GLOBAL WELL-POSEDNESS OF THE FREE-INTERFACE INCOMPRESSIBLE EULER EQUATIONS WITH DAMPING

JIALI LIAN
School of Mathematical Sciences
Xiamen University, Xiamen, Fujian 361005, China

(Communicated by Alberto Bressan)

Abstract. We prove the global well-posedness of the free interface problem for the two-phase incompressible Euler Equations with damping for the small initial data, where the effect of surface tension is included on the free surfaces. Moreover, the solution decays exponentially to the equilibrium.

1. Introduction.

1.1. Eulerian formulation. We consider two distinct, immiscible, inviscid, incompressible fluids evolving in two moving domains \( \Omega_\pm(t) \), respectively, for time \( t \geq 0 \). One fluid (+), called the “upper fluid”, fills the upper domain

\[
\Omega_+(t) = \{ y \in \mathbb{T}^2 \times \mathbb{R} \mid h_-(t, y_1, y_2) < y_3 < \ell + h_+(t, y_1, y_2) \},
\]

and the other fluid (−), called the “lower fluid”, fills the lower domain

\[
\Omega_-(t) = \{ y \in \mathbb{T}^2 \times \mathbb{R} \mid -b < y_3 < h_-(t, y_1, y_2) \}.
\]

Here we assume that the domains are horizontally periodic for \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) the usual 1–torus. The lower boundary is fixed and given by the constant \( b > 0 \), but the two surface functions \( h_\pm \) are free and unknown. The surface \( \Sigma_+(t) = \{ y_3 = \ell + h_+(t, y_1, y_2) \} \) is the moving upper boundary of \( \Omega_+(t) \) where the upper fluid is in contact with the atmosphere, \( \Sigma_-(t) = \{ y_3 = h_-(t, y_1, y_2) \} \) is the moving internal interface between the two fluids, and \( \Sigma_b = \{ y_3 = -b \} \) is the fixed lower boundary of \( \Omega_-(t) \).

For each \( t > 0 \), the two fluids are described by their velocity and pressure functions, which are given by \( u_\pm(t, \cdot) : \Omega_\pm(t) \to \mathbb{R}^3 \) and \( p_\pm(t, \cdot) : \Omega_\pm(t) \to \mathbb{R} \), respectively. For each \( t > 0 \), we require that \( (u_\pm, p_\pm, h_\pm) \) satisfy the following free boundary problem for the two-phase incompressible Euler equations with damping,
with taking into account of the effect of surface tension on the free surfaces,

\[
\begin{align*}
\partial_t u_\pm + u_\pm \cdot \nabla u_\pm + \nabla p_\pm + a_\pm u_\pm &= 0 \quad \text{in } \Omega_\pm(t) \\
\text{div} u_\pm &= 0 \quad \text{in } \Omega_\pm(t) \\
\partial_t h_+ &= u_+ \cdot N_+ \quad \text{on } \Sigma_+(t) \\
\partial_t h_- &= u_- \cdot N_- \quad \text{on } \Sigma_-(t) \\
p_+ &= p_{\text{atm}} - \sigma_+ H_+ \quad \text{on } \Sigma_+(t) \\
p_- &= p_- + \sigma_- H_- \quad \text{on } \Sigma_-(t) \\
u_{3,\pm} &= 0 \quad \text{on } \Sigma_b.
\end{align*}
\]

The third and fourth equations in (1.3) state that the free surfaces \(\Sigma_\pm(t)\) move with the velocities of the two fluids; the fourth equation implies in particular that the normal components of the velocities are continuous across the free interface \(\Sigma_-\).

The first two equations in (1.3) is the incompressible Euler equations with damping (or dissipation), where \(a_\pm > 0\) is the damping coefficient; the equations is used in geophysical models for large-scale processes in atmosphere and ocean [27, 14], where the damping term models the friction due to the bottom of the ocean or the Rayleigh friction (or the Ekman pumping/dissipation) in the planetary boundary layer in the presence of rough boundaries [34, 6, 16]. We may refer to, for instance, [3, 29, 10, 22, 9] for some mathematical results of the incompressible damped Euler equations. \(p_{\text{atm}}\) is the constant pressure of the atmosphere and \(\sigma_\pm > 0\) are the surface tension coefficients. Finally, \(N_\pm = (-\partial_1 h_\pm, -\partial_2 h_\pm, 1)\) are the upward (non-unit) normal vectors to \(\Sigma_\pm(t)\), and \(H_\pm\) are twice the mean curvature of the free surfaces \(\Sigma_\pm(t)\):

\[
H_\pm = \nabla \cdot \left( \frac{\nabla h_\pm}{\sqrt{1 + |\nabla h_\pm|^2}} \right). 
\]

To complete the statement of the problem, we must specify the initial conditions. We suppose that the initial two free surfaces are given by the graphs of the functions \(h_\pm(0) = h_{0,\pm} : T^2 \to \mathbb{R}\), which yield the initial domain \(\Omega_\pm(0)\) on which the initial velocities \(u_\pm(0) = u_{0,\pm} : \Omega_\pm(0) \to \mathbb{R}^3\) are specified. We will assume that \(\ell + h_{0,+} > h_{0,-} > -b\) on \(T^2\).

1.2. Background. The local well-posedness of the two-phase incompressible Euler equations with surface tension, without damping, has been proved by Cheng, Coutand and Shkoller [7]; see also Shatah and Zeng [30, 31], Cheng, Coutant and Shkoller [8] and Pusateri [28]. We may also refer to Ambrose [1] and Ambrose and Masmoudi [2] for the local well-posedness of the irrotational flows. These results of the local well-posedness were established for the large initial data, which in particular implies the short-time structural stability of vortex sheets with surface tension. The presence of surface tension is necessary for the stability of vortex sheets in the two-phase incompressible Euler equations; without surface tension, we have the well-known Kelvin-Helmholtz instability, see Caflisch and Orellana [5] and Ebin [15]. For the short-time structural stability of vortex sheets in the two-phase compressible Euler equations, we refer to Coulombel and Secchi [11, 12] and Stevens [33].

Up to our best knowledge, the global stability of vortex sheets remains open. With including damping, there are no trivial vortex-sheet solutions to the Euler equations, and hence there is no meaning to consider their stability for the Euler
equations with damping. In this paper, we will prove the global well-posedness of the two-phase incompressible Euler equations with damping for the small initial data around the trivial equilibrium; we believe that the damping can not prevent the Kelvin-Helmholtz instability, and hence we need to include the effect of surface tension on the free interface. Note that the global stability of the trivial equilibrium has been established for the two-phase Navier-Stokes equations, where the surface tension is not necessary to be included due to the regularizing effect of the viscosity, see Wang, Tice and Kim [36] for the incompressible fluids and Jang, Tice and Wang [23, 24] for the compressible fluids; the arguments of [36, 24] employed the methods introduced previously by Beale [4] and Guo and Tice [19, 20, 21] for viscous surface waves. The damping effect leads to the global well-posedness for the free boundary problems of the Euler equations with damping is much more involved than that of the case of fixed domains or viscous fluids. Note that without damping the global well-posedness of inviscid water waves has been established under the irrotational assumption, see Germain, Masmoudi and Shatah [17] for instance.

1.3. Reformulation in flattening coordinates. In order to transform the free boundary problem (1.3) to be one in the fixed domain, we will use a flattening transformation as [36, 24] rather than the Lagrangian coordinate transformation. To this end, we define the fixed domain transformation as $[36, 24]$ rather than the Lagrangian coordinate transformation. boundary problem (1.3) to be one in the fixed domain, we will use a flattening 1.3. Reformulation in flattening coordinates. In order to transform the free boundary problem (1.3) to be one in the fixed domain, we will use a flattening transformation as [36, 24] rather than the Lagrangian coordinate transformation. To this end, we define the fixed domain transformation as $[36, 24]$ rather than the Lagrangian coordinate transformation.

$$\Omega_+ := \{0 < x_3 < \ell \} \text{ and } \Omega_- := \{-b < x_3 < 0 \},$$ \hspace{1cm} (1.5)

for which we have written the coordinates as $x \in \Omega_{\pm}$. We shall write $\Sigma_+ := \{x_3 = \ell \}$ for the upper boundary, $\Sigma_- := \{x_3 = 0 \}$ for the internal interface and $\Sigma_b := \{x_3 = -b \}$ for the lower boundary. Throughout the paper we will write $\Omega := \Omega_+ \cup \Omega_- \text{ and } \Sigma = \Sigma_+ \cup \Sigma_-$. We think of $h_{\pm}$ as functions on $\Sigma_{\pm}$ according to $h_+ : \mathbb{R}^+ \times (T^2 \times \{\ell \}) \to \mathbb{R}$ and $h_- : \mathbb{R}^+ \times (T^2 \times \{0 \}) \to \mathbb{R}$, respectively. We will transform the free boundary problem $f \in \Omega_+(t)$ to one in the fixed domains $\Omega_{\pm}$ by using the unknown free surface functions $h_{\pm}$. Indeed, we will flatten the coordinate domains via the following special coordinate transformation:

$$\Omega_{\pm} \ni x \mapsto (x_1, x_2, \varphi(t, x) := x_3 + \eta(t, x)) =: \Phi(t, x) = y \in \Omega_{\pm}(t),$$ \hspace{1cm} (1.6)

Here $\eta = \tilde{b}_1 \mathcal{P}_+ h_+ + \tilde{b}_2 \mathcal{P}_- h_-$, where $\mathcal{P}_{\pm}$ are the Poisson extensions defined by (A.3) and (A.8), respectively, and $\tilde{b}_1 = \tilde{b}_1(x_3), \tilde{b}_2 = \tilde{b}_2(x_3)$ are two smooth functions in $\mathbb{R}$ that satisfy

$$\tilde{b}_1(0) = \tilde{b}_1(\ell) = 0, \tilde{b}_1(\ell) = 1 \text{ and } \tilde{b}_2(\ell) = \tilde{b}_2(-b) = 0, \tilde{b}_2(0) = 1.$$ \hspace{1cm} (1.7)

Note that if $h_{\pm}$ are sufficiently small in an appropriate Sobolev space, then $\partial_3 \varphi = 1 + \partial_3 \eta > 0$ and hence the mapping $\Phi$ is a diffeomorphism. This allows us to transform the free boundary problem to one on the fixed domains $\Omega_{\pm}$. Since the domains $\Omega_{\pm}$ and the boundaries $\Sigma_{\pm}$ are now fixed, we henceforth consolidate notation by writing $f$ to refer to $f_{\pm}$. When we write an equation for $f$ we assume that the equation holds with the subscripts added on the domains $\Omega_{\pm}$ or $\Sigma_{\pm}$, etc.; when we write an equation involving $f$ on $\Sigma_-$, then it means that the equation holds for both $f = f_+$ and $f = f_-$. When necessary to distinguish the two, we will write out the subscripts explicitly. To write the jump conditions on $\Sigma_-$, for a quantity
\[ f = f_\pm, \text{ we define the interfacial jump as} \]
\[ \llbracket f \rrbracket := f_+\{x_3=0\} - f_-\{x_3=0\}. \]  
(1.8)

We define
\[ v(t,x) = u(t, \Phi(t,x)), \quad q(t,x) = p(t, \Phi(t,x)) - p_{atm} \text{ in } \Omega. \]  
(1.9)

Set
\[ \partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_3 \varphi} \partial_3, \quad i = t, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_3 \varphi} \]  
(1.10)

such that
\[ \partial_i u \circ \Phi(t,\cdot) = \partial_i^\varphi v, \quad i = t, 1, 2, 3. \]

Then in the new coordinates, the problem (1.3) becomes
\[
\begin{align*}
\partial_t^\varphi v + v \cdot \nabla^\varphi v + \nabla^\varphi q + av &= 0 \quad \text{in } \Omega \\
\nabla v \cdot v &= 0 \quad \text{in } \Omega \\
\partial_t h = v \cdot N &\quad \text{on } \Sigma \\
q &= -\sigma H &\quad \text{on } \Sigma_+ \\
\llbracket q \rrbracket &= \sigma H &\quad \text{on } \Sigma_- \\
v_3 &= 0 &\quad \text{on } \Sigma_b \\
(v,h)|_{t=0} &= (v_0,h_0).
\end{align*}
\]  
(1.11)

Here we have written \((\nabla^\varphi)_i = \partial_i^\varphi, \quad i = 1, 2, 3, \quad \nabla^\varphi \cdot v = \partial_i^\varphi v_i\) and \(N = (-\partial_1 \eta, -\partial_2 \eta, 1)\). Hereafter, on \(\Sigma_-\) the third equation is assumed to hold for \(v = v_\pm\) both, etc.

