Detecting lower bounds to quantum channel capacities

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(Dated: April 12, 2016)

We propose a method to detect lower bounds to quantum capacities of a noisy quantum communication channel by means of few measurements. The method is easily implementable and does not require any knowledge about the channel. We test its efficiency by studying its performance for most well known single qubit noisy channels and for the generalised Pauli channel in arbitrary finite dimension.
Noise is unavoidably present in any communication channel. In this case the ability of the channel to convey information is lower than in the ideal noiseless case and it can be quantified in terms of channel capacities. Depending on the task to be performed and on the resources available, several kinds of capacities can be defined. The ability of the channel to convey classical information is quantified in terms of the classical channel capacity $C$ \[1\] \[2\], defined as the maximum number of classical bits that can be reliably transmitted per channel use. If the sender and the receiver share unlimited prior entanglement, the capacity of transmitting classical information is quantified in terms of the entanglement-assisted classical capacity $C_E$ \[3\] \[4\]. The ability of the channel to convey classical information privately is quantified in terms of the private channel capacity $P$, defined as the maximum number of classical bits that can be reliably transmitted per channel use in such a way that negligible information can be obtained by a third party \[5\]. The ability of the channel to convey quantum information is quantified in terms of the quantum capacity $Q$ \[6\] \[7\], defined as the maximum number of qubits that can be reliably transmitted per channel use.

In many practical situations a complete knowledge of the kind of noise present along the channel is not available, and sometimes noise can be completely unknown. It is then important to develop efficient means to establish whether in these situations the channel can still be profitably employed for information transmission. A standard method to establish this relies on quantum process tomography, where a complete reconstruction of the CP map describing the action of the channel can be achieved, and therefore all its communication properties can be estimated. This, however, is a demanding procedure in terms of the number of different measurement settings needed, since it scales as $d^4$ for a finite $d$-dimensional quantum system. In this Letter we address the situation where we want to gain some information on the channel ability to transmit quantum information by employing a smaller number of measurements, that scales as $d^2$. We derive a lower bound on the channel capacities that can be experimentally accessed with a simple procedure and can be applied to an unknown quantum communication channel. The efficiency of the method is then studied for many examples of single qubit channels, and for the generalised Pauli channel in arbitrary finite dimension.

In the following we focus on memoryless channels. We denote the action of a generic quantum channel on a single system as $E$ and define $E_N = E^\otimes N$, where $N$ represents the number of channel uses. The quantum capacity $Q$ measured in qubits per channel use is defined as \[5\] \[7\]

$$Q = \lim_{N \to \infty} \frac{Q_N}{N},$$  \hspace{1cm} (1)

where $Q_N = \max_{\rho} I_c(\rho, E_N)$, and $I_c(\rho, E_N)$ denotes the coherent information \[8\]

$$I_c(\rho, E_N) = S(E_N(\rho)) - S_c(\rho, E_N).$$  \hspace{1cm} (2)

In Eq. 2, $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, and $S_c(\rho, E)$ represents the entropy exchange \[9\], i.e.

$$S_c(\rho, E) = S((I_R \otimes E)(|\Psi_\rho\rangle\langle\Psi_\rho|)),$$

where $|\Psi_\rho\rangle$ is any purification of $\rho$ by means of a reference quantum system $R$, namely $\rho = \text{Tr}_R[|\Psi_\rho\rangle\langle\Psi_\rho|]$.

