Binary Darboux transformation for the Sasa–Satsuma equation

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Abstract
The Sasa–Satsuma equation is an integrable higher-order nonlinear Schrödinger (NLS) equation. Higher-order and multicomponent generalizations of the NLS equation are important in various applications, e.g., in optics. One of these equations is the Sasa–Satsuma equation. We present the binary Darboux transformations (BDTs) for the Sasa–Satsuma equation and then construct its quasideterminants solutions by iterating its BDTs. Single-hump, double-hump, breather and resonant two-solitons solutions are given as explicit examples.

Keywords: Sasa–Satsuma equation, binary Darboux transformation, quasideterminants
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(Some figures may appear in colour only in the online journal)

1. Introduction

The celebrated nonlinear Schrödinger (NLS) equation

\[iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0\]  \hspace{1cm} (1.1)

is considered to be one of the fundamental integrable equations admitting an \(n\)-soliton solution. It is proved integrable via the inverse scattering transform \cite{35}. The NLS equation has applications in a wide variety of physical systems such as water waves \cite{3, 4, 36}, plasma physics \cite{37} and nonlinear optics \cite{13, 14}. This equation can be used to model short soliton pulses in optical fibres \cite{18}. The basic phenomena are described by the NLS equation (1.1),
but as the pulses get shorter various additional effects become important. In light of this fact, Kodama and Hasegawa [19, 20] developed a suitable higher-order NLS equation to take these additional effects into account. Their equation takes the form

\[ i q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + i \delta \left[ \beta_1 q_{xxx} + \beta_2 |q|^2 q_x + \beta_3 q \left( |q|^2 \right)_x \right] = 0, \]  

(1.2)

where the \( \alpha_i, \beta_i \) are real constants, \( \delta \) is a real spectral parameter and \( q \) is a complex-valued function. The first three terms (setting \( \delta = 0 \) and \( \alpha_2 = 2\alpha_1 = 1 \)) form the standard NLS equation (1.1). In general, the Kodama–Hasegawa higher-order NLS equation (1.2) is not integrable unless some restrictions are imposed on the real constants \( \beta_i (i = 1, 2, 3) \). With appropriate choices of these real constants, the inverse scattering transform can be applied to verify integrability of the resulting equation. So far it is known that, along with the NLS equation (1.1) itself, there are four cases in which integrability can be proved through the existence of an inverse scattering transform. These are the Chen–Lee–Liu [5] derivative NLS equation \((\beta_1; \beta_2; \beta_3 = 0: 1: 0)\), the Kaup–Newell [16] derivative NLS equation \((\beta_1; \beta_2; \beta_3 = 0: 1: 1)\), the Hirota [15] NLS equation \((\beta_1; \beta_2; \beta_3 = 1: 6: 0)\) and the Sasa–Satsuma [29] NLS equation \((\beta_1; \beta_2; \beta_3 = 1: 6: 3)\).

Sasa and Satsuma [29] consider the case where \( \alpha_1 = \frac{1}{2} \) and \( \alpha_2 = 1 \), that is

\[ i q_t + \frac{1}{2} q_{xx} + |q|^2 q + i \delta \left[ q_{xxx} + 6 |q|^2 q_x + 3q \left( |q|^2 \right)_x \right] = 0. \]  

(1.3)

They introduced variable transformations

\[ u(x, t) = q(X, T) \exp \left\{ -\frac{i}{6\delta} \left( X - \frac{T}{18\delta} \right) \right\}, \quad t = T, \quad x = X - \frac{T}{12\delta}. \]  

(1.4)

Then, setting \( \delta = 1 \), the equation (1.3) is reduced to a complex modified KdV-type equation

\[ u_t + u_{xxx} + 6 |u|^2 u_x + 3u \left( |u|^2 \right)_x = 0 \]  

(1.5)

which is an equivalent version of (1.3). The equation (1.5) is commonly known as the Sasa–Satsuma (SS) equation, and we will denote it as such from now on. The integrability of this equation has been widely studied with various methods such as the inverse scattering scheme [25, 29], the Hirota’s bilinear approach [9, 10], Bäcklund transformation [22, 30] and the Darboux-like transformations [31, 32]. It is proven by these methods that various solutions of the Sasa–Satsuma equation (1.5) exist such as periodic, soliton and rogue wave solutions. The exact rogue wave solutions of the equation (1.5) were first shown by Bendelow and Akhmediev [1, 2]. In [28], Ohta constructed the dark soliton solutions of the Sasa–Satsuma equation (1.5) by using Gram-type determinants. In [31], the authors presented breather, single and double-hump soliton solutions.

In the present paper, we present a systematic approach to the construction of (1.5) by means of a standard binary Darboux transformation (BDT) and written in terms of quasi-determinants [7, 8]. Quasideterminants have various useful properties which play important roles in constructing exact solutions of integrable systems [11, 12, 21, 26, 34].

In this paper, we establish for the first time a standard BDT for the SS equation (1.5). The strategy we employ here is based on the philosophy that the form of the Darboux transformation is universal and the most general form applicable to a particular system can be found by embedding it in a higher dimensional system and then making an appropriate dimensional reduction. The general form of solutions we obtain are written in terms of quasideterminant rather than determinants. We use this formulation since the building blocks for these solutions are noncommutative \((3 \times 2)\) matrices. In the end however, these solutions can be expressed
explicitly as ratios of determinants. Finally, we present a few explicit examples of these solutions.

