Existence and uniqueness results for a class of non linear models

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Abstract - The qualitative analysis of the initial value problem $\mathcal{P}$ related to a non linear third order parabolic equation typical of diffusive models is discussed. Some basic properties of the the fundamental solution of a related linear operator are determined and are applied to an equivalent integro differential formulation of the problem. By the fixed point theorem, existence and uniqueness results are obtained.

Key words: Diffusion models, Partial and integro differential equations, Laplace transform

1 Introduction

Several mathematical models in applied sciences - as Biochemistry, Epidemiology, Population dynamics - deal with higher order parabolic equations typical of diffusive phenomena. The results of experimental analysis show that the diffusion coefficients are generally small and this requires too much time in the transmission of signals. As consequence it necessary to correct the diffusive models by means of reaction-diffusion systems. So, one obtains non linear hyperbolic equations perturbed by viscous terms of higher order.

A typical example is given by equations as

\begin{equation}
Lu \equiv ( \varepsilon \partial_t + c^2 ) u_{xx} - ( \partial_t + a ) u_t = F(x, t, u, u_x),
\end{equation}
which are object of an extensive literature [1]- [5].

When $F$ is linear, numerous contributions are already known [6] - [15]. In particular the fundamental solution $K$ of the operator $L$ has been obtained in [16] where some basic properties has been discussed too.

In this paper, the non linear case of (1.1) will be considered and existence and uniqueness results for the initial value problem $\mathcal{P}$ in all of the space will be obtained.

For this, the equivalence between the problem $\mathcal{P}$ and an integro- differential equation with kernel $K$ is deduced and the fixed point theorem is applied by means of appropriate estimates for $K$.

This analysis can be also applied to obtain the dependence continuously upon the data, stability properties and boundary - layer estimates. More, the study of other boundary value problems (semi infinite or finite media) with Dirichlet or Neumann conditions could be achieved too.

2 Fundamental solution and its properties

If $D = \{(x, t) : x \in \mathbb{R} , \ 0 < t \leq T \}$, let $r = |x|/\varepsilon$, $b = c^2/\varepsilon$. As one can verify, the fundamental solution of the $L-$ operator is simbolically defined by

\[ \hat{K}(r, s) = \frac{e^{-|x|\sqrt{s(a+c^2)/(\varepsilon s + c^2)}}}{2\sqrt{s(a+c^2)/(\varepsilon s + c^2)}}. \]  

If one puts

\[ \hat{G}(r, s) = \frac{e^{-r\sqrt{s+a}/(s+b)}}{2\sqrt{\varepsilon \sqrt{(s+a)(s+b)}}}, \]

obviously one has:

\[ \hat{K}(r, s) = \frac{1}{\sqrt{s}} \hat{G}(r, s) \Rightarrow K(r, t) = \int_0^t G(r, \tau) \frac{d\tau}{\sqrt{\pi(t-\tau)}} \]

where $G(r, t)$ is the inverse tranform of $\hat{G}(r, s)$ given by (2.2).
If \( I_n(z) \) denotes the modified Bessel function of the first kind, the explicit expression of \( G(r, t) \) is given by:

\[
G(r, t) = \frac{r}{4 \sqrt{\pi \varepsilon}} \int_0^t e^{-\frac{r^2}{4v}} e^{-\frac{b(t-v)}{v}} I_0(r \sqrt{(b-a)(t-v)}) \frac{dv}{v},
\]

and the following theorem holds:

**Theorem 2.1** - For all \( r > 0 \), the Laplace integral \( \mathcal{L}_t G(r, t) \) converges absolutely in the half-plane \( \Re s > \max(-a, -b) \), and one has:

\[
\mathcal{L}_t G(r, t) = \hat{G}(r, s) = \frac{e^{-r \sqrt{s(a+b)/(s+b)}}}{2 \sqrt{\varepsilon} \sqrt{(s+a)(s+b)}}.
\]

According to the results established in [16], for the function \( K \) the following properties hold:

**Theorem 2.2** - The function \( K \) defined by (2.3), (2.4) is a \( C^\infty(D) \) solution of the equation \( Lu = 0 \). When \( a < b = \frac{c^2}{\varepsilon} \), \( K \) is never negative in \( D \) and more it results:

\[
0 \leq \int_{-\infty}^{\infty} K(|x-\xi|, t) \, d\xi = \left( \frac{1}{a} \right) (1 - e^{-at}) \leq 1/a,
\]

\[
\lim_{r \to 0} \partial_r K(r, t) = -\frac{e^{-bt}}{2\varepsilon}.
\]

Moreover, one has:

**Corollary 1.2** - By means of the Laplace transforms, it’s easy to verify the following formulae:

\[
\int_{\mathbb{R}} \left( \partial_t + a \right) K(x, t) \, dx = 1,
\]
\begin{align*}
(2.9) & \quad \int_{\mathbb{R}} (\partial_t + b)K(x, t) \, dx = e^{-at} + b(1 - e^{-at})/a , \\
(2.10) & \quad \int_{\mathbb{R}} (\partial_t + a - \varepsilon \partial^2_{xx})K(x, t) \, dx = 1 - e^{-bt} .
\end{align*}

3 Initial value problem and explicit solution in the linear case

Let consider now the initial value problem \( P \) in \( D \):

\begin{equation}
\begin{cases}
Lu \equiv (\varepsilon \partial_t + c^2)u_{xx} - (\partial_t + a)u_t = F(x, t, u, u_x), \quad (x, t) \in D, \\
u(x, 0) = f_0(x), \quad u_t(x, 0) = f_1(x), \\
x \in \mathbb{R},
\end{cases}
\end{equation}

where the initial data \( f_0 \) and \( f_1 \) are arbitrary specified functions.

If \( g(x) \) is a continuous function on \( \mathbb{R} \), consider the convolution:

\begin{equation}
\begin{aligned}
(3.2) \quad u_g &= \int_{\mathbb{R}} g(\xi) K(x - \xi, t) \, d\xi = K * g .
\end{aligned}
\end{equation}

Then, owing to the properties of \( K(x, t) \), it’s possible to prove the following theorem:

**Theorem 3.1** - If \( \alpha, \beta, \gamma \) are three positive constants such that

\begin{equation}
|g(x)| < \alpha e^{\beta|x|^{\gamma+1}} , \quad 0 \leq \gamma < 1,
\end{equation}

then the function \( u_g \) is a \( C^\infty(D) \) solution of the equation \( Lu = 0 \), such that

\begin{equation}
\begin{aligned}
(3.4) \quad \lim_{t \to 0} u_g(x, t) &= 0, \\
& \lim_{t \to 0} \partial_t u_g(x, t) = g(x) ,
\end{aligned}
\end{equation}

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uniformly for all $x$ in a compact subset of $-\infty < x < \infty$.

Observe now that the convolution $K \ast g$ is infinitely differentiable in $D$ and, according to the results of section 2, one has:

$$(3.5) (\partial_t + a - \varepsilon \partial_{xx}^2)u_g = e^{-b t} g(x) + \int_{\mathbb{R}} g(\xi) \left( \partial_t + a - \varepsilon \partial_{xx}^2 \right) K(x - \xi, t) \, d\xi$$

and (2.10) implies:

$$(3.6) \lim_{t \to 0} \int_{\mathbb{R}} g(\xi) \left( \partial_t + a - \varepsilon \partial_{xx}^2 \right) K(x - \xi, t) \, d\xi = 0.$$  

Consequently, one has:

**Theorem 3.2** - If $g(x)$ verifies the hypotheses of Theorem 3.1, then the function

$$(3.7) \quad u^*_g = (\partial_t + a - \varepsilon \partial_{xx}^2) u_g$$

represents a smooth solution of the equation $Lu = 0$, such that

$$(3.8) \quad \lim_{t \to 0} u^*_g(x, t) = g(x), \quad \lim_{t \to 0} \partial_t u^*_g(x, t) = 0$$

uniformly for all $x$ in a compact subset of $-\infty < x < \infty$.