We assume that the initial surface functions satisfy the zero-average conditions
\[ \int_{\Sigma_\pm} h_{0,\pm} = 0. \]  
(1.12)

For sufficiently regular solutions, the conditions (1.12) persist in time, that is,
\[ \int_{\Sigma_\pm} h_{\pm}(t) = 0. \]  
(1.13)

Indeed, by the third, sixth and second equations in (1.11), we have
\[ \frac{d}{dt} \int_{\Sigma_-} h_- = \int_{\Sigma_-} \partial_t h_- = \int_{\Sigma_-} v_- \cdot N_- = \int_{\Sigma_-} \nabla v_\cdot v_- d\mathcal{V}_t = 0, \]  
(1.14)

where \(d\mathcal{V}_t = \partial_3 \varphi dx\). Similarly,
\[ \frac{d}{dt} \int_{\Sigma_+} h_+ = \int_{\Sigma_+} \partial_t h_+ = \int_{\Sigma_+} v_+ \cdot N_+ = \int_{\Sigma_+} \nabla v_\cdot v_+ d\mathcal{V}_t + \int_{\Sigma_-} v_- \cdot N_- = 0. \]  
(1.15)

The conditions (1.13) will allow us to apply Poincaré’s inequality for \(h_{\pm}\) on \(\Sigma_{\pm}\) for all \(t \geq 0\). If it happens that the initial surface functions do not satisfy the zero average conditions (1.12), then we can shift the data and the coordinate system so that (1.12) is satisfied, provided that the extra condition \(\ell + h_{0,\pm} > h_{0,-} > -b\) is satisfied; see [36] for instance.

The problem (1.11) possesses a natural physical energy. For sufficiently regular solutions, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation induced by the damping:
\[ \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} |v|^2 d\mathcal{V}_t + \int_{\Sigma} \sigma \left( \sqrt{1 + |\nabla h|^2} - 1 \right) \right) + \int_{\Omega} a|v|^2 d\mathcal{V}_t = 0. \]  
(1.16)
1. Main results. We write \( H^k(\Omega_\pm) \) with \( k \geq 0 \) and \( H^s(\Sigma_\pm) \) with \( s \in \mathbb{R} \) for the usual Sobolev spaces. We will typically write \( H^0 = L^2 \). If we write \( f \in H^k(\Omega) \), the understanding is that \( f \) represents the pair \( f_\pm \) defined on \( \Omega_\pm \) respectively, and that \( f_\pm \in H^k(\Omega_\pm) \). We employ the same convention on \( \Sigma_\pm \). To avoid notational clutter, we will avoid writing \( H^k(\Omega) \) or \( H^k(\Sigma) \) in our norms and typically write

\[
\|f\|_H^2 = \|f_+\|_{H^2(\Omega_+)}^2 + \|f_-\|_{H^2(\Omega_-)}^2 \quad \text{and} \quad |f|_s^2 = \|f_+\|_{H^s(\Sigma_+)}^2 + \|f_-\|_{H^s(\Sigma_-)}^2. \tag{1.17}
\]

When there is a potential of confusion, we will write explicitly the space.

For a generic integer \( n \geq 3 \), we define the energy as

\[
E_n := \sum_{j=0}^n \|\partial_j v\|^2_{\frac{3}{2}(n-j)} + \|\partial_j q\|^2_{\frac{3}{2}(n-j)-\frac{1}{2}} + \sum_{j=0}^{n-1} \|\partial_j h\|^2_{\frac{3}{2}(n-j)+\frac{1}{2}}. \tag{1.18}
\]

The local well-posedness of the problem (1.11) with surface tension in our energy functional \( E_n \) \((n \geq 3)\) can follow exactly in the same way as that of the problem without damping; indeed, the local well-posedness of the two-phase incompressible Euler equations with surface tension in the energy functional \( E_2 \) \((i.e., \ n = 2)\) was proved by Cheng, Coutand and Shkoller [7], and one could then check step by step that there would not be any essential difficulties to generalize the local well-posedness result in [7] to the case \( n \geq 3 \) or with damping. We may then record the local well-posedness of (1.11) as the following.

**Theorem 1.1.** Let \( n \geq 3 \) be an integer. Assume that \( v_0 \in H^{\frac{3}{2}n}(\Omega) \), \( h_0 \in H^{\frac{3}{2}n+1}(\Sigma) \) satisfy \( \nabla v_0 \cdot v_0 = 0 \) in \( \Omega \), \( \|v_0\| \cdot N_0 = 0 \) on \( \Sigma_- \) and \( v_{0,3} = 0 \) on \( \Sigma_b \). There exists a \( T_0 > 0 \) such that there exists a unique solution \((v, q, h)\) solving (1.11) on \([0, T_0]\) satisfying

\[
\sup_{[0, T_0]} E_n \leq P(E_n(0)). \tag{1.19}
\]

Here \( P \) is a generic polynomial.

Our main results of this paper are stated as follows.

**Theorem 1.2.** Let \( n \geq 3 \) be an integer. Assume that \( v_0 \in H^{\frac{3}{2}n}(\Omega) \), \( h_0 \in H^{\frac{3}{2}n+1}(\Sigma) \) satisfy \( \nabla v_0 \cdot v_0 = 0 \) in \( \Omega \), \( \|v_0\| \cdot N_0 = 0 \) on \( \Sigma_- \), \( v_{0,3} = 0 \) on \( \Sigma_b \) and the zero-average conditions (1.12). There exists a universal constant \( \epsilon_0 > 0 \) such that if \( E_n(0) \leq \epsilon_0 \), then there exists a global unique solution \((v, q, h)\) solving (1.11) on \([0, \infty)\). Moreover, there exists universal constants \( C, \gamma > 0 \) such that for all \( t \geq 0 \),

\[
E_n(t) + \int_0^t E_n(\tau) d\tau \leq C E_n(0) \tag{1.20}
\]

and

\[
E_n(t) \leq C E_n(0) e^{-\gamma t}. \tag{1.21}
\]

**Remark 1.1.** Since \( h \) is such that the mapping \( \Phi(t, \cdot) \), defined in (1.6), is a diffeomorphism for each \( t \geq 0 \). As such, one may change coordinates to \( y \in \Omega(t) \) to produce a global-in-time, exponentially decaying solution to (1.3).

**Remark 1.2.** We can also consider the two-phase Euler equations with different densities under the influence of the gravitational force. In this case, if the upper fluid is heavier than the lower fluid, then the fluids are susceptible to the well-known Rayleigh-Taylor gravitational instability. We may prove that there exists a critical surface tension value \( \sigma_c \) such that if \( \sigma_- > \sigma_c \), then the problem is nonlinearly stable;
if $0 < \sigma_- < \sigma_c$, the the problem is nonlinearly unstable. The critical value $\sigma_c$ is same as the viscous problem \cite{35, 36, 18, 24, 25}. We will report this in a forthcoming paper.

We now present a sketch of main ideas in the proof of Theorem 1.2, and by Theorem 1.1 it suffices to derive the a priori estimates as recorded in Theorem 6.1.

The first step is to utilize the geometric structure of (1.11) and the energy-dissipation structure (1.16) to derive the following temporal energy evolution estimate:

$$
\mathcal{E}_n(t) + \int_0^t \mathcal{D}_n(\tau) d\tau \lesssim \mathcal{E}_n(0) + (\mathcal{E}_n(t))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2} d\tau,
$$

where

$$
\mathcal{E}_n := \sum_{j=0}^n \left( \| \partial_t^j v \|_0^2 + \| \partial_t^j h \|_1^2 \right) \quad \text{and} \quad \mathcal{D}_n := \sum_{j=0}^n \| \partial_t^j v \|_0^2.
$$

The derivation of the estimate (1.22), which is in spirit of the local well-posedness theory in \cite{7, 34}, relies heavily on the definition of $\mathcal{E}_n$ (cf. (1.18)), especially that $\partial_t^{-1} v \in H^{3/2}(\Omega)$ and $\partial_t^n v \in H^{-1/2}(\Sigma)$. For the one-phase incompressible Euler equations [13], one can use either $\partial_t^{-1} v \in H^{3/2}(\Omega)$ or $\partial_t^n v \in H^{-1/2}(\Sigma)$ to close the highest order temporal energy estimate. However, for the two-phase incompressible Euler equations, especially in our flattening coordinates, $\partial_t^n v \in H^{-1/2}(\Sigma)$ is necessary to be included in $\mathcal{E}_n$. Indeed, compared to the one-phase incompressible Euler equations, the new and most technical difficulty is the control of the following boundary integral on $\Sigma_-$:

$$
- \int_{\Sigma_-} \partial_t^{-1} q_- \left[ \partial_t^n v \right] \cdot N = \int_{\Sigma_-} \partial_t^{-1} q_- \left[ \left( n + 1 \right) \| \partial_t^n v \|_0 \cdot N + \| v \| \cdot \partial_t^{n+1} N \right] + \cdots.
$$

(1.24)

Here $+ \cdots$ means a sum of terms that can be controlled easily by $C(\mathcal{E}_n)^{3/2}$. The first integral in the right hand side of (1.24) is controlled by the $H^{1/2}(\Sigma_-) - H^{-1/2}(\Sigma_-)$ dual paring due to that $\partial_t^n v \in H^{-1/2}(\Sigma_-)$. To estimate the second integral we use the third equation with $v_-$ on $\Sigma_-$ to write, upon an integration by parts in $x_*$,

$$
\int_{\Sigma_-} \partial_t^{-1} q_- \left[ \| v \| \right] \cdot \nabla \partial_t^{n+1} N = - \int_{\Sigma_-} \partial_t^{-1} q_- \left[ \| v_* \| \right] \cdot \nabla \partial_t^{n+1} h

= \int_{\Sigma_-} \left[ \| v_* \| \cdot \nabla \partial_t^{n+1} h \right] \cdot \left( \partial_t^n v_- \cdot N - v_+ \cdot \nabla \partial_t^n h \right) + \cdots
$$

(1.25)

It is now crucial to convert the first integral in the right hand side of (1.25) to be an integral over the interior domain $\Omega_-$, by using the divergence theorem,

$$
\int_{\Sigma_-} \left[ \| v_* \| \cdot \nabla \cdot \partial_t^{-1} v_- \right] \cdot N

= \int_{\Omega_-} \xi \left( \left[ \| v_* \| \cdot \nabla \right] \partial_t^{-1} v_- \cdot N + \left[ v_* \right] \cdot \nabla \partial_t^{-1} v_- \cdot \nabla \partial_t v_- \right) dV_n + \cdots,
$$

(1.26)

where $\xi$ is a nonnegative cutoff function so that $\text{supp} \xi \subset \Omega$ and $\xi = 1$ on $\Sigma_-$ and we extend $v_+$ to be defined on $\Omega$. The second integral in the right hand side of (1.26) is controlled by using the second equation of (1.11) in $\Omega_-$. For the first integral, we use the first equation of (1.11) in $\Omega_-$ to write

$$
\int_{\Omega_-} \left[ \| v_* \| \cdot \nabla \right] \partial_t^{-1} v_- \cdot dV_n
$$

(1.27)
\[= \int_{\Omega} ((\nabla \phi) \cdot \nabla \partial_t^{n-1} q_-)
\]
\[= \int_{\Omega} ((\nabla \phi) \cdot \nabla \partial_t^{n-1} q_-)
\]
\[\partial_t^3 \phi \partial_t^{n-1} v_- - \nabla^2 \partial_t^{n-1} q_- \) d\Omega + \cdots,
\]
where \( w = \frac{1}{\partial_t} (v \cdot \mathbf{N} - \partial_t \eta) \). The second integral in the right hand side of (1.27) is controlled by integrating by parts in \( x \). Now the situation is that it remains to estimate the first integral in the right hand side of (1.27) and the second integral in the right hand side of (1.25). However, there are both out of control and can not be estimated by using further the equations. To overcome this difficulty, we will introduce the differential operator \( D^-_t = \partial_t^3 + V_\cdot \nabla \), where \( V_\cdot \) is an extension of \( v_- \) onto \( \Omega \). The key point lies in that when considering \( D^-_t \partial_t^{n-1} \) instead of \( \partial_t^n \), those two troublesome integrals are canceled out; these allow us to conclude the estimate (1.22).

The next goal is to replace \( \mathcal{E}_n \) and \( \mathcal{D}_n \) in the left hand side of (1.22) by the full one \( \mathcal{E}_n \). To derive the estimates for the normal derivatives of \( v \), the natural way is to estimate instead the vorticity \( \text{curl}^\mathcal{E} = \nabla^\mathcal{E} \times v \) to get rid of the pressure term \( \nabla^\mathcal{E} q \), and the conclusion is that
\[
\sup_{[0,T]} \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl} v \right\|_{\frac{2}{3(n-j)-1}}^2 + \int_0^T \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl} v \right\|_{\frac{2}{3(n-j)-1}}^2 \lesssim \mathcal{E}_n(0) + \mathcal{E}_n^2 + \int_0^T \mathcal{E}_n^2. 
\]

We also use the equations of \( \text{curl}^\mathcal{E} v \) to derive the bound for the \( (H^1(\Omega))^\prime \)-norm of \( \text{curl}^\mathcal{E} \partial_t^n v \). The similar estimates for \( \text{div}^\mathcal{E} v \) follow easily by using the incompressible condition. The bound of the \( (H^1(\Omega))^\prime \)-norm of \( \text{curl}^\mathcal{E} \partial_t^n v \) and \( \text{div}^\mathcal{E} \partial_t^n v \) yield in particular that \( \partial_t^n v \in H^{-1/2}(\Sigma) \).

