SURFACES OF GENERAL TYPE WITH $p_g = q = 0$ HAVING A PENCIL OF HYPERELLIPTIC CURVES OF GENUS 3

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Abstract. We prove that the bicanonical map of a surface of general type $S$ with $p_g = q = 0$ is non birational if there exists a pencil $|F|$ on $S$ whose general member is an hyperelliptic curve of genus 3.

Let $S$ be a minimal surface of general type and let $f_g : S \rightarrow B$ be a rational map onto a smooth curve such that the normalization of the general fibre is an hyperelliptic curve of genus $g$. Then the hyperelliptic involution of the general fibre induces a (biregular) involution $\sigma$ on $S$ and one has a map of degree two $\rho : S \rightarrow \Sigma$ onto a smooth ruled surface with ruling induced by $f_g$.

It is known (e.g. [2],[4]) that if $g = 2$ then the bicanonical map $\varphi_{2K}$ of $S$ factors through $\rho$ and moreover, if $K_S^2 \geq 10$ then the non birationality of $\varphi_{2K}$ implies the existence of $f_g$ with $g = 2$ ([2]). On the other hand, we proved in [3] that if $\varphi_{2K}$ factors through a rational map $\rho$ (generically) of degree two onto a ruled surface, then there exists a map $f_g$ where $g \leq 4$ and, more precisely, if $g \neq 2$ then we have $g = 3$ unless $K_S$ is ample and $q(S) = 0, p_g(S) = \frac{1}{2}(d - 3)d + 1, K_S^2 = 2(d - 3)^2$, $d = 4, 5$. Finally, we recall that if $p_g(S) = 0$ and $K_S^2 \geq 3$, then there not exists $f_g$ with $g = 2$ ([3]).

In this note we prove that if $p_g = 0$ then the existence of $f_g$ with $g = 3$ implies $\varphi_{2K}$ non birational. In particular, it follows that if $p_g = 0$ and $K_S^2 \geq 3$ then $\varphi_{2K}$ factors through a map of degree two onto a ruled surface if and only if there exists $f_g$ with $g = 3$.

To motivate this work we notice that, as far as we know, all the examples of surfaces of general type with $p_g = 0$ and non birational bicanonical map have an $f_g$ with $g = 3$, except one case when $K_S^2 = 3$ and $S$ is a double cover of an Enriques surface.

Notation and conventions. We work over the complex numbers. We denote by $K_S$ a canonical divisor, by $p_g = h^0(S, \mathcal{O}_S(K_S)) = 0$ the geometric genus and by $q = h^1(S, \mathcal{O}_S(K_S)) = 0$ the irregularity of a smooth (projective algebraic) surface $S$. The symbol $\equiv$ (resp. $\sim$) will denote the linear (resp. numerical) equivalence of divisors. A curve on a surface has an $[r, r]$-point at $p$ if it has a point of multiplicity $r$ at $p$ which resolves to a point of multiplicity $r$ after one blow up.

1. SURFACES WITH A PENCIL OF CURVES OF GENUS 3

Assumption 1.1. Throughout the end we assume that

a) $S$ is a minimal surface of general type with $p_g = q = 0$ and

b) $f : S \rightarrow \mathbb{P}^1$ is a rational map with connected fibres such that the normalization of general fibre $F$ is a curve of genus $g = 3$.

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Proposition 1.2. Let $F$ be a general member of $|F|$. Then,
- either $F$ has a double point and $F \sim 2K_S$, in this case $K_S^2 = 1$;
- or $F$ is smooth and $F^2 = 2$, in this case $K_S^2 = 1$;
- or $F$ is smooth and $F^2 \leq 1$.

