INPUT REPRESENTATION IN RECURRENT NEURAL NETWORKS DYNAMICS

A PREPRINT

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March 25, 2020

ABSTRACT

Reservoir computing is a popular approach to design recurrent neural networks, due to its training simplicity and its approximation performance. The recurrent part of these networks is not trained (e.g. via gradient descent), making them appealing for analytical studies, raising the interest of a vast community of researcher spanning from dynamical systems to neuroscience. It emerges that, even in the simple linear case, the working principle of these networks is not fully understood and the applied research is usually driven by heuristics. A novel analysis of the dynamics of such networks is proposed, which allows one to express the state evolution using the controllability matrix. Such a matrix encodes salient characteristics of the network dynamics: in particular, its rank can be used as an input-indepedent measure of the memory of the network. Using the proposed approach, it is possible to compare different architectures and explain why a cyclic topology achieves favourable results.

Keywords Reservoir Computing · Recurrent Neural Networks · Dynamical Systems

1 Introduction

Despite of being applied to a large variety of tasks, Recurrent Neural Networks (RNNs) are far from being fully understood and performance improvements are usually driven by heuristics. Understanding how the computation is conducted by the dynamics of the RNNs is an old question [1] which still remains unanswered, even though important progress was recently achieved [2][3]. The introduction of gating mechanisms (such as LSTM [4] and GRU [5]) dramatically improved the performance of RNNs, but the use of complex architectures makes the theoretical analysis harder [6][7][8]. Even for simple networks, we currently lack a sound framework to describe how the signal history is encoded in the state. Another relevant issue is the memory-nonlinearity trade-off [9][10]. Maximizing memory does not necessarily lead to performance (e.g., prediction) maximization [11]. In recent years, a large effort has been devoted to tackle these problems, by studying the dynamical systems underlying RNNs [2][12][13][14][15].

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Reservoir Computing (RC) is a computational paradigm developed independently by Jaeger [16, 17] (Echo State Networks (ESNs)) and Maas [18] (Liquid State Machines (LSMs)) and Tiño [19] (Fractal Predicting Machines (FPMs)). The basic idea is to create a representation of the input signal using an untrained RNN, called the reservoir, and then to use a trainable readout layer to generate the network output. RC demonstrated its effectiveness in various tasks and has risen great interest in the physical computing community due to the underlying idea of natural computation. In particular, photonics [20] and neuromorphic computation [21] are commonly implemented using RC, but also a bucket of water [22] or road traffic [23] have been used as reservoirs. See [24] for a review.

The architecture simplicity makes RC prone to theoretical investigations [25, 26, 27, 15, 12, 28]. These studies have mainly concentrated on questions about computational capabilities of whole classes of dynamical systems [26]: little has been understood about how specific setting of the dynamical system can influence their computational properties [15].

In this work, we propose a novel analysis to describe how an RNN encodes input signals in the reservoir. The analysis consists of representing the network state as a combination of two terms: the controllability matrix $C$, which only depends on the reservoir and input weights, and the network encoded input $s$, which depends on the reservoir weights and the input signal driving the system. Writing the system in this form, allows one to decouple the properties of the reservoir topology – encoded in $C$ – from the specific input driving the network, encoded in $s$. We analyze different reservoir topologies in terms of the nullspace of $C$ and show that the rank of $C$ is a measure for the richness of the representation of the input signal. More specifically, we show that the nullspace of $C$ is linked to the memory forgetting capacity of the network. Based in these results, we demonstrate that a cyclic reservoir topology (forming a ring structure) is optimal. The claim is corroborated by empirical evidence.

Reservoir computing

RC was developed as a tool to explain the brain working principle [18] and as computational paradigm to avoid the complex and expensive training procedure of RNNs [16] based on backpropagation through time [29, 30]. RC training is based on randomly generating the recurrent layer called reservoir, which is fine-tuned only at the hyper-parameter level (e.g. by searching for the best-performing spectral radius of the corresponding weigh matrix). The reservoir reads the input signal through an input layer, resulting in an untrained representation of the input in the network’s state. A readout layer is then trained to produce the desired output. Common tasks involve time-series prediction [31, 32], simulation of dynamical systems [33], and time-series classification [34].

Let $x_k \in \mathbb{R}^n$ be the state of a reservoir of dimension $n$ at time $k$ and let $W \in \mathbb{R}^{n \times n}$ be its reservoir connection matrix and $w \in \mathbb{R}^n$ its input weights vector. When considering left-infinite signal, the time-index $k$ runs from 0 to $-\infty$, so that the driving input signal is $u = (u_0, u_{-1}, u_{-2}, \ldots), u_{-k} \in \mathbb{R}$.