Note that to utilize the vorticity and divergence estimates so as to employ the Hodge-type elliptic estimates of \( \partial_t^j v \) for \( j = 0, \ldots, n-1 \), we need to show that \( \partial_t^j v_3 \in H^{\frac{2}{3}(n-j)-\frac{1}{2}}(\Sigma) \). This will follow from the third equation in (1.11) by showing that \( \partial_t^j h \in H^{\frac{2}{3}(n-j)+\frac{1}{2}}(\Sigma) \) for \( j = 1, \ldots, n \). By chaining the Hodge-type elliptic estimates of \( v \), the regularizing elliptic estimates of \( h \) by the presence of surface tension and the derivation of the estimates of \( q \), we can show an iteration-type argument that if \( \partial_t^j h \in H^{\frac{2}{3}(n-j)+\frac{1}{2}}(\Sigma) \), then \( \partial_t^{j-2} h \in H^{\frac{2}{3}(n-j-2)+\frac{1}{2}}(\Sigma) \), see Proposition 5.1 for more details. On the other hand, by the third equations in (1.11), we have
\[
\left| \partial_t^{n+1} h \right|_{\frac{1}{2}} \lesssim \mathcal{D}_n + \mathcal{E}_n^2. 
\]
So to conclude the estimates, it remains to show that \( \partial_t^n h \in H^1(\Sigma) \). Since \( |\partial_t^n h|_1^2 \leq \mathcal{E}_n \), this implies that for the estimates of \( \mathcal{E}_n \) in the energy, we have
\[
\mathcal{E}_n \lesssim \mathcal{E}_n + \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl} v \right\|_{\frac{2}{3(n-j)-1}}^2 + \mathcal{E}_n^2. 
\]
However, for the estimates of \( \mathcal{E}_n \) in the dissipation, the difficulty is that \( |\partial_t^n h|_1^2 \) is not controlled by \( \mathcal{D}_n \). The most crucial point of our global analysis is the observation of the following:
\[
\int_0^T |\partial_t^n h|_1^2 \lesssim \sup_{[0,T]} \left( |\partial_t^{n-1} h|_1^2 + |\partial_t^n h|_1^2 \right) + \int_0^T \left( |\partial_t^{n-1} h|_{\frac{1}{2}}^2 + |\partial_t^{n+1} h|_{-\frac{1}{2}}^2 \right). 
\]
See (5.24) for the proof of (1.31). Hence, we have the following time-integrated dissipation estimate

\[
\int_0^T E_n \lesssim \sup_{[0,T]} E_n + \int_0^T \left( \| \bar{D}_n + \sum_{j=0}^{n-1} \left\| \partial_j \mathbf{curl} \mathbf{v} \right\|_{L^2}^{2j/(n-j)-1} + (E_n)^2 \right). \tag{1.32}
\]

We remark that the \( L^2 \) norm of \( h \) is controlled by using Poincaré’s inequality since \( \int_\Sigma h = 0 \).

Consequently, combining (1.22), (1.28), (1.30) and (1.32), we can close the a priori estimates for \( E_n \) is small and then the decay estimate can be derived also, as recorded in Theorem 6.1. Hence, the proof of Theorem 1.2 is completed.

1.5. Notation. We use the Einstein convention of summing over repeated indices. Throughout the paper \( C > 0 \) denotes a generic constant that does not depend on the data and time, but can depend on the parameters of the problem, e.g., \( a, b, \sigma \) and \( n \). We refer to such constants as “universal”. Such constants are allowed to change from line to line. We employ \( A_1 \lesssim A_2 \) to mean that \( A_1 \leq CA_2 \) for a universal constant \( C > 0 \).

We write \( \mathbb{N} = \{0, 1, 2, \ldots\} \) for the collection of non-negative integers. When using space-time differential multi-indices, we write \( \mathbb{N}^{1+d} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)\} \) to emphasize that the 0-index term is related to temporal derivatives. For \( \alpha \in \mathbb{N}^{1+d} \), \( \partial^\alpha = \partial^{\alpha_0}_t \partial^{\alpha_1}_1 \cdots \partial^{\alpha_d}_d \). For any differential operator \( \partial^\alpha \), we use the commutators

\[
[\partial^\alpha, f, g] = \partial^\alpha (fg) - f \partial^\alpha g \quad \text{and} \quad [\partial^\alpha, f, g] = \partial^\alpha (fg) - f \partial^\alpha g - \partial^\alpha fg. \tag{1.33}
\]

We denote \( x_* = (x_1, x_2) \) for the horizontal coordinates and \( v_* = (v_1, v_2) \) for the horizontal components of \( v \). We denote \( \nabla_* \) for the horizontal gradient, \( \text{div}_* \) for the horizontal divergence and \( \Delta_* \) for the horizontal Laplacian. For simplify the notations, we still use \( \nabla f \) to mean \( \nabla_* f \), etc., when without confusion, for function \( f \) defined on \( \Sigma \). We omit the differential elements \( dx \) and \( dx_* \) of the integrals over \( \Omega \) and \( \Sigma \), and also sometimes the time differential elements.

2. Preliminary. In this section we record some preliminary results that will be used in the derivation of the a priori estimates for the solutions to (1.11). We will assume throughout the rest of the paper that the solutions are given on the interval \( [0, T] \) and obey the a priori assumption

\[
E_n(t) \leq \delta, \quad \forall t \in [0, T] \tag{2.1}
\]

for an integer \( n \geq 3 \) and a sufficiently small constant \( \delta > 0 \). This implies in particular that

\[
\frac{1}{2} \leq \partial_3 \varphi(t, x) \leq \frac{3}{2}, \quad \forall (t, x) \in [0, T] \times \Omega. \tag{2.2}
\]

(2.1) and (2.2) will be used frequently, without mentioning explicitly.

In order to derive the estimates for the time derivatives of the solutions to (1.11) (essentially for the highest order), as for the local well-posedness, it is natural to utilize the geometric structure of the equations given in (1.11); see also Guo and Tice [19, 20, 21] of utilizing this structure to handle with the pressure term. We
first apply $\partial_{t}^{j}$ for $j = 0, \ldots, n - 1$ to (1.11) to find that
\[
\begin{align*}
\partial_{t}^{j} \partial_{t} v + v \cdot \nabla \phi &+ \nabla \phi \partial_{t} q + a \partial_{t} v = F^{1,j} & \text{in } \Omega \\
\nabla \phi \cdot \partial_{t} v &= F^{2,j} & \text{in } \Omega \\
\partial_{h} \partial_{t} h &= \partial_{t} v \cdot \mathbf{N} + F^{3,j} & \text{on } \Sigma \\
\partial_{t} q &= -\sigma \partial_{t} H & \text{on } \Sigma_{+} \\
\left[ \partial_{t} q \right] &= \sigma \partial_{t} H & \text{on } \Sigma_{-} \\
\partial_{t} v_{3} &= 0 & \text{on } \Sigma_{b},
\end{align*}
\]
where
\[
F^{1,j} = -\left[ \partial_{t}^{j}, \partial_{t}^{j} + v \cdot \nabla \phi \right] v - \left[ \partial_{t}^{j}, \nabla \phi \right] q, \quad (2.4)
\]
\[
F^{2,j} = -\left[ \partial_{t}^{j}, \nabla \phi \right] \cdot v, \quad (2.5)
\]
\[
F^{3,j} = \left[ \partial_{t}^{j}, \mathbf{N} \right] \cdot v. \quad (2.6)
\]
Furthermore, we may write that for $j \geq 1$,
\[
\partial_{t} H = \nabla \cdot \left( \partial_{t} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^{2}}} \right) \right)
\]
\[
= \nabla \cdot \left( \frac{\nabla \partial_{t}^{j} h}{\sqrt{1 + |\nabla h|^{2}}} + \nabla h \partial_{t} \left( \frac{1}{\sqrt{1 + |\nabla h|^{2}}} \right) + \left[ \partial_{t}^{j} \nabla h, \frac{1}{\sqrt{1 + |\nabla h|^{2}}} \right] \right), \quad (2.7)
\]
and
\[
\partial_{t} \left( \frac{1}{\sqrt{1 + |\nabla h|^{2}}} \right) = \partial_{t}^{-1} \partial_{t} \left( \frac{1}{\sqrt{1 + |\nabla h|^{2}}} \right) = -\partial_{t}^{-1} \left( \frac{\nabla h \cdot \nabla \partial_{t} h}{\sqrt{1 + |\nabla h|^{2}}} \right)
\]
\[
= -\frac{\nabla h \cdot \nabla \partial_{t}^{j} h}{\sqrt{1 + |\nabla h|^{2}}} - \left[ \partial_{t}^{-1}, \frac{\nabla h}{\sqrt{1 + |\nabla h|^{2}}} \right] \cdot \nabla \partial_{t} h. \quad (2.8)
\]
It thus follows that
\[
\partial_{t} H = \nabla \cdot \left( \frac{\nabla \partial_{t}^{j} h}{\sqrt{1 + |\nabla h|^{2}}} - \frac{\nabla h \cdot \nabla \partial_{t}^{j} h}{\sqrt{1 + |\nabla h|^{2}}} \nabla h \right) + F^{4,j}, \quad (2.9)
\]
where
\[
F^{4,j} = \nabla \cdot \left( -\left[ \partial_{t}^{-1}, \frac{\nabla h}{\sqrt{1 + |\nabla h|^{2}}} \right] \cdot \nabla \partial_{t} h \nabla h + \left[ \partial_{t}^{j}, \frac{1}{\sqrt{1 + |\nabla h|^{2}}} \nabla h \right] \right). \quad (2.10)
\]
As explained in the introduction we can not simply use the equations (2.3) with taking $j = n$ to derive the estimates for the $n$-th order time derivative of the solutions, and we introduce the following differential operator
\[
\partial_{t}^{n} := D_{t}^{n} - D_{t}^{n-1} \text{ with } D_{t}^{n} = \partial_{t}^{n} + V_{-} \cdot \nabla \phi, \quad (2.11)
\]
where $V_{-}$ is the extension of $v_{-}$, defined in $\Omega_{-}$, onto $\Omega$, with taking the value of $v_{+}$ near $\Sigma_{+}$. Note that $D_{t}^{n}$ is the material derivative operator in $\Omega_{-}$ but not in $\Omega_{+}$; the important advantage of introducing $V_{-}$ not rather using $v$ is that it is continuous across the interface $\Sigma_{-}$. Note also that
\[
D_{t}^{n} = \partial_{t} + v_{n,+} \cdot \nabla \phi \text{ on } \Sigma_{+} \text{ and } D_{t}^{n} = \partial_{t} + v_{n,-} \cdot \nabla \phi \text{ on } \Sigma_{-} \cup \Sigma_{b}. \quad (2.12)
\]
In particular, $D_t^-$ is a tangential derivative on $\partial\Omega_\pm$ and is continuous across $\Sigma_-$.

We now restate the two key points, mentioned in the introduction, of using $\partial_t^n$ instead of $\partial_t^j$. First, it follows from the first equation of (2.3) in $\Omega_-$ with $j = n - 1$ that

$$
\partial_t v = -\nabla_\varphi \partial_t^{n-1} q - a_- \partial_t^{n-1} v + F_{-1,n-1} \text{ in } \Omega_-.
$$

(2.13)

Second, it follows from the third equation of (2.3) with $j = n - 1$ that

$$
\partial_t h = \partial_t^{n-1} v \cdot \mathbf{N} + v \cdot \partial_t^{n-1} \mathbf{N} + \tilde{F}_{3,n-1} = \partial_t^{n-1} v \cdot \mathbf{N} - \varphi_* \nabla_\varphi \partial_t^{n-1} h + \tilde{F}_{3,n-1},
$$

(2.14)

where

$$
\tilde{F}_{3,n-1} = [\partial_t^{n-1} v, \mathbf{N}].
$$

(2.15)

By (2.12), this implies

$$
\partial_t^h = \partial_t^{n-1} v \cdot \mathbf{N} + \tilde{F}_{3,n-1}.
$$

(2.16)

Now applying $D_t^-$ to the equations (2.3) with $j = n - 1$, we find that, by (2.16),

$$
\begin{align*}
\partial_t^2 \partial_t v + v \cdot \nabla \varphi \partial_t^{n-1} v + \nabla \varphi \partial_t q + a \partial_t^q & = F_{1,n} \text{ in } \Omega, \\
\nabla \varphi \cdot \partial_t^q v & = F_{2,n} \text{ in } \Omega, \\
\partial_t^q & = -\sigma \partial_t^H H \text{ on } \Sigma, \\
\left[ \partial_t^q \right] & = \sigma \partial_t^H H \text{ on } \Sigma, \\
\partial_t^3 v & = 0 \text{ on } \Sigma,
\end{align*}
$$

(2.17)

where

$$
\begin{align*}
F_{1,n} & = -[D_t^-, \partial_t^2 \varphi] \partial_t^{n-1} v - [D_t^-, \nabla \varphi] \partial_t^{n-1} q + D_t^- \partial_t^{1,n-1}, \\
F_{2,n} & = -[D_t^-, \nabla \varphi] \cdot \partial_t^{n-1} v + D_t^- F_{2,n-1}, \\
F_{3,n} & = -\varphi_* \nabla_\varphi \partial_t^n h + \tilde{F}_{3,n} \text{ with } \tilde{F}_{3,n} = \partial_t^{n-1} v \cdot D_t^- \mathbf{N} + D_t^- \tilde{F}_{3,n-1}.
\end{align*}
$$

(2.18) - (2.20)

Moreover, applying $D_t^-$ to the equation (2.9) with $j = n - 1$, we have

$$
\partial_t^2 H = \nabla \cdot \left( \frac{\nabla \partial_t^h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla \partial_t^h}{\sqrt{1 + |\nabla h|^2}} \nabla h \right) + F_{4,n},
$$

(2.21)

where

$$
F_{4,n} = \left[ D_t^-, \nabla \cdot \left( \frac{\nabla}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla}{\sqrt{1 + |\nabla h|^2}} \nabla h \right) \right] \partial_t^{n-1} h + D_t^- F_{4,n-1}.
$$

(2.22)

We present the estimates of these nonlinear terms $F_{i,n}$.

