Long-range dispersion and spatial diffusion of fault waves in the Burridge-Knopoff earthquake model.

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The Burridge-Knopoff model of earthquakes has recently gained increased interest for the consistency of the predicted energy released by seismic faults, with the Gutenberg-Richter scaling law. The present work suggests an improvement of this model to account for long-range dispersions and large spatial diffusion of seismic faults. An enhancement of the threshold speed of shock waves driven by translated fault fronts is pointed out and shown to result from the interactions between components of the system situated far away from them and others. Due to the enhanced threshold speed, size of the seismic fault gets increased but a control effect can still be gained from tunable dispersion extent irrespective of the total length of the system. To the viewpoint of the Burridge-Knopoff block-lattice model, this last consideration introduces the possibility of sizable but finite interactions among infinitely aligned massive blocks. Implications on the fault wave propagation are examined by numerical simulations of the improved nonlinear partial differential equation.

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Understanding natural catastrophes is one of the greatest challenges faced by scientists from various fields. Recently, earthquake phenomena attracted a lot of attention to both viewpoints of the statistics of earthquake events and of the dynamics of seismic faults. The first, based on cellular automata models and involving fractal faults, assumes a system which is in perpetual crisis. The main feature of this approach is the prediction of large fluctuations in the avalanche sizes. The second approach follows a deterministic description assuming seismic events as time evolving processes. More explicitly, this last approach is based on spatio-temporal evolutions of the seismic fault considered within equations which in general are of the Klein-Gordon type. The best known of these equations is provided by the Burridge-Knopoff(BK) model intended for two parallel tectonic plates subjected to stick-slip frictions, which can slide one relative to the other under a velocity-weakening driving force. In theory, one considers the motion of massive underground blocks attached to a static surface and interacting them and others via harmonic springs of constant strength.

The dynamic friction force introduces nonlinearity in the fault dynamics so that the corresponding discrete nonlinear equation writes

\[ M\ddot{U}_n = K(U_{n+1} - 2U_n + U_{n-1}) - k_o U_n - \Phi(\dot{U}_n/V_o), \]

where \( U_n \) is the relative sliding amplitude of the \( n^{th} \) block of mass \( M \), \( K \) is the effective coupling strength felt by the block in the presence of two nearest-neighbour blocks, \( k_o \) is the uniform strength of block stick to the surface and dots refer to time derivatives. We assume two tectonic plates at equilibrium, in other words we neglect residual uniform sliding motions of blocks as they are pulled individually forward (or backward) relative to the upper plate. Concentrating ourselves on the sismicity of the model triggered by the dynamic friction force \( \Phi(\dot{U}_n/V_o) \), this force is usually written:

\[ \Phi(\dot{U}_n/V_o) = F_o/(1 + \dot{U}_n/V_o), \]

in which \( F_o \) and \( V_o \) are two constant parameters. Equation 1 exhibits rich physical properties one most relevant being the Gutenberg-Richter scaling law associate to the distribution of the energy released by earthquakes. Otherwise, previous numerical simulations established an agreement of the resulting velocity profiles and wave amplitudes with real earthquake processes and routes to chaos have very recently been investigated.

Discreteness is however another relevant aspect of this equation and constitutes the main subject of the present work. Discreteness connects to dispersion which is an intrinsic property of the system having direct incidence on its stability. Dispersion in the present system is as fundamental as it characterizes the ability of the interacting block lattice to respond to both small and large amplitude excitations propagating along the lattice, namely its fixes the threshold frequency and speed for stable excitations. This problem has been discussed in previous studies in terms of an amplification rate from which the group velocity was derived by linearizing this physical parameter about a characteristic wavector. As it stands, the BK model supposes each block can only see the two nearest-neighbour blocks which means short-range dispersion. Accordingly, the corresponding threshold speed will be appropriate only for short-range(SR) dispersive excitations. However, spring models are flexible in essence and SR descriptions often consist of approximations of the actual physics behind these models.

