Rotation Numbers and Rotation Classes on One-Dimensional Tiling Spaces

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Abstract. We extend rotation theory of circle maps to tiling spaces. Specifically, we consider a one-dimensional tiling space $\Omega$ with finite local complexity and study self-maps $F$ that are homotopic to the identity and whose displacements are strongly pattern equivariant. In place of the familiar rotation number, we define a cohomology class $[\mu] \in \tilde{H}^1(\Omega, \mathbb{R})$. We prove existence and uniqueness results for this class, develop a notion of irrationality, and prove an analogue of Poincaré’s theorem: If $[\mu]$ is irrational, then $F$ is semi-conjugate to uniform translation on a space $\Omega_\mu$ of tilings that is homeomorphic to $\Omega$. In such cases, $F$ is semi-conjugate to uniform translation on $\Omega$ itself if and only if $[\mu]$ lies in a certain subspace of $\tilde{H}^1(\Omega, \mathbb{R})$.

1. Introduction

Since 1885, rotations numbers have been used to understand orientation-preserving self-homeomorphisms $f : S^1 \to S^1$ of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The basic questions of rotation theory are:

(1) When is $f$ the time-one sampling of a uni-directional flow on $S^1$?
(2) When is $f$ conjugate to a uniform rotation on $S^1$?
(3) When is $f$ semi-conjugate to a uniform rotation on $S^1$?

The first two questions are of course equivalent, since any uni-directional flow is conjugate to a constant flow. However, in the more general setting considered in this paper, they will turn out to be different.

Every orientation-preserving homeomorphism $f : S^1 \to S^1$ lifts to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that $F(x + 1) = F(x) + 1$ for all $x \in \mathbb{R}$. Poincaré [31] defined the translation number of $F$ to be $\lim_{n \to \infty} \frac{F^n(x) - x}{n}$. This limit exists for all lifts $F$ and starting points $x$, and does not depend on $x$. Different lifts give translation numbers that differ by integers, and the rotation number of $f$ is their common image in $\mathbb{R}/\mathbb{Z}$. 




Poincaré showed that if the rotation number is irrational, then $f$ is semi-conjugate to a uniform rotation on the circle. Denjoy [15] generated examples where this semi-conjugacy is not a conjugacy. However, he also showed that if $f$ is sufficiently smooth (specifically, if $f'$ has bounded variation, and in particular if $f$ is $C^2$) then the semi-conjugacy must in fact be a conjugacy. If $L$ is any length, the circle $\mathbb{R}/L\mathbb{Z}$ can be viewed as the orbit, under translations, of a tiling that is periodic with period $L$. We can then view an orientation-preserving homeomorphism of $\mathbb{R}/L\mathbb{Z}$ as a map on a space of periodic tilings. The goal of this paper, which can be viewed as an extension of [2], is to study rotation theory of one-dimensional non-periodic tiling spaces from a cohomological point of view.

Tilings and symbolic sequences have been studied extensively. Individual sequences and tilings have interesting properties and applications, such as Morse’s use of the Thue–Morse sequence to study geodesics on Riemann surfaces, or the Penrose tiling’s fivefold symmetry [30]. The latter foreshadowed the discovery of quasicrystals a few years later by Shechtman and his collaborators [35], and non-periodic tilings have been used to study quasicrystals ever since. Aperiodic tilings are also related to numeration systems [7].

Beyond the tilings themselves, spaces of tilings serve as useful models for other structures in mathematics. These tiling spaces are locally homeomorphic to the product of Euclidean space with a Cantor set and have a flat laminated structure [5]. Tiling spaces associated with substitutions are Smale spaces, and every expanding (aka Smale–Williams) attractor is homeomorphic either to a tiling space or to a solenoid [1], providing a connection between tilings and hyperbolic dynamical systems.

A key tool for understanding an aperiodic tiling space $\Omega$ is Čech cohomology, and especially the real-valued first Čech cohomology group $\check{H}^1(\Omega, \mathbb{R})$. This approach was pioneered by Kellendonk [23] in the 1990s and exploded after the seminal work of Anderson and Putnam [1]. See [34] and the references therein for an overview. Among the applications of tiling cohomology are:

1. $\check{H}^\ast(\Omega)$ is a topological invariant that serves to distinguish tiling spaces (up to homeomorphism).
2. The gap labeling group, which can be built either from the (integer-valued) cohomology or the K-theory of the tiling space, constrains the “integrated density of states” of any local operator derived from a tiling. For instance, if the vertices of a tiling are interpreted as atoms acting via finite-range potentials and we compute the energy levels of electrons, then the topology of the tiling space constrains the possible densities of states and hence the possible electrical properties of the material [5,6,8,10].
3. For tilings in $n$ dimensions, $\check{H}^1(\Omega, \mathbb{R}^n)$ parametrizes shape deformations [14]. These are maps between tiling spaces in which the combinatorics of each tiling are preserved, but the shapes and sizes of the tiles are systematically varied.
4. The top cohomology of a tiling controls the rate at which Birkhoff averages converge to their ergodic limits [3,11,33,36].
(5) Every homeomorphism $h : \Omega \to \Omega'$ between tiling spaces can be written as the product of three maps [21]. The first is a self-homeomorphism of $\Omega$ that is homotopic to the identity. The second is a shape change from $\Omega$ to a space $\Omega''$. The third is a sort of local equivalence, called “MLD,” between $\Omega''$ and $\Omega'$. The shape-change component is parametrized (up to MLD) by $\hat{H}^1(\Omega, \mathbb{R}^n)$.

In this paper, we show how, when $n = 1$, $\hat{H}^1(\Omega, \mathbb{R}^1)$ also serves to parametrize the rotation theory of the factor homotopic to the identity. Instead of defining a rotation number $\rho$, we define a rotation class $[\mu]$ in $\hat{H}^1(\Omega, \mathbb{R})$ and define what it means for this class to be irrational. (For a circle $\hat{H}^1(S^1, \mathbb{R}) = \mathbb{R}$, so a rotation class for a circle map is just a number. The usual rotation number $\rho$ turns out to be $1/\mu \mod 1$.) Our main theorems, listed in the next subsection, relate the existence, uniqueness and irrationality of $[\mu]$ to whether our map is the time-one sampling of a flow with certain equivariance properties, and also to whether it is conjugate, or semi-conjugate, to uniform translation.

Extensions of rotation theory to higher dimensions and to non-periodic settings have been considered by many authors in the literature. Misiurewicz and Ziemian [29] extended rotation theory to maps of tori that are homotopic to the identity. There has been extensive work on the study of such maps, and a Poincaré theorem was obtained first for quasiperiodically forced circle maps by Stark and Jäger in [20] and later extended to $\rho$-bounded pseudo-rotations by Jäger in [18]. Extending rotation theory to non-periodic settings goes back perhaps to Johnson and Möser [19] who studied the rotation number of an almost periodic Schrödinger equation. Kwapisz [27] studied the rotation sets for maps of the real line with almost-periodic displacement in the sense of Bohr. Tiling spaces have a similar structure to solenoids, and rotation theory for solenoids was developed by Clark [12]. Later, the first author [2] studied the rotation numbers for maps of the real line with pattern-equivariant displacements and obtained a Poincaré theorem for $\rho$-bounded maps with irrational rotation numbers. In [4], the first author and Jäger obtained a Poincaré theorem that includes at the same time the almost-periodic case studied by Kwapisz and the quasiperiodically forced circle case studied by Stark and Jäger.

### 1.1. Statements of the Main Theorems

Let $\Omega$ be a one dimensional tiling space. We prove the following main theorems under the assumptions that our tiling space $\Omega$ is compact, minimal and uniquely ergodic, and that $F : \Omega \to \Omega$ is self-homeomorphism that is homotopic to the identity and that have strong pattern equivariant displacement (sPE displacement). These terms are defined precisely in Sect. 2. If these assumptions are met, we write $F \in \mathcal{F}(\Omega)$, or sometimes just $F \in \mathcal{F}$ when the space $\Omega$ is clear.

**Theorem 1** (Proposition 3). *If $F \in \mathcal{F}(\Omega)$ is the time-one sampling of a unidirectional strongly pattern-equivariant flow, and if $F$ has no fixed points, then the rotation class $[\mu]$ of $F$ exists.*
In Sect. 6, we exhibit a tiling space \( \Omega \) and a map \( F \in \mathcal{F} \) with no fixed points such that \([\mu]\) does not exist, implying that this \( F \) does not come from an sPE flow.

Rotation classes are defined, using a de-Rham like cohomology theory, via differential forms (rotation forms) meeting certain combinatorial conditions. If \( \mu \) meets these conditions and \( \mu' \) is cohomologous to \( \mu \), then \( \mu' \) also meets these conditions and we say that \([\mu]\) is a rotation class.

**Theorem 2** (Theorem 11). If \( F \in \mathcal{F}(\Omega) \) has rotation forms \( \mu \) and \( \nu \), and if \([\mu]\) is irrational, then \( \mu \) and \( \nu \) are cohomologous.

That is, rotation classes are unique if they are irrational. However, there exist maps that admit multiple rotation classes, all of them rational. We construct such a map in Sect. 6.

**Theorem 3** (Theorem 12 and Corollary 1). If \( F \in \mathcal{F}(\Omega) \) and if the rotation class \([\mu]\) of \( F \) exists and is irrational, then \( F \) is semi-conjugate to the time-one sampling of a strongly PE uni-directional flow on \( \Omega \).

There are two key differences between these results and the classical theory of circle maps. The first difference is that we do not have an analogue of Denjoy’s theorem. While we conjecture that some version of Denjoy’s theorem is still true, the usual proofs fail spectacularly in the setting of tiling spaces.

The second difference is that if \( F \) is (semi-)conjugate to the time-one sampling of a uni-directional (and strongly PE) flow on \( \Omega \), this does not imply that \( F \) is (semi-)conjugate to a uniform translation on \( \Omega \) itself. Instead, it implies that \( F \) can be (semi-)conjugated to a uniform translation on another tiling space \( \Omega' \) that is homeomorphic to \( \Omega \). Whether the translation action on \( \Omega \) is topologically conjugate (up to a uniform rescaling) to the translation action on \( \Omega' \) is a purely cohomological question. Applying results of [21], we show there is a subspace of \( H^1(\Omega, \mathbb{R}) \), denoted \( \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \), such that the translation actions on \( \Omega \) and \( \Omega' \) are topologically conjugate, up to a uniform rescaling, if \([\mu]\) lies in that subspace. This implies that:

**Theorem 4.** Suppose that \( F \in \mathcal{F}(\Omega) \), that \([\mu]\) exists and is irrational, and that \([\mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R}) \). Then, \( F \) is semi-conjugate to a uniform translation on \( \Omega \).

The converse to Theorem 4 is false. Some tiling spaces, such as those that come from substitutions, admit complicated self-homeomorphisms. If \( F \) has irrational rotation class \([\mu]\) and \( G : \Omega \to \Omega \) is such a homeomorphism, then \( F' = G^{-1} \circ F \circ G \) has rotation class \([G^* \mu]\). If \( F \) is semi-conjugate (by a map \( J \)) to uniform translation, then \( F' \) is also semi-conjugate (by \( G^{-1} \circ J \circ G \)) to uniform translation. However, it is possible to have \([\mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R}) \) without having \([G^* \mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R}) \). To avoid the difficulties posed by self-homeomorphisms of \( \Omega \), we must restrict the form of the semi-conjugacy:

**Theorem 5.** Suppose that \( F \in \mathcal{F}(\Omega) \) and that \([\mu]\) exists and is irrational. \( F \) is semi-conjugate to a uniform translation on \( \Omega \), via a semi-conjugacy that sends
each path component of $\Omega$ to itself, if and only if $[\mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R})$.

Theorem 4 is of course a corollary of Theorem 5. We will prove Theorem 5 in Sect. 5.

Theorem 5 is a version of Poincaré’s theorem that replaces the bounded-mean motion hypothesis on $F$ by a cohomological condition that depends only on $[\mu]$. In general, checking whether a given map has bounded mean motion is difficult, while the cohomological condition in Theorem 5 can often be checked with standard tools, especially when the tiling space comes from a substitution or a cut-and-project scheme. The main challenge in applying this theorem lies in developing methods to compute the rotation class of a given map.

2. Background and Precise Definitions

2.1. Tilings

In one dimension, a tile is a pair $t = \{I, \ell\}$, where $I$ is a closed interval and $\ell$ is a label. We assume that the labels are drawn from a finite alphabet and that tiles with the same label always have intervals of the same length. If $t = \{[x_1, x_2], \ell\}$ is a tile, and if $s \in \mathbb{R}$, then we define $t - s = \{[x_1 - s, x_2 - s], \ell\}$. That is, translating a tile means moving its support without changing its label. A tiling is a collection of tiles, intersecting only at their boundaries, whose union is all of $\mathbb{R}$. If $T = \{t_i\}$ is a tiling and $s \in \mathbb{R}$, then $T - s = \{t_i - s\}$. We sometimes denote the action of translations on tilings by $\Gamma$, so $\Gamma_s(T) := T - s$. Tilings whose tiles satisfy the above assumptions are said to have finite local complexity, or FLC.

If $T_1$ and $T_2$ are tilings built on the same set of possible tiles, consider the set of $\epsilon \in (0, 1)$ such that there exist $s_1, s_2 \in \mathbb{R}$ with $|s_i| \leq \epsilon/2$, and such that $T_1 - s_1$ and $T_2 - s_2$ agree exactly on the interval $[-\epsilon^{-1}, \epsilon^{-1}]$. We then define the distance $d(T_1, T_2)$ to be the infimum of such $\epsilon$'s, or 1 if no such $\epsilon$ exists. That is, $d(T_1, T_2) \leq \epsilon$ if $T_1$ and $T_2$ agree on $[-\epsilon^{-1}, \epsilon^{-1}]$ up to translations by up to $\epsilon/2$. With the topology defined by this metric, the set of all possible tilings by the fixed tile set is a compact space on which $\mathbb{R}$ acts by translation, i.e., an abstract dynamical system.

