Non-linear partially massless symmetry in an SO(1,5) continuation of conformal gravity

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Abstract
We construct a non-linear theory of interacting spin-2 fields that is invariant under the partially massless (PM) symmetry to all orders. This theory is based on the SO(1, 5) group, in analogy with the SO(2, 4) formulation of conformal gravity, but has a quadratic spectrum free of ghost instabilities. The action contains a vector field associated with a local SO(2) symmetry which is manifest in the vielbein formulation of the theory. We show that, in a perturbative expansion, the SO(2) symmetry transmutes into the PM transformations of a massive spin-2 field. In this context, the vector field is crucial to circumvent earlier obstructions to an order-by-order construction of PM symmetry. Although the non-linear theory lacks enough first class constraints to remove all helicity-0 modes from the spectrum, the PM transformations survive to all orders. The absence of ghosts and strong coupling effects at the non-linear level are not addressed here.

Keywords: classical theories of gravity, gauge symmetries, conformal gravity

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that local gauge symmetries are crucial for the consistency of the Standard Model of particle physics by ensuring its unitarity and renormalizability. In contrast, the Einstein–Hilbert action does not admit additional gauge symmetries besides the usual diffeomorphisms. Hence it is natural to search for extensions of general relativity with extra gauge symmetries in the hope of improving the quantum behaviour of the theory. Two such theories that have attracted considerable attention are conformal gravity, which is invariant under local...
scale transformations, and partially massless gravity (see below for details). However, while a nonlinear action for conformal gravity is known, it contains higher-order time derivatives and therefore an Ostrogradsky ghost instability. On the other hand, a theory of spin-2 fields with a ‘partially massless’ gauge symmetry is not known to exist beyond the linear level. The goal of this paper is to construct a nonlinear theory with partially massless symmetry that is also related to conformal gravity but avoids its ghost instability, at least to linear order.

In more detail, let us recall that spin-2 fields in de Sitter space fall into one of three representations of its isometry group corresponding to massless, massive, or partially massless (PM) fields [1–7]. The latter are described by the linear Fierz–Pauli theory [8] where the mass of the spin-2 field saturates the Higuchi bound [9],

$$m^2 = \frac{2}{d-1} \Lambda. \tag{1.1}$$

Here $\Lambda$ is the cosmological constant and $d$ is the dimension of spacetime. At this point in parameter space a new gauge symmetry emerges where the spin-2 field $\varphi_{\mu\nu}$ transforms as

$$\delta \varphi_{\mu\nu} = \left( \nabla_{\nu} \nabla_{\mu} + \frac{2}{(d-2)(d-1)} \Lambda \bar{g}_{\mu\nu} \right) \xi(x), \tag{1.2}$$

and $\bar{g}_{\mu\nu}$ is the de Sitter metric. This local symmetry is responsible for removing the helicity-0 component of $\varphi_{\mu\nu}$. Thus, in four dimensions partially massless fields propagate 4 instead of the 5 degrees of freedom that characterize a massive spin-2 field. Going beyond the linear equations, one may ask if an interacting theory exists that is invariant under a generalization of the above transformation.

In general, non-linear ghost-free theories exist for interacting massless and massive spin-2 fields—namely general relativity, massive [10–13], and multi-metric gravity [14–16] (for reviews see [17–19]). However, these theories do not admit extra gauge symmetries and no unitary, non-linear theory of partially massless fields is known. Although specific subclasses of massive [20], bimetric [21–24] and multimetric gravity [25] exhibit interesting PM features, several no-go results suggest that non-linear theories of PM fields containing at most two derivatives do not exist [26–31]. In particular, in the massive gravity approach—with a single PM field in a fixed background—an order by order construction in powers of $\varphi_{\mu\nu}$ encounters obstructions that prevent extension of the PM symmetry (1.2) beyond cubic order [26, 28]. A possible way to circumvent this obstruction is to enlarge the spectrum of the theory with an additional massless spin-2 field that transforms non-trivially under the PM symmetry, as in the bimetric approach. However, as shown in [31, 32], this additional spin-2 field is not sufficient to circumvent the aforementioned obstruction and the PM symmetry cannot be extended beyond terms that are cubic in $\varphi_{\mu\nu}$. This result suggests that additional fields of lower or higher spin are necessary.

In this paper we show that the obstruction to non-linear PM symmetry can be circumvented in the presence of an additional vector field. Indeed, starting from a non-linear action and expanding it in powers of $\varphi_{\mu\nu}$, we show that the PM symmetry can be extended to all orders in the fields. The construction of this theory is motivated by conformal (Weyl) gravity, a non-linear theory featuring both massless and partially massless fields [33]. In conformal gravity the kinetic term of the PM field comes with the wrong sign, which reflects the higher-derivative, non-unitary nature of the theory. A naive analytic continuation of the PM fields $\varphi_{\mu\nu} \rightarrow i\varphi_{\mu\nu}$ will render the quadratic theory ghost free, but will introduce imaginary couplings at odd orders. Given the continued interest in conformal gravity [34, 35], and its close relationship to partial masslessness [24, 32, 33, 36], it proves useful to make the notion of such an ‘analytic continuation’ more precise. For this we recall that in [37], conformal gravity was constructed.
as a gauge theory of the conformal group \( SO(2, 4) \), mirroring a similar construction of Einstein gravity based on \( SO(1, 4) \) [38]. In this approach the spin-2 ghost modes are correlated with the \( (2, 4) \) signature of the group manifold, suggesting that a similar construction based on the \( SO(1, 5) \) group could lead to a better behaved theory. Interestingly, it has been pointed out in [32] that, perturbatively, candidate bimetric PM theories exhibit an \( SO(1, 5) \) global symmetry.

Motivated by these considerations, our starting point in this paper is a gravity action based on the \( SO(1, 5) \) group\(^1\). Our approach is closely related to that of conformal gravity [37], but supplemented by additional elements required by symmetries. Although these elements seem to originate from an extension of the \( \mathfrak{so}(1, 5) \) algebra, in this work we keep to the minimal setup. The outcome is a two-derivative theory that avoids the earlier no-go results and realizes the partially massless symmetry to all orders. While the absence of ghosts at the non-linear level and strong coupling issues are not addressed here, we show that the theory is ghost free at the linear level. Furthermore, we point out that similar kinetic terms have been shown to feature nonlinear constraints that remove propagating modes beyond linear order [44, 45]. Hence it is possible that the \( SO(1, 5) \) theory remains ghost free to all orders.

In more detail, the \( SO(1, 5) \) theory contains two vielbeins \( e^a_\mu \) and \( t^a_\mu \), a Lorentz spin connection \( \omega_{\mu}^{ab} \), and a gauge field \( A_\mu \). The vielbeins lead to two metrics invariant under local Lorentz transformations whose perturbations correspond to linear combinations of a massless and a massive spin-2 field—the minimum field content required for a PM theory coupled to gravity. Generating dynamics without breaking symmetries of the action requires promoting \( \omega_{\mu}^{ab} \) to the spin connection of the complexified Lorentz algebra. Altogether these fields lead to a bimetric theory with non-standard kinetic terms and the specific bimetric potential considered in [21, 22] in connection with PM symmetry.

The key feature of the \( SO(1, 5) \) theory is the additional vector field and its local \( SO(2) = U(1) \) symmetry under which the two vielbeins of the theory are charged. We show that, when re-expressed in terms of canonical spin-2 fields, the gauge transformations of the vector field and the vielbeins are transmuted into the partially massless symmetry of a massive spin-2 field. At linear order the theory describes massless and partially massless spin-2 fields, as well as a massless vector field, along with their respective gauge symmetries, denoted here by \( \text{Diff} \times \text{PM} \times U(1) \). In contrast to conformal gravity, the partially massless field is not a ghost. Furthermore, while the \( \text{Diff} \) symmetry is present to all orders, only the diagonal \( U(1) \) part of the \( \text{PM} \times U(1) \) gauge symmetry is present non-linearly, a fact that is manifest in the vielbein formulation of the theory. This means that the theory loses one of the first class constraints present at linear order, which suggests that not all of the helicity-0 modes decouple.

We call this theory ‘partially massless’ in the sense that it circumvents earlier no-go results and yields an action that is invariant under PM transformations (1.2) to all orders. Consequently, the relation (1.1) between the mass of the spin-2 field and the cosmological constant is preserved. While we succeed in extending the PM symmetry to all orders, the massive spin-2 field does not necessarily carry 4 polarizations\(^2\). This does not mean that the partially massless symmetry is trivially realized, e.g. as in the Stückelberg trick. To the contrary, this is how the massive field must transform in order to render the theory invariant under local \( SO(2) \) transformations. It is natural to

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\(^1\) Our theory is the simplest in a family of four-dimensional theories of charged gravity in de Sitter space. The latter may be realized as gauge theories of \( SO(1, 3) \times SO(n) \subset SO(1, 3 + n) \) mirroring similar constructions for Einstein gravity based on \( SO(1, 4) \) [38], and conformal gravity based on \( SO(2, 4) \) [37]. Note that analogous constructions exist for Einstein and conformal gravity in three dimensions [39–41]. Our approach is also related to that of [42, 43] that construct three-dimensional theories of colored gravity featuring partially massless fields at linearized order.

\(^2\) This may be justified on general grounds: since spin-2 fields with 4 helicities arise only in de Sitter backgrounds, it is expected that a non-linear background independent theory must accommodate massive spin-2 fields (represented as rank-2 symmetric tensors) with 5 polarizations.
conjecture that the partially massless symmetry is a generic feature of theories with charged metrics/vielbeins. Indeed, a similar phenomenon is observed in three dimensions in [42, 43].

