THE q–CHARACTERS OF REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS AND DEFORMATIONS OF W–ALGEBRAS

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Abstract. We propose the notion of q–characters for finite-dimensional representations of quantum affine algebras. It is motivated by our theory of deformed W–algebras.

1. Introduction

Let g be a simple Lie algebra, \( \hat{g} \) be the corresponding non-twisted affine Kac-Moody algebra, and \( U_q\hat{g} \) be its quantized universal enveloping algebra (in this paper, \( q \) is assumed to be generic). Consider the category \( \text{Rep } U_q\hat{g} \), whose objects are the finite-dimensional representations of \( U_q\hat{g} \), and morphisms are homomorphisms of \( U_q\hat{g} \)–modules. Since \( U_q\hat{g} \) is a Hopf algebra, \( \text{Rep } U_q\hat{g} \) is a monoidal tensor category.

An interesting problem is to describe the irreducible objects of \( \text{Rep } U_q\hat{g} \). A complete answer is known in the case when \( g = \mathfrak{sl}_2 \) (it is recalled in Sect. 4.1). For \( g \) other than \( \mathfrak{sl}_2 \) the picture is less clear (see, e.g., [10, 13, 14, 1, 36, 58]). In contrast, when \( q = 1 \), the analogous problem of describing irreducible finite-dimensional representations of \( \hat{g} \) has a simple and elegant solution. Consider the “evaluation homomorphism” \( \phi_a : \hat{g} \to g \) corresponding to evaluating a function on \( \mathbb{C}^\times \) at a point \( a \in \mathbb{C}^\times \). For an irreducible \( g \)–module \( V_\lambda \) with highest weight \( \lambda \), let \( V_\lambda(a) \) be its pull-back under \( \phi_a \) to an irreducible representation of \( \hat{g} \). Then \( V_\lambda(a_1) \otimes \cdots \otimes V_\lambda(a_n) \) is irreducible if \( a_i \neq a_j, \forall i \neq j \), and these are all irreducible finite-dimensional representations of \( \hat{g} \) up to an isomorphism [11]. It is also easy to decompose tensor products of such representations.

The first major difference, which makes the description of irreducible representations difficult in the quantum case is that the evaluation homomorphisms \( \phi_z : \hat{g} \to g \) can not be lifted to homomorphisms \( U_q\hat{g} \to U_q g \) if \( g \) is not \( \mathfrak{sl}_N \). Nevertheless, V. Chari and A. Pressley have shown [10, 13] that for each \( i = 1, \ldots, \ell = \text{rk } g \), and \( z \in \mathbb{C}^\times \), there exists a unique irreducible representation \( V_{\omega_i}(a) \) of \( U_q\hat{g} \) with highest weight \( \omega_i \), when restricted to \( U_q g \subset U_q\hat{g} \). This representation plays the role of \( V_{\omega_i}(a) \) in the case \( q = 1 \), though in general it is bigger than \( V_{\omega_i} \) when restricted to \( U_q g \). Furthermore, Chari and Pressley have shown [13] that any irreducible representation occurs as a subquotient of the tensor product \( V_{\omega_{i_1}}(a_1) \otimes \cdots \otimes V_{\omega_{i_n}}(a_n) \), where the parameters \( (\omega_{i_1}, a_1, \ldots, \omega_{i_n}, a_n) \) are uniquely determined by this representation up to permutation.

This does provide us with a good parametrization of irreducible representations, but does not quite answer the question of describing these representations and their tensor products explicitly. For instance, the only thing that is known about the tensor product...
$V_{\omega_i}(a_1) \otimes V_{\omega_i}(b)$ in general is that it is irreducible provided that $a/b$ does not belong to a countable set (see \[42\]), which is unknown in general. This constitutes the second major difference with the case $q = 1$, when this set consists of a single element, 1.

In order to gain some insights into the problem, we develop in this paper a theory of “characters” for finite-dimensional representations of $U_q \widehat{g}$, which we call the $q$–characters.

Let us recall the situation in the case of finite-dimensional representations of the Lie algebra $g$. Such representations can be integrated to representations of the simply-connected Lie group $G$. Let $\text{Rep} G$ be the Grothendieck ring of finite-dimensional representations of $G$. Denote by $T$ the Cartan subgroup of $G$. We attach to each finite-dimensional $G$–module $V$ its character, the function $\chi_V : T \to \mathbb{C}$, defined by $\chi_V(t) = \text{Tr}_V(t) \forall t \in T$. This way we obtain an injective homomorphism of commutative algebras

$$\chi : \text{Rep} G \to \mathbb{Z}[T] \simeq \mathbb{Z}[y_1^\pm 1, \ldots, y_\ell^\pm 1],$$

where $y_i$ are the fundamental weights (the generators of the lattice of homomorphisms $T \to \mathbb{C}^\times$).

Let $\text{Rep} U_q \widehat{g}$ be the Grothendieck ring of the category $\text{Rep} U_q \widehat{g}$. Using the universal $R$–matrix of $U_q \widehat{g}$, we will construct an injective homomorphism

$$\chi_q : \text{Rep} U_q \widehat{g} \to \mathbb{Y} = \mathbb{Z}[[Y_{i,a_i}]_{i=1,\ldots,\ell}:a_i \in \mathbb{C}^\times]$$

(see also \[19\]). We will show that $\chi_q$ behaves well with respect to the restriction to $U_q g$ and to the quantum affine subalgebras of $U_q \widehat{g}$. We call $\chi_q(V)$ the $q$–character of representation $V$. We hope that the $q$–characters could be used more efficiently than their classical counterparts in describing irreducible $U_q \widehat{g}$–modules and their tensor products. In particular, we will define the screening operators $S_i$ on $\mathbb{Y}$ and conjecture that the image of the homomorphism $\chi_q$ equals the intersection of the kernels of $S_i, i = 1, \ldots, \ell$ (we prove this for $g = sl_2$). Because of this conjecture, we expect that the $q$–character of the irreducible $U_q \widehat{g}$–module with a given highest weight can be found in a purely combinatorial way.

The motivation for our construction of the screening operators comes from our theory of deformed $W$–algebras \[31, 33\]. Let us first briefly describe the analogous picture in the $q = 1$ case. There are essentially three equivalent definitions of the classical undeformed $W$–algebra: as the result of the Drinfeld-Sokolov reduction \[23\], as the center of the completed enveloping algebra $U(\hat{g})$ at the critical level \[27\], and as the algebra of integrals of motion of the Toda field theory \[28\]. According to the last definition, the $W$–algebra is a subalgebra in a Heisenberg algebra, which equals the intersection of the kernels of the screening operators. This definition can be thought of as an explicit description of the center of $U(\hat{g})$ at the critical level (see \[27\]).

Now consider the $q$–version of this picture. We expect that the center of $U_q \widehat{g}$ at the critical level is isomorphic to the $q$–deformed classical $W$–algebra and hence can be described as the intersection of kernels of screening operators. But for $q \neq 1$, we have an injective homomorphism from $\text{Rep} U_q \widehat{g}$ to the center of $U_q \widehat{g}((z))$ \[52, 19\], and this allows us to view $\text{Rep} U_q \widehat{g}$ as the “space of fields” of the $q$–deformed $W$–algebra. This suggests to us that $\text{Rep} U_q \widehat{g}$ can also be described using the screening operators, which are precisely the operators $S_i$. 
In the course of writing this paper, we learned from M. Kashiwara about the work of H. Knight [44]. Knight proposed a character theory for finite-dimensional representations of Yangians using a different approach. Some further results in this direction were subsequently obtained by Chari and Pressley [15]. Knight’s results can be carried over to the case of quantum affine algebras. However, he was unable to show the multiplicative property of characters in this case because a certain result on the structure of comultiplication in $U_q\hat{g}$ was not available at the time. This result is available now (see Lemma [1]), and using it one can extend Knight’s construction to the case of quantum affine algebras. In hindsight, it turns out that Knight’s characters essentially coincide with our $q$–characters. One of the advantages of our definition is that the multiplicative property of the $q$–characters mentioned above follows automatically, because $\chi_q$ is manifestly a ring homomorphism.

The $q$–characters are also closely related to the formulas for the spectra of transfer-matrices in integrable spin chains associated to $U_q\hat{g}$, obtained by the analytic Bethe Ansatz [4, 50, 51, 5, 46]. Our definition of $q$–characters allows us to streamline the rather ad hoc method of writing formulas for these eigenvalues that has been used before (see Sect. 6).

The results of this paper can be generalized in a straightforward way to the Yangians and the twisted quantum affine algebras. Furthermore, it turns out that $\text{Rep} \ U_q\hat{g}$ and $\text{Rep} \ U_t(L\hat{g})$, where $L\hat{g}$ is the Langlands dual affine Kac-Moody algebra to $\hat{g}$ (it is twisted if $g$ is non-simply laced) are two different classical limits of the quantum deformed $W$–algebra $W_{q,t}(g)$. Thus, the fields from $W_{q,t}(g)$ may be viewed as $(q,t)$–characters, which are simultaneous quantizations of the $q$–characters of $U_q\hat{g}$ and the $t$–characters of $U_t(L\hat{g})$. The former appear when $t \to 1$, and the latter appear when $q \to \exp(\pi i/r')$ (see [33]). The precise nature of this duality deserves further study.

The paper is arranged as follows. In Sect. 2 we give the main definitions and recall some of the results of Chari and Pressley on finite-dimensional representations of $U_q\hat{g}$. We define the $q$–characters in Sect. 3. In Sects. 4 and 5 we present some results and a conjecture on the structure of $q$–characters. In particular, we discuss the connection between the $q$–characters and the $R$–matrices. In Sect. 6 we explain the formulas for the spectra of transfer-matrices obtained by the Bethe Ansatz from the point of view of the $q$–characters. In Sect. 7 we define the screening operators and state the conjecture characterizing the $q$–characters. In Sect. 8 we motivate this conjecture by providing a detailed analysis of the connection between $\text{Rep} \ U_q\hat{g}$ and the deformed $W$–algebras.

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2. Background

2.1. Root data. Let \( \mathfrak{g} \) be a simple Lie algebra of rank \( \ell \). We denote by \( I \) the set \( \{1, \ldots, \ell\} \). Let \( h^\vee \) be the Coxeter number of \( \mathfrak{g} \). Let \( \langle \cdot, \cdot \rangle \) be the invariant inner product on \( \mathfrak{g} \), normalized as in [40], so that the square of length of the maximal root equals 2 with respect to the induced inner product on the dual space to the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) (also denoted by \( \langle \cdot, \cdot \rangle \)). Let \( \{\alpha_1, \ldots, \alpha_\ell\} \) and \( \{\omega_1, \ldots, \omega_\ell\} \) be the sets of simple roots and of fundamental weights of \( \mathfrak{g} \), respectively. We have:

\[
\langle \alpha_i, \omega_j \rangle = \frac{\langle \alpha_i, \alpha_i \rangle}{2} \delta_{i,j}.
\]

Let \( r^\vee \) be the maximal number of edges connecting two vertices of the Dynkin diagram of \( \mathfrak{g} \). Thus, \( r^\vee = 1 \) for simply-laced \( \mathfrak{g} \), \( r^\vee = 2 \) for \( B_\ell, C_\ell, F_4, G_2 \), and \( r^\vee = 3 \) for \( D_4 \).

From now on we will use the inner product \( (\cdot, \cdot) = r^\vee \langle \cdot, \cdot \rangle \) on \( \mathfrak{h}^* \). Set

\[
D = \text{diag}(r_1, \ldots, r_\ell),
\]

where

\[
r_i = \frac{\langle \alpha_i, \alpha_i \rangle}{2} = r^\vee \frac{\langle \alpha_i, \alpha_i \rangle}{2}.
\]

All \( r_i \)'s are integers; for simply-laced \( \mathfrak{g} \), \( D \) is the identity matrix.

Now let \( C = (C_{ij})_{1 \leq i,j \leq \ell} \) and \( (I_{ij})_{1 \leq i,j \leq \ell} \) be the Cartan matrix and the incidence matrix of \( \mathfrak{g} \), respectively, so that \( C = 2I - I \). We have:

\[
C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.
\]

Let \( B = (B_{ij})_{1 \leq i,j \leq \ell} \) be the symmetric matrix

\[
B = DC,
\]

i.e.,

\[
B_{ij} = \langle \alpha_i, \alpha_j \rangle = r^\vee \langle \alpha_i, \alpha_j \rangle.
\]

Let \( q \in \mathbb{C}^\times \) be such that \( |q| < 1 \). Set \( q_i = q^{r_i} \).

We will use the standard notation

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

Introduce the \( \ell \times \ell \) matrices \( B(q), C(q), D(q) \) by the formulas

\[
B_{ij}(q) = [B_{ij}]_q, \quad C_{ij}(q) = (q^{r_i} + q^{-r_i})\delta_{ij} - [I_{ij}]_q, \quad D_{ij}(q) = \delta_{ij} [r_i]_q.
\]

We have:

\[
B(q) = D(q)C(q).
\]
2.2. Quantum affine algebras.

Definition 1 ([20, 38]). Let $U_q\hat{g}$ be the associative algebra over $\mathbb{C}$ with generators $x_i^\pm$, $k_i^{\pm1}$ ($i = 0, \ldots, \ell$), and relations:

$$
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,
$$

$$
k_i x_j^\pm k_i^{-1} = q^{\pm B_{ij}} x_j^\pm,
$$

$$
[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},
$$

$$
\sum_{r=0}^{1-C_{ij}} \left[ \frac{1-C_{ij}}{r} \right] q_i (x_i^+)^r (x_i^\pm)^{1-C_{ij}-r} = 0, \quad i \neq j.
$$

We introduce a $\mathbb{Z}$-gradation on $U_q\hat{g}$ by the formulas: $\deg x_0^\pm = \pm 1$, $\deg x_i^\pm = \deg k_i = 0, i \in I = \{1, \ldots, \ell\}$.

Denote the subalgebra of $U_q\hat{g}$ generated by $k_i^{\pm1}, x_i^+$ (resp., $k_i^{\pm1}, x_i^-$), $i = 0, \ldots, \ell$, by $U_q b_+$ (resp., $U_q b_-$).

The algebra $U_q\hat{g}$ is defined as the subalgebra of $U_q\hat{g}$ with generators $x_i^\pm, k_i^{\pm1}$, where $i \in I$.

$U_q\hat{g}$ has a structure of a Hopf algebra with the comultiplication given on generators by:

$$
\Delta(k_i) = k_i \otimes k_i,
$$

$$
\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+,
$$

$$
\Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-.
$$

Remark 2.1. This comultiplication differs from the original one [20, 38]. The difference is essentially accounted for by the automorphism of $U_q\hat{g}$ sending $k_i$ to $k_i^{-1}$, $x_i^+$ to itself and $q$ to $q^{-1}$. The reason for this choice of comultiplication is that it is in terms of this comultiplication that the universal $R$–matrix has the form given in Sect. 3.3. □

The following theorem describes the Drinfeld “new” realization of $U_q\hat{g}$ [21].

Theorem 1 ([21, 43, 48, 6]). The algebra $U_q\hat{g}$ has another realization as the algebra with generators $x_{i,n}^\pm$ ($i \in I = \{1, \ldots, \ell\}, n \in \mathbb{Z}$), $k_i^{\pm1}$ ($i \in I = \{1, \ldots, \ell\}$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z}$), and relations:

$$
\Delta(k_i) = k_i \otimes k_i,
$$

$$
\Delta(x_{i,n}^+) = x_{i,n}^+ \otimes 1 + k_i \otimes x_{i,n}^+,
$$

$$
\Delta(x_{i,n}^-) = x_{i,n}^- \otimes k_i^{-1} + 1 \otimes x_{i,n}^-.
$$

Remark 2.2. This comultiplication differs from the original one [20, 38]. The difference is essentially accounted for by the automorphism of $U_q\hat{g}$ sending $k_i$ to $k_i^{-1}$, $x_i^+$ to itself and $q$ to $q^{-1}$. The reason for this choice of comultiplication is that it is in terms of this comultiplication that the universal $R$–matrix has the form given in Sect. 3.3. □
Remark 2.2. Note that the generators $h_{i,n}$ correspond to $\frac{q_i - q_i^{-1}}{q - q^{-1}} H_{i,n}$ of [10].

Let $Q$ be the root lattice of $\hat{g}$. We introduce the $Q$-gradation on $U_q\hat{g}$ by the formulas:

\[
\deg x_{i,n}^\pm = \pm \alpha_i, \quad \deg k_i = \deg h_{i,n} = \deg e_i = 0.
\]

For any $a \in \mathbb{C}^\times$, there is a Hopf algebra automorphism $\tau_a$ of $U_q\hat{g}$ defined on the generators by the following formulas:

\[
\tau_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm, \quad \tau_a(\phi_{i,r}^\pm) = a^r \phi_{i,r}^\pm,
\]

\[
\tau_a(e_i^{1/2}) = e_i^{1/2}, \quad \tau_a(k_i) = k_i,
\]

for all $i = 1, \ldots, \ell, r \in \mathbb{Z}$. Given a $U_q\hat{g}$-module $V$ and $a \in \mathbb{C}^\times$, we denote by $V(a)$ the pull-back of $V$ under $\tau_a$.

Introduce new variables $\bar{k}_i^\pm, i \in I$, such that

\[
k_j = \prod_{i \in I} \bar{k}_i^{C_{ij}}.
\]

Thus, while $k_i$ corresponds to the simple root $\alpha_i$, $\bar{k}_i$ corresponds to the fundamental weight $\omega_i$. We extend the algebra $U_q\hat{g}$ by replacing the generators $k_i^\pm, i \in I$ with $\bar{k}_i^\pm, i \in I$. From now on $U_q\hat{g}$ will stand for the extended algebra.
2.3. Finite-dimensional representations of $U_q\hat{\mathfrak{g}}$. In this section we recall some of the results of Chari and Pressley [10, 12, 13, 14].

Let $P$ be the weight lattice of $\mathfrak{g}$. A representation $W$ of $U_q\mathfrak{g}$ is said to be of type 1 if it is the direct sum of its weight spaces

$$W_\lambda = \{ w \in W | k_i \cdot w = q^{(\lambda, \alpha_i)} w \}, \quad \lambda \in P.$$ 

If $W_\lambda \neq 0$, then $\lambda$ is called a weight of $W$. A vector $w \in W_\lambda$ is called a highest weight vector if $x_i^+ \cdot w = 0$ for all $i \in I$, and $W$ is a highest weight representation with highest weight $\lambda$ if $W = U_q\mathfrak{g} \cdot w$ for some highest weight vector $w \in W_\lambda$. In that case, $\lambda \in P^+$, the set of dominant weights.

A representation $V$ of $U_q\mathfrak{g}$ is called of type 1 if $c^{1/2}$ acts as the identity on $V$, and if $V$ is of type 1 as a representation of $U_q\mathfrak{g}$. A vector $v \in V$ is called a highest weight vector if

$$x_i^+ \cdot v = 0, \quad \phi_i^+ \cdot v = \psi_i^+ v, \quad c^{1/2} v = v,$$

for some complex numbers $\psi_i^+$. A type 1 representation $V$ is a highest weight representation if $V = U_q\mathfrak{g} \cdot v$, for some highest weight vector $v$. In that case the set $(\psi_i^+)_{i \in I, r \in \mathbb{Z}}$ is called the highest weight of $V$.

If $\lambda \in P^+$, let $\mathcal{P}^\lambda$ be the set of all $I$–tuples $(P_i)_{i \in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1, such that $\deg(P_i) = \lambda(\alpha_i^\vee)$ for all $i \in I$. Such an $I$–tuple is then said to have degree $\lambda$. Set $\mathcal{P} = \bigcup_{\lambda \in P^+} \mathcal{P}^\lambda$.

Set

$$\Phi_i^\pm(u) = \sum_{n=0}^{\infty} \phi_{i,n}^\pm u^{\pm n}, \quad \Psi_i^\pm(u) = \sum_{n=0}^{\infty} \psi_{i,n}^\pm u^{\pm n}.$$

Theorem 2 (10, 13).

1. Every finite-dimensional irreducible representation of $U_q\mathfrak{g}$ can be obtained from a type 1 representation by twisting with an automorphism of $U_q\mathfrak{g}$.

