THE OPEN ALGEBRAIC PATH PROBLEM

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ABSTRACT. The algebraic path problem provides a general setting for the Floyd–Warshall algorithm in optimization and computer science. This work extends the algebraic path problem to networks equipped with input and output boundaries. We show that the algebraic path problem is functorial as a mapping from a bicategory whose composition is gluing of open networks. In particular, we provide an isomorphism relating the solution of the path problem on a composite to the solutions on its components.

The algebraic path problem is a generalization of the shortest path problem to probability, computing, matrix multiplication, and optimization [Tar81, Foo15]. Let $R$ be the rig of positive real numbers $([0, \infty], \min, +)$. A weighted graph is regarded as a matrix weighted in $R$, and the shortest paths of this graph are computed as the transitive closure of this matrix. The algebraic path problem allows $R$ to vary, and gets solutions to other problems of a similar flavor also as the transitive closure of an adjacency matrix. The Floyd–Warshall algorithm and other shortest path algorithms can be extended to compute these transitive closures in a more general setting [HM12].

The algebraic path problem deals only with closed systems, i.e. systems which are isolated from their surroundings. On the other hand, open systems are equipped input and output boundaries, from which they can be composed to form larger and more complicated networks. A research program initiated by Baez, Courser, and Fong aims to provide a theoretical foundation for open systems using cospan formalisms [Fon16, BC19]. For a category of networks $C$, Baez and Courser defined a bicategory which provides a syntax for composition of open systems in $C$ [BC19]. Given an open system $G : X \to Y$, it is of interest to compute some data $D(G)$ about this system which lives in a category $S$. A goal of this line of research is to lift these computations to functors

$$\text{Open}(D) : \text{Open}(C) \to \text{Open}(S)$$

A functor of this sort provides a compositional theory for computing this data on open objects of $C$ [BM20, BP17]. Functoriality of this mapping gives an isomorphism

$$D(G \circ H) \cong D(G) \circ D(H),$$

which gives a way of building the data of $D(G \circ H)$ using only $D(G)$ and $D(H)$. In software engineering, this strategy of building up computations locally is called dynamic programming [Bel66] which in general can reduce the complexity of an algorithm from exponential to linear time.

In this paper, we tell this story for the computation of the algebraic path problem on open graphs weighted by a quantale $R$. In Section 1 we show how the solution

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to the algebraic path problem can be viewed as the left adjoint of an adjunction

\[
\begin{array}{c}
\text{Mat}_R \\
\downarrow F \\
\text{IdMat}_R \\
\uparrow U
\end{array}
\]

between \( R \)-matrices and idempotent \( R \)-matrices. In Section 2, we define open \( R \)-matrices, show how they can be glued together via pushout, and show that the algebra of this gluing forms the structure of a bicategory \( \text{Open}(\text{Mat}_R) \). In Section 3, we show how the left adjoint \( F \) can be lifted to a pseudofunctor

\[
\text{Open}(F) : \text{Open}(\text{Mat}_R) \to \text{Open}(\text{IdMat}_R)
\]

which computes the solution to the algebraic path problem on an open \( R \)-weighted graph. This pseudofunctor provides a formula for the solution of the algebraic path problem on a composite open \( R \)-weighted graph. For a pushout of open \( R \)-graphs \( G +_{LY} H \), its solution to the algebraic path problem is given by

\[
F(G +_{LY} H) \cong F(UF(G) +_{LY} UF(H))
\]

In words, to compute the solution to the path problem on a composite, first compute the solution on each component, glue them together via pushout, and then compute the solution on the result.

1. The Algebraic Path Problem

The algebraic path problem arises from the observation that various optimization problems can be framed in the same way by varying a sufficiently nice sort of rig. The level of generality for this work will be a commutative quantale, which is sufficient to guarantee existence and uniqueness of solutions to these optimization problems.

**Definition 1.1.** A quantale is a monoidal closed preorder with all joins. Explicitly, a quantale is a preorder \( R \) with a associative, unital, and monotone multiplication \( \cdot : R \times R \to R \) such that

- all joins, \( \bigvee_{i \in I} x_i \), exist for arbitrary index set \( I \) and,
- \( \cdot \) preserves all joins, i.e.

\[
a \cdot \bigvee_{i \in I} x_i = \bigvee_{i \in I} a \cdot x_i
\]

for all joins over an arbitrary index set \( I \).

