A Dynamics Behaviour of Two Predators and One Prey Interaction with Competition Between Predators

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Abstract. This research focuses on the dynamical of a Leslie-Gower predator-prey model with competition on predator populations. The model represents an interaction between one prey and two predator populations. The analysis shows that there are four equilibrium points, namely the extinction of predator populations point, the extinction of the first predator population point, the extinction of the second predator and the interior point. The existence of the interior equilibrium point is investigated by using Cardan criteria. Local stability analysis shows that both predator populations have never been extinct together. The second and third equilibrium point is local asymptotically stable under some conditions. Numerical simulations are carried out to investigate the stability of the interior point as well as to show that more than one equilibrium point may be asymptotically stable together for a set of parameter.

1. Introduction
The dynamic relationships between predator-prey of ecological system is one of the most interesting topics in mathematical ecology. The dynamic behaviour of predator-prey model depends on many factors, such as mortality rates, environmental conditions, and competition on predator populations. In this research, a mathematical model is analysed to study the dynamics of a Leslie-Gower predator-prey model with competition on predator populations. Competition describes the situation when two populations use the same limited resource and fight with each other for the resource to survive. According to some biologist, Leslie [1] was the first to propose a prey-predator model with carrying capacity of the predator is proportional to the number of prey. Few models for more than two species [2], proposed Holling type II functional response of two predators one prey system with modified Leslie-Gower. The complex dynamics of a Leslie-Gower model with additive allee effect on prey has a high impact the dynamics, and has a unique positive equilibrium point by Cardan criteria [3]. The model of a Leslie-Gower with additional food for predators, indicating that the solutions of this system by [4] is bounded. It is shown that the model is permanence and has four equilibrium points. A three species predator-prey model, where the third species are omnivores with competition is presented in [5] and employ a different approach to determine the sufficient condition that guarantee the global stability.

The dynamical behaviour of leslie-Gower prey-predator model has investigated the stability of various equilibrium points. The article is organized as follows: in section 2, model formulation using
several variations for Holling type functional responses has been shown. The local stability of equilibria has been discussed in section 3. In section 4 shows the dynamical behaviour the stability of a Leslie-Gower predator-prey model depending on the parameters by performed numerical simulations. Finally, several numerical simulations were completed to support the analitical results.

2. Mathematical Model

There are many researches about competition between two predator is exceptionally critical in environment due changes impact predator can impact dynamical behaviour [6].

2.1. Basic model

Concurring to the model developed by Alebraheem [7] had proposed the taking after the dynamical interaction a prey-predator model where two predators competing on one prey. This show appears the well-known predation rate. Based on [8] the Holling type I, is utilized to relate the feeding of the two predators $y_1$ and $y_2$ on prey $x$. The model can be composed as:

$$
\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) - \beta_{1}xy_{1} - \beta_{2}xy_{2}
$$

$$
\frac{dy_{1}}{dt} = r_{1}y_{1} \left(1 - \frac{y_{1}}{K}\right) - \mu_{1}y_{1} - \sigma_{1}y_{1}y_{2}
$$

$$
\frac{dy_{2}}{dt} = r_{2}y_{2} \left(1 - \frac{y_{2}}{K}\right) - \mu_{2}y_{2} - \sigma_{2}y_{1}y_{2}
$$

(1)

According to Sarwardi, S., Mandal, P. K., & Ray, S. [9], described the dynamical behaviour of a two predator and one prey used Holling-type II functional responses with competition on both of predator.

$$
\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) - \frac{\beta_{1}xy_{1}}{1 + a_{1}x} - \frac{\beta_{2}xy_{2}}{1 + a_{2}x}
$$

$$
\frac{dy_{1}}{dt} = \frac{\alpha_{1}\beta_{1}xy_{1}}{1 + a_{1}x} - \mu_{1}y_{1} - \sigma_{1}y_{1}y_{2}
$$

$$
\frac{dy_{2}}{dt} = \frac{\alpha_{2}\beta_{2}xy_{2}}{1 + a_{2}x} - \mu_{2}y_{2} - \sigma_{2}y_{1}y_{2}
$$

(2)

The Holling type II functional response was introduced in [10] is applied for both predators. Another approach of predation for Holling functional response can found in [11]. Their study applies different functional responses of the two predators on prey. The first predator preys on the prey following Holling type II and the second predator preys on the prey using Holling type III functional responses.

