ON UNRAMIFIED BRAUER GROUPS OF TORSORS OVER TORI

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0. Introduction

In this paper, we use arithmetic information to obtain algebraic ones. Let $G$ be a finite group, and let $M$ be a $G$-lattice. Let $\ell'/\ell$ a finite unramified extension of number fields with Galois group $G$; such an extension exists by [F 62]. Let $T$ be an $\ell$-torus with character group $M$. We have the following isomorphisms

\[(*) \quad \mathbb{I}^2_{\text{cycl}}(G, M) \simeq \mathbb{I}^2(\ell, M) \simeq \mathbb{I}^1(\ell, T)^*;\]

the first isomorphism is Proposition 3.1, the second one follows from Poitou-Tate duality (see §1 and §2 for the notation). In the following, we combine $(*)$ with arithmetic results as well as some theorems of Colliot-Thélène and Sansuc; we illustrate the results with the following example (see §10):

Example. Let $k$ be a field, and let $L = K_1 \times \cdots \times K_n$, where $K_1, \ldots, K_n$ are cyclic extensions of $k$ of prime degree $p$. Let $N_{L/k} : L \to k$ denote the norm map, and let $T_{L/k} = R^{(1)}_{L/k}(\mathbb{G}_m)$ be the $k$-torus defined by

\[1 \to T_{L/k} \to R_{L/k}(\mathbb{G}_m)^{N_{L/k}} \to \mathbb{G}_m \to 1.\]

Let $k'/k$ be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \text{Gal}(k'/k)$. Set $T = T_{L/k}$, let $T^c$ be a smooth compactification of $T$, and let $\text{Br}(T^c)$ be its Brauer group.

Let $a \in k^\times$, and let $X$ be the affine $k$-variety determined by the equation $N_{L/k}(t) = a$; $X$ is a torsor under $T_{L/k}$. Let $X^c$ be a smooth compactification of $X$. We denote by $\text{Br}(X^c)$ the Brauer group of $X^c$.

Theorem. (a) If $G \not\simeq C_p \times C_p$, then

\[\text{Br}(T^c)/\text{Br}(k) = \text{Br}(X^c)/\text{Im(\text{Br}(k))} = 0.\]

(b) If $G \simeq C_p \times C_p$, then

\[\text{Br}(T^c)/\text{Br}(k) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}.\]

(c) If $G \simeq C_p \times C_p$ and if $\text{char}(k) \neq p$, then

\[\text{Br}(X^c)/\text{Im(\text{Br}(k))} \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}.\]

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Parts (a) and (b) are proved in Theorem 10.1 using (*) as well as some (arithmetic) results of [BLP 19]. Part (c) is proved by a combination of arithmetic and algebraic methods. Indeed, $\text{Br}(X^c)/\text{Im}(\text{Br}(k))$ injects into $\text{Br}(T^c)/\text{Br}(k)$ (see §5). We then use algebraic methods to show that if $\text{char}(k) \neq p$, then $\text{Br}(X^c)/\text{Im}(\text{Br}(k))$ contains $n - 2$ elements that are linearly independent over $\mathbb{Z}/p\mathbb{Z}$, hence we obtain part (c) of the above theorem (see Theorems 10.2 and Theorem 10.3).

Further, we also give generators for the group $\text{Br}(X^c)/\text{Im}(\text{Br}(k))$ (see Theorem 10.3), in the spirit of Colliot-Thélène’s results for biquadratic extensions (see [CT 14], §4).

We also obtain some results for tori over number fields. The starting point is the following key observation of Jean-Louis Colliot-Thélène:

**Proposition.** Let $G$ be a finite group, and let $M$ be a $G$-lattice. If for all number fields $k$ and every $k$-torus $T$ with character group isomorphic to the Galois module $M$ via a surjection $\Gamma_k \to G$ one can show that $\text{III}^1(k, T) = 0$, then $\text{III}^2_{\text{cycl}}(G, M) = 0$, and for every number field $k$ every $k$-torus $T$ with group of characters $M$ has weak approximation.

Since every finite group is the Galois group of some unramified extension of number fields, we may realize the purely algebraic group $\text{III}^2_{\text{cycl}}(G, M)$ as the Tate-Shafarevich group of a torus over a number field; this is summarized in (∗). If the module satisfies the hypothesis of the proposition, then $\text{III}^2_{\text{cycl}}(G, M) = 0$, and weak approximation follows from an exact sequence due to Voskresenskii (see 2.4). As we will see, the hypotheses of the proposition can be weakened (see Corollary 4.3).

We thank Jean-Louis Colliot-Thélène for sharing his ideas with us, as well as for several useful suggestions.

1. **Algebraic preliminaries**

Let $k$ be a field, let $k_s$ be a separable closure of $k$ and set $\Gamma_k = \text{Gal}(k_s/k)$. We fix once and for all this separable closure $k_s$, and all separable extensions of $k$ that will appear in the paper will be contained in $k_s$. We use standard notation in Galois cohomology; in particular, if $M$ is a discrete $\Gamma_k$-module and $i$ is an integer $\geq 0$, we set $H^i(k, M) = H^i(\Gamma_k, M)$. A $\Gamma_k$-lattice will be a torsion free $\mathbb{Z}$-module of finite rank on which $\Gamma_k$ acts continuously.

**Lemma 1.1.** Let $M$ be a $\Gamma_k$-lattice, and let $k'/k$ be a finite Galois extension with Galois group $G$ such that $\Gamma_{k'}$ acts trivially on $M$. Then the natural map $H^2(G, M) \to H^2(k, M)$ has trivial kernel.

**Proof.** Since $M$ is isomorphic to the trivial $\Gamma_{k'}$-module $\mathbb{Z}^n$ for some $n$, we have $H^1(k', M) = 0$. Hence the exact sequence of groups $0 \to \Gamma_{k'} \to \Gamma_k \to G \to 0$ induces an exact sequence in Galois cohomology (cf. [S 79], page 118, Remark).

\[
(*) \quad 0 \to H^2(G, M) \to H^2(k, M) \to H^2(k', M)^G.
\]
Therefore the map $H^2(G, M) \to H^2(k, M)$ has trivial kernel, as claimed.

Let $G$ be a finite group. A $G$-lattice is by definition a $\mathbb{Z}$-torsion free $\mathbb{Z}[G]$-module of finite rank. For a $k$-torus $T$, we denote by $\hat{T} = \text{Hom}(T, \mathbb{G}_m)$ its character group; it is a $\Gamma_k$-lattice.

**Proposition 1.2.** Let $M$ be a $G$-lattice. There exists a $k$-torus $T$ such that $\hat{T}$ is isomorphic to the $G$-lattice $M$, regarded as $\Gamma_k$-lattice through the surjection $\Gamma_k \to G$.

**Proof.** See [Bo 91], Chapter III, 8.12.

If $g \in G$, we denote by $\langle g \rangle$ the cyclic subgroup of $G$ generated by $g$. Let $M$ be a $G$-lattice. The cyclic Tate-Shafarevich group $\tilde{\I}_2^\text{cycl}(G, M)$ is the group

$$\tilde{\I}_2^\text{cycl}(G, M) = \text{Ker}[H^2(G, M) \to \prod_{g \in G} H^2(\langle g \rangle, M)].$$

We recall a result of Colliot-Thélène and Sansuc:

**Theorem 1.3.** Let $G$ be a finite group, let $T$ be a $k$-torus, and assume that the character group of $T$ is a $G$-lattice via a surjection $\Gamma_k \to G$. Let $T^c$ be a smooth compactification of $T$. We have $\text{Br}(T^c)/\text{Br}(k) \simeq \tilde{\I}_2^\text{cycl}(G, \hat{T})$.

**Proof.** See [CTS 87], Theorem 9.5 (ii). In [CTS 87], the hypothesis $\text{char}(k) = 0$ is only used to ensure the existence of a smooth compactification of $T$; since this is now known in any characteristic (see [CTHSk 05]), the result holds in general.

Let $Y$ be a smooth projective, geometrically connected $k$-variety, and set $\overline{Y} = Y \times_k k_s$. We have the following spectral sequence

$$H^p(k, H^q(\overline{Y}, \mathbb{G}_m)) \Rightarrow H^{p+q}(Y, \mathbb{G}_m)$$

giving the exact sequence

$$H^2(k, \mathbb{G}_m) \to \text{Ker}[H^2_{\text{et}}(\overline{Y}, \mathbb{G}_m) \to H^2_{\text{et}}(\overline{\overline{Y}}, \mathbb{G}_m)] \to H^1(k, \text{Pic}(\overline{Y})) \to$$

$$\to \text{Ker}[H^3_{\text{et}}(k, \mathbb{G}_m) \to H^3_{\text{et}}(Y, \mathbb{G}_m)].$$

We refer to [CTHSk 03], Section 2 for the following theorem.

