Noncommutative AdS Supergravity in three Dimensions

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Abstract

In this paper we propose a generalization of $\mathcal{N} = 4$ three dimensional AdS supergravity to the noncommutative case. This is a supersymmetric version of the results presented in [1]. We show that it continues to admit an $\mathcal{N} = 4$ supersymmetric solution which is the noncommutative counterpart of $AdS_3$ space. Some other solutions are also discussed.
1 Introduction

As already known, open string theory in presence of a constant background Neveu-Schwarz $B-$ field gives rise, in the field theory limit, to noncommutative gauge theories whose field’s algebra is described by the Moyal product [3]. Even if the particularly good properties of the Moyal product allows to treat these theories as ordinary field theories on a certain smooth manifold, noncommutativity changes drastically the geometrical structure of this base manifold and geometrical objects like the metric become difficult to define. Moreover, the noncommutativity parameters $\theta^{\mu\nu}$ are determined by $B_{\mu\nu}$ which generally behaves as a dynamical field; this leads one to suspect that also noncommutativity (non-locality) could become dynamical. On the other hand, one can imagine a $D-$brane in a background $B-$ field with strings attached on it. These strings could close and escape from the brane giving rise to gravitational interactions. It should then be interesting to understand how field theories couple to gravity, and moreover the problem of a consistent construction of gravity on noncommutative spaces becomes relevant itself. In particular, string theories being supersymmetric, the construction of noncommutative supergravity must be also considered.

Till now, much work has been done to analyze the structure of a noncommutative space-time [2] from a mathematical point of view. It is plausible that if noncommutative gravity were consistently built without reference to its stringy origin, a better understanding of the space-time structure from a physical point of view, could be achieved.

In this paper we define a possible model of noncommutative supersymmetric gravity extending the approach tried in [1] to the supersymmetric case. As in [1] all the fields in our model happen to be real and a metric can be naturally defined. We also find the supersymmetric extension of the invariances of the action which reduce to diffeomorphisms in the commutative limit.

Finally we analyse some solutions like the noncommutative analogue of $AdS_3$ space, and the BTZ black hole with zero mass and zero angular momentum. In particular, the noncommutative $AdS_3$ solution results to be a maximally supersymmetric solution, just like the commutative one.

2 The action and its symmetries

It is well known that three dimensional AdS supergravity, just like the non supersymmetric one, can be written as a Chern-Simons theory [4]. Since Chern-Simons theory is well formulated also in the noncommutative case, we can start from it to produce a noncommutative version of three dimensional supergravity and, for that purpose, the manifestly supersymmetric formulation employing superalgebras and supertraces is particularly well suited. As in the nonsupersymmetric case, use of the star product forces the extension from the usual $su(1, 1) \oplus su(1, 1)$ algebra to $u(1, 1) \oplus u(1, 1)$ [4]. Introducing two new $u(1)$ gauge fields, the simplest supersymmetric extension is now given by the
superalgebra $u(1,1|1) \oplus u(1,1|1)$. The supergravity action thus becomes:

$$S = \kappa \int \text{Str} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) - \kappa \int \text{Str} \left( \bar{\Gamma} \wedge d\bar{\Gamma} + \frac{2}{3} \bar{\Gamma} \wedge \bar{\Gamma} \wedge \bar{\Gamma} \right),$$

(1)

where the 1-forms $\Gamma$ and $\bar{\Gamma}$ are $u(1,1|1)$ super connections and can be written in the following way (cf. appendix A):

$$\Gamma = \begin{pmatrix} A & \psi \\ -i\bar{\psi} & i\bar{g} \end{pmatrix}, \quad \bar{\Gamma} = \begin{pmatrix} \bar{A} & \bar{\psi} \\ -i\bar{\bar{\psi}} & i\bar{\bar{g}} \end{pmatrix};$$

