THE PLURICLOSED FLOW ON NILMANIFOLDS AND TAMED SYMPLECTIC FORMS

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Abstract. We study evolution of (strong Kähler with torsion) SKT structures via the pluriclosed flow on complex nilmanifolds, i.e. on compact quotients of simply connected nilpotent Lie groups by discrete subgroups endowed with an invariant complex structure. Adapting to our case the techniques introduced by Jorge Lauret for studying Ricci flow on homogeneous spaces we show that for SKT Lie algebras the pluriclosed flow is equivalent to a bracket flow and we prove a long time existence result in the nilpotent case. Finally, we introduce a natural flow for evolving tamed symplectic forms on a complex manifold, by considering evolution of symplectic forms via the flow induced by the Bismut Ricci form.

1. Introduction

Let $M$ be a Hermitian manifold of complex dimension $n$. If $M$ is non-Kähler, the Levi-Civita connection is not compatible with the induced $U(n)$-structure and its role is often replaced by other connections having torsion but preserving the Hermitian structure [9]. Although there is no a canonical choice of a Hermitian connection, the Chern and the Bismut connections seem to have a central role. The Chern connection is defined as the unique Hermitian connection $\nabla^C$ for which the $(1,1)$-component of the torsion tensor vanishes, while the Bismut connection has skew-symmetric torsion [2]. Streets and Tian pointed out in [19] that the operator $g \mapsto -S(g)$ defined from the space of Hermitian metrics on a complex manifold $(M, J)$ by the formula

$$S(g)_{ij} = g^{kr} R^C_{i j r k}$$

is strongly elliptic, where $R^C = [\nabla^C, \nabla^C] - \nabla^C_{[\cdot, \cdot]}$ is the curvature of $\nabla^C$. Consequently the standard theory of parabolic equations ensures that the Ricci-type flow

$$\frac{dt}{g} g = -S(g) + L(g)$$

$$g(0) = g_0$$

has a unique maximal solution defined in an interval $[0, T)$, where $L$ is an arbitrary first order differential operator. Moreover, if we choose $L(g)$ to be a suitable
operator $Q(g)$ depending quadratically on the torsion of $\nabla^C$, the flow

\begin{equation}
\begin{aligned}
\frac{dg}{dt} &= -S(g) + Q(g) \\
g(0) &= g_0
\end{aligned}
\end{equation}

is the gradient flow of a functional $F = F(g)$. The flow (1.2) is called the Hermitian curvature flow and preserves both the Kähler and the SKT condition (see [18]). We recall that a Hermitian structure $(J, g)$ is called SKT (strong Kähler with torsion) or pluriclosed if its fundamental form $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ is $\partial\overline{\partial}$-closed. In the SKT case, (1.2) is equivalent to the so-called pluriclosed flow

\begin{equation}
\begin{aligned}
\frac{d}{dt} \omega &= -(\rho^B)^{1,1} \\
\omega(0) &= \omega_0
\end{aligned}
\end{equation}

acting on the space of $J$-compatible non-degenerate real 2-forms, where $(\rho^B)^{1,1}$ denotes the $(1,1)$-part of the Ricci form $\rho^B$ of the Bismut connection (i.e. the $(1,1)$-part of the so-called Bismut Ricci form). In the terminology of [18] an SKT structure $\omega$ is called static if it satisfies the Einstein-type equation

\begin{equation}
r \omega = (\rho^B)^{1,1}
\end{equation}

where $r \in \mathbb{R}$. Static SKT structures seem to be very rare in complex non-Kähler manifolds, since if $\omega$ is static with $r \neq 0$, then $\Omega = \frac{1}{r} \rho^B$ is a symplectic form taming $J$ (i.e. a Hermitian-symplectic structure in the terminology of [18]). If a complex surface admits a Hermitian-symplectic structure $\Omega$, then by [16] the Hermitian metric associated to $\Omega$ is strongly Gauduchon. Indeed, by [16, Lemma 3.2] a complex manifold $(M, J)$ of complex dimension $n$ carries a strongly Gauduchon metric $g$ if and only if there exists a real $d$-closed $C^\infty$ $(2n - 2)$-form $\Omega$ on $M$ such that its component of type $(n-1, n-1)$ is positive on $M$.

It is known that every compact complex surface admitting a Hermitian-symplectic structure is actually Kähler (see [18, 15]) and it is still an open problem to find an example of a compact Hermitian-symplectic manifold not admitting Kähler structures. Some negative results on this context are proved in [6] for nilmanifolds and in [5] for 4-dimensional Lie algebras. In particular, from the results of [6] it turns out that a nilmanifold endowed with an invariant complex structure does not admit Hermitian-symplectic structures unless it is a torus. This last result together with a theorem of [4] implies that complex nilmanifolds cannot admit static SKT structures unless they are tori.

The present paper is divided in two parts. In the first one we investigate the behaviour of solutions of (1.3) on Lie algebras. In particular we prove the following

**Theorem 1.1.** Let $(M = G/\Gamma, J, \omega_0)$ be a nilmanifold endowed with an invariant SKT structure. Then the solution $\omega(t)$ to the pluriclosed flow (1.3) is defined for every $t \in (-\epsilon, \infty)$, where $\epsilon$ is a suitable positive real number.

A key ingredient in the proof of this theorem is a trick introduced by Lauret in [13] for studying the Ricci flow on homogeneous spaces evolving the Lie brackets instead of the Riemannian metrics.

In the second part of the paper we introduce a natural flow for evolving taming symplectic forms on a complex manifold $(M, J)$. Given a symplectic form $\Omega_0$ taming...
we consider the flow
\[
\begin{aligned}
\frac{d}{dt} \Omega &= -\rho^B(\omega), \\
\Omega(0) &= \Omega_0
\end{aligned}
\]
where \(\rho^B(\omega)\) is the Bismut Ricci form of the Hermitian metric associated to \(\omega = \Omega^{1,1}\). For such a flow we prove a short time existence result and a stability theorem involving Kähler-Einstein metrics.

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2. Preliminaries on SKT metrics

Let \((M, g, J)\) be a Hermitian manifold with fundamental form \(\omega\). The form \(\omega\) and the Riemannian metric \(g\) are related
\[
\omega(X, Y) = g(JX, Y).
\]
We denote by \(\nabla^B\) the Bismut connection of \((g, J)\). This connection was introduced by Bismut in \([2]\) and it is the unique Hermitian connection (i.e. \(\nabla^B J = 0, \nabla^B g = 0\)) such that
\[
(2.1) \quad c(X, Y, Z) := g(X, T^B(Y, Z))
\]
is a 3-form, where
\[
T^B(X, Y) = \nabla^B_X Y - \nabla^B_Y X - [X, Y]
\]
denotes the torsion of \(\nabla^B\). This connection induces the curvature tensor
\[
R^B(X, Y) := [\nabla^B_X, \nabla^B_Y] - \nabla^B_{[X,Y]},
\]
the Ricci tensor and the Ricci form given respectively by
\[
\text{ric}^B(X, Y) = g^{kr} R^B(e_k, X, Y, e_r), \quad \rho^B(X, Y) = \frac{1}{2} g^{kr} g(R^B(X, Y)e_k, J_e_r),
\]
with \(\{e_k\}\) an arbitrary local frame. In complex notation we can alternatively write
\[
\text{ric}^B(X, Y) = -ig^{kr} R^B(Z_k, X, Y, Z_r), \quad \rho^B(X, Y) = -ig^{kr} g(R^B(X, Y)Z_k, Z_r),
\]
where \(\{Z_k = \frac{1}{2}(e_k - iJ e_k)\}\) is a local \((1,0)\)-frame and \(Z_T = \frac{1}{2}(e_k + iJ e_k)\).

For a Hermitian manifold \((M, J)\) the Ricci tensor of \(\nabla^B\) and the usual Ricci tensor are related by the following formula
\[
(2.2) \quad \text{ric}^B(X, Y) = \text{ric}^g(X, Y) - \frac{1}{2}(d^* c)(X, Y) - \frac{1}{4} \sum_{i=1}^{2n} g(T^B(X, e_i), T^B(Y, e_i)),
\]
(see \([12]\)), where \(d^*\) is the co-differential operator of \(g\), while \(\rho^B\) is related to the Ricci form \(\rho^C\) of the Chern connection by
\[
(2.3) \quad \rho^B = \rho^C - dd^* \omega.
\]

We recall the following

Definition 2.1. A Hermitian metric \(g\) on a complex manifold \((M, J)\) is called SKT (strong Kähler with torsion) or pluriclosed if the torsion 3-form \(c\) is closed or, equivalently, if its associated fundamental form \(\omega\) satisfies \(\partial \bar{\partial} \omega = 0\).
For a complex surface, an SKT metric is called standard in the terminology of Gauduchon [10]. In the conformal class of any given Hermitian metric on a compact complex manifold there always exists a standard metric. But this property is not anymore true in higher dimensions for SKT metrics.

