FREE BOUNDARY PROBLEM FOR A REACTION-DIFFUSION EQUATION WITH POSITIVE BISTABLE NONLINEARITY

Maho Endo, Yuki Kaneko and Yoshio Yamada

Department of Pure and Applied Mathematics, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

Dedicated to Professor Wei-Ming Ni on the occasion of his 70th birthday

Abstract. This paper deals with a free boundary problem for a reaction-diffusion equation in a one-dimensional interval whose boundary consists of a fixed end-point and a moving one. We put homogeneous Dirichlet condition at the fixed boundary, while we assume that the dynamics of the moving boundary is governed by the Stefan condition. Such free boundary problems have been studied by a lot of researchers. We will take a nonlinear reaction term of positive bistable type which exhibits interesting properties of solutions such as multiple spreading phenomena. In fact, it will be proved that large-time behaviors of solutions can be classified into three types; vanishing, small spreading and big spreading. Some sufficient conditions for these behaviors are also shown. Moreover, for two types of spreading, we will give sharp estimates of spreading speed of each free boundary and asymptotic profiles of each solution.

1. Introduction. We consider a free boundary problem for a reaction-diffusion equation given by:

$$
\begin{cases}
    u_t = du_{xx} + f(u), & t > 0, \ 0 < x < h(t), \\
    u(t, 0) = 0, \quad u(t, h(t)) = 0, & t > 0, \\
    h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
    h(0) = h_0, \quad u(0, x) = u_0(x), & 0 \leq x \leq h_0,
\end{cases}
$$

where $d, \mu$ and $h_0$ are positive constants and $x = h(t)$ represents a free boundary. In (FBP), $f \in C^1([0, \infty))$ satisfies the following properties:

1. $f(u) = 0$ has solutions $u = 0, u^*_1, u^*_2, u^*_3$ ($0 < u^*_1 < u^*_2 < u^*_3$),

2. $f'(0) > 0, f'(u^*_1) < 0, f'(u^*_2) > 0, f'(u^*_3) < 0, \int_{u^*_1}^{u^*_3} f(u)du > 0$

and $f(u) \neq 0$ for $u \notin \{0, u^*_1, u^*_2, u^*_3\}$.

2010 Mathematics Subject Classification. Primary: 35R35; Secondary: 35K57, 35J61, 92D25.
Key words and phrases. Free boundary problem, spreading speed, asymptotic profile, reaction-diffusion equation.

1 Partially supported by Waseda University Grant for Special Research Project (2018K-203, 2018B-103) and Grant-in-Aid for Early-Career Scientists (19K14602).
2 Partially supported by Grant-in-Aid for Scientific Research (C) 16K05244.
* Corresponding author: Yoshio Yamada.
** Current address: Department of Mathematical and Physical Sciences, Japan Women’s University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo 112-8681, Japan.
When \( f \) satisfies (PB), we say that \( f \) is a function of positive bistable type. Initial function \( u_0 \) satisfies
\[
    u_0 \in C^2([0, h_0]), \quad u_0(0) = u_0(h_0) = 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{for} \quad 0 < x < h_0. \tag{1.1}
\]
A free boundary problem like (FBP) was first proposed by Du and Lin \([5]\) to describe the invasion of a new species by putting homogeneous Neumann condition at \( x = 0 \) in place of Dirichlet condition. We denote such a free boundary problem by (FBP-N). Function \( u(t, x) \) stands for the population density of the species over one-dimensional habitat \((0, h(t))\). The free boundary \( x = h(t) \) represents the expanding front of the habitat and its dynamics is determined by the Stefan condition of the form \( h'(t) = -\mu u_x(t, h(t)) \). For the ecological meaning of this condition, see \([2]\).

Du and Lin studied (FBP-N) with logistic nonlinearity \( f(u) = u(a - bu), a, b > 0 \), and established various interesting results such as spreading-vanishing dichotomy and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \) as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution \((u, h)\) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where \( \lim_{t \to \infty} h(t) = 0 \) and asymptotic behaviors of solutions as \( t \to \infty \).
in \([0, \infty)\) and the other is the big spreading of solution \((u, h)\) with \(\lim_{t \to \infty} u(t, \cdot) = u_3^*\) locally uniformly in \([0, \infty)\). Moreover, it was also proved in \([16]\) that under certain circumstances (SWP) does not have a solution, which is a big difference from previous results for other types of nonlinearity. In this sense, positive bistable \(f\) provides us with interesting and significant properties for (FBP-N). Recently, it was proved by Kaneko-Matsuzawa-Yamada \([15]\) that, if (SWP) has no solutions, the corresponding spreading solution approaches a propagating terrace.

Our interest is to investigate the following issues for positive bistable function \(f\):

- What kind of asymptotic behaviors of solutions of (FBP) can be found?
- Are there any differences in asymptotic behaviors between (FBP) and (FBP-N)?
- Is it possible to get precise estimates of \(u(t, x)\) on the whole interval \([0, h(t)]\) when \(h(t) \to \infty\) as \(t \to \infty\)?

As the first step, we will show that any solution \((u, h)\) of (FBP) satisfies one of the following properties:

(I) vanishing: \(\lim_{t \to \infty} h(t) < \infty\) and \(\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0\);

(II) small spreading: \(\lim_{t \to \infty} h(t) = \infty\) and \(\lim_{t \to \infty} u(t, \cdot) = v_1\) locally uniformly in \([0, \infty)\);

(III) big spreading: \(\lim_{t \to \infty} h(t) = \infty\) and \(\lim_{t \to \infty} u(t, \cdot) = v_3\) locally uniformly in \([0, \infty)\).

Here \(v_1\) and \(v_3\) are bounded solutions of the following problem

\[
(SP) \quad \begin{cases}
   dv_{xx} + f(v) = 0, & v > 0 \quad \text{for} \quad 0 < x < \infty, \\
   v(0) = 0
\end{cases}
\]

with \(\lim_{x \to \infty} v_1(x) = u_1^*\) and \(\lim_{x \to \infty} v_3(x) = u_3^*\), respectively. Note that (SP) has no bounded solutions other than \(v_1\) and \(v_3\) (see Proposition 3.1). In order to get better understanding on the above asymptotic behaviors, we will introduce parameter \(\sigma > 0\). Let any \((u_0, h_0)\) satisfying (1.1) be fixed and consider (FBP) with \((u_0, h_0)\) replaced by \((\sigma u_0, h_0)\). We denote such a free boundary problem by \((FBP)_\sigma\).

Let \((u(t, x; \sigma), h(t; \sigma))\) be the solution of \((FBP)_\sigma\). Then it is possible to show the existence of two threshold numbers \(\sigma_1^*\) and \(\sigma_2^*\) \((\sigma_1^* < \sigma_2^*)\) such that the vanishing of \((u(t, \cdot; \sigma), h(t; \sigma))\) occurs for \(0 \leq \sigma \leq \sigma_1^*\), the small spreading of \((u(t, \cdot; \sigma), h(t; \sigma))\) occurs for \(\sigma_1^* < \sigma \leq \sigma_2^*\) and the big spreading of \((u(t, \cdot; \sigma), h(t; \sigma))\) occurs for \(\sigma_2^* < \sigma\).

As the second step, we will derive asymptotic estimates for two types of spreading solutions. Let \((u, h)\) be any big spreading solution of (FBP) and let (SWP) with \(u^* = u_3^*\) admit a unique solution \((c_B, q_B)\). (For the existence and nonexistence of such a solution, see [16]). Then we will prove that \((u, h)\) satisfies

\[
\lim_{t \to \infty} h'(t) = c_B \quad \text{and} \quad \lim_{t \to \infty} (h(t) - c_B t) = H_B
\]

with some \(H_B \in \mathbb{R}\) and

\[
\lim_{t \to \infty} \sup_{ct \leq x \leq h(t)} |u(t, x) - q_B(h(t) - x)| = 0
\]

for any \(c \in (0, c_B)\). In this sense, \((c_B, q_B)\) gives a good approximation of \((u, h)\) near the free boundary \(x = h(t)\) for large \(t\). Moreover, we can also show that for any \(c \in (0, c_B)\)

\[
\lim_{t \to \infty} \sup_{0 \leq x \leq ct} |u(t, x) - v_3(x)| = 0.
\]
For any small spreading solution \((u, h)\), it will be seen that analogous estimates as (1.3)-(1.5) are valid provided that \((u, h)\) satisfies \(\liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} < u^*_2\). Here we should remark that there exists a small spreading solution which does not satisfy this condition. For example, when we take \((u(t, x; \sigma^*_2), h(t; \sigma^*_2))\) which is a borderline solution between the small spreading and the big spreading for (FBP)\(_{\sigma}\), this solution will be proved to satisfy \(\lim_{t \to \infty} \|u(t, \cdot; \sigma^*_2)\|_{C([0, h(t; \sigma^*_2)])} \geq u^*_2 > u^*_1\). This is a new “borderline” behavior which can not be observed in the study of (FBP-N). We have not obtained satisfactory asymptotic estimates for such small spreading solution.

This paper is organized as follows. In Section 2 we will prepare some basic results such as the existence theorem of global solutions, comparison theorem, vanishing theorem and spreading theorem. In Section 3 we study (SP) and related stationary problem by the method of the phase plane analysis. In Section 4 we will investigate large-time behaviors of solutions such as the classification of asymptotic behaviors, sufficient conditions for each behavior and the existence of threshold numbers for (FBP)\(_{\sigma}\) by using parameter \(\sigma \geq 0\). Finally, in Section 5 we will derive precise asymptotic estimates for the spreading speed of the free boundary and sharp estimates for asymptotic profiles of spreading solutions with use of a semi-wave solution of (SWP), which corresponds to the spreading solution.

