A Simple Perceptron that Learns Non-Monotonic Rules†

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Abstract. We investigate the generalization ability of a simple perceptron trained in the off-line and on-line supervised modes. Examples are extracted from the teacher who is a non-monotonic perceptron. For this system, difficulties of training can be controlled continuously by changing a parameter of the teacher. We train the student by several learning strategies in order to obtain the theoretical lower bounds of generalization errors under various conditions. Asymptotic behavior of the learning curve has been derived, which enables us to determine the most suitable learning algorithm for a given value of the parameter controlling difficulties of training.

1 Introduction

Learning from examples has been one of the most attractive problems for computational neuroscientists [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. For a given system, superiority of the learning strategy should be measured by the generalization error, namely the probability of disagreement between the teacher and student outputs for a new example after the student has been trained. Much efforts have been invested into investigations in the case of learnable rules, and it is desirable to construct suitable learning strategies and minimize the residual generalization error even if it is impossible for the student to reproduce the teacher input-output relations perfectly. In the present contribution we investigate the generalization error for such an unlearnable case [11, 12, 13, 14, 15, 16, 17].

In our model system, the student is a simple perceptron whose output is given as $S(u) = \text{sign}(u)$ with $u \equiv \sqrt{N}\langle J \cdot x \rangle / |J|$, where $J$ is the synaptic weight vector and $x$ is a random input vector which is extracted from the $N$-dimensional sphere $|x|^2 = 1$. The teacher is a non-monotonic (or reversed-wedge type) perceptron whose output is represented as $T_a(v) = \text{sign}[v(a - v)(a + v)]$ with $v \equiv \sqrt{N}\langle J^0 \cdot x \rangle$. 

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The weight vector of the teacher has been written as $\mathbf{J}^0$. If $a = 0$ or $a = \infty$, the student can learn the teacher rule perfectly, the learnable case.

If the width $a$ of the reversed wedge is finite, the student can not reproduce the teacher input-output relations perfectly and the generalization error remains non-vanishing even after infinite number of examples have been presented. For this system, when the overlap between the teacher and student is written as $R = (\mathbf{J} \cdot \mathbf{J}^0) / ||\mathbf{J}|| ||\mathbf{J}^0||$, the generalization error $\epsilon_g$ is

$$
\epsilon_g \equiv \langle \Theta(-T_a(v)S(u)) \rangle \\
= 2 \int_{a}^{\infty} Dv H \left( -\frac{Rv}{\sqrt{1-R^2}} \right) + 2 \int_{0}^{a} Dv H \left( \frac{Rv}{\sqrt{1-R^2}} \right) \\
\equiv E(R),
$$

where $H(x) = \int_{x}^{\infty} Dt$ with $Dt = \exp(-t^2/2)/\sqrt{2\pi}$ and $\langle \cdots \rangle$ stands for the averaging over the connected Gaussian distribution:

$$
P_R(u, v) = \frac{1}{2\pi\sqrt{1-R^2}} \exp \left[ -\frac{(u^2 + v^2 - 2Ruv)}{2(1-R^2)} \right].
$$

It is important that this expression is independent of specific learning algorithms. In Fig. 1 we plot $E(R)$ for several values of $a$.

**Fig. 1.** Generalization error as a function of $R$ for $a = \infty$, 2, 1, 0.5 and $a = 0$.

Minimization of $E(R)$ with respect to $R$ gives the theoretical lower bound of the generalization error. In Fig. 2 we show the theoretical lower bound cor-
responding to the minimum value of $E(R)$ in Fig. 1 and in Fig. 3 we plot the corresponding optimal overlap $R_{\text{opt}}$ which gives the bound.

![Graph](image)

Fig. 2. The best possible value (theoretical lower bound) of the generalization error, the residual generalization errors of conventional Hebbian, perceptron and AdaTron learning algorithms are plotted as functions of $a$.

From Fig. 3 we see that one should train the student so that $R$ becomes 1 for $a > a_{c2} = 0.80$. For $a < a_{c2} = 0.80$, the optimal $R$ is not 1 but $R_\ast = -\sqrt{(2\log 2 - a^2)/2\log 2}$. This system shows the first order phase transition at $a = a_{c2}$ and the optimal overlap changes from 1 to $R_\ast$ discontinuously.

In the following sections, we investigate various learning strategies to clarify the asymptotic behavior of learning curves.

