Semileptonic $B \rightarrow D^{**}$ decays in lattice QCD: a feasibility study and first results

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Abstract We compute the decays $B \rightarrow D_0^{(*)}$ and $B \rightarrow D_2^{(*)}$ with finite masses for the $b$ and $c$ quarks. We first discuss the spectral properties of both the $B$ meson as a function of its momentum and the $D_0^{(*)}$ and $D_2^{(*)}$ at rest. We compute the theoretical formulae leading to the decay amplitudes from the three-point and two-point correlators. We then compute the amplitudes at zero recoil of $B \rightarrow D_0^{(*)}$, which turns out not to be vanishing contrary to what happens in the heavy quark limit. This opens the possibility to get better agreement with experiment. To improve the continuum limit we have added a set of data with smaller lattice spacing. The $B \rightarrow D_2^{(*)}$ vanishes at zero recoil and we show a convincing signal but only slightly more than 1 sigma from 0. In order to reach quantitatively significant results we plan to exploit fully smaller lattice spacings as well as another lattice regularisation.

1 Introduction

Understanding the composition of the final state in $B$ meson semileptonic decay into charm meson is of key importance to control the theoretical error on the CKM matrix element $V_{cb}$. The discrepancy between the inclusive determination and the exclusive one, based on $B \rightarrow D^{(*)}l\nu$, is still of the order of 3σ [1]. A significant part of the total width $\Gamma(B \rightarrow X_c l\nu)$ comes from excited states: it was recently argued that the radial excitation $D^*$ might be particularly favoured, implying a suppression of the $B \rightarrow D^*$ form factors as suggested by a study performed using the operator product expansion formalism [2]. Another group of states that contribute to the width, about one quarter of it, is orbital excitations, in other words, positive parity charmed mesons, that we will note $D^{**}$ hereafter. They are not well understood: indeed there seems to be a persistent discrepancy between claims from theory and from experiment [3], while a comparison between semileptonic decay and non-leptonic decay $B \rightarrow D^{**}\pi$, involving the same form factors (at least in the case of the so-called Class I process), is quite confusing on the experimental side [4]. Two types of $D^{**}$ are observed: two “narrow resonances” $D_{3/2}$ and a couple of “broad resonances” $D_{1/2}$, in the same mass region [5]. While experiments point towards a dominance of the broad resonances in semileptonic decays, theory points rather towards the dominance of the narrow resonances: not only a series of sum rules [6, 7] derived from QCD obtains that hierarchy, but also calculations with quark models [8–10] and lattice computations performed in the quenched approximation [11] and with $N_f = 2$ dynamical quarks [12]. However, the main limitation of these results is that they are derived in the heavy quark limit. $1/m_c$ corrections might be pretty large and, before getting any definitive conclusion on the disagreement between theory and experiment in that sector of flavour physics, it is mandatory to reduce the sources of systematic errors on the theory side.

2 Theoretical framework

In this paragraph, all the main formulae up to the differential decay rates will be given for the semileptonic decays of a $B$ heavy meson into the first orbitally excited $D^{**}$ mesons.

We will focus our study on the production of the $|\bar{3}\,P_0\rangle$ (scalar $D_0^{(*)}$) and the $|\bar{3}\,P_2\rangle$ (tensor $D_2^{(*)}$) states.\footnote{We use the $|\bar{2S+1}\,L_J\rangle$ notation of the states, where $S$ is the spin angular momentum, $L = 1$ the orbital angular momentum and $J = L + S$ the total angular momentum of the $D^{**}$ state.}

Finally, we will also give relations in the case where the mass of the lepton cannot be neglected.
2.1 Form factors

In order to derive the decay rates, we need the transition amplitudes. They can be described using six form factors [15]:

\[ P_3 \text{ state} \]
\[ \left\{ P_3 \left( p_{D^*}, \epsilon \left( p_{D^*}, \lambda \right) \right) \mid V_\mu \mid B(p_B) \right\} = i \hat{h}_{\mu\nu\lambda\rho} \epsilon^{(p_{D^*}, \lambda)} p_B \alpha \left( p_B + p_{D^*} \right) \lambda \left( p_B - p_{D^*} \right)^\rho, \]
\[ P_3 \left( p_{D^*}, \epsilon \left( p_{D^*}, \lambda \right) \right) \mid A_\mu \mid B(p_B) \right\} = \hat{K}_{\mu\nu\lambda\rho} \epsilon^{(p_{D^*}, \lambda)} p_B \alpha + \epsilon^{(p_{D^*}, \lambda)} m^{p_B} p_B \rho \mu \]
\[ \times \left[ \hat{b}_+ \left( p_B + p_{D^*} \right) \mu + \hat{b}_- \left( p_B - p_{D^*} \right) \mu \right], \]

\[ P_0 \text{ state} \]
\[ \left\{ P_0 \left( p_{D^*} \right) \mid V_\mu \mid B(p_B) \right\} = 0 \quad \text{(parity invariance)}, \]
\[ P_0 \left( p_{D^*} \right) \mid A_\mu \mid B(p_B) \right\} = \hat{u}_+ \left( p_B + p_{D^*} \right) \mu \]
\[ + \hat{u}_- \left( p_B - p_{D^*} \right) \mu \]  \hspace{1cm} (2.2)

where \( V_\mu \) denotes the vector current \( \bar{e}_\gamma_\mu b \) and \( A_\mu \) the axial current \( \bar{e}_\gamma_\mu y_b \).

\( \epsilon^{(p_{D^*}, \lambda)} \) is the polarisation tensor of the \( P_3 \) state (\( \lambda \) being the projection of the \( J = 2 \) total angular momentum along some quantification axis).

Moreover, the chosen normalisation of the mesonic states is

\( \langle M(p') | M(p) \rangle = (2\pi)^3 \delta^3 (p' - p) \).

Finally, because of parity and time-reversal invariance of the strong interactions, those form factors are real numbers.

2.2 Differential decay rates

The goal is to compute the differential decay width \( d\Gamma(\bar{B} \to D^{**} \ell \bar{\nu}) \) whose general expression is

\[ d\Gamma(\bar{B} \to D^{**} \ell \bar{\nu}) = \frac{1}{2} \frac{d^3 p_{D^{**}}}{(2\pi)^3 2E_{D^{**}}} \frac{d^3 p_B}{(2\pi)^3 2E_B} \frac{d^3 p_\nu}{(2\pi)^3 2E_\nu} (2\pi)^4 \delta^4 (p_B - p_{D^{**}} - p_\nu - p_\ell). \]

In the last equality, \( W_{\mu\nu} \) denotes the hadronic tensor

\[ W_{\mu\nu}(p_B, p_{D^{**}}) = \frac{G_F^2 |V_{cb}|^2}{2} \times \sum_{\text{final spins}} \left( D^{**}(p_{D^{**}}) \mid V_\mu - A_\mu \mid \bar{B}(p_B) \right) \times \left( \bar{B}(p_B) \mid V_\nu - A_\nu \mid D^{**}(p_{D^{**}}) \right), \]

where the transition amplitudes have been given in the preceding paragraph (let us note that there are no summation nor average over the initial spins since the \( \bar{B} \) meson has a spin equal to 0) and \( \ell^{\mu\nu} \) represents the leptonic tensor

\[ \ell^{\mu\nu}(p_\ell, p_\nu) = \sum_s [\bar{u}_\ell(p_\ell, s) \gamma^\mu (1 - \gamma^5) v_\nu(p_\nu) + \bar{u}_\ell(p_\ell, s) \gamma^\nu (1 - \gamma^5) v_\nu(p_\nu)]. \]

In that last formula, \( u_\ell(p_\ell, s) \) is the lepton \( \ell \) spinor (\( s \) denotes the usual projection of its spin), while \( v_\nu(p_\nu) \) represents the antineutrino \( \bar{\nu} \) spinor.

All that remains is to compute the leptonic tensor, and then the hadronic tensor and the measure \( d\Phi \) of the phase space in order to obtain the expressions of the differential decay widths.

2.2.1 Leptonic tensor \( \ell^{\mu\nu} \)

The calculation is classical and straightforward, leading to

\[
\ell^{\mu\nu} = 8 \left[p_\ell^\mu p_\nu^\nu + p_\nu^\mu p_\ell^\nu - (p_\ell \cdot p_\nu) g^{\mu\nu} \right] - i \epsilon^{\mu\nu\sigma\rho} (p_\ell)_\rho (p_\nu)_\sigma.
\]

We notice that the mass of the lepton has vanished, which renders the expression valid in the situations where \( m_\ell = 0 \) as well as \( m_\ell \neq 0 \).

2.2.2 Hadronic tensor \( W_{\mu\nu} \)

By looking at the expressions of the transition amplitudes given above, the general structure of the hadronic tensor can be inferred and put into the form [15]
\[
W_{\mu\nu} = \frac{G_F^2 |V_{cb}|^2}{2} \times \left[ \omega g_{\mu\nu} + \beta_{++}(p_B + p_{D^{**}}) \cdot (p_B + p_{D^{**}}) \right. \\
+ \beta_{+-}(p_B + p_{D^{**}}) \cdot (p_B - p_{D^{**}}) \\
+ \beta_{-+}(p_B - p_{D^{**}}) \cdot (p_B + p_{D^{**}}) \\
+ \beta_{--}(p_B - p_{D^{**}}) \cdot (p_B - p_{D^{**}}) \\
+ i \left[ \gamma \epsilon_{\mu\nu\rho\sigma} (p_B + p_{D^{**}})^\rho (p_B - p_{D^{**}})^\sigma \right]. \tag{2.3}
\]

The coefficients \(\alpha, \beta_{++}, \beta_{+-}, \beta_{-+}, \beta_{--}\) and \(\gamma\) are given in the "Appendix" for the \(3P_0\) and the \(3P_2\) states.

2.2.3 Kinematics and notations

For reasons of simplicity, we now choose to compute the decay rates in the rest frame of the \(\bar{B}\) meson.

We then define two dimensionless parameters \(x\) and \(y\) according to

\[
x m_B = 2 E_\ell \quad \text{as well as} \quad y m_B^2 = (p_B - p_{D^{**}})^2 = (p_\ell + p)^2
\]

where \(E_\ell\) is the energy of the lepton in the \(\bar{B}\) rest frame.

We introduce also the mass ratio \(r_x\)

\[
m_{x} = r_X m_B \quad \text{where} \quad X \text{ is either a } D^{**} \text{ meson or the lepton } \ell.
\]

Many kinematical terms can be expressed with these three parameters, such as

\[
E_{D^{**}} = \frac{m_B}{2} (1 - y + r_{D^{**}}); \quad p_\ell \cdot p_v = \frac{1}{2} m_B^2 (y - r_\ell^2); \\
p_\ell \cdot p_{D^{**}} = \frac{1}{2} m_B^2 (x - y - r_{D^{**}}^2); \quad p_B \cdot p_\ell = \frac{1}{2} m_B^2 x.
\]

2.2.4 Measure \(d\Phi\) of the phase space

The goal is to get the differential widths \(d\Gamma\) with respect to the lepton energy \(E_\ell\) and the momentum transfer \((p_B - p_{D^{**}})^2\), in other words with respect to the variables \(x\) and \(y\): \(d^2\Gamma/dx\,dy\).

