DEFORMATIONAL STRUCTURES ON SMOOTH MANIFOLDS

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Abstract

Abstract deformational structures, in many aspects generalizing standard elasticity theory, are investigated. Within free deformational structures we define algebra of deformations, classify them by its special properties, define motions and conformal motions together with deformational decomposition of manifolds, generalizing isometry of Riemannian spaces and consider some physical examples. In frame of dynamical deformational structures we formulate variational procedure for evolutional and static cases together with boundary conditions, derive dynamical (equilibrium in static case) equations, consider perturbative approach and perform deformational realization of the well known classical field-theoretical topics: strings and branes theories, classical mechanics of solids, gravity and Maxwell electrodynamics.

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1 Introduction

Recent time the strong tendency to inclusion of embedded objects into the scope of theoretical and mathematical physics is observed (see references in [1]). We should relate to the subject all strings and branes models [2, 3], including their supersymmetric and noncommutative generalizations [4], embedding methods of GR [5] and its alternative formulations and generalizations [6], geometrical methods of nonlinear differential equations theory and jets approach [7] and many other things. Probably, such central position of the "embedded objects" in modern physics can’t be accidental: it may reflect either multidimensional nature of physical reality, observed through all its levels, or some "immanent" for us, as observers, means for its description.

At the same time, majority of the field-theoretical models, exploiting embedded objects, reveal amazing and, in our opinion, deep interrelations with some general ideas of elasticity theory of continuous media [8] may be with a number of "nonstandard" properties such as nonlinearity, plasticity, viscosity, anisotropy, internal spin, nematic or smectic structures or memory [9, 10, 11, 12, 13, 14]. Particularly, in papers [15, 16, 17, 18, 19] it has been shown, that Einstein GR and standard classical solids dynamics admit natural formulation in terms of mechanical straining of thin 4D plates and 4D strings (strongly tensed bars) respectively.

Interesting and important problem, arising under such unifying of embedding and elasticity ideas, is to extract and formulate general ideas of continuous media physics in its the most abstract and general form, independent on peculiarities of one or another theory. So, we intend to follow the line of investigations, which can be called general theory of deformational structures (d-structures) with the aim — to formulate and work out universal language for the objects, which are able, in some sense, to be "deformed".

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Although we’ll restrict ourself by the case of real manifolds, majority of statements will take place after suitable complex generalization, which is necessary for constructing of quantum $d$-structures. Moreover, some general concepts “survive” even without smooth structures, but we reserve the more abstract schemes for future.

Present paper is devoted to some first principles of this program. We work out ”deformational terminology” and set some general propositions, statements and relations, which can be recognized within well known theories and which can be used in future works.

Within the first half of the paper (Sec. 2) we consider free $d$-structures, generalizing kinematics of standard elasticity theory and reflecting, mainly, geometrical properties of a number of physical models. The second half (Secs. 3, 4, 5) is devoted to dynamical $d$-structures, which include, apart from kinematics, some dynamical principle and reflect, mainly, physical properties of field-theoretical models. Examples of deformational structures, performed in the paper, involve elasticity theory together with its (generally-)covariant generalization, Hamiltonian formalism, bundle spaces with invariant connection, thermodynamics, strings and branes theories, classical solids dynamics, gravity, Maxwell electrodynamics. Some more subtle technical questions are investigated in Appendices.

Always, when it is possible we use standard notations of sets theory [21], smooth manifolds theory [22] and (almost anywhere) use coordinateless representation of tensor equations. Particularly, we’ll denote by

\begin{align*}
\text{Dom, Im} & \quad \text{domains and images of mappings;} \\
\equiv & \quad \text{equivalence relation } \rho; \\
\text{Dg}(A \times A) & \quad \text{diagonal of a direct product (i.e. set of pairs } (a, a) \in A \times A; \\
\pi_\rho & \quad \text{mapping on quotient space with respect to equivalence } \rho; \\
[a]_\rho & \quad \text{class of equivalence of the element } a \text{ with respect to } \rho; \\
A < B & \quad A \text{ is sub(pseudo)group of (pseudo)group } B; \\
\partial_r X = X_r & \quad \text{partial derivative;} \\
T(r, s) & \quad \text{space of tensors of covariant valency } r \text{ and contravariant valency } s; \\
(, , ) & \quad \text{scalar product in different tensor spaces;} \\
(, ) & \quad \text{pairing of tensors and linear functionals over them;} \\
M_{m \times n}(\mathbb{C}) & \quad \text{module of } m \times n \text{ matrices over ring } \mathbb{C}; \\
\text{Hom}(A, B) & \quad \text{space of linear mappings of modules (linear spaces) } A \to B.
\end{align*}

\section{Free deformational structures}

\subsection{Definitions}

We call \textit{free deformational structure} $\mathcal{D}$ the collection $\langle \mathcal{B}, \mathcal{M}, \mathcal{E}, \Theta \rangle$, where:

- $\mathcal{B}$ and $\mathcal{M}$ — smooth, connected, closed manifolds, $\dim \mathcal{B} = d$, $\dim \mathcal{M} = n \geq d$;
- $\mathcal{E} \subseteq \text{Emb}(\mathcal{B}, \mathcal{M})$ — some subset of all smooth embeddings $\mathcal{B} \hookrightarrow \mathcal{M}$;
- $\Theta \in \Omega^{\otimes p}(\mathcal{M})$ — some smooth real-valued form of degree $p$ on $\mathcal{M}$. In what follows we’ll call: $\mathcal{B}$ — $d$-\textit{body}, $\mathcal{M}$ — $d$-\textit{manifold}, $\Theta$ — $d$-\textit{metrics}, and image $\iota(\mathcal{B}) \equiv \mathcal{S} \subseteq \mathcal{M}$ for some $\iota \in \mathcal{E}$ — $d$-\textit{object} or \textit{deformant}.

Any embedding $\iota$ induces form $(d\iota)^* \Theta \in \Omega^{\otimes p}(\mathcal{B})$, where $(d\iota)^*$ — embedding $\iota$ codifferential\footnote{This is revised and essentially more developed version of [20] which, is, in turn, small part of talk, presented at 5-th Asian-Pacific conference (Moscow, October 2001).} mapping $\Omega^{\otimes p}(\mathcal{M}) \to \Omega^{\otimes p}(\mathcal{B})$. Let consider some another embedding $\iota' \in \mathcal{E}$, which induces its own $d$-object $\iota'(\mathcal{B}) \equiv \mathcal{S}' \subseteq \mathcal{M}$. In $\Omega^{\otimes p}(\mathcal{B})$ we’ll have the form $(d\iota')^* \Theta$. Easily to see, that the composition

\begin{align*}
\iota' \circ \iota^{-1} & \equiv \zeta 
\end{align*}

\footnote{Coordinates in ambient space we denote by small Latin letters with big Latin indexes — $x^A$, $A = 1, \ldots, n$, on embedded $d$-\textit{object} — by Greek letters with Greek indexes — $\xi^\alpha$, $\alpha = 1, \ldots, d$. Such doubling is useful for coordinateless symbolic notations.}

\footnote{We denote by $(d\iota)^*$ mappings $\Omega^{\otimes p}(\mathcal{M}) \to \Omega^{\otimes p}(\mathcal{B})$ for any $p$.}
is diffeomorphism $\mathcal{S} \rightarrow \mathcal{S}' = \zeta(\mathcal{S})$, which we’ll call deformation of $d$-body in $\mathcal{M}$.

Any deformation $\zeta$ has natural local measure — difference of two forms, taken at the same point $b \in \mathcal{B}$:

$$(d\zeta')^*\Theta(b) - (d\zeta)^*\Theta(b) \equiv \Delta_\mathcal{B}(b),$$

where we have introduced notation $\Delta_\mathcal{B}$ for deformation form on $\mathcal{B}$. Using definition (1) and the well known composition property of codifferential:

$$(d(\alpha \circ \beta))^* = (d\beta)^* \circ (d\alpha)^*,$$

we obtain the equivalent representation:

$$\Delta_\mathcal{B} = (d\zeta)^* (\Theta - \Theta),$$

and define the deformation form

$$\Delta_\mathcal{S} \equiv ((d\zeta)^*)^{-1}\Delta_\mathcal{B} = (d\zeta)^*\Theta - \Theta$$

on the deformant $\mathcal{S}$. Note, that the representations (3) and (4) correspond to material and referent descriptions of deformable bodies in classical continuum media dynamics [23].

### 2.2 Algebra of deformations

For any deformation $\zeta$ let define the subsets of $\mathcal{E}$ :

$$\Pr_1(\zeta) = \{ t \in \mathcal{E} \mid \text{Im}(i) = \text{Dom}(\zeta) \}; \quad \Pr_2(\zeta) = \{ t \in \mathcal{E} \mid \text{Im}(i) = \text{Im}(\zeta) \}.$$  

As it follows from the definition (1), the set of all deformations of the $d$-body in $\mathcal{M}$, which we’ll denote $\text{DEF}_{\mathcal{M}}(\mathcal{B})$, can be treated as image of the surjective map $\phi : \mathcal{E} \times \mathcal{E} \rightarrow \text{DEF}_{\mathcal{M}}(\mathcal{B})$, acting by the rule:

$$\phi(t_{\alpha}, t_{\beta}) = t_{\beta} \circ t_{\alpha}^{-1} \equiv \zeta_{\alpha\beta}.$$  

(5)

The following proposition clears the relation between $\mathcal{E} \times \mathcal{E}$ and $\text{DEF}_{\mathcal{M}}(\mathcal{B})$.

**Proposition 1** Fibre $\phi^{-1}(\zeta) = \{ d \in \mathcal{E} \times \mathcal{E} \mid d = (\zeta_{\alpha}, \zeta_{\alpha} \circ i_{\zeta} \circ i_{\alpha} \circ l) \}$, where $\zeta$ — some element of $\text{DEF}_{\mathcal{M}}(\mathcal{B})$, $l$ runs all elements from the $\text{Diff}(\mathcal{B})$, and embedding $i_{\zeta} \in \Pr_1(\zeta)$.

**Proof.** The inclusion $(t_{\zeta} \circ l, \zeta \circ i_{\zeta} \circ i_{\alpha} \circ l) \in \phi^{-1}(\zeta)$ immediately follows from the $\mathcal{B}$. Let the two elements $(t_{\alpha}, t_{\beta})$ and $(t_{\gamma}, t_{\delta})$ of $\mathcal{E} \times \mathcal{E}$ defines the same deformation $\zeta = t_{\beta} \circ t_{\alpha}^{-1} = t_{\delta} \circ t_{\gamma}^{-1}$, images of the firsts $-t_{\alpha}, t_{\gamma}$ and of the seconds $t_{\beta}, t_{\delta}$ embeddings pair-wisely coincide in $\mathcal{M}$ (as domains and images of the same deformation $\zeta$ in $\mathcal{M}$ respectively), i.e.:

$$t_{\alpha}(B) = t_{\gamma}(B) = \mathcal{S} \quad \text{and} \quad t_{\beta}(B) = t_{\delta}(B) = \mathcal{S}'.$$

Then, particularly, it follows, that $t_{\gamma} = t_{\alpha} \circ l$, where $l$ — some diffeomorphism of the $d$-body $\mathcal{B}$. So, if the pairs $(t_{\alpha}, t_{\beta})$ and $(t_{\gamma}, t_{\delta})$ lie in the same fiber $\phi^{-1}(\zeta)$, then they necessary have the form $(t_{\alpha}, \zeta \circ t_{\alpha})$ and $(t_{\alpha} \circ l, \zeta \circ t_{\alpha} \circ l)$ respectively. Simultaneousity of the two inclusions proves the proposition.\]

The map $\phi$ endows $\mathcal{E} \times \mathcal{E}$ the canonical equivalence $\mathcal{E} \subset (\mathcal{E} \times \mathcal{E})$, with quotient space $\pi_D(\mathcal{E} \times \mathcal{E})$, consisting of classes $[(t_{\alpha}, t_{\beta})]_D = \zeta_{\alpha\beta}$, such that $\pi_D^{-1}(\zeta_{\alpha\beta})$ is the fiber of Proposition 1, containing the element $(t_{\alpha}, t_{\beta}) \in \mathcal{E} \times \mathcal{E}$.  

On the set $\pi_D(\mathcal{E} \times \mathcal{E})$ one can introduce the following binary relation:

$$\rho = \{ (\zeta_1, \zeta_2) \in \pi_D(\mathcal{E} \times \mathcal{E}) \times \pi_D(\mathcal{E} \times \mathcal{E}) \mid \text{Pr}_2(\zeta_1) = \text{Pr}_1(\zeta_2) \}. $$

It is easily checked, that $\rho$ is T—reflective and T—antisymmetric, i.e. $(\zeta_1, \zeta^T) \in \rho$, and, if simultaneously $(\zeta_1, \zeta_2) \in \rho$ and $(\zeta_2, \zeta_1) \in \rho$, then $\zeta_2 = \zeta_1^T$. Here $\zeta^T \equiv \zeta_{\alpha\beta}^T \equiv \zeta_{\beta\alpha}$. We’ll call this relation T—tournament.\]

Let denote $Y_\zeta^\pm$ the following subsets:

$$Y_\zeta^- \equiv \{ \zeta' \in \pi_D(\mathcal{E} \times \mathcal{E}) \mid (\zeta', \zeta) \in \rho \}; \quad Y_\zeta^+ \equiv \{ \zeta' \in \pi_D(\mathcal{E} \times \mathcal{E}) \mid (\zeta, \zeta') \in \rho \}. $$

\footnote{Such quotient space is sometimes called twisted multiplication and in our case is denoted $\mathcal{E} \times \text{Diff}(\mathcal{B}) \mathcal{E}$.}  

\footnote{Tournament is reflective and antisymmetric binary relation [24].}
Proposition 2 On the set \( \pi_D(\mathcal{E} \times \mathcal{E}) \) with \( T \)-tournament \( \rho \) there exists pseudogroup structure \( [1] \)

Proof. For any \( \zeta \in \pi_D(\mathcal{E} \times \mathcal{E}) \) and for all \( \zeta' \in Y^- \zeta, \zeta'' \in Y^+ \zeta \) we define left and right pseudogroup multiplications as compositions of deformations:
\[
\zeta' \ast \zeta \equiv \zeta' \circ \zeta, \quad \text{and} \quad \zeta \ast \zeta'' \equiv \zeta \circ \zeta''
\]
respectively. In components:
\[
\zeta' \ast \zeta = [(\zeta', \zeta_1)][(\zeta_1, \zeta_2)]D = [(\zeta', \zeta_2)]D; \quad \zeta \ast \zeta'' = [(\zeta_1, \zeta_2)]D \ast [(\zeta_2, \zeta'')]D = [(\zeta_1, \zeta')]D.
\]
Units elements will be given by the expressions:
\[
e^-_\zeta \equiv [(\zeta, \zeta)]D \in \pi_D(\text{Dg}(\mathcal{E} \times \mathcal{E})), \quad e^+_\zeta \equiv [(\zeta', \zeta_1)]D \in \pi_D(\text{Dg}(\mathcal{E} \times \mathcal{E})),
\]
where \( \zeta \in \text{Pr}_1(\zeta), \zeta'_1 \in \text{Pr}_2(\zeta) \). Finally, for every \( \zeta \in \pi_D(\mathcal{E} \times \mathcal{E}) \) there exist unique inverse element \( \zeta^{-1} \) and it is easily to check in components, that \( \zeta^{-1} = \zeta^T \).

So, the set \( \pi_D(\mathcal{E} \times \mathcal{E}) \equiv \text{DEF}_M(\mathcal{B}) \) — pseudogroup. \( \square \)

2.3 Classification of deformations and Boolean matrix calculus

Let's consider the following formal object:
\[
\mathcal{I} = \begin{pmatrix} \text{Dom} \cap \text{Dom} & \text{Dom} \cap \text{Im} \\ \text{Im} \cap \text{Dom} & \text{Im} \cap \text{Im} \end{pmatrix}.
\]
It can be understood as the mapping: \( \text{DEF}_M(\mathcal{B}) \times \text{DEF}_M(\mathcal{B}) \rightarrow M_{2 \times 2}(\mathcal{B}(\mathcal{M})) \), where \( M_{2 \times 2}(\mathcal{B}(\mathcal{M})) \) — module of \( 2 \times 2 \) matrices over ring of subsets of \( \mathcal{M} \), which form boolean algebra \( \mathcal{B}(\mathcal{M}) \). For every pair \( (\zeta_1, \zeta_2) \in \text{DEF}_M(\mathcal{B}) \times \text{DEF}_M(\mathcal{B}) \), such that \( \zeta_1 : S_1 \rightarrow S'_1 \) and \( \zeta_2 : S_2 \rightarrow S'_2 \) we have:
\[
\mathcal{I}(\zeta_1, \zeta_2) = \begin{pmatrix} S_1 \cap S_2 & S_1 \cap S'_2 \\ S'_1 \cap S_2 & S'_1 \cap S'_2 \end{pmatrix}.
\]
We'll call \( \mathcal{I}(\zeta_1, \zeta_2) \) matrix of intersection of \( \zeta_1 \) and \( \zeta_2 \). For \( \zeta_1 = \zeta_2 = \zeta \) we'll call \( \mathcal{I}(\zeta, \zeta) \) matrix of self-intersection of \( \zeta \). Easily to check, that matrix of intersection is degenerate on any pair of deformations in boolean sense, i.e. \( \det \mathcal{I}(\zeta_1, \zeta_2) \equiv \emptyset \), where determinant is defined as usually, but calculation are carried out with the help of boolean operations \( \cap, \setminus \).

