Minimal diagrams of classical knots

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Abstract

We show that if a classical knot diagram satisfies a certain combinatorial condition then it is minimal with respect to the number of classical crossings. This statement is proved by using the Kauffman bracket and the relation between atoms and knots.

1 The main result

In paper [10], we showed that if a virtual link diagram satisfies two certain conditions (one of them deals with the Kauffman bracket and the other one uses the Khovanov homology) then this diagram is minimal with respect to the number of classical crossings. That result is a generalisation of the famous Kauffman-Murasugi theorem [11].

In the present paper, we show that the Kauffman bracket itself (without Khovanov’s categorification) is indeed a very strong tool to establish minimality of knot diagrams. The condition described in the present paper deals only with some combinatorics of the knot diagram, namely, with so-called atom.

This condition is very easy to check unlike that in [10] where one should be able to calculate (a part of) the Khovanov homology. The condition of the present paper uses only some simple combinatorics.

Note that though the techniques of the present paper uses much from virtual knot theory, the main theorem is stated only for classical knots, i.e., not virtual knots or links and not classical links. The reason is that one important step of the proof deals with the connected summation which is not well defined either for links or for virtual knots.

What remains in the general case of virtual links, is the analogous framed result, which was first proved in [3]. As for virtual knots (not links), one can obtain a similar result in the long category. For long virtual knots, see [8].

We shall give the first definition for the general virtual case, however, to understand the main line of the present paper and the proof of the main theorem one need not know virtual knot theory.

Virtual knots were proposed by Kauffman in [2]. All necessary detailed definitions can be found therein.

The theory of atoms and knots is represented in [6]. Recall the main definitions.

By an atom we mean a pair \((M, \Gamma)\), where \(M\) is a connected 2-manifold and \(\Gamma\) is a 4-valent graph together with a rule for embedding in \(M\) such
that the complement $M \setminus \Gamma$ admits a checkerboard colouring. The graph $\Gamma$ is called the frame of the atom. We also think that for a given atom the colouring is fixed.

The atoms are considered up to natural equivalence, i.e. homeomorphisms mapping the frame to the frame and preserving the colour of edges. Certainly, the atom (up to equivalence) is nothing but its frame together with the rule for attaching black cells at each vertex (the way for attaching white cells is defined automatically together with the structure of opposite edges at vertices).

Let $L$ be a virtual link diagram. Let us construct the atom $V(L)$ as follows. First, we construct the frame $\Gamma$ of $V(L)$. The vertices of the frame are in one-to-one correspondence with classical crossings of the diagram $L$. Classical crossings are connected by arcs which might intersect or have selfintersection at some virtual crossings. In the classical case there are no other crossings, so the branches are just the edges of the shadow of the knot (link). Each classical crossing has four emanating branches. We associate four edges of the atom to these branches.

Then the rule for attaching black 2-cells to the frame is recovered from the diagram $L$. Namely let $X$ be a classical crossing of $L$. Enumerate the four emanating branches by letters $x_1, x_2, x_3, x_4$ in the clockwise direction in such a way that the edges $x_1$ and $x_3$ form an undercrossing, whence the edges $x_2, x_4$ form an overcrossing. Then, the black angles are chosen to be $(x_1, x_2)$ and $(x_3, x_4)$.

Let $L$ be a virtual diagram and let $V(L)$ be the corresponding atom. Each vertex of the atom $V(L)$ is incident to four pieces of cells: two black ones and two white ones. Globally, some of them (e.g. two white ones) may coincide. A diagram $L$ is said to be good if at each vertex of the atom $V(L)$ we have precisely four different cells.

To be concise, we shall say genus (or Euler characteristic) of the diagram $L$ for the genus (resp., Euler characteristic) of the corresponding atom: $\chi(L) = \chi(V(L)), g(L) = g(V(L))$.

The main result of the present paper is the following

**Theorem 1.** Suppose the diagram $L$ of a classical knot is good. Then it is minimal. In other words, if the diagram $L$ has $n$ crossings then for any classical diagram representing the same knot the number of crossing is at least $n$.

**Remark 1.** There is an important conjecture whether a minimal classical diagram is minimal in the virtual category, i.e. whether there exists a minimal classical link $L$ with $n$ crossings admitting a virtual diagram with strictly smaller number of classical crossings.

