How difficult is it to prepare a quantum state?

Davide Girolami
Los Alamos National Laboratory, Theoretical Division, P.O. Box 1663 Los Alamos, NM 87545, USA
Kavli Institute for Theoretical Physics, UCSB, Santa Barbara, CA 93106, USA
(Dated: August 7, 2018)

It is given a quantum system in an input state. One wants to drive it into a target state. Assuming classical states and operations as free resources, I identify a geometric cost function which quantifies the difficulty of the state preparation in terms of how different it is from a classical process. The quantity determines a lower bound to the number of quantum operations, i.e. unitary transformations, required to complete the task. I then discuss the link between the quantumness of a state preparation and the amount of coherence and quantum correlations that are created in the target state.

PACS numbers: 03.65., 03.67.-a

Introduction – Quantum systems can outperform classical devices in information processing protocols [1]. Achieving quantum speed-up requires to prepare a device in a complex configuration, e.g. a highly entangled state. It is then interesting to establish the best way to drive a system into a target state, given a set of available operations. Previous works have identified the time optimal Hamiltonian evolution between two states [2–4], lower bounds to the size of the algorithm implementing a unitary transformation [5–8], and energy efficient driving dynamics for out of equilibrium classical and quantum systems [9–13]. Yet, the difficulty of a computation is not plainly related to the consumption of physical resources. A classical (that is, easy) process can take more time or dissipate more energy than a quantum (difficult) one.

Here I determine the computational difficulty of a quantum state preparation in terms of how different it is from being a classical process. The quantum character of a continuous-time transformation, described via a parametrized completely positive trace-preserving (CPTP) map, is quantified by a geometric index. The best driving strategy is then the input/target transformation which minimizes such cost function. The result provides a design principle for quantum driving of general validity, regardless of the particularities of the physical setting under study. I prove that the geometric cost function yields a lower bound to an operationally interesting quantity, the number of commuting quantum operations (logic gates) a device needs to run to prepare a target state. I then derive quantitative relations between the quantumness of a process and the creation of fundamental computational resources, i.e. coherence and quantum correlations [14, 15].

Quantum state preparation: free resources and cost – It is given a finite dimensional quantum system in a state described by a density matrix \( \rho \). How hard is it to drive the system into a target state \( \tau \)? I formulate the problem in a geometric framework [16, 17]. The dynamics of the system is modeled by a parametrized curve \( \gamma : t \rightarrow \gamma_t \) in the stratified manifold of quantum states \( M \), where \( \gamma_t = \sum_i \lambda_i(t) \ket{i(t)} \bra{i(t)} \), \( \sum_i \lambda_i(t) = 1 \), \( \ket{i(t)} \) \( \bra{j(t)} \) = \( \delta_{ij} \), \( \gamma_0 \equiv \rho \), \( \gamma_T \equiv \tau \), is the spectral decomposition of the system state at time \( t \). To answer the question, the first step is to identify what it is “easy” to obtain and to do. In the parlance of quantum information theory, this is represented by the free states and the free operations, respectively [18, 19]. I aim at associating the difficulty of the computation \( \rho \rightarrow \tau \) with its quantumness. Thus, free state preparations must be classical processes, which are characterized as follows. If an input state \( \rho = \sum_i \lambda_i \ket{i(0)} \bra{i(0)} \equiv \ket{i_R} \bra{i_R} = \delta_{ij} \), is given for free, then any state which is diagonal in the reference basis \( \{|i_R\}\) can be prepared (deterministically or stochastically) via an operation such that the state of the system is at any time described by an element of the commutative submanifold \( M_{R} = \{ \rho = \sum_i \lambda_i \ket{i_R} \bra{i_R} \} \). For example, via a measurement whose outcomes are labeled by \( |i_R\rangle \). The information about the basis is then redundant and the transformation is at any time a classical stochastic process. Hence, the free states are the density matrices in \( M_{R} \). The free operations are the processes such that the state of the system is at any time diagonal in a reference basis, \( \gamma_t = \sum_i \lambda_i(t) \ket{i_R(t)} \bra{i_R(t)} \in M_{R}, \forall t \in [0,T] \). Note that the eigenspaces \( |i_R\rangle \) are not necessarily of multiplicity one, and a state can be free with respect to more than a reference basis. I discuss a few examples to justify these definitions. A transformation between two orthogonal states \( |i\rangle \rightarrow |j\rangle , \langle i| \delta_{ij} \), can be synthesized via a unitary operation, as well as by a classical “amplitude damping” map \( \gamma_t = (1-t/T)|i\rangle \langle i| \langle 1| + t/T|j\rangle \langle j| \), in which the density matrix is diagonal at any time in a basis with elements \( |i,j\rangle \). Hence, it is not necessarily quantum. On the other hand, non-commutativity between input and output density matrices implies that the process is always quantum [20], e.g. one cannot create superpositions \( |i\rangle \rightarrow a|i\rangle + b|j\rangle , a,b \in \mathbb{C} \), classically. Note that the quantumness of a process is independent of the basis in which the states are written. Indeed, a transformation between commuting states displaying coherence in a basis, e.g. \( |a| = \sqrt{1-t/T}|+\rangle \langle +| + \sqrt{1-t/T}|-\rangle \langle -| \rightarrow a|+\rangle \langle +| + b|\rangle \langle -| = |a|+\rangle + b|\rangle \langle -| \). One observes that the free operations of a resource theory are often characterized by the form of their Kraus operators [19], but this is generally not sufficient to signal the quantumness of a transformation. A parametrized Kraus set for the amplitude damping is given by \( K_\pm = |+\rangle \langle +| + \sqrt{1-t/T}|-\rangle \langle -| \), \( K_- = \sqrt{t/T}|+\rangle \langle -| \). Yet,
the very same Kraus set transforms the input $|i\rangle$ into a non-
commuting output. The quantum character of the continuous

