HEIGHT GAP CONJECTURES, $D$-FINITENESS, AND WEAK DYNAMICAL MORDELL–LANG

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Abstract. In previous work, the first author, Ghioca, and the third author introduced a broad dynamical framework giving rise to many classical sequences from number theory and algebraic combinatorics. Specifically, these are sequences of the form $f(\Phi^n(x))$, where $\Phi: X \to X$ and $f: X \to \mathbb{P}^1$ are rational maps defined over $\overline{\mathbb{Q}}$ and $x \in X(\overline{\mathbb{Q}})$ is a point whose forward orbit avoids the indeterminacy loci of $\Phi$ and $f$. They conjectured that if the sequence is infinite, then $\limsup \frac{h(f(\Phi^n(x)))}{\log n} > 0$. They also made a corresponding conjecture for $\liminf$ and showed that it implies the Dynamical Mordell–Lang Conjecture. In this paper, we prove the $\limsup$ conjecture as well as the $\liminf$ conjecture away from a set of density 0. As applications, we prove results concerning the growth rate of coefficients of $D$-finite power series as well as the Dynamical Mordell–Lang Conjecture up to a set of density 0.

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1. Introduction

In [BGS], the authors introduced a broad dynamical framework giving rise to many classical sequences from number theory and algebraic combinatorics. In particular, this construction yields all sequences whose generating functions are \( D \)-finite, i.e., those satisfying homogeneous linear differential equations with rational function coefficients. This class, in turn, contains all hypergeometric series (see, e.g., [WZ92, Gar09]), all series related to integral factorial ratios [Bob09, Sou], generating functions for many classes of lattice walks [DHR18], diagonals of rational functions [Lip88], algebraic functions [Lip89], generating series for the cogrowth of many finitely presented groups [GP17], as well as generating functions of numerous classical combinatorial sequences (see Stanley [Sta99, Chapter 6] and the examples therein). In [BGS], they also stated the so-called lim sup and lim inf Height Gap Conjectures, which if true, would imply both the Dynamical Mordell–Lang Conjecture as well as results concerning the growth rate of coefficients of \( D \)-finite power series. The goal of this paper is to prove a uniform version of the lim sup Height Gap Conjecture and to prove the lim inf version away from a set of density zero. Consequently, we obtain applications to \( D \)-finite power series and a weak version of the Dynamical Mordell–Lang Conjecture.

To state our results, we fix the following notation. Throughout, we let \( \mathbb{N} \) (resp. \( \mathbb{Z}^+ \)) denote the set of all non-negative (resp. positive) integers. Let \( h(\cdot) \) denote the absolute logarithmic Weil height function. We refer the reader to [BG06, Chapter 2] and [Sil07, Chapter 3] for the main properties of height functions. Given an arbitrary rational map \( g \), let \( I_g \) denote its indeterminacy locus. If \( \Phi \) is a rational self-map of a quasi-projective variety \( X \) defined over \( \overline{\mathbb{Q}} \), then we let \( X_\Phi(\overline{\mathbb{Q}}) \) denote the subset of points \( x \in X(\overline{\mathbb{Q}}) \) such that for all \( n \in \mathbb{N} \), the \( n \)-th iterate \( \Phi^n(x) \) avoids \( I_\Phi \); for such an \( x \in X_\Phi(\overline{\mathbb{Q}}) \), we let \( O_\Phi(x) \) denote its forward orbit under \( \Phi \). Lastly, if \( f: X \to \mathbb{P}^1 \) is a rational function, let \( X_{\Phi,f}(\overline{\mathbb{Q}}) \subseteq X_\Phi(\overline{\mathbb{Q}}) \) be the subset of points \( x \) with \( I_f \cap O_\Phi(x) = \emptyset \).

The following conjecture was introduced in [BGS].

**Conjecture 1.1** (lim sup Height Gap Conjecture, cf. [BGS, Conjecture 1.4]). Let \( X \) be a quasi-projective variety, let \( \Phi: X \to X \) be a rational self-map, and let \( f: X \to \mathbb{P}^1 \) be a non-constant rational function, all defined over \( \overline{\mathbb{Q}} \). Then for any \( x \in X_{\Phi,f}(\overline{\mathbb{Q}}) \), either \( f(O_\Phi(x)) \) is finite, or

\[
\limsup_{n \to \infty} \frac{h(f(\Phi^n(x))))}{\log n} > 0.
\]

Our first main result is a simple proof of Conjecture 1.1. This generalizes [BGS, Theorem 1.3], which handled the case where \( \Phi \) and \( f \) are morphisms.

**Theorem 1.2** (lim sup Height Gaps). Conjecture 1.1 is true.
In [BGS], the authors also introduced the following conjecture and showed that it implies Dynamical Mordell–Lang.

**Conjecture 1.3** (lim inf Height Gap Conjecture, cf. [BGS, Conjecture 1.6]). Let $X$, $\Phi$, $f$, and $x$ be as in Conjecture 1.1. If $X$ is irreducible and $\mathcal{O}_\Phi(x)$ is Zariski dense in $X$, then
\[
\liminf_{n \to \infty} \frac{h(f(\Phi^n(x))))}{\log n} > 0.
\]

Generalizing our method of proof of Theorem 1.2 via a more involved technique introduced in Section 3, we obtain a uniform version of the above lim sup height gap result for any subset $T \subseteq \mathbb{N}$ of positive density. See Definition 3.1 for the notion of upper asymptotic density.

**Theorem 1.4** (Uniform lim sup Height Gaps). Let $X$, $\Phi$, $f$, and $x$ be as in Conjecture 1.1. Then either $f(\mathcal{O}_\Phi(x))$ is finite, or there exists an $\epsilon > 0$ such that for any subset $T \subseteq \mathbb{N}$ of positive density, we have
\[
\limsup_{n \in T} \frac{h(f(\Phi^n(x))))}{\log n} > \epsilon.
\]

The significance of our above uniform bound is that it implies the lim inf Height Gap Conjecture 1.3 away from a set of density zero.

**Theorem 1.5** (Weak lim inf Height Gaps). Let $X$, $\Phi$, $f$, and $x$ be as in Conjecture 1.1.\footnote{Note that Conjecture 1.3 is stated only for $X$ is irreducible and $\mathcal{O}_\Phi(x)$ Zariski dense which is necessary as shown in an example of [BGS], however our result holds without these hypotheses.} If $f(\mathcal{O}_\Phi(x))$ is infinite, then there is a constant $C > 0$ and a set $S \subset \mathbb{N}$ of upper asymptotic density zero such that
\[
h(f(\Phi^n(x)))) > C \log n
\]
whenever $n \notin S$, or equivalently,
\[
\liminf_{n \in \mathbb{N} \setminus S} \frac{h(f(\Phi^n(x))))}{\log n} > 0.
\]

As an application of Theorem 1.2, we obtain a simple proof of the univariate version of a result of Bell–Nguyen–Zannier [BNZ] which, in turn, generalized results of van der Poorten–Shparlinski [vdPS96] with the aid of [BC17].

We recall that a power series $F(z) \in \overline{\mathbb{Q}}[[z]]$ is $D$-finite, if it is the solution of a non-trivial homogeneous linear differential equation with coefficients in the rational function field $\overline{\mathbb{Q}}(z)$; this is equivalent to saying that the coefficients of $F(z)$ satisfy certain linear recurrence relations with polynomial coefficients (see [Sta80, Theorem 1.5]).

**Theorem 1.6** (Height gaps for $D$-finite power series). If $\sum_{n \geq 0} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is $D$-finite and
\[
\limsup_{n \to \infty} \frac{h(a_n)}{\log n} = 0,
\]
then the sequence $(a_n)_{n \in \mathbb{N}}$ is eventually periodic.
As an application of Theorem 1.5, we show that the Dynamical Mordell–Lang Conjecture holds away from a set of density zero. We note that this result was obtained in [BGT15, Corollary 1.5] using the upper Banach density function.

