Ideally Statistical Convergence in n-normed Space

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Authors’ contributions
This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information
DOI: 10.9734/JAMCS/2020/v35i730305

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Complete Peer review History: http://www.sdiarticle4.com/review-history/60892

Received: 02 July 2020
Accepted: 07 September 2020
Published: 03 October 2020

Original Research Article

Abstract

The aim of the article is to extend the concept of Ideally statistical convergence from 2 normed spaces to n-normed space. We have also study and prove some important algebraic and topological properties of Ideally-statistical convergence of real sequences in n-normed space. In the last part of this article we obtain a criterion for I-statistically Cauchy sequence in n-normed space to be I-statistically Cauchy with respect to \( \| \|_\infty \).

Keywords: Statistical convergence; Ideal convergence; Filter; statistically Cauchy sequence; real sequences; n-normed space.

2010 Mathematics Subject Classification: 40A05, 40A35, 40G15.

1 Introduction

The concept of statistical convergence was introduced by Steinhaus in 1951 but the extension of convergence of real sequences to statistical convergence was given by Fast [1]. We can find its

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applications in many areas of mathematics like number theory, trigonometric series and summability theory. Also Maddox [2] extended the concept for sequences in any Hausdorff locally convex topological vector spaces. In the case of real sequences, Fridy and Orhan [3] obtained the statistical analogue of the Cauchy criterion for convergence.

Let K be a subset of $\mathbb{N}$. Then the asymptotic density of K is denoted by

$$\delta(K) := \lim_{n \to \infty} \frac{1}{n} |k \leq n : k \in K|$$

where the vertical bars indicate the cardinality of the set.

A sequence $x = (x_k)$ is called Statistically Convergent to L if for $\epsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0.$$

We write this as $st-lim_{k \to \infty} x_k = L$.

It is known that, $I$-convergence (where $I$ stands for ideal) is generalization of the statistical convergence and it was introduced by Kostyrko et al. in [4]. It was further studied by Demirci[5], Das et al. [6], Salt[ et al. [7], and many others.

**Definition 1.1.** A family of sets $I \subseteq 2^X$ (power set of $X$) is said to be an ideal in $X$ if

1) $\phi \in I$.
2) $I$ is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$.
3) $I$ is hereditary i.e $\forall A \in I$ and $B \subseteq A \Rightarrow B \in I$.

A non-trivial ideal $I$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

**Definition 1.2.** A non-empty family of sets $F \subseteq 2^\mathbb{N}$ is said to be filter on $\mathbb{N}$ if and only if

1) $\phi \notin F$,
2) $\forall A, B \in F$ we have $A \cap B \in F$,
3) $\forall A \in F$ and $A \subseteq B \Rightarrow B \in F$.

If $I$ is a proper ideal of $\mathbb{N}$ then the family of sets $F(I) = \{M \subset \mathbb{N}: \exists A \in I : M = \mathbb{N}\setminus A\}$ is a filter of $\mathbb{N}$. It is called as filter associated with the ideal.

Let $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers respectively. The set of all sequences is denoted by $\omega$. Any subset of the $\omega$ is called sequence space. A sequence $(x_k) \in \omega$ is said to be $I$-convergent to a number L if for every $\epsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I-lim_{k \to \infty} x_k = L$.

For more details please see [8] and [9]

The theory of 2-normed spaces was first introduced by Gähler[10] in 1964. Later on it was extended to $n$-normed spaces by Misiak [11]. Since then many mathematicians have worked in this field and obtained many interesting results for instance see Gunawan[12],[13], Gunawan and Mashadi[14], Yamanci and Gürdal [15], E. Savaş([16], [17], [18]) and so on. Let $n \in \mathbb{N}$ and $X^n$ be a linear metric space over the field $\mathbb{K}$ of real or complex numbers of dimension $d$, where $d \geq n \geq 2$.

