SQUARE FUNCTION INEQUALITY FOR OSCILLATORY INTEGRAL OPERATORS SATISFYING HOMOGENEOUS CARLESON-SJÖLIN TYPE CONDITIONS

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Abstract. We establish an improved square function inequality for a certain class of Hörmander-type oscillatory integral satisfying homogeneous Carleson-Sjölin conditions. As a consequence, we further improve the $L^p \to L^q$-local smoothing estimate with $p \in (2,6)$ for Fourier integral operators satisfying cinematic curvature conditions in dimension two. The main ingredients in the argument include multilinear oscillatory integral estimate [3] and decoupling inequality [2].

1. Introduction

In this paper, we continue the study of Fourier integral operators satisfying cinematic curvature conditions via square functions estimates [8]. We start with reviewing a few notations.

Let $Z$ and $Y$ be paracompact manifolds with $\dim Z = 3$ and $\dim Y = 2$. We consider Fourier integral operators $\mathcal{F} \in I^{\sigma - \frac{1}{4}}(Z,Y; \mathcal{C})$, where we adopted the notation in [14] and denote by $\mathcal{C}$ the canonical relation, from $T^*Y \setminus 0$ to $T^*Z \setminus 0$, which is a homogeneous, conic Lagrangian submanifold of $T^*Z \setminus 0 \times T^*Y \setminus 0$ with $\dim \mathcal{C} = 5$.

As in [14], we impose the cone conditions on $\mathcal{C}$. Given $z_0 \in Z$, let $\Pi_{T^*Y}, \Pi_{T^*z_0}Z$ and $H_Z$ be projections from $\mathcal{C}$ to $T^*Y \setminus 0$, $T^*Z \setminus 0$ and $Z$ respectively,

\begin{equation}
\begin{array}{c}
\mathcal{C} \\
\Pi_{T^*Y} \\
\Pi_Z \\
\Pi_{T^*z_0}Z
\end{array}
\quad
\begin{array}{c}
T^*Y \setminus 0 \\
Z \\
T^*Z \setminus 0
\end{array}
\end{equation}

and assume

\begin{equation}
\text{rank } d\Pi_{T^*Y} \equiv 4,
\end{equation}

\begin{equation}
\text{rank } d\Pi_Z \equiv 3.
\end{equation}

Let $\Gamma_{z_0} = \Pi_{T^*z_0}Z(\mathcal{C})$. As a consequence of (1.2)(1.3) and the homogeneity, $\Gamma_{z_0}$ is a conic subset of $T^*Z \setminus 0$. The cone condition imposed on $\mathcal{C}$ is that for every $\zeta \in \Gamma_{z_0}$, there is one principal curvature nonvanishing. We say a Fourier integral operator $\mathcal{F}$ satisfies the cinematic curvature condition, if its canonical relation $\mathcal{C}$ satisfies (1.2)(1.3) and the cone condition.

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This paper is a continuation of our previous work [8] and our purpose is to give a further improvement upon our previous result in dimension two.

**Theorem 1.** Suppose $F \in \mathcal{I}^{\sigma-\frac{1}{4}}(Z,Y; \mathcal{C})$ with $Z,Y$ as above, where the canonical relation $\mathcal{C}$ satisfies the cinematic curvature condition. Then, the following estimate holds
\[
\|Ff\|_{L^p_{\text{loc}}(Z)} \leq C \|f\|_{L^p_{\text{comp}}(Y)}, \text{ for all } \sigma < -\sigma(p),
\]
where $\sigma(p)$ is defined by
\[
\sigma(p) = \begin{cases} 
0, & 2 < p \leq 3, \\
\frac{1}{4} - \frac{3}{4p}, & 3 < p \leq 4, \\
\frac{3}{8} - \frac{5}{4p}, & 4 < p < 6.
\end{cases}
\]

The local smoothing conjecture is formulated initially by Sogge [16] and was found later to have many applications to various of interesting problems in harmonic analysis. We refer to [2, 1, 8] for more information about the motivation and backgrounds. Next, we use our main theorem to derive a few consequences as in [8].

If we write $z = (x,t)$ and let $F_t f(x) = F f(x,t)$, then we have the following maximal theorem under the above assumptions in Theorem 1.

**Corollary 1.1.** Assume that $F \in \mathcal{I}^{\sigma-\frac{1}{4}}(Z,Y; \mathcal{C})$ is a Fourier integral operator satisfying all the same conditions in Theorem 1. Then, if $I \subset \mathbb{R}$ is a compact interval and $Z = X \times I$ such that $X$ and $Y$ are assumed to be compact, then we have
\[
\|\sup_{t \in I} |F_t f(x)|\|_{L^p(X)} \leq C \|f\|_{L^p(Y)},
\]
whenever $\sigma < -\sigma(p) - \frac{1}{p}$ for all $2 \leq p \leq 6$.

Let $M$ be a smooth compact manifold without boundary of dimension $n$, equipped with a Riemmanian metric $g$ and consider the Cauchy problem
\[
\begin{cases} 
(\partial_t^2 - \Delta_g)u(t,x) = 0, & (t,x) \in \mathbb{R} \times M, \\
u(0,x) = f(x), \quad \partial_t u(0,x) = h(x),
\end{cases}
\]
where $\Delta_g$ is the Beltrami-Laplacian associated to a metric $g$. It is a well-known fact that the solution $u$ to this Cauchy problem can be written as
\[
u(x,t) = F_0 f(x,t) + F_1 h(x,t),
\]
where $F_j \in \mathcal{I}^{j-\frac{1}{4}}(M \times \mathbb{R}, M; \mathcal{C})$ with
\[\mathcal{C} = \left\{ (x,t,\xi,\tau, y, \eta) : (x,\xi) = \chi_t(y,\eta), \tau = \pm \sqrt{\sum g^{jk} \xi_j \xi_k} \right\},\]
where $\chi_t : T^* M \setminus 0 \times T^* M \setminus 0$ is the flowing for time $t$ along the Hamilton vector field $H$ associated to $\sqrt{\sum g^{jk} \xi_j \xi_k}$. As a consequence, the convexity condition is automatically verified by $\mathcal{C}$. 

Corollary 1.2. Let \( u \) be the solution to the Cauchy problem (1.7). If \( I \subset \mathbb{R} \) is a compact interval and \( \sigma < -\sigma(p) \) with \( \sigma(p) \) is given by (1.5), then we have
\[
\|u\|_{L^p_{\alpha+\sigma}(M \times I)} \leq C \left( \|f\|_{L^p_{\alpha}(M)} + \|h\|_{L^p_{\alpha-1}(M)} \right), \quad 2 \leq p \leq 6. \tag{1.9}
\]

Let \( \Sigma_{x,t} \subset \mathbb{R}^2 \) be a smooth curve depending smoothly on the parameters \( (x,t) \in \mathbb{R}^2 \times [1,2] \) and \( d\sigma_{x,t} \) denotes the normalized Lebesgue measure on \( \Sigma_{x,t} \). Following the notations in [16, 15], we may assume \( \Sigma_{x,t} = \{ y : \Phi(x,y) = t \} \) where \( \Phi(x,y) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) such that its Monge-Ampère determinant is non-singular
\[
\det \begin{pmatrix} 0 & \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial^2 \Phi}{\partial x \partial y} \end{pmatrix} \neq 0, \quad \text{when } \Phi(x,y) = t. \tag{1.10}
\]
This is referred to as Stein-Phong’s rotational curvature condition.

Define the averaging operator by
\[
Af(x,t) = \int_{\Sigma_{x,t}} f(y) a(x,y) d\sigma_{x,t}(y), \tag{1.11}
\]
where \( a(x,y) \) is a smooth function with compact support in \( \mathbb{R}^2 \times \mathbb{R}^2 \).

If we let \( \mathcal{F} : f(x) \mapsto Af(x,t) \), then \( \mathcal{F} \) is a Fourier integral operator of order \(-1/2\) with canonical relation given by
\[
\mathcal{C} = \{ (x,t,\xi,\tau,y,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi) \}, \tag{1.12}
\]
where \( \chi_t \) is a local symplectomorphism, and the function \( q \) is homogeneous of degree one in \( \xi \) and smooth away from \( \xi = 0 \). Moreover, the rotational curvature condition holds if and only if
\[
\begin{cases}
q(x,\Phi(x,y),\Phi'_x(x,y)) \equiv 1 \\
\text{corank } q''_{\xi \xi} \equiv 1.
\end{cases} \tag{1.13}
\]
In particular, the cinematic curvature condition is fulfilled and one has the following result by Theorem 1.

Corollary 1.3. Let \( A_t \) be an averaging operator defined in (1.11) with \( \Sigma_{x,t} \) satisfying the geometric conditions described as above. Then there exists a constant \( C \) depending on \( p \) such that if \( f \in L^p(\mathbb{R}^2) \), we have
\[
\|Af\|_{L^p_{\gamma}(\mathbb{R}^{2+1})} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \quad 2 \leq p \leq 6, \tag{1.14}
\]
for all \( \gamma < -\sigma(p) + \frac{1}{2} \).

The paper is organized as follows. In Section 2, we introduce the implements that will be used in the subsequent context. This section is similar to that in [8]. The differences focus on the quantitative hypothesis on the phase function which is introduced in [2] and will play a crucial role in the induction argument via multilinear oscillatory estimates of Bennett, Carbery and Tao [3]. In particular, we emphasize the role played among other things by the parabolic rescaling. In Section 3, we show how to deduce the square function estimate from the corresponding multilinear version. This observation on the relationship between linear and multilinear estimates is originated back to Bourgain and Guth [7], and then applied to many important breakthroughs such as Bourgain and Demeter’s proof of \( \ell^2 \)-decoupling theorem [5], Bourgain-Demeter-Guth’s proof of the main conjecture of...
Vinogradov’s mean value theorem [6], as well as the cone multiplier problem by Lee and Vargas [13]. In Section 4, we prove the tri-linear square function estimate by using the multilinear oscillatory integral estimates of [3] and similar argument as we did in [8]. In the last section, we use decoupling inequality to bring in further improvement upon the \( L^4 \) estimate.

**Notations.** If \( a \) and \( b \) are two positive quantities, we write \( a \lesssim b \) when there exists a constant \( C > 0 \) such that \( a \leq Cb \) where the constant will be clear from the context. When the constant depends on some other quantity \( M \), we emphasize the dependence by writing \( a \lesssim_M b \). We will write \( a \approx b \) when we have both \( a \lesssim b \) and \( b \lesssim a \). We will write \( a \ll b \) (resp. \( b \gg a \)) if there exists a sufficiently large constant \( C > 0 \) such that \( Ca \leq b \) (resp. \( a \geq Cb \)).

We adopt the notion of nature numbers \( N \) where \( b \) is a smooth symbol of order \( N \). We will write \( \{z, \eta\} \) to mean a quantity rapidly decreasing in \( \lambda \)

Throughout this paper, \( w_B \) is a weight which is essentially concentrated on a ball \( B \subset \mathbb{R}^3 \) centered at \( c(B) \) with radius \( r(B) \) and rapidly decaying away from the ball.

\[
w_B(z) \lesssim \left( 1 + \frac{|z - c(B)|}{r(B)} \right)^{-N}, \quad N \gg 1.
\]

With the weight, we define the weighted Lebesgue norm \( \|f\|_{L^p(w_B)} \) as follows

\[
\|f\|_{L^p(w_B)} := \left( \int_{\mathbb{R}^3} |f(z)|^p w_B(z) dz \right)^{\frac{1}{p}} 1 \leq p < \infty.
\]

2. **Preliminaries and Reductions**

Given a point \((z_0, \zeta_0, y_0, \eta_0) \in T^*Z \setminus 0 \times T^*Y \setminus 0\), there exists a sufficiently small local conic coordinate patch around it, along with a smooth function \( \phi(z, \eta) \) such that \( \mathcal{C} \) is given by

\[
\{(z, \phi'_z(z, \eta), \phi'_\eta(z, \eta), \eta) : \eta \in (R^2 \setminus 0) \cap \Gamma_{\eta_0}\}
\]

where \( \Gamma_{\eta_0} \) denotes a conic neighborhood of \( \eta_0 \).