Proof. Suppose that $F$ has multiplicity $m_i \geq 2$ at $x_i$ and let $\overline{S} \to S$ be the blow up of $S$ at the $x_i$’s. Denote by $\overline{F} \subset \overline{S}$ the strict transform of $F$. Then, by the adjunction formula we have

$$4 - F^2 - \sum m_i = 2g - 2 - F^2 - \sum m_i = K_S F - \sum m_i = K_S F \geq 1$$

since $K_S$ is nef and $h^0(F) > 0$. Hence, the Hodge Index Theorem says that

$$4 \geq (4 - F^2 - \sum m_i)^2 = (K_S F)^2 \geq K_S^2 F^2 \geq F^2 \geq \sum m_i^2 \geq 4$$

which implies that $F^2 = 4$ and $F$ is numerically equivalent to $2K_S$. In particular, $F$ has exactly a double point.

Analogously, if $F$ is smooth we get that $F^2 \geq 2$ implies $F^2 = 2$ and $K_S^2 \leq 2$ with the equality only if $F$ is numerically equivalent to $K_S$. In the latter case $F - K_S$ is a non zero torsion element $\mu \in \text{Pic}(S)$, and $h^0(F) - h^1(F) + h^2(F) = \chi(F) = \chi(K_S) = 1$ yields $h^1(-\mu) > 0$. Therefore, the unramified covering induced by $\mu$ is irregular. A contradiction, indeed $\pi_1^{alg}(S) \leq 9$ (cfr. [2]).

2. THE HYPERELLIPTIC CASE

Assumption 2.1. From now on we assume that (the normalization of) the general member $F \in |F|$ is hyperelliptic.

2.1. Involutions and double covers. Let $\sigma_F$ be the hyperelliptic involution on $F$ and denote by $\sigma$ the involution induced on $S$. Then, $\sigma$ is biregular since $S$ is minimal of general type, and the fixed locus $Fix(\sigma)$ is union of a smooth curve $R$ and finitely many isolated points $p_1, \ldots, p_\nu$.

Lemma 2.2. The base points of $|F|$ belong to the fixed locus of $\sigma$; moreover if $F^2 < 4$ (i.e. the general $F \in |F|$ is smooth), then they are distinct and isolated.

Proof. Consider $\sigma(q_1)$ for a base point $q_1$ of $|F|$. If $F, F'$ are general members of $|F|$ we have

$$\sigma_F(q_1) = \sigma(q_1) = \sigma_{F'}(q_1)$$

and so $\sigma(q_1) \in F \cap F'$. Therefore, $q_1 = \sigma(q_1)$ if $F^2 = 1$ or 4.

If $F^2 = 2$ denote by $q_2$ the other base point ($q_2 = q_1$ if $F$ and $F'$ are tangent). Then $\sigma(q_1) \in \{q_1, q_2\}$ and the exact sequence

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(F)) \to H^0(F, \mathcal{O}_F(q_1 + q_2)) \to 0$$

implies $h^0(F, \mathcal{O}_F(q_1 + q_2)) = 1$. Therefore, $q_1 = \sigma_F(q_1) \neq q_2$ and $q_1, q_2$ are fixed points of $\sigma$. In particular, $F$ and $F'$ meet transversally at the base points.

Suppose that there exists a component $R_{q_i}$ of $R$ passing through $q_i, i \in \{1, 2\}$, so that locally $\sigma$ is the involution $(x, y) \mapsto (x, -y)$. Notice now that since $F, F'$ meet transversally at $q_i$, they also meet transversally $R_{q_i}$ at $q_i$. A contradiction, since $F, F'$ are $\sigma$-invariant. 

□
Let \( \hat{S} \to S \) be the blow up of \( S \) at \( p_1, \ldots, p_\nu \) and denote by \( \hat{\sigma} \) the induced involution, which is biregular since we are blowing up isolated fixed points. Let \( \rho : \hat{S} \to \Sigma = \hat{S}/\hat{\sigma} \) be the projection onto the quotient. Hence, \( \Sigma \) is a smooth rational surface and \( \rho \) is a (finite) double cover. Denote by \( \hat{B} = \rho(\hat{R}) = \rho(\pi^*(R) + \sum E_i) \) the branch curve of \( \rho \), where \( E_i = \pi^{-1}(p_i), \ i = 1, \ldots, \nu \). Then, \( \hat{B} \) is a smooth curve linearly equivalent to \( 2\hat{\Delta} \) for some \( \hat{\Delta} \in \text{Pic}(\Sigma) \). By [3, Proposition 1.2] we have the following equalities:

\[
(2.1) \quad \nu = K_S.R + 4 = K_S^2 + 4 - 2h^0(\Sigma, 2K_{\Sigma} + \hat{\Delta})
\]

\[
(2.2) \quad K_S^2 = K_{\hat{S}}^2 - \nu
\]

and \( \varphi_{2K_S} \) factors through \( \rho \) if and only if \( h^0(\Sigma, 2K_{\Sigma} + \hat{\Delta}) = 0 \). Notice that \( \nu \geq 4 \) since \( K_S \) is nef.

**Lemma 2.3.** Assume \( K_S^2 = 1 \), then \( \varphi_{2K_S} \) factors through \( \rho \).

**Proof.** Otherwise it would be \( h^0(\Sigma, 2K_{\Sigma} + \hat{\Delta}) > 0 \) and hence \( \nu \leq 3 \). \( \square \)

From now on, in this section we assume \( K_S^2 \geq 2 \). Therefore, \( F \) is smooth and, by Lemma 2.2, \( \Sigma \) is birationally ruled by \( [\Gamma] \), where \( \Gamma \) is the image of \( F \). Let \( \omega : \Sigma \to \mathbb{F}_e, e \geq 0 \) be a birational morphism such that \( \Gamma = \omega_*(\Gamma) \) is a ruling, and consider a factorization \( \omega = \omega_0 \circ \cdots \circ \omega_1 \) where \( \omega_i : \Sigma_{i-1} \to \Sigma_i \) is the blow up at \( q_i \in \Sigma_i, i \geq 1, \Sigma_0 := \Sigma \) and \( \Sigma_d = \mathbb{F}_e \).

Denote by \( E_i, E_i^* \) respectively the exceptional curve of \( \omega_i \) and its total transform on \( \Sigma \) and by \( B = \omega_*(\hat{B}) \) denote the image of \( \hat{B} \) on \( \mathbb{F}_e \). Since \( \hat{B} \equiv 2\hat{\Delta} \) we have \( B \equiv 2\Delta \), where \( \Delta = \omega_*(\hat{\Delta}) \in \text{Pic}(\mathbb{F}_e) \) and hence \( B \equiv 8C_0 + 2b\Gamma \), where \( C_0 \) is the \((-e)\)-section. Recall that \( K_{\mathbb{F}_e} \equiv -2C_0 - (2 + e)\Gamma \) and \( K_\Sigma \equiv \omega^*(K_{\mathbb{F}_e}) + \sum E_i^* \).

Finally, notice that the branch curve \( \hat{B} \) contains exactly \( \nu \) \((-2)\)-curves (they correspond to the isolated fixed points of \( \sigma \)). In particular, the \((-2)\)-curves arising from the base points of \( F \) map to sections of \( \Sigma \to \mathbb{P}_1 \), while each one of the others maps either to a point or to a fibre \( \Gamma \).

**Lemma 2.4.** In the above situation

i) \( \hat{S} \) is the canonical resolution of the double cover of \( \mathbb{F}_e \) branched along \( B \);

ii) we can assume that:

ii.a) the essential singularities of \( B \) are at most \([5,5]\)-points, and

ii.b) if \([x'] \to x\) is a \([5,5]\)-point then the fibre \( \Gamma_x \) through \( x \) belongs to \( B \);

iii) assume (ii), then

iii.a) there is a 1 to 1 correspondence

\[
\begin{align*}
\{ \text{\((-2)\)-curves contained in } \hat{B} \text{ contracted to points} \} & \longleftrightarrow \{ \text{\([r,r]\)-points of } B \}
\end{align*}
\]

and

iii.b) there is a 1 to 1 correspondence

\[
\begin{align*}
\{ \text{\((-2)\)-curves contained in } \hat{B} \text{ which map to fibres} \} & \longleftrightarrow \{ \text{fibres passing through a } [5,5]\text{-point of } B \}
\end{align*}
\]