The first step to classification of deformations is based on the kind of matrix \( \mathcal{I}(\zeta, \zeta) \). We'll say, that deformation \( \zeta : S \rightarrow S' \)
— is parallel, if \( \mathcal{I}(\zeta, \zeta) \) — diagonal in boolean sense (i.e. nondiagonal components are \( \emptyset \));
— is sliding, \( \square \) if \( \mathcal{I}(\zeta, \zeta) = S \cdot \Omega \), where
\[
\Omega = \begin{pmatrix} \mathcal{M} \mathcal{M} \\ \mathcal{M} \mathcal{M} \end{pmatrix}
\]
and multiplication on ”number” \( S \) is component-wise boolean multiplication \( \cap \) of \( S \) on elements of \( \Omega \);
— is stretch of \( S \), if
\[
\mathcal{I}(\zeta, \zeta) = \begin{pmatrix} S \ S \\ S \ S' \end{pmatrix};
\]
\( \square \)

6Let remind, that pseudogroup is a set of elements \( \mathcal{A} \), for which composition \( \ast \) is defined may be on some subset (binary relation) \( \mathcal{U} \subset \mathcal{A} \times \mathcal{A} \) and where the following properties are hold: associativity, for every \( a \in \mathcal{A} \) there exist unique left \( e^a \) and right \( e^a \) units elements (generally speaking depending on \( a \)), lying in \( \mathcal{A} \) and there exists unique inverse element \( a^{-1} \), lying in \( \mathcal{A} \), such that \( e^a \ast a = a \ast e^a = a \) and \( a \ast a^{-1} = e^a, \quad a^{-1} \ast a = e^a \) [21].

7It is useful to differ the following particular cases: total sliding, if \( \zeta \) — sliding with \( S = \mathcal{M} \) and empty sliding, if \( \zeta \) — sliding with \( S = \emptyset \).
— is contraction of $S$, if

$$I(\zeta, \zeta) = \begin{pmatrix} S & S' \\ S' & S'' \end{pmatrix}.$$ 

We'll denote this the *simplest* classes of deformations as

$$\text{Simp} \equiv \{ \text{Par}(S), \text{Sl}(S), \text{Str}(S), \text{Ctr}(S) \}$$

respectively, omitting sometimes argument $S$. We observe, that by the definitions

$$\text{Sl}(M) \equiv \text{DEF}_M(M) \equiv \text{Diff}(M), \text{Str}(S) \cap \text{Ctr}(S) = \text{Sl}(S).$$

Easily to see, that for every $S = \iota(B), \text{Sl}(S), \text{Str}(S), \text{Ctr}(S)$ form subpseudogroup $^8$ of $\text{DEF}_M(B)$, while $\text{Par}(S)$ generally speaking, doesn’t. Obviously, boolean matrix calculus become trivial for $\text{Diff}(M)$, since it is mapped into a single self-intersection (in fact, intersection too) matrix $\Omega$.

Let $\text{DEF}_M(B) \ni \zeta : S \rightarrow S'$ and let $S \cap S' \equiv S_0$ is connected. Then we can define deformations $\zeta_{\pm}$ by the rules:

$$\zeta_+ : S_0 \rightarrow \zeta(S_0) \equiv S_0', \quad \zeta_- : S_0 \rightarrow \zeta^{-1}(S_0) \equiv S_0''.$$ 

We’ll call $S_0$ — zeroth self-intersection, $\zeta_+$ — first direct and $\zeta_-$ — first reverse continuations of $\zeta$. Then we introduce the first direct $S_0 \cap S_0' = S_+$ and first reverse $S_0 \cap S_0'' = S_-$ — intersections and second direct and reverse continuations of $\zeta$ — deformations $\zeta_{\pm}$ :

$$\zeta_{++} : S_+ \rightarrow \zeta(S_+); \quad \zeta_{+-} : S_+ \rightarrow \zeta^{-1}(S_+); \quad \zeta_{-+} : S_- \rightarrow \zeta(S_-); \quad \zeta_{--} : S_- \rightarrow \zeta^{-1}(S_-).$$

Assuming connectedness of $S_{\pm}$ and continuing this procedure, we obtain the chain of self-intersections and corresponding chain of deformational continuations:

$$S_0 \rightarrow \{\zeta_0\} \rightarrow \{S_\pm\} \rightarrow \{\zeta_{n-1}\} \rightarrow \{S_{n-1}\} \rightarrow \{\zeta_n\} \rightarrow \{S_n\},$$

where $\{\alpha_n\}$ denotes collection of $2^n$ binary codes of length $n$ of the kind $1_1 1_2 \ldots 1_n$, $1_k = +, -$. For example, if $S_{\alpha_n}$ — some connected fixed $n$-th self-intersection, then we define by induction:

$$\{\zeta_{\alpha_n} + \} \ni \zeta_{\alpha_n} + : S_{\alpha_n} \rightarrow \zeta(S_{\alpha_n}); \quad \zeta_{\alpha_n} - : S_{\alpha_n} \rightarrow \zeta^{-1}(S_{\alpha_n});$$

$$\zeta_{\alpha_n} = S_{\alpha_n} \cap \zeta(S_{\alpha_n}); \quad \zeta_{\alpha_n} = S_{\alpha_n} \cap \zeta^{-1}(S_{\alpha_n}).$$

Also we get the set of matrices of $n$-th self-intersections as $\{I_{\alpha_n}\} \equiv \{I(\zeta_{\alpha_n}, \zeta_{\alpha_n})\}$.

The following two propositions are basic for classifying of intersected $d$-objects.

**Proposition 3** If $\zeta_{\alpha_n} \in \text{Simp}$, then all continuations of $\zeta_{\alpha_n}$ lie in $\text{Simp}$.

**Proof.** Let $\zeta_{\alpha_n} \in \text{Par}$, then $S_{\alpha_n} = \emptyset$ and all $I_{\alpha_m} = \emptyset_{2 \times 2}$ for $m > n$, so $\zeta_{\alpha_n} \in \text{Sl}(\emptyset) \equiv \text{Par}(\emptyset)$.

Let $\zeta_{\alpha_n} \in \text{Sl}(S_{\alpha_n})$, then $S_{\alpha_n} = S_{\alpha_n-1}$. So, we have $\zeta_{\alpha_n} = S_{\alpha_n-1} \rightarrow S_{\alpha_n-1}$ and $I_{\alpha_m}(\zeta, \zeta) = I_{\alpha_n}(\zeta, \zeta) = S_{\alpha_n-1} \cap \Omega$ for all $m \geq n$.

Let $\zeta_{\alpha_n} \in \text{Str}(S_{\alpha_n})$, then $S_{\alpha_n} = S_{\alpha_n-1}$ and $\zeta_{\alpha_n} \in \text{Str}$, $\zeta_{\alpha_n} \in \text{Ctr}$.

Let $\zeta_{\alpha_n} \in \text{Ctr}(S_{\alpha_n})$, then $S_{\alpha_n} = S_{\alpha_n}(S_{\alpha_n})$ and $\zeta_{\alpha_n} \in \text{Ctr}$, $\zeta_{\alpha_n} \in \text{Str}$.

**Proposition 4** For any $n$ and $k$ $S_{i_1 \ldots i_{k-1} + \ldots i_{k+2} \ldots i_n} = S_{i_1 \ldots i_{k-1} + i_{k+2} \ldots i_n}$.

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$^8$The set $A' \subset A$ is said to be subpseudogroup of pseudogroup $A$, if $A'$ — pseudogroup with respect to composition in $A$. We leave notation $A' \leq A$ from groups theory $^9$.
Proof. Accordingly to inductive definition
\[ \zeta_{i_1\ldots i_{k-1}+} : S_{i_1\ldots i_{k-1}} \to \zeta(S_{i_1\ldots i_{k-1}}); \quad \zeta_{i_1\ldots i_{k-1}-} : S_{i_1\ldots i_{k-1}} \to \zeta^{-1}(S_{i_1\ldots i_{k-1}}). \]
Then
\[ S_{i_1\ldots i_{k-1}+} = S_{i_1\ldots i_{k-1}} \cap \zeta(S_{i_1\ldots i_{k-1}}), \quad S_{i_1\ldots i_{k-1}-} = S_{i_1\ldots i_{k-1}} \cap \zeta^{-1}(S_{i_1\ldots i_{k-1}}). \]
Similarly
\[ \zeta_{i_1\ldots i_{k-1}+-} : S_{i_1\ldots i_{k-1}+} \to \zeta^{-1}(S_{i_1\ldots i_{k-1}+}) ; \quad \zeta_{i_1\ldots i_{k-1}-+} : S_{i_1\ldots i_{k-1}-} \to \zeta(S_{i_1\ldots i_{k-1}-}). \]
Finally, we check
\[ S_{i_1\ldots i_{k-1}+-} = S_{i_1\ldots i_{k-1}+} \cap \zeta^{-1}(S_{i_1\ldots i_{k-1}+}) = \zeta^{-1}(S_{i_1\ldots i_{k-1}}) \cap S_{i_1\ldots i_{k-1}} \cap \zeta(S_{i_1\ldots i_{k-1}}) = S_{i_1\ldots i_{k-1}-+}. \]
So, all continuations of every deformation \( \zeta \) can be depicted by the following commutative branching partially ordered graph \( \Gamma \) of simple self-intersections (Fig.1). Commutativity (convergence of arrows) is guaranteed by proposition \([3]\). Notation \((n, s)\), which is shortening of \( \zeta(n,s) \), includes \( n \) — order of continuation of \( \zeta \) (length of binary code \( \alpha_n \)) and \( s \) — signature of continuation — difference between number of + and − within binary code \( \alpha_n \). Correctness and unambiguity of such notations is again guaranteed by proposition \([3]\). If some arrow \((n_0, s_0)\) belongs to the Simp, then all following arrows \((n, s)\) with \( n > n_0, s_0 - (n - n_0) < s < s_0 + (n - n_0) \) are the simplest accordingly to the proposition \([3]\). So,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{simple_self_intersections.png}
\caption{Graph of simple self-intersections.}
\end{figure}

for every oriented path of graph \( \Gamma \) there are two possible alternatives: either on some step \((n_0, s_0)\) it become simplest, or it can be infinitely prolonged as nonsimplest. In this last case we’ll call order of self-intersection of \( \zeta \) infinite. If any path of \( \Gamma \) become simplest on some step, we say that order of self-intersection of \( \zeta \) is finite. Then we can define type and order of this finite self-intersection, specifying order and type of continued deformation, from which the simplest types begin. Let’s consider some examples.

1. Consider parallel shift of square on \( \mathbb{R}^2 \) along diagonal on its 1/3 part. The deformation and its graph of self-intersection are shown in Fig.2. Beginning with \( n = 3 \) the graph is stabilized and all \( \zeta_{\alpha_3} \) belong to the type Par. So the type of the graph is \((3, \text{Par})\).

2. Let’s consider rotation of square on \( \mathbb{R}^2 \) by angle \( \pi/4 \) around one of its vertexes (Fig.3). Beginning with \( n = 2 \) the graph is stabilized and all \( \zeta_{\alpha_n}, n \geq 2 \) belong to the type Sl(Par). The type of the graph is \((2, \text{Sl}(\text{Par}))\). Sliding set is center of rotation \( \pi/4 \).

3. Consider deformation of \( \mathbb{R}^3 \) in \( \mathbb{R}^2 \), such that \( \mathcal{S}' \) is obtained from \( \mathcal{S} = \mathbb{R}^1 \) by bending \( \mathbb{R}^1 \) at the point 0 and by following constant shift of obtained curve on vector \((a, 0)\) (along \( \mathbb{R}^1 \)) (Fig.4). Easily to
see, that under \( n = 1 \) graph of self-intersection is stabilized. Namely, \( \zeta_+ \in \text{Ctr} \), \( \zeta_- \in \text{Str} \). So, its type is \((1, \text{Ctr}, \text{Str})\).

We have consider the case of simple self-intersections, when every continuation \( \mathcal{S}_{(n,s)} \) is connected. If it is not the case, we need to introduce one additional index \( \gamma_{(n,s)} \), numbering connected components of \( \mathcal{S}_{(n,s)} \) for every pair \((n,s)\):

\[
\mathcal{S}_{(n,s)} = \bigcup_{\gamma_{(n,s)}} \mathcal{S}_{\gamma_{(n,s)}}.
\]

Graph of self-intersection will become more complicated: it acquires additional branching (say in third dimension) due to the possible topological branching of continuation \( \mathcal{S}_{(n,s)} \). However, notions of finite and infinite order of self-intersection remains valid and specifying of finite order and type of self-intersections are well defined.

The more detailed (but more complicated) classification of self-intersections involves analysis of the intersection matrix \( I(\zeta_{(n_1,s_1)}, \zeta_{(n_2,s_2)}) \). We don’t touch this possibility in the present paper.
Let's briefly outline the role of $I(ζ_1, ζ_2)$. Firstly, we observe, that for every $ζ_1 : S_1 → S_2$ and for every $Y_{ζ_1}^{†} : S_2 → S_3$ or $Y_{ζ_1}^{−} : S_3 → S_1$ we have:

$$I(ζ_1, ζ_2) = \left( \begin{array}{cc} S_1 \cap S_2 & S_1 \cap S_3 \\ S_2 & S_2 \cap S_1 \end{array} \right) \text{ or } I(ζ_1, ζ_2) = \left( \begin{array}{cc} S_1 \cap S_0 & S_1 \\ S_2 \cap S_0 & S_2 \cap S_1 \end{array} \right)$$

respectively. It is naturally to call such class of intersection matrices and deformations consequent.

Let we have the pair of consequent deformations $ζ_1 : S_1 → S_1'$ and $ζ_2 : S_1' → S_2$.

**Proposition 5** There is following relations between self-intersection and intersection matrices:

$$I(ζ_1, ζ_1) \cdot I(ζ_2, ζ_2) = (S_1 \cup S_2) \cap S_1' \cdot I(ζ_1, ζ_2),$$

where boolean matrix multiplication is defined as usually (line × column) with the help of boolean operations.

**Proof.** The proposition can be checked directly. □

Particularly, it is follows from (6), that, if $ζ_1$ and $ζ_2$ are both parallel (i.e. $(S_1 \cup S_2) \cap S_1' = \emptyset$), then $I(ζ_1, ζ_1) \cdot I(ζ_2, ζ_2) = \emptyset$.

There is necessary and sufficient matrix criteria for the situation, when two parallel consequent deformations gives parallel composition.

**Proposition 6** Two consequent deformations $ζ_1$ and $ζ_2$ together with their composition $ζ_2 \circ ζ_1$ are parallel, if and only if

$$I(ζ_1, ζ_2) = \begin{pmatrix} \emptyset & \emptyset \\ S_1' & \emptyset \end{pmatrix}$$

**Proof.** Proposition is checked directly in both directions. □

In case of more general situation we have

**Proposition 7** Two consequent deformations give parallel composition, if and only if

$$I(ζ_1, ζ_2) = S_1' \left( \begin{array}{c} S_1 \\ M \end{array} \begin{array}{c} \emptyset \\ S_2 \end{array} \right)$$

**Proof.** Proposition is checked directly in both directions. □

At the end of the subsection we introduce some another special deformations. We'll say, that $ζ : S → S'$ is deformation with invariant (fixed) set $S^3 (S')$, if $ζ|_S : S ∈ S(\tilde{S}) (ζ|_S ∈ S(S\tilde{S})$ for any $S \subseteq S'$).

### 2.4 Homotopies, histories and proper deformations

Let's consider the set $π_H(\mathcal{E})$, consisting of homotopic classes of embeddings $\mathcal{E}$. Here we define strong smooth homotopy of embedding $ι ∈ \mathcal{E}$ as smooth mapping $F : \mathcal{B} × I → \mathcal{M}$, where $I = [0, 1]$, such, that $F(\mathcal{B}, 0) = ι$ and $F(\mathcal{B}, t) = F_1(\mathcal{B}) ∈ \mathcal{E}$ for every $t ∈ I$. The two embeddings $ι$ and $ι'$ are said to be homotopic: $ι ∼ ι'$, if there exist strong homotopy $F$, such that $F_0(\mathcal{B}) = ι$, $F_1(\mathcal{B}) = ι'$. Homotopy relation is equivalence on $\mathcal{E}$ and $π_H(\mathcal{E}) = \mathcal{E} / ∼_H$.

