Generalizations of gravitational Born-Infeld type lagrangians are investigated. Phenomenological constraints (reduction to Einstein-Hilbert action for small curvature, spin two ghost freedom and absence of Coulomb like Schwarzschild singularity) select one effective lagrangian whose dynamics is dictated by the tensors $g_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ (not $R_{\mu\nu}$ or the scalar $R$).

I. INTRODUCTION.

There are numerous suggestions in the literature for modification of the classical Einstein Hilbert (EH) action of general relativity:

$$S_E = \frac{m^2_{Pl}}{16\pi} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right)$$  \hspace{1cm} (1)

where $R$ is the curvature scalar, $g = \det||g_{\mu\nu}||$ is the determinant of the metric tensor, $\Lambda$ the cosmological constant and $m_{Pl}$ is the Planck mass. Many modification attempts (at least in 4-dimensional space-time) are based on the same structure of the action integral with the addition to the Einstein term of some scalar functions of $R$ and/or of combinations of the Ricci, $R_{\mu\nu}$, and/or Riemann, $R_{\mu\nu\rho\sigma}$ tensors ($\int d^4x \sqrt{-g} L(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma})$). Usually, but not necessarily, one considers quadratic in curvature terms proportional to $R^2$, $R_{\mu\nu} R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$. Higher order or even non-local terms may appear as a result of quantum corrections, see e.g. the book [1]. Such a form of the Lagrangian density, i.e. a scalar function multiplied by the determinant of metric tensor, is dictated by the demand of invariance of the action with respect to general coordinate transformation. However, this is not the only way to ensure this invariance. In fact we can build many different unconventional invariants using the properties of the Levi Civita tensor

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g} e_{\mu\nu\rho\sigma}$$  \hspace{1cm} (2)

where $e_{\mu\nu\rho\sigma}$ is the Levi Civita pseudo tensor (or permutation operator, $e_{0123} = 1$), a tensorial density of weight $w = 1$ (note that $e^{\mu\nu\rho\sigma}$ is a tensorial density of weight $w = -1$).  

The determinant of a second rank tensor can be included as one of such cases $^1$. It is easy to show that, taking a generic two index tensor $V_{\mu\nu}$, we can build the following scalar densities of weight $w$:

$$\text{det}||V_{\mu\nu}|| \rightarrow w = -2$$

$$\text{det}||V_{\mu\nu}^\alpha|| \rightarrow w = 2$$

$$\text{det}||V_{\mu}^\nu|| \rightarrow w = 0$$

In this spirit, Born and Infeld (BI) proposed for the electromagnetic action the action [2,11]

$$\int d^4x \sqrt{\text{det}(g_{\mu\nu} + \lambda F_{\mu\nu})}$$  \hspace{1cm} (3)

while, Eddington [3] indicated, as purely affine gravitational action, the term

$$\int d^4x \sqrt{\text{det}||R_{\mu\nu}(\Gamma)||}$$  \hspace{1cm} (4)

whose possible generalizations are analyzed in ref. [4].

Taking into account only a purely gravitational theory without any matter content and considering a purely metric approach, we have at our disposal:

- one scalar $R = g_{\mu\nu} g^{\sigma\rho} R_{\mu\nu\rho\sigma}$
- two (symmetric) tensors : $R_{\mu\nu}$, $g_{\mu\nu}$
- two 4-index tensors: $R_{\mu\nu\rho\sigma}$, $\epsilon_{\mu\nu\rho\sigma}$

We can also build some 4-index combinations of $R_{\mu\nu}$, and $g_{\mu\nu}$, with the same symmetry properties of $R_{\mu\nu\rho\sigma}$:

$$\hat{g}_{\alpha\beta\gamma\delta} \equiv g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}$$  \hspace{1cm} (5)

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv R_{\alpha\gamma\beta\delta} - R_{\alpha\gamma\delta\beta} - R_{\alpha\delta\gamma\beta} + R_{\alpha\delta\beta\gamma}$$  \hspace{1cm} (6)

$$\tilde{R}_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\delta\gamma} - R_{\alpha\gamma\beta\delta} + R_{\alpha\gamma\delta\beta}$$  \hspace{1cm} (7)

All these 4-index tensors are not independent, in fact in four dimension:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \hat{R}_{\mu\nu\rho\sigma} - \frac{R}{6} \hat{g}_{\mu\nu\rho\sigma} + C_{\mu\nu\rho\sigma}$$  \hspace{1cm} (8)

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor or Conformal tensor.