time evolution of a state is independent of reparametrizations of

t, i.e. the numerical value of the parameter. On this hand,

continuous time classical maps seem more appropriate free

operations for state preparation than incoherent operations

[14]. For example, the unitary qubit transformation $e^{-i\omega \sigma, t}$
is a quantum map at any time $t$, but it is a (strictly) incoherent

operation with respect to the basis $\{0,1\}$ for $t = k\pi/2$, creating

coherence otherwise. It is hard to justify why a phase shift

should be a free, operation only for some values of

the parameter $t$, as no experimental challenge emerges to

implement this map at different times.

The difficulty of an input/target transformation can be then

evaluated in terms of how different it is from a free operation,
i.e. a classical process. This cannot be measured by distance

functions, which quantify the ability to distinguish two states

via measurements [1, 21]. For example, two orthogonal states

$|i\rangle, |j\rangle$, are more distinguishable than $|i\rangle$ and any state displaying

coherence $a|i\rangle + b|j\rangle, a, b \neq 0$. In the multipartite case, the
distance between two product states can be greater than their
distance to an entangled state. I search for a function of input

and target states $Q_\rho(\tau)$ which meets a set of desirable proper-
ties: faithfulness, being zero only when a classical computer can

run the transformation, $Q_\rho(\tau) = 0 \iff \tau \in M_{\text{in}}$; invariance

under free operations, taking the same value for all free states,

$Q_\rho(\tau) = Q_\rho(\tau), \forall \rho \in M_{\text{in}}$; contractivity under mixing,

$Q_\rho(\tau) \geq Q_{\rho|\rho}(\Gamma(\tau))$, where $\Gamma$ is a CPTP map. Let us

decompose the state of the system as $\gamma_t = U_t A_t U_t^\dagger, U_0 = I$,

where $A_t$ is a diagonal matrix with the state eigenvalues as

entries. The rate of change of the curve splits into $\dot{\gamma}_t = U_t A_t U_t^\dagger + i[\gamma_t, H_t], H_t = iU_t U_t^\dagger$. For classical processes, only

the first term survives at any time $t$. On the other hand, a uni-
tary transformation $\gamma'^t = U_t A_t U_t^\dagger, \forall t$, is a genuinely quantum

process with no classical analogue. It changes the state eigen-

basis while the spectrum is invariant, so only the second term

appears at any time $t$. For a path corresponding to a general

CPTP map, the two terms coexist. Let us now consider the

energy of a curve at fixed boundaries

$$E^\nu(\rho \rightarrow \tau) := \int_0^T ||\dot{\gamma}_t||^2 dt, \gamma_0 \equiv \rho, \gamma_T \equiv \tau,$$

