T-DUALITY FOR ORIENTIFOLDS AND TWISTED KR-THEORY

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Abstract. D-brane charges in orientifold string theories are classified by the KR-theory of Atiyah. However, this is assuming that all O-planes have the same sign. When there are O-planes of different signs, physics demands a “KR-theory with a sign choice” which up until now has not been studied by mathematicians. We give a definition of this theory and compute it for orientifold theories compactified on $S^1$ and $T^2$. We also explain how and why additional “twisting” is implemented. We show that our results satisfy all possible T-duality relationships for orientifold string theories on elliptic curves, which will be studied further in subsequent work.

1. Introduction

The purpose of this paper is to describe the versions of $K$-theory needed to describe $T$-duality for orientifolds, and to compute and analyze them in a few simple but important cases. By orientifolds we mean spacetimes of the form $\mathbb{R}^k \times X$, where $X$ is a smooth $10-k$ dimensional manifold equipped with an involution, $\iota$, which defines the orientifold structure. The sigma-model of orientifold string theory involves equivariant maps $\varphi: \Sigma \to X$, so that $\iota \circ \varphi = \varphi \circ \Omega$, where $\Sigma$ is the string worldsheet and $\Omega$ is the worldsheet parity operator. (See for example [13]; there some extra twisting data, which we are ignoring for the moment, is also taken into account.)

As described in [35, 30], D-branes are classified by $K$-theory, where the relevant type of $K$-theory depends on the string theory being considered. Since the physics of $T$-dual theories is indistinguishable, the groups classifying stable D-branes in two $T$-dual theories must be isomorphic. This led Bouwknegt, Evslin, and Mathai, and later Bunke and Schick, to describe the $T$-duality between the type IIB theory on a spacetime $X$ that is a circle bundle over base $Z$, with $H$-flux $H$, and the type IIA theory on a dual circle bundle $\tilde{X}$ over $Z$, with dual $H$-flux $\tilde{H}$, as an
isomorphism of twisted $K$-theories:

$K^*(X, H) \cong K^{*+1}(\tilde{X}, \tilde{H})$.

In the above equation

$$c_1(X) = \tilde{\pi}_*(\tilde{H}) \quad \text{and} \quad c_1(\tilde{X}) = \pi_*(H),$$

where $c_1(X) \in H^2(X; \mathbb{Z})$ is the first Chern class and $\pi_*: H^k(X) \to H^{k-1}(Z)$ is the Gysin push-forward map which in terms of de Rham cohomology is defined by integration along the fiber $[9, 12]$. This was later generalized to the case where $X$ is a $T^n$-bundle in $[29]$ and $[11]$.

For orientifolds, $D$-brane charges are classified by $KR$-theory $[35, \S 5.2], [22], [21]$, which we will review in section 2. One benefit of using $KR$-theory is that it can be viewed as a sort of universal $K$-theory. It is “universal” in the sense that the $K$-theories $KU$ for the type II theories, $KO$ for the type I theory, and $KSC$ for the type $\tilde{I}$ theory can all be built out of $KR$. This shows that by keeping track of the appropriate involution $\iota$, one does not need to make a choice of which type of $K$-theory to use and it is already accounted for just by using $KR$-theory. $KR$-theory has some immediate limitations that prevent us from generalizing a topological description of $T$-duality like equation (1) to orientifolds.

The first problem is that it is not immediately clear how to twist $KR$-theory or even what is meant by $H$-flux. Traditionally, $H \in H^3(X; \mathbb{Z})$, so there is no reason to expect $H$ to be equivariant. Another related issue is that orientifold theories involve extra information, which is not just topological, relevant to the stable $O$-plane charges. This issue is already apparent when studying circle orientifolds, even though the dimension of a circle is too low to have to worry about more general twistings.

In section 3 we will review $T$-duality between all possible circle orientifolds and the classification of stable $D$-branes in the different theories. The $T$-dual of the type I theory on a circle (which is a type IIB orientifold on the circle with trivial involution) is a type IIA orientifold on a circle with involution given by reflection (referred to as the type IA or type $I'$ theory). The $T$-dual to the type IIB theory on the circle with the antipodal map (sometimes referred to as the type $\tilde{I}$ theory) is also a type IIA orientifold on the circle with involution given by reflection (often called the type $\tilde{I}A$ theory). The compactification manifolds for both the type IA and $\tilde{I}A$ theories are topologically equivalent, with the difference being the relative charges of the $O$-planes at the two fixed points. There are physical descriptions of the classification of $D$-branes in the two theories $[31, 7]$; however, we are not aware of any mathematical description for the classification of $D$-branes in the type $\tilde{I}A$ theory via $KR$-theory. In fact, a topological invariant such as $KR$-theory cannot pick up the difference between the type IA and $\tilde{I}A$ compactifications since the distinction is non-topological. In section 4 we propose a variant of $KR$-theory, which we call $KR$-theory with a sign choice, that can distinguish between the
two cases, giving a mathematical description of the brane charges in the type $\tilde{I}A$ theory. We then give all possible sign choices for $KR$-theories for orientifolds of 2-tori.

A word about our sign convention: we say that an $O$-plane has positive sign, or is an $O^+$-plane, if the Chan-Paton bundle on it has orthogonal type, and has negative sign, or is an $O^-$-plane, if the Chan-Paton bundle on it has symplectic type. The sign decorations that we attach to $KR$-theory follow the same convention. Since a tensor product of an orthogonal bundle with a symplectic bundle is symplectic, while the tensor product of two symplectic bundles is orthogonal, signs multiply as one would expect. This convention is the same as the one made by Witten in [36], but is the reverse of the convention made by Gao and Hori in [19]. Both sign conventions are in general use, but we feel that the multiplication rule indicates that this one is preferable, even though it means (as Witten points out) that the tadpoles are of opposite sign.

When we move up in dimension to 2-tori, the sign choice is no longer enough to account for all possible orientifold theories. In particular, $KR$-theory with a sign choice cannot describe the type I theory without vector structure [36]. For this we need to include more general twists of $KR$-theory, which will be discussed in section 5. Here we use physics to motivate which $KR$-theories should be isomorphic, and check the results via topology. The twist applied to $KR$-theory is related to the geometry of its $T$-dual theory and is described in [16]. The purpose of the current paper is to describe the relevant twisted $KR$-theories needed to give the geometric interpretation in [16].

One of our motivations for a detailed analysis of $T$-duality via orientifold plane charges in $KR$-theory was the special case of $c = 3$ Gepner models as studied in [6]. The authors of that paper used simple current techniques in CFT to construct the charges and tensions of Calabi-Yau orientifold planes, though a $K$-theoretic interpretation was missing. Although the interpretation of brane charges in $KR$-theory is sensitive to regions of stability, this $K$-theoretic interpretation does not depend on the specific structure of $c = 3$ Gepner models, nor even on a rational conformal field theoretic description. These results should be contrasted with the recent work [17] where a twisted equivariant $K$-theory description of the $D$-brane charge content for WZW models is provided (see also [10] for examples which make explicit the isomorphism with topological $K$-theory in the case of some Gepner models). Work in progress seeks to establish an isomorphism between a suitable (real) variant of twisted equivariant $K$-theory, sufficient to capture orientifold charge content, and our $KR$-theory with sign choices for Gepner models. As a side-effect, such an isomorphism will then permit computation of $KR$-theory for complicated Calabi-Yau manifolds through a simpler computation at the Gepner point.
We would like to thank Max Karoubi for many useful discussions regarding the contents of Section 4, and in particular for suggesting the formulation of Theorem 4, as well as a method of proof for that theorem.

