Switchback Experiments under Geometric Mixing

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Abstract

The switchback is an experimental design that measures treatment effects by repeatedly turning an intervention on and off for a whole system. Switchback experiments are a robust way to overcome cross-unit spillover effects; however, they are vulnerable to bias from temporal carryovers. In this paper, we consider properties of switchback experiments in Markovian systems that mix at a geometric rate. We find that, in this setting, standard switchback designs suffer considerably from carryover bias: Their estimation error decays as $T^{-1/3}$ in terms of the experiment horizon $T$, whereas in the absence of carryovers a faster rate of $T^{-1/2}$ would have been possible. We also show, however, that judicious use of burn-in periods can considerably improve the situation, and enables errors that decay as $\log(T)^{1/2} T^{-1/2}$. Our formal results are mirrored in an empirical evaluation.

1 Introduction

Switchback experiments involve repeatedly toggling a treatment of interest on and off. There are several reasons to consider such experiments. In early work, Brandt [1938] and Cochran et al. [1941] studied the effect of diet on milk yield in dairy cows by alternating different diets for the same cow. Here, the motivation for using switchbacks was variance reduction: Different cows may have vastly different baseline yields, and so using a switchback can improve precision relative to designs that only give a single diet to each animal. More recently, there has been an explosion of interest in using switchbacks for A/B testing in online marketplaces, where a target intervention is toggled on and off at the market level [Bojinov et al., 2023, Chamandy, 2016, Glynn et al., 2020, Kastelman and Ramesh, 2018, Kohavi et al., 2018, Xiong et al., 2023]. When applied at the market level, switchbacks help mitigate bias due to cross-unit interference or spillovers: For example, if the intervention involves a new pricing scheme, then using a market-level switchback avoids market distortions that could arise from simultaneously using two different pricing schemes at the same time.

A key challenge in using switchback experiments, however, is the problem of temporal carryovers or lag effects [Bojinov et al., 2023, Cochran et al., 1941]. Treatments assigned at any specific time point can have not only an immediate effect on the current outcome but also a longer-term effect due to the change of the system’s (potentially latent) state [Glynn et al., 2020]. And any approach to estimation and inference in switchbacks that does not account for temporal carryovers is prone to bias.
As a concrete example of a problem setting where carryovers are likely to matter, consider a switchback used to compare greedy vs. optimized matching strategies in a two-sided market in which jobs arrive sequentially and need to be matched to available workers (consider, e.g., a ride sharing or grocery delivery platform). Suppose that the greedy algorithm matches each incoming job to the nearest available worker, while the optimized strategy seeks to preserve available resources when possible (e.g., if a job could reasonably be matched to one of two available workers and one of them is likely to be in high demand in the future, then the job could be matched to the other one—even if they are slightly further). The optimized strategy promotes good positioning of available workers, and so may cause abnormally good initial performance for the greedy algorithm right after the switch; conversely, the greedy algorithm may lead to anomalously poor initial performance for the optimized strategy. Thus, a switchback analysis that ignores carryovers may, in this setting, severely underestimate the benefit of the optimized strategy relative to the greedy one.

In this paper, we study switchbacks under an assumption that the system we are intervening on is a (non-stationary) Markov decision process with mixing time $t_{\text{mix}}$. Under this assumption, actions taken at time $t$ may affect the state of the system at all times $t' > t$, but the strength of these effects decays as $\exp[-(t' - t)/t_{\text{mix}}]$. We let the state evolution of the system be arbitrarily non-stationary (e.g., the system may respond arbitrarily to the time of the day or exogenous shocks like weather); however, we assume the mixing rate of the system to be uniformly bounded from above (see Section 2 for a formal model).

We first consider the behavior of a standard switchback experiment, i.e., one that toggles a binary treatment at selected time points and then estimates the treatment effect by taking the difference of the average outcome in periods where treatment is on and the average outcome when treatment is off. Under our model, we show that this standard switchback is severely affected by carryover bias: Given a single time series observed for $T$ time periods, the error of the standard switchback cannot be made to decay faster than $1/\sqrt{T}$. This is markedly slower than the $1/\sqrt{T}$ rate of convergence one could have achieved with $T$ time periods and no carryovers.

We also find, however, that judicious use of burn-in periods each time the treatment assignment is switched can alleviate this issue. Specifically, we consider a switchback design that, given a pre-specified burn-in time $b$, throws out all observations that are within $b$ time periods of the last treatment switch. We then show that, using this design, we can estimate the global treatment effect for non-burn-in periods with errors decaying as $\sqrt{\log(T)/T}$ with a simple difference-in-means estimator. Furthermore, we propose a bias-corrected, weighted estimator that can use data from a switchback experiment to estimate the global treatment effect across the entire experiment with errors also decaying as $\sqrt{\log(T)/T}$.

We provide central limit theorems for both estimators that remain valid in non-stationary environments. Finally, in numerical experiments, we also find that—as expected—the use of burn-in periods considerably helps improve the behavior of switchbacks under our Markovian model for carryover effects.

1.1 Related work

The problem of carryover effects in switchbacks has been considered by a number of authors, including Cochran et al. [1941], Bojinov et al. [2023] and Glynn et al. [2020]. Cochran et al. [1941] propose addressing the issue using a regression model, with a regression coefficient that captures the lagged effect of past treatment on future periods. Bojinov et al. [2023] allow for the presence of carryovers, but assume that there is a fixed (known) time horizon...
such that all carryover effects of an action taken at time $t$ disappear by time $t + m$. Glynn et al. [2020] consider a Markov model related to the one used here (although their model is stationary); however, they address the problem of temporal carryovers by fitting the Markov model by maximum likelihood rather than by adapting the switchback.

Relative to existing results, we believe our approach may be helpful in settings where researchers cannot assume that carryover effects will fully vanish after a finite amount of time (as is the case in generic Markovian systems) and do not trust a regression model to capture all carryovers, but still want to use a switchback design to estimate treatment effects. We also emphasize that, although we make Markovian assumptions for analytic purposes, we do not require the researcher to be able observe the full state of the system; we only depend implicitly on this modeling assumption via the mixing time. In contrast, the maximum likelihood approach of Glynn et al. [2020] requires observing the full state in order to compute the estimator. The problem of treatment effect estimation in Markov decision processes under general designs with sequential ignorability is further considered in Liao et al. [2021, 2022], Kallus and Uehara [2020, 2022] and Mehrabi and Wager [2024], but these results again require observing the full state of the system.

Our approach to modeling non-stationarity builds on the well-known Neyman model for finite-population causal inference [Neyman, 1923]. We assume that our switchback is run over $t = 1, \ldots, T$ time periods and—like in the Neyman model where each study participant can be different from the others—we allow for each time period $t$ to be different from the others. All we assume is that the system is Markovian (i.e., memoryless), and that it mixes over time (i.e., the effect of past events washes out over time); see Assumptions 1 and 2 for details. Our central limit theorems build on statistical tools originally built for finite-population causal inference [e.g., Aronow and Samii, 2017, Bojinov et al., 2023, Leung, 2022b, Li and Ding, 2017, Lin, 2013]; however, to our knowledge this paper is the first to use these tools to study non-stationary Markov decision processes.

Finally we note that, at a high level, the switchback design can be regarded as a special case of the cluster randomized experiment [Imbens and Rubin, 2015] in the temporal setting. From this perspective, our work is relevant to the strand of the literature focusing on the optimal design of a clustered randomized experiment without the presence of well-defined clusters [Athey et al., 2018, Harshaw et al., 2021, Leung, 2022a,b, Sävje, 2024, Ugander et al., 2013, Ugander and Yin, 2023, Viviano, 2020]. Much of this literature relies on the existence of some exposure mapping function that fully characterizes the interference structure [Aronow and Samii, 2017, Manski, 2013], with some recent exceptions including Leung [2022a,b] and Sävje [2024] that allow for misspecified exposure mappings and instead only assume, e.g., a rate of decay on the spillover effects. In our Markovian model, the carryover bias never disappears—it only decays over time—and as such our results are closer to these recent papers allowing approximate or misspecified exposure mappings than to analyses that depend on a well-specified exposure mapping. In a recent advance, Jia et al. [2024] build on a preprint version of this paper to develop a clustered switchback design that allows for both spatial spillovers and Markovian carryovers.

2 A Markov Model for Temporal Carryovers

We collect data from following a single system for a time period of length $T$. At each time point $t = 1, \ldots, T$, we assign a binary treatment $W_t \in \{0, 1\}$ to the system and observe an immediate outcome $Y_t \in \mathbb{R}$. We further divide the horizon into blocks of equal length $l$. In
switchback experiments, the treatment is assigned such that $W_t = W_s$ if $t$ and $s$ are from the same block. For simplicity, we assume that there exists $k \in \mathbb{Z}^+$ such that $kl = T$.

We assume that we are in a setting where there exists a (potentially unobserved) state variable $S_t \in \mathcal{S}$, and that the triples $(S_t, W_t, Y_t)$ form a non-stationary Markov decision process with transition operators $P_t(\cdot | \cdot)$. This transition operator can vary arbitrarily across time and captures the influence of all exogenous events on the system. The MDP assumption is illustrated in Figure 1.

**Assumption 1.** For all time points $t = 1, \ldots, T$, the evolution of the underlying state $S_t$ is governed by a pair of (deterministic) state transition distributions $P^0_t(\cdot | \cdot)$ and $P^1_t(\cdot | \cdot)$ that

$$S_{t+1} | \mathcal{F}_t, Y_t \sim P^w_t (\cdot | S_t),$$

where $\mathcal{F}_t = \sigma (S_t, W_t, Y_{t-1}, S_{t-1}, W_{t-1}, \cdots)$ contains all information available from time $-\infty$ to time $t$. Furthermore, there exists a set of functions $\mu_t : \mathcal{S} \times \mathcal{W} \to \mathbb{R}$ such that $Y_t$ is a noisy observation of $\mu_t(S_t, W_t)$, i.e., $Y_t = \mu_t(S_t, W_t) + \epsilon_t$ for some mutually independent noises $\epsilon_t$ satisfying $\mathbb{E} [\epsilon_t | \mathcal{F}_t] = 0$ and $\text{Var} [\epsilon_t | \mathcal{F}_t] \leq \sigma^2 < \infty$.

Next, although the system is not stationary in time, we assume that the system “mixes” at a rate of at least $1/t_{\text{mix}}$ at each time point. Mixing assumptions are common in the literature in contextual bandits and reinforcement learning, and a mixing assumption of the form (2) is used and discussed by, e.g., Van Roy [1998], Even-Dar et al. [2005] and Hu and Wager [2023]. In particular, Hu and Wager [2023] use such an assumption for off-policy evaluation in a partially observed Markov decision process. Unlike us, however, these papers consider mixing assumptions in a stationary environment (i.e., where $P^w_t = P^w$ is the same for all time periods $t$), whereas here we allow the problem dynamics to change arbitrarily from one period to the next.

**Assumption 2.** There exists a mixing time $t_{\text{mix}} < \infty$ such that, for all time points $t = 1, \ldots, T$ and $w \in \{0, 1\}$,

$$\|f'P^w_t - fP^w_t\|_{TV} \leq \exp (-1/t_{\text{mix}}) \|f' - f\|_{TV}$$

(2)
Our next task is to define meaningful causal estimands in the context of our model. In simple randomized controlled trials without interference, it is customary to focus on estimating the average treatment effect [Imbens and Rubin, 2015],

\[
\text{ATE} = \frac{1}{T} \sum_{t=1}^{T} (\mathbb{E} [Y_t | W_t = 1] - \mathbb{E} [Y_t | W_t = 0]).
\]  

(3)

In the presence of temporal carryovers, however, the average treatment effect is no longer well-defined because the distribution \(Y_t\) doesn’t just depend on \(W_t\), but can also depend on past actions \(\{W_{t-1}, W_{t-2}, \ldots \}\).

Figure 2 illustrates difficulties associated with non-stationarity and temporal carryovers in our model. Shaded regions indicate times when treatment is on \((W_t = 1)\) while clear region indicate times when treatment is off \((W_t = 0)\). Meanwhile, dashed vertical lines indicate exogenous shocks to the system unrelated to the treatment; these shocks cause the functions \(P^t(\cdot | \cdot)\) and \(\mu_t(\cdot)\) in Assumption 1 to change. Given this setting, the dashed blue line shows the average behavior we would get if the system were given control assignment throughout, the dotted green one shows average behavior with treatment throughout, and the solid red line shows average behavior with treatment toggled as in a switchback. Thanks to our mixing condition (Assumption 2), the red line eventually converges to the blue or green lines after each time we toggle the treatment—but this change is not instantaneous.

Intuitively, we can think of the average gap between the green and blue curves in Figure 2 as quantifying an average effect we want to estimate. To formalize this notion, for any treatment sequence \(w \in \{0, 1\}^\mathbb{Z}_+\), we write

\[
\mathcal{L}_w = \text{distribution of } Y_t | W_t = w_0, W_{t-1} = w_1, W_{t-2} = w_2, \ldots
\]  

(4)
i.e., \( \mathcal{L}_t^\alpha \) is the long-term outcome distribution at time \( t \) under the treatment assignment sequence \( \alpha \), which is conditioned implicitly on the sequence of transitional kernels \( \{P_t^1, P_t^0, P_{t-1}^1, P_{t-1}^0, \ldots \} \). We use short-hand \( L_t^0 \) and \( L_t^1 \) for distributions indexed by “pure” histories, i.e., where \( \alpha \) is all 0s or all 1s. Given this notation, we define the following estimands.

**Definition 1.** Under Assumption 1, the stable treatment effect at time \( t \) is

\[
\tau_t = \mathbb{E}_{\mathcal{L}_t^1}[Y_t] - \mathbb{E}_{\mathcal{L}_t^0}[Y_t],
\]

and the global average treatment effect (GATE) is

\[
\tau_{\text{GATE}} = \frac{1}{T} \sum_{t=1}^{T} \tau_t.
\]

Meanwhile, for any set of time points \( I \subset \{1, \ldots, T\} \), the filtered average treatment effect (FATE) is

\[
\tau_{\text{FATE}}(I) = \frac{1}{|I|} \sum_{t \in I} \tau_t.
\]

The first estimand, \( \tau_{\text{GATE}} \), is a direct analogue to the estimand considered in Ugander et al. [2013] and Xiong et al. [2023] for the Markov setting. In a cross-sectional randomized trial with spillover effects between the units, GATE is the average difference in outcomes when all units are exposed to treatment versus when all units are exposed to control, conditionally on the group of recruited subjects. Here in the longitudinal setting, our estimand of interest is the average difference in outcomes when the system is always exposed to treatment versus when the system is always exposed to control, conditionally on the time period during which the experiment is conducted. We emphasize that this estimand is only well-defined conditionally on environments before and during the experiment, i.e., on the sequence of transition kernels from time \(-\infty\) to time \( T \). This is also related to the long-run average reward studied in Glynn et al. [2020], except that in our setting the system is never stationary due to the exogenous and arbitrary transition operator. The second estimand, \( \tau_{\text{FATE}} \), gives us more flexibility on defining our causal target, and will effectively let us disregard time periods where our estimator is biased due to carryover effects (for example, it may be convenient to choose \( I \) to correspond to only evenly indexed time points).

## 3 Estimating Treatment Effects with Switchbacks

Our next goal is to establish estimation guarantees for the estimands given in Definition 1 using switchback experiments. To this end, we start by formalizing the switchback design and the associated natural treatment effect estimator as follows.

**Definition 2.** The regular Bernoulli switchback design is characterized by a block-length \( l > 1 \) and a time horizon \( T \), and assigns treatment as

\[
W_t | W_{t-1} = \begin{cases} 
\text{Bernoulli}(0.5) & \text{if } t = (i - 1)l + 1 \text{ for some } i = 1, \ldots, \lfloor T/l \rfloor, \\
W_{t-1} & \text{else}.
\end{cases}
\]

Notice that this is equivalent to a potential outcome specification, where \( Y_t = Y_t(\alpha) \) and \( \alpha \) indexes the full history.
For convenience, we write $Z_i = W_{(i-1)l+1}$ to the treatment given to the $i$-th block, and write $k = \lfloor T/l \rfloor$ for the total number of blocks. The regular switchback estimator takes data from a regular switchback along with an (optional) burn-in length $0 \leq b < l$, and estimates the overall treatment effect via a difference-in-means that discards burn-in periods (if a non-zero burn-in length is used).\footnote{We follow the convention to set $0/0$ to be 0.}

$$
\hat{\tau}^{(l,b)}_{\text{DM}} = \frac{1}{k_1(l-b)} \sum_{\{i: Z_i = 1\}} \sum_{s=b+1}^{l} Y_{(i-1)l+s} - \frac{1}{k_0(l-b)} \sum_{\{i: Z_i = 0\}} \sum_{s=b+1}^{l} Y_{(i-1)l+s},
$$

$$
k_w = \{|i = 1, \ldots, \lfloor T/l \rfloor : Z_i = w\}.
$$

Our definition of the regular switchback design and estimator is closely related to the one used in Bojinov et al. [2023], in that we both consider experiments that may randomly switch treatment according to Bernoulli draws at pre-determined time-points, and both consider estimators that discard observations right after switches to mitigate bias. However, our specification of the estimator (9) differs from the one used in Bojinov et al. [2023] in that it is purely algorithmic. The estimator $\hat{\tau}^{(l,b)}_{\text{DM}}$ does not depend explicitly on the model (Assumptions 1 and 2); rather, it is a function of the block length $l$ and burn-in period $b$ that one could seek to justify from a number of perspectives. In contrast, Bojinov et al. [2023] start by specifying conditions under which carryover effects are guaranteed to vanish, and then consider estimation using the natural Horvitz-Thompson estimator induced by their modeling assumptions. We note that in our setting, i.e., under Assumptions 1 and 2, carryover effects never fully vanish—they simply decay at an exponential rate. Thus, a literal application of the construction of Bojinov et al. [2023] would not be consistent, because no data could ever be used after the first switch. Beyond the work of Bojinov et al. [2023], we are not aware of prior studies of switchback experiments that consider using burn-in periods or other analogous estimation techniques.

