Pseudomonads and Descent

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PhD Thesis | Tese de Doutoramento

September 2017
Acknowledgements

The first year of my PhD studies was supported by a research grant of CMUC, Centre for Mathematics of the University of Coimbra, under the project Pest-C/MAT/UI0324/2011, co-funded by FCT, FEDER and COMPETE.

The research was supported by CNPq, National Council for Scientific and Technological Development – Brazil (245328/2012-2), and by the CMUC – UID/MAT/00324/2013, funded by FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

I would like to thank the members of the CMUC for contributing to create such an inspiring, helpful and productive atmosphere which surely positively influenced me towards my work. I am specially grateful to my supervisor Maria Manuel Clementino for her patience, encouragement, insightful lessons and useful pieces of advice.
Abstract

This thesis consists of one introductory chapter and four single-authored papers written during my PhD studies, with minor adaptations. The original contributions of the papers are mainly within the study of pseudomonads and descent objects, including applications to descent theory, commutativity of weighted bilimits, coherence and (presentations of) categorical structures.

In Chapter 1, we give a glance of the scope of our work and briefly describe elements of the original contributions of each paper, including some connections between them. We also give a brief exposition of our main setting, which is 2-dimensional category theory. In this direction: (1) we give an exposition on the doctrinal adjunction, focusing on the Beck-Chevalley condition as used in Chapter 3, (2) we apply the results of Chapter 5 in a generalized setting of the formal theory of monads and (3) we apply the biadjoint triangle theorem of Chapter 4 to study (pseudo)exponentiable pseudocoalgebras.

Chapter 2 corresponds to the paper Freely generated n-categories, coinserters and presentations of low dimensional categories, DMUC 17-20 or arXiv:1704.04474. We introduce and study presentations of categorical structures induced by \((n+1)\)-computads and groupoidal computads. In this context, we introduce the notion of deficiency and presentations of groupoids via computads. We compare the resulting notions with those induced by monads together with a finite measure of objects. In particular, we find our notions to generalize the usual ones. One important feature of this paper is that we show that several freely generated structures are naturally given by coinserters. After recalling how the category freely generated by a graph \(G\) internal to \(\text{Set}\) is given by the coinserter of \(G\), we introduce higher icons and present the definitions of \(n\)-computads via internal graphs of the 2-category \(n\text{Cat}\) of \(n\)-categories, \(n\)-functors and \(n\)-icons. Within this setting, we show that the \(n\)-category freely generated by an \(n\)-computad is also given by a coinserter. Analogously, we demonstrate that the geometric realization of a graph \(G\) consists of a left adjoint functor \(\mathcal{F}_{\text{Top}_1}: \text{grph} \to \text{Top}\) given objectwise by the topological coinserter. Furthermore, as a fundamental tool to study presentation of thin and locally thin categorical structures, we give a detailed construction of a 2-dimensional analogue of \(\mathcal{F}_{\text{Top}_1}\), denoted by \(\mathcal{F}_{\text{Top}_2}: \text{cmp} \to \text{Top}\). In the case of group presentations, \(\mathcal{F}_{\text{Top}_2}\) formalizes the Lyndon-van Kampen diagrams. Finally, we sketch a construction of the 3-dimensional version \(\mathcal{F}_{\text{Top}_3}\) which associates a 3-dimensional CW-complex to each 3-computad.

Chapter 3 corresponds to the article Pseudo-Kan Extensions and descent theory, arXiv:1606.04999 under review. We develop and employ results on idempotent pseudomonads to get theorems on the general setting of descent theory, which, in our perspective, is the study of the image of pseudomonadic pseudofunctors. After giving a direct approach to prove an analogue of Fubini’s Theorem for weighted bilimits and constructing pointwise pseudo-Kan extensions, we employ the results on pseudomonadic pseudofunctors to get theorems on commutativity of bilimits. In order to use these results as the
main framework to deal with classical descent theory in the context of [25], we prove that the descent category (object) of a pseudocosimplicial category (object) is its conical bilimit. We use, then, this formal approach of commutativity of bilimits to (1) recast classical theorems of descent theory, (2) prove generalizations of such theorems and (3) get new results of descent theory. In this direction, we give formal proofs of transfer theorems, embedding theorems, a pseudopullback theorem, a Galois Theorem and the Bénabou-Roubaud Theorem. We also apply the pseudopullback theorem to detect effective descent morphisms in suitable categories of enriched categories in terms of (the embedding in) internal categories.

Chapter 4 corresponds to the article On Biadjoint Triangles, published in Theory and Applications of Categories, Vol 31, N. 9 (2016). The main contributions are the biadjoint triangle theorems, which have many applications in 2-dimensional category theory. Examples of which are given in this same paper: reproving the Pseudomonadicity characterization of [38], improving results on the 2-monadic approach to coherence of [3, 36, 49], improving results on lifting of biadjoints of [3] and introducing the suitable concept of pointwise pseudo-Kan extension.

Chapter 5 corresponds to the article On lifting of biadjoints and lax algebras, to appear in Categories and General Algebraic Structures with Applications. It can be seen as a complement of the precedent chapter, since it gives further theorems on lifting of biadjoints provided that we can describe the categories of morphisms of a certain domain in terms of weighted (bi)limits. This approach, together with results on lax descent objects and lax algebras, allows us to get results of lifting of biadjoints involving (full) sub-2-categories of the 2-category of lax algebras. As a consequence, we complete our treatment of the 2-monadic approach to coherence via biadjoint triangle theorems.
Resumo

Esta tese consiste em um capítulo introdutório e quatro artigos de autoria única, escritos durante os meus estudos de doutoramento. As contribuições originais dos artigos estão principalmente dentro do contexto do estudo de pseudomônadas e objetos de descida, com aplicações à teoria da descida, comutatividade de bilimites ponderados, coerência e apresentações de estruturas categoriais.

No Capítulo 1, introduzimos aspectos do escopo do trabalho e descrevemos alguns elementos das contribuições originais de cada artigo, incluindo interrelações entre elas. Damos também uma exposição básica sobre o principal assunto da tese, nomeadamente, teoria das categorias de dimensão 2. Nesse sentido, (1) introduzimos adjunção doutrinal, focando na condição de Beck-Chevalley, com a perspectiva adotada no Capítulo 3, (2) aplicamos resultados do Capítulo 5 em um contexto generalizado da teoria formal das mónadas e (3) aplicamos o teorema de triângulos biadjuntos do Capítulo 4 para estudar pseudocoalgebras (pseudo)exponenciáveis.

O Capítulo 2 corresponde ao artigo Freely generated n-categories, coinserters and presentations of low dimensional categories, DMUC 17-20 ou arXiv:1704.04474. Neste trabalho, introduzimos e estudamos apresentações de estruturas categoriais induzidas por \((n+1)\)-computadas e computadas grupoidais. Introduzimos a noção de deficiência de grupóides via computadas. Comparamos, então, as noções resultantes com as noções induzidas por mónadas junto com medidas finitas de objetos. Em particular, concluímos que nossas noções generalizam as noções clássicas. Outras contribuições do artigo consistiram em mostrar que as propriedades universais de várias estuturas livremente geradas podem ser descritas por coinserções. Começamos por relembrar que as categorias livremente geradas são dadas por coinserções de grafos e, então, introduzimos icons de dimensão alta e apresentamos as definições de \(n\)-computadas via grafos internos da 2-categoria \(n\text{-Cat}\) de \(n\)-categorias, \(n\)-functores e \(n\)-icons. Nesse caso, mostramos que a \(n\)-categoria livremente gerada por uma \(n\)-computada é a sua coinserção. Análogamente, demonstramos que a realização geométrica de um grafo \(G\) é parte de um functor adjunto à esquerda \(\mathcal{F}_{\text{Top}}:\text{grph} \to \text{Top}\) definido objeto a objeto pela coinserção topológica. Além disso, como uma ferramenta fundamental para o estudo de estruturas categoriais finas e localmente finas, apresentamos uma construção detalhada de um análogo de dimensão 2 de \(\mathcal{F}_{\text{Top}}:\text{grph} \to \text{Top}\), denotado por \(\mathcal{F}_{\text{Top}}:\text{cmp} \to \text{Top}\). No caso de grupos, \(\mathcal{F}_{\text{Top}}\) formaliza e, portanto, generaliza o diagrama de Lyndon-van Kampen. Finalizamos o capítulo dando uma construção de uma versão em dimensão 3, denotada por \(\mathcal{F}_{\text{Top}}\), que associa um CW-complexo de dimensão 3 para cada 3-computada.

O Capítulo 3 corresponde ao artigo Pseudo-Kan Extensions and descent theory, em revisão para publicação. Desenvolvemos e aplicamos resultados sobre pseudomônadas idempotentes, obtendo teoremas no contexto geral da teoria da descida que, em nossa perspectiva, é o estudo da imagem de pseudofunctores pseudomonádicos. Depois de apresentar uma prova direta do teorema de Fubini...
para bilimites ponderados e de construir pseudo-extensões de Kan, aplicamos os resultados sobre pseudomónadas para provar teoremas sobre comutatividade de bilimites ponderados. Com o objetivo de usar tais resultados como base para lidar com a teoria da descida clássica no contexto de [25], provamos que a categoria (objeto) de descida de uma categoria (objeto) pseudocosimplicial é seu bilimite cónico. Usamos, então, esse tratamento formal de comutatividade de bilimites para (1) recuperar teoremas clássicos da teoria da descida, (2) provar generalizações desses teoremas e (3) obter novos resultados de teoria da descida. Nesse sentido, apresentamos provas formais (de generalizações) de teoremas de transferência, de teoremas de mergulho, do teorema de Galois e do teorema de Bénabou-Roubaud. Provamos também um resultado sobre morfismos de descida efetiva em pseudoproductos fibrados de categorias e o aplicamos para obter morfismos de descida efetiva em algumas categorias de categorias enriquecidas.

O Capítulo 4 corresponde ao artigo \textit{On Biadjoint Triangles}, publicado no \textit{Theory and Applications of Categories}, Vol 31, N. 9 (2016). As contribuições principais são os teoremas de triângulos biadjuntos, os quais possuem muitas aplicações em teoria de categorias de dimensão 2. Apresentamos exemplos de aplicações no próprio artigo: provamos explicitamente o teorema de pseudomonadicidade [38], melhoramos resultados sobre o tratamento 2-monádico do problema de coerência de [3, 36, 49], generalizamos resultados de levantamentos de biadjuntos e introduzimos o conceito de \textit{pseudo-extensões de Kan} para, então, construir as pseudo-extensões de Kan via bilimites ponderados.

O Capítulo 5 corresponde ao artigo \textit{On lifting of biadjoints and lax algebras}, a ser publicado no \textit{Categories and General Algebraic Structures with Applications}. O principal tema deste artigo é a demonstração de teoremas de levantamento de biadjuntos, ao assumir que conseguimos descrever a categoria de morfismos de um domínio em termos de (bi)limites ponderados. Esse tratamento, junto com resultados sobre objetos de descida lassos e álgebras lassas, nos permite obter resultados sobre levantamento de biadjuntos envolvendo sub-2-categorias (plenas) da 2-categoria de álgebras lassas. Como consequência, concluímos nossos resultados de caracterização sobre o tratamento 2-monádico do problema de coerência, via teoremas de triângulos biadjuntos.
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Chapter 1

Introduction

The aim of this chapter is to introduce our main setting, which is 2-dimensional universal algebra, and to give a glimpse of the contributions of this thesis. We start by roughly explaining aspects of the interrelation between pseudomonads and descent objects in Section 1.1. Then, in Section 1.2, we introduce basic notions of 2-dimensional category theory. We take this opportunity to introduce, among other concepts, the notion of colax $T$-morphisms of lax $T$-algebras, which is not introduced elsewhere in this thesis. The notion of colax $T$-morphisms are, then, used in Section 1.7 to relate the formal theory of monads with the problem of lifting of biadjoints studied in Chapter 5 [42]. We also use the concept of colax $T$-morphisms to talk about doctrinal adjuntion in Section 1.4, which is a brief exposition of the main theorem of [27] focusing on the Beck-Chevalley condition as used in our work on the Bénabou-Roubaud Theorem in Chapter 3.

Sections 1.3, 1.5 and 1.6 are dedicated to briefly describe elements of the contributions of the Chapters 2, 3, 4 and 5, which are respectively the papers [44], [43], [41] and [42]. Finally, the last section is an application of the biadjoint triangles of Chapter 4 in the context of exponentiable objects within bicategory theory: we prove that, under suitable hypothesis, a pseudocoalgebra is (pseudo)exponentiable whenever the underlying object is (pseudo)exponentiable.

1.1 Overview: Pseudomonads and the Descent Object

In 2-dimensional category theory, by replacing strict conditions (involving commutativity of diagrams) with pseudo or lax ones (involving a 2-cell plus coherence) we get important notions and problems. We briefly describe two examples of these notions below: namely, descent object and pseudomonad.

Firstly, to give an idea of the role of the descent object in 2-dimensional universal algebra, it is useful to make an analogy with the equalizer: while the equalizer encompasses equality and commutativity of diagrams in 1-dimensional category theory, the descent object and its variations encompass 2-dimensional coherence: structure (2-cell) plus coherence.

One obvious example of the importance of the descent object is within descent theory, as introduced by Grothendieck, initially motivated by the problem of understanding the image of functors induced by fibrations. This theory features a 2-dimensional analogue of the sheaf condition: the (strict) gluing condition, given by an equalizer of sets, is replaced by the descent condition, given by a descent object of a diagram of categories.
Secondly, analogously to the case of monad theory in 1-dimensional universal algebra, \textit{pseudomonad theory} encompasses aspects of 2-dimensional universal algebra, being useful to study many important aspects of 2-dimensional category theory. Again, in the definition of \textit{pseudomonad}, the commutative diagrams of the definition of monad are replaced by invertible 2-cells plus coherence. In this theory, then, we have 2-dimensional versions of the features of monad theory. For instance, adjunctions are replaced by biadjunctions, and we have an Eilenberg-Moore Factorization provided that we consider the 2-category of pseudoalgebras, pseudomorphisms and algebra transformations, as it is shown in Section 5 of Chapter 4 [41].

The main topic of this thesis is the study of 2-dimensional categorical structures, mostly related with descent theory and pseudomonad theory, with applications to 1-dimensional category theory. For instance:

- The contributions of Chapter 2 in this context are within the study of freely generated and finitely presented categorical structures. In particular, for instance, we deal with presentations of domain 2-categories related to the universal property of descent objects. More precisely, presentations of the inclusion of the 2-categories such that Kan extensions along such inclusion gives the (strict) descent object;

- In Chapter 3, we develop an abstract perspective of descent theory in which the fundamental problem is the existence of pseudoalgebra structures over objects: more precisely, the image of pseudomonadic pseudofunctors that induce idempotent pseudomonads. Having this goal, we develop some aspects of biadjoint triangles and lifting of pseudoalgebra structures involving pseudofunctors that induce idempotent pseudomonads and apply it to get results on commutativity of bilimits. We finish the article, then, applying our perspective to the classical context of [24, 25];

- As a (strict) morphism of algebras is given by a morphism plus the commutativity of a diagram, a pseudomorphism between pseudoalgebras is given by a morphism, an invertible 2-cell plus coherence. In Chapters 4 and 5, we show that the coherence aspects of pseudomonad theory are encompassed by descent objects and their variations. More precisely, we show that the category of (lax-)(pseudo)morphisms between (lax-)(pseudo)algebras is given by (lax-)descent objects. As it is proven and explained in Chapter 5, with these results on the category of (lax-)(pseudo)morphisms, we can prove biadjoint triangle theorems and results on lifting of biadjoints.

1.2 2-Dimensional Categorical Structures

Two of the most fundamental notions of 2-dimensional categorical structures are those of \textit{double category} and \textit{2-category}, both introduced by Ehresmann [15]. The former is an example of an internal category (introduced in [15]), while the latter is an example of enriched category (as introduced in [16]).

The study of the dichotomy between the theory of enriched categories and internal categories, including unification theories, is still of much interest. For instance, within the more general setting of
generalized multicategories, we have the introduction of \((T, V)\)-categories [8] (which generalizes enriched categories), \(T\)-categories [4, 21] (which generalizes internal categories) and possible unification theories [5, 10].

Since there is no definitive general framework and the approaches mentioned above are focused on the examples related to multicategories, we do not follow any of them. Instead, we follow the basic idea that internal categories and enriched categories can be seen as monads in suitable bicategories [1, 2].

For simplicity, we use the concept of enriched graphs. This is given in Definition 7.1 of Chapter 2 [44], but we only need to recall that, given a category \(V\), a \(V\)-enriched graph \(G\) is a collection of objects \(G(0) = G_0\) endowed with one object \(G(A, B)\) of \(V\) for each ordered pair of objects \((A, B)\) of \(G_0\).

**Definition 1.2.1.** [Bicategory [1]] A bicategory is a \(\mathbf{CAT}\)-enriched graph \(\mathcal{B}\) endowed with:

- Identities: a functor \(I_A : 1 \to \mathcal{B}(A, A)\) for each object \(A\) of \(\mathcal{B}\);
- Composition: a functor, called composition, \(\circ = \circ_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)\) for each ordered triple \((A, B, C)\) of objects of \(\mathcal{B}\);
- Associativity: natural isomorphisms

\[a_{ABCD} : \circ_{ABD} \left( \circ_{BCD} \times \text{Id}_{\mathcal{B}(A, B)} \right) \Rightarrow \circ_{ACD} \left( \text{Id}_{\mathcal{B}(C, D)} \times \circ_{ABC} \right) ;\]

for every quadruplet \((A, B, C, D)\) of objects of \(\mathcal{B}\);
- Action of Identity: natural isomorphisms

\[e_{AB} : \circ_{ABB} \left( I_B \times \text{Id}_{\mathcal{B}(A, B)} \right) \Rightarrow \text{pro}_{\mathcal{B}(A, B)}^0 ;\]
\[d_{AB} : \circ_{ABB} \left( \text{Id}_{\mathcal{B}(A, B)} \times I_A \right) \Rightarrow \text{pro}_{\mathcal{B}(A, B)}^1 ,\]

in which \(\text{pro}_{\mathcal{B}(A, B)}^0 : \mathcal{B}(A, B) \times 1 \to \mathcal{B}(A, B)\) and \(\text{pro}_{\mathcal{B}(A, B)}^1 : 1 \times \mathcal{B}(A, B) \to \mathcal{B}(A, B)\) are the invertible projections, for each pair \((A, B)\) of objects in \(\mathcal{B}\); such that the diagrams

\[
\begin{align*}
\circ_{ABE} \left( \circ_{BCE} \times \text{Id}_{\mathcal{B}(A, B)} \right) & \xrightarrow{\alpha_{ABCD} + \text{id}} \circ_{ACE} \left( \text{Id}_{\mathcal{B}(C, E)} \times \circ_{ABC} \right) \left( \circ_{CDE} \times \text{Id}_{\mathcal{B}(A, B, C)} \right) \\
\text{id}_{\circ_{ABE}} \times \alpha_{ABCE} + \text{id} & \xrightarrow{\alpha_{ACDE} + \text{id}} \circ_{ADE} \left( \text{Id}_{\mathcal{B}(D, E)} \times \circ_{ACD} \right) \left( \text{Id}_{\mathcal{B}(C, D, E)} \times \circ_{ABC} \right) \\
\end{align*}
\]
when necessary, we implicitly make similar assumptions to those given in Section 1 of Chapter 2 \[44\].

\[ \circ_{ABC}(\circ_{BIC} \times \text{Id}_{B(A,B)}) (\text{Id} \times I_B \times \text{Id}) \xrightarrow{\text{id}_{ABC} \times \text{id}} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

\[ \xrightarrow{\text{id}_{ABC} \times (\text{Id} \times \text{id})} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

\[ \xrightarrow{\text{id}_{ABC} \times (\text{id} \times \text{id})} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

\[ \xrightarrow{\text{id}_{ABC} \times (\text{id} \times \text{id})} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

\[ \xrightarrow{\text{id}_{ABC} \times (\text{id} \times \text{id})} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

\[ \xrightarrow{\text{id}_{ABC} \times (\text{id} \times \text{id})} \circ_{ABC}(\text{Id}_{B(B,C)} \times \circ_{ABB}) (\text{Id} \times I_B \times \text{Id}) \]

commute for every quintuple \((A, B, C, D, E)\) of objects in \(\mathcal{B}\), in which \(\text{Id}_{B(A,B,C)} := \text{Id}_{B(B,C)} \times \text{Id}_{B(A,B)}\), \(\text{pro}_{B(A,B,C)} : \mathcal{B}(B,C) \times 1 \times \mathcal{B}(A,B) \rightarrow \mathcal{B}(B,C) \times \mathcal{B}(A,B)\) is the invertible projection and the omitted subscripts of the identities are the obvious ones.