In ref. [5], Deser and Gibbons proposed the general covariant action:

$$S = \int d^4x \sqrt{\text{det}(g_{\mu\nu} + \lambda R_{\mu\nu} + X_{\mu\nu})}$$  \hspace{1cm} (9)

where $X_{\mu\nu}$ contains terms of second or higher order in the curvature and formulated the minimum physical request that such a theory should satisfy:
1) Reduction to EH action for small curvature;
2) Ghost freedom;
3) Regularization of some singularities (for example the Coulombian like in the Schwarzschild case);
4) Supersymmetrizability †.

All these points will be reanalyzed in chapter V where a candidate lagrangian will be selected.

In ref. [6] there is an extensive analysis of the cosmological behaviors of actions like (9) in Friedmann Robertson Walker (FRW) background. A straightforward generalization of action (9) can be written in the general form of “determinant-action”:

\[ S_{\text{det}} = \int d^4 x \sqrt{\text{det}[G_{\mu\nu}(g_{\alpha\beta}, R, R_{\alpha\beta}, R_{\alpha\beta\gamma})]} \]  

(10)

where \( G_{\mu\nu} \) is a two index covariant tensor, combination of \( g_{\alpha\beta}, R, R_{\alpha\beta} \) and \( R_{\alpha\beta\gamma} \).

Being \( \text{det}[G] \) not necessarily positive definite, \( S_{\text{det}} \) can become imaginary in some portion of the \( g_{\mu\nu} \) space.

A possible solution is given by matrices \( G \) that are product of two other matrices \( M \) and \( N \):

\[ G_{\mu\nu} = M_{\mu}^{\alpha} g_{\alpha\beta} N_{\nu}^{\beta} \]  

(11)

so that

\[ \text{det}[G_{\mu\nu}] = \text{det}[M_{\mu}^{\alpha}] \text{det}[N_{\nu}^{\beta}] \]  

(12)

and in the case \( M_{\mu\nu} = N_{\mu\nu} \) we have \( \text{det}[G] = \text{det}[M]^{2} \) giving

\[ \int d^4 x \sqrt{\text{det}[G]} = \int d^4 x \sqrt{-g} \text{det}[M]\]  

(13)

and the action becomes automatically polynomial §.

Using the properties of the Levi Civita pseudo tensor \( e \), we can rewrite the determinat-action (10) as:

\[ \int d^4 x \sqrt{\frac{1}{4 d^4} e \tau G_{\mu_1 \nu_1} G_{\mu_2 \nu_2} G_{\mu_3 \nu_3} G_{\mu_4 \nu_4}} \]  

(14)

where \( e \tau \) is defined in appendix. This form will be our guideline for the new generalizations of BI type gravity in the next chapter.

II. GENERALIZED BORN-INFELD GRAVITY

The first generalization of the action (14) is obtained inserting an arbitrary number \( n \) of Levi Civita tensors:

\[ \int d^4 x (\frac{e}{2} G_{\mu_1 \nu_1} G_{\mu_2 \nu_2} G_{\mu_3 \nu_3} G_{\mu_4 \nu_4})^{1/2} \rightarrow \int d^4 x (\frac{e}{n} G_{\mu_1 \nu_1} G_{\mu_2 \nu_2} G_{\mu_3 \nu_3} G_{\mu_4 \nu_4})^{1/n} \]  

(15)

then, also the \( G \) tensors, can be taken independents at each insertion:

\[ \int d^4 x (\frac{e}{n} G_{\mu_1 \nu_1} G_{\mu_2 \nu_2} G_{\mu_3 \nu_3} G_{\mu_4 \nu_4})^{1/n} \rightarrow \int d^4 x (\frac{e}{n} G_{\mu_1 \nu_1} G_{\mu_2 \nu_2} G_{\mu_3 \nu_3} G_{\mu_4 \nu_4})^{1/n} \]  

(16)

We stress that \( e \mu\nu\rho\sigma \), being a tensor density of weight \( w = -1 \) generates scalar densities that need to be corrected taking the appropriate power \( \frac{1}{n} \) for the full expression. This allow us also to introduce directly into the lagrangian density higher index tensors, like \( R_{\mu\nu\rho\sigma} \), generalizing the determinant operation to tensors with more than two indices.

