Existence of solutions of $\alpha \in (2, 3]$ order fractional three point boundary value problems with integral conditions

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Abstract

Existence and uniqueness of solutions for $\alpha \in (2, 3]$ order fractional differential equations with three point fractional boundary and integral conditions is discussed. The results are obtained by using standard fixed point theorems. Two examples are given to illustrate the results.

1 Introduction

Recently, the theory on existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of many authors, see for example, [4]-[22] and references therein. Many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multi-point and nonlocal boundary value problems as special cases. The existing literature mainly deals with first order and second order boundary value problems and there are a few papers on third order problems.

Shahed [18] studied existence and nonexistence of positive solution of nonlinear fractional two-point boundary value problem derivative

$$D_{0+}^\alpha u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1; \quad 2 < \alpha < 3,$$

$$u(0) = u'(0) = u'(1) = 0,$$

where $D_{0+}^\alpha$ denotes the Caputo derivative of fractional order $\alpha$, $\lambda$ is a positive parameter and $a : (0, 1) \to [0, \infty)$ is continuous function.

In [11], Ahmad and Ntouyas studied a boundary value problem of nonlinear fractional differential equations of order $\alpha \in (2, 3]$ with anti-periodic type integral boundary conditions:

$$D_{0+}^\alpha u(t) = f(t, u(t)); \quad 0 < t < T; \quad 2 < \alpha \leq 3,$$

$$u^{(j)}(0) - \lambda_j u^{(j)}(T) = \mu_j \int_0^T g_j(s, u(s)) ds, \quad j = 0, 1, 2,$$

where $D_{0+}^\alpha$ denotes the Caputo derivative of fractional order $\alpha$, $u^{(j)}$ denotes $j$-th derivative of $u$, $f, g_0, g_1, g_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \mu_j \in \mathbb{R}$ ($\lambda_j \neq 1$). The same problem for fractional differential inclusions is considered in [12].
Ahmad and Nieto \cite{10} studied existence and uniqueness results for the following general three point fractional boundary value problem involving a nonlinear fractional differential equation of order $\alpha \in (m-1, m]$,

$$D_0^\alpha u(t) = f(t, u(t)); \quad 0 < t < T, \ m \geq 2,$$

$$u(0) = u'(0) = \ldots = u^{(m-2)}(0) = 0, \ u(1) = \lambda u(\eta).$$

However, very little work have been done on the case when the nonlinearity $f$ depends on the fractional derivative of the unknown function. Su and Zhang \cite{19}, Rehman et al. \cite{20} studied the existence and uniqueness of solutions for following nonlinear two-point and three point fractional boundary value problem when the nonlinearity $f$ depends on the fractional derivative of the unknown function.

In this paper, we investigate the existence (and uniqueness) of solution for nonlinear fractional differential equations of order $\alpha \in (2, 3]$}

$$D_0^\alpha u(t) = f \left(t, u(t), D_0^{\beta_1} u(t), D_0^{\beta_2} u(t)\right); \quad 0 \leq t \leq T; \ 2 < \alpha \leq 3 \quad (1)$$

with the three point and integral boundary conditions

$$\begin{align*}
& a_0 u(0) + b_0 u(T) = \lambda_0 \int_0^T g_0(s, u(s))ds, \\
& a_1 D_0^{\beta_1} u(\eta) + b_1 D_0^{\beta_1} u(T) = \lambda_1 \int_0^T g_1(s, u(s))ds, \quad 0 < \beta_1 \leq 1, \quad 0 < \eta < T, \\
& a_2 D_0^{\beta_2} u(\eta) + b_2 D_0^{\beta_2} u(T) = \lambda_2 \int_0^T g_2(s, u(s))ds, \quad 1 < \beta_2 \leq 2,
\end{align*} \quad (2)$$

where $D_0^\alpha$ denotes the Caputo fractional derivative of order $\alpha$, $f, g_0, g_1, g_2$ are continuous functions.

2 Preliminaries

Let us recall some basic definitions \cite{11-13}.

**Definition 1** The Riemann Liouville fractional integral of order $\beta$ for continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined as

$$I_0^\beta f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0$$

provided the integral exists.

**Definition 2** For $n$-times continuously differentiable function $f : [0, \infty) \rightarrow \mathbb{R}$ the Caputo derivative fractional order $\alpha$ is defined as;

$$D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds; \quad n-1 < \alpha < n, \ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integral part of the real number $\alpha$. 

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Lemma 3 Let $\alpha > 0$. Then the differential equation $D^\alpha_{0+} f(t) = 0$ has solutions
\[
f(t) = k_0 + k_1 t + k_2 t^2 + \ldots + k_{n-1} t^{n-1}
\]
and
\[
I^\alpha_{0+} D^\alpha_{0+} f(t) = f(t) + k_0 + k_1 t + k_2 t^2 + \ldots + k_{n-1} t^{n-1},
\]
here $k_i \in \mathbb{R}$ and $i = 1, 2, 3, \ldots, n - 1, n = [\alpha] + 1$.

Caputo fractional derivative of order $n - 1 < \alpha < n$ for $t^\gamma$, is given as
\[
D^\alpha_{0+} t^\gamma = \begin{cases} 
\frac{\Gamma (\gamma + 1)}{\Gamma (\gamma - \alpha + 1)} t^{\gamma - \alpha}, & \gamma \in \mathbb{N} \text{ and } \gamma \geq n \text{ or } \gamma \notin \mathbb{N} \text{ and } \gamma > n - 1, \\
0, & \gamma \in \{0, 1, \ldots, n - 1\}. 
\end{cases}
\]
Assume that $a_i, b_i, \lambda_i \in \mathbb{R}, 0 < \eta < T, \beta_0 = 0, 0 < \beta_1 \leq 1, 1 < \beta_2 \leq 2$ and $\alpha_0 + b_0 \neq 0, a_1 \eta^{1-\beta_1} + b_1 T^{1-\beta_1} \neq 0, a_i \eta^{2-\beta_i} + b_i T^{2-\beta_i} \neq 0$.

