Book inequalities

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Abstract—Information theoretical inequalities have strong ties with polymatroids and their representability. A polymatroid is entropic if its rank function is given by the Shannon entropy of the subsets of some discrete random variables. The book is a special iterated adhesive extension of a polymatroid with the property that entropic polymatroids have \( n \)-page book extensions over an arbitrary spine. We prove that every polymatroid has an \( n \)-page book extension over a single element and over an all-but-one-element spine. Consequently, for polymatroids on four elements, only book extensions over a two-element spine should be considered. F. Matúš proved that the Zhang-Yeung inequalities characterize polymatroids on four elements which have such a 2-page book extension. The \( n \)-page book inequalities, defined in this paper, are conjectured to characterize polymatroids on four elements which have \( n \)-page book extensions over a two-element spine. We prove that the condition is necessary; consequently every book inequality is an information inequality on four random variables. Using computer-aided multiobjective optimization, the sufficiency of the condition is verified up to 9-page book extensions.

Keywords: Entropy; information inequality; polymatroid; adhesive extension.

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I. INTRODUCTION

The entropy function of \( N \) random variables \( (x_i : i \in N) \) maps the non-empty subsets \( I \subseteq N \) to the Shannon entropy \( H(\xi_I) \) of the variable set \( \xi_I = \langle x_i : i \in I \rangle \). The range of the entropy function, a subset of the Euclidean space, is denoted by \( H_N \). The closure \( \overline{H_N} \) (in the usual Euclidean topology) of \( H_N \) is a closed, convex, pointed cone, and \( H_N \) misses only some boundary points as shown in \([10]\).

The region \( \overline{H_N} \) is bounded by linear facets corresponding to some Shannon entropy inequalities. Hyperplanes cutting into the Shannon polyhedron and containing all entropic points on one side are the non-Shannon linear information inequalities. The first such inequality was found by Zhang and Yeung \([16]\). Later the list of such inequalities has been extended significantly, see \([3, 11, 15]\). The method of Zhang and Yeung motivated the definition of adhesive extensions of polymatroids by F. Matúš in \([9]\). An alternative technique for generating non-Shannon inequalities was suggested by K. Makarychev et al \([8]\), which later was found to rely on the same extension property of entropic polymatroids \([5, 12]\).

Section \([11]\) recalls some notation and terminology related to polymatroids; for a detailed account, see \([7]\). Section \([11]\) describes the book, a special iterated adhesive extension. Generalizing results from \([9] \) and \([12]\), we prove that book extensions always exist when the spine of the book has one element, or has all but one elements of the ground set. Sections \([14, 15]\) concentrate on the case \( N = 4 \). Section \([14]\) defines the collection of book inequalities, which is conjectured to characterize polymatroids on four elements which have \( n \)-page book extensions. In Section \([15]\) we prove the necessary part of the conjecture, that is, that book inequalities hold for polymatroids with \( n \)-page book extensions. As entropic polymatroids have this extension property, book inequalities are, consequently, entropy inequalities. The book inequalities contain, among others, one of the infinite lists of Matúš in \([11]\), the list of Dougherty et al \([8]\) Theorem \([10]\), and provide infinitely many new information inequalities. The sufficiency part of the conjecture is left as an open problem.

The collection of book inequalities along with the conjecture that they characterize the book extensions were announced at the First Workshop on Entropy and Information Inequalities held in Hong Kong, April 15–17, 2013. After the conference Randall Dougherty (personal communication) pointed out a misprint in the formulation of inequalities in \([8]\), and supplied a proof for the correct version. In this paper an alternate proof of his result is given along the way inequalities in \([2]\) are proved.

II. DEFINITIONS AND NOTATION

Let \( N \) be a finite set, and \( g \) be a real-valued function on the non-empty subsets of \( N \). The pair \( \langle g, N \rangle \) is a polymatroid if \( g \) is non-negative and non-decreasing: that is, \( 0 \leq g(I) \leq g(J) \) for \( I \subseteq J \subseteq N \); and submodular:

\[
g(I) + g(J) - g(I \cup J) - g(I \cap J) \geq 0, \quad I, J \subseteq N.
\]

Here \( N \) is the ground set, and \( g \) is the rank function. Polymatroids and their rank functions are frequently identified. Shannon inequalities for discrete random variables express the fact that an entropy function is a polymatroid. Polymatroids coming from entropy functions are called entropic, and those in the closure of entropic polymatroids are almost entropic.

For \( I \subseteq N \) let \( \delta_I \) be the \( 2^N - 1 \)-dimensional unit vector whose \( I \)-coordinate is equal to 1, and all other coordinates are 0. Writing

\[
(I, J) \overset{\text{def}}{=} \delta_I + \delta_J - \delta_{I \cup J} - \delta_{I \cap J},
\]

the expression \( (I, J) \cdot g \) can be interpreted as the scalar product of \( (I, J) \) with \( g \), thus the submodularity of \( g \) can be expressed as

\[
(I, J) \cdot g \geq 0, \quad I, J \subseteq N.
\]

We will also use other abbreviations for certain information theoretic expressions:

\[
(I, J \mid K) \overset{\text{def}}{=} \delta_{I \cup K} + \delta_{J \cup K} - \delta_{I \cup J \cup K} - \delta_{I \cap J \cup K},
\]

\[
[I, J, K, L] \overset{\text{def}}{=} -(I, J) + (I, J \mid K) + (I, J \mid L) + (K, L).
\]
For any polymatroid \( g, (I, J | K) \cdot g \geq 0 \) follows from submodularity and monotonicity. \([I, J, K, L] \cdot g \geq 0 \) is the so-called Ingleton inequality\[4\], and it holds when \( g \) is linearly representable over a field, but not necessarily holds when \( g \) is only (almost) entropic.

