Multi-Field Integrable Systems Related to WKI-Type Eigenvalue Problems

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Higher flows of the Heisenberg ferromagnet equation and the Wadati-Konno-Ichikawa equation are generalized into multi-component systems on the basis of the Lax formulation. It is shown that there is a correspondence between the multi-component systems through a gauge transformation. An integrable semi-discretization of the multi-component higher Heisenberg ferromagnet system is obtained.

KEYWORDS: multi-component system, Lax formulation, gauge transformation, Heisenberg ferromagnet equation, WKI equation

§1. Introduction

In recent years there has been a lot of progress in the study of integrable systems with multiple components. Among various approaches, the Lax formulation often helps us to obtain natural and simple multi-field extensions of single-component integrable systems. From this viewpoint, in the present paper, we consider integrable systems derived from the eigenvalue problem

\[
\begin{align*}
\Psi_x &= U \Psi, \\
\Psi_t &= V \Psi,
\end{align*}
\]

where \( U_1 \) is independent of a parameter \( \zeta \). We call this problem the Wadati-Konno-Ichikawa (WKI) type for brevity. As appropriate reductions of the corresponding compatibility condition, \( U_t - V_x + UV - VU = 0 \), we obtain a multi-field generalization of the second flows of the Heisenberg ferromagnet (HF) equation and the Wadati-Konno-Ichikawa (WKI) equation for the first time. As is well-known, there is a gauge transformation between the HF hierarchy and the WKI hierarchy. We show that this correspondence can be generalized for the multi-component case.

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Considering a semi-discrete version of the eigenvalue problem (1.1), we find a space discretization of the coupled system of the second HF flow.

The brief outline of the paper is as follows. In §2, we present a novel generalization of the higher HF equation from the point of view of Lax formulation. A semi-discrete version of the generalized system is also constructed. A multi-field generalization of the higher WKI flow is given in §3. A connection between two systems given in §2 and §3 is clarified by use of a gauge transformation in §4. The last section, §5, is devoted to the concluding remarks.

§2. Heisenberg Ferromagnet System

In this section, we derive a multi-component extension of a higher flow in the HF hierarchy. As a preparation, we begin with the Lax representation of the original HF spin chain.

2.1 Original flow

It is well-known that the original flow in the HF hierarchy is expressed as the compatibility condition of an eigenvalue problem in matrix form. Let us consider the eigenvalue problem

\[ \Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \text{(2.1)} \]

where

\[ U = i\zeta S, \quad V = 2i\zeta^2 S + \zeta SS_x. \quad \text{(2.2)} \]

Here \( \zeta \) is a time-independent parameter and \( S \) is a square matrix which satisfies

\[ S^2 = I, \quad \text{(2.3)} \]

with \( I \) being the identity matrix. Substitution of eq. (2.2) into the compatibility condition of the eigenvalue problem,

\[ U_t - V_x + UV - VU = 0, \quad \text{(2.4)} \]

gives the equation of motion for \( S \),

\[ iS_t = \frac{1}{2}(SS_{xx} - S_{xx}S). \quad \text{(2.5)} \]

The familiar HF spin chain,

\[ S_t = S \times S_{xx}, \quad |S|^2 = 1, \quad S = (s_1, s_2, s_3), \quad \text{(2.6)} \]

is given as the reduction of eq. (2.3) by

\[ S = s_1 \sigma_1 + s_2 \sigma_2 + s_3 \sigma_3 = \sigma \cdot S. \quad \text{(2.7)} \]

Here, \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices:

\[ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad \text{(2.8)} \]
2.2 Generalization of the second flow

In order to investigate a generalization of the higher HF equation, we consider the eigenvalue problem (2.1) with

$$U = i\zeta S, \quad V = 4i\zeta^3 S + 2\zeta^2 SS_x - i\zeta \left( S_{xx} + \frac{3}{2} S_x^2 S \right),$$

and

$$S^2 = I.$$  \hspace{1cm} (2.9)

and

Putting eq. (2.9) into the compatibility condition of the eigenvalue problem (2.4), we obtain the equation of motion for $S$

$$S_t + S_{xxx} + \frac{3}{2} (S_x^2 S)_x = 0.$$  \hspace{1cm} (2.11)

It should be noted that this system is consistent with the condition (2.10), i.e., we can prove $(S^2)_t = S_t S + SS_t = 0$ by use of eqs. (2.11) and (2.10). Of course, we can prove the same fact for the original system (2.5).