**Theorem 1.4.** Let $Y$ be a smooth projective variety defined over $k$ with $\overline{Y}$ birational to the projective space. Then there is an injection

$$\text{Br}(Y)/\text{Im}(\text{Br}(k)) \to H^1(k, \text{Pic}(\overline{Y})).$$

If further $Y(k) \neq \emptyset$, there is an isomorphism $\text{Br}(Y)/\text{Br}(k) \simeq H^1(k, \text{Pic}(\overline{Y}))$.

**Proof.** Since $\overline{Y}$ is birational to the projective space, $H^3_{\text{et}}(\overline{Y}, \mathbb{G}_m) = \text{Br}(\overline{Y}) = 0$ (cf. [CTSk 19], Theorem 5.1.3, Corollary 5.2.6) and we have an injection

$$\text{Br}(Y)/\text{Im}(\text{Br}(k)) \to H^1(k, \text{Pic}(\overline{Y})).$$

If further $Y(k) \neq \emptyset$, $\text{Ker}[H^i(k, \mathbb{G}_m) \to H^i_{\text{et}}(Y, \mathbb{G}_m)] = 0$ for $i = 2, 3$, so that we have an isomorphism

$$\text{Br}(Y)/\text{Br}(k) \simeq H^1(k, \text{Pic}(\overline{Y})).$$
2. Arithmetic preliminaries

Let $k$ be a global field, and let $\Omega_k$ be the set of all places of $k$; if $v \in \Omega_k$, we denote by $k_v$ the completion of $k$ at $v$.

For any $k$-torus $T$, set $\Pi_i^i(k, T) = \text{Ker}(H^i(k, T) \to \prod_{v \in \Omega_k} H^i(k_v, T))$. If $M$ is a $\Gamma_k$-module, set $\Pi_i^i(k, M) = \text{Ker}(H^i(k, M) \to \prod_{v \in \Omega_k} H^i(k_v, M))$, and let $\Pi_{\omega}^i(k, M)$ be the set of $x \in H^i(k, M)$ that map to 0 in $H^i(k_v, M)$ for almost all $v \in \Omega_k$. Recall that by Poitou-Tate duality, we have an isomorphism of finite groups $\Pi^2(k, \hat{T}) \simeq \Pi^1(k, T)^*,$ where $\Pi^1(k, T)^* = \text{Hom}(\Pi^1(k, T), \mathbb{Q}/\mathbb{Z})$ denotes the dual of $\Pi^1(k, T)$. We denote by $G_M$ the kernel of the map $\Gamma_k \to \text{Aut}_k(M)$; set $k_M = (k_\omega)^{G_M}$, and let $G = \Gamma_k/G_M$. If $v \in \Omega_k$, we denote by $G_v$ the decomposition group of a prime $w$ lying over $v$ in the extension $k_m/k$. For various $v$ over $v$, the groups $G_v$’s are conjugate and are subgroups of $G$. Let $\Pi_{\omega}^i(k, M)$ be the set of $x \in H^i(k, M)$ that map to 0 in $H^i(k_v, M)$ for all $v \in \Omega_k$ such that $G_w$ is cyclic for all places $w$ of $k_m$ above $v$.

The following lemmas are well-known:

**Lemma 2.1.** Let $\mathbb{Z}$ be the trivial $\Gamma_k$-module. Then $\Pi_{\omega}^2(k, \mathbb{Z}) = 0$. In particular $\Pi^2(k, \mathbb{Z}) = 0$.

**Proof.** The trivial $\Gamma_k$-module $\mathbb{Q}$ is uniquely divisible, hence $H^i(k, \mathbb{Q}) = 0$ for all $i \geq 1$. Hence the connecting map for the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an isomorphism $H^1(k, \mathbb{Q}/\mathbb{Z}) \simeq H^2(k, \mathbb{Z})$. Thus $\Pi^2(k, \mathbb{Q}/\mathbb{Z}) \simeq \Pi_{\omega}^2(k, \mathbb{Z})$. Since $\Pi_{\omega}^1(k, \mathbb{Q}/\mathbb{Z})$ classifies cyclic extensions of $k$ (together with a generator of the Galois group) which are locally almost everywhere split, by Chebotarev density theorem, $\Pi_{\omega}^1(k, \mathbb{Q}/\mathbb{Z}) = 0$. In particular $\Pi^2(k, \mathbb{Z}) = 0$.

**Lemma 2.2.** Let $M$ be a $\Gamma_k$-module and $G_M$ the kernel of the map $\Gamma_k \to \text{Aut}_k(M)$. Let $G = \Gamma_k/G_M$. The image of the homomorphism $H^2(G, M) \to H^2(k, M)$ contains $\Pi_{\omega}^2(k, M)$.

**Proof.** Let $x \in \Pi_{\omega}^2(k, M)$. Let $k_M = (k_\omega)^{G_M}$. Since $M$ becomes isomorphic $\mathbb{Z}^n$ over $k_M$, by Lemma 2.1, $x$ restricts to zero in $H^2(k_M, M)$. Hence from the exact sequence $(\ast)$, $x$ belongs to the image of $H^2(G, M) \to H^2(k, M)$.

**Lemma 2.3.** Let $M$ be a $\Gamma_k$-module, let $G_M$ be the kernel of the map $\Gamma_k \to \text{Aut}_k(M)$ and let that $\Gamma_k/G_M = G$. Then $\Pi_{\omega}^2(k, M) = \Pi_{\omega}^2(k, M)$.

**Proof.** If $v \in \Omega_k$ is unramified in $k_m/k$, then $G_v$ is cyclic. Hence $\Pi_{\omega}^2(k, M) \subset \Pi_{\omega}^2(k, M)$. We show that $\Pi_{\omega}^2(k, M) \subset \Pi_{\omega}^2(k, M)$. Let $x \in \Pi_{\omega}^2(k, M)$. By Lemma 2.2, there is $y \in H^2(G, M)$ mapping to $x \in H^2(k, M)$. Let $v \in \Omega_k$ be such that $G_v$ is cyclic. By Chebotarev’s density theorem, there exist infinitely many $w \in \Omega_k$ such that $G_w = G_v$. Pick $w \in \Omega_k$ such that $x$ maps to zero in $H^2(k_w, M)$. The image of $x$ in $H^2(k_w, M)$ is the image of $y$ under the composite $H^2(G, M) \to H^2(G_w, M) \to H^2(k_w, M)$. Since $H^2(G_w, M) \to H^2(k_w, M)$ is injective, by $(\ast)$, image of $y$ in $H^2(G_w, M)$ is zero. Thus the image of $y$ in
$H^2(G_v, M)$ is zero and hence its image in $H^2(k_v, M)$ is zero. But this coincides with the image of $x$ in $H^2(k_v, M)$. Thus $x$ maps to zero in $H^2(k_v, M)$. This is true for $v$ with $G_v$ is cyclic so that $x \in \III^2_{cycl}(k, M)$.

Let $\iota : T(k) \to \prod_{v \in \Omega_k} T(k_v)$ be the diagonal embedding, and let $A(T)$ be the quotient of $\prod_{v \in \Omega_k} T(k_v)$ by the closure of the image of $\iota$; the group $A(T)$ is the obstruction to weak approximation on $T$. Set $\III(T) = \III^1(k, T)$; this is the obstruction to the Hasse principle for torsors under $T$.

The following is a reformulation of a result of Voskresenskii:

**Proposition 2.4.** Let $G$ be a finite group, let $T$ be a $k$-torus, and assume that the character group of $T$ is a $G$-lattice via a surjection $\Gamma_k \to G$. We have an exact sequence

$$0 \to A(T) \to \III^2_{cycl}(G, \hat{T})^* \to \III(T) \to 0.$$

**Proof.** Let $T^c$ be a smooth compactification of $T$; by [San 81], Theorem 9.5. (M) we have the exact sequence

$$0 \to A(T) \to \Br_a(T^c)^* \to \III(T) \to 0.$$

Note that since $T_{k_s}$ is split and hence rational, we have $\Br(T^c_{k_s}) = 0$ (see [CTSk 19], Corollary 5.2.6). By Proposition 1.3 we have $\Br(T^c)/\Br(\ell) \simeq \III^2_{cycl}(G, \hat{T})$, hence we get the exact sequence

$$0 \to A(T) \to \III^2_{cycl}(G, \hat{T})^* \to \III(T) \to 0.$$

3. The group $\III^2_{cycl}(G, M)$

Let $G$ be a finite group.

