Computation of critical exponent $\eta$ at $O(1/N^3)$ in the four fermi model in arbitrary dimensions.

J.A. Gracey,
Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

**Abstract.** We solve the conformal bootstrap equations of the four fermi model or $O(N)$ Gross Neveu model to deduce the fermion anomalous dimension of the theory at $O(1/N^3)$ in arbitrary dimensions.
1 Introduction.

The observation that in the neighbourhood of a phase transition in a (renormalizable) quantum field theory there is a conformal or scaling symmetry has proved to be a useful technique in solving models within some expansion scheme, [1, 2, 3]. For instance, one can determine the critical exponents of all the Green’s functions and these govern the quantum properties of the field theory. Also, Polyakov demonstrated that the form of the propagators and the 3-point vertices are fixed up to constants by demanding that the theory is symmetric under the conformal group, [4]. Consequently, with this knowledge various authors formulated a method known as the conformal bootstrap to solve models such as $\phi^3$ theory in $d$-dimensions order by order in some perturbative coupling constant, [5, 6, 7]. The $d$-dimensional exponents could then be compared with the $\epsilon$-expansion of similar exponents in other models to ascertain whether they both lay in the same universality class, [7]. This is important since for physical systems it is useful to understand which class of theories underly and describe the experimental observations and subsequently allows one to make further predictions from the theoretical model. Following the earlier work of [8] the ideas surrounding the conformal bootstrap programme were extended to the $O(N)$ bosonic $\sigma$ model in [8] where $1/N$ replaced the perturbative coupling constant as the expansion parameter with $N$ large. The critical exponent $\eta$ of the fundamental field was deduced at $O(1/N^3)$ by solving a set of bootstrap equations which were derived by using Parisi’s method, [9], since the $\sigma$ model also involves a 3-point vertex. One considers the Dyson equations of the 2-point function and 3-vertex with dressed propagators and vertices. Indeed the $O(1/N^3)$ calculation of [8] built on earlier $O(1/N^2)$ work of [9, 10] where the model was solved completely at this order by similar conformal techniques. In particular, the skeleton Dyson equations for the fields were solved by considering, by contrast, the situation of dressed propagators but undressed vertices. It was the possibility of solving theories in the large $N$ method this way which suggested that the full conformal bootstrap could be exploited to probe the model further.

In this paper, we consider the $O(N)$ Gross Neveu model, [11], and develop the analogous formalism to deduce the anomalous dimension, $\eta$, of the fundamental fermion at $O(1/N^3)$. The viability of such a calculation was also suggested from the $O(1/N^2)$ work of [12] which used the same techniques of [9, 10] in solving the model at the $d$-dimensional critical point of the theory defined as the non-trivial zero of the $\beta$-function where one has a
conformal symmetry. Although the $O(N)$ Gross Neveu model describes the
dynamics of the four fermi interaction the fermionic nature of the fields does
not prevent one from using conformal methods and the four point interac-
tion can be rewritten as a 3-point vertex plus a bosonic auxiliary field which
means it has the same underlying $\phi^3$ type structure which is fundamental
in this area. Indeed one of the motivations for studying such models lies in
its relation to gauge theories. For instance, if we wish to probe four dimen-
sional theories with matter fields to high orders in large $N$ to discover the
structure of the quantum theory we must fully comprehend the way in which
fermions have to be dealt with in the formalism. The $O(N)$ Gross Neveu
model provides an excellent laboratory for testing such ideas since it has a
3-point interaction, in its auxiliary field formulation, which from the critical
point of view mimics the QED interaction. Essentially one is not obstructed
by the tedious complications which occur with the inclusion of a $\gamma$-matrix
at the vertex of integration. Moreover, an analysis of the four fermi model
is necessary and interesting in its own right. For example, there has been
a resurgence of study of the interaction through its relation to providing an
alternative mechanism of mass generation in the standard model. In [13, 14]
it was discussed how the effect of the Higgs boson could be replaced by a
composite bosonic field built out of the binding of two fermions. Further,
as our calculations will be in arbitrary dimensions we will be able to deduce
new and useful information on the structure of the three dimensional model
explicitly. This is currently of interest since four fermi interactions are be-
lieved to contribute to models describing high $T_c$ superconductivity. Thus
it is important to have the basic formalism in place in terms of the values
of the underlying three and four loop Feynman diagrams to ensure that one
can eventually extend the present work to gain predictions for the realistic
models which will involve coupling to a $U(1)$ gauge theory.

The aim therefore of this paper is to provide an extensive analysis of the
four fermi theory at $O(1/N^3)$ in arbitrary dimensions. In [15] the conformal
bootstrap equations which we will solve were derived and checked to ensure
that one could correctly recover the known $O(1/N^2)$ result of [12]. Its ex-
tension to the next order involves the tedious evaluation of several Feynman
diagrams. Unlike the $O(N)$ bosonic $\sigma$ model, [3], there are fewer graphs to
consider since graphs with fermion loops with an odd number of fermions
are trivially zero. However, the structure of fermionic massless Feynman
diagrams means that they can in several instances be much more difficult
to compute than their bosonic counterparts.

The paper is organised as follows. In section 2, we present the basic
background to the $O(N)$ Gross Neveu model and introduce the conformal structure of the fields of the model which are fundamental to the conformal bootstrap programme. The master equation whose solution will yield the critical exponent $\eta$ at $O(1/N^3)$ is derived from the three conformal bootstrap equations of the model in section 3 where we also introduce Polyakov’s conformal triangle approach to formulating the equations. In section 4, we discuss in detail the techniques required for the evaluation of the massless graphs contributing to the vertex function which contains contributions to $\eta$ at $O(1/N^3)$, in a particular limit of the two regularizing parameters which are present. The main result of the analysis is presented in section 5 together with several concluding remarks.

