha)}(\lambda, L) = \ln \langle \alpha | \Omega \rangle \) where \( |\alpha\rangle \) is the unperturbed CFT boundary state, and \( |\Omega\rangle \) is the PCFT vacuum. This will be a power series in the dimensionless quantity \( \lambda L^{2-x_{\varphi}} \); here, we shall obtain the coefficient of the linear term, \( d_{1}^{(\alpha)} \). The calculation follows [10], but is a little different (and in fact simpler) because the theory is unitary, so that the ground state is the conformal vacuum \( |0\rangle \).

First-order perturbation theory implies

\[
|\Omega\rangle = |0\rangle + \lambda \sum_{a} |\psi_{a}\rangle \Omega_{a} + \ldots \tag{8.5}
\]

where the sum is over all states excluding \( |0\rangle \), \( \langle \psi_{a} | 0 \rangle = 0 \) and

\[
\Omega_{a} = \frac{\langle \psi_{a} | \hat{H}_1 |0\rangle}{\langle 0 | \hat{H}_0 |0\rangle - \langle \psi_{a} | \hat{H}_0 | \psi_{a} \rangle}. \tag{8.6}
\]

Since the theory is unitary, \( \langle 0 | \hat{H}_0 |0\rangle - \langle \psi_{a} | \hat{H}_0 | \psi_{a} \rangle = -\langle \psi_{a} | \frac{2\pi}{L} (L_0 + \mathcal{L}_0) | \psi_{a} \rangle \). Using rotational invariance as well,

\[
\lambda \sum_{a} |\psi_{a}\rangle \Omega_{a} = -\frac{\lambda L^{2-x_{\varphi}}}{(2\pi)^{1-x_{\varphi}}} \sum_{a} |\psi_{a}\rangle \langle \psi_{a} | \varphi(1) |0\rangle \langle \psi_{a} | L_0 + \mathcal{L}_0 | \psi_{a} \rangle. \tag{8.7}
\]
Hence
\[ |\Omega\rangle = |0\rangle - \frac{\lambda L^{2-x_\varphi}}{(2\pi)^{1-x_\varphi}} (1-P) \frac{1}{L_0+L_0}(1-P) \varphi(1)|0\rangle + \ldots \] (8.8)
where \( P = |0\rangle\langle 0| \) is the projector onto the ground state. Using the formula
\[ \frac{1}{L_0+L_0} = \int_0^1 q^{L_0+L_0-1} dq , \]
\[ \langle \alpha|\Omega\rangle = \langle \alpha|0\rangle - \frac{\lambda L^{2-x_\varphi}}{(2\pi)^{1-x_\varphi}} \langle \alpha| (1-P) \int_0^1 dq q^{L_0+L_0-1}(1-P) \varphi(1)|0\rangle + \ldots \] (8.9)
Since \( \langle 0|\varphi|0\rangle = 0 \) and \( q^{L_0+L_0}\varphi(1)|0\rangle = q^{L_0+L_0}\varphi(1)q^{-L_0-L_0}|0\rangle = q^{x_\varphi}|0\rangle \) this last expression simplifies to
\[ \langle \alpha|\Omega\rangle = \langle \alpha|0\rangle - \frac{\lambda L^{2-x_\varphi}}{(2\pi)^{1-x_\varphi}} \int_0^1 dq q^{x_\varphi-1}\langle \alpha|\varphi(q)|0\rangle + \ldots \] (8.10)
Now \( \langle \alpha|\varphi(q)|0\rangle \) is a disc amplitude, and by Möbius invariance it is given by
\[ \langle \alpha|\varphi(q)|0\rangle = g_{(\alpha)}^{\varphi} (1 - q^2)^{-x_\varphi} . \] (8.11)
(This is a significant simplification over the nonunitary case discussed in [10], where the corresponding amplitude had to be expressed in terms of hypergeometric functions.)

Taking logarithms,
\[ \ln G_{(\alpha)}(\lambda, L) = \ln g_{(\alpha)} - \frac{\lambda L^{2-x_\varphi}}{(2\pi)^{1-x_\varphi}} \frac{g_{(\alpha)}^{\varphi}}{g_{(\alpha)}} \int_0^1 dq q^{x_\varphi-1}(1 - q^2)^{-x_\varphi} + \ldots \]
\[ = \ln g_{(\alpha)} + d_1^{(\alpha)} L^{2-x_\varphi} + \ldots \] (8.12)
and doing the integral,
\[ d_1^{(\alpha)} = -\frac{1}{2(2\pi)^{1-x_\varphi}} \frac{g_{(\alpha)}^{\varphi}}{g_{(\alpha)}} B(1-x_\varphi, x_\varphi/2) \] (8.13)
where \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \) is the Euler beta function.

