STOCHASTIC VOLterra Equations in Banach Spaces and
Stochastic Partial Differential Equations

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Abstract. In this paper, we first study the existence-uniqueness and large deviation
estimate of solutions for stochastic Volterra integral equations with singular kernels in
2-smooth Banach spaces. Then, we apply them to a large class of semilinear stochastic
partial differential equations (SPDE) driven by Brownian motions as well as by fractional
Brownian motions, and obtain the existence of unique maximal strong solutions (in the
sense of SDE and PDE) under local Lipschitz conditions. Lastly, high order SPDEs in
a bounded domain of Euclidean space, second order SPDEs on complete Riemannian
manifolds, as well as stochastic Navier-Stokes equations are investigated.

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1. Introduction

The aims of this paper are three folds: First of all, we prove the existence and uniqueness of solutions with continuous paths for stochastic Volterra integral equations with singular kernels in a 2-smooth Banach space. Secondly, the large deviation principles (abbrev. LDP) of Freidlin-Wentzell type for stochastic Volterra equations are established under small perturbations of multiplicative noises. Thirdly, we apply them to several classes of semilinear stochastic partial differential equations (abbrev. SPDE). In particular, we give a unified treatment in certain sense for the LDPs of a large class of SPDEs.

In finite dimensional space, stochastic Volterra integral equations with regular kernels and driven by Brownian motions were first studied by Berger and Mizel in [6]. Later, Protter [62] studied the stochastic Volterra equations driven by general semimartingales. Using the Skorohod integral, Pardoux and Protter [57] also investigated the stochastic Volterra equations with anticipating coefficients. The study of stochastic Volterra equations with singular kernels can be found in [18, 20, 78, 45, 53, etc.]. Recently, the present author [82] studied the approximation of Euler type and the LDP of Freidlin-Wentzell type for stochastic Volterra equations with singular kernels. In particular, the kernels in [82] may deal with the fractional Brownian motion kernels as well as the fractional order integral kernels. The study of LDP for stochastic Volterra equations is also referred to [53, 45].

Since the work of Freidlin and Wentzell [26], the theory of small perturbation large deviations for stochastic differential equations (abbrev. SDE) has been studied extensively (cf. [3, 74, etc.]). In the classical method, to establish such an LDP for SDE, one usually needs to discretize the time variable and then prove various necessary exponential continuity and tightness for approximation equations in different spaces by using comparison principle. However, such verifications would become rather complicated and even impossible in some cases, e.g., stochastic evolution equations with multiplicative noises.

Recently, Dupuis and Ellis [24] systematically developed a weak convergence approach to the theory of large deviation. The central idea is to prove some variational representation formula for the Laplace transform of bounded continuous functionals, which will lead to proving a Laplace principle which is equivalent to the LDP. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué-Dupuis [9] and Budhiraja-Dupuis [14]. A simplified proof was given by the present author [81]. This variational representation has already been proved to be very effective for various finite and infinite dimensional stochastic dynamical systems even with irregular coefficients (cf. [64, 65, 15, 82, 66, etc.]). One of the main advantages of this argument is that one only needs to make some simple moment estimates (see Section 4 below).

On the other hand, it is well known that in the deterministic case, many PDE problems of parabolic and hyperbolic types can be written as Volterra type integral equations in Banach spaces by using the corresponding semigroup and the variation-of-constants formula (cf. [27, 37, 58]). An obvious merit of this procedure is that the unbounded operators in PDEs no longer appear and the analysis is entirely analogous to the ODE case. Thus, one naturally expects to take the same advantages for SPDEs in Banach spaces. However, it is not all Banach spaces in which stochastic integrals are well defined. One can only work in a class of 2-smooth Banach spaces. The definition of stochastic integrals in 2-smooth Banach spaces and related properties such as Burkholder-Davis-Gundy’s (abbrev. BDG) inequality, Girsanov’s theorem, stochastic Fubini’s theorem and the distribution of stochastic integrals can be found in [52, 10, 11, 54, etc.]. Thus, similar to the deterministic case, we can develop a parallel theory in 2-smooth Banach spaces for SPDEs. It should be emphasized that besides the usual SPDEs driven by multiplicative Brownian noises, a class of stochastic evolutionary integral equations appearing in viscoelasticity and heat
conduction with memory (cf. [63]) as well as a class of SPDEs driven by additive fractional Brownian noise, can also be written as abstract stochastic Volterra equations in Banach spaces.

In the past three decades, the theory of general SPDEs has been developed extensively by numerous authors mainly based on two different approaches: semigroup method based on the variation-of-constants formula (as said above) (cf. [77, 19, 11, 12, 80, etc.]) and variation method based on Galerkin’s finite dimensional approximation (cf. [56, 44, 68, 43, 50, 61, 83, 31, etc.]). A new regularization method is given in [86]. An overview for the classification and applications of SPDEs are referred to the recent book of Kotelenez [42]. In the author’s knowledge, most of the well known results are primarily concentrated on the mild or weak solutions, even measure-valued solutions. Such notions of solutions naturally appear in the study of SPDEs driven by the space-time white noises, and in this case one cannot obtain any differentiability of the solutions in the spatial variable.

Nevertheless, when one considers an SPDE driven by the spatial regular and time white noises, it is reasonable to require the existence of spatial regular solutions or classical solutions in the sense of PDE. For linear SPDEs, such regular solutions are easy and well known (cf. [44, 68, 25, etc.]). However, for nonlinear SPDEs, there seems to be few results (cf. [43, 18, 81, 86]). A major difficulty to prove the spatial regularity of solutions is that one cannot use the usual bootstrap method in the theory of PDE since there is no any differentiability of solutions with respect to the time variable. The present author [81] (see also [32, 86]) solves this problem by using a non-linear interpolation result due to Tartar [75]. Obviously, for the regularity theory of SPDEs, by using Sobolev’s embedding theorem (cf. [1]), it is natural to consider the $L^p$-solution of SPDEs. This is also why we need to work in 2-smooth Banach spaces. It should be remarked that the $L^p$-theory for SPDEs has been established in [10, 11, 12, 43, 22, 23, 80, etc.]. But, there are few results to deal with the $L^p$-strong solution in the sense of PDE. In the present paper, we shall prove a general result about the existence of strong solutions in the sense of both SDE and PDE (see Theorem 6.9).

We now describe our structure of this paper: In Section 2, we prepare some preliminaries for later use, and divide it into four subsections. In Subsection 2.1, we prove a Gronwall’s lemma of Volterra type under rather weak assumptions on kernel functions. Moreover, two simple examples are provided to show this lemma. In Subsection 2.2, we recall the Itô integral in 2-smooth Banach spaces and Burkholder-Davies-Gundy’s inequality as well as Kolmogorov’s continuity criterion of random fields in random intervals. In Subsection 2.3, we recall the properties of analytic semigroups and prove a local non-linear interpolation lemma, see also [75] for other related non-linear interpolation results. This lemma will play an important role in proving the existence of strong solutions (in the PDE’s sense) in Theorems 7.2 and 8.2 below. In Subsection 2.4, we recall the criterion of Laplace principle established by Budhiraja and Dupuis [9, 14] (see also [84]).

In Section 3, using the Gronwall inequality of Volterra type in Subsection 2.1, we first prove the existence and uniqueness of solutions for stochastic Volterra equations in 2-smooth Banach spaces under global Lipschitz conditions and singular kernels. Next, in Subsection 3.2, we study the regularity of solutions under slightly stronger assumptions on kernels. Moreover, a BDG type of inequality for stochastic Volterra type integral is also proved. In Subsection 3.3, employing the usual localizing method, we prove the existence of a unique maximal solution for stochastic Volterra equation under local Lipschitz conditions. Lastly, in Subsection 3.4, we discuss the continuous dependence of solutions with respect to the coefficients.

In Section 4, using the weak convergence method, we prove the Freidlin-Wentzell large deviation principle for the small perturbations of stochastic Volterra equations under a
compactness assumption and some uniform non-explosion conditions for the controlled equations. We also refer to [17, 60] for the application of weak convergence approach in the LDPs of stochastic evolution equations (the case of evolution triple). In the proof of Section 4, we need to use the Yamada-Watanabe Theorem in infinite dimensional space, which has been established by Ondrejáat [54] (see also [67] for the case of evolution triple). We want to say that although Ondrejáat only considered the case of convolution semigroup, their proofs are also adapted to more general stochastic Volterra equations. Moreover, since we are considering the path continuous solution, the proof in [54] can be simplified.

In Section 5, a simple application in a class of semilinear stochastic evolutionary integral equations is presented, which has been studied in [17, 8, 40, etc.] for additive noises. Such type of stochastic evolution equations appears in viscoelasticity, heat conduction in materials with memory, and electrodynamics with memory [63].

In Section 6, we apply our general results to a large class of semilinear stochastic evolution equations driven by multiplicative Brownian noise and additive fractional Brownian noise. A basic result in semigroup theory states that if $f$ is a Hölder continuous function in the Banach space $X$, then

$$t \mapsto \int_0^t \Sigma_{t-s} f(s) ds$$

is continuous in $D(L)$, where $\Sigma_t$ is an analytic semigroup and $L$ is the generator of $\Sigma_t$. We will use this result to prove the existence of strong solutions (in the sense of PDE) for semilinear SPDEs. Moreover, we also give a simple result about the SPDE driven by additive fractional Brownian noises. The corresponding LDPs are also obtained (see also [71, 59, 15, 66, 47, etc.] for the study of LDPs of stochastic evolution equations). We remark that the skeleton equation for the LDP of SPDEs driven by fractional Brownian motion is a non-convolution type of Volterra integral equation.

In Section 7, high order SPDEs in a bounded domain of Euclidean space are studied. Our stochastic version may be regarded as a parallel result in the deterministic case (cf. [58, p.246, Theorem 4.5]). Moreover, the LDP is also obtained.

In Section 8, we in particular study the second order stochastic parabolic equations on complete Riemannian manifolds. Under one-side Lipschitz and polynomial growth conditions, we obtain the global existence-uniqueness of strong solutions. When the manifold is compact, the LDP also holds in this case. In particular, stochastic reaction diffusion equations with polynomial growth coefficients are included.

In Section 9, we first prove the existence and uniqueness of local $L^p$-strong solutions for stochastic Navier-Stokes equations (SNSE) in any dimensional case. In the two dimensional case, we also obtain the non-explosion of solutions. Moreover, the LDPs for 2-dimensional SNSEs are also established in the case of both Dirichlet boundary and periodic boundary. We remark that the $L^p$-solutions for SNSEs have been studied by Brzezniak and Peszat [13] (bounded domain) and Mikulevicius and Rozovskii [49] (the whole space). The large deviation result for two dimensional SNSEs with additive noise was proved by Chang [16] using Girsanov’s transformation. In [70], the authors also used the weak convergence method to prove the large deviation estimate for two dimensional SNSEs with multiplicative noises. But, it seems that there is a gap in their proofs [70, p.1655 line 6 and p.1658 line 2]. Therein, the $v_n$ only weakly converges to $v$ in $S_M$. This seems not enough to derive their limits.

We conclude this introduction by making the following CONVENTION: Throughout this paper, the letter $C$ with or without subscripts will denote a positive constant, whose value may change from one place to another. Moreover, we also use the notation $E_1 \leq E_2$ to denote $E_1 \leq C \cdot E_2$, where $C > 0$ is an unimportant constant.
2. Preliminaries

2.1. Gronwall’s inequality of Volterra type. Let \( \Delta := \{(t, s) \in \mathbb{R}^2_+ : s \leq t \} \). We first recall the following result due to Gripenberg [33, Theorem 1 and p.88].

**Lemma 2.1.** Let \( \kappa : \Delta \to \mathbb{R}^+ \) be a measurable function. Assume that for any \( T > 0 \)

\[
\int_{0}^{t} \kappa(t, s)ds \in L^\infty(0, T)
\]

and

\[
\limsup_{\epsilon \downarrow 0} \left\| \int_{\cdot}^{\cdot+\epsilon} \kappa(\cdot + \epsilon, s)ds \right\|_{L^\infty(0, T)} < 1.
\]

Define

\[
\begin{align*}
\rho_1(t, s) &:= \kappa(t, s), \\
\rho_{n+1}(t, s) &:= \int_{s}^{t} \kappa(t, u)\rho_n(u, s)du, \quad n \in \mathbb{N}. 
\end{align*}
\]

(2.1)

Then for any \( T > 0 \), there exist constants \( C_T > 0 \) and \( \gamma \in (0, 1) \) such that

\[
\left\| \int_{0}^{t} \rho_n(\cdot, s)ds \right\|_{L^\infty(0, T)} \leq C_T n \gamma^n, \quad \forall n \in \mathbb{N}.
\]

(2.2)

In particular, the series

\[
r(t, s) := \sum_{n=1}^{\infty} \rho_n(t, s)
\]

(2.3)

converges for almost all \((t, s) \in \Delta\), and

\[
r(t, s) - \kappa(t, s) = \int_{s}^{t} \kappa(t, u)\rho(u, s)du = \int_{s}^{t} r(t, u)k(u, s)du
\]

(2.4)

and for any \( T > 0 \)

\[
t \mapsto \int_{0}^{t} r(t, s)ds \in L^\infty(0, T).
\]

(2.5)

The function \( r \) defined by (2.3) is called the resolvent of \( \kappa \). All the functions \( \kappa \) in Lemma 2.1 will be denoted by \( \mathcal{K} \). In what follows, we shall denote by \( \mathcal{K}_0 \) the subclass of \( \mathcal{K} \) with the property that

\[
\limsup_{\epsilon \downarrow 0} \left\| \int_{\cdot}^{\cdot+\epsilon} \kappa(\cdot + \epsilon, s)ds \right\|_{L^\infty(0, T)} = 0.
\]

We also denote by \( \mathcal{K}_{>1} \) the set of all positive measurable functions \( \kappa \) on \( \Delta \) with the property that for any \( T > 0 \) and some \( \beta = \beta(T) > 1 \)

\[
t \mapsto \int_{0}^{t} \kappa^\beta(t, s)ds \in L^\infty(0, T).
\]

(2.6)

It is clear that \( \mathcal{K}_{>1} \subset \mathcal{K}_0 \subset \mathcal{K} \) and for any \( \kappa_1, \kappa_2 \in \mathcal{K}_0 \) (resp. \( \mathcal{K}_{>1} \)) and \( C_1, C_2 > 0 \),

\[
C_1 \kappa_1 + C_2 \kappa_2 \in \mathcal{K}_0 \text{ (resp. } \mathcal{K}_{>1}).
\]

Let \( 0 \leq h \in L^1_{lo}c(\mathbb{R}_+) \). If \( \kappa(t, s) = h(s) \), then \( \kappa \in \mathcal{K}_0 \) and

\[
r(t, s) = h(s) \exp \left\{ \int_{s}^{t} h(u)du \right\}.
\]
if $\kappa(t, s) = h(t - s)$, then $\kappa \in \mathcal{K}_0$ and

$$r(t, s) = a(t - s) := \sum_{n=1}^{\infty} a_n(t - s), \tag{2.7}$$

where

$$a_1(t) = h(t), \quad a_{n+1}(t) := \int_0^t h(t - s)a_n(s)ds.$$ 

When $0 \leq h \in L^1(\mathbb{R}_+)$, a classical result due to Paley and Wiener (cf. [51] p.207, Theorem 5.2) says that

$$a \in L^1(\mathbb{R}_+) \text{ if and only if } \int_0^\infty h(t)dt < 1. \tag{2.8}$$

In this case, $\hat{a}(s) = \hat{h}(s)/(1 - \hat{h}(s))$, where the hat denotes the Laplace transform, i.e.:

$$\hat{h}(s) := \int_0^\infty e^{-st}h(t)dt, \quad s \geq 0.$$

We want to say that (2.8) is useful in the study of large time asymptotic behavior of solutions for Volterra equations. An important extension to nonintegrable convolution kernel can be found in [69, 38] (see also [33]). A simple example is provided in Example 3.2 below.

We now prove the following Gronwall lemma of Volterra type (see also [37] Lemma 7.1.1 for a case of special convolution kernel).

**Lemma 2.2.** Let $\kappa \in \mathcal{K}$ and $r_n$ and $r$ be defined respectively by (2.7) and (2.3). Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ be two measurable functions satisfying that for any $T > 0$ and some $n \in \mathbb{N}$

$$t \mapsto \int_0^t r_n(t, s)f(s)ds \in L^\infty(0, T) \tag{2.9}$$

and for almost all $t \in (0, \infty)$

$$\int_0^t r(t, s)g(s)ds < +\infty. \tag{2.10}$$

If for almost all $t \in (0, \infty)$

$$f(t) \leq g(t) + \int_0^t \kappa(t, s)f(s)ds, \tag{2.11}$$

then for almost all $t \in (0, \infty)$

$$f(t) \leq g(t) + \int_0^t r(t, s)g(s)ds. \tag{2.12}$$

**Proof.** First of all, if we define

$$h(t) := g(t) + \int_0^t r(t, s)g(s)ds,$$

then by (2.4) and (2.10)

$$h(t) = g(t) + \int_0^t \kappa(t, s)h(s)ds \quad \text{for a.a. } t \in (0, \infty).$$

Thus, by (2.11) we have

$$f(t) - h(t) \leq \int_0^t \kappa(t, s)(f(s) - h(s))ds \quad \text{for a.a. } t \in (0, \infty). \tag{2.13}$$
Set \( \tilde{f}(t) := f(t) - h(t) \) and define
\[
\tilde{f}^*(t) := \text{ess sup}_{s \in [0,t]} \tilde{f}(s), \quad t > 0
\]
and
\[
\tau_0 := \inf\{t > 0 : \tilde{f}^*(t) > 0\}.
\]
Clearly, \( t \mapsto \tilde{f}^*(t) \) is increasing and
\[
\tilde{f}(t) \leq 0 \quad \text{for a.a. } t \in [0, \tau_0).
\] (2.14)

We want to prove that \( \tau_0 = +\infty \).

Suppose \( \tau_0 < +\infty \). Iterating inequality (2.13), we obtain
\[
\tilde{f}(t) \leq \int_0^t r_n(t, s) \tilde{f}(s) ds \leq \int_0^t r_n(t, s) f(s) ds, \quad \forall n \in \mathbb{N}.
\]
By (2.9), one knows that \( 0 < \tilde{f}^*(t) < +\infty \) for any \( t > \tau_0 \). Moreover, we have
\[
\tilde{f}(t) \leq \int_{\tau_0}^t r_n(t, s) \tilde{f}(s) ds \leq \tilde{f}^*(t) \int_{\tau_0}^t r_n(t, s) ds, \quad \forall n \in \mathbb{N}.
\]
So, for any \( T > \tau_0 \)
\[
0 < \tilde{f}^*(T) \leq \tilde{f}^*(T) \cdot \left\| \int_{\tau_0}^t r_n(\cdot, s) ds \right\|_{L^\infty(\tau_0, T)} \to 0
\]
as \( n \to \infty \), which is impossible. So, \( \tau_0 = +\infty \). \( \square \)

The following two examples show that (2.12) is sensitive to \( \kappa \in \mathcal{K} \).

**Example 2.3.** For \( C_0 > 0 \), set
\[
\kappa_{C_0}(t, s) := \frac{C_0}{\sqrt{t^2 - s^2}}, \quad s < t.
\]
It is clear that
\[
\int_s^t \kappa_{C_0}(t, u) du = C_0((\pi/2) - \arcsin(s/t)).
\]
From this, one sees that
\[
\begin{cases}
\kappa_{C_0} \notin \mathcal{K}, & \text{if } C_0 \geq 2/\pi; \\
\kappa_{C_0} \in \mathcal{K} \cap \mathcal{K}_0, & \text{if } 0 < C_0 < 2/\pi.
\end{cases}
\]
Consider the following Volterra equation
\[
x(t) = \int_0^t \kappa_{C_0}(t, s)x(s) ds, \quad t \geq 0.
\]
If \( C_0 = 1 \), there are at least two solutions \( x(t) \equiv 0 \) and \( x(t) = t \); if \( C_0 = \frac{2}{\pi} \), there are infinitely many solutions \( x(t) \equiv \text{constant} \); if \( 0 < C_0 < 2/\pi \), by Lemma 2.2 there is only one solution \( x(t) \equiv 0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^+) \).

**Example 2.4.** For \( C_0 > 0 \) and \( \alpha, \beta \in [0, 1) \), set
\[
\kappa_{C_0}^{\alpha, \beta}(t, s) := \frac{C_0}{(t - s)^{\alpha s^\beta}}, \quad s < t.
\]
It is clear that
\[
\int_s^t \kappa_{C_0}^{\alpha, \beta}(t, s) ds = C_0 t^{1-\alpha-\beta} \left( \frac{1}{7} \int_{u/t}^1 \frac{1}{(1-s)^{\alpha s^\beta}} ds \right).
\] (2.15)
From this, one sees that
\[
\begin{aligned}
\kappa_{\alpha, \beta}^{\gamma} &\notin \mathcal{K}, \quad \text{if } \alpha + \beta > 1 \text{ and } C_0 > 0; \\
\kappa_{\alpha, \beta}^{\gamma} &\notin \mathcal{K}, \quad \text{if } \alpha + \beta = 1 \text{ and } C_0 \geq \int_0^1 \frac{1}{(1-s)^{\alpha + \beta}} ds; \\
\kappa_{\alpha, \beta}^{\gamma} &\in \mathcal{K} \cap \mathcal{K}_0, \quad \text{if } \alpha + \beta = 1 \text{ and } C_0 < \int_0^1 \frac{1}{(1-s)^{\alpha + \beta}} ds; \\
\kappa_{\alpha, \beta}^{\gamma} &\in \mathcal{K}_{>1}, \quad \text{if } \alpha + \beta < 1 \text{ and } C_0 > 0.
\end{aligned}
\]

Consider the following Volterra equation
\[
x(t) = \int_0^t k_{\alpha, \beta}^{C_0}(t, s)x(s)ds, \quad t \geq 0.
\]

If \( \alpha + \beta < 1 \), by Lemma 2.2 there is only one solution \( x(t) \equiv 0 \) in \( L_{loc}^\infty(\mathbb{R}_+) \); if \( \alpha = \beta = C_0 = 1/2 \), there are at least two solutions \( x(t) \equiv 0 \) and \( x(t) = \sqrt{t} \).

### 2.2. Itô’s integral in 2-smooth Banach spaces

Throughout this paper, we shall fix a stochastic basis \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\), i.e., a complete probability space with a family of right-continuous filtrations. In what follows, without special declarations, all expectations \( \mathbb{E} \) are taken with respect to the probability measure \( P \).

Let \( \{W^k(t) : t \geq 0, k \in \mathbb{N}\} \) be a sequence of independent one dimensional standard Brownian motions on \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\). Let \( l^2 \) be the usual Hilbert space of all square summable real number sequences, \( \{\varepsilon_k, k \in \mathbb{N}\} \) the usual orthonormal basis of \( l^2 \). Let \( X \) be a separable Banach space, and \( L(l^2; X) \) the set of all bounded linear operators from \( l^2 \) to \( X \). For an operator \( B \in L(l^2; X) \), we also write
\[
B = (B_1, B_2, \cdots) \in X^\mathbb{N}, \quad B_k = B\varepsilon_k.
\]

**Definition 2.5.** An operator \( B \in L(l^2; X) \) is called radonifying if
\[
\text{the series } \sum_k B\varepsilon_k \cdot W^k(1) \text{ converges in } L^2(\Omega; X).
\]

We shall denote by \( L_2(l^2; X) \) the space of all radonifying operators, and write for \( B \in L_2(l^2; X) \)
\[
\|B\|_{L_2(l^2; X)} := \left( \mathbb{E}\|B\varepsilon_k \cdot W^k(1)\|_X^2 \right)^{1/2}. \tag{2.16}
\]

Here and below, we use the convention that the repeated indices will be summed.

The following proposition is well known, and a detailed proof was given in [54, Proposition 2.5].

**Proposition 2.6.** The space \( L_2(l^2; X) \) with norm (2.16) is a separable Banach space.

In order to introduce the stochastic integral of an \( X \)-valued measurable \((\mathcal{F}_t)\)-adapted process with respect to \( W \), in the sequel, we assume that \( X \) is 2-smooth (cf. [60]), i.e., there exists a constant \( C_X \geq 2 \) such that for all \( x, y \in X \)
\[
\|x + y\|_X^2 + \|x - y\|_X^2 \leq 2\|x\|_X^2 + C_X\|y\|_X^2.
\]

Let now \( s \mapsto B(s) \) be an \( L_2(l^2; X) \)-valued measurable and \((\mathcal{F}_t)\)-adapted process with
\[
\int_0^T \|B(s)\|_{L_2(l^2; X)}^2 ds < +\infty, \quad a.s., \quad \forall T > 0.
\]

One can define the Itô stochastic integral (cf. [54, Section 3])
\[
t \mapsto \mathcal{I}_t(B) := \int_0^t B(s) dW(s) = \int_0^t B_k(s) \cdot dW^k(s) \in X
\]
such that \( t \mapsto I_t(B) \) is an \( \mathbb{X} \)-valued continuous local \((\mathcal{F}_t)\)-martingale. Moreover, let \( \tau \) be any \((\mathcal{F}_t)\)-stopping time, then

\[
\int_0^{\tau \wedge t} B(s) dW(s) = \int_0^t 1_{\{s < \tau\}} \cdot B(s) dW(s).
\]

The following BDG inequality for \( I_t(B) \) holds (cf. \cite{54} Section 5).

**Theorem 2.7.** For any \( p > 0 \), there exists a constant \( C_p > 0 \) depending only on \( p \) such that

\[
\mathbb{E} \left( \sup_{t \in [0,T]} \left\| \int_0^t B(s) dW(s) \right\|_X^p \right) \leq C_p \mathbb{E} \left( \int_0^T \left\| B(s) \right\|^2_{L_2(t^2;X)} ds \right)^{p/2}.
\]

The following two typical examples of 2-smooth Banach spaces are usually met in applications.

**Example 2.8.** Let \( \mathbb{X} \) be a separable Hilbert space. Clearly, \( \mathbb{X} \) is 2-smooth. In this case, \( L_2(t^2; \mathbb{X}) \) consists of all Hilbert-Schmidt operators of mapping \( t^2 \) into \( \mathbb{X} \), and

\[
\| B \|_{L_2(t^2; \mathbb{X})} = \left( \sum_{k=1}^{\infty} \| B_k \|_\mathbb{X}^2 \right)^{1/2}.
\]

**Example 2.9.** Let \((E, \mathcal{E}, \mu)\) be a measure space, \( \mathbb{H} \) a separable Hilbert space. For \( p \geq 2 \), let \( L^p(E, \mu; \mathbb{H}) \) be the usual \( \mathbb{H} \)-valued \( L^p \)-space over \((E, \mathcal{E}, \mu)\). Then \( \mathbb{X} = L^p(E, \mu; \mathbb{H}) \) is 2-smooth (cf. \cite{60} \cite{10}). In this case, by BDG’s inequality for Hilbert space valued martingale we have

\[
\| B \|_{L_2(t^2; \mathbb{X})}^2 = \mathbb{E} \left( \int_E \| B_k(x) \cdot W^k(1) \|_{\mathbb{H}}^p \mu(dx) \right)^{2/p} \\
\leq \left( \int_E \mathbb{E} \| B_k(x) \cdot W^k(1) \|_{\mathbb{H}}^p \mu(dx) \right)^{2/p} \\
\leq C_p \left( \int_E \left( \sum_{k=1}^{\infty} \| B_k(x) \|_{\mathbb{H}}^2 \right)^{p/2} \mu(dx) \right)^{2/p} \\
= C_p \| B \|_{L^p(E, \mu; t^2 \otimes \mathbb{H})}^2.
\]

Hence

\[
L^p(E, \mu; t^2 \otimes \mathbb{H}) \hookrightarrow L_2(t^2; \mathbb{X}) = L_2(t^2; L^p(E, \mu; \mathbb{H})).
\]

We also recall the following Kolmogorov continuity criterion, which can be derived directly by Garsia’s inequality (cf. \cite{177}).

