If You Split Something into Two Parts, You Will Get Three Pieces: The Bilateral Binomial Theorem and its Consequences

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Abstract. If a mathematical quantity is taken apart, the binomial expansion will result in three Pascal triangles. Thus a split into two parts generates a mathematical pattern of three pieces. This binomial expansion can now be generalized into a bilateral picture, resulting in an expansion of positive and negative powers as well. In so doing the Binomial Theorem will be generalized into the Bilateral Binomial Theorem, applying the intriguing mathematics of bilateral hypergeometric functions.

If a mathematical quantity is taken apart into two anti-commuting parts, the threefold pattern will triple again. Thus a ninefold symmetry appears. As an identical ninefold symmetry will appear if complex numbers and their conjugates are multiplied, complex conjugation must be seen as a strange, brutal, and illegitimate mathematical trick to model non-commutative structures by using commuting quantities.

1. Introduction I: Remember Sawyer!
Some physicists are looking for an “Equation of Everything”. To be successful one day they surely will also need a “Mathematics of Everything”. Such a mathematics of everything should be able to deliver a single function which contains in itself every function nature has invented in the past, nature invents today and nature will invent in the future. This is a very ambitious goal, and it can be doubted whether mankind with limited brain resources will eventually reach this goal in the far future.

But fortunately mankind has reached an intermediate interim result. We are in possession of “a single function which contained in itself very nearly every function that had previously been studied” [1] – the hypergeometric function. “In fact there must be many universities to-day where 95 per cent, if not 100 per cent, of the functions studied by physicists, engineering, and even mathematics students, are covered by this single symbol \( F(a, b; c; x) \)” [1].

A generalized version of this function, bilateral and with an arbitrary number of variables and coefficients, might well be able to cover the complete range of functions conceivable by human brains. As the hypergeometric function “appears to be the limit of the kind of pattern we are able to recognize at present” [1] this function is our only chance we possess today. “If one goes just beyond its boundaries, everything seems formless” [1]. Therefore we are forced to say: “Ohne hypergeometrische Funktionen geht es nicht” – without hypergeometric functions a unification of physics will not work” [2].

For modeling the laws of physics in a unified frame it is therefore essential to understand the symmetries and non-symmetries behind and inside hypergeometric functions.
And as hypergeometric functions are nothing else than simple, but infinite sums of binomial coefficients or of products of binomial coefficients, it is essential to understand the nature and mathematical structure of binomial coefficients as well.

2. Introduction II: The correct title is much longer
The title of this paper can be considered as incomplete. It should be supplemented by: If you split something into two anti-commuting parts, you will not only get three pieces, but nine. And isn’t it a mistake to think that complex numbers and their conjugates are commutative?

3. Introduction III: Non-lateral, uni-lateral and bilateral binomial theorems
In the comments about Ramanujan’s lost notebook Andrews and Berndt [3] write: “We need one further result, namely, the bilateral binomial theorem. If a and c are complex numbers with Re(c − a) > 1 and if z is a complex number with z = e^iθ, 0 < θ < 2π, then

$$\binom{a}{c, z} = \frac{(1-z)^{a-c-1} \Gamma(1-a) \Gamma(c)}{(-z)^{c-1} \Gamma(c-a)}.$$  (5.2.5)

It would seem that Ramanujan had discovered (5.2.5), but we are unaware of any mention of it by him in his papers or notebooks.”

It is surely true that I am making a lot of mistakes. Nevertheless, the following statement of Andrews and Berndt in [3] is partly wrong, as the equation given in [3] as version (5.2.5) of the Bilateral Binomial Theorem has not been my original formulation: “When the second author and W. Chu gave their proof of Entry 5.1.2 in (now reference [4]), they used a formulation of (5.2.5) given by M.E. Horn (now reference [5]) in 2003. His original formulation is incorrect, but it is corrected in the proof by J.M. Borwein, which follows the statement of the problem, and indeed the correct version (5.2.5) was used by Berndt and Chu in (now reference [4]).”

I love elegance and simplicity. Equation (5.2.5) is not elegant. And it is not simple. At high-school and university we have learned the simple and elegant

non-lateral Binomial Theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad (\text{if } n \in \mathbb{N})$$  (1)

and

uni-lateral Binomial Theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$  (2)

Therefore my original formulation of the Bilateral Binomial Theorem simply has been

$$(x + y)^n = \sum_{k=-\infty}^{\infty} \binom{n}{k} x^k y^{n-k}$$  (3)

which will be explained in later sections in more detail. And this original formulation is correct. And it is elegant and simple.

4. Simplicity and complicatedness
Mathematicians do not like simplicity. They very often like to show their smartness by hiding simple things and by transforming simple and interesting equations into complicated, overrated and very dull equations. This can clearly be seen in the history of the Bilateral Binomial Theorem. Equation (5.2.5) is rather weird and confusing. Koornwinder, who stated this equation in [6], [7], even refused to name it as Bilateral Binomial Theorem and called his version
\[
\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(c)_k} e^{ik\theta} = \sum_{k=-\infty}^{\infty} \frac{\Gamma(a+k) \Gamma(c)}{\Gamma(a) \Gamma(c+k)} e^{ik\theta} = \frac{\Gamma(c) \Gamma(I-a)}{\Gamma(c-a)} e^{i(1-c)(\theta-x)} (1 - e^{i\theta})^{c-a-1} \quad \text{(no number)}
\]

“explicit Fourier series evaluation”. In addition Koornwinder denied it a number in [7].

The English or German word “bi-nomial” expresses the fact that something has been split into two parts and is now written as a sum of exactly two terms (or German “Summe aus zwei Gliedern” [8]). Do you see the two parts in eq. (5.52.5) or in the no-number version of Koornwinder? Mathematical authors very often invest a lot of time to hide these two parts and transform Bilateral Theorems (1) – (3) into expressions like

\[
(1 + z)^n = \binom{n}{k} x^k y^{-k} = \sum_k \binom{n}{k} z^k \quad \text{with } z = \frac{x}{y} \quad \text{(4)}
\]

which still consist of the two parts 1 and z. But readers of mathematical papers often forget about the first part 1 and only notice the second part of variable z. Therefore it is surely helpful to interpret this variable z in a clear way as a ratio z = x/y.

And if the absolute value of this ratio equals one, |z| = 1 or if the absolute values of the two variables x and y are identical |x| = |y|, then the Bilateral Binomial Theorem will converge for sums of terms placed in the first positive Pascal Triangle with n > 0.

5. Some examples

As a physics and mathematics educator teaching engineering students at present, I tell my students that they are allowed to use factorials of fractions (like 0.5! = 0.5 \( \sqrt{\pi} \)) if they talk to each other or to other engineering or natural science students. The exclamation mark is a well-designed, well-defined and useful mathematical symbol already taught at school. We can surely use it in a more generalized way.

