Supersymmetric index on $T^2 \times S^2$ and elliptic genus

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Abstract

We study partition function of four-dimensional $\mathcal{N} = 1$ supersymmetric field theory on $T^2 \times S^2$. By applying supersymmetry localization, we show that the $T^2 \times S^2$ partition function is given by elliptic genus of certain two-dimensional $\mathcal{N} = (0,2)$ theory. As an application, we discuss a relation between 4d Seiberg duality duality and 2d $(0,2)$ triality, proposed by Gadde, Gukov and Putrov. In other examples, we identify 4d theories giving elliptic genera of K3, M-strings and E-strings. In the example of K3, we find that there are two 4d theories giving the elliptic genus of K3. This would imply new four-dimensional duality.
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1 Introduction

Two-dimensional conformal field theories (CFTs) have infinite dimensional symmetry while higher dimensional conformal symmetry is finite dimensional. This is one of reasons why higher dimensional CFTs are less under control compared to the 2d CFTs. It would be nice if some classes of higher dimensional CFTs are related to 2d CFTs in some ways. In this paper, we illustrate that some observables in 4d $\mathcal{N} = 1$ supersymmetric gauge theory at infrared fixed point are described by 2d CFTs. Specifically we study partition function of 4d $\mathcal{N} = 1$ supersymmetric gauge theory with R-symmetry on $T^2 \times S^2$. When we impose appropriate boundary conditions, this partition function is interpreted as the following supersymmetric index \[ Z_{T^2 \times S^2} = \text{Tr} \left[ (-1)^F q^F x^{J_3} \prod_a t_a^{F_a} \right], \] (1.1)

where $F$ is Fermion number, $P$ is momentum along spatial $S^1$ of $T^2$, $J^3$ is angular momentum along $S^2$ and $F_a$ is flavor charge. This index formula is reminiscent of elliptic genus

\[ Z_{T^2} = \text{Tr}_R \left[ (-1)^F q^H \bar{q}^{\bar{H}} \prod_a t_a^{F_a} \right] = \text{Tr}_R \left[ (-1)^F q^P \prod_a t_a^{F_a} \right], \] (1.2)
which is equivalent to partition function of supersymmetric theory on $T^2$ with appropriate boundary conditions. Indeed, the work [1] has shown that the partition function on $T^2 \times S^2$ for theories only with chiral multiplets is exactly the same as elliptic genus of certain $\mathcal{N} = (0, 2)$ theory if we identify $J^3$ in the 4d with a flavor symmetry in the 2d. Here we show that this is true also for theories including vector multiplets by using supersymmetry localization [3]. Namely we will find that the partition function of 4d $\mathcal{N} = 1$ supersymmetric gauge theory on $T^2 \times S^2$ is exactly the same as the one of certain 2d $\mathcal{N} = (0, 2)$ supersymmetric gauge theory on $T^2$, which has been recently studied well in [4, 5, 6]. By using the recent result on the elliptic genus, we find that the $T^2 \times S^2$ partition function is described by Jeffrey-Kirwan residue formula as in [4],

$$Z_{T^2 \times S^2} = \frac{1}{|W|} \sum_{u_* \in M_{\text{sing}}^*} \text{JKRes}_{u_*} (Q(u_*), \eta) \ Z_{1\text{-loop}}(\tau, u, \sigma, \xi_a), \quad (1.3)$$

where we will give several definitions in next section.

Our result shows that if we consider certain 4d supersymmetric gauge theory on $T^2 \times S^2$, then we have corresponding 2d supersymmetric gauge theory on $T^2$, which gives the same partition function. This fact enables us to find nontrivial relations between properties of 4d and 2d supersymmetric gauge theories. For instance, we will discuss that $(0, 2)$ triality [7] in two dimensions, proposed by Gadde-Gukov-Putrov, comes from 4d Seiberg duality [8]. In other examples, we identify 4d theories giving elliptic genera of K3, M-strings and E-strings. In the example of K3, we find that there are two 4d theories giving the elliptic genus of K3. This would imply new four-dimensional duality.

This paper is organized as follows. In section 2 we summarize our formula on supersymmetric partition function on $T^2 \times S^2$. In section 3 we construct 4d $\mathcal{N} = 1$ supersymmetric theory on $T^2 \times S^2$. This section is almost review of the previous works [1, 9, 10]. In section 4 we show that the supersymmetric partition function on $T^2 \times S^2$ is equivalent to elliptic genus of certain 2d $\mathcal{N} = (0, 2)$ theory. In section 5 we discuss extended supersymmetric cases. In section 6 we give several interesting examples. Section 7 is devoted to conclusion and discussions.

Note added

When our paper was ready for submission to the arXiv, there appeared a paper [11] which has some overlaps with ours.

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1 Although the authors of [2] also discussed this case, they did not obtain final expression of the partition function on $T^2 \times S^2$. More concretely, localization locus in our setup is labeled by holonomies along $T^2$ and gaugino zero modes. Hence the partition function is given by integration over the holonomies and gaugino zero modes, and we need to determine the integral contour. However, their analysis ignored the gaugino zero modes and did not reach to final result. Here we obtain final formula for the partition function by fully taking into account the localization locus including the gaugino zero modes.
2 Summary of our main result

In this section we summarize our formula of partition function of 4d $\mathcal{N} = 1$ supersymmetric gauge theory on $T^2 \times S^2$. If we take appropriate boundary conditions, then this partition function is denoted as the supersymmetric index \[ Z_{T^2 \times S^2} = \text{Tr} \left[ (-1)^F q^P x^J \prod_a t_a^F \right], \]
which we will derive in sec. 3.5. The parameters $q, x$ and $t_a$ are given by
\[ q = e^{2\pi i \tau}, \quad x = e^{2\pi i \sigma}, \quad t_a = e^{2\pi i \xi_a}, \tag{2.1} \]
where $(\tau, \sigma)$ and $\xi_a$ are complex structures of $T^2 \times S^2$ and fugacity of flavor symmetry under consideration, respectively. Then we will show that the partition function on $T^2 \times S^2$ is given by
\[ Z_{T^2 \times S^2} = \frac{1}{|W|} \sum_{u_* \in \mathcal{M}_{\text{sing}}^*} \text{JKRes}_{u=\eta_*} (Q(u_*), \eta) \ Z_{1\text{-loop}}(\tau, u, \sigma, \xi_a), \]
where $|W|$ is the order of the Weyl group of gauge group $G$. Several definitions are in order. First $Z_{1\text{-loop}}$ is roughly one-loop determinant around saddle point of localization and $u$ is left-moving component of a holonomy along $T^2$, which takes values in Cartan subalgebra of the gauge group. As we will see, we can rewrite $Z_{1\text{-loop}}$ as a product of one-loop determinants of 2d $\mathcal{N} = (0,2)$ multiplets on $T^2$ with charges $Q(u)$ and fugacities $\sigma, \xi_a$ of certain flavor symmetries. $\mathcal{M}_{\text{sing}}^*$ is set of poles of $Z_{1\text{-loop}}$ satisfying certain conditions. We will also explain these more precisely in next subsection.

$\text{JKRes}_{u=\eta_*} (Q(u_*), \eta)$ denotes a residue operation called the Jeffrey-Kirwan (JK) residue \[ [12, 13], \] which will be defined in sec. 2.2. The parameter $\eta$ takes values in the dual space of Cartan subalgebra and we have a freedom to take $\eta$ in arbitrary nonzero values. Although the each term in the summand depends on $\eta$, we can show that the total expression is independent of choice of $\eta$.

2.1 One-loop determinant and singularities

Here we explain our formula for one-loop determinant in detail. We denote $Z_{1\text{-loop}}$ as
\[ Z_{1\text{-loop}}(\tau, u, \sigma, \xi_a) = Z_V(\tau, u) \prod_i Z_{R_i}^{(r_i)}(\tau, u, \sigma, \xi_a), \tag{2.2} \]
where $Z_V$ is the contribution from 4d $\mathcal{N} = 1$ vector multiplet while $Z_{R_i}^{(r_i)}$ is the one of 4d $\mathcal{N} = 1$ chiral multiplet with the representation $R_i$ of $G$ and magnetic charge $r_i$. The chiral multiplets on $T^2 \times S^2$ generally have the magnetic charge because we need $R$-symmetry background gauge field with monopole configuration in order to keep supersymmetry \[ [9, 11]. \]
We can also turn on magnetic background gauge field of flavor symmetry with integer flux.
as explained in sec. 3.4. Hence if the chiral multiplet has R-charge \( r \) and flavor charge \( q_f \), and turn on the magnetic flux \( g \) of flavor symmetry, then the magnetic charge \( r \) is given by

\[
r = r + q_f g,
\]

which should be integer in order to satisfy quantization condition for the magnetic flux on \( S^2 \). The contribution from the vector multiplet with gauge group \( G \) is simply given by

\[
Z_V(\tau, u) = \left( \frac{2\pi \eta^2(q)}{i} \right)^{|G|} \prod_{\alpha \in G} \frac{i\theta_1(\tau|\alpha(u))}{\eta(q) \prod_{a=1}^{|G|} du_a}, \tag{2.4}
\]

where \( |G| \) and \( \alpha \) are the rank and root of the gauge group \( G \), respectively. Note that this is exactly the same as the contribution coming from 2d \( N = (0, 2) \) vector multiplet in elliptic genus formula. The contribution from the chiral multiplet depends on the magnetic charge:

\[
Z_R^{(r)}(\tau, u, \sigma, \xi) = \begin{cases} 
\prod_{m=-\frac{|r|}{2}}^{\frac{|r|}{2}} Z_{\Lambda, R}(\tau, u, m\sigma + \sum_a q_f^a \xi_a) & \text{for } r > 1 \\
1 & \text{for } r = 1, \\
\prod_{m=-\frac{|r|}{2}}^{\frac{|r|}{2}} Z_{\Phi, R}(\tau, u, m\sigma + \sum_a q_f^a \xi_a) & \text{for } r < 1 
\end{cases} \tag{2.5}
\]

where \( Z_{\Lambda, R} \) and \( Z_{\Phi, R} \) are the same as contributions from 2d \( N = (0, 2) \) Fermi and chiral multiplets, respectively. These are explicitly given by

\[
Z_{\Lambda, R}(\tau, u, y) = \prod_{\rho \in R} \frac{i\theta_1(\tau|\rho(u) + y)}{\eta(q)}, \quad Z_{\Phi, R}(\tau, u, y) = \prod_{\rho \in R} \frac{i\eta(q)}{\theta_1(\tau|\rho(u) + y)}. \tag{2.6}
\]

When we have chiral multiplets with \( r > 1 \), the one-loop determinant has poles. These poles are defined as hyperplanes in \( \mathbb{C}^{|G|} \). Denoting such hyperplane from each \( Z_{\Phi, R} \) as \( H_i \), the hyperplane \( H_i \) is given by

\[
H_i = \{ \rho_i(u) + K_i(\xi) = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}} \}, \tag{2.7}
\]

where \( K_i \) is weight of flavor symmetry group. Using \( H_i \), we define the singular hyperplane

\[
\mathcal{M}_{\text{sing}} = \bigcup_i H_i, \tag{2.8}
\]

and the set of points \( u_s \) in \( \mathcal{M}^*_{\text{sing}} \):

\[
\mathcal{M}^*_{\text{sing}} = \{ u_s \in \mathcal{M}_{\text{sing}} | \text{at least } |G| \text{ linearly independent } H_i \text{'s meet at } u_s \}. \tag{2.9}
\]

It is convenient to introduce the set \( Q(u_s) \) of charges for given \( u_s \in \mathcal{M}^*_{\text{sing}} \) by

\[
Q(u_s) = \{ \rho_i | u_s \in H_i \}. \tag{2.10}
\]

For a technical reason, we assume that the set \( Q(u_s) \) is contained in a half-space of the weight space. This condition is called projective \( [14] \). If the number of hyperplanes intersecting at \( u_s \) is \( |G| \), then the hyperplane arrangement is always projective. Also, even if it is not projective, we can usually deform fugacities to reduce the number of hyperplanes at \( u_s \) and make the arrangement projective.

\[\text{[Footnote]}\]

For simplicity, we assume that non-Abelian part of \( G \) is simply-connected.
2.2 Jefferey-Kirwan residue formula

Here we explain the Jefferey-Kirwan residue operation for projective hyperplane arrangement. Suppose that $n$ linearly independent hyperplanes $H_i$'s meet at $\sum u_i = 0$:

$$H_i = \{ u \in \mathbb{C}^{[G]} | Q_i(u) = 0 \}, \quad (2.11)$$

where $i = 1, \cdots, n$. If this hyperplane arrangement is projective, then JK residue operation is defined by

$$\text{JKRes}_{\sum u_i = 0} (\{Q_i, \eta\}) \frac{d u_1 \wedge \cdots \wedge d u_{[G]}}{Q_{j_1}(u) \cdots Q_{j_{[G]}}(u)} = \begin{cases} |\det(Q_{j_1}, \cdots, Q_{j_{[G]}})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_{[G]}}) \\ 0 & \text{otherwise} \end{cases}, \quad (2.12)$$

where $Q_i$ is the vector, whose $a$-th component is given by coefficient of $u_a$.

For rank-1 case, the JK residue formula is especially simpler:

$$\text{JKRes}_{\sum u_i = 0} (\{q\}, \eta) \frac{d u}{u} = \begin{cases} \text{sign}(q) & \text{if } \eta q > 0 \\ 0 & \text{if } \eta q < 0 \end{cases}, \quad (2.13)$$

Then the partition function is given by

$$Z_{T^2 \times S^2} = \frac{1}{|W|} \sum_{u_+ \in M^+_{\text{sing}}} \frac{1}{2 \pi i} \oint_{u = u_+} Z_{1-\text{loop}} = -\frac{1}{|W|} \sum_{u_- \in M^-_{\text{sing}}} \frac{1}{2 \pi i} \oint_{u = u_-} Z_{1-\text{loop}}, \quad (2.14)$$

where $M^\pm_{\text{sing}}$ is points in $M^*_{\text{sing}}$ coming from charges $\pm$ of the gauge group.

3 Four-dimensional $\mathcal{N} = 1$ theory on $T^2 \times S^2$

In this section, we review\footnote{We can repeat similar argument for generic $u_*$ just by shifting the coordinates.} a construction of 4d $\mathcal{N} = 1$ supersymmetric theory on $T^2 \times S^2$\footnote{We mainly follow a notation of [15].} and discuss some of its properties.

3.1 $T^2 \times S^2$

3.1.1 Definition and complex structures

In this paper we study supersymmetric gauge theory on $T^2 \times S^2$. We regard this space as a quotient one of $\mathbb{C} \times S^2$. Denoting $w$ and $z$ as the complex coordinates of $\mathbb{C}$ and $S^2$, respectively, we define $T^2 \times S^2$ as the following identification:

$$(w, z) \sim (w + 2 \pi, e^{2\pi i \alpha} z) \sim (w + 2 \pi \tau, e^{2\pi i \beta} z), \quad (3.1)$$
where $\tau = \tau_1 + i\tau_2$ is the modular parameter of $T^2$ and $(\alpha, \beta)$ are real parameters with identifications $\alpha \sim \alpha + 1$, $\beta \sim \beta + 1$. We also introduce

$$\sigma = \tau \alpha - \beta.$$  \hfill (3.2)

There are two complex structure moduli $\tau, \sigma$, and we have the choice up to symmetries generated by

$$S : (\tau, \sigma) \mapsto \left( -\frac{1}{\tau}, \frac{\sigma}{\tau} \right), \quad T : (\tau, \sigma) \mapsto (\tau + 1, \sigma),$$

$$U : (\tau, \sigma) \mapsto (\tau, \sigma + \tau), \quad V : (\tau, \sigma) \mapsto (\tau, \sigma + 1).$$  \hfill (3.3)

The metric and frames on $T^2 \times S^2$ are given by

$$ds^2 = dwd\bar{w} + \frac{4}{(1 + z\bar{z})^2}dzd\bar{z}, \quad e^1 = dw, \quad e^2 = \frac{2}{1 + |z|^2}dz.$$  \hfill (3.4)

It is sometimes convenient to use real coordinates $(x, y)$ and $(\theta, \varphi)$:

$$w = x_4 + \tau x_3, \quad z = \tan \frac{\theta}{2}e^{i(\varphi + \alpha x_4 + \beta x_3)}.$$  \hfill (3.5)

Then the metric and frames become

$$ds^2 = (dx_4 + \tau dx_3)^2 + \tau_2^2 dx_2^2 + d\theta^2 + \sin^2 \theta(d\varphi + \alpha dx_4 + \beta dx_3)^2,$$

$$e^1 = dx_4 + \tau dx_3, \quad e^2 = e^{i(\varphi + \alpha x_4 + \beta x_3)}d\theta + i\sin \theta(d\varphi + \alpha dx_4 + \beta dx_3).$$  \hfill (3.6)

### 3.1.2 $T^2 \times S^2$ as a supersymmetric background

As discussed in [16], off-shell supersymmetric field theory with an $R$-symmetry on a curved background can be obtained by freezing configurations of gravity multiplet in off-shell new minimal supergravity (SUGRA) [17, 18]. Bosonic fields in the gravity multiplet in the new-minimal off-shell SUGRA consist of a metric and two auxiliary vector fields $A_\mu$, $V_\mu$ where $A_\mu$ is an $R$-symmetry gauge field, while $V_\mu$ satisfies $\nabla^\mu V_\mu = 0$. In order to keep SUSY in fixed curved background, we should impose vanishing of variation of the gravitino ($\Psi_\mu, \bar{\Psi}_\mu$):

$$\delta \Psi_\mu = (\nabla_\mu - iA_\mu)\zeta + iV_\mu \zeta + iV^\nu \sigma_{\mu\nu} \zeta = 0,$$

$$\delta \bar{\Psi}_\mu = (\nabla_\mu + iA_\mu)\bar{\zeta} - iV_\mu \bar{\zeta} - iV^\nu \bar{\sigma}_{\mu\nu} \bar{\zeta} = 0,$$  \hfill (3.7)

where the variation parameters $\zeta$ and $\bar{\zeta}$ have the $R$-charges as $+1$ and $-1$, respectively.