This is a general result. As a non-trivial check we specialise to the 3-state Potts model, described by the \( A_2 \) scattering theory. There are three possible values A, B and C of the microscopic spin variable, related by an \( S_3 \) symmetry. At criticality the model corresponds to a \( c = 4/5 \) conformal field theory. The primary fields are the identity \( I \), a doublet of fields \( \{ \psi, \psi^\dagger \} \) of dimensions \( \Delta_\psi = \Delta_{\psi^\dagger} = 2/3 \), the energy operator \( \varepsilon \) of dimensions \( \Delta_\varepsilon = \Delta_{\varepsilon^\dagger} = 2/5 \) and a second doublet of fields \( \{ \sigma, \sigma^\dagger \} \) with dimensions \( \Delta_\sigma = \Delta_{\sigma^\dagger} = 1/15 \). The bulk perturbing operator \( \varphi \) which leads to the \( A_2 \) scattering theory is \( \varepsilon \), and so \( x_\varphi = \Delta_\varepsilon + \Delta_\varepsilon = 4/5 \). Boundary conditions and states for the unperturbed model are discussed in [55, 49]. One of
the three ‘fixed’ boundary states, say $|A\rangle$, can be written in terms of $W_3$-Ishibashi states as [55]

$$|A\rangle = |\tilde{I}\rangle \equiv K \left[ |I\rangle + X|\varepsilon\rangle + |\psi\rangle + X|\sigma\rangle + X|\sigma^\dagger\rangle \right]$$ (8.14)

where $K^4 = (5 - \sqrt{5})/30$ and $X^2 = (1 + \sqrt{5})/2$. Hence

$$\ln g|A\rangle = \ln K = -0.5961357674\ldots, \quad g^2|A\rangle/g|A\rangle = X = 1.2720196495\ldots$$ (8.15)

Putting everything into (8.13), the CPT prediction for the coefficient of the first perturbative correction to the $g$-function for fixed boundary conditions in the three-state Potts model is

$$d_1^{(A)} = -3.011357884\ldots$$ (8.16)

The 3-state Potts model also admits ‘mixed’ boundary conditions AB, BC and CA [56]. For later use we recall from [55] that the corresponding boundary state is

$$|AB\rangle = K \left[ X^2|I\rangle - X^{-1}|\varepsilon\rangle + X^2|\psi\rangle + X^2|\psi^\dagger\rangle - X^{-1}|\sigma\rangle - X^{-1}|\sigma^\dagger\rangle \right]$$ (8.17)

so that, for the AB boundary, we find instead

$$d_1^{(AB)} = -\frac{1}{X^4} d_1^{(A)}.$$ (8.18)

The results (8.15) and (8.16) can be compared to the numerical evaluation of our exact $g$-function result, written in terms of $\lambda L^{6/5}$ using Fateev’s formula [51]

$$\kappa = \frac{3\Gamma(4/3)}{\Gamma^2(2/3)} (2\pi)^{5/6} (\gamma(2/5)\gamma(4/5))^{5/12} = 4.504307863\ldots$$ (8.19)

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Setting $x = \lambda L^{6/5} = -(l/\kappa)^{6/5}$ (remembering that $(T - T_c) \propto \lambda < 0$) a fit to our numerical data yielded

$$\ln g(l) = -0.596135768 - 0.49999991 l - 3.0113570 x - 0.90937 x^2 + 0.3982 x^3 + 1.0 x^4 + \ldots$$ (8.20)

which agrees well with (8.15) and (8.16). The match of the constant term in (8.20) with $\ln g|A\rangle$ from (8.15) is guaranteed by our exact formula, and so serves as a check on the accuracy of our numerics. A calculation of the coefficient of the irregular (in $x$) term, proportional to $l$, as in [10], predicts the value 0.5, again in good agreement with our numerical results.

9 One-parameter families and RG flows

Just as the minimal reflection factors were tested in section 7, we do the same now with the one-parameter families of reflection factors. These reflection factors, $R_{a^{(d,C)}}$, depend on the parameter $C$ and are given in (4.6) and (4.10).
9.1 The ultraviolet limit

If we use the reflection factors as input to calculate the $g$-function, eq. (6.7), and take the limit $l \to 0$ then

$$g^{(d)} \equiv g_{d,C}^{(l)}|_{l=0} = g_0 \ T_d,$$

(9.1)

where $T_d = T_d(\theta)|_{l=0}$ is a $\theta$-independent constant. From (6.32) and (6.30) we have

$$T_d = \sqrt{1 + e^{\epsilon_d}}$$

(9.2)

where the $\epsilon_d$ values can be found in [7]. For every ADET theory, eq. (9.1) leads to a possible CFT $g$-function value.