So we must integrate over the antineutrino momentum \(\vec{p}_\ell\), then over all possible orientations of \(\vec{p}_\ell\), so that only the dependence on \(E_\ell\) (i.e. on \(x\)) remains, and finally over all possible directions of the three-vector \(\vec{p}_{D^{**}}\), since we want to keep the dependence on \(E_{D^{**}}\) (i.e. on \(y\)). We finally get:

\[
d\Phi = - \frac{m_B^2}{128\pi^3} dx\,dy\,\theta(1 - x + y - r_{D^{**}}^2)
\]

where \(\theta(z)\) is the usual Heaviside function.

2.2.5 Constraints on \(x\) and \(y\)

The parameters \(x\) and \(y\), that is, the lepton energy \((E_\ell)\) and the \(D^{**}\) meson energy \((E_{D^{**}})\), cannot be arbitrary. They are constrained by two conditions: one which is obvious in the expression of \(d\Phi\) above (the Heaviside function) and another one which appeared during the integration over the direction of \(\vec{p}_\ell\). In other words, we have access to the variation domains of both parameters \(x\) and \(y\) whether we consider \(x = x(y)\) or \(y = y(x)\): they are given in the "Appendix".

2.2.6 Differential decay widths in the \(\bar{B}\) rest frame

Using the definition of \(d\Gamma\) as well as all the preceding results, the construction of the differential decay widths proceeds in the following way:

\[
\frac{d\Gamma}{dx\,dy}(\bar{B} \rightarrow D^{**} \ell \bar{\nu}) = - \frac{\alpha}{256\pi^3} |\vec{A}|^2
\]

where \(|\vec{A}|^2 = W_{\mu\nu}\epsilon^{\mu\nu}\) becomes in this particular frame

\[
|\vec{A}|^2 = 2 G_F^2 |V_{cb}|^2 m_B^2 \left[ - 2 \alpha (y - r_\ell^2) \\
- \beta_{++} m_B^2 [4(x - r_{D^{**}}^2) + (1 - x)(y - x)] \\
+ r_\ell^2 [3y - 3(x + r_{D^{**}}^2) + r_{D^{**}}^2] \\
+ \left( \beta_{-+} + \beta_{++} \right) m_B^2 r_\ell^2 (2(1 - x - r_{D^{**}}^2) + y + r_\ell^2) \\
+ \beta_{--} m_B^2 r_\ell^2 (y - r_\ell^2) \\
- 2 \gamma m_B^2 \left[ y (1 + y - 2x - r_{D^{**}}^2) + r_{D^{**}}^2 (1 + y - r_{D^{**}}^2) \right]\right].
\]

We notice that, for a zero mass lepton, only the coefficients \(\alpha, \beta_{++}\) and \(\gamma\) survive.

The expressions for each \(D^{**}\) are also written in the "Appendix". However, their use requires the knowledge of the momentum dependence of the form factors. In the following, we will focus on a method to obtain such a dependence.

2.3 Extracting the form factors from the transition amplitudes

On the lattice, we compute the transition amplitudes for different momenta of the mesons. But we need the momentum dependence of the form factors in order to calculate the decay rates of the semileptonic decays of the \(B\) to a \(D^{**}\). So we must devise a way to extract the form factors from the lattice transition amplitudes.

2.3.1 Kinematics

We will work in the rest frame of the \(D^{**}\) meson\(^2\) so the \(B\) meson will carry the momentum. Moreover, we will consider the \(B\)'s whose spatial momentum is symmetrical.

\[
p_{D^{**}} = (m_{D^{**}}, \vec{0}) \quad \text{and} \quad p_{\mu}^B = (E_B, p, p, p).
\]

\(^2\) This will greatly simplify the calculations on the lattice.
We will also choose the Minkowski metric: $g_{μν} = \text{Diag}(+, -, -, -)$.

The other piece we need is the expression of the polarisation tensor for the $^3P_0$ state in the $D^{**}$ rest frame, that is, $ε(0, λ)$. We can construct it from the combination of two spin-1 states,

$$ε_{μν}(0, λ) = \sum_{s, s'} (1 1 s s' 2 λ) ε^{μν}(0, s) ε^{ν}(0, s'),$$

where the Clebsch–Gordan coefficients for $1 1 s s' 2 λ$ as well as the polarisation vector $ε^{μν}(0, s)$ of a spin-1 state. The final expressions are gathered in the “Appendix”.

2.3.2 $^3P_0$ form factors

Using the notation

$$\mathcal{J}_μ^{A} \overset{\text{def.}}{=} \{^3P_0 A_μ B(p_B)\},$$

we explicitly get from Eq. (2.2):

$$\mathcal{J}_0^{A} = \vec{u}_+ (E_B + m_{D^{**}}) + \vec{u}_- (E_B - m_{D^{**}})$$

(1 temporal direction),

$$\mathcal{J}_i^{A} = \vec{u}_+ u + \vec{u}_- p \quad \text{(spatial direction).}$$

So it is straightforward to express $\vec{u}_+$ and $\vec{u}_-$ with the $\mathcal{J}_i^{A}$s. The results are presented in the “Appendix”.

2.3.3 $^3P_2$ form factors

In the following, we will adopt the notation

$$\mathcal{J}_μ^{A(λ)} \overset{\text{def.}}{=} \{^3P_2(λ) A_μ B(p_B)\} \quad \text{as well as}$$

$$\mathcal{J}_μ^{V(λ)} \overset{\text{def.}}{=} \{^3P_2(λ) V_μ B(p_B)\}.$$ 

In order to extract one particular form factor, we can choose in Eq. (2.1) either some spatial direction where each coefficient of the other form factors vanishes, or we can construct a linear combination of the $\mathcal{J}_i^{A(λ)}$ and/or the $\mathcal{J}_μ^{V(λ)}$.

This procedure can be carried out by using the expressions for the polarisation tensor and the four-momenta at our disposal and calculating the contribution of the corresponding terms appearing in the matrix elements (2.1) which define the form factors (those contributions are gathered in Table 1).

A few possibilities are collected in the “Appendix”.

2.4 Summary

We have constructed all the theoretical formulae which allow us to calculate the decay widths of the semileptonic $B \to D^{**}$ channels. The strategy to use them is the following:

1. Compute, on the lattice, the transition amplitudes for the $B \to D^{**}$ processes.
2. Extract the form factors from them.
3. Use the formulae in the “Appendix” to obtain the decay widths.

| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |
|----------------|-------------------------|
| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |
| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |

| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |
|----------------|-------------------------|
| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |
| $ε^{μν}_{(λ)}$ | $ε^{μν}_{(λ)} B(p_B)$ |
Since we expect the lattice $^3P_2$ computation to be somewhat tricky, we are first going to estimate the contribution of the $\hat{k}$, $\hat{b}_+,$ $\hat{b}_-$ and $\hat{h}$ form factors to the $B \rightarrow D_s^* \ell \bar{\nu}$ decay width.

2.5 Estimation of the contribution of the form factors to the $^3P_2$ decay width

There are four form factors needed to describe the transition amplitudes from a $B$ to a $^3P_2$ state which increases the difficulty in the lattice computations. So it could be useful to have an idea of each of their contribution to the decay widths.

In order to get a quantitative hint, we will relate these form factors to their infinite mass limit. We notice that the biggest contributions come from the terms where the $\hat{k}$ form factor appears; that is why we will focus on its determination in the actual lattice computation.

2.5.1 Infinite mass limit

In the limit where the heavy quark of the meson has an infinite mass, new symmetries (and thus additional conserved quantities) appear. These new symmetries provide additional relations between the transition amplitudes so that the form factors become dependent. It can be proven [13] that this reduction of the form factors leads to the following relations for the $^3P_2$ state:

$$
\begin{align*}
\hat{h} &= \frac{\sqrt{3}}{2} \frac{1}{m^2} \sqrt{\rho_{12}} \tau_{1/2} \\
\hat{k} &= \sqrt{3} \sqrt{\rho_{12}} (1 + w) \tau_{1/2} \\
\hat{b}_+ &= -\frac{\sqrt{3}}{2} \frac{1}{m^2} \sqrt{\rho_{12}} \tau_{1/2} \\
\hat{b}_- &= \frac{\sqrt{3}}{2} \frac{1}{m^2} \sqrt{\rho_{12}} \tau_{1/2}
\end{align*}
$$

where the parameter $w$ is defined by

$$m_B m_{D^*} w = p_{D^*}^2 \rightarrow y = 1 + r_{D^{**}}^2 - 2 r_{D^{**}} w$$

and $\tau_{1/2}$ is one of the so-called Isgur–Wise functions.

2.5.2 Fit of $\tau_{1/2}$

Using a covariant construction of the transition amplitudes in the infinite mass limit (quark models à la Bakamjian-Thomas), it has been shown [8,14] that the Isgur–Wise function $\tau_{1/2}$ can be well fitted by

$$\tau_{1/2}(w) = \tau_{1/2}(1) \left( \frac{2}{1 + w} \right)^{2 \sigma_{1/2}^2}$$

$$\rightarrow \tau_{1/2}(y) = \tau_{1/2}(1) \left[ \frac{4 \rho_{12}^2}{(1 + r_{12})^2 - y} \right]^{2 \sigma_{1/2}^2}$$

where the accessible phase space domain is given by

$$1 \leq w \leq \frac{m_B^2 + m_{D^{**}}^2}{2 m_B m_{D^{**}}} \rightarrow (1 - r_{12})^2 \geq y \geq 0.$$

We will also take (GI model [15] in [8]):

$$\tau_{1/2}(1) \approx 0.54 \text{ as well as } \sigma_{1/2}^2 \approx 1.50.$$

2.5.3 Quantitative prediction of each contribution to the total width

We are now in a position to estimate the contribution of each form factor to the total width of the $B \rightarrow D_s^* \ell \bar{\nu}$ decay channel. Let us take the case of a zero mass lepton to simplify the calculations. Starting from the expression of $\frac{d^2 \Gamma}{dx dy}$ and with the notations given in the “Appendix”, we can perform both the integrations over $x$ and over $y$ and we get:

| $C_i$ | $C_1 \times \hat{k}^2$ | $C_2 \times \hat{h}^2$ | $C_3 \times \hat{b}^2_+$ | $C_3 \times 2 \hat{k} \hat{b}_+$ | $C_4$ |
|-------|-------------------|-------------------|---------------------|--------------------|------|
| $\int \int C_i \times FF^2$ | $-61.3$ | $-0.86$ | $-4.43$ | $29.0$ | $0$ |

We notice that the biggest contributions come from the terms where the $\hat{k}$ form factor appears; that is why we will focus on its determination in the actual lattice computation.

3 Simulation set up

In our analysis we use gauge ensembles produced by European Twisted Mass Collaboration [16–18] with $N_f = 2$ twisted-mass fermions tuned at maximal twist. Parameters of the simulations are collected in Table 2.

The gauge action is tree-level Symanzik improved [21] and reads:

$$S_G[U] = \frac{\beta}{6} \left( b_0 \sum_{x, \mu \neq 5} \text{Tr}(1 - P^1 \chi(x; \mu, \nu)) + b_1 \sum_{x, \mu \neq 5} \text{Tr}(1 - P^2 \chi(x; \mu, \nu)) \right),$$

where $b_0 = 1 - 8 b_1$ and $b_1 = -1/12$. The fermionic action with two degenerate flavours is Wilson-like with a twisted-mass term and reads [22–24]:

$$S_F[\chi_q, \tilde{\chi}_q, U] = a^4 \sum_x \tilde{\chi}_q(x)(D_W + i \mu_q \gamma_5 \tau_3) \chi(x)_q,$$

where $D_W$ is the massless Wilson–Dirac operator. In the valence sector we add two doublets of charm quarks and “bottom” quarks. Moreover, as we are interested in computing form factors at different momenta we implement $\theta$-boundary conditions [25], using $\tilde{\theta} \equiv (\theta, \theta, \theta)$, for the $b$ doublet:

$$\chi_b(x + L \tilde{\theta}) = e^{i \theta L} \chi_b(x)$$

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This is equivalent to defining the auxiliary field
\[ \tilde{\chi}_b(x) = e^{-i\bar{\theta}\cdot \tilde{x}} \chi_b(x) \]
and the Dirac operator
\[ D(\chi_b, \tilde{\chi}_b, U) \equiv D \left( \tilde{\chi}_b, \tilde{\chi}_b, U \right) \]
with \( U_i^\alpha(x) = e^{i\alpha}U_i(x) \).