2. Basic properties. We first state the global existence result for (FBP).

**Theorem 2.1** (Existence and uniqueness of bounded global solution). Let \(f\) satisfy (PB) and let \(u_0\) satisfy (1.1). Then (FBP) has a unique solution \((u, h)\) satisfying

\[
(u, h) \in \{ C^{1+\alpha, 1+\alpha}(\Omega) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega) \} \times C^{1+\frac{\alpha}{2}}([0, \infty)),
\]

for any \(\alpha \in (0, 1)\) with \(\Omega = \{(t, x) \in \mathbb{R}^2 : t > 0, 0 \leq x \leq h(t)\}\). Moreover, it holds that

\[
u_x(t, x) < 0 \quad \text{for} \quad t > 0, \max\left\{ h_0, \frac{h(t)}{2} \right\} \leq x \leq h(t)
\]

and there exist positive constants \(C_1 = C_1(||u_0||_{C([0, h_0])}, h_0)\) and \(C_2 = C_2(||u_0||_{C([0, h_0])}, h_0)\) such that

\[
0 < u(t, x) \leq C_1 \quad \text{for} \quad t > 0, \quad 0 < x < h(t),
\]

\[
0 < h'(t) \leq \mu C_2 \quad \text{for} \quad t > 0.
\]

Theorem 2.1 has been shown by Du and Lin [5, Theorems 2.1, 2.3 and Lemma 2.2] for Neumann boundary condition at \(x = 0\), by Kaneko and Yamada [12, Theorem 2.7] and [13, Lemma A.1] for Dirichlet boundary condition. In particular \(h'(t) > 0\) implies the existence of \(\lim_{t \to \infty} h(t) \in (h_0, \infty)\).

We define spreading and vanishing of solutions under general situations.

**Definition 2.2.** Let \((u, h)\) be the global solution of (FBP). Then \((u, h)\) is a vanishing solution if

\[
\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0,
\]

and \((u, h)\) is a spreading solution if

\[
\lim_{t \to \infty} h(t) = \infty \quad \text{and} \quad \liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} > 0.
\]

We next give a comparison theorem for (FBP).
Theorem 2.3 (Comparison theorem). Suppose that \( \bar{h} \in C^1([0,T]) \) and \( \bar{u} \in C^{1,2} (D_T) \) with \( T > 0 \) and \( D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, 0 \leq x \leq \bar{h}(t)\} \) satisfy

\[
\begin{cases}
\pi_t \geq d \pi_{xx} + f(\pi), & (t,x) \in D_T, \\
\pi(t,0) \geq 0, \pi(t,\bar{h}(t)) = 0, & t \in (0,T], \\
\bar{h}(t) \geq -\mu \pi_x(t,\bar{h}(t)), & t \in (0,T].
\end{cases}
\]

If \( \bar{h}(0) \geq h_0, \pi(0,x) \geq u_0(x) \) in \([0,h_0]\), then the solution \((u,h)\) of (FBP) satisfies

\[
\bar{h}(t) \geq h(t) \quad \text{for} \quad 0 \leq t \leq T \quad \text{and} \\
\pi(t,x) \geq u(t,x) \quad \text{for} \quad 0 \leq t \leq T, \quad 0 \leq x \leq h(t).
\]

Proof. The proof of this theorem is the same as that of [5, Lemma 3.5]. \( \square \)

Remark 1. The pair \((\pi, \bar{h})\) is called an upper solution of (FBP) when it satisfies the assumptions of Theorem 2.3. Similarly, a lower solution is defined by reversing all inequality signs in the assumptions of Theorem 2.3.

The following result is very useful for the analysis of asymptotic behaviors of the solution of (FBP) (see [12, Theorem 2.11]).

Theorem 2.4 (Spreading theorem). Let \((u,h)\) be the solution of (FBP) with initial data \((u_0,h_0)\) and let \( \phi = \phi(x;\ell) \) be a bounded solution of

\[
(SP-\ell)
\begin{cases}
d\phi_{xx} + f(\phi) = 0, & 0 < x < \ell, \\
\phi(0) = \phi(\ell) = 0
\end{cases}
\]

with positive number \( \ell \). If \( h_0 \geq \ell \) and \( u_0(x) \geq \phi(x;\ell) \) in \([0,\ell]\), then

\[
\lim_{t \to \infty} h(t) = \ell \quad \text{and} \quad \liminf_{t \to \infty} u(t,x) = v^*(x) \quad \text{for} \quad x \geq 0,
\]

where \( v^* = v^*(x) \) is a minimal solution of (SP) satisfying \( v(x) \geq \phi(x) \) in \([0,\ell]\).

We finally give a sufficient condition of the vanishing (see [12, Theorem 2.10]).

Theorem 2.5 (Vanishing theorem). Let \((u,h)\) be the solution of (FBP). If \( \lim_{t \to \infty} h(t) < \infty \), then

\[
\lim_{t \to \infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0.
\]

3. Analysis of stationary problem. To apply the results in Section 2, we study (SP) and \((SP-\ell)\) with nonlinearity \( f \) satisfying (PB) by making use of the phase plane analysis (see for instance [24] and Figure 1). We first give the existence of bounded nonnegative solutions of (SP) without proof.

Proposition 3.1 (Existence of bounded solutions of (SP)). Under assumption (PB), \((SP) \) has three bounded solutions \( v \equiv 0, v_1(x) \) and \( v_3(x) \), where \( v_1 = v_1(x) \) (resp. \( v_3 = v_3(x) \)) is an increasing function satisfying \( \lim_{x \to \infty} v_1(x) = u_1^* \) (resp. \( \lim_{x \to \infty} v_3(x) = u_3^* \)) and \( v_1(x) < v_3(x) \) for \( x > 0 \).

In order to find a solution of \((SP-\ell)\) we consider the following initial value problem

\[
\begin{cases}
dv'' + f(v) = 0, \\
v(0) = 0, \quad v'(0) = P > 0
\end{cases}
\]

Let \( v = v(x;P) \) be a solution of (3.1) and define \( \ell = \ell(P) \) by

\[
\ell(P) := \inf\{x > 0 : v(x;P) = 0\}.
\]
We also define
\[ F(u) := \int_0^u f(s)ds. \]
If \( f \) satisfies (PB), we can choose \( \hat{u} \in (u_2^*, u_3^*) \) such that
\[ F(\hat{u}) = F(u_1^*). \quad (3.2) \]
For functions \( v_1 \) and \( v_3 \) in Proposition 3.1 set \( v_1'(0) =: \omega_1 \) and \( v_3'(0) =: \omega_3 \). Then \( \ell(P) \) is represented by
\[ \ell(P) = \sqrt{2d} \int_0^{v_P} \frac{dv}{\sqrt{F(v_P) - F(v)}} \text{ for } P \in (0, \omega_1) \cup (\omega_1, \omega_3), \quad (3.3) \]
where \( v_P = \inf\{v > 0 : F(v) = dP^2/2\} \). Note that if one can find \( P^* \) satisfying \( \ell(P^*) = \ell \), then \( v(x; P^*) \) becomes a solution of (SP-\( \ell \)). The following result gives an elementary property of \( \ell(P) \).

**Lemma 3.2.** Define \( \ell(P) \) by (3.3). Then \( \ell(P) \) is a continuous function of \( P \in (0, \omega_1) \cup (\omega_1, \omega_3) \) and satisfies
\[ \lim_{P \to 0} \ell(P) = \pi \sqrt{\frac{d}{F'(0)}}, \quad \lim_{P \to \omega_1 - 0} \ell(P) = \lim_{P \to \omega_1 + 0} \ell(P) = \lim_{P \to \omega_3 - 0} \ell(P) = \infty. \]
For the proof of this lemma, see [21].

Lemma 3.2 ensures the existence of a minimum of \( \ell(P) \) in \( (\omega_1, \omega_3) \), namely
\[ \ell^* := \min_{\omega_1 < P < \omega_3} \ell(P). \quad (3.4) \]
We are thus led to the following result on the structure of solutions of (SP-\( \ell \)) by virtue of Lemma 3.2.

**Proposition 3.3 (The structure of solutions for (SP-\( \ell \))).** Assume (PB) and define \( \ell^* \) by (3.4). Then the following properties hold true:

(i) For each \( \ell \in (\pi \sqrt{d/F'(0)}, \infty) \), (SP-\( \ell \)) has a positive solution \( \phi_1 = \phi_1(x; \ell) \) satisfying \( \|\phi_1\|_{\infty} < u_1^* \). Moreover, \( \lim_{\ell \to \pi \sqrt{d/F'(0)}} \|\phi_1\|_{\infty} = 0 \) and \( \lim_{\ell \to \infty} \|\phi_1\|_{\infty} = u_1^* \).
(ii) For each \( \ell \in [\ell', \infty) \), (SP-\( \ell \)) has two positive solutions \( \phi_2 = \phi_2(x; \ell) \) and \( \phi_3 = \phi_3(x; \ell) \) satisfying
\[
\hat{u} < \| \phi_2 \|_{C([0,\ell])} \leq \| \phi_3 \|_{C([0,\ell])} < u_3^*.
\]
\[
\lim_{\ell \to \infty} \| \phi_2 \|_{C([0,\ell])} = \hat{u} \text{ and } \lim_{\ell \to \infty} \| \phi_3 \|_{C([0,\ell])} = u_3^*.
\]
Here \( \hat{u} \in (u_2^*, u_3^*) \) is a constant defined in (3.2). Moreover, \( \phi_1(x; \ell) < \phi_2(x; \ell) \) \( < \phi_3(x; \ell) \) for \( 0 < x < \ell \) when they exist.

4. Asymptotic behaviors of solutions. In this section, we will study asymptotic behaviors of solutions of (FBP) as \( t \to \infty \).

