General Theory of the Quantum Kicked Rotator. I

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This is the first of a series of two papers. We discuss some basic problems of the quantum kicked rotator (QKR) and review some important results in the literature. We point out the flaws in the inverse Cayley transform method to prove dynamic localization. When $\tau/2\pi$, where $\tau$ is the kick period, is very close to a rational number, the localization length is larger than the typical localization length. We analytically prove anomalous localization and confirm it by numerical calculations. We point out open problems that need further work.

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I. PHYSICAL EXPLANATIONS OF QUANTUM KICKED ROTATOR

Nearly thirty years ago, QKR is first studied by G. Casati, B. V. Chirikov, F. M. Izraelev and J. Ford [1]. They discovered by numerical calculations when the kick period $\tau$ is the product of an irrational number and $2\pi$, the rotator localizes in the momentum space. Later, S. Fishman, D. R. Grempel and R. E. Prange explained the localization by transforming QKR into an Anderson localization problem [2]. It does not seem necessary to discuss the basic problems of QKR again. But many of the results are impossible to be solved by present methods. The paper is both a review and a problem list.

The Hamiltonian of QKR is defined as [1]

$$H = H_0 + V(\theta) \sum_{n=1}^{\infty} \delta(t - n\tau)$$

$$= -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \theta^2} - k \cos \theta \sum_{n=1}^{\infty} \delta(t - n\tau),$$

which describes a particle restricted to a ring with free Hamiltonian $H_0 = -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \theta^2}$, and periodically kicked by a homogeneous electric field parallel with the ring plane. The parameters $\tau$ is the kick period and $k$ is the kick strength. In QKR, the interaction between the electric field and the rotator is $V(\theta) = -k \cos \theta$. But this paper sometimes we discuss a more general form of the interaction $V(\theta)$. In the basis $\{|m\rangle = \frac{1}{\sqrt{2\pi}} e^{-im\theta}\}$, the Hamiltonian can be written as

$$H = \sum_{m=-\infty}^{\infty} \frac{m^2}{2} |m\rangle\langle m| - \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} k \left( \frac{1}{2} |m\rangle\langle m+1| + \frac{1}{2} |m+1\rangle\langle m| \right) \delta(t - n\tau).$$

(2)

The following kick system has the same Floquet operator as the Hamiltonian in Eq. (2).

$$H = -\sum_{m=-\infty}^{\infty} \left( \frac{1}{2} |m\rangle\langle m+1| + \frac{1}{2} |m+1\rangle\langle m| \right) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\tau m^2}{2} |m\rangle\langle m| \delta(t - nk).$$

(3)

The physical explanation of the Hamiltonian in Eq. (3) is a particle on a one dimensional lattice. $|m\rangle$ is explained as the $m$-th Wannier state or $m$-th site. The term $-\sum_{m=-\infty}^{\infty} \left( \frac{1}{2} |m\rangle\langle m+1| + \frac{1}{2} |m+1\rangle\langle m| \right)$ is the hopping matrix or the kinetic energy. The particle is periodically delta-kicked by a harmonic potential $\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\tau m^2}{2} |m\rangle\langle m| \delta(t - nk)$. Now $k$ is the kick period or the free diffusion time of the particle, and the potential is defined as $\frac{\tau m^2}{2}$.

Eq. (3) is discovered when we solve the quantum kicked linear rotator (QKLR) [3, 4]. QKLR always localizes except when $\tau = 2\pi \times \text{Integer}$, where Integer is an integer. There is Bloch oscillation phenomenon when a lattice is put in a homogeneous electric potential. It turns out QKLR localization is a general Bloch oscillation. QKLR and the Bloch oscillation are the linear Toeplitz system in two different representations: the rotator representation and the site representation. QKR can also be explained in the site representation.

The site representation of QKR or Eq. (3) can be another implementation in the laboratory compared with the usual implementation Eq. (2) [5, 6]. The classical correspondence of the site explanation is a periodically delta kicked classical random walker on a lattice by the harmonic potential (CKRW). In the rotator representation, the classical correspondence is the standard map. We think CKRW is also very interesting just as the standard map. For example, does CKRW localize when the kick period $\tau$ is the product of an irrational number and $2\pi$? Gong et al independently gave an equation similar to Eq. (3) and proposed QKR can be implemented as a Heisenberg spin chain subjected to a parabolic kicking magnetic field [7, 8]. See [8] for detailed information.

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The general kick system can be defined as

\[ H = H_0 + V \sum_{n=1}^{\infty} \delta(t - n\tau). \]  

(4)

The Floquet operator is \( F = e^{-iV t} e^{-iH_0 t} \). We can always treat \( V \) as the unperturbed Hamiltonian and \( H_0 \) as the perturbation just we reexplain Eq. (2) as Eq. (3). The essence of delta kicked system including QKR is two different Hamiltonians acting on the Hilbert space in turn.

The following Hamiltonian has the same Floquet operator of Eq. (1).

\[
H(t) = \begin{cases} 
-\frac{1}{2}\hbar^2 \frac{\partial^2}{\partial\theta^2} & 0 \leq t \text{Mod}(\tau + k) \leq \tau \\
\cos \theta & \tau \leq t \text{Mod}(\tau + k) \leq \tau + k,
\end{cases}
\]

(5)

where \( t \text{Mod}(\tau + k) \) is the product of the fraction part of \( \frac{t}{\tau} \) and \( \tau + k \).

Apparently \( n \) different Hamiltonians \( \{H_0, H_1, \cdots, H_{n-1}\} \) can act on the wave function in turn. Then the Floquet operator is just \( F = e^{-iH_{n-1} \tau} \cdots e^{-iH_1 \tau} e^{-iH_0 \tau} \). The above explanation also applies classically.

In a laboratory it is impossible to implement Eq. (5) because \( H_0 \) always exists. So the experimental implementation is to increase the interaction strength and decrease the interaction time. When the interaction strength is very large and the interaction time is very small, it is almost a delta function.