**Theorem 2.10.** Let \( \{X(t), t \geq 0\} \) be an \( \mathbb{X} \)-valued stochastic process, and \( \tau \) a bounded random time. Suppose that for some \( C_0, p > 0 \) and \( \delta > 1 \)

\[
\mathbb{E} \| (X(t) - X(s)) \cdot 1_{\{s, \tau \in [0, \tau]\}} \|^p_X \leq C_0 |t - s|^\delta.
\]

Then there exist constants \( C_1 > 0 \) and \( a \in (0, (\delta - 1)/p) \) independent of \( C_0 \) and a continuous version \( \tilde{X} \) of \( X \) such that

\[
\mathbb{E} \left( \sup_{s \neq t \in [0, \tau]} \frac{\| \tilde{X}(t) - \tilde{X}(s) \|^p_X}{|t - s|^{ap}} \right) \leq C_1 \cdot C_0.
\]
2.3. **A local non-linear interpolation lemma.** In what follows, we fix a densely defined closed linear operator \( \mathcal{L} \) on \( X \) for which
\[
S_\phi := \{ \lambda \in \mathbb{C} : 0 < \phi \leq |\arg \lambda| \leq \pi \} \subset \rho(\mathcal{L}),
\]
and for some \( C \geq 1 \)
\[
\| (\lambda - \mathcal{L})^{-1} \|_{L(X)} \leq \frac{C}{1 + |\lambda|}, \quad \lambda \in S_\phi,
\]
where \( \rho(\mathcal{L}) \) denotes the resolvent set of \( \mathcal{L} \). The above operator \( \mathcal{L} \) is also called sectorial (cf. [37, p.18]). It is well known that \( \mathcal{L} \) generates an analytic semigroup
\[
\mathcal{S}_t = e^{-\mathcal{L}t}, \quad t \geq 0.
\]
Moreover, we also assume that \( \mathcal{L}^{-1} \) is a bounded linear operator on \( X \), i.e.,
\[
0 \in \rho(\mathcal{L}).
\]
Thus, for any \( \alpha \in \mathbb{R} \), the fractional power \( \mathcal{L}^\alpha \) is well defined (cf. [37, 58]). For \( \alpha > 0 \), we define the fractional Sobolev space \( X_\alpha \) by
\[
X_\alpha := \mathcal{D}(\mathcal{L}^\alpha)
\]
with the norm
\[
\| x \|_{X_\alpha} := \| \mathcal{L}^\alpha x \|_X.
\]
For \( \alpha < 0 \), \( X_\alpha \) is defined as the completion of \( X \) with respect to the above norm. It is clear that \( X_\alpha \) is still 2-smooth, and \( B \in L_2(l^2; X_\alpha) \) if and only if \( \mathcal{L}^\alpha B \in L_2(l^2; X) \), i.e.,
\[
\| B \|_{L_2(l^2; X_\alpha)} = \| \mathcal{L}^\alpha B \|_{L_2(l^2; X)}.
\]
(2.20)
The following properties are well known (cf. [37, p.24-27] or [58, p.74]).

**Proposition 2.11.** *(i)* \( \mathcal{S}_t : X \to X_\alpha \) for each \( t > 0 \) and \( \alpha > 0 \).

*(ii)* For each \( t > 0 \), \( \alpha \in \mathbb{R} \) and every \( x \in X_\alpha \), \( \mathcal{S}_t \mathcal{L}^\alpha x = \mathcal{L}^\alpha \mathcal{S}_t x \).

*(iii)* For some \( \delta > 0 \) and each \( t, \alpha > 0 \), the operator \( \mathcal{L}^\alpha \mathcal{S}_t \) is bounded in \( X \) and
\[
\| \mathcal{L}^\alpha \mathcal{S}_t x \|_X \leq C_\alpha t^{-\alpha} e^{-\delta t} \| x \|_X, \quad \forall x \in X.
\]

*(iv)* Let \( \alpha \in (0, 1] \) and \( x \in X_\alpha \), then
\[
\| \mathcal{S}_t x - x \|_X \leq C_\alpha t^\alpha \| x \|_{X_\alpha}.
\]

*(v)* For any \( 0 \leq \beta < \alpha \)
\[
\| x \|_{X_\beta} \leq C_{\alpha, \beta} \| x \|^{1 - \frac{\beta}{\alpha}} \| x \|^{\frac{\beta}{\alpha}}_{X_\alpha}, \quad \forall x \in X_\alpha.
\]

We need the following embedding result.

**Proposition 2.12.** For any \( 0 < \theta < 1 \) and \( \alpha > 0 \)
\[
(L_2(l^2; X), L_2(l^2; X_\alpha))_{\theta, 1} \subset L_2(l^2; (X, X_\alpha)_{\theta, 1}) \subset L_2(l^2; X_{\theta \alpha}),
\]
(2.21)
where \( (\cdot, \cdot)_{\theta, 1} \) stands for the real interpolation space between two Banach spaces.

**Proof.** We only prove the first embedding. The second embedding follows from [76, p.101, (d) and (f)], i.e.,
\[
(X, X_\alpha)_{\theta, 1} \subset X_{\theta \alpha}.
\]

Let
\[
B \in (L_2(l^2; X), L_2(l^2; X_\alpha))_{\theta, 1} =: \mathcal{B}_{\theta, 1}.
\]
By the \( K \)-method of real interpolation space, we have (cf. [76, p.24])
\[
\| B \|_{\mathcal{B}_{\theta, 1}} = \int_0^\infty \frac{t^{-\theta} K(t, B)}{t} \, dt,
\]
where ...
where the $K$-function of $B$ is defined by
\[
K(t, B) := \inf_{B = B_1 + B_2} \left\{ \|B_1\|_{L^2(\mathbb{R}; X)} + t\|B_2\|_{L^2(\mathbb{R}; X)} \right\}, \quad t \geq 0.
\]

By Definition 2.5 we have
\[
K(t, B) = \inf_{B = B_1 + B_2} \left\{ \left( \mathbb{E} \|B_1 e_k \cdot W^k(1)\|_{X}^2 \right)^{\frac{1}{2}} + t \left( \mathbb{E} \|B_2 e_k \cdot W^k(1)\|_{X}^2 \right)^{\frac{1}{2}} \right\}
\]
\[
\geq \inf_{B = B_1 + B_2} \left\{ \left| \mathbb{E} \|B_1 e_k \cdot W^k(1)\|_{X} \cdot t + \|B_2 e_k \cdot W^k(1)\|_{X} t\right|^2 \right\}
\]
\[
\geq \left( \mathbb{E} \left[ \inf_{B = B_1 + B_2} \left\{ \|B_1 e_k \cdot W^k(1)\|_{X} + t\|B_2 e_k \cdot W^k(1)\|_{X} \right\} \right]^2 \right)^{\frac{1}{2}},
\]
where
\[
K(t, B_{ek} \cdot W^k(1)) := \inf_{B_{ek} \cdot W^k(1) = x_1 + x_2} \left\{ \|x_1\|_X + t\|x_2\|_{X_{\theta}} \right\}.
\]

Therefore, by Minkowski’s inequality we obtain
\[
\|B\|_{\mathbb{L}_{\varepsilon, 1}} \geq \left( \mathbb{E} \left[ \int_{0}^{\infty} t^{-\theta} K(t, B_{ek} \cdot W^k(1)) \frac{dt}{t} \right]^2 \right)^{\frac{1}{2}}
\]
\[
= \left( \mathbb{E} \|B_{ek} \cdot W^k(1)\|_{X, X_{\theta}}^2 \right)^{\frac{1}{2}} = \|B\|_{L^2(\mathbb{R}; (X, X_{\theta})_{\theta, 1})}.
\]
The result follows. \(\square\)

The following local non-linear interpolation lemma will play a crucial role in the proofs of Theorems 7.2 and 8.2 below. We refer to [75] for some other nonlinear interpolation results.

**Lemma 2.13.** Let $0 \leq \alpha_0 < \alpha_1 \leq 1$ and $0 \leq \alpha_2 < \alpha_3 \leq 1$. Let $\Psi : \mathbb{X}_{\alpha_0} \to L^2(\mathbb{R}; X_{\alpha_2})$ be a locally Lipschitz continuous map, and satisfy that for all $R > 0$ and $x \in \mathbb{X}_{\alpha_1}$ with $\|x\|_{X_{\alpha_0}} \leq R$
\[
\|\Psi(x)\|_{L^2(\mathbb{R}; X_{\alpha_2})} \leq CR(1 + \|x\|_{X_{\alpha_1}}).
\]
Then for any $0 < \theta' < \theta < 1$ and $R > 0$
\[
\sup_{\|x\|_{X_{\alpha_0} + \theta(\alpha_1 - \alpha_0)} \leq R} \|\Psi(x)\|_{L^2(\mathbb{R}; X_{\alpha_2 + \theta'(\alpha_3 - \alpha_2)})} \leq CR.
\]
**Proof.** By (2.20), we may assume that $\alpha_2 = 0$. Fix $R > 0$ and $x \in \mathbb{X}_{\alpha_0 + \theta(\alpha_1 - \alpha_0)}$ with $\|x\|_{X_{\alpha_0 + \theta(\alpha_1 - \alpha_0)}} \leq R$.
Set for $t \geq 0$
\[
K(t, \Psi(x)) := \inf_{\Psi(x) = \Psi_1 + \Psi_2} \left\{ \|\Psi_1\|_{L^2(\mathbb{R}; X)} + t\|\Psi_2\|_{L^2(\mathbb{R}; X_{\alpha_3})} \right\}.
\]
For $\delta > 0$ and $t \in [0, 1]$, noting that
\[
\|\Sigma_{t^\delta} x\|_{X_{\alpha_0}} \leq \|x\|_{X_{\alpha_0}} \leq \|x\|_{X_{\alpha_0 + \theta(\alpha_1 - \alpha_0)}} \leq R,
\]
by the assumptions and (iii) and (iv) of Proposition 2.11 we have
\[
K(t, \Psi(x)) \leq \|\Psi(x) - \Psi(\Sigma_{t^\delta} x)\|_{L^2(\mathbb{R}; X)} + t\|\Psi(\Sigma_{t^\delta} x)\|_{L^2(\mathbb{R}; X_{\alpha_3})}
\]
\[
\leq CR\|\Sigma_{t^\delta} x - x\|_{X_{\alpha_0}} + CRt \cdot (1 + \|\Sigma_{t^\delta} x\|_{X_{\alpha_1}})
\]
\[11\]
\[ C_R t^{\delta \theta (1-\theta)} + C_R t \left( 1 + t^{-\delta (1-\theta)} \right) \| x \|_{C_{\alpha_0+\delta (1-\theta)}} \]\[ \leq C_R (t^{\delta \theta (1-\theta)} + t + t^{1-\delta (1-\theta)}) . \]

Letting \( \delta = \frac{1}{\alpha_1-\alpha_0} \), we obtain that for \( t \in [0,1] \)
\[ K(t, \Psi(x)) \leq C_R (t^\theta + t) \leq C_R t^\theta . \]

Moreover, it is clear that for \( t \geq 1 \)
\[ K(t, \Psi(x)) \leq \| \Psi(x) \|_{L_2(l^2;X)} \leq C_R \| x \|_{\alpha_0} + \| \Psi(0) \|_{L_2(l^2;X)} \leq C_R . \]

Hence, for any \( 0 < \theta' < \theta < 1 \)
\[ \| \Psi(x) \|_{(L_2(l^2;X),L_2(l^2;X_{\alpha_0}))_{\theta',1}} \leq \int_0^\infty \frac{t^{-\theta'} K(t; \Psi(x))}{t} \, dt \leq C_R \left[ \int_0^1 \frac{t^{\theta'-\theta}}{t} \, dt + \int_1^\infty \frac{t^{-\theta'}}{t} \, dt \right] \leq C_R. \]

The result follows by (2.21).

\[ \square \]

2.4. A criterion for Laplace principles. It is well known that there exists a Hilbert space so that \( l^2 \subset \mathbb{U} \) is Hilbert-Schmidt with embedding operator \( J \) and \( \{ W^k(t), k \in \mathbb{N} \} \) is a Brownian motion with values in \( \mathbb{U} \), whose covariance operator is given by \( Q = J \circ J^* \).

For example, one can take \( \mathbb{U} \) as the completion of \( l^2 \) with respect to the norm generated by scalar product
\[ (h, h')_{\mathbb{U}} := \left( \sum_{k=1}^\infty \frac{h_k h'_k}{k^2} \right)^{1/2} , \quad h, h' \in l^2 . \]

For \( T > 0 \) and a Banach space \( \mathbb{B} \), we denote by \( \mathcal{B}(\mathbb{B}) \) the Borel \( \sigma \)-field, and by \( \mathcal{C}_T(\mathbb{B}) \) the continuous function space from \([0,T] \) to \( \mathbb{B} \), which is endowed with the uniform norm. Define
\[ \ell^2_T := \left\{ h = \int_0^T \dot{h}(s) \, ds : \dot{h} \in L^2(0,T;l^2) \right\} \tag{2.22} \]
with the norm
\[ \| h \|_{\ell^2_T} := \left( \int_0^T \| \dot{h}(s) \|_2^2 \, ds \right)^{1/2} , \]
where the dot denotes the generalized derivative. Let \( \mu \) be the law of the Brownian motion \( W \) in \( \mathcal{C}_T(\mathbb{U}) \). Then
\[ (\mathcal{C}_T(\mathbb{U}), \ell^2_T, \mu) \]
forms an abstract Wiener space.

For \( T, N > 0 \), set
\[ \mathbb{D}_N := \left\{ h \in \ell^2_T : \| h \|_{\ell^2_T} \leq N \right\} \]
and
\[ A^T_N := \left\{ h : [0,T] \rightarrow l^2 \text{ is a continuous and } (\mathcal{F}_t) \text{-adapted process, and for almost all } \omega, \ h(\cdot, \omega) \in \mathbb{D}_N \right\} . \tag{2.23} \]

It is well known that with respect to the weak convergence topology in \( \ell^2_T \) (cf. [11]),
\[ \mathbb{D}_N \text{ is metrizable as a compact Polish space.} \tag{2.24} \]

Let \( S \) be a Polish space. A function \( I : S \rightarrow [0,\infty] \) is given.
Definition 2.14. The function $I$ is called a rate function if for every $a < \infty$, the set 
\{f \in \mathbb{S} : I(f) \leq a\} is compact in $\mathbb{S}$.

Let $\{Z_{\epsilon} : \mathbb{C}_{T}(U) \to \mathbb{S}, \epsilon \in (0,1)\}$ be a family of measurable mappings. Assume that there is a measurable map $Z_{0} : \ell_{T}^{2} \to \mathbb{S}$ such that

\[(LD)_{1}\] For any $N > 0$, if a family $\{h^{\epsilon}, \epsilon \in (0,1)\} \subset \mathcal{A}_{N}^{T}$ (as random variables in $\mathbb{D}_{N}$) converges in distribution to $h \in \mathcal{A}_{N}^{T}$, then for some subsequence $\epsilon_{k}$, $Z_{\epsilon_{k}}(h^{\epsilon_{k}} + \frac{h_{k}(1)}{\sqrt{\epsilon_{k}}})$ converges in distribution to $Z_{0}(h)$ in $\mathbb{S}$.

\[(LD)_{2}\] For any $N > 0$, if $\{h_{n}, n \in \mathbb{N}\} \subset \mathbb{D}_{N}$ weakly converges to $h \in \ell_{T}^{2}$, then for some subsequence $h_{n_{k}}$, $Z_{0}(h_{n_{k}})$ converges to $Z_{0}(h)$ in $\mathbb{S}$.

For each $f \in \mathbb{S}$, define
\[
I(f) := \frac{1}{2} \inf_{\{h \in \ell_{T}^{2} : f = Z_{0}(h)\}} \|h\|_{\ell_{T}^{2}}^{2}, \tag{2.25}
\]
where $\inf \emptyset = \infty$ by convention. Then under $(LD)_{2}$, $I(f)$ is a rate function. In fact, assume that $I(f_{n}) \leq a$. By the definition of $I(f_{n})$, there exists a sequence $h_{n} \in \ell_{2}$ such that $Z_{0}(h_{n}) = f_{n}$ and
\[
\frac{1}{2} \|h_{n}\|_{\ell_{T}^{2}}^{2} \leq a + \frac{1}{n}.
\]
By the weak compactness of $\mathbb{D}_{2a+2}$, there exist a subsequence $n_{k}$ (still denoted by $n$) and $h \in \ell_{T}^{2}$ such that $h_{n}$ weakly converges to $h$ and
\[
\|h\|_{\ell_{T}^{2}}^{2} \leq \lim_{n \to \infty} \|h_{n}\|_{\ell_{T}^{2}}^{2} \leq 2a.
\]
Hence, by $(LD)_{2}$ we have
\[
\lim_{k \to \infty} \|Z_{0}(h_{n_{k}}) - Z_{0}(h)\|_{\mathbb{S}} = 0
\]
and
\[
I(Z_{0}(h)) \leq a.
\]

We recall the following result due to [9, 14] (see also [81, Theorem 4.4]).

Theorem 2.15. Under $(LD)_{1}$ and $(LD)_{2}$, $\{Z_{\epsilon}, \epsilon \in (0,1)\}$ satisfies the Laplace principle with the rate function $I(f)$ given by (2.25). More precisely, for each real bounded continuous function $g$ on $\mathbb{S}$:
\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}^{\mu} \left( \exp \left( - \frac{g(Z_{\epsilon})}{\epsilon} \right) \right) = - \inf_{f \in \mathbb{S}} \{g(f) + I(f)\}. \tag{2.26}
\]
In particular, the family of $\{Z_{\epsilon}, \epsilon \in (0,1)\}$ satisfies the large deviation principle in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ with the rate function $I(f)$. More precisely, let $\nu_{\epsilon}$ be the law of $Z_{\epsilon}$ in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, then for any $A \in \mathcal{B}(\mathbb{S})$
\[
- \inf_{f \in A^{c}} I(f) \leq \liminf_{\epsilon \to 0} \epsilon \log \nu_{\epsilon}(A) \leq \limsup_{\epsilon \to 0} \epsilon \log \nu_{\epsilon}(A) \leq - \inf_{f \in A} I(f),
\]
where the closure and the interior are taken in $\mathbb{S}$, and $I(f)$ is defined by (2.25).
3. Abstract stochastic Volterra integral equations

In this section, we consider the following stochastic Volterra integral equation in a 2-smooth Banach space $X$:

$$X(t) = g(t) + \int_0^t A(t, s, X(s))ds + \int_0^t B(t, s, X(s))dW(s),$$

(3.1)

where $g(t)$ is an $X$-valued measurable $(\mathcal{F}_t)$-adapted process, and

$$A : \Delta \times \Omega \times X \to X \in \mathcal{M}_\Delta \times \mathcal{B}(X)/\mathcal{B}(X)$$

and

$$B : \Delta \times \Omega \times X \to L_2(l^2; X) \in \mathcal{M}_\Delta \times \mathcal{B}(X)/\mathcal{B}(L_2(l^2; X)).$$

Here and below, $\Delta := \{(t, s) \in \mathbb{R}_+^2 : s \leq t\}$, and $\mathcal{M}_\Delta$ denotes the progressively measurable $\sigma$-field on $\Delta \times \Omega$ generated by the sets $E \in \mathcal{B}(\Delta) \times \mathcal{F}$ with properties: $1_E(t, s, \cdot) \in \mathcal{F}_s$ for all $(t, s) \in \Delta$, and $s \mapsto 1_E(t, s, \omega)$ is right continuous for any $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

We start with the global existence and uniqueness of solutions of Eq. (3.1) under global Lipschitz conditions and singular kernels.

3.1. Global existence and uniqueness. In this subsection, we make the following global Lipschitz and linear growth conditions on the coefficients:

(H1) For some $p \geq 2$ and any $T > 0$

$$\text{ess sup}_{t \in [0, T]} \int_0^t [\kappa_1(t, s) + \kappa_2(t, s)] \cdot \mathbb{E}\|g(s)\|_X^p ds < +\infty,$$

where $\kappa_1$ and $\kappa_2$ are from (H2) and (H3) below.

(H2) There exists $\kappa_1 \in \mathcal{K}_0$ such that for all $(t, s) \in \Delta$, $\omega \in \Omega$ and $x \in X$

$$\|A(t, s, \omega, x)\|_X \leq \kappa_1(t, s) \cdot (\|x\|_X + 1)$$

and

$$\|B(t, s, \omega, x)\|_{L_2(l^2\times X)}^2 \leq \kappa_1(t, s) \cdot (\|x\|_X^2 + 1).$$

(H3) There exists $\kappa_2 \in \mathcal{K}_0$ such that for all $(t, s) \in \Delta$, $\omega \in \Omega$ and $x, y \in X$

$$\|A(t, s, \omega, x) - A(t, s, \omega, y)\|_X \leq \kappa_2(t, s) \cdot \|x - y\|_X$$

and

$$\|B(t, s, \omega, x) - B(t, s, \omega, y)\|_{L_2(l^2\times X)}^2 \leq \kappa_2(t, s) \cdot \|x - y\|_X^2.$$

We now prove the following basic existence and uniqueness result.

Theorem 3.1. Assume that (H1)-(H3) hold. Then there exists a unique measurable $(\mathcal{F}_t)$-adapted process $X(t)$ such that for almost all $t \geq 0$,

$$X(t) = g(t) + \int_0^t A(t, s, X(s))ds + \int_0^t B(t, s, X(s))dW(s), \quad P\text{-a.s.}$$

(3.2)

and for any $T > 0$ and some $C_{T,p,\kappa_1} > 0$,

$$\mathbb{E}\|X(t)\|_X^p \leq C_{T,p,\kappa_1} \left[ \mathbb{E}\|g(t)\|_X^p + \text{ess sup}_{t \in [0, T]} \int_0^t \kappa_1(t, s) \cdot \mathbb{E}\|g(s)\|_X^p ds \right]$$

(3.3)

for almost all $t \in [0, T]$, where $p$ is from (H1). Moreover, if

$$t \mapsto \int_0^t \kappa_1(t, s)ds \in L^\infty(\mathbb{R}_+),$$

(3.4)
then for almost all $t \geq 0$

\[
E\|X(t)\|_X^p \leq C_{p,\kappa_1} \left( E\|g(t)\|_X^p + \int_0^t \hat{\kappa}_1(t, s) \cdot E\|g(s)\|_X^p ds \right.
\]

\[
+ \left. \int_0^t r_{\hat{\kappa}_1}(t, u) \cdot \left[ \int_0^u \hat{\kappa}_1(u, s) \cdot E\|g(s)\|_X^p ds \right] du \right),
\]

(3.5)

where $\hat{\kappa}_1 = \tilde{C}_{p,\kappa_1} \cdot \kappa_1$, $r_{\hat{\kappa}_1}$ is defined by (2.3) in terms of $\hat{\kappa}_1$, and $C_{p,\kappa_1}, \tilde{C}_{p,\kappa_1}$ are constants only depending on $p, \kappa_1$.

Proof. We use Picard’s iteration to prove the existence. Let $X_1(t) := g(t)$, and define recursively for $n \in \mathbb{N}$

\[
X_{n+1}(t) = g(t) + \int_0^t A(t, s, X_n(s))ds + \int_0^t B(t, s, X_n(s))dW(s).
\]

(3.6)

Fix $T > 0$ below. By (H2), the BDG inequality (2.17) and Hölder’s inequality we have

\[
E\|X_{n+1}(t)\|_X^p \leq E\|g(t)\|_X^p + E \left( \int_0^t \|A(t, s, X_n(s))\|_X ds \right)^p
\]

\[
+ E \left( \int_0^t \|B(t, s, X_n(s))\|_{L^2(p; X)} ds \right)^p
\]

\[
\leq E\|g(t)\|_X^p + E \left( \int_0^t \kappa_1(t, s) \cdot (\|X_n(s)\|_X + 1)ds \right)^p
\]

\[
+ E \left( \int_0^t \|B(t, s, X_n(s))\|_{L^2(p; X)} ds \right)^p
\]

\[
\leq E\|g(t)\|_X^p + \int_0^t \kappa_1(t, s) \cdot E(\|X_n(s)\|_X^p + 1)ds \cdot \left( \int_0^t \kappa_1(t, s)ds \right)^{p-1}
\]

\[
+ \int_0^t \kappa_1(t, s) \cdot E(\|X_n(s)\|_X^p + 1)ds \cdot \left( \int_0^t \kappa_1(t, s)ds \right)^{\frac{p}{p-1}}
\]

\[
\leq E\|g(t)\|_X^p + C_{T,p} \cdot C_T + C_{T,p} \int_0^t \kappa_1(t, s) \cdot E\|X_n(s)\|_X^p ds,
\]

(3.7)

where $C_T := \text{ess sup}_{t \in [0,T]} |\int_0^t \kappa_1(t, s)ds|$ and $C_{T,p} := C_T^{p-1} + C_T^{(p-2)/2}$.

Set

\[
f_m(t) := \sup_{n=1, \ldots, m} E\|X_n(t)\|_X^p.
\]

Then

\[
f_m(t) \leq C_{T,p,\kappa_1} \left( E\|g(t)\|_X^p + 1 \right) + \int_0^t \kappa_1(t, s) \cdot f_m(s)ds,
\]

where $\tilde{\kappa}_1 = C_{T,p,\kappa_1} \cdot \kappa_1$ and the constant $C_{T,p,\kappa_1}$ is independent of $m$.

Let $r_{\tilde{\kappa}_1}$ be defined by (2.3) in terms of $\tilde{\kappa}_1$. Note that by (2.4)

\[
\int_0^t r_{\tilde{\kappa}_1}(t, s) \cdot E\|g(s)\|_X^p ds - \int_0^t \tilde{\kappa}_1(t, s) \cdot E\|g(s)\|_X^p ds
\]

\[
= \int_0^t \left( \int_s^t r_{\tilde{\kappa}_1}(t, u)\tilde{\kappa}_1(u, s)du \right) \cdot E\|g(s)\|_X^p ds
\]

\[
= \int_0^t \tilde{\kappa}_1(t, u) \left( \int_0^u \tilde{\kappa}_1(u, s) \cdot E\|g(s)\|_X^p ds \right) du.
\]

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Hence, by Lemma 2.2 and (H1), we obtain that for almost all \( t \in [0, T] \)

\[
\sup_{n \in \mathbb{N}} \mathbb{E}\|X_n(t)\|^p_X = \lim_{m \to \infty} f_m(t) \leq C_{T,p,\kappa_1} \left( \mathbb{E}\|g(t)\|^p_X + \int_0^t r_{\kappa_1}(t, s) \cdot \mathbb{E}\|g(s)\|^p_X ds \right)
\]

\[
\leq C_{T,p,\kappa_1} \left( \mathbb{E}\|g(t)\|^p_X + \int_0^t \tilde{r}_{\kappa_1}(t, s) \cdot \mathbb{E}\|g(s)\|^p_X ds + \int_0^t r_{\kappa_1}(t, u) \left( \int_u^t \tilde{r}_{\kappa_1}(u, s) \cdot \mathbb{E}\|g(s)\|^p_X ds \right) du \right) \tag{3.8}
\]

\[
\leq C_{T,p,\kappa_1} \left[ \mathbb{E}\|g(t)\|^p_X + \text{ess sup}_{t \in [0, T]} \int_0^t \kappa_1(t, s) \cdot \mathbb{E}\|g(s)\|^p_X ds \right]. \tag{3.9}
\]

On the other hand, set

\[ Z_{n,m}(t) := X_n(t) - X_m(t). \]

As the above calculations, by (H3) we have

\[
\mathbb{E}\|Z_{n+1,m+1}(t)\|^2_X \leq \mathbb{E} \left\| \int_0^t (A(t, s, X_n(s)) - A(t, s, X_m(s))) ds \right\|^2_X
\]

\[
+ \mathbb{E} \left\| \int_0^t (B(t, s, X_n(s)) - B(t, s, X_m(s))) dW(s) \right\|^2_X
\]

\[
\leq \int_0^t \kappa_2(t, s) \cdot \mathbb{E}\|Z_{n,m}(s)\|^2_X ds.
\]

Set

\[ f(t) := \limsup_{n,m \to \infty} \mathbb{E}\|Z_{n,m}(t)\|^2_X. \]

By (3.9), (H1) and using Fatou’s lemma, we get

\[ f(t) \leq \int_0^t \kappa_2(t, s) \cdot f(s) ds. \]

By Lemma 2.2 again, we have for almost all \( t \in [0, T] \)

\[ f(t) = \limsup_{n,m \to \infty} \mathbb{E}\|Z_{n,m}(t)\|^2_X = 0. \]

Hence, there exists an \( X \)-valued \((\mathcal{F}_t)\)-adapted process \( X(t) \) such that for almost all \( t \in [0, T] \)

\[ \lim_{n \to \infty} \mathbb{E}\|X_n(t) - X(t)\|^2_X = 0. \]

Taking limits for (3.6), one finds that (3.2) holds.

Moreover, the estimate (3.3) follows from (3.9). Note that when (3.4) is satisfied, the constant \( C_{T,p} \) in (3.7) is independent of \( T \). Hence, the estimate (3.5) is direct from (3.8). The uniqueness follows by similar calculations as above. \( \square \)

**Example 3.2.** Let for \( \delta > 0 \)

\[ h(s) := \frac{e^{-\delta s}}{s \log^2 s}, \quad t > s \geq 0. \]

It is easy to see that \( h \in L^1(\mathbb{R}_+) \). Consider the following stochastic Volterra equation:

\[ X(t) = x_0 \sqrt{|\log(t \wedge 1)|} + \int_0^t h(t-s)A(X(s))ds + \int_0^t \sqrt{h(t-s)}B(X(s))dW(s), \]

where \( A(X) \) and \( B(X) \) are \( \mathbb{R} \)-valued functions. Hence, we have

\[ \int_0^t \mathbb{E}\|X(t)\|^p_X \leq \int_0^t \mathbb{E}\|X(t)\|^p_X + \int_0^t \mathbb{E}\|X(t)\|^p_X. \]
where \( A : X \to \mathbb{X} \) and \( B : X \to L_2(l^2; \mathbb{X}) \) are global Lipschitz continuous functions. By elementary calculations, one finds that
\[
\sup_{t \geq 0} \int_0^t e^{-\delta(t-s)} |\log(s \land 1)| \frac{e^{\gamma(t-s)}}{(t-s) \log^2(t-s)} ds < +\infty.
\]
So, (H1)-(H3) are satisfied with \( p = 2 \). Moreover, by (2.9) and (3.3), one finds that if \( \delta \) is large enough, then for any \( T > 0 \)
\[
\sup_{t \geq T} \mathbb{E}\|X(t)\|_X^2 < +\infty.
\]
We remark that in this example, \( X(0) = \infty \).

