Graviton Emission from a Schwarzschild Black Hole in the Presence of Extra Dimensions

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Abstract. In this talk I presented a previously published work concerning the evaporation of (4 + n)-dimensional non-rotating black holes into gravitons [1]. In this work we calculated the energy emission rate for gravitons in the bulk obtaining analytical solutions of the master equation satisfied by all three types (S, V, T) of gravitational perturbations and presented graphs for the absorption probability and the energy emission of the black hole in the bulk.

1. Introduction
In the first part of the talk I gave a brief introduction to the brane world scenario. In this higher-dimensional context, which has received a lot of attention in the last few years [2, 3], gravity propagates in $D = 4 + n$ dimensions (Bulk), while all Standard Model degrees of freedom are assumed to be confined on a four-dimensional D-Brane. Within this scenario hierarchy problem can be solved and black holes can be produced in colliders or in cosmic ray interactions.

In the last few years there has been considerable interest in theoretically studying these higher dimensional black holes that may be created in near future experiments, perhaps even LHC (see [4, 5] for a review). An important property of them is Hawking radiation. According to it black holes are not completely black, but they emit radiation in a thermal spectrum characterized by a temperature $T_H$ such as a blackbody, while they deviate from the exact blackbody behavior by a factor called greybody factor. For a spherically symmetric BH

$$\frac{dN^{(s)}(\omega)}{dt} = \sum_{\ell} \sigma_{\ell,n}^{(s)}(\omega) \frac{\omega}{\exp(\omega/T_H) \pm 1} \frac{d^{n+3}k}{(2\pi)^{n+3}}$$

with $\sigma$ being the greybody factor which depends on particle’s energy, spin, angular momentum and on space dimensionality. What is crucial, is that $\sigma$ has been proven to be proportional to the

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1 This work was done in collaboration with S. Creek, P. Kanti and K. Tamvakis.
incoming absorption probability, and the following formulae is found to hold for a spherically
symmetric higher-dimensional black hole
\[ \frac{dE}{dt} = \sum_{\ell} N_{\ell} |A_{\ell}|^2 \frac{\omega}{\exp(\omega/T_H) \pm 1} \frac{d\omega}{2\pi} \] (1)
giving the energy emission rate for a particle of energy \( \omega \), with \( l \) being the angular momentum quantum number, \( A_{\ell} \) the absorption probability and \( N_{\ell} \) a constant counting the multiplicities of states with the same \( l \).

This black hole radiation has been the subject of both analytical and numerical studies in the past. This includes lower spin degrees of freedom \([6, 7, 8, 9, 10, 11]\) as well as graviton emission in the bulk \([12]\). The study of graviton emission in the bulk corresponds to the study of perturbations in a gravitational background. Within this context, we studied the hawking radiation of a Schwarzschild Black hole through gravitons. Gravitons emitted from a black hole will see the entire \( (4+n) \) dimensional space-time (bulk), will account for an energy loss from the brane where the BH is located to the bulk, and their emission rate will depend from the dimensionality of space-time, namely \( n \).

So what we will do is write the equations for graviton emission, analytically solve the equations, compute the absorption probability and from that compute the energy emission rate and its dependence on \( n \).

2. Mathematical Background

The gravitational background for a spherically symmetric, non charged non rotating black hole was found to be \([13]\)
\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{2+n} \]
with
\[ f(r) = 1 - \left( \frac{T_H}{r} \right)^{n+1} \]
and \( d\Omega^2_{n+2} \) the line element in a \( (n+2) \) dimensional unit sphere. The temperature of the black hole associated with Hawking radiation is
\[ T_H = \frac{n + 1}{4\pi r_H} \]

The equations that describe graviton emission from a Schwarzschild black hole in \( (4+n) \) dimensions are known for a little while \([14]\) and the way to produce them is to perturb the metric outside a black hole, use Einstein equations and thus get three independent wave equations describing emitted gravitational waves, analyzed in terms of partial waves with specific angular momentum. These equations are called scalar, vector and tensor perturbation equation respectively and they describe the emitted gravitons that correspond to each type of metric perturbation. All three can be written in the form of a master equation for the partial waves
\[ f \frac{d}{df} \left( f \frac{d\Phi}{dr} \right) + (\omega^2 - V)\Phi = 0 \] (2)
with \( V \) having different form for every perturbation. For the case of tensor and vector type we have:
\[ V_{T,V} = l(l + n + 1) + \frac{n(n+2)}{4} - \frac{k(n+2)^2}{4} \left( \frac{r_H}{r} \right)^{n+1} \] (3)
with \( l \) being the angular momentum quantum number and \( k = -1 \) (3) for tensor (vector) type. For the Scalar case the potential \( V \) is much more complicated but the way to treat it is similar to the tensor-vector type, so we will concentrate on the first case and present the complete results for all three types in the end.

In order to solve the equations we will use an approximate method, valid at the low energy limit of the equations. What we will do is at first solve the equations close to the horizon \( r_{H} \), then solve the equations far away from the horizon and in the end stretch the two solutions and match them in the intermediate zone. We must however take into consideration the boundary condition that nothing can escape from the black hole after it crosses the horizon, so in the solution we will have we must impose the condition that there are no outgoing waves in the limit \( r \to r_{H} \).