**Theorem 1.7** (Weak Dynamical Mordell–Lang). Let $X$ be a quasi-projective variety, $\Phi: X \to X$ a rational self-map, and $Y \subseteq X$ a subvariety of $X$, all defined over $\overline{\mathbb{Q}}$. If $x \in X_{\Phi}(\overline{\mathbb{Q}})$, then \( \{ n \in \mathbb{N} : \Phi^n(x) \in Y \} \) is a union of finitely many arithmetic progressions along with a set of upper asymptotic density zero.

Lastly, we prove a natural generalization of Theorem 1.4 for commuting rational self-maps. Given $m$ commuting rational self-maps $\Phi_1, \ldots, \Phi_m$ of $X$ and $n := (n_1, \ldots, n_m) \in \mathbb{N}^m$, we denote by $\Phi^n$ the composite $\Phi_1^{n_1} \circ \cdots \circ \Phi_m^{n_m}$. Let $X_{\Phi_1, \ldots, \Phi_m}(\overline{\mathbb{Q}})$ denote the subset of points $x \in X(\overline{\mathbb{Q}})$ such that for every $n \in \mathbb{N}^m$, the $n$-th iterate $\Phi^n(x)$ avoids the indeterminacy loci of all $\Phi_1, \ldots, \Phi_m$. For any $x \in X_{\Phi_1, \ldots, \Phi_m}(\overline{\mathbb{Q}})$, we let $\mathcal{O}_{\Phi_1, \ldots, \Phi_m}(x)$ be the set of points of the form $\Phi^n(x)$. Similarly, for a rational function $f: X \to \mathbb{P}^1$, we let $X_{\Phi_1, \ldots, \Phi_m, f}(\overline{\mathbb{Q}}) \subseteq X_{\Phi_1, \ldots, \Phi_m}(\overline{\mathbb{Q}})$ denote the subset of points $x$ with $I_f \cap \mathcal{O}_{\Phi_1, \ldots, \Phi_m}(x) = \emptyset$. We endow $\mathbb{N}^m$ with the 1-norm $\|n\| := n_1 + \cdots + n_m$.

**Theorem 1.8** (lim sup Height Gaps for commuting rational self-maps). Let $X$ be a quasi-projective variety, let $\Phi_1, \ldots, \Phi_m$ be $m$ commuting rational self-maps of $X$, and let $f: X \to \mathbb{P}^1$ be a non-constant rational function, all defined over $\overline{\mathbb{Q}}$. Then for any $x \in X_{\Phi_1, \ldots, \Phi_m, f}(\overline{\mathbb{Q}})$, either $f(\mathcal{O}_{\Phi_1, \ldots, \Phi_m}(x))$ is finite, or there exists an $\epsilon > 0$ such that for any subset $T \subseteq \mathbb{N}$ of positive upper asymptotic density, we have

$$\limsup_{\|n\| \in T} \frac{h(f(\Phi^n(x)))}{\log\|n\|} > \epsilon.$$ 

Theorem 1.8 is shown by induction, where the key step is the base case $m = 1$; note that this base case is precisely Theorem 1.4. In a similar manner to the proof of Theorem 1.5, one can deduce a weak lim inf height gap result for commuting rational maps. In Example 5.2, we show that one cannot expect a version of Theorem 1.8 to hold if the lim sup over $\|n\| \in T$ is replaced by a lim sup over $n \in T$.

In [Lip89], Lipshitz introduced and studied multivariate $D$-finite power series (see Definition 5.3), which extended Stanley’s pioneering work [Sta80] on univariate $D$-finite power series. Recently, the first author, Nguyen, and Zannier proved a height gap result for the coefficients of multivariate $D$-finite power series; see [BNZ, Theorem 1.3(c)]. The reader may be curious to know whether it is possible to deduce their result from Theorem 1.8, analogously to how we deduced the univariate $D$-finiteness result Theorem 1.6 from Theorem 1.2. This appears to be a subtle issue: our proof of Theorem 1.6 relies on the fact that for sufficiently large $n$, the coefficients of a univariate $D$-finite power series are of the form $f(\Phi^n(c))$ for certain choices of $X$, $\Phi$, $f$, and $c$; see [BGT16, Section 3.2.1]. In contrast, we construct
in Example 5.4 a $D$-finite power series in two variables (in fact a rational function) whose
coefficients never arise as $f(\Phi_1^n \circ \Phi_2^m(c))$ for any choices of $X, \Phi_1, \Phi_2, f,$ and $c$.

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2. The $\limsup$ Height Gap Conjecture: Proof of Theorem 1.2

We start by stating Schanuel’s Theorem, which plays a central role in the proofs of Theorems 1.2 and 1.4. It can be regarded as a quantitative version of Northcott’s theorem [Nor49]. Schanuel’s Theorem has a conjectural extension to Fano varieties, known as Manin’s conjecture, which has attracted a lot of attention recently (see the survey [LT19] and references therein).

Theorem 2.1 (Schanuel [Sch79], cf. [BG06, 11.10.5]). Let $K$ be a number field of degree $d$ and let $h(\cdot)$ denote the absolute logarithmic Weil height on $P^n_K$. Then we have

$$\lim_{B \to \infty} \frac{\# \{ P \in P^n_K : h(P) \leq \log B \}}{B^{d(n+1)}} = C_{n,K} > 0,$$

where the positive constant $C_{n,K}$ depends only on $n$ and $K$.

We shall prove Theorem 1.2 via an application of Schanuel’s Theorem 2.1 and the following lemma. Recall that a topological space $U$ is called Noetherian if the descending chain condition holds for closed subsets of $U$, i.e. for every chain of closed sets $Z_1 \supset Z_2 \supset \ldots$, there is some $m \geq 1$ for which $Z_m = Z_n$ for all $n \geq m$.

Lemma 2.2. Let $X$ be a quasi-projective variety, let $\Phi : X \to X$ be a rational self-map, and let $f : X \to \mathbb{P}^1$ be a rational function, all defined over $\overline{\mathbb{Q}}$. Then there exists a constant $\ell \in \mathbb{N}$ with the following property: if $x, y \in X_{\Phi^n}(\overline{\mathbb{Q}})$ and $f(\Phi^n(x)) = f(\Phi^n(y))$ for $0 \leq n \leq \ell$, then $f(\Phi^n(x)) = f(\Phi^n(y))$ for all $n \geq 0$.

Proof. Let $U_n = X \setminus U_{i \leq n}(I_{\Phi} \cup I_{f \circ \Phi})$ and $U = \bigcap_n U_n$. By construction, the $\overline{\mathbb{Q}}$-points of $U$ are precisely those on which $\Phi^n$ and $f \circ \Phi^n$ are well-defined for all $n \geq 0$, i.e. $U(\mathbb{Q}) = X_{\Phi,f}(\overline{\mathbb{Q}})$. We endow $U$ with the subspace topology inherited from $X$ thereby making it a Noetherian topological space. Since

$$U \times U \to U_n \times U_n \to \mathbb{P}^1 \times \mathbb{P}^1$$

is continuous and the image of the diagonal map $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is closed, we see that

$$Z_n := \{(x, y) \in U \times U : f(\Phi_i(x)) = f(\Phi_i(y)) \text{ for } i \leq n\}$$

is a closed subset of $U \times U$. As $U \times U$ is Noetherian, there exists an $\ell \in \mathbb{N}$ such that $Z_n = Z_\ell$ for all $n \geq \ell$. \qed
Proof of Theorem 1.2. Let \( x \in X_{\Phi,f}(\overline{\mathbb{Q}}) \). Without loss of generality, we may assume that \( X \), \( \Phi \), \( f \), and \( x \) are defined over a fixed number field \( K \). Suppose that
\[
\limsup_{n \to \infty} \frac{h(f(\Phi^n(x))))}{\log n} = 0,
\]
i.e., \( h(f(\Phi^n(x)))) = o(\log n) \). We will show that \( f(O_{\Phi}(x)) \) is finite.

Letting \( \ell \) be as in Lemma 2.2, we define
\[
y_i := (f(\Phi^i(x)), f(\Phi^{i+1}(x)), \ldots, f(\Phi^{i+\ell}(x))) \in (\mathbb{P}^1)^{\ell+1}(K)
\]
for \( i \geq 0 \), and let \( S = \{y_i : i \geq 0\} \). Via the Segre embedding, we may view \( S \subseteq \mathbb{P}^{2\ell+1} - 1(K) \).

