Asymptotically Optimal Amplifiers for the Moran Process

Leslie Ann Goldberg†, John Lapinskas†, Johannes Lengler‡, Florian Meier‡, Konstantinos Panagiotou§, and Pascal Pfister‡

Abstract

We study the Moran process as adapted by Lieberman, Hauert and Nowak. This is a model of an evolving population on a graph where certain individuals, called “mutants” have fitness $r$ and other individuals, called “non-mutants” have fitness 1. We focus on the situation where the mutation is advantageous, in the sense that $r > 1$. A family of directed graphs is said to be strongly amplifying if the extinction probability tends to 0 when the Moran process is run on graphs in this family. The most-amplifying known family of directed graphs is the family of megastars of Galanis et al. We show that this family is optimal, up to logarithmic factors, since every strongly-connected $n$-vertex digraph has extinction probability $\Omega(n^{-1})$. Next, we show that there is an infinite family of undirected graphs, called dense incubators, whose extinction probability is $O(n^{-1/3})$. We show that this is optimal, up to constant factors. Finally, we introduce sparse incubators, for varying edge density, and show that the extinction probability of these graphs is $O(n/m)$, where $m$ is the number of edges. Again, we show that this is optimal, up to constant factors.

1 Introduction

We study the Moran process [17] as adapted by Lieberman, Hauert and Nowak [13, 18]. This is a model of an evolving population. There are two kinds of individuals — “mutants” and “non-mutants”. The model has a parameter $r$, which is a positive real number, and is the fitness of the mutants. All non-mutants have fitness 1. The individuals reside at the vertices of a directed graph $G$ — each vertex contains exactly one individual, and it is either a mutant or a non-mutant. In the initial state, one vertex (chosen uniformly at random) contains a mutant. All of the other vertices contain non-mutants. The process evolves in discrete time. At each step, a vertex is selected at random, with probability proportional to its fitness. Suppose that this is vertex $v$. Next, an out-neighbour $w$ of $v$ is selected uniformly at random. Finally, the state of vertex $v$ (mutant or non-mutant) is copied to vertex $w$.

If $G$ is finite and strongly connected then with probability 1, the process will either reach the state where there are only mutants (known as fixation) or it will reach the state where
there are only non-mutants (extinction). If \( G \) is not strongly connected then the process may continue changing forever — thus, it makes sense to restrict attention to strongly-connected digraphs \( G \). We do so for the rest of the paper.

Given a strongly-connected digraph \( G \), we use the notation \( p_r(G) \) to denote the probability that the Moran process (starting from a uniformly-chosen initial mutant) reaches fixation and we use the notation \( \ell_r(G) \) to denote the probability that it reaches extinction. If \( \mathcal{G} \) is a set of digraphs then we use \( \ell_{r,\mathcal{G}}(n) \) to denote \( \max\{\ell_r(G) \mid G \in \mathcal{G} \text{ and } G \text{ has } n \text{ vertices}\} \). (To avoid trivialities, we take the maximum of the empty set to be 0.) The function \( \ell_{r,\mathcal{G}} \) is called the “extinction limit” of the family \( \mathcal{G} \). Lieberman et al. \cite{Lie03} raised the question of whether there exists an infinite family \( \mathcal{G} \) of digraphs for which \( \limsup_{n \to \infty} \ell_{r,\mathcal{G}}(n) = 0 \). We say in this case that \( \mathcal{G} \) is strongly amplifying. They defined two infinite families of strongly-connected digraphs — superstars and metafunnels — which turn out to be strongly amplifying. The most amplifying infinite family of strongly-connected digraphs that is known (in the sense that the extinction limit grows as slowly as possible, as a function of \( n \)) is the family \( \mathcal{Y} \) of megastars from \cite{Gal07}. Galanis et al. show \cite[Theorem 6]{Gal07} that, for every \( r > 1 \) there is an \( n_0 \) (depending on \( r \)) so that, for all \( n \geq n_0 \) and for every \( n \)-vertex digraph \( G \in \mathcal{Y} \), \( \ell_r(G) \leq (\log n)^{2/3}/n^{1/2} \).

The first question addressed by this paper is whether the family of megastars is optimal in the sense that the extinction limit grows as slowly as possible (as a function of \( n \)). We show that this is the case, up to logarithmic factors.

**Theorem 1.** For all \( r > 1 \), any strongly-connected \( n \)-vertex digraph \( G \) with \( n \geq 3 \) satisfies \( \ell_r(G) > 1/(12r n^{1/2}) \).

For undirected graphs, the most amplifying graphs previously known were stars, whose extinction probability tends to \( 1/r^2 \) (as the size of the star grows). In particular, no strongly-amplifying family of undirected graphs was known. In our next result we show that such families do exist, and that they can have extinction probability \( \ell_r(G) = O(n^{-1/3}) \).

**Theorem 2.** For all \( r > 1 \), there exists an infinite family \( \mathcal{D}_r \) of connected graphs with the following property. If \( G \in \mathcal{D}_r \) has \( n \) vertices, then \( \ell_r(G) \leq 71/(r(r-1)^2 n^{1/3}) \).

The graphs in the family \( \mathcal{D}_r \) are called dense incubators. Each such graph is parameterised by a number \( k \), which is the square of an integer. Taking \( \beta \) to be an integer constant depending on \( r \), the graph consists of \( k \) stars, each with \( \lceil r \sqrt{\beta k} \rceil \) leaves, together with a clique of size \( \beta k \). Every centre of every star is connected to every node in the clique. More details are given in Definition \( \ref{def:incubator} \) (this definition also defines sparse incubators, which we will discuss shortly).

It is known \cite[Corollary 7]{Lie03} that extinction probability is monotonic in \( r \) in the sense that if \( 0 < r < r' \) then, for any digraph \( G \), \( \ell_{r'}(G) \leq \ell_r(G) \). Thus, Theorem \ref{thm:extinction_limit} guarantees that, for every \( r' > r \) and every \( n \)-vertex graph in \( \mathcal{D}_r \), we also have \( \ell_{r'}(G) \leq 71/(r(r-1)^2 n^{1/3}) \).

The next question that we address is whether the family \( \mathcal{D}_r \) is optimal (again, in the sense that the extinction limit grows as slowly as possible). We show that this is the case, up to constant factors (depending on \( r \)).

**Theorem 3.** Let \( r > 1 \). Consider any connected \( n \)-vertex graph \( G \) with \( n \geq 3 \). Then \( \ell_r(G) > 1/(2^{15} r^2 n^{1/3}) \).

\footnote{See Section \ref{sec:related_work} for a discussion of simultaneous independent work that also resolves this question.}
Figure 1: The family of $I_{r,b}$ incubators. As $G[V_2, V_3]$ is a biregular graph with $\beta kb(k)^2$ edges, each vertex in $V_2$ sends $\beta b(k)^2$ edges to $V_3$ and each vertex in $V_3$ sends $b(k)^2$ edges to $V_2$.

The reason that dense incubators are called “dense” is that an $n$-vertex dense incubator has $\omega(n)$ edges (more specifically, it has $\Theta(n^{4/3})$ edges). The final question that we address is whether there are sparse families of graphs that are strongly amplifying. Once again, the answer is yes.

Before we present the relevant theorems (Theorems 6 and 7) we define a (parameterised) family of incubators, where the additional parameter controls the edge density. In order to define these, we need some definitions. Given a graph $G = (V, E)$ and subsets $S$ and $T$ of $V$, $E(S, T)$ denotes the set of edges in $E$ with one endpoint in $S$ and the other in $T$. We also use the following standard definition.

**Definition 4.** Let $G = (V, E)$ be a $d$-regular graph with $n$ vertices. $G$ is a small-set expander if

$$\min_{\emptyset \subseteq S \subseteq V, |S| \leq n^{1/3}} \frac{|E(S, V \setminus S)|}{|S|} \geq d/4.$$  

Let $Z_{\geq 1}$ denote the set of positive integers. Given a graph $G = (V, E)$ and disjoint subsets $S$ and $T$ of $V$ we use $G[S]$ to denote the subgraph of $G$ induced by $S$ and we use $G[S,T]$ to denote the graph with vertex set $S \cup T$ and edge set $E[S,T]$. This graph is said to be biregular if all vertices in $S$ have the same degree and also all vertices in $T$ have the same degree. Using Definition 4 we can now define families of incubators (see Figure 1).

**Definition 5.** Let $r > 1$ and let $\beta = 26r^2/(r - 1)$. Let $b : Z_{\geq 1} \rightarrow Z_{\geq 1}$ be any function that satisfies $b(k) \leq \sqrt{k}$ for all $k$. Then a graph $G = (V, E)$ is a member of the family $I_{r,b}$ of incubators with branching factor $b$ if and only if there exists a positive integer $k$ and a partition $V_1, V_2, V_3$ of $V$ such that the following properties hold.
Observation 8. Consider $m/n \leq k$, and $|V_3| = \beta k$.

Proof. By the definition of $\beta b$, we have $n = k[r\beta^{1/2}(b(k))] + k + \beta k$ and $m = k[r\beta^{1/2}(b(k))] + \beta kb(k)^2 + \frac{k\beta(b(k)^2 - 1)}{2}$. The lower bounds on $n$ and $m$ follow immediately, and the upper bounds follow since $\beta \geq 26[r]$. Putting these together gives the bounds on $m/n$. □
1.1 Related work

The Moran process is somewhat similar to a discrete version of directed percolation known as the contact process. There has been a lot of work (e.g., [1, 6, 7, 14, 19]) on the contact process and other related infection processes such as the voter model and SIS epidemic models. We refer the reader to [10, Section 1.4] for a discussion of how these models differ from the Moran process.

Lieberman, Hauert and Nowak [13, 18] introduced the version of the Moran process that we study. They raised the question of strong amplification and defined two infinite families of strongly-connected digraphs — superstars and metafunnels — which turn out to be strongly amplifying. Many papers contributed to determining the fixation probability of these digraphs [13, 3, 12] — see [10, Section 1.4] for a discussion. The first rigorous proof that there is an infinite family of strongly-amplifying digraphs is in [10]. This is the family of megastars discussed in the introduction. The paper also gives lower bounds on the extinction probability of superstars and metafunnels.

The best-known lower bounds on the extinction probability of connected undirected graphs are in [15, 16]. Theorem 1 of [16] shows that there is a constant \( c_0(r) \) such that for every \( \varepsilon > 0 \) the extinction probability is at least \( c_0(r)/n^{3/4+\varepsilon} \).

While this manuscript was under preparation, George Giakkoupis posted simultaneous, independent work [11] also showing that strong undirected amplifiers exist. In the remainder of this section, we discuss this work.

First, consider the model of Lieberman, Hauert and Nowak [13, 18] which we study. Our Theorem 2 shows that there is an infinite family of graphs \( G \) with \( \ell_r(G) \leq 71/(r(r-1)^2 n^{1/3}) \). Theorem 1 of [11] is similar, but weaker by a logarithmic factor — that paper constructs a (similar) family with extinction probability \( \ell_r(G) = O(\log(n)/((r-1)n^{1/3})) \). Our Theorem 6 shows that any connected \( n \)-vertex graph (with \( n \geq 3 \)) has \( \ell_r(G) > 1/(2^{15}r^2n^{1/3}) \). Theorem 2 of [11] is similar, but weaker by a \( (\log n)^{4/3} \) factor — that paper shows that the extinction probability \( \ell_r(G) \) is \( \Omega(1/(r^{5/3}n^{1/3}(\log n)^{4/3})) \).

Our paper is otherwise incomparable to [11]. We give a lower bound on the extinction probability of amplifying directed graphs (Theorem 1) but [11] does not consider digraphs. We also construct sparse families of incubators (Theorem 6) which go all the way down to constant density and are optimally-amplifying up to constant factors (Theorem 7) but [11] does not consider sparse graphs. On the other hand, [11, Theorem 3] constructs a family of suppressors with extinction probability at least \( 1 - O(r^2 \log n/n^{1/4}) \), which is something that we do not study here. Finally, Sood et al. [20] have introduced a variant of the model in which the fitness of a mutant is taken to be a function of the number of vertices of the underlying graph (so as the number of vertices in the graph grows, the fitness of each individual mutant decreases). The results of [11] extend to this model where \( r = 1 + o(1) \), as a function of \( n \). We are not aware of any applications of this model, and we don’t consider it.

1.2 Organisation of the paper

In Section 2 we define some notation that we will use throughout the paper. In Section 3 we show that, as long as \( b(k) \) is eventually sufficiently large, then the set \( I_{r,b} \) is infinitely large. In Section 4 we prove Theorems 2 and 6 which give upper bounds on extinction probability. In Section 5, we prove Theorems 1, 3 and 7 which give lower bounds on extinction probability.
2 Preliminaries

We write $\mathbb{Z}_{\geq 1} = \{1, 2, \ldots \}$. For all $n \in \mathbb{Z}_{\geq 1}$, we write $[n] = \{1, \ldots, n \}$. We write $\log$ for the base-e logarithm and $\lg$ for the base-2 logarithm.

When $G = (V, E)$ is a directed graph and $v \in V$, we write $N_{\text{in}}(v) = \{ w \mid (w, v) \in E \}$, $d_{\text{in}}(v) = |N_{\text{in}}(v)|$, $N_{\text{out}}(v) = \{ w \mid (v, w) \in E \}$, and $d_{\text{out}}(v) = |N_{\text{out}}(v)|$. We view undirected graphs as directed graphs such that for all $u, v \in V$, $(u, v) \in E$ if and only if $(v, u) \in E$. Of course, we use standard conventions when counting edges in undirected graphs. That is, an undirected edge $\{u, v\}$ is only counted as one edge. If $G$ is undirected, we write $N(v) = N_{\text{out}}(v) = N_{\text{in}}(v)$ and $d(v) = d_{\text{out}}(v) = d_{\text{in}}(v)$. If $S \subseteq V$, we write $N(S) = \bigcup_{v \in S} N(v)$.