For $M > 0$ there will be many massive bulk flows as the parameter $C$ is varied. However, it should be possible to tune $C$, as the limit $l \to 0$ is taken, so as to give a massless boundary flow between the conformal $g$-function $g^{(d)}$, corresponding to the UV limit of the reflection factor $R_{d,C}$, and $g$, corresponding to the UV limit of the minimal reflection factor. These flows are depicted in figure 2. Note that the UV $g$-function corresponding to the minimal reflection factors ($g_0$ for the ADE cases and $g_{(1,1)}$ for $T_r$) is the smallest conformal value in all cases so, by Affleck and Ludwig’s $g$-theorem [8], this is a stable fixed point of the boundary RG flow.

![Figure 2: The expected RG flow pattern](image)

For the $T_r$ case we find

$$g^{(d)} = g_{(1,d+1)} \quad \text{for} \quad d = 1, \ldots, r.$$  

(9.3)

This is consistent with a simple pattern of flows

$$(1, d+1) \to (1, 1) \quad \text{for} \quad d = 1, \ldots, r.$$  

(9.4)
The conformal values corresponding to the UV limits of $g^{[d]}$ for the A, D and E cases are given in tables 4 and 5. Again we expect there to be boundary flows

$$g^{[d]} \rightarrow g_0$$

in each case. Notice that in many cases the UV $g$-function values are sums of ‘simple’ CFT values, corresponding to flows from superpositions of Cardy boundary conditions, driven by boundary-changing operators.

$$
\begin{align*}
g^{[1]} &= g^{[r]} = g_{\rho_1} \\
g^{[2]} &= g^{[r-1]} = g_{\rho_2} \\
&\vdots \\
g^{[r/2]} &= g^{[r/2+1]} = g_{\rho_{r/2}} \text{ for } r \text{ even} \\
g^{[h/2]} &= g_{\rho_{h/2}} \text{ for } r \text{ odd}
\end{align*}
$$

Table 4: UV $g$-function values for A and D models

| $A_r$ | $D_r$ |
|-------|-------|
| $g^{[1]} = g^{[r]} = g_{\rho_1}$ | $g^{[i]} = (i+1)g_0$ for $i = 1, \ldots, r-2$ |
| $g^{[2]} = g^{[r-1]} = g_{\rho_2}$ | $g^{[r-1]} = g^{[v]} = g_{\rho_{r-1}} = g_{\rho_r}$ |

Table 5: UV $g$-function values for $E_6$, $E_7$ and $E_8$ models

| $E_6$ | $E_7$ | $E_8$ |
|-------|-------|-------|
| $g^{[1]}$ | $g^{[1]}$ | $g_{\rho_1}$ | $g_{\rho_1}$ | $g_0 + g_{\rho_1}$ |
| $g^{[2]}$ | $g_0 + g_{\rho_1}$ | $g_0 + g_{\rho_1}$ | $2g_0 + g_{\rho_1}$ |
| $g^{[3]}$ | $g_{\rho_1} + g_{\rho_2}$ | $g_{\rho_1} + g_{\rho_2}$ | $2g_0 + 2g_{\rho_1}$ |
| $g^{[4]}$ | $g_0 + g_{\rho_1} + 2g_{\rho_2}$ | $g_0 + 2g_{\rho_1}$ | $3g_0 + 2g_{\rho_2}$ |
| $g^{[5]}$ | $g_0 + 3g_{\rho_1}$ | $5g_0 + 3g_{\rho_1}$ |
| $g^{[6]}$ | $2g_{\rho_1} + 2g_{\rho_2}$ | $5g_0 + 4g_{\rho_1}$ |
| $g^{[7]}$ | $3g_0 + 6g_{\rho_2}$ | $9g_0 + 6g_{\rho_2}$ |
| $g^{[8]}$ | | $16g_0 + 12g_{\rho_1}$ |

The conjectured flow for the three-state Potts model ($A_2$), $g_{\rho_1} \rightarrow g_0$, corresponds to the ‘mixed–to–fixed’ flow ($AB \rightarrow A$) found by Affleck, Oshikawa and Saleur [49], and by Fredenhagen [52].

