Log-Concave Polynomials IV: Approximate Exchange, Tight Mixing Times, and Faster Sampling of Spanning Trees

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Abstract

We prove tight mixing time bounds for natural random walks on bases of matroids, determinantal distributions, and more generally distributions associated with log-concave polynomials. For a matroid of rank \(k\) on a ground set of \(n\) elements, or more generally distributions associated with log-concave polynomials of homogeneous degree \(k\) on \(n\) variables, we show that the down-up random walk, started from an arbitrary point in the support, mixes in time \(O(k \log k)\). Our bound has no dependence on \(n\) or the starting point, unlike the previous analyses \([\text{Ana}+19; \text{CGM}19]\), and is tight up to constant factors. The main new ingredient is a property we call approximate exchange, a generalization of well-studied exchange properties for matroids and valuated matroids, which may be of independent interest.

Additionally, we show how to leverage down-up random walks to approximately sample spanning trees in a graph with \(n\) edges in time \(O(n \log^2 n)\), improving on the almost-linear time algorithm by Schild \([\text{Sch}18]\). Our analysis works on weighted graphs too, and is the first to achieve nearly-linear running time.

1 Introduction

Let \(\mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0}\) be a function on size \(k\) subsets of \([n] = \{1, \ldots, n\}\), defining a distribution \(\mathbb{P}[S] \propto \mu(S)\). The generating polynomial of \(\mu\) is the multivariate \(k\)-homogeneous polynomial defined as follows:

\[
g_\mu(z_1, \ldots, z_n) = \sum_{S \in \binom{[n]}{k}} \mu(S) \prod_{i \in S} z_i.
\]

We say that \(g_\mu\) is log-concave if \(\log(g_\mu)\) is a concave function over \(\mathbb{R}^n_{\geq 0}\). The study of log-concave polynomials has recently enabled breakthroughs on old conjectures about matroids, including the resolution of a conjecture of Mihail and Vazirani \([\text{MV}89]\) on the expansion of the bases-exchange graphs \([\text{Ana}+19]\), and Mason’s ultra-log-concavity conjecture \([\text{Ana}+18; \text{BH}18]\). These results rely on the log-concavity of the generating polynomial for various distributions associated with matroids, most importantly the uniform distribution on the set of bases \([\text{AOV}18]\).
Besides distributions associated with matroids, several other classes of distributions possess a log-concave generating polynomial. An important subclass consists of strongly Rayleigh distributions [BBL09], and the more special subclass of determinantal point processes that have found numerous applications in machine learning [see KT12, for a survey]. A well-studied example belonging to all classes mentioned so far consists of the uniform distribution over spanning trees of a graph $G = (V, E)$. Here $n$ is the number of edges $|E|$ in the graph and $k$ is the number of edges in a spanning tree, i.e., $|V| - 1$. Spanning trees of a graph form bases of a matroid called the graphic matroid [see, e.g., Oxl06] and they can also be viewed as determinantal point processes because of the matrix-tree theorem [BBL09, see, e.g.,] and are consequently strongly Rayleigh.

The motivation behind the conjecture of Mihail and Vazirani [MV89] was to solve the problem of approximately sampling from bases of a matroid. After this conjecture was made, efficient sampling algorithms were developed for various classes of matroids [FM92; Gam99; JS02; Jer+04; Jer06; Clo10; CTY15; GJ18] until Anari, Liu, Oveis Gharan, and Vinzant [Ana+19] showed an efficient approximate sampling algorithm for all matroids. This algorithm used a variant of random walks on the so-called “bases-exchange” graphs of matroids, that coincides with the “down-up” random walks studied in the context of high-dimensional expanders [KM16; DK17; KO20]. For a distribution defined by $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$, the down-up random walk $P$ transitions from a set $S \in \binom{[n]}{k}$ to another set $S' \in \binom{[n]}{k}$ as follows:

- From $S$ choose a subset $T \subseteq S$ of size $k - 1$ uniformly at random.
- From all supersets $S' \supseteq T$, choose one with probability $\alpha(S')$.

$P$ has $\mu$ as its stationary distribution and can be efficiently implemented by probing $\mu$ on at most $n$ different sets each time. Thus assuming oracle access to $\mu$, or in the case of matroids, and independence oracle for the matroid, each step of $P$ takes $O(n)$ time. The challenging part has been establishing the mixing time of $P$, i.e., bounds on the time $t$ such that the distribution after $t$ steps of $P$ applied to the starting set $S_0$ is $\epsilon$-close in total variation distance to the one defined by $\mu$:

$$t_{\text{mix}}(P, S_0, \epsilon) := \min \left\{ t \mid \|P^t(S_0, \cdot) - \mu(\cdot)\|_{\text{TV}} \leq \epsilon \right\}.$$  

Anari, Liu, Oveis Gharan, and Vinzant [Ana+19] proved that when $\mu$ has a log-concave generating polynomial, the spectral gap of the random walk $P$ is at least $1/k$. This immediately implied that

$$t_{\text{mix}}(P, S_0, \epsilon) \leq O\left(k \cdot \left( \log \frac{1}{\mathbb{P}_\mu[S_0]} + \log \frac{1}{\epsilon} \right) \right).$$

Later, Cryan, Guo, and Mousa [CGM19] proved a Modified Log-Sobolev Inequality (MLSI) for the same random walk which resulted in a tighter mixing time:

$$t_{\text{mix}}(P, S_0, \epsilon) \leq O\left(k \cdot \left( \log \log \frac{1}{\mathbb{P}_\mu[S_0]} + \log \frac{1}{\epsilon} \right) \right).$$

These results lead to efficient algorithms assuming that the mass of the starting set, $\mathbb{P}_\mu[S_0]$, is not terribly small; this can often be achieved in practice. For example, for matroids any starting basis $S_0$ will satisfy $\mathbb{P}_\mu[S_0] \geq 1/(\binom{n}{\ell}) \geq n^{-\ell}$, because the number of bases is at most $\binom{n}{\ell}$. Consequently the above bounds turn into $t_{\text{mix}}(P, S_0, \epsilon) \leq O(k(k \log(n) + \log(1/\epsilon)))$ and
For any distribution defined by \( \mu \) with a log-concave generating polynomial, even the special case of determinantal point processes, there is no control on \( \min\{P_\mu[S_0] \mid S_0 \in \text{supp}(\mu)\} \), so one has to rely on clever tricks to find a good starting set \( S_0 \); even then, the best hope is to find a set \( S_0 \) with \( P_\mu[S_0] \gtrsim 1/(\binom{n}{k}) \), which results in a mixing time mildly depending on \( n \).