The orbit of a tiling $T$ is the set $\{T - s | s \in \mathbb{R}\}$. For a fixed tiling $T$, we often identify the orbit of $T$ with a copy of $\mathbb{R}$ by $T - s \leftrightarrow s$. The closure of the orbit of $T$ is called the continuous hull of $T$, or the tiling space of $T$, or simply the orbit closure, and is denoted $\Omega_T$. This is a dynamical system in its own right. A tiling $T'$ is in $\Omega_T$ if and only if every pattern that appears in $T'$ appears somewhere in $T$.

We next consider the local topology of $\Omega_T$. If $T' \in \Omega_T$ and $d(T'', T') < \epsilon$, then $T''$ and $T'$ agree on a big ball around the origin, up to a small translation. A neighborhood of $T'$ is then determined by a small real number (of size

1The FLC condition is usually defined in terms of local patches, but in one dimension it is equivalent to simply having finitely many possible kinds of tiles.
<\epsilon\) describing the translation, and a point in a totally disconnected space describing the possible extensions of the tiling beyond the big ball.

T is said to be repetitive if, for each finite pattern P that appears in T, there is a length LP such that every interval of length L in T contains at least one copy of P. This is equivalent to ΩT being a minimal dynamical system. That is, if T is repetitive and T' ∈ ΩT, then ΩT' = ΩT and the set of patterns that appear in T' (sometimes called the language of T') is the same as the set of patterns that appear in T. Since all the tilings in ΩT have the same orbit closure, we usually denote their common orbit closure as Ω, without any subscripts. If T is repetitive and non-periodic, the totally disconnected set described in the previous paragraph is actually a Cantor set.

In this paper, we only consider tilings that are one-dimensional, have FLC, and are repetitive.

2.2. Pattern-Equivariant Cohomology

Let T be a tiling, let φ : R → R be a continuous function, and let R > 0. We say that φ is pattern equivariant (PE) with radius R with respect to T if φ(x1) = φ(x2) for all pairs (x1, x2) of points such that T − x1 and T − x2 agree exactly on [−R, R]. That is, the value of φ at a point x ∈ R is determined exactly by the pattern of T on [x − R, x + R]. A function φ is called weakly PE (wPE) if it is the uniform limit of PE functions, meaning that for any ϵ > 0 there exists an R > 0 such that φ(x) is determined to within ϵ by the pattern of T on [x − R, x + R]. Here, we will refer to PE functions as strongly PE (sPE) to distinguish them from wPE functions. With respect to the local product structure of Ω, sPE functions are continuous in the R direction and locally constant in the Cantor direction, while wPE functions are merely continuous in both directions.

Using the identification of R with the orbit of a tiling, we extend the ideas of strong and weak pattern equivariance to functions on Ω. Let T be a reference tiling. Suppose g is a map on the orbit of T whose displacement φT(x) := g(T − x) − x is sPE (or wPE). Then, g extends by continuity to a map G on all of Ω, such that G restricted to every reference tiling T has sPE (or wPE) displacement.

In addition to functions with sPE displacement, we can also consider sPE differential forms, either on R with respect to a reference tiling T, or on Ω. Since the derivative of a function with sPE displacement is an sPE differential form, we define the first PE cohomology\(^2\) of T to be

\[ H^1_{PE,T} = \frac{\text{sPE 1-forms}}{d(\text{sPE functions})}. \] (1)

Kellendonk and Putnam [24, 25] (see also [32]) proved that:

**Proposition 1.** If T is a repetitive FLC tiling, then \( H^1_{PE,T} \) is isomorphic to the real-valued \( Č \)ech cohomology \( Č^1(Ω_T, \mathbb{R}) \).

\(^2\)In the literature, “pattern equivariant” usually means sPE. The term “PE cohomology” was defined before the theory of wPE forms was developed.
It may happen that an sPE 1-form $\alpha$ is not the derivative of a function with sPE displacement, but is the derivative of a wPE function. In that case, we say that the class $[\alpha] \in H^1_{PE,T}$ is asymptotically negligible. (This does not depend on which representative we pick for $[\alpha]$.) The asymptotically negligible classes, or more precisely the corresponding elements of $\hat{H}^1(\Omega, \mathbb{R})$, form a subspace of $\hat{H}^1(\Omega, \mathbb{R})$ denoted $H^1_{AN}(\Omega)$. Both $\hat{H}^1(\Omega, \mathbb{R})$ and $H^1_{AN}(\Omega, \mathbb{R})$ are well studied, and many techniques are known for computing them. (See [34] and references therein.)

2.3. Self-homeomorphisms and Related Functions

Let $\Omega$ be a tiling space, and let $T \in \Omega$ be an arbitrary reference tiling. As always, we suppose that $\Omega$ is compact and minimal, or equivalently that $\Omega = \Omega_T$, where $T$ has FLC and is repetitive. Let $F : \Omega \to \Omega$ be a homeomorphism homotopic to the identity. $F$ must then take the path-component of $T$ to itself. However, the path component of $T$ is the same as the orbit of $T$, so $F(T) = T - \Phi(T)$ for some continuous function $\Phi : \Omega \to \mathbb{R}$ (see [28]). There must also be a homeomorphism $f_T : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $F(T - x) = T - f_T(x)$. We furthermore define $\phi_T(x) = f_T(x) - x$. That is, $F$ specifies where a tiling goes, while the displacement function $\Phi$ specifies how far it moves. If we identify the orbit of $T$ with $\mathbb{R}$ by $T - x \leftrightarrow x$, then $F$ becomes $f_T$ and $\Phi$ becomes $\phi_T$. We will typically use capital letters to define maps on $\Omega$ and lower case letters to define maps on $\mathbb{R}$. When the reference tiling $T$ is clear, we will omit the subscripts on $f$ and $\phi$.

In this paper, we only consider maps $F$ for which $\Phi$ is sPE. This then implies that for every $T \in \Omega$, $\phi_T$ is sPE with respect to $T$. We will denote the set of self-homeomorphisms of $\Omega$ with sPE displacement by $F(\Omega)$, or by $\mathcal{F}$ for short. Note that if $\phi_T$ is sPE, then $\phi_T$ is necessarily bounded, so $f_T$ is orientation preserving.

2.4. Rotation Numbers, Rotation Forms, and Rotation Classes

The goal of rotation theory is to understand what happens when you iterate a self-homeomorphism many times. For a lift $F : \mathbb{R} \to \mathbb{R}$ of a circle map $f : S^1 \to S^1$, this is described by a translation number, often denoted $\rho$ and defined as

$$\rho = \lim_{n \to \infty} \frac{F^n(x) - x}{n}. \quad (2)$$

The translation number indicates how far you go on average per unit time, or equivalently how long it takes to travel a certain distance.

Now suppose that $\rho > 0$. If $x_1 \in \mathbb{R}$ and $L$ is a positive integer, then $x_1$ and $x_2 = x_1 + L$ correspond to the same point on $S^1$. If $n_\pm$ are integers such that $n_- < L/\rho < n_+$, then $F^{n_-}(x_1) < x_2 < F^{n_+}(x_1)$. In other words, integrating $dx/\rho$ from $x_1$ to $x_2$ gives sharp upper and lower bounds on how many iterations are required to bring $x_1$ up to and then past $x_2$. Note that these estimates only apply when $x_2 - x_1 \in \mathbb{Z}$. If we imagine the universal cover of $S^1$ as being a periodic tiling with period 1, the estimates apply whenever the patterns around $x_1$ and $x_2$ agree out to distance 1.
For maps $F : \Omega \to \Omega$ of tiling spaces, the analogue of the translation number $\rho$ is easy. We say that $\rho$ is the rotation number of $F$ if, for every tiling $T$ and starting point $x$, \( \lim_{n \to \infty} \frac{f^n_T(x) - x}{n} = \rho \). In [2], the first author showed:

**Theorem 6.** Let $(\Omega, \Gamma)$ be a uniquely ergodic and minimal one dimensional tiling space. Then, for every $F$ in $\mathcal{F}$ with no fixed points, the rotation number $\rho$ of $F$ exists and is different from zero.

Observe that the assumption of unique ergodicity is essential. In Sect. 6, we exhibit a map on a tiling space that meets all of the assumptions of this theorem except unique ergodicity, and for which $\rho$ does not exist. Also observe that in our current context, the assumption of $F$ not having fixed points is equivalent to $\Phi$ not having zeros and (since $\Omega$ is compact) being bounded away from zero. In the case that $\Phi$ has zeros, strong pattern equivariance implies that in each orbit the set of zeros of $\Phi$ is relatively dense, and thus, the rotation number exists and is equal to 0.

We now turn to the analogue of \( \frac{dx}{\rho} \). Let $\mu$ be an sPE $1$-form on $\mathbb{R}$ with respect to a reference tiling $T$. We say that $\mu$ is a rotation form for $F$ if there exists a radius $R$ such that, for all pairs of points $x_1, x_2 \in \mathbb{R}$ such that $T - x_1$ and $T - x_2$ agree on a ball of radius $R$ around the origin, and for all integers $n \pm$ such that $n_- < \int_{x_1}^{x_2} \mu < n_+$, then $x_2$ is strictly between $f^n_{T_1}(x_1)$ and $f^n_{T_1}(x_1)$. This definition is crafted to apply even when $F$ moves points backwards and when $\int_{x_1}^{x_2} \mu$ is an integer. In the typical case where $f_T$ moves points forwards and $\int_{x_1}^{x_2} \mu$ is not an integer, the estimates simplify to

\[
\int_{\lfloor \int_{x_1}^{x_2} \mu \rfloor}^{\lceil \int_{x_1}^{x_2} \mu \rceil+1} f_{T}(x_1) < x_2 < \int_{\lfloor \int_{x_1}^{x_2} \mu \rfloor}^{\lceil \int_{x_1}^{x_2} \mu \rceil+1} f_{T}(x_1),
\]

where $\lfloor \int_{x_1}^{x_2} \mu \rfloor$ denotes the greatest integer less than or equal to $\int_{x_1}^{x_2} \mu$.

If $\mu$ is a rotation form with radius $R$ and $\nu = \mu + dg$ is cohomologous to $\mu$, where $g$ is a function with sPE displacement and radius $R'$, then we claim that $\nu$ is a rotation form with radius $\max(R, R')$. This is because if $x_1$ and $x_2$ are points such that $T - x_1$ and $T - x_2$ agree on a ball of radius $\max(R, R')$, then $\int_{x_1}^{x_2} dg = g(x_2) - g(x_1) = 0$, so $\int_{x_1}^{x_2} \nu = \int_{x_1}^{x_2} \mu$. In other words, every representative of the cohomology class $[\mu] \in H^1_{T, PE} \simeq \tilde{H}^1(\Omega, \mathbb{R})$ is also a rotation form, and we say that $[\mu]$ is a rotation class.

The first question about rotation classes is to understand their basic relationship with rotation numbers. The next statement gives a natural answer when the tiling space is minimal and uniquely ergodic.

**Proposition 2.** Let $(\Omega, \Gamma)$ be a uniquely ergodic and minimal one dimensional tiling space, $F$ be a homeomorphism in $\mathcal{F}$ with no fixed points, and $\mu$ be a rotation form for $F$. Let $T$ be a tiling in $\Omega$ and suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a set of points in $\mathbb{R}$ such that the ball of radius $R$ around the origin in $T - x_n$ agrees with the ball of radius $R$ around the origin in $T$, and such that $\lim_{n \to \infty} x_n = \infty$. Then,

\[
\rho = \lim_{n \to \infty} \frac{x_n}{\int_{0}^{x_n} \mu}. \tag{4}
\]
Proof. We already know, by Theorem 6, that \( \rho \) exists, is unique, and can be computed as
\[
\rho = \lim_{k \to \infty} \frac{f^k_T(0)}{k}.
\] (5)

By the definition of \( \mu \), if \( k = \lfloor \int_0^{x_n} \mu \rfloor \), then \( f^k_T(0) < x_n < f^{k+1}_T(0) \). But then,
\[
\frac{f^k_T(0)}{k+1} < \frac{x_n}{\int_0^{x_n} \mu} < \frac{f^{k+1}_T(0)}{k}.
\] (6)

Taking a limit of (6) as \( n \to \infty \) (and therefore \( k \to \infty \)) and applying (5) then give (4). □

We will see in Example 6 in Sect. 6 that when the tiling space is not uniquely ergodic, it is possible for a map to have a rotation class but no rotation number. Going further, neither the existence nor the uniqueness of the rotation class is obvious. In Sect. 4, we will show that, with the added assumption of irrationality, rotation classes are in fact unique. However, in Sect. 6 we will construct a map for which the rotation class does not exist.

2.5. Collaring, Shape Changes, and Irrationality

Suppose that \( T \) is a one-dimensional tiling with a finite alphabet \( A \). We can generate a new tiling \( T_c \) using the same intervals as the tiles in \( T \), only with a larger alphabet. For each tile \( t \in T \), we replace the label \( \ell \in A \) with a triple \((\ell_-)\ell(\ell_+)\), where \( \ell \) is the original label of \( t \) and \( \ell_\pm \) are the original labels of the tiles preceding and following \( t \). A tile equipped with labels indicating its predecessor and successor is called a collared tile, and the process of relabeling is called collaring. The new alphabet \( A_c \subset A^3 \) is still finite, so the new tiling \( T_c \) still has FLC. Furthermore, \( T_c \) is repetitive if and only if \( T \) is repetitive. The obvious forgetful map \( \Omega_{T_c} \to \Omega_T \) is a homeomorphism that commutes with translation. Since the translation actions on \( \Omega_T \) and \( \Omega_{T_c} \) are conjugate, the spaces \( \Omega_T \) and \( \Omega_{T_c} \) are equivalent for our purposes. One can also repeat the collaring process. For any radius \( R \), it is possible to collar enough times that the label of each (collared) tile indicates the pattern of (ordinary) tiles around it out to distance at least \( R \).

