A Self-similar Solution of Hot Accretion Flow: The Role of the Kinematic Viscosity Coefficient

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Abstract

We investigate the dependency of the inflow-wind structure of a hot accretion flow on the kinematic viscosity coefficient. In this regard, we propose a model for the kinematic viscosity coefficient to mimic the behavior of the magnetorotational instability that would be maximal at the rotation axis. Then, we compare our model with two other prescriptions from numerical simulations of the accretion flow. We solve two-dimensional hydrodynamic equations of hot accretion flows in the presence of thermal conduction. The self-similar approach is also adopted in the radial direction. We calculate the properties of the inflow and the wind such as velocity, density, and angular momentum for three models of the kinematic viscosity prescription. On inspection, we find that in our suggested model the wind is less efficient at extracting the angular momentum outward where the self-similar solutions are applied than it is in two other models. The solutions obtained in this paper might be applicable to hydrodynamical numerical simulations of hot accretion flows.

Unified Astronomy Thesaurus concepts: Accretion (14); Hydrodynamics (1963); Low-luminosity active galactic nuclei (2033)

1. Introduction

Based on the temperature of the flow, black hole accretion disks can be divided to two broad modes: hot and cold. The standard thin-disk and slim-disk models, with relatively high mass-accretion rates, belong to the cold mode (Shakura & Sunyaev 1973; Abramowicz et al. 1988), while the radiative inefficient accretion flow with a low mass-accretion rate, below roughly $2\% M_{\text{Edd}}$, belongs to the hot mode ($M_{\text{Edd}} = 10L_{\text{Edd}}/c^2$) is called the Eddington accretion rate with $L_{\text{Edd}}$ and $c$ being the Eddington luminosity and speed of light, respectively; Narayan & Yi 1994).

Wind is an essential constituent of accretion flows and can impact their dynamics and structure. In addition, the feedback of the wind on the surrounding medium can also play a major role in the formation and evolution of galaxies (Fabian 2012; Kormendy & Ho 2013; Naab & Ostriker 2017). It has been generally acknowledged that winds prevail in hot accretion flows. The fully-ionized winds from hot accretion have been confirmed in recent years by wind observations from low-luminous active galactic nuclei (AGNs) as well as radio galaxies (Tombesi et al. 2010, 2014; Crenshaw & Kraemer 2012; Cheung et al. 2016), the supermassive black hole in the Galactic center (Sgr A*; Wang et al. 2013; Ma et al. 2019), and black hole X-ray binaries in a hard state (Homan et al. 2016; Munoz-Darias et al. 2019).

Three different wind-launching mechanisms have been proposed, namely thermally driven (Begelman et al. 1983; Font et al. 2004; Luketic et al. 2010; Waters & Proga 2012), magnetically driven (Blandford & Payne 1982; Lynden-Bell 1996, 2003), and radiation-driven wind (Murray et al. 1995; Proga et al. 2000; Proga & Kallman 2004; Nomura & Ohsuga 2017). In terms of hot accretion flow the first two mechanisms can play a role in producing wind since the radiation is negligible. Therefore, hot accretion flows are more feasibly simulated. To show the existence of wind in a hot accretion flow, a large number of global hydrodynamical (HD) and magnetohydrodynamical (MHD) numerical simulations have been carried out (e.g., Stone et al. 1999; Igumenshchev & Abramowicz 1999, 2000; Stone & Pringle 2001; Hawley et al. 2001; Machida et al. 2001; Hawley & Balbus 2002; Igumenshchev et al. 2003; Pen et al. 2003; De Villiers et al. 2003, 2005; Yuan & Bu 2010; Pang et al. 2011; McKinney et al. 2012; Narayan et al. 2012; Li et al. 2013; Yuan et al. 2012a, 2012b, 2015; Inayoshi et al. 2018, 2019).

The pioneering analytical studies of hot accretion flow such as Narayan & Yi (1994), Narayan & Yi (1995), Blandford & Begelman (1999), Blandford & Begelman (2004), Xu & Chen (1997), and Xue & Wang (2005) have also claimed that the strong wind must exist. Analytical studies are powerful tools to fully understand the physical properties of inflow and wind from accretion flow. This is mainly because it is very time consuming and difficult to run global numerical simulations of accretion flow with different physics including dissipation and magnetic field terms, while it is much easier to probe the dependency of the results on the input parameters in an analytical approach. Based on radial self-similar approximation, the existence of wind has been investigated in more detail by solving HD and MHD equations of hot accretion flow (e.g., Bu et al. 2009; Jiao & Wu 2011; Mosallanezhad et al. 2014; Samadi et al. 2017; Bu & Mosallanezhad 2018; Kumar & Gu 2018; Zeraatgari et al. 2020). Due to technical difficulties such as the existence of singularity near the rotation axis in these studies, the physical boundary conditions are set at only the equatorial plane and the equations have been integrated from this boundary. Then, the integration stops where the gas pressure or density becomes zero for the HD case (see, e.g., Jiao & Wu 2011) or where the sign of the radial or toroidal components of the magnetic field changes in the MHD case (see, e.g., Mosallanezhad et al. 2016). To get reliable physical
solutions, similar to Narayan & Yi (1995) and Xu & Chen (1997), a second boundary should be considered at the rotation axis and the equations should be solved using a two-point boundary value problem technique.