By splitting \( z = (x, t) \in \mathbb{R}^2 \times \mathbb{R} \) into space-time variables, where we put \( z_0 = 0 \) without loss of generality, any operator \( \mathcal{F} \) in the class \( I^{\sigma-1/4}(Z, Y; \mathcal{C}) \) with \( \mathcal{C} \) satisfying the cinematic curvature condition can be written in an appropriate local coordinates as a finite sum of oscillatory integrals

\[
\mathcal{F} f(x, t) = \int_{\mathbb{R}^n} e^{i\phi(x, t, \eta)} b(x, t, \eta) \hat{f}(\eta) d\eta,
\]

where \( b \) is a smooth symbol of order \( \sigma \). We may assume that the support of the map \( z \to b(z, \eta) \) is contained in a ball \( B(0, \varepsilon_0) \), with \( \varepsilon_0 > 0 \) being sufficiently small and \( \eta \to b(z, \eta) \) is supported in a conic region \( \mathcal{V}_{\varepsilon_0} \), i.e.

\[
b(x, t, \eta) = 0 \text{ if } \eta \not\in \mathcal{V}_{\varepsilon_0} := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus 0 : |\xi_1| \leq \varepsilon_0 \xi_2 \}.
\]

Fix \( \lambda \gg 1 \) and \( \beta \in C_c^\infty(\mathbb{R}) \) which vanishes outside the interval \((1/4, 2)\) and equals one in \((1/2, 1)\). By standard Littlewood-Paley decomposition, one may reduce (2.6) to

\[
\|\mathcal{F}_\lambda f\|_{L^p(\mathbb{R}^3)} \lesssim \lambda^\sigma \|f\|_{L^p(\mathbb{R}^2)}, \quad \sigma > \sigma(p)
\]
where \( \mathcal{F}_\lambda \) is an operator
\[
\mathcal{F}_\lambda f(x, t) = \int e^{i\phi(x, t, \eta)} b^\lambda(x, t, \eta) \hat{f}(\eta) \, d\eta
\]  
(2.3)
and
\[
b^\lambda(z, \eta) = b(z, \eta) \frac{1}{(1 + |\eta|^2)^{\sigma/2}} \beta(\frac{\eta}{\lambda}).
\]  
(2.4)

Let us turn to the strategy of the proof. By interpolation with the trivial \( L^2 \rightarrow L^2 \) estimate and the sharp \( L^6 \rightarrow L^6 \) local smoothing estimate of [2],
\[
\| \mathcal{F} f \|_{L^6_{\text{loc}}(Z)} \leq C \| f \|_{L^6_{\text{comp}}(Y)}, \quad \text{for all } \sigma < -\frac{1}{6},
\]  
(2.5)
one may reduce (1.4) to
\[
\| \mathcal{F} f \|_{L^p_{\text{loc}}(Z)} \leq C \| f \|_{L^p_{\text{comp}}(Y)}, \quad \text{for all } \sigma < -\sigma(p),
\]  
(2.6)
with \( p = 3 \), respectively.

Let \( a(z, \eta) \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^2) \) with compact support contained in \( B(0, \varepsilon_0) \) \( \times \) \( B(e_2, \varepsilon_0) \). Assume \( 1 \leq R \leq \lambda, \ C(e_2, \varepsilon_0) := B(e_2, \varepsilon_0) \cap S^1 \) and make angular decomposition with respect to the \( \eta \)-variable by cutting \( C(e_2, \varepsilon_0) \) into \( N_R \approx R^{1/2} \) many sectors \( \{ \theta_\nu : 1 \leq \nu \leq N_R \} \), each \( \theta_\nu \) spreading an angle \( \approx \varepsilon_0 \ R^{-1/2} \).

Let \( \{ \chi_\nu(\eta) \} \) be a family of smooth cutoff functions associated with the decomposition in the angular direction, each of which is homogeneous of degree 0, such that \( \{ \chi_\nu \}_\nu \) forms a partition of unity on the unit circle and then extended homogeneously to \( \mathbb{R}^2 \setminus 0 \) such that

\[
\left\{ \sum_{0 \leq \nu \leq N_R} \chi_\nu(\eta) \equiv 1, \quad \forall \eta \in \mathbb{R}^2 \setminus 0, \right.
\]
\[
\left. |\partial^\alpha \chi_\nu(\eta)| \leq C_\alpha R^{|\alpha|/2}, \quad \forall \alpha \text{ if } |\eta| = 1. \right.
\]

Define
\[
\mathcal{F}_\lambda f = \sum_\nu \mathcal{F}_\nu^\lambda f, \quad \mathcal{F}_\nu^\lambda f(x, t) = \int e^{i\phi^\lambda(x, t, \eta)} a_\nu^\lambda(x, t, \eta) \hat{f}(\eta) \, d\eta,
\]  
(2.7)
where the rescaled phase function and amplitude read
\[
\phi^\lambda(x, t, \eta) := \lambda \phi(x/\lambda, t/\lambda, \eta), \quad a_\nu^\lambda(x, t, \eta) = \chi_\nu(\eta) a(x/\lambda, t/\lambda, \eta).
\]  
(2.8)

For technical reasons, we assume \( a \) is of the form \( a(z, \eta) = a_1(z) a_2(\eta) \), where \( a_1 \in C^\infty_c(B(0, \varepsilon_0)), \quad a_2 \in C^\infty_c(B(e_2, \varepsilon_0)). \)

The general cases may be reduced to this special one via the following observation
\[
T_\lambda f(z) = \int_{\mathbb{R}^3} e^{i \xi z} \left( \int_{\mathbb{R}^2} e^{i \phi(z, \eta)} \psi(z) \hat{a}(\xi, \eta) f(\eta) \, d\eta \right) \, d\xi,
\]
and that \( \xi \mapsto \hat{a}(\xi, \eta) \) is a Schwartz function, where \( \psi(z) \) is a compactly supported smooth function and equals 1 on \( \text{supp}_z a \).

Moreover, we may reformulate (1.2) (1.3) and the curvature condition as
\[ \text{H}_1 \ \text{rank} \ \partial^2_{\xi \eta} \phi(z, \eta) = 2 \text{ for all } (z, \eta) \in \text{supp}_z a. \]
Define the Gauss map \( G : \text{supp} \, a \to \mathbb{S}^2 \) by 
\[
G_0(x, \eta) := \partial_{n_1, \mathbf{e}} \phi(z, \eta) \land \partial_{n_2, \mathbf{e}} \phi(z, \eta). 
\]
(2.9)

The curvature condition
\[
\text{rank} \, \partial^2_{\eta_0} \langle \partial_z \phi(z, \eta), G(z, \eta_0) \rangle |_{\eta_0 = 1} = 1
\]
holds for all \((z, \eta_0) \in \text{supp} \, a\).

Following the approach in [14] which is used in our previous paper [8], (2.2) is in turn reduced to the following square function estimate

**Theorem 2.** Let the operator \( \mathcal{T}_\lambda \) be as in (2.7) and take \( R = \lambda \). Then we have
\[
\| \mathcal{T}_\lambda f \|_{L^3(\mathbb{R}^3)} \lesssim_{\varepsilon, \phi, \lambda} \lambda^\varepsilon \left( \sum_{\nu} \| \mathcal{T}_{\lambda}^\nu f \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} + \lambda^{-L} \| f \|_{L^2(\mathbb{R}^3)},
\]
(2.11)
\[
\| \mathcal{T}_\lambda f \|_{L^4(\mathbb{R}^3)} \lesssim_{\varepsilon, \phi, \lambda} \lambda^{3/2 + \varepsilon} \left( \sum_{\nu} \| \mathcal{T}_{\lambda}^\nu f \|_{L^3(\mathbb{R}^3)}^2 \right)^{1/2} + \lambda^{-L} \| f \|_{L^2(\mathbb{R}^3)},
\]
(2.12)
where \( L \) is a number which can be taken sufficiently large.

**Remark 2.1.** It is worth noting that the spacial support of the amplitude appearing in the right-hand side of (2.11) and (2.12) is slightly larger than that appearing in the left-hand side.

### 2.1 Normalization of the phase function

Let \( \Upsilon_{x,t} : \eta \to \partial_x \phi(x, t, \eta) \). If \( \varepsilon_0 \) is taken sufficiently small, \( \Upsilon_{x,t} \) is a local diffeomorphism on \( B(\mathbf{e}_2, \varepsilon_0) \). If we denote by \( \Psi_{x,t}(\xi) = \Upsilon_{x,t}^{-1}(\xi) \) the inverse map of \( \Upsilon_{x,t} \), then clearly
\[
\partial_x \phi(x, t, \Psi_{x,t}(\xi)) = \xi. 
\]
(2.13)

Differentiating (2.13) with respect to \( \xi \) on both sides yields
\[
[\partial^2_{x,\mathbf{e}} \phi](x, t, \Psi_{x,t}(\xi)) \partial_\xi \Psi_{x,t}(\xi) = \text{Id}.
\]
(2.14)

This manifests that
\[
\det \partial_\xi \Psi_{x,t}(\xi) \neq 0, \, \forall (x, t) \in B(0, \varepsilon_0), \, \forall \xi \in \Upsilon_{x,t}(B(\mathbf{e}_2, \varepsilon_0)).
\]
(2.15)

In order to facilitate certain kind of induction argument, it is also useful to assume the quantitative conditions on the phase function. Assume supp \( a \subset Z \times \Xi \), where \( Z \subset \mathbb{R}^3 \) is a small neighborhood of \( B(0, \varepsilon_0) \), \( \Xi \subset \mathbb{R}^2 \) is a small open sector around \( \mathbf{e}_2 \). Let \( A = (A_1, A_2, A_3) \in [1, \infty) \).

We adopt the notation from [2] with slight modifications. Datum \((\phi, a)\) is said to be of type \( \mathbf{A} \) if the following quantitative properties are satisfied

\( (H_\mathbf{A}) \)
\[
\phi(x, t, \eta) = \langle x, \eta \rangle + \frac{t}{2} \eta_1^2 / \eta_2 + \eta_2 \mathcal{E}(x, t, \eta_1, \eta_2)
\]
(2.16)

where \( \mathcal{E}(x, t, s) \) obeys
\[
\mathcal{E}(x, t, s) \leq c_{\text{par}} A_1 ((|x| + |t|)^2 |s|^2 + (|x| + |t|)|s|^3).
\]
(2.17)
(D) For some large integer \(N \in \mathbb{N}\), depending only on the fixed choice of \(\varepsilon, L\) and \(p\), one has
\[
\|\partial_\eta^\beta \partial_{\nu}^\alpha \phi\|_{L^\infty(Z \times \Xi)} \leq c_{\text{par}} A_2
\]
for all \((\alpha, \beta) \in \mathbb{N}^3 \times \mathbb{N}^2\) with \(|\alpha| = 2\) and \(1 \leq |\beta| \leq N\).

(M) \text{dist}(\text{supp} a_1, \mathbb{R}^3 \setminus Z \times \Xi) \geq A_3/4.

Remark 2.2. The condition \((H)\) demonstrates that the phase function can be viewed as a minor perturbation of the translation-invariant case \(\langle x, \eta \rangle + \frac{t}{2} \eta^2/\eta_2\). This fact will be useful in the course of verifying the transversality condition. The reason we impose the hypothesis \((D)\) is to bound the higher order derivatives of the phase function in the approximation argument. As above mentioned, the support of amplitude function may be slightly enlarged in the induction. We may partition the support of the amplitude to maintain the margin hypothesis \((M)\).

2.2. Parabolic rescaling. Let \(1 \leq R \leq \lambda\) and \(B_R\) denote a ball of radius \(R\). For convenience, we introduce the \(l^2\) decoupling norm and square function norm respectively as
\[
\|\mathcal{T}_\lambda f\|_{L^p_{\text{Dec}}(Q)} := \left( \sum_{\nu} \|\mathcal{T}_\lambda \nu f\|_{L^p(w_Q)}^2 \right)^{\frac{1}{2}},
\]
\[
\|\mathcal{T}_\lambda f\|_{L^p_{\text{sq}}(Q)} := \left\| \left( \sum_{\nu} |\mathcal{T}_\lambda \nu f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w_Q)}.
\]
Let \(S_{\sigma, \varepsilon}^\lambda(R)\) be the infimum over all \(C\) such that\(^1\)
\[
\|\mathcal{T}_\lambda f\|_{L^p(B_R)} \leq C R^{\sigma + \varepsilon} \|\mathcal{T}_\lambda f\|_{L^p_{\text{sq}}(B_R)} + R^4 \left( \frac{\lambda}{R} \right)^{-2M} \|f\|_{L^2(\mathbb{R}^2)}, \quad \forall R \leq \lambda^{1/\sigma}\]
holds for all type \(A\) data \((\phi, a)\).

