Computationally Efficient Estimation of Factor Multivariate Stochastic Volatility Models

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February 10, 2010

Abstract

An Markov chain Monte Carlo simulation method based on a two stage delayed rejection Metropolis-Hastings algorithm is proposed to estimate a factor multivariate stochastic volatility model. The first stage uses ‘k-step iteration’ towards the mode, with k small, and the second stage uses an adaptive random walk proposal density. The marginal likelihood approach of Chib (1995) is used to choose the number of factors, with the posterior density ordinates approximated by Gaussian copula. Simulation and real data applications suggest that the proposed simulation method is computationally much more efficient than the approach of Chib, Nardari, and Shephard (2006). This increase in computational efficiency is particularly important in calculating marginal likelihoods because it is necessary to carry out the simulation a number of times to estimate the posterior ordinates for a given marginal likelihood. In addition to the Markov chain Monte Carlo method, the paper also proposes a fast approximate EM method to estimate the factor multivariate stochastic volatility. The estimates from the approximate EM method are of interest in their own right but are especially useful as initial inputs to Markov chain Monte Carlo methods, making them more efficient computationally. The methodology is illustrated using simulated and real examples.

Key words: Approximate EM, Adaptive sampling, Delayed rejection, Gaussian copula, marginal likelihood, Markov chain Monte Carlo

JEL Classification: C11, C15, C32
1 Introduction

Factor multivariate stochastic volatility (factor MSV) models are increasingly used in the financial economics literature because they can model the volatility dynamics of a large system of financial and economic time series when the common features in these series can be captured by a small number of latent factors. These models naturally link in with factor pricing models such as the arbitrage pricing theory (APT) model of Ross (1976), which is built on the existence of a set of common factors underlying all asset returns, and the capital asset pricing model (CAPM) of Sharpe (1964) andLintner (1965) where the ‘market return’ is the common risk factor affecting all the assets. However, unlike factor MSV models, standard factor pricing models usually do not attempt to model the dynamics of the volatilities of the asset returns and instead assume the second moments are constant. Factor MSV models have recently been applied to important problems in financial markets such as asset allocation (e.g. Aguilar and West, 2000; Han, 2006) and asset pricing (e.g. Nardari and Scruggs, 2006). Empirical evidence suggests that factor MSV models are a promising approach for modeling multivariate time-varying volatility, explaining excess asset returns, and generating optimal portfolio strategies.

A computationally efficient method of estimating a high dimensional dynamic factor MSV model is necessary if such a model is to be applied to financial problems to make decisions in a timely way. For example, when new information becomes available to financial markets, fund managers need to quickly incorporate it into their portfolio strategies, so the speed with which a factor MSV can be re-estimated is an important practical issue. However, based on results reported in the literature (e.g. Chib et al., 2006) and in our article, estimating a factor MSV using current Bayesian simulation methods can take a considerable amount of time when the number of assets is moderate to large which limits the applicability of factor MSV models in real-time applications. Hence, the main purpose of our article is to develop estimation methods for factor MSV models that are computationally more efficient in order to enhance their applicability.

There are a number of methods to estimate factor MSV models such as quasi-maximum likelihood (e.g. Harvey, Ruiz, and Shephard, 1994), simulated maximum likelihood (e.g. Liesenfeld andRichard 2006; Jungbacker and Koopman 2006), and Bayesian MCMC simulation (e.g. Chib et al. 2006). For high dimensional problems, (e.g. Chib et al. 2006; Han 2006), Bayesian MCMC simulation is the most efficient estimation method, with alternative estimation methods having difficulty handling high dimensions.

The main practical and computational advantage of a factor MSV model is its parsimony, where all the variances and covariances of a vector of time series are modeled by a low-dimensional SV structure governed by common factors. In a series of papers Kim, Shephard, and Chib (1998), Chib, Nardari, and Shephard (2002), Chib et al. (2006), and Omori, Chib, Shephard, and Nakajima (2007), Chib and Shephard and their coauthors consider a variety of univariate and multivariate stochastic volatility (MSV) models whose error distributions range from Gaussian to Student-t, and that allow for both symmetric and asymmetric conditional distributions. In the multivariate case, the correlation between variables is governed by several latent factors. A computationally efficient
estimation method for a factor MSV model depends on how efficiently a univariate SV model is estimated and how efficiently the latent factors and their corresponding coefficients are estimated. Our article first improves the computational efficiency of estimating a univariate SV model and then extends the estimation method to a factor MSV model.

Bayesian Markov chain Monte Carlo (MCMC) simulation is a convenient method to estimate a univariate SV model. The model is transformed to a linear state space model with an error term in the observation equation having a log-chisquared distribution which is approximated by a mixture of normals (e.g. [Kim et al., 1998]). One of the most efficient MCMC methods for estimating a univariate SV model is based on the Metropolis-Hastings method which uses a multivariate t proposal distribution for sampling the model parameters in blocks, with the location and scale matrix obtained from the mode of the posterior distribution (see [Chib and Greenberg, 1995]). This method usually requires numerical optimization and is used in the papers by Chib and Shephard and their coauthors both for univariate SV models and for factor MSV models. We refer to this as the ‘optimization’ method. Although the ‘optimization’ method is a powerful approach for most block-sampling schemes, it is computationally demanding, especially in factor MSV models, because it is necessary to apply it within each MCMC iteration.

Our article provides an alternative Metropolis-Hastings method for sampling the parameters in SV models, which is more efficient than ‘optimization’ method, and is based on a two stage ‘delayed rejection’ algorithm. The first stage consists of k-steps of a Newton-Raphson iteration towards the mode of the posterior distribution, with k small. We refer to this as the ‘k-step iteration’ stage. The second stage consists of an adaptive random walk Metropolis (ARWM) proposal whose purpose is to keep the chain moving in small steps if the first stage proposal is poor in some region of the parameter space. If only the single stage proposal is used then the Markov chain may remain stuck in such a region for many iterations.

We compare our methods with the ‘optimization’ method using simulated data for univariate SV and factor MSV models and show that the ‘delayed rejection’ methods is computationally much more efficient than the ‘optimization’ method when both computing time and inefficiency factors are taken into account. The gains are more than 200% for univariate SV models and more than 500% for factor MSV models.

In addition to MCMC methods, we also consider an approximate Monte Carlo EM method to estimate SV models, which is much faster than the MCMC based methods, and yields good parameter estimates. The speed advantage is especially important in high dimensional factor MSV models. An important use of the Monte Carlo EM method is to provide initial parameter values for the MCMC based methods, especially in high dimensions.

An important practical issue in estimating factor MSV models is determining the number of latent factors. In factor asset pricing models, a popular approach is to rely on intuition and theory as guides to come up with a list of observed variables as proxies of the unobserved theoretical factors. The adequacy of the list of observed variables has crucial effects on the covariance structure of idiosyncratic shocks. We choose the number of factors in factor MSV models by marginal likelihood, based on the approach of [Chib, 1995] and with the conditional posterior ordinates approximated
by Gaussian copulas that are fitted using MCMC iterates. Estimating the marginal posterior
ordinates for a given marginal likelihood usually requires several MCMC runs which makes the
greater computational efficiency of our methods compared to that of the ‘optimization’ method
even more important.

Delayed rejection for Metropolis-Hastings is proposed by Mira (1999) in her thesis and extended
in Green and Mira (2001). ‘k-step iteration’ is proposed by Gamerman (1997) for random effects
models and implemented using iteratively reweighted least squares. The method is refined by
Villani, Kohn, and Giordani (2009) using Newton-Raphson steps and is applied in their paper for
nonparametric regression density estimation. In the second stage of ‘delayed rejection’ we use the
adaptive random walk Metropolis algorithm of Roberts and Rosenthal (2009). The copula based
approach of estimating posterior ordinates is proposed by Nott, Kohn, Xu, and Fielding (2009).

The rest of the paper is organized as follows. Section 2 reviews the “optimization” method
and introduces our approach for estimating univariate SV models. Section 3 extends the MCMC
approach to factor MSV models and also proposes the Monte Carlo EM approach. Section 4 shows
how to approximate the marginal likelihood in complex models using a Gaussian copula approach.
Sections 5 and 6 compare the various estimation methods using simulated and real data. Section
7 discusses the proposed methods for other applications such as GARCH type models. Section 8
concludes the paper.

2 Computational methods for univariate stochastic volatility models

Although there are a number of univariate SV models discussed in literature (see the papers by
Chib and Shephard and their coauthors) our article only deals with one simple model in this family,
which is the basis for the factor MSV models discussed in section 3. The centered parametrization
version of this model (see Pitt and Shephard, 1999) is

\[
\begin{align*}
y_t &= \exp(h_t/2)e_t, \quad e_t \sim N(0, 1), \\
h_t &= \mu + \phi(h_{t-1} - \mu) + \sigma \eta_t, \quad \eta_t \sim N(0, 1),
\end{align*}
\]

where \(y_t\) is the mean-adjusted return of an asset at time \(t\) and \(h_t\) is its log volatility which itself is
governed by an autoregressive (AR) process with mean \(\mu\), persistence parameter \(\phi\) and a Gaussian
noise term with standard deviation \(\sigma\). The two noise terms \(e_t\) and \(\eta_t\) are assumed to be uncorrelated.
We assume that \(|\phi| < 1\) to ensure that the log volatility \(h_t\) is stationary.