In Zeraatgari et al. (2020) and Mosallanezhad et al. (2021), we utilized such a technique to solve the radiation HD equations of supercritical accretion flows and the HD equations of hot accretion flows, respectively. More precisely, in Mosallanezhad et al. (2021), we solved the HD equations of the accretion flow with thermal conduction using the relaxation method. We explained the origin of the discrepancy between previous self-similar solutions of hot accretion flows in the presence of the wind such as Xu & Chen (1997) and Xue & Wang (2005) by extensively studying the energy equation. We showed that in terms of hot accretion flow, thermal conduction would be an essential term for investigating the inflow-wing structure of the flow. Another interesting result was that the hot accretion flow is convectively stable in the presence of the thermal conduction.

It is now widely believed that in a real accretion flow the angular momentum is transferred with the MHD turbulence driven by the magnetorotational instability (MRI; Balbus & Hawley 1991, 1998). Consequently, the MHD equations, rather than HD ones, should be solved. In Mosallanezhad et al. (2021), instead of a magnetic field we considered using the anomalous shear stress to mimic the magnetic stress. For the kinematic viscosity coefficient, following Narayan & Yi (1995), we only adopted the $\alpha$-prescription as $\nu = \alpha c_s^2/\Omega_k$ where $c_s$ and $\Omega_k$ are the sound speed and Keplerian angular velocity, respectively (Shakura & Sunyaev 1973). There are also some MHD numerical simulations which show the dependency of the magnetic stress on the vertical and radial structure of the disk (e.g., Bai & Stone 2013; Penna et al. 2013). Therefore, following numerical simulations of hot accretion flows, we adopt several different viscosity prescriptions and solve the vertical structure of the disk by using the relaxation method. It would be worthwhile to investigate the dependency of the physical quantities of the hot accretion flow, as well as the wind properties such as poloidal velocity or the Bernoulli parameter, on the definition of the kinematic viscosity coefficient.

The remainder of the manuscript is organized as follows. In Section 2, the basic equations, physical assumptions, self-similar solutions, and the boundary conditions will be introduced. The detailed explanations of numerical results to the definition of the viscosity coefficient will be presented in Section 3. Finally, we will provide the summary and discussion in Section 4.

2. Basic Equations and Assumptions

The basic HD equations of a hot accretion flow can be described as

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0,$$

$$\frac{\rho d\mathbf{v}}{dt} = -\rho \nabla \psi - \nabla p_{\text{gas}} + \nabla \cdot \sigma,$$

$$\frac{d\rho}{dt} - \frac{p_{\text{gas}}}{\rho} \frac{d\rho}{dt} = \nabla v : \sigma - \nabla \cdot F_c.$$

In the above equations, $\rho$ is the mass density, $\mathbf{v}$ is the velocity, $\psi = -GM/r$ is the Newtonian potential (where $r$ is the distance from the central black hole, $M$ is the black hole mass, and $G$ is the gravitational constant), $p_{\text{gas}}$ is the gas pressure, $\sigma$ is the viscous stress tensor, $e$ is the gas internal energy, and $F_c$ is the thermal conduction. Here, $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ represents the Lagrangian or comoving derivative. The equation of state of the ideal gas is considered as $p_{\text{gas}} = (\gamma - 1)\rho e$, with $\gamma = 5/3$ being the adiabatic index. In the purely HD limit, like our case, $F_c$ is defined as

$$F_c = -\lambda \nabla T,$$

where $\lambda$ is the thermal diffusivity and $T$ is the gas temperature. We use spherical coordinates $(r, \theta, \phi)$ to solve the above set of equations. In most previous semianalytical studies on hot accretion flow, they assumed that the system is in a steady state and axisymmetric, i.e., $\partial/\partial \phi = \partial/\partial t = 0$. These assumptions imply that all physical quantities are independent of the azimuthal angle, $\phi$, and the time, $t$. Following numerical simulations of hot accretion flow (e.g., Stone et al. 1999; Yuan et al. 2012a), we consider the following components of the viscous stress tensor:

$$\sigma_{\theta \theta} = \rho \nu \left[ \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right],$$

$$\sigma_{r \theta} = \rho \nu \left[ \frac{\partial}{\partial r} \left( \frac{v_r}{r} \right) \right],$$

$$\sigma_{\phi \phi} = \rho \nu \left[ \sin \theta \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) \right],$$

where $\nu$ is called the kinematic viscosity coefficient. By substituting all the above assumptions and definitions into Equations (1)–(3), we obtain following partial differential equations (PDEs) in spherical coordinates. Hence, the continuity equation is reduced to the following form