For convenience, we set
\[
\begin{align*}
\mu^{\beta_1} &:= \frac{\Gamma(3-\beta_1)}{2(a_1 \eta^{2-\beta_1} + b_1 T^{2-\beta_1}),} \\
\mu^{\beta_2} &:= \frac{\Gamma(3-\beta_2)}{2(a_2 \eta^{2-\beta_2} + b_2 T^{2-\beta_2}),} \\
\nu^{\beta_1} &:= \frac{\Gamma(2-\beta_1)}{a_1 \eta^{1-\beta_1} + b_1 T^{1-\beta_1}},
\end{align*}
\]
\[
\omega_0 := \frac{1}{a_0 + b_0}, \quad \omega_1 (t) := \nu^{\beta_1} \left( \frac{b_0}{a_0 + b_0} T - t \right),
\]
\[
\omega_2 (t) := \frac{b_0 T}{a_0 + b_0} \mu^{\beta_2} - \frac{b_0 T}{a_0 + b_0} \frac{\mu^{\beta_2} \mu^{\beta_1}}{\mu^{\beta_1} + \nu^{\beta_1}} t - \mu^{\beta_2} t^2.
\]

Lemma 4 For any $f, g_0, g_1, g_2 \in C ([0, T]; \mathbb{R})$, the unique solution of the fractional boundary value problem
\[
D^\alpha_{0+} u(t) = f(t); \quad 0 \leq t \leq T, \quad 2 < \alpha \leq 3,
\]
\[
\begin{cases}
a_0 u(0) + b_0 u(T) = \lambda_0 \int_0^T g_0(s) ds, \\
a_1 D^{\beta_1}_{0+} u(\eta) + b_1 D^{\beta_1}_{0+} u(T) = \lambda_1 \int_0^T g_1(s) ds, \quad 0 < \eta < T, \quad 0 < \beta_1 \leq 1, \\
a_2 D^{\beta_2}_{0+} u(\eta) + b_2 D^{\beta_2}_{0+} u(T) = \lambda_2 \int_0^T g_2(s) ds, \quad 1 < \beta_1 \leq 2
\end{cases}
\]
is given by
\[
u(t) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma (\alpha)} f(s) ds + \sum_{i=0}^{2} \omega_i (t) b_i \int_0^T \frac{(T - s)^{\alpha-\beta_i-1}}{\Gamma (\alpha - \beta_i)} f(s) ds
\]
\[
+ \sum_{i=1}^{2} \omega_i (t) a_i \int_0^\eta \frac{(\eta - s)^{\alpha-\beta_i-1}}{\Gamma (\alpha - \beta_i)} f(s) ds - \sum_{i=0}^{2} \omega_i (t) \lambda_i \int_0^T g_i(s) ds.
\]
Proof. By Lemma \(3\) for \(2 < \alpha \leq 3\) the general solution of the equation \(D_0^\alpha u(t) = f(t)\) can be written as

\[
u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds - k_0 - k_1 t - k_2 t^2,
\]

where \(k_0, k_1, k_2 \in \mathbb{R}\) are arbitrary constants. Moreover, by the formula \(3\) \(\beta_1\) and \(\beta_2\) order derivatives are as follows:

\[
D_0^{\beta_1} u(t) = I^\alpha - \beta_1 f(t) - k_1 \frac{t^{1 - \beta_1}}{\Gamma(2 - \beta_1)} - 2k_2 \frac{t^{2 - \beta_1}}{\Gamma(3 - \beta_1)},
\]

\[
D_0^{\beta_2} u(t) = I^\alpha - \beta_2 f(t) - 2k_2 \frac{t^{2 - \beta_2}}{\Gamma(3 - \beta_2)}.
\]

Using boundary conditions \(5\), we get the following algebraic system of equations for \(k_0, k_1, k_2\).

\[
- (a_0 + b_0) k_0 - b_0 Tk_1 - b_0 T^2 k_2 = \lambda_0 \int_0^T g_0(s)ds - b_0 \frac{I_0^\alpha}{\mu} f(T),
\]

\[
- \frac{a_1 \eta^{1 - \beta_1} + b_1 T^{1 - \beta_1}}{\Gamma(2 - \beta_1)} k_1 - 2 \frac{a_1 \eta^{2 - \beta_1} + b_1 T^{2 - \beta_1}}{\Gamma(3 - \beta_1)} k_2 = \lambda_1 \int_0^T g_1(s)ds - a_1 \frac{I_0^{\alpha - \beta_1}}{\mu} f(\eta) - b_1 \frac{I_0^{\alpha - \beta_1}}{\mu} f(T),
\]

\[
- \frac{a_2 \eta^{2 - \beta_2} + b_2 T^{2 - \beta_2}}{\Gamma(3 - \beta_2)} k_2 = \lambda_2 \int_0^T g_2(s)ds - a_2 \frac{I_0^{\alpha - \beta_2}}{\mu} f(\eta) - b_2 \frac{I_0^{\alpha - \beta_2}}{\mu} f(T).
\]

Solving the above system of equations for \(k_0, k_1, k_2\), we get the following:

\[
k_2 = b_2 \mu^{\beta_2} \frac{I_0^{\alpha - \beta_2}}{\mu^{\beta_2}} f(T) + a_2 \mu^{\beta_2} \frac{I_0^{\alpha - \beta_2}}{\mu^{\beta_2}} f(\eta) - \lambda_2 \mu^{\beta_2} \int_0^T g_2(s)ds,
\]

\[
k_1 = b_1 \mu^{\beta_1} \frac{I_0^{\alpha - \beta_1}}{\mu^{\beta_1}} f(T) + a_1 \mu^{\beta_1} \frac{I_0^{\alpha - \beta_1}}{\mu^{\beta_1}} f(\eta) - \lambda_1 \mu^{\beta_1} \int_0^T g_1(s)ds
\]

\[
- b_2 \mu^{\beta_2} I_0^{\alpha - \beta_2} \frac{\mu^{\beta_2}}{\mu^{\beta_2}} f(T) - a_2 \mu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} f(\eta) + \lambda_2 \mu^{\beta_1} \mu^{\beta_2} \int_0^T g_2(s)ds,
\]

\[
k_0 = \frac{b_0}{a_0 + b_0} I_0^\alpha f(T) - \frac{\lambda_0}{a_0 + b_0} \int_0^T g_0(s)ds
\]

\[
- \frac{b_0 b_1 \mu^{\beta_1} I_0^{\alpha - \beta_1} f(T)}{a_0 + b_0} - \frac{b_0 a_1 \mu^{\beta_1} I_0^{\alpha - \beta_1} f(\eta)}{a_0 + b_0} + \frac{b_0 \lambda_1 \mu^{\beta_1}}{a_0 + b_0} \int_0^T g_1(s)ds
\]