The paper is organized as follows. In section 2 we construct a gauge theory for the $SO(1, 3)$ subgroup of $SO(1, 5)$ that admits an additional local $SO(2)$ symmetry. In particular, we discuss the constraints that must be obeyed by the vielbeins and the spin connection of the theory. In section 3 we consider the perturbative metric formulation of the theory up to quadratic order in the massive spin-2 field. Therein we show how the local $SO(2)$ symmetry of the vielbeins is transmuted into the partially massless symmetry of the massive graviton. We end with our conclusions and outlook in section 4 where we also comment on the non-abelian generalization of the partially massless symmetry. In appendix we present a generalization of the $so(1, 5)$ algebra that makes our construction more systematic.

2. A gauge theory based on SO(1, 5)

In this section we construct a gauge theory for $SO(1, 3) \times SO(2) \subset SO(1, 5)$. We begin by identifying the appropriate gauge fields, their transformations properties, and their associated field strengths. We then propose an action and the additional constraints necessary to recover the partially massless symmetry in the metric formulation of the theory.

2.1. Fields and curvatures

Our starting point is the $SO(1, 5)$ group. This group is the global symmetry group of the linear theory of a massless and a partially massless spin-2 fields [32]. It is closely related to the conformal group, $SO(2, 4)$, which is the global symmetry group of conformal gravity. Since conformal gravity can be obtained as a gauge theory based on $SO(2, 4)$, it is natural to ask whether a similar construction leads to a non-linear theory with partially massless symmetry. While this turns out to be the case, the analogy is not a perfect one since our theory does not admit a global $SO(1, 5)$ symmetry at non-linear order. What singles out $SO(1, 5)$ is that it admits the direct product of the Lorentz and $SO(2)$ groups as a subgroup. While the former is characteristic of the vielbein formulation of gravitational theories, the latter will be important in the realization of the partially massless symmetry to all orders.

The generators $J_{AB}$ of the $so(1, 5)$ algebra are characterized by the commutation relations,

$$[J_{AB}, J_{CD}] = \eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC},$$

where $A, B \in \{0, \ldots, 5\}$ and $\eta_{AB}$ is the Minkowski metric with signature $(-, +, +, +, +, +)$. It will be convenient to work with a basis where the $so(1, 3) \oplus so(2)$ subalgebra is manifest. Introducing the following notation,

$$P^{(1)}_a = J_{at}, \quad P^{(2)}_a = J_{as}, \quad D = J_{55},$$

where $a, b \in \{0, \ldots, 3\}$, the non-vanishing commutators of the $so(1, 5)$ algebra read,

$$[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac},$$

$$[J_{ab}, P^{(i)}_a] = \eta_{bc}P^{(i)}_c - \eta_{ac}P^{(i)}_b,$$

$$[P^{(i)}_a, P^{(j)}_b] = \epsilon^{ij}\eta_{ab}D - \delta^{ij}J_{ab},$$

$$[D, P^{(i)}_a] = \epsilon^{ij}P^{(j)a},$$
where \( i, j \in \{1, 2\} \), \( \delta_{ij} \) is a Euclidean metric, and \( \epsilon^{12} = -\epsilon^{21} = -1 \). Here the \( a, b \) indices are naturally interpreted as tangent space indices in the vielbein formulation of a gravitational theory while the \( i, j \) indices label vectors of \( SO(2) \).

Following Kaku, Townsend, and Nieuwenhuizen [37] (see also [38, 46–48]) we parametrize the \( SO(1, 5) \) gauge field by the following one-form,

\[
A = \frac{1}{2} \omega^{ab} J_{ab} + \ell^{-1} \epsilon^a P_a^{(1)} + \ell^{-1} \rho^a P_a^{(2)} + AD.
\]  

(2.7)

Here \( \omega^{ab} \) and \( A \) have dimensions of energy while \( \epsilon^a \) and \( \rho^a \) are dimensionless, which explains the presence of the length scale \( \ell \). In order to see that \( \omega^{ab} \) plays the role of the spin connection, while \( \epsilon^a \) and \( \rho^a \) play the role of vielbeins, let us consider their behaviour under infinitesimal \( SO(1, 5) \) transformations\(^3\). If we parametrize the latter by the scalar,

\[
\lambda = \frac{1}{2} \Lambda^{ab} J_{ab} + \chi^a P_a^{(1)} + \zeta^a P_a^{(2)} + \xi D,
\]  

(2.8)

then, under infinitesimal gauge transformations of the form \( \delta_{\lambda} A = d\lambda + [\lambda, A] \), we find,

\[
\delta_{\lambda} \omega^{ab} = D_\omega \Lambda^{ab} + 2\ell^{-1} \chi^{[a} \epsilon^{b]} + 2\ell^{-1} \zeta^{[a} \rho^{b]},
\]  

(2.9)

\[
\delta_{\lambda} \epsilon^a = -\Lambda^a_{\mu} \epsilon^\mu - \xi \rho^a + \ell D_\omega \chi^a + \ell \zeta^a A,
\]  

(2.10)

\[
\delta_{\lambda} \rho^a = -\Lambda^a_{\mu} \rho^\mu + \zeta \epsilon^a + \ell D_\omega \zeta^a - \ell \chi^a A,
\]  

(2.11)

\[
\delta_{\lambda} A = d\xi + \ell^{-1} \chi^a \rho^a - \ell^{-1} \zeta \epsilon^a.
\]  

(2.12)

In these equations we (anti)symmetrize indices with unit weight, e.g. \( \zeta^{[a} \epsilon^{b]} = \frac{1}{2} (\zeta^a \epsilon^b - \zeta^b \epsilon^a) \) and \( D_\omega \) denotes the covariant derivative with respect to the spin connection \( \omega^{ab} \), e.g. \( D_\omega \zeta^a = d\zeta^a + \omega^a_{\mu} \zeta^\mu \).

From equation (2.9) we see that, as expected, \( \omega^{ab} \) transforms as the spin connection under Lorentz transformations. Furthermore it is left invariant under the \( SO(2) \) transformations generated by \( D \). On the other hand, from equations (2.10) and (2.11) we see that both \( \epsilon^a \) and \( \rho^a \) transform homogeneously under Lorentz transformations which motivates their identification as vielbeins. We can also see that these vielbeins form a vector under \( SO(2) \) transformations, i.e.

\[
\delta_{\xi} \left( \begin{array}{c} \epsilon^a \\ \rho^a \end{array} \right) = \left( \begin{array}{cc} 0 & -\xi \\ \xi & 0 \end{array} \right) \left( \begin{array}{c} \epsilon^a \\ \rho^a \end{array} \right) = \xi \left( \begin{array}{c} \epsilon^a \\ \rho^a \end{array} \right).
\]  

(2.13)

In particular, from the \( \epsilon^a \) and \( \rho^a \) vielbeins we can define the following charged metrics which are invariant under local Lorentz transformations\(^4\),

\[
\mathcal{g}_{\mu\nu} = \epsilon^\mu_{\nu} e_{\mu}, \quad \mathcal{f}_{\mu\nu} = \rho^\mu_{\nu} t_{\mu}.
\]  

(2.14)

We can also define a metric that is invariant under both local Lorentz and \( SO(2) \) transformations, namely,

\[
G_{\mu\nu} = \epsilon^\mu_{\nu} e_{\mu} + \rho^\mu_{\nu} t_{\mu}.
\]  

(2.15)

\(^3\)To simplify the notation we express most of our equations using forms, e.g. \( \epsilon^a = \epsilon^a_{\nu} dx^\nu \) and \( \rho^a = \rho^a_{\nu} dx^\nu \) denote vielbein one-forms. However, we will refer to \( \epsilon^a \) and \( \rho^a \) simply as vielbeins.

\(^4\)The presence of two sets of vielbeins and two metrics suggests that the theory describes two interacting spin-2 degrees of freedom, as desired. Whether these fields are propagating, i.e. weakly coupled on the de Sitter background, depends on the choice of spin connection as discussed in detail in the following section.
Note that $A$ transforms as an $SO(2)$ gauge field and as a Lorentz scalar, see equation (2.12). Thus, the $SO(1, 3) \times SO(2)$ subgroup of $SO(1, 5)$ forms a maximal set of symmetries for which the basic ingredients of the theory, namely the spin connection, the vielbeins, and the vector, transform appropriately, i.e. either as tensors or connections. For this reason, our theory will be manifestly invariant only under this subgroup.