Our first result is an error decomposition for regular switchback estimators under our geometric mixing assumptions. The estimator $\hat{\tau}^{(l,b)}_{\text{DM}}$ has two sources of bias. First, it has bias due to the long-term effect carryover from the past treatments. As can be observed from Figure 2, when the treatment condition switches from $w$ to $1-w$, the curve will take time to converge from the mean outcome curve under $w$ to the mean outcome curve under $1-w$. Thus, there is always a bias due to the mixing of the process after each switchback, and the closer it is to the switch point, the larger the bias we will encounter in estimating $\tau_t$. The use of burn-in periods can mitigate carryover bias—but this results in a different source of potential bias due to ignoring outcomes in the burn-in periods.

**Theorem 1.** Under Assumptions 1 and 2, suppose in addition that the conditional expectations are bounded almost surely by a constant $\Lambda$ such that

$$
|\mathbb{E}[Y_t|S_t, W_t]| = |\mu_t(S_t, W_t)| \leq a.s. \, \Lambda,
$$

and that the heterogeneity in treatment effects is bounded by a constant $\Psi$, i.e.,

$$
|\tau_t - \tau_s| \leq \Psi
$$

for arbitrary $t, s \in \{1, \ldots, T\}$. Given the block length $l$ and the burn-in period $b$, define $k = k_1 + k_0 = \lfloor T/l \rfloor$, and let

$$
\mathcal{I}^{(l,b)} = \{t: t - l\lfloor t/l \rfloor > b\}, \quad \tau^{(l,b)}_{\text{FATE}} := \tau_{\text{FATE}} \left( \mathcal{I}^{(l,b)} \right)
$$

(10)
denote the filtered average treatment effect estimator that only considers non-burn-in periods. Then, the bias of the regular switchback estimator \( \hat{\tau}_{DM}^{(l,b)} \) can be bounded as

\[
\left| \mathbb{E} \left[ \hat{\tau}_{DM}^{(l,b)} \right] - \tau_{FATE}^{(l,b)} \right| \leq \frac{4\Lambda}{1 - \exp(-1/t_{max})} \cdot \frac{\exp(-b/t_{max})}{l - b} + O\left(2^{-k}\right),
\]

(11)

and furthermore

\[
\left| \tau_{FATE}^{(l,b)} - \tau_{GATE} \right| \leq \Psi \frac{b}{l}.
\]

(12)

Meanwhile, the variance of \( \tau_{DM}^{(l,b)} \) can be bounded as

\[
\text{Var} \left[ \hat{\tau}_{DM}^{(l,b)} \right] \leq \frac{12\Lambda^2}{k} + \frac{4\sigma^2}{k(l - b)} + \frac{16\Lambda^2 \exp(-b/t_{max})}{(1 - \exp(-1/t_{max}))^2} \cdot \frac{1}{k(l - b)^2} + O\left(\frac{1}{k^2}\right),
\]

(13)

where the three leading terms in the above bound represents the variance from the clustered treatment assignment, the unpredictable outcome noise, and covariance due to carryover effects respectively.

We note that the upper bounds on both the bias and the variance can be decomposed into errors from different sources. If we consider \( \hat{\tau}_{DM}^{(l,b)} \) as an estimator for \( \tau_{GATE} \), then the upper bound on bias captures the tradeoff in tackling the challenge due to carryover effect with burn-in periods. On the one hand, burn-in periods are needed to mitigate mixing bias, i.e., bias arising from switchbacks creating treatment histories that are inconsistent with the always-treat and never-treat policies we are trying to evaluate; but, on the other hand, discarding observations from the burn-in periods will induce burn-in bias because we approximate \( \tau_t \) for \( t \in \text{burn-in} \) with \( \tau_t \) for \( t \notin \text{burn-in} \). However, if we are willing to consider \( \hat{\tau}_{DM}^{(l,b)} \) as an estimator for \( \tau_{FATE} \) instead, then we can ignore the burn-in bias, thus allowing for more favorable error bounds.

The existence of the carryover effect can also inflate the variance by inducing a positive correlation between mean outcomes observed at different times points. In addition to this, the variance of the estimator is also related to how noisy the observations are, and how clustered the treatment assignments are. Nevertheless, as long as \( l(T) - b(T) \) is bounded away from zero, the variance will be dominated by the term from the clustered assignment; this mirrors results available in the context of generic cluster-randomized experiments [Leung, 2022a].

We end this section by giving conditions where the error bounds from Theorem 1 can be sharpened into a central limit theorem. To state our result, we use the following notation: For all blocks \( i = 1, \ldots, k \), let

\[
Y_i = \frac{1}{l - b} \sum_{t=b+1}^{l} Y_{(i-1)l+t},
\]

(14)

where as usual \( l \) denotes the block length. Furthermore, using notation from Pearl [2009], for \( w = 0, 1 \), let \( \bar{Y}_i(w) = \mathbb{Y}_i \left\{ \text{do}(Z_i = w) \right\} \). We also write

\[
\mu_i(w) = \frac{1}{l - b} \sum_{t=b+1}^{l} \mathbb{E}_{Z_i(w)} \left[ Y_{(i-1)l+t} \right], \quad \mathcal{M}_i(w) = \mathbb{E} \left[ \mathbb{Y}_i(w) \right],
\]

(15)
i.e., $\mu_i(w)$ would be the expectation of $Y_i$ in a system that always receives treatment $w$, and $\bar{M}_i(w)$ is the expectation of $Y_i$ in our switchback given $W_i = w$. We note that, following (10), we have

$$\tau_{FATE}^{(l,b)} = \frac{1}{k} \sum_{i=1}^{k} (\mu_i(1) - \mu_i(0)).$$

Finally, for simplicity of exposition we assume that, although our system is non-stationary, relevant empirical variances converge in large samples; similar assumptions are often made in finite population central limit theorems [e.g., Aronow and Samii, 2017, Lin, 2013].

Assumption 3. We consider a regular switchback with $l$ and $b$ chosen such that the following limits exist:

$$\frac{1}{k} \sum_{i=1}^{k} \left( \bar{\mu}_i(0) - \frac{1}{k} \sum_{j=1}^{k} \bar{\mu}_j(0) \right)^2 \rightarrow V_0,$$

$$\frac{1}{k} \sum_{i=1}^{k} \left( \bar{\mu}_i(1) - \frac{1}{k} \sum_{j=1}^{k} \bar{\mu}_j(1) \right)^2 \rightarrow V_1,$$

$$\frac{1}{k} \sum_{i=1}^{k} \left( \bar{\mu}_i(1) - \frac{1}{k} \sum_{j=1}^{k} \bar{\mu}_j(1) \right) \left( \bar{\mu}_i(0) - \frac{1}{k} \sum_{j=1}^{k} \bar{\mu}_j(0) \right) \rightarrow V_{01}. \tag{17}$$

Theorem 2. Under the conditions in Theorem 1, suppose in addition that Assumption 3 holds, that $Y_t$ are upper bounded such that $|Y_t| \leq \Gamma_0$ for some $\Gamma_0 < \infty$, and that there exists a constant $\sigma_0 > 0$ such that $\text{Var} \left[ \epsilon_t \mid F_t \right] \geq \sigma_0^2$. Then provided that we choose the burn-in length $b$ so that $l - b = O(1)$ and

$$k \exp \left(-\frac{2b}{t_{mix}}\right) \rightarrow 0, \tag{18}$$

as $k \rightarrow \infty$, we have

$$\sqrt{k} \left( \tau_{DM}^{(l,b)} - \tau_{FATE}^{(l,b)} \right) \rightarrow_d \mathcal{N}(0, V), \quad V = V_0 + V_1 + 2V_{01} + \Sigma_{\Delta}. \tag{19}$$

where

$$\Sigma_{\Delta} = \mathbb{E} \left[ \left( \frac{1}{0.5} \sum_{i=1}^{k} (Y_i(1) - \bar{M}_i(1)) \frac{Z_i}{0.5} - (Y_i(0) - \bar{M}_i(0)) \frac{1-Z_i}{0.5} \right)^2 \right]. \tag{20}$$

4 Rate-Optimal Switchback Designs

Our next challenge is to give guidance on how to choose the switchback block length $l$ and burn-in length $b$ such as to make the error guarantees for regular switchback estimators obtained in Theorem 1 as good as possible; here, we focus on optimizing mean-squared error. In doing so, however, we first need to specify our target—do we want to target $\tau_{GATE}$, or is it acceptable to target $\tau_{FATE}$ from (10) instead? We start by first giving results for both targets below, and then follow-up with a discussion of trade-offs.

Corollary 3. Under the conditions in Theorem 1, with the choice that

$$l = \frac{(4/3)^{1/3}}{(1 - e^{-1/t_{mix}})^{2/3}} T^{1/3} \quad \text{and} \quad b = 0, \tag{21}$$

as
\( \hat{\tau}_{DM}^{(l,b)} \) achieves the error bound

\[
\mathbb{E} \left[ (\hat{\tau}_{DM}^{(l,b)} - \tau_{GATE})^2 \right] \leq \frac{48^{2/3} \Lambda^2}{(1 - e^{-1/t_{mix}})^{2/3}} T^{-2/3} + o \left( T^{-2/3} \right)
\]  

(22)

in estimating \( \tau_{GATE} \). Furthermore, no regular switchback estimator as given in Definition 2 can guarantee a faster-than-\( O \left( T^{-2/3} \right) \) rate of convergence in this setting.

**Corollary 4.** Under the conditions in Theorem 1, for any bounded constant \( C_1 > 0 \), with the choice that

\[
l = b + C_1 \quad \text{and} \quad b = \frac{t_{mix}}{2} \log T,
\]

(23)

\( \hat{\tau}_{DM}^{(l,b)} \) achieves the error bound

\[
\mathbb{E} \left[ (\hat{\tau}_{DM}^{(l,b)} - \hat{\tau}_{FATE}^{(l,b)})^2 \right] \leq \left( 6\Lambda^2 + \frac{2\sigma^2}{C_1} \right) \cdot t_{mix} \cdot \log T \cdot T^{-1} + o \left( \log T \cdot T^{-1} \right)
\]

(24)

in estimating \( \hat{\tau}_{FATE}^{(l,b)} \).

Given side by side, these two results present a comprehensive characterization of the role of burn-in periods in regular Bernoulli switchbacks as specified in Definition 2. If we insist on targeting \( \tau_{GATE} \), then carryover bias matters, but we cannot use burn-ins to ameliorate the situation. The problem is, essentially, that the burn-in bias from using a non-trivial \( b \) grows faster than the carryover bias decays, thus making the choice of \( b = 0 \) rate optimal. The solution for estimating \( \tau_{GATE} \) using regular Bernoulli switchbacks is then to use a very long block length of \( T^{1/3} \), resulting in an \( O(T^{-1/3}) \) rate of convergence in root-mean squared error.

On the other hand, if we are willing to consider \( \hat{\tau}_{FATE}^{(l,b)} \) as our target estimand, the situation becomes much better. This choice of estimand eliminates burn-in bias, and we can then make aggressive use of burn-in periods to achieve a substantially better rate of convergence. In Corollary 4, we are able to reach the strong benchmark rate of \( O \left( \sqrt{\log(T)/T} \right) \) using shorter blocks (of length \( \log(T) \)). Recall that the best rate of convergence we could have hoped for in the absence of carryovers is \( O(T^{-1/2}) \); and so we are paying a relatively small penalty for the existence of carryovers here.

While the estimand \( \hat{\tau}_{FATE}^{(l,b)} \) may seem surprising at first glance, we argue that it may be a reasonable target in many application areas. In non-stationary settings as considered here, both \( \tau_{GATE} \) and \( \hat{\tau}_{FATE}^{(l,b)} \) share a certain arbitrary nature, in that they both only assess treatment effects during some specific set of time periods specified by the experiment; and, with judicious choices of \( l \) and \( b \), it is likely that \( \hat{\tau}_{FATE}^{(l,b)} \) could be just as relevant as \( \tau_{GATE} \) for downstream decision making. Consider, for example, a hypothetical switchback run by an online marketplace in the first two weeks of November 2021 with a block length \( l = 192 \) minutes and burn-in time \( b = 96 \) minutes (for a total of 105 periods). Then, \( \tau_{GATE} \) is an average over all time periods over the first two weeks of November 2021, while \( \hat{\tau}_{FATE}^{(l,b)} \) would be an average of half of those periods, where each weekday-hour-minute tuple enters into the target set \( I^{(l,b)} \) exactly once (either in the first week or in the second). In this setting, for most purposes, \( \tau_{GATE} \) and \( \hat{\tau}_{FATE}^{(l,b)} \) would be roughly equally informative summaries of the effectiveness of the target intervention; and that the biggest question in considering external validity of the study would be whether, in this online marketplace, the first two weeks of November 2021 are representative of of future time periods where the treatment may be deployed at scale.

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One question that is left open by Corollary 4 is how to precisely choose $l$ and $b$ when targeting $\tau_{\text{FATE}}$. Qualitatively, the insight is that we should use $l \sim \log(T)$, and that “most” periods should be devoted to burn-in, i.e., $b = l - C$ for some constant $C$ that does not scale with $T$. In practice, however, $\log(T)$ may not be materially larger than relevant constants for reasonable values of $T$, and so the optimal choice of $b$ may be a non-trivial multiple of $l$ (e.g., $b = l/2$ may be reasonable). Furthermore, optimal choices of both $b$ and $l$ depend on $t_{\text{mix}}$, which may be challenging to estimate from data; this mirrors the findings of Xiong et al. [2023], which highlights the role of prior information in designing an efficient switchback experiment. In practice, our main recommendation is that analysts using switchbacks to estimate treatment effects in systems that may exhibit carryovers should consider using burn-in periods to mitigate carryover bias. Optimal choices of $l$ and $b$ are likely to depend on specifics of the application setting, and we recommend using a mix of experimental and semi-synthetic validation to pick $l$ and $b$ on a case-by-case basis.

5 A Bias-Corrected Estimator

In the previous section, we found that a regular switchback estimator can only achieve an error rate of $O(T^{-1/3})$ in estimating $\tau_{\text{GATE}}$. We also showed that when considering a design with burn-in periods, it is possible to overcome the challenge brought by carryover bias and achieve a rate of $O(\sqrt{\log T/T})$ in estimating $\tau_{\text{FATE}}$. However, due to the deterministic exclusion of observations from burn-in periods, the use of burn-in periods cannot be used to improve accuracy for $\tau_{\text{GATE}}$ with regular switchback estimators. This observation raises a natural question: Is it fundamentally hard to get good estimates of $\tau_{\text{GATE}}$ using data collected from a switchback design, or could we achieve better results by analyzing the data produced by the switchback using a different estimator?

The answer to this last question turns out to be affirmative: We can also achieve a $O(\sqrt{\log T/T})$ rate of convergence for $\tau_{\text{GATE}}$ by using an inverse-propensity-type bias-corrected estimator that takes data collected from a regular Bernoulli switchback design as its input. Our proposed bias-corrected estimator utilizes the fact that, with Bernoulli randomization, the assigned treatment does not actually change at the beginning of every block, and thus observations within burn-in periods need not always be thrown away. Using shorthand $k_{ww'} = \sum_{i=2}^{k} I(Z_i = w, Z_{i-1} = w')$, we then estimate $\tau_{\text{GATE}}$ as

$$
\hat{\tau}_{\text{BC}}^{(l,b)} = \frac{l - b}{l} \hat{\tau}_{\text{DM}}^{(l,b)} + \frac{1}{k_{11}} \sum_{i:Z_i=Z_{i-1}=1,i \geq 2} \frac{b}{l} \sum_{t=1}^{b} Y_{(i-1)t+t} - \frac{1}{k_{00}} \sum_{i:Z_i=Z_{i-1}=0,i \geq 2} \frac{1}{l} \sum_{t=1}^{b} Y_{(i-1)t+t}.
$$

The intuition behind our proposed estimator is as follows. In a Bernoulli switchback, a switch at the beginning of each block occurs randomly with a probability of 0.5. Each block in a switchback is divided into two parts: a burn-in part and a focal part. The observations from the focal part are always included in the estimation with weight 1, while the observations from the burn-in part are included with a probability of 0.5 (when a switch does not occur), and are given a weight of approximately 2. Hence, on average, observations from all periods are weighted equally, but those from the initial few time periods after a realized switch are still discarded. The following result establishes the claimed rate of convergence for $\tau_{\text{GATE}}$ using this bias-corrected estimator.
Finally, we assume that all cross-products of between the block-averaged potential outcomes for the focal period as in Assumption 3, and let

\[ \text{Var} \left( \hat{\tau}_{BC}^{(l,b)} \right) \leq \frac{28 \Lambda^2}{k} + \frac{16 \Lambda^2 \exp\left(-b/t_{mix}\right)}{(1 - \exp\left(-1/t_{mix}\right))^2} \cdot \frac{1}{kt^2} + \frac{8 \sigma^2}{kt} + \mathcal{O}\left(\frac{1}{k^{2/3}}\right) + \mathcal{O}\left(2^{-k}\right). \]  

Furthermore, for any bounded constant \( C_2 > 0 \), with the choice that

\[ l = b + C_2 \quad \text{and} \quad b = \frac{t_{mix}}{2} \log T \]  

\( \hat{\tau}_{BC}^{(l,b)} \) achieves the error bound

\[ \mathbb{E} \left( \left( \hat{\tau}_{BC}^{(l,b)} - \tau_{GATE} \right)^2 \right) \leq 14 \Lambda^2 t_{mix} \cdot \log T \cdot T^{-1} + \mathcal{O}\left(\log T \cdot T^{-1}\right) \]  

in estimating \( \tau_{GATE} \).