For simplicity, assuming that the structures are implicit, we denote such a bicategory by \((\mathcal{B}, \circ, I, \alpha, \epsilon, \delta)\) or just by \(\mathcal{B}\). For each pair \((A, B)\) of objects of a bicategory \(\mathcal{B}\), if \(f\) is an object of the category \(\mathcal{B}(A,B)\), \(f\) is called an 1-cell of \(\mathcal{B}\) and it is denoted by \(f : A \rightarrow B\). A morphism \(\alpha : f \Rightarrow g\) of \(\mathcal{B}(A,B)\) is called a 2-cell of \(\mathcal{B}\).

**Remark 1.2.2.** In order to take advantage of the context of introducing monads, internal categories, double categories and enriched categories, we define 2-categories via enriched categories below. However, a brief and obvious definition of 2-category is that of a strict bicategory. More precisely, a 2-category is a bicategory such that its natural isomorphisms are identities. We assume this definition herein.

**Remark 1.2.3.** Since the work of this thesis is mainly within the tricategory \(2\text{-CAT}\) of 2-categories, pseudofunctors and pseudonatural transformations (as defined in Section 2 of Chapter 4 \[41\]), the results and definitions on 2-dimensional category theory of this thesis are within the general setting of bicategories up to minor trivial adaptations. Specially in this section, since we should consider the bicategories of Definitions 1.2.9 and 1.2.10, we freely assume these adaptations.

**Remark 1.2.4.** In this chapter, we do not give any further comment on size issues. In this direction, when necessary, we implicitly make similar assumptions to those given in Section 1 of Chapter 2 \[44\] or Section 1 of Chapter 5.

Given a bicategory \(\mathcal{B}\), there are two main duals of \(\mathcal{B}\) which give rise to four duals, including \(\mathcal{B}\) itself. The first dual, denoted by \(\mathcal{B}^{\text{op}}\), comes from getting the dual of the underlying category of \(\mathcal{B}\) (that is to say, the opposite w.r.t. 1-cells), while the other dual, denoted by \(\mathcal{B}^{\text{co}}\), is obtained from getting the duals of the hom-categories (that is to say, the opposite w.r.t. 2-cells). Then, we have \(\mathcal{B}\) itself and \(\mathcal{B}^{\text{coop}} := (\mathcal{B}^{\text{op}})^{\text{co}} \cong (\mathcal{B}^{\text{co}})^{\text{op}}\).

**Remark 1.2.5.** We do not define tricategories \[20\], but we give some independent remarks. For instance, as observed in Section 4.2, \(2\text{-CAT}\) is a tricategory and, specially in the present section, we consider the tricategory \(\text{BICAT}\) of bicategories, pseudofunctors, pseudonatural transformations and modifications as well. The dualizations mentioned above define invertible trifunctors:

\[ (-)^{\text{op}} : \text{BICAT} \cong \text{BICAT}^{\text{co}}, \quad (-)^{\text{co}} : \text{BICAT} \cong \text{BICAT}^{\text{fco}}, \quad (-)^{\text{coop}} : \text{BICAT} \cong \text{BICAT}^{\text{cotco}}, \]

\[ (-)^{\text{op}} : \text{2-CAT} \cong \text{2-CAT}^{\text{co}}, \quad (-)^{\text{co}} : \text{2-CAT} \cong \text{2-CAT}^{\text{fco}}, \quad (-)^{\text{coop}} : \text{2-CAT} \cong \text{2-CAT}^{\text{cotco}}, \]
in which $\mathcal{I}^{\text{co}}$ denotes the dual of the tricategory $\mathcal{I}$ obtained from reversing the 3-cells, and $\mathcal{I}^{\text{co}co} := (\mathcal{I}^{\text{co}})^{\text{co}}$. These isomorphisms are 2-dimensional analogues of the invertible 2-functor $(-)^{\text{op}} : \text{CAT} \cong \mathcal{I}^{\text{co}co}$.

Given a bicategory $\mathcal{B}$, we can consider the bicategory of monads of $\mathcal{B}$. This was introduced in [53, 54] taking the point of [1] that monads in $\mathcal{B}$ are given by lax functors between the terminal bicategory 1 and $\mathcal{B}$. Herein, we introduce monads via lax algebras of the identity pseudomonad. This perspective takes advantage of the concepts introduced in Chapter 5 [42] and it gives a shortcut to understand the role of lifting of biadjoints in the formal theory of monads, which is sketched in Section 1.7. It should be observed that our viewpoint also has connections with the approach of [8] to introduce enriched categories and internal categories as monads.

We assume the definition of pseudomonads of Section 4 of Chapter 5 (or Definition 5.1 of Chapter 4 [41] of pseudocomonads) and Definition 4.1 of Chapter 5 [42] of the bicategory of lax algebras and lax morphisms. Within this context, it is easy to verify that:

**Lemma 1.2.6.** The identity pseudofunctor $\text{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$, with identities 2-natural transformations and modifications, gives a 2-monad and a 2-comonad on $\mathcal{B}$.

**Definition 1.2.7.** [Bicategory of Monads] The bicategory of monads of a bicategory $\mathcal{B}$, denoted by $\text{Mnd}(\mathcal{B})$, is the bicategory of lax $\text{Id}_{\mathcal{B}}$-algebras $\text{Lax-Id}_{\mathcal{B}}$-$\text{Alg}_0$.

Following the notation of lax algebras of Definition 4.1 of Chapter 5 [42], we have that a monad in a 2-category $\mathcal{B}$ is defined by a quadruplet $z = (Z, \alpha_{\mathcal{B}}, \varepsilon, \delta)$ satisfying the condition of lax $\text{Id}_{\mathcal{B}}$-algebra, in which $Z$ is an object of $\mathcal{B}$, $\alpha_{\mathcal{B}} : Z \to Z$ is an endomorphism of $\mathcal{B}$, $\varepsilon : \alpha_{\mathcal{B}} \circ \alpha_{\mathcal{B}} \Rightarrow \alpha_{\mathcal{B}}$ is a 2-cell of $\mathcal{B}$, called the multiplication of $z$, and $\delta : \text{id}_Z \Rightarrow \alpha_{\mathcal{B}}$ is a 2-cell of $\mathcal{B}$, called the unit of the monad $z$. In this case, we say that $z$ is a monad on $Z$.

In order to introduce enriched categories and internal categories as monads, we define bicategories constructed from a suitable categorical structure $V$. These are the bicategory of matrices and the bicategory of spans, denoted respectively by $V$-$\text{Mat}$ and $\text{Span}(V)$.

**Remark 1.2.8.** The bicategory of matrices $V$-$\text{Mat}$ is constructed from a monoidal category $V$. A monoidal category is a bicategory $(\mathcal{B}, \circ, 1, a, e, d)$ which has only one object $\Delta$. The composition is called, in this case, the monoidal product/tensor. The underlying category of a monoidal category is the hom-category $\mathcal{B}(\Delta, \Delta)$. The objects and morphisms of a monoidal category are the objects and morphisms of its underlying category.

Since we are talking about a bicategory with only one object $\Delta$, we actually have only one natural isomorphism of associativity, one identity and two natural isomorphisms of action of identity. That is to say, given a monoidal category $(\mathcal{B}, \circ, 1, a, e, d)$, we denote $a := a_{\Delta\Delta}, d := d_{\Delta\Delta}, e := e_{\Delta\Delta}$ and $I := 1_{\Delta}$. Therefore such a monoidal category is given by a sextuple $(V, \circ, 1, a, e, d)$ in which $V := \mathcal{B}(\Delta, \Delta)$ is the underlying category, $\circ$ is the monoidal product, $1$ is the identity and

$$a : (1 \circ -) \circ - \longrightarrow 1 \circ (1 \circ -), e : (I \circ -) \longrightarrow \text{id}_V, d : (1 \circ I) \longrightarrow \text{id}_V$$

are the respective natural isomorphisms satisfying the axioms of a bicategory with the only object $\Delta$.

For simplicity, letting the natural isomorphisms implicit, a monoidal category is usually denoted by $(V, \circ, 1)$, while $(\circ, 1)$ together with the natural isomorphisms $a, e, d$ is called a monoidal structure.
for $V$. When the monoidal structure is implicit in the context, we denote the monoidal category $(V, \otimes, I)$ by $V$.

A symmetric monoidal category is a monoidal category $(V, \otimes, I)$ endowed with a natural isomorphism, called braiding, $b : - \otimes - \to - \otimes \text{op}$, in which $\otimes \text{op}$ is the composition of the opposite of the bicategory that corresponds to $(V, \otimes, I)$, such that

\[
\begin{align*}
(A \otimes B) \otimes C & \xrightarrow{a_{(A,B,C)}} A \otimes (B \otimes C) \xrightarrow{b_{(A,B,C)}} (B \otimes C) \otimes A \\
B \otimes A & \xrightarrow{b_{(B,A)}} (B \otimes A)
\end{align*}
\]

commute for every triple $(A, B, C)$ of objects of $V$. A symmetric monoidal closed category $(V, \otimes, I)$ is a symmetric monoidal category such that every object of $V$ is exponentiable w.r.t. the monoidal product $\otimes$. In other words, this means that the representable functor $A \otimes - : V \to V$ has a right adjoint for every $A$ of $V$.

If a category $V$ has finite products, it has a natural symmetric monoidal structure called cartesian monoidal structure. That is to say $\otimes := \times$ and the unit is given by the terminal object $1$ of $V$. The natural isomorphisms for associativity, actions of the identity and braiding are given by the universal property of the product. A category with finite products endowed with this monoidal structure is called a monoidal cartesian category or just a cartesian category, and it is denoted by $(V, \times, 1)$.

It would be appropriate to define monoidal categories via 2-dimensional monad theory, as, for instance, it is shown in Remark 5.4.3 [42]. But our interest herein is mostly restricted to the case of cartesian closed categories. Even our result on effective descent morphisms of enriched categories, which is Theorem 9.11 of Chapter 3 [43], is given within the context of cartesian closed categories.

For this reason, we avoid giving further comments on general aspects of monoidal categories. We refer to [39, 40, 45] for the basics on such general aspects.

**Definition 1.2.9.** [Bicategory of Matrices] Let $(V, \otimes, I)$ be a symmetric monoidal closed category with finite coproducts. We define $V$-Mat as follows:

- The objects are the sets;
- A morphism $M : A \to B$ in $V$-Mat is a matrix of objects in $V$, that is to say, a functor $A \times B \to V$, considering $A, B$ as discrete categories;
- The 2-cells are natural transformations. In other words, the category of morphisms for a ordered pair $(A, B)$ of sets is the category of functors and natural transformations $\text{CAT}[A \times B, \text{Set}]$.
– The composition is given by the usual formula of product of matrices. More precisely, given matrices $M : A \times B \to V$ and $N : B \times C \to V$, the composition is defined by

$$N \circ M : A \times C \to V \quad (i, j) \mapsto \sum_{k \in B} M(i, k) \otimes N(k, j)$$

in which $\sum$ denotes the coproduct;

– For each set $A$, the identity on $A$ is the matrix

$$\text{id}_A : A \times A \to V \quad (i, j) \mapsto \begin{cases} I, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

in which $0$ is the initial object;

– The natural isomorphisms for associativity and actions of identities are given by the universal property of coproducts, the isomorphisms of the preservation of the coproducts by $\otimes$ and the isomorphisms of the monoidal structure.

In order to define the bicategory of spans $\text{Span}(V)$ of a category with pullbacks $V$, we denote by $\text{span}$ the category with 3 objects (0, 1 and 2) whose nontrivial morphisms are given by

$$0 \xrightarrow{d^1} 2 \xrightarrow{d^0} 1.$$

**Definition 1.2.10.** [Bicategory of Spans] Let $V$ be a category with pullbacks. The bicategory $\text{Span}(V)$ is defined by

– The objects are the objects of $V$;

– A morphism $M : A \to B$ in $\text{Span}(V)$ is a span in $V$ between $A$ and $B$, that is to say, a functor $M : \text{span} \to V$, such that $M(0) = A$ and $M(1) = B$;

– A 2-cell $f$ between two morphisms $M, K : A \to B$ is a natural transformation $f : M \to K$ such that $f_0 = \text{id}_A$ and $f_1 = \text{id}_B$. That is to say, it is a morphism $f : M(2) \to K(2)$ such that

$$\begin{array}{ccc}
M(2) & \xrightarrow{M(d^1)} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{M(d^0)} & K(2)
\end{array}$$

$$\begin{array}{ccc}
M(2) & \xleftarrow{K(d^1)} & A \\
\downarrow & & \downarrow \\
K(2) & \xleftarrow{K(d^0)} & B
\end{array}$$
commutes in $V$.

- The composition is given by the pullback. More precisely, given a span $M : \text{span} \to V$ between $A$ and $B$ and a span $N : \text{span} \to V$ between $B$ and $C$, the composition is defined by the span

$$
\begin{array}{c}
P(M, N) \\
\downarrow \\
M(2) \\
\downarrow M(d^1) \\
A \\
\end{array}
\quad
\begin{array}{c}
\downarrow N(d^1) \\
\downarrow N(d^0) \\
B \\
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\downarrow \\
C \\
\end{array}
$$

between the objects $A$ and $C$, in which $P(M, N)$ is the pullback $M(2) \times_{(M(d^0), N(d^1))} N(2)$ of $M(d^0)$ along $N(d^1)$ and the unlabeled arrows are the morphisms induced by the pullback;

- For each set $A$, the identity on $A$ is the span $\text{span} \to V$ constantly equal to $A$, taking the morphisms of the domain to the identity on $A$;

- The natural isomorphisms for associativity and actions of identities are given by the universal property of pullbacks.

Assuming that $V$ is a symmetric monoidal closed category with coproducts, a (small) category enriched in $V$ is a monad of the bicategory $V\text{-Mat}$, while, if $V'$ has pullbacks, an internal category of $V'$ is a monad of $\text{Span}(V')$. But 1-cells and 2-cells of $\text{Mnd}(V\text{-Mat})$ and $\text{Mnd}(\text{Span}(V'))$ do not respectively coincide with what should be 1-cells and 2-cells of the 2-category of enriched categories and the 2-category of internal categories. In order to get the appropriate notion of 1-cells, firstly we need to consider co-morphisms of monads and, secondly, we need to consider proarrow equipments.

We introduce co-morphisms between monads via colax morphisms between lax algebras. For short, we use the notation introduced in Definition 4.1 of Chapter 5 [42] and we define the 2-category of lax algebras and colax morphisms, denoted by $\text{Lax}_c^{\mathcal{T}}$, as follows:

**Definition 1.2.11.** [Colax morphisms of lax algebras] Let $\mathcal{T} = (\mathcal{T}, m, \eta, \mu, \tau)$ be a pseudomonad on a 2-category $\mathcal{B}$. We define the 2-category of lax $\mathcal{T}$-algebras and colax $\mathcal{T}$-morphisms, denoted by $\text{Lax}_c^{\mathcal{T}}$, as follows:

1. **Objects:** lax $\mathcal{T}$-algebras as in Definition 5.4.1 [42].

2. **Morphisms:** colax $\mathcal{T}$-morphisms $\xi : y \to z$ between lax $\mathcal{T}$-algebras

$$
y = (Y, \alpha_{q_y}, \overline{\mathcal{T}}_y), z = (Z, \alpha_{q_z}, \overline{\mathcal{T}}_z)
$$

are pairs $\xi = (f, \langle \overline{\mathcal{T}} \rangle)$ in which $f : Y \to Z$ is a morphism in $\mathcal{B}$ and

$$
\langle \overline{\mathcal{T}} \rangle : f \alpha_{q_y} \Rightarrow \alpha_{q_z} \mathcal{T}(f)
$$
1.2 2-Dimensional Categorical Structures

is a 2-cell of \( \mathcal{B} \) such that, defining \( \mathcal{T}(\langle \mathbf{f} \rangle) := t_{\mathcal{T}(\mathbf{f})}^{-1} \mathcal{T}(\langle \mathbf{f} \rangle) t_{\mathcal{T}(\mathbf{f})} \), the equations

\[
\begin{array}{c}
\mathcal{T}(\langle \mathbf{f} \rangle) := t_{\mathcal{T}(\mathbf{f})}^{-1} \mathcal{T}(\langle \mathbf{f} \rangle) t_{\mathcal{T}(\mathbf{f})} \\
\mathcal{T}(\langle \mathbf{f} \rangle) \Rightarrow \mathcal{T}(\langle \mathbf{f} \rangle)
\end{array}
\]

hold. The composition of colax \( \mathcal{T} \)-morphisms \( \mathbf{f} : y \to z \) and \( \mathbf{g} : x \to y \) of lax \( \mathcal{T} \)-algebras is denoted by \( \mathbf{f} \circ \mathbf{g} \) and it is defined by the pair \( \langle \mathbf{fg}, \langle \mathbf{fg} \rangle \rangle \) in which \( \langle \mathbf{fg} \rangle \) is the 2-cell defined by

\[
\begin{array}{c}
\langle \mathbf{fg} \rangle := Y \xrightarrow{\mathcal{T}(\mathbf{f})} \mathcal{T}(\langle \mathbf{fg} \rangle) \\
\langle \mathbf{fg} \rangle \Rightarrow \langle \mathbf{fg} \rangle
\end{array}
\]

3. 2-cells: a \( \mathcal{T} \)-transformation \( m : \mathbf{f} \Rightarrow \mathbf{h} \) between lax \( \mathcal{T} \)-morphisms \( \mathbf{f} = (f, \langle \mathbf{f} \rangle) \), \( \mathbf{h} = (h, \langle \mathbf{h} \rangle) \) is a 2-cell \( m : \mathbf{f} \Rightarrow \mathbf{h} \) in \( \mathcal{B} \) such that the equation below holds.

\[
\begin{array}{c}
\mathcal{T}(h) \circ (m) \Rightarrow \mathcal{T}(f) \\
\mathcal{T}(h) \Rightarrow \mathcal{T}(h)
\end{array}
\]
The compositions of \(\mathcal{T}\)-transformations are defined in the obvious way and these definitions make \(\text{Lax-}\mathcal{T}\text{-Alg}_{cl}\) a 2-category.

