Teleportation in a noisy environment: a quantum trajectories approach

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Abstract

We study the fidelity of quantum teleportation for the situation in which quantum logic gates are used to provide the long distance entanglement required in the protocol, and where the effect of a noisy environment is modeled by means of a generalized amplitude damping channel. Our results demonstrate the effectiveness of the quantum trajectories approach, which allows the simulation of open systems with a large number of qubits (up to 24). This shows that the method is suitable for modeling quantum information protocols in realistic environments.

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The practical implementation of any quantum information protocol has to face the problem of the unavoidable coupling of quantum processors with their environment. Indeed, real systems can never be perfectly isolated from the surrounding world. It is therefore important to understand the impact of the coupling with a noisy environment on the stability of quantum protocols. In particular, the simulation of these protocols including realistic models of noise, promises to give useful insights for the design and future construction of quantum hardware.

As a consequence of the unwanted environmental coupling, a quantum processor becomes, in general, entangled with its environment. Therefore its state is described by a density matrix, whose evolution is ruled, under the assumption that the environment is Markovian, by a master equation. Solving this equation for a state of several qubits is a prohibitive task in terms of memory cost. Instead of doing so, quantum trajectories allow us to store only a stochastically...
evolving state vector. By averaging over many runs we get the same probabilities (within statistical errors) as the ones obtained through the density matrix directly. Therefore quantum trajectories are the natural approach for simulating equations otherwise very hard to solve. It has been pointed out [1] that outside the quantum optics and quantum foundations fields of research, the theory of quantum trajectories is almost unknown. They were used to model continuously monitored open systems [2, 3], in numerical calculations for the study of dissipative processes [4], and in relation to quantum measurement theory [5]. Moreover, without using quantum trajectories explicitly, some studies in this spirit have been previously done also in quantum information [6]. Still, there are only a few cases where they have been used or suggested for calculations in this area [7, 8]. On the other hand, as it will become clear from the present work, this tool can be very powerful for the simulation of quantum information processing.

In this Letter, we apply the quantum trajectories formalism to study the fidelity of the quantum teleportation protocol [9] through a large chain of qubits, in the presence of environmental noise. Quantum teleportation is a scheme by which the state of an arbitrary unknown qubit can be transmitted by means of a shared entangled pair, that is an EPR pair where one particle is with the sender Alice and the other with the receiver Bob. Performing local operations and sending two bits of classical information is enough to accomplish the task. Teleportation is one of the the basic methods of quantum communication, and plays a very important role in a number of quantum computation protocols [10, 11]. In particular, it has been proved that teleportation, together with single qubit operations, is sufficient to construct a universal quantum computation [10]. Several quantum optical experiments demonstrated the teleportation protocol [12, 13], and recently a long distance implementation has been achieved using a 2 Km optical fiber [14]. Besides these rapid developments in the quantum optics arena, other realizations are particularly interesting from the viewpoint of quantum computation. Among them, NMR experiments have been remarkably successful in implementing teleportation [15]. Moreover, there are proposals for teleporting atomic states [16, 17] and using quantum dot systems for electron teleportation [18, 19].

A noiseless quantum channel is required in order for Alice and Bob to share a maximally entangled EPR pair, as required by the teleportation protocol. On the other hand, the available quantum channels are typically noisy, as we must take into account the interactions of the qubits with the external world. Recently, there has been much interest in the study of the fidelity of teleportation through noisy channels [20]. In this work, we assume that the delivery of one of the bits of the EPR pair is done by means of swap gates along a noisy chain of qubits. As described below, this chain is an open system that interacts with the environment through a generalized amplitude damping channel. Using the quantum trajectories approach, we are able to study numerically the fidelity of teleportation, defined as the overlap between the final reduced density ma-
trix of Bob’s qubit and the original unknown state, for chain sizes of up to 24 qubits. We stress that the method, apart from statistical errors, is in principle exact, and allows the treatment of systems with a large number of qubits, not accessible by a direct solution of the master equation. We note that a scheme for quantum teleportation in a large nuclear spin chain has been presented in Ref. [21]. However, in this work the qubits were considered as perfectly isolated from the environment.