Having identified the basic fields, let us now consider the curvatures, or field strengths, from which we can construct an action. If we denote the field strength associated with the $SO(1, 5)$ generator $F$, then, using $F = dA + A \wedge A$ we find,

$$\mathbb{F}^{ab}_J = \frac{1}{2} \left( R^{ab} - \ell^2 e^a \wedge e^b - \ell^2 \phi^a \wedge \phi^b \right),$$  \hspace{1cm} (2.16)

$$\mathbb{F}^{a}_{(1)} = \ell^{-1} \left( D_a e^a + A \wedge e^a \right),$$  \hspace{1cm} (2.17)

$$\mathbb{F}^{a}_{(2)} = \ell^{-1} \left( D_a e^a - A \wedge e^a \right),$$  \hspace{1cm} (2.18)

$$\mathbb{F}_D = dA + \ell^{-2} \phi^a \wedge e_a,$$  \hspace{1cm} (2.19)

where $R^{ab} = d\omega^{ab} + \omega^a \wedge \omega^b$ is the Riemann curvature. In particular, note that the field strengths associated with the $P^{(i)}$ generators are generalizations of torsion associated with each of the vielbeins. The infinitesimal transformation of the curvatures under the action of the $SO(1, 5)$ group mimic the transformation of the gauge fields, namely,

$$\delta \mathbb{F}^{ab}_J = -2\Lambda e^{[ab]} [e^{bc}e_J + \chi^{[ab]} e^{(1)}] + \zeta^{[ab]} e^{(2)},$$  \hspace{1cm} (2.20)

$$\delta \mathbb{F}^{a}_{(1)} = -\Lambda^{ab} e_{(1)}^{ab} + \chi^{[ab]} e_{(2)}^{ab} + \zeta^{[ab]} e_D,$$  \hspace{1cm} (2.21)

$$\delta \mathbb{F}^{a}_{(2)} = -\Lambda_{ab} e_{(2)}^{ab} + \chi^{[ab]} e_{(1)}^{ab} + 2\zeta_J e_D,$$  \hspace{1cm} (2.22)

$$\delta \mathbb{F}_D = -\zeta_{ab} e_{(1)}^{ab} + \chi_{ab} e_{(2)}^{ab},$$  \hspace{1cm} (2.23)

except that now all quantities transform homogeneously under the Lorentz and $SO(2)$ subgroups of $SO(1, 5)$. In fact, both $\mathbb{F}^{ab}_J$ and $\mathbb{F}_D$ are left invariant under $SO(2)$ transformations.

### 2.2. Action and constraints

We now construct an action involving the $SO(1, 5)$ curvatures given in equations (2.16)–(2.19) that preserves parity and is invariant under the local $SO(1, 3) \times SO(2)$ transformations singled out in the previous section. This is analogous to the $SO(2, 4)$ formulation of conformal gravity but our construction differs from the standard approach of [37]; while the spin connection $\omega^{ab}$ was assumed to be real in the previous section, from now on we will regard it as a complex quantity, $\omega^{ab} = \sigma^{ab} + i\sigma^{ab}$. As a consequence the Riemann curvature $R^{ab}$ and the field strength given in equation (2.16) are now complex. The justification for this, and the meaning of the complex connection will be explained in what follows. Hence, with some hindsight we consider the action,

$$I = M_p^2 \ell^2 \left( \text{Re} \left( \mathbb{F}^{ab}_J \wedge \mathbb{F}^{cd}_J \right) \right) \epsilon_{abcd} - M_p^2 \ell^2 \frac{\sigma^2}{2} \int \mathbb{F}_D \wedge \star \mathbb{F}_D,$$  \hspace{1cm} (2.24)

where $\epsilon_{abcd}$ is the totally antisymmetric tensor, $\sigma^2$ is a dimensionless parameter greater than one, and the Hodge dual is defined with respect to the $SO(2)$-invariant metric $G_{\mu \nu}$ given in
equation (2.15). In this action we have not used the curvatures associated with the \( P^{(i)} \) generators of \( SO(1, 5) \). Instead, these will be used to impose constraints on the spin connection that determine it in terms of the vielbeins.

The first term in equation (2.24) is also featured in the gauge theory constructions of Einstein and conformal gravity of [37, 38] with the exception that there \( F_{ab}^{P} \) is strictly real. This is the only term involving any of the curvatures that satisfies the above conditions on symmetry and parity, and which does not require a metric. On the other hand, the second term in equation (2.24) is a non-gravitational action which couples to the \( SO(2) \)-invariant metric. This term must accompany the gravitational action in order to render the theory free of ghosts to linear order around a de Sitter background.

In the action (2.24) we have assumed that the spin connection is complex and, furthermore, that it is subject to a constraint. Let us explain why this must be the case. If we assume that the spin connection is real, then its equations of motion read,

\[
(e^a \wedge F^b_{P^{(1)}} + e^a \wedge F^b_{P^{(2)}}) \epsilon_{abcd} = 0.
\]  

(2.25)

This equation is invariant under \( SO(2) \) transformations, a property that will carry over to the spin connection, in agreement with equation (2.9). Note, however, that the vector field \( A \) does not appear in the above equation and is not a part of the gravitational action. Then the latter will contain at most two spin-2 fields and cannot have PM interactions beyond cubic order [26, 28, 31]. To circumvent the obstructions to non-linear partially massless symmetry we expect the vector field to play a non-trivial role in the theory. Furthermore, while it is difficult to solve for the spin connection in equation (2.25), we can readily find a perturbative solution around a de Sitter background. We then find that, not only does the vector field play no role in the gravitational action but, to linear order in the fields, the latter propagates only a massless spin-2 field\(^5\).

Thus the gravitational action in equation (2.24) is not to be interpreted as a first order action, i.e. one linear in derivatives, but as a second order action where the spin connection obeys a constraint. This is exactly what happens in the gauge theory approach to conformal gravity, based on the \( SO(2, 4) \) group, where one imposes [37],

\[
F^a_K = 0,
\]  

(2.26)

for some \( SO(2, 4) \) generator \( K \) in the complement of \( SO(1, 3) \times SO(1, 1) \). It could be argued that this is also true for Einstein gravity which is based on \( SO(1, 4) \) [38]. Indeed, in this case there is only one generator \( K \) in the complement of \( SO(1, 3) \subset SO(1, 4) \) for which equation (2.26) becomes the torsionless condition which also coincides with the equations of motion derived from the first order action. Note that in contrast to the four dimensional case, the gauge theory approach to Einstein and conformal gravity in three dimensions does not require additional constraints [39–41].

For the \( SO(1, 5) \) group, imposing the constraint in equation (2.26) with \( K \) given by any real linear combination of \( P^{(1)} \) and \( P^{(2)} \) does not lead to an \( SO(2) \)-invariant spin connection. The reason being that the corresponding curvatures, \( F^a_{P^{(1)}} \) and \( F^a_{P^{(2)}} \), form and \( SO(2) \) vector. While imposing \( F^a_{P^{(1)}} = F^a_{P^{(2)}} = 0 \) is \( SO(2) \) invariant, the only solution to these equations is \( \omega^{ab} = 0 \).

On the other hand, \( SO(2) \)-invariant constraints can be easily constructed in a basis of complex vielbeins \( \psi^a \) defined by,

\[
\psi^a = e^a + i r^a.
\]  

(2.27)

\(^5\)This means that the theory is strongly coupled on the de Sitter background. In this calculation we assumed that the vielbeins obey the symmetrization condition discussed below.
Indeed, a constraint that is invariant under SO(2) transformations is given by,

$$F^a_H \equiv F^a_{P(1)} + iS^a_{P(2)} = \ell^{-1} (D\omega - iA) \wedge \psi^a = 0,$$

(2.28)

where $F^a_H$ denotes the curvature associated with the complex vielbein (2.27). We recognize in equation (2.28) the torsionless condition for a complex spin connection which is also invariant under the local $U(1)$ transformations of $\psi^a$. Thus, the justification for extending the SO (1, 5) action to a complex connection stems from the requirement of an SO(2)-invariant $\omega_{ab}$ that satisfies non-trivial constraints. Note that the relevant formulas in section 2.1 can be easily extended to the complex connection case and may be derived by complexifying the Lorentz algebra in equations (2.3)–(2.6), see appendix.

If we assume invertibility of the complex vielbein, the spin connection is then given by,

$$\omega_{ab}^\mu = \psi^\rho_{[a} ^\mu \psi^\sigma_{b]} ,$$

(2.29)

where $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative with respect to $A_\mu$, and $\psi^\rho_{[a}$ satisfies,

$$\psi^\rho_{[a} \psi^\rho_{b]} = \delta^a_b , \quad \psi^\rho_{[a} \psi^\rho_{\nu]} = \delta^\nu_\mu .$$

(2.30)

In particular, note that the spin connection in equation (2.29) reduces to that of general relativity when we turn off the vector field and decouple either one of the vielbeins.

There is an additional constraint we will impose on the vielbeins. In the vielbein formulation of general relativity the local Lorentz symmetry can be used to remove all antisymmetric components from the vielbein, insuring that no antisymmetric rank-2 tensors propagate in the metric formulation of the theory. However, once an additional vielbein is added, the local Lorentz symmetry can be used to remove only one of the antisymmetric rank-2 tensors. Put in a different way, bimetric theories in the vielbein and metric formulations are not equivalent unless we impose the following symmetrization constraint [16, 49] (see also [50])

$$\left( e_\mu^a \quad r_\mu^a \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_{\alpha\nu} \\ t_{\alpha\nu} \end{pmatrix} = 0.$$

(2.31)

As a result of this, the vielbein contribution to the $F_D$ term in the action (2.24) vanishes. We have written the constraint in a manifestly SO(2)-invariant way to highlight that it is compatible with the symmetries of the action. Another consequence of this constraint is that the inverse $\psi_\mu^a$ of the complex vielbein $\psi^a_\mu$ can be easily evaluated as,

$$\psi_\mu^a = G^{\mu\nu} \psi_\nu^a ,$$

(2.32)

where $G^{\mu\nu}$ is the inverse of the SO(2)-invariant metric (2.15).

To summarize, we have constructed a gauge theory based on SO(1, 5) that is invariant under local SO(1, 3) $\times$ SO(2) transformations. The basic fields of this theory are a complex spin connection $\omega_{ab}$, a gauge field $A$, and two vielbeins $e^a$ and $r^a$ from which we can construct charged (2.14) and singlet (2.15) metrics. The theory is described by the action given in equation (2.24) with constraints on the spin connection (2.28) and the vielbeins (2.31).