A quantale is commutative if its multiplication operation, \( \cdot \), is commutative.

A motivating example of such a quantale is the preorder \([0, \infty]\) with + as its monoidal product and with join given by infimum. Note that this preorder is equipped with the reverse of the usual ordering on \([0, \infty]\). Fong and Spivak show how the shortest path problem on this quantale computes the shortest paths between all pairs of vertices in a given \([0, \infty]\)-weighted graph \([FS19, \S 2.5.3]\). Other motivating examples include the Viberti rig (whose algebraic path problem corresponds to most likely path in a Markov chain) and the powerset of the language generated by an alphabet (whose algebraic path problem corresponds to the language decided by a nondeterministic finite automata (NFA)) \([Foo15]\).
Definition 1.2. For a commutative quantale $R$ and sets $X$ and $Y$, an $R$-matrix $M: X \to Y$ is a function $M: X \times Y \to R$. We denote the value $M(i, j)$ with the notation $M_{ij}$. For $R$-matrices $M: X \to Y$ and $N: Y \to Z$, their matrix product $MN$ is defined by the rule

$$MN_{ik} = \bigvee_{j \in Y} M_{ij} N_{jk}$$

If $R$ is a commutative quantale, $R$-matrices form a quantale as well.

Definition 1.3. Let $\text{Mat}_R(X)$ be the set of $X$ by $X$ matrices $M: X \times X \to R$. $\text{Mat}_R(X)$ is equipped with the partial order $M \leq N$ if and only if $M_{ij} \leq N_{ij}$ for all $i, j \in X$.

Proposition 1.4. $\text{Mat}_R(X)$ with matrix product is a quantale.

The proof of this proposition is left to the reader. All the required properties of $\text{Mat}_R(X)$ follow from the analogous properties in $R$.

A square matrix $M: X \times X \to R$ represents an $R$-weighted graph whose vertex set is given by $X$. If $R$ is the quantale $([0, \infty], \inf, +)$ then the entry $M_{ij}$ represents the cost of traveling between vertex $i$ and vertex $j$. In this case, the matrix product $M^2$ has entries of the form

$$M^2_{ij} = \bigvee_{l \in X} M_{il} M_{lj} = \inf_{l \in X} \{M_{il} + M_{lj}\}.$$

If $M_{il}$ and $M_{lj}$ represent the cost of traveling from $i$ to $l$ and from $l$ to $j$, then this infinum computes the cheapest way to travel from $i$ to $j$ while stopping at some $l$ in between. More generally, the entries of $M^n$ for $n \geq 0$ represent the shortest paths between nodes of your graph that occur in $n$ steps. To compute the shortest paths in general, we must take the infimum of the matrices $M^n$ over all $n \geq 0$. This pattern replicates for other choices of quantale. Therefore, the algebraic path problem seeks to compute

$$F(M) = \bigvee_{n \geq 0} M^n \quad (1)$$

where $M$ is an $R$-matrix. The following table summarizes some instances of the algebraic path problem for different choices of $R$. Fink provides an explanation of the algebraic path problems for $([0, \infty], \leq)$ and $\{T, F\}$ and Foote provides an explanation for the quantales $([0, 1], \leq)$ and $(\mathcal{P}(\Sigma), \subseteq)$ [Fin92, Foo15].

| poset | join | multiplication | solution of path problem |
|-------|------|----------------|-------------------------|
| $([0, \infty], \geq)$ | inf | $+$ | shortest paths in a weighted graph |
| $([0, \infty], \leq)$ | sup | inf | maximum capacity in the tunnel problem |
| $([0, 1], \leq)$ | sup | $\times$ | most likely paths in a Markov process |
| $\{T, F\}$ | OR | AND | transitive closure of a directed graph |
| $(\mathcal{P}(\Sigma^*), \subseteq)$ | $\cup$ | concatenation | decidable language of a NFA |

Note that in this table, $\mathcal{P}(\Sigma^*)$ denotes the power set of the language generated by an alphabet $\Sigma$.

Formula (1) is known to category theorists by a different name: the free monoid on $M$. Framing it in this way gives a categorical proof of existence and uniqueness of $F(M)$. 