Then
\[
h(y_i) = \sum_{j=0}^{\ell} h(f(\Phi^{i+j}(x))) = o(\log i).
\]

Next, choose \( 0 < \epsilon < ([K: \mathbb{Q}]2^{\ell+1})^{-1} \). Then there exists \( N_0 \in \mathbb{Z}^+ \) such that for all \( i \geq N_0 \), we have \( h(y_i) < \epsilon \log i \). So, for all \( n \geq N_0 \),
\[
\#\{y_{N_0}, y_{N_0+1}, \ldots, y_n\} \leq \#\{z \in \mathbb{P}^{2\ell+1} - 1(K) : h(z) \leq \log n^\epsilon\} = O(n^{\epsilon[K: \mathbb{Q}]2^{\ell+1}}),
\]
where the equality comes from applying Schanuel’s Theorem 2.1. Choosing \( n \) sufficiently large, we find
\[
\#\{y_{N_0}, y_{N_0+1}, \ldots, y_n\} < n - N_0.
\]
In particular, there exist \( i < j \) for which \( y_i = y_j \). Thus, \( f(\Phi^n(\Phi^i(x))) = f(\Phi^n(\Phi^j(x))) \) for all \( 0 \leq n \leq \ell \), and so Lemma 2.2 implies \( f(\Phi^{n+i}(x)) = f(\Phi^{n+j}(x)) \) for all \( n \geq 0 \). It follows that \( f(\Phi^n(x)) \) is eventually periodic with period dividing \( j - i \). Hence, \( f(O_{\Phi}(x)) \) is finite. \( \square \)

3. Uniform lim sup Height Gap Result: Proof of Theorem 1.4

The main goal of this section is to prove Theorem 1.4 which is the strengthening of Theorem 1.2.

3.1. Preliminary results on sets of positive density.

**Definition 3.1.** Let \( A \) be a subset of \( \mathbb{Z}^+ \). The upper asymptotic (or natural) density \( \overline{d}(A) \) of \( A \) is defined by
\[
\overline{d}(A) := \limsup_{m \to \infty} \frac{|A \cap [1,m]|}{m}.
\]
We frequently refer to \( \overline{d}(A) \) simply as the density of \( A \).

**Remark 3.2.** It is easy to see that the density \( \overline{d}(A) \) of any \( A \subseteq \mathbb{Z}^+ \) is right translation invariant, i.e., \( \overline{d}(A + i) = \overline{d}(A) \) for any \( i \in \mathbb{N} \), where \( A + i := \{a + i : a \in A\} \). Consequently, we can extend the definition of density to any \( A \subseteq \mathbb{Z} \) that is bounded from below.
Remark 3.3. Let $T \subseteq \mathbb{N}$ have positive density and let $L \geq 1$. By the subadditivity of natural density, there exists some $a \in \{0, 1, \ldots, L - 1\}$ such that $T \cap (a + LN)$ has positive density.

Definition 3.4. Given $T \subseteq \mathbb{N}$, the shift set of $T$ is defined to be

$$\Sigma(T) = \{i \in \mathbb{N} : \overline{d}(T \cap (T + i)) > 0\}.$$ 

Our goal in this subsection is to prove that if $T$ has positive density, then $\Sigma(T)$ does as well. We prove this after a preliminary lemma.

Lemma 3.5. Let $T \subseteq \mathbb{N}$ and $N \in \mathbb{Z}^+$ satisfying $\overline{d}(T) > \frac{1}{N}$. Then for any finite subset $F \subseteq \mathbb{N}$ with $|F| \geq N$, there exist $j, k \in F$ with $j > k$ such that $\overline{d}((T + (j - k)) \cap T) > 0$.

Proof. For ease of notation, we let $T_i = T + i$ for any $i \in \mathbb{N}$. By definition of the density function, there is a sequence $0 < m_1 < m_2 < \cdots < m_n < \cdots$ and intervals $I_n = [0, m_n] \subset \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{|T \cap I_n|}{|I_n|} = \overline{d}(T).$$

For each $i \in \mathbb{N}$, we have $|T \cap I_n| - i \leq |T \cap (I_n - i)| \leq |T \cap I_n|$, and so $\lim_{n \to \infty} \frac{|T \cap (I_n - i)|}{|I_n|} = \overline{d}(T)$.

In particular, this holds for each $i \in F$.

Fix an $\epsilon > 0$ with $\frac{1}{N} + \epsilon < \overline{d}(T)$. Then for $n$ sufficiently large,

$$\frac{|T_i \cap I_n|}{|I_n|} = \frac{|T \cap (I_n - i)|}{|I_n|} > \frac{1}{N} + \epsilon$$

for all $i \in F$. Now, suppose to the contrary that $\overline{d}(T_j \cap T_k) = \overline{d}(T_{j-k} \cap T) = 0$ for all distinct $j, k \in F$ with $j > k$. It follows that for $n$ sufficiently large,

$$|T_j \cap T_k \cap I_n| < \frac{2|I_n|}{|F|} \epsilon$$

for all distinct $j, k \in F$. Clearly,

$$|I_n| \geq |I_n \cap \bigcup_{i \in F} T_i| = \left| \bigcup_{i \in F} (T_i \cap I_n) \right|.$$ 

However, the inclusion-exclusion principle asserts that

$$\left| \bigcup_{i \in F} (T_i \cap I_n) \right| \geq \sum_{i \in F} |T_i \cap I_n| - \sum_{j, k \in F, j > k} |T_j \cap T_k \cap I_n|$$

$$> |F|(\frac{1}{N} + \epsilon)|I_n| - \left( \frac{|F|}{2} \right) \frac{2|I_n|}{|F|} \epsilon$$

$$= \left( \frac{|F|}{N} + \epsilon \right)|I_n| \geq (1 + \epsilon)|I_n|,$$

which yields a contradiction and hence Lemma 3.5 follows. \qed
The following result is strengthening of [BGT15, Lemma 2.1].

**Proposition 3.6.** If $T \subseteq \mathbb{N}$ satisfies $\overline{d}(T) > 0$, then $\overline{d}(\Sigma(T)) > 0$.

**Proof.** Choose a positive integer $N$ satisfying $\overline{d}(T) > \frac{1}{N}$, and let $T_i$ denote $T + i$ for any $i \in \mathbb{N}$. If $\Sigma(T) = \mathbb{N}$, then there is nothing to prove. So we may suppose there is some $i \in \mathbb{N}$ such that $\overline{d}(T \cap T_i) = 0$. Consider the set $S$ of those finite subsets $F \subseteq \mathbb{N}$ such that $\overline{d}(T_j - k \cap T) = 0$ for all $j, k \in F$ with $j > k$. Clearly, $S \neq \emptyset$ as $\{1, i + 1\} \in S$. Moreover, by Lemma 3.5, we know $|F| < N$ for any $F \in S$.

Let $\emptyset \neq F_{\text{max}} \subseteq \mathbb{N}$ be any maximal element of $S$ (with respect to inclusion of sets), and let $M$ be the maximum element of $F_{\text{max}}$. Then by our definition of $F_{\text{max}}$, for any integer $n > M$, there exists some $k_n \in F_{\text{max}}$ satisfying $\overline{d}(T_{n-k_n} \cap T) > 0$, i.e., $n - k_n \in \Sigma(T)$.

Since $0 \leq k_n \leq M$, we see $n - M \leq n - k_n \leq n$. In particular, for every $c \geq 2$, we have

$$iM - kcM \in \Sigma(T) \cap [(c-1)M, cM].$$

It thus follows from the definition of density that $\overline{d}(\Sigma(T)) \geq \lim_{c \to \infty} \frac{c-1}{cM} = \frac{1}{M}$.

**Remark 3.7.** Using a similar argument, one can obtain an analogue of Proposition 3.6 where one replaces $\overline{d}$ by upper Banach density.

