Decompounding under Gaussian noise

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Abstract

Assuming that a stochastic process $X = (X_t)_{t \geq 0}$ is a sum of a compound Poisson process $Y = (Y_t)_{t \geq 0}$ with known intensity $\lambda$ and unknown jump size density $f$, and an independent Brownian motion $Z = (Z_t)_{t \geq 0}$, we consider the problem of nonparametric estimation of $f$ from low frequency observations from $X$. The estimator of $f$ is constructed via Fourier inversion and kernel smoothing. Our main result deals with asymptotic normality of the proposed estimator at a fixed point.

Keywords: asymptotic normality, Brownian motion, compound Poisson process, decompounding, kernel density estimation
AMS subject classification: 62G07, 62G20

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1 Introduction

Let \( Y = (Y_t)_{t \geq 0} \) be a compound Poisson process with intensity \( \lambda \) and jump size distribution \( F \), which has a density \( f \). Assume that \( Z = (Z_t)_{t \geq 0} \) is a Brownian motion independent of \( Y \) and consider the stochastic process \( X_t = Y_t + Z_t \). Notice that \( X = (X_t)_{t \geq 0} \) is a Lévy process. Suppose that \( X \) is observed at equidistant time points \( \Delta, 2\Delta, \ldots, n\Delta \). By a rescaling argument, without loss of generality, we may take \( \Delta = 1 \). Given a sample \( X_1, X_2, \ldots, X_n \), the statistical problem we consider is nonparametric estimation of the density \( f \). Notice that the Lévy triplet of the process \( X \) is given by \( (0, 1, \nu) \), where the Lévy measure \( \nu(dx) = \lambda f(x)dx \), see Sato (2004, Example 8.5). Since the Lévy triplet provides a unique means for characterisation of any Lévy process, see e.g. Sato (2004, Chapter 2), inference on the law of \( X \) can be reduced to inference on \( \nu \). Most of the existing literature dealing with estimation problems for Lévy processes is concerned with parametric estimation of the Lévy measure, see e.g. Akritas and Johnson (1981) and Akritas (1982), where a fairly general setting is considered. There are relatively few papers that study nonparametric inference procedures for Lévy processes, and the majority of them assume that high frequency data are available, i.e. either a Lévy process is observed continuously over a time interval \([0, T]\) with \( T \to \infty \), or it is observed at equidistant time points \( \Delta_n, \ldots, n\Delta_n \) and \( \lim_{n \to \infty} \Delta_n = 0, \lim_{n \to \infty} n\Delta_n = \infty \), see e.g. Rubin and Tucker (1959), Basawa and Brockwell (1982) and Figueroa-Lopez and Houdré (2004). On the other hand, high frequency data are not always available and it is interesting to study estimation problems for this case as well. In the particular context of a compound Poisson process we mention Buchmann and Grübel (2003, 2004) and van Es et al. (2007a), where given a sample \( Y_1, \ldots, Y_n \) from a compound Poisson process \( Y = (Y_t)_{t \geq 0} \), nonparametric estimators for the jump size distribution function \( F \) (see Buchmann and Grübel (2003, 2004)) and its density \( f \) (see van Es et al. (2007a)) are proposed and their asymptotics are studied as \( n \to \infty \). This problem is referred to as decompounding. The process \( X_t = Y_t + Z_t \) constitutes a generalisation of the compound Poisson model considered in Buchmann and Grüber (2003, 2004) and van Es et al. (2007a) and is related to Merton’s jump-diffusion model of an asset price, see Merton (1976). Since \( Z \) is a Brownian motion, it is natural to call the estimation problem of \( f \) decompounding under Gaussian noise. Figures 1–4 provide an indication of the difficulty of the problem. Figure 1 on the following page gives a typical path of the Brownian motion, while Figure 2 gives a path of the process \( X \). The difference is at once clear when \( X \) is observed continuously. If this is the case, then one can see all the jumps in the path of \( X \) and the problem of estimating \( f \) is relatively easy, as no decompounding is involved. On the other hand Figures 3 and 4 on the next page provide discretised versions of the typical paths of the Brownian motion \( Z \) and the
process $X$. In this case both plots look similar and given the highly irregular character of Brownian paths, it is difficult to conclude at which time instances jumps occur in the process $X$. The information on $f$ is contained in the jumps and the impossibility to observe them makes the problem of estimation of $f$ much more difficult.

Nonparametric estimation of the Lévy measure of a more general Lévy process than $X$ based on low frequency observations was considered in Watteel and Kulperger (2003) and Neumann and Reiß (2007). However these authors treat the case of estimation of the Lévy measure only (or of the canonical function $K$ in case of Watteel and Kulperger (2003)) and not of its density. Moreover, they study the proposed estimators under the strong moment condition $E[|X_1|^{4+\delta}] < \infty$, where $\delta$ is some strictly positive number. This condition automatically excludes distributions with heavy tails. We refer to those papers for additional details.

Using the stationary independent increments property of a Lévy process, we see that the problem of estimating $f$ from a discrete time sample from $X$ is equivalent to the following: let $X_1, \ldots, X_n$ be i.i.d. observations, where $X_i = Y_i + Z_i$, and $Y_i$ and $Z_i$ are independent. Assume that the unobservable
Y’s are distributed as a random variable
\[ Y = \sum_{j=1}^{N(\lambda)} W_j, \]
where \( N(\lambda) \) has a Poisson distribution with parameter \( \lambda \) and where the \( W \)'s are i.i.d. with distribution function \( F \) and density \( f \) and where by convention a sum over the empty set is understood to be zero. Thus we assume that \( Y \) is a Poisson sum of i.i.d. \( W \)'s. Furthermore, let the random variables \( Z_i \) have a standard normal distribution. Assume that \( \lambda \) is known. The estimation problem is as follows: based on the sample \( X_1, \ldots, X_n \), construct an estimator of \( f \).

In this context one might also think of the \( X \)'s as of measurements of the realisations \( Y \)'s of some quantity of interest, which are corrupted by the noise \( Z \). This way we are in the classical 'signal' plus 'noise' setting and the problem at hand is then related to the deconvolution problem, see e.g. \textit{Wand and Jones (1995)} for an overview, and in particular to its generalisation to the case of an atomic deconvolution, see \textit{van Es et al. (2007b)}.

The method that will be used to construct an estimator for \( f \) is based on Fourier inversion and is similar in spirit to the use of kernel estimators in deconvolution problems, as well as our approach in \textit{van Es et al. (2007a,b)}.

Let \( \phi_X, \phi_Y, \phi_Z \) and \( \phi_f \) denote the characteristic functions of \( X, Y, Z, \) and \( W \), respectively. Then by independence of \( Y \) and \( Z \) and by the fact that
\[ \phi_Y(t) = e^{-\lambda + \lambda \phi_f(t)}, \]
see e.g. \textit{Sato (2004, Chapter 1, Section 4)}, we have
\[ \phi_X(t) = \phi_Y(t)\phi_Z(t) = e^{-\lambda + \lambda \phi_f(t)}e^{-t^2/2}, \] (1)
and therefore
\[ e^{\lambda \phi_f(t)} = \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}}. \] (2)

Notice that \( P(Y = 0) = e^{-\lambda} \). Inverting (2), we get
\[ \phi_f(t) = \frac{1}{\lambda} \text{Log} \left( \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}} \right). \]

Here \( \text{Log} \) denotes the \textit{distinguished logarithm}, called so due to the similarity to the distinguished logarithm as constructed e.g. in \textit{Chow and Teicher (1978, Lemma 1, p. 413)}, \textit{Chung (2001, Theorem 7.6.2)}, \textit{Finkelestein et al. (1997)} and \textit{Sato (2004, Lemma 7.6)}. The difference is that in our case the function \( \exp(\lambda \phi_f(t)) \) equals \( e^\lambda \) at \( t = 0 \) and not 1. The distinguished logarithm of \( \exp(\lambda \phi_f(t)) \) in our case can be \textit{defined} as
\[ \lambda + \text{Log} \left( \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}} \right), \] (3)
where Log denotes the distinguished logarithm as constructed e.g. in Chung (2001, Theorem 7.6.2), or it can be constructed directly.