**Proposition 3.1.** Let $\ell'/\ell$ be a finite Galois extension of global fields with Galois group $G$ which is unramified at all the finite places. Let $M$ be a $G$-lattice regarded as a $\Gamma_k$-module via the surjection $\Gamma_k \to \Gamma_{\ell'}$. Then we have

$$\III^2_{cycl}(G, M) \simeq \III^2(\ell, M).$$

This proposition is an immediate consequence of Proposition 3.2 below:

**Proposition 3.2.** Let $\ell'/\ell$ be a finite Galois extension of global fields with Galois group $G$. Assume that all the decomposition groups of $\ell'/\ell$ are cyclic. Let $M$ be a $G$-lattice, regarded as a $\Gamma_{\ell}$-lattice through the surjection $\Gamma_{\ell} \to G$. Then we have

$$\III^2_{cycl}(G, M) \simeq \III^2(\ell, M).$$

Proposition 3.2 follows from Proposition 3.3 below. We use the notation of the previous section.
Proposition 3.3. Let $\ell$ be a global field, let $M$ be a $\Gamma_\ell$-module, and assume that $\Gamma_\ell/G_M \simeq G$. Then we have

$$\text{III}^2_{\text{cycl}}(G, M) \simeq \text{III}^2_{\text{cycl}}(\ell, M).$$

Proof of Proposition 3.3. Set $\ell' = \ell_M$; note that $\ell'/\ell$ is a Galois extension with group $G$. The homomorphism $f : H^2(G, M) \to H^2(\ell, M)$ induced by the surjection $\Gamma_\ell \to G$ is injective by Lemma 1.1. By Lemma 2.2, the image of $f$ contains $\text{III}^2(\ell, M)$. We next prove that the restriction of $f$ to $\text{III}^2_{\text{cycl}}(G, M)$ maps it into $\text{III}^2_{\text{cycl}}(\ell, M)$. Let $x \in \text{III}^2_{\text{cycl}}(G, M)$. Let $v \in \Omega_\ell$ such that the decomposition group $G_v$ of $\ell'/\ell$ at $v$ is a cyclic subgroup of $G$. Then the restriction of $x \in H^2(G, M)$ to $H^2(G_v, M)$ is zero. The composite $H^2(G, M) \to H^2(\ell, M) \to H^2(\ell_v, M)$ factors as $H^2(G, M) \to H^2(G_v, M) \to H^2(\ell_v, M)$. Hence $f(x)$ maps to zero in $H^2(\ell_v, M)$. Thus $f(x) \in \text{III}^2_{\text{cycl}}(\ell, M)$.

Clearly $\text{III}^2_{\text{cycl}}(G, M) \to \text{III}^2_{\text{cycl}}(\ell, M)$ is injective. We prove that this map is surjective. Let $y \in \text{III}^2_{\text{cycl}}(\ell, M)$ and let $x \in H^2(G, M)$ be such that $f(x) = y$. Let $g \in G$. By Chebotarev’s density theorem, there is a finite place $v \in \Omega_\ell$ and a place $w$ of $\ell'$ above $v$ such that $G_w = \langle g \rangle$. We claim that the restriction of $x$ to $H^2(\langle g \rangle, M) = H^2(G_v, M)$ maps to zero in $H^2(l_w, M)$. In fact this image is the same as the restriction of $y$ to $H^2(l_w, M)$. Since $y \in \text{III}^2_{\text{cycl}}(\ell, M)$ and $G_v$ is cyclic, the image of $y$ in $H^2(l_w, M)$ is zero. By Lemma 1.1 the map $H^2(G_v, M) \to H^2(\ell_v, M)$ is injective. It follows that the restriction of $x$ to $H^2(G_v, M)$ is zero, hence $x$ belongs to $\text{III}^2_{\text{cycl}}(G, M)$. This completes the proof of the proposition.

Proof of Proposition 3.2. Since all the decomposition groups of $\ell'/\ell$ are cyclic, we have $\text{III}^2(\ell, M) = \text{III}^2_{\text{cycl}}(\ell, M)$. By Proposition 3.3 we have $\text{III}^2_{\text{cycl}}(G, M) \simeq \text{III}^2_{\text{cycl}}(\ell, M)$, hence $\text{III}^2_{\text{cycl}}(G, M) \simeq \text{III}^2(\ell, M)$, as claimed.

Proof of Proposition 3.1: Since $\ell'/\ell$ is unramified at all the finite places, all the decomposition groups are cyclic; we conclude by applying Proposition 3.2.

Corollary 3.4. Let $\ell$ be a global field, let $M$ be a $\Gamma_\ell$-module, and assume that $\Gamma_\ell/G_M \simeq G$. Then we have

$$\text{III}^2_{\text{cycl}}(G, M) \simeq \text{III}^2(\ell, M).$$

Proof. This follows from Proposition 3.3 and Lemma 2.3.

Corollary 3.5. Let $M$ be a $G$-lattice. Let $\ell'/\ell$ be as in Proposition 3.2 and let $T$ be an $\ell$-torus with character group $M$. We have

$$\text{III}^2_{\text{cycl}}(G, M) \simeq \text{III}^2(\ell, M) = \text{III}^2(\ell, \hat{T}) \simeq \text{III}^1(\ell, T)^*.$$

Proof. This follows from Proposition 3.2 and from Poitou-Tate duality.

In the following sections, we also need a result of Fröhlich:
Proposition 3.6. (Fröhlich) There exists a Galois extension of number fields with Galois group $G$ that is unramified at all the finite places.

Proof. This is the main result of [F 62].

In the next sections we use Corollary 3.5 together with 3.6 to realize the purely algebraic group $\mathcal{III}^2_{cycl}(G, M)$ as the Tate-Shafarevich group of a torus of over a number field. This makes it possible to apply arithmetic results to obtain algebraic ones.

4. Vanishing results

The aim of this section is to apply the results of §3 and, under an additional hypothesis (condition (C) below) prove some vanishing theorems. Let $G$ be a finite group.

Definition 4.1. Let $M$ be a $G$-lattice. We say that $M$ satisfies condition (C) if there exists a Galois extension $\ell'/\ell$ of number fields with Galois group $G$ such that all the decomposition groups of $\ell'/\ell$ are cyclic, and such that the $\ell$-torus $S$ associated to the Galois lattice $M$ with the Galois group $\Gamma_\ell$ acting through the quotient group $G$ has the property $\mathcal{III}^1(\ell, S) = 0$.

Corollary 4.2. Let $M$ be a $G$-lattice satisfying condition (C). Then $\mathcal{III}^2_{cycl}(G, M) = 0$.

Proof. By Poitou-Tate duality, we have $\mathcal{III}^1(\ell, S) \simeq \mathcal{III}^2(\ell, \hat{S})^* = \mathcal{III}^2(\ell, M)^*$.

Corollary 3.5 implies that the groups $\mathcal{III}^2(\ell, \hat{S}) = \mathcal{III}^2(\ell, M)$ and $\mathcal{III}^2_{cycl}(G, M)$ are isomorphic. Hence $\mathcal{III}^1(\ell, S)$ is dual to the group $\mathcal{III}^2_{cycl}(G, M)$. Since $\mathcal{III}^1(\ell, S) = 0$, we have $\mathcal{III}^2_{cycl}(G, \hat{S}) = \mathcal{III}^2_{cycl}(G, M) = 0$.

Corollary 4.3. Let $k$ be a global field with a surjection $\Gamma_k \rightarrow G$, and let $T$ be a $k$-torus; assume that the character group of $T$ is a $G$-lattice satisfying condition (C). Then $\mathcal{III}^2_\omega(k, \hat{T}) = 0$.

Hasse principle and weak approximation hold for torsors under $T$.

Proof. By Proposition 2.4, we have the exact sequence $0 \rightarrow A(T) \rightarrow \mathcal{III}^2_{cycl}(G, \hat{T})^* \rightarrow \mathcal{III}(T) \rightarrow 0$.

We have $\mathcal{III}^2_{cycl}(G, \hat{T})^* = 0$ by Corollary 4.2 hence $A(T) = \mathcal{III}(T) = 0$. By Lemma 3.4, we have $\mathcal{III}_\omega(k, \hat{T}) = 0$; weak approximation holds for $T$, and Hasse principle holds for torsors under $T$.