Let's define strong homotopy equivalence on $\mathcal{E} × \mathcal{E}$. We'll say, that $(ι_1, ι_2) ∼_H (ι_1', ι_2')$, if simultaneously $ι_1 ∼ ι_1'$ and $ι_2 ∼ ι_2'$. Obviously, the set of classes of the strong homotopic equivalence $π_H(\mathcal{E} × \mathcal{E}) = π_H(\mathcal{E}) × π_H(\mathcal{E})$.

Now we are able to define some special kinds of deformations in $\text{DEF}_M(\mathcal{B})$, using the homotopy relation. Let's consider the set $π_H^{-1}(\text{Dg}(π_H(\mathcal{E} × π_H(\mathcal{E}))))$, i.e. set of pair of homotopic embeddings. The set, after factorization by $π_D$ becomes the subset $\text{DEF}_M(\mathcal{B})_0 ⊆ \text{DEF}_M(\mathcal{B})$, which we'll call proper deformations. Within the classical (nonquantum) $d$-structures it is naturally to restrict ourself only by this type of deformations. Obviously, $\text{DEF}_M(\mathcal{B})_0$ — subpseudogroup of $\text{DEF}_M(\mathcal{B})$. 

8
For every $\zeta \in \text{DEF}_M(B)_0$ by its definition there exists some history — strong homotopy $F(\zeta)_t$, such that $F(\zeta)_0 = \text{Dom} \zeta = S$, $F(\zeta)_1 = \text{Im} \zeta \equiv \zeta(S) = S'$. The set of all histories of the deformation $\zeta$ we’ll denote $\text{Hist}(\zeta)$ and call class of histories of $\zeta$. It is easily to see, that pseudogroup structure on $\text{DEF}_M(B)$ induces composition law for histories: for every $\zeta_1, \zeta_2, \zeta_3$, such that $\zeta_3 = \zeta_2 \circ \zeta_1$, we put

$$F(\zeta_3) = F(\zeta_2 \circ \zeta_1) \equiv F(\zeta_2) \circ F(\zeta_1),$$

where last equation means standard composition of homotopies $\Box$. Similarly, we can define multiplication of classes $\text{Hist}(\zeta_2) \circ \text{Hist}(\zeta_1) = \text{Hist}(\zeta_2) \times \text{Hist}(\zeta_1) \subset \text{Hist}(\zeta_3)$, consisting of all possible compositions of histories from $\text{Hist}(\zeta_1)$ and $\text{Hist}(\zeta_2)$.

Every $\zeta \in \text{DEF}_M(B)_0$ can be classified by the methods of previous section. We’ll say, that history $F(\zeta)_t$ has type $\zeta$ in a strong sense, if $F(\zeta)_t$ has the same type as $\zeta$ on a whole $I$. Now we can introduce the simplest proper deformations as collection

$$\text{Simp}_0 = \text{Simp} \cap \text{DEF}_M(B)_0$$

with histories of corresponding types in the strong sense. Also, we introduce notions of strongly invariant (fixed) subset $S^i \subseteq S = \iota(B)$ ($S^i \subseteq S = \iota(B)$) relatively $F(\zeta)_t$, if

$$F(\zeta)_t(S^i) = S^i \quad (F(\zeta)_t(s) = s \text{ for all } s \in S^i)$$

for all $t \in I$.

### 2.5 Vector fields, motions and generalized Killing equations

Let’s consider some proper deformation

$$\zeta \in \text{DEF}_M(B)_0 : S = \iota(B) \overset{\zeta}{\rightarrow} S' = \iota'(B)$$

and let $F(\zeta)$ will be its some history. Consider the set $\mathcal{M} \supseteq \mathcal{P}_{F(\zeta)} = \bigcup_{t \in I} F(\zeta)_t(B) \equiv \bigcup_{t \in I} S_t$. It can be treated as image of smooth mapping of the smooth manifold $I \times B \rightarrow \mathcal{M}$, which, generally speaking, is not submanifold and even not immersion in $\mathcal{M}$. We’ll call it trace of history $F(\zeta)_t$ in $\mathcal{M}$. Its boundary $\partial \mathcal{P}_{F(\zeta)}$ is $S \cup S' \cup \bigcup_{t \in I} F_t(\partial B)$. Let $d/dt$ — uniquely determined horizontal vector field on $B \times I$, i.e. such that $d\pi_1(d/dt) = 0$, $d\pi_2(d/dt) = d/dt$, where $\pi_1$, $\pi_2$ — projections of $B \times I$ onto $B$ and $I$ respectively. The trace $\mathcal{P}_{F(\zeta)}$ is composed of an integral lines $\{F(\zeta)_t(b)\}_{b \in B}$ of the vector field $\tau = dF(\zeta)(d/dt)$, defined on $\mathcal{P}_{F(\zeta)}$. The family of embeddings $\{F(\zeta)_t(B)\}_{t \in I}$ induces the family of deformation forms $\{\Delta_B^t\}_{t \in I}$ by the following rule:

$$\Delta_B^t = (dF(\zeta)_t)^* \Theta - (dF(\zeta)_0)^* \Theta.$$

We’ll say, that the history $F(\zeta)_t$ is motion of the $d$-body $B$ in $\mathcal{M}$, if $\Delta_B^t = 0$ for any $t \in I$ or, in other words, if the image $(dF(\zeta)_t)^* \Theta$ is constant on $I$. This notion generalizes a concept of absolutely rigid solids in classical mechanics.

**Proposition 8** History $F(\zeta)_t$ is motion, if and only if

$$L_\tau \Theta|_{\mathcal{P}_{F(\zeta)}} = 0,$$

where $L_\tau$ — Lie derivative along the vector field $\tau = dF(\zeta)(d/dt)$ on $\mathcal{P}_{F(\zeta)}$.}

---

*Of course, one can consider not only proper motion. The generalization is obvious and we don’t touch it in present paper.*
in terms of \( \Delta \):

which we'll call \( \pi \) constant mapping and deformationally discrete, \( \pi \) rally to use as configuration space of deformant not theory by the following

\[ \{ R \}_{\alpha} \]

Now for some \( \alpha \) relatively to its \( (B, \Theta, h) \)-coverings of \( S \), \( S \) additionally we have used independence of \( (dF_{(\zeta)})^* \) for any diffeomorphism \( \alpha \), \( \pi \) for any \( \zeta \) — definition of Lie derivative (note, that the mapping \( F_t \circ F_t^{-1} \) maps \( S_t \rightarrow S_{t+s} \)). Since \( (dF_{(\zeta)})^* \) is nondegenerate under every fixed \( t \), then proposition is proved. \( \square \)

The eq. "1" — definition of derivative, in "2" we have used independence of \( (dF_{(\zeta)})^* \) on \( s \), in "3" — property \( (\zeta) \) and identity \( ((d\alpha)^{-1})^* = (d(\alpha^{-1}))^* \) for any diffeomorphism \( \alpha \), \( \zeta \) for any \( \alpha \) and in "4" - definition of Lie derivative (note, that the mapping \( F_t \circ F_t^{-1} \) maps \( S_t \rightarrow S_{t+s} \)). Since \( (dF_{(\zeta)})^* \) is nondegenerate under every fixed \( t \), then proposition is proved. \( \square \)

The equations \( \{ R \} \) we'll call \emph{generalized Killing equations}, and \( \tau \) — generalized Killing vector field.

### 2.6 \( \delta \)-coverings of \( d \)-manifolds

Let \( S = \iota(B) \) will be some fixed deformant and let \( \text{MOT}_M(S) \) — set of all its possible motions in \( M \). Easily to see, that the motions define equivalence \( M \) on \( E \): we'll call the two embeddings \( \iota \) and \( \iota' \) \( M \)-equivalent: \( \iota \cong \iota' \), if there exist history \( F_{(\zeta)} \in \text{Hist}(\zeta) \), where \( \zeta : S \rightarrow S' \), such that \( F_{(\zeta)} \in \text{MOT}_M(S) \). Obviously, the equivalence \( \cong \) is more weak then \( \sim \) and, generally speaking, the set \( \pi_H^{-1}[\iota]_H = \cup_{\alpha} \pi^{-1}_{M}[\iota\alpha]_M \), where \( \{\iota\alpha\} \) — some set of all pair-wisely \( M \)-nonequivalent elements from \( \pi^{-1}_H[\iota]_H \), and \( \pi^{-1}_{M}[\iota\alpha]_M \cap \pi^{-1}_{M}[\iota\beta]_M = \emptyset \) for all \( \alpha \neq \beta \). We'll call \( \pi^{-1}_H[\iota\alpha]_M \) \( \alpha \)-component of \( \pi^{-1}_H[\iota]_H \), and its image

\[ \mathcal{R}(S_\alpha) = \bigcup_{F_{(\zeta)} \in \text{MOT}_M(S_\alpha)} \mathcal{P}_{F_{(\zeta)}} \]

rigidity \( \alpha \)-component of the manifold \( M \) relatively to the embedding \( \iota \). Here \( S_\alpha = \iota_\alpha(B) \). The family \( \{\mathcal{R}(S_\alpha)\} \) forms some covering of \( M \):

\[ \mathcal{M} = \bigcup_{\alpha} \mathcal{R}(S_\alpha), \]

which we'll call \emph{deformational} \( (B, \Theta, h) \)-\emph{covering of the manifold} \( M \) or, more shortly, \( \delta \)-\emph{covering}, where \( \pi_H(E) \ni h = \pi_H(\iota) \).

Within the classical dynamical \( \delta \)-\emph{structures}, which will be considered in the next sections, it is naturally to use as configuration space of deformant not \( \pi^{-1}_H[\iota]_H \), but its factor:

\[ \{ \pi^{-1}_M[\iota]_H / M \} \equiv \{ \pi_M \circ \pi^{-1}_H[\iota]_H \} \cong \{ \mathcal{R}(S_\alpha) \}, \]

that reflects the deformational indistinguishability of those configurations, that are connected by some motion. It will be automatically provided in second half of the paper by formulation of physical action \( \mathcal{S} \) in terms of \( \delta \): \( \mathcal{S} \sim \delta \mathcal{S} \), so that \( \delta \mathcal{S} \sim \delta \Delta \) — vanishes on motions.

We'll call the manifold \( M \) deformationally trivial relatively to its \( (B, \Theta, h) - \delta \)-covering, if \( \pi_M \) — constant mapping and deformationally discrete, if \( \pi_M \) — identical mapping. The manifold \( M \) will be called deformationally homogeneous \( (d \text{-homogeneous}) \), if

\[ \mathcal{R}(S_\alpha) = M \]

for some \( \alpha \) and \emph{completely deformationally homogeneous}, if \( \{\mathcal{R}\} \) is satisfied for all \( \alpha \).

Deformationally trivial manifolds have no significance from the view point of deformational structure theory by the following
**Proposition 9** Any deformationally trivial manifold has:
1) constant d-metric, if $\Theta \in \Omega^0$;
2) zero d-metric, if $\Theta \in \Omega^p$, $p \neq 0$.

**Proof.** For any of the cases deformational triviality by the proposition 8 means $\mathcal{L}_\tau \Theta = 0$ for all smooth vector fields $\tau$. In case $p = 0$ we have $\mathcal{L}_\tau \Theta = \tau \Theta = 0$ and then, in any coordinate system $\{x^A\}$ on $\mathcal{M}$, taking consequently $\tau = \partial_A$, $A = 1, \ldots, n$ we obtain $\partial_A \Theta = 0$, $A = 1, \ldots, n \Rightarrow \Theta = \text{const}$.

In case $p \neq 0$ we have in coordinates:

$$
(\mathcal{L}_\tau \Theta)_{A_1 \ldots A_p} = \tau^B \partial_B \Theta_{A_1 \ldots A_p} + \sum_{j=1}^p (\partial_A \tau^B) \Theta_{A_1 \ldots B \ldots A_p} = 0
$$

(9)

Take as previously $\tau = \partial_A$, $A = 1, \ldots, n$ consequently and obtain that $\Theta$ is constant form (so the first term in the right-hand side of (9) vanishes). Since coordinate system is arbitrary, we conclude, that $\Theta \equiv 0$. □

Riemannian manifold with general metrics $\eta$ is an example of deformationally discrete manifold. Euclidean space $\mathbb{E}^n$ is completely deformationally homogeneous relatively any $(B, \eta, h)$-decomposition, where $\eta$ — Euclidean metric, $B$ — arbitrary d-body, $h$ — arbitrary element $\pi_H(\mathcal{E})$. As an example of deformationally homogeneous, but not completely deformationally homogeneous manifold consider the following situation. Let $\mathcal{M} = \overline{D_\eta^2(0)} \setminus D_\eta^2(0)$ — closed ring on 2D Euclidean plane (as usually, $D_\eta^p(a)$ — $n$-dimensional disk with radius $r$ and center $a$, bar above letter — topological closure), $\Theta = \eta$ — 2D Euclidean metrics, $B = S^1$, $\iota(S^1) = S^1_0 \subset \mathcal{M}$ — circle with radius $R$ and $\pi_H(\iota) = 1$ (for the considered case $\pi_H(\mathcal{E}) \equiv \pi_1(\mathcal{M})$ — fundamental group of $\mathcal{M}$, isomorphic $\mathbb{Z}$.) Then, in case $R < 3r/2$, $\mathcal{R}(S^1_0) = \overline{D_{2R-r}(0)} \setminus D_r^2(0) \neq \mathcal{M}$ and only in case $R = 3r/2$, $\mathcal{R}(S^1_0) = \mathcal{M}$.

Now we formulate two propositions and give an example, all illustrating relation of a free deformational structure theory with isometries of Riemannian spaces.

Let $\text{St}(v) \equiv \{m \in \mathcal{M} | \psi_t(m) = m\}$ will be the set of all stationary points of one-parametric group $\psi_t$, generated by some smooth vector field $v \in T\mathcal{M}$.

**Proposition 10** If manifold $\mathcal{M}$ admits isometry of d-metrics, i.e. if there exists vector field $v \in T\mathcal{M}$, such that $\mathcal{L}_v \Theta = 0$, then $\forall \mathcal{S}$ such that $\mathcal{S} \not\subseteq \text{St}(v)$, there exists nonidentical motion $\psi_{t|\mathcal{S}} \in \text{MOT}_{\mathcal{M}}(\mathcal{S})$ and, by the fact, $\mathcal{M}$ is not deformationally discrete.

**Proof.** The proposition immediately follows from the relation: $\mathcal{L}_v \Theta = 0 \Rightarrow \mathcal{L}_{\tilde{v}} \Theta|_{\mathcal{P}_\mathcal{P}} = 0$, where $\mathcal{S}_t = F_t(\mathcal{B}) \equiv \psi_t|_{\mathcal{S}_t}$, $\tilde{v} \equiv v|_{\mathcal{P}_\mathcal{P}}$. □

**Proposition 11** If manifold $\mathcal{M}$ admits $r$-parametric isometry group $\mathcal{G}$, generated by vector fields $\{v_1, \ldots, v_r\}$, such that $\mathcal{L}_{v_i} \Theta = 0$, $i = 1, \ldots, r$, that acts on $\mathcal{M}$
1) transitively, then $\mathcal{M}$ — completely deformationally homogeneous (relatively any decomposition);
2) intransitively, and if also $\mathcal{S} \cap \text{Orb} \mathcal{G} —$ connected for some orbit $\text{Orb} \mathcal{G}$, then
a) $\text{Orb} \mathcal{G} —$ completely deformationally homogeneous relatively its $(\iota^{-1}(\mathcal{S} \cap \text{Orb} \mathcal{G}), \Theta|_{\text{Orb} \mathcal{G}}, h = \pi_H(\iota))$-decomposition.
b) $\mathcal{G} \subseteq \text{MOT}_{\mathcal{M}}(\mathcal{S} \cap \text{Orb} \mathcal{G})$.

Here, as usually, $\mathcal{S} = \iota(\mathcal{B})$.

**Proof.** 1) Taking any $\mathcal{S}$ and acting by $\mathcal{G}$, we get (by the transitivity property):

$$
\bigcup_{g \in \mathcal{G}} g(\mathcal{S}) = \mathcal{M}.
$$

2) (a) follows from transitivity property of $\mathcal{G}$ on $\text{Orb} \mathcal{G}$. (b) is obvious. □

Particularly, if $\mathcal{S} = \iota(\mathcal{B}) = \text{Orb} \mathcal{G}$, then

$$
\text{MOT}_{\mathcal{M}}(\mathcal{S}) \cap \text{SI}(\mathcal{S}) \neq \emptyset
$$

defines the group of rigid proper slidings.
So, if \( \Theta \) — Riemannian (or any other \( d \)-) metric on \( \mathcal{M} \) and \( \mathcal{M} \) admits isometry, then nontrivial motions of \( d \)-objects always exist. The following example shows, that inverse is not always valid.