**Remark 1.1.** Notice that in general the distinguished logarithm of the non-vanishing characteristic function $\phi(t)$ cannot be reduced to the composition of the principal branch of an ordinary logarithm log with $\phi$. Consider the following trivial example: $\phi(t) = e^{it}$. This characteristic function satisfies the requirements of Chung (2001, Theorem 7.6.2), since it takes its values on the unit circle in the complex plane and hence its distinguished logarithm exists and is given by $\text{Log}(\phi(t)) = it$. On the other hand if one considers the argument of $\log(\phi(t))$, it is easy to see that it jumps whenever $\phi$ crosses the negative real axis, see Figure 5 and compare to the argument of the distinguished logarithm. This fact is not surprising, given that $-1$ lies on the branch cut of the principal branch of an ordinary logarithm.

![Figure 5: Arguments of the principal branch of a logarithm and of the distinguished logarithm.](image)

**Remark 1.2.** Notice that if $\lambda < \log 2$, the distinguished logarithm in (3) reduces to a composition of the principal branch of an ordinary logarithm with $\exp(\lambda \phi_f(t))$. This follows from the fact that

$$\text{Log} \left( e^{\lambda \phi_f(t)} \right) = \text{Log} \left( (e^{\lambda} - 1)\phi_g(t) + 1 \right),$$

where $\phi_g(t) = \phi_{Y|N>0}(t)$, cf. van Es et al. (2007a). It is immediately seen that the condition $\lambda < \log 2$ will then prevent $\exp(\lambda \phi_f(t))$ from taking values on the negative real axis, which constitutes the branch cut for the principal branch of an ordinary logarithm.

Assuming that $\phi_f$ is integrable, by Fourier inversion we obtain

$$f(x) = \frac{1}{2\pi \lambda} \int_{-\infty}^{\infty} e^{-ixt} \text{Log} \left( \frac{\phi_X(t)}{e^{-\lambda e^{-t^2/2}}} \right) dt.$$
This expression will be used as the basis for construction of an estimator of $f$. Let $\phi_{\text{emp}}$ denote the empirical characteristic function of the sample $X_1, \ldots, X_n$,

$$\phi_{\text{emp}}(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j}.$$ 

Furthermore, let $w$ be a symmetric kernel with Fourier transform $\phi_w$ supported on $[-1,1]$ and nonzero there, and let $h > 0$ be a bandwidth. The density $q$ of $X$ can then be estimated by a kernel density estimator

$$q_{nh}(x) = \frac{1}{nh} \sum_{j=1}^{n} w \left( \frac{x - X_j}{h} \right).$$

Its characteristic function $\phi_{q_{nh}}(t) = \phi_{\text{emp}}(t)\phi_w(ht)$ will serve as an estimator of $\phi_X(t)$. For those $\omega$’s from the sample space $\Omega$, for which the distinguished logarithm in the integral below is well-defined, $f$ can be estimated by the following plug-in type estimator,

$$f_{nh}(x) = \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \text{Log} \left( \frac{\phi_{\text{emp}}(t)\phi_w(ht)}{e^{-\lambda e^{-t^2/2}}} \right) dt,$$  

(4)

while for those $\omega$’s, for which the distinguished logarithm cannot be defined, we can assign an arbitrary value to $f_{nh}(x)$, e.g. zero. The distinguished logarithm in (4) can be defined only for those $\omega$’s for which $\phi_{\text{emp}}(t)\phi_w(ht)e^{\lambda e^{t^2/2}}$ as a function of $t$ does not vanish on $[-1/h, 1/h]$. In fact in Section 2 we will prove that as $n \to \infty$, the probability of the exceptional set where the distinguished logarithm is undefined, tends to zero. For technical reasons which will become apparent in the proofs, we also need to truncate $f_{nh}(x)$, and consequently, we define the estimator of $f(x)$ not by the expression above, but by

$$\hat{f}_{nh}(x) = (M_n \wedge f_{nh}(x)) \vee (-M_n),$$  

(5)

where $M = (M_n)_{n \geq 1}$ denotes a sequence of positive numbers converging to infinity at a suitable rate to be specified below.

Concluding this section, we state conditions on the density $f$, the bandwidth $h$ and the truncating sequence $M$ that will be used in Section 2.

**Condition 1.1.** Let the density $f$ be such that $\phi_f$ is integrable.

**Condition 1.2.** Let the kernel $w$ be the sinc kernel, $w(x) = (\sin x) / \pi x$.

The Fourier transform of the sinc kernel is given by $\phi_w(t) = 1_{[-1,1]}(t)$. The sinc kernel has been used successfully in kernel density estimation since a long time, see e.g. Davis (1975, 1977). It is the simplest example among the so-called superkernels, i.e. kernels the Fourier transforms of which are identically 1 in some open neighbourhood of zero. For more information
on the latter class of kernels we refer e.g. to Devroye and Gyorfi (1985), Devroye (1988) or Devroye (1992). An attractive feature of the sinc kernel in ordinary kernel density estimation is that it is asymptotically optimal when one selects the mean square error or the mean integrated square error as the criterion for the performance of an estimator. Notice that the sinc kernel is not Lebesgue integrable, but its square is.

**Condition 1.3.** Let the bandwidth \( h_n \) depend on \( n \) and be such that \( h_n \sim (\log n)^{-\beta} \), where \( \beta < 1/2 \).

Notice that this condition implies \( ne^{-1/h_n^2} \to \infty \). In the sequel we will suppress the subscript used to demonstrate the dependence of \( h \) on \( n \), since no ambiguity will arise.

**Condition 1.4.** Let the truncating sequence \( M = (M_n)_{n \geq 1} \) be such that \( M_n = C \log n \), where \( C > 0 \) is some constant.

The rest of the paper is organised as follows: in Section 2 we show that with probability approaching 1 as \( n \to \infty \), the distinguished logarithm in (4) is well-defined and subsequently we state the main result of the paper concerning the asymptotic normality of \( \hat{f}_{nh} \) at a fixed point \( x \). The section is concluded with a brief discussion of the obtained results. Section 3 contains a simulation example. All the proofs are collected in Section 4.

2 Main result

We first establish that with probability tending to 1 as \( n \to \infty \), the distinguished logarithm in (4) is well-defined. Thus our goal is to find a set \( B_{nh} \), such that on this set the distinguished logarithm might be undefined, while on the set \( B_{nh}^c \) it is well-defined. Fix \( \omega \) from the sample space \( \Omega \) and consider the quantity

\[
\sup_{t \in \left[-\frac{1}{h}, \frac{1}{h}\right]} \left| \frac{\phi_{\text{emp}}(t)}{e^{-\lambda}e^{-t^2/2}} - \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}} \right|,
\]

(6)

Now suppose that there exists a small number \( \delta \), such that

\[
\sup_{t \in \left[-\frac{1}{h}, \frac{1}{h}\right]} e^{1/(2h^2)} \left| \frac{\phi_{\text{emp}}(t)}{e^{-\lambda}} - \frac{\phi_X(t)}{e^{-\lambda}} \right| \leq \delta.
\]

Obviously this implies that (6) is less than \( \delta \). If \( \delta \) is small enough, then since \( \phi_X(t) e^{\lambda t^2/2} = \exp[\lambda \phi_f(t)] \) is bounded away from zero, also \( \phi_{\text{emp}}(t) e^{\lambda t^2/2} \) will be bounded away from zero on \([-1/h, 1/h]\). From this it follows that on this interval one can define the distinguished logarithm of \( \phi_{\text{emp}}(t) e^{\lambda t^2/2} \).