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta \sin \theta) = 0.$$

The three components of the momentum equation are as follows:

$$\rho \left[ v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - v_\phi \frac{v_\phi}{r} \right] = -\frac{GM\rho}{r^2} - \frac{\partial p_{\text{gas}}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (\sin \theta \sigma_{\theta \theta}),$$

$$\rho \left[ v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_r \frac{v_\phi}{r} \right] = -\frac{1}{r^2} \frac{\partial p_{\text{gas}}}{\partial \theta} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r \theta}),$$

$$\rho \left[ v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + v_r \frac{v_\phi}{r} \right] = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{\phi \phi}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sin \theta \sigma_{\phi \phi}) + \frac{1}{r} (\sigma_{r \theta} + \sigma_{\phi \phi} \cot \theta).$$
The energy equation is written as

\[
\rho \left[ V_r + \frac{v_{\theta} \partial e}{r \partial \theta} - \frac{p_{\text{gas}}}{\rho} \left[ V_r + \frac{v_{\theta} \partial \rho}{r \partial \theta} \right] \right] = \frac{\partial \rho}{\partial r} + \frac{\rho v_{\theta} \partial \rho}{r \partial \theta} - \frac{\partial}{\partial r} \left( \sigma_{\theta} \frac{\partial \sigma_{\theta}}{\partial r} + \frac{\partial}{\partial \theta} \left( \sigma_{\theta} \frac{\partial \sigma_{\theta}}{\partial \theta} - v_{\theta} \right) \sigma_{\theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \lambda \sin \theta \partial \theta \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \lambda \frac{\partial \theta}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \lambda \sin \theta \frac{\partial \theta}{\partial \theta} \right).
\]

(12)

Since we consider all three components of the velocity and the thermal conduction in this study, the energy equation differs from previous self-similar solutions such as Narayan & Yi (1995) (see Mosallanezhad et al. 2021 for more details). We follow Tanaka & Menou (2006) and use a standard form of the thermal conduction, where the heat flux depends linearly on the local temperature gradient (see also Khajenabi & Shadmehri 2013 for more details). To preserve self-similarity, the radial dependence of the thermal conductivity coefficient must be a power-law function of the radius as \( \lambda(r) = \lambda_0 r^{-1/2 - n} \), where \( n \) is the density index which will be defined in the self-similar solutions (see Equation (16)). As we explained in the Introduction, the key goal of this study is to check the dependency of the physical properties of the inflow and wind to the kinematic viscosity coefficient. Thus, based on the numerical simulations as well as analytical solutions, we choose three different models of the kinematic viscosity coefficient as

\[
\nu = \alpha \sqrt{\frac{G M r}{c_s}},
\]

(13)

\[
\nu = \alpha (c_s^2 / \Omega_k),
\]

(14)

\[
\nu = \alpha (c_s^2 / \Omega_k) f(\theta),
\]

(15)

where \( \Omega_k(=G M / r^3)^{1/2} \) is the Keplerian angular velocity, \( c_s \) is the sound speed, and \( \alpha \) is the viscosity parameter related to the “alpha” prescription (see Shakura & Sunyaev 1973 for more details). In all of the above definitions, the kinematic viscosity coefficient is proportional to \( \nu(r, \theta) \sim r^{1/2} \). Note here that Equation (13) is similar to the one chosen by Stone et al. (1999) (Model K) and also Yuan et al. (2012a). Equation (14) is similar to that chosen by Shakura & Sunyaev (1973) for thin-disk model. Also, Equation (15) is proposed to mimic the effect of MRI associated with the Maxwell stress. We model the \( f \) parameter as \( f(\theta) = 1/2(1 + |\cos \theta|) \), where 1/2 is the normalization factor. We choose this form because in a real accretion disk there is no viscosity and the angular momentum is transferred by the Maxwell stress associated with the MHD turbulence driven by MRI. The Maxwell stress is provided by the magnetic field. Since the magnetic field is smooth and coherent with the disk height, it can connect to infinity and would not be zero above the disk. In addition, the average stress tensor can be normalized to the gas pressure as \( T_{\text{ref}} = \alpha p_{\text{gas}} \). Moreover, the gas-pressure gradient decreases toward the rotation axis so \( \alpha \) should increase from the equatorial plane to the rotation axis (see, e.g., Figures 3 and 6 of Bai & Stone 2013). We call these three prescriptions of the kinematic viscosity coefficient presented in Equations (13)–(15) the “ST,” “SS,” and “ZE” models, respectively. In the ST model, the viscosity coefficient changes only in the radial direction and its value is constant in the \( \theta \) direction. Meanwhile, in the SS model the viscosity coefficient is proportional to the sound speed squared or equivalently the temperature of the accretion flow, which increases from the equatorial plane toward the rotation axis. Here, we introduce the ZE model, which is more realistic and physical in some senses. In fact, this particular form represents the main features of accretion flow. In reality, the angular momentum of the accretion disk is transferred by the magnetic stress associated with the MHD turbulence driven by MRI. In this HD study, we use the shear stress to mimic the magnetic stress in the dynamics of the disk. In our simple form, i.e., \( \nu \propto \alpha (1 + |\cos \theta|) \), the kinematic viscosity coefficient is at a minimum at the equatorial plane and reaches its maximum at the rotation axis.\(^1\)