(2)

here $A = A^A \tau_A$ and $\bar{A} = \bar{A}^A \tau_A$ are the bosonic $u(1,1)$ 1-form gauge fields, $g$ and $\bar{g}$ are the $u(1)$ 1-forms gauge fields associated to the R-symmetry, whereas $\psi$ and $\bar{\psi}$ are complex spinorial 1-forms. In Eq. (1) $\kappa = -1/(16\pi G)$, where $G$ is the three dimensional Newton constant. In terms of these fields, the action can be written in the following way:

$$S = \kappa \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \wedge A \right) - \kappa \int \text{Tr} \left( \bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A} \wedge \bar{A} \right) - \kappa \int \text{Tr} \left( g \wedge dg + \frac{2i}{3} g \wedge g \wedge g \right) + \kappa \int \text{Tr} \left( \bar{g} \wedge d\bar{g} + \frac{2i}{3} \bar{g} \wedge \bar{g} \wedge \bar{g} \right) + 2i\kappa \int \bar{\psi} \wedge D\psi - 2i\kappa \int \bar{\bar{\psi}} \wedge D\bar{\bar{\psi}},$$

(3)

where $D\psi := d\psi + A \wedge \psi + i\psi \wedge g$ and $D\bar{\bar{\psi}} := d\bar{\bar{\psi}} + \bar{\bar{A}} \wedge \bar{\bar{\psi}} + i\bar{\bar{\psi}} \wedge \bar{\bar{g}}$. We have thus obtained an action which is the sum of the action resulting in [1] plus two Chern-Simons actions for the R-symmetry gauge fields $g$ and $\bar{g}$, and two minimally coupled terms for the gravitino fields $\psi$ and $\bar{\psi}$. The action (1) is clearly invariant under the infinitesimal $u(1,1|1) \oplus u(1,1|1)$ gauge transformations

$$\delta_\Lambda \Gamma = d\Lambda + \Gamma \star \Lambda - \Lambda \star \Gamma,$$

$$\delta_{\bar{\Lambda}} \bar{\Gamma} = d\bar{\Lambda} + \bar{\Gamma} \star \bar{\Lambda} - \bar{\Lambda} \star \bar{\Gamma},$$

$$\delta_{\bar{\Lambda}} \Gamma = \delta_\Lambda \bar{\Gamma} = 0.$$  

(4)

The infinitesimal gauge parameters $\lambda$ and $\bar{\lambda}$ can be written in the explicit form

$$\Lambda = \begin{pmatrix} \lambda^A \tau_A & \epsilon \\ -i\bar{\epsilon} & i\alpha \end{pmatrix}, \quad \bar{\Lambda} = \begin{pmatrix} \bar{\lambda}^A \tau_A & \bar{\epsilon} \\ -i\bar{\bar{\epsilon}} & i\bar{\alpha} \end{pmatrix},$$

(5)

so that the gauge transformations become

$\delta_{\lambda} \Gamma = d\lambda + \Gamma \star \lambda - \lambda \star \Gamma,$

$\delta_{\bar{\lambda}} \bar{\Gamma} = d\bar{\lambda} + \bar{\Gamma} \star \bar{\lambda} - \bar{\lambda} \star \bar{\Gamma},$

$\delta_{\lambda} \Gamma = \delta_{\bar{\lambda}} \bar{\Gamma} = 0.$
\[ \delta_L A = d\lambda + A \star \lambda - \lambda \star A + i(\epsilon \otimes \bar{\psi} - \psi \otimes \epsilon), \]
\[ \delta_L \psi = de + A \star \epsilon - \lambda \star \psi + i\psi \star \alpha - i\epsilon \star g, \]
\[ \delta_L g = d\alpha - \bar{\psi} \star \epsilon + \bar{\epsilon} \star \psi + ig \star \alpha - i\alpha \star g, \]  

(6)

and \( \tilde{A}, \tilde{\psi} \) and \( \tilde{g} \) transform exactly in the same way under \( \tilde{\delta}_\lambda \). The commutation rules for these transformations are given in appendix B.

Following the same argument and notation employed in Refs. [1,10], it is straightforward to show that the action (1) is also invariant under the transformation

\[ \Delta^\star \Gamma := \delta_i^\star \Gamma + i^\star D \Gamma, \]
\[ \Delta^\star \tilde{\Gamma} := \tilde{\delta}_i^\star \tilde{\Gamma} + i^\star D \tilde{\Gamma}, \]  

(7)

where \( D \Gamma = d\Gamma + \Gamma \star \wedge \Gamma \). These transformations reduce to the usual diffeomorphisms \( \mathcal{L}_v \Gamma \) in the \( \theta^{\mu\nu} \to 0 \) limit. This point has been discussed more deeply in the cited references.