\[
+ \frac{b_0 b_2 \mu^{\beta_2} I_0^{\alpha - \beta_2} f(T)}{a_0 + b_0} + \frac{b_0 a_2 \mu^{\beta_1} \mu^{\beta_2} I_0^{\alpha - \beta_2} f(\eta)}{a_0 + b_0} - \frac{b_0 \lambda_2 \mu^{\beta_1} \mu^{\beta_2}}{a_0 + b_0} \mu^{\beta_2} \int_0^T g_2(s)ds
\]

\[
- \frac{b_0 b_2 \mu^{\beta_2} T^2 I_0^{\alpha - \beta_2} f(T)}{a_0 + b_0} - \frac{b_0 a_2 \mu^{\beta_1} T^2 I_0^{\alpha - \beta_2} f(\eta)}{a_0 + b_0} + \frac{b_0 \lambda_2 \mu^{\beta_2} T^2}{a_0 + b_0} \mu^{\beta_2} \int_0^T g_2(s)ds.
\]
Inserting \(k_0, k_1, k_2\) into (1) we get the desired representation for the solution of (4)-(5).

**Remark 5** The Green function of the BVP is defined by

\[
G(t; s) = \begin{cases} 
- \frac{(t-s)^{a-1}}{\Gamma(a)} + G_0(t; s), & 0 \leq s \leq t, \\
G_0(t; s), & 0 \leq t \leq s, 
\end{cases}
\]

where

\[
G_0(t; s) = \sum_{i=0}^{2} \omega_i(t) b_i \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} + \sum_{i=1}^{2} \omega_i(t) a_i \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \chi(\alpha, \eta)(s),
\]

\[
\chi(a, b)(s) := \begin{cases} 
1, & s \in (a, b), \\
0, & s \notin (a, b).
\end{cases}
\]

**Remark 6** For \(\alpha = 3, \beta_1 = 1, \beta_2 = 2\) and \(\eta = 0\), the Green function of (4)-(5) can be written as follows:

\[
G(t; s) = \begin{cases} 
- \frac{(t-s)^{a-1}}{\Gamma(a)} + G_0(t; s), & 0 \leq s \leq t, \\
G_0(t; s), & 0 \leq t \leq s.
\end{cases}
\]

where

\[
G_0(t; s) = \frac{b_0}{a_0 + b_0} \frac{(T-s)^{a-1}}{\Gamma(a)} + \left( \frac{b_0 T}{a_0 + b_0} \frac{b_1}{a_1 + b_1} + \frac{b_1}{a_1 + b_1} t \right) \frac{(T-s)^{a-2}}{\Gamma(a-1)}
+ \left( \frac{b_0}{a_0 + b_0} \frac{b_1}{a_1 + b_1} \frac{b_2}{a_2 + b_2} - \frac{2b_1}{a_1 + b_1} \frac{b_2}{a_2 + b_2} t + \frac{b_2}{2(a_2 + b_2)} t^2 \right) \frac{(T-s)^{a-3}}{\Gamma(a-2)}
\]

Moreover, the case

\[
a_0 = 1, b_0 = 0, a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0
\]

is investigated in [13]. In this case,

\[
G(t; s) = \begin{cases} 
- \frac{(t-s)^{a-1}}{\Gamma(a)} + t \frac{(T-s)^{a-2}}{\Gamma(a-1)}, & 0 \leq s \leq t, \\
t \frac{(T-s)^{a-2}}{\Gamma(a-1)}, & 0 \leq t \leq s.
\end{cases}
\]

### 3 Existence and uniqueness results

In this section we state and prove an existence and uniqueness result for the fractional BVP (1)-(2) by using the Banach fixed-point theorem. We study our problem in the space

\[
C_\beta([0, T]; \mathbb{R}) := \left\{ v \in C([0, T]; \mathbb{R}) : D_0^\beta_1 v, D_0^\beta_2 v \in C([0, T]; \mathbb{R}) \right\}
\]

equipped with the norm

\[
\|v\|_\beta := \|v\|_C + \left\| D_0^{\beta_1} v \right\|_C + \left\| D_0^{\beta_2} v \right\|_C,
\]

where \(\|\cdot\|_C\) is the sup norm in \(C([0, T]; \mathbb{R})\).
The following notations, formulae and estimations will be used throughout the paper.

\[
\mathcal{D}_0^\beta \omega_1(t) = - \frac{\nu^\beta t^{1-\beta}}{\Gamma(2-\beta)} \ldots \mathcal{D}_0^\beta \omega_1(t) = 0, \\
\mathcal{D}_0^\beta \omega_2(t) = \frac{\nu^\beta \mu^{2\beta} t^{2-\beta}}{\mu^\beta \Gamma(3-\beta)} - 2 \frac{\mu^{2\beta} t^{2-\beta}}{\Gamma(3-\beta)}, \quad \mathcal{D}_0^\beta \omega_2(t) = -2 \frac{\mu^{2\beta} t^{2-\beta}}{\Gamma(3-\beta)}.
\]

\[
|\omega_0| = \frac{1}{|a_0 + b_0|} := \rho_0, \quad |\omega_1(t)| \leq |\nu^\beta| (|\omega_0| |b_0| + 1) T := \rho_1, \\
|\omega_2(t)| \leq \frac{|b_0| |\mu^{2\beta}| T^2 + |b_0| |\mu^{2\beta}| |T + |\mu^\beta|| + |\mu^{2\beta}| T^2 := \rho_2, \\
\tilde{\rho}_0 = 0, \quad |\mathcal{D}_0^{\beta_1} \omega_1(t)| \leq \frac{|\nu^\beta| T^{1-\beta_1}}{\Gamma(2-\beta_1)} := \tilde{\rho}_1, \quad |\mathcal{D}_0^{\beta_2} \omega_2(t)| \leq \frac{|\mu^{2\beta}| |\nu^\beta| T^{1-\beta_1}}{\mu^\beta \Gamma(2-\beta_1)} + 2 \frac{|\mu^{2\beta}| T^{2-\beta_2}}{\Gamma(3-\beta_1)} := \tilde{\rho}_2,
\]

\[
\Delta_0 := \frac{T^{\alpha-\tau}}{\Gamma(\alpha)} \left( \frac{1}{\alpha-\tau} \right)^{1-\tau} + \sum_{i=0}^{2} \rho_i \left( |b_i| T^{\alpha-\beta_i} \frac{\eta^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i)} + |a_i| \frac{\eta^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i)} \right) \left( \frac{1}{\alpha-\beta_i} \right)^{1-\tau}, \\
\Delta_1 := \frac{l_f T^{\alpha-\beta_1}}{\Gamma(\alpha-\beta_1 + 1)} + \sum_{i=1}^{2} \tilde{\rho}_i \left( |b_i| \frac{l_f T^{\alpha-\beta_1}}{\Gamma(\alpha-\beta_i + 1)} + |a_i| \frac{l_f \eta^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i + 1)} \right), \\
\Delta_2 := \frac{l_f T^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2 + 1)} + \tilde{\rho}_2 \left( |b_2| \frac{l_f T^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2 + 1)} + |a_2| \frac{l_f \eta^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2 + 1)} \right).
\]

**Theorem 7** Assume that

(H₁) The function \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is jointly continuous.