I only ask my students to be careful when talking to mathematics students or writing a research paper. In these cases they should be able to identify fractional factorials (like 0.5! = \( \Gamma(1.5) \)) by appropriate gamma function values

\[
n! = \Gamma(n+1) \quad \text{or} \quad \Gamma(n) = (n-1)! \quad \text{(5)}
\]

It is then possible to construct binomial coefficients in a very general way:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} \quad \text{(n, k} \in \mathbb{C})
\]

In this way doubtful Roman factorials (like \([-2]! = -1\) or \([-3]! = 0.5\) and Roman coefficients [9], [10] can be avoided. Mathematicians have invented these Roman factorials to prevent themselves from cancelling infinitely high values (like \((-2)! = -2\) \((-3)! = -\infty\)). But I think it makes more sense to consider negative valued binomial coefficients as limits of well-defined expressions like

\[
\binom{-2}{1} = \lim_{h \to 0} \binom{-2+h}{1} = \lim_{h \to 0} \frac{(-2+h)!}{1!} = -2 \quad \text{instead of} \quad \binom{-2}{1} = \frac{(-2)!}{[1]![-3]!} = \frac{-1}{1 \cdot 0.5} = -2.
\]

The strategy of using limits when computing binomial coefficients is more in agreement with Knuth’s view of Gamma coefficients, which can be constructed by considering \( h = 1/\varepsilon \) as an indefinitely small surreal number approaching zero [10]. In contrast Roman’s strategy looks like a tricky nice trick.

Now some simple examples can be analyzed. The first example will be a non-lateral binomial expression in accordance with eq. (1):

\[
(1 + z)^3 = \binom{3}{0} z^0 + \binom{3}{1} z^1 + \binom{3}{2} z^2 + \binom{3}{3} z^3 = 1 z^0 + 3 z^1 + 3 z^2 + 1 z^3 \quad \text{(7)}
\]
Of course, we know this eq. (7) from school. And as mathematicians with a philosophical touch we all know that this binomial expansion in reality is not a non-lateral expansion, but a bilateral expansion

\[(1 + z)^5 = \ldots + \binom{3}{2} z^2 + \binom{3}{1} z + \binom{3}{0} 1 = 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 + \ldots\]  

with zero coefficients for all terms not stated in eq. (7).

In a similar way the second example of the uni-lateral binomial expansion \((1 + z)^{0.5} = \sqrt{1+z}\) can be seen. This binomial expansion is of course identical to the Taylor expansion with \(|z| \leq 1:\)

\[(1 + z)^{0.5} = \binom{0.5}{0} z^0 + \binom{0.5}{1} z^1 + \binom{0.5}{2} z^2 + \binom{0.5}{3} z^3 + \ldots = 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 + \ldots\]  

And as mathematicians with a philosophical touch we again will insist that there are indefinitely many zero coefficients of terms with negative exponents:

\[(1 + z)^{0.5} = \ldots + \binom{0.5}{-2} z^{-2} + \binom{0.5}{-1} z^{-1} + \binom{0.5}{0} z^0 + \binom{0.5}{1} z^1 + \binom{0.5}{2} z^2 + \binom{0.5}{3} z^3 + \ldots\]

These zero coefficients exist. We do not see them and – as physicists with a philosophical touch – we do not measure them, but they are there. And they are everywhere. The complete Pascal plane as the plane of binomial coefficients is filled with binomial coefficients.

And as binomial coefficients

\[\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} = \binom{n}{n-k}\]

are symmetric with respect to the \(\binom{n}{n/2}\) axis with \(k = 0.5 n\) (see fig. 1), it is possible to reverse the binomial expansion \((1 + z)^{0.5} = \sqrt{1+z}\) of eq. (9) and identify this reversed expansion with the Taylor expansion with \(|z| \geq 1:\)

\[(1 + z)^{0.5} = \ldots + \binom{0.5}{-1.5} z^{-1.5} + \binom{0.5}{-0.5} z^{-0.5} + \binom{0.5}{0.5} z^{0.5} + \binom{0.5}{1.5} z^{1.5} + \binom{0.5}{2.5} z^{2.5} + \binom{0.5}{3.5} z^{3.5} + \ldots\]

\[= \ldots + \binom{0.5}{-1.5} z^{-1.5} + \binom{0.5}{-0.5} z^{-0.5} + \binom{0.5}{0.5} z^{0.5} = \ldots + \frac{1}{16} z^{-2.5} - \frac{1}{8} z^{-1.5} + \frac{1}{2} z^{-0.5} + z^{0.5}\]

Now we do not see and do not measure the coefficients of terms with exponents greater than one half. Of course they exist, but they are equal to zero. Thus only some of the non-zero binomial coefficients of eqs. (10) & (12) are shown in fig. 1.

As both binomial expansions of eqs. (10) & (12) are valid for unit magnitudes of the variable \(|z| = 1\), we have a strange identity of the summation results, when the summation is made with index numbers \(k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\) or with \(k = \ldots, -3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5, \ldots\) Shifting the index by 0.5 does not change the sum. This can be generalized: Shifting the index by an arbitrary number does not change the sum. This indeed is the original version and the mathematical core of the Bilateral Binomial Theorem:

\[\sum_{k=-\infty}^{\infty} \binom{n}{k} z^k = \sum_{k=-\infty}^{\infty} \binom{n}{k+a} z^{k+a} = (1+z)^n \iff \frac{d}{da} \sum_{k=-\infty}^{\infty} \binom{n}{k+a} z^{k+a} = 0 \text{ or } \frac{d}{dk} \sum_{k=-\infty}^{\infty} \binom{n}{k} z^k = 0\]
In other words: If \(|z| = 1\), the index \(k\) can be any real number \(k \in \mathbb{R}\) (or even a complex number if you dare). Sums of all terms with distance one of a row of the Pascal plane are identical.

\[
\text{symmetry axis } k = 0.5 n
\]

\[
\begin{array}{cccccccc}
  n = 0 & n = 0.5 & n = 1 & n = 2 & n = 3 & n = 4 \\
  k = n - 3 & k = n - 2 & k = n - 1 & k = n & k = 0 & k = 1 & k = 2 & k = 3 \\
  0.0390625 & 0.0625 & -0.125 & 1 & -0.125 & 0.0625 & 0.0390625 & 0.0390625
\end{array}
\]

\[1 = (1 + z)^0 = \ldots + \left( \frac{0}{k-2} \right) z^{k-2} + \left( \frac{0}{k-1} \right) z^{k-1} + \ldots + \left( \frac{0}{k} \right) z^k + \left( \frac{0}{k+1} \right) z^{k+1} + \ldots
\]

\[= \left( \frac{0}{k} \right) z^k \left[ \ldots - \frac{k}{k-3} z^{-3} + \frac{k}{k-2} z^{-2} - \frac{k}{k-1} z^{-1} + \frac{k}{k+1} z + \frac{k}{k+2} z^2 - \frac{k}{k+3} z^3 + \ldots \right]
\]

With \(z = 1\), \(k = 0.5\), and \(0.5! (-0.5)! = \Gamma(1.5) \Gamma(0.5) = \frac{2}{\pi} = \left( \frac{0}{0.5} \right)^{-1}\) the famous Leibniz formula [12]
\[
\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \right)
\]  
(18)

can be rediscovered. And as briefly indicated at the last line of the previous page it sometimes even makes sense to use complex indices. Inserting \( z = 1, k = i \) and \( i! (− i)! = \Gamma(i+1) \Gamma(−i+1) \) the following bilateral expansion
\[
1 = \ldots + \binom{0}{i-2} + \binom{0}{i-1} + \binom{0}{i} + \binom{0}{i+1} + \binom{0}{i+2} + \ldots
\]  
(19)
can be transformed into
\[
i! (− i)! = \ldots - \frac{i}{i-3} + \frac{i}{i-2} - \frac{i}{i-1} +1 - \frac{i}{i+1} + \frac{i}{i+2} - \frac{i}{i+3} - \ldots
\]  
(20)
which is in agreement with the complex Gamma function reflection formula [13]. And there are even more interesting formulas encoded in the Pascal plane [14].