The whole fermionic action finally reads
\[ S_{val} = S_F[\chi_d, \tilde{\chi}_d, U] + S_F[\chi_c, \tilde{\chi}_c, U] + S_F[\tilde{\chi}_b, \tilde{\chi}_b, U^\dagger]. \]

We use all to all propagators with stochastic sources \( \eta[i] \) diluted in time [26] and improve the variance-to-signal ratio with the one-end trick [27,28]. When it is generalised to \( \theta \)-boundary conditions, it consists in solving the Dirac equations,
\[ \sum_y D[f, r, \phi]_{ab}^{\alpha\beta}(x, y) \phi[i, f, r, \phi, \alpha, \bar{\alpha}, \bar{\alpha}, \bar{\phi}]^{\beta}(y) \]
\[ = \eta[i]^{\alpha}(x) \delta_{a\bar{a}} \delta_{i} \delta_{\tilde{i}}, \]
where \( \tau^3 \chi = r \chi, f \) represents the fermion flavour and
\[ \eta[i]^{\alpha}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \xi[i, \alpha, \bar{\alpha}, \bar{\alpha}, \bar{\phi}]^{\alpha}(x) \xi^*(i, \alpha, \bar{\alpha}, \bar{\phi})^{\beta}(y) \]
\[ = \delta_{a\bar{a}} \eta[i]^{\alpha}(x) \delta_{\tilde{i} \tilde{i}}. \]

We have added a preliminary use of data for \( (\beta = 4.2, a\mu_{sea} = 0.0065) \) and \( m_\pi = 495 \text{ MeV} \). This will be used only for the decay into a scalar charmed meson. The reason is that, with only \( \beta = 3.9 \) and \( \beta = 4.05 \), the extrapolation of the decay amplitude \( B \to D_{l}^{0}l \nu \) to the physical situation produces a result grieved by more than 100% error. The data for \( \beta = 4.2 \), being closer to the continuum, allows for a significant result, as will be seen.

| \( \beta \) | \( L^3 \times T \) | \( \alpha(\text{Im}) \) | # cnfgs | \( \mu_{sea} = \mu_\ell \) | \( \mu_c \) | \( \mu_b \) | \( \theta (\pi/L) \) |
|---|---|---|---|---|---|---|---|
| 3.9 | \( 24^3 \times 48 \) | 0.085(3) | 240 | 0.0085 | 0.215 | 0.3498 | 0.0, 0.99, 1.41 |
| 4.05 | \( 32^3 \times 64 \) | 0.069(2) | 160 | 0.006 | 0.1849 | 0.3008 | 0.0, 1.09, 1.56 |
| 4.2 | \( 32^3 \times 64 \) | 0.054(2) | 300 | 0.0065 | 0.1566 | 0.2548 | 0.0, 0.3525 |

\( \chi_1 S \times \Gamma \chi_2 \), where \( S \) is a path of
We also compute the “neutral” $B$ and $D$ two-point correlators $C_{\bar{b}; S_1 \Gamma_1; S_2 \Gamma_2}^{(2) b l}(t)$, which read [32]

$$
C_{\bar{b}; S_1 \Gamma_1; S_2 \Gamma_2}^{(2) b l}(t) = \frac{1}{2} \sum_{r=\pm 1} \sum_{n=1}^N \left( \text{Tr} \left( \sum_{\tilde{S}_1} \hat{S}_2 \phi^*[n, b, -r, l, S_2, \tilde{\theta}, \tilde{0}, \tilde{\alpha}, \tilde{i}, \tilde{t} + S_3]_\beta \right) \right) \times (\gamma_5 \Gamma_1)_{a \beta} \frac{1}{\Delta_1/2} \left( \sum_{\tilde{S}_1} \hat{S}_2 \phi^*[n, b, -r, l, S_2, \tilde{\theta}, \tilde{0}, \tilde{\alpha}, \tilde{i}, \tilde{t} + S_3]_\beta \right)
$$

where $\langle \ldots \rangle$ stands for the gauge ensemble average and $h \equiv c$ or $b$.

We recall that, in twisted-mass QCD, quark propagators have the hermiticity property:

$$
\mathcal{S}_q(r; \tau; y) = \gamma_y \mathcal{S}_q^\dagger(-r; \gamma; x) \gamma_y.
$$

We also compute the “neutral” $B \rightarrow D$ three-point correlators $C_{\bar{b}; S_1 \Gamma_1; S_2 \Gamma_2}^{(3) b l r}(t, t_s)$, which read

$$
C_{\bar{b}; S_1 \Gamma_1; S_2 \Gamma_2}^{(3) b l r}(t, t_s) = \frac{1}{2} \sum_{r=\pm 1} \left( \text{Tr} \left( \sum_{\tilde{S}_1} \hat{S}_2 \phi^*[n, b, -r, l, S_2, \tilde{\theta}, \tilde{0}, \tilde{\alpha}, \tilde{i}, \tilde{t} + S_3]_\beta \right) \right) \times (\gamma_5 \Gamma_1)_{a \beta} \frac{1}{\Delta_1/2} \left( \sum_{\tilde{S}_1} \hat{S}_2 \phi^*[n, b, -r, l, S_2, \tilde{\theta}, \tilde{0}, \tilde{\alpha}, \tilde{i}, \tilde{t} + S_3]_\beta \right)
$$

in a time range where the contribution from the first excitation is small compared to the statistical error. The stability of the fit is checked by enlarging the time interval and adding a second exponential in the formula, i.e.,

$$
C_{PP}(t, \tilde{\theta}) = \sum_{i=1}^2 \frac{Z^2(i)}{2E_P^i(\tilde{\theta})} \left( e^{-E_P^i(\tilde{\theta}) t} + e^{-E_P^i(\tilde{\theta}) (T-t)} \right),
$$

in which

$$
Z^2(i) = \langle H^i \rangle O_P^{H^i} 0.
$$

4 Masses and energies

We decide to concentrate our effort on the analysis of smeared–smeared two-point correlators because the benefit of such a technique has been already clearly observed in a previous work by ETMC [31]. Masses and energies of pseudoscalar $B$ and $D$ mesons are first extracted from a fit of the form

$$
C_{PP}(t, \tilde{\theta}) = \sum_{i=1}^2 \frac{Z^2(i)}{2E_P^i(\tilde{\theta})} \left( e^{-E_P^i(\tilde{\theta}) t} + e^{-E_P^i(\tilde{\theta}) (T-t)} \right),
$$

in which

$$
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$$
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$$

in which

$$
Z^2(i) = \langle H^i \rangle O_P^{H^i} 0.
$$
The last step in the analysis is to measure the effective energy $E_P \equiv E_P^{(1)}$ of the ground state from the ratio

$$\frac{C_{PP}(t+1, \tilde{\theta})}{C_{PP}(t, \theta)} = \cosh(E_P(\tilde{\theta})) + \sinh(E_P(\tilde{\theta})) \tan\left[E_P(\tilde{\theta})(t-T/2)\right].$$

We show in Fig. 2 examples of plateaus for "$B$"-mesons energies at three different momenta.

We study the dispersion relation to get an idea of the magnitude of cut-off effects. We display in Fig. 3 the $B$ meson energies and compare them to the theoretical formula

$$\sinh^2[a E(\theta)/2] = \sinh^2[a M/2] + 3 \sin^2(\theta/2) \quad \text{where} \quad M \equiv E(0). \quad (4.1)$$

The agreement is good at the two lightest heavy masses but really bad at the heaviest one: cut-off effects are pretty large.

Interpolating fields of the $2^+$ state are given by the formula

$$O^{(\lambda)} = \bar{\psi} \gamma_{\mu} \gamma_{\nu} \gamma_{\lambda} \chi, \quad \lambda = \pm 2, \pm 1, 0.$$

Actually we choose to use linear combinations of the interpolating fields that read

$$\hat{O}^{(1)} = \frac{1}{\sqrt{2}} (O^{(2)} + O^{(-2)}) = \frac{1}{\sqrt{2}} \bar{\chi} (\gamma_1 \nabla_1 - \gamma_2 \nabla_2) \chi_l,$$

$$\hat{O}^{(2)} = O^{(0)} = -\frac{i}{\sqrt{2}} \bar{\chi} (\gamma_1 \nabla_1 + \gamma_2 \nabla_2 - 2 \gamma_3 \nabla_3) \chi_l,$$

$$\hat{O}^{(3)} = \frac{1}{\sqrt{2}} (O^{(1)} - O^{(-1)}) = -\frac{1}{\sqrt{2}} \bar{\chi} (\gamma_2 \nabla_1 + \gamma_1 \nabla_2) \chi_l,$$

$$\hat{O}^{(5)} = \frac{1}{\sqrt{2}} (O^{(1)} + O^{(-1)}) = -\frac{1}{\sqrt{2}} \bar{\chi} (\gamma_1 \nabla_1 + \gamma_3 \nabla_3) \chi_l.$$

The two first interpolating fields live in the $E$ representation of the $O_b$ cubic group symmetry of rotations and inversion in a 3D spatial lattice, while the last three live in the $T_2$ representation of that group [33]. We finally consider the $r$-symmetrized smeared–smeared two-point correlators

$$C_{2^+ \cdot E}^{(2)}(t) = \frac{1}{2} \left[ \sum_{i, \tilde{i}} \hat{O}_S^{(1)}(\tilde{y}, t + \tilde{\tau}) \hat{O}_S^{(1)}(\tilde{x}, \tilde{\tau}) \right]$$

$$+ \left[ \sum_{i, \tilde{i}} \hat{O}_S^{(2)}(\tilde{y}, t + \tilde{\tau}) \hat{O}_S^{(2)}(\tilde{x}, \tilde{\tau}) \right], \quad (4.2)$$

and

$$C_{2^+ \cdot T_2}^{(2)}(t) = \frac{1}{3} \left[ \sum_{i, \tilde{i}} \hat{O}_S^{(3)}(\tilde{y}, t + \tilde{\tau}) \hat{O}_S^{(3)}(\tilde{x}, \tilde{\tau}) \right]$$

$$+ \left[ \sum_{i, \tilde{i}} \hat{O}_S^{(4)}(\tilde{y}, t + \tilde{\tau}) \hat{O}_S^{(4)}(\tilde{x}, \tilde{\tau}) \right]$$

$$+ \left[ \sum_{i, \tilde{i}} \hat{O}_S^{(5)}(\tilde{y}, t + \tilde{\tau}) \hat{O}_S^{(5)}(\tilde{x}, \tilde{\tau}) \right]. \quad (4.3)$$
The masses we extract by studying the ratios \( \frac{C_{ij}^{(2)}(t)}{C_{ij}^{(2)}(t+1)} \) and 
\( \frac{C_{ij}^{(2)}(t)}{C_{ij}^{(2)}(t+1)} \) are in principle equal: any discrepancy comes from cut-off effects. We show in Fig. 4 that, indeed, lattice artefacts are present.