2. Every finite-dimensional irreducible representation of $U_q\mathfrak{g}$ of type 1 is a highest weight representation.

3. Let $V$ be a finite-dimensional irreducible representation of $U_q\mathfrak{g}$ of type 1 and highest weight $(\Psi_i^\pm)_{i \in I, r \in \mathbb{Z}}$. Then, there exists $P = (P_i)_{i \in I} \in \mathcal{P}$ such that

$$\Psi_i^\pm(u) = q_i^{\deg(P_i)} \frac{P_i(ug_i^{-1})}{P_i(uq_i)}.$$ 

as an element of $\mathbb{C}[[u^\pm 1]]$.

Assigning to $V$ the set $\mathcal{P}$ defines a bijection between $\mathcal{P}$ and the set of isomorphism classes of finite-dimensional irreducible representations of $U_q\mathfrak{g}$ of type 1. The irreducible representation associated to $\mathcal{P}$ will be denoted by $V(\mathcal{P})$.

4. If $\mathcal{P} = (P_i)_{i \in I} \in \mathcal{P}$, $a \in \mathbb{C}^\times$, and if $\tau_a^\ast(V(\mathcal{P}))$ denotes the pull-back of $V(\mathcal{P})$ by the automorphism $\tau_a$, we have

$$\tau_a^\ast(V(\mathcal{P})) \cong V(\mathcal{P}^a)$$

as representations of $U_q\mathfrak{g}$, where $\mathcal{P}^a = (P_i^a)_{i \in I}$ and

$$P_i^a(u) = P_i(ua).$$
Let $P, Q \in \mathcal{P}$ be as above, and let $v_P$ and $v_Q$ be highest weight vectors of $V(P)$ and $V(Q)$, respectively. Denote by $P \otimes Q$ the $I$-tuple $(P_iQ_i)_{i \in I}$. Then $V(P \otimes Q)$ is isomorphic to a quotient of the subrepresentation of $V(P) \otimes V(Q)$ generated by the tensor product of the highest weight vectors.

Remark 2.3. An analogous classification result for Yangians has been obtained earlier by Drinfeld [21]. Because of that, the polynomials $P_i(u)$ are sometimes called Drinfeld polynomials.

Remark 2.4. Note that in our notation the polynomial $P_i(u)$ corresponds to the polynomial $P_i(uq_i^{-1})$ in the notation of [10, 13].

For each $i \in I$ and $a \in \mathbb{C}^\times$, we define the irreducible representation $V_{\omega_i}(a)$ as $V(P_a^{(i)})$, where $P_a^{(i)}$ is the $I$-tuple of polynomials, such that $P_i(u) = 1 - au$ and $P_j(u) = 1, \forall j \neq i$. We call $V_{\omega_i}(a)$ the $i$th fundamental representation of $U_q\hat{g}$. Note that in general $V_{\omega_i}(a)$ is reducible as a $U_q\hat{g}$-module.

Theorem 2 implies the following

Corollary 1 ([13]). Any irreducible finite-dimensional representation of $U_q\hat{g}$ occurs as a quotient of the submodule of the tensor product $V_{\omega_1}(a_1) \otimes \ldots \otimes V_{\omega_n}(a_n)$, generated by the tensor product of the highest weight vectors. The parameters $(\omega_1, a_1), \ldots, (\omega_n, a_n)$ are uniquely determined by this representation up to permutation.

Remark 2.5. For $U_q\hat{sl}_N$, V. Ginzburg and E. Vasserot have given an alternative geometric construction of irreducible finite-dimensional representations [36, 58].

2.4. Spectra of $\Phi^\pm(u)$ on finite-dimensional representations. It follows from the defining relations that the operators $\phi^\pm_{i,n}$ commute with each other. Hence we can decompose any representation $V$ of $U_q\hat{g}$ into a direct sum $V = \oplus V(\gamma^\pm_{i,n})$, where

$$V(\gamma^\pm_{i,n}) = \{ x \in V | (\phi^\pm_{i,n} - \gamma^\pm_{i,n})^p \cdot x = 0, \text{ for some } p, \forall i, n \}. $$

Given a collection $(\gamma^\pm_{i,n})$ of eigenvalues, we form the generating functions

$$\gamma^\pm_i(u) = \sum_{n>0} \gamma^\pm_{i,n} u^{i,n}. $$

We will refer to a series $\gamma^\pm_i(u)$ occurring on a given representation $V$ as an eigenvalue of $\Phi^\pm_i(u)$ on $V$. The following result is a generalization of Theorem 2.

**Proposition 1.** The eigenvalues $\gamma^\pm_i(u)$ of $\Phi^\pm_i(u)$ on any finite-dimensional representation of $U_q\hat{g}$ have the form:

$$\gamma^\pm_i(u) = q_i^{\deg Q_i - \deg R_i} Q_i(uq_i^{-1}) R_i(uq_i) Q_i(uq_i^{-1}) R_i(uq_i^{-1}), $$

as elements of $\mathbb{C}[[u]]$ and $\mathbb{C}[[u^{-1}]]$, respectively, where $Q_i(u), R_i(u)$ are polynomials in $u$ with constant term 1.
Proof. Let \( U_q\hat{\mathfrak{g}}_{(i)} \) be the subalgebra of \( U_q\hat{\mathfrak{g}} \) generated by \( k_i^{\pm 1}, h_i, x_{i,n}^\pm, n \in \mathbb{Z} \). This subalgebra is isomorphic to \( U_q\hat{\mathfrak{sl}}_2 \). The eigenvalues of \( \Phi_i^+(u) \) on a finite-dimensional representation \( V \) of \( U_q\hat{\mathfrak{g}} \) coincide with the eigenvalues of \( \Phi_i^+(u) \) on the restriction of \( V \) to \( U_q\hat{\mathfrak{g}}_{(i)} \). Hence it suffices to prove the statement of the proposition when \( \mathfrak{g} = \mathfrak{sl}_2 \).

The finite-dimensional irreducible representations of \( U_q\hat{\mathfrak{sl}}_2 \) have been classified in [12] (see also [13]). The result is recalled in Theorem 4 below. According to this result, each irreducible representation of \( U_q\hat{\mathfrak{sl}}_2 \) is isomorphic to the tensor product \( W_{a_1}(b_1) \otimes \ldots \otimes W_{a_n}(b_n) \). The representation \( W_{a}(b) \) is defined in Sect. 4.1 and the eigenvalues of \( \Phi^\pm(u) \) on it are given by formula (4.2). These eigenvalues are in the form \( [16] \) a more precise formula is proved; that formula has also appeared in [39, 7].

Lemma 1. On representations of \( U_q\hat{\mathfrak{g}} \) on which \( c \) acts as the identity,

\[
\Delta(h_{i,\pm n}) = h_{i,\pm n} \otimes 1 + 1 \otimes h_{i,\pm n} + \text{terms in } U_+ \otimes U_-, \quad n > 0,
\]

where \( U_+ (\text{resp., } U_-) \) is the subalgebra of \( U_q\hat{\mathfrak{g}} \) spanned by elements of positive (resp., negative) \( Q \)-degree.

Let us concentrate on the case of \( \Phi^+(u) \) and hence \( h_n, n > 0 \). The case of \( \Phi^-(u) \) is analyzed in the same way. Let \( V \) and \( W \) be two representations of \( U_q\hat{\mathfrak{sl}}_2 \). We can decompose \( V \) and \( W \) into direct sums \( V = \oplus_{p \in \mathbb{Z}} V_p, W = \oplus_{p \in \mathbb{Z}} W_p \), where

\[
V_p = \{ x \in V | k \cdot x = q^p x \}.
\]

Since \( k \) commutes with all \( h_n \), all \( V_p, W_p \) are \( h_n \)-invariant for all \( n \in \mathbb{Z} \). We can choose bases \( \{ v^p_i \} \) of \( V_p \) and \( \{ w^p_i \} \) in \( W \) with respect to which \( h_n \)'s are upper-triangular for all \( n \). Let us now order the basis \( \{ v^p_i \otimes w^p_j \} \) of \( V \otimes W \) in such a way that \( v^p_\alpha \otimes w^p_\beta < v^p_\alpha' \otimes w^p_\beta' \) if either \( q > s \) or \( q = s \) and \( p < r \). For fixed \( p, q, r, s \) we keep the old orderings on \( \alpha, \beta, \alpha', \beta' \). Formula (2.6) shows that \( \Delta(h_n) \) is upper triangular in this basis of \( V \otimes W \), and the eigenvalues of \( \Delta(h_n) \) are equal to the sums of the eigenvalues of \( h_n \) on \( V \) and \( W \). Using formula (2.2) for \( \Phi^+(u) \) and the formula \( \Delta(k_i) = k_i \otimes k_i \), we obtain that the eigenvalues of \( \Phi^+(u) \) on \( V \otimes W \) are products of its eigenvalues on \( V \) and \( W \), which is what we needed to show.

Remark 2.6. The statement analogous to Proposition 1 in the case of the Yangians has been proved by Knight [44], Prop. 4. Our proof is similar to his proof.

In the same way as above, one can derive from Lemma 1 that the eigenvalues of \( \Phi_i^+(u) \) on the tensor product \( V \otimes W \) are the products of its eigenvalues on \( V \) and \( W \) for any \( U_q\hat{\mathfrak{g}} \).

3. Definition of \( q \)-characters

3.1. Transfer-matrices. The completed tensor product \( U_q\hat{\mathfrak{g}} \otimes U_q\hat{\mathfrak{g}} \) contains a special element \( \mathcal{R} \) called the universal \( R \)-matrix (at level 0). It actually lies in \( U_q\hat{\mathfrak{b}}_+ \otimes U_q\hat{\mathfrak{b}}_- \).
and satisfies the following identities:

\[
\Delta'(x) = R \Delta(x) R^{-1}, \quad \forall x \in U_q \hat{\mathfrak{g}},
\]

(3.1)

\[
(\Delta \otimes \text{id}) R = R^{13} R^{23}, \quad (\text{id} \otimes \Delta) R = R^{13} R^{12}.
\]

(3.2)

Note that the last two equations imply that \( R \) satisfies the Yang-Baxter equation. For more details, see [22, 25].

Now let \((V, \pi_V)\) be a finite-dimensional representation of \( U_q \hat{\mathfrak{g}} \). Define the \( L \)-operator corresponding to \( V \) by the formula

\[
L_V = L_V(z) = (\pi_V(z) \otimes \text{id})(R).
\]

(3.3)

Now define the transfer-matrix corresponding to \( V \) by

\[
t_V = t_V(z) = \text{Tr}_V q^{2n} L_V(z),
\]

(3.4)

where by definition \( q^{2n} = \tilde{k}_1^2 \ldots \tilde{k}_f^2 \). Then for each \( V \),

\[
t_V(z) = \sum_{n \leq 0} t_V[n] z^{-n},
\]

where \( t_V[n] \in U_q \mathfrak{b}_- \). Furthermore, \( t_V[n] \) has degree \( n \) with respect to the \( \mathbb{Z} \)-gradation on \( U_q \mathfrak{b}_- \) (see Definition 1). Thus, for each \( x \in \mathbb{C}^\times \), \( t_V(x) \) lies in the completion \( \hat{U}_q \mathfrak{b}_- \) corresponding to this gradation.

**Lemma 2.**

1. For any pair of finite-dimensional representations, \( V \) and \( W \),

\[
[t_V(z), t_W(w)] = 0;
\]

2. Given a short exact sequence \( 0 \rightarrow V \rightarrow W \rightarrow U \rightarrow 0 \), \( t_W(z) = t_V(z) + t_U(z) \);
3. \( t_V \otimes t_W(z) = t_V(z) t_W(z) \);
4. \( t_{V(a)}(z) = t_{V(z a)}, \forall a \in \mathbb{C}^\times \).

**Proof.** Parts (2)–(4) follow immediately from the definition of \( t_V(z) \).

It is clear that

\[
R_{V,W}(z, w) = (\pi_V(z) \otimes \pi_W(w))(R)
\]

is a well-defined element of \( \text{End}(V \otimes W)[[z, w^{-1}]] \). The Yang-Baxter equation gives:

\[
L_V(z) L_W(w) = R_{V,W}(z, w)^{-1} L_W(w) L_V(z) R_{V,W}(z, w).
\]

Taking the traces and using (2) we obtain: \( t_V(z) t_W(w) = t_W(w) t_V(z) \) as formal power series.

The Lemma implies the following statement.

**Proposition 2.** The linear map \( \nu_q \) sending \( V \in \text{Rep} \ U_q \hat{\mathfrak{g}} \) to \( t_V(z) \in U_q \mathfrak{b}_- [[z]] \) is a \( \mathbb{C}^\times \)-equivariant ring homomorphism from \( \text{Rep} \ U_q \hat{\mathfrak{g}} \) onto a commutative subalgebra of \( U_q \mathfrak{b}_- [[z]] \).
3.2. Analogue of the Harish-Chandra homomorphism. Denote by \( U_q \mathfrak{g} \) the subalgebra of \( U_q \mathfrak{g} \) generated by \( x_{i,n}^\pm, k_i, h_{i,n}, i \in I, n \leq 0 \). As a vector space, \( U_q \mathfrak{g} \) can be decomposed as follows: \( U_q \mathfrak{g} = U_q \mathfrak{n}_- \otimes U_q \mathfrak{h} \otimes U_q \mathfrak{n}_+ \), where \( U_q \mathfrak{n}_\pm \) (resp., \( U_q \mathfrak{h} \)) is generated by \( x_{i,n}^\pm \) (resp., \( k_i, h_{i,n}, i \in I, n \leq 0 \)). Hence

\[
U_q \mathfrak{g} = U_q \mathfrak{g} = U_q \mathfrak{g} \oplus (U_q \mathfrak{g} \cdot (U_q \mathfrak{n}_0) \cdot U_q \mathfrak{g}),
\]

where \((U_q \mathfrak{n}_0) \cdot U_q \mathfrak{g}\) stands for the augmentation ideal of \( U_q \mathfrak{n}_\pm \). Denote by \( h_q \) the projection \( U_q \mathfrak{g} \rightarrow U_q \mathfrak{h} \) along the last two summands. We denote by the same letter its restriction to \( U_q \mathfrak{b}_- \).

**Definition 2.** The map \( \chi_q : \text{Rep} U_q \mathfrak{g} \rightarrow U_q \mathfrak{h}[[z]] \) is the composition of \( \nu_q : \text{Rep} U_q \mathfrak{g} \rightarrow U_q \mathfrak{b}_-[[z]] \) and \( \text{h}_q : U_q \mathfrak{b}_- \rightarrow U_q \mathfrak{h} \).

**Lemma 3.** The map \( \chi_q \) is a ring homomorphisms.

**Proof.** Let \( \mathfrak{z}_q (\mathfrak{g}) \) be the commutative subalgebra of \( U_q \mathfrak{g} \) generated by \( t_V[n], V \in \text{Rep} U_q \mathfrak{g}, n \leq 0 \). Let us show that the restriction of \( \text{h}_q \) to \( \mathfrak{z}_q (\mathfrak{g}) \) is a ring homomorphism \( \mathfrak{z}_q (\mathfrak{g}) \rightarrow U_q \mathfrak{h} \). Since \( \nu_q \) is a ring homomorphism according to Proposition 2, this will prove the statement of the lemma.

Consider two elements \( A, B \in \mathfrak{z}_q (\mathfrak{g}) \). By construction, both of them have degree 0 with respect to the \( Q \)-gradation on \( U_q \mathfrak{g} \). Hence we can write \( A = A_0 + A_1 \), where \( A_0 = \text{h}_q(A) \in U_q \mathfrak{h} \), \( A_1 \in (U_q \mathfrak{n}_0) \cdot U_q \mathfrak{g} \), and \( B = B_0 + B_1 \), where \( B_0 = \text{h}_q(B) \in U_q \mathfrak{h} \), \( B_1 \in U_q \mathfrak{g} \cdot (U_q \mathfrak{n}_0) \). But then \( \text{h}_q(AB) = \text{h}_q(A_0B_0) + \text{h}_q(A_0B_1) + \text{h}_q(A_1B_0) + \text{h}_q(A_1B_1) = A_0B_0 \), which is what we needed to prove. □

3.3. The image of \( \chi_q \). In order to describe the image of \( \chi_q \), we need an explicit formula for the universal \( R \)-matrix. This formula was obtained by Khoroshkin and Tolstoy \[43\] and, by a different method, by Levendorsky-Soibelman-Stukopin \[48\] (in the case of \( U_q \mathfrak{sl}_2 \)) and Damiani \[16\] (for general \( U_q \mathfrak{g} \)).

Denote by \( U_q \mathfrak{n}_\pm \) the subalgebra of \( U_q \mathfrak{g} \) generated by \( x_{i,n}^\pm, i \in I, n \in \mathbb{Z} \). Let \( \tilde{B}(q) \) be the inverse matrix to \( B(q) \) from Sect. \[21\]. The formula for the universal \( R \)-matrix then reads:

\[
R = R^+ R^0 R^- T,
\]

where

\[
R^0 = \exp \left( -(q - q^{-1}) \sum_{n>0} \frac{n}{|n|_q} \tilde{B}_{ij}(q^n) h_{i,n} \otimes h_{j,-n} \right),
\]

\( R^\pm \in U_q \mathfrak{n}_\pm \otimes U_q \mathfrak{n}_\pm \), and \( T \) acts as follows: if \( x, y \) satisfy \( k_i \cdot x = q^{(\lambda,\alpha_i)} x, k_i \cdot y = q^{(\mu,\alpha_i)} y \), then

\[
T \cdot x \otimes y = q^{(-\lambda,\mu)} x \otimes y.
\]

**Remark 3.1.** Note that the above formula for \( T \) differs from Drinfeld’s formula \[22\] by the replacement \( q \rightarrow q^{-1} \). This is because the comultiplication that we use differs in the same way from that of \[22\] (cf. Remark \[21\]). □
The computation of \( \chi_q(V) \) for \( V \in \text{Rep} U_q \wedge \) proceeds along the lines of the computation of Ding and Etingof in Sect. 3 of [19]. First, the projection onto \( U_q \wedge \) eliminates the factor \( R^- \) from the formula, and then taking the trace eliminates \( R^+ \) (recall that \( U_q \wedge \) acts nilpotently on \( V \)). Hence we are left with

\[
\chi_q(V) = \text{Tr}_V \left[ q^{2\rho} \exp \left( -\frac{1}{n} \sum_{n>0} B_{ij}(q^n) z^n \pi_V(h_{i,n}) \otimes h_{j,-n} \right) (\pi_V \otimes 1)(T) \right].
\]

Now let

\[
\tilde{h}_{i,-m} = \sum_{j \in I} q^m - q^{-m} B_{ij}(q^m) h_{j,-m} = \sum_{j \in I} C_{ji}(q^m) h_{j,-m},
\]

where \( C(q) \) is the inverse matrix to \( C(q) \) defined in Sect. 2.1. Set

\[
Y_{i,a} = q^{2(\rho,\omega_i)_{k_i^{-1}}} \exp \left( -\frac{1}{n} \sum_{n>0} \tilde{h}_{i,-n} z^n a^n \right), \quad a \in \mathbb{C}^\times.
\]

We assign to \( Y_{i,a} \) the weight \( \pm \omega_i \).

Before we state our theorem, we need to introduce some more notation.

Let \( \chi : \text{Rep} U_q \wedge \rightarrow \mathbb{Z}[y_{i}^{\pm 1}]_{i \in I} \) be the ordinary character homomorphism, \( \beta \) be the homomorphism \( \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^\times} \rightarrow \mathbb{Z}[y_{i}^{\pm 1}]_{i \in I} \) sending \( Y_{i,a} \) to \( y_{i}^{\pm 1} \), and \( \text{res} : \text{Rep} U_q \wedge \rightarrow \text{Rep} U_q \wedge \) be the restriction homomorphism.

Given a subset \( J \) of \( I \) we denote by \( U_q \wedge J \) the subalgebra of \( U_q \wedge \) generated by \( k_{i,n}, h_{i,n}, x_{i,n}^{\pm} \), \( i \in J, n \in \mathbb{Z} \). Let \( \text{res}_J : \text{Rep} U_q \wedge \rightarrow \text{Rep} U_q \wedge J \) be the restriction map and \( \beta_J \) be the homomorphism \( \mathbb{Z}[Y_{i,a}]_{i \in I} \rightarrow \mathbb{Z}[Y_{i,a}]_{i \in J} \), sending \( Y_{i,a} \) to itself for \( i \in J \) and to 1 for \( i \notin J \).

