Parabolic maps with spin:
Generic spectral statistics with non-mixing classical limit

Grischa Haag¹ and Stefan Keppeler²

Abteilung Theoretische Physik
Universität Ulm, Albert-Einstein-Allee 11
D-89069 Ulm, Germany

We investigate quantised maps of the torus whose classical analogues are ergodic but not mixing. Their quantum spectral statistics shows non-generic behaviour, i.e. it does not follow random matrix theory (RMT). By coupling the map to a spin $1/2$, which corresponds to changing the quantisation without altering the classical limit of the dynamics on the torus, we numerically observe a transition to RMT statistics. The results are interpreted in terms of semiclassical trace formulae for the maps with and without spin respectively. We thus have constructed quantum systems with non-mixing classical limit which show generic (i.e. RMT) spectral statistics. We also discuss the analogous situation for an almost integrable map, where we compare to Semi-Poissonian statistics.

¹E-mail address: grischa.haag@physik.uni-ulm.de
²E-mail address: stefan.keppeler@physik.uni-ulm.de
1. Introduction

A long standing problem in quantum chaology is the precise formulation of the conjecture of Bohigas, Giannoni and Schmit (BGS) [1] which states that, generically, the quantum spectral statistics of systems whose classical analogue is chaotic can be described by the average statistical behaviour of the eigenvalues of large random matrices. In contrast, Berry and Tabor [2] conjectured that for integrable systems the statistical distribution of eigenvalues is that of a Poisson process.

Whereas in the latter case integrability is precisely defined by the Liouville-Arnold theorem [3], there is no general consensus as to what should be sufficient conditions for the BGS conjecture to hold. It is generally believed that hyperbolicity, i.e. positive Lyapunov exponents, is sufficient for observing statistics as in random matrix theory (RMT). However, since this is a very strong property, one might ask whether weaker conditions would suffice. Actually, this question has already been raised by Bohigas, Giannoni and Schmit [1], who originally formulated their conjecture for K-systems. The weakest possible property in the hierarchy of chaos is ergodicity which, roughly speaking, means that almost everywhere in phase space time averages are equal to a phase space average. It is sometimes argued that ergodicity is too weak for implying RMT statistics but that instead one should also require the decay of correlations, i.e. mixing or even the K-property.

In order to shed some light on this problem we investigate parabolic maps of the two-torus, namely Kronecker and skew maps, which are (uniquely) ergodic but satisfy no stronger condition of chaoticity. In particular, they are not mixing, i.e. in general correlations do not decay, and have zero topological entropy and hence zero Kolmogorov-Sinai entropy. Thus, these maps serve as good examples for soft chaos.

The quantised versions of these maps are known to show non-generic spectral statistics. Even worse, it has been proven [4] that some statistical functions do not exist in the semiclassical limit. Although this might be considered as evidence for ergodicity not being a sufficient condition for RMT statistics, we argue that the non-generic behaviour is due to degeneracies in both the quantum spectrum and the spectrum of classical periodic orbits. These degeneracies can be lifted without changing the characteristic properties of the classical system by perturbing the quantum system in higher semiclassical order, namely by coupling the quantum map to a spin $\frac{1}{2}$. We show that in this way one can construct a family of quantum maps which all have a non-mixing classical limit but numerically show RMT statistics. We analyse the lifting of degeneracies both numerically and in terms of semiclassical trace formulae. Thus we provide evidence that generically (which should at least imply the absence of systematic degeneracies both quantum mechanically and semiclassically) ergodicity might be a strong enough condition for implying the BGS conjecture.

For skew maps we also investigate an almost integrable situation. The level spacing distribution for the almost integrable skew map with spin is close to Semi-Poissonian behaviour. This numerical finding is supported by a semi-heuristic analytical calculation of the value of the form factor $K(\tau)$ at $\tau = 0$.

The paper is organised as follows. First we introduce classical and quantised Kronecker maps (section 2 and 3 respectively), which are irrational translations on the two-torus and thus represent the simplest possible ergodic system to be studied classically and quantum mechanically. In section 4 we discuss their non-generic spectral statistics and relate it to degeneracies of classical periodic orbits using a trace formula. The following sections are devoted to the study of Kronecker maps with spin, which show generic spectral statistics.
(section 3). We also discuss how a higher order quantum perturbation lifts the degeneracy of periodic orbits in a semiclassical trace formula (section 4). Then in section 5 we turn our attention to the slightly more complicated skew maps. Quantum skew maps also show non-generic spectral statistics. But again we observe a change to RMT statistics if we add a spin contribution. Almost integrable skew maps are discussed in section 8. We conclude with some remarks in section 9. The proof of the Egorov property for the Kronecker map and the trace formula for the skew map are discussed in two appendices.

2. Kronecker maps

The simplest example for parabolic maps are translations on the two-torus given by

\[ K_{\alpha\beta} : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \]

\[ \left( \frac{p}{q} \right) \mapsto \left( \frac{p + \beta}{q + \alpha} \right) \quad (\text{mod 1}) \]  

(2.1)

with \( \alpha, \beta \in S^1 \). If 1, \( \alpha \) and \( \beta \) are linearly independent over \( \mathbb{Z} \), we have irrational translations on the two-torus which are also called Kronecker maps. In these cases we get uniquely ergodic, but non-mixing maps (see, e.g., [5, 6]). For \( \alpha, \beta \in \mathbb{Q} \) the maps are periodic. In all other cases the torus splits into families of invariant curves.

For later reference it will be important to study the symmetries of \( K_{\alpha\beta} \). There are several symplectic symmetries \( \pi \) fulfilling

\[ \pi \circ K_{\alpha\beta} \circ \pi^{-1} = K_{\alpha\beta} \).

(2.2)

First of all we have invariance under inversion

\[ I : \left( \frac{p}{q} \right) \mapsto \left( -\frac{p}{-q} \right) \quad (\text{mod 1}). \]

(2.3)

Furthermore, there are two continuous families of symmetry operations: the maps commute with both translations in position

\[ T_q(\gamma) : \left( \frac{p}{q} \right) \mapsto \left( \frac{p}{q + \gamma} \right) \quad (\text{mod 1}), \; \gamma \in S^1 \]  

(2.4)

and translations in momentum

\[ T_p(\delta) : \left( \frac{p}{q} \right) \mapsto \left( \frac{p + \delta}{q} \right) \quad (\text{mod 1}), \; \delta \in S^1. \]  

(2.5)

On the other hand, there is no anti-symplectic symmetry \( \tau \) with

\[ \tau \circ K_{\alpha\beta} \circ \tau^{-1} = K_{\alpha\beta}^{-1}, \]

(2.6)

and thus Kronecker maps are not invariant under time reversal.

When discussing the quantisation of the map in the next section, we will also need a generating function of the classical map. Due to the independence of the dynamics in position and momentum, see (2.1), one can only give a generating function in a mixed representation,

\[ G_{\alpha\beta} (q_{n+1}, p_n) = p_n q_{n+1} + \beta q_{n+1} - \alpha p_n, \]

(2.7)

with

\[ p_{n+1} = \frac{\partial G_{\alpha\beta} (q_{n+1}, p_n)}{\partial q_{n+1}} \quad \text{and} \quad q_n = \frac{\partial G_{\alpha\beta} (q_{n+1}, p_n)}{\partial p_n}. \]

(2.8)
3. Quantum mechanics of Kronecker maps

Let us first review the basics of quantum mechanics on the two-torus $\mathbb{T}^2$, see, e.g., [6–11]. Phase space translation operators $T(q,p)$ yield a representation of the Heisenberg group,

$$T(q,p)T(q',p') = e^{i\frac{\hbar}{2}(q'p-pq')} T(q+q',p+p') .$$

(3.1)

Because of the topology of the torus, quantum states $\psi$ have to be periodic up to a phase in both position and momentum, i.e.

$$T(1,0)|\psi\rangle = e^{-2\pi i \kappa_1}|\psi\rangle \quad \text{and} \quad T(0,1)|\psi\rangle = e^{2\pi i \kappa_2}|\psi\rangle .$$

(3.2)

For Kronecker maps arbitrary $\kappa_1$ and $\kappa_2$ are allowed, but the spectrum is not changed by choosing different values. Hence for simplicity we choose $\kappa_1 = \kappa_2 = 0$. The conditions (3.2) require $\hbar = (2\pi N)^{-1}$ with $N \in \mathbb{N}$. Moreover one concludes that a quantum state in position (or in momentum) representation is given by a superposition of $\delta$-distributions

$$\psi(q) = \langle q | \psi \rangle = \frac{1}{\sqrt{N}} \sum_{m \in \mathbb{Z}} \sum_{Q \in \mathbb{Z}_N} \psi(Q) \delta \left( q - \frac{Q}{N} - m \right) ,$$

(3.3)

with $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. Position states are defined by

$$\langle q | Q \rangle := \frac{1}{\sqrt{N}} \sum_{m \in \mathbb{Z}} \delta \left( q - \frac{Q}{N} - m \right) , \quad \text{with} \ Q \in \mathbb{Z}_N ,$$

(3.4)

and momentum states are given by discrete Fourier transformation of the position states with

$$\langle Q | P \rangle = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N}PQ} .$$

(3.5)

We remark that now the action of translations in either momentum or position is given by

$$T \left( \frac{n}{N}, 0 \right) | Q \rangle = | Q + n \rangle , \quad T \left( 0, \frac{m}{N} \right) | Q \rangle = e^{\frac{2\pi i}{N}mQ} | Q \rangle \quad \text{and} \quad T \left( \frac{n}{N}, 0 \right) | P \rangle = e^{-\frac{2\pi i}{N}nP} | P \rangle , \quad T \left( 0, \frac{m}{N} \right) | P \rangle = | P + m \rangle ,$$