(1)

where the norm is induced by a symmetric, semi-positive def-

definite metric on $M$. Up to a constant factor, the quantity is

formally equivalent to the kinetic energy (per unit of time) for

a particle traveling on the manifold [22–24], while being gen-

erally not related to the energy of the system. It is possible to
discriminate between classical and quantum components of

the energy:

$$E^\nu(\rho \rightarrow \tau) = E^c_\rho(\rho \rightarrow \tau) + E^q_\rho(\rho \rightarrow \tau),$$

(2)

$$E^c_\rho(\rho \rightarrow \tau) := \int_0^T ||\dot{A}_t||^2 dt, E^q_\rho(\rho \rightarrow \tau) := \int_0^T ||[\gamma_t, H_t]||^2 dt.$$  

Note that the length functional of a path, and therefore a dis-
tance function, cannot be split into two parts at any time. For

unitary transformations, only the quantum part survives, cap-
turing the sensitivity of the system to phase shifts. This prop-

erty, called asymmetry [25], is the peculiar quantum resource

for phase estimation. It is then justified to extend this inter-

pretation, generalizing the concept of asymmetry to arbitrary

CPTP maps. That is, the basis changing component of the ve-

locity measures the sensitivity of the system in a state $\gamma_t$ to

a map $\Gamma_t$ due to quantum effects. That is, it is the resource

for quantum state preparation. Hence, the quantumness of a

computation $\rho \rightarrow \tau$, i.e. the difficulty of driving the system

into the target state within a time $T$, is given by the minimum

quantum component of the energy over all the possible maps

linking a free state to the target:

$$Q_\rho(\tau) := E_q^\rho(\rho \rightarrow \tau), \quad E_q^\rho(\rho \rightarrow \tau) = \min_{\tilde{\gamma}_t, \tilde{\rho} \in M_{\text{in}}} E_q^\rho(\tilde{\rho} \rightarrow \tau), \quad \tilde{\gamma}_t \neq \gamma_t.$$

(3)

The possibility to split the state rate of change is independent

of the specific metric employed, as it is due to the direct sum

structure of the tangent space to $M$ [17]. A natural choice is

yet the Bures metric, a Riemannian metric contractive under

CPTP maps, which plays an important role in quantum statis-
tics [26–28]. The squared speed of the system at time $t$, corre-

sponding to (one fourth) the symmetric-logarithmic derivative

quantum Fisher information, is given by the sum of the clas-
sical and quantum terms,

$$\|\dot{\gamma}_t\|^2 = \sum_i \lambda_i(t)^2 + \sum_{i < j} \|i(t)[[\gamma_t, H_t]](j)(t)\|^2 \quad \lambda_j(t) + \lambda_j(t).$$

(4)

The first term is the squared norm related to the classical

Fisher metric, while the second one is the quantum con-

tribution. For unitary transformations $\gamma'^t$, only the second

term survives, $E_q^\rho(\rho \rightarrow \tau) = E'\nu(\rho \rightarrow \tau)$ [29, 30].

This is non-negative, vanishing at any time only for classi-
cal processes, and non-increasing under mixing [31, 32].

