SPIN EFFECTS IN GRAVITATIONAL RADIATION BACKREACTION
II. FINITE MASS EFFECTS

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A convenient formalism for averaging the losses produced by gravitational radiation backreaction over one orbital period was developed in an earlier paper. In the present paper we generalize this formalism to include the case of a closed system composed from two bodies of comparable masses, one of them having the spin S.

We employ the equations of motion given by Barker and O’Connell, where terms up to linear order in the spin (the spin-orbit interaction terms) are kept. To obtain the radiative losses up to terms linear in the spin, the equations of motion are taken to the same order. Then the magnitude $L$ of the angular momentum $\mathbf{L}$, the angle $\kappa$ subtended by $\mathbf{S}$ and $\mathbf{L}$ and the energy $E$ are conserved.

The analysis of the radial motion leads to a new parametrization of the orbit.

From the instantaneous gravitational radiation losses computed by Kidder the leading terms and the spin-orbit terms are taken. Following Apostolatos, Cutler, Sussman and Thorne (ACST), the evolution of the vectors $\mathbf{S}$ and $\mathbf{L}$ in the momentary plane spanned by these vectors is separated from the evolution of the plane in space. The radiation-induced change in the spin is smaller than the leading-order spin terms in the momentary angular momentum loss. This enables us to compute the averaged losses in the constants of motion $E$, $L$ and $L_S = L \cos \kappa$. In the latter, the radiative spin loss terms average to zero. An alternative description using the orbital elements $a$, $e$ and $\kappa$ is given.

The finite mass effects contribute terms, comparable in magnitude, to the basic, test-particle spin terms in the averaged losses.

I. INTRODUCTION

Coalescing binaries are important and copious sources of gravitational waves for the projected interferometric gravitational wave detection experiments LIGO, VIRGO and LISA. The requisite signal templates challenge theoreticians to predict the behaviour of such systems under gravitational radiation backreaction. Much progress has been made in this direction mainly by perturbative approaches. In the initial epoch, the separation $r$ of the two bodies forming the binary is large relative to the Schwarzschild radius and the relative motion can be considered slow. This enables one to employ the post-Newtonian expansion parameter $\epsilon \approx v^2 \approx m/r$ or $\epsilon$. Alternatively, with the system of units chosen appropriately, the inverse $1/c$ of the speed of light can be used as an expansion parameter. Computations of instantaneous radiation losses of energy, momentum and angular momentum up to (post)$3/2$-Newtonian order, including the spin-orbit and spin-spin interaction terms, were given by Kidder.

The equations of motion of spinning bodies, disregarding radiation backreaction, were first considered by Barker and O’Connell, then by Thorne and Hartle. The relevant equations can be derived from a generalized Lagrangian depending on relative position, velocity and acceleration. The two-body problem is reduced to a one-body problem by eliminating the center of mass. The masses $m_1$ and $m_2$ still appear in the Lagrangian as reminders of the initial two-body character of the problem. Apostolatos, Cutler, Sussman and Thorne describe the evolution of the spin and orbital angular momentum vectors in the presence of radiation backreaction. They supplement the equations of motion with the leading-order averaged gravitational radiation loss terms, computed by Peters and Mathews.

In the test particle limit the leading spin terms from the averaged losses in energy, magnitude of the orbital angular and spin projection of the orbital angular momentum, respectively has been computed by Ryan for generic orbits. Generalizing his earlier work on circular orbits and Shibata’s work on equatorial orbits, Ryan obtains the first-order spin corrections to the Peters’ equations of radiation-induced change in the orbit parameters. In a previous paper, to be referred to as I, we completed the description of the test particle by giving the averaged losses in terms of unambiguous conserved quantities and by computing the radiation induced change of the remaining orbit elements. Our description relies on the use of Eulerian angle variables, a new parametrization of the orbit and averaging by the residue theorem.

The computation of finite-mass effects can be carried out by choosing a suitable spin supplementary condition (SSC). Three convenient choices of SSC are discussed by Kidder. A treatment of the nonradiative evolution using a noncovariant SSC (A2b) of can be found in Wex’s work, and some radiative losses in Rieth and Schäfer.
The finite mass effects in the spin terms of averaged losses, using the covariant SSC, were considered previously by Kidder, Will and Wiseman [1] and by Kidder [2]. They computed the energy loss and the angular momentum loss for the case of circular orbits. It is the purpose of the present paper to obtain, for generic orbits, the averaged losses of energy, magnitude of the orbital angular momentum and spin projection of the orbital angular momentum.

We employ the second order Lagrangian suitable for the description of the motion of comparable mass bodies in the presence of spin, following Ref. [3]. We want to describe finite-mass effects in binaries consisting of a compact body captured by another, massive, spinning body. The latter is exemplified by a spinning black hole or neutron star. In this way we generalize the picture of the evolution of a system consisting of a black hole and a test particle, which we considered in I. The equations of motion and the spin precession equations valid to the (post)\(^2\)-Newtonian order are reviewed in the second section. The constants of motion are the energy \(E\) and the total angular momentum \(\mathbf{J}\). Since our goal is to get the radiation losses up to the leading terms in the spin, we approximately picture the spin precession by a rigid parallelogram with sides \(L\) and \(S\), rotating about the total angular momentum vector \(J\). In this approach the magnitude \(L\) and the spin projection \(LS\) of the orbital angular momentum are constants of the motion. From their expressions an uncoupled radial equation can be found.

In the third section, we determine the turning points. We then introduce two parameters, the 'eccentric anomaly' and the 'true anomaly'. They have the respective properties, and in the no-spin limit reduce to, the eccentric anomaly and true anomaly of Kepler orbits. The parametrization of the orbit by the eccentric anomaly enables one to compute the orbital period defined as twice the period of time elapsed between consecutive turning points.

In the fourth section, we compute the instantaneous radiation losses. For this purpose, we make the assumption widely used in the literature that the radius \(R\) of the spinning body is comparable with its mass \(m_1\). Then \(S/r^2 \approx m_1RV/r^3 \approx c^2V\) and \(L/r^2 \approx \mu v/r \approx c^{3/2}\). Thus the spin is of \(c^{3/2}\) order smaller than the orbital angular momentum. Under the assumption that the rotational velocity \(V\) is large, Apostolatos, Cutler, Sussman and Thorne [4] estimated the 'S-changing piece of the radiation-reaction torque' (the radiation loss of the spin) to be of order \(rc^{3/2}\). This is smaller by two full post-Newtonian orders than the leading term in \(dJ/dt\) which is of order \(rc^{3/2}\), and by one half order than the spin-orbit terms given by Kidder [2] which are of order \(rc^3V\). At the end of the section we present an estimate of the order of magnitude of the radiation loss of the spin for smaller values of \(V\). We then find that, to the order we are interested in, we may interpret the instantaneous total angular momentum loss as consisting entirely of the loss in the orbital angular momentum. These considerations enable us to compute the loss of magnitude and spin projection of the orbital angular momentum. However the latter will still contain radiative spin loss terms, which are evaluated from the Burke-Thorne potential. Fortunately, introducing suitable new angular variables, all losses can be expressed in terms of the true anomaly parameter alone.