We also provide a central limit theorem analogous to the one given in Theorem 2 for the regular switchback estimator. The bias-corrected estimator has a more complex form, and so we need to require convergence of further moments in addition to Assumption 3.

**Assumption 4.** For \( i = 1, \ldots, k \), let \( \bar{p}_i(w) \) denote average “pure treatment” potential outcomes for the focal period as in Assumption 3, and let

\[ \bar{p}_i^b(w) = \frac{1}{b} \sum_{t=1}^{b} \mathbb{E}_{\bar{L}_{w}^{(i-1)i+t}} \left[ Y_{(i-1)i+t} \right] \]  

denote the analogous quantity for the burn-in periods. We assume that the centered second moment of the block-averaged potential outcomes in the burn-in periods \( (b) \) and focal periods \( (f) \) converge as the number of blocks \( k \) gets large,

\[ \frac{1}{k} \sum_{i=1}^{k} \left( \bar{p}_i^b(w) - \frac{1}{k} \sum_{j=1}^{k} \bar{p}_i^b(w) \right) = \left( \frac{V_w^b}{V_w^b} V_w^{b/f} V_w^f, V_w^{b/f} V_w^{b/f} V_w^f \right), \]  

for both \( w = 0, 1 \). We also assume that the associated cross-terms converge in the same limit:

\[ \frac{1}{k} \sum_{i=1}^{k} \left( \bar{p}_i^b(0) - \frac{1}{k} \sum_{j=1}^{k} \bar{p}_i^b(0) \right) \left( \bar{p}_i^b(1) - \frac{1}{k} \sum_{j=1}^{k} \bar{p}_i^b(1) \right) \rightarrow \left( \frac{V_{01}^b}{V_{01}^b} \frac{V_{01}^{b/f}}{V_{01}^{b/f}} \frac{V_{01}^f}{V_{01}^f} \right). \]  

Finally, we assume that all cross-products of between the block-averaged potential outcomes in one block and those in the burn-in period of the subsequent block \( (s) \) also converge,

\[ \frac{1}{k-1} \sum_{i=1}^{k-1} \left( \bar{p}_i^b(w) - \frac{1}{k} \sum_{j=1}^{k} \bar{p}_i^b(w) \right) \left( \bar{p}_{i+1}^s(w') - \frac{1}{k} \sum_{j=1}^{k} \bar{p}_{i+1}^s(w') \right) \rightarrow \left( \frac{V_{ww}^{s/b}}{V_{ww}^{s/b}} \frac{V_{ww}^{s/f}}{V_{ww}^{s/f}} V_{ww}^f \right), \]  

for all pairs \( w, w' \in \{0, 1\} \).
Theorem 6. Under the assumptions of Theorem 1, suppose in addition that $Y_t$ are upper bounded such that $|Y_t| \leq \Gamma_0$ for some $\Gamma_0 < \infty$, and that Assumption 4 holds. Then provided that we choose $b$ and $l$ so that $l \to \infty$,

$$\frac{k}{l^2} \exp \left(-\frac{2b}{t_{mix}} \right) \to 0,$$

and there exists a constant $\beta \in [0, 1]$ such that $b/l \to \beta$ as $k \to \infty$, we have

$$\sqrt{k} \left( \hat{\tau}^{(l,b)}_{BC} - \tau_{GATE} \right) \to_d N \left( 0, \tilde{V} \right)$$

where

$$\tilde{V} = (1 - \beta)^2 \left( V^f_0 + V^f_1 + 2V^{f}_{01} \right)$$

$$+ \beta^2 \left( 3V^b_0 + 3V^b_1 + 2V^{b}_{01} + 2V^{bs}_{10} + 2V^{bs}_{00} + 2V^{bs}_{11} \right)$$

$$+ 2\beta(1 - \beta) \left( V^{bf}_0 + V^{bf}_1 + V^{bf}_{01} + V^{bf}_{10} + V^{fs}_0 + V^{fs}_{11} + V^{fs}_{01} + V^{fs}_{10} \right).$$

As in the classical Neyman causal inference model [Imbens and Rubin, 2015], the variances (19) and (36) do not in general admit unbiased estimators. This is because estimating the cross-terms appearing in (32) would require observing outcomes under both treatment assignments in the same block, and it is impossible to estimate empirically by design. However, it is still possible to build conservative confidence intervals using the Cauchy-Schwarz inequality such that the resulting confidence intervals have at least as large coverage as the nominal ones. Furthermore, as we will demonstrate in the numerical examples, block-resampling methods such as block jackknife of Künsch [1989] appear to work well in practice.

6 Numerical Experiments

In this section, we show that the asymptotic rates derived in the previous sections approximate well the optimal rates that can be achieved by the estimators in estimating both the average global policy effect and the average filtered policy effect in simulations.

6.1 Simple Illustration

We start with a very simple setting in which all carryovers are mediated by a hidden state variable $H_t \in \{0, 1, \ldots, 20\}$ that evolves according to a random walk with drift. The evolution of $H_t$ is governed by the assigned treatment, as well as an exogenous market condition variable $M_t \in \{1, 2, 3\}$. We generate $M_t$ with the following dynamic process:

$$P(M_{t+1} = m) = \begin{cases} 
2/3, & \text{if } M_t = m, \\
1/6, & \text{otherwise,}
\end{cases}$$

i.e., the market condition will switch with probability 0.5, and once it switches, the new market condition is a uniform random draw from $\{1, 2, 3\}$. Conditionally on the sequence of market conditions, the state variable $H_t$ is generated as follows: If $W_t = 1$,

$$H_{t+1} = \begin{cases} 
\min \{H_t + M_t, 20\}, & \text{with probability 0.7,} \\
\max \{H_t - M_t, 0\}, & \text{with probability 0.3;}
\end{cases}$$

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Figure 3: MSE as a function of \( T \) under different block lengths \( l \) on a log scale across 5000 iterations. The lightest orange corresponds to the case with the smallest \( l \), with a gradient to dark red representing \( l \) increases from the smallest choice to the largest choice gradually. The black dashed lines indicate the convergence rate, with an order of \( T^{-2/3} \) on the left and \( \log(T)/T \) in the middle and on the right.

If \( W_t = 0 \),

\[
H_{t+1} = \begin{cases} 
\min \{ H_t + M_t, 20 \} & \text{with probability 0.3,} \\
\max \{ H_t - M_t, 0 \} & \text{with probability 0.7.}
\end{cases}
\]  

(39)

Given \((H_t, W_t)\), \( Y_t \) is then generated as

\[
Y_t = H_t + 0.5 \cdot W_t \cdot H_t + \epsilon_t,
\]

(40)

where \( \epsilon_t \sim N(0, \sigma^2) \). Throughout, we fix \( \sigma = 3 \), and vary the horizon \( T \) from 400 to 25,600.

First, we evaluate the performance of \( \hat{\tau}_{DM}(l, b) \) in estimating \( \tau_{GATE} \). Motivated by Corollary 3, we fix \( b \) to be always zero, and vary \( l \) from 60 to 480. The left panel of Figure 3 displays how MSE changes with different choices of \( l \) and \( T \) on a log scale across 5000 iterations. In general, \( \hat{\tau}_{DM}(l, 0) \) with a larger \( l \) achieves a quicker rate of convergence in estimating \( \tau_{GATE} \), while \( \hat{\tau}_{DM}(l, 0) \) with a smaller \( l \) starts off with a smaller error. Consequently, as the horizon \( T \) gets larger, the optimal \( l \) also grows gradually from \( l = 60 \) to \( l = 480 \). Nevertheless, the mean-squared error achieved even with the best choices of \( l \) will never surpass a \( T^{-2/3} \) rate of convergence in large samples.

In the middle panel of Figure 3, we plot the performance of \( \hat{\tau}_{DM}^{(l,b)} \) in estimating \( \tau_{FATE}^{(l,b)} \). Motivated by Corollary 4, we vary \( b \) from 10 to 80, and consider the set of \( l \) such that \( l = b + 30 \). We observe a similar pattern in the relationship between the horizon and the length of the burn-in periods, with the optimal \( b \) growing gradually from 10 to 20 as \( T \) increases. However, we can notice that, when the target is to estimate \( \tau_{FATE}^{(l,b)} \), we are now able to achieve a much better error rate with the same estimator \( \hat{\tau}_{DM}^{(l,b)} \). This is especially important if the practitioners are planning for an experiment over a relatively long period of time. Furthermore, comparing the plot to the left panel, we notice that the performance is relatively robust to the choice of \( b \). We also considered the set of \( l \) such that \( l = b + 50 \) and obtained similar results. Our results thus suggest that, in addition to improving asymptotic
behavior relative to standard switchbacks, using burn-in periods (and targeting $\tau_{\text{FATE}}^{(l,b)}$) also makes the performance of the experiment more stable across different choices of tuning parameters.

Next, we investigate the performance of $\hat{\tau}_{\text{BC}}^{(l,b)}$ in estimating $\tau_{\text{GATE}}$, as represented in the right panel of Figure 3. Once again, we vary $b$ from 10 to 80, and consider the set of $l$ such that $l = b + 30$. The overall performance of $\hat{\tau}_{\text{BC}}^{(l,b)}$ in estimating $\tau_{\text{GATE}}$ closely mirrors that of $\hat{\tau}_{\text{DM}}^{(l,b)}$ in estimating $\tau_{\text{FATE}}$, both achieving an error rate of $\log(T) \cdot T^{-1}$ in mean-squared error for estimating their targets.

Finally, we assess the coverage provided by confidence intervals associated with the three estimators: $\hat{\tau}_{\text{DM}}^{(l,0)}$, $\hat{\tau}_{\text{DM}}^{(l,b)}$, and $\hat{\tau}_{\text{BC}}^{(l,b)}$. Throughout this analysis, we set $T = 20,000$ and $l = 200$, while varying $b$ from 50 to 150. To estimate the variance of the difference-in-means estimators, we employ a jackknife resampling procedure [Miller, 1974], which iteratively excludes each block and computes the variance of the estimator based on the remaining observations. For the bias-corrected estimators, we utilize a block-jackknife resampling procedure [Künsch, 1989], which excludes two blocks simultaneously to account for the significant correlation between blocks introduced by weighting with the treatment assigned to the preceding block.

3 We present details of the jackknife variance estimators in Section C.1 of the supplemental materials.

The coverage of 95% confidence intervals around the three estimators can be found in Table 1. We immediately see that the difference-in-means estimator without burn-in periods has zero coverage for $\tau_{\text{GATE}}$. This anomaly arises because of the slow mixing of the Markov chain under study, leading to substantial bias; see Table S1 in the supplementary material. In contrast, the two estimators employing a burn-in period exhibit significantly improved performance and provide reasonable coverage despite the presence of a substantial carryover effect. Moreover, these estimators also demonstrate robustness to variations in the length of the burn-in period. The bias-corrected estimator shows slight under-coverage; however, as seen in Table S1 this is not due to bias, and instead this appears to be due to a finite-sample right-skew of the block-jackknife variance estimate here (see Figure S2). In Section C.2 of the supplemental materials we also consider an analogous-but-easier setting with faster mixing, and verify that all estimators do well in that setting.

### 6.2 A ride-sharing simulator

We also evaluated our estimators using a large-scale ride-sharing simulator adapted from Farias et al. [2022]. The simulator generates drivers and riders based on data from the NYC taxi trip records dataset [Commission, N.D.]. In this simulator, drivers enter the system continuously, each with a fixed capacity of 3 riders. Their initial positions are randomly

| Design | $\hat{\tau}_{\text{DM}}^{(l,0)} (\tau_{\text{GATE}})$ | $\hat{\tau}_{\text{DM}}^{(l,b)} (\tau_{\text{FATE}})$ | $\hat{\tau}_{\text{BC}}^{(l,b)} (\tau_{\text{GATE}})$ |
|--------|----------------|----------------|----------------|
| $T = 20000, b = 50, l = 200$ | 0% | 94.4% | 93.3% |
| $T = 20000, b = 100, l = 200$ | 0% | 95.3% | 92.6% |
| $T = 20000, b = 150, l = 200$ | 0% | 93.1% | 92.5% |

Table 1: Coverage of 95% confidence intervals across 1,000 iterations under various designs.

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3We experimented with excluding different numbers of blocks in the block-jackknife estimator, but did not find this to meaningfully affect performance here.
selected from the trip records dataset, and the duration of their shifts follows an exponential distribution. Once a driver completes their shift, they go offline.

At any given time, a rider may initiate a ride request, with pick-up and drop-off locations randomly drawn from the trip records dataset and their value-for-time parameter (which informs offer acceptance) drawn from a lognormal distribution. When a request is initiated, the system dispatches a driver according to the current dispatching policy. Upon dispatch, an offer is made to the rider based on the expected cost of the dispatched driver’s service. If the dispatched driver is a pool driver, an additional discount of 50% is applied to the offer. The rider then compares the offer with their outside option before deciding whether to accept. If accepted, the dispatched driver’s route is updated accordingly.

As in Farias et al. [2022], we experiment on dispatching policies. Specifically, we examine a class of policies that determine whether to dispatch a pool driver based on the cost comparison with an idle driver. In these policies, a pool driver is dispatched only if the cost of the candidate pool driver is less than $\theta_d$ times the cost of the candidate idle driver, where $\theta_d$ is a threshold parameter determining the dispatching decision. We investigate two policies corresponding to different values of $\theta_d$: $\theta_d = 0.5$ (treatment arm, $W_t = 1$), indicating a more stringent dispatching criterion, and $\theta_d = 1$ (control arm, $W_t = 0$), representing a less stringent criterion.

Upon completion of the ride, both the rider and the ride-sharing company incur costs and payments, respectively. The outcome of interest we study is the aggregated profit, which is defined as the total difference between the price charged to riders and the cost incurred by the ride-sharing company for completed requests. We implement a Bernoulli switchback with a switch-length of 4,000 seconds. We refer the readers to Section C.3 of the supplemental material for more detailed information about the ride-sharing simulator.

The ride-sharing system can be naturally modeled as an MDP, with a large latent state space $S_t$ encompassing drivers’ positions and routes, potential rider’s locations, traffic conditions, riders’ willingness-to-pay, outside options, and more. The outcome $Y_t$, which is the aggregated profit, is a function of both the current treatment $W_t$ and the (partially) unobserved state variable $S_t$. The dispatch policy can have a long-term effect by altering the state variable: If a change in the dispatching policy results in a different driver being dispatched to a request, it will impact this driver’s route, subsequently affecting their future position and the requests they will be dispatched to. This carryover effect may persist indefinitely and may also influence nearby drivers.

We illustrate these carryovers in Figure 4. To this end, we run three coupled simulators, one under the never-treat policy, one under an always-treat policy, and one with a switchback experiment. Each of the coupled simulators has the same random realizations of driver arrivals and rider requests, but then uses different dispatch policies. The figure displays aggregated profit over time from these 3 coupled simulators across a single switch from control to treatment. We see that the red solid curve representing the observed-in-switchback outcomes starts by approximating the never-treat trajectory before the switch; then, after the switch, it slowly diverges from it and eventually starts to approximate the always-treat trajectory. The coupling doesn’t become perfect, though, since the effect of past dispatch decisions on the current system state never fully vanish on any reasonable timescale.

For the switchback experiment, we set the block length to 4,000 seconds and conduct the experiments for a total duration of 400,000 seconds (i.e., $k = 100$). We evaluate three estimators: the difference-in-means estimator $\hat{\tau}_{DM}^{(l,0)}$, the difference-in-means estimator with burn-in periods $\hat{\tau}_{DM}^{(l,b)}$, and the bias-corrected estimator $\hat{\tau}_{BC}^{(l,b)}$. For the latter two estimators,
Figure 4: A sample trajectory of aggregated profit from the ride-sharing simulator under a fixed environment. The red solid curve represents the profit observed in a switchback experiment, while the green and blue curves represent the profit observed in counterfactual scenarios where the treatment assignment is fixed to always treated and always in control, respectively. The green-shaded region indicates the treatment block in the switchback experiment, while the blue-shaded region indicates the control block in the switchback experiment.

| Estimator (Estimand) | Bias | Standard Error | Mean Squared Error | Coverage |
|----------------------|------|----------------|--------------------|----------|
| $\hat{\tau}_{DM}^{(40,0)} (\tau_{GATE})$ | 2.64 | 2.60 | 13.65 | 78% |
| $\hat{\tau}_{DM}^{(40,8)} (\tau_{FATE})$ | 0.56 | 2.69 | 7.46 | 92% |
| $\hat{\tau}_{DM}^{(30,16)} (\tau_{FATE})$ | 0.09 | 2.88 | 8.24 | 90% |
| $\hat{\tau}_{BC}^{(40,8)} (\tau_{GATE})$ | 0.40 | 2.75 | 7.66 | 92% |
| $\hat{\tau}_{BC}^{(30,16)} (\tau_{GATE})$ | 0.09 | 3.09 | 9.47 | 92% |

Table 2: Bias, standard error, mean squared error, and coverage of 95% confidence intervals across 100 iterations. For readability, the superscripts for all estimators and estimands are given in 100s of seconds.

we explore two different lengths of burn-in periods: $b = 800$ seconds and $b = 1,600$ seconds respectively. Additionally, we compute confidence intervals based on the jackknife and block jackknife variance estimators as described in Section 6.1.