Historically, earlier works on a subclass of matroids, called balanced matroids, followed a similar development, where initially a spectral gap result was proved, resulting in a running time\(^1\) of \( O(nk(k \log n + \log(1/\epsilon))) \) followed by MLSI which resulted in a mixing time of \( O(k(\log k + \log \log n + \log(1/\epsilon))) \) [see MT06, for a survey]. Noting that the term \( \log \log n \) seems unnecessary, Montenegro and Tetali [MT06] raised the question of proving a better inequality that would result in a running time of \( O(nk \log(k/\epsilon)) \). They specifically hoped for the possibility of proving a Nash inequality, an advanced type of functional inequality used to derive very tight mixing times for some Markov chains [MT06]. We believe there are barriers to using functional inequalities in general to prove \( O(k \log(k/\epsilon)) \) mixing time for the down-up random walk; we defer an explanation of this to a future version of this paper. However, without proving new functional inequalities, we manage to sidestep this barrier and improve the running time to the conjectured \( O(nk \log(k/\epsilon)) \) for not just balanced matroids, but the class of all matroids.

Our main result is a tight analysis of the mixing time, entirely removing the dependence on \( P_\mu[S_0] \) and \( n \).

**Theorem 1.** For any distribution defined by \( \mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \) with a log-concave generating polynomial \( g_\mu \), the mixing time of the down-up random walk \( P \), starting from any \( S_0 \) in the support of \( \mu \) is

\[
t_{\text{mix}}(P, S_0, \epsilon) \leq O(k \log(k/\epsilon)).
\]

Note that generally we cannot hope for a better mixing time than \( k \log k \); each step of the random walk \( P \) replaces one element of the current set, and by a coupon collector argument, at least \( \approx k \log k \) steps are needed to replace every element of the starting set \( S_0 \). As long as \( k \) is not too close to \( n \), say \( k < 0.99n \), replacing every starting element is needed for sufficient mixing, even for the simple distribution \( \mu \) which is uniform over \( \binom{[n]}{k} \).

Our mixing time bound is an asymptotic improvement over prior work for \( k = O(1) \), or more generally when \( k \) is smaller than \( \log(n)^{\epsilon} \) for all \( \epsilon > 0 \). Another consequence of the new mixing time bound is that it enables the analysis of the down-up random walk when \( n \) is infinitely large; for example, this is the case for continuous determinantal point processes [see, e.g., OR18].\(^2\) To avoid complicating the notation, we do not consider infinitely large ground sets in this paper, but note that the results do generalize to such cases.

Next, we focus on the special case where \( \mu \) is the uniform distribution over spanning trees of a graph \( G \). Much attention has been paid to this special case over the years, starting from the seminal works of Aldous [Ald90] and Broder [Bro89] who proposed a simple routine to extract a random spanning tree from the trace of a random walk on \( G \) itself. Subsequent works introduced improved algorithms [Wil96; CMN96; KM09; MST14; Dur+17a; Dur+17b] until finally Schild

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\(^1\)Note that the running time is \( n \) times the mixing time for the down-up walk.

\(^2\)We note however that one still needs to be able to implement each step of the random walk efficiently when \( n \) is infinitely large. For examples where this is possible see [OR18].
[Sch18] managed to obtain an almost-linear time algorithm running in time $n^{1+o(1)}$ on graphs with $n$ edges. This algorithm and that of several prior works were all based on the original work of Aldous [Ald90] and Broder [Bro89]; they achieved an improved running time by employing several clever, but complicated, tricks to shortcut the trace of a random walk over $G$. Our next result is a wholly different algorithm, based on the down-up random walk, that achieves a nearly-linear running time of $n \log^2(n)$, while being arguably much simpler to describe and implement.

**Theorem 2.** There is an algorithm that takes a (weighted) graph $G$ with $n$ edges and $\epsilon > 0$ as input and outputs a spanning tree $T$ in time $O(n \log(n) \log(n/\epsilon))$; the distribution of $T$ is guaranteed to be $\epsilon$-close in total variation distance to the uniform (weighted) distribution on spanning trees of $G$.

We remark that our algorithm can only approximately sample from the spanning tree distribution. In contrast, some of the prior works, including [Sch18], can sample exactly from this distribution. This is mostly an inconsequential difference in practice, as no polynomial-time user of the algorithm can sense a difference between exact sampling and approximate sampling; one simply needs to set $\epsilon$ to be inverse-polynomially small.

### 1.1 Techniques

In order to prove Theorem 1, our strategy is to combine a new analysis of the initial steps of the down-up random walk with the previously known Modified Log-Sobolev Inequality [CGM19]. Specifically we show that conditioned on having replaced every element of the starting set $S_0$ at least once by time $t$, the set at time $t$ can be used as a warm start for the rest of the steps. Specifically, we show that the density of the set at time $t$ w.r.t. $\mu$, conditioned on this event, is upper-bounded by only a function of $k$.

