SPECIAL FUNCTIONS RELATED TO DEDEKIND TYPE DC-SUMS AND THEIR APPLICATIONS

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Abstract In this paper we construct trigonometric functions of the sum $T_p(h, k)$, which is called Dedekind type DC-(Dahee and Changhee) sums. We establish analytic properties of this sum. We find trigonometric representations of this sum. We prove reciprocity theorem of this sums. Furthermore, we obtain relations between the Clausen functions, Polylogarithm function, Hurwitz zeta function, generalized Lambert series ($G$-series), Hardy-Berndt sums and the sum $T_p(h, k)$. We also give some applications related to these sums and functions.

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1 Introduction, Definitions and Notations

In this section, we give some definitions, notations and results related to the Dedekind sums. Firstly we start with the definition of the classical Dedekind sums.

Let

$$((x)) = \begin{cases} x - [x]G - \frac{1}{2}, & \text{if } x \notin \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z} \end{cases}$$
\([x]_G\) being the largest integer \(\leq x\). Let \(h\) and \(k\) be coprime integers with \(k > 0\), the classical Dedekind sum \(s(h, k)\) is defined as follows

\[
s(h, k) = \sum_{a=1}^{k-1} \left( \left( \frac{a}{k} \right) \left( \frac{ha}{k} \right) \right).
\]

The reciprocity law of the classical Dedekind sums is given by

\[
s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12} \left( \frac{h}{k} + \frac{k}{h} + \frac{1}{hk} \right),
\]

where \((h, k) = 1\) and \(h, k \in \mathbb{N} := \{1, 2, 3, \ldots\}\), and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\).

The classical Dedekind sums \(s(h, k)\) arise in the transformation formula the Dedekind-eta function. By using this transformation formula, Dedekind proved reciprocity law of the classical Dedekind sums cf. [18]. For other proofs of reciprocity law of the classical Dedekind sums, see cf. ([23], [41], [1], [3], [7]-[9], [19], [22], [26]), see also the references cited in each of these earlier works.

In the literature of the Dedekind sums, there are several generalizations of the Dedekind sums that involve higher order Bernoulli functions and Euler functions, the reader should look at [23], [1], [7]-[9], [19], [4], [5] and [36] for references and, see also the references cited in each of these earlier works.

In 1950, Apostol ([1], [3]) generalized Dedekind sums as follows:

\[
S_p(h, k) = \sum_{a \equiv k} \frac{a}{k} \overline{B}_p \left( \frac{ah}{k} \right),
\]

where \((h, k) = 1\) and \(h, k \in \mathbb{N}\) and \(\overline{B}_p(x)\) is the \(p\)-th Bernoulli function, which is defined as follows:

\[
\overline{B}_p(x) = B_p(x - [x]_G)
\]

\[
= -p! (2\pi i)^{-p} \sum_{m=-\infty}^{\infty} \int_{m \neq 0}^{\infty} m^{-p} e^{2\pi imx},
\]

where \(B_p(x)\) is the usual Bernoulli polynomials, which are defined by means of the following generating function

\[
\frac{te^{tz}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}, \quad |t| < 2\pi
\]
where \( B_n(0) = B_n \) is denoted the Bernoulli number cf. [1]-[59].

Observe that when \( p = 1 \), the sums \( S_1(h, k) \) are known as the classical Dedekind sums, \( s(h, k) \).

The following theorem proved by Apostol [1]:

**Theorem 1** Let \( (h, k) = 1 \). For odd \( p \geq 1 \), we have

\[
S_p(h, k) = \frac{p!}{(2\pi i)^p} \sum_{m = 1}^{\infty} \frac{1}{m^p} \left( \frac{e^{2\pi i mh/k}}{1 - e^{2\pi i mh/k}} - \frac{e^{-2\pi i mh/k}}{1 - e^{-2\pi i mh/k}} \right). \tag{3}
\]

In [3], Apostol established a connection between the sums \( S_p(h, k) \) and certain finite sums involving Hurwitz zeta functions. By using this relation, he proved the reciprocity law of the sum \( S_p(h, k) \).

By using same motivation of the Dedekind sums, in this paper, we study on infinite series representation of the Dedekind type DC-sum, reciprocity law of this sum and some special functions.

In [31] and [32], Kim defined the first kind \( n \)-th Euler function \( \overline{E}_m(x) \) as follows:

\[
\overline{E}_m(x) = \frac{2(m!)}{(\pi i)^{m+1}} \sum_{n = -\infty}^{\infty} \frac{e^{(2n+1)\pi i x}}{(2n+1)^{m+1}}, \tag{4}
\]

where \( m \in \mathbb{N} \). Hoffman [25] studied on Fourier series of Euler polynomials. He also expressed the values of Euler polynomials at any rational argument in terms of \( \tan x \) and \( \sec x \).

Observe that if \( 0 \leq x < 1 \), then (4) reduces to the first kind \( n \)-th Euler polynomials \( E_n(x) \) which are defined by means of the following generating function

\[
\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \ |t| < \pi. \tag{5}
\]

Observe that \( E_n(0) = E_n \) denotes the first kind Euler number which is given by the following recurrence formula

\[
E_0 = 1 \text{ and } E_n = - \sum_{k=0}^{n} \binom{n}{k} E_k, \tag{6}
\]
Some of them are given by \(1, -\frac{1}{2}, 0, \frac{1}{2}, \cdots\), \(E_n = 2^n E_n\left(\frac{1}{2}\right)\) and \(E_{2n} = 0\), \((n \in \mathbb{N})\) cf. \([28]-[37], [25], [39], [47], [53], [51]\) and see also the references cited in each of these earlier works.