This paper is organized as follows. In section 1.1 below, we give a brief review on quasideterminants. In section 2, we construct a $3 \times 2$ eigenfunction and corresponding constant $2 \times 2$ square matrix for the eigenvalue problems of the SS equation (1.5) using two symmetries of the Lax pair of the SS equation. In section 3.2, we state a standard binary Darboux theorem for the Sasa–Satsuma system. In section 3.3, we review the reduced BDTs for the SS equation, which can be considered as a dimensional reduction from $(2 + 1)$ to $(1 + 1)$ dimensions. In section 4, we present the quasigrammian solutions of the SS equation obtained by using the BDT. Here, the quasigrammians are written in a general way in terms of arbitrary solutions of linear eigenvalue problems. In section 5, single-hump soliton, double-hump soliton, breather and resonant two-solitons solutions of the SS equation are given for both zero and non-zero seed solutions. The conclusion is given in the final section 6.

1.1. Quasideterminants

In this short section we recall some of the key elementary properties of quasideterminants. The reader is referred to the original papers [7, 8] for a more detailed and general treatment.

Quasideterminants were introduced by Gelfand and Retakh in [7] as a natural generalization of the determinant to matrices with noncommutative entries. Many equivalent definitions of quasideterminants exist, one such being a recursive definition involving inverse minors. Let $M = (m_{ij})$ be an $n \times n$ matrix with entries over an, in general non-commutative, ring $\mathcal{R}$. Then the quasideterminants of $M$ for $i, j = 1, \ldots, n$ are defined by

$$|M|_{ij} = m_{ij} - r_i^j (M^{(j)})^{-1} c_j^i,$$

where $r_i^j$ is the row vector obtained from $i$th row of $M$ with the $j$th element removed, $c_j^i$ is the column vector obtained from $j$th column of $M$ with the $i$th element removed and $M^{(j)}$ is the $(n - 1) \times (n - 1)$ submatrix obtained by deleting the $i$th row and the $j$th column from $M$. Quasideterminants can also be denoted by boxing the entry about which the expansion is made

$$|M|_{ij} = \begin{vmatrix} M^{(j)} & c_j^i \\ r_i^j & \end{vmatrix}.\quad (1.7)$$

If the entries in $M$ happen to commute, then the quasideterminant $|M|_{ij}$ can be expressed as a ratio of determinants

$$|M|_{ij} = (-1)^{i+j} \frac{\det M}{\det M^{(j)}}.\quad (1.8)$$

In this paper, we will consider only quasideterminants that are expanded about a term in the last column, most usually the last entry. For example considering a block matrix $M = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$, where $A$ is an invertible (square) matrix over $\mathcal{R}$ of arbitrary size and $B$, $C$ are column and row vectors over $\mathcal{R}$ of compatible lengths, respectively, and $d \in \mathcal{R}$, the quasideterminant of $M$ is expanded about $d$ is

$$\begin{vmatrix} A & B \\ C & d \end{vmatrix} = d - CA^{-1}B.\quad (1.9)$$

Later we will use the following invariance of quasideterminants which follows immediately from their definition. Let $\alpha$ and $\beta$ be invertible matrices of the same dimensions as $A$. Then
\[
\begin{pmatrix}
\alpha A \beta & \alpha B \\
C \beta & \mathcal{D}
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & \mathcal{D}
\end{pmatrix}
\quad \text{(1.10)},
\]

2. The Lax pair for the Sasa–Satsuma equation

The Lax pair [29] for the Sasa–Satsuma equation (1.5) is given by

\[
L = \partial_t + J \lambda + R,
\]

\[
M = \partial_t + 4J \lambda^3 + 4R \lambda^2 - 2Q \lambda + W,
\]

where \(J, R, Q\) and \(W\) are \(3 \times 3\) matrices such that

\[
J = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i \\
i & 0 & 0
\end{pmatrix},
R = \begin{pmatrix}
0 & 0 & -u \\
0 & 0 & -u^* \\
u^* & u & 0
\end{pmatrix},
Q = \begin{pmatrix}
|u|^2 & u_x^2 & u_x \\
u_x^2 & |u|^2 & u_x^* \\
u_x^* & -2|u|^2 & u_x
\end{pmatrix} \quad \text{(2.3)}
\]

and

\[
W = \begin{pmatrix}
0 & -4u^* |u|^2 - u_{xx} \\
-4u^* |u|^2 + u_{xx} & 0 \\
-4u^* u_{xx} & 0
\end{pmatrix} \quad \text{(2.4)}
\]

Here \(\lambda\) is a spectral parameter and asterisk \(^*\) denotes the complex conjugate. It can be seen that the potential matrix \(R\) in (2.3) has two symmetry properties [17, 33]. One is that it is skew-Hermitian: \(R^\dagger = -R\). The other one is that \(\Sigma R S = R^*\), where

\[
S = S^{-1} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{(2.5)}
\]