In the linear case, the reservoir evolves according to:

$$x_k = W x_{k-1} + w u_k$$

(1)

By recursively applying input history to (1) we get

$$x_0 = W x_{-1} + w u_0 = W^2 x_{-2} + W w u_{-1} + w u_0$$

(2)

$$= W^3 x_{-3} + W^2 w u_{-2} + W w u_{-1} + w u_0$$

(3)

i.e.,

$$x_0 = \sum_{k=0}^{\infty} W^k w u_{-k}$$

(4)

One usually relies on a (trained) linear readout $r \in \mathbb{R}^n$ to generate the desired output $y_{-k}$, so that at time 0

$$y_0 = r \cdot x_0 = r \cdot \sum_{k=0}^{\infty} W^k w u_{-k}$$

(5)
In the seminal paper [16], training is based on a simple least-square regression. More sophisticated techniques where later introduced, including some forms of regularization [35] and online-training procedures [36]. For simplicity, we have only described the case in which the input and the output are uni-dimensional, but the proposed approach can be easily generalized to the multidimensional case. In Fig. 1, a schematic representation of the RC architecture is depicted.

Even though a reservoir layer does not require any training, hyper-parameters tuning must be carried out in order to improve performance. The most studied hyper-parameter is the Spectral Radius (SR) $\rho(W)$ [37] [38] [39] which is the largest absolute value of the eigenvalues of $W$, and is related to the Maximum Singular Value (MSV) $\sigma_{\text{max}}(W)$. Other common hyper-parameters are the input and output scaling factors, the sparsity degree of $W$ and the input signal bias. Moreover, the update equation (1) is usually chosen to be non-linear, i.e., $x_k = \phi(Wx_{k-1} + w u_k)$ and different choices of the nonlinear transfer function $\phi$ can be explored, leading to different behaviors [40].

Many studies were devoted to understand how these hyper-parameters affect the dynamics of the network and its computational capabilities. In particular, it appears that the hyper-parameter space can be divided into a region where the dynamics are “regular” (meaning that they are stable with respect to the inputs driving the system) and another one where they are “disordered” (meaning that they are unstable and do not provide a representation for the input) [41]. The narrow region separating these two is know in the literature as Edge of Chaos or Edge of Criticality (EoC) [42] [43] [44] and appears to be common to a large variety of complex systems beyond RNNs [45] [46] [47].

In the following, we introduce different reservoir architectures that are commonly found in the literature.

**Delay line**

In a delay line each neuron is connected just to another one forming a chain-like structure, so that the reservoir connection matrix for the delay line $W_d$ reads:

$$W_{d,ij} = \delta_{i,j-1}$$

where $\delta$ is the Kronecker delta. Note that the last neuron in the chain is not connected to the first one. Moreover, the input weights vector is $w_d = (1, 0, 0, \ldots, 0)$, meaning that the input enters the network only through the first neuron of the chain. Mathematically, such a model setting corresponds to the $n$-th order AR model.
Figure 2: The different architectures discussed in this work. From left to right: delay line, cyclic, random and Wigner topology. The thickness of the arrow account for the strength of the connection. Notice that it has the same value all the connections in both the delay line and the cyclic reservoir, while it varies for the other two. Note the presence of self loops in random and Wigner architectures.

**Cyclic reservoir**

A reservoir is said to be cyclic when every neuron is connected to another one, in a way that they form a ring. The reservoir matrix of a cyclic reservoir has the form:

$$W_{c,ij} = \delta_{i,j-1}$$  \hspace{1cm} (7)

where, with a little abuse of notation, $\delta_{0,-1} := \delta_{0,n-1}$. Note that:

$$W_{c,ij}^2 = \sum_k W_{c,ik} W_{c,kj} = \sum_k \delta_{i,k-1} \delta_{k,j-1} = \delta_{i,j-2}$$  \hspace{1cm} (8)

and so on for higher power.

**Random reservoir**

In a random reservoir the entries of the reservoir matrix $W_r$ are independent random variables. Among the various possibilities to generate a random matrix (using uniform or bernoulli distributions, random sparsification, etc...) we choose to draw the entries from a Gaussian distribution.

$$W_{r,ij} \sim \mathcal{N}(0, \frac{\rho^2}{n})$$  \hspace{1cm} (9)

This way of creating $W_r$ leads to an expected values for the SR $\langle \rho(W_r) \rangle = \rho$ \cite{13} and MSV $\langle \sigma_{\text{max}}(W_r) \rangle = 2\rho$ \cite{48}.

**Wigner reservoir**

The diagonal elements are distributed as in (9), i.e., $W_{r,ii} \sim \mathcal{N}(0, \frac{\rho^2}{n})$ while the off-diagonal elements are distributed according to:

$$W_{r,ij} = W_{r,ji} \sim \mathcal{N}(0, \frac{\rho^2}{n}), \hspace{1cm} i \neq j$$  \hspace{1cm} (10)

a simple way of constructing such a matrix is to generate $W_r$ according to (9) and then constructing $W_w$ as $W_w = (W_r + W_r^\top)/2$. This way of creating the matrix is known as Gaussian Orthogonal Ensemble and is equivalent to setting $\rho = \rho_2 = 2\rho_1$. We will use this construction for the rest of the paper.