One can show that our background $T^2 \times S^2$ solves the Killing spinor equations (3.7) by

$$V = 0, \quad A = -\frac{i}{2} \frac{\bar{z}dz - zd\bar{z}}{2(1 + z\bar{z})} = \frac{1}{2}(1 - \cos \theta)(d\varphi + \alpha dx_4 + \beta dx_3),$$

---

5 This corresponds to take $\Omega = 1$, $h = 0$ and $c = \frac{2}{1 + |z|^2}$ in the notation of [15].

6 We have used a freedom in choosing $A$ to be real. This corresponds to take $s = 1$ and $\kappa = 0$ in the notation of [15].
\[ \zeta_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{\zeta}^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (3.8)

This fact forces each field to have magnetic charge, whose value is equal to R-charge and hence the R-charges must be integer due to quantization condition for magnetic flux on \( S^2 \).

### 3.2 Vector multiplet

The Lagrangian for the vector multiplet on \( T^2 \times S^2 \) is

\[
\mathcal{L}_{\text{vec}} = \text{Tr} \left[ \frac{1}{4} F_{\mu \nu} F_{\mu \nu} - \frac{1}{2} D^2 + i \lambda \sigma^\mu D_\mu \lambda + i \tilde{\lambda} \tilde{\sigma}^\mu D_\mu \lambda \right],
\] (3.9)

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \), and

\[
D_\mu = \nabla_\mu - iA_\mu - iq_R A_\mu.
\] (3.10)

The R-charges \( q_R \) of the fields \((A_\mu, \lambda, \tilde{\lambda}, D)\) are \((0, 1, -1, 0)\). This action is invariant under

\[
\begin{align*}
\delta A_\mu &= i \zeta \sigma_\mu \tilde{\lambda} + i \tilde{\zeta} \tilde{\sigma}_\mu \lambda, \\
\delta \lambda &= F_{\mu \nu} \sigma^{\mu \nu} \zeta + iD \zeta, \\
\delta \tilde{\lambda} &= F_{\mu \nu} \tilde{\sigma}^{\mu \nu} \tilde{\zeta} - iD \tilde{\zeta}, \\
\delta D &= -\zeta \sigma^\mu D_\mu \tilde{\lambda} + \tilde{\zeta} \tilde{\sigma}^\mu D_\mu \lambda,
\end{align*}
\] (3.11)

where \( \zeta \) and \( \tilde{\zeta} \) are the commuting spinors. Note that \( \lambda \) and \( \tilde{\lambda} \) are independent since we are working in the 4d Euclidean signature. Hence, 4d \( \mathcal{N} = 1 \) SUSY requires that \((A_\mu, D)\) are not hermitian or anti-hermitian a priori. In order to insure the action to be real, we take integral contour of path integral as

\[
(A_\mu, D) = (A_\mu, -D).
\] (3.12)

Namely we practically regard \( A_\mu(D) \) as (anti-)hermitian after computing the SUSY variation.

When we have a \( U(1) \)-part in gauge group, we can also consider the FI-term:

\[
S_{\text{FI}} = -i \zeta \int d^4 x \sqrt{g} D.
\] (3.13)

Note that the Lagrangian for the vector multiplet is \( \delta \)-exact,

\[
\mathcal{L}_{\text{vec}} = \mathcal{L}_{\text{vec}}^{(+)} + \mathcal{L}_{\text{vec}}^{(-)},
\] (3.14)

---

\footnote{As explained in \[3.4\] we can also introduce background vector multiplet for global symmetry, which can give additional magnetic charge. Then the R-charge is not necessary integer because the quantization for the magnetic flux impose only the sum of the magnetic charges by the R-symmetry and global symmetry to be integer.}
where
\[
L_{\text{vec}}^{(+)} = -\delta \zeta \left( \frac{1}{4|\zeta|^2} \text{Tr}(\delta \zeta \lambda)^\dagger \lambda \right) = \text{Tr} \left[ \frac{1}{4} F_{\mu\nu}^{(+)} F^{(+)}_{\mu\nu} - \frac{1}{4} D^2 + \frac{i}{2} \lambda \sigma^\mu D_{\mu} \lambda \right],
\]
\[
L_{\text{vec}}^{(-)} = -\delta \tilde{\zeta} \left( \frac{1}{4|\zeta|^2} \text{Tr}(\delta \zeta \tilde{\lambda})^\dagger \tilde{\lambda} \right) = \text{Tr} \left[ \frac{1}{4} F_{\mu\nu}^{(-)} F^{(-)}_{\mu\nu} - \frac{1}{4} D^2 + \frac{i}{2} \tilde{\lambda} \tilde{\sigma}^\mu D_{\mu} \tilde{\lambda} \right],
\] (3.15)

with
\[
F_{\mu\nu}^{(\pm)} = \frac{1}{2} (F \pm \ast F)_{\mu\nu}.
\] (3.16)

### 3.3 Chiral multiplet

The Lagrangian for the chiral multiplet is
\[
L_{\text{chi}} = D_{\mu} \tilde{\phi} D^\mu \phi + \frac{r}{2} \tilde{\phi} \phi + \tilde{\phi} D \phi - \tilde{F} F + i\tilde{\psi} \tilde{\sigma}^\mu D_{\mu} \psi + i\sqrt{2}(\tilde{\phi} \lambda \psi - \tilde{\psi} \tilde{\lambda} \phi),
\] (3.17)

where we have assigned \(R\)-charges \((r, r-1, r-2, -r, -r+1, -r+2)\) to \((\phi, F, \tilde{\phi}, \tilde{\psi}, \tilde{F})\).

The supersymmetric transformation is
\[
\delta \phi = \sqrt{2} \zeta \psi,
\]
\[
\delta \psi = \sqrt{2} F \zeta + i\sqrt{2}(\sigma^\mu \zeta) D_{\mu} \phi,
\]
\[
\delta F = i\sqrt{2} \tilde{\zeta} \tilde{\sigma}^\mu D_{\mu} \psi - 2i(\tilde{\zeta} \lambda) \phi,
\]
\[
\delta \tilde{\phi} = \sqrt{2} \tilde{\zeta} \tilde{\psi},
\]
\[
\delta \tilde{\psi} = \sqrt{2} \tilde{F} \tilde{\zeta} + i\sqrt{2}(\tilde{\sigma}^\mu \zeta) D_{\mu} \tilde{\phi},
\]
\[
\delta \tilde{F} = i\sqrt{2} \tilde{\zeta} \sigma^\mu D_{\mu} \tilde{\psi} + 2i\tilde{\phi}(\zeta \lambda).
\] (3.18)

Again although \((\phi, F, \tilde{\phi}, \tilde{F})\) are independent complex fields a priori, we take the following integral contour
\[
(\phi, F, \tilde{\phi}, \tilde{F})^\dagger = (\tilde{\phi}, -\tilde{F}, \phi, -F).
\] (3.19)

Note that we can rewrite the Lagrangian as SUSY variation exact:
\[
L_{\text{chi}} = \delta \zeta \left( \frac{1}{2|\zeta|^2} \left[ (\delta \zeta \psi)^\dagger \psi - \tilde{\psi}(\delta \zeta \tilde{\psi})^\dagger \right] + 2i\tilde{\phi} \zeta^\dagger \lambda \phi \right).
\] (3.20)

### 3.4 Background vector multiplet and boundary condition

We can introduce background vector multiplet for global symmetries with keeping supersymmetry. For example, when we consider an Abelian flavor symmetry \(U(1)_f\), supersymmetric configuration for the background vector multiplet is given by
\[
v_{\mu} dx^\mu = v_{\omega} d\omega + v_{\bar{\omega}} d\bar{\omega} - i\frac{\bar{z} dz - zd\bar{z}}{2(1 + z\bar{z})}
\]
\[
= a_4 dx_4 + a_3 dx_3 + \frac{g}{2}(1 - \cos \theta)(d\varphi + adx_4 + \beta dx_3),
\] (3.21)
and

\[ D = \frac{q fg}{2}, \]

(3.22)

where \((a_4, a_3)\) is the flat connection along \(T^2\) direction and this gives the fugacity \(\xi\) of the flavor symmetry introduced in (2.1) by

\[ \xi = \tau a_4 - a_3. \]

(3.23)

Then the Lagrangian for the chiral multiplet with charge \(q_f\) for \(U(1)_f\) slightly changes to

\[ \mathcal{L}_{\text{chi}} = D_\mu \tilde{\phi} D^\mu \phi + \frac{r}{2} \tilde{\phi} \phi + \tilde{\phi} D \phi - \tilde{F} F + i \tilde{\psi} \tilde{\sigma}^\mu D_\mu \psi + i \sqrt{2}(\tilde{\phi} \lambda \psi - \tilde{\psi} \lambda \phi), \]

(3.24)

where the covariant derivatives include the background gauge field \(v_\mu\) and \(r\) is the total magnetic charge defined by (2.3),

\[ r = r + q fg. \]

Because of the quantization condition for the magnetic flux, the total magnetic charge \(r\) has to be integer. Note that when we turn on the background magnetic gauge field of the global symmetry, the R-charge \(r\) can be non-integer depending on the magnetic charges coming from the other global symmetries. Under the identification (3.1), every field \(\Phi\) has the twisted boundary condition:

\[ \Phi \sim e^{i \pi r \alpha} \Phi, \quad \Phi \sim e^{i \pi r \beta} \Phi. \]

(3.25)

### 3.5 Supersymmetry algebra and index formula

Denoting \(\delta = \delta_\zeta + \delta_\bar{\zeta}\), the supersymmetry transformation generates

\[ \{ \delta_\zeta, \delta_\zeta \} = \{ \delta_\bar{\zeta}, \delta_\bar{\zeta} \} = 0, \quad [\delta_\zeta, \delta_K] = [\delta_\bar{\zeta}, \delta_K] = 0, \quad \{ \delta_\zeta, \delta_\bar{\zeta} \} = 2i \delta_K, \]

(3.26)

where

\[ \delta_K = \mathcal{L}_K - i K^\mu A_\mu - i K (q_R A_\mu + q_f v_\mu). \]

\(\mathcal{L}_K\) is a Lie derivative along the vector field \(K\) given by

\[ K = \partial_{\bar{\psi}} = \frac{1}{2i \tau_2} (\tau \partial_4 - \partial_3 - \sigma \partial_\varphi). \]

(3.28)

If we identify the \(x_3\)-direction as “time” circle and \(x_4\)-direction as “spatial” circle, then our partition function on \(T^2 \times S^2\) can expressed as

\[ Z_{T^2 \times S^2} = \text{Tr} \left[ (-1)^F e^{-2\pi H} \right], \]

(3.29)

where \(H\) is Hamiltonian. The SUSY algebra (3.26) implies that the partition function is contributed only by the states satisfying

\[ H = -i (\tau P + \sigma J_3 + (\tau a_4 - a_3)q_f), \]

(3.30)

\[ ^8\text{Here we assume discreteness of the energy spectrum. Otherwise, the partition function would be non-holomorphic as in elliptic genera for non-compact manifolds (see e.g. \cite{19}).} \]
while the other states are canceled between bosonic and fermionic states. Thus our partition function is interpreted as the index

\[ Z_{T^2 \times S^2} = \text{Tr} \left[ (-1)^F q^P x J^3 \prod_a t^F_a \right] . \]

### 3.6 Anomaly cancellation

Since we have the non-trivial background gauge fields, expressions for anomaly are slightly modified:

\[ \partial_\mu J^\mu \propto \text{Tr} \left[ \epsilon^{\mu\nu\rho\sigma} (F_{\mu\nu} + (r - 1) F_{\mu\nu}^R + \sum_a q_f^{(a)} J^{(a)} (F_{\rho\sigma} + (r - 1) F_{\rho\sigma}^R + \sum_a q_f^{(a)} J^{(a)}) \right], \quad (3.31) \]

where \( F_{\mu\nu}, F_{\mu\nu}^R \) and \( J^{(a)} \) are field strengths of gauge symmetry, R-symmetry and flavor symmetries, respectively. Since the background field strengths are nontrivial only along \( S^2 \), the divergence \( \partial_\mu J^\mu \) of the current includes the terms proportional to

\[ \text{Tr} \left[ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right], \quad (r - 1) \text{Tr} \left[ \epsilon^{pq} F_{pq} \right] . \quad (3.32) \]

where \( r = r + \sum_a q_f^{(a)} g^{(a)} \) and indices \( p, q \) denote coordinates of \( T^2 \) direction. The 1st one gives the standard anomaly formula in 4d without nontrivial background while the 2nd one is particular for our setup. Noting the number of zero modes along \( S^2 \) is \( r - 1 \) for 2d positive chirality and \( 1 - r \) for 2d negative chirality, we can easily see that this is the same as anomaly of 2d theory obtained by dimensional reduction along \( S^2 \). Thus, we also require the 2d gauge anomaly cancellation in addition to the standard 4d cancellation condition.

The 2d gauge anomaly cancellation condition becomes simplified after we impose the 4d gauge anomaly cancellation condition. For example, the standard 4d cancellation condition for \( U(1)_R \times G \times G \) type anomaly is

\[ \sum_i (r^{(i)} - 1) T_{R_i} + T_{\text{adj}} = 0, \quad (3.33) \]

while the 2d condition for \( G \times G \) type anomaly is

\[ \sum_i (r^{(i)} - 1) T_{R_i} + T_{\text{adj}} = 0. \quad (3.34) \]

If we use the 4d condition, then the 2d condition becomes

\[ \sum_i \sum_a q_f^{(a)} g^{(a)} T_{R_i} = 0, \quad (3.35) \]

which is nothing but the conservation condition.

---

9 We have some other equivalent ways to get the same index. One way is to consider the identifications \((w, z) \sim (w + 2\pi, z) \sim (w + 2\pi \tau, z)\) and impose the twisted boundary condition \( \Phi(w + 2\pi, z) = x^{-J_3} \prod_a e^{-2\pi i a} \Phi(w, z) \), or \( \Phi(x_3, x_4, z) = e^{-2\pi i a} \prod_a e^{-2\pi i a} \Phi(x_3, x_4, z) \), \( \Phi(x_3, x_4 + 2\pi, z) = e^{-2\pi i a} \prod_a e^{-2\pi i a} \Phi(x_3, x_4, z) \).
4 Partition function on $T^2 \times S^2$ and elliptic genus

In this section we show that the partition function on $T^2 \times S^2$ is the same as the elliptic genus of 2d $\mathcal{N} = (0, 2)$ theory described by zero modes along $S^2$.

4.1 Sketch of derivation

Our proof takes the following steps.

1. We apply localization method to the partition function of the 4d $\mathcal{N} = 1$ SUSY theory on $T^2 \times S^2$. If we denote fields on $T^2 \times S^2$ as $\Phi_{T^2 \times S^2}$, then the SUSY localization tells us that the partition function on $T^2 \times S^2$ is independent of $Q$-exact deformation:

$$Z_{T^2 \times S^2} = \int [D\Phi_{T^2 \times S^2}] e^{-S_{T^2 \times S^2}} = \int [D\Phi_{T^2 \times S^2}] e^{-S_{T^2 \times S^2} - tQ_{T^2 \times S^2}V_{T^2 \times S^2}},$$

where $Q_{T^2 \times S^2}$ is supersymmetric transformation on $T^2 \times S^2$. Here we choose the fermionic functional $V_{T^2 \times S^2}$ such that $Q_{T^2 \times S^2}V_{T^2 \times S^2}$ becomes the action itself without the FI-term. Then we can take the limit $t \rightarrow \infty$ and saddle point method gives exact result.

2. We consider gaussian fluctuation around the saddle point $\Phi_0$ and perform its KK mode expansion along $S^2$. Then we will show that the KK mode expansion can be rewritten as gaussian fluctuation of the action of 2d $\mathcal{N} = (0, 2)$ theory on $T^2$ around $\Phi_0$:

$$Q_{T^2 \times S^2}V_{T^2 \times S^2}[\Phi_{T^2 \times S^2}]_{\text{gaussian}} = \sum_{J=j_0}^{\infty} S_{T^2}^{(J)}[\Phi_0, \Phi_{T^2}^{(J)}]_{\text{gaussian}} + (\cdots),$$

where $J$ is angular momentum along $S^2$ and $j_0$ is the one of zero modes. The symbol “$(\cdots)$” denotes non-zero modes of vector multiplets and gauge-fixing action, whose effect is trivial as shown below. Noting that the supersymmetric action on $T^2$ is also $Q$-exact, we find

$$Z_{T^2 \times S^2} = \lim_{t \rightarrow \infty} \int \prod_{J=j_0}^{\infty} [D\Phi_{T^2}^{(J)}] e^{-tQ_{T^2}V_{T^2}^{(J)}}. \tag{4.3}$$

Thus, the $T^2 \times S^2$ partition function is exactly the same as the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ theory with infinite multiplets.

---

10 More precisely, we choose $tQ_{T^2 \times S^2}V_{T^2 \times S^2} = t_v S_{\text{vec}} + t_c S_{\text{chi}}$ and consider the limits $t_v \rightarrow \infty$ and $t_c \rightarrow \infty$. However, these limits should be taken carefully in our setup contrast to usual analysis by localization. As we will see in next subsection, there is a problem similar to analysis of elliptic genus.

11 Strictly speaking, we will show this equality in a specific gauge and we have already included gauge-fixing actions in each term.
3. Fortunately, we already know formula for the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ theory thanks to the previous studies [4, 5, 6]. Hence, just by using their results, we can obtain exact expression for the $T^2 \times S^2$ partition function. As a result, contributions from non-zero modes along $S^2$ are trivial and we find

$$Z_{T^2 \times S^2} = \lim_{t \to \infty} \int [D\Phi] e^{-tQ_{T^2}V_{T^2}} = Z_{T^2}, \quad (4.4)$$

which is the elliptic genus of 2d $\mathcal{N} = (0, 2)$ theory described by zero modes along $S^2$.