Let us now consider the eigenvalue problems \(L_\phi(M(\phi) = 0\) for the SS equation (1.5), where \(L, M\) are two linear operators given by (2.1)-(2.2) and \(\phi = (\phi_1, \phi_2, \phi_3)^T\) is an eigenfunction. It may be seen that \(\tilde{\phi} = S\phi^* = (\phi_2^*, \phi_3^*, \phi_1^*)^T\) is an eigenfunction for the linear problems \(L_{s\phi}(\phi) = M_{s\phi}(\phi) = 0\), where \(S\) is the symmetry matrix given by (2.5). It follows from \(L(\theta) = M(\theta) = 0\) that

\[
\theta_1 + J \theta \Lambda + R \theta = 0, 
\]

\[
\theta_1 + 4J \theta \Lambda^3 + 4R \theta \Lambda^2 - 2Q \theta \Lambda + W \theta = 0,
\]

where \(\theta\) is a \(3 \times 2\) eigenfunction and \(\Lambda\) a constant \(2 \times 2\) square matrix such that

\[
\theta = \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}, \quad \Lambda = \begin{pmatrix}
\lambda & 0 \\
0 & -\lambda^*
\end{pmatrix} \quad \text{(2.8)}
\]
3. Darboux transformations and dimensional reductions

3.1. Darboux transformation

Let us consider the linear operators

\[ L = \partial_x + \sum_{i=0}^{N} u_i \partial_y^i, \quad M = \partial_t + \sum_{i=0}^{N} v_i \partial_y^i, \]

where \( u_i, v_i \) are \( m \times m \) matrices. The standard approach to Darboux transformations \([6, 23, 24]\) involves a gauge operator \( G_\theta = \theta \partial_y \theta^{-1} \), where \( \theta = \theta(x, y, t) \) is an invertible \( m \times m \) matrix solution to a linear system

\[ L(\phi) = M(\phi) = 0. \]

If \( \phi \) is any eigenfunction of \( L \) and \( M \) then \( \hat{\phi} = G_\theta(\phi) \) satisfies the transformed system \( \hat{L}(\hat{\phi}) = \hat{M}(\hat{\phi}) = 0 \) where \( \hat{L} = G_\theta LG_\theta^{-1} \) and \( \hat{M} = G_\theta MG_\theta^{-1} \) have the same forms as \( L \) and \( M \) but with different coefficients.

3.2. Binary Darboux transformation

The corresponding BDT is given by the operator

\[ B_{\theta, \rho} = I - \theta \Omega(\theta, \rho)^{-1} \partial_y \rho^\dagger, \quad \text{where} \quad \Omega(\theta, \rho)_y = \rho^\dagger \theta, \]

\( \theta \) and \( \rho \) are eigenfunction and adjoint eigenfunction respectively, and \( ^\dagger \) denotes the Hermitian conjugate. It should be noted out that this transformation makes sense for any \( m \times k \) not just invertible \( m \times m \) matrices \( \theta \) and \( \rho \). Here, the only requirement is that the eigenfunction potential \( \theta(\rho, \rho) \) is an invertible square matrix.

Given eigenfunctions \( \theta_i \) and adjoint eigenfunctions \( \rho_i \) of the base Lax pair, the BDT is iterated via the formulae

\[ \phi_{[n+1]} = B_{\theta, \rho} \phi_{[n]} = \phi_{[n]} - \theta_{[n]} \Omega(\theta_{[n]}, \rho_{[n]})^{-1} \Omega(\theta_{[n]}, \rho_{[n]}), \]

\[ \psi_{[n+1]} = B_{\theta, \rho}^\dagger \psi_{[n]} = \psi_{[n]} - \rho_{[n]} \Omega(\theta_{[n]}, \rho_{[n]})^{-1} \Omega(\theta_{[n]}, \rho_{[n]}), \]

where

\[ \theta_{[n]} = \phi_{[n]} |_{y \to \theta_{[n]}}, \quad \rho_{[n]} = \psi_{[n]} |_{y \to \rho_{[n]}}. \]

Using the notation \( \Theta = (\theta_1, \ldots, \theta_n) \) and \( P = (\rho_1, \ldots, \rho_n) \), we can express these results in terms of quasideterminants

\[ \phi_{[n+1]} = \Omega(\Theta, P)_\Theta \Omega(\phi, P)_\Theta, \quad \psi_{[n+1]} = \Omega(\Theta, P)_P \Omega(\Theta, \psi)_P, \]

and

\[ \Omega\left( \begin{array}{c} \phi_{[n+1]} \\ \psi_{[n+1]} \end{array} \right) = \Omega(\Theta, P) \Omega(\phi, P) \Omega(\Theta, \psi). \]

3.3. Dimensional reductions of the BDT

Here, we describe a reduction of the BDT from \((2 + 1)\) to \((1 + 1)\) dimensions. We choose to eliminate the \( y \)-dependence by employing a ‘separation of variables’ technique. The reader is
referred to the paper [27] for the details. We make the $y$-dependence explicit:

$$\phi(x, y, t) = \phi'(x, t)e^{\lambda y}, \quad \theta(x, y, t) = \theta'(x, t)e^{\lambda y},$$

(3.9)

$$\psi(x, y, t) = \psi'(x, t)e^{\mu y}, \quad \rho(x, y, t) = \rho'(x, t)e^{\mu y},$$

(3.10)

where $\lambda, \mu$ are constant scalars and $\Lambda, \Pi$ are $k \times k$ constant matrices. Hence the operator $L$ in (3.1) becomes

$$L' = \partial_x + \sum_{i=0}^n u_i \lambda_i,$$

(3.11)

and $\theta'$ is an $m \times k$ eigenfunction satisfying

$$\theta'_x + \sum_{i=0}^n u_i \theta' \mathcal{N} = 0.$$

(3.12)

Similarly adjoint eigenfunction $\rho'$.