### 3.2. Stable non-periodic dimension.

Given a Noetherian topological space $U$ of finite Krull dimension and continuous map $\Phi: U \to U$, a subset $Y \subseteq U$ is periodic with respect to $\Phi$ if $\Phi^n(Y) \subseteq Y$ for some positive integer $n$; we frequently say $Y$ is $\Phi$-periodic or simply periodic if $\Phi$ is understood from context. It is well known that if $Z \subseteq U$ is a closed subset, let $Z_1, \ldots , Z_r$ denote its irreducible components and consider the set

$$S = \left\{ \bigcup_{i \in I} Z_i : I \subseteq \{1, \ldots , r\} \right\}.$$  

Notice that if $Y_1, Y_2 \in S$ are periodic with respect to $\Phi$, then so is $Y_1 \cup Y_2$. In particular, there is a unique maximal $\Phi$-periodic element of $S$ which we denote by $P_\Phi(Z)$. Notice that $P_\Phi(Z)$ contains all periodic irreducible components of $Z$, but it is possible for $P_\Phi(Z)$ to also contain some non-periodic irreducible components of $Z$ as well. We let $N_\Phi(Z)$ denote the union of the irreducible components of $Z$ that are not contained in $P_\Phi(Z)$.

For each $Z_i \subseteq N_\Phi(Z)$, the sequence $\dim \Phi^n(Z_i)$ is weakly decreasing and converges to some $d_i \in \mathbb{N}$ since $U$ has finite Krull dimension. Let

$$\nu_i := (d_i, \dim Z_i).$$
We put a strict total order \( \prec \) on \( (\mathbb{N} \cup \{-\infty\}) \times (\mathbb{N} \cup \{-\infty\}) \) by declaring \((a, b) \prec (a', b')\) if \(a < a'\), or if \(a = a'\) and \(b < b'\). The relations \(\leq, \succ,\) and \(\succeq\) are then defined in the natural way.

**Definition 3.8.** With notation as above, we define the stable non-periodic dimension \(\nu(Z)\) of \(Z\) to be the maximum \(\nu_i\) with respect to \(\prec\). If \(N_\Phi(Z)\) is empty, we define \(\nu(Z) = (-\infty, -\infty)\).

The following is the main technical result of this subsection.

**Proposition 3.9.** Let \(U\) be a Noetherian topological space of finite Krull dimension and \(\Phi: U \to U\) a continuous map. Suppose that \(T \subseteq \mathbb{N}\) has positive density and \(Z \subseteq U\) is a closed subset with \(N_\Phi(Z) \neq \emptyset\). Then there exist infinitely many \(j \in \Sigma(T)\) with \(\nu(Z) \succ \nu(Z \cap \Phi^{-j}(Z))\).

**Proof.** Let \(Z_1, \ldots, Z_r\) be the irreducible components of \(Z\). After relabelling, we may assume

\[
N_\Phi(Z) = Z_1 \cup \cdots \cup Z_s \quad \text{and} \quad P_\Phi(Z) = Z_{s+1} \cup \cdots \cup Z_r.
\]

Let \(L \in \mathbb{Z}^+\) such that \(\Phi^L(P_\Phi(Z)) \subseteq P_\Phi(Z)\). By Remark 3.3, there is some \(a \in \{0, 1, \ldots, L - 1\}\) such that \(T \cap (a +\mathbb{LN})\) has positive density. Replacing \(T\) by \(T \cap (a +\mathbb{LN})\), we can assume that all elements of \(\Sigma(T)\) are multiples of \(L\).

Fix \(m\) sufficiently large so that

\[
\dim \Phi^m(Z_i) = \lim_{n \to \infty} \dim \Phi^n(Z_i)
\]

for \(i \leq s\), and let \(\nu(Z) = (d, e)\). After relabelling, we may assume that there exist \(1 \leq \ell \leq t \leq s\) such that:

1. \(\dim \Phi^m(Z_i) = d\) and \(\dim Z_i = e\) for \(i \leq \ell\),
2. \(\dim \Phi^m(Z_i) = d\) and \(\dim Z_i < e\) for \(\ell < i \leq t\),
3. \(\dim \Phi^m(Z_i) < d\) for \(t < i \leq s\).

We first claim that for every \(j \in \Sigma(T)\), we have

\[
(3.1) \quad P_\Phi(Z \cap \Phi^{-j}(Z)) \supseteq P_\Phi(Z).
\]

To see this, first note that since \(j\) is a multiple of \(L\), we have \(Z \cap \Phi^{-j}(Z) \supseteq P_\Phi(Z)\). So, it remains to show that every irreducible component \(Z_i\) contained in \(P_\Phi(Z)\) is also an irreducible component of \(Z \cap \Phi^{-j}(Z)\). Since \(Z_i\) is irreducible, it is contained in some irreducible component \(Z_i'\) of \(Z \cap \Phi^{-j}(Z)\). Then \(Z_i \subseteq Z_i' \subseteq Z \cap \Phi^{-j}(Z) \subseteq Z\). As \(Z_i\) is already an irreducible component of \(Z\) and \(Z_i'\) is irreducible, it follows that \(Z_i = Z_i'\) is an irreducible component of \(Z \cap \Phi^{-j}(Z)\).

By (3.1), we necessarily have \(\nu(Z \cap \Phi^{-j}(Z)) \leq \nu(Z)\). Suppose that

\[
(3.2) \quad \nu(Z \cap \Phi^{-j}(Z)) = \nu(Z)
\]
for every sufficiently large \( j \in \Sigma(T) \). We shall derive a contradiction in the remainder of the proof.

We next claim that for every sufficiently large \( j \in \Sigma(T) \), there is some \( i \leq \ell \) such that

\[
Z_i \subseteq \Phi^{-j}(Z).
\]

If \( j \in \Sigma(T) \) is sufficiently large, then by (3.2), there is an irreducible component \( C \) of \( Z \cap \Phi^{-j}(Z) \) not contained in \( P_\Phi(Z \cap \Phi^{-j}(Z)) \) such that \( \dim C = e \) and \( \dim \Phi^m(C) \geq d \) for all \( n \geq 0 \). We have \( C \subseteq Z_i \cap \Phi^{-j}(Z) \) for some \( 1 \leq i \leq r \). By (3.1), we see \( C \not\subseteq P_\Phi(Z) \), and so \( Z_i \) is not an irreducible component contained in \( P_\Phi(Z) \), i.e., \( i \leq s \). Next observe that

\[
d \leq \dim \Phi^m(C) \leq \dim \Phi^m(Z_i \cap \Phi^{-j}(Z)) \leq \dim \Phi^m(Z_i),
\]

and so \( i \leq t \). Moreover, since \( C \subseteq Z_i \cap \Phi^{-j}(Z) \subseteq Z_i \), we see \( \dim Z_i \geq e \) and hence \( i \leq \ell \). By dimension contraints, \( C = Z_i \) which implies (3.3).

Since Proposition 3.6 shows that \( \Sigma(T) \) has positive density, by the subadditivity of natural density, there exists a fixed \( i \in \{1, \ldots, \ell\} \) and a positive density subset \( \Sigma_i \subseteq \Sigma(T) \) such that equation (3.3) holds for all \( j \in \Sigma_i \). Further refining, there exists \( k \in \{1, \ldots, r\} \) and a positive density subset \( \Sigma_{i,k} \subseteq \Sigma_i \) such that

\[
Z_i \subseteq \Phi^{-j}(Z_k)
\]

for all \( j \in \Sigma_{i,k} \).

We next show that \( k \leq s \). If this were not the case, then \( Z_k \subseteq P_\Phi(Z) \) and so \( \Phi^k(Z_i) \subseteq Z_k \subseteq P_\Phi(Z) \). In particular, since \( j \) is a multiple of \( L \), we have \( \Phi^j(Z_i \cup P_\Phi(Z)) \subseteq P_\Phi(Z) \subseteq Z_i \cup P_\Phi(Z) \). By maximality of \( P_\Phi(Z) \), it follows that \( P_\Phi(Z) = Z_i \cup P_\Phi(Z) \), and hence \( Z_i \subseteq P_\Phi(Z) \), a contradiction.

