Chern-Simons theory on the lattice

W. Bietenholz a, J. Nishimura b and P. Sodano c

a Institut für Physik, Humboldt Universität zu Berlin, Invalidenstr. 110, D-10115 Berlin, Germany
b Dept. of Physics, Nagoya University, Nagoya 464-8601, Japan
c Dipartimento di Fisica e Sezione I.N.F.N., Università di Perugia, Via A. Pascoli, 06123 Perugia, Italy

We present new proposals for the representation of a Chern-Simons term on the lattice. In the first part, such a term is constructed from the fermion determinant, and in the second part directly from the Abelian gauge term. In both cases, the parity transformation is modified on the lattice without affecting its continuum limit.

We consider the problem of representing the Chern-Simons (CS) term on a 3d lattice. To this end, we modify the parity transformation on the lattice in a smooth way, similar to Lüscher’s lattice Dirac operator D 1. GW FERMIONS IN 3 DIMENSIONS

In d = 4 the GW relation (6) for a lattice Dirac operator D(4) is well-known by now, \{D(4), \gamma_5\} = aD(4)\gamma_5D(4). In general one also assumes \gamma_5-Hermiticity, D(4)† = \gamma_5D(4)\gamma_5, so that the GW relation can be written as D(4) + D(4)† = aD(4)†D(4). This \gamma_5-independent form can be adapted in d = 3 (or other odd dimensions),

\[ D^{(3)} + D^{(3)†} = aD^{(3)}†D^{(3)}. \tag{1} \]

Such a lattice Dirac operator describes massless fermions in d = 3. In analogy to \gamma_5-Hermiticity, we assume in addition the (natural) property

\[ D^{(3)†}(U_P) = PD^{(3)†}(U)P, \quad \text{where} \]

\[ U_P^\mu(x) = U_\mu(-x - a\hat{\mu})†. \tag{2} \]

P is the (standard) parity operator, and U P is the parity transformed gauge configuration.

As in even dimensions, solutions to the GW relation (6) take the form

\[ D^{(3)} = \frac{1}{a}(1 - V), \quad V \text{ unitary} \tag{3} \]

where \( V = 1 - aD_{cont} + O(a^2) \). The fermion action 
\[ S = a^3 \sum_x \bar{\psi}(x)D^{(3)}(U)\psi(x) \]

is exactly invariant under the lattice modified parity transformation

\[ \psi(x) \rightarrow iP\psi(x), \quad \bar{\psi}(x) \rightarrow i\bar{\psi}(x)P, \quad U_\mu \rightarrow U_\mu^P. \tag{4} \]

The factor V in the transformation of \( \psi \) is a lattice modification in \( O(a) \) (alternatively it could also be attached to \( \bar{\psi} \)).

The measure transforms under the above transformation as

\[ d\bar{\psi}d\psi \rightarrow (\det V)^{-1}d\bar{\psi}d\psi \tag{5} \]

which gives rise to the parity anomaly. The situation is again similar to chiral symmetry in even dimensions: a local, undoubled 3d fermion cannot be P invariant. The Wilson term means a rough P breaking in the action, whereas the GW fermion has a non-trivial transformation term of the modified P symmetry in the measure; both types of symmetry breaking reproduce the correct anomaly (see below).

Explicit GW operators \( D^{(3)} \) are obtained by inserting

\[ V = A/\sqrt{A^1A}, \quad A = 1 - aD_0 \tag{6} \]

into eq. (6), where \( D_0 \) is some lattice Dirac operator (local, free of doubling), e.g. the Wilson operator \( D^{(3)}_w \). Kikukawa and Neuberger further observed the following phase relation

\[ 2 \arg(\det D^{(3)}) = \arg(\det A). \tag{7} \]
Coste and Lüscher studied the continuum limit of the effective action \( \det A \). In particular they considered \( A^{(n)} = (1 - aD_w^3)(1 - 2aD_w^3)^2n, n \in \mathbb{Z} \). In a smooth gauge background, they found
\[
\lim_{m,a \to 0} \arg(\det A^{(n)}) = (2n + 1)\pi e^2 S_{CS}
\]
where \( m, e \) are the fermion mass and charge, and \( S_{CS} \) is the CS action. We see that an infinite set of universality classes (labeled by \( n \)) coexist for the “P anomaly” (which is therefore not an anomaly in the usual sense).

The standard overlap operator, which is obtained from \( A^{(0)} = 1 - D_w^3 \), is in the class \( n = 0 \), as eqs. (3, 8) show. However, it turns out that the 3d GW fermions cover all universality classes, as we see if we vary the terms \( D_0 \) and \( R \) in the overlap solution to the general GW relation
\[
D^{(3)} + D^{(3)} = 2aD^{(3)}RD^{(3)} ,
\]
\[
D^{(3)} = \frac{1}{2a\sqrt{R}} (1 - A/\sqrt{A^2A}) \frac{1}{\sqrt{R}}
\]
\[
A = 1 - 2a\sqrt{R}D_0\sqrt{R}
\]
where the term \( R \) is local and not parity-odd. For example, we arrive at the universality class \( n \) for the choice \( A = A^{(n)} \), \( R_{x,y} = \frac{1}{8m+2}\delta_{x,y} \), relying again on eqs. (3, 8).