In the fifth section we average these expressions by using the residue theorem. We find that the spin-loss terms give no averaged contribution to the losses. The finite-mass backreaction effects are of the same order as the other spin effects. The results obtained in the Lense-Thirring case emerge as smooth limits, although the test particle case is outside the framework of this paper.

The losses are given both in terms of conserved quantities and in terms of 'ellipse parameters' \(a, e\) and the angle \(\kappa\) between the spin vector \(S\) and the orbital angular momentum \(L\). We give also the changes of the geometric orbit elements \(a, e\) and \(\kappa\) due to radiation backreaction. This is carried out in the sixth section.

As in ACST, we separate the motion of the spin and orbital angular momentum vectors in their momentary planes from the motion of the plane. Our approximation in describing the radiation effects on the former motion up to linear terms in the spin.

Since we eliminate the post-Newtonian parameter \(v^2\) from the losses at the very beginning of the averaging process, we keep the gravitational constant \(G\) and the speed of light \(c\) in our formulae. Our post-Newtonian parameter is \(\epsilon \approx v^2/c^2 \approx Gm/c^2r\). The explicit \(c\) dependence allows an easier bookkeeping in the post-Newtonian expansions.

II. THE ORBIT OF THE BINARY SYSTEM

We consider the motion of a bound two-body system with masses \(m_1\) and \(m_2\) and spins \(S\) and 0, respectively. Considering only the leading-order spin-orbit coupling, and adopting the spin supplementary condition of [1], the Lagrangian is:

\[
\mathcal{L} = \frac{\mu v^2}{2} + \frac{Gm\mu}{r} + \delta\mathcal{L},
\]

where \(r = |\mathbf{r}|\) is the relative distance, \(v\) is the relative velocity, and the perturbation term due to rotation effects is

\[
\delta\mathcal{L} = \frac{2(1 + \eta)G\mu}{c^2r^3}\mathbf{v}(\mathbf{r} \times S) + \frac{\eta\mu}{2c^2m}\mathbf{v}(\mathbf{a} \times S).
\]
Here $\mu$ is the reduced mass and $m$ the total mass of the system,
\[
\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad m = m_1 + m_2, \quad \eta = \frac{m_2}{m_1}.
\] (2.3)

The Lagrangian depends on the relative acceleration $a$. The application of a generalized, second-order variational formalism yields the equations of motion:
\[
a = -\frac{Gm}{r^3}r + \frac{G}{c^2 r^3} \left\{\frac{6(1 + \eta)}{r^2} (r \times v) \cdot S - (4 + 3\eta)v \times S + 3(2 + \eta)\frac{\dot{r}}{r} \times S\right\},
\] (2.4)

and the momenta:
\[
q = \frac{\partial L}{\partial a} = \frac{\eta \mu}{2c^2 m} S \times v,
\] (2.5)
\[
p = \frac{\partial L}{\partial v} - \dot{q} = \mu v + \frac{(2 + \eta)G \mu}{c^2 r^3} r \times S.
\] (2.6)

The energy $E$ and the total angular momentum $J$ are constants of motion:
\[
E = p \cdot v + q \cdot a - L = \frac{\mu v^2}{2} - \frac{Gm \mu}{r} + \frac{\eta G \mu}{c^2 r^3} S \cdot (r \times v),
\] (2.7)
\[
J = S + L.
\] (2.8)

Here the orbital angular momentum $L$ is
\[
L = r \times p + v \times q = L_N + L_{SO}
\] (2.9)

with the Newtonian orbital angular momentum
\[
L_N = \mu r \times v
\] (2.10)

and the spin-orbit term
\[
L_{SO} = \frac{\mu}{c^2 m} \left\{(2 + \eta)\frac{Gm}{r^3} [r \times (r \times S)] - \frac{\eta}{2} [v \times (v \times S)]\right\}.
\] (2.11)

Formally the spin terms arising in these expressions are of order $\epsilon S$. Since the spin is of order $S \approx r^2 \epsilon^2 V$, the terms containing the spin in the Lagrangian, the acceleration, the energy and the orbital angular momentum are of order $\epsilon^{3/2} V/c$ higher than the corresponding Newtonian terms.

The spin precession equation [1] is:
\[
\dot{S} = (4 + 3\eta)\frac{G}{2c^2 r^3} L_N \times S.
\] (2.12)

Hence the magnitude $S$ of the spin is constant. Note that up to this point the description is valid up to (post)$^2$-Newtonian order.

Since we are interested in the leading spin terms of the gravitational radiation backreaction losses, we may replace $L_N \rightarrow J$ by inserting terms of higher order in $S$. This is allowed, as we are neglecting terms of order $\epsilon S^2 = \epsilon^3$ in the precessional angular velocity, which are $\epsilon^{3/2}$ order higher then the spin terms that we have kept. (We shall return to the order-of-magnitude estimates at the end of this section.)

From the conservation of total angular momentum (2.8) and the (2.12) precession of $S$ about $J$, the vector $L$ is found to precess about $J$:
\[
\dot{L} = -\dot{S} = (4 + 3\eta)\frac{G}{2c^2 r^3} J \times L.
\] (2.13)

Combining (2.12) (with $J$ instead of $L_N$) and (2.13) it follows that the angle $\kappa$ subtended by $S$ and $L$ is constant. As shown in Appendix A, the angle $\kappa$ is constant to order $\epsilon^{3/2}$. Thus we have the picture of a parallelogram of constant sides $L$ and $S$ rigidly rotating about its diagonal $J$ (Fig.1).
We choose the Cartesian coordinates \( \mathbf{r} = \{x, y, z\} \) of the reduced-mass particle with origin at the center of mass such that \( \mathbf{J} \) points along the \( z \) axis \([17]\). In this approximation we can choose the conserved quantities characterizing the orbital motion as: the energy \( E \) (due to the time-independent nature of the Lagrangian), the magnitude \( L \) of \( \mathbf{L} \) (the vector \( \mathbf{L} \) undergoes a pure rotation in the order we are considering) and \( L_S \), the spin-oriented component of \( \mathbf{L} \) (cf. Appendix A):

\[
\dot{E} = \dot{L}_S = \dot{L} = 0 .
\]  

