Traces of Mirror Symmetry in Nature

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In this work we discuss the place of Veneziano amplitudes (the precursor of string models) and their generalizations in the Regge theory of high energy physics scattering processes. We emphasize that mathematically such amplitudes and their extensions can be interpreted in terms of the Laplace (respectively, multiple Laplace) transform(s) of the generating function for the Ehrhart polynomial associated with some integral polytope $\mathcal{P}$ (specific for each scattering process). Following works by Batyrev and Hibi to each such polytope $\mathcal{P}$ it is possible to associate another (mirror) polytope $\mathcal{P}'$. For this to happen, it is necessary to impose some conditions on $\mathcal{P}$ and, hence, on the generating function for $\mathcal{P}$. Since each of these polytopes is in fact encodes some projective toric variety, this information is used for development of new symplectic and supersymmetric models reproducing the Veneziano and generalized Veneziano amplitudes. General ideas are illustrated on classical example of the pion-pion scattering for which the existing experimental data can be naturally explained with help of mirror symmetry arguments.

Keywords: Veneziano and Veneziano-like amplitudes; Regge theory; Froissart theorem; Ehrhart polynomial for integral polytopes; Duistermaat-Heckman formula; Khovanskii-Pukhlikov correspondence; Lefschetz isomorphism theorem.

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1 Introduction

1.1 Brief history of the Veneziano amplitudes

As is well known, the origins of modern string theory can be traced back to the 4-particle scattering amplitude $A(s, t, u)$ postulated by Veneziano in 1968 [1]. Up to a common constant factor, it is given by

$$A(s, t, u) = V(s, t) + V(s, u) + V(t, u),$$

(1)

where

$$V(s, t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx \equiv B(-\alpha(s), -\alpha(t))$$

(2)

is the Euler beta function and $\alpha(x)$ is the Regge trajectory usually written as $\alpha(x) = \alpha(0) + \alpha'x$ with $\alpha(0)$ and $\alpha'$ being the Regge slope and intercept, respectively. In the case of space-time metric with signature $\{-, +, +, +\}$ the Mandelstam variables $s, t$ and $u$ entering the Regge trajectory are defined by [2]

$$s = -(p_1 + p_2)^2; \quad t = -(p_2 + p_3)^2; \quad u = -(p_3 + p_1)^2.$$  

(3)

The 4-momenta $p_i$ are constrained by the energy-momentum conservation law leading to relation between the Mandelstam variables:

$$s + t + u = \sum_{i=1}^{4} m_i^2.$$  

(4)

Veneziano [1] noticed\(^2\) that to fit experimental data the Regge trajectories should obey the constraint

$$\alpha(s) + \alpha(t) + \alpha(u) = -1$$

(5)

consistent with Eq.(4) in view of the definition of $\alpha(s)$. The Veneziano condition, Eq.(5), can be rewritten in a more general form. Indeed, let $m, n, l$ be some integers such that $\alpha(s)m + \alpha(t)n + \alpha(u)l = 0$. Then by adding this equation to Eq.(5) we obtain, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = -1$, or more generally, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} + \tilde{k} \cdot 1 = 0$. Both equations have been studied extensively.

\(^2\)To get our Eq.(5) from Eq.(7) of Veneziano paper, it is sufficient to notice that his $1-\alpha(s)$ corresponds to ours $-\alpha(s)$.
in the book by Stanley [3] from the point of view of commutative algebra, polytopes, toric varieties, invariants of finite groups, etc. Although this observation is entirely sufficient for restoration of the underlying physical model(s) reproducing these amplitudes, development of string-theoretic models reproducing such amplitudes proceeded historically quite differently. In this work, we abandon these more traditional approaches in favour of taking the full advantage of combinatorial ideas presented in Ref.[3]. This allows us to obtain models reproducing Veneziano amplitudes which are markedly different from those known in traditional string-theoretic literature.

In 1967—a year before Veneziano’s paper was published—the paper [4] by Chowla and Selberg appeared relating Euler’s beta function to the periods of elliptic integrals. The result by Chowla and Selberg was generalized by Andre Weil whose two influential papers [5,6] brought into the picture the periods of Jacobians of the Abelian varieties, Hodge rings, etc. Being motivated by these papers, Benedict Gross wrote a paper [7] in which the beta function appears as period associated with the differential form “living” on the Jacobian of the Fermat curve. His results as well as those by Rohrlich (placed in the appendix to Gross paper) have been subsequently documented in the book by Lang [8]. Perhaps, because in the paper by Gross the multidimensional extension of beta function was considered only briefly, e.g. [7],p.207, the computational details were not provided. These details can be found in our recently published papers, Refs.[9,10,11]. To obtain the multidimensional extension of beta function as period integral, following the logic of papers by Gross and Deligne [12], one needs to replace the Fermat curve by the Fermat hypersurface, to embed it into the complex projective space, and to treat it as Kähler manifold. The differential forms living on such manifold are associated with the periods of Fermat hypersurface. Physical considerations require this Kähler manifold to be of the Hodge type. In his lecture notes [12] Deligne noticed that the Hodge theory needs some essential changes (e.g. mixed Hodge structures, etc.) if the Hodge-Kähler manifolds possess singularities. Such modifications may be needed upon development of our formalism. A monograph by Carlson et al, Ref.[13], contains an up to date exaustive information regarding such modifications, etc. Fortunately, to obtain the multiparticle Veneziano amplitudes these complications are not essential. In Ref.[10] we demonstrated that the period integrals living on Fermat hypersurfaces, when properly interpreted, provide the tachyon-free (Veneziano-like) multiparticle amplitudes whose particle spectrum reproduces those known for both the open and closed bosonic strings. Naturally, the question arises: If this is so, then what kind of models are capable of reproducing such amplitudes? In this paper we would like to discuss some combinatorial properties of the Veneziano (and Veneziano-like) amplitudes sufficient for reproducing at least two of such models: symplectic and supersymmetric. Mathematically, the results presented below are in accord with those by Vergne [14] whose work does not contain practical applications. Before studying these models, we would like to make some comments about the place of Veneziano amplitudes and, hence, of whatever models associated with these amplitudes, within the Regge formalism developed for description of scattering processes in high energy physics. This is
accomplished in the next subsection.

1.2 The Regge theory, theorem by Froissart, quantum gravity and the standard model

As is well known, all information in particle physics is obtainable through proper interpretation of the scattering data. The optical theorem (see below) allows one to connect the imaginary part of the scattering amplitude with the total cross section $\sigma$. By measuring this cross section experimentally one can obtain some information about the scattering amplitudes. Additional useful information can be obtained by collecting data for differential cross sections, by using the dispersion relations, etc. [15]. There is an unproven common belief that in the limit of high energies all scattering processes are adequately described by the Regge theory [16, 17]. The Veneziano amplitude by design is Regge behaving [1]. To our knowledge, the proof that in the limit of high energies scattering amplitudes are Regge behaving had been obtained only for some special cases [16,17], including that of QCD [18]. Since, irrespective to their mathematical nature, all string theories are based on this (generally unproven!) belief of the validity of the Regge theory, they can be as much trusted (even if totally correct mathematically!) as can be the Regge theory.

In the Regge theory the experimental data are presented using the Chew-Frautchi (C-F) plot, Ref.[16], pp. 144-145. On this plot one plots the Regge trajectories. Such trajectories relate particles with the same internal quantum numbers but with different spin (or angular momentum). From the standard string textbook, Ref.[2], it is known that for the open bosonic string the Regge trajectory is given by $\alpha(s) = \alpha(0) + \alpha' s$ (in accord with Eq.(2) above). It is important though that $\alpha(0) = 1$ and $\alpha' = 1/2$ for the open string while $\alpha(0) = 2$ and $\alpha' = 1/4$ for the closed string. In known string-theoretic formulations the numerical values of these parameters cannot be adjusted to fit the available experimental data since their values are deeply connected with the existing string-theoretic formalism [2] and, hence, are not readily adjustable. In the meantime, for high energies currently available it is known, e.g. read Ref.[15], p. 41, that $\alpha(s) = 0.7 + 0.8s$ or $\alpha(s) = 0.44 + 0.92s$ for typical Regge trajectories. Claims made by some string theoreticians that the available range of high energies is not sufficient to test the predictions provided by the existing string theories cannot be justified because of the following.

One of the major reasons for development of string theory, according to Ref.[2], lies in developing of consitent theory of quantum gravity. Indeed, in the case of closed bosonic string the massless (i.e. $s = 0$) spin two graviton occurs in the string spectrum only if $\alpha(0) = 2$. This fact alone fixes the value of the Regge intercept $\alpha(0)$ on the C-F plot to its value : $\alpha(0) = 2$. As plausible as it is, such an identification creates some major problems.

Indeed, in the case of $2 \rightarrow 2$ scattering process the total cross section for the elastic scattering in $s$-channel (in view of the optical theorem, e.g. see Ref.[15],
p. 47) is given by
\[ \sigma(s) \sim s^{-1} \text{Im} A(s, t = 0), \quad (6) \]
where the scattering amplitude \( A(s, t) \) is either postulated (as in the case of Veneziano amplitude) or determined from some model (e.g. the standard string model [2], etc.). The above expression is valid rigorously at any energy. In the limit \( s \to \infty \) the Regge theory provides the estimate for this exact result :
\[ \sigma(s) = cs^{\alpha(t=0)-1}, \quad (7) \]
where \( c \) is some constant. As it is with all processes described by the Regge theory [15-17], physically this result means the following: the analytical behaviour of the amplitude for elastic scattering in the \( s \)-channel is controlled (through the exponent in Eq.(7)) by the resonance in \( t \)-channel. In particular, if the resonance is caused by the graviton this leads the total crosssection to behave as: \( \sigma(s) = cs \). Unfortunately, the obtained result violates the theorem by Froissart. It can be stated as follows (e.g. see Ref.[16], p.53):

**Theorem 1.1.** (Froissart) In the high energy limit : \( s \to \infty \) the total crosssection \( \sigma(s) \) in \( s \)-channel is bounded by \( \sigma(s)_{s \to \infty} \leq \text{const} \log(s/s_0) \) where \( s_0 \) is some (prescribed) energy scale.

Evidently, even if the current efforts (based on commonly accepted formalism) to construct mathematically meaningful string/brane theory eventually might succeed, such a theory will contradict the Froissart theorem for reasons just described. Hence, either this theorem is incorrect and should be reconsidered or the underlying assumptions of string theory regarding gravitons are incorrect.