(3.6)

$n, m \in \mathbb{Z}_N$. Finally the Hilbert-space $H_N \cong \mathbb{C}^N$ on the two-torus has dimension $N$. Classical observables $f(q,p) \in C^\infty(\mathbb{T}^2)$ can be expanded in a Fourier series

$$f(q,p) = \sum_{n,m \in \mathbb{Z}} f_{nm} e^{2\pi i(mq-np)} ,$$

(3.7)

and their Weyl-quantisations are defined as

$$\text{Op}(f) := \sum_{n,m \in \mathbb{Z}} f_{nm} T \left( \frac{n}{N}, \frac{m}{N} \right) .$$

(3.8)

Having reviewed the kinematic aspects of quantised torus maps, the next step is to quantise the dynamics, i.e. to find a suitable quantum time evolution operator $U(K_{\alpha\beta})$. This will be achieved in terms of a Van Vleck-propagator [3] with the generating function (2.7). However, in order to make this quantisation compatible with the discrete lattice, which the wave functions
are supported on, cf. (3.3), one first has to find $N$-dependent rational approximations of $\alpha$ and $\beta$, as has been pointed out by Marklof and Rudnick for quantum skew maps [12]. Following their approach, we approximate the irrational numbers by fractions with denominator $N$, fulfilling

$$\left| \alpha - \frac{a}{N} \right| \leq \frac{1}{2N} \quad \text{and} \quad \left| \beta - \frac{b}{N} \right| \leq \frac{1}{2N}.$$  

(3.9)

Thus, it is ensured that in the semiclassical limit $N \to \infty$ we recover the irrational $\alpha$ and $\beta$. With the generating function (2.7) and the approximations (3.9) we now define the propagator in a mixed representation by

$$\langle Q \mid U(K_{\alpha\beta}) \mid P \rangle := \frac{1}{\sqrt{N}} \left| \frac{\partial^2 G_{\frac{a}{N} \frac{b}{N}}(q,p)}{\partial p \partial q} \right|^{\frac{1}{2}} \exp \left[ 2\pi i N G_{\frac{a}{N} \frac{b}{N}} \left( \frac{P}{N}, \frac{Q}{N} \right) \right],$$

(3.10)

with $Q, P \in \mathbb{Z}_N$. Note that by (2.7) the pre-factor is constant. As can easily be verified using (3.6) we also have the identity

$$U(K_{\alpha\beta}) = T \left( \frac{0}{N}, \frac{b}{N} \right) T \left( \frac{a}{N}, 0 \right).$$

(3.11)

The time evolution operator $U(K_{\alpha\beta})$ is a quantisation of the classical map (2.1) in the sense that in the semiclassical limit $N \to \infty$ it fulfils the Egorov property

$$\left\| U^{-t}(K_{\alpha\beta}) \operatorname{Op}(f) U^t(K_{\alpha\beta}) - \operatorname{Op}(f \circ K_{\alpha\beta}^t) \right\|_{\mathcal{H}_N} = \mathcal{O} \left( \frac{1}{N} \right)$$

(3.12)

for all observables $f \in C^\infty(T^2)$ and all fixed times $t \in \mathbb{N}$. A proof of (3.12) is given in appendix A.

The propagator (3.11) does not preserve all classical symmetries which were discussed in section 2. First of all the Hilbert space structure restricts the continuous families of translations in (2.4) and (2.5) to discrete translations

$$T \left( \frac{c}{N}, 0 \right) \quad \text{and} \quad T \left( 0, \frac{d}{N} \right),$$

(3.13)

with $c, d \in \mathbb{Z}_N$. But not all of these translation operators $\pi := \operatorname{Op}(\pi)$ fulfill

$$\pi U(K_{\alpha\beta}) \pi^{-1} = U(K_{\alpha\beta}).$$

(3.14)

This is only valid for $bc/N \in \mathbb{Z}$ and $ad/N \in \mathbb{Z}$ respectively. Consequently, only translations

$$T \left( \frac{kb}{Db}, 0 \right), \quad \text{with} \quad D_b := \gcd(b, N), \quad k_b \in \mathbb{Z}_{Db},$$

(3.15)

and

$$T \left( 0, \frac{ka}{Da} \right), \quad \text{with} \quad D_a := \gcd(a, N), \quad k_a \in \mathbb{Z}_{Da},$$

(3.16)
are symmetries of the quantum propagator (3.11). Beside these quantum symmetries, which correspond directly to the classical translations (2.4) and (2.5), there are further quantum symmetries,

\[ T \left( \frac{c}{N}, \frac{d}{N} \right), \quad \text{with} \quad \frac{bc - ad}{N} \in \mathbb{Z}, \quad (3.17) \]

which correspond to a combination of classical symmetries. Obviously, the last class of symmetries (3.17) includes the previous ones. The inversion symmetry (2.3) is preserved and the corresponding operator I is given by

\[ [I \psi](Q) = \bar{\psi}(-Q). \quad (3.18) \]

Summarising we emphasise that, although not all symmetries are preserved by the quantisation of Kronecker maps, there are quantum symmetries in position and momentum, which are remnants of the classical translation invariance. However, it depends on the rational approximations for \( \alpha \) and \( \beta \), and thus also on the semiclassical parameter \( N \), which of these phase space translations are conserved. Hence the characteristics of the quantum maps depend on number theoretical properties of \( \alpha \) and \( \beta \).

4. Trace formula and spectral statistics

The purpose of this section is to investigate the spectrum of the propagator \( U(K_{\alpha\beta}) \). The eigenangles \( \vartheta \) and the eigenstates \( |j\rangle \) are defined by

\[ U(K_{\alpha\beta}) |j\rangle = e^{\frac{2\pi i}{N} \vartheta_j} |j\rangle, \quad j \in \mathbb{Z}_N. \quad (4.1) \]

Notice that by this definition the eigenangles are already unfolded, i.e. rescaled to mean density one. One could now explicitly calculate the eigenangles and the eigenvalues of the Kronecker map. However, we will not do so but instead discuss spectral properties in terms of exact and semiclassical trace formulae similar to the Gutzwiller trace formula [13]. This procedure will allow for an immediate generalisation to the case with spin which will be treated in the following sections. To this end recall that the spectral density can be expressed by traces of \( U^l(K_{\alpha\beta}) \),

\[ d(\vartheta) := \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}_N} \delta \left( \frac{2\pi k}{N} (\vartheta - \vartheta_j) - 2\pi j \right) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} e^{\frac{2\pi i}{N} l \vartheta} \text{Tr} \ U^l(K_{\alpha\beta}). \quad (4.2) \]

Thus, in order to derive a trace formula it is sufficient to calculate \( \text{Tr} \ U^l \). One easily verifies by induction that the \( l \)-step propagator in mixed representation is given by, cf. (3.10),

\[ \left< Q \left| U^l(K_{\alpha\beta}) \right| P \right> = \frac{1}{\sqrt{N}} \exp \left[ \frac{2\pi i}{N} G^{l}_{a,b} \left( \frac{P}{N}, \frac{Q}{N} \right) \right], \quad (4.3) \]

where

\[ G^{l}_{a,b}(q,p) = pq + l\beta q - l\alpha p + \frac{l(l-1)}{2} \alpha \beta \quad (4.4) \]
generates the $l$-step Kronecker map via
\[
p_l = \frac{\partial G^l_{\alpha \beta}(q_l, p_0)}{\partial q_l} \quad \text{and} \quad q_0 = \frac{\partial G^l_{\alpha \beta}(q_l, p_0)}{\partial p_0}.
\] (4.5)

In position representation the propagator reads
\[
\langle Q \mid U^l(K_{\alpha \beta}) \mid Q' \rangle = \exp \left[ \frac{2\pi i}{N} \left( lbQ + \frac{l(l - 1)}{2} ab \right) \right] \delta_{Q, Q' + la \mod N},
\] (4.6)

and thus the trace is given by
\[
\text{Tr } U^l(K_{\alpha \beta}) = N e^{\frac{2\pi i}{N} \frac{l(l - 1)}{2} ab} \delta_{0, la \mod N} \delta_{0, lb \mod N}.
\] (4.7)

We only obtain a non-zero trace if both $la$ and $lb$ are multiples of $N$. With $\gcd(a, N) = D_a$ and $\gcd(b, N) = D_b$ we define $M_a = N/D_a$ and $M_b = N/D_b$ respectively. Hence, in order to get a non-vanishing trace, $l$ has to be a multiple of both $M_a$ and $M_b$.

The Kronecker map with translations $a/N$ and $b/N$ has a two-parameter family of periodic orbits $\gamma$. The periodic points are given by all $(q, p) \in \mathbb{T}^2$ and the length of the periodic orbits is determined by
\[
l_\gamma = \text{lcm} (M_a, M_b).
\] (4.8)

Thus the condition for a non vanishing contribution in (4.7) exactly singles out those values of $l$ which correspond to multiples of $l_\gamma$.

