Boundedness and Compactness of Toeplitz operators with $L^1$ symbols on the Bergman Space*

Dieudonne Agbor †

February 1, 2008

Abstract

We characterise the boundedness of a Toeplitz operator on the Bergman space with a $L^1$ symbol. We also prove that the compactness of a Toeplitz operator on the Bergman space with a $L^1$ symbol is completely determined by the boundary behaviour of its Berezin transform. This result extends known results in the cases when the symbol is either a positive $L^1$ function, an $L^\infty$ function or a general $BMO^1$ function.

1 Introduction.

Toeplitz operators are one of the most widely studied classes of concrete operators. The study of their behaviour on Hardy and Bergman spaces has generated an extensive list of results in operator theory and function theory. One of the latest approaches in this area is the use of the Berezin transform as a determining factor of the behaviour of Toeplitz operators (see [1], [2], [5], [7], [8]).

Let $d\lambda$ denote the Lebesgue area measure on the unit disk $\Delta$, normalized so that $\lambda(\Delta) = 1$. For $0 < p < \infty$, the Bergman space $L^p_0(\Delta)$ is the subspace of $L^p(\Delta, d\lambda)$ consisting of functions analytic on the unit disk $\Delta$. Let $P$ be the Bergman projection from $L^2(\Delta, d\lambda)$ onto $L^2_0(\Delta)$ defined by

$$(P g)(z) = \int_{\Delta} g(w) \overline{K_z(w)} d\lambda(w)$$

where $K_z(w) = \frac{1}{(1-\overline{w}z)^2}$ is the Bergman reproducing kernel. The normalized Bergman kernel is $k_z(w) = \frac{1-|z|^2}{(1-\overline{w}z)^2}$. For a function $f$ in $L^1(\Delta, d\lambda)$, the Toeplitz operator $T_f$ on $L^2_0(\Delta)$ is defined by

$$T_fg = P(fg).$$

*Mathematical Subject Classification: 47B35
†Research supported by International Science Program (ISP), University of Uppsala.
Indeed, since the Bergman projection can be extended to $L^1(\Delta, d\lambda)$, the operator $T_f$ is well defined on $H^\infty(\Delta)$, the space of bounded analytic functions on $\Delta$, which is dense in $L^2_0(\Delta)$. Hence, $T_f$ is always densely defined on $L^2_0(\Delta)$. Also, since $P$ is not bounded on $L^1(\Delta, d\lambda)$, $T_f$ can be in general unbounded.

Our first result characterises boundedness of $T_f$ with $f \in L^1(\Delta, d\lambda)$. Precisely we have the following:

**Theorem 1** Let $f \in L^1(\Delta, d\lambda)$. Then $T_f$ extends to a bounded operator on $L^2_0(\Delta)$ if and only if

$$\sup_{z \in \Delta} \|T_f \circ \varphi_z 1\|_2 < \infty \quad \text{and} \quad \sup_{z \in \Delta} \|T_f^* \circ \varphi_z 1\|_2 < \infty.$$ 

For an operator $A$ on $L^2_0(\Delta)$, the Berezin transform of $A$ is the function $\tilde{A}$ on $\Delta$ defined by

$$\tilde{A}(z) = \langle Ak_z, k_z \rangle.$$ 

Here, and elsewhere in this paper, the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_2$ are taken in the space $L^2(\Delta, d\lambda)$. If $A$ is bounded on $L^2_0(\Delta)$, then $\tilde{A}$ is a bounded function. Since the kernel $k_z$ converges weakly to zero in $L^2_0(\Delta)$ as $z$ approaches the unit circle $\partial \Delta$, we have that if $A$ is compact, then $\tilde{A}(z) \to 0$ in $L^2_0(\Delta)$ as $z \to \partial \Delta$. The converse in both cases is not necessarily true, see N. Zorboska [8] for some counterexamples. To simplify the notation, for $f$ in $L^1(\Delta, d\lambda)$, the Berezin Transform $\tilde{T}_f$ of $T_f$ will be denoted $\tilde{f}$ and will be called Berezin transform of $f$. In other words,

$$\tilde{f}(z) = \tilde{T}_f(z) = \int_{\Delta} f(w)|k_z(w)|^2 d\lambda(w).$$

However, we are going to show the converse result when $A$ is a Toeplitz result $T_f$, $f \in L^1(\Delta, d\lambda)$. Precisely we prove the following result:

**Theorem 2** Let $f \in L^1(\Delta, d\lambda)$. Suppose that $T_f$ bounded on $L^2_0(\Delta)$ and that $\tilde{f}(z) \to 0$ as $z \to \partial \Delta$. Then $T_f$ is a compact operator on $L^2_0(\Delta)$.