**Definition 1.3.** A real valued function $\|\cdot, \ldots, \cdot\|$ on $X^n$ satisfying the following four conditions:

1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;
If we take 

\[ \frac{\alpha x_1, x_2, ..., x_n}{\alpha} = \alpha \left\| x_1, x_2, ..., x_n \right\| \text{ for any } \alpha \in \mathbb{K}, \]  

and  

\[ \left\| x + x', x_2, ..., x_n \right\| \leq \left\| x, x_2, ..., x_n \right\| + \left\| x', x_2, ..., x_n \right\|. \]

is called an \( n \)-norm on \( X \) and the pair \( (X, \left\| \cdot, ..., \right\|) \) is called an \( n \)-normed space over the field \( \mathbb{K} \).

**Example 1.4.**  If we take \( X = \mathbb{R}^n \), equipped with Euclidean \( n \)-norm

\[ \left\| x_1, x_2, ..., x_n \right\|_E = \text{vol}(n \text{-dimensional parallelepiped}) \]

spanned by the vectors \( x_1, x_2, ..., x_n \).

It may be given by the formula 

\[ \left\| x_1, x_2, ..., x_n \right\|_E = \text{det}(x_{ij}) \],

where \( x_{ij} = (x_{i1}, x_{i2}, ..., x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, 3, ..., n \).

The standard \( n \)-norm on \( X \), is defined as:

\[ \left\| x_1, x_2, ..., x_n \right\|_E = \sqrt{\left( x_1, x_1 \right) \cdot \cdot \cdot \left( x_1, x_n \right) \cdot \cdot \cdot \left( x_1, x_1 \right)}, \]

\( \langle \cdot, \cdot \rangle \) denotes the inner product on \( X \). If \( X = \mathbb{R}^n \), then this \( n \)-norm is exactly the same as the Euclidean \( n \)-norm \( \left\| x_1, x_2, ..., x_n \right\|_E \) mentioned earlier. For \( n = 1 \) this \( n \)-norm is the usual norm \( \|x\| = (x, x) \frac{1}{2} \).

A sequence \( (x_k) \) in an \( n \)-normed space \( (X, \left\| \cdot, ..., \right\|) \) is said to converge to some \( L \in \mathbb{K} \) if 

\[ \lim_{k \to \infty} \|x_k - L, z_1, ..., z_{n-1}\| = 0 \quad \text{for every } z_1, z_2, ..., z_{n-1} \in X. \]

A sequence \( (x_k) \) in an \( n \)-normed space \( (X, \left\| \cdot, ..., \right\|) \) is said to be Cauchy if 

\[ \lim_{k, p \to \infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0 \quad \text{for every } z_1, z_2, ..., z_{n-1} \in X. \]

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be an \( n \)-Banach space.

**Example 1.5.** Let \( I = I_3 \) where \( I_3 = \{ A \subset \mathbb{N} : \delta(A) = 0 \} \), then \( I_3 \) is an admissible ideal in \( \mathbb{N} \) where ideal convergence coincides with statistical convergence.

Define the sequence \( (x_k) \) in \( n \)-normed space \( (X, \left\| \cdot, ..., \right\|) \) by

\[ x_k = \begin{cases} (0, ..., k), & k = i^2, i \in \mathbb{N}, \\ (0, ..., 0), & \text{otherwise}. \end{cases} \]

let \( x = (0, ..., 0) \). Then for every \( \epsilon > 0 \) and \( x_1, x_2, ..., x_{n-1} \in X \)

\[ \{ k \in \mathbb{N} : \| x_k - x, x_1, x_2, ..., x_{n-1} \| \geq \epsilon \} \subset \{ 1, 4, 9, 16, ..., k^2 \}. \]

Also we have that \( \delta(\{ k \in \mathbb{N} : \| x_k - x, x_1, x_2, ..., x_{n-1} \| \geq \epsilon \}) = 0 \) for every \( \epsilon > 0 \).

This implies that \( I - \lim_{k \to \infty} \| x_k, x_1, x_2, ..., x_{n-1} \| = \| x, x_1, x_2, ..., x_{n-1} \| \) but the sequence \( (x_k) \) is not convergent to \( x \).

## 2 \( \mathcal{I} \)-Statistical Convergence and \( \mathcal{I} \)-Statistically Cauchy Sequence in \( n \)-normed Space

Now we give some useful definitions and examples based on \( \mathcal{I} \)-Statistical convergence and \( \mathcal{I} \)-Statistically Cauchy sequence in \( n \)-normed space.