The univariate SV model can be transformed into linear state space form by writing

\[
\begin{align*}
z_t &= \log(y_t^2) = h_t + \log(e_t^2), \quad e_t \sim N(0, 1), \\
h_t &= \mu + \phi(h_{t-1} - \mu) + \sigma \eta_t, \quad \eta_t \sim N(0, 1).
\end{align*}
\]

Unlike the standard Gaussian linear state space model, the error term \(\log(e_t^2)\) in the measurement
equation (2) is non-Gaussian so that it is not possible to estimate the model parameters by maximum likelihood estimation using the Kalman filter. Harvey et al. (1994) approximate the log-chisquared distribution with one degree of freedom at equation (2) by a Gaussian distribution having the same mean and variance, but the approximation is unreliable. In the Bayesian literature several Gibbs sampling methods are proposed (e.g. Jacquier, Polson, and Rossi 1994), but as discussed in Kim et al. (1998) these methods are computationally inefficient. Carter and Kohn (1997) and Kim et al. (1998) show that a finite mixture of normals can provide a very good approximation the log-chisquare distribution with one degree of freedom. Carter and Kohn (1997) use a mixture with 5 components and Kim et al. (1998) use a normal mixture with 7 components and correct the approximation with a Metropolis-Hastings step. We write such a normal mixture approximation as

\[ p(\epsilon_t) \approx \sum_i \pi_i \phi_N(z_t; m_i, \nu_i), \quad \epsilon_t = \log(e_t^2), \]  

(3)

where \( \phi_N(z; m, \nu) \) is a univariate normal density in \( z \) with mean \( m \) and variance \( \nu \). The weights \( \pi_i \), means \( m_i \) and variances \( \nu_i \) for each normal component are given by Carter and Kohn (1997) and Kim et al. (1998) for their approximations. Our article uses the 5-component approximation in Carter and Kohn (1997).

Using equation (3), we can write equation (2) as a conditionally Gaussian state space model by introducing a sequence of discrete latent variables \( K_t \) each taking the five values \( 1, \ldots, 5 \) such that conditionally on \( h_t \) and \( K_t = i \), \( z_t \) is Gaussian with mean \( h_t + m_i \) and variance \( \nu_i \), i.e. \( z_t | K_t = i \sim \phi_N(z_t; h_t + m_i, \nu_i) \).

The sampling scheme for the univariate SV model at equation (2) and its various extensions in the series of papers by Chib and Shephard and their coauthors are very similar and can be summarized as follows.

2.1 The Optimization MCMC Sampling Method

1. Initialize \( \theta = (\mu, \phi, \sigma)' \) and \( h = (h_1, \ldots, h_T)' \).

2. Sample the indicator variables from \( K \sim p(K|z, h) \), so that

\[ p(K_t = i|z_t, h_t) = \frac{\pi(K_t = i)\phi_N(z_t; h_t + m_i, \nu_i)}{\sum_{j=1}^5 \pi(K_t = j)\phi_N(z_t; h_t + m_j, \nu_j)} . \]

3. Jointly sample \( \theta, h \sim p(\theta, h|z, K) \).

3.1. Sample \( \theta \sim p(\theta|z, K) \) (for convenience, we suppress \( K \) in the densities below) using the Metropolis-Hastings algorithm as

a. Build the proposal distribution \( q_T(\hat{\theta}, \hat{\Sigma}, v) \) for the target, where \( \hat{\theta} \) is the value of \( \theta \) that maximizes \( \log p(z|\theta) \); the density \( p(z|\theta) \) is calculated using the Kalman Filter (see Anderson and Moore 1979 for details). \( \hat{\Sigma} \) is the negative of the inverse of the
Hessian matrix of the objective function evaluated at \( \hat{\theta} \) and \( v \) is the degrees of freedom for the multivariate-t distribution;

b. Sample a candidate value \( \theta^1 \sim q_T(\hat{\theta}, \hat{\Sigma}, v) \);

c. Accept the candidate value \( \theta^1 \) with probability

\[
\alpha(\theta^0 \rightarrow \theta^1) = \min \left\{ 1, \frac{\pi(\theta^1)p(z|\theta^1)q_T(\theta^1|\hat{\theta}, \hat{\Sigma}, v)}{\pi(\theta^0)p(z|\theta^0)q_T(\theta^0|\hat{\theta}, \hat{\Sigma}, v)} \right\},
\]

where \( \theta^0 \) represents the current draw and \( \pi(\theta) \) is the prior. If the candidate value \( \theta^1 \) is rejected, the current value \( \theta^0 \) is retained as the next draw.

3.2. Sample the latent state variable \( h \sim p(h|z, \theta, K) \) using a Forwards-Filtering-Backwards-Sampling (FFBS) algorithm such as Carter and Kohn (1994), Fruwirth-Schnatter (1994) or de Jong and Shephard (1995).

4. Go to Step 2.

Step 3.1.a builds the proposal distribution for block sampling the parameters, and follows the method proposed by Chib and Greenberg (1995) where optimization is carried out at each MCMC iteration to obtain the mode of the log likelihood function. This is the reason that we call this sampling scheme the ‘optimization’ method. One of the main advantages of the ‘optimization’ method is that it jointly samples the parameters \( \theta \) and the latent states \( h \) , which avoids the dependence between the two in an MCMC scheme. Furthermore, it samples \( \theta = (\mu, \phi, \sigma)' \) in one block which also diminishes the dependence in the sampling between the parameters. However, the computational burden of this method is heavy because it is necessary to do the optimization at each MCMC iteration. This problem is more severe for large datasets since the evaluation of the log likelihood at each Newton-Raphson iteration is more time consuming.

2.2 Improved methods for sampling the model parameters

The major computational load of the ‘optimization’ method comes from searching for the mode of the likelihood \( p(z|\theta, K) \) in order to construct the proposal distribution for \( \theta \). One way to reduce the computational burden is to reduce the dimension of the parameter space from three to two by augmenting \( \mu \) to the latent state vector \( h \) and rewriting the univariate SV model at equation (1) in the non-centered form (see Pitt and Shephard 1999) as

\[
\begin{align*}
  z_t &= \log(y_t^2) = (1, 1) \begin{pmatrix} h_t - \mu_t \\ \mu_t \end{pmatrix} + \log(e_t^2), \quad e_t \sim N(0, 1), \\
  \begin{pmatrix} h_t - \mu_t \\ \mu_t \end{pmatrix} &= \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{t-1} - \mu_{t-1} \\ \mu_{t-1} \end{pmatrix} + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \eta_t, \quad \eta_t \sim N(0, 1)
\end{align*}
\] (4)
with initial values

\[
\begin{pmatrix} h_1 - \mu_1 \nabla \mu_1 \end{pmatrix} \sim N \left( \begin{pmatrix} 0 & \sigma^2/(1 - \phi^2) + V_{\mu} & -V_{\mu} \nabla \mu \\ m_{\mu} & -V_{\mu} & V_{\mu} \end{pmatrix} \right).
\]

The prior for \( \mu \) is \( N(m_{\mu}, V_{\mu}) \) and the state vector at time \( t \) is redefined as \( h_t = (h_t - \mu, \mu)' \).

The computational gain by augmenting \( \mu \) to the latent state \( h_t \) is not very significant in the univariate case because optimization is still required. However, it becomes more important in the multivariate case. In this section we propose an alternative sampling method that builds a proposal density for \( \theta = (\phi, \sigma)' \sim p(\theta | z, K) \) but does not require optimization.

\subsection{‘k-step iteration’ sampling method}

Unlike the ‘optimization’ MCMC method where it is necessary to find the mode before building the proposal distribution, the ‘k-step iteration’ method builds the proposal distribution using \( k \) Newton-Raphson iterations towards the mode of \( L(\theta) = \log p(z \mid \theta, K) \), with \( k \) small, typically \( k = 1 \) or 2. For notational convenience we omit to indicate dependence on the indicators \( K \) below.

The iterations based on the second order Taylor series expansion of \( L(\theta) \) at a point \( \hat{\theta} \),

\[
L(\theta) \approx L(\hat{\theta}) + (\theta - \hat{\theta})'L'(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})'L''(\hat{\theta})(\theta - \hat{\theta})
\]

\[
= b(\hat{\theta}) + \frac{1}{2} \Theta'(\hat{\Theta})\Theta
\]

where \( \hat{\theta} \) means equality up to an additive constant that does not depend on \( \theta \), and \( L'(\theta) \) and \( L''(\theta) \) are the first and second partial derivatives of \( L(\theta) \). In equation (5), \( b(\theta) = L'(\theta) - L''(\theta)\theta \) and \( C(\theta) = -L''(\theta) \). The Newton-Raphson iteration is initialized at the current value of \( \theta \), which we call \( \theta^0 \), and at the \( i \)th iterate \( (i < k) \) we expand \( L(\theta) \) about \( \theta^{(i)} = C(\hat{\theta}^{(i-1)})^{-1}b(\hat{\theta}^{(i-1)}) \). Let \( \hat{\theta}^{(k)} \) be the \( \{k\} \)th (last) iterate of \( \theta \). Then the proposal density is the multivariate normal density \( q(\theta | \theta^0) = \text{MVN}(\theta; C(\hat{\theta}^{(k)})^{-1}b(\hat{\theta}^{(k)}), C(\hat{\theta}^{(k)})^{-1}) \). We generate a candidate point \( \theta^1 \) and accept it with probability

\[
\alpha(\theta^c \rightarrow \theta^p) = \min \left\{ 1, \frac{\pi(\theta^1)p(z | \theta^1)q(\theta^1 | \theta^0)}{\pi(\theta^0)p(z | \theta^0)q(\theta^0 | \theta^0)} \right\};
\]

otherwise we take \( \theta^0 \) as the new value of \( \theta \).

At equation (5), the density \( q(\theta | \theta^1) \) is constructed using \( k \) iterates starting at \( \theta^1 \) in exactly the same way that \( q(\theta | \theta^0) \) is constructed starting at \( \theta^0 \). We note that instead of a multivariate normal proposal it is straightforward to use a multivariate-t proposal. In low dimensions, one or two iterations are sufficient to obtain good estimation results with reasonable staring values. Extremely poor initial values will make the ‘k-step iteration’ proposal distribution less appropriate, but the poor initial values will also make it much more difficult for the ‘optimization’ method to converge.