6. The complete Pascal plane
“The gamma function may be continued meromorphically over the whole plane…” [15]. In a similar sense the binomial coefficients may be found for the complete Pascal plane. This complete plane will then have a similar structure as the plane of Knuth’s coefficients [see table 9 in [10], which had been constructed with surreal background support]. And of course the coefficients of the binomial expansion are then identical to the coefficients of the Taylor expansion, e.g.;
\[
(1 + z)^{-2} = \frac{1}{1 + 2z + z^2} = 1 - 2z + 3z^2 - 4z^3 + 5z^4 - 6z^5 + \ldots \quad \text{with } |z| < 1
\]  
(21)
\[
(1 + z)^{-2} = \frac{1}{1 + 2z + z^2} = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + 5z^{-4} - 6z^{-5} + \ldots \quad \text{with } |z| > 1
\]  
(22)

As now in eqs. (21) & (22) and similar equations of this negative half of the Pascal plane the magnitude of \( z \) has to be smaller than one (\(|z| < 1\)) or greater than one (\(|z| > 1\)) but never identical to one, the construction of a bilateral expansion is problematic. Adding up all bilateral terms in this direction will usually give a divergent result if the magnitude of \( z \) equals one (\(|z| = 1\)).

Side remark: There are bold daring mathematicians who try to construct bilateral expressions with \(|z| \neq 1 \) like \( H_3(2n−2m; 2n+1; \sqrt{5}) \) [16], [17], but in this Bregenz paper we will restrict ourselves to expansions which should converge.

It is then clear that we should look on the complete Pascal plane with a different pair of glasses. The old view has been, that the binomial coefficients of the complete Pascal plane should be grouped into six different regions possessing in generalized form different Roman or Knuth coefficients [10].

With our new pair of glasses we do not look at six different regions, but we look at three different directions connected with the complete Pascal plane.

Bilateral summations will then converge if terms of the positive half of the Pascal plane are added up into the conventional direction parallel to the k-axis. And bilateral summations will converge if
terms are added up into southeastern direction at the left side or they will converge if terms are added up into the northwestern direction on the right side of the complete Pascal plane (see fig. 2).

There are three Pascal triangles and thus three directions of summation now. Our starting point has been a split of an entity into two parts. This split results in a pattern with threefold symmetry. And this symmetry is not a usual 120° rotational symmetry. It is something different: It is a binomial symmetry.

![Pascal Plane with directions of summation](image)

**Figure 2.** Binomial coefficients of the complete Pascal plane and the three directions of summation.

This binomial symmetry transforms the binomial coefficients of the complete Pascal plane into its three different species, e.g. the binomial coefficient

$\binom{3}{1} = \frac{3!}{1!2!} = 3$ is transformed into the binomial coefficients $\binom{-2}{2} = 3$ and $\binom{-3}{-4} = -3$

while the binomial coefficient

$\binom{3}{2} = \frac{3!}{1!2!} = 3$ is transformed into the binomial coefficients $\binom{-3}{1} = -3$ and $\binom{-2}{4} = 3$.

In general the positions of transformed binomial coefficients can be found by rotating the position vector of a binomial coefficient $\binom{n}{k}$ by 120° or 240° about the center of rotation. This center of rotation is identical to the the center of the central equilateral triangle at the position of the binomial coefficient.
coefficient \(\binom{-2/3}{-1/3}\) in the middle of fig. 2 (see cross). As Pauli algebra helps to determine the transformed binomial coefficients, the mathematical procedure to find them will be discussed later in section 11.

7. A more symmetric Bilateral Binomial Theorem

Now we will have a closer look at the modified, hypergeometrical formula of the Bilateral Binomial Theorem. It can be found by writing down the bilateral binomial expansion (see eq. 23), by factoring out common terms and then changing signs.

\[
(1 + z)^n = \cdots + \binom{n}{k-2} z^{k-2} + \binom{n}{k-1} z^{k-1} + \binom{n}{k} z^k + \binom{n}{k+1} z^{k+1} + \binom{n}{k+2} z^{k+2} + \cdots
\]

(23)

\[
= \binom{n}{k} z^k \left[ ... + \frac{k(n-k)}{(n-k+1)(n-k+2)} z^{k-2} + \frac{k}{n-k+1} z^{k-1} + \frac{n-k}{k+1} z + \frac{(n-k)(n-k-1)}{(k+1)(k+2)} z^2 + ... \right]
\]

\[
= \binom{n}{k} z^k \left[ ... + \frac{k(n-k)}{(n-k-1)(n-k-2)} z^{k-2} - \frac{k}{n-k-1} z^{k-1} + \frac{n-k}{k+1} z + \frac{(n-k)(n-k-1)}{(k+1)(k+2)} z^2 + ... \right]
\]

Comparing the square bracket with the definition of the bilateral hypergeometric function [18], [19]

\[
_1F_1 \left[ \frac{a}{c}; z \right] = \cdots + \frac{(c-1)(c-2)}{(a-1)(a-2)} z^{a-2} + \frac{c-1}{a-1} z^{a-1} + \frac{a}{c} z + \frac{a(a+1)}{c(c+1)} z^2 + ...
\]

(24)

will result in the following first modified version of the Bilateral Binomial Theorem:

\[
(1 + z)^n = \binom{n}{k} z^k _1F_1 \left[ \frac{k-n}{k+1}; -z \right] = \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} z^k _1F_1 \left[ \frac{k-n}{k+1}; -z \right]
\]

(25)

Of course this version represents again a split into two parts \(z = x/y\) and can be written alternatively as

\[
(x + y)^n = \binom{n}{k} x^k y^{n-k} _1F_1 \left[ \frac{k-n}{k+1}; -\frac{x}{y} \right] = \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} x^k y^{n-k} _1F_1 \left[ \frac{k-n}{k+1}; -\frac{x}{y} \right]
\]

(26)

To get the second, more conventional version (5.2.5) of the Bilateral Binomial Theorem, the variable \(z\) of eq. (25) is taken negative as \((-z)\) and the entries of the binomial coefficients \(n\) over \(k\) are replaced via \(a = k - n\) and \(c = k + 1\) by \(n = c - a - 1\) and \(k = c - 1\). Then eq. (25) can be solved for the following hypergeometric function:

\[
_1F_1 \left[ \frac{a}{c}; z \right] = \frac{(1-z)^{-a-1} \Gamma(1-a) \Gamma(c)}{(z)^{-c} \Gamma(c-a)} = (1-z)^{-a-1}(-z)^{-c} \left( \frac{c-a-1}{c-1} \right)^{-1}
\]