The exponent of (4.7) has to be interpreted as the action of the periodic orbit in the following sense. In general one can pass from the generating function $G^l_{\alpha \beta}(q, p)$ in mixed representation to a generating function $S^l_{\alpha \beta}(q, q')$ in position representation by the Legendre transform
\[
S^l_{\alpha \beta}(q, q') := G^l_{\alpha \beta}(q, p) - pq'.
\] (4.9)

Here, this step is only formal, since we cannot solve $q' = \frac{\partial G^l_{\alpha \beta}(q, p)}{\partial p}$ for $p$ in order to eliminate the initial momentum from (4.9). However, we can use (4.9) to calculate the action
\[
S^l := S^l_{\alpha \beta} \left( \frac{Q + la}{N}, \frac{Q}{N} \right) - lb \frac{Q}{N} \frac{Q}{N} = \frac{l(l - 1)}{2} \frac{a b}{N}.
\] (4.10)

of a periodic orbit, which appears in the trace formula for quantum maps. The last term involves the winding number $lb$ in momentum and thus accounts for the difference of the action $S^l$ on the torus $\mathbb{T}^2$ and the action $S^l_{\alpha \beta \gamma} \left( \frac{Q + la}{N}, \frac{Q}{N} \right)$ on the covering plane $\mathbb{R}^2$. Therefore, the trace of $U^l$ is given by
\[
\text{Tr } U^l(K_{\alpha \beta}) = N e^{2\pi i N S^l} \delta_{0, la \mod M_a} \delta_{0, lb \mod M_b}.
\] (4.11)

Finally, we can express the spectral density (4.2) in terms of the periodic orbits of the classical map which we used to approximate the original map for quantisation,
\[
d(\vartheta) = \frac{N}{2\pi} + \frac{N}{2\pi} \sum_{k \in \mathbb{Z}, k \neq 0} e^{2\pi i kl_\gamma} \theta e^{2\pi i N S^l_{\alpha \beta \gamma}}.
\] (4.12)
Thus we have derived an exact trace formula for quantised Kronecker maps with fixed $N$. In the semiclassical limit the approximations $a$ and $b$ change and so does the length $l_{\gamma}$ of the periodic orbit. For an irrational translation on the two-torus we have

$$l_{\gamma} \xrightarrow{N \to \infty} \infty,$$

(4.13)

as should be expected for a uniquely ergodic map without periodic orbits.

From the spectral density (4.12) it is clear that the Kronecker map does not show generic spectral statistics. Instead the behaviour for fixed $N$ depends crucially on the approximations (3.9). This can be observed explicitly by investigating the form factor

$$K(\tau) := \frac{1}{N} \left| \text{Tr} U^{\tau N} (K_{\alpha \beta}) \right|^2 - N \delta_{\tau,0} = N \delta_{\tau,0 \mod l_{\gamma}},$$

(4.14)

which has peaks if $\tau$ is a multiple of $l_{\gamma}/N$ as shown in figure 1(b). Also the probability density for the spacings $s_j := \vartheta_{j+1} - \vartheta_j$, the level spacing density $P(s)$, defined by

$$\int_a^b P(s) \, ds := \frac{\# \{ s_j \in (a,b) : j \in \mathbb{Z}_N \}}{N},$$

(4.15)

behaves non-universal. In figure 1(a) the level spacing distribution

$$I(s) := \int_0^s P(s') \, ds'$$

(4.16)

is compared to the respective results of the circular orthogonal, unitary and symplectic ensembles (COE, CUE and CSE respectively) of RMT and the Poissonian level spacing distribution.
We only remark that in the limit $N \to \infty$ these statistics have no limit distribution but depend crucially on number theoretical properties of the approximations (3.9), cf. the similar situation for the quantised skew map which was discussed in detail in [1]. This behaviour is related to the ($N$-dependent) quantum symmetries (3.17) which are remnants of the classical translation invariance (2.4) and (2.5). The connection becomes evident in the trace formula (4.12) where all degenerate periodic orbits collectively contribute to the spectral density.

5. Kronecker maps with spin – generic statistics

Having identified the quantum symmetries (and thus the classical symmetries) of the Kronecker map as being responsible for its non-generic spectral statistics, we now aim at breaking these symmetries without changing the characteristic properties of the classical system, i.e. without introducing any stronger chaotic property beyond (unique) ergodicity. To this end we couple the quantised map to a spin $1/2$ [14,15], i.e. we seek a quantum propagator acting on $\mathcal{H}_N \otimes \mathbb{C}^2$ instead of $\mathcal{H}_N$. The explicit coupling procedure can be motivated by a Pauli equation where the translational dynamics is periodically kicked in time [15], yielding

\[
U_{\text{spin}}(K_{\alpha\beta}) = (U(K_{\alpha\beta}) \otimes \mathbb{1}_2) \text{Op} \left( e^{i\sigma \cdot B(q,p)} \right).
\] (5.1)

Here $\mathbb{1}_2$ is the $2 \times 2$ unit matrix, the components of $B(q,p)$ are in $C^\infty(T^2)$ and $\sigma$ denotes the vector of Pauli matrices with

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (5.2)

Thus, $e^{i\sigma \cdot B(q,p)}$ takes values in $SU(2)$ and is quantised by applying the Weyl quantisation (3.8) to its components. The physical meaning of $B(p,q)$ is that of a combination of magnetic fields and spin-orbit coupling terms, and thus $U_{\text{spin}}(K_{\alpha\beta})$ describes iterations of the quantum map alternating with spin precession.

Note that the coupling of the spin degrees of freedom to the quantum map is in non-leading semiclassical order, as can be seen from (5.1), where in contrast to (3.10) there is no pre-factor $N$ in the exponent of the spin part. Therefore the corresponding classical dynamics on the torus is not perturbed but only accompanied by a cocycle taking values in $SU(2)$ as we will point out in the following.

By multiple applications of the Egorov property (3.12) in the semiclassical limit we can write (by a slight abuse of notation we define $U(K_{\alpha\beta}) \otimes \mathbb{1}_2 \equiv U(K_{\alpha\beta})$)

\[
U_{\text{spin}}^t(K_{\alpha\beta}) = \left( U(K_{\alpha\beta}) \text{Op} \left( e^{i\sigma \cdot B} \right) \right)^t = U^t(K_{\alpha\beta}) U^{-(t-1)}(K_{\alpha\beta}) \text{Op} \left( e^{i\sigma \cdot B} \right) \ldots \text{Op} \left( e^{i\sigma \cdot B} \right) U^{-(t-2)}(K_{\alpha\beta}) \ldots \text{Op} \left( e^{i\sigma \cdot B} \right) U^{-(t-1)}(K_{\alpha\beta}) \text{Op} \left( e^{i\sigma \cdot B} \right)
\] (5.3)

\[
= \left( U^t(K_{\alpha\beta}) \text{Op} \left( e^{i\sigma \cdot B \circ K^{t-1}_{\alpha\beta}} \right) \ldots \text{Op} \left( e^{i\sigma \cdot B \circ K_{\alpha\beta}} \right) \text{Op} \left( e^{i\sigma \cdot B} \right) \right) \left( 1 + \mathcal{O}(N^{-1}) \right)
\]

\[
= U^t(K_{\alpha\beta}) \text{Op} \left( d_t \left( 1 + \mathcal{O}(N^{-1}) \right) \right),
\]
where the cocycle \( d_t \) is defined by
\[
d_t(q,p) := e^{i\sigma B(K_{\alpha\beta}^{-1}(q,p))} e^{i\sigma B(K_{\alpha\beta}^{-2}(q,p))} \cdots e^{i\sigma B(q,p)},
\] (5.4)
For later reference we remark that spin dynamics and classical map can be combined to form a skew product on \( T^2 \times SU(2) \) by
\[
Y : \quad T^2 \times SU(2) \rightarrow T^2 \times SU(2)
\]
\[
(q,p,g) \mapsto (K_{\alpha\beta}(q,p), e^{i\sigma B(q,p)}g)
\] (5.5)
thus implying, \( Y^t(q,p,g) = (K_{\alpha\beta}^t(q,p), d_t(q,p)g) \). Notice that \( Y \) leaves the product measure \( \mu = \mu_{T^2} \times \mu_H \) invariant which consists of Lebesgue measure \( \mu_{T^2} \) on the torus and the normalised Haar measure \( \mu_H \) on \( SU(2) \). Observables are now given by hermitian \( 2 \times 2 \) matrices
\[
F(q,p) = \begin{pmatrix}
  f_{11}(q,p) & f_{12}(q,p) \\
  f_{21}(q,p) & f_{22}(q,p)
\end{pmatrix}
\] (5.6)
with \( f^{jk} \in C^\infty(T^2) \) which are quantised component-wise by (3.8). From (3.12) and (5.3) now follows the Egorov property for Kronecker maps with spin.
\[
\|U^t_{spin} \text{Op}(F) U^t_{spin} - \text{Op}(d_t^{-1}) \text{Op}(F \circ K_{\alpha\beta}^t) \text{Op}(d_t)\|_{H_N \otimes \mathbb{C}^2} = O(N^{-1})
\] (5.7)
for all fixed times \( t \in \mathbb{Z} \). Clearly, when applying this Egorov property to a scalar observable, i.e. \( F(q,p) = f(q,p)I_2 \), it reduces to (3.12). Thus the classical dynamics on the torus remains unchanged.

