On Vandiver’s arithmetical function – II

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Abstract: We study more properties of Vandiver’s arithmetical function

\[ V(n) = \prod_{d \mid n} (d + 1), \]

introduced in [2].

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1 Introduction

This is a continuation of the first part [2], where we have considered the Vandiver arithmetical function

\[ V(n) = \prod_{d \mid n} (d + 1), \]

where \( d \) runs through all distinct divisors of \( n \). In the first part we have proved inequalities related to this function; connections with notions of perfect numbers; equations related to \( V(n) \), as well as some open problems and conjectures. The aim of this second part is to offer more properties of this function, and particularly to deduce also certain asymptotic results.

2 Main results

Theorem 2.1. One has

\[ \sum_{d \mid n} \frac{1}{d + 1} < \log \frac{V(n)}{T(n)} < \frac{\sigma(n)}{n} (n \geq 1), \]  

(2.1)

where \( T(n) \) denotes (as in [2]) the product of divisors of \( n \) (see Theorem 7 of [2]).
Proof. Write
\[
\log V(n) = \log \prod_{d|n} (d + 1) = \sum_{d|n} \log(d + 1)
\]
\[
= \sum_{d|n} \left( \log d + \log \left( 1 + \frac{1}{d} \right) \right) = \sum_{d|n} \log d + \sum_{d|n} \log \left( 1 + \frac{1}{d} \right).
\]
Here the first term is \(\log T(n)\). For the second one apply the double inequality
\[
\frac{x}{1 + x} < \log(1 + x) < x \quad (x > 0)
\]
for \(x = \frac{1}{d}\) implying
\[
\frac{1}{d + 1} < \log \left( 1 + \frac{1}{d} \right) < \frac{1}{d}.
\]
Since
\[
\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n},
\]
relation (2.1) follows. \(\square\)

Remark 2.2. As
\[
\sum_{d|n} \frac{1}{d + 1} = \sum_{d|n} \frac{1}{n/d + 1} = \sum_{d|n} \frac{d}{n + d} \leq \sum_{d|n} \frac{d}{n + 1} = \frac{\sigma(n)}{n + 1},
\]
and
\[
\sum_{d|n} \frac{1}{d + 1} \geq \sum_{d|n} \frac{1}{n + 1} = \frac{d(n)}{n + 1},
\]
we get
\[
\frac{d(n)}{n + 1} \leq \sum_{d|n} \frac{1}{d + 1} \leq \frac{\sigma(n)}{n + 1} \leq \frac{d(n)}{2},
\]
where the last inequality is a consequence of relation (9) of [2]. The lower bound in (2.4) may be improved, by using the following inequality due to P. Henrici (see e.g. [1]):
\[
\sum_{i=1}^{k} \frac{1}{1 + x_i} \geq \frac{k}{1 + \sqrt[k]{x_1 \cdots x_k}}, \text{ where } x_i \geq 1(i = 1, k).
\]

By letting \(x_i = d_i = \text{divisors of } n\), and \(k = d(n)\), as \(x_1 \cdots x_k = T(n) = n^{d(n)/2}\), we get from (2.5):
\[
\sum_{d|n} \frac{1}{d + 1} \geq \frac{d(n)}{\sqrt{n + 1}},
\]
which clearly improves the left-hand side of (2.4). We note that a similar result to (2.6) may be obtained by the combined use of the arithmetic-geometric mean inequality, and the right-hand side of (5) from [2]:

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\[
\sum_{d|n} \frac{1}{d+1} \geq d(n) \frac{1}{\prod_{d|n} (d+1)} = \frac{d(n)}{(V(n))^{1/d(n)}} \tag{2.7}
\]

\[
\sum_{d|n} \frac{1}{d+1} \geq \frac{d(n)}{(V(n))^{1/d(n)}} \geq \frac{d(n)}{\sigma(n) / d(n) + 1} = \frac{(d(n))^2}{\sigma(n) + d(n)}. \tag{2.8}
\]

However, the second inequality in (2.8) is weaker than (2.6), according to the known result (see [4])

\[
\sigma(n) / d(n) \geq \sqrt{n}. \tag{2.9}
\]

**Corollary 2.3.** There exists a positive constant \(c > 0\) such that

\[
\frac{V(2^n - 1)}{T(2^n - 1)} < (\log n)^c \quad (n \geq 3). \tag{2.10}
\]

