Stochastic growth of a stable phase from a metastable phase

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(Dated: November 1, 2021)

When a stable phase is adjacent to a metastable phase with a planar interface, the stable phase grows. We propose a stochastic model describing the phase growth accompanying latent heat. The model is based on an energy-conserving Potts model with a kinetic energy term defined on a sparsely random lattice only in one direction. For this model, we calculate the stable and metastable phases exactly using statistical mechanics. Performing numerical simulations, we find that the growth law is qualitatively different from that of the deterministic phase-field model.

I. INTRODUCTION

When a stable phase contacts a metastable phase, the stable phase grows, and eventually, the metastable phase vanishes. This phenomenon is ubiquitously observed in nature [1]. The basic understanding of phase growth is that the propagation velocity of the interface between the two phases is proportional to the difference in free energy densities, where the proportional constant is the mobility [2]. There are cases where propagating interfaces exhibit instabilities leading to a wide variety of patterns [3–5]. Even without the instabilities of interfaces, the phenomenon could be qualitatively affected in realistic situations. For example, latent heat is generated at the interface in growth processes, which diffuses into the bulk regions. The effect may modify the law of the propagation of interfaces.

Among several models describing phase growth, the phase-field model may be the most standard description that takes the effect of latent heat into account [6–11]. The model is given as a set of coupled partial differential equations of the order parameter field and the temperature field. It has been known that the displacement of the planar interface is proportional to the square root of the time interval when the extent of the metastability \( \Delta \) is less than unity [12–14]. Here, \( \Delta \) is defined as

\[
\Delta \equiv \frac{c_p}{T_c(\delta s)}|T_c - T_{ms}|,
\]

where \( T_c \) is the equilibrium transition temperature, \( T_{ms} \) is the temperature of the heat bath in contact with the metastable phase, \( \delta s \) is the entropy jump per unit volume, and \( c_p \) is the specific heat capacity per unit volume under constant pressure. The \( \sqrt{t} \)-dependence was numerically observed [12] and recently it has been derived through a systematic perturbation method [13].

Now, we address whether the phase-field model is appropriate for describing phase growth. First, we briefly review the fundamental assumptions of the model. The starting point is the entropy functional consisting of the spatial integration of the local entropy density and the gradient term [9, 10, 15, 16], where the entropy density is a function of the internal energy density and the number density. Then, following the Onsager principle, one can determine the evolution equation of these densities so that the irreversible currents are given as linear combinations of thermodynamic forces. The obtained equation is equivalent to the phase-field model [9, 10]. This form of the equation was also introduced in the context of the dynamical behavior near the critical point [15, 17]. Because such models derived from the Onsager principle are defined in a mesoscopic regime, thermal noises are inevitable in the description, where the noise intensity is determined by the fluctuation-dissipation relation of the second kind. Thus, it is not evident whether noise effects are irrelevant to the phase growth, which is assumed in the phase-field model.

Even worse, a typical interface width is on the order of \( 10^{-7} \) cm [20]. Thus, precisely speaking, the interface may be out of the mesoscopic description. The interface is interpreted as a unique region where unexpected physical processes occur. In the phase-field model, all such properties are universally represented by only the gradient term. This is another issue concerning the validity of the phase-field model.

Predictions of the phase-field model should be checked by well-controlled experiments or numerical simulations of statistical mechanics models. However, to our knowledge, no experimental report has shown the \( \sqrt{t} \) dependence. Although many statistical mechanics models have been studied in the context of phase growth [18–27], the generation of latent heat is not taken into account. Considering these situations, this paper proposes a statistical mechanics model describing the phase growth accompanying the latent heat.

The model we study is the \( q \)-state Potts model [28] with an additional variable representing the kinetic energy at each site, whose stochastic time evolution satisfies the detailed balance condition at equilibrium [29–31]. Thus, in equilibrium statistical mechanics, we can determine the phase diagram at equilibrium. The model exhibits the order-disorder transition as the temperature decreases from a sufficiently high level. Furthermore, owing to the presence of kinetic energy, the model can describe the conversion from potential energy to kinetic energy, which corresponds to the generation of latent heat. We study the model defined on a two-dimensional lattice, where each site is sparse-randomly connected in one direction and local in the other direction. From this setting, we can precisely identify the metastable phase in addition to the equilibrium properties. The proposal of
the model is one of the claims in this paper.