2. Review of classical $KR$-theory

Let $X$ be a locally compact space (in most physical situations it will be a smooth manifold) with involution $\iota$. Let $E$ be a vector bundle on $X$ and $\Omega$ be the operator that sends $E$ to $\overline{E}$. Physically, we will see that $\Omega$ corresponds to the worldsheet parity operator. If there exists an isomorphism, $\phi$, from the pullback $\iota^* E$ to $\overline{E}$ such that $(\phi \iota^*)^2 = 1$, then $E$ is called an $\iota \cdot \Omega$-equivariant vector bundle. $KR(X)$ is the group of pairs of $\iota \cdot \Omega$-equivariant vector bundles $(E, F)$ (with compact support) modulo the equivalence relation

\[
(E, F) \sim (E \oplus H, F \oplus H),
\]

for any $\iota \cdot \Omega$-equivariant vector bundle, $H$. Note that $KR(X)$ depends on the involution $\iota$ even though it isn’t explicitly stated. The compact support condition means that we can choose $E$ and $F$ to be trivialized off a sufficiently large $\iota$-invariant compact set and for $\phi$ off this compact set to be the standard isomorphism of a trivial bundle with its complex conjugate.

To define the higher $KR$-groups, $KR^{-j}(X)$, we must first introduce some notation. Let $\mathbb{R}^{p,q} = \mathbb{R}^p + i\mathbb{R}^q$, where the involution is given by complex conjugation, and $S^{p,q}$ be the $p + q - 1$ sphere in $\mathbb{R}^{p,q}$. Caution: In this notation, the roles of $p$ and $q$ are the reverse of those in the notation used by Atiyah in [3] but the same as the notation in [28], [22], [7] and [31]. Then we can define

\[
KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q}).
\]

This obeys the periodicity condition

\[
KR^{p,q}(X) \cong KR^{p+1,q+1}(X),
\]

so $KR^{p,q}$ only depends on the difference $p - q$ and we can define

\[
KR^{-p}(X) = KR^{p,q}(X).
\]

$KR^{-j}(X)$ is periodic with period 8.

When $\iota$ is the trivial involution, the equivariance condition is equivalent to $E$ being the complexification of a real bundle. Thus $KR$ gives a classification of real vector bundles and we find

\[
KR^{-j}(X) \cong KO^{-j}(X),
\]

when $\iota$ is trivial [3, p. 371]. Complex $K$-theory can also be obtained from $KR$-theory using

\[
KR^{-j}(X \times S^{0,1}) = KR^{-j}(X \amalg X) \cong K^{-j}(X),
\]
where the involution exchanges the 2 copies of $X$ [3, Proposition 3.3]. And as shown by Atiyah [3, Proposition 3.5], $KR^{-j}(X \times S^{0.2}) \cong KSC^{-j}(X)$, the self-conjugate $K$-theory of Anderson [2] and Green [20], which is periodic with period 4.

In fact, when the involution $\iota$ has no fixed points, there is a spectral sequence \((11)\) below, whose $E_2$-term is 4-periodic, converging to $KR^{-j}(X)$. This motivated Karoubi and Weibel [26, Proposition 1.8] to assert that $KR^{-j}(X)$ is always 4-periodic when the involution is free, but in general this is not the case (unless one inverts the prime 2). The groups $KR^{-j}(S^{0.4})$ provide a counterexample.

When $X$ is compact and $X^\iota$ is non-empty, the inclusion of an $\iota$-fixed basepoint into $X$ is equivariantly split, so the reduced $KR$-groups, $\tilde{KR}^{-j}(X)$, are defined such that

$$KR^{-j}(X) \cong \tilde{KR}^{-j}(X) \oplus KR^{-j}(pt).$$

We will write simply $KR^{-j}$ or $KO^{-j}$ for $KR^{-j}(pt)$. When $Y \subseteq X$ is closed and $\iota$-invariant, we can define the relative $KR$-theory as

$$KR^{-j}(X, Y) \cong \tilde{KR}^{-j}(X/Y).$$

As we will discuss in the following section, this is the relevant quantity for classifying $D$-brane charges.

### 3. Orientifolds on a circle and $T$-duality

In this section we will consider orientifolds of the form $S^1/(\Omega \cdot \iota)$, where $\iota$ is an involution on $S^1$ and $\Omega$ is the worldsheet parity operator. For a circle there are only three distinct possibilities for $\iota^2$ They are the trivial involution corresponding to $S^{2,0}$, reflection corresponding to $S^{1,1}$, and the antipodal map corresponding to $S^{0,2}$. $S^{2,0}$ and $S^{0,2}$ only support the type IIB theory since they are orientation preserving, while $S^{1,1}$ only supports the type IIA theory since it is orientation reversing.

The type IIB theory on $S^{2,0}$ is the type I theory compactified on a circle. It is known to be $T$-dual to the type IIA theory on $S^{1,1}$, sometimes referred to to the type IA or $I'$ theory [31, 7, 22]. The type IIB theory on $S^{0,2}$ is often called the type $\tilde{I}$ theory and is $T$-dual to the type $\tilde{IA}$ theory [36 §6.2], [19 §7.1].

In this section we will review these $T$-duality relations and their $K$-theoretic descriptions. The lack of a mathematical description for the $K$-theory description of the type $\tilde{IA}$ theory will motivate the definition for a variation of $KR$-theory given in Section 4. Before describing the various $T$-dualities we will review how $D$-branes are classified by $K$-theory.

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This is under the assumption that the circle is viewed as the unit circle in $\mathbb{R}^2$ and the involution comes from a linear involution on $\mathbb{R}^2$. 
3.1. **Classification of D-branes by KR-theory.** D-branes on orientifolds of the form $X/(\iota \cdot \Omega)$, where $X$ is a smooth manifold, $\iota$ is an involution on $X$, and $\Omega$ is the world sheet parity operator, are classified by pairs of vector bundles on $X$ (the Chan-Paton bundles) that are equivariant under the action of $\iota \cdot \Omega$ modulo creation and annihilation of charge zero $D$-brane systems (given by equation (3)). So $D$-branes in such orientifolds are classified by $KR$-theory [35].

More generally, when we compactify string theory on an $m$-dimensional space $M$, so that the spacetime manifold is $\mathbb{R}^{10-m,0} \times M$, we are interested in the charges of $D$-branes in the non-compact dimensions. So we want to consider $D$-branes of codimension $9-m-p$ in $\mathbb{R}^{9-m,0}$. These can arise from both $Dp$-branes located at a particular point in $M$ or higher dimensional $D$-branes that wrap non-trivial cycles in $M$. Furthermore, we only want to consider systems with finite energy, so we want to classify systems that are asymptotically equivalent to the vacuum in the transverse space $\mathbb{R}^{9-m-p,0}$. That means that the system must be equivalent to the vacuum on an entire copy of $M$ at infinity. Mathematically this means we want to add a copy of $M$ at infinity (i.e., take the one-point compactification of $\mathbb{R}^{9-m-p,0} \times M$) and consider bundles on $S^{10-m-p,0} \times M$ that are trivialized on the copy of $M$ at infinity. Such bundles are classified by $KR^{-i}(S^{10-m-p,0} \times M, M)$ (the index depends on the string theory and involution being considered). For purposes of calculation it is useful to relate this to the $KR$-theory of $M$.

**Proposition 1.**

$$KR^{-i}(S^{10-m-p,0} \times M, M) \cong KR^{p+m-9-i}(M).$$

**Proof.** Note that

$$KR^{-i}(S^{10-m-p,0} \times M, M) \cong KR^{-i}((S^{10-m-p,0} \setminus \{pt\}) \times M) \cong KR^{-i}(\mathbb{R}^{9-m-p,0} \times M) \cong KR^{p+m-9-i}(M).$$

Thus $Dp$-branes are classified by $KR^{p+m-9-i}(M)$. It is important to keep track of the index. This point is often overlooked when studying $D$-branes in the type II theories on smooth manifolds which are classified by $KU$-theory. Since $KU$-theory has period 2, the distinction is not as essential.

It is also important to note that this correctly takes into account the charged $O$-planes located at the components of the fixed set only when the $O$-planes have charge corresponding to negative $D$-brane charge. When we allow different $O$-planes to have different charges this classification breaks down as we will see below. It is this breakdown that leads us to define $KR$-theory with a sign choice in section 4.

3.2. **The Type I Theory and Its T-dual.** The type I theory compactified on a circle corresponds to the type IIB theory compactified on $S^{2,0}$ and divided out by the action of $\Omega$. Since we will always be dividing out by the action of $\Omega$, we
will not explicitly state it each time. Consider the bosonic fields in the type IIB theory compactified on $S^{2,0}$,
\[ X = X_L + X_R. \]

The worldsheet parity operator reverses the orientation of the string, and so exchanges left-movers and right-movers. This leaves the bosonic fields invariant under $\Omega$ and therefore compatible with the trivial involution.