(H₂) There exists a function \( l_f \in L^+ \left([0, T]; \mathbb{R}^+\right) \) with \( \tau \in (0, \alpha - \beta_2) \) such that

\[
|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq l_f(t) (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),
\]

for each \((t, u_1, u_2, u_3), (t, v_1, v_2, v_3) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \).

(H₃) The function \( g_i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is jointly continuous and there exists \( l_{g_i} \in L^1 \left([0, T]; \mathbb{R}^+\right) \) such that

\[
|g_i(t, u) - g_i(t, v)| \leq l_{g_i}(t) |u - v|, \quad i = 0, 1, 2
\]

for each \((t, u), (t, v) \in [0, T] \times \mathbb{R} \).

If

\[
(\Delta_0 + \Delta_1 + \Delta_2) \|l_f\|_{L_\tau} + \sum_{i=0}^{2} \rho_i |\lambda_i| \|l_{g_i}\|_1 + \sum_{i=1}^{2} \tilde{\rho}_i |\lambda_i| \|l_{g_i}\|_1 + \tilde{\rho}_2 |\lambda_2| \|l_{g_1}\|_1 < 1, \quad (7)
\]

then the problem (1)-(2) has a unique solution on \([0, T]\).
Proof. In order to transform the BVP (1)-(2) into a fixed point problem, we consider the operator \( \mathfrak{F} : C_\beta ([0, T]; \mathbb{R}) \to C_\beta ([0, T]; \mathbb{R}) \) which is defined by

\[
(\mathfrak{F} u) (t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ \sum_{i=0}^{2} \omega_i (t) b_i \int_0^T \frac{(T - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ \sum_{i=1}^{2} \omega_i (t) a_i \int_0^\eta \frac{(\eta - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds - \sum_{i=0}^{2} \omega_i^* \lambda_i \int_0^T g_i(s, u(s)) \, ds,
\]

and take its \( \beta_1 \)-th and \( \beta_2 \)-th fractional derivative to get

\[
D_{0+}^{\beta_1} (\mathfrak{F} u) (t) = \int_0^t \frac{(t - s)^{\alpha - \beta_1 - 1}}{\Gamma(\alpha - \beta_1)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ \sum_{i=1}^{2} D_{0+}^{\beta_1} \omega_i (t) b_i \int_0^T \frac{(T - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ \sum_{i=1}^{2} D_{0+}^{\beta_1} \omega_i (t) a_i \int_0^\eta \frac{(\eta - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds - \sum_{i=1}^{2} D_{0+}^{\beta_1} \omega_i (t) \lambda_i \int_0^T g_i(s, u(s)) \, ds,
\]

and

\[
D_{0+}^{\beta_2} (\mathfrak{F} u) (t) = \int_0^t \frac{(t - s)^{\alpha - \beta_2 - 1}}{\Gamma(\alpha - \beta_2)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ D_{0+}^{\beta_2} \omega_2 (t) b_2 \int_0^T \frac{(T - s)^{\alpha - \beta_2 - 1}}{\Gamma(\alpha - \beta_2)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds
\]

\[+ D_{0+}^{\beta_2} \omega_2 (t) a_2 \int_0^\eta \frac{(\eta - s)^{\alpha - \beta_2 - 1}}{\Gamma(\alpha - \beta_2)} f(s, u(s), D_{0+}^{\beta_1} u(s), D_{0+}^{\beta_2} u(s)) \, ds - D_{0+}^{\beta_2} \omega_2 (t) \lambda_2 \int_0^T g_2(s, u(s)) \, ds.
\]

Clearly, due to \( f, g_0, g_1, g_2 \) being jointly continuous, the expressions (8)-(10) are well defined. It is obvious that the fixed point of the operator \( \mathfrak{F} \) is a solution of the problem (1)-(2). To show existence and uniqueness of the
solution \( \mathfrak{D} \cdot \mathfrak{M} \) we use the Banach fixed point theorem. To this end, we show that \( \mathfrak{F} \) is contraction.

\[
|\langle \mathfrak{F} u \rangle (t) - \langle \mathfrak{F} v \rangle (t) |
\leq \frac{t}{0} \int \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \left| f(s, u(s), \mathcal{D}_{0+}^\beta u(s), \mathcal{D}_{0+}^{\beta^2} u(s)) - f(s, v(s), \mathcal{D}_{0+}^\beta v(s), \mathcal{D}_{0+}^{\beta^2} v(s)) \right| ds
\]

\[
+ \sum_{i=0}^2 |\omega_i(t)| |b_i| \int \frac{(T - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} \left| f(s, u(s), \mathcal{D}_{0+}^\beta u(s), \mathcal{D}_{0+}^{\beta^2} u(s)) - f(s, v(s), \mathcal{D}_{0+}^\beta v(s), \mathcal{D}_{0+}^{\beta^2} v(s)) \right| ds
\]

\[
+ \sum_{i=0}^2 |\omega_i(t)| |\lambda_i| \int \left| g_i(s, u(s)) - g_i(s, v(s)) \right| ds
\]

\[
\leq \|f\|_1/T \int \frac{(1 - \tau)}{(\alpha - \tau)}^{\tau - 1} \|u - v\|_\beta
\]

\[
+ \|f\|_1/T \sum_{i=0}^2 \rho_i \left( |b_i| \frac{T^{\alpha - \beta_i - \tau}}{\Gamma(\alpha - \beta_i)} + |a_i| \frac{\rho^{\alpha - \beta_i - \tau}}{\Gamma(\alpha - \beta_i)} \right) \left( \frac{1 - \tau}{(\alpha - \beta_i - \tau)} \right)^{\tau - 1} + \sum_{i=0}^2 \rho_i |\lambda_i| \|f_i\|_1 \|u - v\|_\beta
\]

\[
= \left( \Delta_0 \|f\|_1/T \sum_{i=0}^2 \rho_i |\lambda_i| \|f_i\|_1 \right) \|u - v\|_\beta.
\]  