When \( M \) is very large the following system is the experimental implementation of Eq. (1).

\[
H(t) = \begin{cases} 
-\frac{1}{2}\hbar^2 \frac{\partial^2}{\partial\theta^2} & 0 \leq t \text{Mod}(\tau + k) \leq \tau \\
H_0 + M \cos \theta & \tau \leq t \text{Mod}(\tau + k) \leq \tau + \frac{k}{M},
\end{cases}
\]

(6)

or

\[
H(t) = \begin{cases} 
-\frac{1}{2}\hbar^2 \frac{\partial^2}{\partial\theta^2} & 0 \leq t \text{Mod}(\tau + \frac{k}{M}) \leq \tau \\
H_0 + kM \cos \theta & \tau \leq t \text{Mod}(\tau + \frac{k}{M}) \leq \tau + \frac{k}{M}.
\end{cases}
\]

(7)

II. FLAWS OF THE INVERSE CAYLEY TRANSFORM METHOD

We study the Schrodinger Equation

\[ i\hbar \frac{\partial}{\partial t} \psi(\theta, t) = H \psi(\theta, t). \]

(8)

The dynamics of a free evolution of period \( \tau \) and a kick is described by the Floquet operator \( F \).

\[ F = e^{-iV \theta} e^{-iH_0 \tau} \]

(9)

The matrix elements of \( F \) are given by

\[ F_{nm} = \langle n | F | m \rangle = \exp(-i\hbar \frac{m^2}{2}) \delta^{m-n} J_{n-m} (k \frac{\theta}{\tau}). \]

(10)

We define one period as a free evolution of period \( \tau \) and a kick, so after \( n \) periods, the QKR wave function is

\[ \psi(n\tau) = F^n \psi(0), \]

(11)

where the unitary operator \( U(n) = F^n \) maps the initial state to the state at the time \( n\tau \).

A. Meaning of dynamic localization

Now we ask the fundamental problem of QKR. If the rotator is initially in the ground state, will its energy \( \sum_{n=-\infty}^{\infty} |c_n|^2 \) increase to infinity when the time runs to infinity? This problem can also be described in another way, such as the wave function \( \sum_{n=-\infty}^{\infty} |c_n|^2 \) is normalizable. The two questions have some nuanced differences. We do not know whether another situation could happen. For example, its energy is infinity, while its wave function is still normalizable. This situation may happen under some conditions, but in this paper we assume, to QKR, the energy finity is equivalent to the wave function normalizability. And the situation of energy finity and wave function normalizability as time runs to the infinity is referred as dynamic localization.

There are close analogies between QKR and some problems in the solid state physics. The quasienergy band of the Floquet operator with rational \( \frac{k}{\pi} \) is analogous to the energy band of the Bloch operator. The irrational case Floquet operator has been transformed into an Anderson localization problem in [2]. But there is one fundamental difference. An Bloch electron can be in the eigenstate of the Bloch operator, while QKR can never be in a Floquet eigenstate, because a Bloch eigenstate has a finite energy, while in the case of QKR the a Floquet eigenstate has an infinite energy. So the rotor state will always be a superposition of many Floquet eigenstates. The absolute values of the coefficients of the eigenstates in the superposition never change, and what changes is only the relative phases between different Floquet eigenstates.

B. Inverse Cayley transform and dynamic localization

One way to study QKR is to find all the eigenvalues and eigenstates of the Floquet operator. If the eigenstates are extended, dynamic delocalization happens and vice versa. Milek et al has formally proved absolutely continuous spectra of the Floquet operator imply delocalization. When the Floquet operator has singular continuous spectra, the usual dynamic delocalization does not happen [3]. Can the Floquet operator have extended eigenstates and singular continuous spectra under some conditions, for example, when \( \frac{k}{\tau} \) is a Liouville number? Jitomirskaya et al proved almost Mathieu equation has singular continuous spectra under some conditions [10].
Yet we assume localized eigenstate is equivalent to the dynamic localization to QKR.

One way to find all the eigenstates of the Floquet operator is to transform the Floquet eigenstate problem into time independent Hermitian operator eigenvalue problem. Especially when \( \frac{2\pi}{\tau} \) is irrational it is transformed into an Anderson problem \([2]\). The Floquet eigenstate problem is

\[
F\phi_\lambda = e^{-iV(\theta)}e^{-iH_0\tau}\phi_\lambda = e^{-i\lambda}\phi_\lambda. \tag{12}
\]

A Hermitian operator \( U \) can be transformed into a unitary operator \( O \) by Cayley transform \( O = \frac{1+iU}{1-iU} \). A unitary operator \( O \) can be transformed into a Hermitian operator by inverse Cayley transform \( U = \frac{i^k - O}{i^k + O} \). In complex analysis, Cayley transform, which is a linear fractional transformation, maps the real line to the unit circle and the inverse Cayley transform maps the unit circle to the real line. Cayley transform and its inverse transform of both the operator form and the complex number form has many applications in different fields of mathematics.

Substitute

\[
e^{-iV(\theta)} = e^{-ik\cos\theta} = \frac{1+iU(\theta)}{1-iU(\theta)}, \tag{13}
\]

where \( U(\theta) = -\tan\frac{V(\theta)}{2} = -\tan\frac{k}{2}\cos\theta \), into Eq. (12). We get

\[
T_m u_{\lambda,m} + \sum_r U_ru_{\lambda,m+r} = 0, \tag{14}
\]

where \( u_\lambda = (1 + e^{i(\lambda - H_0)})\phi_\lambda \), \( U_r = U^{2}_{-r} \) is the Fourier coefficient of \( U(\theta) \), and \( T_m = \tan \frac{k}{2}m^2 \). QKR is transformed into a particle moving on a periodic or non-periodic lattice depending on \( \tau \). The evolution of QKR has no direct relation with the new particle on a lattice. But the eigenstates of the Floquet operator and the eigenstates of the new formed Hermitian operator are related by \( u_\lambda = (1 + e^{i(\lambda - H_0)})\phi_\lambda \). If \( \phi_\lambda \) is extended or localized, then \( u_\lambda \) is also extended or localized and vice versa.