**Controllability matrix and network encoded input**

Here, we develop an alternative, yet convenient representation for the network state evolution based on the Cayley-Hamilton (CH) theorem \cite{49}.
CH theorem states that every real square matrix satisfies its characteristic equation, implying that
\[ W^n = \varphi_{n-1}W^{n-1} + \varphi_{n-2}W^{n-2} + \cdots + \varphi_1W + \varphi_0I \]  
\[ (11) \]
where the \( \varphi_i \) are the opposite of the coefficient of the characteristic polynomial (see Supporting Information (SI) for details). Accordingly, any power of matrix \( W \) can be written as a linear combination of the first \( n - 1 \) powers, where \( n \) is the matrix order (and also the size of the reservoir):
\[ W^k = \sum_{j=0}^{n-1} \phi_j^{(k)}W^j \]  
\[ (12) \]
where the apex \( k \) denotes the fact that the \( n \) coefficients are expansion coefficients of the \( k \)-th power of \( W \). In the SI we also show how the coefficients \( \phi_j^{(k)} \) can be written in terms of \( \varphi_j \) of (11).

By plugging (12) into (4), we obtain:
\[ x_0 = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \phi_j^{(k)}W^j w u_{-k} \]  
\[ = \sum_{j=0}^{n-1} W^j W \sum_{k=0}^{\infty} \phi_j^{(k)} u_{-k} = \sum_{j=0}^{n-1} W^j w s_j \]  
\[ (13) \]
where
\[ s_j := \sum_{k=0}^{\infty} \phi_j^{(k)} u_{-k} \]  
\[ (15) \]
is what we call the network encoded input. It is useful to interpret \( s = [s_j]_{j=0}^{n-1} \) as a vector with \( n \) components, which “encodes” the left-infinite input signal \( u \) in the spatial representation provided by the network. In order for the \( s_j \) terms to exist, the sum must converge; we will discuss this issue in the next section. We emphasize the fact that the sum over \( j \) (the dimensionality of our system) is a finite sum with \( n \) terms, as opposed to the infinite sum over the \( k \) (the time index).

Inspired by well-known tools from control theory \[50\], we define the controllability matrix of the reservoir as
\[ C = [w \; Ww \; W^2w \; \ldots \; W^{n-1}w] \]  
\[ (16) \]
Then the state-update equation (4) becomes
\[ x_0 = C \cdot s \]  
\[ (17) \]
and the output (5) can then be expressed as:
\[ y_0 = r \cdot C \cdot s \]  
\[ (18) \]
where the readout processes the \( s_j \) filtered by the controllability matrix.

**How the network encodes the input signal**

From (5), we see that the possibility for the readout to produce the desired output depends on two distinct elements: the controllability matrix \( C \) (function of \( W \) and \( w \)) and the network encoded input \( s \) (which depends on \( W \) and \( u \)).

In the SI we show that the coefficients \( \phi_i^{(k+1)} \) of (12) can be recursively expressed in terms of \( \phi_i^{(k)} \) as:
\[
\begin{bmatrix}
\phi_0^{(k+1)} \\
\phi_1^{(k+1)} \\
\vdots \\
\phi_n^{(k+1)} \\
\phi_{n-1}^{(k+1)}
\end{bmatrix} = M
\begin{bmatrix}
\phi_0^{(k)} \\
\phi_1^{(k)} \\
\vdots \\
\phi_n^{(k)} \\
\phi_{n-1}^{(k)}
\end{bmatrix}
\]  
\[ (19) \]
where \( M \) is the Frobenius companion matrix of \( W \) (see SI for details). Note that the characteristic polynomial of \( M \) is that of \( W \); as such, the two matrices share the eigenvalues. Thus, the series (15) converges, for bounded inputs, when the companion matrix \( M \) has SR smaller than 1.
The $s$ vector can be written as:

$$
\begin{bmatrix}
  s_0 \\
  s_1 \\
  \vdots \\
  s_{n-2} \\
  s_{n-1}
\end{bmatrix} =
\begin{bmatrix}
  \sum_{k=0}^{\infty} \phi^{(k)}_0 u_{-k} \\
  \sum_{k=0}^{\infty} \phi^{(k)}_1 u_{-k} \\
  \vdots \\
  \sum_{k=0}^{\infty} \phi^{(k)}_{n-2} u_{-k} \\
  \sum_{k=0}^{\infty} \phi^{(k)}_{n-1} u_{-k}
\end{bmatrix}
\tag{20}
$$

In the SI we show that first $n - 1$ terms of the sum are null with the exception of a single element. We also note that terms corresponding to time-step $k = n$ follow from (11). This means that (4) can be written as:

$$
\begin{bmatrix}
  s_0 \\
  s_1 \\
  \vdots \\
  s_{n-2} \\
  s_{n-1}
\end{bmatrix} =
\begin{bmatrix}
  u_0 + u_{-n} \varphi_0 \\
  u_{-1} + u_{-n} \varphi_1 \\
  \vdots \\
  u_{-(n-2)} + u_{-n} \varphi_{n-2} \\
  u_{-(n-1)} + u_{-n} \varphi_{n-1}
\end{bmatrix}
+ \begin{bmatrix}
  \sum_{k=n+1}^{\infty} \phi^{(k)}_0 u_{-k} \\
  \sum_{k=n+1}^{\infty} \phi^{(k)}_1 u_{-k} \\
  \vdots \\
  \sum_{k=n+1}^{\infty} \phi^{(k)}_{n-2} u_{-k} \\
  \sum_{k=n+1}^{\infty} \phi^{(k)}_{n-1} u_{-k}
\end{bmatrix}
+ \begin{bmatrix}
  \sum_{k=0}^{\infty} \phi^{(k)}_{n} u_{-k}
\end{bmatrix}
\tag{21}
$$

All other terms in (21) corresponding to time steps $k > n$ can be computed according to (12).