Similarly, the flow $g_{\rho_1} \rightarrow g_0$ in the tricritical Ising model ($E_7$) corresponds to the ‘degenerate–to–fixed’ (($(d) \rightarrow (-)$ or $(d) \rightarrow (+)$) flows of [53, 52]. Notice that $g_0 + g_{\rho_2} \rightarrow g_0$ in $E_7$ matches the CFT $g$ values for the flow $(-) \oplus (0+) \rightarrow (-)$ conjectured in [52]. However this latter flow is driven by the boundary field with scaling dimension $3/5$, which is inappropriate for the $E_7$ coset description. A more careful analysis shows that our flow, which is driven by a boundary field with scaling dimension $1/10$, must start from either $(+) \oplus (0+)$ or $(-) \oplus (-0)$, and flow to $(+)$ or $(-)$. This serves as a useful reminder that the $g$-function
values alone do not pin down a boundary condition, and that this ambiguity can be physically significant in situations involving superpositions of boundaries.

9.2 On the relationship between the UV and IR parameters

The one-parameter families of boundary scattering theories introduced above should describe simultaneous perturbations of boundary conformal field theories by relevant bulk and boundary operators. The action is

\[ A_{|\alpha\rangle} = A_{BCFT}^{\langle\alpha\rangle} + A_{\text{BULK}}^{\langle\alpha\rangle} + A_{\text{BND}}^{\langle\alpha\rangle} \] (9.6)

where \( A_{BCFT}^{\langle\alpha\rangle} \) is the unperturbed boundary CFT action. We suppose that the boundary condition is imposed at \( x = 0 \); in general it might correspond to a superposition of \( n_{|\alpha\rangle} \) Cardy states. Denoting these by \( |c\rangle \), \( c = 1, 2, \ldots, n_{|\alpha\rangle} \), the boundary is in the state

\[ |\alpha\rangle = \sum_{c=1}^{n_{|\alpha\rangle}} n_{|c\rangle} |c\rangle, \] (9.7)

with \( n_{|c\rangle} \in \mathbb{N} \). The bulk perturbing term is

\[ A_{\text{BULK}}^{\langle\alpha\rangle} = \lambda \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \varphi(x, y) \] (9.8)

while the boundary perturbing part is

\[ A_{\text{BND}}^{\langle\alpha\rangle} = \sum_{c,d=1}^{n_{|\alpha\rangle}} \mu_{\langle c|d\rangle} \int_{-\infty}^{\infty} dy \phi_{\langle c|d\rangle}(y). \] (9.9)

The operators \( \phi_{\langle c|d\rangle} \) live on a single Cardy boundary, while the \( \phi_{\langle c|d\rangle} \) with \( c \neq d \) are boundary changing operators. Since \( \phi_{\langle c|d\rangle} = \phi_{\langle d|c\rangle}^\dagger \), in a unitary theory we also expect (see for example [54])

\[ \mu_{\langle c|d\rangle} = \mu_{\langle d|c\rangle}^\ast. \] (9.10)

Using periodicity arguments to analyse the behaviour of the ground-state energy on a strip with perturbed boundaries as in [37] and [2] (see also (9.15) and (9.16) below), one can argue that the scaling dimensions of the fields \( \phi_{\langle c|d\rangle} \) must, in these integrable cases, be half that of the bulk perturbing operator \( \varphi \):

\[ x_\phi = \frac{x_\varphi}{2} = \frac{2}{h + 2} \] (9.11)

for the ADE systems and

\[ x_\phi = \frac{x_\varphi}{2} = \frac{2 - h}{h + 2} \] (9.12)
for the $T_r$ models. The results recorded in eq. (9.3) and tables 4 and 5 provide information about the conformal boundary conditions associated with the one-parameter families of reflection factors. We see that for $T_r$ and $A_r$ the boundary is always in a pure Cardy state, while for $D_r$, $E_6$ and $E_7$ this is true only in one case per model, and in the $E_8$-related theories the UV boundary always corresponds to a non-trivial superposition of the states $|\pm\rangle$ and $|\text{free}\rangle$. This observation fits nicely with the conformal field theory results for the Ising model [55]: from (9.11) we see that for the $E_8$ coset description the integrable boundary perturbation must have dimension $x_\phi = 1/16$, and indeed the only boundary operators with this dimension in the Ising model are $\phi_{(\pm|\text{free})}$ and $\phi_{|\text{free}|(\pm)}$.