2 Review of the model.

In this section we recall previous conformal approaches to solving the $O(N)$ Gross Neveu model in the large $N$ approximation in arbitrary dimensions. First, we define the lagrangian we use as, \[ L = \frac{i}{2} \bar{\psi}^i \partial^\mu \psi^i + \frac{1}{2} \sigma \bar{\psi}^i \psi^i - \frac{\sigma^2}{2g} \] where $\psi^i$ is the fermion field, $1 \leq i \leq N$, and $1/N$ will be our expansion parameter. The bosonic field $\sigma$ is auxiliary and eliminating it through its equation of motion yields the four fermi interaction explicitly. It is more appropriate though to use the formulation (2.1) since the integration rules to compute massless Feynman integrals are easier to apply to the situation with 3-vertices. In the solution of the model in the usual large $N$ expansion \cite{1} the $\sigma$ field becomes dynamical in the true vacuum of the quantum theory, though classically it is non-propagating. The coupling constant, $g$, is dimensionless in two dimensions, where the model is asymptotically free, \cite{1}. Indeed the three loop $\beta$-function of (2.1) has been deduced perturbatively in $\overline{\text{MS}}$ using dimensional regularization and in $d$-dimensions it is, \cite{16, 17},

\[ \beta(g) = (d-2)g - (N-2)g^2 + (N-2)g^3 + \frac{1}{4}(N-2)(N-7)g^4 \] (2.2)

where the original two loop calculation was carried out in \cite{18}. The result (2.2) is what one obtains before setting $d = 2$ in minimal schemes and there is therefore no $d$-dependence in the higher order coefficients of the coupling constant. Indeed (2.2) is the starting point for analysing (2.1) in
the conformal approach in $d$-dimensions. For $d > 2$ one observes that there is a non-trivial zero of the $\beta$-function at

$$g_c \sim \frac{\epsilon}{(N - 2)}$$

(2.3)

to one loop where $d = 2 + \epsilon$. This corresponds to a phase transition in the theory which has been widely discussed and analysed in three dimensions in recent years, [19-23].

In statistical mechanics it is well known that in the neighbourhood of a phase transition physical systems exhibit special properties. For instance, various (measurable) quantities display a scaling behaviour in that they depend only on a characteristic length scale such as a correlation length raised to a certain power known as the critical exponent or index. This exponent totally characterizes the properties of the critical system. In a completely analogous way, in the neighbourhood of a phase transition of a (renormalizable) quantum field theory one observes that there is a conformal symmetry present. In other words Green’s functions are massless and obey a simple power law behaviour. Further, the exponent of certain Green’s functions such as the propagator can be related to the appropriate critical renormalization group function through a simple analysis of the renormalization group equation. (See, for example, [24].) Indeed the exponent will be a function of the spacetime dimension $d$ and any other internal parameters of the underlying field theory, which for (2.1) will be $N$. (At criticality the coupling constant is not independent but a function of $d$ and $N$.)

Therefore with these simple observations it was possible to solve (2.1) in a conformal approach which allowed one to go beyond the leading order since the masslessness of the problem simplifies the intractable integrals, [12], which would otherwise occur in the conventional large $N$ approach where the fields are massive. We now recall the essential features which are relevant for the $O(1/N^3)$ calculation we discuss here. First, since we will now work in the neighbourhood of $g_c$ given by (2.3) in $d > 2$ dimensions we write down the most general form of the propagators of the theory consistent with Lorentz and conformal symmetry as, [12].

$$\psi(x) \sim \frac{A f}{(x^2)^{\alpha}} , \quad \sigma(x) \sim \frac{B}{(x^2)^{\beta}}$$

(2.4)

as $x \to 0$ in coordinate space. We have chosen a propagating $\sigma$ field since we are solving the theory in the true vacuum. Whilst it is more convenient
to compute in $x$-space one can easily deduce the momentum space forms of (2.4) through the Fourier transform

$$\frac{1}{(x^2)\alpha} = \frac{a(\alpha)}{\pi^\mu 2^{2\alpha}} \int_{k} \frac{e^{ikx}}{(k^2)^{\mu-\alpha}}$$

valid for all $\alpha$ where we set the spacetime dimension $d$ to be $d = 2\mu$ for later convenience. The quantity $a(\alpha)$ is defined by $a(\alpha) = \Gamma(\mu-\alpha)/\Gamma(\alpha)$. In (2.4) the quantities $A$ and $B$ are the amplitudes of the respective fields and are $x$-independent, whilst we have defined the dimensions of the fields to be $\alpha$ and $\beta$. Their canonical dimensions can be determined by a dimensional analysis of the action which is dimensionless. However, quantum mechanically these dimensions do not remain as their engineering values due to quantum fluctuations such as radiative corrections. To allow for this scenario we define,

$$\alpha = \mu + \frac{1}{2} \eta, \quad \beta = 1 - \eta - 2\Delta \tag{2.6}$$

where $\eta$ is the anomalous dimension of the fermion and from an examination of the 3-vertex of (2.1), $2\Delta$ is its anomalous dimension. Both $\eta$ and $\Delta$ are $O(1/N)$ and depend on $d$ and $N$. In a critical point analysis of the renormalization group function, $\eta$ corresponds to the wave function renormalization and $\eta + 2\Delta$ to the mass anomalous dimension. They have both been determined at $O(1/N^2)$ in arbitrary dimensions as,

$$\eta_1 = \frac{2\Gamma(2\mu-1)(\mu-1)^2}{\Gamma(2-\mu)\Gamma(\mu+2)\Gamma(2\mu)} \tag{2.7}$$

$$\Delta_1 = \frac{\mu \eta}{2(\mu-1)} \tag{2.8}$$

$$\eta_2 = \frac{\eta_1^2}{2(\mu-1)^2} \left[ \frac{(\mu-1)^2}{\mu} + 3\mu + 4(\mu-1) + 2(\mu-1)/(2\mu-1) \Psi(\mu) \right] \tag{2.9}$$