This theorem extends known results in the cases when the symbol is either a positive $L^1$ function, an $L^\infty$ function or a general $BMO^1$ function earlier proved respectively by K. Zhu [7], S. Axler and D. Zheng [1] and N. Zorboska [8]. An attempt to prove this theorem was undertaken by N. Zorboska [8]. She proved the following theorem:

**Theorem 3** Let $f \in L^1(\Delta, d\lambda)$ be such that $T_f$ is bounded on $L^2_0(\Delta)$ and that

$$\sup_{z \in \Delta} \|T_f \circ \varphi_z 1\|_p < \infty \quad \text{and} \quad \sup_{z \in \Delta} \|T_f^* \circ \varphi_z 1\|_p < \infty$$

for some $p > 3$. Suppose that $\tilde{f}(z) \to 0$ as $z \to \partial \Delta$. Then $T_f$ is a compact operator on $L^2_0(\Delta)$. 

2
At the end of her paper, N. Zorboska posed the question of extending the latter theorem with the condition \( p > 3 \) replaced by \( p > 2 \). By the way, quite recently, Miao and Zheng\(^5\) have shown that the number \( p \) cannot be improved in general for all bounded operators on \( L^2_a \). Of course, our theorem solves Zorboska’s problem. Our methods of proof are adapted from methods used in both in [1] and [8] which we combine with a result due to D. Luecking [3].

### 2 Useful concepts.

For \( z \in \Delta \), let \( \varphi_z \) be the analytic map of \( \Delta \) onto \( \Delta \) defined by

\[
\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.
\]

A simple calculation shows that \( \varphi_z \circ \varphi_z \) is the identity function on \( \Delta \).

For \( z \in \Delta \), let \( U_z : L^2_a(\Delta) \rightarrow L^2_a(\Delta) \) be the unitary operator defined by

\[
U_zf = (f \circ \varphi_z)\varphi_z'.
\]

Then \( U_z \) is a self-adjoint unitary operator. In particular, for \( g \in L^2_a \) then \( \|U_wg\|_2 = \|g\|_2 \).

We shall use the following lemma:

**Lemma 1** For \( f \) in \( L^1(\Delta, d\lambda) \), if \( T_f \) bounded on \( L^2_a(\Delta) \), then

\[
U_zT_fU_z = T_{f \circ \varphi_z}.
\]

**Proof.** Direct calculation.

The next lemma will also be useful:

**Lemma 2** Let \( f \in L^1(\Delta) \) and \( h \in L^2_a(\Delta) \). Then

1. \((T_fK_z)(u) = K_z(u)P(f \circ \varphi_z)(\varphi_z(u));\)
2. \((T_fK_z)(u) = (T_fK_u)(z);\)
3. \((T_fh)(v) = \int_{\Delta} h(u)(T_fK_u)(v) d\lambda(u)\)
4. \(T_f^*h = T_f^*h \) if \( T_f \) is bounded on \( L^2_a(\Delta);\)

**Proof.** 1. For \( f \in L^1(\Delta, d\lambda) \), we have

\[
(T_fK_z)(u) = \int_{\Delta} f(w)K_z(w)\overline{K_u(w)} d\lambda(w),
\]
which is equal to
\[ \int_{\Delta} f(\varphi_z(v))K_z(\varphi_z(v)\bar{K}_u(\varphi_z(v)))|k_z(v)|^2 d\lambda(v), \]
by making the change of variable \( w = \varphi_z(v) \). Also,
\[ K_z(\varphi_z(v))k_z(v) = \frac{1}{1 - |z|^2} \tag{1} \]
and
\[ \bar{K}_u(\varphi_z(v))k_z(v) = K_{\varphi_z(u)}(v)k_z(u). \]
Thus,
\[ (T_f K_z)(u) = \int_{\Delta} (f \circ \varphi_z)(v) \frac{1}{1 - |z|^2} \bar{K}_{\varphi_z(u)}(v)k_z(u) dA(v) = K_z(u) P(f \circ \varphi_z)(\varphi_z(u)). \]