The $y$-dependence of the potential $\Omega$ can be made explicit by setting $\Omega(\theta, \rho) = e^{\Omega y} \Omega'(\theta', \rho')e^{\lambda y}$. Then $\Omega'$ is defined by the algebraic condition

$$\Pi \Omega'(\theta', \rho') + \Omega'(\theta', \rho') \Lambda = \rho'^\dagger \theta'.$$

(3.13)

From now on, for notational simplicity, we omit the superscript $r$.

### 4. Quasigrammian solutions of the Sasa–Satsuma equation

In this section we determine the effect of the BDT on the operator $L$ given by (2.1). Corresponding results hold for the operator $M$ given by (2.2). The Lax operators $L$ and $M$ are both skew-Hermitian and this allows us to choose adjoint eigenfunctions to be the same as an eigenfunction and in what follows we take $\rho_j = \theta_j$ and $\Pi = -\Lambda$.

The operator $L = \partial_x + J\lambda + R$ is transformed to a new operator $\hat{L}$ in which $J$ is unchanged and

$$\hat{R} = R + \left[ J, \theta \Omega(\theta, \theta)^{-1} \theta^\dagger \right].$$

(4.1)

For notational convenience, we introduce a $3 \times 3$ matrix $Q$ such that $R = [Q, J]$, of the form

$$Q = \frac{1}{2i} \begin{pmatrix} u & u^* \\ u^* & u \end{pmatrix},$$

(4.2)

where the entries left blank are arbitrary and do not contribute to $R$. From (4.1) it follows that $\hat{R} = [\hat{Q}, J]$ where

$$\hat{Q} = Q - \theta \Omega(\theta, \theta)^{-1} \theta^\dagger,$$

(4.3)

in which, from (3.13),

$$\Omega(\theta, \theta) \Lambda - N\Omega(\theta, \theta) = \theta^\dagger \theta,$$

(4.4)

where the $3 \times 2$ eigenfunction $\theta$ and the $2 \times 2$ constant diagonal matrix $\Lambda$ are given by (2.8).

After $n$ applications of the BDT we have

$$Q_{[n+1]} = Q_{[n]} - \theta_{[n]} \Omega(\theta_{[n]}, \theta_{[n]})^{-1} \theta_{[n]}^\dagger,$$

(4.5)
and we may express this in quasigrammian form as

\[ Q_{[n+1]} = Q + \begin{bmatrix} 
\Omega(\Theta, \Theta) & \Theta^T \\
\Theta & 0 
\end{bmatrix} \Theta, \quad (4.6) \]

where \( \Theta \) is a \( 3 \times 2n \) built by concatenating the matrices \( \theta_i \) which are used in each application of the BDT. More explicitly, for \( i \geq 1 \) we index the entries in each \( \theta_i \) as follows (see (2.8)):

\[ \theta_i = \begin{pmatrix} 
\phi_{3i-2} & \phi_{3i-1}^* \\
\phi_{3i-1} & \phi_{3i-2}^* \\
\phi_{3i} & \phi_{3i}^* 
\end{pmatrix}, \tag{4.7} \]

and have

\[ \Theta = (\theta_1, \theta_2, \ldots, \theta_n) = \begin{pmatrix} 
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{pmatrix}, \tag{4.8} \]

where \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) denote the row vectors

\[ \Phi_1 = (\phi_1, \phi_1^*, \ldots, \phi_{3n-2}, \phi_{3n-1}), \tag{4.9} \]

\[ \Phi_2 = (\phi_2, \phi_2^*, \ldots, \phi_{3n-1}, \phi_{3n-2}), \tag{4.10} \]

\[ \Phi_3 = (\phi_3, \phi_3^*, \ldots, \phi_{3n}, \phi_{3n}). \tag{4.11} \]

Thus, we obtain

\[ Q_{[n+1]} = Q + \begin{bmatrix} 
\Omega(\Theta, \Theta) & \Omega(\Theta, \Theta) & \Omega(\Theta, \Theta) \\
\Phi_1 & \Phi_1 & \Phi_1 \\
\Phi_2 & \Phi_2 & \Phi_2 \\
\Phi_3 & \Phi_3 & \Phi_3
\end{bmatrix}, \quad (4.12) \]

Using (4.2) and (4.12) gives two quasigrammian expressions each for the transformed \( u \) and \( u^* \), namely

\[ u_{[n+1]} = u + 2i \begin{pmatrix} 
\Omega(\Theta, \Theta) \\
\Phi_1
\end{pmatrix} = u + 2i \begin{pmatrix} 
\Omega(\Theta, \Theta) \\
\Phi_3
\end{pmatrix}, \quad (4.13) \]

\[ u_{[n+1]}^* = u^* + 2i \begin{pmatrix} 
\Omega(\Theta, \Theta) \\
\Phi_3
\end{pmatrix} = u^* + 2i \begin{pmatrix} 
\Omega(\Theta, \Theta) \\
\Phi_2
\end{pmatrix}. \quad (4.14) \]

The proof that the above four expressions are consistent is given in section 4.1.
4.1. Proof of consistency

In the expressions (4.13)–(4.14), the potential $\Omega(\Theta, \Theta)$ is a $2n \times 2n$ matrix satisfying the relation

$$\Omega(\Theta, \Theta)\Lambda - \Lambda^t\Omega(\Theta, \Theta) = \Theta^t\Theta, \quad (4.15)$$

where $\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_n)$ and $\Lambda_i = \text{diag}(\lambda_i, -\lambda_i^*)$ (see (2.8)).