We also note that if the ‘k-step iteration’ method is continued till convergence, then it is equivalent to the ‘optimization’ method and \( q(\theta | \theta^0) = q(\theta | \theta^1) = \text{MVN}(\theta; \hat{\theta}, \hat{\Sigma}) \), where \( \hat{\theta} \) is the mode of \( L(\theta) \).
and $\hat{\Sigma}$ is the inverse of the negative of the Hessian at the mode.

### 2.2.2 Delayed rejection using ‘$k$-step iteration’ and adaptive MCMC sampling

We refine the ‘$k$-step iteration’ method by using it as a first stage in a ‘delayed rejection’ scheme. If the proposed value of $\theta$ generated by the ‘$k$-step iteration’ method is rejected in the first stage then a second value of $\theta$ is generated from an adaptive random walk Metropolis proposal in the second stage. The motivation for using the second stage is to keep the parameters moving by small increments even when the first stage proposal is poor in certain regions of the parameter space. The acceptance probability in the second stage takes into account the first stage rejection to ensure that the posterior distribution is the invariant distribution of the chain. Our article uses adaptive random walk Metropolis proposal distribution suggested in Roberts and Rosenthal (2009), although there are a number of other adaptive random walk Metropolis proposals in the literature, e.g. Haario, Saksman, and Tamminen (2005).

Let $\theta^0$ be the current value of $\theta$ before ‘delayed rejection’, $q_1$ the ‘$k$-step iteration’ proposal, $\theta^1$ the value of $\theta$ proposed by $q_1$ and $\alpha_1$ the first stage acceptance probability at equation (6). Let $q_2$ be the adaptive random walk Metropolis proposal density, $\theta^2$ the value of $\theta$ proposed by $q_2$ if $\theta^1$ is rejected and let $\alpha_2$ be the second stage acceptance probability. Then,

$$q_2(\theta_0 \rightarrow \theta) = (1 - \delta)\text{MVN}(\theta; \theta_0, 2.38^2\Sigma/d) + \delta \text{MVN}(\theta; \theta_0, 0.1^2I_d/d)$$

and

$$\alpha_2(\theta^0 \rightarrow \theta^1 \rightarrow \theta^2) = \min \left\{ 1, \frac{\pi(\theta^2)p(z|\theta^2)q_1(\theta^1|\theta^2)(1 - \alpha_1(\theta^2 \rightarrow \theta^1))}{\pi(\theta^0)p(z|\theta^0)q_1(\theta^1|\theta^0)(1 - \alpha_1(\theta^0 \rightarrow \theta^1))} \right\},$$

where $\Sigma$ is the sample covariance matrix of $\theta$ estimated from the MCMC iterates obtained thus far, $I_d$ is an identity matrix of dimension $d$, and $\delta$ is a scaling factor which we choose as 0.05. Note that $q_2(\cdot)$ does not enter the second stage acceptance probability because the proposal is symmetric, i.e. $q_2(\theta_2 \rightarrow \theta_0) = q_2(\theta_0 \rightarrow \theta_2)$.

Mira (1999, 2001) proposed the ‘delayed rejection’ method to reduce the number of rejections in Markov chain Monte Carlo when a Metropolis-Hastings proposal is used. In principle the ‘delayed rejection’ method can have more than two stages. In practice, the computational load increases as more stages are added and the acceptance ratios in the later stages are more complex.

### 2.2.3 Monte Carlo EM method

Most approaches in the literature estimate the univariate SV model using MCMC. This section considers a Monte Carlo EM approach to estimate the parameters $\theta = (\mu, \phi, \sigma)'$, using the centered parameterization of the univariate SV model at equation (2).

The EM algorithm is described by McLachlan and Krishnan (2008) and consists of repeated
application of expectation (E) and maximization (M) steps. For our problem, the E-step evaluates

$$Q(\theta, \theta^{\text{old}}) = \int \log p(y, h|\theta)p(h|y, \theta^{\text{old}})dh,$$  \hspace{1cm} (7)

where $\theta^{\text{old}}$ is the current value of $\theta$. However, this integral is intractable because the density $p(h|y, \theta^{\text{old}})$ is non-Gaussian.

To facilitate the computation, we approximate $\log(e_{t}^{2})$ as a five component mixture of normals as at equation (3) and reexpress equation (7) as

$$Q(\theta, \theta^{\text{old}}) = \int \log p(y, h, K|\theta)p(h, K|y, \theta^{\text{old}})d(h, K).$$  \hspace{1cm} (8)

It is difficult to evaluate this integral analytically so we approximate it using the Monte Carlo EM method, discussed in Wei and Tanner (1990), as

$$Q(\theta, \theta^{\text{old}}) \approx \frac{1}{M} \sum_{j=1}^{M} \log p(y, h^{j}, K^{j}|\theta)$$  \hspace{1cm} (9)

where $(h^{j}, K^{j})$, $j = 1, \ldots, M$ are iterates from the joint conditional distribution $p(h, K|y, \theta^{\text{old}})$. Although sampling directly from $p(h, K|y, \theta^{\text{old}})$ is difficult, it is straightforward to sample from $p(K|y, h, \theta^{\text{old}})$ and from $p(h|y, K, \theta^{\text{old}})$, so that we use a Gibbs sampling scheme to obtain a sample from the joint distribution $p(h, K|y, \theta^{\text{old}})$ after some burn-in iterations.

Since $\log p(y, h, K|\theta) = \log p(y|h, K) + \log p(K) + \log p(h|\theta)$, and $\theta$ only enters the last term, the right side of equation (9) is estimated as

$$Q(\theta, \theta^{\text{old}}) \approx -\frac{1}{2M} \sum_{j=1}^{M} \sum_{t=2}^{T} \left( \log(\sigma^{2}) + \frac{(h_{t}^{j} - \mu - \phi(h_{t-1}^{j} - \mu))^{2}}{\sigma^{2}} \right),$$

where $\approx$ means excluding additive constants that do not depend on $\theta$. To maximize $Q(\theta, \theta^{\text{old}})$ with respect to $\mu$, $\phi$ and $\sigma^{2}$, we first reparameterize the centred univariate SV as $h_{t} = \delta + \phi h_{t-1} + \sigma \eta_{t}$ with $\delta = \mu(1 - \phi)$, and then treat the maximization problem as an OLS regression problem for the parameters at the $j$-th iterate. The updated parameters in the M-step are

$$\phi^{\text{new}} = \frac{\sum_{j=1}^{M} \sum_{t=2}^{T} (h_{t}^{j} - \bar{h}_{t})(h_{t-1}^{j} - \bar{h}_{t-1})}{\sum_{j=1}^{M} \sum_{t=2}^{T} (h_{t}^{j} - \bar{h}_{t-1})^{2}}$$
$$\delta^{\text{new}} = \bar{h}_{t} - \phi^{\text{new}} \bar{h}_{t-1}$$
$$(\sigma^{2})^{\text{new}} = \frac{1}{M(T-1)} \sum_{j=1}^{M} \sum_{t=2}^{T} (h_{t}^{j} - \delta^{\text{new}} - \phi^{\text{new}} h_{t-1}^{j})^{2}$$
$$\mu^{\text{new}} = \delta^{\text{new}} / (1 - \phi^{\text{new}})$$  \hspace{1cm} (10)
where
\[
\bar{h}_t = \frac{1}{M(T-1)} \sum_{j=1}^{M} \sum_{t=2}^{T} h_{t}^{j}
\] and \[
\bar{h}_{t-1} = \frac{1}{M(T-1)} \sum_{j=1}^{M} \sum_{t=2}^{T} h_{t-1}^{j}.
\]

A difficulty in using the Monte Carlo EM method is monitoring convergence, since approximation
errors entering in the E-step mean that the usual ways of determining convergence such as stopping
when the change in the parameters is small or the change in the log-likelihood is small may be
unreliable. We propose to monitor convergence by using the bridge sampling approach of Meng and
Schilling (1996) to estimate the likelihood ratio
\[
p(y|\theta^i)/p(y|\theta^{i-1})
\]
at the \(i\)th iteration of the Monte Carlo EM by
\[
R(\theta^i, \theta^{i-1}) = \frac{\sum_{j=1}^{M} \left( p(y, h^{i-1,j}, K^{i-1,j}|\theta^i) / p(y, h^{i-1,j}, K^{i-1,j}|\theta^{i-1}) \right)^{1/2}}{\sum_{j=1}^{M} \left( p(y, h^{i,j}, K^{i,j}|\theta^{i-1}) / p(y, h^{i,j}, K^{i,j}|\theta^i) \right)^{1/2}}
\]
where \((h^{i,j}, K^{i,j}), j = 1, \ldots, M\) are generated from \(p(h, K|y, \theta^i)\) at the \(i\)th Monte Carlo EM iteration. Following Meng and Schilling (1996) we say that Monte Carlo EM iteration has converged
if the plot of \(R(\theta^i, \theta^{i-1})\) against \(i\) converges to 1 from above.