(2.14)

This description is close to the one in the Lense-Thirring approximation. The inclusion of finite mass does not destroy the basic features of the formalism developed in the test particle limit. In computing the expressions of the constants of motion, \( \mathbf{L}_{SO} \cdot \mathbf{S} \) was replaced by \( L_S \mathbf{S} \), which is compatible with the order we are considering. Using polar coordinates, \( x = r \sin \theta \cos \varphi, \ y = r \sin \theta \sin \varphi, \ z = r \cos \theta \), the constants of the motion are:

\[
E = \frac{\mu v^2}{2} - \frac{Gm\mu}{r} + \frac{GL_S S}{c^2 r^3} = \frac{\mu}{2} [r^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \, \dot{\varphi}^2)] - \frac{Gm\mu}{r} + \frac{GL_S S}{c^2 r^3} ,
\]  

(2.15)

\[
L^2 = \mu^2 r^4 (\dot{\theta}^2 + \sin^2 \theta \, \dot{\varphi}^2) - 4 \frac{G\mu L_S S}{c^2 r} + \frac{2\eta}{c^2 m} E L_S S ,
\]  

(2.16)

\[
L_S = L \cdot \frac{\mathbf{S}}{S} = L \cos \kappa .
\]  

(2.17)

From the first expression of the energy, \( v^2 \) can be expressed in terms of constants of motion and \( r \):

\[
v^2 = \frac{2}{\mu} E + \frac{2Gm}{r} - \frac{2GL_S S}{c^2 \mu r^3} .
\]  

(2.18)

From the second expression of the energy in (2.15) and from (2.14), the equation for the radial motion is found to decouple from the angular degrees of freedom:

\[
\dot{r}^2 = \frac{2E}{\mu} + \frac{2Gm}{r} - \frac{L^2}{\mu r^2} + 2\eta \frac{E L_S S}{c^2 m \mu r^2} - 2(2 + \eta) \frac{GL_S S}{c^2 \mu r^3} .
\]  

(2.19)

We conclude this section by considering some order-of-magnitude estimates in the approximation which has led to the picture of a rigidly rotating parallelogram. The neglected time-dependent terms in \( L^2 \) and \( L_S \) are \((4 + 3\eta) \int \mathbf{L} \cdot (\mathbf{L}_{SO} \times \mathbf{S})/r^3 \, dt \approx r^4 e^8 \) and \((4 + 3\eta)/2 \int \mathbf{L} \cdot (\mathbf{S} \times \mathbf{L}_{SO})/r^3 \, dt \approx r^2 e^4 \), respectively. These are of order \( e^{3/2} \) and \( e \) smaller than the terms we keep. In the radial equation (2.19), as declared, we keep terms linear in the spin and drop terms of order \( e^{3/2} \) smaller.

### III. PARAMETERIZATION OF THE ORBIT

In this section, we employ Eq. (2.19) for determining the turning points and then finding convenient parametrizations of the orbit. We describe the procedure only in its outlines, as a detailed exposition was previously given for the Lense-Thirring case in I. The condition for turning points \( \dot{r} = 0 \) yields a cubic equation. By looking for roots slightly different from the roots known for the nonspinning case, one unphysical root and the turning points \( r_{\text{max}} \) are found:

\[
r_{\text{max}} = \frac{Gm\mu}{2E} + \frac{(2 + \eta)G\mu L_S S}{c^2 L^2} \pm \left[ \frac{A_0}{2E} + \frac{(2 + \eta)G^2 m^2 \mu^2 L_S S}{c^2 L^2 A_0} - \eta \frac{E L_S S}{c^2 m \mu A_0} \right] ,
\]  

(3.1)

where \( A_0 \) is the length of the Runge-Lenz vector to the zeroth order in the spin:

\[
A_0 = \sqrt{G^2 m^2 \mu^2 + \frac{2EL^2}{\mu}} .
\]  

(3.2)

As in the Lense-Thirring case, we define the eccentric anomaly parametrization of the Kepler orbits,

\[
r = -\frac{Gm\mu}{2E} + \frac{(2 + \eta)G\mu L_S S}{c^2 L^2} + \left[ \frac{A_0}{2E} + \frac{(2 + \eta)G^2 m^2 \mu^2 L_S S}{c^2 L^2 A_0} - \eta \frac{E L_S S}{c^2 m \mu A_0} \right] \cos \xi ,
\]  

(3.3)

such that \( r_{\text{max}} \) is at \( \xi = \pi \) and \( \xi = 0 \), respectively. The parameter derivative \( d\xi/d\xi \) follows from (3.3). Expressing \( 1/\dot{r} \) from (2.19) and linearizing in \( S \), the expression for \( dt/d\xi \) is found:
\[\frac{dt}{d\xi} = \frac{1}{r} \frac{dr}{d\xi} = \frac{\mu^2 (Gm\mu - A_0 \cos \xi)}{(-2\mu E)^{3/2}} + \frac{(2 + \eta)G^2 m^2 \mu^3 - \eta EL^2) L_S S \cos \xi}{c^2 mL^2 A_0 (-2\mu E)^{1/2}}. \quad (3.4)\]

Integration of \((3.4)\) from 0 to \(2\pi\) gives the orbital period:

\[T = 2\pi \frac{Gm\mu^3}{(-2\mu E)^{3/2}}. \quad (3.5)\]

Note that the expression for the period has the same functional form as in the Lense-Thirring case.