A generic term in \( d \)-dimensions, having \( n \) times \( e\mu\nu\rho\sigma \), \( r \) times \( g_{\mu\nu} \), \( s \) times \( R_{\mu\nu} \), and \( t \) times \( R_{\mu\nu\rho\sigma} \) reads **

\[ \int d^4 x M^d \frac{2(\pm 1)}{n} (\frac{e}{n} g_{\mu\nu} g_{\rho\sigma} R_{\mu\nu} R_{\rho\sigma})^{1/n} \]  

(17)

where the “conservation of the number of indices” requires \( d n = 2 r + 2 s + 4 t \) and \( M \) is a mass scale. The range of variations of \( t \) is : \( 0 \leq t \leq d n/4 \), (when \( r = 0 \) respectively) and the mass scale coefficient varies from \( M^{d - \frac{4}{n}} \) to \( M^{\frac{d}{n}} \).

If we consider conformally flat spaces where \( C_{\mu\nu\rho\sigma} = 0 \) we can simplify the general expression to

\[ \int d^4 x M^d \frac{2(\pm 1)}{n} (\frac{e}{n} g_{\mu\nu} g_{\rho\sigma} R_{\mu\nu} R_{\rho\sigma})^{1/n} \]  

(18)

where now \( d n = 2 r + 2 s \). In this case \( 0 \leq s \leq d n/2 \), (for \( r = d n/2 \) and for \( r = 0 \) respectively) and the dimensionality of the mass coefficient runs from \( M^d \) for \( s = 0 \) to \( M^0 \) for \( s = 2 n \).

We note also that actions with an odd number of indices violate parity and in the FRW or Schwarzschild backgrounds (that are our physical “toy models”), these terms are exactly zero.

We will also reduce our analysis to the four dimensional space time, \( d = 4 \), leaving the extra dimensional spaces to future investigations.

Due to the fact the the number of independent operators is growing very fast with \( n \), we will concentrate on the \( n = 2 \) case and discuss the generalizations with \( n > 2 \) in general terms only.

†This requirement results quite stringent and is probably implemented if gravity descends from String/M-Theory [8,10,11].

‡In ref [7] the case \( M_{\mu\nu} = \delta_{\mu\nu} + \lambda R_{\mu\nu} \) with the lagrangian

\[ \int d^4 x \sqrt{\text{det}[G_{\mu\nu} + 2\lambda R_{\mu\nu} + \lambda^2 R_{\mu\nu} R_{\mu\nu}]} \]  

was studied.

¶We assume a sort of minimal dimensional analysis not considering operators that saturate indices between them self, as \( R_{\mu\nu\rho\sigma} R_{\alpha\beta\rho\sigma} \) and many others. Each single tensor will saturate the respective indices only with the Levi Civita pseudo tensors (see fig.1 for \( n = 2 \) and fig.2 for \( n = 4 \)).
FIG. 1. Here we show the possible saturation patterns for the case $n = 2$. In a) we saturate with four 2-index symmetric tensors; in b) we saturate with two 2-index tensors and one 4-index tensor; in c) we saturate with two 4-index tensors. The black points indicate the four indices for each $\epsilon^{\mu_1 \nu_1 \rho_1 \sigma_1}$ pseudo tensor $(i=1,2)$.

A. Two $\epsilon$

Here we analyze of the case with only two Levi Civita pseudo tensors, $n = 2$; the counting rule of eq.(17) fix the structure

$$
\int d^4 x M^{4-t-s} \left( \sum_{n=2}^{4} \sum_{0 \leq r, s \leq 4} \frac{R^r R^s}{R^2} \right)^{1/2} \left( \begin{array}{c}
\mu \\
\nu \\
\rho \\
\lambda
\end{array} \right) \left( \begin{array}{c}
\mu \\
\nu \\
\rho \\
\lambda
\end{array} \right) = 4! \, \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda}
$$

(21)

(where $\epsilon$ product is defined in appendix).