It is easy to see that \(S_{\sigma, \varepsilon}^\lambda(R)\) is always finite. Indeed, by Hölder’s inequality,
\[
\|\mathcal{T}_\lambda f\|_{L^p(B_R)} = \left\| \sum_{\nu} \mathcal{T}_\lambda \nu f \right\|_{L^p(w_{B_R})} \lesssim R^{\frac{1}{\sigma}} \|\mathcal{T}_\lambda f\|_{L^p_{\text{sq}}(B_R)}
\]
where we have used the fact \(#\{\nu\} \approx R^{1/2}\). Therefore we have
\[
S_{\sigma, \varepsilon}^\lambda(R) \lesssim R^{\frac{1}{\sigma} - \sigma},
\]
which will also serve as a starting point of our induction.

The proof of (2.11) and (2.12) is reduced to
\[
S_{\sigma}^{\lambda(p), \varepsilon}(\lambda, \lambda^{1-\varepsilon/2}) \lesssim_{\varepsilon, M, p} 1, \quad p = 3, 4.
\]
Actually, since the support of \(\text{supp}_\lambda a_\lambda(\cdot, \eta)\) is contained in the ball \(B(0, \lambda)\), we may tile \(B(0, \lambda)\) by a collection of balls of radius \(\lambda^{1-\varepsilon/2}\) and use \(S_{\sigma}^{\lambda(p), \varepsilon}(\lambda, \lambda^{1-\varepsilon/2}) \lesssim_{\varepsilon, M, p} 1\) in each of the balls. By choosing appropriate \(M\) depending \(\varepsilon, L\) and summing over all generated balls, we will obtain the desired result.

\(^1\) we add the term \(R^4\) for technical reasons which will be clear latter.
One key ingredient in the proof will be parabolic rescaling argument. Before the statement of the proposition, we introduce a notion of \( r \)-plate

\[
\Gamma^r := \{ (\eta_1, \eta_2) \in \text{supp}_\eta a : |\eta_1/\eta_2 - \alpha| \leq r \}.
\]

**Proposition 2.3.** Let phase \( \phi \) satisfy the condition \( H_1, H_2 \), amplitude \( a \) satisfy \( \mathbf{M}_A \). Let \( 1 \leq \rho \leq R \leq \lambda \) and \( \mathcal{T}_\lambda \) be defined associated with \( \phi \) and \( a \). If \( f \) is supported on a \( \rho^{-1} \)-plate with \( \rho \) is sufficiently large depending on \( \phi \), then we have

\[
\| \mathcal{T}_\lambda f \|_{L^p(B_R)} \lesssim_{\phi,a} S_1^{\sigma,\varepsilon} \left( \frac{\lambda}{\rho^2}, \frac{R}{\rho^2} \right) \left( \frac{R}{\rho^2} \right)^{\sigma + \varepsilon} \| \mathcal{T}_\lambda f \|_{L^p_{\sup}(B_R)} + \rho^{-2} R^4 \left( \frac{\lambda}{R} \right)^{-2M} \left\| f \right\|_{L^2(\mathbb{R}^2)}.
\]  

(2.25)

**Proof.** The conditions \( H_1, H_2 \) imply that there exists a special coordinate system, so that the phase function \( \phi(z, \eta) \) can be written in a normalized form. More precisely, according to Lemma 3.4 in [12], by appropriate affine transformation we may assume \( G(0, e_2) = e_2 \) and \( \partial_\eta \partial_\eta^2 \phi(0, e_2) = 1 \), then up to multiplying harmless factors to \( \mathcal{T}_\lambda f \) and \( f \), we can write \( \phi \) in this coordinate as

\[
\phi(x, t, \eta) = \langle x, \eta \rangle + \frac{t}{2} \eta_1^2/\eta_2 + \eta_2 \varepsilon(x, t, \eta_1/\eta_2)
\]  

(2.26)

where \( \eta = (\eta_1, \eta_2) \) and \( \varepsilon(x, t, s) \) obeys

\[
\varepsilon(x, t, s) \leq C_\phi((|x| + |t|)^2 |s|^2 + (|x| + |t|) |s|^2).
\]  

(2.27)

provided \( \text{supp} a \) is sufficiently small, otherwise, we may decompose the \( \text{supp} a \) into several small pieces such that (2.26),(2.27) hold. For further details in this direction, one may refer to [4, 10, 9]. We perform to \( a(z, \eta) \) the same transforms which convert \( \phi \) into its normalized form in (2.26), and we denote it still by \( a(z, \eta) \). Let the operator \( T_\lambda \) be defined as follows

\[
T_\lambda f(z, t) := \int e^{i \lambda \phi(z, \eta)} a(z, \eta) f(\eta) d\eta.
\]  

(2.28)

As a result, by changing of variable: \( z \to \lambda z \), we have

\[
\| \mathcal{T}_\lambda f \|_{L^p(B_R)} \lesssim_{\phi} \lambda^\frac{3}{2} \| T_\lambda f \|_{L^p(B_{R/\lambda})}.
\]  

(2.29)

Assume \( f \) is supported in a plate \( \Gamma^\rho_\gamma \). By changing of variable: \( \eta_1 \to \eta_1 + \gamma \eta_2 \), we may rotate the central axis of \( \Gamma^\rho_\gamma \) to the \( e_2 \) axis. We make the associated change of variable in the physical space

\[
\begin{align*}
x_1 + \gamma t &= x^{(1)}_1; \\
\gamma x_1 + x_2 + \frac{1}{2} \gamma^2 t &= x^{(1)}_2; \\
t &= t^{(1)}.
\end{align*}
\]  

(2.30)

For convenience, we will use \( x^{(1)} \) to denote \( x^{(1)} := (x^{(1)}_1, x^{(1)}_2) \). Since the transformation above is a diffeomorphism, we may use \( \Phi^{(1)}(x^{(1)}, t^{(1)}) \) to denote the inverse map of (2.30).

In the new variable system, the phase \( \phi \) is transformed to

\[
\phi^{(1)}(x^{(1)}, t^{(1)}, \eta) = \langle x^{(1)}, \eta \rangle + \frac{1}{2} t^{(1)} \eta_1^2/\eta_2 + \eta_2 \varepsilon_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \gamma),
\]  

(2.31)

where we denote \( \varepsilon_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \gamma) = \varepsilon(\Phi^{(1)}(x^{(1)}, t^{(1)}), \eta_1/\eta_2 + \gamma) \) the error term.
Next make Taylor’s expansion of $E_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \gamma)$ as follows

$$E_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \gamma) = E_1(x^{(1)}, t^{(1)}, \gamma) + \partial_s E_1(x^{(1)}, t^{(1)}, \gamma) \frac{\eta_1}{\eta_2} + \frac{1}{2} \partial_s^2 E_1(x^{(1)}, t^{(1)}, \gamma) \frac{\eta_1^2}{\eta_2} + \frac{1}{2} \int_0^1 \partial_s^3 E_1(x^{(1)}, t^{(1)}, s\frac{\eta_1}{\eta_2} + \gamma) \left(\frac{\eta_1}{\eta_2}\right)^3 (1-s)^2 ds.$$ 

Making change of variables

$$\begin{cases}
  x_1^{(1)} + \partial_s E_1(x^{(1)}, t^{(1)}, \gamma) = x_1^{(2)} \\
  E_1(x^{(1)}, t^{(1)}, \gamma) + x_2^{(1)} = x_2^{(2)} \\
  \frac{1}{2} t^{(1)} + \frac{1}{2} \partial_s^2 E_1(x^{(1)}, t^{(1)}, \gamma) = \frac{1}{2} t^{(2)}.
\end{cases} \quad (2.32)$$

As above, we use $x^{(2)}$ to denote $x^2 := (x_1^{(2)}, x_2^{(2)})$ and $\Phi^{(2)}(x^{(2)}, t^{(2)})$ to denote the inverse map of (2.32).

Accordingly, the phase function in (2.31) is changed to

$$\phi^{(2)}(x^{(2)}, t^{(2)}, \eta) = \phi(x^{(2)}, \eta) + \frac{1}{2} t^{(2)} \frac{\eta_1^2}{\eta_2} + \eta_2 E_2(x^{(2)}, t^{(2)}, \eta_1/\eta_2). \quad (2.33)$$

where

$$E_2(x^{(2)}, t^{(2)}, \eta_1/\eta_2) = \frac{1}{2} \int_0^1 \partial_s^3 E_1(\Phi^{(1)} \circ \Phi^{(2)}(x^{(2)}, t^{(2)}), s \frac{\eta_1}{\eta_2} + \gamma) \left(\frac{\eta_1}{\eta_2}\right)^3 (1-s)^2 ds. \quad (2.34)$$

Define $\Phi = \Phi^{(1)} \circ \Phi^{(2)}$, the amplitude $a$ is changed to

$$a_2(x^{(2)}, t^{(2)}, \eta) = a(\Phi(x^{(2)}, t^{(2)}), \eta_1 + \gamma \eta_2, \eta_2). \quad (2.35)$$

Let $\tilde{T}_\lambda$ be defined as follows

$$\tilde{T}_\lambda f = \int e^{i\lambda \phi^{(2)}(z, \eta)} a_2(z, \eta) f(\eta) d\eta.$$ 

Therefore,

$$\|T_\lambda f\|_{L^p(B_{R/\lambda})} \lesssim \|\tilde{T}_\lambda f\|_{L^p(\Phi^{-1}(B_{R/\lambda}))}. \quad (2.36)$$

Let $\{R_\lambda\}_\lambda \subset \mathbb{R}^3$ be a collection of pairwise disjoint rectangles of sidelength $\rho^{-3/2} \times \rho^{-1} \times \rho^{-2} \frac{R}{\lambda}$ satisfying

$$\Phi^{-1}(B_{R/\lambda}) \subset \bigcup_\lambda R_\lambda. \quad (2.37)$$

By orthogonality of $R_\lambda$, we have

$$\|\tilde{T}_\lambda f\|_{L^p(\Phi^{-1}(B_{R/\lambda}))}^p \lesssim \sum_\Lambda \|\tilde{T}_\lambda f\|_{L^p(R_\lambda)}^p. \quad (2.38)$$

Finally, by scaling $\eta_1 \to \rho^{-1} \eta_1$, the corresponding map for the amplitude

$$\eta \to a_2(z, \rho^{-1} \eta_1, \eta_2)$$

which is contained in

$$\mathcal{D} = \left\{ (\eta_1, \eta_2) \in \mathbb{R}^2 : |\eta_1/\eta_2| \leq 1/10, |\eta_2 - 1| \leq \varepsilon_0 \right\}.$$
For each $R_A$, by changing variables

$$(x, t) \rightarrow (z_A + (\lambda^{-1} \rho^{-1} x_1, \lambda^{-1} \rho^{-2} x_2, \lambda^{-1} t)),$$

with $z_A$ being the center of $R_A$. we have

$$\|\tilde{T}_A f\|_{L^p(R_A)} \lesssim \lambda^{\frac{-2}{p}} \rho^{-\frac{3}{p}} \|\tilde{\mathcal{F}}_{\lambda^{p}} f\|_{L^p(B_{\rho}^{-2} R)},$$

(2.39)

where

$$\tilde{\mathcal{F}}_{\lambda^p} f(z) = \sum_{\nu} \tilde{\mathcal{F}}_{\lambda^p}^\nu f(z), \quad \tilde{\mathcal{F}}_{\lambda^p}^\nu f(z) = \int e^{i \lambda \tilde{\phi}(z, \eta)} \tilde{a}^\nu(z, \eta) f(\eta) d\eta,$$

$$\tilde{\phi}(z, \eta) = x_1 \eta_1 + x_2 \eta_2 + \frac{1}{2} \eta_1^2 / \eta_2 + \rho^2 \eta_2 E_2(z_A + (\lambda^{-1} \rho^{-1} x_1, \lambda^{-1} \rho^{-2} x_2, \lambda^{-1} t), \rho^{-1} \eta_1 / \eta_2),$$

$$\tilde{a}^\nu(z, \eta) = a(z, \eta) \chi(\rho^{-1} \eta_1 + \gamma \eta_2, \eta_2), a(z, \eta) = a_2(z_A + (\lambda^{-1} \rho^{-1} x_1, \lambda^{-1} \rho^{-2} x_2, \lambda^{-1} t), \rho^{-1} \eta_1, \eta_2).$$

If the datum $(\tilde{\phi}, \tilde{a})$ is of type 1, we may apply (2.21) directly to obtain

$$\|\tilde{\mathcal{F}}_{\lambda^{p}} f\|_{L^p(B_{\rho}^{-2} R)} \lesssim S_1^\sigma, \frac{(A_1, B)}{(\rho^2, \rho^2)} \left(\frac{R}{\rho^2}\right) \sigma + \varepsilon \left\|\left(\sum_{\nu} |\tilde{\mathcal{F}}_{\lambda^{p}}^\nu f|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_{\rho}^{-2} R)} + \rho^{-8-4 M} R^4 \left(\frac{A}{R}\right)^{-2 M} \|f\|_{L^2(R^2)}.$$

(2.40)

Reversing the above transforms and summing over $\Lambda$’s, we obtain (2.25).