**Proof.** This follows by the right-hand side of (2.1), and an inequality due to P. Erdős [4]:

\[
\frac{\sigma(2^n - 1)}{2^n - 1} < c \log \log n, \quad n \geq 3, \tag{2.11}
\]

which completes the proof. \(\square\)

**Corollary 2.4.** The right-hand side of (2.1) gives a new proof of Theorem 4 of [2], written equivalently:

\[
\log V(n) \sim \log T(n) \quad \text{as} \quad n \to \infty. \tag{2.12}
\]

**Proof.** It is sufficient to prove that

\[
\frac{\sigma(n)}{nd(n) \log n} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.13}
\]

Indeed, as \(T(n) = n^{d(n)/2}\), one has \(\log T(n) = (d(n) \log n)/2\). Relation (2.13) follows, e.g., by selection (9) of [2], as

\[
\frac{\sigma(n)}{d(n)} \cdot \frac{1}{n \log n} \leq \frac{n + 1}{2n \log n} \to 0
\]

as \(n \to \infty\). \(\square\)

**Theorem 2.5.**

\[
\log \frac{V(n)}{T(n)} = \frac{\sigma(n)}{n} + O(1). \tag{2.14}
\]

**Proof.** By the proof of Theorem 2.1 one has

\[
\log \frac{V(n)}{T(n)} = \sum_{d|n} \log \left(1 + \frac{1}{d}\right).
\]

Since \(\log(1 + x) = x + O(x^2) \quad (x > 0)\), we get

\[
\sum_{d|n} \log \left(1 + \frac{1}{d}\right) = \sum_{d|n} \frac{1}{d} + O\left(\sum_{d|n} \frac{1}{d^2}\right).
\]
By
\[ \sum_{d | n} \frac{1}{d^2} < \sum_{d=1}^{\infty} \frac{1}{d^2} = \frac{\pi^2}{6}, \]
the result follows.

A more concrete proof is based on the double inequality
\[ \frac{\sigma(n)}{n} - \frac{\pi^2}{12} < \log \frac{V(n)}{T(n)} < \frac{\sigma(n)}{n}. \] (2.15)
The left-hand side inequality follows by the logarithmic inequality
\[ \log(1 + x) > x - \frac{x^2}{2} \quad (x > 0) \] (2.16)
(which is stronger than the left-hand side of (2.2)); applied to \( x := \frac{1}{d} \).

Since
\[ \frac{1}{2} \sum_{d | n} \frac{1}{d^2} < \frac{1}{2} \sum_{d=1}^{\infty} \frac{1}{d^2} = \frac{\pi^2}{12}, \]
(2.15) follows. \( \square \)

**Corollary 2.6.** It holds true that
\[ \lim_{n \to \infty} \sup \frac{1}{\log \log n} \cdot \log \frac{V(n)}{T(n)} = e^\gamma, \] (2.17)
where \( \gamma \) is Euler’s constant.

Indeed, by (2.14) one has
\[ \lim_{n \to \infty} \sup \frac{1}{\log \log n} \cdot \log \frac{V(n)}{T(n)} = \lim_{n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = e^\gamma, \]
the last equality is a famous result due to T. H. Gronwall (see e.g. [4]).

**Theorem 2.7.** It holds true that
\[ \lim_{n \to \infty} \frac{V(\sigma(n))}{T(\sigma(n))} = +\infty, \text{ on a set of density one,} \] (2.18)
and
\[ e^{1 - \frac{\pi^2}{12}} \leq \lim_{n \to \infty} \inf \frac{V(\sigma(n))}{T(\sigma(n))} \leq e. \] (2.19)

**Proof.** Applying the left-hand side of (2.15) for \( n := \sigma(n) \), we get
\[ \log \frac{V(\sigma(n))}{T(\sigma(n))} > \frac{\sigma(\sigma(n))}{\sigma(n)} - \frac{\pi^2}{12}. \] (2.20)
Now, by a result of P. Erdős and M. V. Subbarao (see [3]) one has
\[ \frac{\sigma(\sigma(n))}{\sigma(n)} \to \infty \text{ (as } n \to \infty \text{) on a set of density one,} \] (2.21)
so this combined with (2.20) yields (2.18). Particularly

\[
\lim_{n \to \infty} \sup \frac{V(\sigma(n))}{T(\sigma(n))} = +\infty. \tag{2.22}
\]