4.1. Classification of asymptotic behaviors. Our first main result is the classification of asymptotic behaviors.

Theorem 4.1. Assume (PB). Then any solution \((u, h)\) of (FBP) satisfies one of the following properties:

(I) Vanishing: \( \lim_{t \to \infty} h(t) \leq \pi \sqrt{d/f'(0)} \) and \( \lim_{t \to \infty} \| u(t, \cdot) \|_{C([0,h(t)])} = 0 \);

(II) Small spreading: \( \lim_{t \to \infty} h(t) = \infty \) and \( \lim_{t \to \infty} u(t, x) = v_1(x) \) uniformly in \( x \in [0, R] \) for any \( R > 0 \);

(III) Big spreading: \( \lim_{t \to \infty} h(t) = \infty \) and \( \lim_{t \to \infty} u(t, x) = v_3(x) \) uniformly in \( x \in [0, R] \) for any \( R > 0 \),

where \( v_1 \) and \( v_3 \) are bounded increasing functions in Proposition 3.1.

Remark 2. By the standard parabolic regularity, the uniform convergence of \( u(t, x) \) in (II) and (III) of Theorem 4.1 can be replaced by the convergence in the topology of \( C^1([0, R]) \) for any \( R > 0 \).

To prove Theorem 4.1, we will prepare a series of lemmas.

Lemma 4.2. Let \((u, h)\) be the solution of (FBP). If \( \lim_{t \to \infty} h(t) = \infty \), then
\[
v_1(x) \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq v_3(x) \text{ for } x \geq 0,
\]
where \( v_1 \) and \( v_3 \) are given in Proposition 3.1.

Proof. Since \( \lim_{t \to \infty} h(t) = \infty \), there exists a positive number \( T \) such that \( h(T) > \pi \sqrt{d/f'(0)} \). Since \( u(T, x) > 0 \) for \( x \in (0, h(T)) \), Proposition 3.3 allows us to choose \( \ell \in (\pi \sqrt{d/f'(0)}, h(T)) \) such that \( \phi_1(x, \ell) \leq u(T, x) \) for \( x \in [0, \ell] \), where \( \phi_1 \) is the solution to (SP-\( \ell \)). Applying Theorem 2.4 to the solution of (FBP) with initial data \((u(T, x), h(T))\), we obtain
\[
v_1(x) \leq \liminf_{t \to \infty} u(t, x) \text{ for } x \geq 0,
\]
because \( v_1(x) \) is a minimal solution of (SP) satisfying \( v_1(x) \geq \phi_1(x; \ell) \) in \([0, \ell]\).

Next we prove that \( \limsup_{t \to \infty} u(t, x) \leq v_3(x) \) for \( x \geq 0 \). Let \( w(t, x) \) be a solution of
\[
\begin{align*}
&w_t = dw_{xx} + f(w), & t > 0, & x > 0, \\
&w(0, x) = 0, & t > 0, \\
&w(0, x) = w_0(x) := M, & x \geq 0,
\end{align*}
\]
where \( M > 0 \) is a constant such that \( M > \max\{\| u_0 \|_{C([0,h_0])}, u_3^* \} \). Then it follows from the standard comparison principle that
\[
u(t, x) \leq w(t, x) \text{ for } t \geq 0 \text{ and } 0 \leq x \leq h(t).
\]
Moreover, since \( w_0 \) satisfies \( d(w_0)_x+f(w_0) < 0 \) for \( x \geq 0 \), we see from the monotone method (see [23]) that \( w_t(t,x) \leq 0 \) for \( t > 0 \) and \( x > 0 \); that is, \( w(t,x) \) is non-increasing with respect to \( t > 0 \) for each \( x > 0 \). Therefore, there exists a nonnegative function \( \hat{v}(x) \) such that
\[
\lim_{t \to \infty} w(t,x) = \hat{v}(x) \quad \text{for every } x \geq 0.
\]
(4.2)

Note \( w(t,x) \geq 0 \) for \( t \geq 0 \) and \( x \geq 0 \) by the maximum principle. It can be proved that \( \hat{v} \) is a solution of (SP) (see, e.g., [12, Theorem 2.11]). Moreover since \( \hat{w}(x) > v_3(x) \) for \( x \geq 0 \), we see \( w(t,x) \geq v_3(x) \) for \( x \geq 0 \) by the comparison theorem. Combining this fact and (4.2), we conclude that \( \hat{v} \equiv v_3 \). Thus we have shown
\[
\lim_{t \to \infty} w(t,x) = v_3(x) \quad \text{for } x \geq 0.
\]

This equality together with (4.1) implies that
\[
\limsup_{t \to \infty} u(t,x) \leq v_3(x) \quad \text{for } x \geq 0.
\]

This completes the proof. \( \square \)

**Lemma 4.3.** Let \((u,h)\) be the solution of (FBP). If \( \lim_{t \to \infty} h(t) < \infty \), then
\[
\lim_{t \to \infty} h(t) \leq \pi \sqrt{d/f'(0)} \quad \text{and} \quad \lim_{t \to \infty} \|u(t,\cdot)\|_{C(\{0,h(t)\})} = 0.
\]

**Proof.** Since \( \lim_{t \to \infty} h(t) < \infty \), it follows from Theorem 2.5 that
\[
\lim_{t \to \infty} \|u(t,\cdot)\|_{C(\{0,h(t)\})} = 0.
\]
(4.3)

It remains to show that \( \lim_{t \to \infty} h(t) \leq \pi \sqrt{d/f'(0)} \). To prove this, assume that
\( \lim_{t \to \infty} h(t) > \pi \sqrt{d/f'(0)} \). Then there exists \( T > 0 \) such that \( h(T) > \pi \sqrt{d/f'(0)} \). As in the proof of Lemma 4.2, we can show \( v_1(x) \leq \liminf_{t \to \infty} u(t,x) \) for \( x \geq 0 \). This is a contradiction to (4.3). Thus the proof is complete. \( \square \)

The following result can be easily proved by virtue of Lemmas 4.2 and 4.3.

**Corollary 4.4.** Let \((u,h)\) be the solution of (FBP) with initial data \((u_0,h_0)\) satisfying \( h_0 \geq \pi \sqrt{d/f'(0)} \). Then it holds that
\[
\lim_{t \to \infty} h(t) = \infty \quad \text{and} \quad v_1(x) \leq \liminf_{t \to \infty} u(t,x) \leq \limsup_{t \to \infty} u(t,x) \leq v_3(x) \quad \text{for } x \geq 0,
\]
where \( v_1 \) and \( v_3 \) are given in Proposition 3.1.

In order to prove Theorem 4.1, we will make use of the zero number arguments developed by Angenent [1]. Denote by \( Z_I(w) \) the number of zero points of a continuous function \( w \) in an interval \( I \subset \mathbb{R} \). We should recall the following results which are extensions of Angenent’s result. See [7, Lemma 2.2] for Lemma 4.5 and [8, Lemma 2.6] for Lemma 4.6.

**Lemma 4.5.** Let \( \xi(t) \geq 0 \) be a continuous function for \( t \in (t_1,t_2) \) and set \( I(t) := [-\xi(t),\xi(t)] \). Assume that \( w(t,x) \) is a continuous function defined for \( t \in (t_1,t_2) \) and \( x \in I(t) \) and that it satisfies
\[
w_t = dw_{xx} + c(t,x)w \quad \text{for } t \in (t_1,t_2), \; x \in (-\xi(t),\xi(t))
\]
(4.4)
in the classical sense, where \( c \) is a bounded function of \( t \in [t_1,t_2] \) and \( x \in I(t) \). If \( w(t,-\xi(t)) \neq 0 \) and \( w(t,\xi(t)) \neq 0 \) for \( t \in (t_1,t_2) \), then the following properties hold true:

(i) \( Z_I(w(t,\cdot)) < \infty \) for any \( t \in (t_1,t_2) \) and it is non-increasing in \( t \);
(ii) If \( w(s, x) \) has a degenerate zero \( x_0 \in (-\xi(s), \xi(s)) \) at some \( s \in (t_1, t_2) \), then 
\[
\mathcal{Z}_{I(s)}(w(s_1, \cdot)) > \mathcal{Z}_{I(s)}(w(s_2, \cdot)) \quad \text{for any } s_1 \in (t_1, s) \text{ and } s_2 \in (s, t_2).
\]

**Lemma 4.6.** Let \( I \subset \mathbb{R} \) be an open interval and let \( \{w_n(t, x)\}_{n=1}^\infty \) be a sequence of functions which converges to \( w(t, x) \) in \( C^1((t_1, t_2) \times I) \). Assume that for every \( t \in (t_1, t_2) \) and \( n \in \mathbb{N} \), the function \( x \mapsto w_n(t, x) \) has only simple zeros in \( I \) and that \( w(t, x) \) satisfies an equation of the form (4.4) in \( (t_1, t_2) \times I \). Then for every \( t \in (t_1, t_2) \), either \( w(t, x) \equiv 0 \) in \( I \), or \( w(t, x) \) has only simple zeros in \( I \).

We will prove the following convergence property of the solutions of (FBP) by using these zero number arguments and basic properties of the structure of \( \omega \)-limit set.

**Proposition 4.7.** Let \( (u, h) \) be the solution of (FBP). If \( \lim_{t \to \infty} h(t) = \infty \), then 
\[
\lim_{t \to \infty} u(t, \cdot) = v^* \text{ uniformly in } [0, R] \text{ for any } R > 0,
\]
where \( v^* \) is a bounded positive solution of (SP).