Following the usual practice, the union symbol is omitted as well as the curly brackets around singletons. Thus, for example, \( aI \) denotes the set \( \{a\} \cup I \), and

\[
((abcd) + (a,b|c) + (a,c|b) + (b,c|a)) \cdot g \geq 0
\]

is an equivalent form of the Zhang-Yeung inequality\[16\] on the four-element set \( N = \{a, b, c, d\} \). Additionally, we omit the commas in the Ingleton notation \( [a, b, c, d] \) as we did it above, and even the polymatroid \( g \) is omitted when it is clear from the context which polymatroid we are referring to.

The symbol \( \cup^* \) is used to emphasize that the sets whose union is taken are disjoint.

A. Operations on polymatroids

This section recalls some basic operations on polymatroids and their properties.

1) Direct sum: The direct sum of polymatroids \( (g_i, N_i) \) for \( i = 1, \ldots, n \) is the polymatroid \( (g, N) \) where the ground set \( N \) is the disjoint union \( N_1 \cup^* \cdots \cup^* N_n \), and for every \( I_i \subseteq N_i, i = 1, \ldots, n \), the value of \( g \) is defined as

\[
g(I_1 \cup^* \cdots \cup^* I_n) = g_1(I_1) + \cdots + g_n(I_n).
\]

2) Independence: Let \( (g, N) \) be a polymatroid, and \( P_1, P_2, S \) be disjoint subsets of the ground set \( N \). \( P_1 \) and \( P_2 \) are independent over \( S \) when \( (P_1, P_2 | S) = 0 \), that is, when

\[
g(P_1S) + g(P_2S) - g(P_1P_2S) - g(S) = 0.
\]

In matroid terminology, \( (P_1S, P_2S) \) is a modular pair of \( g \). Let \( P_1, \ldots, P_n \) and \( S \) be disjoint subsets of \( N \). The \( P_i \)s are totally independent over \( S \) if for any two disjoint subsets \( \{i_1, i_2, \ldots, i_{n_1}\} \) and \( \{j_1, j_2, \ldots, j_{n_2}\} \) of the indices \( 1, 2, \ldots, n \)

\[
(P_1, P_{i_1}, \ldots, P_{i_{n_1}}, P_{j_1}, P_{j_2}, \ldots, P_{j_{n_2}} | S) = 0.
\]

In this case the collection \( \{P_1S, P_2S, \ldots, P_nS\} \) is called a modular set. We will use the notation \( \langle i \rangle \) to denote the set \( \{i_1, i_2, \ldots, i_{n_1}\} \), and \( P_{\langle i \rangle} \) to denote the disjoint union \( \bigcup \{P_i : i \in \langle i \rangle \} \). Condition \( 1 \) can be written more succinctly as

\[
(P_{\langle i \rangle}, P_{\langle j \rangle} | S) = 0
\]

for disjoint subsets \( \langle i \rangle, \langle j \rangle \) of \( \{1, 2, \ldots, n\} \).

3) Restriction: Restricting the rank function of the polymatroid \( (g, N) \) to the subsets of \( M \subseteq N \) gives the polymatroid \( g | M \), the restriction of \( g \) to \( M \); furthermore, \( g \) is the extension of its restrictions. Restricting an (almost) entropic polymatroid gives an (almost) entropic polymatroid.

4) Pullback: Let \( \varphi \) map \( N' \) into \( N \), and let \( g \) be a polymatroid on \( N \). The pullback \( \varphi^{-1}g \) is the polymatroid defined on the ground set \( N' \) by

\[
(\varphi^{-1}g)(I') = g(\varphi(I')) \quad \text{for all } I' \subseteq N'.
\]

Thus, for example, restricting \( g \) to \( M \subseteq N \) is the same as the pullback \( \text{Id}_M^{-1}g \), where \( \text{Id}_M \) is the identity map on \( M \). Again, the pullback of an (almost) entropic polymatroid is (almost) entropic.

B. A technical lemma

The following lemma describes a polymatroid construction. It will be used in the proof of the main result in Section III.

**Lemma 1.** Let \( (g, N) \) be a polymatroid, \( a \in N \), and \( t \leq g(a) \). Define the function \( h \) on the non-empty subsets \( J \subseteq N \) as follows:

\[
h(J) = \min \{ g(J), g(aJ) - t \}.
\]

Then, \( (h, N) \) is a polymatroid.

**Proof:** The condition \( t \leq g(a) \) gives \( g(aJ) - t \geq 0 \), thus \( h \) is non-negative. As the monotonicity of \( h \) is clear, only the subadditivity needs to be checked. Distinguishing four cases depending on where the minimum is taken in \( h(I) \) and \( h(J) \), in each case the submodularity of \( g \) entails that their sum is at least as large as \( h(I \cup J) + h(I \cap J) \).

C. Tightening

Let \( (g, N) \) be a polymatroid, and \( a \in N \). The polymatroid \( (g\downarrow a, N) \) is defined as follows. For each \( I \subseteq N - \{a\} \),

\[
(g\downarrow a)(I) = g(I),
\]

\[
(g\downarrow a)(aI) = g(aI) - (g(N) - g(N - a)).
\]

Applying Lemma 1 with \( t = g(N) - g(N - a) \), and observing that \( g(J) \leq g(aJ) - t \) by submodularity of \( g \), we see that \( g\downarrow a \) is indeed a polymatroid on \( N \). Moreover, this operation is idempotent: \( (g\downarrow a) \downarrow a = g\downarrow a \), and commutative: \( (g\downarrow a)b = (g\downarrow b)a \). For subsets \( J \subseteq N \) we define \( g\downarrow J \) as follows. If \( J = \{a_1, \ldots, a_k\} \), then we let

\[
g\downarrow J = \langle \cdot \cdot \cdot ((g\downarrow a_1)\downarrow a_2)\downarrow \cdot \cdot \cdot \rangle a_k.
\]

By commutativity, the result depends only on the subset \( J \) and not on the order of its elements. As \( g = g\downarrow a \) if and only if \( g(N) = g(N - \{a\}) \), it follows that \( g = g\downarrow N \) if and only if every co-singleton has full rank. Such polymatroids are called tight in \[12\].