We will now derive a lower bound for the quantum capacity $Q$ that can be easily accessed without requiring full process tomography of the quantum channel. Since for any complete set of orthogonal projectors $\{\Pi_i\}$ one has \[10\]

$S(\rho) \leq S(\sum_i \Pi_i \rho \Pi_i)$, then for any orthonormal basis $\{|\Phi_i\rangle\}$ in the tensor product of the reference and the system Hilbert spaces one has the following bound to the entropy exchange

$$S_c(\rho, E) \leq H(\bar{p}),$$  \hspace{1cm} (3)

where $H(\bar{p})$ denotes the Shannon entropy for the vector of the probabilities $\{p_i\}$, with

$$p_i = \text{Tr}[(I_R \otimes E)(|\Psi_\rho\rangle\langle\Psi_\rho|)|\Phi_i\rangle\langle\Phi_i|].$$  \hspace{1cm} (4)

From Eq. 3 it follows that for any $\rho$ and $\bar{p}$ one has the following chain of bounds

$$Q \geq Q_1 \geq I_c(\rho, E_1) \geq S(E(\rho)) - H(\bar{p}) \equiv Q_{DET}.$$  \hspace{1cm} (5)

A lower bound $Q_{DET}$ to the quantum capacity of an unknown channel can then be detected by the following prescription: prepare a bipartite pure state $|\Psi_\rho\rangle$ and send it through the channel $I_R \otimes E$, where the unknown channel $E$ acts on one of the two subsystems. Then measure suitable local observables on the joint output state to estimate $\bar{p}$ and $S(E(\rho))$ in order to compute $Q_{DET}$. Notice that for a fixed measurement setting, one can infer different vectors of probabilities pertaining to different sets of orthogonal projectors, as will be clarified in the following. In principle, one can even adopt an adaptive detection scheme to improve the bound \[5\] by varying the input state $|\Psi_\rho\rangle$.

We will now be more specific and consider first the case of qubit channels. We assume that only the local observables $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$, and $\sigma_z \otimes \sigma_z$ on the system and on the reference qubits are measured, and we want to optimise the
bound $Q_{DET}$ given these resources. First, we notice that the above measurements allow to measure $\sigma_x$, $\sigma_y$, and $\sigma_z$ on the system qubit alone, by ignoring the statistics of the measurement results on the reference qubit. In this way, a complete tomography of the system output state can be performed, and therefore the term $S(E(\rho))$ in Eq. (5) can be estimated exactly. Furthermore, by denoting the Bell states as

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

(6)

it can be straightforwardly proved that the local measurement settings $\{\sigma_x \otimes \sigma_z, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$ allow to estimate the vector $\vec{p}$ pertaining to the projectors onto the following inequivalent bases

$$B_1 = \{a|\Phi^+\rangle + b|\Phi^-\rangle, -b|\Phi^+\rangle + a|\Phi^-\rangle, c|\Psi^+\rangle + d|\Psi^-\rangle, -d|\Psi^+\rangle + c|\Psi^-\rangle\},$$

(7)

$$B_2 = \{a|\Phi^+\rangle + b|\Phi^-\rangle, -b|\Phi^+\rangle + a|\Phi^-\rangle, c|\Phi^-\rangle + d|\Phi^+\rangle, -d|\Phi^-\rangle + c|\Phi^+\rangle\},$$

(8)

$$B_3 = \{a|\Phi^+\rangle + ib|\Psi^+\rangle, ib|\Phi^+\rangle + a|\Psi^-\rangle, c|\Phi^-\rangle + id|\Psi^+\rangle, id|\Phi^-\rangle + c|\Psi^+\rangle\},$$

(9)

with $a, b, c, d$ real and such that $a^2 + b^2 = c^2 + d^2 = 1$. Actually, the measurements corresponding to the above three bases are achieved by orthogonal projectors of the form

$$\Pi_{\{a|\Phi^+\rangle+b|\Phi^-\rangle\}} = \frac{1}{4} (I \otimes I + \sigma_z \otimes \sigma_z) + \frac{a^2 - b^2}{4} (\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) + \frac{ab}{2} (\sigma_z \otimes I + I \otimes \sigma_z),$$

(10)

$$\Pi_{\{c|\Psi^+\rangle+d|\Psi^-\rangle\}} = \frac{1}{4} (I \otimes I - \sigma_z \otimes \sigma_z) + \frac{c^2 - d^2}{4} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) + \frac{cd}{2} (\sigma_z \otimes I - I \otimes \sigma_z),$$

