Skewed Multivariate Birnbaum–Saunders Distributions

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Abstract

The univariate Birnbaum–Saunders distribution has been used quite effectively to model times to failure for materials subject to fatigue and for modeling lifetime data. In this article, we define a skewed version of the Birnbaum–Saunders distribution in the multivariate setting and derive several of its properties. The proposed skewed multivariate model is an absolutely continuous distribution whose marginals are univariate Birnbaum–Saunders distributions. Estimation of the parameters by maximum likelihood is discussed and the Fisher’s information matrix is determined. A skewed bivariate version for the generalized Birnbaum–Saunders distribution is also introduced. We provide an application to real data which illustrates the usefulness of the proposed multivariate model.

Key words: Birnbaum–Saunders distribution, generalized Birnbaum–Saunders distribution, maximum likelihood estimators, modified moment estimators, multivariate distributions.

1 Introduction

The univariate family of distributions proposed by Birnbaum and Saunders (1969), also known as the fatigue life distributions, has been widely applied for describing fatigue lifetimes. This family was originally derived from a model for which failure follows from the development and growth of a dominant crack. A random variable \( T \) has a Birnbaum–Saunders (BS) distribution if it can be written as

\[
T = \beta \left\{ \alpha Z/2 + \left[ (\alpha Z/2)^2 + 1 \right]^{1/2} \right\}^2,
\]

where \( Z \) is a random variable following the standard normal distribution, i.e. \( Z \sim N(0, 1) \). Its density function is

\[
f_T(t) = \frac{t^{-3/2}(t + \beta)}{2\sqrt{2\pi\alpha\sqrt{\beta}}} \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right], \quad t > 0,
\]

(1)
which depends on two parameters: the shape $\alpha > 0$ and scale $\beta > 0$, which is also the median of the distribution. We have $kT \sim \text{BS}(\alpha, k\beta)$ for any $k > 0$, i.e. the BS distribution is closed under scale transformations. The expected value, variance, skewness and kurtosis of $T$ are, respectively,

$$E(T) = \beta \left( 1 + \frac{1}{2} \alpha^2 \right), \quad \gamma_3 = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}, \quad \gamma_4 = 3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^3}.$$

The density function (1) is right skewed and the skewness decreases with $\alpha$. Notice that both mean and variance increase as $\alpha$ increases. It is also of interest to mention that if $T \sim \text{BS}(\alpha, \beta)$, then $T^{-1} \sim \text{BS}(\alpha, \beta^{-1})$. It implies that the BS distribution also belongs to the family of random variables closed under reciprocation (Saunders, 1974). It then follows that

$$E(T^{-1}) = \beta^{-1} \left( 1 + \frac{1}{2} \alpha^2 \right), \quad \gamma_3 = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}, \quad \gamma_4 = 3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^3}.$$

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The shape of the hazard function of the BS distribution is discussed in Kundu et al. (2008). The authors showed that the hazard rate function is not monotone and is unimodal for all ranges of the parameter values. Some interesting results on improved statistical inference for the BS distribution may be revised in Wu and Wong (2004) and Lemonte et al. (2007, 2008).

The univariate BS distribution has received significant attention over the last few years by many researchers and some generalizations are proposed in Díaz–García and Leiva (2005), Owen (2006), Guiraud et al. (2009), Leiva et al. (2009), Castillo et al. (2011) and Cordeiro and Lemonte (2011), among others. On the other hand, as far as we know, little work has been done to extend the BS distribution to the multivariate case. We can refer to the works by Díaz–García and Domínguez–Molina (2006), Kundu et al. (2010) and Caro–Lopera et al. (2012). In Díaz–García and Domínguez–Molina (2006), the authors defined an independent multivariate BS distribution. By using the bivariate normal distribution function, Kundu et al. (2010) proposed a bivariate BS distribution which is absolutely continuous and has five parameters. Finally, Caro–Lopera et al. (2012) introduced the matrix-variate generalized BS distribution.

As can be observed, little work on multivariate versions for the BS distribution have been published. In this paper, in addition to the existing multivariate BS models, we shall propose the asymmetric (skewed) multivariate BS distribution based on the work of Arnold et al. (2002). The main motivation for introducing this multivariate version of the BS distribution relies on the fact that the practitioners will have a new multivariate BS model to use in multivariate settings, since the formulae related with the new multivariate model are manageable and with the use of modern computer resources and its numerical capabilities, the proposed model may prove to be an useful addition to the arsenal of applied statisticians. Additionally, the new model is quite flexible (see Figure 1 in Section 2) and can be widely applied in analyzing multivariate data. Further, we provide an application to real data in which is showed that the new multivariate model yields a better fit than other multivariate BS distributions available in the literature.
The paper unfolds as follows. The skewed bivariate BS distribution is defined in Section 2 and then several properties are discussed. The multivariate extension is presented in Section 3. In Section 4, we propose different methods for estimating the unknown parameters as well as derive the information matrix and discuss likelihood ratio tests for some hypotheses of interest. In particular, we propose modified moment estimators for the unknown parameters which are explicit in form and can therefore be used effectively as the initial guess in the iterative process for the computation of the maximum likelihood estimators. Further, the asymptotic distribution of the maximum likelihood estimators is derived and thus the asymptotic confidence intervals for the unknown parameters can be constructed. The usefulness of the proposed model is illustrated in an application to real data in Section 5. We also introduce in Section 6 the skewed bivariate generalized BS distribution. Finally, Section 7 closes the paper with some concluding remarks.

2 Skewed bivariate BS distribution

We initially consider the skewed bivariate BS distribution. For each $x \in \mathbb{R}$ and for each $y \in \mathbb{R}$, consider the conditional distributions

$$X|Y = y \sim \text{SN}(\lambda y), \quad Y|X = x \sim \text{SN}(\lambda x),$$

(2)

where $X|Y = y \sim \text{SN}(\lambda y)$ means that given $Y = y$, $X|Y = y$ has skew normal distribution (Azzalini, 1985). The shape parameter $\lambda \in \mathbb{R}$ determines the skewness of the density. From Arnold et al. (2002) and using the conditional distributions in (2), the joint probability density function (pdf) of the random vector $(X, Y)$ takes the form

$$f_{X,Y}(x, y) = 2\phi(x)\phi(y)\Phi(\lambda xy), \quad (x, y) \in \mathbb{R}^2,$$

(3)