By numerically simulating the model, we measure the displacement of the interface \( R(t) \). Let \( T_s \) be the temperature of the heat bath in contact with the stable phase. We find the scaling relation \( R(t) = L_t R(Dt/L_s^2) \), where \( L_t \) is the system size between the two heat baths. The scaling function \( R(z) \) shows \( R(z) \approx z^{0.5} \) for \( z \ll z_c \) and \( R(z) \approx z^\alpha \) for \( z \gg z_c \), where the cross-over value \( z_c \) and the exponent \( \alpha \), \( 0.5 \leq \alpha \leq 1 \), depend on \( T_s \). The scaling relation in the late stage is not observed in the phase-field model. This result indicates that the stochastic phase growth involves a different universality class from that described by the phase-field model.

This paper is organized as follows. In Sec. II, we introduce the model. In Sec. III we analyze the model via equilibrium statistical mechanics. We identify the metastable phase in addition to equilibrium properties. In Sec. IV, we report the results of numerical simulations. In Sec. V we make some concluding remarks. The technical details of the theoretical calculation are summarized in Appendix A, and the value of \( \Delta \) is estimated in Appendix B. Throughout the paper, the Boltzmann constant is set to unity, and \( \beta \) is always connected to the temperature \( T \) via \( \beta = 1/T \).

II. MODEL

A. Hamiltonian

Let \( \Lambda = \{(i = (i_x, i_y)) | 1 \leq i_x \leq L_x, 1 \leq i_y \leq L_y, i_x, i_y \in \mathbb{Z}\} \) be a two-dimensional lattice. For any site \( i \in \Lambda \), a collection of sites connected to site \( i \) is denoted as \( B_i \). We assume that set \( B_i \) is decomposed as

\[
B_i = B_i^- \cup B_i^0 \cup B_i^+,
\]

where \( j_x = i_x - 1 \) for \( j \in B_i^- \), \( j_x = i_x + 1 \) for \( j \in B_i^+ \), and \( j_x = i_x \) for \( j \in B_i^0 \). Note that \( B_i^- = \emptyset \) and \( B_i^+ = \emptyset \). For example, \( B_i^- = \{(i_x - 1, i_y), (i_x, i_y)\} \), \( B_i^+ = \{(i_x + 1, i_y)\} \), and \( B_i^0 = \{(i_x, i_y) \} \) for the square lattice in Fig. 1a. In this paper, we use a sparse-randomly connected lattice defined by \( B_i^- = \{(i_x - 1, i_y), (i_x, i_y)\} \), \( B_i^+ = \{(i_x + 1, i_y)\} \), \( B_i^0 = \{(i_x, i_y) \} \) and \( B_i^0 = \emptyset \), where \( b^+(i_x, i_y) \) is a one-to-one random map from \( \{1, \cdots, L_y\} \) to \( \{1, \cdots, L_y\} \) that satisfies \( \sigma^\tau(i_x, i_y) \neq i_y \) and \( b^-(i_x, i_y) \) is defined by the inverse of the map. This is a special case of the random graphs introduced in [32]. See Fig. 1b for the illustration.

On each site \( i \in \Lambda \), the \( q \)-state variable \( \sigma_i \in \{1, 2, \cdots, q\} \) and the positive valued kinetic energy \( p_i \in \mathbb{R} \) are defined. The collections of variables, \( [\sigma_i]_{i \in \Lambda} \) and \( [p_i]_{i \in \Lambda} \), are simply denoted by \( \sigma \) and \( p \). The Hamiltonian we study is

\[
H(\sigma, p) = -\sum_{i \in \Lambda} \frac{2}{|B_i|} \sum_{j \in B_i} \delta(\sigma_i, \sigma_j) + \sum_{i \in \Lambda} p_i, \tag{3}
\]

where \( \delta(\cdot, \cdot) \) represents the Kronecker delta.

B. Model with energy conservation

Before presenting our model, we first describe a model for a thermally isolated system. We describe the stochastic time evolution in which the stationary distribution is given by the microcanonical ensemble associated with the Hamiltonian \( H(\sigma, p) \). The stochastic process satisfies the detailed balance condition with respect to the uniform distribution on the energy surface \( H(\sigma, p) = E \), where \( E \) is the total energy, which is invariant under time evolution.

We perform the following five procedures at one step:

1. A site \( i \in \Lambda \) is chosen randomly with equal probability.
2. The potential energy difference \( dE \) is calculated for a transition \( \sigma_i \rightarrow \sigma'_i \) chosen randomly with equal probability.
3. The transition \( \sigma_i \rightarrow \sigma'_i \) is accepted if \( p_i - dE \geq 0 \), and then the kinetic energy is changed to \( p'_i = p_i - dE \).
4. A site \( k \) and another site \( j \in B_k \) are chosen randomly.
5. The transition \( (p_k, p_j) \rightarrow (p_k - dp_p, p_j + dp_p) \) is accepted if \( p_k - dp \geq 0 \), where \( dp \) is a numerical parameter.