We obtain approximate large sample standard errors for the parameters using a method by Louis
(see McLachlan and Krishnan, 2008, p. 226). The approximate information matrix of the observed log-likelihood is
\[
I(\theta; y) \approx -\frac{1}{M} \sum_{j=1}^{M} \frac{\partial^2 \log p(y, h^j, K^j|\theta)}{\partial \theta \partial \theta'} - \frac{1}{M} \sum_{j=1}^{M} \left( \frac{\partial \log p(y, h^j, K^j|\theta)}{\partial \theta} \right) \left( \frac{\partial \log p(y, h^j, K^j|\theta)}{\partial \theta'} \right) + \left( \frac{1}{M} \sum_{j=1}^{M} \frac{\partial \log p(y, h^j, K^j|\theta)}{\partial \theta} \right) \left( \frac{1}{M} \sum_{j=1}^{M} \frac{\partial \log p(y, h^j, K^j|\theta)}{\partial \theta'} \right)
\]

We note that Shephard (1993) uses a similar Monte Carlo EM approach to estimate a centered
univariate SV model. The major differences between our treatment and that of Shephard (1993)
are (i) our approach for estimating the right side of equation (8) is more effective than the single-
site Gibbs sampling and Muller’s random walk proposal to sample from \(p(h|y, \theta)\) proposed in
Shephard (1993). The extra efficiency of our approach is especially important in factor MSV models.
(ii) Shephard (1993) monitors convergence using a traditional EM approach instead of using the
likelihood ratios obtained from the bridge sampling method. (iii) Our main focus is on the feasibility
of the Monte Carlo EM approach for factor MSV models discussed in section 3 whereas Shephard
(1993) only considers the univariate case.
3 Computational methods for factor MSV models

There are a number of factor MSV models in the literature. To illustrate the estimation methods in our paper, we consider the following factor MSV model,

\[ y_t = B f_t + S_t^{1/2} e_t, \quad S_t = \text{diag}\{\exp(h_{1,t}), ..., \exp(h_{p,t})\}, \quad e_t \sim N(0, I_p) \]

\[ f_t = V_t^{1/2} u_t, \quad V_t = \text{diag}\{\exp(h_{p+1,t}), ..., \exp(h_{p+k,t})\}, \quad u_t \sim N(0, I_k) \]

where both the latent factors and the idiosyncratic shocks are allowed to follow different stochastic volatility processes, \( B \) is the factor loading matrix, \( p \) and \( k \) are the number of return series and factors, with \( y_t = (y_{1,t}, ..., y_{p,t})' \) and \( f_t = (f_{1,t}, ..., f_{k,t})' \). To identify the model we follow Chib et al. (2006) and set \( b_{ii} = 1 \) for \( i \leq k \) and \( b_{ij} = 0 \) for \( j > i \). We also assume both the idiosyncratic shocks and the latent factors are internally and mutually uncorrelated so that \( S_t \) and \( V_t \) are diagonal matrices. Consequently, the correlation structure of \( y_t \) is governed by the latent factors \( f_t \).

We first review the sampling scheme for estimating the factor MSV model parameters in Chib et al. (2006), which is also used by Han (2006) and Nardari and Scruggs (2006) in their applications. We again refer to it as the ‘optimization’ MCMC method because optimization is executed within each of the MCMC iterations similarly to its use in the univariate SV model.

3.1 Optimization MCMC Sampling Method

1. Initialize \( \theta = (\theta_1, ..., \theta_{p+k}) \) with \( \theta_j = (\phi_j, \sigma_j)' \), \( h = (h_1, ..., h_{p+k})' \) with each \( h_j \) defined as in univariate SV case.

2. Jointly sample \( \beta, f \sim p(\beta, f | y, h) \), where \( \beta \) represents the free parameters in the \( B \) matrix.

   2.1. Sample \( \beta \sim p(\beta | y, h) \)

   a. Sample the candidate value \( \beta^1 \sim \mathcal{N}(\hat{\beta}, \hat{\Sigma}_\beta, v) \), where \( \hat{\beta} \) is the mode maximizing the log-likelihood function \( \log p(y | \beta, h) \), \( \hat{\Sigma}_\beta \) is the corresponding covariance matrix at the mode and \( v \) is degree of freedom for the multivariate-t distribution.

   b. Accept the candidate value \( \beta^1 \) with probability

   \[
   \alpha(\beta^0 \rightarrow \beta^1) = \min\left\{ 1, \frac{\pi(\beta^1)p(y | \beta^1)q_T(\beta^0, \hat{\Sigma}_\beta, v)}{\pi(\beta^0)p(y | \beta^0)q_T(\beta^1, \hat{\Sigma}_\beta, v)} \right\}
   \]

   2.2. Sample the latent space-state vector \( f \sim p(f | y, B, h) \) using a Forwards-Filtering-Backwards-Sampling (FFBS) algorithm.

3. Sample the indicator vector variables \( K \sim p(K | y, B, f, h) \), noting that given \((B, f)\), we can decompose the factor MSV model into the \((p + k)\) univariate SV models

\[
 z_{j,t} = \begin{cases} 
 \log(y_{j,t} - B_{j} f_t)^2 = h_{j,t} + \log(e_{j,t}^2), & j \leq p \\
 \log(f_{j-p,t}) = h_{j-p,t} + \log(u_{j-p,t}^2), & j \geq p + 1 
\end{cases}
\]
where $B_j$ is the $j$-th row of the factor loading matrix $B$. The indicator vectors $K_j$ are now sampled independently through $K_j \sim p(K_j|z_j, h_j)$ for $j = 1, \ldots, p + k$ as for the univariate SV model.

4. Jointly sample $\theta, h \sim p(\theta, h|y, B, f, K)$ from $p+1$ separate series as $\theta_j, h_j \sim p(\theta_j, h_j|z_j, K_j)$ for $j = 1, \ldots, p + k$, each of which can be dealt with as in the univariate SV case. That is we sample $\theta_j \sim p(\theta_j|z_j, K_j)$ using ‘optimization’ MCMC and then sample $h_j \sim p(h_j|z_j, \theta_j, K_j)$.

5. Go to Step 2.

The main advantage of the ‘optimization’ method is that it jointly samples $\beta, f \sim p(\beta, f|y, h)$ and jointly samples $\theta_j, h_j \sim p(\theta_j, h_j|z_j, K_j)$ for $j = 1, \ldots, p + k$. The general Gibbs-sampling method which samples $\beta$ conditional on $f$ and then sample $f$ conditional on $\beta$ is less efficient as demonstrated in various simulation examples discussed in [Chib et al. (2006)]. The ‘optimization’ method is computationally slow, especially when $y$ and $f$ are high dimensional, because now optimization is used for both $\beta$ (which is usually high dimensional) and also for $\theta = (\theta_1, ..., \theta_{p+k})'$ for each univariate SV parameter space.

3.2 ‘$k$-step iteration’ and delayed rejection method

We extend the ‘delayed rejection’ method for sampling parameters in the univariate SV model to the factor MSV model using the same basic ideas. Instead of using numerical optimization to find the proposal distributions based on the mode, we use the ‘delayed rejection’ method discussed in section 2 to build the proposal distributions for $\beta$ and the $\theta_j$ for $j = 1, \ldots, p + k$. That is, we first use the ‘$k$-step iteration’ method to build the proposal after one or two iterations in the first stage, and then use the adaptive random walk Metropolis method in the second stage if the candidate value in the first stage is rejected. The computational load in the factor MSV model is mainly from sampling $\beta \sim p(\beta|y, h)$. The terms $K_j, \theta_j, h_j$, for $j = 1, \ldots, p + k$, are sampled independently of each other as discussed above and, if sufficient computing resources are available, they can be sampled in parallel. If $\beta$ is high dimensional, say 50-dimensional, using the ‘optimization’ method to find the mode of $\log p(y|\beta, h)$ is slow. Although we can easily build the proposal distribution through the ‘$k$-step iteration’ method by only running one or two Newton-Raphson iterations, we may encounter high rejection rates in this stage. Therefore, the efficiency of the Markov chain Monte Carlo relies more on the adaptive random walk Metropolis in the second stage which does not work as well in high dimensions. To reduce problems in high dimensions, we make two suggestions. First, increase the number of iterations in the first stage, e.g. to five or six iterations, but in that case the speed advantage of the ‘delayed rejection’ method is reduced. Second, split $\beta$ into several smaller sub-blocks such as $\beta = (\beta_1', \ldots, \beta_r')'$, with each sub-block containing a relatively small number of parameters. For example, for a 50-dimensional $\beta$, we can use a total of 7 sub-blocks with 8 parameters for the first 6 sub-blocks and 2 parameters for the last sub-block and update $\beta$ using the scheme $\beta_t^{(j)} \sim p(\beta_t|y, h, \beta_t^{(j)}, l < i, \beta_t^{(j-1)}, l > j)$ for $i = 1, \ldots, r$. where $j$ represents the current iteration. By sampling a large dimensional $\beta$ in blocks, the performance of both the ‘optimization’
and ‘delayed rejection’ methods can improve, especially the ‘delayed rejection’ method, where one or two iterations in the first stage may be sufficient to obtain good results.

3.3 Monte Carlo EM Method for factor MSV

We now show how the Monte Carlo EM method discussed in section 2.2.3 for the univariate SV model generalizes in a straightforward manner to the factor MSV. The E-step evaluates the complete log likelihood as

\[
Q(\Theta, \Theta^{\text{old}}) = \int \log p(y, f, h, K|\Theta) p(f, h, K|y, \Theta^{\text{old}}) d(f, h, K),
\]

where \(\Theta = \{\beta, \theta\}, y = (y_1, ..., y_p), f = (f_1, ..., f_k), h = (h_1, ..., h_{p+k})\) and \(K = (K_1, ..., K_{p+k})\) represent matrices with each column series starts from \(t = 1\) to \(T\). Similarly to the univariate SV, we approximate the integral using Monte Carlo simulations as

\[
Q(\Theta, \Theta^{\text{old}}) \approx \frac{1}{M} \sum_{j=1}^{M} \log p(y, f^j, h^j, K^j|\Theta),
\]

where \(\log p(y, f, h, K|\Theta) = \log p(y|f, h, \beta) + \log p(f|h) + \log p(K) + \log p(\theta|\theta)\) with \(\beta\) and \(\theta\) entering the first and last terms. The \((f^j, h^j, K^j)\), for \(j = 1, ..., M\), are generated from the joint posterior \(p(f, h, K|y, \Theta^{\text{old}})\) and are obtained using Gibbs sampling by first generating from \(p(K|y, f, h)\), then from \(p(h|y, f, K, \theta)\) and finally from \(p(f|y, h, \beta)\) after some burn-in iterates.