**Remark 1.2** The way out from this situation was recently developed in our recent work, Ref.[18], where new equivalence principle for gravity is proposed based on known rigorous mathematical results. This new equivalence principle has major implications for the standard model of particle physics [19]. Since physical predictions based this model are in agreement with the Froissart theorem already, the results of Ref.[18] effectively convert the existing standard model into a unified field theory accounting for all four types of known fundamental interactions and being manifestly renormalizable and gauge-invariant.

Incidentally, the intercept \( \alpha(0) = 1 \) for the open string theory does have some physical significance. Indeed, in this case use of Eq.(7) produces \( \sigma(s) = c' \) where \( c' \) is yet another constant. Such high energy behaviour is typical for the pomeron-a hypothetical particle like object predicted by Pomeranchuk- still undiscovered [15,17]. Additional ramifications of Pomeranchuk’s work have lead to the prediction of the companion of the pomeron-the odderon [20].

In addition to the difficulty with the Froissart theorem, just described, the existing string-theoretic models suffer from several no less serious drawbacks.
For instance, the Regge theory in general and the Veneziano amplitude (a precursor of the string model) in particular states that in addition to the leading (parent) Regge trajectory there should be countable infinity of daughter trajectories—all lying below the parent trajectory. Nowhere in string-theoretic literature were we able to find a mention or an explanation of this fact. Experimentally, however, typically for each parent trajectory there are only few daughter trajectories. In this work we shall provide a plausible theoretical explanation of this fact based on the mirror symmetry arguments. We would like to emphasize that since the models reproducing Veneziano amplitudes discussed below differ from those commonly discussed in string-theoretic literature, the numerical values for the slope $\alpha'$ and the intercept $\alpha(0)$ of the Regge trajectories can be readily adjusted to fit the experimental data. This is in accord with the original work by Veneziano [1] where no restrictions on the slope and intercept were imposed.

1.3 Organization of the rest of this paper

The rest of this work is organized as follows. Section 2 begins with some facts revealing the combinatorial nature of Veneziano amplitudes. This is achieved by connecting them with generating function for the Ehrhart polynomial whose properties are described in some detail in the same section. Such a polynomial counts the number of points inside the rational polytope (i.e. polytope whose vertices are located at the nodes of the regular $k$-dimensional lattice) and at its boundaries (faces). In the present case the polytope is a regular simplex which is a deformation retract for the Fermat-type (hyper) surface living in the complex projective space [9,10]. Next, using general properties of generating functions for the Ehrhart polynomials for the rational polytopes we discuss possible generalizations of the Veneziano amplitudes for polytopes other than a simplex. This allows us to use some results by Batyrev [22, 23] and Hibi [24] in order to introduce the mirror symmetry considerations enabling us to exclude the countable infinity of daughter trajectories on the C-F plot using mirror symmetry arguments. General ideas are illustrated on the classical example of the pion-pion scattering [25] for which the existing experimental data can be naturally explained with help of mirror symmetry arguments. Next, in Section 3 we begin our reconstruction of the models reproducing Veneziano and the generalized Veneziano amplitudes. It is facilitated by known connections between the polytopes and dynamical systems [14,26]. Development of these connections is proceeds through Sections 2-4 where we find the corresponding quantum mechanical system whose ground state is degenerate with degeneracy factor being identified with the Ehrhart polynomial. The obtained final result is in accord with that earlier obtained by Vergne [14] whose work does not contain any physical applications. In Section 5 the generating function for the Ehrhart polynomial is reinterpreted in terms of the Poincare’ polynomial. Such a polynomial is used, for instance, in the theory of invariants of finite (pseudo)reflection groups [3,27]. Obtained indentification reveals the topological and group-theoretic nature of the Veneziano amplitudes. To strengthen this point of view, we use some
results by Atiyah and Bott [28] inspired by earlier work by Witten [29] on supersymmetric quantum mechanics. They allow us to think about the Veneziano amplitudes using the terminology of intersection theory [30]. This is consistent with earlier mentioned interpretation of these amplitudes in terms of periods of the Fermat (hyper)surface [9,10]. It also makes computation of these amplitudes analogous to those for the Witten-Kontsevich model [31, 32], whose refinements can be found in our earlier work, Ref.[33]. For the sake of space, in this work we do not develop these connections with the Witten-Kontsevich model any further. Interested reader may find them in Ref.[34]. Instead, we discuss the supersymmetric model associated with symplectic model described earlier and treat it with help of the Lefshetz isomorphism theorem. This allows us to look at the problem of computation of the spectrum for such a model from the point of view of the theory of representations of the complex semisimple Lie algebras. Using some results by Serre [35] and Ginzburg [36] we demonstrate that the ground state for such finite dimensional supersymmetric quantum mechanical model is degenerate with degeneracy factor coinciding with the Ehrhart polynomial. This result is consistent with that obtained in Section 4 by different methods.

2 The extended Veneziano amplitudes, the Ehrhart polynomial and mirror symmetry

2.1 Combinatorics of the Veneziano amplitudes

In view of Eq.(2), consider an identity taken from [37],

\[
\frac{1}{(1-tz_0) \cdots (1-tz_k)} = (1 + t z_0 + (t z_0)^2 + \cdots \cdots (1 + t z_n + (t z_n)^2 + \cdots )
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k_0+\cdots+k_k=n} z_0^{k_0} \cdots z_k^{k_k} \right) t^n. \tag{8}
\]

When \( z_0 = \ldots = z_k = 1 \), the inner sum in the last expression provides the total number of monomials of the type \( z_0^{k_0} \cdots z_k^{k_k} \) with \( k_0 + \cdots + k_k = n \). The total number of such monomials is given by the binomial coefficient\(^3\)

\[
p(k,n) = \frac{(k+n)!}{k!n!} = \frac{(n+1)(n+2) \cdots (n+k)}{k!} = \frac{(k+1)(k+2) \cdots (k+n)}{n!}. \tag{9}
\]

For this special case Eq.(8) is converted to a useful expansion,

\[
P(k,t) \equiv \frac{1}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} p(k,n) t^n. \tag{10}
\]

\(^3\)The reason for displaying 3 different forms of the same combinatorial factor will be explained shortly below.
In view of the integral representation of the beta function given by Eq. (2), we replace \( k+1 \) by \( \alpha(s)+1 \) in Eq. (10) and use it in the beta function representation of the amplitude \( V(s, t) \). Straightforward calculation produces the following known in string theory result [2]:

\[
V(s, t) = -\sum_{n=0}^{\infty} \frac{p(\alpha(s), n)}{\alpha(t) - n}. \tag{11}
\]

The r.h.s. of Eq. (11) is effectively the Laplace transform of the generating function, Eq. (10). Such generating function can be interpreted as a partition function in the sense of statistical mechanics.

The purpose of this work is to demonstrate that such an interpretation is not merely a conjecture and, in view of this, to find the statistical mechanical/quantum model whose partition function is given by Eq. (10).

Our arguments are not restricted to the 4-particle amplitude. Indeed, as we argued earlier [10, 11], the multidimensional extension of Euler’s beta function producing multipartite Veneziano amplitudes (upon symmetrization analogous to the 4-particle case) is given by the following integral attributed to Dirichlet

\[
D(x_1, \ldots, x_k) = \int \cdots \int_{u_1 \geq 0, \ldots, u_k \geq 0, u_1 + \cdots + u_k \leq 1} u_1^{x_1-1} u_2^{x_2-1} \cdots u_k^{x_k-1} (1-u_1-\cdots-u_k)^{x_{k+1}-1} du_1 \cdots du_k. \tag{12}
\]

In this integral let \( t = u_1 + \cdots + u_k \). This allows us to use already familiar expansion Eq. (10). In addition, the following identity

\[
t^n = (u_1 + \cdots + u_k)^n = \sum_{n=(n_1, \ldots, n_k)} \frac{n!}{n_1!n_2!\cdots n_k!} u_1^{n_1} \cdots u_k^{n_k} \tag{13}
\]

with restriction \( n = n_1 + \cdots + n_k \) is of importance as well. This type of identity was used earlier in our work on Kontsevich-Witten model [33]. Moreover, from the same paper it follows that the above result can be presented as well in the alternative useful form:

\[
(u_1 + \cdots + u_k)^n = \sum_{\lambda \vdash k} f^\lambda S_\lambda(u_1, \ldots, u_k), \tag{14}
\]

where the Schur polynomial \( S_\lambda \) is defined by

\[
S_\lambda(u_1, \ldots, u_k) = \sum_{n=(n_1, \ldots, n_k)} K_{\lambda, n} u_1^{n_1} \cdots u_k^{n_k} \tag{15}
\]

with coefficients \( K_{\lambda, n} \) known as Kostka numbers, \( f^\lambda \) being the number of standard Young tableaux of shape \( \lambda \) and the notation \( \lambda \vdash k \) meaning that \( \lambda \) is partition of \( k \). Through such a connection with Schur polynomials one can develop connections with the Kadomtsev-Petviashvili (KP) hierarchy of nonlinear exactly integrable systems on one hand and with the theory of Schubert varieties on another. Although details can be found in our earlier publications.
In this work we shall discuss these issues a bit further in Section 5. Use of Eq. (13) in (12) produces, after performing the multiple Laplace transform, the following part of the multiparticle Veneziano amplitude

\[ A(1, \ldots k) = \frac{\Gamma_{n_1 \ldots n_k}(\alpha(s_{k+1}))}{(\alpha(s_1) - n_1) \cdots (\alpha(s_k) - n_k)}. \]  

(16)

Even though the residue \( \Gamma_{n_1 \ldots n_k}(\alpha(s_{k+1})) \) contains all the combinatorial factors, the obtained result should still be symmetrized (in accord with the 4-particle case considered by Veneziano) in order to obtain the full multiparticle Veneziano amplitude. Since in the above general multiparticle case the same expansion, Eq.(10), was used, for the sake of space it is sufficient to focus on the 4-particle amplitude only. This task is reduced to further study of the expansion given by Eq.(10). Such an expansion can be looked upon from several different angles. For instance, we have mentioned already that it can be interpreted as a partition function. In addition, it is the generating function for the Ehrhart polynomial. The combinatorial factor \( p(k,n) \) defined in Eq.(9) is the simplest example of the Ehrhart polynomial. Evidently, it can be written formally as

\[ p(k,n) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0. \]  

(17)

### 2.2 Some facts about the Ehrhart polynomials

A type of expansion given by Eq.(17) is typical for all Ehrhart-type polynomials. Indeed, let \( \mathcal{P} \) be any convex rational polytope that is the polytope whose vertices are located at the nodes of some \( n \)-dimensional \( \mathbb{Z}^n \) lattice. Then, the Ehrhart polynomial for the inflated polytope \( \mathcal{P} \) (with coefficient of inflation \( k = 1, 2, \ldots \)) can be written as

\[ |k\mathcal{P} \cap \mathbb{Z}^n| = \mathcal{P}(k,n) = a_n(P) k^n + a_{n-1}(P) k^{n-1} + \cdots + a_0(P) \]  