For the proof of (2.19) apply again (2.15), and the following limit

\[
\lim_{n \to \infty} \inf \frac{\sigma(\sigma(n))}{\sigma(n)} = 1. \tag{2.23}
\]

This follows by the inequality \(\frac{\sigma(m)}{m} > 1\) for any \(m > 1\), and the limit due to R. Bojanić (see [4]).

\[
\lim_{p \to \infty, \ p \text{ prime}} \frac{\sigma(2^p - 1)}{2^p - 1} = 1. \tag{2.24}
\]

As for \(n = 2^{p-1}\) one has \(\frac{\sigma(\sigma(n))}{\sigma(n)} = \frac{\sigma(2^p - 1)}{2^p - 1}\), relation (2.23) follows.

\begin{proof}
\end{proof}

\textbf{Remark 2.8.} Relation (2.19) shows that the \(\lim \inf\) of \(\frac{V(\sigma(n))}{T(\sigma(n))}\) is finite, and lies in the interval \([e^{1-\pi^2/12}, e]\). The exact determination of this value is not known to the author.

\textbf{Theorem 2.9.} It holds true that

\[
\lim_{n \to \infty} \sup \frac{V(\phi(n))}{T(\phi(n))} = +\infty. \tag{2.25}
\]

Let

\[
k = \lim_{n \to \infty} \inf \frac{\sigma(\phi(n))}{n}. \tag{2.26}
\]

Then

\[
e^{k-\pi^2/12} \leq \lim_{n \to \infty} \frac{V(\phi(n))}{T(\phi(n))} \leq e^k. \tag{2.27}
\]

\textbf{Proof.} Apply (2.15) for \(n := \phi(n)\). Then one has

\[
\lim_{n \to \infty} \sup \frac{\sigma(\phi(n))}{\phi(n)} = +\infty. \tag{2.28}
\]

Relation (2.28) follows by

\[
\lim_{n \to \infty} \sup \frac{\sigma(\phi(n))}{\phi(n)} = +\infty \tag{2.29}
\]
due to L. Alaoglu and P. Erdős [4] and the remark that

\[
\frac{\sigma(\phi(n))}{\phi(n)} \geq \frac{\sigma(\phi(n))}{n}.\]

Therefore, (2.25) is true. Inequalities (2.27) are consequences of (2.15).

\begin{proof}
\end{proof}

\textbf{Remark 2.10.} The exact value of \(k\) is not known. A. Makovski and A. Schinzel [3] have shown that

\[
k \leq \frac{1}{2} + \frac{1}{2^{34} - 4}, \tag{2.30}
\]

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and conjectured an inequality (which is the famous Makowski–Schinzel conjecture; see, e.g., [3, 4]), namely:

\[ \frac{\sigma(\varphi(n))}{n} \geq \frac{1}{2} \text{ for all } n \geq 1. \] (2.31)

If (2.31) is true, then we get \( k \geq \frac{1}{2} \). We know that \( k > 0 \), and even that \( k > \frac{1}{39.4} \), due to K. Ford (see [3]).

The next result involves the normal order of magnitude of functions. Recall that we say that the normal order of magnitude of arithmetical function \( f(n) \) is \( g(n) \) if for every \( \varepsilon > 0 \), one has

\[ (1 - \varepsilon)g(n) < f(n) < (1 + \varepsilon)g(n) \] (2.32)

for almost all integer \( n \) (i.e. the set of integers not satisfying (2.32) has density zero). Thus \( f(n) \sim g(n) \) as \( n \to \infty \), excepting \( o(n) \) integers.

**Theorem 2.11.** The normal order of magnitude of \( \log \log V(n) \) is

\[ (1 + \log 2) \cdot \log \log n \] (2.33)

The same is true for \( \log \log T(n) \).

**Proof.** We shall use the following lemma.

**Lemma 2.12.** If \( x_n, y_n > 0 \) and \( x_n \sim y_n \ (n \to \infty) \), where \( y_n \to \infty \), then \( \log x_n \sim \log y_n \).

Indeed, as \( \frac{x_n}{y_n} \to 1 \), we get \( \log \left( \frac{x_n}{y_n} \right) \to 0 \), so \( \log x_n - \log y_n \to 0 \), giving \( \frac{\log x_n - \log y_n}{\log y_n} \to 0 \).