(27)

And again this version represents a split into the two parts \(x\) and \(y\), but – Simsalabim und drei mal schwarzer Kater or “in the name of Persil, Omo, Bold, and Safeway’s own brand” [20] – we now do not have a sum \(x + y\), but a difference \(x - y\) in this alternative version:

\[
_1F_1 \left[ \frac{a}{c}; x^{c-1} \right] = \frac{(x-y)^{-a-1} \Gamma(1-a) \Gamma(c)}{x^{-c} \Gamma(c)} = (x-y)^{-a-1} x^{-c} \left( -y \right)^a \left( \frac{c-a-1}{c-1} \right)^{-1}
\]

(28)

In this way pure mathematicians confuse philosophers and the rest of the world: the sum \(x + y\) is hidden behind a difference \(x - y\). (But please do not belief that you live in a hyperbolic world now…)

8
And as we have taken part at a symposium about symmetry in Bregenz, we should not be satisfied with eq. (27) alias (5.2.5). It is surely more beautiful to restate this equation more symmetrically as

\[
\mathcal{H}_1 \left[ a; \frac{1}{z} \right] = \left( \frac{1 - \frac{1}{z}}{1 - z} \right)^{c-1} \frac{\Gamma(1-a) \Gamma(c)}{\Gamma(c-a)} = (1-z)^{c-1} \left( 1 - \frac{1}{z} \right)^{c-1} \left( \frac{c-a-1}{c-1} \right) \tag{29}
\]

or alternatively as

\[
\mathcal{H}_1 \left[ a; \frac{x}{y} \frac{y}{x} \right] = \left( \frac{1 - \frac{x}{y}}{1 - \frac{y}{x}} \right)^{c-1} \left( \frac{c-a-1}{c-1} \right) \tag{30}
\]

But the question still remains: Why should we confuse ourselves with negative signs and differences, if we can simply divide eq. (26) by \( x^k \) and \( y^{n-k} \) and have

\[
\left( 1 + \frac{x}{y} \right)^{n-k} \left( 1 + \frac{y}{x} \right)^k = \binom{n}{k} \mathcal{H}_1 \left[ k-n; a+1; -x y^{-1} \right] \tag{31}
\]

8. Hypergeometric linearity

There is no clear and no unique answer to the last question. As part-time physicist I surely prefer eqs. (3), (26), and (31) because of the simple manner, physical reality can then be connected with mathematical equations.

But I am an unhappy part-time mathematician, too. Thus concepts like exponential linearity

\[
a^x a^y = a^{x+y} \tag{32}
\]

have its own, convincing and inherent strength for me. These concepts show a different sort of beauty which can be seen if eq. (29) is rewritten as

\[
\mathcal{H}_1 \left[ 2a; \frac{1}{z} \right] = \left( \frac{1 - \frac{1}{z}}{1 - z} \right)^{2c-2} \frac{\Gamma(1-2a) \Gamma(2c)}{\Gamma(2c-2a)} \tag{33}
\]

and used to restate the square of the Bilateral Binomial Theorem (29) as:

\[
\mathcal{H}_1 \left[ a; \frac{1}{z} \right] = \left( \frac{1 - \frac{1}{z}}{1 - z} \right)^{2c-2} \frac{\Gamma(1-a)^2 \Gamma(c)^2}{\Gamma(c-a)^2} = \frac{\Gamma(1-a)^2 \Gamma(c)^2}{\Gamma(c-a)^2} \left( 1 - \frac{1}{z} \right)^{c-1} \mathcal{H}_1 \left[ 2a; \frac{1}{z} \right] \tag{34}
\]

The square of a bilateral hypergeometric function can be expressed as a different non-squared bilateral hypergeometric function – without exponent. As a part-time mathematician I like this sort of linearity. But when changing into a part-time physicist again, the identical simple expression

\[
\left( \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} \right)^2 = \sum_{k=0}^{\infty} \binom{2n}{k} x^k y^{2n-k} = (x + y)^{2n} \tag{35}
\]

is as convincing as eq. (34).

In a similar way the square root of the Bilateral Binomial Theorem

\[
\mathcal{H}_1 \left[ a; \frac{1}{z} \right]^{0.5} = \frac{\Gamma(0.5c - 0.5a)}{\Gamma(1 - 0.5a) \Gamma(0.5c)} \left( \frac{\Gamma(1-a) \Gamma(c)}{\Gamma(c-a)} \left( 1 - \frac{1}{z} \right) \right)^{0.5} \mathcal{H}_1 \left[ 0.5a; \frac{1}{z} \right] \tag{36}
\]
or the product of two bilateral hypergeometric functions

\[ \binom{a; z}{c; d} \cdot \binom{b; z}{d; c} = \frac{\Gamma(1-a) \Gamma(1-b) \Gamma(c) \Gamma(d) \Gamma(c+d-a-b)}{\Gamma(c-a) \Gamma(d-b) \Gamma(c+d) \Gamma(1-a-b)} \left(1 - \frac{1}{z}\right)^{-1} \binom{a+b; z}{c+d; z} \] (37)

can be found.

9. A view through another pair of glasses: The second and third bilateral summation

In previous sections the first bilateral summation of terms parallel to the k-axis of fig. 2 was analyzed. In the positive half of the Pascal plane these sums converge and result in \((x + y)^n\).

Now a closer look on the second direction with a southeastern bilateral summation will follow. This time binomial coefficients \(\binom{k}{n}\) are multiplied by identical factors \(z^n\) and added. Then common terms can be factored out and the remaining square bracket can be replaced by the hypergeometric function according to eq. (27). As \(z = 1\) the difference \((1 - z)\) disappears and the result will be zero:

\[
\sum_{k=-\infty}^{\infty} \binom{k}{n} z^n = \ldots + \binom{k}{n-2} z^n + \binom{k}{n-1} z^n + \binom{k}{n} z^n + \binom{k}{n+1} z^n + \binom{k}{n+2} z^n + \ldots \\
= \frac{\Gamma(k+1)}{\Gamma(n+1) \Gamma(k-n+1)} z^n H_1 \left[ \binom{k+1}{k-n+1} \right] \\
= 0
\] (38)

Thus a bilateral summation parallel to the n-axis (in southeastern direction) will always give zero.