This procedure shows that, in general, the inputs from 0 to $n - 1$ steps back in time will always appear in their original form, and the cross-contribution will start only from $u_{-n}$ backwards. We will make use of this behavior to analytically examine the properties of different networks topologies. Moreover, by deriving the expression for the $\phi^{(k)}_j$ we can study how the network is able to recall its past inputs. In general, if the $\phi^{(k)}_j$’s are large then the network will not be able to retrieve the inputs, since the input $u_{-j}$ can only be read through $s_j = u_{-j} + \sum_{k=n}^{\infty} \phi^{(k)}_j u_{-k}$. So, our theory predicts that having large expansion coefficients $\phi^{(k)}_j$ prevents the network from being able to recall its past inputs. We will show in the next section that, when we can derive an analytical expression for the $\phi^{(k)}_j$, then, it is possible to predict how the network recalls its past inputs. Note that, as implied by (18), for a linear network this is deeply related to its expressive power, since the network output is basically a linear combination of past inputs. Their accessibility to the readout is also due to $\mathcal{C}$, which is a property of the network only, since it does not depend on particular input signal.

In the random case, $\mathcal{C}$ can be studied by considering the expected values of the norm of its columns. For instance, consider an $n$-by-$n$ matrix $W = \{ w_{ij} \} \sim \mathcal{N}(0, \frac{\rho^2}{n})$ and a vector with $n$ components $v = \{ v_j \} \sim \mathcal{N}(0, \frac{1}{n})$. Note that the expected value of the squared norm of a random vector is $l(v) = n\langle v_1^2 \rangle$ (which explains the chosen distribution for $v$). If we consider $y := Wv$ we obtain:

$$
\langle z_i^2 \rangle = \langle (Wv)_i^2 \rangle = n\langle v_1^2 \rangle \langle u_i^2 \rangle = \frac{n \rho^2}{n} \frac{1}{n} = \frac{\rho^2}{n}
\tag{22}
$$

where $\langle \cdot \rangle$ is the expected value. This means that $l(z) = n\langle y_1^2 \rangle = \rho^2$ and that the standard deviation is $\sqrt{\langle z_i^2 \rangle} = \frac{\rho}{\sqrt{n}}$.

From the above the first column of $\mathcal{C}$ has euclidean norm $\| w \| = 1$, the second one has norm $\rho$, the third one $\rho^2$; the last one has norm $\rho^{(n-1)}$. Since $\rho$ must be smaller than 1, the last columns of $\mathcal{C}$ tend to get very small. This fact explains the shrinking observed in the columns of $\mathcal{C}$ for the random case in Fig.3 and Fig.4. For the Wigner case the effect is emphasized by the correlations introduced by the symmetry of $W_w$. The controllability matrix $\mathcal{C}$ for the delay line and the cyclic reservoir can instead be described in exact terms (see the SI). A sample of each case is provided in Fig.3 and Fig.4 for case $n = 100$ and $n = 1000$ respectively.

For the delay line a complete analysis of the network output can be carried. As shown in the SI, we can write:

$$
y_0 = r \cdot I \cdot s_0 = \sum_{i=0}^{n-1} r_i u_{-i}
\tag{23}
$$

where $I$ is the identity matrix. This is, as one would expect, simply a regressive model of order $n$.

Also for the cyclic reservoir it is possible to derive an analytical exact expression. We show in the SI that if we define the $i$-time permuted input weights vector as:

$$
w^{(i)} := W^i w
\tag{24}
$$
then, the output of the cyclic reservoir at time zero \( y_0 \) can be written as:

\[
y_0 = r \cdot \tilde{C}_c \cdot \tilde{s}
\]

where

\[
\tilde{s}_j = \sum_{p=0}^{\infty} \rho^{j+p} u_{-j+p+n}
\]

\[
\tilde{C}_c = \begin{bmatrix} w & w^{(1)} & w^{(2)} & \ldots & w^{(n-1)} \end{bmatrix}
\]

The fact that, as suggested in [27], \( w \) should be non-periodic for the network to work at its best, is now evident. If \( w \) is periodic, it means that some columns of \( \tilde{C}_c \) are linearly related and, therefore, the rank degenerates, as supported by theoretical arguments in [15].

The spectral radius controls how fast the memory fades away. Note that \( u_{-k} \) is only readable through the term \( s_{-k} = u_{-k} + \rho^k u_{-(k+n)} + \ldots \) and, in order to do that, it must holds that \( u_{-k} \gg \rho^k u_{-(k+n)} \). This may suggest to choose small spectral radii, but the smaller the spectral radius, the faster the decay of the memory, since \( \tilde{s}_i = \rho^i s_i \). This confirms previous intuitions that by choosing a small spectral radius, the network preserves an accurate representation of recent inputs, at the expense of losing the ability to recall remote ones. Conversely, if one sets a large spectral radius (i.e., close to 1) the network will be able to (partially) recall inputs from the past, but its memory of more recent inputs will decrease.