$$\Delta_2 = \frac{\mu \eta_1^2}{2(\mu-1)^2} \left[ 3\mu(\mu-1)\Theta + (2\mu-1)\Psi - \frac{(2\mu-1)(\mu^2-\mu-1)}{(\mu-1)} \right] \tag{2.10}$$

where $\eta = \sum_{i=1}^{\infty} \eta_i/N^i$, $\Delta = \sum_{i=1}^{\infty} \Delta_i/N^i$, $\Psi(\mu) = \psi(2\mu-1) - \psi(1) + \psi(2-\mu) - \psi(\mu)$, $\Theta(\mu) = \psi'(\mu) - \psi'(1)$ and $\psi(x)$ is the logarithmic derivative of the $\Gamma$-function. The result (2.7) has been given originally in [26] whilst (2.8) has been determined independently in [21]. It is the aim of this paper to deduce $\eta_3$ in arbitrary dimensions. It is worth noting that the $O(1/N^2)$
correction to the critical exponent $\lambda$ which relates to the $\beta$-function of (2.1) has recently been determined too as, [13, 27],

$$\lambda_1 = -(2\mu - 1)\eta_1$$

(2.11)

$$\lambda_2 = \frac{2\mu\eta_1^2}{(\mu - 1)} \left[ \frac{2}{(\mu - 2)^2\eta_1} - \frac{(2\mu - 3)\mu}{(\mu - 2)}(\Phi + \Psi^2) \right]$$

$$+ \Psi \left( \frac{1}{(\mu - 2)^2} + \frac{1}{2(\mu - 2)} - 2\mu^2 - \frac{3}{2} - \frac{1}{2(\mu - 2)} \right)$$

$$+ \frac{3\mu \Theta}{4} \left[ 9 - 2\mu + \frac{6}{(\mu - 2)} \right] + 2\mu^2 - 5\mu - 3 + \frac{5}{4\mu}$$

$$- \frac{1}{4\mu^2} - \frac{7}{2(\mu - 1)} - \frac{1}{(\mu - 1)^2} + \frac{1}{4(\mu - 2)} - \frac{1}{2(\mu - 2)^2} \right]$$

(2.12)

where $\Phi = \psi'(2\mu - 1) - \psi'(2 - \mu) - \psi'(\mu) + \psi'(1)$ which means (2.1) has been solved completely at $O(1/N^2)$. It is important to recognise that these analytic expressions have been deduced merely from the ansätze (2.4) and the solution of the critical Schwinger Dyson equations, which therefore means that the method of [9, 10, 12] is an extremely powerful one.

With (2.4) it is possible to deduce the scaling behaviour of the respective 2-point functions which we will require here. Their coordinate space forms are deduced by first mapping (2.4) to momentum space via (2.5), inverting the propagator before applying the inverse map to coordinate space, [9]. Thus, we have, [12]

$$\psi^{-1}(x) \sim \frac{r(\alpha - 1)x}{A(x^2)^{2\mu - \alpha}} \ , \ \sigma^{-1}(x) \sim \frac{p(\beta)}{B(x^2)^{2\mu - \beta}}$$

(2.13)

where the functions have an analogous scaling structure but the amplitudes are non-trivially related. The functions $r(\alpha - 1)$ and $p(\beta)$ are defined as

$$p(\beta) = \frac{a(\beta - \mu)}{A(2\mu)(\beta)} \ , \ r(\alpha) = \frac{\alpha p(\alpha)}{(\mu - \alpha)}$$

(2.14)

It is the quantities (2.13) and (2.14) which will be necessary to solve the conformal bootstrap equations.

### 3 Conformal bootstrap equations.

In this section, we recall the basic bootstrap equations for the model which will then be solved at $O(1/N^2)$ for $\eta$ and which were given in [15] for (2.1).
The development of a bootstrap programme to solving field theories dates back to the early work of [4, 6]. Basically one simplifies the set of Feynman diagrams contributing to a Green’s function by first using dressed propagators. Further, each vertex of the Feynman graph is replaced by a conformal triangle which was discussed originally in [4] and later in [8]. To understand this concept we first have to introduce the technique known as uniqueness which was first discussed in [28] and is a fundamental tool for evaluating massless Feynman diagrams. The rule for a purely bosonic vertex was given in [28, 10] whilst the development to the style of vertex which arises in the model (2.1) was given in [12] and is illustrated in fig. 1. When the arbitrary exponents \( \beta_i \) are constrained to be their uniqueness value \( \sum_{i=1}^{3} \beta_i = 2\mu + 1 \) then one can compute the integral over the vertex of integration of the left side of fig. 1 to obtain the product of three propagators where \( \nu(\beta_1, \beta_2, \beta_3) = \pi^\mu \prod_{i=1}^{3} a(\beta_i) \). The origin of this uniqueness condition can be understood in several ways. First, if one carries out the explicit integral using Feynman parameters over the vertex of integration one finds that for general \( \beta_i \) the calculation cannot be completed due to the appearance of a hypergeometric function. The analysis can only proceed if one chooses the sum of the exponents to be constrained to \( 2\mu + 1 \), whence the hypergeometric function becomes a simple algebraic function. Alternatively one can make a conformal transformation on the integral representing the graph of the left side of fig. 1. If \( z \) is the location of the integration vertex and the origin is chosen to be at the external end of the bosonic line then one can make the following change of variables, [15],

\[
\frac{z}{z^2} \rightarrow \frac{z^\mu}{z^2} \quad (3.1)
\]

which represents a conformal transformation. The part of the numerator involving fermion propagators transforms as, say,

\[
(\not{x} - \not{z}) \rightarrow - \frac{\not{x}(\not{x} - \not{z})\not{z}}{x^2z^2} \quad (3.2)
\]

This results in a new integral of the same structure but where the exponents of the lines are related to the exponents of the original graph. In particular the exponent of the bosonic line becomes \( 2\mu + 1 - \sum \beta_i \). Therefore choosing this to be zero one obtains a simple chain integral and the result of fig. 1 emerges naturally. We have detailed this type of conformal change of variables (3.1) and (3.2) here since it is fundamental to the problem and is the starting technique to the computation of all our graphs.