The third bilateral summation points into northwestern direction, adding up all terms parallel to the \(\binom{n}{n}\) axis. This time binomial coefficients \(\binom{n+k}{k}\) are multiplied by shifted factors \(z^k\) again and then added. Again common terms can be factored out and the remaining square bracket can be replaced by the hypergeometric function according to eq. (27).

\[
\sum_{k=-\infty}^{\infty} \binom{n+k}{n} z^k = \ldots + \binom{n+k-1}{n} z^{k-1} + \binom{n+k}{n} z^k + \binom{n+k+1}{n} z^{k+1} + \binom{n+k+2}{n} z^{k+2} + \ldots \\
= \binom{n+k}{n} z^k \ldots + \frac{k}{n+k} z^{k-1} + \frac{n+k+1}{k+1} z^k \ldots + \frac{(n+k+1)(n+k+2)}{(k+1)(k+2)} z^{k+2} + \ldots \\
= \frac{\Gamma(n+k+1)}{\Gamma(n+1) \Gamma(k+1)} z^k H_1 \left[ \binom{n+k+1}{k+1} \right] \\
= (-1)^k (1-z)^{-n-1} \frac{\Gamma(n+k+1) \Gamma(-n-k)}{\Gamma(n+1) \Gamma(-n)}
\] (39)

Together with the Gamma function reflection formula [21]

\[
\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(x\pi)} \quad \text{or} \quad \Gamma(-x) \Gamma(1+x) = -\frac{\pi}{\sin(x\pi)} \quad \text{or} \quad x! (-x)! = \frac{x\pi}{\sin(x\pi)}
\] (40)

this third bilateral binomial expansion results in
Thus a bilateral summation in northwestern direction will oscillate and will depend on the summation parameter \( k \). Here bilateral summations somehow act like oscillating mathematical neutrinos.

10. The next step: Anti-commutativity

One of the most important mathematical developments of the last centuries had been the formulation of the theory of extensions by Grassmann [22] in 1844 – a year therefore worth to be called “annis mirabilis” [23] indeed. Grassmann has not only laid the basic foundations of the mathematics of vectors, but he already designed the geometric elements of higher-dimensional spaces: bivectors (oriented area elements), trivectors (oriented volume elements), quadvectors (oriented hyper-volume elements), etc.

\[
\sum_{k=-\infty}^{\infty} \binom{n + k}{n} z^k = (-1)^k (1 - z)^{n-1} \frac{\sin(n\pi)}{\sin((n + k)\pi)}
\]

Figure 3. Pauli matrices represent base vectors. They anti-commute [24], [25], [26], [27].

Now we will not split a scalar into two parts, but a vector. This can be done in two very different geometric ways. Vector \( \mathbf{z} = x \sigma_x + y \sigma_y \) can be split into two parallel, shorter vectors, e.g.: \( \mathbf{z} = 0.5(x \sigma_x + y \sigma_y) + 0.5 (x \sigma_x + y \sigma_y) \). As parallel vectors commute when multiplied, the situation is comparable to the multiplication of scalars. The binomial coefficients at the Pascal plane will then have the structure already described in previous sections. There will be a threefold binomial symmetry with three Pascal triangles again.

But now we will proceed with a different split. The vector will be considered as composed of the two orthogonal vectors \( x \sigma_x \) and \( y \sigma_y \): \( \mathbf{z} = (x \sigma_x) + (y \sigma_y) \). The elements of this new “Pauli Pascal plane” [28] will then be modified binomial coefficients constructed with the help of anti-commuting quantities

\[
(x \sigma_x)(y \sigma_y) = -(y \sigma_y)(x \sigma_x)
\]

They are identical to the coefficients of the Taylor expansions, e.g. the following simple power series of \( \mathbf{z}^3 = (x \sigma_x + y \sigma_y)^3 \). Eqs. (44) show that now a fourth of all terms cancels each other to zero:

\[
\begin{align*}
(x \sigma_x + y \sigma_y)^0 & = 1 x^0 y^0 \\
(x \sigma_x + y \sigma_y)^1 & = 1 x^1 y^0 \sigma_x + 1 x^0 y^1 \sigma_y \\
(x \sigma_x + y \sigma_y)^2 & = 1 x^2 y^0 + 0 x^1 y^1 + 1 x^0 y^2 \\
(x \sigma_x + y \sigma_y)^3 & = 1 x^3 y^0 + x^1 y^1 + 2 x^2 y^2 + 0 x^1 y^3 + 1 x^0 y^4 \\
(x \sigma_x + y \sigma_y)^4 & = 1 x^4 y^0 + 0 x^3 y^1 + 2 x^2 y^2 + 0 x^1 y^3 + 1 x^0 y^4 \\
(x \sigma_x + y \sigma_y)^5 & = 1 x^5 y^0 + 1 x^4 y^1 + 2 x^3 y^2 - 2 x^3 y^3 + 1 x^1 y^4 + 1 x^0 y^5 \sigma_x & + 1 x^4 y^1 \sigma_y & + 2 x^3 y^2 \sigma_x & + 2 x^3 y^3 \sigma_x & + 1 x^1 y^4 \sigma_x & + 1 x^0 y^5 \sigma_x & \\
\end{align*}
\]

In a similar way the coefficients of negative integer exponents can be found for \( x > y \)

\[
\begin{align*}
(x \sigma_x + y \sigma_y)^{-1} & = 1 x^{-1} y^0 \sigma_x + 1 x^{-1} y^1 \sigma_y - 1 x^{-2} y^2 \sigma_x - 1 x^{-1} y^3 \sigma_x + 1 x^{-1} y^4 \sigma_x + ... \cdots \\
(x \sigma_x + y \sigma_y)^{-2} & = 1 x^{-2} y^0 + 0 x^{-3} y^1 - 1 x^{-4} y^2 - 0 x^{-5} y^3 + 1 x^{-6} y^4 + ... \cdots \\
(x \sigma_x + y \sigma_y)^{-3} & = 1 x^{-3} y^0 \sigma_x + 1 x^{-2} y^1 \sigma_y - 2 x^{-3} y^2 \sigma_x - 2 x^{-3} y^3 \sigma_x + 3 x^{-3} y^4 \sigma_x + ... \cdots \\
\end{align*}
\]
and for $x < y$

$$(x \sigma_x + y \sigma_y)^{-1} = 1 x^0 y^{-1} \sigma_x + 1 x^1 y^{-2} \sigma_y - 1 x^2 y^{-3} \sigma_x - 1 x^3 y^{-4} \sigma_y + 1 x^4 y^{-5} \sigma_x + \ldots - \ldots$$

$$(x \sigma_x + y \sigma_y)^{-2} = 1 x^0 y^{-2} + 0 x^1 y^{-3} - 1 x^2 y^{-4} - 0 x^3 y^{-5} + 1 x^4 y^{-6} + \ldots - \ldots$$

$$(x \sigma_x + y \sigma_y)^{-3} = 1 x^0 y^{-3} \sigma_x + 1 x^1 y^{-4} \sigma_y - 2 x^2 y^{-5} \sigma_x - 2 x^3 y^{-6} \sigma_y + 3 x^4 y^{-7} \sigma_x + \ldots - \ldots$$

(46)

The coefficients of these expansions form the three Pauli Pascal triangles shown in fig. 4. These three Pauli Pascal triangles possess a ninefold symmetry. Every element can be identified in three different positions at the three triangles, if minus signs are neglected. For example, nine elements of value $\pm 2$ can be found in fig. 4. Thus a tripled situation is tripled again.