(18)

with coefficients \( a_0, \ldots, a_n \) being specific for a given type of polytope \( \mathcal{P} \). In the case of Veneziano amplitude the polynomial \( p(k,n) \) counts number of points inside the \( n \)-dimensional inflated simplex (with inflation coefficient \( k = 1, 2, \ldots \)). Irrespective to the polytope type, it is known [38] that \( a_0 = 1 \) and \( a_n = Vol\mathcal{P} \), where \( Vol\mathcal{P} \) is the Euclidean volume of the polytope. These facts can be easily checked directly for \( p(k,n) \). To calculate the remaining coefficients of such polynomial explicitly for arbitrary convex rational polytope \( \mathcal{P} \) is a difficult task in general. Such a task was accomplished only recently in [39]. The authors of [39] recognized that in order to obtain the remaining coefficients, it is useful to calculate the generating function for the Ehrhart polynomial. Long before the results of [39] were published, it was known [3,27], that the generating function for the Ehrhart polynomial of \( \mathcal{P} \) can be written in the following universal form

\[ \mathcal{F}(\mathcal{P}, x) = \sum_{k=0}^{\infty} \mathcal{P}(k,n) x^k = \frac{h_0(P) + h_1(P) x + \cdots + h_n(P) x^n}{(1 - x)^{n+1}}. \]  

(19)
The above result leading to possible generalizations/extensions of the Veneziano amplitudes does make physical sense as we shall demonstrate momentarily. Additional details are also presented in Section 5.

The fact that the combinatorial factor \( p(k, n) \) in Eq.(9) can be formally written in several equivalent ways has physical significance. For instance, in particle physics literature, e.g. see [2], the third option is commonly used. Let us recall how this happens. One is looking for an expansion of the factor \((1 - x)^{-\alpha(t)} - 1\) under the integral of beta function, e.g. see Eq.(2). Looking at Eq.(19) one realizes that the Regge variable \( \alpha(t) \) plays the role of dimensionality of \( \mathbb{Z} \)-lattice. Hence, in view of Eq.(8), we have to identify it with \( n \) (or \( k \), in case if Eq.(8) is used) in the second option provided by Eq.(9). This is not the way such an identification is done in physics literature where, in fact, the third option in Eq.(9) is used with \( k = \alpha(t) \) being effectively the inflation factor while \( n \) being effectively the dimensionality of the lattice\(^4\). A quick look at Eq.s(10) and (19) shows that under such circumstances the generating function for the Ehrhart polynomial and that for the Veneziano amplitude are formally not the same. In the first case one is dealing with lattices of fixed dimensionality and is considering summation over various inflation factors at the same time. In the second case (used in physics literature [2]) one is dealing with the fixed inflation factor \( n = \alpha(t) \) while summing over lattices of different dimensionalities. Nevertheless, such arguments are superficial in view of Eq.s(8) and (19) above. Using these equations it is clear that mathematically correct agreement between Eq.s(10) and (19) can be reached only if one is using identification: \( \Psi(k, n) = p(k, n) \), with the second option given by Eq.(7) selected. By doing so no changes in the pole locations for the Veneziano amplitude occur. Moreover, for a given pole the second and the third option in Eq.(9) produce exactly the same contributions into the residue thus making them physically indistinguishable. The interpretation of the Veneziano amplitude as the Laplace transform of the Ehrhart polynomial generating function provides a very compelling reason for development of the alternative string-theoretic formalism. In addition, it allows us to think about possible generalizations of the Veneziano amplitude using generating functions for the Ehrhart polynomials for polytopes other than the \( n \)-dimensional inflated simplex used for the Veneziano amplitudes. As it is demonstrated by Stanley [3,27], Eq.(19) has a group invariant meaning as the Poincaré' polynomial for the so called Stanley-Reisner polynomial ring\(^5\).

This fact alone makes generalization of the Veneziano amplitudes mathematically plausible. From the same reference one can find connections of these results with toric varieties. In view of Ref.[14], this observation is sufficient for restoration of physical models reproducing the Veneziano and Veneziano-like generalized amplitudes. Thus, in the rest of this paper we shall discuss some approaches to the design of these models.

Generalization of the Veneziano amplitudes is justified not only mathematically. It is also needed physically as explained earlier in Subsection 1.2. The

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\(^4\)We have to warn our readers that nowhere in physics literature such combinatorial terminology is used to our knowledge.

\(^5\)In Section 5 we provide some additional details on this topic.
information on Ehrhart polynomials just provided is sufficient for this purpose as we would like to explain now.

2.3 The generalized Veneziano amplitudes and mirror symmetry

As we have explained already in Subsection 1.2., according to the Regge theory [16,17], for each parent trajectory there should be a countable infinity of daughter trajectories—all lying below the parent on the C-F plot. In his original paper [1], page 195, Veneziano took this fact into account and said explicitly that his amplitude is not uniquely defined. Following both the original work by Veneziano and Ref.[15], p.100, we notice that beta function in Eq.(2) given by $B(-\alpha(s), -\alpha(t))$ (which is effectively the unsymmetrized Veneziano amplitude) can be replaced by $B(m - \alpha(s), n - \alpha(t))$ for any integers $m, n \geq 0$. To comply with the Regge theory one should use any linear combination of beta functions just described unless some additional assumptions are made. To our knowledge, the fact that the Veneziano amplitude is not uniquely defined regrettably is not mentioned in any of the existing modern string theory literature. Hence, if the alternative (to ours) formulations of string-theoretic models may finally produce some mathematically meaningful results, these formulations still will be confronted with explanation of the experimental fact that in nature only finite number of daughter trajectories is observed for each parent trajectory. If one accepts the viewpoint of this paper, such experimental fact can be explained quite naturally with help of mirror symmetry arguments. It should be noted, however, that our use of mirror symmetry differs drastically from that currently in use [40,41]. Nevertheless, the initial observations used in the present case do coincide with those used in more popular mirror symmetry treatments [41] since in our case they are also based on the work by Batyrev, Ref.[22]. In turn, Batyrev’s results to some extent have been influenced by the result of Hibi [24] to be used in our work as well.

Following these authors we would like to discuss properties of reflexive (polar (or dual)) polytopes. It is useful to notice at this point that the concept of the dual (polar) polytope was in use in solid state physics literature [42] for quite some time. Indeed both direct and reciprocal (dual) lattices are being used routinely in calculations of physical properties of crystalline solids. The requirement that physical observables should remain the same irrespective to what lattice is used in calculations is completely natural. The same, evidently, should be true in the mirror symmetry calculations used in high energy physics. This is the physical essence of mirror symmetry. In the paper by Greene and Plesser [43], p.26, one finds the following statement: ”Thus, we have demonstrated that two topologically distinct Calabi-Yau manifolds $M$ and $M'$ give rise to the same conformal field theory. Furthermore, although our argument has been based only at one point in the respective moduli spaces $\mathcal{M}_M$ and $\mathcal{M}_{M'}$ of $M$ and $M'$ (namely the point which has a minimal model interpretation and hence
respects the symmetries by which we have orbifolded) the results necessarily extends to all of \( \mathcal{M}_M \) and \( \mathcal{M}_{M''} \).

We would like to explain these statements now using more commonly known terminology. For this purpose we begin with the following

**Definition 2.1.** A subset of \( \mathbb{R}^d \) is considered to be a polytope (or polyhedron) \( P \) if there is a \( r \times d \) matrix \( M \) (with \( r \leq d \)) and a vector \( b \in \mathbb{R}^d \) such that
\[
P = \{ x \in \mathbb{R}^d \mid Mx \leq b \}.
\]
Provided that the Euclidean \( d \)-dimensional scalar product is given by \( \langle x \cdot y \rangle = \sum_{i=1}^{d} x_i y_i \), a rational (respectively, integral) polytope (or polyhedron) \( P \) is defined by the set
\[
P = \{ x \in \mathbb{R}^d \mid \langle a_i \cdot x \rangle \leq \beta_i, i = 1, ..., r \},
\]
where \( a_i \in \mathbb{Q}^d (1-n_d+1) \) and \( \beta_i \in \mathbb{Q} \) for \( i = 1, ..., r \) (respectively \( a_i \in \mathbb{Z}^d \) and \( \beta_i \in \mathbb{Z} \) for \( i = 1, ..., r \)).

Next, we need yet another definition

**Definition 2.2.** For any convex polytope \( P \) the dual polytope \( P^* \) is defined by
\[
P^* = \{ x \in (\mathbb{R}^d)^* \mid \langle a \cdot x \rangle \leq 1, a \in P \}.
\]

Although in algebraic geometry of toric varieties the inequality \( \langle a \cdot x \rangle \leq 1 \) is sometimes replaced by \( \langle a \cdot x \rangle \geq -1 \) [38] we shall use the definition just stated to be in accord with Hibi [24]. According to this reference, if \( P \) is rational, then \( P^* \) is also rational. However, \( P^* \) is not necessarily integral even if \( P \) is integral. This result is of profound importance since the result, Eq.(19), is valid for the integral polytopes only. The question arises: under what conditions is the dual polytope \( P \) integral? The answer is given by the following

**Theorem 2.3.** (Hibi [24]) The dual polytope \( P^* \) is integral if and only if
\[
F(P, x^{-1}) = (-1)^{d+1} x F(P, x)
\]
where the generating function \( F(P, x) \) is defined in Eq.(19).