**Remark 3.1** - When $F \equiv 0$, the results of Theorems 3.1 - 3.2 allow to assert that the function

$$(3.9) \quad u(x, t) = u_{f_1} + (\partial_t + a - \varepsilon \partial_{xx}^2) u_{f_0}$$

represents a smooth solution of the homogeneous initial-value problem (3.1).

Consider now the case that $F$ is not vanishing but it is a linear known function $F = f(x, t)$ and let $u_f$ the volume potential

$$(3.10) \quad u_f = \int_{0}^{t} K \ast f \, d\tau = \int_{0}^{t} d\tau \int_{\mathbb{R}} f(\xi, \tau) K(x - \xi, t - \tau) \, d\xi.$$
It’s possible to prove the following

**Theorem 3.3** - When the source term $f(x, t)$ is continuous and everywhere bounded function in the set $-\infty < \xi < \infty$, $0 < \tau < T$, then $u_f$ represents a smooth solution of the problem $L u_f = -f(x, t)$ with vanishing initial conditions.

Proof - If $||f||_T = \sup_D |f(x, t)|$, by (2.6) one has:

$$
(3.11) \quad |u_f(x, t)| \leq a^{-1} t \cdot ||f||_T \Rightarrow \lim_{t \to 0} u_f = 0.
$$

Further it results:

$$
(3.12) \quad |K \ast f| \leq a^{-1} \left[1 - e^{-a(t-\tau)}\right] ||f||_T
$$

and so

$$
(3.13) \quad \lim_{\tau \to t} \int_{\mathbb{R}} K(x - \xi, t - \tau) f(\xi, \tau) d\xi = 0 \Rightarrow \lim_{\tau \to 0} \partial_t u_f = 0
$$

More, by means of estimates similar to (3.6), one deduces that:

$$
(3.14) \quad \lim_{\tau \to t} \int_{\mathbb{R}} f(\xi, \tau) \left(\partial_t + a - \varepsilon \partial^2_{xx}\right) K(x - \xi, t - \tau) d\xi = 0,
$$

and this, together with the properties of $K$ (theorem 2.2), implies $L u_f = -f(x, t)$. Further, theorems 3.1 - 3.2 imply that $u_f(x, 0) = 0$, $\partial_t u_f(x, 0) = 0$.

At last, by (3.9) and theorem 3.3, it follows:

**Theorem 3.4** - When $F = f(x, t)$ and the data $f$, $f_0$, $f_1$ verify the hypotheses specified in theorems 3.1 - 3.3, then an explicit smooth solution of the problem (3.1) is given by

$$
(3.15) \quad u(x, t) = -u_f + u_{f_1} + (\partial_t + a - \varepsilon \partial^2_{xx}) u_{f_0}
$$
4 Integro differential formulation of the non linear problem

Consider now the non linear case of the problem (3.1), where \( F = F(x, t, u, u_x) \) is defined on the set

\[
\Omega = \{ (x, t, u, p) : (x, t) \in D, \quad -\infty < u < \infty; \quad -\infty < p < \infty \},
\]

with \( D = \{ (x, t) : x \in \mathbb{R}, 0 < t \leq T \} \).

From now on we shall assume for \( F = F(x, t, u(x, t), u_x(x, t)) \) the following Hypotheses H:

- The function \( F(x, t, u, p) \) is defined and continuous on \( \Omega \) and it’s bounded for all \( u \) and \( p \).
- For each \( k > 0 \) and for \( |u|, |p| < k \), the function \( F \) is Lipschitz continuous in \( x \) and \( t \) for each compact subset of \( D \).
- There exists a constant \( \beta_F \) such that:

\[
|F(x, t, u_1, p_1) - F(x, t, u_2, p_2)| \leq \beta_F \{ |u_1 - u_2| + |p_1 - p_2| \}
\]

holds for all \( (u_i, p_i) \) \( i = 1, 2 \).