3.2. **Path continuity of solutions.** In this subsection, in addition to (H2) and (H3), we also assume that

\[ (H1)' \] The process \( t \mapsto g(t) \) is continuous and \((\mathcal{F}_t)\)-adapted, and for any \( p \geq 2 \) and \( T > 0 \)
\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|g(t)\|_X^p \right) < +\infty.
\]

\[ (H4) \] For all \( s < t < t' \), \( \omega \in \Omega \) and \( x \in \mathbb{X} \)
\[
\|A(t', s, \omega, x) - A(t, s, \omega, x)\|_X \leq \lambda(t', t, s) \cdot (\|x\|_X + 1)
\]
and
\[
\|B(t', s, \omega, x) - B(t, s, \omega, x)\|_{L_2(l^2; X)}^2 \leq \lambda(t', t, s) \cdot (\|x\|_X^2 + 1),
\]
where \( \lambda \) is a positive measurable function satisfying that for any \( T > 0 \) and some \( \gamma = \gamma(T), C = C(T) > 0 \)
\[
\int_0^t \lambda(t', t, s) ds \leq C|t' - t|^{\gamma}, \quad 0 \leq t < t' \leq T.
\]

**Theorem 3.3.** Assume that (H1)' and (H2)-(H4) hold, and the kernel function \( \kappa_1 \) in (H2) belongs to \( \mathcal{K}_{\geq 1} \). Then there exists a unique \( \mathbb{X} \)-valued continuous \((\mathcal{F}_t)\)-adapted process \( X(t) \) such that \( P \)-a.s., for all \( t \geq 0 \)
\[
X(t) = g(t) + \int_0^t A(t, s, X(s)) ds + \int_0^t B(t, s, X(s)) dW(s)
\]
and for any \( p \geq 2 \) and \( T > 0 \),
\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|X(t)\|_X^p \right) < +\infty.
\]
Moreover, if for some \( \delta > 0 \) and any \( p \geq 2 \), \( T > 0 \), it holds that
\[
\mathbb{E}\|g(t') - g(t)\|_X^p \leq C_{T,p}|t' - t|^{\delta p},
\]
then, \( t \mapsto X(t) \) admits a H"older continuous modification and for any \( p \geq 2, T > 0 \) and some \( a > 0 \)
\[
\mathbb{E} \left( \sup_{t \neq t' \in [0,T]} \frac{\|X(t') - X(t)\|_X^p}{|t' - t|^{ap}} \right) \leq C_{T,p,a}.
\]

**Proof.** First of all, for any \( p \geq 2 \) and \( T > 0 \), by (H1)' and (3.3) we have
\[
\operatorname{ess} \sup_{t \in [0,T]} \mathbb{E}\|X(t)\|_X^p < +\infty.
\]
Lemma 3.4. We put it here since the proof is similar to Theorem 3.3. The desired conclusions follow from Theorem 2.10. □

and by (H4)

Assume that for all $0 \leq t < t' \leq T$

$$J(t') - J(t) = \int_0^t [B(t', s, X(s)) - B(t, s, X(s))]dW(s)$$

$$+ \int_t^{t'} B(t', s, X(s))dW(s) =: J_1(t', t) + J_2(t', t).$$

In view of $\kappa \in \kappa_1$, (2.6) holds for some $\beta > 1$. Fix $p \geq 2(\beta^* := \beta/(\beta - 1))$. By the BDG inequality (2.17), (H2) and Hölder’s inequality we have

$$\mathbb{E}\|J_2(t', t)\|^p_X \lesssim \mathbb{E}\left(\int_t^{t'} k_\lambda^\beta(t', s)ds\right)^{\frac{p}{\beta}} \mathbb{E}\left(\int_t^{t'} (\|X(s)\|^2_X + 1)ds\right)^{\frac{p}{2}}$$

and by (H4) and Minkowski’s inequality

$$\mathbb{E}\|J_1(t', t)\|^p_X \lesssim \mathbb{E}\left(\int_0^t \lambda(t', s) \cdot (\|X(s)\|^2_X + 1)ds\right)^{\frac{p}{\beta}}$$

$$\lesssim \left(\int_0^t \lambda(t', t) \cdot (\mathbb{E}\|X(s)\|^p_X + 1)ds\right)^{\frac{p}{\beta}}$$

$$\lesssim \left(\int_0^t \lambda(t', t)ds\right)^{\frac{p}{\beta}}$$

$$\lesssim |t' - t|^{\frac{p}{\beta^*}}.$$ 

Hence, for all $0 \leq t < t' \leq T$

$$\mathbb{E}\|J(t') - J(t)\|^p_X \leq |t - t'|^{\frac{p}{\beta}} + |t - t'|^{\frac{p}{\beta^*}}.$$  

Similarly, we may prove that for all $0 \leq t < t' \leq T$ and $p \geq \beta^*$

$$\mathbb{E}\left\|\int_0^{t'} A(t', s, X(s))ds - \int_0^t A(t, s, X(s))ds\right\|^p_X \leq |t - t'|^\gamma + |t - t'|^{\frac{p}{\beta^*}}.$$ 

The desired conclusions follow from Theorem 2.10. □

We conclude this subsection by proving a lemma, which will be used frequently later. We put it here since the proof is similar to Theorem 3.3.

Lemma 3.4. Let $\tau$ be an $(\mathcal{F}_t)$-stopping time and

$$G : \triangle \times \Omega \to L_2(l^2; \mathbb{X}) \in \mathcal{M}_\triangle/\mathcal{B}(L_2(l^2; \mathbb{X})).$$

Assume that for all $0 \leq s < t < t'$ and $\omega \in \Omega$

$$\|G(t, s, \omega)\|^2_{L_2(l^2; \mathbb{X})} \leq \kappa(t, s) \cdot f^2(s, \omega),$$  

(3.14)
\[
\|G(t', s, \omega) - G(t, s, \omega)\|_{L_2(\mathcal{X})}^2 \leq \lambda(t', t, s) \cdot f^2(s, \omega),
\] (3.15)

where \(\kappa \in \mathcal{X}_{> 1}\) and for any \(T > 0\) and some \(\alpha > 1\) and \(\gamma > 0\)
\[
\int_0^t \lambda^\alpha(t', t, s) ds \leq C_T |t' - t|^\gamma, \quad \forall 0 \leq t < t' \leq T,
\]
and \((s, \omega) \mapsto f(s, \omega)\) is a positive measurable process with
\[
\mathbb{E} \left( \int_0^{T \wedge \tau} f^p(s) ds \right) < +\infty, \quad \forall p \geq 2.
\]

Then \(t \mapsto J(t) := \int_0^t G(t, s) dW(s) \in \mathcal{X}\) admits a continuous modification on \([0, \tau]\), and for any \(T > 0\) and \(p\) large enough
\[
\mathbb{E} \left( \sup_{t \in [0, T \wedge \tau]} \left\| \int_0^t G(t, s) dW(s) \right\|_\mathcal{X}^p \right) \leq C_T \mathbb{E} \left( \int_0^{T \wedge \tau} f^p(s) ds \right),
\]
where the constant \(C_T\) is independent of \(f\) and \(\tau\).

**Proof.** Fix \(T > 0\) and write for \(0 \leq t < t' \leq T\)
\[
J(t') - J(t) = \int_t^{t'} G(t', s) dW(s) + \int_0^t [G(t', s) - G(t, s)] dW(s) =: J_1(t', t) + J_2(t', t).
\]
In view of \(\kappa \in \mathcal{X}_{> 1}\) and (2.17), by the BDG inequality (2.17) and Hölder’s inequality we have, for some \(\beta > 1\) and \(p \geq 2(\beta^* = \beta/(\beta - 1))\),
\[
\mathbb{E}\|J_1(t', t) \cdot 1_{\{t' \in [0, \tau]\}}\|_\mathcal{X}^p \leq \mathbb{E} \left( \int_0^{t \wedge \tau} G(t', s) dW(s) \right)^p
\leq \mathbb{E} \left( \int_0^{t \wedge \tau} \|G(t', s)\|_{L_2(\mathcal{X})}^2 ds \right)^{p/2}
\leq \mathbb{E} \left( \int_0^{t \wedge \tau} \kappa(t', s) \cdot f^2(s) ds \right)^{p/2}
\leq \left( \int_t^{t'} \kappa^\beta(t', s) ds \right)^{\frac{p}{2\beta}} \cdot \mathbb{E} \left( \int_0^{t \wedge \tau} f^{2\beta^*}(s) ds \right)^{\frac{p}{2\beta^*}}
\leq |t' - t|^{\frac{p}{2\beta^* - 1}} \cdot \mathbb{E} \left( \int_0^{T \wedge \tau} f^p(s) ds \right)
\]
and for \(p \geq 2(\alpha^* = \alpha/(\alpha - 1))\),
\[
\mathbb{E}\|J_1(t', t) \cdot 1_{\{t' \in [0, \tau]\}}\|_\mathcal{X}^p \leq \mathbb{E} \left( \int_0^{t \wedge \tau} \|G(t', s) - G(t, s)\|_{L_2(\mathcal{X})}^2 ds \right)^{p/2}
\leq \mathbb{E} \left( \int_0^{t \wedge \tau} \lambda(t', t, s) \cdot f^2(s) ds \right)^{p/2}
\leq \left( \int_t^{t'} \lambda^\alpha(t', t, s) ds \right)^{\frac{p}{2\alpha}} \cdot \mathbb{E} \left( \int_0^{t \wedge \tau} f^{2\alpha^*}(s) ds \right)^{\frac{p}{2\alpha^*}}
\leq |t' - t|^{\frac{p}{2\alpha^*}} \cdot \mathbb{E} \left( \int_0^{T \wedge \tau} f^p(s) ds \right).
\]
Hence, for any \( p \geq 2(\alpha^* \vee \beta^*) \) and \( 0 \leq t < t' \leq T \),
\[
\mathbb{E}[(J(t') - J(t)) \cdot 1_{\{t' \in [0, \tau]\}}]^p_x \leq |t' - t|^{(\frac{1}{m+1})(\frac{p}{m})} \cdot \mathbb{E} \left( \int_0^{T \wedge \tau} f^p(s)ds \right).
\]
The desired result now follows by Theorem 2.10.

3.3. Local existence and uniqueness. In this subsection, we assume that

(H2)' For any \( R > 0 \), there exists \( \kappa_{1,R} \in \mathcal{K}_{>1} \) such that for all \( (t, s) \in \Delta, \omega \in \Omega \) and \( x \in \mathbb{X} \) with \( \|x\|_\mathbb{X} \leq R \)
\[
\|A(t, s, x, \omega, x)\|_\mathbb{X} + \|B(t, s, x, \omega, x)\|_{L_2(\mathbb{P}; \mathbb{X})}^2 \leq \kappa_{1,R}(t, s).
\]

(H3)' For any \( R > 0 \), there exists \( \kappa_{2,R} \in \mathcal{K}_0 \) such that for all \( (t, s) \in \Delta, \omega \in \Omega \) and \( x, y \in \mathbb{X} \) with \( \|x\|_\mathbb{X}, \|y\|_\mathbb{X} \leq R \)
\[
\|A(t, s, x, \omega, x) - A(t, s, x, \omega, y)\|_\mathbb{X} \leq \kappa_{2,R}(t, s) \cdot \|x - y\|_\mathbb{X}
\]
and
\[
\|B(t, s, x, \omega, x) - B(t, s, x, \omega, y)\|_{L_2(\mathbb{P}; \mathbb{X})}^2 \leq \kappa_{2,R}(t, s) \cdot \|x - y\|_\mathbb{X}^2.
\]

(H4)' For any \( R > 0 \), there exists a measurable function \( \lambda_R \) satisfying that for any \( T > 0 \) and some \( \gamma, C > 0 \)
\[
\int_0^t \lambda_R(t', t, s)ds \leq C|t' - t|^\gamma, \quad 0 \leq t < t' \leq T,
\]
such that for all \( s < t < t', \omega \in \Omega \) and \( x \in \mathbb{X} \) with \( \|x\|_\mathbb{X} \leq R \),
\[
\|A(t', s, x, \omega, x) - A(t, s, x, \omega, x)\|_\mathbb{X} + \|B(t', s, x, \omega, x) - B(t, s, x, \omega, x)\|_{L_2(\mathbb{P}; \mathbb{X})}^2 \leq \lambda_R(t', t, s).
\]

We first introduce the following notion of local solutions.

**Definition 3.5.** Let \( \tau \) be an \( (\mathcal{F}_t) \)-stopping time, and \( \{X(t); t \in [0, \tau]\} \) an \( \mathbb{X} \)-valued continuous \((\mathcal{F}_t)\)-adapted process. The pair \((X, \tau)\) is called a local solution of **Eq. (3.1)** if \( P \)-a.s., for all \( t \in [0, \tau) \)
\[
X(t) = g(t) + \int_0^t A(t, s, X(s))ds + \int_0^t B(t, s, X(s))dW(s);
\]
\((X, \tau)\) is called a maximal solution of **Eq. (3.1)** if
\[
\lim_{t \uparrow \tau(\omega)} \|X(t, \omega)\|_\mathbb{X} = +\infty \quad \text{on} \quad \{\omega : \tau(\omega) < +\infty\}, \quad P - a.s..
\]
We call \((X, \tau)\) a non-explosion solution of **Eq. (3.1)** if
\[
P\{\omega : \tau(\omega) < +\infty\} = 0.
\]

**Remark 3.6.** The stochastic integral in the above definition is defined on \([0, \tau)\) by
\[
\int_0^t B(t, s, X(s))dW(s) = \lim_{n \rightarrow \infty} \int_0^{t \wedge \tau_n} B(t, s, X(s))dW(s), \quad t < \tau,
\]
where \( \tau_n := \inf\{t > 0 : \|X(t)\|_\mathbb{X} > n\} \nearrow \tau. \)

We now prove the following main result in this section.

**Theorem 3.7.** Under (H1)'-(H4)', there exists a unique maximal solution \((X, \tau)\) for \**Eq. (3.1)** in the sense of Definition 3.3.
Proof. For \( n \in \mathbb{N} \), let \( \chi_n \) be a positive smooth function on \( \mathbb{R}_+ \) with \( \chi_n(s) = 1, s \leq n \) and \( \chi_n(s) = 0, s \geq n + 1 \). Define
\[
A_n(t, s, \omega, x) := A(t, s, \omega, x) \cdot \chi_n(\|x\|_X)
\]
\[
B_n(t, s, \omega, x) := B(t, s, \omega, x) \cdot \chi_n(\|x\|_X).
\]
It is easy to see that for \( A_n \) and \( B_n \), (H2) holds with \( \kappa_{1, n + 1} \), (H4) holds with \( \lambda_{n + 1} \), and (H3) holds with some \( \kappa_{3, n} \in \mathcal{K}_0 \). Thus, by Theorem 3.3 there exists a unique continuous \((\mathcal{F}_t)\)-adapted process \( X_n(t) \) such that for any \( p \geq 2 \) and \( T > 0 \)
\[
\mathbb{E} \left( \sup_{t \in [0, T]} \|X_n(t)\|^p_X \right) \leq C_{T, p, n}
\]
and
\[
X_n(t) = g(t) + \int_0^t A_n(t, s, X_n(s))ds + \int_0^t B_n(t, s, X_n(s))dW(s). \tag{3.16}
\]
We have the following claim:

Let \( \tau \) be any stopping time. The uniqueness holds for (3.16) on \([0, \tau)\).

We remark that when \( \tau = T \) is non-random, it follows from Theorem 3.1. Let \( X_i(t), i = 1, 2 \) be two \( X \)-valued continuous \((\mathcal{F}_t)\)-adapted processes, and satisfy on \([0, \tau)\)
\[
X_i(t) = g(t) + \int_0^t A_i(t, s, X_i(s))ds + \int_0^t B_i(t, s, X_i(s))dW(s), \quad i = 1, 2.
\]
Set
\[
Z(t) := X_1(t) - X_2(t).
\]
Since \( \kappa_{3, n} \in \mathcal{K}_0 \), as the calculations in (3.17), by the BDG inequality (2.17) and (H3) for \( A_n \) and \( B_n \), we have
\[
\mathbb{E}\|Z(t) \cdot 1_{\{t < \tau\}}\|^p_X \leq \mathbb{E} \left( \int_0^{t \wedge \tau} \kappa_{3, n}(t, s) \cdot \|Z(s)\|_X ds \right)^p
\]
\[
+ \mathbb{E} \left( \int_0^{t \wedge \tau} \kappa_{3, n}(t, s) \cdot \|Z(s)\|_X^2 ds \right)^{\frac{p}{2}}
\]
\[
= \mathbb{E} \left( \int_0^t \kappa_{3, n}(t, s) \cdot 1_{\{s < \tau\}} \cdot \|Z(s)\|_X ds \right)^p
\]
\[
+ \mathbb{E} \left( \int_0^t \kappa_{3, n}(t, s) \cdot 1_{\{s < \tau\}} \cdot \|Z(s)\|_X^2 ds \right)^{\frac{p}{2}}
\]
\[
\leq \int_0^t \kappa_{3, n}(t, s) \cdot \mathbb{E} \|Z(s) \cdot 1_{\{s < \tau\}}\|^p_X ds. \tag{3.17}
\]
By Lemma 2.2, we get
\[
\mathbb{E}\|Z(t) \cdot 1_{\{t < \tau\}}\|^p_X = 0 \quad \text{for almost all } t \in [0, T],
\]
which implies by the arbitrariness of \( T \) and the continuities of \( X_i(t), i = 1, 2 \),
\[
X_1(\cdot)|_{[0, \tau)} = X_2(\cdot)|_{[0, \tau)}.
\]
The claim is proved.

Now, for \( n \in \mathbb{N} \), define the stopping times
\[
\tau_n := \inf\{t > 0 : \|X_n(t)\|_X > n\}.\]
and
\[ \sigma_n := \inf\{t > 0 : \|X_{n+1}(t)\|_X > n\}. \]

By the above claim, we have
\[ X_n(\cdot)|_{[0,\tau_n \wedge \sigma_n]} = X_{n+1}(\cdot)|_{[0,\tau_n \wedge \sigma_n]}, \]
which implies
\[ \tau_n \leq \sigma_n \leq \tau_{n+1}, \text{ a.e.} \]

Hence, we may define
\[ \tau(\omega) := \lim_{n \to \infty} \tau_n(\omega) \]
and for all \( t < \tau(\omega) \)
\[ X(t, \omega) := X_n(t, \omega), \text{ if } t < \tau_n(\omega). \]

Clearly, \((X, \tau)\) is a maximal solution of Eq. (3.1) in the sense of Definition 3.5.

We next prove the uniqueness. Let \((\tilde{X}, \tilde{\tau})\) be another maximal solution of Eq. (3.1) in the sense of Definition 3.5. Define the stopping times
\[ \tilde{\tau}_n := \inf\{t > 0 : \|\tilde{X}(t)\|_X > n\} \]
and
\[ \hat{\tau}_n := \tau_n \wedge \tilde{\tau}_n, \quad \hat{\tau} := \tau \wedge \tilde{\tau}. \]

It is clear that
\[ \hat{\tau}_n \nearrow \hat{\tau} \text{ a.s. as } n \to \infty \]
and
\[ 1_{[0,\hat{\tau}_n)}(t) \cdot \tilde{X}(t) = 1_{[0,\hat{\tau}_n)}(t) \cdot g(t) + 1_{[0,\hat{\tau}_n)}(t) \cdot \int_0^t A(t, s, \tilde{X}(s))ds \]
\[ + 1_{[0,\hat{\tau}_n)}(t) \cdot \int_0^t B(t, s, \tilde{X}(s))dW(s) \]
\[ = 1_{[0,\hat{\tau}_n)}(t) \cdot g(t) + 1_{[0,\hat{\tau}_n)}(t) \cdot \int_0^t A_n(t, s, \tilde{X}(s))ds \]
\[ + 1_{[0,\hat{\tau}_n)}(t) \cdot \int_0^t B_n(t, s, \tilde{X}(s))dW(s). \]

By the above claim again, we have
\[ X(\cdot)|_{[0,\hat{\tau}_n]} = \tilde{X}(\cdot)|_{[0,\hat{\tau}_n)}. \]

So
\[ X(\cdot)|_{[0,\hat{\tau}]} = \tilde{X}(\cdot)|_{[0,\hat{\tau}]} = \tilde{\tau}. \]

We have the following simple criterion of non explosion.

\textbf{Theorem 3.8.} Assume that \((H1)'\), \((H2)\) and \((H4)\) hold, and \(\kappa_1\) in \((H2)\) belongs to \(H_{>1}\). Then there is no explosion for Eq. (3.1).

\textbf{Proof.} Let \((X, \tau)\) be a maximal solution of Eq. (3.1). Define
\[ \tau_n := \inf\{t > 0 : \|X(t)\|_X \geq n\}. \]

By the BDG inequality (2.17) and Hölder’s inequality, and using the same method as estimating (3.17), we have, for any \( T > 0 \), some \( \beta > 1 \) and \( p \geq 2(\beta^* = \beta/(\beta - 1)) \)
\[ \mathbb{E}\|X(t) \cdot 1_{\{t \leq \tau_n\}}\|_X^p \leq \mathbb{E}\|g(t)\|_X^p + \mathbb{E}\left(\int_0^{t \wedge \tau_n} \|A(t, s, X(s))\|_X ds\right)^p \]

We have the following simple criterion of non explosion.
\[
+ E \left\| \int_0^{t \land \tau_n} B(t, s, X(s))dW(s) \right\|_p^p \\
\leq E\|g(t)\|_X^p + E \left( \int_0^{t \land \tau_n} \kappa_1(t, s) \cdot (\|X(s)\|_X + 1)ds \right)^p \\
+ E \left( \int_0^{t \land \tau_n} \|B(t, s, X(s))\|_{L_2^p(X)}^2 ds \right)^{\frac{p}{2}} \\
\leq E\|g(t)\|_X^p + E \left( \int_0^{t \land \tau_n} (\|X(s)\|_X^{\beta^*} + 1)ds \right)^{\frac{p}{\beta^*}} \\
+ E \left( \int_0^{t \land \tau_n} (\|X(s)\|_X^{2\beta^*} + 1)ds \right)^{\frac{p}{2\beta^*}} \\
\leq C_{T,p} \left[ E\|g(s)\|_X^p + 1 + \int_0^t E\|X(s)\| \cdot 1\{s \leq \tau_n\}\|_X^p ds \right],
\]
where the constant $C_{T,p}$ is independent of $n$.

By Gronwall’s inequality, we obtain
\[
\sup_{t \in [0, T]} E\|X(t)\|_X \leq C_{T,p}.
\]
Using this estimate, as in the proofs of Theorem 3.3 and Lemma 3.4, we can prove that for any $T > 0$ and $p \geq 2$
\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T \land \tau_n]} \|X(t)\|_X^p \leq C_{T,p}.
\]
Hence,
\[
\lim_{n \to \infty} P\{\tau_n \leq T\} = \lim_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [0, T \land \tau_n]} \|X(t)\|_X \geq n \right\} \\
\leq \lim_{n \to \infty} E \left( \sup_{t \in [0, T \land \tau_n]} \|X(t)\|_X^p \right) / n^p \\
\leq \lim_{n \to \infty} C_{T,p} / n^p = 0,
\]
which produces the non-explosion, i.e., $P\{\tau < \infty\} = 0$. \hfill \Box

**Remark 3.9.** One cannot directly prove
\[
\sup_{n \in \mathbb{N}} E\|X(t \land \tau_n)\|_X^p < +\infty, \quad \forall t \geq 0
\]

**3.4. Continuous dependence of solutions with respect to data.** In this subsection, we study the continuous dependence of solutions for Eq. (3.1) with respect to the coefficients.

Let $\{(g_m, A_m, B_m), m \in \mathbb{N}\}$ be a sequence of coefficients associated to Eq. (3.1). Assume that for each $m \in \mathbb{N}$, $(g_m, A_m, B_m)$ satisfies $(H1)$-$(H4)'$ with the same $\kappa_{1,R}, \kappa_{2,R}$ and $\lambda_R$ as $(g, A, B)$, and for each $p \geq 2$
\[
\lim_{m \to \infty} \sup_{t \in [0, T]} E\|g_m(t) - g(t)\|_X^p = 0 \quad (3.18)
\]
and for each $T, R > 0$,

$$
\lim_{m \to \infty} \sup_{t \in [0, T], ||x|| \leq R} \int_0^t \| A_m(t, s, x) - A(t, s, x) \|_X ds = 0, \quad (3.19)
$$

$$
\lim_{m \to \infty} \sup_{t \in [0, T], ||x|| \leq R} \int_0^t \| B_m(t, s, x) - B(t, s, x) \|^2_{L_2(t; X)} ds = 0. \quad (3.20)
$$

Let $(X_m, \tau_m)$ (resp. $(X, \tau)$) be the unique maximal solution associated with $(g_m, A_m, B_m)$ (resp. $(g, A, B)$). For each $R > 0$ and $m \in \mathbb{N}$, define

$$
\tau_m^R := \inf \{ t > 0 : ||X(t)||_X, ||X_m(t)||_X > R \}.
$$

Suppose that for each $t > 0$

$$
\lim_{R \to \infty} \sup_m \{ \tau_m^R < t \} = 0. \quad (3.21)
$$

Then we have:

**Theorem 3.10.** For each $t > 0$ and $\epsilon > 0$

$$
\lim_{m \to \infty} P \{ ||X_m(t) - X(t)||_X \geq \epsilon \} = 0.
$$

**Proof.** For $R > 0$ and $m \in \mathbb{N}$, set

$$
Z_m^R(t) := (X_m(t) - X(t)) \cdot 1_{\{t \leq \tau_m^R\}}.
$$

Then

$$
Z_m^R(t) = J_{1,m}^R(t) + J_{2,m}^R(t) + J_{3,m}^R(t) + J_{4,m}^R(t) + J_{5,m}^R(t),
$$

where

$$
J_{1,m}^R(t) := 1_{\{t \leq \tau_m^R\}} \cdot [g_m(t) - g(t)],
$$

$$
J_{2,m}^R(t) := 1_{\{t \leq \tau_m^R\}} \cdot \int_0^{t \wedge \tau_m^R} [A_m(t, s, X_n(s)) - A_m(t, s, X(s))] ds,
$$

$$
J_{3,m}^R(t) := 1_{\{t \leq \tau_m^R\}} \cdot \int_0^{t \wedge \tau_m^R} [A_m(t, s, X(s)) - A(t, X(s))] ds,
$$

$$
J_{4,m}^R(t) := 1_{\{t \leq \tau_m^R\}} \cdot \int_0^{t \wedge \tau_m^R} [B_m(t, s, X_m(s)) - B_m(t, s, X(s))] dW(s),
$$

$$
J_{5,m}^R(t) := 1_{\{t \leq \tau_m^R\}} \cdot \int_0^{t \wedge \tau_m^R} [B_m(t, s, X(s)) - B(t, s, X(s))] dW(s).
$$

Fix $T > 0$. Clearly, for any $p \geq 2$ and $t \in [0, T]$

$$
\mathbb{E}[||J_{1,m}^R(t)||_X^p] \leq \sup_{t \in [0, T]} \mathbb{E}[||g_m(t) - g(t)||_X^p] =: \mathcal{J}_{1,m}.
$$

For $J_{2,m}^R(t)$, by $(H3)'$ and Hölder’s inequality we have, for $p$ large enough ($\kappa_{2,R} \in \mathcal{H}_{1}^\kappa$)

$$
\mathbb{E}[||J_{2,m}^R(t)||_X^p] \leq \mathbb{E} \left( \int_0^{t \wedge \tau_m^R} \kappa_{2,R}(t, s) \cdot ||X_m(s) - X(s)||_X ds \right)^p
$$

$$
\leq \left[ \int_0^t \kappa_{2,R}(t, s) ds \right]^\frac{p}{2} \cdot \mathbb{E} \left[ \int_0^t ||Z_m^R(s)||_X^\beta ds \right]^\frac{p}{\beta}
$$

$$
\leq C \int_0^t \mathbb{E}[||Z_m^R(s)||_X^p] ds.
$$
For $J_{3,m}^R(t)$, we have
\[
\mathbb{E}\|J_{3,m}^R(t)\|^p_X \leq \mathbb{E} \left( \sup_{\|x\| \leq R} \int_0^{t \wedge \tau_m^R} \|A_m(t, s, x) - A(t, s, x)\|_X ds \right)^p \\
\leq \left( \sup_{t \in [0,T]} \sup_{\|x\| \leq R} \int_0^t \|A_m(t, s, x) - A(t, s, x)\|_X ds \right)^p =: J_{3,m}^R.
\]
Similarly, by the BDG inequality (2.17) we have, for $p$ large enough
\[
\mathbb{E}\|J_{4,m}^R(t)\|^p_X \leq C \int_0^t \mathbb{E}\|Z_m^R(s)\|^p_X ds
\]
and
\[
\mathbb{E}\|J_{5,m}^R(t)\|^p_X \leq C_p \left( \sup_{t \in [0,T]} \sup_{\|x\| \leq R} \int_0^t \|B_m(t, s, x) - B(t, s, x)\|^2_{L_2(\mathbb{P},\mathbb{F},\mathbb{Q})} ds \right)^{\frac{p}{2}} =: J_{5,m}^R.
\]
Combining the above calculations, we get
\[
\mathbb{E}\|Z_m^R(t)\|^p_X \leq J_{1,m} + J_{3,m}^R + J_{5,m}^R + C \int_0^t \mathbb{E}\|Z_m^R(s)\|^p_X ds.
\]
By Gronwall’s inequality and (3.18)-(3.20) we get, for any $R > 0$ and $p$ large enough
\[
\lim_{m \to \infty} \mathbb{E}\|Z_m^R(t)\|^p_X = 0.
\]
Hence
\[
P\{\|X_m(t) - X(t)\|_X \geq \epsilon\} \leq P\{\|X_m(t) - X(t)\|_X \cdot 1_{\{t \leq \tau_m^R\}} \geq \epsilon\} + P\{\tau_m^R < t\} \\
\leq \mathbb{E}\|Z_m^R(t)\|^p_X / \epsilon^p + P\{\tau_m^R < t\}.
\]
First letting $m \to \infty$, then $R \to \infty$, we then get the desired limit by (3.21). \qed

4. LARGE DEVIATION FOR STOCHASTIC VOLterra EQUATIONS

In this section, we study the large deviation of small perturbations for stochastic Volterra equations. In addition to (H2)', (H3)' and (H4)', we assume that $g$ and $A, B$ are non-random, and

(H1)'' For any $T > 0$ and some $\delta > 0$,
\[
\|g(t) - g(t')\|_X \leq C|t - t'|^\delta, \quad t, t' \in [0, T]
\]
and for some $\alpha > 0$,
\[
\sup_{t \in [0,T]} \|g(t)\|_{X_\alpha} < +\infty.
\]

(H2)'' For the same $\alpha$ as in (H1)'' and any $R > 0$, there exists a kernel function $\kappa_{\alpha,R} \in \mathcal{K}_0$ such that for all $(t, s) \in \Delta$ and $x \in X$ with $\|x\|_X \leq R$
\[
\|A(t, s, x)\|_{X_\alpha} + \|B(t, s, x)\|^2_{L_2(\mathbb{P},\mathbb{F},\mathbb{Q})} \leq \kappa_{\alpha,R}(t, s).
\]

Remark 4.1. If the $\kappa_{\alpha,R}$ in (H2)'' belongs to $\mathcal{K}_{> 1}$, then (H2)'' implies (H2)' in view of $X_\alpha \hookrightarrow X$. 