By combining Eqs.(10) and (19) we obtain the following result for the standard Veneziano amplitude
\[
F(P, x) = \left( \frac{1}{1-x} \right)^{d+1}.
\]

Using this expression in Eq.(22) produces:
\[
F(P, x^{-1}) = \frac{(-1)^{d+1} x^{d+1}}{(1-x)^{d+1}} = (-1)^{d+1} x^{d+1} F(P, x).
\]
This result indicates that scattering processes described by the standard Veneziano amplitudes do not involve any mirror symmetry since, as it is well known \[22,23\] in order for such a symmetry to take place the dual polytope $\mathcal{P}^*$ must be integral. In such a case both $\mathcal{P}$ and $\mathcal{P}^*$ encode (define) the projective toric varieties $X_\mathcal{P}$ and $X_{\mathcal{P}^*}$ which are mirrors of each other and are of Fano-type \[22,23,45,46\]. The question arises: can these amplitudes be modified with help of Eq.(19) so that the presence of mirror symmetry can be checked in nature? To answer this question, let us assume that, indeed, Eq.(19) can be used for such a modification. In this case we must require for the generating function $F(P,x)$ in Eq.(19) to obey Eq.(22). Direct check of such an assumption leads to the desired result provided that $h_{n-i} = h_i$ in Eq.(19). Fortunately, this is the case in view of the fact that these are the famous Dehn-Sommerville equations, Ref.[38], p.16. Hence, at this stage of our discussion, it looks like generalization of the Veneziano amplitudes which takes into account mirror symmetry is possible from the mathematical standpoint. Unfortunately, in physics correctness of mathematical arguments is not sufficient for such generalization since experimental data may or may not support such rigorous mathematics. To check the correctness of our assumptions (at least to a some extent) we would like to discuss now some known in literature results on pion-pion ($\pi\pi$) scattering described, for example, in Refs.[25,46] from the point of view of results we just obtained. By doing so we shall provide the evidence that: a) mirror symmetry does exist in nature (whether or not its validity is nature’s law or just a curiosity remains to be further checked by analysing the available experimental data) and, that b) use of mirror symmetry arguments permits us to eliminate the countable infinity of daughter trajectories allowed by the traditional Regge theory in favour of just several observed experimentally.

Experimentally it is known that, below the threshold, that is below the collisions energies producing more outgoing particles than incoming, the unsymmetrized amplitude $A(s,t)$ for $\pi\pi$ scattering can be written as

$$A(s,t) = -g^2 \frac{\Gamma(1 - \alpha(s))\Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))} = -g^2(1 - \alpha(s) - \alpha(t))B(1 - \alpha(s), 1 - \alpha(t)). \quad (25)$$

This result should be understood as follows. Consider the ”weighted” (still unsymmetrized) Veneziano amplitude of the type

$$A(s,t) = \frac{1}{\lambda} \int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}g(x,s,t) \quad (26)$$

where the weight function $g(x,s,t)$ is given by

$$g(x,s,t) = \frac{1}{2}g^2[(1-x)\alpha(s) + x\alpha(t)]. \quad (27)$$

Upon integration, one recovers Eq.(25). The same result can be achieved if, instead one uses the weight function of the type

$$g(x,s,t) = g^2 x\alpha(t). \quad (28)$$
In early treatments of the dual resonance models (all developed around the Veneziano amplitude) [46] fitting to experimental data was achieved with some ad hoc prescriptions for the weight function \( g(x, s, t) \), e.g. like those given by Eqs (27) and (28). In the case of \( \pi\pi \) scattering such an ad hoc reasoning can be replaced by the requirements of mirror symmetry. Indeed, consider a special case of Eq.(19): \( n=2 \). For such a case we obtain,

\[
F(P, x) = \sum_{k=0}^{\infty} \mathcal{Q}(k, n)x^k = \frac{h_0(P) + h_1(P)x}{(1-x)^{1+1}}
\]  

so that Eq.(22) holds indicating mirror symmetry. At this point, in view of Eqs (26)-(29), one may notice that, actually, for this symmetry to take place in real world, one should replace the amplitude given by Eq.(25) by the following combination

\[
A(s, t) = -g^2 \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))} + g^2 \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}
\]

\[
= -g^2 B(1 - \alpha(s), 1 - \alpha(t)) + g^2 B(-\alpha(s), -\alpha(t)).
\]

Such a combination produces first two terms (with correct signs) of the infinite series as proposed by Mandelstam, Eq.(15) of Ref.[47]. The comparison with experiment displayed in Fig.6.2(a) of Ref.[46], p.321, is quite satisfactory producing one parent and one daughter Regge trajectories. These are also displayed in Ref.[15], p. 41, for the "rho family" of resonances. Thus, at least in the case of \( \pi\pi \) scattering, one can claim that mirror symmetry consideration provides a plausible explanation of the observable data. One hopes, that the case just considered is typical so that mirror symmetry does play a role in Nature.

The rest of this paper is devoted to the reconstruction of physical models reproducing Veneziano and extended Veneziano amplitudes based on mathematical results discussed in these two sections. Additional details can be found in Refs.[10,11,34,48].

### 3 Motivating examples

To facilitate our readers understanding, we would like to illustrate general principles using simple examples. We begin by considering a finite geometric progression of the type

\[
\mathcal{F}(c, m) = \sum_{l=-m}^{m} \exp\{cl\} = \exp\{-cm\} \sum_{l=0}^{\infty} \exp\{cl\} + \exp\{cm\} \sum_{l=-\infty}^{0} \exp\{cl\}
\]

\[
= \exp\{-cm\} \frac{1}{1 - \exp\{c\}} + \exp\{cm\} \frac{1}{1 - \exp\{-c\}}
\]

\[
= \exp\{-cm\} \left[ \frac{\exp\{c(2m + 1)\} - 1}{\exp\{c\} - 1} \right].
\]
The reason for displaying the intermediate steps will be explained shortly below. First, however, we would like to consider the limit \( c \to 0^+ \) of \( \mathcal{F}(c, m) \). It is given by \( \mathcal{F}(0, m) = 2m + 1 \). The number \( 2m + 1 \) equals to the number of integer points in the segment \([-m, m]\) including boundary points. It is convenient to rewrite the above result in terms of \( x = \exp(c) \) so that we shall write formally \( \mathcal{F}(x, m) \) instead of \( \mathcal{F}(c, m) \) from now on. Using such notation, consider a related function

\[
\bar{\mathcal{F}}(x, m) = (-1)^x F\left(\frac{1}{x}, -m\right).
\]

This type of relation (the Ehrhart-Macdonald reciprocity law) is characteristic for the Ehrhart polynomial for rational polytopes discussed earlier. In the present case we obtain,

\[
\bar{\mathcal{F}}(x, m) = (-1)^x \frac{x^{-(2m+1)} - 1}{x^{-1} - 1} x^m.
\]

In the limit \( x \to 1 + 0^+ \) we obtain : \( \bar{\mathcal{F}}(1, m) = 2m + 1 \). The number \( 2m + 1 \) is equal to the number of integer points strictly inside the segment \([-m, m]\). Both \( \mathcal{F}(0, m) \) and \( \bar{\mathcal{F}}(1, m) \) provide the simplest possible examples of the Ehrhart polynomials if we identify \( m \) with the inflation factor \( k \).

These, seemingly trivial, results can be broadly generalized. First, we replace \( x \) by \( x = x_1 \cdots x_d \), next we replace the summation sign in the left hand side of Eq.(31) by the multiple summation, etc. Thus obtained function \( \mathcal{F}(x, m) \) in the limit \( x_i \to 1 + 0^+, i = 1 - d \), produces the anticipated result : \( \mathcal{F}(1, m) = (2m+1)^d \). It describes the number of points inside and at the faces of a \( d \)-dimensional cube in the Euclidean space \( \mathbb{R}^d \). Accordingly, for the number of points strictly inside the cube we obtain : \( \bar{\mathcal{F}}(1, m) = (2m + 1)^d \). Let \( \text{VertP} \) denote the vertex set of the rational polytope. In the case considered thus far it is the \( d \)-dimensional cube. Let \( \{u_1^v, \ldots, u_d^v\} \) denote the orthogonal basis (not necessarily of unit length) made of the highest weight vectors of the Weyl-Coxeter reflection group \( B_d \) appropriate for cubic symmetry [11]. These vectors are oriented along the positive semi axes with respect to the center of symmetry of the cube. When parallel translated to the edges ending at particular hypercube vertex \( v \), they can point either in or out of this vertex. Then, the \( d \)-dimensional version of Eq.(31) can be rewritten in notations just introduced as follows

\[
\sum_{x \in \mathcal{P} \cap \mathbb{Z}^d} \exp\{c \cdot x\} = \sum_{v \in \text{VertP}} \exp\{c \cdot v\} \left[ \prod_{i=1}^d (1 - \exp(-c_i u_i^v)) \right]^{-1}.
\]

The correctness of this equation can be readily checked by considering special cases of a segment, square, cube, etc. The result, Eq.(34), obtained for the polytope of cubic symmetry can be extended to the arbitrary convex centrally symmetric polytope. Details can be found in Ref.[49]. Moreover, the requirement of central symmetry can be relaxed to the requirement of the convexity of \( \mathcal{P} \) only. In such general form the relation given by Eq.(34) was obtained by
Brion [50]. It is of central importance for the purposes of this work: the limiting procedure $c \to 0^+$ produces the number of points inside (and at the boundaries) of the polyhedron $\mathcal{P}$ in the l.h.s. of Eq.(34) and, if the polyhedron is rational and inflated, this procedure produces the Ehrhart polynomial. Actual computations are done with help of the r.h.s. of Eq.(34) as will be demonstrated below.

4 The Duistermaat-Heckman formula and the Khovanskii-Pukhlikov correspondence

Since the description of the Duistermaat-Heckman (D-H) formula can be found in many places, we would like to be brief in discussing it now in connection with earlier obtained results. Let $M \equiv M^{2n}$ be a compact symplectic manifold equipped with the moment map $\Phi : M \to \mathbb{R}$ and the (Liouville) volume form $dV = \left(\frac{1}{2\pi}\right)^n \frac{1}{n!} \Omega^n$. According to the Darboux theorem, locally $\Omega = \sum_{l=1}^n dq_l \wedge dp_l$. We expect that such a manifold has isolated fixed points $p$ belonging to the fixed point set $\mathcal{V}$ associated with the isotropy subgroup of the group $G$ acting on $M$. Then, in its most general form, the D-H formula can be written as [14,26,51]

$$\int_M dV e^\Phi = \sum_{p \in \mathcal{V}} e^{\Phi(p)} \prod_{j} a_{j,p},$$

(35)

where $a_{1,p}, \ldots, a_{n,p}$ are the weights of the linearized action of $G$ on $T_p M$. Using Morse theory, Atiyah [52] and others, e.g. see Ref.[54] for additional references, have demonstrated that it is sufficient to keep terms up to quadratic in the expansion of $\Phi$ around given $p$. In such a case the moment map can be associated with the Hamiltonian for the finite set of harmonic oscillators. In the properly chosen system of units the coefficients $a_{1,p}, \ldots, a_{n,p}$ are just "masses" $m_i$ of the individual oscillators. Unlike truly physical masses, some of $m'_i$s can be negative.