**Proof.** Let \( \omega(u) \) be an \( \omega \)-limit set of \( u(t, \cdot) \) in the topology of \( L^\infty_{loc}([0, \infty)) \), that is, for every \( w \in \omega(u) \) there exists a sequence \( 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots \to \infty \) such that
\[
\lim_{n \to \infty} u(t_n, x) = w(x) \text{ uniformly in } x \in [0, R] \text{ for any } R > 0. \tag{4.5}
\]

By local parabolic regularity estimates, we can replace the topology of \( L^\infty_{loc}([0, \infty)) \) by that of \( C^2_{loc}([0, \infty)) \). Since \( \omega(u) \) is a compact, connected and invariant set, for any \( w \in \omega(u) \) there exists an entire orbit \( \{W(t, x)\}_{t \in \mathbb{R}} \) with \( W(0, x) = w(x) \). This fact implies that for every \( w \in \omega(u) \) there exists \( W(t, x) \) satisfying
\[
\begin{aligned}
W_t &= dW_{xx} + f(W), & t &\in \mathbb{R}, & x &> 0, \\
W(t, 0) &= 0, & t &\in \mathbb{R}, \\
W(0, x) &= w(x) \in \omega(u), & x &> 0,
\end{aligned}
\]
and
\[
\lim_{n \to \infty} u(t + t_n, x) = W(t, x) \text{ in } L^\infty_{loc}(\mathbb{R} \times [0, \infty)). \tag{4.6}
\]
This convergence can be also replaced by the topology of \( C^{1,2}_{loc}(\mathbb{R} \times [0, \infty)) \) on account of parabolic regularity.

Let \( v = v(x) \) be a unique solution of
\[
\begin{aligned}
dv'' + f(v) &= 0, & x &> 0, \\
v(0) &= 0, \\
v'(0) &= w(0).
\end{aligned}
\]

We will investigate intersection points between \( W(t, x) \) and \( v(x) \). Let \( v_1 \) and \( v_3 \) be functions given in Proposition 3.1. Lemma 4.2 gives
\[
v_1(x) \leq w(x) \leq v_3(x) \text{ for } x \geq 0. \tag{4.7}
\]
Since \( v_1(0) = w(0) = v_3(0) \equiv 0 \), we have \( 0 < v_1'(0) \leq w'(0) \leq v_3'(0) \). Therefore, by the phase plane analysis (see Figure 1), it is seen that either
(i) \( v(x) > 0 \) for \( x > 0 \), or
(ii) there exists a positive number \( R \) such that \( v(R) = 0 \) and \( v(x) > 0 \) for \( x \in (0, R) \).
First, we consider the case (i). Let \( \hat{u}(t, x) \) be an odd extension of \( u(t, x) \) for \( t \in (0, \infty) \) and \( x \in [-h(t), h(t)] \); \( \hat{u}(t, x) = -u(t, -x) \) for \( t \in [0, \infty) \) and \( x \in [-h(t), 0] \). Then \( \hat{u} \) is a classical solution of
\[
\hat{u}_t = d\hat{u}_{xx} + \hat{f}(\hat{u}) \quad \text{for } t \in (0, \infty), \ x \in (-h(t), h(t)),
\]
where \( \hat{f} \) is defined by
\[
\hat{f}(u) = \begin{cases} f(u) & \text{for } u \geq 0, \\ -f(-u) & \text{for } u < 0. \end{cases}
\]
Similarly, we also denote by \( \hat{v} \) an odd extension of \( v \) over \( (-\infty, \infty) \). Note that \( \hat{v} \) is also a classical solution of
\[
d\hat{v}'' + \hat{f}(\hat{v}) = 0 \quad \text{for } x \in (-\infty, \infty).
\]
We now set \( \hat{U}(t, x) := \hat{u}(t, x) - \hat{v}(x) \) for \( t \in (0, \infty) \) and \( x \in [-h(t), h(t)] \). Clearly, \( \hat{U} \) satisfies
\[
\hat{U}(t, -h(t)) = -\hat{U}(t, h(t)) = -v(h(t)) < 0 \quad \text{for } t \geq 0
\]
and it is a classical solution of
\[
\hat{U}_t = d\hat{U}_{xx} + C(t, x)\hat{U} \quad \text{for } t \in (0, \infty), \ x \in [-h(t), h(t)],
\]
where \( C(t, x) = \int_0^1 \int_0^{\pi} d\theta \hat{u}(t, x) + (1 - \theta)\hat{v}(x))d\theta \) is a bounded continuous function. Lemma 4.5 asserts that, for all large \( t > 0 \), \( Z_{[-h(t), h(t)]}(\hat{U}(t, \cdot)) < \infty \) and \( \hat{U}(t, \cdot) \) has only simple zeros in \( [-h(t), h(t)] \). Therefore, for any \( t \in \mathbb{R} \), \( \hat{U}(t + t_n, \cdot) \) has only simple zeros provided that \( n \in \mathbb{N} \) is sufficiently large. Moreover, by (4.6),
\[
\lim_{n \to \infty} \hat{U}(t + t_n, x) = \hat{W}(t, x) - \hat{v}(x) \quad \text{in } C^{1,2}_{loc}(\mathbb{R}^2),
\]
where \( \hat{W}(t, x) \) is an odd extension of \( W(t, x) \); \( \hat{W}(t, x) = -W(t, -x) \) for \( (t, x) \in \mathbb{R} \times (-\infty, 0) \). Since \( \hat{W}(t, x) - \hat{v}(x) \) satisfies a parabolic equation of the form (4.4) for any \( (t, x) \in \mathbb{R}^2 \), it follows from Lemma 4.6 that, for every \( t \in \mathbb{R} \), either \( \hat{W}(t, x) - \hat{v}(x) \equiv 0 \) in \( \mathbb{R} \), or \( \hat{W}(t, x) - \hat{v}(x) \) has only simple zeros in \( \mathbb{R} \). However we see that the latter case never occurs because \( \hat{W}(t, 0) - \hat{v}(0) = \hat{W}_x(t, 0) - \hat{v}_x(0) = 0 \) at \( t = 0 \). Therefore, \( \hat{W}(t, x) \equiv \hat{v}(x) \) in \( \mathbb{R} \). Since the right hand side is not dependent on \( t \),
\[
W(t, x) \equiv W(0, x) = w(x) \equiv v(x) \quad \text{for } x \geq 0.
\]
Thus any \( w \in \omega(u) \) is equal to \( v \) which is a bounded positive solution of (SP).

We will next exclude the case (ii). Assume that (ii) holds true. Since \( \lim_{t \to \infty} h(t) = \infty \), there exists a positive number \( T \) such that \( h(t) \geq R \) for \( t \geq T \). By virtue of \( \hat{U}(t, R) \neq 0 \) for \( t > T \), we can repeat the previous argument with \( t \in (0, \infty) \) and \( x \in [-h(t), h(t)] \) replaced by \( t \in (T, \infty) \) and \( x \in [-R, R] \), respectively. Then it is possible to show that, for every \( t \in \mathbb{R} \), either \( \hat{W}(t, x) - \hat{v}(x) \equiv 0 \) for all \( x \in (-R, R) \), or \( \hat{W}(t, x) - \hat{v}(x) \) has only simple zeros in \( (-R, R) \). In the former case, we see that \( w(x) \equiv v(x) \) for \( x \in [0, R] \), which contradicts (4.7). On the other hand the latter case contradicts the fact that \( \hat{W}(t, x) - \hat{v}(x) \) has a degenerate zero \( x = 0 \) at \( t = 0 \). In this way we conclude that the case (ii) never occurs. The proof is complete. \( \square \)

Proof of Theorem 4.1. By Theorem 2.1, either \( \lim_{t \to \infty} h(t) < \infty \), or \( \lim_{t \to \infty} h(t) = \infty \). If \( \lim_{t \to \infty} h(t) < \infty \), then \( \lim_{t \to \infty} h(t) \leq \pi \sqrt{d/f'(0)} \) and \( \lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0 \) by Lemma 4.3. On the other hand, if \( \lim_{t \to \infty} h(t) = \infty \), then it follows from Proposition 4.7 that \( \lim_{t \to \infty} u(t, \cdot) = v^* \) uniformly in \([0, R]\) for any \( R > 0 \) where \( v^* \)}
is a bounded solution of (SP) satisfying $v_1(x) \leq v^*(x) \leq v_3(x)$. By the phase plane analysis, $v^*$ must coincide with $v_1$ or $v_3$. The proof is complete.

4.2. Sufficient conditions for asymptotic behavior. In this subsection we will give some sufficient conditions for (I)-(III) of Theorem 4.1. We first introduce a sufficient condition for the vanishing which can be proved in the same way as [11, Theorem 2.2].

**Theorem 4.8.** Assume $h_0 < \pi \sqrt{d/f''(0)}$. Then there exists a positive function $V^*$ such that, if $u_0(x) \leq V^*(x)$ in $[0, h_0]$, then the solution $(u, h)$ of (FBP) satisfies the vanishing.

The following result gives a sufficient condition for the spreading when $h_0 < \pi \sqrt{d/f''(0)}$:

**Theorem 4.9.** Assume $h_0 < \pi \sqrt{d/f''(0)}$. If

$$\int_0^{h_0} xu_0(x) \, dx > \frac{d}{2\mu} \left( \frac{\pi^2}{f''(0)} - h_0^2 \right) \max \left\{ 1, \frac{\|u_0\|_{C([0,h_0])}}{u_1^*} \right\},$$

then the solution of (FBP) satisfies the spreading.

