A continuous-state nonlinear branching process

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Abstract. A continuous-state nonlinear branching process is constructed as the pathwise unique solution of a stochastic integral equation. The extinction and explosion probabilities and the mean extinction and explosion times are computed explicitly, which are also new in the linear branching case. We present necessary and sufficient conditions for the process to extinguish or explode in finite times. In the critical or subcritical case, we give a construction of the process coming down from infinity. Finally, it is shown that the continuous-state nonlinear branching process arises naturally as the rescaled limit of a sequence of discrete-state processes.

Key words and phrases. Branching process, continuous-state, nonlinear branching, stochastic integral equation, Lamperti transformation, extinction, explosion.

1 Introduction

Branching processes are models for the evolution of populations of particles. Those processes constitute an important subclass of Markov processes. Standard references on those processes with discrete-state space \( \mathbb{N} := \{0, 1, 2, \ldots\} \) are Harris (1963) and Athreya and Ney (1972). As the quantity of particles can sometimes be expressed by other means than by counting, it is reasonable to consider branching stochastic processes with continuous-states; see, e.g., Feller (1951) and Lamperti (1967a, 1967b). In particular, Lamperti (1967a) characterised a class of continuous-state branching processes as the weak limits of sequences of rescaled discrete-state branching processes, and Lamperti (1967b) constructed those processes as random time changed Lévy processes. Continuous-state branching processes can also be constructed in terms of stochastic equations; see Dawson and Li (2006, 2012) and Fu and Li (2010). The transition function \((P_t)_{t \geq 0}\) of a continuous-time branching process satisfies:

\[
P_t(x, \cdot) \ast P_t(y, \cdot) = P_t(x + y, \cdot), \quad x, y \in E,
\]

where \(\ast\) denotes the convolution operation and \(E = \mathbb{N}\) or \([0, \infty)\) is the state space. This is the so-called branching property, which means that different particles act independently of each other. In most realistic situations, however, this property is unlikely to be appropriate. In particular, when the number of particles becomes large or the particles move
with high speed, the particles may interact and, as a result, the birth and death rates can either increase or decrease. Those considerations have motivated the study of nonlinear branching processes.

Let \( \alpha \) and \( b_i, i = 0, 1, \ldots \) be positive constants satisfying \( b_1 = 0 \) and \( \sum_{i=0}^{\infty} b_i \leq 1 \). A **discrete-state nonlinear branching process** is a Markov chain on \( \mathbb{N} \) with \( Q \)-matrix \( (q_{ij}) \) defined by

\[
q_{ij} = \begin{cases} 
\alpha^i \theta b_{j-i+1}, & j \geq i + 1, i \geq 1, \\
-\alpha^i \theta, & j = i \geq 1, \\
\alpha^{i-1} b_0, & j = i - 1, i \geq 1, \\
0, & \text{otherwise.}
\end{cases}
\]  

Observe that \( q_{ij} = i^\theta \rho_{ij} \), where \( (\rho_{ij}) \) is the \( Q \)-matrix of a random walk on the space of integers with jumps larger than \( -1 \). The transition rate of the discrete-state nonlinear branching process is given by the power function \( i^\theta \) and its transition distribution is given by the sequence \( \{b_i : i \geq 0\} \). The process is essentially a particular form of the model introduced by Chen (1997) and it has attracted the interest of many authors; see, e.g. Chen (2002), Chen et al. (2008) and Pakes (2007). When \( \theta = 1 \), the model reduces to a discrete-state linear branching process, which satisfies property [1.1]. We refer to Chen (2004) for the general theory of continuous-time Markov chains.

The purpose of this paper is to introduce and study a continuous-state version of the above model. Let \( C^2_0 [0, \infty) \) be the space of twice continuously differentiable functions on \( [0, \infty) \) which together with their derivatives up to the second order vanish at \( \infty \). By convention, we extend each \( f \) on \( [0, \infty) \) to \( [0, \infty] \) by setting \( f(\infty) = 0 \). Fix a constant \( \theta > 0 \) and let

\[
\mathcal{D}(L) = \left\{ f \in C^2_0 [0, \infty) : \lim_{x \to \infty} x^\theta |f^{(n)}(x)| = 0, n = 0, 1, 2 \right\}.
\]

Let \( a \geq 0, b \in \mathbb{R} \) and \( c \geq 0 \) be constants and \( m(du) \) a \( \sigma \)-finite measure on \( (0, \infty] \) satisfying

\[
\int_{(0,\infty]} (1 \wedge u^2) m(du) < \infty.
\]

For \( x \in [0, \infty) \) and \( f \in \mathcal{D}(L) \) we define

\[
Lf(x) = x^\theta \left[ -af(x) - bf'(x) + cf''(x) + \int_{(0,\infty]} D_z f(x) m(dz) \right],
\]  

where

\[
D_z f(x) = f(x+z) - f(x) - zf'(x) 1_{\{z \leq 1\}}.
\]

It is simple to see that \( Lf(\infty) := \lim_{x \to \infty} Lf(x) = 0 \).

A stochastically continuous Markov process \( (X_t : t \geq 0) \) with state space \( [0, \infty] \) is called a **continuous-state nonlinear branching process** if it has traps \( 0 \) and \( \infty \) and its transition semigroup \( (P_t)_{t \geq 0} \) satisfies the Kolmogorov forward equation

\[
\frac{dP_t f}{dt}(x) = P_t Lf(x), \quad x \in [0, \infty], f \in \mathcal{D}(L).
\]
We call $\theta > 0$ the rate power of the continuous-state nonlinear branching process. The ordinary continuous-state branching process corresponds to the special case $\theta = 1$, which we refer to as the linear branching case; see, e.g., Lamperti (1967a, 1967b). This is the only situation where the branching property (1.1) is satisfied. We say the process has superlinear branching if $\theta > 1$ and sublinear branching if $\theta < 1$. Let $\psi$ be the function on $[0, \infty)$ defined by
\[
\psi(\lambda) = -a + b\lambda + c\lambda^2 + \int_{(0,\infty)} (e^{-\lambda z} - 1 + \lambda z 1_{\{z \leq 1\}})m(dz), \quad \lambda \geq 0.
\] (1.5)

We call $\psi$ the reproduction mechanism of the process. By (1.5) we see
\[
\beta := \psi'(0) = b - \int_{(1,\infty)} zm(dz).
\] (1.6)

In this paper, we always assume that there exists a $\lambda \in (0, \infty)$ such that $\psi(\lambda) > 0$ (i.e., $\psi$ is not the Laplace exponent of a subordinator).

We now present a construction of the continuous-state nonlinear branching process in terms of a stochastic equation with jumps. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Let $(B_t : t \geq 0)$ be an $(\mathcal{F}_t)$-Brownian motion. Let $M(ds, dz, du)$ be an $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty) \times (0, \infty) \times (0, \infty)$ with intensity $dsm(dz)du$ and $\tilde{M}(ds, dz, du)$ the compensated measure. Let $X_0$ be a positive $\mathcal{F}_0$-measurable random variable. We consider positive solutions of the stochastic integral equation
\[
X_t = X_0 + \sqrt{2c} \int_0^t X_s^{\theta/2} dB_s + \int_0^t \int_{(0,1]} \int_0^X z \tilde{M}(ds, dz, du)
- b \int_0^t X_s^\theta ds + \int_0^t \int_{(1,\infty)} \int_0^X z \tilde{M}(ds, dz, du).
\] (1.7)

Here and in the sequel, we understand $\int_a^b = \int_{[a,b]}$ and $\int_a^\infty = \int_{(a,\infty)}$, for $0 \leq a \leq b < \infty$.

By saying $X = (X_t : t \geq 0)$ is a solution of the (1.7) we mean it is a càdlàg $[0, \infty]$-valued $(\mathcal{F}_t)$-adapted process satisfying (1.7) up to time $\zeta_n := \inf\{t \geq 0 : X_t \geq n \text{ or } X_t \leq 1/n\}$ for each $n \geq 1$ and $X_t = \lim_{n \to \infty} X_{\zeta_n-}$ for $t \geq \tau := \lim_{n \to \infty} \zeta_n$.

**Theorem 1.1** (1) For any initial value $X_0 = x \in [0, \infty]$ there exists a pathwise unique solution to (1.7). (2) Let $X^y := (X^y_t : t \geq 0)$ be the solution to (1.7) with $X_0 = x$. Then $y \geq x \in [0, \infty)$ implies $\mathbb{P}(X^y_t \geq X^x_t \text{ for every } t \geq 0) = 1$.

**Theorem 1.2** The solution to (1.7) is a continuous-state nonlinear branching process defined by (1.7).

Let $D$ be the space of càdlàg functions $w : [0, \infty) \to [0, \infty]$ with $0$ and $\infty$ as traps. Let $\rho(x, y) = |e^{-x} - e^{-y}|$ for all $x, y \in [0, \infty)$. We extend $\rho$ to a metric on $[0, \infty]$ making
it homeomorphic to $[0, 1]$. It is easy to see that $\rho(x, y) \leq 1 \wedge |x - y|$ for all $x, y \in [0, \infty)$. Then we define the uniform distance $\rho_D$ on $D$ by
\[
\rho_D(v, w) = \sup_{s \in [0, \infty)} \rho(v(s), w(s)), \quad v, w \in D.
\]
Let $\Lambda$ be the set of increasing homeomorphisms of $[0, \infty)$ into itself and define the metric $d_\infty$ on $D$ by
\[
d_\infty(v, w) := \inf_{\lambda \in \Lambda} \rho_D(v, w \circ \lambda) \vee \|\lambda - I\|, \quad v, w \in D,
\]
where $I$ is the identity and $\|\cdot\|$ is the uniform norm. For $x \in [0, \infty)$ let $P^x$ denote the distribution on $D$ of the process $X^x = (X^x_t : t \geq 0)$ defined by (1.7) with initial value $X_0 = x$.

**Theorem 1.3** The mapping $[0, \infty) \ni x \mapsto P^x$ is continuous by weak convergence.

For any $y \in [0, \infty]$ let $\tau_y = \inf\{t \geq 0 : X^x_t = y\}$. We call $\tau_0$ the extinction time and $\tau_\infty$ the explosion time of $X$. Then $\tau_0 \wedge \tau_\infty = \tau$, which is referred to as the absorbing time. Let $P_x = P(\cdot | X_0 = x)$ be the conditional law given $X_0 = x \in [0, \infty]$. Since $X$ has no negative jump we have $P_x(X^x_{\tau_y} = y) = 1$ for $x \geq y \in [0, \infty)$. Let $q = \inf\{\lambda > 0 : \psi(\lambda) > 0\}$.