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born on or crossing an \( m^{th} \) block relatively far away (i.e. beyond nearest neighbours) from it. Numerous seismic events show clear evidences of an implication of such features and it is of crucial importance to take them into account. Moreover, recent advances on spring models make possible their theoretical descriptions in terms of long-range (LR) dispersions \cite{12} and the associate Klein-Gordon equation is qualitatively similar to the SR one. Nevertheless, the SR coupling strength \( K \) is substituted for a LR dispersion potential such that the system dynamics now obeys the following equation:

\[
M \ddot{U}_n = \sum_{m \neq n} K(m - n)(U_m - U_n) - k_o U_n - \Phi (\dot{U}_n/V_o),
\]

(3)

The LR dispersion force \( K(m - n) \) is thus an adjustable parameter and the equation \( \text{(3)} \) turns to a deformable BK model. One most interesting features of this deformability is the possibility to adjust the model to various situations according to the strength and nature of the interactions among components of the system. For instance, if we take account of the block stick to the upper plate and assume blocks to be relatively strongly attached to the surface, the LR dispersion function should quickly decrease at short distances and an exponentially decreasing function with increasing distance \( m - n \) can be an acceptable approximation. This last consideration is formulated assuming an expression of the form:

\[
K(\ell) = K \frac{1 - r}{r(1 - r^\ell)} |\ell|, \quad \ell = m - n.
\]

(4)

where \( L \) is the spatial extent of the interaction between blocks. By spatial extent we understand the number of neighbour blocks from a given block which can be different from the total number of blocks forming the whole lattice. This precision is of particular importance here since we want a model that allows a control of the spatial extent of the dispersion. In addition to \( L \), the quantity \( r \) is another control parameter but relates instead to the strengths of successive couplings. For an exponential fall-off we need \( r \approx e^{-\lambda} \) where \( \lambda \) is a positive constant. Or also, we can just confine \( r \) in the interval \( 0 \leq r < 1 \).

In this way, the LR dispersion function \( \text{(4)} \) looks quite like the so-called Kac-Baker potential \cite{12, 11, 10, 9}. Nevertheless, unlike the usual Kac-Baker potential the new version \( \text{(4)} \) has finite and constant magnitude for finite values of \( L \). Analytically, this is traduced by the constraint:

\[
\sum_{\ell=1}^{L} K(\ell) = K
\]

(5)

This constraint acquires particular interest in the present context since by fixing the total interaction between blocks to a finite magnitude, we avoid uncontrollable dispersions of the energy carried by the LR excitations which, otherwise, is nothing else but the total energy carried by the fault.

In what follows we examine dispersion properties of the improved model \( \text{(4)} \) and point out some consequences of the account of the LR dispersion on the fault dynamics. Following usual considerations \cite{9, 11}, we linearize this equation assuming small-amplitude motions of blocks about their equilibria. These equilibria correspond to the most stable positions in the lattice after uniformly pulling the blocks, resulting in a uniform equilibrium position \( U_o = F_o/k_o \). Setting \( U_o(t) = U_o + a(q) \exp (n a Q t + \Omega t) \) and keeping only linear terms in \( a \) and \( \dot{a} \), equation \( \text{(4)} \) leads to the following dispersion law:

\[
\Omega^2 (\tilde{Q}) = \Omega_o \left[ 1 \pm \sqrt{1 - \frac{M}{k_o} \left( 1 - \frac{K L(\tilde{Q})}{k_o} \right) \left( \frac{2V_o}{U_o} \right)^2} \right],
\]

(6)

\[
\Omega_o = \frac{F_o}{MV_o}, \quad \tilde{Q} = Qa.
\]

\( \Omega \) in this last relation is the amplification rate for small-amplitude excitations and \( Q \) harmonics associate to their spatial dispersions. The function \( K_L(\tilde{Q}) \) appearing in \( \text{(6)} \) is defined as:

\[
K_L(\tilde{Q}) = 2 \sum_{\ell=1}^{L} K(\ell) \left[ \cos(\tilde{Q} \ell) - 1 \right],
\]

(7)

where \( a \) is the equilibrium separation between two neighbouring blocks. Thus, \( a \) appears as a characteristic length scale of the model. As we will see, the function \( K_L(\tilde{Q}) \) governing spatial dispersion provides another relevant length scale. Instructively, this second length scale was previously considered and called stiffness length (e.g. denoted \( \xi \) in \cite{4}). Below we keep the same viewpoint but introduce more suggestive interpretation in terms of the size of fault wave as it moves in a Galilean frame i.e. with a translated wavefront. To start let us examine the dispersion law obtained in \( \text{(6)} \). The discrete sum in \( K(\tilde{Q}) \) can be calculated analytically using the identity:

\[
V_L(\tilde{Q}) = \sum_{\ell=1}^{L} r^{\ell} \cos(\tilde{Q} \ell) = \frac{r \cos(\tilde{Q}) - r^2 - r^{(L+1)} \left[ \cos((L + 1) \tilde{Q}) - r \cos(L \tilde{Q}) \right]}{1 - 2r \cos(\tilde{Q}) + r^2}.
\]

(8)