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cumulative distribution function (cdf) of the standard normal distribution, respectively. Also, $f_X(x) = \phi(x)$ and $f_Y(y) = \phi(y)$. If $\lambda = 0$ in (3), then $f_{X,Y}(x, y) = \phi(x)\phi(y)$ and hence $X$ and $Y$ become independent. For $\lambda \neq 0$, it can be shown that the correlation between $X$ and $Y$, $\rho(X, Y)$ say, is given by

$$\rho(X, Y) = \text{sign}(\lambda) \times \frac{U(3/2, 2, 1/(2\lambda^2))}{2\lambda^2\sqrt{\pi}},$$

where $U(a, b, z)$ denotes the confluent hypergeometric function, defined as

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}t^{a-1}(1 + t)^{b-a-1}dt,$$

with $b > a > 0$ and $z > 0$, and $\Gamma(\cdot)$ represents the gamma function. Therefore, the parameter $\lambda$ also governs the correlation.
Let \( Z_j \sim N(0, 1) \), for \( j = 1, 2 \), with \( Z_1|Z_2 = z_2 \sim SN(\lambda z_2) \) and \( Z_2|Z_1 = z_1 \sim SN(\lambda z_1) \). Then, taking the transformation

\[
T_j = \beta_j \left[ \frac{\alpha_j}{2} Z_j + \sqrt{\left( \frac{\alpha_j}{2} Z_j \right)^2 + 1} \right]^2, \quad j = 1, 2,
\]

where \( \alpha_j > 0 \) and \( \beta_j > 0 \), the joint pdf of the skewed bivariate BS (SBVBS) distribution takes the form

\[
f_{T_1, T_2}(t_1, t_2) = 2\phi(a_1)\phi(a_2)\Phi(\lambda a_1 a_2)\frac{t_1^{-3/2}(t_1 + \beta_1)}{2\alpha_1 \sqrt{\beta_1}} \frac{t_2^{-3/2}(t_2 + \beta_2)}{2\alpha_2 \sqrt{\beta_2}}, \quad (t_1, t_2) \in \mathbb{R}^2_+,
\]

where

\[
a_j = a_j(\alpha_j, \beta_j) = \frac{1}{\alpha_j} \left[ \left( \frac{t_j}{\beta_j} \right)^{1/2} - \left( \frac{\beta_j}{t_j} \right)^{1/2} \right], \quad j = 1, 2.
\]

The notation used is \((T_1, T_2) \sim SBVBS(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)\). The random variables \(T_1\) and \(T_2\) become independent for \( \lambda = 0 \) in (4) and hence the proposed bivariate model reduces to the independent bivariate model considered by Díaz–García and Domínguez–Molina (2006). So, as remarked, the shape parameter \( \lambda \) also introduces correlation between \(T_1\) and \(T_2\).

Contour plots for the joint pdf (4) are presented in Figure 1. From this figure, note that (4) can take on different shapes and will therefore be useful in analyzing bivariate data. Additionally, notice that (4) can be unimodal or bimodal depending on the value of \( \lambda \).

The following theorem provides the marginal and conditional distributions of the SBVBS distribution.

**Theorem 2.1.** If \((T_1, T_2) \sim SBVBS(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)\), then:

(i) \(T_j \sim BS(\alpha_j, \beta_j)\), for \( j = 1, 2 \).

(ii) The conditional pdf of \(T_1\) given \(T_2 = t_2\) is

\[
f_{T_1|T_2}(t_1|T_2 = t_2) = 2\phi(a_1)\Phi(\lambda a_1 a_2)\frac{t_1^{-3/2}(t_1 + \beta_1)}{2\alpha_1 \sqrt{\beta_1}}.
\]

(iii) The cdf of \(T_1\) given \(T_2 = t_2\) is

\[
Pr(T_1 \leq t_1|T_2 = t_2) = \Phi(a_1) - 2\Upsilon(\alpha_1, \lambda a_2),
\]

where \( \Upsilon(\cdot, \cdot) \) denotes the Owen’s function (Owen, 1956).

**Proof.** Parts (i) and (ii) follow from the definition of the distribution. We have that

\[
Pr(T_1 \leq t_1|T_2 = t_2) = \int_0^{t_1} 2\phi(a_t)\Phi(\lambda a_t a_2)\frac{t^{-3/2}(t + \beta_1)}{2\alpha_1 \sqrt{\beta_1}} \, dt,
\]
Figure 1: Contour plots of the density function \( f \) for some values of \((\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)\):

(a) \((0.5, 0.5, 1.0, 1.0, 0.5)\);  
(b) \((0.2, 0.4, 1.0, 1.0, -1)\);  
(c) \((0.8, 0.5, 1.0, 1.0, 1.5)\);  
(d) \((1.5, 1.5, 1.0, 1.0, 1.5)\);  
(e) \((0.2, 0.2, 1.0, 1.0, 5)\);  
(f) \((0.2, 0.4, 1.0, 1.0, -10)\);
Additionally,

\[ a_t = \frac{1}{\alpha_1} \left[ \left( \frac{t}{\beta_1} \right)^{1/2} - \left( \frac{\beta_1}{t} \right)^{1/2} \right]. \]

Making the change of variable \( u = a_t \), we arrive at

\[ \Pr(T_1 \leq t_1 | T_2 = t_2) = \int_{-\infty}^{a_1} 2\phi(u)\Phi(\lambda u a_2) du. \]

Now, from Azzalini (1985) we can show that \( \Pr(T_1 \leq t_1 | T_2 = t_2) = \Phi(a_1) - 2\Psi(a_1, \lambda a_2) \) and therefore the result (iii) holds.

Some properties of the random vector \((T_1, T_2)\) are provided in the following theorem.

**Theorem 2.2.** If \((T_1, T_2) \sim \text{SBVBS}(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda)\), then:

(i) \((k_1T_1, T_2) \sim \text{SBVBS}(\alpha_1, \alpha_2, k_1\beta_1, \beta_2, \lambda), \quad k_1 > 0.\)

(ii) \((T_1, k_2T_2) \sim \text{SBVBS}(\alpha_1, \alpha_2, \beta_1, k_2\beta_2, \lambda), \quad k_2 > 0.\)

(iii) \((k_1T_1, k_2T_2) \sim \text{SBVBS}(\alpha_1, \alpha_2, k_1\beta_1, k_2\beta_2, \lambda), \quad k_1, k_2 > 0.\)

(iv) \((T_1^{-1}, T_2^{-1}) \sim \text{SBVBS}(\alpha_1, \alpha_2, \beta_1^{-1}, \beta_2^{-1}, \lambda).\)

(v) \((T_1^{-1}, T_2) \sim \text{SBVBS}(\alpha_1, \alpha_2, \beta_1^{-1}, \beta_2, -\lambda).\)

(vi) \((T_1, T_2^{-1}) \sim \text{SBVBS}(\alpha_1, \alpha_2, \beta_1, \beta_2^{-1}, -\lambda).\)

**Proof.** These follow from (4) upon using suitable transformations.

Since the marginal distributions of the bivariate vector \((T_1, T_2)\) are BS distributions, the mean and variance of \(T_1\) and \(T_2\) are obtained directly from these marginals in the forms

\[ \mathbb{E}(T_j) = \beta_j \left( 1 + \frac{1}{2} \alpha_j^2 \right), \quad \mathbb{V}(T_j) = (\alpha_j \beta_j)^2 \left( 1 + \frac{5}{4} \alpha_j^2 \right), \quad j = 1, 2. \]

Additionally,

\[ \mathbb{E}(T_j^{-1}) = \beta_j^{-1} \left( 1 + \frac{1}{2} \alpha_j^2 \right), \quad \mathbb{V}(T_j^{-1}) = \alpha_j^2 \beta_j^{-2} \left( 1 + \frac{5}{4} \alpha_j^2 \right), \quad j = 1, 2. \]

