\( N = 2 \) SUSY QED and nonlinear/linear SUSY relation

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1. Nonlinear Supersymmetric General Relativity (NLSUSY GR)

NLSUSY GR action

is defined in *SGM spacetime*, where the tangent spacetime is specified not only $SO(3,1)$ Minkowski coordinates $x^a$ but also $SL(2,C)$ Grassmanian coordinates $\psi^i_\alpha \sim$ as the coset parameters of $\frac{superGL(4R)}{GL(4R)}$.

$w^a_\mu :$ unified vierbein

$\{x^a, \psi^i_\alpha\}$

$\{x^\mu\}$

$\Lambda$

$w^a_\mu \rightarrow \delta^a_\mu$

( asymptotic )

$w^a_\mu$

$SGM$ spacetime

$\Lambda$
NLSUSY GR action is given by the Einstein-Hilbert (EH) form,

\[ L_{\text{NLSUSY GR}}(w) = \frac{c^4}{16\pi G} |w| \left( \Omega(w) - \Lambda \right), \]  

where

\[ |w| = \det w^a_{\mu} = \det \{ e^a_{\mu} + t^a_{\mu}(\psi^i) \}, \quad t^a_{\mu}(\psi^i) = \frac{\kappa^2}{2i} (\bar{\psi}^i \gamma^a \partial_{\mu} \psi^i - \partial_{\mu} \bar{\psi}^i \gamma^a \psi^i), \]

\[ \Omega(w) \ldots \text{the scalar curvature in terms of } w^a_{\mu}. \]  

In \( L_{\text{NLSUSY GR}} \)

- \( \psi^i_{\alpha} \) can be interpreted as Nambu-Goldstone (NG) fermions associated with the spontaneous breaking of super-\( GL(4R) \) down to \( GL(4R) \).

- SUSY is broken spontaneously from the beginning due to NLSUSY structure of spacetime (as explaines after).
**Symmetries in NLSUSY GR**

$L_{\text{NLSUSY GR}}$ possesses promising large symmetries isomorphic to $SO(N)$ ($SO(10)$) SP group through an invariance under the following NLSUSY transformations,

\[ \delta \zeta \psi^i = \frac{1}{\kappa} \zeta^i - i \kappa \bar{\zeta}^j \gamma^\mu \psi^j \partial_\mu \psi^i, \quad \delta e^a_\mu = 2i \kappa \bar{\zeta}^i \gamma^\rho \psi^i \partial_\mu [e^a_\rho], \tag{3} \]

and a (generalized) local Lorentz invariance, etc.

Therefore, the no-go theorem is overcome (circumvented) in a sense that

"the nontrivial $N$-extended SUSY gravity theory with $N > 8$"

has been constructed in the NLSUSY invariant way.
NLSUSY GR action in Riemann spacetime

\{ SGM spacetime \}

is unstable due to the NLSUSY structure of spacetime, and decays spontaneously (called “Big Decay” of spacetime) to

\{ Riemann spacetime \oplus \text{matter} \} - system

described by the EH action coupled with NG-fermion (superon) matter. Then, the action called the SGM action expanded as follows;

\begin{equation}
L_{SGM}(e, \psi^i) = \frac{c^4}{16\pi G} |e| \{ R(e) - \Lambda + \tilde{T}(e, \psi^i) \},
\end{equation}

where

\begin{align*}
R(e) & \ldots \text{the scalar curvature of ordinary EH action,} \\
\tilde{T}(e, \psi^i) & \ldots \text{the kinetic term and the gravitational interactions of } \psi^i.
\end{align*}
$w^a_\mu$ : unified vierbein

SGM spacetime

$w^a_\mu \rightarrow \delta^a_\mu$

(asymptotic)

Spontaneous SUSY Breaking (Big Decay)