As parity is broken by the twisted-mass action at finite lattice spacing [23] and the states we consider are not made with quarks of the same flavour doublet, contrary to what is discussed in Sect. 5.2 of [24], the scalar \( D \) meson can in principle mix with the pseudoscalar \( D \) meson. We have to build a matrix of correlators \( \{C_{ij}(t)\} \) and solve a generalised eigenvalue problem (GEVP) [34–36]. We study a \( 2 \times 2 \) system whose entries correspond to the interpolating fields with Dirac structures \( \bar{\chi} \gamma^5 \chi \) and \( \bar{\chi} \gamma \chi \):

\[
C_{ij}(t) = \begin{bmatrix}
C_{ij}^{(2)}(\bar{\chi} \gamma^5 \chi; \bar{\chi} \gamma \chi)(t) & C_{ij}^{(2)}(\bar{\chi} \gamma^5 \chi; \bar{\chi} \gamma \chi)(t) \\
C_{ij}^{(2)}(\bar{\chi} \gamma \chi; \bar{\chi} \gamma \chi)(t) & C_{ij}^{(2)}(\bar{\chi} \gamma \chi; \bar{\chi} \gamma \chi)(t)
\end{bmatrix}.
\]

We solve the system

\[
C_{ij}(t) u_{ij}^{(n)}(t, t_0) = \lambda^{(n)}(t, t_0) C_{ij}(t_0) u_{ij}^{(n)}(t, t_0).
\]

We set \( t_0 = 3 \) (\( \beta = 3.9 \)) and 5 (\( \beta = 4.05 \)). \( \lambda^{(n)}(t, t_0) \) and \( u_{ij}^{(n)}(t, t_0) \) are the eigenvalues and eigenvectors of the matrix \( C^{-1}(t_0) C(t) \). The effective mass \( m_{D^*_0} \) of the scalar meson is given by

\[
\lambda^{(2)}(t, t_0) = \frac{\cosh[m_{D^*_0}(T/2 - t)]}{\cosh[m_{D^*_0}(T/2 - t_0)]}
\]

We show in Fig. 5 \( m_{D^*_0} \) for the ensemble (\( \beta = 3.9, \mu_{\text{sea}} = \mu_i = 0.0085 \)). The signal is unfortunately quite short, but still acceptable for our qualitative study.

We collect in the “Appendix” all the masses and energies that we extracted in our analysis. The total error includes the statistical one and the discrepancy of results when we change the time range \([t_{\min}, t_{\max}]\) of the fits by \( t_{\min} \pm 1 \) and

\[
t_{\max} \pm 1, \text{ when we take different } t_0 \text{ in the range } [3, 6] \text{ and, in the case of pseudoscalar } B \text{ mesons, when we perform a two-state exponential fit.}
\]

4.1 \( M_{D^{*}_0} - M_D \) and \( M_{D^{*}_0} - M_D \) in the continuum limit and experiment

In Table 7 in the “Appendix” are given the masses of the \( D^{*}_0, D_0^* \) and \( D \) mesons. It is interesting to perform an extrapolation to the continuum and compare with the experimental data. For the latter we will take the \( c\bar{s} \) mesons. It does not change anything for the tensor meson as compared to the non-strange charmed mesons (\( M_{D^{*}_0}(2460) - M_D \simeq M_{D^{*}_0}(2573) - M_D \)) but for the scalar mesons it does, since the \( M_{D^{*}_0}(2317) \) has a narrow and clear signal which is not the case of the \( D^{*}_0(2400) \) whose signal is very broad due to its S-wave decay into \( D \pi \).

We perform the extrapolation to the continuum limit and compute the error using jackknife. We show our results in
D-wave decay. The possible importance of a non-vanishing of the decay amplitude does vanish at zero recoil whether the mass is taken infinite or finite, thus implying only D-wave decay. The possible importance of a non-vanishing zero recoil amplitude was stressed in [37], where the authors estimated subleading corrections to the infinite mass limit: although subleading in the $\Lambda/m_c$ expansion, the S-wave may not be negligible. Our computation confirms this conclusion as shall be seen.

5.1 Computation of the amplitude ratio ($B \to D_0^*$) over ($B \to D$) at zero recoil

We will compute the ratio of the amplitudes $A(B \to D_0^*)/A(B \to D)$. We take $A(B \to D)$ as a benchmark, since it is experimentally fairly well known, and $D$ being a $J=0$ state as $D_0^*$, it is expected that the momentum dependence of these decays will be rather similar. We recall some formulae neglecting for the moment the $D-D_0^*$ mixing due to the parity violation at finite lattice spacing when using twisted-mass quarks. The matrix elements $\langle B|V_0|D \rangle$ and $\langle B|A_0|D_0^* \rangle$ are given by the following ratio:

$$\langle B|O|H_c \rangle = \frac{C_{BOH}(t_p, t, t_s)\sqrt{Z_{H_c}Z_BZ_O}}{C_{H^0}(t - t_s)(-E_B(t_p - t))/(2E_B)},$$

(5.1)

where $t_s < t < t_p$ are, respectively, the source, current and sink times (cf. Fig. 1).

$Z_0 = Z_V(Z_A)$ when $H_c = D(D_0^*)$. $Z_{H_c}$ and $Z_B$ are defined from the fit of the two-point correlators of the $H_c$ and $B$ mesons, assuming we are far enough from the centre of the lattice to be allowed to neglect the backward exponential in time while the contribution of excited states is small:

$$C_{H^0}(t - t_s) = \frac{Z_B}{2E_B(\bar{t})}\exp(-E_B(\bar{t}));$$

$$C_{H^0}(t, 0) = \frac{Z_{H_c}}{2M_{H_c}}\exp(-M_{H_c}t).$$

(5.2)

Then we compute

$$\frac{\langle B|A_0|D_0^* \rangle}{\langle B|V_0|D \rangle} = \frac{C_{BA_0D_0}(t_p, t, t_s)C_{D_0^*}(t - t_s)Z_A\sqrt{Z_{D_0^*}}}{C_{BV_0D}(t_p, t, t_s)C_{D_0^*}(t - t_s)Z_V\sqrt{Z_D}}.$$

(5.3)

5.2 Taking into account the scalar–pseudoscalar mixing

In Sect. 4 we have detailed the generalised eigenvalue method. As explained there and in Sect. 3, we restrict ourselves to a $2 \times 2$ matrix of smeared and stochastic two-point correlators. The largest (smallest) eigenvalue $\lambda^{(1)}$ ($\lambda^{(2)}$) will be related to the mass of the $D$ ($D_0^*$) state. The corresponding eigenvectors give the linear combination of $\bar{c}\gamma_5 b_1$ and $\bar{c}\gamma_5 b_1$ interpolating fields that have the largest coupling to the $D(D_0^*)$ state. The eigenvectors turn out to be not far from orthogonal
as seen for example in Table 4. One might say that 0.07 is not so that thus to a fair approximation orthonormalise the eigenvectors

\[ v^{(1)} v^{(2)*} = \sum_{k=1,2} v^{(1)}_k v^{(2)*}_k = 0.07 i \simeq 0, \]  

(4.5)
as seen in Table 4. One might say that 0.07 is not so small but this is not surprising, since there are other states in which the B might decay than only the ground state scalar and pseudoscalar that we consider in our analysis. We can thus to a fair approximation orthonormalise the eigenvectors so that

\[ u^{(i)} u^{(j)*} \sim \delta_{i,j}, \]  

(5.5)
without changing the eigenvalues, since in Eq. (4.4) the same factor multiplies both sides of the equation. Thus, we define the two-point correlator of the \( D^*_0 \) meson by

\[ \lambda^{(2)}(t-t_s, t_0) = 2 \sum_{i,j=1} v_i^{(2)*} (t-t_s, t_0) C_{ij}^{(2)} (t_0) v_j^{(2)} (t-t_s, t_0) \]

\[ = \sum_{i,j=1} v_i^{(2)*} (t-t_s, t_0) C_{ij}^{(2)} (t-t_s, t_0), \]

(5.6)
where we fit with

\[ \lambda^{(2)}(t-t_s, t_0) = \exp \left( -M_D^0 (t-t_s) \right) \]

(5.7)
since \( \lambda^{(0)}(t_0, t_0) = 1 \) from Eq. (5.6). We define \( Z_2^{(1)} \) and \( Z_2^{(2)} \) from

\[ v_i^{(2)*} (t-t_s, t_0) C_{ij}^{(2)} (t-t_s) v_j^{(2)} (t-t_s, t_0) \]

\[ = \frac{Z_2^{(2)}}{2M_D^0} \exp \left( -M_D^0 (t-t_s) \right), \]

(5.8)
In Eq. (5.1) we see that the factors \( Z_i \) appear only via their square root.

5.2.1 Symmetry properties of the matrix elements

In the continuum it is obvious by parity conservation that

\[ \langle B | A_0 | D \rangle = \langle B | V_0 | D^*_0 \rangle = 0. \] 

(5.9)
However, parity is not conserved by the twisted-mass quark action at finite lattice spacing. But this action has an exact symmetry [23], the flavour-parity

\[ \mathcal{P} \equiv \mathcal{P} \otimes (\mu_1, \mu_3, \mu_2) \sim (-\mu_1, -\mu_3, -\mu_2) \]

(5.10)
where \( \mathcal{P} \) is the spatial parity, and \( \mu_1, \mu_3, \mu_2 \) are the twisted-mass terms for the light, charm and beauty quarks. We assume we are at maximal twist (vanishing of \( m_{PCAC} \)). Therefore if we use

\[ C^{(3)}_{i,j,k} (t_p, t, t_s) \equiv (1 + \mathcal{P} \mathcal{S}) C^{(3)}_{i,j,k} (t_p, t, t_s), \] 

(5.11)
we get the validity of Eq. (5.9) also for finite lattice spacing. This symmetrisation will be assumed in the following.