Based on the information just provided, we would like to be more specific now. To this purpose, following Vergne [53] and Brion [50], we would like to consider the D-H integral of the form

$$I(k ; y_1, y_2) = \int_{k\Delta} dx_1 dx_2 \exp\{-y_1 x_1 + y_2 x_2\},$$

(36)

where $k\Delta$ is the standard dilated simplex with dilation coefficient $k^6$. Calculation of this integral can be done exactly with the result:

$$I(k ; y_1, y_2) = \frac{1}{y_1 y_2} + \frac{e^{-ky_1}}{y_1(y_1 - y_2)} + \frac{e^{-ky_2}}{y_2(y_2 - y_1)}$$

(37)

Our choice of the simplex as the domain of integration is caused by our earlier made observation [16] that the deformation retract of the Fermat (hyper)surface (on which the Veneziano amplitude lives ) is just the standard simplex. Since such Fermat surface is a complex Kähler-Hodge type manifold and since all Kähler manifolds are symplectic [26,54], our choice makes sense.
consistent with Eq.(35). In the limit: $y_1, y_2 \to 0$ some calculation produces the
anticipated result: $\text{Vol} k \Delta = k^2/2!$ for the Euclidean volume of the dilated
simplex. Next, to make a connection with the previous section, in particular,
with Eq.(34), consider the following sum

$$S(k; y_1, y_2) = \sum_{(l_1, l_2) \in k \Delta} \exp\{-(y_1 l_1 + y_2 l_2)\}$$

$$= \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}} + \frac{1}{1 - e^{y_1}} \frac{e^{-ky_1}}{1 - e^{y_1-y_2}} + \frac{1}{1 - e^{y_2}} \frac{e^{-ky_2}}{1 - e^{y_2-y_1}}$$

(38)

related to the D-H integral, Eq.s(36,37). Its calculation will be explained momentaril y. In spite of the connection with the D-H integral, the limiting procedure: $y_1, y_2 \to 0$ in the last case is much harder to perform. It is facilitated by
use of the following expansion

$$\frac{1}{1 - e^{-s}} = \frac{1}{s} + \frac{1}{2} + \frac{s}{12} + O(s^2).$$

(39)

Rather lengthy calculations produce the anticipated result: $S(k; 0, 0) = k^2/2! + 3k/2 + 1 \equiv |k \Delta \cap Z^2| \equiv \Psi(k, 2)$ for the Ehrhart polynomial. Since generalization of the obtained results to simplicies of higher dimensions is straightforward, the relevance of these results to the Veneziano amplitude should be evident. To
make it more explicit we have to make several steps still. First, we would like to explain how the result, Eq.(38), was obtained. By doing so we shall gain some additional physical information. Second, we would like to explain in some
detail the connection between the integral, Eq.(37), and the sum, Eq.(38). Such
a connection is made with help of the Khovanskii-Pukhlikov correspondence.

We begin with calculations of the sum, Eq.(38). To do this we need a
definition of the convex rational polyhedral cone $\sigma$. It is given by

$$\sigma = \mathbb{Z}_{\geq 0} a_1 + \cdots + \mathbb{Z}_{\geq 0} a_d,$$

(40)

where the set $a_1, \ldots, a_d$ forms a basis (not necessarily orthogonal) of the $d$-
dimensional vector space $V$, while $\mathbb{Z}_{\geq 0}$ are non-negative integers. It is known
that all combinatorial information about the polytope $\mathcal{P}$ is encoded in the complete
fan made of cones whose apexes all having the same origin in common.
Details can be found in literature [26,30]. At the same time, the vertices of $\mathcal{P}$
are also the apexes of the respective cones. Following Brion[50], this fact allows us
to write the l.h.s. of Eq.(34) as

$$f(\mathcal{P}, x) = \sum_{m \in \mathcal{P} \cap \mathbb{Z}^d} x^m = \sum_{\sigma \in \text{Vert} \mathcal{P}} x^\sigma$$

(41)

so that for the dilated polytope the above statement reads as follows [50,55]:

$$f(k \mathcal{P}, x) = \sum_{m \in k \mathcal{P} \cap \mathbb{Z}^d} x^m = \sum_{i=1}^{n} x^{k v_i} \sum_{\sigma_i} x^{\sigma_i}.$$
In the last formula the summation is taking place over all vertices whose location is given by the vectors from the set \{v_1, ..., v_n\}. This means that in actual calculations one can first calculate the contributions coming from the cones \(\sigma_i\) of the undilated (original) polytope \(P\) and only then one can use the last equation in order to get the result for the dilated polytope.

Let us apply these general results to our specific problem of computation of \(S(k; y_1, y_2)\) in Eq.(38). We have our simplex with vertices in x-y plane given by the vector set \{v_1=(0,0), v_2=(1,0), v_3=(0,1)\}, where we have written the x coordinate first. In this case we have 3 cones: 
\[
\sigma_1 = l_2 v_2 + l_3 v_3, \quad \sigma_2 = v_2 + l_1 (v_2 - v_3) + l_2 (v_3 - v_2), \quad \sigma_3 = v_3 + l_3 (v_2 - v_3) + l_1 (v_1 - v_2); \{l_1, l_2, l_3\} \in \mathbb{Z}_{\geq 0}.
\]

In writing these expressions for the cones we have taken into account that, according to Brion, when making calculations the apex of each cone should be chosen as the origin of the coordinate system. Calculation of contributions to the generating function coming from \(\sigma_1\) is the most straightforward. Indeed, in this case we have 
\[
x = x_1 x_2 = e^{-y_1} e^{-y_2}.
\]
Now, the symbol \(x^\sigma\) in Eq.(41) should be understood as follows. Since \(\sigma_i, i = 1-3\), is actually a vector, it has components, like those for \(v_1\), etc. We shall write therefore 
\[
x^\sigma = x_1^{\sigma(1)} \cdots x_d^{\sigma(d)}
\]
where \(\sigma(i)\) is the i-th component of such a vector. Under these conditions calculation of the contributions from the first cone with the apex located at \((0,0)\) is completely straightforward and is given by
\[
\sum_{(l_2, l_3) \in \mathbb{Z}_+^2} x_1^{l_2} x_2^{l_3} = \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}}.
\]
It is reduced to the computation of the infinite geometric progression. But physically, the above result can be looked upon as a product of two partition functions for two harmonic oscillators whose ground state energy was discarded. By doing the rest of calculations in the way just described we reobtain \(S(k; y_1, y_2)\) from Eq.(38) as required. This time, however, we know that the obtained result is associated with the assembly of harmonic oscillators of frequencies \(\pm y_1\) and \(\pm y_2\) whose ground state energy is properly adjusted. The “frequencies” (or masses) of these oscillators are coming from the Morse-theoretic considerations for the moment maps associated with the critical points of symplectic manifolds as explained in the paper by Atiyah [52]. These masses enter into the “classical” D-H formula, Eq.s(36),(37). It is just a classical partition function for a system of such described harmonic oscillators living in the phase space containing critical points. The D-H classical partition function, Eq.(37), has its quantum analog, Eq.(38), just described. The ground state for such a quantum system is degenerate with the degeneracy being described by the Ehrhart polynomial \(P(k, 2)\). Such a conclusion is in formal accord with results of Vergne [14].

Since (by definition) the coefficient of dilation \(k=1,2,\ldots,\) there is no dynamical system (and its quantum analog) for \(k=0\). But this condition is the condition for existence of the tachyon pole in the Veneziano amplitude, Eq.(2). Hence, in view of the results just described this pole should be considered as unphysical and discarded. Such arguments are independent of the analysis made in Ref.[10].
where the unphysical tachyons are removed with help of the properly adjusted phase factors. Clearly, such factors can be reinstated in the present case as well since their existence is caused by the requirements of the torus action invariance of the Veneziano-like amplitudes as explained in [10,26]. Hence, their presence is consistent with results just presented.

Now we are ready to discuss the Khovanskii-Pukhlikov correspondence. It can be understood based on the following generic example taken from Ref.[56].

We would like to compare the integral
\[ I(z) = \int_s^t dxe^{zx} = e^{tx} - e^{sz} \]
with the sum
\[ S(z) = \sum_{k=s}^{t} e^{kz} = \frac{e^{tx}}{1 - e^{-z}} + \frac{e^{sz}}{1 - e^{-z}}. \]

To do so, following Refs[56-58] we introduce the Todd operator (transform) via
\[ Td(z) = \frac{z}{1 - e^{-z}}. \] (44)

Then, it can be demonstrated that
\[ Td(\frac{\partial}{\partial h_1}) Td(\frac{\partial}{\partial h_2}) ( \int_{s-h_1}^{t+h_2} e^{zx} dx ) \mid_{h_1=h_2=0} = \sum_{k=s}^{t} e^{kz}. \] (45)

This result can be now broadly generalized. Following Khovanskii and Pukhlikov [57], we notice that
\[ Td(\frac{\partial}{\partial z}) \exp(\sum_{i=1}^{n} p_i z_i) = Td(p_1, ..., p_n) \exp(\sum_{i=1}^{n} p_i z_i) \]. (46)

By applying this transform to
\[ i(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\xi_1 \cdots \xi_k} \exp(\sum_{i=1}^{k} x_i \xi_i) \] (47)
we obtain,
\[ s(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\prod_{i=1}^{k} (1 - \exp(-\xi_i))} \exp(\sum_{i=1}^{k} x_i \xi_i). \] (48)

This result should be compared now with the individual terms on the r.h.s. of Eq.(34) on one hand and with the individual terms on the r.h.s of Eq.(35) on another. Evidently, with help of the Todd transform the exact "classical" results for the D-H integral are transformed into the "quantum" results of the Brion’s identity, Eq.(34), which is actually equivalent to the Weyl character formula [48].

We would like to illustrate these general observations by comparing the D-H result, Eq.(37), with the Weyl character formula result, Eq.(38). To this
purpose we need to use already known data for the cones $\sigma_i$, $i = 1 \sim 3$, and the convention for the symbol $x^a$. In particular, for the first cone we have already $x^a = x_1^a x_2^a = [\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)]^7$. Now we assemble the contribution from the first vertex using Eq.(37). We obtain, $[\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)]/y_1 y_2$.

Using the Todd transform we obtain,

$$T d(\frac{\partial}{\partial l_1}) T d(\frac{\partial}{\partial l_2}) \frac{1}{y_1 y_2} [\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)] \mid_{l_1=l_2=0} = \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}}. \quad (49)$$

Analogously, for the second cone we obtain: $x_2^a = e^{-ky_1} e^{-l_1 y_1} e^{-l_2 (y_1 - y_2)}$ so that use of the Todd transform produces:

$$T d(\frac{\partial}{\partial l_1}) T d(\frac{\partial}{\partial l_2}) \frac{1}{y_1 (y_1 - y_2)} e^{-ky_1} e^{-l_1 y_1} e^{-l_2 (y_1 - y_2)} \mid_{l_1=l_2=0} = \frac{1}{1 - e^{-y_1}} \frac{e^{-ky_1}}{1 - e^{-y_1 - y_2}}. \quad (50)$$

e tc.