Let us first describe the noiseless protocol. We consider a chain of $n$ qubits, and assume that Alice can access the qubits located at one end of the chain, Bob those at the other end. We assume that initially Alice owns an EPR pair (for instance we take the Bell state $(|00⟩ + |11⟩)/\sqrt{2}$), while the remaining $n-2$ qubits are in a pure state. Thus, the global initial state of the chain is given by

$$
\sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |i_{n-1} \ldots i_2⟩ \otimes \frac{1}{\sqrt{2}}(|00⟩ + |11⟩),
$$

where $i_k = 0, 1$ denotes the down or up state of the qubit $k$. In order to deliver one of the qubits of the EPR pair to Bob, we implement a protocol consisting of $n-2$ swap gates, each one exchanging the states of a pair of qubits:

$$
\sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |i_{n-1} \ldots i_200⟩ + |i_{n-1} \ldots i_211⟩ \rightarrow \sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |i_{n-1} \ldots 0i_200⟩ + |i_{n-1} \ldots 1i_211⟩ \rightarrow \ldots \rightarrow \sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |0i_{n-1} \ldots i_200⟩ + |1i_{n-1} \ldots i_211⟩.
$$

After that, Alice and Bob share an EPR pair, and therefore an unknown state of a qubit ($|ψ⟩ = α|0⟩ + β|1⟩$) can be transferred from Alice to Bob by means of the standard teleportation protocol [9]. In this work, we take random coefficients $c_{i_{n-1},\ldots,i_2}$, that is they have amplitudes of the order of $1/\sqrt{n-2}$ (to assure wave function normalization) and random phases. This ergodic hypothesis models the transmission of a qubit through a chaotic quantum channel.

If the chain interacts with the environment, its state is described by a density operator $ρ$. Under the Markovian assumption, the dynamics of the chain is described by a master equation in the Lindblad form [22]:

$$
\dot{ρ} = -\frac{i}{\hbar}[H_s, ρ] - \frac{1}{2} \sum_k \{L_k^\dagger L_k, ρ\} + \sum_k L_k ρ L_k^\dagger,
$$

where $H_s$ is the system’s Hamiltonian, $\{ , \}$ denotes the anticommutator and $L_k$ are the Lindblad operators, with $k \in [1, \ldots, M]$ (the number $M$ depending on the particular model of interaction with the environment). The first two terms
of this equation can be regarded as the evolution performed by an effective non-hermitian Hamiltonian, \( H_{\text{eff}} = H_s + iK \), with \( K = -\hbar/2 \sum_k L_k^\dagger L_k \). Indeed, we can see that

\[
-\frac{i}{\hbar}[H_s, \rho] - \frac{1}{2} \sum_k \{L_k^\dagger L_k, \rho\} = -\frac{i}{\hbar} [H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger].
\] (4)

The last term in (3) is usually interpreted as the one responsible for the so called quantum jumps. The explanation is as follows. If the initial density matrix describes a pure state \( (\rho(t_0) = |\phi(t_0)\rangle\langle\phi(t_0)|) \), then, after an infinitesimal time \( dt \), it evolves into the following statistical mixture:

\[
\rho(t_0 + dt) = (1 - \sum_k dp_k) |\phi_0\rangle\langle\phi_0| + \sum_k dp_k |\phi_k\rangle\langle\phi_k|,
\] (5)

where \( dp_k = dt \langle\phi(t_0)|L_k^\dagger L_k|\phi(t_0)\rangle \), and the new states are defined by

\[
|\phi_0\rangle = \frac{(1 - iH_{\text{eff}} dt/\hbar)|\phi(t_0)\rangle}{\sqrt{1 - \sum_k dp_k}}
\] (6)

and

\[
|\phi_k\rangle = \frac{L_k |\phi(t_0)\rangle}{\|L_k |\phi(t_0)\rangle\|}.
\] (7)

Then, the quantum jump picture turns out to be clear: with probability \( dp_k \) a jump occurs and the system is prepared in the state \( |\phi_k\rangle \). With probability \( 1 - \sum_k dp_k \) there are no jumps and the system evolves according to the effective Hamiltonian \( H_{\text{eff}} \). We note that the normalization is included also in this case because the evolution is non-hermitian.