For simplicity, we shall only discuss the cases involving a single Cardy boundary, where

$$A_{BND}^{\alpha} = \mu \int_{-\infty}^{\infty} dy \, \phi(y) . \quad \text{(9.13)}$$

As in the Ising and Lee-Yang examples of [26, 2], a simple formula is expected to link the couplings $\lambda$ and $\mu$ of bulk and boundary fields to the parameter $C$ in the reflection factors. However, without a precise identification of the operator $\phi$ it is hard to see how such relation can be determined. Even so, a general argument combined with a numerically-supported conjecture allows the relation formula to be fixed up to a single overall dimensionless constant. This goes as follows. From section 6.3 we know that in all cases

$$G_{(d,C)}^{(0)}(l) = G^{(0)}(l) \, T_d(i\frac{\pi}{h} C) \quad \text{(9.14)}$$

where $T_d$ is the TBA-related T-function, and $G^{(0)}(l)$ is the CPT $G$-function corresponding to the minimal reflection factor, for which there is no boundary perturbation. In addition, $T_d(\theta) \equiv T_d(\theta,l)$ is even in $\theta$, $T_d(\theta) = T_d(-\theta)$, and periodic, $T_d(\theta + i\pi \frac{d}{2h}) = T_d(\theta)$, and so it can be Fourier expanded as

$$T_d(i\frac{\pi}{h} C) = c_0(l) + \sum_{k=1}^{\infty} c_k(l) \cos \left( \frac{2\pi k}{h} C \right) . \quad \text{(9.15)}$$

We can now use the observation of [24] that $T_a(\theta,l)$ admits an expansion with finite domain of convergence in the pair of variables $a_{\pm} = (l e^{\pm \theta})^{\frac{2h}{h+2}}$ to see that

$$c_0(l) = c_0 + O(l^{\frac{4h}{h+2}}) , \quad c_1(l) = c_1 l^{\frac{2h}{h+2}} + O(l^{\frac{4h}{h+2}}) . \quad \text{(9.16)}$$

We also know that the minimal $G$-function has an expansion

$$\ln G^{(0)}(l) = \ln G^{(0)} + \sum_{k=1}^{\infty} g_k l^{k(2-2x_\phi)} \quad \text{(9.17)}$$

while the conformal perturbation theory expansion of $G_{(d,C)}^{(0)}$ has the form (see [10])

$$\ln G_{(d,C)}^{(0)}(\lambda, \mu, L) = \sum_{m,n=1}^{\infty} c_{mn}(\mu L^{1-x_\phi})^m (\lambda L^{2-2x_\phi})^n . \quad \text{(9.18)}$$

34
Comparing (9.14) – (9.17) with (9.18) we conclude that, so long as \( c_{10} \neq 0 \), the relationship between \( C \) and \( \mu \) must have the form

\[
\mu = \tilde{\mu}_0 \cos \left( \frac{2\pi}{h+2} C \right) M^{\frac{2h}{h+2}} = \tilde{\mu}_0 \cos \left( \frac{2\pi}{h+2} C \right) M^{1-x_\phi} \quad (9.19)
\]

where \( \tilde{\mu}_0 \) is an unknown dimensionless constant. However the result (9.19) can only be correct if \( 1 - x_\phi = 2h/(h+2) \). This is true only in the non-unitary \( T_r \) models, and indeed it reproduces the Lee-Yang result of [11] when specialised to \( T_1 \). For the ADE theories, \( 1 - x_\phi = h/(h+2) \) and we conclude that \( c_{10} \), the first \( \mu \)-dependent correction to \( \mathcal{G} \) this correction is proportional to \( \langle \phi \rangle_{\text{disk}} \) = 0. This is not surprising since in a unitary CFT this correction is proportional to \( \langle \phi \rangle_{\text{disk}} \). The first contribution is then at order \( O(\mu^2) = O(M^{2-2x_\phi}) \), and at this order there is an overlap between the expansions of \( T_d(\theta, l) \) and \( \mathcal{G}^{(0)}(l) \). This leads to the less-restricted result

\[
\mu^2 = \hat{k}_0 \left( \hat{z} + \cos \left( \frac{2\pi}{h+2} C \right) \right) M^{\frac{2h}{h+2}} = \hat{k}_0 \left( \hat{z} - \cos \left( \frac{2\pi}{h+2} C \right) \right) M^{2-2x_\phi} \quad (9.20)
\]

where now both \( \hat{k}_0 \) and \( \hat{z} \) are unknown constants. If we now consider the Ising model, then the \( \mu - C \) formula is known [26]. Written in terms of \( C \) it becomes

\[
\left( h^{\text{ref}[26]} \right)^2 = \mu^2 = 2M \left( 1 + \cos \left( \frac{\pi}{2} C \right) \right) \Rightarrow \mu = 2\sqrt{M} \cos \left( \frac{\pi}{2} C \right). \quad (9.21)
\]