For real Lie algebras admitting left-invariant SKT metrics there are some classification results in dimensions four, six and eight. More precisely, 6-dimensional (resp. 8-dimensional) SKT nilpotent Lie algebras have been classified in [7] (resp. in [6] and for a particular class in [17]) and a classification of 4-dimensional SKT solvable Lie algebras has been obtained in [14].

General results are known for nilmanifolds, i.e. for compact quotients of simply connected nilpotent Lie groups \( G \) by discrete subgroups \( \Gamma \). Indeed, in [11] it has been shown that if \( (M = G/\Gamma, J) \) is a nilmanifold (not a torus) endowed with an invariant complex structure \( J \) and an SKT metric \( g \) compatible with \( J \), then the nilpotent Lie group \( G \) must be 2-step nilpotent and \( M \) is a total space of a principal holomorphic torus bundle over a torus.

3. Pluriclosed Flow on Lie algebras

Let \( G \) be a Lie group with a left-invariant SKT structure \( (g_0, J) \) and let \( \Gamma \) be a co-compact lattice in \( G \). We are interested in studying solutions to (1.3) on the compact manifold \( M = G/\Gamma \) endowed with the invariant SKT structure induced by \( (g_0, J) \). Since the pluriclosed flow (1.3) is invariant by biholomorphisms of the complex manifold \( (M, J) \), when \( \omega_0 \) is invariant, the solution \( \omega(t) \) to (1.3) is invariant for every \( t \). Therefore the PDE system (1.3) on \( M = G/\Gamma \) is equivalent to an ODE system on the Lie algebra \( g \) of \( G \). In general neither expect that the flow converges nor that its limit is non-degenerate.

Let \( (g, \mu) \) be a Lie algebra endowed with an SKT structure \( (g, J) \), where \( \mu \) denotes the Lie bracket on \( g \). By an SKT structure on a Lie algebra we means a pair \( (g, J) \), where \( J \) is a complex structure, satisfying the integrability condition

\[
\mu(JX, JY) = J\mu(JX, Y) + J\mu(X, JY) + \mu(X, Y),
\]

for every \( X, Y \in g \) and \( g \) is an inner product compatible with \( J \) and such that \( dc = 0 \), where \( c \) is defined by (2.1). In order to write down a formula for the Bismut Ricci form \( \rho^B \) in this algebraic context, we fix an arbitrary \((1, 0)\) frame \( \{Z_r\} \) of \((g, J)\) with dual frame \( \{\zeta^k\} \). Following the approach of [20] we can write

\[
\rho^B = d\eta,
\]

where \( \eta \) is the real 1-form

\[
\eta(X) = \Im \{ g^{kr} g(\mu(X - iJX, Z_r), Z_k) + ig^{kr} g(\mu(Z_r, Z_k), X) \}.
\]

In complex notation we have

\[
\eta = \eta_a \zeta^a + \eta_\bar{a} \zeta^\bar{a},
\]

where

\[
\eta_a = -ig^{kr} g(\mu(Z_a, Z_r), Z_k) + ig^{kr} g(\mu(Z_r, Z_k), Z_a), \quad \eta_\bar{a} = \eta_a^\bar{a}.
\]

Formula (3.2) can be rewritten in terms of the Lie bracket components \( \mu^k_i \) as

\[
\eta_a = -i\mu^r_a + ig^{kr} \mu^r_i \zeta^a_i, \quad \eta_\bar{a} = \eta_a^\bar{a}.
\]
Therefore, in complex notation we obtain
\[ \rho^B = -i \rho^B_{ij} \zeta^i \otimes \zeta^j - i \rho^B_{hk} \zeta^h \otimes \zeta^k + \frac{i}{2} \rho^B_{ilm} \zeta^i \otimes \zeta^l \otimes \zeta^m, \quad \rho^B_{ij} = g_{ij} R^B_{ij} \Xi, \]
where
\[ \rho^B_{ij} = -i \eta(\rho(Z_i, Z_j)) = -i \mu_a^i \eta_a - i \mu^a_{ij} \eta^a \]
\[ = -\mu^a_{ij} \rho^r + \mu^a_{ij} g^k \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
i.e.,
(3.3) \[ \rho^B_{ij} = -\mu^a_{ij} \mu^r + \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
In the same way
(3.4) \[ \rho^B_{ij} = -\mu^a_{ij} \mu^r + \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
and (3.2) writes as
(3.5) \[ \left\{ \begin{array}{l}
\frac{d}{dt} g_{ij} = -\mu^a_{ij} \mu^r + \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \\
g_{ij}(0) = (g_0)_{ij}.
\end{array} \right. \]

Note that when \((Z_\tau, \xi)\) is a unitary frame we have
(3.6) \[ \rho^B_{ij} = -\mu^a_{ij} \mu^r + \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
(3.7) \[ \rho^B_{ij} = -\mu^a_{ij} \mu^r + \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} \mu^b_{ir} g_{t} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
where the repeated indexes are summed.

If the Lie algebra \(g\) is 2-step nilpotent, i.e. if
\[ \mu(\mu(X, Y), Z) = 0 \]
for every \(X, Y, Z \in g\), then the formulas (3.3) and (3.4) reduce to
(3.8) \[ \rho^B_{ij} = \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
(3.9) \[ \rho^B_{ij} = \mu^a_{ij} g^r \mu_{ik} g_{at} + \mu^a_{ij} g^{k} \mu_{ik} g_{it}, \]
giving the suitable expression
(3.10) \[ \rho^B(X, Y) = -ig^r g(\mu(X, Y), \mu(Z_r, Z_\tau)), \]
for every \(X, Y \in g\).

**Remark 3.1.** We observe that the previous computations hold also in the non-SKT case.

Next we show two examples of SKT Lie algebras in dimension 4 for which the solution \(\omega(t)\) of the pluriclosed flow [1,3] is defined for every \(t \in (-\epsilon, \infty)\), where \(\epsilon\) is a suitable positive real number. The first example is nilpotent and we will show in the next section that this happens for every SKT nilpotent Lie algebra. The second one is solvable and admits a generalized Kähler structure [8, 8].

**Example 3.2.** In dimension 4 the unique nilpotent SKT Lie algebra up to isomorphisms is \(h_3 \oplus \mathbb{R}\), where \(h_3\) is the Lie algebra of the 3-dimensional real Heisenberg Lie group \(H_3(\mathbb{R})\) given by
\[ H_3(\mathbb{R}) = \left\{ \left( \begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right), \quad x, y, z \in \mathbb{R} \right\}. \]
The compact quotient of the corresponding simply-connected $H_3(\mathbb{R}) \times \mathbb{R}$ by the lattice $\Gamma \times \mathbb{Z}$, where $\Gamma$ is the lattice in $H_3(\mathbb{R})$ whose elements are matrices with integer entries, is the so-called Kodaira-Thurston surface. The Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}$ has structure equations $(0, 0, 0, 12)$, where by this notation we mean that there exists a basis of 1-forms \( \{e^i\} \) such that \( de^i = 0, i = 1, 2, 3 \), \( de^4 = e^1 \wedge e^2 \).

Let $J$ be the complex structure on $\mathfrak{g}$ given by

\[
Je_1 = -e_2, \quad Je_3 = -e_4.
\]

Then

\[
Z_1 = \frac{1}{2} (e_1 + ie_2), \quad Z_2 = \frac{1}{2} (e_3 + ie_4)
\]

is a complex basis of type $(1, 0)$ of $(\mathfrak{g}, J)$. Let \( \{\zeta^1, \zeta^2\} \) be its dual frame. Every Hermitian inner product $g$ on $(\mathfrak{g}, J)$ can be written as

\[
g = x \zeta^1 \zeta^1 + y \zeta^2 \zeta^2 + z \zeta^1 \zeta^2 + \overline{z} \zeta^2 \zeta^1.
\]

where $x, y \in \mathbb{R}$, $z \in \mathbb{C}$ satisfy $xy - |z|^2 > 0$ and it is SKT. Since

\[
\mu(Z_1, Z_\overline{1}) = -\frac{1}{2} (Z_2 - Z_\overline{2})
\]

is the only non-vanishing bracket we have

\[
\rho^B = -i \rho^B_{1\overline{1}} \zeta^1\overline{1}
\]

where

\[
\rho^B_{1\overline{1}} = -i \eta ([Z_1, Z_\overline{1}]) = i \frac{1}{2} (\eta_2 - \eta_{\overline{2}}) = - \Im \eta_2.
\]

A direct computation yields

\[
\eta_2 = i \frac{y^2}{2 (xy - |z|^2)}
\]

and

\[
\rho^B_{1\overline{1}} = - \frac{y^2}{2 (xy - |z|^2)}.
\]

