The $C_2$ Heat-Kernel Coefficient in the Presence of Boundary Discontinuities

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Abstract

We consider the heat-kernel on a manifold whose boundary is piecewise smooth. The set of independent geometrical quantities required to construct an expression for the contribution of the boundary discontinuities to the $C_2$ heat-kernel coefficient is derived in the case of a scalar field with Dirichlet and Robin boundary conditions. The coefficient is then determined using conformal symmetry and evaluation on some specific manifolds. For the Robin case a perturbation technique is also developed and employed. The contributions to the smeared heat-kernel coefficient and cocycle function are calculated. Some incomplete results for spinor fields with mixed conditions are also presented.
1 Introduction

In the theory of quantum fields on a curved background space, the short-time expansion of the heat-kernel has been found to be of paramount importance. The coefficients in the expansion are involved in, for example, statistical field theory and the calculation of vacuum energies.

Much effort has been expended in finding explicit expressions for the coefficients but, in view of the rapidly increasing complexity, the evaluation of higher and higher coefficients would seem to reach a point of diminishing returns. Rather than pursue this path, the present work seeks to consider a situation where the domain itself is generalized. We do this within the context of the $C_2$ coefficient which is of particular importance in 4 dimensions being linked to the 1-loop quantum effect on the Einstein field equations, as well as to the conformal anomaly, of relevance to quantum cosmology.

So far this coefficient has been determined in the case where the manifold is closed [1] or has a smooth boundary [2],[3]. In this paper, we determine $C_2$ when the boundary is piecewise smooth. This is an extension of our previous work on $C_{3/2}$ in the piecewise smooth case [4]. Our approach is to find the most general possible expression for the contribution to $C_2$ arising from the discontinuity – a non-trivial problem in itself – and then to confine it using the required conformal symmetries and special-case evaluation.

This knowledge then enables us to find the expression for the change in the effective action under a conformal transformation. The latter provides a follow-up to previous calculations of the effective action on specific manifolds with smooth boundaries [4, 5, 6, 7, 8].

The physical motivation for this analysis is that piecewise smooth boundaries, and manifolds, occur in various idealized situations; for example in simplicial approximations to general relativity and to quantum gravity, [4, 10].

Also, internal vertices (conical singularities), which are the periodic versions of the present structure, appear in the theory of quantum fields on a black hole background and have been the subject of some activity.
2 Background

The quantity of interest is the term $C_2^{(d)}$ in the small-$t$ expansion of the integrated heat kernel on a $d$-manifold $M$ with positive-definite metric

$$G(t) = \int_M d^d x \sqrt{g} \langle x | e^{-t\Delta} | x \rangle = \frac{1}{(4\pi t)^{d/2}} \sum_{k=0, 1/2, \ldots} C_k^{(d)} t^k$$

for the massless scalar field operator

$$\Delta = -\nabla^2 + \xi R$$

on $M$, where the fields are real, and obey Dirichlet conditions at the boundary $\partial M$. $\xi$ is a variable coefficient of coupling to the Ricci scalar $R$. $\Delta$ is conformally covariant if $\xi$ is equal to

$$\xi_d = \frac{d - 2}{4(d - 1)}$$

Although we consider a massless field for simplicity, our results are easily generalised to the non-zero mass case.

In principle, in singular situations, there is the possibility of $\log t$ terms in the expansion. There are none such in the present situation and they are henceforth ignored.

The heat-kernel coefficients can be written as integrals of local geometrical quantities such as the curvature tensor over $M$ and its boundary $\partial M$ if one exists. A traditional method of determining the integrand is to write down the most general possible expression and then use known values of $C_k^{(d)}$ on particular manifolds to confine it. This shall be our approach.

A convenient method of calculating the heat-kernel coefficients in specific cases where the eigenvalues $\lambda_i$ on $M$ are known is via its connection with the generalized zeta function

$$\zeta_M(s) = \sum_i \lambda_i^{-s}$$

where an analytic continuation is involved for $\Re(s) \leq d/2$. $\zeta_M(s)$ can be written as the Mellin transform of the integrated heat-kernel, yielding its connection with the $C_k^{(d)}$: \[C_k^{(d)} = \lim_{s \to 0} (4\pi)^{d/2} \Gamma\left(\frac{d}{2} - k + s\right) \zeta_M\left(\frac{d}{2} - k + s\right)\]
for \( k = 0, 1/2, 1 \cdot \cdot \cdot \). In this way, using the \( \lambda \) on a particular manifold to calculate the zeta function gives us the values of the heat-kernel coefficients on that manifold, and hence information about the general expressions for the coefficients.

If \( \Delta \) and the boundary conditions are conformally covariant, then from the change in the \( \lambda \) brought about by a conformal transformation, it is easily shown that \( \zeta_M(0) \), and hence \( C_{d/2}^{(d)} \), is conformally invariant. In other words, \( C_{k}^{(d)} \) is conformally invariant in \( 2k \) dimensions – this places a large restriction on its form. More generally and more quantitatively, the conformal variation of the heat-kernel coefficients under a transformation \( g_{\mu\nu} \rightarrow e^{2\delta\omega(x)} g_{\mu\nu} \) for small \( \delta \omega \) is

\[
\delta C_{k}^{(d)} = (d - 2k)C_{k}^{(d)}[\delta\omega(x)] + 2C_{k-1}^{(d)}[J\delta\omega(x)]
\]

where

\[
J = (d - 1) (\xi - \xi_d) \nabla^2
\]

and the smeared coefficients \( C_{k}^{(d)}[f(x)] \) are defined as

\[
\int_{\mathcal{M}} d^d x \sqrt{g} \langle e^{-t\Delta} | x f(x) = \frac{1}{(4\pi t)^{d/2}} \sum_{k=0,1/2,\ldots} C_{k}^{(d)}[f(x)] t^k
\]

Although our main interest is in determining the effect of boundary discontinuities on \( C_{2}^{(d)} \), we give the result for the smooth boundary case first. It is found that

\[
C_{2}^{(d)} = \frac{1}{360} \int_{\mathcal{M}} d^d x \sqrt{g} \left[ 2R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2R_{\mu\nu} R_{\mu\nu} + 5(6\xi - 1)^2 R^2 \right] + \frac{1}{360} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{h} \left[ 320 \frac{1}{21} \text{tr} \kappa^3 - \frac{88}{7} \text{tr} \kappa \kappa^2 + \frac{40}{21} \kappa^3 - 4R_{\mu\nu} \kappa_{\mu\nu} \right. \\
\left. - 4\kappa R_{\mu\nu} n_\mu n_\nu + 16 R_{\mu\nu\rho\sigma} n_\mu n_\rho \kappa_{\nu\sigma} + 10(1 - 6\xi) (2R\kappa - 3\nabla n R) \right]
\]

\( n^\mu \) is the inward-pointing normal to the boundary. With the induced metric \( h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \) we then define the extrinsic curvature tensor \( \kappa_{\mu\nu} = \kappa_{\nu\mu} = -h_\alpha^\mu h_\beta^\nu \nabla_\alpha n_\beta \), with \( \kappa = \kappa_\mu^\mu \), \( \text{tr} \kappa^2 = \kappa_{\mu\nu} \kappa^{\mu\nu} \). In our conventions \( \kappa \) is positive on the surface of a ball. As we would expect from (4), (7) is conformally invariant in 4 dimensions if \( \xi = 1/6 \).

We note that the coefficients in (7) are independent of the dimension. This is a general feature, and can be proved by multiplying \( \mathcal{M} \) by a circle. We refer to the book by Gilkey [13] for general information regarding heat-kernel asymptotics.
3 Boundary discontinuities

We shall now consider the situation where $\partial M$ is not smooth, but is made up of a number of boundary parts, $\partial M_i$. The intersection $\mathcal{I}_{ij}$ between two adjacent such parts is a manifold of dimension $d - 2$ (i.e. codimension 2). We also require that the $\mathcal{I}_{ij}$ be closed and smooth. The hemi-ball and ball $\times$ interval are examples of this situation. In general the spectral coefficients will now contain “edge” terms involving integrals over the $\mathcal{I}_{ij}$, as well as volume and boundary terms.

In addition to the boundary normals $n_i^\mu$, we define normals $\hat{n}_{ij}^\mu$ to $\mathcal{I}_{ij}$ pointing into $\partial M_i$. $\mathcal{I}_{ij}$ has induced metric $\gamma_{ij}^{\mu\nu}$. Then on $\mathcal{I}_{ij}$, suppressing the (ij) index for convenience:

$$
\gamma_{\mu\nu} = h_{\mu\nu}^i - \hat{n}_{ij}^\mu \hat{n}_j^\nu = h_{\mu\nu}^j - \hat{n}_{ij}^\mu \hat{n}_j^\nu
$$

$$
h_{\mu\nu}^i = g_{\mu\nu} - n_i^\mu n_i^\nu
$$

$$
h_{\mu\nu}^j = g_{\mu\nu} - n_j^\mu n_j^\nu
$$

(8)

For a dihedral angle $\theta_{ij}$ between $\partial M_i$ and $\partial M_j$, the normals have the relationship

$$
\hat{n}_{ij}^\mu = \hat{n}_i^\mu \cos \theta + n_i^\mu \sin \theta
$$

$$
n_{ij}^\mu = \hat{n}_i^\mu \sin \theta - n_i^\mu \cos \theta
$$

(9)

We limit ourselves to the case where $\theta$ is a constant on each intersection.