We define \( A \) as

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & T_{-2} & U_1 & U_2 & U_3 & \cdots \\
\cdots & U_{-1} & T_{-1} & U_1 & U_2 & U_3 & \cdots \\
\cdots & U_{-2} & U_{-1} & T_0 & U_1 & U_2 & \cdots \\
\cdots & U_{-3} & U_{-2} & U_{-1} & T_1 & U_1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{15}
\]

Then

\[
Au_\lambda + U_0u_\lambda = 0. \tag{16}
\]

One basic observation is the diagonal matrix elements \( T_m \) of \( A \) are pseudorandom numbers if \( \frac{2\pi}{\tau} \) is irrational. \( T_m \) is a Cauchy distribution. Because \( U(\theta) \) looks quite regular here, now we invoke Anderson’s result \([1]\). \( u_\lambda \) is localized. The above analysis gives an analytical proof of dynamic localization discovered in \([1]\). Fishman, Grempe, and Prange’s inspiring result connects the fields of quantum chaos and Anderson localization in disordered matters. It points out the physical mechanism of dynamic localization of QKR. Inverse Cayley transform seems to be the only method to transform the Floquet operator eigenstate problem to the disordered Hamiltonian eigenstate problem.

C. Flaws in the inverse Cayley transform method

Now we check the above argument step by step. The equidistribution of the sequence \( \{an^2Mod1\}_n \) when \( \alpha \) is irrational can be proved from the equidistribution of sequence \( \{an\}_n \) and van der Corput’s Theorem. While the equidistribution of sequence \( \{an^mMod1\}_n \) can be proved by Weyl criterion. So all the sequences \( \{an^mMod1\}_n \), where \( m \) is a positive integer and \( \alpha \) irrational, are equidistributed between 0 and 1. Other equidistribution examples include \( \{x^mMod1\}_n \), where \( x > 1 \). The sequence of all multiples of \( \alpha \) by all prime numbers \( \{2a, 3a, 5a, 7a, 11a, \ldots Mod1\} \) is studied in \([2]\) as the quantum kicked prime number rotator (QKPR). It localizes when \( \alpha \) is irrational because \( \{2a, 3a, 5a, 7a, 11a, \ldots Mod1\} \) are equidistributed between 0 and 1. QKPR also localizes when \( \alpha \) is rational such as \( \frac{1}{2} \) because now it is analogous to a generalized kicked dimer model.

The number theoretic property of a number sequence is used in Shor’s algorithm to find prime factors of a composite number \( S \). If \( S \) is a large number, and we define a rotator with energy levels \( \{p^nModS\}_n \), where \( n \) runs from 1 to \( S \) and \( 1 < p < S \). If \( S \) is a prime number, dynamic localization (To a finite Hilbert space, this is not very rigorous) also happens. If \( S \) is a composite number, dynamic localization generally does not happen. Although we can not find the exact prime factors of \( S \), at least we can judge whether \( S \) is prime.

There is one inherent paradox (Local pseudorandomness paradox) of localization caused by number theoretic randomness. On the one hand, the randomness of the sequence \( \{\frac{m^2}{n^2}\}_n \) with irrational \( \frac{2\pi}{\tau} \) is only meaningful when \( n \) goes to infinity. On the other hand, if the wave is localized, how could the wave feel the randomness of \( \frac{1}{n^2} \) of very large \( n \), where there is randomness? This can be said in another language, how can a remote (global) randomness influence a localized wave or how can a localized wave feel a remote (global) randomness? In Anderson localization, every random variable is independent. So in the place where the wave is localized, the randomness is real randomness. But in number theoretic randomness, there are strong correlations between the local pseudorandomness. For example, the QKR wave function is localized at several localization lengths. Is the pseudorandomness at several localization lengths random-
Randomness can be classified into two categories, the statistical randomness and pseudorandomness caused by chaos or number theoretic origin. The local pseudorandomness paradox is not a problem which only concerns QKR or Eq. (16). It is also not a new question. L. Boltzmann tried to found statistical mechanics on the ergodic theory. A recent example is Bohigas-Giannoni-Schmit conjecture [13].

The first problem of the inverse Cayley transform method to prove localization is to some irrational $\frac{\pi}{4}$, the sequence $\left\{\frac{\pi}{4}n^2\right\}_n$ may be not random enough to cause localization. This is the pseudorandomness problem that has been discussed by many authors, such as [2, 14, 15]. In [10], we give an example of irrational $\frac{\pi}{4}$ which is not irrational enough to cause localization.

Second, Eq. (16) is not really an Anderson localization problem. In Anderson localization, we know the Hamiltonian, and we do not know the eigenvalues and eigenstates. While in Eq. (16), we know the eigenvalue beforehand, which is just $-U_0$. While the unknown $\lambda$ is contained in the diagonal matrix elements $T_m$. This is not an Anderson localization problem. It is rather an inverse Anderson localization problem. We know one special eigenvalue and we need to find what diagonal “disorders” satisfies Eq. (16). Surely there are infinitely many diagonal “disorders” satisfies Eq. (16). Eq. (16) has the form of an eigenvalue problem, while in fact it is a nonlinear equation of $\lambda$ or $e^{-i\lambda}$. One may say to a general irrational $\frac{\pi}{4}$, whatever $\lambda$ is, the sequence $T_m$ is random enough to guarantee the eigenstate localized. In Anderson’s formulation of localization, the diagonal matrix elements are guaranteed to be independent random variables [11]. No independent random variables can satisfy an equation, in which these independent random variables are dependent variables. We use the terms independent and dependent variables in their original sense of Statistics.

Generally an infinite matrix has infinite eigenvalues and eigenstates. For example, we find a $\lambda$ which satisfies Eq. (17). Then the operator $A$ defined in (16) surely has many other eigenvalues besides $-U_0$. Are other eigenstates of the infinite matrix with definite $\lambda$ also localized?