With this basic rule one can define the conformal triangle construction for the vertex of the model. It is illustrated in fig. 2 for the \( \sigma \bar{\psi} \psi \) vertex of
where the internal exponents $a$ and $b$ are defined to be such that each vertex of the triangle is unique 

$$2a + \beta = a + \alpha + b = 2\mu + 1$$  \hspace{1cm} (3.3)$$

It is worth noting that the way the model is defined $2\alpha + \beta = 2\mu + 1 - 2\Delta$ so that the vertex of (2.1) is only unique at leading order in $1/N$, since $\Delta = O(1/N)$. The idea then of [3, 8] is to replace all vertices, which are therefore not conformal, by a triangle which after either conformal transformations or applying the rule of fig. 1 yields the original vertex, up to a (amplitude) factor. In the conformal bootstrap approach all vertices of a Feynman diagram are replaced by conformal triangles. In other words each vertex of this new graph is unique or conformal and it is therefore elementary to observe that such graphs are then proportional to the basic (bare) graph by using conformal transformations which in turn allows one to write down simple equations for the Dyson equations of the Green’s functions.

For example, the 3-point vertex of (2.1) is given by an infinite sum of graphs, where we will use $1/N$ as our ordering parameter, and we have illustrated the first few graphs in its expansion in fig. 3 where the labels $\Gamma_i$ will be used for later purposes. Each vertex of $\Gamma_i$ and the left side are replaced by the construction of fig. 2 and the values of the integral they correspond to will be a function of the exponents $\alpha$ and $\beta$ and the quantity $z$ defined as $z = fA^2B$ where $A$ and $B$ are our earlier amplitudes and $f$ is the amplitude of the triangle. These are the three basic variables of the formalism and to deduce information on each we need three bootstrap equations. The vertex expansion of fig. 3 provides the first. Since each graph of fig. 3 will be a function of $\alpha$, $\beta$ and $z$ and also by our arguments be proportional to the basic vertex, then if we denote by $V$ the sum of all contributions from the graphs of fig. 3 it must be unity. Thus the first bootstrap equation is, [8, 15],

$$V(z, \alpha, \beta; 0, 0) = 1$$  \hspace{1cm} (3.4)$$

where we have displayed the arguments of the vertex function explicitly. The origin of the final two arguments will become apparent later when we have to introduce two regularizing parameters. Thus (3.4) represents the sum of all the contributions of the graphs in fig. 3.

The remaining two bootstrap equations have been derived in [15] by following the analogous construction of [3] for a $\phi^3$ theory in arbitrary dimensions. Rather than repeat that derivation we will record their form and
then make several comments. We have, \[3.5\],

\[
r(\alpha - 1) = zt \frac{\partial}{\partial \epsilon} V(z, \alpha, \beta; \epsilon, \epsilon') \bigg|_{\epsilon = \epsilon' = 0}
\]

\[
p(\beta) = \frac{zt}{N} \frac{\partial}{\partial \epsilon} V(z, \alpha, \beta; \epsilon, \epsilon') \bigg|_{\epsilon = \epsilon' = 0}
\]

where

\[
t = \frac{\pi^4 a^2 (\alpha - 1) a^2 (a - 1) a(b) a(\beta)}{\Gamma(\mu)(\alpha - 1)^2 (a - 1)^2 a(\beta - b)}
\]

The quantities \(\epsilon\) and \(\epsilon'\) are infinitesimal regularizing parameters introduced into the formalism by shifting \(\alpha\) and \(\beta\) respectively by \(\alpha \to \alpha + 2\epsilon', \beta \to \beta + 2\epsilon\). Their introduction is necessary to avoid ill defined quantities in the derivation of the conformal bootstrap equations of the 2-point functions of (3.5) and (3.6). Basically a divergent integral arises which needs to be regularized and this is achieved by the above shift though the integral appears multiplied by either \(\epsilon\) or \(\epsilon'\) so that only the residue of the simple pole, ie \(t\), is required.

The effect of the introduction of the regularization is that the sum of graphs contributing to the vertex function \(V\) become additionally functions of \(\epsilon\) and \(\epsilon'\), ie \(V(\alpha, \beta, z; \epsilon, \epsilon')\). The graphs remain conformal because the internal exponents of the conformal triangle are adjusted (symmetrically) to preserve the uniqueness of each vertex. Thus with the regulators the vertex function is a sum of values which now depend on \(z, \alpha, \beta, \epsilon\) and \(\epsilon'\) and it is this function which appears in (3.4)-(3.6). Consequently, one has to compute the graphs of fig. 3 in the presence of the shift.

With (3.5) and (3.6), it is now possible to deduce a master equation which determines \(\eta_3\). First, we let \(V_1\) denote the contribution to the vertex function from the one loop graph of fig. 3 and \(V_2\) the higher order pieces which will be \(O(1/N)\) relative to \(V_1\). The former will later be expanded in powers of \(1/N\) too. Then taking the quotient of (3.5) and (3.6)

\[
\frac{N r(\alpha - 1)}{p(\beta)} = \left( \frac{\partial V}{\partial \epsilon} \right) \bigg/ \left( \frac{\partial V}{\partial \epsilon'} \right)
\]

On the left side of (3.8) we have a function of the exponents which can be expanded in powers of \(1/N\) and will involve the anomalous exponents \(\eta\) and \(2\Delta\). Expanding to \(O(1/N^2)\) the only unknown there is is \(\eta_3\) since \(\eta_1, \eta_2, \Delta_1\) and \(\Delta_2\) have already been determined. Thus to deduce \(\eta_3\) the right side of
(3.8) must be expanded to $O(1/N^2)$ and this will therefore involve the set of graphs contributing to $V_2$.