To solve the system of the equations in the vertical direction, we adopt the self-similar solutions in the radial direction and present a fiducial radius \( r_0 \) as a power-law form of \( (r/r_0) \). The solutions are introduced as

\[
\rho(r, \theta) = \rho_0 \left( \frac{r}{r_0} \right)^{-n} \rho(\theta),
\]

(16)

\[
v_r(r, \theta) = v_{\theta}(r) \left( \frac{r}{r_0} \right)^{-1/2},
\]

(17)

\[
v_{\theta}(r, \theta) = v_{\phi}(r) \left( \frac{r}{r_0} \right)^{-1/2},
\]

(18)

\[
v_r(r, \theta) = v_{\phi}(r) \left( \frac{r}{r_0} \right)^{-1/2} \tilde{\Omega}(\theta) \sin(\theta),
\]

(19)

\[
c_s(r, \theta) = c_s(\theta) \left( \frac{r}{r_0} \right)^{-1/2},
\]

(20)

where \( r_0, \rho_0 \), and \( v_0 = \sqrt{GM/r_0} \) are the units of length, density, and velocity, respectively. The ordinary differential equations (ODEs) can be derived by substituting the above self-similar solutions into the PDEs (8) and (12) as

\[
\left[ \left( \frac{3}{2} - n \right) \tilde{v}_r + \frac{d\tilde{v}_\theta}{d\theta} + \tilde{v}_\theta \cot \theta \right] \tilde{\rho} + \tilde{v}_\theta \frac{d\tilde{\rho}}{d\theta} = 0,
\]

(21)

\[
\left[ -\frac{1}{2} \tilde{v}_r^2 + \tilde{v}_\theta \frac{d\tilde{v}_\theta}{d\theta} - \tilde{v}_\theta^2 - \frac{\tilde{\Omega}^2}{2} \sin^2 \theta \right] - \rho + (n + 1) \rho_0 + \tilde{\rho} \tilde{v}_r \left( \frac{d^2 \tilde{v}_r}{d\theta^2} - \frac{3}{2} \frac{d\tilde{v}_\theta}{d\theta} \right) + \left( \frac{d\tilde{v}_\theta}{d\theta} - \frac{3}{2} \tilde{v}_\theta \right) \left( \tilde{\rho} + \frac{3}{2} \rho_0 \right) = 0.
\]

(22)

\(^1\) Note that there must be several different forms for the kinematic viscosity coefficient that represent similar behavior but here we adopt the simplest one to mimic its dependency on the \( \theta \) angle. This behavior is consistent with Bai & Stone (2013), where they studied the accretion disk with a strong vertical magnetic field and investigated MRI to transport angular momentum from the accretion disk.
\[ \rho \left[ \frac{1}{2} \dot{v}_r \dot{v}_\theta + \ddot{v}_\theta - \dddot{\Omega} \sin \theta \cos \theta \right] = -\frac{dp}{d\theta} \]

\[ - (n - 2) \rho \ddot{v}_r + \frac{d\dot{v}_r}{d\theta} - \frac{3}{2} \dddot{\Omega}, \quad (23) \]

\[ \rho \left[ \frac{1}{2} \ddot{\Omega} + 2 \dot{\Omega} \cot \theta + \dddot{v}_\theta \right] = \frac{d\ddot{\Omega}}{d\theta} \left[ p \frac{d\rho}{d\theta} + \rho \frac{dp}{d\theta} \right] + \rho \ddot{v} \left[ \frac{3}{2} (n - 2) \dddot{\Omega} + 3 \frac{d\dddot{\Omega}}{d\theta} \cot \theta + \frac{d^2 \dddot{\Omega}}{d\theta^2} \right], \quad (24) \]

\[ \left( n - \frac{1}{\gamma - 1} \right) \rho \ddot{v}_r + \ddot{v}_\theta - (n - 1) \left( \frac{dp}{d\theta} - \gamma \frac{\dddot{v}_\theta}{\rho} \right) = \rho \ddot{v} \left[ \frac{d\dot{v}_r}{d\theta} - \frac{3}{2} \dddot{\Omega} \right] + \frac{9}{4} \dddot{\Omega}^2 \cot \theta + \frac{d^2 \dddot{\Omega}}{d\theta^2} \right], \quad (25) \]

