TAIT’S FIRST CONJECTURE FOR
ALTERNATING WEAVING DIAGRAMS

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ABSTRACT. Many entangled complex networks, like weaving frameworks, can be analyzed from a viewpoint of knot theory to better understand their topology. The number of crossings is in particular a suitable concept to study and classify such structures. In this paper, Tait’s First Conjecture, which states that any reduced diagram of an alternating link has the minimal possible number of crossings, is extended to “Σg-reduced” alternating weaving diagrams, which lie on a surface Σg of genus (g ≥ 1) defined either on the Euclidean plane E2 or the hyperbolic plane H2. A weaving structure, also called weave, has many weaving diagrams on E2 (resp. H2) associated to it, if the weave is constructed using a polygonal tessellation of E2 (resp. H2) as a “scaffold”. The proof of Tait’s First Conjecture for alternating weaves is inspired by the one for classical links from L.H. Kauffman, with an adaptation of the concepts of diagrams and invariants for weaving diagrams in higher genus surfaces, originally defined by S. Grishanov et al. in the case g = 1.

1. INTRODUCTION

The entanglements of multiple networks are useful in many scientific fields and inspire new mathematical developments [15]. A particular case of such frameworks are the weaving structures, also called weaves, which are made of multiple infinite threads interlaced through each other. Although these objects have been known and investigated for so many years by the chemistry and textile communities among others [16], we still do not have a universal study about weaves to identify and classify them. Many attempts have been made to consider weaving structures from a mathematical point of view, and this article is contributing by proposing a new systematic approach, based on low-dimensional topological principles, to describe, construct, and classify weaves. Such a topological approach permits a better understanding of their structure, that is often associated to the physical and functional properties of a material.

To construct weaving networks, different strategies have recently been considered. For example, S.T. Hyde and his coauthors built and analyzed weaves using tiling theory, with a special emphasis on the tilings of the hyperbolic plane [2,3,4,12,26]. In particular, a weaving structure is defined in [26] as the decomposition of a parent net into multiple components and constructed from two dual regular nets of the two-dimensional spaces. The first step is to choose the topology of one of the components and to enumerate all the ways to color the edges of these nets with two colors, using the Dress-Delaney symbol from the combinatorial
tiling theory. E.D. Miro et al. also considered the theory of tilings to study weaving networks in [18][19][28]. They first create a tiling from the repeating of one triangle, and then also color it, using subgroups of the triangle group. Then, new colored vertices and edges are constructed to form two different nets, and thus, a weaving structure is defined there as the union of these two nets associated to a weaving map, which describes the position of the two nets at each crossing points (over or under).

The first aim of this article is to describe a new approach to construct weaves, inspired by the concept of “polyhedral links” defined in [1][11][17][22][23]. It generates a systematic method to build weaving links from tessellations of \( \mathbb{E}^2 \) or \( \mathbb{H}^2 \). Then, by describing the sequence of crossings (over, under,...) of these unclosed links via a weaving map, it is possible to define a weave as a pair of a weaving link and a weaving map. We are in particular interested in weaves built from periodic tessellations, for their periodic and symmetric properties. While studying the topological characteristics of such mathematical objects, one can obviously not avoid to get interested in the properties related to their crossings, and because they seem to share features with knots and links, it is natural to consider knot theory as in [6][7][8].

In the history of knot theory, attention has been paid to link diagrams which have the particularity to be alternating, meaning that the crossings alternate under, over, under, over, and so on, as one travels along each component of the link. This concept is also suitable to describe a class of weaves that shows interesting properties. In the late nineteenth century, P.G. Tait [24] stated several famous conjectures on alternating link diagrams that have been proven one century later. The first conjecture, in which we are interested here, concerns the number of crossings of reduced, alternating link diagrams and has been demonstrated independently by M.B Thistlethwaite [25], K. Murasugi [20], and L.H. Kauffman [10]. The main purpose of this article is to generalize and prove this conjecture, already well-known for classical links, for alternating weaves with diagrams on a surface \( \Sigma_g \) of genus \( g \geq 1 \), defined by the new construction methodology.

**Theorem 1.1. (Tait’s First Conjecture for weaves)**

A connected \( \Sigma_g \)-reduced alternating weaving diagram of minimal size is a minimum diagram of its alternating weave. Moreover, a minimum diagram of a prime alternating weave can only be an alternating diagram. Thus, a non-alternating diagram can never be a minimum diagram of a prime alternating weave.

The proof is inspired by the one for classical links from Kauffman [10], where a link diagram lies on the real plane, and needs an extension of the definition of the Kauffman polynomial from [7], originally defined for the diagrams of textile structures which lie on a torus of genus one. We propose such an extension for the cases where the weaving diagrams lie on higher genus surfaces \( \Sigma_g \), which correspond to hyperbolic weaving diagrams defined on the Poincare disk when \( g \geq 2 \). This theorem implies that the crossing number of a reduced, alternating weave is an invariant of this entangled network. Such an invariant is interesting since it describes in particular the complexity of the structure. As in knot theory, the definition of topological invariants for weaves is a very important step in the study of their classification, which is a long-term project.
This paper starts with an introduction of the construction methodology of the weaving frameworks, based on a “transformation” of a classic tiling into weaving links, as mentioned earlier. The definitions of a weave and their diagrams will also be given in Section 2, as well as important concepts that appear in the main theorem, such as an alternating weave, a $\Sigma_g$-reduced diagram and a crossing. Section 3 recalls the definition and characteristics of the bracket polynomial for a diagram lying on a torus of genus one, developed in [6,7], and gives an extension for higher genus surfaces, using Teichmüller space theory. The proof of Tait’s first conjecture for weaving structures is given in Section 4 and finally, a short outlook concludes this paper in Section 5.

2. Preliminaries: weaving structures and their diagrams

2.1. Construction and definition of weaves.

We start our study by giving a proper definition to our weaving structures in order to distinguish them from other related complex entangled networks. We consider a weave as a 3-dimensional object, laying in the ambient space $X^3 = E^2 \times I$ or $H^2 \times I$, with $I = [-1,1]$, and its planar projection into $X^2 = E^2$ or $H^2$, by a map $\pi : X^3 \rightarrow X^2$, $(x,y,z) \mapsto (x,y,0)$. The construction of these frameworks is directly inspired by the method of “polyhedral links”, used to construct links from polyhedra, as described by W.Y. Qiu et al. in [1,11,17,22,23], and has been extended to “polygonal links”. This allows us to have a systematic method to construct a class of weaving structures.