**Theorem 1.4** (1) For any $y \leq x \in (0, \infty)$ we have
\[
P_x(\tau_y < \infty) = e^{-q(x-y)}. \quad (1.8)
\]
(2) For any $x \in [0, \infty)$ we have
\[
P_x(X_\infty = 0) = e^{-qx}, \quad P_x(X_\infty = \infty) = 1 - e^{-qx}. \quad (1.9)
\]
From the transition semigroup $(P_t)_{t \geq 0}$ of the continuous-state nonlinear branching process $X$ we define its resolvent $(U^\lambda)_{\lambda > 0}$ by
\[
U^\lambda(x, dy) = \int_0^\infty e^{-\lambda t} P_t(x, dy) dt, \quad x, y \in [0, \infty]. \quad (1.10)
\]
The next theorem gives a characterization of the resolvent and plays the key role in the study of the hitting times of $X$.

**Theorem 1.5** For any $\eta > 0$, $\lambda \geq 0$ and $x \in [0, \infty)$ we have
\[
\eta U^\eta e_\lambda(x) - e^{-\lambda x} = \psi(\lambda) \int_{[0, \infty)} y^\theta e^{-\lambda y} U^\eta(x, dy) \quad (1.11)
\]
and
\[
\int_\lambda^\infty l_x(\eta, z)(z - \lambda)^{\theta - 1} dz = \Gamma(\theta) \int_0^\infty e^{-\eta t} dt \int_{(0, \infty)} e^{-\lambda y} P_t(x, dy), \quad (1.12)
\]
where
\[
l_x(\eta, z) = \psi(z)^{-1} [\eta U^\eta e_x(x) - e^{-\lambda x}].
\]
Let $\mathbb{E}_x$ be the expectation with respect to $\mathbb{P}_x$. The following two theorems and their corollaries give explicit expressions of some mean hitting times of $X$.

**Theorem 1.6** For any $x \in (0, \infty)$ we have the moment formulas:

$$
\mathbb{E}_x(\tau_0 : X_\infty = 0) = \frac{1}{\Gamma(\theta)} \int_0^\infty h_x(\lambda + q)\lambda^{\theta-1}d\lambda, \quad (1.13)
$$

$$
\mathbb{E}_x(\tau_\infty : X_\infty = \infty) = \frac{1}{\Gamma(\theta)} \int_0^\infty [h_x(\lambda) - h_x(\lambda + q)]\lambda^{\theta-1}d\lambda. \quad (1.14)
$$

and

$$
\mathbb{E}_x(\tau) = \mathbb{E}_x(\tau_\infty \wedge \tau_0) = \frac{1}{\Gamma(\theta)} \int_0^\infty h_x(\lambda)\lambda^{\theta-1}d\lambda, \quad (1.15)
$$

where $\Gamma$ denotes the Gamma function and

$$
h_x(\lambda) = \frac{e^{-qx} - e^{-\lambda x}}{\psi(\lambda)}. \quad (1.16)
$$

**Theorem 1.7** For any $y \leq x \in (0, \infty)$ we have

$$
\mathbb{E}_x(\tau_\infty \wedge \tau_y) = \frac{e^{-qx}}{\Gamma(\theta)} \int_0^\infty \frac{e^{-(\lambda-q)y} - e^{-(\lambda-q)x}}{\psi(\lambda)}\lambda^{\theta-1}d\lambda. \quad (1.17)
$$

**Corollary 1.8** Suppose that $a = \psi(0) = 0$ and $\psi'(0) \geq 0$. Then for $y \leq x \in (0, \infty)$ we have

$$
\mathbb{E}_x(\tau_y) = \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{e^{-\lambda y} - e^{-\lambda x}}{\psi(\lambda)}\lambda^{\theta-1}d\lambda. \quad (1.18)
$$

The discrete-state versions of (1.13) and (1.14) were proved in Chen (2002) and Pakes (2007), respectively. As far as we know, the discrete-state form of (1.16) has not been established in the literature. One may compare (1.17) with Corollary 9 in Duhalde et al. (2014). It seems other moment formulas are new also for continuous-state linear branching processes.

The next two theorems and their corollaries are about the extinction and explosion probabilities of the process. They generalize the results in Grey (1974) and Kawazu and Watanabe (1971), where the linear branching case $\theta = 1$ was studied.

**Theorem 1.9** Let $\varepsilon > 0$ be a constant so that $\psi(\lambda) > 0$ for $\lambda \geq \varepsilon$. For any $x \in (0, \infty)$, we have $\mathbb{P}_x(\tau_0 < \infty) > 0$ if and only if

$$
\int_\varepsilon^\infty \frac{\lambda^{\theta-1}}{\psi(\lambda)}d\lambda < \infty. \quad (1.19)
$$

In this case, we have $\mathbb{P}_x(\tau_0 < \infty) = \mathbb{P}_x(X_\infty = 0) = e^{-qx}$.
Corollary 1.10 (1) If \( \theta \geq 2 \), for any \( x \in (0, \infty) \) we have \( P_x(\tau_0 < \infty) = 0 \). (2) If \( 0 < \theta < 2 \) and \( c > 0 \), for any \( x \in (0, \infty) \) we have \( P_x(\tau_0 < \infty) > 0 \).

Theorem 1.11 Let \( \varepsilon > 0 \) be a constant so that \( \psi(\lambda) \neq 0 \) for \( 0 < \lambda \leq \varepsilon \). For any \( x \in (0, \infty) \) we have \( P_x(\tau_\infty < \infty) = 0 \) if and only if \( \psi(0) = 0 \) and one of the following two conditions is satisfied: (i) \( \psi'(0) \geq 0 \); (ii) \( \psi'(0) < 0 \) and

\[
\int_0^\varepsilon \frac{\lambda^{\theta-1}}{-\psi(\lambda)} \, d\lambda = \infty. \tag{1.19}
\]

In the case \( P_x(\tau_\infty < \infty) > 0 \), we have \( P_x(\tau_\infty < \infty) = P_x(X_\infty = \infty) = 1 - e^{-qx} \).

Corollary 1.12 In the case \( \theta > 1 \), for any \( x \in (0, \infty) \) we have \( P_x(\tau_\infty < \infty) = 0 \) if and only if \( \psi(0) = 0 \) and \( \psi'(0) \geq 0 \).

Corollary 1.13 In the case \( 0 < \theta \leq 1 \), for any \( x \in (0, \infty) \) we have \( P_x(\tau_\infty < \infty) = 0 \) if and only if \( \psi(0) = 0 \) and one of the following two conditions is satisfied: (i) \( \psi'(0) > -\infty \); (ii) \( \psi'(0) = -\infty \) and \( (1.19) \) holds.

Let \( X^x = (X^x_t : t \geq 0) \) be defined as in Theorem 1.11 and let \( \tau^x_y = \inf\{t \geq 0 : X^x_t = y\} \) for \( x \geq y \in [0, \infty) \). By Theorem 1.11 the mapping \( x \mapsto \tau^x_y \) is increasing in \( x \in [y, \infty) \), thus the limit \( \tau^\infty_y := \lim_{x \to \infty} \tau^x_y \) exists. It is easy to see that the mapping \( y \mapsto \tau^\infty_y \) is decreasing in \( y \in [0, \infty) \). By Corollary 1.12 for any \( y \in (0, \infty) \) we have

\[
E(\tau^\infty_y) = \lim_{x \to \infty} E(\tau^x_y) = \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{e^{-\lambda y}}{\psi(\lambda)} \lambda^{\theta-1} \, d\lambda. \tag{1.20}
\]

Theorem 1.14 Suppose that \( a = \psi(0) = 0 \) and \( \psi'(0) \geq 0 \). Then the following four statements are equivalent:

(i) \( P(\tau^\infty_y < \infty) > 0 \) for each \( y \in (0, \infty) \);

(ii) \( P(\tau^\infty_y < \infty) = 1 \) for each \( y \in (0, \infty) \);

(iii) \( E(\tau^\infty_y) < \infty \) for each \( y \in (0, \infty) \);

(iv) for each \( \varepsilon \in (0, \infty) \) we have

\[
\int_0^\varepsilon \frac{\lambda^{\theta-1}}{-\psi(\lambda)} \, d\lambda < \infty. \tag{1.21}
\]

Under the integrability condition \( (1.21) \), by \( (1.20) \) and dominated convergence we have \( \lim_{y \to \infty} E(\tau^\infty_y) = 0 \). Then we have a.s. \( \lim_{y \to \infty} \tau^\infty_y = \infty \). For each \( n \geq 1 \) let \( X^{(n)} = (X^{(n)}_t : t \geq 0) \) be the unique solution to

\[
X_t = n + \sqrt{2c} \int_0^t X_s^{\theta/2} \, dB_{\tau^\infty + s} + \int_0^t \int_{[0,1]} \int_0^{X_{s-}^n} z \bar{M}(\tau^\infty + ds, dz, du)
\]
\[-b \int_0^t X^\theta_s ds + \int_0^t \int_{[1, \infty]} \int_0^{X^\theta_s} zM(\tau_n^\infty + ds, dz, du).\]

Then we define $X^\infty = (X^\infty_t : t \geq 0)$ by

$$X^\infty_t = \begin{cases} \infty, & t = 0 \\ X^{(n)}_{t-\tau_n^\infty}, & \tau_n^\infty \leq t < \tau_{n-1}^\infty, n \geq 1 \\ 0, & t \geq \tau_0^\infty \end{cases}$$

**Theorem 1.15** Suppose that $a = \psi(0) = 0$, $\psi(0) \geq 0$ and (1.21) holds. Then for each $x \geq 0$ we have $P(X^\infty_t \geq X^x_t$ for every $t \geq 0) = 1$. Moreover, the process $(X^x_t : t \geq 0)$ converges to $(X^\infty_t : t \geq 0)$ a.s. in $(D, d_\infty)$ as $x \to \infty$.

The above theorem shows that $X^\infty$ is formally a solution of (1.7) coming down from $\infty$. This property is not possessed by classical linear branching processes. For coalescent processes and branching models with interaction, however, similar phenomena have been observed and studied by a number of authors; see, for example, Berestycki et al. (2010, 2014), Lambert (2005) and Pardoux (2016) and the references therein.