Table 2 presents the bias, variance, mean squared error, and coverage achieved by those estimators. As expected, we notice that the use of burn-in periods considerably reduces the bias of the treatment effect estimation. Although the variance of the estimators with burn-in periods increases, the decrease in bias still results in a large decrease in the overall mean squared error. Furthermore, including burn-in periods leads to more accurate inferential results, with confidence intervals having coverage close to the nominal level. Again, we observe that the performance of the estimators with burn-in periods remains relatively stable regardless of the chosen length of the burn-in period.
7 Discussion

We studied switchback experiments under a generic, time-heterogeneous Markovian model for carryover effects. We found that, under this model, regular switchback estimators as they are often implemented in practice—that is, without any burn-in periods—suffer from a severe bias problem that limits the best attainable rate of convergence. We also showed that there exist a number of practical solutions to this bias problem. If researchers are willing to change their statistical target to a filtered average treatment effect, then the bias problem can be side-stepped by using burn-in periods with a regular switchback estimator. Meanwhile, researchers who want to target the global average treatment effect can achieve good performance by using a modified, bias-corrected estimator to process the data generated from a Bernoulli switchback design. Whether targeting the global (as opposed to filtered) average treatment effect is worth the extra complexity of using the bias-corrected estimator (as opposed the regular estimator with burn-ins) is likely to vary on a case-by-case basis.

More broadly, our results suggest promise in using dynamic stochastic modeling techniques to understand and improve popular tools for causal inference. The switchback is a simple and intuitive experimental design that can be used without explicitly writing down a stochastic model. However, we found that modeling the underlying system as a Markov decision process enabled us to get new insights on how and why switchback experiments enable accurate causal inference—and these insights can then be put into practice without requiring researchers to make large changes to their analytic approach (and, in particular, without requiring researchers to fit a Markov decision process). It is likely that we could also improve our understanding of other (at face value model-free) causal estimators via a stochastic modeling approach.

Funding

This research was supported by NSF grant SES-2242876.

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Supplemental Materials

A Proof of Theorems

A.1 Proof of Theorem 1

We break the proof of Theorem 1 into the proof of two lemmas on bias and variance, respectively. First, we upper bound the bias of \( E \hat{\tau}^{(l,b)}_{\text{DM}} \) in estimating \( \tau^{(l,b)}_{\text{FATE}} \) and \( \tau^{(l,b)}_{\text{GATE}} \). Recall that \( k = k_1 + k_0 = \lfloor T/l \rfloor \). For simplicity, we write

\[
\hat{\tau}^{(l,b)}_{\text{DM}} = \frac{1}{k} \sum_{s=1}^{k} \frac{1}{l-b} \sum_{t=b+1}^{l} \hat{\tau}_{(t-1)l+t}, \tag{S1}
\]

where

\[
\hat{\tau}_s = \frac{Y_s W_s}{k_1/k} - \frac{Y_s (1 - W_s)}{k_0/k}. \tag{S2}
\]

**Lemma 7.** Under the assumptions in Theorem 1, the bias of \( \hat{\tau}^{(l,b)}_{\text{DM}} \) as an estimator of \( \tau^{(l,b)}_{\text{FATE}} \) can be upper bounded as

\[
\left| E \left[ \hat{\tau}^{(l,b)}_{\text{DM}} \right] - \tau^{(l,b)}_{\text{FATE}} \right| \leq \frac{4\Lambda}{1 - \exp(-1/t_{\text{mix}})} \cdot \frac{\exp(-b/t_{\text{mix}})}{l-b} + O(2^{-k}).
\]

Furthermore,

\[
\left| \tau^{(l,b)}_{\text{FATE}} - \tau^{(l,b)}_{\text{GATE}} \right| \leq \Psi b/l.
\]

**Lemma 8.** Under the assumptions in Theorem 1, the variance of \( \hat{\tau}^{(l,b)}_{\text{DM}} \) can be upper bounded as

\[
\text{Var} \left[ \hat{\tau}^{(l,b)}_{\text{DM}} \right] \leq \frac{12A^2}{k} + \frac{16A^2 \exp(-b/t_{\text{mix}})}{(1 - \exp(-1/t_{\text{mix}}))^2} \cdot \frac{1}{k(l-b)^2} + \frac{4\sigma^2}{k(l-b)} + O \left( \frac{1}{k^2(l-b)} \right) + O(2^{-k}).
\]

Combining Lemma 7 and Lemma 8 gives the result in Theorem 1.

A.2 Proof of Theorem 2

Using notation from (14) and (15), we start by showing that our estimator can be decomposed as follows.

**Lemma 9.** Under the assumptions in Theorem 2, the difference-in-means estimator \( \hat{\tau}^{(l,b)}_{\text{DM}} \) can be decomposed as

\[
\hat{\tau}^{(l,b)}_{\text{DM}} = \tau^{(l,b)}_{\text{FATE}} + \Gamma^{(l,b)} + \Delta^{(l,b)} + O_P \left( \frac{e^{-b/t_{\text{mix}}}}{l-b} \right), \tag{S3}
\]
where
\[
\Gamma^{(l,b)} = \frac{1}{k} \sum_{i=1}^{k} \left[ \left( \bar{\mu}_i(1) - \frac{1}{k} \sum_{i=1}^{k} \bar{\mu}_i(1) \right) Z_i - \left( \bar{\mu}_i(0) - \frac{1}{k} \sum_{i=1}^{k} \bar{\mu}_i(0) \right) \left( 1 - Z_i \right) \right], \quad (S4)
\]
\[
\Delta^{(l,b)} = \frac{1}{k} \sum_{i=1}^{k} \left[ \left\{ \bar{Y}_i(1) - \bar{M}_i(1) \right\} Z_i - \left\{ \bar{Y}_i(0) - \bar{M}_i(0) \right\} \left( 1 - Z_i \right) \right]. \quad (S5)
\]

It remains to verify that \( \Gamma^{(l,b)} \) and \( \Delta^{(l,b)} \) are asymptotically jointly normal and independent. To do this, we note that \( \Gamma^{(l,b)} \) is statistically equivalent to the error term of a difference-in-means estimator in a randomized trial under the Neyman [1923] model. Applying Lindeberg-Feller central limit theorem [Lindeberg, 1922] and a multivariate delta method [van der Vaart, 2000] gives the following result.

**Lemma 10.** Under the assumptions in Theorem 2, as \( k \to \infty \),
\[
\sqrt{k} \Gamma^{(l,b)} \to_d N(0, \Sigma_{\Gamma}), \quad (S6)
\]
where \( \Sigma_{\Gamma} = V_0 + V_1 + 2V_{01} \).

Meanwhile, below, we will use the Rosenblatt central limit theorem for strong mixing sequences [Rosenblatt, 1956, Davis et al., 2011] to verify that a central limit theorem holds for \( \Delta^{(l,b)} \) conditionally on all sequences of possible treatment assignment vectors \( z_k \).

**Lemma 11.** Under the assumptions in Theorem 2, for all sequences \( z_k \) such that \( z_k \in \{0, 1\}^k \) and \( \liminf_{k \to \infty} \sum_{i=1}^{k} z_{i,k}/k > 0 \) as \( k \to \infty \),
\[
\sqrt{k} \frac{\Delta^{(l,b)}}{\sqrt{\Sigma_{\Delta}(z_k)}} \mid Z_{1,k} = z_{1,k}, \ldots, Z_{k,k} = z_{k,k} \to_d N(0, 1), \quad (S7)
\]
where
\[
\Sigma_{\Delta}(z_k) = \mathbb{E} \left[ \left\{ \left( \bar{Y}_i(1) - \bar{M}_i(1) \right) \frac{Z_{i,k}}{k_{1,k}/k} - \left( \bar{Y}_i(0) - \bar{M}_i(0) \right) \frac{(1 - Z_{i,k})}{k_{0,k}/k} \right\}^2 \right] + O \left( \exp \left( -\frac{b}{t_{mix}} \right) \right). \quad (S8)
\]

Since \( k_{1,k}/k \to a.s. 0.5 \) as \( k \to \infty \), by continuous mapping theorem, \( \Sigma_{\Delta}(Z_k) \to_p \Sigma_\Delta \). As a result, \( \Delta^{(l,b)} \) is asymptotically normal and mean-zero conditionally on \( \Gamma^{(l,b)} \). Combining Lemmas 10 and 11 and applying dominated convergence theorem yields the desired result.

### A.3 Proof of Theorem 5

As in Section A.1, we decompose the proof into the proof of two lemmas on bias and variance. To ease the notation, assume that there is one additional block with treatment assignment \( Z_0 \sim \text{Bernoulli}(0.5) \) such that \( W_t = Z_0 \) for \( t = -l, \ldots, 0 \), and consider the following estimator:
\[
\hat{\tau}^{(l,b)}_{BC} = \frac{1}{kl} \sum_{i=1}^{k} \sum_{t=1}^{l} \hat{\tau}^{(i-1)(l+t)}, \quad (S9)
\]
where we define with a slight abuse of notation
\[
\tilde{\tau}_s = \begin{cases} 
\frac{Y_{s,W_{s,k_1}} - Y_{s,(1-W_{s})k_0/k}}{k_1/k}, & \text{if } s \pmod{l} > b, \\
\frac{Y_{s,(1-W_{s})k_0/k} - Y_{s,(1-W_{s})(1-W_{s-1})}}{k_0/k}, & \text{if } s \pmod{l} \leq b.
\end{cases}
\]  
(S10)

For blocks \( i = 1, \ldots, k \), define in addition that
\[
\bar{Y}_i^b = \frac{1}{b} \sum_{t=1}^{b} Y_{i,t} \mid do(Z_i = Z_{i-1} = w) \quad \text{(S11)}
\]
and
\[
\bar{M}_i^b(w) = \mathbb{E} \left[ \bar{Y}_i^b(w) \right]. \quad \text{(S12)}
\]

**Lemma 12.** Under the assumptions in Theorem 5, the bias of \( \hat{\tau}_{BC}^{(l,b)} \) as an estimator of \( \tau_{GATE} \) can be upper bounded as
\[
\left| \mathbb{E} \left[ \hat{\tau}_{BC}^{(l,b)} \right] - \tau_{GATE} \right| \leq \frac{4\Lambda}{1 - \exp(-1/t_{mix})} \frac{\exp(-b/t_{mix})}{l} + \mathcal{O}(2^{-k}). \quad \text{(S13)}
\]
Furthermore,
\[
\left| \mathbb{E} \left[ \hat{\tau}_{BC}^{(l,b)} \right] - \mathbb{E} \left[ \hat{\tau}_{BC}^{(l,b)} \right] \right| \leq \frac{2\Lambda}{k}. \quad \text{(S14)}
\]

**Lemma 13.** Under the assumptions in Theorem 5, the variance of \( \hat{\tau}_{BC}^{(l,b)} \) as an estimator of \( \tau_{GATE} \) can be upper bounded as
\[
\text{Var} \left[ \hat{\tau}_{BC}^{(l,b)} \right] \leq \frac{28\Lambda^2}{k} + \frac{16\Lambda^2 \exp(-b/t_{mix})}{(1 - \exp(-1/t_{mix}))^2} \frac{1}{k l^2} + \frac{8\sigma^2}{k l} + \mathcal{O} \left( \frac{1}{k^2 l} \right) + \mathcal{O}(2^{-k}). \quad \text{(S15)}
\]

Combining Lemmas 12 and 13 gives the result in Theorem 5.

**A.4 Proof of Theorem 6**

Note that
\[
\tau_{GATE} = \frac{1}{k} \sum_{i=1}^{k} \left[ \frac{l-b}{l} \{ \bar{p}_{i}(1) - \bar{p}_{i}(0) \} + \frac{b}{l} \{ \bar{p}_{i}^{b}(1) - \bar{p}_{i}^{b}(0) \} \right]. \quad \text{(S16)}
\]

We first decompose \( \hat{\tau}_{BC}^{(l,b)} \) as follows.

**Lemma 14.** The bias-corrected difference-in-means estimator \( \hat{\tau}_{BC}^{(l,b)} \) with burn-in periods can be decomposed as
\[
\hat{\tau}_{BC}^{(l,b)} = \tau_{GATE} + \hat{\Gamma}^{(l,b)} + \hat{\Delta}^{(l,b)} + \mathcal{O}_p \left( \frac{1}{k} \right) + \mathcal{O}_p \left( \frac{e^{-b/t_{mix}}}{l} \right), \quad \text{(S17)}
\]
\[
\tilde{\Gamma}^{(l,b)} = \frac{l - b}{l} \Gamma^{(l,b)} + \frac{b}{kl} \sum_{i=1}^{k} \left[ \left( \frac{\mu^b(l) - \frac{1}{k} \sum_{i=1}^{k} \mu^b(1)}{k_{11}^b} \right) Z_i Z_{i-1} - \left( \frac{\mu^b(0) - \frac{1}{k} \sum_{i=1}^{k} \mu^b(0)}{k_{00}^b} \right) (1 - Z_i)(1 - Z_{i-1}) \right] \quad (S18)
\]

\[
\tilde{\Delta}^{(l,b)} = \frac{l - b}{l} \Delta^{(l,b)} + \frac{b}{kl} \sum_{i=1}^{k} \left[ \left( \frac{Y_i^b(1) - M_i^b(1)}{k_{11}^b} \right) Z_i Z_{i-1} - \left( \frac{Y_i^b(0) - M_i^b(0)}{k_{00}^b} \right) (1 - Z_i)(1 - Z_{i-1}) \right] \quad (S19)
\]

Again, as in Section A.2, we prove a central limit theorem for \( \tilde{\Gamma}^{(l,b)} \).

**Lemma 15.** Under the assumptions in Theorem 6, as \( k \to \infty \),

\[
\sqrt{k} \tilde{\Gamma}^{(l,b)} \to_d \mathcal{N} \left( 0, \tilde{\Sigma}_\Gamma \right),
\]

(S20)

where

\[
\tilde{\Sigma}_\Gamma = (1 - \beta)^2 \left( V_0^f + V_{11}^f + 2V_{01}^f \right) + \beta^2 (3V_0^b + 3V_1^b + 2V_{01}^b + 2V_{01}^{b0} + 2V_{10}^{b0} + 2V_{00}^{b0} + 2V_{11}^{b0}) + 2\beta(1 - \beta) \left( V_0^{bf} + V_1^{bf} + V_{01}^{bf} + V_{10}^{bf} + V_{00}^{bf} + V_{11}^{bf} + V_{01}^{fs} + V_{10}^{fs} + V_{00}^{fs} \right). \quad (S21)
\]

Finally, we show that \( \tilde{\Delta}^{(l,b)} \) is a smaller-order term that is negligible.