However, it is worth remarking on the functional structure of the graphs themselves. In representing each vertex by a conformal triangle one is replacing a vertex of the theory which is non-unique by a construction where each vertex is unique. The non-uniqueness of this is reflected in the form of the value of the conformal triangle, which can easily be deduced by the method of subtractions discussed in [10, 12]. In particular it will be of the form $1/\Delta$ where the residue is not important for the moment, [29]. In other words the variation from the value of the exponents which make the vertex conformal or unique appears as a pole in the value of the vertex. Therefore, a graph built out of $n$ conformal triangles will have the structure $h(\Delta)/\Delta^n$ where $h(\Delta)$ is an analytic function of $\Delta$. In the regularized theory the variation from the conformal value is $(\Delta - \epsilon)$ and $(\Delta - \epsilon')$ in the vertices which are regularized. Therefore the form of a graph contributing to the vertex functions $V_1$ and $V_2$ will become $h(\Delta, \epsilon, \epsilon'}/[\Delta^n - 2(\Delta - \epsilon)(\Delta - \epsilon')]$. It is this structure which we now consider in the context of (3.8). Differentiating with respect to either regularizing parameters involves differentiating the residue or the simple pole in $(\Delta - \epsilon)$ or $(\Delta - \epsilon')$. In the case of the latter the sum of all such contributions will be $V/(\Delta - \epsilon)$ or $V/(\Delta - \epsilon')$, whilst in the former case it will be the sum of the residue contributions. The upshot of this is that within (3.8) when $\epsilon$ and $\epsilon'$ are set to zero the contribution from the poles cancels and only the residue needs to be considered. In other words, the expansion of each of these quantities to $O(1/N^2)$ which is required for $\eta_3$. More concretely we have

$$ \frac{Nr(\alpha - 1)}{p(\beta)} = 1 + \frac{\Delta_1}{N} \left( \frac{\partial V_{10}}{\partial \epsilon'} - \frac{\partial V_{10}}{\partial \epsilon} \right) + \frac{1}{N^2} \left[ \Delta_2 \left( \frac{\partial V_{10}}{\partial \epsilon'} - \frac{\partial V_{10}}{\partial \epsilon} \right) \right] + \Delta_1 \left( \frac{\partial V_{11}}{\partial \epsilon'} - \frac{\partial V_{11}}{\partial \epsilon} \right) + \Delta_1 \left( \frac{\partial V_{11}}{\partial \epsilon'} - \frac{\partial V_{11}}{\partial \epsilon} \right) $$

$$ + \Delta_1 \left( \frac{\partial V_2}{\partial \epsilon'} - \frac{\partial V_2}{\partial \epsilon} \right) \right] \] (3.10)

where $V_1 = V_{10} + V_{11}/N$ and we now omit any comment on the evaluation symbol and $z_2$ is deduced from (3.4). Thus (3.10) forms the master equation.
to deduce $\eta_3$. It has a two part structure. First, one requires the leading order graph $V_1$ to be expanded to $O(1/N^2)$ and to be computed for non-zero $\epsilon$ and $\epsilon'$, whilst the graphs for $V_2$ must be calculated. In [12] which established the bootstrap equations the vertex function for $V_1$ was determined in order to check that the already known $O(1/N^2)$ results of [12] could be correctly determined. Moreover, the function was given at $O(1/N^2)$ for non-zero $\epsilon$ and $\epsilon'$ and we recall
\begin{equation}
\Gamma_1 = -\frac{Q^3}{\Delta(\Delta - \epsilon)(\Delta - \epsilon')} \exp[F(\epsilon, \epsilon', \Delta)] \tag{3.11}
\end{equation}
where
\begin{align}
F(\epsilon, \epsilon', \Delta) &= \left(5B_\beta - 2B_{\alpha-1} - 3B_0 - \frac{2}{\alpha - 1}\right) \Delta - (B_\beta - B_0)\epsilon \\
&+ \left(B_0 - B_{\alpha-1} - \frac{1}{\alpha - 1}\right) \epsilon' \\
&+ \left(C_{\alpha-1} - \frac{7C_\beta}{2} - \frac{3C_0}{2} - \frac{1}{(\alpha - 1)^2}\right) \Delta^2 \\
&+ \left(C_\beta + C_0 - 2C_{\alpha-1} + \frac{2}{(\alpha - 1)^2}\right) \Delta \epsilon \\
&+ \left(C_0 - C_\beta - 2C_{\alpha-1} + \frac{2}{(\alpha - 1)^2}\right) \Delta \epsilon' \tag{3.12}
\end{align}
with $B_x = \psi(\mu - x) + \psi(x)$, $B_0 = \psi(1) + \psi(\mu)$, $C_x = \psi'(x) - \psi'(\mu - x)$, $C_0 = \psi'(\mu) - \psi'(1)$ and
\begin{equation}
Q = -\frac{\pi^2 a^2(\alpha - 1)a(\beta)}{(\alpha - 1)^2 \Gamma(\mu)} \tag{3.13}
\end{equation}
We have illustrated the full $\Gamma_1$ graph in terms of conformal triangles in fig. 4 to show where the regulators appear in the appropriate exponents which is important for writing down the graphs of $V_2$. It is crucial to note that in (3.12) one should substitute $\alpha = \mu + \frac{1}{2}\eta$ and $\beta = 1 - \eta$ since the calculation of the graph was simplified by having the exponent $\Delta$ of the $\sigma$ line displayed explicitly. With (3.12) it is a trivial matter to deduce $\eta_2$ agrees with (2.9) where the variable $z$ is eliminated through
\begin{equation}
1 = -\frac{zQ^3}{\Delta^3} \tag{3.14}
\end{equation}
at leading order from (3.4).
4 Computation of higher order graphs.