Definition 1.5. An \( R \)-matrix monoid is a monoid in the preorder \( \text{Mat}_R(X) \) for some set \( X \). A matrix monoid homomorphism from \( M \) to \( N \) is an inequality \( M \leq N \) satisfying \( M^2 \leq N^2 \).

Matrix monoids have a more familiar interpretation as idempotent matrices.

**Proposition 1.6.** \( R \)-matrix monoids on \( X \) and their homomorphisms form the full sub-preorder of \( \text{Mat}_R(X) \) consisting of matrices with \( M^2 = M \).

**Proof.** Because \( \text{Mat}_R(X) \) is a preorder, every diagram commutes and a matrix monoid is a matrix \( M \) satisfying \( M \geq 1 \) and \( M^2 \leq M \). Because matrix multiplication is monotone, we have that \( M = 1 \ast M \leq M^2 \) and \( M^2 = M \). A homomorphism of matrix monoids is an inequality \( M \leq N \) satisfying \( M^2 \leq N^2 \). However, because matrix multiplication is monotone, the inequality \( M^2 \leq N^2 \) will always be satisfied. \( \square \)

**Definition 1.7.** Let \( \text{IdMat}_R(X) \) be the full sub-preorder of \( \text{Mat}_R(X) \) consisting of idempotent matrices.

A classical result shows that there is a left adjoint producing free monoids and therefore solutions to the algebraic path problem.

**Proposition 1.8.** There is an adjoint pair

\[
\begin{array}{ccc}
\text{Mat}_R(X) & & \text{IdMat}_R(X) \\
\mapright{F_{R,X}} & & \mapleft{U_{R,X}} \\
\end{array}
\]

where \( F_{R,X} \) is the monotone map which produces the solution to the algebraic path problem on a matrix and \( U_{R,X} \) is the natural forgetful map.

**Proof.** This can be verified directly or by using Theorem 2 of [ML13, §V11]. This theorem says that if a monoidal category \( (C, \otimes, I) \) has countable colimits and the monoidal product preserves these colimits then \( C \) admits a free monoid adjunction as above. Because \( \text{Mat}_R(X) \) is a quantale, it can be regarded as a category satisfying these properties. \( \square \)

Therefore, each matrix valued in \( R \) has a unique solution to the algebraic path problem and this solution can be characterized by a universal property. This adjunction can be extended to matrices over an arbitrary set.

**Definition 1.9.** Let \( \text{Mat}_R \) be the category where objects are square matrices \( M : X \times X \to R \) on some set \( X \) and where a morphism from \( M : X \times X \to R \) to \( N : Y \times Y \to R \) is a function \( f : X \to Y \) satisfying

\[
M_{ij} \leq N_{f(i)f(j)}.
\]

Let \( \text{IdMat}_R \) be the full subcategory of \( \text{Mat}_R \) consisting of only idempotent matrices.

**Proposition 1.10.** The free monoid construction of Proposition 1.8 extends to an adjunction

\[
\begin{array}{ccc}
\text{Mat}_R & & \text{IdMat}_R \\
\mapright{F} & & \mapleft{U} \\
\end{array}
\]
Proof. Let $A : \text{Set}^{\text{op}} \to \text{Cat}$ be the functor which sends a set $X$ to the preorder $\text{Mat}_R(X)$ regarded as a category. For a function $f : X \to Y$, there is a monotone map $\text{Mat}_R(Y) \to \text{Mat}_R(X)$ induced by precomposition with $f \times f$.

Analogously, let $B : \text{Set}^{\text{op}} \to \text{Cat}$ be the functor which sends a set $X$ to the preorder $\text{IdMat}_R(X)$ and sends a function to the monotone map induced by precomposition like before. The functors $F_{R,X}$ form the components of a natural transformation $F_R : A \Rightarrow B$ and the functors $U_{R,X}$ form the components of a natural transformation $U_R : B \Rightarrow A$. Furthermore, these natural transformations form an adjoint pair in the 2-category $[\text{Set}^{\text{op}}, \text{Cat}]$ of functors $\text{Set}^{\text{op}} \to \text{Cat}$, natural transformations between them, and modifications. $F_R$ and $U_R$ are adjoint because an adjoint pair in $[\text{Set}^{\text{op}}, \text{Cat}]$ is a pair of natural transformations which are adjoint in each component.