The following theorem shows that the continuous-state nonlinear branching process $X$ can be obtained as the limit of a sequence of rescaled discrete-state branching processes.

**Theorem 1.16** There exists a sequence of discrete-state nonlinear branching processes $\xi_n = (\xi_n(t) : t \geq 0)$ and a sequence of positive number $\gamma_n$, $n = 1, 2, \ldots$ such that $(n^{-1} \xi_n(\gamma_n t) : t \geq 0)$ converges to $(X_t : t \geq 0)$ weakly in $(D, d_\infty)$.

**Example 1.1** A special continuous-state nonlinear branching process reproduction mechanism $\psi(\lambda) = c\lambda^2$ ($c > 0$) is defined by

$$X_t = X_0 + \sqrt{2c} \int_0^t X^\theta_s/2 dB_s.$$  

In this case, we have a.s. $\tau_\infty = \infty$ and the formulas given above take simple forms. For example, from (1.16) we have

$$E(\tau_\infty^y) = \frac{1}{c\Gamma(\theta)} \int_0^\infty e^{-\lambda y} \lambda^{\theta-3} d\lambda = \frac{\Gamma(\theta-2)}{c\Gamma(\theta)} y^{2-\theta}, \quad y \in (0, \infty),$$

which is finite if and only if $\theta > 2$.

**Example 1.2** Let $0 < \theta < 1$ and let $(z(t) : t \geq 0)$ be the unique positive solution to

$$dz(t) = 2\sqrt{z(t)} dB_t + 2(2-\theta)^{-1}(1-\theta) dt, \quad z(0) = 0. \quad (1.22)$$

From (8.10) in Ikeda and Watanabe (1989, p.236) it follows that

$$E[e^{-\lambda z(t)}] = \frac{1}{(1 + 2\lambda t)^{(1-\theta)/(2-\theta)}}, \quad \lambda \geq 0.$$
By letting $\lambda \to \infty$ in the above equality we see $P(z(t) = 0) = 0$ and hence
\[
E\left( \int_0^\infty 1_{\{z(t) = 0\}} dt \right) = \int_0^\infty P(z(t) = 0) dt = 0.
\]
Then $P(z(t) > 0 \text{ a.e. } t \geq 0) = 1$. Let $x_n(t) = \left( z(t) + 1/n \right)^{1/(2-\theta)}$ for $n \geq 1$. By (1.22) and Itô’s formula,
\[
dx_n(t) = 2(2-\theta)^{-1}(z(t) + 1/n)^{(\theta-1)/(2-\theta)} \sqrt{z(t)} dB_t + 2n^{-1}(2-\theta)^{-2}(1-\theta)(z(t) + 1/n)^{(-3-2\theta)/(2-\theta)} dt.
\]
By letting $n \to \infty$ we see $x(t) = z(t)^{1/(2-\theta)}$ satisfies
\[
dx(t) = \sqrt{2cx(t)^\theta} dB_t, \quad x(0) = 0,
\]
where $c = 2(2-\theta)^{-2}$. It is trivial to see that $x_0(t) \equiv 0$ is another solution to the above equation. Then the requirement of 0 being a trap is necessary to guarantee the pathwise uniqueness of the solution to (1.7).

We present the proofs of the results in the following sections. Section 2 is devoted to the construction of the process. The mean extinction and explosion times are calculated in Section 3. In Section 4, the extinction and explosion probabilities are explored. In Section 5, we prove the construction of the process coming down from $\infty$. The convergence of discrete-state processes is discussed in Section 6.

## 2 Construction of the process

In this section, we construct the continuous-state nonlinear branching process in terms of stochastic equations and random time changes.

**Proof of Theorem 1.7** (1) We prove the result by an approximation argument. For each $n \geq 1$ define
\[
r_n(x) = \begin{cases} n^\theta, & n < |x| < \infty, \\ |x|^\theta, & 1/n < |x| \leq n, \\ n^{2-\theta}|x|^2, & 0 \leq |x| \leq 1/n. \end{cases} \tag{2.1}
\]
By Theorem 9.1 in Ikeda and Watanabe (1989, p.245) there is a pathwise unique solution $\{\xi_n(t) : t \geq 0\}$ to the stochastic equation
\[
\xi_n(t) = x + \int_0^t \sqrt{2cr_n(\xi_n(s))} dB(s) + \int_0^t \int_{(0,1]} \int_{0}^{r_n(\xi_n(s-))} z \tilde{M}(ds,dz,du) \\
- \int_0^t br_n(\xi_n(s)) ds + \int_0^t \int_{[1,\infty]} \int_{0}^{r_n(\xi_n(s-))} (z \wedge n) M(ds,dz,du). \tag{2.2}
\]
Let $\zeta_n = \inf\{t \geq 0 : \xi_n(t) \geq n \text{ or } \xi_n(t) \leq 1/n\}$. Clearly, the sequence of stopping times $\{\zeta_n\}$ is increasing and $\xi_n(t) = \xi_m(t)$ for $t \in [0, \zeta_m \wedge n)$. Let $\tau = \lim_{n \to \infty} \zeta_n$. We define the
process \((X_t : t \geq 0)\) by \(X_t = \xi_n(t)\) for \(t \in [0, \zeta_n]\) and \(X_t = \lim_{n \to \infty} \xi_n(\zeta_n)\) for \(t \in [\tau, \infty)\).

Then \(\zeta_n = \inf\{t \geq 0 : X_t \geq n\) or \(X_t \leq 1/n\}\) and \((X_t : t \geq 0)\) is a solution of (1.7). The pathwise uniqueness of the solution follows from that for (2.2) in the time interval \([0, \zeta_n)\) for each \(n \geq 1\).

(2) Let \(\{\xi^x_n(t) : t \geq 0\}\) denote the solution of (2.2) to indicate its dependence on the initial state. For any \(y \geq x \geq 0\), we can use Theorem 5.5 in Fu and Li (2010) to see \(P(\xi^y_n(t) \geq \xi^x_n(t)\) for every \(t \geq 0\) = 1, and so \(P(X^y_t \geq X^x_t\) for every \(t \geq 0\) = 1.

Proof of Theorem 1.2. For any \(t \geq 0\) let \(P_t(x, \cdot)\) be the distribution of \(X_t\) on \([0, \infty]\) with \(X_0 = x \in [0, \infty]\). By Theorem 1.1 for any \(y \in [0, \infty]\), the mapping \(x \mapsto P_t(x, [0, y])\) is decreasing, so it is Borel measurable. A monotone class argument shows \(x \mapsto P_t(x, A)\) is Borel measurable for each Borel set \(A \subset [0, \infty]\). Then \(P_t(x, dy)\) is a Borel kernel on \([0, \infty]\). For any finite \((\mathcal{F}_t)\)-stopping time \(\sigma\), from the equation (1.7) we have

\[
X_{\sigma+t} = X_{\sigma} + \sqrt{2c} \int_{0}^{t} X_{\sigma+s}^{\theta/2} dB_{\sigma+s} + \int_{0}^{t} \int_{(0,1]} \int_{0}^{X_{\sigma+s}} z M(\sigma + ds, dz, du)
- b \int_{0}^{t} X_{\sigma+s}^{\theta} ds + \int_{0}^{t} \int_{(1,\infty]} \int_{0}^{X_{\sigma+s}} z M(\sigma + ds, dz, du).
\]

Then \(P(X_{\sigma+t} \in \cdot | \mathcal{F}_\sigma) = P_t(X_{\sigma}, \cdot)\). That gives the strong Markov property of the process \((X_t : t \geq 0)\). For \(f \in \mathcal{D}(L)\), we can use (2.2) and Itô’s formula to see

\[
f(\xi_n(t \wedge \zeta_n)) = f(x) + \int_{0}^{t \wedge \zeta_n} f'(\xi_n(s)) br_n(\xi_n(s)) ds + c \int_{0}^{t \wedge \zeta_n} f''(\xi_n(s)) r_n(\xi_n(s)) ds
+ \int_{0}^{t \wedge \zeta_n} \int_{(0,\infty)} \int_{0}^{r_n(\xi_n(s-))} \left[ f(\xi_n(s-)) + z \right] M(ds, dz, du) + \text{martingale}
- f'(\xi_n(s-)) z 1_{\{z \leq 1\}} M(ds, dz, du) + \text{martingale}
= f(x) + \int_{0}^{t \wedge \zeta_n} f'(\xi_n(s)) br_n(\xi_n(s)) ds + c \int_{0}^{t \wedge \zeta_n} f''(\xi_n(s)) r_n(\xi_n(s)) ds
+ \int_{0}^{t \wedge \zeta_n} r_n(\xi_n(s)) ds \int_{(0,\infty)} \left[ f(\xi_n(s)) + z \right] m(dz) + \text{martingale}
- f'(\xi_n(s)) z 1_{\{z \leq 1\}} m(dz) + \text{martingale}
= f(x) + \int_{0}^{t \wedge \zeta_n} Lf(\xi_n(s)) ds + \text{martingale},
\]

where the martingale is locally bounded. Then letting \(n \to \infty\) in the above equality gives

\[
f(X_{t \wedge \tau}) = f(x) + \int_{0}^{t \wedge \tau} Lf(X_s) ds + \text{martingale}.
\]

Since 0 and \(\infty\) are traps for \((X_t : t \geq 0)\) and \(Lf(0) = Lf(\infty) = 0\), it follows that

\[
f(X_t) = f(x) + \int_{0}^{t} Lf(X_s) ds + \text{martingale}. \quad (2.3)
\]
Let \( \tilde{\text{Proposition 2.1}} \)

Let \( Y \) be a spectrally positive \( \text{Lévy process} \) with Laplace exponent \(-\psi\) and initial state \( \tilde{Z}_0 = x \geq 0 \). Note that \( Z \) is absorbed by \( \infty \) after an exponential time with parameter \( a \geq 0 \). Let \( T_y = \inf\{ t \geq 0 : Z_t = y \} \) for \( y \in [0, \infty) \). Let \( T = T_0 \wedge T_\infty \) be the \textit{absorbing time} of \( Z \). Let \( Y_t = Z_{t \wedge T} \) for \( t \geq 0 \). We call \( Y := (Y_t : t \geq 0) \) an absorted spectrally positive \textit{Lévy process}. By the properties of the \textit{Lévy process,} the limit \( Y_\infty := \lim_{t \to \infty} Y_t \) exists a.s. in \([0, \infty)\]; see, e.g., Sato (1999). In fact, on the event \( \{T = \infty\} \) we have a.s. \( Y_\infty = \infty \).