Let \( \mathcal{M} = \mathbb{R}^2 \) with cartesian coordinate system \( \{x_1, x_2\} \), \( \mathcal{B} = I = [0, 1] \in \mathbb{R}, \Omega^1(\mathbb{R}^2) \ni \Theta = (x^1x^2 + \coth x^2)dx^1 \). Let \( \iota(\mathcal{B}) = \mathcal{S} = \{0 \leq x^1 \leq 1, x^2 = 0\} \). By the fact, that \( \Theta|_{x^2=0} = dx^1 = \text{const} \), it is easily to see that the set of homotopies

\[
\{F_t : S \to S_t = (x_1 + t, 0), 0 \leq x_1 \leq 1, -\infty < t < \infty\},
\]

(they are simple rigid translations of units interval along axe \( x^1 \)) lies in \( \text{MOT}_\mathcal{M}(\mathcal{S}) \). Moreover, \( \mathcal{P}_F = \mathbb{R}^1 = \{(x^1, 0)\} \). The related vector field \( \tilde{v}(t, x^1) \), along which \( \mathcal{L}_\tilde{v} \Theta|_{\mathcal{P}_F} = 0 \) is simply \( \partial/\partial x^1 \). It is easily to show, that \( \tilde{v} \) doesn’t admit smooth continuation \( v \) from \( \mathcal{P}_F \) on a whole \( \mathbb{R}^2 \). Really, Killing equations \( \mathcal{L}_v \Theta = 0 \) for this case (under restriction \( v|_{\mathcal{P}_F} = \partial/\partial x^1 \)) ultimately give:

\[
v^2 = -\frac{x^2}{x^1 + \sinh x^2}.
\]

The component has singularity on line \( x^1 = -\sinh x^2 \), which cross any neighborhood of \( \mathcal{P}_F \) in \( \mathbb{R}^2 \).

We’ll call the set

\[
\text{MOT}_\mathcal{M}(\mathcal{B}) = \bigcup_{\iota \in \mathcal{E}} \text{MOT}_\mathcal{M}(\iota(\mathcal{B}))
\]

motions of \( d \)-body in \( \mathcal{M} \). Obviously, \( \text{MOT}_\mathcal{M}(\mathcal{B}) < \text{DEF}_\mathcal{M}(\mathcal{B}) \).

2.7 Conformal motions

Similarly to Riemannian geometry we also define more general (then motions) histories — conformal motions. Infinitesimally, they are defined by the equation:

\[
\frac{d}{dt} \left((dF_{(\zeta)}^*\Theta)\right) = \varphi \cdot (dF_{(\zeta)}^*\Theta),
\]

where \( \varphi : \mathcal{B} \times I \to \mathbb{R} \) — some scalar function. Using calculations similar to proof of proposition 8, it is easily to show, that (10) is equivalent to the following generalized conformal Killing equations:

\[
\mathcal{L}_\tau (\Theta|_{\mathcal{P}_F}) = \varphi \Theta|_{\mathcal{P}_F},
\]

where \( \tau \) — generalized conformal vector field. We’ll denote all possible histories with initial embedding \( \mathcal{S} \), satisfying (10), \( \text{CMOT}_\mathcal{M}(\mathcal{S}) \) and set of such histories for all \( \mathcal{S} \) \( \text{CMOT}_\mathcal{M}(\mathcal{B})_{\varphi} \). Obviously, that \( \text{MOT}_\mathcal{M}(\mathcal{B}) \leq \text{CMOT}_\mathcal{M}(\mathcal{B})_{\varphi} \leq \text{DEF}_\mathcal{M}(\mathcal{B})_0 \) and \( \text{MOT}_\mathcal{M}(\mathcal{B}) = \text{CMOT}_\mathcal{M}(\mathcal{B})_0 \). Similarly to the case of motions, we can define conformal deformational \((\mathcal{B}, \Theta, h)\)—covering of the manifold \( \mathcal{M} \), and conformal generalizations of \( d \)-trivial, \( d \)-discrete and (completely) \( d \)-homogeneous manifolds.

2.8 \( d \)-substructures, compositions and polymetric \( d \)-structures

Lets define isomorphism between free \( d \)-structures. The two \( d \)-structures

\[
\mathcal{D} = (\mathcal{B, M, E, \Theta}) \text{ and } \mathcal{D}' = (\mathcal{B}', \mathcal{M}', \mathcal{E}', \Theta')
\]

will be called isomorphic, if there exist diffeomorphisms

\[
\Psi : \mathcal{B}' \to \mathcal{B}, \Phi : \mathcal{M}' \to \mathcal{M},
\]

such that

\[
\Psi(\mathcal{B}') = \mathcal{B}; \Phi(\mathcal{M}') = \mathcal{M}; (d\Phi)^*\Theta = \Theta'; \mathcal{E}' = \Phi^{-1} \circ \mathcal{E} \circ \Psi.
\]
Proposition 12 Isomorphic \( d \)-structures have isomorphic pseudogroups of deformations and motions\(^{10}\).

**Proof.** Isomorphism

\[ \zeta : \text{DEF}_M(B) \to \text{DEF}_M'(B') \]

is given by the relations:

\[ \zeta(\zeta) = \Phi^{-1} \circ \zeta \circ \Phi, \quad S' = \Phi^{-1}(S_i), \quad i = 1, 2 \]

for every \( \text{DEF}_M(B) \ni \zeta : S_1 \to S_2 \).

If \( F_i \in \text{MOT}_M(B) \), then \( F'_i = \Phi^{-1} \circ F_i \circ \Phi \in \text{MOT}_{M'}(B') \), since

\[ \frac{dt}{dt} \{(dF'_i)^\ast \Theta'\} = \frac{dt}{dt} \{(d(\Phi^{-1} \circ F_i \circ \Phi))^\ast (d\Phi)^\ast \Theta\} = \frac{dt}{dt} \{(dF_i \circ \Phi)^\ast \Theta\} \]

Since \( (d\Phi)^\ast \) — isomorphism, then also

\[ F'_i \in \text{MOT}_{M'}(B') \Rightarrow F_i = \Phi \circ F'_i \circ \Phi^{-1} \in \text{MOT}_M(B), \]

and so \( \text{MOT}_{M'}(B') \cong \text{MOT}_M(B) \). \( \Box \)

We’ll call \( d \)-structure \( \mathcal{D}' = \langle B', M', \mathcal{E}', \Theta' \rangle \) \( d \)-substructure of \( \mathcal{D} = \langle B, M, \mathcal{E}, \Theta \rangle \), if \( B' \subseteq B \) is embedding of \( B' \) in \( B \) and (and) \( M' \subseteq M \) is embedding of \( M' \) in \( M \) and \( \Theta' = \Theta|_{M'} \lor (and) \mathcal{E}' \subseteq \mathcal{E} \). In case "or" some components of \( d \)-structures may be identical. We shall denote this situation as \( \mathcal{D}' \preceq_X \mathcal{D} \), where \( X \) shows restricted elements of \( \mathcal{D} \), for example \( \mathcal{D}' \preceq_B \mathcal{D} \).

**Proposition 13** In case \( \mathcal{D}' \preceq_{B'} \mathcal{D} \) there is homomorphism \( \alpha : \text{DEF}_M(B) \to \text{DEF}_M'(B') \). In case \( \mathcal{D}' \preceq_{M'} \mathcal{D} \) or \( \mathcal{D}' \preceq_{E} \mathcal{D} \), \( \text{DEF}_M'(B) \leq \text{DEF}_M(B) \).

**Proof.** In case \( \mathcal{D}' \preceq_{B'} \mathcal{D} \) homomorphism \( \alpha \) acts by the rule:

\[ \text{DEF}_M(B') \ni \alpha(\zeta) = \alpha([[(t_1, t_2)_D]]) = [[[t_1|B', t_2|B']]|_{B'}], \quad \forall \zeta \in \text{DEF}_M(B), \]

where \( D' \) means factorization by \( \text{Diff}(B)_{B'} < \text{Diff}(B) \) with invariant submanifold \( B' \subseteq B \). In case \( \mathcal{D}' \preceq_{M'} \mathcal{D} \) it is obviously, that if \( \zeta' \in \text{DEF}_{M'}(B) \), then necessarily \( \zeta' \in \text{DEF}_M(B) \). Third case is obvious. One example of the case we already have faced with: *proper substructure* \( \mathcal{D}' \preceq_{E} \mathcal{D} \) with \( \text{DEF}_M(B) \preceq \text{DEF}_M(B) \). \( \Box \)

We’ll say that free \( d \)-structure \( \mathcal{D} = \langle B, M, \mathcal{E}, \Theta \rangle \) is *composition* of the free \( d \)-structures

\[ \mathcal{D}_1 = \langle B_1, M_1, \mathcal{E}_1, \Theta_1 \rangle \text{ and } \mathcal{D}_2 = \langle B_2, M_2, \mathcal{E}_2, \Theta_2 \rangle: \mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2, \]

if

\[ B = B_1 \times B_2, \quad M = M_1 \times M_2, \quad \mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2, \quad \Theta = (d\pi_1)^\ast \Theta_1 \otimes (d\pi_2)^\ast \Theta_2, \]

where \( \pi_1 \) and \( \pi_2 \) — projections of \( M_1 \times M_2 \) onto multipliers.

**Proposition 14** For composite \( d \)-structure \( \mathcal{D} \)

\[ \text{DEF}_M(B) = \text{DEF}_{M_1}(B_1) \times \text{DEF}_{M_2}(B_2), \quad \text{MOT}_M(B) = \text{CMOT}_{M_1}(B_1)_\varphi \times \text{CMOT}_{M_2}(B_2)_{-\varphi}, \]

where \( \varphi = \text{const}_B \).

\(^{10}\) As in case of groups we define *homomorphism between pseudogroups* \( A_1 \) and \( A_2 \) as a mapping \( \alpha : A_1 \to A_2 \), such that for every \( a_1, a_2, a_3 \in A_1 \) connected by the relation \( a_1 \ast a_2 = a_3 \) takes place relation for images \( \alpha(a_1) \ast \alpha(a_2) = \alpha(a_3) \), where \( \ast \) in last expression — pseudogroup multiplication in \( A_2 \). Also, we define *left* and *right kernels* of homomorphism for element \( a \) as the following subsets of \( A_1 \):

\[ \begin{align*}
\text{ker}_{L}^\alpha a & \equiv \{ b \in A_1 \mid (b, a) \in U_1, \quad \alpha(b) = \epsilon_{\alpha(a)}^- \}, \\
\text{ker}_{R}^\alpha a & \equiv \{ b \in A_1 \mid (a, b) \in U_1, \quad \alpha(b) = \epsilon_{\alpha(a)}^+ \}.
\end{align*} \]

In case \( \text{ker}^\alpha_{L,R} a = \epsilon^\alpha_{\alpha(a)} \forall a \in A \), \( \alpha \) — isomorphism of pseudogroups.
Proposition 15. Note, that any pair

\[(\zeta_1, \zeta_2) \in \text{DEF}_{M_1}(B_1) \times \text{DEF}_{M_2}(B_2)\]
defines unique deformation

\[\zeta \in \text{DEF}_{M}(B) : S \rightarrow S',\]

where \(S = S_1 \times S_2, \quad S' = S'_1 \times S'_2\). Inversely, any \(\zeta \in \text{DEF}_{M}(B)\) determines unique \(\zeta_1 = \pi_1 \zeta\) and \(\zeta_2 = \pi_2 \zeta\). We only need to restrict general diffeomorphisms \(\text{Diff}(B)\) on its subgroup \(\text{Diff}(B_1) \times \text{Diff}(B_2)\), conserving \(B_1\) and \(B_2\) in \(B\) and consistent with product structure of \(E\), when define \(\text{DEF}_{M}(B)\) as factor \(E \times E / B\).

To clear out relation between motions pseudogroups, lets calculate the derivative:

\[
\frac{d}{dt} \{(dF_i)^*\Theta\} = \frac{d}{dt} \{(dF_i)^*(d\pi_1)^*\Theta_1 \otimes (dF_i)^*(d\pi_2)^*\Theta_2\},
\]

where \(F_i\) — some history of some deformation \(\zeta = (\zeta_1, \zeta_2)\) in \(M\). Using composition property \(\Theta\), relations \(\pi_1 F_i = F_{1t}, \quad \pi_2 F_i = F_{2t}\), where \(F_{1t} \in \text{Hist}(\zeta_1), \quad F_{2t} \in \text{Hist}(\zeta_2)\) and Leibnitz rule we obtain:

\[
\frac{d}{dt} \{(dF_i)^*\Theta\} = \frac{d}{dt} \{(dF_{1t})^*\Theta_1 \otimes (dF_{2t})^*\Theta_2\} + (dF_{1t})^*\Theta_1 \otimes \frac{d}{dt} \{(dF_{2t})^*\Theta_2\}.
\]

It is easily to see, that if

\[
\frac{d}{dt}((dF_{1t})^*\Theta_1) = \varphi \cdot (dF_{1t})^*\Theta_1, \quad \frac{d}{dt}((dF_{2t})^*\Theta_2) = -\varphi \cdot (dF_{2t})^*\Theta_2,
\]

then previous equations are satisfied and so \(\text{CMOT}_{M_1}(B_1) \times \text{CMOT}_{M_2}(B_2) - \varphi \subseteq \text{MOT}_{M}(B)\). Inversely, let \(d/dt((dF_i)^*\Theta) = 0\). It means, that for any sets of vector fields \(\{u, v\}\) on \(B\): \(u = \{u_1, \ldots, u_{p_1}\} \in (TB_1)^{\times p_1}, \quad v = \{v_1, \ldots, v_{p_2}\} \in (TB_2)^{\times p_2}\), where \(p_1 = \deg \Theta_1, \quad p_2 = \deg \Theta_2\), we have \(d/dt((dF_i)^*\Theta)(u, v) = 0\). Lets denote \((dF_{1t})^*\Theta_1(u) = f_1(u, t), \quad (dF_{2t})^*\Theta_2(v) = f_2(v, t)\). Then

\[
d/dt((dF_i)^*\Theta)(u, v) = \dot{f}_1(u, t)f_2(v, t) + f_1(u, t)\dot{f}_2(v, t) = \frac{d}{dt}(f_1 f_2) = 0,
\]

that gives \(f_1 f_2 = \text{const} \) under any \(u, v\). Then it follows, that

\[
f_1(u, t) = \dot{f}_1(u) \alpha(t), \quad f_2(v, t) = \dot{f}_2(v)/\alpha(t).
\]

Coming back to codifferential, omitting arguments \(u, v\) by its arbitrariness and taking derivatives over \(t\) we obtain

\[
\frac{d}{dt}((dF_{1t})^*\Theta_1) = \varphi(t)(dF_{1t})^*\Theta_1; \quad \frac{d}{dt}((dF_{2t})^*\Theta_2) = -\varphi(t)(dF_{2t})^*\Theta_2,
\]

with \(\varphi(t) = \alpha'/\alpha\). We find that \(\text{MOT}_M(B) \subseteq \text{CMOT}_{M_1}(B_1)_\varphi \times \text{CMOT}_{M_2}(B_2)_{-\varphi}\) with \(\varphi = \varphi(t)\) and so, ultimately, \(\text{MOT}_M(B) = \text{CMOT}_{M_1}(B_1)_\varphi \times \text{CMOT}_{M_2}(B_2)_{-\varphi}\) with \(\varphi = \varphi(t)\). \(\square\)

There is direct generalization of proposition 14.

Proposition 15. For multicomponent d-structure \(D = D_1 \times \ldots \times D_n\)

\[
\text{DEF}_{M}(B) = \bigoplus_{i=1}^{n} \text{DEF}_{M_i}(B_i); \quad \text{MOT}_{M}(B) = \bigoplus_{i=1}^{n} \text{MOT}_{M_i}(B_i)_\varphi;
\]

with \(\varphi_i = (\text{const})_i\) and \(\sum_{i=1}^{n} \varphi_i = 0\).
Note, that in case of composed $d$-structure $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ there is two independent $d$-metrics on $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$: $(d\pi_1)^* \Theta_1$ and $(d\pi_2)^* \Theta_2$, which we have used for constructing universal $d$-metric on $\mathcal{M}$ with “good” properties. The similar situation arises in a more general case, when $d$-manifold possess two (or more) metrics. We’ll call $d$-structure with the set of metrics $\{\Theta_\alpha\}$ on the same $d$-manifold $\mathcal{M}$ *polymetric $d$-structure*. If we consider all of $\Theta_\alpha$ as $d$-metrics, then every $\Theta_\alpha$ burns its own pseudogroup of motions $\text{MOT}_{\mathcal{M}}(\mathcal{B})_{\alpha}$. We can introduce partial order on the set $\{\Theta_\alpha\}$. Namely we define $\Theta_\alpha \preceq \Theta_\beta$, if $\text{MOT}_{\mathcal{M}}(\mathcal{B})_{\alpha} \supseteq \text{MOT}_{\mathcal{M}}(\mathcal{B})_{\beta}$. We’ll say, that in this case the metric $\Theta_\alpha$ is *weaker* then $\Theta_\beta$. Obviously, that if $d$-metric $\Theta_\alpha$ is weaker then $\Theta_\beta$, then $d$-covering induced by $\Theta_\alpha$ is submitted to the $d$-covering, induced by $\Theta_\beta$.

