Collective Modes in a Dilute Bose-Fermi Mixture

S. K. Yip

Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan

Abstract

We here study the collective excitations of a dilute spin-polarized Bose-Fermi mixture at zero temperature, considering in particular the features arising from the interaction between the two species. We show that a propagating zero-sound mode is possible for the fermions even when they do not interact among themselves.

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Recent experimental progress in atomically trapped gases has led to a resurgence of interest in quantum fluids. A particular notable feature is the number of systems available, ranging from single component bose gas in the original experiments where Bose-Einstein Condensation (BEC) was first achieved [1] to binary bose mixture [2], spinor condensate in optical traps [3] and degenerate fermi gas [4,5]. Other systems have also received much recent attention, in particular bose-fermi mixture. This last mentioned system occurs naturally if ‘sympathetic cooling’ is employed to reduce the kinetic energy of the fermions [6]. There have already been several studies on the properties of this system. Questions addressed include stability against phase separation [7,8] and collective excitations [8].

Although Bose-Fermi mixtures have been studied intensively in low temperature physics in the context of $^3$He-$^4$He mixtures [4], the atomically trapped gases offer many additional possibilities. By the choice of atoms, concentration of the various components, or the control of interaction strength among them by external fields [10], one can unmask phenomena previously unobservable. In this paper, we shall study one example of this by considering the density oscillations of a bose-fermi mixture at low temperatures. We shall show that a variety of novel phenomena can arise due to the coupling between the two components for suitable parameters such as the ratio of the sound velocity of the bose gas to the fermi velocity of the fermions. In particular, we shall show that it is possible to have a propagating fermionic sound mode even in the absence of interaction among the fermions themselves. Sound propagation has also been considered in ref [3] which however did not investigate the effects being studied here. We shall comment on this later.

We shall then consider a mixture of weakly interacting bose and fermi gases at zero temperature. Both gases are assumed to be spin-polarized such as would be the case usually in magnetic traps. For a dilute mixture, interaction among the bosons themselves and between the bosons and fermions can be characterised by the scattering lengths $a_{bb}$ and $a_{bf}$ in the s-wave channels. The fermions however do not interact among themselves since they are spin-polarized. For simplicity we shall consider a uniform system. We shall further assume that the gas is stable against phase-separation unless explicitly specified. We are interested in the density waves of this system. As we shall see in general the modes may be damped. Also since the density oscillations are likely to be studied by exciting the systems...
with external potentials, we shall instead consider the density responses of the system under external perturbing potentials. Collective modes of the system will show up as resonances of these responses.

The Hamiltonian density is given by

$$H = \frac{\hbar^2}{2m_b} \nabla \psi_b^\dagger \nabla \psi_b - \mu_b \psi_b^\dagger \psi_b + \frac{\hbar^2}{2m_f} \nabla \psi_f^\dagger \nabla \psi_f - \mu_f \psi_f^\dagger \psi_f$$

$$+ \psi_b^\dagger \psi_b V_b^{\text{ext}} + \psi_f^\dagger \psi_f V_f^{\text{ext}}$$

(1)

where the subscripts $b$ and $f$ denote bosons and fermions respectively, $\psi_f, \psi_b$ are the field operators, $m_b, m_f$ the masses, $\mu_b, \mu_f$ are the chemical potentials, $V_b^{\text{ext}}$ and $V_f^{\text{ext}}$ the external potentials. All $\psi$’s and $V^{\text{ext}}$’s are implicitly at the same physical point $\vec{r}$ in space. The interaction parameters $g_{bf}$ and $g_{bb}$ are related to the scattering lengths $a_{bb}$ and $a_{bf}$ by $g_{bb} = \frac{4\pi\hbar^2 a_{bb}}{m_b}$ and $g_{bf} = \frac{2\pi\hbar^2 a_{bf}}{m_r}$ where $m_r$ is the reduced mass $\frac{1}{m_r} \equiv \frac{1}{m_f} + \frac{1}{m_b}$.