The codimension-2 contribution to $C_2^{(d)}$ will have the form

$$
\sum_{(ij)} \int_{\mathcal{I}_{ij}} d^{d-2}x \sqrt{\gamma} S_{ij}
$$

for some scalar $S_{ij}$ depending on the local geometry. Our first restriction on $S_{ij}$ is that $S_{ij} \rightarrow a^{-2} S_{ij}$ as $g_{\mu\nu} \rightarrow a^2 g_{\mu\nu}$, where $a$ is a constant, to make $C_2^{(4)}$ invariant under a global conformal (scale) transformation. Expressions with this property can be split into two types: intrinsic and extrinsic. Intrinsic terms must be first order in the Riemann tensor, contracted with the various metrics and normals. Extrinsic terms do not involve the Riemann tensor, but contain two $\nabla_\mu$ operators which act on the normals and metrics to produce extrinsic curvatures. It is easily shown using dimensional arguments that these are the only types of term possible.

**Intrinsic terms:**
With intrinsic terms, the objects we have to work with are, on each intersection:

\[ R_{\mu\nu\rho\sigma}^i \hat{R}^j_{\mu\nu\rho\sigma} \hat{R}^j_{\mu\nu\rho\sigma} \hat{R}^i_{\mu\nu\rho\sigma} \]

\[ g_{\mu\nu} \ n^i_{\mu} \ n^j_{\mu} \ \hat{n}^i_{\mu} \hat{n}^j_{\mu} \]

\( \hat{R}^i_{\mu\nu\rho\sigma} \) and \( \tilde{R}^i_{\mu\nu\rho\sigma} \) are the Riemann curvature tensors of \( \partial M_i \) and \( I_{ij} \) respectively, formed using the appropriate induced metrics together with the projected covariant derivative. \( h^i_{\mu\nu} \), \( h^i_{\mu\nu} \) and \( \gamma_{\mu\nu} \) are redundant since from the relations (8) they can be expressed in terms of other quantities.

To construct a general form for \( S_{ij} \), we have to consider, as far as intrinsic terms are concerned, every possible independent contraction of the expressions above which involves a single Riemann tensor. From (9), we never need to mix up different indices \( i \) and \( j \) in any particular term, since any expression which involves a mixture of indices can be written as a linear combination of terms which are pure in \( i \) or \( j \).

The possible contractions are

\[ \rho_1 = R \quad \rho_2^i = \hat{R}_i \quad \rho_3 = \tilde{R} \]

\[ \rho_4^i = R_{\mu\nu} n^\mu_i \hat{n}^\nu_i \quad \rho_5^i = R_{\mu\nu} \hat{n}^\mu_i \hat{n}^\nu_i \quad \rho_6^i = R_{\mu\nu} n^\mu_i n^\nu_i \]

\[ \rho_7^i = \hat{R}_{\mu\nu} \hat{n}^\mu_i \hat{n}^\nu_i \quad \rho_8^i = R_{\mu\nu\rho\sigma} n^\mu_i \hat{n}^\nu_i n^\rho_i \hat{n}^\sigma_i \]

and obviously also \( \rho_2^i \), \( \rho_4^i \) etc. All others are expressible in terms of the above set using the symmetries of the Riemann tensors, or vanish since

\[ \hat{R}^i_{\mu\nu\rho\sigma} n^\mu_i = \tilde{R}^i_{\mu\nu\rho\sigma} \hat{n}^\mu_i = \tilde{R}^i_{\mu\nu\rho\sigma} \hat{n}^\mu_i = 0 \]

In fact, not all of the above quantities are independent, since Gauss’ equations

\[ \hat{R}^\mu_{\nu\rho\sigma} = \kappa^\mu_{\rho\nu\sigma} - \kappa^\mu_{\nu\rho\sigma} + l^\mu_{\sigma} \hat{h}^\rho_{\sigma} \hat{h}^\gamma_{\rho} \hat{R}^\alpha_{\alpha\beta\gamma} \quad (10) \]

and (defining \( \hat{k}^\mu_{\nu\rho} \) to be the extrinsic curvature of \( I_{ij} \) with respect to \( \partial M_i \))

\[ \tilde{R}^\mu_{\nu\rho\sigma} = \hat{k}^\mu_{\rho} \hat{k}^\nu_{\sigma} - \hat{k}^\mu_{\nu} \hat{k}^\rho_{\sigma} + \gamma^\alpha_{\rho} \gamma^\beta_{\sigma} \gamma^\gamma_{\rho} \hat{R}^\alpha_{\alpha\beta\gamma} \quad (11) \]

provide relations between them, assuming we include the required extrinsic terms, which will be dealt with separately. Contracting (10) and (11) with \( g^\mu_{\mu} g^{\nu\sigma} \), and (10) with \( g^\mu_{\mu} \hat{n}^\nu \hat{n}^\sigma \), we get, respectively

\[ 2\rho_6^i = \rho_1 - \rho_2^i + \kappa^2_i - \text{tr} \kappa_i^2 \]

\[ 2\rho_7^i = \rho_2^i - \rho_3 + \hat{k}_i^2 - \text{tr} \hat{k}_i^2 \]

\[ \rho_8^i = \rho_5^i - \rho_7^i + \kappa_i^i \kappa^\mu_{\mu\nu} \hat{n}^\mu_i \hat{n}^\nu_i - \kappa^\mu_{\mu\nu} \kappa^\nu_{\mu\sigma} \hat{n}^\mu_i \hat{n}^\sigma_i \]

\[ (8) \]
A further relation can be found using (9):

\[ \rho_5^i = \rho_4^i \csc^2 \theta - \rho_6^i \cot^2 \theta + 2 \rho_4^i \cot \theta \]

We choose the \( \rho_i \) to \( \rho_4 \) as our basis. \( S_{ij} \) must be symmetric in \( i \) and \( j \), so it has the form

\[ S_{ij} = a_1 R + a_2 (\hat{R}_i + \hat{R}_j) + a_3 \hat{R} + a_4 R_{\mu \nu} (n^\mu_i \hat{n}^\nu_j + n^\nu_i \hat{n}^\mu_j) + \text{extrinsic terms} \quad (12) \]

Note that technically one of the \( R_{\mu \nu} n^\mu_i \hat{n}^\nu_j \) is still superfluous – we could more economically replace the last term by \( a_4 R_{\mu \nu} n^\mu_i n^\nu_j \). However we choose not to do this since the expression we have vanishes on an Einstein space, which will turn out to be convenient.

**Extrinsic terms:**

These terms must contain either two extrinsic curvature tensors, or the derivative of one. In the former case, we have the combinations

\[ \kappa_{\mu \alpha}^i \kappa_{\rho \sigma}^j \quad \kappa_{\mu \alpha}^i \kappa_{\rho \sigma}^j \quad \kappa_{\mu \alpha}^i \kappa_{\rho \sigma}^j \]

plus the expressions created by swapping \( i \) for \( j \). The extrinsic curvature of \( I_{ij} \) with respect to \( \partial \mathcal{M}_i \) is defined as \( \hat{\kappa}_{\mu \nu} = \hat{\kappa}_{\nu \mu} = \gamma_{\alpha \beta} \nabla_\alpha \hat{n}_\beta^i \). We contract the above with

\[ \gamma_{\mu \nu} n^\mu_i n^\nu_j \hat{n}_i^\mu \hat{n}_j^\nu \]

It is not necessary to include terms where all the indices of \( \kappa_{\mu \nu}^i \) or \( \kappa_{\mu \nu}^j \) are contracted with \( \gamma_i \), since these can be rewritten in terms of \( \hat{\kappa}_{\mu \nu}^i \) and \( \hat{\kappa}_{\mu \nu}^j \):

\[ \gamma_{\mu \alpha} \gamma_{\nu \beta} \kappa_{\alpha \beta}^i = -\gamma_{\mu \alpha} \gamma_{\nu \beta} \nabla_\alpha \hat{n}_\beta^i = \hat{\kappa}_{\mu \nu}^i \csc \theta - \hat{\kappa}_{\mu \nu}^i \cot \theta \]

where we have used (9). Additionally, (9) means that we can choose not to contract a \( \kappa_{\mu \nu} \) or \( \hat{\kappa}_{\mu \nu} \) with an \( n^\mu \) or \( \hat{n}^\mu \) of different indices \( i \) and \( j \).