This simple observation shows that on the set

\[
B_{nh}^c = \left\{ \omega : \sup_{t \in \left[-\frac{1}{h}, \frac{1}{h}\right]} e^{1/(2h^2)} \left| \frac{\phi_{\text{emp}}(t)}{e^{-\lambda}} - \frac{\phi_X(t)}{e^{-\lambda}} \right| \leq \delta \right\}
\]
the distinguished logarithm will be well-defined for δ sufficiently small. Thus, what remains to be done is to prove that the probability of the complement of this set converges to zero as \( n \to \infty \). To this end we will make use of the following theorem from Devroye (1994).

**Theorem 2.1.** Let \( X \) be a random variable with characteristic function \( \phi \) and finite first moment, and let \( \phi_n \) be the empirical characteristic function of the i.i.d. sample \( X_1, \ldots, X_n \) drawn from \( X \). Then, for \( \alpha \) and \( \beta \), possibly dependent upon \( n \),

\[
P \left( \sup_{|t| < \alpha} |\phi_n(t) - \phi(t)| > \beta \right) \leq 4 \left( 1 + \frac{8\alpha E[|X|]}{\beta} \right) e^{-n\beta^2/72} + o(1),
\]

where the \( o(1) \) term is uniform over all \( \alpha \) and \( \beta \).

**Remark 2.1.** In our results we need additional information on the \( o(1) \) term in (7). It follows from the proof of Theorem 2.1 that it is bounded by

\[
P \left( \left| \frac{1}{n} \sum_{j=1}^{n} X_j \right| \geq \frac{4}{3} E[|X_1|] \right),
\]

see Devroye (1994). Since the \( X \)'s are not bounded, it is not possible to apply Hoeffding's inequality, see Hoeffding (1963), to show that this probability is exponentially small. At the same time, verification of the moment conditions needed for Bernstein's inequality to hold is difficult in our case and might require strong conditions on \( Y \). Therefore we opt for an unsophisticated application of Chebyshev's inequality to bound this probability.

The following proposition follows from Theorem 2.1.

**Proposition 2.1.** Assume Conditions 1.2 and 1.3 and let \( E[|X|] < \infty \). Then the distinguished logarithm in (4) is well-defined with probability tending to 1 as \( n \to \infty \). Moreover, if \( E[|X|^\rho] < \infty \) for \( 1 < \rho < 2 \), then

\[
P(B_{nh}) = O \left( \frac{1}{n^{\rho-1}} \right),
\]

and if \( E[|X|^\rho] < \infty \) for \( \rho \geq 2 \), then

\[
P(B_{nh}) = O \left( \frac{1}{n^{\rho/2}} \right).
\]

The main result of the paper concerns the asymptotic normality of \( \hat{f}_{nh}(x) \) at a fixed point \( x \). The following theorem holds true.
Theorem 2.2. Suppose that $\lambda$ is known. Let the estimator $\hat{f}_{nh}(x)$ be defined as in (5), and assume that Conditions 1.1–1.4 hold. Furthermore, let $|x|^\rho f(x)$ be integrable with $\rho > 3/2$. Then
\[
\frac{\sqrt{n}}{he^{1/(2h^2)}}(\hat{f}_{nh}(x) - \mathbb{E}[\hat{f}_{nh}(x)]) \overset{D}{\to} \mathcal{N}\left(0, \frac{e^{2\lambda}}{2\pi^2 \lambda^2}\right)
\]
as $n \to \infty$.

Remark 2.2. Notice that the integrability of $|x|^\rho f(x)$ implies $\mathbb{E}[|X|^\rho] < \infty$, see e.g. Sato (2004, Corollary 25.8). Thus the conditions of the Theorem 2.2 cover a large class of distributions with heavy tails.

Remark 2.3. From Theorem 2.2 it follows, that in order to get a consistent estimator, $nh^{-1}e^{-1/h^2}$ has to diverge to infinity. This means that the bandwidth $h$ has to be fairly large, i.e. of order $(\log n)^{-\beta}$, where $\beta \leq 1/2$, thus resulting in a slow, logarithmic rate of convergence of $\hat{f}_{nh}(x)$. This is in sharp contrast with the ordinary decompounding case, where the convergence rate is polynomial, see Section van Es et al. (2007a). On the other hand, the convergence rate of $\hat{f}_{nh}(x)$ is similar to that in the ordinary deconvolution, as well as the deconvolution for an atomic distribution, when the error distribution is assumed to be supersmooth, see e.g. Fan (1991) and van Es et al. (2007b). This fact should not come as a surprise, due to the similar structure of these problems and the presence of Gaussian noise in our model. We also mention that in a recent preprint Neumann and Reiß (2007), under some conditions on the Lévy measure $\nu$, obtained similar logarithmic lower bounds for estimation from low frequency observations of the Lévy measure $\nu$ of a general Lévy process with a Brownian component.

Remark 2.4. Using the estimator $p_{ng}$ from van Es et al. (2007b), an estimator of $\lambda$ can be defined as $\lambda_{ng} = -\log p_{ng}$, of course provided that $p_{ng}$ is strictly positive. However the proof of Theorem 2.2 for the case of unknown $\lambda$ is a highly nontrivial task.

Apart of Theorem 2.2 it is also interesting to study the asymptotic distribution of
\[
\frac{\sqrt{n}}{he^{1/(2h^2)}}(\hat{f}_{nh}(x) - f(x)),
\]
i.e. of the estimator $\hat{f}_{nh}(x)$ centred at the true density $f(x)$. After rewriting the above expression as
\[
\frac{\sqrt{n}}{he^{1/(2h^2)}}(\hat{f}_{nh}(x) - f(x)) = \frac{\sqrt{n}}{he^{1/(2h^2)}}(\hat{f}_{nh}(x) - \mathbb{E}[\hat{f}_{nh}(x)])
\]
\[
+ \frac{\sqrt{n}}{he^{1/(2h^2)}}(\mathbb{E}[\hat{f}_{nh}(x)] - f(x)),
\]
we see that we have to study the behaviour of the bias of the estimator $\hat{f}_{nh}(x)$, which is given by $\mathbb{E}[\hat{f}_{nh}(x)] - f(x)$. It will turn out that the behaviour
of the bias depends on the tail behaviour of the characteristic function of $f$. For our purposes it suffices to distinguish two cases: in the first case we will assume that $\phi_f(t) = O(e^{-|t|^\alpha})$ with $1 < \alpha \leq 2$, and in the second case we will assume that $\phi_f(t) = O(|t|^{-\gamma})$ as $t \to \infty$ with $\gamma > 1$. These two cases find a parallel in deconvolution problems, where a distinction is made between the use of supersmooth or ordinary smooth distributions to model the error distribution, see e.g. Fan (1991).

**Proposition 2.2.** Suppose that $\lambda$ is known. Let the estimator $\hat{f}_{nh}(x)$ be defined as in (5), and assume that Conditions 1.1–1.4 hold. Furthermore, let $f$ have a finite $\rho$th moment, $\rho > 1$.

(i) If $\phi_f(t) = O(e^{-|t|^\alpha})$ as $|t| \to \infty$ for $1 < \alpha \leq 2$, then we have

$$E[\hat{f}_{nh}(x)] - f(x) = O(h^{\alpha-1}e^{-1/h})$$

as $n \to \infty$.