All that remains is the evaluation of the higher order graphs which is far from a trivial exercise. The basic tools to compute the graphs are the conformal transformations (3.1) and (3.2) on each of the vertices of integration of the graph with the vertex joining to the external $\sigma$ line chosen to be the origin and the uniqueness rule of fig. 1 and its bosonic counterpart, [28, 10]. Whilst it is possible to compute each graph with these methods easily when $\epsilon = \epsilon' = 0$ which is required for the subsequent correction to (3.14), it turned out that it was not possible to evaluate the graphs exactly when $\epsilon \neq 0, \epsilon' \neq 0$. It transpired, however, that the difference, i.e.

$$\left. \frac{\partial \Gamma_i}{\partial \epsilon'} \right| - \left. \frac{\partial \Gamma_i}{\partial \epsilon} \right|$$

(4.1)

could be determined in each case, $i = 2, \ldots, 5$. There are four higher order graphs to consider. Whilst $\Gamma_4$ and $\Gamma_5$ are equivalent in the absence of the regulators, in the regularized version they are not equal since the bottom right external vertex of each $\Gamma_i$ is regularized. In the evaluation of $\Gamma_i$ we replace each unregularized conformal triangle with the original vertex and multiply the graph by $-Q/\Delta$ from (3.11), for each such vertex. It is only the regularized vertices which will give the contributions to the residue of $V_2$ which are relevant for $\eta_3$.

Before discussing the determination of $\Gamma_3$, $\Gamma_4$ and $\Gamma_5$ we detail the calculation of $\Gamma_2$ as it will illustrate some elementary steps which occur in all cases. Making a conformal transformation on $\Gamma_2$ in the manner indicated yields a three loop integral which is illustrated in fig. 5 and we have evaluated two elementary chain integrals en route whose integration rule has been given in [12], for example. However, this integral has two unique vertices which leads to the 2-loop integral of fig. 5 where we have now set $\Delta = 0$ since it does not contribute to the residue of $\Gamma_2$ at this order in $1/N$. (The poles in $\Delta$ emerge from factors such as $a(\mu - \Delta)$ which arise from a chain integral and the other from the integration of the two unique vertices.) In the notation of [27] it is $\langle \tilde{\alpha}, \beta - \epsilon', \alpha - \epsilon', \beta, \epsilon + \epsilon' \rangle$ but it cannot be evaluated by uniqueness methods for arbitrary $\epsilon$ and $\epsilon'$. However, if we recall that it is the difference of the derivatives of the residues which is relevant for $\eta_3$ we can make use of the observation that

$$\left( \frac{\partial}{\partial \epsilon'} - \frac{\partial}{\partial \epsilon} \right) \langle \tilde{\alpha}, \beta - \epsilon', \alpha - \epsilon', \beta, \epsilon + \epsilon' \rangle \bigg|_{\epsilon = \epsilon' = 0}$$

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The overall value of the contribution is then given by carrying out the differentiation with respect to the regularizing parameters on the functions obtained after the successive integrations, which is elementary. We find

\[
\frac{\partial \Gamma_2}{\partial \epsilon'} - \frac{\partial \Gamma_2}{\partial \epsilon} = - \frac{2Q^5\nu^2(\alpha - 1, \alpha - 1, \beta)}{(\alpha - 1)^4\Delta^5} \left[ B_\beta - B_{\alpha - 1} - \frac{1}{(\alpha - 1)} \right]
\]

(4.4)

Including the amplitude factor $\epsilon^2$ associated with the graph and using (3.14) we have

\[
\frac{\partial V_{2,2}}{\partial \epsilon'} - \frac{\partial V_{2,2}}{\partial \epsilon} = - \frac{\mu \eta_1}{(\mu - 1)^3N} \left[ (\mu - 1)(2\mu - 1) - (2\mu^2 - 5\mu + 4)\Psi - \frac{5(2\mu^2 - 5\mu + 4)}{2(\mu - 1)} \right]
\]

(4.5)

in an obvious notation where we have substituted for the leading order values of $\alpha$ and $\beta$ with respect to $1/N$.

The treatment of $\Gamma_3$ is equally as straightforward as that of $\Gamma_2$ and we note that

\[
\frac{\partial \Gamma_3}{\partial \epsilon'} - \frac{\partial \Gamma_3}{\partial \epsilon} = - \frac{\pi^4Q^7a(2\mu - 2)a(\mu - 1)}{\Delta^7(\mu - 1)^6} \left[ (\mu - 1)(2\mu - 1) - (2\mu^2 - 5\mu + 4)\Psi - \frac{5(2\mu^2 - 5\mu + 4)}{2(\mu - 1)} \right]
\]

(4.6)

from which we obtain

\[
\frac{\partial V_{2,3}}{\partial \epsilon'} - \frac{\partial V_{2,3}}{\partial \epsilon} = - \frac{\mu \eta_1}{(\mu - 1)^2N} \left[ (\mu - 1)(2\mu - 1) - (2\mu^2 - 5\mu + 4)\Psi - \frac{5(2\mu^2 - 5\mu + 4)}{2(\mu - 1)} \right]
\]

(4.7)

This calculation made use of the elementary chain integral

\[
\int_y \frac{(-\hat{y})(\hat{y} - \hat{x})}{(y^2)^\alpha((x - y)^2)^\beta} = \frac{[2(\mu - \alpha)(\mu - \beta) + 3\mu - \alpha - \beta]}{(\alpha - 1)(\beta - 1)} \times \nu(\alpha - 1, \beta - 1, 2\mu - \alpha - \beta + 2)
\]

