Parallel transport in an entangled ring

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Abstract

This paper defines a notion of parallel transport in a lattice of quantum particles, such that the transformation associated with each link of the lattice is determined by the quantum state of the two particles joined by that link. We focus particularly on a one-dimensional lattice—a ring—of entangled rebits, which are binary quantum objects confined to a real state space. We consider states of the ring that maximize the correlation between nearest neighbors, and show that some correlation must be sacrificed in order to have non-trivial parallel transport around the ring. An analogy is made with lattice gauge theory, in which non-trivial parallel transport around closed loops is associated with a reduction in the probability of the field configuration. We discuss the possibility of extending our result to qubits and to higher dimensional lattices.

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1 Introduction

In lattice gauge theory, the gauge field assigns to every pair of neighboring lattice sites a transformation that tells how to “transport” a vector, representing an internal property such as quark color, from one site to the other. That is, with every ordered pair \(\langle j, k \rangle\) of neighboring sites, or link, one associates a transformation \(U(k, j)\) which is an element of the gauge group. Indeed, this assignment of transformations to links constitutes the configuration of the gauge field. If we apply \(U(k, j)\) to a vector \(v\) associated with site \(j\), we can interpret the image vector, \(v' = U(k, j)v\), as the result of moving \(v\) from site \(j\) to site \(k\). This process is called parallel transport, and the transformation \(U(k, j)\) is sometimes called a parallel transporter.

This paper is not about lattice gauge theory but about actual lattices consisting of simple quantum particles, in which each particle is correlated with its nearest neighbors. As we will see below, for typical states of the lattice, one can use the state itself to specify a notion of parallel transport. Our conception is distinct from the more familiar notion of quantum parallel transport (expressible in terms of gauge fields [1, 2]), in which a quantum particle is physically moved either in actual space or in a parameter space [2, 3, 4]. In our approach there is no physical motion or evolution; rather, the transformation that we associate with a link expresses something about the relationship between the particles joined by that link. The present work is an initial exploration into a possible analogy between the quantum state of a lattice of quantum particles and the configuration of the gauge field in a lattice gauge theory. Roughly, the analogy we are looking for would be along the following lines (it will be spelled out more precisely in later sections of the paper).

A lattice gauge theory assigns a probability distribution to the set of possible field configurations, the probability density of a configuration \(\mathcal{A}\) being proportional to \(\exp[-S(\mathcal{A})]\) where the real function \(S(\mathcal{A})\) is the action. The action depends on the effect of parallel transport around each of the elementary plaquettes of the lattice; that is, it depends on the transformations that the field configuration associates with these elementary closed paths. The more these transformations differ from the identity, the higher the action.

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1It helps to read the arguments of \(U\) from right to left. This notation makes sense when combined with the usual notation for a sequence of operations, in which the operator on the right acts first.
and therefore the lower the probability of that particular field configuration. Let us call this effect “the probability cost of twisting.” I am looking for something similar in a lattice of correlated quantum objects, but instead of a probability cost, I am looking for a “correlation cost.” We will be focusing our attention on states in which nearest neighbors are maximally correlated. The question is whether this maximal correlation must be reduced if there is to be non-trivial parallel transport around closed loops (that is, transport whose net effect is not null), and if so, whether the reduction in correlation becomes more severe as the effect of such parallel transport differs further from the identity. In other words, in an entangled lattice, is there a correlation cost of twisting?

In fact this paper only begins to answer this question. Although we set up the problem for lattices of arbitrary dimension, the one concrete example we work out in detail is the case of a one-dimensional ring. Moreover, for most of the paper we will restrict our attention to the simplest possible quantum object, namely, a rebit, a fictitious object whose state space is a two-dimensional vector space over the real numbers $\mathbb{R}$. Thus we will mostly be analyzing a closed ring of rebits. At the end of the paper I discuss the generalization to qubits and to higher-dimensional lattices.

This research is related to a recent line of work on “entanglement sharing,” concerning the ways in which quantum entanglement can be shared among several objects. A number of authors have found constraints on the sharing of entanglement that follow directly from the structure of quantum mechanics itself and not from any particular Hamiltonian. For example, it has been shown that any entanglement that might exist between a pair of qubits limits the extent to which either of them can be entangled with a third qubit [7, 8]. There are similar limits for $n$ qubits all entangled with each other [9, 10, 11]. Another example is an entangled ring: in a translationally invariant state of a ring of qubits, there is a certain maximum possible entanglement between nearest neighbors [12]. (The measure of entanglement used in all of these studies is the entanglement of formation [13, 14].) In this paper we are putting a different sort of condition on the quantum state of a multipartite system—we are imposing a certain configuration of twists in the nearest-neighbor correlations—and we are asking what constraint this condition places on the strength of the correlations.

I hope that the results of this research will ultimately be useful in analyzing systems of entangled particles on a lattice, such as magnetic systems. If
there are simple laws of quantum correlations that transcend any particular Hamiltonian, then these laws might lead to the identification of interesting generic properties of quantum many-body systems. Arguments along these lines, particularly focusing on entanglement, have appeared recently in the literature [13, 16, 17, 18]. But at least as much of the actual motivation for the present work comes from pure curiosity: I wonder how close an analogy one can draw between the degree of correlation between particles in a quantum lattice and the probability density of a field configuration in lattice gauge theory. Of course there are many connections between lattice gauge theory and the theory of many-body systems—see, for example, Ref. [19] and references cited therein—but I am looking for an analogy along the particular lines traced out above.

The reader may have noticed that in describing the work to be presented here (as opposed to earlier work), I have been using the word “correlation” rather than “entanglement.” Though they are related, the two concepts are not the same. In this paper I focus on correlation because it seems natural in this context and it is easy to work with. But it would also be interesting to explore the same questions using one of the standard measures of entanglement. I might add that the states we will primarily be concerned with are in fact highly entangled; hence the reference to an “entangled ring” in the title.

The paper is organized as follows. First we review briefly those aspects of lattice gauge theory that have suggested our main question. We then define a rebit more precisely and develop our notion of parallel transport. We analyze in some detail the case of a ring of rebits and determine whether non-trivial parallel transport does indeed entail a “correlation cost.” Finally we ask how the problem and the results are likely to change when extended to more complex systems.

2 Lattice gauge theory and a simple analogy

Ideally, lattice gauge theory is done on a four-dimensional lattice representing spacetime, except that the fourth dimension represents imaginary time, so that it acts in many respects like another spatial dimension. The results of a calculation can be interpreted in terms of real time by means of analytic continuation. One consequence of the use of imaginary time is this: in com-
puting the expectation value of an observable, one does not sum up complex
amplitudes associated with different histories; rather, one takes a weighted
average of the observable of interest using real weights \[5\].

As mentioned above, the weighting function is proportional to \( \exp(-S) \),
where the action \( S \) is a real function of the field configuration. More precisely, in a pure
gauge theory, in which there are no matter fields but only the gauge field itself, the
expectation value of an observable \( B \) is

\[
\langle B \rangle = \frac{1}{Z} \int B e^{-S(A)} \prod_{(j,k)} dU(k,j).
\]

(1)

Here \( A \) is the configuration of the gauge field, which assigns a parallel trans-
porter \( U(k,j) \) to each link \((j,k)\). Each such transformation \( U \) is an element
of the gauge group, e.g., \( U(1) \) for electrodynamics or \( SU(3) \) for chromody-
namics. The integral in Eq. (1) is over all field configurations, that is, over all
possible parallel transporters for each link, and \( dU(k,j) \) indicates the invari-
ant measure over the gauge group. In the integral, the parallel transporters
for different links are independent, except that \( U(k,j) = U(j,k)^{-1} \), so that
only one of these two ordered pairs needs to be represented in the integral.
The normalizing constant \( Z \) is simply

\[
Z = \int e^{-S(A)} \prod_{(j,k)} dU(k,j).
\]

(2)

The action \( S \) depends only on the results of parallel transport around
plaquettes of the lattice, e.g., elementary squares in a cubic lattice. Let \( P \) be
a plaquette, which we can think of as a sequence of lattice sites; for a cubic
lattice \( P \) would consist of four sites \( j, k, l, m \). Let \( U(P) \) be the net effect of
parallel transport around plaquette \( P \); in the square example, \( U(P) \) would be
\( U(j,m)U(m,l)U(l,k)U(k,j) \) (the right-most operator acting first). \( S \) is
defined so that it increases as the plaquette transformations \( U(P) \) get farther
from the identity. Various action functions with this property have been used
in the literature; the one originally proposed by Wilson for an \( SU(N) \) gauge
field is \[20\]

\[
S \propto - \sum_P \text{Re}(\text{Tr}(U(P))),
\]

(3)

where the sum is over all plaquettes in the lattice. Note that if \( P' \) consists
of the same set of points as \( P \), but with a different starting point or with
the points taken in the opposite order, then \( \text{Re}(\text{Tr}(U(P'))) = \text{Re}(\text{Tr}(U(P))). \)

Thus we need include in the above sum only one ordered set \( P \) representing each geometric plaquette.

