Conditional displacement operator for traveling fields

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We show that the conditional displacement operator \( \hat{U}_{CD} = \exp[\hat{\beta}^\dagger(\hat{\alpha}^\dagger - \hat{\beta}^\dagger\hat{a})] \) acting upon an arbitrary state of traveling waves can be well approximated by the action of a Kerr medium placed between two beam splitters whose respective second ports are fed by highly excited coherent states. Applications to the generation of nonclassical states and measurement of Wigner function of arbitrary states are also considered.

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I. INTRODUCTION

The conditional displacement operator (CDO) has been extensively used in the literature, e.g., by Milburn and Walls [1] in quantum nondemolition measurements via quantum counting; by Ban [2] in theoretical studies of the photon statistics in the four-wave mixer; and by Avelar et al. [3] in measurements of Wigner characteristic function describing field states of running waves. In cavity QED, Zou et al. [4] have proposed the creation of the CDO through a two-level atom interacting with a single-mode cavity field and driven additionally by an external classical field. The authors used this scheme for the measurements of the Wigner characteristic function. It has been employed also for the generation of nonclassical states [5, 6]. However, to our knowledge, there is no suggestion on how to implement this operator in a reliable way for arbitrary quantum states in running waves.

Here we present a calculation which shows how to displace conditionally arbitrary (pure or mixed) quantum states using a Kerr medium placed between two beam splitters (along the path of the signal beam) whose respective second ports are fed by appropriate, highly excited, coherent states. As applications, we show how to engineer for running waves the even (+) and odd (−) superpositions of an arbitrary single-mode state with its displaced counterparts, \( |\psi\rangle \pm \hat{D}(\beta)|\psi\rangle \).

To this end the CDO device is coupled to one arm of a Mach-Zehnder interferometer (MZI) fed by the vacuum and one-photon states, as applied for Kerr medium by Sanders and Milburn [7] to investigate complementarity in a quantum nondemolition measurement. The present scheme constitutes a modification of another one proposed by Villas-Bôas et al. [8] to measure directly the Wigner function. As a by-product, it is shown that our scheme allows one to measure the Wigner characteristic function of the state \( |\psi\rangle \) through the measurement of photon-detection probabilities in the output of the MZI.

II. ENGINEERING THE CONDITIONAL DISPLACEMENT OPERATOR

A schematic diagram of the procedure is shown in Fig. 1. Both beam splitters (BS1 and BS2) produce the action of the displacement operator \( \hat{D}_a(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \) on a quantum state of the field-mode \( a \), when the second port is fed by highly excited coherent states \( |\gamma\rangle \) and \( |-\gamma\rangle \) as shown in [8]. Thus, after the BS1 the state describing the whole system becomes

\[
|\Psi\rangle_{ab} = \hat{D}_a(\alpha)|\psi\rangle_a|\phi\rangle_b
\]

where \( \alpha = R\gamma \), with \( R \ll 1 \) standing for the reflectance of the BS1.

![FIG. 1: Schematic illustration of the CDO device consisting of a Kerr-medium between two beam-splitters along the path of the signal beam.](image)

The dispersive Kerr interaction between modes \( a \) and \( b \) is described by the Hamiltonian

\[
\hat{H}_K = \hbar K \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}
\]

where \( K \) is proportional to the third-order nonlinear susceptibility \( \chi^{(3)} \). So, the action of the Kerr medium upon (bipartite) field states is represented by the unitary operator,

\[
\hat{U}_K = \exp(-i\theta \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}),
\]

where \( \theta = Kl/v \), \( l \) is the length of the Kerr-medium and \( v \) the velocity of light in the medium. Due to the action of the Hamiltonian [8] upon the modes \( a \) and \( b \), and the action of the BS2 corresponding to a second displacement, the state of the system evolves to

\[
|\Psi''\rangle_{ab} = \hat{D}_a^b(\alpha) e^{-i\theta \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}} \hat{D}_a(\alpha)|\psi\rangle_a|\phi\rangle_b.
\]

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Next, a little algebra furnishes
\[
\hat{D}_a(\alpha)e^{-i\theta\hat{a}^\dagger\hat{a}\hat{b}\hat{b}}\hat{D}_a(\alpha) = e^{-i\theta(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha)\hat{b}\hat{b}}
\]
and for realistic Kerr-media small values of phase shifts \(\theta\) are produced in laboratories; so, when adjusting the device to high values of \(\alpha\) the foregoing equation becomes
\[
\hat{D}_a(\alpha)e^{-i\theta\hat{a}^\dagger\hat{a}\hat{b}\hat{b}}\hat{D}_a(\alpha) \simeq e^{-i\theta|\alpha|^2\hat{b}\hat{b}} \hat{U}_{CD}(\beta),
\]
where \(\beta = -i\theta\alpha\), with \(|\beta| = \theta|\alpha|\) finite, and
\[
\hat{U}_{CD}(\beta) = \exp[\hat{b}\hat{b}(\beta\hat{a}^\dagger - \beta^*\hat{a})]
\]
is the wanted CDO. We emphasize that Eq. (6) is an algebraic operator relation, and so, it does not depend on the input state of the CDO device.

