A simpler approach to obtaining an $O(1/t)$ convergence rate for the projected stochastic subgradient method

Simon Lacoste-Julien, Mark Schmidt, and Francis Bach
INRIA - Sierra project-team
Département d’Informatique de l’Ecole Normale Supérieure
Paris, France

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Abstract

In this note, we present a new averaging technique for the projected stochastic subgradient method. By using a weighted average with a weight of $t + 1$ for each iterate $w_t$ at iteration $t$, we obtain the convergence rate of $O(1/t)$ with both an easy proof and an easy implementation. The new scheme is compared empirically to existing techniques, with similar performance behavior.

1 Introduction

We consider a strongly convex function $f$ defined on a convex set $K$. We denote by $\mu$ its strong convexity constant. Following [1, 2, 3, 4], we consider a stochastic approximation scenario where only unbiased estimates of subgradients of $f$ are available, with the projected stochastic subgradient method.

More precisely, we assume that we have an increasing sequence of $\sigma$-fields $(\mathcal{F}_t)_{t \geq 0}$, such that $w_0 \in K$ is $\mathcal{F}_0$-measurable and such that for all $t \geq 1$,

$$w_t = \Pi_K(w_{t-1} - \gamma_t g_t),$$

where

(a) $\Pi_K$ is the orthogonal projection on $K$,

(b) $\mathbb{E}(g_t | \mathcal{F}_{t-1})$ is almost surely a subgradient of $f$ at $w_{t-1}$ (which we denote $f'(w_{t-1})$),

(c) $\mathbb{E}(\|g_t\|^2) \leq B^2$ (finite variance condition).

We denote by $w^*$ the unique minimizer of $f$ on $K$. 

2 Motivating example

Our main motivating example is the support vector machine (SVM) and its structured prediction extensions [5, 6, 7], where the pairs \((x_t, y_t)\) for \(t \geq 1\) are independent and identically distributed and \(f(w) = E\ell(y, w^\top x) + \frac{\mu}{2}\|w\|^2\), where \(\ell(y, u)\) is a Lipschitz-continuous convex loss function (with respect to the second variable) and \(K\) is the whole space (unconstrained setup). We then have \(g_t = \ell'(y_t, w_{t-1}^\top x_t)x_t + \mu w_{t-1}\), where \(\ell'(y, u)\) denotes any subgradient with respect to the second variable.

If we make the additional assumption that \(E\|x\|^2\) is finite, then this setup satisfies the assumptions above with \(B^2 = 4L^2 E\|x\|^2\), where \(L\) is the Lipschitz constant for \(\ell\). We show this bound in Appendix A.

Alternatively, we can consider \(K\) to be a compact convex subset. This is used in particular in a projected version of the stochastic subgradient method for SVM in [1]. In this case, we can take \(B^2 = (L\sqrt{E\|x\|^2} + \mu \max_{w \in K} \|w\|)^2\).

3 Convergence analysis

Following standard proof techniques [1, 2], we have:

\[
\|w_t - w^*\|^2 \leq \|w_{t-1} - \gamma_t g_t - w^*\|^2 \quad \text{because orthogonal projections contract distances,}
\]

\[
E(\|w_t - w^*\|^2 | \mathcal{F}_{t-1}) \leq \|w_{t-1} - w^*\|^2 + \gamma_t^2 E(g_t^2 | \mathcal{F}_{t-1}) - 2\gamma_t (w_{t-1} - w^*)^\top f'(w_{t-1})
\]

\[
\leq \|w_{t-1} - w^*\|^2 + \gamma_t^2 E(g_t^2 | \mathcal{F}_{t-1}) - 2\gamma_t [f(w_{t-1}) - f(w^*) + \frac{\mu}{2}\|w_{t-1} - w^*\|^2].
\]

The last inequality is obtained from the \(\mu\)-strong convexity of \(f\). Thus, by re-arranging the function values on the LHS and taking expectations on both sides, we get:

\[
2\gamma_t \left[ E(f(w_{t-1}) - f(w^*)) \right] \leq \gamma_t^2 E(g_t^2 | \mathcal{F}_{t-1}) + (1 - \mu \gamma_t) E(\|w_{t-1} - w^*\|^2 - E\|w_t - w^*\|^2)
\]

\[
E(f(w_{t-1}) - f(w^*)) \leq \frac{\gamma_t B^2}{2} + \frac{\gamma_t^2 - \mu}{2} E\|w_{t-1} - w^*\|^2 - \frac{\gamma_t^2}{2} E\|w_t - w^*\|^2.
\] (2)