In order to prove this, we introduce a new property of functions $\mu : [n] \to \mathbb{R}_{\geq 0}$ that we call $\alpha$-approximate exchange. This property says that for every $S, T \in [n]$, and $i \in S$, there exists $j \in T$ such that

$$\mu(S) \mu(T) \leq \alpha \cdot \mu(S - i + j) \mu(T + i - j).$$

Note that when $\mu$ takes values in $\{0,1\}$ and $\alpha \geq 1$, this property becomes equivalent to the famous strong basis exchange axiom of matroids [Oxl06]; if $B = \mu^{-1}(1)$ is the family of sets indicated by $\mu$, this property says that for every $S, T \in B$ and $i \in S$, there exists $j \in T$ such that $S - i + j \in B$ and $T + i - j \in B$. This property can be seen as a quantitative variant of strong basis exchange. Alternatively, it can be viewed as an approximate and multiplicative form of $M^2$-concavity, a cornerstone of discrete convex analysis [MS99]. We prove that every $\mu$ with a log-concave generating polynomial satisfies $2^{O(k)}$-approximate exchange. Crucially, our $\alpha$ does not depend on $n$. We remark that Brändén and Huh [BH18] showed a result that can be thought of as a converse to this. The proved that $M^2$-concavity of $\log \mu$, equivalent to 1-approximate exchange property, implies that the generating polynomial of $\mu$ is log-concave.

To prove Theorem 2, we use mixing time bounds for the down-up walk. While the down-up walk mixes in nearly-linear time, we do not see a way to implement each step of it in polylogarithmic time. Instead, we apply the down-up walk on an equivalent family of sets; the dual of the graphic matroid, known by the names of co-graphic and/or bond matroids, consists of the complements
of spanning trees. We apply the down-up walk to this dual matroid. Each step of the down-up walk on the dual matroid can be viewed as follows:

- Add an edge uniformly at random from those not currently in the spanning tree.
- Remove an edge uniformly at random from the cycle formed by the previous addition.

Classic dynamic tree data structures such as link-cut trees \([ST83]\) can be used to perform every step in amortized \(O(\log n)\) time. After our findings, we became aware of prior work by Russo, Teixeira, and Francisco \([RTF18]\) who proposed the same algorithm. Despite not having the tight mixing time analysis, they empirically observed fast mixing times for the proposed algorithm, and additionally showed how link-cut trees can be used to implement each step.

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2 Preliminaries

We use \([n]\) to denote the set \(\{1, \ldots, n\}\) and \(\binom{[n]}{k}\) to denote the family of size \(k\) subsets of \([n]\). When \(n\) is clear from context, we use \(1_S \in \mathbb{R}^n\) to denote the indicator vector of the set \(S \subseteq [n]\), having a coordinate of 0 everywhere except for elements of \(S\), where the coordinate is 1. We use \(\text{conv}\) to denote the operator that maps a set of points to their convex hull.

2.1 Matroids

In this paper we use one of the many cryptomorphic definitions of a matroid in terms of the polytope of its bases. For equivalence to other prominent definitions of a matroid, and more generally references to facts stated here see \([Oxl06]\).

**Definition 3.** We say that a family \(B \subseteq \binom{[n]}{k}\) is the family of bases of a matroid if the polytope \(\text{conv}\{1_B \mid B \in \mathcal{B}\}\) has only edges of the minimum possible length, namely \(\sqrt{2}\). We call \(k\) the rank of the matroid, and \([n]\) the ground set of the matroid.

A well-known fact about matroids, that can be easily derived from Definition 3, is that the dual of a matroid, defined below, is another matroid.

**Proposition 4.** If \(B \subseteq \binom{[n]}{k}\) is the family of bases of a matroid, then the following is also the family of bases of another matroid, called the dual matroid:

\[
\mathcal{B}^* := \{[n] - B \mid B \in \mathcal{B}\}.
\]

In this paper we will use a famous class of matroids constructed from graphs, called graphic matroids.
Proposition 5. Let \( G = (V, E) \) be a graph. Then the following is the family of bases of a matroid, called the graphic matroid of \( G \):
\[
\{ T \subseteq E \mid T \text{ forms a spanning tree} \}.
\]
Note that the rank of the graphic matroid is \(|V| - 1\) and the ground set is \( E \).

2.2 Log-Concave Polynomials

For a distribution or density function \( \mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0} \) we denote by \( g_\mu \) the generating polynomial of \( \mu \) defined as
\[
g_\mu(z_1, \ldots, z_n) := \sum_{S \in \binom{[n]}{k}} \mu(S) \prod_{i \in S} z_i.
\]
We call a polynomial \( g \in \mathbb{R}[z_1, \ldots, z_n] \) with nonnegative coefficients log-concave when viewed as a function, it is log-concave over the positive orthant, i.e., for \( x, y \in \mathbb{R}^n_{\geq 0} \) and \( \lambda \in (0, 1) \)
\[
g(\lambda x + (1 - \lambda)y) \geq \lambda g(x) \lambda^{-\lambda}.
\]
An important class of polynomials are those associated with uniform distributions over bases of a matroid.

Theorem 6 ([AOV18] based on [AHK18]). If \( B \subseteq \binom{[n]}{k} \) is the family of bases of a matroid, then the following polynomial is log-concave:
\[
g(z_1, \ldots, z_n) := \sum_{B \in B} \prod_{i \in B} z_i.
\]
One of the basic operations preserving log-concavity is composition with a linear map. That is if \( T : \mathbb{R}^m \to \mathbb{R}^n \) is an affine linear map for which \( T(\mathbb{R}^m_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0} \), then \( g \circ T \) is log-concave as well.

We note that all polynomials considered in this paper are homogeneous and multi-affine, i.e., no variable appears with a power more than 1. For a multi-affine polynomial \( g \), its derivatives can be obtained as
\[
\partial_1 g = \lim_{c \to \infty} \frac{g(c, z_2, \ldots, z_n)}{c}.
\]
This shows that the derivatives of a multi-affine log-concave polynomial are limits of log-concave polynomials, which themselves are log-concave. It follows that a multi-affine homogeneous log-concave polynomial satisfies the seemingly stronger notions of strong log-concavity [Gur09] and complete log-concavity [AOV18; Ana+18; BH19]. The latter means that such polynomials are also closed under directional derivatives.