Now suppose that \( \mu \) is a positive 1-form that is sPE with radius \( R \) with respect to a tiling \( T \). Without loss of generality, suppose that the tiles in \( T \) have been collared out to that same distance \( R \). We construct a new tiling \( T' \), with the same alphabet as \( T \), by preserving the labels of the tiles while moving each vertex \( x \) of \( T \) to position \( \int_0^x \mu \). The new length of a tile \( t \) with endpoints \( a \) and \( b \) is \( \int_a^b \mu \). Since \( \mu \) is sPE with radius \( R \), and since the tile labels describe the pattern of \( T \) out to distance \( R \), this integral depends only on the label of \( t \).

Note that this shape change operation does not commute with translation. However, it does map the orbit of \( T \) to the orbit of \( T' \), and the orbit closure \( \Omega_T \) to \( \Omega_{T'} \). The space \( \Omega' \) is called the shape change of \( \Omega \) by \( \mu \). For the theory of shape changes in 1 dimension and higher dimensions, see [13,14].
If \(\mu\) and \(\nu\) are different positive sPE 1-forms, then we can compare the spaces \(\Omega_\mu\) and \(\Omega_\nu\) obtained by doing shape changes by \(\mu\) and \(\nu\), respectively. It turns out that the translation actions on these spaces are topologically conjugate if and only if \(\mu - \nu\) is asymptotically negligible [14]. If \(\mu\) and \(\nu\) are cohomologous, then there is an even stronger equivalence between \(\Omega_\mu\) and \(\Omega_\nu\), called Mutual Local Derivability, or MLD. In short, \(\Omega_\mu\) is determined up to MLD by \([\mu] \in \tilde{H}^1(\Omega, \mathbb{R})\) and is determined up to topological conjugacy by the image of \([\mu]\) in the quotient space \(\tilde{H}^1(\Omega, \mathbb{R}) / \tilde{H}^1_{AN}(\Omega, \mathbb{R})\) [9].

Shape changes by negative 1-forms are defined similarly and are orientation reversing. If \(\mu\) changes sign but has a nonzero average value (where averaging requires unique ergodicity of \(\Omega_T\)), then there is a form \(\bar{\mu}\), cohomologous to \(\mu\), that does not change sign; we can then do a shape change by \(\bar{\mu}\). When \(\Omega_T\) is uniquely ergodic, we can thus do shape changes by all cohomology classes whose average values are nonzero [21].

2.6. \(\rho\)-Boundedness, Irrationality, and Poincaré-like Theorems

If \(f\) is a circle homeomorphism and its rotation number \(\rho\) is irrational, then Poincaré’s theorem asserts that \(f\) is semi-conjugate to a rotation by \(\rho\). In this paper, we wish to understand and state Poincaré-like theorems in terms of the irrationality of the rotation class of a tiling homeomorphism.

Suppose that \([\mu]\) is a cohomology class represented by a positive (or negative) sPE form \(\mu\). Let \(\Gamma_1 : \Omega_\mu \to \Omega_\mu\) be the operator of translation by 1. We say that \([\mu]\) is irrational if \(\Omega_\mu\) is minimal with respect to the action of \(\Gamma_1\). This condition only depends on the cohomology class of \(\mu\), since different representatives of the same class give translation actions that are topologically conjugate.

Recall that a number \(\lambda\) is called a topological eigenvalue of the translation action on \(\Omega_\mu\) if there exists a non-zero continuous function \(\xi : \Omega_\mu \to \mathbb{C}\) such that, for all \(T \in \Omega_\mu\) and all \(x \in \mathbb{R}\), \(\xi(T - x) = \exp(2\pi i \lambda x) \xi(T)\). Minimality of \(\Gamma_1\) is equivalent to the non-existence of rational topological eigenvalues other than 0. Thus, the class \([\mu]\) is irrational if and only if none of the topological eigenvalues of the translation action on \(\Omega_\mu\) lie in \(\mathbb{Q} - \{0\}\).

We also speak of a rotation number \(\rho\) being irrational if \(\Gamma_\rho : \Omega \to \Omega\) is minimal.\(^3\) This is equivalent to \([dx/\rho]\) being an irrational class, since if \(\mu = dx/\rho\), then \(\Gamma_1\) on \(\Omega_\mu\) is conjugate to \(\Gamma_\rho\) on \(\Omega\).

We say that \(F\) is \(\rho\)-bounded if, for every \(T \in \Omega\) and every starting point \(x\), the sequence \(\{f^n_T(x) - x - n\rho\}\) is bounded. This is equivalent to the quantity \(f^n_T(x) - x - n\rho\) being uniformly bounded as a function of \(n\) and \(x\). The main result of [2] is

**Theorem 7.** Let \((\Omega, \Gamma)\) be a uniquely ergodic and minimal one dimensional tiling space. Let \(F : \Omega \to \Omega\) be an orientation preserving homeomorphism with irrational rotation number \(\rho\). Suppose furthermore that \(F\) is \(\rho\)-bounded. Then, \(F\) is semi-conjugate to \(\Gamma_\rho\).

\(^3\)Note that \(\rho\) is a dimensionful quantity, having units of length. As such, the naive definition of \(\rho\) being an irrational number does not make sense.
Thus, $\rho$-boundedness is a crucial property when dealing with Poincaré-like theorems. In practice, $\rho$-boundedness is rather difficult to check. Fortunately, the introduction of rotation classes allows us to understand $\rho$-boundedness in cohomological terms:

**Theorem 8.** Suppose that $F \in \mathcal{F}$ does not have fixed points. Let $\rho$ be the rotation number of $F$, and let $\mu$ be a rotation form for $F$. Then, the following assertions are equivalent:

1. $F$ is $\rho$-bounded.
2. $\mu - \frac{dx}{\rho}$ is asymptotically negligible.

**Proof.** Suppose that $\mu$ is a positive form, the case of negative $\mu$ being similar. Let $\beta = \mu - \frac{dx}{\rho}$. By the Gottschalk–Hedlund theorem (see [26] for a version of this theorem specifically adapted to tiling spaces), $\beta$ is asymptotically negligible if and only if its integral is bounded. More precisely, $\beta$ is asymptotically negligible if and only if the quantity

$$I(x_1, x_2) := \int_{x_1}^{x_2} \beta = \left( \int_{x_1}^{x_2} \mu \right) - \frac{x_2 - x_1}{\rho}$$

is bounded as a function of $x_1$ and $x_2$.

First we show that $\rho$-boundedness of $F$ implies asymptotic negligibility of $\beta$. Let $R$ be the sPE radius of $\mu$. Suppose that $F$ is $\rho$-bounded. Then, there exists a constant $C$ such that $|f^n_T(x) - x - n\rho| < C$ for all $n$. Let $x_1$ and $x_2$ be two real numbers, sufficiently far apart, and suppose without loss of generality that $x_1 < x_2$. Let $x_3$ be such that (a) $T - x_3$ and $T - x_1$ agree on a ball of radius $R$ around the origin, (b) $x_1 < x_3 < x_2$, and (c) there is no other point between $x_3$ and $x_2$ satisfying properties (a) and (b). That is, $x_3$ is the largest return time, smaller than $x_2$, to the pattern of radius $R$ around $x_1$. By repetitivity and FLC, $|x_3 - x_2|$ is uniformly bounded, implying that $I(x_3, x_2)$ is bounded. Since $I(x_1, x_2) = I(x_1, x_3) + I(x_3, x_2)$, all that remains is to bound $I(x_1, x_3)$.

To see that $I(x_1, x_3)$ is bounded, set $n = \lfloor \int_{x_1}^{x_3} \mu \rfloor$, so in particular

$$n - 1 < \int_{x_1}^{x_3} \mu < n + 1. \quad (8)$$

Since $\mu$ is a rotation form, we must also have

$$f^{n-1}(x_1) < x_3 < f^{n+1}(x_1). \quad (9)$$

Combining these estimates, we get

$$I(x_1, x_3) = \left( \int_{x_1}^{x_3} \mu \right) - \frac{x_3 - x_1}{\rho}$$

$$\leq n + 1 - \frac{x_3 - x_1}{\rho}$$

$$< 2 - \frac{f^{n-1}(x_1) - x_1 - (n - 1)\rho}{\rho}$$

$$\leq 2 + \frac{C}{\rho}. \quad (10)$$
Similarly,

\[ I(x_1, x_3) = \left( \int_{x_1}^{x_3} \mu \right) - \frac{x_3 - x_1}{\rho} \]

\[ \geq n - 1 - \frac{x_3 - x_1}{\rho} \]

\[ > -2 - \frac{f^{n+1}(x_1) - x_1 - (n+1)\rho}{\rho} \]

\[ \geq -2 - \frac{C}{\rho}. \]  

(11)

Now we show that asymptotic negligibility of $\beta$ implies $\rho$-boundedness of $F$. We need to show that there exists a $C > 0$ such that

\[ |f^n(x_1) - x_1 - n\rho| < C \]  

(12)

for every natural number $n$ and every starting point $x_1$. For each pair $(x_1, n)$, let $x' = f^n(x_1)$, and let $R$ be the larger of the PE radius of $f$ and that of the rotation form $\mu$. By repetitivity and finite local complexity, there is a radius $R'$ such that every ball of radius $R'$ contains at least one copy of every $T$-patch of radius $R$. In particular, we can find a point $x_2$, with $|x_2 - x'| < R'$, such that the $T$-patches of radius $R$ around $x_1$ and $x_2$ agree.

We will apply the triangle inequality several times. First, since $|x' - x_2|$ is bounded (say, by $C_1$), we have

\[ |f^n(x_1) - x_1 - n\rho| = |x' - x_1 - n\rho| \]

\[ \leq |x' - x_2| + |x_2 - x_1 - n\rho| \]

\[ < C_1 + |x_2 - x_1 - n\rho|. \]  

(13)

Since $\beta$ is asymptotically negligible, $\int \beta$ is bounded by a constant $C_2$ and

\[ |x_2 - x_1 - n\rho| \leq \left| x_2 - x_1 - \rho \int_{x_1}^{x_2} \mu \right| + \rho \left| \int_{x_1}^{x_2} \mu - n \right| \]

\[ = \left| \int_{x_1}^{x_2} \beta \right| + \rho \left| \int_{x_1}^{x_2} \mu - n \right| \]

\[ < C_2 + \rho \left| \int_{x_1}^{x_2} \mu - n \right|. \]  

(14)

To complete the proof, we need to bound $\left| \int_{x_1}^{x_2} \mu - n \right|$. Since $\mu$ is a rotation form for $F$ with radius at most $R$, we have

\[ f^{k-1}(x_1) < x_2 < f^{k+1}(x_1), \]

(15)

where $k = \left| \int_{x_1}^{x_2} \mu \right|$. This implies that $|x_2 - f^k(x_1)|$ is bounded by the maximum value of $\phi$ (the displacement of $f$). Furthermore, $|x_2 - f^n(x_1)| = |x_2 - x'|$ is bounded by $R'$, so $|f^k(x_1) - f^n(x_1)|$ is bounded. Since $\phi$ has a positive minimum value, this in turn bounds $|k - n|$. But $k$ is within 1 of $\int_{x_1}^{x_2} \mu$, so we have bounded $\left| \int_{x_1}^{x_2} \mu - n \right|$. \[ \square \]
3. Flows

In this section, we consider a best-case scenario, when our map \( F : \Omega \to \Omega \) is the time-one sampling of an sPE flow on \( \Omega \). In this situation, we explore the remaining obstructions to \( F \) being conjugate to uniform translation on \( \Omega \). We also show that \( F \) coming from a flow implies the existence of a rotation class. Thus, the non-existence of a rotation class implies that \( F \) does not come from such a flow, and in particular is not conjugate to uniform translation.

Let \( \bar{F} : \Omega \times \mathbb{R} \to \Omega \) be a continuous map. We say that \( \bar{F} \) is a flow with sPE velocity if there exists an sPE function \( V : \Omega \to \mathbb{R} \) such that, for every \((T,t) \in \Omega \times \mathbb{R} \),

\[
\bar{F}(T,t) = T - x_T(t), \quad x_T(0) = 0, \quad \frac{dx_T(t)}{dt} = v_T(x_T(t)) := V(T - x_T(t)).
\]

(16)

For each \( t \in \mathbb{R} \), we define the map \( \bar{F}_t : \Omega \to \Omega \) by \( \bar{F}_t(T) = \bar{F}(T,t) \). We say that \( F \) is the time-one sampling of \( F \) if \( F = F_1 \). To avoid trivialities, we assume that \( F \) has no fixed points, or equivalently that \( V \) has no zeros. As always, we use lower-case letters with subscript \( T \) to denote maps on \( \mathbb{R} \) (or \( \mathbb{R} \times \mathbb{R} \)) associated with maps on \( \Omega \) (or \( \Omega \times \mathbb{R} \)) via a reference tiling \( T \). When the reference tiling \( T \) is clear, we drop the subscript \( T \). In particular, \( \bar{f}_t(x_0) \) is the solution to the differential equation

\[
\frac{dx}{dt} = v(x); \quad x(0) = x_0.
\]

(17)

**Proposition 3.** Let \( F \in \mathcal{F} \). Let \( T \) be a fixed reference tiling, and identify the orbit of \( T \) with \( \mathbb{R} \) via \( T - x \leftrightarrow x \). If the map \( F \) is the time-one sampling of an sPE flow \( \bar{F} \) with velocity function \( V \), then \( \mu = dx/v(x) \) is a rotation form whose PE radius is the same as the PE radius of \( v \).

