Minimax theorems in a fully non-convex setting

Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his 75th birthday

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Abstract. In this paper, we establish two minimax theorems for functions $f : X \times I \to \mathbb{R}$, where $I$ is a real interval, without assuming that $f(x, \cdot)$ is quasi-concave. Also, some related applications are presented.

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The most known minimax theorem ([7]) ensures the occurrence of the equality

$$\sup_Y \inf_X f = \inf_X \sup_Y f$$

for a function $f : X \times Y \to \mathbb{R}$ under the following assumptions: $X$, $Y$ are convex sets in Hausdorff topological vector spaces, one of them is compact, $f$ is lower semicontinuous and quasi-convex in $X$, and upper semicontinuous and quasi-concave in $Y$.

In the past years, we provided some contributions to the subject where, keeping the assumption of quasi-concavity on $f(x, \cdot)$, we proposed alternative hypotheses on $f(\cdot, y)$. Precisely, in [2], we assumed the inf-connectedness of $f(\cdot, y)$ and, at the same time, that $Y$ is a real interval, while, in [5], we assumed the inf-compactness and uniqueness of the global minimum of $f(\cdot, y)$.

In the present paper, we offer a new contribution where the hypothesis that $f(x, \cdot)$ is quasi-concave is no longer assumed.

Let $T$ be a topological space. A function $g : T \to [-\infty, +\infty]$ is said to be relatively inf-compact if, for each $r \in \mathbb{R}$, there exists a compact set $K \subseteq T$ such that $g^{-1}([-\infty, r]) \subseteq K$. Moreover, $g$ is said to be inf-connected if, for each $r \in \mathbb{R}$, the set $g^{-1}([-\infty, r])$ is connected. For the basic notions on multifunctions, we refer to [1].

Our main results are as follows:

**THEOREM 1.** Let $X$ be a topological space, let $I$ be a real interval and let $f : X \times I \to \mathbb{R}$ be a continuous function such that, for each $\lambda \in I$, the set of all global minima of the function $f(\cdot, \lambda)$ is connected. Moreover, assume that there exists a non-decreasing sequence of compact intervals, $\{I_n\}$, with $I = \bigcup_{n \in \mathbb{N}} I_n$, such that for each $n \in \mathbb{N}$, the following conditions are satisfied:

(a) the function $\inf_{\lambda \in I_n} f(\cdot, \lambda)$ is relatively inf-compact;
(b) for each $x \in X$, the set of all global maxima of the restriction of the function $f(x, \cdot)$ to $I_n$ is connected.

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f$$

**THEOREM 2.** Let $X$ be a topological space, let $I$ be a compact real interval and let $f : X \times I \to \mathbb{R}$ be an upper semicontinuous function such that $f(\cdot, \lambda)$ is continuous for all $\lambda \in I$. Assume that:

(a) there exists a set $D \subseteq I$, dense in $I$, such that the function $f(\cdot, \lambda)$ is inf-connected for all $\lambda \in D$;
(b) for each $x \in X$, the set of all global maxima of the function $f(x, \cdot)$ is connected.

Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f$$
REMARK 1. - We want to remark that, in both Theorems 1 and 2, it is essential that $I$ be a real interval. To see this, consider the following example. Take

$$X = I = \{(t,s) \in \mathbb{R}^2 : t^2 + s^2 = 1\}$$

and define $f : X \times I \to \mathbb{R}$ by

$$f(t,s,u,v) = tu + sv$$

for all $(t,s),(u,v) \in X$. Clearly, $f$ is continuous, $f(\cdot, \cdot, u, v)$ is inf-connected and has a unique global minimum, and $f(t,s,\cdot,\cdot)$ has a unique global maximum. However, we have

$$\sup_X \inf_I f = -1 < 1 = \inf_X \sup_I f .$$

The common key tool in our proofs of Theorems 1 and 2 is provided by the following general principle:

THEOREM A ([2], Theorem 2.2). - Let $X$ be a topological space, let $I$ be a compact real interval and let $S \subseteq X \times I$ be a connected set whose projection on $I$ is the whole of $I$.

Then, for every upper semicontinuous multifunction $\Phi : X \to 2^I$, with non-empty, closed and connected values, the graph of $\Phi$ intersects $S$.

