Abstract

We propose a new massive integrable model in quantum field theory. This model is obtained as a perturbed model of the minimal conformal field theories on the hyper-elliptic surfaces by a particular relevant operator $V^{(t)}_{(1,1)}$. The non-local conserved charges of the model and their $q$-deformed algebra are also constructed explicitly.
1 Introduction

During the last years an essential progress has been achieved in the investigation of integrable quantum field theories. Such a success owes much to the fact that these models are characterized by infinite dimensional Hopf algebra symmetries, known as affine quantum group symmetries. These symmetries are generated by non-local conserved currents which in many cases can be constructed explicitly. Such an approach to the quantum field theory permits to obtain non-perturbative solutions in the quantum field theory using algebraic methods [1]–[5]. The situation is analogous to the one taking place in Conformal Field Theory (CFT). In particular, in CFT, as a result of the infinite-dimensional Virasoro algebra (or other extended algebras), exact solutions are successfully obtained with the help of the Ward identities [6].

Explicit currents that generate a $q$-deformation of affine Kac-Moody algebras [7], [8] were constructed for the Sine-Gordon theory and its generalization to imaginary coupling affine Toda theory in [3], and shown to completely characterize the $S$-matrices. At special values of the coupling where these quantum field theories have ordinary Lie group $G$ invariance, the quantum affine symmetry becomes the $G$-Yangian symmetry [2], [9].

The affine quantum group invariance fixes the $S$-matrices up to overall scalar factors, which in turn can be fixed using crossing symmetry, unitarity and analyticity. These quantum group invariant $S$-matrices, which are the specializations of the $R$-matrices satisfy the Yang-Baxter equation.

In the present work a series of new integrable models is identified and its $q$-deformed structure is studied. In particular, the organization of the paper is as follows. In section 2, a brief description of the minimal conformal models on hyper-elliptic surfaces which can be represented as two-sheet coverings of a ramified sphere is given. In section 3, a model of perturbed CFT is proposed; the relevant perturbation is the highest weight vector of the Virasoro algebra at the branching points. The characters of this model are calculated and the existence of an infinite series of Integrals of Motion (IMs) is proved; the integrability of the model is thus established. Furthermore, the $\beta$-function of the model is calculated and it is shown that the theory is massive. In the last section, section 4, the non-local currents are constructed. These are related by non-trivial braiding relations which lead to the $q$-deformed algebra of the conserved charges of the model.
2 CFT on Hyper-Elliptic Surfaces

Conformal field theories on compact Riemann surfaces, and in particular on hyper-elliptic surfaces, have been considered by many authors. One of the pioneering works on hyper-elliptic surfaces was Zamolodchikov’s work for the Ashkin-Teller models \([10]\); another important contribution was Knizhnik’s work \([11]\) on two-loop calculations in string theory. Finally, in \([12]\), the minimal models on hyper-elliptic surfaces were thoroughly discussed.

Let \( \Gamma \) be a compact Riemann surface of genus \( g \geq 1 \). If \( \Gamma \) is a Riemann surface of an algebraic function \( y = y(z) \) given by the equation

\[
R(y, z) = y^n + a_1(z)y^{n-1} + \ldots + a_n(z) = 0 ,
\]

(2.1)

where \( R(y, z) \) is a polynomial of the form shown above, then the affine part of \( \Gamma \) coincides with the complex algebraic curve \((1,1)\) in \( \mathbb{C}^2 \) in case this curve is ordinary (smooth). Of special importance to us is the example of hyper-elliptic curves given by equations of the form

\[
y^2 = P_{2g+1}(z) ,
\]

(2.2)

or

\[
y^2 = P_{2g+2}(z) ,
\]

(2.3)

where \( P_h(z), h = 2g + 1, 2g + 2, \) is a polynomial of degree \( h \) without roots of multiplicity \( h \). In both cases, the genus of the corresponding Riemann surface is \( g \). It is noteworthy that any Riemann surface of genus \( g = 1 \) or \( g = 2 \) has a representation in one of the forms \((2.2)\) or \((2.3)\), while the same statement is not true for surfaces of genus \( g = 3 \). We label the two sheets of the Riemann surface \( \Gamma \) by the numbers \( l = 0, 1; \)

\[
y^{(l)}(z) = e^{i\pi l} P^1_{h}(z) = e^{i\pi l} \prod_{i=1}^{h} (z - w_i)^{1/2} .
\]

(2.4)

Let \( A_a, B_a, a = 1, 2, \ldots, g \) be the basic cycles of the surface. As we encircle the point \( w_i \) along the contours \( A_a, B_a \), in the case of an \( A_a \) cycle we stay on the same sheet, while in the case of a \( B_a \) cycle we pass from the \( l \)-th sheet to the \( (l+1) \)-th one. We shall denote the process of encircling the points \( w_i \) on the cycles \( A_a, B_a \) by the symbols \( \hat{\pi}_{A_a}, \hat{\pi}_{B_a} \) respectively. Here these generators form a group of monodromy that in our case of two-sheet covering of the sphere coincides with the \( \mathbb{Z}_2 \) group.