We now present some numerical results on the spectral statistics of the Kronecker map with spin. For convenience instead of \( \text{Op}(e^{i\sigma B(q,p)}) \) we choose \( \text{Op}(e^{i\sigma B_1(q)}) \text{Op}(e^{i\sigma B_2(p)}) \) with
\[
B_1(q) = \begin{pmatrix}
  \sin(2\pi q) \\
  \sin(4\pi q) \\
  \sin(6\pi q)
\end{pmatrix}
\quad \text{and} \quad
B_2(p) = \begin{pmatrix}
  \sin(2\pi p) \\
  \sin(4\pi p) \\
  \sin(6\pi p)
\end{pmatrix}
\] (5.8)
Due to the group property of \( SU(2) \) we can always find a unique \( B(q,p) \) with \( e^{i\sigma B(q,p)} = e^{i\sigma B_1(q)} e^{i\sigma B_2(p)} \) and by the Moyal product of the Weyl quantisation, see, e.g., [13], we have
\[
\text{Op}(e^{i\sigma B(q,p)}) = \text{Op}(e^{i\sigma B_1(q)}) \text{Op}(e^{i\sigma B_2(p)}) (1 + O(N^{-1}))
\] (5.9)
Therefore, the above semiclassical discussion applies to the numerical situation. Notice that the choice of the fields (5.8) breaks all symmetries of the quantum map \( \text{U}(K_{\alpha\beta}) \). However, since the classical dynamics on the torus is not changed by the spin-coupling, being of higher order in \( N \), the quantum map \( \text{U}_{spin}(K_{\alpha\beta}) \) still corresponds to a non-mixing classical system. If the fields \( B_1 \) and \( B_2 \) generate a sufficiently random dynamics on \( SU(2) \) we may expect the skew product (5.3) to be ergodic, but it can never be mixing since this would also imply the dynamics on the torus, given by \( K_{\alpha\beta} \), to be mixing.

In figure 2 we show the level spacing density and the level spacing distribution for a Kronecker map with spin, observing good agreement with the CUE. This is consistent with the fact that \( \text{U}_{spin}(K_{\alpha\beta}) \) has neither unitary nor anti-unitary symmetries. The form factor 2(b) shows typical fluctuations and when averaging (over an interval of length \( \Delta \tau = 0.2 \) with unit weight) one obtains again nice agreement with the CUE.
Thus we have provided an example in favour of the BGS conjecture, where only the weakest possible chaotic property, i.e. ergodicity, is fulfilled. The only difference to quantised Kronecker maps without spin, which show non-generic spectral statistics, is the absence of quantum mechanical degeneracies, which were discussed in the previous sections. In the following, we will explain that this is related to the lifting of degeneracies of periodic orbits in the semiclassical trace formula. We remark that for different values of $\alpha, \beta \in \mathbb{S}^1 \setminus \mathbb{Q}$ we make similar observations.
6. Trace formula with spin

The purpose of this section is to derive a trace formula for Kronecker maps with spin. Semi-classical trace formulae for flows with spin have been derived in [16, 17] and trace formulae for (perturbed) cat maps with spin have been investigated in [15].

First recall (5.3) that the propagator of the Kronecker map with spin fulfils

$$U_{\text{spin}}^l(K_{\alpha\beta}) = U^l(K_{\alpha\beta}) \text{Op}(d_l) \left(1 + \mathcal{O}(N^{-1})\right),$$

where $d_l$ denotes the cocycle (5.4). Calculating the trace of (6.1) thus reduces to deriving a (semiclassical) trace formula for $U^l(K_{\alpha\beta})$ with matrix elements of the quantum observable $\text{Op}(f)$, $f = \text{tr} d_l$, where $\text{tr}(\cdot)$ denotes the trace on $\mathbb{BV}^2$, i.e. we have

$$\text{Tr}\left(U^l(K_{\alpha\beta}) \text{Op}(f)\right) = \sum_{Q, Q' \in \mathbb{Z}_N} \langle Q | U^l(K_{\alpha\beta}) | Q' \rangle \langle Q' | \text{Op}(f) | Q \rangle.$$ 

With the Weyl quantisation (3.8) and the propagator (4.6) we obtain

$$\text{Tr}\left(U^l(K_{\alpha\beta}) \text{Op}(f)\right) = N \sum_{n, m \in \mathbb{Z}} f_{nm} e^{\frac{i\pi}{N}(mn-l(l-1)ab)} \delta_{n, l\text{mod} N} \delta_{m, l\text{mod} N}.$$

and thus get a non vanishing trace for

$$n \equiv n_{la} \mod N \quad \text{and} \quad m \equiv n_{lb} \mod N.$$ 

Let us discuss the first condition for fixed $l \neq 0$ in the semiclassical limit $N \to \infty$. We distinguish the following two cases:

(i) $n_{la} \equiv 0 \mod N$:

Then (6.4) is clearly fulfilled for $n = 0$ which leads to the same condition as without spin, see (4.11). The next values of $n$ for which (6.4) holds are given by $n = \pm N$. In consequence the smallest non-trivial value of $n$ is given by $\min_{n \neq 0} |n| = N$.

(ii) $n_{la} \neq 0 \mod N$:

The smallest value of $n$ for which (6.4) is fulfilled is given by $\min |n| = \min(la, N - la)$. With the approximation (3.9) we get $\min |n| = \min(lNa + \mathcal{O}(1), N(1 - la) + \mathcal{O}(1))$. Since $l$ is fixed this leads to $\min |n| = \mathcal{O}(N)$.

From (6) and (1) it is clear that in the semiclassical limit the first non-trivial $n$ which yields a contribution to (6.4) is given by

$$\min_{n \neq 0} |n| = \mathcal{O}(N).$$

The second condition in (6.4) is discussed analogously. Hence in the semiclassical limit (6.3) reduces to the term with $n = m = 0$. All other terms are of higher semiclassical order since $f \in C^\infty(\mathbb{T}^2)$ implies that the coefficients $f_{nm}$ of the Fourier series decrease exponentially. We therefore have

$$\text{Tr}\left(U^l(K_{\alpha\beta}) \text{Op}(f)\right) \sim N f_{00} e^{\frac{i\pi}{N}(1-l)ab} \delta_{0, l\text{mod} M_a} \delta_{0, l\text{mod} M_b},$$

(6.6)
which can be written in terms of the periodic orbits of the approximated Kronecker map with length \( l_\gamma = \text{lcm}(M_a, M_b) \) and action \( S^l \) from equation (4.11) yielding

\[
\text{Tr} \left( U^l_{\text{spin}}(K_{\alpha \beta}) \right) \sim N \sum_{k \in \mathbb{Z}} \delta_{l, kl_\gamma} e^{2\pi i NS^l} \int \int d_l(q,p) \, dq \, dp ,
\]

(6.7)

where we have also substituted the observable \( f(q,p) \) by \( \text{tr} d_l(q,p) \). As in the case without spin (4.11) we still have a two-parameter family of periodic orbits with action \( S^{kl_\gamma} \). However, the contribution of each periodic point \((q,p)\in\mathbb{T}^2\) is weighted by the spin trace \( \text{tr} d_{kl_\gamma}(q,p) \), thus breaking the degeneracies semiclassically without changing the periodic orbit structure of the classical torus map.

From the Egorov property (5.7) it was already clear that the spin does not change the classical dynamics on the torus. This can now be nicely illustrated using the trace formula from above. To this end we evaluate the spectral density on a test-function \( g_\varepsilon(L,\vartheta) \),

\[
d_\varepsilon(L) := \int_{-\infty}^{\infty} g_\varepsilon(L,\vartheta) d(\vartheta) \, d\vartheta .
\]

(6.8)

Choosing

\[
g_\varepsilon(L,\vartheta) := \cos(\vartheta L) e^{-\vartheta^2 \varepsilon} ,
\]

(6.9)

we obtain the so-called trace of the cosine-modulated heat-kernel \([20, 21]\), and with the identity (4.2) \( d_\varepsilon(L) \) becomes

\[
d_\varepsilon(L) = \sqrt{\frac{\pi}{\varepsilon}} \left( N e^{-\frac{l^2}{4N}} + \frac{1}{4\pi} \sum_{l=1}^{\infty} e^{-\frac{(l-L)^2}{4\varepsilon}} \text{Tr} U^l \right) .
\]

(6.10)

Therefore, by (6.7), in the semiclassical limit \( d_\varepsilon(L) \) has Gaussian peaks at multiples of the periods of periodic orbits. These can in turn be determined from the quantum eigenvalues. For a strongly chaotic map this is rather uninteresting since for each possible length \( L \in \mathbb{N} \) an exponentially increasing number of orbits contributes to the amplitude. In our situation, however, there are only contributions for lengths \( L \) which are multiples of \( l_\gamma \). Thus, each peak corresponds to a particular number \( k \) of repetitions of the family \( \gamma \) of periodic orbits and we can observe how spin changes their effective multiplicity, i.e. the weight factor which these orbits contribute to the trace formula.

In figure 3 we compare the cosine-modulated heat-kernel (6.10) for the Kronecker map with and without spin \( \frac{1}{2} \). For both maps we have chosen the same parameters \( \alpha \) and \( \beta \) as well as the same value of \( N \). Thus we get identical classical dynamics on the torus for the approximating maps which were used for quantisation. The dimension of the Hilbert space of the map with spin is twice the dimension of the Hilbert space of the map without spin. Thus the Heisenberg time \( T_H^{\text{spin}} \) for the map with spin is twice the Heisenberg time \( T_H = N \) for the map without spin and for the case with spin in (6.10) we simply have to replace \( N \) by \( 2N \).

For small \( L \) we get pronounced peaks at multiples of \( l_\gamma \) for both the map with and without spin. In agreement with the exact trace formula (4.12) for the pure Kronecker map without spin this remains true for all \( L \). For \( U_{\text{spin}}(K_{\alpha \beta}) \), however, when increasing \( L \) we first observe that the height of the peaks decreases due to the spin weight \( \int \int_{\mathbb{T}^2} \text{tr} d_l \, dq \, dp \). Then, for \( L \lesssim T_H^{\text{spin}} \),
For lengths $L \ll T_H^{\text{spin}}$ we get for both maps Gaussian peaks only for $L = kl_\gamma$, $k \in \mathbb{Z}$. The amplitude of the peaks differs for the map with and without spin $1/2$ due to the contribution of the cocycle in the trace formula (6.7).