**Lemma 2.1.** It holds that

$$
\left\| F_{1,n} \right\|^2_0 + \left\| F_{2,n} \right\|^2_{1/2} + \left\| F_{3,n} \right\|^2_1 + \left\| F_{4,n} \right\|^2_2 \lesssim \left( \mathcal{E}_n \right)^2. \tag{2.23}
$$

**Proof.** We expand these commutators in $F_{i,j}$ into a sum of products and then control each product with the highest order derivative term in $H^r$ ($r = 0, 1/2, 1$, accordingly) and the lower order derivative terms in $H^m$ for $m$ depending on $r$, using Lemmas A.2 and A.1, the trace theory along with the definition of $\mathcal{E}_n$ (cf. (1.18)); all of them are bounded by $\mathcal{E}_n$. We remark that it is needed to have included the term $\left| \partial_t^{n+1} h \right|^2_{-1/2}$ in the definition of $\mathcal{E}_n$ so that when estimating $F_{i,n}$, by Lemma A.1,

$$
\left\| \partial_t^{n+1} \varphi \right\|^2_0 = \left\| \partial_t^{n+1} \eta \right\|^2_0 \lesssim \left\| \partial_t^{n+1} \varphi \right\|^2_0 \lesssim \left| \partial_t^{n+1} h \right|^2_{-1/2} \leq \mathcal{E}_n.
$$

(2.24)
The estimate (2.23) follows by summing.

In order to utilize the linear structure of (1.11), it is convenient to write the equations in the linear perturbed form

$$
\begin{cases}
\partial_t v + \nabla q + av = G^1 \quad \text{in } \Omega \\
\text{div} v = G^2 \quad \text{in } \Omega \\
\partial_t h = v_3 + G^3 \quad \text{on } \Sigma \\
q = -\sigma \Delta h - G^4 \quad \text{on } \Sigma_+ \\
\|q\| = \sigma \Delta h + G^4 \quad \text{on } \Sigma_- \\
v_3 = 0 \quad \text{on } \Sigma_b,
\end{cases}
$$

where

$$
\begin{align*}
G^1 &= \partial_t \eta \partial_\xi^3 v + \nabla \eta \partial_\xi^3 q - v \cdot \nabla \partial_\eta^3 v, \\
G^2 &= \nabla \eta \cdot \partial_\xi^3 v, \\
G^3 &= -v \cdot \nabla h, \\
G^4 &= \sigma \nabla \cdot ((1 + |\nabla h|^2)^{-1/2} - 1)\nabla h).
\end{align*}
$$

Here we have used the fact that, by (1.10), $\partial_\xi^i - \partial_i \eta \partial_\xi^3$ for $i = t, 1, 2, 3$.

We present the estimates of these nonlinear terms $G^i$.

**Lemma 2.2.** It holds that

$$
\sum_{j=0}^{n-1} \left( \left\| \partial_\xi^j G^1 \right\|^2_{2(n-j)^{-\frac{1}{2}}} + \left\| \partial_\xi^j G^2 \right\|^2_{2(n-j)^{-1}} + \left\| \partial_\xi^j G^3 \right\|^2_{2(n-j)^{-\frac{1}{2}}} \right) + \sum_{j=0}^{n} \left\| \partial_\xi^j G^4 \right\|^2_{2(n-j)^{-1}} \lesssim (E_n)^2.
$$

**Proof.** We first apply the spatial-time mixed differential operators to $G^i$ to expand them as sums of products and then proceed in the same way as Lemma 2.1, except when estimating $\partial_\xi^j G^4$ we need to use first the structure of $G^4$:

$$
\left\| \partial_\xi^j G^4 \right\|^2 \lesssim \left\| \partial_\xi^j \left( (1 + |\nabla h|^2)^{-1/2} - 1)\nabla h) \right\|^2_0.
$$

The estimate (2.30) follows by summing.

3. **Temporal estimates.** In this section we will derive the energy evolution estimates for the temporal derivatives of the solutions to (1.11). For a generic integer $n \geq 3$, we define the temporal energy by

$$
\mathcal{E}_n := \sum_{j=0}^{n} \left( \left\| \partial_\xi^j v \right\|^2_0 + \left\| \partial_\xi^j h \right\|^2_1 \right)
$$

and the corresponding dissipation by

$$
\mathcal{D}_n := \sum_{j=0}^{n} \left\| \partial_\xi^j v \right\|^2_0.
$$

To derive the estimates for the highest order time derivative of the solutions, we shall use the equations (2.17). A key point is that the various norms of the error between $\hat{\partial}_\xi^i$ and $\partial_\xi^i$ of the solutions are bounded by $C\mathcal{E}_n$. We record the following time-integrated temporal energy evolution estimate.
Proposition 3.1. It holds that
\[ \mathcal{E}_n(t) + \int_0^t \mathcal{D}_n(\tau) \, d\tau \lesssim \mathcal{E}_n(0) + (\mathcal{E}_n(t))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2} \, d\tau. \] (3.3)

Proof. We only present the estimates of the highest order time derivative of the solutions, and the lower orders can be estimated in a much simpler way. Taking the inner product of the first equation in (2.17) with \( \tilde{\partial}_t^n v \) and then integrating by parts over \( \Omega \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\tilde{\partial}_t^n v|^2 \, d\Omega + \int_{\Omega} \nabla \tilde{\partial}_t^n q \cdot \tilde{\partial}_t^n v \, d\Omega = \int_\Sigma \tilde{\partial}_t^n q \nabla \tilde{\partial}_t^n v \cdot \mathbf{N} \, d\Sigma \]
By integrating by parts over \( \Omega \), using the sixth and second equations in (2.17), we obtain
\[ \int_\Sigma \tilde{\partial}_t^n q \nabla \tilde{\partial}_t^n v \cdot \mathbf{N} \, d\Sigma \lesssim \|F^{1,n}\|_0 \|\tilde{\partial}_t^n v\|_0 \lesssim (\mathcal{E}_n)^{3/2}. \] (3.5)
By integrating by parts over \( \Omega \), using the sixth and second equations in (2.17), we obtain
\[ \int_{\Sigma_+} \sigma \tilde{\partial}_t^n H \left( \tilde{\partial}_t \tilde{\partial}_t^n h + v_* \cdot \nabla_* \tilde{\partial}_t^n h - \tilde{F}^{3,n} \right). \] (3.7)
By integrating by parts over \( \Omega \), we obtain
\[ -\int_{\Sigma_-} \sigma \tilde{\partial}_t^n H \tilde{F}^{3,n} \lesssim \left| \tilde{\partial}_t^n H \right|_{-1} \|\tilde{F}^{3,n}\|_1 \lesssim (\mathcal{E}_n)^{3/2}. \] (3.8)
By (2.25), we may write
\[ \int_{\Sigma_-} \sigma \tilde{\partial}_t^n \tilde{\partial}_\tau^n H \sim \int_{\Sigma_-} \sigma \left( \nabla \cdot \left( \frac{\nabla \tilde{\partial}_t^n h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla \tilde{\partial}_t^n h}{\sqrt{1 + |\nabla h|^2}} \nabla h \right) + F^{4,n} \right) \]
\[ \left( \tilde{\partial}_t \tilde{\partial}_t^n h + v_* \cdot \nabla_* \tilde{\partial}_t^n h \right). \] (3.9)
By (2.23), we obtain
\[ \int_{\Sigma_-} \sigma F^{4,n} \left( \tilde{\partial}_\tau^n h + v_* \cdot \nabla_* \tilde{\partial}_t^n h \right) \leq \|F^{4,n}\|_2 \left| \tilde{\partial}_\tau^n h + v_* \cdot \nabla_* \tilde{\partial}_t^n h \right|_{-\frac{1}{2}} \lesssim (\mathcal{E}_n)^{3/2}. \] (3.10)
Integrating by parts in both $x_*$ and $t$, we find that

$$
\int_{\Sigma_-} \sigma \nabla \cdot \left( \frac{\nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \right) \partial_t \tilde{\rho}_t^p h = - \int_{\Sigma_-} \left( \frac{\nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \cdot \nabla \partial_t \tilde{\rho}_t^p h \right)
$$

$$
= - \frac{1}{2} \frac{d}{dt} \int_{\Sigma_-} \sigma \frac{|\nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} + \frac{1}{2} \int_{\Sigma_-} \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\nabla h|^2}} \right) |\nabla \tilde{\rho}_t^p h|^2
$$

$$
\leq - \frac{1}{2} \frac{d}{dt} \int_{\Sigma_-} \sigma \frac{|\nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} + C(\mathcal{E}_n)^{3/2}.
$$

(3.11)

Similarly, we have

$$
- \int_{\Sigma_-} \sigma \nabla \cdot \left( \frac{\nabla h \cdot \nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \right) \partial_t \tilde{\rho}_t^p h \leq \frac{1}{2} \frac{d}{dt} \int_{\Sigma_-} \sigma \frac{|\nabla h \cdot \nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} + C(\mathcal{E}_n)^{3/2}.
$$

(3.12)

Upon an integration by parts in $x_*$, we find that

$$
\int_{\Sigma} \sigma \nabla \cdot \left( \frac{\nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \right) v \cdot \nabla \tilde{\rho}_t^p h = - \int_{\Sigma} \sigma \nabla \cdot \left( \frac{\nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \right) \nabla \cdot (v \cdot \nabla \tilde{\rho}_t^p h)
$$

$$
= - \int_{\Sigma} \sigma \frac{\nabla \cdot \nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \cdot \nabla \cdot (v \cdot \nabla \tilde{\rho}_t^p h) + \frac{1}{2} \int_{\Sigma} \sigma \nabla \cdot \left( \frac{v}{\sqrt{1 + |\nabla h|^2}} \right) |\nabla \tilde{\rho}_t^p h|^2
$$

$$
\lesssim (\mathcal{E}_n)^{3/2}.
$$

(3.13)

Similarly, we have

$$
- \int_{\Sigma} \sigma \nabla \cdot \left( \frac{\nabla h \cdot \nabla \tilde{\rho}_t^p h}{\sqrt{1 + |\nabla h|^2}} \right) v \cdot \nabla \tilde{\rho}_t^p h \lesssim (\mathcal{E}_n)^{3/2}.
$$

(3.14)

Hence, (3.7)--(3.14) implies

$$
I_2 \leq - \frac{1}{2} \frac{d}{dt} \int_{\Sigma_-} \sigma \left( \frac{|\nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} - \frac{|\nabla h \cdot \nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} \right) + C(\mathcal{E}_n)^{3/2}.
$$

(3.15)

Similarly, we have

$$
I_1 \leq - \frac{1}{2} \frac{d}{dt} \int_{\Sigma_+} \sigma \left( \frac{|\nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} - \frac{|\nabla h \cdot \nabla \tilde{\rho}_t^p h|^2}{\sqrt{1 + |\nabla h|^2}} \right) + C(\mathcal{E}_n)^{3/2}.
$$

(3.16)

Now, we estimate $I_3$. By integrating by parts in $t$ and $x_*$, we obtain

$$
I_3 = \frac{d}{dt} \int_{\Sigma_-} \partial_t^{n-1} q_- \left[ \tilde{\rho}_t^p v \right] \cdot N - \int_{\Sigma_-} \partial_t^{n-1} q_- \partial_t \left( \left[ \tilde{\rho}_t^p v \right] \cdot N \right)
$$

$$
+ \int_{\Sigma_-} v_{x_*,-} \cdot \nabla \partial_t^{n-1} q_- \left[ \tilde{\rho}_t^p v \right] \cdot N
$$

$$
\leq \frac{d}{dt} \int_{\Sigma_-} \partial_t^{n-1} q_- \left[ \tilde{\rho}_t^p v \right] \cdot N - \int_{\Sigma_-} \partial_t^{n-1} q_- \left[ D_t \tilde{\rho}_t^p v \right] \cdot N + C(\mathcal{E}_n)^{3/2}.
$$

(3.17)

Here we have used the $H^{1/2}(\Sigma_-) - H^{-1/2}(\Sigma_-)$ dual pairing, the trace theory, Lemma A.3 and the fact that $|\partial_t^{n-1} q_-|_2^{1/2} \lesssim \mathcal{E}_n$. Hereafter, for the conciseness of presentations, we omit to write out explicitly the estimates of many terms which are bounded by
Hence, we obtain, by (2.23),

$$\text{supp}_{\Omega} \subset \Omega$$

By (2.16), we have

Then we obtain, by a further integration by parts in $x_*$,

$$\int_{\Sigma_-} \partial_t \partial_t^n q_- [D_t^{-} \partial_t^n v] \cdot \mathbf{N} \leq -\int_{\Sigma_-} \partial_t^{n-1} q_- [\nabla_* D_t^{-} \partial_t^n h - \nabla_* [D_t^{-} \partial_t^n, \nabla_*] h]. (3.19)$$