$T$-duality leaves the left-moving fields invariant, while reversing the sign of the right-moving fields, so the $T$-dual coordinates are
\[ \tilde{X} = X_L - X_R. \]

Under the action of $\Omega$, the $T$-dual coordinates transform as
\[ \tilde{X} \to -\tilde{X}. \]

This shows that the $T$-dual to the type I theory compactified on a circle must be the type IIA theory (since $T$-duality exchanges types IIA and IIB) mod the action of $\Omega$ combined with the spacetime involution that reflects the compact dimension. This is the type IIA theory compactified on $S^{1,1}$. In the literature it is often referred to as the type IA (or $I'$) theory. One could also show these two theories are $T$-dual to one another by showing there is no momentum, but winding in the $S^{2,0}$ direction, while $S^{1,1}$ has momentum, but no winding.

The type I theory on $S^1$ has a space filling $O9^+$-plane wrapping the compact dimension. The $T$-dual type IA theory has 2 $O8^+$-planes located at the 2 fixed points of $S^{1,1}$. Recall that we use the plus sign to denote that the $O$-planes have negative $D$-brane charge and require the addition of $D$-branes to obtain a zero charge system.

$Dp$-brane charges in the type I theory compactified on a circle are classified by
\[
KR(S^{9-p,0} \times S^{2,0}, S^{2,0}) \cong KO^{p-8}(S^1)
\cong KO^{p-8} \oplus KO^{p-9}.
\]
(6)

The second factor in the last line of equation (6) corresponds to $Dp$-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from branes wrapping $S^1$. The complete brane content is given in Table

Since the type IA theory is obtained from the type I theory compactified on a circle by a $T$-duality, the relevant $KR$-theory is shifted in index by 1. Therefore, $Dp$-brane charges in the type IA theory are classified by
\[
KR^{-1}(S^{9-p,0} \times S^{1,1}, S^{1,1}) \cong KR^{p-9}(S^{1,1})
\cong KO^{p-9} \oplus KO^{p-8},
\]
(7)

where the second factor on the right-hand side corresponds to $Dp$-brane charge coming from unwrapped branes and the first factor corresponds to the charge contribution from wrapped branes. The complete brane content is given in Table
The fact that $T$-duality exchanges wrapped and unwrapped branes is described by the exchanged roles for $KO^{p-8}$ and $KO^{p-9}$ in the two theories.

| $D_p$-brane | $D8$ | $D7$ | $D6$ | $D5$ | $D4$ | $D3$ | $D2$ | $D1$ | $D(−1)$ | type I on $S^1$ | type IIA on $S^{1,1}$ |
|--------------|------|------|------|------|------|------|------|------|---------|----------------|-----------------|
| $KO^{p-8}$   | $Z$  | $Z_2$| $Z_2$| $Z$  | $0$  | $0$  | $0$  | $0$  | $Z_2$   | $(p+1)$-brane wrapping $S^{2,0}$ | unwrapped $p$-brane |
| $KO^{p-9}$   | $Z_2$| $Z_2$| $0$  | $Z$  | $0$  | $0$  | $0$  | $Z$  | $Z_2$   | unwrapped $p$-brane | $(p+1)$-brane wrapping $S^{1,1}$ |

Table 1. $D$-brane charges in the type I theory compactified on a circle and the type IIA theory.

The non-BPS torsion charged branes are not stable at all points of the moduli space. $D0$-brane charge in the type I theory receives an integral contribution from a wrapped BPS $D1$-brane and a $Z_2$ contribution from an unwrapped non-BPS $D0$-brane.

Let us look at the unwrapped $D0$-brane in the type I theory following [7]. The spectrum of open strings beginning and ending on the $D0$-brane is tachyon-free in 10 dimensions. However, when we compactify on a circle of radius $R$, the ground state with winding number 1 will have a classical mass squared given by

$$m^2 = -\frac{1}{2} + R^2,$$

in units with $\alpha' = 1$. It will therefore be tachyonic if the radius of the compactification circle is $R < \frac{1}{\sqrt{2}}$. In this situation, the $D0$-brane will decay into a $D1$-$\overline{D1}$ pair that wraps the $S^1$. The tachyon must have anti-periodic boundary conditions so that above the critical radius it will condense into a stable kink (the $D0$-brane). This requires turning on a $Z_2$ Wilson line on either the $D1$-brane or $\overline{D1}$-brane. The $Z_2$ charge of the unwrapped $D0$-brane corresponds to a $Z_2$ valued Wilson line in its decayed configuration. The same argument shows that the $Z_2$ charge of the unwrapped $D7$- and $D8$-branes correspond to $Z_2$ valued Wilson lines on their decay configurations, wrapped $D8$-$D8$ and $D9$-$\overline{D9}$ pairs respectively. For the $D(−1)$- and wrapped $D0$-brane, you have to compare to the instanton action since they are instantonic.

Under $T$-duality the unwrapped non-BPS $Dp$-branes with $p = 0, 7, 8$ get mapped to wrapped $D(p + 1)$-branes. Since $T$-duality inverts the radius, these develop a tachyon and become unstable when the $T$-dual radius is $\tilde{R} > \sqrt{2}$. For such radii the wrapped $D(p+1)$-branes decay into unwrapped $Dp$-$\overline{Dp}$ systems constrained to the $O8^+$-planes. The non-trivial $Z_2$ Wilson line in the type I theory corresponds to the brane and anti-brane being on different $O$-planes in the type IIA theory. When
a $\mathbb{Z}_2$ charged wrapped $D(p+1)$-brane decays in the type IA theory its charge then corresponds to the $\mathbb{Z}_2$ choice of which $O$-plane the $Dp$-brane is on.

In the region of stability of the type IA theory ($\tilde{R} < \sqrt{2}$) there would, at first glance, seem to be more $\mathbb{Z}_2$ charges than predicted by $K$-theory. Given the above discussion we would expect the $D0$-brane in the type IA theory to get a $\mathbb{Z}_2$ charge contribution from the choice of which $O$-plane to locate an unwrapped $D0$-brane and another $\mathbb{Z}_2$ charge contribution coming from a wrapped $D1$-brane (since we are in the region of stability). However, $K$-theory predicts that there should be only one source of $\mathbb{Z}_2 D0$-brane charge. To understand this, consider a stuck $D0$-brane (half of a $D0$-brane) at one $O$-plane and a wrapped $D1$-brane. This has the same conserved charges as a stuck $D0$-brane at the other $O$-plane and will decay into the latter configuration. In general a stuck $Dp$-brane at one $O$-plane will be transferred to a stuck $Dp$-brane at the other $O$-plane by a wrapped non-BPS $D(p+1)$-brane. This is described in [23] as a non-BPS brane stretched between two $O$-planes switching the type of $O$-plane between an $Op$- and $\bar{Op}$-plane (an $\bar{Op}$-plane can be interpreted as an $Op$-plane with a stuck $Dp$-brane). The above brane transfer operation shows that the $\mathbb{Z}_2$ $D0$-brane charge coming from the wrapped $D1$-brane and the contribution from the choice of which $O$-plane the unwrapped $D0$-brane is located are not distinct sources of charge. Therefore the $K$-theory prediction that there is only one distinct source of $\mathbb{Z}_2 D0$-brane charge is correct.

We have seen that the charge spectrum remains unchanged in and out of the region of stability for the non-BPS branes and that $K$-theory accurately classifies the charges for the entire moduli space.

3.3. The Type $\tilde{I}$ and $\tilde{IA}$ Theories. The type $\tilde{I}$ theory is the type IIB orientifold $(\mathbb{R}^9 \times S^1)/\langle \iota \cdot \Omega \rangle$ where $\iota$ is the spacetime involution that rotates the compact direction $\pi$ radians. In our notation, this is the type IIB theory on $\mathbb{R}^{9,0} \times S^{0,2}$. The $T$-dual of the type $\tilde{I}$ theory is the type $\tilde{IA}$ theory [7]. As we saw in the last section, the type IA theory contains 2 $O8^+$-planes. The type $\tilde{IA}$ theory is obtained from the type IA theory by replacing one of the $O8^+$-planes with an $O8^-$-plane. Here an $O^-$-plane is an $O$-plane with positive $D$-brane charge. (Note if there were $O8^-$-planes at both fixed points, then a charge 0 system would require the addition of anti-branes and wouldn’t be supersymmetric). We will refer to the compactification circle as $S^{0,1}_{(-,+)}$. It is topologically equivalent to a compactification on $S^{1,1}$, in that there are 2 fixed points. However, the net $O$-plane charge is zero.