First, we need to select a polygonal tesselation of $X^2$ that will be used as a “scaffold” to obtain periodic interwoven frameworks. By a polygonal tessellation, we mean a covering of $E^2$ (resp. $H^2$) by Euclidean (resp. hyperbolic) polygons, called tiles, and their images by isometry, such that the interior points of the tiles are pairwise disjoint. We call elements of a polygonal tesselation its tiles, as well as its vertices and edges, where tiles intersect. We say that two elements of a tesselation are symmetric with each other if they are in the same orbit; i.e. they are related to each other by a symmetry of the tesselation. Moreover, it is known that a tesselation is periodic if and only if the number of orbits of the tesselation elements is finite. To be precise in this paper, the tesselations are assumed to be edge to edge, that is, the vertices are corners of all the incident polygons. We also restrict ourselves first to uniform tesselations (only one orbit of vertices), assuming that an extension to more complex cases follows naturally. For more information on tesselations, we refer to the book *Tilings and Patterns* by B. Grünbaum and G.C. Shephard [9].

The following step is the transformation of the vertices and the edges of the chosen tesselation in $X^3$ (see Figure 1). For the “crossed curves and single-line covering” method, each vertex is replaced by a “crossed curves” building block, which consists in making the original edges of the chosen tesselation entangled at each vertex (Figure 1, left). In the two other cases, a polygonal link is constructed from a polygonal tesselation by using a “n-crossed-curves” or “n-branched-curves” building block to cover each vertex (with $n$ the valency of the vertex) and using a “m-twisted double-line” building block — with $m$ either
an odd or even positive integer which represents the number of twists — to replace each edge. Then, the “n-crossed-curves” or “n-branched-curves” blocks are connected to the “m-twisted double-line” blocks in the polygonal tessellation, respectively. Thereby, from one tiling, we can obtain an infinite number of entangled networks and the connection of these building blocks can result in two types of structures (See Figure 2): weaves, consisting of infinite threads in different “directions” entangled to each other, which are the objects of this article; or polycatenanes, consisting of closed rings entangled to each other.

To simplify the notations of a transformation of a polygonal tessellation by any of the three methods described above, we introduce the following new symbols:

**Definition 2.1.** Let \( n \) be the valency of the vertices (\( n \geq 3 \)) of the uniform polygonal tesselation \( \mathcal{P}_T \) of \( \mathbb{E}^2 \) or \( \mathbb{H}^2 \). We say that \( \mathcal{P}_T \) is transformed by the “polygonal links” method \( (\Lambda, m) \), with \( m \in \mathbb{N} \) and \( \Lambda \in \{C_r, nC_r, nB_r\} \), if each vertex of \( \mathcal{P}_T \) and its adjacent edges are transformed by one of the following methods:

\[
(\Lambda, m) = \begin{cases} 
(C_r, 1) & : \text{crossed curves and single-line covering}, \\
(nC_r, m) & : \text{n-crossed curves and m-twisted double-line covering}, \\
(nB_r, m) & : \text{n-branched curves and m-twisted double-line covering}.
\end{cases}
\]

| crossed and single-line covering | n-crossed curves and m-twisted double-line covering | n-branched curves and m-twisted double-line covering |
|---------------------------------|-----------------------------------------------|-----------------------------------------------|
| ![Crossed Curves](image1.png)   | ![n-Crossed Curves](image2.png)               | ![n-Branched Curves](image3.png)              |
| ![Edge](image4.png)            | ![Edge](image5.png)                           | ![Edge](image6.png)                           |

*Figure 1.* Polygonal Links Method: transformation of the vertices and the edges of a polygonal tessellation to obtain an infinite number of weaving links.

From this construction method, we are able to define a weave, or a weaving structure, as a weaving link, made by one of the “polygonal links” methods on an underlying tessellation, together with a weaving map, which defines the sequence of the crossings, meaning when a thread is over or under another one, as explained in the following definitions.

**Definition 2.2.** A thread \( t \), in \( \mathbb{X}^3 = \mathbb{E}^2 \times I \) or \( \mathbb{H}^2 \times I \), with \( I = [-1, 1] \), is a set homeomorphic to \( \mathbb{R} \). A crossing \( c \) is an intersection between the projections on \( \mathbb{X}^2 \) of two threads. Moreover, we say that two threads belong to two different sets of threads if their respective projections on \( \mathbb{X}^2 \) satisfy one of the following conditions:
(1) If $\mathcal{P}_T$ is transformed by $(\Lambda, m) = (C_r, 1)$ or $(nC_r, m)$, there exists at least a “crossed curves”, or a “n-crossed-curves” building block where they intersect.

(2) If $\mathcal{P}_T$ is transformed by $(\Lambda, m) = (nB_r, m)$, there exists at least a “m-twisted double-line” building block where they intersect.

We are interested in characterizing a weave according to the polygonal tessellation used as a “scaffold” to construct it. Therefore, since a uniform periodic polygonal tessellation $\mathcal{P}_T$ has a local structure at a vertex which is, by definition, the same over all vertices, we can describe it by a label $(k_1, k_2, k_3, \ldots)$, called the vertex symbol. Such a vertex symbol is a finite sequence of positive integers $k_1, k_2, k_3, \ldots$ in either clockwise or counter-clockwise order, describing all the polygons which meet any vertex of the tessellation. For example, a vertex with symbol $(k_1, k_2, k_3, k_4)$ meets, in order, a $k_1$-gon, a $k_2$-gon, a $k_3$-gon, and another $k_2$-gon. The vertex symbol $(k_3, k_2, k_1, k_2)$ represents the same tiling since the order is respected, but this would not be the case for $(k_2, k_3, k_1, k_2)$, for example. Of all possible such vertex symbols, we choose by convention the lexicographical order.

**Definition 2.3.** A weaving link in $\mathbb{X}^3$, denoted by $W(k_1, k_2, k_3, \ldots)$, is the union of at least two disjoint sets of threads constructed by the method of ‘polygonal links’ $(\Lambda, m)$, on the uniform polygonal tessellation $\mathcal{P}_T(k_1, k_2, k_3, \ldots)$, with vertex symbol $(k_1, k_2, k_3, \ldots)$.

We can see in the Figure 2 some examples of weaving links constructed from the regular tilings \{4,4\} = $\mathcal{P}_T(4,4,4,4)$ and \{5,4\} = $\mathcal{P}_T(5,5,5,5)$. We notice that for the case of $\mathcal{P}_T(4,4,4,4)$, the weaving links $(C_r, 1)$, $(4C_r, 0)$ and $(4B_r, 1)$ are “equivalent by construction”, assuming that their respective sequence of crossings are the same, since their planar projections are all isotopic. Moreover, the example $(4B_r, 2)$ illustrates a case when the methodology generates polycatenanes. A precise characterization of the conditions to generate these two different structures by this method, as well as the definition of equivalence will be given in a future work.

**Definition 2.4.** A weave $W$ is a pair $(W(k_1, k_2, k_3, \ldots), w(p, q, r, \ldots))$ in $\mathbb{X}^3$, with a weaving link $W(k_1, k_2, k_3, \ldots)$ and a weaving map $w(p, q, r, \ldots)$ such that $w : \Gamma \to (p, q, r, \ldots)$, with $\Gamma$ the set of crossings of a unit cell of the projection of $W(k_1, k_2, k_3, \ldots)$ on $\mathbb{X}^2$, and $(p, q, r, \ldots)$ the finite sequence of positive integers such that, if one travels along any thread of the unit cell with the two following conditions satisfied:

1. starting from a crossing such that at least one of two neighboring crossings on the same thread has a different position (over or under).
2. walking in the opposite direction of this different crossing.