2. We have:
\[ (T_f K_z)(u) = \frac{\overline{T_{\bar{f}} K_z, K_u}}{\langle \bar{f} K_z, K_u \rangle} = \frac{\left\{ \int_{\Delta} \bar{f}(w)K_z(w)K_w(u) dA(w) \right\}}{\int_{\Delta} f(w)K_u(w)K_z(z) dA(w) = (T_f K_u)(z)}. \]

3. Since \( h \in L^2_a(\Delta) \) we have
\[ \int_{\Delta} h(u)(T_f K_u)(v) d\lambda(u) = \int_{\Delta} h(u) \left( \int_{\Delta} f(z)K_u(z)K_z(v) d\lambda(v) \right) d\lambda(u) = \int_{\Delta} f(z)K_z(v) \left( \int_{\Delta} h(u)K_u(z) d\lambda(u) \right) d\lambda(z) = \int_{\Delta} f(z)h(z)K_z(v) d\lambda(z) = (T_f h)(v). \]

4. If \( T_f \) is bounded on \( L^2_a(\Delta) \), then \( T_f^* \) is also bounded on \( L^2_a(\Delta) \). So for \( h \in L^2_a(\Delta) \), we have
\[ (T^*_f h)(z) = \langle T^*_f h, K_z \rangle = \langle h, T_f K_z \rangle = \langle h, f K_z \rangle = T_f^{-1} h(z). \]

Let \( z, w \in \Delta \). Then the Bergman metric \( B(z, w) \) is given by
\[ B(z, w) = \frac{1}{2} \log \left\{ \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} \right\}. \]
For $z \in \Delta$ and $\delta > 0$ we define

$$D(z, \delta) = \{w \in \Delta : B(z, w) < \delta\}.$$  

Then $D(z, \delta)$ is a Euclidean disk with centre $C$ and radius $R$ given by

$$C = \frac{1 - s^2}{1 - s^2|z|^2}z, \quad R = \frac{1 - |z|^2}{1 - s^2|z|^2}R_s$$

where $s = \frac{e^\delta - e^{-\delta}}{e^\delta + e^{-\delta}} = \tanh \delta$.

We will denote the normalized Lebesgue area measure of $D(z, \delta)$ by $\lambda(D(z, \delta))$.

Our next lemma is an application of a result by D. Luecking\cite{3} which gives a necessary and sufficient condition for a positive Borel measure $\mu$ on $\Delta$ to satisfy the following property: the Bergman space $L^p(\Delta, d\lambda)$, the space of analytic functions, is embedded in $L^q(\Delta, d\mu)$ for $p > q > 0$.

Precisely, Luecking’s result is the following:

\textbf{Theorem 4} Suppose $0 < \delta < \frac{1}{2}$ and $0 < q < p$. Define the function $k(w) := \frac{\mu(D(w, \delta))}{\lambda(D(w, \delta))}$, the estimate

$$\left(\int_\Delta |f|^q d\mu\right)^{\frac{1}{q}} \leq C \left(\int_\Delta |f|^p d\lambda\right)^{\frac{1}{p}}$$

for all $f \in L^p(\Delta, \lambda)$ holds if and only if $k$ is in $L^s(\Delta, d\lambda)$, where $\frac{1}{s} + \frac{2}{p} = 1$. Moreover, $C = c\|k\|_s^{\frac{1}{q}}$.

From now on, we shall keep $\delta$ fixed and we shall write $D(z)$ instead of $D(z, \delta)$, for $z \in \Delta$. Furthermore, for $\epsilon > 0$, define an operator $S$ by

$$(Sf)(z) = \int_\Delta |f(v)| \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} d\lambda(v). \tag{2}$$

\textbf{Lemma 3} Let $p > 1$ and $0 < \epsilon < \frac{1}{p'}$ where $p'$ is the conjugate exponent of $p$. Then there exist a constant $C = C(p, \epsilon)$ such that for all $f \in L^1(\Delta)$, the following estimate holds:

$$|Sf(z)| \leq C(K_z(z))^{\epsilon} \|f\|_p.$$  

\textbf{Proof.} Let

$$d\mu(v) = \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} d\lambda(v).$$