If we consider the reduction (2.7), the matrix equation (2.11) is cast into the second flow in the HF hierarchy. It is a vectorial equation for $S$ with the constraint $|S|^2 = 1$. To find a generalization of the second flow with arbitrarily multiple components, we assume that $S$ is expressed as

$$S = \sum_{k=0}^{2m} s_k e_k \equiv S^{(m)},$$

in terms of anti-commutative matrices $\{e_i\}$;

$$\{e_i, e_j\} = e_i e_j + e_j e_i = 2\delta_{ij} I, \quad 0 \leq i, j \leq 2m.$$ \hspace{1cm} (2.12)

Then eq. (2.11) reduces to a simple multi-component system,

$$s_{j,t} + s_{j,xxx} + \frac{3}{2} \left( \sum_{k=0}^{2m} s_{k,x} \cdot s_j \right)_x = 0, \quad j = 0, 1, \ldots, 2m.$$ \hspace{1cm} (2.14a)

The constraint (2.10) is interpreted into

$$\sum_{j=0}^{2m} s_j^2 = 1.$$ \hspace{1cm} (2.14b)

Because $\{e_i\}$ are elements of the Clifford algebra, we can construct their matrix representation. For instance, we can define $S^{(m)}$ recursively by

$$S^{(1)} = \left[ \begin{array}{cc} s_0 & s_1 + is_2 \\ s_1 - is_2 & -s_0 \end{array} \right],$$

$$S^{(m+1)} = \left[ \begin{array}{cc} S^{(m)} & (s_{2m+1} + is_{2m+2}) I_{2m} \\ (s_{2m+1} - is_{2m+2}) I_{2m} & -S^{(m)} \end{array} \right].$$ \hspace{1cm} (2.15a)

Here $I_{2m}$ is the $2^m \times 2^m$ unit matrix. The above statement shows that the system (2.14) has a $2^m \times 2^m$ Lax representation.
2.3 Semi-discretization

A space discretization of the system (2.14) is given in an analogous way to ref. 21. As a semi-discrete version of the eigenvalue problem (2.1) with eq. (2.9), we consider the eigenvalue problem,

\[
\Psi_{n+1} = L_n \Psi_n, \quad \Psi_{n,t} = M_n \Psi_n, \tag{2.16}
\]

where

\[
L_n = I + \lambda S_n, \tag{2.17a}
\]
\[
M_n = \frac{4\lambda^2}{1 - \lambda^2} \cdot S_{n-1}(S_n + S_{n-1})^{-1} + \frac{4\lambda}{1 - \lambda^2} \cdot (S_n + S_{n-1})^{-1}
= 4 \sum_{j=1}^{\infty} \lambda^{2j} \cdot S_{n-1}(S_n + S_{n-1})^{-1} + 4 \sum_{j=1}^{\infty} \lambda^{2j-1} \cdot (S_n + S_{n-1})^{-1}. \tag{2.17b}
\]

Here \(\lambda\) is a time-independent parameter and \(S_n\) is a matrix which satisfies

\[
S_n^2 = I. \tag{2.18}
\]

Substituting eq. (2.17) into the compatibility condition of the eigenvalue problem (2.16),

\[
L_{n,t} + L_n M_n - M_{n+1} L_n = O, \tag{2.19}
\]

we obtain

\[
S_{n,t} + 4(S_n + S_{n-1})^{-1} - 4(S_{n+1} + S_n)^{-1} = O. \tag{2.20}
\]

