SOME SHARP NULL-FORM TYPE ESTIMATES FOR THE KLEIN–GORDON EQUATION

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Abstract. We establish a sharp bilinear estimate for the Klein–Gordon propagator in the spirit of recent work of Beltran–Vega. Our approach is inspired by work in the setting of the wave equation due to Bez, Jeavons and Ozawa. As a consequence of our main bilinear estimate, we deduce several sharp estimates of null-form type and recover some sharp Strichartz estimates found by Quilodrán and Jeavons.

1. INTRODUCTION

Let \( d \geq 2 \) and \( \phi_s(r) = \sqrt{s^2 + r^2} \) for \( r \geq 0 \) and \( s > 0 \). We write \( D \) for the operator \( \sqrt{-\Delta_x} \), that is \( \hat{D}f(\xi) = |\xi| \hat{f}(\xi) \) where \( \hat{\cdot} \) denotes the (spatial) Fourier transform defined by \( \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx \) for appropriate functions \( f \) on \( \mathbb{R}^d \). Additionally, we define \( D_{\pm} \) by \( \tilde{D}_{\pm}f(\tau, \xi) = ||\tau|| \pm ||\xi|| \tilde{f}(\tau, \xi) \), where \( \tilde{\cdot} \) is the space-time Fourier transform of appropriate functions \( f \) on \( \mathbb{R} \times \mathbb{R}^d \). The d’Alembertian operator \( \partial_t^2 - \Delta_x \) will be denoted by \( \Box \), so that \( \Box = D_+ - D_- \). Our main object of interest is the Klein–Gordon propagator given by

\[
e^{it\phi_s(D)}f(x) = \frac{1}{(2\pi)^d} \int_{\hat{\mathbb{R}^d}} e^{i(x \cdot \xi + t\phi_s(|\xi|))} \hat{f}(\xi) \, d\xi
\]

for sufficiently nice initial data \( f \).

As part of the study of sharp bilinear estimates for the Fourier extension operator and inspired by work of Ozawa–Tsutsumi [34], Beltran–Vega [3] very recently presented the following sharp estimate associated to the Klein–Gordon propagator

\begin{equation}
\left\| D^{\frac{1}{2}} \left( e^{it\phi_s(D)} f \right) \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{R})} \\
\leq (2\pi)^{1-3d} \int_{\mathbb{R}^d} |\tilde{f}(\eta_1)|^2 |\tilde{g}(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K^{BV}(\eta_1, \eta_2) \, d\eta_1 \, d\eta_2,
\end{equation}

where

\[
K^{BV}(\eta_1, \eta_2) = \int_{\mathbb{S}^{d-1}} \frac{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)} \left((\eta_1 + \eta_2) \cdot \theta\right)^2 d\sigma(\theta),
\]

Date: January 5, 2022.
2010 Mathematics Subject Classification. 35B45, 42B37, 35A23.
Key words and phrases. Bilinear estimates, Klein–Gordon equation, optimal constants, extremisers, Strichartz estimate.
The estimate \((1.1)\) has some interesting connections to well-known results. For example, as we shall see in more detail later, \((1.1)\) leads to null-form type estimates by appropriately estimating the kernel. In particular, when \(d = 2\) the Strichartz estimate

\[
\|e^{it\phi_s(D)} f\|_{L^4(\mathbb{R}^{2+1})} \leq 2^{-\frac{1}{2}} \|f\|_{H^\frac{1}{2}(\mathbb{R})}
\]

with the optimal constant follows from \((1.1)\), more generally, for \(d \geq 2\) so does the null-form estimate

\[
\|D^{\frac{2-d}{2}} e^{it\phi_s(D)} f\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \frac{\pi^{2-d} |\beta|^{d-2}}{2^d 2} \|\phi_s(D)\|_{L^4(\mathbb{R})}^2 f\|_{L^2(\mathbb{R}^d)},
\]

although the optimality of the constant may be no longer true when \(d \geq 3\). Here, the inhomogeneous Sobolev norm of order \(\alpha\) is defined by

\[
\|f\|_{H^\alpha(\mathbb{R})} := \|\phi_1(D)^\alpha f\|_{L^2(\mathbb{R}^d)}.
\]

The estimate \((1.2)\) with the optimal constant was first obtained by Quilodrán [36]. Bilinear estimates which bear resemblance to \((1.1)\) for the Klein–Gordon equation, as well as the Schrödinger and wave equation, have often arisen in the pursuit of optimal constants for Strichartz estimates and closely related null-form type estimates. As well as the aforementioned work of Beltran–Vega [3], estimates of the form \((1.1)\) for the Klein–Gordon propagator can be found in work of Jeavons [21] (see also [22]). For the Schrödinger equation, in addition to the Ozawa–Tsutsumi estimates in [34], estimates resembling \((1.1)\) may be found in work of Carneiro [13] and Planchon–Vega [35], with a unification of each of these results by Bennett et al. in [5]. For the wave equation, Bez–Rogers [9] and Bez–Jeavons–Ozawa [11] have established estimates resembling \((1.1)\). We also remark that the related literature on sharp Strichartz estimates is large. In addition to the papers already cited, this body of work includes, for example, [10, 11, 13, 15, 17, 20, 28]; the interested reader is referred to the survey article by Foschi–Oliveira e Silva [19] for further information.

Let \(\Gamma(z)\) be the gamma function of \(z\) (with \(\text{Re}(z) > 0\)) and

\[
K^\alpha_\beta(\eta_1, \eta_2) := \frac{(\phi_s(\eta_1)) \phi_s(\eta_2) - \eta_1 \cdot \eta_2 - s^2)^b}{(\phi_s(\eta_1)) \phi_s(\eta_2) - \eta_1 \cdot \eta_2 + s^2)^\pi}.
\]

In the present paper, we establish the following new bilinear estimates for the Klein–Gordon propagator.

**Theorem 1.1.** For \(d \geq 2\) and \(\beta > \frac{1-d}{4}\), we have the estimate

\[
\|\|\| \beta |e^{it\phi_s(D)} f_{e^{it\phi_s(D)}g}\|_{L^2(\mathbb{R}^{d+1})} \leq KG(\beta, d) \int_{\mathbb{R}^{d}} |f(\eta_1)|^2 |\beta(\eta_2)|^2 \phi_s(\eta_1) \|\phi_s(\eta_2)\|K^{\frac{d-2}{2} + \beta} (\eta_1, \eta_2) d\eta_1 d\eta_2,
\]

with the optimal constant

\[
KG(\beta, d) := 2^{-\frac{d-1}{2} + \beta} \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d-1}{2} + \beta\right) / \Gamma\left(d - 1 + 2\beta\right).
\]

In the case when \(s \to 0\), certain sharp bilinear estimates for solutions to the wave equation with the operator \(|\Box|^s\) have been deeply studied by Bez–Jeavons–Ozawa [11]. One may note that, when \(d = 2\), a slightly larger range of \(\beta\) is valid in Theorem \((1.1)\) than one for the corresponding result \((1.14)\) for the wave case in [11]. In order to prove Theorem \((1.1)\), we employ their argument and adapt it into the context of the Klein–Gordon equation. As a consequence of Theorem \((1.1)\), we will generate null-form type estimates of the form

\[
\|\|\| |\Box|^\alpha |e^{it\phi_s(D)} f| \|_{L^2(\mathbb{R}^{d+1})} \leq C \|\phi_s(D)^\alpha f\|_{L^2(\mathbb{R}^d)}
\]

for certain pairs \((\alpha, \beta)\) with the optimal constant.
1. Wave regime. For \( d \geq 4 \), the kernel \( K^{BV} \) can be estimated as

\[
K^{BV}(\eta_1, \eta_2) \leq \frac{|s|^{d-1}}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)} \int_{-1}^{1} \left( 1 - \left| \frac{\eta_1 + \eta_2}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)} \right| \lambda^2 \right)^{-1} (1 - \lambda^2)^{\frac{d-4}{2}} \, d\lambda
\]

for some absolute constant \( C \) since \(|\eta_1 + \eta_2| \leq \phi_s(|\eta_1|) + \phi_s(|\eta_2|)|\). The integral in the first inequality is surely finite as long as \( d \geq 4 \). Then, it follows from the arithmetic-geometric mean that

\[
\phi_s(|\eta_1|)\phi_s(|\eta_2|) K^{BV}(\eta_1, \eta_2) \leq C \phi_s(|\eta_1|) \phi_s(|\eta_2|)^{\frac{1}{2}},
\]

and hence the null-form type estimate

\[
\| D^{\frac{d-2}{2}} (e^{it\phi_s(D)} f e^{it\varphi_s(D)g}) \|_{L^2(\mathbb{R}^d \times \mathbb{R})} \leq C \| \phi_s(D)^{\frac{1}{2}} f \|_{L^2(\mathbb{R}^d)} \| \phi_s(D)^{\frac{1}{2}} g \|_{L^2(\mathbb{R}^d)}
\]

holds. When \( s \to 0 \), the estimate \((1.6)\) yields

\[
\| D^{\beta_0} D_{-}^{\beta_-} D_{+}^{\beta_+} (e^{it\phi_s(D)} f e^{it\varphi_s(D)g}) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{H^{\beta_1-1}} \|g\|_{H^{\beta_2}},
\]

in the case of \((\beta_0, \beta_-, \beta_+, \alpha_-, \alpha_+) = (\frac{2-d}{2}, 0, 0, \frac{1}{2}, \frac{1}{2})\) for the propagator \( e^{itD} \) associated with the wave equation. The estimate \((1.7)\), as well as the corresponding \((+\pm)\) case (while \((1.7)\) is \((-\pm)\) case),

\[
\| D^{\beta_0} D_{-}^{\beta_-} D_{+}^{\beta_+} (e^{it\phi_s(D)} f e^{it\varphi_s(D)g}) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{H^{\beta_1-1}} \|g\|_{H^{\beta_2}}
\]

has found important applications in study of nonlinear wave equations. This type of estimate has been studied back in work of Beals \([2]\) and Klainerman–Machedon \([23, 24, 25]\). A complete characterization of the admissible exponents \((\beta_0, \beta_-, \beta_+, \alpha_-, \alpha_+)\) for \((1.7)\) and \((1.8)\) was eventually obtained by Foschi–Klainerman \([13]\). Such a characterization when the \(L_2^\beta L_r^\alpha\) norm on the left-hand side of \((1.7)\) is replaced by \(L_r^\lambda\) has also drawn great attention. Using bilinear Fourier restriction techniques, Bourgain \([12]\) made a breakthrough contribution, then Wolff \([11]\) and Tao \([9]\) (in the endpoint case; see also Lee \([29]\) and Tataru \([40]\)) completed the diagonal case \(q = r\). For the non-diagonal case we refer readers to \([31]\) due to Lee–Vargas for a complete characterization when \(d \geq 4\) and partial results when \(d = 2, 3\). Soon later Lee–Rogers–Vargas \([39]\) completed \(d = 3\), but a gap between necessary and sufficient conditions still remains when \(d = 2\).

