A note on the $k$-tuple domination number of graphs

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Abstract

In a graph $G$, a vertex dominates itself and its neighbours. A set $D \subseteq V(G)$ is said to be a $k$-tuple dominating set of $G$ if $D$ dominates every vertex of $G$ at least $k$ times. The minimum cardinality among all $k$-tuple dominating sets is the $k$-tuple domination number of $G$. In this note, we provide new bounds on this parameter. Some of these bounds generalize other ones that have been given for the case $k = 2$.

Keywords: $k$-domination; $k$-tuple domination.

1 Introduction

Throughout this note we consider simple graphs $G$ with vertex set $V(G)$. Given a vertex $v \in V(G)$, $N(v)$ denotes the open neighbourhood of $v$ in $G$. In addition, for any set $D \subseteq V(G)$, the degree of $v$ in $D$, denoted by $\deg_D(v)$, is the number of vertices in $D$ adjacent to $v$, i.e., $\deg_D(v) = |N(v) \cap D|$. The minimum and maximum degrees of $G$ will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. Other definitions not given here can be found in standard graph theory books such as [13].

Domination theory in graphs have been extensively studied in the literature. For instance, see the books [10, 11, 12]. A set $D \subseteq V(G)$ is said to be a dominating set of $G$ if $\deg_D(v) \geq 1$ for every $v \in V(G) \setminus D$. The domination number of $G$ is the minimum cardinality among all dominating sets of $G$ and it is denoted by $\gamma(G)$. We define a $\gamma(G)$-set as a dominating set of cardinality $\gamma(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the paper.

In 1985, Fink and Jacobson [5, 6] extended the idea of domination in graphs to the more general notion of $k$-domination. A set $D \subseteq V(G)$ is said to be a $k$-dominating set of $G$ if $\deg_D(v) \geq k$ for every $v \in V(G) \setminus D$. The $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality among all $k$-dominating sets of $G$. Subsequently, and as expected, several variants for $k$-domination were introduced and studied by the scientific community. In two different papers published in 1996 and 2000, Harary and
Haynes [8, 9] introduced the concept of double domination and, more generally, the concept of k-tuple domination. Given a graph $G$ and a positive integer $k \leq \delta(G) + 1$, a $k$-dominating set $D$ is said to be a $k$-tuple dominating set of $G$ if $\deg_D(v) \geq k - 1$ for every $v \in D$. The $k$-tuple domination number of $G$, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all $k$-tuple dominating sets of $G$. The case $k = 2$ corresponds to double domination, in such a case, $\gamma_{\times 2}(G)$ denotes the double domination number of graph $G$.

In this note, we provide new bounds on the $k$-tuple domination number. Some of these bounds generalize other ones that have been given for the double domination number.

2 New bounds on the $k$-tuple domination number

Recently, Hansberg and Volkmann [7] put into context all relevant research results on multiple domination that have been found up to 2020. In that chapter, they posed the following open problem.

**Problem 2.1.** (Problem 5.8, p.194, [7]) Give an upper bound for $\gamma_{\times k}(G)$ in terms of $\gamma_k(G)$ for any graph $G$ of minimum degree $\delta(G) \geq k - 1$.

A fairly simple solution for the problem above is given by the straightforward relationship $\gamma_{\times k}(G) \leq k\gamma_k(G)$, which can be derived directly by constructing a set of vertices $D' \subseteq V(G)$ of minimum cardinality from a $\gamma_k(G)$-set $D$ such that $D \subseteq D'$ and $\deg_{D'}(x) \geq k - 1$ for every vertex $x \in D$. From this construction above, it is easy to check that $D'$ is a $k$-tuple dominating set of $G$ and so,

$$\gamma_{\times k}(G) \leq |D'| = |D| + |D' \setminus D| \leq |D| + (k - 1)|D| = k\gamma_k(G).$$

This previous inequality was surely considered by Hansberg and Volkmann and, in that sense, they have established the previous problem assuming that $\gamma_{\times k}(G) < k\gamma_k(G)$ for every graph $G$ with $\delta(G) \geq k - 1$.

We next confirm their suspicions and provide a solution to Problem 2.1.