If the evolution is time-independent, $U_t = e^{-i\hat{H}t}$, one has

$E_q^\rho(\rho \rightarrow \tau) = T||[H, \rho]||^2$. For pure state transfor-

mations, the quantity equals $T$ times the variance of the Hamil-

tonian, $||[H, \rho]||^2 = \text{Var}_\rho(H) = \text{Tr}(\rho H^2) - \text{Tr}(\rho H)^2$. It follows from

the properties of the quantum Fisher information, i.e. the

instantaneous (squared) speed, that the required constraints

are met. Faithfulness holds because if and only if the target

is a free state, there exists a classical preparation such

that $\|\dot{\gamma}_t\|^2 = \sum_i \lambda_i(t)^2, \forall t$. Invariance under free transfor-

mations of the input state is satisfied by construction. Finally,

defining $\Gamma(\gamma_t) : \Gamma(\rho) \rightarrow \Gamma(t)$ the dynamics of a state sub-

ject at any time to a CPTP map, the quantity is contractive,

$Q_\rho(\tau) \geq E_q^\rho(\gamma(t) \rightarrow \Gamma(t)) \geq Q_{\rho|\rho}(\Gamma(\tau)).$

The definition in Eq. 3 assumes that all the possible dynamics

linking input and target states are implementable in practice,

which is generally not realistic. It is yet possible to derive

an operationally motivated upper bound (see Fig. 1). Con-

sider the case in which only free operations, which are clas-
sical stochastic processes, and unitary transformations, i.e.
Figure 1: A quantum system is in an input state $\rho$ of the commutative submanifold $M_h$. One wants to drive it into a target $\tau$. The optimal map $\tilde{\gamma}$ is the map with the minimum quantum component of the energy $E^\gamma(\tilde{\rho}_t \rightarrow \tau)$.

Proper quantum operations, are available in the experimental toolbox. It is then possible to split a state preparation into two steps: a change of spectrum and change of basis, $\rho \rightarrow \rho^u \rightarrow \tau$, $\rho^u = \sum \lambda_i(T)|i_k\rangle\langle i_k| \in M_h$, where $\lambda_i(T)$ are the eigenvalues of $\tau$. The first step can be completed via a free operation. The second step can be completed via at least one purely quantum, unitary change of basis $\gamma^u : \rho^u \mapsto \tau$.

One then has $E^\gamma(\rho^u \rightarrow \tau) = E^\gamma_{\rho^u}(\rho^u \rightarrow \tau)$. Note that, for a target state of a $d$-dimensional system with eigenvalues having multiplicities $m_r$, there are $d!/(\Pi m_r!)$ isospectral free states which can freely transform into each other via permutations, $\rho^u = P\rho^u P^\dagger$. The minimum energy to complete the second step is then computed by minimizing over the free states which are isospectral to the target. Thus, the difficulty to complete a state preparation with classical operations and unitaries is

$$Q^u(\rho) := E^\gamma_{\rho^u}(\rho^u \rightarrow \tau),$$

$$E^\gamma_{\rho^u}(\rho^u \rightarrow \tau) = \min_{\gamma^u,\rho^u} E^\gamma_{\rho^u}(\rho^u \rightarrow \tau), \gamma^u,\rho^u : \rho^u \rightarrow \tau.$$

Note that $Q^u(\rho) \geq Q(\rho)$, and that this upper bound also means by construction the faithfulness, invariance and contractivity properties, $Q^u(\tau) = 0 \Leftrightarrow \tau \in M_h$, $Q^u(\rho) = Q^u(\rho) \forall \rho \in M_h$, $Q^u(\tau) \geq Q^u(T(\tau))$.

