Long Time Boundedness of Planar Jump Discontinuities for Homogeneous Hyperbolic Systems

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Abstract

Suppose that \( L(\partial_t, \partial_x) \) is a homogeneous constant coefficient strongly hyperbolic partial differential operator on \( \mathbb{R}^{1+d} \) and \( H \) is a characteristic hyperplane. Suppose that in a conic neighborhood of the conormal variety of \( H \), the characteristic variety of \( L \) is the graph of a real analytic function \( \tau(\xi) \) with rank \( \tau_{\xi\xi} \) identically equal to zero or the maximal possible value \( d-1 \). Suppose that the source function \( f \) is compactly supported in \( t \geq 0 \) and piecewise smooth with singularities only on \( H \). Then the solution of \( Lu = f \) with \( u = 0 \) for \( t < 0 \) is uniformly bounded on \( \mathbb{R}^{1+d} \). Typically when rank \( \tau_{\xi\xi} \neq 0 \) on the conormal variety, the sup norm of the jump in the gradient of \( u \) across \( H \) grows linearly with \( t \).

1 Introduction

With \( A_j \in \text{Hom}(\mathbb{C}^k) \) for \( 1 \leq j \leq d \), and \( \xi \in \mathbb{R}^d \), define

\[
L := \frac{\partial}{\partial t} + \sum_{j=1}^{d} A_j \frac{\partial}{\partial x_j}, \quad A(\xi) := \sum_{j} A_j \xi_j.
\] (1.1)

The operator \( L \) maps \( \mathbb{C}^k \) valued functions to themselves.

**Hypothesis 1.1** The operator \( L \) is strongly hyperbolic, that is

\[
\sup_{\xi \in \mathbb{R}^d} \| e^{i \sum_{j=1}^{d} A_j \xi_j} \|_{\text{Hom}(\mathbb{C}^k)} < \infty.
\] (1.2)

Plancherel’s Theorem implies that tempered solutions of \( Lu = 0 \) satisfy for all \( t \in \mathbb{R} \),

\[
\| u(t) \|_{L^2(\mathbb{R}^d)} \leq \sup_{\xi \in \mathbb{R}^d} \| e^{i \sum_{j=1}^{d} A_j \xi_j} \|_{\text{Hom}(\mathbb{C}^k)} \| u(0) \|_{L^2(\mathbb{R}^d)}.
\]

The assertions of the abstract are presented and proved for the hyperplane \( H = \{(t, x) : x_1 = 0\} \). The general case follows by a linear change of variable in \( \mathbb{R}^{1+d} \) that preserves the time \( t \).

**Hypothesis 1.2** The source term \( f \) is compactly supported in time, piecewise smooth, and rapidly decreasing in \( x \). Precisely, \( f \in L^\infty(\mathbb{R}^{1+d}) \) has support in \( \{0 \leq t \leq T < \infty\} \) and the restriction of \( f \) to each each halfplane \( \{(t, x) : \pm x_1 > 0\} \) is infinitely differentiable and for all \( \alpha, \beta \), \( (t, x)^\alpha \partial_t^\beta f \in L^\infty((t, x) : \{(t, x) : \pm x_1 > 0\}) \).
The hyperplane \( \{(t,x) : \pm x_1 > 0\} \) has conormal variety
\[
N^*\left(\{(t,x) : x_1 = 0\}\right) = \{(\tau,\xi) \in \mathbb{R}^{1+d} \setminus \{0\} : \tau = 0 \text{ and } \xi' = 0\}.
\]

**Definition 1.1** A point \((\tau,\xi) \in \text{Char}(L)\) satisfies the smooth variety hypothesis when there is a conic neighborhood of \((\tau,\xi)\) so that in that neighborhood the characteristic variety is equal to the graph of a real analytic function \(\tau(\xi)\).

The function \(\tau(\xi)\) is homogeneous of degree one. Therefore the second derivative in the radial direction vanishes. So, \(\text{rank } \tau_{\xi\xi} \leq d - 1\).

**Hypothesis 1.3** The hyperplane \( \{(t,x) : x_1 = 0\} \) is characteristic for \(L\). This equivalent to \(N^*\left(\{(t,x) : x_1 = 0\}\right) \subset \text{Char } L\) and also to \(\det A_1 = 0\). The conormal variety \(\{(\tau,\xi) \neq 0 : \tau = 0 \text{ and } \xi' = 0\}\) satisfies the smooth variety hypothesis.

**Definition 1.2** Denote by \(\tau(\xi)\) with \(\tau(1,0,\ldots,0) = 0\) the real analytic function so that the characteristic variety is given by \(\{\tau = \tau(\xi)\}\) on a conic neighborhood of \(N^*\left(\{(t,x) : x_1 = 0\}\right)\). Denote by \(v := -\nabla_\xi \tau(1,0,\ldots,0)\) the associated group velocity.

Denote by \(u\) the unique solution of the Cauchy problem
\[
Lu = f, \quad u = 0 \text{ for } t \leq 0. \tag{1.3}
\]

It is proved in [7] that \(u\) is piecewise smooth with singularities only on \(\{(t,x) : x_1 = 0\}\). Moreover, outside the support of \(f\) the jumps in \(u\) are rigidly transported at the group velocity \(v\). They do not decay as \(t \to \infty\) (see Appendix A). This contrasts to the dispersive behavior of solutions that tends to spread out with corresponding amplitudes tending to zero. This paper arose from an attempt, over the last decade by Gués and I, to quantify the dispersive behavior of internal layers of width \(\sim \epsilon\) on time scales \(1/\epsilon\) using ideas from [10]. Our computations yielded a leading amplitude that included a solution \(u\) of an equation like (1.3). The consistency of the expansions required that \(u \in L^\infty(\mathbb{R}^{1+d})\). In the present paper that boundedness is finally proved. We can now return to the problem of the diffraction of internal layers.

Since the operator \(L\) propagates \(H^s\) regularity, one might expect that the solution remains bounded in \(H^s(\{\pm x_1 > 0\})\). The solution would then look like two \(H^s\) solutions side by side. This is not true. In Appendix A it is shown that when rank \(\tau_{\xi\xi} > 0\), the sup norm of the jump in \(\partial u/\partial x_1\) across \(\{x_1 = 0\}\) typically grows linearly with time. Proofs of piecewise smoothness can also be found in [8] Chapter VI Sections 4,5,6, and, [15]. The proof in the appendix benefits from some simplifications from analogous advances in geometric optics.

**Theorem 1.3** Assume Hypothesis 1.1, 1.2, 1.3, and that with \(\tau\) from Definition 1.3 either rank \(\tau_{\xi\xi}(1,0,\ldots,0) = d - 1\) or \(\tau_{\xi\xi}\) is identically equal to zero. Then the unique solution \(u\) of (1.3) satisfies \(u \in L^\infty(\mathbb{R}^{1+d})\).
Example 1.1 Consider compactly supported $f$. Write $f = f_1 + f_2$ where $f_1$ is smooth and $f_2$ is equal to $f$ cutoff to a small neighborhood $x_1 = 0$. The source $f_2$ is small in $L^2(\mathbb{R}^{1+d})$. If the characteristic variety of $L$ has no flat sheets, then the solution with source $f_1$ tends to zero in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$ (Corollary 3.3.2 in [14]). The solution with data $f_2$ on the other hand has $L^\infty(\mathbb{R}^d)$ norm bounded below and small $L^2(\mathbb{R}^d)$ norm. For large $t$, the singularity is a tall island of small $L^2$ norm in a sea of small amplitude that contains most of the $L^2$ norm. The transition from slim peaks to the plateau explains the large derivatives for $|t| \gg 1$.

Remark 1.1 Whether the boundedness of $u$ holds without the rank hypotheses is an open problem. One one hand, the flatter is $\lambda$ near $\xi' = 0$ the more the hyperbolic problem resembles the transport operator $\partial_t + v \cdot \partial_x$ and the easier seem $L^\infty$ estimates. On the other hand, it is conceivable that there is some focussing phenomenon from the high order Taylor polynomials of $\tau$ that leads to amplification of sup norms. I conjecture that the transport idea is the right one and that $u \in L^\infty(\mathbb{R}^{1+d})$ holds generally.

Concerning the pure initial value problem, Theorem 1.3 implies the following. Only the last assertion is new.

Corollary 1.4 Assume Hypothesis 1.1. Suppose that over a neighborhood of $\xi' = 0$ the characteristic variety of $L$ consists of $m \leq d$ real analytic sheets $\tau = \tau_j(\xi)$, $1 \leq j \leq m$. Suppose that for each $j$, rank $\tau_{\xi\xi}^j$ is identically equal to either $d-1$ or 0 on that neighborhood. Suppose that $g(x)$ is piecewise smooth with singularities only on $\{x_1 = 0\}$ and is rapidly decreasing. Then the solution of the Cauchy problem $Lw = 0$, $w(0) = g$ is piecewise smooth on $\mathbb{R}^{1+d}$ with singularities only on the $m$ characteristic hyperplanes passing through $t = 0, x_1 = 0$. In addition, $w \in L^\infty(\mathbb{R}^{1+d})$.

Example 1.2 The hypotheses are satisfied for operators with eigenvalues of constant multiplicity with each sheet of the characteristic variety either convex, concave, or flat. For example, Dirac equations and Maxwell’s equations.

Remark 1.2 The hypothesis is violated only when over $\xi' = 0$ there is either an eigenvalue crossing or a point of a non flat sheet where rank $\tau_{\xi\xi} < d - 1$. Each of these occurrences is a rare event. For example, consider symmetric hyperbolic $L$. The space of such operators is parametrized by $d$ hermitian symmetric matrices, and the hyperplanes by their unit conormals. This is a subset of $\mathbb{R}^M$ described by real polynomial equations. Crossings are given by a vanishing discriminant and low rank $\tau$ by additional real algebraic equations. The exceptional set is a finite union of real algebraic subsets of positive codimension in the set of all problems.

The proof of Theorem 1.3 starts by decomposing $u$ microlocally in $x$ only.

Hypothesis 1.4 $\mathcal{N} \subset \mathbb{R}^{d} \setminus 0$ is an centro-symmetric, convex, open conic neighborhood of $\xi' = 0$ on which $\tau(\xi)$ is real analytic. $\chi \in C^\infty(\mathbb{R}^d \setminus 0)$ satisfies $\chi(\sigma \xi) = \chi(\xi)$ for all $\sigma \in \mathbb{R} \setminus 0$, and, the support of $\chi$ meets $S^{d-1}$ on a compact subset of $\mathcal{N}$. In addition, $\chi$ is equal to one on a conic neighborhood $\mathcal{N}_1$ of $\{\xi' = 0\}$.
Fourier multipliers with symbols independent of $x$ are used throughout. To a symbol $\chi(\xi)$ is associated the operator $\chi(D_x)$, $D_x := i^{-1} \partial_x$ defined using the Fourier Transform $\mathcal{F}$ by

$$\mathcal{F}(\chi(D_x)f) := \chi(\xi)\mathcal{F}f.$$  

Analogous formulas apply to operators in $D_{t,x}$ using the Fourier transform on $\mathbb{R}^{1+d}_{t,x}$.

For the microlocal cutoff $\chi(D_x)$ to a neighborhood of $\xi' = 0$, write

$$u = \chi(D_x)u + (I - \chi(D_x))u.$$  

(1.4)

The second summand is the easy part microlocalized where $\xi' \neq 0$ where $f$ is microlocally smooth. This term is treated in Section 2.

The term $\chi(D_x)u$ is decomposed using the spectral projectors of $A(\xi)$. The easy case of the Kreiss Matrix Theorem (see Appendix 2.I in [14]) asserts that (1.2) holds if and only if the next Conditions A and B hold.

