Asymptotic form of quasi-normal modes of large AdS black holes

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Abstract

We discuss a method of calculating analytically the asymptotic form of quasi-normal frequencies for large AdS black holes in five dimensions. In this case, the wave equation reduces to a Heun equation. We show that the Heun equation may be approximated by a Hypergeometric equation at large frequencies. Thus we obtain the asymptotic form of quasi-normal frequencies in agreement with numerical results. We also present a simple monodromy argument that leads to the same results. We include a comparison with the three-dimensional case in which exact expressions are derived.

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Quasi-normal modes play an important role in the study of black holes and have long been the subject of extensive research primarily in the case of asymptotically flat space-times (for a review see [1, 2]). More recently, the asymptotic form of quasi-normal frequencies was shown to be related to the Barbero-Immirzi parameter of Loop Quantum Gravity (see [3] and references therein). Quasi-normal modes form a discrete spectrum of complex frequencies whose imaginary part determines the decay rate of the fluctuation. They are obtained as solutions to a wave equation subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity.

There has been an extensive investigation of quasi-normal modes in the case of asymptotically AdS space-times [4–20], in conjunction with the AdS/CFT correspondence. In this case the quasi-normal modes should correspond to perturbations of the dual CFT. The potential does not vanish in spatial infinity, so instead of demanding outgoing flux there, as in asymptotically flat space-times, the wavefunction ought to vanish at infinity. Exact results have been obtained in three space-time dimensions [10, 16]. In higher dimensions the wave equation cannot in general be solved analytically and only numerical results on the quasi-normal frequencies have been obtained in four, five and seven dimensions [5, 21–23].

In ref. [24], we discussed an analytic method of calculating the quasi-normal modes of a large AdS black hole. The method was based on a perturbative expansion of the wave equation in the dimensionless parameter $\omega/T_H$, where $\omega$ is the frequency of the mode and $T_H$ is the (high) Hawking temperature of the black hole. This is a non-trivial expansion, for the dependence of the wavefunction on $\omega/T_H$ changes as one moves from the asymptotic boundary of AdS space to the horizon of the black hole. The zeroth-order approximation was chosen to be an appropriate Hypergeometric equation so that higher-order corrections were indeed of higher order in $\omega/T_H$. The first-order correction was also calculated. We thus obtained an approximation to the low-lying quasi-normal frequencies. We showed that our results were in agreement with numerical results [5, 21] in five dimensions where the wave equation reduces to a Heun equation [25].

Here we discuss an approximation to the wave equation which is valid in the high frequency regime instead. In five dimensions we show that the Heun equation reduces to a Hypergeometric equation, as in the low frequency regime [24]. We obtain an analytical expression for the asymptotic form of quasi-normal frequencies in agreement with numerical results [5, 21]. These
expressions may also be easily obtained by considering the monodromies around the singulari-
ties of the wave equation. These singularities lie in the unphysical region. In three dimensions,
they are located at the horizon $r = r_h$, where $r_h$ is the radius of the horizon, and at the black
hole singularity, $r = 0$. In higher dimensions, it is necessary to analytically continue $r$ into the
complex plane. The singularities lie on the circle $|r| = r_h$. The situation is similar to the case
of asymptotically flat space where an analytic continuation of $r$ yielded the asymptotic form
of quasi-normal frequencies [26, 27]. It is curious that unphysical singularities determine the
behavior of quasi-normal modes.

The metric of a $d$-dimensional AdS black hole may be written as

$$ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2$$

$$f(r) = \frac{r^2}{R^2} + 1 - \frac{\omega_{d-1} M}{r^{d-3}}$$

where $R$ is the AdS radius and $M$ is the mass of the black hole. For a large black hole, the metric
simplifies to

$$ds^2 = -\hat{f}(r) \, dt^2 + \frac{dr^2}{\hat{f}(r)} + r^2 ds^2(\mathbb{E}^{d-2})$$