Now we can make the same substitution as in Ref. [1],

\[ A = \omega + \frac{e}{l} + \frac{i}{2} b, \]
\[ \tilde{A} = -\omega - \frac{e}{l} + \frac{i}{2} b, \]  

(8)

(9)

with

\[ \omega = \omega^a \tau_a, \quad e = e^a \tau_a, \quad b = b \mathbb{I} \]  

(10)

so that the supergravity action can be written in the more familiar way

\[ S = -\kappa \int \epsilon_{abc} \left( e^a \wedge R^{bc} + \frac{1}{3!^2} e^a \wedge \bar{e} \wedge e^b \wedge e^c \right) \]
\[ -\frac{l\kappa}{2} \int \left( b \wedge db + \frac{i}{3} b \wedge b \wedge b \right) + \frac{l\kappa}{2} \int \left( \bar{b} \wedge d\bar{b} + \frac{i}{3} \bar{b} \wedge \bar{b} \wedge \bar{b} \right) \]
\[ + \frac{i\kappa}{2} \int \eta_{ab} \left( e^a \wedge \omega^b + \omega^a \wedge \bar{e} \wedge \omega^b \right) \wedge \left( b + \bar{b} \right) \]
\[ + \frac{i\kappa}{2} \int \eta_{ab} \left( \omega^a \wedge \omega^b + \frac{1}{l^2} e^a \wedge e^b \right) \wedge \left( b - \bar{b} \right) + \]
\[ + \kappa \int \text{Tr} \left( g \wedge dg + \frac{2i}{3} g \wedge g \wedge g \right) - \kappa \int \text{Tr} \left( \tilde{g} \wedge d\tilde{g} + \frac{2i}{3} \tilde{g} \wedge \tilde{g} \wedge \tilde{g} \right) + \]
\[ + 2i\kappa \int \bar{\psi} \wedge D\psi - 2i\kappa \int \bar{\tilde{\psi}} \wedge D\tilde{\psi}, \]  

(11)
where we have introduced

\[ R^{ab} = d\omega^{ab} + \frac{1}{2} \left( \omega^a_c \land \omega^{cb} - \omega^b_c \land \omega^{ca} \right) , \]

\[ T^a = de^a + \frac{1}{2} \left( \omega^a_b \land e^b + e^b \land \omega^a_b \right) , \]

(12)

with \( \omega^{ab} = \epsilon^{abc} \omega_c. \)

The natural definition of a metric is

\[ G_{\mu\nu} = e^a_{\mu} \land e^b_{\nu} \eta_{ab} = g_{\mu\nu} + i b_{\mu\nu} , \]

(13)

where

\[ g_{\mu\nu} = \frac{1}{2} \eta_{ab} \{ e^a_{\mu}, e^b_{\nu} \} \star \]

(14)

is real and symmetric, and reduces to the usual expression of the metric in the commutative case, whereas

\[ b_{\mu\nu} = -\frac{i}{2} \eta_{ab} [ e^a_{\mu}, e^b_{\nu} ] \star \]

(15)

is real and antisymmetric, and vanishes for \( \theta^{\mu\nu} = 0. \)

### 3 The BPS solutions

In this section we will look for supersymmetric solutions of our model, obtaining the noncommutative analogues of some classical supersymmetric solutions of commutative AdS\(_3\) supergravity, namely the “fuzzy” AdS\(_3\) and the massless nonrotating BTZ black hole. Finally, we will find a generalization of the massless BTZ black hole, including \( U(1) \) gauge fields \( b \) and \( g. \) To this end, we will follow the noncommutative version of the standard BPS construction.

As in the commutative case, one puts the fermionic fields to zero and looks for pure bosonic solutions of the equations of motion, satisfying the conditions \( \delta_{\epsilon} \psi = \delta_{\tilde{\epsilon}} \tilde{\psi} = 0 \) for some \( \epsilon, \) to end with a residual supersymmetry. At the moment, we can restrict to the untilded sector, the tilded one behaving identically. From the second line of Eq.(13), our conditions read

\[ de + A \land \epsilon - ie \land g = 0 . \]

(16)

From \( d^2 = 0 \) one gets the consistence condition \((dA + A \land A) \land \epsilon = ie \land (dg + ig \land g)\), which is always satisfied on-shell (when \( \psi = 0 \)). So, our noncommutative Killing
equation Eq. (16) seems to be always solvable, at least locally, when the fields $A$ and $g$ are on-shell. Note also from Eq. (3) that when $\psi = 0$, the fields $A$ and $g$ are completely decoupled, resulting in an $AdS_3$ noncommutative gravity (see [1]) plus two $U(1)$ Chern-Simons theories. However from the Killing equation Eq. (16) one sees that supersymmetry conditions restore a dependence between the $g$ and the $A$ fields.