2.3. Geometric interpretation

Let us now discuss the geometrical interpretation of the complexification of the spin connection. We first note that equation (2.28) defines how the covariant derivative acts on the complex vielbein, i.e.
If we let $\omega_{ab} = \tau_{ab} + i\sigma_{ab}$ where $\tau_{ab}$ and $\sigma_{ab}$ are two real one-forms antisymmetric in $a$ and $b$, then equation (2.33) implies that,
\[
D(e^a + t^a) \equiv \left( \frac{dt^a}{dt^b} \right) + \left( \frac{\tau_{ab}}{\sigma_{ab}} \right) \wedge \left( e^b \right).
\] (2.34)

Thus, the complexification of the spin connection introduces an additional real connection $\sigma_{ab}$ that is responsible for mixing the $e^a$ and $t^a$ vielbeins. In particular, this mixing guarantees that the covariant derivative of the vielbeins transforms homogeneously under global SO(2) transformations. To guarantee that the covariant derivative transforms homogeneously under local SO(2) transformations we must also add the vector field in the obvious way, i.e. via a skew-symmetric matrix. The addition of this spin connection is what makes the SO(1,5) theory a bimetric theory of gravity. Indeed, the bimetric theories of [14] also feature two spin connections which, in contradistinction to the SO(1, 5) theory, act diagonally on the corresponding vielbeins [16],
\[
D(e^a + t^a) \equiv \left( \frac{dt^a}{dt^b} \right) + \left( \frac{\tau_{ab}}{\sigma_{ab}} \right) \wedge \left( e^b \right).
\] (2.35)

This property of the covariant derivative is consistent with the fact that the $e^a$ and $t^a$ vielbeins rotate independently of each other under parallel transport. In contrast, in the SO(1, 5) theory the $e^a$ and $t^a$ vielbeins mix linearly with each other as implied by equation (2.34) and illustrated in figure 1 (this is over and above the simple SO(2) rotation generated by the gauge field $A$).

It is important to note that while we do complexify the spin connection in equation (2.7), we do not complexify the Lorentz transformations parametrized by $\Lambda_{ab}$ in equation (2.8). In particular, equation (2.9) implies that, under local Lorentz transformations, the real spin connections transform as,
\[
\delta \Lambda \tau_{ab} = D_\tau \Lambda_{ab}, \quad \delta \Lambda \sigma_{ab} = \sigma_{ac} \Lambda^{cb} + \sigma_{cb} \Lambda^{ac}.
\] (2.36)

Thus, the $\sigma_{ab}$ connection transforms homogeneously, i.e. as a tensor, under local Lorentz transformations. Equation (2.36) guarantees that the covariant derivative defined in equation (2.13) transforms homogeneously as well. In particular, these transformation properties of

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7 The connection $\sigma_{ab}$ may be associated with additional gauge transformations not considered here.
the connections guarantee that the real part of the Riemann curvature, which enters the action defined in equation (2.24), also transforms covariantly under local \( SO(1, 3) \) transformations,

\[
\text{Re} \, R_{ab}(\omega) = R_{ab}(\tau) - \sigma^a \wedge \sigma^b, \quad \delta \Lambda \text{Re} \, R_{ab}(\omega) = -2 \Lambda \epsilon^{[ab]} \text{Re} \, R_{bc}(\omega).
\]  

(2.37)

Here \( R_{ab}(\tau) = d\tau^a \wedge \tau^b + \tau^a \wedge \tau^b \) is the Riemann curvature for the real part of the spin connection. Thus, the theory can be described either in terms of a complex connection, or a real connection on the bundle \( e \oplus t \). This is related to the complexification of the Lorentz algebra as discussed in more detail in appendix.

2.3.1. Analytic continuation of conformal gravity. We conclude this section by discussing the relationship between the \( SO(1, 5) \) theory we have constructed and the \( SO(2, 4) \) theory of [37]. Let us first note that the generator \( H \) associated with the constraint in equation (2.28) is given by,

\[
H_a = P_a^{(1)} - i P_a^{(2)}.
\]  

(2.38)

This is part of the analytic continuation that turns the \( \mathfrak{so}(1, 5) \) algebra described in equations (2.3)–(2.6) into \( \mathfrak{so}(2, 4) \). Indeed, if we let

\[
\tilde{P}_a^{(1)} = H_a, \quad \tilde{P}_a^{(2)} = H^*_a, \quad \tilde{D} = i D,
\]  

(2.39)

we recover the algebra of the conformal group which is used in the construction of conformal gravity [37]. In particular, the constraint (2.28) corresponds to the analytic continuation of the constraint (2.26) imposed in [37], modulo conventions. This observation also extends to the gravitational part of the action given in equation (2.24). Using equation (2.16) and the complex vielbein (2.27), the latter may be written as,

\[
I_G = -\frac{M_p^2}{2} \int \epsilon_{abcd} \left\{ \psi^a \wedge \psi^{*b} \wedge \text{Re} \, R^{cd} - \frac{1}{2l^2} \psi^a \wedge \psi^{*b} \wedge \psi^c \wedge \psi^{*d} \right\}.
\]  

(2.40)

where the higher derivative term \( \int R_{ab} \wedge R^{cd} \epsilon_{abcd} \) is topological and does not contribute to the action. This is the same action obtained in [37] where, instead, \( \psi^a \) and \( \psi^{*a} \) are treated as two independent real vielbein one-forms, and the real spin connection obeys equation (2.28) with \( A \rightarrow -i A \). Thus, the extended \( SO(1, 5) \) theory considered here corresponds to an analytic continuation of the \( SO(2, 4) \) theory that leads to conformal gravity. Note that while it is possible to integrate out, say, \( \psi^{*a} \) in the \( SO(2, 4) \) theory, and thereby obtain conformal gravity, this is no longer the case in the \( SO(1, 5) \) theory since \( \psi^a \) and \( \psi^{*a} \) are complex conjugates of each other. Furthermore, unlike conformal gravity, the \( SO(1, 5) \) theory contains an additional vector field.

2.4. Bimetric formulation

Let us now consider the action (2.24) in more detail. Using equation (2.16), (2.19), and the constraint (2.31), the action can be written concisely in terms of complex vielbeins,

\[
I = -\frac{M_p^2}{2} \int \epsilon_{abcd} \left\{ \psi^a \wedge \psi^{*b} \wedge \left( \text{Re} \, R^{cd} - \frac{1}{2l^2} \psi^a \wedge \psi^{*b} \wedge \psi^d \right) \right\} - M_p^2 l^2 \frac{\sigma^2}{2} \int F \wedge * F,
\]  

(2.41)

where \( F = dA \) is the field strength of the vector field. This action describes a bimetric theory in the vielbein formulation with non-standard kinetic terms but the same potential as the bimetric models studied in [21–24] in connection with PM symmetry.
To see this more explicitly, let us express the action in (2.41) using metric variables. It is easy to verify that when \( e_{\mu\nu}t_{\nu\mu} = 0 \), as implied by the symmetrization constraint (2.31), one can write,

\[
S_0^\mu = (\sqrt{g_{\mu\nu}})^\mu = e_{\mu\nu}t_{\nu\mu}.
\]

Then the action (2.41) can be written as (see [16] for details),

\[
I = M_p^2 \int d^4 x \sqrt{|g|} e_{\mu\nu}e'^{\mu\nu} \text{Re} R_{\mu\nu} + M_p^2 \int d^4 x \sqrt{|f|} t_{\mu\nu}t_{\alpha\beta} \text{Re} R_{\mu\nu}^{\alpha\beta} - 2M_p^2\ell^{-2} \int d^4 x \sqrt{|g|} \sum_{n=0}^{4} \beta_n e_n(S) - M_p^2\ell^2 \sigma^2 \int d^4 x \sqrt{|g|} G^\mu\alpha G^\nu\beta F_{\mu\nu} F_{\alpha\beta},
\]

where the \( g_{\mu\nu}, f_{\mu\nu} \), and \( G_{\mu\nu} \) metrics are defined in equations (2.14) and (2.15). In equation (2.43) the \( \beta_n \) parameters are given by

\[
\beta_0 = 3, \quad \beta_2 = 1, \quad \beta_4 = 3, \quad \beta_1 = \beta_3 = 0,
\]

and \( e_n(S) \) denote the elementary symmetric polynomials of the matrix \( S \). In particular,

\[
e_0(S) = 1, \quad e_2(S) = \frac{1}{2} \left[ \text{tr}(S)^2 - \text{tr}(S^2) \right], \quad e_4(S) = \det S = \sqrt{|g|} \sqrt{|f|}.
\]

The kinetic terms in equation (2.43) are different from those of bimetric gravity which are given by the Einstein–Hilbert action for each of the metrics. Furthermore, by rescaling \( \ell^2 \to \alpha \ell^2 \) and taking the limits \( \sigma \to \infty \) and either \( \alpha \to 0 \) or \( \alpha \to \infty \) with \( \ell^2 \alpha^2 \) finite, we can decouple either one of the metrics in equation (2.43) and recover the Einstein–Hilbert action with a positive cosmological constant. In particular, note that once we decouple the vector field, the limits \( \alpha \to 0 \) and \( \alpha \to \infty \) make the spin connection real and reduce it to that of general relativity for either one of the vielbeins.

On the other hand, the potential in equation (2.43) is the same as that of the ghost-free bimetric model of [21–24], investigated in connection with PM symmetry. This model is singled out among all ghost-free bimetric gravity actions by demanding invariance under a global version of the partially massless symmetry—one given by equation (1.2) with constant \( \xi \)—beyond the linear level. This condition leads to equation (2.44) which uniquely fixes the \( \beta_n \) parameters, up to constant scalings of the metrics. However, as shown in [31], when \( \xi \) is a function of the coordinates, the partially massless symmetry cannot be extended beyond cubic order in the would-be partially massless field.