The obtained results can now be broadly generalized. To this purpose we can formally rewrite the partition function, Eq.(24), in the following symbolic form

$$I(k, f) = \int_{k\Delta} dx \exp(-f \cdot x) \quad (51)$$

valid for any finite dimension $d$. Since we have performed all calculations explicitly for two dimensional case, for the sake of space, we only provide the idea behind such type of calculation\(^7\). In particular, using Eq.(37) we can rewrite this integral formally as follows

$$\int_{k\Delta} dx \exp(-f \cdot x) = \sum_p \exp(p, x(p)) \prod_{i=1}^d h_i^p(f). \quad (52)$$

Applying the Todd operator (transform) to both sides of this formal expression and taking into account Eq.s(49),(50) (providing assurance that such an operation indeed is legitimate and makes sense) we obtain,

$$\int_{k\Delta} dx \prod_{i=1}^d \frac{x_i}{1 - \exp(-x_i)} \exp(-f \cdot x) = \sum_{\nu \in Vert^{\Delta}} \exp(\{f \cdot \nu\}) \prod_{i=1}^d (1 - \exp(-h_i^\nu(f) u_i^\nu))^{-1} = \sum_{x \in P \cap Z^d} \exp(\{f \cdot x\}), \quad (53)$$

where the last line was written in view of Eq.(34). From here, in the limit $f = 0$ we reobtain $p(k, n)$ defined in Eq.(10). Thus, using classical partition

\(^7\)To obtain correct results we needed to change signs in front of $l_1$ and $l_2$. The same should be done for other cones as well.

\(^8\)Mathematically inclined reader is encouraged to read paper by Brion and Vergne, Ref.[59], where all missing details are scrupulously presented.
function, Eq.(51), (discussed in the form of Exercises 2.27 and 2.28 in the book, Ref.[58], by Guillemin) and applying to it the Todd transform we recover the quantum mechanical partition function whose ground state provides us with the combinatorial factor $p(k,n)$.

5 From analysis to synthesis

5.1 The Poincare’ polynomial

The results discussed earlier are obtained for some fixed dilation factor $k$. In view of Eq.(8), they can be rewritten in the form valid for any dilation factor $k$. To this purpose it is convenient to rewrite Eq.(8) in the following equivalent form:

$$\frac{1}{\det(1-Mt)} = \frac{1}{(1-tz_0)\cdots(1-tz_k)} = (1 + tz_0 + (tz_0)^2 + \cdots)(1 + tz_n + (tz_n)^2 + \cdots)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k_0+\cdots+k_k=n} z_0^{k_0} \cdots z_k^{k_k} \right) t^n \equiv \sum_{n=0}^{\infty} \text{tr}(M_n)t^n,$$

(54)

where the linear map from $k+1$ dimensional vector space $V$ to $V$ is given by matrix $M \in G \subset GL(V)$ whose eigenvalues are $z_0,\ldots,z_k$. Using this observation several conclusions can be drawn. First, it should be clear that

$$\sum_{k_0+\cdots+k_k=n} z_0^{k_0} \cdots z_k^{k_k} = \sum_{m \in n\Delta \cap \mathbb{Z}^{k+1}} x^m = \text{tr}(M_n).$$

(55)

Second, following Stanley [3, 27] we would like to consider the algebra of invariants of $G$. To this purpose we introduce a basis $x = \{x_0,\ldots,x_k\}$ of $V$ and the polynomial ring $R = \mathbb{C}[x_0,\ldots,x_k]$ so that if $f \in R$, then $Mf(x) = f(Mx)$. The algebra of invariant polynomials $R^G$ can be defined now as

$$R^G = \{ f \in R : Mf(x) = f(Mx) = f(x) \ \forall M \in G \}.$$

These invariant polynomials can be explicitly constructed as averages over the group $G$ according to prescription:

$$\text{Av}_G f = \frac{1}{|G|} \sum_{M \in G} Mf,$$

(56)

with $|G|$ being the cardinality of $G$. Suppose now that $f \in R^G$, then, evidently, $f \in R^G = \text{Av}_G f$ so that $\text{Av}_G^2 f = \text{Av}_G f = f$. Hence, the operator $\text{Av}_G$ is indepotent. Because of this, its eigenvalues can be only 1 and 0. From here it follows that

$$\dim f_n^G = \frac{1}{|G|} \sum_{M \in G} \text{tr}(M_n).$$

(57)
Thus far our analysis was completely general. To obtain Eq.(9) we have to put $z_0 = ... = z_k = 1$ in Eq.(8). This time, however we can use the obtained results in order to write the following expansion for the Poincaré polynomial [3, 27] which for the appropriately chosen $G$ is equivalent to Eq.(10):

$$P(R^G, t) = \sum_{n=0}^{\infty} \frac{1}{|G|} \sum_{M \in G} tr(M_n)t^n = \sum_{n=0}^{\infty} \dim f^G_n t^n. \quad (58)$$

Evidently, the Ehrhart polynomial $\Psi(k, n) = \dim f^G_n$. To figure out the group $G$ in the present case is easy since, actually, the group is trivial: $G = 1$. This is so because the eigenvalues $z_0, ..., z_k$ of the matrix $M$ all are equal to 1. It should be clear, however, that for some appropriately chosen group $G$ expansion (19) is also the Poincaré polynomial (for the Cohen-Macaulay polynomial algebra [3,27]). This fact provides independent (of Refs. [10,11]) evidence that both the Veneziano and Veneziano-like amplitudes are of topological origin.

### 5.2 Connections with intersection theory

We would like to strengthen this observation now. To this purpose, in view of Eq.(35), and taking into account that for the symplectic 2-form $\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i$ the $n$-th power is given by $\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$, it is convenient to introduce the differential form

$$\exp \Omega = 1 + \Omega + \frac{1}{2!} \Omega \wedge \Omega + \frac{1}{3!} \Omega \wedge \Omega \wedge \cdots.$$  \(59\)

By design, the expansion in the r.h.s. will have only $k$ terms. The form $\Omega$ is closed, i.e. $d\Omega = 0$ (the Liouville theorem), but not exact. In view of the expansion, Eq.(59), the D-H integral, Eq.(51) can be rewritten as

$$I(k, f) = \int_{k\Delta} \exp (\tilde{\Omega}), \quad (60)$$

where, following Atiyah and Bott [28], we have introduced the form $\tilde{\Omega} = \Omega - f \cdot x$. Doing so requires us to replace the exterior derivative $d$ acting on $\Omega$ by $\tilde{d} = d + i(x)$ (where the operator $i(x)$ reduces the degree of the form by one) with respect to which the form $\tilde{\Omega}$ is equivariantly closed, i.e. $d\tilde{\Omega} = 0$. More explicitly, we have $d\tilde{\Omega} = d\Omega + i(x)\Omega - f \cdot dx = 0$. Since $d\Omega = 0$, we obtain the equation for the moment map: $i(x)\Omega - f \cdot dx = 0$ [51,58]. If use of the operator $d$ on differential forms leads to the notion of cohomology, use of the operator $\tilde{d}$ leads to the notion of equivariant cohomology. Although details can be found in the paper by Atiyah and Bott [28], more relaxed pedagogical exposition can be found in the monograph by Guillemin and Sternberg [60]. To make further progress, we would like to rewrite the two-form $\Omega$ in complex notations [51]. To this purpose, we introduce $z_j = p_j + iq_j$ and its complex conjugate. In terms of these variables $\Omega$ acquires the following form: $\Omega = \frac{1}{2} \sum_{j=1}^{k} dz_j \wedge d\bar{z}_j$. Next, recall [61] that for any Kähler manifold the fundamental 2-form $\Omega$ can be written as
\[\Omega = \frac{1}{2} \sum_{ij} h_{ij}(z) dz_i \wedge d\bar{z}_j\] provided that \(h_{ij}(z) = \delta_{ij} + O(|z|^2)\). This means that in fact all Kähler manifolds are symplectic [26,54]. On such Kähler manifolds one can introduce the Chern curvature 2-form which (up to a constant) should look like \(\Omega\). It should belong to the first Chern class [19]. This means that, at least formally, consistency requires us to identify \(x_i\)'s entering the product \(f \cdot x\) in the form \(\tilde{\Omega}\) with the first Chern classes \(c_i\), i.e. \(f \cdot x = \sum_{i=1}^d f_i c_i\). This fact was proven rigorously in the above mentioned paper by Atiyah and Bott [28]. Since in the Introduction we already mentioned that the Veneziano amplitudes can be formally associated with the period integrals for the Fermat (hyper)surfaces \(F\) and since such integrals can be interpreted as intersection numbers between the cycles on \(F\) [13,28,61] (see also Ref.[58], p.72) one can formally rewrite the precursor to the Veneziano amplitude [10] as

\[I = \left(\frac{-\partial}{\partial f_0}\right)^{r_0} \cdots \left(\frac{-\partial}{\partial f_d}\right)^{r_d} \int_{\Delta} \exp(\tilde{\Omega}) |_{f_i=0} \prod_{i}^{d} \int_{\Delta} dx(c_0)^{r_0} \cdots (c_d)^{r_d} \quad (61)\]

provided that \(r_0 + \cdots + r_d = n\) in view of Eq.(13). Analytical continuation of such an integral (as in the case of usual beta function) then will produce the Veneziano amplitudes. In such a language, calculation of the Veneziano amplitudes using generating function, Eq.(60), mathematically becomes almost equivalent to calculations of averages in the Witten-Kontsevich model [31-33]⁹. In addition, as was also noticed by Atiyah and Bott [28], the replacement of the exterior derivative \(d\) by \(\bar{d} = d + i(x)\) was inspired by earlier work by Witten on supersymmetric formulation of quantum mechanics and Morse theory [29]. Such an observation along with results of Ref.[60] allows us to develop calculations of the Veneziano amplitudes using supersymmetric formalism.