Condition A. For all $\xi \in \mathbb{R}^d$, the eigenvalues of $A(\xi)$ are real and the eigenspaces span $\mathbb{C}^k$,

$$\bigoplus_{\lambda \in \text{Spec } A(\xi)} E_\lambda(\xi) = \mathbb{C}^k, \quad E_\lambda(\xi) := \{ v \in \mathbb{C}^k : A(\xi)v = \lambda v \} = \text{Ker } (A(\xi) - \lambda I).$$

For $\lambda \in \text{Spec } A(\xi)$, $\text{Range } (A(\xi) - \lambda I) = \bigoplus_{\nu \in \text{Spec } A(\xi) \setminus \lambda} E_\nu(\xi)$. Denote by $\pi_\lambda(\xi)$ projection along $\text{Range } (A(\xi) - \lambda I)$ onto $\text{Ker } (A(\xi) - \lambda I)$. Then

$$A(\xi) = \sum_{\lambda \in \text{Spec } A(\xi)} \lambda \pi_\lambda(\xi), \quad \sum_{\lambda \in \text{Spec } A(\xi)} \pi_\lambda(\xi) = I, \quad \pi_\lambda(\xi) \pi_\nu(\xi) = 0 \text{ for } \lambda \neq \nu.$$  

(1.5)

Condition B. The function $\xi \mapsto \max_{\lambda \in \text{Spec } A(\xi)} \| \pi_\lambda(\xi) \|$ is uniformly bounded on $\mathbb{R}^d$.

With $\tau(\xi)$ from Definition 1.2 the map $\xi \mapsto \pi_{-\tau(\xi)}$ is real analytic on $\mathcal{N}$ and for all $s \in \mathbb{R} \setminus 0$, $\tau(s\xi) = \tau(\xi)$. The eigenvalue identities read

$$\forall \xi \in \mathcal{N}, \quad (\sum_j A_j \xi_j) \pi_{-\tau(\xi)} = \pi_{-\tau(\xi)} (\sum_j A_j \xi_j) = -\tau(\xi) \pi_{-\tau(\xi)}.$$  

(1.6)

The decomposition of $\chi(D_x)u$ is

$$\chi(D_x)u = \pi_{-\tau(D_x)} \chi(D_x)u + (I - \pi_{-\tau(D_x)})\chi(D_x)u.$$  

(1.7)

The second summand is treated in Section 3 by deriving an equation for it that is microlocally elliptic. The heavy lifting is the analysis of the first summand $\pi_{-\tau(D_x)} \chi(D_x)u$. A scalar hyperbolic equation satisfied by this part is derived in Section 3. The analysis of that equation microlocally at nonstationary points is presented in Section 5. In Section 6 the stationary contributions are written as the sum of a paraxial approximation and an error term,

$$\pi_{-\tau(D_x)} \chi(D_x)u = u_{\text{paraxial}} + (u - u_{\text{paraxial}}).$$

The proof that the paraxial approximation is bounded uses Van de Corput’s Lemma. The proof that the error term is bounded and tends to zero as $t \to \infty$ proceeds by a high/low frequency decomposition in Section 7. The proof that the low frequency term is bounded requires an inequality of stationary phase for test functions with $m$ derivatives with $m > d/2$ but close to $d/2$. In Appendix 9 we present an estimate for the limit point case of exactly $d/2$ derivatives. The estimate, possibly new, is weaker by a factor $|\ln \epsilon|$ than the standard estimate.
2 Analysis away from \( \{ \xi' = 0 \} \)

**Proposition 2.1** Suppose that \( f \) and \( u \) are as in Theorem 1.3. Suppose that \( \beta(\xi) \in C^\infty(\mathbb{R}^d \setminus 0) \) is homogeneous of degree zero and vanishes on a conic neighborhood of \( \xi' = 0 \). Define \( w := \beta(D_x)u \), so, \( w \) is the unique solution of \( Lw = \beta(D_x)f \) that vanishes for \( t \leq 0 \).

i. For any \( s \in \mathbb{R} \), \( \beta(D_x)f \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^d)) \) and is supported in \( 0 \leq t \leq T \).

ii. \( w \) is the unique solution of \( Lw = \beta(D_x)f \) that vanishes for \( t < 0 \).

iii. \( w \in L^\infty(\mathbb{R}^{1+d}) \).

**Proof.** i. Since \( f \) is piecewise smooth one has for all \( s \)

\[
\langle D_{t,x'} \rangle^s f \in L^2(\mathbb{R}^{1+d}).
\]

For \( s \geq 0 \) write

\[
\beta(D_x)f = \left( \beta(D_x)\langle D_{x'} \rangle^{-s} \right) \left( \langle D_{x'} \rangle^s f \right).
\]

Since \( \langle \xi \rangle \lesssim \langle \xi' \rangle \) on the support of \( \chi \) it follows that \( \beta(D_x)\langle D_{x'} \rangle^{-s} \) is bounded from \( L^2(\mathbb{R}^d) \) to \( H^s(\mathbb{R}^d) \). Therefore \( \beta(D_x)f \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^d)) \) and is supported in \( 0 \leq t \leq T \).

ii. Follows from i.

iii. The Duhamel representation

\[
w(t) = \int_0^t e^{i(t-\sigma)\sum A_jD_j} \beta(D_x)f(\sigma) \, d\sigma
\]

implies that

\[
\|w(t)\|_{H^s(\mathbb{R}^d)} \leq \|e^{\sum iA_j\xi_j}\|_{L^\infty(\mathbb{R}^d)} \int_0^T \|\beta(D_x)f\|_{H^s(\mathbb{R}^d)} < \infty.
\]

For \( s > d/2 \), this bound uniform for \( t \in \mathbb{R} \) implies that \( w \in L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^d)) = L^\infty(\mathbb{R}^{1+d}) \). \( \blacksquare \)

3 Proof that \( (I - \pi_{-\tau(D_x)})\chi(D_x)u \in L^\infty(\mathbb{R}^{1+d}) \)

**Proposition 3.1** With Hypothesis \( I.A \), \( q := (I - \pi_{-\tau(D_x)})\chi(D_x)u \in L^\infty(\mathbb{R}^{1+d}) \).

**Proof.** i. The function \( q \) is supported in \( t \geq 0 \) and satisfies two equations,

\[
Lq = (I - \pi_{-\tau(D_x)})\chi(D_x)f, \quad \text{and,} \quad \pi_{-\tau(D_x)}q = 0.
\]

From this pair of equations construct a modified system as follows. Choose \( \chi_1(\xi) \in C^\infty(\mathbb{R}^d \setminus 0) \) homogeneous of degree zero, equal to one on a neighborhood of \( \supp \chi \setminus 0 \) and \( \supp \chi_1 \setminus 0 \) is contained in \( \mathcal{N} \) from Hypothesis \( I.A \). Then

\[
\tilde{L}q = (I - \pi_{-\tau(D_x)})\chi(D_x)f, \quad \tilde{L} := L + \chi_1(D_x)\pi_{-\tau(D_x)}\partial_{x_1}
\]

(3.1)
The last summand is everywhere defined since $\pi_{-\tau(\xi)}$ makes sense wherever $\chi_1 \neq 0$.

The operator $\tilde{L}$ is pseudodifferential in $x$. In contrast to $L$, the hyperplane $x_1 = 0$ is noncharacteristic for $\tilde{L}$. Indeed the symbol at $\tau = 0$, $\xi \in \mathcal{N}_1$ from Hypothesis 1.3 is equal to

$$\tilde{L}(0, \xi) = iA(\xi) + i\pi_{-\tau(\xi)} \xi_1 .$$

This matrix is invertible since $\pi_{-\tau(\xi)}$ is is spectral projection on the kernel of $A(\xi)$. The equation (3.1) is therefore elliptic on the wavefront set of the right hand side.

Choose $\delta > 0$ so that on $\{|\tau| > \delta\} \times \mathcal{N}_1$,

$$\forall (\tau, \xi) \in \{|\tau| > \delta\} \times \mathcal{N}_1, \quad |\det (\tilde{L}(0, \xi)) = iA(\xi) + i\pi_{-\tau(\xi)} \xi_1)| > \delta .$$

II. For $(\tau, \xi) \in \{|\tau| > \delta\} \times \mathcal{N}_1$ define

$$E_1(\tau, \xi) := (\tilde{L}(0, \xi) = iA(\xi) + i\pi_{-\tau(\xi)} \xi_1)^{-1} .$$

The symbol $E_1$ is singular at the origin. Choose $\zeta \in C_0^\infty(\mathbb{R}_{t,x}^{1+d})$, with

$$\zeta_1 = 0 \quad \text{for} \quad |\tau, \xi| < 1, \quad \text{and,} \quad \zeta_1 = 1 \quad \text{for} \quad |\tau, \xi| > 2. $$

Choose $\zeta_2 \in C_0^\infty(\mathbb{R}_{t,x}^{1+d} \setminus 0)$ homogeneous of degree zero so that the support intersects $S^d$ in a compact subset of $\mathcal{N}_1$ and so that $\zeta_2 = 1$ on a conic neighborhood $\mathcal{N}_2$ of $\{\tau = 0, \xi' = 0\}$. Recall the classical space of symbols of order $m \in \mathbb{R}$ that are indenpend of $t, x$,

$$a(\tau, \xi) \in S^m(\mathbb{R}_{t,x}^{1+d}) \iff \forall \alpha, \exists C < \infty, \quad |\partial_{\tau,\xi}^\alpha a| \leq C \langle \tau, \xi \rangle^{m-|\alpha|}, \quad \langle \tau, \xi \rangle := (1 + |\tau, \xi|^2)^{1/2} .$$

In addition $S^{-\infty} := \cap_m S^m$. Define

$$E_2(\tau, \xi) := \zeta_1 \zeta_2 E_1(\tau, \xi) \in S^{-1}(\mathbb{R}_{t,x}^{1+d}), \quad \text{so,} \quad \tilde{L} E_2(D_{t,x}) - I = R_1(D_{t,x}) + R_2(D_{t,x}),$$

with

$$R_1 \in S^{-\infty}(\mathbb{R}_{t,x}^{1+d}), \quad \text{and} \quad R_2 \in S^0(\mathbb{R}_{t,x}^{1+d}) \quad \text{with} \quad R_2(\tau, \xi) = 0 \quad \text{on} \quad \{|\tau| < \delta\} \times \mathcal{N}_1. \tag{3.2}$$

$E_2(D_{t,x})$ is of the form $K*$ with $\text{singsupp} K$ equal to the origin in $\mathbb{R}_{t,x}^{1+d}$. Choose $\gamma \in C_0^\infty(\mathbb{R}_{t,x}^{1+d})$ equal to one on a neighborhood of the origin and supported in $|t, x| \leq 1$ and define

$$E(D_{t,x}) := (\zeta K)*, \quad \text{so,} \quad E - E_2 \in \text{Op}(S^{-\infty}(\mathbb{R}_{t,x}^{1+d})) .$$

With a new $R_1$ satisfying (3.2) one has

$$\tilde{L} E(D_{t,x}) - I = R_1(D_{t,x}) + R_2(D_{t,x}),$$

The Schwartz kernel of $E$ is supported at distance less than one from the diagonal. Since $\tilde{L}$ is local in time, the kernel of the $R_j$ is supported in $\{(t, x), (s, y) : |t - s| \leq 1\}$.