**Remark 1.2.12.** [Identity on a lax \(\mathcal{T}\)-algebra] The identities on a lax \(\mathcal{T}\)-algebra \(y = (Y, \alpha_y, \gamma, \Sigma_0)\) in \(\text{Lax-}\mathcal{T}\text{-Alg}_{cl}\) and in \(\text{Lax-}\mathcal{T}\text{-Alg}_l\) are respectively given by

\[
\begin{pmatrix}
\mathcal{T}Y & \xrightarrow{\alpha_y} & Y \\
\text{id}_Y, \mathcal{T}(\text{id}_Y, \gamma) & = & \text{id}_Y
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\mathcal{T}Y & \xrightarrow{\alpha_y} & Y \\
\text{id}_Y, \mathcal{T}(\text{id}_Y, \gamma) & = & \text{id}_Y
\end{pmatrix}
\]

**Remark 1.2.13.** [Colax algebras and coalgebras] By dualizing, we get below the notions of colax algebras, lax coalgebras and colax coalgebras. The dualization \((-)^{\text{coop}}\) preserves pseudomonads and pseudocomonads, while \((-)^{\text{op}}\) takes pseudomonads to pseudocomonads and pseudocomonads to pseudomonads. In particular, given a pseudomonad \(\mathcal{T}\) on a bicategory \(\mathcal{B}\) and a pseudocomonad \(\mathcal{S}\) on a bicategory \(\mathcal{C}\), we have that \(\mathcal{T}^{\text{co}}\) is a pseudomonad on \(\mathcal{B}^{\text{co}}\) and \(\mathcal{S}^{\text{op}}, \mathcal{S}^{\text{coop}}\) are pseudomonads respectively on \(\mathcal{C}^{\text{op}}\) and \(\mathcal{C}^{\text{coop}}\).

Therefore we can define:

- The bicategories of colax \(\mathcal{T}\)-algebras of the pseudomonad \(\mathcal{T}\) (respectively with colax \(\mathcal{T}\)-morphisms and lax \(\mathcal{T}\)-morphisms):

\[
\text{Colax-}\mathcal{T}\text{-Alg}_{cl} := (\text{Lax-}\mathcal{T}^{\text{co}}\text{-Alg}_{cl})^{\text{co}} \quad \text{and} \quad \text{Colax-}\mathcal{T}\text{-Alg}_l := (\text{Lax-}\mathcal{T}^{\text{co}}\text{-Alg}_l)^{\text{co}};
\]

- The bicategories of lax \(\mathcal{T}\)-coalgebras:

\[
\text{Lax-}\mathcal{T}\text{-CoAlg}_l := (\text{Lax-}\mathcal{T}^{\text{op}}\text{-CoAlg}_l)^{\text{op}} \quad \text{and} \quad \text{Lax-}\mathcal{T}\text{-CoAlg}_{cl} := (\text{Lax-}\mathcal{T}^{\text{op}}\text{-CoAlg}_{cl})^{\text{op}};
\]

- The bicategory of colax \(\mathcal{T}\)-coalgebras:

\[
\text{Colax-}\mathcal{T}\text{-CoAlg}_{cl} := (\text{Lax-}\mathcal{T}^{\text{coop}}\text{-CoAlg}_l)^{\text{coop}} \quad \text{and} \quad \text{Colax-}\mathcal{T}\text{-CoAlg}_l := (\text{Lax-}\mathcal{T}^{\text{coop}}\text{-CoAlg}_{cl})^{\text{coop}}.
\]

**Remark 1.2.14.** A colax \(\mathcal{T}\)-morphism \(\xi = (f, \langle \tilde{f} \rangle)\) is a \(\mathcal{T}\)-pseudomorphism if \(\langle \tilde{f} \rangle\) is an invertible 2-cell. In particular, we have an inclusion 2-functor \(\text{Lax-}\mathcal{T}\text{-Alg} \to \text{Lax-}\mathcal{T}\text{-Alg}_{cl}\), in which \(\text{Lax-}\mathcal{T}\text{-Alg}\) denotes the 2-category of lax \(\mathcal{T}\)-algebras and \(\mathcal{T}\)-pseudomorphisms as introduced in Section 4 of Chapter 5 [42].

**Definition 1.2.15.** [Co-bicategory of monads] The bicategory of monads and co-morphisms of a bicategory \(\mathcal{B}\), called herein the co-bicategory of monads and denoted by \(\text{Mnd}_{\text{co}}(\mathcal{B})\), is the bicategory of lax \(\text{Id}_{\mathcal{B}}\)-algebras and colax \(\text{Id}_{\mathcal{B}}\)-morphisms, that is to say: \(\text{Mnd}_{\text{co}}(\mathcal{B}) := \text{Lax-}\text{Id}_{\mathcal{B}}\text{-Alg}_{cl}\).

The duals of the bicategories of monads are the bicategories of comonads. More precisely, the bicategories are defined by \(\text{CoMnd}(\mathcal{B}) := (\text{Mnd}(\mathcal{B}^{\text{co}}))^{\text{co}}\) and \(\text{CoMnd}_{\text{co}}(\mathcal{B}) := (\text{Mnd}_{\text{co}}(\mathcal{B}^{\text{co}}))^{\text{co}}\).
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Lemma 1.2.16. Given a bicategory $\mathcal{B}$,

$$\text{Mnd}_{\text{co}}(\mathcal{B}) \cong \text{Lax-Id}_{\text{co}}\text{-CoAlg}_{\ell} \cong (\text{Mnd}(\mathcal{B}^{\text{op}}))^{\text{op}} \quad \text{and} \quad \text{Mnd}(\mathcal{B}) \cong \text{Lax-Id}_{\text{co}}\text{-CoAlg}_{\text{cf}}.$$

Herein, we actually do not need to give the full definition of proarrow equipment introduced in [58, 59]. Instead, we can give a much less structured version:

Definition 1.2.17. [Proarrow equipment] A proarrow equipment on a 2-category $\mathcal{B}_0$ is a pseudofunctor $P : \mathcal{B}_0 \to \mathcal{B}_1$ which is the identity on objects and locally fully faithful.

Clearly, the category of proarrow equipments is a subcategory of the category of morphisms of the category of bicategories and pseudofunctors. Similarly, in the context of Remark 1.2.5, there is a tricategory of proarrow equipments which is a sub-tricategory of the tricategory of morphisms of $\text{BICAT}$. Thereby, it is natural to consider pseudomonads on pseudofunctors and on proarrow equipments.

Definition 1.2.18. [Pseudomonad on proarrow equipments] A pseudomonad $\mathcal{T}$ on a pseudofunctor $P : \mathcal{B}_0 \to \mathcal{B}_1$ is a pair $(\mathcal{T}_0, \mathcal{T}_1)$ in which $\mathcal{T}_0 = (T_0, m_0, \eta_0, \mu_0, \iota_0, \tau_0)$ is a pseudomonad on $\mathcal{B}_0$ and $\mathcal{T}_1 = (T_1, m_1, \eta_1, \mu_1, t_1, \tau_1)$ is a pseudomonad on $\mathcal{B}_1$ such that this pair of pseudomonads agrees with $P$, which means that:

$$\mathcal{T}_1 P = P \mathcal{T}_0, \quad m_1 P = P m_0, \quad \eta_1 P = P \eta_0, \quad \mu_1 P = P \mu_0, \quad t_1 P = P t_0, \quad \tau_1 P = P \tau_0.$$

For our purposes, we could define a simpler version of pseudomonads on proarrow equipments on 2-categories. That is to say, we could say that a pseudomonad on $P : \mathcal{B}_0 \to \mathcal{B}_1$ is just a pseudomonad on $\mathcal{B}_1$.

Definition 1.2.19. Given a pseudomonad $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ on a proarrow equipment $P : \mathcal{B}_0 \to \mathcal{B}_1$ on a 2-category $\mathcal{B}_0$, the bicategory of lax $(\mathcal{T}, P)$-algebras, denoted herein by $\text{Lax-} \mathcal{T}_1\text{-Alg}_{\text{cf}}$, is the pullback of $P$ along the forgetful pseudofunctor $\text{Lax-} \mathcal{T}_1\text{-Alg}_{\text{cf}} \to \mathcal{B}_1$ in the category of bicategories and pseudofunctors.

The category of bicategories and pseudofunctors does not have all pullbacks (or equalizers). However, in the context of Definition 1.2.19, the pullbacks always exist. Moreover, $\text{Lax-} \mathcal{T}_1\text{-Alg}_{\text{cf}}$ is always a 2-category. Explicitly, the objects of $\text{Lax-} \mathcal{T}_1\text{-Alg}_{\text{cf}}$ are lax $\mathcal{T}$-algebras $z = (Z, a_{\mathcal{T}, Z}, Z, Z_0)$ and the morphisms between two objects in $\text{Lax-} \mathcal{T}_1\text{-Alg}_{\text{cf}}$ are colax $\mathcal{T}$-morphisms $g = (g, (g))$ between lax $\mathcal{T}$-algebras such that $P(\hat{g}) = g$ for some morphism $\hat{g}$ of $\mathcal{B}_0$. The composition of morphisms $f = (P(\hat{f}), (\hat{f}))$, $g = (P(\hat{g}), (\hat{g}))$ is defined by $f \circ g := (P(\hat{f} \circ \hat{g}), (\hat{f} \circ \hat{g}))$ such that $(\hat{f} \circ \hat{g})$ is
defined by

\[
\begin{align*}
X & \xrightarrow{\text{alg}} \mathcal{TX} \\
\text{alg} & \downarrow \downarrow \downarrow \\
Y & \xrightarrow{\mathcal{T}(\mathcal{f}(\mathcal{g}))} \mathcal{P}(\mathcal{f}(\mathcal{g})) \\
\text{alg} & \downarrow \downarrow \downarrow \\
Z & \xrightarrow{\text{alg}} \mathcal{T}Z
\end{align*}
\]

in which \(\langle \mathcal{f}\mathcal{g}\rangle\) denotes the 2-cell component of the usual composition of the colax \(\mathcal{T}\)-morphisms of \(\mathcal{f}\) and \(\mathcal{g}\).

The 2-category of monads and co-morphisms in a proarrow equipment \(P : \mathfrak{B}_0 \to \mathfrak{B}_1\) is defined, then, by \(\text{Mnd}_{\text{co}}(P) := \text{Lax}(\text{Id}_P, P)\text{-Alg}_{\text{cel}}\).

**Definition 1.2.20.** [Proarrows of Matrices] Given a symmetric closed monoidal category \((V, \otimes, I)\) with coproducts, the bicategory \(V\text{-Mat}\) gives a natural proarrow equipment \(\text{Set} \to V\text{-Mat}\) on the locally discrete bicategory \(\text{Set}\) in which a function \(f : A \to B\) is taken to \(\tilde{f} : A \times B \to V\) defined by

\[
\tilde{f}(i, j) = \begin{cases} 
I, & \text{if } f(i) = j \\
0, & \text{otherwise}
\end{cases}
\]

in which 0 is the initial object.

**Definition 1.2.21.** [Proarrows of Spans] Given a category \(V\) with pullbacks, the bicategory \(\text{Span}(V)\) gives a natural proarrow equipment \(V \to \text{Span}(V)\) on the locally discrete bicategory \(V\) in which a morphism \(f : A \to B\) is taken to the span

\[
A \xleftarrow{id_A} A \xrightarrow{f} B.
\]

**Definition 1.2.22.** [Enriched Categories and Internal Categories] Given a symmetric closed monoidal category \((V, \otimes, I)\) with coproducts, the category of small \(V\)-enriched categories \(V\text{-cat}\) is the underlying category of \(\text{Mnd}_{\text{co}}(\text{Set} \to V\text{-Mat})\).

Given a category with pullbacks \(V'\), the category of \(V'\)-internal categories \(\text{Cat}(V')\) is the underlying category of \(\text{Mnd}_{\text{co}}(V' \to \text{Span}(V'))\).

**Remark 1.2.23.** The introduction of enriched categories via monads of \(V\text{-Mat}\) does not work well for large enriched categories, unless we make tiresome considerations about enlargements of universes/completions. For our setting, however, it is enough to observe that, if \(V\) is a large symmetric monoidal closed category that has large coproducts (indexed by discrete categories), one can consider the bicategory as in Definition 1.2.9 but with discrete categories (objects of \(\text{SET}\)) instead of sets, and matrices indexed by them. By abuse of language, denoting this bicategory as \(V\text{-Mat}\), it is clear that we can consider the category of large \(V\)-enriched categories \(V\text{-Cat}\) defined by the underlying (large) category \(\text{Mnd}_{\text{co}}(\text{SET} \to V\text{-Mat})\).
Explicitly, an internal category of a category with pullbacks $V'$ is a span $A_1 \xleftarrow{d_1} A_2 \xrightarrow{d_0} A_1$, which we denote by $\mathfrak{a}$, together with the multiplication and identity, 2-cells $\mathfrak{a} \circ \mathfrak{a} \Rightarrow \mathfrak{a}$ and $\text{id}_A \Rightarrow \mathfrak{a}$ of $\text{Span}(V')$, satisfying the conditions of monad/lax $\text{Id}_{\text{Cat}}$-algebra of associativity and action of identity described in Definition 4.1 of Chapter 5 [42]. Recall that, by definition, the 2-cells $\mathfrak{a} \circ \mathfrak{a} \Rightarrow \mathfrak{a}$ and $\text{id}_A \Rightarrow \mathfrak{a}$ are just morphisms $\mathfrak{a} : A_2 \times_{(d_0, d_1)} A_2 \to A_1$ and $\mathfrak{a}_0 : A_1 \to A_2$ of $V'$ such that

\[
\begin{array}{ccc}
A_1 & \xleftarrow{d_1} & A_2 \\
\mathfrak{a} & \downarrow & \downarrow \\
A_2 & \xrightarrow{d_0} & A_1
\end{array}
\]

commute. In this case, the object $A_1$ is called the object of objects, the object $A_2$ is called the object of morphisms, $d_1$ is the domain morphism, $d_0$ is the codomain morphism, the morphism $\mathfrak{a}$ is the composition and $\mathfrak{a}_0 : A_1 \to A_2$ is the identity assigning.

**Definition 1.2.24.** [Double Category] The category of double categories is the category of internal categories of $\text{Cat}$, that is to say $\text{Cat}(\text{Cat})$.

A double category $\mathcal{X}$ is, then, a span $\mathcal{X}_1 \xleftarrow{d_1} \mathcal{X}_2 \xrightarrow{d_0} \mathcal{X}_1$ of $\text{Cat}$ with the composition and identity satisfying the usual conditions. Given such a double category, the objects of $\mathcal{X}_1$ are called the objects of the double category, while the morphisms of $\mathcal{X}_1$ are called vertical arrows. The objects $f$ of $\mathcal{X}_2$ are called horizontal arrows (or morphisms) of the double category $\mathcal{X}$ and we denote it by $f : d_1(f) \to d_0(f)$ in which $d_1(f)$ is called the domain of $f$ and $d_0(f)$ is the codomain of $f$. Finally, if $\alpha : f \to g$ is a morphism of $\mathcal{X}_2$, we denote it by

\[
\begin{array}{ccc}
d_1(f) & \xrightarrow{f} & d_0(f) \\
\downarrow & & \downarrow \\
d_1(\alpha) & \xrightarrow{\alpha} & d_0(\alpha) \\
\downarrow & & \downarrow \\
d_1(g) & \xrightarrow{g} & d_0(g)
\end{array}
\]

and we say that $\alpha$ is a 2-cell (or a square) of $\mathcal{X}$.

A 2-category is just a $\text{Cat}$-enriched category. Clearly, if the category of objects of a double category $\mathcal{D}$ is discrete, then it is a 2-category. More generally, the full subcategory of $\text{Cat}(\text{Cat})$ consisting of the double categories without nontrivial vertical arrows is isomorphic to the category of 2-categories and 2-functors $\text{Cat-Cat}$. For suitable categories $V$ instead of $\text{Cat}$, a similar property holds. This generalization is given by Lemma 9.10 of Chapter 3 [43].
1.3 Freely generated n-categories, coinserters and presentations of low dimensional categories

Chapter 2 corresponds to the paper Freely generated n-categories, coinserters and presentations of low dimensional categories [44], DMUC 17-20 or arXiv:1704.04474. As the title suggests, the main subjects of this paper are related to development of a theory towards the study of presentations of low dimensional categories and freely generated categorical structures. Although it was the last paper to be written, it introduces some basic aspects of 2-category theory.

The chapter starts by giving basic aspects of 2-dimensional (weighted) colimits, focusing on coinserter, coequifiers and coinverters. These are the 2-dimensional colimits that have direct applications respectively in the study of adding free morphisms to a category, forcing relations between morphisms and categories of fractions [29, 31]. We are, however, interested in particular cases: freely generated categories, the left adjoint of the inclusion of thin categories and the left adjoint of the inclusion of groupoids.

The category of thin categories $\mathcal{P}d$ and the category of groupoids $\mathcal{G}r$, as defined in 1.1 of Chapter 2, are replete reflective subcategories of $\mathcal{C}at$, with inclusions $M_1 : \mathcal{P}d \to \mathcal{C}at$ and $U_1 : \mathcal{G}r \to \mathcal{C}at$. Hence there is an easy way of characterizing the images of $M_1$ and $U_1$ via universal properties. More precisely, if we denote the reflectors by $M_1 : \mathcal{C}at \to \mathcal{P}d$ and $U_1 : \mathcal{C}at \to \mathcal{G}r$, we have that, given a category $X$ of $\mathcal{C}at$, there is $Y$ of $\mathcal{P}d$ of $\mathcal{G}r$ such that $M_1(Y) \cong X$ ($U_1(Y) \cong X$) if and only if $M_1(X) \cong X$ ($U_1(X) \cong X$). Since $U_1, L_1(X)$ and $M_1, L_1(X)$ are given, respectively, by a coinverter and a coequifier, we get Proposition 2.1.4 and Theorem 2.1.7. Actually, within the context of Chapter 2, the most important fact in this direction is that we can get the category freely generated by a graph $G : \mathcal{G}r^{op} \to \mathcal{S}et$, denoted by $F_1(G)$, via the coinserter of $G$ composed with the inclusion $\mathcal{S}et \to \mathcal{C}at$. This is Lemma 2.2.3. It motivates one of the main points of the paper: to give freely generated categorical structures via coinserters.