(11)

$$\Pi_{\{a|\Phi^+\rangle+b|\Phi^-\rangle\}} = \frac{1}{4} (I \otimes I + \sigma_z \otimes \sigma_z) + \frac{a^2 - b^2}{4} (\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) + \frac{ab}{2} (\sigma_z \otimes I + I \otimes \sigma_z),$$

(12)

$$\Pi_{\{c|\Phi^-\rangle+d|\Phi^+\rangle\}} = \frac{1}{4} (I \otimes I - \sigma_x \otimes \sigma_x) + \frac{c^2 - d^2}{4} (\sigma_z \otimes \sigma_z + \sigma_y \otimes \sigma_y) - \frac{cd}{2} (\sigma_z \otimes I - I \otimes \sigma_z),$$

(13)

$$\Pi_{\{a|\Phi^+\rangle+ib|\Psi^+\rangle\}} = \frac{1}{4} (I \otimes I - \sigma_y \otimes \sigma_y) + \frac{a^2 - b^2}{4} (\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x) - \frac{ab}{2} (\sigma_y \otimes I - I \otimes \sigma_y),$$

(14)

$$\Pi_{\{c|\Phi^-\rangle+id|\Psi^+\rangle\}} = \frac{1}{4} (I \otimes I + \sigma_y \otimes \sigma_y) + \frac{c^2 - d^2}{4} (\sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x) + \frac{cd}{2} (\sigma_y \otimes I + I \otimes \sigma_y),$$

(15)

where $\Pi_{\{a|\Phi^+\rangle+b|\Phi^-\rangle\}}$ denotes the projector onto the state $a|\Phi^+\rangle + b|\Phi^-\rangle$, and analogously for the other projectors. The probability vector $\vec{p}$ for each choice of basis is then evaluated according to Eq. (4). The expectation values for terms of the form $\sigma_z \otimes I$ (or $I \otimes \sigma_z$) can be measured from the outcomes of the observable $\sigma_z \otimes \sigma_z$ by ignoring the measurement results on the second (or first) qubit, and analogously for the other similar terms in the above projectors.

Therefore, in order to obtain the tightest bound in [3], given the fixed local measurements $\{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$, the Shannon entropy $H(\vec{p})$ will be minimised as a function of the bases (7-9), by varying the coefficients $a, b, c, d$ over the three sets. In an experimental scenario, after collecting the outcomes of the measurements $\{\sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$, this optimisation step corresponds to classical processing of the measurement outcomes.

Our procedure can be generalised for arbitrary finite dimension $d$. For simplicity, we will now consider a fixed input maximally entangled state $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$, where $d$ is the finite dimension of each subsystem, and a Bell basis

$$|\Phi_i\rangle = (I_R \otimes U_i)|\Psi\rangle, \quad i = 0, 1, \ldots, d^2 - 1,$$

(16)

with $U_i$ unitary and $\text{Tr}[U_i^\dagger U_i] = d \delta_{ij}$. The detectable bound takes the form

$$Q_{DET} = S\left(E\left(\frac{1}{d}\right)\right) - H(\vec{p}),$$

(17)

with $p_i = \frac{1}{d^2} \sum_j |\text{Tr}[U_i^\dagger A_j]|^2$, where $\{A_j\}$ denotes the Kraus operators of the channel $E(\rho) = \sum_j A_j \rho A_j^\dagger$. The bound of Eq. (17) in this case can be detected by measuring $d^2 - 1$ observables via a local setting and classical processing of the measurement outcomes. Actually, a set of generalised Bell projectors can be written as follows [11]

$$|\Phi^{mn}\rangle\langle\Phi^{mn}| = \frac{1}{d} \sum_{p,q=0}^{d-1} e^{2\pi i (mp - np)} U_{pq} \otimes U_{pq}^\dagger,$$

(18)
where $m,n = 0,1,\ldots,d-1$, and $U_{mn}$ represents the unitary operator $U_{mn} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} km} |k\rangle \langle k+n |$ mod $d$. Hence, a set of measurements on the eigenstates of $U_{mn} \otimes U_{mn}^*$ allows to estimate $Q_{DET}$ in Eq. (17).