It is easy to check that \((S_n^2)_t = S_{n,t} S_n + S_n S_{n,t} = O\) due to eqs. (2.20) and (2.18). Thus, eq. (2.20) is consistent with eq. (2.18). In parallel with the reduction in the continuous case, we set

\[
S_n = \sum_{k=0}^{2m} s_n^{(k)} e_k, \tag{2.21}
\]

and obtain

\[
s_n^{(j)} + 2\left(s_n^{(j)} + s_n^{(j-1)}\right) - \frac{2(s_{n+1}^{(j)} + s_n^{(j)})}{1 + \sum_{k=0}^{2m} s_{n+1}^{(k)} s_{n-1}^{(k)}} = 0, \quad j = 0, 1, \ldots, 2m, \tag{2.22}
\]

with \(\sum_{k=0}^{2m} s_n^{(k)} e_k = 1\). Equation (2.22) with \(m = 1\) was derived in ref. 22. This system is interpreted as a semi-discretization of the coupled system (2.14). In fact, if we expand \(s_n^{(j)}\) in powers of the lattice constant \(\Delta x\),

\[
s_n^{(j)} = s^{(j)} + (\Delta x)s_x^{(j)} + \frac{1}{2}(\Delta x)^2 s_{xx}^{(j)} + \frac{1}{6}(\Delta x)^3 s_{xxx}^{(j)} + \cdots, \tag{2.23}
\]

the system (2.22) is rewritten as

\[
s_x^{(j)} - 2(\Delta x)s_x^{(j)} - \frac{1}{3}(\Delta x)^3 s_{xxx}^{(j)} - \frac{1}{2}(\Delta x)^3 \left(\sum_{k=0}^{2m} s_x^{(k)} e_k \cdot s^{(j)}_x\right) + O(\Delta x^5) = 0. \tag{2.24}
\]
Thus, up to a scaling of $t$ and a Galilei transformation, the semi-discrete system (2.22) coincides with the system (2.14) in the continuum limit. Since (2.22) has a Lax representation, this discretization scheme preserves the complete integrability of the continuous system.

§3. WKI System

In this section, we consider a multi-field generalization of the WKI equation with the linearized dispersion relation $\omega = -k^3$. The generalization is interpreted as the one for the second flow of the WKI hierarchy, because the WKI hierarchy starts from an equation with the linearized dispersion relation $\omega = k^2$. For this purpose, we choose the Lax matrices $U$ and $V$ as

$$U = \zeta \begin{pmatrix} -iI & Q^{(m)} \\ R^{(m)} & iI \end{pmatrix},$$

$$V = 4\zeta^3 f \begin{pmatrix} -iI & Q^{(m)} \\ R^{(m)} & iI \end{pmatrix} + \zeta^2 f^3 \begin{pmatrix} Q_x^{(m)} R^{(m)} - Q^{(m)} R_x^{(m)} & 2iQ_x^{(m)} \\ -2iR_x^{(m)} & R_x^{(m)} Q^{(m)} - R^{(m)} Q_x^{(m)} \end{pmatrix} + \zeta \left\{ f^3 \begin{pmatrix} O & -Q_x^{(m)} \\ -R_x^{(m)} & O \end{pmatrix} \right\}_x.$$  

(3.2)

Here $Q^{(m)}$ and $R^{(m)}$ are $2^{m-1} \times 2^{m-1}$ matrices which satisfy the constraint,

$$Q^{(m)} R^{(m)} = R^{(m)} Q^{(m)} = \sum_{k=1}^{m} q_k r_k \cdot I.$$  

(3.3)

The scalar function $f$ in eq. (3.2) is given by

$$f = \frac{1}{\sqrt{1 - \sum_{k=1}^{m} q_k r_k}}.$$  

(3.4)

An explicit representation of $Q^{(m)}$ and $R^{(m)}$ which satisfy eq. (3.3) is given recursively by

$$Q^{(1)} = q_1, \quad R^{(1)} = r_1,$$

$$Q^{(m+1)} = \begin{pmatrix} Q^{(m)} & q_{m+1} I_{2^{m-1}} \\ r_{m+1} I_{2^{m-1}} & -R^{(m)} \end{pmatrix}, \quad R^{(m+1)} = \begin{pmatrix} R^{(m)} & q_{m+1} I_{2^{m-1}} \\ r_{m+1} I_{2^{m-1}} & -Q^{(m)} \end{pmatrix}.$$  