The eccentric anomaly parametrization \(r = r(\xi)\) was useful in computing the orbital period, but it leads to unnecessarily complicated expressions when instantaneous losses need to be averaged. For the latter purpose we introduce the true anomaly parametrization \(r = r(\chi)\) requiring that it has the following properties:

\[(a) \quad r(0) = r_{\text{min}} \quad \text{and} \quad r(\pi) = r_{\text{max}} \quad (3.6)\]
\[(b) \quad \frac{dr}{d(\cos \chi)} = -(\gamma_0 + S\gamma_1)r^2 \quad (3.7)\]

where \(\gamma_0, \gamma_1\) are constants. Property (b) generalizes Kepler’s second law for the area. The unique parametrization satisfying both (a) and (b) is:

\[r = \frac{L^2}{\mu (Gm\mu + A_0 \cos \chi)} + \frac{2(2 + \eta)GL_SS A_0 (2G^2 m^2 \mu^3 + EL^2) + Gm\mu (2G^2 m^2 \mu^3 + 3EL^2) \cos \chi}{c^2 mL^2 A_0 (Gm\mu + A_0 \cos \chi)^2} \quad (3.8)\]

In the same way as \(dt/d\xi \ [\text{Eq. } (3.4)]\) was obtained, \(dt/d\chi\) is found to have the form

\[\frac{dt}{d\chi} = \frac{1}{r} \frac{dr}{d\chi} = \frac{\mu^2}{L} \left[1 - \frac{L_S S}{c^2 mL^2} \left((2 + \eta)Gm\mu^2 (3Gm\mu + A_0 \cos \chi) - \eta EL^2\right)\right]. \quad (3.9)\]

The true anomaly parametrization \((3.8)\) will be used for the integration over one period of all instantaneous losses. In the next section, all of these losses will be expressed in the form \(F = F(\chi)\). They contain no other \(\chi\) dependent factor in the denominator than \(r^{2+n}\), where \(n\) is a positive integer. Time integration can be replaced by parameter integration;

\[\int_0^T F(t) dt = \int_0^{2\pi} F(\chi) \frac{dt}{d\chi} d\chi, \quad (3.10)\]

where \(dt/d\chi\) is given by \((3.9)\). Here we encounter an advantage of property (b) of our parametrization: the \(\chi\) dependence of the denominator of the integrand is especially simple, \(r^n\).

As in the Lense-Thirring case, the integrals are evaluated by computing the residues enclosed in the circle \(\zeta = e^{i\chi}\). We find, as the second advantageous feature of the parametrization that there is only one pole, at \(\zeta = 0\). Averaging over one period is achieved by dividing the result by the period \(T \ [\text{Eq. } (3.3)]\).

**IV. INSTANTANEOUS RADIATIVE LOSSES**

In this section, we obtain the instantaneous losses in the energy \(E\), magnitude \(L\) and spin projection \(L_S\) of the orbital angular momentum. We then find suitable angular variables such that all losses can be rewritten in terms of the true anomaly parameter \(\chi\).

From the radiative power and the instantaneous total angular momentum losses of the binary system given by Kidder [3, formulae (3.24),(3.28)], we keep the Newtonian and spin-orbit terms:

\[\frac{dE}{dt} = - \frac{8}{15} \frac{G^3 m^2 \mu^2}{c^5 r^4} (12v^2 - 11r^2) - \frac{8}{15} \frac{G^3 m \mu}{c^5 r^6} \left[\mathbf{L}_N \cdot \mathbf{S}\right] \left[27r^2 - 37v^2 - 12\frac{Gm}{r} + \eta (51r^2 - 43v^2 + 4\frac{Gm}{r})\right] \quad (4.1)\]
\[\frac{dJ}{dt} = - \frac{8}{5} \frac{G^2 m \mu}{c^5 r^3} \mathbf{L}_N \left(-3r^2 + 2v^2 + 2\frac{Gm}{r}\right) - \frac{4}{5} \frac{G^2 m \mu}{c^5 r^3} \{ - \frac{2}{3} \frac{Gm}{r} (r^2 - v^2) (1 - \eta) \mathbf{S} - \frac{Gm}{3r^2} \mathbf{r} \times (\mathbf{v} \times \mathbf{S})(7 + 5\eta) \]
We want to compute the instantaneous radiative losses of the constants of motion. As we argued in the Introduction and will show in detail later in this section, the total angular momentum loss can be taken equal to the orbital angular momentum loss, \( \frac{dJ}{dt} = \frac{dL}{dt} \). The loss in the magnitude of the orbital angular momentum is given by

\[
2L \frac{dL}{dt} = \frac{d(L_L L_L)}{dt} = 2L_i \frac{dL_i}{dt}.
\]

(4.3)

The loss in \( L_S \) is given as:

\[
\frac{dL_S}{dt} = \frac{d}{dt} \left( \frac{L \cdot S}{S} \right) = \frac{dL}{dt} \frac{S}{S} + \frac{L}{S} \cdot \frac{dS}{dt} - \left( \frac{L}{S} \cdot \frac{S}{S} \right) \left( \frac{S}{S} \cdot \frac{dS}{dt} \right).
\]

(4.4)

Although the radiation-induced change in the spin \( S \) is \( \epsilon^{1/2} \) order smaller than the spin-orbit part of the change in the orbital angular momentum, due to the vectors \( L/S \approx \epsilon^{-1/2} \), the second and third terms are of comparable order with the first term. We will evaluate the terms depending on the structure of the spinning body later in this section. The first term is found simply by multiplying (4.2) by \( S/S \).

The next step is to find variables in which these radiative losses can be expressed solely in terms of the parameter \( \chi \) and constants of the motion. The polar coordinates \((\theta, \phi)\) are not suitable for this purpose: \( \theta \) is not a monotonous function of time, thus the root of \( \dot{\theta}^2 \) cannot be extracted unambiguously.

We introduce new variables conveniently characterizing the separation vector \( r \), the spin vector \( S \) and the orbital angular momentum vector \( L \). The relative coordinates can be expressed in terms of the time dependent Euler angles \( \Psi, \iota_N, \Phi \) as:

\[
x = r \left( \cos \Phi \cos \Psi - \cos \iota_N \sin \Phi \sin \Psi \right), \\
y = r \left( \sin \Phi \cos \Psi + \cos \iota_N \cos \Phi \sin \Psi \right), \\
z = \dot{r} \sin \iota_N \sin \Psi.
\]

(4.5)

Here \( \iota_N \) is the polar angle and \( \Phi - \pi/2 \) the azimuthal angle of the Newtonian orbital momentum \( L_N \) (Fig. 1). The variable \( \Psi \) is the angle subtended by the momentary position vector and the node line (the intersection of the plane orthogonal to the \( L_N \) with the plane orthogonal to the total angular momentum \( J \)). From the leading terms of the equations of motion (Cf. Appendix of I) it follows that \( \Psi \) is a monotonous function of \( t \). Thus we extract the square root of the equation for \( \dot{\Psi}^2 \), choosing the positive root.