This implies Colliot-Thélène’s observation cited in the introduction:

Corollary 4.4. Let $G$ be a finite group, and let $M$ be a $G$-lattice. If for all number fields $k$ and every $k$-torus $T$ of character group isomorphic to the Galois module $M$ via a surjection of $\Gamma_k \rightarrow G$, one can show that $\mathcal{III}^1(k, T) = 0$, then $\mathcal{III}^2_{cycl}(G, M) = 0$, and for every number field $k$ every $k$-torus $T$ with group of characters $M$ has weak approximation.
5. Unramified Brauer groups

Let $k$ be a field, and let $T$ be a $k$-torus, and let $X$ be a torsor under $T$. Let $T^c, X^c = X \times_T T^c$ be the smooth compactifications; these exists in any characteristic (see [CTHSk 05]). Note that $T^c \simeq X^c$ is birational to the projective space; further $T^c(k) \neq 0$. We therefore have exact sequences, by Theorem 1.4,

\[ \text{Br}(k) \to \text{Br}(X^c) \to H^1(k, \text{Pic}(X^c)) \]

and

\[ 0 \to \text{Br}(k) \to \text{Br}(T^c) \to H^1(k, \text{Pic}(T^c)) \to 0. \]

Since $H^1(k, \text{Pic}(X^c)) \simeq H^1(k, \text{Pic}(T^c))$, we have an injection

\[ \text{Br}(X^c)/\text{Im(Br(k)}) \to \text{Br}(T^c)/\text{Br(k)}. \]

The aim of this section and the following ones is to use the results of the previous sections to obtain information on the quotients $\text{Br}(T^c)/\text{Br(k)}$ and $\text{Br}(X^c)/\text{Im(Br(k)})$. We start with some vanishing results.

Let $G$ be a finite group. Recall that a $G$-lattice $M$ satisfies condition (C) if there exists a Galois extension $\ell'/\ell$ of number fields with Galois group $G$ such that all the decomposition groups of $\ell'/\ell$ are cyclic, and such that the the $\ell$-torus $S$ associated to the Galois lattice $M$ with the Galois group $\Gamma_{\ell}$ acting through the quotient group $G$ has the property $\Pi^1(\ell, S) = 0$ (cf. definition 4.1).

Proposition 5.1. Assume that the character group of $T$ is a $G$-lattice satisfying condition (C). Then $\text{Br}(T^c)/\text{Br(k)} = 0$. If $X$ is a torsor over $T$, we have $\text{Br}(X^c)/\text{Im(Br(k)}) = 0$.

Proof. By Corollary 4.2, we have $\Pi^2_{\text{cyd}}(G, \hat{T}) = 0$. On the other hand, Colliot-Thélène and Sansuc proved that $\text{Br}(T^c)/\text{Br(k)} \simeq \Pi^2_{\text{cyd}}(G, \hat{T})$ (cf. [CTS 87], Proposition 9.5 (ii)), therefore $\text{Br}(T^c)/\text{Br(k)} = 0$. As we saw above, $\text{Br}(X^c)/\text{Im(Br(k)})$ injects into $\text{Br}(T^c)/\text{Br(k)}$, hence this implies that $\text{Br}(X^c)/\text{Im(Br(k)}) = 0$.

6. Norm equations

Let $k$ be a field, and let $L$ be an étale $k$-algebra of finite rank (in other words, a product of a finite number of separable extensions of $k$). Let $T_{L/k} = R^{(1)}_{L/k}(G_m)$ be the $k$-torus defined by

\[ 1 \to T_{L/k} \to R_{L/k}(G_m) \xrightarrow{N_{L/k}} G_m \to 1. \]

Let $a \in k^\times$. Let $X$ be the affine $k$-variety associated to the norm equation

\[ N_{L/k}(t) = a. \]
The variety $X$ is a torsor under $T_{L/k}$; let $X^c$ be a smooth compactification of $X$.

The aim of this section and the next ones is to give some examples of étale algebras for which we apply the results of the previous sections, obtaining information about the unramified Brauer group, Hasse principle and weak approximation (in the case where $k$ is a global field) for the variety $X$.

The first examples concern étale algebras that are products of two fields, finite extensions of the ground field $k$.

**Products of two fields**

We start by introducing some notation that will be used in the two examples of this section. Let $L = K_1 \times K_2$, where $K_1$ and $K_2$ are finite extensions of $k$. Let $k'/k$ be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \text{Gal}(k'/k)$; let $M = \hat{T}_{L/k}$ be the character $G$-lattice of $T_{L/k}$. For $i = 1, 2$, let $H_i$ be the subgroup of $G$ such that $K_i = (k')^{H_i}$. We have the exact sequence of $G$-modules

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G/H_1] \oplus \mathbb{Z}[G/H_2] \rightarrow M \rightarrow 0.$$  

Let $\ell' / \ell$ be an unramified extension of number fields with Galois group $G$.

**Some consequences of Hürlimann’s theorem**

The first example is based on a result of Hürlimann [H 84]. With the notation above, we assume that $K_1/k$ is a cyclic extension.

**Theorem 6.1.** We have $\mathbb{II}^2_{\text{cycl}}(G, M) = 0$.

**Proof.** Set $\ell_i = (\ell')^{H_i}$ for $i = 1, 2$. Let $S$ be the norm torus corresponding to the étale $\ell$-algebra $\ell_1 \times \ell_2$. Hürlimann’s result [H 84], Proposition 3.3 implies that $\mathbb{II}^1(\ell, S) = 0$ (in [H 84], the extension $K_2/k$ is supposed to be Galois, but this is not necessary; see [BLP 19], Proposition 4.1 for a different proof of the general case). We have $\hat{S} \simeq M$ by construction, hence by Proposition 3.1 we have $\mathbb{II}^2_{\text{cycl}}(G, M) = 0$.

**Remark 6.2.** Theorem 6.1 was proved by Sansuc (unpublished) by algebraic methods. His proof is rather involved; the proof presented here, passing from arithmetic to algebra, is simpler, since the proof of the arithmetic result in [BLP 19], Proposition 4.1 is quite short.

**Theorem 6.3.** We have $\text{Br}(T_{L/k})/\text{Br}(k) = 0$, and $\text{Br}(X^c)/\text{Im}(\text{Br}(k)) = 0$.

**Proof.** By Theorem 6.3 and Proposition 5.1, we have $\text{Br}(T_{L/k})/\text{Br}(k) = 0$. The second statement follows from the injection $\text{Br}(X^c)/\text{Im}(\text{Br}(k)) \rightarrow \text{Br}(T^c)/\text{Br}(k)$, cf. §5.

**Linearly disjoint Galois extensions**

This example is based on a result of Pollio and Rapinchuk, [PR 13]. Let $L = K_1 \times K_2$, where $K_1$ and $K_2$ are finite extensions of $k$ such that the Galois closures of $K_1$ and $K_2$ are linearly disjoint.
Theorem 6.4. We have \( \text{Br}(T_{L/k}^c)/\text{Br}(k) = 0 \), and \( \text{Br}(X^c)/\text{Im}(\text{Br}(k)) = 0 \).

Proof. Let \( K_1' \) and \( K_2' \) be the Galois closures of \( K_1 \), respectively \( K_2 \), and let \( H_1', H_2' \) be the subgroups of \( G \) such that \( K_i' = (k')^H_i \), for \( i = 1, 2 \). By hypothesis, the extensions \( K_1' \) and \( K_2' \) are linearly disjoint, hence \( G = H_1'.H_2' \).

Recall that \( \ell'/\ell \) is an unramified extension of number fields with Galois group \( G \). Set \( \ell_i = (\ell')^H_i \) and \( \ell'_i = (\ell')^{H_i} \) for \( i = 1, 2 \). Since \( H_i' \) is a normal subgroup of \( G \) for \( i = 1, 2 \), the extensions \( \ell_i'/\ell \) are Galois, and \( |H_i'| = [\ell' : \ell_i'] \). Since \( G = H_1'.H_2' \), the fields \( \ell_i' \) are linearly disjoint.

Let \( S \) be the norm torus corresponding to the étale \( \ell \)-algebra \( \ell_1 \times \ell_2 \); the main theorem of \cite{PR13} implies that \( \Pi^1(\ell, S) = 0 \). We have \( \hat{S} \simeq M \) by construction, hence by Proposition 3.1 we have \( \Pi^2_{\text{cycl}}(G, M) = 0 \). By Proposition 5.1 we have \( \text{Br}(T_{L/k}^c)/\text{Br}(k) = 0 \). The second statement follows from the injection \( \text{Br}(X^c)/\text{Im}(\text{Br}(k)) \to \text{Br}(T^c)/\text{Br}(k) \), cf. §5.

7. Products of cyclic extensions of prime power degree - statement of results and notation

The proofs of the results of this section will be given in §9. Let \( p \) be a prime number. If \( K/k \) is a cyclic extension of degree a power of \( p \), we denote by \( (K)_{\text{prim}} \) the unique subfield of \( K \) of degree \( p \) over \( k \); if \( E = \prod_{i \in I} K_i \), where \( K_i/k \) is a cyclic extension of degree a power of \( p \) for all \( i \in I \), set \( E_{\text{prim}} = \prod_{i \in I} (K_i)_{\text{prim}} \).