**Proof.** For i) see [3, Lemma 1.3], while for ii) and iii) see [9, Lemmas 6 and 7]. \( \square \)
Proposition 2.5. Assume that \( \omega \) has the property ii) of Lemma 2.4. Denote by \( a_1 \) (resp. \( a_2, a_3 \)) the number of [5, 5]-points (resp. [3, 3]-points, 4 and 5-tuple points) of \( B \). Then

\[
d \geq 2a_1 + 2a_2 + a_3; \quad \nu - F^2 = 2a_1 + a_2
\]

\[
\frac{3}{2} K_S^2 + 12 = a_1 + a_2 + a_3
\]

Proof. Since \( \hat{B} \) is smooth, the inequality is clear, just notice that to resolve a singularity of type \( [r, r] \) we need to blow up twice. The equality \( \nu - F^2 = 2a_1 + a_2 \) follows from ii, b) and iii) of the above lemma.

By Lemma 2.4 and [5], we have \( \hat{B} = \omega^*(B) - \sum 2|\mathcal{E}_i| \) where \( m_i \) is the multiplicity of \( B \) at \( q_i \). Hence, since \( \rho \) is a double cover we get

\[
\chi(\hat{S}) = \chi(\Sigma) + \chi(K_S + \hat{\Delta}) = \frac{1}{2}(K_F + \Delta).\Delta + 2\chi(F) - \frac{1}{2} \sum [m_i]/([m_i] - 1)
\]

\[
K_S^2 = 2(K_S + \Delta)_2 = 2(K_F + \Delta)^2 - \sum [[m_i] - 1]_2
\]

and hence

\[
1 = \frac{1}{2}(6b - 12e - 8) + 2 - \frac{1}{2}(a_18 + a_22 + a_32) = \\
= 3b - 6e - 2 - 4a_1 - a_2 - a_3;
\]

\[
K_S^2 = 2(4b - 8e - 8 - 2a_15 + a_21 + a_31) = \\
= 8b - 16e - 16 - 10a_1 - 2a_2 - 2a_3
\]

because \( B \) has at most [5, 5]-points. Therefore,

\[
3K_S^2 - 8 = 3(8b - 16e - 16 - 10a_1 - 2a_2 - 2a_3) - 8(3b - 6e - 2 - 4a_1 - a_2 - a_3)
\]

and so

\[
\frac{3}{2} K_S^2 + 12 = a_1 + a_2 + a_3
\]

Remark 2.6. Because of the injection \( H^0(\Sigma, \mathbb{C}) \hookrightarrow H^0(\hat{S}, \mathbb{C}) \) induced by \( \rho \) we have \( K_S^2 \geq K_{\hat{S}}^2 \). Therefore, \( d \leq 8 - K_S^2 \leq 8 - K_{\hat{S}}^2 \) since \( \omega \) is a sequence of \( d \) blow ups.

We set \( \hat{d} := 8 - K_S^2 \).

Lemma 2.7. We have \( \frac{\nu - F^2}{2} \geq a_1 \geq \nu - F^2 - \frac{1}{2} \hat{d} \).

Proof. The first inequality follows from Proposition 2.4. For the latter suppose that \( a_1 = \nu - F^2 - \frac{1}{2} \hat{d} - i, i \geq 1 \). Then we have \( a_2 = \nu - F^2 - 2a_1 = \hat{d} + 2i - \nu + F^2 \)

and so \( d \geq 2a_1 + 2a_2 = \hat{d} + 2i > \hat{d} \). A contradiction.