We are left with the contractions

\[ k_1 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \hat{n}_i^\mu \hat{n}_j^\nu \hat{n}_i^\rho \hat{n}_j^\sigma \]
\[ k_2 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \hat{n}_i^\rho \hat{n}_j^\sigma \]
\[ k_3 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \hat{n}_i^\mu \hat{n}_j^\nu \gamma_{\mu \nu} \hat{n}_i^\rho \hat{n}_j^\sigma \]
\[ k_4 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \hat{n}_i^\rho \hat{n}_j^\sigma \]
\[ k_5 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \hat{n}_i^\mu \hat{n}_j^\nu \gamma_{\rho \sigma} \]
\[ k_6 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \gamma_{\rho \sigma} \]
\[ k_7 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \gamma_{\rho \sigma} \]
\[ k_8 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \gamma_{\rho \sigma} \]
\[ k_9 = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \gamma_{\rho \sigma} \]
\[ k_{10} = \kappa_{\mu \nu}^i \kappa_{\rho \sigma}^j \gamma_{\mu \nu} \gamma_{\rho \sigma} \]
as well as \( k_1^i, k_2^j \) etc. Everything else vanishes, since

\[
\kappa_{i\mu\rho} n_1^\nu = \tilde{\kappa}_{i\mu\rho} n_1^\nu = \tilde{\kappa}_{i\mu\rho} \tilde{n}_1^\nu = 0
\]

The fact that we have restricted \( \theta \) to be constant provides an extra constraint

\[
\gamma^{i\alpha j\beta} \gamma_{i\alpha\beta} = 0, \text{ giving } k_4 = -k_2^i = -k_2^j
\]

and we discard \( k_4 \).

In another possible representation, we could use only \( \kappa_{i\mu\nu} \) and \( \kappa_{j\mu\nu} \), since from (9)

\[
\gamma^{i\alpha j\beta} \gamma_{i\alpha\beta} = 0, \text{ giving } \gamma^{i\alpha j\beta} \gamma_{i\alpha\beta} (\kappa_{i\alpha\beta} \cosec \theta + \kappa_{j\alpha\beta} \cot \theta)
\]

However, we prefer some less alien expressions, and will use in our calculations,

\[
k_2^i = k_7^i \cosec^2 \theta - 2k_9^i \cot \theta \cosec \theta + k_5^i \cot^2 \theta + 2 (k_6^i \cosec \theta - k_5^i \cot \theta) + k_1^i
\]

\[
\text{tr} k_2^i = k_8^i \cosec^2 \theta - 2k_10^i \cot \theta \cosec \theta + k_5^i \cot^2 \theta + k_1^i + k_2^i
\]

\[
\kappa_i \kappa_j = - (k_7^i + k_7^j) \cosec \theta \cot \theta + k_9 (\cosec^2 \theta + \cot^2 \theta) + (k_5^i + k_5^j) \cosec \theta
\]

\[
- (k_6^i + k_6^j) \cot \theta + k_3
\]

\[
\kappa_i \tilde{\kappa}_i = k_9 \cosec \theta - k_5^i \cot \theta + k_5^i
\]

\[
\kappa_i \tilde{\kappa}_j = k_7^i \cosec \theta - k_9 \cot \theta + k_6^i
\]

\[
\hat{\kappa}_i \hat{\kappa}_i = k_7^i
\]

\[
\text{tr} \hat{\kappa}_i \hat{\kappa}_i = k_8^i
\]

\[
\hat{\kappa}_i \hat{\kappa}_j = k_9
\]

\[
\text{tr} \hat{\kappa}_i \hat{\kappa}_j = k_{10}
\]

These are manifestly linearly independent (assuming the \( k \)'s are). In fact, since \( k_2^i = k_2^j \) we could discard, for example, \( \text{tr} k_2^i \) or \( \text{tr} k_2^j \) but to keep \( i \leftrightarrow j \) symmetry we do not do this.

**Derivative terms:**

It remains to consider terms which contain the derivative of the extrinsic curvature. The tensors involved are

\[
\nabla_\mu k_{i\nu\rho} \quad \nabla_\mu \tilde{k}_{i\nu\rho}
\]

\[
\nabla_\mu \hat{k}_{i\nu\rho} \quad \nabla_\mu \tilde{\hat{k}}_{i\nu\rho}
\]
where $\bar{\nabla}_\mu$ and $\bar{\nabla}_\mu$ are the covariant derivatives projected onto $\partial M_i$ and $I_{ij}$ respectively:

$$\bar{\nabla}_\mu \kappa^{i\nu}_{\rho\nu} = h^{i\alpha}_\mu h^{j\beta}_\nu h^j_\rho \nabla_\alpha \kappa^{i\beta}_{\gamma\gamma}, \quad \bar{\nabla}_\mu \tilde{\kappa}^i_{\rho\nu} = \gamma^{\alpha\beta\gamma}_\mu \gamma^{\nu\rho\gamma} \nabla_\alpha \kappa^{i\beta}_{\gamma\gamma}$$

Any other projections of the covariant derivative give nothing new, e.g.

$$h^{i\alpha}_\mu n^i_{\nu\nu} \nabla_\mu \kappa^{i\nu}_{\rho\nu} = \kappa^{i\nu}_{\rho\nu} \kappa^{i\nu}_{\alpha\nu}$$

which we have already taken into account. This leaves the possible contractions

$$\delta^i_1 = \tilde{\kappa}^{i\nu}_{\rho\nu} \nabla_\mu \kappa^{i\nu}_{\rho\nu} \quad \delta^i_2 = \tilde{\kappa}^{i\nu}_{\rho\nu} \nabla_\mu \kappa^{i\nu}_{\rho\nu} \quad \delta^i_3 = \tilde{\kappa}^{i\nu}_{\rho\nu} \tilde{\kappa}^{i\nu}_{\rho\nu} \nabla_\mu \kappa^{i\nu}_{\rho\nu}$$

plus $\delta^i_1$ etc. The Codazzi equation

$$\bar{\nabla}_\nu \kappa^i_{\nu} - \bar{\nabla}_\mu \kappa^i_{\nu\nu} = R^i_{\mu\rho\nu} n^\rho_{i}$$

(13)

gives a link between the first two:

$$\delta^i_2 = \delta^i_1 - \rho^i_4$$

and we discard $\delta^i_2$.

Since $I_{ij}$ is closed and smooth, any term which can be written as the divergence of a vector field $\bar{\nabla}_\mu V^\mu$ on $I_{ij}$ can be ignored as its integral over $I_{ij}$ vanishes. The only vector field on $I_{ij}$ with the appropriate scaling we can construct is

$$V^\mu_i = \gamma^{\mu\nu}_{\rho\nu} \kappa^{i\nu}_{\nu\nu} \tilde{\kappa}^i_{\rho}$$

Then

$$\bar{\nabla}_\mu V^\mu_i = \tilde{\kappa}^{i\nu}_{\rho\nu} \tilde{\kappa}^{i\nu}_{\nu\nu} - \kappa^{i\nu}_{\rho\nu} \tilde{\kappa}^{i\nu}_{\nu\nu} + \delta^i_1 - \delta^i_3$$

The first two expressions can be written in terms of the $k$’s, so we can discard $\delta^i_3$.

This concludes our enumeration of the terms involved in the codimension-2 contribution to $C^{(d)}_2$. We have finally

$$S_{ij} = a_1 \bar{R} + a_2 (\tilde{R}_i + \tilde{R}_j) + a_3 \bar{R} + a_4 R_{\mu\nu} \left( n^\mu_{i} \tilde{n}^\nu_{i} + n^\mu_{j} \tilde{n}^\nu_{j} \right) + b_1 (\kappa^2_i + \kappa^2_j) + b_2 (\text{tr} \kappa^2_i + \text{tr} \kappa^2_j) + b_3 (\tilde{\kappa}^2_i + \tilde{\kappa}^2_j) + b_4 (\text{tr} \tilde{\kappa}^2_i + \text{tr} \tilde{\kappa}^2_j) + b_5 (\kappa_i \kappa_j + \kappa_j \kappa_i) + c_1 (\kappa_i \tilde{\kappa}_j + \kappa_j \tilde{\kappa}_i) + c_2 \kappa_i \kappa_j + c_3 \tilde{\kappa}_i \tilde{\kappa}_j + c_4 \text{tr} \tilde{\kappa}_i \tilde{\kappa}_j + d_1 (\nabla_{\tilde{\kappa}_i} \kappa_i + \nabla_{\tilde{\kappa}_j} \kappa_j)$$

(14)

The $a$, $b$, $c$ and $d$ coefficients are functions of $\theta$, but not of the dimension. The $a$’s multiply intrinsic terms while the $b$’s and $c$’s, as we shall see, form two naturally separate groups when $\theta = \pi/2$. The problem is now to evaluate these 14 coefficients by obtaining 14 independent constraints.
4 Conformal invariance in 4 dimensions