### The nullspace of \( C \) and the network memory

By [18] one can understand how the rank of the controllability matrix \( C \) is associated with the degrees of freedom (the effective number of parameters used by the model to solve the task at hand) that can be exploited by the readout (i.e., the “complexity” of the model).

Note from Figures [3] and [4] that the cyclic reservoir always has the highest rank of \( C \), while the Wigner the lowest. The difference increases with the number of neurons [2].

The fact that \( C \) is not full-rank is linked to the presence of the nullspace [3]. This means that there are some network encoded inputs \( s \) which are mapped to 0 by \( C \) (17) and hence are indistinguishable for the readout. In Fig. [5] we plot the rank of \( C \) as a function of the reservoir dimension \( n \). In the experiments using Wigner and cyclic reservoirs, the spectral radius \( \rho \) and the maximum singular value \( \sigma_{\text{max}} \) coincide and their values are set to 0.995 (Fig. [5a]) and 0.9 (Fig. [5b]). For the random reservoir, \( \rho \) and \( \sigma_{\text{max}} \) are distinct, so we design an experiment where the spectral radius is fixed and another one where the maximum singular value is set (we remind the reader that \( \langle \rho \rangle = \frac{1}{2} \langle \sigma_{\text{max}} \rangle \)). But what is the shape of the basis of this this nullspace? We show its basis in two cases (see Fig. [6]). The controllability matrix

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2 Note that this is coherent with the findings in [15], since the \( Q \) defined in that work is simply \( Q = C^\top C \) and the number of motifs is related to the rank of \( Q \) (and so, of \( C \)).

3 What we call the nullspace is practically the effective nullspace detected up to the numerical precision, computed using the Numpy dedicated function [31].
Figure 4: The controllability matrix and its rank for different architectures. The Spectral Radius is \( \rho = 0.99 \) and the reservoirs has \( N = 1000 \) neurons. The four architectures share the same randomly-generated \( w \).

obtained with a cyclic reservoir does not have a nullspace for such a value of \( \rho \), since \( C \) is full-rank. Note that, in order to interpret each vector in Fig. 5 as a time series, one must consider the last inputs seen as the ones closer to the origin. Given this interpretation, we clearly see how the memory is linked to the rank of \( C \): the reservoirs’ ability to recall past inputs depend on the rank of \( C \) because inputs which only differ in the far-away past are mapped to the same final state.

Figure 5: Ranks of the controllability matrix \( C \) as a function of the reservoir dimension \( n \), for \( \rho = 0.995 \) (a) and \( \rho = 0.9 \) (b). Note the saturation of the cyclic reservoir, which happens for numerical reasons.

Let us consider an example. Let \( s_1 \) and \( s_2 \) be two network encoded inputs, which differ only in the last \( n - m \) elements. Denote by \( x_0^n \) the final state of the system after being fed with the signal \( s_i \). We can then write \( s_2 = s_1 + d \), where \( d \) encodes the difference between the two representations. We see that the first \( m \) elements of \( d \) are null. So, we can write:

\[
x_0^n = Cs_2 = C(s_1 + d) = Cs_1 + Cd = Cs_1 + 0 = x_0^n\]

since \( d \) lives in the nullspace of \( C \). This results in the network not being able to distinguish between the two signals.

**Memory curves**

In order to validate our theoretical claims, we tested the network on the classic task of remembering a random i.i.d. input. We generate inputs of length \( T \) and divide them in a training set ranging in \((0, t_0)\) and a test set ranging in \((t_0, T)\). Setting \( T = 1500 \) and \( t_0 = 1000 \) this results in a length for the train set of \( L_{\text{train}} = 1000 \) and \( L_{\text{test}} = 500 \) for the test. The readout is generated using the data in the training set and the performance is then evaluated using the test set. As a measure of the performance, we use the accuracy metric defined as \( \gamma = \max\{1 - \text{NRMSE}, 0\} \), where the
Normalized Root Mean Squared Error (NRMSE) is:

$$\text{NRMSE} := \sqrt{\frac{\sum_{k=t_0}^{T} (y_k - \hat{y}_k)^2}{\sum_{k=t_0}^{T} (y_k - \overline{y})^2}}$$

Here, $y_k$ denotes the desired output at time $k$, $\overline{y} := \sum_{k=t_0}^{T} y_k$ is its average and $\hat{y}_k$ stands for the predicted output. In the task under consideration, the network is trained to reproduce past input (a white noise signal) at a given past-horizon $\tau$, so that $y_k \equiv u_{k-\tau}$. The input signal $u$ is chosen to be Gaussian i.i.d. white noise, $u_k \sim \mathcal{N}(0, 1)$.

In Figure 6 the nullspace basis for a Random (a) and a Wigner (b) reservoir matrices for reservoirs with $n = 100$. In both cases, we set the spectral radius $\rho = 0.99$. The vertical black lines represent the rank of $C$, in each case.