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Definition 2.1. Let $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial ideal in $\mathbb{N}$. The sequence $(x_k)$ of $X$ is said to be \(\mathcal{I}\)-statistically convergent to $\xi$ if for each $\epsilon > 0$, $\exists \delta > 0$ such that for all $z_i \in X, i = 2, 3, \ldots$ the set
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \| x_k - \xi, z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| \geq \delta \right\} \in \mathcal{I}.
\]
or equivalently if for each $\epsilon > 0$
\[
\delta_\mathcal{I}(\epsilon) = \mathcal{I} - \lim \delta_n(\epsilon) = 0
\]
where $(A_n(\epsilon)) = \{ k \leq n : \| x_k - \xi, z_2, z_3, \ldots z_n \| \geq \epsilon \}$ and $\delta_n(\epsilon) = \frac{|A_n(\epsilon)|}{n}$. If $(x_k)$ is $\mathcal{I}$-convergent to $\xi$ then we write $\mathcal{I} - \text{st- lim}_{k \to \infty} \| x_k - \xi, z_2, z_3, \ldots z_n \| = 0$ or $\mathcal{I} - \text{st- lim}_{k \to \infty} \| x_k, z_2, z_3, \ldots z_n \| = \| \xi, z_2, z_3, \ldots z_n \|$. The number $\xi$ is called the $\mathcal{I}$-limit of the sequence $(x_k)$.

Remark 2.2. If $(x_k)$ is a sequence in $X$ and $\xi$ is any element of $X$ then the set
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \| x_k - \xi, z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| \geq \delta \right\} = \emptyset
\]
is any element of $X$, for $i = 2, 3, \ldots$ then $\| x_k - \xi, z_2, z_3, \ldots z_n \| = 0 \not\geq \epsilon$

Definition 2.3. Let $\mathcal{I} \subset 2^\mathbb{N}$ be a non-trivial ideal in $\mathbb{N}$. The sequence $(x_k)$ of $X$ is said to be $\mathcal{I}$-statistically cauchy sequence if for every $\epsilon > 0$, $\delta > 0$ and all nonzero $z_i \in X, i = 2, 3, \ldots$ there exists a number $N$, dependent on $\epsilon$ such that
\[
\delta_\mathcal{I}\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \| x_k - x_N, z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| \geq \delta \right\} = 0,
\]
i.e. for every non zero $z_i \in X$,
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \| x_k - x_N, z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| \geq \delta \right\} \in \mathcal{I}.
\]

3 Main Results

Theorem 3.1 Let $x_k$ be a sequence in $n$-normed space $(X, \| \cdot, \cdot, \cdot \|)$. $\mathcal{I}$ be an admissible ideal and $L, L' \in X$. For each $z_i \in X$, if $\mathcal{I}$-st- $\lim_{k \to \infty} \| x_k, z_2, z_3, \ldots z_n \| = \| L, z_2, z_3, \ldots z_n \|$ and $\mathcal{I}$-st- $\lim_{k \to \infty} \| x_k, z_2, z_3, \ldots z_n \| = \| L', z_2, z_3, \ldots z_n \|$ then $L = L'$.

Proof: Assume $L \neq L'$. Then $L - L' \neq 0$, so there exist nonzero $z_2, z_3, \ldots z_n \in X$, such that $L - L'$ and $z_2, z_3, \ldots z_n$ are linearly independent (such $z_i$ exist as dimension of $X, d \geq n$). Therefore for every $\epsilon > 0$ and $\delta > 0$,
\[
\frac{1}{n} \left| \{ k \leq n : \| L - L', z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| = 2\delta.
\]