Recall that the initial configuration of the Moran process is the configuration in which one vertex is chosen uniformly at random to be a mutant, and the rest of the vertices are non-mutants. We have already defined $\ell_r(G)$, which is the probability that this process reaches extinction. When $G = (V, E)$ is known from the context and $v$ is a vertex of $G$, it will also be useful to define $\ell_r(v)$ to be extinction probability, conditioned on the fact that the initial mutant is $v$. In this case, $\ell_r(G) = \frac{1}{n} \sum_{v \in V} \ell_r(v)$.

3 Infinite sets of incubators

The main result of this Section is Theorem 11, which shows that, as long as $b(k)$ is sufficiently large, then the set $\mathcal{I}_{x, b}$ is infinitely large.

If a graph $G$ has adjacency matrix $A$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, we let $\lambda(G) = \max \{ \lambda_2, -\lambda_n \}$. We use two existing results which, between them, imply that a sparse random regular graph is likely to be a small-set expander.

**Theorem 9.** ([2, Theorem 1.1]) Let $C_0, K > 0$, and let $\alpha = 459652 + 229452 K + \max \{ 30 \sqrt{\frac{3}{2}}, 768 \}$. Let $n, d \in \mathbb{Z}_{\geq 1}$, and suppose $d \leq C_0 n^{2/3}$ and $n \geq 7 + K^2$. Let $G$ be a uniformly random $d$-regular graph on $n$ vertices. Then $\mathbb{P}(\lambda(G) \leq \alpha \sqrt{d}) \geq 1 - n^{-K}$. \qed

The following theorem is well-known, and follows from, e.g., [21, Theorem 8.6.30].

**Theorem 10.** If $G = (V, E)$ is a $d$-regular $n$-vertex graph, and $S$ is a non-empty proper subset of $V$, then $|E(S, V \setminus S)| \geq (d - \lambda(G)) |S| |V \setminus S| / n$. \qed

**Theorem 11.** Let $b : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$. Suppose that for all $k \in \mathbb{Z}_{\geq 1}$, $b(k) \leq \sqrt{k}$. Suppose in addition that there are infinitely many $k$ such that $b(k) \geq 10^7$. Then $\mathcal{I}_{x, b}$ contains infinitely many graphs.

**Proof.** It suffices to prove that for all $k$ such that $b(k) \geq 10^7$, there exists a small-set expander $H_k$ on $\beta k$ vertices with degree $D = \beta b(k)^2 - 1$. Since $b(k) \leq \sqrt{k}$, we have $0 \leq D \leq \beta k - 1$. Since $\beta$ is even, it follows that there exists a regular graph with degree $D$ on $\beta k$ vertices. Let $H$ be a uniformly random such graph, and suppose that $\emptyset < S \subseteq V(H)$ satisfies $|S| \leq (k\beta)^{1/3}$. **Case 1.** If $D \geq 2 (k\beta)^{1/3}$, then we have

$$|E(S, V(H) \setminus S)| \geq \sum_{v \in S} (d(v) - |S|) \geq D|S| - (k\beta)^{1/3}|S| > D|S| / 4.$$

Thus we may take $H_k = H$ with certainty.
Case 2. Suppose instead that $D \leq 2(k\beta)^{1/3}$. Now apply Theorem 9 with $n = \beta k$, $d = D$, $C_0 = 2$ and $K = 1$. This shows that $P(\lambda(H) \leq \alpha \sqrt{d}) \geq 1 - 1/(\beta k)$. If $\lambda(H) \leq \alpha \sqrt{d}$ then using Theorem 10,

$$|E(S, V \setminus S)| \geq (D - \alpha \sqrt{d}) \frac{|S| |k\beta - |S||}{k\beta} \geq (D - \alpha \sqrt{d}) \frac{|S|}{2}.$$ 

To show that $H$ is a small-set expander, we need only show that $\alpha \sqrt{d} \leq D/2$. This follows from the definitions of $\alpha$ and $D$ using $\beta \geq 26$ and $b(k) \geq 10^7$. Since $H$ is a small-set expander with non-zero probability, there exists a small-set expander on $\beta k$ vertices with degree $D$, as required.

Theorem 11 shows that the set $I_{r,b}$ is infinitely large, as long as there are infinitely many $k$ such that $b(k) \geq 10^7$. This is sufficient for our purposes, since our goal is to show that there are infinitely-many incubators, even with constant density. However, the condition that $b(k) \geq 10^7$ is not necessary. The lower bound could be weakened substantially by replacing the use of Theorem 9 with the result of Friedman 9 when $b(k) < 10^7$.

4 Upper bounding the extinction probability of incubators

In this section, we prove Theorems 2 and 6. For this, it will be useful to have a more formal definition of the Moran process, which defines some notation that we will use.

Definition 12. Let $G = (V, E)$ be an $n$-vertex directed graph, let $r > 1$, and let $x_0 \in V$. We define the Moran process $(X_t)_{t \geq 0}$ on $G$ with fitness $r$ and initial mutant $x_0$ inductively as follows, where all random choices are made independently. Let $X_0 = \{x_0\}$. For all $S \subseteq V$, let $W(S) = n + (r - 1)|S|$. Given $X_t$ for some $t \geq 0$, we define $X_{t+1}$ as follows. Randomly choose a vertex $v_t \in V$ with distribution

$$P(v_t = v) = \begin{cases} r/W(X_t) & \text{if } v \in X_t; \\ 1/W(X_t) & \text{if } v \in V \setminus X_t. \end{cases}$$

If $d_{out}(v_t) = 0$, then $X_{t+1} = X_t$. Otherwise, choose $w_t \in N_{out}(v_t)$ uniformly at random. If $v_t \in X_t$, then $X_{t+1} = X_t \cup \{w_t\}$, and we say $v_t$ spawns a mutant onto $w_t$ at time $t + 1$. Otherwise, $X_{t+1} = X_t \setminus \{w_t\}$, and we say $v_t$ spawns a non-mutant onto $w_t$ at time $t + 1$.

If there exists $t$ such that $X_t = \emptyset$, we say the process goes extinct at time $t$, and if there exists $t$ such that $X_t = V$, we say the process fixates at time $t$. In either case, we say the process absorbs at time $t$.

Note that Moran processes are discrete-time Markov chains, and that nothing happens if (e.g.) a mutant is spawned onto a mutant. The notation $v_1, v_2, \ldots$ and $w_1, w_2, \ldots$ is not used outside Definition 12.

Incubators are defined in Definition 5 on page 3. Whenever we discuss a specific graph $G = (V, E) \in I_{r,b}$, we will use the notation $V_1$, $V_2$, $V_3$, $k$ and $\beta$ from Definition 5 without explicitly redefining it. We use $b$ as shorthand for $b(k)$. Because the final theorem assumes $b \geq b_0$ for a constant $b_0$, depending on $r$, it will do no harm to assume that $b$ is sufficiently large. To avoid cluttering the notation, we assume $b \geq 6r$ everywhere, and we mention it explicitly only if we use a stronger bound. For all $v \in V_1$, we write $c(v)$ for the (necessarily) unique neighbour of $v$ in $V_2$. 

7
4.1 Going from a mutant in $V_1$ to many mutants in $V_3$

We first define some stopping times which will be important to our coupling.

**Definition 13.** Let

$$T_{\text{end}} = \min\{t \geq 0 \mid X_t \cap V_1 = \emptyset \text{ or } |X_t \cap V_3| = \lfloor (k\beta)^{1/3} \rfloor + 1\}.$$ 

Note that $T_{\text{end}}$ is finite with probability 1. Define $T_0, T_1, \ldots$ recursively by $T_0 = 0$ and

$$T_i = \min\{\{T_{\text{end}}\} \cup \{t > T_{i-1} \mid X_t \cap V_3 \neq X_{t-1} \cap V_3\}\}.$$ 

If $T_{\text{end}} = 0$ then $T_0, T_1, \ldots$ are all 0. Otherwise, with probability 1, there is a $j$ such that $T_0 < \cdots < T_j$ and, for all $i \geq j$, $T_j = T_{\text{end}}$.

**Lemma 14.** Let $t \geq 0$, let $M \subseteq V$, and suppose $1 \leq |M \cap V_3| \leq (\beta k)^{1/3}$. Then we have

$$\mathbb{P}(|X_{t+1} \cap V_3| = |X_t \cap V_3| + 1 \mid X_t = M) \geq \frac{r|E(V_3 \cap M, V_3 \setminus M)|}{W(M)(\beta b^2 + b^2 - 1)};$$

$$\mathbb{P}(|X_{t+1} \cap V_3| = |X_t \cap V_3| - 1 \mid X_t = M) \leq \left(1 + \frac{r}{\beta}\right) \frac{|E(V_3 \cap M, V_3 \setminus M)|}{W(M)(\beta b^2 + b^2 - 1)}.$$ 

Moreover, if $M \cap V_1 \neq \emptyset$, then

$$\mathbb{P}(X_{t+1} \cap V_1 = \emptyset \mid X_t = M) \leq \frac{1}{W(M)\beta b^2}.$$ 

**Proof.** For brevity, write $M' = M \cap V_3$. For the first equation, note that $|X_t \cap V_3|$ increases whenever a vertex in $M'$ spawns a mutant onto a vertex in $V_3 \setminus M'$. We therefore have

$$\mathbb{P}(|X_{t+1} \cap V_3| = |X_t \cap V_3| + 1 \mid X_t = M) \geq \sum_{v \in M'} \frac{r}{W(M)} \frac{|N(v) \cap (V_3 \setminus M')|}{d(v)}$$

$$= \frac{r|E(M', V_3 \setminus M')|}{W(M)(\beta b^2 + b^2 - 1)},$$

as required.

For the second equation, note that $|X_t \cap V_3|$ decreases precisely when a vertex in $(V_2 \cup V_3) \setminus M$ spawns a non-mutant onto a vertex in $M'$. It follows that

$$\mathbb{P}(|X_{t+1} \cap V_3| = |X_t \cap V_3| - 1 \mid X_t = M) = \sum_{v \in (V_2 \cup V_3) \setminus M} \frac{1}{W(M)} \frac{|N(v) \cap M'|}{d(v)}$$

$$\leq \frac{|E(V_2, M')|}{W(M)(\beta b^2 + \lceil r\beta^{1/2}b \rceil)} + \frac{|E(M', V_3 \setminus M')|}{W(M)(\beta b^2 + b^2 - 1)}.$$ 

Recall from Definition 15 that $G[V_3]$ is a small-set expander, and $1 \leq |M'| \leq |V_3|^3$ by hypothesis, so since $\beta \geq 26$ we have

$$\frac{|E(M', V_3 \setminus M')|}{\beta b^2 + b^2 - 1} \geq \frac{\beta b^2 - 1}{4(\beta b^2 + b^2 - 1)} \geq \frac{|M'|}{5}.$$
Moreover, since every vertex in $V_3$ has degree $b^2$ into $V_2$, 
\[
\frac{|E(V_2, M')|}{\beta b^2 + \lfloor r\beta^{1/2}b\rfloor} \leq \frac{b^2|M'|}{\beta b^2} = \frac{|M'|}{\beta}.
\]
It follows that
\[
P(|X_{t+1} \cap V_3| = |X_t \cap V_3| - 1 \mid X_t = M) \leq \left(1 + \frac{5}{\beta}\right) \frac{|E(M', V_3 \setminus M')|}{W(M)(\beta b^2 + b^2 - 1)}.
\]
as required.

For the third equation, recall that, by hypothesis, $M \cap V_1 \neq \emptyset$. For brevity, write $p = P(X_{t+1} \cap V_1 = \emptyset \mid X_t = M)$. Note that if $|M \cap V_1| > 1$ then $p = 0$. Suppose instead that $M \cap V_1 = \{v_0\}$ for some $v_0 \in V$. Note that $v_0$ can only become a non-mutant if its unique neighbour $c(v_0)$ spawns a non-mutant onto it. Thus, if $c(v_0) \in M$, $p = 0$. If $c(v_0) \notin M$, we have
\[
p = \frac{1}{W(M)} \cdot \frac{1}{\lfloor r\beta^{1/2}b\rfloor + \beta b^2} \leq \frac{1}{W(M)\beta b^2}.
\]
The desired inequality therefore holds in all cases.

Throughout the remainder of the section, we use the following definition.

**Definition 15.** Let $r' = (r + 1)/2$.

We will bound the probability of certain events in terms of $r'$. To facilitate the task, we start with a technical lemma (containing algebraic manipulation which will get used more than once).