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Consider the following small perturbation of stochastic Volterra equation (3.1)

\[
X_\epsilon(t) = g(t) + \int_0^t A(t, s, X_\epsilon(s))ds + \sqrt{\epsilon} \int_0^t B(t, s, X_\epsilon(s))dW(s),
\]

where \(\epsilon \in (0, 1)\). By Theorem 3.7, there exists a unique maximal solution \((X_\epsilon, \tau_\epsilon)\) for Eq. (4.1). Below, we fix \(T > 0\) and work in the finite time interval \([0, T]\), and assume that for each \(\epsilon \in (0, 1)\)

\[\tau_\epsilon > T, \ a.s..\]

By Yamada-Watanabe’s theorem (cf. [54, 67]), there exists a measurable mapping

\[\Phi_\epsilon : C_T(\mathbb{U}) \to C_T(\mathbb{X})\]

such that

\[X_\epsilon(t, \omega) = \Phi_\epsilon(W(\cdot, \omega))(t).\]

It should be noticed that although the equation considered in [54] is a little different from Eq. (3.1), the proof is obviously adapted to our more general equation.

We now fix a family of processes \(\{h^\epsilon, \epsilon \in (0, 1)\}\) in \(A^T_N\) (see (2.23) for the definition of \(A^T_N\)), and put

\[X^\epsilon(t, \omega) := \Phi_\epsilon(W(\cdot, \omega) + \frac{h^\epsilon(\cdot, \omega)}{\sqrt{\epsilon}})(t).\]

Here, we have used a little confused notations \(X_\epsilon\) and \(X^\epsilon\), but they are clearly different. By Girsanov’s theorem (cf. [54, Section 7]), \(X^\epsilon(t)\) solves the following stochastic Volterra equation (also called control equation):

\[
X^\epsilon(t) = g(t) + \int_0^t A(t, s, X^\epsilon(s))ds + \int_0^t B(t, s, X^\epsilon(s))h^\epsilon(s)ds + \sqrt{\epsilon} \int_0^t B(t, s, X^\epsilon(s))dW(s).
\]

Although \(h\) is defined only on \([0, T]\), we can extend it to \(\mathbb{R}_+\) by setting \(h(t) = 0\) for \(t > T\) so that Eq. (4.2) can be considered on \(\mathbb{R}_+\). We shall always use this extension below. Let \(\tau^\epsilon\) be the explosion time of Eq. (4.2). For \(n \in \mathbb{N}\), define

\[\tau^\epsilon_n := \inf\{t \geq 0 : \|X^\epsilon(t)\|_\mathbb{X} > n\}.\]

Then \(\tau^\epsilon_n \not\geq \tau^\epsilon\), and we have:

**Lemma 4.2.** For any \(\alpha_0 \in (0, \alpha)\), there is an \(a > 0\) such that for \(p\) sufficiently large

\[
\sup_{\epsilon \in (0, 1)} \mathbb{E}\left(\sup_{t \neq t' \in [0, T \wedge \tau^\epsilon_n]} \frac{\|X^\epsilon(t') - X^\epsilon(t)\|_{X, \alpha}}{|t' - t|^{a \cdot p}}\right) \leq C_{N,n,T,p,\alpha_0,\alpha}.\]

**Proof.** Note that

\[
\|X^\epsilon(t) \cdot 1_{\{t \leq \tau^\epsilon_n\}}\|_{X, \alpha} \leq \|g(t)\|_{X, \alpha} + \int_0^{t \wedge \tau^\epsilon_n} \|A(t, s, X^\epsilon(s))\|_{X, \alpha} ds
\]

\[+ \int_0^{t \wedge \tau^\epsilon_n} \|B(t, s, X^\epsilon(s))h^\epsilon(s)\|_{X, \alpha} ds
\]

\[+ \sqrt{\epsilon} \left\|\int_0^{t \wedge \tau^\epsilon_n} B(t, s, X^\epsilon(s))dW(s)\right\|_{X, \alpha}
\]

\[=: J_1(t) + J_2(t) + J_3(t) + J_4(t).\]
By (H2) and \([4.3]\) we have
\[
\mathbb{E}|J_2(t)|^p \leq C_n\mathbb{E}\left(\int_0^{t \wedge \tau_n^\varepsilon} \kappa_{n}(t, s) ds\right)^p \leq C_{n,T,p,\kappa_{n,n}},
\]
and by Hölder’s inequality
\[
\mathbb{E}|J_3(t)|^p \leq \mathbb{E}\left(\int_0^{t \wedge \tau_n^\varepsilon} \|B(t, s, X^\varepsilon(s))\dot{h}^\varepsilon(s)\|_{X_\alpha} ds\right)^p
\leq \mathbb{E}\left(\int_0^{t \wedge \tau_n^\varepsilon} \|B(t, s, X^\varepsilon(s))\|_{L_2(t^2; X_\alpha)} \cdot \|\dot{h}^\varepsilon(s)\|_{L_2} ds\right)^p
\leq N^\frac{p}{2}\mathbb{E}\left(\int_0^{t \wedge \tau_n^\varepsilon} \|B(t, s, X^\varepsilon(s))\|_{L_2(t^2; X_\alpha)}^2 ds\right)^\frac{p}{2}
\leq C_{N,n,T,p,\kappa_{n,n}},
\]
where we have used that \(h^\varepsilon \in A^\tau_{D}\).

Similarly, by the BDG inequality \([2.17]\) and (H2) we have
\[
\mathbb{E}|J_4(t)|^p \leq C_p\mathbb{E}\left(\int_0^{t \wedge \tau_n^\varepsilon} \|B(t, s, X^\varepsilon(s))\|_{L_2(t^2; X_\alpha)}^2 ds\right)^\frac{p}{2} \leq C_{n,T,p,\kappa_{n,n}}.
\]
Combining the above calculations, we get
\[
\sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}\|X^\varepsilon(t) \cdot 1_{\{t \leq \tau_n^\varepsilon\}}\|^p_{X_\alpha} \leq C_{N,n,T,p,\kappa_{n,n}}, \quad p \geq 2. \tag{4.4}
\]
Moreover, as in the proofs of Theorem \([3.3]\) and Lemma \([3.4]\) by (H1), (H2) and (H4), for some \(\beta_3 > 1\) and \(p \geq 2(\beta_3/2 - 1)\), we have that for any \(0 \leq t < t' \leq T\)
\[
\sup_{\varepsilon \in (0,1)} \mathbb{E}\|(X^\varepsilon(t') - X^\varepsilon(t)) \cdot 1_{\{t', t \leq \tau_n^\varepsilon\}}\|^p_{X_\alpha} \leq C_{T,p,n} \left(|t - t'|^{\beta_3} + |t - t'|^p + |t - t'|^\frac{p}{2}\right).
\]
Thus, by (v) of Proposition \([2.11]\) and \(4.4\), for any \(\alpha_0 \in (0, \alpha)\) and \(p\) large enough we have
\[
\sup_{\varepsilon \in (0,1)} \mathbb{E}\|(X^\varepsilon(t') - X^\varepsilon(t)) \cdot 1_{\{t', t \leq \tau_n^\varepsilon\}}\|^p_{X_{\alpha_0}}
\leq C_{N,n,T,p,\kappa_{n,n},\alpha_0} \left(|t - t'|^{\beta_3} + |t - t'|^p + |t - t'|^\frac{p}{2}\right)^{1-\frac{\alpha_0}{\alpha}}.
\]
The desired estimate now follows by Theorem \([2.10]\). \hfill \Box

In order to obtain the tightness of the laws of \(\{X^\varepsilon, \varepsilon \in (0,1)\}\) in \(\mathcal{C}_T(\mathbb{X})\), we assume that
\(\text{(C1) } \mathcal{L}^{-1}\) is a compact operator on \(\mathbb{X}\).
\(\text{(C2) } \lim_{n \to \infty} \sup_{\varepsilon \in (0,1)} P\{\omega : \tau_n^\varepsilon(\omega) < T\} = 0.\)
Note that (C2) implies
\(P\{\omega : \tau^\varepsilon(\omega) > T\} = 1.\)

We now prove the following key lemma for the large deviation principle of Eq.\([4.1]\).

**Lemma 4.3.** Under (C1) and (C2), there exist subsequence \(\varepsilon_k \downarrow 0\), a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) and a sequence \(\{(h^k, X^k, \tilde{W}^k)\}_{k \in \mathbb{N}}\) as well as \((h, X^h, \tilde{W})\) defined on this probability space and taking values in \(D_N \times \mathcal{C}_T(\mathbb{X}) \times \mathcal{C}_T(\mathbb{U})\) such that
\(i\) \((h^k, X^k, \tilde{W}^k)\) has the same law as \((h^k, X^\varepsilon, \tilde{W})\) for each \(k \in \mathbb{N}\);
\(ii\) \((h^k, X^k, \tilde{W}^k) \to (h, X^h, \tilde{W})\) in \(D_N \times \mathcal{C}_T(\mathbb{X}) \times \mathcal{C}_T(\mathbb{U})\), \(\tilde{P}\)-a.s. as \(k \to \infty\);
(iii) \((h, X^h)\) uniquely solves the following Volterra equation:

\[
X^h(t) = g(t) + \int_0^t A(t, s, X^h(s))ds + \int_0^t B(t, s, X^h(s))h(s)ds.
\]

(4.5)

In particular, \((LD)\) in Subsection 2.4 holds.

**Proof.** Let \(\alpha_0 \in (0, \alpha)\) and \(a > 0\) be as in Lemma 4.2. For \(R > 0\), set

\[
K_R := \left\{ x \in \mathbb{C}_T(\mathbb{X}) : \sup_{t \in [0, T]} \| x(t) \|_\mathbb{X} + \sup_{s \neq t \in [0, T]} \frac{\| x(t) - x(s) \|_{\mathbb{X}_{\alpha_0}}}{|t - s|^a} \leq R \right\}.
\]

By (C1), \(\mathbb{X}_{\alpha_0} \hookrightarrow \mathbb{X}\) is compact (cf. [37, p.29, Theorem 1.4.8]). Thus, by Ascoli-Arzelà’s theorem (cf. [39]), the set \(K_R\) is compact in \(\mathbb{C}_T(\mathbb{X})\). For any \(\delta > 0\), by (C2) we can choose \(n\) sufficiently large such that

\[
\sup_{\epsilon \in (0, 1)} P\left\{ \omega : \tau_n^\epsilon(\omega) < T \right\} \leq \delta.
\]

By Lemma 4.2 and Chebyshev’s inequality, for any \(R > n\) we have

\[
P\{X^\epsilon(\cdot) \notin K_R\} = P\{X^\epsilon(\cdot) \notin K_R, \tau_n^\epsilon \geq T\} + P\{X^\epsilon(\cdot) \notin K_R, \tau_n^\epsilon < T\} \leq P\left\{ \sup_{s \neq t \in [0, T] \wedge \tau_n^\epsilon} \frac{\| X^\epsilon(t) - X^\epsilon(s) \|_{\mathbb{X}_{\alpha_0}}}{|t - s|^a} \geq R - n \right\} + P\{\tau_n^\epsilon < T\} \leq E \sup_{s \neq t \in [0, T] \wedge \tau_n^\epsilon} \frac{\| X^\epsilon(t) - X^\epsilon(s) \|_{\mathbb{X}_{\alpha_0}}^p}{|t - s|^{ap}} / (R - n)^p + \delta \leq C_{N,n,T,p,\alpha,n,\alpha_0} / (R - n)^p + \epsilon'.
\]

Therefore, for \(R\) large enough we have

\[
\sup_{\epsilon \in (0, 1)} P\{X^\epsilon(\cdot) \notin K_R\} \leq 2\delta.
\]

Thus, by the compactness of \(\mathbb{D}_N\) (see (2.21)), the laws of \((h^\epsilon, X^\epsilon, W)\) in \(\mathbb{D}_N \times \mathbb{C}_T(\mathbb{X}) \times \mathbb{C}_T(\mathbb{U})\) is tight. By Skorohod’s embedding theorem (cf. [39]), the conclusions (i) and (ii) hold.

We now prove (iii). Note that by (i) (cf. [54, Section 8])

\[
\tilde{X}^k(t) = g(t) + \int_0^t A(t, s, \tilde{X}^k(s))ds + \int_0^t B(t, s, \tilde{X}^k(s))\tilde{h}^k(s)ds
+ \sqrt{\epsilon_k} \int_0^t B(t, s, \tilde{X}^k(s))d\tilde{W}^k(s)
= g(t) + J_1^k(t) + J_2^k(t) + J_3^k(t), \ P - a.s.
\]

Set

\[
\tau_n^k := \inf\{t \geq 0 : \| \tilde{X}^k(t) \|_{\mathbb{X}} > n\}.
\]

Then for any \(\delta > 0\), by (i) and (C2) there exists an \(n\) large enough such that

\[
\sup_{k \in \mathbb{N}} \tilde{P}\{\tau_n^k < T\} = \sup_{k \in \mathbb{N}} \tilde{P}\left\{ \sup_{s \in [0, T)} \| \tilde{X}^k(s) \|_{\mathbb{X}} > n \right\} \leq \sup_{k \in \mathbb{N}} P\left\{ \sup_{s \in [0, T]} \| X^\epsilon_k(s) \|_{\mathbb{X}} > n \right\} \leq \sup_{k \in \mathbb{N}} P\{\tau_n^k < T\} \leq \delta.
\]
Hence, for any $\delta' > 0$, by the BDG inequality (2.17) and (H2)', we have
\[
\tilde{P} \left\{ \| J^k_2(t) \|_X \geq \delta' \right\} \leq \tilde{P} \left\{ J^k_2(t) \geq \delta' ; \tau^k_n \geq T \right\} + \tilde{P} \left\{ \tau^k_n < T \right\}
\leq \frac{\mathbb{E} \tilde{P} \| J^k_2(t) \cdot 1_{(t \leq \tau^k_n)} \|_X^2}{\delta'^2} + \delta
\leq \frac{\epsilon_k \cdot C_n \mathbb{E} \tilde{P} \left( \int_0^{t \wedge \tau^k_n} \kappa_{1,n}(t,s)ds \right)}{\delta'^2} + \delta
\leq \frac{\epsilon_k \cdot C_n \cdot n}{\delta'^2} + \delta.
\]
Thus, we get
\[
\lim_{k \to \infty} \tilde{P} \left\{ \| J^k_2(t) \|_X \geq \delta' \right\} = 0.
\]

Let $J_i(t), i = 1, 2$ be the corresponding terms in Eq. (4.5). In order to prove that $X^h$ solves Eq. (4.5), it is now enough to show that for any $t \in [0,T]$ and $y \in X^*$
\[
\lim_{k \to \infty} \mathbb{E} \tilde{P} \left( \| J^k_i(t) - J_i(t), y \|_X \right) = 0, \quad i = 1, 2, \quad \tilde{P} - a.s..
\]

Observe that
\[
\| J^k_2(t) - J_2(t), y \|_X \leq \| y \|_X \cdot \int_0^t \| [B(t,s,X^k(s)) - B(t,s,X^h(s))]\hat{h}^k(s) \|_X ds
\]
\[
+ \left| \int_0^t \mathbb{E} \{ B(t,s,X^h(s))\hat{h}^k(s) - \hat{h}(s), y \} ds \right|
\]
\[
=: \| y \|_X \cdot J^k_{21}(t) + J^k_{22}(t).
\]
By the weak convergence of $\hat{h}^k$ to $\hat{h}$ in $\mathbb{D}_N$, we have
\[
\lim_{k \to \infty} J^k_{22}(t) = 0.
\]
Noting that by (ii), for almost all $\tilde{\omega} \in \tilde{\Omega}$ and some $K(\tilde{\omega}) \in \mathbb{N}$
\[
n(\tilde{\omega}) := \sup_{s \in [0,T]} \| X^h(s,\tilde{\omega}) \|_X \vee \sup_{k \geq K(\tilde{\omega})} \sup_{s \in [0,T]} \| \bar{X}^k(s,\tilde{\omega}) \|_X < +\infty,
\]
we have, by H"older’s inequality and (H3)'
\[
J^k_{21}(t,\tilde{\omega}) \leq \| \hat{h}^k(\tilde{\omega}) \|_{L_1} \cdot \left( \int_0^t \| B(t,s,\bar{X}^k(s,\tilde{\omega})) - B(t,s,X^h(s,\tilde{\omega})) \|_{L_2(t;X)}^2 ds \right)^{1/2}
\leq N \cdot \left( \int_0^t \kappa_{2,n}(\tilde{\omega})(t,s) \cdot \| \bar{X}^k(s,\tilde{\omega}) - X^h(s,\tilde{\omega}) \|_{X}^2 ds \right)^{1/2}
\xrightarrow{(ii)} 0, \quad \text{as } k \to \infty,
\]
where we have used $\tilde{\hat{h}}^k(\tilde{\omega}) \in \mathbb{D}_N$.

Similarly, we have
\[
\lim_{k \to \infty} \| J^k_1(t) - J_1(t) \|_X = 0, \quad \tilde{P} - a.s..
\]
Combining the above estimates, we find that $X^h$ solves Eq. (4.5). \(\square\)

Let $I(f)$ be defined by
\[
I(f) := \frac{1}{2} \inf_{\{ h \in L_2^2: f = \bar{X}^h \}} \| h \|_{L_2^2}^2, \quad f \in \mathbb{C}_T(X), \quad (4.6)
\]
where $X^h$ is defined by Eq. (4.5). In order to identify $I(f)$, we assume that
(C3) For any \( N \in \mathbb{N} \)
\[
\sup_{h \in \mathbb{E}_N} \sup_{t \in [0,T]} \|X^h(t)\|_X < +\infty.
\]
Similar to the proof of Lemma 4.3 we can prove that:

**Lemma 4.4.** Under (C3), (LD)\(_2\) in Subsection 2.4 holds.

Thus, by Theorem 2.15 we have proven:

**Theorem 4.5.** Assume that (H1)\(\prime\)\(\prime\)- (H2)\(\prime\), (H2)\(\prime\)\(\prime\) - (H4)\(\prime\) and (C1) - (C3) hold. Then, \( \{X_\epsilon, \epsilon \in (0,1)\} \) satisfies the large deviation principle in \( C_T(\mathbb{X}) \) with the rate function \( I(f) \) given by (4.7).

**Remark 4.6.** The conditions (C2) and (C3) are satisfied if (H1)\(\prime\), (H2) and (H4) hold, and \( \kappa_1 \) in (H2) belongs to \( \mathcal{X}_{>1} \). In fact, we can prove as the proof of Theorem 3.3
\[
\sup_{n \in \mathbb{N}} \sup_{\epsilon \in (0,1)} \mathbb{E} \left( \sup_{t \in [0,T \land T_\epsilon]} \|X^\epsilon(t)\|_X^p \right) \leq C_{T,p,\kappa_1},
\]
which then implies (C2). The condition (C3) is more direct in this case.

5. **Semilinear stochastic evolutionary integral equations**

In this section, we consider the following semilinear stochastic evolutionary integral equation:

\[
X(t) = x_0 - \int_0^t a(t-s)\mathcal{L}X(s)ds + \int_0^t \Phi(s, X(s))ds + \int_0^t \Psi(s, X(s))dW(s), \quad (5.1)
\]
where \( a : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a measurable function, and
\[
\Phi : \mathbb{R}_+ \times \Omega \times \mathbb{X} \rightarrow \mathbb{X} \in \mathcal{M} \times \mathcal{B}(\mathbb{X})/\mathcal{B}(\mathbb{X})
\]
and
\[
\Psi : \mathbb{R}_+ \times \Omega \times \mathbb{X} \rightarrow L_2(l^2; \mathbb{X}) \in \mathcal{M} \times \mathcal{B}(\mathbb{X})/\mathcal{B}(L_2(l^2; \mathbb{X})).
\]
Here and below, \( \mathcal{M} \) stands for the progressively measurable \( \sigma \)-algebra over \( \mathbb{R}_+ \times \Omega \).

Consider first the following deterministic integral equation:

\[
x(t) = x_0 - \int_0^t a(t-s)\mathcal{L}x(s)ds. \quad (5.2)
\]

The solution of this equation is called the resolvent of \( (a, \mathcal{L}) \), and denoted by \( \mathcal{G}_t x_0 = x(t) \). Note that in general
\[
\mathcal{G}_{t+s} \neq \mathcal{G}_t \circ \mathcal{G}_s.
\]

We make the following assumptions:

**(S1)** The resolvent \( \{\mathcal{G}_t : t \geq 0\} \) is of analyticity type \( (\omega_0, \theta_0) \) in the sense of [63, Definition 2.1], where \( \omega_0 \in \mathbb{R} \) and \( \theta_0 \in (0, \pi/2] \).

**(S2)** For any \( R > 0 \), there exist \( C_R > 0 \) and \( \beta \in (0,1) \) such that for all \( s > 0 \), \( \omega \in \Omega \) and \( x, y \in \mathbb{X} \) with \( \|x\|_X, \|y\|_X \leq R \)
\[
\|\Phi(s, \omega, x)\|_X + \|\Psi(s, \omega, x)\|_{L_2(l^2; \mathbb{X})} \leq \frac{C_R}{(s \wedge 1)^\beta},
\]
and
\[
\|\Phi(s, \omega, x) - \Phi(s, \omega, y)\|_X \leq \frac{C_R}{(s \wedge 1)^\beta} \|x - y\|_X;
\]
\[
\|\Psi(s, \omega, x) - \Psi(s, \omega, y)\|_{L_2(l^2; \mathbb{X})} \leq \frac{C_R}{(s \wedge 1)^\beta} \|x - y\|_X^2.
\]
For all \( s > 0, \omega \in \Omega \) and \( x \in \mathbb{X} \), it holds that
\[
\| \Phi(s, \omega, x) \|_\mathbb{X} \leq \frac{C}{(s \land 1)^\beta} (1 + \| x \|_\mathbb{X}),
\]
\[
\| \Psi(s, \omega, x) \|_{L_2(\mathbb{P}; \mathbb{X})}^2 \leq \frac{C}{(s \land 1)^\beta} (1 + \| x \|_\mathbb{X}^2).
\]

The following property of analytic resolvent \( \{ \mathcal{S}_t : t > 0 \} \) is crucial for the proof of Theorem 5.2 below (cf. [63, Corollary 2.1]).

**Proposition 5.1.** Let \( \mathcal{S}_t \) be an analytic resolvent of type \((\omega, \theta_0)\). Then for any \( T > 0 \)
\[
\sup_{t \in [0,T]} \| \mathcal{S}_t \|_{L(\mathbb{X}; \mathbb{X})} \leq C_T \tag{5.3}
\]
and for any \( t \in (0, T] \)
\[
\| \dot{\mathcal{S}}_t \|_{L(\mathbb{X}; \mathbb{X})} \leq C_T t^{-1}, \tag{5.4}
\]
where the dot denotes the operator derivative and \( \| \cdot \|_{L(\mathbb{X}; \mathbb{X})} \) denotes the norm of bounded linear operators.

By a solution of Eq.(5.1) we mean that \( X(t) \) satisfies the following stochastic Volterra equation:
\[
X(t) = \mathcal{S}_t x_0 + \int_0^t \mathcal{S}_{t-s} \Phi(s, X(s)) ds + \int_0^t \mathcal{S}_{t-s} \Psi(s, X(s)) dW(s). \tag{5.5}
\]

Let us define
\[
A(t, s, \omega, x) := \mathcal{S}_{t-s} \Phi(s, \omega, x), \quad B(t, s, \omega, x) := \mathcal{S}_{t-s} \Psi(s, \omega, x).
\]

We have:

**Theorem 5.2.** Under (S1) and (S2), there exists a unique maximal solution \((X, \tau)\) for Eq. (5.5) in the sense of Definition 3.5. Moreover, if (S3) holds, then \( \tau = +\infty, \) a.s..

**Proof.** First of all, it is easy to see by (5.3) that (H2)' and (H3)' hold with
\[
\kappa_{1,R}(t, s) = \kappa_{2,R}(t, s) = \frac{C_R}{(s \land 1)^\beta} \in \mathcal{K}_{>1}.
\]

For \( 0 \leq s < t < t', \omega \in \Omega \) and \( x \in \mathbb{X} \) with \( \| x \|_\mathbb{X} \leq R \), we have
\[
\| A(t', s, \omega, x) - A(t, s, \omega, x) \|_\mathbb{X} = \| (\mathcal{S}_{t'-s} - \mathcal{S}_{t-s}) \Phi(s, \omega, x) \|_\mathbb{X}
\leq \frac{C_R}{(s \land 1)^\beta} \| (\mathcal{S}_{t'-s} - \mathcal{S}_{t-s}) \|_{L(\mathbb{X}; \mathbb{X})}
\leq \frac{C_R}{(s \land 1)^\beta} \int_{t-s}^{t'-s} \| \dot{\mathcal{S}}_r \|_{L(\mathbb{X}; \mathbb{X})} dr
\leq \frac{C_R}{(s \land 1)^\beta} \int_{t-s}^{t'-s} \frac{1}{r} dr
\leq \frac{C_R}{(s \land 1)^\beta} \log \left( \frac{t'-s}{t-s} \right)
\]
and
\[
\| B(t', s, \omega, x) - B(t, s, \omega, x) \|_{L_2(\mathbb{P}; \mathbb{X})}^2 \leq \frac{C_R}{(s \land 1)^\beta} \log^2 \left( \frac{t'-s}{t-s} \right).
\]

Note that the following elementary inequality holds for any \( \gamma \in (0, 1) \)
\[
\log(1 + s) \leq C s^\gamma, \quad \forall s > 0.
\]
Therefore, for $0 \leq s < t < t'$, $\omega \in \Omega$ and $x \in X$ with $\|x\|_X \leq R$

$$\|A(t', s, \omega, x) - A(t, s, \omega, x)\|_X + \|B(t', s, \omega, x) - B(t, s, \omega, x)\|_{L^2(X)}^2 \leq \frac{C_R(t' - t)^\gamma}{(s \wedge 1)^\beta(t - s)^\gamma} \left[1 + \frac{(t' - t)^\gamma}{(t - s)^\gamma}\right] =: \lambda_R(t', t, s).$$

Thus, we find that (H4)' holds if $\gamma \in (0, (1 - \beta)/2)$.