The third problem is $U(\theta) = -\tan(\frac{\theta}{2} \cos \theta)$ is regular only if $k < \pi$. When $k \geq \pi$, $U(\theta)$ is discontinuous at some points. The first discontinuous point is $\theta = \arccos \frac{\pi}{k}$. At this point $U(\theta)$ is $\infty$. In [11], the interaction is required to be “falling off as the distance $r \rightarrow \infty$ faster than $1/r^3$”. In [11], $r$ is the distance between two lattice sites, while $r$ in $U_r$ is the index. But both two express the interaction strength between different sites. Does $U_r$ fall faster than $1/r^3$? This question can be answered in two ways. First, if $U_r \leq \frac{\pi}{4}$, where $c$ is a constant, then $U(\theta) = \sum_r U_r e^{i\theta r}$ is convergent at every point of $\theta$. So $U_r$ can not fall faster than $1/r^3$. Second, when $k \geq \pi$, $U(\theta)$ is not square integrable. We only need to check whether $U(\theta)^2$ is integrable in the small domain $\arccos \frac{\pi}{k} - \delta < \theta < \arccos \frac{\pi}{k} + \delta$, where $\delta$ is a small number. If $k > \pi$, in the domain, $\frac{k}{2} V(\theta)$ can be approximated by $\frac{\pi}{4} + e\theta$, where $e$ is the derivative of $\frac{k}{2} V(\theta)$ at the point $\theta = \arccos \frac{\pi}{k}$. But $\tan(\frac{\pi}{4} + e\theta)$ is not square integrable in the domain. If $k = \pi$, $\tan(\frac{k}{2} V(\theta))^2$ is approximated by $(\tan(\frac{\pi}{4} - \frac{\theta}{2}))^2 = (\cot(\frac{\pi}{4} + \theta))^2 \approx 1/(\sin \frac{\pi}{4} + \theta)^2 \approx 1/(\frac{\pi}{4})^2$. It is not square integrable either. So when $k \geq \pi$, $U(\theta)$ is not square integrable. $\sum U_r^2$ is divergent due to Parseval’s identity. The worst estimate is $U_r$ falls slower than $1/\sqrt{r}$. The Fourier expansion $U_r$ of $U(\theta)$ does not exist at all, when $k = \pi \times Odd$, where $Odd$ is an odd positive integer such as 1, 3, 5, . . .

So when $k \geq \pi$, we can not invoke Anderson’s result to prove the vectors $u_\lambda$ are localized in Eq. (16). This is a fatal flaw of the inverse Cayley transform method to prove dynamic localization. In one dimension it is easy to localize [17]. So the slowly falling $U_r$ may not destroy localization. But in the conventional theory of condensed matters, the strongest long range force is Coulomb force $\propto \frac{1}{r}$. QKR with a general irrational number $\frac{\pi}{4}$ seems to always localize however large $k$ is. We think the real meaning of the breakdown of the inverse Cayley transform method is there may be a localization-delocalization transition when $k$ increases from $k < \pi$ to $k > \pi$ and $k = \pi$ is the critical point to many kicked systems. In the QKR of [12], there is apparently a localization-delocalization transition in the rational case $\tau = \frac{\pi}{4}$. When $k = 1$ the prime number rotator localizes, while when $k = 5$ it delocalizes.

There are several flaws of the inverse Cayley transform method. Why do not develop an independent localization theory concerning Floquet operators or unitary operators? And the new theory which treats time dependent problems (unitary operators) is parallel with the Anderson localization theory which treats time independent problems (Hermitian operators). This does not seem to be an easy task. The first difficulty is there is not a unitary perturbation theory concerning unitary operators, while to Hermitian operators there are many perturbation methods. We will return to the problem of developing a dynamic localization theory of the Floquet operator in the second paper.

III. ANOMALOUS LOCALIZATION

The inverse Cayley transform method provides a physical picture, because of diagonal pseudorandomness, the eigenstate tends to be localized. But there is another path of the development of QKR theory, which gives a different picture. Casati and Guarneri proposed there is a non-empty set of irrational $\frac{\pi}{4}$, to which dynamic localization can not happen [14]. We must resolve these two conflicting views. Fishman et al also pointed out to the Liouville number $\frac{\pi}{4}$, things are very delicate and they
excluded the Liouville number $\frac{7}{4}$ in their theory [2].

We have to discriminate two kinds of different “delocalizations”. To general irrational $\frac{\pi}{4}$, the localization length $l_\tau$ is estimated to be $\alpha D$ [18, 19], where $D$ is the classical diffusion constant $\frac{\pi^2}{24}$. And the factor $\alpha$ is estimated to be $\frac{1}{2}$ in [20, 21]. The first kind of “delocalization” is $\alpha D < l_\tau < \infty$, which is actually localization with an anomalous localization length. This is referred as anomalous localization. The second delocalization is $l_\tau = \infty$. The anomalous localization can be seen from examples such as $\frac{\pi}{4} = \frac{1}{3} + \frac{\sqrt{5}}{2 \times 10^{-4}}$ or the continued fraction $\frac{\pi}{4} = (0; 3, 10^m, 1, 1, \cdots)$, where $m$ is a large number in both cases.

But we do not know all the irrational numbers $\frac{\pi}{4}$ with anomalous localization or delocalization. We do not know to a special $\frac{\pi}{4}$ whether anomalous localization or delocalization depends on $k$. For example, when $k < \pi$, delocalization does not happen to a $\tau$, while when $k > \pi$ it does. We do not know whether both cases can happen to general Liouville numbers, such as Liouville constant $L = \sum_{n=1}^{\infty} 10^{-n}$. The continued fractional of $L$ is $[0; 9, 11, 99, 1, 10, 9, 9999999999]$. The $n$-th incrementally largest term consisting only of $9$s occurs precisely at position $2^n - 1$, and this term consists of $(n - 1)! \times 9$s [22].

Although we have constructed an irrational $\frac{\pi}{4}$ with delocalization in [10]. There is not one specific irrational $\frac{\pi}{4}$ that has been numerically calculated to be delocalized. It is difficult to do a numerical simulation to these special irrational numbers. It is the periodic structure of the Floquet operator which causes delocalization. So a successful calculation must preserve the periodic structure. To the rational $\frac{\pi}{4} = \frac{4}{9}$ case, the truncated Hilbert space is at least as large as several $q$s. This makes it very difficult to calculate the rational cases with large denominators.