Lemma 7. Let \( g \in \mathbb{R}[z_1, \ldots, z_n] \) be a multi-affine homogeneous polynomial with nonnegative coefficients. If \( g \) is log-concave, then it is completely log-concave as well, which means that for any \( k \in \mathbb{Z}_{\geq 0} \) and directions \( v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0} \), the following polynomial is log-concave:
\[
\partial_{v_1} \cdots \partial_{v_k} g.
\]
We will use the following simple fact about log-concave polynomials:

Proposition 8 ([see Ana+18; BH18]). If \( g \) is a log-concave polynomial with nonnegative coefficients, then \( \nabla^2 g \) evaluated at any point in the positive orthant has at most one positive eigenvalue.
2.3 Down-Up Random Walk

For two distributions \( \nu, \mu \) we define the Kullback-Leibler divergence, KL-divergence for short, \( D_{\text{KL}}(\nu \parallel \mu) \) as

\[
D_{\text{KL}}(\nu \parallel \mu) := \mathbb{E}_{S \sim \mu} \left[ \frac{\nu(S)}{\mu(S)} \log \frac{\nu(S)}{\mu(S)} \right] = \mathbb{E}_{S \sim \nu} \left[ \log \frac{\nu(S)}{\mu(S)} \right],
\]

and the total variation distance between \( \nu \) and \( \mu \) as

\[
\|\nu - \mu\|_{TV} := \frac{1}{2} \sum_{S} |\nu(S) - \mu(S)|.
\]

The two are related by Pinsker’s inequality:

**Proposition 9** ([see, e.g., CT12]). KL-divergence and the total variation distance are related by the following inequality

\[
\|\nu - \mu\|_{TV} \leq \sqrt{\frac{1}{2} D_{\text{KL}}(\nu \parallel \mu)}
\]

Cryan, Guo, and Mousa [CGM19] proved shrinkage of the KL-divergence under the down-up random walk. Coupled with Pinsker’s inequality, this resulted in a mixing time bound.

**Lemma 10.** [CGM19] If \( \nu, \mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0} \) are distributions where \( \mu \) has a log-concave generating polynomial, and \( P \) is the down-up random walk operator whose stationary distribution is \( \mu \), then

\[
D_{\text{KL}}(\nu P \parallel \mu P) = D_{\text{KL}}(\nu P \parallel \mu) \leq (1 - 1/k) D_{\text{KL}}(\nu \parallel \mu).
\]

3 Mixing Time Analysis

In this section we prove **Theorem 1** by analyzing the down-up random walk for distributions \( \mu \) that have a log-concave generating polynomial. As a reminder, in each step, the down-up random walk transitions from a set \( S \in \binom{[n]}{k} \) to \( S' \in \binom{[n]}{k} \) as follows:

- From \( S \) choose a subset \( T \subseteq S \) of size \( k - 1 \) uniformly at random.
- From all supersets \( S' \supseteq T \), choose one with probability \( \propto \mu(S') \).

Notice that the first step above simply drops a uniformly random element, and the second step replaces it with a new one (potentially the same element). Our high-level strategy is to prove that in \( O(k \log k) \) steps, every element of the initial set is replaced at least once, and when this happens the distribution becomes a *warm start* and converges to \( \mu \) in an additional \( O(k \log k) \) steps.

Let \( \tau \) be the first time such that every element in our initial set has been replaced at least once. In other words think of initial elements as unmarked, and every time we replace an element we mark the new element brought in. Then \( \tau \) is the first time that every element is marked.

We will prove the following:
**Lemma 11.** Let $S_t$ be the set at time $t$ in the down-up random walk. Then for any $X \in \binom{[n]}{k}$ and any time $t$,

$$\mathbb{P}[S_t = X | \tau \leq t] \leq 2^{O(k^2)} \mathbb{P}_\mu[X].$$

Note that without $2^{O(k^2)}$, the r.h.s. is simply the stationary distribution. So this statement can be understood to say that as long as we have replaced each element at least once, we cannot be too far off from the stationary distribution.

Before proving Lemma 11, let us see finish the proof of Theorem 1 assuming it.

**Proof of Theorem 1 assuming Lemma 11.** Note that for any fixed time $t$, we can simply bound $\mathbb{P}[\tau > t]$ by $k(1 - 1/k)^t \leq k e^{-t/k}$. In particular this probability rapidly converges to $0$ after about $k \log k$ steps.

Now let $t_1 < t_2$ be two time indices. Let $\nu_t$ denote the distribution of the state of random walk, i.e., $S_t$, at time $t$. Our goal is to bound $\|\nu_t - \mu\|_{TV}$, where for simplicity of notation, we assume $\mu$ is properly normalized to be a probability distribution. Let $\nu_t'$ be the distribution of $S_t$ conditioned on $\tau \leq t$, and let $\nu_{t}''$ be the distribution of $S_t$ conditioned on $\tau \geq t$. Then we can write

$$\nu_{t_1} = \mathbb{P}[\tau \leq t_1] \cdot \nu_{t_1}' + \mathbb{P}[\tau > t_1] \cdot \nu_{t_1}''.$$

If $P$ denotes the random walk operator, then note that $\nu_{t_2} = \nu_{t_1} P^{t_2-t_1}$. So we get

$$\nu_{t_2} = \mathbb{P}[\tau \leq t_1] \nu_{t_1}' P^{t_2-t_1} + \mathbb{P}[\tau > t_1] \nu_{t_1}'' P^{t_2-t_1}.$$

Using the triangle inequality we can bound

$$\|\nu_{t_2} - \mu\|_{TV} \leq \|\nu_{t_1}' P^{t_2-t_1} - \mu\|_{TV} + \mathbb{P}[\tau > t_1].$$

Here we used the fact that $\mathbb{P}[\tau \leq t_1] \leq 1$, and $\|\nu_{t_1}' P^{t_2-t_1} - \mu\|_{TV} \leq 1$; the latter inequality is because $\|\cdot\|_{TV}$ is always upper bounded by $1$.