III. Book extension

The notion of adhesive extension, introduced by F. Matuš in \[9\], captures the essence of the Zhang-Yeung method which can be outlined as follows. Suppose that the rank function \( g \) is given by the Shannon entropy of the subsets of the random variables \( \tilde{x}, \tilde{s} \). Using the terminology of Dougherty et al \[3\], the collection of random variables \( \tilde{y} \) is a copy of \( \tilde{x} \) over \( \tilde{s} \) if \( \tilde{x} \) and \( \tilde{y} \) are independent over \( \tilde{s} \); otherwise, \( \langle \tilde{x}, \tilde{s} \rangle \) and \( \langle \tilde{y}, \tilde{s} \rangle \) have the same distribution. The polymatroid \( h \) defined by the
entropies of the (subsets of the) random variables \(\langle \tilde{x}, \tilde{y}, \tilde{s} \rangle\) extends \(g\) in two different ways: \(g\) can be embedded as \(\langle \tilde{x}, \tilde{s} \rangle\) or as \(\langle \tilde{y}, \tilde{s} \rangle\), and these instances of \(g\) form a modular pair in \(h\). Polymatroids with this special embeddability property are called self-adhesive at \(\tilde{s}\) in \([9]\).

In the above process we could add several independent copies of \(\tilde{x}\) instead of adding just a single copy. The book extension generalizes Matuš’ notion of adhesivity along this line. This generalization, however, does not increase the strength of the iterated method as \(n\) consecutive copy steps over the same set of variables give \(2^n\) many totally independent copies of the pasted variables.

**Definition 2 (Book extension).** Let \(\langle g, P \cup^* S \rangle\) be a polymatroid. \(\langle h, M \rangle\) is an \(n\)-page book extension of \(g\) over \(S\), if the ground set of \(h\) is the disjoint union \(M = P_1 \cup^* \ldots \cup^* P_n \cup^* S\) such that

(i) \(P_1, \ldots, P_n\) are totally independent over \(S\);

(ii) for \(i = 1, 2, \ldots, n\) there are bijections \(\varphi_i : P \cup^* S \leftrightarrow P_i \cup^* S\) which are identity on \(S\) and the pullback of \(h\) along \(\varphi_i\) is \(g = \varphi_i^{-1} h\).

We write \(g \prec_S^h h\) to denote that \(h\) is an \(n\)-page book extension of \(g\) over \(S\).

We use the picturesque name *book* for such an extension \(h\). \(S\) is the spine of the book, and the \(P_i\)'s are its pages. A 2-page extension with spine \(S\) is the same as the adhesive extension at \(S\) in \([9]\). This book is not too interesting as all of its pages are the same, the interesting features come from the interaction between the pages.

A book extension over an empty spine is the same as the direct sum, and when \(S\) is the full ground set, then there is no condition to satisfy. Moreover, as every polymatroid is a 1-page book extension of itself, we always assume that \(n \geq 2\), and the spine \(S\) is a proper, non-empty subset of the ground set of \(g\). The following properties of the book extension follow immediately from the definition.

**Proposition 3.** a) If \(g \prec_S^h h\) and \(h \prec_S^{h'} h'\), then \(g \prec_S^{k} h'\). b) If \(g \prec_S^h\) and \(h' \prec_S^h\) restricted to \(S\) and \(k\) of its pages, then \(g \prec_S^{k} h'\). In particular, if \(g\) has an \(n\)-page book extension, then it has \(\ell\)-page extensions for every \(\ell < n\). \(\square\)

Let \(\langle h, M \rangle\) be an \(n\)-page extension of \(\langle g, N \rangle\) over \(S\) with bijection \(\varphi_i\) between \(PS = P \cup S\) and \(P_iS\). Any permutation \(\pi\) of the page indices \(\{1, 2, \ldots, n\}\) determines a permutation \(\sigma_\pi\) of the ground set \(M\) by keeping \(S\) fixed, and by permuting the pages according to \(\pi:\)

\[
\sigma_\pi(a) = \begin{cases} a & \text{if } a \in S, \\ \varphi_\pi(i) & \text{if } a \in P_i. \end{cases}
\]

Subsets \(I\) and \(J\) of \(M\) are called symmetrical if \(\sigma_\pi(I) = J\) for some permutation \(\pi\) of the pages. This happens if and only if the following two conditions hold: \(I\) and \(J\) intersect the spine in the same set: \(I \cap S = J \cap S\); and the \(n\)-element multisets \(\{\varphi_1^{-1}(P_i \cap J)\}\) and \(\{\varphi_i^{-1}(P_i \cap J)\}\), which consist of subsets of \(P\) with multiplicity, are the same. We call the extension \(h\) symmetrical if symmetrical subsets have the same \(h\)-value.

**Proposition 4.** The polymatroid \(g\) has an \(n\)-page extension if and only if it has such a symmetrical extension.

**Proof:** Let \(h\) be an \(n\)-page book extension of \(g\). For any permutation \(\pi\) of the pages define the polymatroid \(\pi h\) on \(M\) so that \((\pi h)(I) = h(\sigma_\pi(I))\). This polymatroid is also an \(n\)-book extension of \(g\) with the same bijections \(\varphi_i\), and consequently

\[
\frac{1}{n!} \sum \pi h
\]

is again an \(n\)-page book extension of \(g\), which is symmetrical. \(\square\)

The next theorem is a generalization of \([12]\) Theorem 3. It will be used in proving the main result of this section, Theorem 6 and it essentially shows that for book extensions it is enough to consider tight polymatroids.