$$\hat{f}(r) = \frac{r^2}{R^2} - \frac{\omega_{d-1} M}{r^{d-3}}$$

The Hawking temperature is

$$T_H = \frac{d - 1}{4\pi} \frac{r_h}{R^2}$$

where $r_h$ is the radius of the horizon,

$$r_h = R \left[ \frac{\omega_{d-1} M}{R^{d-3}} \right]^{1/(d-1)}$$

The scalar wave equation is

$$\frac{1}{\sqrt{g}} \partial_A \sqrt{g} g^{AB} \partial_B \Phi = m^2 \Phi$$

We are interested in solving this equation in the massless case ($m = 0$) for a wave which is
ingoing at the horizon and vanishes at infinity. These boundary conditions yield a discrete set
of complex frequencies (quasi-normal modes).
We start with a review of the three-dimensional case where the assumption of a large black hole is redundant and the wave equation may be solved exactly [10, 16]. Indeed, in three dimensions ($d = 3$), the metric reads

$$ds^2 = \frac{1}{R^2} \left( r^2 - r_h^2 \right) dt^2 + \frac{R^2}{r^2} \frac{dr^2}{(r^2 - r_h^2)} + r^2 dx^2$$

(6)

independently of the size of the black hole. The wave equation (with $m = 0$) is

$$\frac{1}{R^2 r} \partial_r \left( r^3 \left( 1 - \frac{r_h^2}{r^2} \right) \partial_r \Phi \right) - \frac{R^2}{r^2} \frac{\partial^2 \Phi}{r^2} + \frac{1}{r^2} \partial^2_t \Phi = 0$$

(7)

One normally solves this equation in the physical interval $r \in [r_h, \infty)$. Instead, we shall solve it in the interval $0 \leq r \leq r_h$ (inside the horizon). The solution may be written as

$$\Phi = e^{i(\omega t - px)} \Psi(y), \quad y = \frac{r^2}{r_h^2}$$

(8)

where $\Psi$ satisfies

$$(y(1 - y)\Psi')' + \left( \frac{\dot{\omega}^2}{1 - y} + \frac{\dot{p}^2}{y} \right) \Psi = 0$$

(9)

in the interval $0 < y < 1$, and we have introduced the dimensionless variables

$$\dot{\omega} = \frac{\omega R^2}{2r_h} = \frac{\omega}{4\pi T_H}, \quad \dot{p}^2 = \frac{pR}{2r_h} = \frac{p}{4\pi RT_H}$$

(10)

Two independent solutions are obtained by examining the behavior near the horizon ($y \to 1$),

$$\Psi_{\pm} \sim (1 - y)^{\pm i\dot{\omega}}$$

(11)

where $\Psi_+$ is outgoing and $\Psi_-$ is ingoing. A different set of linearly independent solutions is obtained by studying the behavior at the black hole singularity ($y \to 0$). We obtain

$$\Psi \sim y^{\pm i\dot{p}}$$

(12)

For quasi-normal modes, we demand that $\Psi$ be purely ingoing at the horizon ($\Psi \sim \Psi_-$ as $y \to 1$). By writing

$$\Psi(y) = y^{\pm i\dot{p}}(1 - y)^{-i\dot{\omega}} F(y)$$

(13)

we deduce

$$y(1 - y)F'' + \left\{ 1 \pm 2i\dot{p} - (2 - 2i(\dot{\omega} \mp \dot{p})y \right\} F' + (\dot{\omega} \mp \dot{p})(\dot{\omega} \mp \dot{p} + i) F = 0$$

(14)
whose solution is the Hypergeometric function

$$F(y) = \, _2F_1(1 - i(\hat{\omega} \mp \hat{p}), -i(\hat{\omega} \mp \hat{p}); 1 \pm 2i\hat{p}; y) \tag{15}$$

In general, near the horizon \((y \to 1)\), this solution is a mixture of ingoing and outgoing waves. It also blows up at infinity. To obtain the desired behavior for a quasi-normal mode at \(y = 1\) and \(y \to \infty\), we demand that \(F(y)\) be a Polynomial. This condition implies

$$\hat{\omega} = \pm \hat{p} - in \quad, \quad n = 1, 2, \ldots \tag{16}$$

a discrete set of complex frequencies with negative imaginary part, as expected \([5]\). Notice that we obtained two sets of frequencies, with opposite real parts. Then eq. (15) implies that

$$F(y) = \, _2F_1(1 - n, -n; 1 \pm 2i\hat{p}; y) \tag{17}$$

which is a Polynomial of order \(n - 1\). It is therefore a constant at \(y = 1\), as desired and behaves as \(F(y) \sim y^{n-1} \sim y^{i(\hat{\omega} \mp \hat{p})-1}\) as \(y \to \infty\). Therefore, \(\Psi \sim y^{-1}\) as \(y \to \infty\), as expected.