In the above system of equations, \( \bar{p}_g(\theta) \) is the dimensionless form of the gas pressure defined as\(^2\)

\[ \bar{p}_g(\theta) = \bar{p}(\theta) \frac{c^2_s(\theta)}{\rho}, \quad (26) \]

The dimensionless forms of the kinematic viscosity coefficient for three models will be also written as

\[ \nu(\theta) = \alpha, \quad (27) \]

\[ \nu(\theta) = \alpha c^2_s(\theta), \quad (28) \]

\[ \nu(\theta) = \frac{1}{2} \alpha c^2_s(\theta)(1 + |\cos \theta|). \quad (29) \]

The above ODEs, i.e., Equations (21)–(25), consist of five physical variables: \( v_r(\theta), v_\theta(\theta), \dot{\Omega}(\theta), p(\theta), \) and \( c_s(\theta) \). Following Zeraatgar et al. (2020) and Mosallanezhad et al. (2021), the computational domain will be extended from the equatorial plane, \( \theta = \pi/2 \), to the rotation axis, \( \theta = 0 \). In addition, all physical variables are assumed to be even, symmetric, continuous, and differentiable at both boundaries. We also include the latitudinal component of the velocity, \( \ddot{v}_\theta \), with zero values at both the equatorial plane and the rotation axis. The following boundary conditions will be imposed at \( \theta = \pi/2 \) and \( \theta = 0 \):

\[ \frac{dp}{d\theta} = \frac{d\dddot{c}_s}{d\theta} = \frac{d\dddot{\Omega}}{d\theta} = \frac{d\dddot{v}_r}{d\theta} = \dddot{v}_\theta = 0. \quad (30) \]

To integrate the ODEs, the relaxation method will be adopted with a resolution of 5000 stretch grids as follows: from \( \theta = \pi/2 \) to \( \theta = \pi/4 \), the grid-size ratio is set as \( d\theta_{i+1}/d\theta_i = 1.003 \), while from \( \theta = \pi/4 \) to \( \theta = 0 \) the grid-size ratio is set as \( d\theta_{i+1}/d\theta_i = 0.997 \). The absolute error tolerance is considered to be \( 10^{-15} \). For the appropriate initial guess, we use the Fourier cosine series for all physical variables except \( \dddot{v}_\theta \) where we use the Fourier sine series. Our solutions will satisfy the boundary conditions at both ends which ensures that we have well-behaved solutions in the whole computational domain. In the next section, we will explain in detail the behaviors of the physical variables for the three different viscosity models.

### 3. Numerical Results

The system of ODEs are solved numerically by using the relaxation method, which is utilized in two-point boundary value problems. Equations (21)–(25) are integrated from the equatorial plane (\( \theta = \pi/2 \)), where the maximum density is located, to the rotation axis (\( \theta = 0 \)). Our numerical method might guarantee that the solutions would be mathematically self-consistent in \( r - \theta \) space. The parameters we adopt are \( \alpha = 0.1, n = 0.5, \gamma = 5/3, \) and \( \lambda_0 = 0.02 \). We plotted the latitudinal profiles of the physical variables in Figures 1 and 2.

![Figure 1. Latitudinal profile of radial velocity in units of Keplerian velocity, \( v_K \). The green, blue, and red curves correspond to three models of the kinematic viscosity coefficient: ST, \( \nu = \alpha \sqrt{GM\rho}; \) SS, \( \nu = \alpha (c^2_s/\Omega)^2; \) and ZE, \( \nu = 1/2 \alpha (c^2_s/\Omega)^2(1 + |\cos \theta|) \). The inflow region is in the range of \( 45^\circ < \theta < 90^\circ \) for all models. Here, \( \alpha = 0.1, n = 0.5, \gamma = 5/3, \) and \( \lambda_0 = 0.02 \).](image)