**Theorem 5.** a) Suppose there is an \(n\)-page book extension of \(\langle g, N \rangle\) over \(S\). Then \(g \downarrow M\) also has an \(n\)-page book extension over \(S\). b) Suppose \(\langle h', M \rangle\) is an \(n\)-page book extension of \(g' = g \downarrow N\) over \(S\). Then there is an \(n\)-page book extension \(g \prec_S^h h\) such that \(h_iM = h'\).

**Proof:** Part a) follows by induction on the number of elements in \(N\) from the following claim: if \(a \in N\) and \(g\) has an \(n\)-page book extension \(\langle h, M \rangle\), then so has \(g \downarrow a\). So fix \(a \in N\), and the extension \(g \prec_S^h h\). Let \(\varphi_i\) be the bijection between \(PS\) and \(P_iS\), and let \(\ell = g(N) - g(N-a)\). Consider first the case when \(a \in S\). Define \(h'\) on the subsets of \(M\) by \(h'(J) = \min \{ h(J), h(aJ) - \ell \}\).

This is a polymatroid by Lemma 1 and \(g \downarrow a \prec_S^h h'\). Indeed, the \(\varphi_i\) pullback of \(h'\) is \(g \downarrow a\) trivially. Furthermore, as \(a \in S\), \(h'(P_iS) = h(P_iS) - \ell\) for any subset \(i\) of \(\{1, 2, \ldots, n\}\), thus \((P_i, P_iS) \cdot h = 0\) implies \((P_i, P_iS) \cdot h' = 0\), that is, the pages are totally independent over \(S\) in \(h'\) as well.

In the second case \(a \notin P\). We denote \(\varphi_i(a) \in P_i\) by \(a_i\), and call it the *twin* of \(a\). Let \(h_0 = h\), and define for \(1 \leq \ell \leq n\) the polymatroid \(h_\ell\) on \(M\) as follows:

\[
h_\ell(J) = \min \{ h_{\ell-1}(J), h_{\ell-1}(a_{\ell} J) - \ell \}.
\]

The following holds: the \(\varphi_i\) pullback of \(h_\ell\) is \(g \downarrow a\) when \(i \leq \ell\), and is \(g\) otherwise; and the pages are totally independent over \(S\) in \(h_\ell\). This is true for \(\ell = 0\), and we prove it by induction for all \(\ell \leq n\) below. Thus \(h' = h_n\) is an \(n\)-page extension of \(g \downarrow a\), which completes the induction step for part a).

Suppose the above claim for \(\ell - 1\); pick \(i \neq \ell\) and \(J \subseteq P_i\) arbitrarily. By submodularity and by the induction assumption

\[
h_{\ell-1}(a_{\ell}J) - h_{\ell-1}(J) \geq h_{\ell-1}(a_{\ell}P_iS) - h_{\ell-1}(P_iS) = h_{\ell-1}(a_{\ell}S) - h_{\ell-1}(S) = g(aS) - g(S) \geq t,
\]

which proves \(h_\ell(J) = h_{\ell-1}(J)\), that is, for \(i \neq \ell\) the \(\varphi_i\) pullbacks of \(h_\ell\) and \(h_{\ell-1}\) are the same. The \(\varphi_i\) pullback of \(h_\ell\) is clearly \(g \downarrow a\). Finally, the independence of the pages in \(h_\ell\) follows from their independence in \(h_{\ell-1}\) and from

\[
h_\ell(P_iS) = \begin{cases} h_{\ell-1}(P_iS) & \text{if } \ell \notin \{i\}, \\ h_{\ell-1}(P_iS) - t & \text{if } \ell \in \{i\}. \end{cases}
\]
For part b), let \( a \in N \), \( g' = g \downarrow a \), and suppose \( g' \prec_n h' \). We claim that (i) if \( a \in S \), then \( h \downarrow a = h' \); and (ii) if \( a \in P \), then \( h \downarrow a_1 \ldots a_n = h' \). In case (i) first we check \( h' = h' \downarrow a \). The \( \varphi_t \)-pullback of \( h' \) is \( g' \), \( \varphi_t(N) = g'(N - \{ a \}) \), thus

\[
0 \leq h'(M) - h'(M - \{ a \}) \\
\leq h'(P(S)) - h'(P(S) - \{ a \}) \\
= g'(P(S)) - g'(P(S) - \{ a \}) = 0,
\]

establishing \( h' = h' \downarrow a \). Now let \( t = g(N) - g(N - \{ a \}) \), then \( g' = g \downarrow a \) means that for every \( I \subseteq N - \{ a \}, \) \( g(I) = g'(I) \), and \( g(aI) = g'(aI) + t \). Let us define the polymatroid \( h \) on the ground set \( M \) by

\[
h(J) = h'(J), \\
h(aJ) = h'(aJ) + t, \quad J \subseteq M - \{ a \}.
\]

It is clear that \( g \) is the \( \varphi_t \)-pullback of \( h \) and \( h \downarrow a = h' \downarrow a \). To conclude that \( g \prec_n h \), only the independence of pages in \( h' \) should be checked. To this end let \( \langle i \rangle \) and \( \langle j \rangle \) be two disjoint non-empty subsets of \( \{ 1, 2, \ldots, n \} \). Then

\[
h(P_{\langle i \rangle} S) + h(P_{\langle j \rangle} S) = 2t + h'(P_{\langle i \rangle} S) + h'(P_{\langle j \rangle} S) = 2t + h'(P_{\langle i \rangle} P_{\langle j \rangle} S) + h'(S) = h(P_{\langle i \rangle} P_{\langle j \rangle} S) + h(S).
\]

Here we used the facts that \( a \in S \), and \( P_{\langle i \rangle} \) and \( P_{\langle j \rangle} \) are independent in \( h' \). This concludes part (i).