On the other hand,

\[
|\mathcal{D}_{0+}^\beta (\mathfrak{F} u)(t) - \mathcal{D}_{0+}^\beta (\mathfrak{F} v)(t) |
\leq \frac{t}{0} \int \frac{(t - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} \left| f(s, u(s), \mathcal{D}_{0+}^\beta u(s), \mathcal{D}_{0+}^{\beta^2} u(s)) - f(s, v(s), \mathcal{D}_{0+}^\beta v(s), \mathcal{D}_{0+}^{\beta^2} v(s)) \right| ds
\]

\[
+ \sum_{i=0}^2 |\mathcal{D}_{0+}^\beta \omega_i(t)| |b_i| \int \frac{(T - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} \left| f(s, u(s), \mathcal{D}_{0+}^\beta u(s), \mathcal{D}_{0+}^{\beta^2} u(s)) - f(s, v(s), \mathcal{D}_{0+}^\beta v(s), \mathcal{D}_{0+}^{\beta^2} v(s)) \right| ds
\]

\[
+ \sum_{i=0}^2 |\mathcal{D}_{0+}^\beta \omega_i(t)| |\lambda_i| \int \left| g_i(s, u(s)) - g_i(s, v(s)) \right| ds
\]

\[
\leq \frac{T^{\alpha - \beta_i - \tau}}{\Gamma(\alpha - \beta_i)} \left( \frac{1 - \tau}{\alpha - \beta_i - \tau} \right)^{\tau - 1} \|f\|_1/T \|u - v\|_\beta
\]

\[
+ \sum_{i=0}^2 \rho_i \left( |b_i| \frac{\rho^{\alpha - \beta_i - \tau}}{\Gamma(\alpha - \beta_i)} + |a_i| \frac{\rho^{\alpha - \beta_i - \tau}}{\Gamma(\alpha - \beta_i)} \right) \left( \frac{1 - \tau}{\alpha - \beta_i - \tau} \right)^{\tau - 1} + |\lambda_i| \|f_i\|_1 \|u - v\|_\beta
\]

\[
= \left( \Delta_1 \|f\|_1/T \sum_{i=0}^2 \rho_i |\lambda_i| \|f_i\|_1 \right) \|u - v\|_\beta.
\]  

(11)
Similarly
\[
\left| D_0^{\alpha_2} (\tilde{f}u) (t) - D_0^{\alpha_2} (\tilde{f}v) (t) \right|
\leq \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha - \beta_2)} \left| f(s, u (s), D_0^{\alpha_1} u(s), D_0^{\alpha_2} u(s)) - f(s, v (s), D_0^{\alpha_1} v(s), D_0^{\alpha_2} v(s)) \right| ds
\]
\[+ \left| D_0^{\alpha_2} \omega_2 (t) \right| |b_2| \int_0^{T} \frac{(T-s)^{\alpha_2-1}}{\Gamma(\alpha - \beta_2)} \left| f(s, u (s), D_0^{\alpha_1} u(s), D_0^{\alpha_2} u(s)) - f(s, v (s), D_0^{\alpha_1} v(s), D_0^{\alpha_2} v(s)) \right| ds
\]
\[+ \left| D_0^{\alpha_2} \omega_2 (t) \right| |a_2| \int_0^{T} \frac{(T-s)^{\alpha_2-1}}{\Gamma(\alpha - \beta_2)} \left| f(s, u (s), D_0^{\alpha_1} u(s), D_0^{\alpha_2} u(s)) - f(s, v (s), D_0^{\alpha_1} v(s), D_0^{\alpha_2} v(s)) \right| ds
\]
\[+ \left| D_0^{\alpha_2} \omega_2 (t) \right| |\lambda_2| \int_0^{T} |g_2(s, u (s)) - g_2(s, v (s))| ds
\]
\[\leq \frac{T^{\alpha_2-\gamma}}{\Gamma(\alpha - \beta_2)} \left( \frac{1 - \tau}{\alpha - \beta_2 - \tau} \right)^{1-\tau} \left| f \right|_{1/\tau} \left\| u - v \right\|_{\beta}
\]
\[+ \tilde{\rho}_2 \left( |b_2| \frac{T^{\alpha_2-\gamma}}{\Gamma(\alpha - \beta_2)} + |a_2| \frac{T^{\alpha_2-\gamma}}{\Gamma(\alpha - \beta_2)} \right) \left( 1 - \frac{\alpha - \beta_2 - \tau}{\alpha - \beta_2} \right)^{1-\tau} \left| f \right|_{1/\tau} + \tilde{\rho}_2 |\lambda_2| \left\| g_2 \right\|_1 \left\| u - v \right\|_{\beta}
\]
\[= \left( \Delta_2 \left\| f \right\|_{1/\tau} + \tilde{\rho}_2 |\lambda_2| \left\| g_2 \right\|_1 \right) \left\| u - v \right\|_{\beta}.
\] (13)

Here, in estimations (11) - (13), we used the Hölder inequality
\[
\int_0^t \left| f (s) (t-s)^{\alpha-m-1} \right| ds \leq \left( \int_0^t \left| f (s) \right|^\frac{\tau}{\gamma} ds \right)^{1-\tau} \left( \int_0^t (t-s)^{\alpha-m-1} \frac{1}{\alpha-m-\tau} ds \right)^{\frac{\gamma}{\gamma-\tau}}
\]
\[= \left\| f \right\|_{L^{1/\tau}} \left( \frac{1 - \tau}{\alpha - m - \tau} \right)^{1-\tau} t^{\alpha-m-\tau}, \text{ if } 0 < \gamma < \alpha - m.
\]

From (11) - (13), it follows that
\[
\left\| (\tilde{f}u) - (\tilde{f}v) \right\|_{\beta} \leq \left( \Delta_0 + \Delta_1 + \Delta_2 \right) \left\| f \right\|_{1/\tau} + \sum_{i=0}^2 \rho_1 |\lambda_i| \left\| g_i \right\|_1 + \sum_{i=1}^2 \rho_1 |\lambda_i| \left\| g_i \right\|_1 + \tilde{\rho}_2 |\lambda_2| \left\| g_2 \right\|_1 \left\| u - v \right\|_{\beta}
\]