Thus the boundary magnetic field is an even function of \( C \). It is then tempting to conjecture that \( \hat{z} = 1 \) for all the \( g\{1, C\} \) cases in the \( A_r \) models, and, to preserve the perfect square property, that \( \hat{z} \) is either 1 or −1 in all other ADE single boundary condition situations:

\[
\hat{z} = 1 \quad : \quad \mu = \tilde{\mu}_0 \cos \left( \frac{\pi}{h+2} C \right) M^{1-x_\phi} \leftrightarrow \frac{\mu}{\sqrt{\lambda}} = \tilde{\mu}_0 \kappa^{1-x_\phi} \cos \left( \frac{\pi}{h+2} C \right); \quad (9.22)
\]

\[
\hat{z} = -1 \quad : \quad \mu = \tilde{\mu}_0 \sin \left( \frac{\pi}{h+2} C \right) M^{1-x_\phi} \leftrightarrow \frac{\mu}{\sqrt{\lambda}} = \tilde{\mu}_0 \kappa^{1-x_\phi} \sin \left( \frac{\pi}{h+2} C \right). \quad (9.23)
\]

We now return to the physical picture of flows parametrised by \( C \) depicted in figure 2. At \( \lambda = 0 \) the bulk mass is zero, and the only scale in the problem is that induced by the boundary coupling \( \mu \). The massless boundary flow down the left-hand edge of the diagram therefore corresponds to varying \( |\mu| \) from 0 to \( \infty \). If \( \lambda \) is instead kept finite and nonzero while \( |\mu| \) is sent to infinity, the flow will collapse onto the lower edge of the diagram, flowing from \( g_0 \) in the UV. For the \( g \)-function calculations to reproduce this behaviour, the reflection factor should therefore reduce to its minimal version as \( |\mu| \to \infty \). For (9.22), \( |\mu| \to \infty \) corresponds to \( C \to i\infty \), which does indeed reduce the reflection factor as required. On the other hand, taking \( |\mu| \to \infty \) in (9.23), requires \( C \to (h+2)\pi + i\infty \). While the reduction is again achieved in the limit, real analyticity of the reflection factors is lost at intermediate values of \( \mu \). For this reason option (9.22) might be favoured, but more detailed work will be needed to make this a definitive conclusion.
In fact, the proposal (9.22) can be checked at $\mu = 0$ in the 3-state Potts model ($h = 3$), as follows. Consider the results (8.16) and (8.18) and set

$$
\delta_1 = -\kappa^{-6/5} \left( d_1^{AB} - d_1^{A} \right) = \kappa^{-6/5} \left( 1 + \frac{1}{X^4} \right) d_1^A = -0.683763720 \ldots \quad (9.24)
$$

According to the conclusions of section 9.1 and eq. (9.22),

$$
G_{(0)}^{(l)}(l) = G_{(0)}^{(l)}(l)|_{C=5/2} , \quad (9.25)
$$

and $\delta_1$ should match the coefficients $t_1$ of $l^{6/5}$ in the expansion of the function $T_1(\theta, l)|_{l=0}$ about $l = 0$. Noticing that $T_1(\theta, l) = T_2(\theta, l) = T_{LY}(\theta, l)$ we can use table 6 of [10]: $\ln T_{LY}(i\pi(b + 3)/6) = \varepsilon(i\pi(b + 3)/6)$, $C = 5/2$ corresponds to $b = 2$, $hM^{-6/5} = h_c = -0.6852899839$, and we find

$$
t_1 \sim 0.9977728224 h_c = -0.683763721 \ldots \quad (9.26)
$$

which within numerical accuracy is equal to $\delta_1$.

The study of the cases where the theory is perturbed by boundary changing operators is more complicated. Our results are very preliminary, and we have decided to postpone the discussion to another occasion.

10 Conclusions

Since the initial studies of integrable boundary scattering theories, surprisingly little progress has been made on the identification of the many known boundary reflection factors with particular perturbations of conformal field theories. One aim of this paper has been to use the recently-discovered exact expression for the ground-state degeneracy $g$ to begin to fill this gap. Working within the framework of the minimal ADET models we were able to identify a special class of reflection factors possessing a very simple conformal field theory interpretation: in the ultraviolet limit they describe the reflection of massless particles against a wall with fixed boundary conditions $|i\rangle$ matching the low-temperature vacua.

A collection of one-parameter families of amplitudes was also proposed, and partially justified by a suggested quantum group reduction of the boundary sine-Gordon model at $\beta^2 = 16\pi/(2r + 3)$.

The consistency of these novel families of boundary scattering theories was supported by our $g$-function calculations: they appear to correspond to perturbations of Cardy boundary states, or of superpositions of such states, by relevant boundary operators, with each boundary flow ending at a fixed boundary condition $|i\rangle$.

