We introduce in typical examples new methods for the calculation of massive loop integrals appearing in the radiative correction calculations of the Standard Model.

October 1992

Talk given at the workshop on HEP and QFT, Sochi, Russia, 1992
1 Introduction

In the following we will be concerned with the calculation of massive loop integrals. We will present methods to calculate such integrals for the 1- and 2-loop case. For the 1-loop case these methods allow for the calculation of arbitrary rank tensor integrals for arbitrary n-point functions with arbitrary integer powers of the propagators. The methods for the 2-loop case cover the scalar planar and nonplanar integrals. The tensor cases can be handled by the usual reduction to 1-loop integrals or, in a much more direct approach, by integrating the ‘characteristic integrand’ of the graph under consideration [15].

In this article we give only typical introductory examples of these methods. We refer the reader to the literature to study the methods in detail [9, 10, 7, 11, 12].

2 1-loop Integrals

2.1 The Scalar Function

In this section we calculate the two-point function

\[ I(q^2) = \int d^Dl \frac{1}{P_1 P_2}, \]  

where

\[ P_1 = l^2 - m_1^2 + iq \]
\[ P_2 = (l + q)^2 - m_2^2 + iq. \]

Here we do not specify the masses \( m_1, m_2 \), so we want a result which is valid for the massive as well as for the massless case. We work within dimensional regularization in the sense of [4], \( D = 4 - 2\varepsilon \) is the complex dimension.

There is only one scalar-product involved in the integrand Eq.(1) and so if we choose to be in the rest frame of \( q = q_0 \), we can separate the \( l_0 \)-integration. No non-trivial angular integration is then involved in the \((D-1)\)-dimensional space-like integration. So we can do \( D-2 \) angular integrations trivially, leaving the \( l_0 \) and \( l_\perp \) integrations to be done. We have denoted the modulus of the space-like part of \( l \) by \( l_\perp \).

The integral Eq.(1) then becomes

\[ I(q^2) = 2\pi^{\frac{D-1}{2}} \frac{1}{\Gamma(\frac{D-1}{2})} \int_{-\infty}^{\infty} dl_0 \int_0^\infty dl_\perp l_\perp^{D-2} \frac{1}{P_1 P_2} \]

where

\[ P_1 = l_0^2 - l_\perp^2 - m_1^2 + iq \]
\[ P_2 = (l_0 + q_0)^2 - l_\perp^2 - m_2^2 + iq. \]

The \( l_\perp \)-integration in Eq.(2) can now easily be performed. The result is

\[ I(q^2) = C \int_{-\infty}^{\infty} dl_0 \frac{1}{w_1^2 - w_2^2} \left[ w_1^{1-2\varepsilon} - w_2^{1-2\varepsilon} \right] \]

where

\[ w_1 = \sqrt{-l_0^2 + m_1^2 - iq} \]
\[ w_2 = \sqrt{-(l_0 + q)^2 + m_2^2 - iq} \]
\[ C = \pi^{\frac{D-1}{2}} / (1/2 - \varepsilon) \Gamma(1/2 + \varepsilon). \]
Now the remaining $l_0$-integration becomes
\[ I(q^2) := \frac{C e^{i\pi(1/2-\epsilon)}}{2q} \int_{-\infty}^{\infty} dl_0 \left[ \frac{(l_0^2 - m_1^2 + i\varrho)^{\frac{1}{2} - \epsilon}}{q/2 + M_d + l_0} + \frac{(l_0^2 - m_2^2 + i\varrho)^{\frac{1}{2} - \epsilon}}{q/2 - M_d - l_0} \right] \]
\[ M_d = \frac{m_1^2 - m_2^2}{2q}, \quad (4) \]
where we used translation invariance for the second term in Eq.(3), translation invariance being guaranteed by the very definition of dimensional regularization (Re(\(\epsilon\)) big enough in intermediate steps). The integral Eq.(4) can be determined by various methods. We will do it in the following by introducing $\mathcal{R}$-functions [3, 9, 10] which are the typical ingredient of our 1-loop method.

We modify the integral Eq.(4) as follows.
\[ I = \frac{Ci \exp(i\pi\epsilon)}{2q} \int_0^{\infty} \frac{ds}{\sqrt{s}} \left[ \frac{(q/2 + M_d)(s - m_1^2 + i\varrho)^{\frac{1}{2} - 2\epsilon}}{-(q/2 + M_d)^2 + s} + \frac{(q/2 - M_d)(s - m_2^2 + i\varrho)^{\frac{1}{2} - 2\epsilon}}{-(q/2 - M_d)^2 + s} \right], \quad (5) \]
which gives in terms of $\mathcal{R}$-functions
\[ I \quad = \quad \frac{i \exp(i\pi\epsilon) C}{2q} \int_0^{\infty} \frac{ds}{\sqrt{s}} \left[ (q/2 + M_d)B\left(\frac{1}{2}, \epsilon\right) \mathcal{R}_{-\epsilon}\left(-\frac{1}{2} + \epsilon, 1; -m_1^2 + i\varrho, -(q/2 + M_d)^2\right) + (q/2 - M_d) \times \mathcal{R}_{-\epsilon}\left(-\frac{1}{2} + \epsilon, 1; -m_2^2 + i\varrho, -(q/2 - M_d)^2\right) \right]. \quad (6) \]

