AROUND POISSON–MEHLER SUMMATION FORMULA

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Abstract. We study polynomials in $x$ and $y$ of degree $n+m$ that appeared recently in the following identity: $\gamma_{m,n}(x,y|t,q) = \gamma_{0,0}(x,y|t,q)$ $Q_{m,n}(x,y|t,q)$, where $\gamma_{m,n}(x,y|t,q) = \sum_{i \geq 0} t^i \left[\frac{i}{q}\right] q^i H_{n+i}(x|q) H_{m+i}(y|q)$, $\{H_n(x|q)\}_{n \geq -1}$ are the so-called $q$-Hermite polynomials (qH). In particular we show that the spaces span \( \{Q_{i,n-i}(x,y|t,q) : i = 0, \ldots, n \} \) are orthogonal with respect to a certain measure (two-dimensional $(t,q)$-Normal distribution) on the square \( \{(x,y) : |x|, |y| \leq 2/\sqrt{1-q} \} \). We study structure of these polynomials expressing them with the help of the so-called Al-Salam–Chihara (ASC) polynomials and showing that they are rational functions of parameters $t$ and $q$. We use them in various infinite expansions that can be viewed as simple generalization of the Poisson-Mehler summation formula. Further we use them in the expansion of the reciprocal of the right hand side of the Poisson-Mehler formula.

1. Introduction and auxiliary results

1.1. Preface. We consider various generalizations of the celebrated Poisson–Mehler formula (see e.g. [9], (13.1.24) or [1], (10.11.17)):

(1.1) \[ \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q) = \prod_{j=0}^{\infty} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q), \]

where \( \{H_n\}_{n \geq 0} \) denote $q$–Hermite polynomials and $\omega(x,y|t)$ are certain polynomials symmetric in $x$ and $y$ of degree two. These polynomials as well as symbols $[n]_q!$ and $(\rho^2)_{\infty}$ are defined and explained in Sections 1.2 and 1.3. There exist many proofs of (1.1) (e.g. see [9], [1], [2], [10]). Recently in [17] certain generalization of (1.1) has been proved by the author. It was used in calculating moments of the so called Askey–Wilson distribution.

In the paper we consider functions

(1.2) \[ \gamma_{i,j}(x,y|\rho,q) = \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_{n+i}(x|q) H_{n+j}(y|q), \]

for all $i, j \geq 0$. It was shown by the author in [16] (Lemma 3) that:

(1.3) \[ \gamma_{i,j}(x,y|\rho,q) = Q_{i,j}(x,y|\rho,q) \gamma_{0,0}(x,y|\rho,q), \]

where $Q_{i,j}(x,y|\rho,q)$ is a certain polynomial in $x, y$ of degree $i+j$. Hence (1.3) can be viewed as a generalization of (1.1).

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The main object of the paper is to study the properties and later the rôle of the polynomials $Q_{i,j}(x,y|\rho,q)$ in obtaining various expansions that can be viewed as either generalizations of (1.1) or expansions more or less directly related to this formula.

In particular we find generating function of these polynomials, we express them as linear combinations of polynomials belonging to families of polynomials of one variable.

We also analyze the measure (the so-called $(\rho,q)-2\text{Normal}$ measure) on the square $S(q) \times S(q)$ with the density defined by (2.3) below, that can be easily constructed from the densities of measures that make $q-$Hermite and the so-called Al-Salam–Chihara polynomials orthogonal and which can viewed as a generalization of bivariate Normal distribution. Interval $S(q)$ is defined by (1.5). The probabilistic aspects of this distribution were presented in [14]. We point out the rôle of the polynomials $Q_{n,m}$ in further analysis of this measure. In particular we introduce spaces of functions of two variables

\begin{equation}
\Lambda_n(x,y|\rho,q) = \text{span} \{Q_{i,n-i}(x,y|\rho,q), i = 0, \ldots, n\}, n \geq 0
\end{equation}

and show that they are orthogonal with respect to $(\rho,q)-2\text{Normal}$ measure. Hence these spaces form the direct sum decomposition of the space of functions that are square integrable with respect to $(\rho,q)-2\text{Normal}$ measure.

Further we use these polynomials to obtain various infinite expansions. In particular we obtain an expansion of the reciprocal of the right hand side of (1.1) in an infinite series. In [16], (formula 5.3) one such expansion was presented. The expansions was non-symmetric in $x$ and $y$ (for each finite sum). This time the expansion is symmetric in $x$ and $y$.

Among other possible views one can look at the results of paper as the generalization of the results of the two papers of Van der Jeugt et al. [10], [11]. The authors of these papers introduced convolutions of known families of classical orthogonal polynomials such as Hermite or Laguerre considered at two variables thus obtaining bivariate polynomials. They applied their results in Lie algebra and its generalizations.

Our ”convolutions” concern generalizations of Hermite polynomials ($q$-Hermite, and Al-Salam–Chihara). As possible applications we mean the ones in analysis, two dimensional orthogonal polynomials theory or probability.