Because \(\beta\) only enters \(\log p(y|f, h, \beta)\), maximizing the expected complete log likelihood with respect with \(\beta\) is equivalent to maximizing

\[
Q(\Theta, \Theta^{\text{old}}) = \frac{1}{M} \sum_{j=1}^{M} \log p(y|f^j, h^j, \beta) = \frac{1}{M} \sum_{j=1}^{M} \sum_{i=1}^{p} \log p(y_i|f_i^j, h_i^j, B_i) = \frac{1}{M} \sum_{j=1}^{M} \sum_{i=1}^{p} \sum_{t=1}^{T} \left( h_{i,t}^j + \frac{(y_{i,t} - B_i f_i^j)^2}{\exp(h_{i,t}^j)} \right).
\]

The second line of the equation follows from the first because the \(y_i\) series are independent conditional on \(f\) and \(h\). The rows of the optimal \(\beta\) are

\[
\beta_{\text{new}}^i = \left( \frac{1}{M} \sum_{j=1}^{M} \sum_{t=1}^{T} f_i^j f_t^j \right)^{-1} \left( \frac{1}{M} \sum_{j=1}^{M} \sum_{t=1}^{T} y_{i,t}^j f_t^j \right) \quad \text{for } i = 2, ..., p,
\]

where \(y_{i,t}^* = y_{i,t}, f_{i,t}^j = f_i^j\) for \(i > k\), and \(y_{i,t} = y_{i,t} - B_{i,i:k} f_{i:k,t}^j\), \(f_{t}^{i,j} = f_{i,i-1,t}^j\) for \(2 \leq i \leq k\). With \(f_t = (f_{1,t}, ..., f_{k,t})', B_{i,i,k} = (B_{i,i}, ..., B_{i,k}), f_{i,k,t} = (f_{i,k}, ..., f_{k,t})'\) and \(f_{1,i-1,t} = (f_{1,i-1}, ..., f_{i-1,t})'\).

Because \(\theta\) only enters \(\log p(h|\theta)\), maximizing the expected complete log likelihood respect with
\( \theta \) is equivalent to maximizing

\[
Q(\Theta, \Theta^{\text{old}}) = \frac{1}{M} \sum_{j=1}^{M} \log p(h^j | \theta) = \frac{1}{M} \sum_{j=1}^{M} \sum_{i=1}^{p+k} \log p(h^j_i | \theta_i)
\]

\[
= -\frac{1}{2M} \sum_{j=1}^{M} \sum_{i=1}^{p+k} \sum_{t=2}^{T} \left( \log(\sigma_i^2) + \frac{((h^j_{i,t} - \mu_i) - \phi_i(h^j_{i,t-1} - \mu_i))^2}{\sigma_i^2} \right)
\]

Since each of the series \( h_i, i = 1, \ldots, p+k \) is conditionally independent, each \( \theta_i^{\text{new}} = (\mu_i^{\text{new}}, \phi_i^{\text{new}}, \sigma_i^{\text{new}}) \) is obtained as in the univariate SV case at equation (10).

Convergence is monitored in the univariate SV case as at equation (11), except that we now have the complete log likelihood as \( \log p(y, f, h, K | \Theta) \). The information matrix can be calculated using a similar formula to equation (12) by substituting in the current complete log likelihood.

4 Marginal Likelihood calculation

An important practical issue in estimating factor MSV models is determining the number of latent factors. A Bayesian approach to this problem is to choose the number of factors using marginal likelihood. Usually there is no closed form solution available for the marginal likelihood in the factor MSV model and it is necessary to estimate the marginal likelihood using simulation methods such as Chib (1995), where additional ‘Reduced MCMC’ runs are needed. For a given factor MSV model, an expression for the marginal likelihood is obtained through the identity

\[
p(y) = \frac{p(y | \theta^*, \beta^*) p(\theta^*, \beta^*)}{p(\theta^*, \beta^* | y)},
\]

where \((\theta^*, \beta^*)\) are chosen as high density ordinates such as posterior means or componentwise posterior medians, with \( \theta^* = (\theta^*_1, \ldots, \theta^*_{p+k}) \), \( \beta^*_j = (\mu_{j}^*, \phi_{j}^*, \sigma_{j}^*)' \) for \( j = 1, \ldots, p + k \). Our article uses componentwise posterior medians. Such a choice also has advantages for the copula methods that are discussed later. In particular, we choose \( \beta^* \) as the componentwise posterior median from the MCMC simulation output in the estimation stage and \( \theta^* \) as the componentwise posterior median from later ‘Reduced MCMC’ runs. The likelihood \( p(y | \theta^*, \beta^*) \) in the numerator of equation (16) can be evaluated sequentially by integrating out the latent \( h_t \) using the auxiliary particle filter introduced by [Pitt and Shephard (1999)] and illustrated in detail in [Chib et al. (2006)]. The density \( p(\theta^*, \beta^*) \) in equation (16) is the prior density. The denominator \( p(\theta^*, \beta^* | y) \) in equation (16) is decomposed as

\[
p(\theta^*, \beta^* | y) = p(\beta^* | y) p(\theta^* | y, \beta^*)
\]

where the posterior marginal density \( p(\beta^* | y) \) is approximated by a normal distribution with mean and covariance matrix obtained from the MCMC runs in the estimation stage. (Chib et al., 2006 also use the same approximation). Alternatively, one can use the Gaussian copula method discussed
later to estimate the posterior ordinate. To estimate the second term \( p(\theta^*|y, \beta^*) \), it is necessary to run at least one ‘Reduced MCMC’ to obtain a sampler of \( \theta \) from \( p(\theta|y, \beta^*) \). Generally, this posterior conditional density does not have a closed form and it is necessary to use nonparametric methods such as kernel density estimation (e.g. Terrell, 1990) to evaluate the ordinate. However, the dimension of \( \theta^* \) is usually large, and kernel density estimates may not be accurate enough for such large dimensional cases. One way to obtain a reasonably good estimate is to split \( \theta^* \) into several smaller sub-blocks as in Chib et al. (2006) where two \( \theta^*_j \)’s are put in one sub-block as in

\[
p(\theta^*|y, \beta^*) = p(\theta^*_1, \theta^*_2|y, \beta^*) p(\theta^*_3, \theta^*_4|y, \beta^*, \theta^*_1, \theta^*_2) \times \cdots \times p(\theta^*_{p+k-1}, \theta^*_{p+k}|y, \beta^*, \theta^*_1, \ldots, \theta^*_{p+k-2})
\]  

(17)

By splitting \( p(\theta^*|y, \beta^*) \) into blocks as in equation (17), \( \text{Max}((p + k)/2) \) ‘Reduced MCMC’ runs are necessary, where Max is the smallest integer greater than or equal to \((p + k)/2\); we note that additional sub-blocks of parameters are fixed at successive ‘Reduced MCMC’ runs as in Chib (1995). The computational time necessary to carry out this sequence of ‘Reduced MCMC’ runs is large and usually much greater than the time required for estimating the model.

Our article adopts the copula-based approximation method proposed by Nott et al. (2009) to evaluate posterior ordinates. This method can deal with larger dimensional blocks of parameters than kernel density estimation methods. The Gaussian copula approximation to a multivariate density \( f(\zeta) \), with \( \zeta \) a \( p \times 1 \) vector, is

\[
q(\zeta) = |C|^{-1/2} \exp \left( \frac{1}{2} \eta' \left( I(p) - C^{-1} \right) \eta \right) \prod_{j=1}^{p} f_j(\zeta_j)
\]  

(18)

where \( C \) is the correlation matrix of the Gaussian copula, \( \eta = (\eta_1, \ldots, \eta_p)' \) with \( \eta_j = \Phi^{-1}(F_j(\zeta_j)) \), \( \Phi \) is the standard normal CDF, \( F_j \) is the CDF of the \( j \)th marginal \( \zeta_j \) and \( f_j \) is the corresponding density. When \( \zeta_j \) is the posterior median of the \( j \) marginal, then \( \eta_j = 0 \) and equation (18) simplifies to

\[
q(\zeta) = |C|^{-1/2} \prod_{j=1}^{p} f_j(\zeta_j)
\]  

(19)

Suppose that we wish to approximate \( p(\theta^*|y, \beta^*) \) using one block and that \( \theta^* \) is the componentwise posterior median of the \( \theta \) iterates obtained from \( p(\theta|y, \beta^*) \). Then, from equation (19) the estimate of the posterior ordinate is

\[
p(\theta^*|y, \beta^*) \approx |C|^{-1/2} \prod_{j=1}^{p+k} \prod_{i=1}^{3} f_{j,i}(\theta^*_{j,i}|y, \beta^*)
\]

where \( f_{j,i}(\theta^*_{j,i}|y, \beta^*) \) with \( \theta^*_{j,i} = (\mu^*_j, \phi^*_j, \sigma^*_j)' \) represents the one dimensional marginal density ordinate which can be estimated using kernel density estimation. The copula correlation matrix \( C \) is estimated using order statistics as in Nott et al. (2009).
5 Simulation Study

This section provides several simulation examples for both univariate SV and factor MSV models that illustrate the computational methods discussed in sections 2 and 3. The methods are compared using performance diagnostics.