4.2 Localization locus and danger of naive saddle point analysis

We would like to compute the partition function

$$Z_{T^2 \times S^2} = \int [D\Phi] e^{-S_{FI} - S_{vec} - S_{chi}}. \quad (4.5)$$

If we choose $S_{vec}$ and $S_{chi}$ as the deformation terms of the localization, then we find

$$Z_{T^2 \times S^2} = \lim_{t_v, t_c \to \infty} \int [D\Phi] e^{-S_{FI} - t_v S_{vec} - t_c S_{chi}}. \quad (4.6)$$

Imposing $S_{vec} = 0$ and $S_{chi} = 0$ give the saddle point conditions. For vector multiplet, we find

$$\mathcal{F}_{\mu\nu}^{(+)} = 0, \quad \mathcal{F}_{\mu\nu}^{(-)} = 0, \quad D = 0, \quad \lambda \sigma^\mu D_\mu \tilde{\lambda} = 0, \quad \tilde{\lambda}\tilde{\sigma}^\mu D_\mu \lambda = 0. \quad (4.7)$$

The first and second equations show that saddle point of gauge field is flat connection. The last two equations imply that the saddle points of $\lambda$ and $\tilde{\lambda}$ are zero modes and Cartan valued. Namely $\lambda$ and $\tilde{\lambda}$ are constant proportional to $\zeta$ and $\tilde{\zeta}$, respectively:

$$\lambda = \lambda^{(0)} \zeta, \quad \tilde{\lambda} = \tilde{\lambda}^{(0)} \tilde{\zeta}. \quad (4.8)$$

Saddle point for chiral multiplet is trivial:

$$\phi = 0, \quad \tilde{\phi} = 0, \quad \psi = 0, \quad \tilde{\psi} = 0, \quad F = 0, \quad \tilde{F} = 0. \quad (4.9)$$

In usual story of localization method, we can exactly compute the partition function by using naive saddle point analysis. However, in general situation, such naive saddle analysis sometimes would give ill-defined result and not make sense as seen in recent studies on elliptic genus. We discuss that this is also our case and we need to take the limits $t_v, t_c \to \infty$ more carefully. For finite $t_v$ and $t_c$, the partition function on $T^2 \times S^2$ can be written as

$$Z_{T^2 \times S^2} = \int d^G \tilde{D} \int d^2G \int f_{t_v, t_c}(u, \bar{u}, \tilde{D}) \exp \left[\frac{t_v}{2} \tilde{D}^2 - \zeta \tilde{D}\right], \quad (4.10)$$

where $\tilde{D}$ and $(u, \bar{u})$ are zero modes of $D$ and $A_\mu$ along $T^2 \times S^2$. The function $f_{t_v, t_c}(u, \bar{u}, \tilde{D})$ is the result of the path integral except $\tilde{D}$ and $(u, \bar{u})$ and has a finite value in the limit $t_c \to \infty$. 13
for arbitrary \((u, \bar{u})\) unless we take the limit \(t_v \to \infty\). If we integrate \(\tilde{D}\) out, then the partition function in the \(t_c \to \infty\) limit becomes

\[
Z_{T^2 \times S^2} = \int d^{2[G]}u \ F_{t_v}(u, \bar{u}),
\]  

(4.11)

where

\[
F_{t_v}(u, \bar{u}) = \lim_{t_c \to \infty} \int d^{2[G]}D \ f_{t_v, t_c}(u, \bar{u}, \tilde{D}) \exp \left[ \frac{t_v}{2} \tilde{D}^2 - \zeta \tilde{D} \right]
\]

\[
= C_{t_v}(u, \bar{u}) \int d^{2M*} \phi \ \exp \left[ -t_c \sum_{i=1}^{M_c} |Q_i(u)\phi_i|^2 - \frac{1}{2t_v} \left( \sum_{i=1}^{M_c} |\phi_i|^2 + \zeta \right) \right].
\]  

(4.12)

Here \(\phi_i\) is zero modes of scalar fields in chiral multiplets along \(T^2\), whose eigenvalues \(Q_i(u)\) vanish as approaching \(u \to u_*\). The prefactor \(C_{t_v}(u, \bar{u})\) is contribution from path integral except \((u, \bar{u})\) and \(\phi_i\). We can easily see that the integrations over \(\phi_i\) for \(t_v \to \infty\) diverge for \(u = u_*\) and hence we should take this limit carefully. Thus we decompose the integration over \((u, \bar{u})\) as

\[
\int_{M} d^{2[G]}u = \int_{M-\Delta} d^{2[G]}u + \int_{\Delta} d^{2[G]}u,
\]  

(4.13)

where \(\Delta\) is \(\epsilon\)-neighborhood of \(M\_{\text{sing}}\). Then we take the limit \(\epsilon \to 0\) first for finite \(t_v\) and finally take the \(t_v \to \infty\), namely

\[
Z_{T^2 \times S^2} = \lim_{t_v \to \infty} \lim_{\epsilon \to 0} \left( \int_{M-\Delta} d^{2[G]}u \ F_{t_v}(u, \bar{u}) + \int_{\Delta} d^{2[G]}u \ F_{t_v}(u, \bar{u}) \right).
\]  

(4.14)

In order to perform this procedure, we keep \(t_v\) finite in terms including the zero mode of \(D\). This is equivalent to that we do not regard the zero mode of \(D\) as fluctuation around the saddle point. In subsection \[4.3\] we will consider gaussian fluctuation of the action around the saddle point except the zero mode of \(D\) along \(S^2\) and compute its KK-mode expansion along \(S^2\). Then we will show that the action can be regarded as 2d \(\mathcal{N} = (0, 2)\) theory on \(T^2\) with infinite multiplets.

### 4.3 Gauge fixing

Here we take the gauge

\[
D^p A_p = 0,
\]  

(4.15)

where \(p\) denotes the \(T^2\)-direction. In order to construct gauge fixing action, we introduce BRST transformation as

\[
Q_B A_\mu = D_\mu c_g, \quad Q_B c_g = -\frac{i}{2} [c_g, c_g], \quad Q_B \bar{c}_g = B, \quad Q_B B = 0,
\]  

(4.16)

where \(c_g\) and \(\bar{c}_g\) are ghosts, and \(B\) is the Nakanishi-Lautrup field. Then gauge fixing action is given by

\[
L_{\text{gh}} = Q_B \text{Tr} \left[ \bar{c}_g D^p A_p \right] = B \nabla^p A_p + \bar{c}_g D^p D_p c_g = B \nabla^p A_p + \bar{c}_g D_w D_{\bar{w}} c_g.
\]  

(4.17)
4.4 Gaussian fluctuation around the saddle point

In this subsection, we study quadratic fluctuation around the localization locus. Performing KK-mode expansion along $S^2$, we show that the quadratic fluctuation is the same as the one of 2d $\mathcal{N} = (0, 2)$ supersymmetric theory on $T^2$ except non-zero modes of vector multiplet.

4.4.1 Vector multiplet

Let us expand the action around the saddle point except $D$:

$$A_\mu \to A_\mu^{(0)} + A_\mu, \quad \lambda \to \lambda^{(0)} + \lambda, \quad \tilde{\lambda} \to \tilde{\lambda}^{(0)} + \tilde{\lambda}. \quad (4.18)$$

Then we find the action up to the quadratic fluctuation as

$$\mathcal{L}_{\text{vec}}|_{\text{Gauss}} = \text{Tr} \left[ \frac{1}{4} (F_\mu^{(0)})^2 - \frac{1}{2} D^2 + i \frac{\lambda \sigma^\mu D_\mu^{(0)} \tilde{\lambda}}{2} + i \frac{\tilde{\lambda} \sigma^\mu D_\mu^{(0)} \lambda}{2} \right], \quad (4.19)$$

where $D_\mu^{(0)}$ is covariant derivative in terms of the gauge field at the saddle point and $F_\mu^{(0)} = D_\mu^{(0)} A_\nu - D_\nu^{(0)} A_\mu$.

Next we expand each field by monopole spherical harmonics:

$$A_i = \sum_{\rho=1,2} \sum_J \sum_{m=-J}^J A_{J\rho}^{\rho} C_{i,Jm}^\rho, \quad A_p = \sum_{J=0}^\infty \sum_{m=-J}^J A_{p,Jm} Y_{0,Jm}, \quad D = \sum_{J=0}^\infty \sum_{m=-J}^J D_{Jm} Y_{0,Jm},$$

$$\lambda_\alpha = \sum_{J=1}^\infty \sum_{m=-J}^J \left( \beta_{Jm} Y_{2,Jm} + (-\gamma_{Jm} Y_{0,Jm}) \right), \quad \tilde{\lambda}^\alpha = -\sum_{J=1}^\infty \sum_{m=-J}^J \left( \tilde{\beta}_{Jm} Y_{2,Jm} + \tilde{\gamma}_{Jm} Y_{0,Jm} \right),$$

where $(i, j)$ and $(p, q)$ denote the $S^2$ and $T^2$ directions, respectively. Here $Y_{r,Jm}$ is scalar monopole spherical harmonics with magnetic charge $r$ and $C_{i,Jm}^\rho$ is usual vector spherical harmonics with polarization $\rho$ (see app. D for detail). By some tedious calculations, we find

$$\int dzd\bar{z} \sqrt{g_{S^2}} \mathcal{L}_{\text{vec}}|_{\text{Gauss}} \simeq \text{Tr} \left\{ \sum_{J=0}^\infty \sum_m \left\{ \frac{1}{8} (D_w^{(0)} A_{\bar{w},Jm} - D_{\bar{w}}^{(0)} A_{w,Jm})^2 + \frac{J(J+1)}{2} A_{w,Jm} A_{\bar{w},Jm} \right\} ight. $$

$$+ \sum_{J=1}^\infty \sum_m \left\{ \frac{1}{2} \sum_{\rho=1,2} (D_w^{(0)} A_{Jm}^\rho) (D_{\bar{w}}^{(0)} A_{Jm}^\rho) + \frac{J(J+1)}{2} A_{Jm}^{2\dagger} A_{Jm}^2 \right\} - \frac{1}{2} \sum_{J=0}^\infty \sum_m D_{Jm}^\dagger D_{Jm} $$

$$+ \sum_{J=0}^\infty \sum_m \tilde{\gamma}_{Jm} D_w \gamma_{Jm} + \sum_{J=1}^\infty \sum_m \left( \tilde{\beta}_{Jm} D_w \beta_{Jm} + i \sqrt{J(J+1)} (-\tilde{\gamma}_{Jm} \beta_{Jm} + \tilde{\beta}_{Jm} \gamma_{Jm}) \right) \right\}. \quad (4.22)
In sec. 4.6, we will show that contribution from the non-zero modes \((J \neq 0)\) are canceled by non-zero modes of the ghosts. Hence let us focus on the zero modes:

\[
\int d\bar{z}d\bar{\bar{z}}\sqrt{gS}L_{\text{vec}}|_{\text{zero modes}} = \text{Tr} \left[ \frac{1}{4} (D_p^{(0)} A_q,00 - D_q^{(0)} A_p,00)^2 - \frac{1}{2} D_{00}^2 + \bar{\gamma}_{00} D_{\bar{w}}^{(0)} \gamma_{00} \right],
\]

which is invariant under the supersymmetric transformation

\[
\begin{align*}
\delta A_{w,00} &= \sqrt{2} \bar{\gamma}_{00} - \sqrt{2} \gamma_{00}, \quad \delta A_{\bar{w},00} = 0, \\
\delta \gamma_{00} &= -i \sqrt{2} (D_3^{(0)} A_{4,00} - D_4^{(0)} A_{3,0}) - \frac{i}{\sqrt{2}} D_{00}, \\
\delta \bar{\gamma}_{00} &= +i \sqrt{2} (D_3^{(0)} A_{4,00} - D_4^{(0)} A_{3,0}) - \frac{i}{\sqrt{2}} D_{00}, \\
\delta D_{00} &= -i \frac{i}{\sqrt{2}} D_{\bar{w}}^{(0)} \bar{\gamma}_{00} + i \frac{i}{\sqrt{2}} D_{\bar{w}}^{(0)} \gamma_{00}.
\end{align*}
\]

These are exactly the same as the Lagrangian and SUSY transformation of 2d \(\mathcal{N} = (0,2)\) super Yang-Mills theory described in app. E.3 around the saddle point if we identify

\[
\zeta_+ = \frac{1}{\sqrt{2}}, \quad \bar{\zeta}_+ = \frac{1}{\sqrt{2}}, \quad \lambda_+ = -\gamma_{00}, \quad \bar{\lambda}_+ = -\bar{\gamma}_{00}.
\]

The FI-term becomes

\[
S_{\text{FI}} = -i \zeta \int dwd\bar{w} D_{00},
\]

which is also the same as the FI-term in two dimensions. Non-zero modes are decoupled from other sectors including chiral multiplets.

### 4.4.2 Chiral multiplet

Next, we evaluate the one-loop determinant of the chiral multiplet on \(T^2 \times S^2\). Gaussian fluctuation of the action for chiral multiplet is

\[
\mathcal{L}_{\text{chial}}|_{\text{Gauss}} = D_\mu \bar{\phi} D^\mu \phi + \frac{r}{2} \bar{\phi} \phi + \bar{\phi} D_{00} \phi - \bar{F} F + i \bar{\psi} \bar{\phi} D_\mu \psi + i \sqrt{2}(\bar{\phi} \lambda^{(0)} \zeta \psi - \bar{\psi} \bar{\lambda}^{(0)} \bar{\phi}).
\]

As in vector multiplet, we perform KK-mode expansion along \(S^2\). We expand the bosonic fields as

\[
\begin{align*}
\phi &= \sum_{J = |r|/2} a_{JM} Y_{r,Jm}, \quad \bar{\phi} = \sum_{J = |r|/2} \bar{a}_{JM} Y_{r,Jm}, \\
F &= \sum_{J = |r-2|/2} f_{JM} Y_{r-2,Jm}, \quad \bar{F} = \sum_{J = |r-2|/2} \bar{f}_{JM} Y_{r-2,Jm},
\end{align*}
\]

Note that we can treat this sector by the naive saddle point analysis since this sector is not interacting with \(D_{00}\) and \((\lambda^{(0)}, \bar{\lambda}^{(0)})\).
while mode expansion for fermions depend on \( r \):

\[
\psi^a = \begin{cases} 
\sum_{J=j_0+1} \sum_m \begin{pmatrix} b_{Jm} Y_{rJm} \\ -c_{Jm} Y_{r-2Jm} \end{pmatrix} + \sum_m \begin{pmatrix} 0 \\ -c_{Jm} Y_{r-2Jm} \end{pmatrix} & \text{for } r > 1 \\
\sum_{J=j_0+1} \sum_m \begin{pmatrix} b_{Jm} Y_{rJm} \\ -c_{Jm} Y_{r-2Jm} \end{pmatrix} & \text{for } r = 1 , \\
\sum_{J=j_0+1} \sum_m \begin{pmatrix} b_{Jm} Y_{rJm} \\ -c_{Jm} Y_{r-2Jm} \end{pmatrix} + \sum_m \begin{pmatrix} 0 \\ b_{Jm} Y_{rJm} \end{pmatrix} & \text{for } r < 1 
\end{cases}
\]

\[
\tilde{\psi}^a = \begin{cases} 
- \sum_{J=j_0+1} \sum_m \begin{pmatrix} \tilde{c}_{Jm} Y_{r-2Jm} \\ \tilde{b}_{Jm} Y_{rJm} \end{pmatrix} - \sum_m \begin{pmatrix} \tilde{c}_{Jm} Y_{r-2Jm} \\ 0 \end{pmatrix} & \text{for } r > 1 \\
- \sum_{J=j_0+1} \sum_m \begin{pmatrix} \tilde{c}_{Jm} Y_{r-2Jm} \\ \tilde{b}_{Jm} Y_{rJm} \end{pmatrix} & \text{for } r = 1 , \\
- \sum_{J=j_0+1} \sum_m \begin{pmatrix} \tilde{c}_{Jm} Y_{r-2Jm} \\ \tilde{b}_{Jm} Y_{rJm} \end{pmatrix} - \sum_m \begin{pmatrix} 0 \\ \tilde{b}_{Jm} Y_{rJm} \end{pmatrix} & \text{for } r < 1 
\end{cases}
\]

where

\[
j_0 = \frac{|r-1|}{2} - \frac{1}{2}.
\]

Note that the number of the zero modes and chirality in two dimensions depend on \( r \). This property is important because the non-zero modes do not affect the final formula for the index and only the zero-modes give nontrivial contributions as we will see.