**Proof.** First we consider the case $\|u_0\|_{C([0,h_0])} \leq u_1^*$. Assume $\lim_{t \to \infty} h(t) < \infty$ to derive a contradiction. By the strong maximum principle, $0 < u(t, x) < u_1^*$ for $t > 0$ and $0 < x < h(t)$. Then

$$\frac{d}{dt} \int_0^{h(t)} xu(t, x) \, dx = \int_0^{h(t)} xu_{xx}(t, x) \, dx + \int_0^{h(t)} xf(u(t, x)) \, dx$$

$$> d h(t) u_x(t, h(t))$$

$$= -\frac{d}{2\mu} \{ h(t)^2 \}'.$$

Here we have used $f(u) > 0$ for $u \in (0, u_1^*)$ and $h'(t) = -\mu u_x(t, h(t))$ for $t > 0$. Integrating the above inequality from $0$ to $t$ yields

$$\int_0^{h(t)} xu(t, x) \, dx - \int_0^{h_0} xu_0(x) \, dx \geq -\frac{d}{2\mu} \{ h(t)^2 - h_0^2 \} \text{ for } t \geq 0. \quad (4.9)$$

From $\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0,h(t)])} = 0$ and $\lim_{t \to \infty} h(t) \leq \pi \sqrt{d/f''(0)}$, it follows that

$$\lim_{t \to \infty} \int_0^{h(t)} xu(t, x) \, dx = 0$$

and

$$\lim_{t \to \infty} h(t)^2 \leq \pi^2 \frac{d}{f''(0)}.$$

Letting $t \to \infty$ in (4.9) leads to

$$\int_0^{h_0} xu_0(x) \, dx \leq \frac{d}{2\mu} \left( \frac{\pi^2}{f''(0)} - h_0^2 \right),$$

which contradicts the assumption. Hence $\lim_{t \to \infty} h(t) = \infty$ and the spreading occurs.
We next discuss the case \( \|u_0\|_{C([0,h_0])} > u_1^* \). Let \((u, h)\) be a solution of the following free boundary problem:

\[
\begin{align*}
\frac{u_t}{d} &= d \frac{u_{xx}}{f(u)}, & t > 0, & 0 < x < h(t), \\
u(t, 0) &= u(t, h(t)) = 0, & t > 0, \\
\int h(t) &= -\mu u_x(t, h(t)), & t > 0, \\
h(0) &= h_0, & u(0, x) = u_0(x) := \frac{u_1^*}{\|u_0\|_{C([0,h_0])}}u_0(x), & 0 \leq x \leq h_0.
\end{align*}
\]

(4.10)

Then \((u, h)\) is a lower solution to (FBP). Applying Theorem 2.3 yields \(h(t) \leq h(t)\) and \(u(t, x) \leq u(t, x)\) for \(t \geq 0\) and \(0 \leq x \leq h(t)\). In view of \(\|u_0\|_{C([0,h_0])} = u_1^*\), the preceding result implies the spreading of \((u, h)\) and, therefore, \((u, h)\), provided that \(\int_0^{h_0} xu_0(x)dx > (d/2\mu)(\pi^2d/f'(0) - h_0^2)\). Noting that

\[
\int_0^{h_0} xu_0(x)dx = \frac{u_1^*}{\|u_0\|_{C([0,h_0])}} \int_0^{h_0} xu_0(x)dx,
\]

we complete the proof. \(\square\)

We will show a sufficient condition for the small spreading of solutions.

**Theorem 4.10.** If \(h_0 \geq \pi \sqrt{d/f'(0)}\) and \(\|u_0\|_{C([0,h_0])} < u_2^*\), then the solution of (FBP) satisfies (II) of Theorem 4.1.

**Proof.** By virtue of \(h_0 \geq \pi \sqrt{d/f'(0)}\), it follows from Corollary 4.4 that \(\lim_{t \to \infty} h(t) = \infty\) and \(\liminf_{t \to \infty} u(t, x) \geq v_1(x)\) for \(x \geq 0\).

(4.11)

Let \(\tilde{\pi} = \tilde{\pi}(t, x)\) be the solution of the following problem

\[
\begin{align*}
\frac{\tilde{\pi}_t}{d} &= d \frac{\tilde{\pi}_{xx}}{f(\tilde{\pi})}, & t > 0, & x > 0, \\
\tilde{\pi}(t, 0) &= 0, & t > 0, \\
\tilde{\pi}(0, x) &= \pi_0(x) := M, & x \geq 0,
\end{align*}
\]

where \(M > 0\) is a constant satisfying \(\max\{u_1^*, \|u_0\|_{C([0,h_0])}\} < M < u_2^*\). Since \(\pi_0\) satisfies \(d(\pi_0)_{xx} + f(\pi_0) < 0\), it is possible to prove the monotone decreasing convergence of \(\tilde{\pi}(t, x)\) as \(t \to \infty\) to a solution of (SP) in the same way as the proof of Lemma 4.2. Moreover, since \(\pi_0 \in (u_1^*, u_2^*)\) for \(x \geq 0\), we see that \(\lim_{t \to \infty} \tilde{\pi}(t, x) = v_1(x)\) locally uniformly for \(x \geq 0\). Since \(\tilde{\pi}\) is an upper solution to (FBP), it follows from the comparison principle that \(u(t, x) \leq \tilde{\pi}(t, x)\) for \(t \in [0, \infty)\) and \(x \in [0, h(t)]\).

Hence \(\lim_{t \to \infty} u(t, x) \leq \lim_{t \to \infty} \tilde{\pi}(t, x) = v_1(x)\) for \(x \geq 0\). Combining this fact and (4.11), we conclude

\[
\lim_{t \to \infty} u(t, x) = v_1(x) \text{ locally uniformly in } x \geq 0,
\]

which implies the small spreading of \(u(t, \cdot)\) as \(t \to \infty\). \(\square\)

Finally, we will give a sufficient condition for the big spreading.

**Theorem 4.11.** Let \(\ell^*\) be a positive number defined by (3.4) and assume \(h_0 \geq \ell^*\). If there exists a positive constant \(\ell \in [\ell^*, h_0]\) such that \(u_0(x) \geq \phi_2(x; \ell)\) in \([0, \ell]\), then the solution of (FBP) satisfies (III) of Theorem 4.1, where \(\phi_2(x; \ell)\) is the solution of (SP-\(\ell\)) given in Proposition 3.3.

For the proof of Theorem 4.11, follow the arguments used in [16, Theorem 3.6].
4.3. Sharp threshold numbers. In this subsection we will give a more detailed description on the asymptotic behavior of the solution of (FBP). We introduce a parameter $\sigma \geq 0$ and consider $(\text{FBP})_{\sigma}$ with initial data $(u_0, h_0) = (\sigma \phi, h_0)$ for any fixed $(\phi, h_0)$ satisfying (1.1). Denote by $(u(t, x; \sigma), h(t; \sigma))$ the solution of $(\text{FBP})_{\sigma}$.

It is clear from Theorems 2.1 and 2.3 that, if $\sigma_1 > \sigma_2$, then

$$h(t; \sigma_1) > h(t; \sigma_2) \text{ and } u(t, x; \sigma_1) > u(t, x; \sigma_2) \text{ for } t \geq 0, \ x \in (0, h(t; \sigma_2)].$$

We define two numbers $\sigma^*_1$ and $\sigma^*_2$ by

$$\sigma^*_1 := \sup\{\sigma: \text{ the vanishing occurs for } (u(t, x; \sigma), h(t; \sigma))\}$$

and

$$\sigma^*_2 := \inf\{\sigma: \text{ the spreading occurs for } (u(t, x; \sigma), h(t; \sigma))\}.$$  

Note that $\sigma^*_1 \leq \sigma^*_2$ by the comparison theorem. We begin with the following lemma which gives a condition for $\sigma^*_1 < \infty$:

**Lemma 4.12.** Assume that $(\phi, h_0)$ satisfies (1.1) and

$$\pi^2 \frac{d}{f'(0)} - \frac{2\mu u_1^*}{d} \int_0^{h_0} x\phi(x) \, dx < h_0^2 < \pi^2 \frac{d}{f'(0)}.$$  

(4.15)

Then there exists a positive number $\overline{\sigma}$ such that $(u(t, x; \sigma), h(t; \sigma))$ satisfies the spreading for every $\sigma \geq \overline{\sigma}$.

**Proof.** Let $(\overline{\sigma} \phi, h_0)$ meet the assumption of Theorem 4.9 for some $\overline{\sigma} > 0$, that is to say, $h_0 < \pi \sqrt{d/f'(0)}$ and

$$\overline{\sigma} \int_0^{h_0} x\phi(x) \, dx > \frac{d}{2\mu} \left(\pi^2 \frac{d}{f'(0)} - h_0^2\right) \max \left\{1, \frac{\overline{\sigma} \phi \|C([0, h_0])\|}{u_1^*}\right\}.$$  

If $\overline{\sigma} \phi \|C([0, h_0])\| < u_1^*$, then the above inequality is reduced to

$$\overline{\sigma} \int_0^{h_0} x\phi(x) \, dx > \frac{d}{2\mu} \left(\pi^2 \frac{d}{f'(0)} - h_0^2\right).$$

These relations on $\overline{\sigma}$ are equivalent to

$$\frac{d}{2\mu} \left(\pi^2 \frac{d}{f'(0)} - h_0^2\right) \left(\int_0^{h_0} x\phi(x) \, dx\right)^{-1} < \overline{\sigma} < \frac{u_1^*}{\|\phi\|_{C([0, h_0])}}.$$  

Then assumption (4.15) assures the existence of $\overline{\sigma}$ satisfying the above inequalities and, therefore, Theorem 4.9 implies the spreading of $(u(t, x; \overline{\sigma}), h(t; \overline{\sigma}))$. Hence the comparison principle shows that the spreading of $(u(t, x; \sigma), h(t; \sigma))$ occurs for every $\sigma \geq \overline{\sigma}$.

Using Theorem 4.8, Lemma 4.12 and (12), one can see that $\sigma^*_1$ given in (13) is the threshold number which separates the vanishing and the spreading:

**Theorem 4.13.** Let $(u(t, x; \sigma), h(t; \sigma))$ be the solution of $(\text{FBP})_{\sigma}$ with initial data $(\sigma \phi, h_0)$ for $\sigma > 0$. Then $(u(t, x; \sigma), h(t; \sigma))$ satisfies the vanishing for every $\sigma \leq \sigma^*_1$ and the spreading for every $\sigma > \sigma^*_1$. Moreover, $\sigma^*_1 \in (0, \infty]$ if $h_0 < \pi \sqrt{d/f'(0)}$, $\sigma^*_1 = 0$ if $h_0 \geq \pi \sqrt{d/f'(0)}$, and $\sigma^*_1 \in (0, \infty)$ if $(\phi, h_0)$ satisfies (4.15).