### A. A quantum analogy of Lyapunov exponent equation

There are two different Floquet operators $F$ and $F'$. We calculate the difference between $U(N) = F^N$ and $U(N)' = F'^N$. We define $\delta U(N) = U(N)' - U(N)$, $\delta F = F' - F$ and $\delta \tau = \tau' - \tau$.

$$\delta U(N) = F'U(N - 1)' - FU(N - 1)$$

$$= F'U(N - 1)' - F'U(N - 1) + F'U(N - 1) - FU(N - 1)$$

$$= F'U(N - 1)' - U(N - 1) + \delta FU(N - 1)$$

$$= F'\delta U(N - 1) + \delta FU(N - 1)$$

$$= \sum_{j=1}^{N} U(N - j)' \delta FU(j - 1).$$

In the last step, the mathematical induction method is used. Eq. (17) describes the divergence between unitary operators because of small difference $\delta F$. This is a quantum analogy of the Lyapunov exponent equation. The classical Lyapunov exponent equation describes the divergence between two orbits in the phase space because of small difference between the initial conditions. The stability of quantum systems is indicated by their sensitivity to the perturbation of the Hamiltonian [23]. We think the sensitivity of the Floquet operator to a perturbation is another criteria to judge whether chaos is relevant to a quantum system. In other words, the stability of a periodically driven quantum system is decided by the sensitivity of the Floquet operator to perturbation. There are two kinds of sensitive perturbation to $F$. The first kind is the phase sensitivity. The perturbation rotates the phase of some matrix elements by an angle $\pi$. The second kind is the perturbation greatly modifies the module of some matrix elements. In QKR, $\delta F$ is not sensitive to $\delta k$ or $\delta h$. It is sensitive to $\delta \tau$. We consider $\delta \tau = \tau' - \tau$. The matrix element

$$\delta U(N)_{nm} = \sum_{j=1}^{N} \sum_{l} (U(N - j)'e^{-iv})_{nl} \times (e^{-1/2il^2\tau'} - e^{-1/2il^2\tau})U(j - 1)_{lm}.$$  \hspace{1cm} (18)

For simplicity, we now assume $n = 0$. The summation of $l$ of $U(j - 1)_{lm}$ is effective in a finite bandwidth, which is at most from $-j k$ to $j k$, $|e^{-1/2il^2\tau'} - e^{-1/2il^2\tau}| \approx 1/2l^2\tau$. The row $(U(N - j)'e^{-iv})_{nl}$ and the column $U(j - 1)_{lm}$ are unit vectors. From Cauchy-Schwarz inequality,

$$\left| \sum_{l} (U(N - j)'e^{-iv})_{nl} (e^{-1/2il^2\tau'} - e^{-1/2il^2\tau})U(j - 1)_{lm} \right|^2$$

$$\leq \sum_{l} |(U(N - j)'e^{-iv})_{nl}|^2 \times \sum_{l} |(e^{-1/2il^2\tau'} - e^{-1/2il^2\tau})U(j - 1)_{lm}|^2$$

$$= \sum_{l} |(e^{-1/2il^2\tau'} - e^{-1/2il^2\tau})U(j - 1)_{lm}|^2$$

$$\leq \sum_{l} |1/2l^2\delta \tau U(j - 1)_{lm}|^2$$

$$= \sum_{l} 1/4l^4\delta \tau^2 |U(j - 1)_{lm}|^2$$

$$\leq 1/4(jk)^4 \delta \tau^2.$$  \hspace{1cm} (19)

where the equal sign $=$ in the last step comes from when $|U(j - 1)_{lm}|^2$ concentrates on the boundary $j k$ of the bandwidth, which is actually impossible.

$$\left| \sum_{l} (U(N - j)'e^{-iv})_{nl} (e^{-1/2il^2\tau'} - e^{-1/2il^2\tau}) \right| U(j - 1)_{lm} \leq 1/2(jk)^2 \delta \tau.$$  \hspace{1cm} (20)

After the summation of $j$ in Eq. (18),

$$|\delta U(N)_{nm}| \leq 1/6N^3k^2 \delta \tau.$$  \hspace{1cm} (21)
Before the time \((\frac{6}{e\delta\tau})^{1/3}\), \(|\delta U(N)_{nm}| \leq \epsilon\). Eq. (21) is first got by Casati et al [14]. In fact, generally
\[|\delta U(N)_{nm}| \leq 1/3N^3\varepsilon^2\delta\tau. \tag{22}\]
To the general rational number \(\frac{p}{q}\), the diffusion speed is far slower than the case \(\tau \text{Mod} \pi = 0\). So the effective bandwidth of \(U(j - 1)\) far smaller than \(jk\). The diffusion speed \(v\) of the case \(\tau = 0\) is \(k\). To a rational \(\frac{p}{q}\), the diffusion speed is \(v_{\tau}\). Then the bandwidth of \(U(j - 1)\) is \(v_{\tau}j\). So
\[|\delta U(N)_{nm}| \leq 1/6N^3\varepsilon^2\delta\tau. \tag{23}\]
When \(t/\tau < (\frac{6}{e\varepsilon\delta\tau})^{1/3}\), \(|\delta U(N)_{nm}| \leq \epsilon\). We can define \((\frac{6}{e\varepsilon\delta\tau})^{1/3}\) as the divergence time, after which the \(U(N)\) and \(U(N)\)’ are significantly different from each other. \(v_{\tau}\) is quite small except some strongly resonant cases. To the resonant case \(\frac{p}{q} = \frac{1}{q}\) with \(k = 1\), we estimate \(v_{\tau}\) can be as small as \(c_1e^{-c_2q}\), where \(c_1\) and \(c_2\) are two constants, the relation between which and \(q\) and \(k\) is unclear at present [16].

In the derivation of Eq. (21), the summation of \(l\) is from \(-jk\) to \(jk\). This is the upper limit of diffusion with the fastest diffusion speed, which happens only when \(\tau = 0\) or \(4\pi\). The more exact value of the bandwidth is \(\frac{1}{2}jk\) from the property of Bessel function, where \(e = 2.71828\cdots\). So a factor \(\frac{e}{2\pi}\) is \(1.84726\cdots\) should be added to Eq. (21) and (23). We do not consider the factor, which will cause qualitative differences to our discussion.