We shall treat the interaction $g_{bb}$ and $g_{bf}$ within the Bogoliubov and random phase approximation respectively [11]. The results can be written in the physically transparent form:

$$\delta n_b(q, \omega) = -\chi_b[g_{bf} \delta n_f + V_b^{\text{ext}}]$$

$$\delta n_f(q, \omega) = -\chi_f[g_{bf} \delta n_b + V_f^{\text{ext}}]$$

(2)

expressing the response of the bosons and fermions to the potentials due to the other species and the external perturbations (the terms in the square brackets). Here $\delta n_b(q, \omega), \delta n_f(q, \omega)$ are the deviations of the bosonic and fermionic densities from equilibrium at wavevector $q$ and frequency $\omega$,

$$\chi_b = -\frac{1}{g_{bb}} \left[ \frac{c_b^2 q^2}{\omega^2 - c_b^2 q^2 - (q^2/2m_b)^2} \right]$$

(3)

and

$$\chi_f = N_f \left[ 1 - \frac{\omega}{2v_f q} \ln \frac{\omega + v_f q}{\omega - v_f q} \right]$$

(4)

are the ($q$ and $\omega$ dependent) responses of the pure bosons and fermions systems respectively to effective external potentials. $N_f \equiv p_f m_f/2\pi^2$ is the density of states for the fermions. $(p_f = (6\pi^2 n_f)^{1/3}$ is the fermi momentum, $v_f = p_f/m_f$) For simplicity, in eq (11) I have already left out terms that are small if $q << p_f$. $\omega$ should be interpreted as having a small and positive part.

Eq (3) can be re-arranged as

$$\begin{pmatrix}
1 & g_{bf} \chi_b \\
g_{bf} \chi_f & 1
\end{pmatrix}
\begin{pmatrix}
\delta n_b \\
\delta n_f
\end{pmatrix}
= -
\begin{pmatrix}
\chi_b V_b^{\text{ext}} \\
\chi_f V_f^{\text{ext}}
\end{pmatrix}$$

(5)

Then finally
\[
\begin{pmatrix}
\delta n_b \\
\delta n_f
\end{pmatrix} = - \frac{1}{1 - g_{bf}^2 \chi_b \chi_f} \begin{pmatrix}
1 & -g_{bf} \chi_b \\
-g_{bf} \chi_f & 1
\end{pmatrix}
\begin{pmatrix}
\chi_b V_{b}^{\text{ext}} \\
\chi_f V_{f}^{\text{ext}}
\end{pmatrix}
\] (6)

In the case where \( g_{bf} = 0 \), \( \delta n_b = -\chi_b V_{b}^{\text{ext}} \) and \( \delta n_f = -\chi_f V_{f}^{\text{ext}} \) and the responses thus reduce to those of the pure boson and fermi gases. The corresponding formulas for \( \chi_b \) and \( \chi_f \) were already given in eq (3) and eq (4) above. Before we proceed we shall recall the behavior of these responses \[11\] and thus the collective modes. For simplicity we shall restrict ourselves to small wavevectors, \( i.e. \ q \ll m_b c_b \) and \( p_f \), and without loss of generality \( \omega > 0 \). The bosonic response \( \text{Im}\chi_b \) consists of a delta function at the excitation frequency \( \omega = c_b q \). This is due to the Bogoliubov mode which is purely propagating and undamped. For the fermions however, there is no collective behavior. The absorptive part, \( \text{Im}\chi_f \), is finite for a whole range of frequencies \( |\omega| < v_f q \), known as the particle-hole continuum, arising from the many possibilities of independent particle-hole excitations. \( \text{Re}\chi_b \) is simple. It is given by \( g_{bb}^{-1} \) at \( \omega = 0 \) and diverges to \( \pm \infty \) as \( \omega \to c_b q \) from below and above respectively. \( \text{Re}\chi_f \) is given by \( N_f \) at \( \omega = 0 \). It decreases with increasing \( \omega \), changes sign at around \( \omega \sim 0.83 v_f q \) and approaches \( -\infty \) as \( \omega \to v_f q \) from both above and below. For \( \omega > v_f q \), it remains negative with its magnitude gradually approaching zero as \( \omega \to \infty \).