As a means of comparing our bilinear estimate \((1.4)\) with \((1.1)\), we note that using the trivial bound

\[
\frac{\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 - s^2}{\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 + s^2} \leq 1,
\]

we estimate our kernel as

\[
K^{\frac{d-2}{2}+2\beta}_s(\eta_1, \eta_2) \leq K^{\frac{d-2}{2}+2\beta}_0(\eta_1, \eta_2).
\]

For \(\beta \geq \frac{3-d}{4}\), it follows that

\[
\| \square^{\beta}(e^{it\phi_s(D)} f e^{it\varphi_s(D)g}) \|_{L^2(\mathbb{R}^{d+1})} \leq C \| \phi_s(D)^{\frac{1}{2}} f \|_{L^2(\mathbb{R}^d)} \| \phi_s(D)^{\frac{1}{2}} g \|_{L^2(\mathbb{R}^d)}
\]

for some absolute constant \(C\) (for instance, use \(\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 - s^2 \leq 2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\)), which, as in the discussion for the Beltran–Vega bilinear estimate, places Theorem \((1.1)\) in the framework of null-form type estimates. If we formally set \(\beta = \frac{3-d}{4}\) in \((1.11)\) to get data with regularity whose order is \(\frac{1}{4}\) as in \((1.6)\), the order of “smoothing” from \(\square^{\beta}\) becomes \(2\beta = \frac{3-d}{2}\), which is compatible with \((1.6)\). Unfortunately, \(\frac{3-d}{4}\) is outside the range \(\beta \geq \frac{3-d}{2}\) and, in fact, as we shall see in Proposition \((1.4)\) \(\beta \geq \frac{3-d}{4}\) is a necessary

\[\text{We observe that } d \geq 4 \text{ is important here. For } d = 3, \text{ the estimate } (1.6) \text{ actually does not hold. The counterexample has been given by Foschi \([10]\) for the wave equation, and the same argument appropriately adapted works for the Klein–Gordon propagator.}\]
condition for (1.11). Nevertheless, as an application of Theorem 1.1, one can widen the range to, at least, $\beta \geq \frac{2-d}{4}$ if one considers radially symmetric data. We shall state this result as part of the forthcoming Corollary 1.2. In addition, for a large range of $\beta$ we shall in fact obtain the optimal constant for such null-form type estimates; to state our result, we introduce the constant

$$F(\beta, d) := 2^{d-3+4\beta} \pi^{-\frac{3}{2}} \Gamma(\frac{d}{2}) \Gamma(\frac{d-1}{2} + 2\beta) \frac{\Gamma(\frac{d-3}{2} + 2\beta)}{(d-2 + 2\beta) \Gamma(\frac{3d-3}{2})}.$$ 

**Corollary 1.2.** Let $d \geq 2$, $\beta \geq \frac{2-d}{4}$. Then, there exists a constant $C > 0$ such that (1.11) holds whenever $f$ and $g$ are radially symmetric. Moreover, for $\beta \in [\frac{2-d}{4}, \frac{2-d}{4}] \cup [\frac{2-d}{2}, \infty)$, the optimal constant in (1.11) for radially symmetric $f$ and $g$ is $F(\beta, d)^{\frac{d}{2}}$, but there does not exist a non-trivial pair of functions $(f, g)$ that attains equality.

In the case of the wave propagator when $s \to 0$, the estimate (1.11) becomes

$$\|\Box (e^{itD} f e^{itD} g)\|_{L^2(\mathbb{R}^{d+1})} \leq F(\beta, d)^{\frac{d}{2}} \|f\|_{H^{-\frac{d-3}{2} + s}(\mathbb{R}^d)} \|g\|_{H^{-\frac{d-3}{2} + s}(\mathbb{R}^d)}$$

and, in certain situation, it is known that the constant $F(\beta, d)^{\frac{d}{2}}$ is optimal. In the case $\beta = 0$ and $d = 3$, pioneering work of Foschi [17] established the optimality of the constant $F(0, 3)^{\frac{d}{2}}$. The constant $F(\beta, d)^{\frac{d}{2}}$ is also known to be optimal whenever $(\beta, d) = (0, 4)$ and $(\beta, d) = (0, 5)$; the latter was established by Bez–Rogers [9] building on work of Foschi and obtained via the bilinear estimate

$$\|e^{itD} f e^{itD} g\|_{L^2(\mathbb{R}^{d+1})} \leq W(0, d) \int_{(\mathbb{R}^d)^2} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 |\eta_1| |\eta_2| K_{\beta}^{BR}(\eta_1, \eta_2) \, d\eta_1 \, d\eta_2.$$ 

Here, $W(\beta, d)$ turns out to be $KG(\beta, d)$ and $K_{\beta}^{BR}$ formally coincides with the special case of our kernel $K_{\frac{d}{2}}^{\frac{d-2}{2} + 2\beta}$ when $s = 0$. The optimality of $F(0, 4)^{\frac{d}{2}}$ in (1.12) was proved by Bez–Jeavons [10] by making use of (1.13), polar coordinates and techniques from the theory of spherical harmonics.

Soon later, imposing an additional radial symmetry on the initial data $f$ and $g$, Bez–Jeavons–Ozawa proved the optimality of $F(\beta, d)^{\frac{d}{2}}$ in (1.12) when $d \geq 2$ and $\beta > \beta_d := \max\{\frac{1-d}{4}, \frac{2-d}{4}\}$ by establishing the null-form type bilinear estimate

$$\|\Box (e^{itD} f e^{itD} g)\|_{L^2(\mathbb{R}^{d+1})} \leq W(\beta, d) \int_{(\mathbb{R}^d)^2} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 |\eta_1| |\eta_2| K_{\beta}^{BR}(\eta_1, \eta_2) \, d\eta_1 \, d\eta_2,$$

which again coincides with (1.4) formally substituted with $s = 0$. They accomplished the result by taking advantage of an exceedingly nice structure of the homogeneity that the kernel $K_{\beta}^{\frac{d-2}{2} + 2\beta}$ ($= K_{\beta}^{BR}$) possesses, specifically, when $s = 0$:

$$K_{\frac{d}{2}}^{\frac{d-2}{2} + 2\beta}(r_1 \theta_2, r_2 \theta_2) = (r_1 r_2)^{\frac{d-3}{2} + 2\beta} (1 - \theta_1 \cdot \theta_2)^{\frac{d-3}{2} + 2\beta}, \quad r_1, r_2 > 0, \quad \theta_1, \theta_2 \in S^{d-1},$$

This property completely divides the right-hand side of (1.14) into radial and angular components if the initial data are radial symmetric. In contrast, our concern is the case $s > 0$ and the lack of homogeneity in the kernel causes significant difficulty in this regard. This can be seen as responsible for the gap $(\frac{1-d}{4}, \frac{2-d}{4})$ (as well as the range $(\beta_d, \frac{2-d}{4})$) in Corollary 1.2 for which we also expect (1.11) still holds with $C = F(\beta, d)^{\frac{d}{2}}$.

In the current paper, we prove Corollary 1.2 by first making use of our bilinear estimate (1.4). One can show, however, that it is impossible to obtain the optimality of $F(\beta, d)^{\frac{d}{2}}$ in (1.11) for radial data and any

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2It can also be seen from the form of the sharp constant that widening the range $\beta > \beta_d$ is impossible.

3Because of this fact, we expect that (1.11) is valid with $C = F(\beta, d)^{\frac{d}{2}}$ for $\beta \in (\beta_d, \frac{2-d}{4})$ as well, but we do not pursue this here.
$\beta \in (\frac{3-d}{4}, \frac{5-d}{4})$ once one makes use of (1.4) as a first step; somewhat surprisingly given that (1.4) is sharp. This is a consequence of Lemma 3.2 in the specific setting.

There are some special cases of $\beta$; the endpoints $\frac{3-d}{4}$ and $\frac{5-d}{4}$ of the gap, at which we can remove the radial symmetry hypothesis on the initial data and still keep the optimal constants.

**Corollary 1.3.** Let $d \geq 2$. Then, the estimate (1.3) holds with the optimal constant $C = F(\beta, d)^{\frac{1}{2}}$ for $(\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4})$ and $(\alpha, \beta) = (\frac{1}{2}, \frac{5-d}{4})$, but there are no extremisers. Furthermore, when $(\alpha, \beta) = (1, \frac{5-d}{4})$, we have the refined Strichartz estimate

$$\|e^{it\phi_t(D)}f\|_{L^2(\mathbb{R}^{d+1})} \leq \frac{C}{\sqrt{2}} \left( \frac{C}{\sqrt{2}} \right)^{\frac{1}{2}} \frac{C}{\sqrt{2}} \left( \frac{C}{\sqrt{2}} \right)^{\frac{1}{2}} f \|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} f \|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}},$$

where the constant is optimal and there are no extremisers.

Corollary 1.3 generalizes the following recent results. In the context of the Klein–Gordon equation, Quilodrán [36] appropriately developed Foschi’s argument in [17] and proved the sharp Strichartz estimate

$$\|e^{it\phi_t(D)}f\|_{L^q(\mathbb{R}^{d+1})} \leq H(d, q) \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}$$

for $(d, q) = (2, 4), (2, 6), (3, 4)$, which are the endpoint cases of the admissible range of exponent $q$, namely, $\frac{2(d+2)}{d} \leq q \leq \frac{2(d+1)}{d-1}$. The constant $H(d, q)$ denotes the optimal constant so that (1.16) in the case $(d, q) = (3, 4)$ is recovered by Corollary 1.3 in the case $(\alpha, \beta) = (1, \frac{3-d}{4})$ and $F(0, 3)^{\frac{1}{2}} = H(4, 3)$ holds. In [36], Quilodán also proved that there is no extremiser which attains (1.16) for $(d, q) = (2, 4), (2, 6), (3, 4)$. Later Carneiro–Oliveira e Silva–Sousa [13] further revealed the nature of (1.16) for $d = 1, 2$, by answering questions raised in [36]; in particular, they found the best constant in (1.16) for $(d, q) = (1, 6)$ and absence of the extremisers (the case $(d, q) = (1, 6)$ is the endpoint of the admissible range of $6 \leq q \leq 4$ when $d = 1$). Meanwhile, they also established there exist extremisers in the non-endpoint cases in low dimensions $d = 1, 2$. A subsequent study by the same authors in collaboration with Stovall [15] proved the analogous results in the non-endpoint cases for higher dimensions $d \geq 3$ by using some tools from bilinear restriction theory.

In [21], Jeavons obtained the following refined Strichartz estimate in five spatial dimensions

$$\|e^{it\phi_t(D)}f\|_{L^q(\mathbb{R}^{d+1})} \leq \frac{C}{\sqrt{2}} \left( \frac{C}{\sqrt{2}} \right)^{\frac{1}{2}} \frac{C}{\sqrt{2}} \left( \frac{C}{\sqrt{2}} \right)^{\frac{1}{2}} \frac{C}{\sqrt{2}} f \|_{L^q(\mathbb{R}^5)}^{\frac{1}{2}} f \|_{L^q(\mathbb{R}^5)}^{\frac{1}{2}},$$

which recovers the inequality (1.12) when $(\beta, d) = (0, 5)$ in the limit $s \to 0$. Moreover, by simply omitting the negative second term, it follows that

$$\|e^{it\phi_t(D)}f\|_{L^q(\mathbb{R}^{d+1})} \leq \frac{C}{\sqrt{2}} \left( \frac{C}{\sqrt{2}} \right)^{\frac{1}{2}} \frac{C}{\sqrt{2}} f \|_{H^1(\mathbb{R}^5)},$$

where the constant $F(0, 5)^{\frac{1}{2}} = (24 \pi^2)^{-\frac{1}{2}}$ is still sharp. These results are recovered too by Corollary 1.3 in the case $(\alpha, \beta) = (1, \frac{3-d}{4})$.

