A construction of finite-dimensional faithful representation of Lie algebra

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The Ado theorem is a fundamental fact, which has a reputation to be a 'strange theorem'. We give its natural proof.

1. Construction of faithful representation. Consider a finite-dimensional Lie algebra \( g \). Assume that \( g \) is a semidirect product \( p \ltimes n \) of a subalgebra \( p \) and a nilpotent ideal \( n \). Assume that the adjoint action of \( p \) on \( n \) is faithful, i.e., for any \( z \in p \), there exists \( x \in n \) such that \([z, x] \neq 0\).

Consider the minimal \( k \) such that all the commutators
\[
\ldots [[x_1, x_2], x_3], \ldots, x_k], \quad x_j \in n
\]
are 0.

Denote by \( U(n) \) the enveloping algebra of \( n \). The algebra \( n \) acts on \( U(n) \) by the left multiplications. The algebra \( p \) acts on \( U(n) \) by the derivations
\[
d_z x_1 x_2 x_3 \ldots x_l = [z, x_1] x_2 x_3 \ldots x_l + x_1 [z, x_2] x_3 \ldots x_l + \ldots, \quad \text{where} \ z \in p.
\]
This defines the action of the semidirect product \( p \ltimes n = g \) on \( U(n) \).

Denote by \( I \) the subspace in \( U(n) \) spanned by all the products \( x_1 x_2 \ldots x_N \), where \( N > k + 2 \). Obviously,
1. \( I \) is the two-side ideal in \( U(n) \).
2. Consider the linear span \( A \) of all the elements having the form 1, \( x, x_1 x_2 \in U(n) \). Obviously, \( I \cap A = 0 \).
3. \( I \) is invariant with respect to the derivations \( d_z \).

Obviously, the module \( U(n)/I \) is a finite-dimensional faithful module over \( g \).

2. The Ado theorem.

Lemma 1. Any finite-dimensional Lie algebra \( q \) admits an embedding to an algebra \( g \) such that
a) \( g \) is a semidirect product of a reductive subalgebra \( p \) and a nilpotent ideal \( n \);
b) the action of \( p \) on \( n \) is completely reducible.

Obviously, Lemma 1 implies the Ado theorem. Indeed, \( g \) admits a decomposition
\[
g = p' \oplus (p'' \ltimes n)
\]
where \( p', p'' \) are reductive subalgebras and the action of \( p'' \) on \( n \) is faithful. After this, it is sufficient to apply the construction of p.1.

Remark. The Ado theorem implies Lemma 1 modulo the Chevalley construction of algebraic envelope of a Lie algebra. But Lemma 1 itself can be easily proved directly.

3. Killing lemma. Let \( g \) be a Lie algebra, let \( d \) be its derivation. For an eigenvalue \( \lambda \), denote by \( g_\lambda \) its root subspace \( g_\lambda = \oplus \ker(d - \lambda)^k \); we have \( g = \oplus g_\lambda \). As it was observed by Killing, \( x \in g_\lambda, y \in g_\mu \) implies \([x, y] \in g_{\lambda + \mu}\).

Thus the Lie algebra \( g \) admits the gradation by the eigenvalues of \( d \). Consider the gradation operator \( d_\lambda : g \rightarrow g \) defined by \( d_\lambda v = \lambda v \) if \( v \in g_\lambda \). Obviously, \( d_\lambda \) is a derivation, and \( dd_\lambda = d_\lambda d \). We also consider the derivation \( d_n := d - d_n \), this operator is nilpotent (the equality \( d = d_n + d_n \) is called the Jordan–Chevalley decomposition). Clearly,
\[
\ker d_n \subset \ker d; \quad \ker d_n \subset \ker d; \quad \text{(1)}
\]
\[
im d_n \subset \im d; \quad \im d_n \subset \im d \quad \text{(2)}
\]

4. Elementary expansions. Let \( q \) be a Lie algebra, let \( I \) be an ideal of codimension 1. Let \( x \notin I \). Denote by \( d \) the operator \( \text{Ad}_x : I \rightarrow I \). Consider the corresponding pair of derivations \( d_n, d_n \). Consider the space
\[
q' = \mathbb{C} y + \mathbb{C} z + I
\]
where $y$, $z$ are formal vectors. We equip this space with a structure of a Lie algebra by the rule

$$[y, z] = 0, \quad [y, u] = dsu, \quad [z, u] = dnu,$$

for all $u \in I$

and the commutator of $u, v \in I$ is the same as it was in $I$.

The subalgebra $\mathbb{C}(y + z) \oplus I \subset q'$ is isomorphic $q$. We say that $q'$ is an elementary expansion of $q$.

Obviously, $[q', q'] = [q, q]$.

For a general Lie algebra, the required embedding to a semidirect product can be obtained by a sequence of elementary expansions.

5. Proof of Lemma 1. Let $q$ be a Lie algebra. Let $\mathfrak{h}$ be its Levi part, and $\mathfrak{r}$ be the radical. Denote by $\mathfrak{m}$ the nilradical of $q$, i.e., $\mathfrak{m} = [q, r]$; recall that $\mathfrak{m}$ is a nilpotent ideal, and $[q, q] = \mathfrak{h} \ltimes \mathfrak{m}$ (see [1], 1.4.9).

Consider a nilpotent ideal $\mathfrak{n}$ of $q$ containing the nilradical $\mathfrak{m}$. Consider a subalgebra $\mathfrak{p} \supset \mathfrak{h}$ such that the adjoint action of $\mathfrak{p}$ on $q$ is completely reducible and $\mathfrak{p} \cap \mathfrak{n} = 0$; for instance, the can choice $\mathfrak{n} = \mathfrak{m}$, $\mathfrak{p} = \mathfrak{h}$.

Obviously, the $q$-module $q/(\mathfrak{p} \ltimes \mathfrak{n})$ is trivial. Consider any subspace $I$ of codimension 1 containing $\mathfrak{p} \ltimes \mathfrak{n}$, obviously $I$ is an ideal in $q$. Since the action of $\mathfrak{p}$ on $q$ is completely reducible, there exists a $\mathfrak{p}$-invariant complementary subspace for $I$. Let $x$ be an element of this subspace. Since the $\mathfrak{p}$-module $q/I$ is trivial, $x$ commutes with $\mathfrak{p}$. We apply the elementary expansion to these data.

We obtain the new algebra $q' = \mathbb{C}y + \mathbb{C}z + I$ with the nilpotent ideal $\mathfrak{n}' = \mathbb{C}z + \mathfrak{n}$ and with the reductive subagebra $\mathfrak{p}' = \mathbb{C}y \oplus \mathfrak{p}$ (by (1), $y$ commutes with $\mathfrak{p}$).

It remains to notice that

$$\dim q' - \dim \mathfrak{p}' - \dim \mathfrak{n}' = \dim q - \dim \mathfrak{p} - \dim \mathfrak{n} - 1$$

and we can repeat the same construction.

References

[1] J.Dixmier, *Enveloping algebras*, North.Holland Publ.Co, 1977