Another known proposition which is used in the proof of Theorem 1 is as follows:

PROPOSITION A ([5], Proposition 2.1). - Let $X$ be a topological space, $Y$ a non-empty set, $y_0 \in Y$ and $f : X \times Y \to \mathbb{R}$ a function such that $f(\cdot, y)$ is lower semicontinuous for all $y \in Y$ and relatively inf-compact for $y = y_0$. Assume also that there is a non-decreasing sequence of sets $\{Y_n\}$, with $Y = \bigcup_{n \in \mathbb{N}} Y_n$, such that

$$\sup_{Y_n} \inf_X f = \inf_X \sup_{Y_n} f$$

for all $n \in \mathbb{N}$.

Then, one has

$$\sup_{Y} \inf_X f = \inf_X \sup_{Y} f .$$

A further result which is used in the proofs of Theorems 1 and 2 is provided by the following proposition which, in the given generality, is new:

PROPOSITION 1. - Let $X,Y$ be two topological spaces and let $f : X \times Y \to \mathbb{R}$ be a lower semicontinuous function such that $f(x,\cdot)$ is continuous for all $x \in X$. Moreover, assume that, for each $y \in Y$, there exists a neighbourhood $V$ of $y$ such that the function $\inf_{v \in V} f(\cdot, v)$ is relatively inf-compact. For each $y \in Y$, set

$$F(y) = \left\{ u \in X : f(u,y) = \inf_{x \in X} f(x,y) \right\} .$$

Then, the multifunction $F$ is upper semicontinuous.

PROOF. Let $C \subseteq X$ be a closed set. We have to prove that $F^{-1}(C)$ is closed. So, let $\{y_\alpha\}_{\alpha \in D}$ be a net in $F^{-1}(C)$ converging to some $\tilde{y} \in Y$. For each $\alpha \in D$, pick $u_\alpha \in F(y_\alpha) \cap C$. By assumption, there is a neighbourhood $V$ of $\tilde{y}$ such that the function $\inf_{v \in V} f(\cdot, v)$ is relatively inf-compact. Since the function $\inf_{x \in X} f(x,\cdot)$ is upper semicontinuous, we can assume that it is bounded above on $V$. Fix $\rho > \sup_V \inf_X f$. Then, there is a compact set $K \subseteq X$ such that

$$\left\{ x \in X : \inf_{v \in V} f(x,v) < \rho \right\} \subseteq K .$$

But

$$\left\{ x \in X : \inf_{v \in V} f(x,v) < \rho \right\} = \bigcup_{v \in V} \left\{ x \in X : f(x,v) < \rho \right\} .$$
and so
\[ \bigcup_{v \in V} \{ x \in X : f(x, v) < \rho \} \subseteq K. \tag{1} \]

Let \( \alpha_1 \in D \) be such that \( y_\alpha \in V \) for all \( \alpha \geq \alpha_1 \). Consequently, by (1), \( u_\alpha \in K \) for all \( \alpha \geq \alpha_1 \). By compactness, the net \( \{u_\alpha\}_{\alpha \in D} \) has a cluster point \( \tilde{u} \in K \). Clearly, \((\tilde{u}, \tilde{y})\) is a cluster point in \( X \times Y \) of the net \( \{(u_\alpha, y_\alpha)\}_{\alpha \in D} \). We claim that
\[ f(\tilde{u}, \tilde{y}) \leq \limsup_{\alpha} f(u_\alpha, y_\alpha). \]

Arguing by contradiction, assume the contrary and fix \( r \) so that
\[ \limsup_{\alpha} f(u_\alpha, y_\alpha) < r < f(\tilde{u}, \tilde{y}). \]
Then, there would be \( \alpha_2 \in D \) such that
\[ f(u_\alpha, y_\alpha) < r \]
for all \( \alpha \geq \alpha_2 \). On the other hand, since, by assumption, the set \( f^{-1}([r, +\infty[) \) is open, there would be \( \alpha_3 \geq \alpha_2 \) such that
\[ r < f(u_{\alpha_3}, y_{\alpha_3}) \]
which gives a contradiction. Now, fix \( x \in X \). Then, since \( u_\alpha \in F(y_\alpha) \), we have
\[ f(\tilde{u}, \tilde{y}) \leq \limsup_{\alpha} f(u_\alpha, y_\alpha) \leq \lim_{\alpha} f(x, y_\alpha) = f(x, \tilde{y}). \]
That is, \( \tilde{u} \in F(\tilde{y}) \). Since \( C \) is closed, \( \tilde{u} \in C \). Hence, \( \tilde{y} \in F^{-1}(C) \) and this ends the proof. \( \triangle \)

We now can prove Theorems 1 and 2.