By Theorem 4 of [2], the above Lemma implies

\[ \log \log V(n) \sim \log \log T(n). \] (2.34)

Therefore, it will be sufficient to prove the result for \( \log \log T(n) \). As \( T(n) = n^{d(n)/2} \), we get

\[ \log \log T(n) = \log \frac{d(n)}{2} + \log \log n. \] (2.35)

By a classical result of G. H. Hardy and S. Ramanujan (see [4]), the normal order of magnitude of \( \log d(n) \) is \( (\log 2) \cdot \log \log n \). Clearly, the same is true for \( \log \frac{d(n)}{2} \), therefore by (2.35) the result follows.

**Theorem 2.13.** i) If \( n \geq 6 \) is even, then

\[ V(n) \geq 3(n + 1)(n + 2). \] (2.36)

There is equality only if \( n = 2p \), where \( p \geq 3 \) is a prime.

ii) If \( n \geq 12 \) is divisible by \( 4 \), then

\[ V(n) \geq \frac{15}{4} \cdot (n + 1)(n + 2)(n + 4). \] (2.37)

There is equality only if \( n = 4p \), where \( p \geq 3 \) is a prime.
Proof. i) If $n \geq 6$ is even, then $1, 2, \frac{n}{2}, n$ are distinct divisors of $n$, so

$$V(n) \geq (1 + 1)(2 + 1) \cdot \left(\frac{n}{2} + 1\right)(n + 1) = 3(n + 1)(n + 2).$$

There is equality iff there are no other divisors, i.e., when $\frac{n}{2} = p$ is a prime.

ii) $1, 2, \frac{n}{4}, \frac{n}{2}, n$ are distinct divisors. The proof is similar to the case i).

Corollary 2.14. It holds true that:

1) \[ \lim_{n \to \infty} \sup \frac{V(n - 1)}{V(n)} = +\infty; \quad \lim_{n \to \infty} \sup \frac{V(n + 1)}{V(n)} = +\infty \quad (2.38) \]

2) \[ \lim_{p \to \infty} \sup_{p \text{ prime}} \frac{V(p - 1)}{p^2} = +\infty; \quad \lim_{p \to \infty} \sup_{p \text{ prime}} \frac{V(p + 1)}{p^2} = +\infty \quad (2.39) \]

3) \[ \lim_{p \to \infty} \frac{V(p - 1)}{p^2} = 3, \quad (2.40) \]

if one assumes the existence of infinitely many primes $p$ of the form $p = 2q + 1$, where $q$ is a prime.

4) \[ \lim_{n \to \infty} \sup \frac{V(p + 1)}{p^2} = 3, \quad (2.41) \]

if one assumes the existence of infinitely many primes $p$ of the form $p = 2q - 1$, $q$ prime.

Proof. 1) Let $n = p \geq 7$ be a prime. Then $V(p) = 2 \cdot (p + 1)$, while by (2.36) one has $V(p - 1) \geq 3p(p + 1)$. Similarly, for $p \geq 5$ one has $V(p + 1) \geq 3(p + 2)(p + 3)$, so (2.38) follows.

2) Let $p$ be a prime of the form $4k + 1$. Then, by (2.37) one has

$$V(p - 1) = V(4k) \geq \frac{15}{4} \cdot (k + 1)(k + 2)(k + 4), \quad (k \geq 3),$$

so

$$\frac{V(p - 1)}{p^2} \geq \frac{15(k + 1)(k + 2)(k + 4)}{4 \cdot (4k + 1)^2} \to \infty \quad \text{as} \quad k \to \infty.$$  

The similar proof applies to $\frac{V(p + 1)}{p^2}$.

3) As

$$\frac{V(p - 1)}{p^2} \geq \frac{3p(p + 1)}{p^2} = 3 \cdot \left(1 + \frac{1}{p}\right),$$

clearly

$$\lim_{p \to \infty} \frac{V(p - 1)}{p^2} \geq 3.$$  

On the other hand, if $p = 2q + 1$, then $V(p - 1) = 3 \cdot (2q + 1)(2q + 2)$, so

$$\frac{V(p - 1)}{p^2} = \frac{3 \cdot (2q + 1)(2q + 2)}{2(q + 1)^2} \to 3$$

as $q \to \infty$. This proves (2.40).

4) The proof of (2.41) is similar. \[\square\]
Remark 2.15. The existence of infinitely many primes $p$ of the form $p = 2q + 1$ (or $p = 2q - 1$) is one of the difficult open problems of Number theory (see [3, 4]).