To simulate numerically the evolution of the master equation (3), we use the so-called Monte Carlo wave function approach \cite{4}, actually implementing the above jump picture. We start the time evolution from a pure state \( |\phi(t_0)\rangle \) and, at intervals \( dt \) much smaller than the timescales relevant for the evolution of the density matrix, we choose a random number \( \epsilon \) from a uniform distribution in the unit interval \([0, 1]\). If \( \epsilon \leq dp \), where \( dp = \sum_k dp_k \), the state of the system jumps to one of the states \( |\phi_k\rangle \) (to \( |\phi_0\rangle \) if \( 0 \leq \epsilon \leq dp_1 \), to \( |\phi_2\rangle \) if \( dp_1 < \epsilon \leq dp_1 + dp_2 \), and so on). On the other hand, if \( \epsilon > dp \) the evolution with the non-hermitian Hamiltonian \( H_{\text{eff}} \) takes place and we end up in the state \( |\phi_0\rangle \). We repeat this process as many times as \( n_{\text{steps}} = \Delta t/dt \), where \( \Delta t \) is the total evolution time. This procedure describes a stochastically evolving wave vector, and we say that a single evolution is a quantum trajectory. If we average over different runs, we recover the probabilities obtained using the density operator (see, e.g., Ref. \cite{1}).

Given an operator \( A \), we can write the mean value \( \langle A \rangle_t = \text{Tr}[A \rho(t)] \) as the average over \( N \) trajectories:

\[
\langle A \rangle_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle \phi_i(t)|A|\phi_i(t)\rangle.
\] (8)
We can see immediately the advantage of the quantum trajectories method: we need to store a vector of length \( N \), where \( N = 2^n \) is the dimension of the Hilbert space, rather than a \( N \times N \) matrix. The price to pay is that one has to run many trajectories to get small statistical errors. However, a reasonably small number \( N \) of trajectories is sufficient to obtain a satisfactory convergence (in our case \( N \approx 100 - 500 \), while the maximum value of \( N \) in our simulations is \( N = 2^{24} \)).

We model the coupling with the environment using a generalized amplitude damping channel: a state \( |i_{n-1}\ldots i_0\rangle \) decays with rate \( \Gamma / \hbar \). After an infinitesimal time \( dt \), the possible states of the system are those in which the damping \( |1\rangle \to |0\rangle \) has occurred in one of the qubits, the damping probability being the same for all the qubits [23]. For example, starting from the four-qubit pure state \( \rho(t_0) = |1011\rangle\langle 1011| \), the action of the generalized amplitude damping channel leads, after a time \( dt \), to the statistical mixture

\[
\rho(t_0 + dt) = \left(1 - \frac{\Gamma dt}{\hbar}\right)|1011\rangle\langle 1011| + \frac{\Gamma dt}{3\hbar}(|0011\rangle\langle 0011| + |1001\rangle\langle 1001| + |1010\rangle\langle 1010|).
\]

We would like to stress that this simple model must be understood as a significant example illustrating the power of the quantum trajectories approach and that other kind of environmental noise can be simulated similarly. We assume that our quantum protocol is implemented by a sequence of instantaneous and perfect swap gates, separated by a time interval \( \tau \). We also assume that the only effect of the system’s Hamiltonian \( H_s \) is to generate these swap gates.

Using the quantum trajectories approach, we compute numerically the evolution in time of the initial state vector (1) in presence of the generalized amplitude damping channel. The total evolution time is \( \Delta t = (n-2)\tau \), since \( n-2 \) swap gates are required to transfer a member of the EPR pair from Alice to Bob. Then the standard teleportation protocol [9] is implemented. The fidelity of teleportation is defined by

\[
F = \frac{\langle \psi | \rho_B | \psi \rangle}{\langle \psi | \rho_B | \psi \rangle},
\]

where \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) is the state to be teleported, and \( \rho_B \) is the density matrix of Bob’s qubit at the end of the teleportation protocol, obtained from the final state of the \( n \)-qubit chain after tracing over all the other qubits of the chain. In the quantum trajectories method, we compute the fidelity as

\[
F = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle \psi | (\rho_B)_i | \psi \rangle,
\]

where \( (\rho_B)_i \) is the reduced density matrix of Bob’s qubit, obtained from the final \( n \)-qubit state of the trajectory \( i \).