Let \( u \) be a smooth solution of the problem (3.1) and, referring to the function \( K(x - \xi, t - \tau) u(\xi, \tau) \), consider the operators \( M(x, t, \xi, \tau) \), \( N(x, t, \xi, \tau) \) defined by:

\[
M(x, t, \xi, \tau) = \varepsilon K_{\xi \xi} u_{\xi} - K_{\tau} u_{\tau} + u K_{\tau} - a K u
\]

\[
N(x, t, \xi, \tau) = u (\varepsilon \partial_{\tau} - c^2) K_{\xi} - u_\xi (\varepsilon \partial_{\tau} - c^2) K
\]

Further, let \( \tilde{L} \) be the adjoint operator of \( L \):

\[
\tilde{L} K = -\varepsilon K_{\xi \xi \tau} + c^2 K_{\xi \xi} - K_{\tau \tau} + a K_{\tau}
\]
in order to have:

\begin{equation}
KL u - u \tilde{K} = K F.
\end{equation}

As consequence, one has

\begin{equation}
\partial_t \mathcal{M} + \partial_\xi \mathcal{N} = KF.
\end{equation}

If $u_F$ denotes the non linear potential volume

\begin{equation}
u_F = \int_0^t d\tau \int_\mathbb{R} K(\xi - x, t - \tau) \ F(\xi, \tau, u(\xi, \tau), u_\xi(\xi, \tau), u_{\xi\xi}(\xi, \tau)) \ d\xi,
\end{equation}

it suffices to integrate (4.4) on $D$ in order to have

\begin{equation}
\int_{-\infty}^{\infty} [ \mathcal{M}(x, t, \xi, t) - \mathcal{M}(x, t, \xi, 0) ] \ d\xi = u_F
\end{equation}

where the term with $\partial_\xi \mathcal{N}$ doesn't contribute because $\lim_{|\xi| \to \infty} \mathcal{N} = 0$.

Let assume that $f_0 \in C^2(\mathbb{R})$, $f_1 \in C^1(\mathbb{R})$, and more $f_0, f_0', f_0'', f_1$ let bounded on $\mathbb{R}$. Then, by means of (3.4) - (3.6) and estimates similar to (3.13), (3.14), by (4.6) one obtains the following representation

\begin{equation}
u(x, t) = -u_F + \int_\mathbb{R} f_1(\xi) \ K(\xi - x, t) \ d\xi + \int_\mathbb{R} f_0(\xi) \ K(\xi - x, t) \ d\xi
\end{equation}

Viceversa, suppose that the integral differential equations (4.7) has a solution $u = u(x, t)$ such that $u$ and $u_x$ are continuous and bounded for $x \in \mathbb{R}$ and $0 < t \leq T$. Then the terms depending on the initial data $f_0, f_1$ are differentiable for $t > 0$ with bounded derivatives for $t > 0$. The properties of the fundamental solution $K$ and the hypotheses $H$ allow to prove that also the volume potential $u_F$ and its derivatives are Lipschitz continuous. Then the function $u$ characterized by (4.7) verifies the problem (3.1) everywhere in $D$. 

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The complete equivalence between the problem $P$ and the integro differential equation (4.7) confirms the existence and uniqueness of the solution of (3.1) as soon as when this will be demonstrate for (4.7).

5 Existence and uniqueness

Consider now a time - interval $[0, \eta]$, with $\eta < T$. At first we will show that the solution of the integral differential equation (4.7) exists and is unique for $t \in [0, \eta]$ and then we will extend this for any finite $T$.

For $\eta > 0$, let

\[ D_\eta = \{(x, t) : x \in \mathbb{R}, \ t \in [0, \eta]\} \]

and let $B_\eta$ the space

\[ B_\eta = \{v(x, t) : v, v_x \in C(D_\eta)\} \]

equipped with the norm

\[ ||v||_\eta = \sup_{D_\eta} |v(x, t)| + \sup_{D_\eta} |v_x(x, t)| < \infty. \]

The set is a Banach space. The arguments discussed previously allow to state that the mapping

\[ Fv(x, t) = \int_{\mathbb{R}} f_1(\xi) K(\xi - x, t) d\xi + (\partial_t + a - \varepsilon \partial_{xx}) \int_{\mathbb{R}} f_0(\xi) K(\xi - x, t) d\xi + \]

\[ + \int_0^t d\tau \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, v(\xi, \tau), v_\xi(\xi, \tau)) d\xi \]

maps $B_\eta$ into $B_\eta$.