The above calculation crucially depends on the almost band structure of \(F\), \(F'\), \(U(N)\) and \(U(N)\)’. A band matrix can not be a unitary matrix. The different columns or rows of a unitary matrix are orthonormal with each other. A band matrix can never has its columns or rows orthonormal with each other. For example, let \(b\) denote the end of one row of a band matrix. Then another row of the band matrix has its start at \(b\). These two rows can not be orthonormal with each other. Another thing worth notice is \(e^{-iV}\) is a Toeplitz matrix. \(U\) in Eq. (14) is also Toeplitz. Toeplitz matrices are quite rigid. The quantum kicked harmonic oscillator [23] does not have such a nice property. Casati et al tried to explain dynamic localization from the perspective of band random matrix [22]. The Floquet operator of QKR is neither band nor random. When \(k > \pi\), from the perspective of Anderson localization, \(u_j\) should be delocalized, but it is still localized. The mechanism of the localization of Eq. (16) and QKR is even stronger than the mechanism of Anderson localization. It is this mechanism that QKR theorists need to find.

We can get a more exact estimate than Eq. (21) under some conditions. In Eq. (19), we use Cauchy-Schwarz inequality to drop \(U(N - j)\)'s. We can also drop \(U(j - 1)\). A better estimation is to drop the unitary matrix with a larger bandwidth. For example, if \(\frac{p}{q}\) and \(\frac{r}{s}\) are all irrational numbers, the summation of \(l\) is restricted to the localization length \(l_{\tau}\).
\[|\delta U(N)_{nm}| \leq 1/6N^3\varepsilon^2\delta\tau. \tag{24}\]
So the divergence time between different irrational \(\frac{p}{q}\) are far larger than the divergence time between rational and irrational \(\frac{p}{q}\).

Eq. (24) also holds when a rational \(\frac{p}{q}\) is close to a general irrational number. For example \(|\frac{1}{2\pi} - \frac{\sqrt{2} - 1}{2}| < \epsilon\), where \(\frac{p}{q}\) is rational and \(\epsilon\) small.

**B. Analytical and numerical proof of anomalous localization**

Now we estimate the anomalous localization length. From Eq. (23), before the divergence time
\[t = (\frac{6\varepsilon}{v_{\tau}\delta\tau})^{1/3}, \tag{25}\]
QKR of \(\tau\) and \(\tau'\) will not diverge from each other much. We assume \(\frac{p}{q}\) is rational and \(\frac{r}{s}\) irrational. So QKR of \(\tau\) will at least diffuse to the length \((\frac{6\varepsilon}{\delta\tau})^{1/3}\). We assume \(\epsilon = 1\). Then the localization length \(l_{\tau'}\) is at least \((\frac{6}{\delta\tau})^{1/3}\). Another method is from Eq. (24), if the divergence time is \(N\), then \(N = \frac{6}{\varepsilon\delta\tau}\). And \(l_{\tau'} = Nv_{\tau}\). We get the same result. So
\[l_{\tau'} = \left(\frac{6\varepsilon}{\delta\tau}\right)^{1/3}. \tag{26}\]
We assume \(v_{\tau}(k) = v_{\tau}(k = 1)k\). Then
\[l_{\tau'} = \left(\frac{6\varepsilon(k = 1)k}{\delta\tau}\right)^{1/3}. \tag{27}\]

The anomalous localization is significant when \(v_{\tau}\) is large, and \(\delta\tau\) small. \(v_{\tau}\) is large when \(\tau\) is close to strong resonances and \(k\) large. In our numerical calculation, we take \(\frac{p}{q}\) as the sum of \(\frac{1}{q}\) and a small number. \(\tau = 4\pi\frac{1}{q}\) has the fastest delocalization speed except \(\tau = 0\) or \(4\pi\). In our calculations, \(h = 1\), \(k = 1\) and the truncated Hilbert space is from \([-500\) to \(500]\).

In FIG. 1 and 2, \(\tau = 4\pi\times[0; 3, 100, 1, 1, 1, 1, \cdots]\), where the continued fraction \([0; 3, 100, 1, 1, 1, \cdots] \approx 1/3 - 0.00110064\). The localization length is around 6 from FIG. 1 and 2.

In FIG. 3 and 4, \(\tau = 4\pi\times[0; 3, 1000, 1, 1, 1, 1, \cdots]\), where \([0; 3, 1000, 1, 1, 1, 1, \cdots] \approx 1/3 - 0.000111101. The localization length is around 15 from FIG. 3 and 4. Notice there is staircase like structure in FIG. 3. It is very rough. We do not know what they are.

In FIG. 5 and 6, \(\tau = 4\pi\times[0; 3, 10000, 1, 1, 1, 1, \cdots]\), where \([0; 3, 10000, 1, 1, 1, 1, \cdots] \approx 1/3 - 0.0000111111. The localization length is around 36 from FIG. 5 and 6. Note far outside the localization length, the wave function is not smooth.

It is difficult to tell the exact localization length, which seems to change at different times. Compare FIG. 2, 4.
FIG. 1: QKR wave function at different time. $N$ indicates at time $2^N \tau$. For example, $N = 10$ means at time $2^{10} \tau$. $n$ is the $n$-th basis $|n\rangle = \frac{1}{\sqrt{2}} e^{in \theta}$ and $c_n$ is the base-10 logarithm of the absolute value of the wave function on the $n$-th basis. $k = 1, \tau = 4\pi \times [0; 3, 100, 1, 1, 1, \cdots]$. The initial state is $|0\rangle$.

FIG. 2: QKR wave function at different time. FIG. 1 and FIG. 2 are the same calculation.

FIG. 3: QKR wave function at different time. $k = 1, \tau = 4\pi \times [0; 3, 1000, 1, 1, 1, \cdots]$. The initial state is $|0\rangle$.

FIG. 4: QKR wave function at different time. FIG. 3 and FIG. 4 are the same calculation.

and 6; the general trend of increasing localization length is clear. Compare these localization lengths together.

$6^3 : 15^3 : 36^3 = 1 : 15.6 : 216$. The general trend $l_{\tau'} \propto \delta \tau^{-1/3}$ is confirmed. It seems $\frac{l_{\tau'}}{\delta \tau^{1/3}}$ is a monotonically increasing function of $\delta \tau^{-1/3}$.