Therefore in this case system (3.5) reduces to

\[
\begin{cases}
\dot{x} = \frac{y^2}{2 (xy - |z|^2)} \\
y \equiv y_0, \quad z \equiv z_0 \\
x(0) = x_0,
\end{cases}
\]

and the solution to (4.1) with

\[
\omega_0 = -ix_0 \zeta^1\overline{1} - iy_0 \zeta^2\overline{2} - iz_0 \zeta^1\overline{2} - i\overline{z}_0 \zeta^2\overline{1}
\]

is

\[
\omega(t) = -ix(t) \zeta^1\overline{1} - iy_0 \zeta^2\overline{2} - iz_0 \zeta^1\overline{2} - i\overline{z}_0 \zeta^2\overline{1}
\]

where

\[
x(t) = \frac{1}{y_0} \left( \sqrt{\frac{y_0^2 t + (x_0 y_0 - |z_0|^2)^2}{2}} + |z_0|^2 \right).
\]

For instance if we start from the standard SKT structure

\[
\omega_0 = -i \zeta^1\overline{1} - i \zeta^2\overline{2}
\]
we get
\[ \omega(t) = -i\sqrt{1+t} \bar{\zeta}^T - i\zeta^{2T}. \]

**Example 3.3.** Consider the solvable Lie algebra with structure equations
\[
\begin{align*}
de^1 &= ae^{14} + be^{24}, \\
de^2 &= -be^{14} + ae^{24}, \\
de^3 &= -2ae^{34}, \\
de^4 &= 0,
\end{align*}
\]

endowed with the complex structure given by
\[ Je_1 = e_2, \quad Je_3 = e_4. \]

A compact quotient of the corresponding simply-connected Lie group by a lattice is an Inoue surface of type \( S^0 \) (see [11]). Let \( \{Z_1, Z_2\} \) be the \((1,0)\)-frame
\[ Z_1 = \frac{1}{2}(e_1 - ie_2), \quad Z_2 = \frac{1}{2}(e_3 - ie_4); \]
then a direct computation yields
\[ \mu(Z_1, Z_1) = 0, \quad \mu(Z_1, Z_2) = \lambda Z_1, \quad \mu(Z_2, Z_2) = ai(Z_2 + \bar{Z}_2) \]

where
\[ \lambda = \frac{b + ia}{2}. \]

Consider the \((1,0)\)-coframe
\[ \zeta^1 = e^1 + ie^2, \quad \zeta^2 = e^3 + ie^4 \]
dual to \( \{Z_1, Z_2\} \). Then
\[ d\zeta^1 = -\lambda (\zeta^{12} - \overline{\zeta^{12}}), \quad d\zeta^2 = -ai \zeta^{2\overline{2}}. \]

Let \( g \) be an arbitrary \( J \)-Hermitian metric on \( g \). We can write
\[ g = x \zeta^1 \zeta^1 + y \zeta^2 \overline{\zeta^2} + z\zeta^1 \overline{\zeta^2} + \overline{z} \zeta^2 \zeta^1 \]
where \( x, y \in \mathbb{R}_+, \ z \in \mathbb{C} \) satisfy
\[ xy - |z|^2 > 0. \]

The fundamental form of \( g \) is
\[ \omega = -ix \zeta^1 T - iy \zeta^2 T - iz \zeta^1 T - i\overline{z} \zeta^2 T. \]

Let \( \rho^B = d\eta \) be the Bismut form of \( g \). A direct computation yields
\[ \eta_1 = -\frac{zx}{xy - |z|^2} (i\overline{\lambda} + a) = -\frac{3a + ib}{2} \frac{zx}{xy - |z|^2} \]

and
\[ \eta_2 = i\lambda \frac{(|z|^2 + axy - i|z|^2(\lambda + \overline{\lambda}) + xy(a - i\lambda))}{xy - |z|^2}. \]

In matrix notation we have
\[ (g_\eta) = \begin{pmatrix} x & z \\ \overline{z} & y \end{pmatrix}. \]
and

\[
(g_j^k) = \frac{1}{xy - |z|^2} \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}.
\]

A direct computation yields

\[
\rho^B_{11} = 0, \\
(3.12) \quad \rho^B_{12} = \lambda \theta_1 = \frac{1}{4} (3a^2 + b^2 - 2iab) \frac{xz}{xy - |z|^2}, \\
(3.13) \quad \rho^B_{22} = -\frac{3a^2xy}{xy - |z|^2}.
\]

and the ODEs induced by (3.5) are

\[
\begin{aligned}
&x = \text{const}, \\
&\dot{z} = -\frac{1}{2} (3a^2 + b^2 - 2iab) \frac{xz}{xy - |z|^2}, \\
&\dot{y} = \frac{3a^2xy}{xy - |z|^2}.
\end{aligned}
\]

(3.15)

In particular if we consider as initial SKT structure

\[
\omega_0 = -i\zeta^\mathcal{T}T - i\zeta^\mathcal{F},
\]

then the system (3.15) has solutions

\[
\begin{aligned}
x &= 1 \\
z &= 0 \\
y(t) &= 3a^2t
\end{aligned}
\]

defined for every \(t\) and

\[
\omega(t) = -i\zeta^\mathcal{T}T - i3a^2t\zeta^\mathcal{F}.
\]

4. The pluriclosed flow as bracket flow

We regard the pluriclosed flow (1.3) on Lie algebras as a bracket flow on \(\mathbb{R}^{2n}\) working as in [13]. The idea consists on studying evolution of brackets instead of the Hermitian metrics. We briefly describe the clever trick of [13] adapted to our setting:

Let \((\mathfrak{g}, \mu_0, J, g_0, \omega_0)\) be a almost Hermitian Lie algebra. Then \((\mathfrak{g}, \mu_0, J, \omega_0)\) can be thought as \(\mathbb{R}^{2n}\) equipped with the standard Hermitian structure \((J_0, \omega_0, \langle \cdot, \cdot \rangle)\) and a bracket \(\mu_0\). Consider in this setting the system

\[
\begin{aligned}
&\frac{d}{dt} \omega = - (\rho^B)^{11}(\omega) \\
&\omega(0) = \omega_0
\end{aligned}
\]

(4.1)

where \(\rho^B(\omega)\) is computed with respect to \(\omega\) and \(\mu_0\) using formulae (3.6), i.e.,

\[
(4.2) \quad \rho^B(\omega)(X, Y) = i \sum_{r=1}^{n} \left( g(\mu_0(\mu_0(X, Y), Z_r), Z_r) - g(\mu_0(Z_r, Z_r), \mu_0(X, Y)) \right),
\]

\(g\) is the inner product induced by \((\omega, J_0)\) and \(\{Z_r\}\) is a unitary frame.

Let

\[
V := \Lambda^2(\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}
\]
be the space of skew-symmetric 2-forms on $\mathbb{R}^{2n}$ taking values in $\mathbb{R}^{2n}$ and let
\begin{equation}
\mathcal{A} := \{ \mu \in V : \mu \text{ satisfies the Jacobi identity and } N_\mu = 0 \}
\end{equation}
where $N_\mu$ is the Nijenhuis tensor
\[ N_\mu(X, Y) := \mu(J_0 X, J_0 Y) - J_0 \mu(J_0 X, Y) - J_0 \mu(X, J_0 Y) - \mu(X, Y). \]
$\mathcal{A}$ can be regarded as the space of all $2n$-dimensional Lie algebras equipped with a complex structure. Given a form $\omega \in \Lambda^2(\mathbb{R}^{2n})^*$ compatible with $J_0$, there exists a (non-unique) map $h \in \text{GL}(n, \mathbb{C})$ such that
\begin{equation}
\omega(\cdot, \cdot) = \omega_0(h \cdot \cdot).
\end{equation}
For every $h \in \text{GL}(n, \mathbb{C})$ satisfying (4.4), we can define a new bracket $\mu \in \mathcal{A}$ using the natural relation
\[ \mu := h \cdot \mu_0, \]
where
\[ h \cdot \mu_0(X, Y) = h \mu_0 \left( h^{-1} X, h^{-1} Y \right). \]
Since $h$ belongs to $\text{GL}(n, \mathbb{C})$, $h \cdot \mu \in \mathcal{A}$. Moreover, every $\mu \in \mathcal{A}$ induces a 2-form $\rho_\mu^B \in \Lambda^2(\mathbb{R}^{2n})^*$ defined according to (4.5) as
\begin{equation}
\rho_\mu^B(X, Y) = i \sum_{r=1}^{n} \left( \mu(X, Y), Z_r \right) \left( \mu(Z_r, Z_r), \mu(X, Y) \right),
\end{equation}
where $\{Z_r\}$ be the standard unitary basis on $(\mathbb{R}^{2n}, J_0)$. We denote by $P_\mu$ the endomorphism corresponding to $(\rho_\mu^B)^{1,1}$ and $\mu$ via $\omega_0$, i.e.
\begin{equation}
\omega_0(P_\mu(X), Y) = (\rho_\mu^B)^{1,1}(X, Y).
\end{equation}
By definition $P_\mu$ is $\omega_0$-symmetric and it commutes with $J_0$.
In the same way we denote by $P(\omega)$ the endomorphism corresponding $(\rho^B)^{1,1}(\omega)$ via $\omega_0$, i.e.
\[ \omega(P(\omega)X, Y) = (\rho^B)^{1,1}(\omega). \]
Note that
\[ P_{\mu_0} = P(\omega_0), \]
\[ \rho^B(\omega_0) = \rho_\mu^B. \]
The following lemma describes as the two endomorphisms $P_\mu$ and $P(\omega)$ are related and it can be deduced from the equivalence between the Hermitian Lie algebras $(\mu_0, \omega, J_0)$ and $(\mu, \omega, J_0)$.