After showing these facts, we give further background in Section 2.2. We study basic aspects of freely generated categories. We introduce basic notions of graphs (trees, forests and connectedness) relating with the groupoids and categories freely generated by them. We study reflexive graphs as well. The main importance of reflexive graphs within our context is the fact that the terminal category is freely generated by the terminal reflexive graph, while $F_1$ does not preserve the terminal object. Finally, we also characterize the totally ordered sets that are free categories and show that freeness of groupoids is a property preserved by equivalences.

Then, we give the basic notions of presentations of this paper. On one hand, we show how every monad $T$ induces a natural notion of presentation of $T$-algebras in Section 3 of Chapter 2. In particular, we have that the free category monad induces a notion of presentation of categories. On the other hand, in Section 4 of Chapter 2, we define computads and show how it induces a notion of presentation of categories (and groupoids) with equations between morphisms. We compare both notions of presentations in Theorems 2.4.5 and 2.4.6.

Since we can see computads as free categories together with relations between morphisms, we introduce the suitable variation of the concept of computad to deal with presentation of groupoids: groupoidal computads. This has particular interest in Section 5 of Chapter 2 which deals with the
relation between topology and computads. Moreover, the notion of presentations of groups via groupoidal computads coincides with the usual notion, as explained in Remark 4.19 of Chapter 2.

Section 5 of Chapter 2 ([44]) establishes fundamental connections between topology and computads. We start by showing that the usual association of a topological graph to each graph, usually called geometric realization, consists of a left adjoint functor $\mathcal{F}_{\text{Top}} : \text{grph} \to \text{Top}$ given objectwise by the topological coinserter introduced therein. In this context, we show that there is a distributive law between the monads induced by $\mathcal{F}_1$ and $\mathcal{F}_{\text{Top}}$, which is constructed from the usual notion of concatenation of continuous paths. As a fundamental tool to study presentation of thin and locally thin categorical structures, using the distributive law mentioned above, we give a detailed construction of a 2-dimensional analogue of $\mathcal{F}_{\text{Top}}$, denoted by $\mathcal{F}_{\text{Top}^2} : \text{cmp} \to \text{Top}$, which associates each computad to a topological space.

We introduce the fundamental groupoid functor $\Pi : \text{Top} \to \text{gr}$ using the concept of presentation via computads. More precisely, we firstly associate a computad to each topological space and, then, $\Pi$ is given by the groupoid presented by the associated computad. We prove theorems that relate the fundamental groupoid and freely generated groupoids. In particular, the last results of Section 5 of Chapter 2 state that the fundamental groupoid of $\mathcal{F}_{\text{Top}^2}(g)$ is equivalent to the groupoid presented by $g$. These results show that $\mathcal{F}_{\text{Top}^2}$ formalizes the usual construction of a CW-complex from each presentation of groups, the Lyndon–van Kampen diagrams [26].

In Section 6 of Chapter 2 [44], we introduce our main notions of deficiency. More precisely, we introduce the notion of deficiency of a groupoid w.r.t. presentation of groupoids and the notion of deficiency of a presentation of a $\mathcal{T}$-algebra w.r.t. a finite measure. Under suitable hypotheses, we find both notions to coincide in the particular case of groupoids. They also coincide with the usual notions of deficiency. In this section, mostly using the results of Section 5 of Chapter 2 [44], we also develop a theory towards the study of thin categories and thin groupoids. For instance, we prove that, whenever $g$ is a computad such that $\mathcal{F}_{\text{Top}^2}(g)$ has Euler characteristic smaller than 1, then the groupoid presented by $g$ is not thin. From this fact, we can prove that deficiency of a thin groupoid is 0, recasting and generalizing the result that says that trivial groups have deficiency 0.

Although our definition of computads is equivalent to the original one of [55], we introduce it via a graph satisfying a coincidence property, as it is shown in Remark 8.12 of Chapter 2. The main point of our perspective, besides giving a concise recursive definition, is that it allows us to prove that the 2-category freely generated by a computad $g$ is the coinserter of $g$, when we consider $g$ as a graph internal to an appropriate 2-category of 2-categories: the 2-category of 2-categories, 2-functors and icons [37]. In order to get freely generated $n$-categories via coinserters, we introduce higher dimensional analogues of icons. These concepts and results, including the general result that states that the $n$-category freely generated by an $n$-computad is its coinserter, are given in Sections 7 and 8 of Chapter 2 [44].

We finish the paper studying aspects of presentations of 2-categories. We show simple examples of locally thin 2-categories that are not free and develop a theory to study presentations of locally thin and groupoidal 2-categories. We give efficient presentations of 2-categories related to the strict descent object. In the end of Chapter 2 [44], we sketch a construction of the 3-dimensional analogue of $\mathcal{F}_{\text{Top}^2}$, that is to say $\mathcal{F}_{\text{Top}^3}$, which associates a 3-dimensional CW-complex to each 3-computad.
1.4 Beck-Chevalley

The mate correspondence \cite{33, 52} is a fundamental framework in 2-dimensional category theory. For instance, this correspondence is in the core of the techniques of Chapter 2 to construct $\mathcal{F}_{\text{Top}_2}$ and the distributive law between the monads induced by $\mathcal{F}_1$ and $\mathcal{F}_{\text{Top}_1}$. Another important example is in Chapter 3: the Beck-Chevalley condition, written in terms of a simple mate correspondence, plays an important role in the proof of the Bénabou-Robaud Theorem.

The main aim of this section is to present the Beck-Chevalley condition within the context of 2-dimensional monad theory. In order to do so, we present the most elementary version of mate correspondence in Theorem 1.4.6. We start by defining and giving elementary results on adjunctions in a 2-category.

**Definition 1.4.1.** [Adjunction] An adjunction in a 2-category $\mathcal{A}$ is a quadruplet

$$(f : Y \to Z, g : Z \to Y, \varepsilon : fg \Rightarrow \text{id}_Z, \eta : \text{id}_Y \Rightarrow gf),$$

in which $f, g$ are 1-cells and $\varepsilon, \eta$ are 2-cells of $\mathcal{A}$, such that

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{
Y \ar[r]^{f} & Z \ar[d]_{\eta} \\
Y \ar[r]_{f} & Z
}
\end{array}
\quad
\begin{array}{c}
\xymatrix{
Y \ar[r]^{g} & Z \ar[d]_{\varepsilon} \\
Y \ar[r]_{g} & Z
}
\end{array}
\end{array}$$

are respectively the identity 2-cells $f \Rightarrow f$ and $g \Rightarrow g$. In this case, we denote the adjunction by $(f \dashv g, \varepsilon, \eta) : Y \to Z$. For short, we also denote such an adjunction by just $f \dashv g$ when the counit and unit are implicit.

**Remark 1.4.2.** If $(f \dashv g, \varepsilon, \eta) : Y \to Z$ is an adjunction, $f$ is called left adjoint, $g$ is called right adjoint, $\varepsilon$ is called the counit and $\eta$ is called the unit of the adjunction. Moreover, the equations of Definition 1.4.1 are called triangle identities.

**Remark 1.4.3.** It is clear that adjoints are unique up to isomorphism. More precisely, if $(\tilde{f} \dashv g, \mu, \rho)$ and $(f \dashv g, \varepsilon, \eta) : Y \to Z$ then

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{
Y \ar[r]^{f} & Z \ar[d]_{\eta} \\
Y \ar[r]_{\tilde{f}} & Z
}
\end{array}
\quad
\begin{array}{c}
\xymatrix{
Y \ar[r]^{\tilde{f}} & Z \ar[d]_{\rho} \\
Y \ar[r]_{f} & Z
}
\end{array}
\end{array}$$

is the inverse of

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{
Y \ar[r]^{f} & Z \ar[d]_{\mu} \\
Y \ar[r]_{\tilde{f}} & Z
}
\end{array}
\quad
\begin{array}{c}
\xymatrix{
Y \ar[r]^{\tilde{f}} & Z \ar[d]_{\varepsilon} \\
Y \ar[r]_{f} & Z
}
\end{array}
\end{array}$$

by the triangle identities. In particular, $f \cong \tilde{f}$. 
Let $\mathcal{A}$ be a 2-category. We can construct a category of adjunctions $\mathcal{A}^{\text{adj}}$ of $\mathcal{A}$. The objects of $\mathcal{A}^{\text{adj}}$ are the objects of $\mathcal{A}$, but the morphisms are adjunctions $(f \dashv g, \varepsilon, \eta) : Y \to Z$. The identities are the adjunctions between identities with identities counit and unit. Given adjunctions $(f_2 \dashv g_2, \varepsilon_2, \eta_2) : Y \to Z$ and $(f_1 \dashv g_1, \varepsilon_1, \eta_1) : X \to Y$, the composition is given by $(f_2f_1 \dashv g_1g_2, \varepsilon_3, \eta_3) : X \to Z$ in which $\varepsilon_3$ and $\eta_3$ are defined below.

Of course, 2-functors take adjunctions to adjunctions. However, pseudofunctors do not. Instead, in this case, we can say that left (or right) adjoints are taken to left (or right) adjoints. More precisely, we have Lemma 1.4.5.

In order to prove such result, we use Lemma 1.4.4, a basic result on the image of pasting of 2-cells. Following the notation established in Definition 4.2.1 [41], we have:

**Lemma 1.4.4.** If $L : \mathcal{A} \to \mathcal{B}$ is a pseudofunctor, then:

\[
\begin{align*}
L(W) & \xrightarrow{L(\alpha)} L(X) \\
L(W) & \xrightarrow{L(\gamma)} L(Y)
\end{align*}
\]

Proof. Firstly, by the interchange law, it is clear that the right side of the first equation above is equal to

\[
l_{\eta_3} \cdot (L(\beta) \ast L(\id_Y)) \cdot (L(m) \ast L(n)) \cdot (1_{L(\gamma)} L(\id_Y) \ast L(\beta) L(\id_Y)) \cdot (L id_Y) \ast (L(\alpha) \ast L(\varepsilon_3)) \cdot \Gamma_{\varepsilon_3}^{-1}.
\]

Then, by the naturality of Definition 4.2.1 [41], this is equal to

\[
L(\beta \ast \id_Y) \cdot (L(\mu) \ast L(\gamma)) \cdot (1_{L(\varepsilon_3)} L(\id_Y) \ast L(\eta_3)) \cdot (L id_Y) \ast (L(\alpha) \ast \varepsilon_3).
\]
which is indeed equal to the left side of the equation above, since, by the associativity of Definition 4.2.1 [41],
\[
I_{L(h)} \cdot \left( I_{f(h)} \cdot \left( \text{id}_{L(f)} \ast I_{h(g)} \right) \right) \cdot \left( I_{f(h)} \cdot \left( \text{id}_{L(f)} \ast I_{h(g)} \right) \right)^{-1} = I_{L(h)} \cdot \left( \text{id}_{L(f)} \ast I_{h(g)} \right) \cdot \left( \text{id}_{L(f)} \ast I_{h(g)} \right)^{-1} = \text{id}_{L(h)}
\]
and \( L(\beta \ast \text{id}_x) \cdot L(\text{id}_y \ast \alpha) = L\left( (\beta \ast \text{id}_y) \cdot (\text{id}_y \ast \alpha) \right) \). This proves the first equation. The proof of the second one is analogous. 

Lemma 1.4.5. If \( L : \mathcal{A} \to \mathcal{B} \) is a pseudofunctor and \((f \dashv g, \varepsilon, \eta) : Y \to Z\) is an adjunction in \( \mathcal{A} \), then
\[
(L(f) \dashv L(g), \varepsilon_{L(f)} \cdot L(\varepsilon) \cdot \varepsilon_{L(f)}^{-1} \cdot L(\eta) \cdot \eta_{L(f)})
\]
is an adjunction in \( \mathcal{B} \). Whenever \((f \dashv g, \varepsilon, \eta)\) is implicit, we usually denote the induced adjunction above by \( L(f) \dashv L(g) \).

Proof. By Lemma 1.4.4, we get that:
\[
\begin{array}{ccc}
L & Y & Z \\
\downarrow & \varepsilon & \downarrow \\
Y & & Z
\end{array}
\]

\[
= I_{\text{id}_Z} \cdot \left( I_{\text{id}_Z} \right)^{-1} \cdot \varepsilon_{L(f)} \cdot L(\varepsilon) \cdot \varepsilon_{L(f)}^{-1} \cdot L(\eta) \cdot \eta_{L(f)}
\]

Since, by Equation 2 of Definition 4.2.1 [41],
\[
\varepsilon_{L(f)}^{-1} = \text{id}_{L(f)} \ast \varepsilon_f, \quad \varepsilon_{L(f)} = \varepsilon_f \ast \text{id}_{L(f)} \quad \text{id}_{L(f)} = \varepsilon_f \ast \text{id}_{L(f)} \ast \varepsilon_f, \quad \varepsilon_{L(f)} \ast \eta_{L(f)} = \text{id}_{L(f)} \ast \varepsilon_f
\]

the result follows. 

Theorem 1.4.6 (Mate Correspondence). Let \((f \dashv g) := (f \dashv g, \varepsilon, \eta) : Z \to Y\) and \((l \dashv u) := (l \dashv u, \mu, \rho) : W \to X\) be adjunctions in a 2-category \( \mathcal{A} \). Given 1-cells \( m : X \to Y \) and \( n : W \to Z\) of \( \mathcal{A} \), there is a bijection \( \mathcal{A}(X, Z)(nu, gm) \cong \mathcal{A}(W, Y)(fn, ml)\), given by \( \alpha \mapsto \mathcal{A}^{\ell \dashv \eta}_{l \dashv u} \) in which \( \mathcal{A}^{\ell \dashv \eta}_{l \dashv u} \) is defined.
by:

\[ \alpha : \text{mate of } \alpha \text{ under the adjunction } (l \dashv u, \mu, \rho) \text{ and the adjunction } (f \dashv g, \varepsilon, \eta). \]

**Proof.** The map \( \beta \mapsto \beta^{(f \dashv g)}_{(l \dashv u)} \) defined by

\[ \beta^{(f \dashv g)}_{(l \dashv u)} = \begin{array}{c}
X \\
\downarrow^u \\
W \\
\downarrow^m \\
Y \\
\downarrow^n \\
Z \\
\downarrow^f \\
Y
\end{array} \]

is clearly the inverse of \( \alpha \mapsto \alpha^{(f \dashv g)}_{(l \dashv u)}. \)

Actually, we can say much more about this correspondence. For instance, this is part of an isomorphism of double categories. More precisely, given a 2-category \( \mathcal{A} \), we define two double categories \( \text{RAdj}(\mathcal{A}) \) and \( \text{LAdj}(\mathcal{A}) \). The objects and the horizontal arrows of both double categories are the objects and 1-cells of \( \mathcal{A} \), while the vertical arrows are adjunctions \( (f \dashv g, \varepsilon, \eta) \) of \( \mathcal{A} \). Given vertical arrows \( (f \dashv g, \varepsilon, \eta) : Z \to Y, (l \dashv u, \mu, \rho) : W \to X \) and horizontal arrows \( m : X \to Y, n : W \to Z \), the squares of \( \text{RAdj}(\mathcal{A}) \) are 2-cells \( \alpha : nu \Rightarrow gm \) of \( \mathcal{A} \), while the squares of \( \text{LAdj}(\mathcal{A}) \) are 2-cells \( \beta : fn \Rightarrow ml \). The composition of squares are given by pasting of 2-cells, the composition of horizontal arrows is the composition of 1-cells and the composition of vertical arrows is the composition of adjunctions as in \( \mathcal{A}^{\text{adj}} \). It is clear, then, that the mate correspondence induces an isomorphism between \( \text{RAdj}(\mathcal{A}) \) and \( \text{LAdj}(\mathcal{A}) \). In particular, the mate correspondence respects vertical and horizontal compositions.

As a first application of these observations on the mate correspondence, we give another proof of the statement of Remark 1.4.3. Indeed, in the context of Remark 1.4.3, we take the 2-cells that are mates of the identity

\[ \begin{array}{c}
Z \\
\downarrow^g \\
Y \\
\downarrow^g \\
Y
\end{array} \]
under the adjunctions \((f \dashv g, \mu, \rho)\) and \((f \dashv g, \varepsilon, \eta)\), and under the adjunctions \((f \dashv g, \varepsilon, \eta)\) and \((\tilde{f} \dashv \tilde{g}, \mu, \rho)\). They are respectively denoted by \(\psi : f \text{id}_Y \Rightarrow \text{id}_X \tilde{f}\) and \(\psi' : \tilde{f} \text{id}_Y \Rightarrow \text{id}_X f\) (which actually are the 2-cells of Remark 1.4.3). Since the mate correspondence preserves horizontal composition, the compositions \(\psi' \psi\) and \(\psi \psi'\) are respectively the mates of the 2-cell \(\text{id}_Y \tilde{g} = \text{id}_X g\) above under \(f \dashv g\) and itself, and under \(\tilde{f} \dashv \tilde{g}\) and itself; that is to say, the identity on \(f\) and the identity on \(\tilde{f}\). In particular, \(\psi : f \Rightarrow \tilde{f}\) is an isomorphism.

**Remark 1.4.7.** If \((f \dashv g, \varepsilon, \eta) : Y \rightarrow Z\) is an adjunction in the 2-category \(\text{Cat}\), we know that \(Z(f-, -) \cong Y(-, g-)\). The mate correspondence generalizes this fact since, assuming now that \((l \dashv u, \mu, \rho) : Y' \rightarrow Z'\) is an adjunction in a 2-category \(\mathfrak{A}\), as a particular case of Theorem 1.4.6, we conclude that

\[ \mathfrak{A}(X, Y')(-, u-) \cong \mathfrak{A}(X, Z')(l-, -) \]

for any object \(X\) of \(\mathfrak{A}\). Still, up to size considerations, the Yoneda structure \([57]\) of \(\text{CAT}\) implies that:

given a functor \(\tilde{f} : Y \rightarrow Z\), \(\tilde{f} \dashv \tilde{g}\) in \(\text{Cat}\) if and only if there is a natural isomorphism

\[
\begin{array}{ccc}
\text{CAT}[Y^{op}, \text{SET}] & \xrightarrow{\cong} & \text{CAT}[Z^{op}, \text{SET}] \\
\mathcal{B}_Y & \xrightarrow{(\varphi)} & \mathcal{B}_Z \\
Y & \xrightarrow{\tilde{g}} & Z
\end{array}
\]

in which \(\mathcal{B}_Y, \mathcal{B}_Z\) denote the Yoneda embeddings. Moreover, it should be noted that, if \(\tilde{f} \dashv \tilde{g}\), then

\[
\text{CAT}[\tilde{f}^{op}, \text{SET}] \cong \text{CAT}[\tilde{g}^{op}, \text{SET}]
\]

by the 2-functoriality of \(\text{CAT}[(-)^{op}, \text{SET}] : \text{CAT}^{coop} \rightarrow \text{CAT}\).