**Lemma 16.**
\[
\frac{1 + 5/\beta}{r + 1 + 5/\beta} \leq \frac{1}{1 + r'} - \frac{10}{r\beta b^2}.
\]

Proof.
\[
\frac{1 + 5/\beta}{r + 1 + 5/\beta} \leq \frac{1 + 5(r - 1)/(26r^2)}{1 + r} \leq \frac{1}{1 + r} + \frac{r - 1}{5r^2(r + 1)}.
\]
Moreover, since $r' = (r + 1)/2$ and $b \geq 6$ we have
\[
\frac{1}{1 + r'} - \frac{1}{1 + r} - \frac{10}{r\beta b^2} \geq \frac{2}{3r - 1} - \frac{10(r - 1)}{900r^3} \geq \frac{r - 1}{(r + 1)(r + 3)} - \frac{r - 1}{4(r - 1)}
\]
\[
\geq \frac{41(r - 1)}{45(r + 1)(r + 3)} \geq \frac{41(r - 1)}{180r^2(r + 1)} \geq \frac{r - 1}{5r^2(r + 1)}.
\]

\[\square\]

**Lemma 17.** Let $t \geq 0$, let $j \geq 0$, let $M \subseteq V$, and suppose $1 \leq |M \cap V_3| \leq (\beta k)^{1/3}$ and $M \cap V_1 \neq \emptyset$. Then we have
\[
P(X_{T_{j+1}} \cap V_1 = \emptyset \mid X_t = M, T_j = t \neq T_{\text{end}}) \leq \frac{10}{r\beta b^2},
\]
\[
P(|X_{T_{j+1}} \cap V_3| = |X_{T_j} \cap V_3| - 1 \mid X_t = M, T_j = t \neq T_{\text{end}}) \leq \left(1 - \frac{10}{r\beta b^2}\right) \frac{1}{r' + 1}, \text{ and}
\]
\[
P(|X_{T_{j+1}} \cap V_3| = |X_{T_j} \cap V_3| + 1 \mid X_t = M, T_j = t \neq T_{\text{end}}) \geq \left(1 - \frac{10}{r\beta b^2}\right) \frac{r'}{r' + 1}.
\]
Proof. Write $\mathcal{F}$ for the event that $X_t = M$ and $T_j = t \neq T_{\text{end}}$. Throughout, suppose that $\mathcal{F}$ occurs. First note that this implies
\begin{equation}
T_{j+1} = \min\{t' > t \mid X_{t'} \cap V_3 \neq X_t \cap V_3 \text{ or } X_{t'} \cap V_1 = \emptyset\}.
\end{equation}

Now consider some non-negative integer $i$. For $M' \subseteq V$, let
\begin{equation}
q_i^{M'} = \mathbb{P}(T_{j+1} = t + i + 1 \text{ and } X_{t+i} = M' \mid \mathcal{F}).
\end{equation}

Let $\mathcal{M}_i = \{M' \subseteq V \mid q_i^{M'} \neq 0\}$. For all $M' \in \mathcal{M}_i$, let
\begin{align*}
p_{\text{up},i}^{M'} &= \mathbb{P}(|X_{t+i+1} \cap V_3| = |X_{t+i} \cap V_3| + 1 \mid X_{t+i} = M', T_{j+1} = t + i + 1, \mathcal{F}), \\
p_{\text{down},i}^{M'} &= \mathbb{P}(|X_{t+i+1} \cap V_3| = |X_{t+i} \cap V_3| - 1 \mid X_{t+i} = M', T_{j+1} = t + i + 1, \mathcal{F}), \\
p_{\text{fail},i}^{M'} &= \mathbb{P}(X_{t+i+1} \cap V_1 = \emptyset \mid X_{t+i} = M', T_{j+1} = t + i + 1, \mathcal{F}).
\end{align*}

Note that if $M' \in \mathcal{M}_i$, then $M' \cap V_1 \neq \emptyset$. Also, $M' \cap V_3 = M \cap V_3$, so $1 \leq |M' \cap V_3| \leq (\beta k)^{1/3}$. If $X_{t+i} = M'$, then the three events $|X_{t+i+1} \cap V_3| = |M' \cap V_3| + 1$, $|X_{t+i+1} \cap V_3| = |M' \cap V_3| - 1$ and $X_{t+i+1} \cap V_1 = \emptyset$ are disjoint, and (by (1)) conditioning on $T_{j+1} = t + i + 1$ is precisely equivalent to conditioning on one of the three events occurring. For brevity, write $x(M') = |E(V_3 \cap M', V_3 \setminus M')|$ and $y = \beta b^2 + b^2 - 1$. It follows by Lemma 14 (with Lemma 14’s $t$ equal to our $t + i$ and Lemma 14’s $M$ equal to our $M'$) that for all $M' \in \mathcal{M}_i$,
\begin{equation}
p_{\text{fail},i}^{M'} \leq \frac{1}{r (W(M')/M')} y = \frac{y}{r x(M')/M').
\end{equation}

Recall that $G[V_3]$ is a small-set expander and, for $M' \subseteq \mathcal{M}_i$, $|M' \cap V_3| = |M \cap V_3| \leq |V_3|^{1/3}$, and so
\begin{equation}
x(M') = (\beta b^2 - 1)|M' \cap V_3|/4 \geq \beta b^2 |M \cap V_3|/5 \geq \beta b^2 / 5.
\end{equation}

Moreover, $y \leq 2\beta b^2$. It follows that
\begin{equation}
p_{\text{fail},i}^{M'} \leq \frac{10}{r \beta b^2}.
\end{equation}

Similarly, it follows by Lemma 14 and Lemma 16 that
\begin{equation}
p_{\text{down},i}^{M'} \leq \frac{(1 + 5/\beta) x(M')/(W(M')y)}{r x(M')/(W(M')y) + (1 + 5/\beta) x(M')/(W(M')y)} = \frac{1 + 5/\beta}{r + 1 + 5/\beta} \leq \left(1 - \frac{10}{r \beta b^2}\right) \frac{1}{1 + r'},
\end{equation}

Finally, we note that by (2) and (3),
\begin{equation}
p_{\text{up},i}^{M'} = 1 - p_{\text{fail},i}^{M'} - p_{\text{down},i}^{M'} \geq \left(1 - \frac{10}{r \beta b^2}\right) \frac{r'}{1 + r'}.
\end{equation}

Now, by the law of total probability and (2), we have
\begin{equation}
\mathbb{P}(X_{T_{j+1}} \cap V_1 = \emptyset \mid \mathcal{F}) = \sum_{i \geq 0} \sum_{M' \in \mathcal{M}_i} p_{\text{fail},i}^{M'} q_i^{M'} \leq \frac{10}{r \beta b^2} \sum_{i \geq 0} \sum_{M' \in \mathcal{M}_i} q_i^{M'} = \frac{10}{r \beta b^2},
\end{equation}
and so the first equation of the lemma statement follows. Similarly, the second equation follows from (3) and the third equation follows from (4). □
Lemma 18. Let $t \geq 0$, let $M \subseteq V$, and suppose $M \cap V \neq \emptyset$. Then there exists a stopping time $T^{++} > t$ such that the following hold:

(i) $\mathbb{P}(T^{++} < \infty \mid X_t = M) = 1$;
(ii) $\mathbb{P}(X_{t'} \cap V_1 = \emptyset$ for some $t < t' \leq T^{++} \mid X_t = M) \leq 1/(r\beta b^2)$;
(iii) $\mathbb{P}(|X_{T^{++}} \cap V_3| \geq 1 \mid X_t = M) \geq 1/(6\beta^{1/2} b)$.

Proof. Let $v_0 \in V_1 \cap M$ be arbitrary, and recall that $c(v_0)$ is the unique neighbour of $v_0$. Let $T^+$ be the minimum $t' > t$ such that at time $t'$, either $v_0$ spawns or $c(v_0)$ spawns a non-mutant onto $v_0$. Let $T^{++}$ be the minimum $t' > T^+$ such that at time $t'$, either $c(v_0)$ spawns onto some vertex in $V_3$ or a neighbour of $c(v_0)$ spawns a non-mutant onto $c(v_0)$. Note that since each vertex in $V$ spawns infinitely often with probability 1, (i) holds.

For all $i \geq 0$ and all $M_i \subseteq V$ with $v_0 \in M_i$, we have

$$\mathbb{P}(v_0 \text{ spawns at } t + i + 1 \mid X_{t+i} = M_i) = r/W(M_i),$$

$$\mathbb{P}(c(v_0) \text{ spawns a non-mutant onto } v_0 \text{ at } t + i + 1 \mid X_{t+i} = M_i) \leq 1/(\beta b^2 W(M_i)).$$

Write $p_i^{M'} = \mathbb{P}(T^+ = t + i + 1, X_{t+i} = M' \mid X_t = M)$. Then since $v_0 \in V_1 \cap M$ and $v_0$ remains a mutant throughout $\{t, t+1, \ldots, T^+-1\}$, it follows by the law of total probability applied to $T^+$ that

$$\mathbb{P}(v_0 \in X_{T^+} \text{ and } v_0 \text{ spawns at } T^+ \mid X_t = M) = \mathbb{P}(v_0 \text{ spawns at } X_{T^+} \mid X_t = M) \geq \sum_{i \geq 0} \sum_{\substack{M' \subseteq V \\text{ such that } v_0 \in M'} \atop r/W(M') + 1/(\beta b^2 W(M')).} \frac{r}{r + 1/(\beta b^2)} p_i^{M'} \geq \frac{r}{r + 1/(\beta b^2)} \geq \frac{1}{r \beta b^2}. \quad (5)$$

If $v_0 \in X_{T^+}$ and $v_0$ spawns at $T^+$, it follows that $c(v_0) \in X_{T^+}$. By the definition of $T^{++}$, it therefore follows that $c(v_0) \in X_{t'}$ for all $T^+ \leq t' \leq T^{++} - 1$, and so $c(v_0)$ does not spawn a non-mutant onto $v_0$ at any time in $\{T^+ + 1, \ldots, T^{++}\}$. Hence, (ii) follows from (5).

Now, for all $i \geq 0$ and all $M_i \subseteq V$ with $c(v_0) \in M_i$, since $\beta \geq 26r$ and $b \geq 6r$ we have

$$\mathbb{P}(c(v_0) \text{ spawns into } V_3 \text{ at } t + i + 1 \mid X_{t+i} = M_i) \geq \frac{r}{W(M_i)} \cdot \frac{\beta b^2}{\beta b^2 + |r \beta^{1/2} b|^2} \geq \frac{r}{2W(M_i)}.$$

Moreover, writing $E$ for the event that some $v \in V \setminus X_{t+i}$ spawns onto $c(v_0)$ at time $t + i + 1$, we have

$$\mathbb{P}(E \mid X_{t+i} = M_i) \leq \left[\frac{r \beta^{1/2} b}{W(M_i)} + \frac{\beta b^2}{W(M_i)} \cdot \frac{1}{\beta b^2 + b - 1}\right] \leq \frac{r \beta^{1/2} b + 1}{W(M_i)} + \frac{1}{W(M_i)} \leq \frac{2r \beta^{1/2} b}{W(M_i)}.$$

Now, suppose that $M^+ \subseteq V$ and $t^+ > t$ are such that $\mathbb{P}(X_{t^+} = M^+$ and $T^+ = t^+ \mid X_t = M) \neq 0$ and $c(v_0) \in M^+$. Write

$$q_i^{M^+} = \mathbb{P}(T^{++} = t^+ + i + 1, X_{t+i} = M^{++} \mid T^+ = t^+, X_{t^+} = M^+).$$
Since when $X_t = M^+$ and $T^+ = t^+$, $c(v_0)$ remains a mutant throughout $\{T^+, \ldots, T^{++} - 1\}$, by the law of total probability applied to $T^{++}$ it follows that

$$
\mathbb{P}(X_{T^{++}} \cap V_3 \neq \emptyset \mid T^+ = t^+, X_{t^+} = M^+) \geq \sum_{i \geq 0} \sum_{M^{++} \subseteq V \subseteq M^+} \frac{r/(2W(M^{++}))}{r/(2W(M^{++})) + 2\beta^{1/2}b/W(M^{++})} \mu_{M^{++}}^+
$$

$$
\quad = \frac{1}{1 + 4\beta^{1/2}b} \geq \frac{1}{5\beta^{1/2}b}
$$

It therefore follows from the law of total probability applied to $T^+$ and (5) that

$$
\mathbb{P}(X_{T^{++}} \cap V_3 \neq \emptyset \mid X_t = M) \geq \sum_{t^+ > t} \sum_{M^+ \subseteq V} \mathbb{P}(X_{T^{++}} \cap V_3 \neq \emptyset, T^+ = t^+, X_{t^+} = M^+ \mid X_t = M) \geq \frac{1}{5\beta^{1/2}b} \mathbb{P}(v_0 \in X_{T^+} \mid X_t = M) \geq \frac{1}{5\beta^{1/2}b} - \frac{1}{\beta b^2} \geq \frac{1}{6\beta^{1/2}b}.
$$

Hence (iii) follows.

Lemma 19. Let $t \geq 0$, let $j \geq 0$, let $M \subseteq V$, and suppose $M \cap V_1 \neq \emptyset$ and $M \cap V_3 = \emptyset$. Then we have

(i) $\mathbb{P}(X_{T_{j+1}} \cap V_1 = \emptyset \mid X_t = M, T_j = t \neq T_\text{end}) \leq 6/(r\beta^{1/2}b)$, and

(ii) $\mathbb{P}(X_{T_{j+1}} \cap V_3 \neq \emptyset \mid X_t = M, T_j = t \neq T_\text{end}) \geq 1 - 6/(r\beta^{1/2}b)$.

Proof. Write $\mathcal{F}$ for the event that $X_t = M$ and that $T_j = t \neq T_\text{end}$, and suppose throughout that $\mathcal{F}$ occurs. Thus, by definition, $T_{j+1}$ is precisely the earliest time $t^+ > t$ such that $X_{t^+} \cap V_3 \neq \emptyset$ or $X_{t^+} \cap V_1 = \emptyset$, and (i) and (ii) are equivalent.

Define stopping times $\tau_0, \tau_1, \ldots$ inductively as follows. Let $\tau_0 = t$. If $\tau_i < T_{j+1}$, then we must have $X_{\tau_i} \cap V_1 \neq \emptyset$, so we define $\tau_{i+1}$ to be the stopping time $T^{++}$ obtained by applying Lemma 18 with $t = \tau_i$ and $M = X_{\tau_i}$. Note that in this case $\tau_{i+1} > \tau_i$. If $\tau_i \geq T_{j+1}$, we set $\tau_{i+1} = \tau_i$.