**For \( r > 1 \)**

For \( r > 1 \), the action becomes

\[
\int dzd\bar{z} \sqrt{g} s L_{\chi} \Big|_{\text{Gauss}} = \sum_{J=|r|/2} \sum_m \tilde{a}_{Jm} (-D_p^{(0)} D^{(0)} p + \lambda_{rJ}^2) a_{Jm} - \sum_{J=|r-2|/2} \sum_m \tilde{f}_{Jm} f_{Jm} \\
+ \sum_{J=r/2} \sum_m \tilde{a}_{Jm} D_{00} a_{Jm} + \sum_{J=\frac{r}{2}-1} \sum_m \tilde{c}_{Jm} D^{(0)} c_{Jm} \\
+ \sum_{J=\frac{r}{2}} \sum_m (\tilde{b}_{Jm} D_w^{(0)} b_{Jm} + i \lambda_{rJ} (-\tilde{c}_{Jm} b_{Jm} + \tilde{b}_{Jm} c_{Jm})) \\
+ i \sum_{J=\frac{r}{2}} \sum_m (\tilde{a}_{Jm} \lambda^{(0)} b_{Jm} - \tilde{b}_{Jm} \tilde{\lambda}^{(0)} a_{Jm}),
\]

where

\[
\lambda_{rJ} = \sqrt{(J + \frac{1}{2})^2 - (\frac{r-1}{2})^2}.
\]

We can decompose this as

\[
\int dzd\bar{z} \sqrt{g} s L_{\chi} = \sum_{J=\frac{r}{2}-1}^{\infty} \sum_{m=-J}^{J} L^{(J,m)}_{\Lambda} + \sum_{J=\frac{r}{2}}^{\infty} \sum_{m=-J}^{J} L^{(J,m)}_{\Phi},
\]

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where
\[
\begin{align*}
\mathcal{L}^{(J,m)}_\Lambda &= \hat{c}_{Jm}D_w^{(0)}c_{Jm} - \tilde{f}_{Jm}\bar{f}_{Jm} + \lambda^2_{\ell J}\tilde{a}_{Jm}a_{Jm} + i\lambda_{\ell J}(-\hat{c}_{Jm}b_{Jm} + \tilde{b}_{Jm}c_{Jm}), \\
\mathcal{L}^{(J,m)}_\Phi &= D_p^{(0)}\tilde{a}_{Jm}D_w^{(0)}a_{Jm} + \tilde{a}_{Jm}D_0^0a_{Jm} + \tilde{b}_{Jm}D_0^wb_{Jm} + i\left(\tilde{a}_{Jm}\lambda^{(0)}b_{Jm} - \tilde{b}_{Jm}\lambda^{(0)}a_{Jm}\right).
\end{align*}
\]
(4.34)

These are invariant under the transformations
\[
\begin{align*}
\delta c_{Jm} &= -f_{Jm} + i\lambda_{\ell J}a_{Jm}, & \delta \tilde{c}_{Jm} &= -\tilde{f}_{Jm} - i\lambda_{\ell J}\tilde{a}_{Jm}, \\
\delta f_{Jm} &= -D_w^{(0)}c_{Jm} + i\lambda_{\ell J}b_{Jm}, & \delta \tilde{f}_{Jm} &= -D_w^{(0)}\tilde{c}_{Jm} + i\lambda_{\ell J}\tilde{b}_{Jm}, \\
\delta a_{Jm} &= +b_{Jm}, & \delta \tilde{a}_{Jm} &= -\tilde{b}_{Jm}, & \delta b_{Jm} &= D_0^0a_{Jm}, & \delta \tilde{b}_{Jm} &= -D_0^wb_{Jm},
\end{align*}
\]
(4.35)

which have been derived from the 4d SUSY transformation by using the orthogonal relation of the harmonics. We can easily see that \(\mathcal{L}^{(J,m)}_\Lambda\) and the SUSY transformation (4.35) are exactly the same as the ones of 2d \(\mathcal{N} = (0, 2)\) Fermi multiplet with potential \(E(\tilde{a}_{Jm})\) and \(\tilde{E}(a_{Jm})\) if we identify
\[
\begin{align*}
\zeta_+ &= \frac{1}{\sqrt{2}}, \quad \tilde{\zeta}_+ = \frac{1}{\sqrt{2}}, & \psi_+ &= -c_{Jm}, & \tilde{\psi}_+ &= -\tilde{c}_{Jm}, & G &= f_{Jm}, & \tilde{G} &= \tilde{f}_{Jm}, \\
\psi_- &= -\tilde{b}_{Jm}, & \tilde{\psi}_- &= b_{Jm}, & \phi &= a_{Jm}, & \tilde{\phi} &= \tilde{a}_{Jm}, & E(\phi) &= i\lambda_{\ell J}\phi, & \tilde{E}(\tilde{\phi}) &= -i\lambda_{\ell J}\tilde{\phi}.
\end{align*}
\]
(4.37)

Also, under this identification, \(\mathcal{L}^{(J,m)}_\Phi\) and the SUSY transformation (4.36) are exactly the same as the ones of 2d \(\mathcal{N} = (0, 2)\) chiral multiplet.

**For \(r < 1\)**

The only difference from \(r > 1\) is the zero modes. The result is
\[
\int dzd\bar{z}\sqrt{gS^2}\mathcal{L}_{\text{chi}} = \sum_{J = -r^2 + 1}^{\infty} \sum_{m = -J}^{J} \mathcal{L}^{(J,m)}_\Lambda + \sum_{J = -r^2 + 1}^{\infty} \sum_{m = -J}^{J} \mathcal{L}^{(J,m)}_\Phi.
\]
(4.38)

In this case, the non-trivial contribution only comes from the modes \((a_{Jm}, \tilde{a}_{Jm}, b_{Jm}, \tilde{b}_{Jm})\) with \(J = j_0\).

**For \(r = 1\)**

Similarly, we find
\[
\int dzd\bar{z}\sqrt{gS^2}\mathcal{L}_{\text{chi}} = \sum_{J = \frac{1}{2}}^{\infty} \sum_{m = -J}^{J} \mathcal{L}^{(J,m)}_\Lambda + \sum_{J = \frac{1}{2}}^{\infty} \sum_{m = -J}^{J} \mathcal{L}^{(J,m)}_\Phi.
\]
(4.39)
4.4.3 Gauge fixing term

Let us consider the gauge-fixing term. By expanding

\[ B = \sum_{J=0}^{\infty} \sum_{m} B_{Jm} Y_{0,Jm}, \quad c_g = \sum_{J=0}^{\infty} \sum_{m} c_{g,Jm} Y_{0,Jm}, \quad \bar{c}_g = \sum_{J=0}^{\infty} \sum_{m} \bar{c}_{g,Jm} Y_{0,Jm}, \]  

(4.40)

we find

\[ \int dzd\bar{z} \sqrt{g} S_L^2 \left. L_{gh} \right|_{J=0} = \sum_{J=0}^{\infty} \sum_{m} \text{Tr} \left[ B_{Jm} \nabla^p \tilde{A}_{p,Jm} + \bar{c}_{g,Jm} \nabla_w \nabla_{\bar{w}} c_{g,Jm} \right] \]  

(4.41)

Especially, the \( J = 0 \) part

\[ \int dzd\bar{z} \sqrt{g} S_L^2 \left. L_{gh} \right|_{J=0} = \text{Tr} \left[ B_{00} \nabla^p \tilde{A}_{p,00} + \bar{c}_{g,00} \nabla_w \nabla_{\bar{w}} c_{g,00} \right], \]  

(4.42)

is exactly the same as the gauge fixing action of 2d gauge theory with the gauge-fixing condition

\[ \nabla^p \tilde{A}_{p,00} = 0, \]  

(4.43)

if we identify \( B_{00} \) and \( c_g, \bar{c}_g \) with the Nakanishi-Lautrap field and ghosts in 2d, respectively.

4.5 Twisted boundary condition on the torus

In our setup, the fields \( \Phi_{T_2 \times S^2} \) on \( T^2 \times S^2 \) with the magnetic charge \( r \) satisfy the twisted boundary conditions

\[ \Phi_{T_2 \times S^2}(w + 2\pi, e^{2\pi i \alpha} z) = e^{\pi ir\alpha} \Phi_{T_2 \times S^2}(w, z), \]
\[ \Phi_{T_2 \times S^2}(w + 2\pi \tau, e^{2\pi i \beta} z) = e^{\pi ir\beta} \Phi_{T_2 \times S^2}(w, z). \]  

(4.44)

Since the spherical harmonics \( Y_{r,Jm} \) satisfy \( Y_{r,Jm}(e^{2\pi i \alpha} z) = e^{\pi ir\alpha} e^{2\pi i m \alpha} Y_{r,Jm}(z) \), the fields \( \Phi_{T^2} \) in effective 2d theory on \( T^2 \) should satisfy

\[ \Phi_{T^2}(w + 2\pi) = e^{-2\pi i m \alpha} \Phi_{T^2}(w), \quad \Phi_{T^2}(w + 2\pi \tau) = e^{-2\pi i m \beta} \Phi_{T^2}(w). \]  

(4.45)

4.6 One-loop determinant for non-zero modes in vector multiplet

We show that the non-zero modes of the vector multiplet and gauge-fixing term give trivial one-loop determinant. First, the term including \( (A_{w,Jm}, A_{\bar{w},Jm}) \) is

\[ \int dzd\bar{z} \sqrt{g} S_L^2 \left. L_{\text{vec}} \right|_{\text{Gauss}, (A_{w,Jm}, A_{\bar{w},Jm})} \]
\[ = -\frac{1}{8} (A_{w,Jm}, A_{\bar{w},Jm}) \begin{pmatrix} D_{w}^2 & -D_w D_{\bar{w}} + 2J(J + 1) \\ -D_w D_{\bar{w}} + 2J(J + 1) & D_{\bar{w}}^2 \end{pmatrix} \begin{pmatrix} A_{w,Jm} \\ A_{\bar{w},Jm} \end{pmatrix}. \]  

(4.46)
Hence, the one-loop determinant from \((A_w, J_m, A_{\bar{w}}, J_m)\) is
\[
Z_{(A_w, A_{\bar{w}})} = \text{Det} \left[ D_w D_{\bar{w}} - J(J + 1) \right]^{-1/2},
\]
up to an overall constant. We also find the contributions from \(A^1_{Jm}\) and \(A^2_{Jm}\) in a straightforward manner as
\[
Z_{A^1} = \text{Det} \left[ D_w D_{\bar{w}} \right]^{-1/2}, \quad Z_{A^2} = \text{Det} \left[ D_w D_{\bar{w}} - J(J + 1) \right]^{-1/2}.
\]
Similarly, noting
\[
\int dz d\bar{z} \sqrt{g_{\text{vec}}} \mathcal{L}_{\text{Gauss},(\beta, \gamma)} \left( \beta_{Jm}, \gamma_{Jm} \right) \left( D_w \right) \left( -i \sqrt{J(J + 1)} \right) \left( D_{\bar{w}} \right)
\]
we find
\[
Z_{(\beta, \gamma)} = \text{Det} \left[ D_w D_{\bar{w}} - J(J + 1) \right].
\]
Contribution from the ghosts is
\[
Z_{c_g} = \text{Det} \left[ D_w D_{\bar{w}} \right].
\]
Thus we conclude that the total contribution from the nonzero modes of the vector multiplet and gauge-fixing term is trivial:\[\text{13}\]
\[
Z_{(A_w, A_{\bar{w}})} Z_{A^1} Z_{A^2} Z_{(\beta, \gamma)} Z_{c_g} = 1,
\]
up to an overall constant.

### 4.7 Modular properties

In this subsection, we study modular properties of the \(T^2 \times S^2\) partition function under the \(SL(3; \mathbb{Z})\) transformation
\[
S : (\tau, \sigma) \rightarrow \left( -\frac{1}{\tau}, \frac{\sigma}{\tau} \right), \quad T : (\tau, \sigma) \rightarrow (\tau + 1, \sigma),
\]
\[
U : (\tau, \sigma) \rightarrow (\tau, \sigma + \tau), \quad V : (\tau, \sigma) \rightarrow (\tau, \sigma + 1).
\]
For this purpose, it is sufficient to study the one-loop determinant.

\[\text{13}\] Note that eigenvalues of \(D_w\) and \(D_{\bar{w}}\) are common among all the fields having the same \(m\).
\section*{S-transformation}

Under the S-transformation, the one-loop determinants transform as

\[
S : Z_V \to (-i)^{|G|} \left( \prod_{\alpha} -ie^{\frac{\pi i}{\tau} a^2(u)} \right) Z_V,
\]

\[
S : Z_{\Lambda,R}(-1/\tau, u/\tau, y/\tau, \tau) = \left( \prod_{\rho} -ie^{\frac{\pi i}{\tau}(\rho(u)+y)^2} \right) Z_{\Lambda,R}(\tau, u, y),
\]

\[
S : Z_{\Phi,R}(-1/\tau, u/\tau, y/\tau, \tau) = \left( \prod_{\rho} ie^{-\frac{\pi i}{\tau}(\rho(u)+y)^2} \right) Z_{\Phi,R}(\tau, u, y). \tag{4.53}
\]

Then the one-loop determinant of the 4d chiral multiplet transforms as

\[
Z^{(r>1)}_R(-1/\tau, u/\tau, \sigma/\tau, \xi_a/\tau) = \left( \prod_{m=-\frac{r-1}{2}}^{\frac{r-1}{2}} \prod_{\rho} -ie^{\frac{\pi i}{\tau}(\rho(u)+m\sigma+\sum_a q_a^\rho \xi_a)^2} \right) Z^{(r>1)}_R(\tau, u, \sigma, \xi_a),
\]

\[
Z^{(r<1)}_R(-1/\tau, u/\tau, \sigma/\tau, \xi_a/\tau) = \left( \prod_{m=-\frac{r-1}{2}}^{\frac{r-1}{2}} \prod_{\rho} ie^{-\frac{\pi i}{\tau}(\rho(u)+m\sigma+\sum_a q_a^\rho \xi_a)^2} \right) Z^{(r<1)}_R(\tau, u, \sigma, \xi_a). \tag{4.54}
\]

We find that the exponent of the prefactor is a linear combination of the factors (C.8)-(C.11) appearing in a particular regularization of the one-loop determinant and is related to anomalies in two dimensions (see app. C for detail).

\section*{T-transformation}

The T-transformation acts on each one-loop determinant as

\[
T : Z_V \to e^{\frac{\pi i}{\tau} |A_{\text{adj}}|} Z_V, \quad T : Z_{\Lambda,R} \to e^{\frac{\pi i}{\tau} |R|} Z_{\Lambda,R}, \quad T : Z_{\Phi,R} \to e^{-\frac{\pi i}{\tau} |R|} Z_{\Phi,R}. \tag{4.55}
\]

Hence, the partition function transforms as

\[
T : Z_{T^2 \times S^2} \to e^{\frac{\pi i}{\tau} (|A_{\text{adj}}|+\sum_i (r_i-1)|R_i|)} Z_{T^2 \times S^2}. \tag{4.56}
\]

The prefactor vanishes if we satisfy the 2d gauge anomaly cancellation condition for the 2d zero-mode theory.

\section*{U-transformation}

U-transformation properties of the one-loop determinants are

\[
U : Z_V \to Z_V. \]
\[ U : Z^{(r)}_R \rightarrow \begin{cases} e^{-\frac{i\pi}{12}r(r^2-1)\tau|R|} Z^{(r)}_R & \text{for } r > 1 \\ Z^{(r)}_R & \text{for } r = 1 \\ e^{+\frac{i\pi}{12}|r||r|+1(|r|+2)\tau|R|} Z^{(r)}_R & \text{for } r < 1 \end{cases} \]  

(4.57)

Although one would expect this prefactor can be rewritten in the language of anomalies, we have not found any clear understandings. It would be illuminating if one finds physical implications of this factor. Presumably this would be related to gravitational anomalies.

**V-transformation**

Each one-loop determinant is invariant under the \( V \)-transformation:

\[ V : Z_V \rightarrow Z_V, \quad \begin{array}{c} V : Z^{(r)}_R \rightarrow Z^{(r)}_R. \end{array} \]  

Thus the whole partition function is always invariant under the \( V \)-transformation.

### 4.8 Localization by another deformation term

If we consider another deformation term for localization, then we can obtain another formula for the partition function, which is apparently different from our main formula (1.3) but should be the same. Here let us take the deformation term as

\[ QV = \mathcal{L}_{vec} + \mathcal{L}_{chi} + \mathcal{L}_m, \]  

(4.59)

where

\[ \mathcal{L}_m = \delta \left[ \text{Tr}(\zeta^\dagger \lambda - \tilde{\zeta}^\dagger \tilde{\lambda})(-iD) \right]. \]  

(4.60)

Then bosonic part of \( \mathcal{L}_{vec} \) plus \( \mathcal{L}_m \) is given by

\[ \left. \mathcal{L}_{vec} + \mathcal{L}_m \right|_{bos.} = \frac{1}{2} \mathcal{F}_{ip}^2 + \frac{1}{4} \mathcal{F}_{pq}^2 + \frac{1}{2} (\mathcal{F}_{12} + D)^2. \]  

(4.61)

This leads us to the following condition for the saddle point

\[ \mathcal{F}_{12} = -D, \quad \mathcal{F}_{13} = \mathcal{F}_{14} = \mathcal{F}_{23} = \mathcal{F}_{24} = \mathcal{F}_{34} = 0. \]  

(4.62)

General solution satisfying this condition and compatible with isometry of \( T^2 \times S^2 \) is non-Abelian version of the general background multiplet configuration (3.21). Namely, the gauge field takes monopole configuration on \( S^2 \) and has the holonomies along \( T^2 \). Then, the \( T^2 \times S^2 \) partition function becomes

\[ Z_{T^2 \times S^2} = \sum_{m} e^{-\mathcal{S}_{\text{F}1}(m)} \sum_{u_* \in \mathcal{M}_{\text{sing}}} \text{JKRes}_{u = u_*}(Q(u_*), \eta) \ Z^{(m)}_{1-\text{loop}}(\tau, u, \sigma, \xi). \]  

(4.63)

\[ \text{14 The authors in [11] have claimed this formula by an expectation from their result on } S^1 \times S^2. \]
where \( \mathbf{m} = (m_1, \cdots, m_{|G|}) \) is the monopole charge vector and the factor (sym) denotes the rank of Weyl group of unbroken gauge group. Note that the FI-term is non-zero for general monopole configuration. The one-loop determinant \( Z_{1\text{-loop}}^{(m)} \) is given by

\[
Z_{1\text{-loop}}^{(m)}(\tau, u, \sigma, \xi_a) = Z_V^{(m)}(\tau, u, \sigma) \prod_i Z_{R_i}^{(m, r_i)}(\tau, u, \sigma, \xi_a),
\]

where \( Z_V^{(m)}(\tau, u, \sigma) \) is the contribution from the vector multiplet:

\[
Z_V^{(m)}(\tau, u, \sigma) = \left( \frac{2\pi \eta^2(q)}{i} \right)^{|G|} \prod_{\alpha \in G} i\theta_1(\tau|\alpha(u) + |\alpha(m)|\sigma) \prod_{a=1}^{|G|} du_a,
\]

\( Z_{R_i}^{(m, r_i)}(\tau, u, \sigma, \xi_a) \) is the contribution from the chiral multiplet:

\[
Z_{R_i}^{(m, r_i)}(\tau, u, \sigma, \xi_a) = \prod_{\rho \in R} Z^{(r + \rho(m))}(\tau, \rho(u) + \sum_a q_{f}^a \xi_a, \sigma)
\]

where

\[
Z^{(r)}(\tau, y, \sigma) = \begin{cases} 
\prod_{m=-\frac{r}{2}+1}^{\frac{r}{2}-1} i\theta_1(\tau|y + m\sigma) \eta(q) & \text{for } r > 1 \\
1 & \text{for } r = 1 \\
\prod_{m=-\frac{|r|}{2}}^{\frac{|r|}{2}} \prod_{\rho \in R} i\eta(q) \theta_1(\tau|y + m\sigma) & \text{for } r < 1
\end{cases}
\]

Note that the contribution from \( \mathbf{m} = 0 \) in (4.63) is exactly the same as our main formula (1.3). Therefore, if the both results (1.3) and (4.63) are correct, then the contributions from nonzero monopole configurations must vanish:

\[
\sum_{\mathbf{m} \neq 0} \text{(sym)} \sum_{u_* \in \mathcal{M}^{*}_{\text{sing}}} \text{JKRes}_{u_*} (Q(u_*), \eta) Z_{1\text{-loop}}^{(m)}(\tau, u, \sigma, \xi_a) = 0 \quad \text{(4.68)}
\]

Although the localization procedures by the two different deformation terms lead this equation, we have not shown this equation by explicitly computing the final expression. It is interesting if one can solve this puzzle.