**Figure 4.** The complete Pauli Pascal plane with its three tripled triangles.

There are still many open questions which should be discussed to find a more general, non-commuting version of the Bilateral Binomial Theorems.
\[(x + y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} \]  \[(1 + z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k \] (47)

While
\[(x \sigma_x + y \sigma_y)^n \neq \sum_{k=0}^{\infty} \binom{n}{k} (x \sigma_x)^k (y \sigma_y)^{n-k} \] (48)

has anti-commuting elements (which causes tempting problems), the boring z-version
\[\left(1 + z \sigma_x \sigma_y^{-1}\right)^n = \left(1 + z \sigma_x \sigma_y\right)^n = \sum_{k=0}^{\infty} \binom{n}{k} (z \sigma_x \sigma_y)^k \] (49)

has commuting elements again. Thus forget z! But do not forget eq. (49). We all are a little bit insincere or simple-minded when we naively use complex entities [29], see section 12 of this paper.

And as z of eq. (47) should be a unit element of magnitude one (e.g. ±1, or a unit complex number, or a unit quaternion, or unit vectors like \(\sigma_x, \sigma_y, \sigma_z\) or unit multivectors of higher dimensions) factorials and roots of these unit elements are required to find a more general version of the Bilateral Binomial Theorem.

And please do not hesitate to think about binomial sums with vector-like exponents. There are some useful expressions, for example:
\[(1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \ldots + 0 + 0 + \binom{n}{0} + \binom{n}{1} z + \binom{n}{2} z^2 + \binom{n}{3} z^3 + \ldots \] (50)

\[
= \frac{\sigma_x}{0!} + \frac{\sigma_x}{1!} \sigma_x + \frac{\sigma_x}{2!} \sigma_x^2 + \frac{\sigma_x}{3!} \sigma_x^3 + \ldots \\
= 1 + \sigma_x \frac{1}{2} \left(1 - \sigma_x\right) \left(z^2 - z^3 + z^4 - \ldots\right) \\
= 1 + \sigma_x z + \frac{1}{2} z^2 \frac{1 - \sigma_x}{1 + z} \quad \text{if } |z| < 1
\]

Thus it is
\[1.5 \sigma_x = \frac{13}{12} + \frac{5}{12} \sigma_x \quad \text{with } z = 0.5\] (51)

or
\[0.5 \sigma_x = \frac{5}{4} - \frac{3}{4} \sigma_x \quad \text{with } z = -0.5 \Rightarrow 1.5 \sigma_x = 0.5 \sigma_x = 0.75 \sigma_x = \frac{25}{24} - \frac{7}{24} \sigma_x \quad \text{with } z = -0.25\]

which is a nice result in case you like Pythagorean triangles. And please do not hesitate, go on with
\[(1 + z)^{\sigma_x} = \sum_{k=0}^{\infty} \binom{\sigma_x}{k} n \] (52)

\[
= \frac{\sigma_x}{0! \left(\sigma_x - 0.5\right)!} z^{0.5} \left[\ldots - \frac{(-1)^{-1}}{2 \sigma_x + 1} z^{-1} + \ldots 1 - \frac{1}{2 \sigma_x + 1} z + \frac{7 - 8 \sigma_x}{3.5} z^2 - \frac{17 - 18 \sigma_x}{5.7} z^3 + \ldots \right] \\
= \frac{2 \Gamma(\sigma_x + 1) z^{0.5}}{\Gamma(1.5) \Gamma(\sigma_x + 0.5)} \left[\ldots - \frac{(-1)^{-1}}{2 \sigma_x + 1} (-z)^{-1} + \ldots 0.5 - \frac{1}{1.3} (-z) + \frac{3.5 - 4 \sigma_x}{3.5} (-z)^2 + \frac{8.5 - 9 \sigma_x}{5.7} (-z)^3 + \ldots \right]
\]

and try to find the final result of this bilateral summation.
Applying this strategy of using Pauli algebra exponents I originally intended to find some generalized Bilateral Binomial Theorems of higher order \( m > 1 \) [30]

\[
\sum_{k=-\infty}^{\infty} \binom{n}{k}^m x^k y^{n-k} = \ \sum_{k=-\infty}^{\infty} \binom{n}{k}^m z^k = ?
\] (53)

but I did not succeed. Hopefully you have more luck!

11. Binomial symmetry revisited
Pauli algebra, Dirac algebra, and Geometric Algebra [31], [32] are great stuff! They will be used in the following to describe binomial symmetry in greater detail. To find the three binomial coefficients which are transformed into each other by binomial symmetry, the coordinate system of fig. 2 will be changed to model the associated rotations of 120° and 240° in an appropriate way (see fig. 5).

![Figure 5](image)

*Figure 5.* The complete Pascal plane within a modified, rotation-friendly coordinate system.
The base unit vectors \( e_k \) and \( e_n \) of the old coordinate system are then transformed into Pauli unit vectors as unit vectors \( \sigma_x \) and \( \sigma_y \) of the new Euclidean coordinate system with orthogonal axes via

\[
e_k = \sigma_x \quad \text{or} \quad \sigma_x = e_k
\]

\[
e_n = -\frac{1}{2} \sigma_x - \frac{1}{2} \sqrt{3} \sigma_y \quad \text{or} \quad \sigma_y = -\frac{1}{\sqrt{3}} e_k - \frac{2}{\sqrt{3}} e_n
\]

The position vector of a binomial coefficient \( \binom{n}{k} \) is now given by

\[
r = -\frac{1}{\sqrt{3}} \sigma_y + n e_n + k e_k = (k - 0.5 n) \sigma_x - \left( \frac{1}{2} \sqrt{3} n + \frac{1}{\sqrt{3}} \right) \sigma_y
\]

A counter-clockwise rotation by 120° can be modeled by two succeeding reflections at axes first pointing into the direction of the unit vector \( a = \sigma_x \) and then pointing into the direction of unit vector \( b = \cos 60^\circ \sigma_x + \sin 60^\circ \sigma_y = 0.5 \sigma_x + 0.5 \sqrt{3} \sigma_y \).

The new modified position of a given binomial coefficient can be found with the help of the Sandwich product [33]

\[
r_{\text{new}} = b \ a \ r \ a \ b
\]

\[
= (0.5 \sigma_x + 0.5 \sqrt{3} \sigma_y) \sigma_x ((k - 0.5 n) \sigma_x - \left( \frac{1}{2} \sqrt{3} n + \frac{1}{\sqrt{3}} \right) \sigma_y) \sigma_x (0.5 \sigma_x + 0.5 \sqrt{3} \sigma_y)
\]

\[
= (n - 0.5 k + 0.5) \sigma_x + \left( \frac{1}{2} \sqrt{3} k + \frac{1}{2} \sqrt{3} \right) \sigma_y
\]

\[
= \left( -k - \frac{1}{3} \right) e_n + \left( n - k + \frac{1}{3} \right) e_k = \left( -k - 1 + \frac{2}{3} \right) e_n + \left( n - k + \frac{1}{3} \right) e_k
\]

\[
= (-k - 1) e_n + (n - k) e_k - \frac{1}{\sqrt{3}} \sigma_y
\]