(b) If $L$ becomes larger we get further peaks due to quantum fluctuations for the map with spin, but for $L < T_H^{\text{spin}}$ the peaks at $L = kl_\gamma$ are still dominating.

(c) Finally, for lengths of the order $T_H^{\text{spin}}$ one gets pure quantum fluctuations for the Kronecker map with spin.

Figure 3: Cosine-modulated heat-kernel for a Kronecker map with and without spin. In both cases we have chosen $N = 3000$. Thus the Heisenberg time is given by $T_H = 3000$ and $T_H^{\text{spin}} = 6000$ respectively. For both maps the approximations of $\alpha$ and $\beta$ yield $M_a = 5$ and $M_b = 2$. Therefore the lengths of the periodic orbits are given by multiples of $l_\gamma = 10$.

Additional contributions appear which correspond to higher order corrections to the trace formula (6.7). Finally, for $L \gg T_H^{\text{spin}}$, these corrections dominate the behaviour. Summarising we have confirmed that in the semiclassical regime, i.e. for $L \ll T_H^{\text{spin}}$, the coupling of the map to a spin only changes the amplitudes of the periodic orbit contributions but does not affect the classical dynamics on the torus.

A similar discussion now applies to the spectral form factor, which for $U_{\text{spin}}(K_{\alpha\beta})$ is defined
by
\[
K(\tau) := \frac{1}{T_H^{\text{spin}}} \left| \text{Tr} \left[ U_{\text{spin}}(K_{\alpha\beta}) \right]^{\tau T_H^{\text{spin}}} \right|^2 - 2N\delta_{\tau,0} .
\]  

(6.11)

The smallest value of \( \tau \) for which \( K(\tau) \) does not vanish is given by
\[
\tau_{\text{min}} = \frac{l_{\gamma}}{T_{\text{spin}}/H} = \frac{\text{lcm}(M_a, M_b)}{T_H^{\text{spin}}}.
\]  

(6.12)

If \( \alpha \) is irrational then \( M_a \to \infty \) in the semiclassical limit \( N \to \infty \) \cite[lemma 4.2]{12}, and similarly for \( M_b \). Thus if \( \tau \) tends to zero such that \( \tau < \tau_{\text{min}} \) one concludes that
\[
\lim_{N \to \infty} \lim_{\tau \to 0} K(\tau) = 0 .
\]  

(6.13)

This is, however, the same result as we would have obtained in the case without spin, cf. section \[4\] which is consistent with both the non-generic behaviour in figure \[1\] and the generic behaviour in figure \[2\]. It is interesting to remark that with present methods, even in the regime which is usually dominated by the diagonal approximation \[22,23\], it is not possible to distinguish semiclassically between generic and non-generic behaviour. This unusual observation can be traced back to the fact that the Kronecker map has no periodic orbits and thus in the semiclassical limit, paradoxically, the periodic orbit contributions to the semiclassical trace formula do not determine the behaviour of \( K(\tau) \) in the vicinity of \( \tau = 0 \). Conversely, for small but non-zero \( \tau \), in the semiclassical limit the form factor is dominated by corrections to the trace formula \[6,7\] which manifest themselves in figure \[8\] as additional peaks.

7. Ergodic skew maps

Another interesting class of parabolic maps on the two-torus \( T^2 \) are the so-called skew maps. We will consider skew maps of the form
\[
S_\beta : \quad T^2 \rightarrow T^2 \quad \left( p \atop q \right) \mapsto \left( p + \beta \atop q + 2p \right) \text{ (mod 1)} .
\]  

(7.1)

For \( \beta \in S^1 \smallsetminus Q \) one has again a uniquely ergodic but non-mixing map, for details see \[24\--26\]. If \( \beta \) is rational the map is in general almost integrable. This situation will be discussed in the following section, and is our main motivation for also analysing skew maps in addition to the simplest case of parabolic maps, namely translations of the torus. As in the case of Kronecker maps symmetries play an important rôle. Skew maps also have several symplectic symmetries. First of all there is a translation in momentum
\[
T_p : \left( p \atop q \right) \mapsto \left( p + \frac{1}{q} \atop \frac{p}{q} \right) \text{ (mod 1)} ,
\]  

(7.2)

and a family of translations in position
\[
T_q : \left( p \atop q \right) \mapsto \left( p \atop q + \gamma \right) \text{ (mod 1)} , \quad \gamma \in S^1 .
\]  

(7.3)
Figure 4: Spectral statistics for a skew map with $\beta = (\sqrt{5}-1)/2$ and $N = 3000$, implying $D = \gcd(b,N) = 3$ and $M = N/D = 1000$.

Moreover, skew maps are invariant under two anti-symplectic transformations,

$$\tau : \left( \frac{p}{q} \right) \mapsto \left( -\frac{p+\delta}{q} \right) \pmod{1}, \quad 2\delta \equiv 2\beta \pmod{1},$$

(7.4)

Thus, in contrast to Kronecker maps, we now have a (non-conventional) time reversal symmetry.

The first quantisation of skew maps, which fulfils an Egorov property for all $N \in \mathbb{N}$, cf. (3.12), was given by Marklof and Rudnik [12]. This quantisation is similar to the quantisation of the Kronecker map which we presented in section 3. First of all one has to approximate the translation in momentum by a rational number $b/N$ with

$$\left| \beta - \frac{b}{N} \right| \leq \frac{1}{2N}, \quad b \in \mathbb{Z}.$$  (7.5)

The propagator $U(S_\beta)$ can then be defined similarly to (3.10) by using a generating function $G(q_{n+1}, p_n)$. In mixed representation one obtains

$$\langle Q \mid U(S_\beta) \mid P \rangle = \frac{1}{\sqrt{N}} \exp\left( 2\pi i N G_{\beta} \left( \frac{Q}{N}, \frac{P}{N} \right) \right) = \frac{1}{\sqrt{N}} e^{2\pi i (-P^2 + Q(P+b))},$$

(7.6)

and in momentum representation $U(S_\beta)$ is simply given by

$$\langle P \mid U(S_\beta) \mid P' \rangle = e^{-\frac{2\pi i}{N} P^2} \delta_{P, P'+b}.$$  (7.7)

For this propagator the Egorov property was proven in [12]. Again the quantisation does not preserve all classical symmetries. For our purposes it is not necessary to give a complete list
of all quantum symmetries, instead we only remark that there are always unitary quantum symmetries given by translations in position

$$T \left( \frac{k}{D}, 0 \right), \quad \text{with} \quad D := \gcd(b,N), \quad k \in \mathbb{Z}_D,$$

(7.8)

and an anti-unitary symmetry $\tau := \text{Op}(\tau)$ involving a translation in momentum

$$[\tau \psi](P) = \psi \left( -P + \frac{b}{N} \right).$$

(7.9)

As for Kronecker maps one can calculate the eigenvalues and eigenvectors of the propagator (7.6) explicitly [12]. Analysing the spectrum of skew maps one observes non-generic spectral statistics determined by the value of $D$ defined in (7.8). Furthermore, there is even no limit distribution [4]. The non-generic behaviour of the spectral statistics of a quantum skew map is illustrated in figure 4. For a semiclassical analysis we again use a trace formula. Details of the derivation can be found in appendix B. With (3.25), see also (3.17), we obtain for the spectral density

$$d(\vartheta) = \frac{N}{2\pi} \sqrt{\frac{N}{2\pi}} \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\nu \in \mathbb{Z}_{2k}} \frac{M}{\sqrt{2ikM}} e^{\frac{2\pi i kM}{N}} e^{2\pi i N S_k^\nu},$$

(7.10)

with $M := N/d$. The periods of periodic orbits of the approximating map are given by $kM$, and $\nu$ labels families of periodic points sharing the same action. These families form invariant manifolds $\mathcal{E}_{\nu/kM}(b/N)$, see (8.2) below, which in particular are invariant under translations in position as is the classical map. Again we have high degeneracies in both the quantum spectrum and in the trace formula which are the origin of the non-generic spectral statistics of the quantum map (see figure 4).
As in the case of Kronecker maps we couple the quantised skew map to a spin 1/2 in order to break these symmetries. As we pointed out above the skew map also has symmetries in both position and momentum. Therefore, we use the same magnetic field $B$ as for the Kronecker map (5.8), depending on both $q$ and $p$. The same calculation as in section 5 then yields the Egorov property for skew maps with spin.

In appendix B we also derive a semiclassical trace-formula for the skew map with spin. With (B.25) the spectral density reads,

$$d(\theta) \sim \frac{N}{\pi} + \frac{N}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\nu \in \mathbb{Z}_{2k}} \frac{e^{2\pi i k M \theta}}{\sqrt{2i M}} \int_0^1 \text{tr} d_{k M} \left( q, \tilde{p}^{(\nu)} + \frac{i}{M} j \right) dq,$$

(7.11)

where now in contrast to (7.10) the different periodic orbits of one family, labelled by $\nu$, enter with different spin weights $\text{tr} d$. Hence the magnetic field again breaks the degeneracies in the periodic orbit sum (7.11), but does not change the classical dynamics on the torus in the sense that the set of periodic orbits remains the same and only their amplitudes are altered.