In case (ii), when \( a \in P \), \( h' = h' \downarrow a_1 \ldots a_n \) follows as above. Setting \( t = g(N) - g(N - \{ a \}) \), define the polymatroid \( h \) by

\[
h(J) = h'(J), \\
h(aJ) = h'(aJ) + t \cdot |\langle i \rangle|, \quad J \subseteq M - \{ a_1, \ldots, a_n \},
\]

where \( |\langle i \rangle| \) is the cardinality of the set \( \langle i \rangle \). Now \( h' \downarrow a_1 \ldots a_n = h' \downarrow a_1 \ldots a_n = h' \), and as \( a_i \in P_{\langle i \rangle} \), the independence also holds:

\[
h(P_{\langle i \rangle} S) + h(P_{\langle j \rangle} S) = (h'(P_{\langle i \rangle} S) + t \cdot |\langle i \rangle|) + (h'(P_{\langle j \rangle} S) + t \cdot |\langle j \rangle|) = (h'(P_{\langle i \rangle} P_{\langle j \rangle} S) + t \cdot (|\langle i \rangle| + |\langle j \rangle|)) + h'(S) = h(P_{\langle i \rangle} P_{\langle j \rangle} S) + h(S).
\]

Claim b) follows from (i) and (ii) by induction on the number of the elements of \( N \).

A co-singleton is a subset which misses only one element.

**Theorem 6.** Every polymatroid \( (g, N) \) has an \( n \)-page extension over singletons and co-singletons.

**Proof:** First let \( S = \{ a \} \) and \( P = N - \{ a \} \). Let \( \langle f, N^* \rangle \) be the direct sum of \( n \) disjoint copies of \( (g, N) \) where \( N^* = N_1 \cup \ldots \cup N_n \). Denote \( N_i - \{ a_i \} \) by \( P_i \) where \( a_i \in N_i \) is the copy of \( a \), and let \( M = P_1 \cup \ldots \cup P_n \cup \{ a \} \). Define the map \( \varphi : N^* \to M \) so that \( \varphi(a_i) = a \), otherwise \( \varphi \) is the identity. Applying Lemma [1] to the pullback \( \varphi^{-1} f \) and \( t = (n - 1)g(a) \) gives the polymatroid \( h \) on \( M \), which will be the required extension. The independence of \( P_1, \ldots, P_n \) over \( \{ a \} \) follows from the fact that

\[
h(aP_{\langle i \rangle}) = g(a) + (|\langle i \rangle|)(g(aP) - g(a)).
\]

The restrictions of \( h \) to \( P_i \cup \{ a \} \) are clearly isomorphic to \( g \).

To prove the second claim of the theorem, suppose that \( S \subseteq N \) is a co-singleton and \( P = N - S = \{ a \} \). By Theorem [5] it is enough to show that \( g' = g \downarrow N \) has a \( k \)-page extension. Let \( M = \{ a_1, \ldots, a_n \} \cup S \), and let \( \varphi(a_i) = a \), \( \varphi(S) = 1 \) \( \downarrow |S| \). Define \( h' \) on \( M \) as the pullback \( \varphi^{-1} g' \). Then, \( h' \) is a polymatroid; moreover, \( a_1, \ldots, a_n \) are totally independent over \( S \) as \( h'(a_1) = g'(a) = g'(S) \). Consequently, \( g' \prec_n h' \), which was to be shown.

**IV. Book Inequalities**

From this section on, we concentrate on polymatroids on a four-element ground set \( N \), whose elements will be denoted by the letters \( a, b, c, \) and \( d \). The structure of these polymatroids with a special emphasis on entropic representability have been studied extensively in [1], [2], [3], [6], [12], [13], [14], [15]. According to Theorem [6], every polymatroid on a four-element set has book extensions over singletons and over three-element subsets. Existence of book extensions over the two-element subset \( S = \{ a, b \} \) can be characterized in terms of linear inequalities: such a polymatroid \( g \) has an \( n \)-page book extension over \( S \) if and only if it satisfies a certain collection of linear inequalities. As the existence of a book extension is invariant for permutations of the ground set which keep the spine \( S \) fixed, this characterizing set of inequalities is also invariant under these permutations. In this case the stabilizer of the spine \( S = \{ a, b \} \) is generated by two permutations, those which swap \( a \leftrightarrow b \) and \( c \leftrightarrow d \), respectively; thus, the collection of inequalities is invariant under these swaps of variables. For the 2-page case F. Matuš provided the following characterization.

**Theorem 7 (Matuš [9] Theorem 3).** A polymatroid \( g \) on the four-element set \( \{ a, b, c, d \} \) has a 2-page extension over \( ab \) if and only if the following instances of the Zhang-Yeung inequality, and their \( a \leftrightarrow b \) and \( c \leftrightarrow d \) versions, hold for \( g \):

\[
\begin{align*}
[abcd] + (a, b | c) + (a, c | b) + (b, c | a) & \geq 0, \\
[bdac] + (a, b | d) + (b, d | a) + (a, d | b) & \geq 0.
\end{align*}
\]

Using computer-aided multiobjective optimization, the characterizing collection of linear inequalities were generated for up to 9-page book extensions. Based on these experiments, the collection of \( n \)-page book inequalities is defined below, and it is conjectured to characterize polymatroids which have \( n \)-page book extensions over \( ab \). In Section [M] these inequalities are shown to hold for such polymatroids, thus they are information inequalities. The sufficiency of the characterization is left as an open problem.