Let $I = \max\{i \mid \tau_{i+1} \neq \tau_i\}$. Now for all $i \geq 0$ and $M_i \subseteq V$ with $M_i \cap V_1 \neq \emptyset$ and $M_i \cap V_3 = \emptyset$ and every $t' \geq t$ write $\mathcal{F}_{i, M_i, t'}$ for the event that $\tau_i = t'$, $X_{t'} = M_i$, $t' < T_{j+1}$ and $\mathcal{F}$ occurs. If $\mathcal{F}_{i, M_i, t'}$ has non-zero probability then

$$
\mathbb{P}(I = i \mid \mathcal{F}_{i, M_i, t'}) = \mathbb{P}(T_{j+1} \leq \tau_{i+1} \mid \mathcal{F}_{i, M_i, t'}) \geq \mathbb{P}(|X_{\tau_{i+1}} \cap V_3| \geq 1 \mid \mathcal{F}_{i, M_i, t'}).
$$

Applying Lemma 18(iii) with $t'$ and $M_i$, this probability is at least $1/(6\beta^{1/2}b)$. Thus,

$$
\mathbb{P}(I = i \mid I \geq i, \mathcal{F}) = \sum_{M_i, t'} \mathbb{P}(I = i \mid \mathcal{F}_{i, M_i, t'}) \mathbb{P}(\mathcal{F}_{i, M_i, t'} \mid \mathcal{F}) \geq 1/(6\beta^{1/2}b).
$$

(6)

Hence $\mathbb{P}(I < \infty \mid \mathcal{F}) = 1$. Now, for all $i \geq 0$,

$$
\mathbb{P}(X_{T_{j+1}} \cap V_1 = \emptyset \mid I = i, \mathcal{F}) \leq \mathbb{P}(\exists t' \text{ such that } \tau_i < t' \leq \tau_{i+1} \text{ and } X_{t'} \cap V_1 = \emptyset \mid I = i, \mathcal{F}) = \frac{\mathbb{P}(I = i \text{ and } \exists t' \text{ such that } \tau_i < t' \leq \tau_{i+1} \text{ and } X_{t'} \cap V_1 = \emptyset \mid I \geq i, \mathcal{F})}{\mathbb{P}(I = i \mid I \geq i, \mathcal{F})} \leq \frac{\mathbb{P}(\exists t' \text{ such that } \tau_i < t' \leq \tau_{i+1} \text{ and } X_{t'} \cap V_1 = \emptyset \mid I \geq i, \mathcal{F})}{\mathbb{P}(I = i \mid I \geq i, \mathcal{F})}.
$$

(7)
By Lemma 18(ii), for all $t_i \geq t$ and $M_i \subseteq V$ with $M_i \cap V_1 \neq \emptyset$, we have
\[
P(\exists t' \text{ such that } \tau_i < t' \leq \tau_{i+1} \text{ and } X_{t'} \cap V_1 = \emptyset \mid I \geq i, \tau_i = t_i, X_{t_i} = M_i, \mathcal{F}) \leq 1/(r\beta b^2).
\]
Moreover, if $I \geq i$ then $X_{\tau_i} \cap V_1 \neq \emptyset$ and $X_{\tau_i} \cap V_3 = \emptyset$. It therefore follows from (6) and (7) that
\[
P(X_{T_{i+1}} \cap V_1 = \emptyset \mid I = i, \mathcal{F}) \leq \frac{1/(r\beta b^2)}{1/(6\beta^1/2b)} = \frac{6}{r\beta^1/2b}.
\]
Thus (i) follows from the law of total probability applied to $I$, and (ii) follows from (i).

**Definition 20.** Write $\gamma = [(k\beta)^{1/3}]$. Define $(Y_t)_{t \geq 0}$ to be a discrete-time Markov chain with state space $S_V = \{F, 0, 1, \ldots, \gamma + 1\}$, initial state 0, and the following transition matrix:

\[
p_{F,F} = 1,
p_{0,F} = \frac{6}{r\beta^1/2b}, \quad p_{0,0} = \left(1 - \frac{6}{r\beta^1/2b}\right) \frac{1}{1+r'}, \quad p_{0,1} = \left(1 - \frac{6}{r\beta^1/2b}\right) \frac{r'}{1+r'},
\]

for all $1 \leq i \leq \gamma$,

\[
p_{i,F} = \frac{10}{r\beta^2}, \quad p_{i,i-1} = \left(1 - \frac{10}{r\beta^2}\right) \frac{1}{1+r'}, \quad p_{i,i+1} = \left(1 - \frac{10}{r\beta^2}\right) \frac{r'}{1+r'},
\]

and $p_{i,j} = 0$ for all other $i, j \in S_V$.

**Lemma 21.** Let $i \geq 0$, $t_i \geq 0$, $M \subseteq V$ and $y \geq 0$. Suppose that $y \leq |M \cap V_3| \leq \gamma$ and $M \cap V_1 \neq \emptyset$. Write $\mathcal{F}$ for the event that $X_{t_i} = M$ and $T_i = t_i \neq T_{\text{end}}$. Then, we have
\[
P(X_{T_{i+1}} \cap V_1 = \emptyset \mid \mathcal{F}) \leq p_{y,F}
\]
and likewise
\[
P(|X_{T_{i+1}} \cap V_3| = |X_{T_i} \cap V_3| - 1 \mid \mathcal{F}) \leq \left(1 - \frac{10}{r\beta^2}\right) \frac{1}{r' + 1} = p_{y,y-1},
\]
so the result follows.

Now suppose $y = 0$ and $M \cap V_3 = \emptyset$. Then by Lemma 17 we have
\[
P(X_{T_{i+1}} \cap V_1 = \emptyset \mid \mathcal{F}) \leq \frac{6}{r\beta^1/2b} = p_{0,F}
\]
and clearly
\[
P(|X_{T_{i+1}} \cap V_3| = |X_{T_i} \cap V_3| - 1 \mid \mathcal{F}) = 0,
\]
so the result follows.

Finally, suppose $y = 0$ and $M \cap V_3 \neq \emptyset$. Then by Lemma 17 we have
\[
P(X_{T_{i+1}} \cap V_1 = \emptyset \mid \mathcal{F}) \leq \frac{10}{r\beta^2} \leq \frac{6}{r\beta^1/2b} = p_{y,F}
\]
and
\[
\mathbb{P}(X_{T_{j+1}} \cap V_1 = \emptyset \mid \mathcal{F}) + \mathbb{P}(|X_{T_{j+1}} \cap V_3| = |X_{T_i} \cap V_3| - 1 \mid \mathcal{F})
= 1 - \mathbb{P}(|X_{T_{j+1}} \cap V_3| = |X_{T_i} \cap V_3| + 1 \mid \mathcal{F}) \leq 1 - \left(1 - \frac{10}{r \beta b^2}\right) \frac{r'}{r' + 1}
\leq 1 - \left(1 - \frac{6}{r \beta b^2}\right) \frac{r'}{r' + 1} = 1 - p_{y,y+1}.
\]

Thus, the result follows in all cases. \(\square\)

**Lemma 22.** Suppose \(X_0 \cap V_1 \neq \emptyset\). Then, there exists a coupling \(\Phi(X, Y)\) between \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) such that for all \(i \geq 0\) with \(Y_i \neq \emptyset\), there exists \(t \leq T_i\) such that \(|X_t \cap V_3| \geq Y_i\).

**Proof.** We will construct a coupling \(\Phi(X, Y)\) such that the following properties hold for every non-negative integer \(j\).

1. If \(T_j < T_{\text{end}}\), then either \(Y_j = \emptyset\) or \(|X_{T_j} \cap V_3| \geq Y_j\).
2. If \(T_j = T_{\text{end}}\) and \(j = \min\{s \geq 0 \mid T_s = T_{\text{end}}\}\) and \(|X_{T_j} \cap V_3| = \gamma + 1\), then either \(Y_j = \emptyset\) or \(|X_{T_j} \cap V_3| \geq Y_j\).
3. If \(T_j = T_{\text{end}}\) and \(j = \min\{s \geq 0 \mid T_s = T_{\text{end}}\}\) and \(X_{T_j} \cap V_1 = \emptyset\), then \(Y_j = \emptyset\).

First, we observe that a coupling satisfying Properties 1–3 would satisfy the condition in the statement of the lemma. To see this, consider some non-negative integer \(i\) for which we want to establish the condition in the statement of the lemma (that \(Y_i = \emptyset\) or there exists \(t \leq T_i\) such that \(|X_t \cap V_3| \geq Y_i\)). If \(T_i < T_{\text{end}}\) then this follows from Property 1 with \(j = i\) and \(t = T_i\). If \(T_i = T_{\text{end}}\) and \(i = \min\{s \geq 0 \mid T_s = T_{\text{end}}\}\), then it follows from Properties 2 and 3 with \(j = i\) and \(t = T_i\). Otherwise, there is a non-negative integer \(j < i\) such that \(T_j = T_{\text{end}}\) and \(j = \min\{s \geq 0 \mid T_s = T_{\text{end}}\}\). Properties 2 and 3 guarantee that \(Y_j = \emptyset\) (in which case the definition of \(Y\) ensures that \(Y_i = \emptyset\) so the condition is satisfied) or \(|X_{T_j} \cap V_3| = \gamma + 1\) so (by the definition of \(Y\)) \(Y_i \leq |X_{T_j} \cap V_3|\) and taking \(t = T_j\) satisfies the condition.

In order to construct the coupling, it will be useful to have some notation. Given a coupling \(\Phi(X, Y)\) and a non-negative integer \(j\), let \(\Phi^j\) denote the initial sequence \((X_0, \ldots, X_{T_j}, Y_0, \ldots, Y_j)\). We will construct \(\Phi(X, Y)\) by induction on \(j\), using \(\Phi^j\) (and some randomness) to construct \(\Phi^{j+1}\). To do this, we have to ensure that Properties 1–3 are satisfied, and also that the coupling is valid, in the sense that

- The marginal distribution of \(X_{T_{j+1}}, \ldots, X_{T_j+1}\) is correct, given \(X_{T_j}\) and given whether or not \(T_j < T_{\text{end}}\) (which can be deduced from \(X_0, \ldots, X_{T_j}\)), and
- The marginal distribution of \(Y_{j+1}\) is correct, given \(Y_j\).

Note that \(\Phi^0 = (X_0, 0)\) satisfies Properties 1–3 (for Property 3 it is important that \(X_0 \cap V_1 \neq \emptyset\) and this is guaranteed in the statement of the lemma) so we now show how to construct \(\Phi^{j+1}\), given \(\Phi^j\). In fact, if \(T_{\text{end}} \leq T_j\) then any coupling \(\Phi(X, Y)\) which is consistent with \(\Phi^j\) and satisfies the two marginal distributions is fine (since the three properties are irrelevant for \(T_i\) with \(i > j\)). So we will not consider this case. However, if \(T_j < T_{\text{end}}\) we will show how to construct \(\Phi^{j+1}\).
• **If** $Y_j = F$: The definition of $Y$ guarantees that $Y_{j+1} = F$. This satisfies all three properties, so let $X_{T_j+1}, \ldots, X_{T_{j+1}}$ evolve independently of $Y_{j+1}$ according to its correct marginal distribution, given $X_{T_j}$ and given the fact that $T_j < T_{end}$.

• **If** $Y_j \neq F$: Let $\mathcal{E}_F$ be the event that $X_{T_j+1} \cap V_1 = \emptyset$ and let $p_F$ denote the probability that $\mathcal{E}_F$ occurs in the correct marginal distribution (which depends only on $X_{T_j}$, noting that $T_j < T_{end}$). Let $\mathcal{E}_{down}$ be the event that $|X_{T_j+1} \cap V_3| = |X_{T_j} \cap V_3| - 1$ and let $p_{down}$ be the probability that $\mathcal{E}_{down}$ occurs in the same marginal distribution. Note that $\mathcal{E}_F$ and $\mathcal{E}_{down}$ are disjoint, since $T_j < T_{end}$. Let $\mathcal{E}_{up}$ be the event that $|X_{T_j+1} \cap V_3| = |X_{T_j} \cap V_3| + 1$. In the marginal distribution, this occurs with probability $1 - p_F - p_{down}$. By Property 1 and the definition of $T_{end}$, we have $0 \leq Y_j \leq |X_{T_j} \cap V_3| \leq \gamma$. Now Lemma 21 (with $i = j$, $t_i = T_j$, $M = X_{T_j}$ and $y = Y_j$) shows that $p_F \leq p_{Y_j,F}$ and $p_F + p_{down} \leq 1 - p_{Y_j,Y_j+1}$. The quantity $1 - p_{Y_j,Y_j+1}$ is either $p_{Y_j,F} + p_{Y_j,Y_j-1}$, if $Y_j > 0$, or $p_{Y_j,F} + p_{Y_j,Y_j}$, if $Y_j = 0$. To unify these cases, let $p_{Y_j,down}$ be $p_{Y_j,Y_j-1}$ if $Y_j > 0$ and $p_{Y_j,Y_j}$ if $Y_j = 0$. Then we have

$$p_F \leq p_{Y_j,F} \text{ and } p_F + p_{down} \leq p_{Y_j,F} + p_{Y_j,down}. \quad (8)$$

The coupling is as follows: Choose $X_{T_{j+1}}, \ldots, X_{T_{j+1}}$ according to the correct marginal distribution.

- $\mathcal{E}_F$ happens with probability $p_F \leq p_{Y_j,F}$. When this happens, set $Y_{j+1} = F$.
- $\mathcal{E}_{up}$ happens with probability $p_{up} \geq p_{Y_j,Y_j+1}$. When this happens, with probability $p_{Y_j,Y_j+1}/p_{up}$, set $Y_{j+1} = Y_j + 1$. Let $\xi = p_{up} - p_{Y_j,Y_j+1}$ and $\rho = \min\{p_{Y_j,down}, \xi\}$. With probability $\rho/p_{up}$, set $Y_{j+1} = \max\{Y_j - 1, 0\}$ and with probability $(\xi - \rho)/p_{up}$ set $Y_{j+1} = F$.
- $\mathcal{E}_{down}$ happens with probability $p_{down}$. When this happens, let $\sigma = p_{Y_j,down} - \rho$. Let $Y_{j+1} = \max\{Y_j - 1, 0\}$ with probability $\sigma/p_{down}$ and $Y_{j+1} = F$, with probability $1 - \sigma/p_{down}$.

It is now easy to check that $Y_{j+1} = Y_j + 1$ and $Y_{j+1} = \max\{Y_j - 1, 0\}$ both happen with the correct marginal distribution (so $Y_{j+1} = F$ does as well). Also, Equation (8) guarantees that the probabilities are all well-defined (and non-negative!). Finally, the coupling itself guarantees Properties 1–3.