Now,
\[
\frac{1}{n} \left| \{ k \leq n : \| L - x_k + x_k - L', z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| = 2\delta,
\]
\[
\frac{1}{n} \left| \{ k \leq n : \| (L - x_k) + (x_k - L'), z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| = 2\delta,
\]
\[
\frac{1}{n} \left| \{ k \leq n : \| (x_k - L), z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| + \frac{1}{n} \left| \{ k \leq n : \| (x_k - L'), z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| \geq 2\delta.
\]
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \| (x_k - L), z_2, z_3, \ldots z_n \| \geq \epsilon \} \right| < \delta \right\}
\]
(ii) Let $I$ be an admissible ideal. For each $z_i \in X$,
(i) If $I$-st-$\lim_{k \to \infty} \|x_k, z_2, z_3, \ldots, z_n\| = \|x, z_2, z_3, \ldots, z_n\|$ and $I$-st-$\lim_{k \to \infty} \|y_k, z_2, z_3, \ldots, z_n\| = \|y, z_2, z_3, \ldots, z_n\|$ then $I$-st-$\lim_{k \to \infty} \|x_k + y_k, z_2, z_3, \ldots, z_n\| = \|x + y, z_2, z_3, \ldots, z_n\|
(ii) If $I$-st-$\lim_{k \to \infty} \|ax_k, z_2, z_3, \ldots, z_n\| = \|a x, z_2, z_3, \ldots, z_n\|$ for $a \in \mathbb{R}$.

Proof: (i) Let $I$-st-$\lim_{k \to \infty} \|x_k, z_2, z_3, \ldots, z_n\| = \|x, z_2, z_3, \ldots, z_n\|$ and $I$-st-$\lim_{k \to \infty} \|y_k, z_2, z_3, \ldots, z_n\| = \|y, z_2, z_3, \ldots, z_n\|$ for every nonzero $z_i \in X$, then $\delta_I(K_1) = 0$ and $\delta_I(K_2) = 0$ where

\[
K_1 = K_1(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \left\| (x_k - x, z_2, z_3, \ldots, z_n) \geq \epsilon \right\| \geq \frac{\epsilon}{2} \right\}
\]

and

\[
K_2 = K_2(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \left\| (y_k - y, z_2, z_3, \ldots, z_n) \geq \epsilon \right\| \geq \frac{\epsilon}{2} \right\}
\]

for every $z_i \in X$. Let

\[
K = K(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \left\| (x_k + y_k) - (x + y), z_2, z_3, \ldots, z_n \right\| \geq \epsilon \right\} \geq \frac{\epsilon}{2} \right\}
\]

To prove that $\delta_I(K) = 0$ it is sufficient to show that $K \subset K_1 \cup K_2$. Let $k_0 \in K$. Then

\[
\frac{1}{n} \left\{ k \leq n : \left\| (x_k + y_k) - (x + y), z_2, z_3, \ldots, z_n \right\| \geq \epsilon \right\} \geq \frac{\epsilon}{2}
\]

Suppose to the contrary that $k_0 \notin K_1 \cup K_2$. Then $k_0 \notin K_1$ and $k_0 \notin K_2$. This implies

\[
\frac{1}{n} \left\{ k \leq n : \left\| (x_k - x, z_2, z_3, \ldots, z_n) \geq \epsilon \right\| < \frac{\epsilon}{2}
\]

and

\[
\frac{1}{n} \left\{ k \leq n : \left\| (y_k - y, z_2, z_3, \ldots, z_n) \geq \epsilon \right\| < \frac{\epsilon}{2}
\]

Then, we get

\[
\frac{1}{n} \left\{ k \leq n : \left\| (x_k + y_k) - (x + y), z_2, z_3, \ldots, z_n \right\| \leq \frac{1}{n} \left\{ k \leq n : \left\| x_k - x, z_2, z_3, \ldots, z_n \right\|
\]

\[
+ \frac{1}{n} \left\{ k \leq n : \left\| y_k - y, z_2, z_3, \ldots, z_n \right\|
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]

\[
= \epsilon,
\]

which contradicts (3.1). Hence $k_0 \in K_1 \cup K_2$, that is $K \subset K_1 \cup K_2$.