A counter-clockwise rotation by 240° (or a clockwise rotation by 120°) can be modeled by interchanging the two reflections. The now first reflection at an axis pointing into the direction of unit vector \( b = \cos 60^\circ \sigma_x + \sin 60^\circ \sigma_y = 0.5 \sigma_x + 0.5 \sqrt{3} \sigma_y \) is followed by the second reflection at an axis pointing into the direction of the unit vector \( a = \sigma_x \).

\[
r_{\text{last}} = a \ b \ r \ a \ b
\]

\[
= (k - n - 1) e_n + (-n - 1) e_k - \frac{1}{\sqrt{3}} \sigma_y
\]

Thus the binomial coefficient \( \binom{n}{k} \) is transformed into the new binomial coefficient

\[
\binom{n_{\text{new}}}{k_{\text{new}}} = \binom{-k - 1}{n - k} = \frac{(-k - 1)!}{(n - k)!(n - k - 1)!} = \binom{n}{k} \sin(n\pi) \sin(k\pi)
\]

and the last binomial coefficient
\[
\begin{pmatrix}
    n_{\text{last}} \\
    k_{\text{last}}
\end{pmatrix} = \begin{pmatrix}
    k - n - 1 \\
    -n - 1
\end{pmatrix} = \frac{(k - n - 1)!}{(-n - 1)!k!} = \binom{n}{k} \frac{\sin(n\pi)}{\sin((n-k)\pi)}
\] 

(59)

while applying the reflection formula of eq. (40) at the last step. If \((n \notin \mathbb{Z})\) or \((k \notin \mathbb{Z})\), these three binomial coefficients with binomial symmetry have different values. For example the binomial coefficient with \(n = 3.7\) and \(k = 2.3\) (computed with a short look into the tables of [34])

\[
\begin{pmatrix}
    3.7 \\
    2.3
\end{pmatrix} = \frac{3.7!}{2.3! \cdot 1.4!} = \frac{\Gamma(4.7)}{\Gamma(3.3) \Gamma(2.4)} \approx 4.6295
\]

is transformed into the new binomial coefficient with \(n_{\text{new}} = -2.3 - 1 = -3.3\) and \(k_{\text{new}} = 3.7 - 2.3 = 1.4\)

\[
\begin{pmatrix}
    -3.3 \\
    1.4
\end{pmatrix} = \frac{(-3.3)!}{1.4! \cdot (-4.7)!} = \frac{\Gamma(-2.3)}{\Gamma(2.4) \Gamma(-3.7)} \approx 4.6295 \approx \begin{pmatrix}
    3.7 \\
    2.3
\end{pmatrix} \frac{\sin(3.7\pi)}{\sin(2.3\pi)} = \begin{pmatrix}
    3.7 \\
    2.3
\end{pmatrix}
\]

and the last binomial coefficient with \(n_{\text{last}} = 2.3 - 3.7 - 1 = -2.4\) and \(k_{\text{last}} = 3.7 - 1 = -4.7\)

\[
\begin{pmatrix}
    -2.4 \\
    -4.7
\end{pmatrix} = \frac{(-2.4)!}{(-4.7)! \cdot 2.3!} = \frac{\Gamma(-1.4)}{\Gamma(-3.7) \Gamma(3.3)} \approx 3.9381 \approx \begin{pmatrix}
    3.7 \\
    2.3
\end{pmatrix} \frac{\sin(3.7\pi)}{\sin(1.4\pi)} = 0.8507 \begin{pmatrix}
    3.7 \\
    2.3
\end{pmatrix}
\]

Thus binomial symmetry is connected with two different symmetry operations: A rotation of the complete Pascal plane about 120° or 240° and an oscillation of the binomial values induced by the fractions of sines.

12. Do we cheat when we use complex conjugation?

It is obvious and crystal clear that there is an important difference between the Pascal triangles of fig. 2 or 5 and the Pauli Pascal triangles of fig. 4. Pascal triangles will appear, if a quantity is split into two commuting parts. Pauli Pascal triangles will appear, if a quantity is split into two anti-commuting parts.

Thus if you see some Pascal triangles, you can and will be sure, that the underlying quantity is composed of two multiplicatively commuting parts. And if you see some Pauli Pascal triangles, you likewise can and will be sure, that a quantity is composed of two multiplicatively anti-commuting parts.

So everything is o.k. if we use complex numbers \(z = x + iy\) only and analyze the symmetry properties of different powers of \(z = (x + iy)^n\).

\[
\begin{align*}
    z^0 &= (x + iy)^0 = 1 \\
    z^1 &= (x + iy)^1 = x + 1 iy \\
    z^2 &= (x + iy)^2 = 1 x^2 + 2 i xy - 1 y^2 \\
    z^3 &= (x + iy)^3 = 1 x^3 + 3 i x^2y - 3 xy^2 - 1 i y^3 \\
    z^4 &= (x + iy)^4 = 1 x^4 + 4 i x^3y - 6 x^2y^2 - 4 i xy^3 + 1 y^4 \\
    z^5 &= (x + iy)^5 = 1 x^5 + 5 i x^4y - 10 x^3y^2 - 10 i x^2y^3 + 5 xy^4 + 1 i y^5 \\
    z^6 &= (x + iy)^6 = 1 x^6 + 6 i x^5y - 15 x^4y^2 - 20 i x^3y^3 - 15 x^2y^4 - 6 i xy^5 - 1 y^6
\end{align*}
\]

The coefficients show the usual pattern of the Pascal triangle, if negative signs of the squares of the imaginary unit \(i\) (see the alternating sign structure of pairs of diagonals in fig. 6) are neglected. Thus \(x\) and \(iy\) commute. The multiplication of two complex numbers \(z_1\) and \(z_2\) therefore is a commutative operation – and this is said in most books about complex numbers (e.g. see [35]: \(z_1 z_2 = z_2 z_1\), you virtually find this equation everywhere).

But now we start to cheat: We simply insert some additional minus signs and – Simsalabim und drei mal schwarzer Kater or “in the name of Persil, Omo, Bold, and Safeway’s own brand” [20] – the symmetry properties of the coefficients change completely (see fig. 7).
This process of deliberately introducing additional minus signs is called “complex conjugation”. Using complex conjugated numbers is not mathematics, it is a dirty trick. “Mathematicians, born snobs that they are” [36], only use complex conjugation because they “are exterior algebra-blind” [36] until today and desperately try to avoid the use of anti-commutative or non-commutative quantities.

And here it is, the multiplicative structure of products $z^* z = (x - iy)(x + iy)$ of complex and complex conjugated numbers:

\[
\begin{align*}
    z \cdot z^* &= 1 \\
    z^* z &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
    z^* z^* &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
    zz^* z &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
    zz^* z^* &= \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
    z^4 &= z^3 z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
    z^5 &= z^4 z = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \\
    z^6 &= z^5 z = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}
\end{align*}
\]

Figure 6. The complex Pascal plane showing the usual pattern of binomial coefficients.