Then there are $p$ crossings in which it is over (resp. under) the other threads, followed by $q$ crossings in which it is under (resp. over), followed by $r$ crossings in which it is over (resp. under), and so forth.

2.2. Weaving diagrams.
Figure 2. On the top, examples of weaving links constructed from the Euclidean tiling \{4,4\}. On the bottom, examples of weaving links constructed from the hyperbolic tiling \{5,4\}.

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As mentioned in the previous section, in the same way that for classical knots and links ([21]), we can project a weave $W = \left( W \left( \alpha_{i}, m_{i} \right), w(p, q, r, \ldots) \right)$ onto the plane $X^2$. We call $W_0$ the image of $W$ under the projection map $\pi$ defined earlier. If $Q$ is a point of intersection of $W_0$, then the inverse image $\pi^{-1}(Q)$ of $Q$ in $W$ has exactly two points: so, $Q$ is a double point of $W_0$. We say that $W_0$ is a regular projection.
To each intersection point of the regular projection $W_0$, by recording the extra information of which arc is over and which is under, it defines an infinite planar diagram. But, since our weaves are periodic structures by construction, instead of studying such a planar diagram containing an infinite number of crossings, it is convenient to consider a diagram $D_W$ on a surface $\Sigma_g$ of genus $g \geq 1$, which can be obtained by identifying the sides of a unit cell of the planar diagram by pairs. Regular $4g$-gons of $\mathbb{X}^2$ are natural and easy choices to define the unit cell, but are not the only options. The particular case for $g = 1$, which corresponds to $\mathbb{X}^2 = \mathbb{E}^2$, is detailed in [7] and below, a general approach is described for a closed orientable surface of any genus.

The diagram $D_W$ obtained on the surface $\Sigma_g$ of genus $g \geq 1$ is called a $\Sigma_g$-weaving diagram, or much simpler a $\Sigma_g$-diagram. Such a diagram consists of several closed smooth curves drawn on $\Sigma_g$; each curve is a component of the diagram. Moreover, we assume that $\Sigma_g - D_W$ consists of open discs.

For any infinite periodic planar diagram in $\mathbb{X}^2$, the choice of a unit cell is not unique. For $g = 1$, any parallelogram of unit area which has for sides integer vectors can be a unit cell. For higher genus surfaces, it is not necessary to discuss about the area since two hyperbolic surfaces with the same topology always have the same area. To extend the result of the case $g = 1$, one must consider the Teichmüller Space of the surface $\Sigma_g$ of genus $g \geq 2$, and its Mapping Class Group, whose necessary notions are introduced below. For more details, see [5]. Here a geodesic hyperbolic $4g$-gon in $\mathbb{H}^2$, called a $\Sigma_g$-tile, is considered such that the sum of its interior angles is equal to $2\pi$ and that its edges can be identified pairwise using some labels to obtain a closed marked hyperbolic surface of genus $g$. The Teichmüller space of this surface of genus $g$ can be seen as the space of marked surfaces homeomorphic to it. Moreover, it has been proven that this space is in bijection with the set of equivalence classes of hyperbolic $\Sigma_g$-tiles, when two such tiles are equivalent if they differ by a marked, orientation-preserving isometry and by “pushing the base-point”, which is the point on the surface where all the vertices of a $\Sigma_g$-tile meet after gluing. The details of this bijection are given in the Chapter 10.4.2 [5].

![Figure 3. A (regular) $\Sigma_2$-tile and its corresponding marked hyperbolic surface of genus 2.](image)

Now, it is necessary to demonstrate that there exists an infinite number of $\Sigma_g$-tiles, which means an infinite number of unit cells for any infinite planar diagram. A point in $\text{Teich}(\Sigma_g)$, the Teichmüller space of $\Sigma_g$, represents the equivalence class of a marked surface $\Sigma_g$ of genus
Therefore, from the above bijection, it corresponds to equivalent \( \Sigma_g \)-tiles that can generate isometric tilings, considering such a tile as a fundamental domain. But since any \( \Sigma_g \)-tile can be taken as a unit cell of a hyperbolic \( \Sigma_g \)-diagram, and that such a \( \Sigma_g \)-tile, up to equivalence, corresponds to a unique point in \( \text{Teich}(\Sigma_g) \), different \( \Sigma_g \)-diagrams correspond to different marked surfaces, and therefore are not isometric to the original \( \Sigma_g \)-tile. Moreover, given a marked surface \( \Sigma_g \) of genus \( g \) in \( \text{Teich}(\Sigma_g) \), by the action of an element of the mapping class group of \( \Sigma_g \), denoted by \( MCG(\Sigma_g) \), it is possible to change the marking. The Dehn twists of \( \Sigma_g \) are actually known as the simplest infinite-order elements of \( MCG(\Sigma_g) \). Thus, the same Dehn twist, applied an infinite number of times to the given marked surface \( \Sigma_g \) of \( \text{Teich}(\Sigma_g) \) will never give equivalent marked surfaces, nor isometric \( \Sigma_g \)-tiles. Therefore, there is an infinite number of weaving diagrams in \( \mathbb{X}_g^2 \) corresponding to the same weave, and for a given weave, any \( \Sigma_g \)-weaving diagram can be obtained from an arbitrarily chosen one by a sequence of Dehn twists of \( \Sigma_g \) along its \( \alpha_i, \beta_i \), and \( \gamma_i \) curves [14], which are extensions of the meridians and longitudes of a torus.

As specified in [6], an ambient isotopy of a weaving structure is any continuous deformation that preserves its periodicity. These deformations include homogeneous extensions, shear deformations, translations, rotations in space and, in addition to this, periodical deformations with the period equal to that of the structure. It is important to make that precise that since our weaves lie on surface times interval spaces, \( \mathbb{X}_3 = \mathbb{E}^2 \times I \) or \( \mathbb{H}^2 \times I \), then the ambient isotopy for weaves only allows deformations in such spaces.

For every \( g \geq 1 \), any invariant of weaving structures, which intuitively is a property that holds for all different diagrams of the same weave, must be independent of the Dehn twists of \( \Sigma_g \), so as in [8], with a proof similar to the original one, it can be shown that:

**Theorem 2.5. (Reidemeister Theorem for Weaves):** Two weaving structures in \( \mathbb{X}_3 \) are ambient isotopic if and only if their \( \Sigma_g \)-weaving diagrams be obtained from each other by a sequence of Reidemeister moves \( \Omega_1, \Omega_2, \) and \( \Omega_3 \), isotopies on the surface \( \Sigma_g \) of genus \( g \), and \( \Sigma_g \) twists.

![Figure 4. Reidemeister moves.](image)

In the case of weaving structures, the number of crossings \( C \) means the number of crossings contained in a \( \Sigma_g \)-diagram, so in a unit cell of a planar diagram. The number of crossings may change when Reidemeister moves are applied to the diagram.