The product moments of \((T_1, T_2), \mathbb{E}(T_1 T_2)\) say, are very complicated to be determined algebraically and have to be computed numerically. In the following, we shall derive an expression for \(\mathbb{E}(T_1 T_2)\) which can be of some interest. We can show after some algebra that

\[ \mathbb{E}(T_1 T_2) = \beta_1 \beta_2 \mathbb{E} \left[ \left( \frac{\alpha_1}{2} Z_1 + \sqrt{\left( \frac{\alpha_1}{2} Z_1 \right)^2 + 1} \right) \left( \frac{\alpha_2}{2} Z_2 + \sqrt{\left( \frac{\alpha_2}{2} Z_2 \right)^2 + 1} \right) \right] \]

\[ = \beta_1 \beta_2 \left[ 1 + \frac{1}{2} (\alpha_1^2 + \alpha_2^2) + \frac{1}{4} \alpha_1^2 \alpha_2^2 + 2^{1/2} \pi^{-1/2} \alpha_1 \alpha_2 \lambda \right]. \]
where

$$I = I_{00} + \sum_{i=2}^{\infty} u_i \frac{\alpha_i^2 (4i + 1)!}{2^{2i}} \sum_{m=0}^{2i} \frac{m!(2\lambda)^{2m} I_m}{(2m + 1)! (2i - m)!}$$

$$+ \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} u_i v_k \frac{\alpha_i^2 \alpha_k^2 (4i + 1)!}{2^{2i}} \sum_{m=0}^{2i} \frac{m!(2\lambda)^{2m} I_{ikm}}{(2m + 1)! (2i - m)!},$$

with

$$u_i = (-1)^{i-1} \frac{1 \times 3 \times \cdots \times (2i - 3)}{i! 2^{3i}}, \quad v_k = (-1)^{k-1} \frac{1 \times 3 \times \cdots \times (2k - 3)}{k! 2^{3k}},$$

$$I_{00} = \mathbb{E} \left[ \frac{1 + \frac{\alpha^2 Z_2^2}{1 + \lambda^2 Z_2^2}^{1/2}}{1 + \frac{30 \alpha_1^2}{2^3 (1 + \lambda^2 Z_2^2)^2} \sum_{k=1}^{3} c_k Z_2^{2k}} \right],$$

$$I_{ikm} = \mathbb{E} \left[ \frac{Z_2^{2(m+k+1)}}{(1 + \lambda^2 Z_2^2)^{2i+1/2}} \right], \quad I_{im} = \mathbb{E} \left[ \frac{Z_2^{2(m+1)}}{(1 + \lambda^2 Z_2^2)^{2i+1/2}} \right],$$

being $c_1 = 1$, $c_2 = 4\lambda^2/(3!)$ and $c_3 = 32\lambda^4/(5!)$. For $\lambda = 0$ (independent case), we have immediately that

$$\mathbb{E}(T_1 T_2) = \beta_1 \beta_2 \left[ 1 + \frac{1}{4} (\alpha_1^2 + \alpha_2^2) + \frac{1}{4} \alpha_1^2 \alpha_2^2 \right].$$

### 3 Multivariate extension

We have considered the bivariate case in Section 2, but extensions to higher dimension can be readily accomplished using suitable notation. For a random variable $Z = (Z_1, \ldots, Z_p)^\top$ of dimension $p$, we define the subvectors $Z_{(1)}, \ldots, Z_{(p)}$ of dimensions $(p-1)$ such that, for each $j = 1, \ldots, p$, $Z_{(j)}$ denotes the vector $Z$ with the $j$th coordinate $Z_j$ deleted. Analogously, for a real vector $z = (z_1, \ldots, z_p)^\top$, $z_{(j)}$ is obtained from $z$ with the $j$th coordinate $z_j$ deleted.

By assuming (for each $j = 1, \ldots, p$) that

$$Z_j | Z_{(j)} = z_{(j)} \sim \text{SN} \left( \lambda \prod_{j' \neq j} z_{j'} \right),$$

the joint pdf of $Z = (Z_1, \ldots, Z_p)^\top$ takes the form (Arnold et al., 2002)

$$f_Z(z) = 2 \prod_{j=1}^{p} \phi(z_j) \Phi \left( \lambda \prod_{j=1}^{p} z_j \right), \quad z \in \mathbb{R}^p.$$

Thus, under the transformation

$$T_j = \beta_j \left[ \frac{\alpha_j}{2} Z_j + \sqrt{\left( \frac{\alpha_j}{2} Z_j \right)^2 + 1} \right]^2, \quad j = 1, \ldots, p,$$
where $Z_j \sim N(0,1)$, we obtain the joint pdf of $\mathbf{T} = (T_1, \ldots, T_p)^T$ in the form
\[
 f_{\mathbf{T}}(\mathbf{t}) = 2 \left[ \prod_{j=1}^{p} \phi(a_j) \right] \Phi \left( \lambda \prod_{j=1}^{p} a_j \right) \frac{\prod_{j=1}^{p} t_j^{-3/2} (t_j + \beta_j)}{2 \alpha_j \sqrt{\beta_j}}, \quad \mathbf{t} \in \mathbb{R}_+^p,
\]
where $\alpha_j > 0$, $\beta_j > 0$ and $a_j$ is given in (5), $j = 1, \ldots, p$. Let $\alpha = (\alpha_1, \ldots, \alpha_p)^T$ and $\beta = (\beta_1, \ldots, \beta_p)^T$. If $\mathbf{T} = (T_1, \ldots, T_p)^T$ has skewed multivariate BS distribution, then we use the notation $\mathbf{T} \sim \text{SMVBS}(\alpha, \beta, \lambda)$.

Several properties discussed in the bivariate case hold for this multivariate extension. For example, $T_j \sim \text{BS}(\alpha_j, \beta_j)$ for $j = 1, \ldots, p$, i.e. the marginal distributions are BS distributions; $\lambda = 0$ corresponds to the independent case; for $k_1, \ldots, k_p > 0$, $(k_1T_1, \ldots, k_pT_p) \sim \text{SMVBS}(\alpha, \beta^∗, \lambda)$ with $\beta^∗ = (k_1\beta_1, \ldots, k_p\beta_p)^T$; $(T_1^{-1}, \ldots, T_p^{-1}) \sim \text{SMVBS}(\alpha, \beta^{∗∗}, \lambda)$, where $\beta^{∗∗} = (\beta_1^{-1}, \ldots, \beta_p^{-1})^T$, and so on. In the next section, we shall consider estimation for the unknown parameters of the SMVBS distribution in (6) as well as inference. Thus, from these general results the bivariate case considered in Section 2 can be easily specialized by considering $p = 2$.

4 Estimation and inference

In this section, we address the problem of estimating the unknown parameters of the SMVBS distribution. Let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ denote a random sample of the $\text{SMVBS}(\alpha, \beta, \lambda)$ distribution, where $\mathbf{t}_i = (t_{i1}, \ldots, t_{ip})^T$ and, as before, $\alpha = (\alpha_1, \ldots, \alpha_p)^T$ and $\beta = (\beta_1, \ldots, \beta_p)^T$. Let $\mathbf{\theta} = (\alpha^T, \beta^T, \lambda)^T$ be the parameter vector of interest of dimension $2p + 1$.