C. Casati and Guarneri’s argument

In [14], Casati and Guarneri defined a quantity $R_\tau (n, \psi) = \frac{1}{n} \sum_{j=1}^{n} |(F^n \psi, \psi)|^2$ to measure the recurrent behavior of QKR. They proposed if the Floquet operator $F_\tau$, where $\tau$ is the kick period, has purely continuous
spectrum, then when \( n \to \infty \), \( R_r(n, \psi) \to 0 \). It follows from Eq. (21) that \( |R_r(n, \psi) - R_r(n, \psi)| \leq 2\gamma n^3 \). Then starts Casati and Guarneri’s argument.

\[
R_r(N_n, \psi) = |R_r(N_n, \psi) - R_{\tau_n}(N_n, \psi)| + R_{\tau_n}(N_n, \psi) = \gamma |\tau' - \tau_n| N_n + R_{\tau_n}(N_n, \psi)
\]

(28)

Find the sequences \( \tau_n \) and \( N_n \). They satisfy the conditions, when \( n \to \infty \), \( |\tau' - \tau_n| N_n \to 0 \) (Condition 1) and \( R_{\tau_n}(N_n, \psi) \to 0 \) (Condition 2) at the same time. Then \( R_r(N_n, \psi) \to 0 \). In this way they argue there are some irrational \( \frac{\tau'}{2\pi} \), which have a purely continuous measure.

For a long time, Casati and Guarneri’s surprising argument puzzles QKR theorists [15]. One gap in the argument is whether when \( n \to \infty \), Condition 1 and 2 are true at the same time. \( \tau_n \) is not a static rational number, so \( R_{\tau_n}(N_n, \psi) \to 0 \) is true only if \( N_n \) is large. (See [16] for why \( N_n \) needs to be large.) But now we can not guarantee Condition 1. Is there at least one irrational number \( \frac{\tau'}{2\pi} \) that satisfies Condition 1 and 2? The answer to the question is not a priori true.

In [16], we construct one irrational \( \frac{\tau'}{2\pi} \) with delocalization. Our construction depends on one assumption and three facts. We assume the rational case has a nearly constant diffusion speed \( v_\tau \) from \( t = 0 \) to \( t \to \infty \). At least the rational case diffuses with a nearly constant speed \( v_\tau \) after a threshold time \( t_{LD} \). This is the momentum linear diffusion assumption, which is different from classical diffusion. First, there is an upper limit of momentum diffusion speed. Second, in all \( \frac{\tau'}{2\pi} = \frac{p}{q} \) with different \( p \), the case \( \frac{\tau'}{2\pi} = \frac{1}{q} \) has the lowest diffusion speed. Third, Eq. (21) connects rational and irrational cases. The constructed \( \frac{\tau'}{2\pi} \) is not a general Liouville number. To construct more irrational \( \frac{\tau'}{2\pi} \), we have to exactly know the diffusion speeds of QKR with every rational \( \frac{\tau'}{2\pi} \).

Because some \( \frac{\tau'}{2\pi} = \frac{p}{q} \) are close to \( \frac{p'}{q'} \) with \( q' \ll q \), the diffusion speed \( v_\tau \) can be far faster than other \( \frac{\tau'}{2\pi} = \frac{p''}{q''} \), where \( \frac{p''}{q''} \) is not close to a rational number with smaller denominators. This can be seen from Eq. (21) before the divergence time \( \frac{\tau'}{2\pi} = \frac{5}{7} \) and \( \frac{\tau'}{2\pi} = \frac{5}{7} \) will not depart from each other very much. This is one reason why the diffusion speed \( v_\tau \) and \( v_{\tau'} \) is not significantly different from each other. It is difficult to estimate the diffusion speed of the general \( \frac{\tau'}{2\pi} = \frac{p}{q} \).

### D. Some problems

For all rational numbers \( \frac{\tau'}{2\pi} = \frac{p}{q} \) between 0 and 2, 0 or 2 has the fastest diffusion speed \( v_0 = k \). For \( \frac{\tau'}{2\pi} = 0,2 \), \( U(N) = F^N = e^{-iN\Delta\psi(\theta)} \). For \( \frac{\tau'}{2\pi} = 1 \), the QKR is periodic [1] which is proved in [4]. How to sort all the rational numbers according to their respective diffusion speeds? This sequence is helpful to understand QKR. Is this ordering dependent on \( k \)?

For \( \tau \) close to strong resonances such as \( \left\{ \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\} \times 4\pi \), QKR will diffuse quickly before the time \( \left( \frac{6\pi}{2\pi} \right)^{1/3} \). This is the reason of anomalous localization. Even if the majority of irrational numbers (we assume the Lebesgue measure is 1) are localized, irrational numbers with typical localization length can not have Lebesgue measure 1. All the irrational numbers between, for example, \( [4\pi/3 - 2\epsilon, 4\pi/3 + 2\epsilon] \), where \( \epsilon \) is a small number, do...
not have the typical localization length. From Eq. (26),
\[
\frac{1}{2\pi l} \int_{4\pi/3}^{4\pi/3+2\pi} \int_{\tau}^{\tau'} \frac{d\tau}{d\tau'} = 3 \int_{\tau}^{\tau'} = 4\pi/3+2\pi.
\] (29)

If \(\tau'=4\pi/3+2\pi\) is the typical localization length, the average localization length between \([4\pi/3, 4\pi/3 + 2\pi]\) is apparently not the typical localization length. Is the average localization length \(T = \frac{1}{T} \int_0^{2\pi} I_t \, d\tau\) finite? Even if it is finite, it is not the usual localization length \(aD\).

In Eq. (29), the small domain contributes the factor \(\frac{T}{\pi} T\), which is larger than \(aD\) because of the contribution from the small domain around \(\tau = 4\pi/3\). Does the \(\frac{1}{\pi} \int_0^{4\pi} t, d\tau\) also contribute a factor \(\frac{3}{\pi}\)? \(T = \frac{3}{\pi} aD\)?