### 2.3. Some particular weaving diagrams.

Many definitions that appear in knot theory [10, 21] can be naturally extended for weaving diagrams. A weave \( W = (W_{(\alpha_i, \beta_j, \gamma_k)}^{(\lambda, \mu)}, w(p, q, r, \ldots)) \) in \( \mathbb{X}_3 \), is said to be alternating.
if \((p, q, r, \ldots) = (1, 1)\). This means that over- and under-crossings are alternating with each other while walking along any weaving thread in the weaving diagram. On the other hand, considering every simple closed curve \(\gamma\) on a weaving diagram, if \(\gamma\) meets the diagram planar projection exactly twice transversely away from crossings, such that \(\gamma\) bounds a region of the diagram with no crossings, then the weaving diagram is said to be \textit{prime}. This means that it cannot be written as a \textit{connected sum} of two link diagrams, both of which having at least one crossing.

The notion of a reduced diagram is more complex in the situation of a \(\Sigma_g\)-weaving diagram since it is embedded in a surface of genus \(g \geq 1\).

**Definition 2.6.** A reduced \(\Sigma_g\)-diagram \(D_W\) in \(\Sigma_g \subset \mathbb{R}^2\) is one that does not contain an isthmus, also called nugatory crossing, in its infinite planar diagram. An isthmus is a crossing in the diagram so that two of the four local regions at the crossing are part of the same region in the associated infinite diagram. We say that \(D_W\) is \(\Sigma_g\)-reduced.

Finally, any crossing \(c\) of a \(\Sigma_g\)-diagram \(D_W\) is called \textit{proper} if the four regions around \(c\) in \(D_W\), delimited by the projection of the threads, are all distinct. When every crossing of \(D_W\) is proper, \(D_W\) is said to be \textit{proper}.

We can give an illustration of these last definitions with an example, the alternating weave \((W(4C_r,1), w(1,1))\), based on the square tiling \((4,4)\) with the method \((4C_r,1)\), corresponding to the 4-crossed curves and 1-twisted double-line covering:

- **Figure 5 (a):** infinite reduced (without isthmus) planar diagram.
- **Figure 5 (b):** \(\Sigma_g\)-reduced unproper weaving diagram.
- **Figure 5 (c):** \(\Sigma_g\)-reduced proper weaving diagram.

![Figure 5](image-url)

**Figure 5.** Distinction between proper (all crossings with the four regions distinct) and unproper (some crossings with the four regions not distinct) \(\Sigma_g\)-reduced weaving diagrams \(D_W\).

3. **The Bracket Polynomial**

3.1. **A Kauffman-type weaving invariant.**
This section recalls results from [7] and extends the definition of the bracket polynomial of a torus-diagram in $E^2$ to any diagram of $X^2$, lying on a surface $\Sigma_g$ of genus $g \geq 1$.

**Definition 3.1.** Let $D_W$ be an unoriented $\Sigma_g$-weaving diagram on a surface $\Sigma_g$ of genus $g \geq 1$, for a weave $W$. Let $\langle D_W \rangle$ be the element of the ring $\mathbb{Z}[A,B,d]$ defined recursively by the following identities:

1. $\langle O \rangle = 1$, with $O$ the single unknotted diagram: an essential simple closed curve.
2. $\langle D_W \cup O \rangle = d \langle D_W \rangle$, when adding an isolated circle $O$ to a diagram $D_W$.
3. $\langle \times \rangle = A \langle = \rangle + B \langle \rangle$, for diagrams that differ locally around a single crossing.

This last relation is called the skein relation and $\langle D_W \rangle$ denotes the bracket polynomial.

This polynomial is known to be well defined on link diagrams, but is not invariant under the Reidemeister moves. It is necessary to find a relation between $A$, $B$ and $d$.

Each crossing of an oriented $\Sigma_g$-diagram $D_W$ can be split, via an operation of type $A$ or $B$ (cf. Figure 6 below). Here, a crossing is said to be oriented when the upper thread passes from bottom left to top right [7]. The overall operations can be expressed as a state $S$ of $D_W$, of length $C$, which represents the number of crossings, and consisting of letters $A$ and $B$,

$$S = ABAABB \ldots ABBA.$$  

![Figure 6](image)

**Figure 6.** On the left, the two types of splitting. On the right, an example of a state of a $\Sigma_g$-diagram.

In knot theory, a classical link diagram $D_S$ in a state $S$ is a disjoint union of $c_S$ non-knotted and non-intersecting closed curves, with $c_S$ a positive integer; i.e., each component is isotopic to a circle $O$.

$$D_S = OOOO \ldots O.$$  

So, by Definition 3.1 (2), the bracket of $D_S$, which contains $c_S$ components is,

$$\langle D_S \rangle = d^{c_S-1}.$$  

Moreover, by applying the skein relation recursively, considering $i$, the number of splits of type $A$ and $j$, the number of splits of type $B$, in the state $S$, we can also define for $D_S$ and $S$,

$$\langle D_S/S \rangle = A^i B^j.$$  

So, the total contribution of one given state $S$ to the bracket polynomial of a classic link diagram is given by,

$$P_S = \langle D_S/S \rangle \langle D_S \rangle = A^i B^j d^{c_S-1}.$$
To apply this to a $\Sigma_g$-diagram $D_W$, we need to consider that a trivial $\Sigma_g$-diagram, together with circles, may also contain a set of non-intersecting closed components, which are wound up around the surface $\Sigma_g$. Such a set has been called a winding in [7] and is denoted by $(m', n')$ in the case of a torus of genus one, where $m'$ and $n'$ are the number of intersections of the winding with a torus meridian and longitude, respectively. For example, in the case of the diagram on the right of Figure 6, $(AABBBD) : (m', n') = (0, 2)$. For the general case of $g \geq 1$, the winding is denoted $(m_1', \ldots, m_g', n_1', \ldots, n_g')$, where $m_1', \ldots, m_g', n_1', \ldots, n_g'$ are the number of intersections of the winding with the curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $\Sigma_g$ respectively; see Figure 3 and [14] for more details.

Recall that a winding can only contain curves which have their two ends in opposite sides of the parallelogram, for the torus, or equivalently whose both ends lie on the two edges of the $\Sigma_g$-tile that are identified for higher genus surfaces. Thus, the state $S$ of a $\Sigma_g$-diagram $D_W$ can be represented by,

$$D_{W,S} = OOOO \ldots O \cup (m_1', \ldots, m_g', n_1', \ldots, n_g'), \text{ for every } g \geq 1.$$ 

We can also define the bracket polynomial for a winding,

$$\langle m_1', \ldots, m_g', n_1', \ldots, n_g' \rangle = \langle (m_1', \ldots, m_g', n_1', \ldots, n_g') \rangle, \text{ for every } g \geq 1.$$ 

So, by using the same logic as for the general links above, it is possible to define the bracket polynomial for a $\Sigma_g$-diagram:

**Proposition 3.2.** The bracket polynomial $\langle D_W \rangle$ is uniquely determined on $\Sigma_g$-weaving diagrams by the identities (1), (2), (3) from Definition 3.1, and is expressed by summation over all states of the diagram,

$$\langle D_W \rangle = \sum_S A^i B^j d^{n-1} \langle m_1', \ldots, m_g', n_1', \ldots, n_g' \rangle, \text{ for every } g \geq 1.$$

Now that the bracket polynomial is properly defined for $\Sigma_g$-diagrams, it is possible to study its invariance regarding the Reidemeister moves like in the Lemma 2.3 in [10]:

**Lemma 3.3.** If the three diagrams represent the same projection except in the area indicated, we have $\langle \chi \rangle = AB \langle (\cdot) \rangle + (ABd + A^2 + B^2) \langle \ldots \rangle$.