4.1 Modified moment estimators

First, we shall present modified moment estimators (MMEs) for the unknown parameters by following the approach of Ng et al. (2003). The SMVBS model has $2p + 1$ parameters and the marginal distributions are BS distributions with parameters $(\alpha_j, \beta_j)$, $j = 1, \ldots, p$. Then, the moment estimators for $\alpha_j$ and $\beta_j$ can be obtained by equating $\mathbb{E}(T_j)$ and $\mathbb{V}(T_j)$ to the corresponding sample estimates for $j = 1, \ldots, p$. However, it is known that in the case of univariate BS distribution, the moment estimators may not always exist (Ng et al., 2003). Here, we will use $\mathbb{E}(T_j)$ and $\mathbb{E}(T_j^{-1})$ instead of using $\mathbb{E}(T_j)$ and $\mathbb{V}(T_j)$, and equate them to the corresponding sample quantities. After some algebra, the MMEs for $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_p$ are

\[
 \hat{\alpha}_j = \left[ 2 \left( \frac{\bar{s}_j}{\bar{r}_j} \right)^{1/2} - 1 \right]^{1/2}, \quad \hat{\beta}_j = \left( \bar{s}_j \bar{r}_j \right)^{1/2}, \quad j = 1, \ldots, p,
\]

where
\[
 \bar{s}_j = \frac{1}{n} \sum_{i=1}^{n} t_{ji}, \quad \bar{r}_j = \frac{1}{n} \left( \sum_{i=1}^{n} t_{ji} \right)^{-1}.
\]
The MMEs for $\alpha$ and $\beta$ in (7) are explicit in form and can be used effectively as the initial guess in the iterative process for the computation of the maximum likelihood estimators (MLEs) in the next section.

4.2 Maximum likelihood estimators

The log-likelihood function for the parameter vector $\theta$ (apart from an unimportant constant) is given by

$$
\ell(\theta) = -n \sum_{j=1}^{p} \left[ \log(\alpha_j) + \frac{1}{2} \log(\beta_j) \right] + \sum_{i=1}^{n} \sum_{j=1}^{p} \log(t_{ji} + \beta_j) \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{p} a_{ji}^2 + \sum_{i=1}^{n} \log \left[ \Phi \left( \lambda \prod_{j=1}^{p} a_{ji} \right) \right],
$$

where

$$
a_{ji} = a_{ji}(\alpha_j, \beta_j) = \frac{1}{\alpha_j} \left[ \left( \frac{t_{ji}}{\beta_j} \right)^{1/2} - \left( \frac{\beta_j}{t_{ji}} \right)^{1/2} \right].
$$

The MLEs of the unknown parameters are obtained by maximizing the log-likelihood function in (8) with respect to $\theta$. By taking the partial derivatives of the log-likelihood function in (8) with respect to the parameters $\alpha_j$, $\beta_j$ and $\lambda$, we have (for $j = 1, \ldots, p$)

$$
\frac{\partial \ell(\theta)}{\partial \alpha_j} = -\frac{n}{\alpha_j} + \frac{1}{\alpha_j} \sum_{i=1}^{n} a_{ji}^2 - \frac{\lambda}{\alpha_j} \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji},
$$

$$
\frac{\partial \ell(\theta)}{\partial \beta_j} = -\frac{n}{2\beta_j} + \sum_{i=1}^{n} \frac{1}{\beta_j + t_{ji}} - \frac{1}{2\alpha_j^2 \beta_j} \sum_{i=1}^{n} \left[ \frac{\beta_j}{t_{ji}} - \frac{t_{ji}}{\beta_j} \right]
- \frac{\lambda}{2\alpha_j \beta_j} \sum_{i=1}^{n} w_i d_{ij} \prod_{j' \neq j} a_{j'i},
$$

$$
\frac{\partial \ell(\theta)}{\partial \lambda} = \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji},
$$

where

$$
w_i = w_i(\alpha, \beta, \lambda) = \frac{\phi(\lambda \prod_{j=1}^{p} a_{ji})}{\Phi(\lambda \prod_{j=1}^{p} a_{ji})}, \quad d_{ij} = d_{ij}(\beta_j) = \left( \frac{t_{ji}}{\beta_j} \right)^{1/2} + \left( \frac{\beta_j}{t_{ji}} \right)^{1/2}.
$$

The MLE $\hat{\theta} = (\hat{\alpha}^T, \hat{\beta}^T, \hat{\lambda})^T$ of $\theta = (\alpha^T, \beta^T, \lambda)^T$ can be obtained by solving the likelihood equations

$$
\frac{\partial \ell(\theta)}{\partial \alpha_j} = \frac{\partial \ell(\theta)}{\partial \beta_j} = \frac{\partial \ell(\theta)}{\partial \lambda} = 0, \quad j = 1, \ldots, p.
$$

9
simultaneously. There are no closed form expressions for the MLE and its computation has to be performed numerically using a nonlinear optimization algorithm. The Newton-Raphson iterative technique could be applied to solve the likelihood equations and obtain the estimate $\hat{\theta}$. For computing the MLEs, starting values for the algorithm are required. Since the MMEs for $\alpha_j$ and $\beta_j$ in (7) are explicit, they can be used effectively as the initial guess in the iterative procedure. The Ox\footnote{Ox is freely distributed for academic purposes at \url{http://www.doornik.com}} matrix programming language \cite{Doornik2006} and the R program \cite{RDevelopmentCoreTeam2010} can be used to compute $\theta$ numerically.

We can show from the likelihood equations that, for given $\beta_1, \ldots, \beta_p$, the MLEs of $\alpha_1, \ldots, \alpha_p$ are

$$\hat{\alpha}_j(\beta_j) = \left(\frac{\bar{s}_j}{\bar{f}_j} + \frac{\beta_j}{2}\right)^{1/2}, \quad j = 1, \ldots, p.$$

By replacing $\alpha_j$ by $\hat{\alpha}_j(\beta_j)$ in (8), we obtain the profile log-likelihood function for $\beta$ and $\lambda$ as

$$\ell_p(\beta, \lambda) = -n \sum_{j=1}^p \left[ \log(\hat{\alpha}_j(\beta_j)) + \frac{1}{2} \log(\beta_j) \right] + \sum_{i=1}^n \sum_{j=1}^p \log(t_{ji} + \beta_j)$$

$$- \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^p \hat{a}_{ji}(\beta_j)^2 + \sum_{i=1}^n \log \left[ \Phi \left( \lambda \prod_{j=1}^p \hat{a}_{ji}(\beta_j) \right) \right],$$

where

$$\hat{a}_{ji}(\beta_j) = a_{ji}(\hat{\alpha}_j(\beta_j), \beta_j) = \frac{1}{\hat{\alpha}_j(\beta_j)} \left[ \left( \frac{t_{ji}}{\beta_j} \right)^{1/2} - \left( \frac{\beta_j}{t_{ji}} \right)^{1/2} \right].$$

We can also obtain the MLEs of $\beta$ and $\lambda$ by maximizing the profile log-likelihood function $\ell_p(\beta, \lambda)$ with respect to $\beta$ and $\lambda$. The Newton–Raphson algorithm or some other optimization algorithm to maximize $\ell_p(\beta, \lambda)$ with respect to $\beta$ and $\lambda$ needs to be used, since the MLEs of $\beta$ and $\lambda$ cannot be obtained explicitly. The profile log-likelihood function $\ell_p(\beta, \lambda)$ is not a real log-likelihood function and some of the properties that hold for a genuine log-likelihood do not hold for its profiled version. In particular, there exist score and information biases, both of order $O(1)$.