Consequently by (7), \( \tilde{f} \) is a contraction mapping. As a consequence of the Banach fixed point theorem, we deduce that \( \tilde{f} \) has a fixed point which is a solution of the problem (1)-(2). ■
Remark 8 In the assumptions $(H_2)$ if $f$ is a constant then the condition $[7]$ can be replaced by

$$\frac{l_f T^\alpha}{\Gamma(\alpha + 1)} + l_f \sum_{i=0}^{2} \rho_i \left( |b_i| \frac{T^{\alpha - \beta_i}}{\Gamma(\alpha - \beta_i + 1)} + |a_i| \frac{\eta^{\alpha - \beta_i}}{\Gamma(\alpha - \beta_i + 1)} \right) + l_f T^{\alpha - \beta_i} \sum_{i=1}^{2} \tilde{\rho}_i \left( |b_i| \frac{T^{\alpha - \beta_i}}{\Gamma(\alpha - \beta_i + 1)} + |a_i| \frac{\eta^{\alpha - \beta_i}}{\Gamma(\alpha - \beta_i + 1)} \right) + l_f T^{\alpha - \beta_2} \sum_{i=1}^{2} \tilde{\rho}_2 \left( |b_2| \frac{T^{\alpha - \beta_2}}{\Gamma(\alpha - \beta_2 + 1)} + |a_2| \frac{\eta^{\alpha - \beta_2}}{\Gamma(\alpha - \beta_2 + 1)} \right) + \sum_{i=0}^{2} \rho_i |\lambda_i| \|l_{g_i}\|_1 + \sum_{i=1}^{2} \tilde{\rho}_i |\lambda_i| \|l_{g_i}\|_1 + \tilde{\rho}_2 |\lambda_2| \|l_{g_2}\|_1 < 1.$$  

4 Existence results

To prove the existence of solutions for BVP (1)-(2), we recall the following known nonlinear alternative.

Theorem 9 (Nonlinear alternative) Let $X$ be a Banach space, let $B$ be a closed, convex subset of $X$, let $W$ be an open subset of $B$ and $0 \in W$. Suppose that $F : W \rightarrow B$ is a continuous and compact map. Then either (a) $F$ has a fixed point in $W$, or (b) there exist an $x \in \partial W$ (the boundary of $W$) and $\lambda \in (0,1)$ with $x = \lambda F(x)$.

Theorem 10 Assume that

$(H_1)$ there exist non-decreasing functions $\varphi : [0,\infty) \times [0,\infty) \times [0,\infty) \rightarrow [0,\infty)$, $\psi_i : [0,\infty) \rightarrow [0,\infty)$ and functions $l_f \in L^\infty([0,T],\mathbb{R}^+)$, $l_{g_i} \in L^1([0,T],\mathbb{R}^+)$ with $\tau \in (1,\min(\alpha - \beta_2))$ such that

$$|f(t,u,v,w)| \leq l_f(t) \varphi(|u| + |v| + |w|),$$
$$|g(t,u)| \leq l_{g_i}(t) \psi_i(|u|)$$

$i = 0,1,2$ for all $t \in [0,T]$ and $u,v,w \in \mathbb{R}$;

$(H_5)$ there exists a constant $K > 0$ such that

$$\varphi(K) \|l_f\|_{1/\tau} (\Delta_0 + \Delta_1 + \Delta_2) + \sum_{i=0}^{2} (\rho_i + \tilde{\rho}_i + \tilde{\rho}_2) |\lambda_i| \psi_i(K) \|l_{g_i}\|_1 > 1.$$  

Then the problem (1)-(2) has at least one solution on $[0,T]$.

Proof. Let $B_r := \left\{ u \in C_\beta([0,T];\mathbb{R}) : \|u\|_\beta \leq r \right\}$.

Step 1: We show that the operator $\mathfrak{H} : C_\beta([0,T];\mathbb{R}) \rightarrow C_\beta([0,T];\mathbb{R})$ defined by (5) maps $B_r$ into bounded set.
For each $u \in B_r$, we have

$$
|\left(\mathcal{S}u\right)(t)| \leq \frac{\varphi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s)| \, ds \\
+ \varphi(r) \sum_{i=0}^2 \rho_i |b_i| \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^T (T-s)^{\alpha-\beta_i-1} |f(s)| \, ds \\
+ \varphi(r) \sum_{i=1}^2 \rho_i |a_i| \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^\eta (\eta-s)^{\alpha-\beta_i-1} |f(s)| \, ds \\
+ \sum_{i=0}^2 \rho_i |\lambda_i| \psi_i(r) \int_0^T |g_i(s)| \, ds.
$$

By the H"older inequality, we have

$$
|\left(\mathcal{S}u\right)(t)| \leq \varphi(r) \|f\|_{1/\tau} \left( \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha - \beta_1)} \left( \frac{1-\tau}{\alpha - \beta_1 - \tau} \right)^{1-\tau} + \sum_{i=0}^2 \rho_i |b_i| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha - \beta_i)} \left( \frac{1-\tau}{\alpha - \beta_i - \tau} \right)^{1-\tau} \\
+ \sum_{i=1}^2 \rho_i |a_i| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha - \beta_i)} \left( \frac{1-\tau}{\alpha - \beta_i - \tau} \right)^{1-\tau} \right) + \sum_{i=0}^2 \rho_i |\lambda_i| \psi_i(r) \|g_i\|_1 \\
= \varphi(r) \|f\|_{1/\tau} \Delta_0 + \sum_{i=0}^2 \rho_i |\lambda_i| \psi_i(r) \|g_i\|_1.
$$