(ii) If $\phi_f(t) = O(|t|^{-\gamma})$ as $|t| \to \infty$ for $\gamma > 1$, then

$$E[\hat{f}_{nh}(x)] - f(x) = O(h^{\gamma-1})$$

as $n \to \infty$.

**Remark 2.5.** Despite the fact that the bias of $\hat{f}_{nh}(x)$ asymptotically vanishes, the consequence of Proposition 2.2 is that the asymptotic normality of (9) cannot be established for the symmetric stable densities. Of course, it cannot be established for other densities either, the characteristic functions of which decay algebraically. Examination of the proof of Proposition 2.2 demonstrates that in order to have that (9) is asymptotically normal, one has to assume that, e.g., $\phi_f(t) = e^{-|t|^\alpha}$ with $\alpha > 2$. However if this is the case, then $\phi_f'(0) = \phi_f''(0) = 0$ and consequently the first two moments of $f$ have to vanish. There does not exist a density with such properties.

**Remark 2.6.** It appears that in our case the square bias dominates the variance of the estimator. This is not surprising in view of similar results obtained in Butucea and Tsybakov (2004) for the ordinary deconvolution problem: suppose

$$X = Y + Z,$$

where $Y$ and $Z$ are such that

$$\int_{-\infty}^{\infty} |\phi_Y(t)|^2 \exp(2\alpha|t|^s) \leq 2\pi L,$$

$$b_{\min}|t|^\gamma \exp(-\beta|t|^s) \leq \phi_Z(t) \leq b_{\max}|t|^\gamma' \exp(-\beta|t|^s).$$

Here $\alpha, r, L, b_{\min}, r, b_{\max}$, are strictly positive constants, $\gamma$ are $\gamma'$ are real numbers, and it is assumed that $r < s$. Then the square bias of the deconvolution kernel density estimator, which is based on observations on $X$ and
is evaluated for the sinc kernel, dominates the variance. In our case \( Y \) does not even have a characteristic function which vanishes at plus and minus infinity. The similarity to the model in Butucea and Tsybakov (2004) also holds true when comparing \( \phi_f \) and \( \phi_Z \), as \( \phi_Z \), being the characteristic function of a standard normal random variable, in a certain sense represents an extreme case among characteristic functions of supersmooth distributions.

3 Simulation example

Practical implementation of the estimator (5) is not a straightforward task. The idea we use is similar to that of van Es et al. (2007a). Notice that we can rewrite (4) as

\[
f_{nh}(x) = f_{nh}^{(1)}(x) + f_{nh}^{(2)}(x),
\]

where

\[
f_{nh}^{(1)}(x) = \frac{1}{2\pi\lambda} \int_{0}^{\infty} e^{-itx} \log \left( \frac{\phi_{\text{emp}}(t)}{e^{-\lambda t} e^{-t^2/2}} \right) \, dt,
\]

\[
f_{nh}^{(2)}(x) = \frac{1}{2\pi\lambda} \int_{0}^{\infty} e^{itx} \log \left( \frac{\phi_{\text{emp}}(-t)}{e^{-\lambda t} e^{-t^2/2}} \right) \, dt.
\]

Using the trapezoid rule and setting \( v_j = \eta(j - 1) \), \( f_{nh}^{(1)}(x) \) can be approximated by

\[
f_{nh}^{(1)}(x) \approx \frac{1}{2\pi\lambda} \sum_{j=1}^{N} e^{-iv_j x} \psi(v_j) \eta.
\]

Here we take \( N \) to be some power of 2 and \( \psi \) is defined by

\[
\psi(v_j) = \log \left( \frac{\phi_{\text{emp}}(v_j)}{e^{-\lambda v_j} e^{-v_j^2/2}} \right).
\]

From this point on one can proceed as in van Es et al. (2007a) and evaluate (11) for a set of appropriately selected points \( x_1, \ldots, x_N \) via the Fast Fourier Transform. A similar reasoning applies to \( f_{nh}^{(2)}(x) \).

The general difficulty with implementing the estimator is the computation of the distinguished logarithm, i.e. of function \( \psi \). A way to do this is to take a fine grid of points, evaluate the argument of the ordinary logarithm there and if one sees large jumps of size comparable to \( 2\pi \) between two consecutive points, make appropriate changes to the argument, thus obtaining an approximation to the argument of the distinguished logarithm. Of course this approach works only when \( \phi_{\text{emp}}(t) \) does not vanish on \([-1/h, 1/h]\). The latter fact can be verified in theory only, while in practice this can be done only for a grid of points \( t_1, t_2, \ldots, t_k \), which thus has to be taken rather fine, so that one does not possibly miss the value zero.

Though our emphasis is more on theoretical aspects of decompounding under Gaussian noise, we nevertheless will consider one simulation example in this section. We took \( \lambda = 1 \) and \( f \) the standard normal density and
simulated a sample of size $n = 5000$. The bandwidth $h = 0.5$ was selected by hand. The resulting estimate $f_{nh}$ (bold dotted line) together with the true density $f$ (dashed line) is plotted in Figure 6. We notice that the fit is quite good. Furthermore, notice that

$$P(N(\lambda) \geq 2) = 1 - 2e^{-\lambda} \approx 0.264.$$ 

It turns out that we considered a nontrivial example, since a considerable number among the $Y_i$’s are sums of the $W_j$’s in this case.

We should stress the fact that this simulation example serves as an illustration only and an extensive simulation study is needed to investigate the finite sample performance of our estimator and its behaviour in practice. We have to be very careful when generalising our conclusions concerning this simulation example because of the fact that the empirical characteristic function is oscillatory in its tails. If the integration step size $\eta$ is not small enough, we might miss instances when $\phi_{emp}$ crosses the negative real axis. This will have direct consequences for the argument of the distinguished logarithm. This is especially true for relatively small sample sizes, for which the empirical characteristic function $\phi_{emp}$ might not approximate the true characteristic function $\phi_X$ well enough. The issue of selection of $\eta$ in practice remains open and a thorough simulation study is needed to obtain some practical recommendations how this can be done. Additionally, a data-dependent method of the bandwidth selection has to be created.

![Figure 6: Estimation of the normal density, $n = 5000$.](image-url)

### 4 Proofs

**Proof of Corollary 2.1.** Note that we have

$$P(B_{nh}) \leq 4 \left(1 + \frac{8E[X]}{e^{-\lambda}h} e^{1/(2h^2)}\right) \exp \left(\frac{-e^{-2\lambda^2}n e^{-1/h^2}}{72}\right) + o(1),$$

where...
where we assume that $\delta$ is small enough. This bound follows from Theorem 2.1 with $\alpha = 1/h$ and $\beta = e^{-\lambda \delta e^{-1/(2h^2)}}$. The right-hand side converges to zero as $n \to \infty$ due to Condition 1.3. To prove the corollary, the only additional fact that we need to verify is that the $o(1)$ term from Theorem 2.1, which in the proof of Theorem 2.1 from Devroye (1994) is bounded by (8), is of order $n^{1-\rho}$, if $1 < \rho < 2$, and is of order $n^{-\rho/2}$, if $\rho \geq 2$. In fact, if the inequality
\[
\frac{1}{n} \sum_{j=1}^{n} X_j \geq \frac{4}{3} \mathbb{E} \|X_1\|
\]
holds, then we have
\[
\left| \frac{1}{n} \sum_{j=1}^{n} X_j - \mathbb{E}[X_1] \right| \geq \left| \frac{1}{n} \sum_{j=1}^{n} X_j \right| - |\mathbb{E}[X_1]|
\geq \frac{4}{3} \mathbb{E} \|X_1\| - \mathbb{E} \|X_1\| = \frac{1}{3} \mathbb{E} \|X_1\|.
\]
By Chebyshev’s inequality this implies
\[
P \left( \left| \frac{1}{n} \sum_{j=1}^{n} X_j \right| \geq \frac{4}{3} \mathbb{E} \|X_1\| \right) \leq P \left( \left| \frac{1}{n} \sum_{j=1}^{n} X_j - \mathbb{E}[X_1] \right| \geq \frac{1}{3} \mathbb{E} \|X_1\| \right)
\leq 3^\rho (\mathbb{E} \|X_1\|)^{-\rho} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{j=1}^{n} X_j - \mathbb{E}[X_1] \right) \right]^\rho.
\] (12)

Suppose first that $1 < \rho < 2$. Then it follows from Theorem 4 of von Bahr and Esseen (1965) that the rightmost term in (12) is of order $n^{1-\rho}$. Now suppose $\rho \geq 2$. Then Theorem 2 of Dharmadhikari and Jogdeo (1969) implies that the rightmost term of (12) is of order $n^{-\rho/2}$. For explicit constants we refer to the same papers.