III. Define

$$q_1 := E(D_{t,x}) (I - \pi_{-\tau(D_{t,x})}) \chi(D_{t,x}) f .$$
Since \((I - \pi_{-\tau(D)\pi})\chi(D) f \in L^2(\mathbb{R}^{1+d})\) and \(E \in \text{Op}S^{-1}(\mathbb{R}^{1+d})\), one has \(q_1 \in H^1(\mathbb{R}^{1+d})\). The support property of the kernel of \(E\) implies that \(q_1\) is supported in \(-1 \leq t \leq T + 1\). More generally, for any \(\alpha\),

\[
D_{t,x}^\alpha q_1 = E(D_{t,x}) (I - \pi_{-\tau(D)\pi}) \chi(D) D_{t,x}^\alpha f \in H^1(\mathbb{R}^{1+d})\,.
\]  

(3.3)

The inclusions (3.3) imply, by a Sobolev embedding, that \(q_1 \in L^\infty(\mathbb{R}^{1+d})\). To complete the proof it suffices to show that \(q - q_1 \in L^\infty(\mathbb{R}^{1+d})\).

The function \(q_1\) satisfies

\[
\tilde{L} q_1 = (I - \pi_{-\tau(D)\pi}) \chi(D) f + g_1 + g_2\, , \quad g_j := R_j (D_{t,x}) (I - \pi_{-\tau(D)\pi}) \chi(D) f\, .
\]

The kernel support property of the \(R_j\) implies that \(g_j\) are supported in \(-1 \leq t \leq T + 1\). Define \(v_j\) to be the solutions of,

\[
\tilde{L} v_j = g_j\, , \quad g_j = 0 \text{ for } t \leq -1\, , \quad \text{so, } q = q_1 + v_1 + v_2\, .
\]  

(3.4)

It suffices to show that \(v_j \in L^\infty(\mathbb{R}^{1+d})\).

IV. The Cauchy problem for \(\tilde{L}\) has existence, uniqueness and estimates entirely analogous to those of \(L\). Solutions are explicit on the Fourier transform side and are justified and estimated using the important symbol estimate

\[
\sup_{t \in \mathbb{R}, \xi \in \mathbb{R}^d} \| e^{i t (A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1)} \|_{\text{Hom}(\mathbb{C}^k)} = \sup_{\xi \in \mathbb{R}^d} \| e^{i (A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1)} \|_{\text{Hom}(\mathbb{C}^k)} < \infty\, .
\]  

(3.5)

To prove (3.5), use (1.5) to write

\[
A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1 = (-\tau(\xi) + \chi(\xi) \xi_1) \pi_{-\tau(\xi)} + \sum_{\lambda \in \text{Spec } A(\xi) \setminus \{ -\tau(\xi) \}} \lambda \pi_\lambda(\xi)\, .
\]

With real valued \(a_\lambda(\xi)\) this is an expression of the form

\[
A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1 = \sum_{\lambda \in \text{Spec } A(\xi)} a_\lambda(\xi) \pi_\lambda(\xi)\, .
\]

Therefore

\[
e^{i (A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1)} = \sum_{\lambda \in \text{Spec } A(\xi)} e^{i a_\lambda(\xi) \pi_\lambda(\xi)}\, .
\]

Since \(|e^{i a_\lambda}| = 1\) this yields

\[
\| e^{i (A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1)} \|_{\mathbb{C}^k} \leq \sum_{\lambda \in \text{Spec } A(\xi)} \| \pi_\lambda(\xi) \|_{\mathbb{C}^k}\, .
\]

Condition B from the characterization of strongly hyperbolic operators implies (3.5).

V. Proof that \(v_1 \in L^\infty(\mathbb{R}^{1+d})\). Equation (3.5) implies that for all \(s\), solutions of \(\tilde{L} u = 0\) satisfy for all \(t \in \mathbb{R}\),

\[
\| u(t) \|_{H^s(\mathbb{R}^d)} \leq \sup_{\xi \in \mathbb{R}^d} \| e^{i (A(\xi) + \chi_1(\xi) \pi_{-\tau(\xi)} \xi_1)} \|_{\text{Hom}(\mathbb{C}^k)} \| u(0) \|_{H^s(\mathbb{R}^d)}\, .
\]
The source term \( g_1 \) satisfies
\[
g_1 \in \bigcap_{s} H^s(\mathbb{R}^{1+d}), \quad \text{supp } g_1 \subset \{-1 \leq t \leq T + 1\}.
\]
Therefore for all \( s \in \mathbb{R} \), \( g_1 \in L^1_{\text{compact}}(\mathbb{R}; H^s(\mathbb{R}^d)) \), Duhamel’s formula applied to (3.3) yields
\[
\|v_1(t)\|_{H^s} \leq \sup_{\xi \in \mathbb{R}^d} \left\| e^{i(\mathcal{A}(\xi) + \chi_1(\xi) \pi - \tau(\xi) \xi_1)} \right\|_{\text{Hom}(\mathbb{C}^k)} \int_{-1}^{T+1} \|g_1(\sigma)\|_{H^s(\mathbb{R}^d)} \, d\sigma
\]
\[
\leq \sup_{\xi \in \mathbb{R}^d} \left\| e^{i(\mathcal{A}(\xi) + \chi_1(\xi) \pi - \tau(\xi) \xi_1)} \right\|_{\text{Hom}(\mathbb{C}^k)} \int_{-1}^{T+1} \|g_1(\sigma)\|_{H^s(\mathbb{R}^d)} \, d\sigma.
\]
Therefore \( v_1 \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^d)) \). Taking \( s > d/2 \), yields \( v_1 \in L^\infty(\mathbb{R}^{1+d}) \).

VI. Proof that \( v_2 \in L^\infty(\mathbb{R}^{1+d}) \). The source term \( g_2 \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)) \) is supported in \(-1 \leq t \leq T + 1\), and is in the image of \( R_2(D_{t,x}) \). Therefore, its space time Fourier transform vanishes on \( \{|\tau| \leq \delta \} \times \mathcal{N}_1 \).

On \( \mathbb{R}^{1+d} \setminus (\{|\tau| \leq \delta \} \times \mathcal{N}_1) \), \( (\tau, \xi)^{-2N} \) belongs to \( S^{-2N} \). Therefore there is an element \( M_{-2N}(\tau, \xi) \in S^{-2N}(\mathbb{R}^{1+d}) \) so that \( M_{-2N} = (\tau, \xi)^{-N} \) on the support of \( R_2(\tau, \xi) \). Therefore on that support,
\[
M_{-2N}(\tau, \xi) \left(1 - \tau^2 - \sum_{j \geq 2} \xi_j^2 \right)^N = 1.
\]
The operator version of this identity yields,
\[
g_2 = R_2(D_{t,x}) \, M_{-2N}(D_{t,x}) \, (I - \pi_{-\tau(D_{x})}) \, \chi(D_{x}) \, (1 - \partial_t^2 - \sum_{j \geq 2} \partial_{x_j}^2)^N f \quad (3.6)
\]
In this use that
\[
(1 - \partial_t^2 - \sum_{j \geq 2} \partial_{x_j}^2)^N f \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d)), \quad \text{with support in } \quad 0 \leq t \leq T
\]
to find that
\[
(I - \pi_{-\tau(D_{x})}) \, \chi(D_{x}) \, (1 - \partial_t^2 - \sum_{j \geq 2} \partial_{x_j}^2)^N f \in L^2(\mathbb{R}^{1+d}).
\]
Finally \( R_2(D_{t,x}) M_{-2N}(D_{t,x}) \in \text{Op } S^{-2N}(\mathbb{R}^{1+d}) \) so (3.6) implies that \( g_2 \in H^{2N}(\mathbb{R}^{1+d}) \). In addition, \( g_2 \) is supported in \(-1 \leq t \leq T + 1 \) so \( g_2 \in L^1(\mathbb{R}; H^{2N}(\mathbb{R}^d)) \). The same Duhamel argument as in \( \mathcal{V} \) implies that \( v_2 \in L^\infty(\mathbb{R}^{1+d}) \). Therefore \( q = q_1 + v_1 + v_2 \in L^\infty(\mathbb{R}^{1+d}) \). This completes the proof of Proposition 3.1.

4 Scalar equation for \( \pi_{-\tau(D_{x})} \chi(D_{x}) \) \( u \)

4.1 Derivation of the equation

Lemma 4.1 With the notations of Proposition 2.7, the function \( w := \pi_{-\tau(D_{x})} \chi(D_{x}) u \) is characterized as the unique solution of the scalar pseudodifferential initial value problem
\[
\left( \partial_t - i \tau(D_{x}) \right) w = \pi_{-\tau(D_{x})} \chi(D_{x}) f, \quad w = 0 \quad \text{for } t < 0,
\]
whose spatial Fourier Transform has support in \( \mathcal{N} \).
Proof. Define \( v := \chi(D_x)u \). Then \( v \) is uniquely characterized by
\[
L v = \chi(D_x)f, \quad v = 0 \quad \text{for} \quad t < 0. \tag{4.2}
\]
For any \( t \) both the left and right hand side of (4.2) have Fourier transforms supported in \( \text{supp} \chi \subset \mathcal{N} \) so contained in the domain of analyticity of \( \pi_{r(\cdot)} \). Multiply (4.2) by \( \pi_{-r(D_x)} \) to find
\[
\pi_{-r(D_x)} \left( \partial_t + \sum_j A_j \partial_j \right) \chi(D_x)u = \pi_{-r(D_x)} \chi(D_x)f. \tag{4.3}
\]
The symbol of the operator \( \pi_{-r(D_x)} \left( \partial_t + \sum_j A_j \partial_j \right) \) is \( \pi_{-r(\xi)} \left( \partial_t + \sum_j A_j i\xi_j \right) \). Equation (1.6) implies that
\[
\pi_{-r(\xi)} \left( \partial_t + \sum_j A_j i\xi_j \right) = \left( \partial_t + \sum_j A_j i\xi_j \right) \pi_{-r(\xi)} = \left( \partial_t - i\tau(\xi) \right) \pi_{-r(\xi)}. \tag{x}
\]
Therefore
\[
\pi_{-r(D_x)} \left( \partial_t + \sum_j A_j \partial_j \right) = \left( \partial_t - i\tau(D_x) \right) \pi_{-r(D_x)}. \tag{4.4}
\]
Injecting this is (4.3) yields (4.1) completing the proof. \( \blacksquare \)

4.2 Two simplifications of (4.1)

Definition 4.2 Denote by \( \lambda(\xi) \) the eigenvalue \( -\tau(\xi) \) of the matrix \( A(\xi) \) real analytic on \( \mathcal{N} \) from Hypothesis 1.4. The scalar operator in (4.1) is then \( \partial_t + i\lambda(D_x) \).

4.2.1 Transmission condition and Duhamel

The symbol \( \pi_{\lambda(\xi)} \chi(\xi) \) is homogeneous of degree zero and even in \( \xi \). This implies that satisfies the transmission condition of Boutet de Monvel (5,11) guaranteeing that \( \pi_{\lambda(D_x)} \chi(D_x) \) maps piecewise smooth functions to themselves. The condition requires that the Fourier Transform of the distribution \( \pi_{\lambda(\xi_0,0,...,0)} \chi(\xi_0,0,...,0) \) on \( \mathbb{R}^{\xi_0} \) has smooth extent to each closed half line \( \pi_{\lambda(\xi_0,0,...,0)} \delta(x_1) \) on each half line this extends to the smooth function equal to zero.