5.1 Performance diagnostics

A popular way to inspect the performance of the MCMC samplers is based on the inefficiency factors for the parameters. For any given parameter \( \theta \), the inefficiency factor is defined as

\[
\text{Inefficiency factor} = 1 + 2 \sum_{j=1}^{G-1} \left(1 - \frac{j}{G}\right) \rho(j),
\]

where \( G \) is the number of sample iterates of \( \theta \) and \( \rho(j) \) is the \( j \)-th autocorrelation of the iterates of \( \theta \). When \( G \) is large and \( \rho(j) \) tends to zero quickly, the inefficiency factor is approximated by

\[
1 + 2 \sum_{j=1}^{J} \hat{\rho}(j),
\]

for some given \( J \), with \( \hat{\rho}(j) \) the \( j \)th sample autocorrelation of \( \theta \). We choose \( J = 100 \) in all the empirical analyses because \( \hat{\rho}(j) \) usually decays to zero before 100 lags for almost all the parameters in our simulation examples.

Mathematically, the inefficiency factor is just the ratio of the variance of a posterior mean of the iterates obtained from MCMC sampling to the variance of the posterior mean from independent sampling. The inefficiency factor is interpreted as that multiple of the number of iterates \( G \) that gives the same accuracy as \( G \) independent iterates. A low inefficiency factor is preferred to a higher inefficiency factor.

The inefficiency factor itself may not be informative enough to compare two sampling methods because it does not take computation time into account. Consider, for example, two sampling methods that have very similar inefficiency factor scores, but the first method is computationally faster than the second one. We then say that the first method is relatively more efficient because it needs less computational time to achieve the same accuracy as the second method. To take into account both the inefficiency factor and the computing time we consider the equivalence factor score defined as

\[
\text{Equivalence factor} = \text{inefficiency factor} \times t,
\]

where \( t \) is the computing time per iteration. For a given parameter, the ratio of equivalence factors for two sampling schemes is the ratio of times taken by the two sampling schemes to obtain the same accuracy for a given parameter. We note that although the equivalence factor gives a more complete picture of the performance of a sampler, it is implementation dependent.
5.2 Simulation examples of univariate stochastic volatility models

The univariate SV model at equation (1) is estimated using the ‘optimization’, ‘k-step iteration’, ‘delayed rejection’ and Monte Carlo EM methods discussed in section 2 for two simulated datasets (each having 10 replicates) with sample sizes of 500 and 1,500, respectively. We use \( k = 1 \) steps in the first stage for both the ‘k-step iteration’ and the ‘delayed rejection’ methods. The true parameters for the data sets of 500 and 1,500 observations are \( (\mu = 0.5, \phi = 0.9, \sigma = 0.1) \) and \( (\mu = 1.0, \phi = 0.95, \sigma = 0.15) \). The parameter \( \mu \) is nested in latent the state vector \( h_t \) as in equation (4). The same prior specifications are used for both data sets

\[
\mu \sim N(0, 5), \quad \phi \sim \text{Beta}(8, 0.1), \quad \text{and} \quad \sigma \sim \text{IG}(2, 0.1),
\]

where the persistence parameter \( \phi \) ranges between 0 and 1 because the volatility is positively autocorrelated in most financial time series; \( \text{IG}(a, b) \) is the inverse Gamma distribution with shape parameter \( a \) and scale parameter \( b \) (and with mean \( b/(a - 1) \)). All MCMC simulation results are computed using the last 5,000 draws after discarding the first 1,000 burn-in iterations. For Monte Carlo EM, we set \( M = 100 \) after 10 burn-in iterations. Extensive testing using longer burn-in periods and higher values of \( M \) produced similar results.

Table 1 summarizes the estimation results showing that the three MCMC methods provide accurate parameter estimates for both datasets. The inefficiency factors of the three univariate SV parameters are all less than 20 for all MCMC methods across both datasets. The ‘delayed rejection’ method has the lowest inefficiency factors for the dataset of 500 observations and the ‘optimization’ method has the lowest inefficiencies for the dataset with 1500 observations. However, due to its relatively fast computing speed, the equivalence factors of the ‘delayed rejection’ method are about half those of the ‘optimization’ method. In general, the ‘delayed rejection’ method produces lower inefficiencies and higher acceptance rates than the ‘k-step iteration’ method at little additional computational cost, and it therefore has lower equivalence factor scores than the ‘k-step iteration’ method in the most cases.

We run 25 iterations for each replication of the Monte Carlo EM method. Figure 1 plots the likelihood ratios against the iterate numbers and shows that likelihood ratios stabilizes before 25 iterations. We therefore take the values of the 25-th iterate as the final estimates. Table 1 shows that the Monte Carlo EM method performs well in terms of its accuracy and is promisingly fast in terms of computing speed with only a few minutes of computing time needed compared with hours using the MCMC methods. Therefore, we expect that by taking the Monte Carlo EM parameter estimates as initial values for the MCMC methods we will obtain a chain that converges quickly. The benefits may not be important in the univariate SV model since both the ‘optimization’ and ‘delayed rejection’ methods converge quickly, but using Monte Carlo EM is worthwhile for the factor MSV model.
5.3 Simulation examples of factor multivariate stochastic volatility models

This section fits the factor MSV model at equation (13) to two simulated datasets using 5 replicates for each dataset. The first dataset which we call ‘P5-K1’ has \( p = 5 \) and \( k = 1 \) and the second dataset which we call ‘P10-K2’ has \( p = 10 \) and \( k = 2 \). Both datasets have 500 observations. The three computational methods (with one-step iteration in the first stage for ‘delayed rejection’) discussed in section 3 are used to estimate the model. The true parameter values are \( \theta_j = (0.5, 0.9, 0.1)^T \) for \( j = 1, \ldots, p \) and \( \theta_j = (1.0, 0.95, 0.15)^T \) for \( j = p + 1, \ldots, p + k \). Table 2 gives the elements in the loading matrices \( B \).

The prior specification for both datasets is \( \mu_j \sim N(0, 5), \phi_j \sim \text{Beta}(8, 0.1), \sigma_j \sim \text{IG}(2, 0.1) \), for \( j = 1, \ldots, p + k \) and \( \beta \sim N(0, 10I_d) \) with \( I_d \) the \( d \) dimensional identity matrix.

For model ‘P5-K1’, we use a one block strategy to sample \( \beta \) from \( p(\beta | y, h) \) for both MCMC methods. For the higher dimensional ‘P10-K2’ model, we use a one block strategy for the ‘optimization’ MCMC method but use a sub-block strategy for the ‘delayed rejection’ MCMC method, with 8 parameters in each sub-block, where the last sub-block contains the remaining parameters if \( d \) is not a multiple of 8. All the MCMC simulation estimates are again computed using the last 5,000 draws after discarding the first 1,000 burn-in iterations, and \( M \) is set as 100 after 10 burn-in iterations for Monte Carlo EM. For the Monte Carlo EM method, similar results were obtained for longer burn-in periods and larger values of \( M \).

Table 3 summarizes the estimation results for the ‘P5-K1’ model. The estimates from the two MCMC methods agree for almost all the unknown parameters and are also very close to the true parameter values. The inefficiency factor scores are similar for \( \theta \) but not for \( \beta \), where the inefficiency of the ‘optimization’ method (7.02) is less than half of the value for the ‘delayed rejection’ method (14.53). However, due to the faster sampling speed of the ‘delayed rejection’ method, the ‘optimization’ method has higher equivalence factor scores than ‘delayed rejection’, with the highest value being 48.71 for \( \phi \) compared with the corresponding value of 20.37 using ‘delayed rejection’ (this is almost 2.4 times higher). The parameter estimates from Monte Carlo EM for \( \theta \) are consistent with those obtained by MCMC, but a little lower than the MCMC estimates for \( \beta \). Figure 1 plots the likelihood ratios (see equation (11)) vs iteration number for the ‘P5-K1’ model for all the replicates. The plots suggest that the Monte Carlo EM iterates have converged. From Table 3, the Monte Carlo EM method took 6 minutes compared with the 5.1 hours for the ‘optimization’ MCMC method.

Table 4 summarizes the estimation results of the ‘P10-K2’ model. The parameter estimates of the two MCMC methods are again close to the true parameter values. The inefficiency factors are similar for both MCMC methods, but the ‘delayed rejection’ method has lower equivalence factor scores (by a factor of 5) than the ‘optimization’ method because it is five times faster. The acceptance ratios for \( \theta = (\phi, \sigma)^T \) are almost the same for both MCMC methods. We note that the parameter \( \mu \) does not have an acceptance ratio since it is sampled jointly with the latent state \( h \) from its exact conditional density. However, we observe a significant decrease for the acceptance ratio \( \beta \) from 0.87 to 0.66 in the ‘optimization’ method when a one block strategy is used for both simulation cases. Similar results (that are not reported in the article but that are available from
the authors) are found for ‘delayed rejection’ method, with a decrease from 0.77 to 0.32 for the first stage acceptance ratio if a one block strategy is used in both examples. Tables 3 and 4 show that for the ‘delayed rejection’ method where a one-block approach is used in the ‘P5-K1’ case but several sub-blocks are used in the ‘P10-K2’ case, there is a small decrease, e.g., from 0.77 to 0.72 in the first stage acceptance ratio, but the acceptance ratio is even higher than the value of 0.66 for the ‘optimization’ method where one block is used in both cases. The benefit of the sub-block strategy in sampling high dimensional $\beta$ is clear in this example.