**Theorem 3.**

1. \( \chi_q \) is an injective homomorphism from \( \text{Rep} U_q \wedge \) to \( \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^\times} \subset U_q \wedge[[z]]. \)

2. The diagram

\[
\begin{array}{ccc}
\text{Rep} U_q \wedge & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}]_{i \in I} \\
\downarrow \text{res} & & \downarrow \beta \\
\text{Rep} U_q \wedge & \xrightarrow{\chi} & \mathbb{Z}[y_{i}^{\pm 1}]_{i \in I}
\end{array}
\]

is commutative.
(3) The diagram
\[
\begin{array}{ccc}
\text{Rep } U_q \hat{\mathfrak{g}} & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a_i}]_{i \in I} \\
\text{res} & & | \\
\text{Rep } U_q \hat{\mathfrak{g}} J & \xrightarrow{\chi_{q,J}} & \mathbb{Z}[Y_{i,a_i}]_{i \in J}
\end{array}
\]
is commutative.

Proof. According to Proposition \[\text{I}\], the eigenvalues of \[\pi_V(k_i) \exp \left( (q - q^{-1}) \sum_{n>0} z^n \pi_V(h_{i,n}) \right) = \pi_V(\Phi^+_i(z)) \]
are given by

\[
q^{\deg Q_i - \deg R_i} \frac{Q_i(zq_i^{-1})R_i(zq_i)}{Q_i(zq_i)R_i(zq_i^{-1})},
\]
where

\[
Q_i(z) = \prod_{r=1}^{k_i} (1 - za_{ir}), \quad R_i(z) = \prod_{s=1}^{l_i} (1 - zb_{is}).
\]

This implies that a typical eigenvalue of \[\pi_V(h_{i,n})\] equals

\[
\frac{q^{n} - q^{-n}}{n(q - q^{-1})} \left( \sum_{r=1}^{k_i} a_{ir}^n - \sum_{s=1}^{l_i} b_{is}^n \right), \quad n > 0.
\]

Substituting this into formula \[\text{(3.9)}\] we obtain that \[\chi_q(V)\] is a linear combination of monomials

\[
\prod_{i \in I} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1},
\]
with positive integral coefficients. Furthermore, it follows from the construction that the set of weights of these monomials is the set of weights of \[V\] counted with multiplicities. Thus, the image of \[\chi_q\] lies in \[\mathbb{Z}[Y_{i,a_i}]_{i \in I,a_i \in \mathbb{C}^\times}\], and we obtain part (2) of the theorem. Part (3) is also clear.

It remains to show that \[\chi_q\] is injective. Note that \[\chi_q(V_{\omega_i}(a_i))\] equals \[Y_{i,a_i}\] plus the sum of monomials of lower weight. By Corollary \[\text{I}\] for any finite-dimensional irreducible representation \[V\], \[\chi_q(V)\] equals \[Y_{i_1,a_1} \ldots Y_{i_n,a_n}\], where the set \((i_1, a_1), \ldots, (i_n, a_n)\) is uniquely determined by \[V\] up to permutation, plus the sum of monomials of lower weight. Since \[Y_{i,a_i}\]’s are algebraically independent in \[U_q \mathfrak{h}[[z]]\], this shows that \[\chi_q\] is injective. \[\square\]

Corollary 2. \[\text{Rep } U_q \hat{\mathfrak{g}}\] is a commutative ring that is isomorphic to \[\mathbb{Z}[t_{i,a_i}]_{i \in I,a_i \in \mathbb{C}^\times}\], where \[t_{i,a}\] is the class of \[V_{\omega_i}(a)\].

Remark 3.2. D. Kazhdan and V. Chari have communicated to us two alternative proofs of commutativity of \[\text{Rep } U_q \hat{\mathfrak{g}}\].

The first is based on the fact that for any pair of finite-dimensional representations, \[V\] and \[W\], \[PR_{V,W}(z) : V(z) \otimes W \to W \otimes V(z)\], where \[R_{V,W}(z) = (\pi_V(z) \otimes \pi_W)(\mathcal{R}) \in \mathbb{Z}[[z]]\],
\[ \text{End}(V \otimes W)[[z]] \] and \( P(a \otimes b) = b \otimes a \), is an expansion of a meromorphic function in \( z \), which is an isomorphism for generic \( z \in \mathbb{C} \).

The second proof relies on Proposition 5.1(b) from [14], which states that \( V(\mathcal{P})^* \) is isomorphic to \( V(\mathcal{P}^*) \), where \( P^*_t(u) = P_t(uq^\kappa) \), and \( \kappa \) is a constant (actually, \( \kappa = -r^\alpha h^\alpha \)). Here \( t \) is defined by \( \alpha(t) = -w_0(\alpha_i) \), where \( w_0 \) is the longest element of the Weyl group of \( \mathfrak{g} \). Since \( (V \otimes W)^* = W^* \otimes V^* \), one can use this result to compare the composition factors in tensor products \( V \otimes W \) and \( W \otimes V \).

**Remark 3.3.** The reader may wonder why we do not consider the “naive” character \( \text{Tr}_V \Phi_t^\pm(u) \). In the case of \( U_q\mathfrak{sl}_2 \), explicit computation shows that \( \text{Tr}_V \Phi_t^\pm(u) \) is independent of \( u \), and equals \( \text{Tr}_V k_t^\pm \). This implies that in general \( \text{Tr}_V \Phi_t^\pm(u) = \text{Tr}_V k_t^\pm \). \( \square \)

**Remark 3.4.** The definition of \( q \)-characters carries over to the case of Yangians in a straightforward way. The resulting characters essentially coincide with the characters introduced by Knight [14]. \( \square \)

### 4. THE STRUCTURE OF \( q \)-CHARACTERS

#### 4.1. The case of \( U_q\mathfrak{sl}_2 \)

Let us recall the classification of finite-dimensional representations of \( U_q\mathfrak{sl}_2 \) due to Chari and Pressley [12, 13].

For each \( r \in \mathbb{Z}, r > 0 \), and \( a \in \mathbb{C}^\times \) set

\[ P(u)^{(r)} = \prod_{k=1}^{r} (1 - uaq^{-2k+1}). \]

Denote by \( W_r(a) \) the irreducible representation of \( U_q\mathfrak{sl}_2 \) with highest weight \( P(u)^{(r)} \). Recall that such a representation is unique up to isomorphism. These representations can be constructed explicitly, see [12, 13]. We will need from the construction only the formulas for the spectra of the operators \( \Phi_t^\pm(u) \).

The representation \( W_r(a) \) has a basis \( \{ v_i^{(r)} \}_{i=0,\ldots,r} \), and \( \Phi_t^\pm(u) \) acts on them as follows:

\[ \Phi_t^\pm(u) \cdot v_i^{(r)} = q^r (1 - uaq^{-2i}) (1 - uaq^{r+2}) (1 - uaq^{-2i+2}) (1 - uaq^{-2i+2}) \cdot \frac{1 - uaq^{r+2i}}{1 - uaq^{-2r+2i}} \cdot \frac{1 - uaq^{r+2i-2}}{1 - uaq^{-2r+2i-2}} v_i^{(r)} \]

(we consider the right hand side as a power series in \( u^{\pm1} \)).

Using these formulas we obtain the following expression for the \( q \)-character of \( W_r(a) \):

\[ \chi_q(W_r(a)) = \prod_{k=1}^{r} Y_{aq^{-2k+1}} \left( \sum_{i=0}^{r} \prod_{j=1}^{i} A_{aq^{-2j+2}}^{-1} \right), \]

where

\[ A_a = Y_{aq} Y_{aq^{-1}} = q^2 \Phi^{-1}(z^{-1}a^{-1}). \]
Following [12, 13], call the set \( \Sigma_{a,r} = \{aq^{r-2k+1}\}_{k=1}^{r} \) a \( q \)-segment of length \( r \) and with center \( a \). Two \( q \)-segments are said to be in special position if their union is a \( q \)-segment that properly contains each of them.

**Theorem 4** ([12, 13]). The tensor product \( W_{r_1}(b_1) \otimes \ldots \otimes W_{r_m}(b_m) \) is irreducible if and only if none of the segments \( \Sigma_{r_i}(b_i) \) are in pairwise special position. Further, each irreducible finite-dimensional representation of \( U_q \hat{sl}_2 \) is isomorphic to a tensor product of this form.

Theorem 4 suggests the following construction of the irreducible representation \( V(P) \) of \( U_q \hat{sl}_2 \):

\[
V(P) = \bigotimes_{i=1}^{m} V(b_i)
\]

where each \( V(b_i) \) is the irreducible representation corresponding to \( b_i \). Theorem 4 together with the multiplicative property of \( q \)-characters imply that the \( q \)-character of any irreducible finite-dimensional representation of \( U_q \hat{sl}_2 \) is the product of the \( q \)-characters given by formula (4.3). We conclude that the \( q \)-character of the irreducible representation corresponding to \( P(u) = \prod_{i=1}^{m} (1 - ua_i) \) equals

\[
Y_{a_1} \ldots Y_{a_n} \left( 1 + \sum_{r} M'_r \right)
\]

where each \( M'_r \) is a monomial of the form \( A_{c_1}^{-1} \ldots A_{c_l}^{-1} \) with \( c_j \in \bigcup a_i q^{2z} \).

Therefore we can associate to an irreducible finite-dimensional representation \( V \) of \( U_q \hat{sl}_2 \) an oriented graph \( \Gamma_V \), whose vertices are labeled the monomials appearing in the \( q \)-character of \( V \). We connect the vertices corresponding to monomials \( M_1 \) and \( M_2 \) by an arrow pointing towards \( M_2 \), if \( M_2 = M_1 A_{c}^{-1} \) and assign to this arrow the number \( c \). It is clear that the graph \( \Gamma_V \) is connected.

**Definition 3.** Let \( P(u) \) be a polynomial with the set of inverse roots \( \{a_i\} \) split in a unique way into a union of segments \( \Sigma_{r_i}(b_i) \) that are not in pairwise special position. Then \( P(u) \) is called **irregular** if \( \Sigma_{r_i}(b_i) \subset \Sigma_{r_j}(b_j) \) and \( \Sigma_{r_i}(b_i q^2) \subset \Sigma_{r_j}(b_j) \) at least for one pair \( i \neq j \). Otherwise, \( P(u) \) is called **regular**.

**Definition 4.** Given a ring of polynomials \( \mathbb{Z}[x_{\alpha}^\pm]_{\alpha \in A} \), let us call a monomial in this ring **dominant** if it has the form \( \prod_{k=1}^{n} x_{\alpha_k} \), i.e. it does not contain \( x_{\alpha}^{-1} \).

**Lemma 4.** The irreducible representation \( V(P(u)) \) contains dominant terms other than the one corresponding to the highest weight vector if and only if \( P(u) \) is irregular.

**Proof.** Let \( \bigcup \Sigma_{r_i}(b_i) \) be the splitting of the set \( \{a_i\} \) of inverse roots of \( P(u) \) into a union of segments that are not in pairwise special position. By Theorem 4, \( V(P(u)) \simeq W_{r_1}(b_1) \otimes \ldots \otimes W_{r_m}(b_m) \). According to formula (4.3), \( \chi_q(W_{r}(a)) \) is the sum of the monomials

\[
\prod_{i=0}^{m-1} Y_{aq^{2i-r+1}} \prod_{j=m+1}^{r+1} Y_{aq^{2j-r+1}}, \quad m = 0, \ldots, r.
\]

Suppose that the product \( \chi_q(W_{r_1}(b_1)) \ldots \chi_q(W_{r_m}(b_m)) \) contains a dominant term other the monomial corresponding to the highest weight vector. Then this term is the product
of monomials $M_k$ from each $\chi_q(W_{r_k}(b_k))$, at least one of which is not dominant. Without loss of generality we can assume that $M_1$ is not dominant. But it is clear that if $M_1M_2\ldots M_m$ is dominant, then so is $M_1\tilde{M}_2\ldots \tilde{M}_m$, where $\tilde{M}_k$ is the dominant term of $\chi_q(W_{r_k}(b_k))$, $\tilde{M}_k = \prod_{i=0}^{r_k-1} Y_{aq^i}$. Formula (4.6) shows that for $M_1\tilde{M}_2\ldots \tilde{M}_m$ to be dominant, the intersection between $\Sigma_{r_1}(b_1q^2)$ and the union of the segments $\Sigma_{r_j}(b_j)$, $j \neq 1$, has to be non-empty. Since the segments are not in pairwise special position by our assumption, this immediately implies that $P(u)$ is irregular.

The converse statement is now also clear. $\square$

For instance, if each $a_i$ has multiplicity 1 (i.e., none of the segments $\Sigma_{r_i}(b_i)$ is contained in another), then according to Lemma 4, $\chi_q(V(\prod_{i=1}^{n}(1-ua_i)))$ has no dominant terms other than $Y_{a_1}\ldots Y_{a_n}$.

Lemma 4 implies the following result. Denote by $\mathbb{Z}_+[\{x_a\}_{a \in A}]$ the subset of $\mathbb{Z}[\{x_a^{\pm 1}\}_{a \in A}]$ consisting of all linear combinations of monomials in $x_\alpha^{\pm 1}$ with positive integral coefficients. Let $\text{Rep}_{reg} U_q\hat{sl}_2$ be the abelian subcategory of $\text{Rep} U_q\hat{sl}_2$ with the irreducible objects $V(P(u))$ with regular polynomials $P(u)$. Denote by $\text{Rep}_{reg} U_q\hat{sl}_2$ the Grothendieck ring of this subcategory.

**Corollary 3.** Let $\mathcal{V} \in \text{Rep}_{reg} U_q\hat{sl}_2$ be such that $\chi_q(\mathcal{V}) \in \mathbb{Z}_+[\{Y_a^{\pm 1}\}_{a \in \mathbb{C}^\times}]$. Then $\mathcal{V}$ is a linear combination of irreducible representations of $U_q\hat{sl}_2$ with positive integral coefficients only.

**Proof.** Suppose that there exist irreducible representations $V(P_i), i = 1, \ldots, n$, and $V(Q_j), j = 1, \ldots, m$, where $P_i$’s and $Q_j$’s are regular polynomials, such that

$$
\sum_{i=1}^{n} \chi_q(V(P_i)) = \sum_{j=1}^{m} \chi_q(V(Q_j)).
$$

The left hand side contains dominant terms $Y_{a_1}^{(i)}\ldots Y_{a_n}^{(i)}$, where $P_i(u) = \prod_{k=1}^{n_i}(1-ua_k^{(i)})$. According to Lemma 4, the right hand side contains such monomials if and only if each $P_i(u)$ equals some $Q_j(u)$. Repeating this argument for the right hand side, we see that each representation $V(P_i)$ is isomorphic to a unique representation $V(Q_j)$, and therefore the relation (4.7) is empty. $\square$

Unfortunately, the statement of Corollary 3 is not true if $\mathcal{V}$ is not assumed to lie in $\text{Rep}_{reg} U_q\hat{sl}_2$, as the following counter-example, due to E. Mukhin, shows:

$$
\chi_q(W_1(a) \otimes W_2(aq)) + \chi_q(W_1(a) \otimes W_2(aq^{-1})) - \chi_q(W_1(a)) = Y_a Y_{aq} + Y_a Y_{aq^{-1}} + 2Y_a Y_{aq^2} + Y_a Y_{aq^{-2}} + 2Y_a Y_{aq^2} Y_{aq^{-2}} + 2Y_a Y_{aq^2} Y_{aq^{-2}} + Y_a Y_{aq^2} + Y_a Y_{aq^{-2}}.
$$

This is an element of $\mathbb{Z}_+[\{Y_a^{\pm 1}\}_{a \in \mathbb{C}^\times}]$, which lies in the image of $\chi_q$, but is not the $q$–character of an actual representation of $U_q\hat{sl}_2$.

Finally, note that it is possible to write down a closed formula for the $q$–characters of all irreducible $U_q\hat{sl}_2$–modules. A similar formula in the Yangian case (following Knight’s definition of character [44]) was obtained by Chari and Pressley in [15].
4.2. **General case.** Now we can generalize some of the results of the previous section. Let us set

\[(4.8) \quad A_{i,a} = q^{2k_i^{-1}} \exp \left( -(q - q^{-1}) \sum_{n>0} h_{i,n} z^n a^n \right) = q^{2\Phi^-_i(z^{-1}a^{-1})}, \quad a \in \mathbb{C}^\times.\]

Clearly, \(A_{i,a} \in \mathcal{Y}\). Using formula (3.10), we can express \(A_{i,a}\) in terms of \(Y_{j,b}\)'s:

\[(4.9) \quad A_{i,a} = Y_{i,a} q^i Y_{i,a}^{-1} \prod_{j_i=1} \prod_{j_i=2} Y_{j,a}^{-1} Y_{j,a}^{-1} \prod_{j_i=3} Y_{j,a}^{-1} Y_{j,a}^{-1}.\]

**Proposition 3.** The \(q\)-character of the irreducible finite-dimensional representation \(V(P)\), where

\[(4.10) \quad P_i(u) = \prod_{k=1}^{n_i} (1 - u a_k^{(i)}), \quad i \in I,\]

equals

\[(4.11) \quad \prod_{i \in I} \prod_{k=1}^{n_i} Y_{i,a_k^{(i)}} \left( 1 + \sum M_p' \right),\]

where each \(M_p'\) is a monomial in \(A_{j,c}^{\pm 1}\).

In order to prove this, we consider the \(R\)-matrices

\[R_{V,W}(z) = (\pi_V(z) \otimes \pi_W)(R) = \pi_W(R) \in \text{End}(V \otimes W)[[z]].\]

Let us recall the general result about the \(R\)-matrices (see [35] and [25], Prop. 9.5.3), which follows from their crossing symmetry property.

**Proposition 4.** For any pair of irreducible representations \(V = V(P), W = V(Q)\) of \(U_q\hat{\mathfrak{g}}\),

\[R_{V,W}(z) = f_{V,W}(z) R_{V,W}(z),\]

where the matrix elements of \(R_{V,W}(z)\) are the \(z\)-expansions of rational functions in \(z\) that are regular at \(z = 0\), with \(R_{V,W}(z) \cdot v_P \otimes v_Q = v_P \otimes v_Q\), and \(f_{V,W}(z)\) can be represented in the form

\[f_{V,W}(z) = q^{-(\lambda,\mu)} \prod_{n=1}^{\infty} \rho_{V,W}(z q^{2\nu^\vee \lambda^\vee} n),\]

where \(\lambda\) and \(\mu\) are the degrees of \(P\) and \(Q\), respectively, and \(\rho(z)\) is the expansion of a rational function.

Now observe that

\[(4.12) \quad t_V(z) \cdot v_Q = h_q(t_V(z)) \cdot v_Q = \chi_q(V) \cdot v_Q,\]

so \(v_Q\) is an eigenvector of \(t_V(z)\). According to Theorem 3.1, \(\chi_q(V)\) is a polynomial in \(Y_{i,a}^{\pm 1}\). Recall that \(Y_{i,a}\) is a power series in \(z\), whose coefficients are polynomials in the generators \(k_j, h_{j,n}, j \in I, n < 0\), and rational functions in \(q^n\). We have:

\[Y_{i,a}^{\pm 1} \cdot v_Q = Y_i Q(z a)^{\pm 1} v_Q,\]
where \( Y^Q_i(z) \in \mathbb{C}[[z]] \). Thus, the eigenvalue of \( t_V(z) \) on \( v_Q \) equals \( \chi_q(V) \), in which we substitute each \( Y_{i,a} \) by \( Y^Q_i(za) \). Each \( Y^Q_i(za) \) can be found if we solve the equations (4.9) for \( i \in I \), since we know that the value of \( A_{i,a} = q^{-2}f_i(z^{-1}a^{-1}) \) on \( v_Q \) equals 
\[
q_i^{2+\deg Q_i} \frac{Q_i(z^{-1}a^{-1}q_i^{-1})}{Q_i(z^{-1}a^{-1}q_i)}.
\]

On the other hand, let us choose bases of generalized eigenvectors of \( \Phi^\pm(u) \) in \( V \) and \( V(Q) \), so that the latter includes vector \( v_Q \). The eigenvalue of \( t_V(z) \) on the highest weight vector \( v_Q \in V(Q) = W \) equals the sum of the diagonal entries of the \( R \)-matrix \( R_{V,W}(z) \) written in this basis, which correspond to the vectors \( v \otimes v_Q \), where \( v \) is a basis vector of \( V \) (up to \( q^\rho \)). The proof of Theorem 3 shows that these diagonal entries are in one-to-one correspondence with the monomials appearing in the \( q \)-character \( \chi_q(V) \), namely, the diagonal entry corresponding to the monomial \( Y_{i_1,a_1} \ldots Y_{i_k,a_k} \) equals \( Y^Q_{i_1}(za_1) \ldots Y^Q_{i_k}(za_k) \). This observation allows us to compute \( f_{V,W}(z) \) explicitly.