These properties inspire the following ansatz for a lattice CS term \( S_{CS}^{lat} \),
\[
\exp(iS_{CS}^{lat}) = \frac{\det A^{(0)}}{|\det A^{(0)}|} \quad (9)
\]
(where \( A^{(0)} \) may also be replaced by another operator in the same universality class). The phase is parity odd, the r.h.s. is manifestly gauge invariant, and the normalization is taken so that
\[
S_{CS}^{lat} \to S_{CS}^{lat} + 2\pi \nu
\]
under a gauge transformation with winding number \( \nu \). Considering the above properties, we think that this definition may capture the topology on the lattice (this is somehow similar to the index in even dimensions).

2. A NEW APPROACH TO PURE ABELIAN CS GAUGE THEORY

In this part, we do not introduce any matter fields, but we are only concerned with a suitable discretization of the CS Lagrangian
\[
\mathcal{L}_{CS} = A_\mu \epsilon_{\mu\nu\rho} \partial_\nu A_\rho ,
\]
which represents a topological field theory \[8\]. It is used in condensed matter physics and in polymer physics as a low energy effective action \[1\].

For the naive lattice discretization the functional integral does not exist (not even after gauge fixing) due to the doubling problem, which is typical for linear derivatives \[11\]. More generally, this problem persists for any local, gauge invariant, cubic symmetric and parity-odd action \[11\].

Our new approach to circumvent this problem modifies this time the parity transform of the lattice gauge field. A previous work along this line \[12\] succeeded in constructing a corresponding lattice action, but the modified P transform is unfortunately non-local. Here we insist on its locality, which gives us confidence about a safe continuum limit.

For the lattice action at \( a = 1 \) we write
\[
S[A] = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3p A_\mu(-p)C_{\mu\nu}(p)A_\nu(p) .
\]

Now we follow the fermionic procedure: the perfect lattice action for free fermions was derived analytically \[13\]. If the RG blocking is done with a Gaussian of coefficient \( R^{-1} \), the perfect propagator obeys the GW relation with kernel \( R \), which was then extended to a general criterion for chirality. To adapt this strategy to the current problem, we first constructed a perfect lattice CS action, which is a straightforward application of the RG blocking of non-compact gauge fields \[13\]. The resulting perfect term \( C_{\mu\nu}(p) \) has a property, which is very similar to the GW relation; we denote it as the Chern-Simons-Ginsparg-Wilson relation (CSGWR)
\[
C_{\mu\nu} + C_{\mu\nu}^P = C_{\mu\nu}^P C_{\mu\nu}^P \quad (13)
\]
(where we set the blocking term \( R_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu} \) for simplicity). We only use the perfect action to motivate this relation as a criterion for a smooth parity breaking on the lattice. Now we move on to look for simpler solutions to the CSGWR.

First, the overlap type of solution can also be applied here \[14\]: let \( \Gamma_{\mu\nu} = C_{\mu\nu} - \delta_{\mu\nu} \) so that
the CSGWR reads $\Gamma_{\mu
u}^P \Gamma_{\rho\sigma} = \delta_{\mu\rho}$. Solutions are given by $\Gamma = \Gamma(0)[\Gamma(0) P \Gamma(0)]^{-1/2}$. In analogy to the Wilson fermion, we can choose for instance the Fröhlich-Marchetti action $(\ref{eq:fröhlich-marchetti})$ to fix $\Gamma(0)$. Now we look for even simpler, polynomial solutions to the CSGWR, starting from the ansatz

$$C_{\mu\nu}(p) = L_{\mu\nu}(p)v(p) + M_{\mu\nu}(p)w(p)$$

$$L_{\mu\nu}(p) = -i\epsilon_{\mu\rho\nu}\rho_{\alpha} = \tilde{p}^2 \delta_{\mu\nu} - \tilde{p}_{\mu}\tilde{p}_{\nu}$$

where $\tilde{p}\alpha = 2\sin(p\alpha/2)$. The functions $v, w$ are even, local, lattice isotropic and $v(p) = 1 + O(p^2)$. The Maxwell term $M$ is parity even, so any $w \neq 0$ breaks the anti-parity of $C$ in $O(a^2)$ ($M$ is similar to the Wilson term for fermions). Fröhlich and Marchetti set $v = w = 1$ — which implies a rough symmetry breaking — but here we seek a smooth breaking, which obeys the CSGWR. Due to the identities $L_{\mu\nu}^2 L_{\rho\sigma} = -\tilde{M}_{\mu\nu}$ and $M_{\mu\nu}^2 M_{\rho\sigma} = \tilde{p}^2 \tilde{M}_{\mu\nu}$, the terms reproduce themselves on the r.h.s. of the CSGWR. To solve it we have to require

$$w(p) = \left[1 - \sqrt{1 + \tilde{p}^2 v^2(p)}\right]/\tilde{p}^2.$$  \hspace{1cm} (15)