Then

$$\mu(D(w)) = \int_{D(w)} \frac{1}{1 - |z|^2} \frac{1}{1 - |v|^2} \frac{1}{(1 - |\varphi_z(v)|^2)^{2\epsilon - 1}} d\lambda(v) \leq C \frac{\mu(D(w))}{(1 - |z|^2)(1 - |w|^2)(1 - |\varphi_z(w)|^2)^{2\epsilon - 1}} \lambda(D(w)).$$

5
The last equation comes from the fact that
\[
\frac{1}{(1 - |\varphi_z(v)|^2)} \frac{1}{(1 - |v|^2)} \frac{|1 - zv|^2}{1 - |w|^2} \leq C \frac{1}{(1 - |\varphi_z(w)|^2)} \frac{|1 - zw|^2}{1 - |w|^2}
\]
since \(v \in D(w)\). Also, for \(v \in D(w)\) we have
\[
|1 - zv| \leq |1 - zw| + |w - v| \leq |1 - zw| + C(1 - |w|^2).
\]
This shows that
\[
\frac{|1 - zw|^2}{1 - |w|^2} \leq C
\]
Next, if \(k(w) = \frac{\mu(D(w))}{m(D(w))}\) then
\[
k(w) \leq \frac{C}{(1 - |z|^2)(1 - |w|^2)} \frac{|1 - zw|^2(2\epsilon - 1)}{(1 - |w|^2)^{2\epsilon - 1}(1 - |z|^2)^{2\epsilon - 1}} \leq \frac{C'}{(1 - |w|^2)^{2\epsilon}(1 - |z|^2)^{2\epsilon}} = \frac{C'}{(1 - |w|^2)^{2\epsilon}}K_z(z)^\epsilon.
\]
Furthermore, we see that \(k\) belongs to \(L^{p'}(\Delta, d\lambda)\) if \(0 < \epsilon < \frac{1}{2p'}\). Thus by Luecking\[3\], there exists a constant \(C = C(p, \epsilon) > 0\) such that
\[
\int_{\Delta} |f|d\mu \leq C\|f\|_p(K_z(z))^{2\epsilon}
\]
which proves our lemma.

**Remark 1** For \(\epsilon > 0\) let \(g(v) = (K_v(v))^{\epsilon}\). Then for all \(h \in L^1(\Delta, dA)\) and all \(u \in \Delta\), the following identity is true:
\[
\int_{\Delta} |K_u(v)P(h \circ \varphi_u)(\varphi_u(v))|g(v)d\lambda = (S(P(h \circ \varphi_u))(u)
\]
where the operator \(S\) was defined in \[2\].

**Proof.** The change of variable \(v = \varphi_u(w)\) in the left hand side integral yields that it is equal to
\[
\int_{\Delta} |P(h \circ \varphi_u)(w)| \frac{1}{(1 - |u|^2)(1 - |w|^2)} \frac{1}{(1 - |\varphi_u(w)|^2)^{2\epsilon - 1}} d\lambda(w).
\]
Here we used \[1\].
3 Proof of Theorem \[1\]

Let \( f \in L^1(\Delta) \), \( h \in L^2(\Delta) \) and \( v \in \Delta \). Then by Lemma 2 assertion 3 we have that

\[
(T_fh)(v) = \int_\Delta h(u)(T_fK_u)(v) dA(u)
\]

which implies that \( T_f \) is an integral operator with kernel \((T_fK_u)(v)\). By Schur’s lemma, if there exist a positive measurable function \( g \) on \( \Delta \) and constants \( c_1, c_2 \) such that

\[
\int_\Delta |(T_fK_u)(v)| g(v)^2 d\lambda(v) \leq c_1 g(u)^2 \tag{3}
\]

for all \( u \in \Delta \) and

\[
\int_\Delta |(T_fK_u)(v)| g(u)^2 d\lambda(u) \leq c_2 g(v)^2 \tag{4}
\]

for all \( v \in \Delta \), then \( T_f \) is bounded on \( L^2_\alpha(\Delta) \) and \( \|T_f\| \leq \sqrt{c_1 c_2} \).