Lastly, if (S3) is satisfied, it is clear that (H2) holds with $\kappa_1(t, s) = \frac{C}{(s \wedge 1)^\beta} \in \mathcal{H}_{1}$, and (H4) also holds from the above calculations. The non-explosion now follows from Theorem 3.8.

We now turn to the small perturbation of Eq. (S4) and assume that $\Phi$ and $\Psi$ are non-random. Consider

$$X_\epsilon(t) = \mathcal{S}_t x_0 + \int_0^t \mathcal{S}_{t-s} \Phi(s, X_\epsilon(s)) ds + \sqrt{\epsilon} \int_0^t \mathcal{S}_{t-s} \Psi(s, X_\epsilon(s)) dW(s).$$

In order to use Theorem 4.5 to get the LDP for $\{X_\epsilon, \epsilon \in (0, 1)\}$, we also assume

(S4) Let $\{\mathcal{S}_t : t \geq 0\}$ be an analytic resolvent of type $(\omega_0, \theta_0)$. Assume that for some $\omega_1 > \omega_0$, $0 < \theta_1 < \theta_0$, $C > 0$ and $\alpha_1 > 0$

$$|\hat{a}(\lambda)| \geq C(|\lambda - \omega_1|^{\alpha_1} + 1)^{-1}, \quad \forall \lambda \in \mathbb{C} \text{ with } |\text{arg}(\lambda - \omega)| < \theta_1,$$

where $\hat{a}$ denotes the Laplace transform of $a$. Moreover, we also assume that

$$\int_0^r a(s) ds + \int_0^t |a(r + s) - a(s)| ds \leq C_T |r|^\delta,$$

where $r, t \in [0, T]$ and $T, \delta > 0$.

We have

Theorem 5.3. **Under (S1)-(S4) and (C1), for any $x_0 \in \mathcal{D}(\mathcal{L})$, $\{X_\epsilon, \epsilon \in (0, 1)\}$ satisfies the large deviation principle in $\mathbb{C}_T(X)$ with the rate function $I(f)$ given by (4.6).**

**Proof.** From the proof of Theorem 5.2 it is enough to check (H1)" and (H2)". By (5.6) and [13] p.57, Theorem 2.2 (ii), we have

$$\|\mathcal{L}_t\|_{L^2(X)} \leq C e^{\omega_1 t}(1 + t^{-\alpha_1}), \quad \forall t > 0,$$

which together with (v) of Proposition 2.11 yields that for any $\alpha \in (0, 1)$ and $T > 0$

$$\|\mathcal{L}^\alpha \mathcal{S}_t\|_{L^2(X)} \leq C_T (1 + t^{-\alpha_1}), \quad \forall t \in (0, T].$$

Thus, (H2)" holds by choosing $\alpha = \frac{1 - \beta}{\alpha_1}$, where $\beta$ is from (S3).

For (H1)" since $x_0 \in \mathcal{D}(\mathcal{L}) = X_1$, by (5.3) we have

$$\|\mathcal{L}_t x_0\|_X = \|\mathcal{L}_t \mathcal{L} x_0\|_X \leq C \|\mathcal{L} x_0\|_X.$$

On the other hand, by the resolvent equation (5.2) and (5.7) we have, for any $0 \leq t \leq t' \leq T$

$$\|\mathcal{S}_{t'} x_0 - \mathcal{S}_t x_0\|_X \leq \int_0^t |a(t' - s) - a(t - s)| \cdot \|\mathcal{L} x_0\|_X ds$$

$$+ \int_t^{t'} |a(t' - s)| \cdot \|\mathcal{L} x_0\|_X ds \leq C_T \|\mathcal{L} x_0\|_X \cdot |t' - t|^\delta.$$
Example 5.4. Let \( a \) be a completely monotonic kernel function, i.e.,

\[
a(t) = \int_0^\infty e^{-st}d\rho(s), \quad t > 0,
\]

where \( s \mapsto \rho(s) \) is nondecreasing, and such that \( \int_1^\infty d\rho(s)/s < \infty \). Then the resolvent \( \{S_t : t \geq 0\} \) associated with \( a \) is of analyticity type \((0, \theta)\) for some \( \theta \in (0, \pi/2) \) (cf. [63, p.55, Example 2.2]), i.e., (S1) holds. For (S4), besides (5.8) and (5.7), we also assume that for some \( C, \alpha_1 > 0 \)

\[
C(1 + \lambda)^{-\alpha_1} \leq \int_0^\infty e^{-\lambda t} a(t)dt < +\infty, \quad \forall \lambda > 0,
\]

which implies by [63, p.221, Lemma 8.1 (v)] that (5.6) holds. In particular,

\[
a_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in (0, 1] \]

is completely monotonic, and satisfies (5.7) and (5.9), where \( \Gamma \) denotes the usual Gamma function.

Moreover, for the kernel function \( a_\alpha \), if

\[
1 < \alpha < 2 - \frac{2\phi}{\pi} < 2,
\]

where \( \phi \) comes from (2.19), then \( \mathcal{G}_t \) is analytic (cf. [63, p.55, Example 2.1]). Notice that in [63], \(-\mathcal{L}\) is considered. In this case, (5.6) and (5.7) clearly hold since \( \hat{a}_\alpha(\lambda) = \lambda^{-\alpha}, \Re \lambda > 0 \).

6. Semilinear stochastic partial differential equations

When \( a = 1 \) in Eq. (5.1), one sees that Eq. (5.1) contains a class of semilinear SPDEs. However, it cannot deal with the equation like stochastic Navier-Stokes equation. In this section, we shall discuss strong solutions of a large class of semilinear SPDEs by using the properties of analytic semigroups.

6.1. Mild solutions of SPDEs driven by Brownian motions. Consider the following semilinear stochastic partial differential equation:

\[
dX(t) = [-\mathcal{L}X(t) + \Phi(t, X(t))]dt + \Psi(t, X(t))dW(t), \quad X(0) = x_0.
\]

We study two cases, in application, which correspond to different types of SPDEs. First of all, we introduce the following assumptions on the coefficients:

(M1) For some \( \alpha \in (0, 1) \)

\[
\Phi : \mathbb{R}_+ \times \Omega \times \mathbb{X}_\alpha \to \mathbb{X} \in \mathcal{M} \times \mathcal{B}(\mathbb{X}_\alpha)/\mathcal{B}(\mathbb{X})
\]

and

\[
\Psi : \mathbb{R}_+ \times \Omega \times \mathbb{X}_\alpha \to L_2(l^2; \mathbb{X}_\alpha) \in \mathcal{M} \times \mathcal{B}(\mathbb{X}_\alpha)/\mathcal{B}(L_2(l^2; \mathbb{X}_\alpha)).
\]

(M2) For any \( R > 0 \), there exist \( C_R > 0 \) and \( \beta \in [0, 1) \) with

\[
\alpha + \beta < 1
\]

such that for all \( s > 0, \omega \in \Omega \) and \( x, y \in \mathbb{X}_\alpha \) with \( \|x\|_{\mathbb{X}_\alpha}, \|y\|_{\mathbb{X}_\alpha} \leq R \),

\[
\|\Phi(s, \omega, x)\|_{\mathbb{X}} + \|\Psi(s, \omega, x)\|_{L_2(l^2; \mathbb{X}_\alpha)} \leq \frac{C_R}{(s \wedge 1)^{\beta}}
\]

and

\[
\|\Phi(s, \omega, x) - \Phi(s, \omega, y)\|_{\mathbb{X}} \leq \frac{C_R}{(s \wedge 1)^{\beta}} \|x - y\|_{\mathbb{X}_\alpha},
\]

33
\[ \| \Psi(s, \omega, x) - \Psi(s, \omega, y) \|_{L_2(\mathbb{R}^2; \mathbb{X}_\omega)}^2 \leq \frac{C_R}{(s \wedge 1)^\beta} \| x - y \|_{\mathbb{X}_\alpha}^2. \]

**(M3)** For all \( s > 0, \omega \in \Omega \) and \( x \in \mathbb{X}_\alpha \), it holds that
\[
\| \Phi(s, \omega, x) \|_{\mathbb{X}} \leq \frac{C}{(s \wedge 1)^\beta} (1 + \| x \|_{\mathbb{X}_\alpha}),
\]
\[
\| \Psi(s, \omega, x) \|_{L_2(\mathbb{R}^2; \mathbb{X}_\omega)}^2 \leq \frac{C}{(s \wedge 1)^\beta} (1 + \| x \|_{\mathbb{X}_\alpha}^2).
\]

By a mild solution of equation \((6.1)\) we mean that \( X(t) \) solves the following stochastic Volterra integral equation:
\[
X(t) = \Xi_t x_0 + \int_0^t \Xi_{t-s} \Phi(s, X(s)) ds + \int_0^t \Xi_{t-s} \Psi(s, X(s)) dW(s). \tag{6.2}
\]

**Theorem 6.1.** Under **(M1)** and **(M2)**, for any \( x_0 \in \mathbb{X}_\alpha \) (\( \alpha \) is from **(M1)**), there exists a unique maximal solution \((X, \tau)\) for Eq. \((6.2)\) so that

(i) \( t \mapsto X(t) \in \mathbb{X}_\alpha \) is continuous on \([0, \tau)\) almost surely;

(ii) \( \lim_{t \uparrow \tau} \| X(t) \|_{\mathbb{X}_\alpha} = +\infty \) on \( \{ \omega : \tau(\omega) < +\infty \} \);

(iii) it holds that, \( P\text{-a.s.}, \) on \([0, \tau)\)
\[
X(t) = \Xi_t x_0 + \int_0^t \Xi_{t-s} \Phi(s, X(s)) ds + \int_0^t \Xi_{t-s} \Psi(s, X(s)) dW(s).
\]

Moreover, if **(M3)** holds, then \( \tau = +\infty, \) a.s.

**Proof.** We first consider the following stochastic Volterra integral equation
\[
Y(t) = \mathcal{L}^\alpha \Xi_t x_0 + \int_0^t \mathcal{L}^\alpha \Xi_{t-s} \Phi(s, \mathcal{L}^{-\alpha} Y(s)) ds
\]
\[
+ \int_0^t \mathcal{L}^\alpha \Xi_{t-s} \Psi(s, \mathcal{L}^{-\alpha} Y(s)) dW(s). \tag{6.3}
\]

Define
\[
g(t) := \mathcal{L}^\alpha \Xi_t x_0,
\]
\[
A(t, s, \omega, y) := \mathcal{L}^\alpha \Xi_{t-s} \Phi(s, \omega, \mathcal{L}^{-\alpha} y),
\]
\[
B(t, s, \omega, y) := \mathcal{L}^\alpha \Xi_{t-s} \Psi(s, \omega, \mathcal{L}^{-\alpha} y).
\]

Let us verify **(H1)**\(^{\prime}\)-**(H4)**. Clearly, **(H1)**\(^{\prime}\) holds since \( x_0 \in \mathbb{X}_\alpha \).

By (iii) of Proposition 2.11 and **(M2)**, for all \( t > s > 0, \omega \in \Omega \) and \( x, y \in \mathbb{X} \) with \( \| x \|_{\mathbb{X}}, \| y \|_{\mathbb{X}} \leq R \) we have
\[
\| A(t, s, \omega, x) \|_{\mathbb{X}} + \| B(t, s, \omega, x) \|_{L_2(\mathbb{R}^2; \mathbb{X})} \leq \frac{1}{(t - s)^\alpha} \left( \| \Phi(s, \omega, \mathcal{L}^{-\alpha} x) \|_{\mathbb{X}} + \| \Psi(s, \omega, \mathcal{L}^{-\alpha} x) \|_{L_2(\mathbb{R}^2; \mathbb{X}_\omega)} \right)
\]
\[
\leq \frac{C_R}{(t - s)^\alpha (s \wedge 1)^\beta}, \tag{6.4}
\]
and
\[
\| A(t, s, \omega, x) - A(t, s, \omega, y) \|_{\mathbb{X}} \leq \frac{1}{(t - s)^\alpha} \| \Phi(s, \omega, \mathcal{L}^{-\alpha} x) - \Phi(s, \omega, \mathcal{L}^{-\alpha} y) \|_{\mathbb{X}}
\]
\[
\leq \frac{C_R}{(t - s)^\alpha (s \wedge 1)^\beta} \| \mathcal{L}^{-\alpha} x - \mathcal{L}^{-\alpha} y \|_{\mathbb{X}_\alpha}.
\]
Hence, if we take and (iii) in the theorem.

Set

\[ \kappa \]

as well as

\[ \frac{1}{(t-s)^{\alpha}} \| \Phi(s, \omega, L^{-\alpha}x) - \Phi(s, \omega, L^{-\alpha}y) \|_{L^2(\Omega; X)} \]

\[ \leq \frac{C_R}{(t-s)^{\alpha}(s \land 1)^{\beta}} \| x - y \|_X. \]

Hence, if we take

\[ \kappa_1(t, s) = \kappa_2(t, s) := \frac{C_R}{(t-s)^{\alpha}(s \land 1)^{\beta}} \in \mathcal{K}_{\geq 1}, \]

then (H2)' and (H3)' hold.

Let \( 0 < \gamma < 1 - (\alpha + \beta) \). By (iv) of Proposition \( \underline{2.11} \) and (M2) we have

\[ \| A(t', s, \omega, x) - A(t, s, \omega, x) \|_X = \| (\mathcal{F}_{t-t} - 1) L^{\alpha} \mathcal{F}_{t-s} \Phi(s, \omega, L^{-\alpha}x) \|_X \]

\[ \leq (t' - t)\gamma \| L^{\alpha+\gamma} \mathcal{F}_{t-s} \Phi(s, \omega, L^{-\alpha}x) \|_X \]

\[ \leq \frac{C_R(t' - t)\gamma}{(t-s)^{\alpha+\gamma}(s \land 1)^{\beta}}. \]

and

\[ \| B(t', s, \omega, x) - B(t, s, \omega, x) \|_{L^2(\Omega; X)} \]

\[ \leq \| (\mathcal{F}_{t-t} - 1) L^{\alpha} \mathcal{F}_{t-s} \Phi(s, \omega, L^{-\alpha}x) \|_{L^2(\Omega; X)} \]

\[ \leq (t' - t)\gamma \| L^{\alpha+\gamma/2} \mathcal{F}_{t-s} \Phi(s, \omega, L^{-\alpha}x) \|_{L^2(\Omega; X)} \]

\[ \leq \frac{C_R(t' - t)\gamma}{(t-s)^{\alpha+\gamma/2}(s \land 1)^{\beta}}. \]

So, if we take

\[ \lambda_R(t', t, s) := \frac{C_R(t' - t)\gamma}{(t-s)^{\alpha+\gamma}(s \land 1)^{\beta}}, \]

then (H4)' holds.

Hence, by Theorem \( \underline{3.7} \) there is a unique maximal solution \((Y, \tau)\) for Eq. \( \underline{6.3} \) in the sense of Definition \( \underline{3.5} \). Set

\[ X(t) = L^{-\alpha}Y(t). \]

It is easy to see that \((X, \tau)\) a unique maximal solution for Eq. \( \underline{6.2} \), which satisfies (i), (ii) and (iii) in the theorem.

Lastly, if (M3) is satisfied, then as estimating \( \underline{6.1} \), for the above \( A \) and \( B \), (H2) holds with some \( \kappa_1 \in \mathcal{K}_{\geq 1} \), and also (H4) holds. So, by Theorem \( \underline{3.8} \) we have \( \tau = \infty \) a.s.. \( \square \)

**Remark 6.2.** The solution \((X, \tau)\) in Theorem \( \underline{6.7} \) is clearly a local solution of Eq. \( \underline{6.3} \) in \( X \). However, it may be not a maximal solution in \( X \) because it may happen that

\[ \lim_{t \uparrow \tau(\omega)} \| X(t, \omega) \|_X < +\infty \text{ on } \{ \omega : \tau(\omega) < +\infty \}. \]
Next, we study the large deviation estimate for Eq. (6.1), and assume that $\Phi$ and $\Psi$ are non-random. Consider the following small perturbation of Eq. (6.1):
\[dX^\epsilon(t) = [-\mathcal{L}X^\epsilon(t) + \Phi(t, X^\epsilon(t))]dt + \sqrt{\epsilon}\Psi(t, X^\epsilon(t))dW(t), \quad X^\epsilon(0) = x_0. \tag{6.5}\]

In order to apply Theorem 4.5 to this situation, we need the non-explosion assumptions as (C2) and (C3). For a family of processes $\{h^\epsilon, \epsilon \in (0, 1)\}$ in $A_T^\alpha$ (see (2.23) for the definition of $A_T^\alpha$), consider
\[
X^\epsilon(t) = \mathcal{I}_t x_0 + \int_0^t \mathcal{I}_{t-s} \Phi(s, X^\epsilon(s))ds + \int_0^t \mathcal{I}_{t-s} \Psi(s, X^\epsilon(s))h^\epsilon(s)ds + \sqrt{\epsilon} \int_0^t \mathcal{I}_{t-s} \Psi(s, X^\epsilon(s))dW(s),
\]
and for $h \in \ell_2^1$ (see (2.22))
\[
X^h(t) = \mathcal{I}_t x_0 + \int_0^t \mathcal{I}_{t-s} \Phi(s, X^h(s))ds + \int_0^t \mathcal{I}_{t-s} \Psi(s, X^h(s))h(s)ds.
\]
Below, for $n \in \mathbb{N}$ we define
\[
\tau_n^\epsilon := \inf\{t > 0 : \|X^\epsilon(t)\|_{X_\alpha} > n\}.
\]

Our large deviation principle can be stated as follows:

**Theorem 6.3.** Assume (M1) and (M2). Let $x_0 \in X_\delta$ for some $1 \geq \delta > \alpha$, where $\alpha$ is from (M1). We also assume that $\mathcal{D}(\mathcal{L}) = X_1 \subset X$ is compact, and
\[
\lim_{n \to \infty} \sup_{\epsilon \in (0, 1)} P\{\omega : \tau_n^\epsilon(\omega) < T\} = 0 \tag{6.6}
\]
and for any $N > 0$
\[
\sup_{h \in \mathbb{N}} \sup_{t \in [0, T]} \|X^h(t)\|_{X_\alpha} < +\infty. \tag{6.7}
\]
Then, $\{X_{\epsilon}, \epsilon \in (0, 1)\}$ satisfies the large deviation principle in $C_T(X_\alpha)$ with the rate function $I(f)$ given by
\[
I(f) := \frac{1}{2} \inf_{\{h \in \ell_2^1 : f = X^h\}} \|h\|^2_{\ell_2^1}, \quad f \in C_T(X_\alpha). \tag{6.8}
\]

**Proof.** By Theorem 4.5 it only need to check (H1)" and (H2)" for Eq. (6.3). Since $x_0 \in X_\delta$ with $\delta > \alpha$, by (iv) of Proposition 2.11 (H1)" holds with $\delta' = \delta - \alpha$ and $\alpha' \in (0, \delta - \alpha)$. As the calculations given in (6.4), one finds that (H2)" holds with $\alpha' \in (0, 1 - \alpha - \beta)$. □

**Remark 6.4.** If (M3) is satisfied, one can see that (6.6) and (6.7) hold by Remark 4.0.

We now consider another group of assumptions on the coefficients:

(M1') For some $\alpha \in (0, 1)$
\[
\Phi : \mathbb{R}_+ \times \Omega \times X \to X_{-\alpha} \in \mathcal{M} \times \mathcal{B}(X)/\mathcal{B}(X_{-\alpha})
\]
and
\[
\Psi : \mathbb{R}_+ \times \Omega \times X \to L_2(\ell^2; X_{-\alpha}) \in \mathcal{M} \times \mathcal{B}(X)/\mathcal{B}(L_2(\ell^2; X_{-\alpha})).
\]

(M2') For any $R > 0$, there exist $C_R > 0$ and $\beta \in (0, 1)$ with
\[
\alpha + \beta < 1
\]
such that for all $s > 0$, $\omega \in \Omega$ and $x, y \in X$ with $\|x\|_X, \|y\|_X \leq R$,
\[
\|\Phi(s, \omega, x)\|_{X_{-\alpha}} + \|\Psi(s, \omega, x)\|_{L_2(\ell^2; X_{-\alpha})} \leq \frac{C_R}{(s \wedge 1)^\beta}
\]
and
\[ \| \Phi(s, \omega, x) - \Phi(s, \omega, y) \|_{\mathbb{X}_\alpha} \leq \frac{C_R}{(s \wedge 1)^\beta} \| x - y \|_X, \]
\[ \| \Psi(s, \omega, x) - \Psi(s, \omega, y) \|_{L_2([t, \tau]; \mathbb{X}_{\alpha})}^2 \leq \frac{C_R}{(s \wedge 1)^\beta} \| x - y \|_X^2. \]

(M3)' For all \( s > 0, \omega \in \Omega \) and \( x \in \mathbb{X} \), it holds that
\[ \| \Phi(s, \omega, x) \|_{\mathbb{X}_\alpha} \leq \frac{C}{(s \wedge 1)^\beta} (1 + \| x \|_X), \]
\[ \| \Psi(s, \omega, x) \|_{L_2([t, \tau]; \mathbb{X}_{\alpha})}^2 \leq \frac{C}{(s \wedge 1)^\beta} (1 + \| x \|_X)^2. \]

The following two results are parallel to Theorems 6.1 and 6.3, we omit the details.

**Theorem 6.5.** Under (M1)' and (M2)', for any \( x_0 \in \mathbb{X} \), there exists a unique maximal mild solution \( (X, \tau) \) for Eq. (6.2) in the sense of Definition 3.8. Moreover, if (M3)' holds, then \( \tau = +\infty \), a.s..

**Theorem 6.6.** Assume that (M1)', (M2)' and (C1)-(C3) hold. Let \( x_0 \in \mathbb{X}_\delta \) for some \( \delta > 0 \). Then, \( \{X_\epsilon, \epsilon \in (0, 1)\} \) satisfies the large deviation principle in \( \mathbb{C}_T(\mathbb{X}) \) with the rate function \( I(f) \) given by (4.6).

**Remark 6.7.** Theorem 6.3 is due to Brzeźniak [11]. Compared with Theorem 6.4, the solution in Theorem 6.6 has better regularity, and is in fact a strong solution under a slightly stronger assumption (M4) below.

**6.2. Strong solutions of SPDEs driven by Brownian motions.** In this subsection, following the method used in the deterministic case (cf. [37, 58]), we prove the existence of strong solutions for Eq. (6.1). For this aim, in addition to (M1) and (M2) with \( \beta = 0 \), we also assume

(M4) For any \( R, T > 0 \), there exist \( \delta > 0 \) and \( \alpha' > 1 \) such that for all \( s, s' \in [0, T], \omega \in \Omega \) and \( x \in \mathbb{X}_\alpha \) with \( \| x \|_{\mathbb{X}_\alpha} \leq R \)
\[ \| \Phi(s', \omega, x) - \Phi(s, \omega, x) \|_X \leq C_{T, R}|s' - s|^\delta, \quad (6.9) \]
\[ \| \Psi(s, \omega, x) \|_{L_2([t, \tau]; \mathbb{X}_{\alpha'})} \leq C_{T, R}. \quad (6.10) \]

Let us first recall the following result (cf. [37, Theorem 3.2.2] or [58, p.114, Theorem 3.5]).

**Lemma 6.8.** Let \( [0, T] \ni s \mapsto f(s) \in \mathbb{X} \) be a Hölder continuous function. Then
\[ t \mapsto \int_0^t \mathbb{X}_{t-s}f(s)ds \in C([0, T]; \mathbb{X}_1). \]

Using this lemma, we can prove the existence of strong solutions for Eq. (6.1).

**Theorem 6.9.** Assume that (M1), (M2) and (M4) hold. For any \( x_0 \in \mathbb{X}_1 \), let \( (X, \tau) \) be the unique maximal solution of Eq. (6.2) in Theorem 6.6. Then

(i) \( t \mapsto X(t) \in \mathbb{X}_1 \) is continuous on \( [0, \tau) \) a.s.;
(ii) it holds that in \( \mathbb{X} \)
\[ X(t) = x_0 - \int_0^t \mathbb{L}X(s)ds + \int_0^t \Phi(s, X(s))ds + \int_0^t \Psi(s, X(s))dW(s) \]
for all \( t \in [0, \tau), P\text{-a.s.} \).

We shall call \( (X, \tau) \) the unique maximal strong solution of Eq. (6.1).
Proof. For $n \in \mathbb{N}$, set
\[
\tau_n := \inf\{ t > 0 : \|X(t)\|_{X_2} > n \}
\]
and
\[
G(t, s) := \Xi_{t-s} \Psi(s, X(s)).
\]
Then by (iii) and (iv) of Proposition 2.11 we have
\[
\|G(t, s)\|_{L_2(\mathbb{R}^2;\mathbb{X}_1)} \leq \frac{1}{(t - s)^{2-\alpha'}} \|\Psi(s, X(s))\|_{L_2(\mathbb{R}^2;\mathbb{X}_{2/\alpha'})}^2,
\]
and in view of $\alpha' > 1$
\[
\|G(t', s) - G(t, s)\|_{L_2(\mathbb{R}^2;\mathbb{X}_1)} \leq \frac{(t' - t)^{(\alpha'-1)/2}}{(t - s)^{(3-\alpha')/2}} \|\Psi(s, X(s))\|_{L_2(\mathbb{R}^2;\mathbb{X}_{2/\alpha'})}^2.
\]
Hence, by Lemma 3.4 and (6.10),
\[
t \mapsto \int_0^t \Xi_{t-s} \Psi(s, X(s)) dW(s) \in \mathbb{X}_1
\]
admits a continuous modification on $[0, \tau_n]$.
Moreover, starting from (6.3), as in the proof of Theorem 3.3 there exists an $a > 0$ such that for $p$ sufficiently large
\[
\mathbb{E} \left( \sup_{t \neq t' \in [0, T \wedge \tau_n]} \frac{\|X(t') - X(t)\|_{X_a}^p}{|t' - t|^{ap}} \right) \leq C_{n,T,p}.
\]
Thus, by (M2) and (M4) we know that
\[
s \mapsto \Phi(s, X(s)) \in \mathbb{X} \text{ is H"{o}lder continuous on } [0, T \wedge \tau_n] \text{ P-a.s.}
\]
Therefore, by Lemma 6.8 we have
\[
t \mapsto \int_0^t \Xi_{t-s} \Phi(s, X(s)) ds \in C([0, T \wedge \tau_n], \mathbb{X}_1), \text{ P - a.s.}
\]
Noting that $x_0 \in \mathbb{X}_1$ and
\[
1_{\{t \leq \tau_n\}} \cdot X(t) = 1_{\{t \leq \tau_n\}} \cdot \Xi_t x_0 + 1_{\{t \leq \tau_n\}} \cdot \int_0^t \Xi_{t-s} \Phi(s, X(s)) ds \\
+ 1_{\{t \leq \tau_n\}} \cdot \int_0^t \Xi_{t-s} \Psi(s, X(s)) dW(s), \text{ } \forall t \geq 0, \text{ P - a.s.}
\]
by $\tau_n / \nearrow \tau$, we therefore have that $t \mapsto X(t) \in \mathbb{X}_1$ is continuous on $[0, \tau)$ P-a.s..
Lastly, by stochastic Fubini’s theorem (cf. [54, Section 6]) we have
\[
\int_0^t \mathcal{L} X(s) ds = \int_0^t \mathcal{L} \Xi_{s} x_0 ds + \int_0^t \int_0^s \mathcal{L} \Xi_{s-r} \Phi(r, X(r)) dr ds \\
+ \int_0^t \int_0^s \mathcal{L} \Xi_{s-r} \Psi(r, X(r)) dW(r) ds \\
= x_0 - \Xi_t x_0 + \int_0^t \int_r^t \mathcal{L} \Xi_{s-r} \Phi(r, X(r)) ds dr \\
+ \int_0^t \int_r^t \mathcal{L} \Xi_{s-r} \Psi(r, X(r)) ds dW(r) \\
= x_0 - \Xi_t x_0 + \int_0^t \left[ \Phi(r, X(r)) - \Xi_{t-r} \Phi(r, X(r)) \right] dr
\]
which has the covariance function
\[ C(t) = \begin{cases} \frac{2H(3/2-H)}{3(2-2H)^2} t^{2H} + s^{2H} - \frac{1}{2} s^{H-\frac{1}{2}} f(t/s) \end{cases} 1_{\{s \leq t\}}, \quad s, t \in [0, 1], \]
where \( F(u) := c_H (t/u - H) \int_1^u (r - 1)^H (1 - r^H - \frac{1}{2}) dr. \)

The sequence of independent fractional Brownian motions with Hurst parameter \( H \in (0, 1) \) may be defined by (cf. [21])
\[ W_H^k(t) := \int_0^t K_H(t, s) dW^k(s), \quad k = 1, 2, \cdots, \]
which has the covariance function
\[ R_H(t, s) = \mathbb{E}(W_H^k(t) W_H^k(s)) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}). \]

Consider the following stochastic partial differential equation driven by \( \{W_H^k, k \in \mathbb{N}\} \)
\[ dX(t) = [-\Delta X(t) + \Phi(t, X(t))] dt + \Psi(t) dW_H(t), \quad X(0) = x_0 \in \mathbb{X}, \quad (6.11) \]
where \( \Psi(t) \) is a deterministic function and will be specified in Theorem 6.10.