5.2.2 GEVP on the three-point correlators

In this section we assume \( t_s < t < t_p \) and \( t_0 \leq t - t_s \). Starting from the three-point correlators

\[ C^{(3)}_{B,A_0(V_0)} (t_p, t, t_s) = \langle \bar{\chi}_B Y \chi(t_p) A_0(V_0)(t) O_i(\bar{\chi}_c, \chi, t_s) \rangle, \]

(5.12)
and following the authors [38] as regards their way of extracting the decay constant \( f_B \), we consider the projected three-point correlators

\[ \langle \bar{\chi}_B Y \chi(t_p) A_0(V_0)(t) O_i(\bar{\chi}_c, \chi, t_s) \rangle \]

(5.13)
We recall that the normalisation factor \( Z_2^{(2)} \) cancels between

\[ \langle \bar{\chi}_B Y \chi(t_p) A_0(V_0)(t) \bar{\chi}_c \chi(t_s) v_j^{(1)}(t-t_s, t_0) \rangle \]

(5.14)
and

\[ V_i^{(2)}(t-t_s, t_0) C_{ij}^{(2)} (t-t_s) v_j^{(2)}(t-t_s, t_0) \]

while the factor

\[ \exp \left( -2M_D^0 (t-t_s) \right) \]

(5.15)
compensates the residual time exponential dependence. We do not take into account the contributions \( \propto v_j^{(2)}(t-t_s, t_0) \) to the projected correlators

\[ \propto \exp \left( -2M_D^0 (t-t_s) \right) \]

(5.16)
Table 5 We give the ratios defined in Eq. (5.13) over a plateau corresponding to [6–10] in units of lattice spacing for $\beta = 4.2$. The lattice points at the other $\beta$'s do not match with those at $\beta = 4.2$. To explain our method, let us take a point $x_4$, we take for $\beta = 3.9$ the two points $y_3,\ y_4$ and $y_3,\ y_4 + 1$ such that $y_3,\ y_4/a_4 < x_4 < (y_3,\ y_4 + 1)/a_4$. Then from any function $f$ computed for $3.9$ we define the interpolated function $f_{3.9} = f(y_3,\ y_4 + 1)/a_4 - f(y_3,\ y_4)/a_4$. The b quarks range from the lightest to heaviest from left to right: $m_B \simeq 2.55, 3.18, 3.97$ GeV. The right column corresponds to the extrapolation at the physical $B$ mass: 5.2 GeV. The last line corresponds to the extrapolation to the continuum. The errors are only statistical except on the physical case (bottom-right) where the second error takes into account a systematic error estimated from the dependence of our result on the window choices to compute the masses and on the method in extrapolating to the physical point (in $1/m_B$ or in $1/m_B^2$ (2 GeV)).

| $\beta$ | Ratio $m_{b(1)}$ | Ratio $m_{b(2)}$ | Ratio $m_{b(3)}$ | Ratio at physical $B$ |
|---------|-----------------|-----------------|-----------------|---------------------|
| 3.9     | 0.23 (3)        | 0.17 (2)        | 0.11 (3)        | 0.06 (4)            |
| 4.05    | 0.20 (2)        | 0.14 (2)        | 0.08 (4)        | 0.03 (5)            |
| 4.2     | 0.23 (2)        | 0.19 (3)        | 0.17 (3)        | 0.14 (4)            |
| Continuum | 0.22 (5)   | 0.17 (5)        | 0.17 (7)        | 0.17 (6)            |

The values of the plateaus are reported in Table 5 and show plateaus up to $t = 10$ and we compute the averages to be between 6 and 10, as reported in Table 5. For larger times the signal falls down, presumably because of the parity violation when using twisted masses which induces the scalar $D^0_H$ to “decay” into the pseudoscalar $D$ meson. The error bars are only statistical errors.

6 $B$ decay to the tensor ($J = 2^+$) charmed meson

In this section we want to estimate the amplitudes for $B \rightarrow D^+_s \ell \nu$ decay. In [1] the spin-2 charmed mesons are named $D^+_s(2460)$ ($D^+_s(2573)$). For the initial $B$ meson, we use three “$B_i, \ i = 1, 2, 3$” with increasing masses valued 2.55, 3.18 and 3.97 GeV. As was mentioned before, the “$B_i, \ i = 1, 2, 3$” are moving while the final charmed meson is at rest. We concentrate on the calculation of the form factor $\hat{k}$, since it was shown in Sect. 2.5 that it is, by large, dominant in the decay width.

6.1 Three-point correlators computed for $B \rightarrow D(2^+)$

We start from the formulae recalled in Sect. 2.5. We use a symbolic notation to represent the hadronic matrix elements

$$H_{i,j,k} = \langle B|A_k|D^+_s(\epsilon^{(i)})\rangle \sim A_k V_j D_i.$$ 

The various combinations to extract $\hat{k}$ are collected in Table 6. We consider all of these combinations and average the resulting value for $\hat{k}$. To eliminate artefacts we must also apply the symmetrized result according to Eq. (5.11)
6.2 Subtracting zero momentum three-point correlators

The three-point correlation functions involving a tensor $D$ meson are unfortunately very noisy: hence it is extremely difficult to get a large enough signal-to-noise ratio. We will use a trick\(^3\) which consists in subtracting to every three-point correlator the correlator with the same gauge configuration and the same operators at zero momentum. Indeed we know that the decay $B \rightarrow D_s^* \rightarrow D_s \pi$ vanishes at zero recoil. This is obvious in the continuum limit, since we start with a $B$ meson of vanishing angular momentum $J$. The weak interaction operator (axial current $A_\mu$), having $J = 0$ for $A_0$ and $J = 1$ for $A_i$ ($i = 1, 2, 3$), cannot generate a $J = 2$ state: at zero recoil there is no momentum to generate a higher angular momentum.

However, this vanishing is also exact on a lattice. The proof goes as follows: the three-point correlators which contribute to the $D_s^* \rightarrow B$ are linear combinations of correlators of the type

$$C_{i,j,k}^{(3)}(t_p,t,t_s) = \langle O_B(t_p) A_k(t) O_{D_s} V_j(t_s) \rangle,$$

where $i, j, k \in \{1, 2, 3\}$ may be different or equal. All operators are at rest (zero recoil). We have assumed the $D_s^*$ meson ($B$ meson) interpolating field to be at the source time $t_s$ (sink time $t_p$), and the current at time $t$ with $t_p \geq t \geq t_s$.

Let us choose one of the three spatial directions $\hat{l}$ and consider the rotation $\mathcal{R}_l(\pi)$ of angle $\pi$ around it: the spatial coordinates perpendicular to $\hat{l}$ change sign. All vector operators, $D_i$, $V_j$ and $A_i$, change sign if $i$ is perpendicular to $\hat{l}$, whereas they remain unchanged if $i = l$.

$\mathcal{R}_l(\pi)$ belongs to the 3D cubic symmetry group. The lattice actions are invariant under $\mathcal{R}_l(\pi)$, even the twisted-mass action since $\mathcal{R}_l(\pi)$ is parity even. In Eq. (6.1), there are three operators at three different times. Being at rest, we may assume that their spatial nesting is invariant under $\mathcal{R}_l(\pi)$: it can be a stochastic source, a local operator at the origin of three-space, a smeared operator symmetric around the origin of three-space or a local operator integrated over three-space (for the current). If an odd number among the indices $i, j, k$ are perpendicular to $\hat{l}$, then the correlator in Eq. (6.1) changes sign under $\mathcal{R}_l(\pi)$ and the amplitude must vanish. This happens if $i = j = k, \hat{l}$ being any other direction, or if $i = j \neq k, l = i$.

However, if $i, j, k$ are all different it does not work: any $\mathcal{R}_l(\pi)$ will keep the $C_{i,j,k}^{(3)}$ of Eq. (6.1) unchanged and thus cannot be proven to vanish on the lattice although it should in the continuum limit. This type of term does generate lattice artefacts. A parity operation would change its sign (changing the sign of all three operators) but parity is not an invariant of the twisted-mass action. We must then use correlators symmetrised according to the exact symmetry of the twisted-mass action [23], i.e. we apply Eq. (5.11); the lattice artefact should then disappear and $C_{i,j,k}^{(3)\text{sym}}(t_s, t, t_p) = 0$ on the lattice, at zero recoil.

Since it must vanish at zero recoil on the lattice, we may subtract from the three-point correlator at non-vanishing recoil the same configuration at zero recoil. This reduces some correlated noise, and indeed it turns out that the signal, although still very noisy, is significantly improved. We have computed the three-point functions with both all $\mu$ positive (set $s_{p_0}$) and all opposite in sign (set $s_{p_1}$). It turns out that the real parts of the three-point functions are very similar for both $s_{p_0}$ and $s_{p_1}$ sets, while the imaginary parts are approximatively opposite in signs, from which we can guess that the contributions with $i, j, k$ not all different are dominantly real, while the ones with $i, j, k$ different are dominantly imaginary. This is related to the fact that the terms odd in the $\mu$ have an $i$ with respect to the ones which are

\(^3\) We are indebted to Philippe Boucaud who suggested this trick.

| Combination | Expression |
|------------|------------|
| $p \hat{k} = -\sqrt{3} \mathcal{P}_i^{A(0)}$ | $A_1 V_1 D_1 + A_1 V_2 D_2 - 2 A_1 V_3 D_3$ |
| $p \hat{k} = -\sqrt{3} \mathcal{P}_i^{A(2)}$ | $A_2 V_1 D_1 + A_2 V_2 D_2 - 2 A_2 V_3 D_3$ |
| $p \hat{k} = \sqrt{6}/2 \mathcal{P}_i^{A(0)}$ | $- (A_3 V_1 D_1 + A_3 V_2 D_2 - 2 A_3 V_3 D_3)/2$ |
| $p \hat{k} = \{ \mathcal{P}_i^{A}(1+1) + \mathcal{P}_i^{A}(1-1)\}$ | $(A_1 V_1 D_1 - A_1 V_5 D_2)/2$ |
| $p \hat{k} = i \{ \mathcal{P}_i^{A}(1+1) + \mathcal{P}_i^{A}(1-1)\}$ | $(A_1 V_1 D_2 + A_1 V_2 D_1 - A_1 V_3 D_2 + A_1 V_3 D_3)$ |
| $p \hat{k} = -i \{ \mathcal{P}_i^{A}(1-1) + \mathcal{P}_i^{A}(1+1)\}$ | $-A_3 V_1 D_2 - A_3 V_2 D_1 + A_3 V_5 D_2 + A_3 V_5 D_3$ |
| $p \hat{k} = i \{ \mathcal{P}_i^{A}(2+1) + \mathcal{P}_i^{A}(2-1)\}$ | $(A_1 V_1 D_3 + A_1 V_2 D_1 - A_1 V_3 D_2 - A_1 V_3 D_3)$ |
| $p \hat{k} = -i \{ \mathcal{P}_i^{A}(2-1) + \mathcal{P}_i^{A}(2+1)\}$ | $-A_3 V_1 D_3 - A_3 V_2 D_1 + A_3 V_5 D_2 + A_3 V_5 D_3$ |
| $p \hat{k} = i \{ \mathcal{P}_i^{A}(2+1) - \mathcal{P}_i^{A}(2-1)\}$ | $(A_1 V_3 D_2 + A_1 V_3 D_1 - A_1 V_3 D_2 - A_1 V_3 D_3)$ |
| $p \hat{k} = -i \{ \mathcal{P}_i^{A}(2-1) - \mathcal{P}_i^{A}(2+1)\}$ | $-A_3 V_3 D_2 + A_3 V_3 D_1 + A_3 V_3 D_1 + A_3 V_3 D_3$ |
even, but for the sake of brevity we will skip an exact proof (Fig. 7).