Since in our paper appear kernels built of mostly $q-$Hermite and Al-Salam–Chihara one should remark that some of the technics used in the proofs resemble those used in e.g. [8]. But by no means results are the same.

The paper is organized as follows. In the next two Subsections (i.e. 1.2 and 1.3) we provide simple introduction to $q-$series theory presenting typical notation used and presenting a few typical families of the so called basic orthogonal polynomials. The word basic comes from the base which is the parameter in most cases denoted by $q$. We do this since notation and terminology used in $q-$series theory is somewhat specific and not widely known to those not working within this field. We are also purposely not using notation based on hypergeometric series since it is mostly known to specialists of special functions theory. We believe that the results presented in the paper can be applied in various fields of traditional analysis like the theory of Fourier expansions, theory of reproducing kernels, orthogonal polynomials theory and last but not least probability theory. Then in Section 2 we present
our main results, open questions and remarks are in Section 3 while less interesting
laborious proofs are in Section 4.

1.2. Notation. We use notation traditionally used in the so called $q$–series theory. Since not all readers are familiar with it we will recall now this notation.

$q$ is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. Let us define $[0]_q = 0; [n]_q = 1 + q + \ldots + q^{n-1}, [n]_q! = \prod_{j=1}^{n} [j]_q$, with $[0]_q! = 1$ and

$$[n]_q = \frac{[n]_q!}{[n-k]_q!}$$

$n \geq k \geq 0$, otherwise.

It will be useful to use the so called $q$–Pochhammer symbol for $n \geq 1$:

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

$$(a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n,$$

with $(a; q)_0 = 1$. Often $(a; q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that for $|q| < 1$ we have $(q)_n = (1-q)^n [n]_q!$ and

$$[n]_{k, q} = \left\{ \begin{array}{ll} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0 \\
0 & , \quad otherwise 
\end{array} \right.$$ 

Notice that $[n]_1 = n, [n]_1! = n!, [n]_{k,1} = [k] = (a; 1)_n = (1-a)^n$ and $[n]_0 = \left\{ \begin{array}{ll} 1 & , \quad n \geq 1 \\
0 & , \quad n = 0 
\end{array} \right.$, $[n]_0! = 1$, $[n]_{k,0} = 1$, $(a; 0)_n = \left\{ \begin{array}{ll} 1 & , \quad n = 0 \\
1 - a & , \quad n \geq 1 
\end{array} \right.$.

In the sequel we shall also use the following useful notation:

$$S(q) = \left\{ \begin{array}{ll} -2 \frac{2}{\sqrt{1-q}} \sqrt{1-q} & , \quad |q| < 1 \\
2 & , \quad q = 1 
\end{array} \right.$$ 

$$I_A(x) = \left\{ \begin{array}{ll} 1 & , \quad x \in A \\
0 & , \quad x \notin A 
\end{array} \right.$$ 

1.3. Polynomials.

1.3.1. $q$–Hermite. Let $\{H_n(x|q)\}_{n \geq 0}$ denote the family of the so called $q$–Hermite (briefly qH) polynomials. That is the one parameter family of orthogonal polynomials satisfying the following three term recurrence:

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q),$$

with $H_{-1}(x|q) = 0$ and $H_0(x|q) = 1$. In fact in the literature (see e.g. [4], [9], [12]) function more often the re-scaled versions of these polynomials. Namely more often appear under the name of $q$–Hermite polynomials the following polynomials $\{h_n(x|q)\}_{n \geq 0}$ defined by their three term recurrence:

$$h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q),$$
with \( h_{-1} (x|q) = 0 \) and \( h_0 (x|q) = 1 \). These polynomials are related to one another by the relationship \( \forall n \geq -1: \)

\[
H_n (x|q) = \frac{n}{(1-q)^{n/2}} \frac{h_n (x\sqrt{1-q}/2|q)}{H_n (x|q)},
\]

for \( |q| < 1 \). For \( q = 1 \) \( h_n (x|1) = 2^n x^n \) while \( H_n (x|1) = H_n (x) \), where polynomials \( H_n (x) \) are the so called probabilistic Hermite polynomials i.e. monic polynomials orthogonal with respect to \( \exp (-x^2/2) \). Observe further that \( h_n (x|0) = U_n (x) \) where \( U_n \) denotes the so called Chebyshev polynomial of the second kind (for details see e.g. [1]).

The polynomials \( H_n \) have a nice probabilistic interpretation (see e.g. [17]) and besides constitute really the generalization of the ordinary Hermite polynomials. That is why we will use them in this paper. The results presented here can be easily adopted and expressed in terms of polynomials \( h_n \).