**Lemma 16.** Under the assumptions in Theorem 6, as \( k \to \infty \),

\[
\sqrt{k} \tilde{\Delta}^{(l,b)} \to_p 0.
\]

(S22)
B Proof of Lemmas

Proof of Lemma 7. For \( i = 1, \ldots, k \),

\[
\mathbb{E} \left[ \frac{1}{l-b} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \right] \\
= \mathbb{E} \left[ \frac{1}{l-b} \sum_{t=b+1}^{l} \left( \frac{Y_{(i-1)t+t}Z_i}{k_1/k} - \frac{Y_{(i-1)t+t}(1-Z_i)}{k_0/k} \right) \right] \\
= \frac{1}{l-b} \sum_{t=b+1}^{l} \mathbb{E} \left[ \frac{\mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 1, k_1 \right] \mathbb{P} \left[ Z_i = 1 \mid k_1 \right]}{k_1/k} - \frac{\mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 0, k_1 \right] \mathbb{P} \left[ Z_i = 0 \mid k_1 \right]}{k_0/k} \right] \\
= \frac{1}{l-b} \sum_{t=b+1}^{l} \left( \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 1 \right] - \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 0 \right] \right) + \mathcal{O} \left( 2^{-k} \right),
\]

where the lower order term comes from the special case where \( k_1 \) equals zero or \( k \). By the assumption on mixing time, for \( t = b + 1, \ldots, l \),

\[
\begin{align*}
& \left| \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 1 \right] - \mathbb{E} \mathcal{L}_1 \left[ Y_{(i-1)t+t} \right] \right| \\
& = \mathbb{E} \left[ \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 1, S_{(i-1)t+t} \right] - \mathbb{E} \mathcal{L}_1 \left[ Y_{(i-1)t+t} \mid Z_i = 1, S_{(i-1)t+t} \right] \right] \\
& = \int_{s} \left\{ \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 1, S_{(i-1)t+t} = s \right] \mathbb{P} \left[ S_{(i-1)t+t} = s \mid Z_i = 1 \right] - \mathbb{E} \mathcal{L}_1 \left[ S_{(i-1)t+t} = s \mid Z_i = 1 \right] \right\} ds \\
& \leq \Lambda \int_{s} \left\{ \mathbb{P} \left[ S_{(i-1)t+t} = s \mid Z_i = 1 \right] - \mathbb{P} \mathcal{L}_1 \left[ S_{(i-1)t+t} = s \mid Z_i = 1 \right] \right\} ds \\
& \leq 2\Lambda \exp \left( -t/t_{\text{mix}} \right),
\end{align*}
\]

where the last inequality follows from the mixing assumption and the fact that the two distributions have transitioned under the same sequence of transition operators \( \{ P_{(i-1)t+t}^{1}, P_{(i-1)t+t}^{1}, \ldots, P_{(i-1)t+t}^{1} \} \). Similarly,

\[
\left| \mathbb{E} \left[ Y_{(i-1)t+t} \mid Z_i = 0 \right] - \mathbb{E} \mathcal{L}_0 \left[ Y_{(i-1)t+t} \right] \right| \leq 2\Lambda \exp \left( -t/t_{\text{mix}} \right). \tag{S25}
\]

Thus,

\[
\left| \mathbb{E} \left[ \frac{1}{l-b} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \right] - \frac{1}{l-b} \sum_{t=b+1}^{l} \tau_{(i-1)t+t} \right| \leq \frac{4\Lambda}{l-b} \sum_{t=b+1}^{l} \exp \left( -t/t_{\text{mix}} \right). \tag{S26}
\]
Therefore,
\[
\left| \mathbb{E} \left[ \hat{\tau}_{\text{DM}}^{(l,b)} \right] - \tau_{\text{FATE}}^{(l,b)} \right| \leq \frac{1}{k} \sum_{i=1}^{k} \left| \mathbb{E} \left[ \frac{1}{l - b} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \right] - \frac{1}{l - b} \sum_{t=b+1}^{l} \tau_{(i-1)t+t} \right| 
\leq \frac{4\Lambda}{l - b} \sum_{t=b+1}^{l} \exp \left( - \frac{t}{t_{\text{mix}}} \right) \tag{S27}
\leq \frac{4\Lambda}{1 - \exp \left( - \frac{1}{t_{\text{mix}}} \right)} \frac{\exp \left( - \frac{b}{t_{\text{mix}}} \right)}{l - b}.
\]

Meanwhile, since the maximum difference in treatment effects is bounded by a constant \( \Psi \),
\[
\left| \tau_{\text{FATE}}^{(l,b)} - \tau_{\text{GATE}} \right| = \frac{1}{k} \sum_{i=1}^{k} \left| \frac{1}{l - b} \sum_{t=b+1}^{l} \tau_{(i-1)t+t} - \frac{1}{l} \sum_{t=b+1}^{l} \tau_{(i-1)t+t} \right| 
\leq \frac{1}{kl} \sum_{i=1}^{k} \sum_{t=1}^{b} \tau_{(i-1)t+t} - \sum_{t=b+1}^{l} \frac{b}{l - b} \tau_{(i-1)t+t} \tag{S28}
\leq \Psi \cdot \frac{b}{l}.
\]

\[\square\]

**Proof of Lemma 8.** To start with, we decompose the variance into two parts as
\[
\text{Var} \left[ \hat{\tau}_{\text{DM}}^{(l,b)} \right] = \frac{1}{k^2(l - b)^2} \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \right] 
= \frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right] + \frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\tau}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right]. \tag{S29}
\]

To bound the first term of (S29), note that after conditioning on \( \{W_{1:T}, S_{1:T}\} \), \( \hat{\tau} \) are inde-
pendent of each other, and thus

\[
\frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \tilde{\tau}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right] \\
= \frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \text{Var} \left[ \tilde{\tau}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right] \\
\leq \frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \sigma^2 \left( \frac{kZ_i}{k_1} - \frac{k(1 - Z_i)}{k_0} \right)^2 \right] \\
= \frac{\sigma^2}{k(l - b)} \mathbb{E} \left[ \frac{k^2Z_i^2}{k_1^2} + \frac{k^2(1 - Z_i)^2}{k_0^2} - \frac{2k^2Z_i(1 - Z_i)}{k_1k_0} \right] \\
= \frac{\sigma^2}{k(l - b)} \mathbb{E} \left[ \frac{k^2}{\max(k_1, 1)} \frac{k}{\max(k_0, 1)} + \frac{k}{\max(k_0, 1)} \right] + \mathcal{O}(2^{-k}). \quad (S30)
\]

By a Taylor expansion of \( k/\max(k_1, 1) \) at \( \max(k_1, 1) = 0.5k \),

\[
\mathbb{E} \left[ \frac{k}{\max(k_1, 1)} \right] = \frac{k}{0.5k} + \frac{0.5(1 - 0.5)k^2}{(0.5)^3k^3} + \mathcal{O} \left( \frac{1}{k} \right) + \mathcal{O}(2^{-k}) = 2 + \frac{2}{k} + \mathcal{O} \left( \frac{1}{k} \right). \quad (S31)
\]

Similarly, \( \mathbb{E} [k/\max(k_0, 1)] = 2 + 2/k + \mathcal{O}(1/k) \). Thus,

\[
\frac{1}{k^2(l - b)^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \tilde{\tau}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right] \\
\leq \frac{\sigma^2}{k(l - b)} \mathbb{E} \left[ \frac{k}{\max(k_1, 1)} + \frac{k}{\max(k_0, 1)} \right] + \mathcal{O}(2^{-k}) \\
\leq \frac{4\sigma^2}{k(l - b)} + \mathcal{O} \left( \frac{1}{k^2(l - b)} \right) + \mathcal{O}(2^{-k}). \quad (S32)
\]

To bound the second term of (S29), we start by calculating an upper bound on

\[
\text{Cov} \left[ \mathbb{E} [\tilde{\tau}_{i} \mid W_{1:T}, S_{1:T}], \mathbb{E} [\tilde{\tau}_{t+m} \mid W_{1:T}, S_{1:T}] \right] \quad (S33)
\]

for all \( t \geq 1, m \geq 0 \). There are two cases we need to consider:

1. When \( t \) and \( t + m \) are from two different blocks \( i \) and \( j \). With a slight abuse of notation, we use \( \mu_i := \mathbb{E} [Y_i \mid W_t, S_t] \) to denote the conditional expectation of outcome at time \( t \). In this case,

\[
(S33) = \text{Cov} \left[ \mathbb{E} [\tilde{\tau}_{i} \mid W_{t}, S_{t}], \mathbb{E} [\tilde{\tau}_{t+m} \mid W_{t+m}, S_{t+m}] \right] \\
= \text{Cov} \left[ \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t, \frac{Z_j}{k_1/k} \mu_{t+m} - \frac{1 - Z_j}{k_0/k} \mu_{t+m} \right] \\
= \mathbb{E} \left[ \left( \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right) \left( \frac{Z_j}{k_1/k} \mu_{t+m} - \frac{1 - Z_j}{k_0/k} \mu_{t+m} \right) \right] - \mathcal{O}(2^{-k}) \\
\leq \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right] \mathbb{E} \left[ \frac{Z_j}{k_1/k} \mu_{t+m} - \frac{1 - Z_j}{k_0/k} \mu_{t+m} \right], \quad (S34)
\]
where
\[
\mathbb{E} \left[ \frac{Z_i}{k_1} \mu_t - \frac{1}{k_0} Z_i \right] \left( \frac{Z_j}{k_1} \mu_{t+m} - \frac{1}{k_0} Z_j \mu_{t+m} \right) = \mathbb{E} \left[ \frac{k^2}{k_1} Z_i Z_j \mu_{t+m} \right] + \mathbb{E} \left[ \frac{k^2}{k_0} (1 - Z_i)(1 - Z_j) \mu_{t+m} \right] - \mathbb{E} \left[ \frac{k^2}{k_1 k_0} Z_i (1 - Z_j) \mu_{t+m} \right] - \mathbb{E} \left[ \frac{k^2}{k_1 k_0} (1 - Z_i) Z_j \mu_{t+m} \right].
\] (S35)

Note that
\[
\mathbb{E} \left[ \frac{k^2}{k_1} Z_i Z_j \mu_{t+m} \right] = \mathbb{E} \left[ \frac{k^2}{k_1} P \left[ Z_i = 1, Z_j = 1 \mid k_1 \right] \mathbb{E} [\mu_t \mu_{t+m} \mid Z_i = 1, Z_j = 1, k_1] \right] = \mathbb{E} \left[ \frac{k(k_1 - 1)}{(k - 1)k_1} \mathbb{E} [\mu_t \mu_{t+m} \mid Z_i = 1, Z_j = 1, k_1] \right] + \mathcal{O}(2^{-k})
\leq \mathbb{E} [\mu_t \mu_{t+m} \mid Z_i = 1, Z_j = 1, k_1] + \frac{\Lambda^2}{k - 1} + \mathcal{O}(2^{-k}),
\] (S37)

and
\[
\mathbb{E} \left[ \frac{k}{k_1} Z_i \mu_t \right] = \mathbb{E} \left[ \frac{k}{k_1} \mathbb{E} [Z_i = 1 \mid k_1] \mathbb{E} [\mu_t \mid Z_i = 1, k_1] \right] = \mathbb{E} [\mu_t \mid Z_i = 1, k_1] \mathbb{E} [\mu_{t+m} \mid Z_i = 1, k_1] + \mathcal{O}(2^{-k}).
\] (S38)

Again according to the mixing assumption,
\[
\mathbb{E} [\mu_{t+m} \mid S_t, Z_j = 1, k_1] - \mathbb{E} [\mu_{t+m} \mid Z_j = 1, k_1] \leq 2\Lambda \exp \left( -m/t_{mix} \right),
\] (S39)

for that the two distributions have transited under the same sequence of transition operators \{\(P_t^w, P_{t+1}^w, \ldots, P_{(j-1)}^w, P_{(j-1)+1}^1, \ldots, P_{(j+m-1)}^1\)\}. Then
\[
\mathbb{E} [\mu_t \mu_{t+m} \mid Z_i = 1, Z_j = 1, k_1] = \mathbb{E} [\mu_t \mathbb{E} [\mu_{t+m} \mid S_t, Z_j = 1, k_1] \mid Z_i = 1, Z_j = 1, k_1] - \mathbb{E} [\mu_t \mid Z_i = 1, k_1] \mathbb{E} [\mu_{t+m} \mid Z_j = 1, k_1] \mathbb{E} [\mu_t \mid Z_i = 1, k_1] \mathbb{E} [\mu_{t+m} \mid Z_j = 1, k_1]
\leq \mathbb{E} [\mu_t \mid Z_i = 1, k_1] 2\Lambda \exp \left( -m/t_{mix} \right)
\leq 2\Lambda^2 \exp \left( -m/t_{mix} \right).
\]
Using the mixing assumption as above, we can obtain the same bound on the differences between the other three pairs of terms. Therefore,

\[(S33) \leq 8\Lambda^2 \exp \left( -\frac{m}{t_{\text{mix}}} \right) + \frac{4\Lambda^2}{k-1} + \mathcal{O} \left( 2^{-k} \right). \]  

(S41)

2. When \( t \) and \( t + m \) are from the same block \( i \). Similar to the first case, we can write out

\[(S33) = \text{Cov} \left[ \mathbb{E} [\hat{\tau}_t | W_t, S_t] \mathbb{E} [\hat{\tau}_{t+m} | W_{t+m}, S_{t+m}] \right]

= \mathbb{E} \left[ \left( \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right) \left( \frac{Z_i}{k_1/k} \mu_{t+m} - \frac{1 - Z_i}{k_0/k} \mu_{t+m} \right) \right]

- \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} - \frac{1 - Z_i}{k_0/k} \mu_{t+m} \right], \]  

(S42)

where

\[\mathbb{E} \left[ \left( \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right) \left( \frac{Z_i}{k_1/k} \mu_{t+m} - \frac{1 - Z_i}{k_0/k} \mu_{t+m} \right) \right]

= \mathbb{E} \left[ \frac{k^2}{k_1} Z_i \mu_t \mu_{t+m} \right] + \mathbb{E} \left[ \frac{k^2}{k_0} (1 - Z_i) \mu_t \mu_{t+m} \right]

\leq \Lambda^2 \mathbb{E} \left[ \frac{k}{k_1} \right] + \Lambda^2 \mathbb{E} \left[ \frac{k}{k_0} \right] + \mathcal{O} \left( 2^{-k} \right)

= 4\Lambda^2 + \mathcal{O} \left( 2^{-k} \right)\]

and

\[\mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t - \frac{1 - Z_i}{k_0/k} \mu_t \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} - \frac{1 - Z_i}{k_0/k} \mu_{t+m} \right]

= \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t | Z_i = 1, k_1 \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} | Z_i = 1, k_1 \right] +

\mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t | Z_i = 0, k_1 \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} | Z_i = 0, k_1 \right] -

\mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t | Z_i = 1, k_1 \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} | Z_i = 0, k_1 \right] -

\mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_t | Z_i = 0, k_1 \right] \mathbb{E} \left[ \frac{Z_i}{k_1/k} \mu_{t+m} | Z_i = 1, k_1 \right] + \mathcal{O} \left( 2^{-k} \right)

\leq 4\Lambda^2 + \mathcal{O} \left( 2^{-k} \right).\]

Thus,

\[(S33) \leq 8\Lambda^2 + \mathcal{O} \left( 2^{-k} \right). \]  

(S43)

Now we assemble the bounds on \((S33)\) to obtain an upper bound on the second term of
(S29). By (S41) and (S43),

\[
\frac{1}{k^2(l-b)^2} \text{Var} \left[ \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\tau}(i-1)t+ | W_{1:T}, S_{1:T} \right] \right]
\]

\[
= \frac{1}{k^2(l-b)^2} \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \mathbb{E} \left[ \hat{\tau}(i-1)t+ | W_{1:T}, S_{1:T} \right] \right]
\]

\[
\leq \frac{1}{k^2(l-b)^2} \cdot \left\{ 8k(l-b)^2 \Lambda^2 + \frac{16k^2 \Lambda^2 \exp(-b/t_{\text{mix}})}{(1-\exp(-1/t_{\text{mix}}))^2} + \frac{4\Lambda^2}{k-1} k(k-1)(l-b)^2 \right\} + \mathcal{O}(2^{-k})
\]

\[
= \frac{12\Lambda^2}{k} + \frac{16\Lambda^2 \exp(-b/t_{\text{mix}})}{1-\exp(-1/t_{\text{mix}})^2} \cdot \frac{1}{k(l-b)^2} + \mathcal{O}(2^{-k}).
\]

Thus,

\[
\text{Var} \left[ \hat{\tau}_{\text{DM}}(l,b) \right] \leq \frac{12\Lambda^2}{k} + \frac{16\Lambda^2 \exp(-b/t_{\text{mix}})}{1-\exp(-1/t_{\text{mix}})^2} \cdot \frac{1}{k(l-b)^2} + \frac{4\sigma^2}{k(l-b)} + \mathcal{O} \left( \frac{1}{k^2(l-b)} \right) + \mathcal{O}(2^{-k}).
\]

Proof of Lemma 9. We start by decomposing our estimator as follows:

\[
\hat{\tau}_{\text{DM}}(l,b) = \frac{1}{k} \sum_{i=1}^{k} \left[ \frac{\bar{Y}_i(1)Z_i}{k_1/k} - \frac{\bar{Y}_i(0)(1-Z_i)}{1-k_1/k} \right]
\]

\[
= \frac{1}{k} \sum_{i=1}^{k} \left[ \frac{\bar{\mathcal{M}}_i(1)Z_i}{k_1/k} - \frac{\bar{\mathcal{M}}_i(0)(1-Z_i)}{1-k_1/k} \right] + \Delta_{(l,b)}.
\]

(S46)

Now, by Assumption 2 on mixing time,

\[
|\mathcal{M}_i(w) - \bar{\mathcal{M}}_i(w)| = |\mathbb{E} \left[ Y_i | Z_i = w \right] - \mathbb{E} \left[ Y_i | Z_i = w, Z_{i-1} = w, \cdots \right]|
\]

\[
\leq \frac{1}{l-b} \sum_{t=(i-1)(l+b)}^{il} \left| \mathbb{E} \left[ Y_i | Z_i = w \right] - \mathbb{E} \left[ Y_i | Z_i = w, Z_{i-1} = w, \cdots \right] \right| \]

\[
= \mathcal{O}_P \left( \frac{1}{l-b} \exp(-b/t_{\text{mix}}) \right)
\]

uniformly across \( i = 1, \ldots, k \) and \( w = 0, 1 \). Thus, continuing from (S46) and recalling (16), we see that

\[
\hat{\tau}_{\text{DM}}(l,b) = \frac{1}{k} \sum_{i=1}^{k} \left[ \frac{\bar{\mathcal{M}}_i(1)Z_i}{k_1/k} - \frac{\bar{\mathcal{M}}_i(0)(1-Z_i)}{1-k_1/k} \right] + \Delta_{(l,b)} + \mathcal{O}_P \left( \frac{e^{-b/t_{\text{mix}}}}{l-b} \right),
\]

\[
= \tau_{\text{FATE}} + \Gamma_{(l,b)} + \Delta_{(l,b)} + \mathcal{O}_P \left( \frac{e^{-b/t_{\text{mix}}}}{l-b} \right).
\]