**Lemma 4.1.** The following formula holds
\begin{equation}
P_\mu = h P(\omega) h^{-1}.
\end{equation}
We further consider the bracket flow
\begin{equation}
\begin{cases}
\frac{d}{dt} \mu = \frac{1}{2} \delta_\mu(P_\mu) \\
\mu(0) = \mu_0
\end{cases}
\end{equation}
where $\delta_\mu : \mathfrak{gl}_n(\mathbb{C}) \to V_{2n}$ is defined by
\[ \delta_\mu(\alpha) = \mu(\alpha \cdot, \cdot) + \mu(\cdot, \alpha \cdot) - \alpha \mu(\cdot, \cdot). \]

**Theorem 4.2.** Let $\omega(t)$ be a solution to (4.1) and let $\mu(t)$ be a solution to (4.7). Then there exists a curve $h = h(t) \in \text{GL}(n, \mathbb{C})$ such that:
1. \( \omega = h^*\omega_0; \)
2. \( \mu = h \cdot \mu_0; \)
3. \( \frac{d}{dt} h = -\frac{1}{2} h P(\omega) = -\frac{1}{2} P_{\mu} h. \)

**Proof.** Let \( \omega = \omega(t) \) be a solution to (4.1) and let \( h(t) \) be the solution to the linear ODE
\[
\begin{cases}
\frac{d}{dt} h = -\frac{1}{2} h P(\omega) \\
h(0) = 1.
\end{cases}
\]
If \( \tilde{\omega} = h^*\omega_0, \) then
\[
\frac{d}{dt} \tilde{\omega}(\cdot, \cdot) = \omega_0(h', h) + \omega_0(h, h') = -\frac{1}{2} \omega_0(h P(\omega), h) - \frac{1}{2} \omega_0(h, h P(\omega))
\]
\[
= -\frac{1}{2} \tilde{\omega}(P(\omega), \cdot) - \frac{1}{2} \tilde{\omega}(\cdot, P(\omega))
\]
Since
\[
\frac{d}{dt} \omega(\cdot, \cdot) = -\frac{1}{2} \omega(P(\omega), \cdot) - \frac{1}{2} \omega(\cdot, P(\omega))
\]
\( \omega \) and \( \tilde{\omega} \) solve the same ODE and consequently they coincide, as required.

**Remark 4.3.** Note that the Bismut scalar form \( b(\omega) = g(\rho^B(\omega), \omega) \) reads in terms of bracket as
\[
b_{\mu} := \langle \rho^B_{\mu}, \omega \rangle = \sum_{r,k} \langle \mu(Z_r, Z_r), \mu(Z_k, Z_r) \rangle,
\]
i.e.,
\[
b_{\mu} = -\left\| \sum_{r} \mu(Z_r, Z_r) \right\|^2.
\]
\( \{Z_r\} \) being an arbitrary unitary frame.

**Lemma 4.4.** The bracket flow (4.7) preserves the center of \( \mu_0. \)

**Proof.** Consider on \((\mathbb{R}^{2n}, J_0, \omega_0)\) an arbitrary \( \mu_0 \in \mathcal{A}. \) Let \( \xi_0 \) be the center of \( \mu_0 \) and \( \xi_0^\perp \) its orthogonal complement with respect to \( \langle \cdot, \cdot \rangle. \) Every \( J_0 \)-compatible non-degenerate form \( \omega \) can be decomposed as \( \omega = \alpha' + \alpha \) with respect the splitting \( g = \xi \oplus \xi^\perp, \) where \( \alpha \) is the restriction of \( \omega \) to \( \xi \times \xi^\perp. \) We can write in particular \( \omega_0 = \alpha_0 + \alpha'_0. \) Formula (4.2) implies that \( \rho^B(X, \cdot) \) vanishes for every \( X \in \xi. \) Therefore the solution \( \omega(t) \) of (4.1) can be written as \( \omega(t) = \alpha_0' + \alpha(t) \) and every \( h = h(t) \) satisfying conditions 1,2,3 of Theorem 4.2 preserves \( \alpha_0', i.e, \ h(t)^*(\alpha_0') = \alpha_0' \) for every \( t \). There follows \( h(t)(\xi_0) = \xi_0 \) for any \( t \). The solution \( \mu(t) \) to the bracket
flow (4.7) is defined in terms of \( h \) and \( \mu_0 \) as \( \mu(t) = h(t) \cdot \mu_0 \) and for every \( t \) the kernel of \( \mu(t) \) is \( \xi_t = h(t)\xi_0 \). Hence \( \xi_t \equiv \xi_0 \), as required.

We describe now how the SKT condition reads in terms of brackets: Let \( \mu \) be a bracket in \( A \). Then \( \mu \) induces the differential operator
\[
d_\mu : \Lambda^r(\mathbb{R}^{2n})^* \to \Lambda^{r+1}(\mathbb{R}^{2n})^*
\]
defined in terms of \( \mu \) as
\[
d_\mu \gamma(X_0, X_1, \ldots, X_r) = \sum_{0 \leq i \leq j \leq r} (-1)^{i+j} \gamma(\mu(X_i, X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r).
\]
Furthermore, the complex extension of \( d_\mu \) splits with respect to \( J_0 \) as \( d_\mu = \partial_\mu + \overline{\partial}_\mu \).

We denote by \( \partial, \overline{\partial} \) the usual differential complex operator on \((\mathbb{R}^{2n}, J_0)\). Every SKT Lie algebra can be seen as a Hermitian Lie algebra \((\mathbb{R}^{2n}, J_0, \langle \cdot, \cdot \rangle, \mu)\) whose fundamental form \( \omega_0 \) satisfies\
\[
\overline{\partial}_\mu \partial_\mu \omega_0 = 0.
\]
This motivates the following

**Definition 4.5.** A bracket \( \mu \in A \) is SKT if\
(4.8)\
\[
\overline{\partial}_\mu \partial_\mu \omega_0 = 0.
\]

The identity (4.8) is an algebraic equation in \( \mu \) and, therefore, the set of SKT brackets gives an algebraic subset of \( A \).

5. **Long Time Existence for Nilmanifolds**

In this subsection we focus on SKT structures on nilpotent Lie algebras proving Theorem 1.1.

The following two results will be important in the sequel

**Theorem 5.1.** ([6]) Let \((g, \mu, J, \omega)\) be an SKT nilpotent Lie algebra. Then \( g \) is 2-step and \( J \) preserves the center \( \xi \) of \( g \).

**Theorem 5.2.** ([20]) For a 2-step nilpotent almost Hermitian Lie algebra, the Chern form \( \rho^C \) is always vanishing.

Therefore in the SKT nilpotent case we have to handle 2-step nilpotent Lie algebras. Using the general formula [23]
\[
\rho^B = \rho^C - dd^* \omega
\]
we have that the pluriclosed flow reduces in this case to\
(5.1)\
\[
\begin{cases}
\partial \omega = (dd^* \omega)^{1,1},
\omega(0) = \omega_0.
\end{cases}
\]

Let us consider then an SKT (2-step) nilpotent Lie algebra \((g, \mu, J, \omega)\) with induced metric \( g \). We denote by \( \sharp : g \to g^* \) the duality induced by the inner product \( g \). Given a vector subspace \( W \) of \( g \) we set \( W^\perp := \sharp(W) \) and we denote by \( W^\perp \) the orthogonal complement of \( W \) with respect to \( g \). Finally we denote by \( \theta = -Jd^* \omega \) the Lee form of \((J, \omega)\). We have the following

**Proposition 5.3.** The Lee form \( \theta \) of a nilpotent SKT Lie algebra \((g, \mu, J, g)\) belongs to \((g^\perp)^2 \subseteq \xi^\perp \).
Lemma 5.4. For a nilpotent SKT Lie algebra \((\mathfrak{g}, \mu, J, g)\) the \((1,1)\)-part \((\text{ric}^B)^{1,1}\) of the Ricci tensor of the Bismut connection is symmetric and it is related to \(\rho^B\) by
\[
(\rho^B)^{1,1}(X,Y) = (\text{ric}^B)^{1,1}(X,JY),
\]
for every \(X,Y \in \mathfrak{g}\).