The asymptotic inference for the parameter vector $\theta = (\alpha^T, \beta^T, \lambda)^T$ can be based on the normal approximation of the MLE $\hat{\theta}$ of $\theta = (\alpha^T, \beta^T, \lambda)^T$. Under some regular conditions stated in Cox and Hinkley \cite{CoxHinkley1974, Ch. 9} that are fulfilled for the parameters in the interior of the parameter space, we have $\hat{\theta} \overset{d}{\sim} N_{2p+1}(\theta, \Sigma_\theta^{-1})$, for $n$ large, where $\overset{d}{\sim}$ means approximately distributed and $\Sigma_\theta^{-1}$ is the asymptotic variance-covariance matrix of $\hat{\theta}$. The matrix $\Sigma_\theta$ is given in the Appendix. The multivariate normal $N_{2p+1}(0, \Sigma_\theta^{-1})$ distribution can be used to construct approximate confidence intervals for the parameters $\alpha_j$, $\beta_j$ and $\lambda$, which are given, respectively, by $\hat{\alpha}_j \pm z_{\gamma/2} \times [\hat{\mathcal{V}}(\hat{\alpha}_j)]^{1/2}$, $\hat{\beta}_j \pm z_{\gamma/2} \times [\hat{\mathcal{V}}(\hat{\beta}_j)]^{1/2}$ and $\hat{\lambda} \pm z_{\gamma/2} \times [\hat{\mathcal{V}}(\hat{\lambda})]^{1/2}$, where $\hat{\mathcal{V}}(\cdot)$ is the diagonal element of $\Sigma_\theta^{-1}$ available at $\hat{\theta}$ corresponding to each parameter, and $z_{\gamma/2}$ is the quantile $100(1 - \gamma/2)$% of the standard normal distribution.
Besides estimation of the model parameters, hypotheses tests can be taken into account. Let \( \theta = (\theta_1^T, \theta_2^T)^T \), where \( \theta_1 \) and \( \theta_2 \) are disjoint subsets of \( \theta \). Consider the test of the null hypothesis \( \mathcal{H}_0 : \theta_1 = \theta_{01} \) against \( \mathcal{H}_1 : \theta_1 \neq \theta_{01} \), where \( \theta_{01} \) is a specified vector. Let \( \hat{\theta} \) be the restricted MLE of \( \theta \) obtained under \( \mathcal{H}_0 \). The likelihood ratio (LR) statistic to test \( \mathcal{H}_0 \) is given by \( \omega = 2\{\ell(\hat{\theta}) - \ell(\hat{\theta}_1, \hat{\theta}_2, 0)\} \).

Under \( \mathcal{H}_0 \) and some regularity conditions, the LR statistic converges in distribution to a chi-square distribution with \( \dim(\theta_1) \) degrees of freedom. In particular, the LR statistic to test the null hypothesis \( \mathcal{H}_0 : \lambda = 0 \) against \( \mathcal{H}_1 : \lambda \neq 0 \) takes the form

\[
\omega = 2\{\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) - \ell(\hat{\alpha}, \hat{\beta}, 0)\},
\]

where \( \hat{\alpha} \) and \( \hat{\beta} \) are the restricted MLEs of \( \alpha \) and \( \beta \), respectively, obtained from the maximization of (3) under \( \mathcal{H}_0 : \lambda = 0 \). The limiting distribution of this statistic is \( \chi^2_1 \) under the null hypothesis.

The null hypothesis is rejected if the test statistic exceeds the upper \( 100(1 - \gamma)\% \) quantile of the \( \chi^2_1 \) distribution.

## 5 Application to real data

In this section, for illustrative purposes, we present an empirical application to demonstrate the applicability of the proposed skewed multivariate BS distribution. For the sake of comparison, we also consider the distributions proposed in Díaz–García and Domínguez–Molina (2006) and Kundu et al. (2010). We shall use the data set obtained from Volle (1985), which represent the amount of time (in hours) spent on two categories of activities over 100 days in the year 1976 for 28 individuals. The data are: (115, 175), (100, 115), (130, 160), (115, 180), (119, 143), (100, 150), (960, 132), (150, 115), (142, 870), (180, 125), (152, 122), (174, 119), (140, 100), (147, 840), (105, 700), (950, 600), (130, 600), (105, 800), (117, 650), (850, 400), (102, 450), (100, 960), (920, 640), (128, 860), (102, 122), (107, 730), (860, 580), (940, 580). The first figure represents the amount of time spent on eating and the second figure represents the amount of time spent on watching television. All the computations were done using the Ox matrix programming language (Doornik, 2006).

We now use the SBVBS distribution to model these bivariate data. We obtain from the data \( \bar{s}_1 = 118.14, \bar{s}_2 = 99.43, \bar{r}_1 = 113.40 \) and \( \bar{r}_2 = 84.61 \), and hence the MMEs are \( \hat{\alpha}_1 = 0.2035, \hat{\alpha}_2 = 0.4099, \hat{\beta}_1 = 115.7457 \) and \( \hat{\beta}_2 = 91.7220 \). These values are used as initial guesses for \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \), respectively. An initial guess for \( \lambda \) is also required to start the maximization of the log-likelihood function (8), i.e. to solve the likelihood equations (9) with \( p = 2 \). As initial value for \( \lambda \) we consider \( \hat{\lambda} = 0 \), which corresponds to the independent case. The algorithm converges after 21 steps and the MLEs of \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and \( \lambda \) are \( \hat{\alpha}_1 = 0.2047, \hat{\alpha}_2 = 0.4101, \hat{\beta}_1 = 113.2907, \hat{\beta}_2 = 90.7447 \) and \( \hat{\lambda} = 0.8806 \), respectively. Notice that the MMEs for \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are close to their respective MLEs. We have also considered other initial guesses for \( \lambda \), for example, with the initial values \( \hat{\lambda} = -5, -2, 3 \) and 4, the algorithm converges to the same estimates after 39, 26, 25 and 31 steps, respectively. The 95% asymptotic confidence intervals for \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and \( \lambda \) are
(0.1508, 0.2586), (0.3051, 0.5152), (104.8325, 121.7489), (77.8975, 103.5919) and (0.0349, 1.7263), respectively.

Next, we make use of the LR statistic to test the null hypothesis $H_0 : \lambda = 0$ against $H_1 : \lambda \neq 0$. Here, $\omega = 2\{\ell(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\lambda}) - \ell(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, 0)\}$, where $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1$ and $\tilde{\beta}_2$ are, respectively, the restricted MLEs of $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ obtained under $H_0$ and are given by $\tilde{\alpha}_1 = 0.2035$, $\tilde{\alpha}_2 = 0.4099$, $\tilde{\beta}_1 = 115.7470$ and $\tilde{\beta}_2 = 91.7128$. By a little computation, we have that the LR test statistic ($\omega$) equals 6.6834 (p-value < 0.01). Therefore, the null hypothesis $H_0 : \lambda = 0$ is strongly rejected at the usual significance levels and hence the assumption of the skewness (correlation) is suitable for the current bivariate data. Since the bivariate distribution in Díaz–García and Domínguez–Molina (2006), DG–DM say, and our proposed model are nested models (i.e. the DG–DM model holds for $\lambda = 0$), the null and alternative hypotheses can be rewritten as $H_0$: DG–DM against $H_1$: SBVBS. Thus, based on the LR statistic above, the SBVBS distribution fits the data better than the bivariate DG–DM model.