**Proof of Theorem 1.** Fix \( n \in \mathbb{N} \). Let us prove that
\[ \sup_{I_n} \inf_X f = \inf_X \sup_{I_n} f. \tag{2} \]
Consider the multifunction \( F : I_n \to 2^X \) defined by
\[ F(\lambda) = \left\{ u \in X : f(u, \lambda) = \inf_{x \in X} f(x, \lambda) \right\} \]
for all \( \lambda \in I_n \). Thanks to Proposition 1, \( F \) is upper semicontinuous and, by assumption, its values are non-empty, compact and connected. As a consequence, by Theorem 7.4.4 of [1], the graph of \( F \) is connected. Let \( S \) denote the graph of the inverse of \( F \). So, \( S \) is connected as it is homeomorphic to the graph of \( F \). Now, consider the multifunction \( \Phi : X \to 2^{I_n} \) defined by
\[ \Phi(x) = \left\{ \mu \in I_n : f(x, \mu) = \sup_{\lambda \in I_n} f(x, \lambda) \right\} \]
for all \( x \in X \). By Proposition 1 again, the multifunction \( \Phi \) is upper semicontinuous and, by assumption, its values are non-empty, closed and connected. After noticing that the projection of \( S \) on \( I_n \) is the whole of \( I_n \), we can apply Theorem A. Therefore, there exists \((\tilde{x}, \tilde{\lambda}) \in S \) such that \( \tilde{\lambda} \in \Phi(\tilde{x}) \). That is
\[ f(\tilde{x}, \tilde{\lambda}) = \inf_{x \in X} f(x, \tilde{\lambda}) = \sup_{\lambda \in I_n} f(\tilde{x}, \lambda). \tag{3} \]
Clearly, (2) follows from (3). Now, the conclusion is a direct consequence of Proposition A. \( \triangle \)

**Proof of Theorem 2.** Arguing by contradiction, assume the contrary and fix a constant \( r \) so that
\[ \sup_{I_n} \inf_X f < r < \inf_X \sup_{I_n} f. \]
Let $G: I \to 2^X$ be the multifunction defined by

$$G(\lambda) = \{ x \in X : f(x, \lambda) < r \}$$

for all $\lambda \in I$. Notice that $G(\lambda)$ is non-empty for all $\lambda \in I$ and connected for all $\lambda \in D$. Moreover, the graph of $G$ is open in $X \times I$ and so $G$ is lower semicontinuous. Then, by Proposition 5.7 of [3], the graph of $G$ is connected and so the graph of the inverse of $G$, say $S$, is connected too. Consider the multifunction $\Phi : X \to 2^I$ defined by

$$\Phi(x) = \{ \mu \in I : f(x, \mu) = \sup_{\lambda \in I} f(x, \lambda) \}$$

for all $x \in X$. Notice that $\Phi(x)$ is non-empty, closed and connected, in view of $(b_2)$. By Proposition 1, the multifunction $\Phi$ is upper semicontinuous. Now, we can apply Theorem A. So, there exists $(\hat{x}, \hat{\lambda}) \in S$ such that $\hat{\lambda} \in \Phi(\hat{x})$. This implies that

$$f(\hat{x}, \hat{\lambda}) < r < \inf_{X} \sup_{\lambda \in I} f(\hat{x}, \lambda) = f(\hat{x}, \hat{\lambda})$$

which is absurd. $\triangle$

Here is an application of Theorem 1.