**Theorem 2.2.** Let $k \geq 2$ be an integer. For any graph $G$ with $\delta(G) \geq k - 1$,

$$\gamma_{\times k}(G) \leq k\gamma_k(G) - (k - 1)^2.$$

**Proof.** Let $D$ be a $\gamma_k(G)$-set. As $\gamma_{\times k}(G) \leq |V(G)|$ we assume, without loss of generality, that $|D| - (k - 1)^2 \leq |V(G)|$. Now, let $U = \{u_1, \ldots, u_{k-1}\} \subseteq V(G) \setminus D$, $D' = D \cup U$ and $D_0 = \{v \in D : \deg_{D'}(v) < k - 1\}$. The following inequalities arise from counting arguments on the number of edges joining $U$ with $D_0$ and $U$ with $D \setminus D_0$, respectively.

$$\sum_{v \in D_0} \deg_{D'}(v) \geq \sum_{i=1}^{k-1} \deg_{D_0}(u_i)$$

and

$$|D \setminus D_0|(k - 1) \geq \sum_{i=1}^{k-1} \deg_{D \setminus D_0}(u_i).$$
By the previous inequalities and the fact that $D$ is a $k$-dominating set of $G$, we deduce that
\[
\sum_{v \in D_0} \deg_{D'}(v) + |D \setminus D_0|(k - 1) \geq \sum_{i=1}^{k-1} \deg_{D_0}(u_i) + \sum_{i=1}^{k-1} \deg_{D \setminus D_0}(u_i)
\]
\[
= \sum_{i=1}^{k-1} \deg_D(u_i)
\]
\[
\geq k(k - 1).
\]

Now, we define $D'' \subseteq V(G)$ as a set of minimum cardinality among all supersets $W$ of $D'$ such that $\deg_W(x) \geq k - 1$ for every vertex $x \in D$. Since $\deg_D(x) \geq k - 1$ for every $x \in D \setminus D_0$, the condition on $W$ is equivalent to that every vertex $v \in D_0$ has at least $k - 1 - \deg_{D'}(v)$ neighbours in $W \setminus D$. Hence, by the minimality of $D''$ and the inequality chain above, we deduce that
\[
|D'' \setminus D'| \leq |D_0|(k - 1) - \sum_{v \in D_0} \deg_{D'}(v)
\]
\[
= |D|(k - 1) - \left( \sum_{v \in D_0} \deg_{D'}(v) + |D \setminus D_0|(k - 1) \right)
\]
\[
\leq |D|(k - 1) - k(k - 1).
\]

Moreover, it is easy to check that $D''$ is a $k$-tuple dominating set of $G$ because each vertex in $V(G) \setminus D$ is dominated $k$ times by vertices of $D \subseteq D''$ (recall that $D$ is a $k$-dominating set of $G$) and the construction of $D''$ ensures that each vertex in $D$ is dominated $k$ times by vertices of $D''$. Hence,
\[
\gamma_{\times k}(G) \leq |D''| = |D'| + |D'' \setminus D'|
\]
\[
\leq |D| + k - 1 + |D|(k - 1) - k(k - 1)
\]
\[
= k\gamma_k(G) - (k - 1)^2,
\]
which completes the proof.

The bound above is tight. For instance, it is achieved by any complete bipartite graph $K_{k,k'}$ with $k' \geq k$, as $\gamma_{\times k}(K_{k,k'}) = 2k - 1$ and $\gamma_k(K_{k,k'}) = k$. When $k = 2$, Theorem [2,2] leads to the relationship $\gamma_{\times 2}(G) \leq 2\gamma_2(G) - 1$ given in 2018 by Bonomo et al. [1].

A set $D \subseteq V(G)$ is a 2-packing of a graph $G$ if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in D$. The 2-packing number of $G$, denoted by $\rho(G)$, is the maximum cardinality among all 2-packings of $G$.

The next theorem relates the $k$-tuple domination number with the 2-packing number of a graph. Note that the bounds given in this result are generalizations of the bounds $\gamma_{\times 2}(G) \geq 2\rho(G)$ due to Chellali et al. [3], and $\gamma_{\times 2}(G) \leq |V(G)| - \rho(G)$ due to Chellali and Haynes [2].
Theorem 2.3. Let $k \geq 2$ be an integer. For any graph $G$ of order $n$ and $\delta(G) \geq k$,

$$ k\rho(G) \leq \gamma_{\times k}(G) \leq n - \rho(G). $$

Proof. Let $D$ be a $\rho(G)$-set and $S$ a $\gamma_{\times k}(G)$-set. Since $\deg_S(v) \geq k$ for every $v \in D \setminus S$, and $\deg_S(v) \geq k - 1$ for every $v \in D \cap S$, we deduce that

$$ \gamma_{\times k}(G) = |S| \geq \sum_{v \in D \setminus S} \deg_S(v) + \sum_{v \in D \cap S} (\deg_S(v) + 1) \geq k|D| = k\rho(G), $$

and the lower bound follows.