Remark 2.8. Notice that if \( \omega \) has the property ii) of Lemma 2.4 and \( B \) has a [5, 5]-point at \( p \), then each irreducible component of \( B \) passing through \( p \) is tangent to the fibre which contains \( p \). In particular, if \( F^2 > 0 \) then \( B \) contains \( F^2 \) sections of \( \mathbb{P}_e \rightarrow \mathbb{P}_1 \) and hence it has no [5, 5]-points, i.e \( a_1 = 0 \).
3. THE MAIN RESULTS

Theorem 3.1. Let $S$ be a surface of general type with $p_g(S) = q(S) = 0$. Suppose that there exists a rational map $S \dashrightarrow \mathbb{P}_1$ such that the normalization of the general fibre $F$ is an hyperelliptic curve of genus $3$. Then,

i) the bicanonical map of $S$ is composed with the hyperelliptic involution.

ii) $K_S^2 \leq 8$, and $K_S^2 \leq 3$ if $F^2 = 1$.

Proof. By Lemma 2.3 and Proposition 1.2 we can assume $F^2 \leq 1$. By Proposition 2.5, we have

$$\frac{3}{2}K_S^2 + 12 = a_1 + a_2 + a_3$$

and by the above lemma we can write $a_1 = \nu - F^2 - \frac{1}{2}d + i$, where

$$0 \leq i \leq \left[ \frac{\nu - F^2}{2} \right] - \nu + F^2 + \frac{1}{2}d \leq \frac{1}{2}(d - \nu + F^2)$$

Then, $a_2 = \nu - F^2 - 2a_1 = d - \nu + F^2 - 2i$ and $a_3 \leq d - 2a_1 - 2a_2 = 2i$. Therefore, we get

$$3K_S^2 + 24 = 2\left(\frac{d}{2} - i + a_3\right) \leq \frac{d}{2} + 2i \leq 2d - \nu + F^2 = 16 - 2K_S^2 - \nu + F^2$$

and hence,

$$4 \leq \nu \leq -5K_S^2 - 8 + F^2 = 12 + F^2 - 10h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta})) \leq 13 - 10h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta}))$$

by 2.1 and 2.2. It follows that $h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta})) = 0$ and whence $\varphi_{2K}$ factors through the involution.

Finally, notice that:

a) if $F^2 = 0$ then $K_S^2 + 4 = \nu \leq 12$, i.e. $K_S^2 \leq 8$;

b) if $F^2 = 1$ then $a_1 = 0$, hence

$$12 = 2a_2 + 2a_3 = 2(\nu - 1) + 2a_3 = 2K_S^2 + 6 + 2a_3$$

and so

$$K_S^2 = 3 - a_3 \leq 3$$

In particular, $K_S^2 = 9$ does not occur. \[\square\]

An involution $\iota$ on $S$ is rational if the quotient $S/\iota$ is a rational surface. In the situation of Section 2 we have the following commutative diagram

$\begin{array}{ccc}
\hat{S} & \xrightarrow{\rho} & S \\
\downarrow \hat{\iota} & & \downarrow S/\sigma \\
\hat{\Sigma} & \xrightarrow{\sigma} & S/\sigma
\end{array}$

where $S/\sigma$ is rational and, in particular, $\varphi_{2K}$ factors through $\sigma$ if and only if factors through $\rho$. 
**Theorem 3.2.** Let $S$ be a minimal surface of general type with $p_g = q = 0$. Then the bicanonical map of $S$ factors through a rational involution if and only if there exists a map $f_g : S \rightarrow \mathbb{P}_1$ such that the normalization of the general fibre is an hyperelliptic curve of genus $g \leq 3$ ($g = 3$ if $K_S^2 \geq 3$).

**Proof.** Assume that there exists $f_g$. If $g = 2$ see [2], [4] (in fact, the bicanonical system cuts on the general fibre a subserie of the $g_1^2$). If $g = 3$ then we have Theorem 3.1. Conversely, if the bicanonical map factors through a rational involution then by [3] there exists $f_g$ with $g \leq 3$. Finally, Xiao G. [9, Theoreme 2] proved that if there exists $f_g$ with $g = 2$ then $K_S^2 \leq 2$.

□

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