**Proof.** Suppose that \( x_1 \) and \( x_2 \) are points such that \( T - x_1 \) and \( T - x_2 \) agree on \( B_R(0) \), where \( R \) is the PE radius of \( v \). Let \( s = \int_{x_1}^{x_2} \frac{dx}{v(x)} \). Then, \( \bar{f}_s(x_1) = x_2 \). Since the flow is unidirectional, if \( s \) lies between two integers \( n_- \) and \( n_+ \), then \( x_2 \) lies between \( \bar{f}_{n_-}(x_1) = f^{n_-}(x_1) \) and \( \bar{f}_{n_+}(x_1) = f^{n_+}(x_1) \). \( \square \)

Let \( \mu = dx/v(x) \), and let \( \Omega_{\mu} \) be the tiling space obtained by applying the shape change associated with \( \mu \) to \( \Omega \). The shape-change homeomorphism \( S : \Omega \to \Omega_{\mu} \) conjugates \( \bar{F}_t \) on \( \Omega \) to \( \Gamma_t \) (i.e., translation by \( t \)) on \( \Omega_{\mu} \), and in particular conjugates \( F = \bar{F}_1 \) to \( \Gamma_1 \) on \( \Omega_{\mu} \). However, this does not imply that \( F \) conjugates to a uniform translation on \( \Omega \) itself! That depends on the cohomology class of \( \mu \).

**Theorem 9.** Let \( F : \Omega \times \mathbb{R} \to \Omega \) be a flow with never-zero sPE velocity function \( V : \Omega \to \mathbb{R} \), let \( T \) be a reference tiling, and let \( \mu = dx/v(x) \). Let \( \rho \) be a nonzero real number. The following four conditions are then equivalent.

1. \( \mu = \frac{dx}{\rho} + \beta \), where \( \beta \) is asymptotically negligible.
2. \( F \) is \( \rho \)-bounded.
3. There is a homeomorphism \( H : \Omega \to \Omega \), homotopic to the identity, that conjugates \( \bar{F}_t \) to \( \Gamma_{\rho t} \) for every \( t \in \mathbb{R} \).
There is a homeomorphism $H : \Omega \to \Omega$, homotopic to the identity, that conjugates $F$ to $\Gamma_\rho$.

Note that we have already proven the equivalence of conditions (1) and (2) in Theorem 8. Before proving the rest of Theorem 9, we state and prove a lemma relating conditions (3) and (4).

Lemma 1. Suppose $\Omega$ is the orbit closure of a one-dimensional repetitive tiling with FLC, that $F$ is a flow on $\Omega$, generated by an sPE velocity function $V$, that $F = \bar{F}_1$, and that $G : \Omega \to \Omega'$ is a homeomorphism that conjugates $F$ to $\Gamma_1$. Then, there exists a (possibly different) homeomorphism $\tilde{G} : \Omega \to \Omega'$ that conjugates $\bar{F}_t$ to $\Gamma_t$ for all $t \in \mathbb{R}$.

Proof of Lemma 1. Pick a reference tiling $T$, and let $X$ be the closure of the $F$-orbit of $T$. We will use the notation $T_i$, $T_s$, etc. as shorthand for $\bar{F}_i(T)$, $\bar{F}_s(T)$, etc. Let $S$ be the set of all $t$'s such that $T_i \in X$. We claim that $S$ is a closed subgroup of $\mathbb{R}$ that contains the integers.

Closure comes from the fact that if there is a sequence of $t_i \in S$ converging to $t_\infty$, then $T_{t_\infty}$ is arbitrarily well approximated by the $T_{t_i}$'s, each of which can be arbitrarily well approximated by some $T_{n_i} = F^{n_i}(T)$, where $n_i \in \mathbb{Z}$, so $T_{t_\infty} \in X$ and $t_\infty \in S$.

To see the group property, suppose that $s$ and $t$ are in $S$ and pick an $\epsilon > 0$. We must find an integer $N$ such that $d(T_N, T_{s+t}) < \epsilon$, where $d$ is the tiling metric.

The map $F_t$ is uniformly continuous, so there is a $\delta_1$ such that if two tilings are within $\delta_1$, then their images under $F_t$ are within $\epsilon/2$. Since $s \in S$, we can find an integer $n$ such that $d(T_n, T_s) < \delta_1$. Since $F^n$ is uniformly continuous, there is a $\delta_2$ such that if two tilings are within $\delta_2$, then their images under $F^n$ are within $\epsilon/2$. Since $t \in S$, there exists an integer $m$ such that $d(T_m, T_t) < \delta_2$.

We then have

$$d(T_{n+m}, F^n(T_t)) < \epsilon/2 \quad \text{since} \quad d(T_m, T_t) < \delta_2,$$

$$F^n(T_t) = T_{n+t} = \bar{F}_t(T_n),$$

$$d(\bar{F}_t(T_n), T_{s+t}) < \epsilon/2 \quad \text{since} \quad d(T_n, T_s) < \delta_1,$$

$$d(T_{n+m}, T_{s+t}) < \epsilon < \epsilon$$

by the triangle inequality. (18)

Thus, $s + t \in S$. Since $S$ is a closed subgroup of $\mathbb{R}$ containing $\mathbb{Z}$, either $S = \mathbb{R}$ or $S$ is a cyclic group generated by a fraction $1/q$. We will treat each of these cases in turn.

First suppose that $S = \mathbb{R}$. For each $t \in \mathbb{R}$, let $\chi(t)$ be such that $G(T_t) = G(T) - \chi(t)$.

Since $F = \bar{F}_1$ commutes with $\bar{F}_t$ and conjugates to $\Gamma_1$, we must have $\chi(t+1) = \chi(t) + 1$. We will show that for all $s, t \in \mathbb{R}$, $\chi(s + t) = \chi(s) + \chi(t)$. This, combined with $\chi$ being the identity on $\mathbb{Z}$, implies that $\chi$ is the identity on $\mathbb{Q}$. By continuity, $\chi$ is then the identity on all of $\mathbb{R}$, and $G$ conjugates $\bar{F}_t$ to $\Gamma_t$ on the orbit of $T$. By continuity, $G$ then conjugates $\bar{F}_t$ to $\Gamma_t$ on all of $\Omega$.

Since $S = \mathbb{R}$, we can find an integer $n$ such that $T_n$ approximates $T_t$ arbitrarily well. Since $\bar{F}_s$ is uniformly continuous, this implies that $T_{t+s}$ is
approximated arbitrarily well by $T_{n+s}$. Now, for any tiling $T'$, $G(T'_n)$ is a translate of $G(T')$, and the relative spacing of these two tilings is a continuous function of $T'$. Thus, the displacement between $G(T_{s+t})$ and $G(T_t)$, which is $\chi(t+s) - \chi(t)$, can be arbitrarily well approximated by the relative displacement of $G(T_n)$ and $G(T_{n+s})$, namely $\chi(n+s) - \chi(n)$.

However, $G$ conjugates $F^n$ to translation by $n$, so $\chi(n) = n$ and $\chi(n+s) = n + \chi(s)$. Thus, $\chi(s + t) - \chi(t)$ is arbitrarily well approximated by $\chi(s)$ and must therefore equal $\chi(s)$. This completes the proof in the case that $S = \mathbb{R}$.

Finally, suppose that $S = \frac{1}{q}\mathbb{Z}$. The previous arguments show that $\chi$ is the identity on $\frac{1}{q}\mathbb{Z}$, but say nothing about how $\chi$ behaves on the interval $(0, 1/q)$. However, no two points on this interval are in the same orbit closure, so we are free to pick any orientation-preserving homeomorphism $\chi : [0, 1/q] \to [0, 1/q]$. In particular, we can pick the identity, resulting in $G(T - t) = G(T) - t$ for all $t$. We then extend $G$ to all of $\Omega$ by continuity, and $G$ conjugates $\tilde{F}_t$ to $\Gamma_t$. $\square$

Proof of Theorem 9. The first two conditions are equivalent by Theorem 8.

Next we show that the first and third conditions are equivalent. We apply a shape change by $\mu$ to $\Omega$ to get a new tiling space $\Omega_{\mu}$ and then apply a uniform dilation by $\rho$ to $\Omega_{\mu}$ to get a further space $\Omega_{\rho\mu}$. Let $H_0 : \Omega \to \Omega_{\rho\mu}$ be the resulting shape change map. These shape changes conjugate $\tilde{F}_t$ on $\Omega$ to $\Gamma_t$ on $\Omega_{\rho\mu}$ to $\Gamma_{pt}$ on $\Omega_{\rho pt}$. In particular, $H_0$ conjugates $F$ on $\Omega$ to $\Gamma_0$ on $\Omega_{\rho\mu}$.

However, what we want is to find a homeomorphism $H : \Omega \to \Omega$, homotopic to the identity, that conjugates $\tilde{F}_t$ to $\Gamma_{pt}$. If this exists, then $H_0 \circ H^{-1} : \Omega \to \Omega_{\rho\mu}$ is a topological conjugacy. Conversely, if there exists a topological conjugacy $G : \Omega \to \Omega_{\rho\mu}$, homotopic to $H_0$, then we can take $H = G^{-1} \circ H_0$. Thus, condition (3) is equivalent to the existence of a topological conjugacy $G$ between $\Omega$ and $\Omega_{\rho\mu}$ that is homotopic to $H_0$.

The (non)existence of such a conjugacy is addressed with the results of [21, 22]. Each homeomorphism $G : \Omega \to \Omega'$ of tiling spaces is associated with a class $[G] \in \tilde{H}^1(\Omega, \mathbb{R}^d)$ (where $d$ is the dimension of the tiling space, in our case $d = 1$). Homotopic maps yield the same class, and the class of a shape change is represented by the form generating the shape change, so if $\Omega' = \Omega_{\rho\mu}$ and $G$ is homotopic to $H_0$, then

$$[G] = [H_0] = \rho\mu = dx + \rho\beta.$$  \hspace{1cm} (19)

By a theorem of [22], the spaces $\Omega$ and $\Omega'$ are topologically conjugate, via a map homotopic to $G$, if and only if $[G - dx]$ is asymptotically negligible. Thus, condition (3) is equivalent to the asymptotic negligibility of $\rho\beta$, which is of course equivalent to the asymptotic negligibility of $\beta$.

Thus, the first three conditions are equivalent. By specializing to $t \in \mathbb{Z}$, the third condition implies the fourth. Lemma 1, applied to a uniform rescaling of $\Omega$ by $\rho^{-1}$, shows that the fourth condition implies the third. $\square$
4. Existence, Uniqueness, and Poincaré’s Theorem

4.1. Existence of \( \mu \) and Semi-conjugacy to a Flow

Proposition 3 says that a map \( F \in \mathcal{F}(\Omega) \) that comes from a flow with sPE velocity has an sPE rotation form \( \mu \) and a corresponding rotation class \([\mu]\). But what if \( F \) is merely conjugate, or semi-conjugate, to a map \( F' \) with sPE displacement on another space \( \Omega' \) that comes from a flow with sPE velocity? Does that imply that \( F \) itself comes from a flow with sPE velocity? If not, does it at least imply that \( F \) admits a rotation class?

These questions are surprisingly hard. The difficulty is that a general (semi-)conjugacy does not have to respect the local product structures of \( \Omega \) and \( \Omega' \). If \( H : \Omega \to \Omega' \) (semi-)conjugates a map \( F \in \mathcal{F}(\Omega) \) to a map \( F' \in \mathcal{F}(\Omega') \), and if \( \mu' \) is a rotation form for \( F' \), there is no guarantee that \( H^*\mu' \) is sPE. To make progress on these questions, we must restrict our attention to the subset of (semi-)conjugacies for which \( H^*\mu' \) is in fact sPE.

Suppose that \( F \in \mathcal{F}(\Omega) \), that \( H : \Omega \to \Omega' \) is a map of tiling spaces, and that \( F' \in \mathcal{F}(\Omega') \). If \( F' \circ H = H \circ F \), then we say that \( H \) is a semi-conjugacy of \( F \) to \( F' \). If in addition \( H \) is a homeomorphism, we say that \( H \) is a conjugacy of \( F \) to \( F' \). As usual, we identify the orbits of \( T \in \Omega \) and \( T' = H(T) \in \Omega' \) with \( \mathbb{R} \), resulting in maps \( f \) and \( f' \) on \( \mathbb{R} \), a map \( h \) from the copy of \( \mathbb{R} \) representing the orbit of \( T \) to the copy of \( \mathbb{R} \) representing the orbit of \( T' \), and a flow \( f^t : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). If \( f' \) comes from a flow \( f^t \) with sPE velocity function \( v' \), and if the pullback by \( h \) of the rotation form \( \mu' = dx/v'(x) \) is sPE, then we say that \( H \) is a local (semi-)conjugacy of \( F \) to the time-one sampling of a flow, and we say that \( F \) is locally (semi-)conjugate to \( F' \). (The word “local” here refers to the fact that local semi-conjugacies typically preserve the local product structure of the tiling space \( \Omega \).)

These definitions may sound restrictive, but in fact we will show (Corollary 1) that whenever \( F \) admits an irrational rotation class, it is locally semi-conjugate to uniform motion.

If \( H \) is a local conjugacy, then we can pull \( F' \) back to get a flow \( \bar{F} \) on \( \Omega \), such that \( F \) is the time-one sampling of \( \bar{F} \). Since \( \mu = H^*(dx/v') = dx/v \) is sPE, the function \( v(x) \) is sPE, so \( \mu = dx/v \) is a rotation form for \( F \) by Proposition 3. The following theorem shows that \( F \) admits a rotation form even when \( H \) is only a local semi-conjugacy.

**Theorem 10.** Let \( F \in \mathcal{F}(\Omega) \). Suppose that \( F \) is locally semi-conjugate, via a map \( H : \Omega \to \Omega' \), to the time-one sampling of an sPE flow \( F' \) on \( \Omega' \). Pick reference tilings \( T \in \Omega \) and \( T' = H(T) \in \Omega' \), and identify the orbits of \( T \) and \( T' \) with \( \mathbb{R} \) as usual. Let \( \mu' = dx/v'(x) \) and let \( \mu = h^*(\mu') \). Then, \( \mu \) is an sPE rotation form for \( F \) whose radius (as a rotation form) is the same as its PE radius.