Generally speaking, it is not necessary to consider all metrics from $\{\Theta_\alpha\}$ as $d$-metrics (see example in Sec.5.4). Some of them can be used as $g$-metrics (see Sec.5.4).

### 2.9 Physical realizations of free $d$-structures

Any smooth form $\Theta \in \Omega^{\otimes p}$ defined on arbitrary manifold $\mathcal{M}$ can be viewed as $d$-metric, if one specifies some $d$-body $\mathcal{B}$. So, any $\mathcal{M}$, supported by smooth form can be transformed into some $d$-structures. In physical applications the most often case is $\mathcal{B} = \mathcal{M}$. In this case pseudogroup $\text{DEF}_{\mathcal{M}}(\mathcal{B})$ become *group of deformations of $\mathcal{M}$*, while $\text{MOT}_{\mathcal{M}}(\mathcal{B})$ — its *subgroup of motions of $d$-metric $\Theta$*. Direct physical realizations of the such $d$-structures are following:

1. $\mathcal{M} = \mathcal{B} = \mathbb{E}^3$ — 3D Euclidean space, $\Theta = \eta$ — Euclidean metrics. In this case we obtain kinematics of standard elasticity theory (where $\Delta/2$ — standard strain tensor).

2. $\mathcal{M} = \mathcal{B} = \mathcal{M}_4$ — pseudoeuclidian Minkowski space, $\Theta$ — Minkowski metrics. Such $d$-structure realizes relativistic generalization of 3D elasticity theory $\mathbb{R}$. Here $\text{MOT}_{\mathcal{M}_4}(\mathcal{M}_4) = P^+_+ (1, 3)$ — Poincare group with homogeneous subgroup of proper orthochronal Lorentz transformations.

3. $\mathcal{M} = \mathcal{B} = \mathbb{V}_4$ — arbitrary 4D Riemannian manifold with Riemannian metric $g = \Theta$. Here we have generally covariant 4D elasticity theory, which takes into account gravitational field. The $\text{MOT}_{\mathbb{V}_4}(\mathbb{V}_4)$ is isometry group of $\mathbb{V}_4$. Note, that while in two previous cases $\mathcal{M}$ — completely deformationally homogeneous, $\mathbb{V}_4$ is, if and only if it is homogeneous (in common sense).

Let’s note also the following less direct and obvious realization of free $d$-structures.

4. $\mathcal{M} = 2n$-dimensional phase space of some dynamical system with canonical symplectic 2–form $\omega = \Theta$ $\mathbb{R}_\mathbb{R}$. If $d$-body $\mathcal{B} \subseteq \mathcal{M}$ — some closed subset of initial data, then $\text{MOT}_\mathcal{M}(\mathcal{B})$ — Hamiltonian phase flow, going through $\mathcal{B}$, which is (locally) generated by some Hamiltonian function $h$ (while $\text{DEF}_\mathcal{M}(\mathcal{B})_0$ — group of arbitrary proper diffeomorphisms of $\mathcal{M}$, generally speaking, changing form $\omega$ (see Sec.5.4)). Dynamical systems with constraints $\{f_a (p, q) = 0\}$ are described by $d$-bodies — submanifolds of $\mathcal{M}$ and in this case

$$\text{MOT}_\mathcal{B}(\mathcal{B}) = \text{MOT}_\mathcal{M}(\mathcal{B}) \cap \text{Sl}(\mathcal{B})$$

and is defined by equation of motion for constraints

$$\{f_a, h\} = \sum_a c_a f_a,$$

where $\{ , \}$ — Poissons’s brackets. Quite different physical interpreting of symplectic $d$-structures, $\text{MOT}_\mathcal{M}(\mathcal{B})$ and $\text{DEF}_\mathcal{M}(\mathcal{B})_0$ we’ll consider within dynamical case in Sec.5.4.

5. Let’s $P(\mathcal{B}, G)$ — bundle space with base $\mathcal{B} \xrightarrow{\pi} P$, canonical projection $\pi$, and structural group $G$. Connection on $P$ can be defined by the 1-form $\Theta \in \Omega(\pi, g)$, which maps vector fields on $\pi$.
into Lie algebra \( g \) of \( G \). So, \( \text{MOT}_P(P) \) will consist of all such deformations, which leave \( \Theta \) invariant. Note, that \( \text{MOT}_P(P) \) is always nonempty, since under vertical diffeomorphisms \( P \times G \rightarrow P \) \( \Theta \) is invariant by its definition. If there are additional deformations, which conserve \( \Theta \), it is said, that \( \Theta \) is \textit{invariant connection}. Summary of some results on invariant connections for the case \( \text{MOT}_P(P) \) can be found in \( [1] \). Our approach requires more general consideration in the case \( \text{MOT}_P(B) \) for \( B \neq P \).

Another form, appearing in bundle space is 2-form of curvature: \( \Omega \equiv d\Theta + \Theta \wedge \Theta \). So, formally, we could view on \( \langle P, P, \text{Diff}(P), \{\Theta, \Omega\} \rangle \) as bimetric structure, but easily to show, that \( \Omega \leq \Theta \).

6. Let \( \mathcal{M} \) — space of all thermodynamical parameters, \( \Theta \equiv Q \) — heat form, \( S \equiv B \subset \mathcal{M} \) — some thermodynamical system, described by some set of equations of state \( \{\varphi_\alpha = 0\} \). In other words, \( B \) is some admissible submanifold in \( \mathcal{M} \), which the system can evolve along. Then, \( \text{DEF}_\mathcal{M}(\mathcal{M})_0 \) describes arbitrary continuous evolutions of thermodynamical parameters — some given thermodynamical processes, \( \text{DEF}_\mathcal{M}(B)_0 \), generally speaking, describes continuous changings of properties of the thermodynamical system (parameters of states equation); \( \text{MOT}_\mathcal{M}(\mathcal{M}) \) — continuous processes, conserving heat power, \( \text{MOT}_\mathcal{M}(\mathcal{M}) \cap \text{SI}(B) \) — continuous variations of state of the system, which conserves heat power, \( \text{MOT}_\mathcal{M}(B) \), — continuous variations of properties, conserving heat power. Continuous quasistatic changings of state of the system are exactly proper slidings \( \text{SI}(B) : B \rightarrow B \).

3 Dynamical deformational structures

3.1 Definitions

We have developed the theory of free deformational structures, containing some kinematical aspects of the deformational approach. To consider dynamics it is necessary to supply a free structure \( \mathfrak{D} \) with some \textit{variational principle} \( \mathfrak{A} \). We define \( \mathfrak{A} \) as the triad \( (\mathcal{F}, \mu, \Gamma) \), where \( \mathcal{F} : \Omega^{\otimes p}(B) \times B \rightarrow \mathbb{R} \) — scalar energy density, \( \mu \) — some volume measure on \( B \), \( \Gamma \) — boundary conditions collection. We’ll call \( (\mathfrak{D}, \mathfrak{A}) \) dynamical \textit{d-structure} or simply \textit{d-structure}. Lets discuss every of \( \mathfrak{A} \) components separately.

1. Within standard continuum media physics dependence of \( \mathcal{F} \) on deformable bodies properties, on deformations and on external conditions is defined by, a so called, \textit{material or definitional relation} \( \mathfrak{A} \) and specified either by experiments or by some theoretical considerations, such as \textit{reference frame independence} (see \( [23] \)). In present paper we restrict ourself by those \( d \)-bodies, whose definitional relation 1) does not depend on "past prehistory" of deformations and 2) admits the separation:

\[
\mathcal{F} = \mathcal{F}_0 + \tilde{U},
\]

where \( \mathcal{F}_0 \) — \textit{elastic part} — depends only on deformation measure, \( \tilde{U} : B \rightarrow \mathbb{R} \) — \textit{external potential part} — does not depend on deformation form \( \mathcal{F}_0 \). In analogy with similar common bodies, satisfying the condition 1, we’ll call such \( d \)-bodies \textit{elastic} and satisfying condition 2 — \textit{simple} and corresponding \textit{d-structures} — \textit{elastic} and \textit{simple} respectively. We’ll say, that \textit{d-structure is closed}, if \( \tilde{U} \equiv \text{const}_{B \times I} \), and is \textit{open}, if \( \tilde{U} \neq \text{const}_{B \times I} \). Everywhere below we’ll consider \( \mathcal{F}_0 = \mathcal{F}_0(\Delta_B, \tilde{\Delta}_B) \), that also restricts a wide class of \textit{minimal d-structures} within more general \textit{high ones}, where \( \mathcal{F}_0 \) can depend on high derivatives of \( \tilde{\Delta}_B \).

2. Although the mapping \( (d\mu)^* \) induces form \( (d\mu)^* \Theta \equiv \Theta_B \) on \( B \), there is a problem to build local volume form \( d\mu \) and scalar \( \mathcal{F}_0(\Delta) \), besides the case \( \text{p} = 2 \). We have the following two alternatives: 1) try

\[\text{In } [3], \text{ material relations were defined as expressing instant stress tensor } \sigma^t \text{ through prehistory of system } F_{t'}, \text{ } t' \leq t. \text{ There is no importance how the relation is defined: by } \sigma^t[F_{t'}] \text{ or by local energy } \mathcal{F}[F_{t'}] \text{ in view of the relation } \sigma^t = \delta \mathcal{F}/\delta \Delta^t. \text{ The second will be more convenient for us.} \]

\[\text{In fact, } \tilde{U} = \iota\iota(U), \text{ where } U : \mathcal{M} \rightarrow \mathbb{R}, \iota \in \mathcal{E}, \text{ — potential energy of } B \text{ in external fields on } \mathcal{M}, \text{ which can possess by their own deformational dynamics.} \]

\[\text{Even under } p = 2 \text{ one should check, that } \det \Theta_B \neq 0 \text{ (see Appendix A).} \]
to define somehow form $d\mu$ and scalar $F_0(\Delta)$ in terms of $\Theta_B$ in case of general forms $\Theta_B \in \Omega^\otimes p(B)$:
2) to define $d\mu$ and (or) $F_0(\Delta)$ on $B$ independently on $d$-metrics $\Theta_B$. $d$-structures, realizing the first alternative will be called internal, second — external. Present paper will be mainly concerned with (more economical) internal $d$-structures (see Sec.2.1 and Appendix C). For future purposes we’ll call the metrics, which define scalar products and (or) volume form $g$-metrics, in difference with $d$-metrics, defining deformation measure.

3. Boundary conditions we’ll discuss and specify after derivation of Euler-Lagrange equations in Sec.3.3.

3.2 Static and evolutional cases

In applications of the approach to different physical systems we’ll be faced with the two types of dynamical deformational structures — static and evolutional. To differ them we introduce special index $\epsilon$, which takes value ”1” in case of evolutional structures and ”2” — in case of static ones. Variational functional $\mathfrak{F}$ can be written then as the following universal expression:

$$\mathfrak{F}[\Upsilon_\epsilon] = \int_{\tilde{C}_\epsilon} (F_0(\Delta_\epsilon) + \tilde{U}) \, d\mu_\epsilon,$$

(11)

where in notations of Sec.2.1, 2.4, 3.1

$$\Upsilon_1 = F_1(B), \ C_1 = B \times I, \ \Delta_1 = \{\Delta^1_B, \tilde{\Delta}^1_B\}, \ d\mu_1 = (d\pi_2)^* e(t) dt \wedge (d\pi_1)^* d\mu,$$

$$\Upsilon_2 = F_2(B), \ C_2 = B, \ \Delta_2 = \tilde{\Delta}_B, \ d\mu_2 = d\mu.$$

Here $e(t)$ — some ”metric” on $I, \ \pi_1, \pi_2$ — projections of $B \times I$ on the first and second multipliers respectively. In other words, in case $\epsilon = 1$ we find minimum of $\mathfrak{F}[F_1]$ and vary evolution $F_1$, while ”ends points” $\{F_{\partial I}\}$ hold fixed. In case $\epsilon = 2$ we find minimum of $\mathfrak{F}[\Upsilon'], \ \Upsilon'$ varying final embedding $\Upsilon'$, while initial embedding $\epsilon$ hold fixed.

3.3 Internal $d$-structures and $g$-metrics

The fact of existence of scalar density $F_0(\Delta)$ and variational functional (11) put some restrictions on possible kinds of $d$-metrics $\Theta_B$. Within internal $d$-structures this metric merely should:

1) admit the isomorphism $\Omega^\otimes p(B) \to V^\otimes p(B)$, where $V^\otimes p(B) \subset T(0, p)(B)$ — subspace of contravariant tensor fields of valency $p$ ($p$-vectors). With the help of the isomorphism we are able to build from $\Delta$ scalars of type $(\Delta, \tilde{\Delta}) \equiv \langle \Delta, \tilde{\Delta} \rangle$, where $\tilde{\Delta} \in V^\otimes p$;

2) provide possibility for constructing of invariant volume form $d\mu = \varpi \, d\xi^1 \wedge \ldots \wedge d\xi^d$, where $\varpi = \varpi(\Theta)$ — scalar density of weight $-1$ with respect to coordinate diffeomorphisms on $B$.

In Appendices (A)-(C) we generalize standard square matrix calculus on arbitrary form of even degree $p = 2k$. The results are following:

1. A form $\Theta_B$ of even degree $2k$ admits point-wise isomorphism $\Omega_{\otimes 2k} \to V_{\otimes 2k}$ and has inverse $2k$-vector $\Theta_B^{-1}: \Theta_B^{-1} \cdot \Theta_B = \Theta_B \cdot \Theta_B^{-1} = E$, if in some coordinate system its matrix is point-wise preimage of the isomorphism $\chi_*$ of any nondegenerate section of trivial bundle $B \times M_{2k \times 2k}(\mathbb{R})$:

$$\|\Theta_B\| = \chi_*(M(B)), \ M(b) \in M_{2k \times 2k}(\mathbb{R}), \ \det M(b) \neq 0, \ \forall \ b \in B,$$

where $\chi_*$ and all another notations are introduced in Appendix A.
2. For any natural $d$ and $k$, related by the equation

$$3^k - 2kd^{k-1} = 4m + 1, \ m \in \mathbb{Z}$$

there exists volume form on $\mathcal{B}$ of the kind:

$$\left| \det \Theta_B \right|^{1/2} dx^1 \wedge \ldots \wedge dx^d \equiv \left| \det \chi_x(\Theta_B) \right|^{1/2} dx^1 \wedge \ldots \wedge dx^d,$$

where $\Theta_B$ — any form of degree $2k$, satisfying the existence of $\Theta_B^{-1}$ condition.

3. The form $\Theta_B$ as image $(d\iota_\ast)\Theta_M$ is nondegenerate, if and only if

$$(L_{\Theta(M)} \cup R_{\Theta(M)}) \cap (T^kS)^\otimes k = \emptyset. \quad (12)$$

or in words, when left and right kernels of the form $\Theta_M$ has null intersections with the space of all $k$–vector $V^\otimes k(S)$ on a whole $S$ (see Appendix C).

Everywhere below we assume, that $d$-metric has even valency and satisfies all conditions 1,2,3.

4 $d$-objects dynamical (equilibrium) equations

Now we are going to derive general dynamical equation of $d$-objects. Let's introduce some useful indexless matrix notations, adopted both for static and for evolutional problems.

4.1 Description of embeddings and deformation measure

We'll describe some history $F_t(\mathcal{B})$ by the set of functions $\{x^A(x, \xi, t)\}_{A=1,\ldots,n}$, where

$\{x^A\}_{A=1,\ldots,n}, \ \{\xi^\alpha\}_\alpha=1,\ldots,d$

— coordinates on $\mathcal{M}$ and $\mathcal{B} \times I$ respectively. This multicomponent notation we’ll short as usually to $x = x(\xi, t) \equiv x_t(\xi)$. Corresponding matrix $Dx_t$ for $(dF_t)^\ast$ has the components

$$(Dx_t)^A_\alpha = \frac{\partial x^A_t}{\partial \xi^\alpha}.$$ 

Codifferential $(dF_t)^\ast$ defines induced linear mapping:

$$L_t \equiv (Dx_t)^\otimes p \quad (13)$$

of $d$-metric — representation of $(dF_t)^\ast$ in $(T^*S)^\otimes p$, such that:

$$\Theta^B_t = (dF_t)^\ast \Theta \equiv L_t \Theta_M.$$ 

For measure of deformation we have in evolutional case:

$$\Delta^B_t = (dF_t)^\ast \Theta - (dF_0)^\ast \Theta \equiv L_t \Theta(x_t) - L_0 \Theta(x_0) \equiv \Theta^B_t - \Theta^0_B. \quad \Delta B \equiv \dot{\Theta}^B_t = \frac{d\Theta^B_t}{dt}, \quad (14)$$

where $\Theta^0_B = L_0 \Theta$ — background (initial) metric.

