Confining Phases of a Compact $U(1)$ Gauge Theory from the Sine-Gordon/Massive Thirring Duality

Kentaroh Yoshida
Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan.
E-mail: yoshida@phys.h.kyoto-u.ac.jp
(Dated: November 13, 2018)

We consider the phase structure of a pure compact $U(1)$ gauge theory in four dimensions at finite temperature by treating this system as a perturbative deformation of the topological model. Phases of a gauge theory can be investigated from the phase structure of the topological model. The thermal pressure of the topological model has been calculated, from which its phase structure can be derived. We have obtained phases of a compact $U(1)$ gauge theory. Moreover, the critical-line equation has been explicitly evaluated.

PACS numbers: 11.10.Wx 12.38.Aw 12.38.Lg

A novel scenario to study the confinement is a perturbative deformation of the topological model [6, 7, 8, 10, 11, 12] and it has clarified various properties of the confinement, string tension, phase structure, etc. The phase structure of a gauge theory can be described by that of the topological model. In this scenario we choose a modified gauge fixing which leads to an $OSp(4, 2)$ symmetry in the topological model. This symmetry enables us to apply the Parisi-Sourlas (PS) dimensional reduction [10] for the topological model. Due to the PS reduction the topological model is equivalent to the two-dimensional non-linear sigma model (NLSM$_2$). In particular, the behavior of topological objects in the NLSM$_2$ decides whether a confining string between test particles appears or not. The advantage of this scenario is that the dynamics and mechanics of the confinement are very tractable.

In the case of a compact $U(1)$ gauge theory, the topological model is equivalent to an $O(2)$ NLSM$_2$. It has vortex solutions which form a two-dimensional Coulomb gas (CG). Phases of a compact $U(1)$ gauge theory are determined by the behavior of a CG. It has been shown that the confining phase transition of a compact $U(1)$ gauge theory at zero temperature can be described by the Berezinskii-Kosterlitz-Thouless (BKT) phase transition [10, 11, 12] and obtained that the critical gauge coupling determines by the behavior of a CG. It has been shown that the confining phase transition of a compact $U(1)$ gauge theory (For notations and details, see Ref. [8]). The $\zeta$ is defined by $\zeta \equiv \kappa^{2\pi/g^2}$ where the dimensionless parameter $\kappa$ is defined by $\kappa \equiv R_0/a$ and $R_0$ is a radius for the regularization of vortices in an $O(2)$ NLSM$_2$. This quantity is inherently related to (regularized) Dirac monopoles in four dimensions and important for the existence of the confining phase. The limit $\zeta \to 0$ corresponds to remove the regularization, and the monopole effect vanishes and the confining phase disappears.

The $A \equiv 1/a$ is introduced in a CG through the regularization of the two-dimensional potential,
\begin{equation}
V(r) \sim \ln \frac{r}{a},
\end{equation}
and denotes a cut-off for the small distance. In four dimensions, $A$ corresponds to a cut-off for the short-range, $\Lambda$ corresponds to a cut-off for the short-range, $\Lambda$. In this sense, regularization for monopoles is important for the existence of the confining phase. It is well known that an $O(2)$ NLSM$_2$ is equivalent to some models such as sine-Gordon (SG) model, massive Thirring (MT) model, and XY model. In our previous works [6, 7, 8, 10] we have studied phases of a pure compact $U(1)$ gauge theory at finite temperature using these equivalences and obtained the results consistent to the prediction by Svetitsky and Yaffe [13].

In this letter we will report the existence of the confining phase in a pure compact $U(1)$ gauge theory in four dimensions at finite temperature as in the lattice gauge theory.

Our consideration is based on demonstrating the phase structure of the topological model using the thermal pressure. We can calculate this quantity from the SG/MT duality and the equivalence between the one-dimensional $O(2)$ CG and MT model at high temperature [16]. Moreover, we have obtained the explicit critical-line equation.

We start with the action of the original $U(1)$ gauge theory and treat the theory as a perturbative deformation of the topological model that can be mapped to the SG model [8, 13]. Thus we can obtain the relationship between parameters in the compact $U(1)$ gauge theory and SG model as follows,
\begin{equation}
\lambda = \frac{128\pi^6}{g^4} \zeta A^2, \quad m = \frac{4\pi^{3/2}}{g} \zeta^{1/2} A. \quad (1)
\end{equation}

Here $m$ and $\lambda$ are a mass and a coupling constant in the SG model respectively, and $g$ is a gauge coupling in the compact $U(1)$ gauge theory (For notations and details, see Ref. [8]). The $\zeta$ is defined by $\zeta \equiv \kappa^{2\pi/g^2}$ where the dimensionless parameter $\kappa$ is defined by $\kappa \equiv R_0/a$ and $R_0$ is a radius for the regularization of vortices in an $O(2)$ NLSM$_2$. This quantity is inherently related to (regularized) Dirac monopoles in four dimensions and important for the existence of the confining phase. The limit $\zeta \to 0$ corresponds to remove the regularization, and the monopole effect vanishes and the confining phase disappears.