Estimate now the continuity of $F$. For all $t \in [0, \eta]$, owing to (2.6) and hypotheses $H$, it results
\( |Fv_1(x,t) - Fv_2(x,t)| \leq \beta_F a^{-1} t \|v_1 - v_2\|_\eta \leq \beta_F a^{-1} \eta \|v_1 - v_2\|_\eta. \)

As for the x - derivative, by means of \( \hat{K}_r(z,t) \), one can prove that:

\( | \frac{\partial}{\partial x} Fv_1(x,t) - \frac{\partial}{\partial x} Fv_2(x,t)| \leq \beta_F a^{-1} \frac{1}{\sqrt{\varepsilon(b-a)}} \eta \|v_1 - v_2\|_\eta. \)

As consequence, by (5.3), (5.5),(5.6) it follows

\( \|Fv_1(x,t) - Fv_2(x,t)\|_\eta \leq \beta_F \left( \frac{1}{a} + \frac{1}{\sqrt{\varepsilon(b-a)}} \right) \eta \|v_1 - v_2\|_\eta. \)

If we select \( \eta \) such that

\( \beta_F \left( \frac{1}{a} + \frac{1}{\sqrt{\varepsilon(b-a)}} \right) < 1, \)

then \( F \) is a contraction of \( \mathcal{B}_\eta \) into \( \mathcal{B}_\eta \) and so \( F \) has a unique fixed point \( u(x,t) \in \mathcal{B}_\eta \).

In order to show the existence and uniqueness of the solution in all \([0,T]\) we proceede by induction.

Let assume that the equation (4.7) admits a unique solution \( u \) which is bounded for \( 0 \leq t \leq k\eta \) together with \( u_x \).

For \( k\eta \leq t \leq (k+1)\eta \), let consider the space

\( \bar{\mathcal{B}}_\eta = \{ v(x,t) : v(x,t-k\eta) \in \mathcal{B}_\eta \} \)

and the mapping

\[ F_1 v(x,t) = \int_{\mathbb{R}} f_1(\xi) K(\xi - x, t) d\xi + (\partial_t + a - \varepsilon \partial_{xx}) \int_{\mathbb{R}} f_0(\xi) K(\xi - x, t) d\xi + \]

\[ + \int_0^{k\eta} d\tau \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi + \]

\[ + \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi + \]

\[ + \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi + \]

\[ + \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi + \]

\[ + \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi + \]
\[ + \int_{k \eta}^{t} d\tau \int_{-\infty}^{\infty} K(x - \xi, t - \tau) F(\xi, \tau, v(\xi, \tau), v_x(\xi, \tau)) d\xi, \]

which maps \( \bar{B}_\eta \) into \( \bar{B}_\eta \). Moreover, also the mapping \( F_1 \) satisfies estimates like (5.7).

As consequence, for all \( t \in [0, (k + 1) \eta] \) the mapping \( F_1 \) is a contraction and hence admits an unique fixed point \( u \in \bar{B}_\eta \).

We remark that, as (5.10) shows, the functions \( u \) and \( u_x \) are continuous also on the junction \( t = K\eta \).

In conclusion, the results in sections 4 and 5 allow to state the following theorem:

**Theorem 5.1** - Let the initial data such that \( f_0 \in C^2(\mathbb{R}), f_1 \in C^1(\mathbb{R}) \). Let \( f_0, f'_0, f''_0, f_1 \) bounded on \( \mathbb{R} \). Further, let \( F \) verify the hypotheses \( H \). Then, the nonlinear problem (3.1) admits a unique regular solution in \( D \).

\[ \square \]

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