The Beta-function in Eq.(6) involves a pole in $\epsilon$. Expanding the following $\mathcal{R}$-functions to $\mathcal{O}(\epsilon)$ (see [3]) we can express our result in terms of logarithms. The result is
\[ I(q^2) \quad = \quad i\pi^2 \left\{ \frac{1}{\varepsilon} - \gamma - \log \pi + 2 + \frac{1}{2}(1 + m_1^2/q^2 - m_2^2/q^2) \times \right. \]
\[ \times \left[ z_1 \log \left( \frac{1}{1 + z_1} \right) \right] - \log m_1^2 \right\} \]
\[ + \frac{1}{2}(1 - m_1^2/q^2 + m_2^2/q^2) \times \left[ z_2 \log \left( \frac{1}{1 + z_2} \right) \right] \left\{ \log m_2^2 \right\}, \]
where
\[ z_1 = \sqrt{1 - \frac{4m_1^2q^2}{(q^2 + m_1^2 - m_2^2)^2}}, \]
\[ z_2 = \sqrt{1 - \frac{4m_2^2q^2}{(q^2 - m_1^2 + m_2^2)^2}}. \]

where the upper (lower) sign has to be used if the argument is in the right (left) complex half-plane corresponding to Carlson’s conventions on the scaling law for
$\mathcal{R}$-functions and quadratic transformations on them [3]. This is the well-known result which one also obtains by conventional methods.

2.2 The Tensor Functions

As it is shown in [9] the above reduction allows also to calculate arbitrary tensor integrals. Here we give only the result.

$$I_{jk}(q^2) = \tilde{C} \left( B(\varepsilon - k/2 - [j]/2, 1/2 + [j]/2)(q/2 + M_d)^{i+1-[i]} \right.$$

$$\times (-1)^{j-[j]} \mathcal{R}_{-\varepsilon+k/2+[j]/2}(-1/2 - k/2 + \varepsilon, 1; z_1, y_1)$$

$$+ \sum_{i=0}^{j} \binom{j}{i} (-q)^{j-i} B(\varepsilon - k/2 - [i]/2, 1/2 + [i]/2)(q/2 - M_d)^{i+1-[i]}$$

$$\times \mathcal{R}_{-\varepsilon+k/2+[i]/2}(-1/2 - k/2 + \varepsilon, 1; z_2, y_2) \right)$$

where

$$[m] := \begin{cases} m, m \text{ even} \\ m + 1, m \text{ odd} \end{cases}$$

$$\tilde{C} := \frac{\pi^{D-1}}{1/2 + k/2 - \varepsilon} \frac{i \exp(i\pi\varepsilon)}{2q} \times \frac{(-1)^{k/2}}{2}$$

$$\frac{\Gamma(3/2 - \varepsilon + k/2)\Gamma(1/2 - k/2 + \varepsilon)}{\Gamma(3/2 - \varepsilon)}$$

$$z_1 := -m_1^2 + i\varrho, y_1 = -(q/2 + M_d)^2$$

$$z_2 := -m_2^2 + i\varrho, y_2 = -(q/2 - M_d)^2.$$ 

In the above formula for $I_{jk}(q^2)$ $j$ denotes the number of indices in the parallel space and $2k$ the number of indices in orthogonal space. One then has for a tensor integral

$$I_{\mu_1...\mu_j} = 0, \ j_2 \text{ odd}$$

$$= I_{j_1j_2} N(j_2) T_{\mu_{j_1+1}...\mu_j}, \ j_2 \text{ even},$$

where

$$T_{\mu_{j_1+1}...\mu_j} = g_{\mu_{j_1+1}\mu_{j_1+2}}^{D-1} ... g_{\mu_{j-1}\mu_j}^{D-1} + \text{all permutations of } \{\mu_{j_1+1}...\mu_j\}$$

$$N(j_2) = g^{\mu_{j_1+1}\mu_{j_1+2}} \cdots g^{\mu_{j-1}\mu_j} T_{\mu_{j_1+1}...\mu_j}.$$ 

Here we assumed without loss of generality that the first $j_1$ indices lie in the parallel space. We omitted these indices on the right hand side of the above equation because they all have fixed value $\mu_1 = \ldots = \mu_{j_1} = 0$ (the parallel space is one-dimensional in this simple example). So, the expression on the right hand side is a $j_2$-rank tensor in the orthogonal space. This tensor structure is given by the totally symmetric tensor $T$, constructed as the symmetric product of metric tensors in the orthogonal space. $N(j_2)$ normalizes the tensor $T$. With $g_{\mu\nu}^{D-1}$ we denote this metric tensor in the orthogonal space. One has $g_{\mu\nu}^{D-1} = g_{\mu\nu}^{D-1} g^{\mu\nu} = D - 1$.

In the derivation of this formula the celebrated ’existence theorem of associated functions’ has been used [3].