**THEOREM 3.** - Let $(H, \langle \cdot, \cdot \rangle)$ be a real inner product space, let $K \subset H$ be a compact and convex set, with $0 \notin K$, and let $f : X \to K$ be a continuous function, where

$$X = \bigcup_{\lambda \in \mathbb{R}} \lambda K .$$

Assume that there are two numbers $\alpha, c$, with

$$\inf_{x \in X} \| f(x) \| < c < \| f(0) \| ,$$

such that:

(a) $\{ x \in X : \langle x, f(x) \rangle = \alpha \} \subset \{ x \in X : \| f(x) \| < c \} ;$
(b) $\{ x \in X : c^2 \langle x, f(x) \rangle = \alpha \| f(x) \|^2 \} \subset \{ x \in X : \| f(x) \| \geq c \} .$

Then, there exists $\lambda \in \mathbb{R}$ such that the set

$$\{ x \in X : x = \lambda f(x) \}$$

is disconnected.

**PROOF.** Consider the function $\varphi : X \times \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x, \lambda) = \| x - \lambda f(x) \|^2 - c^2 \lambda^2 + 2\alpha \lambda$$

for all $(x, \lambda) \in X \times \mathbb{R}$. Notice that

$$\varphi(x, \lambda) = \| x \|^2 + (\| f(x) \|^2 - c^2) \lambda^2 - 2(\langle x, f(x) \rangle - \alpha) \lambda .$$

Further, observe that, when $\| f(x) \| \geq c$, in view of $(a)$, we have

$$\sup_{\lambda \in \mathbb{R}} \varphi(x, \lambda) = +\infty$$

as well as

$$\varphi(x, -\lambda) \neq \varphi(x, \lambda)$$

for all $\lambda > 0$. When $\| f(x) \| \geq c$ again, the function $\varphi(x, \cdot)$ is convex and so, by $(6)$, for each $\lambda > 0$, its restriction to $[-\lambda, \lambda]$ it has a unique global maximum. Clearly, $\varphi(x, \cdot)$ has the same uniqueness property also
when \(\|f(x)\| < c\). Now, observe that, for each \(\lambda \in \mathbb{R}\), the function \(\lambda f\) has a fixed point in \(X\), in view of the Schauder theorem. Hence, we have

\[
\sup_{\lambda \in \mathbb{R}} \inf_{x \in X} \varphi(x, \lambda) = \sup_{\lambda \in \mathbb{R}} (-c^2\lambda^2 + 2\alpha\lambda) = \frac{\alpha^2}{c^2} .
\]  

(7)

We claim that

\[
\frac{\alpha^2}{c^2} < \inf_{x \in X} \sup_{\lambda \in \mathbb{R}} \varphi(x, \lambda) .
\]  

(8)

First, observe that, since \(0 \notin K\), every closed and bounded subset of \(X\) is compact. This easily implies that, for each \(\mu > 0\), the function \(x \mapsto \inf_{|\lambda| \leq \mu} \varphi(x, \lambda)\) is relatively inf-compact. Consequently, the sublevel sets of the function \(x \mapsto \sup_{\lambda \in \mathbb{R}} \varphi(x, \lambda)\) (which is finite if \(\|f(x)\| < c\)) are compact. Therefore, there exists \(\hat{x} \in X\) such that

\[
\sup_{\lambda \in \mathbb{R}} \varphi(\hat{x}, \lambda) = \inf_{x \in X} \sup_{\lambda \in \mathbb{R}} \varphi(x, \lambda) .
\]  

(9)

So, by (5), one has \(\|f(\hat{x})\| < c\). Clearly, we also have

\[
\sup_{\lambda \in \mathbb{R}} \varphi(\hat{x}, \lambda) = \|\hat{x}\|^2 + \frac{|\langle \hat{x}, f(\hat{x}) \rangle - \alpha|^2}{c^2 - \|f(\hat{x})\|^2} .
\]  

(10)

Let us prove that

\[
\|\hat{x}\|^2 + \frac{|\langle \hat{x}, f(\hat{x}) \rangle - \alpha|^2}{c^2 - \|f(\hat{x})\|^2} > \frac{\alpha^2}{c^2} .
\]  

(11)

After some manipulations, one realizes that (11) is equivalent to

\[
\frac{1}{c^2 - \|f(\hat{x})\|^2} \left(2\alpha\langle \hat{x}, f(\hat{x}) \rangle - \alpha^2 - \frac{\alpha^2}{c^2}\right) < \|\hat{x}\|^2 .
\]  