(S48)
Proof of Lemma 10. Define \( X_i = (Z_i, Z_i \bar{m}_i(1), (1 - Z_i) \bar{m}_i(0))^\top \). Note that
\[
E \left[ \sum_{i=1}^{k} X_i \right] = \left( \frac{1}{2} \sum_{i=1}^{k} \bar{m}_i(1), \frac{1}{2} \sum_{i=1}^{k} \bar{m}_i(0) \right)^T,
\]
(S49)
By Lindeberg-Feller central limit theorem [Lindeberg, 1922],
\[
\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \left( X_{i,k} - E \left[ \sum_{i=1}^{k} X_{i,k} \right] \right) \to N(0, I),
\]
(S50)
where
\[
\Sigma_{X,k} = \left( \frac{1}{4k} \sum_{i=1}^{k} \bar{m}_i(1) - \frac{1}{4k} \sum_{i=1}^{k} \bar{m}_i(0), -\frac{1}{4k} \sum_{i=1}^{k} \bar{m}_i(1) \bar{m}_i(0) \right).
\]
(S51)
Applying a multivariate delta method [van der Vaart, 2000] yields that
\[
\sqrt{\frac{k}{\Sigma_{\Gamma,k}} \Gamma^{(l,b)}} \to_d N(0, 1),
\]
(S52)
where
\[
\Sigma_{\Gamma,k} = \frac{1}{k^2} \left\{ \sum_i (\bar{m}_i(1) + \bar{m}_i(0)) \right\}^2 + \frac{1}{k} \sum_i \bar{m}_i(1)^2 + \frac{1}{k} \sum_i \bar{m}_i(0)^2 + \frac{2}{k} \sum_i \bar{m}_i(1) \bar{m}_i(0)
\]
\[
\quad - \frac{2}{k^2} \left\{ \sum_i (\bar{m}_i(1) + \bar{m}_i(0)) \right\} \left\{ \sum_i \bar{m}_i(1) \right\} - \frac{2}{k^2} \left\{ \sum_i (\bar{m}_i(1) + \bar{m}_i(0)) \right\} \left\{ \sum_i \bar{m}_i(0) \right\}
\]
\[
\to V_0 + V_1 + 2V_{01}.
\]
□

Proof of Lemma 11. Let \( k_{1,k} = \sum_{i=1}^{k} z_{i,k}, k_{0,k} = k - \sum_{i=1}^{k} z_{i,k} \). Note that, conditionally on \( Z_{1,k} = z_{1,k}, \ldots, Z_{k,k} = z_{k,k}, \)
\[
\Delta^{(l,b)} = \frac{1}{k} \sum_{i=1}^{k} \left[ (\bar{Y}_{i}(1) - \bar{M}_{i}(1)) \frac{z_{i,k}}{k_{1,k}/k} - (\bar{Y}_{i}(0) - \bar{M}_{i}(0)) \frac{(1 - z_{i,k})}{k_{0,k}/k} \right].
\]
(S53)
For \( i = 1, \ldots, k, \) define
\[
D_i = (\bar{Y}_{i}(1) - \bar{M}_{i}(1)) \frac{z_{i,k}}{k_{1,k}/k} - (\bar{Y}_{i}(0) - \bar{M}_{i}(0)) \frac{(1 - z_{i,k})}{k_{0,k}/k},
\]
(S54)
and \( \Delta^{(l,b)} \) can be written as
\[
\Delta^{(l,b)} = \frac{1}{k} \sum_{i=1}^{k} D_i.
\]
To start with, we show that the mean-zero sequence \( \{D_i\}_{i=1,...,k} \) satisfies the strong mixing condition, in the sense that

\[
\sup_{1 \leq i \leq k} \sup_{A \in A_i, B \in B_{i+h}} |P[A \cap B] - P[A]P[B]| \leq d(h),
\]

where \( A_i = \sigma(S_{n_i \leq t}, \epsilon_{n \leq t}) \), \( B_i = \sigma(S_{n_i \geq t}, \epsilon_{n \geq t}) \), and \( d(h) \to 0 \) as \( h \to \infty \). Again by the assumption on mixing time,

\[
\sup_{A \in A_i, B \in B_{i+h}} |P[A \cap B] - P[A]P[B]| \leq \sup_{s \in S, B \in B_{T_{t+h}}} |P[B|S_t = s] - P[B]| \leq O(\exp(-h/t_{mix})).
\]

Thus, \( \{D_i\}_{i=1,...,k} \) satisfies the strong mixing condition.

Secondly, by the assumptions that \( \text{Var} [\epsilon_t | J_t] \geq \sigma_0^2 \) and \( l - b = O(1) \),

\[
\text{Var} \left[ k \Delta^{(l,b)} \right] = \text{Var} \left[ \sum_{i=1}^{k} D_i \right] \geq \frac{k}{l - b} \sigma_0^2 = \Omega(k).
\]

It remains to show that the fourth moment of \( k \Delta^{(l,b)} \) is of order \( O(k^2) \). Note that

\[
k^4\mathbb{E} \left[ (\Delta^{(l,b)})^4 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^{k} D_i \right)^4 \right] = \sum_i \mathbb{E} [D_i^4] + \sum_{i \neq j} \mathbb{E} [D_i^2 D_j^2] + \sum_{i \neq j \neq m} \mathbb{E} [D_i^2 D_j D_m] + \sum_{i \neq j \neq m \neq n} \mathbb{E} [D_i D_j D_m D_n].
\]

To calculate the fourth order terms above, we start by noticing that \( \{Y_t(w) - \bar{M}_t(w)\}_{t}^x = O(1), x \leq 4, w \in \{0,1\} \), due to the assumption that \( |Y_t| \leq \Gamma_0 \). Furthermore, for all \( i < j, w \in \{0,1\} \), similar to the calculation in proof of Lemma 8,

\[
|\mathbb{E} [Y_j(w) | S_{it}] - \mathbb{E} [\bar{M}_j(w) | S_{it}]| = |\mathbb{E} [Y_j(w) | S_{it}] - \mathbb{E} [\bar{Y}_j(w)]|
= O\left( \exp \left( \frac{- (j-i)l + (l-b)}{t_{mix}} \right) \right).
\]

Without loss of generality, we consider the case where \( i < j < m < n \).\(^4\) From the two properties we obtained above,

\[
\mathbb{E} [D_i^4] = O(1),
\]

\[
\mathbb{E} [D_i^2 D_j] = O\left( \exp \left( \frac{- (j-i)l + (l-b)}{t_{mix}} \right) \right),
\]

\(\text{In other cases, it is easy to show that those moments can also be bounded with similar terms of same orders.}\)

\[12\]
\[ E[D_i^2 D_j^2] = E[D_i^2 E[D_j^2 | S_{il}]] = O\left(\exp\left(-\frac{(j-i)t + (l-b)}{t_{\text{mix}}}\right)\right), \]

\[ E[D_i^2 D_j D_m] = E[D_i^2 E[D_j E[D_m | S_{jl}] | S_{il}]] = O\left(\exp\left(-\frac{(m-j)t + (l-b)}{t_{\text{mix}}}\right)\right) \cdot O\left(\exp\left(-\frac{(j-i)t + (l-b)}{t_{\text{mix}}}\right)\right), \]

\[ E[D_i D_j D_m D_n] = E[D_i E[D_j E[D_m E[D_n | S_{ml}] | S_{jl}] | S_{il}]] = O\left(\exp\left(-\frac{(n-m)t + (l-b)}{t_{\text{mix}}}\right)\right) \cdot O\left(\exp\left(-\frac{(m-j)t + (l-b)}{t_{\text{mix}}}\right)\right) \cdot O\left(\exp\left(-\frac{(j-i)t + (l-b)}{t_{\text{mix}}}\right)\right). \]

Putting everything together, we have (S58) \( = O\left(k^2\right) \) for that \( l-b = O\left(1\right) \). It then follows directly from Rosenblatt [1956] and Davis et al. [2011] that, as \( k \to \infty \),

\[ \sqrt{k} \Delta^{(l,b)} / \sqrt{\Sigma_{\Delta}(z_k)} | Z_{1,k} = z_{1,k}, \ldots, Z_{k,k} = z_{k,k} \to_d N(0, 1), \]

where

\[ \Sigma_{\Delta}(z_k) = \frac{1}{k} E \left[ \left( \sum_{i=1}^{k} D_i \right)^2 \right] = E[D_i^2] + \frac{1}{k} \sum_{i=1}^{k} \sum_{j=i+1}^{k} D_i D_j \]

\[ = E[D_i^2] + O\left(\exp\left(-\frac{b}{t_{\text{mix}}}\right)\right). \] (S59)

\[ \sqrt{k} \Delta^{(l,b)} / \sqrt{\Sigma_{\Delta}(z_k)} \]

**Proof of Lemma 12.** Since \( |E[Y_t|S_t, W_t]| = |\mu_t(S_t, W_t)| \leq_{a.s.} \Lambda \),

\[ \left| E\left[\hat{\tau}^{(l,b)}_{\text{BC}}\right] - E\left[\tau^{(l,b)}_{\text{BC}}\right]\right| \leq \frac{1}{kl} \sum_{i=1}^{b} \left| E[\hat{\tau}_i] - E[\tau_i]\right| \leq \frac{2\Lambda}{k} \] (S60)

Note that

\[ E\left[\sum_{i=1}^{k} \frac{1}{k_{11}} \sum_{l=1}^{b} Z_i Z_{i-1} Y_{i-1}|l+t| t\right] \]

\[ = \frac{1}{kl} \sum_{i=1}^{k} \sum_{l=1}^{b} E\left[E\left[Y_{i-1}|l+t| \right] Z_i = Z_{i-1} = 1, k_{11}\right] P[Z_i = Z_{i-1} = 1] \] (S61)

\[ = \frac{1}{kl} \sum_{i=1}^{k} \sum_{l=1}^{b} E\left[Y_{i-1}|l+t| \right] Z_i = Z_{i-1} = 1 + O\left(2^{-k}\right) \]
and by the assumption on mixing time, for \( t = b + 1, \ldots, l, \)
\[
|E \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 1 \right] - E_{L_i} \left[ Y_{(i-1)t+t} \right] | \\
= |E \left[ E \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 1, S_{(i-1)t+t} \right] - E_{L_{i-1}} \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 1, S_{(i-1)t+t} \right] \right] | \\
= \int \left\{ E \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 1, S_{(i-1)t+t} = s \right] P \left[ S_{(i-1)t+t} = s \mid Z_i = Z_{i-1} = 1 \right] - \\
E \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 1, S_t = s \right] P_{L_{i-1}} \left[ S_{(i-1)t+t} = s \mid Z_i = Z_{i-1} = 1 \right] \right\} ds \\
\leq \Lambda \int \left\{ P \left[ S_{(i-1)t+t} = s \mid Z_i = Z_{i-1} = 1 \right] - P_{L_{i-1}} \left[ S_{(i-1)t+t} = s \mid Z_i = Z_{i-1} = 1 \right] \right\} ds \\
\leq 2\Lambda \exp \left( -(l + t)/t_{\text{mix}} \right).
\]
Similarly,
\[
|E \left[ Y_{(i-1)t+t} \mid Z_i = Z_{i-1} = 0 \right] - E_{L_0} \left[ Y_{(i-1)t+t} \right] | \leq 2\Lambda \exp \left( -(l + t)/t_{\text{mix}} \right).
\]
Since
\[
\tau_{GATE} = \frac{l - b}{l} \tau_{FATE} + \frac{1}{b} \sum_{t=1}^{b} \tau_{(i-1)t+t},
\]
combining the results above with Lemma 7 yields
\[
\begin{align*}
& \left| E \left[ \hat{\hat{x}}_{BC}^{(l,b)} \right] \right| - \tau_{GATE} \\
= \left| \frac{l - b}{l} \left( E \left[ \hat{\hat{x}}_{DM}^{(l,b)} \right] - \tau_{FATE} \right) + E \left[ \sum_{i=1}^{k} Z_i Z_{i-1} \sum_{t=1}^{b} \frac{Y_{(i-1)t+t}}{l} \right] - \\
E \left[ \sum_{i=1}^{k} \frac{(1 - Z_i)(1 - Z_{i-1})}{k_00} \sum_{t=1}^{b} \frac{Y_{(i-1)t+t}}{l} \right] - \frac{1}{b} \sum_{t=1}^{b} \tau_{(i-1)t+t} \right| \\
\leq 4\Lambda \left( \frac{1}{l} \sum_{t=b+1}^{l} \exp \left( -t/t_{\text{mix}} \right) + \frac{1}{l} \sum_{t=1}^{b} \exp \left( -(l + t)/t_{\text{mix}} \right) \right) + O \left( 2^{-k} \right) \\
\leq \frac{4\Lambda}{1 - \exp \left( -1/t_{\text{mix}} \right)} \cdot \frac{\exp \left( -b/t_{\text{mix}} \right)}{l} + O \left( 2^{-k} \right).
\end{align*}
\]
\( \square \)

**Proof of Lemma 13.** Similar to (S29),
\[
\text{Var} \left[ \hat{\hat{x}}_{BC}^{(l,b)} \right] \\
= \frac{1}{k^2l^2} E \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\hat{x}}_{(i-1)t+t} + \sum_{i=1}^{k} \sum_{t=1}^{b} \hat{\hat{x}}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right] + \\
\frac{1}{k^2l^2} \text{Var} \left[ E \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \hat{\hat{x}}_{(i-1)t+t} + \sum_{i=1}^{k} \sum_{t=1}^{b} \hat{\hat{x}}_{(i-1)t+t} \mid W_{1:T}, S_{1:T} \right] \right].
\]
To bound the first term of (S64), notice that

$$\frac{1}{k^2l^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \tilde{\tau}_{(i-1)+t} + \sum_{i=2}^{k} \sum_{t=1}^{b} \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right]\right]$$

$$= \frac{1}{k^2l^2} \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \text{Var} \left[ \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right] + \sum_{i=2}^{k} \sum_{t=1}^{b} \text{Var} \left[ \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right] \right].$$

From (S32),

$$\frac{1}{k^2l^2} \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \text{Var} \left[ \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right] \right] \leq \frac{4\sigma^2(l-b)}{kl^2} + \mathcal{O}\left(\frac{1}{k^2l}\right) + \mathcal{O}\left(2^{-k}\right).$$  \hfill (S65)

Moreover,

$$\frac{1}{k^2l^2} \mathbb{E} \left[ \sum_{i=2}^{k} \sum_{t=1}^{b} \text{Var} \left[ \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right] \right]$$

$$\leq \frac{1}{k^2l^2} \mathbb{E} \left[ \sum_{i=1}^{k} \sum_{t=1}^{b} \sigma^2 \left( \frac{kZ_{i-1}}{k_{11}} - \frac{k(1-Z_{i})(1-Z_{i-1})}{k_{00}} \right)^2 \right]$$

$$= \frac{\sigma^2}{kl^2} \mathbb{E} \left[ \frac{k^2Z_{i-1} | k_{11}}{k_{11}^2} + \frac{k^2E \[(1-Z_{i})(1-Z_{i-1}) | k_{00}] \right]$$

$$= \frac{\sigma^2}{kl^2} \mathbb{E} \left[ \frac{k}{\max(k_{11}, 1)} + \frac{k}{\max(k_{00}, 1)} \right] + \mathcal{O}\left(2^{-k}\right).$$

Then by Taylor expansion of $k/\max(k_{11}, 1)$ at max $(k_{11}, 1) = 0.25k$ and of $k/\max(k_{00}, 1)$ at max $(k_{00}, 1) = 0.25k$,

$$\frac{1}{k^2l^2} \mathbb{E} \left[ \sum_{i=2}^{k} \sum_{t=1}^{b} \text{Var} \left[ \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right] \right]$$

$$\leq 8\frac{\sigma^2b}{kl^2} + \mathcal{O}\left(\frac{1}{k^2l}\right) + \mathcal{O}\left(2^{-k}\right),$$

and thus

$$\frac{1}{k^2l^2} \mathbb{E} \left[ \text{Var} \left[ \sum_{i=1}^{k} \sum_{t=b+1}^{l} \tilde{\tau}_{(i-1)+t} + \sum_{i=2}^{k} \sum_{t=1}^{b} \tilde{\tau}_{(i-1)+t} \mid W_{1:T}, S_{1:T} \right]\right]$$

$$\leq 8\frac{\sigma^2}{kl^2} + \mathcal{O}\left(\frac{1}{k^2l}\right) + \mathcal{O}\left(2^{-k}\right). \hfill (S66)$$

To bound the second term of (S64), we need to bound

$$\text{Cov} \left[ \mathbb{E} \left[ \tilde{\tau}_{l+T} | W_{1:T}, S_{1:T} \right], \mathbb{E} \left[ \tilde{\tau}_{l+m} | W_{1:T}, S_{1:T} \right] \right], \hfill (S67)$$
Thus, by (S68) and (S69), now we assemble bounds on (S67) to obtain an upper bound on the second term of (S64). Specifically, for $m \geq 0$, consider the following cases:

1. When $t$ and $t + m$ are from two different blocks $i$ and $j$ where $i \leq j - 1$, $W_{1:t}$ and $W_{t+m-1:T}$ are independent of each other. The derivation follows exactly like the one outlined in the first case of proof of Lemma 8. It is straightforward to obtain that

\[
(S67) \leq 8\Lambda^2 \exp\left(-m/t_{\text{mix}}\right) + \frac{4\Lambda^2}{k-1} + O\left(2^{-k}\right). 
\]  

(S68)

2. When $t$ and $t + m$ are from two different blocks $i$ and $j$, and $j = i + 1$,

(a) if $t + m \pmod{l} > b$, i.e., if $t + m$ is in one of the focal periods, (S68) still holds since $W_{1:t}$ and $W_{t+m:t}$ are independent;

