A NEW RAMANUJAN-LIKE SERIES FOR $1/\pi^2$

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Abstract. Our main results are a WZ-proof of a new Ramanujan-like series for $1/\pi^2$ and a hypergeometric identity involving three series.

1. The WZ-method

We recall that a function $A(n,k)$ is hypergeometric in its two variables if the quotients

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in $n$ and $k$, respectively. Also, a pair of hypergeometric functions in its two variables, $F(n,k)$ and $G(n,k)$ is said to be a Wilf and Zeilberger (WZ) pair [11, Chapt. 7] if

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

In this case, H. S. Wilf and D. Zeilberger [13] have proved that there exists a rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k).$$

The rational function $C(n,k)$ is the so-called certificate of the pair $(F,G)$. To discover WZ-pairs, we use EKHAD [11, Appendix A], a software written by D. Zeilberger. If EKHAD certifies a function, we have found a WZ-pair! Then, if we sum (1) over all $n \geq 0$, we get

$$\sum_{n=0}^{\infty} G(n,k) - \sum_{n=0}^{\infty} G(n,k+1) = -F(0,k) + \lim_{n \to \infty} F(n,k).$$

We will write the functions $F(n,k)$ and $G(n,k)$ using rising factorials, also called Pochhammer symbols, rather than the ordinary factorials. The rising factorial is defined by

$$(x)_n = \begin{cases} x(x+1) \cdots (x+n-1), & n \in \mathbb{Z}^+, \\ 1, & n = 0, \end{cases}$$

or more generally by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}.$$
For $k \in \mathbb{Z} - \mathbb{Z}^-$, (5) coincide with (11). But (5) is more general because it is also defined for all complex $x$ and $k$ such that $x + k \in \mathbb{C} - (\mathbb{Z} - \mathbb{Z}^+)$. To use package EKHAD we will replace groups of rising factorials according to the following equivalences

\[(1 + k)_n = \frac{(n + k)!}{k!}, \quad (1)\]
\[
\left(\frac{1}{2} + k\right)_n = \frac{1}{2^n} \frac{(2n + 2k)!}{(n + k)!(2k)!}, \quad (2)\]
\[
\left(\frac{1}{2} + \frac{k}{2}\right)_n \left(\frac{1}{2} + \frac{k}{2}\right)_n = \frac{1}{2^{2n}} \frac{(4n + 2k)!}{k!}, \quad (3)\]
\[
\left(\frac{1}{3} + \frac{k}{3}\right)_n \left(\frac{1}{3} + \frac{k}{3}\right)_n \left(\frac{2}{3} + \frac{k}{3}\right)_n = \frac{1}{3^{3n}} \frac{(6n + 3k)!}{k!}, \quad (4)\]

which we can derive easily from the properties of the Gamma function.

2. A NEW RAMANUJAN-LIKE SERIES FOR $1/\pi^2$

This paper is originated when we checked that EKHAD certifies the function

\[F(n, k) = \frac{\left(\frac{1}{2}\right)^3_n (1 + k)_n \left(\frac{1}{2} + k\right)_n \left(\frac{1}{2}\right)_k^2}{(1)_n^3 (1 + k)_n^2} \frac{96n^3}{(1)_k^2 2n + k}, \quad (5)\]

giving the companion

\[G(n, k) = \frac{\left(\frac{1}{2}\right)^3_n (1 + k)_n \left(\frac{1}{2} + k\right)_n \left(\frac{1}{2}\right)_k^2}{(1)_n^3 (1 + k)_n^2} \frac{12k(8n^2 + 6kn + 2n + k)}{2n + k}, \quad (6)\]

As $F(0, k) = 0$ and the last limit in (3) is also zero, we get

\[
\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} G(n, k + 1). \quad (7)\]

As a consequence of Weierstrass M-test [12, p. 49], the convergence of this series is uniform. Therefore, the following steps hold

\[
\lim_{k \to \infty} \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} \lim_{k \to \infty} G(n, k)
\]
\[
= 12 \sum_{n=0}^{\infty} \frac{1}{4^n} \frac{(\frac{1}{2})^3_n}{(1)_n^3 (1 + k)_n^2} \frac{6n + 1}{k} \lim_{k \to \infty} \frac{1}{k} \frac{(\frac{1}{2})^2_k}{(1)_k^2} = \frac{48}{\pi^2}, \quad (8)\]

in which we have used the asymptotic approximation $(k)_n \sim k^n$. The series in (13) is a Ramanujan series with sum $4/\pi$, see [2], and we have evaluated the last limit using Stirling’s formula. Hence, we have

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^3_n (1 + k)_n \left(\frac{1}{2} + k\right)_n \left(\frac{1}{2}\right)_k^2}{(1)_n^3 (1 + k)_n^2} \frac{12k(8n^2 + 6kn + 2n + k)}{2n + k} = \frac{48}{\pi^2}. \quad (9)\]
For example, taking \( k = 1 \), we obtain a formula that Maple can evaluate, namely