$e^a_\mu$ : ordinary vierbein

Riemann spacetime $\oplus$ matter

$e^a_\mu \rightarrow \delta^a_\mu$

(asymptotic)
SGM (NLSUSY GR) action in Riemann-flat spacetime

In the asymptotic region (local frame in the Riemann-flat $e^a_\mu \to \delta^a_\mu$),

\[
L_{SGM}(e, \psi^i) \quad (\sim \text{corresponding to } [\text{cosm. const. } \Lambda + \tilde{T}(e, \psi^i)] \text{ terms })
\]

\[
\to \quad L_{NLSUSY}(\psi^i) = -\frac{1}{2\kappa^2}|w| \quad (\Leftrightarrow \text{spacetime volume form}) + \cdots
\]

\[
= -\frac{1}{2\kappa^2} \left( 1 \Leftrightarrow \text{b.g. energy density} + t^a_a \Leftrightarrow \text{kin.} + \frac{1}{2!} t^a_a t^b_b + \cdots \right) + \cdots
\]

\[
= -\frac{1}{2\kappa^2} \left( 1 - i\kappa^2 \bar{\psi}^i \partial \psi^i - \frac{1}{2} \kappa^4 \bar{\psi}^i \partial \psi^i \bar{\psi}^j \partial \psi^j + \cdots \right) + \cdots , \quad (6)
\]

where

\[
\kappa^2 = \left( \frac{c^4 \Lambda}{8\pi G} \right)^{-1} . \quad (7)
\]

\[
\implies \text{Low energy physics for the NLSUSY model } L_{NLSUSY}(\psi^i) ?
\]
2. Cosmology and low energy particle physics in NLSUSY GR

In order to study the low energy particle physics in NLSUSY GR, the NL/L SUSY relation is very important.

**Essential points in NL/L SUSY relation**

1. Component fields $\varphi^I$ in LSUSY theories are uniquely represented as composites of $\psi^i$ in a SUSY invariant way.

2. The NLSUSY action (≈ the massless theory with the NG fermions) is related to the (interacting) LSUSY actions.
As a significant example of the NL/L SUSY relation, we consider **SUSY QED**

for \( N = 2 \) SUSY (realistic in the SGM scenario) and in \( d = 2 \) (for simplicity).

\[ N = 2 \text{ SUSY QED from } N = 2 \text{ NL/L SUSY relation} \]

\[(1) \sim \text{SUSY invariant relations} \]

As for \( N = 2 \) vector supermultiplet which has the following multiplet's structure,

\[
\begin{pmatrix}
+1 \\
+\frac{1}{2}, +\frac{1}{2} \\
0
\end{pmatrix}
\Leftrightarrow
\begin{pmatrix}
v^a \\
\lambda^i \\
A, \phi
\end{pmatrix}
+ [ \text{auxiliary fields } (D, \cdots) ],
\]
the SUSY invariant relations are written as the composite forms of $\psi^i$ in all orders as follows;

\begin{align*}
C &= -\frac{1}{8}\xi\kappa^3 \bar{\psi}^i \psi^i \bar{\psi}^j \psi^j, \\
\Lambda^i &= -\frac{1}{2}\xi\kappa^2 \psi^i \bar{\psi}^j \psi^j (1 - i\kappa^2 \bar{\psi}^k \partial \psi^k), \\
M^{ij} &= \frac{1}{2}\xi\kappa \bar{\psi}^i \psi^j \left(1 - i\kappa^2 \bar{\psi}^k \partial \psi^k - \frac{1}{2}\kappa^4 e^{ab} \bar{\psi}^k \psi^l \partial_a \bar{\psi}^k \gamma_5 \partial_b \psi^l \right), \\
\phi &= -\frac{1}{2}\xi\kappa e^{ij} \bar{\psi}^i \gamma^5 \psi^j \left(1 - i\kappa^2 \bar{\psi}^k \partial \psi^k - \frac{1}{2}\kappa^4 e^{ab} \bar{\psi}^k \gamma_5 \psi^l \partial_a \bar{\psi}^k \partial_b \psi^l \right), \\
v^a &= -\frac{i}{2}\xi\kappa e^{ij} \bar{\psi}^i \gamma^a \psi^j (1 - i\kappa^2 \bar{\psi}^k \partial \psi^k), \\
\lambda^i &= \xi\psi^i |w|, \\
D &= \frac{\xi}{\kappa} |w|, \\
\end{align*}

where $A = M^{ii}$. 