For any local choice of $v$, the locality of $w$ is guaranteed as well. These solutions are gauge invariant, as the identities $\tilde{p}_{\mu} L_{\mu\nu}(p) = 0$ and $\tilde{p}_{\mu} M_{\mu\nu}(p) = 0$ reveal.

The question is now if we find a locally modified $P$ transform which keeps these solutions exactly invariant, i.e.

$$C_{\mu\nu} = -[\delta_{\mu\rho} + X_{\mu\rho}] C_{\rho\sigma}^{(0)}[\delta_{\sigma\nu} + X_{\sigma\nu}].$$ \hspace{1cm} (16)

For the modification term $X = O(a)$ we use the same ansatz as for $C$,

$$X_{\mu\nu}(p) = L_{\mu\nu}(p)x(p) + M_{\mu\nu}(p)y(p),$$ \hspace{1cm} (17)

where $x, y$ are even and lattice isotropic.

Inserting the ansätze $(\ref{eq:fröhlich-marchetti})$ and $(\ref{eq:modified-term})$ into the condition $(\ref{eq:modification-condition})$, and requiring the CSGWR to hold (i.e. implementing eq. $(\ref{eq:fröhlich-marchetti})$), we can express for instance $y, v, w$ in terms of $x$,

$$y = \left[\sqrt{1 + \tilde{p}^2 x^2} - 1\right]/\tilde{p}^2,$$

$$v = -2x\sqrt{1 + \tilde{p}^2 x^2}, \hspace{0.5cm} w = -2x^2.$$ \hspace{1cm} (18)

In particular, any local choice for $x$ leads to local terms $v, w$ and $y$. To provide the right form of $v$ — so that $C$ has the correct continuum limit — we need $x = -1/2 + O(p^2)$. For instance, we can set $x = -1/2$, which implies $y = [\sqrt{1 + \tilde{p}^2}/4 - 1]/\tilde{p}^2$, $v = \sqrt{1 + \tilde{p}^2}/4$, $w = -1/2$.

Therefore, we found explicit, simple solutions to the CSGWR, which are exactly odd under a lattice modified $P$ transform. Both, the action and the transformation term are local. Therefore, we are on safe grounds to recover the correct continuum limit, which is a topological model. However, it is an open question if our lattice formulation is topological already.

To summarize, we note the CS action of the Part 1, $S_{CS}^{a_2}$ in eq. $(\ref{eq:fröhlich-marchetti})$, is not in conflict with the No-Go theorem of Ref. $(\ref{ref:no-go-theorem})$ because it is not bilinear in $A_{\mu}, A_{\nu}$. Part 2 circumvents this theorem by modifying the parity of $A_{\mu}$ on the lattice.

We thank C. Diamantini, K. Jansen, F. Klinkhamer, M. Lüser and U.-J. Wiese for useful comments.

REFERENCES

[1] M. Lüscher, Phys. Lett. B428 (1998) 342.
[2] W. Bietenholz and J. Nishimura, JHEP 07 (2001) 015.
[3] W. Bietenholz and P. Sodano, in preparation.
[4] P. Ginsparg and K. Wilson, Phys. Rev. D25 (1982) 2649.
[5] Y. Kikukawa and H. Neuberger, Nucl. Phys. B513 (1998) 735.
[6] A. Coste and M. Lüser, Nucl. Phys. B323 (1989) 631.
[7] W. Bietenholz, Eur. Phys. J. C6 (1999) 537.
[8] E. Witten, Comm. Math. Phys. 121 (1989) 351.
[9] See e.g. A. Zee, Prog. Theor. Phys. Suppl. 107 (1992) 77; M. Diamantini, P. Sodano and C. Trugenberger, Nucl. Phys. B448 (1995) 585, B474 (1996) 641.
[10] J. Fröhlich and P. Marchetti, Comm. Math. Phys. 121 (1989) 177.
[11] F. Berruto, M. Diamantini and P. Sodano, Phys. Lett. B487 (2000) 366.
[12] C. Fosco and A. Lópex, Phys. Rev. D64 (2001) 025017.
[13] W. Bietenholz and U.-J. Wiese, Phys. Lett. B378 (1996) 222; Nucl. Phys. B464 (1996) 319.
[14] F. Berruto and P. Sodano, private notes.