The latitudinal profile of \( v_r \) in Figure 1 is plotted for three prescriptions of the kinematic viscosity coefficient, i.e., \( \nu = \alpha \sqrt{GM\rho} \) (ST), \( \nu = \alpha (c^2_s/\Omega)^2 \) (SS), and \( \nu = 1/2 \alpha (c^2_s/\Omega)^2(1 + |\cos \theta|) \) (ZE) in green, blue, and red colors, respectively. The radial velocity is negative in the inflow region, which is extended in the range of \( 45^\circ < \theta < 90^\circ \) for all models. The ST prescription shows a higher radial velocity in the inflow and wind region and the wind velocity becomes around twice the Keplerian velocity at the rotation axis. As we mentioned earlier, the ST prescription is similar to the one presented in Stone et al. (1999) (model K) and Yuan et al. (2012b), in which the kinematic viscosity coefficient is only proportional to the radius, therefore it is constant in the vertical direction. As this figure shows, the SS and ZE models in comparison with the ST model have a smaller radial velocity in wind region and it is clear that the smallest radial velocity is for the ZE model around \( \approx 0.7 v_K \). Although we did not include magnetic fields in our study, we mimic the effect of MRI by considering different prescriptions of kinematic viscosity coefficient. Therefore, this behavior of our model comes from the fact that the entire disk becomes magnetically dominated.
where the strong magnetic pressure gradient would suppress the radial velocity of the wind in the region far from the central black hole.

In Figure 2, we plot the latitudinal profile of the four physical variables of latitudinal velocity, $v_\theta$, in units of the Keplerian velocity, $v_K$ (top-left panel), the angular velocity, $\Omega$, in units of the Keplerian angular velocity (top-right panel), density in the unit of density at the equatorial plane (bottom-left panel), and sound speed squared in units of $v_K^2$ (bottom-right panel). As you can see, for all models $v_\theta$ is always negative in the range of $-0.2 < v_\theta < -0.1$ and has a minimum located at $15^\circ < \theta < 30^\circ$. It is zero at $\theta = 0$ and $\theta = \pi/2$ as boundary conditions. The ST model has a minimum value of $v_\theta$ around $-0.2$ at $\theta \sim 24^\circ$. We note that as in our solution, the density index is $0 < n < 3/2$ and the net mass-accretion rate $[M = 2\pi \int_0^\infty \rho(r, \theta) v_\theta(r, \theta) r^2 \sin \theta d\theta]$ becomes zero. In essence, the self-similar solutions are applied far away from the black hole where the gravity of the black hole is not so strong. Therefore, at that region, we would find that the net mass-accretion rate is actually negligible compared to the mass inflow and outflow rates. We can then model this case with zero mass-accretion rate. To find whether the wind can escape from the system and go to infinity we quantitatively calculate some properties of the wind summarized in Table 1.

According to Figure 1 and the top-left panel of Figure 2, the poloidal velocities $|\vec{v}_\theta| = \sqrt{v_\theta^2 + v_r^2}$ in the ST and SS models become greater than those in the ZE model. We define the mass-flux-weighted parameters of the inflow and the wind as

$$q_{\text{inflow}}(r) = \frac{4\pi r^2 \int_0^{\pi/2} \rho \min(v_r, 0) \sin \theta d\theta}{4\pi r^2 \int_0^{\pi/2} \rho \min(v_r, 0) \sin \theta d\theta},$$

Figure 2. Latitudinal profiles of the physical variables. Top left: latitudinal velocity, $v_\theta$, in units of Keplerian velocity, $v_K$. Top right: angular velocity, $\Omega$, in units of Keplerian angular velocity, $\Omega_K$. Bottom left: density in the unit of the density at the equatorial plane. Bottom right: sound speed squared, $c_s^2$, in units of $v_K^2$. The green, blue, and red curves correspond to three models of the kinematic viscosity prescription, “ST,” “SS,” and “ZE”, respectively. Here, $\alpha = 0.1$, $n = 0.5$, $\gamma = 5/3$, and $\lambda_0 = 0.02.$
Table 1

| Model | Kinematic Viscosity Coefficient | $L_{\text{inflow}}$ ($L_K$) | $L_{\text{wind}}$ ($L_K$) | Bernoulli Parameter ($\psi^2$) | $\psi_p$ ($\psi$) |
|-------|---------------------------------|-----------------------------|-----------------------------|-------------------------------|------------------|
| ST    | $\nu = \alpha \sqrt{GMr}$      | 0.54                        | 0.73                        | 1.24                          | 0.75             |
| SS    | $\nu = \alpha (c_s^2 / \Omega_K)$ | 0.63                        | 0.78                        | 0.34                          | 0.52             |
| ZE    | $\nu = \frac{4}{5} \alpha (c_s^2 / \Omega_K)(1 + |\cos \theta|)$ | 0.61                        | 0.72                        | 0.22                          | 0.37             |

where $q$ is a physical quantity. The mass-flux-weighted poloidal velocities of the wind for the three models, ST, SS, and ZE, are shown in the last column of Table 1. Our model, the ZE prescription, presents the minimum poloidal velocity, and ZE, are shown in the last column of Table 1. Our model, the ZE prescription, presents the minimum poloidal velocity, $v_p = 0.37 v_e$, while the ST model shows the maximum one equal to $v_p = 0.75 v_e$. These results are in agreement with the numerical simulation of Yuan et al. (2015).