Proof. We can write
\[\mathfrak{g} = \xi \oplus \xi^\perp,\]
where \(\xi\) is the center of \(\mathfrak{g}\). The 2-step condition implies
\[\{\xi^\perp, \xi^\perp\} \subseteq \xi.\]

Every \(X \in \mathfrak{g}\) can be written accordingly as
\[X = X^\xi + X^\perp,\]
where \(X^\xi \in \xi\) and \(X^\perp \in \xi^\perp\). By [4, Lemma 2.1] we have that
\[
\nabla^B_X Y^\xi = 0, \quad \nabla^B_X Y^\perp \in \xi^\perp, \quad \nabla^B_X Y^\xi \in \xi^\perp
\]
and
\[
\nabla^B_{X^\perp} Y^\perp = \frac{1}{2}([X^\perp, Y^\perp] - [JX^\perp, JY^\perp]) \in \xi.
\]

Therefore
\[
\theta(\nabla^B_X Y) = \frac{1}{2} \theta([X^\perp, Y^\perp] - [JX^\perp, JY^\perp]).
\]

Using the integrability of \(J\) we get
\[
\theta(\nabla^B_X Y) = \frac{1}{2} \theta(-J[X^\perp, Y^\perp] - J[X^\perp, JY^\perp])
\]
\[
= \frac{1}{2} (J\theta)([X^\perp, Y^\perp] + [X^\perp, JY^\perp])
\]
\[
= -\frac{1}{2} d(J\theta)(X^\perp, Y^\perp) - \frac{1}{2} d(J\theta)(X^\perp, JY^\perp).
\]

Taking into account that \(\rho^B = \rho^C + d(J\theta)\) and that \(\rho^C\) in our cases vanishes, we have
\[
\theta(\nabla^B_X Y) = -\frac{1}{2} \rho^B(X^\perp, JY^\perp) - \frac{1}{2} \rho^B(X^\perp, JY^\perp).
\]

Therefore
\[
(\nabla^B_X \theta)(JY) = -\theta(\nabla^B_X JY) = \frac{1}{2} \rho^B(JX^\perp, JY^\perp) - \frac{1}{2} \rho^B(X^\perp, JY^\perp) = -(\rho^B)^{2,0+0,2}(X^\perp, Y^\perp).
\]

Using
\[
\rho^B(X,Y) = \text{ric}^B(X,JY) + (\nabla^B_X \theta)(JY) = \text{ric}^B(X,JY) - (\rho^B)^{2,0+0,2}(X^\perp, Y^\perp),
\]
we get
\[
(\rho^B)^{1,1}(X,Y) = \frac{1}{2} [\text{ric}^B(X,JY) - \text{ric}^B(JX,Y)] = (\text{ric}^B)^{1,1}(X,JY),
\]
as required. □
Theorem 5.5. For a nilpotent SKT Lie algebra \((\mathfrak{g}, \mu, J, g)\) we have
\[
(\text{ric}^B)^{1,1}(X, Y) = 2(\text{ric}^g)^{1,1}(X^\perp, Y^\perp),
\]
for every \(X, Y \in \mathfrak{g}\).

Proof. Using formula (3.10) and Lemma 5.4 we have that
\[
(\text{ric}^B)^{1,1}(X, Y) = (\text{ric}^B)^{1,1}(X^\perp, Y^\perp).
\]
In view of formula (2.2) Lemma 5.4 implies
\[
(\text{ric}^B)^{1,1}(X, Y) = (\text{ric}^g)^{1,1}(X, Y) - \frac{1}{8} \sum_{i=1}^{2n} g(T^B(X, e_i), T^B(Y, e_i) + g(T^B(JX, e_i), T^B(JY, e_i)).
\]
Hence the claim consists on showing that
\[
\frac{1}{8} \sum_{i=1}^{2n} [g(T^B(X^\perp, e_i), T^B(Y^\perp, e_i) + g(T^B(JX^\perp, e_i), T^B(JY^\perp, e_i)] = -2(\text{ric}^g)^{1,1}(X^\perp, Y^\perp).
\]
Let \(S : \xi^\perp \to \xi^\perp\) be the symmetric operator defined by the relation
\[
g(S(X^\perp), Y^\perp) = \text{ric}^g(X^\perp, Y^\perp).
\]
By [3] we have that \(S\) can be written as
\[
S = \frac{1}{2} \sum_{i=1}^{2p} \iota(z_i)^2,
\]
where \(\{z_1, \ldots, z_{2p}\}\) is an orthonormal basis of \(\xi\) and the skew-symmetric map \(\iota(Z) : \xi^\perp \to \xi^\perp\) is defined by
\[
g(\iota(Z)X, Y) = g([X, Y], Z), \quad X, Y \in \xi^\perp,
\]
for \(Z \in \xi\). Equivalently \(\iota(Z)(X) = -(ad_X)^*(Z)\), for every \(X \in \xi^\perp\), where \((ad_X)^*\) is the adjoint of \(ad_X\) with respect to the inner product \(g\). In particular \(S\) is negative definite on \(\xi^\perp\) and
\[
\text{ric}^g(X^\perp, Y^\perp) = \frac{1}{2} \sum_{i=1}^{2p} g(\iota(z_i)^2(X^\perp), Y^\perp) = -\frac{1}{2} \sum_{i=1}^{2p} g(\iota(z_i)(X^\perp), \iota(z_i)(Y^\perp)),
\]
where \(\{z_1, \ldots, z_{2p}\}\) is an orthonormal basis of \(\xi\). By a direct computation we have that, for every \(z_i \in \xi\),
\[
g(T^B(X^\perp, z_i), T^B(Y^\perp, z_i)) = g(\iota(z_i)(JY^\perp), \iota(z_i)(JX^\perp)) = -2\text{ric}^g(JX^\perp, JY^\perp).
\]
On the other hand, if \(v_i \in \xi^\perp\), then
\[
g(T^B(X^\perp, v_i), T^B(Y^\perp, v_i)) = g([Jv_i, JX^\perp], [Jv_i, JY^\perp]).
\]
By Section 2 in [3] for a metric 2-step nilpotent Lie algebra one has
\[
g(R^g(X^\perp, Y^\perp)X^\perp, W) = \frac{3}{4} g(\iota([X^\perp, Y^\perp])X^\perp, W) = \frac{3}{4} g([X^\perp, Y^\perp], X^\perp, W)
\]
for every \(W \in \mathfrak{g}\) and \(X^\perp, Y^\perp, Z \in \xi^\perp\). As a consequence, for every \(v_i \in \xi^\perp\) we have
\[
g(T^B(X^\perp, v_i), T^B(Y^\perp, v_i)) = \frac{4}{3} g(R^g(Jv_i, JX^\perp), Jv_i, JY^\perp).
\]
Moreover, by [3] pag. 622
\[
\sum_{i=1}^{2n-p} g(R^g(v_i, X^\perp)Y^\perp, v_i) = \frac{3}{4} \sum_{k=1}^{2p} g(\iota(z_k)^2 X^\perp, Y^\perp).
\]
Therefore
\[ \sum_{i=1}^{2n-p} g(T^B(X^i, e_i), T^B(Y^1, e_i)) = -\sum_{k=1}^{2p} g(\nu(J z_k)^2 J X^1, J Y^1) = -2\text{ric}^g(J X^1, J Y^1). \]
In this way
\[ (\text{ric}^B)^{1,1}_1(X^i, Y^1) = 2(\text{ric}^g)^{1,1}(X^i, Y^1). \]

\[ \square \]

Remark 5.6. By \[3\] for a metric 2-step Lie algebra \((g, \mu, g)\) one has

1. \(\text{ric}^g(X, Z) = 0\) for all \(X \in \xi\) and \(Z \in \xi^⊥\).
2. \(\text{ric}^g(Z, Z^*) = -\frac{1}{4}\text{tr} i(Z)\mu(Z^*)\) for \(Z, Z^* \in \xi\). In particular \(\text{ric}^g(Z, Z) \geq 0\) for all \(Z \in \xi\) with equality if and only if \(i(Z) = 0\).