Solving (4.15) for $\Omega(\Theta, \Theta)$, we obtain

$$\Omega(\Theta, \Theta) = \begin{pmatrix} \Omega(\theta_1, \theta_1) & \Omega(\theta_2, \theta_1) & \cdots & \Omega(\theta_n, \theta_1) \\ \Omega(\theta_1, \theta_2) & \Omega(\theta_2, \theta_2) & \cdots & \Omega(\theta_n, \theta_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(\theta_1, \theta_n) & \Omega(\theta_2, \theta_n) & \cdots & \Omega(\theta_n, \theta_n) \end{pmatrix}, \quad (4.16)$$

where $\Omega(\theta_i, \theta_j)$ is a $2 \times 2$ matrix satisfying the relation

$$\Omega(\theta_i, \theta_j)\Lambda_i - \Lambda_j^t\Omega(\theta_i, \theta_j) = \theta_i^t\theta_j. \quad (4.17)$$

It follows from this relation that the potential $\Omega$ can be written explicitly as

$$\Omega(\theta_i, \theta_j) = \begin{pmatrix} F_{ij} - G_{ij}^* \\ G_{ij} - F_{ij}^* \end{pmatrix}, \quad (4.18)$$

where $F_{ij} = F_{ij}(x, t, \lambda_i, \lambda_j)$ and $G_{ij} = G_{ij}(x, t, \lambda_i, \lambda_j)$ are the scalar functions

$$F_{ij} = \frac{1}{\lambda_i - \lambda_j^*} \left( \phi_{3i-2}\phi_{3j-2}^* + \phi_{3i-1}\phi_{3j-1}^* + \phi_{3i}\phi_{3j}^* \right), \quad (4.19)$$

$$G_{ij} = \frac{1}{\lambda_i + \lambda_j} \left( \phi_{3i-2}\phi_{3j-1} + \phi_{3i-1}\phi_{3j-2} + \phi_{3i}\phi_{3j} \right). \quad (4.20)$$

Here we observe that $F_{ij}$ and $G_{ij}$ are such that $F_{ij}^* = -F_{ji}$ and $G_{ij} = G_{ji}$, for $i, j = 1, \ldots, n$. Then the $2 \times 2$ potentials $\Omega(\theta_i, \theta_j)$ satisfy the symmetry condition

$$\Omega(\theta_i, \theta_j) + \Omega(\theta_j, \theta_i)^t = 0, \quad (4.21)$$

and the $2n \times 2n$ matrix potential $\Omega(\Theta, \Theta)$, as given by (4.16), is skew-adjoint,

$$\Omega(\Theta, \Theta) + \Omega(\Theta, \Theta)^t = 0. \quad (4.22)$$

Now it is readily seen that

$$\begin{pmatrix} \Omega(\Theta, \Theta) & \Phi_1^t \\ \Phi_1 & 0 \end{pmatrix} = \begin{pmatrix} \Omega(\Theta, \Theta)^t & \Phi_1^t \\ \Phi_3 & 0 \end{pmatrix}, \quad (4.23)$$

$$= -\begin{pmatrix} \Omega(\Theta, \Theta) & \Phi_1^t \\ \Phi_3 & 0 \end{pmatrix}, \quad (4.24)$$

using (4.22), and so the expressions given (4.13) and (4.14) are indeed complex conjugate.

It remains to prove that the two expressions in (4.13) are consistent, i.e. that

$$\begin{pmatrix} \Omega(\Theta, \Theta) & \Phi_1^t \\ \Phi_1 & 0 \end{pmatrix} = \begin{pmatrix} \Omega(\Theta, \Theta) & \Phi_2^t \\ \Phi_3 & 0 \end{pmatrix}. \quad (4.25)$$
Note first that this each side in this equation represents a scalar and so the right hand side can also be written as
\[
\begin{pmatrix}
\Omega(\Theta, \Theta) & \Phi_2^e \\
\Phi_3 & 0
\end{pmatrix}^T =
\begin{pmatrix}
\Omega(\Theta, \Theta)^T & \Phi_3^T \\
\Phi_2^e & 0
\end{pmatrix},
\] (4.26)
where $^T$ denotes the matrix transpose.

Now let $\alpha$ be the $2n \times 2n$ permutation matrix
\[
\alpha = \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\] (4.27)
Pre(post)-multiplying any matrix with $2n$ (rows) columns by $\alpha$ has the effect of interchanging its $2i$th and $(2i-1)$th (rows) columns for $i = 1, \ldots, n$. Hence
\[
\Phi_1 \alpha = \Phi_2^e, \quad \Phi_2 \alpha = \Phi_1^e, \quad \Phi_3 \alpha = \Phi_3^e
\] (4.28)
and
\[
\alpha \Omega(\Theta, \Theta) \alpha = \Omega(\Theta, \Theta)^T.
\] (4.29)

Now using the quasideterminant invariance (1.10), we get
\[
\begin{pmatrix}
\Omega(\Theta, \Theta) & \Phi_2^e \\
\Phi_1 & 0
\end{pmatrix} =
\begin{pmatrix}
\alpha \Omega(\Theta, \Theta) \alpha & \alpha \Phi_3^e \\
\Phi_1 \alpha & 0
\end{pmatrix} =
\begin{pmatrix}
\Omega(\Theta, \Theta)^T & \Phi_3^T \\
\Phi_2^e & 0
\end{pmatrix}.
\]
This completes the proof.