Let \( L \) be a product of \( n \) cyclic extensions of degrees powers of \( p \). With the notation of the previous section, set \( T = T_{L/k} \). Let \( a \in k^\times \), and \( X \) be the affine \( k \)-variety associated to the norm equation \( N_{L/k}(t) = a \); let \( X^c \) be a smooth compactification of \( X \).

Let \( T_{\text{prim}} = T_{L_{\text{prim}}/k} \). If \( a \in k^\times \), we denote by \( X_{\text{prim}} \) the affine \( k \)-variety associated to \( N_{L_{\text{prim}}/k}(t) = a \), and by \( X_{\text{prim}}^c \) a smooth compactification of \( X_{\text{prim}} \).

Theorem 7.1. We have
\[
\text{Br}(T^c)/\text{Br}(k) = 0 \iff \text{Br}(T_{\text{prim}}^c)/\text{Br}(k) = 0.
\]

Let \( k'/k \) be a Galois extension of minimal degree splitting \( T \), and let \( G = \text{Gal}(k'/k) \); similarly, let \( k'_{\text{prim}}/k \) be a Galois extension of minimal degree splitting \( T_{\text{prim}} \), and let \( G_{\text{prim}} = \text{Gal}(k'_{\text{prim}}/k) \).

Theorem 7.2. We have
\[
\Pi^2_{\text{cycl}}(G, \hat{T}) = 0 \iff \Pi^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) = 0.
\]

Products of at least \( p + 2 \) pairwise disjoint cyclic extensions

With the notation above, we now consider the case where \( n \geq p + 2 \).
Theorem 7.3. Assume that $L$ is a product of at least $p + 2$ pairwise disjoint cyclic extensions of degrees powers of $p$. Then we have

(a) $\text{Br}(T^c)/\text{Br}(k) = 0$.

(b) $\text{Br}(X^c)/\text{Br}(k) = 0$.

(c) Suppose that $k$ is a global field. Then $\text{III}_2^2(k, \hat{T}_L/k) = 0$, Hasse principle and weak approximation hold for $X$.

Theorem 7.4. Assume that $n \geq p + 2$. Then we have

$\text{III}^2_\text{cycl}(G, \hat{T}) = 0$.

At least one cyclic factor of degree $p$

In the next results, we assume that $L$ has at least one factor of degree $p$.

Theorem 7.5. Assume that $L$ is a product of $n$ pairwise disjoint cyclic extensions of degrees powers of $p$, and that at least one of these is of degree $p$. Then we have

$\text{Br}(T^c)/\text{Br}(k) \simeq \text{Br}(T^c_{\text{prim}})/\text{Br}(k)$.

Theorem 7.6. Assume that $L$ is a product of $n$ pairwise disjoint cyclic extensions of degrees powers of $p$, and that at least one of these is of degree $p$. Then we have

$\text{III}^2_{\text{cycl}}(G, \hat{T}) \simeq \text{III}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}})$.

Products of cyclic extensions of degree $p$

Finally, we determine $\text{Br}(T^c_{\text{prim}})/\text{Br}(k)$. Let us denote by $C_p$ the cyclic group of order $p$.

Theorem 7.7. (a) If $G_{\text{prim}} \not\cong C_p \times C_p$, then

$\text{Br}(T^c_{\text{prim}})/\text{Br}(k) = \text{Br}(X^c_{\text{prim}})/\text{Im}(\text{Br}(k)) = 0$.

(b) If $G_{\text{prim}} \cong C_p \times C_p$, then $\text{Br}(T^c_{\text{prim}})/\text{Br}(k) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

Theorem 7.8. (a) If $G_{\text{prim}} \not\cong C_p \times C_p$, then $\text{III}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) = 0$.

(b) If $G_{\text{prim}} \cong C_p \times C_p$, then $\text{III}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

In the case where $k$ is a global, we obtain the following corollaries.
Corollary 7.9. Suppose that $k$ is a global field. Assume that $L$ is a product of $n$ pairwise disjoint cyclic extensions of degrees powers of $p$, and that at least one of these is of degree $p$.

(a) If $G_{\text{prim}} \not\cong C_p \times C_p$, then Hasse principle and weak approximation hold for $T$.

(b) If $G_{\text{prim}} \cong C_p \times C_p$, then either Hasse principle holds for torsors over $T$ (and weak approximation for $T$ fails), or weak approximation holds for $T$ (and Hasse principle for torsors over $T$ fails).

Proof. This is a consequence of Theorems 7.8, 7.5 and 2.4.

Corollary 7.10. Assume that $k$ is a global field, and that $L$ is a product of $n$ distinct cyclic extensions of degree $p$. If $G \not\cong C_p \times C_p$, then $\prod_\omega(k, \hat{T}_{L/k}) = 0$.

If $G \cong C_p \times C_p$, then $\prod_\omega(k, \hat{T}_{L/k}) \cong (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

Proof. This follows from Theorem 7.8 and Lemma 3.4.

Remark 7.11. Corollary 7.10 was also proved by Macedo, see [Ma 19], Theorem 4.9 and Corollary 4.10, with different methods (namely, using a generalization of the approach of Drakokhrust and Platonov).

8. Products of cyclic extensions of prime power degree - global fields

We recall some results from [BLP 19]; these will be used in the next section to prove the results of §7. Let $k$ be a global field. We start by recalling some notation from [BLP 19]. If $L$ is an étale algebra of finite rank over $k$ having at least one factor that is a cyclic extension of $k$, the paper [BLP 19] introduces a finite abelian group $\prod(L)$ (see [BLP 19], §5) and proves (see Corollary 5.17) that $\prod(L)^* \cong \prod^1(k, T_{L/k})$; equivalently, by Poitou-Tate duality, we have $\prod(L) \cong \prod^2(k, \hat{T}_{L/k})$.

Let $p$ be a prime number.

Proposition 8.1. Let $L$ be a product of cyclic extensions of degrees powers of $p$. The group $\prod^1(k, T_{L_{\text{prim}}/k})^*$ injects into $\prod^1(k, T_{L/k})^*$.

Proof. This follows from [BLP 19], Lemma 8.7.

Theorem 8.2. Let $L$ be a product of cyclic extensions of degrees powers of $p$. Then we have

$$\prod^1(k, T_{L/k}) = 0 \iff \prod^1(k, T_{L_{\text{prim}}/k}) = 0.$$ 

Proof. This is an immediate consequence of [BLP 19], Theorem 8.1.

Proposition 8.3. Let $L$ be a product of $n$ distinct cyclic extensions of degrees powers of $p$, and assume that at least one of the extensions is of degree $p$. Then $\prod^1(k, T_{L/k})$ is a finite abelian group of type $(p, \ldots, p)$ of order at most $p^{n-2}$. 


Proof. Let us write $L$ as a product $L = K \times K'$, where $K$ is a cyclic extension of $k$ of degree $p$, and $K'$ is a product of $n - 1$ cyclic extensions of degrees powers of $p$. In [BLP 19], 5.1, we construct a finite abelian group $\mathbb{III}(K, K')$ such that when $K$ is cyclic of order $p$, the group $\mathbb{III}(K, K')$ is of type $(p, \ldots, p)$ of order at most $p^{n-2}$. It is shown in [BLP 19], 5.3 that the group $\mathbb{III}(K, K')$ does not depend on the decomposition of $L$ as $K \times K'$, where $K$ is a cyclic extension of $k$, and that $\mathbb{III}(L)^* \simeq \mathbb{III}(k, T_{L/k})$ (see [BLP 19], Corollary 5.17). Hence $\mathbb{III}(k, T_{L/k})$ is a finite abelian group of type $(p, \ldots, p)$ of order at most $p^{n-2}$, as claimed.

Theorem 8.4. Let $L = K_1 \times \cdots \times K_n$, where $K_i$ are cyclic extensions of degree $p$ of $k$.

(a) If $n \leq 2$, or $n \geq p + 2$, or $3 \leq n \leq p + 1$ and $K_1, \ldots, K_n$ are not contained in some field extension of $k$ of degree $p^2$ having all local degrees $\leq p$, then

$$\mathbb{III}(k, T_{L/k}) = 0.$$ 

(b) Assume that $3 \leq n \leq p + 1$ and that the fields $K_1, \ldots, K_n$ are contained in some field extension of $k$ of degree $p^2$ having all local degrees $\leq p$, then

$$\mathbb{III}(k, T_{L/k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}.$$ 

Proof. This follows from [BLP 19], Theorem 8.3 and Corollary 5.17.