### 1.2. Non-wave regime

One may examine the Beltran–Vega bilinear estimate (1.1) from a somewhat different perspective to that taken in our earlier discussion which led to (1.6). For $d \geq 2$ the kernel $K^{BV}$ can be transformed as

$$K^{BV}(\eta_1, \eta_2) = |S|^{d-2} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\phi_s(\eta_1) + \phi_s(\eta_2)}{(\phi_s(\eta_1) + \phi_s(\eta_2))^2} (\sin \theta)^{d-2} \, d\theta$$

$$= |S|^{d-2} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \frac{\tau(\cos \theta)^{d-2}}{(\tau^2 + |\xi|^2)^{\frac{1}{2}} + |\xi|^2 \cos^2 \theta} \, d\theta$$

$$= |S|^{d-2} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\tau(\cos \theta)^{d-2}}{(\tau^2 + |\xi|^2) \tan^2 \theta + \tau^2 \cos^2 \theta},$$
where we denote \( \tau = \phi_s(|\eta_1|) + \phi_s(|\eta_2|) \) and \( \xi = \eta_1 + \eta_2 \) for the sake of convenience. By applying the fact \((\cos \theta)^{d-2} \leq 1\), the change of variables \( \tan \theta \mapsto \frac{\tau}{\sqrt{\tau^2 - |\xi|^2}} x \), it follows that

\[
K^{BV}(\eta_1, \eta_2) \leq \frac{|S^{d-2}|}{\sqrt{\tau^2 - |\xi|^2}} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi |S^{d-2}|}{\sqrt{\tau^2 - |\xi|^2}}.
\]

Hence, another key relation (instead of (1.10));

\[
(1.18) \quad \phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 \geq s^2,
\]

implying

\[
K^{BV}(\eta_1, \eta_2) \leq \frac{\pi |S^{d-2}|}{2s},
\]

with which, as informed earlier, the inequality (1.1) directly yields (1.3). By comparison with (1.6), the regularity level on the initial data has increased to \( H^{\frac{2}{d}} \) but this has allowed for a wider range of \( d \) which, in particular, includes \( d = 2 \) in which case (1.3) coincides with the sharp \( H^{\frac{2}{2}} \rightarrow L^2_1 \) Strichartz estimate (1.2) obtained by Quilodrán. Note that, in the non-wave regime, we are not allowed to let \( s \to 0 \) because of the factor \( s^{-1} \) appearing in the constant.

On the other hand, Theorem (1.1) also yields (1.2) as a special case of the following family of sharp null-form type estimates valid in all dimensions \( d \geq 2 \). Indeed, since we have another kernel estimate

\[
K_0^{d-2+2\beta}(\eta_1, \eta_2) \leq 2^{-\frac{1}{2}} K_0^{d-2+2\beta}(\eta_1, \eta_2) s^{-1}
\]

via (1.18), we immediately deduce the following from Theorem (1.1).

**Corollary 1.4.** Let \( d \geq 2 \). Then the estimate (1.5) holds with the optimal constant

\[
C = \left( \frac{2^{-d+1} \pi^{-\frac{d+2}{2}}}{s \Gamma(\frac{d+2}{2})} \right)^{\frac{1}{2}}
\]

for \( (\alpha, \beta) = (\frac{1}{2}, \frac{d-2}{d}) \), but there are no extremisers. Furthermore, when \( (\alpha, \beta) = (1, \frac{d-2}{d}) \), we have the refined Strichartz estimate

\[
(1.20) \quad \|\Box^{\frac{1-d}{2}} e^{it\phi_s(D)} f \|_{L^2(\mathbb{R}^{d+1})} \leq \left( \frac{2^{-d+1} \pi^{-\frac{d+2}{2}}}{s \Gamma(\frac{d+2}{2})} \right)^{\frac{1}{2}} \left( \|\phi_s(D) f \|_{L^2(\mathbb{R}^d)} - s^2 \|\phi_s(D) \|_{L^2(\mathbb{R}^d)} \right)^{\frac{1}{2}},
\]

where the constant is optimal and there are no extremisers.

One may note that (1.20) provides a sharp form of the following refined Strichartz inequality in the analogous manner of (1.17) when \( d = 4 \):

\[
(1.21) \quad \|e^{it\phi_s(D)} f \|_{L^4(\mathbb{R}^{d+1})} \leq (16\pi)^{-\frac{1}{4}} (\|f \|_{H^1(\mathbb{R}^4)} - \|f \|_{H^{\frac{2}{4}}(\mathbb{R}^4)})^{\frac{1}{2}},
\]

however we are unable to conclude whether the constant \((16\pi)^{-\frac{1}{4}}\) continues to be optimal if we drop the second term on the right-hand side, which is discussed in Section 4.3.

For solutions \( u \) of certain PDE, in addition to the null-form estimates (1.7), estimates which control quantities like \( |u|^2 \) through its interplay with other types of operators have appeared numerous times in the literature. In particular, we note that the approach taken by Beltran–Vega [3], which in turn built on work of Planchon–Vega [35], rested on interplay with geometric operators such as the Radon transform or, more generally, the \( k \)-plane transform. For related work in this context of interaction with geometrically-defined operators, we also refer the reader to work of Bennett et. al [6] and Bennett–Nakamura [8].

\(^4\)Note that the equality holds when \( d = 2 \)
Our approach to proving Theorem 1.1 more closely follows the argument in [11] and does not appear to fit into such a geometric perspective.

Summary of results. Theorem 1.1, a natural generalization of (1.14) in the context of the Klein–Gordon, reproduces several known Strichartz-type inequalities with the sharp constant. The following is the summary of our results and remaining open problems.

- Corollary 1.3 recovers (1.5) when $(\beta, d) = (0, 3)$ due to Quilodrán [36].
- Corollary 1.3 recovers (1.5) when $(\beta, d) = (0, 5)$ due to Jeavons [21].
- Corollary 1.3 recovers (1.17) when $(\beta, d) = (0, 5)$ with $\alpha = \frac{3}{4}$ due to Jeavons [21].
- For $(\beta, d) = (0, 4)$, it remains open whether (1.5) holds with $C = F(0, 4)^{\frac{1}{2}}$ as the sharp constant.
- Corollary 1.4 recovers (1.5) when $(\beta, d) = (0, 2)$ due to Quilodrán [36].
- Corollary 1.4 yields (1.21), an analogous refined Strichartz inequality of (1.17), in the case $(\beta, d) = (0, 4)$.
- Corollary 1.4 recovers (1.5) when $(\beta, d) = (0, 4)$ with the constant $(16\pi)^{-\frac{1}{4}}$, but we do not know whether the constant is sharp.

Notation/Useful formulae. Throughout the paper, we denote $A \gtrsim B$ if $A \geq CB$, $A \lesssim B$ if $A \leq CB$ and $A \sim B$ if $C^{-1}B \leq A \leq CB$ for some constant $C > 0$. The gamma function and the beta function are defined by

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-x} \, dx \quad \text{and} \quad B(z, w) := \int_0^1 \lambda^{z-1}(1-\lambda)^{w-1} \, d\lambda,$$

respectively, for $z, w \in \mathbb{C}$ satisfying $\text{Re}(z), \text{Re}(w) > 0$. Regarding those, we use the following well-known formulae multiple times;

$$(1.22) \quad |S^{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)},$$

and

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$  

Also, it is worth to note here that the inverse Fourier transform of an appropriate function $g$ on $\mathbb{R}^d$ is given by $g^\vee(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi) \, d\xi$ so that the following hold:

- $\hat{g}(\xi) = (2\pi)^{-d} \hat{f} \ast \hat{g}(\xi), \quad \xi \in \mathbb{R}^d$.
- $\|f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-d}\|f\|_{L^2(\mathbb{R}^d)}$ (Plancherel’s theorem).
- $\|\phi_\alpha(D)^\alpha f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-d} \left( \int_{\mathbb{R}^d} \phi_\alpha(|\xi|)^{2\alpha} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$.

Structure of the paper.

- **Section 2** We first prove Theorem 1.1 by adapting the argument of [11].
- **Section 3** We prove (1.11) for radial data and then show that, specifically for $\beta \in \left[\frac{2-d}{4}, \frac{5-d}{4}\right] \cup \left[\frac{5-d}{4}, \infty\right)$, the estimate (1.11) with $C = F(\beta, d)^{\frac{1}{2}}$ holds. We also make an observation that suggests it may be difficult to obtain the optimal constant in (1.11) for $\beta \in \left(\frac{3-d}{4}, \frac{5-d}{4}\right)$ even for radially symmetric data (see Proposition 3.3). At the end of this section, we show $\beta \geq \frac{3-d}{4}$ is necessary for (1.11) to hold for general data.
- **Section 4** The aim of this section is to complete the proof of the corollaries. We first introduce how to deduce the refined form of the Strichartz estimate, and then focus on the sharpness of the constants in Corollary 1.3. Corollary 1.3 and Corollary 1.4. They all are proved by the same method, but it differs from that in [21] or [11], as here we need to deal with the more delicate situation of the non-wave regime. The non-existence of extremisers is also discussed.
Section 5. We end the paper with Section 5 by discussing analogous results for the (++) case. As has already observed, the (++) case is far easier than the (+-) case, and this will become clear from our argument in this section. We employ the null-form $|\Box - (2s)^2|$ instead of $|\Box|$ in order to follow the ideas of the proof of Theorem 1.1 and obtain an analogous bilinear estimate (Theorem 5.1).

Acknowledgment. The first author was supported by JSPS Postdoctoral Research Fellowship (No. 18F18020), and the second author was supported by JSPS KAKENHI Grant-in-Aid for JSPS Fellows (No. 20J11851). Authors express their sincere gratitude to Neal Bez, second author’s adviser, for introducing the problem, sharing his immense knowledge and continuous support. They also wish to thank the anonymous referee for a very careful reading of the manuscript and many valuable suggestions and comments.

2. Proof of Theorem 1.1

Although some steps require additional care due to the extra parameter $s$, broadly speaking Theorem 1.1 can be proved by adapting the argument for wave propagators presented in [1], whose techniques originated in [4] (see also [5]). The key tool here is the following Lorentz transform given by $L$; for fixed $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ such that $\tau > |\xi|$, 

$$L\left(\frac{t}{x}, \frac{\gamma(t - \zeta \cdot x)}{x + \frac{(\tau^2 - |\xi|^2)\zeta \cdot x - \gamma t)}{\zeta}}\right), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $\zeta := -\frac{\xi}{\tau}$ and $\gamma := \frac{\tau}{(\tau^2 - |\xi|^2)^{\frac{1}{2}}}$. It is well known that the measure $\frac{d(\sigma - \phi_s(\eta))}{\phi_s(\eta)}$ for $(\sigma, \eta) \in \mathbb{R} \times \mathbb{R}^d$ is invariant under the Lorentz transform $L$, $|\det L| = 1$, and

$$L\left(\frac{\tau^2 - |\xi|^2)^{\frac{1}{2}}}{0}\right) = \left(\frac{\tau}{\xi}\right).$$

Let us first introduce two lemmas whose proof come later in this section.