2. Operators of dimension 2: $gggR$ ($r=3$, $s=1$) and $ggRR$ ($r=2$, $t=1$)

$$
\epsilon \epsilon \epsilon \epsilon \, g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} g_{\mu_3 \nu_3} g_{\mu_4 \nu_4} = * \hat{G} \times * \hat{G} \times * \hat{G} \times * \hat{G} = 4! \, \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda}
$$

where $\epsilon$ product is defined in appendix.

3. Operators of dimension 4: $ggRR$ ($r=2$, $s=2$), $gRRR$ ($r=1$, $s=1$, $t=1$) and $RRRR$ ($t=2$)

$$
\epsilon \epsilon \epsilon \epsilon \, g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} g_{\mu_3 \nu_3} g_{\mu_4 \nu_4} = * \hat{G} \times * \hat{G} \times * \hat{G} \times * \hat{G} = 4! \, \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda} \times \epsilon_{\mu \nu \rho \lambda}
$$

FIG. 2. The black points represent the indices of the four $\epsilon^{\mu_1 \nu_1 \rho_1 \lambda_1}$ pseudo tensors $(i=1,2,3,4)$ while the colored dashed curves are the way each $G_{\mu_1 \nu_1 \rho_1 \lambda_1}$ tensor saturates the respective pseudo tensors. The first case a) is describing eq.(29) while the case b) is for eq.(30).

B. Four $\epsilon$

When there are four $\epsilon^{\mu \nu \rho \sigma}$ pseudo tensors, the large number of possible saturations generates many invariants. The study or the classification of all of them is beyond the scope of the present letter. My purpose is a short analysis of two possible choices that, from my point of view, are the most symmetric ones.

We define respectively a “determinant” action for a 4-index tensor as

$$
\int d^4 x \left( \det |G_{\mu \nu \rho \sigma}| \right)^{1/4} = \int d^4 x \left( \epsilon \epsilon \epsilon \epsilon \, G_{\mu_1 \nu_1 \rho_1 \sigma_1} \right) \times \epsilon \epsilon \epsilon \epsilon \, G_{\mu_2 \nu_2 \rho_2 \sigma_2} \times \epsilon \epsilon \epsilon \epsilon \, G_{\mu_3 \nu_3 \rho_3 \sigma_3} \times \epsilon \epsilon \epsilon \epsilon \, G_{\mu_4 \nu_4 \rho_4 \sigma_4}
$$

(29)

where $\epsilon \epsilon \epsilon \epsilon \, G$ is defined in appendix, and another interesting saturation pattern given by

$$
\int d^4 x \left( \det |G_{\alpha \beta \gamma \delta}| \right)^{1/4} = \int d^4 x \left( Tr [\hat{G} \hat{G} \hat{G} \hat{G} \hat{G}] \right)^{1/4}
$$

(30)

already investigated in the paper [8] with $\hat{G} = (1 + k \hat{R}) \hat{G} \hat{R} + \hat{R} + \hat{R} +$ higher derivative terms.
In fig.2 it is shown the saturation path corresponding respectively to eq. (29) and to eq. (30).

In order to understand the different physical implications from these two choices, in section III B, we will analyze the respective actions obtained working with $G = M^2 \hat{G} + \mathcal{R}$ in a FRW metric background. This simple exercise can give the feeling of physical implications coming from “higher order” BI generalizations.

III. SMALL CURVATURE LAGRANGIAN EXPANSION

In this chapter we will discuss the small curvature expansion of the most general gravitational lagrangian obtained combining the eqs. (20-28) for $n = 2$ while for $n = 4$ we discuss only the special case with $G = M^2 \hat{G} + \mathcal{R}$ in FRW space.

A. Two $e$ case

The most general combination of eqs. (20-28) gives

$$M^8(1 + \frac{\alpha_1}{M^2} R + \frac{1}{M^4}(\beta_1 R^2 + \beta_2 [R]^2 + \beta_3 [\mathcal{R}]^2) + \frac{1}{M^6}(\lambda_1 R^4 + \lambda_2 R [\mathcal{R}]^2 + ...) + \frac{\gamma}{M^8}(\det||R_{\alpha\beta}||))$$

(31)

Where $M$ is a dimensional mass coefficient and $\alpha_1, \beta_1, \lambda_1$ and $\gamma$ are free parameters.

In order to have some insight about the asymptotic small curvature behavior we can expand around the leading operator, meaning the operator with the lowest dimensionality.