## 5 4d indices and 2d extended supersymmetries

In this section we study 4d theories, whose indices give 2d elliptic genera with larger supersymmetries.

### 5.1 4d \( \mathcal{N} = 2 \) partition function and 2d \( \mathcal{N} = (2, 2) \) elliptic genus

Let us consider 4d theory having \( \mathcal{N} = 1 \) vector multiplet \( V \) and the matters listed in tab. with the superpotential \( W = \tilde{Q}\Phi Q \). It is known that this theory has 4d \( \mathcal{N} = 2 \) supersymmetry. Note that we also turn on the magnetic flux of the \( U(1) \) flavor symmetry
Table 1: Field content of 4d \( \mathcal{N} = 2 \) theory, which gives 2d \( \mathcal{N} = (2, 2) \) elliptic genus.

| Field | \( G \) | \( U(1)_R \) | \( U(1)_{\text{flavor}} \) |
|-------|-------|-----------|-----------------|
| \( \Phi \) | adj. 1 | -1 | |
| \( Q \) | \( R \) | 1/2 | 1/2 - \( q \) |
| \( \tilde{Q} \) | \( \bar{R} \) | 1/2 | 1/2 + \( q \) |
| fugacity | - | - | \( z \) |
| magnetic flux | - | 1 | 1 |

with \( g = 1 \). Since the shifted R-charges should be integer on \( T^2 \times S^2 \), we take \( q \) to be positive integer. One-loop determinants for the fields are given by

\[
Z_V = \left( \frac{2 \pi \eta^3(q)}{i} \right)^{\text{rank}G} \prod_{\alpha \in G} \frac{i \theta_1(\tau | \alpha(u))}{\eta(q)} \prod_{a=1}^{\text{rank}G} du_a, \quad Z_\Phi = \prod_{\rho \in \text{adj}.} \frac{i \eta(q)}{\theta_1(\tau | \rho(u) - z)},
\]

\[
Z_Q = \prod_{m=- \frac{q-1}{2}}^{\frac{q-1}{2}} \prod_{\rho \in \text{R}} \frac{i \eta(q)}{\theta_1(\tau | \rho(u) + (1 - 2q)z/2 + m\sigma)},
\]

\[
Z_{\tilde{Q}} = \prod_{m=- \frac{q-1}{2}}^{\frac{q-1}{2}} \prod_{\rho \in \text{R}} \frac{i \theta_1(\tau) - \rho(u) + (1 + 2q)z/2 + m\sigma)}{\eta(q)}.
\]

Then we easily see that the product

\[
Z_V Z_\Phi = \left( \frac{2 \pi \eta^3(q)}{\theta_1(\tau) - z} \right)^{\text{rank}G} \prod_{\alpha \in G} \frac{\theta(\tau | \alpha(u))}{\theta(\tau | \alpha(u) - z)},
\]

is the same as the one-loop determinant of 2d \( \mathcal{N} = (2, 2) \) vector multiplet \([4, 6]\), if we identify the flavor fugacity \( z \) with the one of \( U(1)_R \) symmetry in 2d. Also the other contribution

\[
Z_Q Z_{\tilde{Q}} = \prod_{m=- \frac{q-1}{2}}^{\frac{q-1}{2}} \prod_{\rho \in \text{R}} \frac{\theta_1(\tau | \rho(u) - (1 + 2q)z/2 + m\sigma)}{\theta_1(\tau | \rho(u) + (1 - 2q)z/2 + m\sigma)},
\]

is the same as the one-loop determinant of 2d \( \mathcal{N} = (2, 2) \) \( q \)-chiral multiplets with the representation \( \text{R} \) and \( \text{R} \)-charge \((1 - 2q)\). Thus the 4d \( \mathcal{N} = 2 \) theory on \( T^2 \times S^2 \) with the appropriate background gives the partition function of the 2d \( \mathcal{N} = (2, 2) \) theory on \( T^2 \).

### 5.2 4d \( \mathcal{N} = 2 \) partition function and 2d \( \mathcal{N} = (0, 4) \) elliptic genus

Let us consider the theory with the same multiplets as in last subsection but different flavor charges listed in tab. [2]. We again have to take \( q \) to be positive integer to satisfy the

\[\text{[15]}\]

The value of \( q \) should be chosen to consistent with gauge anomaly cancellation. The result for negative integer \( q \) is simply obtained by the replacement \( \text{R} \leftrightarrow \bar{\text{R}} \).
Table 2: Field content of 4d $\mathcal{N} = 2$ theory, which gives 2d $\mathcal{N} = (0,4)$ elliptic genus.

| Field | $G$ | $\mathbb{U}(1)_R$ | $U(1)_{\text{flavor}}$ |
|-------|-----|----------------|----------------------|
| $\Phi$ | adj. | 1 | $1 + 2q$ |
| $Q$ | $\mathbb{R}$ | $1/2$ | $-1/2 - q$ |
| $\bar{Q}$ | $\bar{\mathbb{R}}$ | $1/2$ | $-1/2 - q$ |
| fugacity | - | - | $z$ |
| magnetic flux | - | 1 | 1 |

The quantization condition of the magnetic flux on $S^2$. Each one-loop determinants for each field is given by

$$Z_V = \left( \frac{2\pi\eta^2(q)}{i} \right)^{\text{rank} G} \prod_{\alpha \in G} \frac{i\theta_1(\tau | \alpha(u))}{\eta(q)} \prod_{a=1}^{\text{rank} G} du_a,$$

$$Z_\Phi = \prod_{m=-q}^{q} \prod_{\rho \in \text{adj.}} \frac{i\theta_1(\tau | \rho(u) + (1 + 2q)z + m\sigma)}{\eta(q)}.$$

$$Z_Q = \prod_{m=-q}^{q} \prod_{\rho \in \mathbb{R}} \frac{\theta_1(\tau | \rho(u) - (1 + 2q)z/2 + m\sigma)}{i\eta(q)},$$

$$Z_{\bar{Q}} = \prod_{m=-q}^{q} \prod_{\rho \in \mathbb{R}} \frac{\theta_1(\tau | -\rho(u) - (1 + 2q)z/2 + m\sigma)}{i\eta(q)}.$$

First let us consider the combination

$$Z_V Z_\Phi = \left[ \left( \frac{2\pi\eta^2(q)}{i} \right)^{\text{rank} G} \prod_{\alpha \in G} \frac{i\theta_1(\tau | \alpha(u))}{\eta(q)} \prod_{\rho \in \text{adj.}} \frac{i\theta_1(\tau | \rho(u) + (1 + 2q)z + m\sigma)}{\eta(q)} \prod_{a=1}^{\text{rank} G} du_a \right] \times \left[ \prod_{m=1}^{q} \prod_{\rho \in \text{adj.}} \frac{i\theta_1(\tau | \rho(u) + (1 + 2q)z + m\sigma)}{\eta(q)} \frac{i\theta_1(\tau | -\rho(u) + (1 + 2q)z - m\sigma)}{\eta(q)} \right].$$

We easily find that the first factor is the same as the one-loop determinant of $(0,4)$ vector multiplet, while the second is the one of the $(0,4)$ Fermi multiplets. Also, the remaining part

$$Z_Q Z_{\bar{Q}} = \prod_{m=-q}^{q} \prod_{\rho \in \mathbb{R}} \frac{i\eta(q)}{\theta_1(\tau | \rho(u) - (1 + 2q)z/2 + m\sigma)} \frac{i\eta(q)}{\theta_1(\tau | -\rho(u) - (1 + 2q)z/2 + m\sigma)},$$

is the same as the one-loop determinant of $(0,4)$ hyper multiplets.

---

16 Note that the $(0,4)$ vector multiplet consists of the $(0,2)$ vector multiplet and $(0,2)$ Fermi multiplet in adjoint representation. Also, the $(0,4)$ Fermi (hyper) multiplet consists of the $(0,2)$ Fermi (chiral) multiplets with representation $\mathbb{R}$ and $\bar{\mathbb{R}}$. 

25
Table 3: A 4d theory, which gives the elliptic genus of 2d $\mathcal{N} = (4, 4)$ vector multiplet.

| $\Phi_i$ | $G$ | $U(1)_R$ | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ |
|-----|-----|---------|---------|---------|---------|
| $\Phi_1$ | adj. | -1 | 1 | 0 | -1 |
| $\Phi_2$ | adj. | -1 | 1 | 0 | 1 |
| $\Phi_3$ | adj. | 2 | 0 | 0 | 2 |
| fugacity | - | - | $z$ | $\xi_1$ | $\xi_2$ |
| magnetic flux | - | 1 | 1 | 0 | 0 |

Table 4: A 4d theory giving the elliptic genus of 2d $\mathcal{N} = (4, 4)$ hyper multiplet.

| $\Psi$ | $G$ | $U(1)_R$ | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ |
|-----|-----|---------|---------|---------|---------|
| $Q$ | $\mathbb{R}$ | 0 | 0 | 1 | -1 |
| $\tilde{Q}$ | $\bar{\mathbb{R}}$ | 0 | 0 | -1 | -1 |
| $q$ | $\mathbb{R}$ | 3 | -1 | 1 | 0 |
| $\tilde{q}$ | $\bar{\mathbb{R}}$ | 3 | -1 | -1 | 0 |
| fugacity | - | - | $z$ | $\xi_1$ | $\xi_2$ |
| magnetic flux | - | 1 | 1 | 0 | 0 |

5.3 4d partition function and 2d $\mathcal{N} = (4, 4)$ elliptic genus

First we consider the combination of 4d $\mathcal{N} = 1$ multiplet which gives the one-loop determinant of 2d $\mathcal{N} = (4, 4)$ vector multiplet. Suppose theory with $\mathcal{N} = 1$ vector multiplet $V$ and three adjoint chiral multiplets $\Phi_i (i = 1, 2, 3)$ whose charge assignments are listed in tab. 3. The one-loop determinants of chiral multiplets is given by

$$Z_{\Phi_1} = \prod_{\rho \in \text{adj.}} \frac{i \eta(q)}{\theta_1(\tau|\rho(u)+z-\xi_2)}, \quad Z_{\Phi_2} = \prod_{\rho \in \text{adj.}} \frac{i \eta(q)}{\theta_1(\tau|\rho(u)+z+\xi_2)}, \quad Z_{\Phi_3} = \prod_{\rho \in \text{adj.}} \frac{i \theta_1(\tau|\rho(u)+2\xi_2)}{\eta(q)},$$

(5.7)

Then

$$Z_V \prod_{i=1}^3 Z_{\Phi_i} = \left(\frac{2\pi \eta^2(q)}{\rho}\right)^{|G|} \prod_{\alpha \in G} \frac{i \theta_1(\tau|\alpha(u))}{\theta_1(\tau|\rho(u)+z-\xi_2)} \prod_{\rho \in \text{adj.}} \frac{-i \theta_1(\tau|\rho(u)+2\xi_2)}{\theta_1(\tau|\rho(u)+z-\xi_2)\theta_1(\tau|\rho(u)+z+\xi_2)}$$

(5.8)

This agrees with the one-loop determinant of 2d $\mathcal{N} = (4, 4)$ vector multiplet [19].

Next we consider the 4d chiral multiplets, which give the one-loop determinant of 2d $\mathcal{N} = (4, 4)$ hyper multiplet. The field content of the 4d theory is listed in tab. 4. The
|        | $U(1)_1$ | $U(1)_2$ | $U(1)_R$ |
|--------|----------|----------|----------|
| $P$    | -2       | -3       | 2        |
| $X_{1,2}$ | 1        | 0        | 0        |
| $Y_{1,2,3}$ | 0        | 1        | 0        |
| fugacity | -        | -        | $z$      |

Table 5: A 2d $\mathcal{N} = (2, 2)$ theory giving the elliptic genus of K3. This theory has the superpotential $W = Pf(X, Y)$, where $f(X, Y)$ is a homogeneous polynomial of $(X, Y)$ with degree $(2, 3)$.

One-loop determinants are given by

$$
Z_Q = \prod_{\rho \in \mathbb{R}} \frac{i\eta(q)}{\theta_1(\tau | \rho(u) + \xi_1 - \xi_2)}, \quad Z_{\tilde{Q}} = \prod_{\rho \in \mathbb{R}} \frac{i\eta(q)}{\theta_1(\tau | \rho(u) - \xi_1 - \xi_2)},
$$

$$
Z_q = \prod_{\rho \in \mathbb{R}} \frac{i\theta_1(\tau | \rho(u) - z + \xi_1)}{\eta(q)}, \quad Z_{\tilde{q}} = \prod_{\rho \in \mathbb{R}} \frac{i\theta_1(\tau | \rho(u) - z - \xi_1)}{\eta(q)}.
$$

Then we obtain the one-loop determinant of $\mathcal{N} = (4, 4)$ hyper multiplet.

$$
Z_Q Z_{\tilde{Q}} Z_q Z_{\tilde{q}} = \prod_{\rho \in \mathbb{R}} \frac{\theta_1(\tau | \rho(u) - z + \xi_1)}{\theta_1(\tau | \rho(u) + \xi_1 - \xi_2)} \prod_{\rho \in \mathbb{R}} \frac{\theta_1(\tau | \rho(u) - z - \xi_1)}{\theta_1(\tau | \rho(u) - \xi_1 - \xi_2)}.
$$

6 Examples

In this section we present four dimensional theories on $T^2 \times S^2$, whose indices have the same expressions as two dimensional elliptic genera of interesting examples.

6.1 K3

In this subsection we find two 4d theories on $T^2 \times S^2$ giving the elliptic genus of K3. In other words, we show that these two 4d theories have the same partition function via the K3 elliptic genus. This would imply a new four dimensional duality.

6.1.1 Two dimensional description

First we briefly explain 2d SUSY gauge theories giving elliptic genus of K3. This subsection is essentially review of sec. 4.1 in [4]. Suppose 2d $\mathcal{N} = (2, 2) U(1)_1 \times U(1)_2$ gauge theory with the matters listed in tab. 5. One-loop determinant of this theory is

$$
Z_{1\text{-loop}} = \left[ \frac{2\pi \eta(q)^3}{\theta_1(\tau | - z)} \right]^2 \frac{\theta_1(\tau | - 2u_1 - 3u_2)}{\theta_1(\tau | z - 2u_1 - 3u_2)} \left[ \frac{\theta_1(\tau | - z + u_1)}{\theta_1(\tau | u_1)} \right]^2 \left[ \frac{\theta_1(\tau | - z + u_2)}{\theta_1(\tau | u_2)} \right]^3 d\tau_1 \wedge d\tau_2.
$$

(6.1)
Table 6: Another 2d $N = (2, 2)$ theory giving the K3 elliptic genus. This theory has the superpotential $W = Pf(X)$, where $f(X)$ is a homogeneous polynomial of $X$ with degree 4.

Then $\mathcal{M}_{\text{sing}}$ is given by the hyperplanes:

\[ H_P = \{ z - 2u_1 - 3u_2 = 0 \}, \quad H_X = \{ u_1 = 0 \}, \quad H_Y = \{ u_2 = 0 \} \quad (\text{mod } \mathbb{Z} + \tau \mathbb{Z}). \quad (6.2) \]

Hence, we find that $u_*$ is intersections of $(H_P, H_X)$, $(H_P, H_Y)$ and $(H_X, H_Y)$. Note that the each charge covector is $Q_P = (-2, -3)$, $Q_X = (1, 0)$, $Q_Y = (0, 1)$. \quad (6.3)

If we take $\eta = (1, 1)$, which is inside of $\text{Cone}(Q_X, Q_Y)$ but outside of $\text{Cone}(Q_P, Q_X)$ and $\text{Cone}(Q_P, Q_Y)$, then non-zero JK residue comes only from the intersection $(H_X, H_Y)$:

\[ Z_{T^2} = 1 \frac{(2\pi i)^2}{2} \int_{u_1 = u_2 = 0} du_1 du_2 \ Z_{1\text{-loop}}. \quad (6.4) \]

There is another $N = (2, 2)$ theory giving the same elliptic genus. This theory is $U(1)$ gauge theory with the matter listed in tab. 7. By taking $\eta > 0$, we find

\[ Z_{T^2} = \frac{\eta(q)^3}{\Im \tau} \int_{u = 0} \frac{\theta_1(\tau|z - u) - 4u}{\theta_1(\tau|z - 4u)} \left( \frac{\theta_1(\tau|z + u)}{\theta_1(\tau|u)} \right)^4. \quad (6.5) \]

It is known that the K3 elliptic genus in standard form is \[20\]

\[ Z_{T^2} = 8 \left[ \left( \frac{\theta_1(\tau|z + \frac{1}{2})}{\theta_1(\tau|\frac{1}{2})} \right)^2 + \left( e^{\pi i} \frac{\theta_1(\tau|z + \frac{1 + \tau}{2})}{\theta_1(\tau|z)} \right)^2 + \left( e^{\pi i} \frac{\theta_1(\tau|z + \frac{3}{2})}{\theta_1(\tau|\frac{3}{2})} \right)^2 \right]. \quad (6.6) \]

One can easily check that the expressions (6.4) and (6.5) are the same as this standard form.