In order to express the Killing conditions (16) in terms of the gravitational fields, as defined in Eq. (8), it is convenient to introduce the notation
\[
\hat{\delta} \equiv \delta(a_1, a_2) := \delta(a_1 + a_2) + \tilde{\delta}(a_1 - a_2),
\]
and to put $\psi_1 = \psi + \tilde{\psi}, \psi_2 = \psi - \tilde{\psi}, \epsilon = \epsilon_1 + \epsilon_2$ and $\tilde{\epsilon} = \epsilon_1 - \epsilon_2$. Now the generic two-parameters supersymmetric transformations of the fermions can be written as
\[
\hat{\delta}(\epsilon_1, \epsilon_2) \psi_1 = 2d\epsilon_1 + \omega^a \gamma_a \epsilon_1 + l^{-1} e^a \gamma_a \epsilon_2 + \frac{i}{2}(b + \tilde{b}) \epsilon_1 + \frac{i}{2}(b - \tilde{b}) \epsilon_2 - i\epsilon_1 \epsilon_2 \gamma_a \epsilon_2 \gamma_a \epsilon_2
\]
\[
\hat{\delta}(\epsilon_1, -\epsilon_2) \psi_2 = 2d\epsilon_2 + \omega^a \gamma_a \epsilon_2 + l^{-1} e^a \gamma_a \epsilon_1 + \frac{i}{2}(b + \tilde{b}) \epsilon_2 + \frac{i}{2}(b - \tilde{b}) \epsilon_1 - i\epsilon_1 \epsilon_2 \gamma_a \epsilon_1 \gamma_a \epsilon_1
\]
(17)

We now show that there is a maximally supersymmetric solution, which is the noncommutative correspondent of the $AdS_3$ space and that for brevity we will call the “fuzzy” $AdS_3$. To this end let us fix, once for all, the coordinates $(t, r, \phi)$ as parametrizing $\mathbb{R} \times \mathbb{C}$, where $\mathbb{C}$ is a ”fuzzy cylinder”, whose spatial coordinates $(r, \phi)$ satisfy the non commutativity condition
\[
[r, \phi] = i\theta.
\]
(18)

It is then easy to see that the ansatz
\[
e^0 = \sqrt{\frac{r^2}{l^2} + 1} dt =: N(r) dt
\]
\[
e^1 = N(r)^{-1} dr
\]
\[
e^2 = r d\phi
\]
(19)

and
\[
\omega^0 = -N(r) d\phi
\]
\[
\omega^1 = 0
\]
\[
\omega^2 = \frac{r}{l^2} dt,
\]
(20)
is a solution of the equations of motion when all the other fields are set to zero. This solution corresponds to a diagonal symmetric real metric $G_{\mu\nu}$ so that the corresponding imaginary part is $b_{\mu\nu} = 0$ whereas $g_{\mu\nu}$ is the same as in the commutative case
\[
ds^2 = -\left(\frac{r^2}{l^2} + 1\right) dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1} dr^2 + r^2 d\phi^2
\]
(21)

2It would be interesting to see if there really exists fuzzy construction to which this solution corresponds.
To solve eq. (17), it is convenient to consider $\epsilon_1$ proportional to $\epsilon_2$ and so, without loss of generality, treat separately the two independent choices $\epsilon_1 = \alpha \epsilon_2 = \epsilon$, with $\alpha = \pm 1$; this is equivalent to put $\tilde{\epsilon}$ or $\epsilon$ equal to zero respectively (and can be understood in terms of the two possible representations of the Dirac matrices in three dimensions [14]).