### 5.3 Supersymmetry and the Lefshetz isomorphism

We begin with the following observations. Let \(X\) be the complex Hermitian manifold and let \(E^{p+q}(X)\) denote the complex -valued differential forms (sections) of type \((p,q)\), \(p + q = r\), living on \(X\). The Hodge decomposition insures that \(E^r(X) = \sum_{p+q=r} E^{p+q}(X)\). The Dolbeault operators \(\partial\) and \(\bar{\partial}\) act on \(E^{p+q}(X)\) according to the rule \(\partial : E^{p+q}(X) \to E^{p+q+1}(X)\) and \(\bar{\partial} : E^{p+q}(X) \to E^{p+q+1}(X)\), so that the exterior derivative operator is defined as \(d = \partial + \bar{\partial}\). Let now \(\varphi, \psi \in E^p\). By analogy with traditional quantum mechanics we define (using Dirac’s notations) the inner product

\[\langle \varphi | \psi \rangle = \int_M \varphi \wedge *\bar{\psi}, \quad (62)\]

where the bar means the complex conjugation and the star * means the usual Hodge conjugation. Use of such a product is motivated by the fact that the period integrals, e.g. those for the Veneziano-like amplitudes, and, hence, those

⁹This fact is explained in more details in Ref.[34].
given by Eq.(61), are expressible through such inner products [61]. Fortunately, such a product possesses properties typical for the finite dimensional quantum mechanical Hilbert spaces. In particular,

\[ \langle \varphi_p | \psi_q \rangle = C \delta_{p,q} \text{ and } \langle \varphi_p | \varphi_p \rangle \geq 0, \]  

(63)

where \( C \) is some positive constant. With respect to such defined scalar product it is possible to define all conjugate operators, e.g. \( d^* \), etc. and, most importantly, the Laplacians

\[ \Delta = dd^* + d^*d, \]
\[ \Box = \partial \partial^* + \partial^* \partial, \]
\[ \bar{\Box} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}. \]

(64)

All this was known to mathematicians before Witten’s work, Ref.[29]. The unexpected twist occurred when Witten suggested to extend the notion of the exterior derivative \( d \). Within the de Rham picture (valid for both real and complex manifolds) let \( M \) be a compact Riemannian manifold and \( K \) be the Killing vector field which is just one of the generators of isometry of \( M \), then Witten suggested to replace the exterior derivative operator \( d \) by the extended operator

\[ d_s = d + si(K) \]

(65)

briefly discussed earlier in the context of the equivariant cohomology. Here \( s \) is real nonzero parameter conveniently chosen. Witten argues that one can construct the Laplacian (the Hamiltonian in his formulation) \( \Delta \) by replacing \( \Delta \) by \( \Delta_s = d_s d^*_s + d^*_s d_s \). This is possible if and only if \( d^2_s = d^*_s = 0 \) or, since \( d^2_s = s \mathcal{L}(K) \), where \( \mathcal{L}(K) \) is the Lie derivative along the field \( K \), if the Lie derivative acting on the corresponding differential form vanishes. The details are beautifully explained in the much earlier paper by Frankel, Ref.[62]. Atiyah and Bott observed that the auxiliary parameter \( s \) can be identified with earlier introduced \( f \). This observation provides the link between the D-H formalism discussed earlier and Witten’s supersymmetric quantum mechanics.

Looking at Eq.s (64) and following Refs[14,51,58] we consider the (Dirac) operator \( \partial = \bar{\partial} + \partial^* \) and its adjoint with respect to scalar product, Eq.(62). Then use of above references suggests that the dimension \( Q \) of the quatum Hilbert space associated with the reduced phase space of the D-H integral considered earlier is given by

\[ Q = \ker \partial - \ker \partial^*. \]

(66)

Such a definition was also used by Vergne[14]. In view of the results of the previous section, and, in accord with Ref.[14], we make an identification: \( Q = \mathcal{P}(k,n) \).

We would like to arrive at this result using different set of arguments. To this purpose we notice first that according to Theorem 4.7. by Wells, Ref.[61], we have \( \Delta = 2\Box = 2\bar{\Box} \) with respect to the Kähler metric on \( X \). Next, according to the Corollary 4.11. of the same reference \( \Delta \) commutes with \( d, d^*, \partial, \partial^*, \bar{\partial} \text{ and } \bar{\partial}^* \).
From these facts it follows immediately that if we, in accord with Witten, choose \( \Delta \) as our Hamiltonian, then the supercharges can be selected as \( Q^+ = d + d^* \) and \( Q^- = i (d - d^*) \). Evidently, this is not the only choice as Witten also indicates. If the Hamiltonian \( H \) is acting in finite dimensional Hilbert space one may require axiomatically that: a) there is a vacuum state (or states) \( |\alpha> \) such that \( H|\alpha> = 0 \) (i.e. this state is the harmonic differential form) and \( Q^+|\alpha> = Q^-|\alpha> = 0 \). This implies, of course, that \([H,Q^+] = [H,Q^-] = 0\). Finally, once again, following Witten, we may require that \((Q^+)^2 = (Q^-)^2 = H\). Then, the equivariant extension, Eq.(65), leads to \((Q^+)^2 = H+2is\Delta(K)\). Fortunately, the above supersymmetry algebra can be extended. As it is mentioned in Ref.[61], there are operators acting on differential forms living on Kähler (or Hodge) manifolds whose commutators are isomorphic to \( sl_2(C) \) Lie algebra. It is known [63] that all semisimple Lie algebras are made of copies of \( sl_2(C) \). Now we can exploit these observations using the Lefschetz isomorphism theorem whose exact formulation is given as Theorem 3.12 in the book by Wells, Ref.[61]. We are only using some parts of this theorem in our work.

In particular, using notations of this reference we introduce the operator \( L \) commuting with \( \Delta \) and its adjoint \( L^* \equiv \Lambda \). It can be shown, Ref.[61], p.159, that \( L^* = w* L* \) where, as before, \( * \) denotes the Hodge star operator and the operator \( w \) can be formally defined through the relation \( ** = w \), Ref.[61] p.156. From these definitions it should be clear that \( L^* \) also commutes with \( \Delta \) on the space of harmonic differential forms (in accord with p.195 of [61]). As part of the preparation for proving of the Lefschetz isomorphism theorem, it can be shown [61], that

\[
[A, L] = B \quad \text{and} \quad [B, A] = 2\Lambda, \quad [B, L] = -2L. \tag{67}
\]

At the same time, the Jacobson-Morozov theorem, Ref.[36], and results of Ref.[63], p.37, essentially guarantee that any \( sl_2(C) \) Lie algebra can be brought into form

\[
[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha, \quad [e_\alpha, f_\alpha] = h_\alpha \tag{68}
\]

upon appropriate rescaling. The index \( \alpha \) counts thenumber of \( sl_2(C) \) algebras in a semisimple Lie algebra. Comparison between the above two expressions leads to the Lie algebra endomorphism, i.e. the operators \( h_\alpha, f_\alpha \) and \( e_\alpha \) act on the vector space \( \{v\} \) to be described below while the operators \( \Lambda, L \) and \( B \) obeying the same commutation relations act on the space of differential forms. It is possible to bring Eq.s (67) and (68) to even closer correspondence. To this purpose, following Dixmier [64], Ch-r 8, we introduce operators \( h = \sum \alpha h_\alpha, \quad e = \sum \alpha b_\alpha e_\alpha, \quad f = \sum \alpha c_\alpha f_\alpha \). Then, provided that the constants are subject to constraint: \( b_\alpha c_\alpha = a_\alpha \), the commutation relations between the operators \( h, e \) and \( f \) are exactly the same as for \( B, \Lambda \) and \( L \) respectively. To avoid unnecessary complications, we choose \( a_\alpha = b_\alpha = c_\alpha = 1 \).

Next, following Serre, Ref.[35], Ch-r 4, we need to introduce the notion of the primitive vector (or element). This is the vector \( v \) such that \( hv = \lambda v \) but \( ev = 0 \). The number \( \lambda \) is the weight of the module \( V^\lambda = \{ v \in V \mid hv = \lambda v \} \). If the vector space is finite dimensional, then \( V = \sum \lambda V^\lambda \). Moreover, only if \( V^\lambda \) is
finite dimensional it is straightforward to prove that the primitive element does exist. The proof is based on the observation that if \( x \) is the eigenvector of \( h \) with weight \( \lambda \), then \( ex \) is also the eigenvector of \( h \) with eigenvalue \( \lambda - 2 \), etc. Moreover, from the book by Kac [65], Chr.3, it follows that if \( \lambda \) is the weight of \( V \), then \( \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \) is also the weight with the same multiplicity, provided that \( \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \). Kac therefore introduces another module: \( U = \sum_{k \in \mathbb{Z}} V^{\lambda+ka_i} \). Such a module is finite for finite Weyl-Coxeter reflection groups and is infinite for the affine reflection groups associated with the affine Kac-Moody Lie algebras.

We would like to argue that for our purposes it is sufficient to use only finite reflection (or pseudo-reflection) groups. It should be clear, however, from reading the book by Kac that the infinite dimensional version of the module \( U \) leads straightforwardly to all known string-theoretic results. In the case of CFT this is essential, but for calculation of the Veneziano-like amplitudes this is not essential as we are about to demonstrate. Indeed, by accepting the traditional option we loose at once our connections with the Lefschetz isomorphism theorem (relying heavily on the existence of primitive elements) and with the Hodge theory in its standard form on which our arguments are based. The infinite dimensional extensions of the Hodge-de Rham theory involving loop groups, etc. relevant for CFT can be found in Ref.[66]. Fortunately, they are not needed for our calculations. Hence, below we work only with the finite dimensional spaces.

In particular, let \( v \) be a primitive element of weight \( \lambda \) then, following Serre, we let \( v_n = \frac{1}{n!} e^n v \) for \( n \geq 0 \) and \( v_{-1} = 0 \), so that

\[
\begin{align*}
\text{h} v_n &= (\lambda - 2n) v_n \\
\text{e} v_n &= (n + 1) v_{n+1} \\
\text{f} v_n &= (\lambda - n + 1) v_{n-1}.
\end{align*}
\]

Clearly, the operators \( e \) and \( f \) are the creation and the annihilation operators according to the existing in physics terminology while the vector \( v \) can be interpreted as the vacuum state vector. The question arises: how this vector is related to the earlier introduced vector \( | \alpha > \)? Before providing an answer to this question we need, following Serre, to settle the related issue. In particular, we can either: a) assume that for all \( n \geq 0 \) the first of Eq.s(69) has solutions and all vectors \( v, v_1, v_2, \ldots \) are linearly independent or b) beginning from some \( m + 1 \geq 0 \), all vectors \( v_n \) are zero, i.e. \( v_m \neq 0 \) but \( v_{m+1} = 0 \). The first option leads to the infinite dimensional representations associated with Kac-Moody affine algebras just mentioned. The second option leads to the finite dimensional representations and to the requirement \( \lambda = m \) with \( m \) being an integer. Following Serre, this observation can be exploited further thus leading us to crucial physical identifications. Serre observes that with respect to \( n = 0 \) Eq.s(69) possess a ("super")symmetry. That is the linear mappings

\[
\begin{align*}
e^m : V^m &\to V^{-m} \quad \text{and} \quad f^m : V^{-m} &\to V^m
\end{align*}
\]

are isomorphisms and the dimensionality of \( V^m \) and \( V^{-m} \) are the same. Serre provides an operator (the analog of Witten’s \( F \) operator) \( \theta = \exp(f) \exp(e) \exp(-f) \)
such that \( \theta \cdot f = -e \cdot \theta, \theta \cdot e = -\theta \cdot f \) and \( \theta \cdot h = -h \cdot \theta \). In view of such an operator, it is convenient to redefine the operator : \( h \rightarrow \hat{h} = h - \lambda \). Then, for such redefined operator the vacuum state is just \( v \). Since both \( L \) and \( L^* = \Lambda \) commute with the supersymmetric Hamiltonian \( H \) and, because of the group endomorphism, we conclude that the vacuum state \( | \alpha > \) for \( H \) corresponds to the primitive state vector \( v \).