(8)
Also, for $N = 2$ scalar supermultiplet for matter fields which has the following multiplet’s structure,

\[
\begin{pmatrix}
  +\frac{1}{2} \\
  0, 0 \\
  -\frac{1}{2}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
  \chi \\
  B^i \\
  \nu
\end{pmatrix}
+ [ \text{auxiliary fields } F^i ],
\]

the SUSY invariant relations become

\[
\chi = \xi^i \left[ \psi^i |w| + \frac{i}{2} \kappa^2 \partial_a \{ \gamma^a \psi^i \bar{\psi}^j \psi^j (1 - i \kappa^2 \bar{\psi}^k \gamma^a \psi^k) \} \right],
\]

\[
B^i = -\kappa \left( \frac{1}{2} \xi^i \bar{\psi}^j \psi^j - \xi^j \bar{\psi}^i \psi^i \right) |w|,
\]

\[
\nu = \xi^i \bar{\epsilon}^{ij} \left[ \psi^j |w| + \frac{i}{2} \kappa^2 \partial_a \{ \gamma^a \psi^j \bar{\psi}^k \psi^k (1 - i \kappa^2 \bar{\psi}^l \gamma^a \psi^l) \} \right],
\]

\[
F^i = \frac{1}{\kappa} \xi^i \left\{ |w| + \frac{1}{8} \kappa^4 \partial_a \partial^a (\bar{\psi}^j \psi^j \bar{\psi}^k \psi^k) \right\} - i \kappa \xi^j \partial_a (\bar{\psi}^i \gamma^a \psi^j |w|). \tag{9}
\]
Then, (2) the relation between \( N = 2 \) NLSUSY action and \( N = 2 \) SUSY QED action becomes

\[
L_{N=2\text{NLSUSY}}(\psi^i) = -\frac{1}{2\kappa^2}|w| + [\ \text{tot. der.}\ ]
\]

\[
= L_{N=2\text{SUSYQED}} \quad \text{with the SUSY invariant relations}
\]

\[
= -\frac{1}{4}(F_{ab})^2 + \frac{i}{2}\bar{\lambda}^i\phi\lambda^i + \frac{1}{2}(\partial_a A)^2 + \frac{1}{2}(\partial_a \phi)^2 + \frac{1}{2}D^2 - \frac{\xi}{\kappa}D \quad (\Leftrightarrow \mathcal{V}_{\text{kin}})
\]

\[
+ \frac{i}{2}\bar{\chi}\phi\chi + \frac{1}{2}(\partial_a B^i)^2 + \frac{i}{2}\bar{\nu}\phi\nu + \frac{1}{2}(F^i)^2 \quad (\Leftrightarrow \Phi^i_{\text{kin}})
\]

\[
+ f(A\bar{\lambda}^i\lambda^i + \epsilon^{ij}\phi\bar{\lambda}^i\gamma_5\lambda^j - A^2D + \phi^2D + \epsilon^{ab}A\phi F_{ab})
\]

\[
+ e\left\{ iv_a\bar{\chi}\gamma^a\nu - \epsilon^{ij}v^aB^i\partial_a B^j + \bar{\lambda}^i\chi B^i + \epsilon^{ij}\bar{\lambda}^i\nu B^j
\right\}
\]

\[
- \frac{1}{2}D(B^i)^2 + \frac{1}{2}(\bar{\chi}\chi + \bar{\nu}\nu)A - \bar{\chi}\gamma_5\nu\phi\right\} + \frac{1}{2}e^2(v_a^2 - A^2 - \phi^2)(B^i)^2
\]

\[
+ \cdots.
\]

(10)
Cosmology and particle physics from NL/L SUSY relation

\[ L_{SGM}(e, \psi^i) \rightarrow G_{\mu\nu}(e) = \frac{8\pi G}{c^4} \left\{ \tilde{T}_{\mu\nu}(e, \psi^i) - g_{\mu\nu} \frac{c^4 \Lambda}{8\pi G} \right\} \]

\[ \rightarrow \rho_D \sim \frac{c^4 \Lambda}{8\pi G} \]