Moreover, giving a Hermitian structure \((g, J)\) on \(\mathfrak{g}\), for \(X \in \xi\) and \(Y \in \xi^⊥\) we have
\[ g(T^B(X, e_i), T^B(Y, e_i)) = g(\nabla^B_{\xi} e_i - \nabla^B_{\xi} X, \nabla^B_{\xi} Y - [Y, e_i]). \]
If \(e_i \in \xi\) then
\[ g(T^B(X, e_i), T^B(Y, e_i)) = 0. \]
If \(e_i \in \xi^⊥\) we have that \(\nabla^B_{\xi} e_i - \nabla^B_{\xi} X \in \xi^⊥\) and \(\nabla^B_{\xi} e_i - \nabla^B_{\xi} Y - [Y, e_i] \in \xi\). So again
\[ g(T^B(X, e_i), T^B(Y, e_i)) = 0 \]
for every \(X \in \xi\) and \(Y \in \xi^⊥\). There follows that
\[ (\text{ric}^B)^{1,1}_1(X, Y^⊥) = 2(\text{ric}^g)^{1,1}(X^⊥, Y^⊥) \]
for all \(X \in \mathfrak{g}\) and \(Y \in \xi^⊥\), while
\[ (\text{ric}^B)^{1,1}_{\xi \times \xi} \neq 2(\text{ric}^g)^{1,1}_{\xi \times \xi}. \]

Let us consider now the space \(\mathcal{N}\) of all \(2n\)-dimensional nilpotent Lie algebras equipped with a complex structure. Such a space can be seen as a subspace of the space \(\mathcal{A}\) defined in \[13\]. Let \(\mu_0\) be an SKT bracket in \(\mathcal{N}\) and let \(\mu(t)\) be the solution to \[14\] satisfying \(\mu(0) = \mu_0\). Then \(\mu(t)\) is SKT for every \(t\) and we have
\[ \frac{d}{dt}(\mu, \mu) = 2\left( \frac{d}{dt}\mu, \mu \right) = (\delta\mu(P_\mu), \mu) = -4\langle P_\mu, \text{Ric}_\mu \rangle \]
where
\[ \text{Ric}_\mu = -\frac{1}{2}\sum_{i=1}^{2n}(\text{ad}_\mu E_i)^t \text{ad}_\mu E_i + \frac{1}{4}\sum_{i=1}^{2n} \text{ad}_\mu E_i (\text{ad}_\mu E_i)^t \]
is the usual Ricci operator induced by \(\mu\) (see \[13\] Lemma 4.2) (here \(\{E_i\}\) is the canonical basis of \(\mathbb{R}^{2n}\)). Using that \(P^\mu\) is of type \((1,1)\) (i.e. that it commutes with \(J_0\)) and that \(P^\mu\) vanishes along the center of \(\mu\)
\[ \xi_\mu = \{X \in \mathbb{R}^{2n} \text{ s.t. } \mu(X, Y) = 0 \text{ for all } Y \in \mathbb{R}^{2n} \} \]
we have
\[ \frac{d}{dt}(\mu, \mu) = -4\sum_k \langle P^\mu(e_k), (\text{Ric}_\mu)^{1,1}(e_k) \rangle \]
where \( \{e_k\} \) is an arbitrary orthonormal basis to \( \xi^\perp_\mu \). On the other hand Lemma 5.3 and Theorem 5.5 imply
\[
\sum_k \langle P_\mu(e_k), (\text{Ric}_\mu)^{1,1}(e_k) \rangle = 2 \langle P_\mu, P_\mu \rangle
\]
and so
\[
\text{(5.2)} \quad \frac{d}{dt} \langle \mu, \mu \rangle = -8 \langle P_\mu, P_\mu \rangle \leq 0,
\]
which readily implies that in the nilpotent case the unique solution to the system (4.7) is defined for every positive \( t \). This fact, together with Theorem 4.2, implies the statement of Theorem 1.1.

Moreover, we have the following

**Proposition 5.7.** In the nilpotent SKT case the maximal solution to (4.7) converges to the abelian bracket.

**Proof.** Let \( \mu(t) \) be the maximal solution to (4.7). We prove that \( \|\mu(t)\|^2 \) tends to zero when \( t \) tends to infinity. In view of (5.2) we have
\[
\frac{d}{dt} \langle \mu, \mu \rangle = -8 \langle P_\mu, P_\mu \rangle = -2 \sum_k \langle \text{Ric}_\mu^{1,1}(e_k), \text{Ric}_\mu^{1,1}(e_k) \rangle \leq -2 \left( \sum_k \langle \text{Ric}_\mu^{1,1}(e_k), e_k \rangle \right)^2
\]
where \( \{e_k\} \) is an arbitrary orthonormal basis of \( \xi^\perp_\mu \). Since \( \xi^\perp_\mu \) is \( J_\theta \)-invariant, then
\[
\sum_k \langle \text{Ric}_\mu^{1,1}(e_k), e_k \rangle = \sum_k \langle \text{Ric}_\mu(e_k), e_k \rangle
\]
i.e.,
\[
\frac{d}{dt} \langle \mu, \mu \rangle \leq -2 \left( \sum_k \langle \text{Ric}_\mu(e_k), e_k \rangle \right)^2
\]
From the definition of \( \text{Ric}_\mu \) and taking into account that the \( e_k \)'s belong to the orthogonal complement \( \xi^\perp_\mu \) to the center of \( \mu \), we have
\[
\frac{d}{dt} \langle \mu, \mu \rangle \leq -2 \left( \sum_k \langle \text{Ric}_\mu(e_k), e_k \rangle \right)^2 = -\frac{1}{2} \left( \sum_{i,k} \langle \mu(e_i, e_k), \mu(e_i, e_k) \rangle \right)^2 = -\frac{1}{2} \|\mu\|^4
\]
and the claim follows. \( \square \)

**Example 5.8.** Here we study the basic Example 3.11 in terms of bracket flow. The starting bracket is
\[
\mu_0 = -\frac{1}{2} \zeta^1 \wedge \zeta^\top \otimes Z_2 + \frac{1}{2} \zeta^1 \wedge \zeta^\top \otimes Z_\mathbb{T}
\]
which corresponds to the bracket of the Lie algebra \( \mathfrak{h}_3 \oplus \mathbb{R} \). Since the bracket flow preserves the center, we look for a solution \( \mu \) to (4.7) taking value only at \( (Z_1, Z_\mathbb{T}) \), i.e.
\[
\mu = \mu^2_1 \zeta^1 \wedge \zeta^\top \otimes Z_2 + \mu^2_1 \zeta^1 \wedge \zeta^\top \otimes Z_\mathbb{T}.
\]
For such a bracket we have
\[
\rho^B_\mu = -2i |\mu^2_1|^2 \zeta^1 \wedge \zeta^\top
\]
and
\[
P_\mu = -2 |\mu^2_1|^2 \zeta^1 \otimes Z_1 + 2 |\mu^2_1|^2 \zeta^\top \otimes Z_\mathbb{T}.
\]
Therefore
\[ \delta_{\mu}(P_{\mu})(Z_1, Z_T) = 2\mu(P_{\mu}(Z_1), Z_T) = -4|\mu_{1T}|^2 \mu(Z_1, Z_T) \]
and the corresponding bracket flow equation is
\[
\begin{aligned}
\dot{z} &= -2|z|^2 z, \\
z(0) &= -\frac{1}{2} 
\end{aligned}
\]
where \( z = \mu_{1T}^2 \). Since \( \mu_{1T}^2 \) has as solution the real function
\[ z(t) = -\frac{1}{2(t + 1)^2} \]
the solution \( \mu(t) \) of the bracket flow is defined for every positive \( t \) and converges in \( \mathcal{A} \) to the null bracket corresponding to the abelian Lie algebra.

6. Evolution of Tamed Symplectic forms on a complex manifold

Let \((M, J)\) be a complex manifold. We recall that a symplectic form \( \Omega \) on \( M \) tames \( J \) if
\[
\Omega(JX, X) > 0
\]
for every non-zero tangent vector field \( X \) on \( M \). Such a condition is weaker than the compatibility of \( \Omega \) with \( J \) since in this case the positive tensor induced by \( \Omega(JX, X) > 0 \) is not symmetric. A structure \((J, \Omega)\) composed by a complex structure and a taming symplectic form was called in [18] a \textit{Hermitian-symplectic} structure. Such a structure arises considering static solutions of the pluriclosed flow (1.3). Indeed if an SKT form \( \omega \) satisfies the Hermitian-Einstein equation
\[ r \omega = (\rho B)^{1,1}(\omega) \]
with \( r \in \mathbb{R} \) and \( r \neq 0 \), then \( \Omega = \frac{1}{r} \rho B \) is a symplectic form taming \( J \).