4.3. Computation of \( f_{V,W}(z) \). Consider the diagonal matrix element \( f_{V,W}(z) \) of \( R_{V,W}(z) \) corresponding to the vector \( v_P \otimes v_Q \). It has been computed in some cases (see, e.g., [31][1][23]). The following proposition gives a general formula for an arbitrary pair of \( R \)-matrix \( v_Q \) of irreducible representations of \( U_q\mathfrak{g} \).

**Proposition 5.** Let \( P = (P_i)_{i \in I} \), where
\[
P_i(u) = \prod_{k=1}^{n_i} (1 - ua_k^{(i)}), \quad i \in I.
\]
Then
\[
f_{V(P),V(Q)}(z) = q^{-(\lambda,\mu)} \prod_{i \in I} \prod_{k=1}^{n_i} Y^Q_i(za_k^{(i)}).
\]

To find \( Y^Q_i(z) \) explicitly, note first that if \( Q_j(u) = \prod_{l=1}^{m_j} (1 - ub_l^{(j)}) \), then
\[
Y^Q_i(z) = \prod_{j \in I} \prod_{l=1}^{m_j} Y^{(j)}_l(z/b_l^{(j)}), \tag{4.13}
\]
where \( Y^{(j)}_l(z/b) \) corresponds to \( Q = P_b^{(i)} \), where \( (P_b^{(j)})_i = (1 - ub) \), if \( i = j \) and 1, if \( i \neq j \) (this is the case when \( W = V_{\omega_j}(b) \)).

In the same way as in the proof of Theorem 3 we find that the eigenvalue of \( h_{i,-n}, n > 0 \), on the highest weight vector of \( V_{\omega_j}(b) \) equals \( \delta_{i,j}(q^n_j - q^{-n}_j)b^{-n}/n(q - q^{-1}) \). Using formula (3.10), we find that the eigenvalue of \( \tilde{h}_{i,-n} \) on this vector equals
\[
b^{-n}(q^n_j - q^{-n}_j) \tilde{C}_{ji}(q^n). \tag{4.14}
\]
The matrix \( \tilde{C}(x) = C(x)^{-1} \) is known explicitly for \( g \) of classical types (see Appendix C of [33]), and for an arbitrary \( g \) the determinant of \( C(x) \) is known (see [9]). These results imply that \( \tilde{C}_{ij}(x) \) equals \( C_{ij}(x)/(1 - x^{2n}h^{\vee}) \), where \( C_{ij}(x) \) is a polynomial in \( x \) with integral coefficients. When we substitute it into formula (4.14) and then into
formula (3.11), we obtain that each monomial \( \pm x^p \) appearing in \( C_{ij}(x) \) contributes the factor
\[
(4.15) \quad \left( \frac{z q^p q_i q^{-1}}{z q^p q_i q^{2r^0 h^0}} \right)_{\infty}^{\pm 1} \left( \frac{z q^p q_i q^{-1}}{z q^p q_i q^{2r^0 h^0}} \right)_{\infty}^{\mp 1}
\]
to \( Y_{i}^{(j)}(z) \), where
\[
(a; b)_{\infty} = \prod_{n=1}^{\infty} (1 - ab^n).
\]

Thus, for any irreducible representations \( V, W \), the function \( f_{V,W}(z) \) equals \( q^{-(\lambda, \mu)} \) times the product of the factors \( \left( \frac{z q^n q_i q^{-1}}{z q^n q_i q^{2r^0 h^0}} \right)_{\infty}^{\pm 1} \), where \( n \in \mathbb{Z} \). This statement is stronger than the corresponding statement of Proposition 4 in that we claim that the zeroes and poles of \( \rho(x) \) are necessarily integral powers of \( q \). This result can be used to gain insights into the structure of the poles of the \( R \)–matrices.

Following Akasaka and Kashiwara [1], let us denote by \( d_{V,W}(z) \) the denominator of \( R_{V,W}(z) \), i.e., the polynomial in \( z \) of smallest degree, such that \( d_{V,W}(z)R_{V,W}(z) \) has no poles. We normalize it so that its constant term is equal to 1. Using the crossing symmetry of the \( R \)–matrices one can show (see [1], Prop. A.1) that \( f_{V,W}(z) \) satisfies
\[
(4.16) \quad f_{V,W}(z)f_{V,W}^*(z) = c d_{V,W}(z) d_{W,V}(z-1),
\]
where \( c = c' z^n, c' \in \mathbb{C}, n \in \mathbb{Z} \).

As we have shown above, the left hand side is a rational function, whose zeroes and poles are powers of \( q \). Therefore if \( y \) is a pole of \( R_{V,W}(z) \) (i.e., a root of \( d_{V,W}(z) \)) that is not a power of \( q \), then \( y^{-1} \) has to be a pole of \( R_{W,V}(z) \). It is unclear to us at the moment whether one can use this kind of argument to prove that the poles of \( R_{V,W}(z) \) are integral powers of \( q \) as conjectured in [1].

**Remark 4.1.** Here we have used the \( q \)–characters to obtain information about the \( R \)–matrices. Conversely, we can use the information about the \( R \)–matrices to gain insights into the structure of the \( q \)–characters. In fact, \( \chi_q(W) \) can be read off the diagonal entries of the \( R \)–matrices \( R_{W,V\omega_i}(1)(z), i \in I \), corresponding to the highest weight vectors in \( V_{\omega_i}(1) \).

4.4. **Proof of Proposition 3.** Consider the diagonal entry of the \( R \)–matrix corresponding to a monomial
\[
\prod_{i \in I} \prod_{k=1}^{n_i} Y_{i,a_{k}^{(i)}} \cdot \prod_{k=1}^{m} Y_{i, \epsilon_{k}} \cdot m \prod_{k=1}^{m} Y_{i, \epsilon_{k}} (z a_{k})^{\epsilon_{k}}
\]
occuring in \( \chi_q(V) \). Then this entry of the \( R \)–matrix equals
\[
f_{V(P),V(Q)}(z) \cdot \prod_{k=1}^{m} Y_{i, \epsilon_{k}} (z a_{k})^{\epsilon_{k}}.
\]
Proposition 4 then implies that \( \prod_{k=1}^{m} Y_{i, \epsilon_{k}} (z a_{k})^{\epsilon_{k}} \) is a rational function in \( z \) for all \( Q \).

According to formula (4.14), this is equivalent to its being a rational function when
\[ Q = P^{(j)}_b, \forall j \in I, b \in \mathbb{C}^\times. \] The following lemma shows that in that case \( \prod_{k=1}^m Y_{ik,a_k}^{r_k} \) can be written as the product of \( A_{i,c}^{\pm 1} \). This proves Proposition 3.

**Lemma 5.** Suppose that a monomial \( M = \prod_{k=1}^m Y_{ik,a_k}^{r_k} \) is such that \( \prod_{k=1}^m Y_{ik}^{(j)} (za_k)^{r_k} \) is a rational function for all \( j \in I \). Then \( M \) can be written as a product of \( A_{i,c}^{\pm 1} \).

**Proof.** We know that each \( Y_{ik}^{(j)} (za_k)^{r_k} \) is a product of the factors \( 4.15 \). Therefore, without loss of generality, we can assume that each \( a_k = a q_k \) for some \( a \in \mathbb{C}^\times \) and \( l_k \in \mathbb{Z} \). According to formula \( 3.11 \), the above condition on \( M \) then means that the eigenvalue of

\[ a^n \sum_{k=1}^m \zeta_k h_{ik,-n} q_k^{l_k} \]

on \( v_{P_i^{(j)}} \) equals \( a^n \beta(q^n)/n \), where \( \beta(x) \) is a polynomial in \( x^{\pm 1} \) with integral coefficients. In other words, there exist polynomials \( \beta(x) \) and \( \gamma_i(x), i \in I \), such that the eigenvalue of

\[ (q - q^{-1}) \sum_{i \in I} h_{ij,-n} \gamma_i(q^n) \]

is equal to \( \beta(q^n)/n \). But

\[ \sum_{i \in I} h_{ij,-n} \gamma_i(q^n) = (q - q^{-1}) \sum_{i,j \in I} h_{ij,-n} \tilde{C}_{ji}(q^n) \gamma_i(q^n) \]

and since the eigenvalue of \( h_{ij,-n} \) on \( v_{P_i^{(j)}} \) is \( \delta_{i,j} |n| q_i/n \), we obtain that

\[ 4.17 \]

\[ \sum_{i \in I} \tilde{C}_{ji}(x) \gamma_i(x) (x^{r_j} - x^{-r_j}) \]

is a polynomial in \( x^{\pm 1} \). The lemma says that for \( 4.17 \) to be a polynomial for all \( j \in I \), \( \gamma_i(x) \) must have the form

\[ \gamma_i(x) = \sum_{k \in I} C_{ik}(x) R_k(x), \quad i \in I, \]

where \( R_k(x) \) are some polynomials in \( x^{\pm 1} \).

It is clear that if we allow \( \gamma_i(x) \) to be a rational function, and we want \( 4.17 \) to be equal to a polynomial \( R_i^{(j)}(x) \) for all \( j \in I \), then \( \gamma_i(x) \) can be represented in the form

\[ 4.18 \]

\[ \gamma_i(x) = \sum_{k \in I} C_{ik}(x) \frac{R_k'(x)}{x^{r_k} - x^{-r_k}}, \quad i \in I. \]

It remains to show that \( 4.18 \) is a polynomial for all \( i \in I \) if and only if each \( R_i'(x) \) is divisible by \( x^{r_k} - x^{-r_k} \).

Since \( \det C(\pm 1) \neq 0 \), \( R_i'(x) \) is divisible by \( x - x^{-1} \). This proves the result for simply laced \( \mathfrak{g} \), for which \( r_k = 1, \forall k \in I \). For non-simply laced \( \mathfrak{g} \), we can now replace \( 1/(x^{r_k} - x^{-r_k}) \) in formula \( 4.18 \) with \( (x - x^{-1})/(x^{r_k} - x^{-r_k}) \). After that the result is easy to establish by inspection. \( \square \)
4.5. The conjecture. The following conjecture is motivated by explicit examples of the $q$-characters (see Sects. 4.1, 5.4).

**Conjecture 1.** The $q$-character of the irreducible finite-dimensional representation $V(P)$, where $P$ is given by formula (4.10), can be represented in the form (4.11) where each $M'_p$ is a monomial in $A^{-1}_{j,c}, c \in \mathbb{C}^\times$.

Conjecture 1 holds for $U_q \widehat{sl}_N$. In this case the $q$-characters of $V_{\omega_i}(a)$ can be found (see Remark 4.1) from the explicit formulas for the $R$-matrices $R_{V_{\omega_i}(z), V_{\omega_j}(w)}, i,j = 1, \ldots, N - 1$, which are known in the literature (see [17, 11]). For other classical $\mathfrak{g}$ and $\mathfrak{g} = G_2$, Conjecture 1 can probably also be derived from a case by case analysis based on what is known about the structure of the $R$-matrices of the fundamental representations and the decompositions of their tensor products. The expected formulas for the $q$-characters of the fundamental representations for these algebras are given in [31, 33] (see also Sect. 5.4 below) for classical $\mathfrak{g}$, and [45, 9] for $\mathfrak{g} = G_2$.

Now we give two corollaries to Conjecture 1.

**Corollary 4.**

1. $\chi_q(V_{\omega_i}(a))$ equals $Y_{i,a}(1 + \sum_p M'_p)$, where each $M'_p$ is a monomial in $A^{-1}_{j,q^p}, n \in \mathbb{Z}$, and $Y_{i,a}$ is the only dominant monomial in $\chi_q(V_{\omega_i}(a))$.

2. The monomials $M'_p$ occurring in $\chi_q(V(P))$, where $P$ is given by formula (4.10), are products of $A^{-1}_{j,c}$, where $c \in \bigcup a_k(i) q^Z$.

**Proof.** Let us consider $U_q \widehat{sl}_N$ as a module over $\mathbb{C}(q)$. It then has an algebra automorphism that sends $q$ to $q^{-1}$, $k_i$ to $k_i^{-1}$ and leaving the generators $x_i^\pm$ unchanged. If we apply this automorphism to $R$, we obtain $(S \otimes \text{id})(R)$, where $S$ is the antipode. This means that for any representation $V$ of $U_q \widehat{sl}_N$, if we replace in $\chi_q(V)$ each $Y_{i,a}$ by $Y_{i,-a}$, where $\tau$ is obtained from $a$ by replacing $q$ by $q^{-1}$, then we obtain $\chi_q(V^\tau)$.

On the other hand, [14], Proposition 5.1(b) implies that $V_{\omega_i}(a)^* \simeq V_{\omega_i}(ap^{-1})$, where $p = q^{\nu \cdot h^\vee}$ and $\omega_i = -w_0(\omega_i)$, $w_0$ being the longest element of the Weyl group of $\mathfrak{g}$.

Now, according to Proposition 3 we can write

\[
\chi_q(V_{\omega_i}(a)) = Y_{i,a}(1 + \sum_r M'_r), \quad \chi_q(V_{\omega_i}(a)) = Y_{i,a}(1 + \sum_r M''_r),
\]

where each $M'_r, M''_r$ is a monomial in $A^{-1}_{j,c}$, and

\[
Y_{i,a}^{-1} Y_{i,ap^{-1}}^{-1} = M'_r M''_r.
\]

(4.19) Here $M'_r$ is obtained from $M'_p$ by replacing each $A_{j,c}$ by $A_{j,\tau} - c$.

But Conjecture 1 tells us that both $M'_r$ and $M''_r$ are monomials in $A_{j,c}^{-1}$ only. Therefore they can only be monomials in $A_{j,q^n}, n \in \mathbb{Z}$, for otherwise formula (4.19) can not hold.

This implies that $\chi_q(V_{\omega_i}(a))$ equals $Y_{i,a}(1 + \sum_p M'_p)$, where each $M'_p$ is a monomial in $A_{j,q^n}, n \in \mathbb{Z}$.

Let us now prove that the only dominant term in $\chi_q(V_{\omega_i}(a))$ is $Y_{i,a}$.

In the same way as in the proof of Proposition 3 we can show that this statement is equivalent to the following. If $R_k(x), k \in I$, are polynomials in $x^{\pm 1}$ with non-negative...
integral coefficients, such that

\[ - \sum_{k \in I} C_{jk}(x) R_k(x) + \delta_{ij} \]

is a polynomial with non-negative integral coefficients for all \( j \in I \), then \( R_k(x) = 0, \forall k \in I \) (the matrix \( C_{ij}(x) \) is given in Sect. 2.1). Suppose that this is not so. Let \( l_k \) and \( h_k \) be the lowest and highest degrees of those \( R_k(x) \) that are non-zero. Choose the smallest, \( l_s \), among \( l_k \)'s, and the largest, \( h_t \), among \( h_k \)'s. If there are several \( R_k(x) \) with the same lowest (resp., highest) degree, we pick the \( s \) (resp. \( t \)) that corresponds to the largest value of \( r_k \) (i.e., to the longer root). Then we obtain that \(- \sum_{k \in I} C_{sk}(x) R_k(x) \) contains the monomial \( x^{l_s-r_s} \) with a negative coefficient. This monomial can only be compensated by \( \delta_{is} \). But then \( l_s = r_s \). Applying the same argument to \( h_t \), we obtain that \( h_t = -r_t \), and so \( h_t < l_s \), which is a contradiction. This completes the proof of part (1) of the corollary.

Part (2) follows from the fact that all other representations can be obtained as subfactors of the tensor products of the fundamental representations.

**Corollary 5.** The tensor product \( V_{\omega_1}(a_1) \otimes \ldots \otimes V_{\omega_n}(a_n) \) is irreducible if \( a_j/a_k \notin q^Z \), \( \forall i \neq j \).

**Proof.** If \( V_{\omega_1}(a_1) \otimes \ldots \otimes V_{\omega_n}(a_n) \) is reducible, then \( \chi_q(V_{\omega_1}(a_1)) \ldots \chi_q(V_{\omega_n}(a_n)) \) should contain a dominant term other than the product of the highest weight terms. But for that to happen, for some \( j \) and \( k \), there have to be cancellations between some \( Y^{-1}_{p,aj} q^n \) appearing in \( \chi_q(V_{\omega_j}(a_j)) \) and some \( Y_{r,a_k} q^m \) appearing in \( \chi_q(V_{\omega_k}(a_k)) \). These cancellations may only occur if \( a_j/a_k \in q^Z \).

**Remark 4.2.** Corollary 5 has been conjectured earlier by Akasaka and Kashiwara [1]. Furthermore, they conjectured that \( V_{\omega_i}(a) \otimes V_{\omega_j}(b) \) is reducible only when \( b = a q^n \), where \( n \in Z, |n| \leq r^V h^V \). This can also be derived from Conjecture [1], because it is easy to see that \( \chi_q(V_{\omega_i}(a)) \) is a linear combination of monomials in \( Y_k^{\pm 1} q^n \), where \( n \in Z, 0 \leq n \leq r^V h^V \). We thank Kashiwara for a discussion of these conjectures.

5. **Combinatorics of \( q \)-characters**

5.1. **Interpretation in terms of the joint spectra of \( \Phi^{\pm}_i(u) \).** For \( i \in I, a \in C^x \), we define special polynomials \( R_{ij}^a(u) \) and \( Q_{ij}^a(u) \). Consider the matrix element \( C_{ji}(q) \) of the matrix \( C(q) \) defined in Sect. 2.1. This is a combination of powers of \( q \) with coefficients \( \pm 1 \). We define \( R_{ij, a}(u) \) (resp., \( Q_{ij, a}(u) \)) as the polynomial in \( u \) with constant term 1, whose zeroes are \( a \) times the powers of \( q \) appearing in \( C_{ji}(q) \) with coefficient 1.
$\pm$ eigenvalues of $\Phi_j$ for some $h$

Finally, let us set $V$ the basis $\{x_i\}$, which is an eigenvector of $\Phi_j$.

According to formula (5.1), we can rewrite this as

$$Q_i^a(u) = \frac{P_j^i(u) - \prod_{k=1}^m P_{j,k}^i(u)_{\pm 1}^{\pm 1}}{P_i(u q_i) \prod_{k=1}^m P_{j,k}^i(u q_i)_{\pm 1}^{\pm 1}},$$

for some $c_1, \ldots, c_m \in \mathbb{C}^\times$ (up to obvious overall power of $q$ factors). Explicit calculation shows that

$$\frac{P_j^i(u q_i^{-1})}{P_j^i(u q_i)} = \frac{1 - u a q^{(\alpha_i, \alpha_j)}}{1 - u a q^{-(\alpha_i, \alpha_j)}}.$$}

There is an interesting connection between the eigenvalues of $\Phi_j^\pm(u)$ (and hence the $q$-characters) and the action of the generators $x_{i,n}$ on the finite-dimensional $U_q\hat{g}$-modules.