2 Off-line learning

We first investigate the generalization ability of the student in off-line (or batch) mode following the minimum error algorithm. The minimum error algorithm is a natural learning strategy to minimize the total error for $P$ sets of examples $\{\xi^\mu\}$

$$E(J|\{(\xi^\mu)\}) = \sum_{\mu=1}^{P} \Theta(-T^\mu \cdot u^\mu)$$

(3)
where we set $u^\alpha = (J \cdot x^\alpha) / \sqrt{N}$. From the energy defined by Eq. (3), the partition function with the inverse temperature $\beta$ is given by

$$Z(\beta) = \int dJ \delta(|J|^2 - N) \exp \left( -\beta E(J|\{\xi_P\}) \right)$$

$$= \int dJ \delta(|J|^2 - N) \prod_{\mu=1}^P \left[ e^{-\beta} + (1 - e^{-\beta}) \Theta(-J^\mu \cdot u^\mu) \right]$$

(4)

There exists weight vectors that reproduce input-output relations completely if $\alpha = P/N$ is smaller than a critical capacity $\alpha_c$. Therefore, we can calculate the learning curve (LC) below $\alpha_c$ by evaluating the logarithm of the Gardner-Derrida volume $V_{GD} = Z(\infty)$ as

$$\frac{\log V_{GD}}{N} = \frac{\langle \log Z(\infty) \rangle_{\{\xi_P\}}}{N} = \frac{1}{N \lim_{n \to 0} \langle Z^n(\infty) \rangle_{\{\xi_P\}} - 1}.$$  

(5)

On the other hand, at $\alpha = \alpha_c$, $V_{GD}$ shrinks to zero and for $\alpha > \alpha_c$, we cannot find the solution in the weight space. Then, we treat the next free energy

$$-f = \lim_{\beta \to \infty} \frac{\langle \log Z(\beta) \rangle_{\{\xi_P\}}}{N \beta} = \lim_{\beta \to \infty} \lim_{n \to 0} \frac{\langle Z^n(\beta) \rangle_{\{\xi_P\}} - 1}{N \beta n}$$

(6)

to find the solution weight $J$ which gives a minimum error for $\alpha > \alpha_c$. Introducing the order parameters $R_{\alpha} = (J^\alpha \cdot J_{\alpha}) / N$ and $q_{\alpha \beta} = (J_{\alpha} \cdot J_{\beta}) / N$ and using the

**Fig. 3.** The optimal overlap $R$ which gives the best possible value and overlaps which give the residual errors in Fig. 2 for Hebbian, perceptron and AdaTron learning algorithms.
replica symmetric approximation $R_\alpha = R$ and $q_{\alpha \beta} = q$, Eq. (5) is evaluated as

$$\text{ext}_{(R,q)} \left\{ 2\alpha \int Dt \Omega(R/\sqrt{q} : t) \log q : t + \frac{1}{2} \log(1 - q) + \frac{q - R^2}{2(1 - q)} \right\}$$

with

$$\Omega(R : t) = \int Dz \left[ \Theta(-z\sqrt{1 - R^2} - Rt - a) + \Theta(z\sqrt{1 - R^2} + Rt) - \Theta(z\sqrt{1 - R^2} + Rt - a) \right],$$

$$\Xi(q : t) = \int Dz \Theta(z\sqrt{1 - R^2} + t\sqrt{q}).$$

And Eq. (6) is evaluated as

$$\text{ext}_{(R,x)} \left\{ -2\alpha \left[ \int_{-\infty}^{0} Dt \Omega(R : t) \left\{ \Theta(-t - \sqrt{2x}) + \frac{t^2}{2x} \Theta(t + \sqrt{2x}) \right\} + \frac{1 - R^2}{2x} \right\}$$

where we have set $x = \beta(1 - q)$ to find a non-trivial solution in the limit of $\beta \to \infty$ and $q \to 1$. By solving the saddle point equation from Eqs. (7) and (10), we found that the LC is classified into the following five types depending on the parameter $a$.