A great deal of information about the coefficients can be derived by using conformal invariance. Under a conformal transformation \( g_{\mu\nu} \rightarrow e^{2\omega(x)} g_{\mu\nu} \), we have

\[
R_{\mu\nu} \rightarrow R_{\mu\nu} - g_{\mu\nu} \nabla^2 \omega + (d - 2) \left( \nabla_\mu \omega \nabla_\nu \omega - g_{\mu\nu} \nabla_\rho \omega \nabla^\rho \omega - \nabla_\mu \nabla_\nu \omega \right) \tag{15}
\]

\[
\tilde{R}_i^{\mu\nu} \rightarrow \tilde{R}_i^{\mu\nu} - h_i^{\mu\nu} \tilde{\nabla}^2 \omega + (d - 3) \left( \tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu \omega - h_i^{\mu\nu} \tilde{\nabla}_\rho \omega \tilde{\nabla}^\rho \omega - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega \right) \tag{16}
\]

\[
\tilde{R}_i^{\mu\nu} \rightarrow \tilde{R}_i^{\mu\nu} - \gamma_{\mu\nu} \tilde{\nabla}^2 \omega + (d - 4) \left( \tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu \omega - \gamma_{\mu\nu} \tilde{\nabla}_\rho \omega \tilde{\nabla}^\rho \omega - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega \right) \tag{17}
\]

\[
\kappa_i^{\mu\nu} \rightarrow e^\omega \left( \kappa_i^{\mu\nu} - h_i^{\mu\nu} \nabla_{n_i} \omega \right) \tag{18}
\]

\[
\tilde{\kappa}_i^{\mu\nu} \rightarrow e^\omega \left( \tilde{\kappa}_i^{\mu\nu} - \gamma_{\mu\nu} \nabla_{\tilde{n}_i} \omega \right) \tag{19}
\]

We use the above to first order in \( \omega \) to work out the change in \( S_{ij} \) under a small conformal transformation \( g_{\mu\nu} \rightarrow e^{2\omega(x)} g_{\mu\nu} \). With general \( \xi \), there is an added complication in that \( C_i^{(d)} \) is not in fact conformally invariant, and we must use equation (5). This involves the codimension-2 contribution to \( C_1^{(d)}[f] \), which is known to be [7]

\[
C_1^{(d)} \big|_I = \sum_{(ij)} \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} \frac{1}{6} \left( \frac{\pi^2 - \theta^2}{\theta} \right) f(x) \tag{20}
\]

The change in \( C_i^{(d)} \) then turns out to have the form

\[
\left( \delta C_i^{(d)} - 2C_1^{(d)}[J\delta \omega] \right) \big|_I = O(d - 4) + \sum_{(ij)} \delta \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} S_{ij}
\]

\[
-\eta_d \sum_{(ij)} \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} \left( \kappa_i^{\mu\nu} \tilde{n}_i^\mu \delta \omega_\nu - \kappa_i \nabla_{\tilde{n}_i} \delta \omega + \kappa_j^{\mu\nu} \tilde{n}_j^\mu \delta \omega_\nu - \kappa_j \nabla_{\tilde{n}_j} \delta \omega \right)
\]

\[
- \sum_{(ij)} \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} \frac{1}{3} \left( \frac{\pi^2 - \theta^2}{\theta} \right) J \delta \omega \tag{21}
\]

where \( J \) is given beneath equation (5). The extra term, which vanishes in the smooth boundary case, is equal to the boundary divergence which comes from the conformal variation of the volume and boundary parts of \( C_i^{(d)} \). We keep the coefficient \( \eta_d \) general since we wish briefly to discuss spin-1/2 later.

The right-hand side of (21), which we must set to zero in 4 dimensions, is ex-
pressible in terms of the 7 linearly independent quantities

\[
\begin{align*}
(\kappa_i \nabla_{n_i} + \kappa_j \nabla_{n_j}) \delta \omega &= (\kappa_i \nabla_{n_i} + \kappa_j \nabla_{n_j}) \delta \omega \\
(\tilde{\kappa}_i \nabla_{n_i} + \tilde{\kappa}_j \nabla_{n_j}) \delta \omega &= (\tilde{\kappa}_i \nabla_{n_i} + \tilde{\kappa}_j \nabla_{n_j}) \delta \omega \\
(n_i^{\mu} n_i^{\nu} + n_j^{\mu} n_j^{\nu}) \delta \omega_{\mu\nu} &= n_i^{\mu} n_j^{\nu} \delta \omega_{\mu\nu} \\
(\kappa^{\mu}_{\mu} \tilde{\kappa}_i^{\nu} + \kappa^{\nu}_{\mu} \tilde{\kappa}_j^{\mu}) \delta \omega^{\nu}
\end{align*}
\]

where indices on \( \delta \omega \) denote covariant derivatives on \( \mathcal{M} \). Any other quantity which occurs can be written in terms of these, using (9) and the relations between Laplacians on \( \mathcal{M}, \partial M_i, \partial M_j \) and \( \mathcal{I}_{ij} \). \( \tilde{\nabla}^2 \delta \omega \) can be ignored since it is a total divergence on \( \mathcal{I} \). Setting the coefficients of these to zero at \( d = 4 \) gives us 7 constraints on the \( a, b, c \) and \( d \). We find

\[
\begin{align*}
(4a_2 + 6b_1 + 2b_2) \sin \theta + 2b_5 + (2c_1 + d_1) \cos \theta &= \eta_4 \cos \theta \\
2b_5 \cos \theta + 3c_2 \sin \theta + 2c_1 + d_1 &= \eta_4 \\
(4b_3 + 2b_4) \cos \theta + 3c_1 \sin \theta + 2c_3 + c_4 &= 0 \\
-6\pi_1 - 8a_2 + 4b_3 + 2b_4 + 3b_5 \sin \theta + (2c_3 + c_4) \cos \theta &= 0 \\
6\pi_1 + 4a_2 \left(1 + \cos^2 \theta\right) + (2a_4 + 3d_1) \sin \theta \cos \theta &= 0 \\
(6\pi_1 + 8a_2) \cos \theta + (2a_4 + 3d_1) \sin \theta &= 0 \\
3d_1 &= \eta_4 
\end{align*}
\]

where \( \pi_1 \) is a correction for non-conformal coupling:

\[
\pi_1 = a_1 + \frac{1}{6} \left(\xi - \frac{1}{6}\right) \left(\frac{\pi^2 - \theta^2}{\theta}\right)
\]

Also, from conformally varying (7) we find that

\[
\eta_d = \frac{d - 6}{90} \Rightarrow \eta_4 = -\frac{1}{45}
\]

5. The smeared coefficient and conformal invariance in 6 dimensions

There is a further conformal invariance we can use, in that

\[
\delta C^{(2k+2)}_k[f] = 0 \text{ as } g_{\mu\nu} \rightarrow e^{2\delta \omega} g_{\mu\nu}, \text{ } f \rightarrow e^{-2\delta \omega} f
\]
as long as $\Delta$ is conformally covariant in $2k + 2$ dimensions.

To implement this, it is necessary to calculate the smeared coefficient $C_2^{(d)}[f]$ using (3). We do not give it here since it is rather involved. Applying the conformal variation above in 6 dimensions with $\xi = 1/5$ then yields the following independent terms in the integrand of the codimension-2 part of $\delta C_2^{(6)}[f]$: 

$$f(\kappa_i \nabla_{n_i} + \kappa_j \nabla_{n_j}) \delta \omega$$ 

$$f(\tilde{\kappa}_i \nabla_{n_i} + \tilde{\kappa}_j \nabla_{n_j}) \delta \omega$$ 

$$f(n_i^\mu n_i^\nu + n_j^\mu n_j^\nu) \delta \omega_{\mu\nu}$$ 

$$f(\kappa_{\mu\nu} \tilde{n}_i^\mu + \kappa_{\mu\nu} \tilde{n}_j^\mu) \delta \omega^{\nu}$$ 

$$f(\nabla_{n_i} f \nabla_{n_i} + \nabla_{n_j} f \nabla_{n_j}) \delta \omega$$ 

All other terms can be written in terms of these, up to divergences on $I_{ij}$. Somewhat surprisingly, it turns out that the volume and boundary parts of $\delta C_2^{(6)}[f]$ contain no divergences, so that the codimension-2 part vanishes by itself. We therefore set the coefficients of all the above terms to zero. The first seven simply give us the equations we have already derived from 4-dimensional conformal invariance. From the remaining three, we get, respectively

$$6\pi_1 + 8a_2 + 3a_3 = 0$$  \hspace{1cm} (32) 

$$-12\pi_1 - 16a_2 - 5(b_1 + b_2) \sin^2 \theta + 8b_3 \left(1 + \cos^2 \theta\right)$$ 

$$+ 4b_5 \sin \theta + 8c_3 \cos \theta + \left(4c_1 - \frac{11}{2}d_1 - \frac{5}{54}\right) \cos \theta \sin \theta = 0$$  \hspace{1cm} (33) 

$$(-12\pi_1 - 16a_2 + 16b_3 + 9b_5 \sin \theta) \cos \theta$$ 

$$+ 4c_3 - \left(9c_1 + 5c_2 \sin \theta - \frac{1}{27} - 3d_1\right) \sin \theta = 0$$  \hspace{1cm} (34) 

### 6 The lune

Having extracted everything we can from conformal invariances, we now turn to specific manifolds. Evaluation of the zeta function on lunes will turn out to give us some more information. Lunes are particularly helpful since the angle between adjacent boundary parts is arbitrary. To define a $d$-lune, we start with a 2-lune, which is the region $0 \leq \phi \leq \theta$ on a 2-sphere of unit radius, $\phi$ being the azimuthal angle. Higher-dimensional lunes are then defined inductively by $ds_{d-\text{lune}}^2 = d\chi^2 + \sin^2 \chi ds_{(d-1)-\text{lune}}^2$, $0 \leq \chi \leq \pi$. 