(b) if $t + m \pmod{l} \leq b$, i.e., if $t + m$ is in one of the burn-in periods, $W_{1:t}$ and $W_{t+m-1:T}$ are correlated, which is similar to case 2 of proof of Lemma 8. It can be verified that

\[
(S67) \leq 12\Lambda^2 + O\left(2^{-k}\right). 
\]  

(S69)

3. When $t$ and $t + m$ are from the same block $i$ and $j$, we still bound (S67) with (S69).

Now we assemble bounds on (S67) to obtain an upper bound on the second term of (S64). By (S68) and (S69),

\[
\frac{1}{k^2l^2} \text{Var}\left[\mathbb{E} \left[\sum_{t=1}^{k} \sum_{t=t+1}^{l} \hat{\tau}_{(i-1)t+t} \big| W_{1:T}, S_{1:T}\right]\right]
\]

\[
= \frac{1}{k^2l^2} \text{Var} \left[\mathbb{E} \left[\sum_{i=1}^{k} \sum_{t=1}^{l} \hat{\tau}_{(i-1)t+t} \big| W_{1:T}, S_{1:T}\right]\right]
\]

\[
\leq \frac{1}{k^2l^2} \cdot \left\{ \frac{24kl^2\Lambda^2 + 16k\Lambda^2 \exp\left(-b/t_{\text{mix}}\right)}{\left(1 - \exp\left(-1/t_{\text{mix}}\right)\right)^2} + \frac{4\Lambda^2}{k-1} \cdot k(k-2) \right\} + O\left(2^{-k}\right) 
\]

\[
\leq \frac{28\Lambda^2}{k} + \frac{16\Lambda^2 \exp\left(-b/t_{\text{mix}}\right)}{\left(1 - \exp\left(-1/t_{\text{mix}}\right)\right)^2} \cdot \frac{1}{k^2l^2} + O\left(2^{-k}\right). 
\]

(S70)

Thus,

\[
\text{Var}\left[\hat{\tau}_{(i, b)}^{\text{DM}}\right] \leq \frac{28\Lambda^2}{k} + \frac{16\Lambda^2 \exp\left(-b/t_{\text{mix}}\right)}{\left(1 - \exp\left(-1/t_{\text{mix}}\right)\right)^2} \cdot \frac{1}{k^2l^2} + \frac{8\sigma^2}{kl} + O\left(\frac{1}{k^2l}\right) + O\left(2^{-k}\right).
\]

(S71)
Proof of Lemma 14. We start by decomposing our estimator as follows:

\[ \hat{\zeta}^{(l,b)}_{BC} = \hat{\zeta}^{(l,b)}_{BC} + \mathcal{O}_P \left( \frac{1}{k} \right) \]

\[ = \frac{1}{k} \sum_{i=1}^{b} \left[ \frac{l-b}{l} \cdot \frac{\hat{M}_i(1)Z_i}{k_1/k} - \frac{l-b}{l} \cdot \frac{\hat{M}_i(0)(1-Z_i)}{1-k_1/k} + \frac{b}{l} \cdot \frac{\hat{M}_i(1)Z_iZ_{i-1}}{k_{11}/k} - \frac{b}{l} \cdot \frac{\hat{M}_i(0)(1-Z_i)(1-Z_{i-1})}{k_{00}/k} \right] + \hat{\Delta}^{(l,b)} + \mathcal{O}_P \left( \frac{1}{k} \right), \]

(S72)

Now, by Assumption 2 on mixing time,

\[ \frac{b}{l} \left| \hat{M}_i(w) - \hat{\mu}_i^b(w) \right| = \frac{b}{l} \left| \mathbb{E} \left[ \hat{Y}_i^b \mid Z_i = Z_{i-1} = w \right] - \mathbb{E} \left[ \hat{Y}_i^b \mid Z_i = w, Z_{i-1} = w, \ldots \right] \right| \]

\[ \leq \frac{1}{l} \left( \sum_{t=(i-1)t+1}^{(i-1)t+b} \left| \mathbb{E} \left[ Y_t \mid Z_i = Z_{i-1} = w \right] - \mathbb{E} \left[ Y_t \mid Z_i = w, Z_{i-1} = w, \ldots \right] \right| \right) \]

\[ = \mathcal{O}_P \left( \frac{1}{l} \exp \left( -l/t_{\text{mix}} \right) \right), \]

(S73)

uniformly across \( i = 1, \ldots, k \) and \( w = 0, 1 \). Recalling (S16) and (S47), we see that

\[ \hat{\zeta}^{(l,b)}_{BC} = \hat{\Delta}^{(l,b)} + \mathcal{O}_P \left( \frac{1}{k} \right) + \frac{l-b}{kl} \sum_{i=1}^{k} \left[ \frac{\hat{\mu}_i(1)Z_i}{k_1/k} - \frac{\hat{\mu}_i(0)(1-Z_i)}{1-k_1/k} \right] + \mathcal{O}_P \left( \frac{e^{-b/t_{\text{mix}}}}{l} \right) \]

\[ + \frac{b}{kl} \sum_{i=1}^{k} \left[ \frac{\hat{\mu}_i^b(1)Z_iZ_{i-1}}{k_{11}/k} - \frac{\hat{\mu}_i^b(0)(1-Z_i)(1-Z_{i-1})}{k_{00}/k} \right] + \mathcal{O}_P \left( \frac{e^{-l/t_{\text{mix}}}}{l} \right) \]

\[ = \tau_{\text{GATE}} + \hat{\tau}^{(l,b)} + \hat{\Delta}^{(l,b)} + \mathcal{O}_P \left( \frac{1}{k} \right) + \mathcal{O}_P \left( \frac{e^{-b/t_{\text{mix}}}}{l} \right). \]

(S74)

Proof of Lemma 15. Define

\[ \hat{X}_i = (Z_i, Z_iZ_{i-1}, (1-Z_i)(1-Z_{i-1}), Z_i\hat{\mu}_i(1), (1-Z_i)\hat{\mu}_i(0), Z_iZ_{i-1}\hat{\mu}_i^b(1), (1-Z_i)(1-Z_{i-1})\hat{\mu}_i^b(0))^T \]

(S75)

and \( \hat{\Sigma}_{X,k} = \text{Cov} \left[ \frac{1}{\sqrt{k}} \sum X_i \right] \). Note that

\[ \mathbb{E} \left[ \sum_{i=1}^{k} X_i \right] = \left( \frac{k}{2} \frac{k}{4} \frac{k}{4} \frac{k}{4} \sum_{i=1}^{k} \hat{\mu}_i(1), \frac{1}{2} \sum_{i=1}^{k} \hat{\mu}_i(0), \frac{1}{4} \sum_{i=1}^{k} \hat{\mu}_i^b(1), \frac{1}{4} \sum_{i=1}^{k} \hat{\mu}_i^b(0) \right)^T, \]

(S76)
and $\text{Cov} \left[ \sum_{i=1}^{k} X_i \right]$ can be calculated by noticing that

$$\text{Var} \left[ \sum_{i=1}^{k} Z_i \right] = \frac{k}{4}$$

$$\text{Var} \left[ \sum_{i=1}^{k} Z_i Z_{i-1} \right] = \frac{5k}{16} + O(1)$$

$$\text{Var} \left[ \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \right] = \frac{5k}{16} + O(1)$$

$$\text{Var} \left[ \sum_{i=1}^{k} Z_i \mu_i(1) \right] = \frac{1}{4} \sum_{i=1}^{k} \mu_i(1)^2$$

$$\text{Var} \left[ \sum_{i=1}^{k} (1 - Z_i) \mu_i(0) \right] = \frac{1}{4} \sum_{i=1}^{k} \mu_i(0)^2$$

$$\text{Var} \left[ \sum_{i=1}^{k} Z_i Z_{i-1} \mu_i^2(1) \right] = \frac{3}{16} \sum_{i=1}^{k} \mu_i^2(1)^2 + \frac{1}{8} \sum_{i=1}^{k-1} \mu_i^2(1) \mu_{i+1}^2(1)$$

$$\text{Var} \left[ \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \mu_i^2(0) \right] = \frac{3}{16} \sum_{i=1}^{k} \mu_i^2(0)^2 + \frac{1}{8} \sum_{i=1}^{k-1} \mu_i^2(0) \mu_{i+1}^2(0),$$

and that

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} Z_i Z_{i-1} \right] = \frac{k}{4} + O(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \right] = -\frac{k}{4} + O(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} Z_i \mu_i(1) \right] = \frac{1}{4} \sum_{i=1}^{k} \mu_i(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} (1 - Z_i) \mu_i(0) \right] = -\frac{1}{4} \sum_{i=1}^{k} \mu_i(0)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} Z_i Z_{i-1} \mu_i^2(1) \right] = \frac{1}{4} \sum_{i=1}^{k} \mu_i^2(1) + O(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i, \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \mu_i^2(0) \right] = -\frac{1}{4} \sum_{i=1}^{k} \mu_i^2(0) + O(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \right] = -\frac{3k}{16} + O(1)$$

$$\text{Cov} \left[ \sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} Z_i \mu_i(1) \right] = \frac{1}{4} \sum_{i=1}^{k} \mu_i(1) + O(1)$$

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\[
\begin{align*}
\text{Cov} \left[ \sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} (1 - Z_i) \overline{\mu}_i(0) \right] &= -\frac{1}{4} \sum_{i=1}^{k} \overline{\mu}_i(0) + \mathcal{O}(1) \\
\text{Cov} \left[ \sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} Z_i Z_{i-1} \overline{\mu}_i(1) \right] &= \frac{5}{16} \sum_{i=1}^{k} \overline{\mu}_i^b(1) + \mathcal{O}(1) \\
\sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \overline{\mu}_i(0) &= -\frac{3}{16} \sum_{i=1}^{k} \overline{\mu}_i(0) + \mathcal{O}(1) \\
\sum_{i=1}^{k} Z_i Z_{i-1}, \sum_{i=1}^{k} Z_i \overline{\mu}_i(1) &= -\frac{1}{4} \sum_{i=1}^{k} \overline{\mu}_i(1) + \mathcal{O}(1) \\
\sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}), \sum_{i=1}^{k} Z_i \overline{\mu}_i(0) &= \frac{1}{4} \sum_{i=1}^{k} \overline{\mu}_i(0) + \mathcal{O}(1) \\
\sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}), \sum_{i=1}^{k} Z_i Z_{i-1} \overline{\mu}_i(1) &= -\frac{3}{16} \sum_{i=1}^{k} \overline{\mu}_i^b(1) + \mathcal{O}(1) \\
\sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}), \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \overline{\mu}_i(0) &= \frac{5}{16} \sum_{i=1}^{k} \overline{\mu}_i^b(0) + \mathcal{O}(1) \\
\sum_{i=1}^{k} Z_i \overline{\mu}_i(1), \sum_{i=1}^{k} (1 - Z_i) \overline{\mu}_i(0) &= -\frac{1}{4} \sum_{i=1}^{k} \overline{\mu}_i(0) \overline{\mu}_i(1) \\
\sum_{i=1}^{k} Z_i \overline{\mu}_i(1), \sum_{i=1}^{k} Z_i Z_{i-1} \overline{\mu}_i(1) &= \frac{1}{8} \sum_{i=1}^{k} \overline{\mu}_i(1) \overline{\mu}_i^b(1) + \frac{1}{8} \sum_{i=1}^{k-1} \overline{\mu}_i(1) \overline{\mu}_i^{b+1}(1) \\
\sum_{i=1}^{k} Z_i \overline{\mu}_i(1), \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \overline{\mu}_i(0) &= -\frac{1}{8} \sum_{i=1}^{k} \overline{\mu}_i(1) \overline{\mu}_i^b(0) - \frac{1}{8} \sum_{i=1}^{k-1} \overline{\mu}_i(1) \overline{\mu}_i^{b+1}(0) \\
\sum_{i=1}^{k} (1 - Z_i) \overline{\mu}_i(0), \sum_{i=1}^{k} Z_i Z_{i-1} \overline{\mu}_i(1) &= -\frac{1}{8} \sum_{i=1}^{k} \overline{\mu}_i(0) \overline{\mu}_i^b(1) - \frac{1}{8} \sum_{i=1}^{k-1} \overline{\mu}_i(0) \overline{\mu}_i^{b+1}(1) \\
\sum_{i=1}^{k} (1 - Z_i) \overline{\mu}_i(0), \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \overline{\mu}_i(0) &= \frac{1}{8} \sum_{i=1}^{k} \overline{\mu}_i(0) \overline{\mu}_i^b(0) + \frac{1}{8} \sum_{i=1}^{k-1} \overline{\mu}_i(0) \overline{\mu}_i^{b+1}(0) \\
\sum_{i=1}^{k} Z_i Z_{i-1} \overline{\mu}_i(1), \sum_{i=1}^{k} (1 - Z_i)(1 - Z_{i-1}) \overline{\mu}_i(0) &= -\frac{1}{16} \sum_{i=1}^{k} \overline{\mu}_i^b(1) \overline{\mu}_i^b(0) - \frac{1}{16} \sum_{i=2}^{k} \overline{\mu}_i^b(1) \overline{\mu}_i^{b-1}(0) - \frac{1}{16} \sum_{i=1}^{k-1} \overline{\mu}_i^b(1) \overline{\mu}_i^{b+1}(0).
\end{align*}
\]
Notice that
\[
\hat{\Gamma}^{(l,b)} = \frac{t+b}{T} \frac{1}{k} \sum_i Z_i \hat{\mu}_i(1) - \frac{l-b}{l} \frac{1}{k} \sum_j \hat{\nu}_j(1) \\
- \frac{t+b}{T} \frac{1}{k} \sum_i (1-Z_i) \hat{\mu}_i(0) + \frac{l-b}{l} \frac{1}{k} \sum_j \hat{\nu}_j(0) \\
+ \frac{b}{k} \frac{1}{k} \sum_i Z_i Z_{i-1} \hat{\mu}_i^2(1) - \frac{b}{k} \sum_j \hat{\nu}_j^2(1) \\
- \frac{b}{k} \frac{1}{k} \sum_i (1-Z_i)(1-Z_{i-1}) \hat{\mu}_i^2(0) + \frac{b}{k} \sum_j \hat{\nu}_j^2(0). \tag{S77}
\]

By standard results developed in Stein’s method framework with dependency graphs [see, e.g., Ross, 2011, Theorem 3.5] together with a multivariate delta method [van der Vaart, 2000], it follows that
\[
\sqrt{k} \hat{\Sigma}_{\Gamma,k} \rightarrow_d \mathcal{N}(0,1), \tag{S78}
\]
where
\[
\Sigma_{\Gamma,k} = (\nabla \hat{\Gamma}^{(l,b)})^\top \Sigma_{\mathcal{X},k} \nabla \hat{\Gamma}^{(l,b)} \tag{S79}
\]
with
\[
\nabla \hat{\Gamma}^{(l,b)} = \left(-2 \frac{l-b}{kl} \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)), -4 \frac{b}{kl} \sum_i \hat{\mu}_i^2(1), \right. \\
\left. 4 \frac{b}{kl} \sum_i \hat{\mu}_i^2(0), 2 \frac{l-b}{l}, -2 \frac{l-b}{l}, 4 \frac{b}{l}, -4 \frac{b}{l} \right)^\top, \tag{S80}
\]
and thus
\[
\Sigma_{\Gamma,k} = \left(\frac{l-b}{k^2 l^2} \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0))^2 \left\{ \frac{5b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(1) \right\}^2 + \frac{5b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(0) \right)^2 \\
+ \left(\frac{l-b}{k^2 l^2} \sum_i \hat{\mu}_i(1)^2 + \frac{(l-b)^2}{k^2 l^2} \sum_i \hat{\mu}_i(0)^2 \right)^2 \\
+ \frac{3b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(1)^2 + \frac{2b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(1) \hat{\mu}_i^2(1+1) + \frac{3b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(0)^2 + \frac{2b^2}{k^2 l^2} \sum_i \hat{\mu}_i^2(0) \hat{\mu}_i^2(1+1) \\
+ \frac{4(l-b)b}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i^2(1) \right\} + \frac{4(l-b)b}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i^2(0) \right\} \\
- \frac{2(l-b)^2}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i(1) \right\} - \frac{2(l-b)^2}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i(0) \right\} \\
- \frac{4(l-b)b}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i^2(1) \right\} + \frac{4(l-b)b}{k^2 l^2} \left\{ \sum_i (\hat{\mu}_i(1) + \hat{\mu}_i(0)) \right\} \left\{ \sum_i \hat{\mu}_i^2(0) \right\}.
\]
\[
\begin{align*}
&+ \frac{6b^2}{k^2 l^2} \left( \sum_i \pi_i^h(1) \right) \left( \sum_i \pi_i^h(0) \right) \\
&- \frac{4(l - b)b}{k^2 l^2} \left( \sum_i \pi_i(1) \right) \left( \sum_i \pi_i^h(1) \right) - \frac{4(l - b)b}{k^2 l^2} \left( \sum_i \pi_i(0) \right) \left( \sum_i \pi_i^h(1) \right) \\
&- \frac{10b^2}{k^2 l^2} \left( \sum_i \pi_i^h(1) \right)^2 - \frac{6b^2}{k^2 l^2} \left( \sum_i \pi_i^h(1) \right) \left( \sum_i \pi_i^h(0) \right) \\
&- \frac{4(l - b)b}{k^2 l^2} \left( \sum_i \pi_i(1) \right) \left( \sum_i \pi_i^h(0) \right) - \frac{4(l - b)b}{k^2 l^2} \left( \sum_i \pi_i(0) \right) \left( \sum_i \pi_i^h(0) \right) \\
&+ \frac{2(l - b)^2}{k l^2} \sum_i \pi_i(1) \pi_i(0) \\
&+ \frac{2(l - b)b}{k l^2} \sum_i \pi_i(1) \pi_i^h(1) + \frac{2(l - b)b}{kl} \sum_i \pi_i(1) \pi_i^h_{i+1}(1) \\
&+ \frac{2(l - b)b}{k l^2} \sum_i \pi_i(0) \pi_i^h(0) + \frac{2(l - b)b}{kl} \sum_i \pi_i(0) \pi_i^h_{i+1}(0) \\
&+ \frac{2(l - b)b}{k l^2} \sum_i \pi_i(0) \pi_i^h(1) + \frac{2(l - b)b}{kl} \sum_i \pi_i(0) \pi_i^h_{i+1}(1) \\
&+ \frac{2b^2}{k l^2} \sum_i \pi_i^h(1) \pi_i^h(0) + \frac{2b^2}{k l^2} \sum_i \pi_i^h(1) \pi_i^h_{i-1}(0) + \frac{2b^2}{k l^2} \sum_i \pi_i^h(1) \pi_i^h_{i+1}(0) \\
&\rightarrow (1 - \beta)^2 \left( V_0^f + V_1^f + 2V_0^f \right) \\
&+ \beta^2 \left( 3V_0^b + 3V_1^b + 2V_1^b + 2V_0^b + 2V_{10}^b + 2V_{00}^b + 2V_{11}^b \right) \\
&+ 2\beta(1 - \beta) \left( V_0^f + V_1^f + V_0^f + V_{10}^f + V_{00}^f + V_{11}^f + V_{01}^f + V_{00}^f + V_{10}^f \right).
\end{align*}
\]