The $A \equiv 1/a$ is introduced in a CG through the regularization of the two-dimensional potential,
interaction between monopoles. Moreover, the string tension \( \sigma_{st} \) in a gauge theory (at zero temperature) is given by

\[
\sigma_{st} \simeq \frac{\zeta}{a^2} = \zeta \Lambda^2.
\]

(3)

We cannot obtain the string tension at finite temperature yet, but we can guess even in the finite-temperature case that the behavior of a CG would determine whether the confining string appears or not. The \( \Lambda \) plays an important role and decides the scale of the theory (The \( \Lambda \) has been missed until this paper. The expression in this paper is correct.). Note that Eq. (1) does not depend on the temperature and holds at any temperature. It has been shown that the duality between the SG and MT model holds at zero temperature \([20]\) and even finite temperature \([21, 22]\) if the following relation

\[
\frac{4\pi m^2}{\lambda} = 1 + g^2_{\text{MT}} \frac{\pi}{\lambda}, \quad m^4 = \rho m_{\text{MT}},
\]

(4)

is satisfied. Here \( m_{\text{MT}} \) and \( g^2_{\text{MT}} \) are a renormalized mass and a dimensionless coupling constant, respectively. Here the renormalization scale is set as \( \rho = m_{\text{MT}} \). Combining Eq. (1) with Eq. (4), we can write parameters of the MT model by those of the gauge theory as

\[
m_{\text{MT}} = \sqrt{2\zeta} \Lambda, \quad g^2_{\text{MT}} = \frac{g^2}{2\pi} - \pi.
\]

(5)

Here we should remark that the physical temperature \( T \) is common in the SG model, MT model and gauge theory by derivation in our scenario.

The equivalence between the one-dimensional CG and MT model at high temperature has been shown in the dimensional reduction (DR) regime \([13]\)

\[
T \gg m_{\text{MT}}, \quad g^2_{\text{MT}} > 0,
\]

(6)

\[
T \gtrsim m_{\text{MT}}, \quad g^2_{\text{MT}}/\pi \gg 1.
\]

(7)

A one-dimensional CG system is an exactly solvable and has two phases, which are a molecule phase and a plasma phase \([23, 24]\). Thus a one-dimensional CG system undergoes a BKT-like phase transition at certain temperature of the CG system. This phase transition can be explained by the intensity of the thermal pressure,

\[
P_{\text{CG}}(z, \theta, \sigma) = 2\pi \sigma^2 \gamma_0(\hat{z}), \quad \hat{z} = \frac{z \theta}{2\pi \sigma^2},
\]

(8)

where \( z \) and \( \theta \) are the fugacity and temperature of the CG system respectively, and \( \sigma \) is a charge of the particle forming the CG. The \( \gamma_0 \) is the highest eigenvalue of Mathieu’s differential equation

\[
\left[ \frac{d^2}{d\phi^2} + 2 \hat{z} \cos \phi \right] y(\phi) = \gamma y(\phi)
\]

(9)

FIG. 1: The \( \hat{z} \) is numerically plotted as a function of \( T/m_{\text{MT}} \) and \( g^2_{\text{MT}}/\pi \). A cliff and a slope exist. A cliff exists in the strong-coupling and low-temperature region. There is a slope in the negative-coupling region of the MT model. This slope appears as another confining phase in a gauge theory. However, we cannot rely the result in this region as noted later.

with \( y(\phi + 2\pi) = y(\phi) \). The thermal pressure of the MT model can be written as \([19]\)

\[
P_{\text{MT}}(T, m_{\text{MT}}, g_{\text{MT}}) = \frac{\pi T^2}{6} + P_{\text{CG}}(T, m_{\text{MT}}, g_{\text{MT}}),
\]

(10)

\[
P_{\text{CG}}(T, m_{\text{MT}}, g_{\text{MT}}) = \frac{2\pi T}{1 + g^2_{\text{MT}}/\pi} \gamma_0(\hat{z}).
\]

(11)

where \( \hat{z} \) is defined by

\[
\hat{z} = \frac{m^2_{\text{MT}}}{4\pi T^2} \left( 1 + \frac{g^2_{\text{MT}}}{\pi} \right)^2 \left( \frac{T}{m_{\text{MT}}} \right)^{1+g^2_{\text{MT}}/\pi-1}.
\]

(12)

The first term in Eq. (10) is the finite-size effect and the dominant CG contribution of the thermal pressure comes from \( \gamma_0(\hat{z}) \). Since \( \gamma_0(\hat{z}) \) is a monotonously increasing function, \( \hat{z} \) becomes the order-parameter of the one-dimensional CG and MT model due to the behavior of \( \gamma_0(\hat{z}) \). If \( \hat{z} \ll 1 \), then the CG is in a molecule phase and the MT model is in the chirally symmetric phase. If \( \hat{z} \gg 1 \), then the CG is in a plasma phase and the MT model is in the chirally broken phase. The numerical plot of \( \hat{z} \) is shown in FIG. 1.