![Fig. 1. The orientation of the orbit and the angular momenta.](image)
In the absence of spin, the orbital plane is at rest, and $\Psi$ is the usual polar angle, satisfying the area law. Thus the time derivatives of the angles are given by

$$ \dot{\Psi} = \frac{L}{\mu r^2} + S \dot{\Psi}_1, \quad \dot{\Phi} = S \dot{\Phi}_1, \quad i_N = i_1. $$

We will not need the explicit expressions of the first-order (in $S$) terms $\Psi_1, \Phi_1$ and $i_1$ for parameterizing the instantaneous losses. Rather it will be sufficient for us to find how they are interrelated. First, by computing the square $L^2_N$ of the Newtonian orbital angular momentum in two different ways:

$$ L^2_N = (\mu r \times v)^2 = L^2 - 2L_N \cdot L_{SO}, $$

where $2L_N \cdot L_{SO}$ are the spin terms from (2.16), the following relation is found:

$$ \dot{\Psi}_1 = -\cos i_N \dot{\Phi}_1 + \cos \kappa \frac{2Gm\mu - \eta Er}{c^2m\mu^3}. $$

Second, we compute the $z$ component of $L_N$ in two independent ways:

$$ (L_N)_z = (\mu r \times v)_z = (L^2 - 2L_N \cdot L_{SO}) \frac{i}{\sin i_1}. $$

Linearizing the second equality of (4.9) in the spin, we obtain

$$ \dot{\Phi}_1 = \frac{\tan \Psi}{\sin i_1} \dot{i}_1. $$

When the relations (4.5) and (4.10) are inserted in the instantaneous losses of energy, magnitude and spin projection of the orbital angular momentum, all terms with $\Psi_1, \Phi_1, i_1$ will cancel.

The parallelogram spanned by the spin and orbital angular momentum is next described in terms of the new variables (Fig.1). We denote the polar and azimuthal angles of $S$ by $\alpha, \beta$ and $\iota, \beta + \pi$. During the first period when $J$ lies on the $z$ axis, $\kappa = \iota + \alpha$. Similarly as with $L$ and $L_N$, the azimuthal and polar angles differ only by first order terms in the spin. As a result, no dependence on the azimuthal angles $\Phi$ and $\beta$ remains in the losses.

As we mentioned in the Introduction, the spin is of order smaller than the orbital angular momentum. This implies that the side $S$ of the parallelogram shrinks. As the angles $\iota, i_N$ and $\alpha$ are present in the losses only in the spin-terms, we need only their zeroth order expressions:

$$ i_N = \iota = 0, \quad \alpha = \kappa. $$

The nonradiative evolution of the angles $\iota$ and $i_N$ is discussed in Appendix B.

From (2.18) and (2.19), $v^i$ and $i^2$ can be replaced by expressions containing solely $r$ and constants. Making use of the foregoing observations, we obtain the instantaneous losses of energy, magnitude and spin component of the orbital angular momentum:

$$ \frac{dE}{dt} = -\frac{8G^3m^2}{15c^3r^6} (2\mu Er^2 + 2Gm\mu^2r + 11L^2) $$

$$ + \frac{8G^3mLS\cos\kappa}{15c^3\mu^3r^8} [20\mu Er^2 - 12Gm\mu^2r + 27L^2 + \eta (6\mu Er^2 - 18Gm\mu^2r + 51L^2)] $$

$$ \frac{dL}{dt} = +\frac{8G^2mL}{5c^3\mu^3r^5} (2\mu Er^2 - 3L^2) $$

$$ + \frac{8G^2S\cos\kappa}{15c^3\mu^2r^7} \left(12Gm^3\mu^3r^3 + 3\mu^2 (6EL^2 + G^2m^2\mu^3) - 11Gm\mu^2L^2r $$

$$ + \eta [2\mu^2E^2r^4 + 12Gm^3\mu^3r^3 + 3\mu r^2 (5EL^2 + G^2m^2\mu^3) - 5Gm\mu^2L^2r + 15L^4] \right) $$

$$ \frac{dL_s}{dt} = +\frac{8G^2mLS\cos\kappa}{5c^3\mu^5} (2\mu Er^2 - 3L^2) + \frac{2G^2S}{15c^3\mu^2r^7} \left[48Gm^3\mu^3r^3 + 12\mu^2 (6EL^2 + G^2m^2\mu^3) - 44Gm\mu^2L^2r $$

$$ + 4\eta [2\mu^2E^2r^4 + 12Gm^3\mu^3r^3 + 3\mu^2 (5EL^2 + G^2m^2\mu^3) - 5Gm\mu^2L^2r + 15L^4] $$

$$ + \sin^2\kappa \left[-24Gm^3\mu^3r^3 + 6\mu r^2 (6EL^2 - G^2m^2\mu^3) + 72Gm\mu^2L^2r - 90L^4 \right] $$

7
estimate of ACST \[7\] for Burke-Thorne potential \[18,3\] to the Newtonian order and we compute them from the radiation-reaction potential. To lowest order, this is the cases, the approximation above, equating \(d\) \(V\) direction. When the body is rapidly rotating (the spinning body. The torque \(\tau\) is perpendicular to the spin \(S\) \(I\) where \(\Theta\) \(j_k\) The torque induces the radiative change in the spin. The volume integral can be expressed with the tensor of inertia \(\Theta\) \(jk\) \(1\)\(35\) \((\Theta\) \(\Theta\) \(\delta\) \(y_jy_j)\) \(dV\):

\[
\tau_i = \frac{2G}{5c^3} \epsilon_{i k l} I_j^{(5)} j_l \int \rho(y) y_k y_j dV .
\]

(4.14) The torque induces the radiative change in the spin. The volume integral can be expressed with the tensor of inertia \(\Theta_{jk} = \int \rho(y) y_j y_k dV\):

\[
\frac{dS_i}{dt} = \tau_i = \frac{2G}{5c^3} \epsilon_{i k l} I_j^{(5)} \Theta_{j k} .
\]

(4.15) For an axisymmetrically spinning body the tensor of inertia is \(\Theta_{jk} = diag(\Theta, \Theta, \Theta')\) in the system of principal axes with \(S\) on the \(z\) axis. The magnitude of the spin is \(S = \Theta \Omega\), where \(\Omega\) is the angular velocity of the body. Thus

\[
\frac{1}{S} \frac{dS_i}{dt} = \frac{2G}{5c^3} \left( \frac{\Theta}{\Theta'} - 1 \right) \epsilon_{i j k} I_{j l}^{(5)} \hat{S}_k \hat{S}_l
\]

(4.16) is perpendicular to the spin \(S\). Here the quantity \(\delta = \Theta/\Theta' - 1\) is the deviation from sphericity and \(\hat{S}\) the spin direction. When the body is rapidly rotating \((V \gg c^{1/2})\), it is centrifugally flattened \((\delta\) nonnegligible\) and the estimate of ACST \[8\] for \(dS/dt\) holds. For slow rotation, \(V \approx c^{1/2}\), the deviation from sphericity is \(\delta\) small. In both cases, the approximation above, equating \(dJ/dt\) with \(dL/dt\) is valid.