### 6.1.2 Four dimensional description

Next we find two 4d theories on $T^2 \times S^2$, whose partition functions are the same as the K3 elliptic genus. Let us consider 4d $U(1)_1 \times U(1)_2$ gauge theory with matters listed in tab. 7 and the superpotential

\[ W = \sum_{i=1}^2 \tilde{P}_i \Phi_1 P + \sum_{i=1}^3 \tilde{X}_i' \Phi_1 X_i' + \sum_{i=1}^2 \tilde{Y}_i' \Phi_2 Y_i'. \quad (6.7) \]
Table 7: A 4d theory giving the elliptic genus of K3. This theory has the same one-loop determinant as the 2d theory described in tab. 5 for special fugacities.

|                  | $U(1)_1$ | $U(1)_2$ | $U(1)_R$ | $U(1)_f$ | $U(1)_P$ | $U(1)_X$ | $U(1)_Y$ |
|------------------|----------|----------|----------|----------|----------|----------|----------|
| $\Phi_{1,2}$    | 0        | 0        | 1        | -1       | 0        | 0        | 0        |
| $P'$             | -2       | -3       | 1/2      | -1/2     | 1        | 0        | 0        |
| $\tilde{P}'$    | +2       | +3       | 1/2      | 3/2      | -1       | 0        | 0        |
| $X'_{1,2}$      | +1       | 0        | 1/2      | -1/2     | 0        | 1        | 0        |
| $\tilde{X}'_{1,2}$ | -1     | 0        | 1/2      | 3/2      | 0        | -1       | 0        |
| $Y'_{1,2,3}$    | 0        | +1       | 1/2      | -1/2     | 0        | 0        | 1        |
| $\tilde{Y}'_{1,2,3}$ | 0     | -1       | 1/2      | 3/2      | 0        | 0        | -1       |
| fugacity        | -        | -        | z        | $\xi_P$  | $\xi_X$  | $\xi_Y$  |
| magnetic flux    | -        | -        | 1        | 1        | 0        | 0        | 0        |

Table 8: Another 4d theory giving the elliptic genus of K3. This theory has the same one-loop determinant as the 2d theory explained in tab. 6 for special fugacities.

|                  | $U(1)$ | $U(1)_R$ | $U(1)_f$ | $U(1)_P$ | $U(1)_X$ |
|------------------|--------|----------|----------|----------|----------|
| $\Phi$           | 0      | 1        | -1       | 0        | 0        |
| $P'$             | -4     | 1/2      | -1/2     | 1        | 0        |
| $\bar{P}'$       | +4     | 1/2      | 3/2      | -1       | 0        |
| $X'_{1,2,3,4}$   | +1     | 1/2      | -1/2     | 0        | 1        |
| $\bar{X}'_{1,2,3,4}$ | -1  | 1/2      | 3/2      | 0        | -1       |
| fugacity         | -      | -        | z        | $\xi_P$  | $\xi_X$  |
| magnetic flux     | -      | 1        | 1        | 0        | 0        |

The one-loop determinant is given by

$$Z_{1-\text{loop}} = \left[ \frac{2\pi \eta(q)^3}{\theta_1(\tau - z)} \right]^2 \frac{\theta_1(\tau - 3z/2 - 2u_1 - 3u_2 + \xi_P)}{\theta_1(\tau - z/2 - 2u_1 - 3u_2 + \xi_P)} \left[ \frac{\theta_1(\tau - 3z/2 + u_1 + \xi_X)}{\theta_1(\tau - z/2 + u_1 + \xi_X)} \right]^2 \left[ \frac{\theta_1(\tau - 3z/2 + u_2 + \xi_Y)}{\theta_1(\tau - z/2 + u_2 + \xi_Y)} \right]^3 du_1 \wedge du_2. \quad (6.8)$$

If we take $\xi_P = 3z/2, \xi_X = z/2, Y = z/2$, then this becomes the same as the one-loop determinant (6.1) of the theory in tab. 5 and hence gives the K3 elliptic genus.

Let us consider another 4d $U(1)$ gauge theory, whose matters are listed in tab. 8 and superpotential is given by

$$W = \bar{P}' \Phi P + \sum_{i=1}^{4} \bar{X}'_i \Phi X'_i. \quad (6.9)$$
One-loop determinant is given by
\[ Z_{1-\text{loop}} = \frac{2\pi \eta(q)^3}{\theta_1(\tau - z) \theta_1(\tau - z/2 - 4u + \xi_p)} \left( \frac{\theta_1(\tau - 3z/2 + u + \xi_X)}{\theta_1(\tau - z/2 + u + \xi_X)} \right)^4. \] (6.10)

Taking \( \xi_p = 3z/2 \) and \( \xi_X = z/2 \), this becomes the integrand of (6.5) and leads also the K3 elliptic genus.

Thus we find that the two 4d theories have the same partition function. This implies that there is a new type of duality between the two theories. It is interesting if we further test this relation in other observables or find any physical reasons for that.

6.2 Elliptic genus of E-strings from 4d index

E-strings [21] are M2-branes suspended between M5 and M9 branes in M-theory description. It is discussed in [22] that low-energy dynamics of \( N \) E-strings is described by the 2d \( \mathcal{N} = (0, 4) \) \( O(N) \) gauge theory with the field content

- \( \mathcal{N} = (0, 4) \) vector multiplet
- \( \mathcal{N} = (0, 4) \) hyper multiplet in symmetric representation
- Four \( \mathcal{N} = (0, 4) \) Fermi multiplets in fundamental representation,

at its IR fixed point. Let us consider the elliptic genus of this theory defined by

\[ Z = \text{Tr}_{\mathbb{R}_R} \left[ (-1)^F q^H L \bar{q}^H R e^{2\pi i \epsilon_1(J_1 + J_1)} e^{2\pi i \epsilon_2(J_2 + J_1)} \prod_{\ell=1}^8 e^{2\pi i m_\ell F_\ell} \right]. \] (6.11)

Here \( J_{1,2} \) is Cartan of \( SO(4) = SU(2) \times SU(2) \) associated with rotational symmetry in four-directions, where NS5 and D8-O8 spread out. \( F_\ell \) is flavor symmetry of the eight \( (0, 2) \) Fermi multiplets, or equivalently Cartan of \( SO(16) \) symmetry. One-loop determinant of the \( (0, 4) \) theory is given by

\[
Z_V = \left( \frac{2\pi \eta(q)^2}{i} \right)^{|O(n)|} \prod_{\alpha \in \text{root}} \frac{i\theta_1(\tau | \alpha(u))}{\eta(q)} \prod_{\rho \in \text{anti-sym}} \frac{i\theta_1(\tau | \epsilon_1 + \epsilon_2 + \rho(u))}{\eta(q)},
\]

\[
Z_{\text{hyper}} = \prod_{\rho \in \text{sym}} \frac{\eta(q)}{\theta_1(\tau | \epsilon_1 + \rho(u)) \theta_1(\tau | \epsilon_2 + \rho(u))}, \quad Z_{\text{Fermi}} = \prod_{\ell=1}^8 \prod_{\rho \in \text{fund}} \frac{i\theta_1(\tau | m_\ell + \rho(u))}{\eta(q)}. \] (6.12)

We can engineer 4d theory on \( T^2 \times S^2 \) giving the elliptic genus of the E-strings. Suppose 4d \( O(n) \) gauge theory with the field content summarized in tab. Then one-loop determinant is given by

\[
Z_V = \left( \frac{2\pi \eta(q)^2}{i} \right)^{|O(n)|} \prod_{\alpha \in \text{root}} \frac{i\theta_1(\tau | \alpha(u))}{\eta(q)}, \quad Z_{\Phi} = \prod_{\rho \in \text{anti-sym}} \frac{i\theta_1(\tau | \rho(u))}{\eta(q)},
\]

30
Comparing the one-loop determinants, we easily see that the partition function of this theory is the same as the elliptic genus of the E-strings with

\[ \epsilon_1 = -\epsilon_2 = \frac{1}{2}\sigma. \]  (6.14)

It is interesting if we find any physical implications of this correspondence. For example, in F-theory setup, the E-strings arise by wrapping D3-branes on \( \mathbb{P}^1 \) and hence the elliptic genus of the \( N \) E-strings would be described by the partition function \( \mathcal{N} = 4 \, U(N) \, \text{SYM} \) on \( T^2 \times \mathbb{P}^1 \). We expect that this point of view would give some insights on this correspondence.

### 6.3 M-strings

M-strings [23] are M2-branes suspended between parallel adjacent M5-branes. It is expected that the low-energy dynamics of \( N \)-tuple M-strings is described [23] (see also [24]) by the 2d \( \mathcal{N} = (0, 4) \) \( U(N) \) gauge theory with

- \( \mathcal{N} = (0, 4) \) vector multiplet
- \( \mathcal{N} = (0, 4) \) hyper multiplet in adjoint representation
- \( \mathcal{N} = (0, 4) \) hyper multiplet in fundamental representation
- \( \mathcal{N} = (0, 4) \) Fermi multiplet in fundamental representation

at its IR fixed point. Here we consider the elliptic genus for the M-strings defined by

\[ \text{Tr} \left[ (-1)^F \, q^{H_L} \, \bar{q}^{H_R} \, e^{2\pi i e_1(J_1+J_2+J_4)} \, e^{2\pi i e_2(-J_1+J_2+J_4)} \, e^{2\pi i m J_3} \right], \]  (6.15)

where \( J_1, J_2, J_3, J_4 \) is the Cartan part of generator of \( SU(2) \) flavor symmetry (\( SU(2)^3 \) R-symmetry), respectively. The one-loop determinants of \( \mathcal{N} = (0, 4) \) multiplets are

\[ Z_{\text{vec.}} = \left( \frac{2\pi \eta^2(q)}{i} \right)^N \prod_{i \neq j} i \theta_1(\tau | u_i - u_j |) \frac{\eta(q)}{\eta(q)} \prod_{i,j=1}^N \frac{i \theta_1(\tau | \epsilon_1 + \epsilon_2 + u_i - u_j |) \eta(q)}{\eta(q)}, \]
Table 10: A 4d theory giving elliptic genus of $N$ M-strings.

\begin{tabular}{|c|c|c|c|}
\hline
 & $U(N)$ & $U(1)_R$ & $U(1)_f$ \\
\hline
$\Phi_V$ & adj & 2 & 0 \\
$\Phi_C$ & adj & -1 & 0 \\
$Q$ & fund. & 0 & 0 \\
$\bar{Q}$ & a-fund. & 0 & 0 \\
$Q'$ & fund. & 2 & -1 \\
$\bar{Q}'$ & a-fund. & 2 & -1 \\
\hline
fugacity & - & - & $m$ \\
magnetic flux & - & 1 & 0 \\
\hline
\end{tabular}

We can identify 4d theory giving the elliptic genus of this 2d theory. Let us consider 4d $U(N)$ supersymmetric gauge theory with the matters listed in tab. 10. Then one-loop determinant is given by

\begin{align*}
Z_{\text{adj-hyp}} &= \prod_{i,j=1}^{N} \frac{i\eta(q)}{\theta_1(\tau|\epsilon_1 + u_i - u_j) \theta_1(\tau|\epsilon_2 + u_i - u_j)}, \\
Z_{\text{fund-hyp}} &= \prod_{i=1}^{N} \frac{i\eta(q)}{\theta_1(\tau| - \frac{m+\epsilon_2}{2} + u_i) \theta_1(\tau| - \frac{m+\epsilon_1}{2} - u_i)}, \\
Z_{\text{Fermi}} &= \prod_{i=1}^{N} \frac{i\theta_1(\tau| - m + u_i) \theta_1(\tau| - m - u_i)}{\eta(q) \eta(q)}. \quad (6.16)
\end{align*}

Comparing these with (6.16), we find that this is the same as the elliptic genus of the M-strings if we make the identification

$$\epsilon_1 = -\epsilon_2 = \frac{1}{2} \sigma. \quad (6.18)$$

It is attractive if we find any physical implications of this correspondence.
Table 11: Matter content for the 2d $\mathcal{N} = (0, 2)$ $U(N_c)$ SQCD with $N_c = (N_1 + N_2 - N_3)/2$. This theory has the superpotential $\Phi \Gamma P$.

### 6.4 4d Seiberg duality and 2d $(0, 2)$ triality

We discuss that 4d Seiberg duality [8] for $T^2 \times S^2$ partition function gives 2d $(0, 2)$ triality [7] for elliptic genus [17].

#### 6.4.1 2d $(0, 2)$ triality

First we briefly introduce the $(0, 2)$ triality proposed by Gadde-Gukov-Putrov [7]. Let us consider the 2d $\mathcal{N} = (0, 2)$ $U(N_c)$ SQCD with

$$N_c = \frac{N_1 + N_2 - N_3}{2},$$

whose matter content is summarized in tab. [11] The fields $\Gamma$ and $\Omega$ are required to cancel gauge anomalies. Then the authors in [7] have conjectured the $(0, 2)$ triality, which states that the SQCD at an infrared fixed point is invariant under the replacements

$$(N_1, N_2, N_3) \to (N_2, N_3, N_1) \to (N_3, N_1, N_2).$$

(6.20)

This conjecture has been checked for some observables [7, 25, 26].

Let us explicitly check the $(0, 2)$ triality for the elliptic genus as in [7]. First, one-loop determinant from each field is

$$Z_V = \left(\frac{2\pi \eta^2(q)}{i}\right)^{N_c} \prod_{i \neq j} \frac{i\theta_1(\tau|u_i - u_j)}{\eta(q)},$$

$$Z_P = \prod_{i=1}^{N_c} \prod_{\alpha=1}^{N_1} \frac{i\eta(q)}{\theta_1(\tau|u_i - \xi_\alpha)},$$

$$Z_\Phi = \prod_{i=1}^{N_c} \prod_{\beta=1}^{N_2} \frac{i\eta(q)}{\theta_1(\tau|u_i + \eta_\beta)},$$

$$Z_\Psi = \prod_{i=1}^{N_c} \prod_{\gamma=1}^{N_3} \frac{i\theta_1(\tau|u_i + \zeta_\gamma)}{\eta(q)}.$$
\[ Z_T = \prod_{\alpha=1}^{N_1} \prod_{\beta=1}^{N_2} \frac{i\theta_1(\tau|\xi_{\alpha} - \eta_{\beta})}{\eta(q)}, \quad Z_\Omega = \prod_{s=1}^{2} \frac{i\theta_1(\tau|\sum_i u_i + \lambda_s)}{\eta(q)}, \quad (6.21) \]

where the fugacities satisfy \( \sum_{\alpha=1}^{N_1} \xi_{\alpha} = 0, \sum_{\beta=1}^{N_2} \eta_{\beta} = 0, \sum_{\gamma=1}^{N_3} \zeta_{\gamma} = 0 \) and \( \sum_{s=1}^{2} \lambda_s = 0 \). Hence, \( \mathcal{M}_{\text{sing}} \) is given by

\[ H_{i\alpha}^p = \{u_i = \xi_{\alpha}\}, \quad H_{i\beta}^q = \{-u_i = -\eta_{\beta}\}. \quad (6.22) \]

If we take \( \eta = (1, \cdots, 1) \), then we have contributions only from \( H_{i\alpha}^p \) and get

\[ Z_{T^2} = \sum_{I \in \mathcal{C}(N_c, N_1)} \left[ \prod_{\beta \in I} \frac{i\eta(q)}{\theta_1(\tau|\eta_{\beta} - \xi_{\alpha})} \right] \left[ \prod_{\alpha \neq I} \frac{i\theta_1(\tau|\xi_{\alpha} - \eta_{\beta})}{\eta(q)} \right] \times \left[ \prod_{\alpha \in I} \frac{i\theta_1(\tau|\xi_{\alpha} + \zeta_{\gamma})}{\eta(q)} \right] \left[ \prod_{s=1}^{2} \frac{i\theta_1(\tau|\sum_{\alpha \in I} \xi_{\alpha} + \lambda_s)}{\eta(q)} \right]. \quad (6.23) \]

Also, taking \( \eta = (-1, \cdots, -1) \) and picking up contributions from \( H_{i\beta}^q \) lead us to

\[ Z_{T^2} = \sum_{I \in \mathcal{C}(N_c, N_1)} \left[ \prod_{\beta \in I} \frac{i\eta(q)}{\theta_1(\tau|\eta_{\beta} - \xi_{\alpha})} \right] \left[ \prod_{\alpha \neq I} \frac{i\theta_1(\tau|\xi_{\alpha} - \eta_{\beta})}{\eta(q)} \right] \times \left[ \prod_{\beta \in I} \frac{i\theta_1(\tau|\eta_{\beta} + \zeta_{\gamma})}{\eta(q)} \right] \left[ \prod_{s=1}^{2} \frac{i\theta_1(\tau|\sum_{\beta \in I} \eta_{\beta} + \lambda_s)}{\eta(q)} \right]. \quad (6.24) \]

Let us introduce \( \tilde{I} \in \mathcal{C}(N_1 - N_c, N_1) \). Then, by using \( \prod_{\alpha \in \tilde{I}} = \prod_{\alpha \neq \tilde{I}} \), we find

\[ Z_{T^2} = \sum_{\tilde{I} \in \mathcal{C}(N_1 - N_c, N_1)} \left[ \prod_{\alpha \in \tilde{I}} \frac{i\eta(q)}{\theta_1(\tau|\eta_{\beta} - \xi_{\alpha})} \right] \left[ \prod_{\alpha \neq \tilde{I}} \frac{i\theta_1(\tau|\xi_{\alpha} + \zeta_{\gamma})}{\eta(q)} \right] \times \left[ \prod_{\alpha \in \tilde{I}} \frac{i\theta_1(\tau|\eta_{\beta} + \zeta_{\gamma})}{\eta(q)} \right] \left[ \prod_{s=1}^{2} \frac{i\theta_1(\tau|\sum_{\alpha \in \tilde{I}} \eta_{\beta} + \lambda_s)}{\eta(q)} \right]. \quad (6.25) \]

Comparing this with (6.23), we find that the elliptic genus is invariant under the replacements \((N_1, N_2, N_3) \rightarrow (N_3, N_1, N_2)\) and \((\xi_{\alpha}, \eta_{\beta}, \zeta_{\gamma}) \rightarrow (-\zeta_{\gamma}, \xi_{\alpha}, \eta_{\beta})\). If we repeat the same analysis, then we can obtain the elliptic genus with the replacements \((N_2, N_3, N_1) \rightarrow (N_3, N_1, N_2)\) and \((-\zeta_{\gamma}, \xi_{\alpha}, \eta_{\beta}) \rightarrow (-\eta_{\beta}, -\zeta_{\gamma}, \xi_{\alpha})\). Thus we have confirmed the \((0, 2)\) triality for the elliptic genus.

### 6.4.2 Engineering the 2d \( \mathcal{N} = (0, 2) \) SQCD from 4d SQCD

We discuss that there are 4d SQCDs on \( T^2 \times S^2 \) giving the elliptic genus of the 2d \( \mathcal{N} = (0, 2) \) SQCD described in tab. \( \boxed{1} \) Let us consider the 4d \( \mathcal{N} = 1 \) \( U(N_c) \) SQCD with
Table 12: Matter content of the 4d SQCD giving the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ SQCD. This theory has the rank $N_c = (N_1 + N_2 - N_3)/2$ and superpotential $W = M_i \tilde{Q}_1 Q_i$.