In a similar manner,

$$
|\left(\mathcal{D}_{0+}^\beta \left(\mathcal{S}u\right)\right)(t)| \leq \varphi(r) \|f\|_{1/\tau} \left( \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha - \beta_1)} \left( \frac{1-\tau}{\alpha - \beta_1 - \tau} \right)^{1-\tau} + \sum_{i=1}^2 \rho_i |b_i| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha - \beta_i)} \left( \frac{1-\tau}{\alpha - \beta_i - \tau} \right)^{1-\tau} \\
+ \sum_{i=1}^2 \rho_i |a_i| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha - \beta_i)} \left( \frac{1-\tau}{\alpha - \beta_i - \tau} \right)^{1-\tau} \right) + \sum_{i=0}^2 \rho_i |\lambda_i| \psi_i(r) \|g_i\|_1 \\
= \varphi(r) \|f\|_{1/\tau} \Delta_1 + \sum_{i=1}^2 \rho_i |\lambda_i| \psi_i(r) \|g_i\|_1,
$$

and

$$
|\left(\mathcal{D}_{0+}^{\beta_2} \left(\mathcal{S}u\right)\right)(t)| \leq \varphi(r) \|f\|_{1/\tau} \left( \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha - \beta_2)} \left( \frac{1-\tau}{\alpha - \beta_2 - \tau} \right)^{1-\tau} + \tilde{\rho}_2 \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha - \beta_2)} \left( \frac{1-\tau}{\alpha - \beta_2 - \tau} \right)^{1-\tau} \\
+ \tilde{\rho}_2 \frac{\eta^{\alpha-\beta_2-\tau}}{\Gamma(\alpha - \beta_2)} \left( \frac{1-\tau}{\alpha - \beta_2 - \tau} \right)^{1-\tau} \right) + \tilde{\rho}_2 |\lambda_2| \psi_2(r) \|g_2\|_1 \\
= \varphi(r) \|f\|_{1/\tau} \Delta_2 + \tilde{\rho}_2 |\lambda_2| \psi_2(r) \|g_2\|_1.
$$

Thus

$$
|\left(\mathcal{S}u\right)|_\beta \leq \varphi(r) \|f\|_{1/\tau} (\Delta_0 + \Delta_1 + \Delta_2) + \sum_{i=0}^2 (\rho_i + \tilde{\rho}_i + \tilde{\rho}_i) |\lambda_i| \psi_i(r) \|g_i\|_1.
$$
Step 2: The families \( \{ \tilde{\mathcal{G}}u : u \in B_r \} \), \( \{ \mathcal{D}_{0+}^{\beta_1} (\tilde{\mathcal{G}}u) : u \in B_r \} \), \( \{ \mathcal{D}_{0+}^{\beta_2} (\tilde{\mathcal{G}}u) : u \in B_r \} \) are equicontinuous. Because of continuity of \( \omega_i (t) \) and assumption (H), we have

\[
| (\tilde{\mathcal{G}}u)(t_2) - (\tilde{\mathcal{G}}u)(t_1) | \leq \frac{1}{\Gamma(\alpha)} \varphi (r) \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} l_f (s) ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \varphi (r) \int_0^{t_1} \left( (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} \right) l_f (s) ds
\]

\[
+ \varphi (r) \sum_{i=0}^{2} | \omega_i (t_2) - \omega_i (t_1) | \| \beta_i \| \int_0^{T} \frac{(T - s)^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)} l_f (s) ds
\]

\[
+ \varphi (r) \sum_{i=0}^{2} | \omega_i (t_2) - \omega_i (t_1) | \| \lambda_i \| \psi_i (r) \| l_{g_i} \|_1
\]

\[
\to 0 \quad \text{as} \quad t_2 \to t_1.
\]

Therefore, \( \{ \tilde{\mathcal{G}}u : u \in B_r \} \) is equicontinuous. Similarly, we may prove that \( \{ \mathcal{D}_{0+}^{\beta_1} (\tilde{\mathcal{G}}u) : u \in B_r \} \) and \( \{ \mathcal{D}_{0+}^{\beta_2} (\tilde{\mathcal{G}}u) : u \in B_r \} \) are equicontinuous.

Hence, by the Arzela–Ascoli theorem, the sets \( \{ \tilde{\mathcal{G}}u : u \in B_r \} \), \( \{ \mathcal{D}_{0+}^{\beta_1} (\tilde{\mathcal{G}}u) : u \in B_r \} \), \( \{ \mathcal{D}_{0+}^{\beta_2} (\tilde{\mathcal{G}}u) : u \in B_r \} \) are relatively compact in \( C ([0, T] ; \mathbb{R}) \). Therefore, \( \tilde{\mathcal{G}}(B_r) \) is a relatively compact subset of \( C_{\beta} ([0, T] ; \mathbb{R}) \). Consequently, the operator \( \tilde{\mathcal{G}} \) is compact.

Step 3: \( \tilde{\mathcal{G}} \) has a fixed in \( \overline{W} = \left\{ u \in C_{\beta} ([0, T] ; \mathbb{R}) : \| u \|_{\beta} < K \right\} \).

We let \( u = \lambda (\tilde{\mathcal{G}}u) \) for \( 0 < \lambda < 1 \). Then for each \( t \in [0, T] \),

\[
\| u \|_{\beta} = \| \lambda (\tilde{\mathcal{G}}u) \|_{\beta} \leq \varphi \left( \| u \|_{\beta} \right) \| l_f \|_{1/\tau} \left( \Delta_0 + \Delta_1 + \Delta_2 \right) + \sum_{i=0}^{2} (\rho_i + \tilde{\rho}_i + \tilde{\rho}_i) | \lambda_i \| \psi_i \left( \| u \|_{\beta} \right) \| l_{g_i} \|_1.
\]

In other words,

\[
\varphi \left( \| u \|_{\beta} \right) \| l_f \|_{1/\tau} \left( \Delta_0 + \Delta_1 + \Delta_2 \right) + \sum_{i=0}^{2} (\rho_i + \tilde{\rho}_i + \tilde{\rho}_i) | \lambda_i \| \psi_i \left( \| u \|_{\beta} \right) \| l_{g_i} \|_1 \leq 1.
\]

According to the assumptions, we know that there exists \( K > 0 \) such that \( K \neq \| u \|_{\beta} \). The operator \( \tilde{\mathcal{G}} : \overline{W} \to C_{\beta} ([0, T] ; \mathbb{R}) \) is continuous and compact. From Theorem 9 we can deduce that \( \tilde{\mathcal{G}} \) has a fixed point in \( \overline{W} \).

**Remark 11** Notice that analogues of Theorem 7 and 10 for the case \( f(t, u, v, w) = f(t, u) \) were considered in \( [12] \). Thus our results are generalization of \( [12] \).