Let \( (p_i^0)_{i \in \Lambda} \) be a collection of the initial value of kinetic energy for each site. From the evolution rule, \( p_i \) is written as \( p_i = p_i^0 + n_i + m_i dp \), where \( n_i, m_i \in \mathbb{Z} \). The set of all possible values of \( p_i \) is denoted by \( P_i \). The phase space of the model \( \Sigma \) as a function of \( E \) is then expressed by the discrete set

\[
\Sigma(E) = \{(\sigma, p) | H(\sigma, p) = E, \sigma_i \in \{1, \cdots, q\}, p_i \in P_i, i \in \Lambda\}. \tag{4}
\]

The transition probability determined by the procedures is expressed as \( W(\sigma', p'|\sigma, p) \). Let \( t \in \mathbb{Z} \) be the discrete time and \( P_t(\sigma, p) \) be the probability of taking the state \( (\sigma, p) \) at \( t \)-step. Then, we have

\[
P_{t+1}(\sigma, p) = \sum_{\sigma', p'} W(\sigma, p|\sigma', p') P_t(\sigma', p'). \tag{5}
\]
Because \( W(\sigma', p'|\sigma, p) = W(\sigma, p|\sigma', p') \) holds, the stationary distribution is given by the microcanonical form

\[
P_{\text{mc}}(\sigma, p) = \frac{1}{|\Sigma(E)|} \delta(H(\sigma, p), E),
\]

where \(|\Sigma(E)|\) denotes the number of elements of set \(\Sigma(E)\).

### C. Model with two heat baths

In the model, we attach two heat baths, one on the left side \(i_x = 1\) and one on the right side \(i_x = L_x\) of the system introduced in the previous subsection. The temperatures of the left and right heat baths are denoted by \(T_L\) and \(T_R\), respectively. The stochastic time evolution of the model is given by imposing an additional rule at the boundaries. We perform the following procedures when a site \(i\) with \(i_x = 1\) or \(i_x = L_x\) is chosen in procedure 1 of the time evolution described in the previous subsection:

1. The potential energy difference \(dE\) is calculated for a transition \(\sigma_i \rightarrow \sigma_i'\) chosen randomly.
2. The transition \(\sigma_i \rightarrow \sigma_i'\) is accepted with the probability \(w(\sigma_i \rightarrow \sigma_i')\), where
   \[
   w(\sigma_i \rightarrow \sigma_i') = \frac{1}{2} \left( 1 - \tanh \left( \frac{dE}{2T} \right) \right) \tag{7}
   \]
   with \(T = T_L\) for \(i_x = 1\) and \(T = T_R\) for \(i_x = L_x\).
3. The value of \(p_i\) is replaced with a new one sampled from the distribution
   \[
P(p_i) = \frac{1}{T} e^{-\frac{p_i}{T}} \tag{8}
   \]
   with \(T = T_L\) for \(i_x = 1\) and \(T = T_R\) for \(i_x = L_x\).

Note that (7) satisfies the detailed balance condition

\[
\frac{w(\sigma \rightarrow \sigma')}{w(\sigma' \rightarrow \sigma)} = e^{-\frac{dE}{2T}}. \tag{9}
\]

The stochastic rule in the previous subsection is used except at the boundaries \(i_x = 1\) and \(i_x = L_x\). The transition probability determined by the procedures is expressed as \(\tilde{W}(\sigma', p'|\sigma, p)\). Let \(t \in \mathbb{Z}\) be the discrete time and \(\tilde{P}_t(\sigma, p)\) be the probability of taking state \((\sigma, p)\) at each \(t\)-step. Then, we have

\[
\tilde{P}_{t+1}(\sigma, p) = \sum_{\sigma', p'} \tilde{W}(\sigma, p|\sigma', p') \tilde{P}_t(\sigma', p'). \tag{10}
\]

When \(T_L = T_R = T\), the detailed balance condition

\[
\frac{\tilde{W}(\sigma \rightarrow \sigma')}{\tilde{W}(\sigma' \rightarrow \sigma)} = e^{-\frac{dE}{2T}} \tag{11}
\]

holds. Thus, the stationary distribution is given by the canonical form

\[
P_{\text{can}}(\sigma, p) = \frac{1}{Z_{\text{tot}}(\beta)} e^{-\beta H(\sigma, p)}, \tag{12}
\]

where \(Z_{\text{tot}}(\beta) = \sum_{\sigma, p} e^{-\beta H(\sigma, p)}\).