Thus, the bracket is invariant for the Reidemeister move $\Omega_2$ for all diagrams if $AB = 1$ and $d = -A^2 - A^{-2}$.

Moreover, this implies also the invariance of the bracket for the Reidemeister move $\Omega_3$, which allows a partial conclusion about the invariance of the polynomial:

**Lemma 3.4.** The bracket invariance for the Reidemeister move $\Omega_2$ implies the bracket invariance for Reidemeister move $\Omega_3$. Thus, the bracket polynomial is an invariant of regular isotopy for unoriented weaves.
Indeed, it is not possible to conclude at this point, that the bracket polynomial is an invariant of ambient isotopy, since we do not have the invariance for the Reidemeister move $\Omega_1$. The following proposition gives an identity for this first move, [10]:

**Proposition 3.5.** If $AB = 1$ and $d = -A^2 - A^{-2}$, then, for the Reidemeister move $\Omega_1$, we have

\[
\begin{align*}
\langle \gamma \rangle &= (-A^3)\langle \longrightarrow \rangle, \\
\langle \gamma \rangle &= (-A^{-3})\langle \longleftarrow \rangle.
\end{align*}
\]

To define an ambient isotopy invariant, we need to have the invariance for this first Reidemeister move. The idea is to use the *writhe* $w_r(D_W)$ of a $\Sigma_g$-diagram $D_W$, which is the sum of the signs of all the crossings where each crossing is given a sign of plus or minus 1, as in Figure 7 below.

![Figure 7. Sign convention.](image)

For any weave, a diagram $D_W$ consists of $c_S$ components, each denoted by $K_i$, that can be oriented in an arbitrary way. We call $D^i_W$ the part of diagram $D_W$ that corresponds to the component $K_i$. Then we have in $D_W$,

\[
(3.2) \quad w_r(D_W) = \sum_{i=1}^{c_S} w_r(D^i_W).
\]

We can now define a polynomial constructed from the bracket: for every $g \geq 1$, we set

\[
(3.3) \quad f[D_W] = (-A)^{-3w_r(D_W)}D_W, \\
= (-A)^{-3w_r(D_W)}\left(\sum_S A^{i-j}(-A^2-A^{-2})^{c_S-1}\langle m_1^1, \ldots, m_i^g, n_1^1, \ldots, n_g^g \rangle\right).
\]

**Theorem 3.6.** The polynomial $f[D_W] \in \mathbb{Z}[A]$ defined above is an ambient isotopic invariant for oriented weaves.

**Proof.** From Lemma 3.4, we already have the invariance of $f[D_W]$ for the Reidemeister moves $\Omega_2$ and $\Omega_3$. Then, by combining the behavior of the writhe defined above under the Reidemeister move $\Omega_1$ with the previous relation of the bracket for $\Omega_1$ in *Proposition 3.5,*
it follows that $f[D_W]$ is invariant under $\Omega_1$ type moves. Thus, $f[D_W]$ is invariant under all three moves, and is therefore an invariant of ambient isotopy.

Nevertheless, this polynomial still depends on the choice of the unit cell because the multipliers $\langle m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s \rangle$, associated with the windings, depend on the Dehn twists. As seen earlier, to have a weaving invariant, we also need the independence of the polynomial for the Dehn twists of $\Sigma_g$. Once again, the particular case of the torus of genus one is described in [7] and is extended below to $g \geq 1$.

**Theorem 3.7.** The polynomial $f[D_W]$, when $(m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s)$ is independent of the Dehn twists, or in canonical form, defines a Kauffman-type weaving invariant.

**Proof.** To construct an invariant independent of the twists of $\Sigma_g$, it is necessary to consider the set $\{v_s\} = \{(m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s)\}$ of windings for every state $S$. Indeed, since this set $\{v_s\}$ depends on the twists of $\Sigma_g$, one must transform it to the canonical form by using the Dehn twists, in order to make it invariant.

Dehn twists are known to be elements of the Mapping Class Group of a surface of genus $g$. Indeed, the Dehn–Lickorish theorem states that it is sufficient to select a finite number of Dehn twists to generate the Mapping Class Group $\text{MCG}(\Sigma_g)$ of a surface $\Sigma_g$ of genus $g$. Moreover, since the symplectic representation $\varphi : \text{MCG}(\Sigma_g) \to \text{Sp}(2g, \mathbb{Z})$ is surjective for $g \geq 1$ (Chapter 6.3.2, [5]), then it means that the images of the Dehn twists generate $\text{Sp}(2g, \mathbb{Z})$. Besides, recall that the determinant of every symplectic matrix $A \in \text{Sp}(2g, \mathbb{Z})$ is one and that for $g = 1$, $\text{Sp}(2g, \mathbb{Z}) = \text{SL}_2(\mathbb{Z})$. Thus, it is possible to use the same reasoning as in [7] (Section “Component Number and Related Invariants”) and to represent the transformation of any winding $v_s = (m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s)$ by a sequence of Dehn twists of $\Sigma_g$ as a product of $v_s$ by a matrix $U \in \text{Sp}(2g, \mathbb{Z})$, $v' = v_sU$, considering the canonical matrix multiplication on $\text{Sp}(2g, \mathbb{Z})$.

To define the canonical form of a set $V = \{v_s\}$, we associate a quadratic functional $Q$ with $V$,

$$(3.4)\quad Q(V) := \sum_{s}^{N} |v_s|^2.$$ 

A sequence of twists, defined by a symplectic matrix $U$, converts the set $V = \{v_s\}$ to a set $V' = \{v'_s\}$, with $v'_s = v_sU$ and the value of $Q$ changes to:

$$(3.5)\quad Q(V') = \sum_{s}^{N} |v'_s|^2 = \sum_{s}^{N} v_sUU^Tv_s^T.$$ 

So for a given set $V = \{v_s\}$, the idea is to find a sequence of twists represented by a matrix $U$ of determinant 1, that minimizes the value of $Q$,

$$(3.6)\quad Q(V') = \sum_{s}^{N} |v'_s|^2 = \sum_{s}^{N} v_sUU^Tv_s^T \rightarrow \min, U \in \text{Sp}(2g, \mathbb{Z})$$.
This equation always has a unique non-trivial solution \( U_0 \), which can be shown considering the following setting: let \( M = UU^T \), then \( M \) is a symmetric definite positive matrix, and let \( x = v_s \) and let \( f(x) = xMs^T \), then \( f(0) = 0 \) and for all \( x \neq 0 \), \( f(x) > 0 \). Thus, there is an orthonormal basis \( \{ e_1, \ldots, e_d \} \) such that for all \( i \) in \( \{ 1, \ldots, d \} \), \( e_i \) is an eigenvector of \( M \). We denote the corresponding eigenvalue \( \lambda_i \) and we show that \( f \) is strictly convex.