For the proof of this theorem, see [6, Theorem 5.2] or [16, Theorem 3.7].

Next we will show that $\sigma^*_2$ defined as (4.14) is the threshold number which separates the small spreading and the big spreading:
Theorem 4.14. Let \((u(t, x; \sigma), h(t; \sigma))\) be the solution of \((\text{FBP})_\sigma\) with initial data \((\sigma_0, h_0)\) for \(\sigma > 0\). Then \((u(t, x; \sigma), h(t; \sigma))\) satisfies the small spreading for every \(\sigma \in (\sigma_1^*, \sigma_2^*)\) and the big spreading for every \(\sigma \in (\sigma_2^*, \infty)\). Moreover, \(\sigma_2^* \in (\sigma_1^*, \infty)\) if \(h_0 > \ell^*\), where \(\ell^*\) is a positive constant given in Proposition 3.3.

Remark 3. In Theorem 4.14, \(\sigma_2^*\) may be infinite. But we cannot find any satisfactory condition for \(\sigma_2^* < \infty\) in the case that \(h_0\) is small.

Proof. The proof of this theorem is similar to that of [16, Theorem 3.8]. Assume \(\sigma_2^* < \infty\). We will divide the proof into three steps.

Step 1. The big spreading occurs for every \(\sigma > \sigma_2^*\).

Take any \(\sigma > \sigma_2^*\). By the definition of \(\sigma_2^*\), there exists a positive number \(\hat{\sigma} \in (\sigma_2^*, \sigma)\) such that
\[
\lim_{t \to \infty} h(t; \hat{\sigma}) = \infty \quad \text{and} \quad \lim_{t \to \infty} u(t; x; \hat{\sigma}) = v_3(x) \text{ uniformly in } x \in [0, R] \tag{4.16}
\]
for any \(R > 0\). By \(\sigma > \hat{\sigma}\), it follows from (4.12) that
\[
\lim_{t \to \infty} h(t; \sigma) = \infty \quad \text{and} \quad \liminf_{t \to \infty} u(t; x; \sigma) \geq v_3(x) \text{ for } x \geq 0.
\]
On the other hand, Lemma 4.2 implies \(\limsup_{t \to \infty} u(t, x; \sigma) \leq v_3(x)\) for \(x \geq 0\). Therefore, \(\lim_{t \to \infty} u(t, x; \sigma) = v_3(x)\) uniformly in \(x \in [0, R]\) for any \(R > 0\); so that the big spreading occurs for \(\sigma > \sigma_2^*\).

Step 2. The small spreading occurs at \(\sigma = \sigma_2^*\).

First we assume that the vanishing occurs for \((u(t, x; \sigma_2^*), h(t; \sigma_2^*))\) and there exists \(T > 0\) such that \(u(T, x; \sigma_2^*) < u_2^*\) for \(0 \leq x \leq h(T; \sigma_2^*)\). By the continuous dependence of the solution of \((\text{FBP})\) on initial data, \(u(T, x; \sigma_2^* + \varepsilon) < u_2^*\) for \(0 \leq x \leq h(T; \sigma_2^* + \varepsilon)\) if \(\varepsilon > 0\) is sufficiently small. Applying Theorems 2.3 and 4.10, we can show \(\limsup_{t \to \infty} u(t, x; \sigma_2^* + \varepsilon) \leq u_1^*(\leq u_2^*)\) for \(x \geq 0\), which contradicts the result of Step 1.

We next assume that the big spreading occurs for \((u(t, x; \sigma_2^*), h(t; \sigma_2^*))\). Then for each \(\ell > \ell^*\), there exists \(T > 0\) such that
\[
h(T; \sigma_2^*) > \ell \quad \text{and} \quad u(T, x; \sigma_2^*) > \phi_2(x; \ell) \text{ in } (0, \ell),
\]
where \(\phi_2(x; \ell)\) is a function given in Proposition 3.3.

Indeed we have \(u(t, x; \sigma_2^*) \to v_3(x)\) as \(t \to \infty\) in the topology of \(C^1((0, \ell])\) (see Remark 2). Noting that \(\phi_2(x; \ell) < v_3(x)\) for \(0 < x \leq \ell\), \(\phi_2(0; \ell) < v_4(0, 0)\) and \(\phi_2(0; \ell) = v_3(0) = 0\), we get the desired inequality by choosing sufficiently large \(T > 0\). Making use of the continuous dependence of the solution on initial data again, we can choose a sufficiently small number \(\varepsilon > 0\) such that
\[
h(T; \sigma_2^* - \varepsilon) > \ell \quad \text{and} \quad u(T, x; \sigma_2^* - \varepsilon) > \phi_2(x; \ell) \text{ in } (0, \ell).
\]

It follows from Theorem 4.11 that the big spreading occurs for \((u(t, x; \sigma_2^* - \varepsilon), h(t; \sigma_2^* - \varepsilon))\). This is a contradiction to the definition of \(\sigma_2^*\).

Thus Theorem 4.1 enables us to conclude that the small spreading occurs for \((u(t, x; \sigma_2^*), h(t; \sigma_2^*))\).

Step 3. The small spreading occurs for \(\sigma \in (\sigma_1^*, \sigma_2^*)\).

Clearly, the preceding arguments show \(\sigma_1^* < \sigma_2^*\). It follows from Theorem 4.13 that \((u(t, x; \sigma), h(t; \sigma))\) satisfies the spreading for \(\sigma > \sigma_1^*\). By the definition of \(\sigma_2^*\) and Theorem 4.1, for each \(\varepsilon > 0\) there exists \(\bar{\varepsilon} \in (0, \varepsilon)\) such that \((u(t, x; \sigma_2^* - \bar{\varepsilon}), h(t; \sigma_2^* - \bar{\varepsilon}))\) satisfies the small spreading. Application of Theorem 2.3 shows that the small spreading occurs for every \(\sigma \in (\sigma_1^*, \sigma_2^* - \bar{\varepsilon})\). Since one can take arbitrary \(\varepsilon > 0\), we see that \((u(t, x; \sigma), h(t; \sigma))\) satisfies the small spreading for every \(\sigma \in (\sigma_1^*, \sigma_2^*)\).
Remark 4. If we consider (FBP-N), then the transition occurs at $\sigma = \sigma^*_2$ when $\sigma^*_1$ and $\sigma^*_2$ are defined by (4.13) and (4.14) (see, [16, Theorem 3.8]). This fact means that the transition is a borderline behavior between the small spreading and big spreading in the case of zero Neumann boundary condition at $x = 0$.

We will study more whether one can find a special borderline behavior concerned with the small spreading for $\sigma \in (\sigma^*_1, \sigma^*_2)$.

**Theorem 4.15.** Let $(u, h; \sigma) := (u(t, x; \sigma), h(t; \sigma))$ be the solution of (FBP)$_\sigma$ with initial data $(\sigma \phi, h_0)$ for $\sigma > 0$ and assume $\sigma^*_2 < \infty$. Then there exists a positive number $\sigma^*_1, \sigma^*_2 \in (\sigma^*_1, \sigma^*_2]$ such that $(u, h; \sigma)$ satisfies the small spreading and $\liminf_{t \to \infty} \|u(t, \cdot; \sigma)\|_{C([0, h(t; \sigma)])} < u^*_2$ for every $\sigma \in (\sigma^*_1, \sigma^*_2)$, while $(u, h; \sigma)$ satisfies the small spreading and $\liminf_{t \to \infty} \|u(t, \cdot; \sigma)\|_{C([0, h(t; \sigma)])} \geq u^*_2$ for every $\sigma \in [\sigma^*_1, \sigma^*_2]$.

**Proof.** We define

$$\sigma^*_1, \sigma^*_2 : = \sup \{ \sigma \in [\sigma^*_1, \sigma^*_2] : \text{the small spreading occurs for } (u, h; \sigma) \text{ and } \liminf_{t \to \infty} \|u(t, \cdot; \sigma)\|_{C([0, h(t; \sigma)])} < u^*_2 \}.$$ 

We will show that $\sigma^*_1 > \sigma^*_2$. Since $(u, h; \sigma^*_1)$ satisfies the vanishing, there exists a positive constant $T > 0$ such that $u(t, x; \sigma^*_1) < u^*_2$ for $t \geq T$ and $0 \leq x \leq h(t; \sigma^*_1)$. By the continuous dependence of the solution of (FBP)$_\sigma$ on initial data, it is possible to see that

$$u(T, x; \sigma^*_1 + \varepsilon) < u^*_2 \text{ for } 0 \leq x \leq h(T; \sigma^*_1 + \varepsilon)$$

if $\varepsilon > 0$ is sufficiently small. Applying Theorem 4.10, we also see that

$$\lim_{t \to \infty} \|u(t, \cdot; \sigma^*_1 + \varepsilon)\|_{C([0, h(t; \sigma^*_1 + \varepsilon)])} = u^*_1 < u^*_2.$$ 

We thus conclude $\sigma^*_1, \sigma^*_2 \geq \sigma^*_1 + \varepsilon$.

We will prove that, if $\sigma = \sigma^*_1, \sigma^*_2$, then $\liminf_{t \to \infty} \|u(t, \cdot; \sigma^*_1, \sigma^*_2)\|_{C([0, h(t; \sigma^*_1, \sigma^*_2)])} \geq u^*_2$. Assume that $\liminf_{t \to \infty} \|u(t, \cdot; \sigma^*_1, \sigma^*_2)\|_{C([0, h(t; \sigma^*_1, \sigma^*_2)])} < u^*_2$. Repeating the above arguments one can derive

$$\lim_{t \to \infty} \|u(t, \cdot; \sigma^*_1, \sigma^*_2 + \varepsilon)\|_{C([0, h(t; \sigma^*_1, \sigma^*_2 + \varepsilon)])} < u^*_2$$

provided that $\varepsilon > 0$ is sufficiently small. This contradicts the definition of $\sigma^*_1, \sigma^*_2$. By the comparison theorem, the same small spreading behavior occurs for every $\sigma \in [\sigma^*_1, \sigma^*_2]$. Moreover, since $(u, h; \sigma^*_2)$ satisfies the small spreading and $\liminf_{t \to \infty} \|u(t, \cdot; \sigma^*_2)\|_{C([0, h(t; \sigma^*_2)])} \geq u^*_2$, we obtain $\sigma^*_1, \sigma^*_2 \leq \sigma^*_2$. The proof is finished.