By (2.16), we have

$$D_t^{-} \partial_t^n h = \partial_t^n v \cdot \mathbf{N} + \tilde{\mathbf{F}}^{3,n}. (3.20)$$

Hence, we obtain, by (2.23),

$$-\int_{\Sigma_-} [\nabla_* \partial_t^{n-1} q_- D_t^{-} \partial_t^n \mathbf{N}] \leq \int_{\Sigma_-} [\nabla_* \partial_t^{n-1} q_- D_t^{-} \partial_t^n \mathbf{N}] + C(\mathcal{E}_n)^{3/2}. (3.22)$$

Now we convert the remaining integral in the right hand side of (3.22) to be one over the interior domain $\Omega_-$. For this, let $\xi$ be a nonnegative cutoff function so that $\text{supp} \xi \subset \Omega$ and $\xi = 1$ on $\Sigma_-$. Let $V_+$ be an extension of $v_+$, defined in $\Omega_+$, onto $\Omega_-$. By using the divergence theorem, we have

$$\int_{\Omega_-} [\nabla_* \partial_t^{n-1} q_- \nabla_* \partial_t^n v_- \cdot \mathbf{N}] = \int_{\Omega_-} \nabla_* \cdot \left(\xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \partial_t^n v_-ight) d\mathcal{V}_t$$

$$\leq \int_{\Omega_-} \left(\xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \partial_t^n v_- d\mathcal{V}_tight)$$

$$+ \int_{\Omega_-} \xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \nabla_* \partial_t^n v_- d\mathcal{V}_t + C(\mathcal{E}_n)^{3/2}. (3.23)$$

By using the second equation of (2.17) and (2.23), we obtain

$$\int_{\Omega_-} \xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \nabla_* \partial_t^n v_- d\mathcal{V}_t$$

$$\leq \| (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \|_0 \| F^{2,n} \|_0 \lesssim \mathcal{E}_n^2. (3.24)$$

By integrating by parts in $x_*$ and using the first equation of (2.17) and (2.23), we have

$$\int_{\Omega_-} (\xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^{n-1} q_- \partial_t^n v_- d\mathcal{V}_t)$$

$$\leq -\int_{\Omega_-} \nabla_* \partial_t^{n-1} q_- \cdot \xi (V_+ - v_-)_* \cdot \nabla_* \partial_t^n v_- d\mathcal{V}_t + C(\mathcal{E}_n)^{3/2}$$

$$= \int_{\Omega_-} \nabla_* \partial_t^{n-1} q_- \cdot \xi (V_+ - v_-)_* \cdot \nabla_* \left(\nabla_* \partial_t^{n-1} q_- + a_- \partial_t^{n-1} v_- - F^{1,n-1}_- \right) d\mathcal{V}_t + C(\mathcal{E}_n)^{3/2}$$
Accordingly, integrating by parts in $\Omega_-$, we obtain

$$
\int_{\Omega_-} (\xi(V_+ - v_-)_* \cdot \nabla_*) \nabla^2 \partial_t^{-1} q_- dV_i
$$

where

$$
\partial_t^{-1} q_- \nabla^2 \partial_t^{-1} q_- dV_i = -\frac{1}{2} \int_{\Omega_-} \text{div_*} (\partial_3 \xi (V_+ - v_-)_*) |\nabla^2 \partial_t^{-1} q_-|^2 \lesssim (\mathcal{E}_n)^{3/2}.
$$

Hence, (3.17), (3.20) and (3.22)–(3.26) imply

$$
\mathcal{I}_3 \leq \frac{d}{dt} \int_{\Sigma} \partial_t^{-1} q_- \left[ \bar{\partial}_t^n v \right] \cdot N + C(\mathcal{E}_n)^{3/2}.
$$

Finally, we turn to estimate $\mathcal{I}_4$. By integrating by parts in $t$, we obtain

$$
\int_{\Omega} \partial_t^n q F^{2,n} dV_i = \int_{\Omega} (\partial_t^n q F^{2,n} + (D_t^- - \partial_t) \partial_t^{-1} q F^{2,n}) dV_i
$$

$$
\leq \frac{d}{dt} \int_{\Omega} \partial_t^{-1} q F^{2,n} dV_i - \int_{\Omega} \partial_t^{-1} q \partial_t F^{2,n} dV_i + C(\mathcal{E}_n)^{3/2}.
$$

To estimate the second integral in (3.28), since $\nabla^2 v = \text{div}_* v_* + \frac{1}{\partial_3 \xi} \partial_3 v \cdot N = 0$, we may use the following decomposition

$$
F^{2,n} = -\partial_3 \xi \tilde{F}^{2,n} \text{ with } \tilde{F}^{2,n} = \sum_{i=1}^{5} \tilde{F}_i^{2,n},
$$

where

$$
\tilde{F}_1^{2,n} = n \partial_t N \cdot \partial_t^{-1} \partial_3 v,
$$

$$
\tilde{F}_2^{2,n} = n \partial_t \partial_3 \eta \partial_t^{-1} \text{div}_* v_*,
$$

$$
\tilde{F}_3^{2,n} = \partial_t^n N \cdot \partial_3 v,
$$

$$
\tilde{F}_4^{2,n} = \partial_t^n \partial_3 \eta \text{div}_* v_*,
$$

$$
\tilde{F}_5^{2,n} = \sum_{\ell=2}^{n-1} C_n \left( \partial_t^n N \cdot \partial_t^{-\ell} \partial_3 v + \partial_t^n \partial_3 \eta \cdot \partial_t^{-\ell} \text{div}_* v_* \right).
$$

Accordingly,

$$
- \int_{\Omega} \partial_t^{-1} q \partial_t F^{2,n} dV_i \leq \sum_{i=1}^{5} \int_{\Omega} \partial_t^{-1} q \partial_t \tilde{F}_i^{2,n} + C(\mathcal{E}_n)^{3/2}.
$$

We easily have

$$
\int_{\Omega} \partial_t^{-1} q \partial_t \tilde{F}_5^{2,n} \lesssim \|\partial_t^{-1} q\|_0 \|\partial_t \tilde{F}_5^{2,n}\|_0 \lesssim (\mathcal{E}_n)^{3/2}.
$$

Integrating by parts in $x_3$, we obtain, using $|\partial_t^n v|_{-1/2}^2 \lesssim \mathcal{E}_n$,

$$
\int_{\Omega} \partial_t^{-1} q \partial_t \tilde{F}_1^{2,n} = \int_{\Sigma} \partial_t^{-1} q n \partial_t N \cdot \partial_t^n v - \int_{\Sigma} n \partial_t N \cdot \left[ \partial_t^n v \partial_t^{-1} q \right]
$$

$$
+ \int_{\Omega} \partial_t (n \partial_t N \partial_t^{-1} q) \cdot \partial_t^n v \lesssim (\mathcal{E}_n)^{3/2}.
$$

(3.37)
Similarly, we have
\[
\int_{\Omega} \partial_t^{n-1} q \partial_t \left( F_{2,n}^2 + F_{3,n}^2 + F_{4,n}^2 \right) \lesssim (E_n)^{3/2}. \tag{3.38}
\]
Hence, (3.28) and (3.35)–(3.38) imply that
\[
\mathcal{I}_4 \leq \frac{d}{dt} \int_{\Omega} \partial_t^{n-1} q F^{2,n} \, dV_t + C(E_n)^{3/2}. \tag{3.39}
\]
Consequently, in light of the estimates (3.5), (3.15), (3.16), (3.27) and (3.39), we deduce from (3.4) that, by integrating in time from 0 to \( t \),
\[
\begin{align*}
\| \partial_t^n v(t) \|_0^2 + \| \nabla \partial_t^n h(t) \|_0^2 + \int_0^t \| \partial_t^n v(\tau) \|_0^2 \, d\tau \\
\lesssim \| \partial_t^n v(0) \|_0^2 + \| \nabla \partial_t^n h(0) \|_0^2 + B_n(t) - B_n(0) + \int_0^t (E_n(\tau))^{3/2} \, d\tau,
\end{align*}
\tag{3.40}
\]
where
\[
B_n = \frac{1}{2} \int_{\Sigma} \left( \sigma \left( 1 - \frac{1}{\sqrt{1 + |\nabla h|^2}} \right) \| \nabla \partial_t^n h \|^2 + \frac{|\nabla h \cdot \nabla \partial_t^n h|^2}{\sqrt{1 + |\nabla h|^2}} \right) + \int_{\Sigma_-} \partial_t^{n-1} q - \left[ \partial_t^n v \right] \cdot \mathbf{N} + \int_{\Omega} \partial_t^{n-1} q F^{2,n} \, dV_t. \tag{3.41}
\]
Recalling the definition (2.11) of \( \partial_t^n \), so
\[
\partial_t^n - \partial_t^n = -\partial_t \varphi \partial_t^2 \partial_t^{n-1} + V_- \cdot \nabla \varphi \partial_t^{n-1}. \tag{3.42}
\]
Then as Lemma 2.1, we may bound
\[
\| (\partial_t^n - \partial_t^n) v \|_0^2 + \| \nabla (\partial_t^n - \partial_t^n) h \|_0^2 \lesssim (E_n)^2. \tag{3.43}
\]
On the other hand, by the trace theory and using again that \( \| v \| \cdot \mathbf{N} = 0 \) on \( \Sigma_- \), we obtain
\[
\begin{align*}
\int_{\Sigma_-} \partial_t^{n-1} q - \left[ \partial_t^n v \right] \cdot \mathbf{N} \lesssim |\partial_t^{n-1} q_-|_{1/2} \left[ \left| \partial_t^n v \right| \cdot \mathbf{N} \right]_{-1/2} \\
\lesssim \| \partial_t^{n-1} q_- \|_{1} \left[ \left| \partial_t^n v, \mathbf{N} \right| \cdot \| v \| \right]_{-1/2} \lesssim (E_n)^{3/2}. \tag{3.44}
\end{align*}
\]
Also, by Lemma 2.1, we have
\[
\int_{\Omega} \partial_t^{n-1} q F^{2,n} \, dV_t \lesssim \| \partial_t^{n-1} q \|_0 \| F^{2,n} \|_0 \lesssim (E_n)^{3/2}. \tag{3.45}
\]
Therefore, by the estimates (3.43), (3.44) and (3.45), we deduce from (3.40) that
\[
\begin{align*}
\| \partial_t^n v(t) \|_0^2 + |\partial_t^n h(t)|_1^2 + \int_0^t \| \partial_t^n v(\tau) \|_0^2 \, d\tau \\
\lesssim (E_n(0)) + (E_n(t))^{3/2} + \int_0^t (E_n(\tau))^{3/2} \, d\tau,
\end{align*}
\tag{3.46}
\]
where we have used Poincaré’s inequality since \( \int_{\Sigma} h = 0 \). Deriving the same estimates for the lower order time derivatives in a much simpler way, the estimate (3.3) then follows.
4. Vorticity and divergence estimates. In this section we will derive the estimates for the normal derivatives of $v$. The natural way is to estimate instead the vorticity $\text{curl}^\phi v = \nabla^\phi \times v$ to get rid of the pressure term $\nabla^\phi q$; applying $\text{curl}^\phi$ to the first equation in (1.11), we have
\[
\partial_t^\phi \text{curl}^\phi v + v \cdot \nabla^\phi \text{curl}^\phi v + a\text{curl}^\phi v = -[\text{curl}^\phi, v] \cdot \nabla^\phi v \text{ in } \Omega_\pm.
\] (4.1)

Note that the third and sixth equations in (1.11) implies that the equations (4.1) are characteristic in each of $\Omega_\pm$, and the estimates follow in the same way as the one-phase incompressible Euler equations. For the completeness, we still repeat the procedure. We derive the following time-integrated energy estimates of the vorticity.