Remark. More generally, one can treat the case where the $K_i$’s are field extensions of degree $p$, with at least one of them cyclic (see [BLP 19], Proposition 8.5).

Theorem 8.5. Let $L$ be a product of $n$ pairwise disjoint cyclic extensions of degrees powers of $p$, and assume that at least one of the extensions is of degree $p$. Then we have

$$\mathbb{III}(k, T_{L/k}) \simeq \mathbb{III}(k, T_{L_{\text{prim}}/k}).$$ 

Proof. By Theorem 8.4, we have either $\mathbb{III}(k, T_{L_{\text{prim}}/k}) = 0$ or $\mathbb{III}(k, T_{L_{\text{prim}}/k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$. If $\mathbb{III}(k, T_{L_{\text{prim}}/k}) = 0$, then by Theorem 8.2 we have $\mathbb{III}(k, T_{L/k}) = 0$. Assume now that $\mathbb{III}(k, T_{L_{\text{prim}}/k}) \neq 0$; then by Theorem 8.4 we have $\mathbb{III}(k, T_{L_{\text{prim}}/k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$, therefore the order of $\mathbb{III}(k, T_{L_{\text{prim}}/k})$ is equal to $p^{n-2}$. By Proposition 8.3, this implies that the order of $\mathbb{III}(k, T_{L/k})$ is at least $p^{n-2}$. On the other hand, since at least one of the factors of $L$ is of order $p$, Proposition 8.3 implies that $\mathbb{III}(k, T_{L/k})$ is a finite abelian group of type $(p, \ldots, p)$ of order at most $p^{n-2}$. Hence the order of $\mathbb{III}(k, T_{L/k})$ is equal to $p^{n-2}$, and this completes the proof of the Theorem.
9. Products of cyclic extensions of prime power degree - proofs

We keep the notation of §7. In particular, $p$ is a prime number, $L$ is a product of $n$ cyclic extensions of degrees powers of $p$, $k'/k$ is a Galois extension of minimal degree splitting $T = T_{L/k}$, and $G = \text{Gal}(k'/k)$. Let $M = \hat{T}$ be the $G$-lattice of characters of $T$. Let us write $L = \prod_{i \in I} K_i$, and let $H_i$ be the subgroup of $G$ such that $K_i = (k')^{H_i}$. We have the exact sequence of $G$-modules

$$0 \to \mathbf{Z} \to \bigoplus_{i \in I} \mathbf{Z}[G/H_i] \to M \to 0.$$

Recall that $k'_{\text{prim}}/k$ is a Galois extension of minimal degree splitting $T_{\text{prim}}$, and that $G_{\text{prim}} = \text{Gal}(k'_{\text{prim}}/k)$. Let $M_{\text{prim}}$ be the $G_{\text{prim}}$-lattice of characters of $T_{\text{prim}}$. Let $(H_i)_{\text{prim}}$ be the subgroup of $G_{\text{prim}}$ such that $(K_i)_{\text{prim}} = (k'_{\text{prim}})^{(H_i)_{\text{prim}}}$. We have the exact sequence of $G_{\text{prim}}$-modules

$$0 \to \mathbf{Z} \to \bigoplus_{i \in I} \mathbf{Z}[G_{\text{prim}}/(H_i)_{\text{prim}}] \to M_{\text{prim}} \to 0.$$

Set $\Gamma_i = (G/H_i)/(G_{\text{prim}}/(H_i)_{\text{prim}})$, and note that $(K_i)_{\text{prim}} = K_{\Gamma_i}^{T_i}$.

Let $\ell'/\ell$ be an extension of number fields with Galois group $G$ which is unramified at all the finite places. For all $i \in I$, let $L_i$ be the fixed field of $H_i$ in $\ell'$, and set $E = \prod_{i \in I} L_i$. The character lattice of the torus $T_{E/\ell}$ is isomorphic to the $G$-lattice $M$. We have $(L_i)_{\text{prim}} = L_{\Gamma_i}^{T_i}$. Note that $\mathfrak{I}^2_{\text{cycl}}(G, M) \simeq \mathfrak{I}^2(\ell, \hat{T}_{E/\ell})$ and that $\mathfrak{I}^2_{\text{cycl}}(G_{\text{prim}}, M_{\text{prim}}) \simeq \mathfrak{I}^2(\ell, \hat{T}_{E_{\text{prim}}/\ell})$.

**Proof of Theorem 7.2** By Theorem 8.2 we have

$$\mathfrak{I}^1(\ell, T_{E/\ell}) = 0 \iff \mathfrak{I}^1(\ell, T_{E_{\text{prim}}/\ell}) = 0,$$

therefore, since $\hat{T}_{E/\ell} \simeq M$ and $\hat{T}_{E_{\text{prim}}/\ell} \simeq M_{\text{prim}}$, we have

$$\mathfrak{I}^2(\ell, M) = 0 \iff \mathfrak{I}^2(\ell, M_{\text{prim}}) = 0.$$

By Proposition 3.1 we have

$$\mathfrak{I}^2_{\text{cycl}}(G, M) \simeq \mathfrak{I}^2(\ell, M)$$

and

$$\mathfrak{I}^2_{\text{cycl}}(G_{\text{prim}}, M_{\text{prim}}) \simeq \mathfrak{I}^2(\ell, M_{\text{prim}}).$$

Since $M = \hat{T}$ and $M_{\text{prim}} = \hat{T}_{\text{prim}}$, we obtain

$$\mathfrak{I}^2_{\text{cycl}}(G, \hat{T}) = 0 \iff \mathfrak{I}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) = 0,$$

as claimed.

**Proof of Theorem 7.1** Theorems 7.2 and 7.1 are equivalent by Theorem 1.3.

**Proof of Theorem 7.8** Note that $G_{\text{prim}}$ is an elementary abelian $p$-group, with $|G_{\text{prim}}| = p$ if $n = 1$, and $|G_{\text{prim}}| \geq p^2$ if $n \geq 2$. We may assume that $n \geq 2$. 
Assume first that $G_{\text{prim}} \not\cong C_p \times C_p$. Then the étale $\ell$-algebra $E$ is not contained in a field extension of degree $p^2$, therefore Proposition 8.4 implies that $\text{III}^1(\ell, T) = 0$, and by Corollary 4.2 this implies that $\text{III}^2_{\text{cycl}}(G_{\text{prim}}, M) = 0$.

Assume now that $G_{\text{prim}} \cong C_p \times C_p$; this implies that $\ell'$ is a degree $p^2$ extension of $k$ containing all the factors of $E$. Since $\ell'/\ell$ is unramified at all the finite places, the local degrees are $\leq p$. By Proposition 8.4, this implies that $\text{III}^1(\ell, T) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$. By Poitou-Tate duality, $\text{III}^2(\ell, \hat{T}) \simeq \text{III}^1(\ell, T)^*$, hence $\text{III}^2(\ell, \hat{T}) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$. Since the $G$-lattices $\hat{T}$ and $M$ are isomorphic, by Proposition 3.5 we have $\text{III}^2_{\text{cycl}}(G, M) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

**Proof of Theorem 7.7.** Theorem 7.8 and Theorem 1.3 imply part (b) of Theorem 7.7 and that $\text{Br}(T_{\text{prim}}^c)/\text{Br}(k) = 0$ if $G_{\text{prim}} \not\cong C_p \times C_p$. We have seen in §5 that $\text{Br}(X_{\text{prim}}^c)/\text{Im(\text{Br}(k)))$ injects into $\text{Br}(T_{\text{prim}}^c)/\text{Br}(k)$, hence we get (a).

**Proof of Theorem 7.4.** By Theorem 7.8 we have $\text{III}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) = 0$, and by Theorem 7.2 this implies that $\text{III}^2_{\text{cycl}}(G, \hat{T}) = 0$.

**Proof of Theorem 7.3.** Theorems 7.4 and 7.3 (a) are equivalent by Theorem 1.3. Theorem 7.3 (b) follows by the injection $\text{Br}(X_{\text{prim}}^c)/\text{Im(\text{Br}(k))) \to \text{Br}(T_{\text{prim}}^c)/\text{Br}(k)$, cf. §5, and (c) by Proposition 3.4.

**Proof of Theorem 7.6.** By Theorem 8.5, we have

$$\text{III}^1(\ell, T_{E/\ell})^* \simeq \text{III}^1(\ell, T_{E_{\text{prim}}/\ell})^*,$$

and by Poitou-Tate duality this implies that $\text{III}^2(\ell, \hat{T}_{E/\ell}) \simeq \text{III}^2(\ell, \hat{T}_{E_{\text{prim}}/\ell})$. Since $\hat{T}_{E/\ell} \simeq M \simeq \hat{T}$, and $\hat{T}_{E_{\text{prim}}/\ell} \simeq M_{\text{prim}} \simeq \hat{T}_{\text{prim}}$, applying Proposition 3.5 we obtain $\text{III}^2_{\text{cycl}}(G_{\text{prim}}, \hat{T}_{\text{prim}}) \simeq \text{III}^2_{\text{cycl}}(G, \hat{T})$, as claimed.