In figure 5(a) we show the level spacing distribution and the form factor for an ergodic skew map with spin 1/2. As in the case of the Kronecker map we observe good agreement with the CUE. Hence, we have a further example of a system which is only ergodic, but obeys the BGS conjecture.

By using the same arguments as in section 6 and the semiclassical trace formula (7.11) we conclude that for the form factor of ergodic skew maps

$$\lim_{N \to \infty \tau \to 0} K(\tau) = 0$$

(7.12)

holds. But again, as in the case of Kronecker maps, the behaviour of $K(\tau)$ for small but non-zero $\tau$ cannot simply be determined by a semiclassical analysis.

8. Almost integrable skew maps

We now turn our attention to the interesting case of rational skew maps

$$S_\beta : \mathbb{T}^2 \rightarrow \mathbb{T}^2$$

$$ \left( \begin{array}{c} p \\ q \end{array} \right) \mapsto \left( \begin{array}{c} p + \beta \\ q + 2p \end{array} \right) \pmod{1},$$

(8.1)

with $\beta \in \mathbb{S} \cap \mathbb{Q}$, i.e. we can write $\beta = B/M$ were $M \in \mathbb{N}$ and $B \in \{0,1,\ldots,M-1\}$ are relatively prime. One immediately finds that invariant sets of $S_\beta$ are given by $M$ disjoint circles

$$E_{p_0}(\beta) := \left\{ (q,p) \in \mathbb{T}^2 \mid p = p_0 + \frac{j}{M}, \ j \in \mathbb{Z}_M \right\}.$$

(8.2)

If $\beta = 0$ then $E_{p_0}(\beta) \cong \mathbb{S}$ and thus the map is integrable. In all other cases one has $M > 1$ and the map restricted to $E_{p_0}(\beta)$ can be represented as an interval-exchange transformation (for interval-exchange transformations see, e.g., [22] and references therein). Therefore, we will call this situation almost integrable or pseudointegrable. We only remark that for $M \to \infty$...
we approach the situation of the last section, i.e. when increasing $M$, starting from $M = 1$, we observe a transition from integrable to ergodic.

For the quantisation of $S_\beta$ we can use the same procedure as in section 7, with the important difference that we will not approximate $\beta$ but instead restrict $N$ to multiples of $M$, i.e. $N = DM$, $D \in \mathbb{N}$. As a consequence, the Egorov property, cf. (3.12), is not merely an asymptotic relation but an identity. Following the notation of section 7 we will also write $\beta = b/N$ with $b = BD$.

The spectral statistics of the skew map depends sensitively on the value of $D$, which according to (7.8) determines the number of quantum symmetries. In [4] it was shown that for $\beta \notin \mathbb{Q}$ subsequences of quantisations with fixed $D$ lead to different limit distributions as $N \to \infty$ for various spectral statistics. In [12, corollary 5.2] it was shown that the multiplicity of eigenphases of $U(S_\beta)$ is bounded from above by $c(\epsilon)D^{1/2+\epsilon}$ for each $\epsilon > 0$, where $c(\epsilon)$ does not depend on $D$, i.e. for larger values of $D$ one can obtain more degenerate spectra. Whereas for $\beta \in \mathbb{Q}$ we have $D = N/M$, i.e. $D$ grows linearly with $N$, for $\beta \notin \mathbb{Q}$ it can be shown [4] that as $N \to \infty$ there exist subsequences with fixed $D$ for any finite value of $D$. For example there is always a subsequence with $D = 1$. If $\beta \notin \mathbb{Q}$ is badly approximable, $D$ is bounded from above by $c(\epsilon)N^{1/2+\epsilon}$ for each $\epsilon > 0$ [12, lemma 4.2]. Thus, in general, for rational skew maps we obtain much larger values of $D$, which typically lead to higher degeneracies. This is nicely illustrated in figure 6(a), where $I(0)$ is larger than in figure 6(b). Roughly speaking, quantisations of skew maps with rational $\beta$ are even more degenerate than those with $\beta$ irrational.

We remark that one directly obtains an analytic expression for the non-generic form factor in figure 6(b) from the trace formula for the skew map, cf. appendix 4. Moreover, the mean of the form factor for $\tau \gg 1$ gives the average of the degeneracies, which now is 13 in contrast to $5/3$ for the ergodic example in the previous section.

When now coupling a spin $1/2$ to the map as in the previous cases, we again expect that this procedure lifts the degeneracies of periodic orbits in the trace formula and the degenera-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Spectral statistics of an almost integrable skew map with $N = 3000$ and $\beta = 3/5$, i.e. $D = 600$ and $M = 5$.}
\end{figure}
Figure 7: Spectral statistics of an almost integrable skew map with spin $1/2$. We have chosen the same parameters as in figure 6. Therefore we have $N = 3000$, $\beta = 3/5$, $D = 600$ and $M = 5$.

The intermediate level spacing distribution of almost integrable – or pseudointegrable – systems, cf., e.g., [28, 29], is often associated with so-called Semi-Poisson statistics [30, 31] which yields

$$
P_{SP}(s) = 4s e^{-2s} \quad \text{and} \quad I_{SP}(s) = 1 - (2s + 1)e^{-2s}.
$$

At a first glance this appears to be consistent with figure 8. However, when looking more carefully, in figure 8 we still observe fluctuations about $I_{SP}(s)$ when only slightly changing $N$. The value of $K(0)$ also changes and, thus, from our data we cannot judge whether the statistics will converge to a limit distribution as $N \to \infty$. However, considering our previous discussions it is not clear whether one should expect convergence. We have argued that in order to observe generic statistics one has to break the degeneracies of the quantum skew map. Since these degeneracies grow with $D$ this can be nicely achieved for irrational skew maps where we obtain CUE statistics. For rational skew maps, on the other hand, the quickly
(a) The level spacing distribution varies with $N$. We get fluctuating about the Semi-Poissonian distribution.

(b) The form factor also varies and even the value for $\tau = 0$ changes slightly with $N$.

Figure 9: Comparison of spectral statistics for an almost integrable skew map at different values of $N$. We show the level spacing distribution and the smoothed form factor where we averaged over $\Delta \tau = 0.2$.

growing $D$ might prevent us from restoring genericity and we might only observe a tendency towards Semi-Poissonian statistics.

However, based on the semiclassical trace formula (8.25) for the skew map with spin we can give a semi-heuristic argument yielding the value to which $K(0)$ should converge in the semiclassical limit. If we insert (8.25) in the definition of the form factor,

$$K(\tau) = \frac{1}{2N} \left| \text{Tr} U_{\text{spin}}(S_{\beta}) \right|^2 - 2N\delta_{0l}, \quad \tau = \frac{l}{2N},$$

and perform the diagonal approximation [22,23], i.e. in the double sum over families of periodic orbits, labelled by $\nu$, we keep only terms with like actions, we obtain

$$K(\tau) \sim \frac{1}{2} \sum_{k \in \mathbb{Z}} \delta_{l,kM} \frac{1}{2kM} \sum_{\nu \in \mathbb{Z}_{2k}} M^2 \left[ \langle \text{tr} d_{l}(q,p) \rangle_{E_{p(\nu)}(\beta)} \right]^2,$$

where

$$\langle f(q,p) \rangle_{E_{p(\nu)}(\beta)} := \frac{1}{M} \sum_{j \in \mathbb{Z}_M} \int_{0}^{1} f \left(q, \tilde{p}^{(\nu)} + \frac{j}{M} \right) dq$$

denotes the average over the invariant manifold $E_{p(\nu)}(\beta)$. Due to the anti-symplectic symmetries (7.4) there are additional degeneracies in the actions. However, assuming that the corresponding spin weights $\text{tr} d_{l}$ are sufficiently uncorrelated we do not take correlations between these terms into account. Thus there appears no additional factor of 4 in (8.5). In order to determine $K(0)$ we average $K(\tau)$ over an interval $(0,\tau^*)$ and simultaneously perform the limit $N \to \infty$ and $\tau^* \to 0$ such that $\tau^*N \to \infty$. Since then the dominating contributions come

21
from large $l$ we are interested in the behaviour of \((8.5)\) as $l \to \infty$. In this limit we can replace the sum over $\nu$ by an integral obtaining
\[
K(\tau) \sim \frac{1}{2} \sum_{k \in \mathbb{Z}} \delta_{l,kM} M \bar{a}_l
\tag{8.7}
\]
with the combined average
\[
\bar{a}_l := M \int_{0}^{1/M} \left[ \langle (\text{tr} d_l(q,p))_{E_{p'}(\beta)} \rangle \right]^2 dp'.
\tag{8.8}
\]
Performing the $\tau$-average or $l$-average, respectively, as described above, yields
\[
K(0) \sim \frac{1}{2} \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \bar{a}_{kM},
\tag{8.9}
\]
and, since due to $|\text{tr} d_l| \leq 2$ we have $0 \leq \bar{a}_l \leq 4$, the value of $K(0)$ is restricted to the interval $[0, 2]$.