The generalized LR statistic ($T_{LR,NN}$) presented in Vuong (1989) can be used for discriminating among non-nested models, which is a distance between the two models measured in terms of the Kullback–Liebler information criterion. Then, our proposed model and the bivariate model in Kundu et al. (2010) can be compared by using $T_{LR,NN}$. For strictly nonnested models, $T_{LR,NN}$ converges in distribution to a standard normal distribution under the null hypothesis of equivalence of the models and the null hypothesis is not rejected if $|T_{LR,NN}| \leq z_{\gamma/2}$, where $z_{\gamma/2}$ is the quantile 100(1 – $\gamma/2$)% of the standard normal distribution. On the other hand, we reject at significance level $\gamma$ the null hypothesis of equivalence of the models in favor of the SBVBS model being better (or worse) than the model in Kundu et al. (2010) if $T_{LR,NN} > z_\gamma$ (or $T_{LR,NN} < -z_\gamma$). The generalized LR test statistic ($T_{LR,NN}$) equals 4.0903 (p-value < 0.01). Therefore, the proposed model is significantly better than the model in Kundu et al. (2010) according to the generalized LR statistic to model the current data.

A natural question at this point is whether SBVBS model fits the current data satisfactorily. Here, in order to verify it, we computed the modified Cramér-von Mises ($W^*$) and Anderson-Darling ($A^*$) statistics for the fitted marginals, i.e. BS(0.2047, 113.2907) and BS(0.4101, 90.7447). The statistics $W^*$ and $A^*$ are described in details by Chen and Balakrishnan (1995). The values of these statistics are 0.0971 (p-value > 0.1) and 0.5680 (p-value > 0.1), and 0.0513 (p-value > 0.1) and 0.3145 (p-value > 0.1), respectively. Therefore, based on the marginals, we have that the SBVBS distribution can be used effectively in this case. Although it does not guarantee that the bivariate real data will have SBVBS distribution, at least it gives an indication that the SBVBS model may be used to analyze this bivariate data.
6 Skewed bivariate generalized BS distribution

The univariate generalized BS (GBS) distribution was proposed in Díaz–García and Leiva (2005), which is a highly flexible lifetime model that admits different degrees of kurtosis and asymmetry and possesses unimodality and bimodality. The GBS distribution is related to standard symmetrical distributions in \( \mathbb{R} \), also known as elliptically contoured univariate distributions. The reader is referred to Fang et al. (1990) and Gupta and Varga (1993) for more details about symmetrical distributions. For the univariate case, elliptical distributions correspond to all the symmetric distributions in \( \mathbb{R} \). Specifically, a random variable \( X \) has an elliptical distribution if its probability density function is given by \( f_X(x) = c g((x - \mu)/\phi^2), x \in \mathbb{R} \), where \( \mu \in \mathbb{R} \) is a location parameter and \( \phi > 0 \) is a scale parameter. The function \( g : \mathbb{R} \to [0, \infty) \) corresponds to the kernel of the density of \( X \) and \( c \) is the normalization constant such that \( f_X(x) \) is a density. The function \( g(\cdot) \) is typically known as density generator. We then write \( X \sim E(\mu, \phi^2; g) \).

The notation \( Z \sim E(0, 1; g) \) or \( Z \sim E(g) \) is used for a random variable \( Z \) that follows a standard elliptical distribution in \( \mathbb{R} \). The pdf and cdf of \( Z \) are denoted by \( f(\cdot) \) and \( F(\cdot) \), respectively, where \( f(z) = c g(z^2) \) and \( F(z) = \int_{-\infty}^{z} f(z) \, dz \). The density generator of the normal, Cauchy, Student-\( t \), generalized Student-\( t \), type I logistic, type II logistic and power exponential are, respectively, given by \( g(u) = (2\pi)^{-1/2} \exp(-u/2) \), \( g(u) = \{\pi(1 + u)\}^{-1} \), \( g(u) = \nu^{\nu/2} B(1/2, \nu/2)^{-1}(\nu + u)^{-\nu/2} \), \( g(u) = \nu^{\nu/2} B(1/2, \nu/2)^{-1}(\nu + u)^{-\nu/2} \), \( g(u) = c e^{-u}(1 + e^{-u})^{-2} \), where \( c \approx 1.484300029 \) is the normalizing constant obtained from \( \int_{0}^{\infty} u^{-1/2} g(u) \, du = 1, g(u) = e^{-\sqrt{u}}(1 + e^{-\sqrt{u}})^{-2} \) and \( g(u) = c(k) \exp(-\frac{1}{2}u^{1/(1+k)}) \), \(-1 < k \leq 1 \), where \( c(k) = \Gamma(1 + (k + 1)/2)2^{1/(1+k)}/ \).

In the following, we shall introduce the skewed bivariate GBS (SBVGBS) distribution. A random variable \( Y \) follows a standard skew-elliptical distribution in \( \mathbb{R} \) if its pdf takes the form

\[
    f_Y(y) = 2f(y)F(\lambda y), \quad y \in \mathbb{R}.
\]

We use the notation \( Y \sim SE(\lambda; g) \). If \( \lambda = 0 \) in (10), then the standard elliptical distribution holds, i.e. \( Y \sim E(g) \). Now, let \( Z_j \sim E(g) \), for \( j = 1, 2 \), with \( Z_1|Z_2 = z_2 \sim SE(\lambda z_2; g) \) and \( Z_2|Z_1 = z_1 \sim SE(\lambda z_1; g) \). Additionally, consider the transformation

\[
    T_j = \beta_j \left[ \frac{\alpha_j}{2} Z_j + \sqrt{\frac{\alpha_j}{2} Z_j} \right]^2, \quad j = 1, 2,
\]

where \( \alpha_j > 0 \) and \( \beta_j > 0 \). Then, from the above transformation and using results due to Arnold et al. (2002), the joint pdf of the SBVGBS distribution is given by

\[
    f_{T_1, T_2}(t_1, t_2) = 2f(a_1)f(a_2)F(\lambda a_1 a_2) \frac{t_1^{3/2}(t_1 + \beta_1) t_2^{-3/2}(t_2 + \beta_2)}{2\alpha_1 \sqrt{\beta_1} 2\alpha_2 \sqrt{\beta_2}}, \quad (t_1, t_2) \in \mathbb{R}^2_+,
\]

where \( a_j \) is defined in (5). If \( (T_1, T_2) \) follows the SBVGBS distribution, the notation used is \( (T_1, T_2) \sim SBVGBS(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda; g) \). Notice that the joint pdf (4) is a special case of (11). All extra parameters are considered as known or fixed in (11). For example, the degrees of freedom for the Student-\( t \)
model. The main motivation for this generalization of the SBVBS model presented in Section 2 is based on the search for bivariate distributions that are more flexible than the SBVBS model in analyzing bivariate data.

Some properties for this bivariate class of distributions are presented in the following theorem.

**Theorem 6.1.** If \((T_1, T_2) \sim \text{SBVGBS}(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda; g)\), then:

(i) \(T_j \sim \text{GBS}(\alpha_j, \beta_j; g)\), for \(j = 1, 2\).

(ii) \((k_1 T_1, T_2) \sim \text{SBVGBS}(\alpha_1, \alpha_2, k_1 \beta_1, \beta_2, \lambda; g)\), \(k_1 > 0\).

(iii) \((T_1, k_2 T_2) \sim \text{SBVGBS}(\alpha_1, \alpha_2, \beta_1, k_2 \beta_2, \lambda; g)\), \(k_2 > 0\).