Let us consider again the $U_q\hat{g}_2$-module $W_r(a)$. The action of the generators $x_n^-$ in the basis $\{v_i|^{(r)}\}$ is given by a very simple formula:

$$x_n^- \cdot v_i^{(r)} = \gamma^n x_0^- \cdot v_i^{(r)} = \gamma^{r-2i} x_0^- \cdot v_i^{(r)}.$$}

Now suppose, more generally, that $v$ is a vector in a finite-dimensional $U_q\hat{g}$-module $V$, which is an eigenvector of $\Phi_j^\pm(u), \forall i \in I$ with the eigenvalues $\Psi_j^\pm(u)$, and that we have:

$$x_j^- \cdot v = \gamma^n x_0^- \cdot v, \quad n \in \mathbb{Z},$$

for some $j \in I$ and $\gamma \in \mathbb{C}^\times$. In that case, using the commutation relations between $h_{i,m}$ and $x_{j,n}$, we obtain that $x_j^- \cdot v$ is also an eigenvector of $\Phi_j^\pm(u), \forall i \in I$, with the eigenvalues

$$\Psi_j^\pm(u) q^{(\alpha_i, \alpha_j)} \frac{1 - u \gamma q^{(\alpha_i, \alpha_j)}}{1 - u \gamma q^{-(\alpha_i, \alpha_j)}}.$$}

According to formula (5.1), we can rewrite this as

$$\Psi_j^\pm(u) q^{(\alpha_i, \alpha_j)} \frac{P_j^\gamma(u q_i^{-1})}{P_j^\gamma(u q_i)},$$
Empirical evidence suggests that in that case the action of the operators $V$ means that the restriction of the form (5.2). Certainly, not all representations have this property: for one thing, it means that the restriction property of the $U$ that one can use them to prove Conjecture 1. Assuming that Conjecture 1 is true.

Therefore if $M$ is the monomial in $Y_{i,a}^{±1}$ that corresponds to vector $v$ in $\chi_q(V)$, then the monomial corresponding to the vector $x_{j,0}^±v$ is $M \cdot A_{j,γ}^{−1}$. This observation clarifies the statements of Proposition 3 and Conjecture 1 in the case when $V$ is spanned by vectors obtained by the action of $x_{j,0}$ on the highest weight vector. Empirical evidence suggests that in that case the action of the operators $x_{n,j}$ indeed has the form (5.2). Certainly, not all representations have this property: for one thing, it means that the restriction of $V$ to $U_q\hat{\mathfrak{g}}$ is irreducible, which is not always the case. Even in the case of $U_q\hat{\mathfrak{g}}$, the modules $W_r(a), r > 0$, seem to be the only representations with this property. Still, such representations apparently exist in general, and it is plausible that one can use them to prove Conjecture 1.

In the following two sections we discuss the combinatorial structure of $q$–characters assuming that Conjecture 1 is true.

5.2. Reconstructing the $q$–character from the highest weight. Recall the restriction property of the $q$–characters from Theorem 3(3): if we apply the homomorphism $\operatorname{res}_{(i)}$ to $\chi_q(V)$, i.e., replace all $Y_{j,b,j} \notin I$ in $\chi_q(V)$ by 1, we obtain the $q_i$–character of the semi-simplification of the restriction of $V$ to $U_q\hat{\mathfrak{g}}(i) \simeq U_q\hat{\mathfrak{g}}_2$.

Given an irreducible representation $W$ of $U_q\hat{\mathfrak{g}}_2$ we write its $q$–character in the form (4.5). We then replace each $A_{i,c}^{−1}$ by $A_{i,c}^{−1}$ and denote the resulting element of $Z[Y_{j,a}^{±1}]_{j \in I}$ by $\chi_q^{(i)}(W)$. It is clear that $\chi_q^{(i)}$ extends linearly to a homomorphism $\operatorname{Rep} U_q\hat{\mathfrak{g}}_2 \to Z[Y_{i,a}^{±1}]$, and its image equals $Z[Y_{i,b} + Y_{i,b}A_{i,bq}]_{b \in \mathbb{C}^×}$.

In view of Theorem 3(3) and Conjecture 1 it is natural to expect that for any $V \in \operatorname{Rep} U_q\hat{\mathfrak{g}}$,

$$\chi_q(V) \in \bigcap_{i \in I} R_i, \tag{5.3}$$

where

$$R_i = Z[Y_{j,a}^{±1}]_{j \neq i,a \in \mathbb{C}^×} \otimes Z[Y_{i,b} + Y_{i,b}A_{i,bq}]_{b \in \mathbb{C}^×}. $$

If $V$ is an actual representation, then $\chi_q(V) \in \bigcap_{i \in I} R_{i,+}$, where $R_{i,+} = R_i \cap Z_{+}[Y_{j,a}^{±1}]_{j \in I}$.

It is natural to conjecture that any element of $\bigcap_{i \in I} R_{i,+}$ equals $\chi_q(V)$ for some representation $V$ of $U_q\hat{\mathfrak{g}}$. In other words, if an element of $Z[Y_{i,a}]_{i \in I,a \in \mathbb{C}^×}$ has good restrictions, i.e., it restricts to $q$–characters of $U_q\hat{\mathfrak{g}}(i)$ for each $i \in I$, then it is necessarily a $q$–character of $U_q\hat{\mathfrak{g}}$. This conjecture is essentially equivalent to Conjecture 2 that we state in Sect. 7. Some evidence for Conjecture 2 coming from the theory of $W$–algebras is presented in Sect. 8.

If this conjecture is true, then the $q$–character of the irreducible representation $V(P)$, where $P_i(u) = \prod_{k=1}^{a_i}(1 - u \alpha_k^{(i)})$, can be constructed combinatorially as follows.

We start with the monomial $M = \prod_{i \in I} \prod_{k=1}^{a_i} Y_{i,a_k^{(i)}}$ corresponding to the highest weight vector and try to reconstruct the $q$–character of $V(P)$ from it. We know that any element of $R_{i,+}$ that contains a monomial of the form $N \cdot \prod_{k=1}^{a_i} Y_{i,a_k^{(i)}}$, where $N \in Z[Y_{j,a}^{±1}]_{j \neq i}$, also contains $N \cdot \chi_q^{(i)}(V(P)_{k=1}^{a_i}(1 - u \alpha_k^{(i)}))$. Thus, the first step is to replace
the monomial $M$ with the product
\[
\prod_{i \in I} \chi_q^{(i)} \left( V \left( \prod_{k=1}^{n_i} (1 - ua_k^{(i)}) \right) \right).
\]

Next, we continue by induction following the rule: whenever $\chi_q(V)$ contains a monomial
\[
\prod_{k=1}^{m} Y_{i,k}^{(i)} N,
\]
where $N \in \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i}$, it should be part of a combination
\[
\chi_q^{(i)} \left( V \left( \prod_{k=1}^{m} (1 - ub_k^{(i)}) \right) N. \right.
\]

Since we know that the representation $V(P)$ is finite-dimensional, the inductive process
should stop after a finite number of steps. The result should be $\chi_q(V(P))$. At each
step we use only our knowledge of the $q$–characters of $U_q \widehat{\mathfrak{sl}}_2$.

**Remark 5.1.** It is clear from the proof of Theorem 3 that if $V \in \text{Rep} U_q \widehat{\mathfrak{g}}$ is a linear
combination of irreducible representations of $U_q \widehat{\mathfrak{g}}$ with positive integral coefficients,
then $\chi_q(V) \in \mathbb{Z}_+[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}$. The counter-example given in Sect. 111 shows that
the converse is not true. However, it is possible that a weaker version still holds.  

5.3. **The graph $\Gamma_V$**. Now we attach to each irreducible finite-dimensional representation $V$ of $U_q \widehat{\mathfrak{sl}}_2$, an oriented colored graph $\Gamma_V$. Its vertices are labeled by the monomials
appearing in the $q$–character of $V$.

Denote the monomial $\prod_{i \in I} \prod_{k=1}^{n_i} Y_{i,k}^{(i)} M_p$ by $M_p$. Two vertices corresponding to
monomials $M_1$ and $M_2$ are connected by an arrow pointing towards $M_2$, if $M_2 = M_1 A_{i,c}^{-1}$
for some $c$, and in $\beta_{(i)}(\chi_q(V))$, the monomial $\beta_{(i)}(M_2)$ does not correspond to a highest
weight vector of the semi-simplification of $V|_{U_q \widehat{\mathfrak{sl}}_{(i)}}$. We then assign to this arrow the
color $i$ and the number $c$.

It follows from the construction that if we erase in $\Gamma_V$ all arrows but those of color $i$, we obtain the graph $\Gamma_q^{(i)}$ of the semi-simplification of $V|_{U_q \widehat{\mathfrak{sl}}_{(i)}}$.
Furthermore, the graphs $\Gamma_V$ are compatible with restrictions to all of its quantum affine subalgebras: if $J$ is a subset of $I$, then the graph of the semi-simplification of the restriction of $V$ to $U_q \widehat{\mathfrak{sl}}_2$ can be obtained from the graph $\Gamma_V$ by erasing the arrows of colors $i \not\in J$.

Conjecture 1 implies that the graphs of the fundamental representations $V_{\omega_i}(a)$ are connected, and moreover, there exists an oriented path from the vertex corresponding to
the highest monomial $Y_{i,a}$ to any other vertex. Indeed, we have shown in the proof
of Corollary 5 that $\chi_q(V_{\omega_i}(a))$ is the sum of monomials of the form $Y_{i,a} \prod_k A_{i,k}^{-1} A_{i,k.q} A_{i,k.q}$, and the only dominant term in $\chi_q(V_{\omega_i}(a))$ is $Y_{i,a}$. Now let $M$ be an arbitrary monomial
in $\chi_q(V_{\omega_i}(a))$ different from $Y_{i,a}$. Then it is not dominant. Suppose that there are
no incoming arrows for this vertex. But then $\beta_{(i)}(M)$ corresponds to a highest weight
vector in the semi-simplification of $V|_{U_q \widehat{\mathfrak{sl}}_{(i)}}$ for each $i \in I$. Hence $M$ is dominant, which
is a contradiction.

Now we see that $M$ has an incoming arrow, we can move up along this arrow. The
weight of the monomial on the other end of this arrow equals that of $M$ plus $\alpha_i$.
Continuing by induction, we arrive at the highest weight monomial $Y_{i,a}$. Thus, we find
an oriented path from the vertex corresponding to the highest monomial $Y_{i,a}$ to the vertex corresponding to a monomial $M$.

We conjecture that the graph $\Gamma_V$ is connected for any irreducible representation $V$ of $U_q\hat{g}$. If this is true, then the reason for irreducible $U_q\hat{g}$–modules being reducible when restricted to $U_qg$ essentially lies in that happening already for each of the $U_q\hat{sl}_2$ subalgebras of $U_q\hat{g}$.

\textbf{Remark 5.2.} The graph $\Gamma_V$ is similar to the crystal graph of $V$. However, there is an important difference: while the arrows of a crystal graph are labeled by the simple roots of $\hat{g}$, i.e., from 0 to $\ell$, the arrows of $\Gamma_V$ are labeled by the simple roots of $g$, i.e., from 1 to $\ell$, and there is also a number attached to each row. The crystal graph is designed so that it respects the subalgebras $U_q\hat{sl}_2$, corresponding to the Drinfeld-Jumbo realization of $U_q\hat{g}$, while $\Gamma_V$ respects the affine subalgebras $U_q\hat{sl}_2$ in the Drinfeld “new” realization.

It would be interesting to develop a theory analogous to the theory of crystal basis [41] for the graphs $\Gamma_V$. $\square$

\section{Examples.}

Here we give examples of $q$–characters and graphs associated to the first fundamental representation of $U_q\hat{g}$, where $g$ is of classical type. These $q$–characters can be obtained (see Remark [4.1]) from the explicit formulas for the $R$–matrices $R_{V\omega_i(z),V\omega_j(w)}$ that are known in the literature: see [17, 1] for $A_\ell$, [1] for $C_\ell$, and [18] for $B_\ell$ and $D_\ell$. They also agree with the formulas for the eigenvalues of the transfer-matrices [50, 51, 5, 46].

In all cases,

$$\chi_q(V\omega_1(a)) = \sum_{i \in J} \Lambda_{i,a}.$$ 

Additional examples can be found in [31, 33, 45].

\subsection{The $A_\ell$ series.}

$J = \{1, \ldots, \ell + 1\}$.

$$\Lambda_{i,a} = Y_{i,aq^{i-1}}Y_{i-1,aq^i}^{-1}, \quad i = 1, \ldots, \ell + 1.$$ 

Equivalently,

$$\Lambda_{1,a} = Y_{1,a},$$

$$\Lambda_{i,a} = \Lambda_{i-1,a}A_{i-1,aq^{i-1}}^{-1}, \quad i = 2, \ldots, \ell.$$ 

\subsection{The $B_\ell$ series.}

$J = \{1, \ldots, \ell, 0, \ell, \ldots, \ell\}$.
\[ \Lambda_{i,a} = Y_{i,a} q^{2i-2} Y^{-1}_{i-1,a} q^{2i}, \quad i = 1, \ldots, \ell - 1, \]
\[ \Lambda_{\ell,a} = Y_{\ell,a} q^{2\ell-3} Y_{\ell-1,a} q^{2\ell-1}, \]
\[ \Lambda_{0,a} = Y_{\ell,a} q^{2\ell-3} Y_{\ell,a} q^{2\ell+1}, \]
\[ \Lambda_{\ell-1,a} = Y_{\ell-1,a} q^{2\ell-2} Y_{\ell-1,a} q^{2\ell+1}, \]
\[ \Lambda_{i,a} = Y_{i-1,a} q^{4\ell-2i-2} Y_{i,a} q^{4\ell-2i}, \quad i = 1, \ldots, \ell - 1. \]

Equivalently,
\[ \Lambda_{1,a} = Y_{1,a}, \]
\[ \Lambda_{i,a} = \Lambda_{i-1,a} q^{-1} Y^{-1}_{i-1,a}, \quad i = 2, \ldots, \ell, \]
\[ \Lambda_{0,a} = \Lambda_{\ell,a} q^{-1} Y_{\ell,a}, \]
\[ \Lambda_{\ell-1,a} = \Lambda_{0,a} q^{-1} Y_{\ell-1,a}, \]
\[ \Lambda_{i,a} = \Lambda_{i+1,a} q^{-1} Y_{i,a} q^{2(\ell-i-1)}, \quad i = 1, \ldots, \ell - 1. \]

5.4.3. The \( C_{\ell} \) series. \( J = \{1, \ldots, \ell, \ell, \ldots, 1\} \).
\[ \Lambda_{i,a} = Y_{i,a} q^{i-1} Y^{-1}_{i-1,a}, \quad i = 1, \ldots, \ell, \]
\[ \Lambda_{\ell,a} = Y_{i-1,a} q^{2\ell-2i+2} Y_{i,a} q^{2\ell-2i+3}, \quad i = 1, \ldots, \ell. \]

Equivalently,
\[ \Lambda_{1,a} = Y_{1,a}, \]
\[ \Lambda_{i,a} = \Lambda_{i-1,a} A_{i-1,a}^{-1}, \quad i = 2, \ldots, \ell, \]
\[ \Lambda_{\ell,a} = \Lambda_{\ell,a} A_{\ell,a}^{-1}, \]
\[ \Lambda_{i,a} = \Lambda_{i+1,a} A_{i,a}^{-1} q^{2\ell-2i-2}, \quad i = 1, \ldots, \ell - 1. \]
5.4.4. The $D_\ell$ series. $J = \{1, \ldots, \ell, 7, \ldots, 1\}$.

\[
\Lambda_{i,a} = Y_{i,aq^{-1}}Y_{i-1,aq^{-1}}^{-1}, \quad i = 1, \ldots, \ell - 2,
\]
\[
\Lambda_{\ell-1,a} = Y_{\ell,aq^{-2}}Y_{\ell-1,aq^{-2}}^{-1}Y_{\ell-2,aq^{-1}},
\]
\[
\Lambda_{\ell,a} = Y_{\ell,aq^{-2}}Y_{\ell-1,aq^{-1}},
\]
\[
\Lambda_{\ell-1,a} = Y_{\ell-1,aq^{-2}}Y_{\ell,aq^{-1}},
\]
\[
\Lambda_{\ell,a} = Y_{\ell,aq^{-2}}Y_{\ell-1,aq^{-1}},
\]
\[
\Lambda_{\ell-1,a} = Y_{\ell-2,aq^{-1}}Y_{\ell-1,aq^{-1}},
\]
\[
\Lambda_{\ell,a} = Y_{i-1,aq^{2i-2}}Y_{i,aq^{2i-3}}, \quad i = 1, \ldots, \ell - 2.
\]

Equivalently,

\[
\Lambda_{1,a} = Y_{1,a},
\]
\[
\Lambda_{i,a} = \Lambda_{i-1,a}A_{1-1,aq^{-1}}, \quad i = 2, \ldots, \ell,
\]
\[
\Lambda_{\ell,a} = \Lambda_{\ell-1,a}A_{\ell,aq^{-1}},
\]
\[
\Lambda_{\ell-1,a} = \Lambda_{\ell,a}A_{\ell-1,aq^{-1}},
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\Lambda_{\ell,a} = \Lambda_{\ell-1,a}A_{\ell,aq^{-1}},
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\Lambda_{\ell-1,a} = \Lambda_{\ell,a}A_{\ell-1,aq^{-1}},
\]
\[
\Lambda_{\ell,a} = \Lambda_{\ell-1,a}A_{\ell,aq^{-1}}, \quad i = 1, \ldots, \ell - 2.
\]

6. Connection with Bethe Ansatz

As we have observed earlier [31, 33], the formulas for the $q$–characters have the same structure as the formulas for the spectra of transfer-matrices on finite-dimensional representations of $U_q\hat{g}$ obtained by the analytic Bethe Ansatz method [50, 51, 5]. In this section we explain this connection in more detail.
6.1. Bethe Ansatz formulas. Let us recall the idea of the Bethe Ansatz method (see the original works \cite{57,54}). The problem is to find the eigenvalues of the transfer-matrices $t_V(z), V \in \text{Rep} \ U_q \hat{g}$, defined by formula \eqref{1.9}, on a finite-dimensional representation $W$. This problem arises naturally, particularly, when $W = U^\otimes N$, in the study of quantum spin chains, such as the XXZ model (see \cite{57,51,50}). Conjecturally, for each $V$, all eigenvalues of $t_V(z)$ can be represented in a uniform way, and the eigenvalues are parametrized by finite sets of complex numbers, which are solutions of the so-called Bethe Ansatz equations (see \cite{50,51,5,46} for details).

The Analytic Bethe Ansatz hypothesis is a statement that all other eigenvalues of $t_V(z)$ on $\otimes_{j=1}^N V(P_j)$ have similar structure.

**Hypothesis.** Each eigenvalue of $t_V(z)$ on the tensor product of finite-dimensional representations $\otimes_{i=1}^N V(P_i)$ can be written as $\chi_q(V)$, in which we substitute $Y_{i,a}$ by $Y_{i,a}^0(z)$.

We want to give an interpretation of these formulas from the point of view of the $q$-characters. Suppose for simplicity that $W = V(P)$ is irreducible. Then it has a unique highest weight vector $v_P$. Due to the property \eqref{2.3}, we obtain that $v_P$ is an eigenvector of $t_V(z)$, whose eigenvalue equals $\chi_q(V)$, in which we substitute each $Y_{i,a}$ by $Y_{i,a}^0(z)$. We recall from Sect. 4.2 that $Y_{i,a}^0(z)$ can be found if we solve the equations \eqref{1.9} for $i \in I$, in which we set

$$A_{i,a} = q_i^{2 + \deg P_i} P_i(z^{-1} a^{-1} q_i^{-1}) / P_i(z^{-1} a^{-1} q_i).$$

The last condition can be written as a system of algebraic equations on the zeroes $w_{j}^{(i)}$, which are called the Bethe Ansatz equations. It turns out that the pole cancellation condition places such a stringent constraint on the polynomial $\chi_q(V)$ that together with other natural requirements, it suffices to reconstruct $\chi_q(V)$ completely in many examples, see \cite{50,51,5,46}.