Now the coefficients very clearly show the pattern of the Pauli Pascal triangle. Thus the product of a complex conjugated number $z_1^*$ and another complex number $z_2$ is not a commutative product $z_1^* z_2 \neq z_2 z_1^*$ but a non-commutative one: $z_1^* z_2 \neq z_2 z_1^*$. 

\[
\begin{align*}
    1 \\
    1 & 1 \\
    1 & 0 & 1 \\
    1 & 1 & 1 & 1 \\
    1 & 0 & 2 & 1 \\
    1 & 1 & 2 & 1 \\
    1 & 0 & 3 & 0 \\
    1 & 0 & 3 & 0 & 1
\end{align*}
\]

Figure 7. These coefficients very clearly show the pattern of the Pauli Pascal triangle.
Together with complex conjugation, complex numbers are indeed very often used (or very often misused or unconsciously misused) to model anti-commutative structures by using commuting quantities. And I really do not know why mathematicians do that! At least most of them claim to have read about and understood Grassmann and his ideas.

It is very easy to translate Grassmann's base units into complex numbers or to translate complex numbers back into Grassmann's base units [37]. As Grassmann had already invented the foundations of Pauli algebra and Dirac algebra [38], this is a convincing starting point.

A two-dimensional Pauli algebra (or Geometric Algebra [24], [25], [31], [39], [40] if you like) is based on linear combinations of two anti-commuting base elements, e.g. $\sigma_x$ and $\sigma_y$ of eqs. (42). These base elements – sometimes called Pauli matrices – represent base vectors of Euclidean space.

The Geometric Product of two vectors $a = a_0 \sigma_x + a_1 \sigma_y$ and $b = b_0 \sigma_x + b_1 \sigma_y$ can be transformed into a product of complex numbers in the following way:

$$a \ b = (a_0 \sigma_x + a_1 \sigma_y) (b_0 \sigma_x + b_1 \sigma_y)$$

$$= (a_0 \sigma_x + a_1 \sigma_y) \sigma_x^2 \ (b_0 \sigma_x + b_1 \sigma_y)$$

$$= (a_0 \sigma_x^2 + a_1 \sigma_y \sigma_x) (b_0 \sigma_x^2 + b_1 \sigma_x \sigma_y)$$

$$= (a_0 - a_1 \sigma_y \sigma_x) (b_0 + b_1 \sigma_x \sigma_y)$$

$$= (a_0 - a_1 i) (b_0 + b_1 i) = z_a^* z_b$$

Thus the Geometric Product of vectors $a = a_0 \sigma_x + a_1 \sigma_y$ and $b = b_0 \sigma_x + b_1 \sigma_y$ is identical to the product of the complex conjugated number $z_a^* = a_0 - a_1 i = b_0 - b_1 \sigma_x \sigma_y$, and the second complex number $z_b = b_0 + b_1 i = b_0 + b_1 \sigma_x \sigma_y$.

There is nothing dubious about this identity of eq. (62). A factor of one $\sigma_x^2 = 1$ is middle-multiplied between $a$ and $b$ and the unit bivector $\sigma_x \sigma_y$ is identified with the imaginary unit $i = \sigma_x \sigma_y$ which can be done as the unit bivector is an imaginary quantity squaring to minus $1$:

$$\sigma_x \sigma_y^2 = i^2 = -1$$

The conclusion is clear: The Geometric Product and the product of a complex conjugated number with a second complex number are identical. The Geometric Product and this complex conjugated product are equal. They have the same mathematical structure. (And physicist will say: They have the same real world interpretation as parallelogram [41] or rotor [42].) But again: The Geometric Product and the product of a complex conjugated number with a second complex number are identical. You cannot run away from this fact. They are identical: $a \ b = z_a^* z_b$. They have the same structure. They always show the same result. They are equal. Everything of these two objects $a \ b$ and $z_a^* z_b$ is equal. Thus the symmetry properties of these two objects are identical, too. They model non-commutative multiplications. And they do not model commutative multiplications!

Inserting additional minus signs via complex conjugation destroys the commutativity of multiplication. It means cheating symmetry. Fig. 7 unambiguously shows that the product $z^*: z$ is anti-commutative. And the product $z^* z_b$ clearly is non-commutative.

In a similar way it can be seen that inserting additional minus signs via quaternionic complex conjugation means cheating symmetry as well [37]. The scalar part of a quaternion must not be seen as a commutative factor, if a conjugated quaternion is multiplied by another quaternion. Hamilton had not understood this symmetry of quaternionic conjugation. Thus he failed to model rotations correctly and he never made it to a reasonable theory of quaternions – in contrast to Grassmann who had seen and had described the correct strategy in his own, powerful style [22], [43], [44].

So please stop cheating! And I am only able to repeat the words at the end my paper [29] which mathematics journals do not dare to publish: “Do mathematicians really again and again want to commit a crime against symmetry by using complex conjugation? Do mathematicians really want to go on with cheating the world by expressing anti-commutative structures by commutative quantities?
Do mathematicians really want to lead the world of symmetry into chaos and confusion by hiding simple symmetries behind conjugated quantities? And do they really want to mislead poor physicists who desperately try to understand the mathematical meaning of quantum mechanical expressions like $\psi^*\phi$? How can mankind understand the physical behavior of quantum mechanical probability densities one day if mathematicians are doing everything to hide the relevant symmetries? At present mathematicians are throwing red herrings! Therefore a word of advice at the end: Trash complex conjugation and throw it into the didactical waste basket – right next to present-day schoolroom vector algebra” [29].

13. Outlook
At present the mathematics of hypergeometric and of bilateral hypergeometric functions is still a highly abstract topic. But there are more and more areas of application which are modeled by generalized, higher-dimensional binomial relations and hypergeometric functions.

Figure 8. The trinomial coefficients of the complete three-dimensional Pascal space form four Pascal pyramids.
One of these interesting applications is the n-dimensional n-person chess game described by Kyppö [45]. To understand this game and other higher-dimensional applications, the mathematics of multinomial expansions should be analyzed. And it is possible to find a Bilateral Multinomial Theorem [46]. For example, generalized trinomial coefficients

\[
(x, y, z) = \frac{(x + y + z)!}{x! y! z!} = \frac{\Gamma(x + y + z + 1)}{\Gamma(x + 1) \Gamma(y + 1) \Gamma(z + 1)} \quad x, y, z \in \mathbb{C}
\]

(64)
can be arranged in four Pascal pyramids of a complete Pascal space (see fig. 8).

Therefore it can be said: If we split something into three parts, we will get four pieces. And it is also possible to generalize all this by using anti-commutative base elements again. This way the Pascal hyper-pyramids of [46] can be generalized into the Pauli Pascal hyper-pyramids of [47].

And the final statement therefore will be: If we split something into three anti-commuting parts, we will get 16 pieces. There will be 16 copies of every element in this trinomial symmetry.

14. The story goes on
There are two sides to everything: As said in the abstract and in section 12, complex conjugation must be seen as a strange, brutal, and illegitimate mathematical trick to model anti-commutative or non-commutative structures by using commuting quantities.

And it is of some interest to look at the other, second side of this trick: It is also possible to model commutative structures by using anti-commuting quantities if complex conjugation is applied [48].

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