In \([32]\) and \([31]\), by using Fourier transform for the Euler function, Kim derived some formulae related to infinite series and the first kind Euler numbers. For example, \([4]\), and

\[
\sum_{n=1}^{\infty} \frac{1}{(2n+1)^{2m+2}} = \frac{(-1)^{m+1} n^{2m+2} E_{2m+1}}{4(2m+1)!}.
\]  

(7)

Kim \([31]\) gave the following results:

\[
\sec hx = \frac{1}{\cosh x} = \frac{2e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2},
\]

(8a)

where \(E_n^*\) is denoted the second kind Euler numbers. By \((5)\) and \((8a)\), it is easy to see that

\[
E_m^* = \sum_{n=0}^{m} \binom{m}{n} 2^n E_n,
\]

and

\[
E_{2m}^* = -\sum_{n=0}^{m-1} \binom{2m}{2n} E_{2n}^* \text{ cf. [31].}
\]

From the above \(E_0^* = 1, E_1^* = 0, E_2^* = -1, E_3^* = 0, E_4^* = 5, \cdots\), and \(E_{2m+1}^* = 0, (m \in \mathbb{N})\).

The first and the second kind Euler numbers are also related to \(\tan z\) and \(\sec z\).

\[
\tan z = -\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{e^{2iz}}{2i} \left(\frac{2}{e^{2iz} + 1}\right) - \frac{e^{-2iz}}{2i} \left(\frac{2}{e^{-2iz} + 1}\right).
\]
By using (5) and Cauchy product, we have

\[
\tan z = \frac{1}{2i} \sum_{n=0}^{\infty} E_n \frac{(2iz)^n}{n!} \sum_{n=0}^{\infty} \frac{(2iz)^n}{n!} - \frac{1}{2i} \sum_{n=0}^{\infty} E_n \frac{(-2iz)^n}{n!} \sum_{n=0}^{\infty} \frac{(-2iz)^n}{n!}
\]

\[
= \frac{1}{2i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_k \frac{(2iz)^k}{k!} \frac{(2iz)^{n-k}}{(n-k)!} - \frac{1}{2i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_k \frac{(-2iz)^k}{k!} \frac{(-2iz)^{n-k}}{(n-k)!}
\]

\[
= \frac{1}{2i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{E_k}{k!(n-k)!} (2i)^n z^n - \frac{1}{2i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{E_k}{k!(n-k)!} (-2i)^n z^n
\]

\[
= \sum_{j=0}^{\infty} (-1)^n 2^{2j+1} \left( \sum_{k=0}^{2j+1} \binom{2j+1}{k} E_k \right) \frac{z^{2j+1}}{(2j+1)!}
\]

By using (6), we find that

\[
\tan z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} E_{2n+1}}{(2n+1)!} z^{2n+1}, \quad |z| < \pi.
\] (9)

**Remark 2** The other proofs of (9) has also given the references cited in each of these earlier work. In [31], Kim gave another proof of (9). We shall give just a brief sketch as the details are similar to those in [37].

\[
i \tan z = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}
\]

\[
= 1 - \frac{2}{e^{2iz} - 1} + \frac{4}{e^{4iz} - 1}.
\] (10)

From the above

\[z \tan z = \sum_{n=1}^{\infty} (-1)^n \frac{4^n (1 - 4^n) B_{2n}}{(2n)!} z^{2n}.
\]

By using the above, we arrive at (9). Similarly Kim [31] also gave the following relation:

\[\sec z = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} z^{2n}, \quad |z| < \frac{\pi}{2}.
\]

Kim [36] defined the Dedekind type DC (Dahee-Changhee) sums as follows:
**Definition 3** Let $h$ and $k$ be coprime integers with $k > 0$. Then

\[
T_m(h, k) = 2 \sum_{j=1}^{k-1} (-1)^{j-1} \frac{j}{k} E_m\left(\frac{hj}{k}\right),
\]

where $E_m(x)$ denotes the $m$-th (first kind) Euler function.

The sum $T_m(h, k)$ gives us same behavior of the Dedekind sums. Several properties and identities of the sum $T_m(h, k)$ and Euler polynomials were given by Kim [36]. By using these identities, Kim [36] proved many theorems. The most fundamental result in the theory of the Dedekind sums, Hardy-Berndt sums, Dedekind type DC and the other arithmetical sums is the reciprocity law. The reciprocity law can be used as an aid in calculating these sums.

The reciprocity law of the sum $T_m(h, k)$ is given as follows:

**Theorem 4** ([36]) Let $(h, k) = 1$ and $h, k \in \mathbb{N}$ with $h \equiv 1 \mod 2$ and $k \equiv 1 \mod 2$. Then we have

\[
k^p T_p(h, k) + h^p T_p(k, h) = 2 \sum_{u=0}^{k-1} \left(kh(E + \frac{u}{k}) + k(E + h - \frac{hu}{k})\right)^p + (hE + kE)^p + (p + 2)E_p,
\]

where

\[
(hE + kE)^{n+1} = \sum_{j=1}^{n+1} \binom{n+1}{j} h^j E_j k^{n+1-j} E_{n+1-j}.
\]

We summarize the result of this paper as follows:

In Section 2, we construct trigonometric representation of the sum $T_p(h, k)$. We give analytic properties of the sum $T_p(h, k)$.

In Section 3, we give some special functions and their relations. By using these functions, we obtain relations between the sum $T_p(h, k)$, Hurwitz zeta function, Lerch zeta function, Dirichlet series for the polylogarithm function, Dirichlet’s eta function and Clausen functions.