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Our construction of $q$-characters so far explains the Bethe Ansatz hypothesis only for the vector $v_P$. However, such an explanation can be given along the lines of our construction \cite{30} of the spectra of the hamiltonians of the Gaudin system, which are certain limits of the transfer-matrices $t_V(z)$ as $q \to 1$. Recall that in \cite{30} we constructed the eigenvectors of the Gaudin hamiltonians using the Wakimoto modules at the critical level over the affine Kac-Moody algebra $\hat{g}$. A generalization of this construction, in which $\hat{g}$ is replaced by $U_q \hat{g}$, should give us eigenvectors of $t_V(z)$ with eigenvalues of the above form. The eigenvectors can be constructed from intertwining operators between $U_q \hat{g}$-modules at the critical level (see Prop. 2.3 of \cite{19}). To construct such intertwiners explicitly, one has to use special singular vectors in Wakimoto modules, which should exist precisely when the Bethe Ansatz equations are satisfied, just as in the case of the Gaudin model \cite{30}. We plan to discuss this in more detail in a separate publication.
6.2. The case of $U_q\hat{\mathfrak{sl}_2}$. According to formula (4.14), we have:

$$Y_{i,a}Y_{i,a^{-1}} = A_{i,a} = q^2\Phi^-(z^{-1}a^{-1})$$

(this is the equation (4.19) in this case). Note that the value of $a$ in our formulas is not important, since we can always set it to be 1 by applying the automorphism $z \mapsto za^{-1}$. Hence from now on we will set $a = 1$.

Let $v_0^{(r)}$ be the highest weight vector of $W_r(b)$. According to formula (4.11), we have

$$\Phi^-(u) \cdot v_0^{(r)} = q^r \frac{1 - ub^{-r}}{1 - ub^{-r}q^{-1}} v_0^{(r)}.$$  

Therefore $Y^{(r)}(z) = Y^{P^{(r)}}(z)$ satisfies:

$$Y^{(r)}(zq)Y^{(r)}(zq^{-1}) = q^{2r+1} \frac{1 - z^{-1}bq^{-r}}{1 - z^{-1}bq^{-r}} = q^{2r} \frac{1 - zb^{-1}q^{-r}}{1 - zb^{-1}q^{-r}},$$

as a formal power series in $z$. This can be solved as follows:

$$Y^{(r)}(z) = q^{-\frac{1}{2}} \frac{(zb^{-1}q^{r+1}; q_4^2)\infty (zb^{-1}q^{-r+3}; q_4^4)\infty}{(zb^{-1}q^{r+3}; q_4^4)\infty (zb^{-1}q^{-r+1}; q_4^4)\infty} \mu^{(r)}(zb^{-1}),$$

where

$$\mu^{(r)}(z) = \frac{(zb^{-1}q^{r+2}; q_4^4)\infty}{(zb^{-1}q^{-r+2}; q_4^4)\infty}.$$

The hypothesis above means in this case that the eigenvalues of $t_{W_1(1)}$ on $\otimes_{j=1}^N W_{r_j}(b_j)$ are given by

$$q^\Delta \frac{\mu(zq^{-1})}{\mu(zq)} \frac{Q(zq^{-1})}{Q(zq)} + q^{-\Delta} \frac{\mu(zq^3)}{\mu(zq)} \frac{Q(zq^3)}{Q(zq)},$$

where

$$\mu(z) = \prod_{j=1}^N (zb_j^{-1}q^{-r_j+2}; q_4^4)\infty = \prod_{j=1}^N \mu^{(r_j)}(zb_j^{-1}),$$

$$Q(z) = \prod_{k=1}^m (1 - zw_k^{-1}),$$

and $\Delta = 2m + \sum_j (1 - r_j / 2)$. Here $w_k$’s are complex numbers subject to the equations:

$$q^\Delta \frac{\mu(w_k q^{-2})}{\mu(w_k)} \prod_{s \neq k} \frac{1 - w_k w_s^{-1}q^{-2}}{1 - w_k w_s^{-1}} + q^{-\Delta} \frac{\mu(w_k q^2)}{\mu(w_k)} \prod_{s \neq k} \frac{1 - w_k w_s^{-1}q^2}{1 - w_k w_s^{-1}} = 0,$$

which mean that the expression (6.4) has no singularities at the points $z = w_kq^{-1}$.

These equations are equivalent to the equations

$$\prod_{j=1}^N q^{r_j} \frac{w_k - bq^{-r_j}}{w_k - bq^{-2}} = -q^{-N} \prod_{s \neq k} \frac{w_k - w_s q^{-2}}{w_k - w_s q^2}.$$

These are the Bethe Ansatz equations for $\mathfrak{g} = \mathfrak{sl}_2$, which ensure that the eigenvalues (6.4) have no poles at $z = w_kq^{-1}$. 

Formula (6.4) first appeared in the works of R. Baxter (see [3]), and polynomials $Q(z)$ are often called Baxter’s polynomials.

Note that $Q(z) = 1 - zw^{-1}$ is actually $\mu^{(r)}(z)$ in the special case when $r = -2$. This reflects the special role that the Wakimoto modules with highest weights $-\alpha_i$ play in the construction of the eigenvectors (cf. [30]).

6.3. General case. For a general Lie algebra $\mathfrak{g}$ we find the Bethe Ansatz equations by looking at the possible cancellations of poles at $z = w^{(i)} a^{-1} q_i^{-1}$ in the sum of the expressions (6.1). We find that there are indeed possible pairwise cancellations between such cancellations give rise to an equation, which says that the sum of the residues of the expressions (6.1) corresponding to $z$ by looking at the possible cancellations of poles at $M$ of the expressions (6.1) corresponding to $\chi$ above Hypothesis. The following system of equations (as before, we set $W = 0$. Explicit calculation similar to the one presented in the previous section gives us the system of equations (6.6) corresponding to an arbitrary collection of Drinfeld polynomials $P_{j,i}$, $j = 1, \ldots, N; i = 1, \ldots, \ell$:

$$\prod_{j=1}^{N} q_i \deg P_{j,i} \frac{P_{j,i}(q_i^{-1}/w_i^{(i)})}{P_{j,i}(q_i/w_i^{(i)})} = -q^N \prod_{s \neq k} \frac{w_k^{(i)} - w_s^{(i)} q_i^{-2}}{w_k^{(i)} - w_s^{(i)} q_i} \prod_{l \neq i} \frac{w_l^{(i)} - w_s^{(l)} q^{-C_{il}}}{w_l^{(i)} - w_s^{(i)} q^{C_{il}}}. \tag{6.6}$$

This is the most general system of Bethe Ansatz equations corresponding to an arbitrary collection of Drinfeld polynomials $P_{j,i}, j = 1, \ldots, N; i = 1, \ldots, \ell$.

Because they come from “local” cancellations (those occurring between monomials of the form $M$ and $MA_i^{-1}$), these equations are the same for each choice of the “auxiliary” space $V$ (which gives rise to the transfer matrix $t_V(z)$ acting on the “physical” space $W$). Conjecturally, each solution gives rise to a common eigenvector of all transfer-matrices $t_V(z), V \in \text{Rep } U_q(\mathfrak{g})$. The eigenvalues are conjecturally given by the linear combinations of the expressions (6.1), corresponding to the $q$-characters $\chi_q(V)$, as stated in the above Hypothesis. Moreover, we expect that in a generic situation this Bethe Ansatz is “complete”; that is, the set of all solutions of the system (6.6) (modulo the obvious action of the product of symmetric groups) is in one-to-one correspondence with the set of common eigenvectors (or eigenvalues, since we expect the spectrum to be simple, at least, in the generic situation) of the transfer-matrices $t_V(z)$ on $W = \bigotimes_{j=1}^{N} V(P_j)$.

For $\mathfrak{g} = \mathfrak{s}l_2$ and $V(P_j) = W_{r_j}(b)$, we have

$$q^{\deg P_{j,i}} \frac{P_{j,i}(uq^{-1})}{P_{j,i}(aq)} = q^{r_j} \frac{1 - ubq^{-r_j}}{1 - ubq^{r_j}},$$

and formula (6.10) gives us formula (6.3).

7. The screening operators

In this section we define certain operators on $Y = \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i \in I, a_i \in \mathbb{C}^\times}$, which we call the screening operators. The terminology is explained by the fact that these operators...
are certain limits of the screening operators used in the definition of the deformed $W$–algebras (see the next section).

7.1. **Definition of the screening operators.** Consider free $\mathcal{Y}$–module, which is the direct sum of vector spaces

$$\tilde{\mathcal{Y}}_i = \bigoplus_{x \in \mathbb{C}^\times} \mathcal{Y} \otimes S_{i,x}.$$ 

Let $\mathcal{Y}_i$ be the quotient of $\tilde{\mathcal{Y}}_i$ by the submodule generated by elements of the form

$$S_{i,xq} = A_{i,xq} S_{i,x}.$$ 

Clearly,

$$\mathcal{Y}_i \simeq \bigoplus_{x \in (\mathbb{C}^\times / q^2)} \mathcal{Y} \otimes S_{i,x},$$

and so $\mathcal{Y}_i$ is also a free $\mathcal{Y}$–module.

Now define a linear operator $\tilde{S}_i : \mathcal{Y} \to \tilde{\mathcal{Y}}_i$ by the formula

$$\tilde{S}_i \cdot Y_{j,a} = \delta_{i,j} Y_{i,a} S_{i,a},$$

and the Leibniz rule: $\tilde{S}_i \cdot (ab) = (\tilde{S}_i \cdot a)b + a(\tilde{S}_i \cdot b)$. In particular,

$$\tilde{S}_i \cdot Y_{j,a}^{-1} = -\delta_{i,j} Y_{i,a}^{-1} S_{i,a}.$$ 

Finally, let $S_i : \mathcal{Y} \to \mathcal{Y}_i$ be the composition of $\tilde{S}_i$ and the projection $\tilde{\mathcal{Y}}_i \to \mathcal{Y}_i$. We call $S_i$ the $i$th screening operator.

**Conjecture 2.** The image of the homomorphism $\chi_\mathcal{Y}$ equals the intersection of the kernels of the operators $S_i$, $i \in I$.

One can check explicitly that for all $\mathfrak{g}$ of classical types, the $q$–character of $V_{\omega_1(a)}$ lies in the intersection of the kernels of the operators $S_i$, $i \in I$. In fact, in [33] that has been proved for the more general $(q,t)$–characters. This means that the subring of $\text{Rep} U_q \hat{\mathfrak{g}}$ generated by $t_1(a) = \chi_\mathcal{Y}(V_{\omega_1(a)})$ lies in the intersection of the kernels of the operators $S_i$, $i \in I$. If $\mathfrak{g} = A_\ell$ or $C_\ell$, then this subring coincides with $\text{Rep} U_q \hat{\mathfrak{g}}$. For the $B_\ell$ and $D_\ell$ series, one needs to check that the $q$–characters of the spinor representations also belong to the kernel of $S_i$’s. Explicit formulas for the latter are given in [33].

For $x \in \mathbb{C}^\times$, denote by $\mathcal{Y}(x)$ the subring $\mathbb{Z}[Y_{i,xq^n}]_{i \in I; n_i \in \mathbb{Z}}$ of $\mathcal{Y}$. Then we have:

$$\mathcal{Y} \simeq \bigoplus_{x \in (\mathbb{C}^\times / q^2)} \mathcal{Y}(x),$$

and it follows from the definition of the operators $S_i$ that

$$\bigcap_{i \in I} \text{Ker}_\mathcal{Y} S_i \simeq \bigoplus_{x \in (\mathbb{C}^\times / q^2) \in I} \bigcap_{i \in I} \text{Ker}_{\mathcal{Y}(x)} S_i.$$ 

Conjecture 2 implies that

$$\text{Rep}(x) U_q \hat{\mathfrak{g}} \simeq \bigcap_{i \in I} \text{Ker}_{\mathcal{Y}(x)} S_i,$$

where $\text{Rep}(x) U_q \hat{\mathfrak{g}} = \mathbb{Z}[t_{i,xq^n}]_{i \in I; n_i \in \mathbb{Z}}$. 
Therefore we expect that any irreducible representation $V$ of $U_q\mathfrak{sl}_2$, considered as an element of $\text{Rep}U_q\mathfrak{g}$, is the tensor product of irreducible representations that belong to $\text{Rep}^+(U_q\mathfrak{g})$, so that the structure of $\text{Rep}U_q\mathfrak{g}$ is contained in the structure of $\text{Rep}^+(U_q\mathfrak{g})$ (which are isomorphic to each other for different $x$).

7.2. **Proof of Conjecture** [2] for $U_q\mathfrak{sl}_2$. In this case $\text{Rep}U_q\mathfrak{sl}_2 \cong \mathbb{Z}\lbrack t \rbrack_{a \in \mathbb{C}^\times}$, where $t_a$ is the class of $V(a)$, which is the same as $W_1(a)$ from Sect. 4.1. The results of that section give us the following formula:

$$\chi_q(V) = Y_a + Y^{-1}_{aq^2}.$$  

Thus, the image of $\chi_q$ is $\mathbb{Z}[Y_a + Y^{-1}_{aq^2}]_{a \in \mathbb{C}^\times}$.

The action of the operator $S$ is given by the formula

$$S \cdot Y_a^{\pm 1} = \pm Y_a^{\pm 1} S,$$

and the Leibniz rule. We also have the following relation

$$S_{aq^2} = S_{q^2} Y_a Y_{aq^2}.$$

Now it is clear that

$$\text{Ker}_y S = \bigotimes_{x \in (\mathbb{C}^\times/q^{2\mathbb{Z}})} \text{Ker}_y(x), S,$$

where $y(x) = \mathbb{Z}[Y_{xq^{2n}}]_{n \in \mathbb{Z}}$.

On the other hand, it follows from Theorem 4 that

$$\text{Rep}U_q\mathfrak{sl}_2 = \bigotimes_{x \in (\mathbb{C}^\times/q^{2\mathbb{Z}})} \text{Rep}^+(x)U_q\mathfrak{sl}_2,$$

where $\text{Rep}^+(x)U_q\mathfrak{sl}_2 = \mathbb{Z}\lbrack t_{xq^{2n}} \rbrack_{n \in \mathbb{Z}}$.

Hence it suffices to show that

$$\mathbb{Z}\lbrack t_{xq^{2n}} \rbrack_{n \in \mathbb{Z}} = \text{Ker}_y(x), S.$$

Without loss of generality we can set $x = 1$. To simplify notation, we denote $Y_{q^{2n}}$ by $y_n$ and $S_{q^{2n}}$ by $s_n$. We need to prove that

$$\mathbb{Z}[y_n + y_{n+1}^{-1}]_{n \in \mathbb{Z}} = \text{Ker} S,$$

where $S : \mathbb{Z}[y_n^{\pm 1}]_{n \in \mathbb{Z}} \to y^{(1)} = \bigoplus_{m \in \mathbb{Z}} \mathbb{Z}[y_n^{\pm 1}]_{n \in \mathbb{Z}} \otimes s_m/(s_m - y_m y_{m-1} s_{m-1})$

is given by

$$S \cdot y_n^{\pm 1} = \pm y_n^{\pm 1} s_n$$

and the Leibniz rule. Note that $S$ commutes with the shift $y_n \to y_{n-k}$ for any integer $k$. Given an element in the kernel of $S$, we can apply to it a shift with sufficiently large $k$ to make it into an element of $\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0}$, which also belongs to the kernel of $S$. Therefore without loss of generality we can restrict ourselves to $\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0}$.

Furthermore, we can identify $y^{(1)}$ with $\mathbb{Z}[y_n^{\pm 1}]_{n \in \mathbb{Z}} \cdot s_0$ and hence with $\mathbb{Z}[y_n^{\pm 1}]_{n \in \mathbb{Z}}$.

After this identification, $S$ becomes the derivation

$$S \cdot y_n^{\pm 1} = \begin{cases} 
\pm y_n^{\pm 1} \prod_{i=1}^{n} y_i y_{i-1}, & n \geq 0 \\
\pm y_n^{\pm 1} \prod_{i=0}^{n+1} y_i^{-1} y_{i-1}, & n < 0.
\end{cases}$$
In particular, we see that $\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0}$ is $S$-invariant, and we want to show that the kernel of $S$ on $\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0}$ equals $\mathbb{Z}[t_n]_{n \geq 0}$, where $t_n = y_n + y_{n+1}^{-1}$.

Let us write

$$\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0} = \mathbb{Z}[t_n, y_n]_{n \geq 0} / (t_n y_{n+1} - y_n y_{n+1} - 1).$$

Consider the set of monomials

$$t_n \ldots t_n y_{m_1} \ldots y_{m_l},$$

where all $n_i \geq 0$ and are lexicographically ordered, all $m_i \geq 0$ are lexicographically ordered, and also $m_j \neq n_i + 1$ for all $i$ and $j$. We call these monomials reduced. It is easy to see that the set of reduced monomials is a basis of $\mathbb{Z}[y_n, y_{n+1}^{-1}]_{n \geq 0}$. Now let $P$ be an element of the kernel of $S$. Let us write it as a linear combination of the reduced monomials. We can then represent $P$ as $y_N^a Q + R$. Here $N$ is the largest integer, such that $y_N$ is present in at least one of the basis monomials appearing in its decomposition; $a$ is the largest power of $y_N$ in $P$; $Q$ does not contain $y_N$, and $R$ is not divisible by $y_N^a$; we assume here that both $y_N Q$ and $R$ are linear combinations of reduced monomials.

When we apply $S$ to $P$ we obtain

$$(7.2) \quad a y_N^{a+1} y_{N-1} \prod_{i=1}^{N-1} y_i y_{i-1} Q$$

plus the sum of terms that are not divisible by $y_N^{a+1}$. Of course, (7.2) may not be in reduced form. But $Q$ does not contain $t_{N-1}$. Therefore when we rewrite it as a linear combination of reduced monomials, that linear combination will still be divisible by $y_N^{a+1}$. On the other hand, no other terms in $S \cdot P$ will be divisible by $y_N^{a+1}$. Hence for $P$ to be in the kernel, (7.2) has to vanish, which can only happen if $P$ does not contain $y_N$'s at all, i.e., $P \in \mathbb{Z}[t_n]_{n \geq 0}$, which is what we wanted to prove.

In the same way we obtain the following statement.

**Proposition 6.** The kernel of $S_i : \mathcal{Y} \to \mathcal{Y}_i$ equals

$$\mathcal{R}_i = \mathbb{Z}[Y_{j,a_j}]_{j \neq i, a_j \in \mathbb{C}^*} \otimes \mathbb{Z}[Y_i, b + Y_i, b A_i^{-1}, b_{i,b}]_{b \in \mathbb{C}^*}.$$

Conjecture 2 can therefore be interpreted as saying that the image of $\chi_q$ in $\mathcal{Y}$ equals $\bigcap_{i \in I} \mathcal{R}_i$ (cf. Sect. 5.2).

### 8. The connection with the deformed $\mathcal{W}$–algebras

Our motivation for the definition of the screening operators $S_i$ and for Conjecture 2 comes from the theory of deformed $\mathcal{W}$–algebras. In this section we will explain this connection.

#### 8.1. The representation ring and the center at the critical level

We start by recalling the connection between $\text{Rep} \ U_q \hat{\mathfrak{g}}$ and the center of $U_q \hat{\mathfrak{g}}$ at the critical level $-\hbar^\vee$ (minus dual Coxeter number).

Following [52], [19], for each $V \in \text{Rep} U_q \hat{\mathfrak{g}}$, we define in addition to the $L$–operator $L_V(z)$ given by formula (3.3), the opposite $L$–operator

$$L_V^-(z) = (\pi_V(z) \otimes \text{id})(\sigma(\mathcal{R})).$$
where $\sigma(a \otimes b) = b \otimes a$, and the total $L$–operator
\[ L_{V_{tot}}(z) = L_{V}(zq^{2h^\vee})L_{\overline{V}}(z). \]

Let
\[ T_{V}(z) = \text{Tr}_{V} \left[ (q^{2\rho} \otimes 1)L_{V_{tot}}(z) \right]. \]

Note that $T_{V}(z)$ is a formal power series, whose coefficients lie in a completion of $U_q\widehat{\g}$ and their action is well-defined on any $U_q\widehat{\g}$–module on which the action of the subalgebra $U_q\mathfrak{b}_+$ is locally finite. We will consider the projections of these elements onto $U_q\widehat{\g}_{cr} = U_q\widehat{\g}/(c - q^{-h^\vee})$. Let $Z_q(\g)$ be the center of $U_q\widehat{\g}_{cr}$.

**Theorem 5 ([52] [19]).**

1. For each $V \in \text{Rep} U_q\widehat{\g}$, all Fourier coefficients of $T_{V}(z)$ lie in $Z_q(\g)$.
2. The map $V \rightarrow T_{V}(z)$ is a $C_{x^+}$–equivariant ring homomorphism $\tilde{\nu}_q : \text{Rep} U_q\widehat{\g} \rightarrow Z_q(\g) \otimes C(z)$.

