Likelihood Methods and Classical Burster Repetition

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We develop a likelihood methodology which can be used to search for evidence of burst repetition in the BATSE catalog, and to study the properties of the repetition signal. We use a simplified model of burst repetition in which a number \( N_r \) of sources which repeat a fixed number of times \( N_{\text{rep}} \) are superposed upon a number \( N_{\text{nr}} \) of non-repeating sources. The instrument exposure is explicitly taken into account. By computing the likelihood for the data, we construct a probability distribution in parameter space that may be used to infer the probability that a repetition signal is present, and to estimate the values of the repetition parameters. The likelihood function contains contributions from all the bursts, irrespective of the size of their positional errors — the more uncertain a burst’s position is, the less constraining is its contribution. Thus this approach makes maximal use of the data, and avoids the ambiguities of sample selection associated with data cuts on error circle size. We present the results of tests of the technique using synthetic data sets.

INTRODUCTION

Classical gamma-ray burst repetition is one of the most startling possibilities suggested by analyses of the BATSE 1B catalog. Quashnock & Lamb (1) searched for evidence of strong clustering of burst positions by studying the distribution of nearest-neighbor separations, and reported an excess of bursts clustered on \( 5^\circ \) scales with a significance of \( 2.5 \times 10^{-4} \). Wang & Lingenfelter (2,3) have studied clustering of bursts in the 1B catalog in both angle and time, and found clustering on scales of 4 days and \( 4^\circ \) with a significance of \( 2 \times 10^{-5} \). Quashnock (4) finds the odds favoring repetition over no repetition to be about 4:1, based on a counts-in-cells likelihood technique.

On the other hand, the data in the BATSE 2B − 1B catalog (that is, those bursts seen in the 2B catalog after the end of the 1B epoch) show no such significant repetition signal (5). This is not unexpected, given the lower sky exposure due to the failure of the tape recorders aboard CGRO. Neverthe-
less, the absence of confirmation of the 1B signals, combined with the only moderate significance of those signals, have fueled a controversy over whether the evidence for burst repetition in the 1B catalog is merely a statistical fluctuation, perhaps amplified by choice of bin intervals (in the case of the time-angle analysis) or of cuts in error circle size (in the case of the nearest-neighbor analysis). These criticisms have been answered convincingly through analysis of the effect of data cuts [1] and of choice of binsize [2]. Nevertheless the evidence for repetition is not universally regarded as conclusive.

Under the circumstances, there is a premium on the development of techniques to probe for burst repetition that are more sensitive than existing ones — to achieve higher significance detections — and that do not rely on either binning or data cuts. In this work we present such a technique, founded upon the likelihood function for a simple model of burst repetition. We also exhibit the results of tests of the method on synthetic data sets, which allow us to gauge its sensitivity and to understand potential systematic effects.

**THE LIKELIHOOD FUNCTION**

In order to calculate the likelihood function, we require a model of burst repetition. The model we have adopted is one in which \( N_r \) repeating sources are superposed upon \( N_{nr} \) non-repeating sources. Each repeating source bursts exactly \( N_{rep} \) times, although not all these bursts are observed because of the limited sky exposure. This simplified model captures the essential features of the phenomenon — number of repeating sources and repetition rate — while permitting practical calculation. The model parameters are \( N_r \) and \( N_{rep} \). \( N_{nr} \) is set by the data when \( N_r \) and \( N_{rep} \) are chosen, and the sky exposure \( E \) is fixed at the value appropriate for the epoch of the BATSE catalog under study.

The likelihood function calculation requires that we identify likely “partitions” \( \pi \) of the data. A partition is an assignment of some bursts to each of the repeating sources. The likelihood function \( L(N_r, N_{rep}) \) is then given by

\[
L(N_r, N_{rep}) = \sum_{\pi} (N_r + N_{nr})! \times \left\{ \prod_{j=1}^{N_r} \frac{N_{rep}!}{n_j!(N_{rep} - n_j)!} E^{n_j}(1 - E)^{N_{rep} - n_j} \times L_j \right\}, \hspace{1cm} (1)
\]

where \( n_j \) is the observed number of bursts from the \( j \)th source (according to the partition \( \pi \)), \( E \) is the sky exposure, and \( L_j \) is the “cluster likelihood” for the \( j \)th source of \( \pi \). \( L_j \) acts as a measure of the plausibility of the assignment of the \( n_j \) bursts to the source, and is in fact the expression for the Bayesian odds favoring the hypothesis that the \( n_j \) bursts were produced by the same source over the hypothesis that they each had a separate source. Assuming
that the scatter of the observed positions about the true source positions is described by the Fisher distribution, $L_j$ is given by the following expression:

$$L_j = \frac{\sum_j 2 \sinh \Sigma_j^2}{\prod_k \frac{1}{\sigma_{jk}^2} [\sinh \sigma_{jk}^2 - 2]^2} ; \quad \Sigma_j \equiv \left[ \sum_k \frac{\bar{x}_{jk}^2}{\sigma_{jk}^2} \right]^{-1/2}. \quad (2)$$

Here, $\bar{x}_{jk}$ is the unit vector describing the position on the sky of the $k$th burst of the $j$th source, and $\sigma_{jk}$ is the error in that position. $\Sigma_j$ is the error in the maximum likelihood position of the $j$th source. The maximum likelihood position itself is $\bar{z}_j = \frac{\bar{x}_j}{\sigma_j^2}$.