Let \( 0 < \mu < 1 \) and consider \( f(\mu x + (1 - \mu)y) \), with \( x \neq y \). Then,

\[
f(\mu x + (1 - \mu)y) = \langle \mu x + (1 - \mu)y, M(\mu x + (1 - \mu)y) \rangle,
\]

(3.7)

\[= \left( \sum_{i=1}^{d} \mu x_i e_i + (1 - \mu)y_i e_i, M \sum_{i=1}^{d} \mu x_i e_i + (1 - \mu)y_i e_i \right),\]

\[= \sum_{i=1}^{d} \lambda_i (\mu x_i + (1 - \mu)y_i)^2.\]

Moreover, \( x^2 \) is strictly convex and for some \( i, x_i \neq y_i \), thus,

(3.8)

\[
\sum_{i=1}^{d} \lambda_i (\mu x_i + (1 - \mu)y_i)^2 < \sum_{i=1}^{d} \lambda_i (\mu x_i^2 + (1 - \mu)y_i^2).
\]

Therefore, since \( f \) is strictly convex and as a limit at infinity, it has a unique minimizer, which concludes our proof. So, for every state \( S \), the canonical form of \( V = \{ v_s \} \), with the winding as coordinates \( (m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s) \) is an invariant and thus, \( f[D_W] \) too.

3.2. The case of alternating weaving diagrams.

Now, we study the bracket polynomial for the case of alternating weaves. It is known \(^{[21]} \) that the degree of a polynomial is the most important aspect of the polynomial as an invariant.

**Proposition 3.8.** Let \( D_W \) be an alternating weaving diagram that is connected and \( \Sigma_g \)-reduced. Let \( D_W \) be colored so that all the regions labelled “A” are white and all the regions labelled “B” are black. Let \( C \) be the number of crossings, \( W \) be the number of white regions and \( B \) be the number of black ones. Then,

(3.9)

\[
\max \deg(\langle D_W \rangle) = C + 2W - 2,
\]

\[
\min \deg(\langle D_W \rangle) = -C - 2B + 2,
\]

with \( \max \deg(P) \) and \( \min \deg(P) \) are respectively the maximal and the minimal degree of any polynomial \( P \) in \( \mathbb{Z}[A, B, d] \).

**Proof.** Since \( D_W \) is alternating, it has a canonical checkerboard coloring, which means that two edge-adjacent regions always have different colors. Let \( S = S_A \) be the state obtained by splitting every crossing in the diagram \( D_W \) in the \( A \)-direction. Then we have \( \langle D_W / S \rangle = \)
A^C$, and since the number of components $c_S$ is equal to $W$, thus as seen earlier, the total contribution of the state $S$ to the bracket polynomial is given by:

$$P_S = \langle D_W/S \rangle d^{c_S-1}\langle m_1^1, \ldots, m_s^g, n_1^1, \ldots, n_s^g \rangle$$

(3.10)

$$= A^C d^{w-1}\langle m_1^1, \ldots, m_s^g, n_1^1, \ldots, n_s^g \rangle, \quad g \geq 1.$$  

And since $d = -A^2 - A^{-2}$, then $\max deg(P_S) = C + 2W - 2$, which is the desired relation.

Now let $S' \neq S_A$, be any another state and verify that $\deg(P_{S'}) \neq \deg(P_S)$. Then $S'$ can be obtained from $S = S_A$ by switching some splittings of $S$. Thus, a sequence of states can be defined: $S(0), S(1), \ldots, S(n)$ such that $S = S(0), S' = S(1)$, and for every positive integer $i$, $S(i+1)$ is obtained from $S(i)$ by switching one splitting from type $A$ to type $B = A^{-1}$. Then, since a splitting of type $B = A^{-1}$ contributes a factor of $A^{-1}$ in the polynomial

$$\langle D_W/S(i+1) \rangle = A^{-2} \langle D_W/S(i) \rangle.$$ 

(3.11)

We need now to distinguish two cases.

Case 1: The weaving diagram $D_W$ is $\Sigma_d$-reduced and proper. Then, $c_{S(i+1)} \leq c_{S(i)} - 1$, since switching one splitting can change the component number by at most one. Thus, $\max deg(P_{S(i+1)}) \leq \max deg(P_{S(i)})$. Moreover, let $c$ be the crossing point for which we change the $A$-splice into the $B$-splice from $S(0)$ to $S(1)$. Since $D_W$ is proper, the crossing $c$ is proper. Thus, we can use the following lemma, (Lemma 3.2, [13]).

**Lemma 3.9.** Let $D_W$ be an alternating weaving diagram, and let $S_A$ (or $S_B$, resp.) be the state of $D_W$ obtained from $D_W$ by doing an $A$-splice (resp. $B$-splice) for every crossing. For a crossing $c$ of $D_W$, let $R_1(c)$ and $R_2(c)$ be the closed regions of $S_A$ (or $R'_1(c)$ and $R'_2(c)$ be the closed regions of $S_B$) around $c$. If $c$ is a proper crossing, then

$$R_1(c) \neq R_2(c) \text{ and } R'_1(c) \neq R'_2(c).$$

**Proof.** Since $c$ is a proper crossing, the four closed regions of $D_W$ appearing around $c$ are all distinct. Since $D_W$ is alternating, it has a canonical checkerboard coloring and there is a one-to-one correspondence:

$$\{ \text{the closed regions of } S_A \} \cup \{ \text{the closed regions of } S_B \} \rightarrow \{ \text{the closed regions of } D_W \}$$

Then $R_1(c), R_2(c), R'_1(c)$ and $R'_2(c)$ correspond to the four distinct closed regions of $D_W$ around $c$. This concludes the proof. \[\square\]

Thus, from this lemma, since $S(1)$ is obtained from $S(0)$ by changing an $A$-splice to a $B$-splice at $c$, two distinct regions $R_1(c)$ and $R_2(c)$ become a single region. Hence $c_{S(1)} = c_S - 1$. To conclude, the term of maximal degree in the entire bracket polynomial is contributed by the state $S = S_A$, and is not cancelled by terms from any other state, so we arrive at

$$\max deg(\langle D_W \rangle) = C + 2W - 2.$$  

The proof is similar for $\min deg(\langle D_W \rangle) = -C - 2B + 2$.