In [6] it was observed that Hermitian-symplectic structures are actually special STK structures. This is because given a symplectic form \( \Omega \) taming \( J \) and considering the decomposition of \( \Omega \) in complex-type forms
\[ \Omega = \omega + \beta + \overline{\beta} \in \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \]
one has that \( d\Omega \) vanishes if and only if \( \beta \) solves
\[
\begin{aligned}
\mathcal{D}_{\Omega}^{1,1} &= -\partial \beta \\
\mathcal{D}_{\beta} &= 0.
\end{aligned}
\]
In the sequel of the paper we are going to take into account the following evolution equation
\[
\begin{aligned}
\frac{d}{dt}\Omega &= -\rho B(\omega) \\
\Omega(0) &= \Omega_0,
\end{aligned}
\]
which we will call the \textit{Hermitian-symplectic (or simply HS) flow}.

**Proposition 6.1.** Let \( \Omega_0 \) be a tamed symplectic form on a compact complex manifold \((M, J)\). Then short-time existence of a solution \( \Omega(t) \) of (6.3) is guaranteed. Moreover, \( \Omega(t) \) is a symplectic form taming \( J \) for every \( t \).

**Proof.** We can write \( \Omega_0 = \omega_0 + \beta_0 + \overline{\beta}_0 \) and the Hermitian-symplectic flow decomposes in its \((1,1)\)-part
\[
\begin{aligned}
\frac{d}{dt}\omega &= -(\rho B)^{1,1}(\Omega^{1,1}) \\
\omega(0) &= \omega_0
\end{aligned}
\]
and the $(2,0)$-part

\begin{equation}
\begin{cases}
\frac{d}{dt} \beta = -(\rho B)^{2,0}(\omega) \\
\beta(0) = \beta_0.
\end{cases}
\end{equation}

(6.5)

Since (6.4) is the “usual” pluriclosed flow, it admits a solution $\omega(t)$ defined in an interval $[0, \varepsilon)$, for $\varepsilon$ small enough. On the other hand, since $(\rho B)^{2,0}(\omega)$ does not depend on $\beta$, $\beta(t) = \beta_0 + \int_0^t (\rho B)^{2,0}(\omega)(s) \, ds$ is a solution to (6.5) and $\Omega(t) := \omega(t) + \beta(t)$ provides the unique solution to (6.3).

We finally observe that the taming condition is preserved by the flow. Indeed, $\omega(t)$ is positive since it is a solution to the pluriclosed flow and $\Omega(t)$ is closed since

\[ \frac{d}{dt} (d\Omega(t)) = d \left( \frac{d}{dt} \Omega(t) \right) = -d\rho B = 0, \]

and then $d\Omega(t)$ is constant. □

The previous result says that the pluriclosed flow preserves the Hermitian-symplectic condition. Indeed, a Hermitian-symplectic structure can be defined as an SKT structure $(\omega_0, J)$ together a solution $\beta$ to (6.2). As a consequence of Proposition 6.1 we have that if an SKT form $\omega_0$ admits a solution $\beta_0$ to (6.2), then the solution $\omega(t)$ to the pluriclosed flow with initial condition $\omega_0$ has a solution $\beta(t)$ for every $t$.

We recall the following stability theorem for the Hermitian curvature flow (1.2) obtained by Streets and Tian

**Theorem 6.2.** ([19]) Let $(M, \tilde{g}, J)$ be a complex manifold with a Kähler-Einstein metric $\tilde{g}$ and $c_1(M) < 0$ or $c_1(M) = 0$. Then there exists $\epsilon = \epsilon(\tilde{g})$ so that if $g_0$ is a $J$-Hermitian metric on $M$ and $\|\tilde{g} - g_0\|_{C^\infty} < \epsilon$, then the solution to (1.2) with initial condition $g_0$ exists for all time and converges exponentially to a Kähler-Einstein metric.

**Corollary 6.3.** In the hypothesis of Theorem 6.2, let $\Omega_\omega$ be a symplectic form on $M$ taming $J$ and such that $\|\tilde{g} - g_0\|_{C^\infty} < \epsilon$, where $g_0$ is the Hermitian metric of $\Omega_\omega^{1,1}$. Then the solution $\Omega(t)$ of flow (6.3) with initial condition $\Omega(0) = \Omega_\omega$ is defined for every $t \in [0, \infty)$ and converges to a symplectic form whose $(1,1)$-component induces a Kähler-Einstein metric.

**Proof.** Let $\omega_\omega = \Omega^{1,1}_\omega$. Then using Theorem 6.2 we have that the equation

\[ \begin{cases}
\frac{d\omega}{dt} = -(\rho B)^{1,1}(\omega) \\
\omega(0) = \Omega^{1,1}_\omega
\end{cases} \]

has a unique solution $\omega(t)$ defined in $[0, \infty)$ and converging exponentially to a Kähler-Einstein structure $\omega_\infty$. Since $\omega(t)$ is defined in $[0, \infty)$, the system

\[ \begin{cases}
\frac{d\beta}{dt} = -(\rho B)^{2,0}(\omega) \\
\beta(0) = \Omega^{2,0}_\omega
\end{cases} \]

has a solution $\beta(t)$ in $[0, \infty)$ which can be written as

\[ \beta(t) = \int_0^t f(s) \, ds + \Omega^{2,0}_\omega \]
We claim that \( f(s) \) converges exponentially to 0. This last assertion can be proved as follows: Let \( g \) be an arbitrary \( J \)-Hermitian metric on \((M, J)\) with fundamental form \( \omega \). Then a standard computation yields that in local complex coordinates we have
\[
(\rho^{B,0}(\omega)) = -\frac{i}{2} \partial z_a(g^{a\overline{b}}(g_{\overline{b}l,k} - g_{kl,b})) dz^a \wedge d\overline{z}^b,
\]
Therefore we have the estimates
\[
\|(\rho^{B,0}(\omega))\|_{C^k} \leq C_k \sum_{i+j=k+1} \|\omega\|_{C^i} \|\partial \omega\|_{C^j},
\]
where all the \( C^k \)-norms are computed with respect to \( \tilde{g} \). Now, since \( \omega(t) \) converges exponentially to \( \omega_\infty \) and \( \omega_\infty \) is closed, we have that \( \partial \omega(t) \) converges exponentially to 0 in the \( C^\infty \)-norm. On the other hand
\[
\|\omega(t)\|_{C^k} \leq \tilde{C}_k e^{-\lambda_k t} + \|g_\infty\|_{C^k}
\]
for a suitable constants \( \tilde{C}_k \) and \( \lambda_k \). It follows that \( f(s) \) converges to 0 in \( C^\infty \)-norm, i.e. for every positive integer \( k \) there exists suitable constants \( B_k \) and \( \mu_k \) such that
\[
\|f(s)\|_{C^k} \leq B_k e^{-\mu_k t}.
\]
Therefore \( \beta(t) \) converges in \( C^\infty \)-norm to
\[
\beta_\infty := \int_0^\infty f(s) \, ds + \Omega^{2,0}_0.
\]
and (6.3) has a unique solution \( \Omega(t) \) for \( t \in [0, \infty) \) converging to
\[
\Omega_\infty := \omega_\infty + \beta_\infty + \Omega_\infty.
\]
Finally, since \( \Omega(t) \) is closed for every \( t \), its limit is a symplectic form, as required.

**Remark 6.4.** Generically we do not expect that \( \beta(t) \) converges to zero. A trivial counterexample is the following: consider the standard complex torus \( \mathbb{T}^{2n} = \mathbb{C}^n / \mathbb{Z}^{2n} \) with the standard flat Kähler structure \( \omega_0 = -i \sum dz^i \wedge d\overline{z}^i \). Then \( \Omega_0 = \omega_0 + dz^1 \wedge d\overline{z}^2 + d\overline{z}^1 \wedge dz^2 \) is a Hermitian-symplectic structure and \( \Omega(t) = \Omega_0 \) solves the flow (6.3).