As above, we consider the mild solution:
\[ X(t) = \mathcal{T}_t x_0 + \int_0^t \mathcal{T}_{t-s} \Phi(s, X(s)) ds + \int_0^t \mathcal{T}_{t-s} \Psi(s) dW_H(s). \quad (6.12) \]

Here the stochastic integral is defined by the integration by parts formula as
\[ \int_0^t \mathcal{T}_{t-s} \Psi(s) dW_H(s) := \Psi(t) W^H(t) + \int_0^t W^H(s) [\mathcal{L} \mathcal{T}_{t-s} \Psi(s) - \mathcal{T}_{t-s} \Psi(s)] ds \]
\[= \int_0^t B(t, s) dW(s), \]
where
\[ B(t, s) := \Psi(t) K_H(t, s) + \int_s^t K_H(u, s) \mathcal{L} \mathcal{T}_{t-u} \Psi(u) \mathcal{T}_{t-s} \Psi(s) du. \]

We also define
\[ A(t, s, x) := \mathcal{T}_{t-s} \Phi(s, x). \]

Then we have:

**Theorem 6.10.** *Assume that \( \Phi \) satisfies (M1)' and (M2)' and \( \Psi \) satisfies for some \( \gamma > 0 \)
\[ \|\Psi(t') - \Psi(t)\|_{L^2(\mathcal{T}; \mathcal{X})} \leq C |t' - t|^\gamma, \quad t, t' \in [0, 1] \]
and for some \( \delta \in (0, 1) \)
\[ \sup_{t \in [0, 1]} \left( \|\Psi(t')\|_{L^2(\mathcal{T}; \mathcal{X}_\delta)} + \|\Psi(t)\|_{L^2(\mathcal{T}; \mathcal{X}_{\delta-1})} \right) < +\infty, \quad t \in [0, 1]. \]
Then for any \( x_0 \in \mathbb{X} \), there exists a unique maximal solution for Eq. \((6.12)\) in the sense of Definition \(3.7\). In particular, if \( \Phi \) also satisfies \((M3)’\), then there is no explosion for Eq. \((6.12)\).

**Proof.** As the proof of Theorem \(6.1\) one can check that \( A \) satisfies \((H2)’-(H4)’\). In order to finish the proof by Theorem \(3.7\) we need to verify that \( B \) also satisfies \((H2)\) and \((H4)\). We first check that for some \( \gamma’ > 0 \)

\[
\int_0^t \|B(t', s) - B(t, s)\|_{L_2(\mathbb{R}^2; \mathbb{X})}^2 ds \leq |t - t'|^{\gamma’}, \quad 0 \leq t < t' \leq 1. \tag{6.15}
\]

Noting that

\[
K_H(t, s) \leq C|t - s|^{H - \frac{1}{2}} + Cs^{-|H - \frac{1}{2}|} \tag{6.16}
\]

and

\[
\int_0^t [K_H(t, s) - K_H(t', s)]^2 ds \leq R_H(t, t) - 2R_H(t, t') + R_H(t', t') \tag{6.13}
\]

by \((6.13)\) we have

\[
\int_0^t \|\Psi(t')K_H(t', s) - \Psi(t)K_H(t, s)\|^2_{L_2(\mathbb{R}^2; \mathbb{X})} ds \leq |t' - t|^{2H \wedge \gamma'}.
\]

Observe that

\[
\int_0^t \left\| \int_s^{t'} K_H(u, s)\mathcal{L}\xi_{t-u} \Psi(u) du \right\|^2_{L_2(\mathbb{R}^2; \mathbb{X})} ds \\
\leq \int_0^t \left[ \int_s^{t'} K_H(u, s) \cdot \|\mathcal{L}\xi_{t-u} \Psi(u)\|_{L_2(\mathbb{R}^2; \mathbb{X})} du \right]^2 ds \\
+ \int_0^t \left[ \int_s^{t'} K_H(u, s) \cdot \|\mathcal{L}(\xi_{t-u} - \xi_{t-u}) \Psi(u)\|_{L_2(\mathbb{R}^2; \mathbb{X})} du \right]^2 ds \\
=: J_1 + J_2.
\]

By \((6.16)\), (iii) (iv) of Proposition \(2.11\) and \((6.14)\), we have

\[
J_1 \leq \int_0^t \left[ \int_s^{t'} [(u - s)^{H - \frac{1}{2}} + s^{-|H - \frac{1}{2}|}] \cdot (t' - u)^{\delta - 1} du \right]^2 ds \\
\leq \int_0^t \left[ (t - s)^{H - \frac{1}{2}} + s^{-|H - \frac{1}{2}|} \right]^2 \cdot \left[ \int_s^{t'} (t' - u)^{\delta - 1} du \right]^2 ds \\
\leq |t' - t|^\delta
\]

and

\[
J_2 \leq \int_0^t \left[ \int_s^{t'} [(u - s)^{H - \frac{1}{2}} + s^{-|H - \frac{1}{2}|}] \cdot (t' - t)^{\frac{\delta}{2}} \cdot (t - u)^{\frac{\delta}{2} - 1} du \right]^2 ds \\
\leq (t' - t)^{\delta} \int_0^t \left[ \int_s^{t'} (u - s)^{H - \frac{1}{2}} (t - u)^{\frac{\delta}{2} - 1} du \right]^2 ds
\]
\[(t' - t)^{\delta} \int_0^t s^{-2|H - \frac{1}{2}|} \left| \int_s^t (t - u)^{\frac{H}{2} - 1} du \right|^2 ds \]
\[= J_{21} + J_{22}.\]

It is clear that
\[J_{22} \leq (t' - t)^{\delta} \int_0^t s^{-2|H - \frac{1}{2}|} (t - s)^{\delta} ds \leq (t' - t)^{\delta}.\]

For \(J_{21}\), let us make the following elementary estimation:
\[\int_0^t \left\| \int_s^t (u - s)^{H - \frac{1}{2}} (t - u)^{\frac{H}{2} - 1} du \right\|^2 ds \]
\[\leq \int_0^t \left\| \int_s^{t + \frac{H - \frac{1}{2}}{2}} (u - s)^{H - \frac{1}{2}} (t - u)^{\frac{H}{2} - 1} du \right\|^2 ds \]
\[+ \int_0^t \left\| \int_s^{t - \frac{H - \frac{1}{2}}{2}} (u - s)^{H - \frac{1}{2}} (t - u)^{\frac{H}{2} - 1} du \right\|^2 ds \]
\[\leq \int_0^t (t - s)^{\delta - 2} \left\| \int_s^{t + \frac{H - \frac{1}{2}}{2}} (u - s)^{H - \frac{1}{2}} du \right\|^2 ds \]
\[+ \int_0^t (t - s)^{2H - 1} \left\| \int_s^{t - \frac{H - \frac{1}{2}}{2}} (u - s)^{\frac{H}{2} - 1} du \right\|^2 ds \]
\[\leq \int_0^t (t - s)^{2H + \delta - 1} ds \leq C.\]

Hence
\[J_1 + J_2 \leq C(t' - t)^{\delta}.\]

Similarly, we have
\[\int_0^t \left\| \int_s^t K_H(u, s) \Sigma_{t-u} \Psi(u) du - \int_s^t K_H(u, s) \Sigma_{t-u} \Psi(u) du \right\|_{L_2(\mathbb{R}; \mathcal{X})}^2 ds \]
\[\leq C(t' - t)^{\delta}.\]

Summing up the above calculations, we get (6.15). Thus, \(B\) satisfies (H4)'.

Moreover, from the above calculations, one can see that
\[\|B(t, s)\|^2_{L_2(\mathbb{R}; \mathcal{X})} \leq C \left( |t - s|^{2H - 1} + s^{-[2H - 1]} + |t - s|^{2H + \delta - 1} \right) =: \kappa(t, s).\]

So, \(B\) satisfies (H2)' with \(\kappa \in \mathcal{K}_{\geq 1}.\)

Lastly, if \(\Phi\) also satisfies (M3)', then the non-explosion follows from Theorem 3.8. □

Consider the following small perturbation of Eq. (6.12):
\[X_\epsilon(t) = \Sigma_t x_0 + \int_0^t \Sigma_{t-s} \Phi(s, X_\epsilon(s)) ds + \sqrt{\epsilon} \int_0^t \Sigma_{t-s} \Psi(s) dW_H(s).\]

A direct application of Theorem 4.5 yields that

**Theorem 6.11.** Keep the same assumptions as Theorem 6.10, where \(\Phi\) satisfies (M1)'- (M3)'. Then for any \(x_0 \in \mathcal{X}_\alpha (\alpha > 0)\), \(\{X_\epsilon, \epsilon \in (0, 1)\}\) satisfies the large deviation principle in \(\mathcal{C}_1(\mathcal{X})\) with the rate function \(I(f)\) given by
\[I(f) := \frac{1}{2} \inf_{\{h \in \ell^2_1 : \int_X h^2 \}} \|h\|^2_{\ell^2_1}, \quad f \in \mathcal{C}_1(\mathcal{X}),\]
where $X^h$ solves the following integral equation

$$X^h(t) = \mathfrak{T}_t x_0 + \int_0^t \mathfrak{T}_{t-s} \Phi(s, X^h(s)) ds + \int_0^t B(t, s) \hat{h}(s) ds.$$ 

**Remark 6.12.** Let $\Psi_0 \in L_2(l^2, \mathcal{X}_\delta)$ for some $\delta \in (0, 1)$. Then $\Psi(t) := \mathfrak{T}_t \Psi_0$ satisfies (6.13) and (6.14) by Proposition 2.11. Moreover, under stronger assumptions on $\Psi(t)$, we can also prove the existence of strong solutions for Eq. (6.12) as Theorem 6.9.

7. Application to SPDEs in bounded domains of $\mathbb{R}^d$

Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^d$ with smooth boundary. For $m \in \mathbb{N}$, by $C^m(\mathcal{O})$ (resp. $C_0^m(\mathcal{O})$) we denote the set of all $m$-times continuously differentiable functions in $\mathcal{O}$ (resp. with compact support in $\mathcal{O}$). For $u \in C^m(\mathcal{O})$ and $p \geq 1$ we define

$$\|u\|_{m,p} := \left( \sum_{j=0}^{m} \int_{\mathcal{O}} |D^j u(x)|^p dx \right)^{1/p},$$

where $D^j$ is the usual derivative operator. The Sobolev spaces $W^{m,p}(\mathcal{O})$ and $W_0^{m,p}(\mathcal{O})$ are defined respectively as the completions of $C^m(\mathcal{O})$ and $C_0^m(\mathcal{O})$ with respect to the norm $\| \cdot \|_{m,p}$.

Let $\mathcal{A}(x, D)$ be a strongly elliptic differential operator in $\mathcal{O}$ of the form (cf. [27, 58]):

$$\mathcal{A}(x, D) u := \sum_{k=0}^{2m} \sum_{\alpha_1+\cdots+\alpha_d=k} a_{\alpha_1 \cdots \alpha_d}(x) D_{\alpha_1}^1 \cdots D_{\alpha_d}^d u, \quad m \geq 1,$$

where $a_{\alpha_1 \cdots \alpha_d}(x) \in C^\infty(\mathcal{O})$, and $D_{\alpha_j}^j$ is the $\alpha_j$-order derivative with respect to the $j$-th variable. We consider the following stochastic partial differential equation:

$$\begin{cases}
\frac{du(t, x)}{dt} = \left[ \mathcal{A}(x, D) u(t) + \varphi(t, x, u, Du, \cdots, D^{2m-1}u) \right] dt \\
\quad + \psi(t, x, u, Du, \cdots, D^{m-1}u) dW(t), \\
\frac{\partial^j u(t, x)}{\partial \nu^j} = 0, \quad j = 0, 1, \cdots, m-1, \quad x \in \partial \mathcal{O}, \\
u(0, x) = u_0(x),
\end{cases} \tag{7.1}$$

where $\frac{\partial^j}{\partial \nu^j}$ denotes the $j$-th outward normal derivative, $\varphi$ and $\psi$ are two measurable functions with the entries:

$$\varphi: \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \cdots \times \mathbb{R}^{(2m-1)d} \to \mathbb{R},
\psi: \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \cdots \times \mathbb{R}^{(m-1)d} \to l^2.$$ 

Define for $p > 1$ and $\lambda > 0$

$$\mathcal{L}_p u := \lambda u - \mathcal{A}(x, D) u$$

with

$$u \in \mathcal{D}(\mathcal{L}_p) := W_0^{2m,p}(\mathcal{O}) \cap W^{m,p}(\mathcal{O}).$$

It is well known that for $u \in \mathcal{D}(\mathcal{L}_p)$ (cf. [58, p. 212, Theorem 3.1] or [76])

$$\|u\|_{2m,p} \leq \|\mathcal{L}_p u\|_{L^p} + \|u\|_{L^p}, \tag{7.2}$$

and $(\mathcal{L}_p, \mathcal{D}(\mathcal{L}_p))$ is a sectorial operator on $X^0_0 = L^p(\mathcal{O})$ with $0 \in \rho(\mathcal{L}_p)$ for $\lambda$ large enough (cf. [58, p. 213, Theorem 3.5]). Below we shall write for $p > 1$ and $\alpha \geq 0$

$$X_0^p := \mathcal{D}(\mathcal{L}_p).$$

We first recall the following well known result (cf. [58, p.243]).
Lemma 7.1. For any $p > 1$, $j < 2m$ and any $0 \leq \alpha' < \frac{d}{2m} < \alpha \leq 1$ we have

$$\|\mathcal{L}_p^\alpha u\|_{L_p} \leq \|u\|_{j,p} \leq \|\mathcal{L}_p^\alpha u\|_{L_p}, \quad u \in \mathcal{D}(\mathcal{L}_p^\alpha).$$  \hspace{1cm} (7.3)

Moreover,

$$\mathbb{X}_\alpha^p \hookrightarrow W^{k,q} \quad \text{for} \quad k - \frac{d}{q} < 2m\alpha - \frac{d}{p}, \quad q \geq p$$

and

$$\mathbb{X}_\alpha^p \hookrightarrow C^\nu(\mathcal{O}) \quad \text{for} \quad 0 \leq \nu < 2m\alpha - \frac{d}{p},$$  \hspace{1cm} (7.4)

where $C^\nu(\mathcal{O})$ is the usual Hölder space (cf. [1, 18]).

In this section, we fix

$$p > d \quad \text{and} \quad \frac{2m - 1 + \frac{d}{\nu}}{2m} < \alpha_0 < \alpha < 1$$

so that

$$m(1 - \alpha)^2 < (\alpha - \alpha_0).$$  \hspace{1cm} (7.5)

Suppose that

(F1) For any $T, R > 0$, there exist $\delta > 0$ and $C_{R,T} > 0$ such that for all $s, t \in [0, T]$, $x \in \mathcal{O}$ and $U, V \in \mathbb{R}^{m(2m-1)d+1}$ with $|U|, |V| \leq R$

$$|\varphi(t, x, U) - \varphi(s, x, V)| \leq C_{R,T}(|t - s|^\delta + |U - V|).$$

Moreover, $\sup_{x \in \mathcal{O}} |\varphi(0, x, 0)| < +\infty$.

(F2) For each $t \in \mathbb{R}_+$, $\psi(t, \cdot) \in C^{\nu+1}(\mathcal{O} \times \mathbb{R}^{m(m+1)d/2 + d+1}; L^p)$. Here and below, the asterisk stands for the rest variables.

(F3) For each $u \in \mathbb{X}_{\alpha_0}^p$,

$$\psi(t, \cdot, u, Du, \ldots, D^{m-1}u) \in \mathbb{X}_{\frac{\alpha}{\nu}}^p.$$

(F4) For any $T > 0$, there exist constant $C_T > 0$ and $\lambda_0 \in L^p(\mathcal{O})$ such that for all $t \in [0, T]$, $x \in \mathcal{O}$ and $U \in \mathbb{R}^{m(2m-1)d+1}$

$$|\varphi(t, x, U)| \leq C_T(\lambda_0(x) + |U|),$$  \hspace{1cm} (7.7)

and

$$\psi(t, x, u, Du, \ldots, D^{m-1}u) = \sum_{j=0}^{m-1} g_j(t) \cdot D^j u + \psi_0(t, x), \quad m \geq 2,$$

for some $\delta > 0$ and each $r \in \mathbb{R}$, $\supp(\psi(t, \cdot, r)) \subset \mathcal{O}_\delta$,

$$\psi(t, \cdot) \in C^2(\mathcal{O} \times \mathbb{R}^{d+1}; L^2), \quad \|\partial_r \psi(t, x, r)\|_{L^2} \leq C_T,$$

$$\|D_x \psi(t, x, r)\|_{L^2 \times \mathbb{R}^d} + \|\psi(t, x, r)\|_{L^2} \leq C(f_0(t, x) + |r|)$$

where $\mathcal{O}_\delta \subset \mathcal{O}_\delta \subset \mathcal{O}$ is an open subset, and for each $j = 0, \ldots, m - 1$,

$$t \mapsto g_j(t) \in L^2 \times \mathbb{R}^d,$$

$$t \mapsto \psi_0(t, \cdot) \in L^2 \times \mathbb{X}_\alpha^p,$$

$$t \mapsto f_0(t, \cdot) \in L^p(\mathcal{O})$$

are bounded measurable functions.
Theorem 7.2. Let \( p > d \) and \( \alpha, \alpha_0 \) satisfy (F3) and (F6). Assume that (F1)-(F3) hold. For any \( u_0 \in X^p_\alpha \), there exists a unique maximal strong solution \( (u, \tau) \) for Eq. (7.11) so that

(i) \( t \mapsto u(t) \in X^p_1 \) is continuous on \([0, \tau)\) almost surely;
(ii) \( \lim_{t \to \tau} ||u(t)||_{X^p_\alpha} = +\infty \) on \( \{\omega: \tau(\omega) < +\infty\} \);
(iii) it holds that in \( L^p(\Omega) \)

\[
\begin{align*}
\tau &= +\infty, \ a.s..
\end{align*}
\]

Proof. We only need to verify that (M1)-(M4) hold for \( \Phi \) and \( \Psi \) defined by (7.9) and (7.10). In virtue of (7.5), by (7.4) we have

\[
\begin{align*}
\Phi(t, u)(x) &= \varphi(t, x, u, Du, \cdots, D^{2m-1}u) + \lambda u, \\
\Psi(t, u)(x) &= \psi(t, x, u, Du, \cdots, D^{m-1}u).
\end{align*}
\]

(7.9) (7.10)

Then the system (7.11) can be written as the following abstract form:

\[
\begin{align*}
du(t) &= [-\mathcal{L}_p u + \Phi(t, u(t))]dt + \Psi(t, u(t))dW(t), \quad u(0) = u_0.
\end{align*}
\]

(7.11)

Using Theorem 6.9, we have the following result.

**Theorem 7.2.** Let \( p > d \) and \( \alpha, \alpha_0 \) satisfy (F3) and (F6). Assume that (F1)-(F3) hold. For any \( u_0 \in X^p_\alpha \), there exists a unique maximal strong solution \( (u, \tau) \) for Eq. (7.11) so that

(i) \( t \mapsto u(t) \in X^p_1 \) is continuous on \([0, \tau)\) almost surely;
(ii) \( \lim_{t \to \tau} ||u(t)||_{X^p_\alpha} = +\infty \) on \( \{\omega: \tau(\omega) < +\infty\} \);
(iii) it holds that in \( L^p(\Omega) \)

\[
\begin{align*}
\tau &= +\infty, \ a.s..
\end{align*}
\]

Proof. We only need to verify that (M1)-(M4) hold for \( \Phi \) and \( \Psi \) defined by (7.9) and (7.10). In virtue of (7.5), by (7.4) we have

\[
||D^j u||_{C(\bar{\Omega})} \preceq ||u||_{X^p_\alpha}, \quad j = 0, 1, \cdots, 2m - 1.
\]

(7.12)

It is easy to see by (F1) that \( \Phi \) given by (7.9) is locally Lipschitz continuous and locally bounded with respect to \( u \) on \( X^p_\alpha \), and is \( \delta \)-order Hölder continuous with respect to \( t \).

Note that by the chain rule, for \( j = 1, \cdots, m + 1 \)

\[
\begin{align*}
D^j \Psi(t, u) &= (\partial_{D^{m-1}u} \psi)(t, x, u, Du, \cdots, D^{m-1}u) \cdot D^{m-1+j}u \\
&\quad + \psi_j(t, x, u, Du, \cdots, D^{m-2+k}u),
\end{align*}
\]

(7.13)

where \( \psi_j \) is an \( l^2 \)-valued continuously differentiable function of all its variables with the exception of the \( t \)-variable. For any \( u, v \in X^p_\alpha \) with \( ||u||_{X^p_\alpha}, ||v||_{X^p_\alpha} \leq R \), by (F2) and (F3) we have

\[
\begin{align*}
||\nabla^{2}(\Psi(t, u) - \Psi(t, v))||_{L^2([t,T], X^p_\alpha)}^2 &\leq ||\nabla^{2}(\Psi(t, u) - \Psi(t, v))||_{L^p(\Omega)}^2 \\
&\leq \sum_{k=1}^{m} \sum_{j=0}^{m} ||D^j(\Psi_k(t, u) - \Psi_k(t, v))||_{L^p(\Omega,L^2)}^2 \\
&\leq \sum_{k=1}^{m} \sum_{j=0}^{m} ||D^j(\Psi_k(t, u) - \Psi_k(t, v))||_{C(\bar{\Omega})}^2
\end{align*}
\]

(7.14) (7.15)
\[ C_R \sum_{j=0}^{2m-1} \| D^j (u - v) \|_{C(\bar{O})}^2 \leq C_R \sum_{j=0}^{2m-1} \| D^j (u - v) \|_{C(\bar{O})}^2 \]

\[ C_R \| u - v \|_{\mathbb{X}^p_{\alpha_0}}^2 \leq C_R \| u - v \|_{\mathbb{X}^p_{\alpha}}^2. \]

Thus, (M2) holds.

We next look at (M4). As above, by (7.13) and (7.12) we have

\[ \| D^{m+1} \psi(t, u) \|_{L^p(\bar{O}, l^2)} \leq C_R (1 + \| D^{2m} u \|_{L^p}) \leq C_R (1 + \| u \|_{\mathbb{X}^p_{\alpha}}) \]

for all \( u \in \mathbb{X}^p_{\alpha} \) with \( \| u \|_{\mathbb{X}^p_{\alpha}} \leq R \). By (7.6), we may choose \( 1 < \alpha' < \alpha'' < \frac{m+1}{m} \)

such that

\[ \theta := \frac{\alpha - \alpha_0}{1 - \alpha_0} > \frac{\alpha' - \alpha}{\alpha'\alpha''} =: \theta'. \]

Thus, for all \( u \in \mathbb{X}^p_{\alpha} \) with \( \| u \|_{\mathbb{X}^p_{\alpha}} \leq R \), we have

\[ \| \psi(t, u) \|_{L^2(\mathbb{R}^2, \mathbb{X}^p_{\alpha''})} \leq \| \mathcal{F}^{\alpha''} \psi(t, u) \|_{L^2(\bar{O}, l^2)} \]

\[ \leq \sum_{j=0}^{m+1} \| D^j \psi(t, u) \|_{L^2(\bar{O}, l^2)} \]

\[ \leq \sum_{j=0}^{m} \sum_{k=1}^{\infty} \| D^j \psi_k(t, u) \|_{C(\bar{O})} + \| D^{m+1} \psi(t, u) \|_{L^2(\bar{O}, l^2)} \]

\[ \leq C_R (1 + \| u \|_{\mathbb{X}^p_{\alpha}}^2). \]

Using Lemma 2.13 with the data \( \alpha_0, \theta \) and \( \theta' \) as above and \( \alpha_1 = 1, \alpha_2 = \frac{\alpha}{2}, \alpha_3 = \frac{\alpha''}{2} \),

we obtain that for all \( u \in \mathbb{X}^p_{\alpha} \) with \( \| u \|_{\mathbb{X}^p_{\alpha}} \leq R \)

\[ \| \psi(t, u) \|_{L^2(\mathbb{R}^2, \mathbb{X}^p_{\alpha''})} = \| \psi(t, u) \|_{L^2(\mathbb{R}^2, \mathbb{X}^p_{\alpha_0 - \alpha_2} + \alpha_2)} \leq C_R. \]

Thus, (M4) holds.

We now verify (M3) under (F4). First of all, by the linear growth of \( \varphi(t, x, \cdot) \) with respect to \( \cdot \), we have

\[ \| \Phi(t, u) \|_{\mathbb{X}^p_{\alpha}} = \| \varphi(t, \cdot, u, Du, D^2 u, \ldots, D^{2m-1} u) \|_{L^p} \]

\[ \leq 1 + \sum_{j=0}^{2m-1} \| D^j u \|_{L^p} \]

\[ \leq 1 + \| u \|_{2m-1, p} \]

\[ \leq 1 + \| u \|_{\mathbb{X}^p_{\alpha}}. \]
For $\Psi$, we only consider the case of $m = 1$, and have

$$
\|\Psi(t, u)\|_{L^2(T^2, \mathbb{R}^p)}^2 \leq \|\omega^2_t \Psi(t, u)\|_{L^p(T^2)}^2 \\
\leq \|\Psi(t, u)\|_{L^p(T^2)}^2 + \|D\Psi(t, u)\|_{L^p(T^2)}^2.
$$

Noting that

$$
D\Psi(t, u) = (D_x \psi)(t, x, u) + (\partial_u \psi)(t, x, u)Du
$$

by (7.8) we have

$$
\|D\Psi(t, u)\|_{L^p(T^2)}^2 \leq 1 + \|u\|_{L^p} + \|Du\|_{L^p} \leq 1 + \|u\|_{\mathcal{X}_\alpha^p}.
$$

So

$$
\|\Psi(t, u)\|_{L^2(T^2, \mathbb{R}^p)}^2 \leq 1 + \|u\|_{\mathcal{X}_\alpha^p}.
$$

Thus, (M3) holds.

Consider the small perturbation of equation (7.11):

$$
du_t(t) = [-\omega^2_t u_t(s) + \Phi(t, u_t(t))]dt + \sqrt{\epsilon} \Psi(t, u_t(t))dW(t), \quad u_t(0) = u_0. \tag{7.15}
$$

Using Theorem 6.3 we have

**Theorem 7.3.** Let $p > d$ and $\alpha, \alpha_0$ satisfy (7.5) and (7.6). Assume that (F1) and (F4) hold. Let $u_0 \in \mathcal{X}_\alpha^p$. Then $\{u_\epsilon, \epsilon \in (0, 1)\}$ satisfies the large deviation principle in $\mathcal{C}_T(\mathcal{X}_\alpha^p)$ with the rate function $I(f)$ given by (6.5).

8. Application to SPDEs on complete Riemannian manifolds

Let $(M, g)$ be a $d$-dimensional complete Riemannian manifold without boundary. The Riemannian volume is denoted by $d_g x$. Let $\nabla$ denote the gradient or covariant derivative associated with $g$, $\Delta$ the Laplace Beltrami operator, $T(M)$ the tangent bundle. Let $L^p(M, d_g x)$ be the usual real $L^p$-space on $M$ with respect to $d_g x$. It is well known that the symmetric heat semigroup $(\mathcal{F}_t)_{t \geq 0}$ associated with $\Delta$ is strongly continuous and contracted on $L^p(M, d_g x)$ for $1 \leq p < +\infty$, which is also contracted on $L^\infty(M, d_g x)$ (cf. Strichartz [73, Theorem 3.5]). Therefore, for each $1 < p < +\infty$, $(\mathcal{F}_t)_{t \geq 0}$ forms an analytic semigroup on $L^p(M, d_g x)$ (cf. Stein [72, p.67 Theorem 1]). The Bessel spaces over $M$ are defined by

$$
\mathbb{H}^p_\alpha := (I - \Delta)^{-\alpha/2}(L^p(M, d_g x)).
$$

In this section, we make the following geometric assumptions:

- **(G)$_n$:** The Ricci curvature $\text{Ric}_g$ and curvature $R_g$ tensors together with their covariant derivatives up to $n$-th order are bounded.
- **(G)$_{inj}$:** The injectivity radius of $(M, g)$ is strictly positive.