We will now give a heuristic argument why it is reasonable to expect $K(0) \sim \frac{1}{2}$. To this end consider $\langle (\text{tr} d_l(q,p))_{E_{p'}(\beta)} \rangle^2$. Since the skew map $S_{\beta}$ restricted to $E_{p'}(\beta)$ is ergodic for almost all $p'$ we can replace $\langle \cdot \rangle_{E_{p'}(\beta)}$ by some time average $\langle \cdot \rangle_{t}$ along an orbit with initial condition $(q_0, p_0) \in E_{p'}(\beta)$. We now assume that for large $l$ the spin weights $\text{tr} d_l(S_{\beta}^l(q_0, p_0))$ are uncorrelated, which for the magnetic fields \((5.8)\) is a reasonable assumption. Therefore, we have $\langle (\text{tr} d_l)^2 \rangle = \langle \text{tr} d_l \rangle^2$, which again due to the ergodicity of $S_{\beta}$ on $E_{p'}(\beta)$, implies
\[
\bar{a}_l = M \int_{0}^{1/M} \langle (\text{tr} d_l(q,p))^2 \rangle_{E_{p'}(\beta)} dp'.
\tag{8.10}
\]
If we also assume that the skew product $Y^l : (q, p, g) \mapsto (S_{\beta}^l(q, p), d_l(q,p)g)$, cf. \((5.5)\), is ergodic on $E_{p'}(\beta) \times \text{SU}(2)$ for almost all $p'$ (which, again referring to the fields \((5.8)\), is a good assumption) we obtain, cf. \([15\ 18]\),
\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \bar{a}_{kM} = \lim_{L \to \infty} \frac{1}{L} \sum_{l=1}^{L} \bar{a}_l = \int_{\text{SU}(2)} (\text{tr} g)^2 d\mu_H(g) = 1,
\tag{8.11}
\]
where $\mu_H$ denotes the normalised Haar measure on $\text{SU}(2)$. Thus we find $K(0) \sim \frac{1}{2}$, which is in agreement with the Semi-Poisson distribution \([30\ 31]\). However one should note that both in the light of our previous discussion about the slow numerical convergence and considering the various assumptions made in the last paragraph this should not be viewed as a derivation but rather as a semiclassical illustration of our numerical findings. A final decision about whether the quantised skew map with spin shows Semi-Poissonian statistics requires further investigations, both numerical and semiclassical.
9. Conclusions

We have investigated quantised parabolic maps on the torus which have highly degenerate spectra showing non-generic spectral statistics. An outstanding property of their classical limit is the high degeneracy of periodic orbits occurring in continuous families. For ergodic maps we have shown numerically that both the classical and quantum degeneracies can be lifted by coupling the map to spin $1/2$, thus leading to generic spectral statistics following RMT predictions. Our numerical findings are supported by a semiclassical analysis. Hence we have given evidence that in generic situations ergodicity might be a sufficiently strong condition in order to observe RMT statistics, shedding new light on the weakest possible requirements for the BGS conjecture.

Moreover, we have investigated an almost integrable map which also has a highly degenerate spectrum. Again by coupling the map to a spin $1/2$ we numerically observed a transition to intermediate statistics. We have also given a preliminary semiclassical discussion on whether this situation can be described by the so-called Semi-Poisson statistics. However, a final decision on this particular issue is beyond the scope of the present paper.

Acknowledgement

We would like to thank Jens Bolte, Roman Schubert and Jan Wiersig for helpful discussion. We also acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) under contracts no. Ste 241/9-2 and Ste 241/10-1.

A. Egorov theorem for the Kronecker map

For the sake of completeness we give a proof of the Egorov property (3.12) for the quantised Kronecker map, cf. [12]. To this end first consider the Weyl quantisation of a time evolved observable $f(q,p)$,

$$\text{Op}(f \circ K_t^{\alpha \beta}) = \text{Op} \left( \sum_{n,m \in \mathbb{Z}} f_{nm} e^{2\pi i t (m(q+\alpha) - n(p+\beta))} \right)$$

$$= \sum_{n,m \in \mathbb{Z}} f_{nm} e^{2\pi i t (ma - n\beta)} T \left( \frac{n}{N}, \frac{m}{N} \right),$$

with $t \in \mathbb{Z}$. On the other hand, by virtue of the commutation relations (3.1), for the quantum time evolution we have

$$U^{-t}(K_{\alpha \beta}) \text{Op}(f) U^t(K_{\alpha \beta}) = \sum_{n,m \in \mathbb{Z}} f_{nm} e^{2\pi i t \frac{ma-n\beta}{N}} T \left( \frac{n}{N}, \frac{m}{N} \right).$$

Thus, the discrepancy between classical and quantum mechanical time evolution is given by

$$\Delta f(t) := \left\| U^{-t}(K_{\alpha \beta}) \text{Op}(f) U^t(K_{\alpha \beta}) - \text{Op}(f \circ K_t^{\alpha \beta}) \right\|_{H_N}$$

$$= \left\| \sum_{n,m \in \mathbb{Z}} f_{nm} \left( e^{2\pi i t \frac{ma-n\beta}{N}} - e^{2\pi i t (ma-n\beta)} \right) T \left( \frac{n}{N}, \frac{m}{N} \right) \right\|_{H_N}$$

$$\leq \sum_{n,m \in \mathbb{Z}} |f_{nm}| \left\| e^{2\pi i t \frac{ma-n\beta}{N}} - e^{2\pi i t (ma-n\beta)} \right\|_{H_N} T \left( \frac{n}{N}, \frac{m}{N} \right).$$
With \[\|T\left(\frac{N}{N}, \frac{N}{N}\right)\|_{\mathcal{H}_N} = 1\], and an application of
\[|e^{ixz} - e^{iyz}| \leq |z||x - y|, \quad x, y, z \in \mathbb{R},\]
we have
\[
\Delta f(t) \leq 2\pi|t| \sum_{n,m \in \mathbb{Z}} \left(\left|\frac{a}{N} - \alpha\right||n| + \left|\frac{b}{N} - \beta\right||n|\right)|f_{nm}|
\]
\[
\leq \frac{1}{N\pi|t|} \sum_{n,m \in \mathbb{Z}} (|n| + |m|) |f_{nm}|,
\]
which concludes the proof, since \(f \in C^\infty(T^2)\) implies that \(f_{nm}\) decays exponentially as \(n, m \to \infty\).

B. Trace formula for skew maps

Before calculating the trace formula for the skew map we will summarise the structure of periodic orbits \(\gamma\) for a skew map with a rational translation \(b/N\). From the definition of the map we get the condition for periodic orbits
\[
\left(\begin{array}{c}
p_l \\
q_l
\end{array}\right) = \left(\begin{array}{c}
p_0 + \frac{l b}{N} \\
q_0 + 2lp_0 + \frac{l(l-1)b}{N}
\end{array}\right) \equiv \left(\begin{array}{c}
p_0 \\
q_0
\end{array}\right) \pmod{1}.
\]  
(B.1)

The period \(l_\gamma\) is determined by the condition in momentum. With
\[
D := \gcd(b, N) \quad \text{and} \quad M := \frac{N}{D}
\]  
(B.2)
we get
\[
l_\gamma = kM, \quad \text{with} \quad k \in \mathbb{N}.
\]  
(B.3)

The other equation gives the momenta of the periodic orbits,
\[
p_0 = \frac{\nu}{2kM}, \quad \text{with} \quad \nu \in \mathbb{Z}_{2kM}.
\]  
(B.4)

There is no restriction on the initial position \(q_0\) and thus the periodic orbits occur in one-parameter families.

After this preliminary remark we now turn to the calculation of the trace formula. Starting point is the one-step propagator \((14)\) which in position representation reads
\[
\langle Q \left| U(S_\beta) \right| Q' \rangle = \frac{1}{N} \sum_{P \in \mathbb{Z}_N} \exp \left[2\pi i N \left(G_{b/N} \left(\frac{Q}{N}, \frac{P}{N}\right) - \frac{Q'}{N} \frac{P'}{N}\right)\right].
\]  
(B.5)

One easily verifies by induction that the \(l\)-step propagator is given by
\[
\langle Q \left| U^l(S_\beta) \right| Q' \rangle = \frac{1}{N} \sum_{P \in \mathbb{Z}_N} \exp \left[2\pi i N \left(G^l_{b/N} \left(\frac{Q}{N}, \frac{P}{N}\right) - \frac{Q'}{N} \frac{P'}{N}\right)\right],
\]  
(B.6)
Therefore, the initial momentum $p^{(\nu)}$ of an orbit from $Q'/N$ to $Q/N + \nu$ with length $l$ is given by

$$p^{(\nu)} = \left( \frac{Q - Q'}{2lN} + \frac{\nu}{2l} + \frac{l - 1}{2l} \right) \pmod{1},$$

(B.15)
where \( \nu \) labels the \( 2l \) different orbits from \( Q'/N \) to \( Q/N \). Hence the propagator in position representation is given by the sum over orbits from \( Q'/N \) to \( Q/N \). We can now calculate the trace of the propagator (B.11),

\[
\text{Tr} U^l(S_\beta) = \sum_{Q \in \mathbb{Z}_N} \langle Q | U^l(S_\beta) | Q \rangle .
\]  

(B.16)

By using eq. (B.13) we get

\[
\text{Tr} U^l(S_\beta) = \sqrt{\frac{N}{2i}} \sum_{k \in \mathbb{Z}} \delta_{l,kM} \sum_{\nu \in \mathbb{Z}_{2l}} \exp \left[ 2\pi iNS_l^\nu \right] ,
\]  

(B.17)

with

\[
S_l^\nu := S_{l/N}^\nu (q + \nu, q) - \frac{l}{N} q .
\]  

(B.18)

One easily verifies that the right-hand side of (B.18) does not depend on the position \( q \). Notice that \( l/b/N \) is the winding number in momentum. Thus for periodic orbits with fixed momentum \( p \) we have the same action \( S_l^\nu \) for all \( q \in S^1 \).

