Advection Dominated Accretion Flows.
A Toy Disk Model∗

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ABSTRACT

A toy model of a disk undergoing steady state accretion onto a black hole is presented. The disk is in a hydrostatic equilibrium for all radii \( r > r_{\text{in}} \), with the inner disk radius located between the marginally stable and marginally bound orbits: \( r_{\text{ms}} > r_{\text{in}} > r_{\text{mb}} \). Matter flows from the disk through a narrow cusp at \( r_{\text{ms}} \) and falls freely into the black hole, carrying with it no thermal energy. At radii larger than \( r_{\text{out}} \) the disk is assumed to radiate away all locally generated heat, and therefore the disk is geometrically thin for \( r > r_{\text{out}} \). We assume that no heat generated in the inner disk, with \( r_{\text{out}} > r > r_{\text{in}} \) can be radiated away, i.e. the disk is 100% advective, and it becomes geometrically thick in this range of radii. All enthalpy of the thick disk is used up to press the inner disk radius towards the marginally bound orbit, and to lower the efficiency of conversion of accreted mass into radiation generated only for \( r > r_{\text{out}} \), by assumption.

Conservation laws of mass, angular momentum and energy make it possible to calculate the inner thick disk radius \( r_{\text{in}} \) for any specified value of its outer radius \( r_{\text{out}} \). As the nature of disk viscosity is not known there is some freedom in choosing the shape of the thick disk, subject to several general conditions, which include the hydrostatic equilibrium everywhere for \( r > r_{\text{in}} \). The main purpose of this toy model is to emphasize the effect the disk thickness has on lowering the energetic efficiency of a black hole accretion.

Key words: accretion, accretion disks – black hole physics

1 Introduction

Following the publication of the first paper on the advection dominated accretion flows by Narayan and Yi (1994) over one hundred papers appeared on this subject, with ever more sophisticated physics and ever more sophisticated treatment of geometry (cf. Gammie and Popham 1998, and references therein). However, there is no fully two dimensional treatment published

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so far, and there is no clear link between the recent papers and those written about two decades ago on the theory of thick disks (e.g. Jaroszyński, Abramowicz and Paczyński 1980, Paczyński and Wiita 1980, and references therein). The purpose of this paper is to present a toy model of a disk accreting onto a black hole but not radiating, i.e. advection dominated, presented in the spirit of the early 1980s, which seems to be simpler and more transparent than the spirit dominating the late 1990s.

2 Thin Pseudo-Newtonian Disk

I shall use pseudo-Newtonian gravitational potential (Paczyński and Wiita 1980):

\[ \Psi = -\frac{GM}{R - R_g}, \quad \Psi' = \frac{d\Psi}{dR} = \frac{GM}{(R - R_g)^2}, \quad R_g = \frac{2GM}{c^2}, \quad (1) \]

which has the property that a test particle has a marginally stable orbit at \( R_{ms} = 3R_g \), and marginally bound orbit at \( R_{mb} = 2R_g \), just as it is the case with particles orbiting Schwarzschild black hole. \( R \) is the spherical radius. Throughout this paper cylindrical coordinates \((r, z)\) will be used, with \( R = (r^2 + z^2)^{1/2} \).

In the following we consider a thin accretion disk fully supported against gravity by the centrifugal acceleration. The disk is in the equatorial plane of the coordinate system. It follows the motion of test particles on circular orbits, with the rotational velocity \( v(r) \), angular velocity \( \Omega(r) \), specific angular momentum \( j(r) \), and total specific energy \( e(r) \) given as

\[ v = (r\Psi')^{1/2} = \left( \frac{GM}{r} \right)^{1/2} \left[ \frac{r}{r - r_g} \right], \quad (2a) \]

\[ \Omega = \frac{v}{r} = \left( \frac{r\Psi'}{r^3} \right)^{1/2} = \left( \frac{GM}{r^3} \right)^{1/2} \left[ \frac{r}{r - r_g} \right], \quad (2b) \]

\[ j = vr = \left( r^3\Psi' \right)^{1/2} = (GMr)^{1/2} \left[ \frac{r}{r - r_g} \right], \quad (2c) \]

\[ e = \Psi + \frac{v^2}{2} = \left( -\frac{GM}{2r} \right) \left[ \frac{(r - 2r_g)r}{(r - r_g)^2} \right], \quad (2d) \]
where we adopted $r_g \equiv R_g$. It follows that
\[
\frac{de}{dr} = \Omega \frac{dj}{dr}.
\] (3)