(iv) \((k_1 T_1, k_2 T_2) \sim \text{SBVGBS}(\alpha_1, \alpha_2, k_1 \beta_1, k_2 \beta_2, \lambda; g)\), \(k_1, k_2 > 0\).

(v) \((T_1^{-1}, T_2^{-1}) \sim \text{SBVGBS}(\alpha_1, \alpha_2, \beta_1^{-1}, \beta_2^{-1}, \lambda; g)\).

(vi) \((T_1^{-1}, T_2) \sim \text{SBVGBS}(\alpha_1, \alpha_2, \beta_1^{-1}, \beta_2, -\lambda; g)\).

(vii) \((T_1, T_2^{-1}) \sim \text{SBVGBS}(\alpha_1, \alpha_2, \beta_1, \beta_2^{-1}, -\lambda; g)\).

**Proof.** Using suitable transformations in (11), these results follow.

From (11), several new SBVGBS distributions can be obtained. For example, the joint pdf of the skewed bivariate BS Student-\(t\) model takes the form

\[
 f_{T_1, T_2}(t_1, t_2) = 2 \prod_{j=1}^{2} \frac{\Gamma([\nu_j] + 1/2)}{\nu_j \pi^{1/2} \Gamma(\nu_j/2)} \left(1 + \frac{a_j^2}{\nu_j} \right)^{-\nu_j + 1/2} \frac{t_j^{-3/2}(t_j + \beta_j)}{2 \nu_j \beta_j} \left[1 + I_{\nu_j} \left(\frac{1}{2}, \nu_j \frac{\nu_j}{2}\right)\right],
\]

where \(\nu_j\) is the degrees of freedom, \(q_j = (\lambda a_1 a_2)^2 / [(\lambda a_1 a_2)^2 + \nu_j]\) and \(I_x(r, s)\) is the incomplete beta ratio function. The skewed bivariate BS Cauchy distribution is a special case of the joint pdf above when \(\nu_1 = \nu_2 = 1\). It is evident that other bivariate models can be obtained as, for example, the skewed bivariate BS type I (type II) logistic model, skewed bivariate BS power exponential model, and so on. Further, extensions to higher dimension can be derived and MLE of the unknown parameters can also be considered. These problems can be developed in a future research.

7 Concluding remarks

The univariate BS model has many attractive properties and has found several applications in the literature including lifetime, survival and environmental data analysis (see, for example, Leiva et al., 2008, 2009). As mentioned before, little work has been done to extend the BS model for the multivariate case. In this article, we have introduced the skewed multivariate BS distribution. The new distribution is very general, quite flexible and widely applicable. The new model is an absolutely continuous multivariate distribution whose marginals are univariate BS distributions. We have discussed
several properties of this new class of distributions and the estimation of parameters is approached
by the method of maximum likelihood. The observed and expected information matrices are deter-
mined and likelihood ratio tests for some hypotheses of interest are also considered. The skewed
bivariate BS distribution is discussed and we have shown that the additional shape parameter (λ) in-
troduces skewness, correlation and bimodality to this distribution. These interesting properties make
this bivariate model a quite flexible distribution to model bivariate data. Other bivariate BS models
have been introduced and are given in Díaz–García and Domínguez–Molina (2006) and Kundu et al.
(2010), KBJ say. The DG–DM model is an independent bivariate model and hence does not consider
correlation between the random bivariate vector. The KBJ model considers correlation between the
random bivariate vector, but does not allow bimodality. As remarked, the skewed bivariate BS model
proposed in this article can be skewed, correlated and bimodal, and therefore is much more flexible
than the other bivariate BS models available in the literature for analyzing bivariate data. This is sup-
ported in an application to real data in which we show that the skewed bivariate BS model provides
consistently better fit than the DG–DM and KBJ models. Finally, we have also introduced in this pa-
per the skewed bivariate generalized BS distribution and discussed some of its properties. Although
we have discussed the generalized BS distribution in bivariate settings, the skewed multivariate gen-
eralized BS distribution can be introduced along the same lines. This problem can be developed in a
future research.

Acknowledgments

We gratefully acknowledge grants from FAPESP (Brazil) and Mobility Program of the Universidad
Industrial de Santander (Colombia).

Appendix. Fisher information matrix

We present the elements of the Fisher information matrix Σθ. First, we shall compute the elements
of the Hessian matrix

\[ \mathbf{\tilde{L}}_{\theta\theta} = \frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^\top} = \begin{bmatrix} \tilde{L}_{\alpha\alpha} & \tilde{L}_{\alpha\beta} & \tilde{L}_{\alpha\lambda} \\ \tilde{L}_{\alpha\beta}^\top & \tilde{L}_{\beta\beta} & \tilde{L}_{\beta\lambda} \\ \tilde{L}_{\alpha\lambda}^\top & \tilde{L}_{\beta\lambda}^\top & \tilde{L}_{\lambda\lambda} \end{bmatrix}, \]

with

\[ \tilde{L}_{\alpha\alpha} = \frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \alpha^\top} = ((\tilde{L}_{\alpha_j, \alpha_j})), \quad \tilde{L}_{\alpha\beta} = \frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \beta^\top} = ((\tilde{L}_{\alpha_j, \beta_j})), \]

\[ \tilde{L}_{\alpha\lambda} = \frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \lambda} = (\tilde{L}_{\alpha_1, \lambda}, \ldots, \tilde{L}_{\alpha_p, \lambda})^\top, \quad \tilde{L}_{\beta\beta} = \frac{\partial^2 \ell(\theta)}{\partial \beta \partial \beta^\top} = ((\tilde{L}_{\beta_j, \beta_j})), \]

\[ \tilde{L}_{\beta\lambda} = \frac{\partial^2 \ell(\theta)}{\partial \beta \partial \lambda} = (\tilde{L}_{\beta_1, \lambda}, \ldots, \tilde{L}_{\beta_p, \lambda})^\top, \quad \tilde{L}_{\lambda\lambda} = \frac{\partial^2 \ell(\theta)}{\partial \lambda^2}. \]
where $j, j' = 1, \ldots, p,$

$$
\tilde{L}_{\alpha_j \alpha_j} = \frac{n}{\alpha_j^2} - \frac{3}{\alpha_j^2} \sum_{i=1}^{n} a_{ji}^2 + \frac{2\lambda}{\alpha_j^2} \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji} - \frac{\lambda^2}{\alpha_j^2} \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji}^2,
$$

$$
\tilde{L}_{\lambda} = \frac{\lambda}{\alpha_j \alpha_j'} \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji} - \lambda \alpha_j \alpha_j' \sum_{i=1}^{n} a_{ji}^3 - \frac{\lambda^2}{\alpha_j} \sum_{i=1}^{n} w_i \prod_{j=1}^{p} a_{ji}^2,
$$

$$
\tilde{L}_{\beta_j \beta_j} = \frac{1}{\alpha_j^2} \sum_{i=1}^{n} \left[ \beta_j - t_{ji} - \frac{\beta_j}{\beta_j} \right] + \frac{2\lambda}{\alpha_j^2} \sum_{i=1}^{n} w_i d_{ij} \prod_{j' \neq j} a_{ji},
$$