Since \( \Sigma_{i,k} \) is infinite, there exist \( a, b \in \Sigma_{i,k} \) with \( b - a, a > m \). We write \( b = a + Lc \) with \( c > 0 \). Since \( Z_i \subseteq \Phi^{-a}(Z_k) \), we have \( \Phi^{a+Lc}(Z_i) = \Phi^{Lc}(\Phi^a(Z_i)) \subseteq \Phi^{Lc}(Z_k) \), and hence \( \Phi^{a+Lc}(Z_i) \subseteq \Phi^{Lc}(Z_k) \). As \( a + Lc, Lc > m \), we see \( \dim \Phi^{Lc}(Z_k) \leq d = \dim \Phi^{a+Lc}(Z_i) \). Then by irreducibility of \( Z_k \), we have \( \Phi^{a+Lc}(Z_i) = \Phi^{Lc}(Z_k) \). On the other hand, \( b \in \Sigma_{i,k} \), so \( \Phi^{a+Lc}(Z_i) \subseteq Z_k \), which implies

\[
\Phi^{Lc}(Z_k) \subseteq \Phi^{Lc}(Z_k) = \Phi^{a+Lc}(Z_i) \subseteq Z_k.
\]

So, \( Z_k \) is periodic and hence contained in \( P_\Phi(Z) \), contradicting the fact that \( k \leq s \). \( \square \)

**Lemma 3.10.** Let \( U \) be a non-empty Noetherian topological space of Krull dimension \( d \). Suppose that

\[
Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m
\]

is a descending chain of non-periodic closed subsets of \( U \) such that \( \nu(Z_0) > \nu(Z_1) > \cdots > \nu(Z_m) \). Then \( m < (d + 1)^2 \).
Proof. We necessarily have \( \nu(Z_0) \leq (d, d) \). Write \( \nu(Z_i) = (d_i, e_i) \). Then by definition of \( \prec \), we have \( d \geq d_0 \geq d_1 \geq \cdots \geq d_m \). For \( s \in \{0, 1, \ldots, d\} \), let \( A_s = \{i : d_i = s\} \). Then \( A_s \) is an interval. Notice that if \( A_s = \{j, j+1, \ldots, j+\ell\} \), then since \( d_j = \cdots = d_{j+\ell} = s \), we must have \( e_j > e_{j+1} > \cdots > e_{j+\ell} \). Since \( e_j \leq d \), we see that \( \ell \leq d \). Hence \( |A_s| \leq d+1 \) for each \( s \in \{0, 1, 2, \ldots, d\} \). Then since \( \{0, 1, 2, \ldots, m\} = A_0 \cup A_1 \cup \cdots \cup A_d \), we see that \( m+1 \leq (d+1)^2 \), as required.

3.3. Finishing the proof of Theorem 1.4.

Proof of Theorem 1.4. We divide the proof into several steps.

Step 1. We shall start with the following set-up. Let

\[
U = X \setminus \bigcup_{n \in \mathbb{N}} (I_{\varphi^n} \cup I_{f \varphi^n}),
\]

which may not be open. We endow \( U \) with the subspace topology, thereby making it a Noetherian topological space. Clearly, \( U(\overline{\mathbb{Q}}) = \mathbb{X}_{\varphi,f}(\overline{\mathbb{Q}}) \). If \( U \) does not have any \( \overline{\mathbb{Q}} \)-points, then the theorem is vacuously true, so we assume that there is an \( x \in U(\overline{\mathbb{Q}}) \) such that \( f(\mathcal{O}_\varphi(x)) \) is infinite. We may also assume that \( X, \Phi, f \) and \( x \) are all defined over a fixed number field \( \mathbb{K} \). By construction, \( \Phi|_U \) is a regular self-map of \( U \) and \( f|_U : U \to \mathbb{P}^1 \) is regular; by abuse of notation, we denote these restriction maps by \( \Phi \) and \( f \), respectively. Finally, replacing \( U \) by the Zariski closure of the orbit \( \mathcal{O}_\varphi(x) \) in \( U \), we may assume \( \mathcal{O}_\varphi(x) \) is Zariski dense in \( U \); see also the end of the introduction of [BGS].

So, from now on, we may assume:

1. \( U \subseteq X \) is a Noetherian topological space;
2. \( \Phi \) is regular on \( U \) and \( \Phi(U) \subseteq U \);
3. \( f : U \to \mathbb{P}^1 \) is regular;
4. \( x \in U(\overline{\mathbb{Q}}) \) whose orbit \( \mathcal{O}_\varphi(x) \) is Zariski dense in \( U \);
5. \( f(\mathcal{O}_\varphi(x)) \) is infinite.

Step 2. Let \( T \subseteq \mathbb{N} \) be a subset of positive density, \( d = \dim(U \times U) \), and

\[
Z_0 = \{(u, v) \in U \times U : f(u) = f(v)\}.
\]

Note that \( Z_0 \) is a closed subset of \( U \times U \) since it is the inverse image of the diagonal \( \Delta_{\mathbb{P}^1} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \) under the product map \( (f, f) : U \times U \to \mathbb{P}^1 \times \mathbb{P}^1 \). Applying Proposition 3.9 to \( Z_0 \subseteq U \times U \), the product map \( (\Phi, \Phi) \), and \( T \), we see that there is some \( i_0 \in \Sigma(T) \) such that \( T_1 := T \cap (T + i_0) \) has positive density and \( Z_1 := Z_0 \cap (\Phi, \Phi)^{-i_0}(Z_0) \) satisfies \( \nu(Z_1) \prec \nu(Z_0) \). If \( Z_1 = P_{\varphi,\varphi}(Z_1) \) is periodic under \( (\Phi, \Phi) \), then let \( m = 1 \). Otherwise, applying Proposition 3.9 to \( Z_1 \) yields an element \( i_1 \in \Sigma(T_1) \subseteq \Sigma(T) \) with \( i_0 < i_1 \) such that
$T_2 := T_1 \cap (T_1 + i_1)$ has positive density and $Z_2 := Z_1 \cap (\Phi, \Phi)^{-i_1}(Z_1)$ satisfies $\nu(Z_2) \prec \nu(Z_1)$. Proceeding in this manner, we obtain a sequence of integers

$$0 < i_0 < i_1 < \cdots < i_m$$

and a descending chain of closed subsets $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m$ such that $\nu(Z_i) > \nu(Z_{i+1})$ and $Z_m = P(\Phi, \Phi)(Z_m)$, i.e., $Z_m$ is periodic. Furthermore, by construction, if

$$S := \left\{ \sum_{i \in I} i : I \subseteq \{i_0, \ldots, i_m\} \right\}$$

then

$$T' := \bigcap_{s \in S} (T + s) \subseteq T$$

has positive density,

$$Z_m = \bigcap_{s \in S} (\Phi, \Phi)^{-s}(Z_0)$$

and there is some $L \in \mathbb{Z}^+$ such that

$$(\Phi, \Phi)^L(Z_m) \subseteq Z_m.$$ 

Lastly, Lemma 3.10 implies

$$|S| \leq 2^{m+1} \leq 2^{(d+1)^2}.$$ 

Notice that $Z_m \neq \emptyset$ since the diagonal $\Delta_U \subseteq U \times U$ is contained in $Z_0$ and $(\Phi, \Phi)^{-n}(\Delta_U) \supseteq \Delta_U$ for every $n \geq 0$.

**Step 3.** By Schanuel’s Theorem 2.1, there exists a positive real number $\kappa > 0$ depending only on the number field $K$ such that for all sufficiently large $B$, we have

$$\# \{ y \in \mathbb{P}^1(K) : h(y) \leq \log B \} \leq B^\kappa.$$ 

Choose an $\epsilon$ independent of $T$ such that $0 < \epsilon < (2^{(d+1)^2+1}\kappa)^{-1}$. We shall prove that this choice of $\epsilon$ satisfies the conclusion of Theorem 1.4. Suppose to the contrary that

$$\limsup_{n \in T} \frac{h(\Phi^n(x)))}{\log n} \leq \epsilon.$$ 

In particular, there is a positive integer $N_0$ such that $h(\Phi^n(x))) \leq 2\epsilon \log n$ for all $n \in T$ with $n \geq N_0$.