We consider the energy-momentum tensor with representation \( T^{(l)}(z) \) on each of these sheets. The above definition of the monodromy properties along the cycles \( A_a, B_a \) implies that the following boundary conditions should be satisfied by the energy-momentum
tensor:

\[ \hat{\pi}_A T^{(l)} = T^{(l)}, \quad \hat{\pi}_B T^{(l)} = T^{(l+1)}. \]  

(2.5)

It is convenient to pass to a basis, in which the operators \(\hat{\pi}_A, \hat{\pi}_B\) are diagonal

\[ T = T^{(0)} + T^{(1)}, \quad T^- = T^{(0)} - T^{(1)}, \tag{2.6} \]

\[ \hat{\pi}_A T = T, \quad \hat{\pi}_A T^- = T^- \tag{2.7} \]

\[ \hat{\pi}_B T = T, \quad \hat{\pi}_B T^- = -T^- \tag{2.8} \]

The corresponding operator product expansions (OPEs) of the \(T, T^-\) fields can be determined by taking into account the OPEs of \(T^{(l)}\), \(T^{(l')}\). On the same sheet, the OPEs of the two fields \(T^{(l)}(z_1)T^{(l)}(z_2)\), are the same as that on the sphere, while on different sheets they do not correlate, i.e. \(T^{(l)}(z_1)T^{(l+1)}(z_2) \sim \text{reg} \). Thus, in the diagonal basis the OPEs can be found to be

\[ T(z_1)T(z_2) = \frac{c}{2z_{12}^4} + \frac{2 T(z_2)}{z_{12}^2} + \frac{T'(z_2)}{z_{12}} + \text{reg}, \]  

(2.9)

\[ T^-(z_1)T^-(z_2) = \frac{c}{2z_{12}^4} + \frac{2 T(z_2)}{z_{12}^2} + \frac{T'(z_2)}{z_{12}} + \text{reg}, \]  

(2.10)

\[ T(z_1)T^-(z_2) = \frac{2}{z_{12}^2} T^-(z_2) + \frac{T''(z_2)}{z_{12}} + \text{reg}, \]  

(2.11)

where \(c = 2\hat{c}\), and \(\hat{c}\) is the central charge in the OPE of \(T^{(l)}(z_1)T^{(l)}(z_2)\). It is seen from (2.11) that \(T^-\) is primary field with respect to \(T\). To write the algebra (2.9)-(2.11) in the graded form we determine the mode expansion of \(T\) and \(T^-\):

\[ T(z)V_{(k)}(0) = \sum_{n \in \mathbb{Z}} z^{n-2} L_{-n} V_{(k)}(0), \]  

(2.12)

\[ T^-(z)V_{(k)}(0) = \sum_{n \in \mathbb{Z}} z^{n-2-k/2} L^-_{-n-k/2} V_{(k)}(0), \]  

(2.13)

where \(k\) ranges over the values 0,1 and determines the parity sector in conformity with the boundary conditions (2.7) and (2.8). Standard calculations lead to the following algebra for the operators \(L_{-n}\) and \(L^-_{-n+k/2}\):

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{m,0}, \]  

\[ [L^-_{m+k/2}, L^-_{n+k/2}] = (m - n) L_{n+m+k} + \frac{c}{12} [(m + k/2)^3 - (m + k/2)] \delta_{n+m+k,0}, \]  

(2.14)

\[ [L_m, L^-_{n+k/2}] = [m - n - k/2] L^-_{n+m+k/2}. \]
The operators $L_n, L_{m+k/2}, L_n^{-}, L_{m+k/2}^{-}$ satisfy the same relations and $L_n, L_{m+k/2}, L_n^{-}, L_{m+k/2}^{-}$ commute with $L_n, L_{m+k/2}, L_n^{-}, L_{m+k/2}^{-}$.

To describe the representations of the algebra (2.14), it is necessary to consider separately the non-twisted sector with $k = 0$ and the twisted sector sector with $k = 1$. In order to write the $[V_{(k)}]$ representation of the algebra (2.14) in a more explicit form, it is convenient to consider the highest weight states. In the $k = 0$ sector, the highest weight state $|\Delta, \Delta^-\rangle$ is determined with the help of a primary field $V_{(0)}$ by means of the formula

$$|\Delta, \Delta^-\rangle = V_{(0)}|\emptyset\rangle .$$

(2.15)

Using the definition of vacuum, it is easy to see that

$$L_0|\Delta, \Delta^-\rangle = \Delta|\Delta, \Delta^-\rangle , \quad L_0^-|\Delta, \Delta^-\rangle = \Delta^-|\Delta, \Delta^-\rangle ,$$

$$L_n|\Delta, \Delta^-\rangle = L_n^-|\Delta, \Delta^-\rangle = 0 , \quad n \geq 1 .$$