$$
\tilde{L}_{\alpha_j \lambda} = -\frac{\lambda}{2\alpha_j} \sum_{i=1}^{n} w_i d_{ij} \left[ -1 + \frac{\lambda^2}{\alpha_j} \prod_{j=1}^{p} a_{ji}^2 + \lambda w_i \prod_{j=1}^{p} a_{ji} \right] \prod_{j' \neq j} a_{ji},
$$

$$
\tilde{L}_{\beta_j \lambda} = -\frac{\lambda}{2\alpha_j} \sum_{i=1}^{n} w_i d_{ij} \left[ -1 + \frac{\lambda}{\alpha_j} \prod_{j=1}^{p} a_{ji} \prod_{j' \neq j} a_{ji} \right] \prod_{j' \neq j} a_{ji},
$$

$$
\tilde{L}_{\lambda \lambda} = -\sum_{i=1}^{n} w_i \left( \lambda \prod_{j=1}^{p} a_{ji}^2 + \lambda \prod_{j=1}^{p} a_{ji} \right) \prod_{j=1}^{p} a_{ji}.
$$
The Fisher information matrix is given by

\[
\Sigma_\theta = -\mathbb{E}(\tilde{\mathbf{L}}_{\theta\theta}) = \begin{bmatrix}
\Sigma_{\alpha\alpha} & \Sigma_{\alpha\beta} & \Sigma_{\alpha\lambda} \\
\Sigma_{\alpha\beta}^\top & \Sigma_{\beta\beta} & \Sigma_{\beta\lambda} \\
\Sigma_{\alpha\lambda}^\top & \Sigma_{\beta\lambda}^\top & \Sigma_{\lambda\lambda}
\end{bmatrix},
\]

where

\[
\Sigma_{\alpha\alpha} = ((\Sigma_{\alpha_j\alpha_j})), \quad \Sigma_{\alpha\beta} = ((\Sigma_{\alpha_j\beta_j})), \quad \Sigma_{\alpha\lambda} = (\Sigma_{\alpha_1\lambda}, \ldots, \Sigma_{\alpha_p\lambda})^\top,
\]

\[
\Sigma_{\beta\beta} = ((\Sigma_{\beta_j\beta_j})), \quad \Sigma_{\beta\lambda} = (\Sigma_{\beta_1\lambda}, \ldots, \Sigma_{\beta_p\lambda})^\top,
\]

for \( j, j' = 1, \ldots, p, \)

\[
\Sigma_{\alpha_j\alpha_j} = \frac{2n}{\alpha_j^2} + \frac{\lambda^3}{\alpha_j} \sum_{i=1}^n \mathbb{E} \left[ w_i \prod_{j=1}^p a_{ji}^3 \right] + \frac{\lambda^2}{\alpha_j^2} \sum_{i=1}^n \mathbb{E} \left[ w_i^2 \prod_{j=1}^p a_{ji}^2 \right],
\]

\[
\Sigma_{\alpha_j\beta_j} = \frac{\lambda^3}{2\alpha_j^2 \beta_j} \sum_{i=1}^n \mathbb{E} \left[ w_i \prod_{j=1}^p a_{ji}^3 \right] + \frac{\lambda^2}{\alpha_j \beta_j} \sum_{i=1}^n \mathbb{E} \left[ w_i^2 \prod_{j=1}^p a_{ji}^2 \right], \quad j' \neq j,
\]

\[
\Sigma_{\alpha_j\lambda} = -\frac{\lambda}{\alpha_j} \sum_{i=1}^n \mathbb{E} \left[ w_i \left( \prod_{j=1}^p a_{ji}^2 + \prod_{j=1}^p a_{j'j'} \right) \prod_{j' \neq j} a_{j'j} \right],
\]

\[
\Sigma_{\beta_j\beta_j} = \frac{n}{\alpha_j^2 \beta_j^2} + \frac{nK(\alpha_j)}{\alpha_j^2 \beta_j^2} + \frac{\lambda^2}{4\alpha_j^2 \beta_j^2} \sum_{i=1}^n \mathbb{E} \left[ w_i d_{ij}^2 \left( w_i + \lambda w_i \prod_{j=1}^p a_{ji} \right) \prod_{j' \neq j} a_{j'j}^2 \right],
\]

\[
\Sigma_{\beta_j\lambda} = -\frac{\lambda}{2\alpha_j \beta_j} \sum_{i=1}^n \mathbb{E} \left[ w_i d_{ij} \left( \prod_{j=1}^p a_{ji}^2 + \prod_{j=1}^p a_{j'j'} \right) \prod_{j' \neq j} a_{j'j} \right],
\]

\[
\Sigma_{\lambda\lambda} = \sum_{i=1}^n \mathbb{E} \left[ w_i \left( \prod_{j=1}^p a_{ji}^2 + \prod_{j=1}^p a_{j'j'} \right) \prod_{j=1}^p a_{j'j} \right].
\]

All the expected values above are obtained numerically. Also, \( K(\alpha_j) = [\alpha_j - \sqrt{\pi}K^*(\alpha_j)/\sqrt{2}]/2, \)

\[K^*(\alpha_j) = [1 - \text{erf}(\sqrt{2}/\alpha_j)]\exp(2/\alpha_j^2), \]

for \( j = 1, \ldots, p, \) where \( \text{erf}(\cdot) \) is the error function.
given by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Details on $\text{erf}(\cdot)$ can be found in Gradshteyn and Ryzhik (2007). For small values of $\alpha$ (Abramowitz and Stegun, 1970, p. 298)

$$K^*(\alpha_j) \approx \frac{\alpha_j}{\sqrt{2\pi}} \left( 1 - \frac{\alpha_j^2}{4} + \frac{3\alpha_j^4}{16} \right).$$

(12)

For numerical evaluation we recommend the use of (12) when $\alpha < 0.5$.

For $\lambda = 0$, which corresponds to the independent case, we obtain the Fisher information matrix

$$\Sigma_{\theta\theta} = n \text{ block-diag} \{ \Sigma_{\alpha\alpha}, \Sigma_{\beta\beta}, 2/\pi \},$$

where $\Sigma_{\alpha\alpha} = 2 \text{ diag} \{ \alpha_1^{-2}, \ldots, \alpha_p^{-2} \}$, $\Sigma_{\beta\beta} = \text{ diag} \{ b_1, \ldots, b_p \}$, with $b_j = [\alpha_j K(\alpha_j) + 1]/(\alpha_j^2 \beta_j^2)$ for $j = 1, \ldots, p$. It can be shown that

$$|\Sigma_{\theta\theta}| = \frac{2^{p+1} n^{2p+1}}{\pi} \prod_{j=1}^p \frac{[\alpha_j K(\alpha_j) + 1]}{\alpha_j^2 \beta_j^2} \neq 0.$$

Therefore, the Fisher information matrix is not singular at $\lambda = 0$.

Finally, it is well known that under some mild regularity conditions, the asymptotic behavior remains valid if $\Sigma_{\theta}$ is approximated by $-\ddot{L}_{\theta\theta}$, where $-\ddot{L}_{\theta\theta}$ is the $(2p + 1) \times (2p + 1)$ observed information matrix evaluated at $\hat{\theta}$, obtained from $\ddot{L}_{\theta\theta}$. So, in order to avoid numerical integrations, one can use $-\ddot{L}_{\theta\theta}$ instead of $\Sigma_{\theta}$ to make inference.

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