Assume again that $1 < \rho < 2$. To complete the proof of the corollary, we have to verify that
\[
e^{1/(2h^2)} \frac{1}{h} \exp \left( -e^{-2\lambda \delta^2 n e^{-1/(2h^2)}} \right) n^{\rho-1} \to 0.
\] (13)

To this end we take the logarithm of the left-hand side to obtain
\[
\frac{1}{2h^2} - \log h - \frac{e^{-2\lambda \delta^2 n e^{-1/(2h^2)}}}{72} + (\rho - 1) \log n.
\] (14)

The first term here is of order $(\log n)^{23}$ and is negligible compared to $\log n$. The second term is of order $\log \log n$ and is thus negligible, while the third term dominates $\log n$. Therefore (14) diverges to minus infinity and consequently (13) holds. The proof for the case $\rho \geq 2$ is virtually identical and therefore it is omitted. □
The following lemma will be used in the proof of Theorem 2.2.

Lemma 4.1. Assume the conditions of Theorem 2.2. Let

\[ f_{n,h}^*(x) = \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \phi_{\text{emp}}(t) \frac{1}{e^{-t^2/2}} dt. \]

Then

\[ \frac{\sqrt{n}}{he^{1/(2h^2)}} f_{n,h}^*(x) - E[f_{n,h}^*(x)] \to N\left(0, \frac{e^{2\lambda}}{2\pi^2 \lambda^2}\right) \]

as \( n \to 0. \)

Proof. The proof is a minor variation of the proof of Theorem 2.1 of van Es and Uh (2004). The arguments of van Es and Uh (2004) are applicable, because they only use the existence and continuity of the density \( q \) of \( X \), which is still true in our case.

Let \( S \) denote a random variable, independent of the \( X \)'s and with a density

\[ f_S(x) = \frac{1}{c(h)} e^{x^2/(2h^2)} 1_{[0,1]}, \]

where the normalisation constant \( c(h) = \int_0^1 e^{x^2/(2h^2)} dx \). Furthermore, let \( E_n = (S - 1)/h^2 \) and let \( E \) be a standard exponential random variable. If for a random variable \( S \) we set

\[ E \left( \cos \left( \frac{S}{h} (X_j - x) \right) \big| X_j \right) = E_S \left[ \cos \left( \frac{S}{h} (X_j - x) \right) \right], \]

then \( f_{n,h}^*(x) - E[f_{n,h}^*(x)] \) can be written as

\[
\begin{align*}
& f_{n,h}^*(x) - E[f_{n,h}^*(x)] = \frac{c(h)}{\pi n h} \sum_{j=1}^n \left( \cos \left( \frac{X_j - x}{h} \right) E_E \left[ \cos \left( -hE(X_j - x) \right) \right] \\
& \quad - \sin \left( \frac{X_j - x}{h} \right) E_E \left[ \sin \left( -hE(X_j - x) \right) \right] \right) \\
& \quad - E \left[ \cos \left( \frac{X_j - x}{h} \right) E_E \left[ \cos \left( -hE(X_j - x) \right) \right] \right] \\
& \quad + E \left[ \sin \left( \frac{X_j - x}{h} \right) E_E \left[ \sin \left( -hE(X_j - x) \right) \right] \right] \\
& \quad + O_P \left( \frac{1}{\sqrt{n}} h^3 e^{1/(2h^2)} \right),
\end{align*}
\]

cf. equation (31) of van Es and Uh (2004).

A straightforward computation yields

\[ E_E \left[ \cos \left( -hE(X_j - x) \right) \right] = \frac{1}{1 + h^2 (X_j - x)^2} = w_1(h(X_j - x)) \]
and
\[ E_E \left[ \sin \left( -h^E(X_j - x) \right) \right] = -\frac{h(X_j - x)}{1 + h^2(X_j - x)^2} = w_2(h(X_j - x)), \]
where
\[ w_1(u) = \frac{1}{1 + u^2} \quad \text{and} \quad w_2(u) = -\frac{u}{1 + u^2}. \]
Define the random variables \( V_{n,j} \) as
\[ V_{n,j} = \cos \left( \frac{X_j - x}{h} \right) w_1(h(X_j - x)) - \sin \left( \frac{X_j - x}{h} \right) w_2(h(X_j - x)) \]
\[ = \cos(Y_{h,j})w_1(h(X_j - x)) - \sin(Y_{h,j})w_2(h(X_j - x)), \tag{16} \]
where \( Y_{h,j} = (X_j - x)/h \mod 2\pi \). Then
\[ f_{nh}^*(x) - E[f_{nh}(x)] = \frac{c(h)}{\pi h} \sum_{j=1}^{n} (V_{n,j} - E[V_{n,j}]) + O_P \left( \frac{1}{\sqrt{n}} h^3 e^{1/(2h^2)} \right). \tag{17} \]
Note that by the inequality \(|a + b|^p \leq 2^p (|a|^p + |b|^p), p \geq 0\), we have
\[ E[(V_{n,j} - E[V_{n,j}])^4] \leq 16(E[V_{n,j}^4] + (E[V_{n,j}])^4). \tag{18} \]
Since the characteristic function \( \phi_X \) is integrable, by Chung (2001, Theorem 6.2.3) the density \( q \) of \( X \) is continuous and bounded. Hence \( X \) satisfies the conditions of Lemma 3.1 of van Es and Uh (2004), and we have \( (hX, Y_{h,j}) \overset{D}{\to} (0, U) \). It also holds that
\[ E[V_{n,j}] = E[\cos(Y_{h,j})w_1(h(X_j - x)) - \sin(Y_{h,j})w_2(h(X_j - x))] \]
\[ \to E[\cos(U)w_1(0) - \sin(U)w_2(0)] = 0 \]
and
\[ E[V_{n,j}^4] = E[(\cos(Y_{h,j})w_1(h(X_j - x)) - \sin(Y_{h,j})w_2(h(X_j - x)))^4] \]
\[ \to E[(\cos(U)w_1(0) - \sin(U)w_2(0))^4] = E[(\cos(U))^4] = \frac{3}{8}, \]
because the cosine is a bounded and continuous function. The asymptotic variance of \( V_{n,j} \) is given by
\[ \text{Var}[V_{n,j}] \to E[(\cos(U)w_1(0) - \sin(U)w_2(0))^2] = E[(\cos(U))^2] = \frac{1}{2}. \]
It follows from (18) that
\[ \frac{E[|V_{n,j} - E[V_{n,j}]|^4]}{n(\text{Var}[V_{n,j}])^2} = \frac{O(1)}{n(\frac{1}{2} + o(1))^2} \to 0. \tag{19} \]
Consequently, Lyapunov’s condition with \( \delta = 2 \) is satisfied for \( V_{n,j} \), and hence both \( 1/n \sum_{j=1}^{n} (V_{n,j} - E[V_{n,j}]) \) and \( c(h)/(\pi h n) \sum_{j=1}^{n} (V_{n,j} - E[V_{n,j}]) \) are asymptotically normal. The asymptotic variance of the latter is given by

\[
\text{Var} \left[ \frac{c(h)}{\pi h} \frac{1}{n} \sum_{j=1}^{n} (V_{n,j} - E[V_{n,j}]) \right] = \frac{1}{n} \frac{c(h)^2}{\pi^2 h^2} \text{Var}[V_{n,1}] \sim \frac{1}{n} \frac{1}{2\pi^2 h^2} e^{1/h^2},
\]

which follows from Lemma 2.1 of van Es and Uh (2004). This completes the proof of the lemma.