To summarize, we have a pair of adjoint natural transformations

$$
\begin{array}{ccc}
\text{Set}^{\text{op}} & \overset{F}{\underset{U}{\leftrightarrow}} & \text{Cat} \\
& A \downarrow \uparrow & \\
& \downarrow \uparrow & B
\end{array}
$$

A restriction of the Grothendieck construction [Bor94] is a 2-functor

$$
\int : [\text{Set}^{\text{op}}, \text{Cat}] \to \text{CAT}
$$

where $\text{CAT}$ is the 2-category of large categories. Because every 2-functor preserves adjunctions, the above diagram maps to an adjunction

$$
\int A \overset{\int F_R}{\leftrightarrow} \int B.
$$

The result follows from the equivalences $\int A \cong \text{Mat}_R$ and $\int B \cong \text{IdMat}_R$. The desired functors $F$ and $U$ are obtained by composing $\int F_R$ and $\int U_R$ with these equivalences.

Idempotency shows up twice in this adjunction. Besides being the free idempotent matrix adjunction, the adjunction itself is idempotent.

Proposition 1.11.

$$
\begin{array}{ccc}
\text{Mat}_R & \overset{F_R}{\leftrightarrow} & \text{IdMat}_R \\
& \downarrow \uparrow & \\
& \downarrow \uparrow & \\
\text{Mat}_R & \overset{U_R}{\leftrightarrow} & \text{IdMat}_R
\end{array}
$$

is an idempotent adjunction.

Proof. Every adjunction between posets is idempotent. Therefore the smaller adjunctions $F_{R,X} \dashv U_{R,X}$ are idempotent. Because $F$ and $U$ are stitched together using these adjunctions, it is idempotent as well. $\square$
2. Open Weighted Matrices

The machinery of [BC19] can be used to define a syntax for $R$-matrices equipped with input and output boundaries. To do this, we need a notion of a discrete weighted matrix on a set. The map sending a set to its discrete $R$-matrix is a functor and a left adjoint.

**Proposition 2.1.** Let $R : \text{Mat}_R \to \text{Set}$ be the functor which sends a weighted graph to its underlying set of vertices and sends a morphism to its underlying function. Then $R$ has a left adjoint

$$L : \text{Set} \to \text{Mat}_R$$

which sends a set $X$ to the $R$-weighted graph $L(X)$ with $L(X)_{ij} = 0$ for all $i$ and $j$ in $X$. $F$ sends a function $f : X \to Y$ to the morphism of $R$-matrices which has $f$ as its underlying function between vertices.

**Proof.** The natural isomorphism $\text{Hom}(L(X), G) \cong \text{Hom}(X, R(G))$ can be seen by noting that a morphism $L(X) \to G$ is uniquely determined by its underlying function on vertices and every such function obeys the inequality in Definition 1.9. □

A weighted graph can be opened up to its environment by equipping it with inputs and outputs.

**Definition 2.2.** An **open $R$-matrix** is a cospan in $\text{Mat}_R$ of the form

$$\begin{array}{c}
M \\
LX \quad LY
\end{array}$$

The idea is that the maps of this cospan point to input and output nodes of the matrix $M$. Pushouts can be used to glue two open $R$-matrices together.

**Proposition 2.3.** $\text{Mat}_R$ has pushouts.

**Proof.** This is a consequence of Proposition 2.4 of [Wol74] after noting that $\text{Mat}_R$ is the category of $R$-graphs, the generating data for $R$-enriched categories. For concreteness and practicality, we offer an explicit construction of pushouts here. First we define the coproduct of $R$-matrices. For $R$-matrices $G : X \times X \to R$ and $H : Y \times Y \to R$, their coproduct

$$G + H : (X + Y) \times (X + Y) \to R$$

is given by

$$G + H(x, y) = \begin{cases}
G(x, y) & \text{if } x \in X \text{ and } y \in X \\
H(x, y) & \text{if } x \in Y \text{ and } y \in Y \\
0 & \text{otherwise}
\end{cases}$$