**Definition 23.** Write $S_Z = \{0, 1, \ldots, \gamma\}$, and suppose $z \in S_Z$. Then we define $(Z^z_t)_{t \geq 0}$ to be a discrete-time Markov chain with state space $S_Z$, initial state $z$, and the following transition matrix:

$$p_{0,0}^t = 1/(1 + r'), \quad p_{0,1}^t = r'/(1 + r'),$$

for all $i \in [\gamma - 1]$, $p_{i,i+1}^t = 1/(1 + r')$, $p_{i,i-1}^t = r'/(1 + r')$, $p_{i,i}^t = 1$, and $p_{i,j}^t = 0$ for all other $i, j \in S_Z$.

The following analysis of the classical gambler’s ruin problem is well-known. See, for example, [8] Chapter XIV].
Lemma 24. Consider a random walk on \( \mathbb{Z}_{\geq 0} \) that absorbs at 0 and \( a \) (for some positive integer \( a \)), starts at \( z \in \{0, \ldots, a\} \), and from each state in \( \{1, \ldots, a-1\} \) has probability \( p \neq 1/2 \) of increasing (by 1) and probability \( q = 1 - p \) of decreasing (by 1). Then, the probability of reaching state \( a \) is
\[
\frac{1 - (q/p)^z}{1 - (q/p)^a}.
\]
Moreover, if \( p > 1/2 \), then the expected number of transitions before absorption is at most
\[
\frac{a}{p - q} \cdot \frac{1 - (q/p)^z}{1 - (q/p)^a}.
\]
\[
\square
\]

Lemma 25. Suppose \( b \geq (1/\lg r') + 1)^3 \), and write \( T^Z = \min\{i \mid Z^0_i = [b^{1/3}]\} \). Then
\[
E(|\{0 \leq i < T^Z \mid Z^0_i = 0\}|) \leq 2r'(r' - 1), \text{ and}
\]
\[
E(T^Z) \leq 6[b^{1/3}]/r'(r' - 1).
\]

Proof. By Lemma 24, the probability of reaching \([b^{1/3}]\) before 0 in \( Z^1 \) is
\[
\frac{1 - 1/r'}{1 - (1/r')^{b^{1/3}}} \geq \frac{r' - 1}{r'}.
\]

Thus in \( Z^0 \), the probability of reaching \([b^{1/3}]\) before returning to 0 is at least \( p^0_{0,1}(r' - 1)/r' = (r' - 1)/(r' + 1) \). Thus, the number of steps \( Z^0 \) spends at 0 before reaching \([b^{1/3}]\) is dominated from above by a geometric variable with parameter \((r' - 1)/(r' + 1)\), and so
\[
E(|\{0 \leq i < T^Z \mid Z^0_i = 0\}|) \leq \frac{r' + 1}{r' - 1} \leq \frac{2r'}{r' - 1},
\]
as required.

Now, by Lemma 24, the expected number of transitions that it takes for \( Z^1 \) to reach either 0 or \([b^{1/3}]\) is at most
\[
\frac{[b^{1/3}](r' + 1)}{r' - 1} \cdot \frac{1 - 1/r'}{1 - (1/r')^{b^{1/3}}} \leq \frac{[b^{1/3}](r' + 1)}{r' - 1} \cdot 2(1 - 1/r') = 2[b^{1/3}]/r'.
\]

So the expected number of transitions that it takes \( Z^0 \) to return to 0 or reach \([b^{1/3}]\) is at most
\[
1 + p^0_{0,1} \cdot \frac{2[b^{1/3}](r' + 1)}{r'} = 1 + 2[b^{1/3}] \leq 3[b^{1/3}],
\]

By Wald’s equation, it follows that
\[
E(T^Z) \leq \left(\frac{2r'}{r' - 1}\right) 3[b^{1/3}],
\]
and so the result follows.
\[
\square
\]

Lemma 26. Suppose \( b \geq \max\{(1/\lg r') + 1)^3, 120\} \). Then the probability that \( Y \) reaches state \([b^{1/3}]\) is at least \( 1 - 25/(\beta^{1/2}b(r - 1)) \).

16
Proof. Let $T^Z = \min\{i \mid Z^0_i = [b^{1/3}]\}$ and let $T^Y = \min\{i \geq 0 \mid Y_i \in \{[b^{1/3}], F\}\}$. Then we have
\[
\mathbb{P}(Y_{TV} = F) = \sum_{i=0}^{\infty} \sum_{x=0}^{[b^{1/3}]-1} \mathbb{P}(Y_i = x \text{ and } T^Y > i)p_{x,F}
= p_{0,F} \sum_{i=0}^{\infty} \mathbb{P}(Y_i = 0 \text{ and } T^Y > i) + p_{1,F} \sum_{i=0}^{\infty} \sum_{x=1}^{[b^{1/3}]-1} \mathbb{P}(Y_i = x \text{ and } T^Y > i).
\]
Since $[b^{1/3}] \leq \gamma$, the definitions of $Y$ and $Z^0$ show that the following are equivalent.

- $(y_0, \ldots, y_i)$ is a possible value of $(Y_0, \ldots, Y_i)$ which implies $Y_i = x$ and $T^Y > i$.
- $(y_0, \ldots, y_i)$ is a possible value of $(Z^0_0, \ldots, Z^0_i)$ which implies $Z^0_i = x$ and $T^Z > i$.

Moreover, for all $0 \leq i \leq [b^{1/3}] - 1$ and all $0 \leq j \leq [b^{1/3}]$ we have $p_{i,j} \leq p_{i,j}'$. It follows that
\[
\mathbb{P}(Y_{TV} = F) \leq p_{0,F} \sum_{i=0}^{\infty} \mathbb{P}(Z^0_i = 0 \text{ and } T^Z > i) + p_{1,F} \sum_{i=0}^{\infty} \sum_{x=1}^{[b^{1/3}]-1} \mathbb{P}(Z^0_i = x \text{ and } T^Z > i)
\leq p_{0,F} \cdot \mathbb{E}(|\{0 \leq i < T^Z \mid Z^0_i = 0\}|) + p_{1,F} \cdot \mathbb{E}(T^Z).
\]
It follows by Lemma 25 and the fact that $b \geq 120$ (and hence $b^{2/3} \geq 120/5$) that
\[
\mathbb{P}(Y_{TV} = F) \leq \frac{2r'}{r^{1/2}b} + \frac{10}{r^2b^2} \frac{6[b^{1/3}]r'}{r'^{-1}} \leq \frac{r'}{r(r' - 1)\beta^{1/2}b} \left(12 + \frac{60}{b^{3/2}\beta^{1/2}}\right)
\leq \frac{2}{(r - 1)\beta^{1/2}b} \left(12 + \frac{1}{2}\right) \leq \frac{25}{\beta^{1/2}b(r - 1)},
\]
as required. \qed

Lemma 27. Suppose $b \geq \max\{((1/{\lg} r') + 1)^3, 120\}$ and $X_0 \cap V_i \neq \emptyset$. Then with probability at least $1 - 25/(\beta^{1/2}b(r - 1))$, there exists $t \geq 0$ such that $|X_t \cap V_3| \geq [b^{1/3}]$.

Proof. By Lemma 26, the probability that $Y$ reaches state $[b^{1/3}]$ is at least $1 - 25/(\beta^{1/2}b(r - 1))$. The result therefore follows from Lemma 22. \qed

4.2 Going from mutants in $V_3$ to fixation

Lemma 28. Suppose $t \geq 0$ and $M \subseteq V$. Let $z \geq 1$, and suppose $z \leq \min\{|M \cap V_3|, \gamma\}$. Let $I = \min\{i \mid Z^*_i = 0\}$. Then, conditioned on $X_t = M$, there exists a coupling $\Psi(X, Z)$ between $(X_{t'})_{t' \geq t}$ and $(Z^*_{t'})_{t' \geq 0}$ such that, for all $i < I$, there is a $t' \geq t + i - 1$ such that $|X_{t'} \cap V_3| \geq Z^*_i$.

Proof. Following Definition 13 let $T_{end} = \min\{\bar{t} \geq t \mid X_{\bar{t}} \cap V_3 = \emptyset \text{ or } X_{\bar{t}} = V\}$. Note that $T_{end}$ is finite with probability 1. Define $T_0, T_1, \ldots$ recursively by $T_0 = t$ and
\[
T_i = \min\{T_{end} \cup \{\bar{t} > T_{i-1} \mid X_{\bar{t}} \cap V_3 \neq X_{\bar{t}-1} \cap V_3\}\}.
\]

17
Consider any \( t_i \geq t \) and \( M_i \subseteq V \) with \( 1 \leq |M_i \cap V_3| \leq \gamma \). Write \( \mathcal{F}_i \) for the event that \( T_i = t_i \neq T_{\text{end}} \), \( X_i = M_i \), and \( \Xi_i = \Xi \). For \( t'_i \geq t_i \) and \( M'_i \subseteq V \), we have \( P^{M'_i}_{t'_i} = P(X_{t'_i} = M'_i \text{ and } T_{t'_i+1} = t'_i + 1 \mid \mathcal{F}_i) \). Then we have
\[
P(|X_{T_{t_i} \cup t_i} \cap V_3| = |X_{T_i} \cap V_3| - 1 \mid \mathcal{F}_i)
\]
\[
= \sum_{t'_i \geq t_i} \sum_{M'_i \subseteq V \text{ s.t. } M'_i \cap V_3 = M_i \cap V_3} P(|X_{t'_i+1} \cap V_3| = |X_{t'_i} \cap V_3| - 1 \mid X_{t'_i} = M'_i, T_{t'_i+1} = t'_i + 1, \mathcal{F}_i) \cdot P^{M'_i}_{t'_i}.
\]

Note that since \( 1 \leq |M'_i \cap V_3| \leq \gamma \leq |V_3| - 1 \), the conditioning on \( T_{t'_i+1} = t'_i + 1 \) in the above expression is precisely equivalent to conditioning on \( |X_{t'_i+1} \cap V_3| = |X_{t'_i} \cap V_3| \pm 1 \). Moreover, by Lemma 14, writing \( \kappa = |E(V_3 \cap M'_3, V_3 \setminus M'_3)|/(W(M'_3)(\beta b^2 + b^2 - 1)) \), when \( M'_i \cap V_3 = M_i \cap V_3 \) we have
\[
P(|X_{t'_i+1} \cap V_3| = |X_{t'_i} \cap V_3| + 1 \mid X_{t'_i} = M'_i, \mathcal{F}_i) \geq r\kappa,
\]
\[
P(|X_{t'_i+1} \cap V_3| = |X_{t'_i} \cap V_3| - 1 \mid X_{t'_i} = M'_i, \mathcal{F}_i) \leq (1 + 5/\beta)\kappa.
\]

It therefore follows from Lemma 16 that
\[
P(|X_{T_{t_i+1}} \cap V_3| = |X_{T_{t_i}} \cap V_3| - 1 \mid \mathcal{F}_i) \leq \frac{1 + 5/\beta}{1 + 5/\beta + r} \sum_{t'_i \geq t_i} \sum_{M'_i \subseteq V \text{ s.t. } M'_i \cap V_3 = M_i \cap V_3} P^{M'_i}_{t'_i} \leq \frac{1}{1 + r^2}.
\]

Let \( I' = \min\{i \mid T_i = T_{\text{end}}\} \). We are now in a position to define a coupling \( \Psi(X, Z^z) \) such that
\[
\text{for all } j \leq I', \ |X_{T_j} \cap V_3| \geq Z_j^z.
\]
(10)

We first observe that such a coupling would satisfy the condition in the statement of the lemma. Consider \( i < I \). We wish to show that there is a \( t' \geq t+i-1 \) such that \( |X_{t'} \cap V_3| \geq Z_i^z \).

There are two cases to consider.

- If \( i < I' \), then, since \( T_i < T_{\text{end}} \), we have \( t+i-1 < T_i \). But from (10), \( |X_{T_i} \cap V_3| \geq Z_i^z \).

So we can take \( t' = T_i \).

- Suppose instead that \( i \geq I' \). From the definition of \( I' \), \( T_{I'} = T_{\text{end}} \), so from (10), we have \( |X_{T_{\text{end}}} \cap V_3| \geq Z_{I'}^z \). But since \( I' < i < I \), \( Z_{I'}^z > 0 \), so since \( X_{T_{\text{end}}} \cap V_3 \) is non-empty, the definition of \( T_{\text{end}} \) implies that the process fixates by time \( T_{\text{end}} \). Thus, for any \( t' \geq T_{\text{end}} \), we have \( |X_{t'} \cap V_3| = |V_3| \), and this is at least \( Z_j^z \) for any \( j \) (since the state space of \( Z^z \) only goes up to \( \gamma \)) so it suffices to take any \( t' \geq \max\{T_{\text{end}}, t+i-1\} \).

Given a coupling \( \Psi(X, Z^z) \) and a non-negative integer \( j \), let \( \Psi^j \) denote the initial sequence \( (X_0, X_{T_j}, Z_0^j, \ldots, Z_j^j) \). We will first construct the sequence \( \Psi^0, \Psi^1, \ldots \) by induction on \( j \), using \( \Psi^j \) (and some randomness) to construct \( \Psi^{j+1} \). We will continue this process until, for some \( j > 0 \), we obtain a \( \Psi^j \) which implies \( T_j = T_{\text{end}} \). (Note that \( P(I' < \infty \mid X_t = M) = 1 \).) We will then complete the coupling by allowing \( X_{T_{\text{end}}+1}, X_{T_{\text{end}}+2}, \ldots \) and \( Z_{I'+1}, Z_{I'+2}, \ldots \) to evolve independently according to their marginal distributions (which vacuously satisfy (10)). Note that \( \Psi^0 = (M, z) \) satisfies (10) since \( z \leq |M \cap V_3| \). Suppose we are given \( \Psi^j \) satisfying (10) with \( T_j < T_{\text{end}} \). We will now construct \( \Psi^{j+1} \).
• If $|X_{T_j} \cap V_3| \geq \gamma + 1$: We let $X_{T_{j+1}}, \ldots, X_{T_{j+1}}$ and $Z^*_{j+1}$ evolve independently according to their correct marginal distributions. Note that $|X_{T_{j+1}} \cap V_3| \geq |X_{T_j} \cap V_3| - 1 \geq \gamma \geq Z^*_{j+1}$, so (10) is satisfied for $j + 1$.