**Optimal path, interplay with algorithmic complexity and quantum resources** — An important question is what is the best strategy, i.e. the best unitary path $\tilde{\gamma}$, to reach the target from an isospectral free state. The map between two states which minimizes the energy $E^\gamma(\rho \rightarrow \tau)$ is the length minimizer at constant speed [22]. A distance function is $D(\rho, \tau) := \min_{\gamma^u,\rho^u \mapsto \tau} \int_0^T ||\dot{\gamma}|| dt$. Specifically, the one related to the Bures metric is the Bures angle $D_B(\rho, \tau) = \cos^{-1}[\sqrt{\rho^u \tau}]$. The energy minimizing map from a pure free state $\rho^u = |\psi^u\rangle\langle\psi^u|$ to a pure target $\tau = |\psi\rangle\langle\psi|$ is the length minimizing unitary, and the Bures angle reduces to the Fubini-Study distance $D_F(\rho^u, \tau) = \cos^{-1}[\langle\psi^u|\psi\rangle]$. The closest free pure state $\rho^u$ to the target is then the one with maximal overlap. The explicit form for the length/energy minimizing constant speed path is $\tilde{\gamma}^u_t = |\psi^u\rangle\langle\psi^u|\langle\psi^u|\langle\psi^u|\psi^u\rangle\rangle = (\cos \theta - \sin \theta / \tan d)|\psi^u\rangle + (\sin \theta / \tan d)|\psi^u\rangle, \theta = d \tau / T, d := D_B(\rho^u, \psi)$. This is obtained by the expression for length minimizing path [33–35], and by noting that the minimizer of the energy is unique up to affine changes of parametrization $\tau' = at + b, a, b \in \mathbb{R}$. Finding the optimal unitary for preparing mixed target states is more challenging, while necessary conditions for the shortest unitary path between isospectral states have been found [36]. However, one can use the result for pure states to obtain a lower bound to $Q^u(\tau)$ for arbitrary target states. The distance between two density matrices corresponds to the minimum distance between their purifications [33]. The closest isospectral free state to the target is then the one with the closest purification $|\psi^u_{\text{purif}}\rangle$ to a target purification $|\psi^u\rangle$. The two closest purifications have a compact expression [35], which in this case reads $|\psi^u_{\text{purif}}\rangle = \sum |\psi_{\text{purif}}\rangle \otimes |i_k\rangle, |\psi^u_{\text{purif}}\rangle = \sum 1/\sqrt{p^u} \sqrt{\sqrt{p^u} \tau / \sqrt{p^u}|i_k\rangle} \otimes |i_k\rangle$. The length/energy minimizing (generally not unitary) path between two mixed states is obtained by partial trace along the shortest (unitary) path between the closest purifications. Thus, one has $Q^u(\tau) \geq Q^u_{\text{purif}}(\rho^u_{\text{purif}})$. The inequality is saturated for pure targets. As an illustrative example, consider driving a qubit from an input state with Bloch form $\rho = 1/2(I + q_i\sigma_i)$ to a target $\tau = 1/2(I + \tilde{r}_i\tilde{\sigma}_i)$. The closest isospectral free state to the target is identified by $q_i = \tilde{r}_i$. One has $Q^u_{\text{purif}}(\rho^u_{\text{purif}}) = \{ \cos^{-1}[(\sqrt{r_1} + \sqrt{r_2})/2]^2]/T, f_z = 1 + |r_1| r_2 \pm 1 + |r_2| r_1 - (\sqrt{r_1} - \sqrt{r_2})^2 - 1, f_z = 1 + |r_1| r_2 \pm 1 + |r_2| r_1 - (\sqrt{r_1} - \sqrt{r_2})^2 - 1\}$. The process is classical for $r_1 = r_2 = 1/|\tau| = |r_1|$, while the maximum energy $\tau^2/(16T)$ is required to prepare a pure state, $r_1^2 + r_2^2 = 1, r_1 = 0$.

The quantumness of a preparation $Q^u(\tau)$ lower bounds the size of any algorithm implementing the input/target transformation. Suppose a unitary map $\gamma^u = \rho^u \mapsto \tau$ is synthesized by N commuting unitary operations, e.g. logic gates, $\gamma^u_i = U_i P^u_i U_i^\dagger, \tau = e^{-iH_t} = \sum_1^N H_{i_t}[H_{i_t}, H_{k_t}] = 0, \forall l, k$. The scenario is generally applicable to state preparation schemes of multipartite systems, e.g. the phase imprinting step in parallel estimation protocols [28], or the preparation of highly entangled symmetric states, $(a(0) + b(1)) \otimes |0\rangle^{N} \rightarrow a(0) |0\rangle^{N+1} + b(1) |1\rangle^{N+1}, a, b \in \mathbb{C}$, via controlled gates between the first and $l + 1$-th qubit. Consider the seminorm of each Hamiltonian $|H_i| = h_{i,M} - h_{i,p}$ being the difference between its largest and smallest eigenvalues [37]. The quantity measures the complexity of $H_i$, as it depends on the number of gates implementing the Hamiltonian, and the size of the correlations they induce [37, 38]. Also, it bounds the value of the physical energy, $|H_i| \geq \langle H_i - h_{i,M} \rangle, \forall p$. Since $|H| \leq \sum_{i, |\langle H_i|}\rangle$.
one has \( E^\tau(\rho \to \tau) = \int_0^T ||[H, \rho]||^2 dt = T||[H, \rho]||^2 \leq TV \rho(H) \leq TH^2/4 \leq T(\sum_{i=1}^N |H_i|^2)/4 = TN^2[H^2]/4 \), where \( |H|^2 \) is the average squared seminorm over all the generators \( H_i \). By assuming every Hamiltonian to have the same seminorm \( |H_i| = h, \forall i \), the number of quantum operations to prepare a target state is bounded as