If for example the leading operator is the dimension zero one, then the expanded lagrangian becomes

$$\int d^4x \sqrt{-g} \left( M^4 + \frac{\alpha_1}{2} M^2 R + \left( \frac{\beta_1}{2} - \frac{\beta_2}{4} \right) R^2 + \frac{\beta_2}{2} [R]^2 + \frac{\beta_3}{2} [\mathcal{R}]^2 + ... \right)$$

(32)

and if we want to take care of the ghost problem we need the constraint $-4 \beta_3 = \beta_2$ (see section V).

In the case the leading operator is the two dimensional one, we get

$$\int d^4x \sqrt{-g} \left( M^2 \sqrt{\mathcal{R}} + \frac{M}{2 \sqrt{\mathcal{R}}} (\beta_1 R^2 + \beta_2 [R]^2 + \beta_3 [\mathcal{R}]^2 + ...) \right)$$

(33)

For leading dimension four operator, in order to have the leading EH term, we need $\beta_2 = \beta_3 = 0$ so that

$$\int d^4x \sqrt{-g} \left( M^2 R + \frac{M}{2 R} (\beta_1 R^2 + \beta_2 [R]^2 + \beta_3 [\mathcal{R}]^2 + ...) \right)$$

(34)

etc...

In order to get a phenomenological acceptable model we will follow the usual attitude to cancel “by hand” the cosmological constant with the following trick

$$\int d^4x \sqrt{-g} M^4 \left( \sqrt{1 + \frac{\alpha}{M^2} R + \frac{1}{M^4} - 1} \right)$$

(35)

with no justification from the symmetry point of view.

B. Four $e$ case

As already discussed before we decided to work out this case with two models in FRW metric. Taking $G = M^2 \hat{G} + \mathcal{R}$ the small curvature expansion is around a non zero cosmological constant value for both the eqs. (29,30).

The next to leading operators are for eq.(29)

$$\int d^4x \sqrt{-g} M^2 e e e \hat{G} \hat{G} \mathcal{R} \sim \int dt a^3 M^2 (3 H^2 + H') = 0$$

(36)

while for eq. (30)

$$\int d^4x \sqrt{-g} M^2 \hat{G} * \hat{G} * \hat{G} * \mathcal{R} \sim \int d^4x \sqrt{-g} M^2 R$$

(37)

The next to next to leading operators correspond to

$$\int d^4x \sqrt{-g} \hat{G} \hat{G} \mathcal{R} \mathcal{R} \sim \int dt a^3 H^4$$

(38)

from eq.(29) while eq.(30) generates

$$\int d^4x \sqrt{-g} Tr[\hat{G} * \hat{G} * \hat{G} * \mathcal{R}] = \int d^4x \sqrt{-g} \mathcal{E} = 0$$

(39)

It is evident that for this particular choice of $G$ eq.(29) fails to match EH at small curvature and predict ghosts (see next chapter), while eq.(30) fit the EH constraint and results ghost free (see next chapter), being in such a way an interesting phenomenological theory of gravity (see [8]).

IV. SYMMETRIES

The existence of some guiding symmetry principle to build our action will strongly improve the prediction of our approach. We know for example that the effective gravity lagrangians derived by String Theory to the fourth order in curvature expansion are ghost free [15]. This it means that in second curvature order we have the Gauss Bonnet combination ($\mathcal{E} = R^2 - 4[R]^2 + [\mathcal{R}]^2$).
In higher order we find cubic corrections for bosonic string theories while the supersymmetric extensions predict zero cubic and non zero quartic corrections \([16]\). Because string inspired effective lagrangians are computed evaluating on shell graviton amplitudes \(R_{\mu\nu} = 0\), the terms which contains at least twice the Ricci tensor or the Ricci scalar are non properly included.

Supersymmetric BI type generalizations of Weyl supergravity action are given in \([10]\).

Since there is no decisive hint about the correct string model, we will attempt a phenomenological approach guessing some ad hoc principles and testing the possible implications.