Table 4 also shows that the estimates from the Monte Carlo EM method are similar to those obtained by both MCMC methods for most of the parameters. Figure 1 plots the likelihood ratios vs iteration number for the ‘P10-K2’ model for all the replicates. The figure suggests that the Monte Carlo EM iterations have converged. Figure 2 plots the iterates of the first four elements of $\beta$ for one replicate of ‘P5-K1’ example. Both MCMC methods seem to converge quickly, with the ‘optimization’ method converging almost immediately and the ‘delayed rejection’ method also converging in less than 100 iterations. Figure 3 is a similar plot for the ‘P10-K2’ example. In general, the ‘optimization’ method converges faster than the ‘delayed rejection’ method which takes around 250 iterations to converge. We did not plot the iterates for $\theta$, since they generally mix very well even in the higher dimensional ‘P10-K2’ example. Figure 3 also plots the iterates when we set the parameter estimates from the Monte Carlo EM method as initial values for the ‘delayed rejection’ method. In this case the chain converges almost immediately. In results for the ‘P10-F2’ example that are not reported in the article, where a one-block strategy for sampling $\beta$ is used for ‘delayed rejection’, it sometimes takes more than 1,000 iterations to obtain convergence which demonstrates the benefits of the sub-block strategy.

The results suggest that the estimates from the Monte Carlo EM method can be used effectively as starting values for the MCMC methods because Monte Carlo EM is much faster than either of the MCMC methods.

5.4 Determining the number of factors

The ‘delayed rejection’ method is applied to fit the two factor MSV simulation examples with $k = 1, \ldots, 3$ factors and evaluate their marginal likelihoods. We use the Gaussian copula approximation method discussed in section 4 to evaluate the posterior ordinate $p(\theta^*|y, \beta^*)$ using one ‘Reduced MCMC’ simulation. A similar simulated example in Nott et al. (2009) shows that the posterior ordinate $p(\theta^*|y, \beta^*)$ from a Gaussian copula approximation does not change too much for different numbers of sub-block strategies. More importantly, it agrees with the evaluation from the benchmark joint kernel smoothing method using the many sub-blocks strategy in Chib et al. (2006). Therefore, we adopt a one block strategy by only running one ‘Reduced MCMC’ simulation. We run 5,000 iterations in sampling $\theta$ from $p(\theta^*|y, \beta^*)$, and 10,000 particles for $M$ and 20,000 particles for $R$ in the auxiliary particle filter used to calculate the likelihood.

Table 5 reports the marginal likelihood evaluation results for two simulated examples using 5 replicates. The true number of factors in the ‘P5-K1’ and the ‘P10-K2’ cases are 1 and 2 respectively.
The marginal likelihoods suggest one (two) factors for the first (second) examples for all 5 replicates. We did not evaluate the marginal likelihood as in [Chib et al., 2006] because it is very slow due to the multiple sub-blocks needed when applying kernel density estimation, with the ‘optimization’ method used for each ‘Reduced MCMC’ sub-block (for evidence see [Nott et al., 2009]). However, we believe that based on the results of [Nott et al., 2009], ‘delayed rejection’ combined with the copula-based approximation method gives similar results to those of [Chib et al., 2006], but is more practical in terms of computational load.

5.5 Faster computing speed

The computational burden for the factor MSV model usually becomes heavier as the dimension increases. However, given $B$ and $f$, the model can be decomposed into several independent univariate SV models, so that parallel computing can be used in following areas.

I. Sampling $K_j, \theta_j$, and $h_j$ for $(p+k)$ univariate SV equations in both MCMC based methods. This can be applied in both the estimation stage and later in the ‘Reduced MCMC’ needed to compute the marginal likelihood.

II. Calculating the gradient corresponding to $\beta$ in the factor MSV model, which is needed for all three methods discussed in section 3.

III. Sampling $K_i^j$ and $h_i^j$ for $i = 1, ..., p+k$, and evaluating the complete log likelihood $p(y, f^j, h^j, K^j|\Theta)$ within each Monte Carlo simulation $j = 1, ..., M$, for the Monte Carlo EM method.

IV. Calculating the importance weights in the auxiliary particle filter, which can be applied twice since the original auxiliary particle filter needs two re-sampling steps.

5.6 Computing details

All the algorithms in the article are coded in the Matlab M-language running on a PC with Intel®Core 2 Quad CPU (3.0 GHz) under the Matlab®2009a framework with the job assigned to 4 local workers. We believe that the algorithms can be speeded up if coded in C, as in [Chib et al., 2006].

6 Real data application

The factor MSV model is now fitted to a real dataset containing 18 international stock indices that cover three major regions: America, Europe and Asian Pacific and both developed and emerging markets. Specifically, they are USA, Canada, Mexico, and Chile in America; UK, Germany, France, Switzerland, Spain, Italy, and Norway in Europe; and Japan, Hong Kong, Australia, New Zealand, Malaysia, Singapore, and Indonesia in Asia-Pacific. We use weekly continuously compounded returns (Wednesday to Wednesday) in nominal local currency from January 10th, 1990 to December
27th, 2006, giving a total of 886 weekly observations. The dataset is obtained from DataStream Morgan Stanley Capital International (MSCI) indices.

The ‘delayed rejection’ method with one iteration in the first stage is applied to estimate the model, using sub-blocks with 8 parameters in each sub-block, to sample $\beta$ from $p(\beta|y, h)$. We use marginal likelihood estimated using a Gaussian copula to select the number of factors, with two sub-blocks used to calculate the ordinate $p(\theta^*|y, \beta^*)$. Table 6 shows the log marginal likelihoods for $k = 1, \ldots, 5$ factors and suggests that the four factor model is the best with the Bayes factors of the 4-factor model compared with the other models all greater than 100, providing strong evidence according to Jeffrey’s scale. We do not list the parameter estimates due to space considerations, but the estimation results show good inefficiency levels: e.g. for the estimation of the 4-factor model we have average inefficiency factor scores of $\phi(13.2)$, $\sigma(20.3)$ and $\beta(32.3)$, and acceptance ratios $(\phi, \sigma)(0.57)$ and $\beta(0.62)$ in the first stage.

7 Other Applications

The ‘optimization’ method discussed in Chib and Greenberg (1995) is a powerful approach for MCMC simulation when it is necessary to form a Metropolis-Hastings proposal, either as a single block approach or using multiple sub-blocks. However, it can be quite slow. The ‘delayed rejection’ method discussed in our article can in principle be applied instead. This section discusses several such applications.

7.1 GARCH model

This section applies the ‘delayed rejection’ method to the GARCH volatility model developed by Engle (1982) and generalized by Bollerslev (1986). We first consider the univariate GARCH model which forms the basic building block of the factor multivariate GARCH model discussed in section 7.1.2.

7.1.1 Univariate model

The popular univariate Gaussian-GARCH(1,1) model is

\[ y_t = \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_t), \]
\[ \sigma^2_t = \omega + \alpha \epsilon^2_{t-1} + \beta \sigma^2_{t-1}. \]  \hspace{1cm} (22)

Maximum likelihood estimation (MLE) is a convenient tool to estimate this model, but it may have trouble in more sophisticated GARCH type models such as the factor Multivariate-GARCH (factor-MGARCH) model. Asai (2005) surveys work on Bayesian inference for the univariate GARCH model and compares several existing estimation methods.

We use two simulated examples to compare the performance of the ‘delayed rejection’ method to the ‘Griddy Gibbs’ method (e.g. Bauwens and Lubrano 1998) the ‘optimization’ method, and to the MLE. Each example uses 1500 observations and 10 replications. The true parameters are
\( \omega = 0.1, \alpha = 0.25, \beta = 0.70 \) for example 1 and \( \omega = 0.1, \alpha = 0.05, \beta = 0.90 \) for example 2. The volatility persistence parameter \( \beta \) is moderate for example 1 and high for example 2.

The priors used for the Bayesian methods are

\[
\omega \sim \text{IG}(2, \text{Var}(y) \times (1 - 0.95)), \quad \alpha \sim \text{Beta}(1, 8), \quad \beta \sim \text{Beta}(8, 1),
\]

where \( \text{Var}(y) \) is the unconditional variance of \( y \). For the ‘Griddy-Gibbs’ method we follow Bauwens and Lubrano (1998), and use 33 grid points with parameter ranges:

\[
0 < \omega < \text{Var}(y) \times (1 - 0.8), \quad 0 < \alpha < 0.3, \quad 0.35 < \beta < 1.
\]

Table 7 reports the estimation results for the two simulated examples. The parameter estimates are quite accurate for the ‘optimization’ and ‘delayed rejection’ and the MLE methods, but not for the ‘Griddy-Gibbs’ method, whose parameter estimates are far from the true values, especially in the second example where the volatility persistence parameter is high. One reason for the poor performance of ‘Griddy-Gibbs’ could be that the range of \( \beta \) is too wide in this case, and as suggested in Bauwens and Lubrano (1998), we may further restrict its range while allowing most of the posterior density to fit within this range.

The two block sampling MCMC methods (‘optimization’ and ‘delayed rejection’) usually have lower inefficiency factor scores than the single Gibbs based method, and the ‘delayed rejection’ method compares favourably with the ‘optimization’ method in terms of both inefficiency factors and equivalence factor scores.

### 7.1.2 Factor-MGARCH model

Although maximum likelihood estimation is much faster than Bayesian methods in the univariate case, Bayesian methods become more attractive for estimating factor multivariate GARCH (factor-MGARCH) models because it is much more difficult to apply the maximum likelihood estimation in this case. See, for example, the discussion in Harvey, Ruiz, and Sentana (1992) who need to make approximation in order to calculate the likelihood of a closely related model. Under suitable assumptions on the factor MGARCH model, parts of the computation can be decomposed into working with several independent univariate GARCH models and the ‘delayed rejection’ can be used to sample both the loading matrix and the parameters in each univariate GARCH model.

### 7.2 Heavy tailed models

Chib et al. (2006) consider a factor MSV model with Student-t distributions for the idiosyncratic shocks, with \( e_{j,t} \sim St(0, 1, v_j) \) for \( j = 1, ..., p \) in equation (13), where \( v_j \) is the degrees of freedom parameter in the Student-t distribution.