\[
N \geq \frac{2}{h} \left( \frac{Q^\rho(\tau)}{T} \right)^{1/2}.
\]

The bound is saturated for superpositions of the largest and smallest eigenvalues of \( H, |\psi_0\rangle = (|h_M\rangle + e^{i\theta}|h_0\rangle)/\sqrt{2} \), which is the most sensitive input to the map.

The difficulty to perform a state transformation is also related to the creation of computational resources. The quantumness of a preparation is linked to how much coherence the target displays with respect to the reference basis \([14, 39, 40]\), being quantified by the distance to the set of incoherent states \( C^\rho_B(\tau) := \min_{\rho \in \mathcal{M}_i} \langle \rho, \tau \rangle \). One has \( Q^\rho_B(\tau) \geq D_B(\rho, \tau) \geq (C^\rho_B(\tau))^2/T \geq Q^\rho(\tau) / T \), \( Q^\rho(\tau) = 0 \Rightarrow C^\rho_B(\tau) = 0 \). For pure states, the geometric measure of coherence induced by the Bures angle is \( C^\rho_B(\psi) = \cos^{-1} \max_{ik} \langle \rho_{ik} \rangle \) \([41]\), which implies \( D_B(\psi) = C^\rho_B(\psi) \). In the multipartite case, it is possible to link the quantumness of a state preparation to the creation of quantum correlations in the target whenever the reference basis is local or multi-local. The most general form of bipartite quantum correlations, quantum discord \([15]\), can be measured by the minimum coherence over all the bi-local bases, \( D_B(\rho_{12}) := \min_{i,j} C^\rho_B(\rho_{ij12}, |i_j12\rangle \langle i_j12|) \). I am here employing the symmetric discord \([42, 43]\), but the argument applies to the original asymmetric definition as well. Consider the set of free states being the zero discord states \( \rho_{12} = \sum_{ij} p_{ij}^2 |i_j12\rangle \langle i_j12| \), \( \sum_{ij} p_{ij}^2 = 1, \rho_{12} \in \mathcal{M}_{i,j} \). That is, the reference basis is the bi-local basis \( |i_j12\rangle \). One has \( Q^\rho(\rho_{12}) = D_B(\rho_{12})/T \). Yet, a quantum preparation does not necessarily create discord, \( D_B(\rho_{12}) = 0 \Rightarrow Q^\rho(\rho_{12}) = 0 \). For example, the qutrit-qubit map \( \rho(0)(0) \langle 0| + (1-p)\rho(1)\langle 1| \langle 1| \langle 1| + \rho(2)\langle 2| \langle 2| \rangle \langle 2| \langle 1| \langle 1| = \rho(p) \langle 0| \langle 0| + (1-p)/2(\langle 1| + \langle 2|) \langle 1| + \langle 2| \rangle \langle 1| \langle 1| ) \) does not create discord, but it generates coherence with respect to the reference basis \([0, 1, 2]\) \([44]\). Note that if the reference basis is not bi-local, e.g. the Bell basis for two-qubit states, creating entanglement from scratch, \( I_{12} \to \rho_{12} = (00)_{12} + (11)_{12} )/ \sqrt{2} \) is a free operation, and the entangled state is a free state \([45]\). The bound can be extended to the multipartite scenario. The amount of coherence and correlations of different orders, i.e. describing information shared between more than two parties, depend on the Hilbert space partition \([46–50]\). Specifically, it is possible to build an hierarchy of measures of genuine multipartite correlations of different orders. Given an \( N \)-local reference basis \( |i_1 \ldots i_N\rangle \), observing that the coarse grained bases containing up to \( k \)-local terms read \( \{k\} := \{i_{2+k}\} \equiv \{i_{2+k}| i_{k+1+k} \ldots i_{k+1+k} \ldots i_{k+1+k}, \sum_{j} k_j = N, k \geq \}, V \rangle \). I define the Bures quantum discord of “order higher than \( k \)” in an \( N \)-partite target state \( \tau_{1 \ldots N} \) as \( D^k_B(\tau_{1 \ldots N}) := \min_{\rho \in \mathcal{M}_{i,j}} C^\rho_B(\tau_{1 \ldots N}) \). Suppose the free states to be the incoherent states in a coarse grained basis \( \bar{i}_k, \rho_{1 \ldots N} = \sum_{\rho} \rho |\bar{i}_k\rangle \langle \bar{i}_k| \), i.e. a subset of the states without quantum discord of order higher than \( k \). One has \( Q^\rho_{\bar{i}_k \in \mathcal{M}_{i,j}}(\tau_{1 \ldots N}) \geq (C^\rho_B(\tau_{1 \ldots N}))^2/T \geq \rho \rho \). The result holds for correlations of any order \( k \). A bound for the total amount of quantum correlations, i.e. the sum of correlations at any order, is obtained by defining the free states to be the incoherent states \( \rho_{1 \ldots N} = \sum_{\rho} \rho |\bar{i}_k\rangle \langle \bar{i}_k| \). Then, \( Q^\rho_{\bar{i}_k \in \mathcal{M}_{i,j}}(\tau_{1 \ldots N}) \geq (D_B^k(\tau_{1 \ldots N}))^2/T \), where \( D_B^k(\tau_{1 \ldots N}) \) is the distance of the target to the set of classically correlated (and uncorrelated) states. Note that consistently, by defining the genuine \( k \)-partite discord as \( D^k_B(\tau_{1 \ldots N}) := D_B^{k-1-N}(\tau_{1 \ldots N}) - D_B^N(\tau_{1 \ldots N}) \), the total quantum discord is the sum of the correlations at any order, \( D^{k-1-N}(\tau_{1 \ldots N}) = \sum_{k=2}^N D^k_B(\tau_{1 \ldots N}) \). Also, as a geometric measure of discord is always an upper bound to a geometric measure of entanglement \([51]\), being equal to it for pure states, the quantumness of the preparation map is also an upper bound to the Bures measure of multipartite entanglement of any order in the target state.