- $N_1$ fundamental multiplets $Q_i$ with R-charge $r_f^{(i)}$
- $N_1$ anti-fundamental multiplets $\tilde{Q}_i$ with R-charge $r_a^{(i)}$
- 2 chiral multiplets $\Omega_a$ in det representation with R-charge 2
- $N_1$ singlet chiral multiplets $M_i$ with R-charge $1 + N_2$ and the superpotential $W = M_i \tilde{Q}_1 Q_i$

We have included the matters in the det representation in order to cancel mixed anomaly between $U(1)_R$ and $U(1)$ part of the gauge group. Conditions for all the gauge anomaly cancellations are boiled down to the following single equation

$$\sum_{i=1}^{N_1} (r_f^{(i)} + r_a^{(i)}) = 2N_1 - 2N_c.$$ (6.26)

Because we do not turn on magnetic flux of flavor symmetries here, we have to take the R-charges to be integers satisfying this condition. Here let us take the R-charges as (the setup is summarized in tab. 12)

$$r_f^{(i)} = 0, \quad r_a^{(1)} = 1 - N_2, \quad r_a^{(2)} = 1 + N_3, \quad r_a^{(i)} = 1 \quad (i = 3, \cdots, N_1).$$ (6.27)

Recalling our formula, we easily see that corresponding zero mode theory of the 4d SQCD on $T^2 \times S^2$ is the 2d $\mathcal{N} = (0, 2)$ SQCD on $T^2$. Indeed there are simple correspondences among the one-loop determinants:

$$\prod_{i=1}^{N_1} Z_{Q_i} = Z_P, \quad Z_{\tilde{Q}_1} = Z_{\Phi}\big|_{\eta_\beta = \{m_\sigma\}}, \quad Z_{\tilde{Q}_2} = Z_{\Psi}\big|_{\zeta_\gamma = \{m_\sigma\}},$$

$$Z_{\tilde{Q}_{i=3, \cdots, N_1}} = 1, \quad \prod_{i=1}^{N_1} Z_{M_i} = Z_{\Gamma}\big|_{\eta_\beta = \{m_\sigma\}}.$$ (6.28)
Table 13: Matter content for the Seiberg dual of the 4d SQCD in tab. 12 which gives the elliptic genus for the triality pair of the 2d $\mathcal{N} = (0, 2)$ SQCD. This theory has the rank $(N_3 + N_1 - N_2)/2$ and superpotential $W = M_i'Q_i'$. 

| $Q_{i=1,\ldots,N_1}$ | $U(N_1 - N_c)$ | $U(1)_R$ | $SU(N_1)$ | $SU(2)$ |
|-----------------------|----------------|----------|------------|----------|
| a-fund                | 0              | fund     | 1          |          |
| fund                  | $1 + N_2$      | 1        | 1          |          |
| fund                  | $1 - N_3$      | 1        | 1          |          |
| fund                  | 1              | 1        | 1          |          |
| fund                  | 1              | $1 + N_3$| a-fund     | 1        |
| $\tilde{Q}_{i=1,\ldots,N_2}$ | a-fund | 0        | fund       | 1        |
| $\tilde{Q}_{i=1,\ldots,N_2}$ | fund | 1        | 1          |          |
| $M_{i=1,\ldots,N_1}$ | 1              | 1        | a-fund     | 1        |
| $\Omega_{s=1,2}$     | det            | 2        | 1          | fund     |
| fugacities            | -              | -        | $\xi_\alpha$| $\lambda_\alpha$ |
| magnetic flux         | -              | 1        | 0          | 0        |

Table 14: Matter content of another 4d SQCD giving the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ SQCD. This is $U(N_c)$ gauge theory with $N_2$ flavors. 

| $Q_{i=1,\ldots,N_1}$ | $U(N_c)$ | $U(1)_R$ | $SU(N_2)$ | $SU(2)$ |
|-----------------------|----------|----------|------------|----------|
| fund                  | 1        | $1 - N_1$| 1          | 1        |
| fund                  | $1 + N_3$| 1        | 1          |          |
| fund                  | 1        | 1        | 1          |          |
| a-fund                | 0        | fund     | 1          |          |
| $M_{i=1,\ldots,N_1}$ | 1        | 1        | a-fund     | 1        |
| $\Omega_{s=1,2}$     | det      | 2        | 1          | fund     |
| fugacities            | -        | -        | $\xi_\alpha$| $\lambda_\alpha$ |
| magnetic flux         | -        | 1        | 0          | 0        |

Thus, the $T^2 \times S^2$ partition function of the 4d SQCD is the same as the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ SQCD with the special fugacities.

As in usual Seiberg duality [3], let us consider the 4d $U(N_1 - N_c)$ SQCD with the matter content listed in tab. 13. Then we easily find that the partition function of this theory is the same as the elliptic genus of the 2d SQCD with $(N_1, N_2, N_3) \rightarrow (N_3, N_1, N_2)$ and $(\xi_\alpha, \eta_\beta, \zeta_\gamma) \rightarrow (\{m\sigma\}, \xi_\alpha, \{m\sigma\})$. Since the 2d SQCD elliptic genus enjoys the $(0, 2)$ triality, this indicates that the 4d SQCD partition function on $T^2 \times S^2$ also enjoys the Seiberg duality.

How can we get the remaining part of the triality $(N_1, N_2, N_3) \rightarrow (N_2, N_3, N_1)$? For this purpose, let us consider the 4d $U(N_c)$ SQCD with $N_2$ flavors described in tab. 14. We easily find that this theory gives the same partition function as the 4d theory in tab. 13 and elliptic genus of the 2d SQCD in tab. 11. If we consider the Seiberg dual of this theory in tab. 15, then its zero-mode theory becomes the 2d SQCD with $(N_1, N_2, N_3) \rightarrow (N_2, N_3, N_1)$ and hence the 4d partition function is the same as the elliptic genus of the 2d SQCD. Thus
Table 15: The Seiberg dual of the 4d SQCD described in tab.\[14\] giving the elliptic genus of the 2d $\mathcal{N} = (0, 2)$ SQCD with $(N_1, N_2, N_3) \rightarrow (N_2, N_3, N_1)$. This theory has the rank $(N_2 + N_3 - N_1)/2$ and the superpotential $W = M'Q'_iQ'_1$.

the 2d $(0, 2)$ triality guarantees the 4d duality for the partition function on $T^2 \times S^2$ and the $(0, 2)$ triality for the elliptic genus comes from the Seiberg duality for the partition function on $T^2 \times S^2$. It is interesting if we further test this in other observables or find more physical arguments as in connection between 4d and 3d dualities [27].

### 7 Conclusion and discussions

In this paper we have studied the partition function of 4d $\mathcal{N} = 1$ supersymmetric gauge theory on $T^2 \times S^2$. We have shown by supersymmetry localization that the partition function on $T^2 \times S^2$ is given by elliptic genus of 2d $\mathcal{N} = (0, 2)$ gauge theory and obtained the exact formula. This result is natural extension of the previous study [1] in theory with only chiral multiplets. Although [2] also discussed theories with vector multiplets, they did not taken the gaugino zero mode into account and did not obtained final formula. We have appropriately treated the gaugino zero mode by relating our analysis to the careful analysis of the elliptic genus [4].

Our result shows that if we consider certain 4d SUSY gauge theory on $T^2 \times S^2$, then we have corresponding 2d SUSY gauge theory on $T^2$, which gives the same partition function. This fact enables us to find nontrivial relations between properties of 4d and 2d supersymmetric gauge theories. Indeed we have shown that the 2d $(0, 2)$ triality [7] for the elliptic genus comes from the 4d Seiberg duality [8] for the partition function on $T^2 \times S^2$. Another possible attractive direction, which we have not pursued here, is symmetry. We have shown that the $T^2 \times S^2$ partition function is given only by the zero-modes along $S^2$, which is described by the 2d $\mathcal{N} = (0, 2)$ supersymmetric theory. This fact implies that the sub-sector of the 4d $\mathcal{N} = 1$ theory would have hidden infinite dimensional symmetry at infrared fixed point. It is interesting if we can relate our result to recent arguments on hidden symmetries in 4d [28, 29, 30, 31]. Also, some 2d CFTs have higher spin symmetry and are expected...
to be dual to Vasiliev theory on $AdS_3$ (see e.g. [32]). It would be illuminating if we can engineer 4d theories on $T^2 \times S^2$, which give elliptic genera of 2d supersymmetric CFTs with higher spin symmetries [33, 34, 35, 36].

It would be interesting to study the partition function of $\mathcal{N} = 2$ Gaiotto theory [37] on $T^2 \times S^2$, which is obtained by compactification of the 6d $\mathcal{N} = (2, 0)$ theory on Riemann surface, and find $T^2 \times S^2$ version of the AGT relation [38]. For this purpose, previous studies on 4d superconformal index [39] would be helpful. It is known that the 4d superconformal indices of class $S$ theories correspond to correlation functions of 2d TQFT on the Riemann surface [40]. We expect that the $T^2 \times S^2$ partition functions of the class $S$ theories have also similar structures.

One of remaining questions is on factorization of supersymmetric partition functions [41]. It is known that partition functions of supersymmetric theories on some spaces exhibit structures like Heegaard decomposition of the spaces. For example, 3d $\mathcal{N} = 2$ theories on squashed $S^3$, $S^2 \times S^1$ and $S^3/\mathbb{Z}_k$ can be interpreted as particular gluings of partition functions on $D^2 \times S^1$ [42, 43, 44]. There is much evidence for this obtained by integration of Coulomb branch localization formula [41, 45, 46, 47] and direct derivation by Higgs branch localization [48, 49] for some theories (see also another argument [50] and similar structures in 2d [51, 52, 53, 54]). Similar structure also appears on 4d superconformal index, which is partition function on $S^3 \times S^1$ [55, 56]. It is natural to wonder if the partition function on $T^2 \times S^2$ can be also interpreted as a gluing of partition functions on $T^2 \times D^2$. Indeed if we further add the deformation term

$$L_H = \delta \left[ \text{Tr}(\zeta \lambda - \bar{\zeta}^\dagger \bar{\lambda}) h(\phi) \right], \quad \text{with } h(\phi) = \frac{i}{2} \left( \phi \bar{\phi} - \text{const.} \times 1 \right),$$

(7.1)

then bosonic part of $L_{\text{vec}} + L_H$ is given by

$$L_{\text{vec}} + L_H|_{\text{bos.}} = \frac{1}{2} F_{\text{ip}}^2 + \frac{1}{4} F_{\text{pq}}^2 + \frac{1}{2} (F_{12} + i h(\phi))^2 - \frac{1}{2} (D - i h(\phi))^2.$$  

(7.2)

Combined with the action of the chiral multiplet, we can show that the saddle point of localization is described by supersymmetric vortex solution with one supercharge. Therefore we can perform Higgs branch localization by using this deformation term in principle. However one of the vortex equations given by (7.2) exists not only on the north or south pole on $S^2$, but also exists on every point on $S^2$. Moreover the vortex preserves only one supercharge. These are different from usual story of the factorization, where the vortex equation (anti-vortex equation) preserves two supercharges, appears only on the north pole (the south pole) in saddle point and each of their world volume theory is captured by 2d $\mathcal{N} = (0, 2)$ theory or its dimensional reductions. Since the vortex preserves only one supercharge for our $T^2 \times S^2$ case, its world volume theory seems to be 2d $\mathcal{N} = (0, 1)$ theory, whose partition function has been less studied. Thus, if factorization occurs also for $T^2 \times S^2$, then its structure would be slightly different from the usual story. It is interesting to further pursue this direction.
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A Convention

Here we summarize our convention of spinors, which is based on [10, 15]. We have two Weyl spinors in representations of the rotational group \( SO(4) = SU(2)_+ \times SU(2)_- \): \( \zeta_\alpha \) in \( SU(2)_+ \) doublet and \( \tilde{\zeta}^{\dot{\alpha}} \) (\( \dot{\alpha} = 1, 2 \)) in \( SU(2)_- \) doublet with \( \alpha, \dot{\alpha} = 1, 2 \). We define contraction, upper and lower indices as

\[
\zeta_{\chi} = \zeta^{\alpha} \chi_{\alpha}, \quad \tilde{\zeta}_{\tilde{\chi}} = \tilde{\zeta}^{\dot{\alpha}} \tilde{\chi}^{\dot{\alpha}}, \quad \zeta_\alpha = \epsilon^{\alpha\beta} \zeta_\beta, \quad \tilde{\zeta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\zeta}^{\dot{\beta}}, \tag{A.1}
\]

with \( \epsilon^{12} = +1, \quad \epsilon_{12} = -1 \). We also define Hermitian conjugates as

\[
(\zeta^\dagger)^\alpha = (\zeta_\alpha)^*, \quad (\tilde{\zeta}^\dagger)^{\dot{\alpha}} = (\tilde{\zeta}_{\dot{\alpha}})^* \tag{A.2}
\]

The sigma matrices are given by

\[
\sigma^{\hat{\mu}}_{\dot{\alpha}\dot{\alpha}} = (\tilde{\sigma}, -i), \quad \tilde{\sigma}^{\hat{\mu}}_{\dot{\alpha}\dot{\alpha}} = (-\tilde{\sigma}, -i), \tag{A.3}
\]

where \( \hat{\mu}, \dot{\nu} = 1, \cdots, 4 \) are local Lorentz indices and \( \tilde{\sigma} \) denotes the Pauli matrices. These satisfy the following useful identities

\[
\begin{align*}
\sigma^{\hat{\mu}} \tilde{\sigma}^{\dot{\nu}} + \sigma^{\dot{\nu}} \tilde{\sigma}^{\hat{\mu}} &= -2\delta^{\hat{\mu}\dot{\nu}}, \\
\tilde{\sigma}^{\hat{\mu}} \sigma^{\dot{\nu}} + \sigma^{\dot{\nu}} \tilde{\sigma}^{\hat{\mu}} &= -2\delta^{\hat{\mu}\dot{\nu}}, \\
\frac{1}{2} \epsilon^{\hat{\mu}\dot{\nu}\dot{\rho}\dot{\lambda}} \sigma^{\hat{\rho}\dot{\lambda}} &= \sigma^{\hat{\mu}\dot{\nu}}, \\
\frac{1}{2} \epsilon^{\hat{\mu}\dot{\nu}\dot{\rho}\dot{\lambda}} \tilde{\sigma}^{\hat{\rho}\dot{\lambda}} &= -\tilde{\sigma}^{\hat{\mu}\dot{\nu}}, \tag{A.4}
\end{align*}
\]

where \( \epsilon_{1234} = 1 \) and

\[
\begin{align*}
\sigma^{\hat{\mu}}_{\dot{\nu}} &= \frac{1}{4}(\sigma^{\hat{\mu}} \tilde{\sigma}^{\dot{\nu}} - \sigma^{\dot{\nu}} \tilde{\sigma}^{\hat{\mu}}), \\
\tilde{\sigma}^{\hat{\mu}}_{\dot{\nu}} &= \frac{1}{4}(\tilde{\sigma}^{\hat{\mu}} \sigma^{\dot{\nu}} - \sigma^{\dot{\nu}} \tilde{\sigma}^{\hat{\mu}}). \tag{A.5}
\end{align*}
\]

B Eta function and theta function

Here we briefly summarize properties of the Dedekind eta function and Jacobi theta function.

B.1 Eta function

The Dedekind eta function is defined by

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{B.1}
\]
where

\[ q = e^{2\pi i \tau}. \]  

(B.2)

This has the following properties

\[ \eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \]  

(B.3)

### B.2 Theta function

The Jacobi theta function is defined by

\[ \theta_1(\tau|z) = -iq^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{k=1}^{\infty} (1 - q^k)(1 - yq^k)(1 - y^{-1}q^{k-1}), \]  

(B.4)

where

\[ y = e^{2\pi iz}. \]  

(B.5)

This satisfies some transformation properties

\[ \theta_1(\tau|z + a + b\tau) = (-1)^{a+b} e^{-2\pi b z - i\pi b^2 \tau} \theta_1(\tau|z) \quad \text{with } a, b \in \mathbb{Z}, \]

\[ \theta_1(\tau + 1|z) = e^{\pi i \frac{z}{\tau} \frac{1}{\tau}} \theta_1 \left( \frac{1}{\tau} \right), \quad \theta_1 \left( \frac{-1}{\tau} \right) = -i \sqrt{-i\tau} e^{\frac{\pi iz^2}{\tau}} \theta_1(\tau|z). \]  

(B.6)

The following formula is useful for picking up residues

\[ \frac{1}{2\pi i} \oint_{u=a+b\tau} du \frac{1}{\theta_1(\tau|u)} = \frac{(-1)^{a+b} e^{i\pi b^2 \tau}}{2\pi \eta^3(\tau)} \quad \text{with } a, b \in \mathbb{Z}. \]  

(B.7)

### C Zeta function regularization and scheme dependent factor

The one-loop determinants are expressed in terms of the following infinite product

\[ \prod_{n_1, n_2 \in \mathbb{Z}} (n_1 + n_2 \tau + z). \]  

(C.1)

Since this infinite product is divergent, we have to specify a regularization scheme. We evaluate the infinite product by the two different ways and see the scheme dependent factor.

First, we split the infinite product as

\[ \prod_{n_1, n_2 \in \mathbb{Z}} (n_1 + n_2 \tau + z) = \prod_{n_1 \in \mathbb{Z}} (n_1 \tau + z) \prod_{n_2=1}^{\infty} \left[ 1 - \frac{(n_1 \tau + z)^2}{n_2^2} \right] \]

\[ = \prod_{n_1 \in \mathbb{Z}} 2(-1)^{\frac{1}{2}} \sin \pi(n_1 \tau + z) \]  

(C.2)
From the first line to the second line in the above equation, we have used \( \pi z \prod_{n \in \mathbb{Z}} (1 - z^2/n^2) = \sin \pi z \) and \( e^{\sum_{n=1}^{\infty} \log(-n^2)} = e^{\zeta(0) \log(-1) - 2\zeta'(0)} = 2\pi(-1)^{\frac{1}{2}} \). We split the above divergent infinite product as
\[
\prod_{n \in \mathbb{Z}} 2(-1)^{\frac{1}{2}} \sin(n\tau + z) = 2(-1)^{\frac{1}{2}} \sin \pi z \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau} e^{2\pi i z}) (1 - e^{-2\pi i n \tau + 2\pi i z})
\]
(C.3)

Again we have used the zeta function regularization \( \prod_{n=1}^{\infty} x^{2} = e^{\zeta(0) \log x^2} = x^{-1} \) and \( \prod_{n=1}^{\infty} e^{nx} = e^{x\zeta(-1)} = e^{\frac{1}{12}} \). Then we obtain a regularized infinite product as
\[
\prod_{n,m \in \mathbb{Z}} (n + m\tau + z) = \frac{\theta_1(\tau|z)}{\eta(\tau)}.
\]
(C.4)

In our paper, we adapt this regularization scheme and also include fugacity independent over all constant \( i \) in front of the one-loop determinant which is not fixed by the localization argument.