**Remark 12** Since the number \( (\alpha - \beta_2 - 1) \) can be negative, the function \( (T - s)^{\alpha - \beta_2 - 1} \notin L^\infty ([0, T] ; \mathbb{R}) \). That is why in Theorem 7 and 10 it is assumed that \( l_f \in L^\tau, \tau \in (0, \min(1, \alpha - \beta_2)) \).
5 Examples

Example 1. Consider the following boundary value problem of fractional differential equation:

\[
\begin{cases}
D_0^{5/2} u(t) = l_f \left( \frac{|u(t)|}{1 + |u(t)|} + \frac{|D_0^{1/2} u(t)|}{1 + |D_0^{1/2} u(t)|} + \tan^{-1}\left( D_0^{3/2} u(t) \right) \right), & 0 \leq t \leq 1, \\
u(0) + u(1) = \int_0^1 u(s) \frac{1}{(1 + s)^2} \, ds, \\
D_0^{1/2} u\left(\frac{1}{4}\right) + D_0^{1/2} u(1) = \frac{1}{2} \int_0^1 \left( \frac{e^s u(s)}{1 + 2e^s} + \frac{1}{2} \right) \, ds, \\
D_0^{3/2} u\left(\frac{1}{4}\right) + D_0^{3/2} u(1) = \frac{1}{3} \int_0^1 \left( \frac{u(s)}{1 + e^s} + \frac{3}{4} \right) \, ds.
\end{cases}
\]

(14)

Here

\[\alpha = \frac{5}{2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{2}, T = 1, a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = 1,\]
\[\eta = \frac{1}{10}, \lambda_0 = 1, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_0 = \lambda_1 = \lambda_2 = 1,\]

and

\[f(t, u, v, w) := \frac{u}{1 + u} + \frac{v}{1 + v} + \tan^{-1}(w),\]
\[g_0(t, u) := \frac{u}{(1 + t)^{2}}, \quad g_1(t, u) := \frac{e^t u}{1 + 2e^t} + \frac{1}{2},\]
\[g_2(t, u) := \frac{u}{1 + e^t} + \frac{3}{4}.\]

Since \(1.77 < \Gamma\left(\frac{5}{2}\right) < 1.78; 0.88 < \Gamma\left(\frac{3}{2}\right) < 0.89; 1.32 < \Gamma\left(\frac{3}{4}\right) < 1.33\) and \(3.32 < \Gamma\left(\frac{5}{4}\right) < 3.33\) with simple calculations we show that

\[\Delta_0 = 2.34, \quad \Delta_1 = 0.19, \quad \Delta_2 = 0.15,\]
\[\rho_0 = 0.5, \quad \rho_1 = 1.01, \quad \rho_2 = 1.2,\]
\[\tilde{\rho}_0 = 0, \quad \tilde{\rho}_1 = 0.76, \quad \tilde{\rho}_2 = 0.9,\]
\[\hat{\rho}_0 = \hat{\rho}_1 = 0, \quad \hat{\rho}_2 = 0.51\]

Furthermore,

\[(\Delta_0 + \Delta_1 + \Delta_2) \|f\|_{1/\tau} + \sum_{i=0}^{2} \rho_i |\lambda_i| \|l_{y_i}\|_1 + \sum_{i=1}^{2} \tilde{\rho}_i |\lambda_i| \|l_{y_i}\|_1 + \hat{\rho}_2 |\lambda_2| \|l_{y_2}\|_1 < 2.7l_f + 0.75 < 1.\]

Therefore, we can choose

\[l_f < \frac{0.25}{2.7}.\]

Thus, all the assumptions of Theorem 7 are satisfied. Hence, the problem (14) has a unique solution on \([0, 1]\).
Example 2. Consider the following boundary value problem of fractional differential equation:

\[
\begin{aligned}
\mathcal{D}^{5/2}_{0^+} u(t) &= \frac{|u(t)|^3}{9(|u(t)|^3 + 3)} + \frac{\sin \mathcal{D}^{1/2}_{0^+} u(t)}{9|\sin \mathcal{D}^{1/2}_{0^+} u(t)| + 1} + \frac{1}{12}, \quad t \in [0, 1], \\
\mathcal{D}^{1/2}_{0^+} \left( \frac{u(s)}{10} \right) + \mathcal{D}^{1/2}_{0^+} u(1) &= \frac{1}{2} \int_0^1 \frac{e^s u(s)}{3(1 + e^s)^2} ds, \\
\mathcal{D}^{3/2}_{0^+} \left( \frac{u(s)}{10} \right) + \mathcal{D}^{3/2}_{0^+} u(1) &= \frac{1}{3} \int_0^1 \frac{u(s)}{3(1 + e^s)^2} ds,
\end{aligned}
\]

where \( f \) is given by

\[
f(t, u, v, w) = \frac{|u|^3}{10(|u|^3 + 3)} + \frac{|\sin v|}{9|\sin v| + 1} + \frac{1}{12}.
\]

We have

\[
|f(t, u, v, w)| \leq \frac{|u|^3}{9(|u|^3 + 3)} + \frac{|\sin v|}{9|\sin v| + 1} + \frac{1}{12} \leq \frac{11}{36}, \quad u \in \mathbb{R}.
\]

Thus

\[
\|f\| \leq \frac{11}{36} = l_f(t) \varphi(K), \quad \text{with} \quad l_f(t) = \frac{1}{3}, \varphi(K) = \frac{11}{12}.
\]

Moreover

\[
\alpha = \frac{5}{2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{2}, T = 1, a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = 1,
\]

\[
\eta = \frac{1}{10}, \quad \lambda_0 = 1, \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{3}, \quad l_{g_0} = l_{g_1} = l_{g_2} = \frac{1}{3},
\]

\[
\Delta_0 = 2.34, \quad \Delta_1 = 0.19, \quad \Delta_2 = 0.15, \\
\rho_0 = 0.5, \quad \rho_1 = 1.01, \quad \rho_2 = 1.2, \\
\tilde{\rho}_0 = 0, \quad \tilde{\rho}_1 = 0.76, \quad \tilde{\rho}_2 = 0.9, \\
\hat{\rho}_0 = \hat{\rho}_1 = 0, \quad \hat{\rho}_2 = 0.51,
\]

and

\[
g_0(t, u) := \frac{u}{3(1 + t)^2}, \quad g_1(t, u) := \frac{e^t u}{3(1 + e^t)^2}, \quad g_2(t, u) := \frac{u}{3(1 + e^t)^2}, \quad \psi_i(K) = K.
\]

From the condition

\[
\frac{K}{\varphi(K) \|f\|_{1/2} (\Delta_0 + \Delta_1 + \Delta_2) + \sum_{i=0}^{2} (\rho_i + \tilde{\rho}_i + \hat{\rho}_i) |\lambda_i| \psi_i(K) \|g_i\|_1} > 1
\]

we find that

\[
K > 9.8.
\]

Thus, all the conditions of Theorem \ref{thm:main} are satisfied. So, there exists at least one solution of problem \ref{eq:example2} on \([0, 1]\).
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