We can bound the second term in the above inequality by $ke^{-t_1/k}$ as stated before. For the first term, note that the KL-divergence between $\nu_{t_1}'$ and $\mu$ is at most $O(k^2)$ by Lemma 11. This is because

$$D_{KL}(\nu_{t_1}' \| \mu) = \mathbb{E}_{S \sim \nu_{t_1}'} \left[ \log \frac{\nu_{t_1}'(S)}{\mu(S)} \right] \leq \log(2^{O(k^2)}) = O(k^2).$$

So by Lemma 10 in $t_2 - t_1$ steps this KL-divergence decreases to $(1 - 1/k)^{t_2-t_1}O(k^2) = O(k^2 e^{-(t_2-t_1)/k})$. By Pinsker’s inequality, Proposition 9, we get that

$$\|\nu_{t_1}' P^{t_2-t_1} - \mu\|_{TV} \leq O(ke^{-(t_2-t_1)/2k}).$$

So in the end we get the following bound

$$\|\nu_{t_2} - \mu\|_{TV} \leq O(ke^{-(t_2-t_1)/2k} + ke^{-t_1/k}).$$

In order for this to be at most $\epsilon$, it is enough to make sure that $\min\{t_1, t_2 - t_1\} = \Omega(k \log k + k \log \frac{1}{\epsilon})$. So we can simply let $t_1 = t_2/2$, and then make sure that $t_2 = \Omega(k \log(k/\epsilon))$.  

\[\square\]
As the main tool we use to prove Lemma 11, we introduce a new inequality for log-concave polynomials, that we call approximate exchange. We state the inequality below and defer its proof to Section 4.

Lemma 12. Any \( \mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0} \) with a log-concave generating polynomial satisfies a \( 2^{O(k)} \)-exchange property. That is, for every \( S, T \in \binom{[n]}{k} \) and \( i \in S \) there exists \( j \in T \) such that

\[
\mu(S) \mu(T) \leq 2^{O(k)} \mu(S-i+j) \mu(T+i-j).
\]

Armed with Lemma 12, let us prove Lemma 11.

Proof of Lemma 11. Let’s look at the down-up walk process with orders. This means that we start with some elements \( e_1, \ldots, e_k \) that together form the starting set. In each time step we replace one of the \( e_i \)’s. But we keep track of the ordering and do not convert these to sets. So we can talk about \( e_i^t \) as the \( i \)-th element at time \( t \). In particular \( S_t \) is simply the unordered collection \( \{e_1^t, \ldots, e_k^t\} \). Let’s say that \( X = \{f_1, \ldots, f_k\} \). Then to have \( S_t = X \), there must be some permutation of \( f_1, \ldots, f_k \) that equals \( e_1^t, \ldots, e_k^t \). We will show that for any such permutation the promised bound in Lemma 11 holds. Since there are \( k! = 2^{O(k \log k)} \) many permutations, this extra factor of \( k! \) can be absorbed into the factor of \( 2^{O(k^2)} \) without any loss. So we fix an arbitrary permutation, w.l.o.g. the identity permutation, and try to bound the following

\[
P[e_1^t = f_1, \ldots, e_k^t = f_k \mid \tau \leq t].
\]

Since we are conditioning on \( \tau \leq t \), note that there must be some time \( \tau_i \leq t \), which is the last time before \( t \) where the \( i \)-th element gets replaced by the down-up random walk. We will bound the above probability, even conditioned on \( \tau_1, \ldots, \tau_k \) having any set of fixed values up to \( t \). Note that the index of the element that gets replaced in every step is uniformly random and independent of everything else that happens in the random walk, in particular the identity of the elements that come in as replacements. In the rest of the proof, we condition on the indices of the elements that get replaced at every step up to time \( t \); note that this also uniquely determines \( \tau_1, \ldots, \tau_k \), so we assume \( \tau_1, \ldots, \tau_k \) are some fixed time indices. W.l.o.g. assume that \( \tau_1 < \tau_2 < \cdots < \tau_k \). We will use induction to prove the following statement for \( i = 0, \ldots, k \):

\[
P[e_1^{\tau_1} = f_1, e_2^{\tau_2} = f_2, \ldots, e_i^{\tau_i} = f_i \mid \text{replacement indices}] \leq 2^{O(i)} P_{U \sim \mu}[f_1, \ldots, f_i \in U].
\]

Notice that for \( i = 0 \), both sides are trivially equal to 1, and for \( i = k \), this inequality is the main statement we want to prove.

It remains to show the inductive step. We will show that going from \( i - 1 \) to \( i \), the l.h.s. gets multiplied by a smaller quantity compared to the r.h.s. It is not hard to see that the factors that get multiplied on each side are the two sides of the following inequality:

\[
P[e_i^{\tau_i} = f_i \mid e_1^{\tau_1} = f_1, \ldots, e_{i-1}^{\tau_{i-1}} = f_{i-1} \text{ and replacement indices}] \leq 2^{O(i)} P_{U \sim \mu}[f_i \in U \mid f_1, \ldots, f_{i-1} \in U].
\]