**Proof of Theorem 2.2.** Write \( \zeta_n(h) = \sqrt{nh}^{-1} e^{-1/(2h^2)} \). We have

\[
\zeta_n(h)(\hat{f}_{nh}(x) - E[\hat{f}_{nh}(x)]) = \zeta_n(h)(\hat{f}_{nh}(x) - f(x) + \zeta_n(h)(f(x) - E[\hat{f}_{nh}(x)])
\]

\[
= \zeta_n(h)((\hat{f}_{nh}(x) - f(x))1_{B_{nh}} - E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}}])
\]

\[
+ \zeta_n(h)((\hat{f}_{nh}(x) - f(x))1_{B_{nh}^c} - E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}^c}]),
\]

where the set \( B_{nh} \) is defined as in Section 2. Now notice that, for an arbitrary constant \( \eta > 0 \), by Chebyshev’s inequality we have

\[
P(\zeta_n(h)((\hat{f}_{nh}(x) - f(x))1_{B_{nh}} - E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}}]) > \eta)
\]

\[
\leq \frac{2}{\eta} \zeta_n(h) E[|\hat{f}_{nh}(x) - f(x)|1_{B_{nh}}]. \tag{20}
\]

Since \( \phi_f \) is integrable, it follows that \( |f(x)| \leq C \), where \( C \) is some constant.

It then follows that the probability at the left-hand side of (20) is bounded by

\[
\frac{2}{\eta} \zeta_n(h)(M_n + C) P(B_{nh}). \tag{21}
\]

Now we apply the bound of Proposition 2.1 to \( P(B_{nh}) \). To prove that (21) converges to zero, it is sufficient to verify that

\[
\frac{\sqrt{n}}{he^{1/(2h^2)}} (M_n + C) \frac{1}{n^{\rho-1}} \rightarrow 0 \tag{22}
\]

for \( 3/2 < \rho < 2 \). This is obviously true due to Condition 1.3 and 1.4. The proof that (20) converges to zero for \( \rho \geq 2 \) is likewise straightforward. Therefore

\[
\zeta_n(h)((\hat{f}_{nh}(x) - f(x))1_{B_{nh}} - E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}}]) \overset{P}{\rightarrow} 0.
\]

Hence by Slutsky’s theorem, see van der Vaart (1998, Lemma 2.8), this term can be neglected and it suffice to consider

\[
\zeta_n(h)((\hat{f}_{nh}(x) - f(x))1_{B_{nh}^c} - E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}^c}]).
\]
We have that
\[
\log \left( \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}} \right),
\]
i.e. the real part of the distinguished logarithm
\[
\text{Log} \left( \frac{\phi_X(t)}{e^{-\lambda}e^{-t^2/2}} \right)
\]
is bounded. On the set \( B_{n\mu}^c \), if \( \delta \) is selected small enough, \( \phi_{\text{emp}}(t)e^\lambda e^{t^2/2} \) is arbitrarily close to \( \phi_X(t)e^\lambda e^{t^2/2} \) and also stays bounded away from zero at a positive distance. Therefore
\[
\int_{-1/h}^{1/h} \log \left( \frac{\phi_{\text{emp}}(t)}{e^{-\lambda}e^{-t^2/2}} \right) dt \leq C \frac{1}{h},
\]
where \( C \) is a constant. This grows slower than \( M_n \) and hence \( M_n \) will eventually dominate. Now we turn to the imaginary part. Let \( \psi : \mathbb{R} \to \mathbb{C} \), where
\[
\psi(t) = \phi_X(t)e^\lambda e^{t^2/2} = e^{\lambda \psi(t)}.
\]
By the Riemann-Lebesgue theorem \( \psi(t) \) converges to 1 as \( |t| \to \infty \) and hence there exists \( t^* > 0 \), such that
\[
|\psi(t) - 1| < \frac{1}{2}, \quad |t| > t^*.
\]
Furthermore, we have
\[
|\psi(t)| \geq e^{-\lambda}, \quad t \in \mathbb{R}.
\]
Since \( f \) has a finite first moment, by Schwartz [1966, Theorem 1, p. 182] \( \phi_f \) and consequently \( \psi \) are continuously differentiable. Therefore, the path \( \psi : [-t^*, t^*] \to \mathbb{C} \) is rectifiable, i.e. has a finite length. In view of this fact and (24), \( \psi : [-t^*, t^*] \to \mathbb{C} \) cannot spiral infinitely many times around zero and for \( |t| > t^* \) it cannot make a turn around zero at all because of (24). Consequently, for the same reason as we gave above for the real part of the distinguished logarithm, the truncation on this set becomes unimportant for the argument as well and we have
\[
\hat{f}_{n\mu}(x)1_{B_{n\mu}^c} = f_{n\mu}(x)1_{B_{n\mu}^c}.
\]
Thus we have to consider
\[
\zeta_n(h)\{(f_{n\mu}(x) - f(x))1_{B_{n\mu}^c} - \mathbb{E}[(f_{n\mu}(x) - f(x))1_{B_{n\mu}^c}]\}.
\]
Plugging in the expressions for \( f_{nh}(x) \) and \( f(x) \), we obtain that the above expression is equal to

\[
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \log \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) dt \right\} 1_{B_{nh}^c} - E \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \log \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) dt \right] 1_{B_{nh}^c} \\
- \frac{1}{2\pi} \left\{ \int_{-\infty}^{-1/h} e^{-itx} \phi_f(t) dt + \int_{1/h}^{\infty} e^{-itx} \phi_f(t) dt \right\} (1_{B_{nh}^c} - E [1_{B_{nh}^c}]). \tag{27}
\]

First notice that

\[
\left| \int_{-\infty}^{-1/h} e^{-itx} \phi_f(t) dt + \int_{1/h}^{\infty} e^{-itx} \phi_f(t) dt \right| \leq \int_{-\infty}^{\infty} |\phi_f(t)| dt < \infty.
\]

Consequently, the last term in (27) converges to zero in probability if

\[
\zeta_n(h)(1_{B_{nh}^c} - E [1_{B_{nh}^c}]) \xrightarrow{P} 0.
\]

This in turn is equivalent to

\[
\zeta_n(h)(1_{B_{nh}^c} - E [1_{B_{nh}^c}]) \xrightarrow{P} 0,
\]

because \( 1_{B_{nh}^c} = 1 - 1_{B_{nh}} \). By Chebyshev’s inequality it is sufficient to prove that \( \zeta_n(h) P(B_{nh}) \to 0 \). However, this follows from (21) and (22).