We can translate phases in the CG and MT model as those of a gauge theory using Eq. (3). The order-parameter \( \hat{z} \) can be rewritten as

\[
\hat{z} = \frac{g^2}{2\pi} \left( \frac{2T^2}{\zeta \Lambda^2} \right)^{\sigma^2/\sigma^2-1}.
\]

(13)

The phase of a gauge theory at high temperature is determined by a one-dimensional CG \([7, 13]\). Therefore, if \( \hat{z} \ll 1 \), then it is in a deconfining phase, and if \( \hat{z} \gg 1 \), then it is in a confining phase. We have numerically plotted Eq. (13) as shown in FIG. 2. We can see two precipices which corresponds to confining phases. One corresponds to the traditional confining phase predicted in the lattice gauge theory \([7, 13]\). Another is an unpredicted confining phase, though it is out of the region that the result is valid as noted later.
Moreover we can evaluate the critical-line equation by setting \( \hat{z} \simeq 1 \), because \( \gamma_0(\hat{z}) \) increases very rapidly more than \( \hat{z} \simeq 1 \). As a result, we obtain

\[
T \simeq \frac{\zeta^{1/2} \Lambda}{\sqrt{2}} \left( \frac{2\pi^3}{g^2} \right)^{g^2/(\pi^2-g^2)}.
\] (14)

We have also numerically plotted Eq. (14) as shown in FIG. 3. This result explicitly shows the critical behaviors of the traditional confining phase and another confining phase. In the strong-coupling region, \( g^2 \gg \pi^2 \), the critical-line equation (14) becomes

\[
T \simeq \frac{\zeta^{1/2} \Lambda}{2\pi^{3/2}} g.
\] (15)

This equation (15) is identical with the asymptotic form of the critical-line equation obtained by the calculation of the one-loop effective potential in the SG model [9]. This coincidence surely confirms our results in this paper.

Here we should comment on the validity of our results. Recall that we have used the equivalence between the one-dimensional CG and MT model at high temperature in our derivation. This equivalence is valid in the DR regime where the thermo-dynamical limit exists. This regime in a gauge theory is expressed by

\[
T \gg \sqrt{2\zeta \Lambda}, \quad g^2 > 2\pi^2,
\] (16)

\[
T \gtrsim \sqrt{2\zeta \Lambda}, \quad g^2 \gg 4\pi^2.
\] (17)

The above constraints for the temperature have no problem, but those for the gauge coupling are obstacles to propose the existence of another confining phase. As discussed in Ref. [19], we may formally take the gauge coupling \( g \) arbitrary from the viewpoint of the gauge theory and the CG, although extra renormalizations at least would be required from the standpoint of the MT model. However, it is quite well known that the density of the magnetic monopoles decreases rapidly as the coupling constant gets smaller [28]. That, the monopole effect would almost vanish in this region. Therefore another

confining phase would not exist. Also, the existence of this phase depends on the value of \( \kappa \). If we take \( \kappa > 1 \), then this phase disappears.

Also, the traditional confining phase is consistent as shown in FIG. 3 and we can also see the universal behavior near the critical coupling \( g_{cr} = \pi \). This region is also out of the DR regime. Nevertheless, the critical-line equation behaves as expected. The reason is unknown.

In summary we have investigated the phase structure of a pure compact \( U(1) \) gauge theory at finite temperature using the scenario of a perturbative deformation of the topological model. The topological model has been mapped to the two-dimensional MT model through the PS dimensional reduction and SG/MT duality. Due to the equivalence between the one-dimensional CG and MT model at high temperature, we have obtained the thermal pressure of the topological model. In conclusion our results suggest that the confining phase would exist and we propose the phase structure of a pure compact \( U(1) \) gauge theory in four dimensions at finite temperature as shown in FIG. 4. The confining phase at weak-coupling and high temperature region has not been predicted in
Ref. [17, 18] and our results are invalid in this region. Therefore this phase would be an error in our calculation. Moreover we have explicitly evaluated the critical-line equation. This result shows the traditional confining phase as known in the lattice [17, 18], and also includes the result obtained in our previous work [8]. The detailed analysis will be done in the forthcoming paper [26].

The phase transition of a gauge theory could be translated to the chiral symmetry restoration in the two-dimensional MT model at finite temperature. It is expected to have something to do with the monopole condensation of a gauge theory in four dimensions. It can be also described by the behavior of a one-dimensional CG, that is a BKT-like phase transition. Concerning with these descriptions, we may consider that topological objects in the SG model should play an important role in the thermal phase transition of a gauge theory. This perspective is also attractive.

In addition we can investigate the phase structure of a gauge theory by calculating the Gaussian effective potential (GEP) in the SG model and derive the critical-line equation. In this approach, we would be able to study the low-temperature region and evaluate the critical-line equation more precisely [27]. We have almost calculated and obtained confirmed results.

The author would like to thank W. Souma for useful discussion and valuable comments. He also acknowledges H. Aoyama and K. Sugiyama for their support in his working.

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