It is clear that the images of the mates by 2-functors are the mates of the images. In the case of pseudofunctors, it follows from Lemma 1.4.4 that:

**Lemma 1.4.8.** Let \(E : \mathfrak{A} \rightarrow \mathfrak{B}\) be a pseudofunctor and \((f \dashv g, \varepsilon, \eta), (l \dashv u, \mu, \rho)\) adjunctions in \(\mathfrak{A}\).

Given a 2-cell \(\alpha : nu \Rightarrow gm\),

\[
E\left(\alpha_{l-\mu} \dashv \eta\right) = E(\alpha)_{E(f) \dashv E(g)},
\]

in which \(E\left(\alpha_{l-\mu} \dashv \eta\right) := \varepsilon_{mu}^{-1} \cdot E\left(\alpha_{l-\mu} \dashv \eta\right) \cdot \varepsilon_{fn}\) and \(E(\alpha) := \varepsilon_{nu}^{-1} \cdot E(\alpha) \cdot \varepsilon_{mn}\). In other words,

\[
\varepsilon_{mu}^{-1} \cdot E\left(\alpha_{l-\mu} \dashv \eta\right) \cdot \varepsilon_{fn}
\]

is the mate of \(\varepsilon_{nu}^{-1} \cdot E(\alpha) \cdot \varepsilon_{mn}\) under \(E(l) \dashv E(u)\) and \(E(f) \dashv E(g)\).

We also have an important result relating mates and pseudonatural transformations:
Lemma 1.4.9. Let \( \lambda : E \rightarrow L \) be a pseudonatural transformation between pseudofunctors. If \((f \dashv g, \varepsilon, \eta) : Y \rightarrow Z\) is an adjunction of \(\mathcal{A}\), then the mate of

\[
\begin{array}{c}
E(Y) \\
\downarrow \lambda_Y \\
E(Z) \\
\downarrow \lambda_f \\
L(Y) \\
\downarrow \lambda_L \\
L(Z)
\end{array}
\]

under the adjunction \((E(f) \dashv E(g), \varepsilon_f, \varepsilon^{-1}_g, E(\varepsilon) \cdot \varepsilon_f)\) and \((L(f) \dashv L(g), \Gamma^{-1}_Z \cdot L(\varepsilon), \Gamma^{-1}_Y \cdot L(\eta))\) is equal to \(\lambda^{-1}_g\).

Proof. In order to simplify the terminology, we prove it for a pseudonatural transformation \(\lambda : E \rightarrow L\) between 2-functors. The proof in the case of pseudofunctors is analogous.

The proof consists in verifying that the mate \(\lambda \cdot L(f) \cdot L(g)\) composed with \(\lambda_g\) is equal to the identity \(L(g) \lambda_z \Rightarrow L(g) \lambda_z\). Firstly, observe that this composition is equal to

\[
\begin{array}{c}
E(Z) \\
\downarrow \lambda_Z \\
E(f) \\
\downarrow \lambda_f \\
E(Y) \\
\downarrow \lambda_Y \\
L(f) \\
\downarrow \lambda_L \\
L(g) \\
\downarrow \lambda_g \\
L(Z) \\
\downarrow \lambda_g \\
L(Y)
\end{array}
\]

Since \(\lambda_{LZ} = \text{id}_{\lambda Z}\), by Equations 1 and 3 of Definition 4.2.1 [41] we get that this composition is equal to

\[
\begin{array}{c}
E(Z) \\
\downarrow \lambda_Z \\
E(Y) \\
\downarrow \lambda_Y \\
L(Y) \\
\downarrow \lambda_g \\
L(Y)
\end{array}
\]

which is clearly equal to the identity \(L(g) \lambda_z \Rightarrow L(g) \lambda_z\), since \((L(f) \dashv L(g), L(\varepsilon), L(\eta))\) is an adjunction.

\(\square\)
Definition 1.4.10. [Beck-Chevalley condition] Let \((f \downarrow g) := (f \downarrow g, \varepsilon, \eta) : Z \to Y\) and \((l \downarrow u) := (l \downarrow u, \mu, \rho) : W \to X\) be adjunctions in a 2-category \(\mathcal{A}\). Assume that \(\alpha : nu \Rightarrow gm\) is a 2-cell of \(\mathcal{A}\). We say that

\[
\begin{array}{c}
X \\
\downarrow m \\
\downarrow n \\
W \\
\downarrow g \\
\downarrow u \\
Y \\
\end{array}
\]

satisfies the Beck-Chevalley condition if the mate of \(\alpha\) under \(l \downarrow u\) and \(f \downarrow g\) is an invertible 2-cell.

Outside any context, the meaning of the Beck-Chevalley condition might seem vacuous. Even when some context is provided, it is many times considered as an isolated technical condition. In this thesis, however, this condition is always applied in the context of doctrinal adjunction. More precisely, our informal perspective is that, whenever the Beck-Chevalley condition plays an important role, we can frame our problem in terms of 2-dimensional monad theory, getting a problem of lifting of adjunctions of the base 2-category to the 2-category of pseudoalgebras. The Beck-Chevalley condition is precisely the obstruction condition to this lifting. The most important example of this approach in this thesis is Chapter 3 or, more specifically, the proof of the Bénabou-Robaud Theorem presented therein.

Below, we briefly explain the Beck-Chevalley condition within the context of 2-dimensional monad theory. This section can be considered, then, as prerequisite to the understanding of Section 7 of Chapter 3 [43], since herein we do not assume familiarity with the doctrinal adjunction. We also show in 1.4.17 how, within our context, Kock-Zöberlein pseudomonads encompass the situation of “the Beck-Chevalley condition always holding”.

We start by showing the most elementary version of an important bijection between colax and lax \(\mathcal{T}\)-structures in adjoint morphisms. Again, the mate correspondence is the basic technique to introduce this bijection.

Let \(\mathcal{T}\) be a pseudomonad on a 2-category \(\mathcal{B}\) and \(g : Z \to Y\) a morphism of \(\mathcal{B}\). Given lax \(\mathcal{T}\)-algebras \(y = (Y, \text{lax}\text{-}\text{alg}_g, \underline{Y}, \underline{Y}_0)\) and \(z = (Z, \text{lax}\text{-}\text{alg}_z, \underline{Z}, \underline{Z}_0)\), the collection of lax \(\mathcal{T}\)-structures for \(g : Z \to Y\) w.r.t. \(z\) and \(y\), denoted by \(\text{lax}\text{-}\text{Alg}_g(z, y)_\mathcal{T}\), is the pullback of the inclusion of \(g\) in the category of morphisms \(\mathcal{B}(Z, Y)\), \(1 \to \mathcal{B}(Z, Y)\), along the functor \(\text{lax}\text{-}\text{Alg}_g(z, y) \to \mathcal{B}(Z, Y)\) induced by the forgetful 2-functor. Analogously, given a morphism \(f : Y \to Z\) of \(\mathcal{B}\), \(\text{lax}\text{-}\text{Alg}_f(y, z)\) is the pullback of the inclusion of \(f\) into \(\mathcal{B}(Y, Z)\) along the forgetful functor \(\text{lax}\text{-}\text{Alg}_f(y, z) \to \mathcal{B}(Y, Z)\).

It is clear, then, that a lax \(\mathcal{T}\)-morphism in \(\text{lax}\text{-}\text{Alg}_f(z, y)\) corresponds to a 2-cell \(\langle \bar{g} \rangle : \text{alg}_y\mathcal{T}(g) \Rightarrow \text{alg}_z\mathcal{T}(f)\) of \(\mathcal{B}\) satisfying the axioms of Definition 5.4.1 [42], while a colax \(\mathcal{T}\)-morphism in \(\text{colax}\text{-}\text{Alg}_f(y, z)\) corresponds to a 2-cell \(\langle \bar{f} \rangle : \text{colax}\text{-}\text{alg}_y\mathcal{T}(f) \Rightarrow \text{colax}\text{-}\text{alg}_z\mathcal{T}(g)\) of \(\mathcal{B}\) satisfying the axioms of Definition 1.2.11.

Moreover, we can consider the category of lax \(\mathcal{T}\)-structures for \(f\) w.r.t. lax \(\mathcal{T}\)-algebras which is the pullback of the inclusion of the morphism \(f\) into \(\mathcal{B}\), \(2 \to \mathcal{B}\), along the forgetful 2-functor \(\text{lax}\text{-}\text{Alg}_f \to \mathcal{B}\). Finally, the category of colax \(\mathcal{T}\)-structures for \(g\) w.r.t. lax \(\mathcal{T}\)-algebras is the pullback of the inclusion of the morphism \(g\) into \(\mathcal{B}\) along \(\text{lax}\text{-}\text{Alg}_f \to \mathcal{B}\).
**Theorem 1.4.11** (Colax and lax structures in adjoints). Let $\mathcal{T}$ be a pseudomonad on $\mathcal{B}$ and

$$(f \dashv g, \varepsilon, \rho) : Y \to Z$$

an adjunction in $\mathcal{B}$. Given lax $\mathcal{T}$-algebras $y = (Y, \text{alg}_y, \{\mathcal{Y}_y\})$ and $z = (Z, \text{alg}_z, \{\mathcal{Z}_z\})$, the mate correspondence under the adjunction $(\mathcal{F}(f) \dashv \mathcal{F}(g), t_y^{-1} \cdot \mathcal{F}(\varepsilon) \cdot t_f, t_f^{-1} \cdot \mathcal{F}(\rho) \cdot t_y)$ and the adjunction $(f \dashv g, \varepsilon, \rho)$ induces a bijection

$$\diamond : \text{Lax-}\mathcal{T}\text{-Alg}_l(z, y)_g \cong \text{Lax-}\mathcal{T}\text{-Alg}_{cl}(y, z)_f.$$

These bijections induce an isomorphism between the category of lax $\mathcal{T}$-structures for $g : Z \to Y$ and category of colax $\mathcal{T}$-structures for $f : Y \to Z$ w.r.t. lax $\mathcal{T}$-algebras.

**Proof.** Assume that $(f \dashv g, \varepsilon, \rho) : Y \to Z$ is an adjunction in $\mathcal{B}$ and $(\mathcal{F}, m, \eta, \mu, \iota, \tau)$ is a pseudomonad on $\mathcal{B}$ as in the hypothesis. Given 2-cells $\langle \tilde{g} \rangle : \text{alg}_y \mathcal{F}(g) \Rightarrow \text{gal}_y$ and $\langle \tilde{f} \rangle : \text{gal}_z \mathcal{F}(f) \Rightarrow \text{alg}_z \mathcal{F}(f)$ that are mates under the adjunctions $(\mathcal{F}(f) \dashv \mathcal{F}(g), t_y^{-1} \cdot \mathcal{F}(\varepsilon) \cdot t_f, t_f^{-1} \cdot \mathcal{F}(\rho) \cdot t_y)$ and $(f \dashv g, \varepsilon, \rho)$, we have that:

1. By Lemmas 1.4.8 and 1.4.9, the 2-cells

\[
\begin{array}{ccc}
\mathcal{T}Z & \overset{\text{alg}_x}{\longrightarrow} & Z \\
\mathcal{F}(f) & \overset{\tilde{f}}{\longleftarrow} & \mathcal{F}(g) \\
\mathcal{T}^2Z & \overset{\text{alg}_y}{\longrightarrow} & \mathcal{T}Y \\
\mathcal{T}^2Y & \overset{\text{alg}_z}{\longrightarrow} & \mathcal{T}Z
\end{array}
\]

are respectively the mates of

\[
\begin{array}{ccc}
\mathcal{T}Z & \overset{\text{alg}_x}{\longrightarrow} & Z \\
\mathcal{F}(g) & \overset{\tilde{g}}{\longleftarrow} & \mathcal{F}(f) \\
\mathcal{T}Y & \overset{\text{alg}_y}{\longrightarrow} & \mathcal{T}Y \\
\mathcal{T}Z & \overset{\text{alg}_z}{\longrightarrow} & \mathcal{T}Z
\end{array}
\]

under the adjunctions $\mathcal{T}^2(f) \dashv \mathcal{T}^2(g)$ and $f \dashv g$. 

2. By Lemma 1.4.9, the 2-cells

\[
\begin{array}{c}
Z & \xrightarrow{f} & Y \\
\eta_z & & \eta_f \\
\mathcal{F}Z & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}Y \\
\alpha_{\mathcal{F}z} & \xleftarrow{\eta_f} & \alpha_{\mathcal{F}y} \\
Z & \xrightarrow{f} & Y
\end{array}
\begin{array}{c}
Z & \xrightarrow{g} & Y \\
\eta_z & & \eta_g \\
\mathcal{F}Z & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}Y \\
\alpha_{\mathcal{F}z} & \xleftarrow{\eta_g} & \alpha_{\mathcal{F}y} \\
Z & \xrightarrow{g} & Y
\end{array}
\]

are respectively the mates of

\[
\begin{array}{c}
Z & \xrightarrow{g} & Y \\
\eta_z & & \eta^{-1}_g \\
\mathcal{F}Z & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}Y \\
\alpha_{\mathcal{F}z} & \xleftarrow{\eta_g} & \alpha_{\mathcal{F}y} \\
Z & \xrightarrow{g} & Y
\end{array}
\begin{array}{c}
Z & \xrightarrow{g} & Y \\
\eta_z & & \eta^{-1}_g \\
\mathcal{F}Z & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}Y \\
\alpha_{\mathcal{F}z} & \xleftarrow{\eta_g} & \alpha_{\mathcal{F}y} \\
Z & \xrightarrow{g} & Y
\end{array}
\]

under \(f \dashv g\) and itself.

Therefore \(\varphi : \alpha_{\mathcal{F}z} \mathcal{F}(g) \Rightarrow g \alpha_{\mathcal{F}y}\) corresponds to a lax \(\mathcal{F}\)-morphism in \(\text{Lax-}\mathcal{F}\text{-Alg}_f(z, y)_g\) if and only if \(\langle \overline{\tau} \rangle : f \alpha_{\mathcal{F}y} \Rightarrow \alpha_{\mathcal{F}z} \mathcal{F}(f)\) corresponds to a colax \(\mathcal{F}\)-morphism in \(\text{Lax-}\mathcal{F}\text{-Alg}_{c\ell}(y, z)_f\).

\[\square\]

**Remark 1.4.12.** Clearly, we have dual results. For instance, given colax \(\mathcal{F}\)-algebras \(z, y\) and an adjunction \((f \dashv g, \epsilon, \rho) : Y \rightarrow Z\) in the base 2-category \(\mathcal{B}\), we can analogously define the collections \(\text{Colax-}\mathcal{F}\text{-Alg}_{c\ell}(y, z)_f\) and \(\text{Colax-}\mathcal{F}\text{-Alg}_f(z, y)_g\) of colax \(\mathcal{F}\)-structures for \(f\) and lax \(\mathcal{F}\)-structures for \(g\). The mate correspondence induces a bijection between such collections.

Now, we can introduce the Beck-Chevalley condition in the context of lax \(\mathcal{F}\)-morphisms, being a particular case of Definition 1.4.10. The relevance of this case is demonstrated in Theorem 1.4.14.

**Definition 1.4.13.** [Beck-Chevalley within 2-dimensional monad theory] Let \(\mathcal{F}\) be a pseudomonad on \(\mathcal{B}\) and \((f \dashv g, \epsilon, \rho) : Y \rightarrow Z\) an adjunction in \(\mathcal{B}\). Assume that \(g = (g, \langle \overline{\varphi} \rangle) : z \rightarrow y\) is a lax \(\mathcal{F}\)-morphism between the lax \(\mathcal{F}\)-algebras \(z = (Z, \alpha_{\mathcal{F}z}, z, z_0)\) and \(y = (Y, \alpha_{\mathcal{F}y}, y, y_0)\).

We say that \(g\) satisfies the Beck-Chevalley condition if the corresponding colax \(\mathcal{F}\)-morphism \((f, \Diamond \langle \varphi \rangle) : z \rightarrow y\) via the bijection of Theorem 1.4.11 is a \(\mathcal{F}\)-pseudomorphism. That is to say, \(g\)
satisfies the Beck-Chevalley condition if

\[
\diamond \langle g \rangle = T Y (f) \cdot \mathsf{alg}_{\epsilon} \cdot T (g) \cdot \mathsf{alg}_{\rho} \cdot \mathsf{alg}_{\varphi} 
\]

is an invertible 2-cell.

Given a 2-monad \( \mathcal{T} \) on a 2-category \( \mathcal{B} \), while the forgetful 2-functors \( \mathcal{T}\text{-Alg}_s \to \mathcal{B} \) and \( \mathsf{Ps}\text{-} \mathcal{T}\text{-Alg} \to \mathcal{B} \) respectively reflect isomorphisms and equivalences, the forgetful 2-functor

\[
\mathsf{Lax}\text{-} \mathcal{T}\text{-Alg}_\ell \to \mathcal{B}
\]

reflects right adjoints that satisfy the Beck-Chevalley condition. More generally, the \textit{Doctrinal Adjunction} characterizes when the unit and the counit of an adjunction satisfy the condition of being a \( \mathcal{T} \)-transformation (given in in Eq. 3 of Definition 5.4.1 [42]).

\textbf{Theorem 1.4.14 (Doctrinal Adjunction).} \textit{Let \( \mathcal{T} \) be a pseudomonad on \( \mathcal{B} \) and

\[
g = (g, \langle g \rangle) : z \to y, f = (f, \langle f \rangle) : y \to z
\]

lax \( \mathcal{T} \)-morphisms. Assume that \( (f \dashv g, \varepsilon, \rho) : Y \to Z \) is an adjunction in \( \mathcal{B} \). The 2-cells \( \varepsilon, \rho \) give \( \mathcal{T} \)-transformations \( \tilde{\varepsilon} : fg \Rightarrow \mathsf{id}_y \) and \( \tilde{\rho} : \mathsf{id}_z \Rightarrow g \mathsf{f} \) if and only if \( \langle f \rangle \) is invertible and \( \diamond \langle g \rangle = \langle f \rangle^{-1} \). In this case, \( (f \dashv g, \tilde{\varepsilon}, \tilde{\rho}) \) is an adjunction in \( \mathsf{Lax}\text{-} \mathcal{T}\text{-Alg}_\ell \).