(ii) Let $I$-st-$\lim_{k \to \infty} \|x_k, z_2, z_3, \ldots, z_n\| = \|x, z_2, z_3, \ldots, z_n\|$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \left\| (x_k - x, z_2, z_3, \ldots, z_n) \geq \frac{\epsilon}{|\alpha|} \right\| \geq \frac{\epsilon}{2} \right\} \in I.
\]
Then since \( \|\alpha x_k - \alpha x, z_2, z_3, ..., z_n\| = |\alpha| \|x_k - x, z_2, z_3, ..., z_n\| \), we have

\[
\begin{align*}
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \|x_k - x, z_2, z_3, ..., z_n\| \geq \frac{\epsilon}{|\alpha|} \right\} \geq \delta \right\} & \subseteq \mathcal{I}, \\
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : |\alpha| \|x_k - x, z_2, z_3, ..., z_n\| \geq \epsilon \right\} \geq \delta \right\} & \subseteq \mathcal{I}, \\
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \|\alpha x_k - \alpha x, z_2, z_3, ..., z_n\| \geq \epsilon \right\} \geq \delta \right\} & \subseteq \mathcal{I}.
\end{align*}
\]

Hence from equation (3.2) we get \( \mathcal{I} \)-st-lim \( k \to \infty \) \( \|\alpha x_k, z_2, z_3, ..., z_n\| = \|\alpha x, z_2, z_3, ..., z_n\| \), for every non zero \( z_i \in X, i = 2, 3, ..., n \).

Recall that we assume \( X \) to have dimension \( d \), where \( 2 \leq n \leq d < \infty \), unless otherwise stated. Let \( u = \{u_1, ..., u_{(n - 1)}\} \) to be a basis for \( X \). Then we have the following:

**Theorem 3.3** Let \( \mathcal{I} \) be an admissible ideal. A sequence \( (x_k) \in X \) is \( \mathcal{I} \)-statistically convergent to \( x \in X \) if and only if \( \mathcal{I} \)-st-lim \( k \to \infty \) \( \|x_k - x, u_1, u_2, ..., u_i\| = 0 \) for every \( i = 1, ..., (n - 1) \).

**Proof:** Let \( (x_k) \in X \) is \( \mathcal{I} \)-statistically convergent to \( x \in X \). Then by the definition of \( \mathcal{I} \)-st-convergence, we have
\[
I - st \lim_{k \to \infty} \|x_k - x, z_2, z_3, ..., z_n\| = 0
\]

Then \( \mathcal{I} \)-st-lim \( k \to \infty \) \( \|x_k - x, u_1, u_2, ..., u_i\| = 0 \) for every \( i = 1, ..., (n - 1) \), is trivial since every \( z_i \) can be expressed as a linear combination of \( u_i \), for \( i = 1, 2, ..., (n - 1) \)

Next we prove the result conversely.

Let us assume
\[
\mathcal{I} - st \lim_{k \to \infty} \|x_k - x, u_1, u_2, ..., u_i\| = 0 \text{ for every } i = 1, ..., (n - 1), \tag{3.3}
\]

We want to show that
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : \|x_k - x, z_2, z_3, ..., z_n\| \geq \epsilon \right\} \geq \delta \right\} \tag{3.4}
\]

For this, Let us consider the n-norm
\[
\|x_k - x, z_2, z_3, ..., z_n\|
\]

Also as \( u = \{u_1, ..., u_{(n - 1)}\} \) is a basis of \( X \), then we have
\[
z_2 = \sum_{i=1}^{n-1} \alpha_i^2 u_i, \quad z_3 = \sum_{i=1}^{n-1} \alpha_i^3 u_i, \quad ..., \quad z_n = \sum_{i=1}^{n-1} \alpha_i^n u_i
\]

This implies,
\[
\|x_k - x, z_2, z_3, ..., z_n\| = \|x_k - x, \sum_{i=1}^{n-1} \alpha_i^2 u_i, \sum_{i=1}^{n-1} \alpha_i^3 u_i, ..., \sum_{i=1}^{n-1} \alpha_i^n u_i\|
\]

As \( n \) is any positive integer, we have
\[
\|x_k - x, z_2, z_3, ..., z_n\| \leq \|n(x_k - x)\|, \quad \sum_{i=1}^{n-1} \alpha_i^2 u_i, \sum_{i=1}^{n-1} \alpha_i^3 u_i, ..., \sum_{i=1}^{n-1} \alpha_i^n u_i
\]