From \[1,16\], the last term in the \(L_S\)-loss vanishes, and the other term containing the radiative spin loss is

\[
\frac{L}{S} \frac{dS}{dt} = \frac{2G^2 m L \sin^2 \kappa}{5c^5 \mu^2 \Omega r^2} \left( \frac{\Theta}{\Theta'} - 1 \right) \left\{ \mu r \dot{r} \sin 2\Psi \left( 12 \mu \dot{r} r^2 + 20 Gm \mu^2 r + 45 L^2 \right) + 4 L \cos 2\Psi \left( 18 \mu \dot{r} r^2 + 20 Gm \mu^2 r - 15 L^2 \right) \right\} .
\]

(4.17) The losses in the energy and in the magnitude of angular momentum contain constants of the motion and they depend only on \(r = r(\chi)\). These expressions will take the suitable form for averaging, after inserting the parametrization \[8\]. The loss \(dL_S/dt\) depends also on the zeroth order expressions of the angle variable \(\Psi\) and of \(\dot{r}\), which are:

\[
\Psi = \Psi_0 + \chi , \quad \dot{r} = \frac{A_0}{L} \sin \chi .
\]

(4.18) With these expressions, the task of parametrizing is completed: the loss in \(L_S\) can also be expressed in terms of the true anomaly parameter \(\chi\).
Solving the system (3.1) and (6.1), one has:

\[ \langle \frac{L}{S}, \frac{dS}{dt} \rangle = 0. \]  

(5.1)

The averaged losses in the constants of motion are:

\[
\langle \frac{dE}{dt} \rangle = -\frac{G^2 m (-2E\mu)^{3/2}}{15c^3 L^7} (148E^2 L^4 + 732G^2 m^2 \mu^3 E L^2 + 425G^4 m^4 \mu^6) \\
+ \frac{G^2 (-2E\mu)^{3/2} S \cos \kappa}{10c^7 L^10} \left[ 520E^3 L^6 + 10740G^2 m^2 \mu^3 E^2 L^4 + 24990G^4 m^4 \mu^6 E L^2 + 12579G^6 m^6 \mu^9 \right] \\
+ \eta (256E^3 L^6 + 6660G^2 m^2 \mu^3 E^2 L^4 + 16660G^4 m^4 \mu^6 E L^2 + 8673G^6 m^6 \mu^9) \\
\]

(5.2a)

\[
\langle \frac{dL}{dt} \rangle = -\frac{4G^2 m (-2E\mu)^{3/2}}{5c^3 L^3} (14EL^2 + 15G^2 m^2 \mu^3) \\
+ \frac{G^2 (-2E\mu)^{3/2} S \cos \kappa}{15c^3 L^7} \left[ 1188E^2 L^4 + 6756G^2 m^2 \mu^3 E L^2 + 5345G^4 m^4 \mu^6 \right] \\
+ \eta (772E^2 L^4 + 4476G^2 m^2 \mu^3 E L^2 + 3665G^4 m^4 \mu^6) \\
\]

(5.2b)

\[
\langle \frac{dS}{dt} \rangle = -\frac{4G^2 m (-2E\mu)^{3/2} \cos \kappa}{5c^3 L^3} (14EL^2 + 15G^2 m^2 \mu^3) + \frac{G^2 (-2E\mu)^{3/2} S \cos \kappa}{15c^3 L^7} \left[ 1188E^2 L^4 + 6756G^2 m^2 \mu^3 E L^2 + 5345G^4 m^4 \mu^6 \right] \\
- \frac{\sin^2 \kappa}{2} \left[ 3516E^2 L^4 + 17676G^2 m^2 \mu^3 E L^2 + 12975G^4 m^4 \mu^6 \right] \\
+ \eta (2428E^2 L^4 + 12216G^2 m^2 \mu^3 E L^2 + 9125G^4 m^4 \mu^6) \\
+ \sin^2 \kappa \cos (2\Psi_0) \left[ (26EL^2 + 33G^2 m^2 \mu^3)(12EL^2 + 6G^2 m^2 \mu^3) \right] \\
+ \eta (119EL^2 + 156G^2 m^2 \mu^3)(2EL^2 + G^2 m^2 \mu^3)) \right) . \\
\]

(5.2c)

The expressions (5.2a) and (5.2b) agree with the available results of [15] for the averaged power and \( < dL/dt > \), despite the differences encountered in the higher-order terms of the definition of the energy and angular momentum.

The averaged loss in \( < dS/dt > \) contains terms proportional to \( \cos(2\Psi_0) \). Ryan [10] argues that in most cases such terms are averaged out by the Schwarzschild precession.

V. AVERAGED RADIATIVE LOSSES

VI. AVERAGED LOSSES IN TERMS OF ORBIT PARAMETERS

As in \( \mathbf{I} \), we define the generalized semimajor axis and eccentricity imposing the conditions:

\[ r_{max} = a(1 \pm e) \]  

(6.1)

Solving the system (3.1) and (6.1), one has:

\[ a = -\frac{Gm \mu}{2E} \left[ 1 - \frac{2ELsS}{c^2 m L^2}(2 + \eta) \right] , \]

\[ 1 - e^2 = -\frac{2EL^2}{G^2 m^2 \mu^3} \left[ 1 + \frac{8LsS}{c^2 m L^3}(EL^2 + G^2 m^2 \mu^3) + \frac{2\eta LsS}{c^2 m L^4}(EL^2 + 2G^2 m^2 \mu^3) \right] . \]

(6.2)

From these relations one can express the constants of motion in terms of the orbit parameters:

\[ E = -\frac{Gm \mu}{2a} \left( 1 + \frac{GS \cos \kappa(2 + \eta)}{c^2 a^{3/2} \sqrt{Gm(1 - e^2)}} \right) , \]

\[ L^2 = Gm \mu^2 a(1 - e^2) \left( 1 - \frac{2GS \cos \kappa}{c^2 a^{3/2} \sqrt{Gm(1 - e^2)}} \right) \left( 3 + e^2 + 2\eta \right) . \]