### III. EQUILIBRIUM STATISTICAL MECHANICS

In this section, for reference in studying dynamical processes, we confirm some results of equilibrium statistical mechanics using the canonical ensemble

\[
P_{\text{can}}(\sigma) = \frac{1}{Z(\beta)} e^{-\frac{\beta}{2} \sum_{i \in A} \sum_{j \in B_i} \delta(\sigma_i, \sigma_j)} \tag{13}
\]

for the configuration space of \(\sigma\), where \(P_{\text{can}}(\sigma) = \sum_{p} P_{\text{can}}(\sigma, p)\). We study

\[
m = \lim_{|A| \to \infty} \sum_{\sigma} P_{\text{can}}(\sigma) \frac{\sum_{j \in B_i} \delta(\sigma_i, 1)}{|A|} \tag{14}
\]

with an infinitely small external potential \(-\sum_i h_{\text{ext}}(\sigma_i, 1)\) in the Hamiltonian, and the free energy density defined as

\[
f = -\lim_{|A| \to \infty} \frac{T}{|A|} \log Z. \tag{15}
\]

It should be noted that the free energy density \(f\) and the partition function \(Z\) are defined in the configuration space of \(\sigma\).

In the calculation of \(m\) and \(f\), we conjecture that the contribution from loops in the lattice can be ignored in the large-size limit \([32, 33]\). On the basis of this conjecture, the thermodynamic phase can be determined using the model on a Cayley tree with three branches corresponding to coordination number 4. Concretely, it has been known that \(m\) and \(f\) are calculated from the probability of the state \(\sigma \in \{1, \cdots, q\}\) at a site connected with a cavity site, which is denoted as \(u(\sigma)\). As shown in Sec. \(A\), we first have the self-consistent equation for \(u(\sigma)\),

\[
u(\sigma) = \frac{[\gamma u(\sigma) + 1]^3}{\sum_{\sigma} [\gamma u(\sigma) + 1]^3}, \tag{16}
\]

with \(\gamma = e^\beta - 1\). Using the solutions of (16), we express the order parameter \(m(\beta)\) and the free energy density \(f(\beta)\) as

\[
m(\beta) = \frac{[\gamma u(1) + 1]^4}{\sum_{\sigma} [\gamma u(\sigma) + 1]^4}, \tag{17}
\]

\[
f(\beta) = -\beta^{-1} \log \frac{\sum_{\sigma} [\gamma u(\sigma) + 1]^4}{[\gamma u(\sigma) + 1]^4}. \tag{18}
\]

Here we notice that (16) has the trivial solution \(u_0(\sigma) \equiv 1/q\) for any \(\beta\). There exists a temperature \(\beta_{sp}\) beyond which (16) has another solution denoted as \(u_*(\sigma)\), where

\[
u_*(\sigma) = \begin{cases} \frac{c_*}{1 - c_*(\beta)} & (\sigma = 1) \\ \frac{c_*}{1 - c_*(\beta)} & (2 \leq \sigma \leq q) \end{cases}
\]

with \(c_* \neq 1/q\). The temperature \(\beta_{sp}\) is called the spinodal point. Using these two solutions \(u_0\) and \(u_*\),
we have \((m_0(\beta), m_*(\beta))\) and \((f_0(\beta), f_*(\beta))\). We display \((m_0(\beta), m_*(\beta))\) in Fig. 2a and \((f_0(\beta), f_*(\beta))\) in Fig. 2b. The equilibrium transition temperature \(\beta_c\) is identified as \(f_0(\beta_c) = f_*(\beta_c)\).