The calculation of higher powers of propagators can now be easily done by applying appropriate differential operators with respect to masses to the result expressed in $\mathcal{R}$-functions [3].
The next step in our examination of one-loop integrals is the calculation of the three-point functions. These form factor functions have infrared and on-shell singularities. They are, therefore, a more sophisticated test of our method. Nevertheless, they can be calculated following the same route as for the 2-point functions \[10\]. The same is true for arbitrary \(n\)-point functions \[11\, 12\].

In the following we shortly recapitulate the calculation of the 3-point functions. We use the following notation for the one-loop three-point scalar integral

\[
I = \int d^D l \frac{1}{P_0 P_1 P_2},
\]

where

\[
P_0 = l_0^2 - l_1^2 - l_\perp^2 - m_0^2 + i\eta, \\
P_1 = (l_0 + q_{10})^2 - l_1^2 - m_1^2 + i\eta, \\
P_2 = (l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_\perp^2 - m_2^2 + i\eta,
\]

so that \(l_0\) and \(l_1\) span the parallel space (the now two-dimensional linear span of the exterior momenta), while \(l_\perp\), as before, is the modulus of the loop-momentum in the orthogonal space.

The resulting non-trivial integrations are

\[
I = C \int_{-\infty}^{+\infty} dl_0 \int_{-\infty}^{+\infty} dl_1 \int_{0}^{+\infty} dl_\perp \frac{i^{D-3}}{P_0 P_1 P_2},
\]

where \(C = 2\pi \frac{\rho_{\Pi}}{\Gamma(D-2)}\).

With the help of a partial fraction we have three summands to calculate. The \(l_\perp\)-integration is easy to do. The next integration can be done via the residue theorem and the remaining integrals will again be interpreted as \(\mathcal{R}\)-functions.

Introducing the abbreviations

\[
a_{01} = -2q_{10}, \quad b_{01} = 0, \quad c_{01} = m_1^2 - m_0^2 - q_{10}^2, \\
a_{02} = -2q_{20}, \quad b_{02} = 2q_{21}, \quad c_{02} = m_2^2 - m_0^2 - q_{20}^2 + q_{21}^2, \\
a_{11} = 2q_{10}, \quad b_{11} = 0, \quad c_{11} = -m_2^2 + m_0^2 - q_{10}^2, \\
a_{12} = 2(q_{10} - q_{20}), \quad b_{12} = 2q_{21}, \quad c_{12} = -m_1^2 + m_0^2 - (q_{10} + q_{20})^2 + q_{21}^2, \\
a_{21} = 2q_{20}, \quad b_{21} = -2q_{21}, \quad c_{21} = -m_3^2 + m_0^2 - q_{20}^2 + q_{21}^2, \\
a_{22} = 2(q_{20} - q_{10}), \quad b_{22} = -2q_{21}, \quad c_{22} = -m_2^2 + m_0^2 - (q_{10} + q_{20})^2 + q_{21}^2,
\]

we find the final result to be

\[
I = C\mathcal{B}(2\varepsilon, 1)B(1 - \varepsilon, \varepsilon) \sum_{i=0}^{2} \sum_{l=1}^{4} \alpha_l \frac{A_{i,l}}{C_{i,l}}, \quad \mathcal{R}_{-2\varepsilon}(\varepsilon, \varepsilon, 1; -y_{i,l}^+, y_{i,l}^-, D_{i,l}/C_{i,l})
\]

where

\[
y_{i,l}^+ = -\frac{B_{i,l}}{2A_{i,l}} + \sqrt{\frac{B_{i,l}^2}{4A_{i,l}^2} - \frac{m_l^2 - i\eta}{A_{i,l}}}, \\
y_{i,l}^- = -\frac{B_{i,l}}{2A_{i,l}} - \sqrt{\frac{B_{i,l}^2}{4A_{i,l}^2} - \frac{m_l^2 - i\eta}{A_{i,l}}}, \\
\alpha_l = \begin{cases} +1, l = 1, 2 \\ -1, l = 3, 4 \end{cases}
\]
that we have to modify the three steps as follows:

1. The $l_1$-integration, followed by a shift in $l_0, l_1$ to have the $z_i$ in a standard form.

2. A further shift $l_1 \to l_1 = l_1 + l_0$ linearizes the $z_i$’s and allows for an application of the residue-theorem to do the $l_0$-integration.

3. Interpretation of the remaining integration as a representation of a $\mathcal{R}$ function.

Arbitrary tensor integrals involve additional powers $l_0^{n_1} l_1^{n_2} l_2^{n_3}$ in the numerator so that we have to modify the three steps as follows:

1. The $l_1$-integration now gives $-B(1 + n_3 - \varepsilon, -n_3 + \varepsilon)(-z_i)^{-\varepsilon + n_3}$. The shift in $l_0$ and $l_1$ now introduces a (double) binomial sum of powers from $l_0^0 l_1^{n_2}$ to $l_0^{n_1} l_1^{n_2}$ in the numerator.

2. The further shift $l_1 \to l_0 + l_1$ changes the powers in the numerator to have values from $l_0^0 l_1^{n_2}$ to $l_0^{n_1 + n_2} l_1^{n_1}$. The application of the residue theorem inserts the residue also in the numerator, so that the remaining $l_1$ integration involves powers $l_1^{0} \ldots l_1^{n_1+2n_2}$ in the numerator.