For static problem we formally put:

$$\Delta_B = (dF_1)^\ast \Theta - (dF_0)^\ast \Theta \equiv L_1 \Theta(y) - \Theta^0_B,$$

where $y \equiv x_1(\xi)$.

\textsuperscript{14}They are often called in literature embedding variables.
4.2 Equations of motion (evolutional case)

Let's begin from a more general evolutional case. Accordingly to Sec.3.2 (case $\epsilon = 1$) full action has the following kind:

$$\mathcal{S}[x_i(\xi)] = \int_{B \times I} (\mathcal{F}_0 + \mathcal{U}) v dt \wedge d\mu_\xi,$$

where $\mathcal{F}_0 = \mathcal{F}_0(\Delta_B^1, \Delta_B^2) -$ internal elastic part of energy of (generally speaking, nonhomogeneous\textsuperscript{15}) \textit{d-body} $B, \mathcal{U}(x_i(\xi)) = \mathcal{U}(\xi, t) -$ external potential part of energy, $v \equiv e(t) \cdot \varpi(\Theta_B^1), e dt -$ metric on $I$, $\varpi d\mu_\xi -$ volume form on $B$, induced by $\Theta_B^1, d\mu_\xi \equiv d\xi_1 \wedge \ldots \wedge d\xi_d$.

First variation of (15) over $x_i(\xi)$ takes the form:

$$\delta \mathcal{S} = \int_{B \times I} \{ (\delta \mathcal{F} + \delta \mathcal{U}) v + (\mathcal{F} + \mathcal{U}) \delta v \} dt \wedge d\mu_\xi.$$

Everywhere below in our derivation we’ll omit $B$ and $t$ at the bottom and top of $\Delta$ and of other values. Using the relations and definitions:

$$\delta \mathcal{F} = \langle \mathcal{F}_1, \delta \Delta \rangle + \langle \mathcal{F}_2, \delta \Delta \rangle \equiv \langle \sigma, \delta \Delta \rangle + \langle \pi, \delta \Delta \rangle; \quad \delta \mathcal{U} = \langle \mathcal{U}_1, \delta x \rangle; \quad \delta v = \langle \nu, \delta \Delta \rangle,$$

where we have introduced \textit{stress tensor} $\sigma$ and \textit{surface momentum density tensor} $\pi$, have used $(\ , \ )$ for coordinateless representation of summation as "linear functional" over variations in corresponding spaces and have taken into account, that by (14) $\delta \Delta_B^1 = \delta \Theta_B^1$.

After integrating by parts over $t$ we have:

$$\delta S = \int_{B \times I} \{ (\sigma v - \frac{d}{dt} \langle \pi v \rangle + (\mathcal{F} + \mathcal{U}) v | \Delta \rangle, \delta \Delta \rangle + \langle \mathcal{U}_1 x, \delta x \rangle \} dt \wedge d\mu_\xi + \int_{B \times \partial I} \langle \pi, \delta \Delta \rangle v d\mu_\xi.$$

The first triangle bracket within volume term can be transformed by the following way:

$$\sigma v - \frac{d}{dt} \langle \pi v \rangle + (\mathcal{F} + \mathcal{U}) v | \Delta \rangle = \sigma v - \hat{\sigma} v - \pi v | \Delta \rangle (\mathcal{F} + \mathcal{U}) v | \Delta \rangle = (\sigma - \hat{\sigma}) v - \mathcal{T} v | \Delta \rangle \equiv \nu \Sigma,$$

where we have introduced

$$\hat{T} = \pi \otimes \Delta - (\mathcal{F} + \mathcal{U}) \hat{I}$$

— deformational energy-momentum affinnor, $\hat{I} -$ identical linear operator in $T(0, p): \hat{I} v | \Delta \rangle = v | \Delta \rangle$;

$$\Sigma \equiv (\sigma - \hat{\sigma} - \mathcal{T} (\ln \varpi)|_{\Delta \rangle}$$

— generalized stress tensor.

Simple calculation with using (13) and (14) gives:

$$\delta \Delta = \delta \langle \mathcal{L} \Theta \rangle = \delta \mathcal{L} \Theta + \mathcal{L} \delta \Theta \equiv \langle \Theta, \delta D x \rangle + \langle (\Theta|_x) B, \delta x \rangle,$$

where

$$\Theta_{\alpha_1 \ldots \alpha_p} \equiv \left( \frac{\partial \mathcal{L}}{\partial (D^x_i)} \right)_{\alpha_1 \ldots \alpha_p} \equiv \sum_{s=1}^{p} \Theta_{B \alpha_1 \ldots \alpha_{s-1} \alpha_{s+1} \ldots \alpha_p} \equiv \Theta \Theta_{|x} B \equiv \mathcal{L} \Theta|_x.$$

Substituting all into $\delta \mathcal{S}$ and integrating by parts over $\xi$, we have:

$$\delta \mathcal{S} = \int_{B \times I} \{ (-\partial_\xi (\nu \Sigma) + \nu \Sigma, (\Theta|_x)_B + \mathcal{U}_1 x, \delta x \} dt \wedge d\mu.$$

\textsuperscript{15}Such nonhomegeneous \textit{d-body} possess different elastic properties at different points. We should denote it by using apparent dependency of $\mathcal{F}$ on $\xi$, but for the brevity don’t do it.
\[
+ \int_{\mathcal{B} \times \partial I} \langle \pi, \delta \Delta \rangle v \, d\mu_\xi + \int_{\partial \mathcal{B} \times \partial I} \langle \overline{\Sigma}, \delta x \rangle v \, dt \wedge d\mu_\xi',
\]

where we use notation \( \overline{\Sigma} = \langle \Sigma, \overline{\Theta} \rangle \) and in last boundary integral \( d\mu_\xi' \) symbolizes elements of the sets \( \{d\xi^{\alpha_1} \wedge \ldots \wedge d\xi^{\alpha_{d-1}}\} \) of \( d-1 \) coordinate boundary hypersurface volume form.

Extremality condition \( \delta \overline{\mathfrak{F}} = 0 \) gives the following equations of motion:

\[
\text{div } \Sigma + f_{\Theta} + f_{\text{ext}} = 0,
\]

where

\[
\langle \text{div } \Sigma \rangle_A \equiv \frac{1}{v} \partial_{\alpha_s} \left( v\Sigma^{\alpha_1 \ldots \alpha_s \ldots \alpha_p} \overline{\Theta}_{\alpha_1 \ldots \alpha_p} A \right) \equiv \frac{1}{v} \partial_{\alpha_s} \left( v\Sigma^A_{\alpha_1 \ldots \alpha_p} A \right)
\]

— operator of divergence,

\[f_{\Theta} \equiv -\langle \Sigma, (\Theta_{|x})_B \rangle\]

— \( \Theta \)-gravity force density, induced by nonhomogeneity of \( \Theta \) in \( \mathcal{M} \) (it vanishes, when \( \Theta \) — constant form),

\[f_{\text{ext}} \equiv -\mathcal{U}_{|x}\]

— external force density, induced by external fields (it vanishes in case of closed \( d \)-structures).

### 4.3 Boundary conditions

Under derivation of dynamical equations we have obtained boundary conditions (18) of the following general kind:

\[
\int_{\Gamma_a} \langle X_a, \delta x \rangle \, d\mu_a, \quad a = 1, 2, 3,
\]

having sense of vanishing of ”average work” on variations \( \delta x \) at boundary. Here boundary\(^{16}\)

\[
\Gamma = \partial (\mathcal{B} \times I) = (\partial \mathcal{B} \times I) \cup (\mathcal{B} \times \partial I) \equiv \Gamma_1 \cup \Gamma_2; \quad \Gamma_3 = \Gamma_1 \cap \Gamma_2 = \partial \mathcal{B} \times \partial I;
\]

\[
X_1 = \overline{\Sigma}; \quad X_2 = -\text{div } \pi + \langle \pi, (\Theta_{|x})_B \rangle; \quad X_3 = \pi \equiv \langle \pi, \overline{\Theta} \rangle;
\]

\[
d\mu_1 = v \, dt \wedge d\mu_\xi; \quad d\mu_2 = v \, d\mu_\xi; \quad d\mu_3 = v \, d\mu_\xi'.
\]

Here we consider only the most known and widely used boundary conditions (generalizing ones in standard elasticity theory):

1. **Pinned boundaries** (\( P \)). In this case \( \delta x|_{\Gamma} = 0 \) and all equations (20) are satisfied identically.

2. **Free boundaries** (\( F \)). In this case variations \( \delta x \) are arbitrary on \( \Gamma \) and boundary conditions takes the form:

\[
(F) : \quad X_a|_{\Gamma_a} = 0 \quad a = 1, 2, 3.
\]

One only should check consistency of this independent equations on \( \Gamma_3 \).

\(^{16}\)The term with \( \Gamma_3 \) arises after integrating by parts of term with \( \Gamma_2 \) with using (17).
3. **Sliding boundaries** ($S$). Let $\delta x_\tau \equiv \delta F_\tau(\Gamma) —$ variational homotopy of $\Gamma$. We’ll relate $\delta x_\tau$ to a class of variations of *sliding type* $\delta x_\|$, if $d\delta F_\tau(d/d\tau) \in \mathbb{T}$. Then sliding boundary conditions takes the form:

$(S) : \langle X_a, \delta x_\| \rangle = 0, \quad a = 1, 2, 3.$

If $\{\eta^a\}$ — coordinates on $\Gamma$, then its image $\iota(\Gamma)$ in $\mathcal{M}$ can be described by the set of functions $\{x(\eta)\}$. The set $\{\partial_\eta x\} \subset \mathcal{T}M$ forms collection of basis vector fields on $\iota(\Gamma)$. Then coordinate form of sliding boundary conditions will be

$(S) : \langle X_a, \partial_\eta x \rangle = 0, \quad a = 1, 2, 3.$

One should only check it consistency on $\Gamma_3$.

4. **Boundaries with given variations** ($R$). If $\delta x|_{\Gamma_a} \equiv \varphi_a(\eta)$ — some fixed functions on $\Gamma$, such that $\varphi_3 = \varphi_2|_{\Gamma_3} = \varphi_1|_{\Gamma_3}$, then we come back to general conditions (20) and get:

$(R) : \int_{\Gamma_a} \langle X_a, \varphi_a \rangle \, d\mu_a = 0, \quad a = 1, 2, 3.$

### 4.4 Static case

In the static case accordingly to the Sec.3.2 (case $\epsilon = 2$) we start from the action:

$$\mathcal{S}[y(\xi)] = \int_{\mathcal{B}} (\mathcal{F}_0 + \mathcal{U}) \, \omega d\mu_\xi.$$  

Then we should carry out similar to the evolutional case manipulations, that lead to the particular case of evolutional equations (19) and boundary conditions (20), taken under

$$\pi = 0, \quad I = \{0, 1\}, \quad \partial I = \emptyset, \quad v \to \omega.$$  

So, in static case we obtain (19) as equilibrium equations, with

$$\tilde{T} = -(\mathcal{F}_0 + \mathcal{U}) I; \quad \Sigma = \sigma - \tilde{T}(\text{ln}(\omega))|_\Delta$$

and with the only boundary condition:

$$\int_{\partial \mathcal{B}} \langle \Sigma, \delta x \rangle \, \omega d\mu_\xi' = 0.$$  

### 4.5 Perturbative elasticity theory

Since deformational energy density $\mathcal{F}_0(\Delta)$ is scalar\[^{17}\] it can depend on $\Delta$ only through the following combinations:

$$\Delta^{(i)} \equiv \text{Tr} (\tilde{\Delta}^i),$$  

(21)

where $\tilde{\Delta} \equiv \Delta \cdot (\Theta^0)^{-1} \in T(p/2, p/2)$ and matrix degree, multiplication and trace operation are understood in the sense of the corresponding operations of its $\chi^*$-images (see Appendix A) in some coordinate system. Since $\dim \mathcal{B} = d$, then there exists no more then $d$ functionally-independent scalars $\Delta^{(i)}$, which can be ordered by increasing $i$. Let $\{\Delta^{(i)}\}_{1 \leq i \leq I; \ 1 \leq i \leq s \leq d}$ will be collection of the first $s$ such independent scalars.

---

\[^{17}\]Here we don’t differ $\Delta$ and $\tilde{\Delta}$ since they possess similar algebraic structure. So, our consideration in this paragraph touches pure cases $\mathcal{F}_0(\Delta_a)$ or $\mathcal{F}_0(\tilde{\Delta})_B$, but general mixed case $\mathcal{F}_0(\Delta, \tilde{\Delta})$ can be considered by the similar manner
To compare equations of the Sec.13.4 with well known equations of standard field theory it is necessary to go to decomposition of the energy $\mathcal{F}_0$ over power of $\Delta$. We’ll see, that the most part of modern field-theoretical models can be described by the first members of the decomposition — the so called $\Delta^1$ and $\Delta^2$-structures (see below). So we need investigate the structure of the following formal row:

$$\mathcal{F}_0(\Delta) = \sum_{i=1}^{\infty} \frac{1}{i!} \frac{\partial^i \mathcal{F}_0}{\partial \Delta^i} \bigg|_{\Delta = 0} \Delta^i.$$  

(22)

The symbolic Macloren row (22) with using notations (21) can be rewritten as follows:

$$\mathcal{F}_0 = \sum_{i=0}^{\infty} \sum_{(\vec{k},\vec{i})=i} \mu_{k_1 \ldots k_i} (\Delta^{(i_1)})^{k_1} (\Delta^{(i_2)})^{k_2} \ldots (\Delta^{(i_k)})^{k_k},$$  

(23)

where in the second sum there is summation over all vectors $\vec{k} = (k_1, \ldots, k_i)$ of $s$-dimensional integer-valued lattice, whose nonnegative coordinates satisfy the equation of atomic hyperplane $(\vec{k}, \vec{i}) = i$. Parenthesis denote Euclidean scalar product in $\mathbb{E}^s$, the vector $\vec{i} = (i_1, i_2, \ldots, i_s)$. Scalar coefficients $\{\mu_{k_1 \ldots k_i}\}$ characterize "elastic properties" of the $d$-body. Similarly to standard elasticity theory we’ll call it generalized Lamé coefficients.

We’ll call deformational structure $\mathcal{D}$ with energy density $\mathcal{F}_0$ as exact finite sum of powers of $\Delta$ with highest term of order $i$ in (23) $\Delta^i$-structure.

Let’s consider in more details $\Delta^3$-structure, assuming that the scalars $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}$ are nonzero and independent.

1) $i = 0$. There is one Lame coefficient $\mu_0 = \mathcal{F}_0(0)$, which represent background (null) energy and practically always can be annihilated by constant shift of $\mathcal{F}_0$.

2) $i = 1$. There is one Lame coefficient $\mu_1$. The corresponding term of finite sum is:

$$\mu_1 \Delta^{(1)}.$$  

(24)

Within the standard elasticity theory the term is responsible for energy of strongly tensed bars and plates (strings and membranes) and heat expanding of isotropic bodies.

3) $i = 2$. There is two Lame coefficient $\mu_{01}^2$ and $\mu_{20}^2$ and two terms in $\mathcal{F}_0$ respectively:

$$\mu_{01}^2 \Delta^{(2)} + \mu_{20}^2 (\Delta^{(1)})^2.$$  

(25)

The expression is well known Hooks law of linear elasticity theory (where $\mu_{01}^2 = \mu$ — shift modulus, $\mu_{20}^2 = \lambda/2$ — second independent Lamé coefficient).

4) $i = 3$. Three nonzero Lamé coefficient $\mu_{001}^3, \mu_{110}^3, \mu_{300}^3$ gives the following terms in $\mathcal{F}_0$:

$$\mu_{001}^3 \Delta^{(3)} + \mu_{110}^3 \Delta^{(1)} \Delta^{(2)} + \mu_{300}^3 (\Delta^{(1)})^3.$$  

(26)

This part describes nonlinear corrections to the linear models within elasticity and field theory. In present paper we’ll not touch it.