Choose \( \zeta \in C_c(\mathbb{R}^d) \) with \( \zeta(\xi) = 0 \) on \( |\xi| < 1/2 \) and \( \zeta(\xi) = 1 \) on \( |\xi| > 1 \). Write
\[
\pi_{-r(D_x)} \chi(D_x)f = \zeta(D_x) \pi_{-r(D_x)} \chi(D_x)f + (I - \zeta(D_x)) \pi_{-r(D_x)} \chi(D_x)f := F_1 + F_2.
\]
The \( F_j \) are supported in \( 0 \leq t \leq T \) and have spatial Fourier transform supported in \( \text{supp} \chi \). Denote by \( w_j \) the solutions vanishing for \( t \leq 0 \) with source terms \( F_j \) The source \( F_2 \) has compactly supported Fourier Transform and \( F_2 \in \cap_{s} L^\infty(\mathbb{R}; H^s(\mathbb{R}^d)) \). Therefore, \( w_2 \in \cap_{s} L^\infty(\mathbb{R}; H^s(\mathbb{R}^d)) \subset L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^d)) = L^\infty(\mathbb{R}^{1+d}) \). It remains to show that \( w_1 \in L^\infty(\mathbb{R}^{1+d}) \).

The equation for \( w_1 \) is
\[
(\partial_t + i\lambda(D_x))w_1 = F_1, \quad w_1 = 0 \quad \text{for} \quad t < 0
\]
with \( F_1 \) piecewise smooth, rapidly decreasing, supported in \( 0 \leq t \leq T \). Duhamel’s formula implies that to show that \( w_1 \in L^\infty(\mathbb{R}^{1+d}) \) it suffices to prove Proposition 4.3.
Proposition 4.3 For any $g(x)$ that is piecewise smooth with singularities in $\{x_1 = 0\}$, and rapidly decreasing, the solution of the initial value problem with $\text{supp} \tilde{v}(t) \subset \text{supp} \chi$,

$$ (\partial_t + i\lambda(D_x))v = 0, \quad v(0) = \chi(D_x)g $$

satisfies $v \in L^\infty(\mathbb{R}^{1+d})$.

4.2.2 Simpler right hand side with the same jump

Lemma 4.4 Suppose that $\phi(\xi_1)$ is smooth, even, identically equal to one for $|\xi_1| > 2$, and identically equal to zero for $|\xi_1| \leq 1$. Then Proposition 4.3 holds provided that for all $a \in S(\mathbb{R}^{d-1})$,

$$ P.V. \int \frac{\chi(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{i(x\xi + \lambda(\xi)t)} d\xi \in L^\infty(\mathbb{R}^{1+d}) . $$

(4.5)

Proof of Lemma 4.4 Denote by $a(x') \in S(\mathbb{R}^{d-1})$ the jump of $g$ from left to right at $x'$. Then the function

$$ \bar{g} := e^{-x_1} a(x') \mathbf{1}_{x_1 > 0} $$

has the same jump as $g$. It follows that for all $j \leq 1$ and $\alpha$

$$ \partial_x^\alpha \partial_x^{j}(g - \bar{g}) \in L^2(\mathbb{R}^d) . $$

The solution $v$ of (4.4) is the sum of the solution with initial data $\chi(D_x) \bar{g}$ and that with data $\chi(D_x)(g - \bar{g})$. By inhomogeneous Sobolev the latter solution belongs to $L^\infty(\mathbb{R}^{1+d})$. The former is equal to

$$ P.V. \int \frac{\chi(\xi) \phi(\xi_1) a(\xi')}{1 + i\xi_1} e^{i(x\xi + \lambda(\xi)t)} d\xi \quad . $$

(4.6)

The solution given by (4.6) differs from

$$ P.V. \int \frac{\chi(\xi) \phi(\xi_1) a(\xi')}{1 + i\xi_1} e^{i(x\xi + \lambda(\xi)t)} d\xi $$

(4.7)

by the inverse Fourier transform of functions uniformly in $L^1(\mathbb{R}^d)$. In particular by an element of $L^\infty(\mathbb{R}^{1+d})$. It suffices to show that the solution given by formula (4.7) belongs to $L^\infty(\mathbb{R}^{1+d})$. Similarly,

$$ P.V. \int \frac{\chi(\xi) \phi(\xi_1) a(\xi')}{1 + i\xi_1} e^{i(x\xi + \lambda(\xi)t)} d\xi - \frac{1}{i} P.V. \int \frac{\chi(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{i(x\xi + \lambda(\xi)t)} d\xi $$

has spatial Fourier transform belonging to $L^1(\mathbb{R}^d)$ uniformly in $t$. This completes the proof of Lemma 4.4.

The next three sections are devoted to proving (4.5).
5 Nonstationary phase bounds

Definition 5.1 On \( \mathbb{R}^{1+d}_x \times \mathcal{N} \) define the real valued phase
\[
\psi(t, x, \xi) := x\xi + t\lambda(\xi).
\] (5.1)
A point \( t, x, \xi \) with \( 0 \neq \xi \) and \( \xi' = 0 \) is stationary if
\[
0 = \nabla_\xi \psi(t, x, \xi) = x + t\lambda(\xi).
\] (5.2)
The stationary points with \( \xi' = 0 \) are exactly those so that \( x = vt \). First treat the easier case of nonstationary points with \( \xi' = 0 \).

Hypothesis 5.1 From here on \( \phi \) and \( a \) are as in Lemma 4.4.

Proposition 5.2 Suppose that \( t, x, \xi \in (\mathbb{R}^{1+d}_x \setminus 0) \times (\mathbb{R}^d \setminus 0) \) with \( \xi' = 0 \) and that \( \nabla_\xi \psi(t, x, \xi) \neq 0 \). Then there are open conic neighborhoods \( \mathcal{G} \) of \( \xi \) in \( \mathbb{R}^d \setminus 0 \) and \( \mathcal{M} \) of \( t, x \) in \( \mathbb{R}^{1+d} \setminus 0 \) so that if \( \gamma(\xi) \) smooth and homogenous of degree one and whose support intersects \( S^{d-1} \) in a compact subset of \( \mathcal{G} \). Then for all \( \alpha \in \mathbb{N}^{1+d} \) and \( n \in \mathbb{N} \),
\[
P.V. \int \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{i(x\xi + t\lambda(\xi))} \, d\xi \in \langle t, x \rangle^{-n} L^\infty(\mathcal{M}).
\] (5.3)

Proof. For \(|t| \leq 1 \) the estimate is easy. Choose \( \mathcal{G} \) and \( \mathcal{M} \) so that \( \lambda \) is smooth on \( \mathcal{G} \) and so that there is an \( \eta > 0 \) so that
\[
|\nabla \psi| > \eta|t, x| \quad \text{on} \quad \mathcal{M} \times \mathcal{G}.
\]

I. Define on \( \mathcal{M} \times \mathcal{G} \)
\[
L := -\frac{1}{|\nabla_\xi \psi|^2} \nabla_\xi \psi \cdot D \xi,
\]
so,
\[
L e^{i\psi} = e^{i\psi}.
\]
Using the smoothness of \( \lambda \), denote by \( L^\dagger \) the transposed operator
\[
L^\dagger w := D_\xi \left( \frac{\nabla_\xi \psi}{|\nabla_\xi \psi|^2} w \right).
\]
The coefficients of \( L^\dagger \) are \( O(|t, x, \xi|^{-1}) \) on the support \( \phi(\xi_1) \). An integration by parts yields
\[
P.V. \int \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} L e^{i\psi} \, d\xi = P.V. \int e^{i\psi} L^\dagger \left( \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} \right) \, d\xi
\]
The right hand integral has integrand with \( L^1(\mathbb{R}^d_\xi) \) norm that is \( O(|t, x|^{-1}) \). Therefore
\[
P.V. \int \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{i\psi(\xi)} \frac{1}{\xi_1} \, d\xi \in |t, x|^{-1} L^\infty(\mathcal{M}).
\]

II. Repeated integration by parts yields for \( n \geq 1 \),
\[
P.V. \int \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} L e^{i\psi} \, d\xi = \int e^{i\psi} \left( L^\dagger \right)^n \left( \frac{\gamma(\xi) \phi(\xi_1) a(\xi')}{\xi_1} \right) \, d\xi.
\]
The operator \( (L^\dagger)^n \) in this expression has coefficients that are \( O(|t, x, \xi|^{-n}) \) on the support of the integrand. The right hand integral has integrand with \( L^1(\mathbb{R}^d_\xi) \) norm that is \( O(|t, x|^{-1}) \) implying (5.3) for \(|t| \geq 1\).
Remark 5.1 In addition, for all $\alpha \in \mathbb{N}^{1+d}$ and $n \in \mathbb{N}$,
\[ D^\alpha_{t,x} \left( P.V. \int \frac{\gamma(\xi) \phi(\xi_1)}{\xi_1} a(\xi') e^{i\psi(\xi)} d\xi \right) \in \langle t, x \rangle^{-n} L^\infty(M). \]
This is not needed below. The omitted proof follows the strategy above.

6 Paraxial approximation for (4.5)

Use the homogeneity $\lambda(\xi_1, \xi') = \xi_1 \lambda(1, \xi'/\xi_1)$. Taylor expansion about $\xi' = 0$ yields
\[ \lambda(1, \xi') = \lambda(1, 0) + \xi' \nabla_\xi' \lambda(1, 0) + Q(\xi', \xi') + \text{h.o.t}, \]
with
\[ Q(\xi', \xi') := \frac{1}{2} \sum_{2 \leq i, j \leq d-1} \frac{\partial^2 \lambda(1, 0)}{\partial \xi_i' \partial \xi_j'} \xi_i \xi_j. \]
Therefore,
\[ \lambda(1, \xi'/\xi_1) = \lambda(1, 0) + (\xi'/\xi_1) \nabla_\xi' \lambda(1, 0) + Q(\xi'/\xi_1, \xi'/\xi_1) + O(|\xi'|^3/\xi_1^3). \]

Multiply by $\xi_1$. Using $\lambda(1, 0) = \partial \lambda(1, 0)/\partial \xi_1$ in the third line yields,
\[ \lambda(\xi_1, \xi') = \xi_1 \lambda(1, \xi'/\xi_1) \\
= \xi_1 \lambda(1, 0) + \xi' \nabla_\xi' \lambda(1, 0) + Q(\xi', \xi')/\xi_1 \xi_1 + O(|\xi'|^3/\xi_1^3) \]
\[ = \xi \nabla_\xi \lambda(1, 0) + Q(\xi', \xi')/\xi_1 + O(|\xi'|^3/\xi_1^3) \]
\[ = v \cdot \xi + Q(\xi', \xi')/\xi_1 + O(|\xi'|^3/\xi_1^3). \]

Injecting in the definition of the solution $u$ yields the paraxial approximation,
\[ u_{\text{paraxial}} := P.V. \int \frac{\chi(\xi)}{\xi_1} \phi(\xi_1) a(\xi') e^{i\xi \cdot \xi} e^{it(\xi \cdot v + Q(\xi', \xi')/\xi_1)} d\xi. \]

Remark 6.1 The paraxial approximation satisfies the differential equation
\[ \partial_{\xi_1} \left( \partial_t + v \cdot \partial_x \right) u_{\text{paraxial}} = Q(D'_{x}, D'_{x}) u_{\text{paraxial}}. \]

Equation (6.3) is classical in diffractive geometric optics with sources whose spectrum is broad. For example, [7], [8], [9], [10]. If $\partial_{\xi_1}$ were replaced by $i$ this would be a Schrödinger equation. The operators $\partial_{\xi_1}$ and $i$ are both antiselfadjoint. The two equations share many properties.

Theorem 6.1 Suppose that $0 \neq \xi, \xi' = 0$, and $t, x, \xi$ is stationary. Then the paraxial approximation defined by (6.2) satisfies $u_{\text{paraxial}} \in L^\infty(\mathbb{R}^{1+d}_{t,x}).$

Proof of Theorem 6.1 The factor $e^{it\cdot v \xi}$ induces a translation in $x$ by $vt$. Thus it suffices to consider the integral with $v = 0$. 