For \( \epsilon > 0 \), take \( g(v)^2 = (K_u(v))^\epsilon \). Then assertion 1 of Lemma 2 implies that the left hand side of (3) is equal to

\[
\int_\Delta |K_u(v)P(f \circ \varphi_u)(\varphi_u(v))| g(v)^2 d\lambda(v). \tag{5}
\]

According to (5) and Remark 1, the left hand side of (3) can be written as

\[
(S(P(h \circ \varphi_u))(u).
\]

Now for \( 0 < \epsilon < \frac{1}{4} \) we obtain from Lemma 3 that there exists a constant \( C > 0 \) such that

\[
(S(P(f \circ \varphi_u))(u) \leq C\|P(f \circ \varphi_u)\|_2(K_u(u))^\epsilon
\]

\[
\leq C \sup_{u \in \Delta} \|P(f \circ \varphi_u)\|_2(K_u(u))^\epsilon.
\]

This gives estimate (3) with \( c_1 = C \sup_{u \in \Delta} \|P(f \circ \varphi_u)\|_2 \).

Let us next prove the estimate in (4). We use assertion 2 of Lemma 2 to get that the left hand side of (4) is at most

\[
\int_\Delta |T_fK_v(u)| g(u) d\lambda(u) = \int_\Delta |K_v(u)P(\bar{f} \circ \varphi_v)(\varphi_v(u))| g(u) d\lambda(u)
\]

\[
= S(P(\bar{f} \circ \varphi_v))(v).
\]

If \( 0 < \epsilon < \frac{1}{4} \), then Lemma 3 implies there exists a constant \( C > 0 \) such that

\[
SP(\bar{f} \circ \varphi_v)(v) \leq C \sup_{v \in \Delta} \|P(\bar{f} \circ \varphi_v)\|_2 g(v).
\]
This gives estimate (11) with \( c_2 = C \sup_{v \in \Delta} \| P(\overline{f} \circ \varphi_v) \|_2 \).

Thus by Schur’s Lemma \( T_f \) is bounded on \( L^2(\Delta) \).

Conversely, if \( T_f \) is bounded we have using Lemma 1
\[
\sup_{z \in \Delta} \| T_{f \circ \varphi_z} 1 \|_2 = \sup_{z \in \Delta} \| U_z T_f U_z 1 \|_2 = \sup_{z \in \Delta} \| T_f U_z 1 \|_2 \leq \| T_f \|_2 < \infty.
\]

Since \( T_f \) bounded implies \( \overline{T}_f \) is bounded this also shows in a similar manner that \( \sup_{z \in \Delta} \| T_{f \circ \varphi_z} 1 \|_2 \) is finite.

4 Compactness of \( T_f, f \in L^1(\Delta) \)

We start by presenting some necessary results that will be useful in our prove of the main theorem in this section.

Theorem 5 Let \( A \) be a bounded operator on \( L^2_{\alpha}(\Delta) \) and let \( U_z \) be the unitary operator on \( L^2_{\alpha}(\Delta) \). Then the property \( \tilde{A}(z) \to 0 \) as \( z \to \partial \Delta \) implies that \( U_z A U_z 1 \to 0 \) weakly in \( L^2_{\alpha}(\Delta) \) as \( z \to \partial \Delta \).

Proof. We shall represent the operator \( U_z A U_z \) simply by \( A_z \). To prove this result, it suffices to show that \( \langle A_z 1, w^p \rangle \to 0 \) as \( z \to \partial \Delta \) for any non-negative integer \( p \).

Firstly, we have that \( \tilde{A} \circ \varphi_z = \tilde{A_z} \). Indeed, For \( w \in \Delta \), we have
\[
\tilde{A_z}(w) = \langle A_z k_w, k_w \rangle = \langle A U_z k_w, U_z k_w \rangle.
\]
Also, since
\[
U_z k_w(v) = \frac{-(1 - |\varphi_z(w)|^2)(1 - z \bar{w})^2}{(1 - v \varphi_z(w))^2(1 - z \bar{w})^2}
\]
we see that
\[
\langle A U_z k_w, U_z k_w \rangle = \langle A k_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle.
\]
We shall also make use of the power series representation of the normalised Bergman kernel, precisely
\[
k_z(w) = (1 - |z|^2) \sum_{m=0}^{\infty} (m + 1) z^m w^m.
\]
For \( z \in \Delta \) we have
\[
\tilde{A}(z) = (1 - |z|^2)^2 \sum_{m,n=0}^{\infty} (m + 1)(n + 1) \langle A w^m, w^n \rangle \bar{z}^m z^n
\]
by first multiplying both sides (6) by \( A \) and then take the inner product with \( k_z \). Thus we have immediately that
\[
\tilde{A_z}(v) = (1 - |v|^2)^2 \sum_{m,n=0}^{\infty} (m + 1)(n + 1) \langle A_z w^m, w^n \rangle \bar{v}^m v^n
\]
Fix $r \in (0,1)$ and multiply both sides (5) by $\frac{\overline{\nu}}{(1-|\nu|^2)^2}$ and integrate over $r\Delta$ to obtain