Now let $U_q\widehat{\mathfrak{n}}_+$ be the subalgebra of $U_q\widehat{\g}$ generated by $x_i^+, i = 0, \ldots, \ell$. We will keep the same notation $U_q\widehat{\mathfrak{n}}_+$ and $U_q\mathfrak{b}_-$ for the projections of these subalgebras onto $U_q\widehat{\g}_{cr}$. Then we have the decomposition
\[ U_q\widehat{\g}_{cr} = U_q\mathfrak{b}_- \otimes U_q\widehat{\mathfrak{n}}_+, \]
as a vector space, and so
\[ U_q\widehat{\g}_{cr} = U_q\mathfrak{b}_- \oplus (U_q\mathfrak{b}_+ \otimes (U_q\widehat{\mathfrak{n}}_+)_0), \]
where $(U_q\widehat{\mathfrak{n}}_+)_0$ is the augmentation ideal of $U_q\widehat{\mathfrak{n}}_+$. Denote by $p$ be the corresponding projection onto $U_q\mathfrak{b}_-$.

We find that
\[ p(T_{V}(z)) = \text{Tr}_{V} q^{2\rho}L_{V}(z)T, \]
where $T$ is defined by formula (3.8). Thus, up to the inessential factor of $T$, $p(T_{V}(z))$ equals the transfer-matrix $t_{V}(z)$ defined in formula (3.4).

Moreover, let $\mathfrak{z}_q(\g)$ be the subalgebra of $U_q\mathfrak{b}_-$ generated by the Fourier coefficients $t_{V}[n]$ of $t_{V}(z)$, $V \in \text{Rep} U_q\widehat{\g}$. It is then easy to show that the restriction of $p$ to $Z_q(\g)$ is a homomorphism of commutative algebras $Z_q(\g) \rightarrow \mathfrak{z}_q(\g)$, and we have the following commutative diagram (up to the factor $T$ in (8.1)):
8.2. **Free field realization of** $U_q\widehat{\mathfrak{g}}$ **and** $q$-**characters.** Recall that in Sect. 3.2 we defined the $q$-characters using the Harish-Chandra homomorphism $h_q$. A natural question is how to construct an analogue of this homomorphism for $Z_q(\widehat{\mathfrak{g}})$. For that one needs a free field (or Wakimoto) realization of $U_q\widehat{\mathfrak{g}}$. By this we understand a family of homomorphisms $\mathcal{F}_k, k \in \mathbb{C}$, from $U_q\widehat{\mathfrak{g}}$ to the tensor product of a Heisenberg algebra $A_q$ and the Heisenberg algebra $U_q\widehat{\mathfrak{h}}_{k+\hbar^\vee}$, sending $c \in U_q\widehat{\mathfrak{g}}$ to $q^k$. Here we denote by $U_q\widehat{\mathfrak{h}}_\alpha$ the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $c, k_i^\pm, h_{i,n}, i \in I, n \in \mathbb{Z}\setminus\{0\}$, modulo the relation $c = q^\alpha$.

Such a realization has been constructed in [3] for $U_q\widehat{\mathfrak{sl}}_N$. Let us assume for a moment that it exists for any $U_q\widehat{\mathfrak{g}}$.

At the critical level $k = -\hbar^\vee$ the algebra $U_q\widehat{\mathfrak{h}}_0$ is the center of $A_q \otimes U_q\widehat{\mathfrak{h}}_0$, and therefore the image of the center $Z_q(\widehat{\mathfrak{g}}) \subset U_q\widehat{\mathfrak{g}}_{cr}$ under $\mathcal{F}_{-\hbar^\vee}$ should lie in $U_q\widehat{\mathfrak{h}}_0$. Furthermore, the commutative algebras $Z_q(\widehat{\mathfrak{g}})$ and $U_q\widehat{\mathfrak{h}}_0$ have natural Poisson structures, and the resulting Harish-Chandra type homomorphism $\overline{h}_q : Z_q(\widehat{\mathfrak{g}}) \to U_q\widehat{\mathfrak{h}}_0$ should preserve these Poisson structures. In the case of $U_q\widehat{\mathfrak{sl}}_N$, when the Wakimoto realization is available, this homomorphism was analyzed in detail in [31], where it was called the $q$-deformation of the Miura transformation.

Comparing the results of Sect. 3 with the results of [31] in the case of $U_q\widehat{\mathfrak{sl}}_N$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
Z_q(\widehat{\mathfrak{g}}) & \overset{\overline{h}_q}{\longrightarrow} & U_q\widehat{\mathfrak{h}}_0 \\
\downarrow p & & \downarrow p \\
\tilde{Z}_q(\widehat{\mathfrak{g}}) & \overset{h_q}{\longrightarrow} & U_q\widehat{\mathfrak{h}}
\end{array}
$$

(8.2)

where $p : U_q\widehat{\mathfrak{h}}_0 \to U_q\widehat{\mathfrak{h}}$ is the homomorphism that sends $h_{i,n}, n > 0$, to 0, $h_{i,n}$ to $-h_{i,n}, n < 0$, and $k_i$ to $k_i^{-1}$. In particular, $p \circ \overline{h}_q(T_v(z)) = h_q(t_v(z)) = \chi_q(V)$ (in which we replace $Y_{i,a}$ by $Y_{i,a}^{-1}$).

We conjecture that the same is true in general. Thus, we expect that the $q$-character homomorphism is a truncation of the free field realization of the center $Z_q(\widehat{\mathfrak{g}})$ of $U_q\widehat{\mathfrak{g}}$ at the critical level.

The next step is to identify $Z_q(\widehat{\mathfrak{g}})$ with the classical limit of the deformed $W$-algebra.

8.3. **The algebra** $W_{q,t}(\mathfrak{g})$. Let us recall the definition of the deformed $W$-algebra $W_{q,t}(\mathfrak{g})$ and its free field realization from [33].

Let $\mathcal{H}_{q,t}(\mathfrak{g})$ be the Heisenberg algebra with generators $a_i[n], i = 1, \ldots, \ell; n \in \mathbb{Z}$, and relations

$$
[a_i[n], a_j[m]] = \frac{1}{n} (q^n - q^{-n})(t^n - t^{-n}) B_{ij}(q^n, t^n) \delta_{n,-m},
$$

(8.3)

where

$$
B_{ij}(q,t) = [r_i]_q \left( (q^{r_i}t^{-1} + q^{-r_i}t)\delta_{i,j} - [I_{ij}]_q \right).
$$

(8.4)
There is a unique set of elements \( y_i[n], i = 1, \ldots, \ell; n \in \mathbb{Z} \), that satisfy:

\[
[a_i[n], y_j[m]] = \frac{1}{n} (q^{r_i n} - q^{-r_i n})(t^n - t^{-n}) \delta_{i,j} \delta_{n,-m}.
\]

Introduce the generating series:

\[
A_i(z) = q^{2n_i[0]} : \exp \left( \sum_{m \neq 0} a_i[m] z^{-m} \right) :,
\]

\[
Y_i(z) = q^{2n_i[0]} : \exp \left( \sum_{m \neq 0} y_i[m] z^{-m} \right) :.
\]

Next we define the formal power series \( S_i^+(z), i = 1, \ldots, \ell \), of linear operators acting between Fock representations of \( \mathcal{H}_{q,t}(\mathfrak{g}) \) that satisfy the difference relations

\[
S_i^+(zq^{-r_i}) = : A_i(z) S_i^+(zq^{r_i}) :.
\]

These are the screening currents. Let \( S_i^+ \) be the 0th Fourier coefficient of \( S_i^+(z) \).

**Remark 8.1.** There is another set of screening currents: \( S_i^-(z), i = 1, \ldots, \ell \), introduced in [33], but we will not need these currents here. \( \square \)

Let \( \mathbf{H}_{q,t}(\mathfrak{g}) \) be the vector space spanned by formal power series of the form

\[
\partial_s^{n_1} Y_{i_1}(z q^{j_{i_1} t k_{i_1}}) \ldots \partial_s^{n_\ell} Y_{i_\ell}(z q^{j_{i_\ell} t k_{i_\ell}}) :,
\]

where \( \epsilon = \pm 1 \). The pair \( \mathbf{H}_{q,t}, \pi_0 \), where \( \pi_0 \) is the Fock representation of \( \mathcal{H}_{q,t}(\mathfrak{g}) \) with highest weight 0 (see [33]) is a deformed chiral algebra (DCA) in the sense of [32, 33].

We defined in [33] the DCA \( \mathcal{W}_{q,t}(\mathfrak{g}) \) as the maximal subalgebra of \( \mathbf{H}_{q,t}, \pi_0 \), which commutes with the operators \( S_i^+, i = 1, \ldots, \ell \), i.e., the subspace of \( \mathbf{H}_{q,t} \), which consists of all fields that commute with these operators. We also defined in [32] the deformed \( \mathfrak{w} \)-algebra \( \mathcal{W}_{q,t}(\mathfrak{g}) \) as the associative algebra, topologically generated by the Fourier coefficients of fields from \( \mathcal{W}_{q,t}(\mathfrak{g}) \) (in the case of \( \mathfrak{s} \mathfrak{l}_N \), the deformed \( \mathfrak{w} \)-algebra had been previously constructed in [55, 29, 2], see also [31, 49]). All elements of the algebra \( \mathcal{W}_{q,t}(\mathfrak{g}) \) act on the Fock representations \( \pi_\lambda \) and commute with the screening operators.

8.4. The classical limit of \( \mathcal{W}_{q,t}(\mathfrak{g}) \). Now let us consider the limit of the algebras \( \mathcal{H}_{q,t}(\mathfrak{g}), \mathcal{W}_{q,t}(\mathfrak{g}) \) and the operators \( S_i^+ \) as \( t \to 1 \). The algebra \( \mathcal{H}_{q,t}(\mathfrak{g}) \) becomes commutative with the Poisson structure given by

\[
\{A, B\} = \lim_{t \to 1} \frac{1}{2 \log t} [A, B]_t.
\]

Using this formula, we obtain the following Poisson brackets between the generators:

\[
\{a_i[n], a_j[m]\} = (q^{B_i n} - q^{-B_i n}) \delta_{n,-m},
\]

This formula shows that the map sending \( a_i[n] \) to \(-h_{i,n}\) is an isomorphism \( \mathcal{H}_{q,1}(\mathfrak{g}) \simeq \mathfrak{u}_q \hat{\mathfrak{sl}}_N \) of Poisson algebras.

The algebra \( \mathcal{W}_{q,1}(\mathfrak{g}) \) is the Poisson subalgebra of \( \mathcal{H}_{q,1}(\mathfrak{g}) \) that consists of the elements Poisson commuting with the operators \( S_i^+, i = 1, \ldots, \ell \). Let us recall the following conjecture from [33], which has been proved in the case of \( \mathfrak{u}_q \hat{\mathfrak{sl}}_N \) in [31].
Conjecture 3. The Poisson algebra $\mathcal{W}_{q,1}(\mathfrak{g})$ is isomorphic to the Poisson algebra $Z_q(\tilde{\mathfrak{g}})$. The embedding $\mathcal{W}_{q,1}(\mathfrak{g}) \rightarrow \mathcal{H}_{q,1}(\mathfrak{g}) \simeq U_{q,0}$ coincides with the free field realization homomorphism of $Z_q(\tilde{\mathfrak{g}})$.

An analogous statement is true in the $q = 1$ case as discussed in the next section.

Now let us compute the Poisson brackets with $S_i^+$. Formula (8.6) now reads:

\[ S_i^+(zq^{-r_i}) = A_i(z)S_i^+(zq^{r_i}) \]

Combining it with (8.9) we obtain:

\[ \{ S_i^+(z), Y_j(w) \pm 1 \} = \pm \delta \left( \frac{wq_i}{z} \right) S_i^+(z)Y_j(w) \pm 1 \delta_{i,j} = \pm \delta \left( \frac{w}{zq_i} \right) Y_j(w) \pm 1 S_i^+(w) \delta_{i,j} \]

Taking the 0th Fourier coefficient in $z$ we obtain:

\[ \{ S_i^+, Y_j(w) \pm 1 \} = \pm Y_j(w) \pm 1 S_i(w) \delta_{i,j} \]

Now let $\tilde{\mathcal{H}} = \mathbb{C}[Y_j(wq^{2n}) \pm 1]_{j=1,...,\ell, n \in \mathbb{Z}}$ and

\[
\tilde{\mathcal{H}}_i = \left( \bigoplus_{m \in \mathbb{Z}} \mathbb{C}[Y_{j}(wq^{2n}) \pm 1] \otimes S_i(wq_i^{2m}) \right) / (S_i^+(zq_i^{-1}) - A_i(z)S_i^+(zq_i)) \\
\simeq \mathbb{C}[Y_j(wq^{2n}) \pm 1] \otimes S_i(w)
\]

Then $\{ S_i^+, \cdot \}$ is a linear operator $\tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}_i$.

The Poisson algebra $\mathcal{H}_{q,1}(\mathfrak{g})$ consists of the Fourier coefficients of elements of $\tilde{\mathcal{H}}$. Let $\mathcal{H}_{q,1}(\mathfrak{g})_i$ be the span of the Fourier coefficients of elements of $\tilde{\mathcal{H}}_i$. Then $\{ S_i^+, \cdot \}$ gives rise to an operator $\mathcal{H}_{q,1}(\mathfrak{g}) \rightarrow \mathcal{H}_{q,1}(\mathfrak{g})_i$. Conjecture 3 then tells us that

\[ Z_q(\tilde{\mathfrak{g}}) \simeq \bigcap_{i=1,...,\ell} \text{Ker}_{\mathcal{H}_{q,1}(\mathfrak{g})} \{ S_i^+, \cdot \} \]

On the other hand, we can consider $\tilde{\mathcal{H}}$ itself as the classical limit of the deformed chiral algebra $\mathcal{H}_{q,1}(\mathfrak{g})$ – the “space of fields” of $\mathcal{H}_{q,1}(\mathfrak{g})$, and its subspace

\[ \bigcap_{i=1,...,\ell} \text{Ker}_{\mathcal{H}_{q,1}(\mathfrak{g})} \{ S_i^+, \cdot \} \]
as the classical limit of \( W_{q,t}(\mathfrak{g}) \) – the “space of fields” of the \( W \)-algebra \( W_{q,1}(\mathfrak{g}) \). Then Conjecture 3 and the commutative diagrams (8.1), (8.2) suggest the following isomorphism

\[
\bigcap_{i=1,\ldots,\ell} \text{Ker} Y_i \{ S_i^+, \cdot \} \simeq \text{Rep}^{(w)} U_q \hat{\mathfrak{g}}.
\]

If we identify \( Y_j(wq^{2n}) \) with \( Y_j^{-1}(wq^{2n}) \), then \( \hat{Y} \) and \( \hat{Y}_i \) gets identified with \( Y \) and \( Y_i \), respectively, and the operator \( \{ S_i^+, \cdot \} \) becomes the operator \( S_i \) introduced in Sect. 7.1. Hence (8.12) is equivalent to Conjecture 2.

8.5. **Comparison with the case** \( q = 1 \). Now we explain the analogous picture in the case of affine Lie algebras, i.e., when \( q = 1 \). In this case most of the conjectures of the previous subsections become theorems, except that for \( q = 1 \) there is no obvious connection between the center \( Z(\hat{\mathfrak{g}}) \) and \( \text{Rep} \hat{\mathfrak{g}} \).

8.5.1. **The definition of conformal \( W \)-algebras.** Let \( \mathcal{H}_\beta(\mathfrak{g}) \) be the Heisenberg algebra with generators \( a_i[n], i=1,\ldots,\ell; n \in \mathbb{Z} \), and relations:

\[
[a_i[n], a_j[m]] = nB_{ij} \delta_{n,-m}.
\]

The conformal screening currents satisfy the differential equations

\[
\partial_z S_i^+(z) = -\frac{1}{r_i} : A_i(z) S_i^+(z) :,
\]

where

\[
A_i(z) = \sum_{n \in \mathbb{Z}} a_i[n] z^{-n-1}.
\]

The conformal screening operators are given by

\[
S_i^\pm = \int S_i^\pm(z) dz.
\]

Let \( \pi_0 \) be the vertex operator algebra (VOA) associated to the Heisenberg algebra \( \mathcal{H}_\beta(\mathfrak{g}) \) (see [28]). For generic \( \beta \) the VOA \( W_\beta(\mathfrak{g}) \) is defined [27, 28] as the vertex operator subalgebra of the VOA \( \pi_0 \), which is the intersection of kernels of the screening operators \( S_i^-, i=1,\ldots,\ell \):

\[
W_\beta(\mathfrak{g}) = \bigcap_{i=1,\ldots,\ell} \text{Ker} \pi_0 S_i^+.
\]

Thus, \( W_\beta(\mathfrak{g}) \) consists of the fields that commute with \( S_i^+ \). The \( W \)-algebra \( W_\beta(\mathfrak{g}) \) is defined as the associative (or Lie) algebra generated by the Fourier coefficients of these fields.

The \( W \)-algebra \( W_\beta(\mathfrak{g}) \) can be obtained from \( W_{q,t}(\mathfrak{g}) \) as \( q \to 1 \) with \( t = q^\beta \) (see [33]).
8.5.2. Classical limit. The classical limit of $\mathcal{W}_\beta(g)$ as $\beta \to 0$ coincides with a limit of $\mathcal{W}_q(g)$ as $q \to 1$. In this limit, the algebra $\mathcal{H}_\beta(g)$ becomes commutative and it inherits a Poisson structure, which is given on the generators by the formula

$$\{a_i[n], a_j[m]\} = nB_{ij}\delta_{n,-m}. \tag{8.15}$$

We rewrite this as

$$\{a_i[n], y_j[m]\} = n\delta_{i,j}\delta_{n,-m}. \tag{8.18}$$

Introduce new variables $y_i[n]$, such that

$$\{a_i[n], y_j[m]\} = n\delta_{i,j}\delta_{n,-m}. \tag{8.17}$$

We rewrite this as

$$\{A_i(z), Y_j(w)\} = -\delta_{i,j}w^{-1}\partial_z\delta\left(\frac{w}{z}\right). \tag{8.16}$$

In the limit $\beta \to 0$, formula (8.14) becomes

$$\partial_zS_i^+(z) = -\frac{1}{r_i}A_i(z)S_i^+(z) \tag{8.17}$$

Thus, we can consider $S_i^+(z)$ as the Fourier coefficients of fields from $\mathcal{H}_0(g)$, which defines on $\mathcal{H}_0(g)$ a Poisson algebra [8] (also called vertex Poisson algebra in [24]).

Classical limit.

Let $\pi_0 = \mathbb{C}[a_i[n]]_{n \leq -1}$ be the Fock representation of $\mathcal{H}_0(g)$, on which $a_i[n]$ acts as multiplication by itself for $n \leq -1$ and by 0 for $n \geq 0$. We can identify $\pi_0$ naturally with $\mathcal{U} = C[\partial_w^n A_i(z)]_{n \geq 0, i=1,\ldots,m}$ by sending $P(\partial_w^n A_i(z))$ to $P(\partial_w^n A_i(z)) \cdot 1_{z=0} \in \pi_0$. This identification, which is the remnant of the VOA structure on $\pi_0$ for $\beta \neq 0$ is actually a ring isomorphism sending $a_i[n], n \leq -1$, to $\partial_z^{-n-1} A_i(z)/(-n-1)!$.

Furthermore, the $\beta$–linear term of the VOA structure defines on $\pi_0$ the structure of coisson algebra [8] (also called vertex Poisson algebra in [24]) and the classical limit of the VOA $\mathcal{W}_\beta(g), \mathcal{W}_0(g) \subset \pi_0$, can be viewed as a coisson subalgebra of $\pi_0$. The coisson structure encodes the Poisson structures in $\mathcal{H}_0(g)$ and $\mathcal{W}_0(g)$, which are spanned by the Fourier coefficients of fields from $\pi_0$ and $\mathcal{W}_0(g)$, respectively.