\textit{Proof.} Assume that \( \varepsilon, \rho \) give \( \mathcal{T} \)-transformations \( \tilde{\varepsilon} : fg \Rightarrow \mathsf{id}_y \) and \( \tilde{\rho} : \mathsf{id}_z \Rightarrow g \mathsf{f} \) in \( \mathsf{Lax}\text{-} \mathcal{T}\text{-Alg}_\ell \). That is to say, by hypothesis, \( \varepsilon, \rho \) satisfy Eq. 3 of Definition 5.4.1 [42]. Denoting \( t^{-1}_f \cdot \mathcal{T}(\rho) \cdot t_y \) by \( \mathcal{T}(\rho) \),
we have that \( f \cdot \langle \diamond \langle g \rangle \rangle \) and \( \langle \diamond \langle g \rangle \rangle \cdot f \) are respectively equal to

\[
\text{Id}_Y \Longrightarrow f \langle \text{alg}_y \rangle \qquad \text{Id}_Z \Longrightarrow g \langle \text{alg}_z \rangle.
\]

which, by Eq. 3 of Definition 5.4.1 and the triangle identities (of Definition 1.4.1), are respectively equal to the identities \( f \text{alg}_y \Rightarrow f \text{alg}_y \) and \( \text{alg}_z T(f) \Rightarrow \text{alg}_z T(f) \). Therefore the proof of the first part is complete.

Reciprocally, assume now that \( \diamond \langle g \rangle = \langle f \rangle ^{-1} \). We prove below that \( \rho \) gives a \( \mathcal{T} \)-transformation \( \tilde{\rho} : \text{id}_y \Rightarrow gf \). On one hand, the mate of \( \rho * \text{id}_{\text{alg}_y} \) under the adjunction of identities and the adjunction \( f \dashv g, \varepsilon, \rho \) is equal to \( \langle f \rangle \cdot (\diamond \langle g \rangle) \) which by hypothesis is equal to \( \text{id}_f \text{alg}_y \). On the other hand, the mate of \( \rho * \text{id}_{\text{alg}_y} \) under the adjunction of identities and the adjunction \( f \dashv g, \varepsilon, \rho \) is also equal to \( \text{id}_f \text{alg}_y \) by the triangle identity. Therefore, by the mate correspondence (Theorem 1.4.6), we conclude that the left side of the equation above is equal to
\( \rho \ast \text{id}_{\text{alg} \ Y} \). Therefore

\[
\begin{array}{c}
\text{Alg}_Y \\
\text{Alg}_Y \\
\text{Alg}_Y
\end{array}
\begin{array}{c}
\mathcal{Y} \\
\mathcal{Y} \\
\mathcal{Y}
\end{array}
\begin{array}{c}
Y \\
Y \\
Y
\end{array}
\xymatrix{
\mathcal{Y} \ar[r]^(0.5){\text{alg}_Y} & Y & \\
\mathcal{Y} \ar[r]_(0.5){\text{alg}_Y} & Y & \\
\mathcal{Y} \ar[r] & Y
}
\]

which, by Remark 1.2.12, shows that \( \rho \) satisfies Eq. 3 of Definition 5.4.1 [42]. This proves that indeed \( \rho \) gives a \( \mathcal{T} \)-transformation \( \tilde{\rho} : \text{id} \to g \mathcal{f} \). The proof for \( \varepsilon \) is analogous. \( \square \)

**Corollary 1.4.15.** Let \( U : \text{Lax}-\mathcal{T}-\text{Alg}_\ell \to \mathcal{B} \) be the forgetful 2-functor. Given a lax \( \mathcal{T} \)-morphism \( \mathcal{f} : y \to z \):

- \( \mathcal{f} \) is left adjoint in \( \text{Lax}-\mathcal{T}-\text{Alg}_\ell \) if and only if \( U(\mathcal{f}) \) is left adjoint and \( \mathcal{f} \) is a \( \mathcal{T} \)-pseudomorphism;
- \( \mathcal{f} \) is right adjoint if and only if \( U(\mathcal{f}) \) is right adjoint and \( \mathcal{f} \) satisfies the Beck-Chevalley condition.

In the case of pseudomorphisms, the second condition remains equally, but, for the case of lifting of left adjoints, we still need to assure that the right adjoint is going to be a pseudomorphism. More precisely:

**Corollary 1.4.16.** Let \( U : \text{Lax}-\mathcal{T}-\text{Alg}_\ell \to \mathcal{B} \) be the forgetful 2-functor. Given a \( \mathcal{T} \)-pseudomorphism \( \mathcal{f} = (f, \langle \mathcal{T} \rangle) : y \to z \):

- \( \mathcal{f} \) is left adjoint in \( \text{Lax}-\mathcal{T}-\text{Alg}_\ell \) if and only if \( U(\mathcal{f}) \) is left adjoint and \( \circ^{-1} \langle \mathcal{T} \rangle \) is an invertible 2-cell;
- \( \mathcal{f} \) is right adjoint if and only if \( U(\mathcal{f}) \) is right adjoint and \( \mathcal{f} \) satisfies the Beck-Chevalley condition.

### 1.4.17 Kock-Zöberlein pseudomonads

The concept of Kock-Zöberlein doctrine was originally introduced by Kock [35] and Zöberlein [60]. We adopt the natural extended notion of Kock-Zöberlein pseudomonad [46], called herein lax idempotent pseudomonad.

Furthermore, since, in our context, the most important property of a lax idempotent pseudomonad \( \mathcal{T} \) is the fact that the forgetful 2-functor \( \text{Lax}-\mathcal{T}-\text{Alg}_\ell \to \mathcal{B} \) is fully faithful, we get a shortcut, defining lax idempotent pseudomonads via this property:

**Definition 1.4.18.** A pseudomonad \( \mathcal{T} \) on a 2-category \( \mathcal{B} \) is called a lax idempotent if the forgetful 2-functor \( \text{Lax}-\mathcal{T}-\text{Alg}_\ell \to \mathcal{B} \) is fully faithful (meaning that it is locally an isomorphism).

It should be noted that our definition is actually equivalent to the usual Kock-Zöberlein adjoint property as stated below: the proof of this fact for the strict case is originally given in [30]. Since the Kock-Zöberlein adjoint property has no important role in our observation, we avoid the proof.
Proposition 1.4.19. A pseudomonad \((\mathcal{T}, m, \eta, \mu, 1, \tau)\) is lax idempotent if and only if it satisfies the Kock-Zöberlein adjoint structure property: that is to say, there is a modification \(\gamma : \text{Id}_{\mathcal{T}} \Rightarrow (\eta \mathcal{T})(m)\) such that \((m \dashv \eta \mathcal{T}, 1, \gamma)\) is an adjunction.

In some situations, it can be easier to verify whether a pseudomonad \(\mathcal{T}\) satisfies the Kock-Zöberlein adjoint property than to verify whether the forgetful 2-functor \(\text{Lax-}\mathcal{T}\text{-Alg}_\ell \to \mathcal{B}\) is fully faithful. However our main observation on lax idempotent pseudomonads relies on the last property. More precisely:

Remark 1.4.20. Given a lax idempotent pseudomonad \(\mathcal{T}\), with a forgetful 2-functor \(U : \text{Lax-}\mathcal{T}\text{-Alg}_\ell \to \mathcal{B}\), it is clear that:

- The forgetful 2-functor \(\text{Ps-}\mathcal{T}\text{-Alg}_\ell \to \mathcal{B}\) is fully faithful as well;
- Given an object \(Z\) of \(\mathcal{B}\), if there is a lax \(\mathcal{T}\)-algebra \(z\) such that \(U(z) = Z\), it is unique up to isomorphism;
- For every adjunction \((f \dashv U(g), \varepsilon, \rho) : Y \to Z\) in \(\mathcal{B}\), there is an adjunction \((f \dashv g, \bar{\varepsilon}, \bar{\rho})\) in \(\text{Lax-}\mathcal{T}\text{-Alg}\) such that \(U(\bar{\varepsilon}) = f\);
- For every adjunction \((U(\bar{f}) \dashv g, \varepsilon, \rho) : Y \to Z\) in \(\mathcal{B}\), there is an adjunction \((f \dashv g, \bar{\varepsilon}, \bar{\rho})\) in \(\text{Lax-}\mathcal{T}\text{-Alg}\) such that \(U(g) = g\).

By Corollaries 1.4.15, 1.4.16 and, by Remark 1.4.20, we get that, for every lax \(\mathcal{T}\)-morphism \(g : z \to y\) such that \(U(g)\) is right adjoint, \(g\) satisfies the Beck-Chevalley condition. More precisely:

Corollary 1.4.21. Assume that \(\mathcal{T}\) is a lax idempotent pseudomonad, \(U : \text{Lax-}\mathcal{T}\text{-Alg}_\ell \to \mathcal{B}\) is the forgetful 2-functor and \((f \dashv g, \varepsilon, \rho) : U(y) \to U(z)\) is an adjunction in \(\mathcal{B}\).

- There is only one lax \(\mathcal{T}\)-morphism \(\bar{f} : y \to z\) such that \(U(\bar{f}) = f\). Furthermore, \(\bar{f}\) is a \(\mathcal{T}\)-pseudomorphism which is left adjoint in \(\text{Lax-}\mathcal{T}\text{-Alg}_\ell\);
- There is only one lax \(\mathcal{T}\)-morphism \(g : z \to y\) such that \(U(g) = g\). Furthermore, \(g\) is right adjoint to \(\bar{f}\) in \(\text{Lax-}\mathcal{T}\text{-Alg}_\ell\) and \(g\) satisfies the Beck-Chevalley condition.

This shows how Kock-Zöberlein pseudomonads encompass situations when “the Beck-Chevalley conditions always hold”. In other words, given such a lax idempotent pseudomonad, whenever \(g\) is a right adjoint between objects in the base 2-category that can be endowed with lax \(\mathcal{T}\)-algebra structure, the unique lax \(\mathcal{T}\)-structure for \(g\) always satisfies the Beck-Chevalley condition.

We can now work on examples of lax idempotent pseudomonads. Besides the idempotent pseudomonads of Chapter 3 [43], the examples we should mention are the cocompletion pseudomonads [32, 50], which motivated the definition of Kock-Zöberlein doctrines. Our aim is to show how the elementary result that says that left adjoints preserve colimits can be stated in our context.

The definition of cocompletion 2-monads for enriched categories is given in [50] and, by direct verification, via Definition 1.4.18 or via the Kock-Zöberlein adjoint property, one can see that cocompletion 2-monads are lax idempotent pseudomonads. We are more interested on the particular case of the Kock-Zöberlein pseudomonad of cocompletion on \(\text{CAT}\). In order to work out our example, we need the list of well known properties of the cocompletion pseudomonad below. Although we do not present proofs, they can be found in any of the main references [28].
1. There exists a lax idempotent pseudomonad \( (P, m, \eta, \mu, \iota, \tau) \) on CAT such that \( PX \) is the free cocompletion of \( X \) for every category \( X \);

2. The \( P \)-pseudoalgebras (as the lax \( P \)-algebras) are the cocomplete categories;

3. Clearly, since \( P \) is lax idempotent, the lax \( P \)-morphisms are just functors between cocomplete categories;

4. If \( f = (f, \langle f \rangle) : y \to z \) is a lax \( P \)-morphism, \( \langle f \rangle \) is given by the natural comparisons of the colimit of the image and the image of the colimit of diagrams. In particular, the \( P \)-pseudomorphisms are exactly the cocontinuous functors (which means that they preserve all the colimits).

By Corollary 1.4.21, we get that right adjoint functors between cocomplete categories always satisfy the Beck-Chevalley condition w.r.t. the pseudomonad \( P \). Or, in other words, left adjoints between cocomplete categories always induce \( P \)-pseudomorphisms.

**Corollary 1.4.22.** Left adjoint functors between cocomplete categories are cocontinuous.

This shows how our result on Beck-Chevalley condition for Kock-Zöberlein pseudomonads can be seen as a generalization of this elementary result. Of course, left adjoints in general preserve colimits. But we can see this fact as a consequence of Corollary 1.4.22. More precisely, given an adjunction \( (f \dashv g) : Y \to Z \) in CAT, by Yoneda embedding, there is an extension

\[
(\tilde{f} \dashv \tilde{g}) : (\text{CAT}[Y, \text{SET}])^{\text{op}} \to (\text{CAT}[Z, \text{SET}])^{\text{op}}
\]

which commutes up to isomorphism with the Yoneda embeddings. This means in particular that we have an invertible natural transformation

\[
\begin{array}{ccc}
\text{CAT}[Y, \text{SET}]^{\text{op}} & \xrightarrow{\tilde{f}} & \text{CAT}[Z, \text{SET}]^{\text{op}} \\
\tilde{y}_{Y^{\text{op}}}^{\text{op}} & \cong & \tilde{y}_{Z^{\text{op}}}^{\text{op}} \\
Y & \xrightarrow{f} & Z
\end{array}
\]

which completes the argument that \( \tilde{f} \) preserves colimits, since the Yoneda embeddings \( \tilde{y}_{Y^{\text{op}}}, \tilde{y}_{Z^{\text{op}}} \) preserve and reflect colimits and \( \tilde{f} \) preserves colimits by Corollary 1.4.22.

Of course, the usual argument works very similarly. That is to say, by Remark 1.4.7,

\[
\text{CAT}[\tilde{f}^{\text{op}}, \text{SET}] \to \text{CAT}[\tilde{g}^{\text{op}}, \text{SET}].
\]

We have that \( \text{CAT}[\tilde{f}^{\text{op}}, \text{SET}] \) preserves limits, since the the limits of \( \text{CAT}[Y^{\text{op}}, \text{SET}], \text{CAT}[Z^{\text{op}}, \text{SET}] \) are constructed pointwise. By the natural isomorphism \( \varphi \) of Remark 1.4.7, since \( \tilde{Y}, \tilde{Z} \) preserve and reflect limits, we conclude that \( \tilde{g} \) preserves limits as well.
1.5 Pseudo-Kan Extensions and Descent Theory

Chapter 3 is the article *Pseudo-Kan Extensions and descent theory* [43], under review. We give a formal approach to descent theory, framing classical descent theory in the context of idempotent pseudomonads. Within this perspective, we recast and generalize most of the classical results of the context of [24, 25], including transfer results, embedding results and the Bénabou-Roubaud Theorem.

The chapter starts by giving an outline of the setting, presenting basic problems and results of the classical context of descent theory. We give an outline of the classical results that are proved and generalized in that paper, including the results mentioned above.

In Section 2 of Chapter 3 [43], we prove theorems on pseudoalgebra structures and biadjoint triangles, always focusing in the case of idempotent pseudomonads. The main advantages on focusing our study on idempotent pseudomonads are the following: the pseudoalgebra structures w.r.t. an idempotent pseudomonad are easier to study. In this case, if a pseudoalgebra structure over an object \( X \) exists, it is unique (up to isomorphism) and, moreover, the pseudoalgebra structure over an object \( X \) exists if and only if the unit of the pseudomonad on \( X \) is an equivalence. This fact allows us to study situations when we “almost have” a pseudoalgebra structure over an object \( X \), which correspond to the situations when the component of the unit on \( X \) is faithful or fully faithful. This is important, later, to study descent and almost descent morphisms.

The results on pseudoalgebra structures and biadjoint triangles give the formal account to study descent theory. In order to study classical descent theory in the context of [25], the first step was to give results on commutativity of bilimits. More precisely, we firstly give a direct approach to prove an analogue of Fubini’s Theorem for weighted bilimits. This allows us to construct pointwise pseudo-Kan extensions and prove the basic results about them. Secondly, since we prove that

\[
[t, \mathfrak{B}]_{PS} : [\mathfrak{A}, \mathfrak{B}]_{PS} \to [\mathfrak{A}, \mathfrak{B}]
\]

is pseudomonadic and induces an idempotent pseudomonad whenever \( t \) is locally fully faithful and \( \mathfrak{A} \) is a small 2-category, we are able to get results on commutativity of bilimits as direct consequences of our results on pseudomonadic pseudofunctors.

Section 4 of Chapter 3 [43] introduces the descent objects, giving key results to finally frame the classical context of descent theory. The main result of this section is that the conical bilimit of a pseudocosimplicial object is its descent object. In other words, it shows that our definition of descent object coincides with the usual definition (as, for instance, given in [56]). More concisely, within the language of pseudo-Kan extensions, we prove that

\[
\text{PsRan}_j \mathcal{A}(0) \simeq \text{PsRan}_j \mathcal{A} t_3
\]

in which \( \mathcal{A} : \Delta \to \mathcal{I} \) is a pseudofunctor, \( \mathcal{I} \) is a bicategorically complete 2-category, \( j : \Delta \to \hat{\Delta} \) is the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions, \( t_3 : \Delta_3 \to \Delta \) is the inclusion of the 2-category given by the faces and degeneracies

\[
\begin{array}{ccc}
1 & \overset{d^0}{\leftarrow} & 2 \\
\downarrow d^1 & & \downarrow d^1 \\
2 & \overset{d^0}{\leftarrow} & 3
\end{array}
\]
1.6 Biadjoint Triangles and Lifting of Biadjoints

into $\Delta$, and $j_3$ is the inclusion of $\Delta_3$ into the 2-category

$$
\begin{array}{ccccccc}
0 & \overset{d}{\rightarrow} & 1 & \overset{d^0}{\rightarrow} & 2 & \overset{d^0}{\rightarrow} & 3 \\
& \overset{d^1}{\downarrow} && \overset{d^1}{\downarrow} & \overset{\partial^2}{\downarrow} & \overset{\partial^2}{\downarrow} & \\
\end{array}
$$

which is the 2-category obtained from the addition of an initial object to $\Delta_3$. This proves in particular that the definition of descent category via biased descent data on objects, which corresponds to $\text{Ps Ran}_j \mathcal{A}_1$, is equivalent to the definition of the descent category via unbiased descent data on objects, which corresponds to the case $\text{Ps Ran}_j \mathcal{A}(0)$. But the main points of this result are (1) this gives a very simple universal property of the descent category/object and (2) this gives a way of easily fitting the descent object in our language.

After this detailed work on descent objects, we turn to elementary and known examples. The Eilenberg-Moore objects and the monadicity of functors also fit easily in our context of weighted bilimits/pseudoalgebra structures, once we follow the ideas of [51]. This is explained in Section 6 of Chapter 3.

Finally, in Section 7 of Chapter 3, we show how our perspective on the Beck-Chevalley condition (as explained in Section 1.4) allows us to get results on pseudoalgebra structures/commmutativity of bilimits and monadicity. This leads to our first result of the type of Bénabou-Roubaud Theorem. After that, we finally show how our results work in the context of classical descent theory. We recast and generalize classical results as direct consequences of our previous work.

We then give refinements of our results on commutativity of bilimits. In the context of descent theory, this allows us to give better results on effective descent morphisms of weighted bilimits of 2-categories. It also gives the Galois result of [23] as a direct consequence.