To study the free energy landscape, we define
\[
\tilde{f}(\beta, c) = -\beta^{-1} \log \frac{\sum \sigma [\gamma \tilde{u}(\sigma) + 1]^4}{[\gamma \sum \tilde{u}^2(\sigma) + 1]^2},
\]
where
\[
\tilde{u}(\sigma) = \begin{cases} 
c & (\sigma = 1) 
\frac{1-c}{q-1} & (2 \leq \sigma \leq q).
\end{cases}
\]
Setting \(c_0 = 1/q\), we have \(f_0(\beta) = \tilde{f}(\beta, c_0)\) and \(f_*(\beta) = \tilde{f}(\beta, c_*)\). Furthermore, by straightforward calculation, we confirm
\[
\frac{\partial}{\partial c} \left( e^{-\beta \tilde{f}(\beta, c)} \right) = 0
\]
at \(c = c_0\) and \(c = c_*\). See Appendix A.4 for the derivation. Therefore, displaying \(f(\beta, c)\) in Fig. 3 we see that the solution \(u_*\) appears at the spinodal point \(\beta = \beta_{sp}\), and the equilibrium transition occurs at \(\beta = \beta_c\). We also find that \(\partial^2 \tilde{f}/\partial c^2 = 0\) at \(c = c_0\) and \(\beta = \beta_{un}\), which means that the trivial solution \(u_0\) loses stability. Letting \(F(u)\) be the right-hand side of (16), the solutions of the self-consistent equation are given by the fixed points \(u_{n+1} = F(u_n)\). The temperature \(\beta_{un}\) is also characterized by the onset of the instability of the trivial solution for this iteration equation. For the case \(q = 10\), we find that \(\beta_{sp} \simeq 1.34\), \(\beta_c \simeq 1.39\), and \(\beta_{un} \simeq 1.78\).
Then, we prepare a metastable ordered phase of the temperature $T_L$ in $1 \leq i_x \leq L_x / 2$ and a stable disordered phase of the temperature $T_R$ in $L_x / 2 + 1 \leq i_x \leq L_x$. Here, we fix $T_L = 1.01 T_c$ so that the ordered phase is metastable, and $T_R$ is assumed to be greater than $T_c$ so that the disordered phase is stable. We prepare configurations at $t = 0$ via the following steps. First, we remove the coupling between $i_x = L_x / 2$ and $i_x = L_x / 2 + 1$ by setting $B_i = 0$ for $i_x = L_x / 2$. Second, we prepare a configuration where $\sigma_i = 1$ and $p_i = T_L$ for any $i$ satisfying $i_x \leq L_x / 2$, while each $\sigma_i$ is randomly chosen with equal probability and $p_i = T_R$ for any $i$ satisfying $i_x > L_x / 2 + 1$. Third, we make the system evolve up to 50 MCS. The reached configuration is set to be an initial state at $t = 0$ in the following analysis. Note that s MCS corresponds to the $t = s L_x L_y$ step in the procedure described in Sec. III.

The stable disordered phase grows from the initial configuration. To describe the phase growth, we observe an order parameter profile defined as

$$\phi(x, t) \equiv \frac{q m(x, t) - 1}{q - 1}$$

for $1 \leq x \leq L_x$. We define the value $\phi(x, t)$ for any real $x$ via linear interpolation. An example of $\phi(x, t)$ is displayed in Fig. 5a. We identify the interface position $R(t)$ from $\phi(x, t)$ such that $\phi(R(t), t) = 0.5$. $R(t)$ is uniquely determined because $\phi$ crosses 0.5 only once as far as we observe. In Fig. 5b, we show an example of configuration $\sigma$ with the interface position.

In Fig. 6, we show the statistical average of $R(t)$ for the case $T_R = 3 T_c > T_{sp}$. The five graphs correspond to results for different systems sizes $(L_x, L_y) = (100, 100), (200, 100), (300, 100), (100, 200), \text{and} (200, 200)$. Because the interface eventually reaches the left boundary and does not move anymore, all the curves become flat in the long-time limit. The guideline represents $R(t) = At^{0.5}$ with $A \approx 0.18$. We also confirmed that the $L_y$ dependence is hardly visible for this system size. From these results, we reasonably conjecture that the behavior in the large-$L_x$ limit is described by the phase-field model because the extent of the metastability $\Delta$ is evaluated as 0.05, as shown in Appendix B.

FIG. 4. Circle (green) symbols represent the simulation result for the model given in Sec. II with $T_L = T_R = \beta^{-1}$. The solid (purple) curve is the theoretical result obtained in Sec. III.

FIG. 5. (a) $\phi(x, t)$ as a function of $x$ at $t = 5 \times 10^6$ MCS for the system with $L_x = 200, L_y = 100$. (b) Configuration of $\sigma$ at $t = 5 \times 10^6$ MSC for the system with $L_x = 200, L_y = 100$. The dots represent sites where $\sigma_i = 1$. The solid line represents the interface position.

FIG. 6. $R(t)$ versus $t$ for $T_R = 3 T_c$. The average of 100 samples was taken for the system with different system sizes. The solid line represents $R(t) = At^{0.5}$ with $A \approx 0.18$. 
An unexpected behavior of $R(t)$ is observed for the case $T_R = 1.2T_c > T_{sp}$. As shown in Fig. 7a, four graphs for different system sizes, $(L_x, L_y) = (100,100), (200,100), (300,100)$, and $(400,100)$ do not overlap, while the $L_y$ dependence is hardly visible. We then consider a finite size scaling to plot $R/L_x$ as a function of $t/L_x^2$. The result is displayed in Fig. 7b. We find the scaling relation

$$\frac{R(t)}{L_x} = \mathcal{R}\left(\frac{Dt}{L_x^2}\right) \quad (25)$$

works well, where $\mathcal{R}$ is a scaling function. The scaling function indicates the cross-over from $\mathcal{R}(z) \simeq z^{0.5}$ to $\mathcal{R}(z) \simeq z^\alpha$ with $0.5 \leq \alpha \leq 1$. It should be noted that the scaling relation in the late stage is not observed in the phase-field model.