In Section 4, we prove reciprocity law of the sum $T_p(h, k)$.

In Section 5, we find relation between $G$-series (generalized Lambert series) and the sums $T_{2y}(h, k)$.

In Section 6, we investigate relations between Hardy-Berndt sums, the sums $T_{2y}(h, k)$ and the other sums.
2 Trigonometric Representation of the DC-sums

In this section we can give relations between trigonometric functions and the sum $T_p(h, k)$. We establish analytic properties of the sum $T_p(h, k)$. We give trigonometric representation of the sum $T_p(h, k)$.

We now modify (4) as follows:

$$\frac{(\pi i)^{m+1}}{2(m!)} E_m(x) = \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi ix}}{(2n+1)^{m+1}}, \quad \text{if } m + 1 \text{ is odd}$$

$$\frac{2(m!)}{(\pi i)^{m+1}} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x)}{(2n+1)^{m+1}}, \quad \text{if } m + 1 \text{ is even}.$$  (12)

From the above, we have

$$\bar{E}_m(x) = \begin{cases} 
\frac{2(m!)}{(\pi i)^{m+1}} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x)}{(2n+1)^{m+1}}, \text{ if } m + 1 \text{ is odd} \\
\frac{2(m!)}{(\pi i)^{m+1}} \sum_{n=1}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^{m+1}}, \text{ if } m + 1 \text{ is even.} 
\end{cases}$$  (13)

If $m+1$ is even, then $m$ is odd, consequently, (13) reduces to the following relation:

For $m = 2y - 1, \ y \in \mathbb{N}$, 

$$\bar{E}_{2y-1}(x) = 4(-1)^y \frac{(2y-1)!}{\pi^{2y}} \sum_{n=1}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^{2y}}.$$  

If $m + 1$ is odd, then $m$ is odd, hence (13) reduces to the following relation:

For $m = 2y, \ y \in \mathbb{N}$, 

$$\bar{E}_{2y}(x) = 4(-1)^y \frac{(2y)!}{\pi^{2y+1}} \sum_{n=1}^{\infty} \frac{\sin((2n+1)\pi x)}{(2n+1)^{2y+1}}.$$  

Hence, from the above, we arrive at the following Lemma.
Lemma 5 Let $y \in \mathbb{N} \setminus \{1\}$ and $0 \leq x \leq 1$; $y = 1$ and $0 < x < 1$. Then we have

$$E_{2y-1}(x) = \frac{(-1)^y 4(2y - 1)!}{\pi^{2y}} \sum_{n=1}^{\infty} \frac{\cos((2n + 1)\pi x)}{(2n + 1)^{2y}}$$

and

$$E_{2y}(x) = \frac{(-1)^y 4(2y)!}{\pi^{2y+1}} \sum_{n=1}^{\infty} \frac{\sin((2n + 1)\pi x)}{(2n + 1)^{2y+1}}.$$  

In Lemma 5 substituting $0 \leq x < 1$, thus $E_{2y-1}(x)$ and $E_{2y}(x)$ reduce to the Euler polynomials, which are related to Clausen functions, given in Section 3, below.

We now modify the sum $T_m(h, k)$ for odd and even integer $m$. Thus, by (11), we define $T_{2y-1}(h, k)$ and $T_{2y}(h, k)$ sums as follows:

Definition 6 Let $h$ and $k$ be coprime integers with $k > 0$. Then

$$T_{2y-1}(h, k) = 2 \sum_{j=0}^{k-1} (-1)^{j-1} \frac{j}{k} E_{2y-1} \left( \frac{hj}{k} \right),$$  

and

$$T_{2y}(h, k) = 2 \sum_{j=0}^{k-1} (-1)^{j-1} \frac{j}{k} E_{2y} \left( \frac{hj}{k} \right),$$

where $E_{2y-1}(x)$ and $E_{2y}(x)$ denote the Euler functions.

By substituting equation (16) into (17), we have

$$T_{2y-1}(h, k) = \frac{-8(-1)^y (2y - 1)!}{k \pi^{2y}} \sum_{j=1}^{k-1} (-1)^j j \sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi hj(2n + 1)}{k} \right)}{(2n + 1)^{2y}}$$

From the above we have

$$T_{2y-1}(h, k) = \frac{-8(-1)^y (2y - 1)!}{k \pi^{2y}} \sum_{n=1}^{\infty} \frac{1}{(2n + 1)^{2y}} \sum_{j=1}^{k-1} (-1)^j j \cos \left( \frac{\pi hj(2n + 1)}{k} \right).$$  

$$8$$
We next recall from [10] and [21] that

$$
\sum_{j=1}^{k-1} je^{\frac{(2n+1)\pi hj}{k}} = \begin{cases} \\
\frac{k}{e^{(2n+1)\pi hj/k} - 1} \\
\frac{k(k-1)}{2} \text{ if } 2n + 1 \equiv 0(k), \\
\end{cases}
$$

From the above, it is easy to get

$$
\sum_{j=1}^{k-1} (-1)^j je^{\frac{(2n+1)\pi hj}{k}} = \frac{k}{e^{(k+(2n+1)h)\pi k} - 1}.
$$

By using an elementary calculations, we have

$$
\sum_{j=1}^{k-1} (-1)^j j \cos\left(\frac{(2n+1)\pi hj}{k}\right) = -\frac{k}{2}, \quad (20)
$$

and

$$
\sum_{j=1}^{k-1} (-1)^j j \sin\left(\frac{(2n+1)\pi hj}{k}\right) = \frac{k \tan\left(\frac{\pi h(2n+1)}{2k}\right)}{2}, \quad (21)
$$

where $2n+1 \not\equiv 0(k)$. By substituting (20) into (19) and after some elementary calculations, we obtain

$$
T_{2y-1}(h, k) = \frac{8(-1)^y(2y-1)!}{k \pi^{2y}} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^{2y}}.
$$

By substituting (7) into the above, we easily arrive at the following theorem.