\textbf{Proposition 2.1} Let \( \alpha(t) = \int_0^t Y_{s-\theta} ds \) and \( \eta(t) = \inf\{ s \geq 0 : \alpha(s) > t \} \) for \( t \geq 0 \). Then \( J_\theta(Y) := (Y_{\eta(t)} : t \geq 0) \) solves \((1.7)\) on an extension of the original probability space.

\textbf{Proof.} Let \((W_t : t \geq 0)\) be a Brownian motion and let \( N_0(ds, dz) \) be a Poisson random measure on \((0, \infty) \times (0, \infty)\) with intensity \( dsdz \). Then a realization of the \textit{Lévy process} \( Z := (Z_t : t \geq 0) \) is defined by

\[
Z_t = x - bt + \sqrt{2cW_t} + \int_0^t \int_{[0,1]} z\tilde{N}_0(ds, dz) + \int_0^t \int_{[1,\infty]} zN_0(ds, dz),
\]

where \( \tilde{N}_0(ds, dz) = N_0(ds, dz) - dsm(dz) \). Let \( \{(s_i, z_i) : i = 1, 2, \ldots\} \) be an enumeration of the atoms of \( N(ds, dz) \). On an extension of the original probability space, we can construct a sequence of \((0,1)\)-valued i.i.d. random variables \( \{u_i\} \) independently of \( \{W(t)\} \) and \( \{N(ds, dz)\} \) such that \( P(u_i \in du) = du \). Then

\[
M_0(ds, dz, du) := \sum_{i=1}^{\infty} \delta_{(s_i, z_i, u_i)}(ds, dz, du)
\]

defines a Poisson random measure on \((0, \infty) \times (0, \infty) \times (0, \infty)\) with intensity \( dsdzdu \). Let \( \tilde{M}_0(ds, dz, du) = M_0(ds, dz, du) - dsm(dz)du \). Then we have

\[
Z_t = x - bt + \sqrt{2cW(t)} + \int_0^t \int_{[0,1]} \int_0^1 z\tilde{M}_0(ds, dz, du)
\]

\[
+ \int_0^t \int_{[1,\infty]} \int_0^1 zM_0(ds, dz, du).
\]

Let \( Y := (Y_t : t \geq 0) \) be the absorsted process associated with \( Z \) and let \( X_t = Y_{\eta(t)} \) for \( t \geq 0 \). Let \( \zeta_n = \inf\{ t \geq 0 : X_t \geq n \text{ or } X_t \leq 1/n \} \) and \( \zeta = \lim_{n \to \infty} \zeta_n \). Then we have

\[
X_{t \wedge \zeta_n} = x + \sqrt{2cW(\eta(t \wedge \zeta_n))} + \int_0^{\eta(t \wedge \zeta_n)} \int_{[0,1]} \int_0^1 z\tilde{M}_0(ds, dz, du)
\]
By the definition of \( \alpha(t) \) we have \( d\alpha(t) = Y_{t^{-}} \gamma_{\theta} \, dt \) for \( 0 \leq t < T \) and \( d\eta(t) = Y_{\eta(t)-} \, dt = X_{t^{-}} \theta_{-} \, dt \) for \( 0 \leq t < \tau \). It follows that

\[
\eta(t \wedge \zeta_{t}) = \int_{0}^{t \wedge \zeta_{t}} X_{s^{-}} \gamma_{\theta} 1_{\{s < \tau\}} \, ds = \int_{0}^{t \wedge \zeta_{t}} X_{s^{-}} \theta \, ds. \tag{2.6}
\]

By representation of continuous martingales, there is a Brownian motion \( \{B(t)\} \) on an extension of the original probability space so that

\[
W(\eta(t \wedge \zeta_{t})) = \int_{0}^{t \wedge \zeta_{t}} X_{s^{-}}^{\theta/2} dB(s).
\]

On the extended probability space, we can take another independent Poisson random measure \( \{M_{1}(ds,dz,du)\} \) on \( (0, \infty) \times (0, \infty) \times (0, \infty) \) with intensity \( ds \, d\gamma_{\theta}(dz,du) \) and define the random measure

\[
M(ds,dz,du) = 1_{\{s<\tau, u \leq X_{s^{-}}^{\theta}\}} M_{0}(d\eta(s),d\gamma_{\theta}(dz,du)) + 1_{\{u > X_{s^{-}}^{\theta}\}} M_{1}(ds,dz,du).
\]

Using (2.6) one can see \( \{M(ds,dz,du)\} \) has the deterministic compensator \( ds \, d\gamma_{\theta}(dz,du) \), so it is a Poisson random measure. Now (1.7) follows from (2.5). \( \square \)

By Proposition 2.1 for the solution \( X := (X_{t} : t \geq 0) \) to (1.7) the limit \( X_{\infty} := \lim_{t \to \infty} X_{t} \) exists a.s. in \( [0, \infty] \).

**Proposition 2.2** Let \( \gamma_{\theta}(t) = \int_{0}^{t} X_{s}^{\theta} \, ds \) and \( \beta(t) = \inf\{s \geq 0 : \gamma_{\theta}(s) > \theta\} \) for \( t \geq 0 \). Then \( L_{\theta}(X) := (X_{\beta(t)} : t \geq 0) \) is an absorbed spectrally positive Lévy process.

**Proof.** Without loss of generality, we assume \( X_{0} = x \in [0, \infty) \) is deterministic. Let \( Y_{t} = X_{\beta(t)} = X_{\beta(t)\wedge \tau} \) for \( t \geq 0 \) and let \( T = \inf\{t \geq 0 : Y_{t} = 0 \text{ or } Y_{t} = \infty\} \). We have

\[
Y_{t} = x + \sqrt{2c} \int_{0}^{t \wedge \tau} X_{s}^{\theta/2} dB(s) + \int_{0}^{t \wedge \tau} \int_{[0,1]} \int_{0}^{X_{s}^{\theta}} z M(ds,dz,du) - b \int_{0}^{t \wedge \tau} X_{s}^{\theta} ds + \int_{0}^{t \wedge \tau} \int_{[1,\infty]} \int_{0}^{X_{s}^{\theta}} z M(ds,dz,du)
\]

\[
= x + \sqrt{2c} \int_{0}^{t \wedge \tau} X_{\beta(s)}^{\theta/2} 1_{\{\beta(s) \leq \tau\}} dB(\beta(s)) + \int_{0}^{t \wedge \tau} \int_{[0,1]} \int_{0}^{X_{\beta(s)}^{\theta}} z 1_{\{\beta(s) \leq \tau\}} M(d\beta(s),dz,du)
\]

\[
- b \int_{0}^{t \wedge \tau} X_{\beta(s)}^{\theta} 1_{\{\beta(s) \leq \tau\}} d\beta(s) + \int_{0}^{t \wedge \tau} \int_{[1,\infty]} \int_{0}^{X_{\beta(s)}^{\theta}} z 1_{\{\beta(s) \leq \tau\}} M(d\beta(s),dz,du)
\]
\[ Y_t = x + \sqrt{2c} \int_0^t Y_{s-}^{\theta/2} 1_{\{\theta(s) \leq \tau\}} dB(\theta(s)) + \int_0^t \int_{(0,1]} \int_0^{Y_{s-}^{\theta}} z 1_{\{\theta(s) \leq \tau\}} \tilde{M}(d\theta(s), dz, du) \]

\[- b \int_0^t Y_{s-}^{\theta} 1_{\{\theta(s) \leq \tau\}} d\theta(s) + \int_0^t \int_{[1,\infty]} \int_0^{Y_{s-}^{\theta}} z 1_{\{\theta(s) \leq \tau\}} M(d\theta(s), dz, du). \quad (2.7)\]

By the definition of \( \gamma(t) \) and \( \beta(t) \) we have \( d\gamma(t) = X_t^\theta \, dt \) for \( 0 \leq t < \tau \) and \( d\beta(t) = X_{\beta(t)}^- \, dt = Y_t^- \, dt \) for \( 0 \leq t < T \). Thus

\[
\int_0^t Y_{s-}^{\theta} 1_{\{\theta(s) < \tau\}} d\beta(s) = \int_0^t 1_{\{s < \tau\}} ds = t \wedge T.
\]

It follows that

\[ W_0(t) := \int_0^t Y_{s-}^{\theta/2} 1_{\{\theta(s) < \tau\}} dB(\theta(s)) \]

defines a continuous local martingale with \( \langle W_0 \rangle(t) = t \wedge T \). Then we can extend \( \{W_0(t)\} \) to a Brownian motion \( \{W(t)\} \). Now define the random measure \( \{N_0(ds, dz)\} \) on \( (0, \infty) \times (0, \infty] \) by

\[ N_0((0, t] \times (a_1, a_2]) = \int_0^t \int_a^{a_2} \int_0^{Y_{s-}^\theta} 1_{\{\theta(s) < \tau\}} M(d\theta(s), dz, du), \]

where \( t \geq 0 \) and \( a_1, a_2 \in (0, \infty] \). It is easy to check that \( \{N_0(ds, dz)\} \) has predictable compensator \( Y_{s-}^{\theta} \, 1_{\{\theta(s) < \tau\}} d\beta(s)(m)(dz) = 1_{\{s < \tau\}} dsm(dz) \). Then we can extend \( \{N_0(ds, dz)\} \) to a Poisson random measure \( \{N(ds, dz)\} \) on \( (0, \infty)^2 \) with intensity \( dsm(dz) \); see, e.g., Ikeda and Watanabe (1989, p.93). From (2.7) it follows that

\[ Y_t = x + \sqrt{2c} W(t \wedge T) + \int_0^{t \wedge T} \int_{(0,1]} \int_0^{Y_{s-}^\theta} z \tilde{N}(ds, dz) - b(t \wedge T) + \int_0^{t \wedge T} \int_{[1,\infty]} \int_0^{Y_{s-}^\theta} z N(ds, dz). \]

Then \( \{Y_t\} \) is an absorbed spectrally positive Lévy process. \( \square \)

We call \( L_\theta \) a generalized Lamperti transformation and \( J_\theta \) the inverse generalized Lamperti transformation. In the particular case \( \theta = 1 \), they reduce to the classical transformations introduced by Lamperti (1967a, 1967b).