6.3 Extracting the matrix element

An estimate of $p \tilde{k}$ is thus given by

$$R_{i,D_2^*}(t_p, t, t_s) = \frac{C^{(3)}_{B(\vec{0})_A} D_{D_2^*}^{(2)}(t_p, t, t_s) \sqrt{Z_B Z_{D_2^*} Z_{A}}}{C^{(2)}_{D_2^*}(t - t_s) C^{(2)}_{B(\vec{0})}(t_p - t)},$$

(6.2)

where the three-point correlators are the combinations of correlators listed in Table 6, symmetrised according to Eq. (5.11) and with the corresponding zero recoil three-point correlators subtracted. Let us recall that $\vec{p} = \vec{0}\pi/L$ where $L$ is the spatial length of the lattice. We have used systematically $\vec{\theta}_x = \vec{\theta}_y = \vec{\theta}_z = \theta$ whence $|\vec{p}| = \sqrt{3}\theta\pi/L$. In fact we prefer to present another ratio. Since the estimate of $B \to D_2^*$ in the infinite mass limit is rather successful, in good agreement with experiment, we will compute the ratio of the three-point correlators divided by the one which is derived from the infinite mass limit formula [8]:
\[ \tau_{3/2}(w) = \tau_{3/2}(1) \left( \frac{2}{1 + w} \right)^{2\gamma_{3/2}}. \]  

Here the fit gives \( \sigma_{3/2}^2 \simeq 1.5 \) and \( \tau_{3/2}(1) \simeq 0.54 \) and the formula in Sect. 2.5.1

\[ \tilde{k}_{\text{inf}} = \sqrt{3} \sqrt{r_{D_s^0}^2} (1 + w) \tau_{3/2}(w), \]  

with \( r_{D_s^0} = M_{D_s^0}/M_B \). This \( \tilde{k}_{\text{inf}} \) will be used as a benchmark for \( \tilde{k} \) extracted from our present calculations. From our benchmark \( \tilde{k}_{\text{inf}} \) we compute the benchmark three-point correlator, the \( D_s^* \) meson being created at time \( t_s \) the current inserted at time \( t_p \) and the \( B \) annihilated at time \( t_p \):

\[ C^{(3)}_{\text{inf}}(t_p, t, t_s) = \langle \bar{b} | \mathcal{C}^{(2)}_{D_s^*}(t - t_s) \mathcal{C}^{(2)}_{B}(t_p - t) Z_A/\sqrt{Z_B Z_{D_s^*}} \rangle. \]  

We thus consider the ratio

\[ \tilde{k} = \frac{C^{(3)}_{B(t) A(t)} D_s^*(t_p, t, t_s)}{C^{(3)}_{\text{inf}}(t_p, t, t_s)}. \]  

To increase the signal we take the average of the 11 expressions for \( \tilde{k} \) in Table 6, for all three masses of the \( B \) meson, for \( \beta = 3.9 \) and \( \beta = 4.05 \). For the two-point functions in Eq. (6.5) we have used the numerical values. As can be seen all these plots show similar shapes. There is a positive noisy signal, of the order 1 for \( \beta = 4.05 \). For \( \beta = 3.9 \) the ratio is about one order of magnitude smaller. We do not understand this feature.

To conclude we may claim that there is a hint of a signal for \( B \to D_s^* \ell\nu \) with finite \( m_c, b \). But the size of the statistical and systematic errors do not allow us to provide any quantitative result. It is clear that improving the signal is a major goal. This can be performed using larger statistics, using the points at \( \beta = 4.2 \), using different times for the sink, using other lattice actions and trying to find better interpolating fields.

7 Conclusions and prospects

The major goal of this paper concerned the orbitally excited states of the charmed mesons and the semileptonic decay of the \( B \) meson into the latter. We have concentrated on \( D_s^* \) and \( D_0^* \) (see [39] for a study of the mass spectrum including the spin-1 particles). We have considered three “\( B \) mesons” with, respectively, masses of 2.55, 3.18 and 3.97 GeV. We have used only two lattice spacings, 0.085 and 0.069 fm, with the addition of data with 0.054 fm for \( D_0^* \). Concerning the spectroscopy, we have noted a discrepancy between the masses of the \( D_s^* \) states which are in the \( E \) (\( D_l V_j \)) discrete group and the ones in the \( T_2 \) (\( D_l V_j; j \neq i \)). This is, of course, a lattice artefact. We have also studied the \( B \) meson energy as a function of the momentum. There is a clear departure from the theoretical formula for the heaviest meson. This is presumably also an artefact. The \( D_0^* \) state can decay into a \( D \) meson due to the parity violation when using twisted quarks. It was necessary to use the GEVP method to overcome this difficulty. The mass differences between the \( D_s^* \) (\( D_0^* \)) with the \( D \) meson mass, extrapolated to the continuum, do not agree well with experiment. There are indications that reducing the cut-off effects improves this result.

To compute the form factors and branching fractions we have derived all the needed theoretical formulae necessary to estimate any form factor from lattice calculation.

Concerning \( B \to D_0^* \) we have, up to now, only considered the zero recoil quantity. Our result is that, contrary to the case of infinite \( b \) and \( c \) masses, the zero recoil amplitude does not vanish. This should increase drastically the ratio of \( B \to D_0^* \) branching fraction over the \( B \to D_s^* \) one, as compared to the infinite mass case. We estimate the ratio of zero recoil amplitudes \( (B \to D_0^*\ell\nu)/(B \to D\ell\nu) \) = 0.17(6)(6). The corresponding ratio of branching fractions should be around 0.02, with very large errors. Some experimental semileptonic branching ratios seem to indicate for this ratio a figure of the order of 0.1, but the experimental situation is far from clear and our theoretical estimate still needs much work.

The \( B \to D_s^* \) is treated by a subtraction of the zero recoil contribution which we prove to be theoretically vanishing. There is a signal, although still very noisy. We take the infinite mass result as a benchmark. The ratio to the infinite mass prediction is around 1 for \( \beta = 4.05 \) but around 0.1 for \( \beta = 3.9 \), which indicates that beyond the very large statistical errors the systematic effects are not yet well understood.

Altogether, this paper has to be taken as a preliminary study. To our knowledge it is the first study of semileptonic decays to orbitally excited charmed mesons with finite masses for the \( b \) and \( c \). The considered process is very noisy and it is already rewarding that we got signals which seem to make sense, although the uncertainty is still much too large.

To improve the situation it seems that the path to follow is to further analyse the data set of ETMC at \( \beta = 4.2 \) (\( a \simeq 0.055 \) fm) and check against another lattice regularisation. The extrapolation to the continuum will thus be on a much safer ground. An increase of the statistics might also help.

We also stress that \( B_s \) and \( D_s \) sectors are presumably interesting to examine. Indeed, the \( D_s^{*0}(2317) \) and \( D_s^*(2460) \) states stand below the \( DK \) and \( D^*K \) thresholds; hence they are narrow. At LHCb, according to a phenomenological study [44], about 100 events in the channel \( B_s \to D_s^{*0} \ell\nu \) are
expected with 1 fb\(^{-1}\) of integrated luminosity. It is an encour-
gagement to extend our effort of measuring the form factors of \(B \rightarrow D^{**}\) semileptonic decays in the heavy-scat-
tor.

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**Appendix**

Coefficients of the hadronic tensor \(W_{\mu\nu}\)

In order to compute explicitly the coefficients \(\alpha, \beta_{++}, \beta_{+-},
\beta_{-+}, \beta_{-}\) and \(\gamma\) given in (2.3), we will have to evaluate
the possible summation over the \(D^{**}\) spins. Scalar meson We
have no summation and we get

\[
\begin{align*}
\alpha & = 0 \\
\beta_{++} & = \tilde{u}_+^2 \\
\beta_{-+} & = 0 \\
\beta_{-} & = \tilde{u}_-^2.
\end{align*}
\]

Tensor meson We have \(J = 2\); the polarisation tensor \(e_{\mu\nu}^{(p)}\) satisfies [41, 42]:

\[
\sum_s e_{\mu\nu}^{(p)s} e_{\rho\sigma}^{(p)s} = -\frac{1}{3} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left( g_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) + \frac{1}{2} \left( g_{\mu\rho} - \frac{p_\mu p_\rho}{p^2} \right) \left( g_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2} \right) + \frac{1}{2} \left( g_{\mu\sigma} - \frac{p_\mu p_\sigma}{p^2} \right) \left( g_{\nu\rho} - \frac{p_\nu p_\rho}{p^2} \right).
\]

After calculation, we obtain

\[
\begin{align*}
\beta_{++} & = 2 \tilde{k}_+ \tilde{b}_+ \frac{2 \tilde{k}_+ \tilde{b}_+ \tilde{D}_+^2}{3 m_{D^*}^4} \left( (p_B \cdot p_{D^*})^2 - m_B^2 m_{D^*}^2 \right) \\
\beta_{-+} & = 2 \tilde{k}_+ \tilde{b}_- \frac{2 \tilde{k}_+ \tilde{b}_- \tilde{D}_-^2}{3 m_{D^*}^4} \left( (p_B \cdot p_{D^*})^2 - m_B^2 m_{D^*}^2 \right) \\
\beta_{-} & = 2 \tilde{k}_+ \tilde{b}_- \frac{2 \tilde{k}_+ \tilde{b}_- \tilde{D}_-^2}{3 m_{D^*}^4} \left( (p_B \cdot p_{D^*})^2 - m_B^2 m_{D^*}^2 \right) \\
\gamma & = \frac{k \cdot h}{m_{D^*}^2} \left( (p_B \cdot p_{D^*})^2 - m_B^2 m_{D^*}^2 \right).
\end{align*}
\]
Variation domains of $x$ and $y$

**First type of constraints: $x = x(y)$**

Non-zero mass lepton

We have

\[
\begin{align*}
(m_\ell \neq 0) & \quad \begin{cases}
    x_{\text{max}} = \frac{1}{2} \left[ 1 + y - r_{p^{+}}^2 + r_{\ell}^2 \left( 1 + \frac{1}{y} \left( 1 - r_{p^{+}}^2 \right) \right) \right] + \left( 1 - \frac{r_{\ell}^2}{y} \right) \sqrt{\left( y - (1 - r_{p^{+}}^2) \right)^2 \left[ y - (1 + r_{p^{+}}^2) \right]} , \\
    x_{\text{min}} = \frac{1}{2} \left[ 1 + y - r_{p^{+}}^2 + r_{\ell}^2 \left( 1 + \frac{1}{y} \left( 1 - r_{p^{+}}^2 \right) \right) \right] - \left( 1 - \frac{r_{\ell}^2}{y} \right) \sqrt{\left( y - (1 - r_{p^{+}}^2) \right)^2 \left[ y - (1 + r_{p^{+}}^2) \right]} ,
\end{cases}
\end{align*}
\]

with $r_{\ell}^2 \leq y \leq (1 - r_{p^{+}}^2)^2$.

Zero mass lepton We have

\[
\begin{align*}
(m_\ell = 0) & \quad \begin{cases}
    y_{\text{max}} = \frac{1}{2} \left[ 1 + y - r_{p^{+}}^2 + \sqrt{\left( y - (1 - r_{p^{+}}^2) \right)^2 \left[ y - (1 + r_{p^{+}}^2) \right]} \right] , \\
    y_{\text{min}} = \frac{1}{2} \left[ 1 + y - r_{p^{+}}^2 - \sqrt{\left( y - (1 - r_{p^{+}}^2) \right)^2 \left[ y - (1 + r_{p^{+}}^2) \right]} \right] ,
\end{cases}
\end{align*}
\]

with $0 \leq y \leq (1 - r_{p^{+}}^2)^2$.

**Second type of constraints: $y = y(x)$**

Non-zero mass lepton We have

\[
\begin{align*}
(m_\ell \neq 0) & \quad \begin{cases}
    y_{\text{max}} = \frac{1}{2} \left( x - \frac{r_{p^{+}}^2 (x - 2r_{\ell}^2)}{1 - x + r_{\ell}^2} \right) + \left( 1 - \frac{r_{p^{+}}^2}{1 - x + r_{\ell}^2} \right) \sqrt{x^2 - 4r_{\ell}^2} , \\
    y_{\text{min}} = \frac{1}{2} \left( x - \frac{r_{p^{+}}^2 (x - 2r_{\ell}^2)}{1 - x + r_{\ell}^2} \right) - \left( 1 - \frac{r_{p^{+}}^2}{1 - x + r_{\ell}^2} \right) \sqrt{x^2 - 4r_{\ell}^2} ,
\end{cases}
\end{align*}
\]

with $2r_{\ell} \leq x \leq 1 - r_{p^{+}}^2 + r_{\ell}^2$.

Zero mass lepton We have

\[
\begin{align*}
(m_\ell = 0) & \quad \begin{cases}
    y_{\text{max}} = x \left( 1 - \frac{r_{p^{+}}^2}{1 - x} \right) , \\
    y_{\text{min}} = 0,
\end{cases}
\end{align*}
\]

with $0 \leq x \leq 1 - r_{p^{+}}^2$.