First, by equation (3.4), we have

$$\# \{ f(\Phi^n(x)) \in \mathbb{P}^1(K) : n \in T, N_0 \leq n \leq N \} \leq N^{2\kappa}$$

for $N$ sufficiently large. Let

$$T'' = \bigcap_{s \in S} (T - s) \subseteq T.$$
Since \(d(T') > 0\) and \(T'' = T' - (i_0 + \cdots + i_m)\), we see \(d(T'') > 0\). By construction, for any \(j \in T''\), we have \(j + s \in T\) for every \(s \in S\). In particular, equation (3.5) implies

\[
\# \left\{ \left( f(\Phi^{j+s}(x)) \right)_{s \in S} : j \in T'', \ N_0 \leq j \leq N - (i_0 + \cdots + i_m) \right\} 
\leq \prod_{s \in S} \# \left\{ f(\Phi^{j+s}(x)) \in \mathbb{P}^1(\mathbb{K}) : j \in T'', \ N_0 - s \leq j \leq N - s \right\}
\leq \prod_{s \in S} \# \left\{ f(\Phi^{j+s}(x)) \in \mathbb{P}^1(\mathbb{K}) : j + s \in T, \ N_0 \leq j + s \leq N \right\}
\leq N^{2\nu\kappa |S|}.
\]

Equation (3.6)

Step 4. Let \(L\) be as in Step 2. Since \(d(T'' \cap [N_0, \infty)) = d(T'') > 0\), by subadditivity of natural density, there exists an integer \(a \in [0, L]\) such that \(\{j \in T'' : j \geq N_0, \ j \equiv a \ (mod \ L)\}\) has positive density. Then by the definition of natural density (see Definition 3.1), there exist a subsequence \((n_\ell)_{\ell \in \mathbb{Z}^+}\) of positive integers and a positive real number \(\delta > 0\), such that

\[
\# \left\{ j \in T'' : N_0 \leq j \leq n_\ell, \ j \equiv a \ (mod \ L) \right\} \geq \delta n_\ell
\]

for sufficiently large \(\ell\). Now, replacing \(\delta\) by a smaller positive number if necessary, we can further assume that

\[
\# \left\{ j \in T'' : N_0 \leq j \leq n_\ell - (i_0 + \cdots + i_m), \ j \equiv a \ (mod \ L) \right\} \geq \delta n_\ell,
\]

for sufficiently large \(\ell\).

Recall from Step 2 that \(|S| \leq 2^{m+1} \leq 2^{(d+1)^2}\) and hence \(2\kappa \cdot |S| < 1\). Therefore, we can choose \(\ell\) large enough such that \(\delta n_\ell > n_\ell^{2\nu\kappa |S|}\). Combining equation (3.7) with (3.6) where \(N = n_\ell\), a direct counting argument yields that there exist positive integers \(i, j \in T''\) with \(i < j\) such that

\[
f(\Phi^{j+s}(x)) = f(\Phi^{j+s}(x)) \text{ for all } s \in S \quad \text{and} \quad i \equiv j \ (mod \ L).
\]

Then by definition, \((\Phi^i(x), \Phi^j(x)) \in Z_m\) (see Step 2 for the construction and properties of \(Z_m\)). Since \((\Phi, \Phi)^L(Z_m) \subseteq Z_m\), we have \((\Phi^{kL+i}(x), \Phi^{kL+j}(x)) \in Z_m\) for every \(k \in \mathbb{N}\). As \(Z_m \subseteq Z_0\), the definition of \(Z_0\) yields

\[
f(\Phi^{kL+i}(x)) = f(\Phi^{kL+j}(x)).
\]

It thus follows from the fact that \(i \equiv j \ (mod \ L)\) that the sequence \(\{f(\Phi^{kL}(\Phi^i(x))) : k \in \mathbb{N}\}\) is periodic. In particular, the orbit \(O_{\Phi^L}(\Phi^i(x))\) of \(\Phi^i(x)\) under \(\Phi^L\) is contained in finitely many fibers \(F_1, \ldots, F_s\) of \(f\). Note that

\[
O_{\Phi}(\Phi^i(x)) = \bigcup_{t=0}^{L-1} O_{\Phi^L}(\Phi^{i+t}(x)) = \bigcup_{t=0}^{L-1} \Phi^t(O_{\Phi^L}(\Phi^i(x))) \subseteq \bigcup_{t=0}^{L-1} \Phi^t(F_1 \cup \cdots \cup F_s).
\]
Therefore, the full forward orbit $O_\Phi(x)$ is contained in some proper closed subset of $U$, which contradicts our assumption in Step 1 that $O_\Phi(x)$ is Zariski dense in $U$. We thus complete the proof of Theorem 1.4.

4. Applications of Theorems 1.2 and 1.4

4.1. Weak lim inf Height Gaps. Theorem 1.5, which asserts that Conjecture 1.3 holds away from a set of density zero, is an immediate consequence of Theorem 1.4.

Proof of Theorem 1.5. Suppose that $f(O_\Phi(x))$ is infinite. Let $\epsilon > 0$ be the positive real number as in Theorem 1.4 and

$$S := \left\{ n \in \mathbb{N} : \frac{h(f(\phi^n(x))))}{\log n} \leq \frac{\epsilon}{2} \right\}.$$

Then by Theorem 1.4, $S$ has density zero, which concludes the proof. \qed

4.2. Height gaps for $D$-finite power series. We apply Theorem 1.2 to obtain a simple proof of Theorem 1.6 recovering the univariate case of [BNZ, Theorem 1.3(c)].

Proof of Theorem 1.6. If $\sum_{n \geq 0} a_n z^n \in \mathbb{Q}[[z]]$ is a $D$-finite power series, then there is a rational self-map $\Phi: \mathbb{P}^d \to \mathbb{P}^d$ for some $d \geq 2$, a point $c \in \mathbb{P}^d(\mathbb{Q})$, and a rational map $f: \mathbb{P}^d \to \mathbb{P}^1$ such that $a_n = f(\Phi^n(c))$ for $n \gg 0$; see e.g., [BGT16, Section 3.2.1]. So, Theorem 1.2 immediately implies Theorem 1.6. \qed

4.3. Weak Dynamical Mordell–Lang. As mentioned before in the introduction, the lim inf Height Gap Conjecture 1.3 would imply the Dynamical Mordell–Lang Conjecture. Similarly, we deduce Theorem 1.7 as an application of Theorem 1.5.

Proof of Theorem 1.7. For any $n \in \mathbb{N}$, we denote by $Z_{\geq n}$ the Zariski closure of $\{\Phi^i(x) : i \geq n\}$ in $X$. Since $X$ is a Noetherian topological space, there is some $m \in \mathbb{N}$ such that $Z_{\geq n} = Z_{\geq m}$ for every $n \geq m$. Denote $Z_{\geq m}$ by $Z$ and $\Phi^m(x)$ by $x_1$. It then suffices to prove Theorem 1.7 for $(Z, \Phi|_Z, x_1, Y \cap Z)$.

Let $Z_1, \ldots, Z_r$ denote the irreducible components of $Z$ and let $Y_i := Y \cap Z_i$. Then $x_1 \in Z_i$ for some $i$. After relabeling, we may assume that $x_1 \in Z_1$. For each $i = 2, \ldots, r$, choose an arbitrary $x_i \in O_\Phi(x_1) \cap Z_i$; the intersection is non-empty since $O_\Phi(x_1)$ is dense in $Z$ by definition. We claim that $\Phi|_Z$ cyclically permutes the irreducible components $Z_i$ of $Z$. To see this, first note that $\Phi|_Z$ is a dominant rational self-map of $Z$, so it permutes the $Z_i$. Suppose that $(Z_{i_1}, \ldots, Z_{i_s})$ is a cycle under $\Phi|_Z$ with $1 \leq s \leq r$, and consider the forward orbit $O_\Phi(x_{i_1})$ of $x_{i_1} \in Z_{i_1}$. Clearly, the closure of $O_\Phi(x_{i_1})$ in $X$ is contained in the union of $Z_{i_1}, \ldots, Z_{i_s}$. On the other hand, $x_{i_1} = \Phi^n(x)$ for some $n \geq m$, and so the closure of $O_\Phi(x_{i_1})$ in $X$ is $Z_{\geq n} = Z$. It follows that $s = r$ and hence $\Phi|_Z$ is a cyclic permutation $(Z_{i_1}, \ldots, Z_{i_r})$. Hence $\Phi^s(Z_i) \subseteq Z_i$ for each $i$. Moreover, after relabeling, we may assume that $\Phi(Z_i) \subseteq Z_{i+1}$
for $i = 1, \ldots, r - 1$ and $\Phi(Z_r) \subseteq Z_1$. So, for $i = 2, \ldots, r$, our $x_i$ could be taken to be $\Phi^{i-1}(x_1)$. Therefore, by subadditivity of natural density, it suffices to show Theorem 1.7 for $(Z_i, \Phi^i|Z_i, x_i, Y_i)$ for each $i = 1, \ldots, r$.