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The strategy is to integrate $d\xi_1$ with $\xi'$ fixed. This yields integrals,

$$\lim_{M \to \infty} \int a(\xi') \left( \int_{-M}^{M} \chi(\xi_1, \xi') \phi(\xi_1) e^{i(x_1 \xi_1 + \Lambda/\xi_1)} \right) \frac{1}{\xi_1} d\xi', \quad \Lambda := tQ(\xi', \xi'). \quad (6.4)$$

Parity in $\xi_1$ shows that the inner integral is equal to

$$\int_{0}^{M} \chi(\xi_1, \xi') \phi(\xi_1) \frac{\sin(x_1 \xi_1 + \Lambda/\xi_1)}{\xi_1} d\xi_1. \quad (6.5)$$

Thanks to the rapid decay of $a(\xi')$, to bound the quantity in $(6.4)$ it suffices to show that there is a constant $C$ so that the integral in $(6.5)$ is bounded in absolute value by $C(\xi)$ with constant independent of $\Lambda$.

The next Lemma is the heart of the proof of Theorem 6.1. It is proved with a $\Lambda$-dependent high/low frequency decomposition. Van der Corput’s Lemma treats the low frequency part.

**Lemma 6.2** For all $0 < k \in \mathbb{R}$ there is a constant $C(k)$ so that for all real $x, \Lambda$ and all bounded intervals $I \subset ]0, \infty[$ one has

$$\left| \int_{I} \frac{\sin(x \eta + \Lambda/\eta^k)}{\eta} d\eta \right| \leq C(k).$$

**Proof of Lemma 6.2** By continuity in $x, \Lambda$ it suffices to treat the case where $x \neq 0$ and $\Lambda \neq 0$. Changing the sign of both $x$ and $\Lambda$ multiplies the integral by $-1$ so it suffices to consider $\Lambda > 0$.

The domain of integration is divided into two intervals $I \cap ]0, \Lambda^{1/k}[ \text{ and } I \cap ]\Lambda^{1/k}, \infty[$. The first and more interesting is empty when $0 < \Lambda < 1$.

**I. Estimate for $I \cap ]0, \Lambda^{1/k}[$.** Change variable to $\xi := e^u$ so,

$$u = \ln \xi, \quad \xi = e^u, \quad \frac{d\xi}{\xi} = du.$$ 

Then,

$$x\xi + \Lambda/\xi^k = \psi(u), \quad \text{with} \quad \psi(u) := xe^u + \Lambda e^{-ku}.$$ 

The integral is transformed to

$$\int_{J} \sin \psi(u) \ du, \quad J := \ln I \subset ]-\infty, \ln(\Lambda^{1/k})[.$$ 

With $' = d/du$,

$$\psi' = xe^u - k\Lambda e^{-ku}, \quad \psi'' = xe^u + k^2\Lambda e^{-ku}, \quad \text{and,} \quad \psi''' = xe^u - k^3\Lambda e^{-ku}.$$ 

Since $\Lambda > 0$, the function $u \mapsto \Lambda e^{-ku}$ is decreasing on $]-\infty, \infty[$. On $J$ it is no smaller than its value at $u = \ln(\Lambda^{1/k})$. At $u = \ln(\Lambda^{1/k})$, its value is equal to $1$.

If $x$ and $\Lambda$ have the same sign, the summands yielding $\psi'''$ are both positive. Thus $\psi'''$ is bounded below by the second summand so on $J$, $\psi''' \geq k^2$. Van der Corput’s Lemma (see [16]) bounds the integral.
If $x$ and $\Lambda$ have opposite signs the two summands yielding $\psi'''$ are both negative. The second summand is $\leq -k^3$ on $J$. Therefore $\psi''' \leq -k^3$ on $J$. Van der Corput’s Lemma bounds the integral. Alternatively, $\psi' \leq -k$ and $\psi''$ is of one sign. Therefore $\psi'$ is monotone in $J$ and again Van der Corput’s Lemma bounds the integral.

II. Estimate for $I \cap ]\Lambda^{1/k}, \infty[$. First show that

$$\int_{I \cap ]\Lambda^{1/k}, \infty[} \frac{\sin x\xi}{\xi} \, d\xi = x \int_{I \cap ]\Lambda^{1/k}, \infty[} \frac{\sin x\xi}{x\xi} \, d\xi$$

is bounded. Since changing the sign of $x$ multiplies the integral by $-1$ it suffices to treat $x > 0$. In that case, the change of variables $v = x\xi$ with $d\xi = x^{-1}dv$ transforms the integral to

$$\int_K \frac{\sin v}{v} \, dv, \quad K := x \left( I \cap ]\Lambda^{1/k}, \infty[ \right) \subset [0, \infty[.$$

The function $v^{-1}\sin v$ in $v > 0$ consists of a sequence of hills of alternating signs and decreasing areas. The hill of largest area has area equal to $\int_0^\pi \frac{\sin v}{v} \, dv$.

The integral over $K$ usually starts and ends with partial hills both bounded by the largest area. The middle is then an alternating decreasing series of hills whose sum is also bounded by the largest. Therefore the absolute value of the integral over $K$ can be no larger than $3 \int_0^\pi \frac{\sin v}{v} \, dv$.

It then suffices to bound

$$\left| \int_{I \cap ]\Lambda^{1/k}, \infty[} \frac{\sin(x\xi + \Lambda/\xi^k)}{\xi} - \frac{\sin x\xi}{\xi} \, d\xi \right|$$

Passing the absolute value inside the integral and using the fact that the derivative of $\sin$ is never larger than one yields the bound

$$\int_{I \cap ]\Lambda^{1/k}, \infty[} \frac{1}{\xi} \frac{\Lambda}{\xi^k} \, d\xi \leq \int_\Lambda \frac{\Lambda}{\xi^{k+1}} \, d\xi = \frac{\Lambda}{-k} \xi^{-k} \bigg|_{\Lambda^{1/k}}^{\infty} = \frac{1}{k}.$$

This completes the proof of Lemma 6.2.

End of Proof of Theorem 6.1. It suffices to prove the estimate in italics after (6.5). Define $f_\Lambda \in C^1([0, \infty[)$ by

$$f'_\Lambda(\eta) = \frac{\sin(x\eta + \Lambda/\eta)}{\eta}, \quad f_\Lambda(1) = 0.$$

The case $k = 1$ of Lemma 6.2 implies that $f_\Lambda \in L^\infty([0, \infty[)$ with bound independent of $\Lambda$. Define

$$g(\xi) = g(\xi_1, \xi') := \chi(\xi) \phi(\xi_1), \quad h(\xi) := \partial g/\partial \xi_1.$$

For $x_1 \neq 0$, the change of variable

$$\eta = x_1 \xi_1, \quad \xi_1 = \frac{\eta}{x_1}, \quad d\xi_1 = \frac{d\eta}{x_1}$$

yields

$$\int_0^M \chi(\xi_1, \xi') \phi(\xi_1) \frac{\sin(x_1\xi_1 + \Lambda/\xi_1)}{\xi_1} \, d\xi_1 = \int_0^{Mx_1} \chi(\eta/x_1, \xi') \phi(\eta/x_1) \frac{\sin(x_1 + \Lambda/\eta)}{\eta} \, d\eta$$

$$= \int_0^{Mx_1} g(\eta/x_1, \xi') f'_\Lambda(\eta) \, d\eta.$$
An integration by parts yields

\[ = h(M, \xi') f_\Lambda(Mx_1) - \int_0^{Mx_1} \frac{d}{dn} \left( g(\eta/x_1, \xi') \right) f_\Lambda(\eta) \ d\eta. \]

The first summand is bounded independent of \( M, \Lambda, x_1, \xi' \). The second summand is equal to

\[- \int_0^{Mx_1} \frac{1}{x_1} \left( \frac{d}{d\xi_1} g(\xi_1, \xi') \right) \big|_{\xi_1 = \eta/x_1} f_\Lambda(\eta) \ d\eta = - \int_0^{Mx_1} \frac{1}{x_1} h(\eta/x_1, \xi') f_\Lambda(\eta) \ d\eta. \]

The absolute value of this quantity is bounded by

\[ \|f_\Lambda\|_{L^\infty(\mathbb{R}_n)} \left\| \frac{1}{x_1} h(\eta/x_1, \xi') \right\|_{L^1(\mathbb{R}_n)}. \]

Changing back to the variable \( \xi_1 \) shows that

\[ \left\| \frac{1}{x_1} h(\eta/x_1, \xi') \right\|_{L^1(\mathbb{R}_n)} = \int_0^\infty \left| h(\xi_1, \xi') \right| \ d\xi_1. \]

It suffices to show that

\[ \int_0^\infty \left| h(\xi_1, \xi') \right| \ d\xi_1 \lesssim \langle \xi' \rangle. \] (6.6)

The function \( h \) is continuous and uniformly bounded on \( \mathbb{R}^d \). Therefore it suffices to prove the same bound for the integral with lower limit equal to 2. In the range \( 2 \leq \xi_1 < \infty \), the function \( \phi \) is constant. Therefore in \( |\xi_1| \geq 2 \), \( h \) is homogeneous of degree minus one. In addition in this range \( h(\xi_1, 0) = 0 \). Therefore \( h = h(\xi_1, \xi') - h(\xi_1, 0) \). The gradient of \( h \) is homogeneous of degree minus two. Estimating the increment in \( h \) by a bound for the derivative times the change in the argument yields \( |h| \lesssim |\xi'|/|\xi|^2 \). This implies that

\[ \left| \int_0^\infty \left| h(\xi_1, \xi') \right| \ d\xi_1 \lesssim |\xi'|. \]

With the earlier estimate this yields (6.6). This completes the proof of Theorem 6.1.

\[ \]

7 Bound for the stationary contributions

It is here that the inequality of stationary phase is required for test functions with fractional derivatives. If one used only integer derivatives it would lead to an unnatural lower bound on the dimension. The sharp limit point inequality of Appendix B is more than sufficient.

**Proposition 7.1** Suppose that \( \xi' = 0 \) and that \( \lambda \) is stationary at \( t, x, \xi \). Then there is a conic open neighborhood of \( \mathcal{G} \) of \( \xi \) and a conic neighborhood \( \mathcal{M} \) of \( t, x \) so that if \( \beta \in C^\infty(\mathbb{R}^d \setminus 0) \) is homogeneous of degree whose support in \( \xi \neq 0 \) is contained in \( \mathcal{G} \),

\[ P.V. \int \frac{\beta(\xi') \phi(\xi_1)}{\xi_1} a(\xi') e^{ix\xi} e^{it\lambda(\xi)} \ d\xi \in L^\infty(\mathcal{M}). \]

In addition the left hand side differs from its paraxial approximation by a term that decays algebraically. That is, there is a \( \mu > 0 \) so that

\[ \text{PV} \int \frac{\beta(\xi') \phi(\xi_1)}{\xi_1} e^{i(\xi_1 + t\lambda(\xi))} d\xi - \text{PV} \int \frac{\beta(\xi) \phi(\xi_1)}{\xi_1} e^{i(\xi_1 + t(x+Q(\xi', \xi)/\xi_1))} d\xi \in t^{-\mu} L^\infty(\mathcal{M}). \]
Theorem 6.1 shows that the high frequency bound holds for $G$ and $\lambda$. Therefore, I. High frequency bound, only used in the low frequency bound. For $\mu > 0$ to be chosen, write the integral as $P.V. \int \beta(\xi) e^{ix\xi} e^{it(\nu |\xi| + Q(\xi'))/\xi_1} \phi(\xi) a(\xi') \frac{1}{\xi_1} d\xi \in L^\infty(\mathbb{R}^{d+1})$.

