Practical scheme for the optimal measurement in quantum interferometric devices

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We apply a Kennedy-type detection scheme, which was originally proposed for a binary communication system, to interferometric sensing devices. We show that the minimum detectable perturbation of the proposed system reaches the ultimate precision bound which is predicted by Neyman-Pearson hypothesis testing. To provide concrete examples, we apply our interferometric scheme to phase shift detection by using coherent and squeezed probe fields.

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It is well known that the ultimate sensing precision of interferometric devices is limited by the quantum mechanical properties of the probing field \( |1 \rangle \). Precision limit analysis has conventionally been studied in the context of a quantum estimation problem \([1, 2, 3]\). The problem was also recently treated as a binary decision problem based on the Neyman-Pearson criterion \([2, 3]\). This criterion is often applied to the problem of detecting small, low-rate perturbations, such as gravitational waves. The criterion is given by

\[
\text{P}_{\text{F}} = \frac{\text{P}_{\text{D}}}{(1 - \text{P}_{\text{D}})} = \frac{1}{2},
\]

where \( \text{P}_{\text{D}} \) is the probability of detecting a small parameter shift to be detected. Here we consider a small perturbation modeled by a unitary operator \( U_p(g) \), and the pure probe state \( |\psi_0 \rangle \). The small parameter shift to be detected is given by \( g \). In this restricted case, the maximum detection probability has been analytically derived as

\[
P_{\text{D}} = \frac{1}{2} \left( P_{\text{DE}} + \sqrt{1 - P_{\text{DE}}(1 - \kappa)} \right)^2,
\]

where \( \kappa = \langle \psi_0 | \psi_1 \rangle^2 = |\langle \psi_0 | U_p(g) | \psi_0 \rangle|^2 \) with the perturbed state \( |\psi_1 \rangle = U_p(g) |\psi_0 \rangle \). This general result has been applied to derive minimum detectable perturbation \( g_{\text{M}} \). Since the minimum threshold for \( P_{\text{D}} \) to detect perturbation is given by

\[
P_{\text{D}}(g_{\text{M}}; P_{\text{DE}}) = \frac{1}{2},
\]

one can figure out the value of \( g_{\text{M}} \) for given probe states from Eqs. (1) and (2).

Although these results can be used to predict the bounds of ultimate precision limits for given probe states, they tell us nothing about how to design optimal measurement devices in practice. A practical measurement scheme has only been reported for a certain entangled probe field \( \left| \alpha \right\rangle \). In this paper, we discuss the practical implementation of optimal measurement based on the so-called Kennedy scheme \([5]\), which was originally proposed for semi-optimal detection strategy in terms of the average error probability for the binary phase-shift keyed coherent states \( \{ |\alpha \rangle, | - \alpha \rangle \} \). The scheme consists of the displacement operation \( D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \) and the photodetection operation \( \{ |0 \rangle \langle 0|, |0 \rangle |0 \rangle \} \) discriminating \( |2\alpha \rangle \) and \( |0 \rangle \). Since the signal \( |0 \rangle \) is perfectly projected into the second measurement operator, in principle, the total bit error rate performance is greatly enhanced. It is indeed semi-optimal to the criterion developed by Helstrom \([2]\).

We will now apply this concept to interferometric sensing devices. The outline of our scheme is in Fig. (a). The set of measurement operators is given by the POVM

\[
\begin{align*}
\hat{\Pi}_0 &= |\psi_0 \rangle \langle \psi_0| \\
\hat{\Pi}_1 &= |\hat{1} - \hat{\Pi}_0|
\end{align*}
\]

Since \( \langle \psi_0 | \hat{\Pi}_0 | \psi_0 \rangle = 1 \), the false-alarm probability \( P_{\text{F}} \) is always zero. The detection probability \( P_{\text{D}} \) is given by

\[
P_{\text{D}} = \langle \psi_1 | (\hat{1} - \hat{\Pi}_0) | \psi_1 \rangle = 1 - \kappa.
\]

Comparing these probabilities to the predicted bound in Eq. (4), we can see that our scheme achieves the optimal POVM for Neyman-Pearson hypothesis testing where the false-alarm probability is zero, i.e. Eq. (4) equals the \( P_{\text{D}}(P_{\text{DE}} = 0) \) of Eq. (4). Obviously, the minimum detectable perturbation \( g_{\text{M}} \) derived from Eq. (4) achieves the ultimate limit predicted by the Neyman-Pearson approach.