- $a = 0, \infty$ (learnable case)
  The solutions of the saddle point equation are thermodynamically stable and the LC behaves asymptotically as \cite{18, 19}
  $$\epsilon_g \sim 0.624 \alpha^{-1}. \quad (11)$$

- $a > a_{c0} \sim 1.53$
  The order parameter $R$ monotonically increases to 1 as $\alpha \to \infty$. The LC behaves asymptotically as
  $$\epsilon_g - \epsilon_{\text{min}} \sim \alpha^{-1}. \quad (12)$$

- $a_{c0} > a > a_{c1}$
  A first order phase transition from the poor generalization phase to the good generalization phase is observed at $\alpha \sim O(1)$ in this parameter region (see Fig. 4). In the limit $\alpha \to \infty$, $R$ approaches to 1 which achieves the global minimum of the generalization error in this parameter region and the asymptotic LC is identical to Eq. (12).

- $a_{c1} > a > a_{c2}$
  The first order phase transition is observed similarly to the previous parameter region of $a$ (see Fig. 5). However, the spinodal point $\alpha_{sp}$ becomes infinity. The asymptotic form of the LC for this parameter region of $a$ is the same as Eq. (12).
\( a_{c2} > a > 0 \)

In this parameter region \( E(R) \) is minimized not at \( R = 1 \) but at \( R = R_* \). Therefore, the solution \( (R, x) = (R_*, 0) \) is the global minimum of the free energy for all values of \( \alpha \) and there is no phase transition. The LC decays to its minimum as

\[
\epsilon_g - \epsilon_{\text{min}} \sim \alpha^{-2/3}.
\]

This result implies that the non-monotonic teacher with small \( a \) is more difficult for a simple perceptron to learn than that with large \( a \) [15]. We conclude that minimum error algorithm can lead to the best possible value of the generalization error (see Fig. 2) for all values of \( a \). Watkin and Rau [11] also investigated the LC for the same system as ours, however, they investigated only \( \mathcal{O}(1) \) range of \( \alpha \). In this section, we investigated the LC for all ranges of \( \alpha \).

**Fig. 4.** The learning curve for the case of \( a = 1.3 \). A first order phase transition appears at \( \alpha_{\text{th}} \approx 14.7 \). The spinodal point is at \( \alpha_{\text{sp}} \approx 24.2 \).

### 3 On-line learning dynamics

#### 3.1 Conventional on-line learning algorithms

The on-line learning dynamics we investigate in this work is generally written as follows.

\[
J^{m+1} = J^m + gf(T_a(v), u)x,
\]

(14)
Fig. 5. The learning curve for the case of $a = 1.0$. A first order phase transition appears at $\alpha_{th} \approx 47$ and $\epsilon_g$ changes discontinuously from the branch 1 to the branch 2. The spinodal point $\alpha_{sp}$ has gone to infinity.

where $m$ is the number of the presented patterns and $g$ is the learning rate. In the limit of large $N$, the recursion relation Eq. (14) of the $N$-dimensional vector $J^m$ is reduced to a set of differential equations for $R$ and $l = |J|/\sqrt{N}$:

\[
\frac{dl}{d\alpha} = \frac{1}{2l} \ll g^2 f^2(T_a(v), u) + 2gf(T_a(v), u)ul \gg \\
\frac{dR}{d\alpha} = \frac{1}{l^2} \ll -\frac{R}{2} g^2 f^2(T_a(v), u) - (Ru - v)gf(T_a(v), u)l \gg
\] (15)

(16)

where $\alpha$ is the number of presented patterns per system size $m/N$. In the present subsection we set $g = 1$. We now restrict ourselves to the following well-known algorithms:

- Perceptron learning : $f = -S(u)\Theta(-T_a(v)S(u))$
- Hebbian learning : $f = -T_a(v)$
- AdaTron learning : $f = -u \Theta(-T_a(v)S(u))$.

For the above three learning strategies, asymptotic forms of the generalization error for the learnable case are given as [2, 3]:

- Perceptron learning : $\epsilon_g \sim \alpha^{-1/3}$
- Hebbian learning : $\epsilon_g \sim \alpha^{-1/2}$
- AdaTron learning : $\epsilon_g \sim \alpha^{-1}$. 
Fig. 6. Generalization errors of the AdaTron, perceptron and Hebbian learning algorithms for the case $a = 2.0$. The AdaTron learning became the worst algorithm among the three.

On the other hand, for the unlearnable case, the generalization error converges exponentially to $a$-dependent non-zero values both for perceptron and AdaTron learnings. Unfortunately, these residual errors are not necessarily the best possible value as seen in Fig. 2. From this figure, we see that for the unlearnable case the AdaTron learning is not superior to the perceptron learning, although the AdaTron learning is regarded as the most sophisticated learning algorithm for the learnable case [20]. In Fig. 6 we plot the generalization error of the perceptron, Hebbian and AdaTron learnings for the unlearnable case ($a = 2.0$).