Next, let us proceed to prove that $V(G) \setminus D$ is a $k$-tuple dominating set of $G$. Since $\delta(G) \geq k$, $N(D) \cap D = \emptyset$ and $\deg_D(x) \leq 1$ for every $x \in V(G) \setminus D$, we deduce that $\deg_{V(G) \setminus D}(v) \geq k$ for every $v \in D$ and $\deg_{V(G) \setminus D}(v) \geq k - 1$ for every $v \in V(G) \setminus D$. Hence, $V(G) \setminus D$ is a $k$-tuple dominating set of $G$, as desired.

Therefore, $\gamma_{\times k}(G) \leq |V(G) \setminus D| = n - \rho(G)$, which completes the proof. \(\square\)

Let $\mathcal{H}$ be the family of graphs $H_{k,r}$ defined as follows. For any pair of integers $k,r \in \mathbb{Z}$, with $k \geq 2$ and $r \geq 1$, the graph $H_{k,r}$ is obtained from a complete graph $K_{kr}$ and an empty graph $rK_1$ such that $V(H_{k,r}) = V(K_{kr}) \cup V(rK_1)$, $V(K_{kr}) = \{v_1, \ldots, v_{kr}\}$ and $V(rK_1) = \{u_1, \ldots, u_r\}$ and $E(H_{k,r}) = E(K_{kr}) \cup (\bigcup_{i=0}^{r-1}\{u_{i+1}v_{ki+1}, \ldots, u_{i+1}v_{ki+k}\})$.

Figure 1 shows a graph of this family. Observe that $|V(H_{k,r})| = r(k+1)$, $\gamma_{\times k}(H_{k,r}) = kr$ and $\rho(H_{k,r}) = r$ for every $H_{k,r} \in \mathcal{H}$. Therefore, for these graphs the bounds given in Theorem 2.3 are tight, i.e., $\gamma_{\times k}(H_{k,r}) = k\rho(H_{k,r}) = |V(H_{k,r})| - \rho(H_{k,r})$.

![Figure 1: The graph $H_{4,2} \in \mathcal{H}$.](image)

In [9], Harary and Haynes showed that $\gamma_{\times k}(G) \geq \frac{2km-2m}{k+1}$ for any graph $G$ of order $n$ and size $m$ with $\delta(G) \geq k - 1$. The next result is a partial refinement of the bound above because it only considers graphs with minimum degree at least $k$.

Proposition 2.4. Let $k \geq 2$ be an integer. For any graph $G$ of order $n$ and size $m$ with $\delta(G) \geq k$,

$$ \gamma_{\times k}(G) \geq \frac{(\delta(G) + k)n - 2m}{\delta(G) + 1}. $$
Proof. Let $S$ be a $\gamma_{xk}(G)$-set and $\overline{S} = V(G) \setminus S$. Hence,
\[
2m = \sum_{v \in S} \deg_S(v) + 2 \sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{V(G)}(v)
\]
\[
= \sum_{v \in S} \deg_S(v) + \sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{V(G)}(v)
\]
\[
\geq (k - 1)|S| + k(n - |S|) + \delta(G)(n - |S|)
\]
\[
= (k - 1)|S| + (\delta(G) + k)(n - |S|)
\]
\[
= (\delta(G) + k)n - (\delta(G) + 1)|S|,
\]
which implies that $|S| \geq \frac{(\delta(G) + k)n - 2m}{\delta(G) + 1}$. Therefore, the proof is complete.

The bound above is tight. For instance, it is achieved for the join graph $G = K_k + C_k$ obtained from the complete graph $K_k$ and the cycle graph $C_k$, with $k \geq 3$. For this case, we have that $\gamma_{xk}(G) = k$, $|V(G)| = 2k$, $\delta(G) = k + 2$ and $2|E(G)| = 3k^2 + k$. Also, it is achieved for the complete graph $K_n$ ($n \geq 3$) and any $k \in \{2, \ldots, n - 1\}$.

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