![Fig. 7](image)

**FIG. 7.** (a) $R(t)$ versus $t$ for $T_R = 1.2T_c$. The average of 100 samples was taken for the system with different system sizes. (b) $R(t)/L_x$ as a function of $z = t/L_x^2$. The two guidelines are $A z^{0.5}$ and $A' z^{0.75}$, where $A \simeq 0.04$ and $A' \simeq 0.03$, respectively.

The exponent $\alpha$ does not seem universal but depends on the value of $T_R$. Decreasing $T_R$ further, we find that the late stage becomes dominant and $\alpha$ increases. For the case $T_R = T_L = 1.01T_c < T_{sp}$, $\alpha$ turns out to be close to unity, as shown in Fig. 8.

![Fig. 8](image)

**FIG. 8.** $R/L_x$ as a function of $t/L_x^2$ for the case $T_R = 1.01T_c$. The guideline represents $A z^{0.95}$ with $A \simeq 8.2 \times 10^{-4}$.

**V. CONCLUDING REMARKS**

We have proposed a stochastic model that describes phase growth. We propose a $q$-state Potts model with an additional variable representing the kinetic energy at each site and define a stochastic process satisfying the detailed balance condition. By designing a sparse-randomly connected lattice in one direction, we have explicitly calculated the thermodynamic properties via equilibrium statistical mechanics. Using this model, we have numerically observed that the interface between the stable and metastable phases moves following the scaling relation \[ (25) \] with the scaling function $\mathcal{R}(z)$. The scaling function shows a cross-over from $\mathcal{R}(z) \simeq z^{0.5}$ to $\mathcal{R}(z) \simeq z^\alpha$, where the cross-over value of $z$ and the exponent $\alpha$ depends on the temperature of the heat bath attached to the stable phase.

These results suggest that the interface motion qualitatively changes when stochastic processes are considered. When the driving force originating from the difference in free energy densities is the main factor in the interface motion, the result is consistent with the solution of the phase-field model. Contrary to this case, when the driving force is weak, the behavior cannot be described by the phase-field model, presumably because of strong fluctuation effects. We do not yet have a theoretical understanding of the observed behavior. Nevertheless, it is reasonable to interpret the observation as renormalization of the interface motion even apart from the critical point. To our knowledge, this aspect has never been studied. Therefore, it is a challenging problem to theoretically derive our observation.

It should be noted that the results are not specific to the model on the tailored lattice. For example, in Figs. 9a, 9b, and 9c, we show the results for the $q$-state Potts model on the square lattice whose parameter is the...
same as that of the model on the sparsely random lattice. The observed behavior is qualitatively similar to that reported in the previous section. Therefore, we expect that our discovery will be observed in experiments. It is interesting to observe such new fluctuation effects in macroscopic dynamics.

ACKNOWLEDGMENTS

This work was supported by KAKENHI (Grant Nos. 17H01148, 19H05795, and 20K20425).

Appendix A: Derivation of the formulas in Sec. III

In this section, we derive formulas (16), (17), (18), and (22) in Sec. III.

1. Derivation of (16)

We study a Cayley tree with a root site connected with four sites in the first generation. Each site in the n-th generation (n ≥ 1) is connected with three sites in the n + 1-th generation. See Fig. 10 for the illustration of the Cayley tree.

Let $Z$ be the partition function of the Potts model on the lattice. We consider the partition function of a system in which a root site is replaced by the cavity and the state of a site in the first generation is fixed as $\sigma' \in \{1, \cdots, q\}$, which is denoted by $\tilde{Z}_1(\sigma')$. $Z$ is then the partition function of the model expressed as

$$Z = \sum_{\sigma} \left( \sum_{\sigma'} e^{\beta \delta(\sigma, \sigma')} \tilde{Z}_1(\sigma') \right)^4. \quad (A1)$$

A graphical representation is displayed in Fig. 11.

By setting

$$\gamma \equiv e^{\beta} - 1, \quad (A2)$$

$$G_1 = \sum_{\sigma} \tilde{Z}_1(\sigma), \quad (A3)$$

FIG. 10. Illustration of the Cayley tree.