The inclusions $X \to X + Y$ and $Y \to X + Y$ make up the inclusions of $G + H$ in $\text{Mat}_R$. That this indeed is a coproduct is left as an exercise. Let

$$\begin{array}{c}
G \\
K \quad H
\end{array}$$

be two open $R$-matrices connected by pushouts. □
be a diagram in \( \text{Mat}_R \) with

\[
\begin{array}{c}
X & \xleftarrow{g} & Y \\
\downarrow{f} & & \downarrow{f} \\
Z
\end{array}
\]

as the underlying diagram of sets. The pushout \( G +_K H \) has an underlying set given by the pushout of sets \( X +_Z Y \). It will be helpful to have an explicit description of this pushout of sets. \( X +_Z Y \) can be regarded as the coproduct \( X + Y \) modulo an equivalence relation \( \sim \). Here we define \( a \sim b \) if and only if there exists a \( z \in Z \) such that \( f(z) = a \) and \( g(z) = b \). An element of \( X +_Z Y \) is denoted by \([a]\) where \( a \) is some representative of its equivalence class. Define \( G +_K H : X +_Z Y \times X +_Z Y \to R \) by the rule

\[
G +_K H([a], [b]) = \bigvee_{a_i \in [a], b_j \in [b]} G + H(a_i, b_j)
\]

This does indeed define a pushout in \( \text{Mat}_R \). Suppose we have a commutative diagram of \( R \)-matrices as follows:

\[
\begin{array}{c}
\text{L} \searrow & G +_K H & \swarrow \text{K.} \\
& G & \\
\downarrow{f} & & \downarrow{g} \\
\text{X} & \xleftarrow{c_1} & \xrightarrow{c_2} \text{Y} \\
\text{Z.}
\end{array}
\]

then the underlying diagram of sets induces a unique function \( u \)

\[
\begin{array}{c}
\text{C} \searrow & \uparrow u & \swarrow \text{c_1} \\
& \text{X +_Z Y} & \\
\downarrow{f} & & \downarrow{g} \\
\text{X} & \xleftarrow{c_1} & \xrightarrow{c_2} \text{Y} \\
\text{Z.}
\end{array}
\]

commuting suitable with \( c_1 \) and \( c_2 \). The map \( u \) is certainly unique, it remains to show that it is well-defined i.e. it satisfies the inequality

\[
G +_K H([a], [b]) \leq L(u[a], u[b])
\]

Let \([a]\) be the set \( \{a_1, a_2, \ldots \} \) and \([b]\) be the set \( \{b_1, b_2, \ldots \} \). Then because \( c_1 \) and \( c_2 \) are morphisms of \( R \)-matrices, each pair satisfies \( G + H(a_i, b_j) \leq L(u[a_i], u[b_j]) \). Therefore, the maximum of such pairs satisfies this inequality as well. \( \square \)

An \( R \)-matrix \( M : X \times X \to R \) can represent a graph with vertex set \( X \) weighted in \( R \). Similarly, an open \( R \)-matrix, represents an \( R \)-weighted graph equipped with
inputs and outputs. For example, the $[0, \infty]$-matrix
\[
\begin{bmatrix}
1 & 2 & .1 \\
3 & 0 & .2 \\
\infty & 1 & .2
\end{bmatrix}
\]
on the set \{a, b, c\} can be regarded as an open $[0, \infty]$-matrix with left input set \{1, 2\} and right input set \{3\}. The mappings of the cospan are given by $1 \mapsto a$, $2 \mapsto b$ and $3 \mapsto c$. This can be drawn as an open weighted graph

where a tuple labeling an edge indicates the weights on that edge in both directions. Similarly, we define an open $[0, \infty]$-matrix on \{d, e\}
\[
\begin{bmatrix}
6 & \infty \\
0 & 9
\end{bmatrix}
\]
with left input set given by \{3\} and right input set given by \{4\}. The mappings in the cospan for this open $[0, \infty]$-matrix are given by the assignments $3 \mapsto d$ and $4 \mapsto e$. This open $[0, \infty]$-matrix is drawn as

The pushout of these two $[0, \infty]$-matrices is represented by

where edges are omitted if their weight is infinite in both directions. The matrix on the apex of this pushout is given by
\[
\begin{bmatrix}
1 & 2 & .1 & \infty \\
3 & 0 & .2 & \infty \\
\infty & 1 & .2 & \infty \\
\infty & \infty & 0 & 9
\end{bmatrix}
\]
whose values are determined the formula given in equation 2. Note that the only nontrivial join in this formula is performed on the vertex which was common to both open weighted graphs. This pushout operation is the horizontal composition in the structure of a bicategory.