**Conclusion** – I have quantified the difficulty of preparing a quantum system in a target state from a predetermined input in terms of the process quantumness. The optimal driving dynamics is obtained by solving the geometric problem of minimizing the quantum contribution to the energy of the associated curve. The result highlights the usefulness of geometric methods to establish fundamental limits of quantum information processing, which may be employed in finding optimal algorithms to solve other information theoretic tasks. Geometric bounds could provide a benchmark to evaluate the performance of numerical methods, e.g. automated design of quantum algorithms \([52]\), which is currently of renewed interest due to the applicability of machine learning techniques. It also suggests that the resource theory approach can be fruitful to solve critical problems in quantum control \([53]\), e.g. developing optimal time/energy protocols for quantum state and quantum dynamics engineering.

**Acknowledgements.** – I thank F. Anzì, I. Bengtsson, C. Caffaro, P. Gibilisco, A. Jencova, S. Luo, R. Maity, V. Moretti, B. Yadim and K. Zyczkowski for fruitful discussions. I acknowledge support from LANL through the LDRD project 20180702PRD1 and the LDRD Rapid Response project “Unraveling Entanglement in a black box quantum computer”. Also, this research was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958, as part of the work was carried out at the KITP in S. Barbara.

---

* Electronic address: davegirolami@gmail.com

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, New York, 2000).