Here we will introduce a sort of “selection rule” such that only some tensors generate the gravitational dynamics. Our possible choices are in between \(\mathcal{G}, \hat{R}, \hat{\mathcal{R}}\) and \(\mathcal{R}\). If for example, only the tensor \(\mathcal{R}_{\mu\nu,\rho} (t = 2)\) is present, the lagrangian results:

\[
\int d^4x \sqrt{-g} M^2 \sqrt{R^2 - 4|R|^2 + \mathcal{R}^2} \tag{40}
\]

In the case with only the tensors \(\hat{\mathcal{G}}_{\mu\nu,\rho}\) and \(\mathcal{R}_{\mu\nu,\rho}\) \((r = 4) + (r = 2, t = 1) + (t = 2)\) we get:

\[
\int d^4x \sqrt{-g} M^4 \sqrt{\hat{\mathcal{G}} \hat{\mathcal{G}} + \frac{\alpha'}{M^2} \hat{\mathcal{G}} \hat{\mathcal{R}} + \frac{\beta'}{M^4} \hat{\mathcal{R}} \hat{\mathcal{R}}} = \int d^4x \sqrt{-g} M^4 \sqrt{1 + \frac{\alpha''}{M^2} R + \frac{\beta''}{M^4} \mathcal{E}} \tag{41}
\]

and it fits exactly the constraints for ghost freedom and the EH asymptotic (see cap.V).

While in the opposite case where no \(\mathcal{R}_{\mu\nu,\rho}\) can enter \((r = 4) + (r = 3, s = 1) + (r = 2, s = 2) + (r = 1, s = 3) + (s = 4)\), we have the lagrangian:

\[
\int d^4x \sqrt{-g} M^4 \sqrt{1 + \frac{\alpha}{M^2} R + \frac{\beta}{M^4} (R^2 - |R|^2) + \frac{\delta}{M^6} (R^3 - 3 R |R|^2 + 2 |R|^3) + \frac{\gamma}{M^8} \det||R_{\mu\nu}||} \tag{42}
\]

where ghosts show up (see cap.V).

When only \(\mathcal{R}_{\mu\nu,\rho}\) and \(R_{\mu\nu}\) tensors are present \((t = 2) + (t = 1, s = 2) + (s = 4)\):

\[
\int d^4x \sqrt{-g} M^4 (\mathcal{E}) + M^2 (...) + \gamma \det||R_{\mu\nu}|| \tag{43}
\]

Finally we have only one operator with \(R_{\mu\nu}\) tensors \((s = 4)\) that turns out to be also local Weyl invariant:

\[
\int d^4x \sqrt{\det||R_{\mu\nu}||} \tag{44}
\]

with no EH matching.

V. PHENOMENOLOGICAL MODEL

The only phenomenologically interesting lagrangian coming from the above selection rules is certainly \(^{11}\)

\[
\int d^4x \sqrt{-g} M^4 \left( 1 - \sqrt{1 - \frac{\alpha R}{M^2} + \frac{\beta}{M^4} (R^2 - 4|R|^2 + \mathcal{R}^2)} \right) \tag{45}
\]

where we have applied the cancellation mechanism of cosmological constant.

At this point, we will reanalyze more carefully the above lagrangian to the light of the physical criteria suggested by Deser and Gibbons in [5].

1. EH small curvature limit

The small curvature limit of eq.(45) results:

\[
\int d^4x \sqrt{-g} \left( \frac{\alpha M^2}{2} R + \frac{\alpha^2}{8} R^2 - \frac{\beta}{4} (R^2 - 4|R|^2 + \mathcal{R}^2) + \frac{\alpha R}{16 M^2} \right) \tag{46}
\]

So, the requirement of a correct EH leading gravitational operator fix \(\frac{\alpha M^2}{2} = \frac{m_{pl}^2}{16\pi}\) while the coefficient of the \(R^2\) operator results \(\frac{\alpha^2}{8} = \frac{m_0^4}{512\pi^2 M^2}\) generating an extra scalar degree of freedom with mass \(m_0 = \sqrt{\frac{16\pi M^2}{m_{pl}}}\).

The exchange of such a scalar between two test particles changes the \(1/r\) static gravitation potential slope to \(\frac{1}{r} (1 + \frac{1}{2} e^{-m_0 r})\). Using the results of ref. [12] with the strength parameter equal to 1/3 we can obtain a lower bound on \(m_0\) of \(\sim 2 \times 10^{-2} eV\) corresponding to a value for the mass parameter \(M \geq 250\) GeV.