The generating function of these polynomials is given by the following formula that is in fact adapted to our setting formula (14.6.1) of [12]

\[
\varphi_H (x|\rho, q) = \sum_{n \geq 0} \frac{\rho^n}{\binom{n}{\lfloor n/2 \rfloor}} H_n (x|q) = \frac{1}{\prod_{j=0}^{\infty} v (x\sqrt{1-q}/2|q^{j} \sqrt{1-q})},
\]

convergent for \( |\rho(1-q)| < 1, x \in S (q) \), where we denoted

\[
v (x|t) = 1 - 2xt + t^2.
\]

Let us observe that \( \forall x \in [-1, 1], t \in \mathbb{R}: v (x|t) \geq 0. \)

Besides adapting formula (14.6.2) of [12] to our setting we have:

\[
\int_{S (q)} H_n (x|q) H_m (x|q) f_N (x|q) dx = [n|q] \delta_{mn},
\]

with

\[
f_N (x|q) = \frac{\sqrt{(1-q)(4-(1-q)x^2)}(q^{\infty})}{2\pi} \prod_{j=1}^{\infty} l (x\sqrt{1-q}/2|q^j),
\]

for \( x \in S (q) \), where

\[
l (x|a) = (1+a)^2 - 4ax^2.
\]

One shows that

\[
\lim_{q \rightarrow 1^-} f_N (x|q) = \frac{1}{\sqrt{2\pi}} \exp (-x^2/2),
\]

\[
\lim_{q \rightarrow 1^-} \frac{1}{\prod_{j=0}^{\infty} v (x|\rho q^j)} = \exp (x\rho - x^2/2).
\]

Apart from \( q \)–Hermite polynomials we will need the so called big \( q \)–Hermite (briefly bqH) polynomials \( \{H_n (x|a, q)\}_{n \geq 1} \) with \( a \in \mathbb{R} \). They are defined through their three term recurrence:

\[
H_{n+1} (x|a, q) = (x - aq^n) H_n (x|a, q) - [n|q] H_{n-1} (x|a, q),
\]

with \( H_{-1} (x|a, q) = 0, H_0 (x|a, q) = 1 \). To support intuition let us remark that \( H_n (x|a, 1) = H_n (x - a) \) and \( H_n (x|a, 0) = U_n (x/2) - aU_{n-1} (x/2) \).

\[i.e. \text{polynomials with leading coefficient equal to 1.}\]
One knows its relationship with the $q$–Hermite polynomials: 

$$H_n(x|a,q) = \sum_{k=0}^{n} \binom{n}{k} q^k (-a)^{k} H_{n-k}(x|q),$$

and that (see e.g. [12], (14.18.2) with an obvious modification for polynomials $H_n$):

$$\int_{S(q)} H_n(x|a,q) H_m(x|a,q) f_{bN}(x|a,q) \, dx = [n]_q! \delta_{mn},$$

where

$$f_{bN}(x|a,q) = f_N(x|q) \varphi_H(x|a,q).$$

We will need the following Lemma concerning another relationship between polynomials $H_n(x|q)$ and $H_n(x|a,q)$.

**Lemma 1.** Let us define for all $n \geq 0$: $x \in S(q); (1-q)t^2 < 1$ such that $H_n(x|t,q) = \sum_{j \geq 0} \frac{t^n}{[n]_q!} H_j(x|q)$. Then

$$\eta_n(x|t,q) = H_n(x|t,q) \varphi_H(x|t,q),$$

where $H_n(x|t,q)$ is the bqH polynomial defined by (1.17).

**Proof.** In a version with continuous $q$–Hermite polynomials $h$ defined by (1.8) and $h_n(x|t,q)$ the big $q$–Hermite polynomials as defined in [12] (14.18.4) this formula has been proved as a particular case in [20] (2.1). We notice that $\eta_0(x|t,q)$ is elementary.

Remark 1. Let us remark that Carlitz in [7] considered similar shifted characteristic functions of the form $\sum_{j \geq 0} \frac{t^n}{[n]_q!} w_{n+j}(x|q)$ with Rogers–Szegő polynomials $w_n$ (see discussion below following formula (2.3)). From this result of Carlitz one can also deduce assertion of Lemma 1.

1.3.2. Al-Salam–Chihara. Next family of polynomials that we are going to consider depends on 2 (apart from $q$) parameters denoted by $a$ and $b$, that satisfy the following three term recurrence (see e.g. [12], (14.8.4)):

$$A_{n+1}(x|a,b,q) = (2x-(a+b)q^n)A_n(x|a,b,q) - (1-abq^{n-1})(1-q^n)A_{n-1}(x|y,\rho,q),$$

with $A_{-1}(x|a,b,q) = 0$, $A_0(x|a,b,q) = 1$. These polynomials will be called Al-Salam–Chihara polynomials $\{A_n(x|a,b,q)\}_{n \geq -1}$ (briefly ASC). It follows from Favard’s theorem that the measure that makes these polynomials orthogonal is positive if $\forall n \geq 1 : (1-abq^{n-1})(1-q^n) \geq 0$, which for $|q| \leq 1$ is reduced to $|ab| \leq 1$.