**Proof.** Without loss of generality, supposed that \( \bar{F} \) is a forwards flow, so that \( t_1 < t_2 \) implies that \( \bar{f}_{t_1}(y) < \bar{f}_{t_2}(y) \) for every \( y \in \mathbb{R} \).

Let \( R \) be the PE radius of \( \mu \), and suppose that \( T - x_1 \) and \( T - x_2 \) agree out to radius \( R \). Then, \( h^{x_2}(x_1) \mu' \) is the travel time from \( h(x_2) \) to \( h(x_1) \) under
the flow $\tilde{f}'$. Suppose that $n_-, n_+ \in \mathbb{Z}$ and that $n_- < \int_{x_1}^{x_2} \mu < n_+$. Then

$$n_- < \int_{h(x_1)}^{h(x_2)} \mu' < n_+. \quad (20)$$

Thus, for any $y$,

$$\tilde{f}_{n_-}(y) < \tilde{f}_{h(x_1)}^{h(x_2)} \mu'(y) < \tilde{f}_{n_+}(y), \quad (21)$$

so

$$\tilde{f}_{n_-}(h(x_1)) < \tilde{f}_{h(x_1)}^{h(x_2)} \mu(h(x_1)) < \tilde{f}_{n_+}(h(x_1)), \quad (22)$$

$$\tilde{f}_{n_-}(h(x_1)) < h(x_2) < \tilde{f}_{n_+}(h(x_1)). \quad (23)$$

Since $h \circ f = \tilde{f}_1 \circ h$,

$$h \circ f^{(n_-)}(x_1) < h(x_2) < h \circ f^{(n_+)}(x_1). \quad (24)$$

Note that although $x < y$ only implies $h(x) \leq h(y)$, it is still the case that $h(x) < h(y)$ implies $x < y$. Therefore,

$$f^{(n_-)}(x_1) < x_2 < f^{(n_+)}(x_1). \quad (25)$$

Since $x_1$ and $x_2$ were arbitrary points whose neighborhoods agreed to distance $R$, $\mu$ is a rotation form with radius $R$. \hfill \Box

### 4.2. Uniqueness of the Rotation Class

**Theorem 11.** Suppose that the map $F : \Omega \to \Omega$ has rotation forms $\mu$ and $\nu$, and that $[\mu]$ is irrational. Then, $\mu$ and $\nu$ are cohomologous. That is, the rotation class is unique.

**Proof.** We do a shape change $S : \Omega \to \Omega_\mu$ by $\mu$. Using $S$ and $S^{-1}$, we transfer quantities from $\Omega$ to $\Omega_\mu$. In the process, $\mu$ becomes $(S^{-1})^* \mu = dx$, $\nu$ becomes $\nu' = (S^{-1})^* (\nu)$, and $F$ becomes $F' = S \circ F \circ S^{-1}$. By construction, $dx$ and $\nu'$ are rotation forms on $\Omega_\mu$ for $F'$. We will show that $dx$ and $\nu'$ are cohomologous on $\Omega_\mu$, which will imply that $\mu$ and $\nu$ are cohomologous on $\Omega$.

Let $T' \in \Omega_\mu$ be a reference tiling, and as usual, we identify the translational orbit of $T'$ with $\mathbb{R}$. Let $x_1$ and $x_2$ be corresponding points in sufficiently large identical patches in $T'$. Since $dx$ is a rotation class, if there is an integer $n$ such that $n < x_2 - x_1 < n + 1$, then $(f')^n(x_1) < x_2 < (f')^{n+1}(x_1)$. Similarly, if $m < \int_{x_1}^{x_2} \nu' < m + 1$ for some integer $m$, then $(f')^m(x_1) < x_2 < (f')^{m+1}(x_2)$. That is, $m$ must equal $n$, so whenever $x_2 - x_1$ is not an integer, we must have $[\int_{x_1}^{x_2} \nu'] = [x_2 - x_1]$.

Since the integral of $\nu' - dx$ is bounded (by one) on returns to similar patches, and since these patches appear with bounded gaps, the integral of $\nu' - dx$ is bounded, so $\nu' - dx$ is asymptotically negligible. Thus, there is a function $G$ with wPE displacement on $\Omega_\mu$, and a corresponding function $g$ with wPE displacement on $\mathbb{R}$, such that $\nu' = dx + dg$. Our goal is to show that $g$ has sPE displacement; hence, that $dx$ and $\nu'$ are cohomologous.

If the displacement of $g$ is not sPE, then there exists a positive $\epsilon$ and corresponding points $x_3$ and $x_4$ in identical large patches such that $g(x_4) - g(x_3) > \epsilon$. Since the action of $\Gamma_1$ is minimal on $\Omega_\mu$, we can find an integer
such that $n + x_3$ is less than $\epsilon/4$ to the right of a point $x_5$, such that an arbitrarily large patch around $x_5$ agrees with the corresponding patch around $x_4$. Since $g$ is continuous, by making this patch large enough we can ensure that $|g(x_5) - g(x_4)| < \epsilon/2$. Thus, we have $g(x_5) \geq g(x_3) + \epsilon/2$ and $n - \epsilon/4 < x_5 - x_3 < n$ for some integer $n$. But then $\int_{x_3}^{x_5} \nu' = x_5 - x_3 + g(x_5) - g(x_3)$ is slightly greater than $n$, and the integer parts of $x_5 - x_3$ and $\int_{x_3}^{x_5} \nu'$ do not agree, which is a contradiction.

We conclude that $g$ does in fact have sPE displacement, so $\nu'$ and $dx$ are cohomologous, so $\mu$ and $\nu$ are cohomologous. □

The assumption that $\mu$ is irrational cannot be removed. In Sect. 6, we construct a map on a tiling space that admits multiple rotation classes, all of them rational.

### 4.3. Existence and Irrationality Imply Semi-conjugacy

**Theorem 12.** Suppose that the map $F : \Omega \to \Omega$ has a rotation class $[\mu]$ that is irrational. Then, $F$ is semi-conjugate to uniform translation by 1 on $\Omega_{\mu}$.

**Proof.** The proof follows the same idea as the Proof of Theorem 11. We start by doing a shape change $S : \Omega \to \Omega_{\mu}$ by $\mu$. Pulling everything back to $\Omega_{\mu}$, we check that $\mu$ becomes $dx$, $F$ becomes $F' = S \circ F \circ S^{-1}$ and that $dx$ is a rotation form for $F'$. Since $[\mu]$ is irrational, translation by 1 on $\Omega_{\mu}$ is minimal, that is, 1 is $\Omega_{\mu}$-irrational. By Theorem 6, $F'$ is $\rho$-bounded with $\rho = 1$, which in turn implies that $F'$ preserves orientation. Since $F'$ satisfies the assumptions of Theorem 7 with $\rho = 1$, there is a new map $H' : \Omega_{\mu} \to \Omega_{\mu}$ which semi-conjugates $F'$ to translation by 1. Hence, $F$ is semi-conjugate, by $H' \circ S$, to translation by 1 on $\Omega_{\mu}$. □

Note that this theorem proves the existence of a semi-conjugacy, but does not prove the existence of a local semiconjugacy. The results of this section can be summarized as follows:

(1) If $\mu$ does not exist, then $F$ is not locally semi-conjugate to the time-one sampling of an sPE flow.

(2) If $\mu$ exists and is irrational, then $F$ is semi-conjugate to the time-one sampling of an sPE flow. In the next section, we will see that this semi-conjugacy is in fact local.

(3) If $\mu$ exists and is rational, there is no reason to expect $F$ to be semi-conjugate to the time-one sampling of an sPE flow, or even of any flow. In fact, by modifying examples for circle maps, it is easy to construct examples of $F$’s that are not.

### 5. Exploring the Semi-conjugacy

In this section, we assume throughout that $\mu$ exists and is irrational and hence that $F$ is semi-conjugate to the time-one sampling of a flow. We examine the possible form of this semi-conjugacy. First, we show that the semi-conjugacy
must be a local map. This allows for a strengthening of Theorem 12, completing the Proof of Theorem 3. We then address Theorems 4 and 5, concerning whether our map \( F \) is semi-conjugate to uniform translation on \( \Omega \) itself. Finally, we consider the barriers to showing that the semi-conjugacy is actually a conjugacy. That is, we set up the framework for the analogue of Denjoy’s theorem and explain why the usual proofs for circle maps cannot be carried over to tilings. Stating and proving the analogue of Denjoy’s theorem are an intriguing problem for future work.

Let \( F \) be a map on \( \Omega \) with irrational rotation form \( \mu \). By applying a shape change by \( \mu \), let \( \Omega_\mu \) be the space whose rotation class is exactly \( dx \). By Theorem 12, irrationality of \( \mu \) gives us that \( F \) is semi-conjugate to the time-one sampling of an sPE flow with rotation class \( dx \) on \( \Omega_\mu \). We will use the convention that maps and tilings within \( \Omega_\mu \) are denoted with \( \tilde{\cdot} \). Thus, \( \tilde{F} \) denotes the time-one sampling map on \( \Omega_\mu \), where \( \tilde{F}(\tilde{T}) = \tilde{T} - \tilde{\Phi}(\tilde{T}) \). As \( \tilde{F} \) is iterated under \( n \), let \( \tilde{\Phi}^n(\tilde{T}) = \tilde{T} - \tilde{\Phi}^n(\tilde{T}) \), and for \( \tilde{T} \in \Omega_\mu \), let \( \tilde{\Psi}(\tilde{T}) = \lim \sup_{n \in \mathbb{N}} \{ \tilde{\Phi}^n(\tilde{T}) - n \} \) and \( \tilde{J}(\tilde{T}) = \tilde{T} - \tilde{\Psi}(\tilde{T}) \) (26), so that \( \tilde{\Psi} : \Omega_\mu \to \mathbb{R} \) and \( \tilde{J} : \Omega_\mu \to \Omega_\mu \).

For a fixed reference tiling \( \tilde{T} \in \Omega_\mu \) whose orbit is identified with \( \mathbb{R} \), we abbreviate \( \tilde{f} \tilde{T} \) as \( \tilde{f} \), so that for all \( x \in \mathbb{R} \), \( \tilde{F}(\tilde{T} - x) = \tilde{T} - f(x) \). We also take \( \tilde{f}(x) = x + \tilde{\phi}(x) \), as usual. For \( x \in \mathbb{R} \), let

\[
\tilde{\psi}(x) = \lim \sup_{n \in \mathbb{N}} \{ \tilde{\phi}^n(x) - n \}, \quad \tilde{j}(x) = x + \tilde{\psi}(x),
\]

(27) where \( \tilde{j} : \mathbb{R} \to \mathbb{R} \) (where \( \mathbb{R} \) is marked as the tiling \( \tilde{T} \)) and \( \tilde{\psi} \) is the displacement of \( \tilde{f} \). As with \( \tilde{\Phi}^n \), \( \tilde{\phi}^n(x) \) represents the displacement of \( \tilde{f}^n(x) \), rather than the \( n \)th composition of \( \tilde{\phi} \) with itself.

The Proof of Theorem 7 (see [2]) shows that \( \tilde{\Psi} \) and \( \tilde{j} \) are continuous on the orbit of \( \tilde{T} \) (as a dense subspace of \( \Omega_\mu \)) and that \( \tilde{j} \) extends by continuity to a semi-conjugacy on all of \( \Omega_\mu \). Since \( \tilde{f} \) is monotonic, the function \( \tilde{j} \) is non-decreasing. If \( \tilde{j} \) is strictly increasing, then \( \tilde{j} \) is a conjugacy. However, if there are intervals where \( \tilde{j} \) is constant, or equivalently where \( \tilde{\psi} \) has derivative \(-1\), then \( \tilde{j} \) is not injective. Distinguishing between these two cases boils down to understanding \( \tilde{\psi} \).

5.1. Strong Pattern Equivariance and Locality

**Theorem 13.** The function \( \tilde{\psi} \) is sPE. That is, there exists a constant \( R \) such that if \( \tilde{T} - x \) and \( \tilde{T} - y \) agree out to distance \( R \), then \( \tilde{\psi}(x) = \tilde{\psi}(y) \).

**Proof.** Let \( R_0 \) be the radius of the rotation form \( dx \). By the definition of a rotation form, if \( \tilde{T} - x \) and \( \tilde{T} - y \) agree out to distance \( R_0 \), and if \( y - x \) is not an integer, then \((\tilde{f})^L(x) < y < (\tilde{f})^{L+1}(x)\), where \( L = \lfloor y - x \rfloor \).
Suppose that $\tilde{T} - x$ and $\tilde{T} - y$ agree to distance $R_0$, and that $k$ is an integer such that $y > (\tilde{f})^k(x)$. Then,
\begin{align*}
(f)^n(y) &> (f)^{n+k}(x), \\
y + \tilde{\phi}^n(y) &> x + \tilde{\phi}^{n+k}(x), \\
\tilde{\phi}^n(y) - n &> \tilde{\phi}^{n+k}(x) - n - y + x, \\
&= \tilde{\phi}^{n+k}(x) - (n + k) - (y - k - x). \tag{28}
\end{align*}
As a result,
\begin{align*}
\limsup_{n \in \mathbb{N}} \{(\tilde{\phi})^n(y) - n\} &> \limsup_{(n+k) \in \mathbb{N}} \{(\tilde{\phi})^{n+k}(x) - (n+k)\} - (y - k - x), \\
\tilde{\psi}(y) &> \tilde{\psi}(x) - (y - k - x). \tag{29}
\end{align*}
If $y - x$ is not an integer, then we can take $k = \lfloor y - x \rfloor$, so
\begin{align*}
\tilde{\psi}(x) - \tilde{\psi}(y) &< y - x - \lfloor y - x \rfloor. \tag{30}
\end{align*}

The function $\tilde{\psi}$ comes from a continuous function on our tiling space $\Omega_{\mu}$, and so is wPE. If $\tilde{\psi}$ is not sPE, then there exist points $x_1$ and $x_2$ such that $\tilde{T} - x_1$ and $\tilde{T} - x_2$ agree to distance $R_0$, and such that $\tilde{\psi}(x_1) \neq \tilde{\psi}(x_2)$. By swapping $x_1$ and $x_2$ if necessary, we can assume that $\tilde{\psi}(x_1) = \tilde{\psi}(x_2) + \epsilon$, where $\epsilon > 0$. We will show that this is impossible.