With help of the analytical expression of \( \Omega(\tilde{Q}) \) derived from this identity an explicit formulation of the dispersion property of the LR system becomes trivial. In this goal, note the presence of two signs \( \pm \) in \( \text{(6)} \) which relate to shock fronts moving (translating) respectively to the left (backward) and to the right (forward) in the block lattice. To clearly see these two distinct polarities in the dispersion properties of the fault wave, on figure \( \text{1} \) we plot the two amplification rates \( \Omega_{\pm} \) as function of the wavector \( Q/\pi \) for the infinite-extent dispersion and for some arbitrary values of the LR parameter \( r \). The zero dispersion mode \( Q = 0 \) appears as lying inside a gap of
finite width $\Delta \Omega = |\Omega_+ - \Omega_-|_{Q \to 0}$. The effect of this gap on the overall lattice dispersion is to lift the degeneracy of the dispersion spectrum giving rise to two separate sub-spectra associate to backward and forward dispersion modes. However, both sub-spectra possess common characteristic(sound) speed which is approximately the slope of the linear part of their dispersion curves and consequently can be estimated from:

$$C_L = (1/M) \partial_Q \Omega_{\pm}(Q)|_{Q \to 0}, \quad (9)$$

$C_L$ is plotted on fig. 2 as a function of $r$ assuming different values $L$. A striking feature in this last figure is an enhancement of the threshold speed as the number of interacting neighbours increases. The infinite increase of the characteristic speed become quite clear if we consider excitations with long wavelengths compared to the characteristic length scale $a$, then readily assumed as an ultra-violet cut-off. These are continuum-limit processes of the equation (9) and hence can be described by the continuum nonlinear Klein-Gordon equation:

$$\ddot{U}_n = C_L^2 U_{xx} - \omega_o^2 U - \frac{1}{M} \Phi(\dot{U}_n/V_o), \quad \omega_o^2 = \frac{k_o}{M}. \quad (10)$$

where $n \to x$. To see the meaning of $C_L$ to the system dynamics, we perform Galilean transformation $U(s, t) \to U(s \pm vt)$ setting: $z = \pm \frac{vt}{\xi_L}$, $\gamma^2 = 1 - v^2/C_L^2$ and find:

$$Y_{zz} + Y - 1/(1 + \alpha Y_z) = 0, \quad \xi_L^2 = \frac{C_L^2}{\omega_o^2}, \quad \alpha = \frac{vU_o}{\xi_L V_o}. \quad (11)$$

with $Y = U/U_o$. Eq. (11) reveals that $\xi_L$ is precisely the characteristic topological size of the seismic fault, hence a second characteristic length scale of the model. The linear dependence of this quantity on $C_L$ is suggestive enough as for the effect of the LR dispersion on this second length scale.

To end, let us look at the consequence of an account of the LR dispersion on the spatial diffusion of large-amplitude(soliton-like) fault waves. For this purpose, the continuum nonlinear equation (11) was solved(in $U$) approximating the diffusion operator(second-order spatial derivatives) within a central-difference scheme and applying combined fourth-order Runge-Kutta and fifth-order Runge-Kutta rules(so-called Runge-Kutta-Fehlberg algorithm) with error controls on time-dependent variables. Arbitrary values of the model parameters were chosen except $\xi_L$ whose dependence on the two control parameters $r$ and $L$ were carefully treated. To this last point, according to fig. 2 the effect of increasing the dispersion extent $L$ keeping $r$ fixed is qualitatively similar to the effect of an increase of $r$ for a fixed value of $L$. Therefore results displayed below assume two different but equivalent qualitative interpretations. On figs. 3, 4, 5 and 6 we present some velocity profiles as the topological size $\xi_L$ is increased by tuning either $r$ fixing $L$, or $L$ fixing $r$. Initial conditions are $(V, \dot{V}) = (-1.0, 0)$. The velocity profile in fig. 4 can serve as a reference since it corresponds to the SR solution($r = 0$, $L \to \infty$). The three next figures are spatio-temporal shapes of the velocity profile for $r = 0.2$ 0.4 and 0.6 respectively(with $L = 40$ nearest neighbours). As one notes, the wave carrying the seismic fault gradually spreads all over the space as $r$ increases. This suggests an increasingly large diffusion of the fault due to a growth of the topological size of this wave as $r$ increases. Therefore, numerical simulations are in excellent agreement with our analytical predictions.

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FIG. 3: Velocity profile for $r = 0$.

FIG. 4: Velocity profile for $r = 0.2$.

FIG. 5: Velocity profile for $r = 0.4$.

FIG. 6: Velocity profile for $r = 0.6$.

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