For the Hebbian learning, the generalization error converges to $2H(a)$ for $a > a_{c1} = \sqrt{2\log 2}$ and to $1 - 2H(a)$ for $a < a_{c1}$ as $a^{-1/2}$. For $a > a_{c1}$, this residual error $2H(a)$ corresponds to the optimal value. However, for $a < a_{c1}$, the generalization error of the Hebbian learning exceeds 0.5 and, in addition, an over training is observed (Figs. 2, 3).

This difficulty can be avoided partially by allowing the student to select suitable examples [21]. If the student uses only examples which lie in the decision boundary, that is, if examples satisfy $u = 0$, the generalization error converges to the optimal value as $a^{-1/2}$ except only for $a_{c2} < a < a_{c1}$. 
3.2 Optimization of learning rate

We next regard the learning rate $g$ as a function of $\alpha$ and construct an algorithm by optimizing $g$. In order to decide the optimal rate $g_{\text{opt}}$ we maximize the right hand side of equation (14) with respect to $g$. This procedure is somewhat similar to the processes of determining the annealing schedule. This optimization procedure is different from the method of Kinouchi and Caticha [22].

We apply this technique to the case of the perceptron, the Hebbian and the AdaTron learning algorithms. For the perceptron learning, this optimization procedure leads to the asymptotic form of generalization error as

$$\epsilon_g = \frac{4}{\pi \alpha}$$

(17)

for the learnable case and to

$$\epsilon_g = 2H(a) + \frac{\sqrt{4\pi H(a)}}{\pi(1 - 2\Delta)} \alpha^{-1/2}$$

(18)

for the unlearnable case, where $2H(a)$ is the optimal value for $a > a_{c2}$. In the asymptotic region $\alpha \to \infty$, the learning rate $g_{\text{opt}}$ behaves as $g_{\text{opt}} \sim 1/\alpha$. This learning strategy thus seems to work well for $a > a_{c2}$. However, at $a = a_{c1}$, this optimization procedure fails to reach the best possible value of the generalization error and the generalization ability deteriorates to 0.5 (which is equal to the result by the random guess) [16]. The reason is that for $a = a_{c1}$ the optimal learning rate $g_{\text{opt}}$ vanishes.

For the AdaTron learning, this type of optimization procedure gives the generalization ability as

$$\epsilon_g = \frac{4}{3\alpha}$$

(19)

for the learnable case and

$$\epsilon_g = 2H(a) + \frac{\sqrt{2}}{\pi} \left[ \frac{2\pi H(a) + \sqrt{2\pi a \Delta}}{4a^2 \Delta} \right] \frac{1}{\sqrt{\alpha}}$$

(20)

for the unlearnable rule. Fortunately, for the AdaTron learning, the optimal learning rate does not vanish even at $a = a_{c1}$, and therefore this optimization procedure works effectively for $a > a_{c2}$ [17].

On the other hand, for the Hebbian learning, the above optimization procedure does not change the asymptotic form of the generalization error [16]. Nevertheless, if we introduce the optimal learning rate $g_{\text{opt}}$ into the Hebbian learning with queries, we get the very fast convergence of generalization error as

$$\epsilon_g = 2H(a) + \frac{\sqrt{c}}{\pi} \exp\left(-\frac{\alpha}{\pi}\right),$$

(21)

where $c$ is a positive constant.

The present optimization procedure does not work effectively for $a < a_{c2}$ because the key point of this method consists in pushing the student toward the state $R = 1$ and this state is not optimal for $a < a_{c2}$ (see Fig. 2).
4 Remarks

In the present work, we have found that the off-line learning obtain the best possible value of the generalization error for the whole range of $a$. On the other hand, the conventional on-line learning algorithm should be improved. We could improve the conventional on-line learning strategies by introducing the time-dependent optimal learning rate, and queries. We could obtain the theoretical lower bound of the generalization error for the whole parameter range in the on-line mode. As our optimal learning rate contains the parameter $a$ unknown to the student, the result can be regarded only as a lower bound of the generalization error. However, if one uses the asymptotic form of $g_{\text{opt}}$, the parameter independent learning algorithm can be formulated and the same generalization ability as the parameter dependent case can be obtained [16, 17].

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