III. APPLICATIONS

As interesting applications of the foregoing scheme we will use it to prepare the superposition states \(|\psi\rangle \pm |D(\beta)|\psi\rangle\) and to measure Wigner characteristic function of \(|\psi\rangle\), where \(|\psi\rangle\) is an arbitrary state incoming in the CDO.

A. Engineering the superposed state: \(|\psi\rangle \pm |D(\beta)|\psi\rangle\)

The superposed state of the kind \(|\psi\rangle \pm |D(\beta)|\psi\rangle\) in traveling waves has interesting applications in the literature. For example, setting \(|\psi\rangle = |\alpha\rangle\) and \(\beta = -2\alpha\) one obtains the even (+) and odd (−) Schrödinger’s cat states \(|\alpha\rangle \pm | - \alpha\rangle\); setting \(|\psi\rangle = |0\rangle\) one obtains the superposition of the vacuum state and a coherent state, \(|0\rangle \pm |\beta\rangle\), with the CDO playing the role of the optical quantum switch [11]. Other interesting family of states can be got from \(|\psi\rangle = |n\rangle\), yielding the superposition \(|n\rangle \pm |D(\beta)|n\rangle\), which depends on the availability of a Fock state \(|n\rangle\).

To prepare the superposition \(|\psi\rangle \pm |D(\beta)|\psi\rangle\) one needs a MZI associated with an auxiliary CDO and a phase shifter (PS), both inserted in one arm (mode \(b\)) of the MZI. The schematic setup is depicted in the Fig. 2. The CDO couples one of the internal modes of the MZI (mode \(b\)) with an external mode (mode \(a\)) where a field in the state \(|\psi\rangle\) is injected. Measurements of the probability of photon detection in the output of the MZI allow us the preparation of the superposition \(|\psi\rangle \pm |D(\beta)|\psi\rangle\) in the mode \(a\), as follows.

Consider the states \(|0\rangle_b\) and \(|1\rangle_c\) entering the ideal 50/50 symmetric BS1 of the MZI, whose action upon them, such as that of the BS2, is described by the following transformations [12]
\[
|0\rangle_b|1\rangle_c \rightarrow (|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c)/\sqrt{2},
\]
\[
|1\rangle_b|0\rangle_c \rightarrow (|1\rangle_b|0\rangle_c + i|0\rangle_b|1\rangle_c)/\sqrt{2}.
\]
FIG. 2: Schematic illustration of the MZI, including a CDO device in one arm, coupling the internal mode \(b\) with the signal beam \(a\).

The PS is assumed to add a phase \(e^{i\eta}\) to the field crossing it. Thus, just after the BS1 and the PS the (entangled) state of the whole system reads,
\[
|\Psi\rangle_{abc} = \frac{1}{\sqrt{2}} (e^{i\eta}|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c) |\psi\rangle_a,
\]
where the state \(|\psi\rangle_a\) in the second member stands for the initial state entering the mode \(a\).

Now, the action of the CDO device on the state \(|\Psi\rangle_{abc}\) is obtained from the relations
\[
e^{-i\theta|\alpha|^2\hat{b}\hat{b}} |\Psi\rangle_{abc} |0\rangle_b \rightarrow |\psi\rangle_a |0\rangle_b,
\]
\[
e^{-i\theta|\alpha|^2\hat{b}\hat{b}} |\Psi\rangle_{abc} |1\rangle_b \rightarrow |\psi\rangle_a |1\rangle_b,
\]
and, after the BS2, the state of the whole system can be written as
\[
|\Psi\rangle_{abc} = \frac{1}{2} e^{-i\theta|\alpha|^2} \left( |0\rangle_b|1\rangle_c \left[ e^{i\xi - \hat{D}_a(\beta)} |\psi\rangle_a \right.ight.
\]
\[
\left. + i|1\rangle_b|0\rangle_c \left[ e^{i\xi + \hat{D}_a(\beta)} |\psi\rangle_a \right]\right),
\]
with \(\xi = \eta + \theta|\alpha|^2\). Note that in this scheme the PS turns irrelevant the action of the additional factor \(e^{-i\theta|\alpha|^2\hat{b}\hat{b}}\) accompanying the CDO in the Eq. (6).