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Using triangle inequality and distributing corresponding components for each \( z_i \) over \( n \)-norm we get

\[
\|x_k - x, z_2, z_3, ..., z_n\| \leq \|(x_k - x), \alpha_1 u_1, \alpha_1 u_1, ..., \alpha_1 u_1\|_{(n-1)} + \|(x_k - x), \alpha_2 u_2, \alpha_2 u_2, ..., \alpha_2 u_2\|_{(n-1)} + ... + \|(x_k - x), \alpha_n u_n, \alpha_n u_n, ..., \alpha_n u_n\|_{(n-1)}.
\]

Let \( \max \alpha_i = \alpha_1 \), \( \forall i = 2, 3, 4, ..., n \), similarly let \( \max \alpha_{(n-1)} = \alpha_{(n-1)} \), \( \forall i = 2, 3, 4, ..., n \).

Substituting these values in above equation, we get

\[
\|x_k - x, z_2, z_3, ..., z_n\| \leq \|(x_k - x), \alpha_1 u_1, \alpha_1 u_1, ..., \alpha_1 u_1\| + \|(x_k - x), \alpha_2 u_2, \alpha_2 u_2, ..., \alpha_2 u_2\| + ... + \|(x_k - x), \alpha_{(n-1)} u_{(n-1)}, \alpha_{(n-1)} u_{(n-1)}, ..., \alpha_{(n-1)} u_{(n-1)}\|
\]

By our assumption in equation (3.3) we have,

\[
\mathcal{I}\text{-}\lim_{k \to \infty} \|x_k - x, u_1, u_1, ..., u_1\| = 0 \text{ for every } i = 1, ..., (n-1), \text{ which implies,}
\]

\[
\left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_1, u_1, ..., u_1 \| \geq \frac{\epsilon}{\alpha_1} \right\} \geq \delta \right\} \subset \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_1, u_1, ..., u_1 \| \geq \frac{\epsilon}{\alpha_1} \right\} \geq \delta \right\} \cup \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_2, u_2, ..., u_2 \| \geq \frac{\epsilon}{\alpha_2} \right\} \geq \delta \right\} \cup ... \cup \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)} \| \geq \frac{\epsilon}{\alpha_{(n-1)}} \right\} \geq \delta \right\}
\]

which gives,

\[
\left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| x_k - x, z_2, z_3, ..., z_n \| \geq \epsilon \right\} \geq \delta \right\} \subset \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_1, u_1, ..., u_1 \| \geq \frac{\epsilon}{\alpha_1} \right\} \geq \delta \right\} \cup \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_2, u_2, ..., u_2 \| \geq \frac{\epsilon}{\alpha_2} \right\} \geq \delta \right\} \cup ... \cup \left\{ n \in N : \frac{1}{n} \left\{ k \leq n : \| (x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)} \| \geq \frac{\epsilon}{\alpha_{(n-1)}} \right\} \geq \delta \right\}
\]
In the light of lemma 3.4, we can define a norm on \(X\), denoted by \(\|x\|\), if and only if
\[
\{z \in X : k \leq n : \{\|(x_k-x), (u_{n-1}), u_{(n-1)}, \ldots, u_{(n-1)}\| \geq \frac{\epsilon}{|\alpha_{(n-1)}|} \} \geq \delta \}
\]
Since the right hand side of the above inclusion belongs to ideal, so does the left hand side. Consequently, we get \(I\)-st- \(\lim_{k \to \infty} \|x_k - x, z_2, z_3, \ldots, z_n\|\) for every nonzero \(z_i \in X\). This proves the result.

Following theorem 3.3, we have the next lemma:

**Lemma 3.4** Let \(I\) be an admissible ideal. A sequence \((x_k)\) in \(X\) is \(I\)-statistically convergent to \(x\) in \(X\) if and only if \(I\)-st- \(\lim_{k \to \infty} \max\|x_k - x, u_i, u_i, \ldots, u_i\| = 0, \forall i = 1, \ldots, (n-1)\).