(2.16)

In the $k = 1$ sector, we define the vector of highest weight $|\Delta\rangle$ of the algebra to be

$$|\Delta\rangle = V_{(1)}|\emptyset\rangle ,$$

(2.17)

where $V_{(1)}$ is a primary field with respect to $T$. In analogy with the non-twisted sector we obtain

$$L_0|\Delta\rangle = \Delta|\Delta\rangle , \quad L_0^-|\Delta\rangle = L_{n-1/2}^-|\Delta\rangle = 0 , \quad n \geq 1 .$$

(2.18)

Thus, the Verma module over the algebra (2.14) is obtained by the action of any number of $L_m$ and $L_{-m+k/2}^-$ operators with $n, m > 0$ on the states (2.13) and (2.17). As was shown in ref. [12] by means of GKO (coset construction) method, the central charge of a reducible unitary representation of the algebra (2.14) has the form

$$c = 2 - \frac{12}{p(p + 1)} = 2\hat{c} , \quad p = 3, 4, \ldots .$$

(2.19)

Using ref. [13], Dotsenko and Fateev [14] gave the complete solution for the minimal model correlation functions on the sphere. They were able to write down the integral representation for the conformal blocks of the chiral vertices in terms of the correlation functions of the vertex operators of a free bosonic scalar field $\Phi$ coupled to a background charge $\alpha_0$. This construction has become known as the Coulomb Gas Formalism (CGF). In the present case, this approach is also applicable by considering a Coulomb gas for
each sheet separately but coupled to the same background charge:

\[ T^{(l)} = -\frac{1}{4}(\partial_z \Phi^{(l)})^2 + i\alpha_0 \partial_z^2 \Phi^{(l)}, \quad (\Phi^{(l)}(z)\Phi^{(l)}(z')) = -\delta^{ll'} \ln |z - z'|^2, \]

where \( c = 2 - 24\alpha_0^2 \) or \( \alpha_0^2 = 1/2(p+1) \).

Passing to the basis which diagonalizes the operators \( \hat{\pi}_{A_a} \partial_z \Phi \), \( \hat{\pi}_{B_a} \partial_z \Phi \), i.e.

\[ \Phi = \Phi^{(0)} + \Phi^{(1)}, \quad \Phi^- = \Phi^{(0)} - \Phi^{(1)}, \]

\[ \hat{\pi}_{A_a} \partial_z \Phi = \partial_z \Phi, \quad \hat{\pi}_{B_a} \partial_z \Phi = -\partial_z \Phi, \]

we finally obtain the bosonization rule for the operators \( T \), \( T^- \) in the diagonal basis

\[ T = -\frac{1}{4}(\partial_z \Phi)^2 + i\alpha_0 \partial_z^2 \Phi - \frac{1}{4}(\partial_z \Phi^-)^2, \]

\[ T^- = -\frac{1}{2}\partial_z \Phi \partial_z \Phi^- + i\alpha_0 \partial_z^2 \Phi^- . \]

In conventions of ref. [12], the vertex operator with charges \( \alpha, \beta \) in the \( k = 0 \) (non-twisted) sector is given by

\[ V_{\alpha\beta}(z) = e^{i\alpha \Phi + i\beta \Phi^-}, \]

with conformal weights \( \Delta = \alpha^2 - 2\alpha_0 \alpha + \beta^2 \) and \( \Delta^- = 2\alpha \beta - 2\alpha_0 \beta \).

In the \( k = 1 \) (twisted) sector the situation is slightly different. Here we have an antiperiodic bosonic field \( \Phi^- \), i.e. \( \Phi^-(e^{2\pi i}z) = -\Phi^- \); this leads to the deformation of the geometry of space-time. If we recall that the circle is parametrized by \( \Phi^- \in S^1[0, 2\pi R] \), the condition \( \Phi^- \sim -\Phi^- \) means that pairs of points of \( S^1 \) have been identified. Thus, \( \Phi^- \) lives on the orbifold \( S^1/\mathbb{Z}_2 \); under the identification \( \Phi^- \sim -\Phi^- \) the two points \( \Phi^- = 0 \) and \( \Phi^- = \frac{1}{2}(2\pi R) \) are fixed points. One can try to define the twist fields \( \sigma_\epsilon(z), \epsilon = 0, 1 \), for the bosonic field \( \Phi^- \), with respect to which \( \Phi^- \) is antiperiodic. Notice that there is a separate twist field for each fixed point. The OPE of the current \( I^- = i\partial_z \Phi^- \) with the field \( \sigma_\epsilon \) is then

\[ I^-(z)\sigma_\epsilon(0) = \frac{1}{2}z^{-1/2}\sigma_\epsilon(0) + \ldots, \]