One can now understand how the \( \text{SO}(1, 5) \) theory avoids this obstruction. In bimetric gravity, the potential satisfying equation (2.44) is locally \( \text{SO}(2) \)-invariant in the vielbein formulation.

However, the kinetic terms explicitly break this local symmetry of the potential. In the present construction, the constraint (2.28) introduces an additional vector field that allows us to construct locally \( \text{SO}(2) \)-invariant kinetic terms. The presence of the vector field drastically modifies the structure of the cubic and higher order interactions, thus circumventing the known obstructions to non-linear PM symmetry. As will be shown later, the PM symmetry is present to all orders and is simply a transmutation of the local \( \text{SO}(2) \) symmetry of the \( \text{SO}(1, 5) \) theory.

Let us conclude this section with comments on the possible ghost instabilities of the theory. While the potential in (2.43) does not introduce ghost instabilities in the standard bimetric gravity, bimetric gravity for either one of the vielbeins.

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8. The \( \text{SO}(2) \) invariance of the bimetric potential, and its possible significance to PM symmetry, was first noticed by Latham Boyle and investigated with Kurt Hinterbichler and Angnis Schmidt-May [52].
gravity [15], this has only been proved when the kinetic terms of the latter are described by the Einstein–Hilbert action for each of the metrics. Thus, the non-standard kinetic terms of the $SO(1, 5)$ theory have the potential to introduce ghosts, unless additional constraints exist that remove these modes. Below we will show that the theory is indeed ghost free at linear order. Furthermore, note that similar kinetic terms have been considered recently in the literature [53–56] and these have been shown to contain nonlinear constraints that remove propagating modes also beyond linear order [44, 45]. This opens up the possibility that the $SO(1, 5)$ theory remains ghost free to all orders.

Finally, note that the coupling to matter fields via the $SO(2)$-invariant metric $G_{\mu\nu}$, or even via the charged metrics, has the potential to introduce ghost instabilities as well. We will see that perturbatively around a de Sitter background, the $SO(2)$-invariant metric contains only the massless spin-2 mode and does not induce couplings between matter and the massive spin-2 field. While coupling matter to the massless mode is not ghost free in bimetric gravity [57], the appearance of ghosts at the non-linear level depends on the structure of the kinetic terms, which are different in the $SO(1, 5)$ theory and not investigated here.

3. Perturbative metric formulation

In this section we expand the action around an off-shell, $SO(2)$-invariant metric and express the result in terms of spin-2 variables. We find that to linear order around a de Sitter background the theory propagates a massless spin-2 field, a partially massless graviton, and a massless vector field. We then show that the partially massless symmetry found at linear order in the metric formulation is nothing but the local $SO(2)$ symmetry manifest to all orders in the vielbein formulation of the theory.

3.1. Background solutions

We begin by considering the background solutions to the equations of motion. First, it is important to note that imposing the symmetrization constraint (2.31) before and after varying the action can lead to different equations of motion. The reason is that terms linear in the constraint give contributions to the equations of motion that are not proportional to the constraint itself. In the $SO(1, 5)$ theory we must impose the symmetrization constraint directly in the action. However, this is not an easy task due to the non-linear structure of the kinetic term which mixes the $e^a$ and $t^a$ vielbeins. It is nevertheless possible to derive the equations of motion consistently provided we complement the variation of the action by appropriate ‘counterterms’. Thus, in order to take care of possible terms linear in the constraint, we determine the equations of motion from the following variation of the action,

$$\delta \tilde{I} = \delta I - \int d^4 x \frac{\delta I}{\delta Q_{\mu\nu}} \delta Q_{\mu\nu}, \quad \text{(3.1)}$$

where $I$ is the action of the $SO(1, 5)$ theory given in equation (2.41) and $Q_{\mu\nu} = \epsilon_{\mu\nu}^{\rho\sigma} e_\rho^a$ is the symmetrization constraint given in equation (2.31). The modified variation of the action (3.1) allows us to impose $Q_{\mu\nu} = 0$ either before or after we obtain the equations of motion.

The variation of the action is then given by,

$$\delta \tilde{I} = -M_p^2 \int d^4 x \left( X^a_{\mu} \delta e^{\mu}_a + Y_{a\mu} \delta t^{\mu}_a + Z^a_{\mu} \delta A_{\mu} + \frac{1}{M_p^2} \frac{\delta I}{\delta Q_{\mu\nu}} \delta Q_{\mu\nu} \right), \quad \text{(3.2)}$$

where the functions $X^a_{\mu}$, $Y_{a\mu}$, and $Z^a_{\mu}$ read,
\[ X^\mu = \epsilon_{ijkl} \epsilon^{\lambda \rho \alpha \beta} \left[ i \frac{1}{2} \text{Re} R_{\alpha \beta}^{cd} - \ell^{-2} \left( \epsilon_c^\alpha \epsilon_d^\beta + \epsilon_d^\alpha \epsilon_c^\beta \right) \right] + \frac{1}{4} \left( \delta \omega_{\nu}^{ij} \mathcal{D}_{\nu} + \delta \omega_{\nu}^{* ij} \mathcal{D}^*_{\nu} \right) \left( \epsilon^\alpha e^\beta_{\lambda} + \epsilon^\beta e^\alpha_{\lambda} \right) \right] = \frac{1}{M_p^2} \frac{\delta I_A}{\delta \mu}, \tag{3.3} \]

\[ Y^\mu = \epsilon_{ijkl} \epsilon^{\lambda \rho \alpha \beta} \left[ i \frac{1}{2} \text{Re} R_{\alpha \beta}^{cd} - \ell^{-2} \left( \epsilon_c^\alpha \epsilon_d^\beta + \epsilon_d^\alpha \epsilon_c^\beta \right) \right] + \frac{i}{4} \left( \delta \omega_{\nu}^{ij} \mathcal{D}_{\nu} \mathcal{D}_{\nu}^* \mathcal{D}_{\nu}^* \mathcal{D}_{\nu}^* \right) \left( \epsilon^\alpha e^\beta_{\lambda} + \epsilon^\beta e^\alpha_{\lambda} \right) \right] = \frac{1}{M_p^2} \frac{\delta I_A}{\delta \mu}, \tag{3.4} \]

\[ Z^\mu = \epsilon_{ijkl} \epsilon^{\lambda \rho \alpha \beta} \left( \psi^{[\nu} [\psi^{\lambda]} \mathcal{D}_{\nu} - \psi^{\nu} [\psi^{\lambda} \mathcal{D}^*_{\nu}] \right) \left( \epsilon^\alpha e^\beta_{\lambda} + \epsilon^\beta e^\alpha_{\lambda} \right) - \frac{2}{M_p^2} \frac{\delta I_A}{\delta \mu}. \tag{3.5} \]

In these equations \( I_A = -M_p^2 \ell^2 \frac{\text{d}^2}{\text{d} t^2} + \frac{\text{d}}{\text{d} t} \int F \wedge F, \) whose variations are proportional to \( A \) and will not be needed in what follows. Furthermore, in equations (3.3) and (3.4) \( \delta \omega^\mu_{\nu}/\delta \psi_{\lambda}^* \) is an operator that contains \( U(1) \) covariant derivatives \( \mathcal{D}_{\mu} = \partial_{\mu} - iA_{\mu} \) acting on terms to its right, and \( \mathcal{D}_{\mu} \) is the covariant derivative with respect to the complex spin connection.

In close analogy to bimetric gravity [21], the \( SO(1, 5) \) theory admits proportional background solutions where

\[ \eta^\mu = \bar{\eta}^\mu, \quad t^\mu = c \bar{t}^\mu, \quad A = 0, \tag{3.6} \]

and \( c \) is an arbitrary constant. For this ansatz it is not necessary to evaluate \( \delta I/\delta Q^{\mu \nu} \) explicitly and the equations of motion become,

\[ X^\mu_{\nu} = 0, \quad Y^\mu_{\nu} = 0, \quad Z^\mu = 0. \tag{3.7} \]

Furthermore, for the parametrization given in equation (3.6) the spin connection is real and satisfies the torsionless condition with respect to the background vielbein \( \bar{\eta}^\mu \),

\[ D \bar{\eta}^\mu = d \bar{\eta}^\mu + \omega^ab \wedge \bar{\eta}^b = 0 \Rightarrow \omega^ab = \bar{\eta}^\mu \partial_{\mu} \bar{\eta}^b - \bar{\eta}^\mu \partial_{\mu} \bar{\eta}^b - \bar{\eta}^\mu \partial_{\mu} \bar{\eta}^b. \tag{3.8} \]

This implies that the second line in each of equation (3.3) and (3.4) vanish, while the \( Z^\mu \) equation of motion is satisfied precisely. The remaining equations of motion are equal to each other and reduce to,

\[ 0 = \epsilon_{abcd} \epsilon^{\mu \nu \alpha \beta} \bar{\eta}^b \left[ R_{\alpha \beta}^{cd} - 2 \ell^{-2} \left( 1 + c^2 \right) \bar{\epsilon}^\alpha_{\alpha} \bar{\epsilon}^\beta_{\beta} \right], \tag{3.9} \]

where \( R_{\alpha \beta}^{cd} \) is the Riemann curvature for the real spin connection given in equation (3.8). Equation (3.9) is equivalent to Einstein’s equation \( 0 = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda \bar{g}_{\mu \nu} \) where \( \bar{g}_{\mu \nu} = \bar{\eta}^a \bar{\epsilon}_{ab} \) and the cosmological constant is given by,

\[ \Lambda = 3 \ell^{-2} \left( 1 + c^2 \right). \tag{3.10} \]

Thus, the \( SO(1, 5) \) theory admits asymptotically de Sitter backgrounds.
3.2. Expansion of the action

To identify the physical content of the theory we can now expand the action perturbatively around a de Sitter background. However, before doing so, it is convenient to first obtain an expansion around a generic off-shell SO(2)-invariant configurations.