Instead of conditioning only on \( f_1, \ldots, f_{i-1} \) being chosen at the appropriate times on the l.h.s., we will refine the conditioning and condition on the history of the random walk up to time \( \tau_i - 1 \). This means we can in particular assume that the elements \( e_{i+1}^{\tau_{i+1}}, \ldots, e_k^{\tau_k} \) are fixed, that \( e_1^{\tau_1} = f_1, \ldots, e_{i-1}^{\tau_{i-1}} = f_{i-1} \), and the only uncertain thing is what the \( i \)-th element is being replaced by at time \( \tau_i \).
Let $S = \{f_1, \ldots, f_i, e^T_{i+1}, \ldots, e^T_k\}$. Then the conditional probability of choosing $f_i$ at time $\tau_i$ is:

$$\Pr[e^T_i = f_i | e^T_1 = f_1, \ldots, e^T_{i-1} = f_{i-1} \text{ and replacement indices}] = \frac{\mu(S)}{\sum_{V \supset S-f_i} \mu(V)}.$$ 

On the other hand

$$\Pr_{U \sim \mu}[f_i \in U | f_1, \ldots, f_{i-1} \in U] = \frac{\sum_{U \ni f_1, \ldots, f_i \mu(U)}{\sum_{T \ni f_1, \ldots, f_{i-1}} \mu(T)}}.$$ 

So we have to show the following:

$$\mu(S) \left( \sum_{T \ni f_1, \ldots, f_{i-1}} \mu(T) \right) \leq 2^{O(k)} \left( \sum_{V \supset S-f_i} \mu(V) \right) \left( \sum_{U \ni f_1, \ldots, f_i} \mu(U) \right).$$ 

We will give an injection from the terms on the l.h.s. to the terms in the expanded form of the r.h.s. Choose some set $T \ni f_1, \ldots, f_{i-1}$. Apply Lemma 12 to $S$ and $T$ with the element $f_i \in S$. We get that there must be some element $e \in T$ such that

$$\mu(S)\mu(T) \leq 2^{O(k)}(S-f_i+e)\mu(T+f_i-e).$$

Note that $V := S-f_i+e$ contains $S-f_i$, and $U := T+f_i-e$ contains $\{f_1, \ldots, f_i\}$. So $\mu(U)\mu(V)$ appears on the r.h.s. of the desired inequality. So for each $T$ appearing on the l.h.s. of the desired inequality we produced a pair of $U$ and $V$. Note that this mapping from $T$ to $(X,Y)$ is injective. This is because given $(X,Y)$, we can recover $T$ as the xor/symmetric difference of the other three sets, that is $T = S \Delta U \Delta V$. \hfill \Box

## 4 Approximate Exchange Property

In this section we prove Lemma 12.

**Definition 13.** We say that $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ has an $\alpha$-approximate exchange property, or $\alpha$-exchange for short, if for every $S, T \in \binom{[n]}{k}$ and every $i \in S$, there exists $j \in T$ such that

$$\alpha \cdot \mu(S-i+j)\mu(T+i-j) \geq \mu(S)\mu(T).$$

Note that if $\mu$ is the indicator of bases of a matroid, then it has a 1-exchange property, also known as the strong basis exchange property [Oxl06].

Although we do not directly need it, we give another example where approximate exchange can be proven by elementary means. This is the class of $k$-determinantal point processes [BBL09; KT12].

**Proposition 14.** Suppose that $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ is defined as

$$\mu(S) = \det([v_i]_{i \in S})^2,$$

for some vectors $v_1, \ldots, v_n \in \mathbb{R}^k$. Then $\mu$ has a $k^2$-exchange property.
Proof. It is enough to consider the case where $S$ and $T$ are disjoint; otherwise, the problem can be reduced to lower values of $k$ by taking out the intersection, and projecting all vectors on the orthogonal complement of the space spanned by the intersection.

Define the number $\beta_j$ as $\sqrt{\mu(S - i + j)\mu(T + i - j)}$ and let $\alpha$ be $\sqrt{\mu(S)\mu(T)}$. The Plücker relations for the Grassmanian [see, e.g., Abe80] say that a signed sum of $\alpha$ and $\beta_j$ is zero:

$$\alpha + \sum_{j \in T} \pm \beta_j = 0.$$ 

This means that there is at least one $j$ such that $|\beta_j| \geq \frac{1}{k}\alpha$, and this concludes the proof. 

Next we take steps to prove Lemma 12, namely that if $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ has a log-concave generating polynomial $g_\mu$, then $\mu$ has a $2^{O(k)}$-exchange property. We conjecture that a $k^{O(1)}$-exchange property should hold, but even if true, this will not improve the mixing time results in this paper beyond constants hidden in the $O(\cdot)$ notation.

Our strategy is to prove the case of $k = 2$ of Lemma 12 by using log-concavity of $g_\mu$ (note that $k = 1$ is trivial). We will then use an induction to prove the general case. We remark that this type of induction is a standard procedure used in many other places, such as in the context of proving Plücker relations and $M^3$-concavity [MS18].

Before delving into the proof, note that we can always assume $S \cap T = \emptyset$. This is because we can always condition the distribution $\mu$ on having any set of elements, and then throwing out those elements; this operation corresponds to taking partial derivatives of $g_\mu$ which results in a log-concave polynomial by Lemma 7. In particular, we can condition $\mu$ on having $S \cap T$, and then throwing out $S \cap T$ from the ground set.

Proof of Lemma 12 for $k = 2$. When $k = 2$, we might as well assume that $n = 4$, because no element outside of $S \cup T$ is important, and we can condition the distribution $\mu$ on not having those elements. This corresponds to substituting 0 for variables outside $S \cup T$ in $g_\mu$ which preserves log-concavity.

So our goal now is to show that for a log-concave quadratic polynomial in four variables

$$g_\mu = \sum_{\{i,j\} \in \binom{[4]}{2}} \mu(\{i,j\})z_iz_j,$$

we have an $O(1)$-exchange property. W.l.o.g. assume that $S = \{1,2\}$ and $T = \{3,4\}$.