A gauge transformation associates with each lattice point \( j \) a group element \( \Lambda(j) \), and under such a transformation each parallel transporter \( U(k, j) \) transforms according to

\[
U(k, j) \to \Lambda(k)U(k, j)\Lambda(j)^{-1}.
\] (4)

It is easy to see that, though a gauge transformation changes the field configuration, it does not change any of the plaquette transformations \( U(P) \). Therefore it does not affect the action \( S \) and so has no physical consequences.

This invariance under gauge transformations provides a simple analogy between the configuration of a gauge field and the state of a lattice of quantum particles. Consider, for example, a lattice of qubits. Rotating each of the individual qubits separately is analogous to a gauge transformation. The state of the lattice changes under such rotations, but certain physical properties do not change. In particular, any reasonable measure of the degree of entanglement or correlation between two qubits does not change. So at least in this one modest respect, the degree of correlation in a quantum lattice is similar to the action or the probability density of a field configuration in a lattice gauge theory. We want to see whether the similarity goes any further than this.

### 3 What is a rebit?

As we have said, the quantum object we will mostly be concerned with in this paper is the rebit. We now define this object more precisely.

A pure state \( |\psi\rangle \) of a single rebit is simply a normalized vector in a two-dimensional real vector space. A mixed state of a rebit is a mixture of pure states:

\[
\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,
\] (5)

where \( p_i > 0 \) and \( \sum_i p_i = 1 \). Equivalently, a mixed state can be represented as a real, symmetric \( 2 \times 2 \) matrix with unit trace and no negative eigenvalues.

Of course any rebit state is also a qubit state, and for our purposes it will be helpful to think of rebits simply as restricted qubits. On the Bloch sphere,
the restriction to real density matrices becomes a restriction to the $x$-$z$ plane. But it will be more useful to change the representation by rotating the Bloch sphere. Let us rotate all states by $90^\circ$ in the left-handed sense around the positive $x$ axis, so that our rebit states now lie in the $x$-$y$ plane. A general mixed state lying in this plane can be written as

$$\rho = \frac{1}{2}[I + a\sigma_x + b\sigma_y],$$

(6) where the $\sigma$’s are Pauli matrices and the real numbers $a$ and $b$ satisfy $a^2 + b^2 \leq 1$. Let us call this representation of rebit states the “horizontal representation,” as opposed to the original “real-number representation.” Given a rebit density matrix $\rho$ expressed in the horizontal representation, we can always re-express it as a real density matrix $\rho_{\text{real}}$ simply by reversing the rotation around the $x$ axis:

$$\rho_{\text{real}} = U \rho U^\dagger,$$

(7) where

$$U = e^{-i(\pi/4)\sigma_z} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$ (8)

Here we have written the matrix in the basis of eigenstates of $\sigma_z$, $\{|\uparrow\rangle, |\downarrow\rangle\}$. Note that $U\sigma_y U^\dagger = \sigma_z$ and $U\sigma_x U^\dagger = \sigma_x$.

In the real-number representation, a pure state of $n$ rebits is a real vector in $2^n$ dimensions, and a mixed state of such a system is a real, symmetric $2^n \times 2^n$ density matrix. But again we will usually work in the horizontal representation, in which each rebit has been rotated by $90^\circ$ around the $x$ axis. It will be helpful to have a simple way of recognizing whether a given $n$-qubit state is a legitimate $n$-rebit state expressed in the horizontal representation. Conceptually, the test is straightforward: apply the above transformation $U$ to each qubit—that is, rotate each qubit by $90^\circ$ (in the right-handed sense) around the positive $x$ axis—and see whether the resulting state has only real components. Thus for a pure state $|\Psi\rangle$, we insist that $U^{\otimes n}|\Psi\rangle$ be real in the standard up-down basis. But this is the same as saying that $U^{\otimes n}|\Psi\rangle = (U^{-1})^{\otimes n}|\Psi^*\rangle$, where the asterisk indicates complex conjugation in the standard basis. Multiplying both sides of this equation by $U^{\otimes n}$ and noting that $U^2 = -i\sigma_x$, we arrive at the following criterion:

$$(-i\sigma_x)^{\otimes n}|\Psi\rangle = |\Psi^*\rangle.$$ (9)
In practice it will be simplest if we also allow ourselves to use state vectors of the form \( \exp(i\alpha)|\Psi\rangle \), where \( \alpha \) is real and \( |\Psi\rangle \) satisfies Eq. (9). Though such state vectors do not become real when they are transformed by \( U^{\otimes n} \), their density matrices do become real. Allowing this possibility leads to the following weaker condition on an \( n \)-qubit state \( |\Psi\rangle \).

\[
\sigma_x^{\otimes n}|\Psi\rangle = e^{i\beta}|\Psi^*\rangle,
\]

\( \beta \) being any real phase. Note that the matrix \( \sigma_x \) simply interchanges \( |\uparrow\rangle \) and \( |\downarrow\rangle \). Thus we can recognize a pure \( n \)-qubit state \( |\Psi\rangle \) as a legitimate horizontal representation of an \( n \)-rebit state by checking to see that the coefficient of each basis state, e.g., \( |\uparrow\uparrow\downarrow\uparrow\rangle \), is the complex conjugate of the coefficient of the opposite state, in this case \( |\downarrow\downarrow\uparrow\downarrow\rangle \), multiplied by a phase factor that is the same for all basis states. Let us call Eq. (10) the “rebit condition” for pure states. The corresponding test for mixed states can be obtained by a similar argument; one finds that a density matrix \( \rho \) of \( n \) qubits is the horizontal representation of a legitimate \( n \)-rebit state if and only if

\[
\sigma_x^{\otimes n}\rho\sigma_x^{\otimes n} = \rho^*,
\]

the complex conjugation again being in the standard basis.

We conclude this section with a word about rotations of a rebit. Viewing the states of a rebit as qubit states confined to the equatorial plane of the Bloch sphere, we could take as the allowed rotations all the unitary transformations that represent rotations around the \( z \) axis, that is, all transformations of the form

\[
R = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}.
\]

However, our definition of parallel transport will not be able to distinguish unitary transformations that are different only by an overall phase factor; so we will call such transformations identical. For definiteness we pick a standard representative from each of the resulting equivalence classes: a rotation by an angle \( \xi \) around the \( z \) axis, with \( 0 \leq \xi < 2\pi \), will be represented by the matrix

\[
R = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\xi} \end{pmatrix}.
\]
4 Parallel transport in a lattice of rebits

We now consider a lattice of rebits, on which we want to define a notion of parallel transport. For now the structure of the lattice does not matter, as long as any two rebits are identified as either being neighbors or not. Thus the lattice is simply a graph. The mathematical object that is to be parallel transported is a pure state of a single rebit, which we can picture as a direction in the horizontal plane. The parallel transporter associated with a link in the lattice will be a rebit rotation—represented in the form (13)—which we can identify with an element of U(1). We want the assignment of parallel transporters to links to be determined by the quantum state of the lattice itself.

We begin with the following scenario. Consider a pair of neighboring rebits labeled $j$ and $k$. They are in some joint state $\rho^{(jk)}$, a $4 \times 4$ density matrix obtained from the state of the whole lattice by tracing over all the other particles. Imagine performing an arbitrary orthogonal measurement on particle $j$. In the standard von Neumann model this measurement brings particle $j$ into a certain pure state—one of the eigenstates of the measurement—and it also brings particle $k$ into some state, typically a mixed state, determined as follows. Let $|\psi^{(j)}\rangle$ be the eigenstate into which particle $j$ is brought by the measurement. Then the post-measurement state of particle $k$ is the $2 \times 2$ density matrix

$$\omega^{(k)} = \frac{1}{P} \langle \psi^{(j)} | \rho^{(jk)} | \psi^{(j)} \rangle,$$

where $P = \langle \psi^{(j)} | \rho^{(jk)} | \psi^{(j)} \rangle$ is the probability with which that particular outcome occurs. In Eq. (14) the matrix operations are done only in the space of particle $j$. To be more explicit, if $r = 0, 1$ and $s = 0, 1$ are indices associated with particles $j$ and $k$ respectively, we can write the components of $\omega^{(k)}$ in terms of the components of $|\psi^{(j)}\rangle$ and $\rho^{(jk)}$ as

$$\omega^{(k)}_{ss'} = \frac{1}{P} \sum_{r,r'} \psi^{(j)*}_{r} \rho^{(jk)}_{rs,s' \prime} \psi^{(j)}_{r'},$$