3.1 Classical analysis

With \(\gamma_t = \frac{1}{\mu \ell}\), then inequality (2) becomes

\[
E(f(w_{t-1}) - f(w^*)) \leq \frac{B^2}{2\mu \ell} + \frac{\mu(t - 1)}{2} E\|w_{t-1} - w^*\|^2 - \frac{\mu t}{2} E\|w_t - w^*\|^2,
\]

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and by summing from \( t = 1 \) to \( t = T \), we obtain:

\[
\mathbb{E} f\left( \frac{1}{T} \sum_{t=1}^{T} w_{t-1} \right) - f(w^*) \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} f(w_{t-1}) - f(w^*) \\
\leq \frac{B^2}{2\mu T} \sum_{t=1}^{T} \frac{1}{t} + \frac{\mu}{2T} \left[ 0 - T\mathbb{E}\|w_T - w^*\|^2 \right] \leq \frac{B^2}{2\mu T} (1 + \log T).
\]

The first line used the convexity of \( f \); the second line is obtained from a telescoping sum. We also obtain \( \mathbb{E}\|w_T - w^*\|^2 \leq \frac{B^2}{\mu^2 T} (1 + \log T) \).

### 3.2 New analysis

With \( \gamma_t = \frac{2}{\mu(t+1)} \) and multiplying inequality (2) by \( t \), we obtain:

\[
t \left[ \mathbb{E} f(w_{t-1}) - f(w^*) \right] \leq \frac{tB^2}{\mu(t+1)} + \frac{\mu}{4} \left[ t(t-1)\mathbb{E}\|w_{t-1} - w^*\|^2 - t(t+1)\mathbb{E}\|w_t - w^*\|^2 \right] \\
\leq \frac{B^2}{\mu} + \frac{\mu}{4} \left[ t(t-1)\mathbb{E}\|w_{t-1} - w^*\|^2 - t(t+1)\mathbb{E}\|w_t - w^*\|^2 \right].
\]

By summing from \( t = 1 \) to \( t = T \) these \( t \)-weighted inequalities, we obtain a similar telescoping sum, but this time the term with \( B^2 \) stays constant across the sum:

\[
\sum_{t=1}^{T} t \left[ \mathbb{E} f(w_{t-1}) - f(w^*) \right] \leq \frac{TB^2}{\mu} + \frac{\mu}{4} \left[ 0 - T(T+1)\mathbb{E}\|w_T - w^*\|^2 \right].
\]

Thus

\[
\mathbb{E} f\left( \frac{2}{T(T+1)} \sum_{t=0}^{T-1} (t+1)w_t \right) - f(w^*) + \frac{\mu}{2} \mathbb{E}\|w_T - w^*\|^2 \leq \frac{2B^2}{\mu(T+1)}.
\]

which implies

\[
\mathbb{E} f\left( \frac{2}{T(T+1)} \sum_{t=0}^{T-1} (t+1)w_t \right) - f(w^*) \leq \frac{2B^2}{\mu(T+1)}
\]

and

\[
\mathbb{E}\|w_T - w^*\|^2 \leq \frac{4B^2}{\mu^2(T+1)}.
\]

So by using the weighted average \( \bar{w}_T \doteq \frac{2}{(T+1)(T+2)} \sum_{t=0}^{T} (t+1)w_t \) instead of a uniform average, we get a \( O\left(\frac{1}{T}\right) \) rate instead of \( O\left(\frac{\log T}{T}\right) \). Note that these averaging schemes are efficiently implemented in an online fashion as:

\[
\bar{w}_t = (1 - \rho_t)\bar{w}_{t-1} + \rho_tw_t.
\]
For the proposed weighted averaging scheme, $\rho_t = 2/(t + 2)$ (compare with $\rho_t = 1/(t + 1)$ for the uniform averaging scheme).

4 Experiments

To test the empirical performance of the averaging scheme, we performed a series of experiments using the support vector machine optimization problem

$$\min_w \frac{\lambda}{2} \|w\|^2 + \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i w^\top x_i\},$$

where $x_i$ is in an Euclidean space and $y_i \in \{-1, 1\}$.

We performed experiments on a set of freely available benchmark binary classification data sets. The quantum ($n = 50000, p = 78$) and protein ($n = 145751, p = 74$) data sets were obtained from the KDD Cup 2004 website,\(^1\) the sido data set ($n = 12678$, $p = 4932$) was obtained from the Causality Workbench website,\(^2\) while the rcv1 ($n = 20242$, $p = 47236$), covertype ($n = 581012$, $p = 54$), and news ($n = 19996$, $p = 1355191$) data sets were obtained from the LIBSVM data website.\(^3\) We added a (regularized) bias term to all data sets, and for dense features we standardized so that they would have a mean of zero and a variance.