In Figure 7 the memory curves, introduced in [16], are plotted for three different architectures. The Random and the Wigner architectures appear to have a short memory. Their performance dramatically decreases as $\tau$ grows. The behavior of the cyclic reservoir appears to be radically different. As described in [15], the performance does not decrease gradually, but remains almost constant for some time and then abruptly decreases. The drop in performance occurs when $\tau = n$, with $n$ being the dimension of the network. This is coherent with the theory we developed and with the findings in [27].

It is important to note how the SR affects the performance variation with $\tau$: when the SR is closer to one, the accuracy is lower for recent inputs (i.e., smaller $\tau$) but higher for more distant in time ones. In other words, choosing a large $\rho$ allows the network to better remember the distant past, at the price of compromising its ability to deal with the recent one. This fact confirms our theory, since it is a direct implication of (21) and (26): increasing the SR also amplifies the contribution of distant inputs over the recent ones, since this property is controlled by $\rho^{j+pn}$ (see SI for more details).

We explore the impact of the spectral radius on memory in Figure 8 where the accuracy of the three architectures in recalling a past input (for various $\tau$) is plotted as functions of the SR. According to our prediction, a larger SR is required to correctly recall inputs that are further in past (but for which $\tau < N$), since the SR controls the magnitude of the $\phi_j^{(k)}$, i.e., the permanence of $u_k$ on the state. We notice that the Random and the Wigner architectures show a similar behavior, with the former displaying a superior performance than the latter. Instead, the Cyclic network has the same growing behavior as the SR increases, but display an abrupt fall as it approaches 1. This is coherent with the theory we developed, since for $\rho \approx 1$ the power of $\rho$ appearing in (56) will not converge anymore and the network state will just be an unreadable superposition of all the past outputs.

Conclusions

In this paper, we proposed a methodology for explaining how linear reservoirs represent inputs. The methodology consists in expressing the system state in terms of the controllability matrix $C$ and the network encoded input $s$. The analysis of $C$ allows us to compare different connectivity patterns for the reservoir in a quantitative way by using the rank of $C$. Our results show that reservoirs with a cyclic topology give the richest possible representation of input signals, yet they also offer one the most parsimonious reservoir parametrization. To the best of our knowledge, our contribution pioneers the rigorous study of how specific coupling patterns for the recurrent layer and individual setting
Figure 7: Memory curves for the random, Wigner and cyclic reservoir with different values of $\rho$, for a reservoir of $N = 100$ neurons. The values plotted are averages over 10 repetitions, with the shaded area accounting for the standard deviation. Note that having a high ability to reconstruct recent inputs ($k < N$) compromises the capacity to remember the more distant ones.

...of the dynamical system influence computational properties (e.g., memory), providing deeper insights about phenomena that so far have been observed only empirically in the literature.

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Figure 8: Accuracy in remembering an i.i.d. past-input as functions of the spectral radii. All the networks have $N = 100$ neurons. The values plotted are averages over 10 repetitions, with the shaded area accounting for the standard deviation.

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Accordingly, it is possible to show that the \( \lambda \) where

\[
\text{The CH Theorem allow one to describe the edge of criticality in binary echo state networks.}
\]

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**Cayley-Hamilton Theorem**

The CH Theorem allow one to describe the \( n \)-th power of a matrix in term of the first \( n - 1 \)-powers (including the zero power, which is the identity).

Let \( W \in \mathbb{R}^{n \times n} \) be a squared matrix. Its characteristic polynomial is defined as:

\[
\det (\lambda I - W) = 0 \quad \rightarrow \quad \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 = 0
\]

where \( \lambda \) is an eigenvalue of \( W \) and the \( \alpha_k \) are the coefficients of the characteristic polynomial.

**Theorem 1 (Cayley-Hamilton)** Every real square matrix satisfies its characteristic equation.

\[
W^n + \alpha_{n-1}W^{n-1} + \cdots + \alpha_1W^1 + \alpha_0I = 0
\]

Accordingly, it is possible to show that the \( n \)-th power of the matrix can be represented as a linear combination of the lower powers:

\[
W^n = -\alpha_{n-1}W^{n-1} - \cdots - \alpha_1W^1 - \alpha_0I
\]

For a matrix \( W \), Theorem 1 states:

\[
W^n = \varphi_{n-1}W^{n-1} + \varphi_{n-2}W^{n-2} + \cdots + \varphi_1W + \varphi_0I
\]

Here, \( W \) is a \( n \times n \) matrix, \( I \) is the \( n \times n \) identity matrix, and \( \varphi_k = -\alpha_k \) are the opposites of the coefficients of the characteristic polynomial of \( W \). It holds true that

\[
W^m = \phi_{n-1}^{(m)}W^{n-1} + \phi_{n-2}^{(m)}W^{n-2} + \cdots + \phi_1^{(m)}W + \phi_0^{(m)}I
\]

implying that any power \( m \geq n \) of \( W \) can be specified by \( W \) and scalars \((\phi_{n-1}^{(m)}, \ldots, \phi_0^{(m)})\). The apexes denote the fact that the \( n \) coefficients are those proper of the \( m \)-th power for the \( \phi_{n-1}^{(m)} \) coefficients. Note that, for \( m = i < n \), we have