The average localization length \(T\) is very difficult to calculate. The question can be answered only after we have totally understood delocalization of all the rational \(\frac{p}{q}\). Surely if we did so, we would totally understand QKR. We guess the average localization length could be infinity. Is \(\int_{\tau_1}^{\tau_2} I_t/d\tau/|\tau_1 - \tau_2|\) finite, where \(\tau_1\) and \(\tau_2\) are boundaries of a very small domain? We guess it could be also infinity. \(T\) may be \(\infty\), but the Lebesgue measure of finite length \(l\) is still almost 1. The probability distribution of \(P(l)\) is surely interesting. Our guess is inspired by Anderson. Anderson emphasized in disordered matters the average localization length may be infinite, nevertheless the Lebesgue measure of finite localization length is still large. It is the \(P(l)\) rather than \(T\) that are really relevant to QKR. But the problem whether \(T\) of QKR is finite is still interesting.

Although dynamic localization happens to irrational numbers, the key to understand localization rests on understanding delocalization of rational numbers, especially the delocalization speeds. In the paper, we used the top-down method from the irrational cases to irrational cases to understand irrational cases. Now we outline the bottom-up method from the irrational cases to understand rational cases. We use Eq. (21) in reverse. For a typical irrational number with localization, \(|\frac{\pi}{2\pi} - \frac{c}{q}| < \frac{c}{q^{2/3}}\), where \(c\) is a trivial constant, according to Dirichlet’s approximation theorem on diophantine approximation. So \(\frac{\pi}{2\pi} = \frac{c}{q}\) is still localized before the time approximately \(q^{2/3}\). If we want stronger result, we have to prove there is a typical localized irrational \(\frac{\pi}{2\pi}, |\frac{\pi}{2\pi} - \frac{c}{q}| < \frac{c}{q^{2/3}}\). In this way we may increase the time to approximately \(q^{4/3}\). This is petitio principii. But the essence is both rational and irrational numbers are dense within each other. If typical irrational number with typical localization length has a Lebesgue measure approximately 1, why is there not one such number in \(\{\frac{p}{q} - \frac{c}{q^n}, \frac{p}{q} + \frac{c}{q^n}\}\)?

In the finite time of an experiment in a laboratory, \(\frac{4\pi}{3} = \frac{1}{3} + \frac{1}{2\times10^m}\), where \(m\) is large, can diffuse much faster than an rational \(\frac{4\pi}{3}\), which has a large denominator \(q\) and \(\frac{c}{q}\) is far from any strong resonance. Another interesting case is when \(\tau\) close to 0 or 4\pi. Before the diverge time between \(\tau\) and 0 (or 4\pi), QKR will diffuse with a large speed. This is discussed from classical perspective in [27]. For small time \(t\), the energy grows quadratically. When \(t\) is large and not too large, the energy growth will stop because of big denominator of \(\frac{4\pi}{3}\) or \(2 - \frac{4\pi}{3}\) [10]. This is referred as dynamical freezing in [27]. It is difficult to estimate the energy linear growth at the intermediate time from our discussion here.

IV. CONCLUSIONS

In the paper, we discuss some basic problems of QKR theory. We point the flaws of the proof of dynamic localization in Fishman et al’s method [2]. We emphasize the physical mechanism of Anderson localization can not totally explain dynamic localization of QKR. We emphasize it is necessary to understand the delocalization of all the rational \(\frac{T}{2\pi}\)s. In [16], we have numerically calculated the delocalization of \(\frac{T}{2\pi}\). Yes, it delocalizes. But the delocalization time is around \(3^{10}\) or \(7^{15}\). In [16], we constructed an irrational \(\frac{T}{2\pi}\) with delocalization. In the paper, we theoretically prove anomalous localization and numerically confirm it. These three phenomena tell us the QKR theory is not just localization with irrational \(\frac{T}{2\pi}\) and delocalization with rational \(\frac{T}{2\pi}\). The whole picture is more complete only if when we take these three facts into consideration. What we have touched is only a fraction of the QKR theory. We point out the open problems and hope the readers can solve them.

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[1] G. Casati, B. V. Chirikov, F. M. Izraelev, and J. Ford, vol. 93 of Lecture Notes in Physics (Springer, Berlin, 1979).
[2] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982).
[3] D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. 49, 833 (1982).
[4] T. Ma, quant-ph/0709.2494.
[5] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Phys. Rev. Lett. 73, 2974 (1994).
[6] F. L. Moore, J. C. Robinson, C. Bharucha, B. Sundaram, and M. G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).
[7] J. Gong and P. Brumer, Phys. Rev. A 75, 032331 (2007).
[8] J. Gong and J. Wang, Phys. Rev. E 76, 036217 (2007).
[9] B. Milek and P. Seba, Phys. Rev. A 42, 3213 (1990).
[10] S. Jitomirskaya and B. Simon, Commun. Math. Phys 165, 201 (1994).
[11] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[12] T. Ma, nlin/0709.2735.
[13] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett 52, 1 (1984).
[14] G. Casati and I. Guarneri, Commun. Math. Phys. 95, 121 (1984).
[15] G. Casati, F. M. Izrailev, and V. V. Sokolov, Phys. Rev. Lett. 80, 640 (1998).
[16] T. Ma, nlin/0709.2395.
[17] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
[18] B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. C 2, 209 (1981).
[19] B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Physica D 33, 77 (1988).
[20] D. L. Shepelyansky, Phys. Rev. Lett. 56, 677 (1986).
[21] S. Fishman, R. E. Prange, and M. Griniasty, Phys. Rev. Lett. 39, 1628 (1989).
[22] J. O. Shallit, J. Number Theory 2, 228 (1982).
[23] A. Peres, Phys. Rev. A 30, 1610 (1984).
[24] G. P. Berman, V. Y. Rubaev, and G. M. Zaslavsky, Nonlinearity 4, 543 (1991).
[25] G. Casati, L. Molinari, and F. Izrailev, Phys. Rev. Lett. 64, 1851 (1990).
[26] P. W. Anderson, Rev. Mod. Phys. 50, 191 (1978).
[27] M. Sadgrove, S. Wimberger, S. Parkins, and R. Leonhardt, Physical Review Letters 94, 174103 (2005).