**Proposition 4.1.** It holds that
\[
\sup_{[0,T]} \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 + \int_0^T \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 \lesssim \mathcal{E}_n(0) + \sup_{[0,T]} (\mathcal{E}_n)^2 + \int_0^T (\mathcal{E}_n)^2. 
\] (4.2)

**Proof.** Let $j = 0, \ldots, n-1$. Apply $\mathcal{D}_j$, where $\mathcal{D}_j$ denotes for any $\partial^\alpha$, $\alpha \in \mathbb{N}^{1+3}$ with $\alpha_0 = j$ and $\alpha_1 + \alpha_2 + \alpha_3 \leq \left\lfloor \frac{3}{2} (n-j) \right\rfloor - 1$, to (4.1) to find that
\[
\partial_t^\phi \mathcal{D}_j \text{curl}^\phi v + v \cdot \nabla^\phi \mathcal{D}_j \text{curl}^\phi v + a\mathcal{D}_j \text{curl}^\phi v = \mathfrak{g}^j,
\] (4.3)
where
\[
\mathfrak{g}^j = -[\mathcal{D}_j, \partial_t^\phi + v \cdot \nabla^\phi] \text{curl}^\phi v - [\mathcal{D}_j, \text{curl}^\phi, v] \cdot \nabla^\phi v.
\] (4.4)

Denote $r_j = \frac{3}{2} (n-j) - \left\lfloor \frac{3}{2} (n-j) \right\rfloor \in [0, 1)$. It is easy to have
\[
\left\| \mathfrak{g}^j \right\|_{L^2_r} \lesssim (\mathcal{E}_n)^2.
\] (4.5)

Now the standard $H^r$ energy estimates on (4.3) yields that, by (4.5),
\[
\sup_{[0,T]} \left\| \mathcal{D}_j \text{curl}^\phi v \right\|_{L^2_r}^2 + a \int_0^T \left\| \mathcal{D}_j \text{curl}^\phi v \right\|_{L^2_r}^2 \lesssim \left\| \mathcal{D}_j \text{curl}^\phi v(0) \right\|_{L^2_r}^2 + \int_0^T \left\| \mathfrak{g}^j \right\|_{L^2_r}^2 \lesssim \left\| \mathcal{D}_j \text{curl}^\phi v(0) \right\|_{L^2_r}^2 + \int_0^T (\mathcal{E}_n)^2.
\] (4.6)

This implies that, by summing over such $\alpha$ in $\mathcal{D}_j$,
\[
\sup_{[0,T]} \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 + a \int_0^T \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 \lesssim \left\| \partial_t^\phi (\text{curl}^\phi - \text{curl}) v \right\|_{L^2(\Omega_\pm)}^2 + \int_0^T (\mathcal{E}_n)^2.
\] (4.7)

Since $\left\| \partial_t^\phi (\text{curl}^\phi - \text{curl}) v \right\|_{L^2(\Omega_\pm)}^2 \lesssim (\mathcal{E}_n)^2$ as Lemma 2.1, we deduce from (4.7) that
\[
\sup_{[0,T]} \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 + a \int_0^T \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl}^\phi v \right\|_{L^2(\Omega_\pm)}^2 \lesssim \mathcal{E}_n(0) + \sup_{[0,T]} (\mathcal{E}_n)^2 + \int_0^T (\mathcal{E}_n)^2.
\] (4.8)

This yields the estimate (4.2). $\square$
We now record an estimate for the divergence.

**Proposition 4.2.** It holds that

\[
\sum_{j=0}^{n-1} \left\| \partial^j_t \text{div} v \right\|_{L^2(\Omega)}^2 \lesssim \mathcal{E}^2_n. \tag{4.9}
\]

**Proof.** It follows directly by the second equation of (2.25) and (2.30).

Finally, we record the $H^{-1/2}(\Sigma)$-estimate of $\partial^n_t v$.

**Proposition 4.3.** It holds that

\[
\left\| \partial^n_t v \right\|_{L^2(\Omega)}^2 \lesssim \left\| \partial^n_{t-1} \text{curl} v \right\|_{L^2(\Omega)}^2 + \mathcal{E}^2_n. \tag{4.10}
\]

**Proof.** It follows from (4.1) that

\[
\partial_t \text{curl}^2 v + (v_\ast \cdot \nabla_\ast + w_3) \text{curl}^2 v + acur^2 v = -[\text{curl}^2, v] \cdot \nabla^2 v, \tag{4.11}
\]

where $w = \frac{1}{\partial_\Sigma^2} (v \cdot \mathbf{N} - \partial_\Sigma \varphi)$. Applying $\partial_{t-1}^{n-1}$ to (4.11) yields

\[
\partial_t^n \text{curl} v = \nabla \varphi \times \partial_t^n \varphi v - (v_\ast \cdot \nabla_\ast + w_3) \partial_t^{n-1} \text{curl}^2 v + F^n, \tag{4.12}
\]

where

\[
F^n = \left[ \partial_t^n, \nabla \varphi \times \partial_t^n \varphi \right] v - \left[ \partial_t^{n-1}, v_\ast \cdot \nabla_\ast + w_3, \text{curl}^2 v \right] - a\partial_t^{n-1} \text{curl}^2 v + \partial_t^{n-1} \left[ \text{curl}^2, v \right] \cdot \nabla^2 v. \tag{4.13}
\]

It is easy to see that $\|F^n\|_0 \lesssim \mathcal{E}_n + \|\partial_t^{n-1} \text{curl} v\|_0$. We now take $\psi \in H^1(\Omega)$, then we have

\[
\int_{\Omega} \partial_t^n \text{curl} v \psi = \int_{\Omega} (\nabla \varphi \times \partial_t^n \varphi v - (v_\ast \cdot \nabla_\ast + w_3) \partial_t^{n-1} \text{curl}^2 v + F^n) \psi. \tag{4.14}
\]

By integrating by parts over $\Omega_-$, we have

\[
\int_{\Omega_-} \nabla \varphi \times \partial_t^n \varphi v \psi = -\int_{\partial_\Omega_-} \nabla \varphi \times \partial_t^n \varphi v \psi - \int_{\Omega_-} \partial_3 \left( \frac{\nabla \varphi}{\partial_3 \varphi} \psi \right) \times \partial_t^n v \psi \lesssim \mathcal{E}_n \|\psi\|_1. \tag{4.15}
\]

and

\[
\int_{\Omega_-} (v_\ast \cdot \nabla_\ast + w_3) \partial_t^{n-1} \text{curl}^2 v \psi \lesssim \mathcal{E}_n \|\psi\|_1. \tag{4.16}
\]

Hence, we obtain

\[
\int_{\Omega_-} \partial_t^n \text{curl} v \psi \lesssim \mathcal{E}_n \|\psi\|_1, \tag{4.17}
\]

which, together with the similar arguments on $\Omega_+$, implies that

\[
\|\partial_t^n \text{curl} v\|_{L^2(\Omega)} \lesssim \mathcal{E}_n + \|\partial_t^{n-1} \text{curl} v\|_0. \tag{4.18}
\]

Similarly, we have

\[
\|\partial_t^n \text{div} v\|_{L^2(\Omega)} \lesssim \mathcal{E}_n. \tag{4.19}
\]

By the trace estimates (A.14), (4.18) and (4.19), we conclude (4.10).
5. Elliptic regularity. In this section we will chain the Hodge-type elliptic estimates of \(v\), the regularizing elliptic estimates of \(h\) by the presence of surface tension and the derivation of the estimates of \(q\) by using directly the fourth, fifth and first equations in (2.25) to derive the full energy estimates.

We first present the result in the “energy”.

**Proposition 5.1.** It holds that

\[
\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n + \sum_{j=0}^{n-1} \| \partial^j_t \text{curl } v \|_{L^2}^2 h_{\bar{\mathcal{E}}_n} + (\mathcal{E}_n)^2. \quad (5.1)
\]

**Proof.** We first derive some preliminary estimates resulting from the control of \(\bar{D}_n\) (weaker than \(\bar{\mathcal{E}}_n\)). By the normal trace estimate (A.13), the second equation in (2.17) and (2.23), we obtain

\[
\sum_{j=0}^{n} \left| \partial^j_t v \cdot N \right|_{\dot{H}^2}^2 \lesssim \sum_{j=0}^{n} \left( \| \partial^j_t v \|_{L^2}^2 + \| \nabla^2 \partial^j_t v \|_{L^2}^2 \right) \\
= \sum_{j=0}^{n} \left( \| \partial^j_t v \|_{L^2}^2 + \| F^{2,j} \|_{L^2}^2 \right) \lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.2)
\]

By the third equation in (2.17), (2.23) and (5.2), we have

\[
\sum_{j=1}^{n+1} \left| \partial^j_t h \right|_{\dot{H}^2}^2 \lesssim \sum_{j=1}^{n+1} \left( \| \partial^{j-1} v \cdot N \|_{\dot{H}^2}^2 + \| F^{3,j-1} \|_{\dot{H}^2}^2 \right) \lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.3)
\]

By the fourth and fifth equations in (2.25), (5.3) and (2.30), we obtain

\[
\sum_{j=1}^{n} \left( \| \partial^j q \|_{H^{-\frac{1}{2}}(\Sigma_+)}^2 + \| \partial^j q \|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \right) \lesssim \sum_{j=1}^{n} \left( \| \partial^j h \|_{H^{-\frac{1}{2}}(\Sigma_+)}^2 + \| \partial^j G^1 \|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \right) \\
\lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.4)
\]

On the other hand, by the first equation in (2.25) and (2.30), we have

\[
\| \partial^n_t \nabla q \|_{L^2(\Sigma_+)}^2 \lesssim \| \partial^n_t v \|_{L^2}^2 + \| \partial^n_t v \|_{L^2}^2 + \| \partial^n_t G^1 \|_{L^2}^2 \lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.5)
\]

Then by the Poincaré type inequality and the trace theory, (5.5) and (5.4), we obtain

\[
\| \partial^n_t q \|_1^2 \lesssim \| \partial^n_t \nabla q \|_{L^2(\Sigma_+)}^2 + \| \partial^n_t q \|_{H^{-\frac{1}{2}}(\Sigma_+)}^2 + \| \partial^n_t q \|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \\
\lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.6)
\]

By the standard elliptic regularity of \(h\) of the fourth and fifth equations in (2.25), the trace theory, (2.30), (5.3) and (5.6), we have

\[
\| \partial^n_t h \|_{\dot{H}^\frac{1}{2}(\Sigma_+)}^2 \lesssim \| \partial^n_t h \|_{\dot{H}^\frac{1}{2}}^2 + \| \partial^n_t q \|_{\dot{H}^\frac{1}{2}}^2 + \| \partial^n_t G^1 \|_{\dot{H}^\frac{1}{2}}^2 \\
\lesssim \| \partial^n_t h \|_{\dot{H}^\frac{1}{2}}^2 + \| \partial^n_t q \|_1^2 + (\mathcal{E}_n)^2 \lesssim \bar{D}_n + (\mathcal{E}_n)^2. \quad (5.7)
\]

Next, we will prove an iteration type argument: we assume the control of \(\| \partial^j_t h \|_{\dot{H}^\frac{1}{2}(\Sigma_+)}^2 \) for \(j = 1, \ldots, n\), then we derive the estimates of \(\| \partial^{j+1} h \|_{\dot{H}^\frac{1}{2}(\Sigma_+)}^2 \) for \(j = 1, \ldots, n\), and we continue this process until \(n = n_0\).
(for $j - 2 \geq 0$). We write compactly

$$X_n := \tilde{\mathcal{D}}_n + \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl} v \right\|_{\frac{1}{2}(n-j)-\frac{1}{2}}^2 + \sum_{j=0}^{n-2} \left\| \partial_t^j v \right\|_{\frac{1}{2}(n-j)-\frac{3}{2}}^2. \quad (5.8)$$

By the third equation in (2.25) and (2.30), we have

$$\left| \partial_t^{j-1} v_3 \right|_{\frac{1}{2}(n-j)-\frac{1}{2}}^2 \leq \left| \partial_t^{j-1} h \right|_{\frac{1}{2}(n-j)-\frac{1}{2}}^2 + \left| \partial_t^{j-1} G \right|_{\frac{1}{2}(n-j)-\frac{1}{2}}^2 \lesssim \partial_t^j h \left| v \right|_{\frac{1}{2}(n-j)+1} + (E_n)^2. \quad (5.9)$$

Then employing the Hodge-type elliptic estimates (A.15), by (4.9), (5.8) and (5.9), we obtain

$$\left\| \partial_t^{j-1} v \right\|_{\frac{1}{2}(n-j)-1}^2 \lesssim \left\| \partial_t^{j-1} v \right\|_{0}^2 + \left\| \partial_t^{j-1} \text{curl} v \right\|_{\frac{1}{2}(n-j)-1}^2 + \left\| \partial_t^{j-1} \text{div} v \right\|_{\frac{1}{2}(n-j)-1}^2 + \left| \partial_t^{j-1} v_3 \right|_{\frac{1}{2}(n-j)-\frac{1}{2}}^2 \lesssim \partial_t^j h \left| v \right|_{\frac{1}{2}(n-j)+1} + X_n + (E_n)^2. \quad (5.10)$$

By the first equation in (2.25), (5.10), (5.8) and (2.30), we have (for $j - 2 \geq 0$)

$$\left\| \partial_t^{j-2} \nabla q \right\|_{\frac{1}{2}(n-j)-1}^2 \lesssim \left\| \partial_t^{j-2} v \right\|_{\frac{1}{2}(n-j)-1}^2 + \left\| \partial_t^{j-2} v \right\|_{\frac{1}{2}(n-j)-1}^2 \lesssim \partial_t^j h \left| v \right|_{\frac{1}{2}(n-j)+1} + X_n + (E_n)^2. \quad (5.11)$$

We then use Poincaré’s inequality to estimate $\left\| \partial_t^{j-2} q \right\|_0^2$. If $j - 2 \geq 1$, then we use the boundary estimate (5.4); by the Poincaré type inequality and the trace theory, (5.11) and (5.4), we have that for $j - 2 \geq 1$,

$$\left\| \partial_t^{j-2} q \right\|_{\frac{1}{2}(n-j-2)-\frac{1}{2}}^2 \lesssim \left\| \partial_t^{j-2} \nabla q \right\|_{\frac{1}{2}(n-j)-1}^2 + \left\| \partial_t^{j-2} q \right\|_{H^{-\frac{1}{2}}(\Sigma_+)}^2 + \left\| \partial_t^{j-2} q \right\|_{H^{-\frac{1}{2}}(\Sigma_-)}^2 \lesssim \partial_t^j h \left| v \right|_{\frac{1}{2}(n-j)+1} + X_n + (E_n)^2. \quad (5.12)$$