The difference of this integral and the desired integral is bounded above by $C \int \frac{1}{|\xi_1|} \frac{t^3}{|\xi_1|^2} d\xi$. For $\mu > 0$ to be chosen, write the integral as $t^{-\mu} \int \frac{1}{|\xi_1|} \frac{t^{1+\mu}}{\xi_1^2} |\xi_1|^3 d\xi$. In the high frequency region, $t \leq |\xi_1|^{1/(\delta+1/2)}$, $\frac{t^{1+\mu}}{\xi_1^2} \leq |\xi_1|^{(1+\mu)/(\delta+1/2)-2}$. Choose $\mu > 0$ so that $(1 + \mu)/(\delta + 1/2) < 2$. Define $0 < \nu := 2 - (1 + \mu)/(\delta + 1/2)$. Then (7.1) is bounded by $t^{-\mu}$ times $C \int \frac{1}{|\xi_1|} \frac{|\xi_1|^3}{|\xi_1|^\nu} d\xi$. Since $a(\xi') |\xi'|^3 \in L^1(\mathbb{R}^{d-1}_{\xi'})$ and $\phi(\xi) |\xi_1|^{-1-\nu} \in L^1(\mathbb{R}_{\xi_1})$, this integral is absolutely convergent. This completes the high frequency bound.

II. Low frequency bound, $|\xi_1| \leq t^{1/2+\delta}$, $0 < \delta < 1/2$.

For $t$ fixed, the domain of integration is compact and the integrand is smooth on that domain. No principal value is needed in

$$\int_{|\xi_1| \leq t^{1/2+\delta}} \frac{\beta(\xi) \phi(\xi) a(\xi')}{\xi_1} e^{ix\xi} e^{it\lambda(\xi)} d\xi.$$ (7.2)

Introduce polar coordinates

$$\xi = r\omega, \quad |\omega| = 1.$$ (7.3)
Also introduce

$$w := x/t \quad \text{so,} \quad x\xi = trw\omega. \quad (7.4)$$

Stationarity implies that at $t, x, \xi$, $w = v$ is the group velocity at $\xi$.

The region of integration for the low frequency region lies in $0 < c < r < t^{1/2+\delta}$. The homogeneity of $x\xi + t\lambda(\xi)$ in $\xi$ implies that the integral (7.2) is equal to

$$\int_c^{t^{1/2+\delta}} \left( \int_{S^d-1} e^{itr(\omega\omega + \lambda(\omega))} \frac{\beta(\omega \omega) \phi(r\omega_1) a(r\omega')}{r\omega} d\omega \right) r^{d-1} dr, \quad \xi = r\omega. \quad (7.5)$$

For $w = v$ the phase $S^d-1 \ni \omega \mapsto w\omega + \lambda(\omega)$ is stationary at $\omega = (1, 0, \ldots, 0)$. Choose $\mathcal{G}$ so small that $\lambda$ is smooth on a neighborhood of the closure of $\mathcal{G} \cap S^d-1$, and there is an $\eta > 0$ so that on $\mathcal{G} \cap S^d-1$, $|\nabla_\xi (w\omega + \lambda(\omega))| \geq \eta|\omega'|$.

Since rank $\lambda_\xi(1, 0) = d - 1$, that stationary point is nondegenerate on $S^d-1$. The implicit function theorem implies that for $w$ in an open neighborhood $\Omega$ of $v/\|v\|$ there is a unique nondegenerate stationary point in $S^d-1$ that lies close to $(1, 0, \ldots, 0)$. Choose $\mathcal{M}$ so small that for $t, x \in \mathcal{M}$ this critical point is the only one in $\mathcal{G}$ and lie in compact subset of $\mathcal{G} \cap S^d-1$.

For $w$ in $\mathcal{M}$ and $r \in [c_1, c_2 t^{1/2+\delta}]$, the integral $d\omega$ is a stationary phase integral with unique nondegenerate stationary point that is close to $(1, 0, \ldots, 0)$. With the interpolation spaces $Y^m$ defined in Appendix B, Theorem 3.4 yields the bound uniform in $w$,

$$\left| \int_{S^d-1} e^{itr(\omega\omega + \lambda(\omega))} \frac{1}{r\omega_1} \phi(r\omega_1) a(r\omega') d\omega \right| \lesssim |tr|^{-(d-1)/2} \ln(1 + |tr|) \left\| \frac{\beta(\omega \omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{(L^\infty \cap Y^{(d-1)/2})(S^d-1)}. \quad (7.6)$$

The sup norm satisfies,

$$\left\| \frac{\beta(\omega \omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{L^\infty(S^d-1)} \lesssim r^{-1}. \quad (7.6)$$

$L^1(S^d-1)$ norms are smaller. First there is the scaling by $r$ of all $d - 1$ variables that yields,

$$\left\| \frac{\beta(\omega \omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{L^1(S^d-1)} = r^{-(d-1)} \left\| \frac{\beta(\zeta \zeta) \phi(\zeta') a(\zeta')}{\zeta_1} \right\|_{L^1(r S^d-1)}. \quad (7.7)$$

Recall that $\zeta = (\zeta_1, \zeta')$. Therefore on $r S^d-1$ with $r$ bounded away from zero, there is a constant $c > 0$ so that

$$c \dist(\zeta, (r, 0, 0)) \leq |\zeta'| \leq c^{-1} \dist(\zeta, (r, 0, 0)).$$

The rapid decay of $a$ yields for $\zeta \in r S^d-1, |a(\zeta')| \lesssim \dist(\zeta, (r, 0, 0))^{-N}$ for all $N$. Therefore the norm on the right of (7.7) is equal to

$$\int_{r \Gamma} \left| \frac{\beta(\zeta \zeta) \phi(\zeta') a(\zeta')}{\zeta_1} \right| d\sigma \lesssim \int_{r \Gamma} \frac{\dist(\zeta, \mathbb{R}(1, 0, 0))^{-N} \dist(\zeta, \mathbb{R}(1, 0, 0))^{-N}}{r} d\sigma \lesssim r^{-1}. \quad (7.7)$$

Equation (7.7) then yields

$$\left\| \frac{\beta(\omega \omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{L^1(S^d-1)} \lesssim r^{-d}. \quad (7.7)$$
The $r$ in $\xi = r\omega$ appears as a prefactor when one differentiates with respect to $\omega$ yielding

$$\left\| \frac{\partial^\alpha}{\partial \omega^\beta} \frac{\phi(r\omega) a(r\omega')}{r\omega_1} \right\|_{L^1(S^{d-1})} \lesssim r^{-d} r^{\left\vert \alpha \right\vert}. \quad (7.8)$$

It follows that when $d$ is odd so $(d-1)/2$ is an integer,

$$\left\| \frac{\beta(r\omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{Y^{(d-1)/2}(S^{d-1})} \lesssim r^{-d} r^{(d-1)/2} = r^{-(d+1)/2}. \quad (7.9)$$

Equation (7.9) then holds by interpolation when $d$ is even.

The $L^\infty$ contribution to the norm in $L^\infty \cap Y^{(d-1)/2}$ is dominant yielding

$$\left\| \frac{\beta(r\omega) \phi(r\omega_1) a(r\omega')}{r\omega_1} \right\|_{(L^\infty \cap Y^{(d-1)/2})(S^{d-1})} \lesssim r^{-1}. \quad (7.10)$$

Therefore, the absolute value of the integral (7.5) is bounded above by

$$\lesssim \int_c^{t^{1/2+\delta}} |tr|^{-(d-1)/2} \ln(1 + |tr|) r^{-1} r^{d-1} dr$$

$$\quad = t^{-(d-1)/2} \int_c^{t^{1/2+\delta}} \ln(1 + |tr|) r^{-1} r^{(d-1)/2} dr.$$

Estimate

$$\ln(1 + |tr|) \lesssim \ln \left(1 + t \left( t^{1/2+\delta} \right) \right) \lesssim \ln(1 + |t|).$$

The absolute value of the integral (7.5) is therefore

$$\lesssim t^{-(d-1)/2} \ln(1 + |t|) \int_c^{t^{1/2+\delta}} r^{-1} r^{(d-1)/2} dr \lesssim t^{-(d-1)/2} \ln(1 + |t|) \left( t^{1/2+\delta} \right)^{(d-1)/2}.$$

When $\delta < 1/2$ this tends to zero as fast as a negative power of $t$ as $t \to \infty$. This completes the proof of Proposition 7.1.

8 Proofs of Proposition 4.3, Theorem 1.3, and Corollary 1.4

This section combines the preceding results to prove the main Theorems.

**Proof of Proposition 4.3.** Thanks to Lemma 4.4 it suffices to prove (4.5).

1. If $\xi = (1, 0, \ldots, 0)$ and $\zeta \in \mathbb{R}^{1+d}$ is a unit vector, then there is an open neighborhood $U \subset S^d$ of $\zeta$ and $V \subset S^{d-1}$ of $\xi$ so that if $\beta \in C^\infty(\mathbb{R}^d \setminus 0)$ is homogeneous of degree zero so that $S^{d-1} \cap \text{supp } \beta \subset V$ and $\mathcal{M} \subset \mathbb{R}^{1+d}$ is the open cone on $U$ then

$$\text{P.V.} \int \frac{\beta(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{ix\xi} e^{it\lambda(\xi)} d\xi \in L^\infty(\mathcal{M}_{\xi_1}).$$

If $(\zeta, \xi)$ is nonstationary, the result follows from Proposition 5.2. If $(\zeta, \xi)$ is stationary, the result follows from Proposition 7.1. This completes the proof of 1.
II. For any $\zeta \in S^d$, choose $U_\zeta \subset S^d$ and $V_\zeta \subset S^{d-1}$ and $M_\zeta$ as in I. Choose a finite subcover $S^d \subset \bigcup_{k=1}^K U_\zeta$. Let $W_1 := \cap_{k=1}^K V_\zeta$ and define $\mathcal{M}$ to be the cone on $W_1$, a conic neighborhood of $\xi' = 0$. Choose an open $W_2 \subset S^d$ so that $W_2 \subset \{\xi' \neq 0\}$ and $W_1, W_2$ cover $S^{d-1}$. Choose a smooth partition of unity $1 = \psi_1 + \psi_2$ on $S^{d-1}$ with supp $\psi_j \subset W_j$. Define $\gamma_j(\xi) := \psi_j(\xi/|\xi|)$. The result from I implies that

$$P.V. \int \frac{\chi(\xi) \gamma_1(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{ix\xi} e^{it\lambda(\xi)} d\xi \in L^\infty(\mathcal{M}_{k}), \quad 1 \leq k \leq K. \quad (8.1)$$

Since the $M_\zeta$ cover $\mathbb{R}_+^{1+d} \setminus 0$ this implies that

$$P.V. \int \frac{\chi(\xi) \gamma_1(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{ix\xi} e^{it\lambda(\xi)} d\xi \in L^\infty(\mathbb{R}_+^{1+d}). \quad (8.2)$$

Proposition 5.2 implies that

$$P.V. \int \frac{\chi(\xi) \gamma_2(\xi) \phi(\xi_1) a(\xi')}{\xi_1} e^{ix\xi} e^{it\lambda(\xi)} d\xi \in L^\infty(\mathbb{R}_+^{1+d}). \quad (8.3)$$

Since $\gamma_1 + \gamma_2 = 1$, adding (8.1) and (8.3) proves (1.5).

**Proof of Theorem 1.3.** Begin with (1.4). That $(I - \chi(D_x)) u \in L^\infty(\mathbb{R}_+^{1+d})$ is proved in Proposition 2.1.

The remaining part $\chi(D_x) u$ is decomposed in (1.7). That the summand $(I - \pi_{-\tau(D_x)}) \chi(D_x) f$ belongs to $L^\infty(\mathbb{R}_+^{1+d})$ is proved in Proposition 3.1.