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In our calculations we consider only the case \( \theta = \pi/q \), where \( q \) is an integer, and analytically continue our results to all \( \theta \). The calculational procedure can be extended to \( \theta = p\pi/q \), with a great deal of added complexity. We are assuming that the spectral coefficients are smooth functions of \( \theta \) – this is the case for \( C_1^{(d)} \). Some work has been done by Cook \([14]\) on the case where \( \theta/\pi \) is irrational. The results agree with analytic continuation.

We will consider \( 2 \leq d \leq 5 \). Since \( S_{ij} \) is quadratic in \( d \) on a spherical domain, we can derive at most three independent constraints, and stopping at \( d = 4 \) turns out to be sufficient. \( d = 5 \) provides a check of the equations. The procedure is to use the eigenvalues and degeneracies to calculate the zeta function and derive the heat-kernel coefficients via \([4]\). The zeta function on orbifolded spheres, of which the lune is an example, has been calculated \([15]\), and we do not go into detail here.

For a general coupling \( \xi \), we find

\[
C_2^{(2)} = \frac{\pi}{720q} \left( 8q^4 + 20q^2 - 7 \right) + \frac{\pi \alpha^2}{6q} \left( 2q^2 - 1 \right) + \frac{\pi \alpha^4}{q}
\]

\[
C_2^{(3)} = \frac{\pi^2 \alpha^2}{3q} \left( q^2 - 1 \right) + \frac{\pi^2 \alpha^4}{2q}
\]

\[
C_2^{(4)} = \frac{\pi^2}{360q} \left( 51 - 60q^2 - 8q^4 \right) + \frac{\pi^2 \alpha^2}{3q} \left( 2q^2 - 3 \right) + \frac{2\pi^2 \alpha^4}{3q}
\]

\[
C_2^{(5)} = \frac{\pi^3}{45q} \left( 11 - 10q^2 - q^4 \right) + \frac{\pi^3 \alpha^2}{3q} \left( q^2 - 2 \right) + \frac{\pi^3 \alpha^4}{4q}
\]

where

\[
\alpha^2 = d(d-1) \left[ \frac{d-1}{4d} - \xi \right]
\]

In using this data to obtain information about \( S_{ij} \), we need to separate off the codimension-2 contribution to \( C_2^{(d)} \). On the lune all extrinsic curvatures vanish, as does \( \nabla_n R \), so the boundary contribution is zero. Since the volume part is proportional to \( 1/q \), we have

\[
C_2^{(d)}(q) \bigg|_\mathcal{I} = C_2^{(d)}(q) - \frac{1}{q} C_2^{(d)}(q = 1)
\]

The boundary intersection \( \mathcal{I} \) on a lune is a \( (d-2) \)-sphere of unit radius (2 points, each of content 1, in the case \( d = 2 \)). The boundaries are \( (d-1) \)-hemispheres of unit radius. The only nonvanishing geometrical quantities are

\[
R = d(d-1) \quad \tilde{R}_i = \tilde{R}_j = (d-1)(d-2) \quad \tilde{R} = (d-2)(d-3)
\]

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$R_{\mu\nu}n_i^\mu \tilde{n}_i^\nu$ and $R_{\mu\nu}n_j^\mu \tilde{n}_j^\nu$ vanish since the lune is an Einstein space $R_{\mu\nu} = (d - 1)g_{\mu\nu}$ and the two normals are orthogonal.

Equating equation (14) with our expressions for $C_2^{(d)}|_I$ on the $d$-lune, we obtain the constraints

\[
\begin{align*}
a_1 &= \frac{\pi}{36} (6\xi - 1) \left( \frac{\theta}{\pi} - \frac{\pi}{\theta} \right) + \frac{\pi}{360} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right) \\
3a_1 + 2a_2 &= \frac{\pi}{12} (6\xi - 1) \left( \frac{\theta}{\pi} - \frac{\pi}{\theta} \right) \\
6a_1 + 6a_2 + a_3 &= \frac{\pi}{6} (6\xi - 1) \left( \frac{\theta}{\pi} - \frac{\pi}{\theta} \right) + \frac{\pi}{360} \left( \frac{\theta}{\pi} - \frac{\pi^3}{\theta^3} \right)
\end{align*}
\]

from 2, 3 and 4 dimensions respectively. In each case we can conveniently replace $a_1$ by $\overline{a}_1$ and remove the $(6\xi - 1)$ term. We now have $a_1$, $a_2$ and $a_3$ for all $\theta$:

\[
\begin{align*}
\overline{a}_1 &= \frac{\pi}{360} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right) \quad (39) \\
a_2 &= -\frac{\pi}{240} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right) \quad (40) \\
a_3 &= \frac{\pi}{180} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right) \quad (41)
\end{align*}
\]

From 5 dimensions we get

\[
10\overline{a}_1 + 12a_2 + 3a_3 = -\frac{\pi}{180} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right)
\]
as a check of (34), (40) and (41).

The conformal variation equations (26) and (27) give $3\overline{a}_1 + 2a_2 = 0$, in agreement with the above. In addition, the lune results agree with equation (32). This is encouraging, although we have less information than we hoped for, i.e. lunes have only given us 1 new constraint.

We now have all the $a$ coefficients for general $\theta$, since from (27), (28)

\[
a_4 = \frac{1}{90} + \frac{\pi}{120} \left( \frac{\pi^3}{\theta^3} - \frac{\theta}{\pi} \right) \cot \theta \quad (42)
\]

7 A further constraint

We can in fact make the situation a bit simpler by noticing that $k_i^2$ is conformally invariant for all $d$, so we can consistently set this quantity to zero. This restriction
makes sense since $k^i_2$ vanishes on most spaces we are liable to deal with, e.g. the hemiball, lune and cylinder. $k^i_2$ measures the rate at which $n_i$ rotates in the direction of $\mathbf{n}_i$ as we move around on $I_{ij}$ – an example of a boundary where $k^i_2$ does not vanish is a rectangle twisted so that each edge forms a helix.

If we set $k^i_2 = k^j_2 = 0$, there are now only 13 degrees of freedom, and we can remove any one of the terms with $b$ or $c$ coefficients from (14). We choose to get rid of $\kappa_i\hat{\kappa}_j$ and $\kappa_j\hat{\kappa}_i$, so we set $b_5$ to zero.

In the case where this quantity did not vanish, due to its conformal invariance in all dimensions we would not be able to find the extra degree of freedom in the coefficients except by calculating the zeta function on a manifold where the measure of “twist” we have described is non-zero. As far as we can see, this would present a very difficult task and for the moment it is necessary to make this constraint if we wish to complete the calculation.

8 Right-angled edges and product manifolds

We can go further if we limit ourselves to the case $\theta = \pi/2$. Again, many of the manifolds we come across have this property. We also maintain the constraint we have made above, that $k^i_2 = 0$. For $\theta = \pi/2$, this condition takes the simple form

$$(\kappa_i - \hat{\kappa}_j)^2 = \text{tr}\kappa_i^2 - \text{tr}\hat{\kappa}_j^2$$

(43)

Our 11 equations, now for only 13 coefficients, become

$$a_1 = \frac{\pi}{48}, \quad a_2 = -\frac{\pi}{32}, \quad a_3 = \frac{\pi}{24}, \quad a_4 = \frac{1}{90}$$

$$3b_1 + b_2 = \frac{\pi}{16}, \quad 2b_3 + b_4 = -\frac{\pi}{16}, \quad b_1 + 5b_2 - 8b_3 = \frac{\pi}{4}$$

$$2c_1 + 3c_2 = -\frac{2}{135}, \quad 3c_1 + 2c_3 + c_4 = 0, \quad d_1 = -\frac{1}{135}$$

$$9c_1 + 5c_2 + 4c_3 = \frac{2}{135}$$

(44)

We can derive the last two constraints we need by considering a product manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, with an operator $\Delta = -(\nabla^2_1 + \nabla^2_2) + \xi(R_1 + R_2)$. $\mathcal{M}_1$ and $\mathcal{M}_2$ have smooth boundaries. $\mathcal{M}$ has a codimension-2 submanifold $\partial\mathcal{M}_1 \times \partial\mathcal{M}_2$ with $\theta = \pi/2$, and obeys (43).