**Theorem 7** Let $y \in \mathbb{N}$, then we have

$$
T_{2y-1}(h, k) = 4E_{2y-1}.
$$

By substituting equation (15) into (17), we have

$$
T_{2y}(h, k) = \frac{8(-1)^y(2y)!}{k \pi^{2y+1}} \sum_{j=1}^{k-1} (-1)^j j \sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n+1)hj\pi}{k}\right)}{(2n+1)^{2y+1}}.
$$

By substituting (21) into the above, after some elementary calculations, we arrive at the following theorem.

9
Theorem 8 Let $h$ and $k$ be coprime positive integers. Let $y \in \mathbb{N}$, then we have
\[
T_{2y}(h, k) = \frac{4(-1)^y(2y)!}{\pi^{2y+1}} \sum_{n=1 \atop 2n+1 \not\equiv 0(\text{mod } k)}^{\infty} \tan\left(\frac{h\pi(2n+1)}{2k}\right) \frac{1}{(2n+1)^{2y+1}}.
\]  

3 DC-sums related to special functions

In this section, we give relations between DC-sums and some special functions.

In [58], Srivastava and Choi gave many applications of the Riemann zeta function, Hurwitz zeta function, Lerch zeta function, Dirichlet series for the polylogarithm function and Dirichlet’s eta function. In [24], Guillera and Sandow obtained double integral and infinite product representations of many classical constants, as well as a generalization to Lerch’s transcendent of Hadjicostas’s double integral formula for the Riemann zeta function, and logarithmic series for the digamma and Euler beta functions. They also gave many applications. The Lerch transcendent $\Phi(z, s, a)$ (cf. e.g. [58, p. 121 et seq.], [24]) is the analytic continuation of the series
\[
\Phi(z, s, a) = \frac{1}{a^s} + \frac{z}{(a+1)^s} + \frac{z}{(a+2)^s} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},
\]
which converges for $(a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z| < 1; \Re(s) > 1$ when $|z| = 1)$ where as usual
\[
\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}, \mathbb{Z}^- = \{-1, -2, -3, \ldots\}.
\]

$\Phi$ denotes the familiar Hurwitz-Lerch Zeta function (cf. e.g. [58, p. 121 et seq.]). Relations between special function and the function $\Phi$ are given as follows [24]:

Special cases include the analytic continuations of the Riemann zeta function
\[
\Phi(1, s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1,
\]
the Hurwitz zeta function
\[ \Phi(1, s, a) = \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \Re(s) > 1, \]
the alternating zeta function (also called Dirichlet’s eta function \( \eta(s) \))
\[ \Phi(-1, s, 1) = \zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \]
the Dirichlet beta function
\[ \Phi(-1, s, \frac{1}{2}) \frac{2^s}{2^s} = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s}, \]
the Legendre chi function
\[ z\Phi(z^2, s, \frac{1}{2}) = \chi_s(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)^s}, (|z| \leq 1; \Re(s) > 1), \]
the polylogarithm
\[ z\Phi(z, n, 1) = \text{Li}_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^m} \]
and the Lerch zeta function (sometimes called the Hurwitz-Lerch zeta function)
\[ L(\lambda, \alpha, s) = \Phi(e^{2\pi i \lambda}, s, \alpha), \]
which is a special function and generalizes the Hurwitz zeta function and polylogarithm cf. ([2], [6], [16], [15], [14], [13], [24], [55], [56], [58], [29]) and see also the references cited in each of these earlier works.

By using (12), we give relation between the Legendre chi function \( \chi_s(z) \), and the function \( E_m(x) \) by the following corollary:

**Corollary 9** Let \( m \in \mathbb{N} \). Then we have
\[ E_m(x) = \frac{2(m!)^m}{(\pi i)^{m+1}} \left( (-1)^{m+1} \chi_{m+1}(e^{-\pi i x}) + \chi_{m+1}(e^{\pi i x}) \right). \]
In [56, p. 78, Theorem B], Srivastava proved the following formulae which are related to Hurwitz zeta function, trigonometric functions and Euler polynomials:

\[ E_{2y-1}(\frac{p}{q}) = (-1)^y \frac{4(2y-1)!}{(2q\pi)^2y} \sum_{j=1}^{q} \zeta(2y, \frac{2j-1}{q}) \cos(\frac{\pi p(2j-1)}{q}), \]

where \( y, q \in \mathbb{N}, p \in \mathbb{N}_0, 0 \leq p \leq q, \) and

\[ E_{2y}(\frac{p}{q}) = (-1)^y \frac{4(2y)!}{(2q\pi)2y+1} \sum_{j=1}^{q} \zeta(2y+1, \frac{2j-1}{2q}) \sin(\frac{\pi p(2j-1)}{q}), \]

where \( y, q \in \mathbb{N}, p \in \mathbb{N}_0, 0 \leq p \leq q \) and \( \zeta(s, x) \) denotes the Hurwitz zeta function. By substituting \( p = 0 \) in the above, then we have

\[ E_{2y-1}(0) = (-1)^y \frac{4(2y-1)!}{(2q\pi)^2y} \sum_{j=1}^{q} \zeta(2y, \frac{2j-1}{q}). \]

By using the above equation, we modify the sum \( T_{2y-1}(h, k) \) as follows:

**Corollary 10** Let \( y, q \in \mathbb{N}. \) Then we have

\[ T_{2y-1}(h, k) = (-1)^y \frac{4(2y-1)!}{(2q\pi)^2y} \sum_{j=1}^{q} \zeta(2y, \frac{2j-1}{q}). \]

In [17], Choi et al. gave relations between the Clausen function, multiple gamma function and other functions. The higher-order Clausen function \( Cl_n(t) \) (see [58], [17, Eq-(4.15)]) be defined, for all \( n \in \mathbb{N} \setminus \{1\}, \) by

\[ Cl_n(t) = \begin{cases} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k^n} \text{ if } n \text{ is even}, \\ \sum_{k=1}^{\infty} \frac{\cos(kt)}{k^n} \text{ if } n \text{ is odd}. \end{cases} \]

The following functions are related to the higher-order Clausen function (cf. [55], [13, Eq-(5) and Eq-(6)])

\[ S(s, x) = \sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{(2n+1)^s} \quad (24) \]
and
\[ C(s, x) = \sum_{n=1}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^s}. \] (25)

In [55], Srivastava studied on the functions \( S(s, x), C(s, x) \). When \( x \) is a rational multiple of \( 2\pi \), he gave the functions \( S(s, x), C(s, x) \) in terms of Hurwitz zeta functions. In [13 Eq-(5) and Eq-(6)], Cvijovic studied on the functions \( S(s, x), C(s, x) \). He gave many applications of this function.

Espinosa and Moll [20] gave relation between the functions \( S(s, x), C(s, x) \) and \( Cl_{2n}(t) \) as follows:
For \( 0 \leq q \leq 1 \)
\[ S(2m+2, q) = Cl_{2m+2}(2\pi q), \]
\[ S(2m+1, q) = \frac{(-1)^{m+1}(2\pi)^{2m+1}}{2(2m+1)!} B_{2m+1}(q), \]
and
\[ C(2m+1, q) = Cl_{2m+1}(2\pi q), \]
\[ C(2m+2, q) = \frac{(-1)^{m}(2\pi)^{2m+2}}{2(2m+2)!} B_{2m+2}(q). \]

Setting \( x = \frac{h\pi}{k} \) and \( s = 2y \) and \( s = 2y + 1 \) in (24) and (25) respectively, than combine (18) and (22), we obtain the next corollary.

**Corollary 11** Let \( h \) and \( k \) be coprime positive integers. Let \( y \in \mathbb{N} \). Then we have
\[ T_{2y-1}(h, k) = \frac{-8(-1)^{y}(2y-1)!}{k\pi^{2y}} \sum_{j=1}^{k-1} (-1)^{j} j C(2y, \frac{hj\pi}{k}), \]
and
\[ T_{2y}(h, k) = \frac{8(-1)^{y}(2y)!}{k\pi^{2y+1}} \sum_{j=1}^{k-1} (-1)^{j} j S(2y+1, \frac{hj\pi}{k}). \]

Trickovic et al. [60, p. 443, Eq-(3)] gave relations between the Clausen function and polylogarithm \( Li_\alpha(z) \). They also gave the following relations:
For \( \alpha > 0 \)
\[ S(\alpha, x) = \frac{i}{2} \left( \left( Li_\alpha(e^{-ix}) - \frac{1}{2\alpha} Li_\alpha(e^{-2ix}) \right) - \left( Li_\alpha(e^{ix}) - \frac{1}{2\alpha} Li_\alpha(e^{2ix}) \right) \right) \] (26)
and
\[
\mathcal{C}(\alpha, x) = \frac{1}{2} \left( \left( \text{Li}_\alpha(e^{-ix}) - \frac{1}{2\alpha} \text{Li}_\alpha(e^{-2ix}) \right) + \left( \text{Li}_\alpha(e^{ix}) - \frac{1}{2\alpha} \text{Li}_\alpha(e^{2ix}) \right) \right).
\]

By substituting \( x = \frac{h j \pi}{k} \) into (27) and (26); and combine (18) and (22), respectively; after some elementary calculations, we easily find the next results.

**Corollary 12** Let \( h \) and \( k \) be coprime positive integers. Let \( y \in \mathbb{N} \). Then we have
\[
T_{2y-1}(h, k) = -\frac{4(-1)^y(2y - 1)!}{k \pi^{2y}} \sum_{j=1}^{k-1} (-1)^j j \times
\left( \text{Li}_{2y}(e^{-\frac{h j \pi}{k}}) + \text{Li}_{2y}(e^{\frac{h j \pi}{k}}) - \frac{\text{Li}_{2y}(e^{\frac{2h j \pi}{k}}) + \text{Li}_{2y}(e^{-\frac{2h j \pi}{k}})}{2^{2y}} \right),
\]
and
\[
T_{2y}(h, k) = \frac{4i(-1)^y(2y)!}{k \pi^{2y+1}} \sum_{j=1}^{k-1} (-1)^j j \times
\left( \text{Li}_{2y+1}(e^{-\frac{h j \pi}{k}}) - \text{Li}_{2y+1}(e^{\frac{h j \pi}{k}}) + \frac{\text{Li}_{2y+1}(e^{\frac{2h j \pi}{k}}) - \text{Li}_{2y+1}(e^{-\frac{2h j \pi}{k}})}{2^{2y+1}} \right).
\]

## 4 Reciprocity Law

The first proof of reciprocity law of the Dedekind sums does not contain the theory of the Dedekind eta function related to Rademacher [40]. The other proofs of the reciprocity law of the Dedekind sums were given by Grosswald and Rademacher [23]. Berndt [7]-[10] gave various types of Dedekind sums and their reciprocity laws. Berndt’s methods are of three types. The first method uses contour integration which was first given by Rademacher [40]. This method has been used by many authors for example Isaki [26], Grosswald [22], Hardy [43], his method is a different technique in contour integration. The second method is the Riemann-Stieltjes integral, which was invented by Rademacher [41]. The third method of Berndt is (periodic) Poisson summation formula. For the method and technique see also the references cited in each of these earlier works.
The famous property of the all arithmetic sums is the reciprocity law. In this section, by using contour integration, we prove reciprocity law of (23). Our method is same as [7] and also for example cf. ([40], [7]-[10], [22], [23]).