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\(^1\)http://osmot.cs.cornell.edu/kddcup
\(^2\)http://www.causality.inf.ethz.ch/home.php
\(^3\)http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets
of one. We set the regularization parameter $\lambda$ to $1/n$, although we found that the relative performance of the methods was not particularly sensitive to this choice. We didn’t use any projection ($K$ is the whole space). Our experiments compared the following averaging strategies:

- **0**: No averaging.
- **1**: Averaging all iterates with uniform weight.
- **0.5**: Averaging the second half of the iterates with uniform weight, as proposed in [4].
- **D**: Averaging all iterates since the last iteration that was a power of 2 with uniform weight (the ‘doubling trick’), also proposed in [4].
- **W**: Averaging all iterates with a weight of $t + 1$, as discussed in this note.
- **W^2**: Averaging all iterates with a weight of $(t+1)^2$, which puts even further emphasis on recent iterations.

We plot the performance of these different averaging strategies in Figure 1, which shows the objective function against the number of effective passes through the data (the number of iterations divided by $n$). This figure uses a step size of $1/\mu t$ for all methods as we found this gave better performance than a step size of $2/\mu(t+1)$, although we include the performance of $W$ with the latter step-size for comparison. In Figure 1, we observe the following trends:

- **0**: Not averaging at all is typically among the worst strategies. However, this proved to be the best strategy on the *sido* data set. This may be because the method is still far from the solution after 50 passes through the data.
- **1**: Uniform averaging of all iterates is always the worst strategy.
- **0.5**: Uniform averaging of the second half of the iterates is typically among the best strategies, provided we are in fact in the second half of the iterates.
- **D**: The doubling trick typically gave among the best performance across the methods.
- **W**: The proposed weighting typically performed between the doubling trick and not averaging.
- **W^2**: Weighting the iterates by $(t+1)^2$ always outperformed weighting them by $t + 1$.

## 5 Discussion

- We note that the averaging of linear approximations of $f$ (rather than the iterates) by $t + 1$ is also used in the optimization strategy of Nesterov [8], which achieves an optimal $O(1/t^2)$ convergence rate for optimizing (deterministic) objectives with Lipschitz-continuous gradients (see step 3 for their Equation 3.11).
There are previous approaches to removing the log \( t \) term [3, 4], but the one presented in this note is arguably somewhat simpler to implement and analyze. Rakhlin et al. propose in [4] the ‘1/2-suffix averaging’ scheme (and the ‘doubling trick’ that we used in the experiments). Their proof technique requires separately bounding \( E \| w_t - w^* \|^2 \) and then controlling the sum of inequalities in (2) by using that only the last half of the iterations is averaged (the ‘1/2-suffix’). Hazan and Kale propose in [3] the epoch-GD scheme, which uses a similar averaging schedule as in the ‘doubling trick’ of [4], but using a fixed step-size within each geometrically sized ‘epoch’ of averaging, as well as using the previous average as the initialization for an epoch.

We note that all the schemes presented in the experiments can have their convergence rate proven. Schemes 0 and 1 have \( O((\log t)/t) \) rate whereas the schemes 0.5, D, W and \( W^2 \) have \( O(1/t) \) rate. We can show the \( O(1/t) \) rate for general weighted averaging schemes (with weight \( t^k \) for iterate \( t \) for some fixed \( k \geq 1 \)) as well as step-sizes of the form \( \gamma_t = c/(t + b) \) for \( c > 1/2 \) and \( b \geq 0 \). The proof becomes longer though as the nice telescoping sum in (3) doesn’t cancel out in these cases. One has to use instead a bound on \( E \| w_t - w^* \|^2 \) such as in Lemma 1 in [4] to control the non-canceling terms. The overall rate is still \( O(1/t) \), but with different constants depending on \( c \) and \( k \).

At the same time that we first posted this note, Shamir and Zhang independently proposed a similar weighted average scheme in [9] which they call ‘polynomial-decay averaging’. They consider a running average scheme as in (4), but with the more general \( \rho_t = \frac{1+\eta}{t+1+\eta} \), where the integer \( \eta \geq 0 \) parameterizes the different schemes.\(^4\) \( \eta = 0 \) yields the standard uniform averaging scheme, whereas \( \eta = 1 \) yields the simple weighted average analyzed in Section 3.2. The general \( \eta \) gives a weight of \( O(t^{\eta}) \) for each iterate, similar to what was mentioned in the previous paragraph, but with a different exact formula. They provide in [9] a proof of a rate of \( O(1/t) \) for \( \eta \geq 2 \). The proof that we give in Section 3.2 can be seen as complementary and is especially much simpler (as well as giving a tighter constant). We also note that the rate of \( O((\log t)/t) \) for the last iterate \( w_t \) (scheme 0 above) is proven for the first time in [9].