The inner edge of a thin accretion disk is at
\[
r_{in} = r_{ms} = 3r_g,
\] (4)
where the binding energy and the specific angular momentum reach their minima:
\[
e_{ms} = -\frac{c^2}{16},
\] (5)
\[
j_{ms} = 1.5 \times 6^{1/2} \times \frac{GM}{c} \approx 3.674 \frac{GM}{c}.
\] (6)
The matter falls freely into the black hole once it crossed the $r_{ms}$, conserving its angular momentum and total energy.

The total thin disk luminosity is given with the formula:
\[
L_d = \dot{M}e_{in} = \dot{M}e_{ms} = \left(-\dot{M}\right) \frac{c^2}{16},
\] (thin disk), (7)
where I adopt a convention that $\dot{M} < 0$ for the accretion flow.

In a steady state accretion of a thin disk the equations of mass and angular momentum conservation may be written as
\[
\dot{M} = \text{const.}
\] (8a)
\[
\dot{J} = \dot{M}j + g = \text{const.}
\] (8b)
where $g$ is the torque acting between two adjacent rings in the disk. With no torque at the inner edge of the disk at $r_{in} = r_{ms}$ we have
\[
g = \left(-\dot{M}\right) (j - j_{in}) = \left(-\dot{M}\right) (j - j_{ms}).
\] (9)

The rate of energy flow in the disk is given as
\[
\dot{E} = \dot{M}e + g\Omega.
\] (10)
This is not constant, as the disk must radiate energy dissipated by viscous stresses in order to remain thin. The energy balance equation may be written as
\[
2F \times 2\pi r = -\frac{d\dot{E}}{dr} = g \left(-\frac{d\Omega}{dr}\right) = \left(-\dot{M}\right) (j - j_{in}) \left(-\frac{d\Omega}{dr}\right).
\] (11)
where $F$ is the energy radiated per unit disk area from each of its two surfaces. The eq. (11) may be integrated from $r_{ms}$ to infinity to obtain the same result as that given with the eq. (7).

Note, that disk viscosity was not specified. All we needed were the conservation laws and the assumption of a steady state accretion of a geometrically thin disk. Small geometrical thickness implied that no significant internal energy could be stored within the disk.

3 Thick Pseudo-Newtonian Disk

I shall consider now geometrically thick, axisymmetric disk, with the surface given with the relation $z_s(r)$, where $(r, z)$ are the two cylindrical coordinates, and we have

$$R = \left(r^2 + z^2\right)^{1/2}.$$  

The disk is assumed to be in a hydrostatic equilibrium. This implies that the three vectors have to balance at its surface: gravitational acceleration, centrifugal acceleration, and the pressure gradient divided by gas density. This implies that angular velocity $\Omega_s$ at the disk surface is given as (cf. Jaroszyński et al. 1980, Paczyński and Wiita 1980):

$$\Omega_s^2 = \left(\frac{\Psi_s'}{R_s}\right) \left(\frac{z_s}{r} \frac{dz_s}{dr} + 1\right),$$  

which is equivalent to

$$\frac{de_s}{dr} = \Omega_s \frac{dj_s}{dr},$$

where subscript ‘s’ indicates that the corresponding quantities are defined at the disk surface.

Thick disk has a large radial pressure gradient, and it remains in a hydrostatic equilibrium for $r > r_{in}$, and it has a cusp at $r_{in}$. The matter falls freely into the black hole inwards of $r_{in}$, which is located between the marginally stable and marginally bound orbits, i.e.

$$r_{mb} < r_{in} < r_{ms}$$

where a no torque inner boundary condition is applied. I shall not analyze the transition from a sub-sonic accretion flow at $r > r_{in}$ to a supersonic flow at $r < r_{in}$ (cf. Loska 1982).
The total disk luminosity is given with the formula similar to eq. (7):

\[ L_d = \dot{M} e_{\text{in}} = \left( -\dot{M} \right) \frac{GM (r_{\text{in}} - 2r_g)}{2(r_{\text{in}} - r_g)^2}, \quad \text{thick disk}. \]  

(15)

The specific binding energy at \( r_{\text{in}} \) has the range

\[ 0 > e_{\text{in}} > -\frac{c^2}{16} \quad \text{for} \quad r_{mb} < r_{\text{in}} < r_{ms}. \]  

(16)

Note, that while the total luminosity of a thick disk may be larger than that of a thin disk, the energy radiated per each gram of matter accreted into the black hole is smaller in the thick disk case.