We want to use this measurement scenario to associate with the two-particle state $\rho^{(jk)}$ a simple rotation $U(k, j)$. First, let $\mathcal{M}$ (for “measurement”) be the mapping defined by Eq. (13), which takes each pure state of particle $j$ for which $P \neq 0$ into a pure or mixed state of particle $k$; that is,
\( \mathcal{M}(\psi^{(j)}) = \omega^{(k)} \). If \( P = 0 \), let us say for definiteness that \( \mathcal{M}(\psi^{(j)}) = \text{Tr}_j \rho^{(jk)} \), though it will not actually matter in what follows. For an arbitrary rebit rotation \( R \), we define a function \( F(R) \) by

\[
F(R) = \frac{\int |\langle \psi | R^\dagger \mathcal{M}(\psi) R | \psi \rangle| P d\psi}{\int P d\psi}.
\]

(16)

Here \( P \) is the probability given above, and \( d\psi \) indicates the uniform measure over the circle of pure rebit states, normalized so that \( \int d\psi = 1 \). Thus \( F(R) \) is an average fidelity of \( \mathcal{M}(\psi) \), not with respect to \( |\psi\rangle \) itself but with respect to a rotated version of \( |\psi\rangle \). As we will see in the next paragraph, depending on the density matrix \( \rho^{(jk)} \), one of the following two conditions will hold:

(i) \( F(R) \) is independent of \( R \), or
(ii) there is a unique rotation \( R = U \) that maximizes \( F(R) \). In case (i), we say that there is no correlation between particles \( j \) and \( k \). In case (ii), we take the special rotation \( U \) that maximizes \( F(R) \) to be the parallel transporter associated with the link \( \langle j, k \rangle \). In a certain sense, \( U \) is the rotation that most closely approximates the action of \( \mathcal{M} \).

Combining Eqs. (14) and (16) and the definition of \( P \), we have

\[
F(R) = \frac{\int (|\psi\rangle \otimes |\psi\rangle R^\dagger) \rho^{(jk)} (|\psi\rangle \otimes R |\psi\rangle) d\psi}{\int |\psi\rangle \text{Tr}_k \rho^{(jk)} |\psi\rangle d\psi}.
\]

(17)

It is not hard to show that the denominator is always 1/2. Also, in the horizontal representation, the integral in the numerator involves rotating \( |\psi\rangle \) around the \( z \) axis, so that we can rewrite Eq. (17) as

\[
F(R) = 2(|\psi_0\rangle \otimes |\psi_0\rangle R^\dagger) \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{i\gamma S_z} \rho^{(jk)} e^{-i\gamma S_z} d\gamma \right] (|\psi_0\rangle \otimes R |\psi_0\rangle),
\]

(18)

where \( S_z = (\sigma_z^{(j)} + \sigma_z^{(k)})/2 \) and \( |\psi_0\rangle \) is some fixed reference state which for definiteness we take to be \( |\psi_0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \). Now, a general two-particle density matrix satisfying the rebit condition (11) is of the form

\[
\rho^{(jk)} = \begin{pmatrix}
  a & x_1 & x_2 & x_3 \\
  x_1^* & b & c & x_2 \\
  x_2^* & c^* & b & x_1 \\
  x_3^* & x_2^* & x_1^* & a
\end{pmatrix},
\]

(19)
the representation being in the standard basis \{\ket{\uparrow\uparrow}, \ket{\uparrow\downarrow}, \ket{\downarrow\uparrow}, \ket{\downarrow\downarrow}\}. The average over \(\gamma\) in Eq. (18) has the effect of replacing the \(x_i\)'s in \(\rho^{(jk)}\) with zero and leaving the matrix elements \(a, b,\) and \(c\) unchanged. To evaluate \(F(R)\), we write \(R\) explicitly as a rotation around the \(z\) axis by some angle \(\xi\):

\[
R = \begin{pmatrix}
  1 & 0 \\
  0 & e^{i\xi}
\end{pmatrix}.
\]

(20)

Inserting this matrix into Eq. (18) we find that

\[
F(R) = \frac{1}{2}[1 + 2|c| \cos(\xi - \phi)],
\]

(21)

where \(\phi\) is the phase of the matrix element \(c\); that is, \(c = |c|e^{i\phi}\). If \(c\) is zero, then we find ourselves in case (i) mentioned above: \(F(R)\) is independent of \(R\). Otherwise \(F\) is maximized when the angle of rotation \(\xi\) is equal to the phase \(\phi\). According to our prescription, then, the parallel transporter \(U(k, j)\) is the rotation given in Eq. (20) with \(\xi = \phi\).

We will take as our measure of the degree of correlation between particles \(j\) and \(k\) the quantity \(2|c|\), which ranges from 0 to 1. Eq. (21) makes it clear that \(2|c|\) measures the degree of angular correlation between the two particles. We can also interpret this quantity in terms of the more standard correlation matrix \(\tau_{\mu\nu} = \text{Tr}[\rho^{(jk)}(\sigma_\mu \otimes \sigma_\nu)]\), in which both \(\mu\) and \(\nu\) take as values the axis labels \(x\) and \(y\). One finds that \((2|c|)^2 = \left[\left(\tau_{xx} + \tau_{yy}\right)^2 + (\tau_{xy} - \tau_{yx})^2\right]/4 = \left[2 \det \tau + \text{Tr}(\tau^T \tau)\right]/4\).

Note that the single complex number \(c\) determines both the parallel transporter (through its phase) and the degree of correlation (through its magnitude). Typically we will be trying to maximize the magnitude of \(c\) for a given value of its phase. The fact that it is possible for a link \(\langle j, k \rangle\) to have an undefined parallel transporter \(U(k, j)\) (case (i) above, where \(c = 0\)) will not cause any difficulties for the problem we will be studying. We will be considering the set of all states that are consistent with a given specification of the parallel transporters, \textit{i.e.}, the phases of the \(c\)'s. If a link has \(c = 0\), then we simply say that that link is consistent with any specified phase.

Our notion of parallel transport has a particularly simple interpretation if \(\rho^{(jk)}\) has the form (14) with all the \(x_i\)'s equal to zero. This will happen, for example, if the state of the lattice is invariant under identical rotations of all the rebits. If \(\rho^{(jk)}\) has this form, then Eq. (14) yields

\[
\omega^{(k)} = p U(k, j)\ket{\psi(j)}\bra{\psi(j)} U(k, j)^\dagger + (1 - p)(I/2),
\]

(22)
where $I$ is the $2 \times 2$ identity and $p = 2|c|$. Thus the post-measurement state of particle $k$ is simply a rotated and partially depolarized version of $|\psi^{(j)}\rangle$, and the weight $p$ of the pure rotated state is our measure of correlation.

Let us now imagine transporting, mathematically, a rebit state around a closed loop in accordance with the above prescription. As we will see, the final state in such a process need not be the same as the initial state but could be rotated by some angle $\theta$. One might ask: How does one interpret physically this process of transport, and what is the meaning of the rotation angle $\theta$? I regard our concept of parallel transport primarily as a mathematical notion; nothing is being physically transported. However, in the special case considered above, in which the state of the lattice is rotationally invariant, one can extract from our definition a simple physical interpretation of the net rotation angle. Consider a closed loop of $n$ lattice sites $j_0, j_1, \ldots, j_{n-1}$. At each of the sites $j_0, \ldots, j_{n-2}$, that is, at all but the last site, perform an orthogonal measurement, with outcomes labeled ‘0’ and ‘1’, choosing the measurement at $j_m$ so that it maximizes the probability of getting the same outcome (0 or 1) as at site $j_{m-1}$. That is, we are trying to minimize the expected number of flips from 0 to 1 or from 1 to 0 as we go around the loop. Now, at the last site, $j_{n-1}$, one is faced with a dilemma: there will be a measurement that maximizes the probability of agreement between $j_{n-2}$ and $j_{n-1}$, and there will be a (possibly different) measurement that maximizes the probability of agreement between $j_{n-1}$ and $j_0$. The angle between these two measurements, that is, between their ‘0’ eigenstates, is the angle $\theta$ associated with parallel transport around the loop. We can think of this angle as measuring the net “twist” in the nearest-neighbor correlations.

When the state of the lattice is not rotationally invariant, the angle between the two competing optimal measurements at the last site (the two being reckoned optimal from different directions) may depend on the choice of the initial measurement on the first particle. So the interpretation in this case is not as simple. Still, it is reasonable to think of the net rotation angle as a measure of the net twist in the correlations. In the following section where we analyze the case of a rebit ring, we will find that the optimal states, which are the states most relevant to our problem, are in fact rotationally invariant, so that the above interpretation applies.
5 Analysis of a rebit ring

So far we have not made any assumptions about the structure of the lattice. In order to obtain a concrete result, we now specialize to the simplest possible lattice for which our general question can be addressed, namely, a closed one-dimensional ring. Let the ring consist of \( n \) rebits with \( n \geq 2 \), labeled by \( j = 0, 1, \ldots, n - 1 \); the labeling is mod \( n \), so that \( j = n \) is the same as \( j = 0 \).