2. Ghost Cancellation

The analysis of the particle content in higher derivative lagrangians of the form

\[
\int d^4x \sqrt{-g} \ F[R, |R|^2, \mathcal{R}] \tag{47}
\]

shows the existence of massless gravitons plus new degrees of freedom. In general there is a massive spin zero field \((m_0\) mass\) and a massive spin two field \((m_2\) mass\) with a wrong sign of kinetic term: a ghost fields. Due

\(^{11}\)Note that the expression under square root corresponds to the Lovelock lagrangian in four dimensions \([13]\).
to the fact that the mass of such particles and their potential ghostlike may be very different around different vacuum states we give the full set of eqs that fix such a parameters around solutions characterized by a constant curvature $R = R_0$ in a maximally symmetric background [14], $R_{\mu\nu\rho\sigma} = \frac{R_0}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$.

The equations of motion that fix $R_0$ are:

$$F - \frac{R}{2}F_R - \frac{R^2}{4}F_P - \frac{R^2}{6}F_Q \bigg|_{R_0} = 0 \quad (48)$$

where $P \equiv [R]^2$ and $Q \equiv \mathcal{R}^2$ and $F_R = \frac{\partial \mathcal{F}}{\partial R}$, $F_P = \frac{\partial \mathcal{F}}{\partial P}$, $F_Q = \frac{\partial \mathcal{F}}{\partial Q}$ and so on.

To each solution of (48) it corresponds a cosmological constant [14] given by

$$\Lambda = -\frac{8\pi}{m_{pl}^2} \left( F - F_R R + \frac{1}{2} F_{RR} R^2 \right) \bigg|_{R_0} \quad (49)$$

with an effective Planck mass

$$\frac{m_{pl}^2}{16\pi} = (F - F_{RR} R) \bigg|_{R_0} \quad (50)$$

and the masses of the two extra degrees of freedom $m_0$ and $m_2$:

$$\frac{m_{pl}^2}{96\pi m_0^2} = \frac{1}{2} F_{PP} + \frac{1}{6} F_{PQ} + \frac{1}{18} F_{QQ} \bigg|_{R_0} \quad (51)$$

$$\frac{m_{pl}^2}{32\pi m_2^2} = \frac{1}{2} F_P + 2 F_Q \bigg|_{R_0} \quad (52)$$

where we always take $R = R_0$ and $P = \frac{R_0^2}{4}$, $Q = \frac{R_0^2}{16}$.

A ghost free spectrum (as required for example from string theory [15]) is realized when $m_2 \to \infty$ and this request is automatically satisfied for lagrangians of the form

$$F[R, \mathcal{R}^2 - 4[R]^2] = f[R, \mathcal{E}] = f[(*\mathcal{G}, \mathcal{R}), (\star \mathcal{R} \ast \mathcal{R})] \quad (53)$$

that fit the structure of eq.(45).

For this particular lagrangian (45) we have two possible background solutions: one flat background $R_0 = 0$ with zero cosmological constant, that corresponds to the small curvature EH limit previously described; one with $R_0 \neq 0$ (we can give analytical results only in the small $\alpha$ limit corresponding to $m \geq m_{pl}$ : $m_0^2 = 8\pi\alpha m^2 + O(\alpha^3)$, $R_0 \sim \frac{\alpha}{4m_{pl}^2} m_0^2 + O(\alpha^3)$, a non zero cosmological constant $\Lambda \sim \frac{4\pi\alpha}{m_{pl}^2} m_0^2 + O(\alpha^3)$, and an extra scalar degree of mass $m_0^2 \sim \frac{\alpha}{8m_{pl}^2} m_0^2 + O(\alpha^3)$). In fig.3 we plot in $(\alpha, \beta)$ parameter space the zero limit for $m_0^2$ and $\Lambda$. Only the region $m_0^2 > 0$ is stable, while $\Lambda$ can have both signs [17](note that only positive $\alpha, \beta$ values are allowed).

Many generalizations of EH action induce corrections to Schwarzschild metric which could have interesting consequences. In the class of Born-Infeld action there are some studies about spherically symmetric Schwarzschild solutions (see [8,9]) . In this paper we are not interested in a full analysis of new solutions because we can reduce our action to the one studied in [9] in some portion of parameter space.