The main theorem of this paper does not give even a partial answer to this theorem: we say that a classical diagram is minimal only among classical ones.

To prove this theorem, we shall use some auxiliary lemmas.

By $\text{span} X$ for a one-variable (Laurent) polynomial $X$ we mean the difference between its leading degree and lowest degree. For a polynomial in many variables, we also may define $\text{span}$ with respect to any of these variables.
Lemma 1. Let $L$ be a virtual link diagram. Then the following inequality holds:

$$\text{span}(L) \leq 4n + 2(\chi(L) - 2),$$

whence if $L$ is a good diagram then this inequality becomes a strict equality.

The proof can be found in e.g. [6] or [10].

We shall use the operation of taking $k$ parallel copies of the link $L \rightarrow D_k(L)$. This operation is well defined only in the framed category. Framed (virtual) links are equivalence classes of virtual link diagrams modulo (generalised Reidemeister moves), where we do not allow the first classical Reidemeister move and replace it by the double twist move (addition/removal of two loops having opposite signs), for more details, see, e.g., [6].

More precisely, the following lemma holds.

Statement 1. If $L, L'$ are equivalent virtual link diagrams so that the writhe numbers (framings) of the corresponding components for $L$ and $L'$ coincide then for each natural $m$, the diagrams $D_m(L), D_m(L')$ represent equivalent virtual links.

Lemma 2. Suppose a virtual link diagram $L$ is good. Then for any natural $k$, the diagram $D_k(L)$ is good as well.

The proof can be found in [8]. The main idea is that to each cell of the atom $V(L)$, there correspond precisely $k$ “parallel” cells of the atom $V(D_k(L))$. If a cell $C_1, C_2$ of $V(D_k(L))$ touches itself at a crossing $X$ of $D_k(L)$ then the corresponding cell in $L$ touches itself at the crossing corresponding to $X$: for each crossing of $L$, we have $k^2$ corresponding crossings of $D_k(L)$.

For any virtual link diagram $L$, its mirror diagram $\bar{L}$ is defined to be the diagram obtained from $L$ by switching all classical crossings (over-crossings are replaced by undercrossings and vice versa).

Obviously, the atom $V(\bar{L})$ is obtained from $V(L)$ by changing the colour of the cells. Thus, if $L$ is a good diagram, then so is $\bar{L}$.

We shall also use the notion of connected sum $K_1 \# K_2$ for two oriented classical knots or two oriented long virtual knots. Note that the connected sum is not well defined for links; it is not well defined for compact virtual knots, either: it depends on the choice of break points.

Nevertheless, for any two virtual link diagrams $L_1$ and $L_2$ we can take any of its connected sums. We shall use the notation $K_1 \# K_2$ only for the classical connected sum, which is well defined. In this case, the following lemma holds.

Lemma 3. Let $K$ be a good diagram of a virtual link. Then the diagram $K \# \bar{K}$ is good as well.

Proof. Assume the contrary. Suppose that $l$ is a cell of the atom $V(K \# \bar{K})$ (say, black) that touches itself at some crossing $X$. The boundary $\partial l$ of this cell is a cycle on the frame of the atom. On the atom $V(K \# \bar{K})$, we have two edges $e_1, e_2$, “separating” $V(K_1)$ from $V(K_2)$. Choose points $X_1, X_2$ on these edges. These points divide $\partial l$ into two parts. One part of
the cycle $\partial l$ generates a black cell of the atom $V(K)$ whence the other one generates a black cell for $V(\bar{K})$. By definition, the diagram containing the vertex $X$ ($K$ or $\bar{K}$) is not good. This means that the diagram $K$ is not good. The contradiction completes the proof.

The following fact is evident

**Statement 2.** If two classical knot diagrams $K_1, K_2$ are isotopic, then the diagrams $K_1 \# \bar{K}_1$ and $K_2 \# \bar{K}_2$ are framed equivalent.