Proof of Lemma 16. Define \( Y_t(w) = Y_t \mid do(W_t = w), M_t(w) = E[Y_t], \) and

\[
\tilde{D}_t = \begin{cases} \\
\frac{\{Y_t(1) - M_t(1)\}}{k_{11}/k} W_t - \frac{\{Y_t(0) - M_t(0)\}}{1 - k_{11}/k} (1 - W_t), & \text{if } t \pmod l > b, \\
\frac{\{Y_t^h(1) - M_t^h(1)\}}{k_{11}/k} W_{t-l} - \frac{\{Y_t^h(0) - M_t^h(0)\}}{k_{00}/k} (1 - W_t)(1 - W_{t-l}), & \text{if } t \pmod l \leq b.
\end{cases}
\]
Note that, if \( t \mod l > b \),

\[
\mathbb{E} \left[ \hat{D}_t^2 \right] = \mathbb{E} \left[ \left( \frac{\{Y_t(1) - M_t(1)\} W_t}{k_1/k} - \{Y_t(0) - M_t(0)\} (1 - W_t) \right)^2 \right]
\leq \Gamma_0 \mathbb{E} \left[ \left( \frac{W_t}{k_1/k} - \frac{1 - W_t}{1 - k_1/k} \right)^2 \right] = \mathcal{O}(1),
\]

and if \( t \mod l \leq b \),

\[
\mathbb{E} \left[ \hat{D}_t^2 \right] = \mathbb{E} \left[ \left( \frac{\{Y_t^b(1) - M_t^b(1)\} W_t W_{t-l}}{k_{11}/k} - \frac{\{Y_t^b(0) - M_t^b(0)\} (1 - W_t)(1 - W_{t-l})}{k_{00}/k} \right)^2 \right]
\leq \Gamma_0 \mathbb{E} \left[ \left( \frac{W_t W_{t-l}}{k_{11}/k} - \frac{(1 - W_t)(1 - W_{t-l})}{k_{00}/k} \right)^2 \right] = \mathcal{O}(1).
\]

Furthermore, for \( m > 0 \), by mixing assumption,

\[
\mathbb{E} \left[ \hat{D}_t \hat{D}_{t+m} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \hat{D}_t \hat{D}_{t+m} \mid W_{1:T} \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ \hat{D}_t \mathbb{E} \left[ \hat{D}_{t+m} \mid S_t, W_{1:T} \right] \mid W_{1:T} \right] \right] = \mathcal{O}(\exp(-m/t_{\text{mix}})).
\]

Putting everything together

\[
k \mathbb{E} \left[ \left( \hat{\Delta}^{(l,b)} \right)^2 \right] = k \mathbb{E} \left[ \left( \frac{1}{kl} \sum_t \hat{D}_t \right)^2 \right]
= \frac{1}{kl^2} \left( \sum_t \mathbb{E} \left[ \hat{D}_t^2 \right] + \sum_{m=1}^{T-t} \sum_t \mathbb{E} \left[ \hat{D}_t \hat{D}_{t+m} \right] \right)
= \mathcal{O} \left( \frac{1}{l} \right).
\]

Since \( l \to \infty \) as \( k \to \infty \), \( k \mathbb{E} \left[ \left( \hat{\Delta}^{(l,b)} \right)^2 \right] \to 0 \), and thus \( \sqrt{k} \hat{\Delta}^{(l,b)} \to_p 0 \) as \( k \to \infty \).

\[\square\]

C Additional Details of Numerical Experiments

C.1 Jackknife variance estimation

In the numerical examples, we calculate the confidence intervals of the three estimators of interest using a Jackknife variance estimation procedure. Here, we provide details on how the confidence intervals are constructed.
Algorithm 1 Jackknife Variance Estimation

Require: $Y = \{Y_1, Y_2, \ldots, Y_T\}$, $Z = \{Z_1, Z_2, \ldots, Z_k\}$, $\hat{\tau}^{(l)}_{\text{DM}}$, $(l, b)$. 

1: Let $k = \lfloor T/l \rfloor$ be the number of blocks.
2: for $i = 1$ to $k$ do
3:    Remove $Y_{(i-1)l+1}, \ldots, Y_{il}$ and $Z_i$ to obtain $Y_{-i}$ and $Z_{-i}$.
4:    Calculate $\hat{\tau}^{(l)}_{\text{DM},-i}$ using $Y_{-i}$ and $Z_{-i}$ according to equation (9).
5: end for
6: Compute the jackknife variance estimator: $\hat{V}^{(l,b)}_{\text{DM}} = \frac{(k-1)}{k} \sum_{i=1}^{k} (\hat{\tau}^{(l)}_{\text{DM},-i} - \hat{\tau}^{(l)}_{\text{DM}})^2$
7: return $\hat{V}^{(l,b)}_{\text{DM}}$

Figure S1: An illustration of the block Jackknife variance estimation. The burn-in periods are in gray while the focal periods are in white. The arrows above/below the display show different (overlapping) regions that will in turn be removed with the block jackknife.

For the two difference-in-means estimators $\hat{\tau}^{(l,0)}_{\text{DM}}$ and $\hat{\tau}^{(l,b)}_{\text{DM}}$, we note that this can be regarded as a special case of estimating variance with resampling methods using clustered data. The treatment assignments are perfectly correlated within blocks and independent across blocks, while the outcomes are weakly correlated with an exponentially decaying correlation. Thus, we follow the standard practice and resample at the cluster level, i.e., we iteratively exclude one block at a time. The details of the Jackknife variance estimation are described in Algorithm 1. We then construct the confidence intervals as $\hat{\tau}^{(l,b)}_{\text{DM}} \pm \frac{z_{\alpha/2}}{\hat{V}^{(l,b)}_{\text{DM}}}$, where $z_{\alpha/2}$ is the critical value from the standard normal distribution corresponding to the desired level of confidence.

For the bias-corrected estimator $\hat{\tau}^{(l,b)}_{\text{BC}}$, we utilize a block jackknife resampling procedure [Künsch, 1989], which excludes two blocks simultaneously to account for the significant correlation between blocks introduced by weighting with the treatment assigned to the preceding block. To ensure that data preceding and following the excluded blocks are (almost) independent of each other, we iteratively remove focal periods in blocks $i, i+1$ and burn-in periods in blocks $i+1, i+2$. We provide an illustration of this strategy in Figure S1, and details of the block jackknife variance estimation in Algorithm 2.

C.2 Additional results for the simple illustration

In Section 6.1, we demonstrate that the coverage of the two estimators employing burn-in periods, $\hat{\tau}^{(l,b)}_{\text{DM}}$ and $\hat{\tau}^{(l,b)}_{\text{BC}}$, closely approximates the nominal level, while the naive difference-in-means estimator, $\hat{\tau}^{(l,0)}_{\text{DM}}$, exhibits zero coverage. To elucidate the rationale behind this observation, we analyze the bias and variance of the three estimators across various designs, as outlined in Table S1. We observe a disproportionately large bias in estimating $\tau$ with $\hat{\tau}^{(l,0)}_{\text{DM}}$, relative to the variance of the estimators. This confirms our conjecture that the undercoverage of the difference-in-means estimator stems from its substantial bias.

In addition to the MDP discussed in Section 6.1, we explore a simpler MDP characterized by faster mixing dynamics. In this alternate setting, the state transition is governed by the
Algorithm 2 Block Jackknife Variance Estimation

Require: \( Y = \{Y_1, Y_2, \ldots, Y_T\}, Z = \{Z_1, Z_2, \ldots, Z_k\}, \hat{\tau}_{BC}^{(l,b)} , (l, b) \)

1: Let \( k = \lfloor T/l \rfloor \) be the number of blocks
2: for \( i = 1 \) to \( k - 2 \) do
3: Remove \( Y_{(i-1)l+b+1}, \ldots, Y_{(i+1)l+b} \) and \( Z_{i-1} \) to obtain \( Y_{-i,2} \) and \( Z_{-i,2} \)
4: Calculate \( \hat{\tau}_{BC, -i} \) using \( Y_{-i,2} \) and \( Z_{-i,2} \) according to equation (25)
5: end for
6: Compute the jackknife variance estimator:
\[
\hat{\nu}_{BC}^{(l,b)} = \frac{(k-2)^2}{d(k-2)^2} \sum_{i=1}^{k-2} (\hat{\tau}_{BC, -i}^{(l,b)} - \hat{\tau}_{BC}^{(l,b)})^2
\]
7: return \( \hat{\nu}_{BC}^{(l,b)} \)

Table S1: Bias and standard error over 1,000,000 iterations under the original setup with \( T = 20,000 \) and \( l = 200 \).

| Estimator (Estimand) | Bias       | Standard Error |
|----------------------|------------|----------------|
| \( \hat{\tau}_{DM}^{(l,b)} (\tau_{GATE}) \) | \( b = 50 \) | \( b = 100 \) | \( b = 150 \) | \( b = 50 \) | \( b = 100 \) | \( b = 150 \) |
| 1.170                | 1.170      | 1.170          |
| \( \hat{\tau}_{DM}^{(l,b)} (\tau_{FATE}) \) | 0.007      | 0.005          | 0.004          |
| 0.199                | 0.239      | 0.332          |
| \( \hat{\tau}_{BC}^{(l,b)} (\tau_{GATE}) \) | 0.0001     | 0.004          | 0.003          |
| 0.202                | 0.226      | 0.248          |

following rules:

\[
H_{t+1} = \begin{cases} 
\min \{H_t + M_t, 20\} & \text{with probability 0.7,} \\
0 & \text{with probability 0.3;}
\end{cases} \tag{S81}
\]

if \( W_t = 1 \), and

\[
H_{t+1} = \begin{cases} 
\min \{H_t + M_t, 20\} & \text{with probability 0.3,} \\
0 & \text{with probability 0.7.}
\end{cases} \tag{S82}
\]

if \( W_t = 0 \). Tables S2 and S3 present the coverage of 95% confidence intervals, as well as biases and variances of the three estimators under this easier setup. In this fast-mixing environment, the bias associated with the difference-in-means estimator is significantly reduced, leading to a much higher coverage rate for its confidence interval.

In Figure S2, we plot the distributions of the ratios between the estimated variance and true variance for the three estimators under the original and the easier setups, respectively. We notice that there is a relatively large variability in the estimated variance, especially under the original setup, potentially leading to the slight undercoverage in Table 1. As we move to the easier, rapid mixing setup, the tails become thinner, and we observe a closer-to-nominal-level coverage, as presented in Table S2. This is especially the case with the two estimators utilizing the burn-in periods, for that the outcome observed during the burn-in periods might have a large variation under the original setup.

In Figure S2, we plot the distributions of the ratios between the estimated variance and true variance for three estimators under both the original and easier setups. We notice that there is considerable variability in the estimated variance, particularly evident in the original setup, which may contribute to the slight undercoverage observed in Table 1. Transitioning
Table S2: Coverage of 95% confidence intervals over 1,000 iterations under an easier setup.

| Design | \( \hat{\tau}_{DM}^{(l,0)}(\tau_{GATE}) \) | \( \hat{\tau}_{DM}^{(l,b)}(\tau_{FATE}) \) | \( \hat{\tau}_{BC}^{(l,b)}(\tau_{GATE}) \) |
|--------|------------------------------------------|------------------------------------------|------------------------------------------|
| \( T = 20000, b = 50, l = 200 \) | 95% | 95.6% | 94.6% |
| \( T = 20000, b = 100, l = 200 \) | 95% | 95.1% | 94.1% |
| \( T = 20000, b = 150, l = 200 \) | 95% | 95.5% | 93.5% |

Table S3: Bias and standard error over 1,000 iterations under an easier setup with \( T = 20,000 \) and \( l = 200 \).

| Estimator (Estimand) | Bias | Standard Error |
|----------------------|------|----------------|
| \( \hat{\tau}_{DM}^{(l,0)}(\tau_{GATE}) \) | \( b = 50 \) | \( b = 100 \) | \( b = 150 \) | \( b = 50 \) | \( b = 100 \) | \( b = 150 \) |
| \( \hat{\tau}_{DM}^{(l,b)}(\tau_{FATE}) \) | 0.030 | 0.030 | 0.030 | 0.175 | 0.175 | 0.175 |
| \( \hat{\tau}_{BC}^{(l,b)}(\tau_{GATE}) \) | 0.001 | 0.001 | 0.013 | 0.200 | 0.245 | 0.345 |
| \( \hat{\tau}_{BC}^{(l,b)}(\tau_{GATE}) \) | 0.003 | 0.003 | 0.005 | 0.202 | 0.222 | 0.242 |

C.3 Additional details on the ride-sharing simulator

In this section, we provide further details on the ride-sharing simulator. To generate drivers and ride requests, we rely on the NYC Yellow Taxi Trip Records data from the year 2015 [Commission, N.D.]. Our simulation design is adapted from Farias et al. [2022].

Ride Request Generation. Request times are randomly generated, with interarrival times drawn from an exponential distribution with a rate of \( 0.1\gamma_{\text{taxi}} \), where \( \gamma_{\text{taxi}} \) is the average rate at which requests are observed in the dataset. The pickup and dropoff location for each request are randomly drawn with replacement from the full trip records dataset. We further generate, for each rider, a value-of-time parameter as a random draw from a lognormal distribution with a mean of 0.003 and a variance of 1. Each request is for a single rider.

Driver Generation. Drivers are created by drawing pickup locations with replacement from the same requests dataset as used above, with the interarrival time following an exponential distribution with a rate of \( 0.003\gamma_{\text{taxi}} \). The driver then remains in the system until the end of their shift, with a fixed capacity of 3 riders. The duration of the shift follows an exponential distribution with a mean of 30000 seconds.

Dispatch. When a ride request is initiated, the dispatcher selects a driver according to the following procedure:

1. Fetch the top 40 nearest drivers and sort them based on their estimated time of arrival.
2. Categorize the drivers into two groups: idle drivers and pool drivers. Consider the top 10 pool drivers and the top 1 idle driver as candidates for dispatching.

3. Calculate the cost of adding the new ride request to the route of each candidate driver.

4. If the cost of a candidate pool driver is less than $\theta_d$ times the cost of the idle driver, dispatch the pool driver with the lowest cost. Otherwise, dispatch the idle driver.

Whenever a driver is dispatched to a ride request, their current route is updated by inserting the pickup and dropoff locations based on Dijkstra’s algorithm that finds the shortest path.

**Offer.** After dispatching a driver, we extend an offer to the rider based on the anticipated cost of providing the ride. This cost is calculated as the sum of multiplying the total distance (in kilometers) by 0.6 and the total time (in seconds) by 0.01, covering driver earnings, maintenance, fuel, and all other expenses. The resulting cost is then multiplied by 1.5 to determine the price offered to the rider. If the dispatched driver is a pool driver, the rider receives an additional 50% discount on the offer.

**Response.** Riders have access to an outside option allowing them to travel directly from the pickup to the dropoff location after a 15-minute wait. The price of this outside option is calculated as the sum of multiplying the total distance (in kilometers) by 0.6 and the total time (in seconds, including the waiting time) by 0.01. To determine whether to accept a ride offer, riders compare it with their outside option. This comparison is based on their disutilities associated with each option, calculated as the estimated time to their destination multiplied by their value-of-time parameter, plus the price of the option (before any discount).

**Completion.** If the offer is accepted, the dispatched driver’s route is updated accordingly. Upon completion of the ride, both the rider and the ride-sharing company incur payments.
and costs, respectively. The net platform profit is calculated as the difference between the payment made by the customer and the cost of providing the ride.