• If $Z^*_j = \gamma$: We let $X_{T_{j+1}}, \ldots, X_{T_{j+1}}$ and $Z^*_{j+1}$ evolve independently according to their correct marginal distributions. Note that by (10), $|X_{T_{j+1}} \cap V_3| \geq |X_{T_j} \cap V_3| - 1 \geq Z^*_j - 1 = Z^*_j$, so (10) is again satisfied for $j + 1$.

• If $|X_{T_j} \cap V_3| \leq \gamma$ and $Z^*_j < \gamma$: Note that since $T_j < T_{\text{end}}$, in this case we also have $|X_{T_j} \cap V_3| \geq 1$. Let $\mathcal{E}_{\text{down}}$ be the event that $|X_{T_{j+1}} \cap V_3| = |X_{T_j} \cap V_3| - 1$, and let $p_{\text{down}}$ be the probability that $\mathcal{E}_{\text{down}}$ occurs in the correct marginal distribution (which depends only on $X_{T_j}$). Let $\mathcal{E}_{\text{up}}$ be the event that $|X_{T_{j+1}} \cap V_3| = |X_{T_j} \cap V_3| + 1$, and let $p_{\text{up}} = 1 - p_{\text{down}}$. Then (9) shows that $p_{\text{down}} \leq 1/(1 + r')$. The coupling is as follows.

- Choose $X_{T_{j+1}}, \ldots, X_{T_{j+1}}$ according to the correct marginal distribution.
- $\mathcal{E}_{\text{down}}$ occurs with probability $p_{\text{down}} \leq 1/(1 + r')$. When this happens, set $Z^*_{j+1} = \max\{Z^*_j - 1, 0\}$.
- $\mathcal{E}_{\text{up}}$ occurs with probability $p_{\text{up}} \geq r'/(1 + r')$. When this happens, set $Z^*_{j+1} = Z^*_j + 1$ with probability $r'/(1 + r')$, and set $Z^*_{j+1} = \max\{Z^*_j - 1, 0\}$ with probability $1 - r'/(1 + r')$.

It is now easy to check that $Z^*_{j+1} = Z^*_j + 1$ and $Z^*_{j+1} = \max\{Z^*_j - 1, 0\}$ both happen with the correct marginal distribution. Moreover, the coupling itself guarantees that $|X_{T_{j+1}} \cap V_3| \geq Z^*_{j+1}$, so (10) is satisfied for $j + 1$.

We use the following Lemma from [3] Theorem 9 (which applies to all graphs).

**Lemma 29.** For all $M \subseteq V$, the expected absorption time of $X$ from state $M$ is at most $r|V|^3/(r - 1)$.

**Lemma 30.** There exists $b_0$ depending only on $r$ such that the following holds whenever $b \geq b_0$. Suppose $t \geq 0$ and $M \subseteq V$ with $|M \cap V_3| \geq [b^{1/3}]$. Then we have

$$
P(X \text{ fixes } X_t = M) \geq 1 - 1/(\beta^{1/2}b(r - 1)).
$$

**Proof.** Recall that $\gamma = [(k\beta)^{1/3}]$. Let $\xi = [b^{1/3}]$ and $T = [(r')(\gamma - 1)/2]$, and let $b_0$ be such that $b_0 \geq \max\{\beta/r, 120, ((1/\log r') + 1)^3\}$ and, for all $b \geq b_0$,

$$
1/(r')^\xi + T/(r')^{\gamma - 1} + 16r^k\beta^4b^4/(r - 1)(T - 1) \leq 1/\beta^{1/2}b(r - 1).
$$

(Note that $b \geq b_0$ implicitly gives a lower bound on $k$ since $b = b(k) \leq \sqrt{k}$.)

By Lemma 24, the probability that $Z^\xi$ reaches $\gamma$ before zero is

$$
1 - (1/r')^\xi \geq 1 - 1/(r')^\xi.
$$
Moreover, Lemma 24 also shows that the probability that $Z_\xi$ reaches 0 on any given sojourn from $\gamma$ is at most

$$1 - \frac{1}{1 - \left(\frac{r}{r'}\right)^{\gamma-1}} \leq \frac{1}{(r')^{\gamma-1}}.$$

Thus the probability that $Z_\xi$ never reaches zero when it makes $T$ transitions from state $\gamma$ is at least

$$\left(1 - \frac{1}{(r')^{\gamma-1}}\right)^T \geq 1 - \frac{T}{(r')^{\gamma-1}}.$$

Thus the probability that $Z_\xi$ reaches zero from state $\xi$ within $T$ transitions is at most

$$\frac{1}{(r')^{\xi}} + \frac{T}{(r')^{\gamma-1}}.$$

If $Z_\xi$ does not reach zero within $T$ transitions and we couple it with $X$ according to Lemma 28, noting that $T < I$, then there is a $t' \geq t + T - 1$ such that $X_{t'}$ is non-empty. Thus,

$$\mathbb{P}(X_{t+T-1} = \emptyset \mid X_t = M) \leq \frac{1}{(r')^{\xi}} + \frac{T}{(r')^{\gamma-1}}.$$

(12)

Now, by Lemma 29 combined with Markov’s inequality, we have

$$\mathbb{P}(X_{t+T-1} \notin \{\emptyset, V\} \mid X_t = M) \leq \frac{r|V|^4}{(r-1)(T-1)} \leq \frac{r(2kr\beta^{1/2}b)^4}{(r-1)(T-1)} = \frac{16r^5k^4\beta^2b^4}{(r-1)(T-1)}.$$

(Here the upper bound on $|V|$ follows from Observation 8.) Hence by (12) and a union bound, it follows that

$$\mathbb{P}(X_{t+T-1} \neq V \mid X_t = M) \leq \frac{1}{(r')^{\xi}} + \frac{T}{(r')^{\gamma-1}} + \frac{16r^5k^4\beta^2b^4}{(r-1)(T-1)}.$$

The result therefore follows from (11).

Lemma 31. There exists $b_0$ depending only on $r$ such that the following holds whenever $b \geq b_0$. If $X_0 \cap V_1 \neq \emptyset$, then $(X_t)_{t \geq 0}$ fixates with probability at least $1 - 26/(\beta^{1/2}b(r-1)).$

Proof. By Lemma 27 and Lemma 30 when $b$ is sufficiently large we have

$$\mathbb{P}(X \text{ fixates} \mid X_0 \cap V_1 \neq \emptyset) \geq 1 - \frac{25}{\beta^{1/2}b(r-1)} - \frac{1}{\beta^{1/2}b(r-1)},$$

so the result follows.

Lemma 32. There exists $b_0$ depending only on $r$ such that, whenever $b \geq b_0$, for all $x_0 \in V_3$,

$$\mathbb{P}(X \text{ fixates} \mid X_0 = \{x_0\}) \geq 1 - 2/r.$$

Proof. Let $T = \min\{t > 0 \mid |X_t \cap V_3| = \gamma\}$. By Lemma 24, the probability that $Z^1$ reaches $\gamma$ before 0 is at least $1 - 1/r'$. Thus by Lemma 28 we have

$$\mathbb{P}(T < \infty \mid X_0 = \{x_0\}) \geq 1 - 1/r'.$$

(13)
Moreover, by Lemma 30 when $b$ is sufficiently large, for all $t > 0$ and all $M \subseteq V$ with $|M \cap V_3| \geq \gamma$, we have

$$P(X \text{ fixes } | T = t, X_t = M, X_0 = \{x_0\}) \geq 1 - \frac{1}{\beta^{1/2}b(r-1)}.$$  

Summing over all possible values of $t$ and $M$, we obtain

$$P(X \text{ fixes } | T < \infty, X_0 = \{x_0\}) \geq 1 - \frac{1}{\beta^{1/2}b(r-1)}.$$ 

Thus by (13), taking $b_0 \geq (r+1)\sqrt{r-1}$, it follows that

$$P(X \text{ fixes } | X_0 = \{x_0\}) \geq 1 - \frac{2}{r+1} - \frac{1}{5r(r+1)} \geq 1 - \frac{2}{r}.$$ 

\[4.3\]

Putting it all together

We can now prove Theorem 6, which we restate for convenience.

**Theorem 6.** Let $r > 1$. There is a constant $b_0$ depending only on $r$ such that the following holds. Let $b : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ be any function that satisfies $b(k) \leq \sqrt{k}$ for all $k$. Consider a graph $G \in I_{r, b}$ with branching factor $b(k) \geq b_0$. Let $n$ be the number of vertices of $G$ and $m$ be the number of edges of $G$. Then $\ell_r(G) \leq 2^{14} r n / ((r-1)^2 m)$.

**Proof.** By Lemmas 31 and 32 when $b(k)$ is sufficiently large we have

$$\ell_r(G) \leq \frac{|V_3|}{n} \cdot \frac{2}{r} + \frac{|V_2|}{n} + \frac{|V_1|}{n} \left( \frac{26}{\beta^{1/2}b(k)(r-1)} \right) \leq \frac{3\beta k}{rn} + \frac{26}{\beta^{1/2}b(k)(r-1)}.$$  

Since Observation 8 implies that $n \geq kr\beta^{1/2}b(k)$, it follows that

$$\ell_r(G) \leq \frac{3\beta^{1/2}}{r^2 b(k)} + \frac{26}{\beta^{1/2}b(k)(r-1)} \leq \frac{4\beta^{1/2}}{r^2 b(k)}. \quad (14)$$

Since Observation 8 implies that $m/n \leq \beta^{3/2}b(k)/r$, it follows that

$$\ell_r(G) \leq \frac{\beta^{3/2}b(k)n}{rm} \cdot \frac{4\beta^{1/2}}{r^2 b(k)} = \frac{4\beta^2 n}{r^3 m}.$$ 

Since $r^2/(r-1) > 1$, we have $\beta \leq 52 r^2/(r-1)$ and hence

$$\ell_r(G) \leq \frac{2^{14} r n}{(r-1)^2 m},$$

as required.  

Finally, we prove Theorem 2.

**Theorem 2.** For all $r > 1$, there exists an infinite family $D_r$ of connected graphs with the following property. If $G \in D_r$ has $n$ vertices, then $\ell_r(G) \leq 71 / (r(r-1)^2 m)^{1/5}$.  

21
Proof. Let \( b(k) = \lfloor \sqrt{k} \rfloor \). Consider any \( r > 1 \) and let the constant \( b_0 \) (depending on \( r \)) be the one from the statement of Theorem 6. Define

\[ D_r = \{ G \in \mathcal{I}_{r,b} \mid \text{The parameter, } k, \text{ of } G \text{ is the square of an integer and } b(k) \geq b_0 \} \]

Note that in the definition of \( \mathcal{I}_{r,b} \) (Definition 5), when \( k \) is a square integer, \( G[V_2, V_3] \) is a complete bipartite graph and \( G[V_3] \) is a clique. Thus \( D_r \) is an infinite family.

Consider any \( G \in D_r \). Note that by Observation 8, \( n \leq 2k^{3/2}r^{\beta/2} \), and hence \( k^{1/2} \geq (n/(2r^{\beta/2}))^{1/3} \). Moreover, as in (14), we have \( \ell_r(G) \leq 4\beta^{1/2}/(r^2b(k)) \). It follows that

\[ \ell_r(G) \leq \frac{4\beta^{1/2}}{r^2} \cdot \frac{2^{1/3}r^{1/3} \beta^{1/6}}{n^{1/3}} = \frac{2^{7/3} \beta^{2/3}}{r^{5/3}n^{1/3}} \leq \frac{71}{r^{1/3}(r-1)^{2/3}n^{1/3}}, \]

and so the result follows. (The final inequality uses \( \beta \leq 52r^2/(r-1) \) as in the proof of Theorem 6.)

5 Lower bounds on extinction probability

In this section, we prove Theorems 1, 3 and 7 which give lower bounds on extinction probability. The proofs of our theorems rely on the following quantity, which has also been studied in the undirected case in [15, 16].

Definition 33. Given a digraph \( G = (V, E) \), we define the danger of any vertex \( v \) as

\[ Q_v = \sum_{u \in N_{in}(v)} \frac{1}{d_{out}(u)}. \]

Note that the danger of \( v \) is essentially the rate at which \( v \) dies when all of its in-neighbours are non-mutants. Let \( G = (V, E) \) be a graph and let \( v \) be a vertex in \( V \).

We start with a lemma giving an upper bound on the danger of a vertex. The lemma is based on exploring several steps of a dominating process. For other related dominations, see Theorem 1 of [15].

Lemma 34. Let \( G = (V, E) \) be a strongly-connected digraph with \( |V| \geq 3 \), and suppose that \( u \in V \). Consider the Moran process on \( G \) with fitness \( r \geq 1 \). Then \( \ell_r(u) \geq Q_u/(r + Q_u) \). Moreover, if \( \ell_r(u) \leq 1/(4r) \), then

\[ Q_u \leq \frac{2^5r\ell_r(u)}{d_{out}(u)} \sum_{v \in N_{out}(u)} \frac{r}{r + Q_v}. \]

Proof. The first part of the result is immediate, since \( Q_u/(r + Q_u) \) is precisely the probability that \( u \) dies before spawning a mutant. For the second part, suppose \( \ell_r(u) \leq 1/(4r) \). For \( X \subseteq V \), let \( \mathcal{E}_X \) be the event that from state \( X \), the process goes extinct before any of the following events occur:

(E1) \( u \) spawns a mutant onto a vertex \( v \) with \( Q_v \leq 2r \); or

(E2) \( u \) spawns a mutant while there is another mutant in the process; or

(E3) some vertex other than \( u \) spawns a mutant.
Write $p_X = \mathbb{P}(E_X)$. If $E_{(u)}$ occurs then extinction occurs from $u$, so $\ell_r(u) \geq p_{(u)}$. Note that since $\ell_r(u) \leq 1/(4r) \leq 1/2$, we have $Q_u \leq r$ by the first part of the result.