Following the lines of [9], we can neglect, in the action, terms proportional to $R$ and $R_{\mu\nu}$, due to that fact that we are looking for solutions similar to the Schwarzschild one ($R_{\mu\nu} \sim 0$). Only the presence of terms proportional to the Weyl tensor can, in principle, remove the black hole singularity at the origin. This observation reduce our action to

$$\int d^4x \sqrt{-g} M^4 \left( 1 - \sqrt{1 + \frac{\beta}{2M^2} \mathcal{R}^2} \right) \quad (54)$$

and this exact form is studied in chapter four of [9] (see there for details) \textsuperscript{§§}. The main results in [9] are the existence of solutions which behave asymptotically as black holes and becomes spaces of constant $\mathcal{R}^2$ at small radii. In some portion of parameters space there is not even an event horizon with the presence of a bare mass instead of a black hole (bare in the sense that is not hidden behind an event horizon) without a naked singularity.

\textsuperscript{§§}Note that in this case we need a negative $\beta$ parameter that cancel the stability of the background $R_0 \neq 0$ solution (see chapter V 2) leaving only the $R_0 = 0$ background.
VI. CONCLUSIONS

In this paper we have discussed some generalization of determinant gravity following the steps:

\[
\int d^4 x \sqrt{\det |G_{\alpha\beta}|} = \int d^4 x \left( \frac{1}{2} e e e \right)^{1/2} \rightarrow \\
\int d^4 x \left( \frac{1}{n} G_{1} G_{2} G_{3} G_{4} \right)^{1/2} \rightarrow \\
\int d^4 x \left( \frac{1}{n} G_{1} G_{2} G_{3} G_{4} \right)^{1/n} \rightarrow \\
\int d^4 x \left( \frac{1}{n} G_{1} G_{2} G_{3} G_{4} \right)^{1/n} \rightarrow \\
\int d^4 x \left( \frac{1}{n} G_{1} G_{2} G_{3} G_{4} \right)^{1/n} \rightarrow \\
\int d^4 x \left( \frac{1}{n} G_{1} G_{2} G_{3} G_{4} \right)^{1/n} \rightarrow \\
\int \sqrt{-g} M^4 \left( 1 - \frac{\alpha R}{M^2} + \frac{\beta}{M^2} (R^2 - 4[R]^2 + R^2) \right)
\]

with \( 4n = 2r + 2s + 4t \).

Then we analyzed all the possible operators obtained in the case \( n = 2 \). Selecting as guide lines the following requests:

1) Reduction to EH action for small curvature;
2) Ghost freedom;
3) Regularization of some singularities;
4) Supersymmetrizability.

We selected the lagrangian

\[
\int d^4 x \sqrt{-g} M^4 \left( 1 - \frac{\alpha R}{M^2} + \frac{\beta}{M^2} (R^2 - 4[R]^2 + R^2) \right)
\]

which has an EH leading term in the small curvature limit, it results ghost free and for some parameter space show indications for the cancellation of the Coulomb like Schwarzschild singularity.

For what concern the possible terms with \( n \geq 4 \), there are many interesting “determinants” definitions that have to be physically investigated and to cover all of them requires some more effort.

Acknowledgements

I would like to thanks A. Dolgov, A. Riotto and in particular M. Pietroni for stimulating discussions.

VII. APPENDIX

Some definitions:

\[
e^{\mu_1 \nu_1 \rho_1 \sigma_1} e^{\nu_2 \nu_2 \rho_2 \sigma_2} \equiv e e e \quad \quad (55)
\]

\[
e^{\mu_1 \nu_1 \rho_1 \sigma_1} e^{\nu_2 \nu_2 \rho_2 \sigma_2} e^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \equiv e e e e \quad \quad (56)
\]

\[
[R]^2 = R^\mu_\nu R^\nu_\mu , \quad [R \partial R] = R^\mu_\nu \partial^\rho_\sigma R^\rho_\sigma \quad \quad (57)
\]

\[
\frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} R_{\alpha \beta \rho \sigma} \equiv \ast R^\mu_\nu \rho \sigma \quad \quad (58)
\]

\[
\frac{1}{4} \varepsilon^{\mu \nu \alpha \beta} R_{\alpha \beta \gamma \delta} \varepsilon^{\gamma \delta \rho \sigma} \equiv \ast R^\mu_\nu \rho \sigma \quad \quad (59)
\]