The description of the book inequalities is rather involved. The set of non-negative integers \( \{ 0, 1, \ldots, n \} \) is denoted by \( N \). Among the finite subsets of the non-negative lattice points \( N \times N \), the following subsets will be of particular interest for integers \( n \geq 2 \):

\[
\begin{align*}
u_n & \triangleq \{(k, 0) \in N \times N : k \leq n - 2\}, \\
t_n & \triangleq \{(k, \ell) \in N \times N : k + \ell \leq n - 2\}.
\end{align*}
\]
For \( k, \ell \in \mathbb{N} \) the three-dimensional integer vector \( v_{k,\ell} \) is defined as
\[
v_{k,\ell} = \left( \begin{array}{c} k + \ell \\ k \end{array} \right) (1, k + 1, \ell).
\]

For example, \( v_{0,0} = \langle 1, k + 1, 0 \rangle \), \( v_{0,1} = \langle 1, 1, \ell \rangle \) and \( v_{-1,1} = \langle k, k^2, k \rangle \), \( v_{1,-1} = \langle \ell, 2\ell, \ell^2 - \ell \rangle \). For a finite subset \( s \) of the lattice points let \( v_s \) be the sum of the vectors \( v_{k,\ell} \) when \( \langle k,\ell \rangle \) runs over \( s \):
\[
v_s \stackrel{\text{def}}{=} \sum \{v_{k,\ell} : \langle k,\ell \rangle \in s\}.
\]

This value, computed for the subsets \( u_n, v_n \) and \( t_n \), gives
\[
\begin{align*}
v_{u_n} &= \sum_{k \leq n} v_{k,0} = \langle n - 1, n(n - 1)/2, 0 \rangle, \\
v_{v_n} &= \sum_{\ell \leq n} v_{0,\ell} = \langle n - 1, n - 1, (n - 1)(n - 2)/2 \rangle, \\
v_{t_n} &= (2n - 1)^2 - 1, (n - 1)^2 n - 2, (n - 2)2^{n-2} + 1.
\end{align*}
\]

The value \( v_s \) is symmetrical in the following sense: if \( v_s = \langle x_s, y_s, z_s \rangle \) and \( s^T \) is the transpose of \( s \), then \( v_s^T = \langle x_s, y_s + x_s, y_s - x_s \rangle \).

The subset \( s \) of the lattice points is downward closed if from \( \langle k,\ell \rangle \in s \) and \( 0 \leq k' \leq k, \ 0 \leq \ell' \leq \ell \), it follows that \( \langle k',\ell' \rangle \in s \). In particular, \( u_n, v_n \) and \( t_n \) are downward closed sets. For \( n \geq 2 \) let us define \( S_n \) as the collection of downward closed subsets of \( t_n \):
\[
S_n \stackrel{\text{def}}{=} \{s \subseteq t_n : s \text{ is downward closed}\}.
\]

\( S_n \) is a subset of \( S_{n+1} \), and \( u_n, v_n, t_n \) are elements of \( S_n \). The family \( S_2 \) has a single one-element subset \( \{0,0\} \), which is the same as \( u_2 = v_2 = t_2 \). \( S_3 \) has three additional subsets: \( u_3, v_3 \), and \( t_3 \). Elements of \( S_4 - S_3 \) are depicted on Figure 1; the subsets \( u_4, v_4 \), and \( t_4 \) are marked.

**Definition 8 (Book inequalities).** Let \( n \geq 2 \). The collection \( B_n \) of \( n \)-page book inequalities is the following set of inequalities on polymatroids on the four-element set \( abcd \):
\[
x_s[abcd] + (a, b | c) + y_s((a, c | b) + (b, c | a)) \\
+ z_s((a, d | b) + (b, d | a)) \geq 0,
\]
where \( \langle x_s, y_s, z_s \rangle = v_s \) and \( s \) runs over the set \( S_n \); plus the inequalities
\[
\ell [bdac] + (a, b | d) + \frac{\ell(\ell + 1)}{2} ((b, d | a) + (a, d | b)) \geq 0, \quad (3)
\]
where \( \ell = 1, 2, \ldots, n - 1 \).

The collection of \( n \)-page book inequalities is increasing: every inequality in \( B_n \) is in \( B_{n+1} \) as well. \( B_2 \) consists of the two inequalities which appeared in Theorem 2; this is so as \( S_2 \) has the only element \( \{0,0\} \), and \( v_{0,0} = \langle 1, 1, 0 \rangle \). The sequence \( B_n \) contains two previously identified infinite lists of entropy inequalities. Setting \( s = u_n \in S_n \), inequality 3 becomes
\[
(n - 1)[abcd] + (a, b | c) + \frac{n(n - 1)}{2} ((a, c | b) + (b, c | a)) \geq 0,
\]
which is one of the (implicit) infinite families of new entropy inequalities from [11] Theorem 2. When \( s = t_n \in S_n \), then (2) becomes
\[
\begin{align*}
(2n - 1)^2 [abcd] &+ (a, b | c) \\
+ (n - 1)2^{n-2} ((a, c | b) + (b, c | a)) \\
+ ((n - 2)2^{n-2} + 1) ((a, d | b) + (b, d | a)) \geq 0,
\end{align*}
\]
which is the inequality of [3] Theorem 10. Another interesting infinite family of inequalities arises from \( v_n \in S_n \):
\[
\begin{align*}
(n - 1)[abcd] &+ (a, b | c) \\
+ (n - 1) ((a, c | b) + (b, c | a)) \\
+ \frac{(n - 1)(n - 2)}{2} ((a, d | b) + (b, d | a)) \geq 0,
\end{align*}
\]
and several others can be constructed easily. Some of the inequalities in \( B_n \) are redundant: they are consequences of others. For example, \( B_4 \) contains 12 inequalities, eight of them come from the downward closed subsets depicted on Figure 1. The \( \langle x_s, y_s, z_s \rangle \) coefficients in the order above are \( (3,3,3), (4,5,3), (6,9,5), (7,12,5), (6,11,3), (4,7,1), (3,6,0), (5,8,3) \) and \( (5,8,3) \). The inequality coming from the last two triplets is a consequence of the others, as it is just the average of the inequalities coming from the triplets \( (4,5,3) \), and \( (6,11,3) \). It is not difficult to eliminate the redundant inequalities from \( B_n \) but their description is cumbersome, so we skipped this step. Figure 2 shows nodes \( \langle y_s/x_s, z_s/x_s \rangle \) on a logarithmic scale, where \( \langle x_s, y_s, z_s \rangle \) are coefficients in non-redundant inequalities. Two such nodes are connected by a straight line when the corresponding subsets \( s \in S_n \) differ by a single element only. Nodes on the horizontal and vertical bounding lines come from the sets \( u_n, v_n, \) respectively; \( t_n \) gives the nodes along the diagonal. The symmetry of the figure comes from the symmetry of \( v_s \) observed earlier.