In the sequel in fact we will consider these polynomials with complex parameters forming a conjugate pair and also re-scaled. Namely we will take $a = \frac{\sqrt{1-q}}{2}\rho (y - i\sqrt{1-y^2})$, $b = \frac{\sqrt{1-q}}{2}\rho (y + i\sqrt{1-y^2})$, with $y \in S(q)$ and $|\rho| < 1$. More precisely we will consider polynomials $\{P_n(x|y,\rho,q)\}_{n \geq 0}$ defined by:

$$A_n\left(x \frac{\sqrt{1-q}}{2}|a,b,q\right)/(1-q)^{n/2} = P_n(x|y,\rho,q).$$
One can easily notice that \( a + b = \rho y \sqrt{1 - q} \), \( ab = \rho^2 \) and thus that the polynomials \( P_n \) satisfy the following three term recurrence:

\[
P_{n+1}(x|y, \rho, q) = (x - \rho q^2) P_n(x|y, \rho, q) - [n]_q(1 - \rho^2 q^{n-1}) P_{n-1}(x|y, \rho, q),
\]

with \( P_1(x|y, \rho, q) = 0, \) \( P_2(x|y, \rho, q) = 1. \)

**Proposition 1.**

In the following sections many times. It has been formulated and proved in [20].

\[
\varphi_P(x|y, \rho, t, q) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} P_n(x|y, \rho, q) = \prod_{j=0}^{\infty} \frac{v(y \sqrt{1 - q}/2 | q \sqrt{1 - q})}{v(x \sqrt{1 - q}/2 | q \sqrt{1 - q})},
\]

convergent for \( |t\sqrt{1 - q}|, |\rho| < 1, x, y \in S(q). \)

We also have (see e.g. [17]) or:

\[
\int_{-1}^{1} P_n(x|y, \rho, q) P_m(x|y, \rho, q) f_{CN}(x|y, \rho, q) = \delta_{nm} [n]_q!(\rho^2)_n,
\]

where

\[
f_{CN}(x|y, \rho, q) = f_N(x|q) \prod_{j=0}^{\infty} \omega(x \sqrt{1 - q}/2, y \sqrt{1 - q}/2 | q \sqrt{1 - q}),
\]

with

\[
\omega(x, y|\rho) = (1 - \rho^2)^2 - 4\rho(1 + \rho^2)xy + 4\rho^2(x^2 + y^2).
\]

**Remark 2.**

To support intuition let us remark (following e.g. [17]) that \( P_n(x|y, \rho, 1) = H_n \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) (1 - \rho^2)^{n/2}. \) On the other hand \( P_n(x|y, \rho, 0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2)\) where \( U_n(x) \) denotes Chebyshev polynomial of the second kind.

It is known see e.g. [12], (formula (14.8.13) adapted to our setting), [6] or [17] that the polynomials \( P_n \) have the following generating function:

\[
\varphi_P(x|y, \rho, t, q) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} P_n(x|y, \rho, q) = \prod_{j=0}^{\infty} \frac{v(y \sqrt{1 - q}/2 | q \sqrt{1 - q})}{v(x \sqrt{1 - q}/2 | q \sqrt{1 - q})},
\]

We end up this section by recalling an auxiliary simple result that will be used in following sections many times. It has been formulated and proved in [20].

**Proposition 1.**

Let \( \sigma_n(\rho|q) = \sum_{j \geq 0} \frac{\rho^j}{[j]_q!} \xi_{n+j} \) for \( |\rho| < 1, -1 < q \leq 1 \) and certain sequence \( \{\xi_m\}_{m \geq 0} \) such that \( \sigma_n \) exists for every \( n. \) Then

\[
\sigma_n(\rho q^n|q) = \sum_{k=0}^{m} (-1)^k \left[ \frac{m}{k} \right]_q q^{\binom{k}{2}} (1 - q)^k \rho^k \sigma_{n+k}(\rho|q).
\]
Remark 4. Notice that this Proposition is trivially true also for both \( q = 0 \) and \( q = 1 \).

2. Main Results

One of our main interests in this paper are the generalizations of the Poisson-Mehler formula \((1.1)\).

It is well known that convergence in \((1.1)\) takes place for \( x, y \in S(q) \), \( |\rho| < 1 \) and for \( |q| < 1 \) is uniform. For \( q = 1 \) we have almost uniform convergence.

As a immediate corollary of Proposition 1 we have:

**Corollary 1.** For \( |q| < 1 \) we have:

\[
\begin{align*}
(2.1) \quad \gamma_{i,j}(x, y|\rho q^m, q) &= \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} q(k)(1-q)^{k} \rho^{k} \gamma_{i+k,j+k}(x, y|\rho, q), \\
(2.2) \quad H_{i}(x|q) H_{j}(y|q) &= \sum_{k \geq 0} (-1)^{k} q(k) \rho^{k} \gamma_{i+k,j+k}(x, y|\rho, q),
\end{align*}
\]

where \( \gamma_{i,j}(x, y|\rho, q) \) is defined by \((1.3)\). \((2.2)\) is also true trivially for \( q = 1 \).