An initial value problem satisfied by the other summand $\pi_{-\tau(D_x)} \chi(D_x) u$ is derived in Lemma 4.1. In Subsection 4.2.1 it is proved that the boundedness of that summand is a consequence of Proposition 4.3 that has just been proved. This completes the proof of Theorem 1.3.

**Proof of Corollary 1.4.** All but the last assertion are proved by constructing such a solution as a sum of progressing waves as in 8. To prove boundedness choose a smooth cutoff function $\rho(t)$ vanishing for $t < 1$ and equal to 1 for $t \geq 2$. Define $u := \rho(t) u$. Then $Lu = F$ with $F$ piecewise smooth, supported in $1 \leq t \leq 2$, and, rapidly decreasing. The singularities are on the parts of the $m$ characteristic hyperplanes that lie in $1 \leq t \leq 2$. Write the source term as $F = \sum_{j=1}^m F_j$ with $F_j$ carrying the singularities on the $j^{th}$. Then $u = \sum u_j$ where $u_j$ is the solution to $Lu_j = F_j$ that vanishes in $t \leq 0$. Apply Theorem 1.3 adapted to sources with singularities on the $j^{th}$ hyperplane to complete the proof.

**A Propagation of jumps across flat discontinuities**

Recall that $\{x_1 = 0\}$ is characteristic, that is det $A_1 = 0$. The spectral projection associated to $\xi = (1, 0, \ldots, 0)$ and the eigenvalue zero has been denoted $\pi_0(1, 0, \ldots, 0)$.

**Definition A.1** Denote by $\pi_-$ the projector $\pi_0(1, 0, \ldots, 0)$.

**Definition A.2** For a piecewise smooth $f(t, x)$ with singularities only on $\{x_1 = 0\}$ denote by $J^p_j(t, x')$ the jump

$$J^p_j(t, x') := \frac{\partial^p f}{\partial x_1^n}(t, 0^+, x') - \frac{\partial^p f}{\partial x_1^n}(t, 0^-, x').$$
Then for all $M \geq 0$

$$f - \sum_{n=0}^{M} \frac{x_1^n}{n!} J_n^f 1_{\mathbb{R}^+} \in C^M(\mathbb{R}^{1+d}).$$

This relation is abbreviated as

$$f = \sum_{n=0}^{\infty} \frac{x_1^n}{n!} J_n^f 1_{\mathbb{R}^+} + C^\infty(\mathbb{R}^{1+d}).$$

Given such a source, Courant and Lax [7] construct a piecewise smooth solution $u$ to

$$Lu - f \in C^\infty(\mathbb{R}^{1+d}). \quad (A.1)$$

Direct computation yields

$$A_1 \frac{\partial u}{\partial x_1} = A_1 J_0^u \delta(x_1) + \sum_{n=0}^{\infty} A_1 J_{n+1}^u \frac{x_1^n}{n!} 1_{\mathbb{R}^+} + C^\infty(\mathbb{R}^{1+d}).$$

**Definition A.3** Denote the tangential part of $L$ by

$$L_{tan}(\partial_t, \partial_{x'}) := \frac{\partial}{\partial t} + \sum_{j=2}^{d} A_j \frac{\partial}{\partial x_j}, \quad \text{so,} \quad L = L_{tan} + A_1 \frac{\partial}{\partial x_1}.$$  

Then $L_{tan}$ maps piecewise smooth functions to piecewise smooth functions and

$$L_{tan} u = \sum_{n=0}^{\infty} (L_{tan} J_n^u) \frac{x_1^n}{n!} 1_{\mathbb{R}^+} + C^\infty(\mathbb{R}^{1+d}).$$

Therefore, in order that (A.1) be satisfied it is necessary and sufficient that

$$A_1 J_0^u = 0,$$ \hspace{1cm} (A.2)

and

$$\forall n \geq 0, \quad A_1 J_{n+1}^u + L_{tan} J_n^u = J_f^n.$$ \hspace{1cm} (A.3)

**Definition A.4** Define a partial inverse $Q$ to $A_1$ by

$$A_1 \pi = 0, \quad Q A_1 (I - \pi) = I - \pi.$$ \hspace{1cm} (A.4)

**Proposition A.5** Suppose that $f(t, x)$ is piecewise smooth with singularities only on $\{x_1 = 0\}$ and $f = 0$ for $t \leq 0$. Then there are uniquely determined jumps $J_n^u \in C^\infty(\mathbb{R}^{d}_{t,x'})$ satisfying the sequence of equations (A.2) and (A.3).

**Proof.** The key to deciphering the equations is to observe that equations (A.2) hold if and only if their projections by $\pi$ and $I - \pi$ hold. The $\pi$ projection eliminates the $A_1$ term. Thus Equations (A.3) hold if and only if

$$\forall n \geq 0, \quad \pi L_{tan} J_n^u = \pi J_f^n,$$ \hspace{1cm} (A.5)
\[ \forall n \geq 0, \quad (I - \pi)A_1J_u^{n+1} = (I - \pi)(J_f^u - L_{\tan}J_u^n). \quad (A.6) \]

Equation (A.3) is rewritten by writing \[ J_u^n = \pi J_u^n + (I - \pi)J_u^n \] to find
\[ \forall n \geq 0, \quad \pi L_{\tan} \pi J_u^n = \pi J_u^n - \pi L_{\tan} (I - \pi)J_u^n. \quad (A.7) \]

Equation (A.6) is between vectors in the range of \( I - \pi \). The equation is equivalent to the same equation multiplied by \( Q \). Using \( (1 - \pi)A_1 = A_1(1 - \pi) \), (A.6) is equivalent to
\[ \forall n \geq 0, \quad (I - \pi)J_u^n + 1 = Q(J_f^u - L_{\tan}J_u^n). \quad (A.8) \]

Summarizing, the jumps satisfy (A.2) and (A.3) if and only they satisfy (A.2), (A.7), and (A.8).

In the next discussion, \( (A.7)_n \) means the case \( n \) of equation (A.7). The jump \( J_u^0 \) is first determined uniquely by (A.2) and (A.7)0. The jumps \( J_u^n \) with \( n \geq 1 \) are determined from (A.7)n and (A.8)n−1. Conversely, with these determinations the equations (A.2)n−1, (A.7)n, and (A.8) are satisfied.

First consider \( J_u^0 \). Equation (A.2) implies that \( (I - \pi)J_u^0 = 0 \). The case \( n = 0 \) of (A.7) reads \[ \pi L_{\tan} \pi J_u^0 = \pi J_u^0. \] Since \( \pi A_1 = 0 \) one has \( \pi L_{\tan} = \pi L \). The smoothness of \( \lambda \) on a neighborhood of \((1,0,\ldots,0)\) implies the key transport identity of geometric optics (Proposition 5.4.1 in [14]),
\[ \pi L_{\tan} \pi = \pi L = \partial_t + \mathbf{v} \cdot \partial_x, \quad \mathbf{v} := \nabla_x \lambda(1,0,\ldots,0). \quad (A.9) \]

Then \( J_u^0 = \pi J_u^0 \) is determined by
\[ \left( \partial_t + \mathbf{v} \cdot \partial_x \right) \pi J_u^0 = \pi J_u^n, \quad J_u^n = 0 \text{ for } t \leq 0. \quad (A.10) \]

Conversely, this equation together with (A.2) imply the case \( n = 0 \) of (A.7) are satisfied.

For \( n \geq 1 \), the jump \( J_u^n \) is uniquely determined \( (A.8)_{n-1} \) and (A.7)n. Begin by replacing \( (I - \pi)J_u^n \) in (A.7)n using (A.8)n−1. Then using (A.9) yields
\[ \forall n \geq 1, \quad \left( \partial_t + \mathbf{v} \cdot \partial_x \right) \pi J_u^n = \pi J_u^n - \pi L_{\tan} Q L_{\tan} J_u^{n-1}, \quad J_u^n = 0 \text{ for } t \leq 0. \quad (A.11) \]

Solving this determines \( \pi J_u^n \) and \( (A.8)_{n-1} \) determines \( (1 - \pi)J_u^n \). Conversely when (A.11)n together with (A.8)n−1 hold, one recovers (A.7)n.

**Theorem A.6** Suppose that \( f(t,x) \) is a piecewise smooth function on \( \mathbb{R}^{1+d} \) with singularities only on \( \{x_1 = 0\} \) and that \( f = 0 \) for \( t \leq 0 \). Then the unique solution of \( Lu = f \) that vanishes for \( t \leq 0 \) is also piecewise smooth with singularities only on \( \{x_1 = 0\} \).

**Proof.** Determine jumps \( J_u^n(t,x') \) using Proposition A.5. Choose a piecewise smooth \( v \) vanishing in \( t \leq 0 \) whose jumps are equal to the functions \( J_u^n \). Then for all \( M \), one has \( L_{\tan} - f \in C^M(\mathbb{R}^{1+d}) \). Define \( g := L_{\tan} - f \). Then \( g \in C^\infty(\mathbb{R}^{1+d}) \) and vanishes for \( t \leq 0 \). Define \( w \in C^\infty(\mathbb{R}^{1+d}) \) to be the solution vanishing for \( t \leq 0 \) to \( Lw = -g \). Then \( u := v + w \) is piecewise smooth and
\[ Lu = L_{\tan} + Lw = (g + f) - g = f. \]

Therefore \( u = v + w \) is the unique solution of the Initial value problem. Since \( u \) has the desired properties, this completes the proof.
Corollary A.7 If $f$ is compactly supported then $J^0_u \in L^\infty(\mathbb{R}^{d+1}_{t,x'})$.

**Proof.** Choose $R > 0$ so that $|t| + |x| > R \Rightarrow f = 0$. The recipe for $J^0_u$ implies that for $|t| > R$

$$(\partial_t + v \cdot \partial_x) J^0_u = 0.$$ 

Therefore

$$\| J^0_u \|_{L^\infty(\mathbb{R}^{d+1})} \leq \| J^0_u \|_{L^\infty(\{|t| \leq R\} \times \mathbb{R}^{d-1}_{x'})}.$$ 

In $|t| \leq R$, $J^0_u$ is smooth and compactly supported so bounded. This shows that the right hand side is finite completing the proof. 

**Example A.1** Tangential derivatives $\partial^\alpha_{t,x'} u$ satisfy an equation entirely analogous to that satisfied by $u$. It follows that the jumps in these derivatives belong to $L^\infty(\mathbb{R}^{d+1}_{t,x'})$.

Nontangential derivatives need not be bounded. Beyond the support of $f$,

$$(\partial_t + v \cdot \partial_x) \pi J^1_u = -\pi L_{\tan} Q L_{\tan} J^0_u.$$ 

In this range,

$$(\partial_t + v \cdot \partial_x) (\pi L_{\tan} Q L_{\tan} J^0_u) = 0.$$ 

Therefore either $\pi L_{\tan} Q L_{\tan} J^0_u$ is identically equal to zero beyond the support of $f$, or $J^1_u$ grows linearly in time.

Since $\pi J^0_u = J^0_u$ and $\pi A_1 = A_1 \pi = 0$,

$$\pi L_{\tan} Q L_{\tan} J^0_u = \pi L_{\tan} Q L_{\tan} \pi J^0_u = \pi L Q L \pi J^0_u.$$ 

The fundamental identity of diffractive geometric optics \[12\] Proposition 3.2, and, \[12\] reads

$$-\pi L Q L \pi = \frac{1}{2} \pi \sum \frac{\partial^2 \lambda(1,0,\ldots,0)}{\partial \xi_\mu \partial \xi_\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} := P(\partial_{x'})$$

Therefore beyond the support of $f$

$$(\partial_t + v \cdot \partial_x) \pi J^1_u = P(\partial_{x'}) J^0_f = J^0_{P(\partial_{x'}) f}.$$ 

The right hand side is constant on integral curves of $\partial_t + v \cdot \partial_x$. When that constant is not zero, $J^1_u$ grows linearly along the integral curve.