**Proof of Theorem 7.5.** Theorems 7.6 and 7.5 are equivalent by Theorem 1.3.

10. Unramified Brauer groups and products of cyclic extensions

Let $p$ be a prime number, and let $L = K_1 \times \cdots \times K_n$, where $K_1, \ldots, K_n$ are distinct cyclic extensions of $k$ degree $p$. Let $T_{L/k} = R^{(1)}_{L/k}(G_m)$ be the $k$-torus defined by

$$1 \to T_{L/k} \to R_{L/k}(G_m)^{N_{L/k}} \to G_m.$$

Let $k'/k$ be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \text{Gal}(k'/k)$. Let $M$ be the $G$-lattice of characters of $T_{L/k}$.

Let $a \in k^\times$, and let $X$ be the affine $k$-variety determined by the equation $N_{L/k}(t) = a$, and note that $X$ is a torsor under $T = T_{L/k}$. Let $X^c$ be a smooth compactification of $X$.

The following result is an immediate consequence of Theorem 7.7.
Theorem 10.1. (a) If $G \not\cong C_p \times C_p$, then
$$\text{Br}(T^c)/\text{Br}(k) = \text{Br}(X^c)/\text{Im(Br(k))} = 0.$$  
(b) If $G \cong C_p \times C_p$, then $\text{Br}(T^c)/\text{Br}(k) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$. 

Assume now that the characteristic of $k$ is not $p$. In this case, we determine the structure of $\text{Br}(X^c)/\text{Im(Br(k))}$, as follows

Theorem 10.2. Suppose that $\text{char}(k) \neq p$, and that $G \cong C_p \times C_p$. Then we have
$$\text{Br}(X^c)/\text{Im(Br(k))} \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}.$$ 

The proof of this theorem is a combination of arithmetic and algebra. By Theorem 10.1 (b), we already know that $\text{Br}(T^c)/\text{Br}(k) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$. Recall from §5 that $\text{Br}(X^c)/\text{Im(Br(k))}$ injects into $\text{Br}(T^c)/\text{Br}(k)$. On the other hand, Theorem 10.3. shows that $\text{Br}(X^c)/\text{Im(Br(k))}$ is of dimension at least $n - 2$ over $(\mathbb{Z}/p\mathbb{Z})$.

In order to prove this theorem, we obtain a more precise result, namely we give generators for the group $\text{Br}(X^c)/\text{Im(Br(k))}$.

Set $I = \{1, \ldots, n\}$. We consider the norm polynomials $N_{K_i/k}(t_i)$ for $i \in I$ as elements of $k(X)_i$. For all $i \in I$, set $N_i = N_{K_i/k}(t_i)$ and let $\sigma_i$ be a generator of $\text{Gal}(K_i/k)$. Let $\bar{K}_n \in H^1(k, \mathbb{Z}/p\mathbb{Z})$ be the element associated to the pair $(K_n, \sigma_n)$. The variety $X$ is defined by $N_1 N_2 \ldots N_n = a$ in the affine space $k[t_i, 1 \leq i \leq n]$. Let $(N_i, \bar{K}_n)$ denote the cyclic algebra of degree $p$ over $k(X)^c$ associated to $[N_i] \in H^1(k(X), \mu_p)$ and $\bar{K}_n \in H^1(k(X), \mathbb{Z}/p\mathbb{Z})$.

Theorem 10.3. Suppose that $\text{char}(k) \neq p$, and that $G \cong C_p \times C_p$. Then $\text{Br}(X^c)/\text{Im(Br(k))}$ is generated by the $n - 2$ linearly independent elements 
$$(N_i, \bar{K}_n), \ i = 1, \ldots, n - 2.$$ 
in $\text{Br}(k(X))$.

We begin with the following lemma:

Lemma 10.4. Let $K/k$ be a cyclic extension of degree $n$ with $(n, \text{char}(k)) = 1$. Let $\sigma$ be a generator of $\text{Gal}(K/k)$ and let $A$ be the cyclic algebra over $k$ defined by $(K, \sigma, c)$ for some $c \in k^\times$. Let $X$ be the variety $N_{K/k}(t) = c$. Then the kernel of $\text{Br}(k) \to \text{Br}(k(X))$ is generated by the class of $A$.

Proof. Let $Y_A$ be the Severi-Brauer variety of $A$. Since $A$ is split by $k(Y_A)$, the element $c$ is a norm from the extension $Kk(Y_A)/k(Y_A)$. Thus $X$ has a rational point over $k(Y_A)$ and the map $\text{Br}(k(Y_A)) \to \text{Br}(k(Y_A)(X))$ has trivial kernel. We have 
$$\text{Ker(Br(k) \to Br(k(X)))} \subset \text{Ker(Br(k) \to Br(k(X)(Y_A)))} = \text{Ker(Br(k) \to Br(k(Y_A)))} = \text{Ker(Br(k) \to Br(k(Y_A)(X)))} = \langle [A] \rangle$$
by a theorem of Amitsur. Since $[A]$ is zero in $\text{Br}(k(X))$, it follows that $\text{Ker(Br(k) \to Br(k(X)))} = \langle [A] \rangle$. 


In the following proof of Theorem 10.3 we use the fact that the Brauer group of $X^c$ is the unramified Brauer group of $X^c$ (cf. [CT14], Section 2), namely the subgroup of $\text{Br}(k(X^c))$ consisting of all elements which are unramified at all discrete valuations of $k(X^c)$. This is a consequence of the purity results of Cesnavius [C19], Theorem 1.2.

**Proof of Theorem 10.3** The strategy of the proof is the following. We first show that the algebras $(N_i, \tilde{K}_n), \ i = 1, \ldots, n - 2$ belong to $\text{Br}(X^c)$. By Theorem 10.1 (b), we know that $\text{Br}(X^c)/\text{Br}(k) \cong (\mathbb{Z}/p\mathbb{Z})^{n-2}$. Moreover, $\text{Br}(X^c)/\text{Im}(\text{Br}(k))$ injects into $\text{Br}(T^c)/\text{Br}(k)$ (see §5). We next show that the elements $(N_i, \tilde{K}_n), \ i = 1, \ldots, n - 2$, are linearly independent over $\mathbb{Z}/p\mathbb{Z}$, and this yields the desired result.

Identifying $G$ with $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, let $\sigma(i, j) \in G$ correspond to $(i, j) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $0 \leq i \leq p - 1, 0 \leq j \leq p - 1$. We assume $K_1 = (k')^\sigma(0, 1)$ and $K_n = (k')^\sigma(1, 0)$. Pick a generator $\sigma_1$ of $\text{Gal}(K_1/k)$ and let $\tilde{K}_1 = [(K_1, \sigma_1)] \in H^1(k, \mathbb{Z}/p\mathbb{Z})$. For each $j \in I$, one can choose a generator $\sigma_j$ of $\text{Gal}(K_j/k)$ such that for $j \geq 2$, we have $\tilde{K}_j = \sigma_i\tilde{K}_1 + \tilde{K}_n$ for some $l_j$ with $1 \leq l_j \leq p - 1$. In fact for $j \geq 2$, we have $K_j = (k')^\sigma(1, j)$ and $\sigma_j$ is determined by $\sigma_1$ and $\sigma_n$.

Since $N_i \in N(K_i, k(X^c)/k(X^c))$, we get $(N_i, \tilde{K}_n)_j = 0$. Hence we have $(N_i, \tilde{K}_j) = (N_i, i\tilde{K}_1 + \tilde{K}_n) = (N_i, \tilde{K}_n)$ for every $j \in I, j \geq 2$.

We show that the cyclic algebras $(N_i, \tilde{K}_n)$ are unramified on $X^c$ for all $i \in I$, and that $(N_1, \tilde{K}_n), \ldots, (N_{n-2}, \tilde{K}_n)$ are linearly independent in $\text{Br}(X^c)/\text{Br}(k)$.

Let $R$ be a discrete valuation ring containing the field $k$, with fraction field of $R$ equal to $k(X^c)$. We prove that the algebras $(N_i, \tilde{K}_n)$ are unramified with respect to the valuation $v_R$. Let us denote by $\kappa$ the residue field of $R$, and let $\partial_R : \text{Br}(k(X^c)) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ be the residue map.