**Proof.** First assertion we get by applying directly \((1.23)\) by setting \( \sigma_{i,j} = \gamma_{i,j} \). Second assertion we get by passing in the first one with \( m \)

where \( \gamma_{i,j}(x, y|\rho, q) \) is defined by \((1.3)\). It was shown in \([17]\) that for all \(-1 < q \leq 1, |\rho| < 1, x, y \in \mathbb{R}\):

\[
(2.3) \quad Q_{i,j}(x, y|\rho, q) = \sum_{s=0}^{j} (-1)^{s} q(s) \rho^{s} H_{j-s}(y|q) P_{i+s}(x|y, \rho, q) / (\rho^{2})^{i+s},
\]

and \( Q_{i,j}(x, y|\rho, q) = Q_{j,i}(y, x|\rho, q) \).

**Remark 5.** It has to be remarked that Carlitz in \([17]\) considered the sum \( \xi_{k,j}(x, y|\rho, q) = \sum_{n \geq 0} \frac{d^{n}}{(q)^{n}} w_{n+k}(x|q) w_{n+j}(y|q) \), where \( w_{n}(x|q) \) are the so called Rogers–Szegö polynomials related to polynomials \( h_{n}(x|q) \) by the formula: \( h_{n}(x|q) = e^{n\theta} w_{n}(e^{-2i\theta}|q) \) with \( x = \cos \theta, i-\text{imaginary unit} \). Indeed it turned out that functions \( \xi_{k,j} \) also have the property that

\[
\xi_{k,j}(x, y|\rho, q) = \nu_{k,j}(x, y|\rho, q) \xi_{0,0}(x, y|\rho, q),
\]

where \( \nu_{k,j} \) are polynomials of degree \( k + j \) in \( x \) and \( y \). However to show that \( \nu_{k,j}(e^{-id}, e^{-im}|\rho, q) \) can be expressed as \( Q_{k,j}(\cos \theta, \cos \eta|\rho, q) \) is not an easy task. Discussion on this subject is in \([18]\). In particular see the proof of Proposition 5.

In particular we have

\[
(2.4) \quad Q_{k,0}(x, y|\rho, q) = P_{k}(x|y, \rho, q) / (\rho^{2})_{k}.
\]

To analyze further properties of polynomials \( Q_{k,j} \) let us introduce the following 2 dimensional density defined for \( S^{2}(q) \)

\[
(2.5) \quad f_{2D}(x, y|\rho, q) = f_{CN}(x|y, \rho, q) f_{N}(y|q).
\]
Measure that has density \( f_{2D} \) will be called \((\rho, q)-\text{bivariate Normal (briefly (\rho, q)-2N)}\). Obviously \( f_{2D}(x, y|\rho, q) = \gamma_{0,0}(x, y|\rho, q) f_N(x|q) f_N(y|q) \). Its applications in theories of probability and Markov stochastic processes have been presented in \([14]\) and \([15]\).

Here below we give another interpretation of the polynomials \( Q_{n,m} \) in particular its connection with the big \( q \)-Hermite polynomials.

**Proposition 2.** For \(|q| < 1, |\rho| < 1, x, y \in \mathbb{R}\) we have:

i) \( \forall i, j, m, k, i + j \neq m + k : \)

\[
\int_{S(q)} Q_{i,j} (x, y|\rho, q) Q_{m,k} (x, y|\rho, q) f_{2D} (x, y|\rho, q) \, dx \, dy = 0,
\]

\[
\int_{S(q)} Q_{n-j,j} (x, y|q) Q_{n-k,k} (x, y|\rho, q) f_{2D} (x, y|\rho, q) \, dx \, dy =
\]

\[
(-1)^{k-j} \frac{\rho^{k-j} q_h^{(2)}}{(\rho^2)_n} \sum_{s=0}^{\infty} q^{(s-1)+ns} \left[ \frac{k}{k-j+s} \right] \left[ \frac{n-j+s}{s} \right] \rho^{2s} (\rho^2 q^{n-s})_{j-s}.
\]

ii) \( \sum_{n,m \geq 0} n^m q^{mn} Q_{n,m} (x, y|\rho, q) = \frac{f_{BH}(x, t|q) f_{BH}(y, s|q)}{f_{2D}(x, y|\rho, q)} \sum_{k \geq 0} \rho^k H_k (x|t, q) H_k (y|s, q), \)

where function \( f_{BH} \) is defined by \([1, 18]\). The above mentioned formulae are also true for \( q = 1 \).