The comparison between the quantum trajectories approach and the direct solution of the master equation (3) is shown in Fig. 1, where we compute the fidelity of the teleportation protocol as a function of the dimensionless damping
Figure 1: Fidelity $\bar{F} = F - F_\infty$ of the teleportation of the state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ as a function of the dimensionless damping rate $\gamma$, for $n = 9$ (circles) and $n = 22$ qubits (filled circles). The curve is obtained by direct solution of the master equation at $n = 9$, while the circles give the results of the quantum trajectories approach with $N = 400$ trajectories. Inset: the same but with a logarithmic scale for $\bar{F}$. The error bars give the size of the statistical errors.

As an example, we consider the teleportation of the state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Since the amplitude damping channel is such that the system ends up, at times much longer that $\hbar/\Gamma$, in the state $|0\ldots0\rangle$, we expect that for this state the fidelity drops to the value $F_\infty = 1/2$. This is confirmed by the numerical data of Fig. 1, where we can see that the quantum trajectories method correctly reproduces the decay of $\bar{F} \equiv F - F_\infty$. We point out that already with a rather small number of trajectories, $N = 400$, statistical errors are sufficiently small to capture the relevant physical features of the model. It can be seen from Fig. 1 that the agreement with the results obtained from the direct integration of the master equation (3) is remarkably good. The most important point is that the quantum trajectories approach allows us to simulate a number of qubits much larger than those accessible by direct solution of the master equation, which, due to memory restrictions in a classical computer, is possible only up to $n \approx 10$ qubits. The decay of the fidelity with the length $n$ of the qubit chain is shown in Fig. 2. Again we note that the agreement between quantum trajectories and direct numerical solution of the master equation is satisfactory. However, with the first approach we are able to simulate much longer spin chains with up to 24 qubits. We note that, as we will discuss in detail elsewhere, the non exponential decay of the fidelity with the damping rate and the number of qubits is a feature of the generalized amplitude damping channel.
We would like to stress that this non trivial decay is correctly reproduced by the quantum trajectories approach.

In summary, we have studied the fidelity of the teleportation protocol in a noisy environment by means of the quantum trajectories method. Our studies demonstrate the ability of this approach to model quantum information protocols with a large number of qubits. This opens up many possibilities for future studies. Theoretical predictions for the behavior of the fidelity and of other relevant quantities with respect to the system size and different kinds of environment can now be explored with the help of numerical simulations. It will be also possible to include the effects of realistic internal Hamiltonians. Since various quantum protocols can be easily modeled, the scope of the present approach can be extended to the study of their stability. Finally, quantum trajectories offer a very convenient framework to model experiments. In this context, we point out the ability of a single quantum trajectory to provide a good illustration of an individual experimental run \cite{4}. Therefore quantum trajectories promise to become a very valuable tool for quantum hardware design and to determine optimal regimes for the operability of quantum processors.

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Figure 2: Fidelity as a function of the number of qubits in the chain, for $\gamma = 0.5$. The state to be teleported is $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The results are obtained from the quantum trajectories method with $N = 400$ (circles) and from direct integration of the master equation (triangles). Inset: the same but with a logarithmic scale for $F$. 
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[23] It is possible to work out a compact expression for the Lindblad operators $L_k$ in our model. Their matrix elements in the basis $|i\rangle \equiv |i_{n-1} \ldots i_0\rangle$, with $i \equiv \sum_{l=0}^{n-1} i_l 2^l$, are given by

$$[L_k]_{i,j} = \begin{cases} \sqrt{\frac{\Gamma}{h} \sum_{l=0}^{j-1} i_l}, & \text{for } j \geq 2^k, \ i = j - 2^k, \ i_l = 1, \\ 0, & \text{otherwise}. \end{cases} \quad (11)$$

There are $n$ operators $L_k$ ($k = 0, 1, \ldots, n - 1$), where the index $k$ singles out which qubit undergoes the transition $|1\rangle \rightarrow |0\rangle$. 