As mentioned above, the advantage of this procedure is to require $d^2 - 1$ measurement settings with respect to a complete process tomography, where $d^2 - 1$ observables have to be measured.

We want to point out that all detectable bounds we are providing also give lower bounds to the private information $P$, since $P \geq Q_1$ [10]. Moreover, we can also derive a detectable lower bound to the entanglement-assisted classical capacity. Actually, this is defined as $C_E = \max_{\rho} I(\rho, \mathcal{E}_1)$, where $I(\rho, \mathcal{E}_1) = S(\rho) + I_c(\rho, \mathcal{E}_1)$. By considering the procedure outlined above, where a maximally entangled state $|\Psi\rangle$ is considered as input, we then have the lower bound $C_E \geq \log_2 d + Q_{DET}$.

In the following we will analyse explicit examples of noisy quantum channels that are typically encountered in practical implementations, and analyse in detail the efficiency of the proposed procedure. We will consider in particular the most well known channels for qubits, i.e. the dephasing, the depolarising and the more general Pauli channel, the erasure and the amplitude damping channel. The Pauli channels are also generalised to arbitrary dimension. We finally consider a family of qubit channels with two Kraus operators. We will always consider input states with maximal entanglement between system and reference.

A dephasing channel for qubits with unknown probability $p$ can be written as $\mathcal{E}(\rho) = (1 - \frac{p}{2}) \rho + \frac{p}{2} \sigma_z \rho \sigma_z$. Since it is a degradable channel, its quantum capacity coincides with the one-shot single-letter quantum capacity $Q_1$, and one has $Q = Q_1 = 1 - H_2(\frac{p}{2})$, where $H_2(x) \equiv -x \log_2 x - (1-x) \log_2(1-x)$ is the binary Shannon entropy. More generally, we can consider the channel $\mathcal{E}(\rho) = (1 - \frac{p}{2}) \rho + \frac{p}{2} U \rho U^\dagger$, for any dimension $d$, with $U$ as unitary and traceless operator. The von Neumann entropy of the output state $\mathcal{E}(\frac{1}{d}) = \frac{1}{d}$ is given by $S(\mathcal{E}(\frac{1}{d})) = \log_2 d$. Using the Bell basis [18] one obtains the detectable bound

$$Q_{DET} = \log_2 d - H_2(\frac{p}{2}) .$$

This detectable quantum capacity clearly coincides with the quantum channel capacity for $d = 2$.

The depolarising channel with probability $p$ for qubits is given by [10] $\mathcal{E}(\rho) = (1 - p) \rho + \frac{p}{4} \sum_{x,y,z} \sigma_x \rho \sigma_z$. The quantum capacity is still unknown, although one has the upper bound [12] $Q \leq 1 - 4p$, thus showing that $Q = 0$ for $p \geq \frac{1}{4}$. On the other hand, by random coding the following hashing bound [13] has been proved

$$Q \geq 1 - H_2(p) - p \log_2 3 .$$

This lower bound coincides with our detectable bound of Eq. (17). In Fig. 1 we plot the lower bound [20], along with the upper bound $Q \leq 1 - 4p$, versus the probability $p$. As we can see, our procedure allows to detect $Q(p) \neq 0$ as long as $p < 0.1892$.