3. The remaining integrals are of the form $\int_0^\infty dy \, y^{\mu_2 + \nu_2 + \mu_3 + \nu_3} \mathcal{R}^{-2\varepsilon + 2n_3 + k}(\varepsilon - n_3, \varepsilon - n_3, 1; -y_+, -y_-, s/r)$ so that we will end up with results of the form $\sim B(k + 1, 2\varepsilon - 2n_3 - k) \frac{u^{-\varepsilon + n_3}}{r} \mathcal{R}^{-2\varepsilon + 2n_3 + k}(\varepsilon - n_3, \varepsilon - n_3, 1; -y_+, -y_-, s/r)$.

Applying reduction formulas for $\mathcal{R}$-functions all tensor integrals are now expressible in a standard set of well-known functions [10]. Integrals involving higher powers of propagators can again be obtained via differentiation formulas.

### 2.3 The Singular Case

In evaluating the three- and higher $n$-point functions one counters more than just the $\text{UV}$-singularities. There are on-shell and infrared singularities as well. These divergences always appear as parallel-space divergences. Here they enter the calculations in the form of endpoint singularities. So they can be systematically handled by applying Hadamard’s finite part to them [10].
3 2-loop Functions

In this section we start with the calculation of massive scalar 2-loop functions. We start with the 2-point case as a more detailed example and present the results for the two different topologies which appear in the 3-point case.

3.1 The 2-point Function

Let us first introduce some notation. The expression for the normalized two-loop two-point function is

\[ I(q^2) = -\frac{q^2}{\pi^2} \int d^4l_1 \int d^4l_2 \prod_{i=1}^5 \frac{1}{P_i}. \]

According to our method of separating parallel and orthogonal space variables we rewrite the integral as

\[ I(q^2) = -\frac{q^2}{\pi^2} \int d^4l_1 \int d^4l_2 \prod_{i=1}^5 \frac{1}{P_i}, \]

(10)

where the inverse propagators \( P_i \) are now defined by

\[
\begin{align*}
P_1 &= l_{10}^2 - l_{1\perp}^2 - m_1^2 + i\varrho \\
P_2 &= (l_{10} + q_0)^2 - l_{2\perp}^2 - m_2^2 + i\varrho \\
P_3 &= (l_{10} + l_{20})^2 - l_{1\perp}^2 - l_{2\perp}^2 - 2l_{1\perp}l_{2\perp}z - m_3^2 + i\varrho \\
P_4 &= (l_{20} - q_0)^2 - l_{2\perp}^2 - m_3^2 + i\varrho \\
P_5 &= l_{20}^2 - l_{2\perp}^2 - m_4^2 + i\varrho,
\end{align*}
\]

and we choose to be in the rest frame of the exterior momentum \( q \equiv q_0 \).

The loop momenta \( l_1 \) and \( l_2 \) satisfy

\[
\begin{align*}
l_1 &= (l_{10}, \vec{l}_{1\perp}) \\
l_2 &= (l_{20}, \vec{l}_{2\perp}) \\
l_{1\perp} &= |\vec{l}_{1\perp}| \\
l_{2\perp} &= |\vec{l}_{2\perp}| \\
\vec{l}_1 \cdot \vec{l}_2 &= l_{1\perp}l_{2\perp}z,
\end{align*}
\]

where the space-like \( \vec{l}_{1\perp} \) and \( \vec{l}_{2\perp} \) span the orthogonal space.

Note that we have to introduce an angular variable for the relative angle between \( \vec{l}_{1\perp} \) and \( \vec{l}_{2\perp} \). We have to do so because we are not allowed to choose \( \vec{l}_{1\perp} \cdot \vec{l}_{2\perp} = 0 \). This choice would fail to give the same results for DR and ordinary Riemann integration in the case of finite integrals.

The integration over the orthogonal space involves six integrations. Three of them give the product of the volumes of the unit spheres \( S^1 \) and \( S^2 \) as a prefactor because only one non-trivial angular integration is involved in the scalar-product \( \vec{l}_1 \cdot \vec{l}_2 \). As a consequence we are left with only five non-trivial integrations, two \( (l_{10} \text{ and } l_{20}) \) from the parallel space and three \( (l_{1\perp}, l_{2\perp}, \text{ and } z) \) coming from the orthogonal space. We are left with the the following integrations to perform

\[ I(q^2) = -\frac{8q^2}{\pi^2} \int_{-\infty}^{\infty} dl_{10} \int_{-\infty}^{\infty} dl_{20} \int_0^\infty dl_{1\perp} \int_0^\infty dl_{2\perp} l_{1\perp}^2 l_{2\perp} \int_{-1}^1 dz \prod_{i=1}^5 \frac{1}{P_i}. \]

(11)
As usual $\varrho$ is small and strictly positive and the integrations are well defined for $\varrho \neq 0$. Because we do not Wick rotate we keep the $i\varrho$-prescription throughout the whole calculation. At the end of the calculation we will find the limit $\varrho \to 0$ to be well-behaved in every case.