Lemma 2.1. For $\eta_1, \eta_2 \in \mathbb{R}^d$ and $\beta > \frac{1-d}{4}$, define

$$J^{2\beta}(\eta_1, \eta_2) := \int_{\mathbb{R}^{2d}} \frac{|\phi_s(\eta_1)\phi_s(\eta_2) - \eta_1 \cdot \eta_2 - s^2|^{2\beta}}{\phi_s(\eta_1)\phi_s(\eta_2)} \delta\left(\tau = \phi_s(\eta_1) - \phi_s(\eta_2)\right) \eta_1 \cdot \eta_2 \eta_1 \cdot \eta_2,$$

where $\tau = \phi_s(\eta_1) + \phi_s(\eta_2)$ and $\zeta = \eta_1 + \eta_2$. Then, we have

$$J^{2\beta}(\eta_1, \eta_2) = (2\pi)^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{d-1}{2} + 2\beta\right)}{\Gamma\left(d - 1 + 2\beta\right)} \frac{2^{\frac{d+1}{2}}}{{\frac{d+1}{2} + 2\beta}} (\eta_1, \eta_2).$$

Lemma 2.2. Let $\eta_1, \eta_2 \in \mathbb{R}^d$. Set

$$\xi = \eta_1 + \eta_2, \quad \tau = \phi_s(\eta_1) + \phi_s(\eta_2)$$

and $\eta \in \mathbb{R}^d$ satisfying

$$2\phi_s(\eta) = (\tau^2 - |\xi|^2)^{\frac{1}{2}}.$$

Then, there exists $\omega_s \in \mathbb{S}^{d-1}$ depending only on $\eta_1, \eta_2$ and $|\eta|$ such that

$$\left(\phi_s(\eta_1) - \frac{\eta}{\eta_1}\right) \cdot L\left(\frac{\phi_s(\eta_2)}{\eta}\right) - s^2 = |\eta|^2 \left(1 + \frac{\eta}{|\eta|} \cdot \omega_s\right).$$
Proof of Theorem 1.1. Let \( u(t, x) = e^{it\phi_\omega(D)}f(x) \) and \( v(t, x) = e^{it\phi_\omega(D)}g(x) \). By the expressions \( \tilde{u}(\tau, \xi) = 2\pi\delta(\tau - \phi_\omega(|\xi|^2)f(\xi) \) and \( \tilde{v}(\tau, \xi) = 2\pi\delta(\tau + \phi_\omega(|\xi|^2)g(\xi) \), Plancherel’s theorem, and appropriately relabeling the variables, one can deduce

\[
\frac{(2\pi)^{3d-1}}{|\omega|^2} \| |\xi|^\beta (u(x)) \|_{L^2_{\mathbb{R}^{d+1}}}^2 \\
= (2\pi)^{-4} \int_{\mathbb{R}^{d+1}} |\tau - |\xi|^2|^{2\beta} \tilde{u}(\xi, \tau) \tilde{v}(\xi, \tau) d\tau d\xi \\
= \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |\tau - |\xi|^2|^{2\beta} \tilde{f}(\eta_1) \tilde{g}(-\eta_2) \tilde{f}(\eta_3) \tilde{g}(-\eta_4) \\
\quad \times \delta \left( \tau - \phi_\omega(\eta_1) + \phi_\omega(\eta_2) \right) \delta \left( \tau - \phi_\omega(\eta_3) + \phi_\omega(\eta_4) \right) d\tau d\xi d\eta_1 d\eta_2 d\eta_3 d\eta_4 \\
= 2^{2\beta} \int_{\mathbb{R}^{d+1}} |\phi_\omega(\eta_1)| \phi_\omega(\eta_2) - \eta_1 \cdot \eta_4 - s^2\frac{F(\eta_1, \eta_2)F(\eta_3, \eta_4)}{(\phi_\omega(\eta_1)\phi_\omega(\eta_2)\phi_\omega(\eta_3)\phi_\omega(\eta_4))^\frac{1}{2}} \\
\quad \times \delta \left( \phi_\omega(\eta_1) + \phi_\omega(\eta_2) - \phi_\omega(\eta_3) - \phi_\omega(\eta_4) \right) d\eta_1 d\eta_2 d\eta_3 d\eta_4.
\]

Here, the change of variables; \((\eta_2, \eta_4) \mapsto (-\eta_4, -\eta_2)\) has been performed in the last step and

\[
F(\eta_1, \eta_2) := \tilde{f}(\eta_1) \tilde{g}(\eta_2) \phi_\omega(\eta_1)|^\frac{3}{2} \phi_\omega(\eta_2)|^\frac{1}{2}.
\]

If we define \( \Psi = \Psi_\omega(\eta_1, \eta_2, \eta_3, \eta_4) := \left( \frac{\phi_\omega(\eta_1)\phi_\omega(\eta_2)}{\phi_\omega(\eta_3)\phi_\omega(\eta_4)} \right)^\frac{1}{2} \), then by the arithmetic-geometric mean we have

\[
|F(\eta_1, \eta_2)F(\eta_3, \eta_4)| \leq \frac{1}{2} \left( |F(\eta_1, \eta_2)|^2 \Psi + |F(\eta_3, \eta_4)|^2 \psi^{-1} \right)
\]

so that

\[
|F(\eta_1, \eta_2)| \leq \frac{1}{2} \left( \frac{|F(\eta_1, \eta_2)|^2}{\phi_\omega(\eta_1)\phi_\omega(\eta_2)} \right)^\frac{1}{2} \left( \frac{|F(\eta_3, \eta_4)|^2}{\phi_\omega(\eta_3)\phi_\omega(\eta_4)} \right)^\frac{1}{2}.
\]

The equality holds if and only if

\[
\phi_\omega(\eta_1)\phi_\omega(\eta_2)\tilde{f}(\eta_1)\tilde{g}(\eta_2) = \phi_\omega(\eta_3)\phi_\omega(\eta_4)\tilde{f}(\eta_3)\tilde{g}(\eta_4)
\]

almost everywhere on the support of the delta measure, which is satisfied by, for instance, \( f = g = f_a \) with \( a > 0 \) that is given by

\[
f_a(\xi) = \frac{e^{-a\phi_\omega(|\xi|)}}{\phi_\omega(|\xi|)}.
\]

Therefore,

\[
\left( (2\pi)^{-3d+1} 2^{2\beta} \right)^{-1} \| |\xi|^\beta (u(x)) \|_{L^2_{\mathbb{R}^{d+1}}}^2 \\
\leq \frac{1}{2} \left[ \int_{\mathbb{R}^{2d}} F(\eta_1, \eta_2)F(\eta_3, \eta_4) d\eta_1 d\eta_2 + \int_{\mathbb{R}^{2d}} F(\eta_2, \eta_3)F(\eta_1, \eta_4) d\eta_1 d\eta_2 \right],
\]

which implies (1.4) by applying Lemma 2.1. One may note that the constant in (1.4) is sharp since we only apply the inequality (2.3) in the proof. \( \square \)

We now prove the aforementioned lemmas.
Proof of Lemma 2.4. Let \( \tau = \phi_s(|\eta_1|) + \phi_s(|\eta_2|) \) and \( \xi = \eta_1 + \eta_2 \). Recall the Lorentz transform \( L \). The change of variables \( \left( \begin{array}{c} \eta_j' \\ \eta_j \end{array} \right) \mapsto L \left( \begin{array}{c} \eta_j' \\ \eta_j \end{array} \right) \) for \( j = 1, 2 \) gives

\[
J^{2\beta}(\eta_1, \eta_2) = \int_{\mathbb{R}^{d+1}} \left( \frac{\phi_s(|\eta_1|)}{-\eta_1} - \frac{\phi_s(|\eta_2|)}{-\eta_2} \right) \cdot \left( \frac{\sigma_4}{\eta_1} - \frac{\sigma_4}{\eta_2} \right) \cdot s^2 \, \delta(\sigma_4 - \phi_s(|\eta_1|)) \, d\sigma_3 d\sigma_4 d\eta_3 d\eta_4
\]

\[
= \int_{\mathbb{R}^{d+1}} \left( \frac{\phi_s(|\eta_1|)}{-\eta_1} - \frac{\phi_s(|\eta_2|)}{-\eta_2} \right) \cdot \frac{L(\sigma_4)}{\phi_s(|\eta_1|)} \cdot \left( \frac{\sigma_4}{\eta_1} - \frac{\sigma_4}{\eta_2} \right) \cdot s^2 \, \delta(\sigma_4 - \phi_s(|\eta_1|)) \, d\sigma_3 d\sigma_4 d\eta_3 d\eta_4
\]

\[
= \int_{\mathbb{R}^{d+1}} \left( \frac{\phi_s(|\eta_1|)}{-\eta_1} - \frac{\phi_s(|\eta_2|)}{-\eta_2} \right) \cdot \frac{L(\sigma_4)}{\phi_s(|\eta_1|)} \cdot \left( \frac{\sigma_4}{\eta_1} - \frac{\sigma_4}{\eta_2} \right) \cdot s^2 \, \delta(\sigma_4 - \phi_s(|\eta_1|)) \, d\sigma_3 d\sigma_4 d\eta_3 d\eta_4
\]

By Lemma 2.2 and switching to polar coordinates,

\[
J^{2\beta}(\eta_1, \eta_2) = \left( \int_{S^{d-1}} (1 + \theta \cdot \omega_s)^{2\beta} \, d\sigma(\theta) \right) \left( \int_{0}^{\infty} \frac{r^{4\beta}}{\phi_s(r)^2} \delta(2\phi_s(r) - (\tau^2 - |\xi|^2)^{\frac{1}{2}}) \, r^{d-1} \, dr \right).
\]

The first integral can be further simplified as

\[
\int_{S^{d-1}} (1 + \theta \cdot \omega_s)^{2\beta} \, d\sigma(\theta) = |S^{d-2}| \int_{-1}^{1} (1 + \lambda)^{2\beta} (1 - \lambda^2)^{\frac{d-1}{2}} \, d\lambda = 2^{d-2+2\beta}|S^{d-2}|B(d-1 \, 2, \beta, d-\frac{1}{2})
\]

by using the beta function \( B \). For the remaining radial integration, one can perform the change of variables \( 2\phi_s(r) \mapsto \nu \) in order to get

\[
\int_{0}^{\infty} \frac{r^{4\beta}}{\phi_s(r)^2} \delta(2\phi_s(r) - (\tau^2 - |\xi|^2)^{\frac{1}{2}}) \, r^{d-1} \, dr = 2^{-d+2-4\beta} \int_{\frac{\nu}{\phi_s(r)}}^{\infty} \left( \nu^2 - 4s^2 \right)^{\frac{d-2}{2}+2\beta} \delta(\nu - (\tau^2 - |\xi|^2)^{\frac{1}{2}}) \, d\nu
\]

and hence

\[
J^{2\beta}(\eta_1, \eta_2) = 2^{d+\frac{1}{2}} |S^{d-2}|B(d-\frac{3}{2}, 2\beta, d-\frac{1}{2}),
\]

Finally, simplifying the constant by the formula

\[
B(d-\frac{1}{2}, 2\beta, d-\frac{1}{2}) = \frac{\Gamma(d-\frac{1}{2} + 2\beta)\Gamma(d-\frac{1}{2})}{\Gamma(d-1 + 2\beta)},
\]

we are done.