(12)

Now, for each \(y \in X \setminus \{0\}\), \(t \in \mathbb{R}\), set

\[
I(y, t) = \{x \in H : \langle x, y \rangle = t\} .
\]

Consider the inequality

\[
\frac{1}{c^2 - \|y\|^2} \left(2\alpha t^2 - \frac{\alpha^2}{c^2}\right) < \frac{t^2}{\|y\|^2} .
\]  

(13)

After some manipulations, one realizes that (13) is equivalent to

\[
(\alpha\|y\|^2 - \alpha t^2)^2 > 0 .
\]

So, (13) is satisfied if and only if

\[
\alpha\|y\|^2 \neq \alpha t^2 .
\]  

(14)

Observe that

\[
\frac{|t|}{\|y\|} = \text{dist}(0, I(y, t)) \leq \text{dist}(0, I(y, t) \cap X) .
\]  

(15)

Therefore, if (14) is satisfied, for each \(x \in I(y, t) \cap X\), in view of (13) and (15), we have

\[
\frac{1}{c^2 - \|y\|^2} \left(2\alpha\langle x, y \rangle - \|x\|^2 \right) < \|x\|^2 .
\]  

(16)

At this point, taking into account that \(c^2\langle \hat{x}, f(\hat{x}) \rangle \neq \alpha\|f(\hat{x})\|^2\) (by (b)), we draw (12) from (16) since \(\hat{x} \in I(f(\hat{x}), \langle \hat{x}, f(\hat{x}) \rangle)\). Summarizing: taking \(I = \mathbb{R}\) and \(I_n = [-n, n] (n \in \mathbb{N})\), the continuous function \(\varphi\) satisfies \((a_1)\) and \((b_1)\) of Theorem 1, but, in view of (7) – (11), it does not satisfy the conclusion of
that theorem. As a consequence, there exists $\tilde{\lambda} \in \mathbb{R}$ such that the set of all global minima of $\varphi(\cdot, \tilde{\lambda})$ is disconnected. But such a set agrees with the set of all solutions of the equation $x = \tilde{\lambda}f(x)$, and the proof is complete. △

REMARK 2. - We do not know whether Theorem 3 is still true when $0 \in K$ and $(b)$ is (necessarily) changed in

$$\{x \in X : f(x) \neq 0, \ c^2(x, f(x)) = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}.$$ 

However, the proof of Theorem 3 shows that the following is true:

THEOREM 4. - Let $(X, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real Hilbert space and let $f : X \to X$ be a continuous function with bounded range. Assume that there are two numbers $\alpha, c$, with

$$\inf_{x \in X} \|f(x)\| < c < \|f(0)\|,$$

such that:

(a') $\{x \in X : \langle x, f(x) \rangle = \alpha\} \subset \{x \in X : \|f(x)\| < c\}$;
(b') $\{x \in X : f(x) \neq 0, \ c^2(x, f(x)) = \alpha\|f(x)\|^2\} \subset \{x \in X : \|f(x)\| \geq c\}.$

Then, there exists $\tilde{\lambda} \in \mathbb{R}$ such that the set

$$\{x \in X : x = \tilde{\lambda}f(x)\}$$

is disconnected.

Finally, we present two applications of Theorem 2.

THEOREM 5. - Let $X$ be a Banach space, let $\varphi \in X^* \setminus \{0\}$ and let $\psi : X \to \mathbb{R}$ be a Lipschitzian functional whose Lipschitz constant is equal to $\|\varphi\|_{X^*}$. Moreover, let $[a, b]$ be a compact real interval, $\gamma : [a, b] \to [-1, 1]$ a convex (resp. concave) and continuous function, with $\text{int}(\gamma^{-1}([-1, 1])) = \emptyset$, and $c \in \mathbb{R}$. Assume that

$$\gamma(a)\psi(x) + ca \neq \gamma(b)\psi(x) + cb$$

for all $x \in X$ such that $\psi(x) > 0$ (resp. $\psi(x) < 0$).