Now we return to the bose-fermi mixture. The response \( \delta n_b \) to an external potential \( V_{b}^{\text{ext}} \) acting on the bosons only is given by \( \chi_b/(1 - g_{bf}^2 \chi_b \chi_f) \). The existence and the dispersion of the bosonic collective mode are determined by the solution to the equation \((\chi_b)^{-1} - g_{bf}^2 \chi_f = 0\), \( i.e. \)

\[
[-\omega^2 + c_b^2 q^2 + \left(\frac{q^2}{2m_b}\right)^2] - \left(\frac{g_{bf}}{g_{bb}}\right)^2 (c_b^2 q^2) \chi_f = 0
\] (7)

It will be convenient to discuss the normalized response

\[
\tilde{\chi}_b \equiv g_{bb} \chi_b/(1 - g_{bf}^2 \chi_b \chi_f)
\] (8)

\( \tilde{\chi}_b = 1 \) in the static limit \( (\omega = 0, q \to 0) \) when there is no boson-fermion interaction \( (g_{bf} = 0) \).

Similarly the fermionic response to an external potential acting on the fermions alone is \( \chi_f/(1 - g_{bf}^2 \chi_b \chi_f) \). We shall discuss the behavior of the normalized quantity

\[
\tilde{\chi}_f \equiv N_f^{-1} \chi_f/(1 - g_{bf}^2 \chi_b \chi_f)
\] (9)

The normalization is chosen such that \( \tilde{\chi}_f = 1 \) in the static limit \( (\omega = 0, q \to 0) \) when there is no boson-fermion interaction \( (g_{bf} = 0) \).

Before proceeding let us first examine the responses at \( \omega = 0 \). Stability requires that the density responses \( \chi_b/(1 - g_{bf}^2 \chi_b \chi_f) \) and \( \chi_f/(1 - g_{bf}^2 \chi_b \chi_f) \) be positive. Using the \( \omega = 0 \) values of \( \chi_b \) and \( \chi_f \) above, these necessary conditions can be rewritten as \( g_{bb} > 0 \) and \( W \equiv N_f g_{bf}^2 / g_{bb} < 1 \). Using the expression of \( N_f \) given earlier, the last inequality gives \( n_f^{1/3} g_{bf}^2 < \frac{2}{3} A g_{bb} \), where \( A \equiv \frac{\hbar^2}{2m_f} (6\pi^2)^{2/3} \) as defined in \[7\]. These conditions were derived earlier in \[7\] and \[8\] using slightly different considerations. For bosons and fermions with similar masses, we shall see shortly that \( W \), a dimensionless parameter, serves as a useful measure of the coupling between the bosons and fermions. If the bosons and fermions have
similar masses, $|W|$ is of order $|a_{bf}^2/a_{bb}n_f^{-1/3}|$ and thus typically small for dilute gases unless $|a_{bf}| >> |a_{bb}|$. We shall limit ourselves only to the cases where $|W|$’s are small.

We shall discuss now the behavior of $\tilde{\chi}_b$ and $\tilde{\chi}_f$ in turn. The results are qualitatively different depending on whether $c_b > c_f$. The velocity ratio $u \equiv c_b/v_f$ can be re-expressed as $u = \frac{m_f}{m_b} \frac{(4/3)^{-1/3} (\alpha a_{bb})^{1/2}}{n_f^{1/2}}$. The value of $u$ can basically be arbitrary without violating any stability criterion (not only the linear stability condition above but also others derived in [2])

**Bosonic Response:**

1. $c_b > v_f$: In this regime a propagating bosonic mode exists. It can be easily verified (e.g., graphically) that the mode frequency $\omega$ satisfies $\omega > c_b q$ ($> v_f q$). The original bosonic mode at $\omega = c_b q$ is pushed upwards by the particle-hole ‘modes’ lying below. Some examples are shown in Fig 1. This mode ‘repulsion’ is generally expected (c.f. coupled harmonic oscillators). It is however of interest to examine the microscopic nature of the mode. At the mode frequency both $\chi_b$ and $\chi_f$ are negative. Thus, e.g., if $g_{bf} > 0$, $\delta n_b$ and $\delta n_f$ are of the same sign (see eqn (5)). The repulsion between the two species provides the enhanced restoring force and oscillation frequency. This frequency shift is typically small since usually $W << 1$.