Now it remains to show the datum $(\tilde{\phi}, \tilde{a})$ is of type 1, note that (2.33) and (2.34), by replacing the $\tilde{\phi}$ with $\tilde{\phi} - \rho^2 \eta_2 E_2(\Phi(z_A, \rho^{-1} \eta_1 / \eta_2), H_1, D_1)$ can be readily verified by choosing $\rho$ sufficiently large depending on $\phi$. It is an issue that the hypothesis $M_1$ may break down, however, this can be resolved by decomposing $\text{supp}_2 a$ into finite many small balls and translating to the origin, as a results, one may obtain a number of operators each defined associated with type 1 datum. It follows by using (2.21) on each of ball, then summing over all the generated balls. \hfill \square

**Remark 2.4.** Throughout the proof of Proposition 2.3, one may reduce type A datum to type 1 datum. Hence from now on, we shall always assume $(\phi, a)$ is of type 1 without specified clarification.

3. **Linear v.s. Multilinear square function estimates**

In this section, we establish the relation between linear and multilinear square function estimates. In order to describe the setup, it requires the notion of transversality.

**Definition 3.1.** Let $T_\lambda = (\mathcal{F}_1^\lambda, \mathcal{F}_2^\lambda, \mathcal{F}_3^\lambda)$ be 3-tuple of oscillatory integral operators satisfying the homogeneous type of Carleson-Sj"olin condition, where $\mathcal{F}_j^\lambda$ has associated phase $\phi_j^\lambda$, amplitude $a_j^\lambda$ and generalised Gauss map $G_j$ for $1 \leq j \leq 3$. Then $T_\lambda$ is said to be $\Delta$ transverse for some $0 < \Delta \leq 1$ if

$$\left| \bigwedge_{j=1}^3 G_j(z, \eta_j) \right| \geq \Delta \quad \text{for all} \quad (z, \eta_j) \in \text{supp} \ a_j, \quad 1 \leq j \leq 3.$$  

(3.1)
Let \( 2 \leq p < \infty, 1 \leq R \leq \lambda^{1-\varepsilon/2} \), \( \mathbf{T}_\lambda = (\mathcal{F}_\lambda^1, \mathcal{F}_\lambda^2, \mathcal{F}_\lambda^3) \) are \( \Delta \) transverse, we use \( \mathbf{MS}_1^{\sigma, \varepsilon}(\lambda, R) \) to denote the sharp constant such the following trilinear square function estimate

\[
\left\| \prod_{j=1}^3 [\mathcal{F}^j_\lambda f]^{1 \over 3} \right\|_{L^p(B_R)} \leq \mathbf{MS}_1^{\sigma, \varepsilon}(\lambda, R) R^{\alpha+\varepsilon} \prod_{j=1}^3 \left\| \mathcal{F}^j_\lambda f \right\|_{L^p_{\text{sq}}(B_R)} + R^2 \left( \frac{\lambda}{R} \right)^{-3M} \| f \|_{L^2}, \tag{3.2}
\]

holds for all type \( 1 \) data \( (\phi_j, a_j), j = 1, 2, 3 \).

In the seminal paper \cite{7}, Bourgain and Guth initiated the multilinear approach towards Fourier restriction conjectures and oscillatory integral estimates based on a fundamental result of \cite{3}. This strategy was later adapted to the cone multiplier problem by Lee and Vargas in \cite{13}. Motivated by these previous works, we adopt the similar idea to deduce linear square function estimates from the multilinear one. The following lemma can be established by using Bourgain-Guth’s method and we omit the details (see also \cite{13} for details).

**Lemma 3.2.** Let \( 1 \ll K_1 \ll K_2 \ll R \) and \( \{\theta_\mu\}_\mu, \{\theta_\tau\}_\tau \) be a family of plates of aperture \( K_1^{-1} \) and \( K_2^{-1} \), respectively. We make decomposition of \( \mathcal{F}_\lambda f \) as follows

\[
\mathcal{F}_\lambda f = \sum_{\mu: \theta_\mu \in \{\theta_\mu\}} \mathcal{F}_\mu f = \sum_{\tau: \theta_\tau \in \{\theta_\tau\}} \mathcal{F}_\tau f, \tag{3.3}
\]

where

\[
\mathcal{F}_\mu f(z) := \int e^{i\phi_\lambda(x,t,\eta)} a^\mu_\lambda(x,t,\eta) f(\eta)d\eta, \quad a^\mu_\lambda(z,\eta) = \sum_{\nu: \theta_\nu \supset \theta_\mu} a^\nu_\lambda(z,\eta), \tag{3.4}
\]

\[
\mathcal{F}_\tau f(z) := \int e^{i\phi_\lambda(x,t,\eta)} a^\tau_\lambda(x,t,\eta) f(\eta)d\eta, \quad a^\tau_\lambda(z,\eta) = \sum_{\nu: \theta_\nu \supset \theta_\tau} a^\nu_\lambda(z,\eta). \tag{3.5}
\]

Then for any \( z \in \text{supp } a_\lambda \), we have

\[
|\mathcal{F}_\lambda f(z)| \leq C \max_{\mu} |\mathcal{F}_\mu f(z)| + CK_1 \max_{\tau} |\mathcal{F}_\tau f(z)| + \frac{K_2^{50}}{2} \max_{(\tau_1, \tau_2, \tau_3) \in \text{Tr}(\tau)} \prod_{j=1}^3 |\mathcal{F}_{\tau_j}^{\tau_j} f(z)|^{1/3}, \tag{3.6}
\]

where \( \text{Tr}(\tau) := \{(\tau_1, \tau_2, \tau_3) : \theta_{\tau_j} \in \{\theta_\tau\}, j = 1, 2, 3, \text{ and } \text{Ang}(\theta_{\tau_j}, \theta_{\tau_{j'}}) \geq K_2^{-1}, j \neq j'\} \).

Based on the Lemma 3.2, we relate the trilinear estimate (3.2) to (2.21).

**Proposition 3.3.** Suppose \( \mathbf{MS}_1^{\sigma, \varepsilon}(\lambda, R) \lesssim \varepsilon \) 1 for all \( (\lambda, R) \) satisfying \( 1 \leq R \leq \lambda^{1-\varepsilon/2} \). Then

\[
\mathbf{S}_1^{\sigma, \varepsilon}(\lambda, R) \lesssim \varepsilon \, 1,
\]

whenever \( (\lambda, R) \) fulfills the condition \( 1 \leq R \leq \lambda^{1-\varepsilon/2} \).

**Proof.** If \( 1 \leq R \leq 100 \), then by (2.23), we have \( \mathbf{S}_1^{\sigma, \varepsilon}(\lambda, R) \leq C \). We proceed by induction argument. Suppose

\[
\mathbf{S}_1^{\sigma, \varepsilon}(\lambda', R') \lesssim \varepsilon \, 1 \text{ for all } R' \leq R/2, R' \leq (\lambda')^{1-\varepsilon/2}, \tag{3.7}
\]
it suffices to show
\[ S_1^{\sigma,2\varepsilon}(\lambda, R) \lesssim \varepsilon 1 \text{ for all } R \leq \lambda^{1-\varepsilon/2}. \] (3.8)

To this end, using Lemma 3.2, we have
\[ \|\mathcal{T}_\lambda f\|_{L^p(B_R)}^p \leq C \sum_{\mu, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\mu f\|_{L^p(B_R)}^p + CK_1^p \sum_{\tau, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\tau f\|_{L^p(B_R)}^p \\
+ K_2^{50p} \sum_{(\tau_1, \tau_2, \tau_3) \in \text{Tr}(\tau)} \left(\prod_{j=1}^3 \|\mathcal{T}_\tau f\|_{L^p(B_R)}^{1/3}\right)^p. \] (3.9)

By Proposition 2.3, we have
\[ \|\mathcal{T}_\lambda f\|_{L^p(B_R)} \leq CS_1^{\sigma,2\varepsilon}(\frac{\lambda}{K_1^2}, \frac{R}{K_1^2}) \left(\frac{R}{K_1^2}\right)^{\sigma+2\varepsilon} \left(\sum_{\mu, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\mu f\|_{L(wB_R)}^{3/2}\right) \]
\[ + K_1^{-1} R^4(\frac{\lambda}{R})^{-2M} \|f\|_{L^2}. \] (3.10)

Raising to \( p \)-th power and summing over all \( \mu \)'s, we get
\[ \left(\sum_{\mu, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\mu f\|_{L^p(B_R)}^p\right)^{\frac{1}{p}} \leq CS_1^{\sigma,2\varepsilon}(\frac{\lambda}{K_1^2}, \frac{R}{K_2^2}) \left(\frac{R}{K_1^2}\right)^{\sigma+2\varepsilon} \left(\sum_{\mu, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\mu f\|_{L(wB_R)}^{3/2}\right) \]
\[ + K_1^{-1} R^4(\frac{\lambda}{R})^{-2M} \|f\|_{L^2}. \] (3.12)

Similarly, we obtain the corresponding estimate for the second term in (3.9).
\[ \left(\sum_{\tau, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\tau f\|_{L^p(B_R)}^p\right)^{\frac{1}{p}} \leq CK_1 S_1^{\sigma,2\varepsilon}(\frac{\lambda}{K_1^2}, \frac{R}{K_2^2}) \left(\frac{R}{K_2^2}\right)^{\sigma+2\varepsilon} \left(\sum_{\nu, \theta, \tau \in \{\theta, \tau\}} \|\mathcal{T}_\nu f\|_{L(wB_\lambda)}^{3/2}\right) \]
\[ + K_1 K_2^{-1} R^4(\frac{\lambda}{R})^{-2M} \|f\|_{L^2}. \] (3.13)

It remains to estimate (3.10). In order to use the trilinear square function estimate (3.2) with \( MS_1^{\sigma,\varepsilon}(\lambda, R) \lesssim 1 \), it requires the verification of the transversality condition.

Let \( (\tau_1^0, \tau_2^0, \tau_3^0) \in \text{Tr}(\tau) \) be such that
\[ \left\|\prod_{j=1}^3 |\mathcal{T}_\lambda f|^{1/3}\right\|_{L^p(B_R)}^p = \max_{(\tau_1, \tau_2, \tau_3) \in \text{Tr}(\tau)} \left\|\prod_{j=1}^3 |\mathcal{T}_\lambda f|^{1/3}\right\|_{L^p(B_R)}^p. \]

Then (3.10) is bounded by
\[ K_2^{O(1)} \left\|\prod_{j=1}^3 |\mathcal{T}_\lambda f|^{1/3}\right\|_{L^p(B_R)}^p, \] (3.14)

since the cardinality of the set \( \text{Tr}(\tau) \) is at most \( K_3^2 \). By change of variable \( z \to \lambda z \), (3.14) becomes
\[ \lambda^3 K_2^{O(1)} \left\|\prod_{j=1}^3 |\mathcal{T}_\lambda f|^{1/3}\right\|_{L^p(B_{R/\lambda})}^p, \] (3.15)
where for $j = 1, 2, 3$,
\[
T^{\tau_0}_f(x, t) = \int e^{i\lambda\phi(x, t, \eta)} a^{\tau_0}(x, t, \eta) f(\eta) d\eta, \quad a^{\tau_0}(x, t, \eta) = a^{\tau_0}(\lambda(x, t), \eta).
\]

Let $\eta^j = (\eta_1^j, \eta_2^j)$ be the center of $\theta_{\tau_0}^j$ with $j = 1, 2, 3$ and set $\alpha_j = \eta_1^j / \eta_2^j$. Then, for every $\eta \in \theta_{\tau_0}^j$, we have
\[
\left| \frac{\eta_1}{\eta_2} - \alpha_j \right| \leq K_2^{-1}, \ j = 1, 2, 3.
\]

Due to the angular separation condition involved in $\text{Tr}(\tau)$, we may assume $\alpha_1 > \alpha_2 > \alpha_3$ after a rearrangement if necessary. Denote
\[
\alpha := \min\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3\}. \quad (3.16)
\]
Then we have $\alpha \geq K_2^{-1}$.