Physical vacuum with \( U(1) \) ... stable and the lightest massive fermion \( \lambda^i \Leftrightarrow \nu \)

\[ \rightarrow m_\nu^2 \sim \sqrt{\frac{c^4 \Lambda}{8\pi G}} \quad \text{(at } - f \xi \sim \mathcal{O}(1)) \]
Therefore, SGM (NLSUSY GR) predicts the observed value of the (dark) energy density of the universe $\rho_{D}^{\text{obs}}$ and naturally explains the mysterious numerical relations between $m_\nu$ and $\rho_{D}^{\text{obs}}$:

$$\rho_{D}^{\text{obs}} \sim (10^{-12}\text{GeV})^4 \sim m_\nu^4.$$ (11)
3. Systematics of NL/L SUSY relation and $N = 2$ SUSY QED

The SUSY invariant relations
\[ \Rightarrow \]
are systematically obtained in the superfield formulation.

### Linearization of NLSUSY in the $d = 2$ superfield formulation

- **General superfields** are given for the $N = 2$ vector supermultiplet by

\[
\mathcal{V}(x, \theta^i) = C(x) + \bar{\theta}^i \Lambda^i(x) + \frac{1}{2} \bar{\theta}^i \theta^j M^{ij}(x) - \frac{1}{2} \bar{\theta}^i \theta^i M^{jj}(x) + \frac{1}{4} \epsilon^{ij} \bar{\theta}^i \gamma_5 \theta^j \phi(x)
\]

\[
- \frac{i}{4} \epsilon^{ij} \bar{\theta}^i \gamma_a \theta^j v^a(x) - \frac{1}{2} \bar{\theta}^i \theta^j \bar{\theta}^j \chi^j(x) - \frac{1}{8} \bar{\theta}^i \theta^i \bar{\theta}^j \theta^j D(x),
\]

and for the $N = 2$ scalar supermultiplet by

\[
\Phi^i(x, \theta^i) = B^i(x) + \bar{\theta}^i \chi(x) - \epsilon^{ij} \bar{\theta}^i \nu(x) - \frac{1}{2} \bar{\theta}^j \theta^j F^i(x) + \bar{\theta}^i \theta^j F^j(x) - i \bar{\theta}^i \partial B^j(x) \theta^j
\]

\[
+ \frac{i}{2} \bar{\theta}^j \theta^j (\bar{\theta}^i \partial \chi(x) - \epsilon^{ik} \bar{\theta}^k \partial \nu(x)) + \frac{1}{8} \bar{\theta}^j \theta^j \bar{\theta}^k \theta^k \partial_a \partial^a B^i(x).
\]
Consider the general superfields on the following $\psi^i$-dependent specific supertranslations,

$$x'^a = x^a + i\kappa \bar{\theta}^i \gamma^a \psi^i, \quad \theta'^i = \theta^i - \kappa \psi^i,$$

(14)

and we denote the general superfields on $(x'^a, \theta'^i)$ by

$$\tilde{\mathcal{V}}(x^a, \theta^i; \psi(x)) = \mathcal{V}(x'^a, \theta'^i), \quad \tilde{\Phi}(x^a, \theta^i; \psi(x)) = \Phi(x'^a, \theta'^i).$$

(15)

Then,

- **SUSY invariant constraints** of the component fields in $\tilde{\mathcal{V}}$ and $\tilde{\Phi}$,

$$\tilde{C} = \tilde{\Lambda}^i = \tilde{M}^{ij} = \tilde{\phi} = \tilde{v}^a = \tilde{\lambda}^i = 0, \quad \tilde{D} = \frac{\xi}{\kappa},$$

$$\tilde{B}^i = \tilde{\chi} = \tilde{\nu} = 0, \quad \tilde{F}^{i} = \frac{\xi_i}{\kappa},$$

(16)

give the SUSY invariant relations.
**Actions in the $d = 2$, $N = 2$ NL/L SUSY relation**

By changing the integration variables $(x^a, \theta^i) \rightarrow (x'^a, \theta'^i)$, we can confirm systematically what LSUSY actions reduce to in NLSUSY representation.