It remains to show that the structure of the trace formula remains the same when the coupling to the spin degrees of freedom is introduced. To this end we have to calculate (cf. section 6 and [15])

\[
\text{Tr} U_{\text{spin}}^l(S_\beta) = \text{Tr} (U(S_\beta) \text{Op}(f))
\]  

(B.19)

with \( f = \text{tr} d_l \). With (3.8) and (B.12) we have

\[
\text{Tr} U_{\text{spin}}^l(S_\beta) = \sum_{Q \in \mathbb{Z}_N} \frac{1}{\sqrt{2iN}} \sum_{\nu \in \mathbb{Z}_{2l}} \sum_{m,n \in \mathbb{Z}} \exp \left( 2\pi iNS_l^\nu \left( \frac{Q}{N} - \nu, \frac{Q - m}{N} \right) \right) \times f_{nm} \exp \left( \frac{1}{N} \frac{nm}{\pi} + 2\pi i \frac{Qn}{N} \right) .
\]  

(B.20)

For \( f \in C^\infty(\mathbb{T}^2) \) we may expand the action as \( N \to \infty \) (cf. [15]),

\[
S_{l/N}^l \left( \frac{Q}{N} - \nu, \frac{Q - m}{N} \right) \sim S_{l/N}^l \left( \frac{Q}{N} - \nu, \frac{Q}{N} \right) - \frac{\partial S_{l/N}^l}{\partial q_0} \left( \frac{Q}{N} - \nu, \frac{Q}{N} \right) \frac{m}{N}
\]  

(B.21)

Now by the same considerations as in section 4 for \( l \) fixed and \( N \) large the sum over the positions \( Q \) becomes

\[
\sum_{Q \in \mathbb{Z}_N} \exp \left( \frac{2\pi i}{N} (lb + n)Q \right) \sim \delta_{n0} \sum_{k \in \mathbb{Z}} N \delta_{l,kM} ,
\]  

(B.22)

i.e. it selects the periods \( kM \) of periodic orbits of the skew map. Thus, the trace of the propagator reads

\[
\text{Tr} U_{\text{spin}}^l(S_\beta) \sim \sum_{k \in \mathbb{Z}} \delta_{k,lM} \sqrt{\frac{N}{2i}} \sum_{\nu \in \mathbb{Z}_{2l}} e^{2\pi iNS_l^\nu} \sum_{m \in \mathbb{Z}} f_{0m} e^{2\pi i mp(\nu)} .
\]  

(B.23)
By (3.7) the sum over \( m \) is given by \( \int_{0}^{1} f(p^{(\nu)}, q) \, dq \), i.e. resubstituting \( f = \text{tr} \, d_{l} \) we obtain

\[
\text{Tr} \, \mathcal{U}_{\text{spin}}^{l}(S_{\beta}) \sim \sum_{k \in \mathbb{Z}} \delta_{k,M} \sqrt{\frac{N}{2l}} \sum_{\nu \in \mathbb{Z}_{2l}} e^{2\pi i N S_{\nu}^{l}} \int_{0}^{1} \text{tr} \, d_{l}(q, p^{(\nu)}) \, dq.
\] (B.24)

Observing that \( S_{\nu}^{l} = S_{\nu}^{l} \), if \( p^{(\nu)} \in \mathcal{E}_{p_{0}^{(b/N)}} \), cf. (8.2), which can be checked by direct computation, instead of summing over the momenta \( p^{(\nu)} \) of periodic points we can sum over the invariant manifolds \( \mathcal{E}_{p_{0}^{(b/N)}} \) and their disconnected components. Finally we have derived the trace formula

\[
\text{Tr} \, \mathcal{U}_{\text{spin}}^{l}(S_{\beta}) \sim \sum_{k \in \mathbb{Z}} \delta_{l,k,M} \sqrt{\frac{N}{2l}} \sum_{\nu \in \mathbb{Z}_{2k}} e^{2\pi i N S_{\nu}^{l}} \sum_{j \in \mathbb{Z}_{M}} \int_{0}^{1} \text{tr} \, d_{l}(q, \tilde{p}^{(\nu)} + \frac{j}{M}) \, dq,
\] (B.25)

where, \( \tilde{p}^{(\nu)} \in \mathcal{E}_{p_{0}^{(b/N)}} \) can be chosen such that \( 0 \leq \tilde{p}^{(\nu)} < 1/M \). Now the right-hand side of (B.25) has to be interpreted as follows: We have obtained a sum over all periodic orbits of the (approximating) classical skew map. The periodic orbits appear in families labelled by \( \nu \) and the spin weights \( \text{tr} \, d_{l} \) are integrated over the corresponding invariant manifold \( \mathcal{E}_{p_{0}^{(b/N)}} \).

References

[1] O. Bohigas, M.-J. Giannoni and C. Schmit: *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52** (1984) 1–4.

[2] M. V. Berry and M. Tabor: *Level clustering in the regular spectrum*, Proc. R. Soc. London Ser. A **356** (1977) 375–394.

[3] V. I. Arnold: *Mathematical Methods of Classical Mechanics*, Springer-Verlag, (1978).

[4] A. Bäcker and G. Haag: *Spectral statistics for quantized skew translations on the torus*, J. Phys. A. **32** (1999) L393–L398.

[5] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai: *Ergodic Theory*, Springer-Verlag, New York, (1982).

[6] A. Katok and B. Hasselblatt: *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, (1995).

[7] J. H. Hannay and M. V. Berry: *Quantization of linear maps on the torus - fresnel diffraction by a periodic grating*, Physica D **1** (1980) 267–290.

[8] M. Degli Esposti: *Quantization of the orientation preserving automorphism of the torus*, Ann. Inst. Henri Poincaré **58** (1993) 323–341.

[9] M. Degli Esposti, S. Graffi and S. Isola: *Classical limit of the quantized hyperbolic toral automorphisms*, Commun. Math. Phys. **167** (1995) 471–505.

[10] S. De Bièvre, M. Degli Esposti and R. Giachetti: *Quantization of a class of piecewise affine transformations on the torus*, Commun. Math. Phys. **176** (1996) 73–94.
[11] A. Bouzouina and S. De Bièvre: *Equipartition of the eigenfunctions of quantized ergodic maps on the torus*, Commun. Math. Phys. 178 (1996) 83–105.

[12] J. Marklof and Z. Rudnick: *Quantum unique ergodicity for parabolic maps*, Geom. funct. anal. 10 (2000) 1554–1578.

[13] M. C. Gutzwiller: *Periodic orbits and classical quantization conditions*, J. Math. Phys. 12 (1971) 343–358.

[14] R. Scharf: *Kicked rotator for a spin 1/2*, J. Phys. A. 22 (1989) 4223–4242.

[15] S. Keppeler, J. Marklof and F. Mezzadri: *Quantum cat maps with spin 1/2*, Nonlinearity 14 (2001) 719–738.

[16] J. Bolte and S. Keppeler: *Semiclassical Time Evolution and Trace Formula for Relativistic Spin-1/2 Particles*, Phys. Rev. Lett. 81 (1998) 1987–1991.

[17] J. Bolte and S. Keppeler: *A semiclassical approach to the Dirac equation*, Ann. Phys. (N.Y.) 274 (1999) 125–162.

[18] J. Bolte and S. Keppeler: *Semiclassical form factor for chaotic systems with spin 1/2*, J. Phys. A. 32 (1999) 8863–8880.

[19] G. B. Folland: *Harmonic Analysis in Phase Space*, Princeton University Press, (1989).

[20] R. Aurich, M. Sieber and F. Steiner: *Quantum chaos of the Hadamard-Gutzwiller model*, Phys. Rev. Lett. 61 (1988) 483–487.

[21] R. Aurich and F. Steiner: *Staircase functions, spectral rigidity and a rule for quantizing chaos*, Phys. Rev. A 45 (1992) 583–592.

[22] J. H. Hannay and A. M. Ozorio de Almeida: *Periodic orbits and a correlation function for the semiclassical density of states*, J. Phys. A. 17 (1984) 3429–3440.

[23] M. V. Berry: *Semiclassical theory of spectral rigidity*, Proc. R. Soc. London Ser. A 400 (1985) 229–251.

[24] H. Furstenberg: *Strict ergodicity and transformation of the torus*, Amer. J. Math. 83 (1961) 573–601.

[25] L. M. Abramov and V. A. Rohlin: *The entropy of a skew product of measure-preserving transformations*, Amer. Math. Soc. Transl. 48 (1965) 255–265.

[26] M. Courbage and D. Hamdan: *Decay of correlation and mixing properties in a dynamical system with zero K-S entropy*, Ergod. Th. & Dynam. Sys. 17 (1997) 87–113.

[27] B. Hasselblatt and A. Katok: *Principal Structures*, in: *Handbook of Dynamical Systems A* (Eds. B. Hasselblatt and A. Katok), Elsevier, (in press), http://www.tufts.edu/~bhasselb/research_info.html.

[28] P. J. Richens and M. V. Berry: *Pseudointegrable systems in classical and quantum mechanics*, Physica D 2 (1981) 495–512.
[29] A. Shudo, Y. Shimizu, P. Šeba, J. Stein, H.-J. Stöckmann and K. Życzkowski: Statistical properties of spectra of pseudointegrable systems, Phys. Rev. E 49 (1994) 3748–3756.

[30] E. B. Bogomolny, U. Gerland and C. Schmit: Models of intermediate spectral statistics, Phys. Rev. E 59 (1999) R1315–R1318.

[31] E. B. Bogomolny, U. Gerland and C. Schmit: Short-range plasma model for intermediate spectral statistics, Eur. Phys. J. B 19 (2001) 121–132.