(4.8)
The analysis of both $\Gamma_4$ and $\Gamma_5$ turned out to be extremely intricate and involved and it is worth discussing several of the intermediate steps in one due to the presence of a common difficult integral in each. We consider $\Gamma_4$ which after various transformations and elementary integrals leads us to consider the derivative with respect to $\delta$ of the two loop integral of fig. 6 where there is a trace over the lower right and one of the top fermion propagators. This is, in fact, a remnant of the fermion loop of the original graph. To evaluate this graph to $O(\delta)$ we restricted attention to the leading order values since that leads to an integral where most of the exponents are unity which is convenient for taking a fermion trace. For instance, taking the trace over the open fermion propagators and dividing by the appropriate factor $\text{tr}1$, fig. 6 is equivalent to

\[
\frac{1}{2}\text{tr}[\langle 1, 2, \tilde{1}, 1 - \delta, \mu - 1 \rangle - \langle 0, 2, \tilde{1}, 1 - \delta, \mu - 1 \rangle - \langle 1, \tilde{1}, \tilde{1}, 1 - \delta, \mu - 1 \rangle] \tag{4.9}
\]

The second term is trivial to deduce whilst the third becomes trivial after making the transformation $\leftarrow$ in the notation of [10]. The hardest part of the integral lurks within the first term, which after performing the trace explicitly yields one trivial graph and

\[
\langle 1, 2, 1, 1 - \delta, \mu - 2 \rangle - \langle 1, 1, 1 - \delta, \mu - 1 \rangle \tag{4.10}
\]

The $\delta$ expansion of an integral similar to the second term of (4.10) arises in the analogous $O(1/N^3)$ calculation in the pioneering work of [8]. Unfortunately it is not possible to give a closed form for the $O(\delta)$ correction in terms of elementary functions like $\Psi$ and $\Theta$ for all dimensions. Instead one is forced to leave the $O(\delta)$ correction defined as the quantity $I(\mu)$. Thus

\[
\langle \mu - 1, 1, 1 - \delta, \mu - 1, \mu - 1 + \delta \rangle = \text{ChT}(1, 1)[1 + \delta I(\mu) + O(\delta^2)] \tag{4.11}
\]

where we have used the result $\text{ChT}(1, 1 - \delta) = \text{ChT}(1, 1 + \delta)$ and the function $\text{ChT}(\alpha, \beta)$ is defined to be $\langle \alpha, \mu - 1, \mu - 1, \beta, \mu - 1 \rangle$ for all $\alpha$ and $\beta$ in the notation of [10] where it was evaluated exactly as [10]

\[
\text{ChT}(\alpha, \beta) = \frac{\pi^{2\mu}a(2\mu - 2)}{\Gamma(\mu - 1)} \left[ \frac{a(\alpha)a(2 - \alpha)}{(1 - \beta)(\alpha + \beta - 2)} + \frac{a(\beta)a(2 - \beta)}{(1 - \alpha)(\alpha + \beta - 2)} \right. \\
\left. + \frac{a(\alpha + \beta - 1)a(3 - \alpha - \beta)}{(\alpha - 1)(\beta - 1)} \right] \tag{4.12}
\]

Whilst a closed form for $I(\mu)$ is not available it can be analysed using the Gegenbauer polynomial techniques of [30] to obtain a set of double sums
over \( \Gamma \)-functions and its \( \epsilon \)-expansion near \( d = 2 + \epsilon \) can be given to several orders. Then with the definition

\[
I(\mu) = -\frac{2}{3(\mu - 1)} + \Xi(\mu)
\]  

we have

\[
\Xi(\mu) = \frac{\zeta(3)}{6} - \frac{\zeta(4)\epsilon^3}{8} + \frac{13\zeta(5)\epsilon^4}{48} + O(\epsilon^5)
\]

where the first few terms of (4.14) were given in \([10]\) and later ones in \([31, 32]\). It turns out that in three dimensions an exact expression can be deduced as \([10]\),

\[
I(\frac{3}{2}) = 2 \ln 2 + 3 \psi''(\frac{1}{2}) \quad (4.15)
\]

The remaining integral of (4.10) also contains \( I(\mu) \) which can be deduced by various transformations of bosonic two loop integrals given in \([10]\) and recursion relations of \([33]\). One result which was required in this and which is worth recording is

\[
\frac{\partial}{\partial \epsilon}\langle \mu - 1 + \epsilon, 2, 1, \mu - 1, \mu - 2 \rangle = \pi^{2\mu} a(1)a(2\mu - 2) \left[ \frac{(2\mu - 3)(\mu - 3)}{(\mu - 2)^2} - \frac{2(\mu - 1)}{(\mu - 2)} + \frac{(2\mu - 3)\Psi}{(\mu - 2)} \right] + \frac{(2\mu - 3)}{2(\mu - 1)(\mu - 2)} - \mu \left( \Theta + \frac{1}{(\mu - 1)^2} \right) - 3(\mu - 2)I(\mu) \left( \Theta + \frac{1}{(\mu - 1)^2} \right)
\]

Consequently we have that the coefficient of the \( O(\delta) \) term of (4.9) is

\[
- \frac{\pi^{2\mu} a(1)a(2\mu - 2)}{2(\mu - 1)} \left[ 3(\mu - 1)\Theta \Xi + \frac{3\Xi}{(\mu - 1)} + (\mu - 5)\Theta + \frac{2\mu \Psi}{(\mu - 1)} + \frac{(2\mu - 3)}{(\mu - 1)^2} \right]
\]

Collecting all the contributions to the integral and evaluating the derivative of the residue gives the relatively simple result

\[
\frac{\partial V_{2A}}{\partial \epsilon} - \frac{\partial V_{2A}}{\partial \epsilon} = \frac{\mu}{N} \left[ \frac{3\Theta \Xi}{2} + \frac{3\Xi}{2(\mu - 1)^2} + \frac{\mu \Psi}{(\mu - 1)^2} \right] + \frac{(\mu - 8)\Theta}{2(\mu - 1)} + \frac{(2\mu - 3)}{(\mu - 1)^3}
\]