This ‘art’ of calculating complicated two-loop functions can be described as transforming the integrand so that it can be recognized as an integral representation of special functions [1]. Usual methods use Wick rotations, so an analytical continuation to physical values at the end of the calculation is always understood. As a consequence, to have sensible results, the final expressions must be given in terms of known special functions in order to manage the analytic continuation back. The $i\varrho$-prescription is then re-installed in order to make this continuation unique. In our approach we avoid altogether the need to Wick rotate. No analytic continuation at the end of the calculation is necessary and sensible results can be obtained by numerical evaluation of the final integral representation.

Let us proceed with the integrations (a detailed discussion can be found in [11]). To do so, we first integrate $z$, the cosine of the angle between $\vec{l}_1$ and $\vec{l}_2$.

The integral becomes

$$I(q^2) = \frac{-8q^2}{\pi^2} \int dl_{10} dl_{20} dl_{1\perp} dl_{2\perp} \frac{l_{1\perp} l_{2\perp}}{2} \left( \frac{1}{P_1 P_2 P_3 P_5} \right) \times \left[ \log((l_{10} + l_{20})^2 - m_3^2 + i\varrho - (l_{1\perp} + l_{2\perp})^2) - \log((l_{10} + l_{20})^2 - m_3^2 + i\varrho - (l_{1\perp} - l_{2\perp})^2) \right].$$

Expanding the logarithms and using symmetry properties of the integrand one finds with the help of the residue theorem the simple result

$$I(q^2) = C \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(w_1^2 - w_2^2)(w_4^2 - w_5^2)} \times \left[ \mathcal{L}_{325} - \mathcal{L}_{315} + \mathcal{L}_{314} - \mathcal{L}_{324} \right]$$

$$C = \frac{4q^2}{1},$$

$$\mathcal{L}_{ijk} = \log(w_i + w_j + w_k),$$

$$w_1^2 - w_2^2 = -m_1^2 + m_3^2 - q^2 - 2l_{10} q,$$

$$w_4^2 - w_5^2 = -m_4^2 + m_5^2 + q^2 - 2l_{20} q,$$

or in dimensionless variables $x = l_{10}/q$, $y = l_{20}/q$

$$I(q^2) = C' \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(w_1^2 - w_2^2)(w_4^2 - w_5^2)} \times \left[ \mathcal{L}_{325} - \mathcal{L}_{315} + \mathcal{L}_{314} - \mathcal{L}_{324} \right]$$

$$C' = 4,$$

$$\mathcal{L}_{ijk} = \log(w_i + w_j + w_k),$$

$$w_1^2 - w_2^2 = -z_1^2 + z_2^2 - 1 - 2x,$$

$$w_4^2 - w_5^2 = -z_4^2 + z_5^2 + 1 - 2y,$$

where $z_i^2 := m_i^2/q^2$ replaces $m_i^2$ in each $w_i$:  

$$w_1 = \sqrt{x^2 - z_1^2 + i\varrho}$$
\[
\begin{align*}
  w_2 &= \sqrt{(x + 1)^2 - z_2^2 + i\varrho} \\
  w_3 &= \sqrt{(x + y)^2 - z_3^2 + i\varrho} \\
  w_4 &= \sqrt{(y - 1)^2 - z_4^2 + i\varrho} \\
  w_5 &= \sqrt{y^2 - z_5^2 + i\varrho}.
\end{align*}
\]

Note that for each \( w_i \) we have \( 0 < \arg(w_i) < \pi/2 \) so that the limit \( \varrho \to 0 \) turns out to be well defined. A detailed discussion of this result can be found in [1]. This result was tested and confirmed by analytical and numerical checks and was widely used in comparison with other numerical routines [2, 8] and asymptotic expansions [5].

### 3.2 The 3-point Functions

In the following we will present a integral representation for 2-loop 3-point functions similar to the one for the 2-loop 2-point function.

In the 3-point case, we have two scalar master diagrams according to the two possible different topologies, Fig. (1) and Fig. (2). We will find the amazing result that for both topologies a threefold integral representation can be given over the same simple functions as in the 2-point case. Only the number of terms is different (from 36 for Fig. (2) to 80 for Fig. (1)).

For 3-point functions additional problems may arise from on-shell singularities. As we will see our method offers a very systematic handling for these divergences. Because only endpoint singularities arise in our calculations, it seems very appropriate to extract these singularities via principle value prescriptions. A brief introduction to this method is given in [12].