Let us consider $\nabla^2 g_\mu$. This is a constant matrix, which has at most one positive eigenvalue by Proposition 8. On the other hand it is a matrix with nonnegative entries, so it must have at least one nonnegative eigenvalue as well. Analyzing the possible signs of the eigenvalues, we see that their product, i.e., the determinant is nonpositive:

$$\det(\nabla^2 g_\mu) \leq 0.$$ 

This determinant can be written in a special way. Let us define:

$$A := \mu(\{1,2\})\mu(\{3,4\}),$$

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Lemma 12

We now complete the proof by inducting on \(k\). We will apply approximate exchange a second time. The sets \(S\) which we have already proven. By this exchange property, we have

\[
\mu(S)\mu(S - i - i' + j + j') \leq 2^{O(1)} \max \{ \mu(S - i + j)\mu(S - i' + j'), \mu(S - i + j')\mu(S - i' + j) \}. \tag{2}
\]

In particular one of \(\sqrt{B}\) and \(\sqrt{C}\) must be at least \(\frac{1}{2}\sqrt{A}\). This proves that \(\mu\) satisfies a \(2^2 = 4\)-approximate exchange property for \(S = \{1, 2\}\) and \(T = \{3, 4\}\).

We now complete the proof by inducting on \(k\).

Proof of Lemma 12 for the general case. We can assume that for any \(S, T\) such that \(|S \cap T| \geq 1\), we have a \(2^{O(k - |S \cap T|)}\)-approximate exchange property. This is because by the arguments we had, such nonempty intersections can be reduced to smaller values of \(k\) by conditioning and throwing out \(S \cap T\).

Now let \(S \cap T = \emptyset\) and let \(i \in S\) be given. Our goal is to find \(j\) such that

\[
\mu(S)\mu(T) \leq \mu(S - i + j)\mu(T + i - j).
\]

Let \(i' \neq i\) be another, arbitrary, element of \(S\). We will exchange \(i'\) with an element \(j' \in T\) and use induction on \(S - i' + j'\) and \(T\). We need to be careful how we choose \(j'\) though. Let us choose \(j'\) to be the element of \(T\) that maximizes the expression \(\mu(T + i - j')\mu(S - i' + j')\). The reason for this choice will become apparent in the rest of he proof.

Then the sets \(S - i' + j'\) and \(T\) have an intersection of one element, so by induction we know an approximate exchange property for them. Therefore, there must be a \(j \in T\) such that

\[
\mu(S - i' + j')\mu(T) \leq 2^{O(k - 1)} \mu(S - i - i' + j + j')\mu(T + i - j). \tag{1}
\]

We will apply approximate exchange a second time. The sets \(S\) and \(S - i - i' + j + j'\) have a very large intersection. In particular their exchange property reduces to the case of \(k = 2\) of Lemma 12, which we have already proven. By this exchange property, we have

\[
\mu(S)\mu(S - i - i' + j + j') \leq 2^{O(1)} \max \{ \mu(S - i + j)\mu(S - i' + j'), \mu(S - i + j')\mu(S - i' + j) \}. \tag{2}
\]

Notice that approximate exchange for \(S, T\) any any \(i \in S\) is equivalent to saying that \(A \leq O(1) \cdot \max \{B, C\}\). We can write \(\det(\nabla^2 g_{ij}) = A^2 + B^2 + C^2 - 2(AB + AC + BC)\). So we get the inequality

\[
A^2 + B^2 + C^2 \leq 2(AB + AC + BC).
\]

This is the same as

\[
(A - B - C)^2 \leq 4BC.
\]

Taking square-roots we get

\[
A - B - C \leq 2\sqrt{BC},
\]

which is the same as saying

\[
A \leq (\sqrt{B} + \sqrt{C})^2.
\]

Taking square-roots again we get

\[
\sqrt{A} \leq \sqrt{B} + \sqrt{C}.
\]

Proof of Lemma 12 for the general case. We can assume that for any \(S, T\) such that \(|S \cap T| \geq 1\), we have a \(2^{O(k - |S \cap T|)}\)-approximate exchange property. This is because by the arguments we had, such nonempty intersections can be reduced to smaller values of \(k\) by conditioning and throwing out \(S \cap T\).

Now let \(S \cap T = \emptyset\) and let \(i \in S\) be given. Our goal is to find \(j\) such that

\[
\mu(S)\mu(T) \leq \mu(S - i + j)\mu(T + i - j).
\]

Let \(i' \neq i\) be another, arbitrary, element of \(S\). We will exchange \(i'\) with an element \(j' \in T\) and use induction on \(S - i' + j'\) and \(T\). We need to be careful how we choose \(j'\) though. Let us choose \(j'\) to be the element of \(T\) that maximizes the expression \(\mu(T + i - j')\mu(S - i' + j')\). The reason for this choice will become apparent in the rest of he proof.

Then the sets \(S - i' + j'\) and \(T\) have an intersection of one element, so by induction we know an approximate exchange property for them. Therefore, there must be a \(j \in T\) such that

\[
\mu(S - i' + j')\mu(T) \leq 2^{O(k - 1)} \mu(S - i - i' + j + j')\mu(T + i - j). \tag{1}
\]

We will apply approximate exchange a second time. The sets \(S\) and \(S - i - i' + j + j'\) have a very large intersection. In particular their exchange property reduces to the case of \(k = 2\) of Lemma 12, which we have already proven. By this exchange property, we have

\[
\mu(S)\mu(S - i - i' + j + j') \leq 2^{O(1)} \max \{ \mu(S - i + j)\mu(S - i' + j'), \mu(S - i + j')\mu(S - i' + j) \}. \tag{2}
\]

The reason for this choice will become apparent in the rest of he proof.