(4.18)
The procedure for $\Gamma_5$ is equally as tedious but does not merit extensive coverage since the key techniques have already been covered in the discussion of $\Gamma_4$. A similar integral to fig. 6 occurs but this time the regulator is not on the line with exponent $\beta$ but on the other bosonic line appearing as $(\mu - \beta - \epsilon)$. Its $O(\epsilon)$ contribution is

$$- \frac{\pi^2 a(1)a(2\mu - 2)}{2(\mu - 1)} \left[ 3(\mu - 1)\Theta \Xi + \frac{3\Xi}{(\mu - 1)} - (3\mu + 2)\Theta + \frac{2\mu \Psi}{(\mu - 1)} \right]$$

(4.19)

as an intermediate check on the calculation. Overall we find

$$\left( \frac{\partial V_{2.5}}{\partial \epsilon'} - \frac{\partial V_{2.5}}{\partial \epsilon} \right) = \frac{\mu}{N} \left[ \frac{3\Theta \Xi}{2} + \frac{3\Xi}{2(\mu - 1)^2} + \frac{\mu \Psi}{(\mu - 1)^2} - \frac{4\Theta}{(\mu - 1)} - \frac{(2\mu - 5)(\mu - 2)}{2(\mu - 1)^3} \right]$$

(4.20)

This completes the discussion of the contributions to $V_2$. However, it is worth recording the values of each of the $\Gamma_i$ when $\epsilon = \epsilon' = 0$ since they are required for the higher order corrections to (3.14) which determines $z_2$. We found that

$$\Gamma_2 = - \frac{Q^5 \nu^2(\alpha - 1, \alpha - 1, \beta)}{\Delta^6} \left( \alpha - 1 \right)^4$$

(4.21)

$$\Gamma_3 = - \frac{Q^7 \nu^3(\alpha - 1, \alpha - 1, \beta)}{\Delta^2} \nu(\beta, \beta, 2\mu - 2\beta)|2(\mu - \beta)(\mu - \beta - 1) + \mu|$$

(4.22)

$$\Gamma_4 = \Gamma_5 = \frac{\pi^2 a(1)a(2\mu - 2)Q^7 \nu^2(\alpha - 1, \alpha - 1, \beta)}{2\Delta^7(\alpha - 1)^4} \left[ 3\Theta - \frac{(2\mu - 3)}{(\mu - 1)^2} \right]$$

(4.23)

5 Discussion.

With the explicit values of the $V_2$ contributions to (3.10) calculated it is now possible to deduce $\eta_3$ although the remainder of the calculation involves a substantial amount of tedious algebra. We find

$$\eta_3 = \eta_1^3 \left[ \mu \Theta \left( \frac{3\mu - 1}{4(\mu - 1)^2} + \frac{\mu(\mu - 16)}{4(\mu - 1)^2} - \frac{1}{4\mu} + \frac{6\mu}{(\mu - 1)^2} \right) + \frac{(2\mu - 1)^2 \Phi}{2(\mu - 1)^2} + \frac{3\mu^2 \Xi}{2(\mu - 1)^3} + \frac{3\mu^2 \Theta \Xi}{2(\mu - 1)} + \frac{3\mu^2 \Theta \Psi}{(\mu - 1)} + \frac{3(2\mu - 1)^2 \Psi^2}{2(\mu - 1)^2} \right]$$

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\[ + \Psi \left( \frac{\mu^3}{(\mu - 1)^3} - \frac{\mu(2\mu - 1)}{2(\mu - 1)^3} + \frac{\mu^2(2\mu^2 - 5\mu + 4)}{2(\mu - 1)^3} \right) - 2\mu - 3 + \frac{3}{2\mu} \]
\[ + \frac{35}{2(\mu - 1)} + \frac{18}{(\mu - 1)^2} + \frac{9}{2(\mu - 1)^3} + \frac{1}{2\mu^2} - 4 - \frac{3}{\mu} - \frac{7}{(\mu - 1)} \]
\[ + \frac{13}{2(\mu - 1)^2} + \frac{8}{(\mu - 1)^3} + \frac{2}{(\mu - 1)^4} - \frac{\mu^2(2\mu - 1)(\mu - 3)}{(\mu - 1)^3} \]

(5.1)

which is the main result of this paper and represents the first \(O(1/N^3)\) analysis in this model in arbitrary dimensions. (Previous \(O(1/N^3)\) analysis was strictly two dimensional and examined the corrections to the \(\sigma\) field mass, [34].) We can evaluate (5.1) in three dimensions to obtain

\[ \eta_3 = \frac{256}{27\pi^6} \left[ \frac{47\pi^2}{12} + 9\pi^2 \ln 2 - \frac{189\zeta(3)}{2} - \frac{167}{9} \right] \]

(5.2)

which has a similar structure to the analogous quantity in the \(O(N)\) bosonic \(\sigma\) model in terms of the appearance of numbers such as \(\ln 2\) and \(\zeta(3)\).

We conclude with several remarks. One important point of the analysis we have described here is the adaptation of conformal techniques and the conformal bootstrap programme to a model with fermions. Whilst the interaction is relatively simple it ought now to be possible to perform an analogous calculation in a gauge theory such as QED where the only essential difference is the appearance of a \(\gamma\)-matrix at each vertex. This is partly motivated by the fact that \(\eta_2\) could be deduced via the self consistency approach of [10] in [35] and it ought therefore to be possible to construct analogous bootstrap equations for that model to allow it to be probed as far as is analytically possible.

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**Note added.** Whilst in the final part of this work we received a preprint, [36], where \(\eta_3\) is also recorded and we note that both results are in agreement.
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Figure Captions.

Fig. 1. Uniqueness rule for a fermionic vertex.

Fig. 2. Conformal triangle.

Fig. 3. Expansion of 3-vertex.

Fig. 4. Regularized 3-vertex graph.

Fig. 5. Intermediate graphs in the evaluation of $\Gamma_2$.

Fig. 6. $\Gamma_4$ after several integrations.