The initial different proof of the following reciprocity theorem is due to Kim [36], who first defined $T_y(h, k)$ sum.

**Theorem 13** Let $h, k, y \in \mathbb{N}$ with $h \equiv 1 \mod 2$ and $k \equiv 1 \mod 2$ and $(h, k) = 1$. Then we have

$$kh^{2y+1}T_{2y}(h, k) + hk^{2y+1}T_{2y}(k, h) = (-1)^y\pi^{2y-1}\Gamma(2y+1)E_{4y+1} + 4\pi^2(2y)! \sum_{a=0}^{y-1} \frac{E_{2a+1}E_{2y-2a-1}h^{2a+2}k^{2y-2a}}{(2a+1)!(2y-2a+1)!},$$

where $\Gamma(n+1) = n!$ and $E_n$ denote Euler gamma function and first kind Euler numbers, respectively.

**Proof.** We shall give just a brief sketch as the details are similar to those in [7 see Theorem 4.2], [10 see Theorem 3], [22] or [23]. For the proof we use contour integration method. So we define

$$F_y(z) = \frac{\tan \pi h z \tan \pi k z}{z^{2y+1}}.$$

Let $C_N$ be a positive oriented circle of radius $R_N$, with $1 \leq N < \infty$, centred at the origin. Assume that the sequence of radii $R_N$ is increasing to $\infty$. $R_N$ is chosen so that the circles always at a distance greater than some fixed positive integer number from the points $\frac{m}{2h}$ and $\frac{n}{2k}$, where $m$ and $n$ are integers.

Let

$$I_N = \frac{1}{2\pi i} \int_{C_N} \frac{\tan \pi h z \tan \pi k z}{z^{2y+1}} dz.$$

From the above, we get

$$I_N = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\tan (\pi h R_N e^{i\theta}) \tan (\pi k R_N e^{i\theta})}{(R_N e^{i\theta})^{2y}} d\theta.$$

By $C_N$, if $R_N \to \infty$, then $\tan (R_N e^{i\theta})$ is bounded. Consequently, we easily see that

$$\lim_{N \to \infty} I_N = 0 \text{ as } R_N \to \infty.$$
Thus, on the interior $C_N$, the integrand of $I_N$ that is $F_y(z)$ has simple poles at $z_1 = \frac{2m+1}{2h}$, $-\infty < m < \infty$, and $z_2 = \frac{2n+1}{2k}$, $-\infty < n < \infty$. If we calculate the residues at the $z_1$ and $z_2$, we easily obtain respectively as follows

\[-\frac{2^{2y+1}k^{2y}}{\pi(2m+1)^{2y+1}} \tan\left(\frac{(2m+1)\pi h}{2k}\right), \quad -\infty < m < \infty\]

and

\[-\frac{2^{2y+1}h^{2y}}{\pi(2n+1)^{2y+1}} \tan\left(\frac{(2n+1)\pi k}{2h}\right), \quad -\infty < n < \infty.\]

If $h$ and $k$ is odd integers, then $F_y(z)$ has double poles at $z_3 = \frac{2j+1}{2}$, $-\infty < j < \infty$. Thus the residue is easily found to be

\[-\frac{(2y + 1)2^{2y+1}}{2(2j+1)^{4y+2}\pi^2hk}, \quad -\infty < j < \infty.\]

The integrand of $I_N$ has pole of order $2y + 1$ at $z_4 = 0$, $y \in \mathbb{N}$. Recall the familiar Taylor expansion of $\tan z$ in (9). By straight-forward calculation, we find the residues at the $z_4$ as follows

\[- \frac{(2y+1)2^{2y+1}}{2(2j+1)^{4y+2}\pi^2hk}, \quad -\infty < j < \infty.\]

Now we are ready to use residue theorem, hence we find that

\[
I_N = -\frac{2^{2y+1}h^{2y}}{\pi} \sum_{\frac{2m+1}{2h} < R_N} \tan\left(\frac{(2m+1)\pi k}{2h}\right) - \frac{2^{2y+1}k^{2y}}{\pi} \sum_{\frac{2n+1}{2k} < R_N} \tan\left(\frac{(2n+1)\pi k}{2h}\right) - \frac{(2y + 1)2^{2y+1}}{2(2j+1)^{4y+2}\pi^2hk} \sum_{j=-\infty}^{\infty} \frac{1}{(2j+1)^{4y+2}} + (-1)^y (2\pi)^{2y+2} \sum_{a=0}^{y-1} E_{2a+1} E_{2y-2a-1} h^{2a+1} k^{2y-2a-1} \frac{1}{(2a+1)!}\]

By using (7) and letting $N \to \infty$ into the above, after straight-forward calculations, we arrive at the desired result. 