It was proved by Yoshida [79] that under $(G)_n$, an equivalent norm of $\mathbb{H}^p_\alpha$ is given by the covariant derivatives up to $n$-th order, i.e., there are two positive constants $C_1$ and $C_2$ such that for any $u \in C_0^\infty(M)$

$$
C_1 \sum_{k=0}^n \|\nabla^k u\|_{L^p} \leq \|(I - \Delta)^{n/2} u\|_{L^p} \leq C_2 \sum_{k=0}^n \|\nabla^k u\|_{L^p}, \tag{8.1}
$$

where $\nabla^k$ denotes the $k$-th covariant derivative. As an example, the components of $\nabla u$ in local coordinates are given by $(\nabla u)_i = \partial_i u$, while the components of $\nabla^2 u$ in local coordinates are given by

$$
(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma^k_{ij} \partial_k u,
$$
where \( \Gamma_{ij}^k \) are Christoffel symbols. By definition one has that
\[
|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k)_{i_1 \cdots i_k} (\nabla^k)_{j_1 \cdots j_k},
\]
where \( g_{ij} = g(\partial_i, \partial_j) \) and \( (g^{ij}) \) denotes the inverse matrix of \( (g_{ij}) \).

We remark that when \( n = 1 \), \( (8.1) \) was first proved by Bakry \([4]\) under the assumption that Ricci curvature is bounded from below.

The following embedding result was proved in \([83]\). We refer to \([2, 35, 36]\) for a detailed study of integer order Sobolev spaces over \( M \).

**Theorem 8.1.** Under \((G)_{n+1}\) and \((G)_{inj}\), for \( \alpha \in (0,1) \) and \( p > d/\alpha \) we have
\[
\mathbb{H}^p_{n+\alpha} \hookrightarrow C^n_\alpha(M),
\]
where \( C^n_\alpha(M) \) denotes the Banach space of all \( n \)-times continuously differentiable functions on \( M \) with
\[
\|u\|_{C^n_\alpha} := \sup_{x \in M} \sum_{k=0}^n |\nabla^k u(x)| < +\infty.
\]

Consider the following SPDE:
\[
\begin{aligned}
du(t, x) &= [\Delta u(t, x) + \varphi(t, x, u(t), g(Y(x), \nabla u(t)))] dt \\
&\quad + \psi(t, x, u(t, x))dW(t), \\
u(0, x) &= u_0(x),
\end{aligned}
\]
where \( Y: M \to T(M) \) is a measurable vector field with
\[
\sup_{x \in M} g(Y(x), Y(x)) < +\infty \tag{8.3}
\]
and
\[
\varphi: \mathbb{R}_+ \times \Omega \times M \times \mathbb{R}^2 \to \mathbb{R} \in \mathcal{M} \times \mathcal{B}(M) \times \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}),
\]
\[
\psi: \mathbb{R}_+ \times \Omega \times M \times \mathbb{R} \to \ell^2 \in \mathcal{M} \times \mathcal{B}(M) \times \mathcal{B}(\mathbb{R})/\mathcal{B}(\ell^2).
\]

In this section, we fix
\[
p > d \text{ and } 3/2 - 3d/2p < \alpha < 1. \tag{8.4}
\]
Assume that
\[(\text{R1})\] For each \( T, R > 0 \), there exist constants \( C_{R,T}, \delta > 0 \) and \( \lambda_{R,T}^1, \lambda_{R,T}^2 \in L^p(M, d\sigma x) \) such that for all \( s, t \in [0, T], \omega \in \Omega, x \in M \) and \( |u_1|, |v_1|, |u_2|, |v_2| \leq R \)
\[
\begin{align*}
|\varphi(t, \omega, x, u_1, u_2) - \varphi(t, \omega, x, v_1, v_2)| \\
\leq C_{R,T}(\lambda_{R,T}^1(x) \cdot |t - s|^{\delta} + |u_1 - v_1| + |u_2 - v_2|)
\end{align*}
\]
and
\[
|\varphi(t, \omega, x, u_1, u_2)| \leq \lambda_{R,T}^2(x).
\]
\[(\text{R2})\] For each \( (t, \omega) \in \mathbb{R}_+ \times \Omega, \psi(t, \omega, \cdot, \cdot) \in C^2(M \times \mathbb{R}; \ell^2) \). For each \( T, R > 0 \), there exist constant \( C_{R,T} > 0 \) and \( \lambda_{R,T}^\psi \in L^p(M, d\sigma x) \) such that for all \( t \in [0, T], \omega \in \Omega, x \in M \) and \( |u| \leq R \)
\[
\|\nabla_x \partial_u \psi(t, \omega, \cdot, u)\|_{\ell^2} + \|\partial^2_u \psi(t, \omega, \cdot, u)\|_{\ell^2} \leq C_{R,T}, \quad j = 1, 2
\]
and
\[
\|\psi(t, \omega, \cdot, u)\|_{\ell^2} + \|\nabla_x^2 \psi(t, \omega, \cdot, u)\|_{\ell^2} \leq \lambda_{R,T}^\psi(x), \quad j = 1, 2.
\]
Theorem 8.2. Let $p > d$ and $\alpha$ satisfy (8.4). Under (G) 2-(G) inj and (R1)-(R2), for each $u_0 \in \mathbb{H}_2^p$, there exists a unique maximal strong solution $(u, \tau)$ for Eq. (8.2) so that

(i) $t \mapsto u(t) \in \mathbb{H}_2^p$ is continuous on $[0, \tau)$ almost surely;
(ii) $\lim_{t \uparrow \tau} \|u(t)\|_{\mathbb{H}_2^p} = +\infty$ on $\{\omega : \tau(\omega) < +\infty\};$
(iii) it holds that, $\bar{P}$-a.s., on $[0, \tau)$

$$u(t) = u_0 + \int_0^t \left[ \Delta u(s) + \varphi(s, \cdot, u(s), g(Y(\cdot), \nabla u(s))) \right] ds + \int_0^t \psi(s, \cdot, u(s)) dW(s) \in L^p(M, d\bar{g}x).$$

Proof. Choose $\alpha_0$ such that

$$\frac{1}{2} + \frac{d}{2p} < \alpha_0 < 3\alpha - \alpha^2 - 1 < \alpha < 1. \quad (8.5)$$

Let $u, v \in \mathbb{H}_{2\alpha_0}^p$ with $\|u\|_{\mathbb{H}_{2\alpha_0}^p}, \|v\|_{\mathbb{H}_{2\alpha_0}^p} \leq R$. By Theorem 8.1 we have

$$\|u\|_{C_b^1} + \|v\|_{C_b^1} \leq C_R. \quad (8.6)$$

Set

$$\Phi(t, \omega, u) := \varphi(t, \omega, \cdot, u, g(Y(\cdot), \nabla u)),$$
$$\Psi(t, \omega, u) := \psi(t, \omega, \cdot, u).$$

By (R1) and (8.1) (8.3) (8.6), we have

$$\|\Phi(t, \omega, u) - \Phi(s, \omega, v)\|_{L^p} \leq |t - s|^{\delta} + \|u - v\|_{L^p} + \|\nabla(u - v)\|_{L^p} \leq |t - s|^{\delta} + \|u - v\|_{\mathbb{H}_1^p} \leq |t - s|^{\delta} + \|u - v\|_{\mathbb{H}_2^{p}}$$

and

$$\|\Phi(t, \omega, u)\|_{L^p} \leq C_R.$$

Note that

$$\nabla_x \Psi(t, \omega, u) = (\nabla_x \psi)(t, \omega, \cdot, u) + (\partial_u \psi)(t, \omega, x, u) \nabla_x u$$

and

$$\nabla_x^2 \Psi(t, \omega, u) = (\nabla_x^2 \psi)(t, \omega, \cdot, u) + 2(\nabla_x \partial_u \psi)(t, \omega, \cdot, u) \otimes \nabla_x u + (\partial_u \psi)(t, \omega, x, u) \nabla_x u \otimes \nabla_x u + (\partial_u \psi)(t, \omega, x, u) \nabla_x^2 u. \quad (8.7)$$

By (R2) and (8.6) we have

$$\|\Psi(t, \omega, u) - \Psi(t, \omega, v)\|_{L^p} \leq \|u - v\|_{L^p}$$

and by (8.1) and (8.7)

$$\|\nabla(\Psi(t, \omega, u) - \Psi(t, \omega, v))\|_{L^p} \leq \|u - v\|_{\mathbb{H}_1^p}.$$

Hence,

$$\|\Psi(t, \omega, u) - \Psi(t, \omega, v)\|_{\mathbb{H}_2^p} \leq \|\Psi(t, \omega, u) - \Psi(t, \omega, v)\|_{\mathbb{H}_1^p} \leq \|u - v\|_{\mathbb{H}_2^{p_0}}.$$

Moreover, by (R2) and (8.1) (8.7) (8.8),

$$\|\Psi(t, \omega, u)\|_{\mathbb{H}_1^p} \leq C_{R,T}$$

and

$$\|\Psi(t, \omega, u)\|_{\mathbb{H}_2^p} \leq C_{R,T}(1 + \|u\|_{\mathbb{H}_1^p}).$$
Lemma 8.4. Then we have:

\[\alpha_3 = \alpha_1 = 1, \quad \alpha_2 = \frac{\alpha}{2}, \quad \frac{\alpha - \alpha_0}{1 - \alpha_0} =: \theta > \theta' > \frac{1 - \alpha}{2 - \alpha},\]

we find that for all \(u \in X^p_\alpha\) with \(\|u\|_{X^p_\alpha} \leq R\)

\[\|\Psi(t, \omega, u)\|_{L_2(\mathbb{P}^\alpha)} = \|\Psi(t, \omega, u)\|_{L_2(\mathbb{P}^{\alpha_3, \alpha_2})} \leq C_R,\]

where \(\alpha' = 2\theta'(\alpha_3 - \alpha_2) + 2\alpha_2 > 1\). Thus, (M2) and (M4) hold, and the theorem follows from Theorem 7.9. \(\square\)

For the non-explosion, we assume that

(R3) For each \(T > 0\), there exist \(\lambda_i \in L^p(M, d_\theta x), i = 0, 1, 2\) and \(k \in \mathbb{N}\) such that for all \((t, \omega) \in [0, T] \times \Omega, u, v \in \mathbb{R}\) such that

\[u \cdot \varphi(t, \omega, x, u, v) \leq C_T|u| \cdot (|u| + |v| + \lambda_0(x)),\]

\[|\varphi(t, \omega, x, u, v)| \leq C_T(|u|^k + |v| + \lambda_1(x))\]

and

\[\|\partial_u \varphi(t, \omega, u, v)\|_{L^2} \leq C_T,\]

\[\|\varphi(t, \omega, u, v)\|_{L^2} + \|\nabla u \varphi(t, \omega, u, v)\|_{L^2} \leq C_T(|u| + \lambda_2(x)).\]

The following theorem will follow from the proof of Lemma 8.4 below.

Theorem 8.3. Keep the same assumptions as in Theorem 8.2 and also assume (R3). Let \((u, \tau)\) be the unique maximal strong solution of Eq. (8.2) in Theorem 8.2. Then \(\tau = +\infty\) a.s.

Let \(\varphi\) and \(\psi\) be independent of \(\omega\). Consider now the small perturbation of Eq. (8.2):

\[
\begin{cases}
    du_\epsilon(t, x) = \left[ \Delta u_\epsilon(t, x) + \varphi(t, x, u_\epsilon(t), g(Y(x), \nabla u_\epsilon(t))) \right] dt \\
    \quad + \sqrt{\epsilon} \psi(t, x, u_\epsilon(t)) dW(t), \\
    u_\epsilon(0, x) = u_0(x) \in H^p_2,
\end{cases}
\]

as well as the control equation

\[
\begin{cases}
    du'(t, x) = \left[ \Delta u'(t, x) + \varphi(t, x, u'(t), g(Y(x), \nabla u'(t))) \right] dt \\
    \quad + \psi(t, x, u'(t)) h'(s) ds + \sqrt{\epsilon} \psi(t, x, u'(t)) dW(t), \\
    u'(0, x) = u_0(x) \in H^p_2,
\end{cases}
\]

where \(h' \in A^T_N\) (see (2.23) for the definition of \(A^T_N\)), and \(T > 0\) is fixed below.

Let \((u', \tau')\) be the unique maximal strong solution of Eq. (8.13). Define

\[\tau'_n := \inf \{ t : \|u'(t)\|_{H^p_2} > n \}.
\]

Then we have:

Lemma 8.4. Assume (R3). Then

\[
\lim_{n \to \infty} \sup_{\epsilon \in (0, 1)} \mathbb{P}\{ \omega : \tau'_{n}(\omega) < T \} = 0.
\]

Proof. For the simplicity of notations, we drop the superscript \(\epsilon\) in \(u'\) in the following. First of all, note that (cf. [2])

\[u_0 \in H^p_2 \subset \cap_{q > 1} L^q(M, d_\theta x).
\]

For \(q, r \geq 2\), by the usual Itô formula (cf. [12, Theorem A.2]), we have

\[\|u(t)\|_{L^r_\theta}^q = \|u_0\|_{L^r_\theta}^q + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t)\]

\[J_6(t) = \int_0^t \int \psi(t, x, u'(t)) \frac{\partial^2 u'(t, x)}{\partial x^k} \frac{\partial^2 u'(t, x)}{\partial x^l} \frac{\partial^2 u'(t, x)}{\partial x^m} \frac{\partial^2 u'(t, x)}{\partial x^n} dW(t, x, u'(t)) dW(t, x, u'(t)) dW(t, x, u'(t)) dW(t, x, u'(t)).\]
on $[0, \tau_n^\epsilon]$, where

$$J_1(t) := rq \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)|^{q-2} u(s), \Delta u(s)\|_{L^2}\,ds,$$

$$J_2(t) := rq \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)|^{q-2} u(s), \varphi(s, \cdot, u(s), g(Y(\cdot), \nabla u(s)))\|_{L^2}\,ds,$$

$$J_3(t) := rq \sum_k \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)|^{q-2} u(s), \psi_k(s, \cdot, u(s))\|_{L^2}\,dW^k_s,$$

$$J_4(t) := \frac{rq(q - 1)}{2} \sum_k \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)|^{q-2} \|\psi_k(s, \cdot, u(s))\|^2\|_{L^2}\,ds,$$

$$J_5(t) := \frac{q^2 r(r - 1)}{2} \sum_k \int_0^t \|u(s)\|_{L^q_t}^{(r-2)q} \|u(s)|^{q-2} u(s), \psi(s, \cdot, u(s))\|_{L^2}^2\,ds,$$

$$J_6(t) := rq \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)|^{q-2} u(s), \psi(s, \cdot, u(s))\|_{L^2}\,ds.$$

For $J_1(t)$ we have

$$J_1(t) = -rq(q - 1) \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \int_M |u(s)|^{q-2} |\nabla u(s)|^2 d\mathcal{g}_s\,ds.$$

For $J_2(t)$, by (8.3) and Young’s inequality we have

$$J_2(t) \leq rq \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \int_M |u(s)|^{q-1} (|u(s)| + |g(Y, \nabla u(s))| + \lambda_0) d\mathcal{g}_s\,ds$$

$$\leq -\frac{J_1(t)}{2} + C \int_0^t (\|u(s)\|_{L^q}^{rq} + 1)\,ds.$$

Similarly, by (8.12) we have

$$J_4(t) + J_5(t) \leq C \int_0^t (\|u(s)\|_{L^q_t}^{rq} + 1)\,ds,$$

and by Young’s inequality

$$J_6(t) \leq \int_0^t \|u(s)\|_{L^q_t}^{(r-1)q} \|u(s)\|_{L^q_t}^q \|u(s)\|_{L^q_t}^{q-1} \|\hat{h}(s)\|_{L^2}\,ds$$

$$\leq N \left( \int_0^t \|u(s)\|_{L^q_t}^{2(q-1)q} \|u(s)\|_{L^q_t}^q \|u(s)\|_{L^q_t}^{q-1} \|\hat{h}(s)\|_{L^2}\,ds \right)^{1/2}$$

$$\leq N \sup_{s \in [0,t]} \|u(s)\|_{L^q_t}^{rq/2} \cdot \left( \int_0^t (\|u(s)\|_{L^q_t}^{rq} + 1)\,ds \right)^{1/2}$$

$$\leq \frac{1}{2} \sup_{s \in [0,t]} \|u(s)\|_{L^q_t}^{rq} + C \int_0^t (\|u(s)\|_{L^q_t}^{rq} + 1)\,ds.$$

Combining the above calculations, we obtain

$$\sup_{s \in [0,t]} \|u(s \wedge \tau_n^\epsilon)\|_{L^q}^{rq} \leq 2\|u_0\|_{L^q}^{rq} + 2 \sup_{s \in [0,t]} J_3(s \wedge \tau_n^\epsilon) + C \int_0^{t \wedge \tau_n^\epsilon} (\|u(s)\|_{L^q}^{rq} + 1)\,ds.$$

Set

$$f_1(t) := \mathbb{E} \left( \sup_{s \in [0,t]} \|u(s \wedge \tau_n^\epsilon)\|_{L^q}^{rq} \right).$$
By BDG’s inequality and as \([8.11]\) we have
\[
\mathbb{E} \left( \sup_{s \in [0,t]} |J_3(s \wedge \tau_n^\epsilon)| \right) \leq \mathbb{E} \left( \int_0^{t \wedge \tau_n^\epsilon} \|u(s)\|^2_{L_q} \left( \|u(s)\|_{L_q}^2 + 1 \right)^2 ds \right)^{1/2} \leq \frac{1}{2} f_1(t) + C \mathbb{E} \left( \int_0^{t \wedge \tau_n^\epsilon} (\|u(s)\|_{L_q}^q + 1) ds \right).
\]
Therefore,
\[
f_1(t) \leq 4\|u_0\|_{L_q}^q + C_N \mathbb{E} \int_0^{t \wedge \tau_n^\epsilon} (\|u(s)\|_{L_q}^q + 1) ds \leq 4\|u_0\|_{L_q}^q + C_N \int_0^t (f_1(s) + 1) ds,
\]
which yields by Gronwall’s inequality that
\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|u(t \wedge \tau_n^\epsilon)\|_{L_q}^q \right) \leq C_{T,N}. \tag{8.15}
\]
Here and below, the constant \(C_{T,N}\) is independent of \(n\) and \(\epsilon\).

Set
\[
\xi_n^\epsilon(t) := t \wedge \tau_n^\epsilon
\]
and for \(q \geq 2\)
\[
f_2(t) := \mathbb{E} \left( \sup_{t' \leq \xi_n^\epsilon(t)} \|u(t')\|_{H_{2\alpha}}^q \right).
\]
Note that
\[
u(t) = \mathfrak{T}_t u_0 + \int_0^t \mathfrak{T}_{t-s} \varphi(s, u(s), \mathbf{g}(Y(s), \nabla u(s))) ds + \int_0^t \mathfrak{T}_{t-s} \psi(s, u(s)) ds + \sqrt{\epsilon} \int_0^t \mathfrak{T}_{t-s} \eta(s, u(s)) \, dW(s)
\]
which yields
\[
\mathbb{E} \left( \sup_{t' \in [0,\xi_n^\epsilon(t)]} \|\mathcal{J}_1(t)\|_{H_{2\alpha}}^q \right) \leq \mathbb{E} \left( \int_0^{\xi_n^\epsilon(t)} \|\varphi(s, \cdot, u(s), \mathbf{g}(Y(s), \nabla u(s)))\|_{L_p} ds \right)^q \leq \mathbb{E} \left( \int_0^{\xi_n^\epsilon(t)} \|\varphi(s, \cdot, u(s), \mathbf{g}(Y(s), \nabla u(s)))\|_{L_p}^q ds \right) \leq \mathbb{E} \left( \int_0^{\xi_n^\epsilon(t)} (1 + \|u(s)\|_{L_p}^q + \|\nabla u(s)\|_{L_p}^q) ds \right) \leq 4 \int_0^t (f_2(s) + 1) ds.
\]

On the other hand, by \([8.1], \[2.16\] and \(R3\) we have, for \(u \in H_{1}^p\)
\[
\|
\psi(s, \cdot, u)\|_{L_2(t; \mathbb{R}^p)}^2 \leq \|
\psi(s, \cdot, u)\|_{L_p(M; t^2)}^2 + \|
\nabla \psi(s, \cdot, u)\|_{L_p(M; t^2)}^2
\]
for \(1 \leq q \leq 51\).
Thus, as above, by (iii) of Proposition 2.11 and Hölder’s inequality we have, for $q > \frac{2}{1-\alpha}$

\[
\mathbb{E} \left( \sup_{t' \in [0, \xi_n(t)]} \| \mathcal{J}_2(t') \|_{H_{2\alpha}^q} \right) 
\leq \mathbb{E} \left( \sup_{t' \in [0, \xi_n(t)]} \int_0^t \| \mathcal{S}_{t-s} \psi(s, \cdot) u(s) \|_{H_{2\alpha}^q} ds \right)^q 
\leq N \mathbb{E} \left( \sup_{t' \in [0, \xi_n(t)]} \int_0^t \| \mathcal{S}_{t-s} \psi(s, \cdot) u(s) \|_{L_2(t^2, H_{2\alpha}^0)}^2 ds \right)^{q/2} 
\leq C_N \mathbb{E} \left( \sup_{t' \in [0, \xi_n(t)]} \int_0^t \frac{1}{(t-s)^\alpha} \| \psi(s, \cdot) u(s) \|_{L_2(t^2, H_{2\alpha}^0)}^2 ds \right)^{q/2} 
\leq C_{T,N} \mathbb{E} \left( \int_0^{\xi_n(t)} (\| u(s) \|_{H_{2\alpha}^q} + 1) ds \right) 
\leq C_{T,N} \int_0^t (f_2(s) + 1) ds. 
\]

Set \( G(t, s) := \sqrt{\mathcal{S}_{t-s} \psi(s, \cdot) u(s)} \). Then by (iii) and (iv) of Proposition 2.11 we have

\[
\| G(t, s) \|_{H_{2\alpha}^q}^2 \leq \frac{C}{(t-s)^\alpha} \| \psi(s, \cdot) u(s) \|_{L_2(t^2, H_{2\alpha}^0)}^2 
\]

and for \( \gamma \in (0, (1 - \alpha)/2) \)

\[
\| G(t', s) - G(t, s) \|_{H_{2\alpha}^q}^2 \leq \frac{|t' - t|^\gamma}{(t-s)^{\alpha + 2\gamma}} \| \psi(s, \cdot) u(s) \|_{L_2(t^2, H_{2\alpha}^0)}^2. 
\]

Therefore, using Lemma 3.24 for \( q \) large enough, we have

\[
\mathbb{E} \left( \sup_{t' \in [0, \xi_n(t)]} \| \mathcal{J}_3(t') \|_{H_{2\alpha}^q} \right) = \mathbb{E} \left( \sup_{t' \in [0, T \wedge \xi_n(t)]} \left\| \int_0^{t'} G(t', s) dW(s) \right\|_{H_{2\alpha}^q}^q \right) 
\leq C_{T,N} \mathbb{E} \left( \int_0^{T \wedge \xi_n(t)} \| \psi(s, \cdot) u(s) \|_{L_2(t^2, H_{2\alpha}^0)}^q ds \right) 
\leq C_{T,N} \int_0^t (f_2(s) + 1) ds. 
\]

Combining the above estimates, we get

\[
f_2(t) \leq C \| u_0 \|_{H_2}^q + C_{T,N} \int_0^t (f_2(s) + 1) ds. 
\]

By Gronwall’s inequality again, we find

\[
\mathbb{E} \left( \sup_{t \in T \wedge T_0} \| u(t) \|_{H_{2\alpha}^q}^q \right) = f_2(T) \leq C_{T,N},
\]
which in turn implies that
\[ \lim_{n \to \infty} \sup_{\epsilon \in (0,1)} P\{\tau_n^\epsilon < T\} = 0. \]

The proof is complete. \(\square\)

Moreover, under (R3), similar to the above lemma, we can check that (6.7) holds. Thus, using Theorem 6.3 we obtain

**Theorem 8.5.** Let \((M, g)\) be a compact Riemannian manifold, and \(p > d, \alpha\) satisfy (8.4). Let \(u_0 \in H^p_0\). Under (R1)-(R3), \(\{u_\epsilon, \epsilon \in (0,1)\}\) satisfies the large deviation principle in \(C_T(H^p_{2\alpha})\) with the rate function \(I(f)\) given by

\[ I(f) := \frac{1}{2} \inf_{\{h \in \ell^2_p: f = u^h\}} \|h\|_{\ell^2_p}^2, \quad f \in C_T(H^p_{2\alpha}), \]

where \(u^h\) solves the following equation:

\[ u^h(t) = u_0 + \int_0^t \left[ \Delta u^h(s) + \varphi(s, \cdot, u^h(s), g(Y(\cdot), \nabla u^h(s))) \right] ds \]

\[ + \int_0^t \psi(s, \cdot, u^h(s)) \dot{h}(s) ds. \]

9. **Application to stochastic Navier-Stokes equations**

9.1. **Unique maximal strong solution for SNSEs.** Let \(\mathcal{O}\) be a bounded smooth domain in \(\mathbb{R}^d (d \geq 2)\), or the whole space \(\mathbb{R}^d\), or \(d\)-dimensional torus \(\mathbb{T}^d\). Let

\[ W^{m,p}(\mathcal{O}) := (W^{m,p}(\mathcal{O}))^d, \quad W_0^{m,p}(\mathcal{O}) := (W_0^{m,p}(\mathcal{O}))^d \]

and

\[ C_{0,\sigma}^\infty(\mathcal{O}) := \{u \in (C^\infty(\mathcal{O}))^d : \text{div}(u) = 0\}. \]

Notice that \(W^{m,p}(\mathbb{R}^d) = W_0^{m,p}(\mathbb{R}^d)\) and \(W^{m,p}(\mathbb{T}^d) = W_0^{m,p}(\mathbb{T}^d)\).

Let \(L^p_\sigma(\mathcal{O})\) be the closure of \(C_{0,\sigma}^\infty(\mathcal{O})\) with respect to the norm in \(L^p(\mathcal{O}) := (L^p(\mathcal{O}))^d\).

Let \(\mathcal{P}_2\) be the orthonormal projection from \(L^2(\mathcal{O})\) to \(L^2_\sigma(\mathcal{O})\). It is well known that \(\mathcal{P}_2\) can be extended to a bounded linear operator from \(L^p(\mathcal{O})\) to \(L^p_\sigma(\mathcal{O})\) (cf. [23]) so that for every \(u \in L^p(\mathcal{O})\)

\[ u = \mathcal{P}_2 u + \nabla \pi, \quad \pi \in (L^p_{\text{loc}}(\mathcal{O}))^d. \]

The Stokes operator is defined by

\[ A_p u := -\mathcal{P}_2 \Delta u, \quad \mathcal{D}(A_p) := H^p_2 \cap L^p_\sigma(\mathcal{O}), \quad (9.1) \]

where

\[ H^p_2 := W^{2,p}(\mathcal{O}) \cap W^{1,p}_0(\mathcal{O}) = \mathcal{D}(I - \Delta_p) \]

and \(\Delta_p\) is the Laplace operator on \(L^p(\mathcal{O})\).

It is well known that \((A_p, \mathcal{D}(A_p))\) is a sectorial operator on \(L^p_\sigma(\mathcal{O})\) (cf. [29]). It should be noticed that when \(\mathcal{O} = \mathbb{R}^d\) or \(\mathbb{T}^d\), since the projection \(\mathcal{P}_p\) can commute with \(\nabla\) (cf. [H6] p.84)), we have

\[ A_p u = -\Delta \mathcal{P}_p u = -\Delta u, \quad u \in \mathcal{D}(A_p). \]

That is, the Stokes operator is just the restriction of \(-\Delta_p\) on \(W^{2,p}(\mathcal{O}) \cap L^p_\sigma(\mathcal{O})\), where \(\mathcal{O} = \mathbb{R}^d\) or \(\mathbb{T}^d\).