Let $[\tilde{K}_i]$ denote the image of $\tilde{K}_i$ in $H^1(\kappa, \mathbb{Z}/p\mathbb{Z})$. We have

$$\partial_R(N_i, \tilde{K}_n) = [\tilde{K}_n]^{v_R(N_i)}.$$  

If $K_n \subset \kappa$, we have $[\tilde{K}_n] = 0$ and $\partial_R(N_i, \tilde{K}_n) = 0$ for $1 \leq i \leq n - 2$. Suppose that $\partial_R(N_1, \tilde{K}_n) \neq 0$. Then $\tilde{K}_n$ is not contained in $\kappa$. In this case, $\tilde{K}_n\kappa$ is a degree $p$ cyclic extension of $\kappa$. The extension $k(X^c)K_n/k(X^c)$ is cyclic of degree $p$, and has residual degree $p$, hence is unramified at $R$. Further $N_n \in k(X^c)$ is a norm from the extension $k(X^c)K_n/k(X^c)$. Hence the valuation $v_R(N_n)$ is divisible by $p$. Since $N_1, \tilde{K}_j) = (N_1, \tilde{K}_n)$ for $j \geq 2$, we have $\partial_R(N_1, \tilde{K}_j) = 0$, and $K_j$ is not contained in $\kappa$. Repeating the above argument, one we see that $p$ divides $v_R(N_j)$ for all $j \geq 2$. Since $a = N_1 \ldots N_n$, $v_R(a) = 0$, and $p$ divides $v_R(N_j)$ for $2 \leq j \leq n$, and it follows that $p$ divides $v_R(N_1)$. This implies that $\partial_R(N_1, \tilde{K}_n) = (\tilde{K}_n)^{v_R(N_1)} = 0$, contradicting the assumption that $\partial_R(N_1, \tilde{K}_n) \neq 0$. This implies that $\partial_R(N_1, \tilde{K}_n) = 0$. A similar argument, interchanging 1 and i, with $i \leq n - 1$, gives that $\partial_R(N_i, \tilde{K}_n) = 0$. Hence the elements $(N_i, \tilde{K}_n)$ are unramified at $R$ for every discrete valuation ring $R$ with field of fractions $k(X^c)$. By purity for $\text{Br}(X^c)$, we have $(N_i, \tilde{K}_n) \in \text{Br}(X^c)$.  


Let us check that the algebras \((N_1, \tilde{K}_n), \ldots, (N_{n-2}, \tilde{K}_n)\) are linearly independent in \(\text{Br}(k(X^c))/\text{Br}(k)\). Let us project \(X^c\) to the \(d\)-dimensional affine space, where \(d = (n - 1)p\), corresponding to the coordinates involving the first \(n - 1\) norm polynomials. Let \(M\) be the function field of this affine space; we have \(k \subset M \subset k(X^c)\). Note that \(N_1, \ldots, N_{n-1} \in M\). We have \((N_i, \tilde{K}_n) \in \text{Br}(M)\) for all \(i = 1, \ldots, n - 1\).

We want to show that the algebras \((N_1, \tilde{K}_n), \ldots, (N_{n-2}, \tilde{K}_n)\) are linearly independent in \(\text{Br}(k(X^c))/\text{Br}(k)\). If not, then there exist \(r_1, \ldots, r_{n-2} \in \mathbb{Z}\) with \(0 \leq r_i \leq p - 1\) such that \(\sum_{i=1}^{n-2} r_i(N_i, \tilde{K}_n)_{k(X^c)} = \alpha \in \text{Br}(k)\) for some \(\alpha \in \text{Br}(k)\).

The kernel of the natural homomorphism \(\text{Br}(M) \to \text{Br}(k(X^c))\) is generated by the class of the algebra \((a^{-1}N_1, \ldots, N_{n-1}, \tilde{K}_n)\). Hence there exists \(s \in \mathbb{Z}\) with \(0 \leq s \leq p - 1\) such that

\[
\sum_{i=1}^{n-2} r_i(N_i, \tilde{K}_n)_M - \alpha = s(a^{-1}N_1 \ldots N_{n-1}, \tilde{K}_n)_M
\]

in \(\text{Br}(M)\). Take residue on both sides at the valuation \(v_{N_i}\) corresponding to the irreducible polynomial \(N_i\) for all \(i = 1, \ldots, n - 2\). The residue of the left side is \(r_i[\tilde{K}_n]\), and the right side is \(s[\tilde{K}_n]\).

**Claim.** \([\tilde{K}_n] \neq 0\) in the residue field \(\kappa(v_{N_i})\) for all \(i = 1, \ldots, n - 2\).

Assume that the claim holds. Then we have \(r_i = s\) for all \(i = 1, \ldots, n - 2\), and we get \(s(a^{-1}N_{n-1}, \tilde{K}_n) = \alpha\). With respect to the valuation \(v_{N_{n-1}}\), we have the residue \(\partial(a^{-1}N_{n-1}, \tilde{K}_n) = [\tilde{K}_n] \neq 0\) by the claim, and this leads to a contradiction.

It remains to prove the claim. Let us show that \(K_n\) is not contained in \(\kappa(v_{N_i})\) for all \(i = 1, \ldots, n - 2\). Let \(M_i\) be the function field of the \(k\)-variety determined by the polynomial \(N_i\). Since \(\kappa(v_{N_i})\) is a purely transcendental extension of \(M_i\), it suffices to show that \(K_n\) is not contained in \(M_i\). Suppose that \(K_n\) is a subfield of \(M_i\). Let us base change to \(K_i : \) the field \(K_iK_n\) is a subfield of \(K_iM_i\). After base change to \(K_i\), the polynomial \(N_i\) is transformed to the product \(X_1 \ldots X_p\) for some variables \(X_1, \ldots, X_p\). Hence \(K_iM_i\) is a product of rational function fields over \(K_i\); therefore it cannot contain \(K_n\), thereby leading to contradiction.

11. **Some consequences for semi-global fields**

Let \(K\) be a complete discrete valued field with valuation ring \(O\) and residue field \(\kappa\). Let \(X/K\) be a normal projective, geometrically integral curve over \(K\) and let \(F = \bar{K}(X)\). We call \(F\) a semi-global field. In \([\text{CTPS} 16]\), \S 2.3, certain higher reciprocity obstructions were constructed to study the failure of the Hasse principle for varieties over \(F\) with respect to discrete valuations of \(F\) centered on a regular proper model \(\mathcal{X}/O\) of the curve \(X/K\). The question was whether these obstructions suffice to detect the failure of the Hasse principle on principal homogeneous spaces under reductive groups defined over \(F\). Using
an example of a torus constructed in [Su 19], Corollary 7.12, we prove that the higher obstructions constructed in [CTPS 16] do not suffice to detect failure of Hasse principle.

We recall the following construction from [Su 19], Corollary 7.12. Let 

\[ K = \mathbb{C}(t) \]  

and 

\[ F = \mathbb{C}(\frac{Y}{X(Y + Y - 1)(Y - 2) - t}). \]  

Let 

\[ L_1 = F((XY)^{1/n}, (Y(X + Y - 1))^{1/n}) \]  

and 

\[ L_2 = F((XY\theta_1)^{1/n}, (Y(X + Y - 1)\theta_2)^{1/n}), \]  

where \( \theta_1 = (X-2)(X-2+XY(X+Y-1)) \) and \( \theta_2 = (Y-2)(Y-2+XY(X+Y-1)). \) Then \( L_1 \) and \( L_2 \) are Galois extensions of \( F \) which are linearly disjoint over \( F \) (c.f [Su 19], Corollary 7.12). Let 

\[ T = \mathbb{R}_{L_1 \times L_2/F}(\mathbb{G}_m) \]  

be the associated norm one torus. It is proved in [Su 19], Corollary 7.12, there is principal homogeneous space under \( T \) which fails the Hasse principle with respect to all discrete valuations of \( F \). The proof invokes \( R \)-equivalence of tori to prove that Hasse principle fails in the patching setting of Harbater-Hartmann-Krashen. The second step is to prove that failures of Hasse principle in the patching setting implies the failure of the Hasse principle with respect to all discrete valuations of \( F \). Since the cohomological dimension of \( F \) is 2, the only higher obstruction of [CTPS 16] in this case is the one coming from the Brauer group of \( X^c \). In view of Theorem 6.4, we have \( \text{Br}(X^c)/\text{Br}(k) = 0 \), and the obstruction vanishes. However, Hasse principle fails for \( X \). Hence we get the following

**Theorem 11.1.** Higher reciprocity obstructions constructed in [CTPS 16] are not sufficient to detect failure of Hasse principle for principal homogeneous spaces under tori over semi-global fields.

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