2.2. Model formulation

Motivated by above studies, in this work, we construction from the system (1) and system (2). In this paper we have presented the dynamical behaviour of three populations a Leslie-Gower predator-prey model, where two predators competing on one prey. The model consists of one prey, the first predator and the second predator. The Leslie Gower equation with Holling type I functional response can be written as:

$$
\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) - \beta_{1}xy_{1} - \beta_{2}xy_{2}
$$

$$
\frac{dy_{1}}{dt} = y_{1}r_{1} \left(1 - \frac{y_{1}}{x}\right) - \alpha_{1}y_{1}y_{2}
$$

$$
\frac{dy_{2}}{dt} = y_{2}r_{2} \left(1 - \frac{y_{2}}{x}\right) - \alpha_{2}y_{1}y_{2}
$$

(3)

where $x, y_1$ and $y_2$ correspond respectively to the population of prey, the first predator population and the second predator population at each instant of time. We have eight parameters, all parameters of the system (3) are positive, with the initial conditions: $x(0) > 0$, $y_{1}(0) > 0$, $y_{2}(0) > 0$.

2.2.1. Assumption. Before we introduce the mathematical model, let us describe the basic assumptions that we made to formulate it. The first equation shows that the population of prey grows logistically with the carrying capacity $k$ parameter and the intrinsic growth rate of prey is $r$ describe in the absence of
predation is given by the factor \(1 - \frac{x}{k}\). The growth rate of the first and the second predator are described by parameter \(r_2\) dan \(r_3\) which the carrying capacity proportional to the prey populations size \(x\). The term \(\frac{y_1}{x}\) and \(\frac{y_2}{x}\) of this equation are called the Leslie–Gower term. The parameters \(\beta_1\) and \(\beta_2\) mean the rate of the first predator \(y_1\) and the second predator \(y_2\) feeding prey upon \(x\). The both of predator have competition parameters are \(\alpha_1\) and \(\alpha_2\).

2.2.2. Scalling. The model can be written in non-dimensional form to reduce the number of parameters, we choose: \(\tilde{x} = \frac{x}{k}\), \(\tilde{y}_1 = \frac{y_1}{k}\), \(\tilde{y}_2 = \frac{y_2}{k}\), \(\tilde{a} = \frac{a}{r}\), \(\tilde{b} = \frac{b}{r}\), \(\tilde{s} = \frac{s}{r}\), \(\tilde{\alpha} = \frac{\alpha}{r}\), \(\tilde{p} = \frac{p}{r}\), \(\tilde{\beta} = \frac{\beta}{r}\). The form of system (3) can rewrite the simplified:

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - x) - axy - bxz \\
\frac{dy}{dt} &= sy \left(1 - \frac{y}{x}\right) - axz \\
\frac{dz}{dt} &= p z \left(1 - \frac{z}{x}\right) - \beta yz
\end{align*}
\]

where \(x\), \(y\) and \(z\) constitute population density of prey, the first predator population and the second predator population at time \(t\), respectively.

3. Local stability of the equilibrium point
In this section, we explore the stabilities of a system (4) of all nonnegative equilibrium points. As well known, the stabilities are established by the real parts of the eigenvalues of the Jacobian matrix at each equilibrium point must be negative. The analysis the local stability of the positive equilibrium and existence of the system (4) are as follows.

3.1. Equilibrium point
It is easy to verify all possible equilibrium point of the system (4). By the definition of equilibria, the equilibria of a system (4) are satisfied when

\[
\begin{align*}
\frac{dx}{dt} &= \frac{dy}{dt} = \frac{dz}{dt} = 0
\end{align*}
\]

or

\[
\begin{align*}
x(1 - r - ay - bz) &= 0 \quad 4.1a \\
y(s - \frac{sy}{x} - az) &= 0 \quad 4.1b \\
z(p - \frac{pz}{x} - \beta y) &= 0 \quad 4.1c
\end{align*}
\]

Solution equation (4) is indicated by \((x^*, y^*, z^*)\). Because the equilibrium point is a real value solution, we have four equilibrium points of the system (4) are described in the following.