In order to verify the transversality condition, we need to make another transform with respect to $\eta_1$ to ensure the angular separation $\approx 1$ independent of $\varepsilon_0$. To this end, one may modify the argument in the proof of Proposition 2.3.

We make a change of variable $\eta_1 \rightarrow \eta_1 + \alpha_2 \eta_2$ in each oscillatory integral $T^{\tau_0}_f$ with respect to the frequency space for all $j = 1, 2, 3$. In the mean time, we make an affine transform in (3.15) with respect to the $(x, t)$-variables
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\alpha_2 x_1 + x_2 + \frac{1}{2} \alpha_2^2 t = x_2^{(1)} \\
t = t^{(1)}
\end{array}
\right.
\end{aligned}
\]
(3.17)

We denote $x^{(1)} := (x_1^{(1)}, x_2^{(1)})$ and use $\Phi^{(1)}(x^{(1)}, t^{(1)})$ to denote the inverse map of (3.17). In the new variable system, the phase function $\phi$ is transformed to
\[
\phi^{(1)}(x^{(1)}, t^{(1)}, \eta) = \langle x^{(1)}, \eta \rangle + \frac{1}{2} t^{(1)} \eta_1^2 / \eta_2 + \eta_2 \mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1 / \eta_2 + \alpha_2),
\]
(3.18)
where
\[
\mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1 / \eta_2 + \alpha_2) = \mathcal{E}(\Phi^{(1)}(x^{(1)}, t^{(1)}), \eta_1 / \eta_2 + \alpha_2).
\]
Taylor expanding $\mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1 / \eta_2 + \alpha_2)$ as follows
\[
\mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1 / \eta_2 + \alpha_2) = \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) + \partial_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) \frac{\eta_1}{\eta_2}
+ \frac{1}{2} \partial_s^2 \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) \left( \frac{\eta_1}{\eta_2} \right)^2
+ \frac{1}{2} \int_0^1 \partial_s^3 \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) \left( \frac{\eta_1}{\eta_2} \right)^3 (1 - s)^2 ds.
\]
Making an additional change of variables

\[
\begin{cases}
  x_1^{(1)} + \partial_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) = x_1^{(2)}, \\
  \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) + x_2^{(1)} = x_2^{(2)}, \\
  \frac{1}{2} t^{(1)} + \frac{1}{2} \partial_s^2 \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_2) = \frac{1}{2} t^{(2)},
\end{cases}
\tag{3.19}
\]

the phase function in (3.18) is changed to

\[
\phi^{(2)}(x^{(2)}, t^{(2)}, \eta) = \langle x^{(2)}, \eta \rangle + \frac{1}{2} t^{(2)} \eta_1^2/\eta_2 + \eta_2 \mathcal{E}_2(x^{(2)}, t^{(2)}, \eta_1/\eta_2),
\tag{3.20}
\]

where

\[
\mathcal{E}_2(x^{(2)}, t^{(2)}, \eta_1/\eta_2) = \frac{1}{2} \int_0^1 \partial_s^3 \mathcal{E}(\Phi^{(1)} \circ \Phi^{(2)}(x^{(2)}, t^{(2)}), s \frac{\eta_1}{\eta_2} + \alpha_2) \left( \frac{\eta_1}{\eta_2} \right)^3 (1 - s)^2 ds,
\tag{3.21}
\]

with \(x^{(2)} := (x_1^{(2)}, x_2^{(2)})\) and \(\Phi^{(2)}(x^{(2)}, t^{(2)})\) denoting the inverse map of (3.19).

Setting \(\Phi = \Phi^{(1)} \circ \Phi^{(2)}\), the amplitude \(a^{\gamma^0}_j\) is changed by the above affine transforms to

\[
a^{\gamma^0}_j(\Phi(x^{(2)}, t^{(2)}), \eta_1 + \alpha_2 \eta_2, \eta_2) := a^{\gamma^0}_2(x^{(2)}, t^{(2)}, \eta).
\tag{3.22}
\]

We introduce an operator \(\widetilde{T}^{\gamma^0}_\lambda\)

\[
\widetilde{T}^{\gamma^0}_\lambda f(x^{(2)}, t^{(2)}) = \int e^{i \lambda \phi^{(2)}(x^{(2)}, t^{(2)}, \eta)} a^{\gamma^0}_2(x^{(2)}, t^{(2)}, \eta) f(\eta) d\eta.
\]

Therefore,

\[
\left\| \prod_{j=1}^3 \left| \widetilde{T}^{\gamma^0}_\lambda f \right|^{\frac{1}{3}} \right\|_{L^p(B_{R/\lambda})} \lesssim \left\| \prod_{j=1}^3 \left| T^{\gamma^0}_\lambda f \right|^{\frac{1}{3}} \right\|_{L^p(\Phi^{-1}(B_{R/\lambda}))},
\tag{3.23}
\]

Let \(\{R_\lambda\}_\Lambda \subset \mathbb{R}^3\) be a pairwise disjoint collection of rectangles of sidelength \(\alpha^3 R \times \alpha^4 \frac{R}{\lambda} \times \alpha^2 \frac{R}{\lambda}\), where the first two components correspond to the \(x\)-direction, such that

\[
\Phi^{-1}(B_{R/\lambda}) \subset \bigcup \Lambda R_\lambda.
\tag{3.24}
\]

By almost orthogonality, we have

\[
\left\| \prod_{j=1}^3 \left| \widetilde{T}^{\gamma^0}_\lambda f \right|^{\frac{1}{3}} \right\|_{L^p(B_{R/\lambda})} \lesssim \sum \Lambda \left\| \prod_{j=1}^3 \left| T^{\gamma^0}_\lambda f \right|^{\frac{1}{3}} \right\|_{L^p(R_\lambda)}.
\tag{3.25}
\]

Finally, under scaling \(\eta_1 \to \alpha \eta_1\), the corresponding amplitude function becomes

\[
(\eta_1, \eta_2) \to a^{\gamma^0}_2(x^{(2)}, t^{(2)}, (\alpha \eta_1, \eta_2))
\]

which is supported respectively in \(\mathcal{D}_j\) with

\[
\begin{align*}
\mathcal{D}_1 &= \{(\eta_1, \eta_2) \in \mathbb{R}^2 : |\eta_1/\eta_2 - \beta_1| \leq 1/10, |\eta_2 - 1| \leq \varepsilon_0\}, \\
\mathcal{D}_2 &= \{(\eta_1, \eta_2) \in \mathbb{R}^2 : |\eta_1/\eta_2| \leq 1/10, |\eta_2 - 1| \leq \varepsilon_0\}, \\
\mathcal{D}_3 &= \{(\eta_1, \eta_2) \in \mathbb{R}^2 : |\eta_1/\eta_2 - \beta_3| \leq 1/10, |\eta_2 - 1| \leq \varepsilon_0\},
\end{align*}
\]

where \(\beta_k = (\alpha_k - \alpha_2) \alpha^{-1}\) for \(k = 1, 3\). From (3.16), it is easy to see \(|\beta_k| \geq 1/2\) for \(k = 1, 3\).
By changing variables \((x^{(2)}, t^{(2)}) \rightarrow z_\Lambda + \left( \lambda^{-1} \alpha x_1, \lambda^{-1} \alpha^2 x_2, \lambda^{-1} t \right)\) with \(z_\Lambda\) being the center of \(R_\Lambda\), we have

\[
\left\| \prod_{j=1}^{3} \left| T_\lambda^{\tau_0^j} f \right|^{\frac{1}{2}} \right\|_{L^p(B_\alpha^2 R)} \leq C \lambda^{\frac{1}{2}} \alpha^{\frac{3}{2}} \left\| \prod_{j=1}^{3} \left| \tilde{T}_{\lambda^2}^{\tau_0^j} f \right|^{\frac{1}{2}} \right\|_{L^p(B_\alpha^2 R)},
\]

(3.26)

where \(B_{\alpha^2 R}\) is a ball of radius \(\alpha^2 R\) centered at the origin and

\[
\tilde{T}_{\lambda^2}^{\tau_0^j} f(x, t) = \int e^{i \lambda^2 \tilde{\phi}(x, t, \eta)} \tilde{a}^{\tau_0^j} (x, t, \eta) f(\eta) d\eta,
\]

\[
\tilde{\phi}(x, t, \eta) = x_1 \eta_1 + x_2 \eta_2 + \frac{1}{2} t \eta_1^2 / \eta_2 + \alpha^{-2} \eta_2 \mathcal{E}_2(\Phi(z_\Lambda + (\lambda^{-1} \alpha x_1, \lambda^{-1} \alpha^2 x_2, \lambda^{-1} t), \alpha \eta_1 / \eta_2),
\]

\[
\tilde{a}^{\tau_0^j} (x, t, \eta) = a_2^{\tau_0^j} (z_\Lambda + (\lambda^{-1} \alpha x_1, \lambda^{-1} \alpha^2 x_2, \lambda^{-1} t), (\alpha \eta_1, \eta_2)).
\]

Noting that \(z_\Lambda + (\lambda^{-1} \alpha x_1, \lambda^{-1} \alpha^2 x_2, \lambda^{-1} t) \in B(0, 2 \varepsilon_0) \subset \mathbb{R}^3\), we have by direct calculation for \(\varepsilon_0\) small enough

\[
\partial_x \tilde{\phi}(x, t, \eta) = \eta + O(\varepsilon_0 |\eta|^2) + O(\varepsilon_0 |\eta|^3),
\]

\[
\partial_t \tilde{\phi}(x, t, \eta) = \frac{1}{2} \eta_1^2 / \eta_2 + O(\varepsilon_0 |\eta|^2) + O(\varepsilon_0 |\eta|^3).
\]

Therefore, it is easy to obtain the formula for \(G^j_0, j = 1, 2, 3\)

\[
G^j_0(x, t, \eta) = \left( -\frac{\eta_1}{\eta_2}, \frac{1}{2} \frac{\eta_1^2}{\eta_2}, 1 \right) + O(\varepsilon_0), \quad \eta \in \mathcal{D}_j.
\]

(3.27)

From this, we obtain

\[
|G^1_0 \land G^2_0 \land G^3_0| \geq \frac{1}{8} |\beta_1| |\beta_3| |\beta_1 - \beta_3| + O(\varepsilon_0) \geq \frac{1}{32} |\beta_1 - \beta_3|^2 + O(\varepsilon_0),
\]

(3.28)

for all \((x, t, \eta) \in B(0, \varepsilon_0) \times \mathcal{D}_j, j = 1, 2, 3\). Since \(G^j := \frac{G^j_0(z, \eta)}{|G^j_0(z, \eta)|}\), we have

\[
|G^1 \land G^2 \land G^3| \approx |\beta_1 - \beta_3|^2 |G^1_0 \land G^2_0 \land G^3_0| \geq c > 0.
\]

(3.29)

then the transversality condition is guaranteed. Following the approach in the proof of Proposition 2.3, one may verify that the corresponding phase and amplitude \((\tilde{\phi}, \tilde{a})\) is of type 1 datum.