First consider a state $\{u, v\}$, where $v \in N_{out}(u)$ and $Q_v > 2r$ (if such a state exists). Write

$$Q_u = \sum_{w \in N_{out}(u) \setminus \{v\}} \frac{1}{d_{out}(w)} \quad \text{and} \quad Q_v = \sum_{w \in N_{in}(v) \setminus \{u\}} \frac{1}{d_{out}(w)}.$$

Since $(E2)$ occurs if $u$ spawns a mutant and $(E3)$ occurs if $v$ spawns a mutant, we have

$$p_{(u,v)} = \frac{Q_u}{2r + Q_u + Q_v} p_{(v)} + \frac{Q_v}{2r + Q_u + Q_v} p_{(u)}.$$

Note that $Q_v - Q_u = 1/d_{out}(u) \leq 1$, so since $Q_v \geq 2r \geq 2$ we have $Q_v \geq Q_v/2$. Moreover, $Q_u \leq Q_u \leq r$. Since the second summand of the right hand side of (15) is increasing in $Q_v$ and decreasing in $Q_u$, it follows that

$$p_{(u,v)} \geq \frac{Q_v/2}{2r + r + Q_v/2} p_{(u)} = \left(1 - \frac{6r}{6r + Q_v}\right) p_{(u)} \geq \left(1 - \frac{6r}{r + Q_v}\right) p_{(u)}.$$

Now consider the state $\{u\}$. Here, $(E1)$ occurs if $u$ spawns a mutant onto a vertex with danger at most 2r, and $E_{(u)}$ occurs if $u$ dies. It follows from (16) that

$$p_{(u)} = \frac{Q_u}{r + Q_u} + \frac{r}{r + Q_u} \sum_{v \in N_{out}(u)_{Q_v \geq 2r}} p_{(u,v)}$$

$$\geq \frac{Q_u}{2r} + \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)_{Q_v \geq 2r}} \left(1 - \frac{6r}{r + Q_v}\right) p_{(u)}$$

$$\geq \frac{Q_u}{2r} + \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)_{Q_v \geq 2r}} \left(1 - \frac{6r}{r + Q_v}\right) p_{(u)} - Q_u p_{(u)}.$$

Note that when $Q_v \leq 2r$, $1 - 6r/(r + Q_v) < 0$, and so

$$\frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)_{Q_v \geq 2r}} \left(1 - \frac{6r}{r + Q_v}\right) \geq \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)} \left(1 - \frac{6r}{r + Q_v}\right)$$

$$= 1 - \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)} \frac{6r}{r + Q_v}.$$

It follows from (17) and (18) that

$$p_{(u)} \geq \frac{Q_u/2}{Q_u + \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)} \frac{6r}{r + Q_v}}.$$

Also, $\frac{Q_u/2}{Q_u} = 1/(4r) \geq \ell_r(u) \geq p_{(u)}$, so combining and rearranging yields

$$Q_u \leq \frac{1}{d_{out}(u)} \sum_{v \in N_{out}(u)} \frac{6r}{r + Q_v}.$$
From (19) and the fact that \( \ell_r(u) \geq p_{(u)} \), we therefore obtain

\[
\ell_r(u) \geq \frac{Q_u/(4r)}{d_{out}(u) \sum_{v \in N_{out}(u)} \frac{br}{r+Q_v}}.
\]

The result therefore follows. (Note that we bound 24 < 25 for readability.) \( \square \)

The following lemma gives an upper bound on the total danger of a set of vertices with low extinction probability. Throughout the rest of the section, this lemma will be the main point of interaction between our arguments and the definition of the Moran process — the remainder of our arguments will focus on how vertex dangers and extinction probabilities can be distributed.

**Lemma 35.** Let \( G \) be a strongly-connected \( n \)-vertex digraph with \( n \geq 3 \). Consider the Moran process on \( G \) with fitness \( r \geq 1 \). Let \( S \subseteq V(G) \). Suppose that, for some \( \alpha \leq 1/(4r) \), every vertex \( v \in S \) has \( \ell_r(v) \leq \alpha \). Then \( \sum_{v \in S} Q_v \leq 2^5 r^2 \alpha |N_{out}(S)| \) and \( \sum_{v \in S} Q_v \leq 2^5 r^2 n \alpha \ell_r(G) \).

**Proof.** By applying Lemma 34 to all \( v \in S \),

\[
\sum_{v \in S} Q_v \leq \sum_{v \in S} \frac{2^5 r^2 \alpha}{d_{out}(v)} \sum_{w \in N_{out}(v)} \frac{r}{r+Q_w} = \sum_{w \in N_{out}(S)} \frac{2^5 r^2 \alpha}{r+Q_w} \sum_{v \in N_{in}(w) \cap S} \frac{1}{d_{out}(v)} \leq 2^5 r^2 \alpha \sum_{w \in N_{out}(S)} \ell_r(w),
\]

where the final inequality follows by the first part of Lemma 34. The first part of the result follows by bounding \( \ell_r(w) \leq 1 \), and the second part of the result follows since

\[
\sum_{w \in N_{out}(S)} \ell_r(w) \leq \sum_{w \in V} \ell_r(w) = n \ell_r(G).
\]

\( \square \)

We can now prove Theorem 1, which we restate here for convenience. Note that when \( r = 1 \), we have \( \ell_r(G) = 1 - 1/n \) for all \( n \)-vertex graphs \( G \) (see Lemma 1 of [4]). So from now on, we will take \( r > 1 \).

**Theorem 1.** For all \( r > 1 \), any strongly-connected \( n \)-vertex digraph \( G \) with \( n \geq 3 \) satisfies \( \ell_r(G) > 1/(12 r n^{1/2}) \).

**Proof.** Let \( V \) be the vertex set of \( G \). Suppose that \( \ell_r(G) \leq 1/(8r) \), or we are done. Note that \( Q_v \geq 1/n \) for all \( v \in V \). Let \( A = \{ v \in V \mid \ell_r(v) \leq 2 \ell_r(G) \} \), and note that \( \ell_r(G) > (|V \setminus A|/n) \cdot 2 \ell_r(G) \) and hence \( |A| > n/2 \). Applying Lemma 35 to \( A \) with \( \alpha = 2 \ell_r(G) \leq 1/(4r) \) yields

\[
\frac{1}{2} \leq \sum_{v \in A} Q_v \leq 2^6 r^2 n \ell_r(G)^2,
\]

from which the result follows. \( \square \)
In the proof of Theorem 3 we used the fact that, in an $n$-vertex graph, every vertex $v$ with “low” extinction probability has $Q_v \geq 1/n$. For undirected graphs, where we want to prove a stronger result, this bound is too loose. Instead we must account for the vertices with low extinction probability that have high danger. We consider undirected graphs in the rest of this section. In order to prove Theorem 3 we define a useful partial order.

**Observation 37.** If $U$ is a finite set of positive integers and $f : U \to \mathbb{R}_{>0}$ then $\preceq_f$ is a partial order on $U$ with a non-empty set $\mathcal{M}_f$ of maximal elements. For all $i \in U \setminus \mathcal{M}_f$, there exists an element $f^*(i) \in \mathcal{M}_f$ such that $f(i) \leq 4^{-|i-f^*(i)|}f(f^*(i))$.

The proof of Theorem 3 will follow easily from two lemmas.

**Lemma 38.** Let $r > 1$. Consider any connected $n$-vertex graph $G = (V, E)$ with $n \geq 3$ and $\ell_r(G) \leq 1/(8r)$. Then there exists a non-empty subset $B$ of $V$ such that $\sum_{v \in B}d(v) \geq n/(2^{15}r^2\ell_r(G))$ and, for all $v \in B$, $d(v) \geq 1/(2^{12}r^2\ell_r(G))^2$.

**Proof.** Let $A = \{v \in V \mid \ell_r(v) \leq 2\ell_r(G)\}$. As in the proof of Theorem 1, note that $|A| > n/2$. We now apply Lemma 35 with $S = A$ and $\alpha = 2\ell_r(G)$. Note that $\alpha \leq 1/(4r)$. This shows that $\sum_{v \in A}Q_v \leq 2^{5r^2}\ell_r(G)^2$. Let $A' = \{v \in A \mid Q_v < 2^{5r^2}\ell_r(G)^2\}$. Then

$$2^{8r^2}\ell_r(G)^2|A \setminus A'| \leq \sum_{v \in A \setminus A'}Q_v \leq \sum_{v \in A}Q_v \leq 2^{6r^2}\ell_r(G)^2;$$

so $|A \setminus A'| \leq n/4$ and $|A'| \geq |A| - |A \setminus A'| \geq n/2 - n/4 = n/4$.

Let $K = 16$, and let $I = \lceil \log_K(2^{8r^2}\ell_r(G)^2) \rceil$. Before proceeding, we prove that $I \geq 1$. Since $|A'| \geq n/4$, $A'$ contains some vertex $v$. Since $G$ is connected and has more than one vertex, $v$ has a neighbour $w$ and the degree of $w$ is less than $n$ (since all degrees are less than $n$). Hence by the definition of $Q_v$ and $A'$, $\frac{1}{2} < Q_v < 2^{5r^2}\ell_r(G)^2$. Thus, $2^{8r^2}\ell_r(G)^2 > 1$ and $\log_K(2^{8r^2}\ell_r(G)^2) > 0$. Since $I$ is an integer, it follows that $I \geq 1$.

For $i \in [I]$, let $Q_i = K^{i-1}/n$ and let $A_i = \{v \in A' \mid Q_i \leq Q_v < Q_{i+1}\}$. As we already observed, every vertex $v \in V$ has $Q_v > 1/n = Q_1$. Also, if $Q_v \geq Q_{i+1} = K^i/n \geq 2^{5r^2}\ell_r(G)^2$, then from the definition of $A'$, $v$ is not in $A'$. Thus, $A_1, \ldots, A_i$ partition $A'$.

For each $i \in [I]$, let $B_i = N(A_i)$. Since every vertex $v \in N(A_i)$ is joined to some vertex $w \in A_i$, we have $1/d(v) \leq Q_w < Q_{i+1}$ and hence

$$\text{for all } i \in [I], \text{ for all } v \in B_i, \ d(v) > 1/Q_{i+1}. \quad (20)$$

We now apply Lemma 35 with $S = A_i$ and $\alpha = 2\ell_r(G)$ to show that $\sum_{v \in A_i}Q_v \leq 2^{5r^2}\alpha |N(A_i)| = 2^{6r^2}\ell_r(G)|B_i|$. Since, by the definition of $A_i$, we have $Q_i |A_i| \leq \sum_{v \in A_i}Q_v$, we conclude that $|B_i| \geq Q_i |A_i|/(2^{5r^2}\ell_r(G))$. For each $i \in [I]$, let $B'_i$ be an arbitrary subset of $B_i$ such that $|B'_i| = |Q_i |A_i|/(2^{5r^2}\ell_r(G))|$. Before we progress further, we give the intuition behind the argument. Suppose that we tried to take the set $B$ in the statement of the lemma to be $B = B'_1 \cup \cdots \cup B'_I$. Then by (20),

$$\sum_{i \in [I]} \sum_{v \in B'_i}d(v) \geq \sum_{i \in [I]} \frac{Q_i |A_i|}{2^{6r^2}\ell_r(G)|B'_i|} = \frac{1}{2^{10r^2}\ell_r(G)} \sum_{i \in [I]} |A_i| = \frac{|A'|}{2^{10r^2}\ell_r(G)} \geq \frac{n}{2^{12r^2}\ell_r(G)}.$$
However, this would not suffice to prove the first requirement of the lemma, as the sets $B'_1, \ldots, B'_j$ need not be disjoint. Note that if the sizes of $A_1, \ldots, A_j$ were not too different, say if $|A_i| \geq 4^{-|i-j|}|A_j|$ for all $i, j \in [I]$, then $|B'_i|, \ldots, |B'_j|$ would grow geometrically. Thus we could make $B'_1, \ldots, B'_j$ disjoint (at the cost of a constant factor) by simply removing the smaller sets from the larger ones. It will turn out that if $|A_j| \leq 4^{-|i-j|}|A_i|$, then we may safely remove $B'_j$ from consideration without affecting the sum too much. We will obtain our set $B$ by combining these two ideas.

To make the intuition rigorous, let $I = \{i \mid B_i \neq \emptyset\}$, let $f : I \to \mathbb{R}_{>0}$ be defined by $f(i) = |B'_i|/Q_i$ and consider the relation $\lesssim_f$ from Definition 36 and the set $\mathcal{M}_f \subseteq I$ of maximal elements from Observation 37. Let $f^* : I \setminus \mathcal{M}_f \to \mathcal{M}_f$ be the function defined in Observation 37.

If $i, j \in I$ do not satisfy $i \lesssim_f j$, then we have $|B'_i|/Q_i \geq 4^{-|i-j|}|B'_j|/Q_j$. So when $i, j \in \mathcal{M}_f$ and $j < i$, we have

$$|B'_j| \leq \left(\frac{Q_i}{Q_j}\right)^{4^{-j}}|B'_i| = \left(\frac{4}{K}\right)^{i-j}|B'_i| = 4^{i-j}|B'_i|. \quad (21)$$

Let $B''_i = B'_i \setminus \bigcup_{j \in [i-1]} B'_j$, and let $B = \bigcup_{i \in \mathcal{M}_f} B''_i$. The set $B$ is the one promised in the statement of the lemma. We can immediately prove the second inequality in the statement of the lemma since every vertex in $B$ is in some $B''_i \subseteq B'_i \subseteq B$ so by Equation (20), it has degree greater than $1/Q_{I+1} = n/K^{1} \geq 1/(2^qK^2\ell_{v}(G)^2)$.