Since $\tilde{\psi}$ is wPE, there is a radius $R_1 > R_0$ such that if $\tilde{T} - y_1$ and $\tilde{T} - y_2$ agree to distance $R_1$, then $\tilde{\psi}(y_1)$ and $\tilde{\psi}(y_2)$ are within $\epsilon/3$ of each other.

Let $V_1$ be the set of $x$’s such that $\tilde{T} - x$ agrees with $\tilde{T} - x_1$ out to distance $R_1$, and let $V_2$ be the set of $x$’s such that $\tilde{T} - x$ agrees with $\tilde{T} - x_2$ out to $R_1$. Then, for every $y_1 \in V_1$ and $y_2 \in V_2$, $\hat{\psi}(y_1)$ is within $\epsilon/3$ of $\tilde{\psi}(x_1)$ and $\tilde{\psi}(y_2)$ is within $\epsilon/3$ of $\tilde{\psi}(x_2)$. Since $\tilde{\psi}(x_1) - \tilde{\psi}(x_2) = \epsilon$, we must have $\tilde{\psi}(y_1) - \tilde{\psi}(y_2) > \epsilon/3$.

Since $dz$ is irrational, the set of integer translates of $\tilde{T} - y_1$ is dense in $\Omega_{\mu}$. In particular, there is an integer $k$ such that $\tilde{T} - (y_1 + k)$ agrees with $\tilde{T} - (x_2 - \frac{\epsilon}{6})$ on a ball of radius at least $R_1 + \frac{\epsilon}{6}$, up to a translation by less than $\frac{\epsilon}{6}$. That is, there is a point $y_2 \in V_2$ such that
\begin{align*}
y_1 + k < y_2 < y_1 + k + \frac{\epsilon}{3}. \tag{31}
\end{align*}
However, $\tilde{T} - y_1$ and $\tilde{T} - y_2$ agree to distance at least $R_0$, so $\tilde{\psi}(y_1) - \tilde{\psi}(y_2) < \frac{\epsilon}{3}$ by equation (30), which is a contradiction. \hfill \square

The following corollary completes Theorem 3:

**Corollary 1.** Suppose that the map $F : \Omega \to \Omega$ has a rotation class $[\mu]$ that is irrational. Then, $F$ is locally semi-conjugate to uniform translation by 1 on $\Omega_{\mu}$.

**Proof.** By Theorem 12, $F$ is semi-conjugate to uniform translation on $\Omega_{\mu}$. By Theorem 13, this semi-conjugacy is the composition of a shape change and translation by the sPE function $\tilde{\Psi}$. Since both shape changes and translations by sPE functions preserve the local product structure of tiling spaces, the
pullback of $\tilde{\mu} = dx$ by the composition of these maps is an sPE form, so our semi-conjugacy is local.

5.2. Proof of Theorems 4 and 5

We now turn to the proofs of Theorems 4 and 5, which we restate for easy reference.

Theorem 14 (Theorem 4). Suppose that $F \in \mathcal{F}(\Omega)$, that $[\mu]$ exists and is irrational, and that $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R}) \subset H^1(\Omega, \mathbb{R})$. Then, $F$ is semi-conjugate to a uniform translation on $\Omega$.

Theorem 15 (Theorem 5). Suppose that $F \in \mathcal{F}(\Omega)$ and that $[\mu]$ exists and is irrational. $F$ is semi-conjugate to a uniform translation on $\Omega$, via a semi-conjugacy that sends each path component of $\Omega$ to itself, if and only if $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R})$.

Both proofs rely on the following result of [21].

Lemma 2. Let $S : \Omega \to \Omega_\mu$ be a shape change map according to the sPE form $\mu$ on $\Omega$. There is a map, homotopic to $S^{-1}$, that conjugates a fixed translation on $\Omega_\mu$ to a fixed translation on $\tilde{\Omega}$ (possibly by a different distance) if and only if $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R})$.

Proof of Theorem 4. Suppose that $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R})$. We have already established that $F$ is semi-conjugate to uniform translation on $\Omega_\mu$. However, by Lemma 2, uniform translation on $\Omega_\mu$ is conjugate to uniform translation on $\tilde{\Omega}$ itself, so $F$ is semi-conjugate to uniform translation on $\tilde{\Omega}$. □

Proof of Theorem 5. We begin with the “if” part of Theorem 5. The map $\tilde{J}$, constructed above, maps each path component of $\Omega_\mu$ to itself and semi-conjugates $\tilde{F} = S \circ F \circ S^{-1}$ to uniform translation on $\Omega_\mu$. If $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R})$, then uniform translation on $\Omega_\mu$ is conjugate to uniform translation on $\tilde{\Omega}$, via a homeomorphism $S_1^{-1}$ that is homotopic to $S^{-1}$. The map $J := S_1^{-1} \circ \tilde{J} \circ S : \Omega \to \Omega$ then sends each path component of $\Omega$ to itself and semi-conjugates $F$ to uniform motion on $\Omega$.

The “only if” direction is more difficult. By assumption, $[\mu]$ exists and is irrational and $F$ is semi-conjugate to a translation map on $\tilde{\Omega}$. We also know that $\tilde{F}$ is semi-conjugate to a translation map on $\Omega_\mu$. We will show that the translations on $\Omega$ and $\Omega_\mu$ are conjugate and use Lemma 2 to show that $[\mu] \in \mathbb{R}dx \oplus H^1_{AN}(\Omega, \mathbb{R})$.

Let $J : \Omega \to \Omega$ be a map that semi-conjugates $F$ to a fixed translation on $\Omega$. That fixed translation is of course the time-one sampling of a (uniform translation) flow $K_t$ on $\Omega$. Let $\tilde{K}_t = S \circ K_t \circ S^{-1}$, and let $\tilde{J}' = S \circ J \circ S^{-1}$. Meanwhile, let $\tilde{J}$ be the lim sup map for $\tilde{F}$. We then have that $\tilde{J}$ semi-conjugates $\tilde{F}$ to translation by 1, while $\tilde{J}'$ semi-conjugates $\tilde{F}$ to $\tilde{K}_1$. Let $\Psi$ and $\tilde{\Psi}'$ be the displacements of $\tilde{J}$ and $\tilde{J}'$, respectively. Picking a reference tiling $T$, we similarly consider the induced maps $\tilde{f}$, $\tilde{j}$, $\tilde{J}'$, $\tilde{\Psi}$, and $\tilde{\Psi}'$.

Lemma 3. The map $\tilde{J}'$ factors through $\tilde{j}$. That is, if $x_1, x_2 \in \mathbb{R}$ and $\tilde{j}(x_2) = \tilde{j}(x_1)$, then $\tilde{J}'(x_2) = \tilde{J}'(x_1)$. 
Proof of lemma. We can assume without loss of generality that $x_1 < x_2$ (since if $x_1 = x_2$ there is nothing to prove) and that $\tilde{j}(x_2) = \tilde{j}(x_1)$. That is,

$$\limsup \left( \tilde{f}^n(x_1) - n \right) = x_1 + \limsup \left( \tilde{\phi}^n(x_1) - n \right)$$

$$= \tilde{j}(x_1)$$

$$= \tilde{j}(x_2)$$

$$= x_2 + \limsup \left( \tilde{\phi}^n(x_2) - n \right)$$

$$= \limsup \left( \tilde{f}^n(x_2) - n \right).$$

(32)

Since $\tilde{f}$ is orientation-preserving, for every $n$ we have $\tilde{f}^n(x_1) < \tilde{f}^n(x_2)$, and hence, $\tilde{f}^n(x_1) - n < \tilde{f}^n(x_2) - n$. Pick $\epsilon > 0$. Since $\tilde{f}^n(x_1) - n$ is within $\epsilon/2$ of $\tilde{j}(x_1)$ infinitely often, and since $\tilde{f}^n(x_2) - n$ is eventually bounded by $\tilde{j}(x_1)+\epsilon/2$, there are (infinitely many) values of $n$ for which $\tilde{f}^n(x_2) - \tilde{f}^n(x_1) < \epsilon$.

Since $\tilde{j}$ is continuous, this means that there exist values of $n$ for which $\tilde{j}((\tilde{f})^n(x_1))$ and $\tilde{j}((\tilde{f})^n(x_2))$ are arbitrarily close, so the time it takes to flow by $\tilde{k}_t$ from $\tilde{j}((\tilde{f})^n(x_1))$ to $\tilde{j}((\tilde{f})^n(x_2))$ is arbitrarily small. However, this is exactly the same as the time it takes to flow from $\tilde{j}(x_1)$ to $\tilde{j}(x_2)$, since $\tilde{j}$ semi-conjugates $\tilde{f}$ to $\tilde{k}_t$. Thus, we must have $\tilde{j}(x_1) = \tilde{j}(x_2)$. \qed

We return to the Proof of Theorem 5. Since $\tilde{j}$ (or $\tilde{J}$) identifies all points that are identified by $\tilde{j}$ (or $\tilde{J}$), there must be a continuous $\tilde{H} : \Omega_\mu \to \Omega_\mu$ such that $\tilde{J}' = \tilde{H} \circ \tilde{J}$, and a corresponding map $\tilde{h} : \mathbb{R} \to \mathbb{R}$. To show that $\tilde{H}$ is a homeomorphism, we must show that $\tilde{J}'$ only identifies points that are identified by $\tilde{J}$.

We now let $x_1 < x_2$ be points such that $\tilde{j}(x_1) < \tilde{j}(x_2)$. Let $y_1 = \tilde{j}(x_1)$ and $y_2 = \tilde{j}(x_2)$. Note that $\tilde{h}$ semi-conjugates $\tilde{\Gamma}_1$ (that is, translation by 1 on $\Omega_\mu$) to $\tilde{k}_1$, so for all integers $n$ and $y \in \mathbb{R}$,

$$\tilde{h}(y + n) = \tilde{k}_n(\tilde{h}(y)).$$

(33)

If $\tilde{h}$ identifies $y_1$ and $y_2$, $\tilde{h}$ must also identify $y_1 + n$ and $y_2 + n$. However, by the irrationality of $dx$, \( \{ T - (y_1 + n) \} \) is dense in $\Omega_\mu$. Since $\tilde{H}$ is continuous, this means that $\tilde{h}$ must identify every point $y$ with $y + (y_2 - y_1)$, and so must collapse the entire real line to a point. Since this is a contradiction, we conclude that $\tilde{h}(y_2) \neq \tilde{h}(y_1)$. Therefore, $\tilde{j}(x_1) = \tilde{j}(x_2)$ iff $\tilde{j}'(x_1) = \tilde{j}'(x_2)$. That is, $\tilde{H}$ is a homeomorphism.

Since $\tilde{H}$ conjugates $\tilde{\Gamma}_1$ to $\tilde{K}_1$, and since $S^{-1}$ conjugates $\tilde{K}_1$ to $K_1 = \Gamma_1$, $S^{-1} \circ \tilde{H}$ conjugates uniform translation on $\Omega_\mu$ to uniform translation on $\Omega$. Since $\tilde{H}$ preserves translational orbits, $\tilde{H}$ is homotopic to the identity, so $S^{-1} \circ \tilde{H}$ is homotopic to $S^{-1}$. By Lemma 2, $[\mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R})$. \qed

5.3. Steps Toward Denjoy’s Theorem

Ideally, we would like to prove an analogue of Denjoy’s theorem, that if our map $F$ is smooth enough (say, $F_T$ being $C^2$ for each reference tiling $T$), and if $[\mu]$ is irrational, then $F$ is conjugate, and not merely semi-conjugate, to uniform translation on $\Omega_\mu$. 
Recall the usual proof of Denjoy’s theorem for circle maps. If \( f : S^1 \to S^1 \) is an orientation-preserving homeomorphism with irrational winding number, then there is a map \( \Psi : S^1 \to S^1 \) that semi-conjugates \( f \) to an irrational rotation. The map \( \Psi \) may collapse intervals to points. Such intervals are called wandering. If \( U \) is a wandering interval, then \( U_n := f^n(U) \) is also a wandering interval. If \( n \neq m \) and \( U_n \cap U_m \neq \emptyset \), then we must have \( U_n = U_m \), in which case the left endpoint of \( U_n \) is a periodic point of \( f \), which contradicts the irrationality of the rotation number. Thus, the intervals \( \{U_n\} \) are disjoint. Since the total length of the intervals is bounded by the circumference of \( S^1 \), the length of \( U_n \) must go to zero as \( n \to \pm \infty \) and the ratio \( \frac{|U_0|^2}{|U_n||U_{-n}|} \) must go to \( \infty \). However, that ratio is bounded by a quantity called the distortion. If \( f \) is \( C^2 \) and the first derivative \( f' \) is never zero, then the distortion is in turn bounded by the integral of \( |f''|/f' | \) over the entire circle. This is a contradiction, so wandering intervals cannot exist, so \( \Psi \) must be a genuine conjugacy.\(^4\)

Theorem 13 is a very powerful tool, since it says that the intervals that are collapsed by \( \tilde{\psi} \) (and play a role similar to wandering intervals) can be identified locally as the intervals where \( \tilde{\psi} \) has derivative \(-1\). If \( R \) is the greater of the sPE radius of \( \tilde{f} \) and the sPE radius of \( \phi \), then we can consider an approximant \( X_R \) to \( \Omega_\mu \) obtained by collaring out to distance \( R \). That is, each point in \( X_R \) describes a pattern out to distance \( R \), and there is a natural map from \( \Omega_\mu \) to \( X_R \) that sends each tiling to its pattern around the origin. Since \( \tilde{\Psi} \) is sPE, \( \tilde{\Psi} \) can be viewed as a function on \( X_R \). We call the intervals where \( \tilde{\psi}' = -1 \) collapsing intervals.