\[ I^-(z)\sigma'_\epsilon(0) = \frac{1}{2}z^{-3/2}\sigma_\epsilon(0) + 2z^{-1/2}\sigma'_\epsilon(0) + \ldots. \]
The twist fields $\sigma_\epsilon$ and $\hat{\sigma}_\epsilon$ are primary fields for the $T_{\text{orb}} = -\frac{1}{4}(\partial_z \Phi^-)^2$ with dimensions $\Delta_\epsilon = 1/16$ and $\hat{\Delta}_\epsilon = 9/16$ respectively. So, in the twisted sector the highest weight vectors (or primary fields) can be written as follows

$$V_{\gamma \epsilon}(t) = e^{i\gamma \Phi} \sigma_\epsilon, \quad \Delta^{(t)} = \gamma^2 - 2\alpha_0 \gamma + \frac{1}{16}. \quad (2.25)$$

In ref. [12], the anomalous dimensions of the primary fields of the minimal models for the algebra (2.14) were obtained both in the non-twisted and twisted sectors in conformity with the spectrum of the central charge (2.19); in particular, it was found that the charges $\alpha, \beta, \gamma$ of the primary fields corresponding to $k = 0$ and $k = 1$ sectors have the form:

$$\alpha_{n,m'}^{nm} = \frac{2-n-n'}{2} \alpha_+ + \frac{2-m-m'}{2} \alpha_-, \quad \beta_{n,m'}^{nm} = \frac{n-n'}{2} \alpha_+ + \frac{m-m'}{2} \alpha_-, \quad \gamma_{nm} = \frac{2-n}{2} \alpha_+ + \frac{2-m}{2} \alpha_- \quad (2.26)$$

where the constants $\alpha_\pm$ are expressed in terms of the background charge $\alpha_0$:

$$\alpha_\pm = \alpha_0/2 \pm \sqrt{\alpha_0^2/4 + 1/2}. \quad (2.27)$$

We denote the corresponding fields by $V_{n'm'}^{nm}$, $V^{(t)}_{nm}$ and their conformal weights by $\Delta_{nm}^{n'm'}$, $\Delta_{nm}^{(t)}$.

We can thus represent the CFT on a hyper-elliptic surface as a CFT on the plane with an additional symmetry, exactly as described by the algebra (2.14). The corresponding highest weight vectors of the algebra are given by (2.23) and (2.25); finally, the central charge is given by (2.19).

We will confine ourselves to the minimal models on hyper-elliptic surfaces as presented above; keeping this in mind we pass to the construction of perturbed models of these CFTs.

### 3 Perturbation by $V^{(t)}_{nm}$ and Integrals of Motion

Let $S_p$ be the action the $p$-th conformal minimal model on the hyper-elliptic surface $\Gamma$

$$S_p[\Phi, \Phi^-] \sim \int d^2z \left( \partial_z \Phi \partial_{\bar{z}} \Phi^- - i\alpha_0 R \Phi \right) + \int d^2z \partial_z \Phi^- \partial_{\bar{z}} \Phi^- \quad (3.1)$$
We now consider the perturbation of this conformal field theory by the degenerate relevant operator $V_{nm}^{(t)}$.

$$S_\lambda = S_\mu[\Phi, \Phi^+] + \lambda \int d^2 z e^{i\gamma_{nm}(z,\bar{z})} \sigma(z,\bar{z}).$$ \hspace{1cm} (3.2)

The parameter $\lambda$ is a coupling constant with conformal weight $(1 - \Delta_{nm}^{(t)}, 1 - \Delta_{nm}^{(t)})$.

Obviously, for a generic perturbation the new action $S_\lambda$ does not describe an integrable model. We are going to choose the perturbation in a way that the corresponding field theory is integrable. To prove the integrability of this massive (this claim is proved at the end of the present section) theory, one must calculate the characters of the modules of the identity $I$ and $V_{nm}^{(t)}$.

The “basic” currents $T(z)$ and $T^-(z)$ generate an infinite-dimensional vector subspace $\Lambda$ in the representation space. This subspace can be constructed by successive applications of the generators $L_{-n}$ and $L_{-m}$ with $n, m > 0$ to the identity operator $I$. $\Lambda$ can be decomposed to a direct sum of eigenspaces of $L_0$.

$$\Lambda = \bigoplus_{s=0}^\infty \Lambda_s, \quad L_0 \Lambda_s = s \Lambda_s.$$ \hspace{1cm} (3.3)

The space $\Lambda$ contains the subspace $\Lambda' = \partial_2 \Lambda$. Therefore, in order to separate the maximal linearly independent set, one must take the factor space $\hat{\Lambda} = \Lambda / (L_{-1} \Lambda \oplus L_{-2} \Lambda)$ instead of $\Lambda$. The space $\hat{\Lambda}$ admits a similar decomposition as a direct sum of eigenspaces of $L_0$.