If $j - 2 = 0$, then we need to estimate for $|q_+|^2_{H^{-\frac{1}{2}}(\Sigma_+)} + ||q||^2_{H^{-\frac{1}{2}}(\Sigma_-)}$ in a different way. For this, we recall here that $|h|^2_1 \leq E_n$. By the fourth and fifth equations in (2.25) and (2.30), we obtain

$$|q_+|^2_{H^{-\frac{1}{2}}(\Sigma_+)} + ||q||^2_{H^{-\frac{1}{2}}(\Sigma_-)} \lesssim |h|^2_1 + |G|^2_{-\frac{1}{2}} \lesssim |h|^2_1 + (E_n)^2. \quad (5.13)$$

Then by the Poincaré type inequality and the trace theory, (5.11) and (5.4), we have

$$\left\| q \right\|_{\frac{1}{2}(n-1)}^2 \lesssim \| \nabla q \|_{\frac{1}{2}(n-1)}^2 + |q_+|^2_{H^{-\frac{1}{2}}(\Sigma_+)} + ||q||^2_{H^{-\frac{1}{2}}(\Sigma_-)} \lesssim |h|^2_1 + \partial_t^j h \left| v \right|_{\frac{1}{2}(n-2)+1} + X_n + (E_n)^2. \quad (5.14)$$
By the elliptic regularity of the fourth and fifth equations in (2.25), the trace theory, (2.30), (5.12) and (5.14), we have that for \(j - 2 \geq 1\),
\[
|\partial^j \bar{h}|^2_{\frac{3}{2}(n-(j-2))+1} \lesssim |\partial^j \bar{q}|^2_{\frac{3}{2}(n-(j-2))-1} + |\partial^j G|^2_{\frac{3}{2}(n-(j-2))-1}
\]
\[
\lesssim \left\| \partial^j \bar{q} \right\|^2_{\frac{3}{2}(n-(j-2))-\frac{1}{2}} + (E_n)^2
\]
\[
\lesssim |\partial_t \bar{h}|^2_{\frac{3}{2}(n-j)+1} + X_n + (E_n)^2
\]  \(\text{(5.15)}\)
and that
\[
|h|^2_{\frac{3}{2}n+1} \lesssim |q|^2_{\frac{3}{2}n-1} + |G|^2_{\frac{3}{2}n-1} \lesssim \|q\|_{\frac{3}{2}n-\frac{1}{2}}^2 + (E_n)^2
\]
\[
\lesssim |h|^2_{\frac{3}{2}n} + |\partial_t^2 \bar{h}|^2_{\frac{3}{2}(n-2)+1} + X_n + (E_n)^2.
\]  \(\text{(5.16)}\)
We have thus arrived at the iterated estimates.

By a simple inductive argument on (5.15), combining with (5.16), we have
\[
\sum_{j=0}^{n-2} |\partial^j \bar{h}|^2_{\frac{3}{2}(n-j)+1} \lesssim |\partial_t^n \bar{h}|^2_1 + |\partial_t^{n-1} \bar{h}|^2_{\frac{3}{2}} + |h|^2_{\frac{3}{2}} + X_n + (E_n)^2.
\]  \(\text{(5.17)}\)
This together with (5.7) and (5.3) implies
\[
\sum_{j=0}^{n+1} |\partial^j \bar{h}|^2_{\frac{3}{2}(n-j)+1} \lesssim |\partial_t^n \bar{h}|^2_1 + |h|^2_{\frac{3}{2}} + X_n + (E_n)^2.
\]  \(\text{(5.18)}\)
We then obtain that by (5.10) and (5.18),
\[
\sum_{j=0}^{n} |\partial^j \bar{v}|^2_{\frac{3}{2}(n-j)} \lesssim |\partial_t^n \bar{h}|^2_1 + |h|^2_{\frac{3}{2}} + X_n + (E_n)^2
\]  \(\text{(5.19)}\)
and that by (5.6), (5.12), (5.14) and (5.18),
\[
\sum_{j=0}^{n-1} |\partial^j \bar{q}|^2_{\frac{3}{2}(n-j)-1} \lesssim |\partial_t^n \bar{h}|^2_1 + |h|^2_{\frac{3}{2}} + X_n + (E_n)^2.
\]  \(\text{(5.20)}\)

Consequently, combining (5.18)–(5.20) and recalling (5.8), by the definition of \(\tilde{E}_n\) and (4.10), we have
\[
E_n \lesssim |\partial_t^n \bar{h}|^2_1 + |h|^2_{\frac{3}{2}} + X_n + (E_n)^2
\]
\[
\lesssim \tilde{E}_n + \sum_{j=0}^{n-1} \left\| \partial^j \text{curl} \bar{v} \right\|^2_{\frac{3}{2}(n-j)-1} + \sum_{j=0}^{n-2} \left\| \partial^j \bar{v} \right\|^2_{\frac{3}{2}(n-j)-\frac{1}{2}} + (E_n)^2.
\]  \(\text{(5.21)}\)
We may use the Sobolev interpolation to improve the above to be
\[
E_n \lesssim \tilde{E}_n + \sum_{j=0}^{n-1} \left\| \partial^j \text{curl} \bar{v} \right\|^2_{\frac{3}{2}(n-j)} + \sum_{j=0}^{n-2} \left\| \partial^j \bar{v} \right\|^2_{0} + (E_n)^2
\]
\[
\lesssim \tilde{E}_n + \sum_{j=0}^{n-1} \left\| \partial^j \text{curl} \bar{v} \right\|^2_{\frac{3}{2}(n-j)-1} + (E_n)^2.
\]  \(\text{(5.22)}\)
This is the estimate (5.1).

Now we present the time-integrated result in the “dissipation”.

\[\square\]
Proposition 5.2. It holds that
\[
\int_0^T \mathcal{E}_n \leq \sup_{[0,T]} \mathcal{E}_n + \int_0^T \left( \mathcal{D}_n + \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl} v \right\|_{\mathfrak{S}(n-j-1)}^2 + (\mathcal{E}_n)^2 \right). \tag{5.23}
\]

**Proof.** We remark that we may not be able to show a pointwise-in-time estimate of \(\mathcal{E}_n\) in the dissipation as that in the energy of Proposition 5.1 since we can not control \(|\partial_t^n h|^2\) by \(\mathcal{D}_n\). Our crucial observation is that we can show an \(L^1\)-in-time estimate of \(|\partial_t^n h|^2\) by the control of the \(L^1\)-in-time bound of \(|\partial_t^{n+1} h|_{-1/2}^2 + |\partial_t^n h|^2_{5/2}\) and \(L^\infty\)-in-time bound of \(|\partial_t^n h|^2 + |\partial_t^{n-1} h|_{1}^2\). Recall that \(|\partial_t^n h|^2 + |\partial_t^{n-1} h|_{1}^2 \leq \mathcal{E}_n\).

Also, since \(|h|_{1}^2\) is not controlled by \(\mathcal{D}_n\), we need to modify those estimates that involve \(|h|_{1/2}^2\) in Proposition 5.1.

We first derive the \(L^1\)-in-time estimate of \(|\partial_t^n h|^2\). We can still use the estimates (5.2)–(5.7). By Parseval’s theorem, we have that, by (5.3) with \(j = n + 1\) and (5.7), using Cauchy’s inequality,
\[
\int_0^T |\partial_t^n h|^2 = \int_0^T \sum_{\xi \in \mathbb{Z}^2} (1 + |\xi|^2) \left| \partial_t^n \hat{h} \right|^2 \\
= \int_0^T \sum_{\xi \in \mathbb{Z}^2} \partial_t \left((1 + |\xi|^2) \partial_t^{n-1} \hat{h} \partial_t^n \hat{h}\right) - \sum_{\xi \in \mathbb{Z}^2} (1 + |\xi|^2) \partial_t^{n-1} \hat{h} \partial_t^{n+1} \hat{h} \\
= \sum_{\xi \in \mathbb{Z}^2} \left((1 + |\xi|^2) \partial_t^{n-1} \hat{h} \partial_t^n \hat{h}\right) (T) - \sum_{\xi \in \mathbb{Z}^2} \left((1 + |\xi|^2) \partial_t^{n-1} \hat{h} \partial_t^n \hat{h}\right) (0) \\
- \int_0^T \sum_{\xi \in \mathbb{Z}^2} (1 + |\xi|^2)^{5/2} \partial_t^{n-1} \hat{h} (1 + |\xi|^2)^{-1/2} \partial_t^{n+1} \hat{h} \\
\lesssim \sup_{[0,T]} \left( |\partial_t^{n-1} h|_{1}^2 + |\partial_t^n h|_{1}^2 \right) + \int_0^T \left( |\partial_t^{n-1} h|_{1/2}^2 + |\partial_t^{n+1} h|_{-1/2}^2 \right) \\
\lesssim \sup_{[0,T]} \mathcal{E}_n + \int_0^T \left( \mathcal{D}_n + (\mathcal{E}_n)^2 \right). \tag{5.24}
\]

Now we continue to proceed with the estimates in Proposition 5.1. We can use the estimates (5.8)–(5.12). But we can not use the estimate (5.13), and we need to employ a different argument. Indeed, by the trace theory and (5.11) with \(j = 2\), we obtain
\[
|\nabla^* q|_{2(n-2)}^2 \lesssim \|\nabla q\|_{2(n-1)}^2 \lesssim |\partial_t^2 h|_{2(n-2)+1}^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.25}
\]

Since \(\int_{\Sigma} q = 0\) and \(\int_{\Sigma} \|q\| = 0\) by the fourth and fifth equations of (1.3), by Poincaré’s inequality and (5.25), we have
\[
\|q\|_{2(n-1)}^2 \lesssim \|\nabla q\|_{2(n-1)}^2 \lesssim |\partial_t^2 h|_{2(n-2)+1}^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.26}
\]

Hence, we have removed \(|h|_{-1/2}^2\) from the estimate (5.14). We can still use the estimate (5.15), but we can not use the estimate (5.16). Indeed, by (5.26), the fourth and fifth equations of (2.25) and since \(\int_{\Sigma} h = 0\), we have
\[
|h|_{2(n+1)}^2 \lesssim |q|_{2(n-1)}^2 + |G^2|_{2(n-1)}^2 \lesssim \|q\|_{2(n-1)}^2 + (\mathcal{E}_n)^2.
\]
In light of (5.27), we can remove $|h|^2 \frac{\cdot}{3}$ from the estimates (5.17)–(5.20) and then improve the estimate (5.21) to be

$$E_n \lesssim |\partial_t h|^2 + \mathcal{X}_n + (\mathcal{E}_n)^2.$$  (5.28)

Consequently, combining (5.28) and (5.24), we deduce that

$$\int_0^T E_n \lesssim \int_0^T \left( |\partial_t h|^2 + \mathcal{X}_n + (\mathcal{E}_n)^2 \right) \lesssim \sup_{[0,T]} \mathcal{E}_n + \int_0^T (\mathcal{X}_n + (\mathcal{E}_n)^2).$$  (5.29)

This thus implies (5.23) as did in Proposition 5.1.

6. Global energy estimates. Now the proof of Theorem 1.2 follows, in a standard way, by the local well-posedness theory as illustrated after Remark 1.2 in the introduction section, a continuity argument and the following a priori estimates.

**Theorem 6.1.** Let $n \geq 3$ be an integer. There exists a universal constant $\delta > 0$ such that if

$$E_n(t) \leq \delta, \quad \forall t \in [0, T],$$  (6.1)

then

$$E_n(t) + \int_0^t E_n(\tau) \, d\tau \leq E_n(0), \quad \forall t \in [0, T].$$  (6.2)

Moreover, there exists a universal constant $\gamma > 0$ such that

$$E_n(t) \lesssim E_n(0) e^{-\gamma t}, \quad \forall t \in [0, T].$$  (6.3)

**Proof.** Combining Propositions 3.1 and 4.1, by (6.1), we have

$$\sup_{0 \leq \tau \leq T} \left( \mathcal{E}_n(\tau) + \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl } v(\tau) \right\|^2 \frac{\cdot}{2(n-j)-1} \right)$$

$$+ \int_0^T \left( \overline{D}_n(\tau) + \sum_{j=0}^{n-1} \left\| \partial_t^j \text{curl } v(\tau) \right\|^2 \frac{\cdot}{2(n-j)-1} \right) \, d\tau$$

$$\lesssim E_n(0) + \sup_{0 \leq \tau \leq T} (\mathcal{E}_n(\tau))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2} \, d\tau,$$  (6.4)

This together with Propositions 5.1 and 5.2 yields that

$$\sup_{0 \leq \tau \leq T} E_n(\tau) + \int_0^T E_n(\tau) \, d\tau \lesssim E_n(0) + \sup_{0 \leq \tau \leq T} (\mathcal{E}_n(\tau))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2} \, d\tau$$

$$\lesssim E_n(0) + \sqrt{\delta} \left( \sup_{0 \leq \tau \leq T} E_n(\tau) + \int_0^t E_n(\tau) \, d\tau \right).$$  (6.5)

This implies (6.2) since $\delta$ is small.