(6.3)
Inserting (6.3) in the equations (5.2a) we get the averaged radiation losses in terms of $a$, $e$ and $\kappa$:

$$\langle \frac{dE}{dt} \rangle = -\frac{G^4m^3\mu^2}{15c^5a^6(1-e^2)^{7/2}}(37e^4 + 292e^2 + 96)$$

$$+ \frac{G^{9/2}m^{3/2}\mu^2S\cos\kappa}{30c^7a^{13/2}(1-e^2)^{5}} \left[ 191e^6 + 5694e^4 + 6584e^2 + 1168 + \eta(355e^6 + 5316e^4 + 7248e^2 + 1200) \right]$$

$$\langle \frac{dL}{dt} \rangle = -\frac{4G^{7/2}m^{5/2}\mu^2}{5c^8a^{7/2}(1-e^2)^{2}}(7e^2 + 8)$$

$$+ \frac{G^4m^2\mu^2S\cos\kappa}{15c^5a^6(1-e^2)^{7/2}} \left[ 549e^4 + 1428e^2 + 488 + \eta(403e^4 + 1366e^2 + 456) \right]$$

$$\langle \frac{dS}{dt} \rangle = -\frac{4G^{7/2}m^{5/2}\mu^2\cos\kappa}{5c^8a^{7/2}(1-e^2)^{2}}(7e^2 + 8) + \frac{G^4m^2\mu^2S}{30c^7a^{13/2}(1-e^2)^{5}} \left\{ 2[549e^4 + 1428e^2 + 488 + \eta(403e^4 + 1366e^2 + 456)] \right.$$  \left.$$- \sin^2\kappa[1383e^4 + 4368e^2 + 240 + \eta(1027e^4 + 3922e^2 + 1296)] \right.$$  \left.$$+ \sin^2\kappa\cos(2\Psi_0)e^2[156e^2 + 240 + \eta(119e^2 + 193)] \right\}.$$  \tag{6.4}

The losses of $\kappa$, $a$ and $e$ follow by taking the time derivatives of (2.17) and of (5.2):

$$\langle \frac{d\kappa}{dt} \rangle = \frac{G^{7/2}m^{3/2}\mu}{{3c^7a^{11/2}(1-e^2)^{4}}} \left\{ 285e^4 + 1512e^2 + 488 + \eta(221e^4 + 1190e^2 + 384) \right.$$  \left.$$- e^2\cos(2\Psi_0)[156e^2 + 240 + \eta(119e^2 + 193)] \right\}$$

$$\langle \frac{da}{dt} \rangle = -\frac{2G^3m^2\mu}{15c^8a^9(1-e^2)^{7/2}}(37e^4 + 292e^2 + 96)$$

$$+ \frac{G^{7/2}m^{3/2}\mu\cos\kappa}{{15c^7a^{9/2}(1-e^2)^{5}}} \left[ 363e^6 + 3510e^4 + 7936e^2 + 2128 + \eta(291e^6 + 4224e^4 + 7924e^2 + 1680) \right]$$

$$\langle \frac{de}{dt} \rangle = -\frac{G^{3/2}m^2\mu}{15c^8a^9(1-e^2)^{5/2}} e(121e^2 + 304)$$

$$+ \frac{G^{7/2}m^{3/2}\mu\cos\kappa}{{30c^7a^{11/2}(1-e^2)^{4}}} e \left[ 1313e^4 + 5592e^2 + 7032 + \eta(1097e^4 + 6822e^2 + 6200) \right].$$  \tag{6.5}

From Eq. (6.3), the angle $\kappa$ increases. Hence we find that the vectors $L$ and $S$ of a finite-mass binary system tend to antialign as in $I$. Substituting $\eta = 0$ for a test particle with negligible mass, we reproduce the Lense-Thirring losses of $I$.

The evolution of the angles $\epsilon$ and $\chi_\nu$ under radiation backreaction is determined by $< dL/dt >$. This computation, relegated to Appendix B, involves higher-order terms in $S$ which are, however of (post)$^{3/2}$-order.

**VII. CONCLUDING REMARKS**

The computation of finite-mass effects has proved to be much more difficult than the corresponding description of a test particle. We have been helped by several fortunate circumstances in overcoming the obstacles in computing the radiative losses up to the leading spin terms. First, in this approximation, one can still decouple a radial equation of motion. Furthermore, a generalization of the true anomaly parametrization $\chi$, found in $I$ for the Lense-Thirring case, exists with the inclusion of the finite mass contributions. Although the dynamics of the system is much more complicated, we succeeded in describing all quantities entering in the radiative losses in terms of the parameter $\chi$.

We were able to separately describe the relative evolution of the spin and orbital angular momentum vectors, up to first order in spin, under radiation backreaction. In the loss of the spin projection of the orbital angular momentum two terms, containing the radiative spin loss, appear. One of them is known to vanish (ACST). We computed the other term, and have shown that it averages out to nothing. We did not consider no-spin post-Newtonian effects here. In the equations of motion, we did not keep terms of higher order than $e^{3/2}$, which, however, are comparable or
even larger than the radiative losses. They are needed for a complete characterization of the orbit, but they do not contribute to the radiative losses in the order to which we are computing them.

In the test-particle (\(\eta = 0\)) and constant-spin case our results agree with those of \(I\). The Lense-Thirring description implies \(S \gg L\). These pictures arise as different limits of a more general description in which both \(S\) and \(L\) are arbitrary. Such a description is subject to further investigation as it will go beyond considering only the linear terms in \(S\).

VIII. ACKNOWLEDGMENTS

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APPENDIX A: NONRADIATIVE EVOLUTION OF \(L_S\) AND \(\kappa\)

In this Appendix, we consider the nonradiative evolution of \(L_S\) and the angle \(\kappa\) to linear order in the spin, and we show that these changes are smaller than order-\(\epsilon^3/2\).