One particular result obtained from our setting of commutativity of bilimits is the pseudopullback theorem. It gives conditions to get effective descent morphisms (w.r.t. basic fibration) of well behaved pseudopullbacks of categories. We finish this chapter applying this result to detect effective descent morphisms in categories of enriched categories. Firstly, we prove that, for suitable cartesian categories $V$, we have an embedding $V\text{-Cat} \to \text{Cat}(V)$ that is actually induced by a pseudopullback of categories. Then, using the pseudopullback theorem, we prove that such embedding reflects effective descent morphisms.

1.6 Biadjoint Triangles and Lifting of Biadjoints

Chapter 4 corresponds to the article *On Biadjoint Triangles* [41], published in *Theory and Applications of Categories, Vol 31, N. 9* (2016). The main contributions are the biadjoint triangle theorems, which can be seen as 2-dimensional analogues of the adjoint triangle theorem of [13]. As mentioned in Section 1.1, in order to prove the main results, we use the fact that the category of pseudomorphisms between two pseudooalgebras has the universal property of the descent object. More precisely, assuming that

$$
\begin{array}{ccc}
\mathcal{A} & \overset{J}{\rightarrow} & \mathcal{B} \\
\mathcal{C} & \overset{\sim}{\leftarrow} & \mathcal{B} \\
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{A} & \overset{R}{\rightarrow} & \mathcal{C} \\
\mathcal{C} & \overset{U}{\rightarrow} & \mathcal{B} \\
\end{array}
$$
is a pseudonatural equivalence, in which \( R, J, U \) are pseudofunctors, \( U \) is pseudopremonadic and \( R \) has a right biadjoint. We prove that \( J \) has a left biadjoint \( G \), provided that \( \mathcal{A} \) has some needed codescent objects. We also study the unit and the counit of the obtained biadjunction: we give sufficient conditions under which the unit and counit are pseudonatural equivalences. Finally, we show that, under suitable conditions, it is possible to construct a (strict) left 2-adjoint.

Similarly to the case of adjoint triangles in 1-dimensional category theory, the biadjoint triangles have many applications in 2-dimensional category theory. Examples of which are given in this same paper:

- **Pseudomonadicity characterization**: without avoiding pseudofunctors, using the biadjoint triangle theorem and the results on counit and unit, we give a explicit proof of the pseudomonadicity characterization due to Le Creurer, Marmolejo and Vitale in [38];

- **2-monadic approach to coherence**: as immediate consequences of our main theorems, we recast and improve results on the 2-monadic approach to coherence developed in [3, 36, 38, 49]. More precisely, we characterize when there is a left 2-adjoint (and biadjoint) to the inclusion of the 2-category of strict algebras into the 2-category of pseudoalgebras,

\[ \mathcal{T} \text{-Alg}_s \rightarrow \text{Ps} \cdot \mathcal{T} \text{-Alg} \]

of a given 2-monad \( \mathcal{T} \). We also characterize when the unit of such biadjunction is a pseudonatural equivalence;

- **Lifting of biadjoints**: the biadjoint triangle theorem gives biadjoints to algebraic pseudofunctors, that is to say, lifting of biadjoints. These results recover and generalize, for instance, results of Blackwell, Kelly and Power [3].

- **Pointwise pseudo-Kan extension**: We originally introduce the notion of pseudo-Kan extension and, using the results on lifting of biadjoints, in the presence of weighted bilimits, we construct pseudo-Kan extensions with them. This result, hence, gives the notion of pointwise pseudo-Kan extension. It also gives a way of recovering the construction of weighted bilimits via descent objects, cotensor (bi)products and (bi)products given originally in [56], if we assume the construction of the pointwise pseudo-Kan extension via Fubini’s Theorem for weighted bilimits given in Chapter 3.

Similarly to the pointwise Kan extension in 1-dimensional category theory, the concept of pointwise pseudo-Kan extension plays a relevant role in 2-dimensional category theory. As mentioned in Section 1.5, one instance of application of this concept given in this thesis is within the study of commutativity of weighted bilimits of Chapter 3. Other examples are within the study of 2-dimensional flat pseudofunctors of [11] and within the study of formal aspects of 2-dimensional category theory via Gray-categories [12].

Chapter 5 corresponds to the article *On lifting of biadjoints and lax algebras* [42], to appear in *Categories and General Algebraic Structures with Applications*. It gives further theorems on lifting of biadjoints provided that we can describe the categories of morphisms of a certain domain 2-categories.
In terms of weighted (bi)limits. This gives an abstract account of the main idea of some proofs of Chapter 4. Still, we show that this setting allows us to get results outside of the context of Chapter 4.

In particular, this approach, together with results on lax descent objects and lax algebras, allows us to give results on lifting of biadjoints involving (full) sub-2-categories of the 2-category of lax algebras. This gives biadjoint triangle theorems involving the 2-category of lax algebras. As an immediate consequence, we complete our treatment of the 2-monadic approach to coherence via biadjoint triangle theorems.

Remark 1.6.1. Unlike in the case of Chapter 5, our results on biadjoint triangles involving the 2-category of lax algebras, Theorems 5.5.2 and .5.5.3 [42], lack the study of the counit and the unit of the obtained biadjunctions. The study of counit and the unit in this setting could lead to new applications. One example is given in Remark 1.7.8.

1.7 Lifting of Biadjoints and Formal Theory of Monads

In this section, we talk about applications of the results of Chapter 5 in the context of the formal theory of monads. In order to do so, we assume most of the prerequisites of that chapter, including the concept of weighted limits and colimits in a 2-category w.r.t. the $\text{Cat}$-enrichment, usually called 2-limits and 2-colimits. In this direction, we adopt the terminology and definitions of Section 1 of Chapter 2 [44].

Every adjunction induces a monad. This was originally shown in [22] for the 2-category $\text{Cat}$. However, it works for any 2-category. Indeed, given an adjunction $(f \dashv g, \varepsilon, \eta): Y \rightarrow Z$ in a 2-category $\mathcal{C}$, $(Y, gf, \text{id}_Y \ast \varepsilon \ast \text{id}_F, \eta)$ is a monad on $Y$, that is to say, a lax $\text{Id}_{\mathcal{C}}$-algebra structure on $Y$.

Remark 1.7.1. Given a monad $y = (Y, \text{alg}_y, Y_0, \text{id}_Y)$ in a 2-category $\mathcal{C}$, we define the category of $y$-adjunctions $y$-adj$(\mathcal{C})$ as follows:

- The objects of $y$-adj$(\mathcal{C})$ are adjunctions $(f \dashv g, \varepsilon, \eta): Y \rightarrow Z$ that induce $y$;
- A morphism of $y$-adj$(\mathcal{C})$ between two adjunctions $(f \dashv g, \varepsilon, \eta): Y \rightarrow Z$ and $(\tilde{f} \dashv \tilde{g}, \tilde{\varepsilon}, \tilde{\eta}): \tilde{Y} \rightarrow \tilde{Z}$ is a morphism $j: Z \rightarrow \tilde{Z}$ such that $jf = \tilde{f}$ and $g = \tilde{g}j$.

If there exists, the terminal object of $y$-adj$(\mathcal{C})$ is called an Eilenberg-Moore adjunction for the monad $y$. In this case, the domain of the right adjoint of such adjunction is called the Eilenberg-Moore object of $y$ and denoted by $Y^y$. Dually, the initial object (if it exists) is called the Kleisli adjunction for the monad $y$. In this case, the domain of the right adjoint of such adjunction is called the Kleisli object of $y$ and denoted by $Y_y$.

There is an Eilenberg-Moore adjunction and a Kleisli adjunction for each monad in $\text{Cat}$. These results were shown respectively in [17] and [34]. However, it is easy to construct counterexamples of 2-categories not having all the Eilenberg-Moore (or Kleisli) adjunctions. In order to give a non-artificial easy to check example, we consider bicategories. More precisely, we can consider the suspension of the monoidal cartesian category $(\text{Set}, \times, 1)$, that is to say, we see such a monoidal category as a bicategory with only one object $\Delta$ as in Remark 1.2.8. A monad in such bicategory is the same as a (classical) monoid. There are plenty nontrivial monoids, while the suspension of $(\text{Set}, \times, 1)$ has only the trivial adjunction.
Remark 1.7.2. Clearly there is a bijection between monads in $\mathcal{B}$ and monads in $\mathcal{B}^{\text{op}}$. More precisely, the contravariant 2-functor $\mathcal{B} \to \mathcal{B}^{\text{op}}$ takes monads in $\mathcal{B}$ to monads in $\mathcal{B}^{\text{op}}$. So, by abuse of language, if $y$ is a monad in $\mathcal{B}$, we denote by $y$ the corresponding monad in $\mathcal{B}^{\text{op}}$.

We can, then, give precise meaning to the fact that the notion of Eilenberg-Moore objects is dual to the notion of Kleisli objects. Indeed, the Kleisli object for a monad $y$ in a 2-category $\mathcal{B}$ is, if it exists, the Eilenberg-Moore object of $y$ in $\mathcal{B}^{\text{op}}$.

In [54], it is observed that the Eilenberg-Moore object has a concise universal property. Namely, given a 2-category $\mathcal{B}$, there is an inclusion 2-functor $\mathcal{B} \to \operatorname{Mnd}(\mathcal{B})$ which takes each object $Z$ to the monad $(Z, \text{id}_Z, \text{id}_Z, \text{id}_Z)$. The Eilenberg-Moore object of a monad $y = (Y, \text{alg}_y, \gamma, \eta)$ is given by the right 2-reflection of $y$ along $\mathcal{B} \to \operatorname{Mnd}(\mathcal{B})$, if it exists. In particular, a morphism $f : X \to Y$ corresponds to a pair $\bar{f} = (\bar{f}, \langle \bar{f} \rangle)$ in which $\bar{f} : X \to Y$ is a morphism and $\langle \bar{f} \rangle : \text{alg}_Y \bar{f} \Rightarrow \bar{f}$ is a 2-cell, such that the equations

$$
\begin{array}{ccc}
\text{Y} & \xrightarrow{\text{alg}_Y} & \text{Y} \\
\downarrow{\bar{f}} & \Downarrow{\gamma} & \downarrow{\bar{f}} \\
\text{Y} & \xrightarrow{\text{alg}_Y} & \text{Y}
\end{array}
$$

hold. Since the Kleisli object of $y$ is the Eilenberg-Moore object of $y$ in $\mathcal{B}^{\text{op}}$, we have that the Kleisli object is given by the right 2-reflection of $y$ along $\mathcal{B}^{\text{op}} \to \operatorname{Mnd}(\mathcal{B}^{\text{op}})$ which is the same as the left 2-reflection of $\mathcal{B} \to (\operatorname{Mnd}(\mathcal{B}^{\text{op}}))^{\text{op}} \cong \operatorname{Mnd}_{\text{co}}(\mathcal{B})$.

Moreover, [54] generalizes the Eilenberg-Moore and the Kleisli constructions. More precisely, if $X$ is a category, [54] constructs the right 2-adjoint to the inclusion $[X, \operatorname{Cat}] \to [X, \operatorname{Cat}]_{\text{Lax}}$ of the 2-category of lax functors $X \to \operatorname{Cat}$, lax natural transformations and modifications into the 2-category of 2-functors, 2-natural transformations and modifications. In [54], Street also constructs the left 2-adjoint to the inclusion $[X, \operatorname{Cat}] \to [X, \operatorname{Cat}]_{\text{Lax}}$, in which $[X, \operatorname{Cat}]_{\text{Lax}}$ denotes the 2-category of lax functors, colax natural transformations and modifications.

In order to verify that these 2-adjoints actually are generalizations of the Eilenberg-Moore and Kleisli objects, we should observe that, considering the inclusions $\operatorname{Cat} \to \operatorname{Mnd}_{\text{co}}(\operatorname{Cat})$ and $\operatorname{Cat} \to \operatorname{Mnd}(\operatorname{Cat})$, we actually have isomorphisms $\operatorname{Mnd}(\operatorname{Cat}) \cong [1, \operatorname{Cat}]_{\text{Lax}}$ and $\operatorname{Mnd}_{\text{co}}(\operatorname{Cat}) \cong [1, \operatorname{Cat}]_{\text{Lax}}$ such that the diagrams

$$
\begin{array}{ccc}
\text{Cat} & \xrightarrow{=} & [1, \text{Cat}] \\
\downarrow{\operatorname{Mnd}(\operatorname{Cat})} & \downarrow{\operatorname{Mnd}_{\text{co}}(\operatorname{Cat})} & \downarrow{\operatorname{Mnd}_{\text{co}}(\operatorname{Cat})} \\
[1, \text{Cat}]_{\text{Lax}} & \xrightarrow{=} & [1, \text{Cat}]_{\text{Lax}}
\end{array}
$$
1.7 Lifting of Biadjoints and Formal Theory of Monads

commute. More generally, in our context, this is given by the fact that, given a 2-category \( \mathcal{B} \), the diagrams

\[
\begin{array}{ccc}
\text{Lax-Id}_{\mathcal{B}} \cdot \text{CoAlg}_{\mathcal{B}_{\text{cl}}} & \simeq & \text{Id}_{\mathcal{B}} \cdot \text{CoAlg}_{\mathcal{B}} \\
\downarrow & & \downarrow \\
\mathcal{B} & & \text{Lax-Id}_{\mathcal{B}} \cdot \text{Alg}_{\mathcal{B}}
\end{array}
\]

are naturally isomorphic, in which the horizontal arrows are the obvious inclusions while the non-horizontal arrows are the forgetful 2-functors. In particular, the inclusion \([1, \mathcal{B}] \to [1, \mathcal{B}]_{\text{Lax}}\) is actually the inclusion \(\text{Id}_{\mathcal{B}} \cdot \text{CoAlg}_{\mathcal{B}_{\text{cl}}} \simeq \text{Id}_{\mathcal{B}} \cdot \text{Alg}_{\mathcal{B}} \to \text{Lax-Id}_{\mathcal{B}} \cdot \text{CoAlg}_{\mathcal{B}_{\text{cl}}}\).

More generally, if \( \mathfrak{A} \) is any 2-category, denoting by \( \mathfrak{A}_0 \) the discrete 2-category of objects, the inclusion \( \mathfrak{A}_0 \to \mathfrak{A} \) induces a restriction 2-functor \( [\mathfrak{A}, \mathcal{B}] \to [\mathfrak{A}_0, \mathcal{B}] \). If \( \mathcal{B} \) has suitable weighted limits and colimits and \( \mathfrak{A} \) is small, this restriction has right and left 2-adjoints given by the (global) pointwise right and left Kan extensions. Assuming that this restriction has right and left 2-adjoints, we have a 2-monad \( \mathcal{T}an \) and a 2-comonad \( \mathcal{T}an \) on the 2-category \([\mathfrak{A}_0, \mathcal{B}]\). In this case, the diagrams

\[
\begin{array}{ccc}
\text{Lax-}\mathcal{T}an \cdot \text{CoAlg}_{\mathcal{B}_{\text{cl}}} & \simeq & \mathcal{T}an \cdot \text{CoAlg}_{\mathcal{B}} \\
\downarrow & & \downarrow \\
[\mathfrak{A}_0, \mathcal{B}] & & \text{Lax-}\mathcal{T}an \cdot \text{Alg}_{\mathcal{B}}
\end{array}
\]

are naturally isomorphic. These observations immediately show how the results of Chapter 5 generalize the construction: they actually characterize when it is possible to get such constructions. More precisely, using the techniques of that chapter, we are able to study the existence of the right 2-adjoint to \( \mathcal{T}an \cdot \text{CoAlg}_{\mathcal{B}} \to \text{Lax-}\mathcal{T}an \cdot \text{CoAlg}_{\mathcal{B}_{\text{cl}}} \) for any given 2-comonad \( \mathcal{T}an \), or, equivalently, a left 2-adjoint to \( \mathcal{T}an \cdot \text{Alg}_{\mathcal{B}} \to \text{Lax-}\mathcal{T}an \cdot \text{Alg}_{\mathcal{B}_{\text{cl}}} \) for any given 2-monad \( \mathcal{T}an \). So it is clear that this generalizes the constructions of [54]. Since we do not explicitly deal with colax morphisms in Chapter 5, we briefly describe below how we get the results for our context. We omit most of the proofs, since some of them are slight variations of the proofs on lax morphisms of Chapter 5, while the rest of the proofs follow directly from results of that chapter.
Definition 1.7.3. \([t^c : \Delta^c_\ell \to \hat{\Delta}^c_\ell]\) We denote by \(\hat{\Delta}^c_\ell\) the 2-category generated by the diagram

\[
\begin{array}{ccc}
0 & \overset{d}{\rightarrow} & 1 \\
\downarrow{d^0} & & \downarrow{d^0} \\
1 & \overset{d^1}{\rightarrow} & 2 \\
\downarrow{d^1} & & \downarrow{d^1} \\
2 & \overset{d^2}{\rightarrow} & 3
\end{array}
\]

with the 2-cells:

\[
\begin{align*}
\sigma_{00} : & \partial^0 d^0 \Rightarrow \partial^1 d^0, & n_0 : & \id_1 \Rightarrow s^0 d^0, \\
\sigma_{01} : & \partial^0 d^1 \Rightarrow \partial^2 d^0, & n_1 : & \id_1 \Rightarrow s^0 d^1, \\
\sigma_{21} : & \partial^2 d^1 \Rightarrow \partial^1 d^1, & \vartheta : & d^0 d \Rightarrow d^1 d,
\end{align*}
\]

satisfying

- **Associativity:**

\[
\begin{array}{ccc}
0 & \overset{d}{\rightarrow} & 1 \\
\downarrow{d^0} & & \downarrow{d^0} \\
1 & \overset{d^1}{\rightarrow} & 2 \\
\downarrow{d^1} & & \downarrow{d^1} \\
2 & \overset{d^2}{\rightarrow} & 3
\end{array}
\quad = 
\quad
\begin{array}{ccc}
3 & \overset{\partial^1}{\rightarrow} & 2 \\
\downarrow{d^0} & & \downarrow{d^0} \\
2 & \overset{\partial^2}{\rightarrow} & 1 \\
\downarrow{d^1} & & \downarrow{d^1} \\
1 & \overset{d}{\rightarrow} & 0
\end{array}
\]

- **Identity:**

\[
\begin{array}{ccc}
0 & \overset{d}{\rightarrow} & 1 \\
\downarrow{d^0} & & \downarrow{d^0} \\
1 & \overset{d^1}{\rightarrow} & 2 \\
\downarrow{d^1} & & \downarrow{d^1} \\
2 & \overset{d^2}{\rightarrow} & 3
\end{array}
\quad = 
\quad
\begin{array}{ccc}
0 & \overset{d}{\rightarrow} & 1 \\
\downarrow{d^0} & & \downarrow{d^0} \\
1 & \overset{d^1}{\rightarrow} & 2 \\
\downarrow{d^1} & & \downarrow{d^1} \\
0 & \overset{d}{\rightarrow} & 1
\end{array}
\]

The 2-category \(\Delta^c_\ell\) is, herein, the full sub-2-category of \(\hat{\Delta}^c_\ell\) with objects 1, 2 and 3. We denote the inclusion by \(t^c : \Delta^c_\ell \rightarrow \hat{\Delta}^c_\ell\).