Let \( c_j \) be the matrix element \( c \) of Eq. (19) when the two particles in question are particles \( j \) and \( j + 1 \). The quantity \( K = \frac{2}{n} \sum_j |c_j| \) will be the measure of average nearest-neighbor correlation that we will be trying to maximize; note that \( 0 \leq K \leq 1 \). The product \( \prod_j (c_j/|c_j|) \), which we will call \( e^{i\theta} \), is the net phase factor associated with transport around the whole ring, and \( \theta \) (defined only mod \( 2\pi \)) is the net rotation angle. Our question is this: what is the maximum possible value of \( K \) for a fixed value of \( \theta \)? Let us call this maximum value \( K_{\text{max}}(\theta) \). If \( K_{\text{max}}(\theta) \) decreases as \( e^{i\theta} \) gets farther from unity, then we can say that there is a correlation cost associated with non-trivial parallel transport.

As we have stated the problem so far, the phases of the different \( c_j \)'s, that is, the phases that define the individual parallel transporters, need not be the same for all links in the ring. However, for our purpose there is no loss of generality in assuming that these phases are all equal. This is because if they were not equal, we could always apply local rotations to the individual rebits (analogous to a gauge transformation) so as to make them equal. Local rotations can change neither the magnitude of any \( c_j \) nor the overall phase factor \( e^{i\theta} \). So the restriction to equal phases does not eliminate any states that might change the answer to our question. That is, states with maximal \( K \) for any given \( \theta \) are still represented in the restricted set.

With this restriction, we can simplify our problem by expressing the average correlation \( K \) and the overall phase \( \theta \) in terms of creation and annihilation operators. Let \( a_j \) be an operator on particle \( j \) defined by \( a_j |\uparrow\rangle = |\downarrow\rangle \) and \( a_j |\downarrow\rangle = 0 \). Then for the link between the \( j \)th and the \( (j + 1) \)st rebits, we can write \( c_j \) as

\[
c_j = \text{Tr}(\rho a_{j+1}^\dagger a_j),
\]

(23)

\( \rho \) being the density matrix of the ring. Now if we define the operator \( \Gamma \) to be

\[
\Gamma = \frac{1}{n} \sum_{j=0}^{n-1} a_{j+1}^\dagger a_j
\]

(24)
and write $\langle \Gamma \rangle = \text{Tr} \rho \Gamma$, it follows (assuming that each $c_j$ has the same phase) that

$$K = 2|\langle \Gamma \rangle|,$$

(25)

and that the overall phase $\theta \pmod{2\pi}$ is simply

$$\theta = n \arg(\langle \Gamma \rangle).$$

(26)

$\Gamma$ is not a Hermitian operator—so its eigenvalues may be complex and indeed must be complex if $\theta$ is to take any non-trivial value—but $\Gamma$ does commute with its adjoint, which implies that eigenvectors corresponding to distinct eigenvalues are orthogonal. Note also that $\Gamma$ commutes with the total $z$ component of spin; so the eigenstates of $\Gamma$ can be taken to be states with a definite number $u$ of up spins. Our immediate goal is to find the eigenvalues of $\Gamma$, from which we will be able to determine the set of possible values of $\langle \Gamma \rangle$, which in turn will give us $K_{\text{max}}(\theta)$.

The $a$ operators are not quite fermionic creation and annihilation operators, because the operators associated with different sites commute with each other rather than anticommute. However, we can use a standard trick \[21\] to define genuinely fermionic operators $b_j$:

$$b_j = \exp \left[ i\pi \sum_{k=0}^{j-1} a_k^\dagger a_k \right] a_j.$$

(27)

The operators $b_j$ satisfy the usual fermionic anticommutation relations:

$$\{b_j, b_k\} = \{b_j^\dagger, b_k^\dagger\} = 0; \quad \{b_j, b_k^\dagger\} = \delta_{jk}.$$

(28)

We now express $\Gamma$ in terms of the $b$’s. The expression depends on whether $u$, the number of up spins in the ring, is even or odd; that is, the expression is different in different subspaces. For odd values of $u$, $\Gamma$ looks the same in terms of the $b$’s as it does in terms of the $a$’s:

$$\Gamma = \frac{1}{n} \sum_{j=0}^{n-1} b_{j+1}^\dagger b_j.$$

(29)

For even values of $u$, there is a sign change in the last term:

$$\Gamma = \frac{1}{n} \left[ \sum_{j=0}^{n-2} b_{j+1}^\dagger b_j - b_0^\dagger b_{n-1} \right].$$

(30)
In either case, we can diagonalize $\Gamma$ and find its exact single-fermion eigenvalues. From these we can obtain the eigenvalues of $\Gamma$ for an arbitrary value of $u$ by summing $u$ of the single-fermion eigenvalues.

For odd $u$, the single-fermion eigenstates of $\Gamma$ are $|\omega_m\rangle = d_m^\dagger |0\rangle$, $m = 0, \ldots, n - 1$, where $|0\rangle$ is the vacuum state, that is, the state with all spins down, and the creation operator $d_m^\dagger$ is given by

$$d_m^\dagger = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{-2m\pi ij/n} b_j^\dagger.$$  \hfill (31)

The corresponding eigenvalues of $\Gamma$ are

$$g_m = (1/n)e^{2m\pi i/n}. \hfill (32)$$

For even $u$, the creation operators for the single-fermion eigenstates are

$$d_m^\dagger = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{-(2m+1)\pi ij/n} b_j^\dagger, \quad m = 0, \ldots, n - 1,$$  \hfill (33)

and the corresponding eigenvalues are

$$g_m = (1/n)e^{(2m+1)\pi i/n}. \hfill (34)$$

Regardless of the value of $u$, the single-fermion eigenvalues of $\Gamma$ are complex numbers of length $1/n$, with phases uniformly spaced around the complex plane.

The eigenvalues of $\Gamma$ corresponding to a system of $u$ fermions (that is, $u$ up spins) are all the possible sums of $u$ of the $g_m$'s. That is, we can write each such eigenvalue as

$$G(M) = \sum_{m \in M} g_m,$$  \hfill (35)

where $M$ is a set of exactly $u$ integers chosen from the set $\{0, \ldots, n-1\}$. (We cannot use the same value of $m$ twice in this sum because no two identical fermions can be in the same state, and all the single-fermion eigenvalues are non-degenerate.)

As we will see, the most important eigenvalues $G(M)$ for our purpose will be the ones with the greatest magnitude. For even values of $n$, these are the ones for which $u = n/2$ and the set $M$ consists of a string of consecutive
integers (mod \( n \)), so that the corresponding values \( g_m \) constitute a “fan” of complex numbers spread out over half of the complex plane. Performing the sum in Eq. (35), we find that these extreme eigenvalues are

\[
G_r = \frac{e^{2r\pi i/n}}{n \sin(\pi/n)}, \quad r = 0, \ldots, n-1, \quad \text{(even } n) \]

an equation that holds for all even values of \( n \) (even though \( u = n/2 \) might be even or odd). For odd values of \( n \), the eigenvalues with largest magnitude are obtained by setting \( u \) equal to either \((n + 1)/2\) or \((n - 1)/2\), and again letting \( M \) consist of a string of consecutive integers mod \( n \). For either of these choices of \( u \), the eigenvalues thereby obtained are

\[
G_r = \frac{e^{2r\pi i/n}}{n \sin(\pi/n)} \cos(\pi/(2n)), \quad r = 0, \ldots, n-1. \quad \text{(odd } n) \]

For either even or odd \( n \), each eigenstate corresponding to one of the eigenvalues \( G_r \) can be written as

\[
|\Omega_{r,u}\rangle = d_{m_0}^\dagger d_{m_0+1}^\dagger \cdots d_{m_0+u-1}^\dagger |0\rangle, \quad (38)
\]

where \( m_0 = r - \lfloor u/2 \rfloor \) mod \( n \). (Here and below, addition in the subscript of \( d \) is always mod \( n \).) This value of \( m_0 \) places \( \exp(2r\pi i/n) \) in the center of the fan of complex eigenvalues \( g_{m_0}, \ldots, g_{m_0+u-1} \). For later convenience, we define the following density matrices based on these eigenstates. For even \( n \),

\[
\rho_r = |\Omega_{r,n/2}\rangle \langle \Omega_{r,n/2}|, \quad (39)
\]

and for odd \( n \),

\[
\rho_r = \frac{1}{2} \left[ |\Omega_{r,(n+1)/2}\rangle \langle \Omega_{r,(n+1)/2} | + |\Omega_{r,(n-1)/2}\rangle \langle \Omega_{r,(n-1)/2} | \right]. \quad (40)
\]

As we will see shortly, both of these density matrices satisfy the rebit condition, even though \( |\Omega_{r,(n+1)/2}\rangle \) and \( |\Omega_{r,(n-1)/2}\rangle \) do not. Note also that for both even and odd \( n \), \( \text{Tr} \rho_r \Gamma = G_r \).