The BPS equations are then

\[
2\partial_t \epsilon - \frac{r}{l^2} \gamma_2 \star \epsilon + \frac{\alpha}{l} N(r) \gamma_0 \star \epsilon = 0 \\
2\partial_r \epsilon + \frac{\alpha}{l} N(r)^{-1} \gamma_1 \star \epsilon = 0 \\
2\partial_\phi \epsilon - N(r) \gamma_0 \star \epsilon + \frac{r}{l} \gamma_2 \star \epsilon = 0 .
\]

By means of the relations $\gamma_0 \gamma_1 \gamma_2 = 1$ and $(N(r) + 1)^{\frac{1}{2}}(N(r) - 1)^{\frac{1}{2}} = \xi$, it can be seen that the Killing spinors are

\[
\epsilon = \left[(N(r) + 1)^{\frac{1}{2}} - \alpha(N(r) - 1)^{\frac{1}{2}} \gamma_1 \right] \left[\cos \left(\frac{\phi}{2} - \frac{\alpha t}{2l}\right) - \sin \left(\frac{\phi}{2} - \frac{\alpha t}{2l}\right) \gamma_0 \right] \xi 
\]

for every choice of $\alpha$ and constant spinor $\xi$. As in the commutative case [14] there are four independent generators of supersymmetries.

Next we generalize to the case of a static BPS black hole. Let us make the ansatz

\[
e^0 = e^{A(r)} dt , \quad e^1 = e^{B(r)} dr \quad e^2 = e^{C(r)} d\phi .
\]

We consider again the case in which $b_{\mu \nu} = 0$, and $g_{\mu \nu}$ is given by

\[
ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + e^{2C(r)} d\phi^2 .
\]

Taking all the other fields but the spin connection, equal to zero one finds

\[
\omega^0 = -e^{-B} C' e^2 , \quad \omega^1 = 0 , \quad \omega^2 = -e^{-B} A' e^0 .
\]

The fuzzy $AdS_3$ found above clearly belongs to this more general class. In order to study the BPS equations (17), we closely follow [14] and assume that $\epsilon$ depends on $r$ only. In this way we are selecting particular coordinate systems so that for example the fuzzy $AdS_3$ case is excluded by this condition, as it is clear from Eq.(23). It would be interesting to study the possible equivalence of different solutions under the transformations (3).

Under the previous assumption, the BPS equations (17) read

\[
2\partial_t \epsilon + \frac{1}{l} e^1 \gamma_1 \epsilon = 0 , \\
-e^{-B} A' \gamma_2 \epsilon + \frac{1}{l} \gamma_0 \epsilon = 0 , \\
-e^{-B} C' \gamma_0 \epsilon + \frac{1}{l} \gamma_2 \epsilon = 0 .
\]
The first equation of (27) requires \( \epsilon^\pm = Q(r)\epsilon^{(0)}_\pm \) where \( Q(r) \) is a scalar function of \( r \) and \( \epsilon^{(0)}_\pm \) is a constant spinor satisfying
\[
\gamma_1 \epsilon^{(0)}_\pm = \pm \epsilon^{(0)}_\pm .
\]
(28)

Multiplying the second and the third equations of (27) by \( \gamma_2 \) and \( \gamma_0 \) respectively and choosing \( \alpha \) appropriately, it is now easy to obtain as a complete solution\(^3\)
\[
e^0 = \frac{r}{l} dt , \quad e^1 = \frac{l}{r} dr \quad e^2 = r d\phi ,
\]
(29)
which corresponds to the metric
\[
ds^2 = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2
\]
(30)
and has supersymmetry generators
\[
\epsilon^\pm = \sqrt{\frac{r}{l}} \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) .
\]
(31)

This is a fuzzy version of the BTZ black hole with zero mass and zero angular momentum\(^{14}\). Note that to obtain this solution one has to make peculiar choices of \( \alpha \): requiring a positive sign for \( e^B(r) \) needs \( \alpha \) to have a sign which is opposite to the one of the given eigenvalue of \( \gamma_1 \); as before, this is equivalent to choose only one of the two possible representations of the Dirac matrices in three dimensions\(^{14}\).