5. Particular solutions

In order to construct particular solutions for the Sasa–Satsuma equation (1.5), we consider the quasi-Grammian solution given by (4.13)
\[
u_{n+1} = u + 2i \begin{pmatrix}
\Omega(\Theta, \Theta) & \Phi_1^e \\
\Phi_1 & 0
\end{pmatrix},
\] (5.1)
where $\Phi_1$ and $\Phi_2$ denote the first and third rows repectively of a $3 \times 2n$ matrix eigenfunction $\Theta$, and the potential $\Omega(\Theta, \Theta)$ is the $2n \times 2n$ matrix with entries defined by (4.16) and (4.18)–(4.20).

Let us consider the spectral problem $L(\phi) = M(\phi) = 0$ with eigenvalue $\lambda_j$ ($j = 1, \ldots, n$), where $\phi = (\phi_{3j-2}, \phi_{3j-1}, \phi_{3j})^T$ and $L$ and $M$ are given by (2.1)–(2.2) so that
\[
\phi_j + J \phi_j \lambda_j + R \phi = 0,
\] (5.2)
\[
\phi_j + 4J \phi_j \lambda_j^2 + 4R \phi \lambda_j^2 - 2Q \phi \lambda_j + W \phi = 0,
\] (5.3)
where $J$, $R$, $Q$ and $W$ are given by (2.3)–(2.4).
Case \((n = 1)\). In this case \(\phi = (\phi_1, \phi_2, \phi_3) \) is a solution of the spectral problem
\[ L(\phi) = M(\phi) = 0 \]
with eigenvalue \(\lambda_0\). Thus, from (5.1), we derive the following explicit solution
\[ u_{[2]} = u + 2i \begin{pmatrix} F_{11} - G_{11}^* \phi_3^* \\ G_{11} F_{11} - \phi_3 \phi_3^* \end{pmatrix} \]
which can be expanded as
\[ u_{[2]} = u - 2i \left( \phi_1 \phi_3^* + \phi_2 \phi_3 \right) F_{11} + \phi_1 \phi_3 G_{11}^* - \phi_2^* \phi_3^* G_{11} \]
\[ \frac{F_{11}^2 + |G_{11}|^2}{F_{11}^2 + |G_{11}|^2} \]
(5.5)
in which
\[ F_{11} = \frac{1}{\lambda_1 - \lambda_0} \left( |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \right) \]
(5.6)
\[ G_{11} = \frac{1}{2\lambda_1} \left( 2\phi_1 \phi_2 + \phi_2^2 \right) \]
(5.7)
By letting \(\lambda_1 = \xi + i\eta\), this solution can be written as
\[ u_{[2]} = u + 4\eta \frac{H_1 \phi_1 \phi_3^* + H_2 \phi_2 \phi_3}{\xi^2 r_1^2 + \eta^2 \left( \xi^2 + 2 \right) |\phi_1|^2 - |\phi_3|^2} \]
(5.8)
where
\[ H_1 = \xi^2 r_1 + \eta^2 r_2 - i\xi \eta \left( 2 |\phi_2|^2 - |\phi_3|^2 \right) \]
(5.9)
\[ H_2 = \xi^2 r_1 - \eta^2 r_2 + i\xi \eta \left( 2 |\phi_1|^2 - |\phi_3|^2 \right) \]
(5.10)
in which \(r_1 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\) and \(r_2 = |\phi_1|^2 - |\phi_2|^2\) are real functions. Here, for simplicity, we may choose \(\lambda_1 \in i\mathbb{R}\) such that \(\lambda_1 = i\eta\). Then, the solution (5.8) can be written as the following simplified form
\[ u_{[2]} = u + 4\eta \frac{\text{Re}}{f^2 + 2 |g|^2} \]
(5.11)
where \(f = |\phi_1|^2 - |\phi_2|^2\) and \(g = \phi_1 \phi_3^* - \phi_2^* \phi_3\) are real and complex valued functions respectively.