One can check that \( \rho_r \) is translationally invariant and thus is in accord with the assumption we made earlier, that the matrix element \( c_j \) has the same phase for each link of the ring. But we also want to check that \( \rho_r \) is a
legitimate rebit state, i.e., that it satisfies Eq. (11). Let us do this first for even values of \( n \), in which case we are dealing with a pure state \(|\Omega_{r,n/2}\rangle\).

We begin by noting that
\[
\sigma_x \otimes b_j^\dagger = (-1)^j b_j \sigma_x \otimes n,
\] (41)
which can be seen directly from the definition of \( b_j \). It follows that
\[
\sigma_x \otimes n d_{m_1}^\dagger \cdots d_{m_u}^\dagger |0\rangle = [d_{m_1+n/2} \cdots d_{m_u+n/2} \sigma_x \otimes n |0\rangle]^*,
\] (42)
where \( m_1, \ldots, m_u \) are any distinct values chosen from the set \( \{0, \ldots, n-1\} \). (The addition of \( n/2 \) in the subscripts on the right-hand side comes from \((-1)^j\) in the preceding equation.) In general, a state of the form \( d_{m_1}^\dagger \cdots d_{m_u}^\dagger |0\rangle \) will not satisfy the rebit condition, even if \( u = n/2 \). However, for the special case in which \( m_1, \ldots, m_u \) are \( n/2 \) consecutive integers, as they are in the definition of \( |\Omega_{r,n/2}\rangle \), the subscripts on the right-hand side of Eq. (42) are precisely those elements of \( \{0, \ldots, n-1\} \) that are not included in \( \{m_1, \ldots, m_u\} \).

Therefore, when those annihilation operators are applied to the all-up-spin state \( \sigma_x \otimes n |0\rangle \), which within a phase factor is the same as \( d_0^\dagger d_1^\dagger \cdots d_{n-1}^\dagger |0\rangle \), the resulting state, again up to an overall phase factor, is \( d_{m_1}^\dagger \cdots d_{m_u}^\dagger |0\rangle \). We have thus shown that
\[
\sigma_x \otimes n |\Omega_{r,n/2}\rangle = e^{i\beta} |\Omega_{r,n/2}'\rangle
\] (43)
for some phase \( \beta \), so that \( \rho_r \) satisfies the rebit condition (10) for even values of \( n \).

Turning now to the case of odd \( n \), one can use an argument like the one in the preceding paragraph to show that
\[
\sigma_x \otimes n |\Omega_{r,n/2}\rangle \langle \Omega_{r,n/2}| \sigma_x \otimes n = |\Omega_{r,(n+1)/2}\rangle \langle \Omega_{r,(n+1)/2}|,
\] (44)
and vice versa, so that
\[
\sigma_x \otimes n \rho_r \sigma_x \otimes n = \rho_r^*.
\] (45)
So \( \rho_r \) satisfies the rebit condition for odd values of \( n \) as well.

We are now in position to find the set—call it \( \mathcal{G} \)—of possible values of \( \langle \Gamma \rangle \), from which we will be able to determine \( K_{\text{max}}(\theta) \). The complex numbers \( G_r \) given by Eq. (36) or Eq. (37), being values of \( \langle \Gamma \rangle \) corresponding to the legitimate rebit states \( \rho_r \), are elements of \( \mathcal{G} \). By taking mixtures of these states, we can obtain other possible values of \( \langle \Gamma \rangle \). Let
\[
\rho = \sum_r q_r \rho_r,
\] (46)
where the \( q \)'s are non-negative numbers summing to 1. For this state we have
\[
\langle \Gamma \rangle = \text{Tr} \rho \Gamma = \sum_r q_r G_r. \tag{47}
\]
The complex numbers \( G_r \) are the vertices of a regular \( n \)-gon in the complex plane, and Eq. (47) shows that this \( n \)-gon and its interior are contained in \( G \).

In fact it is easy to see that \( G \) contains no other points. Any complex number \( \langle \Gamma \rangle \) in \( G \) must be a weighted average of eigenvalues of \( \Gamma \):
\[
\langle \Gamma \rangle = \sum_M q_M G(M). \tag{48}
\]
But one can show that each eigenvalue \( G(M) \), regardless of the value of \( u \), lies on or inside the \( n \)-gon defined by the special eigenvalues discussed above. Therefore it is impossible for the average to get outside this region.

For the special case \( n = 2 \), the interior of the "\( n \)-gon" is simply a segment of the real axis, running from \(-1/2\) to \(+1/2\). Thus the only possible phases of \( \langle \Gamma \rangle \) are zero and \( \pi \), so that according to Eq. (48) the only possible value of \( \exp(i\theta) \) is 1. (This is because there is no real loop to traverse; to return to the starting place, one has to retrace one's steps.) For all other values of \( n \), all values of \( \theta \) from 0 to \( 2\pi \) are possible. To see this, it is enough to consider a single side of the \( n \)-gon; let us take the side consisting of the line segment joining the point \( G_0 \), on the positive real axis, with the point \( G_1 = e^{2\pi i/n} G_0 \). As we travel along this segment, the phase of \( \langle \Gamma \rangle \) varies from 0 to \( 2\pi/n \), so that \( \theta \) varies from 0 to \( 2\pi \). The range of values of \( |\langle \Gamma \rangle| \) as a function of \( \theta \) is the same for each of the other sides of the \( n \)-gon.

It thus becomes a simple geometric problem to find \( K_{\text{max}}(\theta) \). For \( n > 2 \), consider the line segment just described, connecting \( G_0 \) to \( G_1 \), and note that for any \( \theta \) in the range \( 0 \leq \theta \leq 2\pi \), \( K_{\text{max}}(\theta) \) is twice the magnitude of the unique point along this segment whose phase is \( \theta/n \). Doing the geometry, and using the values of \( G_r \) given in Eqs. (36) and (37), one finds that for even \( n \),
\[
K_{\text{max}}(\theta) = \frac{2 \cos(\pi/n)}{n \sin(\pi/n) \cos[(\pi - \theta)/n]}, \tag{49}
\]
and for odd \( n \),
\[
K_{\text{max}}(\theta) = \frac{2 \cos(\pi/n) \cos(\pi/(2n))}{n \sin(\pi/n) \cos[(\pi - \theta)/n]}. \tag{50}
\]
It is clear both from the geometric picture and from Eqs. (49) and (50) that $K_{\text{max}}$ is largest at $\theta = 0$ and $\theta = 2\pi$ and smallest at $\theta = \pi$. Indeed, the value of $K_{\text{max}}$ becomes smaller the more $e^{i\theta}$ differs from unity. In this sense there is a correlation cost of non-trivial parallel transport around the ring.

It is not hard to interpret the states $\rho_r$ physically. The state $\rho_0$, which entails no twisting as one goes around the ring, is the ground state, or in the case of odd $n$ an equal mixture of the two degenerate ground states, of the ferromagnetic XY model [21] on a one-dimensional ring, whose Hamiltonian is $H = -\sum_j (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) = -n(\Gamma^\dagger + \Gamma)$. The state $\rho_r$ can be obtained from $\rho_0$ by rotating each rebit, the rotation angle at site $j$ being $2\pi r j/n$. These rotations do not change the strength of the nearest-neighbor correlations, but for each site $j$ they change the phase of the matrix element $c_j$ from zero to $2\pi r/n$. Still, this does not change the overall phase factor $e^{i\theta}$ associated with the whole ring. When one creates a mixture of two of these differently rotated states, e.g., $\rho_0$ and $\rho_1$, the resulting matrix element $c_j$ is an average of two complex numbers with different phases. It is this averaging process that allows the possibility of a non-trivial net phase change around the ring.

In the case of even $n$, where $\rho_r$ represents the pure state $|\Omega_{r,n/2}\rangle$, one could achieve the same averaging effect by creating coherent superpositions of eigenstates of $\Gamma$ rather than incoherent mixtures. I have chosen to use mixtures because superpositions of these eigenstates are not necessarily translationally invariant. However, just to demonstrate that it is possible for a translationally invariant pure state to have a non-zero rotation angle $\theta$ associated with parallel transport around the ring, I offer the following example for $n = 6$:

$$|\Psi\rangle = \frac{i}{\sqrt{12}}\left[e^{i\phi/2}(|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\rangle + \cdots) + e^{-i\phi/2}(|\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle + \cdots)\right].$$  \hspace{1cm} (51)$$

Here each ellipsis stands for all possible translations of the given state. (Though we are not paying particular attention to the overall phase factors of pure states, I have chosen the overall phase in Eq. (51) to satisfy Eq. (9).) One finds that the matrix element $c_j$ for each link in this ring is $(1/6)e^{i\phi}$, so that the overall rotation angle is $\theta = 6\phi$ and the average correlation is $K = 1/3$. Thus any value of $\theta$ can be realized with a translationally invariant pure state. But the value $K = 1/3$ is not optimal. To obtain the optimal value $K_{\text{max}}(\theta)$ in a translationally invariant state, one
must typically use mixed states rather than pure states.