From (11), it is easily shown that, for scalar fields with Dirichlet conditions

$$C^\mathcal{M}_2 = C^\mathcal{M}_0 C^\mathcal{M}_2 + C^\mathcal{M}_1 C^\mathcal{M}_0 C^\mathcal{M}_2 + C^\mathcal{M}_1 C^\mathcal{M}_2 + C^\mathcal{M}_3 C^\mathcal{M}_2 + C^\mathcal{M}_4 C^\mathcal{M}_2$$

(45)
The codimension-2 part of the left hand side is, in terms of geometrical quantities on $\mathcal{M}_1$ and $\mathcal{M}_2$

$$\int_{\partial \mathcal{M}_1 \times \partial \mathcal{M}_2} d^{d-2}x \sqrt{h_1 h_2} \left[ (a_1 + a_2)(R_1 + R_2) + (a_2 + a_3) \left( \hat{R}_1 + \hat{R}_2 \right) 
+ (b_1 + b_3) (\kappa_1^2 + \kappa_2^2) + (b_2 + b_4) \left( \text{tr}\kappa_1^2 + \text{tr}\kappa_2^2 \right) + (2c_1 + c_2 + c_3) \kappa_1 \kappa_2 \right]$$

The expressions for $C_k^{(d)}$ for $k = 0, 1/2, 1, 3/2$ are well-known \[11, 16\]:

$$C_0^{(d)} = \int_{\mathcal{M}} d^d x \sqrt{g}$$

$$C_{1/2}^{(d)} = -\frac{\sqrt{\pi}}{2} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{h}$$

$$C_1^{(d)} = \left( \frac{1}{6} - \xi \right) \int_{\mathcal{M}} d^d x \sqrt{g} R + \frac{1}{3} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{h} \kappa$$

$$C_{3/2}^{(d)} = \frac{\sqrt{\pi}}{192} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{h} \left[ 6\kappa R - 3\kappa^2 + 12(8\xi - 1)(R_1 + R_2) - 4 \left( \hat{R}_1 + \hat{R}_2 \right) \right]$$

The codimension-2 part of the right-hand side of (43) then comes from the middle three terms, and is

$$\int_{\partial \mathcal{M}_1 \times \partial \mathcal{M}_2} d^{d-2}x \sqrt{h_1 h_2} \left[ \frac{1}{9} \kappa_1 \kappa_2 - \frac{\pi}{384} \left[ 6 \left( \text{tr}\kappa_1^2 + \text{tr}\kappa_2^2 \right) - 3 \left( \kappa_1^2 + \kappa_2^2 \right) + 12(8\xi - 1)(R_1 + R_2) - 4 \left( \hat{R}_1 + \hat{R}_2 \right) \right] \right]$$

Equating the two sides, we find

$$\bar{a}_1 + a_2 = -\frac{\pi}{96} \quad a_2 + a_3 = \frac{\pi}{96}$$

$$b_1 + b_3 = \frac{\pi}{128} \quad b_2 + b_4 = -\frac{\pi}{64}$$

$$2c_1 + c_2 + c_3 = \frac{1}{9}$$

The first two of these agree with (44). The second two give one new constraint, completing the information for the $b$'s, and one in agreement with the rest. The last completes the information for the $c$'s. Our final result is

$$C_2^{(d)} \bigg| = \sum_{i,j} \int_{\mathcal{I}_{ij}} d^{d-2}x \sqrt{\gamma} \left\{ \frac{\pi}{384} \left[ 24(1 - 4\xi)R - 12 \left( \hat{R}_i + \hat{R}_j \right) + 16 \tilde{R} 
+ 24 \left( \text{tr}\kappa_i^2 + \text{tr}\kappa_j^2 \right) + 3 \left( \hat{\kappa}_i^2 + \hat{\kappa}_j^2 \right) - 30 \left( \text{tr}\hat{\kappa}_i^2 + \text{tr}\hat{\kappa}_j^2 \right) \right] 
+ \frac{1}{270} \left[ 3R^{\mu\nu} \left( n^\mu \tilde{n}_i^\nu + n^\mu \tilde{n}_j^\nu \right) - 344 (\kappa_i \hat{\kappa}_i + \kappa_j \hat{\kappa}_j) + 228 \kappa_i \kappa_j 
+ 490 \hat{\kappa}_i \hat{\kappa}_j + 52 \text{tr}\hat{\kappa}_i \hat{\kappa}_j - 2 \left( \nabla_{\tilde{n}_i} \kappa_i + \nabla_{\tilde{n}_j} \kappa_j \right) \right] \right\}$$

(51)
We note that this, together with equation (7), can be extended to the non-zero mass case, where the operator becomes

\[ \Delta = -\nabla^2 + \xi R - m^2 \]

simply by replacing \( \xi R \) by \( \xi R - m^2 \).

9 The smeared coefficient and cocycle function

For completeness, we give the smeared coefficient with \( \theta = \pi/2 \), where (43) applies. The expression for the smooth boundary case is well-known. Using (7), (51) and (5), we find

\[
C^{(d)}[f] = \int_M d^d x \sqrt{g} \left[ U f + \frac{1}{6} \left( \frac{1}{5} - \xi \right) R \nabla^2 f \right] + \sum_i \int_{\partial M_i} d^{d-1} x \sqrt{h} \left[ T f + \frac{1}{3} \left( \xi - \frac{3}{20} \right) R \nabla_n f + \frac{1}{15} \kappa \nabla^2 f \right]
\]

\[
- \frac{1}{210} \kappa^2 \nabla_n f + \frac{1}{42} \text{tr} \kappa^2 \nabla_n f - \frac{1}{12} \nabla_n \nabla^2 f \right]
+ \sum_{(ij)} \int_{I_{ij}} d^{d-2} x \sqrt{\gamma} \left\{ S_{ij} f + \frac{\pi}{64} \left[ 4 \left( \kappa_i \nabla_n f + \kappa_j \nabla_n f \right) f - 9 \left( \tilde{\kappa}_i \nabla_n f + \tilde{\kappa}_j \nabla_n f \right) f + 4 \left( n^i_\mu n^\nu_j + n^i_\mu n^\nu_j \right) f_{\mu\nu} \right] - 9 \left( \tilde{\kappa}_i \nabla_n f + \tilde{\kappa}_j \nabla_n f \right) f - 146 \left( \tilde{\kappa}_i \nabla_n f + \tilde{\kappa}_j \nabla_n f \right) f
\]

\[
- 2n^i_\mu n^\nu_j f_{\mu\nu} - 5 \left( \kappa^i_{\mu\nu} \tilde{n}^\mu_i + \kappa^j_{\mu\nu} \tilde{n}^\mu_j \right) f_{\mu\nu} \right]\} \quad (52)
\]

where \( U \) and \( T \) are the volume and boundary parts of the standard smooth expression for \( C^{(d)}[1] \), (7).

We can use this to calculate the change in the effective action, or cocycle function, in 4 dimensions for a conformal field theory (\( \xi = 1/6 \)). It can be shown using zeta function techniques and \( W = -\frac{1}{2} \zeta'(0) \) that under a small conformal variation,

\[
\delta W = -\frac{1}{(4\pi)^{d/2}} C^{(d)}_{d/2} [\delta \omega]
\]

The difference in effective action for two metrics \( g \) and \( \overline{g} = e^{2\omega} g \) is then given by

\[
W[\overline{g}] - W[g] = -\frac{1}{(4\pi)^{d/2}} \int_{u=0}^{1} C^{(d)}_{d/2} \left[ e^{2\omega} g; \omega du \right] \quad (53)
\]
Again the volume and boundary parts are known [17], the result being

\[
(W [\mathcal{g}] - W[g])_{V,B} = \frac{1}{2880\pi^2} \int_M d^4x \sqrt{g} \left[ -180U\omega - \omega \nabla^2 R \
- 2R^\mu\nu\omega_\mu\omega_\nu + 4\omega^\mu\omega_\mu \nabla^2 \omega + 2(\omega^\mu\omega_\mu)^2 + 3(\nabla^2 \omega)^2 \right] \\
+ \frac{1}{5760\pi^2} \sum_i \int_{\partial M_i} d^3x \sqrt{h} \left[ -360T\omega - 2\omega \nabla_n R \
+ \nabla_n \omega \left( \frac{12}{7}\kappa^2 - \frac{60}{7}\text{tr}\kappa^2 + 12\nabla^2 \omega + 8\omega^\mu\omega_\mu \right) \right] \\
- \frac{4}{7}\kappa(\nabla_n \omega)^2 - \frac{16}{21}(\nabla_n \omega)^3 - 24\nabla^2 \omega - 4\kappa^\mu\nu\omega^\mu\omega_\nu \\
- 20\kappa\omega^\mu\omega_\mu + 30\nabla_n(\nabla^2 \omega + \omega^\mu\omega_\mu) \right) 
\]