**Remark 5.** The notion of small spreading in Theorem 4.1 is defined by $\lim_{t \to \infty} h(t) = \infty$ and $\lim_{t \to \infty} u(t, x) = v_1(x)$ in $[0, R]$ for any $R > 0$. It may be classified into two sub-cases: (i) $\liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} < u^*_2$, (ii) $\liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} \geq u^*_2$. In particular, case (ii) implies that $u(t, x)$ has a peak at $x = x^*(t)$ satisfying $u(t, x^*(t)) \geq u^*_2$ for sufficiently large $t$. This is an interesting phenomenon, but we have no further information on this kind of small spreading. The phenomenon of case (ii) may correspond to the "transition", which is a borderline solution between small spreading and big spreading for solutions of (FBP-N).
5. Spreading speeds and profiles of solutions. In this section we will discuss
an asymptotic spreading speed of the free boundary and an asymptotic profile of
any spreading solution of (FBP). It was shown by Du and Lou [6] that the analysis
of asymptotic spreading speed and profile of the solution for (FBP) is closely related
with the semi-wave problem:

\[
\begin{aligned}
&d_{qz} - c q + f(q) = 0, 
&q(z) > 0, 
&z > 0, 
&q(0) = 0, 
&\mu q_z(0) = c, 
&\lim_{z \to \infty} q(z) = u^*
\end{aligned}
\]

with \( u^* = u^*_1 \) or \( u^*_2 \). The idea and method for the existence of a semi-wave, that
is, a solution \((c, q(z))\) to (SWP), have been established by Du and Lou [6] for the
typical monostable, bistable and combustion types of \( f \). When \( f \) satisfies (PB),
Kawai and Yamada [16] have shown the following existence and uniqueness of the
solution of (SWP) by applying the phase plane method (see [16, Theorem 4.1]).

**Theorem 5.1.** The following properties hold true.

(i) For \( u^* = u^*_1 \), (SWP) has a unique solution \((c, q) = (c_S, q_S)\) for each \( \mu > 0 \).

(ii) For \( u^* = u^*_2 \), either Case A or Case B holds true;

Case A: (SWP) has a unique solution \((c, q) = (c_B, q_B)\) for each \( \mu > 0 \),

Case B: there exists a positive number \( \mu^* \) such that (SWP) has a unique
solution \((c, q) = (c_B, q_B)\) for each \( \mu \in (0, \mu^*) \), whereas (SWP) has no
solution for \( \mu \geq \mu^* \).

Hereafter, we sometimes use the same notation \((c^*, q^*)\) for any solution of (SWP)
when it is not necessary to distinguish \((c_S, q_S)\) and \((c_B, q_B)\).

5.1. Asymptotic spreading speed. In this subsection, we discuss the asymptotic
speed of the free boundary for the spreading solution of (FBP).

**Theorem 5.2.** Let \((u, h)\) be the solution of (FBP) and let \( c_S \) and \( c_B \) be positive
constants given in Theorem 5.1.

(i) If \((u, h)\) is the small spreading solution and
\[\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t))]} < u^*_2,\]
then
\[\lim_{t \to \infty} \frac{h(t)}{t} = c_S.\]

(ii) If \((u, h)\) is the big spreading solution and (SWP) with \( u^* = u^*_3 \) has a unique
solution, then
\[\lim_{t \to \infty} \frac{h(t)}{t} = c_B,\]
and, if (SWP) with \( u^* = u^*_3 \) has no solution, then
\[\lim_{t \to \infty} \frac{h(t)}{t} = c_S.\]

In order to prove this theorem, it is sufficient to follow the arguments used in the
proof of [13, Theorem 2]. See also [16, Theorem 4.2].

5.2. Asymptotic profiles of spreading solutions. We will show sharp estimates
of spreading speed and profile to each spreading solution for \( h(t)/2 \leq x \leq h(t) \).

**Theorem 5.3.** Let \((u, h)\) be any small spreading solution of (FBP) satisfying
\[\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t))]} < u^*_2.\] Then there exists a constant \( H_S \in \mathbb{R} \) such that
\[\lim_{t \to \infty} (h(t) - c_S t) = H_S \quad \text{and} \quad \lim_{t \to \infty} h'(t) = c_S.\]
Moreover, it holds that
\[ \lim_{t \to \infty} \sup_{h(t)/2 \leq x \leq h(t)} |u(t, x) - q_s(h(t) - x)| = 0. \]

Here \((c_S, q_S)\) is a unique solution of (SWP) with \(u^* = u_1^*\).

**Theorem 5.4.** Let \((u, h)\) be any big spreading solution of (FBP) and assume that (SWP) with \(u^* = u_3^*\) has a unique solution \((c_B, q_B)\). Then there exists a constant \(H_B \in \mathbb{R}\) such that
\[ \lim_{t \to \infty} (h(t) - c_B t) = H_B \quad \text{and} \quad \lim_{t \to \infty} h'(t) = c_B. \]
Moreover, it holds that
\[ \lim_{t \to \infty} \sup_{h(t)/2 \leq x \leq h(t)} |u(t, x) - q_B(h(t) - x)| = 0. \]

**Proofs of Theorems 5.3 and 5.4.** These theorems can be proved essentially in the same way as [13, Theorem 3] (see also [13, Theorems 5 and 6]). \(\square\)

In the other parts of the interval, we get the convergence to the stationary solutions.

**Theorem 5.5.** Let \((u, h)\) be any small spreading solution of (FBP) satisfying
\[ \liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)))} < u_2^*. \]
Then for any \(c \in (0, c_S)\), it holds that
\[ \lim_{t \to \infty} \sup_{0 \leq x \leq ct} |u(t, x) - v_1(x)| = 0, \]
where \(v_1\) is the solution of (SP) given in Proposition 3.1. Similarly, let \((u, h)\) be any big spreading solution of (FBP) and assume that (SWP) with \(u^* = u_3^*\) has a unique solution \((c_B, q_B)\). Then for any \(c \in (0, c_B)\), it holds that
\[ \lim_{t \to \infty} \sup_{0 \leq x \leq ct} |u(t, x) - v_3(x)| = 0, \]
where \(v_3\) is the solution of (SP) given in Proposition 3.1.

Before the proof, we will give two lemmas.

**Lemma 5.6.** ([13, Lemma 3]) Let \((u, h)\) be a small spreading or a big spreading solution of (FBP) which satisfies assumptions of Theorem 5.5. Let \((c^*, q^*)\) be the corresponding semi-wave for (SWP). If \(c \in (0, c^*)\) is sufficiently close to \(c^*\), then there exist positive numbers \(T, M\) and \(\delta\) such that
\[ u(t, x) \geq u^* - Me^{-\delta t} \quad \text{for} \quad t \geq T \text{ and } h(t)/2 \leq x \leq ct. \]

**Lemma 5.7.** Under the same conditions as Lemma 5.6, there exist positive numbers \(T, M\) and \(\delta\) such that
\[ u(t, x) \leq u^* + Me^{-\delta t} \quad \text{for} \quad t \geq T \text{ and } 0 \leq x \leq h(t). \]

**Proof.** Since \(f'(u^*) < 0\), for any \(\delta \in (0, -f'(u^*))\) there exists a number \(\rho = \rho(\delta) \in (0, 1)\) such that \(f(u) \leq \delta (u^* - u)\) for \(u \in [u^*, u^* + \rho]\).

First, we will show that the big spreading solution \((u, h)\) satisfies
\[ u(t, x) \leq u_3^* + Me^{-\delta t} \quad \text{for} \quad t \geq T, \quad 0 \leq x \leq h(t). \]

Consider a solution of the following initial value problem:
\[
\begin{cases}
\eta_1'(t) = f(\eta_1), & t > 0, \\
\eta_1(0) = \|u_0\|_{C([0, h(t)])} + u_3^*.
\end{cases}
\]
Since $\eta_1$ is an upper solution of $u$ in $[0, h(t)]$, the usual comparison theorem assures $u(t, x) \leq \eta_1(t)$ for $t \geq 0$ and $0 \leq x \leq h(t)$. Moreover, $f(u) < 0$ for $u > u_2^*$; so that $\eta_1$ is monotone decreasing in $t$ and converges to $u_2^*$ as $t \to \infty$. Thus there exists $\ell > 0$ such that $\eta_1(t) = f(\eta_1) \leq \delta(u_2^* - \eta_1)$ for $t \geq T$. Therefore
\[ u(t, x) \leq \eta_1(t) \leq u_2^* + \rho e^{-\delta(t-T)} \quad \text{for} \ t \geq T, \ 0 \leq x \leq h(t). \]

We next prove that if the small spreading solution $(u, h)$ satisfies
\[
\liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} < u_2^*,
\]
then it holds that
\[
u(t, x) \leq u_1^* + Me^{-\delta t} \quad \text{for} \ t \geq T, \ 0 \leq x \leq h(t).
\]

It is sufficient to consider the following problem:
\[
\begin{cases}
\eta_2'(t) = f(\eta_2), \ t > 0, \\
\eta_2(0) \in \left( \max \left\{ u_1^*, \liminf_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} \right\}, u_2^* \right)
\end{cases}
\]
and repeat the previous argument to get the conclusion. □

**Proof of Theorem 5.5.** Take any $\varepsilon > 0$ and fix it. For each $c \in (0, c^*)$, consider the following problem:
\[
\begin{cases}
dq'' - cq' + f(q) = 0, \\
q(0) = 0, \ q'(0) = c^*/\mu.
\end{cases}
\]
Let $q = q_c(z)$ be the solution of (5.1). By the phase plane analysis, there exists a unique number $z_c > 0$ such that
\[ q_c'(z) > 0 \quad \text{for} \ z \in [0, z_c), \ q_c'(z_c) = 0. \]
Set $Q_c = q_c(z_c)$; then we can show that $\lim_{c \to c^*} Q_c = u^*$. Moreover, from Proposition 3.3 there exist a unique number $x_c > 0$ and a solution $v_c = v_c(x)$ of (SP-\ell) with $\ell = 2x_c$ such that
\[ v_c'(x) > 0 \quad \text{for} \ x \in (0, x_c), \ v_c'(x_c) = 0 \quad \text{and} \quad v_c(x_c) = Q_c, \]
when $c \in (0, c^*)$ is sufficiently close to $c^*$.