Hence, by Slutsky’s theorem we have to consider the first term of (27),

\[
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \log \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) dt \right\} 1_{B_{nh}^c} - E \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \log \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) dt \right] 1_{B_{nh}^c} \\
- \frac{1}{2\pi} \left\{ \int_{-\infty}^{-1/h} e^{-itx} \phi_f(t) dt + \int_{1/h}^{\infty} e^{-itx} \phi_f(t) dt \right\} (1_{B_{nh}^c} - E [1_{B_{nh}^c}]). \tag{27}
\]

Rewrite this as

\[
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right) dt \right\} 1_{B_{nh}^c} \\
- E \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right) dt \right] 1_{B_{nh}^c} \\
+ \zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} R_{nh}(t) dt \right\} 1_{B_{nh}^c} - E \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} R_{nh}(t) dt \right] 1_{B_{nh}^c}, \tag{28}
\]

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where
\[ R_{nh}(t) = \log \left( 1 + \left\{ \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right\} \right) - \left\{ \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right\}. \] (29)

Notice that on the set \( B_{c}^{c} \) we have
\[ \left| \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right| < \frac{1}{2}, \]
if \( \delta \) is small enough. Indeed, it suffices to choose \( \delta \) in such a way that \( e^{\lambda \delta} < \frac{1}{2} \).

From the inequality \(| \log(1 + z) - z | \leq |z|^2\), valid for \( z \) sufficiently small, it follows that
\[ |R_{nh}(t)| \leq \left| \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right|^2. \]

Consequently, to prove that the second term in (28) asymptotically vanishes, it is sufficient to prove that
\[ \zeta_n(h) \frac{1}{2\pi \lambda} \mathbb{E} \left[ \int_{-1/h}^{1/h} |R_{nh}(t)| dt 1_{B_{c}^{c}} \right] \leq \zeta_n(h) \frac{1}{2\pi \lambda} \mathbb{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right|^2 \right] \rightarrow 0. \] (30)

Since \( |\phi_Y(t)|^{-1} \leq e^{2\lambda} \), we have
\[ \mathbb{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} - 1 \right|^2 \right] \leq C e^{1/h^2} \mathbb{E} \left[ \int_{-\infty}^{\infty} |\phi_{\text{emp}}(t)\phi_w(ht) - \phi_X(t)\phi_w(ht)|^2 dt \right], \]
where \( \phi_w \) is the characteristic function of the sinc kernel and \( C \) is a constant.

By Parseval’s identity the expectation on the right-hand side equals
\[ \frac{1}{2\pi} \mathbb{E} \left[ \int_{-\infty}^{\infty} (q_{nh}(x) - q \ast w_h(x))^2 dx \right]. \]

This in turn equals the integrated variance of a kernel estimator \( q_{nh} \), which is of order \((nh)^{-1}\), see Tsybakov (2004, Proposition 1.7). Thus we have to show that
\[ \frac{\sqrt{n}}{h e^{1/(2h^2)}} \frac{1}{n} \frac{e^{1/(2h^2)}}{h^2 \sqrt{n}} = \frac{e^{1/(2h^2)}}{h^2 \sqrt{n}} \rightarrow 0. \]

The result follows from Condition 1.3 and can be verified by taking the logarithm of the left-hand side of the above expression and concluding that it diverges to minus infinity. We obtain
\[ \frac{1}{2h^2} - \log h^2 - \frac{1}{2} \log n \rightarrow -\infty, \]
because $h^{-2} = (\log n)^{2\beta}$ and $2\beta < 1$, and hence the dominating term on the left-hand side in the above expression is the last one.

We deal with the first summand in (28). Rewrite it as

$$
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} \, dt 1_{B_{nh}^c} 
- \mathbb{E} \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} \frac{e^{-itx} \phi_{\text{emp}}(t)}{\phi_X(t)} \, dt 1_{B_{nh}^c} \right] \right\}

- \zeta_n(h) \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \, dt (1_{B_{nh}^c} - \mathbb{E} [1_{B_{nh}^c}]). \tag{31}
$$

We want to show that the second summand in this expression converges to zero in probability. Notice, that it is bounded by

$$
C \zeta_n(h) \frac{1}{h} |1_{B_{nh}^c} - \mathbb{E} [1_{B_{nh}}]|,
$$

because $1_{B_{nh}^c} = 1 - 1_{B_{nh}}$. Here $C$ is some constant. By Chebyshev's inequality it is sufficient to prove that

$$
\zeta_n(h) \frac{1}{h} \mathbb{P}(B_{nh}) \to 0.
$$

This is obviously true thanks to (20) and (21).

Thus, by Slutsky’s theorem, instead of (31) we may consider

$$
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} \, dt 1_{B_{nh}^c} - \mathbb{E} \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} \frac{e^{-itx} \phi_{\text{emp}}(t)}{\phi_X(t)} \, dt 1_{B_{nh}^c} \right] \right\}.
$$

Note that for $|t| \leq 1/h$

$$
|\phi_X(t)| = \left| e^{-t^2/2} e^{-\lambda + \lambda \phi_f(t)} \right| \geq e^{-1/(2h^2)} e^{-2\lambda}
$$

holds. Consequently, we have

$$
\zeta_n(h) \mathbb{E} \left[ \left| \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} \, dt 1_{B_{nh}} \right| \right]

\leq \zeta_n(h) \frac{1}{2\pi \lambda} \frac{2}{h} e^{1/(2h^2)} e^{2\lambda} \mathbb{P}(B_{nh}), \tag{32}
$$

which converges to zero thanks to the fact that $\mathbb{P}(B_{nh}) = O(n^{1-\rho})$ for $3/2 < \rho < 2$, and $\mathbb{P}(B_{nh}) = O(n^{-\rho/2})$ for $\rho \geq 2$, see Proposition 2.1.

Hence by Slutsky’s theorem we may consider

$$
\zeta_n(h) \left\{ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} \, dt - \mathbb{E} \left[ \frac{1}{2\pi \lambda} \int_{-1/h}^{1/h} e^{-itx} \frac{\phi_{\text{emp}}(t)}{\phi_X(t)} \, dt \right] \right\}.
$$
By (1) the expression above can be rewritten as

\[
\zeta_n(h) \frac{e^\lambda}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{\phi_{\text{emp}}(t)}{e^{-t^2/2}} - \frac{\phi_X(t)}{e^{-t^2/2}} \right) dt \\
+ \zeta_n(h) \frac{e^\lambda}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{\phi_{\text{emp}}(t)}{e^{-t^2/2}} - \frac{\phi_X(t)}{e^{-t^2/2}} \right) \left( e^{-\lambda \phi_f(t)} - 1 \right) dt. \tag{33}
\]

By Lemma 4.1 the first summand in this expression is asymptotically normal with zero mean and variance given by \( \sigma^2 = e^{2\lambda}/(2\pi^2\lambda^2) \).