It follows that the formula of the character for $\hat{\Lambda}$ takes the form

$$\chi_0 = (1 - q)^2 \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^2}.$$ \hspace{1cm} (3.4)

The dimensionalities of the subspaces $\hat{\Lambda}_s$ can be determined from the character formula

$$\sum_{s=0}^\infty q^s \dim(\hat{\Lambda}_s) = (1 - q) \chi_0 + q.$$ \hspace{1cm} (3.5)

On the other hand, the module $V$ of the primary field $V_{nm}^{(t)}$ can be constructed by successively applying the generators $L_{-k}$ and $L_{1/2-l}$ with $k, l > 0$ to the primary field $V_{nm}^{(t)}$. This space $V$ and the corresponding factor space $\hat{V} = V/L_{-1}V$ may also be decomposed in a direct sum of $L_0$ eigenspaces:

$$V = \bigoplus_{s=0}^\infty V_s^{(t)}, \quad L_0 V_s^{(t)} = s V_s^{(t)}.$$ \hspace{1cm} (3.6)
The dimensionalities of $V_s(t)$ in this factor space associated with the relevant field

$$V_{(1,1)}^{(t)} = e^{i \frac{\alpha_0}{2} \Phi} \sigma_{\epsilon}$$

are given by the character formula

$$\sum_{s=N/2}^{+\infty} q^{s+\Delta_{(1,1)}^{(t)}} \dim(\hat{V}_s^{(t)}) = \chi_{\Delta_{(1,1)}^{(t)}}(1 - q),$$

where

$$\chi_{\Delta_{(1,1)}^{(t)}} = q^{\Delta_{(1,1)}^{(t)}} \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)(1 - q^{n-1/2})},$$

$$\Delta_{(1,1)}^{(t)} = \frac{1}{16} \left( 1 - \frac{6}{p(p+1)} \right).$$

When the dimensionalities of $\hat{V}_s^{(t)}$ (calculated from (3.8), (3.9)) are compared to those of $\hat{\Lambda}_{s+1}$, we see that for $s = 1, 3, 5, \ldots$ the $\dim(\hat{\Lambda}_{s+1})$ exceeds the $\dim(\hat{V}_s^{(t)})$ at least by the unity, i.e. $\dim(\hat{\Lambda}_{s+1}) > \dim(\hat{V}_s^{(t)}), s = 1, 3, 5, \ldots$. This proves that the model

$$S_\lambda = S_p + \lambda \int d^2 z e^{i \frac{\alpha_0}{2} \Phi(z, \overline{z})} \sigma_{\epsilon}(z, \overline{z})$$

possesses an infinite set of non-trivial IMs. We note here that there are no such IMs for perturbations by the operators $V_{nm}^{(t)}$ with $n, m > 1$.

We now briefly study the renormalization group flow behaviour in the vicinity of the fixed point.

Solving the Callan-Symanzik equation [13] up to third order, one can obtain the $\beta$-function

$$\beta = \epsilon g \left( 1 + \frac{Y}{6} g^2 \right) + O(g^4).$$

In the above equation, we have denoted

$$\epsilon = 1 - \Delta_{(1,1)}^{(t)}$$

and

$$Y = \int d^2 z_1 \int d^2 z_2 \langle V_{(1,1)}^{(t)}(z_1, \overline{z}_1) V_{(1,1)}^{(t)}(z_2, \overline{z}_2) V_{(1,1)}^{(t)}(1, 1) V_{(1,1)}^{(t)}(0, 0) \rangle.$$  

Since $Y > 0$, we conclude that there is no reason to expect the existance of any non-trivial zeros of the $\beta$-function. In the absence of zeros, the field theory described by the action (3.11) has a finite correlation length $R_c \sim \lambda^{-1/2\epsilon}$ and the spectrum consists of particles with non-zero mass of order $m \sim R_c^{-1}$. In this case, the IMs force the scattering of the particles to be factorizable, i.e. there is particle production, the set of particle momenta is preserved, the $n$-particle $S$-matrix is a product of 2-particle $S$-matrices etc.
4 Infinite Quantum Group Symmetry

In this section we briefly review the method developed in ref. [3] and then we apply it to our model.

We consider a CFT perturbed by a relevant operator with zero Lorentz spin. The Euclidean action is given by

$$S_\lambda = S_{\text{CFT}} + \frac{\lambda}{2\pi} \int d^2z V_{\text{pert}}(z, \overline{z}) ,$$

(4.1)

where the perturbation field can be written as $V_{\text{pert}}(z, \overline{z}) = V_{\text{pert}}(z)\overline{V_{\text{pert}}(\overline{z})}$ (or a sum of such terms but in our case this is irrelevant). Let us assume that for the conformal invariant action $S_{\text{CFT}}$ there exist the chiral currents $J(z)$, $\overline{J}(\overline{z})$ satisfying equations $\partial_z J(z) = 0$, $\partial_{\overline{z}} \overline{J}(\overline{z}) = 0$. Then for the action (4.1) $S_\lambda$, the perturbed currents, which are local with respect to the perturbing field, up to the first order, are given by Zamolodchikov’s equations [16]

$$\partial_z J(z, \overline{z}) = \lambda f_z \oint \frac{d\omega}{2\pi i} V_{\text{pert}}(\omega, \overline{z}) J(z) ,$$
$$\partial_{\overline{z}} \overline{J}(z, \overline{z}) = \lambda f_{\overline{z}} \oint \frac{d\omega}{2\pi i} V_{\text{pert}}(z, \omega) \overline{J}(\overline{z}) .$$