One can easily obtain a further solution, which is similar to (30), but has nonvanishing \( U(1) \) gauge fields. Let us take
\[
b = \tilde{b} = 2 g = 2 \tilde{g} = b_0[\phi dr + \beta r d\phi]
\]
(32)
with \( b_0 \) and \( \beta \) constants. Then it is easy to see that all the equations of motion are satisfied if \( \beta = 1/(1 + \theta b_0) \) and that a new solution is given by
\[
e^0 = \frac{r}{l} dt , \quad e^1 = \frac{l}{r} dr \quad e^2 = r d\phi ,
\]
\[
b = \tilde{b} = b_0 \phi dr + \frac{b_0}{1 + \theta b_0} r d\phi ,
\]
\[
g = \tilde{g} = \frac{b_0}{2} \phi dr + \frac{b_0}{1 + \theta b_0} \frac{r}{2} d\phi ,
\]
(33)
with supersymmetry generators:
\[
\epsilon^\pm = \left( \frac{r}{l} \right)^{\frac{1}{2+\theta b_0}} \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) .
\]
(34)
In the commutative case this solution reduces to a zero mass BTZ black hole plus four completely decoupled $U(1)$ gauge sectors given by

$$b = \tilde{b} = 2g = 2\tilde{g} = b_0[\phi dr + rd\phi] = -ie^{-ib_0\phi r}de^{ib_0\phi r};$$

(35)

the latter can then be written at least locally, as pure gauge solutions but not globally, and can thus be considered as non trivial statical field configurations. For example, the Wilson loop of $b$ along the closed curve $\gamma : \pi \mapsto (t_0, r_0, \phi)$ where $t_0$ and $r_0$ are constants and $\phi \in [0, 2\pi]$, is given by

$$W_\gamma[b] = \mathcal{P}e^{i\frac{\theta}{\pi} b} = \exp(ib_02\pi r_0),$$

(36)

so that the parameter $b_0$ can be considered as a coordinate in the moduli space of this kind of solutions. In the noncommutative case these $U(1)$ gauge fields acquire pure noncommutative couplings with the supergravity sector of the theory which forces the fields $b, \tilde{b}$ and $g, \tilde{g}$ to be equal, whereas in the commutative limit they are completely decoupled and need only satisfy the zero curvature condition separately. Moreover noncommutativity requires a modification of the dependence from the moduli coordinate $b_0$. The fact that for these solutions the noncommutative curvature of each $U(1)$ is equal to zero allows to apply the prescription given by Alekseev and Bytsko to calculate the noncommutative monodromies defined in [15], which reduce to ordinary Wilson loops in the case of $U(1)$ commutative gauge theory

$$M = G' \star G^{-1},$$

(37)

where

$$G = Ae^{\frac{i\theta}{2\pi r}\phi r}$$

(38)

satisfies the condition $dG = iG \star b$, $G'(r, \phi) := G(r, \phi + 2\pi)$ and $A$ is a constant that must be chosen in order to have $G' \star G = 1$. For example, at second order in $\theta$ we have

$$A = 1 + \frac{\theta^2b_0^2}{2} + O(\theta^3)$$

(39)

and so, after a straightforward calculation up to second order in $\theta$, the noncommutative monodromy, associated to the Wilson loop (36), results in:

$$M = e^{2\pi ib_0r} \left[1 + \theta^2 \left(\frac{i}{2} \pi b_0^3 r - \frac{1}{2} \pi^2 b_0^4 r^2\right)\right] + O(\theta^3).$$

(40)

Note that these field configurations have singularities for $b_0\theta^{-1} = -1, -2$: first of all this shows that there are solutions of the commutative theory which have no noncommutative counterpart, but as in the case of Yang-Mills theories [3], this also suggests that, in general,

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\text{\footnote{Here we have adapted the definition so as to take care of the fact that in our case the gauge action is on the right, contrarily to [13]}}$

singular solutions can arise, with a $\theta$ dependent singularity and that one should expect these to have no analogue in the correspondent commutative theory.

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A Conventions

An element $M$ of the Lie algebra $u(1,1)$ is a complex $2 \times 2$ matrix satisfying

$$M^j_i = -\eta_{jk}(M^k_i)^*\eta^{li},$$  \hspace{1cm} (41)

where the $*$ denotes complex conjugation and $\eta = \text{diag}(-1,1)$. We choose as $u(1,1)$ generators

$$\tau_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  
$$\tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \hspace{1cm} (42)$$