So, we have the following general kind of $\mathcal{F}_0$ within $\Delta^3$-structure:

$$\mathcal{F}_0(\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}) = \mu^0 + \mu_1 \Delta^{(1)} + \mu_{01}^2 \Delta^{(2)} + \mu_{20}^2 (\Delta^{(1)})^2 + \mu_{001}^3 \Delta^{(3)} + \mu_{110}^3 \Delta^{(1)} \Delta^{(2)} + \mu_{300}^3 (\Delta^{(1)})^3.$$  

(26)

Lets calculate stress tensor $\sigma$ for $\Delta^3$-structure. Using its definition in (16) and decomposition (26), we have:

$$\sigma = \frac{\partial \mathcal{F}_0}{\partial \Delta} = \sum_{i=1}^{3} \frac{\partial \mathcal{F}_0}{\partial \Delta^{(i)}} \frac{\partial \Delta^{(i)}}{\partial \Delta} = (\mu_1 + 2\mu_{20}^2 \Delta^{(1)} + \mu_{110}^3 \Delta^{(2)} + 3\mu_{300}^3 (\Delta^{(1)})^2)(\Theta^0)^{-1}$$

$$+ 2(\mu_{01}^2 + \mu_{110}^3)(\Theta^0)^{-1} \cdot \Delta \cdot (\Theta^0)^{-1} + 3\mu_{001}^3 (\Theta^0)^{-1} \cdot \Delta \cdot (\Theta^0)^{-1}.$$  

(27)

The expression (26), (27) generalize in many aspects well known expression for elastic energy and stresses tensor within standard linear elasticity theory.

\[\text{Footnote: Surface momentum density can be obtained by changing } \Delta \rightarrow \dot{\Delta}.\]
5 Examples of dynamical deformational structures

Now we consider some examples of classical $d$-structures, which can be observed within the well known theories. We leave without attention those examples, which concern either with standard elasticity theory — starting point of our generalizations, or with its development in $M_4$ or in $V_4$, mentioned in Sec.2.3, since we are intending to devote them special papers in future.

5.1 Example 1: The theory of classical $d-1$-brane

Let $M = M_{N+4}$ be pseudoeuclidean space with metric $\Theta$. There is unique scalar invariant:

$$\Delta^{(1)} = \text{Tr}[(\Delta \cdot (\Theta^0)^{-1}) = \text{Tr}(Dy^T \cdot \Theta \cdot Dy - \Theta^0) \cdot (\Theta^0)^{-1}] = \Theta \otimes (\Theta^0)^{-1}(Dy, Dy) - d \equiv |Dy|^2 - d,$$

where $\Theta \otimes (\Theta^0)^{-1}$ contracts $Dy$ as vectors in $TM_{N+4}$ and as forms in $T^*B$. Assuming $\mu_0^1 = T/2$ in (24), we obtain the action of the following kind [24]:

$$\mathfrak{F} = \frac{T}{2} \int (|Dy|^2 - d) \sqrt{\text{det} \Theta^0} d\mu_\xi. \quad (28)$$

This expression coincides with well known Polyakov’s action for classical $d-1$-brane with special cosmological term [2]. In a difference with string and brane models the metric $\Theta^0$ is considered here as fixed (background). In accordance with string and brane ideology variation of the (28) over $(\Theta^0)^{-1}$ leads to the constraint:

$$\Theta(Dy, Dy) - \frac{1}{2} (|Dy|^2 - d) \Theta^0 = 0.$$

Its contraction with $(\Theta^0)^{-1}$ leads to the relation:

$$|Dy|^2 = \frac{d^2/2}{d/2 - 1},$$

which under $d = 2$ (string case) gives inconsistent constraint $d = 0$. This arguments, typical for original string theory, are not so catastrophic within deformational approach, since true dynamical variables are not components of metric $\Theta^0$, but embedding variables $x_0(\xi)$. If we minimize $\mathfrak{F}$ with respect to both final and initial position of $d$-object, we obtain the following consistent system:

$$\text{div } Dy = 0; \text{ div } (Dx_0 - 2\Theta \otimes (\Theta^0)^{-1}(Dy, Dx_0)Dy) = 0,$$

where the first is obtained by $y-$variation (it is identical to the string theory equation) and second — by $x_0-$variation of action (28). We do not write here boundary conditions. Note also, that cosmological term $-d \cdot T/2$ can be absorbed by suitable choice of $F_0(0)$.

5.2 Example 2: Classical solids dynamics as $\Delta_1^1 + \Delta_2^2$-structure

Let $\mathcal{M} = M_4$, $\Theta$ — Minkowski metric, $\mathcal{B} \subset M_4$ — thin 4D time-like bar, i.e. body, whose size along time-like direction much more then in space-like. Within approach, been proposed in [18], it performs the so called absolute ("objective") history of thing, while its space-like sections, observing from the point of view of some reference frame, performs relative ("subjective") history of the thing. Then, we have endowed the bar by some linear elastic properties, described by Lamé coefficients $\mu_0^2 = \mu$ and $\lambda = 2\mu_0^2$, (see Hooks law (25)) and have generalized standard elasticity theory of common bars in Euclidean 3D space on the 4D case. Analysis of the theory has led to the following curious conclusions:

19Here we omit external potential energy $\mathcal{U}$. 

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• Classical mechanics can be formulated within 4D static deformatonal picture in terms of straining of the thin strongly tensed bars (strings) without special notion of mass (it has 4D force nature). In terms of deformatonal structures such theory should be related to the anisotropic $\Delta^1_\parallel + \Delta^2_\perp$ structure, where symbols "\parallel" and "\perp" differ time-like and space-like directions within our 4D objects — 4D strings. General formulation of such anisotropic $d-$structures is obvious, but goes beyond the scope of the present paper.

• Only third Newton’s law\(^{20}\) remains independent, while first and second appear as its consequences. Curiously, that second Newton’s law can be viewed as 1D Laplace formula for strings, similar to 2D case for membrane.

• The approach reveals, that classical Newton laws are, in fact, result of some extremely exact ”tunings” in mechanical structure of Universe, which itself can be imagined as twisted and strongly tensed net.

• In principle, there is possibility of violation of Newton dynamics in some special situation: rapid rotations, large accelerations, beginning and end of absolute history of some 3D body and others additionally to relativistic effects.

• The approach reveals fundamental role of observer as not only ”spectator” but ”participants” of formation of physical laws even within classical mechanics (see also ([26])).

Similar ideas, revealing connections of elasticity and inertia has been discussed in [27]. We hope that the deformational picture of classical mechanics will provide useful means for its more deep understanding.

5.3 Example 3: Einstein gravity as $\Delta^2$-structure

Let as in Example 1 $\mathcal{M} = M_{N+4} —$ pseudoeuclidian space with metric $\Theta$ and $\mathcal{B} \subset M_{N+4} —$ thin 4D plate, i.e. body, whose sizes along some four dimensions (one — timelike and three space-like) are much more then in other ones. In the works [13, 14, 17] some generalization of standard elasticity theory for common (2D in $\mathbb{E}^3$) plate has been applied for the 4D plate equilibrium problem. We have endowed $\mathcal{B}$ with elastic constants $\lambda = 2\mu^2_{20}$ and $\mu = \mu^2_{01}$ and derived energy of bending $\overline{\mathcal{F}}_b$ ([13, 14]) and stretching $\overline{\mathcal{F}}_s$ ([17]) by integrating over extradimensions and 4D directions within static $\Delta^2$-structure. We had found, that:

• Theory of straining of 4D plates in $M_{N+4}$ can describe space-time (this is plate itself!) and matter (this is special stresses of the plate) dynamics in unified language. Namely, pure bending energy $\overline{\mathcal{F}}_b$, calculated within $\Delta^2$-structure (up to a dimensional constant) generalizes linearized Gilbert-Einstein’s action for gravity. On the other hand, pure stretch energy $\overline{\mathcal{F}}_s$ plays role of action of 4D matter field, living on the plate.

• Within the deformatonal approach physical essence of Einstein equations becomes very clear. They express vanishing of total 4D stresses on the plate, induced by bending and stretching. In other words, Einstein equations says, that true dynamic of space-time is realized as locally unstressed states of space-time.

• This strange (from the view point of common plate theory) fact is originated from the ”wrong” variational procedure, used in GR. From the viewpoint of deformatonal approach true variational variables are not Riemannian metric components $\{g_{\alpha\beta}(x)\}$, but embedding variables $\{y(\xi)\}$. Varying $\overline{\mathcal{F}}_s$ over $y(\xi)$ we have obtained in [17] ”right” plate equilibrium equations for $y(\xi)$ and have proved that they possess more generality, then Einstein equations.

\(^{20}\)As it has been cleared in [28] third Newton’s law follows only from assumption of additivity of force function $f$ on bodies of mechanical Universe $\Omega$: $f(U_1 \cup U_2) = f(U_1) + f(U_2)$, $\forall \ U_1, U_2 \subset \Omega$.\[24\]
• More detailed analysis shows, that $\mathcal{G}_0$ is reduced to an exact linearized Einstein-Gilbert action when Poissons coefficient $\sigma_F$ of the plate is $1/2$. In this case variational derivative of $\mathcal{G}_0$ (over $g$) transforms into linearized purely geometrical Einstein tensor, whose divergence vanishes by Bianchi identities. So, from the viewpoint of the deformational approach matter equations of motion follows from the field equations due to special elastic properties of space-time.

• Curiously, that the Einstein case $\sigma_F = 1/2$ is degenerate from the viewpoint of the deformational approach, since $F_0 \, d\mu$ becomes exact form relatively to embedding variables $x(\xi)$ (but not $g$). In physical deformational language plate’s cylindrical stiffness factors $\{D_m\}_{m=1,\ldots,N}$ in all $N$ extradimensions vanish under $\sigma_F = 1/2$.

• Dimensional manipulations leads to the following relation between Einstein gravitational constant $\kappa$ and elastic parameters of the plate in extradimensions. Assuming $h \sim l_{P1}$, $N \sim 1$ we obtain $\ln E(Pa) \sim 10^2$ — huge stiffness of space-time!

Some another interesting topics, involving thermodynamics, origin of hyperbolicity of space-time, lagrangian formalism and boundary conditions have been discussed in cited papers. Cosmological implication of the theory in the simplest case $N = 1$ has been considered in [19].

5.4 Example 4: Maxwell electrodynamics as symplectic bimetric $\Delta^2$-structure.

Let $\mathcal{M} = \text{symplectic manifold (dim} \mathcal{M} = 2n)$ with $\Theta = \omega \in \Lambda^2(\mathcal{M})$ — symplectic form, which is closed ($d\omega = 0$) and nondegenerate. As usually we define the mapping $i_z : \Lambda^2(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$, where $z \in T\mathcal{M}$, by the relation:

$$i_z \omega(u) \equiv \omega(z, u), \quad (29)$$

for all $u \in T\mathcal{M}$.

Let $\mathcal{B} = \mathcal{M}$ and let $F_i(\mathcal{M})$ — some diffeomorphism $\mathcal{M} \rightarrow \mathcal{M}$, which we consider as a history of some deformation. It induces corresponding vector field $A = dF(d/dt) \in T\mathcal{M}$. Lets calculate local measure of the deformation. Using rule of action of Lie derivatives on external forms [22] $L_z = i_z \circ d + d \circ i_z$, we have:

$$\omega \equiv \mathcal{L}_A \omega = (i_A \circ d + d \circ i_A) \omega = di_A \omega = dA = \hat{F},$$

where closeness of $\omega$ has been used. It is naturally to associate $\hat{F} = d\hat{A}$ with Faradey-Maxwell 2-form and $\hat{A} = i_A \omega$ — with electromagnetic potential 1-form, whose deformational nature become clear. Following to the ideology of (unimetric) deformational structures, $\Delta^2$-structure should be based on the lagrangian:

$$F^\omega_0 = \mu(F, F)_{\omega} + \frac{\lambda}{2}(\text{Tr}_{\omega}(F))^2,$$

where notations $\omega$ and $\text{Tr}_{\omega}$ remind us, that they are defined relatively to $\omega$ as both $d-$metric and $g-$metric. Easily to check (for example, using Darboux theorem and going to canonical form of $\omega : dx^1 \wedge dx^2 + \ldots + dx^{2n-1} \wedge dx^{2n}$), that this lagrangian is not maxwellian. To get Maxwell electrodynamics we need to introduce Minkowski metric $\eta$ and, so, go to bimetric structure. Obviously, that $\text{Tr}_{\eta}(F) \equiv 0$, and we have:

$$F^\eta_0 = \mu(F, F)_{\eta}$$

— standard maxwellian lagrangian with $\mu = -1/16\pi$ in Gauss units. Note also, that $d\mu_{\eta} = d\mu_{\omega} \sim \omega^{\wedge n}$. 25
It is easily to understand the role of $\text{MOT}_{\mathcal{M}}(\mathcal{M})$ within the considered model. It just generates gauge transformations of $\hat{A}$. More exactly, let $\Phi_t(\mathcal{M}) \in \text{MOT}_{\mathcal{M}}(\mathcal{M})$ and let $v = d\Phi(d/dt)$. Then by interrelations of $\text{MOT}_{\mathcal{M}}(\mathcal{M})$ and Hamilton vector fields (see Sec.2.4 and [25]), $v = \text{grad} \, h$, where $h$ — some (local) Hamiltonian function, generating motion $\Phi_t$ and vector field $v$. By uniqueness theorem there is isomorphism (up to a constant) between $\text{MOT}_{\mathcal{M}}(\mathcal{M})$ and set of all Hamilton functions \{h\}, defined by equation $d\Phi(d/dt) = \text{grad} \, h$. Then we define gauge mapping:

$$\phi_h : \text{DEF}_{\mathcal{M}}(\mathcal{M})_0 \to \text{DEF}_{\mathcal{M}}(\mathcal{M})_0$$

by the rule:

$$d(\phi_h(F)) = \hat{A}_h = dF(d/dt) + \text{grad} \, h = A + \text{grad} \, h.$$

In terms of forms, we’ll have

$$i_{\hat{A}_h} \omega = i_A \omega + i_{\text{grad} \, h} \omega = \hat{A} + dh$$

— gauge transformation of electromagnetic potential.

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A The problem of $\Theta_B^{-1}$

Our approach to the question of existence of $\Theta_B^{-1}$ will be based on some well known facts of standard square matrix algebra. Namely, in case of $d$-metrics, taken as bilinear quadratic forms, we know robust criteria, which provides existence of both inverting and scalar density of weight $-1$:

1) Metric $g$ admits point-wise isomorphism $\Omega^\otimes 2 \to V^\otimes 2$, if and only if $\text{det} \|g\| \neq 0$, where $\|g\|$ — matrix of the form $g$ in any basis. The element of bivector space, isomorphic to $g$ will be $g^{-1} \in V^\otimes 2$, which in basis, dual to basis of $\Omega^\otimes 2$ has inverse to $\|g\|$ matrix;

2) Let $\|L\|$ — is matrix of nondegenerate linear transformation in the same vector space, where form $g$ is acting. Then, as well known, matrix of the form is transformed by the rule:

$$\|g\| = \|L\|^T \cdot \|g\|' \cdot \|L\|.$$  \hspace{1cm} (30)

Taking determinant of the both sides and square root we get: $|\text{det} \|g\'||^{1/2} = |\text{det} \|g\||^{1/2} / \text{det}\|L\|$ — required scalar density of weight $-1$.