### 4 Thick Advection Dominated Disk

A thin accretion disk has to radiate in order to remain thin. If the disk is not radiating away the energy which is generated by viscous stresses then it must be thick in order to accommodate this energy in its interior. Let us consider a disk which is thin for \( r > r_{\text{out}} \), but is thick for \( r_{\text{out}} > r > r_{\text{in}} \). Note, that the disk is also thin at its cusp, i.e. at \( r = r_{\text{in}} \). Our new disk radiates energy only where it is thin, i.e. for \( r > r_{\text{out}} \), where the disk surface brightness \( F \) is given with the eq. (11), and the total disk luminosity is given by the formula similar to the eq. (7) and (15):

\[ L_d = \dot{M} e_{\text{in}}. \]  

(17)

All this luminosity is radiated by the thin disk at \( r > r_{\text{out}} \), and none is radiated by the thick disk at \( r_{\text{out}} > r > r_{\text{in}} \), by assumption.

The structure of the thick disk, extending from \( r_{\text{in}} \) to \( r_{\text{out}} \) is of interest for us, as by assumption the disk is 100% advection dominated in this range of radii, i.e. the mass, angular momentum, and energy entering it at \( r_{\text{out}} \) must come out of it at \( r_{\text{in}} \):

\[ \dot{M}_{\text{in}} = \dot{M}_{\text{out}}. \]  

(18a)

\[ \dot{J}_{\text{in}} = \left( \dot{M} j + g \right)_{\text{in}} = \left( \dot{M} j + g \right)_{\text{out}} = \dot{J}_{\text{out}}. \]  

(18b)

\[ \dot{E}_{\text{in}} = \left( \dot{M} e + g\Omega \right)_{\text{in}} = \left( \dot{M} e + g\Omega \right)_{\text{out}} = \dot{E}_{\text{out}} \]  

(18c)

with \( j \) and \( \Omega \) having their ‘Keplerian’ values (cf. eqs. 2) at the two ends, as the disk is thin at both ends, and there is no torque at \( r_{\text{in}} \), i.e. \( g_{\text{in}} = 0 \).
The torque at $r_{\text{out}}$ follows from the conservation of angular momentum (18b):

$$g_{\text{out}} = \left(-\dot{M}\right) (j_{\text{out}} - j_{\text{in}}),$$

which, together with the energy conservation law give us:

$$e_{\text{out}} - e_{\text{in}} = (j_{\text{out}} - j_{\text{in}}) \Omega_{\text{out}}.$$

Note, that all quantities in the eq. (20) are unique functions of either $r_{\text{in}}$ or $r_{\text{out}}$ (cf. eqs. 2), and therefore the eq. (20) gives the relation between $r_{\text{in}}$ and $r_{\text{out}}$. If the outer radius of the advection dominated disk is specified, then the inner radius of the disk can be calculated with the eq. (20), and the eq. (17) gives the total luminosity of the thin disk which is assumed to extend from $r_{\text{out}}$ to infinity. The relation between $r_{\text{out}}$ and $r_{\text{in}}$ is shown in Figure 1. The larger is $r_{\text{out}}$, the smaller is $r_{\text{in}}$, and the less energy is radiated per one gram of accreted matter.
The variation of the inner thick disk radius $r_{in}$, and the binding energy at the inner thick disk radius $e_{in}$ with the outer thick disk radius $r_{out}$ is shown in Fig. 1. Notice, that for very large $r_{out}$ we have asymptotically

$$\frac{r_{in}}{r_g} \approx 2 + 3 \frac{r_g}{r_{out}}, \quad \frac{e_{in}}{e_{ms}} \approx 12 \frac{r_g}{r_{out}}, \quad \text{for} \quad \frac{r_{out}}{r_g} \gg 1. \quad (21)$$