iii) \( \forall m \geq 0 : \)

\[
S_{n} \frac{m^{-1} \omega (x \sqrt{1-q/2}, y \sqrt{1-q/2} | \rho q^i)}{\sum_{k=0}^{\infty} (-1)^k \frac{m!}{k!} q^{(2)} (1-q)^k \rho^k Q_{i+k,k+k} (x, y|\rho, q)},
\]

where polynomial \( \omega \) is defined by \([1, 22]\). In particular we have:

\[
\prod_{j=0}^{n-1} \omega (x \sqrt{1-q/2}, y \sqrt{1-q/2} | \rho q^i) = \rho^2 \frac{\sum_{k=0}^{\infty} (-1)^k \frac{n!}{k!} q^{(2)} (1-q)^k \rho^k Q_{k,k} (x, y|\rho, q)},
\]

and

\[
q^{(2)} \rho^n (1-q)^n Q_{n,n} (x, y|\rho, q) = \sum_{k=0}^{\infty} (-1)^k q^{(n-k)} \left[ \frac{n!}{k!} \right] \prod_{j=0}^{k-1} \omega (x \sqrt{1-q/2}, y \sqrt{1-q/2} | \rho q^i),
\]

with understanding that \( \prod_{j=0}^{k-1} \) for \( k = 0 \) is equal to 1.

**Proof:** Is shifted to section \([4]\). \( \square \)

Our main results follow in fact directly the results presented above.

**Theorem 1.** Either for \(|q| < 1; x, y \in S(q); |\rho| < 1 \) we have:

i) \( H_i (x|q) H_j (y|q) \prod_{k=0}^{\infty} \omega (x \sqrt{1-q/2}, y \sqrt{1-q/2} | \rho q^k) \frac{(\rho^2)^k}{(\rho^2)_\infty} = \sum_{k=0}^{\infty} (-1)^k q^{(2)} \frac{\rho^k}{k!} Q_{i+k,j+k} (x, y|\rho, q). \)
In particular we get:

\[ 1/ \sum_{n \geq 0} \rho^n [n]_q! H_n (x|q) H_n (y|q) = \prod_{k=0}^{\infty} \frac{\omega \left( x \sqrt{1-q}/2, y \sqrt{1-q}/2 | \rho^k \right)}{(\rho^2)_\infty \omega (1, 1 | \rho^k)} = \sum_{k=0}^{\infty} (-1)^k q^{(\frac{k}{2})} \sum_{|k|_q} Q_{k,k} \left( x, y | \rho, q \right). \]

(2.10)

The last formula is valid also for \( x, y \in \mathbb{R} \), \( q = 1 \) and \( |\rho| < 1/2 \).

**Proof.** To get i) we pass in (2.6) with \( m \to \infty \) noting by (2.8) and (1.20) that \( Q_{n,m}(x,y;0,q) = H_n (x|q) H_m (y|q) \). On the way we observe that \( \lim_{m \to \infty} [n]_q! \frac{1}{(q)_n} = (1-q)^{-k} [k]_q! \). As far as the case \( q = 1 \) is concerned denote by \( g_N(x,y,\rho) \) density of the bivariate Normal density with parameters \( \sigma_1 = \sigma_2 = 1 \), correlation coefficient \( \rho \). Then notice that function \( \exp(-\frac{1}{2}(x^2+y^2)) / g_N(x,y,\rho) \) is square integrable on the plane with respect to \( g_N(x,y,\rho) \) if \( |\rho| < 1 \). \( \square \)

3. Open problems and comments

**Remark 6.** The non-symmetric kernels constructed of \( bqH \) polynomials were given in [13]. Formula ii) of Proposition 2 gives its new interpretation. Besides, recall that these kernels were expressed using basic hypergeometric function \( _3\phi_2 \). Expansion on the left hand side of Proposition 2ii) gives new outlook on the properties of this function.

Notice also that for \( q = 1 \) we have \( \eta(x|t,1) = \exp(x/t - t^2/2) \), \( H_n (x|t,1) = H_n (x-t) \) and

\[ \sum_{n \geq 0} \frac{\rho^n}{n!} H_n (x) H_n (y) = \exp\left(\frac{x^2}{2} - \frac{(x-\rho y)^2}{2(1-\rho^2)}\right), \]

hence characteristic function of polynomials \( Q_{i,j} \) can be calculated explicitly.

Similarly for \( q = 0 \) we have \( \eta(x|t,0) = 1 - xt + t^2/2 \) (characteristic function of the Chebyshev polynomials) and \( H_n (x|t,0) = U_n (x/2) - tU_{n-1} (x/2) \) (see e.g. [19]) hence also in this case we can get explicit form of the characteristic function of polynomials \( Q_{i,j} \).

**Remark 7.** First of all notice that the left hand side of (2.10) is equal to \( 1/\gamma_0,0 (x, y|\rho, q) = f_N(x|q) / f_{CN}(x|y,\rho, q) \) and that it is a symmetric (with respect to \( x \) and \( y \)) function. In [16] there was presented (formula 5.3) an expansion of this function involving polynomials \( P_n \) and certain polynomials related to \( q-Hermite \) ones. The expansion was non-symmetric for every partial sum. Thus we get another expansion of known important special function.