While our results indicate that these models all have interesting interpretations as perturbed boundary conformal field theories, the checks performed so far should be considered as preliminary, and more work will be needed to put these results on a firmer footing. Two immediate open problems are fixing completely
the boundary UV/IR relations, and determining the boundary bound-state content of each ADER model. The effort involved should be rewarded both by the well-established role in condensed matter physics taken by some of these theories, and by the mathematical insights into general integrable boundary theories their further study should bring.

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A Diagonal $\hat{g}_1 \times \hat{g}_1/\hat{g}_2$ coset data

In general, to extract the $\hat{g}/\hat{h}$ coset conformal theory from the $\hat{g}$ WZW model, we need to decompose the representations, $\mu$, of $\hat{g}$ into a direct sum of representations, $\nu$, of $\hat{h}$:

$$\mu \mapsto \bigoplus_{\nu} b_{\mu \nu} \nu. \quad (A.1)$$

This decomposition corresponds to the character identity

$$\chi_{\mu}(\tau) = \tilde{\chi}_{\nu}(\tau)b_{\mu \nu}(\tau) \quad (A.2)$$

where $\chi_{\mu}$ and $\tilde{\chi}_{\nu}$ are the characters of the $\hat{g}$ and $\hat{h}$ representations $\mu$ and $\nu$ respectively and the branching function, $b_{\mu \nu}(\tau)$, is the character of the coset theory. Let $\Pi$ denote the projection matrix giving the explicit projection of every weight of $g$ onto a weight of $h$. Clearly for a coset character to be non zero we must have

$$\Pi \mu_0 - \nu_0 \in \Pi Q \quad (A.3)$$

where $Q$ is the root lattice of $g$. For our diagonal cosets, since $\Pi(Q \oplus Q) = Q$ this selection rule is particularly simple:

$$\mu_0 + \nu_0 - \rho_0 \in Q \quad (A.4)$$
where $\mu_0, \nu_0$ and $\rho_0$ are weights of $\hat{g}$.

The group of outer automorphisms of $\hat{g}$, $O(\hat{g})$, permutes the fundamental weights in such a way as to leave the extended Dynkin diagram invariant. The action of an element, $A \in O(\hat{g})$, on a modular S-matrix element of $\hat{g}$ at some level $l$ is

$$S_{(A\mu)}\nu = S_{\mu\nu}e^{2\pi i(AA_0,\mu_0')}.$$  \hfill (A.5)

The modular S-matrix for a diagonal coset theory has the form

$$S_{\{\mu, \nu; \rho\}}\{\mu', \nu'; \rho'\} = S_{\{\mu, \nu; \rho\}}S_{\{\mu, \nu; \rho\}}S_{\{\mu, \nu; \rho\}}.$$  \hfill (A.6)

Under the $O(\hat{g})$ action it transforms as

$$S_{\{A\mu, A\nu, A\rho\}}\{\mu', \nu'; \rho'\} = S_{\{\mu, \nu; \rho\}}S_{\{\mu, \nu; \rho\}}S_{\{\mu, \nu; \rho\}}.$$  \hfill (A.7)

since $\mu_0' + \nu_0' - \rho_0' \in \mathbb{Q}$ [57]. This suggests that $S$ is a degenerate matrix, which cannot be the case since it represents a modular transformation. We must therefore remove this degeneracy by identifying the fields

$$\{A\mu, A\nu; A\rho\} \equiv \{\mu, \nu; \rho\} \quad \forall A \in O(\hat{g}).$$  \hfill (A.8)

For these diagonal cosets, the orbit of every element $A$ of $O(\hat{g})$ has the same order, $N$, which is simply the order of the global symmetry group of the model. With this field identification, we find that every field in the theory appears with multiplicity $N$. To remedy this, the partition function must be divided by this multiplicity, which has the effect of introducing $N$ into the coset modular S-matrix [58]:

$$S_{\{\mu, \nu; \rho\}}\{\mu', \nu'; \rho'\} = NS_{\mu\nu}S_{\nu\rho}S_{\rho\mu}.$$  \hfill (A.9)

More information on field identifications can be found, for example, in [59].