In a Bayesian framework, the Student-t error terms can be expressed in a conditionally normal form so that we can use ‘delayed rejection’ or the simpler ‘k-step iteration’ method to build the proposal density in sampling the degrees of freedom parameters, instead of building the proposal
densities using the ‘optimization’ method. Similar ideas can be applied in the t-GARCH model with heavy tails.

8 Conclusions

Factor MSV models provide a parsimonious representation of a dynamic multivariate system when the time varying variances and covariances of the time series can be represented by a small number of fundamental factors. Factor MSV models can be used in many financial and economic application such as portfolio allocation, asset pricing, and risk management. MCMC methods are the main tool for estimating the parameters of factor MSV models and determining the number of factors in these models. In particular, the ‘optimization’ method is widely used in the literature to estimate the parameters of stochastic volatility models. Its main shortcoming is its computational expense, because numerical optimization is required to build Metropolis-Hastings proposals in each MCMC iteration. The computational cost is especially heavy for high dimensional models containing many parameters. Evaluating marginal likelihoods to determine the number of factors is even more computationally expensive than model estimation because it may require several reduced MCMC runs for each marginal likelihood evaluation. We propose an alternative MCMC estimation method which is based on ‘delayed rejection’ that is substantially faster than the ‘optimization’ method and is more efficient when measured in terms of equivalence factors. We also propose a fast EM based approximation method for factor MSV models which can be used either on its own or to provide initial parameter estimates to increase the speed of convergence of MCMC methods. We also note that we do not have a good way to determine the number of factors using the Monte Carlo EM approach, whereas the Bayesian approach can use marginal likelihood. Our article also simplifies the marginal likelihood calculation for determining the number of latent factors in the factor MSV model by using Gaussian copula approximations to the marginal ordinates instead of using kernel density estimates. We also show how that our approach also applies in GARCH-type models and compares favorably with existing MCMC estimation methods. The MCMC estimation methods and the fast EM based approximation method proposed in our article reduce the computational cost of estimating factor MSV models considerably, which is important for certain real-time applications in financial markets.

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Table 1: **Summary output of the univariate SV simulated examples.** The table reports the average values of posterior estimates and diagnostic measures across 10 replicates. ‘Mean’, ‘Stdev’, ‘Ineff’, ‘Equiv’, and ‘Acr’ denote the posterior mean, posterior standard deviation, inefficiency factor, equivalence factor and acceptance ratio, respectively. Time is time in hours (h) and minutes (m). For ‘delayed rejection’ MCMC, there are two-stage acceptance ratios.

|                | Mean  | Stdev | Ineff | Equiv | Acr  | Time  | Mean  | Stdev | Ineff | Equiv | Acr  | Time  |
|----------------|-------|-------|-------|-------|------|-------|-------|-------|-------|-------|------|-------|
| **Optimization-** |       |       |       |       |      |       |       |       |       |       |      |       |
| MCMC-            |       |       |       |       |      |       |       |       |       |       |      |       |
| \(\mu\)         | 0.469 | 0.143 | 2.35  | 2.40  | 0.37 | 1.7(h)| 0.993 | 0.088 | 2.15  | 7.78  | 0.75 | 6.0(h) |
| \(\phi\)        | 0.890 | 0.069 | 14.04 | 13.78 |      |      | 0.946 | 0.022 | 5.13  | 18.60 |      |       |
| \(\sigma\)      | 0.117 | 0.035 | 8.18  | 8.33  |      |      | 0.130 | 0.028 | 6.12  | 22.19 |      |       |
| **k-step**       |       |       |       |       |      |       |       |       |       |       |      |       |
| Iteration-       |       |       |       |       |      |       |       |       |       |       |      |       |
| MCMC-            |       |       |       |       |      |       |       |       |       |       |      |       |
| \(\mu\)         | 0.471 | 0.143 | 2.33  | 1.05  | 0.33 | 0.8(h)| 0.994 | 0.088 | 2.14  | 3.02  | 0.67 | 2.4(h) |
| \(\phi\)        | 0.897 | 0.066 | 15.77 | 7.15  |      |      | 0.947 | 0.021 | 7.37  | 10.41 |      |       |
| \(\sigma\)      | 0.113 | 0.034 | 10.31 | 4.67  |      |      | 0.129 | 0.027 | 8.60  | 12.13 |      |       |
| **Delayed-**     |       |       |       |       |      |       |       |       |       |       |      |       |
| Rejection-       |       |       |       |       |      |       |       |       |       |       |      |       |
| MCMC             |       |       |       |       |      |       |       |       |       |       |      |       |
| \(\mu\)         | 0.469 | 0.159 | 2.07  | 1.12  | 0.32 | 0.9(h)| 0.993 | 0.088 | 2.10  | 3.23  | 0.67 | 2.6(h) |
| \(\phi\)        | 0.896 | 0.066 | 12.09 | 6.54  | 0.15 |      | 0.946 | 0.022 | 6.60  | 10.14 | 0.06 |       |
| \(\sigma\)      | 0.115 | 0.035 | 8.08  | 4.34  |      |      | 0.130 | 0.028 | 8.00  | 12.27 |      |       |
| **Monte Carlo-** |       |       |       |       |      |       |       |       |       |       |      |       |
| \(\mu\)         | 0.454 | 0.061 |       | 1.4(m)|       |       | 0.991 | 0.066 |       |       |      | 13.1(m)|
| \(\phi\)        | 0.892 | 0.062 |       |       |       |       | 0.949 | 0.011 |       |       |      |       |
| \(\sigma\)      | 0.115 | 0.010 |       |       |       |       | 0.121 | 0.007 |       |       |      |       |

Table 2: Values of the constrained (in bold) and unconstrained elements of the \(B\) matrix

|       | \(y_1\) | \(y_2\) | \(y_3\) | \(y_4\) | \(y_5\) |
|-------|---------|---------|---------|---------|---------|
| **P5-K1: Factor-Loadings** \(B\) \(f_1\) | \(1\) | \(-1.5\) | \(1.5\) | \(-1.5\) | \(1.5\) |
|       | \(y_1\) | \(y_2\) | \(y_3\) | \(y_4\) | \(y_5\) | \(y_6\) | \(y_7\) | \(y_8\) | \(y_9\) | \(y_{10}\) |
| **P10-K2: Factor-Loadings** \(B\) \(f_2\) | \(0\) | \(1\) | \(0.5\) | \(-0.5\) | \(0.5\) | \(-0.5\) | \(0.5\) | \(-0.5\) | \(0.5\) | \(-0.5\) | \(0.5\) | \(-0.5\) |
Table 3: **Summary output of factor MSV simulated examples for the ‘P5-K1’ example.**
In the table ‘P5-K1’ means $p = 5$ and $k = 1$. The table reports the average values of the posterior estimates across 5 replicates. The inefficiency factor and equivalence factor scores are the average values for corresponding parameters across 5 replicates: e.g. the reported inefficiency factor for $\phi$ is the average value of inefficiency factors $(\phi_1, \ldots, \phi_5)$ with each $\phi_j$ representing the average value of inefficiency factors $(\phi_{j,1}, \ldots, \phi_{j,p+k})$ for $j = 1, \ldots, 5$. The 5.9(m) computing time for Monte Carlo EM does not include the cost of calculating the standard error.
Table 4: **Summary output of the factor MSV ‘P10-K2’ simulated example.** ‘P10-K2’ means $p = 10$ and $k = 2$. The table reports the average values of the posterior estimates across 5 replicates. The 8.8(m) computing time for Monte Carlo EM does not include the cost of calculating the standard errors.
### Table 5: Log Marginal likelihood estimates for the two factor MSV simulated examples

|          | Simulation Example of $p = 5$ | Simulation Example of $p = 10$ |
|----------|-------------------------------|-------------------------------|
|          | $k = 1$ (true)                | $k = 2$                        |
|          | $k = 3$                        | $k = 1$                        |
|          | $k = 2$ (true)                | $k = 3$                        |
| Replicate-1 | -5,001.85                    | -5,012.58                     |
| Replicate-2 | -4,967.76                    | -4,974.77                     |
| Replicate-3 | -5,042.42                    | -5,051.48                     |
| Replicate-4 | -5,060.63                    | -5,069.60                     |
| Replicate-5 | -4,958.36                    | -4,966.16                     |

Table 6: Log Marginal likelihood estimates for the real example

|          | Sample from 01/1990 ~ 12/2006 |
|----------|--------------------------------|
|          | $k = 1$                        |
|          | $k = 2$                        |
|          | $k = 3$                        |
|          | $k = 4$                        |
|          | $k = 5$                        |
| Log Marginal-Likelihood | -34,097.05 | -33,722.09 | -33,610.93 | -33,554.69 | -33,594.61 |

### Table 7: Summary output of the univariate GARCH simulated examples.

The table reports the average values of the posterior estimates across 10 replicates. For ‘delayed rejection’ MCMC, there are two-stage acceptance ratios.
Figure 1: Plots of likelihood ratio against iteration number for Monte Carlo EM. The left column of panels plots the likelihood ratios for each replicate of the first univariate SV simulated example (USV-Exp(1)). The middle column plots the likelihood ratios for each replicate of the second univariate SV simulated example (USV-Exp(2)). The right column plots the likelihood ratios for each replicate of the factor MSV models ‘P5-F1’ example (‘MSV-P5F1’) and ‘P10-F2’ example (‘MSV-P10F2’). The likelihood ratios for the first two iterates are left out because sometimes they are very large, especially in the factor MSV case.

Figure 2: Plots of the iterates for $\beta_1$ to $\beta_4$ for the ‘P5-F1’ simulation example
Figure 3: Plots of the iterates for $\beta_1$ to $\beta_4$ for the ‘P10-F2’ simulation example