$$
\int_{r\Delta} \frac{\tilde{A}(\varphi_z(v))\overline{\nu}}{(1-|\nu|^2)^2} d\lambda(v) = \sum_{m,n=0}^{\infty} (m+1)(n+1) \langle A_z w^m, w^n \rangle \int_{r\Delta} \overline{\nu}^{m+p} v^n d\lambda(v) = \sum_{m=0}^{\infty} (m+1) \langle A_z w^m, w^{m+p} \rangle r^{2m+2p+2} = r^{2p+2}(\langle A_z 1, w^p \rangle + \sum_{m=1}^{\infty} (m+1) \langle A_z w^m, w^{m+p} \rangle r^{2m}).
$$

We also have that for all $w \in \Delta$ kept fixed, $\varphi_z(w) \to \partial \Delta$ as $z \to \partial \Delta$, thus $\tilde{A}(\varphi_z(w)) \to 0$ as $z \to \partial \Delta$. Letting $z \to \partial \Delta$, the left side of the equation above tends to 0, since the integrand is bounded by $\|A\| \|\overline{\nu}\| w^m \|w^{m+p}\| r^{2m}$ which is independent of $v$ and $z$ which justifies the passage of the limit. Dividing the right side of the equation by $r^{2p+2}$ we conclude that

$$
\langle A_z 1, w^p \rangle + \sum_{m=1}^{\infty} (m+1) \langle A_z w^m, w^{m+p} \rangle r^{2m} \to 0
$$

as $z \to \partial \Delta$ for each $r \in (0,1)$. Note that

$$
|\sum_{m=1}^{\infty} (m+1) \langle A_z w^m, w^{m+p} \rangle r^{2m}| \leq \sum_{m=1}^{\infty} (m+1) \|A\| \|w^m\| \|w^{m+p}\| r^{2m} \leq \|A\| \sum_{m=1}^{\infty} r^{2m} = \|A\| \frac{r^2}{1-r^2}.
$$

Thus, given $\epsilon > 0$, we can choose $r = r(\epsilon) \in (0,1)$ such that

$$
|\sum_{m=1}^{\infty} (m+1) \langle A_z w^m, w^{m+p} \rangle r^{2m}| < \frac{\epsilon}{2}
$$

for all $z \in \Delta$. Fix such an $r$ then there exist $R \in (0,1)$ such that for $R < |z| < 1$, we have

$$
\frac{1}{r^{2p+2}} \int_{r\Delta} \frac{\tilde{A}(\varphi_z(v))\overline{\nu}}{(1-|\nu|^2)^2} dA(v) < \frac{\epsilon}{2}
$$

and hence

$$
|\langle A_z 1, v^n \rangle| < \epsilon.
$$

**Theorem 6** Let $f \in L^1(\Delta)$. Suppose that

1. $T_f$ bounded on $L^2_0(\Delta)$;
2. $\overline{f}(z) \to 0$ as $z \to \partial \Delta$.

Then $\|T_{f \circ \varphi_z}1\|_q \to 0$ as $z \to \partial \Delta$ for $1 \leq q \leq 2$.

Proof. It follows from property 1 and Lemma 1 that $\sup_{z \in \Delta} \|T_{f \circ \varphi_z}1\|_2 < \infty$. From Theorem 5 we see that $T_{f \circ \varphi_z}1 \to 0$ weakly in $L^2_\alpha(\Delta)$ as $z \to \partial \Delta$. Thus $T_{f \circ \varphi_z}1 \to 0$ uniformly on any compact subsets of $\Delta$, e.g. on $r\Delta$, $r \in (0,1)$. First write:

$$\|T_{f \circ \varphi_z}1\|_q^q = \int_{r\Delta} |T_{f \circ \varphi_z}1(w)|^q d\lambda(w) + \int_{\Delta/r\Delta} |T_{f \circ \varphi_z}1(w)|^q d\lambda(w).$$