We finish this section by giving asymptotic expressions for $K_{\text{max}}(\theta)$ as the number of rebits in the ring gets very large. For even $n$, Eq. (49) to order $1/n^2$ becomes (for $0 \leq \theta \leq 2\pi$)

$$K_{\text{max}}(\theta) = \frac{2}{\pi} \left[ 1 + \frac{1}{n^2} \left( \frac{\pi^2}{6} - \pi \theta + \frac{\theta^2}{2} \right) \right],$$

while for odd $n$ we have

$$K_{\text{max}}(\theta) = \frac{2}{\pi} \left[ 1 + \frac{1}{n^2} \left( \frac{\pi^2}{24} - \pi \theta + \frac{\theta^2}{2} \right) \right].$$

Thus the correlation cost of non-trivial parallel transport becomes smaller as the size of the ring increases.

6 Other lattices

Let us now think about how the above problem might be generalized to a finite or infinite lattice of higher dimension (still using rebits as our basic objects). We can state the problem as follows. As in Section 4, let $\rho^{(j,k)}$ be the density matrix of the pair of rebits at the neighboring sites $j$ and $k$, and let $c_{j,k}$ be the coefficient of $|\uparrow\downarrow\rangle\langle\downarrow\uparrow|$ in this density matrix. (This $c_{j,k}$ is analogous to the $c_j$ of the preceding section.) Now suppose that the phases of all the $c$’s, for all the links $\langle j, k \rangle$, are specified. We will call this complete specification $\mathcal{A}$, since it is analogous to a field configuration in the $U(1)$ gauge theory. Given this specification, we have two questions: (i) Is it possible to find a lattice state for which the numbers $c_{j,k}$ are all non-zero and have the chosen phases? (One can always find a state in which all the $c_{j,k}$’s are zero, making the state consistent with any phases, but such a state is not very interesting.) (ii) What is the maximum possible value of

$$K = \frac{2}{L} \sum_{(j,k)} |c_{j,k}|,$$

consistent with the specification $\mathcal{A}$? Here $L$ is the number of links in the lattice. In the case of an infinite lattice, $K$ can be defined as a limit over a sequence of finite lattices. Let us call the maximum value $K_{\text{max}}(\mathcal{A})$.  

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For definiteness let us consider a specific lattice, namely, an infinite square lattice in two dimensions. Let us consider first the configuration $\mathcal{A}_0$ in which the phases of all the $c_{j,k}$’s are zero. In this case it is again helpful to invoke the Hamiltonian of the ferromagnetic $XY$ model:

$$H = - \sum_{\langle j,k \rangle} (a_k^\dagger a_j + a_j^\dagger a_k).$$

(55)

Here the sum is over all links in the square lattice. The optimal value of $K$ is the infinite-lattice limit of $(-E_0/L)$, $E_0$ being the minimum eigenvalue of this Hamiltonian. In the thermodynamic limit, the ground state of the $XY$ model on a square lattice breaks the $SO(2)$ symmetry of the problem and picks out a preferred direction of magnetization in the $x$-$y$ plane [22, 23], which can be characterized by a single angle $\alpha$. But if we choose to do so (in order to simplify the interpretation of parallel transport), we can easily generate a rotationally invariant state with the same energy—or in our context, with the same degree of correlation—simply by averaging the ground state density matrix $\rho(\alpha)$ over all angles: $\rho_0 = (1/2\pi) \int \rho(\alpha) d\alpha$. The ground-state energy has been evaluated numerically [24, 25, 26], and one finds that $K_{\text{max}}(\mathcal{A}_0) = 0.549$. Notice that this value is smaller than the corresponding value for a rebit ring in the limit $n \to \infty$ (see Eqs. (52) and (53) or Ref. [21]), which is $2/\pi = 0.637$.

Given a different specification of the phases, it is not immediately obvious whether there exists a state that has all non-zero $c_{j,k}$’s—let us call such a state “fully connected”—and that is also consistent with the given phases. Consider, for example, the configuration $\mathcal{A}$ in which all the phases are zero except at a specific link $\langle j, k \rangle$, where the phase is required to be $\phi$. Can one find a fully connected state of the lattice consistent with these phases? The following method will work, though it is not likely to be optimal. Start with the state $\rho_0$ defined in the preceding paragraph, in which all the phases are zero. Construct the following sum:

$$\rho(j, k, \phi) = \frac{1}{3} \left[ V_j \rho_0 V_j^\dagger + V_k^\dagger \rho_0 V_k + (V_j^\dagger \otimes V_k) \rho_0 (V_j \otimes V_k^\dagger) \right],$$

(56)

where $V_j$ is the matrix

$$V_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\xi} \end{pmatrix}$$

(57)
applied to particle $j$ and $V_k$ is the same matrix applied to particle $k$. Here $\xi$ will be a function of $\phi$ to be determined later. Because the rotations in Eq. (56) affect only particles $j$ and $k$, all links not involving either of these particles will continue to have zero phase. Moreover, any link involving only one of the two special sites will likewise have its phase unchanged. Consider, for example, the link $\langle j, l \rangle$ where $l \neq k$. The value of $c_{j,l}$ is

$$c_{j,l} = \frac{1}{3}(c_0 e^{-i\xi} + c_0 + c_0 e^{i\xi}) = \left(\frac{1 + 2 \cos(\xi)}{3}\right)c_0,$$  

so that the phase has not been affected. On the other hand, the value of $c$ associated with the link $\langle j, k \rangle$ is

$$c_{j,k} = \left(\frac{2e^{-i\xi} + e^{2i\xi}}{3}\right)c_0,$$  

which has a phase that can be made equal to $\phi$ by a proper choice of the value of $\xi$.

An even simpler strategy, which is surely not optimal, shows that any phase configuration $A$ can be realized in a fully connected quantum state of the lattice. Let us imagine the two dimensions of the lattice to be horizontal and vertical. Start with a state in which each vertical column of lattice sites is in the ground state of the ferromagnetic XY model for an infinite chain. Call this state $|\phi_V\rangle$. Now rotate each of the rebits in each of these chains so as to achieve the desired phases for the vertical links. This can be done with no loss of correlation, because we are simply performing local rotations. Similarly, consider the state $|\phi_H\rangle$ in which each horizontal row is in the XY ground state, and rotate the rebits so as to achieve the desired phases for the horizontal links. Let $|\phi'_V\rangle$ and $|\phi'_H\rangle$ be the states resulting from these rotations. Then the mixed state

$$\rho = \frac{1}{2}(|\phi'_V\rangle\langle\phi'_V| + |\phi'_H\rangle\langle\phi'_H|)$$  

(60)

completely matches the phase configuration $A$. We can even compute the value of $K$ for the state $\rho$: it is equal to half of $K_{\text{max}}$ for the infinite chain, independent of the configuration $A$. That is, $K = 1/\pi = 0.318$. For the special configuration considered above, in which only one link has non-zero phase, this value is smaller than what one can achieve with the specialized
method of Eq. (57). Nevertheless, the method we have just described does answer our first question: all configurations $\mathcal{A}$ can be achieved without making any $c_{j,k}$ vanish. Notice also that this construction gives us a lower bound on $K_{\text{max}}(\mathcal{A})$ for all configurations $\mathcal{A}$: $K_{\text{max}}(\mathcal{A}) \geq 1/\pi$.

Actually finding $K_{\text{max}}(\mathcal{A})$, even for simple configurations $\mathcal{A}$, is probably a very hard problem. If one uses strategies similar in spirit to the one given in Eq. (57), then it would seem that the value of $K_{\text{max}}(\mathcal{A})$ must decrease in order to achieve non-trivial parallel transport around a loop. (One can see the decrease in the values of $|c_{j,l}|$ and $|c_{j,k}|$ in Eqs. (58) and (59).) But it is conceivable that a completely different strategy could do much better; so we must leave this basic question unanswered.