Let $x$ be the coordinate of the compact direction in the type $\tilde{I}$ theory. Considering the circle as $\mathbb{R}/\mathbb{Z}$, we see that $S^{0,2}$ is the circle mod the involution

$$x \mapsto x + \frac{1}{2}.$$
Under $T$-duality this becomes the dual circle mod the involution
\[ \tilde{x} \mapsto -\tilde{x} + \frac{1}{2}. \]

In the language of [27], we should consider $S^{0.2}$ as a circle with a crosscap attached and then we see that the $T$-dual of a crosscap is an $O^+ - O^-$ plane pair.

$Dp$-brane charges in the type $\tilde{I}$ theory are classified by
\[
KR(S^{0-p,0} \times S^{0.2}, S^{0.2}) \cong KSC^{p-8}.
\]

$KSC$ doesn’t split into pieces from wrapped and unwrapped branes as in the previous case. The authors of [7] were still able to determine which charges come from wrapped and unwrapped branes using what we know about $T$-duality, the type IA theory and $O^{8\pm}$-planes. We will follow their argument here.

Since the type $\tilde{I}A$ theory is $T$-dual to the type $\tilde{I}$ theory, $Dp$-brane charges in the type $\tilde{I}A$ theory must also be classified by $KSC^{p-8}$. It is important to note that there is no mathematical description for this that we are aware of. There is only the physical reasoning, which requires the assumption of $T$-duality, that we are currently following. We will give a mathematical description in the next section.

We saw in section 3.2 that unwrapped $Dp$-branes near an $O^{8+}$-plane are classified by $KO^{p-8}$ (see Table 1). Conversely, $O^{8-}$-planes are quantized with symplectic gauge bundles, corresponding to $\Omega^2 = -1$. Symplectic gauge bundles are classified by $KSp(X) = KO^{-4}(X)$. Therefore, unwrapped $Dp$-brane charges near the $O^{8-}$-plane are classified $KSp^{p-8}$. See Table 2 for a list of unwrapped $Dp$-brane charges near $O^{8\pm}$-planes in a type IIA orientifold.

| $Dp$-brane | $D8$ | $D7$ | $D6$ | $D5$ | $D4$ | $D3$ | $D2$ | $D1$ | $D0$ | $D(-1)$ |
|------------|------|------|------|------|------|------|------|------|------|--------|
| $KO^{p-8}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $0$ | $\mathbb{Z}$ | $0$ | $0$ | $0$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ |
| $KSp^{p-8}$ | $\mathbb{Z}$ | $0$ | $0$ | $0$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $0$ | $\mathbb{Z}$ | $0$ |

Table 2. Unwrapped $D$-brane charges near $O^{8+}$- and $O^{8-}$-planes.

The complete brane content for the type $\tilde{I}$ and $\tilde{I}A$ theories is listed in Table 3. To see this, let us first consider BPS branes. Table 3 shows BPS $D8$-branes, but tadpole cancellation in the type $\tilde{I}A$ theory will require the net $D8$-brane charge to be zero. Additionally, Table 3 shows that there are unwrapped BPS $Dp$-branes for $p = 0, 4$. The $\mathbb{Z}$ contribution to these charges coming from both $KO$ and $KSp$ (see Table 2) are equated because 2 half $D0$-branes on the $O^{8+}$-plane form a $D0$-brane in the bulk which can then be interpreted as a $D0$-brane on the $O^{8-}$-plane and similarly a $D4$-brane on the $O^{8+}$-plane can be considered as 2 half $D4$-branes on the $O^{8-}$-plane. Now half $Dp$-branes can only live on one of the $O8$-planes and it no longer makes sense to have a brane transfer operation. Therefore there is no longer the $\mathbb{Z}_2$ charge contribution coming from a choice of $O8$-plane that we saw in the type IIA theory on $S^{1,1}$. It is important to note that the BPS branes are
stable near both $O^+$-planes and $O^-$-planes as is apparent from the fact that there are integral contributions coming from both the $KO$ and $KSp$ terms.

We will now consider the unwrapped non-BPS $Dp$-branes. Unlike the BPS case, we cannot just look locally at stable branes near the different $O8$-planes, but need to take into account global aspects. Table 2 correctly predicts the $Z_2$ charge contribution coming from unwrapped non-BPS $Dp$-branes for $p = -1, 3, 7$, but it also seems to predict $Z_2$ charge contributions from unwrapped non-BPS $Dp$-branes for $p = 2, 6$ that don’t appear in table 3. This is because the $D6$-brane ($D2$-brane) is stable at the $O8^+$-plane ($O8^-$-plane), but unstable near the $O8^-$-plane ($O8^+$-plane), so not globally stable. Table 3 lists only those charges that are globally stable. Let us look a little closer at why the $D6$-brane is not globally stable. The $D6$-brane can be viewed as $D6$-brane together with its mirror $\overline{D6}$-brane. Near the $O8^+$-plane the orientifold action projects out the tachyon in the system. Near the $O8^-$-plane the projection is different and the tachyon is not removed.

The wrapped brane charges in the type $\tilde{I}A$ theory are determined in [7] by considering unwrapped brane charges in the type $\tilde{I}$ theory, and then using $T$-duality arguments. We will not repeat their description here, but the results are listed in Table 3.

| $Dp$-brane | $KSC_{p-8}$ | Region of Stability | Type $\tilde{I}$ | Type $\tilde{IA}$ |
|------------|--------------|---------------------|------------------|------------------|
| $D8$       | $\mathbb{Z}$ | stable for all radii | wrapped $D9$-brane | unwrapped $D8$-brane |
| $D7$       | $Z_2$        | $R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\tilde{IA}} > \sqrt{2}$ | wrapped $D8$-brane | unwrapped $D7$-brane |
| $D6$       | 0            | $R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\tilde{IA}} < \sqrt{2}$ | unwrapped $D7$-brane | wrapped $D8$-brane |
| $D5$       | $\mathbb{Z}$ | stable for all radii | unwrapped $D5$-brane | doubly wrapped $D6$-brane |
| $D4$       | $\mathbb{Z}$ | stable for all radii | wrapped $D5$-brane | unwrapped $D4$-brane |
| $D3$       | $Z_2$        | $R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\tilde{IA}} > \sqrt{2}$ | wrapped $D4$-brane | unwrapped $D3$-brane |
| $D3$       | $Z_2$        | $R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\tilde{IA}} < \sqrt{2}$ | unwrapped $D3$-brane | wrapped $D4$-brane |
| $D2$       | 0            |                      |                  |                  |
| $D1$       | $\mathbb{Z}$ | stable for all radii | unwrapped $D1$-brane | doubly wrapped $D2$-brane |
| $D0$       | $\mathbb{Z}$ | stable for all radii | wrapped $D1$-brane | unwrapped $D0$-brane |
| $D(-1)$    | $Z_2$        | $R_{\tilde{I}} < \frac{1}{\sqrt{2}}, R_{\tilde{IA}} > \sqrt{2}$ | wrapped $D0$-brane | unwrapped $D(-1)$-brane |
| $D(-1)$    | $Z_2$        | $R_{\tilde{I}} > \frac{1}{\sqrt{2}}, R_{\tilde{IA}} < \sqrt{2}$ | unwrapped $D(-1)$-brane | wrapped $D0$-brane |

Table 3. $D$-brane charges in the type $\tilde{I}$ and type $\tilde{IA}$ theories.

There is no mathematical description for the classification of $D$-brane charges in the type $\tilde{IA}$ theory, so it requires the assumption of $T$-duality. Since the underlying topological space for the type $\tilde{IA}$ theory is $S^{1,1}$, we should be able to classify
$D$-brane charges by some twisted $KR$-theory of $S^{1,1}$. This idea motivates the
definition given in the following section.