Let consider the set $\Omega^{\otimes p}(b)$ of all forms of degree $p$ at some fixed point $b$ of $d$-body. The set, after fixing some basis, can be naturally identified with the space of $p$-cubic real matrices $M_{d^p \times p}$ of dimension $d$. Let $p = 2k$, $k \in \mathbb{N}$ and let some fixed division of all vector arguments of the $2k$—forms on two set with $k$ elements is given. Without loss of generality we can relate the first $k$ arguments to the first set, and remaining $k$ — to the second. Let, then, $\chi : (\mathbb{Z}_d^+)^{\times k} \to \{1, 2, \ldots, d^k\}^k$ — some ordering of $(\mathbb{Z}_d^+)^{\times k}$, where $\mathbb{Z}_d^+$ — the set of positive integer numbers from $1$ to $d$. This ordering induces the isomorphism (depending on the ordering) $\chi_* : M_{d^k \times 2k} \to M_{d^k \times d^k}$ between spaces of $p$-cubic matrices and square matrices of dimension $d^k$, which maps every matrix element $A_{\alpha_1 \ldots \alpha_k, \alpha_{k+1} \ldots, \alpha_{2k}}$ into matrix element $\chi_*(A)_{ab}$ by the following rule$^{21}$

$$\chi_*(A)_{ab} = A_{\chi^{-1}(a) \chi^{-1}(b)}.$$

$^{21}$For example $a$ and $b$ can be taken as $k$—digits numbers of $d$-adic system of calculus of respective halves groups of indexes:

$$a = \alpha_1 d^0 + \alpha_2 d^1 + \ldots + \alpha_k d^{k-1}; \quad b = \alpha_{k+1} d^0 + \alpha_{k+2} d^1 + \ldots + \alpha_{2k} d^{k-1}. $$
The isomorphism lets to pull-back all operations of standard matrix algebra from $M_{d^k×d^k}$ to $M_{d×2k}$. Namely, let the following operations are given on $M_{d^k×d^k}$: $\alpha: M_{d^k×d^k} → M_{d^k×d^k}$, $\beta: M_{d^k×d^k} → \mathbb{R}$ and $* : M_{d^k×d^k} × M_{d^k×d^k} → M_{d^k×d^k}$. Then this operations by the isomorphism $\chi_*$ induce the operations $\bar{\alpha}$, $\beta$ and $*$ in $M_{d×2k}$ by the rules:

\[\bar{\alpha} = \chi_*^{-1} \circ \alpha \circ \chi_* \quad (i); \quad \bar{\beta} = \beta \circ \chi_* \quad (ii); \quad \bullet \circ \bullet = \chi_*^{-1}(\chi_* \bullet \chi_*) \quad (iii).\]

Let $*$ — is standard matrix multiplication in $M_{d^k×d^k}$. Then (iii) gives the rule for multiplication of matrices in $M_{d×2k}$:

\[(A*B)_{\alpha_1...\alpha_k\beta_1...\beta_k} = \sum_{\beta_1...\beta_k=1}^d A_{\alpha_1...\alpha_k\beta_1...\beta_k} B_{\beta_1...\beta_k\alpha_{k+1}...\alpha_{2k},}\]

$\forall A, B \in M_{d×2k}$. Preimage of matrix unit $I_{d^k×d^k} \equiv e$ will be matrix $\chi_*^{-1}(e) = I_{d×2k} \equiv E$ with components

\[E_{\alpha_1...\alpha_k\alpha_{k+1}...\alpha_{2k}} = \delta_{\alpha_1\alpha_{k+1}}\delta_{\alpha_2\alpha_{k+2}}\cdots\delta_{\alpha_k\alpha_{2k}}.\]

Let $\alpha$ — inversion operation in $M_{d^k×d^k}$. Then by (i), if inverse matrix exists for image $\chi_*(A)$, it will always exist for its preimage and $A^{-1} \equiv \chi_*^{-1}(\chi_*(A))^{-1}$ for every $A \in M_{d×2k}$. Let $\beta \equiv \det$. Then det-operation is well defined in $M_{d×2k}$. Namely, by (ii) it follows that $\det A \equiv \det(\chi_*(A))$.

Now we clame that the matrix equation

\[A\bar{X} \equiv A \cdot X = E\]

in $M_{d×2k}$ has solution, if $\det A \neq 0$. The solution we call matrix $A^{-1}$, inverse to $A$. Identifying $E$ in the fixed coordinate system with mixed tensor in $(TB)^\otimes k \otimes (T^*B)^\otimes k$

(in components $E_{\alpha_1...\alpha_k\beta_1...\beta_k} = \delta_{\alpha_1\alpha_{k+1}}\cdots\delta_{\alpha_k\alpha_{2k}}$),

$A$ — with form $[\Theta]_B \equiv (d\ell)^*\Theta$ of degree $2k$, we go to the statement 1 of Sec 3.3.

**B. The problem of internal $d\mu$ ($d$-structures)**

Let $j$ — Jacobi matrix of some smooth nondegenerate coordinate transformation $\xi \to \xi'(\xi)$ on $B$:

\[j_\beta^\alpha = \frac{\partial \xi'^\alpha}{\partial \xi^\beta},\]

and $j^{-1}$ — its inverting. In the space $\Omega^\otimes 2k(B)$ this transformation induces $2k$-cubic matrix $J^{-1} \in M_{d×2k}$, such that

\[\Theta^\alpha\beta_{\alpha_1...\alpha_k\beta_1...\beta_k} = (J^{-1})_{\alpha_1...\alpha_k\beta_1...\beta_k} \Theta^\beta_{\beta_1...\beta_k\beta_{k+1}...\beta_{2k}} (J^{-1})_{\alpha_{k+1}...\alpha_{2k} \beta_{k+1}...\beta_{2k}}. \quad (31)\]

Obviously, that $J^{-1} = (j^{-1})^\otimes k$. The expression (31) has image in $M_{d^k×d^k}$:

\[\chi_*(\Theta') = \chi_*(\Theta)' = \chi_*(J^{-1})^T \chi_*(\Theta) \chi_*(J^{-1}),\]

— the formula similar to the $[30]$. Taking determinant of the both sides we get:

\[\det \chi_*(\Theta') = \det \chi_*(\Theta) |\det \chi_*(J^{-1})|^2 = \frac{\det \chi_*(\Theta)}{[\det \chi_*(J)]^2}. \quad (32)\]

where the relation $\chi_*(J^{-1}) = (\chi_*(J))^{-1}$ has been used, which, in turn, is direct consequence of the (i). From the [12] we see, that scalar density of weight $-1$ exists when the expression $[\det \chi_*(J)]^2$ is

\[\text{Here and below we omit for brevity } || \text{ and identify tensors with matrices, which represent them in some coordinate system.}\]
some degree of det $j$. It means, that degrees of $[\det \chi_*(J)]^2$ and det $j$, viewed as homogeneous polynomial relatively to derivatives $\partial \xi'/\partial \xi$, should be connected by the relation:

$$\deg(\partial \xi'/\partial \xi)[\det \chi_*(J)]^2 = l \cdot \deg(\partial \xi'/\partial \xi)\det j,$$

where $l \in \mathbb{R}$. Since

$$\deg(\partial \xi'/\partial \xi)\det j = d, \quad \deg(\partial \xi'/\partial \xi)[\det \chi_*(J)]^2 = 2 \cdot \deg J \cdot \deg \det|_{M^k \times M^k} = 2kd^k,$$

we go to the condition:

$$l = 2kd^{k-1},$$

which means, that the expression

$$|\det \chi_*(\Theta)|^{1/l} = \left|\det \Theta^{1/l} = \left|\det \Theta\right|^{1/2kd^{k-1}}$$

is the candidate on the scalar density of weight $-1$ relatively to general coordinate transformation on $B$. As it follows from (35), the case of forms of degree 2 is peculiar, since under $k = 1$ volume form takes the standard kind: $|\det \Theta|^{1/2} dx^1 \wedge \ldots \wedge dx^d$ and dependency on dimension of $B$ disappears.

The condition (33) and its consequences (34) and (35) are necessary but not sufficient for existence $d\mu$, since one should check that the homogeneous polynomial $|\det \chi_*(J)|^{2/l}$ with right degree $d$ is exactly equal to det $j$. Let consider the transformation $\xi_{\mu,\nu}: J \rightarrow \tilde{J}$ in $M^d \times \mathbb{R}^s$, which permutes any matrix element of $J$, up indexes of which contains $\mu$ and (or) $\nu$ with the elements, which have on the same positions indexes $\nu$ and (or) $\mu$ respectively. The transformation, in turn, induces transformation $(\xi_{\mu,\nu})_*: \chi_*(J) \rightarrow \chi_*(J)$, acting by the rule: $(\xi_{\mu,\nu})_*\chi_*(J) \equiv \chi_*(\xi_{\mu,\nu}J)$. It pair-wisely permutes lines in matrix $\chi_*(J)$, whose numbers $\alpha$ has preimages $\chi_{\alpha}^{-1}(a) = a_1 \ldots a_k$, containing in their sequences numbers $\mu$ and (or) $\nu$. Total number of such permutations in matrix $\chi_*(J)$ is equal:

$$P = \sum_{i=1}^{k} 2^{-i}i^k = (3^k - 1)/2. \quad (36)$$

So, under any permutation of two lines of Jacobi matrix $j$, (for columns all statements remains the same), det $\chi_*(J)$ considered as homogeneous polynomial with respect to $\partial \xi'/\partial \xi$ is transformed by the rule det $(\xi_{\mu,\nu})_*\chi_*(J) = (-1)^P \det \chi_*(J)$. It means, that det $\chi_*(J)$ up to a constant factor is $P + 2m$-th ($m$ — any integer) degree of det $j$, which is the unique function of $\partial \xi'/\partial \xi$ with required antisymmetry property. By the kind of isomorphism $\chi_*$ (identifying of elements), and by the tensor product structure $j^{\otimes k}$ of matrix $J$, the constant multiplier can not be dependent on the matrix. The fact, that it is equal unity can be directly checked by calculation of determinant of image of identical coordinate transformation:

$$\det \chi_*(E) = \det e = +1.$$ 

Now comparing the expressions

$$\det \chi_*(J) = (\det j)^{P + 2m}$$

with (33) and (35), we get their general consequence: $P + 2m = l/2$ or (using (36) and (34)):

$$3^k - 2kd^{k-1} = 4m + 1, \quad (37)$$

which should be considered as equation, relating dimensions of $d$-body and admissible degree of a $d$-metric within internal $d$-structures. All solutions of the equation can be parametrized by the three integer numbers $(m, k, d)$. For $-5 \leq m \leq 5$ there are the following solutions of (37):

$$(0, 1, d), (0, 2, 2), (1, 2, 1), (1, 2, 3), (2, 2, 0), (2, 2, 4), (3, 2, 5), (4, 4, 2), (4, 2, 6), (5, 3, 1), (5, 2, 7).$$

The first parenthesis says, that forms of degree 2 can be $d$-metrics of internal $d$-structures on manifolds with any dimensions. Easily to check, that for $k = 2$ and $k = 4$ there are no restrictions on $d$ too. In case $k = 3$ dimension $d$ can not be even.
C Restrictions on $\Theta$ in $\mathcal{M}$

To the moment we have considered the form $\Theta_B$, as initially defined on $\mathcal{B}$. Lets clear what conditions on the form $\Theta_M$ and embedding $\iota$ to be satisfied, when $\Theta_B = (d\iota)^*\Theta$ possesses nondegeneracy property as induced $d$-metric. We'll use $\chi^*_*$-representation for proving of statement 3 in Sec 4.3. Lets turn to the diagram (38).

\[
\begin{array}{c}
\Theta_M(s) \xrightarrow{(d\iota)^*(s)} \Theta_B(b) \\
\chi^{(n)} \Downarrow \chi^{(d)} \\
M_{n^k \times n^b} \xrightarrow{\chi^{(n)}(d\iota)^*} M_{d^k \times d^b}
\end{array}
\]

(38)

It shows, that after fixing some coordinates system, codifferential $(d\iota)^*$ at every point $s = \iota(b)$ of deformat $\mathcal{S}$ can be isomorphically represented by the linear operator $\chi^{(d)}(d\iota)^*(s) \equiv \chi^d \circ (d\iota)^*(s) \circ (\chi^*_*)^{-1}$, lying in $\text{Hom}(M_{n^k \times n^b}, M_{d^k \times d^b})$, where real linear spaces of matrices $M_{n^k \times n^b}$ and $M_{d^k \times d^b}$ represent $\chi^{(n)}(\Theta_M)$ and $\chi^{(d)}(\Theta_B)$ respectively in the fixed basis. In compact matrix form we have:

\[\Lambda_b = A^T \Lambda_s A,\]

where $\Lambda_b \equiv \chi^{(d)}(\Theta_B)(b)$, $\Lambda_s \equiv \chi^{(n)}(\Theta_M)(s)$, $A \equiv \chi^{(nd)}((d\iota)^*)$ and $M_{n^k \times d^k} \ni \chi^{(nd)}((d\iota)^*)_b \equiv ((Dx)^{\otimes k})_b(\chi^{(n)}(s))^{-1}(b)$ which we can interpret as element of $\text{Hom}(\mathbb{R}^{n^k}, \mathbb{R}^{d^k})$. Here we identify $\mathbb{R}^{n^k}$ and $\mathbb{R}^{d^k}$ with $(T_s\mathcal{M})^{\otimes k}$ and $(T_b\mathcal{B})^{\otimes k}$ respectively in our fixed coordinate system. So, we can put the problem at the point in language of real vector spaces. Let remind some definitions 29.

Consider $\mathcal{V}_1$ and $\mathcal{V}_2$ — some real linear vector spaces and $\mathcal{V}_1^*$ and $\mathcal{V}_2^*$ — their dual spaces (of linear functionals). Let $\Lambda : \mathcal{V}_1 \times \mathcal{V}_1 \to \mathbb{R}$ — bilinear form in $\mathcal{V}_1$. The set $L_A \subset \mathcal{V}_1$ is called left kernel of $\Lambda$, if $\Lambda(l, x) = 0 \forall x \in \mathcal{V}_1 \text{ and } l \in L_A$. Similarly, $R_A \subset \mathcal{V}_2$ is right kernel of $\Lambda$, if $\Theta(x, r) = 0 \forall x \in \mathcal{V}_1 \text{ and } r \in R_A$. The form $\Lambda$ is called nondegenerate 23 if $L_A = R_A = 0$. Let $A \in \text{Hom}(\mathcal{V}_1^*, \mathcal{V}_2)$ — some linear mapping of dual spaces. It has dual conjugated mapping $A^* \in \text{Hom}(\mathcal{V}_2^*, \mathcal{V}_1^*)$, defined by the rule:

\[(Au)(z) = u(A^*z)\]

for all $u \in \mathcal{V}_1^*$, $z \in \mathcal{V}_2$. Its kernel

\[\ker A^* = \{ w \in \mathcal{V}_2 | A^*w = 0 \}.\]

We denote $\text{Im} A^* \equiv \mathcal{V}_2^A$. The mapping $A$ induces mapping $A^2 : \mathcal{V}_1^{\otimes 2} \to \mathcal{V}_2^{\otimes 2}$ by the rule

\[A^2(z, w) \equiv A^2 \Theta(z, w) = \Theta(A^*z, A^*w),\]

(40)

for all $z, w \in \mathcal{V}_2$. If we fix some basises in $\mathcal{V}_1$ and $\mathcal{V}_2$, then (40) takes the following matrix form:

\[A^2 = A^T \Lambda A.\]

When $\Lambda^2$ will be nondegenerate? Let $L_{\Lambda^A} = R_{\Lambda^A} = 0$, then for $z, w \in \mathcal{V}_2$ where $z$ any fixed and $w$ runs whole $\mathcal{V}_2$ we have:

\[\Lambda^A(w, z) = \Lambda^A(z, w) = 0 \Rightarrow z = 0.\]

By (40) it means, that for any fixed $z$ and any $w$ we have:

\[A(A^*w, A^*z) = \Lambda(A^*z, A^*w) = 0 \Rightarrow A^*z = 0.\]

In other words, the narrowing $|\Lambda|_{\mathcal{V}_2^A}$ of the form $\Lambda$ must be nondegenerate. It means, that

\[(L_A \cup R_A) \cap \mathcal{V}_2^A = \emptyset.\]

23Now we use notations $\chi^{(n)}$ and $\chi^{(d)}$ for ordering of different sets $(Z_1^k)^{\times k}$ and $(Z_2^k)^{\times k}$ respectively.

24More generally, the form $\Lambda$ can be nondegenerate from the left and from the right.
Inversely, if $\Lambda|_{V_2^A}$ is nondegenerate, then for any $u$ running whole $V_2^A$ and for any fixed $v \in V_2^A$ we have

$$\Lambda(u,v) = \Lambda(v,u) = 0 \Rightarrow v = 0.$$  

By (39) we can symbolically express $u$ and $v$ as

$$u = A^*(w + \ker A^*), \quad v = A^*(z + \ker A^*),$$

for some $w, z \in V_2$. Then by (40) we have:

$$\Lambda(A^*(w + \ker A), A^*(z + \ker A^*)) = \Lambda(A^*(w, z)) = 0 \Rightarrow z \in \ker A^*.$$  

Similar conclusion takes place for left kernel of $\Lambda^A$. So, necessary (sufficient) condition of $\Lambda^A$ nondegeneracy is nondegeneracy of $\Lambda|_{V_2^A}$ (the condition $\ker A^* = 0$).

Now we come back to our initial problem. Identifying:

$$V_1 \equiv (T_sM)^{\otimes k}; \quad V_2 \equiv (T_bB)^{\otimes k}; \quad A \equiv (d\iota)^{\ast}; \quad A^* \equiv d\iota; \quad V_{A^2} \equiv (T_sS)^{\otimes k},$$

and observing, that $\ker d\iota \equiv 0$, since $\iota$ — embedding, we go to the expression (12) of Sec.3.3, which is exactly (11), where $L_{\Theta(M)}$ and $R_{\Theta(M)}$ are subsets of $V^{\otimes k}(M)$, such that at every point $m \in M$

$$L_{\Theta(m)} \equiv (\chi_n^{(n)})^{-1}(L_{\chi_n^{(n)}(\Theta(m))}); \quad R_{\Theta(m)} \equiv (\chi_n^{(n)})^{-1}(R_{\chi_n^{(n)}(\Theta(m))}).$$

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