Expressions for the various decay widths

\[
\frac{d^2\Gamma}{dx\,dy} \quad \text{differential decay width}
\]
\( ^3P_0 \) states

Non-zero mass lepton:

We have

\[
\frac{d^2 \Gamma}{dx \, dy} = -\frac{G_F^2 |V_{cb}|^2}{128 \pi^3} \frac{m_B^5}{m_b} \left\{ -\bar{u}_+ \left[ 4 \left( x r_{\nu_0}^2 + (1-x)(y-y) \right) + r_\ell^2 \left[ 3 y - 4 (x + r_{\nu_0}^2) + r_\ell^2 \right] \right] + 2 \bar{u}_+ \bar{e}_- r_\ell^2 \left[ 2 (1-x - r_{\nu_0}^2) + y + r_\ell^2 \right] + \bar{u}_+^2 r_\ell^2 (y - r_\ell^2) \right\} .
\]

Zero mass lepton: We have

\[
\frac{d^2 \Gamma}{dx \, dy} = \frac{G_F^2 |V_{cb}|^2}{32 \pi^3} \frac{m_B^5}{m_b} \bar{u}_+ \left[ x r_{\nu_0}^2 + (1-x)(y-y) \right] .
\]

\( ^3P_2 \) states

Non-zero mass lepton: We have

\[
\frac{d^2 \Gamma}{dx \, dy} = -\frac{m_B}{256 \pi^3} \frac{G_F^2 |V_{cb}|^2}{2} \left\{ C_1 \bar{k}^2 + C_2 \bar{b}^2 + C_3 \bar{b}_-^2 + C_4 \bar{b}_+^2 + 2 C_5 \bar{k} \bar{b}_- + 2 C_6 \bar{k} \bar{b}_+ + 2 C_7 \bar{b}_- \bar{b}_+ + 2 C_8 \bar{k} \bar{k} \right\}
\]

where the \( C_i \) coefficients are given by

\[
C_1 = \frac{m_B^8}{3 \, m_b} \left\{ \left[ y^2 - \left( 2 + r_{\nu_0}^2 \right) \right] \left[ y + \left( 1 - r_{\nu_0}^2 \right) \right] \left[ 2 (1-x)(y-y) + r_\ell^2 \left[ 3 y - 4 (x + r_{\nu_0}^2) + r_\ell^2 \right] \right] - 3 y^2 r_{\nu_0}^4 \right\}
\]

\[
-\bar{r}_\ell^2 \left[ (1-2x+y) \right] \left[ 2 (1-y)^2 - 3 r_{\nu_0}^2 (1+y) \right] - r_{\nu_0}^4 \left[ 2 x - r_{\nu_0}^2 \right] \right\}
\]

\[
-2 \bar{r}_\ell^4 \left[ y^2 - 2 \left( 1 + r_{\nu_0}^2 \right) y + 1 - r_{\nu_0}^2 + r_{\nu_0}^4 \right] .
\]

\[
C_2 = \frac{m_B^8}{2 \, m_b} \left[ y - (1-r_{\nu_0}^2)^2 \right] \left[ y - (1+r_{\nu_0}^2)^2 \right] \times \left[ 2 (1-x + r_{\nu_0}^2)(1-x - r_{\nu_0}^2) \right] - (1-y + r_{\nu_0}^2)(1+y - 2x - r_{\nu_0}^2) \right] + r_{\nu_0}^2 \left[ (1+y - r_{\nu_0}^2)(1+y - 2x - r_{\nu_0}^2) + 2 r_{\nu_0}^2 \right] .
\]

\[
C_3 = \frac{m_B^8}{6 \, m_b} \left[ y - (1-r_{\nu_0}^2)^2 \right] \left[ y - (1+r_{\nu_0}^2)^2 \right] \left[ 4 x \left( 1-x - r_{\nu_0}^2 \right) - 4 y (1-x) + r_\ell^2 \left[ 4 \left( x + r_{\nu_0}^2 \right) - 3 y - r_\ell^2 \right] \right] .
\]
where the

\[ C_4 = \frac{m_n}{6 r^4_{\nu_2} r_\ell^2} (y - r_\ell^2)^2 \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right], \]

\[ C_5 = \frac{m_n}{3 r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \times \left[ 2 \left( 1 - y - r_\nu^2 \right) \left( 1 - x \right) \left( x - y - x r_\nu^2 \right) - r_\ell^2 \left[ \left( 1 - y + r_\nu^2 \right) \left( 1 - 3 x + 2 y - r_\nu^2 + r_\ell^2 \right) + 2 x r_\nu^2 \right] \right], \]

\[ C_6 = \frac{m_n}{3 r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \times \left[ \left( 1 - y + r_\nu^2 \right) \left( 1 - x + r_\nu^2 \right) + 2 r_\nu^2 \left( x - 2 \right) + r_\ell^2 \left( 1 - y + r_\nu^2 \right) \right], \]

\[ C_7 = \frac{m_n}{6 r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \left[ 2 \left( 1 - y + r_\nu^2 \right) + y + r_\ell^2 \right], \]

\[ C_8 = - \frac{m_n}{r^2_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \left[ y \left( 1 + y - 2 x - r_\nu^2 \right) + r_\ell^2 \left( 1 + y - r_\nu^2 \right) \right]. \]

Zero mass lepton: We notice that the coefficients of \( C_4, \) \( C_6 \) and \( C_7 \) cancel in this limit, leading to

\[
\frac{d^3 \Gamma}{dx dy} = - \frac{m_n}{256 \pi^3} \frac{G_F^2 |V_{cb}|^2}{2} \left[ C_1 \bar{k}^2 + C_2 \bar{b}^2 + C_3 \bar{b}^\downarrow + 2 C_3 \bar{k} b^+ + 2 C_8 \bar{k} \bar{k} \right]
\]

where the \( C_1 \) coefficients are given by

\[ C_1 = \frac{m_n}{3 r^4_{\nu_2} r_\ell^2} \left[ y^2 - \left( 2 + r_\nu^2 \right) y + \left( 1 - r_\nu^2 \right)^2 \right] \left[ 2 \left( 1 - x \right) \left( x - y \right) + r_\nu^2 \left( 3 y - 2 x \right) \right] - 3 y^2 r^4_{\nu_2}, \]

\[ C_2 = \frac{m_n}{r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \times y \left[ 2 \left( 1 - x - r_\nu^2 \right) \left( 1 - x - r_\nu^2 \right) - \left( 1 - y + r_\nu^2 \right) \left( 1 + y - 2 x - r_\nu^2 \right) \right], \]

\[ C_3 = \frac{m_n}{6 r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \left[ 4 x \left( 1 - x - r_\nu^2 \right) - 4 y \left( 1 - x \right) \right], \]

\[ C_5 = \frac{m_n}{3 r^4_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] \left[ 2 \left( 1 - y - r_\nu^2 \right) \left[ \left( 1 - x \right) \left( x - y \right) - x r_\nu^2 \right] \right], \]

\[ C_8 = - \frac{m_n}{r^2_{\nu_2} r_\ell^2} \left[ y - \left( 1 - r_\nu^2 \right)^2 \right] \left[ y - \left( 1 + r_\nu^2 \right)^2 \right] y \left( 1 + y - 2 x - r_\nu^2 \right). \]
\( \frac{d\Gamma}{dy} \) differential decay width

The form factors depend only on the \( y \) parameter, but in an unknown way. So, the integration over the \( x \) variable can be done through the use of the expressions of the type \( x = x(y) \).

**\( ^3P_0 \) states**

Non-zero mass lepton: We have

\[
\frac{d\Gamma}{dy} = - \frac{G_F^2 |V_{cb}|^2}{128\pi^3} \frac{m^5}{m_{\bar{b}}^2} \left[ D_1 \tilde{u}_+ + 2 D_2 \tilde{u}_+ \tilde{u}_- + D_3 \tilde{u}_-^2 \right]
\]

where the \( D_i \) coefficients are functions of \( y \) and are given by

\[
D_1 = \frac{1}{3} y^3 \left( y - r^2 \right)^2 \left[ y - \left( 1 - rD_0^* \right)^2 \right]^{1/2}
\]

\[
\times \left\{ 2 y \left[ y - \left( 1 - rD_0^* \right)^2 \right] \left[ y - \left( 1 + rD_0^* \right)^2 \right] + r^2 \left[ y^2 - 2 y \left( 1 + rD_0^* \right) + 4 \left( 1 - r^2 \right)^2 \right] \right\},
\]

\[
D_2 = \frac{1}{y^2} r^2 \left( y - r^2 \right)^2 \left[ y - \left( 1 - r^2 \right)^2 \right]^{1/2},
\]

\[
D_3 = \frac{1}{y} r^2 \left( y - r^2 \right)^2 \left[ y - \left( 1 - rD_0^* \right)^2 \right]^{1/2}.
\]

Let us recall that, in that expression of \( \frac{d\Gamma}{dy} \), the \( y \) parameter belongs in the interval: \( r^2 \leq y \leq (1 - rD_0^*)^2 \)

Zero mass lepton: We have

\[
\frac{d\Gamma}{dy} = - \frac{G_F^2 |V_{cb}|^2}{256\pi^3} \frac{m^2}{m_{\bar{b}}^2} \tilde{u}_+^2 D_1
\]

where

\[
D_1 = \frac{2}{3} \left[ y - \left( 1 - rD_0^* \right)^2 \right] \left[ y - \left( 1 + rD_0^* \right)^2 \right]^{3/2},
\]

since the other coefficients, \( D_2 \) and \( D_3 \), give zero in this case.

**\( ^3P_2 \) states**

Non-zero mass lepton: We have

\[
\frac{d\Gamma}{dy} = - \frac{m_{\bar{b}}}{256\pi^3} \frac{G_F^2 |V_{cb}|^2}{2} \left\{ D_1 \tilde{k}^2 + D_2 \tilde{h}^2 + D_3 \tilde{h}_+^2 + D_4 \tilde{h}_-^2 + 2 D_5 \tilde{k} \tilde{h} + 2 D_6 \tilde{k} \tilde{h}_- + 2 D_7 \tilde{h}_+ \tilde{b}_- + 2 D_8 \tilde{h} \tilde{h}_- \right\}
\]
where the $D_i$ coefficients are given by

\[
D_1 = \frac{m_\mu^4}{\bar{r}^4 r_{\nu_2}^2} \frac{1}{9y^3} (y-r_\ell^2)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{3/2} \times \left\{ y \left[ y^2 - 2y \left( 1 - 4r_{\nu_2}^2 \right) + \left( 1 - r_{\nu_2}^2 \right)^2 \right] + r_\ell^2 \left[ 2y^2 - y \left( 4 - r_{\nu_2}^2 \right) + \left( 1 - r_{\nu_2}^2 \right)^2 \right] \right\},
\]

\[
D_2 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{3y^2} (y-r_\ell^2)^2 (2y + r_\ell^2) \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2},
\]

\[
D_3 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{18y^2} \left( y-r_\ell^2 \right)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2} \times \left\{ 2y \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] + r_\ell^2 \left[ y^2 - 2y \left( 1 + r_{\nu_2}^2 \right) + \left( 1 - r_{\nu_2}^2 \right)^2 \right] \right\},
\]

\[
D_4 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{6y} y^2 r_\ell (y-r_\ell^2)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2},
\]

\[
D_5 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{18y^2} \left( y-r_\ell^2 \right)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2} \times \left\{ 2y \left( 1 - y - r_{\nu_2}^2 \right) + r_\ell^2 \left( 4 - y - 4r_{\nu_2}^2 \right) \right\},
\]

\[
D_6 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{6y} y^2 r_\ell (y-r_\ell^2)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2},
\]

\[
D_7 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{1}{6y} y^2 r_\ell (y-r_\ell^2)^2 \left( 1 - r_{\nu_2}^2 \right)^2 \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2},
\]

\[
D_8 = 0.
\]

We recall that, in those formulae, the $y$ parameter varies inside the domain $r_\ell^2 \leq y \leq (1 - r_{\nu_2}^2)^2$.