**Definition 3.1.** In the light of lemma 3.4, we can define a norm on \(X\), denoted by \(\|x\|_{\infty}\), with respect to the basis \(u = u_1, \ldots, u_d\), by
\[
\|x\|_{\infty} = \max\{\|x, u_i, \ldots, u_i\| : i = 1, 2, \ldots, d\}
\]
Using the derived norm \(\|x\|_{\infty}\), Lemma 3.4 now reads:

**Lemma 3.5** Let \(I\) be an admissible ideal. A sequence \((x_k)\) in \(X\) is \(I\)-statistically convergent to \(x \in X\) if and only if \(I\)-st- \(\lim_{k \to \infty} \max\|x_k - x\|_{\infty} = 0\).

Associated to the derived norm \(\|\|_{\infty}\), we can define the balls \(B(x, \epsilon)\) centered at \(x\) having radius \(\epsilon\) by
\[
B(x, \epsilon) := \{y : \|x - y\|_{\infty} \leq \epsilon\}
\]
where \(\|x - y\|_{\infty} = \max\|x - y, u_i, u_i, \ldots, u_i\|\).

Using these balls, Lemma 3.5 becomes:

**Lemma 3.6** Let \(I\) be an admissible ideal. A sequence \((x_k)\) in \(X\) is \(I\)-statistically convergent to \(x \in X\) if and only if \(\delta_I(A_n(\epsilon)) = 0\), where \(A_n(\epsilon) = \{k \leq n : x_k \notin B(x, \epsilon)\}\).

**Theorem 3.7** Any \(I\)-st-Cauchy sequence \((x_k)\) in an \(n\)-normed space \((X, || \cdot ||, || \cdot ||, \ldots, || \cdot ||)\) is \(I\)-st-convergent if and only if any \(I\)-st-Cauchy sequence is \(I\)-st-convergent with respect to \(\|\|_{\infty}\).

**Proof:** From previous result, it is clear that \(I\)-st-convergence in \(n\)-norm is equivalent to that in \(\|\|_{\infty}\). That is for all \(z_i \in X, i = 2, 3, \ldots, n\)
\[
I \text{- st} \lim_{k \to \infty} \|x_k - x, z_2, z_3, \ldots, z_n\| = 0
\]
\[
\iff I \text{- st} \lim_{k \to \infty} \|x_k - x\|_{\infty} = 0.
\]
It is sufficient to show that \((x_k)\) is \(I\)-st-Cauchy sequence with respect to \(n\)-norm if and only if it is \(I\)-st-Cauchy sequence with respect to \(\|\|_{\infty}\). Let \((x_k)\) is \(I\)-st-Cauchy sequence with respect to \(n\)-norm. Then there exists \(N \in \mathbb{N}\) such that for \(k, m \geq N\) we have
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \{k \leq n : \|x_k - x_m, z_2, z_3, \ldots, z_n\| \geq \epsilon\} \geq \delta \right\} \in I
\]

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Consider,
\[ \| x_k - x_m, z_2, z_3, ..., z_n \| \geq \epsilon, \]

Then from lemma 3.3, we have \( \| x_k - x_m, u_i, u_i, ..., u_i \| \geq \epsilon \) for all \( i = 1, 2, ..., n \).
Hence \( \max \| x_k - x_m, u_i, u_i, ..., u_i \| \geq \epsilon \) for all \( i = 1, 2, ..., n \).
By definition, it gives \( \| x_k - x_m \|_{\infty} \geq \epsilon \).
Therefore \((x_k)\) is \( I\)-st-Cauchy with respect to \( \| \|_{\infty} \).

4 Conclusion

In this study, we develop the concept of Ideally-statistical convergence of sequences over n-normed space. Related algebraic and topological properties have been proved. The construction of max-norm of sequences over n-normed spaces lead to develop criterion for \( I\)-st-Cauchy sequence to be \( I\)-st-convergent.

Acknowledgement

The authors are thankful to the anonymous reviewers and the editor, whose helpful suggestions added greatly to improving the quality of this paper.

Competing interests

The authors declare that they have no competing interest.

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