Then the sets \(S - i' + j'\) and \(T\) have an intersection of one element, so by induction we know an approximate exchange property for them. Therefore, there must be a \(j \in T\) such that

\[
\mu(S - i' + j')\mu(T) \leq 2^{O(k - 1)} \mu(S - i - i' + j + j')\mu(T + i - j). \tag{1}
\]

We will apply approximate exchange a second time. The sets \(S\) and \(S - i - i' + j + j'\) have a very large intersection. In particular their exchange property reduces to the case of \(k = 2\) of Lemma 12, which we have already proven. By this exchange property, we have

\[
\mu(S)\mu(S - i - i' + j + j') \leq 2^{O(1)} \max \{ \mu(S - i + j)\mu(S - i' + j'), \mu(S - i + j')\mu(S - i' + j) \}. \tag{2}
\]
Figure 1: The down-up random walk on spanning trees. An edge of the spanning tree is uniformly at random selected and removed, and then an edge from those connecting the resulting two components is selected uniformly at random and added.

If the first term in Eq. (2) achieves the maximum, then we are done, because multiplying Eqs. (1) and (2) yields

\[ \mu(S - i' + j') \mu(T) \mu(S) \mu(S - i - i' + j + j') \leq 2^{O(k)} \mu(S - i - i' + j + j') \mu(T + i - j) \mu(S - i + j) \mu(S - i' + j'), \]

which simplifies to

\[ \mu(S) \mu(T) \leq 2^{O(k)} \mu(S - i + j) \mu(T + i - j), \]

showing that \( i \) can be exchanged for \( j \).

So assume that the second term in Eq. (2) achieves the maximum. We will show that in this case \( i \) can be exchanged for \( j' \). Multiplying Eqs. (1) and (2) yields

\[ \mu(S - i' + j') \mu(T) \mu(S - i - i' + j + j') \leq 2^{O(k)} \mu(S - i - i' + j + j') \mu(T + i - j) \mu(S - i + j') \mu(S - i' + j), \]

which simplifies to

\[ \mu(S) \mu(T) \leq 2^{O(k)} \cdot \mu(S - i + j') \mu(T + i - j') \frac{\mu(T + i - j) \mu(S - i' + j)}{\mu(T + i - j') \mu(S - i' + j')} \]

Notice that by our choice of \( j' \), the fraction appearing on the r.h.s. is \( \leq 1 \). So we can conclude that

\[ \mu(S) \mu(T) \leq 2^{O(k)} \mu(S - i + j') \mu(T + i - j'). \]

\[ \square \]

5 Sampling Spanning Trees

In this section we prove Theorem 2.\(^3\)

\(^3\)We remark that reliance on Theorem 1 in this section is not mandatory and the results of this section would have been possible even without Theorem 1.
For simplicity of exposition, we first explain the algorithm for sampling uniformly random spanning trees from unweighted graphs $G = (V, E)$, and then explain simple modifications that result in weighted distributions.

Given that the uniform distribution over spanning trees has a log-concave generating polynomial, one could be tempted to run the down-up walk on the set of spanning trees. This is depicted in Fig. 1. The random walk is guaranteed to come within an $\epsilon$ distance of the target distribution in $O(|V| \log(|V|/\epsilon))$ steps by Theorem 1. However the main problem is implementing each step. The first half of the down-up walk is easy; we just need to remove a uniformly random edge from the current spanning tree. However in the second step where we add an edge, we have to potentially look at close to all edges in the graph to see which ones can reconnect the two connected components, and form a spanning tree again. So the overall time complexity would be $O(|E| \cdot |V| \log(|V|/\epsilon))$, which is quadratic in the size of the graph.

Fortunately, the complements of spanning trees also form a matroid, known as the cographic or bond matroid; see Propositions 4 and 5. Moreover, choosing the complement of a spanning tree uniformly at random is the same as choosing a spanning tree uniformly at random. Note that the cographic matroid is of rank $k = |E| - |V| + 1 \leq |E|$ and is over a ground set of $|E|$ elements. By Theorem 1, the down-up random walk on it, depicted in Fig. 2, mixes to within an $\epsilon$ distance of the uniform distribution in time $O(|E| \log(|E|/\epsilon))$. However here, each step of the down-up random walk can be implemented much more efficiently. In the first half step, we need to select an edge uniformly at random from those not in the spanning tree, and add it to the spanning tree. This can be simply done by maintaining an array of edges not currently in the spanning tree. The more complicated step is detecting the cycle formed by new edge and the tree, and picking a uniformly random edge from it. Fortunately this step can be done in amortized $O(\log|E|)$ time using dynamic tree data structures such as link-cut trees [ST83].

Fortunately this random walk has already been studied by Russo, Teixeira, and Francisco [RTF18], and details of the link-cut tree operation can be found there. Russo, Teixeira, and Francisco [RTF18] simply did not have a tight mixing time bound for the random walk, which we provided here. Since each step takes amortized time $O(\log|E|)$ and the overall mixing time is $O(|E| \log(|E|/\epsilon))$, the whole algorithm takes time $O(n \log(n) \log(n/\epsilon))$ where $n$ is the number of edges in the graph.
**Weighted Graphs.** In the case of weighted graphs where there is a weight of \( w_e \) over every edge \( e \), the weighted spanning tree distribution is defined to be

\[
P[T] \propto \prod_{e \in T} w_e.
\]

In order to sample from this distribution, it is enough to sample the complement of a spanning tree \( E - T \) with inverted edge weights, that is sampling \( E - T \) in such a way that

\[
P[E - T] \propto \prod_{e \in E - T} \frac{1}{w_e}.
\]

So the down-up random walk for the complement of the spanning tree is modified as follows:

- To the current spanning tree \( T \), add edge \( e \notin T \) selected uniformly at random.
- From the edges on the cycle created, select an edge \( f \) with probability \( \propto 1/w_f \) and remove it.

We simply need to modify the sampling procedure to handle the second step above. Fortunately link-cut trees can already do this in the same amortized \( O(\log|E|) \) time [RTF18].

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