**Conjecture 9 (Book conjecture).** A polymatroid \( g \) on the four elements set \( abcd \) has an \( n \)-page book extension at \( ab \) if and
only if \( g \) satisfies the \( n \)-page book inequalities in \( \mathcal{B}_n \), and their versions where the variables \( a \leftrightarrow b \) and \( c \leftrightarrow d \) are swapped.

The condition of this conjecture is necessary; this will be proved in the next section as Theorem 10. Sufficiency has been checked by a computer program for \( n \leq 9 \). The technique used can be outlined as follows. The ground set of the \( n \)-page book extension of \( abcd \) has \( 2 + 2n \) elements, thus the polymatroid is an element of the \( 2^{2+2n} - 1 \)-dimensional Euclidean space. The region of polymatroids is a convex polyhedral cone bounded by half-planes corresponding to submodularity and monotonicity. The collection of \( n \)-page book extensions is a sub-cone \( \mathcal{P} \) cut out by the requirement that all pullbacks are isomorphic, and the pages are independent over the spine. These requirements can also be expressed as linear constraints, thus \( \mathcal{P} \) is also polyhedral. A polymatroid on \( abcd \) has an \( n \)-page book extension if and only if it is in the projection of \( \mathcal{P} \) to the \( 15 \)-dimensional subspace corresponding to the non-empty subsets of \( abcd \). The characterizing inequalities are just the equations of the facets of the projection. Finding these facets is the subject of multiobjective optimization. To be applicable in practice, the problem dimension should be reduced significantly. This reduction comes from several sources. By Proposition 4 we can assume the book extension be symmetric, this alone drops the dimension of \( \mathcal{P} \) significantly from \( 2^{2+2n} - 1 \) to around \((n+1)^3\). Further reduction is achieved from the independence of pages, from the \( a \leftrightarrow b \) and \( c \leftrightarrow d \) symmetries, from the sufficiency of considering tight polymatroids only, and by cutting \( \mathcal{P} \) into several well-chosen pieces. Table 1 shows, as a function of \( n \), the size of the reduced problem: its dimension and the number of linear constraints in that dimension. The last column contains the running time in hours, minutes and seconds required to generate the facets of the projection on a stand-alone workstation running a highly optimized algorithm.

We conjecture that the book inequalities do give a sufficient condition for the existence of an \( n \)-page book extension.

V. Necessity of book inequalities

The aim of this section is to prove that the condition in the book conjecture is necessary.

**Theorem 10.** Suppose the polymatroid \( g \) on the four element set \( abcd \) has an \( n \)-page book extension \( h \) at \( ab \). Then \( g \) satisfies all inequalities in \( \mathcal{B}_n \) and their versions where the variables \( a \leftrightarrow b \) and \( c \leftrightarrow d \) are swapped.

**Proof:** As it was remarked earlier, it is enough to show that \( g \) satisfies the inequalities in \( \mathcal{B}_n \) as the symmetric versions follow by applying \( \mathcal{B}_n \) to the permuted instances of \( g \).

As \( h \) is an \( n \)-page book extension of \( g \), the ground set of \( h \) is the disjoint union \( M = P_1 \cup \ldots \cup P_n \cup \{a, b\} \). We let \( P_i = \{c_i, d_i\} \), where \( c_i, d_i \) are the twins of \( c, d \), respectively. For non-negative integers \( k, \ell \) and \( m \) where \( k + \ell + m \leq n \), let \( c^k \ell d^m \) denote the following subset of \( M \):

\[
\{c_1, \ldots, c_k, c_{k+\ell+1}, \ldots, c_{k+\ell+m},
\}

\[
d_{k+1}, \ldots, d_{k+\ell}, d_{k+\ell+1}, \ldots, d_{k+\ell+m},
\]

that is, we pick \( c_i \) from the first \( k \) pages, \( d_i \) from the next \( \ell \) pages, and both \( c_i \) and \( d_i \) from the following \( m \) pages. When any of \( k, \ell \), or \( m \) is zero, we leave out the corresponding term from the notation. According to Proposition 4 \( h \) can be assumed to be symmetric, that is, the value of \( h(I) \) depends only on whether \( a \) and \( b \) are in \( I \), and in how many pages \( I \) intersects \( P_i \) in the empty set, in \( c_i \), in \( d_i \), or in \( c_i, d_i \). Consequently \( h(I) \) is equal to one of the values \( h(X), h(aX), h(bX) \) or \( h(abX) \), where \( X = c^k d^\ell (cd)^m \) for some triplet \( k, \ell, m \).

To simplify the notation, in the rest of this section we omit the symbols \( g \) and \( h \) before the subsets of \( abcd \) and \( M \); any subset also denotes the value of the corresponding polymatroid. As \( g \) and \( h \) agree on subsets of \( ab \), this convention is unambiguous.

First we prove some easy propositions.