4. $KR$-theory with a Sign Choice

The compactification manifolds for the type IA and $\tilde{I}A$ theories are topologically
equivalent, even taking the involution $\iota$ into account. Therefore, $KR$-theory cannot
differentiate between them. These two physical theories are differentiated by the
signs of the $O$-planes located at their fixed sets, so we must enhance $KR$-theory
with this information.

Along with the space $X$ and the action of a group $G$ (in our case $\mathbb{Z}_2$), we must
also include a sign choice, $\alpha$, on the components of the fixed set. Physically this
sign choice determines the type of $O$-plane at the different components of the fixed
set. In other words, it is a choice of orthogonal or symplectic Chan-Paton bundles
on the different components. Recall our convention that a $+$ choice corresponds to
an orthogonal Chan-Paton bundle, and a $-$ choice to a symplectic one. Note that
the fixed sets for the type IA and $\tilde{I}A$ theories both have 2 components, each a point.
The type IA theory is the sign choice $\alpha = (+, +)$, while the type $\tilde{I}A$ theory is the
sign choice $\alpha = (+, -)$. We define an extension of $KR$-theory that contains this
information. We start with a preliminary definition that explains the basic idea,
and then give a rigorous definition of a theory that has the necessary properties.
Note that (as in [37] and the literature on twisted $K$-theory, for example) this
rigorous definition requires noncommutative geometry.

Definition 1 (preliminary). Let $X$ be a compact Real space (that is, a $G$-space,
where $G = \mathbb{Z}_2$) with an assignment of signs $\alpha$ to the components of the fixed set.
The $KR$-theory with sign choice $\alpha$, $KR_\alpha(X)$, is defined to be the Grothendieck
group of complex vector bundles over $X$ with a Real structure compatible with the
involution on $X$ everywhere except on the restriction to components of the fixed
set with negative sign, where the bundles have a symplectic structure instead.

Since it is not clear that Definition 1 really makes sense, because the structure
of the bundle changes discontinuously as we approach the components of the fixed
set with negative sign, what we really do is to define $KR_\alpha(X)$ to be the topological
$K$-theory of a certain noncommutative Banach algebra. In what follows, $K$ and
$KR$ denote the algebras of compact operators on an infinite-dimensional separable
complex Hilbert space and an infinite-dimensional separable real Hilbert space,
respectively.

Definition 2. Let $X$ be a Real locally compact space with an assignment of signs
$\alpha$ to the components of the fixed set. Also, let $A_\alpha(X)$ be a real continuous-trace
algebra whose complexification $A_\alpha(X) \otimes \mathbb{C}$ has spectrum $X$ and trivial Dixmier-
Douady invariant, and for which the induced action $\sigma$ of $Gal(\mathbb{C}/\mathbb{R})$ on $X$ is the
given involution on $X$. We also require that the quotient of $A_\alpha(X)$ associated to
any component $Y^+$ of $X^G$ with positive sign choice to be Morita equivalent (over $\mathbb{R}$) to $C^0_{\mathbb{R}}(Y^+)$, and that the quotient of $\mathcal{A}_\alpha(X)$ associated to any component $Y^-$ of $X^G$ with negative sign choice is Morita equivalent (over $\mathbb{R}$) to $C^0_{\mathbb{H}}(Y^-)$, where $\mathbb{H}$ denotes the quaternions. We define $KR^*_\alpha(X)$ to be the topological $K$-theory of $\mathcal{A}_\alpha(X)$ (in the sense of [33, §3]).

Since neither existence nor uniqueness of the algebra $\mathcal{A}_\alpha(X)$ is obvious, we will give an explicit construction. To make the definition more precise, we will from now on always assume that this particular algebra has been chosen. But we will also show that the conditions in Definition 2 suffice at least to determine $KR^*_\alpha(X)$ up to extensions.

**Theorem 1.** Let $(X, \iota)$ be a Real locally compact space with an assignment of signs $\alpha$ to the components of the fixed set. Then an algebra $\mathcal{A}_\alpha(X)$ satisfying Definition 2 exists.

**Proof.** Let $Y^+$ be the union of the components of the fixed set with + sign choice, $Y^-$ be the union of the components of the fixed set with − sign choice, and $Z = X \setminus (Y^+ \cup Y^-)$, which is the open subset of $X$ on which the involution $\iota$ acts freely. Let $\mathcal{A}(Z)$ denote the commutative real $C^*$-algebra $\mathcal{A}(Z) = \{ f \in C_0(Z) \mid f(\iota(x)) = \overline{f(x)} \}$. (Recall that the $K$-theory of $\mathcal{A}(Z)$ is identical to $KR^*(Z)$.) First we will show that there is a a spectrum-fixing isomorphism of real $C^*$-algebras

$$\varphi: \mathcal{A}(Z) \otimes_{\mathbb{R}} \mathcal{K}_\mathbb{R} \xrightarrow{\cong} \mathcal{A}(Z) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathcal{K}_\mathbb{R}.$$  
(The induced isomorphism on complexifications is equivariant for the involution $\iota$.) Then we can define $\mathcal{A}_\alpha(X)$ by “clutching.”

The algebra $\mathcal{A}(Z) \otimes_{\mathbb{R}} \mathcal{K}_\mathbb{R}$ is, as explained in [33, §3], the algebra of sections (vanishing at infinity on $Z$) of a bundle over $\mathcal{Z} = Z/\iota$ of real $C^*$-algebras with fibers $\mathcal{K}$ and structure group $PU'$, the projective infinite-dimensional unitary/antiunitary group. This group is a semidirect product of $PU$ by $\mathbb{Z}_2$ (acting by complex conjugation), and the bundle is induced from the $\mathbb{Z}_2$-bundle $Z \to \mathcal{Z}$ defined by the free involution $\iota$. Now $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$, so $\mathcal{A}(Z) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathcal{K}_\mathbb{R}$ is also the algebra of sections of a bundle over $\mathcal{Z}$ with fibers $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ and the same structure group, and since the bundle came from the original $PU'$-bundle (via tensoring with $\mathbb{H}$) and induces the same covering map $Z \to \mathcal{Z}$, the bundles are isomorphic (as $PU'$-bundles). This guarantees existence of the desired isomorphism $\varphi$.

Now, let $X^+ = Z \cup Y^+$, $X^- = Z \cup Y^-$, which are both open subsets of $X$. $\mathcal{A}(Z)$ is an ideal in each of the commutative real $C^*$-algebras $\mathcal{A}(X^\pm) = \{ f \in C_0(X^\pm) \mid f(\iota(x)) = \overline{f(x)} \}$. We can construct $\mathcal{A}_\alpha(X)$ as the algebra of sections of a bundle of algebras obtained by clutching the stabilized bundles for $\mathcal{A}(X^+)$ and for $\mathcal{A}(X^-) \otimes \mathbb{H}$ together over $\mathcal{Z}$ via (the bundle isomorphism associated to) $\varphi$, i.e., we construct $\mathcal{A}_\alpha(X)$ by gluing $\mathcal{A}(X^+) \otimes_{\mathbb{R}} \mathcal{K}_\mathbb{R}$ (which represents $KR^*(X^+)$)
to $\mathcal{A}(X^-) \otimes \mathbb{R} \mathbb{H} \otimes \mathbb{R} \mathbb{K} \mathbb{R}$ (which represents $KR_0^j(X^-) \cong KSp^*(X^-)$) over $\mathbb{Z}$ using $\varphi$. It remains to show that we can choose $\varphi$ so that the Dixmier-Douady invariant of $\mathcal{A}_\alpha(X) \otimes \mathbb{R} \mathbb{C}$ vanishes. This follows from the Mayer-Vietoris sequence for the diagram

$$
\begin{array}{ccc}
X^+ & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
X^- & \xrightarrow{\varphi} & X
\end{array}
$$

since the Dixmier-Douady invariant is trivial over $X^+$ and $X^-$ (by construction) and thus the Dixmier-Douady invariant in $H^3(X)$ comes from the Phillips-Raeburn invariant of $\varphi$ in $H^2(\mathbb{Z})$ via the Mayer-Vietoris boundary map. (See the discussion in Remark 1 below for more details.)