Proof of Lemma 2.2. Observe first that

\[
L \left( \frac{\phi_s(|\eta|)}{\eta} \right) = \frac{1}{2} \left( 2\eta + \xi \left( 1 + \left( \frac{\xi}{\phi_s(|\eta|)} \right) \right) \right).
\]

Then, a direct calculation gives

\[
\left( \frac{\phi_s(|\eta_1|)}{-\eta_1} \right) \cdot L \left( \frac{\phi_s(|\eta|)}{\eta} \right) = \left( \phi_s(|\eta|) \right)^2 \left( 1 + \frac{\eta}{|\eta|} \cdot |\xi| \right),
\]

where

\[
z = \frac{[\phi_s(|\eta|) + \phi_s(|\eta_1|)]\eta_2 - [\phi_s(|\eta|) + \phi_s(|\eta_2|)]\eta_1}{\phi_s(|\eta|)^2[\phi_s(|\eta_1|) + \phi_s(|\eta_2|)] + 2\phi_s(|\eta|)}.
\]
Since we have the relation \(2\phi_s(\eta) = \phi_s(\eta_1)\phi_s(\eta_2) - \eta_1 \cdot \eta_2 + s^2\), the numerator of \(z\) can be simplified by taking the absolute value of \(\phi_s(\eta_1)\phi_s(\eta_2) - \eta_1 \cdot \eta_2 + s^2\) and using the fact that \(\phi_s(\eta_1)\phi_s(\eta_2) - \eta_1 \cdot \eta_2 + s^2\) is positive. Moreover, for \(\beta \geq \frac{2-d}{4}\) and an explicit constant \(C < \infty\); for \(\beta = (\frac{2-d}{4}, \frac{3-d}{4})\), this explicit constant coincides with \(F(\beta, d)^{\frac{1}{2}}\). In order to complete the proof of Corollary 3.2, we need to show the sharpness of \(F(\beta, d)^{\frac{1}{2}}\) for \(\beta \in [\frac{2-d}{4}, \frac{3-d}{4}] \cup [\frac{3-d}{4}, \infty)\), and the non-existence of extremisers; for these arguments, we refer the reader to Section 4.

**Lemma 3.1.** Let \(a + b > -1\), \(b > -1\) and \(\kappa \in [0, 1]\). Define

\[
h^{a,b}(\kappa) := \int_{-1}^{1} (1 - \kappa \lambda)^a (1 - \lambda^2)^b d\lambda.
\]

Then,

\[
\sup_{\kappa \in [0, 1]} h^{a,b}(\kappa) < \infty.
\]

Moreover, for \(\alpha \in (-\infty, 0] \cup [1, \infty)\)

\[
\sup_{\kappa \in [0, 1]} h^{a,b}(\kappa) = h^{a,b}(1) = 2^{a+2b+1} B(a+b+1, b+1).
\]

**Proof.** By the Lebesgue dominated convergence theorem,

\[
\frac{d}{d\kappa} h^{a,b}(\kappa) = -ak \int_{-1}^{1} (1 - \kappa \lambda)^{a-1} \lambda (1 - \lambda^2) d\lambda
\]

\[
= ak \int_{0}^{1} ((1 + \kappa \lambda)^{a-1} - (1 - \kappa \lambda)^{a-1}) \lambda (1 - \lambda^2)^b d\lambda
\]
Thus,
\[
\begin{cases}
\frac{d}{d\kappa} h^{a,b}(\kappa) \geq 0 & \text{if } \kappa \in (-\infty, 0] \cup [1, \infty), \\
\frac{d}{d\kappa} h^{a,b}(\kappa) < 0 & \text{if } \kappa \in (0, 1).
\end{cases}
\]

For \( a \in (-\infty, 0] \cup [1, \infty), \)
\[
\sup_{\kappa \in [0,1]} h^{a,b}(\kappa) = h^{a,b}(1) = \int_{-1}^{1} (1 - \lambda)^a(1 - \lambda^2)^b d\lambda
\]
and the change of variables \( 1 + \lambda \mapsto 2\lambda \) gives
\[
\int_{-1}^{1} (1 - \lambda)^a(1 - \lambda^2)^b d\lambda = 2^{a+2b+1} B(a + b + 1, b + 1) < \infty
\]
if \( a + b > 0 \) and \( b > -1. \) Similarly, for \( a \in (0, 1), \)
\[
h^{a,b}(\kappa) \leq h^{a,b}(0) = 2^{2b+1} B(b + 1, b + 1) < \infty
\]
if \( b > -1. \)

**Proof of Corollary 1.2.** Let \( f, g \) be radially symmetric. By Theorem 1.1 we have
\[
\|\|\| \beta(e^{i\phi_0(\sqrt{-1})} f e^{i\phi_0(\sqrt{-1})} g)\|\|_{L^2(\mathbb{R}^{d+1})}^2 \leq KG(\beta, d) \int_{0}^{\infty} \int_{0}^{\infty} |\hat{f}(r_1)|^2|\hat{g}(r_2)|^2 \phi_s(r_1) \frac{4^d+2b}{\phi_s(r_2)} \frac{d+2b}{\phi_s(r_2)} \Theta_{\phi_s}^{\frac{d+2b}{2}}(r_1, r_2) \frac{d}{r_1^d} d\lambda_1 d\lambda_2,
\]
where
\[
\Theta_{\phi_s}^b(r_1, r_2) := \int_{(S^{d-1})^2} \left( \frac{1 - \frac{r_1 r_2 \phi_s(r_1) \phi_s(r_2)}{\phi_s(r_1) \phi_s(r_2)}}{1 - \frac{r_1 r_2 \phi_s(r_1) \phi_s(r_2)}{\phi_s(r_1) \phi_s(r_2)}} \right)^b d\sigma(\phi_1) d\sigma(\phi_2).
\]
We divide the range of \( \beta \) into \( \beta \in [\frac{2-d}{2}, \frac{3-d}{2}] \) and \( \beta \in [\frac{5-d}{2}, \infty) \) and treat these cases differently. First, let us consider \( \beta \in [\frac{5-d}{2}, \infty) \) as the easier case. By applying the fundamental kernel estimate \( 1.10 \), we have
\[
\Theta_{\phi_s}^{\frac{d+2b}{2}}(r_1, r_2) \leq \Theta_0^{\frac{d+2b}{2}}(r_1, r_2) = |S^{d-1}| |S^{d-2}| h_{\frac{d+2b}{2}}(\kappa) \Sigma_b^{\frac{d+2b}{2} \frac{d-2}{2}}(\kappa)
\]
with \( \kappa = \frac{r_1 r_2}{\phi_s(r_1) \phi_s(r_2)} \). Since \( d + 2b \geq 1 \), Lemma 3.1 implies that
\[
\sup_{\kappa \in [0,1]} h_{\frac{d+2b}{2}}^{\frac{d-2}{2}}(\kappa) = h_{\frac{d+2b}{2}}^{\frac{d-2}{2}}(1),
\]
and hence
\[
\sup_{r_1, r_2 > 0} \Theta_{\phi_s}^{\frac{d+2b}{2}}(r_1, r_2) \leq 2^{\frac{d+2b}{2}} |\Sigma_b^{\frac{d-1}{2}}| |\Sigma_b^{d-2}| B \left( d - 2 + 2b, \frac{d-1}{2} \right),
\]
which yields \( 1.11 \) with \( C = F(\beta, d) \frac{1}{2}. \)

For \( \beta \in [\frac{2-d}{2}, \frac{3-d}{2}], \) in which case \( \frac{d-2}{2} + 2b \in [0, \frac{1}{2}], \) the basic idea of our argument is the same as above but it requires a few more steps. Let
\[
\Xi(\nu, \nu) := \int_{-1}^{1} \frac{(1 - \nu - \sqrt{1 - \nu^2} \nu^2) \frac{d+2b+2b}{2}}{(1 + \nu - \sqrt{1 - \nu^2} \nu^2) \nu^2} d\mu(\lambda)
\]
with \( \nu \) and \( \nu^2 \) satisfying
\[
\nu, \nu^2 \in [0, 1], \quad v^2 \leq 1 - \nu^2,
\]
and \( d\mu(\lambda) = (1 - \lambda^2)^{\frac{d+2}{2}} d\lambda. \) Then, from (3.1), it suffices to show
\[
\Xi(\nu, \nu) \leq \Xi(0, 0) \leq \Xi(0, 0).
\]
In order to show the first inequality of (3.2), we establish monotonicity in \( \nu \in \left[ 0, \sqrt{\frac{1-\nu^2}{2}} \right] \), and calculate directly for \( \nu \in \left[ \sqrt{\frac{1-\nu^2}{2}}, \sqrt{1-\nu^2} \right] \). Indeed, it simply follows that

\[
\partial_{\nu} \Xi(\nu, v) \leq - \frac{1}{2} \int_0^1 \frac{(1 - \nu - \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d-4}{2}}}{(1 + \nu - \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d}{2}}} \left( 1 - \frac{\nu}{\sqrt{1 - \nu^2 - \nu^2 \lambda}} \right) d\mu(\lambda)
\]

which is non-positive since \( \beta \geq \frac{2-d}{4} \) and

\[
1 - \frac{\nu}{\sqrt{1 - \nu^2 - \nu^2 \lambda}} \geq 0
\]

for \( \nu \in \left[ 0, \sqrt{\frac{1-\nu^2}{2}} \right] \). On the other hand, for \( \nu \in \left[ \sqrt{\frac{1-\nu^2}{2}}, \sqrt{1-\nu^2} \right] \), which imposes \( 0 \leq \sqrt{1 - \nu^2} \leq \nu \), it follows that

\[
\Xi(\nu, v) = \int_0^1 \frac{(1 - \nu - \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d-4}{2}}}{(1 + \nu - \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d}{2}}} d\mu(\lambda) + \int_0^1 \frac{(1 - \nu + \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d-4}{2}}}{(1 + \nu + \sqrt{1 - \nu^2 - \nu^2 \lambda})^{\frac{d}{2}}} d\mu(\lambda)
\]

Here, the first inequality is justified as long as \( \beta \geq \frac{2-d}{4} \) and the second inequality is given by the arithmetic-geometric mean:

\[
\frac{1}{2} \left( (1 - \sqrt{1 - \nu^2 \lambda})^{\frac{d-2}{2}} + (1 + \sqrt{1 - \nu^2 \lambda})^{\frac{d-2}{2}} \right) \geq (1 - (1 - \nu^2 \lambda)^{\frac{d}{2}})^{\frac{d}{4}} \geq 1.
\]

Since the second inequality of (3.2) can be readily proved by Lemma 3.1, we have (1.11) with \( C = F(\beta, d)^{\frac{1}{4}} \) for \( \beta \in \left[ \frac{2-d}{4}, \frac{3-d}{4} \right] \) as well.

### 3.2. Threshold of our argument for \( \beta \in (\frac{3-d}{4}, \frac{5-d}{4}) \)

Although \( C = F(\beta, d)^{\frac{1}{4}} \) will be shown to be optimal for \( \beta \in [\frac{2-d}{4}, \frac{3-d}{4}] \cup [\frac{5-d}{4}, \infty) \) in the case of radial data, it remains unclear whether this continues to be true for \( \beta \in (\frac{3-d}{4}, \frac{5-d}{4}) \) here we establish that there is no way to obtain the constant \( F(\beta, d)^{\frac{1}{4}} \) if one first makes use of Theorem 1.1. In order to show that, we shall invoke the following useful result for the beta function due to Agarwal–Barnett–Dragmir

**Lemma 3.2** (II). Let \( m, p \) and \( k \in \mathbb{R} \) satisfy \( m, p > 0 \), and \( p > k > -m \). If we have

(3.3) \[ k(p - m - k) > 0 \]

then

\[
B(p, m) \geq B(p - m, m + k)
\]
With this in hand, we establish the following.