They are normalized according to

$$\text{Tr}(\tau_A\tau_B) = \frac{1}{2}\eta_{AB},$$  \hspace{1cm} (43)

where $(\eta_{AB}) = \text{diag}(-1,1,1,-1)$ is the inner product on the Lie algebra. The generators (42) satisfy the relation (41). Further, if $a,b,c$ assume the values $0,1,2$, then the following relations hold:

$$[\tau_a, \tau_b] = -\epsilon_{abc}\tau^c, $$  \hspace{1cm} (44)$$
$$[\tau_a, \tau_3] = 0, $$  \hspace{1cm} (45)$$

$$\tau_a \tau_b = \frac{1}{2} \epsilon_{abc} \tau^c - \frac{i}{2} \eta_{ab}\tau_3, $$  \hspace{1cm} (46)$$

$$\text{Tr}(\tau_a\tau_b\tau_c) = -\frac{i}{4} \epsilon_{abc}, $$  \hspace{1cm} (47)$$

$$\text{Tr}(\tau_a\tau_b\tau_3) = \frac{i}{4} \eta_{ab}, $$  \hspace{1cm} (48)$$

where $(\eta_{ab}) = \text{diag}(-1,1,1)$. Furthermore we defined $\epsilon_{012} = 1$.

A general element of the superalgebra $\Omega \in u(1,1|1)$ is a $3 \times 3$ supermatrix which must satisfy the condition

$$\Omega^a_b = -\eta_{bc}(\Omega^c_d)^*\eta^{da}. $$  \hspace{1cm} (49)$$

and can be written as

$$\Omega = \begin{pmatrix} M & \zeta \\ -i\bar{\zeta} & i\gamma \end{pmatrix}. $$  \hspace{1cm} (50)$$
where $M \in u(1, 1)$ and $\gamma$ constitute the bosonic part of $\Omega$, while $\zeta$ is a complex fermionic 2-dimensional vector in the spinorial representation of $so(2, 1) \simeq su(1, 1)$ given by the following representation of the three dimensional Clifford algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$:

$$\gamma_a := 2\tau_a \Rightarrow \gamma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

which satisfies $\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0$, $\gamma_a^\dagger \gamma_a = 1$ and $[\gamma_a, \gamma_b] = -2\epsilon_{abc} \gamma^c$. We have also introduced the notation $\tilde{\zeta} = \zeta^\dagger \gamma_0$.

## B The supersymmetry algebra

We can restrict to analyze the transformation of $\Gamma$, the “tilded” sector being identical; the gauge transformations (4) obey the commutation rules $[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = -2\epsilon_{abc} \gamma^c$. In terms of the new fields introduced in Eq. (50), they can be written in the following way:

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] A = -d[\lambda_1, \lambda_2]_* + i d(\epsilon_1 \otimes \bar{\epsilon}_2 - \epsilon_2 \otimes \bar{\epsilon}_1) + [[\lambda_1, \lambda_2]_*, A]_* - i[(\epsilon_1 \otimes \bar{\epsilon}_2 - \epsilon_2 \otimes \bar{\epsilon}_1), A]_* - i(\lambda_1 \epsilon_2 - \lambda_2 \epsilon_1) \otimes \bar{\psi} + i\psi \otimes (\bar{\epsilon}_1 \lambda_2 - \bar{\epsilon}_2 \lambda_1) + (\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1) \otimes \bar{\psi} - \bar{\psi} \otimes (\alpha_1 \bar{\epsilon}_1 - \alpha_2 \bar{\epsilon}_2),$$

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] \psi = -d(\lambda_1 \epsilon_2 - \lambda_2 \epsilon_1) - i d(\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1) + [\lambda_1, \lambda_2]_* \psi - i(\epsilon_1 \otimes \bar{\epsilon}_2 - \epsilon_2 \otimes \bar{\epsilon}_1) \psi + i(\lambda_1 \epsilon_2 - \lambda_2 \epsilon_1) \gamma^g - (\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1) \gamma^g - A \gamma(\lambda_1 \epsilon_2 - \lambda_2 \epsilon_1) - i\psi(\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1) + \psi \gamma(\alpha_1 \epsilon_2 - \alpha_2 \epsilon_1) + \gamma \psi(\alpha_1 \alpha_2 - \alpha_2 \alpha_1),$$

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] g = d(\epsilon_1 \epsilon_2 - \bar{\epsilon}_2 \epsilon_1) - i d[\alpha_1, \alpha_2]_* - (\epsilon_1 \lambda_2 - \bar{\epsilon}_2 \lambda_1) \gamma \psi - i(\alpha_1 \bar{\epsilon}_1 - \alpha_2 \bar{\epsilon}_2) \gamma \psi - \gamma \psi(\lambda_1 \epsilon_2 - \lambda_2 \epsilon_1) + \gamma \psi(\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1).$$
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