![FIG. 1. Detectable quantum capacity (thick line) for the depolarising channel with error probability $p$ (which coincides with the hashing bound [20]) versus $p$. The dashed line represents the known upper bound $Q \leq 1 - 4p$.](image)

In arbitrary dimension $d$ the depolarising channel takes the form

$$\mathcal{E}(\rho) = (1 - p \frac{d^2}{d^2 - 1}) \rho + p \frac{d^2}{d^2 - 1} I .$$

Hence, the detectable bound is simply generalised to

$$Q \geq Q_{DET} = \log_2 d - H_2(p) - p \log_2 (d^2 - 1) ,$$

and can be detected by estimating $\tilde{p}$ pertaining to the Bell projectors [18].
Similarly to the depolarising channel, for a generic Pauli channel \( \mathcal{E}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i \), the hashing bound \[13\] provides a lower bound to the quantum capacity \( Q \geq 1 - H(\vec{p}) \), which coincides with our detectable bound \[17\] by using a maximally entangled input state and estimating \( \vec{p} \) for the Bell basis. In dimension \( d \) one can consider the generalised channel \( \mathcal{E}(\rho) = \sum_{m,n=0}^{d-1} p_{mn} U_{mn} \rho U_{mn}^\dagger \), and one has

\[
Q \geq Q_{DET} = \log_2 d - H(\vec{p}) ,
\]

\( \vec{p} \) being now the \( d^2 \)-dimensional vector of probabilities \( p_{mn} \) pertaining to the generalised Bell projectors in Eq. \[18\]. We consider now an erasure channel \[14, 15\] with erasure probability \( p \), where the optimal vector of probabilities is given by

\[
\vec{p} = (1 - p) \rho \otimes |e\rangle \langle e| \text{Tr}[\rho] ,
\]

where \( |e\rangle \) denotes the erasure flag which is orthogonal to the system Hilbert space. Since it is a degradable channel, its quantum capacity coincides with the one-shot single-letter quantum capacity \( Q \), for \( p < 1/2 \), and \( Q = 0 \) for \( p \geq 1/2 \). The output of any maximally entangled state \( |\Psi\rangle \) can be written as

\[
\mathcal{E}(|\Psi\rangle\langle\Psi|) = (1 - p)|\Psi\rangle\langle\Psi| \oplus \frac{p}{d} (I_R \otimes |e\rangle \langle e|).
\]

A basis constructed by the union of the projectors on \( |i\rangle \otimes |e\rangle \) (with \( i = 0, 1, \ldots, d - 1 \)) and Bell projectors (where one of them corresponds to \( |\Psi\rangle\langle\Psi| \)) gives a vector of probability \( d \) with \( d \) elements equal to \( p/d \) and one element \( (1 - p) \), while all other terms are vanishing. We then have \( H(\vec{p}) = H_2(p) + p \log_2 d \). The von Neumann entropy of the reduced output state \( \mathcal{E}(\frac{1}{d}) = (1 - p)\frac{1}{d} \otimes |e\rangle \langle e| \) is given by \( S(\mathcal{E}(\frac{1}{d})) = H_2(p) + (1 - p) \log_2 d \). It then follows that the detectable bound \( Q_{DET} \) for the erasure channel coincides with \( Q \) in Eq. \[25\].

The amplitude damping channel for qubits has the form \[10\]

\[
\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger ,
\]

where \( A_0 = |0\rangle\langle 0| + \sqrt{1 - \gamma}|1\rangle\langle 1| \) and \( A_1 = \sqrt{\gamma}|0\rangle\langle 1| \). Since it is a degradable channel \[16\], its quantum capacity coincides with the one-shot single-letter quantum capacity \( Q \), and one has

\[
Q = Q_1 = \max_{q \in [0,1]} H_2((1 - \gamma)q) - H_2(\gamma q) ,
\]

for \( \gamma \leq 1/2 \), and \( Q = 0 \) for \( \gamma \geq 1/2 \). For an input Bell state \( |\Phi^\pm\rangle \) the output is given by

\[
I_R \otimes \mathcal{E}(|\Phi^\pm\rangle\langle\Phi^\pm|) = \frac{1}{4} (1 + \sqrt{1 - \gamma})^2 |\Phi^\pm\rangle\langle\Phi^\pm| + \frac{1}{4} (1 - \sqrt{1 - \gamma})^2 |\Phi^\mp\rangle\langle\Phi^\mp| + \frac{\gamma}{2} (|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|)
\]

\[
+ \frac{\gamma}{2} (|\Phi^\mp\rangle\langle\Phi^\mp| - |\Phi^+\rangle\langle\Phi^+|) + |\Phi^-\rangle\langle\Phi^-| - |\Phi^\mp\rangle\langle\Phi^\mp| + |\Phi^-\rangle\langle\Phi^-| - |\Phi^-\rangle\langle\Phi^-|).
\]