Zero mass lepton: We have

\[
\frac{d\Gamma}{dy} = -\frac{m_\mu}{256\pi^3} \frac{G_F^2 |V_{cb}|^2}{2} [D_1 \bar{k}^2 + D_2 \bar{h}^2 + D_3 \bar{b}_+^2 + 2D_5 \bar{k} \bar{b}_+ + 2D_8 \bar{h} \bar{k}]
\]

where the $D_i$ coefficients are given by

\[
D_1 = \frac{m_\mu^4}{\bar{r}^4 r_{\nu_2}^2} \frac{1}{9} \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{3/2} \times \left\{ y^2 - 2y \left( 1 - 4r_{\nu_2}^2 \right) + \left( 1 - r_{\nu_2}^2 \right)^2 \right\},
\]

\[
D_2 = \frac{m_\mu^8}{\bar{r}^8 r_{\nu_2}^4} \frac{2}{3} y \left[ \left[ y - \left( 1 - r_{\nu_2}^2 \right) \right] \left[ y - \left( 1 + r_{\nu_2}^2 \right) \right] \right]^{5/2},
\]
\[
D_3 = \frac{m_y^6}{4 r_\pi^4} \frac{1}{9} \left[ \left( y - \left( 1 - r_\pi^2 \right)^2 \right) \left[ y - \left( 1 + r_\pi^2 \right)^2 \right] \right]^{7/2},
\]
\[
D_5 = \frac{m_y^6}{4 r_\pi^4} \frac{1}{9} \left( 1 - y - r_\pi^2 \right) \left[ \left( y - \left( 1 - r_\pi^2 \right)^2 \right) \left[ y - \left( 1 + r_\pi^2 \right)^2 \right] \right]^{5/2},
\]
\[
D_8 = 0.
\]

Here, the \( y \) parameter lies in the domain \( 0 \leq y \leq (1 - r_\pi^2)^2 \).

\[ d\Gamma \, dx \] differential decay width

It is impossible to give general expressions for the leptonic spectra \( d\Gamma \, dx \), since the integration over \( y \) cannot be performed because we do not know the dependence of the form factors on \( y \).

Nevertheless, the procedure to do those calculations is the following:

1. We start from the expressions of the \( \frac{d^2\Gamma}{dx\,dy} \) decay widths given above.
2. We use the constraints of the type \( y = y(x) \) in order to perform the integration over \( y \) from \( y_{\text{min}} \) to \( y_{\text{max}} \) (expressions given also above). Incidentally, we must not forget that the maximum of \( E_{\mu\nu} \) corresponds to the minimum of \( y \) and vice versa. So, to integrate over \( E_{\mu\nu}^{\,\text{min}} \) from \( E_{\mu\nu}^{\,\text{max}} \), we have to integrate equivalently over \( y \) from \( y_{\text{max}} \) to \( y_{\text{min}} \):

\[
\frac{d\Gamma}{dx} = \int_{y_{\text{min}}}^{y_{\text{max}}} \frac{d^2\Gamma}{dx\,dy} \, dy
\]

with

\[
\begin{align*}
y_{\text{max}} & = \frac{1}{2} \left[ x - r_\mu^2 \left( x - 2 r_\mu^2 \right) \right] + \left( 1 - \frac{r_\mu^2}{1 - x + r_\mu^2} \right) \sqrt{x^2 - 4 r_\mu^2}, \\
y_{\text{min}} & = \frac{1}{2} \left[ x - r_\mu^2 \left( x - 2 r_\mu^2 \right) \right] - \left( 1 - \frac{r_\mu^2}{1 - x + r_\mu^2} \right) \sqrt{x^2 - 4 r_\mu^2}.
\end{align*}
\]

Once again, \( r_\mu = 0 \) gives the variation domain in the case of a zero mass lepton.

Total decay width \( \Gamma \)

The problem, mentioned for the leptonic spectra, pops up here again because, in order to get \( \Gamma \), we will have to integrate over \( y \) at some point. So we will have to follow the same procedure.

Polarisation tensor for the \( 3P_2 \) state

Using expressions for the spin-1 polarisation vector found in [43] for instance and the values of the Clebsch–Gordan coefficient from the Particle Physics Booklet, we get

\[
\varepsilon_{\mu\nu}^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix},
\]

\[
\varepsilon_{\mu\nu}^{(-2)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & 1 \end{pmatrix},
\]

\[
\varepsilon_{\mu\nu}^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & i \end{pmatrix},
\]

\[
\varepsilon_{\mu\nu}^{(-1)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -i & 0 \end{pmatrix},
\]

\[
\varepsilon_{\mu\nu}^{(0)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

We dropped the \( \hat{0} \) in the notation.

\( r_\mu = 0 \) gives the relations in the case of a zero mass lepton.

3. The last free parameter \( x \) lies in the domain

\[
2 r_\mu \leq x \leq 1 - r_\mu^2 + r_\mu^2.
\]
Extraction of the form factors

The following expressions are not exhaustive.

Note that it is possible to recover the momentum transfer

\[ y m_B^2 = (p_B - p_{\rho^*})^2 \]

using

\[ E_B = \frac{m_B}{2 r_{\rho^*}} [1 - y + r_{\rho^*}^2] \quad \text{and} \quad p^2 = \frac{m_B^2}{12 r_{\rho^*}^2} [y - (1 - r_{\rho^*})^2] [y - (1 + r_{\rho^*})^2], \]

leading to

\[ E_B + m_{\rho^*} = \frac{m_B}{2 r_{\rho^*}} (1 - y + 3 r_{\rho^*}^2) \]

\[ E_B - m_{\rho^*} = \frac{m_B}{2 r_{\rho^*}} (1 - y - r_{\rho^*}^2). \]

\[ 3 \ P_0 \ form \ factors \]

With \( \mathcal{F}_\mu^A \ \text{def.} = \langle 3 \ P_0 A_\mu B(p_B) \rangle \):

- form factor \( \tilde{u}_+ \)

\[
\tilde{u}_+ = -\frac{1}{2 m_{\rho^*}} \left[ \frac{E_B - m_{\rho^*}}{p} \mathcal{F}_\mu^A - \mathcal{F}_\mu^0 \right] \\
= -\frac{1}{2 m_{\rho^*}} \left[ \frac{E_B - m_{\rho^*}}{3 p} (\mathcal{F}_1^A + \mathcal{F}_2^A + \mathcal{F}_3^A - \mathcal{F}_0^A) \right],
\]

- form factor \( \tilde{u}_- \)

\[
\tilde{u}_- = \frac{1}{2 m_{\rho^*}} \left[ \frac{E_B + m_{\rho^*}}{p} \mathcal{F}_\mu^A - \mathcal{F}_\mu^0 \right] \\
= \frac{1}{2 m_{\rho^*}} \left[ \frac{E_B + m_{\rho^*}}{3 p} (\mathcal{F}_1^A + \mathcal{F}_2^A + \mathcal{F}_3^A - \mathcal{F}_0^A) \right].
\]

\[ 3 \ P_2 \ form \ factors \]

With \( \mathcal{F}_{\mu(\lambda)}^A \ \text{def.} = \langle 3 \ P_2(\lambda) A_\mu B(p_B) \rangle \) and \( \mathcal{F}_{\mu(\lambda)}^V \ \text{def.} = \langle 3 \ P_2(\lambda) V_\mu B(p_B) \rangle \):

- form factor \( \tilde{k} \)

\[
\tilde{k} = -\frac{\sqrt{6}}{p} \mathcal{F}_{1(0)}^A = -\frac{\sqrt{6}}{p} \mathcal{F}_{2(0)}^A = \frac{\sqrt{6}}{2 p} \mathcal{F}_{3(0)}^A \\
= \frac{1}{p} \left[ \mathcal{F}_1^{A(2)} + \mathcal{F}_2^{A(-2)} \right] = -\frac{1}{p} \left[ \mathcal{F}_2^{A(2)} + \mathcal{F}_1^{A(-2)} \right].
\]
Masses and energies

We collect in Table 7 the masses and energies that we extract in our analysis.

| Meson | $\beta = 3.9$ | $\beta = 4.05$ | $\beta = 4.2$ |
|-------|---------------|---------------|---------------|
| $D$   | $0.76 (1)$    | $0.62 (1)$    | $0.66 (1)$    |
| $D_0^*$| $1.01 (4)$    | $0.80 (2)$    | $1.01 (1)$    |
| $D_2^*$ | $1.14 (2)$    | $0.93 (2)$    | $1.25 (1)$    |
| $B(\mu_{h_1})$ | $1.00 (1)$    | $0.93 (2)$    | $1.04 (1)$    |
| $B(\mu_{h_2})$ | $0.99 (2)$    | $0.81 (1)$    | $1.04 (1)$    |
| $B(\mu_{h_3})$ | $1.24 (1)$    | $1.13 (1)$    | $1.13 (1)$    |
| $B(\mu_{h_4})$ | $1.51 (1)$    | $1.67 (1)$    | $1.52 (1)$    |
| $B(\mu_{h_5})$ | $1.41 (1)$    | $1.58 (1)$    | $1.25 (1)$    |
| $B(\mu_{h_6})$ | $1.26 (1)$    | $1.17 (1)$    | $1.04 (1)$    |
| $B(\mu_{h_7})$ | $1.15 (1)$    | $1.28 (1)$    | $1.04 (1)$    |
| $B(\mu_{h_8})$ | $2.02 (1)$    | $2.13 (1)$    | $1.04 (1)$    |
| $B(\mu_{h_9})$ | $2.46 (1)$    | $2.74 (1)$    | $1.08 (1)$    |
| $B(\mu_{h_{10}})$ | $3.01 (1)$    | $3.39 (1)$    | $1.31 (1)$    |
| $B(\mu_{h_{11}})$ | $2.50 (1)$    | $2.76 (1)$    | $0.92 (1)$    |
| $B(\mu_{h_{12}})$ | $3.05 (1)$    | $3.40 (1)$    | $1.11 (2)$    |
| $B(\mu_{h_{13}})$ | $3.73 (1)$    | $4.21 (1)$    | $1.34 (1)$    |
| $B(\mu_{h_{14}})$ | $2.92 (1)$    | $3.23 (1)$    | $0.95 (2)$    |
| $B(\mu_{h_{15}})$ | $3.56 (1)$    | $3.97 (1)$    | $1.15 (2)$    |
| $B(\mu_{h_{16}})$ | $4.36 (1)$    | $4.91 (1)$    | $1.38 (2)$    |
| $B(\mu_{h_{17}})$ | $3.66 (1)$    | $4.04 (1)$    | $1.00 (3)$    |
| $B(\mu_{h_{18}})$ | $4.46 (1)$    | $4.97 (1)$    | $1.22 (3)$    |
| $B(\mu_{h_{19}})$ | $5.46 (1)$    | $6.15 (1)$    | $1.45 (1)$    |

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