6.1. **Flow (6.3) on Lie algebras.** Let \((g, \mu)\) be a Lie algebra endowed with a complex structure \( J \). Let \( \{Z_i\} \) an arbitrary \((1,0)\)-frame with dual frame \( \{\zeta^i\} \). Every Hermitian inner product \( g \) on \((g, \mu, J)\) can be written as
\[
g = g_{r\overline{s}} \zeta^r \overline{\zeta}^s,
\]
for some real coefficients \((g_{r\overline{s}})\). The inner product \( g \) induces the fundamental form
\[
\omega = -i g_{r\overline{s}} \zeta^r \wedge \overline{\zeta}^s.
\]
Therefore an arbitrary non-degenerate 2-form \( \Omega \) dominating \( J \) can be written as
\[
\Omega = -i g_{r\overline{s}} \zeta^r \wedge \overline{\zeta}^s + \beta_{ij} \zeta^i \wedge \zeta^j + \overline{\beta}_{ij} \zeta^i \wedge \overline{\zeta}^j
\]
Using equations (3.6), we get that the problem (6.3) is equivalent to the following system

\begin{align*}
\frac{d}{dt}g_{ij} &= -\mu_{ij}^a \mu_{ar}^g + \mu_{ij}^r \eta_{ar}^g + \mu_{ij}^r \eta_{ar}^g - \mu_{ij}^r \eta_{ar}^g \mu_{kr}^g g_{kl} \\
\frac{d}{dt}\beta_{ij} &= -i\mu_{ij}^a \mu_{ar}^g + i\mu_{ij}^r \eta_{ar}^g + i\mu_{ij}^r \eta_{ar}^g - i\mu_{ij}^r \eta_{ar}^g \mu_{kr}^g g_{kl} \\
g_{ij}(0) &= h_{ij} \\
\beta_{ij}(0) &= h_{ij}
\end{align*}

(6.6)

where

\[\Omega_0 = -ih_{ij} \xi^i \wedge \xi^j + h_{rs} \xi^r \wedge \xi^s + h_{lm} \xi^l \wedge \xi^m\]

is the starting symplectic form taming \(J\) and

\[\Omega = -ig_{ij} \xi^i \wedge \xi^j + \beta_{rs} \xi^r \wedge \xi^s + \beta_{lm} \xi^l \wedge \xi^m\]

is the solution to (6.3).

In real dimension four, the equations (6.6) can be simplified by writing every \(J\)-Hermitian inner product on \(g\) in matrix notation as

\[(g_{ij}) = \begin{pmatrix} x & z \\ z & y \end{pmatrix}\]

where \(x, y\) are positive real numbers and \(z \in \mathbb{C}\) satisfying

\[xy - |z|^2 > 0.\]

In this way the inverse of \(g\) is

\[(g_{ij})^{-1} = \frac{1}{xy - |z|^2} \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}.\]

**Example 6.5.** Consider the solvable Lie algebra \(\mathfrak{g}\) with structure equations (24, -14, 0, 0) endowed with the complex structure \(J(e_1) = e_2, \ J(e_3) = e_4\).

Let \(\{Z_1, Z_2\}\) be the (1, 0)-frame

\[Z_1 = \frac{1}{2}(e_1 - ie_2), \quad Z_2 = \frac{1}{2}(e_3 - ie_4)\]

then

\[[Z_1, Z_T] = [Z_2, Z_T] = 0, \quad [Z_1, Z_2] = -\frac{1}{2}Z_1, \quad [Z_T, Z_2] = -\frac{1}{2}Z_T, \quad [Z_1, Z_2] = \frac{1}{2}Z_1.\]

Consider the (1, 0)-coframe

\[\zeta^1 = e^1 + ie^2, \quad \zeta^2 = e^3 + ie^4\]

dual to \(\{Z_1, Z_2\}\). Then

\[d\zeta^1 = -i\zeta^1 \wedge e^4, \quad d\zeta^2 = 0.\]

There follows

\[d\zeta^1 = -\frac{1}{2}\zeta^{12} + \frac{1}{2}\zeta^{17},\]

i.e.

\[\partial\zeta^1 = -\frac{1}{2}\zeta^{12}, \quad \overline{\partial}\zeta^1 = \frac{1}{2}\zeta^{17}.\]
The generic 2-form taming the complex structure \( J \) is
\[
\tilde{\Omega} = -ix^2 \zeta^1 - iy^2 \zeta^2 - iz \zeta^1 + w \zeta^1 + \overline{w} \zeta^2,
\]
where \( x, y \in \mathbb{R} \) and \( z, w \in \mathbb{C} \) satisfy
\[
x^2y^2 - |z|^2 > 0,
\]
Moreover, the closure of \( \tilde{\Omega} \) implies \( iz = w \), i.e.
\[
\tilde{\Omega} = -ix^2 \zeta^1 - iy^2 \zeta^2 - iz \zeta^1 + w \zeta^1 + \overline{w} \zeta^2.
\]
The Bismut Ricci form with respect to \( \tilde{\omega} := \tilde{\Omega} \), is then given by
\[
\rho^B(\tilde{\omega}) = i \frac{zx^2}{4(x^2y^2 - |z|^2)} \zeta^1 - i \frac{zx^2}{4(x^2y^2 - |z|^2)} \zeta^1 + i \frac{yx^2}{4(x^2y^2 - |z|^2)} \zeta^1 + i \frac{yx^2}{4(x^2y^2 - |z|^2)} \zeta^1.
\]
The HS flow reduces to
\[
\dot{x} = \dot{y} = 0,
\]
\[
\dot{z} = -\frac{zx^2}{4(x^2y^2 - |z|^2)},
\]
with initial conditions \( x(0) = x_0, y(0) = y_0, z(0) = z_0 \). In particular \( x \) and \( y \) have to be constant and our system reduces to
\[
\dot{z} = -\frac{zx^2}{4(x_0^2y_0^2 - |z|^2)}, \quad z(0) := z_0.
\]
This last equation is radial in the sense that its solutions \( z = \rho e^{i\theta} \) have \( \theta \) constant and our problem reduces to
\[
\dot{\rho} = -\frac{\rho x^2}{4(x_0^2y_0^2 - \rho^2)}, \quad \rho(0) = \rho_0
\]
in terms of an unknown real function \( \rho \). If \( \rho_0 \) is vanishing, then (6.7) has solution \( \rho \equiv 0 \); otherwise its solution \( \rho \) is defined and strictly positive in \([0, \infty)\) and satisfies
\[
\frac{\rho^2}{2x_0^2} - y_0^2 \log(\rho) - \frac{\rho_0}{x_0} + \frac{y_0^2}{\rho_0^2} = \frac{t}{4}.
\]
This last relation ensures that \( \rho \) tends to zero when \( t \) tends to infinity. Therefore we have
\[
\begin{cases}
  z_\infty = 0 \\
  w_\infty = 0,
\end{cases}
\]
and thus
\[
\Omega_\infty = -ix^2 \zeta^1 - iy^2 \zeta^2.
\]

REFERENCES

[1] V. Apostolov, M. Gualtieri, Generalized Kähler manifolds, commuting complex structures, and split tangent bundles, Comm. Math. Phys. 271 (2007), no. 2, 561–575.
[2] J. M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (1989), no. 4, 681–699.
[3] P. Eberlein, Geometry of 2-step nilpotent Lie groups, Annales Scientifiques de l’É.N.S. (1994), 611-660
[4] N. Enrietti, Static SKT metrics on Lie groups, Manuscripta Math. 140 (2013), 557–571.
[5] N. Enrietti, A. Fino, Special Hermitian metrics and Lie groups, Differential Geom. Appl. 29 (2011), suppl. 1, 211–219.
[6] N. Enrietti, A. Fino and L. Vezzoni, Hermitian Symplectic structures and SKT metrics, J. Symplectic Geom. 10, n. 2 (2012), 203–223.
[7] A. Fino, M. Parton and S. Salamon, Families of strong KT structures in six dimensions, Comment. Math. Helv. 79 (2004), no. 2, 317–340.
[8] A. Fino, A. Tomassini, Non-Kähler solvmanifolds with generalized Kähler structure, J. Symplectic Geom. 7 (2009), no. 2, 1–14.
[9] P. Gauduchon, Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 257–288.
[10] P. Gauduchon, La 1-forme de torsion d’une variété hermitienne compacte, Math. Ann. 267 (1984), 495–518.
[11] M. Inoue, On surfaces of class VII0, Invent. Math. 24 (1974), 269–310.
[12] S. Ivanov, G. Papadopoulos, Vanishing theorems and string backgrounds, Classical Quantum Gravity 18 (2001), no. 6, 1089–1110.
[13] J. Lauret, The Ricci flow for simply connected nilmanifolds. Comm. Anal. Geom. 19 (2011), no. 5, 831–854.
[14] T. Madsen and A. Swann, Invariant strong KT geometry on four-dimensional solvable Lie groups, J. Lie Theory 21 (2011), no. 1, 55–70.
[15] T.-J. Li and W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, Comm. Anal. Geom., 17 (2009), no. 4, 651–683.
[16] D. Popovici, Limits of Projective Manifolds Under Holomorphic Deformations, preprint arXiv:1003.3605.
[17] F. A. Rossi and A. Tomassini, On strong Kähler and astheno-Kähler metrics on nilmanifolds, Adv. Geom. 12 (2012), no. 3, 431–446.
[18] J. Streets and G. Tian, A parabolic flow of pluriclosed metrics, Int. Math. Res. Not. IMRN 2010, no. 16, 3101–3133.
[19] J. Streets and G. Tian, Hermitian curvature flow, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 3, 601–634.
[20] L. Vezzoni, A note on Canonical Ricci forms on 2-step nilmanifolds, Proc. Amer. Math. Soc. 141 (2013), no. 1, 325–333.

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