**Claim 11.**

a) If \( k + \ell \leq n \), then

\[
abc^kd^\ell = ab + k \cdot (abc - ab) + \ell \cdot (abd - ab);
\]

b) if \( k + \ell \leq n - 1 \), then

\[
abc^kd^\ell (cd)^1 = ab(cd)^1 + k \cdot (abc - ab) + \ell \cdot (abd - ab).
\]

**Proof:** By induction on \( k + \ell \). Both statements are true when \( k = \ell = 0 \). Assume \( k + \ell < n \). As \( c_n \) is independent of \( c^{k+1}d^\ell \) over \( ab \), that is, \( (c_n, c^{k+1}d^\ell | ab) = 0 \), and as \( abc_n = abc \), \( abc_n c^{k+1} d^\ell = abc^{k+1}d^\ell \), we know that

\[
abc^{k+1}d^\ell - abc^kd^\ell = abc - ab,
\]

and similarly for the other three cases. This concludes the induction step.

**Claim 12.** If \( k + \ell \leq n \), then

\[
k \cdot (ac - a) + \ell \cdot (ad - a) + a \geq ac^kd^ \ell,
\]

\[
k \cdot (bc - b) + \ell \cdot (bd - b) + b \geq bc^kd^\ell.
\]

**Proof:** The claims are true with equality when \( k = \ell = 0 \). As \( (c_n, c^{k+1}d^\ell | a) \geq 0 \) and \( ac_n = ac \), \( c_n c^{k+1}d^\ell = c^{k+1}d^\ell \), we know that \( ac - a \geq ac^{k+1}d^\ell - ac^{k+1}d^\ell \). Using this fact and three other similar inequalities we arrive at the claim by induction on \( k + \ell \).

**Claim 13.** If \( k + \ell < n \), then

\[
c^{k+1}d^\ell (cd)^1 \geq cd + k(abc - ab) + \ell(abd - ab),
\]

\[
bd^\ell (cd)^1 \geq bcd + \ell(abd - ab),
\]

\[
acd^\ell \geq ac + \ell(abd - ab).
\]
Proof: By submodularity, $c^k d^l (cd)^1 - (cd)^1 \geq abc^k d^l (cd)^1 - ab(cd)^1$. This, and part b) of Claim 11 give the first inequality. The other inequalities can be proved in a similar way.

The next lemma describes the crucial inequality that allows us to prove that $g$ satisfies the inequalities in $B_n$. The symbols $C, D$ will be used to denote the following entropy expressions:

$C = (a, c | b) + (b, c | a)$,

$D = (a, d | b) + (b, d | a)$.

Lemma 14. For non-negative integers $k$ and $\ell$ where $k + \ell < n$,

$$[abcd] + kC + \ell D + (a, b | c^k d^l) \geq$$

$$(a, b | c^k d^{l+1}) + (a, c | d^k) + (b, d_n | c_n d^k),$$

and

$$[bdac] + \ell D + (a, b | d^\ell) \geq$$

$$(a, b | d^{\ell+1}) + (a, c | d^\ell) + (b, d_n | c_n d^\ell).$$

Before proving this lemma, let us see how it implies Theorem 10. Denote the inequality (4) by $I(k, \ell)$, and the inequality (5) by $J(\ell)$. Let $s \in S_n$, and $x_s = \{x_s, y_s, z_s\}$, we want to show inequality (2), which can be written as

$$x_s [abcd] + (a, b | c) + y_s C + z_s D \geq 0.$$ 

Consider the following combination of the inequalities in (4) over the elements of the downward closed set $s$:

$$\sum_{(k, \ell) \in S} \binom{k + \ell}{k} I(k + 1, \ell).$$

On the left hand side of the $\geq$ sign we have $x_s$, $y_s$, and $z_s$, many instances of $[abcd], C,$ and $D$, respectively; and we also have $(a, b | c)$ from $I(1, 0)$. If $k + \ell \geq 1$, then $(a, b | c^k d^l)$ occurs $\binom{k + \ell}{k}$ many times on the left hand side, and, as $s$ is downward closed, $\binom{k-1 + \ell}{k-1} + \binom{\ell + 1}{k-1}$ times on the right hand side, thus they cancel out. All remaining items on the right hand side are non-negative, which proves inequality (6).

The book inequality (3) requires us to show

$$\ell [bdac] + (a, b | d) + \frac{\ell (\ell + 1)}{2} D \geq 0$$

(7)

for every $\ell < n$. Summing up the inequalities $J(1), J(2), \ldots, J(\ell)$ we get an inequality where the left hand side equals that of (7), and where terms on the right hand side are non-negative. ■

Proof of Lemma 14. To arrive at inequality (4) sum up the inequalities in the list below, and rearrange. The last column indicates why the inequality holds: SM stands for submodularity, and numbers refer to the corresponding Claim:

$ac - a + ac^k d^l \geq ac^{k+1} d^l,$

$bd - b + bc^k d^l \geq bc^{k+1} d^l,$

$-abd + ab - abc d^l \geq -abc^{k+1} d^l,$

$-abc - k(abc - ab) - \ell (ab - a b) \geq -abc^{k+1} d^l,$

$ad + k(\ell a - a) + \ell (a - a) \geq ac^{k+1} d^l,$

$bc + k(bc - b) + \ell (bd - b) \geq bc^{k+1} d^l,$

$-cd - k(ab - a) - \ell (a b - a b) - c^k d^l (cd)^1.$

Similarly, inequality (5) follows from the sum of the inequalities in the list below:

$$cd - d \geq d^l (cd)^1 - d^{l+1},$$

$$bd - b + bd \ell \geq bd^{l+1},$$

$$-abd + ab - abd \ell = -abd^{l+1},$$

$$ad + \ell (a - a) \geq ad^{l+1},$$

$$bc + \ell (bd - b) \geq bcd^\ell,$$

$$-ac - \ell (ab - a b) \geq -acd^\ell,$$

$$-bcd - \ell (ab - a b) \geq -bd^l (cd)^1.$$ ■

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