(4.2)

The condition for the conservation of the currents up to first order in perturbation theory is that the residues of OPEs appearing in the above contour integrals are total derivatives:

$$\text{Res}\left(V_{\text{pert}}(\omega)J(z)\right) = \partial_z h(z) ,$$
$$\text{Res}\left(\overline{V}_{\text{pert}}(\omega)\overline{J}(\overline{z})\right) = \partial_{\overline{z}} \overline{h}(\overline{z}) .$$

(4.3)

Then Zamolodchikov’s equations for the currents are written in the form

$$\partial_z J(z, \overline{z}) = \partial_z H(z, \overline{z}) ,$$
$$\partial_{\overline{z}} \overline{J}(z, \overline{z}) = \partial_{\overline{z}} \overline{H}(z, \overline{z}) ,$$

(4.4)

where the fields $H$, $\overline{H}$ are

$$H(z, \overline{z}) = \lambda [h(z)\overline{V}_{\text{pert}}(\overline{z}) + \ldots] ,$$
$$\overline{H}(z, \overline{z}) = \lambda [V_{\text{pert}}(z)\overline{h}(\overline{z}) + \ldots] .$$

(4.5)
where the dots represent contributions coming from terms in the OPEs which are more singular than the residue term. The conserved charges following from the conserved currents (4.4) are

\[ Q = \int \frac{dz}{2\pi i} J + \int \frac{d\bar{z}}{2\pi i} H, \]

\[ \overline{Q} = \int \frac{d\bar{z}}{2\pi i} \overline{J} + \int \frac{dz}{2\pi i} \overline{H}. \]

(4.6)

Using the non-trivial braiding relations between the conserved currents, one can obtain the \( q \)-deformed affine Lie algebra for the conserved charges (4.6).

We are now going to implement the above construction of non-local charges for the theory described by the action (3.11). We will thus derive the \( q \)-deformed Lie algebra underlying the theory. Using the construction explained above, we can show that the action (3.11) admits the following non-local conserved quantum currents:

\[ \partial \bar{z} J = \partial z \overline{H}, \]

\[ \partial z \overline{J} = \partial \bar{z} H, \]

(4.7)

where

\[ J = e^{i a \varphi(z)} e^{i b \varphi^-(z)} \sigma(z), \]

\[ \overline{J} = e^{i a \overline{\varphi}(\bar{z})} e^{i b \overline{\varphi}^-(\bar{z})} \overline{\sigma(\bar{z})}, \]

(4.8)

\[ H(z, \bar{z}) = \lambda A : e^{i(a + \alpha_0/2) \varphi(z)} e^{i(b + k) \varphi^-(z)} \sigma(z) e^{i\alpha_0 \overline{\varphi}(\bar{z})}:, \]

\[ \overline{H}(z, \bar{z}) = \lambda A : e^{i(a + \alpha_0/2) \overline{\varphi}(\bar{z})} e^{i(b + k) \overline{\varphi}^-(\bar{z})} \overline{\sigma(\bar{z})} e^{i\alpha_0 \varphi(z)}:, \]

and

\[ a = -(15/8 + k^2)/(\alpha_0 + 4k^2/\alpha_0), \]

\[ b = 2ka/\alpha_0, \]

\[ A = \alpha_0/2(a + \alpha_0/2). \]

(4.9)

In the derivation of (4.8), we used the OPEs

\[ \sigma(z) \sigma(x) = (z - x)^{k^2-1/8} : e^{ik \varphi^-(x)}: + \ldots, \]

(4.10)

\[ \overline{\sigma}(\bar{z}) \overline{\sigma}(\bar{x}) = (\bar{z} - \bar{x})^{\bar{k}^2-1/8} : e^{i \bar{k} \varphi^-(\bar{x})} + \ldots. \]
From the continuity equations (4.7) we define the conserved charges

\[ Q = \int \frac{dz}{2\pi i} J + \int \frac{dz}{2\pi i} H , \]

\[ \overline{Q} = \int \frac{dz}{2\pi i} \overline{H} + \int \frac{dz}{2\pi i} \overline{J} . \]  

To find the commutation relations between the charges \( Q \) and \( \overline{Q} \), we must first derive the braiding relations of the non-local conserved currents \( J, \overline{J} \). To this end we will make use of the well known identity

\[ e^{iA} e^{B} = e^{B} e^{[A,B]} , \quad [A, [A, B]] = [B, [A, B]] = 0 . \]  