Then (with the convention $\sup \emptyset = -\infty$), one has

$$\sup_{\lambda \in \gamma^{-1}([-1, 1])} \inf_{x \in X} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda) = \inf_{x \in X} \sup_{\lambda \in [a, b]} (\varphi(x) + \gamma(\lambda)\psi(x) + c\lambda).$$

PROOF. Consider the continuous function $f : X \times [a, b] \to \mathbb{R}$ defined by

$$f(x, \lambda) = \varphi(x) + \gamma(\lambda)\psi(x) + c\lambda$$

for all $(x, \lambda) \in X \times [a, b]$. By Theorem 2 of [4], for each $\lambda \in \gamma^{-1}([-1, 1])$, the function $f(\cdot, \lambda)$ is inf-connected and unbounded below. Also, notice that $\gamma^{-1}([-1, 1])$, by assumption, is dense in $[a, b]$. Now fix $x \in X$. If $\psi(x) > 0$ (resp. $\psi(x) < 0$) the function $f(x, \cdot)$ is convex and, by assumption, $f(x, a) \neq f(x, b)$. As a consequence, the unique global maximum of this function is either $a$ or $b$. If $\psi(x) \leq 0$, the function is concave and so, obviously, the set of all its global maxima is connected. Now, the conclusion follows directly from Theorem 2. △

Let $(T, F, \mu)$ be a $\sigma$-finite measure space, $E$ a real Banach space and $p \geq 1$.

As usual, $L^p(T, E)$ denotes the space of all (equivalence classes of) strongly $\mu$-measurable functions $u : T \to E$ such that $\int_T \|u(t)\|^p \, d\mu < +\infty$, equipped with the norm

$$\|u\|_{L^p(T, E)} = \left(\int_T \|u(t)\|^p \, d\mu\right)^{\frac{1}{p}}.$$
A set $D \subseteq L^p(T, E)$ is said to be decomposable if, for every $u, v \in D$ and every $A \in F$, the function

$$t \rightarrow \chi_A(t)u(t) + (1 - \chi_A(t))v(t)$$

belongs to $D$, where $\chi_A$ denotes the characteristic function of $A$.

A real-valued function on $T \times E$ is said to be a Carathéodory function if it is measurable in $T$ and continuous in $E$.

THEOREM 6. - Let $(T, F, \mu)$ be a $\sigma$-finite non-atomic measure space, $E$ a real Banach space, $p \in [1, +\infty[$, $X \subseteq L^p(T, E)$ a decomposable set, $[a, b]$ a compact real interval, $\gamma : [a, b] \rightarrow \mathbb{R}$ a convex (resp. concave) and continuous function. Moreover, let $\varphi, \psi, \omega : T \times E \rightarrow \mathbb{R}$ be three Carathéodory functions such that, for some $M \in L^1(T)$, $k \in \mathbb{R}$, one has

$$\max\{|\varphi(t, x)|, |\psi(t, x)|, |\omega(t, x)|\} \leq M(t) + k\|x\|^p$$

for all $(t, x) \in T \times E$ and

$$\gamma(a) \int_T \psi(t, u(t))d\mu + a \int_T \omega(t, u(t))d\mu \neq \gamma(b) \int_T \psi(t, u(t))d\mu + b \int_T \omega(t, u(t))d\mu$$

for all $u \in X$ such that $\int_T \psi(t, u(t))d\mu > 0$ (resp. $\int_T \psi(t, u(t))d\mu < 0$).

Then, one has

$$\sup_{\lambda \in [a, b]} \inf_{u \in X} \left( \int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t))) + \lambda\omega(t, u(t))d\mu \right) =$$

$$\inf_{u \in X} \sup_{\lambda \in [a, b]} \left( \int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t))) + \lambda\omega(t, u(t))d\mu \right).$$

PROOF. The proof goes on exactly as that of Theorem 5. So, one considers the function $f : X \times [a, b] \rightarrow \mathbb{R}$ defined by

$$f(u, \lambda) = \int_T (\varphi(t, u(t)) + \gamma(\lambda)\psi(t, u(t))) + \lambda\omega(t, u(t))d\mu$$

for all $(u, \lambda) \in X \times [a, b]$, and realizes that it satisfies the hypotheses of Theorem 2. In particular, for each $\lambda \in [a, b]$, the inf-connectedness of the function $f(\cdot, \lambda)$ is due to [6], Théorème 7. \hfill $\triangle$

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