It is an amazing consequence of our method that the crossed diagram Fig. (1) is
The generic function is a function of nine variables (six masses and three kinematical variables). First we give the explicit expressions for the propagators:

\[ P_1 = (l_0 + q_{10})^2 - l_1^2 - l_2^2 - m_1^2 + i\eta \]
\[ P_2 = (l_0 - q_{20})^2 - (l_1 - q_{21})^2 - l_2^2 - m_2^2 + i\eta \]
\[ P_3 = (l_0 + k_0)^2 - (l_1 + k_1)^2 - l_2^2 - k_2^2 - 2l_1 k_1 z - m_3^2 + i\eta \]
\[ P_4 = (k_0 - q_{10})^2 - k_1^2 - k_2^2 - m_3^2 + i\eta \]
\[ P_5 = (k_0 + q_{20})^2 - (k_1 + q_{21})^2 - k_2^2 - m_4^2 + i\eta \]
\[ P_6 = k_0^2 - k_1^2 - k_2^2 - m_5^2 + i\eta . \]

Again \( z \) denotes the angle between \( l_\perp \) and \( k_\perp \) in the orthogonal space. The expression for the whole integral is then

\[
I = C \int_{-\infty}^{+\infty} dl_0 dl_1 dk_0 dk_1 \int_{0}^{\infty} dl_\perp dk_\perp \int_{-1}^{+1} dz \frac{l_1 k_\perp}{\sqrt{1 - z^2}} P_1 P_2 P_3 P_4 P_5 P_6. \tag{13}
\]

We first integrate \( z \) and then make the shifts \( l_0 \to l_0 + l_1 \) and \( k_0 \to k_0 + k_1 \) which locates both the cuts in \( l_1 \) and \( k_1 \) of the square-root in the result of the \( z \)-integration in either the upper or lower halfplanes, depending on the sign of \( (l_0 + k_0) \). Furthermore, all propagators become linear in \( l_0 \) (resp. \( k_0 \)). So we can do without any difficulties the \( l_1 \) and \( k_1 \) integrations next using the residue theorem (details of all these calculations are in [12]).

The explicit form of the remaining propagators is now:

\[ P_1 = (l_0 + l_1 + q_{10})^2 - l_1^2 - l_2^2 - m_1^2 + i\eta \]
\[ P_2 = (l_0 + l_1 - q_{20})^2 - (l_1 - q_{21})^2 - l_2^2 - m_2^2 + i\eta \]
\[ P_4 = (k_0 + k_1 - q_{10})^2 - k_1^2 - k_2^2 - m_3^2 + i\eta \]
\[ P_5 = (k_0 + k_1 + q_{20})^2 - (k_1 + q_{21})^2 - k_2^2 - m_4^2 + i\eta \]
\[ P_6 = (k_0 + k_1)^2 - k_1^2 - k_2^2 - m_5^2 + i\eta . \]

Again we use a partial fraction to simplify the structure of the above product of propagators. Each of the above propagators involve either \( l \) or \( k \) variables, but not both. So we partial fractionize the \( l \) and \( k \) propagators separately:

\[
\frac{1}{P_1 P_2 P_4 P_5 P_6} = \frac{1}{P_1 - P_2} \left[ \frac{1}{P_2} - \frac{1}{P_1} \right] \times
\]
This decoupling of $l$ and $k$-variables reflects the nested loop structure of the planar topology of our graph. For the crossed ladder graph we will not have this decoupling of variables.

Again all possible differences $P_i - P_j$ are linear monomials in $l_1$ or $k_1$ with poles on the real axis (if all $\eta$’s are assumed to be equal).

All of these poles give $i\pi$ contributions to the corresponding integrations. But there is a further pole coming from the one linearized propagator which remains in each partial fraction summand. This propagator has a pole which is not located on the real axis. This pole contributes if it is in the interior of the contour of integration, and this in turn is determined by the sign of $(l_0 + k_0)$. For example, if $(l_0 + k_0) > 0$, we have to close the contour in the upper halfplane for both the $l_1$ and $k_1$-integrations. Then the pole of $P_1$ in $l_1$ contributes for $l_0 < -q_{10}$. This gives a further splitting in the $l_0$-integration.

A list of all the resulting monomials, sufficient for the construction of the full result, together with the corresponding REDUCE program, will be published elsewhere [14]. It can be also obtained from the author on request.

Having done the $l_1$ and $k_1$ integrations, one ends with expressions of the following form for the $l_2^0$ and $k_2^0$ dependence,

$$
\int_0^\infty dl_2^0 dk_2^0 \frac{1}{\sqrt{\text{Poly}(l_2^0, k_2^0)}} \frac{1}{(a l_2^0 + b_l)(a k_2^0 + b_k)} 
$$

where $\text{Poly}(l_2^0, k_2^0)$ is a polynomial expression in $l_2^0, k_2^0$ of second degree

$$
\text{Poly}(x, y) = a(x - x_+(y))(x - x_-(y)).
$$

Let us do the $l_2^0$-integration next. Expressing the integrand in terms of $R$-functions and applying a Landen transformation [3] one can express the result in terms of logarithms and square roots [2]

$$
R_{-1}(1/2, 1/2, 1; z_+, z_-, z_0) = R_{-1}(1, 1; u_+, u_-) = \frac{\log(u_+) - \log(u_-)}{u_+ - u_-},
$$

where $u_+, u_-$ are given by

$$
u_+ = \sqrt{z_0^2 - z_0(z_+ + z_-) + z_+ z_- + z_0 \sqrt{z_+ z_-}}
$$

$$
u_- = \sqrt{z_0^2 - z_0(z_+ + z_-) + z_+ z_- - z_0 \sqrt{z_+ z_-}}.
$$

where the $z_i$ variables are polynomial expressions in the remaining integration variables.