By our assumption, we have \(\text{MS}^{\alpha, \varepsilon}_1(\lambda \alpha^2, R \alpha^2) \lesssim \varepsilon_1\). This yields

\[
\left\| \prod_{j=1}^{3} \left| \tilde{T}_{\lambda^2}^{\tau_0^j} f \right|^{\frac{1}{2}} \right\|_{L^p(w_{B_\alpha^2 R})} \leq C K_2^{O(1)} R^{\sigma + \varepsilon} \prod_{j=1}^{3} \left\| \left( \sum_{\nu_j} \left| \tilde{T}_{\lambda^2}^{\tau_0^j} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w_{B_\alpha^2 R})}
\]

\[
+ K_2^{O(1)} R^2 \left( \frac{\lambda}{R} \right)^{-3M} \left\| f \right\|_{L^2(\mathbb{R}^2)}.
\]

Undoing the transform and summing over \(\Lambda\), we arrive at
\[ \left\| \prod_{j=1}^{3} |\mathcal{F}_{\lambda} f|^{\frac{4}{3}} \right\|_{L^p(w_{BR})}^{p} \leq C_\varepsilon K_2^{O(1)} R^{(\sigma+\varepsilon)p} \left( \sum_{\nu} |\mathcal{F}_{\lambda} \nu f|^2 \right)^{\frac{1}{2}} \left\| f \right\|_{L^p(w_{BR})}^{p} + K_2^{O(1)} R^{2p} \left( \frac{\lambda}{R} \right)^{-3M_p} \|f\|_{L^2(\mathbb{R}^2)^{\frac{1}{2}}} \right\|_{L^p(w_{BR})}, \] 

(3.30)

Collecting (3.9)(3.10)(3.12) (3.13) (3.30), we obtain

\[ \| \mathcal{T}_\lambda f \|_{L^p(B_R)} \leq \left( C(S_1^{\sigma,2\varepsilon}(\lambda/K_1^2,R/K_1^2)(1/K_1^2)^{\sigma+2\varepsilon} + C_\varepsilon K_2^{O(1)} R^{-\varepsilon} \right. \]

\[ + C K_1^{O(1)} S_1^{\sigma,2\varepsilon}(\lambda/K_2^2,R/K_2^2)(1/K_2^2)^{\sigma+2\varepsilon} \right) R^{\sigma+2\varepsilon} \left( \sum_{\nu} |\mathcal{F}_{\lambda} \nu f|^2 \right)^{\frac{1}{2}} \left\| f \right\|_{L^p(w_{BR})}^{p} + \left( K_1^{-1} + K_1 K_2^{-1} + K_2^{O(1)} R^{-M} R^{-2} \right) R^{4} \left( \frac{\lambda}{R} \right)^{-2M} \|f\|_{L^2}. \]

Choose \( 1 \ll K_1 \) such that \( R/K_1^2 \leq R/2 \). Under the assumption \( R \leq \lambda^{1-\varepsilon}/2 \), we have \( R/K_1^2 \leq (\lambda/K_1^2)^{1-\varepsilon}/2 \).

By assumption (3.7), \( S_1^{\sigma,2\varepsilon}(\lambda/K_1^2,R/K_1^2) \leq C_\varepsilon \). In view of \( 1 \ll K_1 \ll K_2 \ll R \), we have \( C S_1^{\sigma,2\varepsilon}(\lambda/K_1^2,R/K_1^2) \leq C_\varepsilon \). Finally, we obtain

\[ \| \mathcal{T}_\lambda f \|_{L^p(B_R)} \leq C_\varepsilon \left( \sum_{\nu} |\mathcal{F}_{\lambda} \nu f|^2 \right)^{\frac{1}{2}} \left\| f \right\|_{L^p(w_{BR})}^{p} + R^{4} \left( \frac{\lambda}{R} \right)^{-2M} \|f\|_{L^2}, \]

which completes the proof. \( \square \)

4. \( L^3 \)-variable coefficient square function estimate via multilinear approach

In this section, we prove for \( 1 \leq R \leq \lambda^{1-\varepsilon}/2 \)

\[ \left\| \prod_{j=1}^{3} |\mathcal{F}_{\lambda} f|^{\frac{4}{3}} \right\|_{L^3(B_R)} \leq C_\varepsilon R^\varepsilon \left\| \left( \sum_{\nu} |\mathcal{F}_{\lambda} \nu f|^2 \right)^{1/2} \right\|_{L^3(w_{BR})} + \lambda^{-4M} \|f\|_{L^2(\mathbb{R}^2)}, \] 

(4.1)

which implies \( MS_1^{0,\varepsilon}(\lambda,R) \lesssim \varepsilon \).

The main ingredient of the proof is the Bennett-Carbery-Tao’s multilinear oscillatory integral estimates below.

**Theorem 4.1** ([3]). Under the transversality condition described as above, then for each \( \varepsilon > 0 \) the datum \((\phi,a)\) is of type \(A\), then there is a constant \( C > 0 \), depending only on \( \varepsilon, \Delta, A_2 \), for which

\[ \left\| \prod_{j=1}^{3} |\mathcal{F}_{\lambda} f_j|^{\frac{1}{3}} \right\|_{L^3(\mathbb{R})} \lesssim \varepsilon, \Delta, A_2 R^\varepsilon \prod_{j=1}^{3} \|f_j\|_{L^2(\mathbb{R}^2)}, \] 

(4.2)
As in [8], we may reduce (4.1) to a discretized version by exploiting a standard transference argument based on locally constant property. For this purpose, it is necessary to employ an additional decomposition of the oscillatory integral along the radial direction.

Let \( \rho \in C_c^\infty(\mathbb{R}^2) \) and satisfy

\[
\sum_{l \in \mathbb{Z}^2} \rho(\eta - l) \equiv 1, \forall \eta \in \mathbb{R}^2. \tag{4.3}
\]

Let \( \varepsilon > 0 \) be small and \( Q = \{Q_k\}_k \) be a mesh of cubes of sidelength \( R^{1/2-\varepsilon} \), which are centered at lattices belong to \( R^{1/2-\varepsilon}\mathbb{Z}^{2+1} \) with sides parallel to the axis and form a tiling of \( \text{supp}_z a_\lambda(\cdot, \eta) \). For each \( Q_k \in Q \), let \( z_k \) be the center of \( Q_k \) and set

\[
\mathcal{T}_k^{\nu_j,l_j} f(z) = \int e^{i\phi(z,\eta)} a^{\nu_j,l_j}_\lambda(z,\eta)f(\eta)d\eta,
\]

\[
a^{\nu_j,l_j}_\lambda(z,\eta) = a^{\nu_j}(z,\eta)\rho(R^{1/2}\partial_x\phi(z_k,\eta) - l_j).
\]

Obviously

\[
\sum_{l \in \mathbb{Z}^2} \rho(R^{1/2}\partial_x\phi(z_k,\eta) - l) \equiv 1, \forall \eta \in \mathbb{R}^2. \tag{4.4}
\]

The support of \( \eta \to a^{\nu_j,l_j}_\lambda(z,\eta) \) is essentially contained in the cube of sidelength comparative to \( R^{-1/2} \) centered at \( \eta^{\nu_j,l_j}_k \), which is denoted as \( \mathcal{D}^{\nu_j,l_j}_k \). Ultimately we have

\[
\mathcal{T}_k^{\nu_j,l_j} f(z) \bigg|_{z \in Q_k} = \sum_{\nu_j} \mathcal{T}_k^{\nu_j} f(z), \quad \mathcal{T}_k^{\nu_j} f(z) \bigg|_{z \in Q_k} = \sum_{l_j} \mathcal{T}_k^{\nu_j,l_j} f(z). \tag{4.5}
\]

**Proposition 4.2.** Let \( Q_k \in Q \) be as above defined and each triplet \( (\nu_1, \nu_2, \nu_3) \) satisfies angular separation condition in the sense that

\[
\text{Ang}(\theta_{\nu_i}, \theta_{\nu_j}) \geq \varepsilon_0, \quad \text{for} \quad 1 \leq i < j \leq 3. \tag{4.6}
\]

Then

\[
\left\| \prod_{j=1}^3 \left| \sum_{\nu_j,l_j} e^{i\phi(z,\eta^{\nu_j,l_j}_k)} c_{\nu_j,l_j} \right|^{1/3} \right\|_{L^3(Q_k)} \lesssim R^{1/2} \prod_{j=1}^3 \left( \sum_{\nu_j,l_j} |c_{\nu_j,l_j}|^2 \right)^{1/2}. \tag{4.7}
\]

**Proof.** Without loss of generality, we may assume the center of \( B \) is \( 0 \) and normalise the phase function by setting \( \psi^\lambda(z,\eta) = \phi^\lambda(z,\eta) - \phi^\lambda(0,\eta) \). Let

\[
b_k^{\nu_j,l_j}(z,\eta_j) = \left( \int_{\mathcal{D}^{\nu_j,l_j}_k} e^{i[\psi^\lambda(z,\eta) - \psi^\lambda(z,\eta^{\nu_j,l_j}_k)]} d\eta_j \right)^{-1} \times \chi_{\mathcal{D}^{\nu_j,l_j}_k}(\eta_j). \tag{4.8}
\]

It is easy to see\(^2\)

\[
|b_k^{\nu_j,l_j}(z,\eta_j)| \leq R, \quad |\partial_\eta^\alpha b_k^{\nu_j,l_j}(z,\eta_j)| \leq C_\alpha R^{1-\frac{\text{dim}}{2}}, \quad \forall z \in Q_k. \tag{4.9}
\]

After multiplying \( c_{\nu_j,l_j} \) with a harmless factor \( e^{i\phi^\lambda(0,\eta^{\nu_j,l_j}_k)} \), we may evaluate the \( L^3 \) norm of the quantity in (2.31) by

\[
\left\| \prod_{j=1}^3 \left| \int e^{i\psi^\lambda(z,\eta_j)} \sum_{\nu_j,l_j} b_k^{\nu_j,l_j}(z,\eta_j) c_{\nu_j,l_j} d\eta_j \right|^{1/3} \right\|_{L^3(Q_k)}. \tag{4.10}
\]

\(^2\)See the proof of Lemma 4.6 in [14] for a similar fact.
Let
\[ B_k(z, \eta_1, \eta_2, \eta_3) = \prod_{j=1}^{3} \sum_{\nu_j, l_j} b_k^{\nu_j, l_j}(z, \eta_j) c_{\nu_j, l_j}. \]  
(4.11)

By the fundamental theorem of calculus,
\[
B_k(z, \eta_1, \eta_2, \eta_3) = B_k(0, \eta_1, \eta_2, \eta_3) + \int_0^{x_1} \partial u_1 B_k(u_1, 0, \eta_1, \eta_2, \eta_3) \, du_1 \\
+ \cdots + \int_0^{x_1} \int_0^{x_2} \int_0^{t} \frac{\partial^3 B_k}{\partial u_1 \partial u_2 \partial u_3}(u_1, \eta_1, \eta_2, \eta_3) \, du_1 \, du_2 \, du_3. 
\]  
(4.12)

We only handle \( B_k(0, \eta_1, \eta_2, \eta_3) \) for the others terms can be handled in the same way.

The explicit formula of \( B_k(0, \eta, \eta') \) reads
\[ B_k(0, \eta_1, \eta_2, \eta_3) = \prod_{j=1}^{3} \sum_{\nu_j, l_j} b_k^{\nu_j, l_j}(0, \eta_j) c_{\nu_j, l_j}. \]  
(4.13)

The contribution of (4.13) to (4.10) is bounded by
\[
\left\| \prod_{j=1}^{3} \int e^{i\psi^\lambda(z, \eta_j)} \sum_{\nu_j, l_j} c_{\nu_j, l_j} b_k^{\nu_j, l_j}(0, \eta_j) \, d\eta_j \right\|_{L^3}^{\frac{1}{3}}. 
\]  
(4.14)

Following the same approach in the proof of the Proposition 3.3, one may verify the transversality condition (3.1) for the triple product in (4.14). Thus, by Theorem 4.1, we have
\[
(4.14) \lesssim \prod_{j=1}^{3} \left\| \sum_{\nu_j, l_j} c_{\nu_j, l_j} b_k^{\nu_j, l_j}(0, \eta_j) \right\|_{L^2(d\eta_j)} \lesssim R^{1/2} \prod_{j=1}^{3} \left( \sum_{\nu_j, l_j} |c_{\nu_j, l_j}|^2 \right)^{1/6}. 
\]

The proof is complete. \( \square \)

We next introduce the locally constant property as we did in our previous paper.

**Definition 4.3** (locally constant property). Assume \( d \geq 1 \), given a function \( F : \mathbb{R}^d \to [0, \infty) \), we say \( F \) satisfies the locally constant property at scale of \( \rho \) if \( F(x) \approx F(y) \) whenever \( |x - y| \leq C_0 \rho \). Here the implicit constant in “≈” could depend on the structure constant \( C_0 \).