The number $V(n - 1)$ behaves differently as $V(n)$, it was shown by the case $n = \text{prime}$. As

$$\frac{\log \log (n - 1)}{\log (n - 1)} \sim \frac{\log \log n}{\log n},$$

by (2.24) of [2], we can write also

$$\lim_{n \to \infty} \sup \log \log V(n - 1) \frac{\log \log n}{\log n} = \log 2. \quad (2.42)$$

One has:

Theorem 2.16. It holds true that:

$$\lim_{p \to \infty} \log \log V(p - 1) \cdot \frac{\log \log p}{\log p} > 0. \quad (2.43)$$

Proof. By the left-hand side of relation (5) of [2] we can write

$$\log \log V(p - 1) > \log d(p - 1) + \log \log(\sqrt{p - 1} + 1). \quad (2.44)$$

Now, by a result of K. Prachar [3] there exists $c > 0$ such that

$$\log d(q - 1) > c \cdot \frac{\log q}{\log \log q} \quad (2.45)$$

for infinitely many primes $q$. This combined with (2.44) implies (2.43). \qed

Theorem 2.17. The series $\sum_{n=1}^{\infty} \frac{1}{V(n)}$ is divergent, while $\sum_{n=1}^{\infty} \frac{1}{(V(n))^{1+a}}$ is convergent for any $a > 0$.

One has the asymptotic formula

$$\sum_{n \leq x} \frac{1}{V(n)} = \frac{1}{2} \log \log x + O(1). \quad (2.46)$$

Proof. As $V(p) = 2(p + 1)$, and

$$\sum_{n=1}^{\infty} \frac{1}{V(n)} > \frac{1}{2} \sum_{p \text{ prime}} \frac{1}{p + 1} > \frac{1}{4} \sum_{p \text{ prime}} \frac{1}{p},$$

the divergence follows by the known divergence of the series $\sum_{p} 1/p$.

Let now consider the series of general term $\frac{1}{(V(n))^{1+a}}$. Clearly,

$$\sum_{n \geq 1} \frac{1}{(V(n))^{1+a}} = \sum_{p \text{ prime}} \frac{1}{(2(p + 1))^{1+a}} + \sum_{n \text{ composite}} \frac{1}{(V(n))^{1+a}}.$$

Now, as $\frac{1}{(2(p + 1))^{1+a}} < \frac{1}{p^{1+a}}$, remark that the series $\sum_{p \text{ prime}} \frac{1}{p^{1+a}}$ is known to be convergent. This follows, e.g., by the remark that, if $p_k$ denotes the $k$-th prime, then $p_k > k$, so

$$\sum_{k \geq 1} \frac{1}{p_k^{1+a}} < \sum_{k \geq 1} \frac{1}{k^{1+a}} = \zeta(1 + a) < \infty$$

for $a > 0$. 717
On the other hand,

\[
\sum_{n \text{ composite}} \frac{1}{(V(n))^{1+a}} \leq \sum_{n \text{ composite}} \frac{1}{(\sqrt{n} + 1)^{d(n)(1+a)}} \leq \sum_{n} \frac{1}{(\sqrt{n} + 1)^{\beta(1+a)}}
\]

\[
< \sum_{n} \frac{1}{n^{\beta(1+a)/2}} < \infty \quad \text{as} \quad 3(1+a)/2 > 1.
\]

For the proof of (2.46) remark that

\[
\sum_{n \leq x} \frac{1}{V(n)} = \frac{1}{2} \sum_{p \text{ prime}} \frac{1}{p+1} + \sum_{n \text{ composite}} \frac{1}{V(n)} \tag{2.47}
\]

As above, it is immediate that the second term of (2.47) is \( < C \), where \( C \) is a positive constant. For the first term of (2.47) however, we will use the known fact that

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1) \tag{2.48}
\]

(see [4]). Now

\[
\sum_{p \leq x} \left( \frac{1}{p} - \frac{1}{p+1} \right) = \sum_{p \leq x} \frac{1}{p(p+1)} < \sum_{p \leq x} \frac{1}{p^2} < \sum_{p \leq x} \frac{1}{n^2} < \pi^2/6,
\]

so

\[
\sum_{p \leq x} \frac{1}{p+1} = \sum_{p \leq x} \frac{1}{p} + O(1),
\]

and by (2.47) the result follows. \( \square \)

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