**Proposition 3.3.** Let $d \geq 2$ and $\beta \in (\frac{3-d}{4}, \frac{3-d}{4})$. Then there exist radially symmetric $f$ and $g$ such that

\[
\KG(\beta, d) \int_{\mathbb{R}^d} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K_{\frac{d-2}{4} + 2\beta}^2(\eta_1, \eta_2) \, d\eta_1 d\eta_2 \\
> F(\beta, d) \|\phi_s(\delta)\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta} f(\delta) \|\phi_s(\delta)\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta} g(\delta) \|\phi_s(\delta)\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta}
\]

holds.

**Proof.** Let $0 < \delta \ll 1$ and

\[\begin{align*}
A &= \left\{ \xi \in \mathbb{R}^d : \frac{1}{2} < |\xi| < 2 \right\}.
\end{align*}\]

Define $f_A$ and $g_A$ so that for $\xi \in \mathbb{R}^d$

\[
\hat{f}_A(\xi) = \chi_A\left(\frac{\xi}{\delta}\right) \quad \text{and} \quad \hat{g}_A(\xi) = \chi_A(\delta \xi),
\]

where $\chi_A$ is the characteristic function of $A$. By use of polar coordinates,

\[
\int_{\mathbb{R}^d} |\hat{f}_A(\eta_1)|^2 |\hat{g}_A(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K_{\frac{d-2}{4} + 2\beta}^2(\eta_1, \eta_2) \, d\eta_1 d\eta_2
\]

\[
= \int_{O_\delta} |\hat{f}_A(r_1)|^2 |\hat{g}_A(r_2)|^2 (\phi_s(r_1)\phi_s(r_2))^{\frac{d-1}{4} + 2\beta} \Theta_{\frac{d-2}{4} + 2\beta}^2(r_1, r_2) r_1^{d-1} d r_1 d r_2.
\]

Here, the set $O_\delta$ is defined by (see also Figure 1)

\[O_\delta = \left\{ (r_1, r_2) : \frac{1}{2\delta} < r_1 < \frac{2}{\delta}, \frac{\delta}{2} < r_2 < 2\delta \right\}.
\]

Now, for $(r_1, r_2) \in O_\delta$, by taking the limit $\delta \to 0$ represented by $\phi_s(r_1) \to \infty$ and $\phi_s(r_2) \to s$, it follows that

\[
\Theta_{\frac{d-2}{4} + 2\beta}^2(r_1, r_2) \to |S^{d-1}|^2.
\]

Therefore, for sufficiently small $\delta > 0$,

\[
\KG(\beta, d) \int_{\mathbb{R}^d} |\hat{f}_A(\eta_1)|^2 |\hat{g}_A(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K_{\frac{d-2}{4} + 2\beta}^2(\eta_1, \eta_2) \, d\eta_1 d\eta_2
\]

\[
= (2\pi)^{2d} \KG(\beta, d) \|\phi_s(\sqrt{\Delta})\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta} f_A \|\phi_s(\sqrt{\Delta})\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta} g_A \|\phi_s(\sqrt{\Delta})\|_{L^2(\mathbb{R}^d)}^{\frac{d-1}{4} + \beta},
\]

\[\text{Figure 1. The set } O_\delta \text{ along the curve } r_1 = r_2^{-1}.\]
and it is enough to show
\[(2x)^{2d} |G_\beta(d, d)| > F(d, d)\]
for \(d \geq 2\) and \(\beta \in (\frac{3d}{4}, \frac{5d}{4})\). By the formula (1.22) and the definitions of constants, this can be simplified as
\[B(\frac{3d-5}{4} + \beta, \frac{3d-3}{4} + \beta) > B(d - 2 + 2\beta, \frac{3d}{4})\]
which, if fact, follows from Lemma 3.2 by letting \(p = \frac{3d-5}{4} + \beta\), \(m = \frac{3d-3}{4} + \beta\) and \(k = \frac{3d}{4} - \beta\) for \(d \geq 2\) and \(\beta \in (\frac{3d}{4}, \frac{5d}{4})\). Note that for the specific triple \((p, m, k)\), the hypothesis (3.3) is equivalent to considering \(\beta\) from the gap \((\frac{3d}{4}, \frac{5d}{4})\) when \(d \geq 2\). 

3.3. Contributions of radial symmetry. Here, we observe for general (not necessarily radially symmetric) data \(f\) and \(g\) the inequality (1.11) holds only if \(\beta \geq \frac{3d}{4}\), in other words, the radial symmetry condition on \(f\) and \(g\) widens the range of the regularity parameter \(\beta\). The proof is based on the Knapp type argument in [13] where they proved \(\beta \geq \frac{3d}{4}\) is necessary for (1.7) to hold.

Proposition 3.4. Let \(\beta < \frac{3d}{4}\). For any \(\beta < \frac{3d}{4}\), there exists \(f, g \in H^{\frac{2d+1}{2} + \beta}(R^d)\) such that
\[
\left\| \int \left( e^{it\phi_1(D)} f(x) e^{it\phi_2(D)} g(x) \right) \right\|^2_{L^2(R^{d+1})} > \frac{1}{C_d} \int \int \frac{1}{\left\| \phi_1(D) \phi_2(D) \right\|_{L^2(R^{d+1})}} \left( |\phi_1(D)f(x)|^2 + |\phi_2(D)g(x)|^2 \right) \right\|_{L^2(R^{d+1})},
\]
where \(\Phi_\beta(x, t; \eta_1, \eta_2) = x \cdot (\eta_1 - \eta_2) + t(\phi_1(|\eta_1|) - \phi_2(|\eta_2|))\).

Now, we follow the idea of Knapp’s example to derive a lower bound. From the setting we have \(|\eta_1| \sim |\eta_2| \sim |\phi_1(|\eta_1|)| \sim \phi_1(|\eta_1|) \sim |\eta_1 + \eta_2| \sim L, |\eta_{1(1)} - \eta_{2(1)}| \sim \theta \sim L^{-1}, |\phi_1(|\eta_1|) - \eta_{1(1)}| \sim |\eta_{1(2)}^4| \sim \theta^2 \sim L^{-1}, \) \(|\phi_1(|\eta_1|) - \eta_{2(1)}| \sim \theta^2 \sim L^{-1}, \) and \(|\eta_1' + \eta_2'| \leq 1\) for \((\eta_1, \eta_2) \in \mathcal{S} \times \mathcal{G}\). Then, it follows that
\[
(\phi_1(|\eta_1|) \phi_2(|\eta_2|))^2 \sim (\eta_1 \cdot \eta_2 - s^2)^2 \sim s^2 |\eta_1 + \eta_2|^2 + |\eta_1|^2 |\eta_2|^2 \sin^2 \theta \sim L^2
\]
and hence
\[
K_{\beta, \eta_1, \eta_2} \sim \frac{(\phi_1(|\eta_1|) \phi_2(|\eta_2|))^2 - (\eta_1 \cdot \eta_2 - s^2)^2}{\phi_1(|\eta_1|) \phi_2(|\eta_2|) + \eta_1 \cdot \eta_2 + s^2} \sim 1.
\]
Moreover, for the phase, then it follows that
\[
|\Phi_\beta(x, t; \eta_1, \eta_2)| = |t(\phi_1(|\eta_1|) - \eta_{1(1)} - \phi_2(|\eta_2|) + \eta_{2(1)} + (x_1 + t)(\eta_{1(1)} - \eta_{2(1)}) + x' \cdot (\eta_1' + \eta_2')| \\
\leq |t|L^{-1} + |x_1 + t|L^{-1} + |x'| < \frac{\pi}{3}
\]
for \((x, t) = (x_1, x', t)\) in a slab \(\mathcal{R} = [-L^{-1}, L^{-1}] \times [-1, 1]^{d-1} \times [-L, L]\) whose volume is the order of 1. Hence,
\[
|\square \beta (e^{it\phi_1(D)} f(x) e^{it\phi_2(D)} g(x))| \geq |\mathcal{S}| |\mathcal{G}| \chi_{\mathcal{R}}(x, t)
\]
and so
\[\|\| \sqrt{-\Delta}^{d-1/4} f\|_L^2 \geq \|\| \sqrt{-\Delta}^{d-1/4} g\|_L^2 \geq L^{d-1+4\beta}\|\| F\|_G\].

On the other hand, we have
\[\|\| \sqrt{-\Delta}^{d-1/4} f\|_L^2 \geq \|\| \sqrt{-\Delta}^{d-1/4} g\|_L^2 \leq L^{d-1+4\beta}\|\| F\|_G\].

Therefore, it is implied that
\[\|\| F\|_G\| \geq L^{d-1+4\beta}\|\| F\|_G\|.

The fact \[\|\| F\|_G\| \sim L\] and \[\|\| F\|_G\| \sim L\] result in the desired necessary condition
\[\frac{3}{4} - \beta = 0.

4. Sharpness of constants

Let us begin with some supplemental discussions on the refined Strichartz estimates (1.15) and (1.20). Then, we focus on completing our proof of Corollaries 1.2, 1.3 and 1.4 by proving that the stated constants are optimal and non-existence of extremisers. We achieve optimality of constants by considering the functions \[f_a\] given by (2.4); this is a natural guess given that such functions are extremisers for (1.4), as shown in our proof of Theorem 1.1. Before proceeding, we introduce the following useful notation.

\[L_a(\beta) := \int_{4\alpha a} e^{-\rho} \int_{(2a)^{-1}(\rho^2-(4a)^2)^{\frac{d-1}{4}}} (\rho^2 - (2ar)^2)^{\frac{d-1}{4}} d\rho d\rho,

and

\[R_a(\beta, b(\beta)) := \int_{2a} e^{-\rho\beta} (\rho^2 - (2as)^2)^{\frac{d-2}{4}} d\rho.

4.1. On the refined Strichartz estimates. It is straightforward that the estimates (1.15) with claimed constants in Corollaries 1.3 and 1.4 when \((\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4})\) and \((\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4})\) coincide with the results obtained by applying the kernel estimates (1.10) and (1.19) to (1.4), respectively. To obtain the estimates (1.15) and (1.20), we require the additional fact that
\[\int_{\mathbb{R}^d} f(x)f(y)x \cdot y dx dy \geq 0.

Figure 2. The sets \(\mathfrak{F}\) and \(\mathfrak{G}\), which are sent away from the origin along \(\eta_{1}\)-axis.
Indeed, in the wave regime, after we apply the kernel estimate (1.10) to (1.4), it follows that
\[
\int_{\mathbb{R}^{d+1}} |\tilde{f}(\eta_{1})|^2 |\tilde{f}(\eta_{2})|^2 \phi_{s}(|\eta_{1}|)\phi_{s}(|\eta_{2}|) K_0^s(\eta_{1}, \eta_{2}) \, d\eta_{1} \, d\eta_{2} \\
\leq \int_{\mathbb{R}^{d+1}} |\tilde{f}(\eta_{1})|^2 |\tilde{f}(\eta_{2})|^2 \phi_{s}(|\eta_{1}|)\phi_{s}(|\eta_{2}|) (\phi_{s}(\eta_{1})\phi_{s}(\eta_{2}) - s^2) \, d\eta_{1} \, d\eta_{2},
\]
which immediately yields (1.15). Similarly, one can deduce (1.20) in the non-wave regime. Finally, the estimate (1.5) with \(C = F(\frac{d-1}{4}, d)^{\frac{1}{2}}\) when \((\alpha, \beta) = (1, \frac{5-d}{4})\) is obtained by further estimating the kernel of (1.15)
\[
\phi_{s}(\eta_{1})\phi_{s}(|\eta_{2}|) - s^2 \leq \phi_{s}(\eta_{1})\phi_{s}(\eta_{2}).
\]
Again, we will see the sharpness of constants below. Of course, by a similar argument to the above, one can easily obtain the estimate (1.5) with
\[
C = 2^{-d+1} \pi^{-\frac{d+3}{2}} \frac{\Gamma(\frac{d-1}{2} + 2\beta)}{s \Gamma(\frac{d-1}{2})}
\]
when \((\alpha, \beta) = (1, \frac{5-d}{4})\) from (1.20) in the non-wave regime, and it is natural to hope that the constant is still optimal. We do not, however, know whether or not the constant (4.1) is optimal, which will become clear from the following argument on the sharpness of constants.