We also have

$$e_{out} \approx -\frac{c^2}{4} \frac{r_g}{r_{out}}, \quad e_{in} \approx 3e_{out} \approx -\frac{3c^2}{4} \frac{r_g}{r_{out}}, \quad \text{for} \quad \frac{r_{out}}{r_g} \gg 1. \quad (22)$$

So far we used only conservation laws to constrain our thick advective disk. If we want to find the disk shape we must make some additional assumptions. There are two general inequalities which must be satisfied by the matter at the disk surface (cf. Jaroszyński et al. 1980, Paczyński and Wiita 1980):

$$\frac{dj_s}{dr} > 0, \quad \frac{d\Omega_s}{dr} < 0, \quad (23)$$

supplemented with the condition of hydrostatic equilibrium at the thick disk surface, as expressed with the eq. (13b), and the conditions that the disk must be geometrically thin at $r_{in}$ and $r_{out}$.

There is a lot of freedom in choosing thick disk structure that satisfies all the conditions listed in the previous paragraph. For our toy model we adopted

$$j_s = j_{in} \left[ 1 + b \left( \frac{x_s}{x_{in}} - 1 \right)^{a} + b \left( \frac{x_s}{x_{in}} - 1 \right)^{3a} \right]^{1.5/a}, \quad (24)$$

and

$$e_s = e_{in} + \int_{r_{in}}^{r_s} \Omega_s \frac{dj_s}{dr} dr = e_{in} + \int_{r_{in}}^{r_s} \frac{1}{2r^2} \frac{dj^2_s}{dr} dr. \quad (25)$$

Thin disk conditions at $r_{in}$ are satisfied automatically with the eqs. (24) and (25), and we have

$$j_{in} = \frac{(GM_{in}^3)}{r_{in} - r_g}^{1/2}, \quad e_{in} = -\frac{GM(r_{in} - 2r_g)}{2(r_{in} - r_g)^2}. \quad (26)$$

The parameters $a$ and $b$ have to be adjusted so that thin disk conditions are satisfied at $r_{out}$, where the eqs. (24) and (25) must give

$$j_{out} = \frac{(GM_{out}^3)}{r_{out} - r_g}^{1/2}, \quad e_{out} = -\frac{GM(r_{out} - 2r_g)}{2(r_{out} - r_g)^2}. \quad (27)$$
Figure 2: A cross section of the toy disk model accreting into a black hole. All energy generated inwards of $r_{\text{out}} = 100 \, r_g$ is assumed to be advected with the accretion flow, and none is radiated away. At large distance, $r > r_{\text{out}}$, all energy generated by viscosity is assumed to be radiated away, hence the disk is geometrically thin for $r > r_{\text{out}}$. The inability to radiate away energy for $r < r_{\text{out}}$ forces the disk to become thick, and pushes its inner disk radius, $r_{\text{in}} = 2.026 \, r_g$, close to the marginally bound orbit, $r_{\text{mb}} = 2 \, r_g$, where the binding energy is only $|e_{\text{in}}| \approx 0.006 \, c^2$.

As an example, a toy disk model with $r_{\text{out}} = 100 \, r_g$ is shown in Fig. 2 and Fig. 3. The inner disk radius $r_{\text{in}} = 2.026031 \ldots \, r_g$ was obtained from the condition given with the eq. (20). The eqs. (27) were satisfied for $a = 1.32048586889 \ldots$ and $b = 0.000000442466655 \ldots$.

5 Discussion

The shape of our toy disk as shown in Fig. 2 is an artifact of the strong assumptions made in this paper. In particular, a rapid transition from a thin disk to a thick disk at $r_{\text{out}}$ is a direct consequence of the assumption that 100% of all energy dissipated locally is radiated away locally for $r > r_{\text{out}}$, while none is radiated away for $r < r_{\text{out}}$. In any realistic disk there will be
Figure 3: The inner parts of the toy disk model presented in Fig. 2. The dots on the equatorial axis indicate the location of the marginally bound orbit at \( r_{mb} = 2r_g \), and the marginally stable orbit at \( r_{ms} = 3r_g \). The disk becomes geometrically thin at \( r_{in} = 2.026r_g \), and cold matter falls freely into the black hole at \( r < r_{in} \). No thermal energy is advected into the black hole in this model.

partial radiation at all radii, and the variation of the efficiency is likely to change gradually, with no abrupt changes in the disk thickness.