For the level 1 representations of simply laced affine Lie algebras, the characters have a particularly simple form [60, 61]:

$$\chi_{\mu}(q) = \frac{1}{\eta^\theta(q)}\theta_{\mu}(q)$$  \hfill (A.10)

where $q = \exp(2\pi i\tau)$ and the generalised $\theta$ and Dedekind $\eta$ functions are

$$\theta_{\mu}(q) = \sum_{\mu_0 \in \mathbb{Q}} q^{\frac{1}{2}(\mu_0, \mu_0)}$$  \hfill (A.11)

$$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty} (1-q^n).$$  \hfill (A.12)

Under the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ the theta function transforms as

$$\theta_{\mu} \rightarrow (-i\tau)^{\theta^2/2} \frac{1}{\sqrt{|P/Q|}} \sum_{\mu_0 \in P/Q} e^{-2\pi i(\nu_0, \mu_0)}\theta_{\nu}.$$  \hfill (A.13)
where $P$ and $Q$ are the weight lattice and root lattice respectively \cite{61,62}, and the $\eta$ function becomes
\[ \eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau). \] (A.14)

The modular S-matrix for level 1 is therefore
\[ S_{\mu\nu}^{(1)} = \frac{1}{\sqrt{|P/Q|}} e^{-2\pi i (\nu_0, \mu_0)}. \] (A.15)

For the coset description of the $g$-function, eq.(7.20), we need to compute $S_{0\mu}^{(1)}$ which is simply
\[ S_{0\mu}^{(1)} = \frac{1}{\sqrt{|P/Q|}}, \] (A.16)

It is important to note that $P/Q$ is isomorphic to the centre of the group under consideration \cite{63,59}. The group of field identifications of the coset is also isomorphic to the centre of this group so consequently eq.(A.16) becomes
\[ S_{0\mu}^{(1)} = \frac{1}{\sqrt{N}}, \text{ for all } \mu. \] (A.17)

The $g$-function can now be written in terms of the level 2 modular S-matrices only:
\[ g_{(\mu,\nu,\rho)} = \frac{S_{0\rho}^{(2)}}{\sqrt{S_{00}^{(2)}}}. \] (A.18)

As we can see from (A.18), identifying the conformal $g$-function value will only pin down the level 2 representation. For $E_8$, since there is only one possible level 1 representation (with Dynkin labels $n_i = 0$, $i = 1, \ldots, 8$), by specifying the level 2 representation we fix the coset field. However, for other cases, although the coset selection and identification rules (A.4),(A.8) do constrain the possible coset representations we are still left with some ambiguity in general. For example, for the $A_r$ cases the selection and identification rules are quite simple and the possible cosets, given a fixed level 2 weight, are shown in table 6. The notation is as follows: $\mu_j, \nu_j$ are level 1 weights and $\rho_j$ is the level 2 weight with Dynkin labels $n_i = 1$ for $i = j$ and $n_i = 0$, $i = 1, \ldots, r$ otherwise. The labels $\mu_0, \nu_0$ and $\rho_0$ now represent $0$ with Dynkin labels $n_i = 0$, $i = 1, \ldots, r$.

The coset fields for the $A_2$ (three-state Potts model) and $E_7$ (tricritical Ising model) cases, along with the corresponding boundary condition labels from \cite{49} and \cite{50} respectively, are given in tables 7 and 8 as concrete examples.

It is useful to note that for the $A_r$, $D_r$, $E_6$ and $E_7$ models, $S_{00} = S_{0\rho_0}$ only when $\rho = A0$ for some $A \in O(\hat{g})$ \cite{64} and for each such $\rho$ there is a unique coset field, so the number of fields with $g$-function equal to $g_{(\mu,\nu,0)}$ is equal to the size of the orbit of $0$. On the other hand, $E_8$ has no diagram symmetry, but it is also exceptional in that $S_{00} = S_{0\rho_2}$, where $\rho_2$ has Dynkin labels $n_2 = 1$, all other $n_i = 0$. Physically, this is to be expected as the two fields correspond to
Table 6: $A_r$ coset fields, indicating the number of distinct fields for each level 2 weight; in the first column, $[x]$ denotes the integer part of $x$.

Table 7: $A_2$ coset fields with the corresponding boundary labels and level 2 weight labels from table 2. The weights are given in terms of Dynkin labels $[n_0,n_1,n_2,...]$.

the two fixed boundary conditions (−) and (+), which clearly must have equal $g$-function values. (This exceptional equality is discussed from a more mathematical perspective in, for example, [64].)

From (4.5) it is clear that $S_{0(A_0)} = S_{0}$ is also true for $\rho \neq 0$. For the $A$ and $E$ models at level 2, we find that these are the only cases where $S_{0\rho_i} = S_{0\rho}$; for the $D_r$ models there is more degeneracy.
Table 8: $E_7$ coset fields with the corresponding boundary labels and level 2 weight labels from table 2. The weights are given in terms of Dynkin labels $[n_0, n_1, n_2, \ldots]$

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