The reduced output state is given by \( \mathcal{E}(\frac{1}{d}) = \frac{1}{2} (I + \gamma \sigma_z) \), hence it has von Neumann entropy \( S(\mathcal{E}(\frac{1}{d})) = H_2(\frac{1 - \gamma}{2}) \). By performing the local measurement of \( \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \) and \( \sigma_z \otimes \sigma_z \) and optimising \( \vec{p} \) over the bases \[4, 9\], one can detect the bound

\[
Q \geq Q_{DET} = H_2\left(\frac{1 - \gamma/2}{2}\right) - H(\vec{p})
\]

\[
= H_2\left(\frac{1 - \gamma}{2}\right) - H_2\left(\frac{\gamma}{2}\right) ,
\]

where the optimal vector of probabilities is given by \( \vec{p} = (1 - \gamma/2, 0, 0, \gamma/2) \), and it corresponds to the basis in Eq. \[7\], with \( a = \frac{1 + \sqrt{1 - \gamma}}{\sqrt{2(1 - \gamma)}}, b = \frac{\gamma}{(1 + \sqrt{1 - \gamma})\sqrt{2(1 - \gamma)}}, \) and \( c = d = \frac{1}{\sqrt{2}} \). This basis is clearly made of projectors on the eigenstates of the output state \[29\]. It turns out that, as long as \( \gamma < 1/2 \), a non-vanishing quantum capacity is detected. Indeed the difference \( Q - Q_{DET} \) never exceeds 0.005. We notice that the Bell basis \[6\] does not provide the minimum value of \( H(\vec{p}) \). Actually, in such case one has

\[
\vec{p} = \frac{1}{4} \left( (1 + \sqrt{1 - \gamma})^2, (1 - \sqrt{1 - \gamma})^2, \gamma, \gamma \right) ,
\]
and by using this value of $\vec{p}$ a non-vanishing quantum capacity is detected only for $\gamma < 0.3466$. In Fig. 2 we plot the detectable bound from Eq. (30) [which is indistinguishable from the quantum capacity (28)], along with the bound obtained by the probability vector (31) pertaining to the Bell projectors, versus the damping parameter $\gamma$. The difference of the curves shows how the optimisation of $Q_{DET}$ over the bases (7) is crucial to achieve the optimal bound.

![Graph showing the detectable bound and the Bell projectors bound versus $\gamma$.](image)

**FIG. 2.** Amplitude damping channel with parameter $\gamma$: detected quantum capacity with maximally entangled input and estimation of $\vec{p}$ for the eigenstates of (29) and for the Bell basis (solid and dashed line, respectively).

We finally consider the following set of channels, characterised by just two Kraus operators [17–19],

$$\mathcal{E}(\rho) = \sum_{i=1}^{2} A_i \rho A_i^\dagger,$$

where $A_1 = \cos \alpha |0\rangle\langle 0| + \cos \beta |1\rangle\langle 1|$ and $A_2 = \sin \beta |0\rangle\langle 1| + \sin \alpha |1\rangle\langle 0|$, with $\alpha, \beta \in \mathbb{R}$. Details on these channels are given in the Supplemental Material [20]. Our detectable quantum capacity can be written as

$$Q \geq Q_{DET} = H_2((\cos^2 \alpha + \sin^2 \beta)/2) - H_2((\sin^2 \alpha + \sin^2 \beta)/2).$$

(33)

We checked numerically that $Q - Q_{DET} < 0.005$ for all values of $\alpha$ and $\beta$. The positive region of the detected capacity $Q_{DET}$ is plotted in Fig. 3.