While this paper focuses on the non-smooth case, it is still interesting to relate results to the smooth case (see, e.g., [10] and references therein), where in the strongly convex case, averaging with longer step sizes—i.e., of the form \( t^{-\alpha} \) with \( \alpha \in (1/2,1) \)—leads to better and more robust rates. Can larger step sizes improve results for the non-smooth case?

### A Finite variance bound for SVM

We derive here the finite variance bound \( E\|g_t\|^2 \leq 4L^2 \| x \|^2 = B^2 \) for the general SVM-like objective considered in Section 2 and update rule (1). To see this, we consider the more general case of \( f(w) = \mathbb{E} h(z, w) + \frac{\nu}{2} \| w \|^2 \), where \( h(z, w) \) is convex in \( w \) for each \( z \)\(^4\).

\(^4\)We note that the index \( t \) is shifted by one between this note and their paper as their initial point is \( w_1 \) whereas ours is \( w_0 \). We also note that they use the misnomer ‘gradient descent’ for their algorithm despite using subgradients which don’t necessarily yield a descent direction.
(for SVM, \(z = (x, y)\) and \(h(z, w) = \ell(y, w^\top x)\)). We make a Lipschitz-like (in expectation) assumption on \(h\) that \(\mathbb{E}||h'(z, w)||^2 \leq L^2\), where \(h'(z, w)\) denotes any subgradient with respect to the second variable (note that \(L^2 = L_2^2\mathbb{E}||x||^2\) for SVM). With \(g_t = h'(z_t, w_{t-1}) + \mu w_{t-1}\), Leibniz rule yields \(\mathbb{E}(g_t|w_{t-1}) = f'(w_{t-1})\), as required by our setup (see (1.3) in [2] for some regularity conditions for this to be true). Given this definition of \(g_t\), we use the Minkowski inequality on the norm function\(^5\) to get:

\[
\sqrt{\mathbb{E}||g_t||^2} \leq \sqrt{\mathbb{E}||h'(z_t, w_{t-1})||^2} + \mu \sqrt{\mathbb{E}||w_{t-1}||^2} \leq L + \mu \sqrt{\mathbb{E}||w_{t-1}||^2}.
\]

We can then obtain the required bound of \((2L)^2\) on \(\mathbb{E}||g_t||^2\) by showing that \(\sqrt{\mathbb{E}||w_{t-1}||^2} \leq L/\mu\). This can easily be proven by induction, with the assumption that \(\gamma_t \leq 1/\mu\) and either \(\gamma_1 = 1/\mu\) or \(\sqrt{\mathbb{E}||w_0||^2} \leq L/\mu\) (these assumptions are satisfied by the step sizes considered in this note). To see this, we use the subgradient update (1) applied to this form of \(f(w)\):

\[w_t = (1 - \mu \gamma_t)w_{t-1} - \gamma_t h'(z_t, w_{t-1}).\]

Applying Minkowski inequality again, we get

\[
\sqrt{\mathbb{E}||w_t||^2} \leq (1 - \mu \gamma_t)\sqrt{\mathbb{E}||w_{t-1}||^2} + \gamma_t \sqrt{\mathbb{E}||h'(z_t, w_{t-1})||^2} \\
\leq (1 - \mu \gamma_t)\sqrt{\mathbb{E}||w_{t-1}||^2} + \mu \gamma_t \frac{L}{\mu}.
\]

The first line above used the assumption that \(\gamma_t \leq 1/\mu\) to ensure that \((1 - \mu \gamma_t)\) is non-negative. The assumption \(\gamma_1 = 1/\mu\) or \(\sqrt{\mathbb{E}||w_0||^2} \leq L/\mu\) then yields the base case of \(t = 1\). Plugging in the induction hypothesis then yields:

\[
\sqrt{\mathbb{E}||w_t||^2} \leq (1 - \mu \gamma_t)\frac{L}{\mu} + \mu \gamma_t \frac{L}{\mu} = \frac{L}{\mu},
\]

which completes the proof.

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\(^5\)Minkowski inequality says that \(\sqrt{\mathbb{E}(X + Y)^2} \leq \sqrt{\mathbb{E}X^2} + \sqrt{\mathbb{E}Y^2}\) for scalar random variables \(X\) and \(Y\). If we have \(a = b + c\) for some random vectors \(a, b, c\), by the triangle inequality, we have \(\|a\| \leq \|b\| + \|c\|\) and so \(\sqrt{\mathbb{E}\|a\|^2} \leq \sqrt{\mathbb{E}(\|b\| + \|c\|)^2} \leq \sqrt{\mathbb{E}\|b\|^2 + \sqrt{\mathbb{E}\|c\|^2}}\) by using Minkowski on the norm of the random vectors.
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