Let us define the extension operator \( E \) to be
\[
Ef(x, t) := \int_{\mathbb{R}^2} e^{i(x_\eta + th(\eta))} a_2(\eta) f(\eta) \, d\eta, 
\]  
(4.15)

where \( h(\eta) \) is a smooth away from origin and homogeneous of degree 1 with
\[
\text{rank } \partial_{\eta_\eta}^2 h = 1, \quad \text{for all } \eta \in \text{supp } a_2. 
\]  
(4.16)

Assume \( r \geq 1 \) and \( f \) is supported in a \( r^{-1} \) neighborhood of \( \eta_0 \in \text{supp } a_2 \), then \( \widehat{Ef} \) is contained in a ball of radius \( r^{-1} \), by uncertainty principle, one may roughly view \( |Ef| \) essentially as a constant at a scale of \( r \). However, that is not the case for the oscillatory operator \( \mathcal{F}_\lambda \), since \( \mathcal{F}_\lambda f \) is not necessarily compactly supported. One may nevertheless, up to phase rotation and a negligible term, recover the locally constant property.
Lemma 4.4 ([9]). Let $\mathcal{F}_\lambda$ be given by (2.7). There exists a smooth, rapidly decreasing function $\varrho : \mathbb{R}^3 \to [0, \infty)$ with the following property: supp$\varrho \subset B(0,1)$ such that if $\varepsilon > 0$ and $f$ is supported in a $r^{-1}$-cube centered at $\bar{\eta}$ with $1 \leq r \leq \lambda^{1-\varepsilon}$, then
\[
e^{-i\phi(z,\bar{\eta})} \mathcal{F}_\lambda f = \left[ e^{-i\phi(z,\bar{\eta})} \mathcal{F}_\lambda f \right] * \varrho_r + \text{RapDec}(\lambda) \|f\|_{L^2(\mathbb{R}^2)}
\] (4.17)
holds, where $\varrho_r(z) = r^{-3}\varrho(z/r)$.

One may further choose $\varrho$ to satisfy the locally constant property at the scale of 1. Correspondingly, one may view $\varrho_r$ as a constant at scale of $r$.

Now, we may complete the proof of (4.1). We start with proving the following estimate

Lemma 4.5. For $z \in Q_k$, if we denote by
\[
\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f(z) = \left( e^{-i\phi(z,\eta_k^{\nu_j,l_j})} \mathcal{F}_{\lambda,k}^{\nu_j,l_j} f \right)(z),
\]
then we have
\[
\sum_{Q_k \in \mathcal{Q}} \left\| \prod_{j=1}^3 \left( \sum_{\nu_j,l_j} e^{i\phi(z,\eta_k^{\nu_j,l_j})} (\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f) * \varrho_R \right) \right\|_{L^3(Q_k)} ^{\frac{1}{3}} \leq \lambda^{-N} \|f\|_{L^2}.
\] (4.18)

By combining (4.18) with the following lemma established in our previous paper [8], we complete the proof of (4.1).

Lemma 4.6.
\[
\left\| \left( \sum_{\nu_j,l_j} |\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^3(wQ_k)} \lesssim \left\| \left( \sum_{\nu_j} |\mathcal{H}_{\lambda}^{\nu_j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^3(wQ_k)} + \lambda^{-N} \|f\|_{L^2}.
\] (4.19)

Indeed, applying (4.19) to the right hand side of (4.18), summing over $Q_k \in \mathcal{Q}$ and applying Cauchy-Schwarz’s inequality, we obtain (4.1) and hence (2.11) by Proposition 3.3. Thus, it remains to prove Lemma 4.5.

Proof of Lemma 4.5. By Minkowski’s inequality and the locally constant property at scale $R$ enjoyed by $\varrho_R$, we have
\[
\left\| \prod_{j=1}^3 \left( \sum_{\nu_j,l_j} e^{i\phi(z,\eta_k^{\nu_j,l_j})} (\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f) * \varrho_R \right) \right\|_{L^3(Q_k)} ^{\frac{1}{3}}
\] (4.20)
is bounded by
\[
\iint \left\| \prod_{j=1}^3 \left( \sum_{\nu_j,l_j} e^{i\phi(z,\eta_k^{\nu_j,l_j})} (\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f(y_j))^\ast \varrho_R(\bar{z} - y_j) \right) \right\|_{L^3(Q_k)} ^{\frac{1}{3}} \prod_{j=1}^3 \varrho_R(\bar{z} - y_j) dy_j,
\] (4.21)
whenever $\bar{z} \in Q_k$. If we use Proposition 4.2, we obtain the following bound
\[
(4.21) \lesssim R^{1/2} \prod_{j=1}^3 \int \left( \sum_{\nu_j,l_j} |\mathcal{H}_{\lambda,k}^{\nu_j,l_j} f(\bar{z} - y_j)|^2 \right)^{\frac{1}{6}} \varrho_R(y_j) dy_j
\] (4.22)
which is bounded by, after averaging over $Q_k$ in $\tilde{z}$–variable, and neglecting RapDec($\lambda$) terms

$$\prod_{j=1}^{3} \int \left\| \left( \sum_{\nu_j,l_j} |\mathcal{F}_{\lambda,k}^{\nu_j,l_j} f(\tilde{z} - y_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^3(\bar{Q}_k)}^3 \theta_R(y_j) dy_j$$

(4.23)

By Hölder’s inequality, we have

$$\int \left\| \left( \sum_{\nu_j,l_j} |\mathcal{F}_{\lambda,k}^{\nu_j,l_j} f(\tilde{z} - y)|^2 \right)^{\frac{1}{2}} \right\|_{L^3(\bar{Q}_k)}^3 \theta_R(y) dy \lesssim \left\| \left( \sum_{\nu_j,l_j} |\mathcal{F}_{\lambda,k}^{\nu_j,l_j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^3(\bar{Q}_k)}^3$$

where we have used the following fact,

$$\int_{\mathbb{R}^3} w_{Q_k}(z + y) \theta_R(y) dy \lesssim w_{Q_k}(z).$$

Summing over $Q_k \in \mathcal{Q}$ and using Hölder’s inequality, we conclude the proof of (4.18). □

5. $L^4$–variable coefficient square function estimate via decoupling

In this section, we will use decoupling theorem and the induction on scale argument to establish (2.12).

Let us start with a few notations. Let $\{\theta_k\}$ to be a family of sectors each stretching an angle $\approx R^{-1/4}$ and define

$$\mathcal{P}_\lambda f := \sum_{\nu, \theta \in \theta_k} \mathcal{P}_\lambda^{\nu} f.$$ (5.1)

Assume $\delta > 0$ and $1 \leq K \leq \lambda^{1/2-\delta}$. For a given point $\tilde{z} \in \text{supp}_z a_\lambda$, we take Taylor expansion of $\phi^\lambda$ around $\tilde{z}$ and make a change of variable: $\eta \to \Psi(\tilde{z} / \lambda, \eta) := \Psi(\tilde{z} / \lambda, \eta)$ to write

$$\mathcal{P}_\lambda f(z) = \int_{\mathbb{R}^d} e^{i\langle z - \tilde{z}, \partial_v \phi^\lambda(\tilde{z}, \Psi(\tilde{z} / \lambda, \eta)) \rangle + i\epsilon^\lambda(z - \tilde{z}, \eta)} a_{\lambda, \tilde{z}}(z, \eta) f_\tilde{z}(\eta) d\eta, \quad \text{for } |z - \tilde{z}| \leq K,$$ (5.2)

where $f_\tilde{z} := e^{i\phi^\lambda(\tilde{z}, \Psi(\tilde{z}, \cdot))} f \circ \Psi(\tilde{z}, \cdot)$ and

$$a_{\lambda, \tilde{z}}(z, \eta) = a_\lambda(z, \Psi(\tilde{z}, \eta)) |\det \partial_\eta \Psi^\lambda(\tilde{z}, \eta)|,$$

$$\epsilon^\lambda(z, \eta) = \frac{1}{\lambda} \int_0^1 (1 - s) \langle (\partial_{\tilde{z}}^2 \phi)(\tilde{z} + sv, \lambda, \Psi^\lambda(\tilde{z}, \eta)) v, v \rangle ds, \quad \text{for } |v| \leq K.$$

For $\lambda \gg 1$ and thanks to the assumption (D\_\lambda), we have

$$\sup_{(v, \eta) \in B(0, K) \times \text{supp}_\eta a_{\lambda, \tilde{z}}} |\partial_\eta^\beta \epsilon^\lambda(z, \eta)| \leq 1, \quad \text{for } |v| \leq K,$$ (5.3)

where $\beta \in \mathbb{N}^2$ and $|\beta| \leq N$. By using (2.13), we obtain

$$\langle z, \partial_\eta \phi^\lambda(\tilde{z}, \Psi^\lambda(\tilde{z}, \eta)) \rangle = x \eta + h_\tilde{z}(\eta),$$ (5.4)

with $h_\tilde{z}(\eta) := (\partial_t \phi^\lambda)(\tilde{z}, \Psi^\lambda(\tilde{z}, \eta)).$
Since we assume \( a(z, \eta) = a_1(z)a_2(\eta) \), by neglecting the influence of spatial variables, we may approximate \( \mathcal{T}_\lambda \) in a suitable manner by extension operators \( E_{\bar{z}} \) at a sufficiently small neighborhood of \( \bar{z} \), where

\[
E_{\bar{z}} g(z) := \int_{\mathbb{R}^2} e^{i(x\eta + t\bar{z}(\eta))} a_{2,\bar{z}}(\eta) g(\eta) d\eta,
\]
with \( a_{2,\bar{z}}(\eta) = a_2(\Psi^\lambda(\bar{z}, \eta)) |\det \partial_{\eta} \Psi^\lambda(\bar{z}, \eta)| \). It is clear that \( h_{\bar{z}}(\eta) \) is homogeneous of degree 1 and satisfying

\[
\text{rank } \partial_{\eta}^2 h_{\bar{z}} = 1, \quad \text{for all } \eta \in \text{supp } a_{2,\bar{z}}.
\]
Due to the compactness of the support of \( a \), we may assume the nonvanishing eigenvalue of \( \partial_{\eta}^2 h_{\bar{z}}(\eta) \) is comparable to 1 and is independent of \( \bar{z} \).

Based on Proposition 3.3, it suffices to prove its multilinear version. First, One may carry over \textit{mutis mutandis} the approach in [2] to prove the following variable variant of \( l^2 \) decoupling theorem.

**Theorem 5.1.** Let \( 1 \leq R \leq \lambda \) and \( \mathcal{T}_\lambda \) be defined as above, for \( 2 \leq p \leq \infty \), for all \( \varepsilon > 0 \) we have

\[
\| \mathcal{T}_\lambda f \|_{L^p(B_R)} \lesssim_{\varepsilon, \phi, N, a} R^{\alpha(p) + \varepsilon} \| \mathcal{T}_\lambda f \|_{L^{p,R}(w_{B_R})} + \lambda^{-N} \| f \|_{L^2},
\]

where

\[
\alpha(p) = \begin{cases} 0, & 2 \leq p \leq 6, \\ \frac{1}{4} - \frac{3}{2p}, & 6 \leq p \leq \infty. \end{cases}
\]

**Remark 5.2.** The specific form of Theorem 5.1 did not appear in [2], however [2] genuinely provided a mechanism for transferring Bourgain-Demeter’s \( l^2 \)-decoupling theorem to the variable coefficient setting with an additional convexity assumption on the phase function. The key point is that one may approximate the oscillatory integrals of Hörmander’s type at sufficiently small spatial scale by \( E_{\bar{z}} \). The convexity condition is superfluous in this paper since we only consider the two dimensional case.

On one hand, by Hölder’s inequality, one may obtain the following

\[
\| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)} \leq \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}^\gamma \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}^{1-\gamma} \leq C \lambda R^\gamma \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}, \quad \frac{1}{p} = \frac{\gamma}{p_1} + \frac{1-\gamma}{p_2},
\]

for \( 0 \leq \gamma \leq 1 \) and \( 1 \leq p, p_1, p_2 \leq \infty \) where \( \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p} \) is defined in the same fashion as in (2.19).

On the other hand, the reverse direction of (5.8) fails to hold in general. Nevertheless, Bourgain and Demeter [5] observed that if a function is spectrally supported in a neighborhood of a curved hypersurface, then based on wave-packet decompositions, it is possible to establish reverse Hölder’s inequalities for such a function by losing a \( \varepsilon \)-power of scales. In particular, one may establish the following reverse Hölder inequality for conic surfaces. See [5, 11] for more details.

**Lemma 5.3** (Reverse Hölder inequality [11]). Let the extension operator \( E_{\bar{z}} \) be defined as in (5.5), then

\[
\| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}^\gamma \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}^{1-\gamma} \leq Cz R^\gamma \| E_{\bar{z}} f \|_{L_{p,\text{Dec}}^p(B_R)}, \quad \frac{1}{p} = \frac{\gamma}{p_1} + \frac{1-\gamma}{p_2},
\]

for \( 0 \leq \gamma \leq 1, 1 \leq p, p_1, p_2 \leq \infty \).
The following lemma is established in [2].