Now we will show that the second term in (33) asymptotically vanishes in probability. By Chebyshev’s inequality it suffices to show

\[
(\zeta_n(h))^2 \mathbb{E} \left[ \left( e^{\lambda} \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{\phi_{\text{emp}}(t)}{e^{-t^2/2}} - \frac{\phi_X(t)}{e^{-t^2/2}} \right) \left( e^{-\lambda \phi_f(t)} - 1 \right) dt \right)^2 \right] \\
= (\zeta_n(h))^2 \mathbb{V} \mathbb{A} \mathbb{R} \left[ \int_{-1/h}^{1/h} e^{-itx} \phi_{\text{emp}}(t) \left( e^{-\lambda \phi_f(t)} - 1 \right) dt \right] \rightarrow 0.
\]

Using the independence of the \( X_i \)'s, after further simplification we obtain that we have to prove that

\[
\frac{1}{he(1/2h^2)} \int_{-1/h}^{1/h} e^{t^2/2} |e^{-\lambda \phi_f(t)} - 1| dt \rightarrow 0.
\]

Thus we have to prove that

\[
\frac{1}{he(1/2h^2)} \int_{-1/h}^{1/h} e^{t^2/2} |e^{-\lambda \phi_f(t)} - 1| dt \rightarrow 0.
\]

From van Es et al. (2007a) we have

\[
|e^{-\lambda \phi_f(t)} - 1| \leq C_\lambda |\phi_f(t)|,
\]

where the constant \( C_\lambda \) depends on \( \lambda \) only. Therefore it suffices to prove

\[
\frac{1}{he(1/2h^2)} \int_{-1/h}^{1/h} e^{t^2/2} |\phi_f(t)| dt \rightarrow 0. \tag{34}
\]

This can be done either via an application of L’Hôpital’s rule or via the method similar to the one used in the proof of Lemma 5 in van Es and Uh (2005). We follow the latter path. It is enough to consider the integral over \([0, 1/h]\) as the integral over \([-1/h, 0]\) can be dealt with in a similar fashion. After the change of integration variable \( v = (1 - ht)/h^2 \), we obtain

\[
h \int_0^{1/h^2} e^{-\frac{1-v^2}{2\pi^2}} \phi_f \left( \frac{1 - vh^2}{h} \right) dv = he \frac{1}{2\pi^2} \int_0^{1/h^2} e^{-v^2 + \frac{2\lambda^2}{2\pi^2}} \phi_f \left( \frac{1 - vh^2}{h} \right) dv.
\]
By the Riemann-Lebesgue theorem \( \lim_{|u| \to \infty} \phi_f(u) = 0 \), and therefore by the dominated convergence theorem the above expression is of order \( o(h \epsilon^{1/(2h^2)}) \).

The dominated convergence theorem is applicable, because

\[
(e^{-v/2} - e^{-v+v^2h^2/2})1_{[0,1/h^2]} \geq 0
\]

and hence \( e^{-v/2} \) can be taken as the dominating function. Consequently \( (34) \) vanishes as \( h \to 0 \) and this argument concludes the proof of the theorem.

**Proof of Proposition 2.2.** We will prove both parts of the statement simultaneously. Write

\[
E[\hat{f}_{nh}(x)] - f(x) = E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}}] + E[(\hat{f}_{nh}(x) - f(x))1_{B^c_{nh}}].
\]

Notice that for some \( C > 0 \),

\[
\left| E[(\hat{f}_{nh}(x) - f(x))1_{B_{nh}}] \right| \leq (M_n + C) P(B_{nh}).
\]  

Here we used the fact that \( f \) is bounded, because \( \phi_f \) is integrable. Due to Theorem 2.1, Corollary 2.1 and Conditions 1.3 and 1.4, we see that \( (36) \) converges to zero as \( n \to \infty \). Moreover, this term is negligible compared to \( h^{\alpha-1}e^{-1/h^\alpha} \) (case (i)) or \( h^{\gamma-1} \) (case (ii)), since \( P(B_{nh}) = O(n^{-\rho/2}) \), depending whether \( 1 < \rho < 2 \) or \( \rho \geq 2 \), see Proposition 2.1.

Now we turn to the second summand in \( (35) \). By selecting \( \delta \) small enough, on the set \( B^c_{nh} \) truncation in the definition of \( \hat{f}_{nh}(x) \) becomes unimportant, see the arguments that led to \( (26) \). Hence we have to deal with \( E[(f_{nh}(x) - f(x))1_{B^c_{nh}}] \). Using expressions for \( f_{nh}(x) \) and \( f(x) \), we see that this term equals

\[
E \left[ \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \log \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) dt 1_{B^c_{nh}} \right] - \frac{1}{2\pi} \int_{-\infty}^{1/h} e^{-itx} \phi_f(t) dt P(B^c_{nh}) - \frac{1}{2\pi} \int_{1/h}^{\infty} e^{-itx} \phi_f(t) dt P(B^c_{nh}).
\]  

The last two terms in this expression can be treated similarly and therefore we consider only the second one. It will turn out that these are the leading terms in the bias expansion. Notice that

\[
\frac{1}{2\pi} \int_{1/h}^{\infty} e^{-itx} \phi_f(t) dt P(B^c_{nh}) \to 0
\]

as \( n \to 0 \), because \( \phi_f \) is integrable. Moreover, if \( \phi_f(t) = O\left( e^{-|t|^{\alpha}} \right) \), \( \alpha > 1 \), then

\[
\int_{1/h}^{\infty} |\phi_f(t)| dt = O\left( h^{\alpha-1}e^{-1/h^\alpha} \right).
\]
This fact can be proved using the same type of arguments as in Casella and Berger (2002, Example 3.6.3, p. 123). Furthermore, if φ(t) = O (|t|−γ), then
\[ \int_{1/h}^{\infty} |φ(t)| dt ≤ C \int_{1/h}^{\infty} t^{-γ} dt = O(h^{γ-1}). \]

Now we turn to the first term in (37). Rewrite it as
\[
E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right] \]
\[ + E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} R_{nh}(t) dt \right], \quad (38)
\]
where \( R_{nh}(t) \) is defined as in (29). Consider the first term in this expression. Rewrite it as
\[
E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right] \]
\[ = E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right] \]
\[ - E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right].
\]

The first summand here is equal to zero. As far as the second summand is concerned, notice that
\[
\left| \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right| ≤ C \frac{1}{h} e^{1/(2h^2)},
\]
where \( C \) is some constant. This inequality follows from the facts that for \( t ∈ [-1/h, 1/h] \),
\[ \left| \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right| ≤ \left| \frac{φ_{emp}(t)}{φ_X(t)} \right| + 1,
\]
\[ \left| \frac{φ_{emp}(t)}{φ_X(t)} \right| ≤ e^{2λ} e^{1/(2h^2)},
\]
because \( φ_X(t) = φ_Y(t)e^{-t^2/2} \) and \( |φ_Y(t)| ≥ e^{-2λ} \). Consequently
\[
\left| E \left[ \frac{1}{2πλ} \int_{-1/h}^{1/h} e^{-itx} \left( \frac{φ_{emp}(t)}{φ_X(t)} - 1 \right) dt \right] \right| \leq C \frac{1}{h} e^{1/(2h^2)} P(B_{nh}).
\]
This term will converge to zero as \( n \to \infty \) due to Theorem 2.1 and Proposition 2.1. Moreover, due to the same facts, it is negligible compared to \( h^{\gamma - 1} \) or to \( h^{\alpha - 1} e^{-1/h^\alpha} \).

Now we consider the second term in (38). Notice that this term is of order \( (nh)^{-1} \), which was shown in the proof of Theorem 2.2 in the arguments concerning (30). Consequently it will be negligible compared to \( h^{\gamma - 1} \) or to \( h^{\alpha - 1} e^{-1/h^\alpha} \). This completes the proof of the proposition.

Proof of Remark 2.5. We have to study the behaviour of

\[
\zeta_n(h) h^{\alpha - 1} e^{-1/h^\alpha}.
\]

(39)

After taking the logarithm, we obtain

\[
\log \left( \frac{1}{2\pi \alpha} \right) + \frac{1}{2} \log n - \log h - \frac{1}{2} h^2 + (\alpha - 1) \log h - \frac{1}{h^\alpha}.
\]

(40)

Dominating terms here are the second, the fourth and the last one. Now note that the fourth and the last terms equal \(- (1/2) (\log n)^{2\beta} \) and \(- (\log n)^{\alpha \beta} \), respectively. In view of \( 2\beta < 1 \) and \( \alpha \beta < 1 \), these terms are dominated by \( \log n \) and hence (10) diverges to plus infinity. It follows that so does (39).

The case of \( \zeta_n(h) h^{\gamma - 1} \to \infty \) is trivial given Condition 1.3.

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