We then obtain the following braiding relations

\[ e^{ia\varphi(z)} e^{ib\varphi(z')} = e^{\mp i\pi ab} e^{ib\varphi(z')} e^{ia\varphi(z)} , \quad z \leq z' , \]

\[ e^{ia\varphi^{-}(z)} e^{ib\varphi^{-}(z')} = e^{\mp i\pi ab} e^{ib\varphi^{-}(z')} e^{ia\varphi^{-}(z)} , \quad z \leq z' , \]

\[ e^{ia\overline{\varphi}(z)} e^{ib\overline{\varphi}(z')} = e^{\pm i\pi ab} e^{ib\overline{\varphi}(z')} e^{ia\overline{\varphi}(z)} , \quad \overline{z} \leq \overline{z}' , \]

\[ e^{ia\overline{\varphi}^{-}(z)} e^{ib\overline{\varphi}^{-}(z')} = e^{\pm i\pi ab} e^{ib\overline{\varphi}^{-}(z')} e^{ia\overline{\varphi}^{-}(z)} , \quad \overline{z} \leq \overline{z}' , \]

\[ e^{ia\varphi(z)} e^{ib\varphi(z')} = e^{i\pi ab} e^{ib\varphi(z')} e^{ia\varphi(z)} , \quad \forall z, z' , \]

\[ e^{ia\varphi^{-}(z)} e^{ib\varphi^{-}(z')} = e^{i\pi ab} e^{ib\varphi^{-}(z')} e^{ia\varphi^{-}(z)} , \quad \forall z, z' . \]  

Using the representation of the twist fields \( \sigma, \overline{\sigma} \) in terms of scalar bosonic fields which was proposed in ref. [17], we can derive the following braiding relations:

\[ \sigma(z)\sigma(z') = e^{\mp i\pi/8} \sigma(z')\sigma(z) , \quad z \leq z' , \]

\[ \overline{\sigma}(z)\overline{\sigma}(z') = e^{\pm i\pi/8} \overline{\sigma}(z')\overline{\sigma}(z) , \quad \overline{z} \leq \overline{z}' , \]

\[ \sigma(z)\overline{\sigma}(z') = e^{i\pi/8} \overline{\sigma}(z')\sigma(z) , \quad \forall z, \overline{z}' . \]  

Consequently the non-local conserved currents have the non-trivial braiding relations

\[ J(x, t)\overline{J}(y, t) = q^{\nu} \overline{J}(y, t)J(x, t) , \]  

\[ J(x, t)J(y, t) = q^{\nu} J(y, t)J(x, t) , \]  

\[ J(x, t)\overline{J}(y, t) = q^{\nu} \overline{J}(y, t)J(x, t) . \]
where
\[ q = e^{-i\pi}, \quad \nu = 1/8 - aa - bb. \] (4.16)

Using the above braiding relations and the expressions (4.8), one finds that the conserved charges satisfy the relations
\[
Q\overline{Q} - q^\prime Q\overline{Q} = \frac{\lambda}{2\pi i} \int_t (dz \partial_z + d\overline{z} \partial_{\overline{z}}) A e^{i(a+a_0/2)\varphi(z)} e^{i(b+k)\varphi^-(z)} \times
\]
\[ \times A e^{i(a+a_0/2)\overline{\varphi}(\overline{z})} e^{i(b+k)\overline{\varphi}^-(\overline{z})}. \] (4.17)

Now let us recall that the scalar field \( \varphi^- \) lives on the orbifold \( S^1/\mathbb{Z}_2 \) and hence the momentum \( k \) must be quantized. Therefore, the above relations must be transformed to
\[
\tilde{Q}, \tilde{Q}^\tau - q^\prime \tau \tilde{Q}^\tau \tilde{Q}^\prime = \frac{\lambda}{2\pi i} \sum (A_L^{nm} A_R^{nm}) \int_t (dz \partial_z + d\overline{z} \partial_{\overline{z}}) \times
\]
\[ \times e^{i(a_L^{nm}+a_0/2)\varphi(z)+i(a_R^{nm}+a_0/2)\overline{\varphi}(\overline{z})} \times
\]
\[ \times e^{i(b_L^{nm}+k_L^{nm})\varphi^-(z)+i(b_R^{nm}+k_R^{nm})\overline{\varphi}^-(\overline{z})}, \] (4.18)

where
\[
\nu_\epsilon = 1/8 - a_L^{nm} a_R^{nm} - b_L^{nm} b_R^{nm},
\]
\[
k_L^{nm} = k_L^{nm} (\epsilon, \epsilon') = \frac{n}{R} + \left( m + \frac{\epsilon+\epsilon'}{2} \right) \frac{R}{2},
\] (4.19)
\[
k_R^{nm} = k_R^{nm} (\tau, \epsilon') = \frac{n}{R} - \left( m + \frac{\epsilon+\epsilon'}{2} \right) \frac{R}{2}.
\]
The constants \( a_L^{nm}, a_R^{nm}, b_L^{nm}, b_R^{nm}, A_L^{nm}, A_R^{nm} \) are obtained from the relations (4.9) and \( \epsilon, \tau, \epsilon' \in \{0, 1\} \).