**Remark 8.** Assertion i) of Proposition 2 states that polynomials \( Q_{n,m} \) and \( Q_{i,j} \) are orthogonal with respect to the two dimensional measure \( \mu_{2D} \) with the density given by (2.5) if only the \( n + m \neq i + j \). Let us define space \( \mathcal{L} = L_2 (S^2(q), B, \mu_{2D}) \) of functions \( f : S^2(q) \to \mathbb{R} \) square integrable with respect to the measure \( \mu_{2D} \). Do polynomials \( Q_{m,n} \) constitute a base of this space? It seems that yes. We can define subspaces of \( \Lambda_m = \text{span} \{Q_{m,0}, \ldots, Q_{0,m}\} \) of polynomials that are linear combinations of polynomials \( Q_{i,j} \) such that \( i + j = m \). Subspaces \( \Lambda_m \) are mutually orthogonal. Besides following argument that polynomials are dense in \( \mathcal{L} \) we deduce
that $\mathcal{L} = \bigoplus_{n=0}^{\infty} \Lambda_n$. What is the orthogonal base of $\mathcal{L}$? We have calculated covariances between polynomials $Q_{i,j}$ from $\Lambda_m$ following (1.21) and (2.3). Thus we can follow Gram-Schmidt orthogonalization procedure within the spaces $\Lambda_m$. Is the union of orthogonal bases of $\Lambda_m$ an orthogonal base of $\mathcal{L}$? Again it seems that yes. It would be interesting to find this base.

**Remark 9.** In 2001 Wünsche in [22] considered Hermite and Laguerre polynomials on the plane. He has not however related his Hermite polynomials to any particular measure on the plane. In particular he defined Hermite polynomials depending on parameters forming a $2 \times 2$ matrix. This matrix is however not connected in any way to the covariance matrix of the measure with respect to which these polynomials are supposed to be orthogonal.

On the other hand definition of polynomials $Q_{i,j}$ depends heavily on the measure with the density $f_{2D}$. For $q = 1$ following (2.3), we have

$$Q_{i,j} (x, y|\rho, 1) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} H_{j-k} (y) H_{k+i} \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) / \left( \sqrt{1 - \rho^2} \right)^{k+i}.$$

Hence polynomials $Q_{i,j} (x, y, \rho, 1)$ are in fact another (different from that of Wünsche’s) family of two dimensional generalization of Hermite polynomials.

### 4. Proofs

**Proof of Proposition 2**

i) We use (2.3), assume that $i > m$. We have:

$$\int_{S^2(\rho)} Q_{i,j} (x, y|q) Q_{m,k} (x, y|\rho, q) f_{2D} (x, y|\rho, q) \, dx \, dy =$$

$$\sum_{s=0}^{j} \sum_{t=0}^{k} (-1)^{s+t} q^j \binom{s}{q} \binom{k}{t} \rho^{s+t} \frac{1}{(\rho^2)^{s+t}} \int_{S(q)} H_{j-s} (y) H_{k-t} (y|q) f_N (y|q) \, dy \times$$

$$\int_{S(q)} P_{i+s} (x|y, \rho, q) P_{m+t} (x|y, \rho, q) f_{2D} (x|y, \rho, q) \, dx \, dy =$$

$$(-1)^{i-m} \rho^{i-m} \sum_{s=0}^{j} \sum_{t=0}^{k} q^j \binom{s}{q} \binom{k}{t} \rho^{s+t} \int_{S(q)} H_{j-s} (y) H_{k-t} (y|q) f_N (y|q) \, dy = 0,$$

if $j - s \neq k + m - i - s$ i.e. if $j + i \neq k + m$. 


Assuming that \( k \geq j \) we get

\[
\int_{S^2(q)} Q_{n-j,j} (x,y|q) Q_{n-k,k} (x,y|\rho, q) f_{2D} (x,y|\rho, q) \, dx dy =
\]

\[
\sum_{s=0}^{j} \sum_{t=0}^{k} (-1)^{s+t} q^{(s+1)} (\gamma) \left( \begin{array}{c} j \\ s \end{array} \right) \left( \begin{array}{c} k \\ t \end{array} \right) \rho^{s+t} \frac{1}{(\rho^2)_{n-j+s} (\rho^2)_{n-k+t}} \int_{S(q)} H_{j-s} (y) H_{k-t} (y|q) f_N (y|q) \, dx dy =
\]

\[
\times \int_{S(q)} P_{n-j+s} (x|\rho, q) P_{n-k+t} (x|\rho, q) f_{C_N} (x|\rho, q) f_{2D} (x,y|\rho, q) \, dx dy =
\]