![Graph showing the positive region of the detected quantum capacity.](image)

**FIG. 3.** Positive region of the detected quantum channel capacity (33) for the two-Kraus channel in Eq. (32).

In conclusion, we have proposed a method to detect lower bounds to capacities of quantum communication channels, specifically to the quantum capacity, the entanglement assisted capacity, and the private capacity. The procedure does not require any a priori knowledge about the quantum channel and relies on a number of measurement settings that scales as $d^2$. It is therefore much cheaper than full process tomography and it can be easily accessed in the lab without posing any particular restriction on the nature of the physical system under consideration. In particular, for quantum optical systems it is easily implementable with present-day technologies [21]. We tested the method for significant qubit channels and it turned out to give extremely good results for various forms of noise. The method can be successfully applied also to correlated Pauli and amplitude damping channels acting on two qubits [22].

We thank Antonio D’Arrigo for valuable suggestions.
Appendix: Supplemental material

The following set of channels
\[ \mathcal{E}(\rho) = \sum_{i=1}^{2} A_i \rho A_i^\dagger, \]
where \( A_1 = \cos \alpha |0\rangle \langle 0| + \cos \beta |1\rangle \langle 1| \) and \( A_2 = \sin \beta |0\rangle \langle 1| + \sin \alpha |1\rangle \langle 0| \), with \( \alpha, \beta \in \mathbb{R} \), is characterised by just two Kraus operators \([17, 19]\), and represents the normal form of equivalence classes, since two channels have the same capacity if they differ merely by unitaries acting on input and output. Notice that for \( \alpha = \beta \) the channel is dephasing, and for \( \beta = 0 \) it is amplitude damping. These channels are shown to be degradable \([18]\) for \( \cos(2\alpha) / \cos(2\beta) > 0 \), hence \( Q = Q_1 \). On the other hand, they are antidegradable for \( \cos(2\alpha) / \cos(2\beta) \leq 0 \), thus with \( Q = 0 \).

The coherent information is maximised by diagonal input states, and in the region \( \cos(2\alpha) / \cos(2\beta) > 0 \) the quantum capacity is given by \([18]\)
\[ Q = \max_{p \in [0,1]} H_2(p \cos^2 \alpha + (1-p) \sin^2 \beta) - H_2(p \sin^2 \alpha + (1-p) \sin^2 \beta) \] 
(A.2)

For a detection scheme with the maximally entangled input state \( |\Phi^+\rangle \), the output state can be shown to be diagonal on the basis in Eq. (7), with
\[ a = \frac{\cos \beta - \cos \alpha}{\sqrt{2(\cos^2 \alpha + \cos^2 \beta)}}, \quad b = \frac{\cos \alpha + \cos \beta}{\sqrt{2(\cos^2 \alpha + \cos^2 \beta)}}, \]
\[ c = \frac{\sin \beta - \sin \alpha}{\sqrt{2(\sin^2 \alpha + \sin^2 \beta)}}, \quad d = \frac{\sin \alpha + \sin \beta}{\sqrt{2(\sin^2 \alpha + \sin^2 \beta)}}, \]
and eigenvalues \( \{0, (\cos^2 \alpha + \cos^2 \beta)/2, 0, (\sin^2 \alpha + \sin^2 \beta)/2\} \). The optimal vector of probabilities \( \vec{p} \) for the detectable bound \( Q_{\text{DET}} \) corresponds to these eigenvalues. The output entropy of the reduced state is then given by \( S(\mathcal{E}(\frac{1}{2})) = H_2((\cos^2 \alpha + \sin^2 \beta)/2) \), hence the detectable quantum capacity can be written as
\[ Q \geq Q_{\text{DET}} = H_2((\cos^2 \alpha + \sin^2 \beta)/2) - H_2((\sin^2 \alpha + \sin^2 \beta)/2). \] 
(A.3)

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