The meaning of Eq. (2) may be clarified by taking the small-angle limit, in which all the $\sigma$ are much less than one. In this limit the Fisher distribution becomes the two-dimensional, symmetric Gaussian distribution, and the expression for the $j$th cluster likelihood becomes

$$L_j \approx \frac{1}{\prod_k \frac{1}{\sigma_{jk}^2}} \times \exp \left[ -\frac{1}{2} \sum_k \frac{(\bar{x}_{jk} - \bar{z})_j^2}{\sigma_{jk}^2} \right] ; \quad \Sigma_j \approx \left[ \sum_k \frac{1}{\sigma_{jk}^2} \right]^{-1/2}. \quad (3)$$

From Eq. (3) we see that $L_j$ is a product of two factors: the exponential factor, which penalizes (decreases) the likelihood if the $j$th cluster of $\pi$ is too dispersed, and the rational factor, which rewards (increases) the likelihood for making $n_j$ as large as possible (recall that $\sigma_{jk} \ll 1$). These two factors compete with each other, since the addition of a burst to a cluster may well provoke a harsher penalty from the exponential factor (if the burst is implausibly distant from the rest of the cluster) than the reward it reaps from the rational factor. This fact allows $L_j$ to be used as a tool in identifying “optimum” clusters of bursts. This is an important observation, since the sum over partitions $\pi$ in the likelihood clearly ranges over an impossibly large number of configurations. Fortunately, most partitions contribute negligibly to the likelihood, as they assign bursts to improbably dispersed clusters. We may thus restrict our attention to partitions that make appreciable contributions to the likelihood by stringing together into partitions only “optimum” clusters identified using $L_j$. This is a key ingredient of our technique.

Note that the dependence of $L_j$ on the burst positions is strongest for bursts with small error circles and weakest for bursts with large error circles. This is the reason that the inclusion of bursts with poorly determined positions does not weaken the repetition signal — such bursts are individually unable to seriously constrain inferences based on the likelihood. On the other hand, even bursts with large error circles bear some information, particularly if there are many of them. Our approach allows that information to be exploited to the fullest extent possible.

We use Bayes theorem to interpret the likelihood function as a probability distribution in $(N_r, N_{rep})$ parameter space. This distribution is the basis for
FIG. 1. Sky map resulting from a simulation with \( N_r = 5, N_{rep} = 15 \), and a sky exposure \( E = 0.33 \). Note that more than three bursts are observed from only two of the five repeating sources, even though \( N_{rep} = 15 \), because the sky exposure is limited.

all our statistical inferences. By comparing the contribution of the likelihood function to the entire parameter space to the value of the likelihood for the non-repeating model \( (N_r = 0) \), we may compute the odds favoring the repetition hypothesis over the no-repetition hypothesis, thus obtaining a sensitive test for the presence of a repetition signal. Furthermore, the detailed shape of the distribution allows us to estimate the maximum likelihood parameter values, and to infer credible regions for those parameters.

TESTING THE TECHNIQUE

We have tested the method on synthetic data, simulating both repeating and non-repeating \( (N_r = 0) \) models, in order to test its sensitivity and power, as well as to gauge the reliability of the parameter estimates that it produces.

Fig. 1 shows a sky map resulting from a simulation assuming \( N_r = 5, N_{rep} = 15 \), and sky exposure \( E = 0.33 \). There are 25 bursts from the 5 repeating sources, and 50 bursts from all sources. The assumed total error \( \sigma_{circ} \) — that is, the radius of the 68% circular confidence region including both systematic and statistical error — is 5° for all bursts. \( \sigma_{circ} \) is related to the Fisher distribution parameters \( \sigma_{jk} \) above by \( \sigma_{jk}^2 = 0.871(1 - \cos \sigma_{circ}) \).

Fig. 2 shows the probability distribution in the \( (N_r,N_{rep}) \)-plane that results from the analysis of this simulated data set. The distribution is sharply peaked near the true parameter values. Also shown are the 1-, 2-, and 3-\( \sigma \) credible regions for the model parameters.

An interesting question that arises is whether the credible regions shown in Fig. 2 are well-calibrated, in the sense that they bracket the true parameters with frequency equal to the stated probability contained within the contour.
To begin addressing this question, we analyzed 20 simulated data sets with the same parameter values as those given above. We found that the 68% contour included the true parameters in 18 out of 20 simulations, which by the binomial distribution is quite consistent with the 68% interpretation of the contour. We are carrying out analyses on many more simulations in order to refine the contour calibration.

The odds may be computed by ascribing equal a priori probability to the repeating and the non-repeating hypothesis, and, within the repeating hypothesis, equal a priori probability to each of the discrete parameter values \((N_r, N_{rep})\) at which the likelihood is evaluated. This last assignment dilutes the value of the odds if the parameter space is very large, and accounts naturally for the “number of attempts” correction to the statistical significance of the detection.

We calculated the odds for each of the 20 simulated data sets discussed above, and found that the mean and standard deviation log odds were \(\log O = 22.5 \pm 7.9\). We also simulated 20 data sets assuming no repetition, and calculated the odds for each of those. The result was \(\log O = 0.5 \pm 2.0\). Clearly, then, the likelihood function approach is quite capable of distinguishing between no repetition and the kind of repetition simulated here, and thus offers a promise of a sensitive and powerful statistic for the detection of repetition signals in burst data.

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