The top-right panel of Figure 2 shows the behavior of the angular velocity, $\Omega$, in the latitudinal direction. As is clear, the angular velocity in all models is sub-Keplerian. From the equatorial plane to the rotation axis, it has an increasing trend. Since the maximum angular velocity, for all models, occurs in the polar region, it is strongly suggested that the wind can vertically transport the angular momentum to the outside of the system. As it was noted before, we propose the ZE model for our self-similar solution. For instance, when the gas approaches the inner boundary, its angular momentum is enough to produce the wind at that radius. However, the amount of angular momentum that is released in the polar region is smaller than that on the equatorial plane by several (two, three, or even more) orders of magnitude.

The top-right panel of Figure 2 is plotted. To have a similar density scale as Xu & Chen (2005); we normalized the density to the maximum density at the equatorial plane. In all models, the density decreases from the disk midplane to the rotation axis. More precisely, based on the density curves, the densities reach maximum values at the rotation axis, which are (0.1–0.3)$\rho_0$ (2). Note that based on our self-similar solution, the mass-density profile increases by about one order of magnitude, which is consistent with the previous self-similar studies such as Narayan & Yi (1995; Figure 1 therein) and Tanaka & Menou (2006; Figure 3 therein). However, the numerical simulations of hot accretion flow (see, e.g., Figures 1 and 5 of Yuan & Bu 2010) show that the mass density close to the polar axis is lower than that on the equatorial plane by several (two, three, or even more) orders of magnitude.

The latitudinal profile of the sound speed, $c_s^2$, is shown in the bottom-right panel of Figure 2. This plot shows that the sound speed in both the SS and ZE models is almost in the same range and small, while it is much higher in the ST model in the wind region. It shows that in the ST model the flow is easily heated to a temperature about the virial. This is in good agreement with MHD simulations of Yuan et al. (2012a).

To calculate the wind properties, we define the Bernoulli parameter as

$$Be(r) = \frac{\gamma^2}{2} + \frac{\gamma p}{(\gamma - 1) \rho} - \frac{GM}{r},$$

where on the right-hand side of this equation, the first, second, and third terms correspond to the kinetic energy, enthalpy, and gravitational energy, respectively. The mass-flux-weighted Bernoulli parameter of the wind is given in the fifth column of Table 1 for the three models of kinematic viscosity prescription. For all models, the Bernoulli parameter of the wind is positive, which shows the flow has enough energy to overcome the gravity of the central object and take it away from the system. As we mentioned earlier, the density index is set to $n = 0.5$ in our current study, which causes the net mass-accretion rate to become zero. At the region far away from the black hole, the wind can be produced at any radius based on our self-similar solution. For instance, when the gas approaches a given radius, i.e., $r = r_{0s}$, the release of the gravitational energy is enough to produce the wind at that radius. However, our self-similar model cannot be applied to observations of high velocity outflow (such as ultrastar outflows) arguably produced very close to the black hole.

To investigate the momentum balance in different models of kinematic viscosity prescription, we plotted latitudinal profiles of various terms in the radial and vertical components of the momentum equation (Equations (22) and (23)) in Figures 3 and 4, respectively. In the left panels of Figure 3, radial components of the inertial (thick solid lines) and the centrifugal (dashed-dotted lines) terms are plotted for three models of kinematic viscosity prescriptions, ST, SS, and ZE, based on the radial component of the momentum equation (see the left-hand side of Equation (22)). In the right panels of this figure, the gravitational (thin solid lines), pressure-gradient (dashed lines), and viscous (dotted lines) terms are calculated from the right-hand side of the $r$-momentum. As you can see, in the left panels of Figure 3, the absolute value of the radial centrifugal term is larger than the inertial term at the equatorial plane in all models. The inertial term becomes significant in the polar region and the maximum amount is for the ST model. From the right panels it is clear that the gravitational term plays an important role in the equatorial plane while it drops in the polar
Figure 3. Latitudinal profiles of the terms in the $r$-component of the momentum equation for three models of kinematic viscosity prescription, ST (top panels), SS (middle panels), and ZE (bottom panels), in which the left and right panels correspond to the left- and right-hand sides of this Equation (Equation [22]).
Figure 4. Latitudinal profiles of terms in the \( \theta \) component of the momentum equation for three models of kinematic viscosity prescription, ST (top panels), SS (middle panels), and ZE (bottom panels), in which the left and right panels correspond to left- and right-hand sides of Equation (23)).
region where the density reaches its minimum. The pressure-gradient and viscous terms become substantial in the wind region for all models. Comparing the three models, the maximal amount of the viscous term is shown in the ST model. The amount and behavior of the pressure gradient is nearly the same in the SS and ZE models. it decreases from the equatorial plane toward the rotation axis, while it is high at both the equatorial plane and the rotation axis for the ST model. Further, the pressure-gradient term in the ST model should also at its highest, which could balance the large viscous term in this model at the rotation axis.