(a) The kinetic (free) action with the Fayet-Iliopoulos (FI) $D$ term for the $N = 2$ vector supermultiplet $\mathcal{V}$ reduces to $S_{N=2\text{NLSUSY}}$;

$$
S_{\mathcal{V}_{\text{free}}} = \int d^2 x \left\{ \int d^2 \theta^i \frac{1}{32} (\bar{D}^i \mathcal{W}^{jk} D^i \mathcal{W}^{jk} + \bar{D}^i \mathcal{W}^{j5}_{5} D^i \mathcal{W}^{j5}_{5}) + \int d^4 \theta^i \frac{\xi}{2\kappa} \mathcal{V} \right\}_{\theta^i = 0} = \xi^2 S_{N=2\text{NLSUSY}},
$$

(17)

where

$$
\mathcal{W}^{ij} = \bar{D}^i D^j \mathcal{V}, \quad \mathcal{W}^{ij}_{5} = \bar{D}^i \gamma_5 D^j \mathcal{V}.
$$

(18)

(Note) The FI $D$ term gives the correct sign of the NLSUSY action.
(b) Yukawa interaction terms for $\mathcal{V}$ vanish, i.e.

\[
S_{\mathcal{V}f} = \frac{1}{8} \int d^2 x \ f \left[ \int d^2 \theta^i \ \mathcal{W}^{jk}(\mathcal{W}^{jl} \mathcal{W}^{kl} + \mathcal{W}_5^{jl} \mathcal{W}_5^{kl}) + \int d\bar{\theta}^i d\theta^j \ 2\{\mathcal{W}^{ij} (\mathcal{W}^{kl} \mathcal{W}^{kl} + \mathcal{W}_5^{kl} \mathcal{W}_5^{kl}) + \mathcal{W}^{ik} (\mathcal{W}^{jl} \mathcal{W}^{kl} + \mathcal{W}_5^{jl} \mathcal{W}_5^{kl})\}\right]_{\theta^i = 0}
= 0, \tag{19}
\]

by means of cancellations among four NG-fermion self-interaction terms.
(c) The most general gauge invariant action for $V$ coupled with $\Phi^i$ reduces to $S_{N=2\text{NLSUSY}}$;

$$S_{\text{gauge}} = -\frac{1}{16} \int d^2x \int d^4\theta^i e^{-4eV(\Phi^j)^2}$$

$$= -(\xi^i)^2 S_{N=2\text{NLSUSY}}, \quad (20)$$

Here the $U(1)$ gauge interaction terms proportional to the gauge coupling constant $e$ becomes four NG-fermion self-interaction terms as

$$S_e(\text{for the minimal off shell multiplet}) = \int d^2x \left\{ \frac{1}{4} e\kappa \xi (\xi^i)^2 \bar{\psi}^j \psi^j \bar{\psi}^k \psi^k \right\}, \quad (21)$$

which are absorbed in the SUSY invariant relation of the auxiliary field $F^i = F^i(\psi)$ by adding four NG-fermion self-interaction terms as

$$F''^i(\psi) = F^i(\psi) - \frac{1}{4} e\kappa^2 \xi^i \bar{\psi}^j \psi^j \bar{\psi}^k \psi^k. \quad (22)$$
Therefore, under the SUSY invariant relations, which are obtained systematically, the $N = 2$ NLSUSY action $S_{N=2\text{NLSUSY}}$ is related to $N = 2$ SUSY QED action defined by

$$S_{N=2\text{NLSUSY}} = S_{N=2\text{SQED}} \equiv S_{\nu\text{free}} + S_{\nu f} + S_{\text{gauge}}$$

when $\xi^2 - (\xi^i)^2 = 1$.

This NL/L SUSY relation gives the relation between the cosmology and the low energy particle physics in NLSUSY GR as explained before.
4. Discussions

- The similar results are anticipated in $d = 4$ as well.

- The similar arguments in SUSY QCD are also interesting problems in NL/L SUSY relation.

- The extension to large $N$, especially to $N = 5$ is important for superon quintet hypothesis of SGM scenario with $N = 10 = 5 + 5^*$ for equipping the $SU(5)$ GUT structure and to $N = 4$ may shed new light on the mathematical structures of the anomaly free non-trivial $d = 4$ field theory.

- Linearizing SGM action $L_{SGM}(e, \psi^i)$ on curved spacetime, which elucidates the topological structure of spacetime, is a challenging problem.