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^4}{(1)^n} \frac{8n^2 + 8n + 1}{(n+1)^2} = \frac{16}{\pi^2},
\]

which is an example of series which converge slowly to the constant \( 1/\pi^2 \). To obtain more interesting series, we replace \( k \) with \( k + n \) in \( F(n, k) \). Then, we have the new function

\[
F(n, k) = U(n, k) \frac{96n^3}{3n + k},
\]

where

\[
U(n, k) = \frac{27}{64} n \left(\frac{1}{2}\right)_n^3 \left(1 + \frac{k}{3}\right)_n \left(\frac{1}{2} + \frac{k}{3}\right)_n \left(\frac{2}{3} + \frac{k}{3}\right)_n \left(\frac{2}{3}\right)_k^2
\]

\[
\left(1\right)_n (1 + k)_n \left(1 + \frac{k}{2}\right)_n \left(1 + \frac{k}{2}\right)_n^2 \left(1\right)_k^2.
\]

Package EKHAD gives the companion

\[
G(n, k) = U(n, k) \frac{n(2n+1)^2(74n^2 + 27n + 3) + kP(n, k)}{(n + \frac{k}{3})(2n + k + 1)^2},
\]

where

\[
P(n, k) = (2n+1)(296n^3 + 164n^2 + 26n + 1)
\]

\[
+ (480n^3 + 360n^2 + 78n + 5)k
\]

\[
+ (176n^2 + 80n + 8)k^2
\]

\[
+ (24n + 4)k^3.
\]

If we observe the steps in (13), we see that again we have

\[
\sum_{n=0}^{\infty} G(n, k) = \frac{48}{\pi^2}.
\]

Finally, taking \( k = 0 \), we obtain

\[
\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)_n^3 \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{2}{3}\right)_n \left(\frac{2}{3}\right)_n^2 (74n^2 + 27n + 3)}{(1)^n} = \frac{48}{\pi^2}.
\]

Although the convergence of this series is not very fast, it seems to us very interesting. The reason is that it is a new formula which belongs to a family of series for \( 1/\pi^2 \) discovered by the author. See [3], [6], [7], [8] and [2], [3], [14]. Until now the unique existing proofs, and only for some of these series, are based on the WZ-method. However, it would be a major achievement to find a modular-like theory which can explain all these kind of formulas; see [4], [15] and [10].
3. An Hypergeometric identity

If \( F(n, k) \) and \( G(n, k) \) is a WZ-pair then obviously \( F_x(n, k) = F(n + x, k) \) and \( G_x(n, k) = G(n + x, k) \) is also a WZ-pair for every value of \( x \). If the last limit in \([1]\) is equal to zero then, if we repeat the proof in \([1]\) we see that

\[
\sum_{n=0}^{\infty} G_x(n, 0) = \sum_{n=0}^{\infty} G_x(n, 1) + F_x(0, 0) = \sum_{n=0}^{\infty} G_x(n, 2) + F_x(0, 1) + F_x(0, 0)
\]

\[
= \sum_{n=0}^{\infty} G_x(n, 3) + \sum_{k=0}^{2} F_x(0, k) = \sum_{n=0}^{\infty} G_x(n, 4) + \sum_{k=0}^{3} F_x(0, k) = \cdots .
\]

Therefore, as in \([1]\), we arrive to

\[
\sum_{n=0}^{\infty} G_x(n, 0) = \lim_{k \to \infty} \sum_{n=0}^{\infty} G_x(n, k) + \sum_{k=0}^{\infty} F_x(0, k).
\]

This is the formula we used to obtain the formulas in \([9]\). Observe that for \( x = 0 \) the last sum is zero. If we now apply the formula to the WZ-pair of functions \([15]\) and \([16]\), we obtain the following hypergeometric identity:

\[
\frac{1}{48} \sum_{n=0}^{\infty} \left( \frac{27}{64} \right)^n \frac{\left( \frac{1}{2} \right)^n}{n} + \left( \frac{1}{3} \right)^n \left( \frac{2}{3} \right)^n \frac{\left( 74(n + x)^2 + 27(n + x) + 3 \right)}{(1 + x)^{n+x}}
\]

\[
= \frac{1}{4\pi} \sum_{n=0}^{\infty} \left( \frac{1}{4} \right)^n \frac{\left( \frac{1}{2} \right)^n}{n} + \left( \frac{1}{3} \right)^n \left( \frac{2}{3} \right)^n \frac{\left( 6(n + x) + 1 \right)}{(1 + x)^{n+x}}
\]

\[
+ 2x^3 \left( \frac{27}{64} \right)^x \frac{\left( \frac{1}{2} \right)^x}{n} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)^n}{n} (1 + x)^n \frac{1}{k + 3x}.
\]

or equivalently

\[
\frac{1}{48} \sum_{n=0}^{\infty} \left( \frac{27}{64} \right)^n \frac{\left( \frac{1}{2} + x \right)^n}{n} \frac{\left( \frac{1}{3} + x \right)^n}{n} \frac{\left( \frac{2}{3} + x \right)^n}{n} \frac{(74(n + x)^2 + 27(n + x) + 3)}{(1 + x)^{n+x}}
\]

\[
= \frac{2x}{\pi} \left( \frac{16}{27} \right)^x \frac{\left( \frac{1}{3} \right)^x}{n} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)^n}{n} (1 + x)^n \frac{1}{n + 3x}.
\]

where we have used \([11]\) Iden. 1]. Taking \( x = 1/2 \), we get

\[
\sum_{n=0}^{\infty} \left( \frac{27}{64} \right)^n \frac{\left( \frac{1}{2} \right)^n}{n} \frac{\left( \frac{1}{3} + x \right)^n}{n} (74n^2 + 101n + 35)(6n + 1)
\]

\[
= \frac{16\pi^2}{3},
\]

which is a new formula for \( \pi^2 \).
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