1. The equilibrium point of the extinction of both predators \(E_1 = (1,0,0)\) is always unstable.

2. The equilibrium of the first predator extinction \(E_2 = \left(\frac{1}{1+b}, 0, 0\right)\) is stable.

3. The equilibrium of the second predator extinction \(E_3 = \left(0, \frac{1}{1+a}, 0\right)\) is stable.

4. The interior equilibrium point \(E_4 = (x^*, y^*, z^*)\)

The existence of positive the interior equilibrium point is investigated by using Cardan criteria. We mainly focus on the existence of positive roots of the cubic of the system (4)

\[
x^3 + 3p_1x^2 + 3p_2x + p_3 = 0 \quad 4.2
\]

By the transformation \(z = x + \rho_1\), Eq. 4.2 is reduced to

\[
h(z) = z^3 + 3rz + q = 0 \quad 4.3
\]

where

\[
r = \rho_2 - \rho_1^2
\]
\[
q = \rho_3 - 3\rho_1\rho_2 + 2\rho_1^3
\]

\[
\]
In the following, we consider the existence of positive roots of Eq. (4.3).

Lemma 2.1. (a) If $q < 0$, Eq. (4.3) has single positive root.
(b) Suppose $q > 0$ and $r < 0$, then: if $q^2 + 4r^3 = 0$, Eq. (4.3) has a positive root of multiplicity two, if $q^2 + 4r^3 < 0$, Eq. (4.3) has two positive roots;
(c) If $q = 0$ and $r < 0$, Eq. (4.3) has a unique positive root.

With simple computations show that, when $s = 1.2, p = 1.5, a = 1.1, b = 1.4, \alpha = 2.4, \beta = 2.18$ we get $q = \rho_3 - 3\rho_1\rho_2 + 2\rho_1^3 = 2.34 > 0$, and $r = \rho_2 - \rho_1^2 = -1.11 < 0$. We have the following Lemma 2.1 regarding, it is shown that the conditions of the existence of the positive equilibria in a system (4) has two positive roots.

3.2. Local stability

The local asymptotic stability of each equilibrium point is analyzed by calculating the Jacobian matrix and finding the eigenvalues assessed at each equilibrium point. The real parts of the eigenvalues of the Jacobian matrix must be negative for the stability of the equilibrium points.

The system (4) is a nonlinear autonomous. The stability of the nonlinear autonomous can be obtained by linearising the system (4) with the Jacobian matrix is given by:

$$ J(x^*, y^*, z^*) = \begin{bmatrix} -ay - bz - 2x + 1 & -ax & -bx \\ sy^2 & s - 2sy/x & -az \\ pz^2 & -\beta y & p - 2pz/x - \beta y \end{bmatrix} $$

The characteristic equation is given by $\det(J(x^*, y^*, z^*) - \lambda I) = 0$ where $(x^*, y^*, z^*)$ is the equilibrium of (4). The local dynamical behaviour of the equilibrium point is explored where the result is established by calculating the eigenvalue of the matrix from each the equilibrium point.

Stability analysis for Equilibrium point $E_1 = (1, 0, 0)$

Matrix of $J(E_1)$ can be expressed as

$$ \begin{bmatrix} -1 - \lambda & -a & -b \\ 0 & s - \lambda & 0 \\ 0 & 0 & p - \lambda \end{bmatrix} = 0 $$

The characteristic equation of the matrix $J(E_1)$ are $(-1 - \lambda)(s - \lambda)(p - \lambda) = 0$

It has always one negative eigenvalue $\lambda_1 = -1$, both of the other eigenvalue always positive $\lambda_2 = s$ and $\lambda_3 = p$. Therefore the equilibrium point $(1, 0, 0)$ is unstable around $E_1$ which is, in fact, a saddle-node.