\[
(-1)^{k-j} \rho^{k-j} \sum_{s=0}^{j} q^{(s+1)} (\gamma) \left( \begin{array}{c} j \\ s \end{array} \right) \sum_{t=0}^{k} q^{(t+1)} (\gamma) \left( \begin{array}{c} k \\ t \end{array} \right) \rho^{s+t} \frac{[n-j+s]_q!}{(\rho^2)_{n-j+s} [j-s]_q!} \int_{S(q)} H_{j-s} (y) H_{k-t} (y|q) f_N (y|q) \, dy
\]

we use here \( s(s-1)/2 + (s+n)(s-1+n)/2 - s(s-1) - n(n-1)/2 = ns \)

ii) We have

\[
\sum_{i \geq 0, j \geq 0} \frac{s^t j^i}{[i]_q! [j]_q!} \gamma_{i,j} (x,y|\rho, q) = \frac{1}{\gamma_{0,0} (x,y|\rho, q)} \sum_{i \geq 0, j \geq 0} \frac{[i]_q! [j]_q!}{n^i q^n} \sum_{n \geq 0} \rho^n H_{i+n} (x|q) H_{n+j} (y|q)
\]

\[
= \sum_{n \geq 0} \rho^n \sum_{j=0}^{\infty} \frac{s^t j^i}{[j]_q!} H_{j+s} (y|q) \sum_{j=0}^{\infty} \frac{t^i}{[i]_q!} H_{n+j} (x|q).
\]

Now we use Lemma 1 twice and get

\[
\sum_{i \geq 0, j \geq 0} \frac{s^t j^i}{[i]_q! [j]_q!} \gamma_{i,j} (x,y|\rho, q) = \frac{\varphi_H (x|t,q) \varphi_H (y|s,q)}{\gamma_{0,0} (x,y|\rho, q)} \sum_{n \geq 0} \rho^n H_n (x|t,q) H_n (y|s,q)
\]

\[
= \frac{1}{(\rho^2)^\infty} \prod_{j=0}^{\infty} \omega (x^2 \sqrt{1-\frac{q}{2}}, y^2 \sqrt{1-\frac{q}{2}}(pq)^i) v (x^2 \sqrt{1-\frac{q}{2}}, y^2 \sqrt{1-\frac{q}{2}}(pq)^i) \sum_{n \geq 0} \rho^n H_n (x|t,q) H_n (y|s,q).
\]

iii) First we notice that from (1.3) it follows that for \( x, y \in S(q) ; \rho^2 < 1, -1 < q \leq 1 \):

\[
\gamma_{i,j} (x,y|\rho q^m, q) = Q_{i,j} (x,y|\rho q^m, q) \prod_{l=0}^{m} \omega (x^2 \sqrt{1-\frac{q}{2}}, y^2 \sqrt{1-\frac{q}{2}}(pq)^l) \gamma_{0,0} (x,y|\rho, q),
\]

and also that \( \gamma_{0,0} (x,y|\rho, q) = \prod_{l=0}^{m} \omega (x^2 \sqrt{1-\frac{q}{2}}, y^2 \sqrt{1-\frac{q}{2}}(pq)^l) \gamma_{0,0} (x,y|\rho, q) \). Then we apply (2.1) to \( \gamma_{i,j} \) above and then use (1.22) and cancel out \( \gamma_{0,0} \) on both sides of (2.4). Finally we observe that on both sides we have polynomials hence one can extend the identity for all values of the variables. To get other formula of this assertion we argue by induction checking that the equality is true for \( n = 0 \). Then we put (2.8) into (2.9).
and get:

\[
\sum_{k=0}^{n} (-1)^{k} q^{\binom{n-k}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\binom{k-1}{2}} \frac{1}{(\rho^2)_{2k}} \sum_{i=0}^{k-1} \omega \left( x \sqrt{1-q/2}, y \sqrt{1-q/2} \right) \phi_i^{(1)} = \\
\sum_{k=0}^{n} (-1)^{k} q^{\binom{n-k}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] \sum_{j=0}^{k} (-1)^{j} \left[ \begin{array}{c} k \\ j \end{array} \right] q^{\binom{j}{2}} \rho^{j} Q_{j,j}(x, y | \rho, q) = \\
\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} \left[ \begin{array}{c} n \\ j \end{array} \right] \rho^{j} Q_{j,j}(x, y | \rho, q) \sum_{k=j}^{n} (-1)^{k} q^{\binom{n-k}{2}} \left[ \begin{array}{c} n-j \\ k-j \end{array} \right] q^{\binom{m+j}{2}} \left[ \begin{array}{c} n-j \\ m \end{array} \right] = \\
q^{\binom{j}{2}} \rho^{n} (1-q)^{n} Q_{n,n}(x, y | \rho, q)
\]

since \( \forall n \geq 1 : \sum_{i=0}^{n} (-1)^{i} q^{\binom{i}{2}} \left[ \begin{array}{c} n \\ i \end{array} \right] q = 0. \)

\[\square\]

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