**Proof of Theorem 1.3**. Let \( Z = (Z_t : t \geq 0) \) be the Lévy process starting at 0 with Laplace exponent \( -\psi \). Let \( Z_t^x = x + Z_t \) for \( x \in [0, \infty) \). Let \( T_0^x = \inf\{t \geq 0 : Z_t^x = 0\} \) and \( T_\infty^x = \inf\{t \geq 0 : Z_t^x = \infty\} \). Then \( x \mapsto T_0^x \) is a.s. increasing. By Theorem 3.12 in Kyprianou (2006, p.81), for any \( \lambda > 0 \) we have

\[ \mathbb{E}(e^{-\lambda T_0^{x}}) = \mathbb{E}(e^{-\lambda T_0^{y}} 1_{\{T_0^{y} < T_\infty^{x}\}}) = \exp\{-\psi^{-1}(\lambda)x\}, \]

where \( \psi^{-1}(\lambda) = \inf\{z \geq 0 : \psi(z) > \lambda\} \). Then \( \lim_{y \to x} T_0^y = T_0^x \) first in distribution and then almost surely. Let \( Y^x = (Z_t^{x} : t \geq 0) \). For \( x < y \in [0, \infty) \) we have

\[
\rho(Z_t^{x}, Z_t^{y}) \leq \rho(Z_{t \wedge T_0^{x}}^{x}, Z_{t \wedge T_0^{y}}^{y}) + \rho(Z_{t \wedge T_0^{y}}^{y}, Z_{t \wedge T_0^{y}}^{y}) \leq |Z_{t \wedge T_0^{x}}^{x} - Z_{t \wedge T_0^{y}}^{y}| + \sup_{s \in [T_0^{x}, T_0^{y})} \rho(0, Z_s^{y})
\]
\[ \leq |x - y| + \sup_{s \in [T_x^0, T_y^0]} Z_s^y. \]

By the right-continuity and quasi-left-continuity of the Lévy process we have a.s. \( \lim_{y \to x} d_\infty(Y^y, Y^x) = \lim_{x \to y} d_\infty(Y^y, Y^x) = 0 \). By Proposition 2.1 the process \( X^x := J_\theta(Y^x) \) is a solution to (1.7). A modification of the proof of Proposition 5 in Caballero et al. (2009) shows that the transformation \( J_\theta \) is continuous on \((D, d_\infty)\). Then we have a.s. \( \lim_{y \to x} d_\infty(X^y, X^x) = \lim_{x \to y} d_\infty(X^y, X^x) = 0 \). That proves the desired result. \( \square \)

### 3 Mean extinction and explosion times

In this section we prove the results on the mean hitting times of the continuous-state nonlinear branching process. We shall see that the relations established in Theorem 1.5 play important roles in the proofs.

**Proposition 3.1** Let \( e_\lambda(x) = e^{-\lambda x} \) for \( \lambda \in (0, \infty) \) and \( x \in [0, \infty] \). Then: (i) \( t \mapsto P_t e_\lambda(x) \) is decreasing if \( 0 < \lambda \leq q \); (ii) \( t \mapsto P_t e_\lambda(x) \) is increasing if \( q \leq \lambda < \infty \); (iii) \( \lim_{t \to \infty} P_t e_\lambda(x) = e_q(x) \) for all \( 0 < \lambda < \infty \); (iv) \( e_q(x) \) is an invariant function of \( (P_t)_{t \geq 0} \).

**Proof.** It is easy to see that \( e_\lambda \in \mathcal{D}(L) \) and \( L e_\lambda(x) = x^\theta \psi(\lambda) e_\lambda(x) \). By (1.4) we have

\[ \frac{d}{dt} P_t e_\lambda(x) = \psi(\lambda) \int_{[0, \infty)} y^\theta e^{-\lambda y} P_t(x, dy). \] (3.1)

Then (i) and (ii) hold. By Theorem 1.4(ii) we get (iii), from which (iv) follows. \( \square \)

**Proof of Theorem 1.5.** Taking the Laplace transform in both sides of (3.1) and using integration by parts we get

\[
\psi(\lambda) \int_{[0, \infty)} y^\theta e^{-\lambda y} U_\eta(x, dy) = \int_0^\infty e^{-\eta t} \frac{d}{dt} P_t e_\lambda(x) dt
= e^{-\eta t} P_t e_\lambda(x) \big|_{t=0}^t + \eta \int_0^\infty e^{-\eta t} P_t e_\lambda(x) dt
= -e^{-\lambda x} + \eta \int_0^\infty e^{-\eta t} P_t e_\lambda(x) dt.
\]

Then we get (1.11). It follows that

\[
l_x(\eta, z) = \int_{[0, \infty)} y^\theta e^{-zy} U_\eta(x, dy) = \int_0^\infty e^{-\eta t} dt \int_{[0, \infty)} y^\theta e^{-zy} P_t(x, dy).
\]

Multiplying the above equation by \((z - \lambda)^{\theta-1}\) and integrating both sides, we have

\[
\int_{\lambda}^\infty l_x(\eta, z)(z - \lambda)^{\theta-1} dz = \int_{\lambda}^\infty (z - \lambda)^{\theta-1} dz \int_0^\infty e^{-\eta t} dt \int_{[0, \infty)} y^\theta e^{-zy} P_t(x, dy).
\]
Lemma 3.2 For any $\lambda \geq 0$ and $x \in [0, \infty)$ we have
\[
e^{-qx} - e^{-\lambda x} = \psi(\lambda) \int_0^\infty dt \int_{(0,\infty)} y^\theta e^{-\lambda y} P_t(x, dy)
\]
and
\[
\int_\lambda^\infty h_x(z)(z - \lambda)^{\theta - 1} dz = \Gamma(\theta) \int_0^\infty dt \int_{(0,\infty)} e^{-\lambda y} P_t(x, dy).
\]

Proof. By Proposition 3.1 as $t \uparrow \infty$ we have $e^{-\lambda x} \leq P_t e_{\lambda}(x) \uparrow e^{-qx}$ for $q < \lambda \leq \infty$ and $e^{-\lambda x} \leq P_t e_{\lambda}(x) \downarrow e^{-q x}$ for $0 < \lambda < q$. Since
\[
\eta U^n e_{\lambda}(x) = \int_0^\infty \eta e^{-\eta t} P_t e_{\lambda}(x) dt = \int_0^\infty e^{-t} P_t e_{\lambda}(x) dt,
\]
we see that $e^{-\lambda x} \leq \eta U^n e_{\lambda}(x) \uparrow e^{-qx}$ for $q < \lambda \leq \infty$ and $e^{-\lambda x} \leq \eta U^n e_{\lambda}(x) \downarrow e^{-q x}$ for $0 < \lambda < q$ as $n \downarrow 0$. Then we use monotone convergence to get (3.2) and (3.3) by letting $\eta \rightarrow 0$ in (1.11) and (1.12), respectively.

Proof of Theorem 1.6. Observe that, for any $\eta \geq 0$,
\[
\int_{(0,\infty)} e^{-\eta z} P_t(x, dz) = \frac{1}{\Gamma(\theta)} \int_{(0,\infty)} e^{-\eta z} P_t(x, dz) \int_0^\infty y^{\theta - 1} e^{-y} dy
\]
\[
= \frac{1}{\Gamma(\theta)} \int_{(0,\infty)} \int_0^\infty z^\theta e^{-\eta z} P_t(x, dz) \int_0^\infty \lambda^{\theta - 1} e^{-\lambda z} d\lambda
\]
\[
= \frac{1}{\Gamma(\theta)} \int_0^\infty \lambda^{\theta - 1} d\lambda \int_{(0,\infty)} z^\theta e^{-(\lambda + \eta) z} P_t(x, dz).
\]

By Proposition 3.1(iv), the function $x \mapsto e^{-q x} = P_x(X_\infty = 0) = P_x(\tau_\infty > t, X_\infty = 0)$ is invariant for the transition semigroup of $X$. By (3.4),
\[
P_x(\tau_0 > t, X_\infty = 0) = P_x(X_\infty = 0) - P_x(\tau_0 \leq t, X_\infty = 0)
\]
\[
= e^{-q x} - P_t(x, \{0\}) = \int_{(0,\infty)} e^{-q x} P_t(x, dz)
\]
\[
= \frac{1}{\Gamma(\theta)} \int_0^\infty \lambda^{\theta - 1} d\lambda \int_{(0,\infty)} z^\theta e^{-(\lambda + \eta) z} P_t(x, dz)
\]
and
\[
P_x(\tau_\infty > t, X_\infty = \infty) = P_x(\tau_\infty > t) - P_x(\tau_\infty > t, X_\infty = 0)
\]
\[
= P_t(0, (0, \infty)) - e^{-q x} = \int_{(0,\infty)} (1 - e^{-q x}) P_t(x, dz)
\]
Proposition 3.3 For any \( \eta > 0 \) let \( (P_t^{(y)})_{t \geq 0} \) denote the transition semigroup of the stopped process \( X_t^{(y)} = (X_{t \wedge \tau_y} : t \geq 0) \) with state space \( [y, \infty] \). For any \( \eta > 0, \lambda \geq 0 \) and \( x \in [y, \infty) \) we have

\[
\eta \int_0^\infty e^{-\eta t}P_t^{(y)}e_\lambda(x)dt = e^{-\lambda x} + \psi(\lambda) \int_0^\infty e^{-\eta t}dt \int_{(y, \infty)} z^\theta e^{-\lambda z}P_s^{(y)}(x, dz). \tag{3.5}
\]

Proof. Since \( x \geq y \in [0, \infty) \), we have \( P_x(\tau_y < \tau_0, X_{\tau_y} = y) = 1 \). Then \( \text{(2.3)} \) implies

\[
e^{-\lambda x_{t \wedge \tau_y}} = e^{-\lambda x} + \psi(\lambda) \int_0^{t \wedge \tau_y} X_s^{\theta} e^{-\lambda X_s}ds \text{ + martingale}
\]

\[
= e^{-\lambda x} + \psi(\lambda) \int_0^t X_s^{\theta} e^{-\lambda X_s}1_{(y, \infty)}(X_{s \wedge \tau_y})ds \text{ + martingale.}
\]

Taking the expectation in both sides yields

\[
P_t^{(y)}e_\lambda(x) = e^{-\lambda x} + \psi(\lambda) \int_0^t ds \int_{(y, \infty)} z^\theta e^{-\lambda z}P_s^{(y)}(x, dz).
\]

Thus we have

\[
\eta \int_0^\infty e^{-\eta t}P_t^{(y)}e_\lambda(x)dt = e^{-\lambda x} + \eta \psi(\lambda) \int_0^\infty e^{-\eta t}dt \int_0^t ds \int_{(y, \infty)} z^\theta e^{-\lambda z}P_s^{(y)}(x, dz).
\]
Proof of Theorem 1.9. in this section we give the proofs of the results on the extinction and explosion probabilities.