5.1. Solutions for zero seed

For \(u = 0\), the eigenvalue problems (5.2)–(5.3) transform into the first-order linear system
\[ \phi_x + J(\phi) \lambda = 0 \]
(5.12)
\[ \phi_x + 4J(\phi) \phi \lambda = 0 \]
(5.13)
which has solution \(\phi = (\phi_1, \phi_2, \phi_3)^T\), in which
\[ \phi_1 = c_1 e^{-i\lambda(x + 4k\xi)}, \phi_2 = c_2 e^{-i\lambda(x + 4k\xi)}, \phi_3 = c_3 e^{i\lambda(x + 4k\xi)} \]
(5.14)
where \(c_1, c_2\) and \(c_3\) are arbitrary complex constants.
Case \((n = 1)\). By substituting (5.14) into (5.8), we obtain
\[
 u_{[2]} = 4\eta^2 \frac{R_1 - i\xi_0 R_2}{R_3 - 2\eta^2 R_4},
\]
(5.15)
where
\[
 R_1 = \xi^2 \left( K e^{i\beta_1} + \left| c_3 \right|^2 e^{-i\beta_1} \right) f_1 - \eta^2 K e^{i\beta_1} f_2,
\]
(5.16)
\[
 R_2 = \left| c_3 \right|^2 e^{-i\beta_1} f_2 + 2c_1 c_2 \left| e^{i\beta_1} \right|^2 f_2^* ,
\]
(5.17)
\[
 R_3 = \left( \xi^2 K_1^2 + \eta^2 K_2^2 \right) e^{2i\beta_1} + \left| c_3 \right|^4 \xi^2 e^{-2i\beta_1} + 2 \left| c_3 \right|^2 K_1 \left( \xi^2 + \eta^2 \right),
\]
(5.18)
\[
 R_4 = K e^{2i\beta_1} + K e^{-2i\beta_1},
\]
(5.19)
in which
\[
 f_1 = c_2^* c_3 e^{i\beta_1} + c_1 c_3^* e^{-i\beta_1},
\]
(5.20)
\[
 f_2 = c_2^* c_3 e^{i\beta_1} - c_1 c_3^* e^{-i\beta_1},
\]
(5.21)
\[
 \beta_1 = 2\eta \left[ x + 4 \left( 3\xi^2 - \eta^2 \right) r \right],
\]
(5.22)
\[
 \beta_2 = 2\xi \left[ x + 4 \left( \xi^2 - 3\eta^2 \right) r \right],
\]
(5.23)
and \(K_1 = |c_1|^2 + |c_2|^2, K_2 = 2|c_1|^2 - |c_2|^2, K_3 = c_1^* c_2^* c_3^2 \).

If the constants \(c_1\) and \(c_2\) are both nonzero and \(c_3 = c_2\), the solution (5.15) is a breather. The particular choice \(c_3 = c_2 = 2c_1\) is plotted in the figure 1. For the case of \(c_3 = c_1\), we observe that (5.15) can be reduced to a real solution which is just a solution to the mKdV equation.

On the other hand, by letting \(c_2 = 0\) and \(c_3 = c_1\), (5.15) can be reduced the following soliton solution:
\[
 \left| u_{[2]} \right|^2 = 16\eta^2 \left( \xi^2 + \eta^2 \right) \left( 2\xi^2 + \eta^2 \right) \cosh(2\beta_1) + \eta^2 \sinh(2\beta_1) + 2\xi^2 \left( 2\xi^2 + \eta^2 \right) \cosh(2\beta_1) + \eta^2 \sinh(2\beta_1) + 2 \left( \xi^2 + \eta^2 \right)^2 ,
\]
(5.24)
which has been first presented by Sasa–Satsuma [29]. Analysing this solution, it can be observed the appearance of single and double-hump solitons, depending on the values of parameters \(\xi\) and \(\eta\). For the case \(|\eta| > |\xi|\), we obtain a double-hump soliton solution. This solution is plotted in the figure 2. For the case \(|\eta| \leq |\xi|\), we obtain a single-hump soliton solution. This solution is plotted in the figure 3.

5.2. Solutions for non-zero seed

Here we consider the seed solution \(u = k\), a real nonzero constant solution of the Sasa–Satsuma equation (1.5). Substituting \(u = k\) into the linear system (5.2)–(5.3) and then solving for the eigenfunction \(\phi = (\phi_{y-2}, \phi_{y-1}, \phi_y)^T\), we obtain
\[
 \phi_{y-2} = -\frac{1}{2} c_1 e^{-i\Delta x^2 (x + 4\Delta y^2 r)} + c_2 e^{i\Delta x^2 (x + 4\Delta y^2 r)} + c_3 e^{-i\Delta x^2 (x + 4\Delta y^2 r)} ,
\]
(5.25)
\[ \phi_{j-1} = \frac{1}{2} c_1 e^{-i \lambda_j (x + 4 t \eta^2)} + c_2 e^{i \sqrt{\lambda_j^2 + 2 k^2} [x + 4 (\lambda_j^2 - k^2) t]} + c_3 e^{-i \sqrt{\lambda_j^2 + 2 k^2} [x + 4 (\lambda_j^2 - k^2) t]}, \]  
\[ \phi_j = K_1 e^{i \sqrt{\lambda_j^2 + 2 k^2} [x + 4 (\lambda_j^2 - k^2) t]} + K_2 e^{-i \sqrt{\lambda_j^2 + 2 k^2} [x + 4 (\lambda_j^2 - k^2) t]}, \]  
where \( K_1 = \frac{1}{\lambda_j + \sqrt{\lambda_j^2 + 2 k^2}}, \) \( K_2 = \frac{1}{\lambda_j - \sqrt{\lambda_j^2 + 2 k^2}} \) and \( c_1, c_2, c_3 \) are arbitrary complex constants.