We next apply Hölder’s inequality to the second integral and we obtain:

$$\int_{\Delta/r\Delta} |T_{f \circ \varphi_z}1(w)|^q d\lambda(w) \leq \left\{ \int_{\Delta/r\Delta} |T_{f \circ \varphi_z}1(w)|^2 d\lambda(w) \right\}^{\frac{q}{2}} (1 - r^2)^{\frac{2-q}{2}}$$

This yields the inequality:

$$\|T_{f \circ \varphi_z}1\|_q^q \leq \int_{r\Delta} |T_{f \circ \varphi_z}1(w)|^q d\lambda(w) + \|T_{f \circ \varphi_z}1\|_2^q (1 - r^2)^{\frac{2-q}{2}}$$

where $C = \sup_{z \in \Delta} \|T_{f \circ \varphi_z}1\|_2^q < \infty$ according to property 2.

Now, given $\epsilon > 0$, choose $r = r_\epsilon \in (0,1)$ such that $C(1 - r^2)^{\frac{2-q}{2}} < \frac{\epsilon}{2}$. Fix such an $r$; then there exists $R = R(\epsilon) \in (0,1)$ such that for $R < |z| < 1$, we have:

$$\int_{r\Delta} |T_{f \circ \varphi_z}1(w)|^2 d\lambda(w) < \frac{\epsilon}{2}.$$ 

Hence, for all $\epsilon > 0$, there exists $R = R(\epsilon) \in (0,1)$ such that $R < |z| < 1$, we have:

$$\|T_{f \circ \varphi_z}1\|_q^q < \epsilon.$$

In other words, $\lim_{z \to \partial \Delta} \|T_{f \circ \varphi_z}1\|_q = 0$.

5 Proof of Theorem 2

Let $h \in L^2_\alpha(\Delta)$. Then for $f \in L^1(\Delta, \lambda)$ such that $T_f$ is bounded on $L^2_\alpha(\Delta)$, we get:

$$(T_f h)(v) = \langle T_f h, K_v \rangle = \langle h, T^*_f K_v \rangle.$$ (9)

But according to assertions 3 and 4 of Lemma 2 we have:

$$(T^*_f K_v)(u) = T_{f^*}K_u(v).$$ (10)
Substituting (10) in (9) gives:

\[
(T_f h)(v) = \int_\Delta h(u)(T_f K_u)(v) d\lambda(u) = \int_\Delta h(u)(T_f K_u)(v) d\lambda(u).
\]

For \( r \in (0, 1) \), define an operator \( T_f^r \) on \( L_2^a(\Delta) \) by

\[
(T_f^r h)(v) = \int_{r\Delta} h(u)(T_f K_u)(v) d\lambda(u).
\]

Then \( T_f^r \) is an integral operator with kernel \((T_f K_u)(v)\chi_{r\Delta}(u)\). This operator is a Hilbert Schmidt operator, since

\[
\int_{r\Delta} \int_\Delta |(T_f K_u)(v)\chi_{r\Delta}(u)|^2 d\lambda(v) d\lambda(u) = \int_{r\Delta} |T_f K_u|^2 d\lambda(v) \\
\leq \|T_f\|^2 \int_{r\Delta} \frac{1}{(1-|u|^2)^2} d\lambda(u)
\]

which is finite since \( T_f \) is bounded on \( L_2^2(\Delta) \) and the last integral is over a compact set. This proves that \( T_f^r \) is a compact operator on \( L_2^2(\Delta) \).

Furthermore, we only need to show that

\[
\|T_f - T_f^r\|^2 \to 0 \text{ as } r \to 1^-
\]

where \( \|T_f - T_f^r\| \) denotes the norm operator of \( T_f - T_f^r \) as an operator from \( L_2^a(\Delta) \) to itself. Now, for \( h \in L_2^a \) we have

\[
(T_f - T_f^r)h(v) = \int_\Delta \chi_{\Delta/r\Delta}(u)h(u)(T_f K_u)(v) d\lambda(u).
\]

This implies \( T_f - T_f^r \) extends to an integral operator on \( L^2(\Delta, d\lambda) \) with kernel

\[
(T_f K_u)(v)\chi_{\Delta/r\Delta}(u).
\]