4 Extinction and explosion probabilities

Proof of Corollary 1.8. By Theorem 1.4 we see

Thus we have

That implies (1.16) by a formula for the expectation.

Proof of Theorem 1.7. By Theorem 1.4, as \( \eta \to 0 \) we have

From (3.5) it follows that

Thus we have

That implies (1.16) by a formula for the expectation.

Proof of Corollary 1.8. By Theorem 1.4 we see \( P_x (\tau_y < \infty) = 1 \) and hence \( P_x (\tau_y < \tau_{\infty}) = 1 \). Then (1.17) follows from (1.16).
Therefore we can find a constant \( C = C(s) > 0 \) such that

\[
\int_{\varepsilon+1}^{\infty} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda \leq \frac{C}{\eta \rho_x(\eta)} \int_{\varepsilon}^{\infty} \eta U^\eta e_{\lambda}(x) - e^{-\lambda x} \psi(\lambda)^{\theta-1} d\lambda.
\]

By (1.12) we see the right hand side of is finite, and hence (1.18) holds.

(2) Suppose that \( P_x(\tau_0 < \infty) = 0 \) for some \( x \in (0, \infty) \). Then \( P_t(x, \{0\}) = 0 \) for every \( t \geq 0 \). By Theorem 1.4 (2), for \( \varepsilon > q \) we have

\[
\lim_{t \to \infty} \int_{(0,\infty)} e^{-\varepsilon t} P_t(x, dz) = \lim_{t \to \infty} \int_{(0,\infty)} e^{-\varepsilon t} P_t(x, dz) = e^{-q x} > 0.
\]

By (3.3) it follows that

\[
\int_{\varepsilon}^{\infty} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda \geq \int_{\varepsilon}^{\infty} h_x(\lambda)(\lambda - \varepsilon)^{\theta-1} d\lambda = \Gamma(\theta) \int_{0}^{\infty} dt \int_{(0,\infty)} e^{-\varepsilon t} P_t(x, dz) = \infty.
\]

(3) Suppose that \( P_x(\tau_0 < \infty) > 0 \) for some \( x \in (0, \infty) \). We only need to prove \( P_x(\tau_0 = \infty, X_\infty = 0) = 0 \) if (1.18) holds. In this case, we have \( P_x(\tau_0 = \infty) < 1 \), and hence \( \alpha := P_x(\tau_0 > v) = P_x(X_v > 0) < 1 \) for some \( v > 0 \). By Theorem 1.1 we have \( P_y(X_v > 0) < \alpha \) for \( y \leq x \). Let \( \sigma_0 = 0 \) and \( \sigma_n = \inf\{t > \sigma_{n-1} + v : X_t \leq x\} \) for \( n \geq 1 \). It is easy to see that \( X_{\sigma_n} \leq x \). By the strong Markov property, for any \( n \geq 1 \) we have

\[
P_x(\tau_0 = \infty, X_\infty = 0) \leq P_x\left( \bigcap_{k=1}^{n} \{\sigma_k < \infty, X_{\sigma_k+v} > 0\} \right)
\leq \mathbb{E}_x \left[ \prod_{k=1}^{n-1} 1_{\{\sigma_k < \infty, X_{\sigma_k+v} > 0\}} 1_{\{\sigma_n < \infty\}} P_x(X_{\sigma_n+v} > 0 | \mathcal{F}_{\sigma_n}) \right]
\leq \mathbb{E}_x \left[ \prod_{k=1}^{n-1} 1_{\{\sigma_k < \infty, X_{\sigma_k+v} > 0\}} 1_{\{\sigma_n < \infty\}} P_{X_{\sigma_n}}(X_v > 0) \right]
\leq \alpha \mathbb{E}_x \left[ \prod_{k=1}^{n-1} 1_{\{\sigma_k < \infty, X_{\sigma_k+v} > 0\}} 1_{\{\sigma_n < \infty\}} \right]
\leq \alpha P_x\left( \bigcap_{k=1}^{n} \{\sigma_k < \infty, X_{\sigma_k+v} > 0\} \right) \leq \cdots \leq \alpha^n.
\]

Then the left-hand side vanishes.

\( \square \)

**Proof of Corollary 1.10.** (1) By the Taylor expansion, we see \( e^{-\lambda u} - 1 + \lambda u \leq \lambda^2/2 \). In view of (1.5) we have

\[
\psi(\lambda) \leq |b| \lambda + c \lambda^2 + \frac{1}{2} \lambda^2 \int_{(0,1]} u^2 m(du) - \lambda \int_{(1,\infty]} (1 - e^{-\lambda u}) m(du).
\]

Then there is a constant \( C > 0 \) so that \( \psi(\lambda) \leq C \lambda^2 \) for \( \lambda \geq \varepsilon \). If \( \theta \geq 2 \), then

\[
\int_{\varepsilon}^{\infty} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda \geq \frac{1}{C} \int_{\varepsilon}^{\infty} \lambda^{\theta-3} d\lambda = \infty.
\]
By Theorem 1.9 the process does not hit 0.

(2) If $c > 0$, we can take $\varepsilon > 0$ so that $\psi(\lambda) \geq c\lambda^2/2$ for $\lambda \geq \varepsilon$. When $0 < \theta < 2$, we have
\[
\int_{\varepsilon}^{\infty} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda \leq \frac{2}{c} \int_{\varepsilon}^{\infty} \lambda^{\theta-3} d\lambda < \infty,
\]
so the process hits 0 by Theorem 1.9.

Proof of Theorem 1.11. (1) In the case $\psi(0) < 0$, we can let $\lambda \to 0$ in (1.11) to see
\[
\eta \int_{0}^{\infty} e^{-\eta t} r_L(x, [0, \infty)) dt = \eta U^\eta(x, [0, \infty)) = 1 + \psi(0) \int_{[0, \infty)} z U^\eta(x, dz) < 1.
\]
Then for some $t > 0$ we have $P_t(x, [0, \infty)) < 1$ and so $P_x(\tau_\infty \leq t) = P_t(x, \{\infty\}) > 0$.

(2) Suppose that $\psi(0) = 0$ and $\psi'(0) \geq 0$. By the convexity of $\psi$ we have $\psi(\lambda) > 0$ for each $\lambda > 0$. Then (1.11) implies
\[
\eta \int_{0}^{\infty} e^{-\lambda z} U^\eta(x, dz) = \eta \int_{0}^{\infty} e^{-\eta t} dt \int_{[0, \infty)} e^{-\lambda z} P_t(x, dz) > e^{-\lambda x}.
\]
By letting $\lambda \to 0$ on the both sides we see $\eta U^\eta(x, [0, \infty)) = 1$. Then $P_x(\tau_\infty > t) = P_t(x, [0, \infty)) = 1$ for every $t > 0$. That implies $P_x(\tau_\infty = \infty) = 1$.

(3) Consider the case with $\psi(0) = 0$ and $\psi'(0) < 0$. (i) Suppose that (1.19) holds but $P_x(\tau_\infty < \infty) > 0$. Then $P_t(x, [0, \infty)) = P_x(\tau_\infty > t) < 1$ for sufficiently large $t \geq 0$. For any $\eta > 0$ we have
\[
\varepsilon(x) := 1 - \eta \int_{0}^{\infty} e^{-\eta t} P_t(x, [0, \infty)) dt > 0. \tag{4.1}
\]
By continuity there exists an $\varepsilon \in (0, q]$ such that $\psi(\lambda) < 0$ and
\[
e^{-\lambda x} - \eta U^\eta e_\lambda(x) \geq \frac{1}{2} \varepsilon(x) > 0, \quad 0 < \lambda \leq \varepsilon.
\]
By (1.19) we have
\[
\int_{0}^{\varepsilon} l_x(\eta, \lambda) \lambda^{\theta-1} d\lambda = \int_{0}^{\varepsilon} \frac{\eta U^\eta e_\lambda(x) - e^{-\lambda x}}{\psi(\lambda)} \lambda^{\theta-1} d\lambda \geq \frac{\varepsilon(x)}{2} \int_{0}^{\varepsilon} \lambda^{\theta-1} d\lambda = \infty.
\]
Then (3.3) implies
\[
\int_{0}^{\infty} e^{-\eta t} P_t(x, (0, \infty)) dt = \frac{1}{\Gamma(\theta)} \int_{0}^{\infty} l_x(\eta, \lambda) \lambda^{\theta-1} d\lambda = \infty,
\]
which is in contradiction to (1.11). (ii) Conversely, suppose that (1.19) does not hold. Then we have
\[
\int_{0}^{\varepsilon} \frac{\lambda^{\theta-1}}{-\psi(\lambda)} d\lambda < \infty.
\]
Using the convexity of $\psi$ we know $\psi'(q) > 0$, and so
\[
\lim_{\lambda \to q} \frac{e^{-(\lambda-q)y} - e^{-(\lambda-q)x}}{\psi(\lambda)} = \frac{(x-y)}{\psi'(q)}.
\]
Since $\lim_{\lambda \to \infty} \psi(\lambda) = \infty$, by Theorem 1.7 we see
\[
E_x(\tau_\infty : \tau_y = \infty) \leq \frac{e^{-qx} - e^{-(\lambda-q)x}}{\psi(\lambda)} \int_0^\infty \lambda^{\theta-1} d\lambda < \infty.
\]
It follows that $P_x(\tau_\infty < \infty, X_\infty = \infty) \leq P_x(\tau_\infty = \infty) = e^{-q(x-y)} > 0$.