**Theorem 2.4.** For a quantale $R$, there is a bicategory $\text{Open}(\text{Mat}_R)$ where

- objects are sets,
- morphisms are open $R$-matrices $M: X \to Y$,
- composition is given by pushout, i.e. given two horizontal open $R$-matrices

\[
\begin{array}{ccc}
P & \xrightarrow{\text{LY}} & Q \\
\downarrow & & \downarrow \\
LX & \xrightarrow{\text{LY}} & LZ
\end{array}
\]

their composite is given by this cospan from $LX$ to $LZ$:

\[
\begin{array}{ccc}
P & \xrightarrow{\text{LY}} & Q \\
\downarrow & & \downarrow \\
P +_{\text{LY}} Q & \xrightarrow{\text{LX}} & LZ
\end{array}
\]

where the diamond is a pushout square.

- A 2-morphism from $M: X \to Y$ to $N: X \to Y$ is a commutative diagram

\[
\begin{array}{ccc}
M \\
\downarrow & & \downarrow \\
LX & \xleftarrow{\text{LY}} & LY \\
\downarrow & & \downarrow \\
N
\end{array}
\]

**Proof.** Corollary 2.4 of [BC19] constructs this category as long as $\text{Mat}_R$ has pushouts and $L$ preserves pushouts. The former has been proved in Proposition 2.1 and the latter has been proved in Proposition 2.3.

\[\square\]

3. The Open Algebraic Path Problem as a Functor

In this section we show how the algebraic path problem functor $F: \text{Mat}_R \to \text{IdMat}_R$ extends to a psuedofunctor

$$\text{Open}(F): \text{Open}(\text{Mat}_R) \to \text{Open}(\text{IdMat}_R).$$

An important consequence of this is that the solution of the algebraic path problem on a composite $F(M \circ N)$ can be obtained by taking the pushout of the solution on its components $F(M)$ and $F(N)$. To define this psuedofunctor, we first need to define its codomain.

**Proposition 3.1.** There is a bicategory $\text{Open}(\text{IdMat}_R)$ where

- objects are sets,
• morphisms are open idempotent $R$-matrices, i.e. cospans of $R$-matrices

$$M \quad \text{LX} \quad \text{LY}$$

where the apex is idempotent.

• Composition is given by pushout, i.e. given two open idempotent $R$-matrices,

$$\text{P \quad LY} \quad \text{Q \quad LY} \quad \text{LZ}$$

their composite is given by this cospan from LX to LZ:

$$\text{P +}_{FLY} \quad \text{Q}$$

where the diamond is a pushout square.

• A 2-morphism from $M: X \to Y$ to $N: X \to Y$ is a commutative diagram

$$\text{M \quad LX \quad LY} \quad \text{N \quad LX \quad LY \quad LZ}$$

Proof. First note that for a set $X$, LX is an idempotent matrix and can be regarded as an object of $\text{IdMat}_R$. Also, because $F \dashv U$ is an idempotent adjunction, $FULX = LX$. Therefore, to construct the above bicategory, we may apply Corollary 2.4 of [BC19] to the composite left adjoint $F \circ L: \text{Set} \to \text{IdMat}_R$. To use this corollary, we also need $\text{IdMat}_R$ to have pushouts. This follows from Corollary 2.14 of [Wol74] after recognizing that $\text{IdMat}$ is the category of $R$-categories, i.e. categories enriched in $R$. □

So far we have the commutative diagram of functors

$$\text{Mat}_R \quad \text{IdMat}_R \quad \text{Set}$$

The definition of $\text{Open}$ is functorial with respect to this sort of diagram.

**Theorem 3.2.** There is a pseudofunctor

$$\text{Open}(F): \text{Open}(\text{Mat}_R) \to \text{Open}(\text{IdMat})$$

which is
• the identity on objects,
• an open $R$-matrix

\[
\begin{array}{ccc}
M & \xrightarrow{LX} & LY \\
\downarrow & & \downarrow \\
FLX & \xleftarrow{FM} & FLY \\
\end{array}
\]

is sent to the solution of its algebraic path problem

\[
\begin{array}{ccc}
M & \xrightarrow{LX} & LY \\
\downarrow & & \downarrow \\
FLX & \xleftarrow{FM} & FLY \\
N & \xrightarrow{FN} & N \\
\end{array}
\]

and,

• a 2-morphism of open $R$-matrices

\[
\begin{array}{ccc}
M & \xrightarrow{LX} & LY \\
\downarrow & & \downarrow \\
FLX & \xleftarrow{FM} & FLY \\
\end{array}
\]

Proof. Section 4 of [BC19] proves functoriality of the “Open” construction in a different context. The proof of coherence for this pseudofunctor follows from very similar arguments. Because $F$ preserves pushouts, the pseudofunctor $\text{Open}(F)$ is indeed pseudofunctorial.