**Remark 14** We also recall from [42, pp. 20, Eq-(11.2)-(11-3)] that

\[
\tan z = \sum_{k=1}^{\infty} T_k \frac{z^{2k-1}}{(2k-1)!},
\]

\[\text{(28)}\]
where
\[ T_k = (-1)^{k-1} \frac{B_{2k}}{(2k)} (2^{2k} - 1) 2^{2k}. \]

The integrand of \( I_N \) has pole of order \( 2y + 1 \) at \( z_4 = 0, \ y \in \mathbb{N} \). Recall the familiar Taylor expansion of \( \tan z \) in (28). By straighforward calculation, we find the residues at the \( z_4 \) as follows
\[
\pi^{2y} \sum_{a=0}^{y+1} \frac{T_a T_{y-a+1}}{(2a-1)!(2y-2a-1)!} h^{2a-1} k^{2y-2a+1}.
\]

Thus we modify Theorem 13 as follows:
\[
kh^{2y+1} T_{2y}(h, k) + hk^{2y+1} T_{2y}(k, h) = 
\frac{(-1)^y \pi^{2y} \Gamma(2y + 1)}{2\Gamma(4y + 2)} E_{4y+1} + \frac{(-1)^y (2y)!}{4^y} \sum_{a=0}^{y+1} \frac{T_a T_{y-a+1}}{(2a-1)!(2y-2a-1)!} h^{2a-1} k^{2y-2a+1}.
\]

We now give relation between Hurwitz zeta function, \( \tan z \) and the sum \( T_{2y}(h, k) \).

Hence, substituting \( n = rk + j, \ 0 \leq r \leq \infty, \ 1 \leq j \leq k \) into (23), and recalling that \( \tan(\pi + \alpha) = \tan \alpha \), then we have
\[
T_{2y}(h, k) = \frac{4(-1)^y (2y)!}{\pi^{2y+1} (2k)^{2y+1}} \sum_{j=1}^{k} \tan(\frac{\pi h (2j + 1)}{2k}) \sum_{r=0}^{\infty} \frac{1}{(r + \frac{2j+1}{2k})^{2y+1}}.
\]

where \( \zeta(s, x) \) denotes the Hurwitz zeta function. Thus we arrive at the following theorem:

**Theorem 15** Let \( h \) and \( k \) be coprime positive integers. Let \( y \in \mathbb{N} \). Then we have
\[
T_{2y}(h, k) = \frac{4(-1)^y (2y)!}{(2k\pi)^{2y+1}} \sum_{j=1}^{k} \tan(\frac{\pi h (2j + 1)}{2k}) \zeta(2y + 1, \frac{2j+1}{2k}). \tag{29}
\]
5 \textbf{G-series (Generalized Lambert series) related to DC-sums}

The main purpose of this section is to give relation between G-series and the sums $T_{2y}(h,k)$.

By using (10), we have

$$i \tan z = \frac{e^{2iz}}{1 + e^{2iz}} - \frac{e^{-2iz}}{1 + e^{-2iz}}.$$  \hfill (30)

We recall in [44] that

$$\frac{e^{2iz}}{1 + e^{2iz}} = i \tan z + \frac{e^{-2iz}}{1 + e^{-2iz}}.$$  \hfill (31)

Hence setting $2iz = \frac{h \pi i}{k}$, with $(h,k) = 1, n \in \mathbb{N}$ in (30) with (23), we obtain the following corollary:

\textbf{Corollary 16} \textbf{Let} $h$ \textbf{and} $k$ \textbf{be coprime positive integers. Let} $y \in \mathbb{N}$, \textbf{then we have}

$$T_{2y}(h,k) = \frac{4i(-1)^{y+1}(2y)!}{(2k\pi)^{2y+1}} \sum_{n=1}^{\infty} \frac{1}{(2n + 1)^{2y+1}} \left( e^{\frac{h \pi i (2n+1)}{k}} - e^{\frac{-h \pi i (2n+1)}{k}} \right),$$

where $i = \sqrt{-1}$.

The above corollary give us the sums $T_{2y}(h,k)$ are related to G-series.

In [61], Trahan defined the \textit{G-series (or Generalized Lambert series)} as follows:

$$G(z) = \sum \frac{a_n z^n}{1 + c_n z^n},$$ \hfill (33)

where the coefficients $a_n$ and $c_n$ are complex numbers and $a_n c_n \neq -1$. A G-series is a \textit{power series} if, for all $n$, $c_n = 0$ and a \textit{Lambert series} if, if, for all $n$, $c_n = -1$. In the literature a G-series is usually considered as a generalized Lambert series. The Lambert series, first studied by J. H. Lambert, is analytic at the origin and has a power series expansion at the origin. For $|z| < 1$ J. H. Lambert found that

$$\sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} \tau_n z^n = z + 2z^2 + 2z^3 + 3z^4 + 2z^5 + 4z^6 + \ldots,$$
where $\tau_n$ is the number of divisors of $n$ (cf., e.g., [61]) and see also the references cited in each of these earlier works.

**Theorem 17** ([61, p. 29, Theorem A and Theorem B])

a) If $|z| < \frac{1}{\lim_{n \to \infty} \sqrt{|c_n|}}$, then the $G$-series $\sum \frac{a_n z^n}{1 + c_n z^n}$ converges if and only if the power series $\sum a_n z^n$ converges.

b) If $|z| > \frac{1}{\lim_{n \to \infty} \sqrt{|c_n|}}$ and $c_n \neq 0$, then the $G$-series $\sum \frac{a_n z^n}{1 + c_n z^n}$ converges if and only if the power series $\sum \frac{a_n}{c_n}$ converges.