(4) Let us consider the case of $P_x(\tau_\infty < \infty) > 0$ for some $x \in (0, \infty)$. We only need to prove $P_x(\tau_\infty = \infty, X_\infty = \infty) = 0$ for each $x \in (0, \infty)$. Fix $x \in (0, \infty)$ and choose sufficiently large $v > 0$ so that $\alpha := P_x(\tau_\infty > v) = P_x(X_v < \infty) < 1$. By Theorem 1.1 we have $P_y(X_v < \infty) < \alpha$ for $y \geq x \in [0, \infty)$. Let $\sigma_1 = 0$ and $\sigma_n = \inf\{t > \sigma_{n-1} + v : X_t \geq x\}$ for $n \geq 1$. As in the proof of Theorem 1.9 one sees
\[
P_x(\tau_\infty = \infty, X_\infty = \infty) \leq P_x\left(\bigcap_{k=1}^n \{\sigma_k < \infty, X_{\sigma_k+v} < \infty\}\right) \leq \alpha^n.
\]
for every $n \geq 1$. Then we must have $P_x(\tau_\infty = \infty, X_\infty = \infty) = 0$. \hfill \Box

Proof of Corollary 1.12. By Theorem 1.11 we have $P_x(\tau_\infty < \infty) > 0$ if $\psi(0) < 0$, and $P_x(\tau_\infty < \infty) = 0$ if $\psi(0) = 0$ and $\psi'(0) \geq 0$. Since $\theta > 1$, when $\psi(0) = 0$ and $\psi'(0) < 0$, we have
\[
\int_0^\epsilon \frac{\lambda^{\theta-1}}{-\psi(\lambda)} d\lambda < \infty.
\]
Then $P_x(\tau_\infty < \infty) > 0$ by Theorem 1.11. \hfill \Box

Proof of Corollary 1.13. It suffices to consider the case with $\psi(0) = 0$ and $\psi'(0) > -\infty$. In this case, since $0 < \theta \leq 1$ and $\psi(\lambda) = \psi'(0)\lambda + o(\lambda)$ as $\lambda \to 0$, we have
\[
\int_0^\epsilon \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda = \infty.
\]
Then $P_x(\tau_\infty < \infty) = 0$ by Theorem 1.11. \hfill \Box

5 The property of coming down from infinity

In this section, we establish the construction of the continuous-state nonlinear branching process coming down from $\infty$.

Proof of Theorem 1.14. Obviously we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). By (1.20) we see (iii) $\Leftrightarrow$ (iv). To show that (i) $\Rightarrow$ (iii), assume that $P(\tau_y^\infty < \infty) > 0$ for some $y > 0$. Then
there exists $t > 0$ such that $\alpha := P(\tau^y > t) < 1$. By Theorem 1.1 we see for each $x \geq y \in [0, \infty)$ we have $P(\tau^x > t) \leq \alpha < 1$. By the Markov property, for $n \geq 1$,

$$P(\tau^y > nt) = \lim_{x \to \infty} P(\tau^x > nt)$$

$$= \lim_{x \to \infty} E(1_{\{\tau^y > nt\}} 1_{\{\tau^y > (n-1)t\}})$$

$$= \lim_{x \to \infty} E[1_{\{\tau^y > (n-1)t\}} E(1_{\{\tau^y > (n-1)t\}} | \mathcal{F}_t)]$$

$$= \lim_{x \to \infty} E[1_{\{\tau^y > (n-1)t\}} E(1_{\{\tau^y > (n-1)t\}})]$$

$$\leq E[1_{\{\tau^y > (n-1)t\}} E(1_{\{\tau^y > (n-1)t\}})]$$

$$= \alpha P(\tau^y > (n-1)t).$$

Then $P(\tau^y > nt) \leq \alpha^n$ by induction. That implies (iii). \hfill \Box

**Proposition 5.1** Suppose that $a = \psi(0) = 0$ and $\psi'(0) \geq 0$. Then for each $x \in (0, \infty)$ we have $\lim_{y \to x} d_\infty(X^x, X^y) = 0$.

**Proof.** By the assumption we may rewritten (1.7) as

$$X_t = X_0 - \beta \int_0^t X^\theta_s ds + \sqrt{2c} \int_0^t X^\theta_s dB_s + \int_0^t \int_{(0, \infty]} \int_0^t z \tilde{M}(ds, dz, du),$$

where $\beta$ is defined as in (1.6). For each $n \geq 1$ define the function $r_n$ as in (2.1). For each $x > 0$ let $\{\xi_n^x(t) : t \geq 0\}$ be the unique solution to the following equation

$$\xi_n(t) = x + \int_0^t \sqrt{2cr_n(\xi_n(s-))} dB(s) - \int_0^t \beta r_n(\xi_n(s-)) ds$$

$$+ \int_0^t \int_{(0, \infty]} \int_0^{r_n(\xi_n(s-))} (z \wedge n) \tilde{M}(ds, dz, du).$$

For $0 < y < x$ define $\zeta_n^{x,y} = \inf\{t \geq 0 : X^x_t \geq n \text{ or } X^y_t \leq 1/n\}$. Since $1/n \leq X^y_t = \xi_n^y(t) \leq X^x_t = \xi_n^x(t) \leq n$ for $0 \leq t < \zeta_n^{x,y}$, the trajectory $t \mapsto \xi_n^x(t)$ and $t \mapsto \xi_n^y(t)$ have no jumps larger than $n$ on the time interval $[0, \zeta_n^{x,y}]$. Then we have

$$\xi_n^x(t \wedge \zeta_n^{x,y}) - \xi_n^y(t \wedge \zeta_n^{x,y}) = x - y - \beta \int_0^{t \wedge \zeta_n^{x,y}} [\xi_n^x(s-)^\theta - \xi_n^y(s-)^\theta] ds$$

$$+ \sqrt{2c} \int_0^{t \wedge \zeta_n^{x,y}} [\xi_n^x(s-)^{\theta/2} - \xi_n^y(s-)^{\theta/2}] dB_s$$

$$+ \int_0^{t \wedge \zeta_n^{x,y}} \int_{(0, \infty]} \int_0^{\xi_n^x(s-)} (z \wedge n) \tilde{M}(ds, dz, du).$$

By applying Doob’s inequality to the martingale terms and applying Hölder’s inequality to the drift term in above we have

$$E \left[ \sup_{0 \leq s \leq t \wedge \zeta_n^{x,y}} |\xi_n^x(s-) - \xi_n^y(s-)|^2 \right]$$
Obviously $x \mapsto x^\theta$ and $x \mapsto x^{\theta/2}$ are Lipschitz functions on $[1/n, n]$. Then for $t < k$ there exists a constant $C_{n,k}$ such that

$$
E\left[ \sup_{0 \leq s \leq t \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)|^2 \right] \leq 4|x - y|^2 + C_{n,k} \int_0^t E(|\xi_n^x(s) - \xi_n^y(s)|^2) ds
$$

$$
\leq 4|x - y|^2 + C_{n,k} \int_0^t E\left[ \sup_{0 \leq u \leq s \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)|^2 \right] ds.
$$

Then by Gronwall’s inequality for $t \leq k$ we see

$$
E\left[ \sup_{0 \leq s \leq t \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)|^2 \right] \leq 4|x - y|^2 \exp[C_{n,k} t].
$$

It follows that

$$
\lim_{y \uparrow x} E\left[ \sup_{0 \leq s \leq k \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)|^2 \right] = 0,
$$

which yields

$$
\lim_{y \uparrow x} \sup_{0 \leq s \leq k \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)| = 0 \quad \text{a.s.}
$$

By Theorem 1.3 and Theorem 3.1 in Ethier and Kurtz (1986), we see $\lim_{y \uparrow x} \tau_0^y = \tau_0^x$. Since each trajectory of $X^x$ has no negative jumps, for any $\varepsilon > 0$ there exists a $0 < T < \tau_0^x$ such that $X^x_t < \varepsilon$ for $t > T$. Obviously, we have $\lim_{y \uparrow x} \lim_{n \to \infty} \zeta_n^{x,y} = \tau_0^x$. Then for sufficiently large $k, n$ and $y < x$ we have $k \wedge \zeta_n^{x,y} > T$, and thus

$$
d_{\infty}(X^x, X^y) \leq \sup_{0 \leq s \leq k \wedge \zeta_n^{x,y}} |X^x_{s^-} - X^y_{s^-}| + \varepsilon
$$

$$
= \sup_{0 \leq s \leq k \wedge \zeta_n^{x,y}} |\xi_n^x(s) - \xi_n^y(s)| + \varepsilon.
$$

Since $\varepsilon > 0$ can be arbitrarily small, we have $\lim_{y \uparrow x} d_{\infty}(X^x, X^y) = 0$. □

**Proof of Theorem 1.15.** By Theorem 1.1 we see

$$
\Omega_n := \{ n = X_n^x \geq X_n^y \} \subset \{ X_{n+t}^x \geq X_{n+t}^y \text{ for every } t \geq 0 \}.
$$
Under the assumption, by using Theorem 1.14 and (1.20) we see 
\[ \lim_{n \to \infty} \tau_n = 0. \]
Then
\[ 1 = P\left( \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \Omega_m \right) = P(X_t^\infty \geq X_t^x \text{ for every } t \geq 0). \]

That gives the first part of the theorem. Let \( m > n \) be positive integers. Then \( X_t^\infty \geq X_t^m \geq n \) for \( t \in [0, \tau_m^m) \). It follows that 
\[ d_{\infty}(X^\infty, X^m) \leq e^{-n} + d_{\infty}((X_{\tau_m^m+t}^\infty : t \geq 0), (X_{\tau_m^m+t}^m : t \geq 0)). \]
(5.1)

Since \( \lim_{m \to \infty} \tau_m^m = \tau_\infty^m \) increasingly, we have \( \lim_{m \to \infty} X_{\tau_m^m}^\infty = X_\infty^\infty = n = X_{\tau_m^m}^m \) by the quasi-left-continuity of \( X^\infty \). By Proposition 5.1, the second term on the right-hand side of (5.1) tends to 0 as \( m \to \infty \). Thus \( \lim_{m \to \infty} d_{\infty}(X^\infty, X^m) = 0. \) \( \square \)

6 Convergence of discrete-state processes

In this section, we study the convergence of rescaled discrete-state nonlinear branching processes to continuous-state ones. Let us consider a sequence of generating functions \( g_n, n = 1, 2, \ldots \) given by
\[ g_n(s) = \sum_{i=0}^{\infty} b_i^{(n)} s^i, \quad s \in [0, 1], \]
(6.1)
where \( \{b_i^{(n)} : i = 0, 1, \ldots, \infty\} \) is a discrete probability distribution. Let \( \{\gamma_n : n = 1, 2, \ldots\} \) be a sequence of positive numbers. We define the two sequences of functions \( \{\psi_n\} \) and \( \{\phi_n\} \) by
\[ \phi_n(\lambda) = \gamma_n[g_n(e^{-\lambda/n}) - e^{-\lambda/n}], \quad \lambda \geq 0 \]
(6.2)
and
\[ \psi_n(\lambda) = \gamma_n[g_n(1 - \lambda/n) - (1 - \lambda/n)], \quad 0 \leq \lambda \leq n. \]
(6.3)

**Proposition 6.1** The sequence \( \{\phi_n\} \) defined by (6.2) is Lipschitz uniformly on each bounded interval \( [\alpha, \beta] \subset (0, \infty) \) if and only if so is the sequence \( \{\psi_n\} \) defined by (6.3). In this case, we have \( \lim_{n \to \infty} |\psi_n(\lambda) - \phi_n(\lambda)| = 0 \) uniformly on each bounded interval \( [\alpha, \beta] \subset (0, \infty) \).