The latitudinal profiles of the left- and right-hand side terms in the $\theta$ momentum are shown in Figure 3. It also shows a comparison between three different prescriptions of the kinematic viscosity coefficient: ST (top panels), SS (middle panels), ZE (bottom panels). As you can see, the vertical component of the centrifugal (dashed–dotted lines) and inertial (solid lines) terms are shown in the left panels, while the pressure-gradient (dashed line) and viscous (dotted lines) terms are plotted in the right panels. From the left panels, it is clear that the vertical component of the inertial term is not important in any of the models. The right panels indicate that the viscous term does not show a dominant role anywhere in the flow in the SS and ZE models. Instead, in the ST model, this term is considerable everywhere in the flow, excluding at the equatorial and rotation axis where it is zero. This figure points out that in the vertical component of the momentum equation, the latitudinal pressure gradient, projected centrifugal force and viscosity would balance each other. Moreover, the most important feature of Figures 3 and 4 is that through the momentum balance analysis, one cannot determine the inflow and wind regions, so the winds can be specified through energetics treatment.

Therefore, to describe the wind, we examined the energy balance in three different models of the kinematic viscosity prescription. Figure 5 represents the latitudinal profiles of the advection (left panel), viscous (middle panel), and conduction (right panel) terms in the energy equation. The energy advection is on the left-hand side of Equation (25) while the viscosity and the thermal conduction are on the right-hand side. As you can see, the viscous term is positive from the equatorial plane toward the rotation axis and heats the flow. At the equatorial plane, the conduction term is positive and is a heat source, so the advection heating acts as a cooling term with a positive sign. In contrast, near the rotation axis the amount of the viscous term declines so that the advection term has to balance the thermal conduction. As the sign that the conduction term is negative and a cooling term, the advection works as a heating term despite being negative. From these results, it is suggested that to launch wind from the system near the rotation axis there should be a significant amount of cooling so that advection shows heating behavior. It should be noted that the larger amounts of all the energy terms in the ST model with respect to the other models is only because the scales are different.

4. Summary and Discussion

In this work, we have solved the HD equations of hot accretion flows associated with thermal conduction in spherical coordinates $(r, \theta, \phi)$ and in two dimensions using self-similar solutions in the radial direction. We assumed the flow is axisymmetric and in a steady state. We also considered three components of the viscous stress tensor, i.e., $r\theta$, $r\phi$, and $\theta\phi$. We examined three different models for kinematic viscosity coefficient: the ST model, where the kinematic viscosity coefficient depends only on $r$; the SS model, where the kinematic viscosity coefficient is proportional to the sound speed squared; and the ZE model, where the kinematic viscosity coefficient is proportional to $(1 + \cos \theta)$. In the ZE model that we proposed, the kinematic viscosity coefficient is at a minimum at the equatorial plane and reaches its maximum at the rotation axis to satisfy MHD numerical simulations which predict that $\alpha$ increases with decreasing $\theta$. We made use of the relaxation method and integrated the ODEs from the equatorial plane to the rotation axis. Our motivation was to see which model of the kinematic viscosity coefficient could produce strong wind far away from the black hole in the hot accretion flow. The physical variables of the inflow and wind were obtained for three models of the kinematic viscosity coefficient. The positive Bernoulli parameter of the inflow and wind were attained in the range of $45^\circ < \theta \leq 90^\circ$. In addition, our model illustrates that the radial velocity of the wind near the rotation axis is lower than the two other models, mainly due to the existence of a strong magnetic pressure gradient at the surface of the disk. We found that the poloidal velocity in the ST model is higher than that in the two other models. Additionally, the gas rotates at sub-Keplerian velocities for all three models.
Comparing the angular velocity profile in the three models clearly demonstrates that in the model we suggested, compared to the two other models, wind is less efficient at outwardly extracting the angular momentum due to the behavior of MRI. We treated the momentum balance in the radial and polar directions for the three models of the kinematic viscosity coefficient and found the pressure gradient and also the viscous term in both directions in the ST model is stronger than those in the two other models. It is imperative that the inflow and wind regions cannot be distinguished by momentum balance analysis. We note that we need a energetics examination for a detailed description of the winds. In this regard, we investigated the energy balance in three different prescriptions of the kinematic viscosity coefficient. Subsequently, our results implied that in the inflow region where the viscous term is large enough, the conduction is a heat source so advection acts as a cooling term. While near the rotation axis where the viscosity decreases, the conduction is a cooling term and advection acts as a heating term. As a consequence, to produce wind in the high latitudes of the accretion flow there should be sufficient cooling such that the advection acts as a heating term. The solutions obtained in this paper might be applicable for numerical HD simulations of hot accretion flow. Therefore, it is worthwhile to be revisit the numerical simulations.

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