4.2. Wave regime. Recall \(\beta_d = \max\{\frac{1+d}{4}, \frac{5-d}{4}\}\). We shall consider (1.5) with \((\alpha, \beta) = (\frac{d+1}{4} + \beta, \beta)\) for \(\beta \in (\beta_d, \infty)\). For \(f_a\) given by (2.4), one can observe that
\[
\|\| \|e^{it\phi_s(D)} f_a\|_{L^2(\mathbb{R}^{d+1})}^2 = 2^{-\frac{3d+7}{2} - 2\beta} |\mathcal{KG}(\beta, d)(2a)^{-2d+5-4\beta} L_a(\beta)|
\]
and
\[
\|\phi_s(D)^{\frac{d+1}{4} + \beta} f_a\|_{L^4(\mathbb{R}^d)}^4 = (2\pi)^{-2d} |S^{d-1}| (2a)^{-3d+5-4\beta} R_a(\beta, \frac{d+3}{2} + 2\beta),
\]
and so it is enough to show
\[
\lim_{a \to 0} \frac{\|\phi_s(D)^{\frac{d+1}{4} + \beta} f_a\|_{L^4(\mathbb{R}^d)}^4}{\|\phi_s(D)^{\frac{d+1}{4} + \beta} f_a\|_{L^2(\mathbb{R}^{d+1})}^2} = \lim_{a \to 0} (2\pi)^d C(\beta, d) \frac{L_a(\beta)}{R_a(\beta, \frac{d+3}{2} + 2\beta)} = F(\beta, d),
\]
where
\[
C(\beta, d) = 2^{-2(d-2)} \pi^{-\frac{d+1}{2}} \frac{\Gamma(\frac{d+1}{2} + 2\beta)}{\Gamma(d-1 + 2\beta)}.
\]
Since we have, by appropriate change of variables,
\[
L_a(\beta) = e^{-4as}(2a)^{-d} \int_0^\infty e^{-\rho} \rho^{\frac{d-1}{2} + 2\beta} (\rho + 8as)^{\frac{d-1}{2} + 2\beta} \int_0^1 \frac{(1 - \nu^2)(d+2\beta)(d+1)}{(\rho + 4as)^2(1 - \nu^2) + (4as)^2\nu^2} \, d\nu \, d\rho
\]
and
\[
R_a(\beta, \frac{d+3}{2} + 2\beta) = e^{-4as} \left( \int_0^\infty e^{-\rho} (\rho + 2as)^{\frac{d+3}{2} + 2\beta} \rho^{\frac{d-2}{2}(\rho + 4as)^2\nu^2} \, d\nu \right)^2,
\]
one may deduce
\[
\lim_{a \to 0} (2\pi)^d \frac{L_a(\beta)}{R_a(\beta, \frac{d+3}{2} + 2\beta)} = \frac{\Gamma(3d - 5 + 4\beta) B(d - 2 + 2\beta, \frac{d}{2})}{2\Gamma(\frac{2d+3}{2} + 2\beta)^2},
\]
which leads to (4.3).

In order to show the constant \(F(\frac{5-d}{4}, d)^{\frac{1}{2}}\) is sharp in (1.15), we apply a similar calculation. In particular, one may note that the right-hand side of (1.15) can be written as
\[
(2\pi)^{-2d} |S^{d-1}| (2a)^{-2d} [R_a(\frac{5-d}{4}, 1) - (2as)^2 R_a(\frac{5-d}{4}, 0)],
\]

(4.4)
instead of (4.2). One can also see the second term is negligible in the sense of the optimal constant since it vanishes while \( a \) tends to 0.

4.3. Non-wave regime. Let \( f_a \) satisfy (2.4). Note that in the non-wave regime the right-hand side of (1.5) for the pair \((\frac{d}{4} + \beta, \beta)\) is expressed as

\[
\| \phi_a(D) \frac{d}{4} + \beta \|_{L^2(\mathbb{R}^d)} = (2\pi)^{-\frac{d}{2}} |\mathcal{S}^{d-1}|^2 (2a)^{-3d+4-4\beta} R_a(\beta, \frac{d}{4} - 2 + 2\beta).
\]

Then, as we have done above, reform \( L_a(\beta) \) and \( R_a(\beta, \frac{d}{4} - 2 + 2\beta) \) as follows by some appropriate change of variables:

\[
L_a(\beta) = e^{-4a(2a)\frac{d}{4} - 4 + 2\beta} \int_0^\infty e^{-\rho} \rho^{\frac{d}{2} - 2 + 2\beta} \left( \frac{\rho}{2a} + 4s \right)^{\frac{d}{2} - 2 + 2\beta} \int_0^1 \left( \frac{\rho}{2a} + 2s \right)^2 (1 - \nu^2) + (2s)^2 \frac{\nu^2}{\rho^2} \, d\nu \, d\rho
\]

and

\[
R_a(\beta, \frac{d}{4} - 2 + 2\beta) = e^{-4a(2a)\frac{d}{4} - 4 + 2\beta} \left( \int_0^\infty e^{-\nu}(\frac{\rho}{2a} + s)^{\frac{d}{2} - 2 + 2\beta} \rho^{\frac{d}{2} - 2 + 2\beta} (\frac{\rho}{2a} + 2s)^{\frac{d}{2} - 2 + 2\beta} \, d\rho \right)^2.
\]

First, we shall consider (1.5) with \((\alpha, \beta) = (0, \frac{2-d}{4})\). By a similar argument to the wave regime above, one can easily check that

\[
\lim_{a \to \infty} (2a)^{d+1} \frac{L_a(\frac{2-d}{4}, 0)}{R_a(\frac{2-d}{4}, 0)} = 2^{d-3} s^{-1}
\]

holds, from which it follows that

\[
\lim_{a \to \infty} \frac{\| [D]^{\frac{d}{4}} e^{it\phi_a(D)} f_a \|^2_{L^2(\mathbb{R}^{d+1})}}{\| \phi_a(D) \frac{d}{4} + \beta \|_{L^2(\mathbb{R}^d)}^2} = \lim_{a \to \infty} (2a)^{d+1} C(\frac{d}{4}, d) \frac{L_a(\frac{2-d}{4}, 0)}{R_a(\frac{2-d}{4}, 0)} = \frac{2^{d+1} \pi^{\frac{d+2}{4}}}{\Gamma(\frac{d}{4})}.
\]

We now turn to (1.20), where \((\alpha, \beta) = (1, \frac{4-d}{4})\), and the argument goes almost the same as above. In this case, we have

\[
C(\frac{4-d}{4}, d) = \frac{2^{d+2} \pi^{\frac{d+2}{4}}}{\Gamma(\frac{d+2}{4})},
\]

\[
L_a(\frac{4-d}{4}) = e^{-4a(2a)^{-1}} \int_0^\infty e^{-\rho} \rho^d \left( \frac{\rho}{2a} + 4s \right)^d \left( \int_0^\infty \left( \frac{\rho}{2a} + 2s \right)^2 (1 - \nu^2) + (2s)^2 \frac{\nu^2}{\rho^2} \, d\nu \right) \, d\rho
\]

and

\[
((2a)^{d-2} e^{-4as})^{-1} \left( R_a(\frac{4-d}{4}, 1) - (2as)^2 R_a(\frac{4-d}{4}, 0) \right) = \left( \int_0^\infty e^{-\nu}(\rho + 2as)^{\frac{d}{2} - 2} \frac{\rho^{\frac{d}{2} - 2}}{\rho^{\frac{d}{2} - 2}} \, d\rho \right)^2 - \left( 2as \right) \int_0^\infty e^{-\nu} \rho^{\frac{d}{2} - 2} \frac{\rho^{\frac{d}{2} - 2}}{\rho^{\frac{d}{2} - 2}} \, d\rho = (2a) \left( \int_0^\infty e^{-\rho} \rho^{\frac{d}{2} - 2} \left( \frac{\rho}{2a} + 2s \right)^2 \, d\rho \right) \left( \int_0^\infty e^{-\rho} \rho^d \left( \frac{\rho}{2a} + 2s \right)^{\frac{d}{2} - 2} \, d\rho \right).
\]
Hence, one can easily check that
\[
\lim_{a \to \infty} \frac{\|\phi_s(D)f_a\|^4_{L^2(B^d, \mathbb{R}^d)}}{\|\phi_s(D)^2f_a\|^2_{L^2(\mathbb{R}^d)}} = \lim_{a \to \infty} (2a)^{d+1} C(\frac{d-1}{4}, d) \frac{L_a(\frac{4-d}{4}, 1) - (2as)^2 R_a(\frac{4-d}{4}, 0)}{2^{-d+2}\pi^{-\frac{d-2}{2}} \Gamma(d+1)} \int_0^\infty (1 - \nu^2) \frac{\nu^{d-1}}{s}\Gamma(\frac{d}{2}) \frac{d\nu}{\Gamma(\frac{d}{2} + 1)}
\]
(4.6)

by noticing \(\int_0^\infty (1 - \nu^2) \frac{\nu^{d-1}}{s}\Gamma(\frac{d}{2}) \frac{d\nu}{\Gamma(\frac{d}{2} + 1)} = \frac{1}{2} B(\frac{d}{2} + 1, \frac{d}{2}).\)

In contrast to the wave regime, the squared right-hand side of (1.15) without the constant can be expressed as
\[
(2\pi)^{-2d}\nu^{d-1} \beta (2a)^{-2d}(R_a(\frac{4-d}{4}, 1) - (2as)^2 R_a(\frac{4-d}{4}, 0)).
\]

Unlike the wave regime, however, \(a\) is sent to \(\infty\) (instead of 0) to derive (4.6) and the second term of (1.15) does not vanish. Hence, we cannot follow the argument for the wave regime and do not know whether the constant (4.1) is still optimal for (1.5) when \((\alpha, \beta) = (1, \frac{4-d}{4}).\)

4.4. Non-existence of an extremiser. Suppose there were non-trivial \(f\) and \(g\) that satisfy any of the statements in Corollary [1.4] with equality. From our proof via Theorem [1.1] it would be required that
\[
\int_{\mathbb{R}^d} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K^{d-2+2\beta}(\eta_1, \eta_2) \, d\eta_1 \, d\eta_2 = 2^{-\frac{1}{2}} s^{-1} \int_{\mathbb{R}^d} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K^{d-2+2\beta}(\eta_1, \eta_2) \, d\eta_1 \, d\eta_2
\]
holds. Then,
\[
\int_{\mathbb{R}^d} |\hat{f}(\eta_1)|^2 |\hat{g}(\eta_2)|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) \left( K^{d-2+2\beta}(\eta_1, \eta_2) - 2^{-\frac{1}{2}} s^{-1} K^{d-2+2\beta}(\eta_1, \eta_2) \right) \, d\eta_1 \, d\eta_2 = 0
\]
would hold. Since \(f\), \(g\) are assumed to be non-trivial \(\hat{f}, \hat{g} \neq 0\) on some set \(\mathcal{G} \times \mathcal{G} \subseteq \mathbb{R}^{2d}\) with \(|\mathcal{G}||\mathcal{G}| > 0\), it would be deduced that
\[
K^{d-2+2\beta}(\eta_1, \eta_2) - 2^{-\frac{1}{2}} s^{-1} K^{d-2+2\beta}(\eta_1, \eta_2) = 0
\]
on \((\mathcal{G} \times \mathcal{G}) \setminus \mathcal{R}\) where \(\mathcal{R} \subseteq \mathbb{R}^{2d}\) is a null set. However, (4.7) would hold only on the diagonal line \(\{(\eta_1, \eta_2) : \eta_1 = \eta_2\}\) (the equality condition of (1.19)), which is a null set and so is \(\{(\eta_1, \eta_2) : \eta_1 = \eta_2\} \cap (\mathcal{G} \times \mathcal{G})\). This is a contradiction.