Now we prove the decay estimate (6.3). Following the proof of (6.2), we may deduce that

$$E_n(t) + \int_s^t E_n(\tau) \, d\tau \lesssim E_n(s), \quad \forall t \geq s \geq 0.$$  (6.6)

If we define

$$V(s) = \int_s^t E_n(\tau) \, d\tau,$$  (6.7)
then by (6.6), we have that there exists a universal constant $\gamma > 0$ such that
\[
2\gamma V(s) \leq \mathcal{E}_n(s) = -\frac{d}{ds}V(s).
\] (6.8)

Applying the Gronwall lemma to (6.8), we obtain
\[
V(s) \leq V(0)e^{-2\gamma s} \leq \frac{1}{2\gamma}\mathcal{E}_n(0)e^{-2\gamma s}.
\] (6.9)

Now integrating (6.6) in $s$ from $t/2$ to $t$ implies that, by (6.9),
\[
\frac{t}{2}\mathcal{E}_n(t) \lesssim \int_{t/2}^{t} \mathcal{E}_n(s) ds = V\left(\frac{t}{2}\right) \leq \frac{1}{2\gamma}\mathcal{E}_n(0)e^{-\gamma t}.
\] (6.10)

This together with (6.2) yields (6.3). \qed

**Appendix A. Analytic tools.**

**A.1. Poisson extensions.** We will now define the appropriate Poisson sums that allow us to extend $h_\pm$, defined on the surfaces $\Sigma_\pm$, to functions defined on $\Omega$, with “good” boundedness.

Suppose that $\Sigma_+ = \mathbb{T}^2 \times \{\ell\}$. We define the Poisson sum in $\mathbb{T}^2 \times (-\infty, \ell)$ by
\[
P_{-,-}f(x) = \sum_{\xi \in (2\pi\mathbb{Z}) \times (2\pi\mathbb{Z})} e^{i\xi \cdot x'}e^{i\xi(x_3-\ell)}\hat{f}(\xi),
\] (A.1)

where for $\xi \in (2\pi\mathbb{Z}) \times (2\pi\mathbb{Z})$ we have written
\[
\hat{f}(\xi) = \int_{\mathbb{T}^2} f(x')e^{-i\xi \cdot x'}dx'.
\] (A.2)

Here “$-\ell$” stands for extending downward and “$\ell$” stands for extending at $x_3 = \ell$, etc. We can then extend $h_+$ to be defined on $\Omega$ by
\[
P_+h_+(x',x_3) := P_{-,\ell}h_+(x',x_3), \text{ for } x_3 \leq \ell.
\] (A.3)

Similarly, for $\Sigma_- = \mathbb{T}^2 \times \{0\}$ we define the Poisson sum in $\mathbb{T}^2 \times (-\infty, 0)$ by
\[
P_{-,-}f(x) = \sum_{\xi \in (2\pi\mathbb{Z}) \times (2\pi\mathbb{Z})} e^{i\xi \cdot x'}e^{i\xi x_3}\hat{f}(\xi).
\] (A.4)

This allows us to extend $h_-$ to be defined on $\Omega_-$. However, we do not extend $h_-$ to the upper domain $\Omega_+$ by the reflection since this will result in the discontinuity of the partial derivatives in $x_3$ of the extension. We instead to do the extension through the following. Let $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_m < \infty$ for $m \in \mathbb{N}$ and define the $(m+1) \times (m+1)$ Vandermonde matrix $V(\lambda_0,\lambda_1,\ldots,\lambda_m)$ by $V(\lambda_0,\lambda_1,\ldots,\lambda_m)_{ij} = (-\lambda_j)^{i} \text{ for } i,j = 0,\ldots,m$. It is well-known that the Vandermonde matrices are invertible, so we are free to let $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_m)^T$ be the solution to
\[
V(\lambda_0,\lambda_1,\ldots,\lambda_m)\alpha = q_m, \quad q_m = (1,1,\ldots,1)^T.
\] (A.5)

Now we define the specialized Poisson sum in $\mathbb{T}^2 \times (0, \infty)$ by
\[
P_{+,0}f(x) = \sum_{\xi \in (2\pi\mathbb{Z}) \times (2\pi\mathbb{Z})} e^{i\xi \cdot x'}\sum_{j=0}^{m} \alpha_j e^{-|\xi|\lambda_j x_3}\hat{f}(\xi).
\] (A.6)

It is easy to check that, due to (A.5), $\partial_0^\alpha P_{+,0}f(x',0) = \partial_0^\alpha P_{-,0}f(x',0)$ for all $0 \leq l \leq m$ and hence
\[
\partial_0^\alpha P_{+,0}f(x',0) = \partial_0^\alpha P_{-,0}f(x',0), \forall \alpha \in \mathbb{N}^3 \text{ with } 0 \leq |\alpha| \leq m.
\] (A.7)
These facts allow us to extend $h_-$ to be defined on $\Omega$ by

$$\mathcal{P}_- h_-(x', x_3) := \begin{cases} \mathcal{P}_+ h_-(x', x_3), & x_3 > 0 \\ \mathcal{P}_- h_-(x', x_3), & x_3 \leq 0. \end{cases} \quad (A.8)$$

We may assume that $m$ is sufficiently large for our use.

We have the following boundedness of $\mathcal{P} = \mathcal{P}_\pm$.

**Lemma A.1.** It holds that

$$\|\mathcal{P} f\|_s \lesssim |f|_{s - \frac{1}{2}}, \quad s \in \mathbb{R}. \quad (A.9)$$

**Proof.** It follows by the definition of $\mathcal{P}$. \hfill $\square$

A.2. **Estimates in Sobolev spaces.** We will need some estimates of the product of functions in Sobolev spaces.

**Lemma A.2.** Let the domain be either $\Omega$ or $\Sigma$, and $d$ be the dimension.

1. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_1 > d/2$. Then

$$\|f g\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \quad (A.10)$$

2. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + d/2$. Then

$$\|f g\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \quad (A.11)$$

**Proof.** These results are standard and may be derived, for example, by use of the Fourier characterization of the $H^s$ spaces and the extension if the domain is $\Omega$. \hfill $\square$

We also need the following $|\cdot|_{-1/2}$ product estimates.

**Lemma A.3.** Let $m > 2$. Then

$$|fg|_{-\frac{1}{2}} \lesssim |f|_{-\frac{1}{2}} |g|_m. \quad (A.12)$$

**Proof.** It follows by the duality and (A.11) with $r = s_1 = 1/2, \; d = 2$ and $s_2 = m > 2$. \hfill $\square$

A.3. **Trace estimates.** We record the following $H^{-1/2}$ boundary estimates for functions satisfying $v \in L^2$ and $\nabla \varphi \cdot v \in L^2$.

**Lemma A.4.** Assume that $\|\nabla \varphi\|_{L^\infty} \leq C$, then

$$|v \cdot N|_{-\frac{1}{2}} \lesssim \|v\|_0 + \|\nabla \varphi \cdot v\|_0. \quad (A.13)$$

**Proof.** It follows by an easy dual argument. \hfill $\square$

It is crucial to use the following trace estimates:

**Lemma A.5.** It holds that

$$|v|_{-1/2} \lesssim \|v\|_0 + \|\text{div} v\|_{(H^1(\Omega))'} + \|\text{curl} v\|_{(H^1(\Omega))'} \quad (A.14)$$

**Proof.** It follows from combining the normal and tangential trace estimates, see the statement after Theorem 3.1 in [7]. \hfill $\square$

A.4. **Elliptic estimates.** Our derivation of high order energy estimates for the velocity $v$ is based on the following Hodge-type elliptic estimates.

**Lemma A.6.** Let $r \geq 1$, then it holds that for $v \cdot v = 0$ on $\Sigma_0$,

$$\|v\|_r \lesssim \|v\|_0 + |\text{curl} v|_{r - 1} + \|\text{div} v\|_{r - 1} + |v_3|_{r - 1/2}. \quad (A.15)$$

**Proof.** The estimate is well-known and follows from the identity $-\Delta v = \text{curl} \text{curl} v - \nabla \text{div} v$. \hfill $\square$
Acknowledgments. The author is deeply grateful to the referees for the valuable comments and suggestions.

REFERENCES

[1] D. M. Ambrose, Well-posedness of vortex sheets with surface tension, SIAM J. Math. Anal., 35 (2003), 211–244.
[2] D. M. Ambrose and N. Masmoudi, Well-posedness of 3D vortex sheets with surface tension, Commun. Math. Sci., 5 (2007), 391–430.
[3] V. Barcilon, P. Constantin and E. S. Titi, Existence of solutions to the Stommel-Charney model of the gulf stream, SIAM J. Math. Anal., 19 (1988), 1355–1364.
[4] J. T. Beale, Large-time regularity of viscous surface waves, Arch. Ration. Mech. Anal., 84 (1983/84), 307–252.
[5] R. E. Caflisch and O. F. Orellana, Singular solutions and ill-posedness for the evolution of vortex sheets, SIAM J. Math. Anal., 20 (1989), 293–307.
[6] J. G. Charney, The Gulf Stream as an inertial boundary layer, Proc. Nat. Acad. Sci. USA, 41 (1955), 731–740.
[7] C.-H. A. Cheng, D. Coutand and S. Shkoller, On the motion of vortex sheets with surface tension in three-dimensional Euler equations with vorticity, Comm. Pure Appl. Math., 61 (2008), 1715–1752.
[8] C.-H. A. Cheng, D. Coutand and S. Shkoller, On the limit as the density ratio tends to zero for two perfect incompressible fluids separated by a surface of discontinuity, Comm. Partial Differential Equations, 35 (2010), 817–845.
[9] V. Chepyzhov and S. Zelik, Infinite energy solutions for dissipative Euler equations in $\mathbb{R}^2$, J. Math. Fluid Mech., 17 (2015), 513–532.
[10] P. Constantin and F. Ramos, Inviscid limit for damped and driven incompressible Navier-Stokes equations in $\mathbb{R}^2$, Comm. Math. Phys., 275 (2007), 529–551.
[11] J.-F. Coulombel and P. Secchi, The stability of compressible vortex sheets in two space dimensions, Indiana Univ. Math. J., 53 (2004), 941–1012.
[12] J.-F. Coulombel and P. Secchi, Nonlinear compressible vortex sheets in two space dimensions, Ann. Sci. Éc. Norm. Supér. (4), 41 (2008), 85–139.
[13] D. Coutand and S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, J. Amer. Math. Soc., 20 (2007), 829–930.
[14] Y. Guo and I. Tice, Decay of viscous surface waves without surface tension in horizontally infinite domains, Anal. PDE, 6 (2013), 1429–1533.
[15] A. A. Ilyin, The Euler equations with dissipation, Mat. Sb., 182 (1991), 1729–1739.
[16] J. H. Jang, I. Tice and Y. J. Wang, The compressible viscous surface-internal wave problem: Local well-posedness, SIAM J. Math. Anal., 48 (2016), 2602–2673.
[17] J. H. Jang, I. Tice and Y. J. Wang, The compressible viscous surface-internal wave problem: Stability and vanishing surface tension limit, Comm. Math. Phys., 343 (2016), 1039–1113.
[18] R. H. Pan and K. Zhao, The 3D compressible Euler equations with damping in a bounded domain, J. Differential Equations, 246 (2009), 581–596.
[19] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1979.
[28] F. Pusateri, On the limit as the surface tension and density ratio tend to zero for the two-phase Euler equations, *J. Hyperbolic Differ. Equ.*, 8 (2011), 347–373.

[29] J.-C. Saut, Remarks on the damped stationary Euler equations, *Differ. Integral Equ.*, 3 (1990), 801–812.

[30] J. Shatah and C. C. Zeng, A priori estimates for fluid interface problems, *Comm. Pure Appl. Math.*, 61 (2008), 848–876.

[31] J. Shatah and C. C. Zeng, Local well-posedness for fluid interface problems, *Arch. Ration. Mech. Anal.*, 199 (2011), 653–705.

[32] T. C. Sideris, B. Thomases and D. H. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, *Comm. Partial Differential Equations*, 28 (2003), 795–816.

[33] B. Stevens, Short-time structural stability of compressible vortex sheets with surface tension, *Arch. Ration. Mech. Anal.*, 222 (2016), 603–730.

[34] H. Stommel, The westward intensification of wind-driven ocean currents, *Trans. Amer. Geophys. Union*, 29 (1948), 202–206.

[35] Y. J. Wang and I. Tice, The viscous surface-internal wave problem: Nonlinear Rayleigh-Taylor instability, *Comm. Partial Differential Equations*, 37 (2012), 1967–2028.

[36] Y. J. Wang, I. Tice and C. Kim, The viscous surface-internal wave problem: Global well-posedness and decay, *Arch. Rational Mech. Anal.*, 212 (2014), 1–92.

Received December 2018; 1st revision July 2019, 2nd revision September 2019.

E-mail address: jiali.lian@163.com