Stability analysis for Equilibrium point $E_2 = \left( \frac{1}{1+b}, 0, \frac{1}{1+b} \right)$

The Jacobian matrix of $J(E_2)$ are

$$ \begin{bmatrix} 1 - \frac{b}{1+b} - \frac{a}{1+b} - \lambda & -a & -b \\ 0 & s - \frac{a}{1+b} - \lambda & 0 \\ p & -\beta & p - \lambda \end{bmatrix} = 0 $$

The characteristic equation of equilibrium point $E_2$ is $\lambda^2 - \text{trace}(J(E_2))\lambda + \text{det}(J(E_2)) = 0$. Obviously, they can be expressed as $\lambda_{2,3} = \frac{\text{trace}(J(E_2)) \pm \sqrt{(\text{trace}(J(E_2)))^2 - 4\text{det}(J(E_2))}}{2}$, when $\text{trace}(J(E_2)) < 0$ and $\text{det}(J(E_2)) > 0$. The eigenvalue of matrix $J(E_2)$ are $\lambda_1 = \frac{b\alpha - a + s}{b + 1}$, $\lambda_2 = \frac{\alpha \beta - \alpha b + b + 1}{2(b + 1)}$, and $\lambda_3 = \frac{\alpha \beta - \alpha b + b + 1}{2(b + 1)}$.
All parameters are positive and the eigenvalue $\lambda_2$ and $\lambda_3$ must be negative. The eigenvalue $\lambda_1$ is negative if it satisfies $\alpha > s(b+1)$. The equilibrium point $E_2 = \left(\frac{1}{1+b}, 0, \frac{1}{1+b}\right)$ stabilizes the local asymptotic by fulfilling the condition. Meanwhile, if the reverse applies then $E_2$ will become unstable.

Stability analysis for equilibrium point $E_3 = \left(\frac{1}{1+a'}, \frac{1}{1+a'}, 0\right)$

The Jacobian matrix of system (4) evaluated at the equilibrium point $E_3$ is given by:

$$
\begin{pmatrix}
1 - \frac{a}{1+a} - \frac{2}{1+a} - \lambda & -\frac{a}{1+a} - \frac{b}{1+a} - \lambda & 0 \\
0 & -s - \lambda & -\frac{a}{1+a} - \lambda \\
0 & 0 & (p - \frac{\beta}{1+a} - \lambda)
\end{pmatrix} = 0
$$

The eigenvalues of matrix $J(E_3)$ are $\lambda_1 = \frac{ap+\beta}{a+1}$, $\lambda_2 = -\frac{1}{2} \frac{(as+s+1)+\sqrt{a^2s^2-4a^2s+2as^2-6as+s^2-2s+1}}{a+1}$ dan $\lambda_3 = -\frac{1}{2} \frac{(as+s+1)-\sqrt{a^2s^2-4a^2s+2as^2-6as+s^2-2s+1}}{a+1}$

All of the parameters in system (4) are positive, then the eigen values $\lambda_2$, $\lambda_3$ should be negative. If the eigenvalue $\lambda_1$ is a negative, the equilibrium point $E_3 = \left(\frac{1}{1+a'}, 0, \frac{1}{1+a'}\right)$ is locally stable provided the following condition is satisfied $\beta > p(a+1)$.

Stability analysis for equilibrium point $E_4 = (x^*, y^*, z^*)$

We suppose that $E_4 = (x^*, y^*, z^*)$, then we can rewrite $J(E_4)$ as following:

$$
J(x^*, y^*, z^*) = \begin{pmatrix}
-ay^* - bz^* - 2x^* + 1 & -ax^* & -bx^* \\
\frac{sy^*^2}{x^*^2} & s - \frac{2sy^*}{x^*} - az^* & -ay^* \\
\frac{pz^*^2}{x^*^2} & -\beta y^* & p - \frac{2pz^*}{x^*} - \beta y^*
\end{pmatrix}
$$

The characteristic equation of the linearized system (4) at the interior equilibrium $E_4 = (x^*, y^*, z^*)$ is

$$
\lambda^3 + \varphi_1 \lambda^2 + \varphi_2 \lambda + \varphi_3 = 0
$$

We use the Routh–Hurwitz criterion to investigate the stability, which has a negative real part if and only if $\varphi_3 > 0$, $\varphi_3 > 0$, and $\varphi_4 \varphi_3 - \varphi_3 > 0$, the coexistence steady-state $E_4$ is locally asymptotically stable. We need a further analysis for the existence of $E_4$ is asymptotically stable under certain condition while $E_3$ is unstable, $E_2$ and $E_3$ are also asymptotically stable on some condition. We also confirm the dynamical behaviour near $E_3$ by simulating numeric with different initial value of parameter $\alpha$ and $\beta$.