Below, we write

\[ \mathcal{L}_p := I + A_p \]

and

\[ H^p_\alpha := \mathcal{D}(\mathcal{L}^{\alpha/2}_p). \]
Giga [30] proved that for \( \alpha \in [0, 1] \),

\[
H^p_\alpha = [L^p_\alpha(O), \mathcal{D}(A_p)]_\alpha = \mathbb{H}^p_\alpha \cap L^p_\alpha(O),
\]  

(9.2)

where \( \mathbb{H}^p_\alpha = [L^p(O), \mathbb{H}^p_\alpha]_\alpha \) and \([\cdot, \cdot]_\alpha\) stands for the complex interpolation space between two Banach spaces. In particular, the following embedding results hold (see (7.3) and (7.4)):

for \( p > 1 \) and \( 0 \leq \alpha' < \frac{1}{2} < \alpha \leq 1 \)

\[
\|u\|_{H^p_{2\alpha}} \leq \|u\|_{1,p} \leq \|u\|_{H^p_{\alpha}}, \quad u \in H^p_{\alpha},
\]  

(9.3)

and for \( q \geq p \), \( k - \frac{d}{q} < 2\alpha - \frac{d}{p} \)

\[
H^p_{2\alpha} \hookrightarrow W^{k,q}(O),
\]  

(9.4)

and for \( \alpha > \frac{d}{p} \)

\[
H^p_{\alpha} \hookrightarrow C_b(O).
\]  

(9.5)

In what follows, we fix

\[
p > d, \quad \frac{1}{2} < \alpha < 1,
\]  

(9.6)

and consider the following stochastic Navier-Stokes equation with Dirichlet boundary (only for bounded smooth domain):

\[
\begin{align*}
\frac{du(t)}{dt} &= \left[ \Delta u(t) + (u(t) \cdot \nabla)u(t) + \nabla \pi(t) \right] dt \\
& \quad + F(t, u(t)) dt + \Psi(t, u(t)) dW(t) \\
\mathbf{u}(t, \cdot)|_{\partial \Omega} &= 0, \quad \nabla \cdot u(t) = 0, \\
u(0, x) &= u_0(x),
\end{align*}
\]  

(9.7)

where \( u \) and \( \pi \) are unknown functions, and

\[
F : \mathbb{R}^+ \times H^p_{2\alpha} \to H^p_0 \quad \text{and} \quad \Psi : \mathbb{R}^+ \times H^p_{2\alpha} \to H^p_0
\]

are two measurable functions.

We assume that

(N1) For each \( T, R > 0 \), there exist \( \delta > 0 \) and \( C_{T,R,\delta} > 0 \) such that for all \( t, s \in [0, T] \) and \( u, v \in H^p_{2\alpha} \) with \( \|u\|_{H^p_{2\alpha}}, \|v\|_{H^p_{2\alpha}} \leq R \)

\[
\|F(t, u) - F(s, v)\|_{H^p_0} \leq C_{T,R,\delta} \left( |t - s|^\delta + \|u - v\|_{H^p_{2\alpha}} \right).
\]

(N2) For each \( T, R > 0 \), there exist \( \alpha' > 1 \) and \( C_{T,R} > 0 \) such that for all \( t \in [0, T] \) and \( u, v \in H^p_{2\alpha} \) with \( \|u\|_{H^p_{2\alpha}}, \|v\|_{H^p_{2\alpha}} \leq R \)

\[
\|\Psi(t, u) - \Psi(t, v)\|_{L^2(p; H^p_0)} \leq C_{T,R} \|u - v\|_{H^p_{2\alpha}}.
\]

and

\[
\|\Psi(t, u)\|_{L^2(p; H^p_{2\alpha})} \leq C_{T,R}.
\]

(9.8)

Set

\[
\Phi(t, u) := u + \mathcal{P}_p[(u \cdot \nabla)u] + F(t, u).
\]

(9.9)

Then Eq. (9.7) can be written as the following abstract form:

\[
du(t) = [-\mathcal{L}_p u(t) + \Phi(t, u)] dt + \Psi(t, u) dW(s), \quad u(0) = u_0.
\]

(9.10)

**Theorem 9.1.** Let \( p > d \) and \( \frac{1}{2} < \alpha < 1 \). Under (N1) and (N2), for any \( u_0 \in H^p_2 \), there exists a unique maximal strong solution \((u, \tau)\) for Eq. (9.10) so that

(i) \( t \mapsto u(t) \in H^p_2 \) is continuous on \([0, \tau)\) a.s.;

(ii) \( \lim_{t \to \tau} \|u(t)\|_{H^p_{2\alpha}} = \infty \) on \( \{\tau < +\infty\} \).
(iii) it holds that in $L^p_\alpha(O) = H^p_\alpha$

$$u(t) = u_0 + \int_0^t [-\mathcal{L}_p u(s) + \Phi(s, u(s))]ds + \int_0^t \Psi(s, u(s))dW(s)$$

$$= u_0 + \int_0^t [A_p u(s) + \mathcal{P}_p((u(s) \cdot \nabla)u(s))]ds$$

$$+ \int_0^t F(s, u(s))ds + \int_0^t \Psi(s, u(s))dW(s),$$

for all $t \in [0, \tau)$, $P$-a.s..

Proof. In view of (9.6), (9.3) and (9.5), for any $u, v \in H^p_\alpha$, we have

$$\|\mathcal{P}_p[(u \cdot \nabla)u - (v \cdot \nabla)v]\|_{L^p_\alpha} \leq \|(u \cdot \nabla)u - (v \cdot \nabla)v\|_{L^p}$$

$$\leq \|u - v\|_{L^\infty} \cdot \|\nabla u\|_{L^p} + \|v\|_{L^\infty} \cdot \|\nabla(u - v)\|_{L^p}$$

$$\leq \|u - v\|_{H^\alpha_\alpha} \cdot \|u\|_{H^\alpha_\alpha} + \|v\|_{H^\alpha_\alpha} \cdot \|u - v\|_{H^\alpha_\alpha},$$

Thus, by (N1) and (N2), it is easy to see that (M2) and (M4) hold for the above $\Phi$ and $\Psi$. The result now follows by Theorem 6.9.

We now give two concrete functionals so that (N1) and (N2) are satisfied. Let $f : R_+ \times O \times R^d \to R^d$ be a measurable function, and satisfy that: for any $T, R > 0$, there exist constants $\delta, C_{T,R} > 0$ and $\lambda^f_{R,T} \in L^p(O)$ such that for all $t, s \in [0, T], x \in O$ and $u, v \in R^d$ with $|u|, |v| \leq R$

$$|f(t, x, u) - f(s, x, v)| \leq C_{T,R}(\lambda^f_{R,T}(x) \cdot |t - s|^{\delta} + |u - v|).$$

Let $g : R_+ \times O \times R^d \to L^2 \times R^d$ be a measurable function, and satisfy that:

$$g(t, x, u) = c(t)u + g_2(t, x),$$

$$\exists \alpha > 1 \ s.t. \ \sup_{t \in [0, T]} (|c(t)| + \|g_2(t, \cdot)\|_{H^\alpha_\alpha}) \leq C_{T},$$

$$\lambda^g_{R,T} \in L^p(O).$$

We define

$$F(t, u) := \mathcal{P}_p(f(t, \cdot, u)) \quad (9.11)$$

and

$$\Psi(t, u) := \mathcal{P}_p(g(t, \cdot, u)) \quad (9.12).$$

One can see that (N1) and (N2) hold. Indeed, for $u, v \in H^p_\alpha$, with $\|u\|_{H^\alpha_\alpha}, \|v\|_{H^\alpha_\alpha} \leq R$, we have

$$\|F(t, u) - F(s, v)\|_{L^p_\alpha} \leq \|F(t, \cdot, u) - F(s, \cdot, v)\|_{L^p}$$

$$\leq C_{T,R}(|t - s|^{\delta} + \|u - v\|_{L^p})$$

$$\leq C_{T,R}(|t - s|^{\delta} + \|u - v\|_{H^\alpha_\alpha}).$$
Thus, (N1) holds. For (N2), let us look at the case of $\mathcal{O} = \mathbb{R}^d$ or $\mathbb{T}^d$. Since $\Delta_p$ can commute with $\mathcal{P}_p$, we have, for $u, v \in H_{2\alpha}^p$ with $\|u\|_{H_{2\alpha}^p}, \|v\|_{H_{2\alpha}^p} \leq R$

$$\|\Psi(t, u) - \Psi(t, v)\|_{L_2(\mathbb{R}^d; H_{2\alpha}^p)}^2 \leq \|\mathcal{P}_p \Psi(t, u) - \mathcal{P}_p \Psi(t, v)\|_{L_2(\mathcal{O}; \mathbb{R}^d)}^2 \leq \|\mathcal{P}_p \Psi(t, u) - \mathcal{P}_p \Psi(t, v)\|_{L_2(\mathcal{O}; \mathbb{R}^d)}^2 \leq \sum_k \|\mathcal{P}_p \Psi(t, u) - \mathcal{P}_p \Psi(t, v)\|_{L_2(\mathcal{O}; \mathbb{R}^d)}^2 \leq C_R \|u - v\|_{H_{2\alpha}^p}^2.$$  

Using Lemma 2.13 as the calculations given in Theorem 8.2 one can verify that (9.8) holds under (8.4). Thus, (N2) holds.

9.2. Non-explosion and large deviation for 2D SNSEs. In this subsection, we study the non-explosion and large deviation for SNSE in the case of two dimension. For this aim, in addition to (N1) and (N2), we also suppose that

(N3) For any $T > 0$, there exists $C_T > 0$ such that for all $t \in [0, T]$ and $u \in H_{2\alpha}^p$

$$\|F(t, u)\|_{H_2^p} \leq C_T (\|u\|_{H_2^p} + 1),$$

and for $i = 0, 1$

$$\|\Psi(s, u)\|_{L_2(\mathbb{R}^d; H_{2\alpha}^p)} \leq C_T (1 + \|u\|_{H_2^p}), \]

$$\|\Psi(s, u)\|_{L_2(\mathbb{R}^d; H_{2\alpha}^p)} \leq C_T (1 + \|u\|_{H_{2\alpha}^p})$$

where $p$ and $\alpha$ satisfy (9.6).

We remark that $F$ and $\Psi$ defined by (9.11) and (9.12) satisfy (N3) when $f$ satisfies

$$|f(t, x, u)| \leq C_T (\|u\| + \lambda_0(x))$$

and $g$ satisfies ($\mathcal{O} = \mathbb{R}^2$ or $\mathbb{T}^2$)

$$\|\partial_x g(t, x, u)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)} \leq C_T,$$

$$\|g(t, x, u)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)} + \|\nabla_x g(t, x, u)\|_{L_2(\mathbb{R}^d; \mathbb{R}^d)} \leq C_T (\|u\| + \lambda_1(x)),$$

where $\lambda_0, \lambda_1 \in L^p(\mathcal{O})$.

We have the following result, the proof will be given in Lemma 9.7 below.

**Theorem 9.2.** Let $p > d$ and $\frac{1}{2} < \alpha < 1$. Assume that (N1)-(N3) hold. Let $(u, \tau)$ be the unique maximal solution of Eq. (9.13) in Theorem 9.1. Then $\tau = +\infty$ a.s..

We now consider the small perturbation for 2D stochastic Navier-Stokes equation:

$$du(t) = [ - \mathcal{L}_p u(t) + \Phi(t, u(t)) + \sqrt{\epsilon} \Psi(t, u(t)) \] dW(t), \quad u(0) = u_0$$

as well as the control equation:

$$d\hat{u}(t) = [ - \mathcal{L}_p \hat{u}(t) + \Phi(t, \hat{u}(t)) + \Psi(t, \hat{u}(t)) \hat{h}(t) ] dt + \sqrt{\epsilon} \Psi(t, \hat{u}(t)) dW(t), \quad \hat{u}(0) = u_0,$$

where $\hat{h} \in \mathcal{A}_V^T$ (see (2.23) for the definition of $\mathcal{A}_V^T$), and $T > 0$ is fixed below.

Let $(u^\epsilon, \tau^\epsilon)$ be the unique maximal strong solution of Eq. (9.13) with the properties:

$$\lim_{t \uparrow \tau^\epsilon} \|u^\epsilon(t)\|_{H_{2\alpha}^p} = +\infty \text{ on } \{\tau^\epsilon < \infty\},$$

and $t \mapsto u^\epsilon(t) \in H_{2\alpha}^p$ is continuous on $[0, \tau^\epsilon]$.

Before proving the non-explosion result (Lemma 9.7), we first prepare a series of lemmas.
Lemma 9.3. There exists a constant $C_T > 0$ such that for any $t \in [0, T]$ and $u \in H_2^2$

\[
\langle u, -\mathcal{L}_2 u + \Phi(s, u) \rangle_{H_0^6} \leq -\frac{1}{2} \|u\|_{H^1_0}^2 + C_T(\|u\|_{H_0^6}^2 + 1), \tag{9.14}
\]

\[
\langle \mathcal{L}_2 u, -\mathcal{L}_2 u + \Phi(s, u) \rangle_{H_0^6} \leq C\|u\|_{H_0^6}^2\|u\|_{H_0^6}^4 + C_T (1 + \|u\|_{H_1^2}^2) \tag{9.15}
\]

and

\[
\|\Phi(t, u)\|_{H_0^6} \leq C_T (1 + \|u\|_{H_1^2}) \cdot (1 + \|u\|_{H_2^\alpha}). \tag{9.16}
\]

Proof. Let $u \in H_2^2$. Noting that

\[
\langle u, \mathcal{P}_2((u \cdot \nabla)u) \rangle_{H_0^6} = \langle u, (u \cdot \nabla)u \rangle_{L^2} = \frac{1}{2} \int \mathcal{O} u(x) \cdot \nabla |u(x)|^2 dx = 0,
\]

by (N3) and Young’s inequality we have

\[
\langle u, -\mathcal{L}_2 u + \Phi(s, u) \rangle_{H_0^6} = \langle u, \Phi(s, u) \rangle_{H_0^6} \leq -\|u\|_{H_1^2}^2 + \langle u, u + F(t, u) \rangle_{H_0^6} \leq -\frac{1}{2} \|u\|_{H_1^2}^2 + C_T (\|u\|_{H_0^6}^2 + 1).
\]

Thus, (9.14) is proved.

For (9.15), noting that by Gagliado-Nirenberge’s inequality (cf. [27, p.24 Theorem 9.3]) and (9.2)

\[
\|u\|_{L^\infty} \leq \|u\|_{H_2^2} \cdot \|u\|_{H_0^2} \leq \|u\|_{H_2^2} \cdot \|u\|_{H_0^6},
\]

by Young’s inequality we have

\[
\langle \mathcal{L}_2 u, \mathcal{P}_2((u \cdot \nabla)u) \rangle_{H_0^6} \leq \frac{1}{4} \|u\|_{H_2^2}^2 + \|\mathcal{P}_2((u \cdot \nabla)u)\|_{H_0^6}^2 \leq \frac{1}{4} \|u\|_{H_2^2}^2 + C \|u \cdot \nabla u\|_{L^2}^2 \leq \frac{1}{4} \|u\|_{H_2^2}^2 + C \|u\|_{L^\infty}^2 \cdot \|\nabla u\|_{L^2}^2 \leq \frac{1}{4} \|u\|_{H_2^2}^2 + C \|u\|_{H_0^6} \cdot \|u\|_{H_2^2} \cdot \|u\|_{H_1^2}^2 \leq \frac{1}{2} \|u\|_{H_2^2}^2 + C \|u\|_{H_0^6} \cdot \|u\|_{H_1^2}^4
\]

and by (N3)

\[
\langle \mathcal{L}_2 u, F(s, u) \rangle_{H_0^6} \leq \frac{1}{2} \|u\|_{H_2^2}^2 + C_T (1 + \|u\|_{H_1^2}^2).
\]

Thus, (9.15) holds.

Let

\[
p < q < \frac{d}{1 + \frac{2}{p} - 2\alpha}, \quad q^* = \frac{qp}{q - p}.
\]

By Hölder’s inequality we have

\[
\|\mathcal{P}_p(u \cdot \nabla)u\|_{H_0^6} \leq \|u \cdot \nabla u\|_{L^p} \leq \|u\|_{L^p} \cdot \|\nabla u\|_{L^q} \leq \|u\|_{H_2^2} \cdot \|u\|_{H_2^\alpha}.
\]

The estimate (9.16) now follows by (N3). \[\square\]

Below, set for $n \in \mathbb{N}$

\[
\tau_n^\epsilon := \inf \left\{ t \geq 0 : \|u^\epsilon(t)\|_{H_2^\alpha} > n \right\}.
\]

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Lemma 9.4. There exists a constant $C_T > 0$ such that for all $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$

$$
\mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_n^\epsilon]} \|u^\epsilon(s)\|_{H_0^2}^2 \right) + \mathbb{E} \left( \int_0^{T \wedge \tau_n^\epsilon} \|u^\epsilon(s)\|_{H_1^2}^2 ds \right) \leq C_T.
$$

Proof. By Ito’s formula we have

$$
\text{Lemma 9.4.}
$$

and similarly, we also have

$$
\text{First of all, noting that by (9.14) we have}
$$

(43)

By (N3) and Young’s inequality we have

$$
\mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_n^\epsilon]} J_1(s) \right) + \mathbb{E} \left( \int_0^{T \wedge \tau_n^\epsilon} \|u^\epsilon(s)\|_{H_1^2}^2 ds \right) \leq C_T \int_0^t (f(s) + 1) ds.
$$

By (N3) and Young’s inequality we have

$$
\mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_n^\epsilon]} J_2(s) \right) \leq \mathbb{E} \left( \int_0^{T \wedge \tau_n^\epsilon} \|u^\epsilon(s)\|_{H_0^2} \cdot \|\Psi(s, u^\epsilon(s))\|_{L_2(t^2, H_2^0)} \cdot \|\dot{h}^\epsilon(s)\|_{L_2} ds \right)
$$

$$
\leq \mathbb{E} \left( \int_0^{T \wedge \tau_n^\epsilon} \|u^\epsilon(s)\|_{H_0^2} \cdot \|\Psi(s, u^\epsilon(s))\|_{L_2(t^2, H_2^0)}^2 ds \right)^{1/2}
$$

$$
\leq \frac{1}{4} f(t) + C_N \mathbb{E} \left( \int_0^{T \wedge \tau_n^\epsilon} (1 + \|u^\epsilon(s)\|_{H_2^0}^2) ds \right)
$$

$$
\leq \frac{1}{4} f(t) + C_N \int_0^t (1 + f(s)) ds.
$$

Similarly, we also have

$$
\mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_n^\epsilon]} J_3(s) \right) \leq \frac{1}{4} f(t) + C \int_0^t (1 + f(s)) ds
$$

and

$$
\mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_n^\epsilon]} J_4(s) \right) \leq C \int_0^t (1 + f(s)) ds.
$$
Combining the above calculations we get
\[ f(t) + 2E \int_0^{t \wedge \tau_n^\epsilon} \| u'(s) \|^2_{H_1^2} ds \leq 2\| u_0 \|^2_{H_0^2} + C_N + C_N \int_0^t (1 + f(s)) ds. \]
The desired estimate follows by Gronwall’s inequality. \( \square \)

Set for \( n \in \mathbb{N} \)
\[ \eta_n^\epsilon(t) := \int_0^{t \wedge \tau_n^\epsilon} \| u'(s) \|^2_{H_1^2} \cdot \| u'(s) \|^2_{H_0^2} ds + t \]
\[ = \int_0^t \| u'(s) \|^2_{H_1^2} \cdot \| u'(s) \|^2_{H_0^2} \cdot 1_{[0, \tau_n^\epsilon]}(s) ds + t \]
and
\[ \theta_n^\epsilon(t) := \inf \{ s \geq 0 : \eta_n^\epsilon(s) \geq t \}. \]
Clearly, \( t \mapsto \eta_n^\epsilon(t) \) is a continuous and strictly increasing function, and the inverse function of \( t \mapsto \theta_n^\epsilon(t) \) is just given by \( \eta_n^\epsilon \). Moreover, since \( \eta_n^\epsilon(t) > t \), we have
\[ \theta_n^\epsilon(t) < t. \]

**Lemma 9.5.** For any \( K > 0 \), there exists a constant \( C_{K,N} > 0 \) such that for all \( \epsilon \in (0,1) \) and \( n \in \mathbb{N} \)
\[ \mathbb{E} \left( \sup_{s \in [0,\theta_n^\epsilon(K) \wedge \tau_n^\epsilon]} \| u'(s) \|^2_{H_1^2} \right) \leq C_{K,N}. \]

**Proof.** Consider the following evolution triple
\( H_2 \subset H_1 \subset H_0^2 \).
By Ito’s formula (cf. [68]), we have
\[ \| u'(t) \|^2_{H_1^2} = \| u_0 \|^2_{H_1^2} + 2 \int_0^t \langle \mathcal{L}_2 u'(s), -\mathcal{L}_2 u'(s) + f(s, u'(s)) \rangle_{H_0^2} ds \]
\[ + 2 \int_0^t \langle \mathcal{L}_2 u'(s), \Psi(s, u'(s)) \rangle_{H_0^2} ds \]
\[ + 2 \sqrt{\epsilon} \sum_k \int_0^t \langle u'(s), \Psi_k(s, u'(s)) \rangle_{H_1^2} dW^k(s) \]
\[ + \epsilon \sum_k \int_0^t \| \Psi_k(s, u'(s)) \|^2_{H_1^2} ds \]
\[ =: \| u_0 \|^2_{H_1^2} + J_1(t) + J_2(t) + J_3(t) + J_4(t). \]
Set
\[ f(t) := \mathbb{E} \left( \sup_{s \in [0,t]} \| u'(\theta_n^\epsilon(s) \wedge \tau_n^\epsilon) \|^2_{H_1^2} \right) \]
\[ = \mathbb{E} \left( \sup_{s \in [0,\theta_n^\epsilon(t) \wedge \tau_n^\epsilon]} \| u'(s) \|^2_{H_1^2} \right). \]
For \( J_1(t) \), by (9.15) we have, for \( t \in [0, K] \)
\[ J_1(\theta_n^\epsilon(t) \wedge \tau_n^\epsilon) \leq \int_0^{\theta_n^\epsilon(t) \wedge \tau_n^\epsilon} \left[ C \| u'(s) \|^4_{H_0^2} + C_K (1 + \| u'(s) \|^2_{H_1^2}) \right] ds \]
By (iii) of Proposition 2.11, Hölder’s inequality and Lemma 9.16, we have, for all 
\[ \epsilon \]
Thus, we get
\[ \text{Lemma 9.6.} \]
Using the same trick as used in Lemma 9.4 and by (N3), we also have
\[ \mathbb{E}\left( \sup_{s \in [0,t]} J_i(\theta_n^\epsilon(s) \wedge \tau_n^\epsilon) \right) \leq C \int_0^t f(s)ds + C_K. \]

Thus, we get
\[ f(t) \leq 2\|u_0\|_{H_1^2}^2 + C_{N,K} \int_0^t (f(s) + 1)ds, \quad i = 2, 3, 4. \]

which yields the desired estimate by Gronwall’s inequality.

Set for \( M > 0 \)
\[ \zeta_n^\epsilon(M) := \inf \left\{ t \geq 0 : \|u^\epsilon(t \wedge \tau_n^\epsilon)\|_{H_1^2} \geq M \right\}. \]

Lemma 9.6. For any \( M > 0 \) and \( q \geq 2 \), there exists a constant \( C_{T,M,N} > 0 \) such that for all \( \epsilon \in (0,1) \) and \( n \in \mathbb{N} \)
\[ \mathbb{E}\left[ \sup_{t \in [0,T] \wedge \tau_n^\epsilon \wedge \zeta_n^\epsilon(M)} \|u^\epsilon(t)\|_{H_2^\alpha}^q \right] \leq C_{T,M,N}. \]

Proof. Set for \( t \in [0,T] \)
\[ \xi_n^\epsilon(t) := t \wedge \tau_n^\epsilon \wedge \zeta_n^\epsilon(M) \]
and for \( q \geq 2 \)
\[ f(t) := \mathbb{E}\left[ \sup_{t' \in [0,\xi_n^\epsilon(t)]} \|u^\epsilon(t')\|_{H_2^\alpha}^q \right]. \]

Note that
\[ u^\epsilon(t) = \mathcal{T}_t u_0 + \int_0^t \mathcal{T}_{t-s} \Phi(s, u^\epsilon(s))ds + \int_0^t \mathcal{T}_{t-s} \Psi(s, u^\epsilon(s)) \dot{h}^\epsilon(s)ds + \sqrt{\epsilon} \int_0^t \mathcal{T}_{t-s} \Psi(s, u^\epsilon(s))dW(s). \]

By (iii) of Proposition 2.11 Hölder’s inequality and Lemma 9.16 we have, for \( q > \frac{1}{1-\alpha} \)
\[ \mathbb{E}\left[ \sup_{t' \in [0,\xi_n^\epsilon(t)]} \left\| \int_0^{t'} \mathcal{T}_{t'-s} \Phi(s, u^\epsilon(s))ds \right\|_{H_2^\alpha}^q \right] \]
\[ \leq \mathbb{E}\left[ \sup_{t' \in [0,\xi_n^\epsilon(t)]} \left( \int_0^{t'} \frac{1}{(t'-s)^\alpha} \|\Phi(s, u^\epsilon(s))\|_{H_0^\alpha}ds \right)^q \right] \]
\[ \leq \mathbb{E}\left[ \int_0^{\xi_n^\epsilon(t)} \|\Phi(s, u^\epsilon(s))\|_{H_0^\alpha}^q ds \right] \]
Lemma 9.7. It holds that
\[ \lim_{n \to \infty} \sup_{\epsilon \in (0,1)} P \left\{ \omega : \tau_n^\epsilon(\omega) \leq T \right\} = 0. \quad (9.17) \]

Proof. First of all, for any \( M, K > 0 \) we have
\[
P \{ \zeta^\epsilon_n(M) < T \} \leq P \{ \zeta^\epsilon_n(M) < T; \theta^\epsilon_n(K) \geq T \} + P \{ \theta^\epsilon_n(K) < T \}
\]
\[
= P \left\{ \sup_{t \in [0,T]} \|u^\epsilon(t \wedge \tau_n^\epsilon)\|_{H^2} > M; \theta^\epsilon_n(K) \geq T \right\} + P \left\{ \sup_{s \in [0,T]} \eta^\epsilon_n(s) > K \right\}
\]
\[ \begin{align*}
&\leq \mathbb{P}\left\{ \sup_{t \in [0, \theta_n^K \land \tau_n^\epsilon]} \|u^f(t)\|_{H^2_{2\alpha}} > M \right\} + \mathbb{P}\{\eta_n^\epsilon(T) > K\} \\
&\leq \mathbb{E}\left( \sup_{t \in [0, \theta_n^K \land \tau_n^\epsilon]} \|u^f(t)\|^2_{H^2_{2\alpha}} \right)/M^2 + \mathbb{E}\left( \eta_n^\epsilon(T) \right)/K.
\end{align*} \]

Hence, by Lemmas 9.4 and 9.5 we have
\[ \lim_{M \to \infty} \sup_{n, \epsilon} \mathbb{P}\{\zeta_n^\epsilon(M) < T\} = 0. \]

Secondly, we also have
\[ P\{\tau_n^\epsilon < T\} \leq P\{\tau_n^\epsilon < T; \zeta_n^\epsilon(M) > T\} + P\{\zeta_n^\epsilon(M) < T\}. \quad (9.18) \]

For the first term, by Lemma 9.6 we have
\[ P\{\tau_n^\epsilon < T; \zeta_n^\epsilon(M) > T\} = P\left\{ \sup_{t \in [0, T \land \tau_n^\epsilon]} \|u^f(t)\|_{H^2_{2\alpha}} > n; \zeta_n^\epsilon(M) > T \right\} \]
\[ \leq P\left\{ \sup_{t \in [0, T \land \tau_n^\epsilon]} \|u^f(t)\|_{H^2_{2\alpha}} > n; \zeta_n^\epsilon(M) > T \right\} \]
\[ \leq P\left\{ \sup_{s \in [0, T \land \zeta_n^\epsilon(M) \land \tau_n^\epsilon]} \|u^f(t)\|_{H^2_{2\alpha}} > n \right\} \]
\[ \leq \mathbb{E}\left( \sup_{s \in [0, T \land \zeta_n^\epsilon(M) \land \tau_n^\epsilon]} \|u^f(t)\|^q_{H^2_{2\alpha}} \right)/n^q \]
\[ \leq \frac{C_{T,M,N}}{n^q}, \]

where \( C_{T,M,N} \) is independent of \( \epsilon \) and \( n \). The desired limit now follows by taking limits for (9.18), first \( n \to \infty \), then \( M \to \infty \).

Thus, using Theorem 6.3 we get:

**Theorem 9.8.** Let \( \mathcal{O} = \mathbb{T}^2 \) or a bounded smooth domain in \( \mathbb{R}^2 \). Under (N1)-(N3), for \( u_0 \in H^p_{2\alpha} \), \( \{u_\epsilon, \epsilon \in (0, 1)\} \) satisfies the large deviation principle in \( C_T(H^p_{2\alpha}) \) with the rate function \( I(f) \) given by
\[ I(f) := \frac{1}{2} \inf_{\{h \in \ell_2^n: f = u^h\}} \|h\|^2_{\ell_2^n}, \quad f \in C_T(H^p_{2\alpha}), \]
where \( u^h \) solves the following equation:
\[ u^h(t) = u_0 + \int_0^t \Delta u^h(s)ds + \int_0^t \mathcal{P}_p((u^h(s) \cdot \nabla)u^h(s))ds \]
\[ + \int_0^t F(s, u^h(s))ds + \int_0^t \Psi(s, u^h(s))\dot{h}(s)ds. \]

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