We claim further that for each $i$, the forward orbit $O_{\Phi^r}(x_i)$ of $x_i \in Z_i$ under $\Phi^r$ is dense in $Z_i$. In fact, if we denote the irreducible decomposition of the closure of $O_{\Phi^r}(x_i)$ by $W_{i,1}, \ldots, W_{i,r_i}$, then

$$\bigcup_{i=1}^{r} Z_i = Z = \overline{O_{\Phi^r}(x_1)} = \bigcup_{i=1}^{r} \overline{O_{\Phi^r}(x_i)} = \bigcup_{i=1}^{r} \bigcup_{j=1}^{r_i} W_{i,j}.$$ 

Since $Z_i$ is irreducible, $Z_i \subseteq W_{k,j}$ for some $1 \leq k \leq r$ and $1 \leq j \leq r_k$. However, we note that $W_{k,j} \subseteq \overline{O_{\Phi^r}(x_k)} \subseteq Z_k$. As $Z_1, \ldots, Z_r$ are the irreducible components of $Z$, we must have $k = i$. The claim $O_{\Phi^r}(x_i) = Z_i$ thus follows.

We shall prove that either $O_{\Phi^r}(x_i) \subseteq Y_i$, or the set

$$A_i := \{ n \in \mathbb{N} : \Phi^{rn}(x_i) \in Y_i \}$$

has zero density, thereby proving Theorem 1.7 for $(Z_i, \Phi^r|Z_i, x_i, Y_i)$. If $Y_i = Z_i$ or $Y_i = \emptyset$, then the result is immediate. Thus we may assume, without loss of generality, that $Y_i$ is a non-empty proper subvariety of $Z_i$. We pick a non-constant morphism $f_i : Z_i \to \mathbb{P}^1$ such that $f_i(Y_i) = 1$; one can accomplish this by choosing a non-constant rational function $F_i$ vanishing on $Y_i$ and then letting $f_i := F_i + 1$. In particular, if $\Phi^{rn}(x_i) \in Y_i$, then $h(f_i(\Phi^{rn}(x_i))) = 0$. On the other hand, as $O_{\Phi^r}(x_i)$ is dense in $Z_i$, $f_i(O_{\Phi^r}(x_i))$ is necessarily infinite. So, by Theorem 1.5, there is a positive constant $C$ and a set $S \subset \mathbb{N}$ of zero density such that for any $n \in \mathbb{N} \setminus S$, the height of $f_i(\Phi^{rn}(x_i))$ is greater than $C \log n > 0$; in particular, for such an $n$, $\Phi^{rn}(x_i) \notin Y_i$. It follows that $A_i \subseteq S$ has zero density, as required. We hence complete the proof of Theorem 1.7.

\[ \square \]

5. Height gaps for commuting rational self-maps: Proof of Theorem 1.8

We begin this section by proving Theorem 1.8.

\textbf{Proof of Theorem 1.8.} We fix a number field $\mathbb{K}$ such that $X, \Phi_1, \ldots, \Phi_m, f$ and $x$ are defined over $\mathbb{K}$. We shall prove this theorem by induction on $m$. When $m = 1$, it reduces to Theorem 1.4. Let us assume $m \geq 2$. Suppose that Theorem 1.8 holds true for the semigroup $S_{m-1}$ generated by $\Phi_1, \ldots, \Phi_{m-1}$. Assume that $f(O_{\Phi_1,\ldots,\Phi_{m-1}}(x))$ is infinite. If $f(O_{\Phi_1,\ldots,\Phi_{m-1}}(x))$ is infinite, then by the induction hypothesis, we have

$$\limsup_{|n| \in T} \frac{h(f(\Phi^n(x))))}{\log |n|} \geq \limsup_{|n| \in T} \frac{h(f(\Phi^n(x))))}{\log |n|} > \epsilon_1,$$
where \( \epsilon_1 > 0 \) depends only on \( \mathbb{K} \) and \( X \). Thus we may assume that \( f(\mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x)) \) is finite.

As in Step 1 of the proof of Theorem 1.4, there exists a Noetherian topological space \( U \) such that \( U(\mathcal{O}) = X_{\Phi_1,\ldots,\Phi_{m-1}}(\overline{\mathcal{O}}) \), the \( \Phi_i \) are regular self-maps of \( U \), \( f: U \to \mathbb{P}^1 \) is also regular, and \( x \in U(\mathcal{O}) \). Now, let \( Z \) denote the Zariski closure of the orbit \( \mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x) \) of \( x \) under the semigroup \( S_{m-1} = \langle \Phi_1, \ldots, \Phi_{m-1} \rangle \) in \( U \), and let \( Z_1, \ldots, Z_r \) be the irreducible components of \( Z \). By assumption, \( \mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x) \) is contained in finitely many fibers of \( f: U \to \mathbb{P}^1 \), then so is its closure \( Z \). In particular, \( f \) is constant on each \( Z_i \). Note that

\[
\limsup_{\|n\| \to \infty} \frac{h(f(\phi^n(x)))}{\log \|n\|} = \limsup_{\|n\| \to \infty} \frac{h(f(\phi^n(x)))}{\log \|n\|} = \limsup_{n \in \mathbb{N}^m} \frac{h(f(\phi^n(z_i)))}{\log n_m} > \epsilon_2,
\]

where \( \epsilon_2 > 0 \) from Theorem 1.4 depends only on \( \mathbb{K} \) and \( X \).

Case 2. There is some fixed \( i_0 \) with \( 1 \leq i_0 \leq r \) and some fixed \( j_0 \geq 1 \) such that the map induced by \( f \) from the closure of \( \Phi_{m-1}^{j_0}(Z_{i_0}) \) to \( \mathbb{P}^1 \) is dominant. Note that

\[
\Phi_{m-1}^{j_0}(\mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x)) \supseteq \Phi_{m-1}^{j_0}(U_{i=1}^r Z_i) = \bigcup_{i=1}^r \Phi_{m-1}^{j_0}(Z_i).
\]

So the restriction of \( f \) to the closure of \( \Phi_{m-1}^{j_0} \circ S_{m-1}(x) = \Phi_{m-1}^{j_0}(\mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x)) \) in \( U \) is dominant. But \( \Phi_m \) commutes with \( S_{m-1} \) and so letting \( x_{j_0} = \Phi_{m-1}^{j_0}(x) \), we see that the restriction of \( f \) to the closure of \( \mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x_{j_0}) \) is dominant. In particular, \( f(\mathcal{O}_{\Phi_1,\ldots,\Phi_{m-1}}(x_{j_0})) \) is infinite. Now
by the induction hypothesis, we also get
\[
\limsup_{\|n\| \in T} \frac{h(f(\Phi^n(x)))}{\log \|n\|} \geq \limsup_{\|n\| \in T} \frac{h(f(\Phi^n(x)))}{\log \|n\|} = \limsup_{\|n\| \in T_{\neq j_0}} \frac{h(f(\Phi^n(x_{j_0})))}{\log \|n\|} > \epsilon_1.
\]

Set \(\epsilon := \min\{\epsilon_1, \epsilon_2\}\). We thus complete the proof of Theorem 1.8 by induction. \(\square\)

Remark 5.1. If we denote the corresponding constant in Theorem 1.4 for each \(\Phi_i\) by \(\epsilon_i\), then it follows readily from the proof that the \(\epsilon\) in Theorem 1.8 is the minimum of the \(\epsilon_i\).