Take any $c \in (0, c^*)$ such that $c$ is sufficiently close to $c^*$ and it satisfies
\[ Q_c \in (u^* - \varepsilon/2, u^*) \quad \text{and} \ c > (c^* + \varepsilon)/2. \]
As in the proof of [13, Theorem 2], we define $(u, h)$ by
\[
\begin{align*}
h(t) := & \quad ct + x_c + z_c, \\
u(t, x) := & \begin{cases}
v_c(x), & 0 \leq x \leq x_c, \\
Q_c, & x_c \leq x \leq h(t) - z_c, \\
q_c(h(t) - x), & h(t) - z_c \leq x \leq h(t).
\end{cases}
\end{align*}
\]
Then it is possible to choose $T_0 > 0$ such that
\[ h(0) = x_c + z_c \leq h(T_0), \ u(0, x) \leq u(T_0, x) \quad \text{for} \ 0 \leq x \leq h(0) \]
because we have $h(t) \to \infty$, $u(t, \cdot) \to v^*$ in $C^1([0, R])$ for any $R > x_c + z_c$ as $t \to \infty$, $v^*(x) > u(0, x)$ for $0 < x \leq h(0)$ and $(v^*)'(0) > u_x(0, 0) = v_x'(0)$. Here $v^* = v^*(x)$ denotes the solution $v_1$ or $v_3$ of (SP). Hence the comparison principle shows
\[ h(t) \leq h(t + T_0) \quad \text{for} \ t \geq 0 \quad \text{and} \ u(t, x) \leq u(t + T_0, x) \quad \text{for} \ t \geq 0, \ 0 \leq x \leq h(t). \]
Thus Theorem 2.3 and (5.2) imply that
\[ u(t, x) \geq Q_c > u^* - \varepsilon/2 \quad \text{for} \ t \geq T_0, \ x_c \leq x \leq c(t - T_0) + x_c. \]
Since $v^*$ is an increasing function such that $\lim_{x \to \infty} v^*(x) = u^*$, it is possible to take a positive number $R > x_c$ such that
\[ v^*(x) \in (u^* - \varepsilon/2, u^*) \quad \text{for } x \geq R. \tag{5.4} \]

By (5.2) we see that there exists $T_1 \geq T_0$ such that
\[ R \leq h(t)/2 \leq c(t - T_0) + x_c \quad \text{for } t \geq T_1. \tag{5.5} \]

Moreover, the uniform convergence of $\lim_{t \to \infty} u(t, x) = v^*(x)$ in $x \in [0, R]$ implies the existence of a constant $T_2$ such that
\[ |u(t, x) - v^*(x)| < \varepsilon \quad \text{for } t \geq T_2, \ 0 \leq x \leq R. \tag{5.6} \]

It follows from Lemmas 5.6 and 5.7 that
\[ |u(t, x) - u^*| < \varepsilon/2 \quad \text{for } t \geq T_3, \ h(t)/2 \leq x \leq ct, \tag{5.7} \]
\[ u(t, x) - u^* < \varepsilon/2 \quad \text{for } t \geq T_3, \ R \leq x \leq h(t)/2 \tag{5.8} \]

with sufficiently large $T_3$. Combining (5.3)-(5.8), one can conclude that there exists a positive number $T > \max\{T_1, T_2, T_3\}$ such that
\[ |u(t, x) - v^*(x)| < \varepsilon \quad \text{for all } t \geq T \text{ and } 0 \leq x \leq ct. \]

Thus the proof is complete. \hfill \Box

**Remark 6.** Consider any big spreading solution $(u, h)$ of (FBP) when (SWP) with $u^* = u^*_1$ has no solutions. In this case, one can apply the result of [15, Proposition 4.2] to show
\[ \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_S, \]
where $c_S$ is a semi-wave speed of solution $(c_S, q_S)$ of (SWP) with $u^* = u^*_1$. On the other hand, Theorem 2.3 implies
\[ \liminf_{t \to \infty} \frac{h(t)}{t} \geq c_S. \]

Therefore, it holds that
\[ \lim_{t \to \infty} \frac{h(t)}{t} = c_S. \]

When we discuss a big spreading solution of (FBP-N) in the case (SWP) with $u^* = u^*_3$ has no solutions, we already know from [15] that it possesses a propagating terrace. This is composed of a semi-wave corresponding to a small spreading solution and a traveling wave connecting $u^*_1$ and $u^*_3$. So we infer that any big spreading solution also has a similar propagating terrace. We will study this issue elsewhere.

**Acknowledgments.** The authors would like to thank referees for careful readings and helpful suggestions.

**REFERENCES**

[1] S. Angenent, The zero set of a solution of a parabolic equation, *J. Reine Angew. Math.*, **390** (1988), 79–96.

[2] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, **7** (2012), 583–603.

[3] W. Choi and I. Ahn, Non-uniform dispersal of logistic population models with free boundaries in a spatially heterogeneous environment, *J. Math. Anal. Appl.*, **479** (2019), 283–314.

[4] W. Ding, R. Peng and L. Wei, The diffusive logistic model with a free boundary in a heterogeneous time-periodic environment, *J. Differential Equations*, **263** (2017), 2736–2779.
Y. Du and Z. G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, 42 (2010), 377–405; *SIAM J. Math. Anal.*, 45 (2013), 1995–1996 (erratum).

Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.*, 17 (2015), 2673–2724.

Y. Du, B. Lou and M. Zhou, Nonlinear diffusion problems with free boundaries: Convergence, transition speed, and zero number arguments, *SIAM J. Math. Anal.*, 47 (2015), 3555–3584.

Y. Du and H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, *J. Eur. Math. Soc.*, 12 (2010), 279–312.

Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.*, 46 (2014), 375–396.

J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differential Equations*, 24 (2012), 873–895.

Y. Kaneko, K. Oeda and Y. Yamada, Remarks on spreading and vanishing for free boundary problems of some reaction-diffusion equations, * Funkcial. Ekvac.*, 57 (2014), 449–465.

Y. Kaneko and Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology, *Adv. Math. Sci. Appl.*, 21 (2011), 467–492.

Y. Kaneko and Y. Yamada, Spreading speed and profiles of solutions to a free boundary problem with Dirichlet boundary conditions, *J. Math. Anal. Appl.*, 465 (2018), 1159–1175.

Y. Kaneko, H. Matsuzawa, Spreading and vanishing in a free boundary problem for nonlinear diffusion equations with a given forced moving boundary, *J. Differential Equations*, 265 (2018), 1000–1043.

Y. Kaneko, H. Matsuzawa and Y. Yamada, Asymptotic profiles of solutions and propagating terrace for a free boundary problem of reaction-diffusion equation with positive bistable nonlinearity, to appear in *SIAM J. Math. Anal*.

Y. Kawai and Y. Yamada, Multiple spreading phenomena for a free boundary problem of a reaction-diffusion equation with a certain class of bistable nonlinearity, *J. Differential Equations*, 261 (2016), 538–572.

C. Lei, H. Matsuzawa, R. Peng and M. Zhou, Refined estimates for the propagation speed of the transition solution to a free boundary problem with a nonlinearity of combustion type, *J. Differential Equations*, 265 (2018), 2897–2920.

X. Liu and B. Lou, Asymptotic behavior of solutions to diffusion problems with Robin and free boundary conditions, *Math. Model. Nat. Phenom.*, 8 (2013), 18–32.

X. Liu and B. Lou, On a reaction-diffusion equation with Robin and free boundary conditions, *J. Differential Equations*, 259 (2015), 423–453.

B. Lou, Convergence in time-periodic quasilinear parabolic equations in one space dimension, *J. Differential Equations*, 265 (2018), 3952–3969.

D. Ludwig, D. G. Aronson and H. F. Weinberger, Spatial patterning of the spruce budworm, *J. Math. Biol.*, 8 (1979), 217–258.

H. Matsuzawa, A free boundary problem for the Fisher-KPP equation with a given moving boundary, *Commun. Pure Appl. Anal.*, 17 (2018), 1821–1852.

D. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.*, 21 (1972), 979–1000.

J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations*, 39 (1981), 269–290.

N. Sun, B. Lou and M. Zhou, Fisher-KPP equation with free boundaries and time-periodic advections, *Calc. Var. Partial Differential Equations*, 56 (2017), Art. 61, 36 pp.

R. H. Wang, L. Wang and Z. C. Wang, Free boundary problem of a reaction-diffusion equation with nonlinear convection term, *J. Math. Anal. Appl.*, 467 (2018), 1233–1257.

Y. Zhao and M. X. Wang, A reaction-diffusion-advection equation with mixed and free boundary conditions, *J. Dynam. Differential Equations*, 30 (2018), 743–777.

Received January 2019; 1st revision June 2019; 2nd revision August 2019.

E-mail address: e-maho@toki.waseda.jp
E-mail address: y. kaneko@aoni.waseda.jp (current: kanekoy@fc.jwu.ac.jp)
E-mail address: yamada@waseda.jp