\[
(\phi_{n-1}^{(i)}, \ldots, \phi_0^{(i)}) = (0, \ldots, 0, 1, 0, \ldots, 0),
\]

where the only non-null term is the \( i \)-th. Moreover, note that for \( m = n \), we have

\[
(\phi_{n-1}^{(m)}, \ldots, \phi_0^{(m)}) = (\varphi_{n-1}, \ldots, \varphi_0).
\]
For each \( m > n \), we can derive the scalars in recursive way by noting that:

\[
W^{m+1} = W^m W
\]

\[
= (\phi_{n-1}^{(m)} W^{n-1} + \phi_{n-2}^{(m)} W^{n-2} + \cdots + \phi_1^{(m)} W + \phi_0^{(m)} I) W
\]

\[
= \phi_{n-1}^{(m)} W^n + \phi_{n-2}^{(m)} W^{n-1} + \cdots + \phi_1^{(m)} W^2 + \phi_0^{(m)} W
\]

\[
= \phi_{n-1}^{(m)} (\phi_{n-1} W^{n-1} + \phi_{n-2} W^{n-2} + \cdots + \phi_1 W + \phi_0 I) + \phi_{n-2}^{(m)} W^{n-1} + \cdots + \phi_1^{(m)} W^2 + \phi_0^{(m)} W
\]

\[
= (\phi_{n-1} \phi_{n-1}^{(m)} W^{n-1} + \phi_{n-2}^{(m)} W^{n-2} + \cdots + \phi_0^{(m)} W) + \phi_{n-1}^{(m)} W^{n-1} + \cdots + \phi_1^{(m)} W^2 + \phi_0^{(m)} W
\]

(38)

which implies

\[
\begin{align*}
\left\{ \begin{array}{l}
\phi_0^{(m+1)} = (\varphi_0 \phi_{n-1}^{(m)}) \\
\phi_1^{(m+1)} = (\varphi_1 \phi_{n-1}^{(m)}) + \phi_0^{(m)} \\
\vdots \\
\phi_{n-2}^{(m+1)} = (\varphi_{n-2} \phi_{n-1}^{(m)}) + \phi_{n-3}^{(m)} \\
\phi_{n-1}^{(m+1)} = (\varphi_{n-1} \phi_{n-1}^{(m)}) + \phi_{n-2}^{(m)} \\
\end{array} \right.
\]

(40)

Eq. 40 can be thought as a linear system:

\[
\begin{bmatrix}
\phi_0^{(m+1)} \\
\phi_1^{(m+1)} \\
\vdots \\
\phi_{n-2}^{(m+1)} \\
\phi_{n-1}^{(m+1)}
\end{bmatrix} = M
\begin{bmatrix}
\phi_0^{(m)} \\
\phi_1^{(m)} \\
\vdots \\
\phi_{n-2}^{(m)} \\
\phi_{n-1}^{(m)}
\end{bmatrix}
\]

(41)

where \( M \) is defined as:

\[
M = \begin{bmatrix}
0 & \cdots & 0 & \varphi_0 \\
1 & 0 & \cdots & 0 & \varphi_1 \\
0 & 1 & \cdots & 0 & \varphi_{n-2} \\
0 & 0 & \cdots & 1 & \varphi_{n-1}
\end{bmatrix}
\]

(42)

Note that the characteristic polynomial of \( M \) is equal to the one of \( W \), so that they also share the same eigenvalues. In fact, \( M \) is also know as the Frobenius companion matrix of \( W \).

The network encoded input

From the text we see that the possibility for the readout to produce the desired output depends on two distinct elements: the controllability matrix \( C \) (which depends on \( W \) and \( w \)) and the \( s \) vector (which depends on both \( W \) and the signal \( u \)). Here we show how \( s \) is obtained from \( u \).

Under the assumption of bounded inputs \( u_{-k} \in [-U, U], \forall k \), we see that

\[
|s_j| = \sum_{k=0}^{\infty} \phi_j^{(k)} u_{-k} \leq U \sum_{k=0}^{\infty} |\phi_j^{(k)}|
\]

allowing us to focus on the properties of the \( \phi_j^{(k)} \).

These terms are the element of \( s \), which we now write as follows:

\[
\begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{n-2} \\
s_{n-1}
\end{bmatrix} = \begin{bmatrix}
\sum_{k=0}^{\infty} \phi_0^{(k)} u_{-k} \\
\sum_{k=0}^{\infty} \phi_1^{(k)} u_{-k} \\
\vdots \\
\sum_{k=0}^{\infty} \phi_{n-2}^{(k)} u_{-k} \\
\sum_{k=0}^{\infty} \phi_{n-1}^{(k)} u_{-k}
\end{bmatrix}
\]