While the detailed shape of the thick disk must be uncertain as long as we have no quantitative understanding of disk viscosity, the formation of a thick disk and pushing its inner radius towards the marginally bound orbit is a very general property. It was noticed two decades ago with supercritical accretion disks of Jaroszyński et al. (1980) and Paczyński and Wiita (1980). In our toy model the inability of the disk to radiate energy dissipated within \( r_{out} > r > r_{in} \) forces the disk to become thick, and pushes its inner cusp towards \( r_{mb} \), lowering the efficiency with which rest mass is converted into radiation. The farther out the \( r_{out} \) is assumed to be the closer \( r_{in} \) is pushed towards the \( r_{mb} \) in order to lower the efficiency.

While we retain the term ‘advective’ to describe our thick disk, it should be stressed that as we require the disk to be thin at its inner cusp at \( r_{in} \), there
Figure 4: The variation of angular momentum with radius for the surface of the thick disk presented in Figures 2 and 3 is shown with the solid line \( (j_s) \). The variation of ‘Keplerian’ angular momentum with radius is shown with the dashed line \( (j_K) \). Note that thick disk angular momentum is almost constant for \( r < 10 \, r_g \).

is no advection of any heat into the black hole, as all enthalpy of the thick disk has been used up to press its inner radius towards the \( r_{mb} \). However, the kinetic energy of thick disk matter at \( r_{in} \) is much larger than is the kinetic energy of a thin disk, which must have its inner radius located at \( r_{ms} \). In the pseudo-Newtonian potential the ratio of the two is \( \frac{v_{mb}^2}{v_{ms}^2} = \frac{8}{3} \) (cf. eq. 2a).

There is a somewhat paradoxical aspect of our toy model. Some kind of viscosity has to reduce angular momentum from \( j_{out} \) at the outer disk boundary to \( j_{in} \) at its inner boundary, and this must generate either heat or some other form of internal energy which puffs up the disk, and it would seem that this energy has to be advected through \( r_{in} \) into the black hole. Figure 4 shows the distribution of angular momentum for our thick disk model (solid line), and for the ‘Keplerian’ angular momentum given with the eq. (2c) (dashed line). It is clear that thick disk angular momentum is almost constant for \( r < 10 \, r_g \), and that all dissipation takes place only in
the outer parts of the thick disk. All the same the entropy must be higher at $r_{in}$ than it is at $r_{out}$. However, high entropy does not have to imply high internal energy if the gas density is low at $r_{in}$.

Obviously, our requirement for the disk to be thin at $r_{in}$ is ad hoc, motivated by our desire to have as simple structure as possible. Our toy model may require uninterestingly low accretion rate to satisfy all the conditions that are imposed at its inner boundary: hydrostatic equilibrium, low geometrical thickness, high entropy and low internal energy. It is very likely that at the accretion rate of any interest some of these conditions are broken. It is likely that the cusp at $r_{in}$ opens up considerably, the speed of sound is not negligible compared to rotational velocity, and the transonic flow carries a non trivial amount of internal energy into the black hole (Loska 1982). However, the conservation laws do not require the advected thermal energy to be large, as demonstrated by our toy model.

A generic feature of any thick disk is the formation of a narrow funnel along the rotation axis. It is far from clear how realistic is the presence of the funnel, as some instability might fill it in, and make the accretion nearly spherical near the black hole. On the other hand, if the funnel forms and lasts, it may help collimate a jet-like outflow. Unfortunately, our lack of quantitative understanding of viscous processes in accretion flows makes it impossible to prove what topology of solutions is realistic under which conditions. The main virtue of the toy model presented in this paper is the set of assumptions that was made; this set is very different from the assumptions which are used in the recently booming industry of advection dominated accretion flows. While the assumptions adopted in this paper are ad hoc, so are the assumptions adopted in any thick disk models. In particular, there is nothing ‘natural’ about the popular assumption that the accretion flow is self-similar. Another popular assumption: constant ‘alpha’ parameter is ad hoc as well. It is useful to read the old paper by Galeev, Rossner and Vaiana (1979) to realize that the very concept of the ‘alpha’ parameter is ad hoc. Disk properties which depend on any ad hoc assumptions should not be taken seriously.

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