Case 2: The weaving diagram is $\Sigma_d$-reduced but not proper. Then, there exists at least one crossing which is not proper. If we change an $A$-splice to a $B$-splice at a crossing $c$ that is proper, then the conclusion is the same than before. Now, if we change an $A$-splice to a
B-splice at a crossing $c'$ that is not proper, then some white regions would touch both sides of a crossing. In this case, the number of split components does not decrease from $S(0)$ to $S(1)$: $c_{S(1)} = c_{S(0)}$. But, as seen before, $\langle D_W/S(1) \rangle = A^{-2}(D_W/S(0))$ and there is no isthmus in the diagram, so the number of components either decreases or is constant. Therefore,

$$\max deg(P_{S(1)}) \leq \max deg(P_{S(0)}).$$

Thus, once again, the term of maximal degree in the entire bracket polynomial is contributed by the state $S = S_A$, and is not cancelled by terms from any other state and therefore,

$$\max deg(\langle D_W \rangle) = C + 2W - 2.$$

The proof is similar for $\min deg(\langle D_W \rangle) = -C - 2B + 2$. \hfill \square

Now it is possible to define a relation between the closed regions of $D_W$ and the regions of the diagram after splitting as in Section 3 \cite{13}. Let $S_A$ (resp. $S_B$) be again the state obtained by splitting every crossing in the diagram in the $A$ (resp. $B$)-direction, and $D_W$ be colored so that all the regions labelled “$A$” are white (grey on the Figure 8) and all the regions labelled “$B$” are black.

![Diagram](image)

**Figure 8.** Example of $D_W$ (left), $S_A = A...A$ (middle) and $S_B = B...B$ (right).

Therefore, we have the following correspondences,

$\{\text{the closed regions of } S_A\} \rightarrow \{\text{the closed regions of } D_W \text{ in black regions } B\}$,

$\{\text{the closed regions of } S_B\} \rightarrow \{\text{the closed regions of } D_W \text{ in white regions } W\}$.

So, we have the following bijection,

$\{\text{the closed regions of } S_A\} \cup \{\text{the closed regions of } S_B\} \rightarrow \{\text{the closed regions of } D_W\}$.

And when “closing” a diagram on a $4g$-gon to have a surface of genus $g$, using the Euler characteristic and the fact that such a diagram is 4-valent:

**The number of closed regions of $D_W$ is equal to $C + 2 - 2g$, for every $g \geq 1$.**

We have seen in Section 2 that if we project a weave $W$ into the plane $\mathbb{R}^2$, we obtain the regular projection $W_0$, which is an infinite connected graph. We notice that such a regular
projection is also a periodic tiling by convex polygons, up to isotopy. Therefore, \( W_0 \) possesses a set of prototiles that characterizes it. Recall that a prototile is one of the shapes of a tile in a tessellation such that some of the tiles of the tiling are congruent to it. This notion is essential to define our notion of size of a diagram.

**Definition 3.10.** The size of a \( \Sigma_g \)-weaving diagram, for a fixed \( g \geq 1 \), is the number of edge-adjacent prototiles contained in the projection of this \( \Sigma_g \)-diagram on its associated \( 4g \)-gon of \( \mathbb{X}^2 \).

In Figure 2, the weaves \( W_1 = (W(C_r,1,4,4,4,4),w(1,1)), W_2 = (W(4C_r,0,4,4,4,4),w(1,1)) \) and \( W_3 = (W(4B_r,1,4,4,4,4,4g),w(1,1)) \) have for regular projection the square tiling, up to isotopy. Regarding the choice of the metric, many unit cells of different sizes can be considered for a \( \Sigma_g \)-diagram. Figure 9 shows some projections of \( \Sigma_1 \)-diagrams on a square (4-gon) associated to a torus of genus one.

\[
\text{size} = 0 \quad \text{size} = 1 \quad \text{size} = 4
\]

**Figure 9.** A weave whose projection is a square tiling (up to isotopy) and three projections of \( \Sigma_1 \)-diagrams on a 4-gon with different sizes.

Thus, it is possible to extend the proof of the following theorem (Theorem 2.10) from [10] for \( \Sigma_g \)-reduced diagrams, considering diagrams of same size, according to a chosen metric:

**Theorem 3.11.** The number of crossings \( C \) in a simple alternating projection of a \( \Sigma_g \)-weaving diagram \( D_W \) is a topological invariant of its associated weave \( W \). Therefore any two \( \Sigma_g \)-reduced connected alternating projections of a given weave have the same number of crossings.

**Proof.** Let \( \text{span}(D_W) \) defined by

\[
\text{span}(D_W) = \max \deg(\langle D_W \rangle) - \min \deg(\langle D_W \rangle).
\]

Then we have, \( \text{span}(D_W) = 2C + 2(W + B) - 4 = 2C + 2(C + 2 - 2g) - 4 \).

So finally, \( \text{span}(D_W) = 4C - 4g \), for every \( g \geq 1 \).  

\[\square\]
4. The Jones Polynomial and Tait’s First Conjecture for Weaves

4.1. The Jones Polynomial of a Weave.

The Jones polynomial is defined by the following identities in the Section 2 of [10],

\[ V_O = 1, \]
\[ t^{-1}V_\times - tV_\times = (\sqrt{t} - \frac{1}{\sqrt{t}})V_\times. \]

And it is related to the weaving invariant defined above by the following relation:

**Theorem 4.1.** The Jones polynomial \( V_W \) of a weave \( W \) is related to the bracket-type polynomial, for every \( g \geq 1 \), by the expression:

\[ V_W(t) = (-1)^{-3w_r(D_W) + 1}t^{-3w_r(D_W)} \left( \sum_S t^{i-j}(-t^2-t^{-2})^{c_S-1}\langle m^1_s, \ldots, m^g_s, n^1_s, \ldots, n^g_s \rangle \right). \]

**Proof.** By the skein relation:

\[ \langle \times \rangle = A \langle \times \rangle + A^{-1} \langle \circ \rangle, \]
\[ \langle \times \rangle = A^{-1} \langle \circ \rangle + A \langle \times \rangle. \]

Thus, we have \( A \langle \times \rangle - A^{-1} \langle \circ \rangle = (A^2 - A^{-2}) \langle \times \circ \rangle. \)

If we consider the writhe \( w_r(D_W) \) of the weaving diagram in the bracket on the right side of the equation, then the other two diagrams on the left have writhes \( (w_r(D_W) + 1) \) and \( (w_r(D_W) - 1) \) respectively. Thus, by multiplying the previous equation by the appropriate writhe, we obtain

\[ A^4f[\times \circ] - A^{-4}f[\times \circ] = (A^{-2} - A^2)f[\circ \circ]. \]

4.2. Tait’s First Conjecture for Weaves.

Before stating the main result of this paper, it is necessary to give a last essential definition, that is particular to the case of \( \Sigma_g \)-diagrams of infinite weaving structures.

**Definition 4.2.** The crossing number of a weave \( W \) is defined as the minimum number of crossings in a \( \Sigma_g \)-diagram \( D_W \) of minimal size of \( W \), for a fixed \( g \geq 1 \):

\[ C(W) = \min\{ C(D_W) : D_W \text{ is of minimal size} \}. \]

Any such weaving diagram of \( W \) which has exactly \( C(W) \) crossings is called minimal.