The remainder of the proof gives a lower bound on $\sum_{v \in B} d(v)$ to establish the first inequality in the statement of the lemma. The sets in $\{B''_i \mid i \in \mathcal{M}_f\}$ are disjoint by construction, so by (21),

$$\sum_{v \in B} d(v) \geq \sum_{i \in \mathcal{M}_f} \frac{|B''_i|}{Q_{i+1}}. \quad (22)$$

By the definition of $B''_i$ and Equation (21), we have that

$$\text{for all } i \in \mathcal{M}_f, \quad |B''_i| \geq |B'_i| - \sum_{j \in [i-1] \setminus \mathcal{M}_f} |B'_j| \geq |B'_i| - \sum_{t=1}^{i-1} 4^{-t}|B'_t| \geq |B'_i|/2.$$ 

Thus by (22),

$$\sum_{v \in B} d(v) \geq \sum_{i \in \mathcal{M}_f} \frac{|B'_i|}{2Q_{i+1}} = \sum_{i \in [I]} \frac{|B'_i|}{2Q_{i+1}} \geq \sum_{i \in \mathcal{M}_f} \sum_{f^*(j) = i} \frac{|B'_j|}{2Q_{j+1}}. \quad (23)$$

For all $i \in \mathcal{M}_f$, we have (by the definition of $\mathcal{M}_f$)

$$\sum_{j \in [I] \setminus \mathcal{M}_f} \frac{|B'_j|}{2Q_{j+1}} \leq \sum_{j \in (I \setminus \mathcal{M}_f) \cap \mathcal{M}_f} \frac{|B'_j|}{2Q_{j+1}} \leq 2 \sum_{t=1}^{\infty} 4^{-t} \frac{|B'_t|}{2Q_{t+1}} = \frac{|B'_i|}{3Q_{i+1}}. \quad (24)$$

(Here the factor of 2 in the final inequality is to account for the possibility that, for example, both $i - 1$ and $i + 1$ are elements of $I \setminus \mathcal{M}_f$.) It follows by (23) that

$$\sum_{v \in B} d(v) \geq \sum_{i \in [I]} \frac{|B'_i|}{2Q_{i+1}} - \sum_{i \in \mathcal{M}_f} \frac{|B'_i|}{3Q_{i+1}} > \sum_{i \in [I]} \frac{|B'_i|}{8Q_{i+1}}.$$
Then using the definition of $B_i'$, we have

$$
\sum_{v \in B} d(v) \geq \sum_{i \in [l]} \frac{Q_i|A_i|}{2^{2r^2}t_r(G)S_{Q_i+1}} = \frac{1}{2^{2r^2}t_r(G)} \sum_{i \in [l]} |A_i|.
$$

Since the $A_i$’s partition $A'$, which has size at least $n/4$, we get $\sum_{v \in B} d(v) \geq n/(2^{15r^2}t_r(G))$, which is the desired inequality in the statement of the lemma. \hfill \Box

Lemma 38 already makes it easy to prove Theorem 7, which we restate here for convenience.

**Theorem 7**. Let $r > 1$. Consider any connected graph $G$ with $n \geq 3$ vertices and $m$ edges. Then

$$
\ell_r(G) \geq \frac{n}{2^{16r^2}m}.
$$

**Proof**. Let $G = (V, E)$. Note that since $G$ is connected and $n \geq 3$, we have $m \geq n - 1 > n/2$. Thus if $\ell_r(G) \geq 1/(8r) > n/(2^{16r^2}m)$, then the result holds. If not, we may apply Lemma 38 to $G$ to obtain a subset $B$ of $V$. We then have

$$
|E| \geq \frac{1}{2} \sum_{v \in B} d(v) \geq \frac{n}{2^{16r^2}t_r(G)}.
$$

The result follows. \hfill \Box

The second lemma used in the proof of Theorem 3 is the following.

**Lemma 39**. Let $r > 1$ and $\delta = 1/(2^{15r^2})$. Consider any connected $n$-vertex graph $G = (V, E)$ with $n \geq 3$ and $\ell_r(G) \leq \delta n^{-1/3}$. Then there exists no non-empty subset $B$ of $V$ such that $\sum_{v \in B} d(v) \geq n^{4/3}/2$ and, for all $v \in B$, $d(v) \geq 2^{5r^2}n^{2/3}$.

**Proof**. Suppose, for a contradiction, that such a non-empty subset $B$ of $V$ does exist. For every positive integer $i$, let $d_i = 2^{5r^2}n^{2/3}4^{i-1}$ and let $B_i = \{v \in B \mid d_i \leq d(v) < d_{i+1}\}$. Let $I = [\log_4(n^{1/3}/(2^{5r^2}))]$. Note that $B$ is non-empty, so there is a vertex with degree at least $2^{5r^2}n^{2/3}$. Since all degrees are less than $n$, this implies that $2^{5r^2} < n^{1/3}$. Hence, $I \geq 1$. Every vertex in $B$ has degree at least $d_1$. Also, $d_{i+1} \geq n$, so $B_1, B_2, \ldots, B_I$ is a partition of $B$. Let $I = \{i \in [I] \mid |B_i| > 0\}$ and $D = \sum_{i=1}^{I} d_i |B_i|$. Then

$$
D = \sum_{i \in I} d_i |B_i| \geq \frac{1}{4} \sum_{v \in B} d(v) \geq \frac{n^{4/3}}{8}.
$$

For any positive integer $i$, let $\ell_i = d_i |B_i|/(4^i r^2 D)$ and let $X_i = \{v \in V \mid \ell_r(v) \leq \ell_i\}$. We next give an upper bound on the number of edges from $B_i$ to $X_i$. By Lemma 38 applied with $S = X_i$ and $\alpha = \ell_i \leq 1/(4r)$, for all $i \in [I]$ we have

$$
\sum_{v \in X_i} Q_v \leq 2^{6r^2}t_r |\delta n^{2/3} / 4^{i-1} D| = \frac{2^3 d_i |B_i| \delta n^{2/3}}{4^{i-1} D} = \frac{|B_i| n^{4/3}}{2^2 D}.
$$

It follows by (24) that

$$
\sum_{v \in X_i} Q_v \leq |B_i|/16.
$$

(25)
On the other hand, using the definition of $Q_v$ and then the definitions of $B_i$ and $d_i$,
\[
\sum_{v \in X_i} Q_v \geq \sum_{v \in X_i} \sum_{w \in B_i \cap N(v)} \frac{1}{d(w)} = \sum_{w \in B_i} \frac{|N(w) \cap X_i|}{d(w)} \geq \sum_{w \in B_i} \frac{|N(w) \cap X_i|}{d_i + 1} \geq \frac{|E(B_i, X_i)|}{4d_i}.
\]
Thus by (25), $|E(B_i, X_i)| \leq d_i|B_i|/4$.

Now let $C_i = N(B_i) \setminus X_i$. Note that there are at least $\frac{1}{2} \sum_{v \in B_i} d(v)$ edges incident to $B_i$, where the factor of $\frac{1}{2}$ is to avoid over-counting edges with both endpoints in $B_i$. It follows that
\[
|E(B_i, C_i)| = |E(B_i, V \setminus X_i)| \geq \frac{1}{2} \sum_{v \in B_i} d(v) - |E(B_i, X_i)| \geq d_i|B_i|/4. \quad (26)
\]
When $i \in I$, $B_i \neq \emptyset$ and so there is some vertex $v \in B_i$ that sends at least as many edges into $C_i$ as the average over $B_i$. Since the degree of every vertex is an integer, it follows that $|C_i| \geq [d_i/4]$. Now, for all $i \in I$, let $C_i'$ be an arbitrary subset of $C_i$ such that $|C_i'| = [d_i/4]$. Thus for all $i \in I$, since $d_i \geq d_i \geq 25 \geq 12$,
\[
d_i/4 \leq |C_i'| \leq d_i/4 + 1 \leq d_i/3. \quad (27)
\]
Let $C_i'' = C_i' \setminus \bigcup_{j \in [i-1] \cap I} C_j'$. Then all sets $C_i''$ are disjoint by construction, and by (27) we have
\[
|C_i''| \geq \frac{d_i}{4} - \sum_{j \in [i-1] \cap I} |C_j'| \geq \frac{d_i}{4} - \sum_{j \in [i-1]} d_j/3 \geq \frac{d_i}{4} - \frac{1}{3} \sum_{j=1}^\infty d_j 4^{-j} = \frac{d_i}{8},
\]
Since $C_i'' \subseteq C_i' \subseteq C_i = N(B_i) \setminus X_i$, we have $C_i'' \cap X_i = \emptyset$. So by the definition of $X_i$, for all $v \in C_i''$, we have $\ell_v(v) > \ell_i$. It follows that
\[
\ell_r(G) = \frac{1}{n} \sum_{v \in V} \ell_v(v) > \frac{1}{n} \sum_{i \in I} |C_i''| \ell_i \geq \frac{1}{n} \sum_{i \in I} \frac{d_i}{8} \cdot \frac{|B_i|}{4d_i/2} \cdot \frac{d_i/4}{D} = \frac{\sum_{i \in I} d_i |B_i|}{4n^{1/3}D} = \frac{1}{n^{1/3}} > \delta n^{-1/3}.
\]
But this contradicts the hypothesis of the lemma, so $B$ cannot exist. $\square$

We can now prove Theorem 3 which we restate here for convenience.

**Theorem 3.** Let $r > 1$. Consider any connected $n$-vertex graph $G$ with $n \geq 3$. Then $\ell_r(G) > 1/(2^{15}n^{1/3})$.

**Proof.** Let $\delta = 1/(2^{15}r^2)$. Consider a connected $n$-vertex graph $G = (V, E)$ with $n \geq 3$. If $\ell_r(G) > 1/(8r)$ then since $1/(8r) > \delta/n^{1/3}$, we are finished. So suppose $\ell_r(G) \leq 1/(8r)$. By Lemma 28, there is a non-empty subset $B$ of $V$ such that $\sum_{v \in B} d(v) \geq \delta n/\ell_r(G)$ and, for all $v \in B$, $d(v) \geq 2^3 \delta/\ell_r(G)^2$.

If $\ell_r(G) > \delta/n^{1/3}$, we are finished, so suppose for contradiction that $\ell_r(G) \leq \delta/n^{1/3}$. Then $\sum_{v \in B} d(v) \geq \delta n/\ell_r(G) \geq n^{4/3}$. Also, for all $v \in B$, $d(v) \geq 2^3 \delta/\ell_r(G)^2 \geq 2^{18} r^2 n^{2/3}$. This contradicts Lemma 39. $\square$
References

[1] Carol Bezuidenhout and Geoffrey Grimmett. The Critical Contact Process Dies Out. *Ann. Probab.*, 18(4):1462–1482, 10 1990.

[2] N. A. Cook, L. Goldstein, and T. Johnson. Size biased couplings and the spectral gap for random regular graphs. *ArXiv e-prints*, 1510.06013, October 2015.

[3] J. Díaz, L. A. Goldberg, G. B. Mertzios, D. Richerby, M. J. Serna, and P. G. Spirakis. On the Fixation Probability of Superstars. *Proc. R. Soc. A*, 469(2156):20130193, 2013.

[4] J. Díaz, L. A. Goldberg, G. B. Mertzios, D. Richerby, M. J. Serna, and P. G. Spirakis. Approximating Fixation Probabilities in the Generalised Moran Process. *Algorithmica*, 69(1):78–91, 2014.

[5] Josep Díaz, Leslie Ann Goldberg, David Richerby, and Maria J. Serna. Absorption time of the Moran process. *Random Struct. Algor.*, 49(1):137–159, 2016.

[6] Richard Durrett and Jeffrey E. Steif. Fixation Results for Threshold Voter Systems. *Ann. Probab.*, 21(1):232–247, 01 1993.

[7] Rick Durrett. Some features of the spread of epidemics and information on random graphs. *PNAS*, 107(10):4491–4498, 2010.

[8] W. Feller. *An Introduction to Probability Theory and its Applications*, volume I. Wiley, 3rd edition, 1968.

[9] J. Friedman. A Proof of Alon’s Second Eigenvalue Conjecture. In *Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing*, STOC ’03, pages 720–724, 2003.

[10] Andreas Galanis, Andreas Göbel, Leslie Ann Goldberg, John Lapinskas, and David Richerby. Amplifiers for the Moran Process. In *Automata, Languages, and Programming - 43rd International Colloquium, ICALP 2016*, pages 62:1–62:13, 2016.

[11] G. Giakkoupis. Amplifiers and Suppressors of Selection for the Moran Process on Undirected Graphs. *ArXiv e-prints*, November 2016.

[12] A. Jamieson-Lane and C. Hauert. Fixation probabilities on superstars, revisited and revised. *J. Theor. Biol.*, 382:44–56, 2015.

[13] E. Lieberman, C. Hauert, and M. A. Nowak. Evolutionary dynamics on graphs. *Nature*, 433(7023):312–316, 2005.

[14] Thomas M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, 1999.

[15] George B. Mertzios, Sotiris E. Nikoletseas, Christoforos Raptopoulos, and Paul G. Spirakis. Natural models for evolution on networks. *Theor. Comput. Sci.*, 477:76–95, 2013.

[16] George B. Mertzios and Paul G. Spirakis. Strong Bounds for Evolution in Networks. In *Automata, Languages, and Programming - 40th International Colloquium, ICALP 2013*, pages 669–680, 2013.
[17] P. A. P. Moran. Random Processes in Genetics. *Proc. Cambridge Philos. Soc.*, 54(1):60–71, 1958.

[18] Martin A. Nowak. *Evolutionary Dynamics: Exploring the Equations of Life*. Harvard University Press, 2006.

[19] Devavrat Shah. Gossip Algorithms. *Found. Trends Netw.*, 3(1):1–125, January 2009.

[20] V. Sood, Tibor Antal, and S. Redner. Voter models on heterogeneous networks. *Phys. Rev. E*, 77:041121, Apr 2008.

[21] Douglas Brent West. *Introduction to graph theory*. Prentice Hall, Upper Saddle River (N. J.), 2001.