**Proof.** Clearly, the sequences \( \{\phi_n\} \) or \( \{\psi_n\} \) are Lipschitz uniformly on some interval \( [\alpha, \beta] \subset (0, \infty) \) if and only if the sequences of derivatives \( \{\phi'_n\} \) or \( \{\psi'_n\} \) are bounded uniformly on the interval. From (6.2) and (6.3) we have
\[ \phi'_n(\lambda) = n^{-1} \gamma_n e^{-\lambda/n} [1 - g'_n(e^{-\lambda/n})], \quad \lambda \geq 0 \]
and
\[ \psi'_n(\lambda) = n^{-1} \gamma_n [1 - g'_n(1 - \lambda/n)], \quad 0 \leq \lambda \leq n. \]
Then \(\{\phi_n\}\) is uniformly bounded on each bounded interval \([\alpha, \beta] \subset (0, \infty)\) if and only if so is \(\{\psi_n'\}\). That proves the first assertion. We next assume \(\{\phi_n\}\) is Lipschitz uniformly on each bounded interval \([\alpha, \beta] \subset (0, \infty)\). Observe that

\[
\phi_n(\lambda) - \psi_n(\lambda) = \gamma_n [g_n(e^{-\lambda/n}) - e^{-\lambda/n} - g_n(1 - \lambda/n) + (1 - \lambda/n)].
\]

By the mean-value theorem, for \(n \geq \beta\) and \(\alpha \leq \lambda \leq \beta\) we have

\[
\phi_n(\lambda) - \psi_n(\lambda) = \gamma_n [g_n'(\eta_n) - 1] (e^{-\lambda/n} - 1 + \lambda/n),
\]

(6.4)

where \(1 - \lambda/n \leq \eta_n := \eta_n(\lambda) \leq e^{-\lambda/n}\). Choose sufficiently large \(n_0 \geq \beta\) so that \(e^{-2\beta/n_0} \leq 1 - \beta/n_0\). For \(n \geq n_0\) we have \(e^{-2\beta/n} \leq 1 - \beta/n \leq 1 - \lambda/n\). It follows that \(e^{-2\beta/n} \leq \eta_n \leq e^{-\alpha/n}\) for \(\alpha \leq \lambda \leq \beta\). By the monotonicity of \(z \mapsto g'(z)\),

\[
n^{-1} \gamma_n |g_n'(\eta_n) - 1| \leq \sup_{\alpha \leq \lambda \leq 2\beta} n^{-1} \gamma_n |g_n'(e^{-\lambda/n}) - 1| = \sup_{\alpha \leq \lambda \leq 2\beta} e^{\lambda/n} |\phi_n'(\lambda)|.
\]

Then \(\{n^{-1} \gamma_n |g_n'(\eta_n) - 1| : n \geq n_0\}\) is a bounded sequence. Since \(\lim_{n \to \infty} n(e^{-\lambda/n} - 1 + \lambda/n) = 0\) uniformly on \([\alpha, \beta]\), the desired result follows by (6.4). \(\square\)

**Proposition 6.2** For any function \(\psi\) on \([0, \infty)\) with representation (1.5) there is a sequence \(\{\phi_n\}\) in form (6.2) so that \(\lim_{n \to \infty} \phi_n(\lambda) = \psi(\lambda)\) for \(\lambda \geq 0\).

**Proof.** By Proposition 6.1 it is sufficient to construct a sequence \(\{\psi_n\}\) in form (6.3) that is Lipschitz uniformly on \([\alpha, \beta]\) and \(\lim_{n \to \infty} \psi_n(\lambda) = \psi(\lambda)\) uniformly on \([0, \beta]\) for any \(\beta > \alpha > 0\). In view of (1.5), we can write

\[
\psi(\lambda) = -a + b_n \lambda + c\lambda^2 + \int_{(0,\infty)} (e^{-\lambda u} - 1 + \lambda u 1_{u \leq \sqrt{n}}) m(du),
\]

where

\[
b_n = b - \int_{(1, \sqrt{n})} um(du).
\]

Observe that \(|b_n| \leq |b| + m(1, \infty)\sqrt{n}\). Let \(\gamma_{1,n} = n\) and \(g_{1,n}(z) = (1 - n^{-2}a)z\). Let \(\psi_{1,n}(\lambda)\) be defined by (6.3) with \((\gamma_n, g_n)\) replaced by \((\gamma_{1,n}, g_{1,n})\). Then we have \(\psi_{1,n}(\lambda) = -a(1 - \lambda/n)\). Following the proof of Proposition 4.4 in Li (2011, p.93) one can find a sequence of positive numbers \(\{\alpha_{2,n}\}\) and a sequence of probability generating functions \(\{g_{2,n}\}\) so that the function \(\psi_{2,n}(\lambda)\) defined by (6.3) from \((\alpha_{2,n}, g_{2,n})\) is given by

\[
\psi_{2,n}(\lambda) = b_n \lambda + \frac{1}{2n} (|b_n| - b_n) \lambda^2 + c\lambda^2 + \int_{(0, \sqrt{n})} (e^{-\lambda u} - 1 + \lambda u) m(du).
\]

Let \(\gamma_n = \gamma_{1,n} + \gamma_{2,n}\) and \(g_n(z) = \gamma_n^{-1} [\gamma_{1,n} g_{1,n}(z) + \gamma_{2,n} g_{2,n}(z)]\). Then the sequence \(\psi_n(\lambda)\) defined by (6.3) is equal to \(\psi_{1,n}(\lambda) + \psi_{2,n}(\lambda)\), which clearly possesses the required properties. \(\square\)
Proof of Theorem 1.16. By Proposition 2.2, the generalized Lamperti transform $Y = L_\theta(X)$ is a Lévy process with Laplace exponent $-\psi$ stopped at $0$. By Proposition 6.2, there is a sequence $\{\phi_n\}$ in form $6.2$ so that $\lim_{n \to \infty} \phi_n(\lambda) = \psi(\lambda)$ for $\lambda \geq 0$. By adjusting the parameters, we may assume the probability distribution $\{b_1^{(n)} : i = 0, 1, 2, \ldots, \infty\}$ satisfies $b_1^{(n)} = 0$. Let $Z_n = (Z_n(t) : t \geq 0)$ be a compound Poisson process on the state space $\{0, \pm 1, \pm 2, \ldots, \infty\}$ with $Q$-matrix defined by

$$
\rho_n(i, j) = \begin{cases} 
    b_{i+1}^{(n)}, & i + 1 \leq j < \infty, \\
    -1, & i = j < \infty, \\
    b_0^{(n)}, & i - 1 = j < \infty, \\
    0, & \text{otherwise.}
\end{cases}
$$

Then $Z_n$ has Laplace exponent $e^\lambda[e^{-\lambda} - g_n(e^{-\lambda})]$. Let $T_n = \inf\{t \geq 0 : Z_n(t) = 0\}$ and let $Y_n = (Z_n(t \wedge T_n) : t \geq 0)$ be the stopped process. Set $Z^{(n)}(t) = n^{-1}Z_n(\gamma_nt)$. The rescaled compound Poisson process $Z^{(n)} = (Z^{(n)}(t) : t \geq 0)$ has Laplace exponent

$$
e^\lambda/n\phi_n(\lambda) = n\gamma_n e^{\lambda/n}[e^{-\lambda/n} - g_n(e^{-\lambda/n})].$$

Let $T^{(n)} = \inf\{t \geq 0 : Z^{(n)}(t) = 0\}$ and let $Y^{(n)} = (Z^{(n)}(t \wedge T^{(n)}) : t \geq 0)$ be the stopped process. The inverse generalized Lamperti transforms $X_n := J_\theta(Y_n)$ and $X^{(n)} := J_\theta(Y^{(n)})$ can be defined similarly as in the introduction. By a simple extension of Theorem 2.1 in Chen et al. (2008), one can see $X_n$ is a discrete-state nonlinear branching process with $Q$-matrix given by

$$q_n(i, j) = \begin{cases} 
i^\theta b_{i+1}^{(n)}, & i + 1 \leq j < \infty, i \geq 1, \\
-1, & 1 \leq i = j < \infty, \\
i^\theta b_0^{(n)}, & 0 \leq i - 1 = j < \infty, \\
0, & \text{otherwise.}
\end{cases}$$

Then $X^{(n)}$ is a rescaled discrete-state nonlinear branching process. Since

$$\lim_{n \to \infty} e^{\lambda/n}\phi_n(\lambda) = \lim_{n \to \infty} \phi_n(\lambda) = \psi(\lambda), \quad \lambda \geq 0,$$

by Proposition 6 in Caballero et al. (2009) we see $Y^{(n)} \to Y$ weakly in $(D, d_\infty)$. By a slight generalization of Proposition 5 in Caballero et al. (2009), one can see the transformation $J_\theta$ is continuous on $(D, d_\infty)$. Then $X^{(n)} \to X = J_\theta(Y)$ in $(D, d_\infty)$. \qed

Proof of Theorem 1.4. Let $X = J_\theta(Y)$ be constructed as in Proposition 2.1. Let $T_y = \inf\{t \geq 0 : Y_t = y\}$. By Theorem 3.12 in Kyprianou (2006, p.81) we have $P(T_y < \infty) = e^{-\alpha(x-y)}$ for $0 \leq y \leq x$. Since $Y = (Y_t : t \geq 0)$ has no negative jumps, we have $Y_t > y$ for $0 \leq t < T_y$. By Proposition 2.1, for $0 < y \leq x$ we have $\tau_y = \alpha(T_y) < \infty$ if and only if $T_y < \infty$. Then (1.8) holds. Clearly, on the event $\{T_0 < \infty\}$ we have $X_\infty = 0$. By the property of the Lévy process, on the event $\{T_0 = \infty\}$ we have $\lim_{t \to \infty} Y_t = \infty$ and hence $X_\infty = \infty$. Then we get (1.9). \qed

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