(54)

The calculation of the edge part is fairly easy, and we find

\[
(W [\mathcal{g}] - W[g])_{E} = \frac{1}{16\pi^2} \sum_{(ij)} \int_I d^2x \sqrt{g} \left\{ S_{ij}\omega + \frac{\pi}{64} \left[ 4 \left( \kappa_i \nabla_n i + \kappa_j \nabla_n j \right) \omega \\
- 9 \left( \kappa_i \nabla_n i + \kappa_j \nabla_n j \right) \omega + 4 \left( n_i^\mu n_i^\nu + n_j^\mu n_j^\nu \right) \omega_{\mu\nu} \\
+ 3 \left[ (\nabla_n i)^2 + (\nabla_n j)^2 \right] - \frac{8}{3}\omega \nabla^2 \omega \right] \\
+ \frac{1}{270} \left[ 119 \left( \kappa_i \nabla_n i + \kappa_j \nabla_n j \right) \omega - 146 \left( \kappa_i \nabla_n i + \kappa_j \nabla_n j \right) \omega \\
- 2n_i^\mu n_j^\nu \omega_{\mu\nu} - 5 \left( \kappa_i^{\mu\nu} \kappa_j^{\mu\nu} + \kappa_i^{\mu\nu} \kappa_j^{\mu\nu} \right) \omega_{\mu\nu} - 58\nabla_n i \omega \nabla_n j \omega \right] \right\} 
\]

(55)

This formula can be used to find the effective action on a manifold if the value on a manifold to which it is conformally related is known. Previously, in 4 dimensions, we could only consider spaces with smooth boundaries – we can now do calculations where an edge is present.

For example, a hemicap of a 4-sphere of unit radius (\(\mathcal{g}\)) is related to the hemiball (\(g\)) by the conformal transformation

\[
\mathcal{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \quad \omega = \ln \left( \frac{2}{1 + r^2} \right) 
\]

\(r\) being the radial coordinate on the hemiball, where the radius \(a\) of the hemicap and the colatitude \(\Theta\) of the hemicap boundary are related by

\[
a = \tan \frac{\Theta}{2} 
\]
If \( a = 1 \), then the hemicap is a quarter-sphere. We find

\[
W_{\text{hemiball}}(a) - W_{\frac{1}{4}\text{-sphere}} = \frac{19}{1440} \ln a - \frac{1}{360} \ln 2 - \frac{6751}{241920}
\]

\[
W_{\text{hemicap}}(\Theta) - W_{\frac{1}{4}\text{-sphere}} = \frac{1}{96} \ln (1 + \cos \Theta) + \frac{1}{1440} \left[ 19 \ln \tan \Theta / 2 + \frac{5}{8} \cos \Theta (139 + 63 \cos \Theta) - \frac{1399}{168} \cos^3 \Theta \right]
\]

Expression for the quartersphere effective action have been derived in [4, 6].

10 Spin-1/2

A similar calculation should be possible if \( \Delta \) is the squared Dirac operator

\[
\Delta = (i \gamma^\mu \nabla_\mu)^2 = -g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{1}{4} R
\]

as long as the boundary conditions are local. A suitable set are the mixed conditions [18, 19]

\[
P_\pm \psi \big|_{\partial M} = 0, \quad \left( \nabla_n - \frac{1}{2} \kappa \right) \psi \big|_{\partial M} = 0
\]

for the projection operators \( P_\pm = \frac{1}{2} (1 \pm i \gamma^5 \gamma^\mu n_\mu) \).

In this case the volume and boundary parts of \( C_2 \) are known [20, 21, 22]. \( S_{ij} \) has the form (14), and the equations we derive from conformal invariance in 4 dimensions are identical to (22) to (28), with the exception that

\[
\eta_d = \frac{2d + 3}{180} \mathcal{D}
\]

where \( \mathcal{D} \) is the dimension of the spinor space, and \( \overline{a}_1 = a_1 \) since there is no variable coupling to take into account.

From direct calculation on lunes we find

\[
C_{2}^{(2)} = -\frac{\pi \mathcal{D}}{720 q} \left( 7 q^4 + 10 q^2 + 7 \right)
\]

\[
C_{2}^{(3)} = 0
\]

\[
C_{2}^{(4)} = \frac{\pi^2 \mathcal{D}}{360 q} \left( 7 q^4 + 30 q^2 + 51 \right)
\]

\[
C_{2}^{(5)} = \frac{\pi^3 \mathcal{D}}{360 q} \left( 7 q^4 + 40 q^2 + 88 \right)
\]
so that (from the first 3 of these)

\[ a_1 = -\frac{\pi D}{2880} \left( \frac{7\pi^3}{\theta^3} + 10\frac{\pi}{\theta} - 17\frac{\theta}{\pi} \right) \]  
\[ a_2 = \frac{\pi D}{1920} \left( \frac{7\pi^3}{\theta^3} + 10\frac{\pi}{\theta} - 17\frac{\theta}{\pi} \right) \]  
\[ a_3 = \frac{7\pi D}{1440} \left( \frac{\theta}{\pi} - \frac{\pi^3}{\theta^3} \right) \]  

\( \text{(61)} \)
\( \text{(62)} \)
\( \text{(63)} \)

\( \)From 5 dimensions we get

\[ 10a_1 + 12a_2 + 3a_3 = \frac{\pi D}{1440} \left( \frac{\pi^3}{\theta^3} + 40\frac{\pi}{\theta} - 47\frac{\theta}{\pi} \right) \]

in agreement with the other relations. As for the scalar case, we have \( 3a_1 + 2a_2 = 0 \), in agreement with the conformal equations. Additionally,

\[ a_4 = -\frac{11D}{360} - \frac{\pi D}{960} \left( \frac{7\pi^2}{\theta^2} + 10\frac{\pi}{\theta} - 17\frac{\theta}{\pi} \right) \cot \theta \]  
\[ \text{(64)} \]

Unfortunately, this is as far as we can go with the techniques used in this paper. Conformal invariance of the smeared coefficient in 6 dimensions does not apply – this is to do with the fact that the squared Dirac operator is not in fact conformally covariant; it is only a power of a conformally covariant operator. Additionally, on the product of two manifolds with boundaries, the heat-kernel does not turn out to be a simple product, i.e. equation (45) does not apply. This is a result of the boundary conditions.

### 11 Robin boundary conditions

A conformally invariant boundary condition, alternative to the Dirichlet case, is the Robin condition

\[ (\nabla_n \phi - \psi \phi) \bigg|_{\partial M} = 0 \]  
\[ \text{(65)} \]

for some boundary function \( \psi \), where under a conformal transformation we set

\[ \psi \to e^{-\omega} \left[ \psi - \frac{1}{2}(d-2)\nabla_n \omega \right] \]  
\[ \text{(66)} \]

so that (65) is conformally invariant.
A general expression for the integrand of the codimension-2 contribution is

\[ S_{ij} = S_{ij}^N + u_1 (\kappa_i \psi_j + \kappa_j \psi_i) + u_2 (\hat{\kappa}_i \psi_j + \hat{\kappa}_j \psi_i) + u_3 (\psi_i^2 + \psi_j^2) + v_1 (\kappa_i \psi_j + \kappa_j \psi_i) + v_2 (\hat{\kappa}_i \psi_j + \hat{\kappa}_j \psi_i) + v_3 \psi_i \psi_j + w_1 (\nabla_{\hat{n}_i} \psi_i + \nabla_{\hat{n}_j} \psi_j) \]  

(67)

where the Neumann (\( \psi_i = \psi_j = 0 \)) expression \( S_{ij}^N \) has the same form as the Dirichlet expression (14), although in general the \( a, b, c \) and \( d \) constants will have different values.