**Table 1. Stability condition of all equilibria in system (3)**

| Equilibrium point | Type of stability | Local stability condition |
|-------------------|-------------------|--------------------------|
| $E_1 = (1,0,0)$   | unstable          | unstable                  |
| $E_2 = \left(\frac{1}{1+b}, 0, \frac{1}{1+b}\right)$ | asymptotically stable | $s(b+1) < \alpha$ |
| $E_3 = \left(\frac{1}{1+a'}, \frac{1}{1+a'}, 0\right)$ | asymptotically stable | $p(a+1) < \beta$ |
| $E_4 = (x^*, y^*, z^*)$ | asymptotically stable | |
4. Numerical simulation

The last performed, using the fourth order Runge-Kutta method by software Matlab, we solved the predator-prey system. We observed the effect on the dynamic behaviour of the system (4) by considering the different value of competition parameter $\alpha$ and $\beta$. The fixed the other parameters values are taken from Table 2. In our numerical outcomes confirmed the analytical results, and the dynamics of the predator-prey model around the positive interior the steady-state.

| Table 2 Parameter Value of Simulation |
|---------------------------------------|
| Parameter | Simulation 1 | Simulation 2 | Simulation 3 | Simulation 4 | Simulation 5 |
|-----------|--------------|--------------|--------------|--------------|--------------|
| $s$       | 0.22         | 0.22         | 0.22         | 0.22         | 0.22         |
| $p$       | 0.27         | 0.27         | 0.27         | 0.27         | 0.27         |
| $a$       | 0.76         | 0.76         | 0.76         | 0.76         | 0.76         |
| $b$       | 0.12         | 0.12         | 0.12         | 0.12         | 0.12         |
| $\alpha$  | 0.33         | 0.33         | 0.23         | 0.23         | 0.33         |
| $\beta$   | 0.39         | 0.57         | 0.57         | 0.39         | 0.41         |

Now, talking the same set of parametric values, we presented dynamical behaviour for each the equilibrium point by considering the different value of parameter $\alpha$ and $\beta$. Numerical simulated are represented as phase portraits to draw the stability of equilibrium point, and some figures to support the theoretical findings.

4.1. The Dynamical behaviour for the equilibrium point $E_2$

The first simulation, based on table 2, we choose the value of parameter $s = 0.22$, $p = 0.27$, $a = 0.76$, $b = 0.12$, $\alpha = 0.33$ and $\beta = 0.39$ so that the equilibrium point $E_4$ does not exist, while the equilibrium point $E_1$, $E_2$, $E_3$ exists and $E_2$ is asymptotically stable because it satisfies the stability requirements of $E_2$ namely $s(b + 1) = 0.2464 < 0.33 = \alpha$.

![Figure 1. The Dynamical Behaviour for the equilibrium point $E_2$.](image)

Numerical simulations are shown through several different initial values, namely $N_1 = (1.1, 0.8, 1.2)$, $N_2 = (0.2, 0.2, 0.1)$, $N_3 = (0.9, 0.7, 0.3)$, and $N_4 = (0.8, 0.1, 0.2)$. The dynamic behaviour in the model with this parameter can be seen in Figure 1. It can be seen that all solutions to the equilibrium point tend $E_2$. Indicate that the prey population and the second predator will survive while the first predator population will be extinct. Simulation results show compatibility with the results of the analysis.

4.2. The dynamical behaviour for the equilibrium point $E_2$ and $E_3$

From our numerical study we have obtained two value for the competition parameter $\alpha = 0.33$ and $\beta = 0.57$. Further analysis shows that, there are three equilibrium points exist, that is $E_0=(1,0,0)$, $E_1=(1.72,0,0)$, and $E_2=(0.1,1.28,2.0304)$. In this results $E_2$ and $E_3$ in the stability requirements being met $s(b + 1) = 0.2464 < 0.33 = \alpha$ and $p(a + 1) = 0.4752 < 0.57 = \beta$. 
Figure 2. The Phase Portrait of Equilibrium point $E_2$ and $E_3$.