At this point we can see that if the detector D1 fires (does not fire) while D2 does not fire (fires) this corresponds to the output state \(|1\rangle_b|0\rangle_c\) (\(|0\rangle_b|1\rangle_c\)) in the BS2, and the mode−\(a\) is projected onto the state
\[
|\psi\rangle_a^{\pm} out = \frac{1}{2} e^{-i\theta|\alpha|^2} \left[ e^{i\xi_0 \pm \hat{D}_a(\beta)} |\psi\rangle_a \right).
\]
Finally, up to a global phase, our wanted state \(|\psi\rangle \pm |D(\beta)|\psi\rangle\) is obtained from the Eq. (14): this can be achieved by adjusting the PS such that \(\eta = -\theta|\alpha|^2\).

B. Measuring the Wigner function

As discussed previously in [3], a MZI supplemented by a nonlinear medium allows us to measure the Wigner characteristic functions of field states in traveling waves, yielding the reconstruction of the Wigner functions themselves. The scheme in [3] employs a MZI associated with an auxiliary (specific) four-wave mixer that entangles the field state in one
the characteristic function \( \chi \) two points (namely, characteristic function. So, one gets These two measurements actually lead to values of \( W \) leading to

\[
\Delta P(\beta, \xi_0) = P_{10}(\beta, \xi_0) - P_{01}(\beta, \xi_0) = \text{Re}[e^{-i\xi_0} \chi_a(\beta)],
\]

where the relation \( \chi_a(\beta) = \text{Tr} [\hat{\rho}_a \hat{D}_a(\beta)] \) has been employed \([13]\). As shown in \([3]\) the measurement of \( \Delta P(\beta, 0) \) \((\Delta P(\beta, \pi/2))\) furnishes the real (imaginary) part of the characteristic function. So, one gets

\[
\chi_a(\beta) = \Delta P(\beta, 0) + i \Delta P(\beta, \pi/2).
\]

These two measurements actually lead to values of \( \chi_a(\beta) \) at two points (namely, \( \beta \) and \( -\beta \)) owing to the property \( \chi_a^*(\beta) = \chi_a(\beta) \).

Now, since the Wigner function is the Fourier transform of the characteristic function \( \chi_a \), namely,

\[
W(z) = \frac{1}{\pi^2} \int d^2 \beta \chi_a(\beta) \exp \left( z \beta^* - z^* \beta \right),
\]

the determination of \( \chi_a(\beta) \) for a reasonable set of values permits the reconstruction of the Wigner function of the original state of the external mode \( a \). We note that \( \beta \) is a free-parameter in the Eq. \([13]\) and, since \( \beta = -i\theta a \) where \( |a| \) was assumed to be large and \( \theta \) takes small values in the experiment, then \( \beta \) can be varied in the range of interest through the control of the parameters \( \alpha \) and \( \theta \).

### IV. CONCLUSIONS

In summary, this report introduces a method to engineer the conditional displacement operator in running fields via a Kerr-medium placed between two beam-splitters. This device, when conveniently inserted in one arm of a Mach-Zehnder interferometer, allows one to prepare states of the kind \( |\psi(\beta)\rangle \pm D(\beta)|\psi(\beta)\rangle \), with the interesting applications mentioned in the Sect. 2.A, and the measurement of their Wigner functions (Sect.2.B). The procedure is also valid for mixed states. Concerning with the reliability of the scheme, it is supported by recent technological advances which have achieved photodetectors with efficiency near 100\% \([14]\) and stable single-photons sources yielding many single photons, e.g., using: single quantum dots \([15]\); single molecules \([16]\); diamond colour centers \([17]\); atoms \([18]\); turnstile devices \([19]\); and also traditional parametric down conversion methods \([20]\).

With respect to the Wigner function, it is worth mentioning a pertinent comparison between the present scheme and the Ref. \([5]\); in \([5]\) the Wigner function is measured directly and in this aspect it is advantageous. However, \([6]\) requires a very large nonlinear susceptibility yielding \( \theta = \pi \), whose experimental implementation is very hard till now. Here, our goal is achieved via the use of small values of phase-shifts \( \theta \), available in laboratories through realistic Kerr-medium, which constitutes a remarkable result from the experimental point of view, a feature having no guarantee in Ref. \([3]\). In fact, following various trials to get large Kerr nonlinearities \([22]\), a recent result by Munro et al. \([23]\) has achieved phase shifts \( \theta \approx 0.01 \), with residual absorption rates less than 1\%. Finally, we mention that decoherence effects, not considered here, could be included via the phenomenological operator approach, as implemented in \([15]\).

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