7 Lattices of qubits

In our definition of parallel transport for rebits, developed in Section 4, we implicitly made use of the fact that for rebits, there exists a two-particle state $\rho^{(+)}$ such that if particles $j$ and $k$ are in this state, and if particle $j$ is measured and found to be in the state $|\psi\rangle$, then particle $k$ will always be brought to the same state $|\psi\rangle$. Mathematically,

$$\omega = \frac{1}{P} \langle \psi|\rho^{(+)}|\psi\rangle = |\psi\rangle\langle\psi|.$$  

(61)

The state $\rho^{(+)}$ is in fact $|\Psi^{(+)}\rangle\langle\Psi^{(+)}|$, with $|\Psi^{(+)}\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$. So for this state, the parallel transporter is the identity and the degree of correlation is 1. In effect, our definition of parallel transport compares other two-particle states to this special state; when the parallel transporter $U$ is not the identity, it is because one of the particles has been rotated (and possibly distorted in other ways as well) compared to the standard state $\rho^{(+)}$.

For qubits, there is no two-particle state with this property. The most closely analogous state is the singlet state $|\Psi^{(-)}\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. It has the property that if a measurement on one of the particles brings it to the state $|\psi\rangle$, the other particle will be brought to the orthogonal state $|\tilde{\psi}\rangle = \sigma_y|\psi^*\rangle$, where the complex conjugation is in the standard basis. In defining parallel transport for qubits, we will take the singlet state as our standard state, for which the parallel transporter is defined to be the identity. All other states will then be compared with this one. A general parallel
transporter will be a rotation of the Bloch sphere, that is, an element of SO(3); as in the case of rebits, our definition will not allow us to distinguish overall phases. We will usually represent such a rotation as a $2 \times 2$ unitary matrix, keeping in mind that the overall phase is irrelevant.

We start again with Eq. (14):

$$\omega^{(k)} = \frac{1}{P} \langle \psi^{(j)} | \rho^{(jk)} | \psi^{(j)} \rangle,$$

and again let $\mathcal{M}$ be the map that takes $|\psi^{(j)}\rangle$ to $\omega^{(k)}$. But now we define our generalized fidelity as

$$F(R) = \frac{\int \langle \tilde{\psi} | R^\dagger \mathcal{M}(\psi) R | \tilde{\psi} \rangle P d\psi}{\int P d\psi},$$

where $d\psi$ refers to the uniform measure over the surface of the Bloch sphere. Note that $F$ compares $\mathcal{M}(\psi)$ with a rotated version of $|\tilde{\psi}\rangle$ rather than a rotated version of $|\psi\rangle$. If there is a unique rotation $R = U$ (up to an overall phase) that maximizes $F(R)$, then we will take this $U$ to be the parallel transporter for the link $\langle j, k \rangle$. Thus, if $\rho^{(jk)}$ happens to be the singlet state, we have $U(k,j) = I$.

Carrying out the integrals in Eq. (63), we find that

$$F(R) = \frac{1}{3} + \frac{2}{3} (\Psi^{-}) (I \otimes R^\dagger) \rho^{(jk)} (I \otimes R) (\Psi^{-})^\dagger.$$

Now, it is a fact that any maximally entangled state of two qubits can be written as $(I \otimes R) |\Psi^{-}\rangle$. So maximizing $F(R)$ over all rotations $R$ is the same as finding the maximally entangled state that has the greatest overlap with $\rho^{(jk)}$. The quantity

$$f = \max_R \langle \Psi^{-} | (I \otimes R^\dagger) \rho^{(jk)} (I \otimes R) |\Psi^{-}\rangle$$

has been called the “fully entangled fraction” of $\rho^{(jk)}$ \cite{Ekert1991}. It ranges from $1/4$ (for the completely mixed state) to 1 (for a maximally entangled state) \cite{EPR-1935}. We will take $(4f - 1)/3$, which ranges from 0 to 1, as our measure of the degree of correlation between particles $j$ and $k$, and define $K_q$ (q for “qubit”) to have a value of $f$ greater than $1/4$.

\footnote{Even though the word “entangled” appears in the description of $f$, it is not a proper measure of entanglement; a separable state can have a value of $f$ greater than $1/4$.}
be the average of this quantity over all the links of the lattice. As has been mentioned, the rotation $R$ that achieves this maximum value, if it is unique, will be our parallel transporter $U(k, j)$. We would like to maximize $K_q$ for a fixed set of parallel transporters.

Our notion of parallel transport for qubits is particularly simple if $\rho^{(jk)}$ happens to be a “twisted Werner state,” that is, a state of the form [27]

$$p(I \otimes V)|\Psi(^{-})\rangle\langle\Psi(^{-})| (I \otimes V^\dagger) + (1 - p)(I/4).$$

(66)

Here $0 < p \leq 1$, $|\Psi(^{-})\rangle$ is the singlet state, $I$ is the $4 \times 4$ identity matrix, and $V$ is a unitary transformation acting on particle $k$. If $\rho^{(jk)}$ is of this form, then the parallel transporter $U(k, j)$ works out, not surprisingly, to be the transformation $V$. Moreover, the weight $p$ appearing in Eq. (66) is none other than our measure of correlation $(4f - 1)/3$. Thus both the parallel transporter and the degree of correlation are particularly easy to interpret in this case. One can show that the six-rebit state of Eq. (51), reinterpreted as the state of a six-qubit ring, has the property that each pair of nearest neighbors is of the twisted Werner form.

A more interesting example of a qubit ring exhibiting non-trivial parallel transport—but whose pairs are not necessarily of the twisted Werner form—is given by the following state of six qubits.

$$|\psi\rangle = \alpha(|\uparrow\uparrow\downarrow\downarrow\downarrow\rangle - \cdot \cdot \cdot) + \beta[e^{-i\xi}(\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle - \cdot \cdot \cdot] + e^{i\xi}(\uparrow\uparrow\downarrow\downarrow\uparrow\rangle - \cdot \cdot \cdot) + \gamma(|\downarrow\uparrow\uparrow\downarrow\downarrow\rangle - |\uparrow\downarrow\uparrow\downarrow\downarrow\rangle).$$

(67)

Here each ellipsis indicates all the translations of the given state, but with alternating signs. For example, the coefficient $\alpha$ multiplies

$$|\uparrow\uparrow\downarrow\downarrow\downarrow\rangle - |\uparrow\downarrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle.$$

(68)

The coefficients $\alpha$, $\beta$ and $\gamma$ are real and positive; their values will be specified shortly. Note that each pair of nearest neighbors in this state has the same density matrix, so that each has the same degree of correlation and the

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3In the case of qubits, in contrast to that of rebits, it is possible for $R$ not to be unique even when the correlation $(4f - 1)/3$ is not zero. For example, if $\rho^{(jk)} = (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)/2$, then any rotation around the $z$ axis maximizes $F(R)$, and yet the correlation $(4f - 1)/3$ has the value 1/3. However, as in the case of rebits, this sort of ambiguity does not cause any difficulties for the problem we are considering.
same parallel transporter. Because the state $|\psi\rangle$ is an eigenstate of $S_z$ with eigenvalue zero, the density matrix of each pair is of the form (19) with all the $x_i$'s equal to zero. (It is convenient to introduce a negative sign in the off-diagonal elements since our standard state is now the singlet.)

\[
\rho^{(jk)} = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & -c & 0 \\
0 & -c^* & b & 0 \\
0 & 0 & 0 & a
\end{pmatrix}.
\]  \tag{69}

Carrying out the trace of $|\psi\rangle\langle\psi|$ over the other particles, one finds that $a = 2\alpha^2 + 2\beta^2$, $b = \alpha^2 + 4\beta^2 + \gamma^2$, and $c = 2\beta[(\alpha + \gamma)e^{i\xi} + \beta e^{-2i\xi}]$. The correlation $K_q$ is $(4f - 1)/3$, where

\[
f = b + |c| = \alpha^2 + 4\beta^2 + \gamma^2 + 2\beta |(\alpha + \gamma)e^{i\xi} + \beta e^{-2i\xi}|,
\]  \tag{70}

and the parallel transporter is

\[
U = \begin{pmatrix}
1 & 0 \\
0 & e^{i\phi}
\end{pmatrix},
\]  \tag{71}

where $\phi$ is the phase of $c$, that is, the phase of $(\alpha + \gamma)e^{i\xi} + \beta e^{-2i\xi}$. The fact that $K_q$ depends on the matrix element $b$, while the analogous quantity $K$ for rebits depended only on $c$, ultimately comes from the fact that in Eq. (63) we average over the entire surface of the Bloch sphere and not just over the equator.