FIG. 11. Graphical representation of (A1).
we rewrite (A1) as

\[ Z = \sum_{\sigma} \left( \gamma \tilde{Z}_1(\sigma) + G_1 \right)^4. \] (A4)

Defining \( \tilde{Z}_n(\sigma) \) and \( G_n \) similarly, we have the iterative equation

\[ \tilde{Z}_n(\sigma) = \left( \gamma \tilde{Z}_{n+1}(\sigma) + G_n \right)^3, \] (A5)

whose graphical representation is shown in Fig. 12.

![Fig. 12. Graphical representation of (A5).](image)

We define \( u_n(\sigma) \) as

\[ u_n(\sigma) \equiv \frac{\tilde{Z}_n(\sigma)}{G_n}, \] (A6)

which corresponds to the probability of the state \( \sigma \) of the cavity-connected site in the \( n \)-th generation. By substituting (A6) into (A5), we obtain

\[ G_n u_n(\sigma) = G_{n+1}^3 \left[ \gamma u_{n+1}(\sigma) + 1 \right]^3. \] (A7)

We also have

\[ G_n = G_{n+1}^3 \sum_{\sigma} \left[ \gamma u_{n+1}(\sigma) + 1 \right]^3 \] (A8)

using \( \sum_{\sigma} u_n(\sigma) = 1 \). From (A7) and (A8), we derive the iterative equation for \( u_n(\sigma) \),

\[ u_n(\sigma) = \frac{\left[ \gamma u_{n+1}(\sigma) + 1 \right]^3}{\sum_{\sigma} \left[ \gamma u_{n+1}(\sigma) + 1 \right]^3}. \] (A9)

Assuming homogeneity in the equilibrium state, \( u_n(\sigma) \) is independent of \( n \) in the large-size limit. This provides (16).

2. Derivation of (17)

The order parameter \( m \) for the model is calculated by the expectation value of \( \delta(\sigma, 1) \) at the root site. That is,

\[ m = \frac{1}{Z} \sum_{\sigma} \delta(\sigma, 1) \left[ \gamma \tilde{Z}_1(\sigma) + G_1 \right]^4. \] (A10)

Using the expression given in (A4), we have

\[ m = \frac{\left[ \gamma u_1(1) + 1 \right]^4}{\sum_{\sigma} \left[ \gamma u_1(\sigma) + 1 \right]^4}. \] (A11)

By replacing \( u_1 \) with the solution of the self-consistent equation (16), we obtain (17).

3. Derivation of (18)

To derive the free energy density, we use a tactical method manipulating graphs. We first remove one edge connected to the root site. The partition function of this system with \( \sigma \) at the root site and \( \sigma' \) at the other site connected by the removed edge is \( \tilde{Z}_0(\sigma)\tilde{Z}_1(\sigma') \). See Fig. 13. We thus express the partition function \( Z \) as

\[ Z = \sum_{\sigma, \sigma'} e^{\beta \delta(\sigma, \sigma')} \tilde{Z}_0(\sigma)\tilde{Z}_1(\sigma') \] (A12)

\[ = G_0 G_1 \left[ \gamma \sum_{\sigma} u_0(\sigma) u_1(\sigma) + 1 \right], \] (A13)

where \( G_0 \equiv \sum_{\sigma} \tilde{Z}_0(\sigma) \) and \( u_0(\sigma) \equiv \tilde{Z}_0(\sigma)/G_0 \). Note that \( u_0(\sigma) \) also satisfies (A9).

![Fig. 13. By removing one edge, we get two rooted graphs. \( \times \) represents the cavity.](image)

Next, we prepare four independent systems. The partition function of the total system is \( Z^4 \). We remove one edge connected to the root site for each graph. Then, we combine four graphs with the root site by adding one site. See Fig. 14. The partition function of this system, \( \tilde{Z}_0 \), is expressed as

\[ \tilde{Z}_0 = G_0^4 \sum_{\sigma} \left[ \gamma u_0(\sigma) + 1 \right]^4. \] (A14)
This is rewritten as
\[ \text{difference in free energy is equal to } 2f \]
That is,
\[ \text{new system is the original system with two sites added.} \]
The same free energy density in the large-size limit and the free energy density, because the two systems have the same free energy density in the large-size limit and the new system is the original system with two sites added. That is,
\[ -T \log \tilde{Z}_0 \tilde{Z}_1 + T \log Z^4 = 2f. \]
This is rewritten as
\[ e^{-\beta f} = \left( \frac{\tilde{Z}_0 \tilde{Z}_1}{Z^4} \right)^{1/2} \]
\[ = \left( \frac{\sum_{\sigma'} [\gamma u_0(\sigma') + 1]^4 \sum_{\sigma''}[\gamma u_1(\sigma'') + 1]^4}{[\gamma \sum_{\sigma} u_0(\sigma)u_1(\sigma) + 1]^4} \right)^{1/2}. \]
By replacing \( u_0 \) and \( u_1 \) with the solution of the self-consistent equation \( \text{[16]} \), we obtain \( \text{[18]} \).