**Corollary 1.** $KR_0^j$ has periodicity with period 8.

**Proof.** This is immediate from Bott periodicity for topological $K$-theory of real Banach algebras. □

**Remark 1.** One has to be cautious; even though Theorem 1 guarantees existence of $\mathcal{A}_\alpha(X)$, it does not guarantee uniqueness, since the isomorphism $\varphi$ is only determined up to an automorphism of the $PU'$-bundle over $\mathbb{Z}$. Such an automorphism, which (if $\mathbb{Z}$ is connected) we can assume is in the connected component of the identity in the automorphism group, is simply a section of the bundle of topological groups $B_{PU} = (\mathbb{Z} \times_{\mathbb{Z}} PU) \to \mathbb{Z}$, where the covering group $\mathbb{Z}_2$ acts on $U$ and thus on $PU$ by complex conjugation. The automorphism will not affect the $K$-groups if it is inner, i.e., comes from a section of $B_U = (\mathbb{Z} \times_{\mathbb{Z}} U) \to \mathbb{Z}$. From the exact sequence in sheaf cohomology for the exact sequence of sheaves of groups

$$1 \to B_T \to B_U \to B_{PU} \to 1,$$

where $B_T = (\mathbb{Z} \times_{\mathbb{Z}} \mathbb{T}) \to \mathbb{Z}$ and we identify bundles of topological groups with their sheaves of sections, and from the fact that the sheaf $B_U$ is fine since $U$ is contractible, we see that the obstruction to an automorphism being inner lies in $H^1(\mathbb{Z}, B_T) = H^2(\mathbb{Z}, \mathbb{T})$, where $\mathbb{T}$ is the sheaf of local sections of $B_T$. The obstruction group is via the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 1$$

identifiable with $H^2(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ is the locally constant sheaf with stalks $\mathbb{Z}$ and twisting given by the covering map $\mathbb{Z} \to \mathbb{Z}$. The obstruction is what we can call the *twisted Phillips-Raeburn invariant* (cf. [33 §1]). It vanishes when $H^2(\mathbb{Z}, \mathbb{Z}) = 0,$
and in particular when dim $Z = 1$, so in this case we have uniquely defined $\mathcal{A}_\alpha(X)$.

While we will always use the particular algebra $\mathcal{A}_\alpha(X)$ constructed in the proof of Theorem I, any algebra satisfying Definition 2 gives the same groups $KR^*_\alpha(X)$ up to extensions.

**Theorem 2.** Any choice satisfying Definition 2 fits into the long exact sequence

$$
\cdots \to KR^{-i}(Z) \to KR^{-i}_\alpha(X) \to KO^{-i}(Y^+) \oplus KSp^{-i}(Y^-) \to KR^{-i+1}(Z) \to \cdots,
$$

and the groups $KR^*_\alpha(X)$ are uniquely determined at least up to extensions.

**Proof.** The long exact sequence

$$
\cdots \to KR^{-i}_\alpha(X) \to KR^{-i}_\alpha(X \setminus Z) \to KR^{-i}_\alpha(Y^+) \oplus KR^{-i}_\alpha(Y^-) \to \cdots.
$$

follows from the long exact $K$-theory sequence of the extension of real $C^*$-algebras associated to the open inclusion $Z \subseteq X$ (see for example [33, equation (*), p. 376]). Since the action on $Z$ is free, $KR^{-i}_\alpha(Z) \cong KR^{-i}(Z)$.

$$
KR^{-i}_\alpha(X \setminus Z) \cong KR^{-i}_\alpha(Y^+) \oplus KR^{-i}_\alpha(Y^-).
$$

By definition, $KR_\alpha \cong KR$ if the space only has $+$ sign choices. Therefore,

$$
KR^{-i}_\alpha(Y^+) \cong KO^{-i}(Y^+),
$$

since $Y^+$ has trivial involution.

For a space $M$ where all the components of $MG$ have $-$ sign choice, the only difference is that the quotient of $\mathcal{A}_\alpha(M)$ associated to any component of $MG$ is Morita equivalent (over $\mathbb{R}$) to $C^*_\alpha(MG)$ (instead of $C^*_0(MG)$). This defines a symplectic structure (instead of a real structure). In this case, $KR$-theory with a sign choice reduces to what is sometimes referred to as $KSp$- or $KH$-theory, which is just ordinary $KO$-theory with a shift in index by 4. Therefore,

$$
KR^{-i}_\alpha(Y^-) \cong KO^{-i+4}(Y^-) \cong KSp^{-i}(Y^-).
$$

Putting this all together gives the long exact sequence (9).

Since the connecting maps

$$
KO^{-i}(Y^+) \to KR^{-i+1}(Z) \quad \text{and} \quad KSp^{-i}(Y^-) \to KR^{-i+1}(Z)
$$

in (10) are determined by the $KR$-theories of $X^+$ and $X^-$, respectively, we conclude that regardless of what choice one makes of $\mathcal{A}_\alpha(X)$ satisfying Definition 2, the groups $KR^*_\alpha(X)$ are uniquely determined at least up to extensions. \qed
Sometimes it is possible to make a very explicit choice of $\mathcal{A}_\alpha(X)$, even when $Y^+$ and $Y^-$ are both non-empty, in which the complexified algebra $\mathcal{A}_\alpha(X) \otimes \mathbb{C}$ is just $C_0^\infty(X) \otimes M_2(\mathbb{C})$. We will illustrate with a case we will consider shortly, $X = S^{11}_{(+,-)}$. Let $c$ be complex conjugation and

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

so that $I$, $J$, $K$ satisfy $I^2 = J^2 = K^2 = -1$ and $IJ = K$, $JK = I$, $KI = J$, so the $\mathbb{R}$-span of $1$, $I$, $J$, and $K$ is $\mathbb{H}$, the quaternions. The trick is to properly define the conjugation operator $\sigma$ so that this operator is $c$ over $Y^+$ and $c \circ \text{Ad}(J)$ over $Y^-$. $\text{Ad}(J)$ fixes $1$ and $J$ and reverses the signs of $I$ and $K$, but since these two matrices are purely imaginary and $1$ and $J$ are real, the fixed-point algebra of $c \circ \text{Ad}(J)$ is precisely $\mathbb{H}$, and thus over $Y^-$ we get the algebra $C_0^\infty(Y^-)$, whose topological $K$-theory is just $KSp^*(Y^-)$. Similarly, over $Y^+$, the fixed-point algebra for $\sigma = c$ is just $C_0^\infty(Y^+) \otimes M_2(\mathbb{R})$, which is Morita equivalent to $C_0^\infty(Y^+)$, whose topological $K$-theory is just $KO^*(Y^+)$. Finally, over $Z = X \setminus X^G$, we take $\sigma$ to be any conjugate-linear involution which sends the fiber $M_2$ over a point $x$ to the fiber $M_2$ over $\iota(x)$ ($\iota$ is the free involution on $Y$) via $c \circ \text{Ad}(u(x))$, where $u: X \to U(2)$ is a function whose image as a map $X \to PU(2)$ is continuous. (We can think of $u$ as taking values in the projective unitary group since changing $u$ by a scalar matrix doesn’t change $\text{Ad}u$.) Note that $u$ is not unique, but it is required to have the properties that $u \equiv 1$ on $X^G_f$, $u \equiv J$ on $X^G$ (here $\equiv$ means “equal in $PU$”), and $u(x)u(\iota(x)) \equiv 1$ (so that $\sigma^2 = 1$).