(3.5a)

(3.5b)

It is proved by induction that eq. (3.3) is satisfied for integers $m \geq 1$. Substituting eqs. (3.1) and (3.2) with eqs. (3.4)–(3.5) into the compatibility condition (2.4), we obtain a coupled version of the second WKI flow,

$$q_{j,t} + \left\{ 1 - \sum_{k=1}^{m} q_k r_k \right\}^{\frac{3}{2}} q_{j,x}^{xx} = 0,$$

$$r_{j,t} + \left\{ 1 - \sum_{k=1}^{m} r_k q_k \right\}^{\frac{3}{2}} r_{j,x}^{xx} = 0,$$  

(3.6)  

$j = 1, 2, \ldots, m.$
As is clear from the above discussion, the Lax formulation for the system (3.6) is given in terms of $2^m \times 2^m$ matrices.

§4. Gauge Transformation

In previous sections, we have found new integrable multi-field systems (2.14) and (3.6), which are related to the WKI-type eigenvalue problem (1.1). It was shown that there is a gauge transformation between the system (2.14) with $m = 1$ and the system (3.6) with $m = 1$. In fact, the gauge transformation is applicable to the multi-component systems. In what follows, we shall prove this fact. For the system (2.14), we perform a transformation of the independent variables,

$$\xi = \int_{x_0}^x s_0(x',t)dx', \quad \tau = t. \quad (4.1)$$

Here, we assume the boundary condition, $s_j(x) \to 0$ as $x \to x_0$ for $j = 0, 1, \ldots, 2^m$. From the transformation, we obtain

$$\partial_x = s_0 \partial_\xi, \quad \partial_t = \partial_\tau - \left( s_0 s_0^2 \xi + s_0^2 s_0 \xi \xi + \frac{3}{2} s_0^{2^m} \sum_{k=0}^{2^m} s_k^2 \xi \right) \partial_\xi. \quad (4.2)$$

Then the Lax formulation for the system (2.14) in §2.2 is transformed into

$$\Psi_\xi = U' \Psi, \quad \Psi_\tau = V' \Psi, \quad (4.3)$$

where

$$U' = i \zeta X, \quad X \equiv \frac{1}{s_0} S, \quad (4.4a)$$

$$V' = 4i \zeta^3 s_0 X + \zeta^2 s_0^3 (XX_\xi - X_\xi X) - i \zeta(s_0^3 X_\xi) \xi. \quad (4.4b)$$

Due to eq. (2.10), $X$ satisfies the constraint,

$$X^2 = \frac{1}{s_0^2} I. \quad (4.5)$$

The compatibility condition of the transformed eigenvalue problem, $U'_\tau - V'_\xi + U'V' - V'U' = O$, yields

$$X_\tau + (s_0^3 X_\xi) \xi \xi = O. \quad (4.6)$$

If we take an appropriate representation of the anti-commutative matrices $\{e_i\}$ introduced in §2, the matrix $X$,

$$X = e_0 + \sum_{k=1}^{2^m} \left( \frac{s_k}{s_0} \right) e_k \equiv X^{(m)}, \quad (4.7)$$

is expressed as

$$X^{(m)} = \begin{bmatrix} -I_{2^{m-1}} & -iQ^{(m)} \\ -iR^{(m)} & I_{2^{m-1}} \end{bmatrix}. \quad (4.8)$$
Here $Q^{(m)}$ and $R^{(m)}$ are given recursively by

$$Q^{(1)} = s_1 + i s_2, \quad R^{(1)} = -\frac{s_1}{s_0} + i \frac{s_2}{s_0}, \quad (4.9a)$$