(43)
As discussed above, the first \( n - 1 \) terms of the sum are null but a single element. This implies that the first \( n - 1 \) time steps are simply the inputs:

\[
\begin{bmatrix}
  s_0 \\
  s_1 \\
  \vdots \\
  s_{n-2} \\
  s_{n-1}
\end{bmatrix}
= \begin{bmatrix}
  u_0 + \sum_{k=n}^{\infty} \phi_0^{(k)} u_{-k} \\
  u_{-1} + \sum_{k=n}^{\infty} \phi_1^{(k)} u_{-k} \\
  \vdots \\
  u_{-(n-2)} + \sum_{k=n}^{\infty} \phi_{n-2}^{(k)} u_{-k} \\
  u_{-(n-1)} + \sum_{k=n}^{\infty} \phi_{n-1}^{(k)} u_{-k}
\end{bmatrix}
= \begin{bmatrix}
  u_0 \\
  u_{-1} \\
  \vdots \\
  u_{-(n-2)} \\
  u_{-(n-1)}
\end{bmatrix}
+ \begin{bmatrix}
  \sum_{k=n}^{\infty} \phi_0^{(k)} u_{-k} \\
  \sum_{k=n}^{\infty} \phi_1^{(k)} u_{-k} \\
  \vdots \\
  \sum_{k=n}^{\infty} \phi_{n-2}^{(k)} u_{-k} \\
  \sum_{k=n}^{\infty} \phi_{n-1}^{(k)} u_{-k}
\end{bmatrix}
\] (44)

Then, we observe that the terms corresponding to time-step \( k = n \) follow from Eq. [33].

Successive terms corresponding to time steps \( k > n \) can be computed by using (40).

This procedure shows that, in general, the inputs from 0 to \( n - 1 \) time steps in the past will always appear in their original form, and the “mixing” will begin starting from the \( n \)-th time step in the past.

**Delay line**

It is easy to see that, by applying \( W_d \) to a vector \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) results in a vector

\[
\mathbf{v}' := W_{d_{ij}} \mathbf{v} = (0, v_1, \ldots, v_n - 1)
\]

and because of the associativity of the matrix product, we see that applying \( W_{d_{ij}} \) to a vector \( k \) times results in permuting the vector \( k \) times and the substituting the first \( k \) elements with the same number of 0s. So, the controllability matrix for the delay line is:

\[
\mathbf{C}_d = [w_d \ W_d w_d \ \ldots \ W_d^{n-1} w_d]
\] (46)

which would be a lower diagonal matrix for a generic \( \mathbf{v} \) but for \( w_d = [1, 0, \ldots, 0] \) is just the identity.

Now, consider the fact that

\[
W_d^n = 0
\] (47)

The CH theorem implies that any higher power will be as well. So we simply have:

\[
s_0 = u_0
\]
\[
s_1 = u_1
\]

and so on, because all the \( \phi_j^{(m)} \) for \( m > n \) are null. So, if we define \( s_d := (u_0, u_{-1}, u_{-2}, \ldots, u_{-(n-1)}) \) we see that:

\[
\mathbf{y}_0 = \mathbf{r} \cdot \mathbf{C}_d \cdot \mathbf{s}_d = \mathbf{r} \cdot \mathbf{I} \cdot \mathbf{s}_d = \sum_{i=0}^{n-1} r_i u_{-i}
\] (48)

which is, as expected, simply a regressive model of order \( n \).
Cyclic reservoirs

The characteristic polynomial of $W_c$ is $\lambda^n = 1$ so that the CH Theorem implies:

$$W_c^n = I \quad (49)$$

Meaning that, for all $m > n$,

$$W_c^m = \sum_{j=0}^{n-1} \phi_j^{(m)} W_c^j = W_c^\mu \quad (50)$$

where $\mu := m \mod n$. Note that, in general:

$$(aW_c)^m = a^m W_c^\mu \quad (51)$$

So, if in our reservoir we fix $W = \rho W_c$ (where $\rho$ is a parameter controlling the spectral radius) we obtain a number of simplifications. First of all, the elements of $s$ assume a regular form. For example:

$$s_0 = u_0 + \rho^n u_{-n} + \rho^{2n} u_{-2n} + \ldots$$
$$s_1 = u_{-1} + \rho^n u_{-(n+1)} + \rho^{2n} u_{-(2n+1)} + \ldots$$

so that their general form is

$$s_j = \sum_{k=0}^{\infty} \phi_j^{(k)} u_{-k} = \sum_{p=0}^{\infty} \rho^{pn} u_{-j+pn} \quad (52)$$

Moreover, the controllability matrix $\mathcal{C}_c$ assumes a simple form. If we define the $i$-time permuted input weight vector as:

$$w^{(i)} := W_c^{i} w \quad (53)$$

we obtain:

$$\mathcal{C}_c = [w \quad \rho w^{(1)} \quad \rho^2 w^{(2)} \ldots \quad \rho^{n-1} w^{(n-1)}] \quad (54)$$

so that:

$$y_0 = (r_0, r_1, \ldots, r_{n-1}) \mathcal{C}_c \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{pmatrix} \quad (55)$$

The output can be written in compact form by defining:

$$\tilde{s}_j = \sum_{p=0}^{\infty} \rho^{j+pn} u_{-j+pn} \quad (56)$$

$$\tilde{\mathcal{C}}_c = [w \quad w^{(1)} \quad w^{(2)} \ldots w^{(n-1)}] \quad (57)$$

so that, finally:

$$y_0 = r \cdot \tilde{\mathcal{C}}_c \cdot \tilde{s} \quad (58)$$