Besides the vector field \( A \), our initial variables are the two vielbeins \( e^a \) and \( t^a \), subject to the symmetrization constraint given in equation (2.31). From these vielbeins it is possible to define two intermediate metric variables that transform non-trivially under \( \text{SO}(2) \)—these are the \( g_{\mu \nu} \) and \( f_{\mu \nu} \) metrics given in equation (2.14). The only \( \text{SO}(2) \)-invariant quantity is the \( G_{\mu \nu} \) metric given in equation (2.15). For convenience we introduce the following rescaled version of the \( \text{SO}(2) \)-invariant metric,

\[
G_{\mu \nu} = g_{\mu \nu} + f_{\mu \nu} = (1 + c^2) \tilde{g}_{\mu \nu}.
\]  

(3.11)

The metrics \( g_{\mu \nu} \) and \( f_{\mu \nu} \) can now be parametrized in terms of a final, physical set of metric variables, namely

\[
g_{\mu \nu} = \tilde{g}_{\mu \nu} - \phi_{\mu \nu}, \quad f_{\mu \nu} = c^2 \tilde{g}_{\mu \nu} + \phi_{\mu \nu}.
\]  

(3.12)

Here \( \tilde{g}_{\mu \nu} \) is the spacetime metric whose perturbations around a de Sitter background describe a massless spin-2 field. On the other hand, by expanding the action in powers of \( \phi_{\mu \nu} \) and diagonalizing the quadratic terms, we will see that \( \phi_{\mu \nu} \) is a massive spin-2 field. Also note that up to a normalization equation (3.12) is the same expansion used in the bimetric theory of [21–24].

In order to write the action (2.41) using metric variables we need an explicit parametrization of the vielbeins in terms of \( \tilde{g}_{\mu \nu} \) and \( \phi_{\mu \nu} \). We use

\[
e^a = \tilde{e}^a + \delta e^a, \quad t^a = c \left( \tilde{e}^a + \delta t^a \right),
\]  

(3.13)

where \( \tilde{e}^a \) is the \( \text{SO}(2) \)-invariant vielbein associated with the \( \text{SO}(2) \)-invariant metric via

\[
\tilde{g}_{\mu \nu} = \tilde{e}_a^\mu \tilde{e}_a^\nu.
\]  

(3.14)

In particular, if we let \( \delta e^a \to 0, \delta t^a \to 0, \) and \( A \to 0 \), then \( \tilde{g}_{\mu \nu} \) obeys Einstein’s equations with a positive cosmological constant, as shown in the previous section. Note that the \( \text{SO}(2) \) transformations of \( \delta e^a \) and \( \delta t^a \) in equation (3.13) can be worked out from equation (2.13) provided that we treat \( \tilde{e}^a \) as an \( \text{SO}(2) \) singlet. It is also obvious that these fields are not independent of each other and, in fact, are non-linear in \( \phi_{\mu \nu} \). Indeed, from equations (3.13) and (3.12) we find that

\[
\phi_{\mu \nu} = - \left( \tilde{e}_a^\mu \delta e_{\nu a} + \tilde{e}_a^\nu \delta e_{\mu a} + \delta e^\mu_\alpha \delta e_{\alpha a} \right) = c^2 \left( \tilde{e}_a^\mu \delta t_{\nu a} + \tilde{e}_a^\nu \delta t_{\mu a} + \delta t^\mu_\alpha \delta t_{\alpha a} \right).
\]  

(3.15)

Thus, equation (3.15) provides a non-linear relation between \( \delta e^a \) and \( \delta t^a \). Another non-linear relationship between \( \delta e^a \) and \( \delta t^a \) follows from the symmetrization constraint on the vielbeins (2.31), which is responsible for removing an antisymmetric rank-2 tensor from the metric formulation of the theory.

We now solve the symmetrization constraint (2.31) and equation (3.15) perturbatively in the field \( \phi_{\mu \nu} \). First, the vielbein symmetrization constraint is solved to quadratic order by,

\[
\delta e^\mu_\mu = \frac{1}{2} \delta E_{\mu \nu} \tilde{e}^{\nu a} = - \frac{1}{8} \delta E_{\mu}^\alpha \delta T_{\alpha \nu} \tilde{e}^{\nu a} + \ldots,
\]

\[
\delta t^\mu_\mu = \frac{1}{2} \delta T_{\mu \nu} \tilde{e}^{\nu a} = - \frac{1}{8} \delta T_{\mu}^\alpha \delta E_{\alpha \nu} \tilde{e}^{\nu a} + \ldots.
\]  

(3.16)
where all spacetime indices are raised with the $\tilde{g}_{\mu\nu}$ metric, and the fields $\delta E_{\mu\nu}$ and $\delta T_{\mu\nu}$ are symmetric rank-2 tensors. In these equations we have used the local Lorentz symmetry of the theory to make $\delta E_{\mu\nu}$ symmetric. Then, to satisfy the symmetrization constraint $\delta T_{\mu\nu}$ must be symmetric as well. In terms of these variables the solution to equation (3.15) is,

$$\delta E_{\mu\nu} = -\phi_{\mu\nu} - \left(\frac{1 + c^2}{4c^2}\right) \phi^\alpha_{\mu} \phi_{\alpha\nu} + \ldots,$$

$$\delta T_{\mu\nu} = \frac{1}{c^2} \left[ \phi_{\mu\nu} - \left(\frac{1 + c^2}{4c^2}\right) \phi^\alpha_{\mu} \phi_{\alpha\nu} \right] + \ldots \quad (3.17)$$

Equations (3.16) and (3.17) allow us to express the vielbeins in terms of the physical metric variables order by order in $\phi_{\mu\nu}$.

Finally, to express the action entirely in terms of metric variables and their perturbations we need to define an appropriate covariant derivative compatible with the $\tilde{g}_{\mu\nu}$ metric. We then recall that in the absence of perturbations, i.e. for $\epsilon^a = \tilde{\epsilon}^a$, $\epsilon^a = \tilde{\epsilon}^a$, and $\Lambda = 0$, the action (2.41) reduces to the Einstein–Hilbert action for the vielbein $\tilde{e}^a$. Furthermore, in this case $\omega^a_{\mu\nu}$ becomes the spin connection given in equation (3.8) for the vielbein $\tilde{e}^a$. Thus, it is natural to define the covariant derivative via the vielbein postulate for $\tilde{e}^a$,

$$\nabla_\mu \tilde{e}^a = \partial_\mu \tilde{e}^a + \omega^a_{\mu\nu} \tilde{e}^\nu - \Gamma^\lambda_{\mu\nu} \tilde{e}_\lambda = 0 \Rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \tilde{g}^{\lambda\alpha} \left( \partial_\mu \tilde{g}_{\nu\alpha} + \partial_\nu \tilde{g}_{\mu\alpha} - \partial_\alpha \tilde{g}_{\mu\nu} \right),$$

(3.18)

where $\Gamma^\lambda_{\mu\nu}$ is the Christoffel connection for $\tilde{g}_{\mu\nu}$. In particular, note that equation (3.18) is consistent provided that $\tilde{e}^a$ is a singlet under $SO(2)$ transformations.

With these ingredients in place we can now write the vielbein action (2.41) in a perturbative expansion of the metric formulation of the theory. Up to quadratic order in $\phi_{\mu\nu}$ we find,

$$I = (1 + c^2) M_p^2 \int d^4x \sqrt{|\tilde{g}|} \left\{ \mathcal{R} - 2\Lambda + \frac{1}{2c^2} \mathcal{L}_{\text{PM}}(\phi)^2 - \frac{2}{c} \left( \phi_{\mu\nu} - \tilde{g}_{\mu\nu} \phi \right) \nabla^\mu A^\nu + 6A_{\mu} A^\mu \right\}$$

$$- \frac{M_p^2 \epsilon^2}{2} \sigma^2 \int d^4x \sqrt{|\tilde{g}|} F_{\mu\nu} F^{\mu\nu},$$

(3.19)

where $\mathcal{R}$ is the Ricci scalar of $\tilde{g}_{\mu\nu}$ and $\phi = \tilde{g}^{\mu\nu} \phi_{\mu\nu}$. In equation (3.19), $\mathcal{L}_{\text{PM}}(\phi)$ is the Lagrangian of a partially massless field defined on an arbitrary metric $\tilde{g}_{\mu\nu}$,

$$\mathcal{L}_{\text{PM}}(\phi) = -\frac{1}{2} \nabla_\rho \phi_{\mu\nu} \nabla^\rho \phi_{\mu\nu} + \frac{1}{2} \nabla_\sigma \phi_{\mu\nu} \nabla^\rho \phi_{\sigma\rho} - \partial_\sigma \phi_{\mu} \nabla_\rho \phi_{\sigma\rho} + \nabla_\mu \phi_{\nu} \nabla_\rho \phi_{\mu\rho} + \frac{2\Lambda}{3} \phi_{\mu\nu} \phi^{\mu\nu}$$

$$- \frac{\Lambda}{6} \phi^2 + (\mathcal{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu}) \left( -\frac{1}{2} \tilde{g}_{\mu\nu} \phi_{\sigma\rho} \phi_{\sigma\rho} + \frac{1}{4} \tilde{g}_{\mu\nu} \phi^2 + 2\phi^\rho_{\mu} \phi^\rho_{\nu} - \phi \phi_{\mu\nu} \right),$$

(3.20)

where $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} R$ is the Einstein tensor. In particular, in a de Sitter background this Lagrangian reduces to the Fierz–Pauli Lagrangian [8] describing a massive spin-2 field where the mass of the graviton saturates the Higuchi bound [9], see equation (1.1).