4. Derivation of \( \text{[22]} \)

For later convenience, we set
\[ \tilde{c} \equiv \frac{1 - c}{q - 1}. \]
From the definition of \( \tilde{f} \) given in \( \text{[20]} \), the left-hand side of \( \text{[22]} \) is calculated as
\[ \frac{\partial}{\partial c} e^{-\beta \tilde{f}(\beta,c)} \]
\[ = 4\gamma \frac{(\gamma c + 1)^3}{(\gamma c^2 + \gamma (q - 1)c^2 + 1)^2} \]
\[ - 4\gamma (c - \tilde{c}) \frac{(\gamma c + 1)^4 + (q - 1)(\gamma \tilde{c} + 1)^4}{(\gamma c^2 + \gamma (q - 1)c^2 + 1)^3}. \]
The self-consistent equation \( \text{[16]} \) is expressed as
\[ (\gamma \tilde{c} + 1)^3 = \frac{\tilde{c}}{c} (\gamma c + 1)^3. \]
Thus, the right-hand side of \( \text{[22]} \) for the special values of \( c \) satisfying \( \text{[21]} \) is calculated as
\[ 4\gamma \frac{(\gamma c + 1)^3(c - \tilde{c})}{c(\gamma c^2 + \gamma (q - 1)c^2 + 1)^2} \]
\[ \times \left( 1 - \frac{\gamma^2 c^2 + \gamma(q - 1)c^2 + c + (q - 1)\tilde{c}}{\gamma c^2 + \gamma (q - 1)c^2 + 1} \right), \]
which turns out to be zero from \( \text{[19]} \).

Appendix B: Estimation of \( \Delta \)

In this section, we estimate the value of \( \Delta \) defined by \( \text{[1]} \) for the model we study.

We first calculate the energy density \( h \) defined as
\[ h = \lim_{|\mathcal{A}| \to \infty} \frac{1}{|\mathcal{A}|} \sum_{\sigma,p} P_{\text{can}}(\sigma,p)H(\sigma,p), \]
where \( P_{\text{can}}(\sigma,p) \) is given in \( \text{[12]} \). Using the free energy density \( f \) calculated in Sec. \( \text{[11]} \) we express the energy density \( h \) as
\[ h = T + g, \]
where \( g \) is the potential energy density given by
\[ g(\beta) \equiv \frac{\partial}{\partial \beta} (\beta f(\beta)). \]
As with the free energy density, \( g_0(\beta) \) and \( g_* \) denote the potential energy densities corresponding to the trivial solution \( u_0 \) and the nontrivial solution \( u_* \) of \( \text{[13]} \), respectively. In Fig. \( \text{[15a]} \), \( g_0(\beta) \) and \( g_*(\beta) \) are displayed. Then, the latent heat per unit volume \( T_c \delta s \) at the equilibrium transition temperature is determined by the entropy jump defined as
\[ \delta s \equiv \beta_*(g_0(\beta_c) - g_*(\beta_c)). \]
For the model with \( q = 10 \), we obtain \( T_c \delta s \approx 1.07 \). Next, we consider the heat capacity per unit volume \( C \) expressed as
\[ C(\beta) \equiv \frac{\partial h}{\partial T} = 1 + \frac{\partial g}{\partial T}. \]
Using similar notations, we obtain \( C_0(\beta) \) and \( C_*(\beta) \) from \( g_0(\beta) \) and \( g_*(\beta) \). These are shown at the bottom of Fig. \( \text{[15b]} \). For the cases \( q = 10 \) and \( T_L = 1.01T_c \), we obtain \( C_*(T_L) \approx 6.95 \). Therefore, for the model we numerically study, we have
\[ \Delta \approx 0.05, \]
which is less than unity. Note that in the stochastic model studied in this paper, \( C \) corresponds to \( c_p \) in the phase-field model.
FIG. 15. (a) Potential energy density $g$ as a function of $\beta$. The solid (purple) line represents $g = g_*$ in the range $\beta > \beta_{ap}$. The dashed (green) line represents $g = g_0$ in the range $\beta < \beta_{ap}$. The black line indicates $\beta = \beta_c$. (b) Heat capacity per unit volume $C$ as a function of $\beta$. The styles (colors) of lines correspond to the graphs in (a).

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