The orbit under \( \tilde{F} \) of any tiling the induces a path on \( X_R \). We would like to construct a map \( \tilde{F}_R \) from \( X_R \) to itself induced by \( \tilde{F} \). As with circle maps, such a \( \tilde{F}_R \) would have to send each collapsing interval onto another such interval.

Unfortunately, the resemblance to circle maps ends there. Relative to a reference tiling \( T \), let \( \hat{U}_0 \subset \mathbb{R} \) be an interval collapsed by \( \hat{\psi} \), let \( \hat{U}_n = (\hat{f})^n(\hat{U}_0) \), and let \( U_n \) be the image of \( \hat{U}_n \) in \( X_R \). Of course, the intervals \( \hat{U}_n \) are disjoint, but they march off to infinity, so there is no a priori reason why their total length should be finite. As for the collapsing intervals \( U_n \), they don’t have to be disjoint! If \( U_n = U_0 \), this merely says that the patches around \( \hat{U}_n \) and \( \hat{U}_0 \) are the same. It does not imply that \( \hat{U}_{2n} \) will be similar, or that we have a periodic orbit in \( X_R \), and does not contradict the irrationality of \( \mu \). \( X_R \) is a branched manifold, and there is no single-valued map \( \tilde{F}_R \). When we reach branch points, the image in \( X_R \) of a \( \hat{f} \)-orbit in \( \mathbb{R} \) can pick either branch and does so in a way that reflects the underlying non-periodicity of our reference tiling \( T \). Depending on one’s point of view, one can say that \( \tilde{F}_R \) is multi-valued or that \( \tilde{F}_R \) does not actually exist. Either way, we cannot apply to \( X_R \) the sorts of arguments that work on \( S^1 \).

This is not to say that Denjoy’s theorem is false for tiling spaces. We conjecture that when \( f \) is \( C^2 \) and \( \mu \) is irrational, \( F \) will indeed be conjugate

\(^4\)When \( f \) is \( C^2 \) but \( f' \) is zero at isolated points, the argument is technically more complicated, but follows the same overall strategy.
to uniform motion. However, the proof of this conjecture will not be straightforward. It will either require properties of strong pattern equivariance such as stronger versions of Theorem 13, or it will require a strategy qualitatively different from the usual proof for circle maps.

6. Examples

In this section, we examine a number of different tiling spaces and maps on these spaces to get a feel for the meaning of our main theorems.

Example 1 (The Fibonacci tiling). The Fibonacci tiling has alphabet \( A = \{a, b\} \), and the allowed (bi-infinite) sequences of tile labels are generated by the substitution \( \sigma : a \rightarrow ab, b \rightarrow a \).\(^5\) We pick two arbitrary positive constants \( L_a \) and \( L_b \) to be the lengths of the \( a \) and \( b \) tiles and denote by \( \Omega_{Fib} \) the resulting space of tilings.

The cohomology of \( \Omega_{Fib} \) does not depend on \( L_a \) and \( L_b \). Regardless of the tile lengths, \( \tilde{H}^1(\Omega_{Fib}, \mathbb{R}) = \mathbb{R}^2 \). Let \( i_a \) be an sPE 1-form that integrates to 1 on each \( a \) tile and to 0 on each \( b \) tile, and let \( i_b \) integrate to 1 on each \( b \) tile and to 0 on each \( a \) tile. Since the sequences associated with the Fibonacci tiling are Sturmian, the form \( i_a - \phi i_b \), where \( \phi = (1 + \sqrt{5})/2 \), is asymptotically negligible. Meanwhile, \( dx \) is cohomologous to \( L_a i_a + L_b i_b \).

Since \( [dx] \) and \( [i_a - \phi i_b] \) span \( \tilde{H}^1(\Omega_{Fib}, \mathbb{R}) \), every sPE 1-form can be written as a multiple of \( dx \) plus something asymptotically negligible. In particular, if \( \bar{F} \) is any unidirectional flow on \( \Omega_{Fib} \) with sPE velocity function \( V \), and if \( \mu = dx/v \), then there is a nonzero constant \( \rho \) and an asymptotically negligible form \( \beta \) such that \( \mu = \rho dx + \beta \). By Theorem 9, \( \bar{F} \) is then topologically conjugate to uniform translation on \( \Omega_{Fib} \) at speed \( \rho \).

The key points are that \( \tilde{H}^1(\Omega_{Fib}, \mathbb{R})/\tilde{H}^1_{AN}(\Omega_{Fib}, \mathbb{R}) = \mathbb{R} \) and that this group is generated by \([dx]\). This is reminiscent of the situation for circle maps, where \( H^1(S^1, \mathbb{R}) = \mathbb{R} \) is already generated by \([dx]\), and there are no nontrivial asymptotically negligible classes. For any tiling space (meeting the usual assumptions) where \( \tilde{H}^1/\tilde{H}^1_{AN} = \mathbb{R}, \) an sPE self-homeomorphism \( F \) that is homotopic to the identity is conjugate to a uniform translation if and only if it is the time-one sampling of an sPE flow.

Example 2 (A non-Pisot substitution). The situation is different when the substitution matrix is not Pisot. Consider a space \( \Omega_{nP} \) of tilings generated by the substitution \( a \rightarrow ab, b \rightarrow aaa \), with tile lengths \( L_a = L_b = 1 \). As with the Fibonacci tiling, \( \tilde{H}^1(\Omega_{nP}, \mathbb{R}) = \mathbb{R}^2 \) is generated by \([i_a]\) and \([i_b]\). However, the substitution matrix \( M = (1 3 \\ 0) \) has eigenvalues \((1 \pm \sqrt{13})/2\), and hence is expansive. Any (nontrivial) linear combination of \( i_a \) and \( i_b \) gives large results when integrated over supertiles of sufficiently large order. Thus, \( \tilde{H}^1_{AN}(\Omega_{nP}, \mathbb{R}) \) is trivial.

\(^5\)Applying the substitution \( \sigma \) recursively, one obtains a list of increasingly long words \( \sigma^n(a) \) and \( \sigma^n(b) \). The allowed bi-infinite words are those for which every finite sub-word is found in one of these \( n \)-times substituted letters.
Suppose that we have a flow $\bar{F}$ on $\Omega_{nP}$ with sPE velocity that takes 1 second to cross each $a$ tile and 2 seconds to cross each $b$ tile. Then, $\mu$ will be cohomologous to $i_a + 2i_b$, while $dx$ is cohomologous to $i_a + i_b$. Since $\mu$ is not cohomologous to a multiple of $dx$ plus something asymptotically negligible, $\bar{F}$ cannot be conjugated to a uniform translation on $\Omega_{nP}$.

Note that if we do a shape change by $\mu$ to $\Omega_{nP}$, then we obtain a space $\Omega_{\mu}$ of tilings with $L_a = 1$ and $L_b = 2$. On $\Omega_{\mu}$, the time it takes to cross a tile is exactly equal to its length, and the flow $\bar{F}_t$ has been conjugated to $\Gamma_t$. However, $\Gamma_t$ on $\Omega_{\mu}$ is not conjugate to any $\Gamma_{\rho t}$ on $\Omega_{nP}$.

To summarize the last two examples, $\tilde{H}^1(\Omega_{Fib}, \mathbb{R}) = \tilde{H}^1_{AN}(\Omega_{Fib}, \mathbb{R}) \oplus \mathbb{R}$, so every shape change on $\Omega_{Fib}$ is conjugate to a uniform dilation, and every unidirectional flow on $\Omega_{Fib}$ with sPE velocity is conjugate to uniform translation on $\Omega_{Fib}$ at some speed $\rho \neq 0$. However, $\tilde{H}^1(\Omega_{nP}, \mathbb{R}) \neq \tilde{H}^1_{AN}(\Omega_{nP}, \mathbb{R}) \oplus \mathbb{R}$, so there are shape changes on $\Omega_{nP}$ that change the translational dynamics. Every flow on $\Omega_{nP}$ with sPE velocity can be conjugated to a uniform flow on some shape-changed space $\Omega_{\mu}$, but that is different from a uniform flow on the original space $\Omega_{nP}$.

**Example 3 (A Denjoy-like construction).** Denjoy [15] constructed a circle map with irrational rotation number that is semi-conjugate, but not conjugate, to a uniform rotation. Here we adapt Denjoy’s construction to get a map of tiling spaces, with irrational rotation class, that is semi-conjugate but not conjugate to uniform translation.

We first recall Denjoy’s construction. Start with a circle $S^1 = \mathbb{R}/\mathbb{Z}$ of length 1, an irrational number $\alpha$, and a countable collection $\{I_n\}, n \in \mathbb{Z}$ of closed intervals, disjoint from each other and from $S^1$, whose total length is finite. Let $r : S^1 \to S^1$ be rotation by $\alpha$. Pick a point $x_0 \in S^1$, and let $x_n = r^n(x_0)$.

Now construct a new circle $\bar{S}^1$ by replacing each point $x_n \in S^1$ by the interval $I_n$. The total length of $\bar{S}^1$ is finite, and we can rescale to give it total length 1. We define a new map $f : \bar{S}^1 \to \bar{S}^1$ that maps each $I_n$ homeomorphically (say, linearly) onto $I_{n+1}$, and that equals $r$ on the complement of the inserted intervals. Note that $f$ is not minimal, since the orbit of a point in the complement of the inserted intervals does not contain any points in $I_n$, its closure does not contain any points in the interior of $I_n$.

Since $f$ is not minimal, it cannot be topologically conjugate to an irrational rotation. Since $f$ has no periodic points, it cannot be topologically conjugate to a rational rotation. Thus, $f$ is not conjugate to any rotation. However, $f$ is semi-conjugate to a rotation by $\alpha$, since if $\pi : \bar{S}^1 \to S^1$ is the obvious projection, then $\pi \circ f = r \circ \pi$.

We now transfer this construction to the category of one-dimensional FLC tiling spaces. Let $\Omega$ be such a tiling space whose tiles all have integer length. (Say, all of length 1). Then, there is a map $\pi : \Omega \to S^1$ that sends
Each tiling to the common location of all its vertices (mod 1). This gives \( \Omega \) the structure of a Cantor bundle over \( S^1 \).

Now pick an irrational \( \alpha \in (0,1) \) and construct \( f \) as before. Let \( F_0 : \mathbb{R} \to \mathbb{R} \) be a lift of \( f \) such that, for all \( x \in \mathbb{R}, x < F_0(x) < x + 1 \) and \( F_0(x + 1) = F_0(x) + 1 \). We now define a self-homeomorphism \( F : \Omega \to \Omega \). For each tiling \( T \in \Omega \) with vertices at integer points, and for each \( x \in \mathbb{R} \), let \( F(T - x) = T - F_0(x) \). Since \( (T + n) - F_0(x + n) = T - F_0(x) \), \( F \) is well defined. \( F \) is not conjugate to a uniform translation, but the projection \( \pi : \bar{S}^1 \to S^1 \) lifts to a self-map of \( \Omega \) that semi-conjugates \( F \) to \( \Gamma_\alpha \).

**Example 4 (A map with multiple rotation classes).** In Theorem 11, we showed that if a map \( F \in \mathcal{F}(\Omega) \) has an irrational rotation class, then this class is unique. Here we show that a map on a tiling space can have multiple rotation classes, all of them rational.

Pick an \( \epsilon \in (0,1/10) \) and consider a Fibonacci tiling space \( \Omega \) with tile lengths \( L_a = 1 + \epsilon \) and \( L_b = 1 - \phi \epsilon \), where \( \phi = (1 + \sqrt{5})/2 \) is the golden mean. The length of \( n \) consecutive tiles is close to \( n \), with the difference being bounded by \( \phi^2 \epsilon < 3 \epsilon < 0.3 \), but is never exactly \( n \).

On this space, let \( F = \Gamma_1 \). Manifestly, \( dx \) is a rotation class for \( F \) and is cohomologous to \( (1 + \epsilon)i_a + (1 - \phi \epsilon)i_b \). Note that the \( F \)-orbit of a tiling with a vertex at the origin always has a vertex within 0.3 of the origin, so this orbit is not dense and \( dx \) is rational.

Pick an arbitrary \( s \in (0,1) \) and let \( \mu_s = (1 + s \epsilon)i_a + (1 - s \epsilon \phi)i_b \). If \( x_1 \) and \( x_2 \) are corresponding points in large patches, then they are separated by the lengths of \( n \) consecutive tiles. Since \( \int_{x_1}^{x_2} \mu_s = s(x_2 - x_1) + (1 - s)n \), the integral \( \int_{x_1}^{x_2} \mu_s \) is strictly between \( n \) and \( x_2 - x_1 \) and therefore has the same integer part (either \( n \) or \( n - 1 \) as \( x_2 - x_1 \). Since the estimates that characterize a rotation form \( \mu \) only depend on whether or not \( \int_{x_1}^{x_2} \mu \) is an integer, and on the integer part of that integral, \( \mu_s \) is a rotation form, and since \([i_a]\) and \([i_b]\) are a basis for \( H^1(\Omega_{fib}) = \mathbb{Z}^2 \), \([\mu_s] \neq [dx] \) as rotation classes.

**Example 5 (Maps without rotation classes).** We now construct a family of tiling spaces and maps on those tiling space, meeting the usual technical assumptions, for which rotation forms do not exist. By the contrapositive to Proposition 3, these maps are not time-one samplings of flows with sPE velocity, and in particular are not conjugate to translations on \( \Omega \) or on any shape change of \( \Omega \).