**Theorem 18** ([61, p. 30, Theorem 2 and Theorem 3])

a) Assume $|z| < \frac{1}{\lim_{n \to \infty} \sqrt{|a_n|}}$. If there is no subsequence of $\{c_n z^n\}$ which has limit $-1$, then the $G$-series $\sum \frac{a_n z^n}{1 + c_n z^n}$ converges absolutely.

b) If $|z| > \frac{1}{\lim_{n \to \infty} \sqrt{|c_n|}}$ and $c_n \neq 0$ for all $n$, then the $G$-series $\sum \frac{a_n z^n}{1 + c_n z^n}$ converges (absolutely) if and only if the series $\sum \frac{a_n}{c_n}$ converges (absolutely).

By setting $a_n = \frac{1}{n^{2y+1}}, y \in \mathbb{N}, c_n = 1$ in (33) and using Theorem 17 and Theorem 18, we obtain the following relation:

$$G(e^{\pi i z}) - 2^{-2y} G(e^{2\pi i z}) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^{2y+1}} \left( \frac{e^{(2n+1)\pi i z}}{1 + e^{(2n+1)\pi i z}} \right),$$

(34)

and

$$G(e^{-\pi i z}) - 2^{-2y} G(e^{-2\pi i z}) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^{2y+1}} \left( \frac{e^{-(2n+1)\pi i z}}{1 + e^{-(2n+1)\pi i z}} \right).$$

(35)

By substituting (34) and (35) into (32), we arrive at the following theorem. The next theorem give us relation between $T_{2y}(h, k)$ sum and $G$-series.

**Theorem 19** Let $h$ and $k$ be coprime positive integers. If $y \in \mathbb{N}$, then we have

$$T_{2y}(h, k) = \frac{4i(-1)^{y+1}(2y)!}{(2\pi)^{2y+1}} \left( G\left(\frac{h \pi i}{k}\right) - G\left(-\frac{h \pi i}{k}\right) + G\left(e^{-\frac{2h \pi i}{k}}\right) - G\left(e^{\frac{2h \pi i}{k}}\right) \right).$$

**Remark 20** We recall from [44], [45] and [46] that relations between Dedekind sums, Hardy-Berndt sums and Lambert series were given by the author and see also the references cited in each of these earlier works.
6 Some Applications

In (23) if \( h \) and \( k \) are odd and \( y = 0 \), then \( T_{2y}(h, k) \) reduces to the Hardy-Berndt sum \( S_5(h, k) \).

Recently Hardy sums (Hardy-Berndt sums) have been studied by many mathematicians ([43], [10], [21], [59], [49], [44]) and see also the references cited in each of these earlier works. Hardy-Berndt sum \( s_5(h, k) \) is defined as follows:

Let \( h \) and \( k \) be integers with \( (h, k) = 1 \). Then

\[
s_5(h, k) = \sum_{j=1}^{k} (-1)^{j+\lfloor \frac{hj}{k} \rfloor} G((\frac{j}{k})).
\]  

(36)

From the above, recall from [10] that, we have

\[
s_5(h, k) = \sum_{j=1}^{k} (-1)^{j} \frac{j}{k} (-1)^{\lfloor \frac{hj}{k} \rfloor}.
\]  

(37)

By using the well-known Fourier expansion

\[
(-1)^{[x]} G = \frac{4}{\pi} \sum_{n=0}^{\infty} \sin((2n+1)\pi x) \frac{2n+1}{2n+1} \text{ cf. (}[10], [21]\)
\]

into (37), we get

\[
s_5(h, k) = \frac{4}{k\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{j=1}^{k} (-1)^{j} j \sin\left(\frac{(2n+1)\pi hj}{k}\right).
\]

By substituting (22) into the above, we immediately find the following result.

**Lemma 21** Let \( h \) and \( k \) are odd with \( (h, k) = 1 \). Then we have

\[
T_0(h, k) = 2s_5(h, k).
\]

By using Lemma 21 and Theorem 8, we arrive at the following theorem.

**Theorem 22** Let \( h \) and \( k \) are odd with \( (h, k) = 1 \). Then we have

\[
S_5(h, k; y) = \frac{T_{2y}(h, k)}{2}.
\]
**Remark 23** Substituting $y = 0$ into Theorem 22, we get $s_5(h, k) = 2S_5(h, k; 0)$. Consequently, the sum $T_{2y}(h, k)$ give us generalized Hardy-Berndt sum $s_5(h, k)$.

In [52], the author defined that

$$Y(h, k) = 4k s_5(h, k),$$

where $h$ and $k$ are odd with $(h, k) = 1$. Thus from Lemma 21 we have the following corollary.

**Corollary 24** Let $h$ and $k$ are odd with $(h, k) = 1$. Then we have

$$T_0(h, k) = \frac{Y(h, k)}{2k}.$$ 

Observe that the sum $T_{2y}(h, k)$ also give us generalization of the sum $Y(h, k)$.

**Remark 25** Elliptic Apostol-Dedekind sums have been studied by many authors. Bayad [4], constructed multiple elliptic Dedekind sums as an elliptic analogue of Zagier’s sums multiple Dedekind sums. In [54], Simsek et al. defined elliptic analogue of the Hardy sums. By using same method in [4], elliptic analogue of the sum $T_m(h, k)$ may be defined. In this paper, we do not study on elliptic analogue of the sum $T_m(h, k)$. By using p-adic q-Volkenborn integral, in [27] and [28], Kim defined p-adic q-Dedekind sums. In [49], [48], [52], [53], we defined q-Dedekind type sums, q-Hardy-Berndt type sums and p-adic q-Dedekind sums. By using same method, p-adic q-analogue of the sum $T_m(h, k)$ may be defined.

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