By Schur’s lemma, if there exist a positive measurable function \( g \) on \( \Delta \) and constants \( c_1, c_2 \) such that

\[
\int_\Delta |(T_f K_u)(v)\chi_{\Delta/r\Delta}(u)| g(v) d\lambda(v) \leq c_1 g(u) \quad (11)
\]

for all \( u \in \Delta \) and

\[
\int_\Delta |(T_f K_u)(v)\chi_{\Delta/r\Delta}(u)| g(u) d\lambda(u) \leq c_2 g(v) \quad (12)
\]
for all \( v \in \Delta \), then
\[
\|T_f - T_f^{[r]}\|_2^2 \leq c_1 c_2.
\]

For \( \epsilon > 0 \), take \( g(v) = (K_v(v))^\epsilon \). Then using the same argument as in the prove of Theorem 1 we see that the left hand side of (11) can be written as
\[
\chi_{\Delta/r\Delta}(u)(S(P(h \circ \varphi_u))(u).
\]

Now for \( 1 \leq q < 2 \) and for \( 0 < \epsilon < \frac{1}{2q} \) where \( q' \) is the conjugate exponent of \( q \), we obtain from Lemma 3 that there exists a constant \( C > 0 \) such that
\[
\chi_{\Delta/r\Delta}(u)(S(P(f \circ \varphi_u))(u) \leq C \chi_{\Delta/r\Delta}(u)\|P(f \circ \varphi_u)\|_q(K_v(u))^\epsilon.
\]

This gives estimate (11) with \( c_1 = C \sup_{|u| > r} \|P(f \circ \varphi_u)\|_q \).

Similarly we get that the left side of (12) is atmost
\[
C \sup_{v \in \Delta} \|P(f \circ \varphi_v)\|_2 g(v).
\]

This gives estimate (12) with \( c_2 = C \sup_{v \in \Delta} \|P(f \circ \varphi_v)\|_2 \).

Finally, since \( \min(\frac{1}{2q'}, \frac{1}{3}) = \frac{1}{2q'} \), the estimates (11) and (12) both for \( g(v) = (K_v(v))^\epsilon \) with \( \epsilon < \frac{1}{2q} \). By Schur’s lemma, we therefore get:
\[
\|T_f - T_f^{[r]}\|_2^2 \leq c_1 c_2
\]

where
\[
c_1 = C \sup_{|u| > r} \|P(f \circ \varphi_u)\|_q \quad \text{and} \quad c_2 = C' \sup_{v \in \Delta} \|P(f \circ \varphi_v)\|_2.
\]

Since by Theorem 4, \( c_1 \to 0 \) as \( r \to 1 \) and \( c_2 < \infty \), the conclusion
\[
\|T_f - T_f^{[r]}\|_2^2 \to 0 \quad \text{as} \quad r \to 1^-
\]

follows.

**Acknowledgements** I will wish to thank Professor David Békollé for introducing me to the area of Toeplitz operators and for his constant assistance during this work. My gratitude also goes to Lars Filipsson for the useful discussions we had and also to the Department of Mathematics, Stockholm University, for their hospitality during my stay in Stockholm.
References

[1] Axler, S., Zheng, D.: *Compact operators via Berezin transforms*, Indiana University Mathematical Journal 47 (1998), 387-400.

[2] Berger, C., Coburn, L., Zhu, K.: *Function theory on Cartan domains and the Berezin-Toeplitz symbol calculus*, Amer. J. Math. 110 (1988), 921-953.

[3] Luecking, D.: *Multipliers of Bergman spaces into Lebesgue spaces*, Proc. Edinburgh Math. Soc. (2) 29 (1986), no 1, 125-131.

[4] D. Luecking.: *Characterisation of certain classes of Hankel operators on Bergman spaces of the unit disk*, J. funct. Anal. 110 (1992), 271-274.

[5] Miao, J., Zheng, D.: *Compact operators on Bergman spaces*, Integr. Equ. and Oper. Theory 48 (2004), 61-79.

[6] Zhu, K.: *Operator theory in function spaces*, Marcel Dekker, New York (1990).

[7] Zhu, K.: *Positive Toeplitz operators on weighted Bergman spaces of symmetric domains*, Journal of Operator Theory 20 (1988), vol.2, 329-357.

[8] Zorboska, N.: *Toeplitz operators with BMO symbols and the Berezin transform*, International Journal of Mathematics and Mathematical Sciences 46 (2003), 2926-2945.

Address
Agbor Dieudonne Agbor
Dept. of Mathematics,
Faculty of Science,
University of Buea,
P.O BOX 63, Buea
Cameroon