The above quasi-normal frequencies may also be deduced from a simple monodromy argument. Let \(\mathcal{M}(y_0)\) be the monodromy around the singular point \(y = y_0\) computed along a small circle centered at \(y = y_0\) running counterclockwise. For \(y = 1\), we obtain

$$\mathcal{M}(1) = e^{2\pi \hat{\omega}} \tag{18}$$

whereas for \(y = 0\), we have

$$\mathcal{M}(0) = e^{\mp 2\pi \hat{p}} \tag{19}$$

Since the function vanishes at infinity, the two contours around the two singular points \(y = 0, 1\) may be deformed into each other without encountering any singularities. This implies that

$$\mathcal{M}(1) \mathcal{M}(0) = 1 \tag{20}$$

hence \(e^{2\pi(\hat{\omega} \mp \hat{p})} = e^{2\pi in} \quad (n \in \mathbb{Z})\), which leads to the same set of quasi-normal frequencies as before, if we demand \(\text{Im}\hat{\omega} < 0\). Notice that the monodromy argument is much simpler here than in the case of an asymptotically flat space-time \([27]\). This is because of a simpler boundary condition at infinity \((y \to \infty)\).
In five dimensions \((d = 5)\), the wave equation \((5)\) with \(m = 0\) reads

\[
\frac{1}{r^3} \partial_r \left( r^5 \hat{f}(r) \partial_r \Phi \right) - \frac{R^4}{r^2 \hat{f}(r)} \partial_r^2 \Phi - \frac{R^2}{r^2} \nabla^2 \Phi = 0
\]

(21)

where

\[
\hat{f}(r) = 1 - \frac{r_h^4}{r^4}
\]

(22)

The solution may be written as

\[
\Phi = e^{i(\omega t - \vec{p} \cdot \vec{x})} \Psi(r)
\]

(23)

Upon changing the coordinate \(r\) to \(y\),

\[
y = \frac{r^2}{r_h^2}
\]

(24)

the wave equation becomes

\[
(y^2 - 1) (y(y^2 - 1)\Psi')' + \left( \frac{\hat{\omega}^2}{4} y^2 - \frac{\hat{p}^2}{4} (y^2 - 1) \right) \Psi = 0
\]

(25)

where we have introduced the dimensionless variables

\[
\hat{\omega} = \frac{\omega R^2}{r_h} = \frac{\omega}{\pi T_H}, \quad \hat{p} = \frac{|\vec{p}| R}{r_h} = \frac{|\vec{p}|}{\pi R T_H}
\]

(26)

Two independent solutions are obtained by examining the behavior near the horizon \((y \to 1)\),

\[
\Psi_\pm \sim (y - 1)^{\pm i \hat{\omega}/4}
\]

(27)

where \(\Psi_+\) is outgoing and \(\Psi_-\) is ingoing. A different set of linearly independent solutions is obtained by studying the behavior at large \(r\) \((y \to \infty)\). We obtain

\[
\Psi \sim y^{h \pm}, \quad h \pm = 0, -2
\]

(28)

so one of the solutions contains logarithms. For quasi-normal modes, we are interested in the analytic solution which behaves as \(\Psi \sim y^{-2}\) as \(y \to \infty\). By considering the other (unphysical) singularity at \(y = -1\), we obtain another set of linearly independent wavefunctions behaving as

\[
\Psi \sim (y + 1)^{\pm \hat{\omega}/4}
\]

(29)

near \(y = -1\). Following the discussion in the three-dimensional case, we shall isolate the behavior at the two singularities \(y = \pm 1\) and write the wavefunction as

\[
\Psi(y) = (y - 1)^{-i \hat{\omega}/4}(y + 1)^{\pm \hat{\omega}/4} F(y)
\]

(30)
The two sets of modes have the same imaginary parts, but opposite real parts, as we shall see (similarly to the $d = 3$ case (eq. (16))).