In the spirit of our main problem, we would like to choose $\alpha$, $\beta$ and $\gamma$ so as to maximize $K_q$. Let us do this extremization for the special case $\xi = 0$, for which the parallel transporter given in Eq. (71) is the identity. In this case one finds that the optimal values are $\alpha = (130 + 34\sqrt{13})^{-1/2}$, $\beta = (1/2)(3 + \sqrt{13})\alpha$, and $\gamma = (4 + \sqrt{13})\alpha$, which satisfy the normalization condition $6\alpha^2 + 12\beta^2 + 2\gamma^2 = 1$. In what follows we will assume that $\alpha$, $\beta$ and $\gamma$ have these values. One finds then that for $\xi = 0$, each pair of nearest neighbors is in a Werner state, and the correlation $K_q$ is $(2 + \sqrt{13})/9 = 0.623$. Let us call this value $K_q^{(0)}$.

\[\text{By no accident, the state } |\psi\rangle \text{ with these values of the coefficients is the ground state of the antiferromagnetic Heisenberg model for a ring of six qubits.} \]
How do $K_q$ and $U$ for the state $|\psi\rangle$ change as $\xi$ departs from zero? Let us first look at $U$. To lowest order in $\xi$, the angle $\phi$, which is the rotation angle associated with parallel transport across a link, is

$$\phi = \arg[(\alpha + \gamma)(1 + i\xi) + \beta(1 - 2i\xi)] = \left(\frac{\alpha + \gamma - 2\beta}{\alpha + \gamma + \beta}\right)\xi.$$  \hspace{1cm} (72)

Note that $\alpha + \gamma - 2\beta > 0$, so that this linear contribution to $\phi$ does not vanish. Meanwhile, the correlation $K_q$ diminishes by an amount proportional to the square of $\xi$:

$$K_q = K_q^{(0)} - \left(\frac{12\beta^2(\alpha + \gamma)}{\alpha + \gamma + \beta}\right)\xi^2.$$  \hspace{1cm} (73)

Letting $\theta = 6\phi$ be the net rotation associated with parallel transport around the whole ring, we can see how $K_q$ depends on $\theta$ to lowest order:

$$K_q = K_q^{(0)} - \left(\frac{\beta^2(\alpha + \gamma)(\alpha + \gamma + \beta)}{3(\alpha + \gamma - 2\beta)^2}\right)\theta^2 = 0.623 - (0.369)\theta^2.$$  \hspace{1cm} (74)

Notice that the reduction in correlation is of second order in $\theta$, whereas in the case of rebits it is of first order as seen in Eqs. (52) and (53). Of course we have not done the thorough optimization for qubits that we have done for rebit rings, but this example indicates that there is a significant difference between the two cases.

### 8 Discussion

We have shown, first of all, that there is a sense in which certain quantum states exhibit non-trivial parallel transport around a closed loop, which is to say that the nearest-neighbor correlations exhibit a net twist as one goes around the loop. One might regard this result as somewhat surprising, since there is, after all, only a single quantum state for the whole loop, and one might think that the local twists would therefore have to cancel each other out. We have also shown that in the case of the rebit ring, there is a loss of nearest-neighbor correlation associated with non-trivial parallel transport around the ring. For other lattices or for lattices of qubits, we do not know whether there is such a correlation cost, though it is certainly plausible that there would be.
As we have seen, there is a close relationship between our rebit problem and the XY model, a model that has been very well studied. Studies of spin stiffness in this model (see, for example, Ref. [26]) have a certain similarity with the problem we have been considering in that in both cases one enforces a twist between neighboring spins. What distinguishes the present work is the fact that we have not actually specified any Hamiltonian. Though it has been helpful for us to use an operator similar to the Hamiltonian of the XY model, we are really working with what might be called the *kinematics* of quantum mechanics. We are asking what correlation properties of quantum states follow from certain other correlation properties—specifically, we are asking what one can say about the *strength* of correlations given some information about the *twist* in the correlations—and this question is independent of any considerations of energy.

I introduced the subject by relating it to lattice gauge theory. To what extent, then, have we found an analogy between the state of a quantum lattice and the configuration of a lattice gauge field? In a qualitative sense, the reduction in correlation that we have observed in a rebit ring can be compared to the reduction in probability that one finds in a lattice gauge theory. But for this rebit case, the analogy must be regarded as quite rough, because there is a significant lack of congruence in the details. In gauge theory, the initial decrease in the probability is of second order in the net rotation angle $\theta$ associated with transport around a plaquette. In the U(1) theory, for example, the function $\exp(-S)$, with $S$ given by Eq. (3), decreases in proportion to $\theta^2$ for small values of $\theta$. In contrast, in the rebit ring, the average correlation $K$, as given by Eqs. (49) and (50), decreases in proportion to $\theta$ itself. The second-order dependence is in fact important in lattice gauge theory for taking the continuum limit. Moreover, our first-order dependence makes $K_{\text{max}}(\theta)$ a non-analytic function, since near $\theta = 0$ it takes the form $a - b|\theta|$. So this difference is not trivial.

On the other hand, we have just seen that the *qubit* correlation as we have defined it does seem to diminish quadratically in $\theta$, at least for a ring of six qubits. It is interesting to ask whether in the case of a two-dimensional or higher-dimensional lattice, the dependence of $K_{\text{max}}(A)$ is of second order in the rotation angles. In the one relevant example we have considered for a rebit lattice, namely, the strategy given in Eq. (57), the value of $K$ decreases as $\phi^{2/3}$, which is an even sharper dependence than in the rebit ring. But we have not explored at all fully the range of possible states that one might
consider for these higher-dimensional lattices.

It is worth commenting on the fact that in the case of qubits, our approach makes the parallel transporters elements of SO(3), whereas one might have expected SU(2). The nature of our definition does not allow us to pick out a relative phase in the relation between neighboring qubits. For example, we cannot distinguish between the identity operation and a rotation by $2\pi$, even though a pure qubit state experiencing the latter rotation picks up a phase factor of $-1$. It is conceivable that by taking into account the density matrix of an entire loop, in addition to the density matrices of the neighboring pairs, one might be able to make sense of this distinction as it applies to the net rotation associated with the loop as a whole.

There are other ways in which one might modify the problem we have been considering. One could use a different measure of correlation or entanglement. Moreover, even if one continues to use the quantity $2|c|$ for rebits and the quantity $(4f - 1)/3$ for qubits as the measure of nearest-neighbor correlation, one could combine the correlations from all the links in a different way. For example, in the case of a ring it would make some sense to consider the product of the individual correlations rather than the average; this measure has the pleasing feature that it vanishes if any of the links in the ring is broken.

Again, our main conclusion is this: in the one example we have worked out in detail, if there is a non-trivial twist in the nearest-neighbor correlations, that is, a twisting that cannot be undone by local rotations, then there is a corresponding reduction in the maximum possible magnitude of these correlations. That is, in this one example at least, twisted correlations are weaker correlations. This conclusion follows from the structure of the quantum state space and is true irrespective of the system’s Hamiltonian.

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References

[1] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).

[2] For a guide to the literature see J. Anandan, J. Christian, and K. Walenik, Am. J. Phys. 65, 180 (1997).

[3] M. V. Berry, Proc. R. Soc. London Ser. A 392, 45 (1984).

[4] Y. Aharonov and J. S. Anandan, Phys. Rev. Lett. 58, 1593 (1987).

[5] For an introduction, see G. Münster and M. Walzl, hep-lat/0012003.

[6] C. M. Caves, C. A. Fuchs, and P. Rungta, quant-ph/0009063.

[7] D. Bruß, Phys. Rev. A 60, 4344 (1999).

[8] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).

[9] M. Koashi, V. Bužek, and N. Imoto, Phys. Rev. A 62, 050302 (2000).

[10] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).

[11] Phys. Rev. A 65, 010301 (2001).

[12] K. M. O’Connor and W. K. Wootters, Phys. Rev. A 63, 052302 (2001).

[13] C. H. Bennett, D. P. DiVincenzo, J. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).

[14] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996).

[15] T. J. Osborne and M. A. Nielsen, quant-ph/0109024.

[16] J. Preskill, J. Mod. Opt. 47, 127 (2000).

[17] P. Zanardi and X. Wang, quant-ph/0201028.

[18] T. J. Osborne and M. A. Nielsen, “Entanglement in a simple quantum phase transition,” in preparation.
[19] F. Berruto, G. Grignani, G. W. Semenoff, and P. Sodano, Ann. Phys. (N.Y.) 275, 254 (1999).

[20] K. G. Wilson, Phys. Rev. D 10, 2445 (1974).

[21] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).

[22] T. Kennedy, E. H. Lieb, and B. S. Shastry, Phys. Rev. Lett. 61, 2582 (1988).

[23] K. Kubo and T. Kishi, Phys. Rev. Lett. 61, 2585 (1988).

[24] C. J. Hamer, T. Hövelborn, and M. Bachhuber, J. Phys. A 32, 51 (1999).

[25] S. Zhang and K. J. Runge, Phys. Rev. B 45, 1052 (1992).

[26] A. W. Sandvik and C. J. Hamer, Phys. Rev. B 60, 6588 (1999).

[27] Werner states are discussed, for example, in K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A 64, 062307 (2001).