2. $c_b < v_f$: In this case the original bosonic mode lies inside the particle-hole continuum of the fermions. The bosonic mode is thus Landau damped. For weak-coupling the damping, thus the width of the response, can be estimated easily using eq(4) to be

$$\sim \frac{\pi N_f g_{bf}^2}{4 g_{bb}} \left[ \frac{c_b}{v_f} \right] (c_b q).$$

Examples are shown in Fig 2. There is a small shift of the mode due to $\text{Re} \chi_f$. The shift is towards higher frequency for $u$ sufficiently close to 1 but opposite otherwise ($\text{Re} \chi_f < (>)$ 0 for $\omega/v_f q > (<) 0.83$.)

3. It is also of interest to study the bosonic mode for $g_{bb} < 0$. This is in fact the case for the $^6\text{Li}-^7\text{Li}$ mixture investigated in ref [4], where the $^7\text{Li}$ bosons have a negative scattering length of $\approx -1.5 \text{nm}$. In this case the original bosonic system is unstable, and the Bogoliubov mode has an imaginary frequency for sufficiently small wavevector ($q < q_c = 2 m_b |c_b|/\hbar$, here $|c_b| \equiv |g_{bb} n_b / m_b|^{1/2}$). Since $N_f g_{bf}^2 > 0 > g_{bb}$, the system is still unstable in the presence of the fermions [6] (see also above). Of interest is the effect of the fermions on the unstable mode. Now for imaginary frequencies $\omega = i \alpha$, $\chi_f(q, i \alpha) = N_f \{ 1 - \frac{g_{bf}}{g_{bb}} \left[ \frac{q^2}{2} - \tan^{-1} \frac{g_{bf}}{g_{bb}} q_{i \alpha} \right] \}$ is purely real and positive. $\chi_f$ decreases monotonically with $\alpha$ from $\chi_f = N_f$ at $\alpha = 0$ to 0 as $\alpha \rightarrow \infty$. It can be easily verified that there is a real solution for $\alpha$ to the dispersion relation (c.f. eq (7))

$$[\alpha^2 - |c_b|^2 q^2] - \left( \frac{g_{bf}}{g_{bb}} \right) (|c_b|^2 q^2) \chi_f(q, i \alpha) = 0$$

(10)

for sufficiently small $q$ (which includes the physically most relevant region where $\alpha$ attains its maximum, i.e., the fastest growing instability). Thus the instability is not damped by the particle-hole degree of freedom. In fact it can be verified easily that, for given $q$, $\alpha$ is increased in the presence of the fermions. The system has become even more unstable. This mode has $\delta n_b$ and $\delta n_f$ of opposite signs and corresponds to phase-separation as expected.

**Fermionic Response:**

1. $c_b > v_f$: In this case the fermionic response for $0 < \omega < q v_f$ is only slightly modified.
A new feature appears near $\omega \sim c_b q > v_f q$ due to the coupling to the bosonic mode. An example is as shown in Fig. 3.

2. $c_b < v_f$: In this regime there are two important features of the fermionic response. If $u = \frac{c_b}{v_f}$ is sufficiently close to 1, the imaginary part contains a sharp resonance at $\omega$ above the particle-hole continuum (Fig 4). There are two ways of understanding this mode. It can be regarded as the continuation of the situation from $c_b > v_f$, i.e., it is due to the bosonic mode which is itself slightly pushed up in frequency (c.f., Fig 3, note in particular the result for $u = 1$). Alternatively, this mode can be considered as a zero-sound mode induced by the bosons. The form for $\tilde{\chi}_f$ in eq (9) is precisely that of an interacting fermi gas with s-wave interaction $g_{ff}$ (and therefore necessary with more than one spin species, where the response is given by $\chi_f / (1 + g_{ff} \chi_f)$) though with an effective frequency dependent interaction $g_{ff} \rightarrow -g_{bf}^2 \chi_b$, i.e. an effective s-wave Landau parameter given by $F_0 \rightarrow W/[(\frac{1}{2})^2 - 1]$. The bosonic mode $\omega \sim c_b q$ for $c_b$ sufficiently close to but below $v_f$ will thus induce a zero-sound mode for the fermions just like an interaction among the fermions will. [12] Note however there cannot be a real s-wave interaction among the fermions as they are of equal spins. Thus this mode cannot be obtained by considering the effective interaction among the fermions as in ref [3].