Finally, the topological charge for the model (3.11) is defined as follows:
\[
T_{\text{top}} = \int_{-\infty}^{+\infty} dx \partial_x \Phi(x) + \int_{-\infty}^{+\infty} dx \partial_x \Phi^-(x)
\]
\[
= \int_{-\infty}^{+\infty} dx \partial_x (\varphi + \overline{\varphi}) + \int_{-\infty}^{+\infty} dx \partial_x (\varphi^- + \overline{\varphi}^-)
\]
\[
= T_{\text{top}} + T_{\text{top}}^- + T_{\text{top}}^- + T_{\text{top}}^-, \] (4.20)

where \( \Phi, \Phi^- \) and the quasi-chiral components \( \varphi, \overline{\varphi}, \varphi^-, \overline{\varphi}^- \) are related by the following
equations:
\[
\varphi(x, t) = \frac{1}{2} \left( \Phi(x, t) + \int_{-\infty}^{x} dy \partial_t \Phi(y, t) \right),
\]
\[
\varphi(x, t) = \frac{1}{2} \left( \Phi(x, t) - \int_{-\infty}^{x} dy \partial_t \Phi(y, t) \right),
\]
\[
\varphi^-(x, t) = \frac{1}{2} \left( \Phi^-(x, t) + \int_{-\infty}^{x} dy \partial_t \Phi^-(y, t) \right),
\]
\[
\varphi^-(x, t) = \frac{1}{2} \left( \Phi^-(x, t) - \int_{-\infty}^{x} dy \partial_t \Phi^-(y, t) \right),
\]
\[(4.21)\]

These equations guarantee that \( \Phi = \varphi + \varphi^- \) and \( \Phi^- = \varphi^- + \varphi^+ \). Taking into account all these, the right hand side of the equation (4.17) can be reexpressed in terms of the usual topological charges charge in (4.20):
\[
\hat{Q}_{\epsilon} \hat{Q}_{\tau} - q^{\tau} \hat{Q}_{\epsilon} \hat{Q}_{\epsilon} = \lambda \frac{\lambda}{2\pi i} \sum A_{L}^{nm} A_{R}^{nm} \left[ 1 - e^{i(a_{L}^{nm} + \alpha_{0}/2)T_{\text{top}} + i(a_{R}^{nm} + \alpha_{0}/2)\overline{T}_{\text{top}}} \times \right.
\]
\[
\times e^{i(b_{L}^{nm} + k_{L}^{nm})T_{\text{top}}^{-} + i(b_{R}^{nm} + k_{R}^{nm})\overline{T}_{\text{top}}^{-}} \right].
\]
\[(4.22)\]

Then, one can easily calculate the commutators
\[
[T_{\text{top}}, Q_{\epsilon}^{nm}] = a_{L}^{nm} Q_{\epsilon}^{nm}, \quad [T_{\text{top}}, \overline{Q}_{\tau}^{nm}] = a_{R}^{nm} \overline{Q}_{\tau}^{nm},
\]
\[
[T_{\text{top}}^{-}, Q_{\epsilon}^{nm}] = b_{L}^{nm} Q_{\epsilon}^{nm}, \quad [T_{\text{top}}^{-}, \overline{Q}_{\tau}^{nm}] = b_{R}^{nm} \overline{Q}_{\tau}^{nm}.
\]
\[(4.23)\]

Thus, these commutation relations (4.23) together with the relations (4.22) constitute the algebra, to the lowest non-trivial order in perturbation theory, which is the symmetry of the \( S \)-matrix of the theory.

Unfortunately, the isomorphism between the algebra (4.22), (4.23) and the Hopf algebra has not been established yet, and, hence, the universal \( R \)-matrix of this hidden Hopf algebra has not been studied. However, we are going to make some additional comments about these open questions in the near future.

5 Conclusions

To summarize, in the present paper we have introduced a new integrable model in quantum field theory. The novelty of the model resides in the fact that it is built on a hyper-elliptic surface instead of the usual Euclidean plane. The quantum symmetry of the model has been identified in terms of the non-local conserved charges. This has led to a generalization
of the method first introduced by Bernard and LeClair \cite{3} for the affine Toda field theories where only boson fields are involved. As is understood very well by now, the quantum non-local conserved charges provide a quantum field theoretic basis for understanding quantum groups. Unfortunately, the mapping from the physical algebra satisfied by the non-local charges to the $q$-deformed Lie algebra has not been discovered yet. If this mapping is found, one will be able to study the universal $R$-matrix and consequently uncover the structure of the $S$-matrix.

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