The portrait of the solution phase in Figure 2 using the same initial values parameter in the first simulation by considering the different value of parameter $\alpha$ and $\beta$. In addition, the conditions for the existence of points $E_4$, and not fulfilled, so that these points do not exist. For various initial values, the numerical solution of system (4) converges to $E_2$ and $E_3$. The results of simulation 2 indicate that only the predator population survives, while the prey population and competitors will become extinct.

4.3. The dynamical behaviour for the equilibrium point $E_3$

The parameter values by considering the different value of parameter $\alpha = 0.23$ and $\beta = 0.57$ used in simulation 3 cause the existence conditions of equilibrium points $E_1=(1,0,0)$, $E_2=(0.8928, 0, 0.8928)$, and $E_3=(0.5682, 0.5682, 0)$ to be exist.

Figure 3. The Dynamic behaviours of the solution system in Equilibrium Point $E_3$.

The stability requirements of $E_4$ are not fulfilled. Point $E_3$ is asymptotically stable because the stability requirement $p(\alpha + 1) < \beta$ is met. This is shown by the portrait of the solution phase in Figure 3 just to the point $E_3$. In this case, the population of prey and the first predators will coexist within certain initial values $N_1=(0.3, 0.9, 0.7)$, $N_2=(1.1, 0.7, 0.5)$, $N_3=(0.4, 0.3, 1)$, and $N_4=(0.8, 0.1, 0.2)$.

4.4. The dynamical behaviour for the equilibrium point $E_4$

Based on simulation numeric, by considering the different value of parameter $\alpha = 0.23$ and $\beta = 0.39$ it can be seen that the interior equilibrium point $E_4$ is locally asymptotically stable so that all equilibrium points exist.
Figure 4. The dynamic behaviour that the interior equilibrium point $E_4$ stable. Indicated by a portrait of the solution phase in Figure 4 with several initial values $N_1 = (1.6, 0.9, 1.3)$, $N_2 = (0.4, 0.8, 0.9)$, $N_3 = (0.3, 0.2, 0.1)$, and $N_4 = (1.4, 0.7, 0.3)$, converge to the interior equilibrium point $E_4$. The results of numerical simulation indicate that all populations can survive.

4.5. The dynamical behaviour for the equilibrium point $E_2$ and $E_4$

To demonstrate the existence of phenomenon near of equilibrium point $E_4$ and $E_2$, take a set value of parameter $\alpha = 0.33$ and $\beta = 0.41$ on the last numerical simulation.

Figure 5. Phase trajectories of the system (4) with $\alpha = 0.33$ and $\beta = 0.41$.

Clearly, as shown figure 5, the system (4) approaches asymptotically to the positive equilibrium point $E_4$ and $E_2$ from different initial point. That all equilibrium points $E_1 = (1,0,0)$, $E_2 = (0.892, 0.892)$, $E_3 = (0.568, 0.568, 0)$ and $E_4 = (0.649, 0.425, 0.229)$ exist. It depicts that the stability of equilibrium point $E_4$ and $E_2$ are fulfilled.

5. Conclusion

In this research, we are investigated with the local stability in a Leslie-Gower predator-prey model with competition on both of the predator population. The model has four equilibrium point and several possible stable, $E_1$ is unstable, $E_2$, $E_3$, and $E_4$ are local asymptotically stable with certain conditions. The term competition on predator populations is an important factor for the biological control of the predator-prey population. Numerical simulations are carried out for illustrating our analysis and gives to the concluding remarks. The effect of the competitions parameter $\alpha$ and $\beta$ of the system (4) are drowned by phase trajectories. According to these figures, has five possible stable equilibrium point that is $E_2$, $E_3$, $E_2$ and $E_3$, $E_4$, $E_2$ and $E_4$. The above research supports that phenomenon exhibited by a Leslie-Gower predator-prey model. The next research, we will explore the global stability analysis behaviour of the interior equilibrium point.
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