In the above mentioned programs the explicit expressions for $\text{Poly}(l_2^0, k_2^0)$ are generated. From them, one can easily determine the resulting $u_+$ and $u_-$ expressions. So we end with a representation as a threefold integral for the generic case of six different masses. In all cases the $l_0$ (resp. $k_0$)-axis is divided in at most two
pieces through the splitting of domains which means that we have four domains to investigate. We have six summands from the partial fraction and six (three from the \( k_1 \) integral times two from the \( l_1 \) integral) contributing residues in each domain. This gives 36 terms, each one having its own domain of contribution.

The crossed function is regarded to be much more difficult because of its non-planar topology. But we will see in a moment that again, with partial fractions and shifting of momenta, we can express it in terms of the same function as for the planar one. Only the number of terms increases. The crucial observation is that we can do a partial fraction decomposition in the two propagators involving both loop momenta. Let us first list the propagators explicitly:

\[
P_1 = (l_0 + k_0 - q_{10})^2 - (l_1 + k_1)^2 - l_1^2 - k_1^2 - 2l_\perp k_\perp z - m_1^2 + i\eta
\]

\[
P_2 = (l_0 + k_0 + q_{20})^2 - (l_1 + k_1 + q_{21})^2 - l_1^2 - k_1^2 - 2l_\perp k_\perp z - m_2^2 + i\eta
\]

\[
P_3 = (l_0 - q_{10})^2 - l_1^2 - l_1^2 - m_3^2 + i\eta
\]

\[
P_4 = (k_0 + q_{20})^2 - (k_1 + q_{21})^2 - l_1^2 - m_4^2 + i\eta
\]

\[
P_5 = l_0^2 - l_1^2 - l_1^2 - m_5^2 + i\eta
\]

\[
P_6 = k_0^2 - k_1^2 - k_1^2 - m_6^2 + i\eta.
\]

Then we have

\[
\frac{1}{P_1 P_2} = \frac{1}{P_1 - P_2} \left[ \frac{1}{P_1} - \frac{1}{P_2} \right]
\]

\[
P_1 - P_2 = -2(q_{10} + q_{20})(l_0 + k_0) + 2q_{21}(l_1 + k_1) - m_1^2 + m_2^2. \quad (19)
\]

In the difference \( P_1 - P_2 \) the orthogonal space dependence cancels out completely.

In each summand we can make a shift so as to have the same structure after the \( z \)-integration just as for the planar case. Then we can proceed exactly as for the planar diagram. The one crucial difference lies in the extra pole \( 1/(P_1 - P_2) \). This is a pole which mixes \( l \) and \( k \) variables. Nevertheless, the \( l_1, k_1 \) and \( l_2^2 \) integrations can be done in a similar manner, only the book keeping of the contributing expressions and the relevant splittings in the \( l_0 \) and \( k_0 \) become rather tedious. This book keeping is done again with the help of REDUCE and is included in [14].

It is remarkable that we end up with integral representations of the same kind as in the planar case while only the number of terms increases. Explicitly, we have three possible residues after the \( l_1 \) integration. Inserting the \( P_1 - P_2 \) pole, we have four poles in the following \( k_1 \) integration. The other two poles in \( l_1 \) only lead to three poles only by the following \( k_1 \) integration. So we end with \( 4 + 3 + 3 = 10 \) contributing residues. For the crossed function, we have \( 2 \times 2 \times 2 = 8 \) summands from the partial fractions (instead of \( 2 \times 3 = 6 \) in the planar case). Altogether we have 80 terms which contribute.

Let us mention here that possible on-shell divergences which may appear will be handled by using Hadamard’s finite part. A detailed demonstration of this method will be presented elsewhere [13] (see also [2, 10, 12]).

Future work will be devoted to implement this Hadamard subtraction prescription in our programs. Note that for the singularities that have been extracted one fixes one or two integration variables, so that we have only one- or twofold integral representations for the coefficient of the singularity, which can be evaluated at least numerically. A list of the relevant coefficients in the on-shell case (with the ladder (resp. cross-ladder) propagators assumed massless) is already included in [14].

11
4 Conclusions

We have presented in examples methods which apply to all loop integrals which may appear in the Standard Model up to the 2-loop level. We hope that these methods add to the set of tools used in multiloop calculations. Apart from a remarkable systemization of 1-loop functions we have derived integral representations valid for planar and non-planar massive 2-loop 2- and 3-point functions. We calculated these results for the generic case, that is as functions of nine variables \( q_0^2, q_1^2, q_0 \cdot q_1, \) six masses) in the three-point case. Accordingly, the results turn out to be complicated. Nevertheless they are found to have a very systematic structure which reflects itself in a repeated pattern of typical terms. The extraction of possible infrared singularities is reduced to a book keeping of end-point singularities for both topologies. In future work, one must investigate how much CPU time is necessary for all the book keeping. We hope that the above integral representations turn out to be manageable with respect to these practical considerations.

Also one can try to find easier integral representations if one treats simpler mass cases. There are enormous possibilities for the application of these results. Almost every calculation in the electro-weak sector involves integrals with different masses, but typically one can restrict ones attention to three different masses. So, if the above integral representation allows for an evaluation in a reasonable CPU time, this may open the door to a new era for calculations devoted to a test of the Standard Model at two-loop order.