As a concrete example, let us now discuss an ordinary interferometer which detects small phase shifts given by the operator \( \hat{U}_p = \exp(i n \varphi) \). Here, \( n \) is the photon number operator and \( \varphi \) is the parameter for small phase shift. Needless to say, this is the most conventional interferometric device commonly used in various sensing applications.

First, let us consider a coherent state \( |\psi_0 \rangle = |\alpha \rangle \) as a probe field quantum state. Without loss of generality, we can assume that \( \alpha \) is real. Figure (b) has a
perturbation
Kennedy

\[ \text{Eq. (2), we can find the minimum detectable phase order of a coherent probe field. Expanding Eq. (5) into the second Pearson optimization procedure, which means that this can then be calculated as} \]

\[ \text{where} \]

\[ \text{As is well known, the power for the probe field is fully used for} \]

\[ \text{When the power transmission} \]

\[ \text{Then, from Eqs. (2), (3) and (4), the minimum detectable phase shift} \]

\[ \text{For small} \varphi, \kappa \text{can be approximated to} \]

\[ \text{Thus, comparing performances by coherent and squeezed probe fields, under the power constraint condition} \]

\[ \text{The probe field is the so-called “ideal squeezed state} \]

\[ \text{The inference probabilities} \]
In the limit of large $\langle n \rangle$, $\varphi_M$ is proportional to $1/\langle n \rangle$. This is similar to that of previous predictions [1, 2]. From Fig. 3 and Eq. (12), we can conclude that when $\langle n \rangle$ is large, $\varphi_M^{\text{opt}}$ is also proportional to $1/\langle n \rangle$. Nevertheless, our results clearly show that when we are only allowed to use extremely weak probe field, an optimization of the power distribution certainly improves the precision limit compared to not only coherent state, but also the squeezed vacuum state.

On the other hand, from a practical point of view, it is very difficult to prepare a squeezed state with large $\bar{m}$ while large coherent amplitude can easily be generated. If we assume that $\bar{n} \gg \bar{m}$, i.e., $\bar{m}$ is significantly smaller than $1/\varphi_M$, $\varphi_M^{\text{sq}}$ can simply be calculated by

$$\varphi_M^{\text{sq}} = \frac{1}{e^\varphi_M^{\text{sq}}} \sqrt{\ln 2 / \bar{n}},$$

which indicates how squeezing enhances the precision limit of small phase shift detection in a bright squeezed probe field.

We also need to note that even though our scheme can directly be applied to the two-mode squeezed state probing, it has no advantages compared to the single mode squeezed state. This is because the advantage of using the two-mode squeezed state, i.e. entanglement, instead of the single-mode squeezed state for interferometric devices is understood as its stability against the technical phase fluctuations [3], while our scheme requires the use of local oscillators. However, our scheme only requires devices that can presently be obtained and the photodetector restrictions are less severe than with the scheme utilizing entanglement, especially in the extremely weak probe field region, which has the squeezing power that is currently available. While the entanglement scheme assumes to detect the difference of the photon number between two modes [3] which means detectors need to resolve the number of photons, our scheme only requires a detector that can discriminate between zero or non-zero photons. This kind of photodetection is possible by extending the current technology e.g., by using an avalanche photodiode (APD) operating in the Geiger mode. In practice, APDs are parametrized by quantum efficiency $\eta$ and dark current $I_d$ and the latter causes serious false-alarm probability and also decreases the detection probability. Typical quantities for the best devices that are commercially available at present are $\eta \sim 80\%$ and $I_d \sim 50$ counts per second, for example. We therefore need to pursue further quantitative improvements in detecting devices. Nevertheless, we believe that our scheme still represents a straightforward extension of current photodetection technology.

To summarize, we applied the concept of a Kennedy detection scheme, which has been studied in the field of communications theory, to interferometric sensing devices. We showed that the ultimate precision of our physically realizable scheme reaches the ultimate precision bound predicted by Neyman-Pearson hypothesis testing. It allows us to design concrete optimal detection apparatus for various given probe sources, e.g., coherent or squeezed states. These are useful in various applications where very small signals must reliably be detected, especially in regions where only weak probe power is available.

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