It is not obvious that such a $u$ should exist for every assignment of signs to the components of $X^G$, but when such a $u$ exists, it is often unique up to homotopy. Consider $X = S^{11}_{(+,-)}$. Identify $X$ with the unit circle in the complex plane and assume for sake of definiteness that we have sign $+$ at the point $+1$ and sign $-$ at the point $-1$. Then define

$$u(e^{i\theta}) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad 0 \leq \theta \leq \pi.$$

Note that $u(1) = 1$ and $u(-1) = J$. Meanwhile take

$$u(e^{-i\theta}) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad 0 \leq \theta < \pi.$$

Since $\iota(e^{i\theta}) = e^{-i\theta}$, the necessary compatibility condition is satisfied. At first sight there is a problem with continuity at $-1$, since $u(e^{-i\theta}) \to -J$ while $u(e^{i\theta}) \to +J$ as $\theta \to \pi^-$, but this is OK since $J$ and $-J$ agree in $PU$. In this example, $u$ could be constructed out of any path from $1$ to the class of $J$ in $PU(2)$ (giving $u$ on the top half of the circle; $u$ on the bottom half is determined by $u$ on the top half). Such paths are unique up to homotopy modulo an element of $\pi_1(PU(2)) \cong \mathbb{Z}_2$. Homotopic choices for $u$ certainly give identical $KR_\alpha$-theories, but in this case it’s
also easy to see that both homotopy classes give the same groups, since when we
tensor with $K_R$, the paths become homotopic since $PU(H)$ is simply connected.

$KR$-theory with a sign choice can also be computed using a generalization of
the equivariant Atiyah-Hirzebruch spectral sequence of \cite[(A.2)]{26}

\[ E_2^{p,q} = H^p_G(X; KR^q) \Rightarrow KR^{p+q}(X), \]

where $KR^*$ is the Bredon coefficient system for $G$ associated to $KR$.

As described in \cite{26}, $KR^*(G) = K^*$ and $KR^*(pt) = KO^*$. We are now allowing
for different components of the fixed set to have symplectic or orthogonal bundles,
corresponding to the coefficient system being $KO$ or $KSp$.

**Theorem 3.** There is a spectral sequence

\[ E_2^{p,q} = H^p_G(X; KR_{\alpha}^q) \Rightarrow KR_{\alpha}^{p+q}(X). \]

where $KR_{\alpha}^*(G) \cong K^*$ and

\[ KR_{\alpha}^{-i}(pt_j) = \begin{cases} KO^{-i}, & \text{if } \alpha_j = +, \\
KSp^{-i}, & \text{if } \alpha_j = -. \end{cases} \]

*Proof. The proof is quite similar to the case handled in \cite{26}. We filter $K^*_\alpha(X)$
using the equivariant skeletal filtration, but with fixed cells separated into two
types. Then this is just the spectral sequence associated to this filtration.

The picture of the coefficient system is as in Figure 1 (right side). \qed

If we remove the 2 fixed points from $S^{1,1}$ we are left with 2 copies of $\mathbb{R}$
that are exchanged by the involution. This gives $KR_{\alpha}^*(S^{1,1} \setminus \text{fixed points}) \cong K^{*-1}$
by (5). The type IA theory has $O8^+$-planes (hence orthogonal bundles) at both
fixed points, so has $KO^*$ at both fixed points (see Figure 1) and matches with
the spectral sequence as described in \cite{26}. While motivated by physics, we are
just decorating $G$-$CW$-complexes with some extra information on the equivariant
cells of the form $(G/G) \times e^n$ that we have called “sign.” We show in \cite{16}
that the sign can be given a geometric interpretation in the $T$-dual theory, thus giving a
completely mathematical description of $T$-duality.

Note that flipping the sign of every component of the fixed set exchanges $KO$
and $KSp$ and so just results in a shift of the index by 4. For example, type IIA
theory on $S^{1,1}$ with 2 $O8^-$-planes does not make physical sense since it is not
supersymmetric, and it is mathematically uninteresting since it is just the usual
theory with an index shift.
We can now turn our attention to the only case of $KR$-theory of a circle with a non-trivial sign choice, corresponding to the type $IA$ theory.

4.1. The Type $IA$ Theory. As noted in Section 3, the compactification manifold for the $IA$ theory is $S^{1,1}$, but with an $O8^-$-plane at one fixed point and an $O8^+$-plane at the other. Therefore, $Dp$-branes are classified by $KR^{p-9}_{(+,-)}(S^{1,1})$. The index is determined as a shift by one from the $T$-dual theory $KR^{p-9}(S^{0,2})$.

Recall that

$$S^{1,1} \setminus S^{1,0} \cong \mathbb{R}^{1,0} \times S^{0,1},$$

with an involution that exchanges the 2 copies of $\mathbb{R}$. Therefore

$$KR^i_\alpha(S^{1,1} \setminus S^{1,0}) \cong KR^{-(i)}_\alpha(\mathbb{R}^{1,0} \times S^{0,1})$$

$$\cong K^{-i}(\mathbb{R})$$

$$\cong K^{-i-1},$$

for all $\alpha$. For $\alpha = (+, -)$,

$$KR^i_\alpha(S^{1,0}) \cong KO^{-i} \oplus KSp^{-i}.$$  

Plugging these into equation (9) we get the long exact sequence

$$\cdots \rightarrow K^{-i-1} \rightarrow KR^{-i}_{(+,-)}(S^{1,1}) \rightarrow KO^{-i} \oplus KSp^{-i} \rightarrow \delta \rightarrow K^{-i} \rightarrow \cdots.$$  

The map $\delta$ is complexification on the first summand and doubling on the second summand, since symplectic bundles contain 2 complex bundles. Furthermore, the long exact sequence splits into 2 parts

$$0 \rightarrow KR^0_{(+,-)}(S^{1,1}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \delta \rightarrow K^{-3}R^{(mod \ 4)}_{(+,-)}(S^{1,1}) \rightarrow 0.$$  

$$0 \rightarrow KR^{-2}_{(+,-)}(S^{1,1}) \rightarrow \mathbb{Z}_2 \gamma \rightarrow \mathbb{Z} \rightarrow \sigma \rightarrow KR^{-1}_{(+,-)}(S^{1,1}) \rightarrow \mathbb{Z}_2 \rightarrow 0.$$  

The map $\delta$ is $(m, n) \mapsto m + 2n$, which is surjective. This means $KR^0_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$ and $KR^{-2}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}_2$. $\gamma$ must be 0, showing $KR^{-2}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}_2$. This gives us an extension problem

$$0 \rightarrow \mathbb{Z} \rightarrow \sigma \rightarrow K^{-1}_{(+,-)}(S^{1,1}) \rightarrow \mathbb{Z}_2 \rightarrow 0.$$  

However, since removing the fixed point with the $-$ sign from $S^{1,1}_{(+,-)}$ leaves $\mathbb{R}^{0,1}$, we also have an exact sequence

$$0 = KSp^{-2} \rightarrow KR^{-1}(\mathbb{R}^{0,1}) \rightarrow KR^{-1}_{(+,-)}(S^{1,1}) \rightarrow KSp^{-1} = 0,$$
and since $KR^{-1}(\mathbb{R}^{0,1}) \cong KO_0 = \mathbb{Z}$, we see that $KR^{-1}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$. The corresponding argument where we remove the point with the + sign instead shows that $KR^{-5}_{(+,-)}(S^{1,1}) \cong \mathbb{Z}$. Putting this all together we find

$$KR^{-i}_{(+,-)}(S^{1,1}) \cong KSC^{-i+1},$$

which has periodicity with period 4.

Now we can see $T$-duality between the type $\tilde{I}$ and $\tilde{I}A$ theories as an isomorphism

$$KR^{-i}(S^{0,2}) \cong KR^{-i-1}_{(+,-)}(S^{1,1}).$$

The fact that we need to include a charge in the $T$-dual of the IIB theory on $S^{0,2}$ is contained in the geometry of $S^{0,2}$ in a way that is explored in [16]. Now let us turn to the different possibilities of sign choices for 2-torus orientifolds.

4.2. Torus Orientifolds with a Sign Choice. For a 2-torus with an involution the possible fixed point sets are empty, 1 copy of $S^1$, 2 disjoint copies of $S^1$, 4 isolated points, or the entire copy of $T^2$. Obviously, when the fixed set is empty there are no possible sign choices. Also, when the fixed set is a single copy of $S^1$ or the entire 2-torus, then the fixed set has only a single component. Therefore, there is only one possible sign choice giving either ordinary $KR$-theory (+ sign choice) or an index shift by 4 of ordinary $KR$-theory (− sign choice). The only cases that do not immediately reduce to ordinary $KR$-theory are when the fixed point set is either 2 disjoint copies of $S^1$ or 4 isolated points.