Next, we evaluate the infinite product by another regularization scheme. We introduce the following regularized infinite products by the double gamma function \( \Gamma_2(z|\omega_1,\omega_2) \) or double zeta function \( \zeta(s,z|\omega_1,\omega_2) \) as
\[
\prod_{n_1,n_2 \geq 0} (n_1\omega_1 + n_2\omega_2 + z) := \Gamma_2(z|\omega_1,\omega_2)^{-1} := \exp(-\zeta'(s=0,z|\omega_1,\omega_2))
\]
(C.5)

with
\[
\zeta(s,z|\omega_1,\omega_2) := \sum_{n_1,n_2 \geq 0} (n_1\omega_1 + n_2\omega_2 + z)^{-s}.
\]
(C.6)

Then we define a regularization of (C.1) in terms of the double gamma function as
\[
\prod_{n_1,n_2 \in \mathbb{Z}} (n_1 + n_2\tau + z) = \Gamma_2(z|1,\tau)\Gamma_2(z - |1,-\tau)\Gamma_2(1 - z|1,-\tau)\Gamma_2(1 + \tau - z|1,\tau)^{-1}
\]
\[= e^{\frac{2\pi}{\tau}(z^2 - z + \frac{1}{6})} \frac{\theta_1(\tau|z)}{\eta(\tau)}. \]
(C.7)

From the first line to the second line, we have used proposition 2 in [57].

In the second regularization scheme an additional factor \( e^{\frac{2\pi}{\tau}(z^2 - z + \frac{1}{6})} \) appeared in (C.7). This factor breaks invariance under the large gauge transformation or integer shifts \( u \rightarrow u + n, (n \in \mathbb{Z}) \). Although, it is not clear that this factor have physical meaning or it can be eliminated by local counter terms in 4d, we briefly study the cancellation conditions of this anomalous factor. The coefficients of quadratic terms for fugacities \( u \) and \( \xi \) and \( \sigma \) in the exponential are proportional to
\[
u^a u^b : \text{Tr}_{\text{adj}}(H^a H^b) + \sum_i (r_i - 1) \text{Tr}_{R_i}(H^a H^b),
\]
(C.8)
\[ u^a \xi^b : \sum_i (r_i - 1) \text{Tr}_{R_i}(H^a) q^{(i,b)}_f, \quad (C.9) \]
\[ \xi^a \xi^b : \sum_i (r_i - 1) q^{(i,a)}_f q^{(i,b)}_f, \quad (C.10) \]
\[ \sigma^2 : \sum_i \sum_{r_i > \frac{r}{2} + 1} m^2 - \sum_i \sum_{r_i < \frac{|r|}{2} - 1} m^2. \quad (C.11) \]

Here \( \{H^a\}_{a=1,\cdots,|G|} \) is the generator of Cartan subalgebra of Lie algebra of \( G \). (C.8) and (C.9) are same as the coefficients of gauge-gauge and gauge-flavor anomaly, respectively and also (C.10), (C.11) are the one of flavor-flavor anomalies. When the gauge-gauge and gauge-flavor anomalies in two dimensions are canceled, the one-loop determinant is invariant under the integer shift. Next linear terms in the exponential is written as
\[ u^a : \sum_i (r_i - 1) \text{Tr}_{R_i}(H^a), \quad \xi^a : \sum_i (r_i - 1) q^{(i,a)}_f. \quad (C.12) \]

The coefficient of \( u^a \) in (C.12) is proportional to axial anomaly in two dimensions. The fugacity independent factor is given by \( e^{\frac{\pi i}{6} \tau (|G|^2 + \sum_i (r_i - 1)|R_i|)} \).

**D Monopole spherical harmonics on \( S^2 \)**

In this appendix, we briefly summarize properties of monopole spherical harmonics on \( S^2 \).

**D.1 Scalar Monopole spherical harmonics**

Laplacian on \( S^2 \) is given by
\[ \Delta_{S^2} = - (1 + z \bar{z})^2 \partial_z \partial_{\bar{z}} - \frac{r}{2} (1 + z \bar{z}) \left( z \partial_z - \bar{z} \partial_{\bar{z}} - \frac{r}{2} \right) - \frac{r^2}{4}. \quad (D.1) \]

The scalar monopole spherical harmonics \( Y_{r,Jm} \) satisfies
\[ \Delta_{S^2} Y_{r,Jm} = \left( J(J + 1) - \frac{r^2}{4} \right) Y_{r,Jm}, \quad J^2 Y_{r,Jm} = J(J + 1) Y_{r,Jm}, \quad J_3 Y_{r,Jm} = m Y_{r,Jm}, \]
\[ \int d\!zd\!\bar{z} \sqrt{g_{S^2}} Y_{r,Jm} \bar{Y}_{r,Jm'} = \delta_{J,J'} \delta_{mm'}, \quad (D.2) \]

where
\[ J = \frac{|r|}{2}, \frac{|r|}{2} + 1, \cdots. \quad (D.3) \]
D.2 Spinor Monopole spherical harmonics

Dirac operator on $S^2$ in our notation is

$$\left(\tilde{\sigma}^1 D_1 + \tilde{\sigma}^2 D_2\right)\dot{\alpha}_\alpha = -\begin{pmatrix} 0 & (1 + z\bar{z})\partial_{\bar{z}} + \frac{r-2}{2} \bar{z} \\ (1 + z\bar{z})\partial_{z} - \frac{r-2}{2} z & 0 \end{pmatrix}.$$ \hfill (D.4)

Then, the spinor harmonics satisfy

$$\left(\tilde{\sigma}^1 D_1 + \tilde{\sigma}^2 D_2\right)\dot{\alpha}_\alpha \psi^{\pm}_{r-1,Jm} = \pm i \sqrt{\left(J + \frac{1}{2}\right)^2 - \frac{(r-1)^2}{4}} \psi^{\pm}_{r-1,Jm},$$ \hfill (D.5)

where $J = |r| - \frac{1}{2}, \frac{|r|}{2} + \frac{1}{2}, \cdots$, and

$$(\psi^{\pm}_{r-1,Jm})_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm Y_{r,Jm} \\ -i Y_{r-2,Jm} \end{pmatrix}.$$ \hfill (D.6)

This relation leads the useful identities

$$\left(\tilde{\sigma}^1 D_1 + \tilde{\sigma}^2 D_2\right) \begin{pmatrix} 0 \\ Y_{r-2,Jm} \end{pmatrix} = -i \sqrt{\left(J + \frac{1}{2}\right)^2 - \frac{(r-1)^2}{4}} \begin{pmatrix} Y_{r,Jm} \\ 0 \end{pmatrix},$$ \hfill (D.7)

D.3 Vector spherical harmonics

Here we need only usual vector spherical harmonics. According to notation of [54], the harmonics satisfy

$$\Delta_{S^2} C_{i,Jm}^\rho = - (J(J+1) - 1) C_{i,Jm}^\rho, \quad D^{i(0)} C_{i,Jm}^1 = -\sqrt{J(J+1)} Y_{0,Jm},$$
$$D^{i(0)} C_{i,Jm}^2 = 0, \quad D^{(0)} C_{i,Jm}^1 = 0, \quad D^{(0)} C_{j,Jm}^2 = -\sqrt{J(J+1)} Y_{0,Jm},$$
$$\int dzd\bar{z} \sqrt{g_{S^2}} g_{S^2}^{ij} c_{i,Jm}^\rho c_{j,J'm'}^{\rho'} = \delta_{\rho\rho'} \delta_{J J'} \delta_{m m'}.$$ \hfill (D.8)

E 2d supersymmetric gauge theory on $T^2$

In this appendix we write down actions and supersymmetric transformations of 2d $\mathcal{N} = (2, 2)$ vector multiplet, $\mathcal{N} = (2, 2)$ chiral multiplet, $\mathcal{N} = (0, 2)$ vector multiplet, $\mathcal{N} = (0, 2)$ chiral multiplet and $\mathcal{N} = (0, 2)$ Fermi multiplet on $T^2$.
E.1 $\mathcal{N} = (2, 2)$ vector multiplet

Lagrangian for $\mathcal{N} = (2, 2)$ SYM can be obtained by dimensional reduction of 4d $\mathcal{N} = 1$ SYM on flat space along (1, 2)-direction:

$$\mathcal{L}_{\text{vec}} = \text{Tr} \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} D^2 + \frac{i}{2} \lambda^\mu D_\mu \lambda + \frac{i}{2} \tilde{\lambda}^\mu D_\mu \tilde{\lambda} \right], \quad (E.1)$$

where $F_{ij} = -i[A_i, A_j]$, $F_{pi} = D_p A_i$, $D_i (\cdot) = -i [A_i, (\cdot)]$. The action is invariant under

$$\delta A_\mu = i \zeta^{\alpha} \lambda + i \tilde{\zeta}^{\dot{\alpha}} \tilde{\lambda}, \quad \delta \lambda = F_{\mu\nu} \sigma^{\mu\nu} \zeta + i D \zeta,$$
$$\delta \tilde{\lambda} = F_{\mu\nu} \tilde{\sigma}^{\mu\nu} \tilde{\zeta} - i D \tilde{\zeta}, \quad \delta D = -\zeta^{\mu} D_\mu \lambda + \tilde{\zeta}^{\mu} D_\mu \tilde{\lambda}. \quad (E.2)$$

E.2 $\mathcal{N} = (2, 2)$ chiral multiplet

The Lagrangian for $\mathcal{N} = (2, 2)$ chiral multiplet is also obtained by dimensional reduction of 4d $\mathcal{N} = 1$ chiral multiplet:

$$\mathcal{L}_{\text{chi}} = D_\mu \tilde{\phi} D^{\mu} \phi + \tilde{\phi} D \phi - \tilde{F} F + i \tilde{\psi} \tilde{\sigma}^{\mu} D_\mu \psi + i \sqrt{2} (\tilde{\phi} \lambda \psi - \tilde{\psi} \lambda \phi), \quad (E.3)$$

and the supersymmetric transformation is

$$\delta \phi = \sqrt{2} \zeta \psi, \quad \delta \psi = \sqrt{2} F \zeta + i \sqrt{2} (\tilde{\sigma}^{\mu} \zeta) D_\mu \phi,$$
$$\delta F = i \sqrt{2} \tilde{\zeta} \tilde{\sigma}^{\mu} D_\mu \psi - 2 i (\tilde{\zeta} \lambda) \phi,$$
$$\delta \tilde{\phi} = \sqrt{2} \tilde{\zeta} \tilde{\psi}, \quad \delta \tilde{\psi} = \sqrt{2} \tilde{F} \tilde{\zeta} + i \sqrt{2} (\tilde{\sigma}^{\mu} \zeta) D_\mu \tilde{\phi},$$
$$\delta \tilde{F} = i \sqrt{2} \zeta \sigma^{\mu} D_\mu \tilde{\psi} + 2 i \tilde{\phi} (\zeta \lambda). \quad (E.4)$$

We can also add superpotential term:

$$\mathcal{L}_{\text{pt}} = F_i W_i + \tilde{F} \tilde{W}_i - \frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} \tilde{W}_{ij} \tilde{\psi}^i \tilde{\psi}^j. \quad (E.5)$$

E.3 $(0, 2)$ SYM

We can get Lagrangian and SUSY transformation of $(0, 2)$ SYM by taking

$$\zeta_\alpha = \begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix}, \quad \tilde{\zeta}^{\dot{\alpha}} = \begin{pmatrix} \tilde{\zeta}_+ \\ 0 \end{pmatrix}, \quad \lambda_\alpha = \begin{pmatrix} 0 \\ \lambda_+ \end{pmatrix}, \quad \tilde{\lambda}^{\dot{\alpha}} = \begin{pmatrix} \tilde{\lambda}_+ \\ 0 \end{pmatrix}. \quad (E.6)$$

Then we find

$$\mathcal{L}_T = \text{Tr} \left[ \frac{1}{4} F_{pq} F_{pq} - \frac{1}{2} D^2 + \tilde{\lambda}_+ D_\omega \lambda_+ \right], \quad (E.7)$$
\[ \delta A_w = -2\zeta_+ \bar{\lambda}_+ + 2\tilde{\zeta}_+ \lambda_+ , \quad \delta A_{\bar{w}} = 0 , \]
\[ \delta \lambda_+ = i\mathcal{F}_{34} \zeta_+ + iD\zeta_+ , \quad \delta \tilde{\lambda}_+ = i\mathcal{F}_{34} \tilde{\zeta}_+ - iD\tilde{\zeta}_+ , \]
\[ \delta D = i\zeta_+ D_{\bar{w}} \bar{\lambda}_+ - i\tilde{\zeta}_+ D_{\bar{w}} \lambda_+ . \quad (E.8) \]

### E.4 \((0, 2)\) decomposition of \(\mathcal{N} = (2, 2)\) chiral multiplet

Let us take
\[ \zeta_\alpha = \begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix}, \quad \tilde{\zeta}_\dot{\alpha} = \begin{pmatrix} \tilde{\zeta}_+ \\ 0 \end{pmatrix}, \quad \lambda_\alpha = \begin{pmatrix} 0 \\ \lambda_+ \end{pmatrix}, \quad \tilde{\lambda}_\dot{\alpha} = \begin{pmatrix} \tilde{\lambda}_+ \\ 0 \end{pmatrix}, \quad (E.9) \]
and decompose the matter fermions as
\[ \psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \quad \tilde{\psi}_\dot{\alpha} = \begin{pmatrix} \tilde{\psi}_+ \\ \tilde{\psi}_- \end{pmatrix}. \quad (E.10) \]

Then, we can decompose \(\mathcal{N} = (2, 2)\) chiral multiplet into \((0, 2)\) chiral and Fermi multiplet. Lagrangian for \((0, 2)\) chiral multiplet is given by
\[ L_\Phi = \bar{D}_p \tilde{\phi} D^p \phi + \tilde{\phi} D_\phi \psi_- - \bar{\psi}_- D_\psi \psi_- + i\sqrt{2}(\bar{\phi} \lambda_+ \psi_- + \bar{\psi}_- \lambda_+ \phi). \quad (E.11) \]

This is invariant under
\[ \delta \phi = +\sqrt{2}\zeta_+ \psi_- , \quad \delta \tilde{\phi} = +\sqrt{2}\tilde{\zeta}_+ \tilde{\psi}_-, \]
\[ \delta \psi_- = \sqrt{2}\zeta_+ D_{\bar{w}} \phi , \quad \delta \tilde{\psi}_- = \sqrt{2}\tilde{\zeta}_+ D_{\bar{w}} \tilde{\phi}. \quad (E.12) \]

Lagrangian for Fermi multiplet without potential is
\[ L_{\text{fermi}} = +\tilde{\psi}_+ D_{\bar{w}} \psi_+ - \bar{F} F , \quad (E.13) \]
which is invariant under
\[ \delta \psi_+ = \sqrt{2}F \zeta_+ , \quad \delta \tilde{\psi}_+ = \sqrt{2}\bar{F} \tilde{\zeta}_+, \]
\[ \delta F = +\sqrt{2}\tilde{\zeta}_+ D_{\bar{w}} \psi_+, \quad \delta \bar{F} = +\sqrt{2}\zeta_+ D_{\bar{w}} \tilde{\psi}_+. \quad (E.14) \]

We can also add potential in supersymmetric way:
\[ L_{\text{fermi}} = +\tilde{\psi}_+ D_{\bar{w}} \psi_+ - \bar{F} F + E F + \tilde{E} \bar{F} - \psi_+ \psi_- F + \tilde{\psi}_- \tilde{\psi}_- E, \quad (E.15) \]
where
\[ E = E(\phi) , \quad \tilde{E} = \tilde{E}(\tilde{\phi}) , \quad \psi_- = \frac{\partial E}{\partial \phi^i} \psi_-^i , \quad \tilde{\psi}_- = \frac{\partial \bar{E}}{\partial \tilde{\phi}^i} \tilde{\psi}_-^i. \quad (E.16) \]

Note that this potential terms themselves are \(\delta\)-exact:
\[ \delta(\psi_+ E) = \sqrt{2}\zeta_+ (F E - \psi_+ \psi_- E) , \quad \delta(\tilde{\psi}_+ \tilde{E}) = \sqrt{2}\tilde{\zeta}_+ (\bar{F} \bar{E} - \tilde{\psi}_- \tilde{\psi}_- E) . \quad (E.17) \]
Hence elliptic genus should be independent of parameters in $E$ and $\tilde{E}$. Redefining

$$G = F - \tilde{E}, \quad \tilde{G} = \tilde{F} - E,$$

we rewrite the Lagrangian as

$$\mathcal{L}_{\text{fermi}} = +\tilde{\psi}_+ D_\omega \psi_+ - \tilde{G} G + \tilde{E} E - \psi_+ \psi_+^E - \tilde{\psi}_+ \tilde{\psi}_+^E,$$

where supersymmetric transformation is given by

$$\delta \psi_+ = \sqrt{2}(G + \tilde{E})\zeta_+ + \sqrt{2}(\tilde{G} + E)\tilde{\zeta}_+, \quad \delta \tilde{\psi}_+ = \sqrt{2}(\tilde{G} + E)\tilde{\zeta}_+, \quad \delta G = +\sqrt{2}\zeta_+ D_\omega \psi_+ - \sqrt{2}\tilde{\zeta}_+ \psi_+^E, \quad \delta \tilde{G} = +\sqrt{2}\tilde{\zeta}_+ D_\omega \tilde{\psi}_+ - \sqrt{2}\psi_+ \psi_+^E.$$  \hspace{1cm} (E.20)

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