It is easily deduced from eqs. (25) and (30) that the function $F(y)$ satisfies the Heun equation

$$y(y^2 - 1)F'' + \left\{ \left( 3 - i \frac{1}{2} \hat{\omega} \right) y^2 - i \frac{1}{2} \hat{\omega} y - 1 \right\} F' + \left\{ \hat{\omega} \left( \pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) \right\} F = 0$$

(31)

We wish to solve this equation in a region in the complex $y$-plane containing $|y| \geq 1$, which includes the physical regime $r > r_h$. For large $\hat{\omega}$, the constant terms in the respective Polynomial coefficients of $F'$ and $F$ in (31) are small compared with the other terms, so they may be dropped. Eq. (31) may then be approximated by the Hypergeometric equation

$$(y^2 - 1)F'' + \left\{ \left( 3 - i \frac{1}{2} \hat{\omega} \right) y - i \frac{1}{2} \hat{\omega} \right\} F' + \hat{\omega} \left( \pm \frac{i\hat{\omega}}{4} \mp 1 - i \right) F = 0$$

(32)

in the asymptotic limit of large frequencies $\hat{\omega}$. The analytic solution of (32) is the Hypergeometric function

$$F_0(x) = _2F_1(a_+, a_-; c; (y + 1)/2)$$

(33)

where

$$a_\pm = 1 - \frac{i\hat{\omega}}{4} \pm 1 \quad , \quad c = \frac{3}{2} \pm \frac{1}{2} \hat{\omega}$$

(34)

For proper behavior at $y \to \infty$, we demand that $F$ be a Polynomial. This requires

$$a_+ = -n \quad , \quad n = 1, 2, \ldots$$

(35)

Then $F$ is a Polynomial of order $n$, so at infinity it behaves as $F \sim y^n \sim y^{-a_+}$. The behavior of $\Psi$ is then deduced from (30) to be

$$\Psi \sim y^{-i\hat{\omega}/4} y^{+\hat{\omega}/4} y^{-a_+} \sim y^{-2}$$

(36)

as expected. The quasi-normal frequencies from eq. (35) are then easily found to be given by

$$\hat{\omega} = 2n(\pm 1 - i)$$

(37)

in agreement with numerical results [21].
These frequencies may also be obtained by a monodromy argument similar to the one in \(d = 3\). If the function has no singularities other than \(y = \pm 1\), the contour around \(y = +1\) may be unobstructedly deformed into the contour around \(y = -1\). Hence

\[
\mathcal{M}(1)\mathcal{M}(-1) = 1
\]

(38)

Since \(\mathcal{M}(1) = e^{i\hat{\omega}/2}, \mathcal{M}(-1) = e^{-i\hat{\omega}/2}\), the expression (37) is easily deduced (also using \(\text{Im}\hat{\omega} < 0\)).

The above approximation (33) to the exact wavefunction satisfying the Heun equation (32) may be used as the basis for a systematic calculation of corrections to the asymptotic form (37) of quasi-normal frequencies. To this end, let us write the Heun equation (32) as

\[
(\mathcal{H}_0 + \mathcal{H}_1)F = 0
\]

(39)

where

\[
\mathcal{H}_0 = (1 - y^2)\frac{d^2}{dy^2} + \left\{ \frac{i \pm 1}{2} \hat{\omega} - \left( 3 - \frac{i \mp 1}{2} \hat{\omega} \right) y \right\} \frac{d}{dy} - \frac{\hat{\omega}}{2} \left( \pm \frac{i\hat{\omega}}{4} \mp 1 - i \right)
\]

(40)

and

\[
\mathcal{H}_1 = \frac{1}{y} \left( \frac{d}{dy} + (i \mp 1)\frac{\hat{\omega}}{4} + \frac{\hat{p}^2}{4} \right)
\]

(41)

The zeroth-order equation,

\[
\mathcal{H}_0 F_0 = 0
\]

(42)

is the approximation we discussed above (the Hypergeometric eq. (32)). By treating \(\mathcal{H}_1\) as a perturbation, we may expand the wavefunction,

\[
F = F_0 + F_1 + \ldots
\]

(43)

and solve eq. (39) perturbatively. Corrections to the quasi-normal frequencies (37) may then be obtained once an explicit expression for the corrections to \(F_0\) have been calculated. Details of this calculation will appear elsewhere [28].

In higher dimensions, the wave equation possesses more than two singularities on the circle \(|r| = r_h\) in the complex \(r\)-plane. Thus, a simple monodromy argument such as the one discussed above in three and five dimensions (eqs. (20) and (38), respectively) does not appear to be applicable. Work in this direction is in progress.
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