For Corollary [1.2] Corollary [1.3] similar arguments above can be carried. In particular, for equality in the wave regime, the formula (4.7) might be replaced by
\[
K^{d-2+2\beta}(\eta_1, \eta_2) - K^{d-2+2\beta}(\eta_1, \eta_2) = 0
\]
on \((\mathcal{G} \times \mathcal{G}) \setminus \mathcal{R}\), which would only occur when \(s = 0\) (the equality condition of (1.9)).
5. Analogous Results for \((++\)) Case

Here we note the analogous versions of our main results in the \((++\)) case. Here we observe an interesting phenomenon; the null-form \(|\square - (2s)^2|\) instead of \(\square\) somehow fits into the estimate in this framework, which does not occur in similar discussions for the wave propagators in \([11]\).

**Theorem 5.1.** For \(d \geq 2\) and \(\beta > \frac{2-d}{4}\), we have the sharp estimate

\[
\|(\square - (2s)^2)^{\beta}(e^{it\phi_s(D)} f e^{it\phi_s(D)} g)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq KG_{(++)}(\beta, d) \int_{\mathbb{R}^d} \left| \hat{f}(\eta_1) \right|^2 \left| \hat{g}(\eta_2) \right|^2 \phi_s(|\eta_1|) \phi_s(|\eta_2|) K \frac{d^2 - \beta}{2} \phi_s(\eta_1, \eta_2) d\eta_1 d\eta_2,
\]

where

\[
KG_{(++)}(\beta, d) := 2^{-\frac{5d+1}{2} + 2\beta} \frac{\Gamma\left(\frac{4-d}{4}\right)}{\Gamma(d-1)}.
\]

Moreover, the constant \(KG_{(++)}(\beta, d)\) is sharp.

One may note that by invoking the Legendre duplication formula; \(\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1 - z} \pi^{\frac{1}{2}} \Gamma(2z)\),

\[
KG_{(++)}(\beta, d) = 2^{-\frac{5d+1}{2} + 2\beta} \frac{\pi^{-\frac{5d+2}{4}}}{\Gamma\left(\frac{4-d}{4}\right)}
\]

holds, which is the same constant introduced in \([11]\).

An extra symmetry in the \((++\)) case allows \((5.1)\) a wider range of \(\beta\) than \(\beta > \frac{1-d}{4}\) for \((1.4)\). Indeed, the condition \(\beta > \frac{1-d}{4}\) is no longer imposed because of the form of the sharp constant \(KG_{(++)}(\beta, d)\), and the alternative lower bound of \(\beta\) emerges from the kernel \(K \frac{d^2 - \beta}{2}\). To see this, let us consider an extremiser \(f = g = f_1\) in \((2.4)\) and after applying polar coordinates to the right-hand side of \((5.1)\) (without the constant) we obtain

\[
\int e^{-\phi_s(r_1) + \phi_s(r_2)} (\phi_s(r_1) \phi_s(r_2))^{-1} (r_1 r_2)^{d-1}
\]

\[
\times \int_{-1}^{1} \left( \phi_s(r_1) \phi_s(r_2) - s^2 - r_1 r_2 \lambda \right) \frac{d^2 + 2\beta}{2} - \lambda \left(1 - \lambda^2\right)^{\frac{d-2}{2}} d\lambda d\eta_1 d\eta_2.
\]

By \((1.18)\), one may observe that \((5.2)\) is bounded (up to some constant) by

\[
s^{-3} \int e^{-\phi_s(r_1) + \phi_s(r_2)} (\phi_s(r_1) \phi_s(r_2))^{\frac{d^2 - \beta}{2} + 2\beta} \int_{-1}^{1} \left(1 - \frac{r_1 r_2}{\phi_s(r_1) \phi_s(r_2) - s^2} \right) \frac{d^2 + 2\beta}{2} - \lambda \left(1 - \lambda^2\right)^{\frac{d-2}{2}} d\lambda d\eta_1 d\eta_2
\]

and finite whenever \(\beta > \frac{3-d}{4}\) by applying Lemma 3.1.

In order to state the various results which follow from Theorem 5.1, here we introduce the following constant for the wave regime

\[
F_{(++)}(\beta, d) = 2^{-1 + 4\beta} \pi^{\frac{d+1}{2}} \frac{\Gamma(d - 2 + 2\beta)}{\Gamma\left(\frac{4-d}{4} + 2\beta\right)}.
\]

**Corollary 5.2.** Let \(d \geq 2\), \(\beta \geq \frac{2-d}{4}\). Then, there exists a constant \(C > 0\) such that

\[
\|\square - (2s)^2\|^2(e^{it\phi_s(D)} f e^{it\phi_s(D)} g)\|_{L^2(\mathbb{R}^{d+1})} \leq C \|\phi_s(D)\|^2 \frac{d^2 + 2\beta}{2} \|f\|_{L^2(\mathbb{R}^{d+1})} \|g\|_{L^2(\mathbb{R}^{d+1})}
\]

holds whenever \(f\) and \(g\) are radially symmetric. Moreover, for \(\beta \in \left[\frac{2-d}{4}, \frac{3-d}{4}\right] \cup \left[\frac{5-d}{4}, \infty\right)\), the optimal constant in \((5.3)\) for radially symmetric \(f\) and \(g\) is \(F_{(++)}(\beta, d)^{\frac{1}{2}}\), but there does not exist a non-trivial pair of functions \((f, g)\) that attains equality.
We remark that, following the argument in Section 3.2 once we apply Theorem 5.1 as a first step, it is not possible to obtain the constant \( F_{(\cdot\cdot)}(\beta, d)^{\frac{\beta}{\alpha}} \) for \( \beta \in (\frac{3-d}{4}, \frac{5-d}{4}) \).

**Corollary 5.3.** Let \( d \geq 2 \).

(i) The estimate

\[
\|\Box - (2s)^{2\beta}(e^{it\phi_s(D)}f)^2\|_{L^2(\mathbb{R}^{d+1})} \leq C\|\phi_s(D)^\alpha f\|_{L^2(\mathbb{R}^d)}^2
\]

holds with the optimal constant \( C = F_{(\cdot\cdot)}(\beta, d)^{\frac{\beta}{\alpha}} \) for \( (\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4}) \) and \( (\alpha, \beta) = (1, \frac{3-d}{4}) \), but there are no extremisers. Furthermore, when \( (\alpha, \beta) = (1, \frac{5-d}{4}) \), we have the refined Strichartz estimate

\[
\|\Box - (2s)^{2\beta}(e^{it\phi_s(D)}f)^2\|_{L^2(\mathbb{R}^{d+1})} \leq \left(2^{2-d} \pi^{\frac{-d+1}{2}} \Gamma(\frac{d-1}{2}) \right) \left(2\|\phi_s(D)f\|_{L^2(\mathbb{R}^d)} - s^2\|\phi_s(D)^{\frac{\beta}{\alpha}} f\|_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}},
\]

where the constant is optimal and there are no extremisers.

(ii) The estimate (5.4) holds with the optimal constant

\[
C = \left(2^{2-d} \pi^{\frac{-d+1}{2}} \Gamma(\frac{d-1}{2}) \right) \left(2\|\phi_s(D)f\|_{L^2(\mathbb{R}^d)} - s^2\|\phi_s(D)^{\frac{\beta}{\alpha}} f\|_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}},
\]

for \( (\alpha, \beta) = (\frac{1}{2}, \frac{2-d}{4}) \), but there are no extremisers. Furthermore, when \( (\alpha, \beta) = (1, \frac{2-d}{4}) \), we have the refined Strichartz estimate

\[
\|\Box - (2s)^{2\beta}(e^{it\phi_s(D)}f)^2\|_{L^2(\mathbb{R}^{d+1})} \leq \left(2^{2-d} \pi^{\frac{-d+1}{2}} \Gamma(\frac{d-1}{2}) \right) \left(2\|\phi_s(D)f\|_{L^2(\mathbb{R}^d)} - s^2\|\phi_s(D)^{\frac{\beta}{\alpha}} f\|_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}},
\]

where the constant is optimal and there are no extremisers.

Theorem [5.1] follows from adapting the calculation by Jeavons [21] without any major difficulty. Indeed, he has computed by the Cauchy–Schwarz inequality that

\[
|\tilde{uv}(\tau, \xi)| \leq J^0(\tau, \xi) \int_{|\xi|\neq 0} |F(\eta_1, \eta_2)|^2 \delta \left(\frac{|\tau - \phi_s(\eta_1)|}{\xi - \eta_1} - \phi_s(\eta_2)\right) d\eta_1 d\eta_2.
\]

Here, \( u(t, x) = e^{it\phi_s(\sqrt{-1})}f(x) \), \( v(t, x) = e^{it\phi_s(\sqrt{-1})}g(x) \),

\[
F(\eta_1, \eta_2) = \hat{f}(\eta_1)\hat{g}(\eta_2)\psi_s(\eta_1)^{\frac{\beta}{\alpha}} \psi_s(\eta_2)^{\frac{\beta}{\alpha}},
\]

and \( J^\beta \) is given by (2.2). In terms of the Lorentz invariant measure \( d\sigma_s(t, x) = \frac{4(t - \phi_s(x))}{\phi_s(x)} dzdt \), \( J^0(\tau, \xi) \) is also written as

\[
J^0(\tau, \xi) = \sigma_s * \sigma_s(\tau, \xi).
\]

Invoking Lemma 1 in [21], we have

\[
J^0(\tau, \xi) = \frac{|\tilde{uv}|^{\frac{\beta}{\alpha}}}{2^{d-2}} \left(\tau^2 - |\xi|^2 - (2s)^{\frac{d-2}{\alpha}}\right)^{\frac{\beta}{2}}
\]

so that

\[
\|\Box - (2s)^{2\beta}(e^{it\phi_s(D)}f e^{it\phi_s(D)}g)^2\|_{L^2(\mathbb{R}^{d+1})}^2
\]

\[
= (2\pi)^{-d+1} \int_{\mathbb{R}^{d+1}} |\tilde{uv}|^{2\beta} \left(\tau^2 - |\xi|^2 - (2s)^{\frac{d-2}{\alpha}}\right)^{\frac{\beta}{2}} d\xi dx
\]

\[
\leq \frac{2^{-d+2\beta + 2\pi^{\frac{-d+1}{2}}}}{\Gamma(\frac{d-1}{2})} |\tilde{uv}|^{\frac{\beta}{\alpha}} \left(\tau^2 - |\xi|^2\right)^{\frac{\beta}{2}} d\eta_1 d\eta_2,
\]
which is what we desired.

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