Our examples are all fusion tilings [16]. These are hierarchical tilings that generalize substitution tilings. In our examples, there are three species of tiles, labeled \( a, b, \) and \( c \), that group into two kinds of clusters, \( A_1 \) and \( B_1 \). The \( A_1 \) and \( B_1 \) clusters then group into larger clusters \( A_2 \) and \( B_2 \), that group into larger clusters \( A_3 \) and \( B_3 \), and so on. We will show that every sPE 1-form \( \mu \)

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6It is always possible to do a shape change to make all the tiles have integer length, so every tiling space is homeomorphic to a Cantor bundle over a circle. Moreover, every higher dimensional minimal FLC tiling space is homeomorphic to a Cantor bundle over a torus. [37].
has the property that, for all sufficiently large \( j \), \( \int_{A_j} \mu = \int_{B_j} \mu \). However, our \( F \) will be such that the number of steps needed to cross an \( A_j \) tile is different from the number of steps needed to cross a \( B_j \) tile. In fact, the difference in those crossing times can be made arbitrarily large! This implies that the arbitrary form \( \mu \) is not a rotation form, so rotation forms (and rotation classes) for these maps do not exist.

We begin our construction with the tiles \( a, b, \) and \( c \). Let \( c \) be a tile of length \( L_c = 1 \), and let \( a \) and \( b \) have lengths greater than 1. (For example, \( L_a = L_b = \pi \).) Pick a smooth function \( \phi \) on the disjoint union of the three tiles \( a, b, \) and \( c \), such that \( \phi(x) = 1 \) on the endpoints of the three tiles, and such that for any two points \( x_1 < x_2 \) within the same tile, \( x_1 + \phi(x_1) < x_2 + \phi(x_2) \). This extends to a function on all the points of any tiling constructed from \( a, b, \) and \( c \) tiles. We now define a map

\[
F : \Omega \to \Omega, \quad F(T) = T - \phi(T(0)),
\]

where \( T(0) \) means the origin of the tiling \( T \). That is, our displacement map \( \Phi : \Omega \to \mathbb{R} \) is given by \( \Phi(T) = \phi(T(0)) \), and the function \( \phi_T \) is just the restriction of \( \phi \) to the points of \( T \) itself.

Now pick integers \( n_2, n_3, n_4, \ldots \) that grow sufficiently rapidly. (The precise rate required will depend on \( \phi \).) We construct our “supertiles” \( A_j, B_j \) recursively, as follows:

\[
A_1 = ac, \quad B_1 = bc, \quad A_j = (A_{j-1}B_{j-1})^{n_j} \text{ if } j > 1, \quad B_j = A_{j-1}^{n_j}B_{j-1}^{n_j} \text{ if } j > 1.
\]

That is, an \( A_j \) is an \( A_{j-1} \) followed by a \( B_{j-1} \) followed by an \( A_{j-1} \) followed by a \( B_{j-1} \), etc., with each kind of \((j-1)\)-supertile appearing \( n_j \) times, while a \( B_j \) consists of \( n_j \) \( A_{j-1} \)’s followed by \( n_j \) \( B_{j-1} \)’s. Note that every \( A_j \) and \( B_j \) supertile begins with an \( A_{j-1} \) and ends with a \( B_{j-1} \). Thus, every \( A_j \) and \( B_j \) is preceded by a \( B_{j-1} \) and followed by an \( A_{j-1} \). Also note that the lengths \( |A_j| \) of \( A_j \) and \( |B_j| \) of \( B_j \) are equal.

Let \( \alpha \) be an sPE 1-form with some radius \( R \). Then there is a positive integer \( j - 2 \) such that \( |A_{j-2}| = |B_{j-2}| > R \). This implies that for every \( k \) with \( k > j - 2 \), \( \int_{A_k} \alpha \) is the same for every \( A_k \) cluster, regardless of where that cluster sits in the tiling. Likewise, \( \int_{B_k} \alpha \) is the same for every \( B_k \) cluster. But then, for every \( k \geq j \),

\[
\int_{A_k} \alpha = n_k \int_{A_{k-1}} \alpha + n_k \int_{B_{k-1}} \alpha = \int_{B_k} \alpha.
\]

That is, integrating sPE forms cannot distinguish between \( A \) and \( B \) supertiles of sufficiently high degree.

Next we consider the dynamics of \( F \). Note that \( F \) takes the left endpoint of each \( c \) tile to the right endpoint. The specific dynamics of \( F \) on \( a \) and \( b \) induce maps \( F_{1,a} : c \to c \) and \( F_{1,b} : c \to c \). For \( x \in c \), consider iterates of \( x \) under \( F \). If \( c \) is followed by \( ac \), then let \( F_{1,a}(x) \) be the first return of \( x \) in \( c \).
In this way $F_{1,a} : c \rightarrow c$. Identifying the endpoints of $c$ to form a circle, $F_{1,a}$ is a circle map, and hence has a rotation number $\rho_{1,a}$. Likewise, let $F_{1,b}$ be the first return on $c$ when followed by $bc$, and let the induced rotation number on $c$ be $\rho_{1,b}$. For definiteness, we take both $\rho_{1,a}$ and $\rho_{1,b}$ to lie in $[0, 1)$. We also let $\rho_{1,ab}$ be the rotation number of $F_{1,b} \circ F_{1,a}$. Note that $\rho_{1,ab}$ is generically not congruent to $\rho_{1,a} + \rho_{1,b} \pmod{1}$.

The number of steps needed to cross an $A_1$ is always one of two consecutive integers $m_{a,1}$ and $m_{a,1} + 1$. Similarly, the number of steps needed to cross a $B_1$ is either $m_{b,1}$ or $m_{b,1} + 1$. If $N$ is a large integer, then the number of steps needed to cross $A_1^N$ is approximately $N(m_{1,a} + 1 - \rho_{1,a})$, and the number of steps needed to cross $B_1^N$ is approximately $N(m_{1,b} + 1 - \rho_{1,b})$. (In both cases, it takes at most $N(m + 1)$ steps, but we save a step every time the circle map crosses back from the end of $c$ to the beginning, which happens a fraction $\rho$ of the time.)

As a result, if $n_2$ is large, the number of steps needed to cross $B_2 = A_1^{n_2}B_1^{n_2}$ is approximately (that is, within one of) $n_2(m_{1,a} + m_{1,b} + 2 - \rho_{1,a} - \rho_{1,b})$. Meanwhile, the number of steps needed to cross $A_2 = (A_1B_1)^{n_2}$ is approximately $n_2(m_{1,a} + m_{1,b} + 2 - \rho_{1,ab})$. (Depending on whether it is sometimes possible to cross $A_1 B_1$ in $m_{1,a} + m_{2,b}$ steps, we may need to take $\rho_{1,ab}$ in $[1, 2)$ for this to apply.) Since $\rho_{1,ab}$ is not equal to $\rho_{1,a} + \rho_{1,b} \pmod{1}$, if we take $n_2$ large enough we can get the crossing times to differ by an arbitrarily large amount.

The crossing of $A_2$ and $B_2$ generates their own return maps from the copy of $c$ immediately preceding the supertile to the one at the end of the supertile. Call these $F_{2,a}$ and $F_{2,b}$, with rotation numbers $\rho_{2,a}$ and $\rho_{2,b}$. As before, the rotation number of $F_{2,b} \circ F_{2,a}$, which we denote $\rho_{2,ab}$, is generically not equal to $\rho_{2,a} + \rho_{2,b} \pmod{1}$. By taking $n_3$ large enough, we can make the crossing time for $B_3 = A_2^{n_3}B_2^{n_3}$, which is approximately $n_3(m_{2,a} + m_{2,b} + 2 - \rho_{2,a} - \rho_{2,b})$ differ by an arbitrarily large amount from the crossing time for $A_3 = (A_2B_2)^{n_3}$, which is approximately $n_3(m_{2,a} + m_{2,b} + 2 - \rho_{2,ab})$.

We repeat the process, using the fact that generically $\rho_{j-1,ab} \neq \rho_{j-1,a} + \rho_{j-1,b}$ to pick $n_j$ big enough so that the crossing times for $A_n$ and $B_n$ differ by more than 1. The end result is a tiling space $\Omega$ and a map $F$ on $\Omega$ with sPE displacement that has no rotation class.

**Example 6 (A map with no rotation number).** Finally, we construct an example that has a rotation class, but that doesn’t have a rotation number. We consider a fusion tiling with two tile types, $a$, and $b$, each of length 1. The fusion rule is

$$A_n = A_{n-1}^{10^n}B_{n-1}, \quad B_n = AB_{n-1}^{10^n}, \tag{40}$$

where $A_0 = a$ and $B_0 = b$. (This is a small modification of Example 3.7 in [16].) The translation flow on the resulting tiling space $\Omega$ is minimal but not uniquely ergodic. Rather, there are two ergodic measures, $\mu$ and $\nu$, that describe the frequencies of patches in a high-order $A$-supertile (where roughly 90% of the tiles, 99% of the 1-supertiles, and 99.9% of the 2-supertiles are of type $A$), and the frequencies of patches in a high-order $B$-supertile (where only 10% of the tiles, 1% of the 1-supertiles, and 0.1% of the 2-supertiles are of type $A$).
Let \( v_0(x) \) be exactly 1 on every \( a \) tile and 2 on every \( b \) tiles. Convolve \( v_0 \) with a smooth bump function of narrow support to get a smooth sPE velocity function \( v(x) \). Let \( F \) be the time-one sampling of the flow by \( v \). By the results of Sect. 3, \( dx/v \) is a rotation form, and \([dx/v]\) is a rotation class. However, the limit

\[
\lim_{n \to \pm\infty} \frac{f_n^F(x) - x}{n}
\]

(41)
depends on the reference tiling \( T \) and may not even exist.

If \( T \) is in the support of \( \mu \), then roughly 90% of the tiles are type \( a \), and take about 1 unit of time to cross, while 10% are type \( b \), and take about half a unit of time to cross. The time needed to cross a distance \( L \) averages to approximately

\[
0.9L + \frac{0.1L}{2} = 0.95L
\]

(42)
so the distance traveled per unit time is about \( 1/0.95 \approx 1.05 \). In this case, both limits in (41), as \( n \to \infty \) and as \( n \to -\infty \), will be around 1.05.

If \( T \) is in the support of \( \nu \), then only 10% of the tiles are of type \( a \), 90% are of type \( b \), and the time needed to cross a distance \( L \) is, on average,

\[
0.1L + \frac{0.9L}{2} = 0.55L.
\]

(43)
In this case, both limits in (41) will be around \( 1/0.55 \approx 1.8 \). If \( T \) is neither in the support of \( \mu \) nor in the support of \( \nu \), then the limits in (41) may not exist, or may not agree. For instance, if \( T \) consists of an infinite-order \( A \)-supertile for \( x > 0 \) and an infinite-order \( B \) supertile for \( x < 0 \), then the limit as \( n \to +\infty \) will be around 1.05, while the limit as \( n \to -\infty \) will be around 1.8.

Although this map comes from a flow and is conjugate to a uniform translation on a tiling space \( \Omega_{[dx/v]} \), the concepts of rotation number and \( \rho \)-boundedness simply make no sense on \( \Omega \) itself and none of the results of [2] apply. This example shows that the assumption of unique ergodicity in Theorem 6, and in our main theorems, is essential.

7. Conclusions and Open Problems

In this paper, we have studied rotation theory in the category of FLC tiling spaces and maps with sPE displacement. In this setting, we have shown that

1. Instead of having a rotation number, we need to study the rotation class

\[ [\mu] \in H^1(\Omega, \mathbb{R}). \]

2. This class doesn’t always exist and isn’t always unique. However, irrational rotational classes are unique, and maps where the rotation class does not exist do not come from time-one samplings of flows with sPE velocity.

3. Unlike with circle maps, there is a difference between coming from a flow with sPE velocity and being conjugate to a uniform translation. If a map \( F : \Omega \to \Omega \) comes from a flow with sPE velocity, then \( F \) is conjugate to
uniform translation on a different space $\Omega_{\mu}$, where $\Omega_{\mu}$ is obtained from $\Omega$ by doing a shape change by the rotation class $\mu$. Whether or not $F$ is conjugate to uniform translation on $\Omega$ itself, via a map that preserves path components, is purely a cohomological question.

(4) If a map $F \in \mathcal{F}(\Omega)$ has a rotation class $[\mu]$ that is irrational, then $F$ is semi-conjugate to uniform translation on $\Omega_{\mu}$. $F$ is then semi-conjugate to uniform translation on $\Omega$ itself, via a map that preserves path components, if and only if $[\mu] \in \mathbb{R} dx \oplus H^1_{AN}(\Omega, \mathbb{R})$.

One natural open question is whether a version of Denjoy’s theorem applies in this situation. Is there a condition on the smoothness of $F$ that guarantees that the resulting semi-conjugacies are in fact conjugacies? We believe the answer to be “yes,” but the usual approaches to proving Denjoy’s theorem for circle maps do not carry over to tiling spaces. A genuinely new approach is needed.

Another natural open question is to study what happens when we relax the assumptions that our maps have sPE displacement and that our flows have sPE velocity. For instance, we could ask when a map $F$ with sPE displacement on a tiling space $\Omega$ with FLC is the time-one sampling of a flow whose velocity function is not necessarily sPE. The trouble is that, if such a flow existed, the resulting form $\mu = dx/v$ would not be sPE, so doing a shape change by $\mu$ would change $\Omega$ into a space of tilings that no longer have FLC. This problem is not really approachable using the existing theory of FLC tilings.

Relatively little is known about tilings with infinite local complexity, or ILC. (See [17] for some recent results.) The Čech cohomology of the tiling space is well-defined, but does not directly correspond to a theory of PE forms. An alternate approach is to study the cohomology of wPE forms, without regard to Čech cohomology, but this “weak PE cohomology” is infinitely generated even for simple tilings like the Fibonacci tiling.

The upshot is that once we leave the category of FLC tilings and of sPE functions, flows and maps, the problem stops reflecting the fairly rigid structure of tilings. Instead, it becomes a problem about “matchbox manifolds” is general. That is, about foliated spaces with one-dimensional leaves. Such problems are interesting and well worth studying, but require a whole new toolkit.

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