It is important to recall at this point that any \( \Sigma_g \)-diagram \( D_W \) must describe the crossing sequence of the different sets of threads and the periodicity of its associated weave \( W \). For example, on Figure 10, the picture on the right is a \( \Sigma_1 \)-diagram of \( W = (W_{4,4,4,4,1}, w(1,1)) \) but not the one on the left. Moreover, as seen earlier, a minimal diagram of a weave is not unique by construction.
Theorem 4.3. (Tait’s First Conjecture for weaves)
A connected $\Sigma_g$-reduced alternating weaving diagram of minimal size is a minimum diagram of its alternating weave. Moreover, a minimum diagram of a prime alternating weave can only be an alternating diagram. Thus, a non-alternating diagram can never be the minimum diagram of a prime alternating weave.

Proof. Since $V_W(t) = f[D_W](t^{-1/4})$ and span$(D_W) = 4C - 4g$, for every $g \geq 1$, thus,

\[
\text{span}(V_W(t)) = \max \deg (\langle V_W(t) \rangle) - \min \deg (\langle V_W(t) \rangle) = C - g.
\]

(4.1)

And the number of crossings is an invariant thus, it is fixed here for a $\Sigma_g$-reduced (connected) alternating weaving diagram of minimal size. Moreover, we have a generalization of the previous result for the general case, not necessary alternating, that can be proven in a similar way than in the proof of the generalization of Proposition 2.9 in [10],

\[
\text{span}(D_W) \leq 4C - 4g, \text{ for every } g \geq 1.
\]

(4.2)

Thus, the number of crossing points cannot decrease below span$(V_W(t))$. We conclude that $D_W$ must be a minimum diagram. Then span$(V_W(t)) = C - g$ is not true for a non-alternating weaving diagram of a prime alternating weave; this can be demonstrated in a similar way than in the proof of Theorem 3 in [27]. Thus, such a diagram cannot be a minimum diagram. \hfill \Box

Finally, it is important to notice at this point that our main theorem implies a dependence between a weave and its associate unit cells, according to the construction methodology described in Section 2. For this reason, we are interested in generalizing our study for periodic and symmetric weaves without being limited to our methodology.

Definition 4.4. The fundamental domain $F$ of a weave $W$ is defined as a subset of $W$ which contains exactly one point from each of its entanglement types. Such a type is defined by the way two threads of a same or different sets of threads are entangled with each other (over or under).

Sometimes, such a fundamental domain $F$ in a surface $\Sigma_g$ would not represent the periodicity of the weaving and cannot be used for a $\Sigma_g$-diagram. Moreover, it may represent the
periodicity but would not admit a canonical checkerboard coloring, and thus, cannot be used for an alternating $\Sigma_g$-diagram. We assume that a sufficient (but not necessary) condition to have such a property is the existence of at least one proper crossing in its planar projection, which means that we have at least four distinct regions around this crossing, as specified in Section 2. When it is the case, it is possible to create a $\Sigma_g$-reduced alternating diagram from $F$ by copying it, considering some symmetry arguments, and therefore to express its number of crossings. Such a diagram will not necessarily be minimal.

From any weaving fundamental domain $F$, we can indeed obtain different information to construct a weave $W$ using some symmetry and periodicity arguments. Indeed, $F$ gives the number of different sets of threads, the different type of entanglements between threads from different sets, and also from the same one. We use the following notations:

- $C_0$ the number of entanglement types of $F$ for two threads of different sets.
- $C_t$ the (desired) number of twists for two threads of a same set.
  ($F$ would contain only one twist in this case by definition.)
- $d_t$ the number of the different sets of threads concerned by this twisting pattern.

Therefore, we can express explicitly the number of crossings $C$ of an alternating weave $W$, by constructing an alternating $\Sigma_g$-reduced diagram from a weaving fundamental domain:

$$C = 2 \times C_0 + d_t \times C_t$$

Indeed, the minimal number of regions that can be found in a fundamental domain of an alternating weave is two, since it would represent at least one crossing over and one under for a weave with two sets of threads. Therefore, having two copies of such a fundamental domain is the natural way to obtain a diagram on $\Sigma_g$ with a proper crossing, which could therefore be $\Sigma_g$-reduced. Moreover, the addition of the twists of two threads of a same set would not affect this property.

In Figure 2, the weaves $W_1 = (W(\frac{C_r}{4,4,4,4}, 1, 1), w(1, 1)), W_2 = (W(\frac{4C_r}{4,4,4,4}, 0, 1, 1))$ and $W_3 = (W(\frac{4B_r}{4,4,4,4}, 1, 1), w(1, 1))$ have fundamental domains with $C_0 = 2$. Indeed, these weaves consist of two different sets of threads (the set of blue threads and the one of red threads in the pictures), and there are only two entanglement types: a red thread over a blue one, or a blue thread over a red one. Here, a minimal alternating $\Sigma_g$-reduced diagram can be constructed from a weaving fundamental domain, as seen in Figure 9. Therefore, the crossing number is 2 for these three weaves. On the other hand, $W_4 = (W(\frac{4C_r}{4,4,4,4}, 1, 1), w(1, 1))$ and $W_5 = (W(\frac{4C_r}{4,4,4,4}, 2, 1, 1))$ have fundamental domains with $C_0 = 4$. In these cases, there are also two sets of threads with the same entanglement types than previously and two more: a blue thread over another blue thread and a red thread over another red thread. Thus, for these two weaves, we have $C_t = 1$ for $W_4$ (resp. 2 for $W_5$), and with $d_t = 2$, it is possible to construct an example of alternating $\Sigma_g$-reduced diagram from a weaving fundamental domain. In this case, the number of crossings is equal to 6 for $W_4$ (resp. 8 for $W_5$).
5. OUTLOOK

A new methodology to construct and define Euclidean and hyperbolic weaving structures has been introduced in this paper. Moreover, the problem of classification of such frameworks has also been discussed with a viewpoint of knot theory. A polynomial invariant, originally defined for knot and link diagrams, and then adapted to Euclidean “textile” diagrams on a torus, has been extended here for weaving diagrams lying on higher genus surfaces. This invariant opens the door of classification of Euclidean and hyperbolic weaves according to the number of crossings of their minimal diagrams. The study has been limited to weaves constructed from uniform polygonal tessellations and will be generalized to more complex structures.

It is interesting to notice at this point that weaves differ from other doubly periodic entangled frameworks. For example, links made by a periodic union of knots, like polycatenanes (mentioned in Section 2) or “fishing net” structures, would consist of a collection of embeddings of topological circles, linked with each other in the three-dimensional space, which do not intersect. Nevertheless, a “closed weave”, where each end of the threads could be connected in pairs, would form a link. Other examples which would deserve a clear mathematical definition and a comparison to weaves are the knitting structures, which are sometimes defined as analogous to weaves in mathematical papers, but are actually different from an engineering point of view.

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