Let us note that equation (3.20) is the same quadratic Lagrangian recovered from bimetric gravity in [31] and which had been used in [32] to analyze the global symmetries of partially massless gravity. However, this similarity between the bimetric and $SO(1,5)$ theories does not extend to higher orders. Indeed, already at cubic order and ignoring contributions from the vector field, we find that the action of the $SO(1,5)$ theory disagrees with the candidate PM bimetric theory studied in [31, 32]. This means that the realization of partially massless gravity...
discussed in the next section cannot be obtained simply by adding a vector field to bimetric gravity. As stressed previously, it is not only the vector field who plays an important role in the theory, but also the kinetic terms used in (2.41) which render the action $SO(2)$ invariant.

3.3. Partially massless symmetry

Let us now recover the sought-after partially massless symmetry by diagonalizing the quadratic action given in equation (3.19). We begin by expanding the metric $\tilde{g}_{\mu\nu}$ around a de Sitter background,

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$  \hspace{1cm} (3.21)

and performing the following field redefinition

$$\phi_{\mu\nu} = \varphi_{\mu\nu} - 6c\Lambda \nabla(\mu A_\nu).$$  \hspace{1cm} (3.22)

While it is possible to diagonalize the action away from the de Sitter background, this requires an additional redefinition of the metric that removes higher derivative terms induced by equation (3.22) at higher orders. These field redefinitions leave our results unchanged at quadratic order so we can safely ignore them.

Using the field redefinition given in equation (3.22) the action (3.19) reduces to

$$I = \frac{1}{M_p^2} \int d^4x \sqrt{|\bar{g}|} \left\{ (1 + c^2) \left[ L_{FP}(h) + \frac{1}{2c^2} L_{PM}(\varphi) \right] - \ell^2 \frac{\sigma^2}{2} F_{\mu\nu} F^{\mu\nu} \right\},$$  \hspace{1cm} (3.23)

where we must choose $\sigma^2 > 1$ in order to keep the action ghost-free to quadratic order in the fields. In equation (3.23) $L_{FP}(h)$ is the Fierz–Pauli Lagrangian [8] for a massless spin-2 field, while $L_{PM}(\varphi)$ is given by equation (3.20) restricted to the de Sitter background. We thus find that, to linear order on a de Sitter background, the $SO(1, 5)$ theory propagates a massless spin-2 field, a partially massless graviton, and a massless vector field. In particular, the partially massless symmetry of $\varphi_{\mu\nu}$ is nothing but the local $SO(2)$ symmetry of the vielbein action (2.41). In order to see this let us consider the behavior of $h_{\mu\nu}$, $A_\mu$, and $\varphi_{\mu\nu}$ under infinitesimal $SO(2)$ transformations.

From equations (2.13) and (3.13) we find that perturbations of the vielbeins transform as

$$\delta e_a = \xi (\tilde{e}_a + \delta \tilde{e}_a), \hspace{1cm} (\delta e^a)' = \delta e^a - c \xi (\tilde{e}^a + \delta \tilde{e}^a),$$  \hspace{1cm} (3.24)

Then, using equation (3.15), the $SO(2)$ transformations of the fields appearing in the perturbative formulation of the theory prior to diagonalization are given by

$$\delta \xi h_{\mu\nu} = 0,$$  \hspace{1cm} (3.25)

$$\delta \xi A_\mu = \partial_\mu \xi,$$  \hspace{1cm} (3.26)

$$\delta \xi \varphi_{\mu\nu} = 2c\xi g_{\mu\nu} + 2c\xi h_{\mu\nu} + \frac{1 - c^2}{c} \xi \phi_{\mu\nu} + \ldots,$$  \hspace{1cm} (3.27)

where we have ignored higher order corrections in $\delta \xi \phi_{\mu\nu}$ that depend on both $\phi_{\mu\nu}$ and $h_{\mu\nu}$. In contrast, the transformations of $h_{\mu\nu}$ and $A_\mu$ given in equation (3.25) and (3.26) are exact. That $\delta \xi h_{\mu\nu} = 0$ is valid to all orders follows from the $SO(2)$ invariance of $\tilde{g}_{\mu\nu}$.

On the other hand, the fields that diagonalize the action via (3.22) transform as,$^9$

$^9$ We have checked that such field redefinitions exist at least up to cubic order terms in the action.
\[ \delta \xi h_{\mu\nu} = \mathcal{O}(\varphi), \quad (3.28) \]
\[ \delta \xi A_\mu = \partial_\mu \xi, \quad (3.29) \]
\[ \delta \xi \varphi_{\mu\nu} = \frac{6c}{\Lambda} \left( \nabla_\mu \nabla_\nu + \frac{\Lambda}{3} g_{\mu\nu} \right) \xi + \mathcal{O}(\varphi), \quad (3.30) \]

where \( \delta \xi h_{\mu\nu} \) receives corrections from the (higher order) field redefinitions of the metric we have ignored. In equation (3.30) we recognize the gauge transformation that characterizes a partially massless field (1.2). Thus, in the perturbative metric formulation of the \( SO(1, 5) \) theory the local \( SO(2) \) rotations of the vielbeins are realized as the partially massless symmetry of a massive graviton. This is reminiscent of how diffeomorphisms in three-dimensional gravity correspond to gauge transformations in the Chern–Simons formulation of the theory [40] (see also [58]).

Crucially, since the PM transformation is a consequence of the manifest \( SO(2) \) invariance of the theory, we have indirectly established that the \( \text{PM symmetry exists to all orders in the fields} \). The higher order terms we have neglected in equations (3.28) and (3.30) guarantee the invariance of the action order by order in \( \varphi_{\mu\nu} \). In particular, the cubic and higher order terms depend non-trivially on the gauge field \( A \), thereby avoiding the obstructions to PM symmetry encountered in previous constructions [26, 28, 31, 32]. Clearly, the non-linear vielbein formulation is the appropriate set up to verify that the partially massless symmetry is realized to all orders.

There is a price to pay for this non-linear realization of the partially massless symmetry, however. While the quadratic action (3.23) admits independent PM and \( U(1) \) transformations, only the diagonal part of the \( \text{PM} \times U(1) \) symmetry survives non-linearly. This can be readily established in the vielbein formulation of the theory and is reflected in equations (3.29) and (3.30). This implies that we lose one of the first class constraints manifest in the quadratic theory that is responsible for removing one of the helicity-0 modes from either the massive graviton or the massless vector field. Thus, without a proper Hamiltonian analysis, it is not clear whether all of the helicity-0 modes decouple from the theory\(^{10}\).

Nevertheless, the fact that only the diagonal version of the \( \text{PM} \times U(1) \) gauge symmetry survives non-linearly does not mean that the partially massless symmetry is trivially realized, e.g. as in the Stückelberg trick. Rather, it is the precise form of the transformation given in equation (3.30), along with all its non-linear corrections, that guarantee invariance of the action under the local \( SO(2) \) transformation of the vielbeins. In particular, note that the diagonal part of \( \text{PM} \times U(1) \) is sufficient to maintain the relationship between the mass of the spin-2 field and the cosmological constant, see equation (1.1). Given that a similar phenomenon is found in colored/charged theories of three-dimensional gravity (at quadratic order) [42, 43], it is natural to conjecture that the partially massless symmetry is a generic feature of theories with charged metrics/vielbeins.

We conclude this section by pointing out another feature of the \( SO(1, 5) \) theory. Unlike the gauge theory approach to Einstein and conformal gravity, where the gauge groups used in the construction of the theory describe the global symmetries of the latter, the \( SO(1, 5) \) theory does not admit a global \( SO(1, 5) \) symmetry group. Indeed, while the quadratic action (3.23) does admit such a symmetry [32], at non-linear order only the \( SO(1, 4) \) symmetries of the de Sitter background survive. A similar result was found in [32] who studied the theory  

\(^{10}\) It may be possible that the non-linear theory has an additional pair of second class constraints which become first class upon linearization, resulting in the \( \text{PM} \times U(1) \) symmetry of the quadratic theory.
of a massless spin-2 field and a partially massless graviton order by order in the PM field. In that case, the loss of the $SO(1, 5)$ symmetry at non-linear order can be understood as a consequence of the obstruction to the partially massless symmetry at higher orders [31]. In contrast, the reason why the $SO(1, 5)$ theory does not admit such a global symmetry group is that only the diagonal part of the $PM \times U(1)$ gauge symmetry survives non-linearly.

4. Conclusions and outlook

In this paper we constructed a bimetric theory that realizes the partially massless symmetry to all orders. The starting point was a gauge theory based on the $SO(1, 5)$ group, manifestly invariant under its $SO(1, 3) \times SO(2)$ subgroup, supplemented by additional constraints. The resulting theory may be interpreted as an analytic continuation of conformal gravity which is based on the $SO(2, 4)$ group. The outcome is a bimetric theory with non-standard kinetic terms and an additional vector field which, along with a specific potential of bimetric gravity, render the theory invariant under local $SO(2)$ transformations. Unlike conformal gravity, the linear spectrum of the theory is free of ghost instabilities. More importantly, we showed that in a perturbative formulation of the theory the local $SO(2)$ symmetry is transmuted into the non-linear partially massless symmetry of a massive spin-2 field.