$$Q^{(m+1)} = \begin{bmatrix} Q^{(m)} & \left( \frac{s_2m+1}{s_0} + i \frac{s_2m+2}{s_0} \right) I_{2m-1} \\ \left( -\frac{s_2m+1}{s_0} + i \frac{s_2m+2}{s_0} \right) I_{2m-1} & -R^{(m)} \end{bmatrix}, \quad (4.9b)$$

$$R^{(m+1)} = \begin{bmatrix} R^{(m)} & \left( \frac{s_2m+1}{s_0} + i \frac{s_2m+2}{s_0} \right) I_{2m-1} \\ \left( -\frac{s_2m+1}{s_0} + i \frac{s_2m+2}{s_0} \right) I_{2m-1} & -Q^{(m)} \end{bmatrix}. \quad (4.9c)$$

It should be noticed that the representation of $\{e_i\}$, which we have chosen here, differs from that given by eqs. (2.12) and (2.15). If we introduce a new set of variables $\{q_k\}$ and $\{r_k\}$ by $q_k = (s_{2k-1} + i s_{2k})/s_0$, $r_k = (-s_{2k-1} + i s_{2k})/s_0$, eq. (4.6) is reduced to

$$q_{j,\tau} + \left( \frac{s_0}{s_0} q_{j,\xi} \right) \xi = 0, \quad j = 1, 2, \ldots, m, \quad (4.10a)$$

$$r_{j,\tau} + \left( \frac{s_0}{s_0} r_{j,\xi} \right) \xi = 0, \quad j = 1, 2, \ldots, m,$$

where

$$s_0 = \pm \left\{ 1 + \sum_{k=1}^{2m} \left( \frac{s_k}{s_0} \right)^2 \right\}^{-\frac{1}{2}} = \pm \frac{1}{\sqrt{1 - \sum_{k=1}^{m} q_k r_k}}. \quad (4.10b)$$

The system (4.10) essentially coincides with the coupled WKI system (3.6). Further, it is easily checked that the transformed Lax representation (4.3) with eqs. (4.4), (4.8) and (4.9) agrees with that for the coupled WKI system in §3. This shows that eq. (2.14) and eq. (3.6), or their Lax representations are connected by the gauge transformation.

§5. Concluding Remarks

In this paper, we have obtained new coupled systems in the Heisenberg ferromagnet (HF) hierarchy and the Wadati-Konno-Ichikawa (WKI) hierarchy. These two systems have been proved to be connected with each other. It is noteworthy that Lax representations by use of the Clifford algebra yield coupled systems (2.14) and (3.6) in a simple manner. The technique has been shown to be effective for some soliton equations. Meanwhile just a replacement of scalar variables in $2 \times 2$ Lax matrices by vectors does not give a consistent equation of motion for the WKI-type eigenvalue problems as far as we have examined. We have found an integrable semi-discretization of the coupled system (2.14), i.e., eq. (2.23), by considering a semi-discrete version of the eigenvalue problem. For systems (2.14), (2.23) and (3.6), an infinite set of conservation laws can be obtained recursively on the basis of the Lax pairs. We do not give an explicit derivation of the conservation laws in this paper.
For the original HF equation,
\[ S_t = S \times S_{xx}, \quad |S|^2 = 1, \quad S = (s_1, s_2, s_3), \] (5.1)
and the WKI equation with the linearized dispersion relation \( \omega = k^2 \),
\[ i q_t + \left( \frac{q}{\sqrt{1 - qr}} \right)_{xx} = 0, \]
\[ i r_t - \left( \frac{r}{\sqrt{1 - rq}} \right)_{xx} = 0, \] (5.2)
we have not found any simple generalization with multiple components so far. It remains an open problem to find a simple multi-field generalization of eq. (5.1) or eq. (5.2). The system (2.5) with the constraint (2.3) has a Lax representation (2.1) with (2.2), regardless of the size of \( S \).

However, it is difficult to find an interesting reduction, except for the \( 2 \times 2 \) matrix case. If there is no simple generalization of eq. (5.1) or eq. (5.2), it should be explained why the higher flows do have a generalization and the original flows do not.

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