### 3. Solving the Equations - Computing the Absorption Coefficient

We will solve equation (2) for the potential (3), first at the near horizon regime. We change the variables from \( r \) to 
\[
\Phi = f^{\kappa}(1 - f)^{\lambda}F
\]
and taking the limit \( r \to r_{H} \), the equation (2) can be brought to the following form
\[
f(1 - f) \frac{d^{2}F}{df^{2}} + [c - (1 + a + b)] \frac{dF}{df} - abF = 0
\]
for the following choice of the parameters:
\[
a = \kappa + \lambda + \frac{(n + 2)}{2(n + 1)} + G \quad b = \kappa + \lambda + \frac{(n + 2)}{2(n + 1)} - G
\]
\[
c = 1 + 2\kappa \quad G^{(r,V)} = \frac{(1 + k)(n + 2)}{4(n + 1)}
\]
\[
\lambda = \frac{1}{2(n + 1)} \left[ -1 - \sqrt{(2l + n + 1)^{2} - 4\omega^{2}r_{H}^{2}} \right] \quad \kappa = -\frac{i\omega r_{H}}{n + 1}
\]
Equation (4) has known solutions, the hypergeometric functions, so the near horizon solution for tensor-vector mode gravitons is:
\[
\Phi_{NH} = A_{1}f^{\kappa}(1 - f)^{\lambda}F(a, b, c; f) + A_{2}f^{-\kappa}(1 - f)^{\lambda}F(a - c + 1, b - c, 2 - c; f)
\]
By expanding this solution in the limit \( r \to r_{H} \) and taking the boundary condition we get \( A_{2} = 0 \), and
\[
\Phi_{NH} = A_{1}f^{\kappa}(1 - f)^{\lambda}F(a, b, c; f)
\]
We then solve the equation in the far field regime, that is far away from the horizon. The solution after taking \( r >> r_{H} \) is easily obtained in terms of the Bessel functions
\[
\Phi_{FF} = B_{1}\sqrt{r}J_{l+(n+1)/2}(\omega r) + B_{2}\sqrt{r}Y_{l+(n+1)/2}(\omega r)
\]
We now have to match the two solutions in the intermediate zone. So if we take the limit \( r >> r_{H} \) in the near horizon solution, the limit \( \omega r \ll 1 \) in the far field solution and look in the low energy regime \( \omega r_{H} \ll 1 \), we see that the two solutions actually match and we
can compute the relation between the constants $B_1$ and $B_2$ of equation (5). Thus we have constructed a solution for the whole domain of $r$, valid for low energies, which we can use to compute the absorption coefficient. We expand the far field solution for $r \to \infty$ in terms of ingoing/outgoing spherical waves, and we only need to compute the ratio of their amplitudes. The result for the absorption coefficient put in a compact form, is:

$$|A_l|^2 = 4\pi \left( \frac{\omega r_H}{2} \right)^{2l+n+2} \frac{\Gamma \left( l + \frac{l+1}{n+1} - G \right)^2 \Gamma \left( l + \frac{l+1}{n+1} + G \right)^2}{\Gamma \left( l + \frac{n+3}{2} \right)^2 \Gamma \left( 1 + \frac{2l}{n+1} \right)^2}$$

We found that the same formula holds for the scalar type as well, with the only dependence from the type of perturbation encoded in $G$, which is different for every type. In figure 1 there are plots of the absorption probability for all three types of perturbation, for several combinations of $n,l$.

We can now proceed in computing the energy emission rates, but before that we have to know $N_l$ of equation (1), which is the number counting the state multiplicities: there is a number of different states that correspond to the same angular momentum number $l$. This number depends on $l,n$ and fortunately it has been computed in the literature for every type of perturbation [15]

$$N_{T_l} = \frac{n(n+3)(\ell + n + 1)(\ell - 1)(2\ell + n + 1)(\ell + n - 1)!}{2(\ell + 1)!(n+1)!}$$

$$N_{V_l} = \frac{(\ell + n + 1)\ell(2\ell + n + 1)(\ell + n - 1)!}{(\ell + 1)!n!}$$

$$N_{S_l} = \frac{(2\ell + n + 1)(\ell + n)!}{\ell!(n+1)!}$$

Now we can compute the energy emission rates for all three types and for every $n$ and plot the results. In figure 2 there are the energy emission rates for the case of 1 and 6 extra dimensions.
4. Results-Discussion
In this work I presented, we studied graviton emission in the bulk from a spherically symmetric (4+n) dimensional Schwarzschild black hole. Our results, valid in the low-energy regime, are complementary to existing studies in the intermediate energy regime [12] and are in agreement with two papers [16, 17] that appeared in the literature the same time with ours. These results show that vector perturbations are the dominant mode emitted in the bulk for all values of n. The relative emission rates of the subdominant scalar and tensor modes depend on n, with scalars foremost at small n and tensors more prevalent at high n. The absence of the \( l = 0 \), 1 partial waves, dominant in the low-energy regime, from all gravitational spectra causes even the total gravitational emission rate to be subdominant to that of scalar fields in the bulk. Finally, as previously found for bulk scalar fields [6, 8], the energy emission rates for all types of gravitational perturbations are suppressed with the number of extra dimensions in the entire low-energy regime.

The differential equations for the graviton emission were analytically solved for low energies using a matching technique, according to which the equation solutions were found in the near horizon and far field regime and were stretched and matched in the intermediate zone. We thus computed the absorption probability for all 3 types of perturbations, and written down the equations describing the black hole’s Hawking radiation spectrum. Although the radiation emitted in the bulk is not directly observable, it determines the energy left for emission on our brane. In this context, our results, in addition to their theoretical interest, would be of particular use to experiments developed to detect the low-energy spectrum of radiation emitted from a higher-dimensional black hole.

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