The example below explains why we have to consider sets \(T = \{\|n\| : n \in T\}\) of positive density in Theorem 1.8, rather than sets \(T \subseteq \mathbb{N}^m\) of positive density.\(^2\)

Example 5.2. We define two self-maps \(\Phi_1\) and \(\Phi_2\) of \(X = \mathbb{A}^3\) as follows:
\[
\Phi_1(x, y, z) = (2x, y + 1, z) \quad \text{and} \quad \Phi_2(x, y, z) = (xz, y, z + 1).
\]

It is easy to verify that \(\Phi_1 \circ \Phi_2(x, y, z) = \Phi_2 \circ \Phi_1(x, y, z) = (2xz, y + 1, z + 1)\). Now let us consider the point \((1, 0, 0)\). First, for any \(n_2 \in \mathbb{Z}^+\) and \(n_1 \in \mathbb{N}\), we have \(\Phi_2^{n_2}(1, 0, 0) = (0, 0, n_2)\) and \(\Phi_1^{n_1}(0, 0, n_2) = (0, n_1, n_2)\). So,
\[
\Phi_1^{n_1, n_2}(1, 0, 0) = \begin{cases} (0, n_1, n_2) & \text{if } n_2 > 0, \\ (2^{n_1}, n_1, 0) & \text{if } n_2 = 0. \end{cases}
\]

Let \(f(x, y, z) = x\) be the projection to the \(x\)-coordinate. Then \(f(\Phi_1^{n_1, n_2}(1, 0, 0)) = 0\) if \(n_2 > 0\), and \(2^{n_1}\) if \(n_2 = 0\). It follows that if \(T \subseteq \mathbb{N}^2\), then
\[
\limsup_{(n_1, n_2) \in T} \frac{h(f(\Phi_1^{n_1, n_2}(1, 0, 0)))}{\log(n_1 + n_2)} > 0
\]
only if \(T\) contains infinitely many points from the ray \(R := \{(n_1, n_2) \in \mathbb{N}^2 : n_2 = 0\}\) which has density zero. On the other hand, the \(\liminf\) of the above quantity is zero as long as \(T\) has positive density. Moreover, the only set \(T\) over which the \(\liminf\) is positive is a subset of the above ray \(R\) plus finitely many points.

Before we move to a concluding remark on Theorem 1.8, let us recall the precise definition of multivariate \(D\)-finite power series due to Lipshitz.

\(^2\)Generalizing Definition 3.1, the upper asymptotic (or natural) density of \(T \subseteq \mathbb{N}^m\) is defined by \(d(T) := \limsup_{n \to \infty} |T \cap [0, n]^m|/(n + 1)^m\).
**Definition 5.3 ([Lip89]).** A formal power series \( F(z) = F(z_1, \ldots, z_m) \in \mathbb{C}[[z]] \) is said to be \( D \)-finite over \( \mathbb{C}(z) \), if the set of all derivatives \( (\partial/\partial z_1)^{i_1} \cdots (\partial/\partial z_m)^{i_m} \) with \( i_j \in \mathbb{N} \) span a finite-dimensional \( \mathbb{C}(z) \)-vector space. Equivalently, for each \( i \in \{1, \ldots, m\}, F(z) \) satisfies a nontrivial linear partial differential equation of the form

\[
\left\{ p_{i,n_1}(z) \left( \frac{\partial}{\partial z_1} \right)^{n_1} + p_{i,n_1-1}(z) \left( \frac{\partial}{\partial z_1} \right)^{n_1-1} + \cdots + p_{i,0}(z) \right\} F(z) = 0,
\]

where the \( p_{i,j}(z) \in \mathbb{C}(z) \).

At the end of the introduction, we mentioned that it appears to be a subtle issue to deduce a multivariate \( D \)-finiteness result (e.g., [BNZ, Theorem 1.3(c)]) from our Theorem 1.8. In the univariate case the coefficients of a \( D \)-finite power series arise as \( f(\Phi^n(c)) \) for certain choices of \( X, \Phi, f, \) and \( c \); see [BGT16, Section 3.2.1]. However, in Example 5.4 below, we construct a rational function in two variables whose coefficients never arise as \( f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c)) \) for any choices of \( X, \Phi_1, \Phi_2, f, \) and \( c \). It is well known that all algebraic functions are \( D \)-finite (see [Lip89, Proposition 2.3]).

**Example 5.4.** Let us consider the following rational function

\[
F(z_1, z_2) := \frac{1}{(1 - z_1 z_2)(1 - z_1)} = \sum_{n_2 \geq 0} \sum_{n_1 \geq n_2} z_1^{n_1} z_2^{n_2}.
\]

We let \( a_{n_1, n_2} = 1 \) if \( n_1 \geq n_2 \geq 0 \) and \( a_{n_1, n_2} = 0 \) if \( n_2 > n_1 \geq 0 \), so that

\[
F(z_1, z_2) = \sum_{n_1, n_2 \in \mathbb{N}} a_{n_1, n_2} z_1^{n_1} z_2^{n_2}.
\]

Our goal is to show that there is no choice of algebraic variety \( X \), commuting rational self-maps \( \Phi_1, \Phi_2: X \to X \), rational function \( f: X \to \mathbb{P}^1 \) all defined over \( \overline{\mathbb{Q}} \), and a point \( c \in X_{\Phi_1, \Phi_2, f}(\overline{\mathbb{Q}}) \) such that

\[
f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c)) = a_{n_1, n_2},
\]

for sufficiently large \( n_1 \) and \( n_2 \).

Suppose to the contrary that such choices exist. First, as in the proof of Theorem 1.8 or in Step 1 of the proof of Theorem 1.4, we can find a Noetherian topological space \( U \) such that \( \Phi_1 \) and \( \Phi_2 \) are continuous regular self-maps of \( U \), \( f: U \to \mathbb{P}^1 \) is regular and continuous, and \( c \in U(\overline{\mathbb{Q}}) \). Let \( Z_{i,j} \) denote the closure of the set of \( \{ \Phi_1^{n_1} \circ \Phi_2^{n_2}(c) : n_1 \geq i, n_2 \geq j \} \) in \( U \).

Then, as in the proof of Lemma 2.2, there is some \( N \) such that \( Z_{N,N} = Z_{i,j} \) for all \( i > N \) and \( j > N \). Let \( c' := \Phi_1^{N} \circ \Phi_2^{N}(c) \) and note that the orbit \( \mathcal{O}_{\Phi_1, \Phi_2}(c') \) is dense in \( Z := Z_{N,N} \).

We now choose a sufficiently large \( N \) such that

\[
f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c')) = 1 \text{ if } n_1 \geq n_2 \geq 0 \text{ and } 0 \text{ if } n_2 > n_1 \geq 0.
\]

Let \( T_1 \) (resp. \( T_0 \)) denote \( f^{-1}(1) \cap Z \) (resp. \( f^{-1}(0) \cap Z \)). Then \( c' \in T_1 \). Since the orbit of \( c' \) is dense in \( Z \) and also contained in the closed set \( T_0 \cup T_1 \), we have \( Z = T_0 \cup T_1 \).
Let $T_{0,1}, \ldots, T_{0,r}$ denote the irreducible components of $T_0$ and let $T_{1,1}, \ldots, T_{1,s}$ denote the irreducible components of $T_1$. Clearly, for each $T_{0,i}$, the intersection $\mathcal{O}_{\Phi_i}(c') \cap T_{0,i} \neq \emptyset$; pick up an arbitrary $c'_i$ in it. Then by (5.1) there exists some fixed $L_i \in \mathbb{Z}^+$ large enough such that $f(\Phi_{L_i}(c'_i)) = 1$, or $\Phi_{L_i}(c'_i) \in T_1$. It thus follows from the irreducibility of $T_{0,i}$ that $\Phi_{L_i}(c'_i) \subset T_1$. Let $L$ be the maximum of the finite $L_i$. By (5.1) again, we have $f(\Phi_{L+1}(c'_i)) = 0$ which yields that $\Phi_{L+1}(c'_i) \in T_{0,i}$ for some $i$. Hence, $\Phi_{L} \circ \Phi_{L+1}(c'_i) \in \Phi_{L}(T_{0,i}) \subset T_1$ by the choice of $L_i$. On the other hand, as $L_i < L + 1$, we have $\Phi_{L} \circ \Phi_{L+1}(c'_i) \in T_0$, which is absurd.

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