Remark 1.7.4. [Colax descent object and category] If \(\mathcal{A} : \Delta^c_\ell \rightarrow \mathcal{B}\) and \(\mathcal{B} : (\Delta^c_\ell)^{\text{op}} \rightarrow \mathcal{B}\) are 2-functors, if it exists, the weighted limit \(\hat{\Delta}^c_\ell(0, t^c-), \mathcal{A}\) is called the strict colax descent object of \(\mathcal{A}\), while the weighted colimit \(\Delta^c_\ell(0, t^c-) \ast \mathcal{B}\) is called the strict colax codescent object of \(\mathcal{B}\) (if it exists).

In the case of strict colax descent categories, we have a result similar to that described by Remark 5.3.4 [42]. More precisely, if \(\mathcal{D} : \Delta^c_\ell \rightarrow \text{Cat}\) is a 2-functor, then

\[
\{ \hat{\Delta}^c_\ell(0, t^c-), \mathcal{D} \} \cong [\Delta^c_\ell, \text{Cat}] (\Delta^c_\ell(0, t^c-), \mathcal{D}) .
\]
Thereby, we can describe the strict colax descent object of $\mathcal{D} : \Delta_1^s \rightarrow \text{Cat}$ explicitly as follows:

1. Objects are 2-natural transformations $f : \Delta(0, -) \rightarrow \mathcal{D}$. We have a bijective correspondence between such 2-natural transformations and pairs $(f, \langle \mathcal{F} \rangle)$ in which $f$ is an object of $\mathcal{D}1$ and $\langle \mathcal{F} \rangle : \mathcal{D}(d^0)f \rightarrow \mathcal{D}(d^1)f$ is a morphism in $\mathcal{D}2$ satisfying the following equations:
   - Associativity:
     $$(\mathcal{D}(\sigma_{01})_f) \left( \mathcal{D}(\mathcal{F})((\mathcal{F})(\langle \mathcal{F} \rangle)) \right) \left( \mathcal{D}(\sigma_{21})_f \right) \left( \mathcal{D}(\mathcal{F})((\langle \mathcal{F} \rangle)) \right) = \mathcal{D}(\mathcal{F})((\langle \mathcal{F} \rangle)) \left( \mathcal{D}(\sigma_{00})_f \right)$$
   - Identity:
     $$(\mathcal{D}(\mathcal{F})((\langle \mathcal{F} \rangle))) \left( \mathcal{D}(n_0)_f \right) = \mathcal{D}(n_1)_f$$

If $f : \Delta(0, -) \rightarrow \mathcal{D}$ is a 2-natural transformation, we get such pair by the correspondence $f \mapsto (f_1(d), f_2(\mathcal{F})).$

2. The morphisms are modifications. In other words, a morphism $m : f \rightarrow g$ in $\mathcal{D}1$ such that $\mathcal{D}(d^1)(m)(\langle \mathcal{F} \rangle) = \langle h \rangle \mathcal{D}(d^0)(m)$.

Similarly to Proposition 4.5.5 [41] and 5.4.5 [42], we clearly have:

**Proposition 1.7.5.** Let $\mathcal{F} = (\mathcal{F}, m, \eta, \mu, t, \tau)$ be a pseudomonad on a 2-category $\mathcal{B}$. Given lax $\mathcal{F}$-algebras $y = (Y, \eta_y, \alpha, \tau_y)$, $z = (Z, \alpha_z, \eta_z, \tau_z)$ the category $\text{Lax-}\mathcal{F}\text{-Alg_{cf}}(y, z)$ is the strict colax descent object of the diagram $T_2^\mathcal{F} : \Delta_1 \rightarrow \text{Cat}$

\[
\begin{array}{ccc}
\mathcal{B}(\mathcal{F}U_y, \eta_y, \alpha_y) & \xrightarrow{\eta_{U_y, U_z}} & \mathcal{B}(U_y, U_z) \\
\mathcal{B}(U_y, U_z) & \xrightarrow{\mathcal{B}(m_{U_y, U_z})} & \mathcal{B}(\mathcal{F}U_y, U_z) \\
\mathcal{B}(\mathcal{F}U_y, \eta_y, \alpha_y) & \xleftarrow{\mathcal{B}(t_{U_y, U_z})} & \mathcal{B}(U_y, U_z) \\
\end{array}
\]

in which $U : \text{Lax-}\mathcal{F}\text{-Alg_{cf}} \rightarrow \mathcal{B}$ denotes the forgetful 2-functor and

\[
T_2^\mathcal{F}(\sigma_{01})_f := \left( \text{id}_{\alpha_z} \ast t_{\langle \mathcal{F}(\sigma_{01}) \rangle} \right) \\
T_2^\mathcal{F}(\sigma_{21})_f := \left( \text{id}_{\alpha_y} \ast \tau \right) \\
T_2^\mathcal{F}(n_0)_f := \left( \text{id}_{\alpha_z} \ast m \right) \\
T_2^\mathcal{F}(n_1)_f := \left( \text{id}_{\alpha_y} \ast \eta \right)
\]

Furthermore, the strict descent object of $T_2^\mathcal{F}$ is $\text{Lax-}\mathcal{F}\text{-Alg}(y, z)$.

**Remark 1.7.6.** Similarly to the case of Remark 5.4.6 [42], we can actually define a pseudofunctor $T^\mathcal{F} : \Delta_1^s \times \text{Lax-}\mathcal{F}\text{-Alg} \rightarrow \text{Cat}$ in which $T^\mathcal{F}(-, z) := T_2^\mathcal{F}$, since the morphisms defined above are actually pseudonatural in $z$ w.r.t. $\mathcal{F}$-pseudomorphisms and $\mathcal{F}$-transformations. More importantly to our context, if $\mathcal{F}$ is a 2-monad, the restriction of $T^\mathcal{F}$ to $\Delta_1^s \times \text{Lax-}\mathcal{F}\text{-Alg}_{sf}$, in which $\text{Lax-}\mathcal{F}\text{-Alg}_{sf}$ denotes the locally full sub-2-category of lax algebras and (strict) $\mathcal{F}$-morphisms and lax algebras, is actually a 2-functor.

By Remark 1.7.6 and Proposition 1.7.5, as a consequence of the results of Chapter 5 [42], we conclude as a particular case that:
**Theorem 1.7.7.** Let \( \mathcal{T} = (\mathcal{T}, m, \eta) \) be a 2-monad on \( \mathcal{B} \) and \( (E \dashv R, \varepsilon, \eta) : \mathcal{B} \to \mathcal{T} \text{-Alg}_s \) the Eilenberg-Moore 2-adjunction induced by \( \mathcal{T} \). The inclusion \( J : \mathcal{T} \text{-Alg}_s \to \text{Lax-\mathcal{T} \text{-Alg}_{gel}} \) has a left 2-adjoint if and only if \( \mathcal{T} \text{-Alg}_s \) has the strict colax codescent object of

\[
\begin{array}{ccc}
E\varepsilon_U y & \xrightarrow{E\varepsilon_U y(E(\eta y)_{E y}))} & E\varepsilon y U y \\
\downarrow & & \downarrow \\
E\varepsilon_U y y(E(\eta y)_{E y}) & \xrightarrow{E\varepsilon_U y(E(\eta y)_{E y}))} & E\varepsilon y U y
\end{array}
\]

(with omitted 2-cells) in which \( U : \text{Lax-\mathcal{T} \text{-Alg}_{gel}} \to \mathcal{B} \) is the forgetful 2-functor. In this case, the left 2-adjoint is given by \( G y = \Delta^C \cdot (0, t^f -) \ast \mathcal{B}_y \).

As a corollary, if \( \mathcal{T} \) is a 2-monad on \( \mathcal{B} \), the inclusion \( \mathcal{T} \text{-Alg}_s \to \text{Lax-\mathcal{T} \text{-Alg}_{gel}} \) has a left 2-adjoint if \( \mathcal{B} \) has and \( \mathcal{T} \) preserves strict colax codescent objects. Dually, if \( \mathcal{T} \) is a 2-comonad on \( \mathcal{B} \), \( \mathcal{T} \text{-CoAlg}_c \to \text{Lax-\mathcal{T} \text{-CoAlg}_{gel}} \) has a right 2-adjoint if \( \mathcal{B} \) has and \( \mathcal{T} \) preserves strict colax descent objects.

Since the 2-monad \( \mathcal{T} \text{an} \) on \( [\mathcal{A}, \mathcal{B}] \) preserves strict colax codescent objects whenever \( \mathcal{A} \) is small and \( \mathcal{B} \) has 2-colimits, our result gives a left 2-adjoint to the inclusion \( [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{B}]_{\text{Lax}} \). Dually, the 2-comonad \( \mathcal{T} \text{an} \) on \( [\mathcal{A}, \mathcal{B}] \) preserves strict colax descent objects whenever \( \mathcal{B} \) has 2-limits, hence our result gives the right 2-adjoint of \( [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{B}]_{\text{Lax}} \). These facts explain how our results generalize greatly the constructions of [54]. In particular, unlike [54], our setting includes the case of lax actions of monoidal categories (called graded monads): or, more precisely, the case in which \( \mathcal{A} \) is a bicategory with only one object (see Remark 5.4.3 [42] and [18]).

Applying to the very special case of the classical theory of monads, we get that the Eilenberg-Moore category of a monad (seen as a lax coalgebra) \( y = (Y, \text{coalg}_y, \eta, \mu) \) is the colax descent category of

\[
\begin{array}{ccc}
\text{coalg}_y & \xrightarrow{\text{id}_y} & \text{coalg}_y \\
\downarrow & & \downarrow \\
\text{id}_y & \xrightarrow{\text{id}_y} & \text{id}_y
\end{array}
\]

in which \( \mathcal{D}(\sigma_{00}) = \eta, \mathcal{D}(\sigma_{01}) = \eta, \mathcal{D}(\sigma_{21}) = \eta, \mathcal{D}(n_1) = \eta \) and the images of \( \sigma_{01}, \sigma_{21} \) and \( n_1 \) are the identities.

**Remark 1.7.8.** The article [54] also recasts the original universal properties w.r.t. adjunctions. That is to say, it generalizes the (universal property) of the (generalized) Eilenberg-Moore and Kleisli adjunctions to its setting. In order to do so, it relies on the study of the counit and unit of the 2-adjunctions \( [X, \mathcal{B}] \to [X, \mathcal{B}]_{\text{Lax}} \) and \( [X, \mathcal{B}] \to [X, \mathcal{B}]_{\text{Lax}} \) constructed therein. This fact shows that the study of counit and unit of the obtained biadjunctions (and 2-adjunctions) in the context of Chapter 5 could be interesting to recast the Eilenberg-Moore and Kleisli adjunctions in our context, in order to generalize the setting of [54].

### 1.8 Pseudoexponentiability

There is a vast literature on exponentiability of objects and morphisms within 1-dimensional category theory [47, 48]. As mentioned in Section 1.2.8, we could consider exponentiability of objects w.r.t.
other monoidal structures in $V$, but we are particularly interested in the case of exponentiation w.r.t.
the cartesian structure.

An object $A$ of a cartesian category $V$ is exponentiable if the functor $A \times -: V \to V$ is left adjoint.
In this case, the right adjoint is usually denoted by $[A, -]$. A morphism $f : A \to B$ is exponentiable if
it is an exponentiable object of the comma category $V/B$ defined by:

- The objects are morphisms with $B$ as codomain;
- A morphism $f \to g$ is a morphism $h$ of $V$ between the domains of $f$ and $g$ such that

\[
A \xrightarrow{f} A' \xleftarrow{g} B
\]

commutes in $V$;
- The composition and identities are given by the composition and identities of $V$.

The main problem is to characterize objects and morphisms that are exponentiable in a given
category $V$ of interest. For instance, the characterization of the exponentiable morphisms (functors)
of $\text{Cat}$ is given in [9, 19], while in [6, 7] the exponentiable morphisms (enriched functors) between
$V$-enriched categories are characterized, for suitable monoidal categories $V$.

As many concepts of 1-dimensional category theory, exponentiability is very strict to most of the
cases within bicategory theory. Hence we should consider a weaker version: pseudoexponentiability.
Firstly, we consider bicategorical products instead of products. Secondly, we consider a biadjunction
instead of an adjunction. That is to say:

**Definition 1.8.1.** An object $Y$ of a 2-category $\mathcal{B}$ with bicategorical products is *pseudoexponentiable*
if the pseudo functor $Y \times -: \mathcal{B} \to \mathcal{B}$ has a right biadjoint, while $Y$ is exponentiable if $Y \times -$ is a
2-functor and it has a right 2-adjoint.

As briefly mentioned in Section 1.6, the adjoint triangle theorem of [13] has many applications
in 1-dimensional category theory. In particular, it is very useful within the study of exponentiable
objects and morphisms. For instance, some of the results of the theory developed in [14] can be seen
as applications of the adjoint triangle theorem.

This fact suggests the possibility of applying the biadjoint triangle theorem proved in Chapter 4 to
develop an analogue theory for pseudoexponentiable objects and morphisms. We however do not do
this here. Instead, we finish the chapter giving what seems to be a folklore result on exponentiability
of coalgebras. Then, employing the biadjoint triangle theorem, we give the bicategorical analogue
that studies the (pseudo)exponentiability of pseudocoalgebras.

**Theorem 1.8.2 (Exponentiability of Coalgebras).** Let $\mathcal{F}$ be a comonad on a finitely complete category
$\mathcal{B}$. If $\mathcal{F}$ preserves finite limits, then the forgetful functor $L : \mathcal{F}\text{-CoAlg} \to \mathcal{B}$ reflects exponentiable
objects.
Proof. In this setting, \(L\) creates finite limits. In particular, given an \(\mathcal{S}\)-coalgebra \(y = (Y, \text{coalg}_y)\), we get that

\[
\begin{array}{cccc}
\mathcal{S}\text{-CoAlg}_s & \xrightarrow{y \times -} & \mathcal{S}\text{-CoAlg}_s \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{(L y) \times -} & \mathcal{B}
\end{array}
\]

commutes. If \((L y) \times - \dashv [(L y), -]\), we have that \((L y) \times L(-) \dashv U [(L y), -]\). Since \(L\) is comonadic, by the adjoint triangle theorem of Dubuc, we conclude that \(y \times -\) has a right adjoint. \(\square\)

Within the context of the theorem above, given an \(\mathcal{S}\)-coalgebra \(y\), by the Beck theorem, it is clear that \(L\) induces a functor \(\mathcal{S}\text{-CoAlg}_s/y \to \mathcal{B}/L(y)\) which creates limits and is comonadic as well. Therefore:

**Corollary 1.8.3.** Let \(\mathcal{S}\) be a comonad on a finitely complete category \(\mathcal{B}\). If \(\mathcal{S}\) preserves finite limits, then the forgetful functor \(L : \mathcal{S}\text{-CoAlg}_s \to \mathcal{B}\) reflects exponentiable morphisms.

**Remark 1.8.4.** An elementary application is given by the category of functors. For instance, if \(X, Y\) are categories such that \(X\) is small and \(Y\) is complete, there exists a (global) pointwise right Kan extension \(\text{Cat}[X_0, Y] \to \text{Cat}[X, Y]\). Since the forgetful functor \(\text{Cat}[X, Y] \to \text{Cat}[X_0, Y]\) creates equalizers, we conclude that \(\text{Cat}[X, Y] \to \text{Cat}[X_0, Y]\) is comonadic.

By Theorem 1.8.2, we conclude that \(\text{Cat}[X, Y] \to \text{Cat}[X_0, Y]\) reflects exponentiable objects. More generally, a natural transformation is exponentiable in \(\text{Cat}[X, Y]\) whenever it is objectwise exponentiable.

From an argument entirely analogous to that given in the proof of Theorem 1.8.2, using the enriched version of the adjoint triangle theorem as presented in Section 4.1 [41], we get that:

**Theorem 1.8.5 (Exponentiability of Strict Coalgebras).** Let \(\mathcal{S}\) be a 2-comonad on a 2-category \(\mathcal{B}\). If \(\mathcal{B}\) has and \(\mathcal{S}\) preserves products and equalizers, then the forgetful 2-functor \(L : \mathcal{S}\text{-CoAlg}_s \to \mathcal{B}\) reflects exponentiable objects.

Recall the definition of (strict) descent objects given in Section 4.3 [41]. Employing the strict version of the biadjoint triangle theorem given in Theorem 4.5.10 [41], we can study the exponentiability of pseudocoalgebras.

**Theorem 1.8.6 (Exponentiability of Pseudocoalgebras).** Let \(\mathcal{S}\) be a 2-comonad on a 2-category \(\mathcal{B}\). If \(\mathcal{B}\) has and \(\mathcal{S}\) preserves products and strict descent objects, then the forgetful 2-functor \(L : \mathcal{S}\text{-CoAlg} \to \mathcal{B}\) reflects exponentiable objects.

Finally, by the biadjoint triangle theorem given in Theorem 5.10 of Chapter 4 [41], we conclude that:

**Theorem 1.8.7 (Pseudoexponentiability of Pseudocoalgebras).** Let \(\mathcal{S}\) be a pseudocomonad on a 2-category \(\mathcal{B}\). If \(\mathcal{B}\) has and \(\mathcal{S}\) preserves biproducts and descent objects, then the forgetful 2-functor \(L : \mathcal{S}\text{-CoAlg} \to \mathcal{B}\) reflects pseudoexponentiable objects.
**Remark 1.8.8.** Similarly to the case of Remark 1.8.4, we can use the results above to study exponentiability and pseudoexponentiability of the 2-category of 2-functors $[\mathcal{A}, \mathcal{B}]$ (with 2-natural transformations and modifications) and in the 2-category of pseudofunctors $[\mathcal{A}, \mathcal{B}]_{PS}$ (with pseudonatural transformations and modifications) as defined in Section 2 of Chapter 4 [41].

For instance, if $\mathcal{A}$ is small and $\mathcal{B}$ is 2-complete, we conclude that a 2-functor is exponentiable in $[\mathcal{A}, \mathcal{B}]$ if it is objectwise exponentiable. Moreover, using the pointwise pseudo-Kan extension constructed in 4.9.2 [41] (or in 3.3.5, Section 3 of Chapter 3), we get an analogous result for pseudofunctors. More precisely, assuming that $\mathcal{B}$ is bicategorically complete and $\mathcal{A}$ is small, we get that a pseudofunctor in $[\mathcal{A}, \mathcal{B}]_{PS}$ is pseudoexponentiable whenever it is objectwise pseudoexponentiable.
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