**Proof.** Indeed, the two knots in question are isotopic and have the same framing equal to zero.

Let us now prove the main theorem of this paper. Let $K$ be a good classical knot diagram having $n$ crossings. Denote the Euler characteristic of the diagram $K \# \bar{K}$ by $\chi$. Set $N = 2n$. Suppose a classical diagram $K'$ having $n'$ crossings ($n' < n$) generates the same knot as $K$. Denote the genus of the diagram $K'$ by $\chi'$.

Let $m$ be a positive integer. It follows from Statement 2 that $D_m(K \# \bar{K})$ and $D_m(K' \# \bar{K}')$ generate isotopic knots. Denote $D_m(K \# \bar{K})$ by $D_m$ and denote $D_m(K' \# \bar{K}')$ by $D'_m$. By definition we have $\langle D_m \rangle = \langle D'_m \rangle$. Also, set $\chi = \chi(K \# \bar{K}), \chi' = \chi(K' \# \bar{K}')$.

The diagram $K \# \bar{K}$ is good. Thus, so are all diagrams $D_m$ for arbitrary positive integers $m$:

$$\text{span}(D_m) = 4m^2 N + 2(\chi_m - 2),$$

where $\chi_m = \chi(D_m)$. The atom $V(D_m)$ has $m^2 N$ vertices, $2m^2 N$ edges and $m \Gamma$ 2-cells, where $\Gamma = N + \chi$ is the number of the 2-cells of the atom $K \# \bar{K}$. Thus,

$$\text{span}(D_m) = 2(m^2 + m)N + 2m\chi - 4.$$  \hspace{1cm} (3)

Analogously, for the diagram $D'_m$ the following inequality holds

$$\text{span}(D'_m) \leq 2(m^2 + m)N' + 2m\chi' - 4.$$ \hspace{1cm} (4)

Here we can not say whether the exact equality takes place, since we do not know whether the diagram $D'_m$ is good.

Comparing the right-hand sides of (3) and (4), we get

$$(\chi' - \chi) \geq (m + 1)(N - N').$$ \hspace{1cm} (5)

According to the assumption, we have $n - n' > 0$; thus $N - N' > 0$. Since $m$ is chosen arbitrarily, we get to a contradiction: the quantity $\chi' - \chi$ (which is fixed and does not depend on $m$) should exceed any preassigned positive integer number. The contradiction completes the proof of the theorem.
2 The general case of virtual links

The proof given above works neither for classical links nor for virtual knots because of the connected summation. The trick using the connected sum is indeed needed to compare the diagrams \( D_m(K_1) \) and \( D_m(K_2) \). But since we do not know whether framings of the knot \( K_1 \) (whose minimality is being tested) and \( K_2 \) coincide, we cannot say whether \( D_m(K_1) \) and \( D_m(K_2) \) generate isotopic links. In order to avoid the problem with framing, we have to take the connected sum of the initial knot with its inverse, thus restricting ourselves only for the case of classical knots.

However, the trick using the connected sum with the inverse image is unnecessary, if we deal with framed knots. This leads to the following

**Theorem 2.** Let \( L_1 \) be a good diagram of a framed virtual link. Then it is minimal in the framed category.

This theorem was proved in [6].

Also, if we deal with long virtual knots (i.e. virtual knots with fixed endpoints), we have a well-defined connected sum operation.

This leads to the following

**Theorem 3.** Let \( K_1 \) be a long virtual knot diagram such that the corresponding compact virtual knot diagram \( Cl(K_1) \) is good. Then the diagram \( K_1 \) is minimal in the long category.

The proof literally repeats the proof of the main theorem of this paper.

3 Examples

Actually, it is not difficult to construct a non-alternating classical knot whose minimality can be detected by the main theorem of this paper. For instance, so are knots represented by closures of positive braids with arbitrary number of strands, where we use only exponents of standard generators \( \sigma_j^i \) for \( j \geq 2 \) (one should just make sure that the obtained diagram is a knot, not link).

**Remark 2.** Note that if we can apply a Reidemeister move decreasing the number of crossings to a classical knot diagram \( L \) then the condition of the theorem evidently fails. Thus, we do not lose generality: evidently non-minimal diagrams could not be minimal. However, if we can apply a third Reidemeister move to a diagram \( L \), then the diagram does not satisfy the condition of the main theorem. Thus, this theorem works only for “fixed” diagrams, i.e. those for which any Reidemeister move should increase the number of crossings.

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