The frequency of this propagating mode can be estimated by using the well-known dispersion relation of the zeroth sound $\omega/v_f q \approx 1 + 2e^{-2(1 + \frac{1}{c_b})}$ with the effective $F_0 \rightarrow W/[(\frac{1}{2})^2 - 1]$ as suggested above. In order for the velocity of the mode to be say 1% above $v_f$, then $c_b$ has to be within around 7% of $v_f$ if $W = 0.1$. This estimate agrees with the numerical results of Fig 4.

The second interesting feature is that near the original bosonic mode frequency $\omega \sim c_b q$, there is a reduction in the absorptive part $\text{Im} \tilde{\chi}_f$ (see Fig 3). In fact $\text{Im} \tilde{\chi}_f \rightarrow 0$ as $\omega \rightarrow c_b q$. This, as well as the corresponding behavior of $\text{Re} \tilde{\chi}_f$, can be seen easily mathematically from eq(5) due to the resonance nature of $\chi_b$ at this frequency. Physically this can be regarded as due to mode-mode repulsion – the bosonic mode has pushed away the particle-hole ‘modes’ near $\omega \sim c_b q$. This feature is present even for small coupling $W$. A larger $W$ mainly increases the width of this ‘transparent’ region. Thus in fact the frequency dependence of $\text{Re} \tilde{\chi}_f$ is actually stronger for smaller $W$’s.

The energy absorption by the bose-fermi mixture from an external perturbation acting on the fermions is thus substantially reduced for frequencies within this ‘transparent region’. The width of this region can be estimated by using the observation that the fermionic response is roughly reduced by the factor $1 + W^2 (c_b q)^2$ for these frequencies. For the fermionic response to be reduced to say 1/2 of its bare value, then $|\omega - c_b q|/c_b q < W/2$. This estimate agrees very roughly with the numerical results in Fig. 4.

In conclusion, I have investigated the collective modes of a Bose-Fermi mixture, and have shown that there is important mode-mode coupling effects, especially if $v_f \sim c_b$.

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[12] One can check (e.g., graphically) that there is a real solution for \( \omega \) to the equation \( \chi_f^{-1} - g_{bf}^2 \chi_b \), i.e., the mode is undamped.
FIG. 1. Dimensionless bosonic responses $\text{Re}\tilde{\chi}_b$ and $\text{Im}\tilde{\chi}_b$ for $u \equiv c_b/v_f > 1$, $W = 0.01$. Lines for the imaginary parts are decorated with circles.

FIG. 2. Same as Fig. 1 but for $u \equiv c_b/v_f < 1$, $W = 0.01$. 
FIG. 3. Dimensionless fermionic responses $\text{Re}\tilde{\chi}_f$ and $\text{Im}\tilde{\chi}_f$ for $u \equiv c_b/v_f \geq 1$, $W = 0.1$. The imaginary parts (lines decorated with circles) contain the particle-hole continua $\omega < v_f q$ and sharp spikes at the bosonic mode frequencies.

FIG. 4. Dimensionless fermionic response $\text{Im}\tilde{\chi}_f$ for $u \equiv c_b/v_f < 1$ showing the zero-sound modes induced by the bosons. $W = 0.1$. Also shown is $\text{Im}\chi_f$ for the pure fermi gas ($g_{bf} = 0$) for comparison.
FIG. 5. Same as Fig. 4 except that now \( \text{Re} \tilde{\chi}_f \) is shown.

FIG. 6. Imaginary part of dimensionless fermionic response \( \text{Im} \tilde{\chi}_f \) for \( u \equiv c_b/v_f < 1 \), showing mainly the region \( \omega < v_f q \).