Now we are ready to apply yet another isomorphism following Ginzburg [36], Ch-r. 4, pp 205-206\(^{10}\). To this purpose we make the following identification

\[
e_i \rightarrow t_{i+1} \frac{\partial}{\partial t_i}, \quad f_i \rightarrow t_i \frac{\partial}{\partial t_{i+1}}, \quad h_i \rightarrow 2 \left( t_{i+1} \frac{\partial}{\partial t_{i+1}} - t_i \frac{\partial}{\partial t_i} \right),
\]

\(^{11}\)The physical meaning of \( h \) is discussed in some detail in our earlier work, Ref.[11].
References

[1] G.Veneziano, Construction of crossing symmetric, Regge behaved, amplitude for linearly rising trajectories, 
*Il Nuovo Chimento* 57A (1968) 190-197.

[2] M.Green, J.Schwarz, E.Witten, *Superstring Theory*, vol.1, 
(Cambridge U.Press, Cambridge, UK, 1987).

[3] R.Stanley, *Combinatorics and Commutative Algebra*, 
Birkhäuser, Boston, MA, 1996.

[4] S.Chowla, A.Selberg, On Epstein’s Zeta function, 
*J. Reine Angew.Math.* 227 (1967) 86-100.

[5] A.Weil, Abelian varieties and Hodge ring, *Collected Works*, 
vol.3, Springer-Verlag, Berlin, 1979.

[6] A.Weil, Sur les periods des integrales Abéliennes, 
*Comm.Pure Appl. Math.* 29 (1976) 81-819.

[7] B.Gross, On periods of Abelian integrals and formula of Chowla and Selberg, 
*Inv.Math.* 45 (1978) 193-211.

[8] S.Lang, *Introduction to Algebraic and Abelian Functions*, 
Springer-Verlag, Berlin, 1982.

[9] A.Kholodenko, New string amplitudes from old Fermat (hyper)surfaces, 
*IJMP* A19 (2004) 1655-1703.

[10] A.Kholodenko, New strings for old Veneziano amplitudes I. Analytical treatment, 
*J.Geom.Phys.* 55 (2005) 50-74, arXiv: hep-th/0410242.

[11] A.Kholodenko, New strings for old Veneziano amplitudes II. Group-theoretic treatment, 
*J.Geom.Phys.* (2005) in press, arXiv: hep-th/0411241.

[12] P.Deligne, Hodge cycles and Abelian varieties, 
*LNM.* 900 (1982) 9-100.

[13] J.Carlson, S.Muller-Stach, C.Peters, *Period Mapping and Period Domains* 
Cambridge U.Press, Cambridge, UK, 2003.

[14] M.Vergne, Convex polytopes and quantization of symplectic manifolds, 
*PNAS* 93 (1996) 14238-14242.

[15] S.Donachie,G.Dosch, P.Landshoff,O.Nachtmann, 
*Pomeron Physics and QCD*, 
Cambridge U.Press, Cambridge, UK, 2002.

[16] P.Collins, *An Introduction to Regge Theory and High Energy Physics*, 
Cambridge U.Press, Cambridge, UK, 1977.

[17] V.Gribov, *The Theory of Complex Angular Momenta*, 
Cambridge U.Press, Cambridge, UK, 2003.

[18] J.Forshaw, D.Ross, *Quantum Chromodynamics and the Pomeron*, 
Cambridge U.Press, Cambridge, UK, 1997.

[19] A.Kholodenko, E.Ballard, From Ginzburg-Landau to Hilbert Einstein via Yamabe, 
Reviews in Mathematical Physics (2006) to be published, arxiv: gr-qc/0410029.

[20] M.Herrero, The standard model, arxiv: hep-ph/9812242.
[21] C.Ewerz, The odderon in quantum chromodynamics, arxiv: hep-ph/0306137.
[22] V.Batyrev, Variation of the mixed Hodge structure of affine hypersurfaces in algebraic tori, Duke Math.J. 69 (1993) 349-409.
[23] V.Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J.Alg.Geom. 3 (1994) 493-535.
[24] T.Hibi, Dual polytopes of rational convex polytopes, Combinatorica 12 (1992) 237-240.
[25] V.De Alfaro, S.Fubini,G.Furlan, C.Rossetti, Currents in Hadron Physics, Elsevier Publishing Co. Amsterdam, 1973.
[26] M.Audin, Torus Actions on Symplectic Manifolds, Birkhäuser, Boston, MA, 2004.
[27] R.Stanley, Invariants of finite groups and their applications to combinatorics, BAMS 1 (1979) 475-511.
[28] M.Atiyah, R.Bott, The moment map and equivariant cohomology, Topology 23 (1984) 1-28.
[29] E.Witten, Supersymmetry and Morse theory, J.Diff.Geom. 17 (1982) 661-692.
[30] W.Fulton, Introduction to Toric Varieties, Princeton U. Press, Princeton, 1993.
[31] E.Witten, Two dimensional gravity and intersection theory on moduli space, Surv.Diff.Geom.1 (1991) 243-310.
[32] M.Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm.Math.Phys. 147 (1992) 1-23.
[33] A.Kholodenko, Kontsevich-Witten model from 2+1 gravity: new exact combinatorial solution, J.Geom.Phys. 43 (2002) 45-91.
[34] A.Kholodenko, New strings for old Veneziano amplitudes IV. Combinatorial treatment, in preparation.
[35] J-P.Serre, Algèbres de Lie Simi-Simples Complexes, Benjamin, Inc. New York, 1966.
[36] V.Ginzburg, Representation Theory and Complex Geometry, Birkhäuser-Verlag, Boston, 1997.
[37] F.Hirzebruch, D.Zagier, The Atiyah-Singer Theorem and Elementary Number Theory, Publish or Perish, Berkeley, CA, 1974.
[38] V.Buchstaber, T.Panov, Torus actions and Their Applications in Topology and Combinatorics, AMS Publishers, Providence, RI, 2002.
[39] R.Dias, S.Robins, The Ehrhart polynomial of a lattice polytope, Ann.Math. 145 (1997) 503-518.
[40] K.Hori, S.Katz, A.Klemm, R.Phadharipande,R.Thomas, C.Vafa, R.Vakil, E.Zaslow, Mirror Symmetry, AMS Publishers, Providence, RI, 2003.
[41] D.Cox,S.Katz, Mirror Symmetry and Algebraic Geometry, AMS Publishers, Providence, RI, 2003.
[42] N.Ashcroft, D.Mermin, Solid State Physics, Saunders College Press, Philadelphia, PA, 1976.
[43] B.Greene, M.Plesser, Duality in Calabi-Yau moduli space,
Nucl.Phys. B338 (1990) 15-37.

[44] O.Debarre, Fano varieties, in
Higher Dimensional Varieties and Rational Points,
pp.93-132, Springer-Verlag, Belin, 2003.

[45] C.Haase, I.Melnikov, The reflexive dimension of a lattice polytope,
arxiv: math.CO/0406485.

[46] P.Frampton, Dual Resonance Models, W.A.Benjamin, Inc.,
Reading, MA, 1974.

[47] S.Mandelstam, Veneziano formula with trajectories spaced by two units,
Phys.Rev.Lett. 21 (1968) 1724-1728.

[48] A.Kholodenko, New strings for old Veneziano amplitudes III.
Symplectic treatment, J.Geom.Phys.(2005) in press,
arxiv: hep-th/0502231.

[49] A.Barvinok, Computing the volume, counting integral points,
and exponential sums, Discr.Comp.Geometry 10 (1993) 123-141.

[50] M.Brion, Points entiers dans les polyedres convexes,
Ann.Sci.Ecole Norm. Sup. 21 (1988) 653-663.

[51] V.Guillemin, V.Ginzburg, Y.Karshon, Moment Maps,
Cobordisms, and Hamiltonian Group Actions,
AMS Publishers, Providence, RI, 2002.

[52] M.Atiyah, Convexity and commuting Hamiltonians,
London Math.Soc.Bull. 14 (1982)1-15.

[53] M.Vergne, in E.Mezetti, S.Paycha (Eds),
European Women in Mathematics, pp 225-284,
World Scientific, Singapore, 2003.

[54] M.Atiyah, Angular momentum, convex polyhedra
and algebraic geometry,
Proc. Edinburg Math.Soc. 26 (1983)121-138.

[55] A.Barvinok, A Course in Convexity,
AMS Publishers, Providence, RI, 2002.

[56] M.Brion, M.Vergne, Lattice points in simple polytopes,
J.AMS 10 (1997) 371-392.

[57] A.Khovanskii, A.Pukhlikov, A Riemann–Roch theorem
for integrals and sums of quasipolynomials over virtual
polytopes, St.Petersburg Math.J. 4 (1992) 789-812.

[58] V.Guillemin, Moment Maps and Combinatorial
Invariants of Hamiltonian T^n Spaces,
Birkhäuser, Boston, MA, 1994.

[59] M.Brion, M.Vergne, An equivariant Riemann-Roch theorem
for complete simplicial toric varieties,
J.Reine Angew.Math. 482 (1997) 67-92.

[60] V.Guillemin, S.Sternberg, Supersymmetry and Equivariant
de Rham Theory, Springer-Verlag, Berlin, 1999.

[61] R.Wells, Differential Analysis on Complex Manifolds,
Springer-Verlag, Berlin, 1980.

[62] T.Frankel, Fixed points and torsion on Kähler manifolds,
Ann.Math.70 (1959) 1-8.
[63] J.Humphreys, Introduction to Lie Algebras and Representations Theory, Springer-Verlag, Berlin, 1972.
[64] J.Dixmier, Enveloping Algebras, Elsevier, Amsterdam, 1977.
[65] V.Kac, Infinite Dimensional Lie Algebras, Cambridge U. Press, 1990.
[66] A.Huckelberry, T.Wurzbacher, Infinite Dimensional Kahler Manifolds, Birkhäuser, Boston, 1997.