One may expect a PM theory with an additional vector field to propagate $2 + 4 + 2$ degrees of freedom corresponding to a massless spin-2 graviton, a partially massless field, and the massless vector field. While this is indeed the case at linear order, we have found that only the diagonal $PM \times U(1)$ symmetry survives non-linearly. This suggests that additional degrees of freedom may propagate beyond linear order. Thus, it would be interesting to analyze the constrained Hamiltonian of the theory and establish whether the theory possesses enough constraints to propagate a total of 8 degrees of freedom non-linearly. Otherwise, a helicity-0 mode will be strongly coupled.

We have also seen that our construction requires the complexification of the spin connection. As shown in appendix, this spin connection is naturally associated with the complexified Lorentz algebra, pointing towards an enlargement of the starting $SO(1, 5)$ group. One possibility is to consider a gauge theory based on the complexified $so(1, 5)$ algebra. We hope to report on this in the near future.

The structure of the kinetic terms used in the $SO(1, 5)$ theory, where the spin connection obeys the constraint equation (2.28), also deserves further study. This is necessary to determine if ghosts propagate non-linearly. It would also be interesting to determine whether a bimetric theory with such kinetic terms and a generic bimetric potential, i.e. one where the all the $\beta_n$ parameters in equation (2.43) are arbitrary, leads to a consistent theory.

It is natural to expect that a generalization of the $SO(1, 5)$ theory leads to a non-abelian generalization of the partially massless symmetry. If instead of $SO(1, 5)$ one considers $SO(1, 3 + n)$, it is possible to construct gauge theories for the $SO(1, 3) \times SO(n)$ subgroups of the latter. These theories are characterized by $n$ vielbeins $e_i^{(j)}$ with $i = 1, \ldots, n$, that transform homogeneously under Lorentz transformations and as a vector under $SO(n)$ rotations. In particular, if one imposes the symmetrization condition (2.31) for all the possible pairs of vielbeins, the generalization of the action (2.24) leads to

$$I = -\frac{M_p^2}{2} \int \epsilon_{abcd} \left\{ \sum_{i=1}^{n} e_i^{(a)} \wedge e_i^{(b)} \wedge R^{cd} - \frac{1}{2 \ell^2} \left( \sum_{i=1}^{n} e_i^{(a)} \wedge e_i^{(b)} \right) \wedge \left( \sum_{j=1}^{n} e_j^{(a)} \wedge e_j^{(b)} \right) \right\}$$

$$- M_p^2 \frac{\sigma^2}{2} \int \text{tr} (F \wedge \star F), \quad (4.1)$$
where \( F = dA + A \wedge A \) is the field strength of the \( SO(n) \) gauge field \( A \). In analogy to the \( SO(1, 5) \) theory, the Hodge dual in equation (4.1) is defined with respect to the \( SO(n) \) invariant metric given by \( G_{\mu \nu} = \sum_{i=1}^{n} e^{(i)a} \epsilon_{(i)ab} \). The action (4.1) is manifestly \( SO(1, 3) \times SO(n) \) invariant provided that the spin connection does not transform under \( SO(n) \), as required from the \( so(1, 3+n) \) algebra.

Note, however, that the non-trivial part in the construction of the \( SO(1, 5) \) theory is the constraint (2.28) for the spin connection. It is this constraint that forces us to make the latter complex, contrary to what one would naively expect from gauging the \( so(1, 5) \) algebra. Likewise, the non-trivial step that is necessary in the construction of \( SO(1,3+n) \) theories is the identification of an appropriate constraint for the spin connection. The non-abelian nature of \( SO(n) \) makes this a difficult task that we leave for future study.

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### Appendix. Enhanced algebra

In this section we consider a generalization of the \( so(1,5) \) algebra that does not require the complexification of the spin connection introduced in section 2. This algebra contains an additional antisymmetric generator \( M_{ab} \), enlarging the algebra to a total of 21 generators, and satisfies

\[
[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \tag{A.1}
\]

\[
[M_{ab}, J_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}, \tag{A.2}
\]

\[
[M_{ab}, M_{cd}] = -\left( \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} \right), \tag{A.3}
\]

\[
[J_{ab}, P_{c}^{(i)}] = \delta^{ij} \left( \eta_{bc}P_{(j)a} - \eta_{ac}P_{(j)b} \right), \tag{A.4}
\]

\[
[M_{ab}, P_{c}^{(i)}] = -\epsilon^{ij} \left( \eta_{bc}P_{(j)a} - \eta_{ac}P_{(j)b} \right), \tag{A.5}
\]

\[
[P_{a}^{(i)}, P_{b}^{(j)}] = \epsilon^{ij} \eta_{ab}D - \delta^{ij}J_{ab}, \tag{A.6}
\]

\[
[D, P_{a}^{(i)}] = \epsilon^{ij}P_{(j)a}, \tag{A.7}
\]

while all other commutators vanish. It is interesting to note that the equations (A.1)–(A.3) can be obtained from a complexification of the Lorentz algebra, i.e. by letting \( J_{ab} \rightarrow \frac{1}{2} \left( J_{ab} + iM_{ab} \right) \). Thus, in some sense the complexification of the spin connection is unavoidable. More importantly, the algebra described by equations (A.1)–(A.7) is not a Lie algebra since the Jacobi identity is not satisfied for all the possible combinations of the generators. Indeed, it is not
difficult to check that $[M, [P^{(i)}, P^{(j)}]] + \text{cyclic permutations} \neq 0$. Nevertheless, this algebra may be embedded into a larger Lie algebra where the Jacobi identity is satisfied for all the generators. One example of this is the complexification of the $\mathfrak{so}(1, 5)$ algebra given by equations (2.3)–(2.6).

Assuming that the algebra described by equations (A.1)–(A.7) is the truncation of a consistent Lie algebra, the corresponding gauge field is now parametrized by

$$A = \frac{1}{2} \tau^{ab} I_{ab} + \frac{1}{2} \sigma^{ab} M_{ab} + \ell^{-1} e^a P_a^{(1)} + \ell^{-1} e^a P_a^{(2)} + AD + \ldots$$

(A.8)

If we focus on the $SO(1, 3) \times SO(2)$ symmetries generated by this algebra we then find that $\tau^{ab}$ and $\sigma^{ab}$ transform as

$$\delta \lambda \tau^{ab} = D_\tau \Lambda^{ab}, \quad \delta \lambda \sigma^{ab} = \sigma^a_c \Lambda^{cb} + \sigma^b_c \Lambda^{ac},$$

(A.9)

where $D_\tau$ is the covariant derivative with respect to the Lorentz connection $\tau^{ab}$. On the other hand the transformation of the vielbeins and the vector under $SO(1, 3) \times SO(2)$ transformations remain unchanged. Note that the transformation of the connections $\tau^{ab}$ and $\sigma^{ab}$ given in equation (A.9) is the same transformation inferred from the complexification of the spin connection, see equation (2.36).

Let us now consider the field strengths associated with each of the generators of the enhanced algebra. We find,

$$F_M^{ab} = D_\tau \sigma^{ab},$$

(A.10)

$$F_J^{ab} = \frac{1}{2} \left( R^{ab}_\tau - \sigma^a_c \Lambda^{cb} - \ell^{-2} e^a \Lambda^{cb} - \ell^{-2} e^b \Lambda^{ca} \right),$$

(A.11)

$$F_{P^{(1)}}^{ab} = \ell^{-1} \left( D_\tau e^a - \sigma^a_c \Lambda^{cb} + A \Lambda^{cb} \right),$$

(A.12)

$$F_{P^{(2)}}^{ab} = \ell^{-1} \left( D_\tau e^a + \sigma^a_c \Lambda^{cb} - A \Lambda^{ca} \right),$$

(A.13)

$$F_D = dA + \ell^{-2} e^a \Lambda_{ab},$$

(A.14)

where $R^{ab}_\tau = d\tau^{ab} + \tau^{a}_{c} \Lambda^{cb}$. In particular, the first two terms in equation (A.11) reproduce the real part of the complex curvature $R^{ab}_\tau = d\omega^{ab} + \omega^{a}_{c} \Lambda^{cb}$ that directly contributes to the action, see equation (2.37). Also note that the constraint

$$F_{P^{(1)}}^{ab} + i F_{P^{(2)}}^{ab} = 0,$$

(A.15)

reproduces the constraint given in equation (2.28) for the complex vielbein $\omega^{ab} = \tau^{ab} + i \sigma^{ab}$. Furthermore, it is not difficult to check that all these curvatures transform homogeneously under local $SO(1, 3) \times SO(2)$ transformations and that, in fact, the $F_M^{ab}$, $F_{P^{(1)}}^{ab}$, and $F_D$ curvatures are left invariant under $SO(2)$ rotations. Thus the enhanced algebra presented in this section reproduces the desired curvatures, constraints, and transformations properties deduced from the complexification of the spin connection introduced in section 2. Let us conclude by noting that the additional curvature term $F_M^{ab}$ is important in the construction of the action. Indeed, this term guarantees that the higher derivative term $\int R^{ab}_{\tau} R^{cd}_{\tau} \Lambda_{abcd}$, which is part of the gauge theory action given in equation (2.24), is a topological invariant that does not contribute to the vielbein action (2.41).
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