Acknowledgements

Thanks are due to many friends and colleagues, especially to David Broadhurst, Andrei Davydychev, Junpei Fujimoto San and Vladimir Smirnov for lots of stimulating discussions. The author likes to thank the organizing committee of the Workshop on High Energy Physics and Quantum Field Theory (Sochi, Russia, Oct. 1992) for the hospitality and friendly atmosphere during this conference.

This work was supported in part by the Deutsche Forschungsgemeinschaft.

A Hadamard’s Finite Part

We give a short summary of the properties of Hadamard’s finite part. Our presentation follows [10]. We use this distribution to have a systematic regularization for on-shell and infrared singularities.

Consider the integral

\[ F(a, b, \epsilon) = \int_{a+\epsilon}^{b} f(t)\phi(t)dt, \quad b > a, \epsilon > 0, \quad (20) \]

where \( \phi \) belongs to an appropriate set of test-functions. Let \( f(t) \) be locally integrable but singular in the neighborhood of \( t = a \).

Let us assume that it is possible to subtract from \( F(a, b, \epsilon) \) a finite linear combination of negative powers of \( \epsilon \) and positive powers of \( \log(\epsilon) \) such that the remaining function of \( \epsilon \) allows for a well-defined finite limit \( \epsilon \to 0_+ \). Then, this limit is called Hadamard’s finite part of \( f \). It defines a distribution \( H(f) \). The functions \( f \) which allow for such a subtraction of the singularities are called pseudofunctions.

To subtract the singularities in \( \epsilon \) explicitly, one uses a Taylor expansion of the test-function \( \phi(t) = \phi(a) + \phi^{(1)}(a)(t-a) + \ldots \) at \( a \). For a pseudofunction \( f \), only a
finite number of terms is not integrable (in the limit $\epsilon \to 0_+$) when the expansion of $\phi$ is inserted in Eq. (20). Let us assume that the first $k$ terms in the Taylor expansion are ill-defined. Then the finite part is given by

$$F_{\text{fin}}(a,b,0) = \int_a^b f(t)(\phi(t) - \phi(a) - \ldots - \phi^{(k)}(a)(t - a)^k/k!\) \, dt, \quad (21)$$

while the singular part is a divergent combination of negative powers of $\epsilon$ and $\log(\epsilon)$ given by the integral of the first $k$ terms.

The only property of the test-function, that is used in the above definition of Hadamard’s finite part is the fact that it is integrable in the interval $(a,b)$. Therefore, we can define a regularization scheme by the use of the above procedure. It is applicable for every integrand which belongs to the class of pseudofunctions. This is sufficient for our applications because the $i\eta$-prescription guarantees every Feynman integrand to be a pseudofunction.

The scheme assigns a linear combination of negative powers of $\epsilon$ and positive powers of $\log(\epsilon)$ to the singular part. For the finite part, its result is Hadamard’s finite part. The generalization to iterated integrals is straightforward, one mainly has to replace the above Taylor expansion in one variable by an expansion in several variables.

References

[1] D. Broadhurst, Z.Phys.C-47 (1990) 115
[2] D. Broadhurst, Which Feynman Diagrams Are Algebraically Calculatable?, talk given at the AI-92 conference (L’Agelonde, France, Jan. 1992)
[3] B. C. Carlson, Special functions of applied mathematics, Academic Press, 1977
[4] J. Collins, Renormalization, Cambridge monographs on math. physics, (1984), Cambridge Univ. Press
[5] A. I. Davydychev, J. B. Tausk Leiden preprint INLO-PUB 11/92
[6] J. Franzkowski, diploma thesis, Univ. Mainz, Sept. 1992 (available via email request (in german))
[7] J. Franzkowski, D. Kreimer, One-loop integrals revisited III, -the four- and higher n-point functions, in preparation
[8] J. Fujimoto, Y. Shimizu, K. Kato, Y. Oyanagi, Prog. Th. Phys. 87, 1233 (1992) Numerical Approach to Loop Integrals, talk given at the AI92 conference, (L’Agelonde, France, Jan. 1992)
[9] D. Kreimer, Z. Phys. C54, 667 (1992)
[10] D. Kreimer, One-loop integrals revisited II, -the three-point functions, MZ-TH/92-20, acc. for publ. in Int. J. of Mod. Phys. A
[11] D. Kreimer, Phys. Lett. B273, 277, (1991)
[12] D. Kreimer, Phys. Lett. B292, 341, (1992)
[13] D. Kreimer, *Dimensional Regularization in the Standard Model*, thesis Univ. Mainz (1992) (available via email request)

[14] D. Kreimer, *A program for the calculation of the 3-point 2-loop massive master functions*, in preparation. A preliminary version written in REDUCE is available via e-mail (kreimer@vpmza.physik.uni-mainz.de). Connect the author in case of interest.

[15] D. Kreimer, K. Schilcher, *On-Shell Wave Function Renormalization - Flavour Changing $\mathcal{O}(g^2) \times \mathcal{O}(\alpha_s)$ corrections*, in preparation

[16] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Mc Graw-Hill, (1965)