**Lemma 5.4.** Let $0 < \delta < 1/2$, if $1 \leq R \leq \lambda^{1/2} - \delta$, provided $N$ is sufficiently large depending on $\delta, p$, then

\[
\begin{align*}
\left\| \mathcal{T}_z^\nu f \right\|_{L^p(w_{BR})} &\lesssim N \left\| E_z^\nu f \right\|_{L^p(w_{BR})} + \lambda^{-\delta N/2} \left\| f \right\|_{L^2}, \\
\left\| E_z^\nu f \right\|_{L^p(w_{BR})} &\lesssim N \left\| \mathcal{T}_z^\nu f \right\|_{L^p(w_{BR})} + \lambda^{-\delta N/2} \left\| f \right\|_{L^2}.
\end{align*}
\]  

(5.10)

(5.11)

where $\bar{z}$ is the center of $B_R$ and the operator $E_z^\nu$ is defined by

\[
E_z^\nu f(x, t) := \int_{\mathbb{R}^2} e^{i(x, \eta + th_z(\eta))} a_{z, \nu}^\nu(\eta) f(\eta) d\eta,
\]

\[a_{z, \nu}^\nu(\eta) = a_{z}^\nu(\Psi^\lambda(\bar{z}, \eta)) |\det \partial_\eta \Psi^\lambda(\bar{z}, \eta)|.
\]

We will also need the following locally orthogonality property.

**Lemma 5.5.** Let the operator $E_z$ be defined as above, then

\[
\left( \sum_{\nu} \left\| E_{z}^\nu f \right\|_{L^2(w_{B\sqrt{R}})}^2 \right)^{1/2} \leq \left\| E_z f \right\|_{L^2(w_{B\sqrt{R}})} + R^{-\varepsilon N} \left\| f \right\|_{L^2}.
\]  

(5.12)

where $E_z f := \sum_{\nu} E_{z}^\nu f$.

**Proof.** Due to the fast decay of the weight $w_{\sqrt{R}}$ away from $|z| \geq R^{1/2 + \varepsilon/4}$, it suffices to consider

\[
\left( \sum_{\nu} \left\| 1_{\{|x| \leq R^{1/2 + \varepsilon/4}\}} (E_{z}^\nu f) \right\|_{L^2(w_{B\sqrt{R}})}^2 \right)^{1/2}.
\]  

(5.13)

Freeze time $t_0$ and note that

\[
E_{z}^\nu f(x, t_0) = \int_{\mathbb{R}^2} e^{i(x, \eta + th_z(\eta))} a_{z, \nu}^\nu(\eta) f(\eta) d\eta
\]

where

\[
E_z f(x, t_0) = \int_{\mathbb{R}^2} e^{i(x, \eta + th_z(\eta))} a_{z}^\nu(\eta) f(\eta) d\eta, \quad \chi_{z, \nu}(\eta) = \chi_{\nu}(\Psi^\lambda(\bar{z}, \eta)).
\]

We further decompose

\[
E_z f(\cdot, t_0) = 1_{\{|x| \leq R^{1/2 + \varepsilon/2}\}} \cdot E_z f(\cdot, t_0) + 1_{\{|x| > R^{1/2 + \varepsilon/2}\}} \cdot E_z f(\cdot, t_0).
\]  

(5.14)

(5.15)

It remains to estimate

\[
\int e^{i(x, \eta)} \chi_{z, \nu}(\eta) \left( 1_{\{|x| \leq R^{1/2 + \varepsilon/2}\}} \cdot E_z f(\cdot, t_0) \right)^\wedge(\eta) d\eta.
\]

In fact for $|\bar{x}| \leq R^{1/2 + \varepsilon/2}$, the contribution of the second term in (5.15) to (5.14) equals

\[
\int \tilde{\chi}_{z, \nu}(\bar{x} - y) 1_{\{|x| > R^{1/2 + \varepsilon/2}\}}(y) E_z f(y, t_0) dy \leq R^{-\varepsilon N} \left\| f \right\|_{L^2}.
\]  

(5.16)
Now unfreezing $t_0$, by Plancherel’s theorem, we have
\[
\left( \sum_{\nu} \left\| \int e^{i(x, \eta)} \chi_{\nu}(\eta) \left( 1_{\{ |\xi| \leq R^{\frac{1}{3} + \frac{1}{2}} \}} (\cdot) e^{i(\cdot, t)} \right)(\cdot) d\eta \right\|_{L^{2}(w_{B_{R}^{\sqrt{\eta}}})}^{2} \right)^{\frac{1}{2}} \lesssim \left\| E_{\xi} f \right\|_{L^{2}(w_{B_{R}^{\sqrt{\eta}}})}.
\]
This completes the proof of Lemma 5.5. \qed

The proof of (2.12) amounts to showing
\[
S^{\frac{1}{10} + \varepsilon}_{1}(\lambda, R) \lesssim_{\varepsilon} 1,
\]
which is deduced from the following bootstrapping argument.

**Lemma 5.6.** If there exists an $\alpha \geq 0$ such that $S^{\alpha + \varepsilon}_{1}(\lambda, R) \lesssim 1$ holds for all $1 \leq R \leq \lambda^{1-\varepsilon/2}$, then $S^{3\alpha + \varepsilon}_{1}(\lambda, R) \lesssim 1$, with $\beta = \frac{1}{24} + \frac{\alpha}{3} + 2\varepsilon$.

**Proof.** By Proposition 3.3, it suffices to show $MS^{\frac{1}{10} + \frac{1}{3} + 2\varepsilon}_{1}(\lambda, R) \lesssim_{\varepsilon} 1$ for all $1 \leq R \leq \lambda^{1-\varepsilon/2}$.

Modifying the approach in the proof of (4.1), one may obtain
\[
\left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(B_{R})} \leq C_{\varepsilon} R^{-\frac{1}{3} + \varepsilon} \prod_{j=1}^{3} \left\| \mathcal{T}^{j}_{\lambda} f \right\|_{L^{6,R}(B_{R})}^{\frac{1}{2}} + \lambda^{-N} \left\| f \right\|_{L^{2}},
\]
Indeed, it just needs to replace $L^{3}$ norm with $L^{2}$ norm in (4.23).

Using Theorem 5.1, we have
\[
\left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(B_{R})} \leq C R^{\varepsilon} \prod_{j=1}^{3} \left\| \mathcal{T}^{j}_{\lambda} f \right\|_{L^{6,R}(B_{R})}^{\frac{1}{2}} + \lambda^{-N} \left\| f \right\|_{L^{2}},
\]
For the sake of brevity, we neglect the influence of all the fast decreasing error terms in the following estimates.

Let $\{Q_{k}\}$ be a class of finitely-overlapping cubes of sidelength comparative to $R^{1/2}$ that together form a cover of $B(0, R)$. By interpolation
\[
\left( \sum_{k} \left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(Q_{k})}^{4} \right)^{\frac{1}{4}} \leq \left( \sum_{k} \left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(Q_{k})}^{2} \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right)^{\frac{1}{4}} \left( \sum_{k} \left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(Q_{k})}^{2} \right)^{\frac{1}{4}}
\]
Using the Lemma 5.4, we have
\[
\left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{6}(Q_{k})} \lesssim R^{\frac{1}{3} + \varepsilon} \prod_{j=1}^{3} \left\| \mathcal{T}^{j}_{\lambda} f \right\|_{L^{6,R}(Q_{k})}^{\frac{1}{2}} \lesssim R^{\frac{1}{3} + \varepsilon} \prod_{j=1}^{3} \left\| \mathcal{T}^{j}_{\lambda} f \right\|_{L^{6,R}(Q_{k})}^{\frac{1}{2}}.
\]
Similarly, by (5.19) and Lemma 5.5, we obtain
\[
\left\| \prod_{j=1}^{3} |\mathcal{T}^{j}_{\lambda} f|^{\frac{1}{2}} \right\|_{L^{2}(Q_{k})} \leq C_{\varepsilon} R^{-\frac{1}{3} + \varepsilon} \prod_{j=1}^{3} \left\| E^{j}_{2k} f \right\|_{L^{2,R}(Q_{k})}^{\frac{1}{2}}
\]
\[
\leq C_{\varepsilon} R^{-\frac{1}{3} + \varepsilon} \prod_{j=1}^{3} \left\| E^{j}_{2k} f \right\|_{L^{2,R}(Q_{k})}^{\frac{1}{2}}.
\]

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Plugging (5.22) and (5.23) into (5.21) and using reverse Hölder’s inequality (5.9) yield

\[ \left( \sum_k \left| \prod_{j=1}^{3} |\mathcal{T}_j f|_2^4 \right|^{\frac{1}{4}} \right)^\frac{1}{3} \]

\[ \lesssim \varepsilon R^{-\frac{1}{3} + \frac{2}{3}\varepsilon} \left( \sum_k \left( \prod_{j=1}^{3} \|E_{z_k} f\|_{L_{Dec}^2(Q_k)}^2 \|E_{z_k}^j f\|_{L_{Dec}^2(Q_k)}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

\[ \lesssim \varepsilon R^{-\frac{1}{3} + \frac{2}{3}\varepsilon} \left( \sum_k \left( \prod_{j=1}^{3} \|E_{z_k} f\|_{L_{Dec}^2(Q_k)}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

\[ \lesssim \varepsilon R^{-\frac{1}{3} + \frac{2}{3}\varepsilon} \left( \sum_k \left( \prod_{j=1}^{3} \|E_{z_k} f\|_{L_{Dec}^2(Q_k)}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

\[ \lesssim \varepsilon R^{-\frac{1}{3} + \frac{2}{3}\varepsilon} \left( \sum_k \left( \prod_{j=1}^{3} \|E_{z_k} f\|_{L_{Dec}^2(Q_k)}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

(5.24)

Owing to the orthogonality property and Lemma 5.4, we have

\[ \left( \sum_k \|E_{z_k} f\|_{L_{Dec}^2(Q_k)}^4 \right)^\frac{1}{4} \lesssim \left( \sum_k \left( \sum_{\nu_j} \|E_{z_k}^\nu f\|_{L^2(w(Q_k))}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

\[ \lesssim \left( \sum_k \left( \sum_{\nu_j} \|\mathcal{T}_{\lambda} f\|_{L^2(w(Q_k))}^2 \right)^{\frac{1}{4}} \right)^{\frac{1}{3}} \]

\[ \lesssim R^3 \|\mathcal{T}_{\lambda} f\|_{L_{S_0}^4(B_R)} \]  

(5.25)

Now we turn to estimate the remained term in (5.24). By Hölder’s inequality, Proposition 2.3, \( \ell^2 \subset \ell^4 \) and stability lemma 5.4, we have

\[ \left( \sum_k \|E_{z_k} f\|_{L_{Dec}^2(w(Q_k))}^4 \right)^\frac{1}{4} \lesssim R^{\frac{1}{16}} \left( \sum_k \sum_{\nu_j} \|E_{z_k}^\nu f\|_{L^4(w(Q_k))}^4 \right)^\frac{1}{4} \]

\[ \lesssim R^{\frac{1}{16}} S_1^{\alpha,\varepsilon} \left( \frac{\lambda}{R^2}, \frac{R}{R^2} \right) R^{2\frac{1}{8} + \frac{2\varepsilon}{3}} \left( \sum_k \sum_{\nu_j, \theta_{\nu_j} \subseteq \theta_{\varepsilon j}} \| \mathcal{T}_{\lambda}^\nu f \|_{L^4(w(Q_k))}^2 \right)^\frac{1}{2} \]

\[ \lesssim R^{\frac{1}{16} + \frac{1}{8} + \frac{2\varepsilon}{3}} \|\mathcal{T}_{\lambda} f\|_{L_{S_0}^4(B_R)} \]  

(5.26)

where \( E_{z_k} f = \sum_{\nu_j, \theta_{\nu_j} \subseteq \theta_{\varepsilon j}} E_{z_k}^\nu f \).

Inserting (5.25),(5.26) into (5.24), discarding the rapid decay term, finally we obtain

\[ \left\| \prod_{j=1}^{3} |\mathcal{T}_j f|_2 \right\|_{L^4(B_R)} \leq C \varepsilon R^{\frac{1}{16} + \frac{1}{8} + 2\varepsilon} \prod_{j=1}^{3} \|\mathcal{T}_{\lambda} f\|_{L_{S_0}^4(B_R)} \]  

(5.27)

This completes the proof of Lemma 5.6. \( \square \)
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