Case (n = 1). The solution (5.11) becomes
\[ u_{(2)} = k + 4 \eta \frac{f g}{f^2 + 2 |g|^2}. \]
where \( f = |\phi_1|^2 - |\phi_2|^2 \) and \( g = \phi_1 \phi_3^* - \phi_2^* \phi_3 \) in which

\[
\phi_1 = -\frac{1}{2} c_1 e^{i(\xi - 4\eta^2 x)} + c_2 e^{iD[x - 4(\xi + \eta^2 x)]} + c_3 e^{-iD[x - 4(\xi + \eta^2 x)]},
\]

(5.29)

\[
\phi_2 = \phi_1 + c_1 e^{i(\xi - 4\eta^2 x)},
\]

(5.30)

\[
\phi_3 = -\frac{c_2}{k} (\eta - iD) e^{iD[x - 4(\xi + \eta^2 x)]} - \frac{c_3}{k} (\eta + iD) e^{-iD[x - 4(\xi + \eta^2 x)]},
\]

(5.31)

where \( D = \sqrt{2k^2 - \eta^2} \). Here, for simplicity, we have chosen \( \lambda_1 \in \mathbb{R} \) such that \( \lambda_1 = i\eta \).

**Case** \( D^2(\eta) = 2k^2 - \eta^2 > 0 \). For the case \( D^2 = 2k^2 - \eta^2 > 0 \), (5.28) can be written as

\[
u_{(2)} = k - 2k \eta c^2 Ke^{2i\alpha} + c^2 K^* e^{-2i\alpha} + 2\eta |c|^2 - 2\eta S[(c e^{i\alpha} + c^* e^{-i\alpha}) e^{-\beta}]
\]

\[
\frac{\eta}{(c^2 K e^{2i\alpha} + c^2 K^* e^{-2i\alpha}) + 4k^2 |c|^2 + 2k^2 |S^2 e^{-2i\beta}}.
\]

(5.32)

where \( \alpha = D[x - 4(k^2 + \eta^2)x], \beta = \eta(x - 4\eta^2 x), \ K = \eta - iD, \ S = \frac{2i}{k}(|c_2|^2 - |c_3|^2)D \)

and \( c = c_1^2 + c_1 c_3^* \neq 0 \). If we choose \( c_1 = c_2 = c_3 \) then \( c = |c_2|^2 - |c_3|^2 \) and \( S = \frac{2i}{k}cD \).

Then, the solution (5.32) can be written in the following form

\[
u_{(2)} = k - 2k \eta \frac{-\eta \cos(2\alpha) + D \sin(2\alpha)}{\eta[\eta \cos(2\alpha) + D \sin(2\alpha)]} + \eta - 4iD[\cosh(\beta + \sin(\beta\beta)] \cos \alpha
\]

\[
\eta[\eta \cos(2\alpha) + D \sin(2\alpha)] + 2k^2 + 4D^2[\cosh(2\beta) - \sinh(2\beta)].
\]

(5.33)

This solution is plotted in the figure 4.
Case $D^2(\eta) = 2k^2 - \eta^2 < 0$. For the case $D^2 = 2k^2 - \eta^2 < 0$, (5.28) can be written as

$$u_{[2]} = k - \frac{\eta (e^{2\gamma} + e^{-2\gamma}) - E(e^{2\gamma} - e^{-2\gamma}) + 2\eta + 4Eh_1^{-1}h_2(e^{\gamma} + e^{-\gamma})e^{-\beta}}{\eta^2(e^{2\gamma} + e^{-2\gamma}) - \eta E(e^{2\gamma} - e^{-2\gamma}) + 4k^2 - 8E^2h_1^{-2}h_2^2e^{-2\beta}},$$

(5.34)

where $E = \sqrt{\eta^2 - 2k^2}$, $\gamma = E(x - 4(k^2 + \eta^2)t)$, $\beta = \eta(x - 4\eta^2t)$ and $h_1 = c_1c_2^* + c_1^*c_2 \neq 0$, $h_2 = c_2c_3^* - c_2^*c_3$ such that $h_1 \in \mathbb{R}$ and $h_2 \in i\mathbb{R}$. If we choose...
\[ c_3 = -i c_1 \text{ so that } h_2 = i h_1. \] Then the solution (5.34) can be written as
\[ u_{[2]} = -k - 2kE \frac{E \left( 1 - 4e^{-2\beta} \right) + 4i\gamma \cosh \gamma e^{-\beta}}{\eta^2 \cosh(2\gamma) - \eta E \sinh(2\gamma) + 2k^2 + 4E^2e^{-2\beta}}. \] (5.35)

This is resonant two-soliton solution [32]. This solution is plotted in the figure 5. In the case of non-zero background, the solution which ought to be a 1-soliton solution exhibits a resonance-type behaviour and for this reason we call them resonant two-soliton solutions.

6. Conclusion

In this paper, we have presented a standard BDT for the SS equation (1.5) and using this we have constructed a wide family of solutions in quasigrammian form. These quasigrammians are expressed in terms of solutions of the linear partial differential equations given by (2.6)–(2.7). Moreover, single and double-hump solitons, breather and resonant two-solitons solutions for zero and non-zero seeds have been given as particular examples for the SS equation. Examples of these solutions are plotted in the figures 1–5 for particular choices of parameters.

One should notice that we have chosen \( u = k \), where \( k \) is a real constant, as a seed solution of the SS equation. This is the simplest non-zero seed. However, one might also choose the seed \( u_k e^{-\alpha t} \), where \( \alpha \), \( k \) \( \in \mathbb{R} \), to construct various rich explicit solutions including those we present here. Furthermore, in the present paper, we only consider the case \( n = 1 \) for constructing explicit solutions. It can be obtained more explicit solutions by considering other cases such as \( n = 2 \). Finally, it should be pointed out that the BDT technique is a universal instrument that allows us to construct exact solutions for other integrable systems.

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