Knowing that $\text{IdMat}_R$ has pushouts and knowing how to construct them are two different matters. In general, pushouts in a category of monoids are computed via the transfinite construction of free algebras [Kel80]. The idea behind this construction is that colimits in a category of monoids can be constructed by first taking the colimit of their underlying objects, taking the free monoid on that colimit, and then quotienting out by the equations in your original monoids. However, in $\text{IdMat}$ this process simplifies a little.

**Proposition 3.3.** For a diagram $D: C \to \text{IdMat}_R$, its colimit is given by the formula

$$\colim_{c \in C} D(c) \cong F(\colim_{c \in C} U(D(c)))$$

Proof. It suffices to show that $F(\colim_{c \in C} U(D(c)))$ satisfies the universal property of $\colim_{c \in C} D(c)$. Let $\alpha: \Delta_d \Rightarrow D$ be a cocone from an object $d \in \text{IdMat}_R$ to our
diagram $D$. Because $\alpha$ can be regarded as a cocone in $\text{Mat}_R$, the universal property of colimits induces a unique map

$$\text{colim}_{c \in C} U(D(c)) \to U(d)$$

of $R$-matrices. Applying $F$ to this morphism gives a map

$$F(\text{colim}_{c \in C} U(D(c))) \to FU(d) = d$$

where the last equality follows either from elementary considerations or from the adjunction $F \dashv U$ being idempotent. The above map is a unique morphism satisfying the universal property for $\text{colim}_{c \in C} D(c)$. □

With this description of colimit, the pseudofunctoriality of $\text{Open}(F)$ can be expressed in a friendlier form.

**Corollary 3.4.** Given open $R$-matrices $M: X \to Y$ and $N: Y \to Z$, the solution of the algebraic path problem on their composite $M +_{LY} N$ is given by

$$F(M +_{LY} N) \cong F(UF(M) +_{LY} UF(N))$$ (3)

**Proof.** This follows directly from functoriality of $\text{Open}(F)$ and the description of colimits in Proposition 3.3. □

Pouly and Kohlas present a similar formula in the context of valuation algebras. [PK12, §6.7]. For matrices $M$ and $N$ representing weighted graphs on vertex sets $s$ and $t$ respectively, the solution to the algebraic path problem on the union of their vertex sets is given by

$$F(M) \otimes F(N) = F\left(F(M)^\uparrow_{s \cup t} \lor F(N)^\uparrow_{s \cup t}\right)$$

In this formula, $\uparrow_{s \cup t}$ indicates that the matrix is trivially extended to the union of the vertex sets. This formula is less general than isomorphism (3). It corresponds to the special case of Isomorphism (3.4) when the legs of the open $R$-matrices are inclusions.

4. Conclusion

The Floyd–Warshall algorithm computes the solution to the algebraic path problem with complexity $\Theta(n^3)$ where $n$ is the number of vertices in your weighted graph [Flo62]. Isomorphism (3) suggests a strategy for computing the solution to the algebraic path problem which reduces this complexity. First cut your weighted graph into smaller chunks, compute the solution to the algebraic path problem on those chunks, then combine their solutions using isomorphism (3). The success of this strategy relies on choosing the cuts so that the last application of $F$ stabilizes in a small number of steps. In general, this application can take just as long as computing $F(M +_{LY} N)$ directly. However, if the cuts are chosen so that there aren’t many paths which zig-zag across the cut, then the last application of $F$ will require relatively few operations. Determining the viability of this strategy will require further exploration and experimentation. Other works which employ a similar strategy provide encouraging results. In [STV92], Sairam, Tamassia, and Vitter show how choosing one way separators as cuts in a graph, allow for an efficient divide and conquer parallel algorithm for computing shortest paths. In [RSS14] Rathke, Sobocinski, and Stephens show how the reachability problem on a 1-safe Petri net can be computed more efficiently by cutting it up into more manageable pieces. We
hope that similar heuristics and algorithms can be developed in the general setting of the algebraic path problem.

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