Let $\mathcal{U}$ be the free $\mathcal{U}$–module with generator $S_i^+(w)$. We extend the action of $\partial_w$ to $\mathcal{U}$ using formula (8.17). Define the operator $S_i^+: \mathcal{U} \to \mathcal{U}$ by formulas (8.18), (8.17), the Leibniz rule and the property that it commutes with the action of $\partial_w$ on both spaces. Then by definition

$$\mathcal{W}_0(g) = \bigcap_{i=1,\ldots,\ell} \text{Ker}_\mathcal{U}\{S_i^+, \cdot\}. \tag{8.18}$$
\[ (8.19) \quad W_0(\mathfrak{g}) = \bigcap_{i=1, \ldots, \ell} \ker \mathcal{H}_0(\mathfrak{g}) \{ S^+_i, \cdot \}. \]

**Remark 8.2.** What we denote by \( W_0(\mathfrak{g}) \) here is really \( W_{-\sqrt{\gamma}}(\mathfrak{g}) \) in the notation of [28], where \( L_\mathfrak{g} \) stands for the Langlands dual Lie algebra to \( \mathfrak{g} \). The reason for the appearance of \( L_\mathfrak{g} \) is the factor of \( 1/r_i = 2\sqrt{\gamma}/(\alpha_i, \alpha_i) \) in formula (8.14). While generators \( \mathfrak{a}_i[n] \) correspond to the simple roots of \( \mathfrak{g} \), the rescaled generators \( \mathfrak{a}_i'[n] = \mathfrak{a}_i[n]/r_i \) correspond to the coroots of \( \mathfrak{g} \) and hence to the roots of \( L_\mathfrak{g} \).

In particular, what we denote by \( W_0(\mathfrak{g}) \) here is really \( W(\mathfrak{g}) \) of [28], which is the Poisson algebra of integrals of motion of the Toda field theory associated to \( L_\mathfrak{g} \). Again, this is due to the factor \( 1/r_i \) in formula (8.17), which makes the sum

\[ \sum_{i=1}^\ell S^+_i = \sum_{i=1}^\ell \int \exp(-\Phi_i'(z))dz \]

the hamiltonian of the Toda field theory of \( L_\mathfrak{g} \), see [28] for more details. This Poisson algebra can also be obtained by the Drinfeld-Sokolov reduction [23] from the dual space to the Lie algebra \( \widehat{L}_\mathfrak{g} \), see [27]. The embedding \( W_0(\mathfrak{g}) \to \mathcal{H}_0(\mathfrak{g}) \) is the classical Miura transformation.

**8.5.3. The center.** Now consider the affine Kac-Moody algebra \( \widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus CK \) and the completion \( \widehat{U}(\widehat{\mathfrak{g}})_k \) of \( U(\widehat{\mathfrak{g}})/\mathcal{K} + k \) introduced in [27]. Let \( Z(\widehat{\mathfrak{g}}) \) be the center of \( \widehat{U}(\widehat{\mathfrak{g}})_n = \widehat{U}(\widehat{\mathfrak{g}})_{-h^\vee} \). We have the following result.

**Theorem 6 ([27]).** \( Z(\widehat{\mathfrak{g}}) \) is isomorphic to \( W_0(\mathfrak{g}) \) as a Poisson algebra.

This theorem and (8.19) imply that

\[ Z(\widehat{\mathfrak{g}}) \simeq \bigcap_{i=1, \ldots, \ell} \ker \mathcal{H}_0(\mathfrak{g}) \{ S^+_i, \cdot \}. \]

The classical Miura transformation \( W_0(\mathfrak{g}) \to \mathcal{H}_0(\mathfrak{g}) \) can be interpreted in terms of the free field (Wakimoto) realization of \( \widehat{\mathfrak{g}} \). Recall [59, 26] that this is a homomorphism from \( \widehat{U}(\widehat{\mathfrak{g}})_k \) to the tensor product of the Heisenberg algebras \( A \) and \( U\hat{h}_{k,h^\vee} \). The latter is the completion of the universal enveloping algebra of the homogeneous Heisenberg subalgebra \( \hat{h} = h[t, t^{-1}] \oplus CK \) of \( \widehat{\mathfrak{g}} \) in which we set the central element \( K \) to be equal to \( k + h^\vee \). In particular, when \( k = -h^\vee \), it becomes commutative, and inherits a Poisson structure. The resulting Poisson algebra is isomorphic to \( \mathcal{H}_0(\mathfrak{g}) \).

Under the free field homomorphism, the image of \( Z(\widehat{\mathfrak{g}}) \subset \widehat{U}(\widehat{\mathfrak{g}})_{ct} \) lies in \( U\hat{h}_0 \). The resulting homomorphism \( \hat{h} : Z(\widehat{\mathfrak{g}}) \to U\hat{h}_0 \) preserves the Poisson structures and we have the following commutative diagram (see [27]):

\[ \begin{array}{ccc}
Z(\widehat{\mathfrak{g}}) & \xrightarrow{\hat{h}} & U\hat{h}_0 \\
\downarrow & & \downarrow \\
W_0(\mathfrak{g}) & \xrightarrow{h} & \mathcal{H}_0(\mathfrak{g})
\end{array} \]
where the isomorphism $U\tilde{\mathfrak{h}}_0 \to \mathcal{H}_0(\mathfrak{g})$ sends $h_{i,n}$ to $-a_i[n]$. The bottom arrow is the Miura transformation $W_0(\mathfrak{g}) \to \mathcal{H}_0(\mathfrak{g})$, see [27, 31] for more detail.

Finally, there is an analogue of the commutative diagram (8.2). Let $\tilde{\mathfrak{h}} = \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}] \subset \mathfrak{h}[t, t^{-1}] = \tilde{\mathfrak{h}}$, and $p$ be the quotient of $U\tilde{\mathfrak{h}}_0$ by the ideal generated by $\mathfrak{h}[t]$. Denote $\tilde{\mathfrak{g}}_- = \mathfrak{g} \otimes t^{-1} \mathbb{C}[t^{-1}]$. In the same way as in the quantum case we define the projections $p : \tilde{U}(\tilde{\mathfrak{g}}) \to U\tilde{\mathfrak{g}}_-$ and $h : U\mathfrak{b}_- \to U\tilde{\mathfrak{h}}$. Let $\tilde{\mathfrak{s}}(\tilde{\mathfrak{g}})$ be the commutative subalgebra of $U\tilde{\mathfrak{g}}_-$ that is the image of $Z(\tilde{\mathfrak{g}})$ under $p$. Then we have:

$\begin{align*}
Z(\tilde{\mathfrak{g}}) & \xrightarrow{p} U\tilde{\mathfrak{h}}_0 \\
\tilde{\mathfrak{s}}(\tilde{\mathfrak{g}}) & \xrightarrow{h} U\tilde{\mathfrak{h}}
\end{align*}$

(8.21)

up to the automorphism of $U\tilde{\mathfrak{h}}$ that sends $h_{i,n}$ to $-h_{i,n}$.

Consider now the homomorphism of coisson algebras $U\tilde{\mathfrak{h}} \to \pi_0$ sending $h_{i,n}$ to $a_i[n]$. Theorem 3 can be rephrased as saying that the corresponding map $\tilde{\mathfrak{s}}(\tilde{\mathfrak{g}}) \to \pi_0$ is an isomorphism onto $W_0(\mathfrak{g}) \subset \pi_0$.

8.5.4. Example. The case of $\tilde{\mathfrak{s}}l_2$. Let $\{e, h, f\}$ be the standard basis of $\mathfrak{s}l_2$. Set $a_n = a \otimes t^n$ and

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}.$$

Introduce the generating function of the Sugawara operators $\tilde{S}_n$ by formula

$$\tilde{S}(z) = \sum_{n \in \mathbb{Z}} \tilde{S}_n z^{-n-2} = 1 + h(z)^2 : + \frac{1}{2} : e(z) f(z) : + \frac{1}{2} : f(z) e(z) :.$$

It is well-known that $\tilde{S}_n \in Z(\tilde{\mathfrak{s}}l_2)$, and $Z(\tilde{\mathfrak{s}}l_2)$ is topologically generated by $\tilde{S}_n$, $n \in \mathbb{Z}$.

The Lie subalgebra $\tilde{\mathfrak{g}}_-$ of $\tilde{\mathfrak{s}}l_2$ is spanned by $e_n$, $h_n$, $f_n$, $n < 0$. We find:

$$p(\tilde{S}_n) = \begin{cases} S_n, & n \leq -2 \\ 0, & n > -2 \end{cases},$$

where

$$S_n = \frac{1}{4} \sum_{k + m = n; k, m < 0} h_m h_k + \frac{1}{2} \sum_{k + m = n; k, m < 0} (e_k f_m + f_m e_k).$$

Hence we obtain:

$$(8.22) \quad h(S_n) = \frac{1}{4} \sum_{k + m = n; k, m \leq 0} h_m h_k - \frac{1}{2} (n + 1) h_n, \quad n \leq -2.$$

In fact, $\tilde{\mathfrak{s}}(\tilde{\mathfrak{s}}l_2) = \mathbb{C}[S_n]_{n \leq -2}$ and formula (8.22) defines its embedding into $U\tilde{\mathfrak{h}} = \mathbb{C}[h_n]_{n \leq -1}$.

On the other hand, let $\chi_n, n \in \mathbb{Z}$, be the generators of $\tilde{\mathfrak{h}}$. Then one finds (see [31]):

$$(8.23) \quad \tilde{h}(\tilde{S}(z)) = \frac{1}{4} \chi(z)^2 - \frac{1}{2} \partial_z \chi(z).$$
Formulas (8.22) and (8.23) show that the diagram (8.21) is commutative up to the automorphism of $U_n^\h$ sending $h_n$ to $-h_n$.

Now consider the $\mathcal{W}$–algebra $\mathcal{W}_0(\mathfrak{sl}_2)$, which is in this case the classical Virasoro algebra. We have: $\mathcal{U} = \mathbb{C}[\partial_z^m Y(z)]_{m \geq 0}$ and $\mathcal{U}_1 = \mathcal{U} \otimes S^+(z)$. The operator $\{S^+, \cdot\} : \mathcal{U} \to \mathcal{U}_1$ is given by the formula
$$\{S^+, \partial_z^n Y(z)\} = \partial_z^n S^+(z),$$
the relation
$$\partial_z S^+(z) = -2Y(z)S^+(z),$$
and the Leibniz rule.

One finds [28] that $\mathcal{W}_0(\mathfrak{sl}_2) = \text{Ker}\{S^+, \cdot\} = \mathbb{C}[\partial_z^n T(z)]_{n \geq 0}$, where
\begin{equation}
(8.24) \quad T(z) = Y(z)^2 + \partial_z Y(z).
\end{equation}

Formulas (8.23) and (8.24) show the commutativity of diagram (8.20) (note that $Y(z)$ goes to $-\frac{1}{2} \chi(z)$).

8.6. **Deformed $\mathcal{W}$–algebra as a quantization of Rep $U_{q, t^{\prime}} \mathfrak{g}$.** According to Sect. 8.3, the DCA $\mathcal{W}_{q, t}(\mathfrak{g})$ can be viewed as a quantization (along the $t$ variable) of the representation ring $\text{Rep} U_{q, t^{\prime}} \mathfrak{g}$, while the $\mathcal{W}$–algebra $\mathcal{W}_{q, t}(\mathfrak{g})$ is a quantization of the center $Z_{q, t^{\prime}} \mathfrak{g}$.

The structure of the DCA $\mathcal{W}_{q, t}(\mathfrak{g})$ is similar to that of the representation ring $\text{Rep} U_{q, t^{\prime}} \mathfrak{g}$. In particular, each finite-dimensional representation $V$ of $U_{q, t^{\prime}} \mathfrak{g}$ should give rise to a $(q, t)$–character $\chi_{q, t}(V) \in \mathcal{W}_{q, t}(\mathfrak{g})$, which becomes $\chi_q(V)$ at $t = 1$. These $(q, t)$–characters should be linear combinations of normally ordered monomials in $Y_i(z q^m t^n)$ whose coefficients are rational functions in $q$ and $t$ taking non-negative integral values at $t = 1$ and $q = \epsilon, 1$ (see [33]). The $(q, t)$–characters for the first fundamental representation of $U_{q, t^{\prime}} \mathfrak{g}$, where $\mathfrak{g}$ is of classical type are given in [33]; for fundamental representations of $U_{q, t^{\prime}} \mathfrak{g}$, they are given in [9] (see also [45]).

Tensor product structure of $\text{Rep} U_{q, t^{\prime}} \mathfrak{g}$ is reflected in the pole structure of the fusion of the $(q, t)$–characters. Namely, in all known examples (see [32, 33, 9]), whenever $U$ appears in the decomposition of the tensor product $V \otimes W$, there exists an integer $a$, such that $\text{Res}_{z = w^{a+}} \chi_{q, t}(V)(z) \chi_{q, t}(W)(w) dz/z$ equals $\chi_{q, t}(U)(w)$ up to a constant multiple. It is natural to conjecture that this is true in general.

Now consider other classical limits of $\mathcal{W}_{q, t}(\mathfrak{g})$ and $\mathcal{W}_{q, t}(\mathfrak{g})$ (see [33]).

If $\mathfrak{g}$ is simply-laced, then the other classical limit is $q \to 1$. But $\mathcal{W}_{q, t}(\mathfrak{g})$ is actually invariant under the replacement $q \to t^{-1}, t \to q^{-1}$ in this case, and so the structure of $\mathcal{W}_{1, t}(\mathfrak{g})$ is essentially the same as that of $\mathcal{W}_{q, 1}(\mathfrak{g})$.

If $\mathfrak{g}$ is non-simply laced, then there are two interesting limits: $q \to 1$, and $q \to \epsilon = \exp(\pi i/r^{\vee})$.

According to the [33], the algebra $\mathcal{W}_{\epsilon, t}(\mathfrak{g})$ is not commutative, but it contains a large center $\mathcal{W}_{\epsilon, t}(\mathfrak{g})$. Conjecture 4 of [33] states that $\mathcal{W}_{\epsilon}(\mathfrak{g})$ is isomorphic to the center of $U_{\epsilon}(\mathfrak{g})$ at the critical level, where $\mathfrak{g}$ is the twisted affine algebra that is Langlands dual to $\mathfrak{g}$ (its Dynkin diagram is obtained from the Dynkin diagram of $\mathfrak{g}$ by reversing the arrows).
Alternatively, one can say that the limit $q \to \epsilon$ of $W_{q,t}(g)$ contains a commutative subalgebra that is isomorphic to $\text{Rep} U_t(\hat{L} g)$. In that sense, $W_{q,t}(g)$ appears to be a simultaneous quantization of $\text{Rep} U_q \hat{g}$ and $\text{Rep} U_t(\hat{L} g)$. Some evidence for this is presented in [33], where the $(q,t)$–characters $\chi_{q,t}(V_{\omega_1}(z))$ are given explicitly for $g$ of classical type. If we set $t = \epsilon$ in those formulas, we indeed obtain the $t$–characters of representations of $U_t(\hat{L} g)$, which can be seen by comparing them with the corresponding formulas for the spectra of transfer-matrices [51, 47]. The $(q,t)$–characters for the fundamental representations in the case $g = G_2$ obtained in [9] also agree with the conjecture.

Note that our conjecture above on the residues in the operator product expansion of $(q,t)$–characters suggests that even the tensor structures of $\text{Rep} U_q \hat{g}$ and $\text{Rep} U_t(\hat{L} g)$ are related.

The analysis of the above formulas also shows that in the limit $q \to 1$ they look like characters of some algebra closely related to $U_t(\hat{L} g)$ (see [33]). The limit $q \to 1$ can be described alternatively using the difference Drinfeld-Sokolov reduction, which we now recall.

8.7. Difference Drinfeld-Sokolov reduction. According to Conjecture 3 of [33], $W_{1,t}(g)$ and $W_{1,t}(g)$ can be obtained by the $t$–difference Drinfeld-Sokolov reduction from the loop group of $G$ [34, 53]. On the other hand, as we remarked above, if $g$ is simply-laced, then $\text{Rep} U_t(\hat{g})$ equals $W_{1,t}(g)$. Therefore for simply-laced $g$, $\text{Rep} U_q \hat{g}$ can be obtained by the $q$–difference Drinfeld-Sokolov reduction. This gives one an alternative method to find the $q$–characters of irreducible representations with a given highest weight.

Consider for example the case of $\mathfrak{sl}_N$ (see [34]). Let $M_{n,q}$ be the vector space of first order difference operators $D + A(s)$, where $A(s)$ is an element of the formal loop group $\text{LSL}_n = SL_n((s))$ of $SL_n$, and $D$ is the difference operator $(D \cdot f)(s) = f(s^q)$. The group $\text{LSL}_n$ acts on this manifold by the $q$–gauge transformations

\[(8.25)\quad g(s) \cdot (D + A(s)) = g(sq^2)(D + A(s))g(s)^{-1},\]

i.e. $g(s) \cdot A(s) = g(sq^2)A(s)g(s)^{-1}$.

Now we consider the submanifold $M_{n,q}^J \subset M_{n,q}$ which consists of operators $D + A(s)$, where $A(s)$ is of the form

\[(8.26)\quad \begin{pmatrix}
* & * & * & \ldots & * & * \\
-1 & * & * & \ldots & * & * \\
0 & -1 & * & \ldots & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \ldots & * & * \\
0 & 0 & 0 & \ldots & -1 & *
\end{pmatrix},\]

It is preserved under the $q$–gauge action of the group $LN$. 
Lemma 6. The action of $LN$ on $M^J_{n,q}$ is free and each orbit contains a unique operator of the form

\[
\Lambda = D + \begin{pmatrix}
  t_1 & t_2 & t_3 & \ldots & t_{n-1} & 1 \\
  -1 & 0 & 0 & \ldots & 0 & 0 \\
  0 & -1 & 0 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & 0 & 0 \\
  0 & 0 & 0 & \ldots & -1 & 0 
\end{pmatrix}.
\]

(8.27)

For other groups, the analogous statement has been proved by Semenov-Tian-Shansky and Sevostyanov [53].

Now let $\mathcal{F}$ be the vector space the $q$–difference operators

\[
\Lambda = D + \begin{pmatrix}
  \lambda_1(s) & 0 & \ldots & 0 & 0 \\
  -1 & \lambda_2(sq^{-2}) & \ldots & 0 & 0 \\
  0 & 0 & \ldots & \lambda_{n-1}(sq^{-2n+4}) & 0 \\
  0 & 0 & \ldots & -1 & \lambda_n(sq^{-2n+2}) 
\end{pmatrix},
\]

where $\prod_{i=1}^{n} \lambda_i(sq^{-2i+2}) = 1$.

Let $\mu_q : \mathcal{F} \to M$ be the composition of the embedding $\mathcal{F} \to M^J_{n,q}$ and $\pi_q : M^J_{n,q} \to M^J_{n,q}/LN \simeq M$. Using the definition of $\pi_q$ above, one easily finds that for $\Lambda$ given by (8.28), $\mu_q(\Lambda)$ is the operator (8.27), where $t_i(s)$ is given by

\[
t_i(s) = \sum_{j_1 < \ldots < j_i} \lambda_{j_1}(s)\lambda_{j_2}(sq^{-2})\ldots\lambda_{j_i}(sq^{-2i+2}).
\]

(8.29)

This coincides with the formula for the $q$–character of the $i$th fundamental representation of $U_q\mathfrak{sl}_N$, if we make the replacement $\lambda_i(s) \to \Lambda_{i,s}$, where $\Lambda_{i,s}$ is defined in Sect. 5.4.1. Thus, if we consider $\mu_q$ as a homomorphism

\[
\mathbb{C}[t_i(s)]_{i=1,\ldots,n-1; s \in \mathbb{C}^*} \to \mathbb{C}[\lambda_i(s)]_{i=1,\ldots,n; s \in \mathbb{C}^*} / \left( \prod_{i=1}^{n} \lambda_i(sq^{-2i+2}) - 1 \right),
\]

then it coincides with the $q$–character homomorphism $\chi_q$. For $q = 1$, this result is due to Steinberg [56].

Now we see that the problem of reconstructing the $q$–character from the dominant monomial $Y_{i_1,a_1} \ldots Y_{i_k,a_k}$ is equivalent to finding the minimal combination of monomials in $\lambda_i(sq^{2n_i})$ with positive integral coefficients, which lies in the image of the homomorphism $\mu_q$. One can probably use the geometry of the orbit space $M^J_{n,q}/LN$ to study this question. This method can also be applied to other simply-laced $\mathfrak{g}$.

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