By hypothesis the matrix of second derivatives of $\lambda$ has rank $d - 1 > 0$ so the operator $P$ is not identically equal to zero.

Equation (A.12) asks one to integrate the compactly supported jump in $P f$ along integral curves of $\partial_t + v \cdot \partial_x$. The constant in the preceding paragraph vanishes if and only if the integration yields answer zero. The value zero is a rare occurence. For generic $f$, $\pi J^0_u$ is nonzero and constant on almost all (in the sense of Lebesgue measure) integral curves that touch the support of $J^0_f$. For those $f$ and integral curves, $J^1_u$ grows linearly in time.
B Limit case stationary phase inequality

B.1 Non stationary phase lemmas

Lemma B.1 (Lemma of Nonstationary Phase). Suppose that $\Omega \subset \mathbb{R}^d$ is open, $m \in \mathbb{N}$, and $C_1 > 1$. Then there is a constant $C > 0$ so that for all $f \in C^m_0(\Omega)$, and $\phi \in C^m(\Omega; \mathbb{R})$ satisfying

$$\forall |\alpha| \leq m + 1, \|\partial^\alpha \phi\|_{L^\infty(\Omega)} \leq C_1,$$

and

$$\forall x \in \Omega, \quad C_1^{-1} \leq |\nabla_x \phi| \leq C_1,$$

one has,

$$\left| \int e^{i\phi(x)/\epsilon} f(x) \, dx \right| \leq C \epsilon^m \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^1(\mathbb{R}^d)}.$$  \hspace{1cm} (B.1)

Proof. Introduce the differential operator of order one with smooth coefficients

$$L := \frac{\nabla \phi}{i|\nabla \phi|} \cdot \nabla_x, \quad \text{so,} \quad L e^{i\phi/\epsilon} = \epsilon^{-1} e^{i\phi/\epsilon}.$$

Write

$$\int e^{i\phi(x)/\epsilon} f(x) \, dx = \epsilon^m \int L^m e^{i\phi/\epsilon} f(x) \Delta x = \epsilon^m \int e^{i\phi/\epsilon} \left(L^\dagger\right)^m f \, dx,$$

where $L^\dagger$ denotes the transposed operator. Since

$$\|\left(L^\dagger\right)^m f\|_{L^1(\Omega)} \lesssim \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^1(\Omega)},$$

with constant depending only on $C_1$ and $m$. The result follows.

Need the preceding result for fractional values of $m$. Define for $0 \leq m \in \mathbb{N}$ Banach spaces

$$W^{m,1}(\mathbb{R}^d) := \{ f \in L^1(\mathbb{R}^d) : \forall |\alpha| \leq m, \quad \partial^\alpha f \in L^1(\mathbb{R}^d) \}, \quad \|f\|_{W^{m,1}(\mathbb{R}^d)} := \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^1(\mathbb{R}^d)}.$$

The norms of dilated functions satisfy

$$\|f(r(\cdot))\|_{L^1(\mathbb{R}^d)} = r^{-(d-1)} \|f\|_{L^1(\mathbb{R}^d)}, \quad \text{and,} \quad \|\partial^\alpha [f(r(\cdot))]\|_{L^1(\mathbb{R}^d)} = r^{\left|\alpha\right|} r^{-(d-1)} \|f\|_{L^1(\mathbb{R}^d)}.$$

They imply that for $f \in W^{m,1}(\mathbb{R}^d)$ and $r > 0$ the function $x \mapsto f(rx)$ also belongs to $W^{m,1}(\mathbb{R}^d)$ and there is a constant independent of $r, f$ so that

$$\|f(r(\cdot))\|_{W^{m,1}(\mathbb{R}^d)} \leq C(m) r^m r^{-(d-1)} \|f\|_{W^{m,1}(\mathbb{R}^d)}. \hspace{1cm} (B.2)$$

For $m \geq 1/2$ with $m - 1/2 \in \mathbb{N}$, $[W^{m-1/2,1}(\mathbb{R}^d), W^{m+1/2,1}(\mathbb{R}^d)]_{1/2}$ denotes the complex interpolation space

Lemma B.2 With $\Omega, f, \phi$ as above and $m \in \mathbb{N} + 1/2$,

$$\left| \int e^{i\phi/\epsilon} f(x) \, dx \right| \leq C \epsilon^m \|f\|_{[W^{m-1/2,1}(\mathbb{R}^d), W^{m+1/2,1}(\mathbb{R}^d)]_{1/2}}.$$
**Proof.** Follows by interpolation from the cases $m - 1/2$ and $m + 1/2$ proved in Lemma B.1.

**Definition B.3** Spaces $Y^m$ are defined for $0 \leq m \in \mathbb{N}/2$ as follows.

$$
Y^m(\mathbb{R}^d) := W^{m,1}(\mathbb{R}^d) \quad \text{for} \quad 0 \leq m \in \mathbb{N},
$$

$$
Y^m(\mathbb{R}^d) := [W^{m-1/2,1}(\mathbb{R}^d), W^{m+1/2,1}(\mathbb{R}^d)]_{1/2} \quad \text{for} \quad 0 < m \in \mathbb{N} + 1/2.
$$

Lemmas B.1 and B.2 assert that for nonnegative $0 \leq m \in \mathbb{N}/2$,

$$
\left| \int e^{i\phi/\epsilon} f(x) \, dx \right| \leq C(m) \epsilon^m \|f\|_{Y^m(\mathbb{R}^d)}. \quad (B.3)
$$

**B.1.1 Limit case inequality of stationary phase**

**Definition.** A point $\underline{x}$ in an open subset $\Omega \subset \mathbb{R}^d$ is a stationary point of $\phi \in C^\infty(\Omega; \mathbb{R})$ when $\nabla_x \phi(\underline{x}) = 0$. It is a nondegenerate stationary point when the matrix of second derivatives at $\underline{x}$ is nonsingular.

When $\underline{x}$ is a nondegenerate stationary point, the map $x \mapsto \nabla_x \phi(x)$ has nonsingular jacobian at $\underline{x}$. It follows that the map is a local diffeomorphism and in particular the stationary point is isolated.

Taylor’s Theorem shows that

$$
\nabla_x \phi(x) = \frac{1}{2} \nabla^2_x \phi(\underline{x})(x - \underline{x}) + O(|x - \underline{x}|^2).
$$

If $\omega \subset \subset \Omega$ contains $\underline{x}$ and no other stationary point, there is a constant $C > 0$ so,

$$
\forall \underline{x} \in \omega, \quad C^{-1} |x - \underline{x}| \leq |\nabla_x \phi(x)| \leq C |x - \underline{x}|. \quad (B.4)
$$

The following stationary phase inequality in the borderline regularity case follows a proof I learned from G. Météver for the case of more regular $f$ (see Theorem 3.II.1 in [14]).

**Theorem B.4** Suppose that $\Omega \subset \mathbb{R}^n$ is bounded and open and $m$ is the smallest integer less than or equal to $n/2$. For any $c_1 > 0$ there is a constant $C$ so that for all $f \in C^\infty_0(\Omega)$ and phase functions $\phi \in C^\infty(\Omega; \mathbb{R})$ there a point $\underline{x} \in \Omega$ so that for all $x \in \Omega$

$$
\forall x \in \Omega, \quad \frac{1}{c_1} |x - \underline{x}| \leq |\nabla \phi| \leq c_1 |x - \underline{x}|, \quad \text{and}, \quad \forall |\alpha| \leq m + 1, \quad \|\partial^\alpha \phi\|_{L^\infty(\Omega)} \leq c_1.
$$

one has for all $0 < \epsilon < 1$

$$
\left| \int e^{i\phi(x)/\epsilon} f(x) \, dx \right| \leq C \epsilon^{n/2} \ln(1 + \epsilon^{-1}) \left( \|f\|_{L^\infty(\Gamma)} + \|f\|_{Y^{n/2}(\Omega)} \right).
$$

**Lemma B.5** There is a nonnegative $\chi \in C^\infty_0(\mathbb{R}^d \setminus 0)$ so that for all $x \neq 0$, $\sum_{k=-\infty}^{\infty} \chi(2^k x) = 1$. 

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Proof of Lemma. Choose nonnegative $g \in C^\infty_c(\mathbb{R}^d \setminus \{0\})$ so that $g \geq 1$ on $\{1 \leq |x| \leq 2\}$. Define the locally finite sum

$$G(x) := \sum_{k=-\infty}^{\infty} g(2^k x), \quad \text{so} \quad G(2^k x) = G(x).$$

Then $G \in C^\infty(\mathbb{R}^d \setminus \{0\})$, and $G \geq 1$. The function $\chi := g/G$ has the desired properties.

Proof of Theorem. Translating coordinates it suffices to consider $x = 0$. Choose $\chi$ as in the Lemma B.5. Write

$$\int e^{i\phi(x)} f(x) \, dx = \sum_{k=-\infty}^{\infty} \int \chi(2^k x) e^{i\phi(x)} f(x) \, dx := \sum_{k=-\infty}^{\infty} I(k).$$

The half sum $\sum_{k<0} \chi(2^k x)$ is a smooth function on $\mathbb{R}^d$ that vanishes on a neighborhood of the origin and is identically equal to 1 outside a large ball. Inequality B.3 yields

$$\left| \int e^{i\phi(x)} \left( \sum_{k<0} \chi(2^k x) \right) f(x) \, dx \right| \leq C \epsilon^{d/2} \|f\|_{Y^{d/2}(\mathbb{R}^d)}.$$

The sum $\sum_{2^{k+1/2} \geq 1} \chi(2^k x)$ is a bounded function supported in a ball $|x| \leq C \epsilon^{1/2}$ so

$$\left| \int e^{i\phi(x)} \left( \sum_{2^{k+1/2} \geq 1} \chi(2^k x) \right) f(x) \, dx \right| \leq C \epsilon^{d/2} \|f(x)\|_{L^\infty(\Omega)}.$$

There remains the sum over $1 \leq 2^k < \epsilon^{-1/2}$. The change of variable $y = 2^k x$ yields

$$I(k) = 2^{-kd} \int \chi(y) e^{i\phi_k(y)/(2^{2k} \epsilon)} f(2^{-k} y) \, dy, \quad \phi_k(y) := 2^{2k} \phi(2^{-k} y).$$

It follows from B.4 that there is a constant $C_1 > 0$ so that on the support of $\chi$,

$$C_1^{-1} \leq |\nabla \phi_k| \leq C_1.$$

In addition there are constants $C(\alpha)$ independent of $k \geq 0$ so that $|\partial^\alpha \phi_k| \leq C(\alpha)$. Inequality B.3 shows that there is a constant independent of $k \geq 0$ so that

$$\left| \int \chi(y) e^{i\phi_k(y)/(2^{2k} \epsilon)} f(2^{-k} y) \, dy \right| \leq C \left( 2^{2k} \epsilon \right)^{d/2} \|f\|_{Y^{d/2}(\Omega)} = C \epsilon^{kd} \|f\|_{Y^{d/2}(\mathbb{R}^d)}.$$

Used to estimate $I(k)$, the powers of $2^{\pm kd}$ cancel yielding

$$\sum_{1 \leq 2^k < \epsilon^{-1/2}} |I(k)| \leq C \epsilon^{d/2} \sum_{1 \leq 2^k < \epsilon^{-1/2}} \|f\|_{Y^{d/2}(\mathbb{R}^d)}.$$

The number of summands is $\lesssim \ln(1 + \epsilon^{-1})$. This completes the proof.
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