From the conservation of the total angular momentum \(J\) and the spin precession equation (2.12), the time derivative of \(L_S = \mathbf{L} \cdot \hat{S}\) has the form:

\[
\dot{L}_S = \frac{4 + 3\eta}{2r^3} \frac{G^2\mu^2 S}{c^4} \left\{ \left( \mathbf{r} \cdot \mathbf{v} \right)^2 + \frac{2 + \eta r^3}{2Gm} \left( \mathbf{v} \cdot \hat{S} \right)^2 \right\} - \left( \mathbf{r} \cdot \hat{S} \right) \left( \mathbf{r} \cdot \mathbf{S} \right) \left( \frac{2 + \eta}{r} + \frac{\eta r}{2Gm} \right). \tag{A1}
\]

The multiplicative factor of \(S\) needs to be evaluated only to Newtonian order. We write the non-radiative change in \(L_S\) in terms of the radial and Euler-angle coordinates (4.5):

\[
\dot{L}_S = -\frac{GS(4 + 3\eta) \sin^2 \kappa}{4c^3 m \mu r^5} \left[ L(-E\eta \mu r^2 + 2Gm \mu^2 r + L^2 \eta) \sin(2\Psi) - L^2 \dot{r} \eta \mu r \cos(2\Psi) \right]. \tag{A2}
\]

After parameterizing by \(\chi\), the averaged expression over one period is:

\[
\langle \dot{L}_S \rangle = -\frac{G(-2E\mu)^{3/2} \mu A_0^2 \sin^2 \kappa \sin(2\Psi_0)}{16c^3 L^4 m} (15\eta^2 + 26\eta + 8). \tag{A3}
\]

This expression is of order \(r\epsilon^{9/2}V\), comparable with the leading radiative loss in \(L\) given by Peters and Mathews, corresponding to 2.5PN relative order. However we pursue our description to the 1.5PN relative order. If further terms were considered, e.g. \(\epsilon^5/2\), they would generate additional terms in \(L\), as well as in \(\dot{L}_S\). It is simple to check that these terms would be of the same order as (A3). Thus the change in \(L_S\) is negligible in our approximation.

Similarly, the change in \(\cos \kappa\) is negligible, as follows from \(\langle \cos \kappa \rangle = \dot{L}_S/L\).

APPENDIX B: EVOLUTION OF \(\iota\) AND \(\iota_N\)

In this Appendix, we discuss the evolution, both in the absence of radiation and under radiation backreaction, of the angles

\[
\cos \iota = \frac{\mathbf{L} \cdot \mathbf{J}}{L}, \quad \cos \iota_N = \frac{\mathbf{L}_N \cdot \mathbf{J}}{L_N}, \tag{B1}
\]

where \(J\) and \(L_N\) are the magnitude of the total angular momentum \(\mathbf{J}\) and \(\mathbf{L}_N\), respectively.

An expansion of the expressions on the right hand sides of (B1) to 1.5PN order yields

\[
\cos \iota = \cos \iota_N = 1 - \frac{1}{2} \sin^2 \kappa \left( \frac{S}{L} \right)^2 - \cos \kappa \sin^2 \kappa \left( \frac{S}{L} \right)^3. \tag{B2}
\]

By this relation, both the radiative and the nonradiative changes of \(\cos \iota\) and \(\cos \iota_N\) are given in terms of the corresponding changes in \(\kappa\), \(S\) and \(L\).
\[
\delta \cos \iota = \delta \cos \iota_N = - \left( \frac{S}{L} \right)^2 \left[ \cos \kappa \delta \cos \kappa + \sin^2 \kappa \left( \frac{\delta S}{S} - \frac{\delta L}{L} \right) \right]
\]
\[
- \left( \frac{S}{L} \right)^3 \left[ (1 - 3 \cos^2 \kappa) \delta \cos \kappa + 3 \cos \kappa \sin^2 \kappa \left( \frac{\delta S}{S} - \frac{\delta L}{L} \right) \right].
\] (B3)

The first and second terms on the right hand side are of 1PN order and of 1.5PN order, respectively.

We have shown in Appendix A that the nonradiative evolution of the angle \( \kappa \) is negligible in our 1.5PN approximation. Because \( L \) and \( S \) are also constants, \( \cos \iota = \cos \iota_N = 0 \) under the nonradiative evolution.

Finally we turn our attention to the radiative evolution of the angles \( \iota \) and \( \iota_N \). The spin term correction in \( B.4 \) to the leading order radiative loss of \( L \) and the loss \( B.3 \) of the angle \( \kappa \) are of 1.5PN order, as they originate in the spin-orbit term \( L_{SO} \). Thus these terms do not contribute to the losses \( B.3 \). Similarly the loss in the magnitude of the spin \( S \) was seen to vanish by \( B.14 \). Hence the only loss contributing to the radiative changes in \( \iota \) and \( \iota_N \) is the leading term in the loss of \( L \):

\[
\frac{d \cos \iota}{dt} = \frac{d \cos \iota_N}{dt} = \left( \frac{S}{L} \right)^2 \left( 1 + 3 \cos \kappa \frac{S}{L} \right) \frac{1}{L} \frac{dL}{dt}.
\] (B4)

As the factors multiplying \( dL/dt \) are constant during a period of revolution, the averaged rates can be simply obtained by use of the leading terms in Eq. \( B.3 \) for \( \langle dL/dt \rangle \).

[1] L.Kidder, C.Will and A.Wiseman, Phys. Rev. D47, 4183 (1993)
[2] L.Kidder, Phys. Rev. D52, 821 (1995)
[3] L.Blanchet, Phys. Rev. D55, 714 (1997)
[4] L.Blanchet, T.Damour and B.Iyer, Phys. Rev. D51, 5360 (1995)
[5] B.M.Barker and R.F.O’Connell, Gen.Relativ.Gravit. 11, 149 (1979)
[6] K.S.Thorne and J.Hartle, Phys. Rev. D31, 1815 (1985)
[7] T.A.Apostolatos, C.Cutler, G.J.Sussman and K.S.Thorne, D49, 6274 (1994)
[8] P.C.Peters and S.Mathews, Phys. Rev. 131, 435 (1963)
[9] P.C.Peters, Phys. Rev. 136, B1224 (1964)
[10] F.Ryan, Phys. Rev. D53, 3064 (1996)
[11] F.Ryan, Phys. Rev. D52, R3159 (1995)
[12] M.Shibata, Phys. Rev. D50, 6297 (1994)
[13] L.Gergely, Z.Perjés and M.Vasúth, paper I, (1997)
[14] N. Wex, Class. Quantum Grav. 12, 983 (1995)
[15] R. Rieth and G. Schäfer, Class. Quantum Grav. 14, 2357 (1997)
[16] We adopt the convention that the derivatives with respect to the time parameter \( t \) occurring in the equations of motion (Sec.II) are denoted by an overdot but we reserve the notation \( d/dt \) for time derivatives due to radiation backreaction.
[17] The gravitational backreaction will slowly rotate out the vector \( J \) from the \( z \) direction. In our perturbative treatment, this effect, as well as the changes in the other constants of motion are small during the period of averaging, thus neglected.
[18] W.L.Burke, J. Math. Phys. 12, 401 (1971)