Going through the procedure we have detailed for the Dirichlet case, this time sticking with \( \theta = \pi/2 \), yields

\[
\begin{align*}
\bar{a}_1 &= \frac{\pi}{48} & a_2 &= -\frac{\pi}{32} & a_3 &= \frac{\pi}{24} & a_4 &= \frac{1}{90} \\
 b_1 &= \frac{\pi}{16} & b_2 &= \frac{\pi}{8} & b_3 &= -\frac{5\pi}{128} & b_4 &= -\frac{7\pi}{64} \\
u_1 &= -\frac{\pi}{2} & u_2 &= \frac{\pi}{4} & u_3 &= \frac{\pi}{2} & v_3 &= 4
\end{align*}
\]

(68)

with 7 equations for the remaining 8 constants:

\[
\begin{align*}
2c_1 + 3c_2 + d_1 + v_1 &= -\frac{1}{45} \\
3c_1 + 2c_3 + c_4 + v_2 &= 0 \\
3d_1 + w_1 &= -\frac{1}{45} \\
3v_1 + 2v_2 + v_3 + w_1 &= 0 \\
9c_1 + 5c_2 + 4c_3 - 3d_1 + \frac{9}{2}v_1 + 4v_2 + v_3 - \frac{1}{6}w_1 &= \frac{1}{27} \\
2c_1 + c_2 + c_3 &= \frac{1}{9} \\
v_1 + v_2 &= -\frac{2}{3}
\end{align*}
\]

(69-75)

with checks.

To acquire the final piece of information, we consider the specific case of a square with Neumann conditions on three sides and a non-zero boundary function on the fourth. While this analysis would be difficult for general \( \psi \), it is possible to calculate the eigenvalues and zeta function for a small perturbation \( \psi = \epsilon f, \epsilon \ll 1 \). Our results will be correct to first order in \( \epsilon \). This is sufficient since we will be able to obtain the coefficient of the \( \nabla_{\hat{n}_i} \psi_i \) term in \( C_2 \).
It can be shown that under such a perturbation, in general the eigenvalues become
\[ \lambda = \lambda_N + \epsilon \eta, \quad \eta = \int_{\partial M} d^{d-1}x \sqrt{h} |\phi_N|^2 f \]  
(76)
where the \( \lambda_N \) and \( \phi_N \) are the eigenvalues and normalized eigenfunctions for \( \epsilon = 0 \) (Neumann conditions).

In the case of a square of side \( \pi \), with \( \psi = \epsilon f(y) \) on the \( x = 0 \) boundary,
\[ \eta_{mn} = \frac{4}{\pi^2} \int_0^\pi dy f \cos ny, \quad \eta_{0n} = \frac{1}{2} \eta_{mn} \]
\[ \eta_{m0} = \frac{2}{\pi^2} \int_0^\pi dy f, \quad \eta_{00} = \frac{1}{2} \eta_{m0}, \quad m, n > 0 \]
where the \( m, n \) label the eigenfunctions \( \phi_N \sim \cos mx \cos ny \) with the eigenvalues \( m^2 + n^2 \).

We will find it convenient to write
\[ \eta_{mn} = A + \frac{B}{n^2} + O \left( \frac{1}{n^4} \right) \]
where, from integration by parts,
\[ A = \frac{2}{\pi^2} \int_0^\pi dy f, \quad B = \frac{1}{2\pi^2} \left[ f'(\pi) - f'(0) \right] \]
Then to first order in \( \epsilon \),
\[ \zeta(s) = \zeta_N(s) - \frac{1}{2} \epsilon \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left[ A + \frac{B}{n^2} + O \left( \frac{1}{n^4} \right) \right] (m^2 + n^2)^{-(s+1)} \]
\[ -2\epsilon \eta_{00} \zeta_R(2s + 2) + (\eta_{00} \epsilon)^{-s} \]

The evaluation of the sum is an old, well-known procedure [23] used in calculating the zeta function on cylinders – see for example [1, 24]. The standard result is
\[ \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} g(n) (m^2 + n^2)^{-u} = \frac{\Gamma(u - 1/2) \sqrt{\pi}}{\Gamma(u)} \sum_{n=1}^{\infty} g(n)n^{1-2u} \]
\[ + \frac{2\sqrt{\pi}}{\Gamma(u)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\infty} dt t^{u-3/2} g(n)e^{-nt^2} e^{-m^2\pi^2/t} \]  
(77)
Since we are interested in \( C_2 = -4\pi \zeta(-1) \) we wish to evaluate the above expression for \( u = 0 \). If \( g(n) \) is a finite sum of powers of \( 1/n^2 \), the second term is zero, since the sum over \( m \) and \( n \) converges and \( \lim_{u \to 0} \Gamma(u) = 1/u \). The first picks out
the $1/n^2$ term from $g(n)$ since $\lim_{u \to 0} \zeta_R(1+2u) = 1/2u$, and $\zeta_R$ is finite for all other arguments. We are left with

$$
\zeta(-1) = \zeta_N(-1) - \frac{1}{2} \epsilon B \pi + 2 \epsilon \eta_0 \zeta_R(0) + \epsilon \eta_0 \\
\Rightarrow C_2 = C_2^N + \epsilon \left[ f'(\pi) - f'(0) \right]
$$

since $\zeta_R(0) = -1/2$. Comparing with (67) immediately gives us the required number

$$
w_1 = -1
$$

We note that there may possibly be functions $f(y)$ for which this analysis does not apply since the sum in (77) diverges. However it is of course sufficient to consider only one special case – certainly our arguments hold for eg. polynomial $f(y)$, where the expansion of $g(n)$ terminates.

Other heat-kernel coefficients may be evaluated in this way. For example, setting $s = 0$, it is the constant term in $g(n)$ which contributes, and

$$
\zeta(0) = \zeta_N(0) - \frac{\pi}{4} \epsilon A + 1
$$

In the Neumann case there is a zero mode which is not included in the zeta function, so that $C_1^N = 4\pi (\zeta_N(0) + 1)$, $C_1 = 4\pi \zeta(0)$. Therefore

$$
C_1 = C_1^N - 2 \int_{\partial M} d^{d-1}x \sqrt{h} \psi
$$

– a well-known result.

Finally, we find

$$
C_2^{(d)} \big|_Z = C_2^{(d)N} \big|_Z + \sum_{(ij)} \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} \left\{ \frac{\pi}{4} \left[ -2 (\kappa_i \psi_j + \kappa_j \psi_i) + (\tilde{\kappa}_i \psi_j + \tilde{\kappa}_j \psi_i) \right] + 2 \left( \psi_i^2 + \psi_j^2 \right) \right\} \\
+ 12 \psi_i \psi_j - 3 \left( \nabla_{\tilde{\kappa}_i} \psi_i + \nabla_{\tilde{\kappa}_j} \psi_j \right) \right\}
$$

$$
C_2^{(d)N} \big|_Z = \sum_{(ij)} \int_{I_{ij}} d^{d-2}x \sqrt{\gamma} \left\{ \frac{\pi}{384} \left[ 24(1 - 4\xi) R - 12 \left( \tilde{R}_i + \tilde{R}_j \right) + 16 \tilde{R} \right] + 24 \left( \kappa_i^2 + \kappa_j^2 \right) + 48 \left( \tr \kappa_i^2 + \tr \kappa_j^2 \right) + 15 \left( \tilde{\kappa}_i^2 + \tilde{\kappa}_j^2 \right) - 42 \left( \tr \tilde{\kappa}_i^2 + \tr \tilde{\kappa}_j^2 \right) \right\} \\
+ \frac{1}{270} \left[ 3 R^{\mu\nu} \left( \eta_i^\mu \eta_j^\nu + \eta_j^\mu \eta_i^\nu \right) - 434 (\kappa_i \tilde{\kappa}_i + \kappa_j \tilde{\kappa}_j) + 408 \kappa_i \kappa_j \right] + 490 \tilde{\kappa}_i \tilde{\kappa}_j + 52 \tr \tilde{\kappa}_i \tilde{\kappa}_j + 88 \left( \nabla_{\tilde{\kappa}_i} \kappa_i + \nabla_{\tilde{\kappa}_j} \kappa_j \right) \right\}
$$
12 Conclusions

We have not been able to obtain the codimension-2 integrand, $S_{ij}$, in full for general $\theta$, except when all extrinsic curvatures vanish. Relaxing this restriction seems to be the most desirable extension of our work, and would probably involve a complex problem in specific-case evaluation.

Another interesting extension would be to find $S_{ij}$ for general mixed boundary conditions in terms of the $P_{\pm}$ operators, thus completing the calculations started in section 10. As we have noted there, with mixed boundary conditions the heat kernel on a product manifold is not in general a product. Work on mixed conditions has been done in [2] for the smooth boundary case.

A problem related to ours is that of conical singularities. These exist on, for example, a lune with periodic boundary conditions, so that there is a codimension-2 submanifold, but not one of codimension-1. Fursaev [25] has evaluated the heat-kernel coefficients in this situation, and has derived expressions similar to the intrinsic part of ours.

The perturbative approach implemented in section 11 may be useful elsewhere in the ongoing work of determining heat-kernel coefficients. As an alternative to perturbing the boundary function, a perturbation in the coordinates of the boundary, and hence in $\kappa$, yields a simple expression for the first-order effect on the eigenvalues, possibly enabling terms in the $C_k$ to first-order in $\kappa$ to be calculated.

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