Let us first consider the orientifold of the 2-torus with 4 fixed points, corresponding to when the involution is reflection. Topologically our orientifold is $S^{1,1} \times S^{1,1}$. There are 3 supersymmetric sign choices for the 4 fixed points: $\alpha = (+, +, +, +), (+, +, -, -)$, or $(+ ,+ ,+ ,-)$. (The non-supersymmetric cases $(- ,-, -, -)$ and $(- ,-, -, +)$ can be obtained from $(+, +, +, +)$ and $(+, +, +, -)$ by an index shift.) Constructing an explicit algebra satisfying the requirements of Definition 2 is obvious for $(+, +, +, +)$ (which gives usual $KR$-theory) and for $(+, +, -, -)$, since we have already constructed an explicit noncommutative algebra giving rise to $S^{1,1}_{(+, -)}$, so the algebra for $(+, +, -, -)$ is just the tensor product of this algebra with the (commutative) algebra for $S^{1,1}_{(+, +)}$.

The case of $(+, +, +, -)$ is considerably more subtle, though as shown by Witten [36], the case of four $O$-planes, three with a + charge and one with a − charge, does indeed occur in physics. The construction of a noncommutative algebra giving rise to $KR^*_y(+)_{(+, +, +, -)}(S^{1,1} \times S^{1,1})$ is not so straightforward but of course follows from Theorem 1. As explained in Remark 1 there is a potential problem with non-uniqueness, but in this case the definition is unambiguous since deleting the fixed points gives a Real space equivariantly contractible to a space of dimension $< 2$.

---

3Since this is what’s needed for physics, we are assuming the torus can be identified with a complex smooth curve of genus 1, and the involution is either holomorphic or anti-holomorphic. This will be explained further in [16].
To determine $KR_\alpha$ in each case, we first need the following result.

**Proposition 2.** Let $X$ be $T^2$ (realized as $\mathbb{R}^2/\mathbb{Z}^2$) with involution given by multiplication by $-1$. Let $Y$ be the set of 4 points fixed by the involution. Then

$$KR^{-i}(X \setminus Y) \cong KSC^{-i-1} \oplus K^{-i-1} \oplus K^{-i-1},$$

for all $\alpha$.

**Proof.** Consider the fundamental domain of the $T^2$ shown in Figure 2. When we remove the four fixed points and the dashed lines along the boundary of the fundamental domain, what remains retracts onto the square with vertices at $(\pm \frac{1}{4}, \pm \frac{1}{4})$ shown in color. This square is, as a Real space, just $S^{0,2}$, so $X \setminus Y \setminus$ (dashed lines) is as a Real space just $S^{0.2} \times \mathbb{R}^{1.0}$. The eight dashed lines in the figure, when divided out by the action of $\mathbb{Z}^2$, become four lines in $T^2$, and $\iota$ permutes them in two pairs. Thus we have an equivariant decomposition

$$X \setminus Y \cong (S^{0.2} \times \mathbb{R}^{1.0}) \amalg (\mathbb{R}^{1.0} \times S^{0.1}) \amalg (\mathbb{R}^{1.0} \times S^{0.1}).$$

Therefore we get an exact sequence

$$\cdots \to KR^{-i}(X \setminus Y) \to KR^{-i}(\mathbb{R}^{1.0} \times S^{0.1}) \oplus KR^{-i}(\mathbb{R}^{1.0} \times S^{0.1}) \to \cdots.$$

Since $\iota$ acts freely on $X \setminus Y$, $KR^{-i}(X \setminus Y)$ is independent of the sign choice $\alpha$ and reduces to ordinary $KR$-theory. Furthermore, $KR^{-i}(S^{0.2} \times \mathbb{R}^{1.0}) \cong KSC^{-i-1}$ and $KR^{-i}(\mathbb{R}^{1.0} \times S^{0.1}) \cong K^{-i-1}$. So sequence (22) simplifies to

$$\cdots \to KR^{-i}(X \setminus Y) \to K^{-i-1} \oplus K^{-i-1} \xrightarrow{\delta} KSC^{-i} \to KR^{-i+1}(X \setminus Y) \to \cdots.$$

When the sign choices are all positive, we get the exact sequence

$$\cdots \to KR^{-i}(X \setminus Y) \to KR^{-i}(X) \to KR^{-i}(Y) \to \cdots.$$

$X = S^{1.1} \times S^{1.1}$ as a Real space, so from the equivariant decomposition

$$S^{1.1} \times S^{1.1} = (S^{1.1} \times \{\text{pt}\}) \amalg (S^{1.1} \times \mathbb{R}^{0.1})$$
we get
\begin{equation}
KR^{-i}(X) \cong KR^{-i}(S^{1,1}) \oplus KR^{-i+1}(S^{1,1}) \cong KO^{-i} \oplus KO^{-i+1} \oplus KO^{-i+2}.
\end{equation}

$KR^{-i}(Y) \cong (KO^{-i})^4$, since $Y$ is 4 points with trivial involution and positive sign choice. Putting this into sequence (24) immediately gives $KR^{-6}(X \setminus Y) \cong Z$ and $KR^{-5}(X \setminus Y) \cong Z^2$. Plugging $KR^{-5}(X \setminus Y)$ into (23) gives the exact sequence
\[
0 \to Z^2 \to Z^2 \to Z_2 \to KR^{-4}(X \setminus Y) \to 0,
\]
showing that $KR^{-4}(X \setminus Y)$ must be 0 or $Z_2$. From (24) we see that $KR^{-4}(X \setminus Y)$ cannot be 0, and therefore must be $Z_2$. Plugging $KR^{-6}(X \setminus Y)$ into (23) shows that $\delta_{-7} = 0$, giving the short exact sequence
\[
0 \to Z \to KR^{-7}(X \setminus Y) \to Z^2 \to 0.
\]
Therefore we see that $KR^{-7}(X \setminus Y) \cong Z^3$. The sequence (24) shows that $KR^{-2}(X \setminus Y)$ must have free rank 1 (plus possibly some torsion). However, the sequence (23) shows that $KR^{-2}(X \setminus Y)$ can have at most 1 generator. Therefore $KR^{-2}(X \setminus Y) \cong Z$. Putting this into (23) shows that $KR^{-3}(X \setminus Y) \cong Z^3$. Sequence (23) implies that $KR^{-1}(X \setminus Y)$ is torsion free. This means that the map $(KO^{-2})^4 \cong (Z_2)^4 \to KR^{-1}(X \setminus Y)$ in (24) is 0, and we get the exact sequence
\[
0 \to KR^{-1}(X \setminus Y) \to Z^2 \oplus Z_2 \to (Z_2)^4 \to \cdots.
\]
This shows that $KR^{-1}(X \setminus Y)$ must have free rank 2, and we find $KR^{-1}(X \setminus Y) \cong Z^2$. Finally, plugging this into sequence (23) shows that $KR^0(X \setminus Y)$ must be 0 or $Z_2$, but sequence (24) shows that it cannot be 0. Therefore $KR^0(X \setminus Y) \cong Z_2$.

Putting this all together, we see that $KR^{-i}_\alpha(X \setminus Y)$ has periodicity with period 4 and is given by
\begin{equation}
KR^{-i}_\alpha(X \setminus Y) = \begin{cases} 
Z_2, & i = 0 \\
Z^2, & i = 1 \\
Z, & i = 2 \\
Z^3, & i = 3.
\end{cases}
\end{equation}

\[ \square \]

Alternate proof using the spectral sequence. We can also prove Proposition 2 using the spectral sequence in Theorem 3. Since $\iota$ acts freely on $X \setminus Y$, the spectral sequence reduces to the one studied by Karoubi and Weibel [26, Example A.3]. Let $W = (X \setminus Y)/\iota$, which is diffeomorphic to $S^2 \setminus \{4 \text{ points}\}$. The map $(X \setminus Y) \to W$ is a 2-to-1 covering map. The spectral sequence has $E^{p,q}_2 = 0$ for $q$ odd, and reduces to $H^p_\iota(W, Z(i)) \Rightarrow KR^{p+2}_\alpha(X \setminus Y)$, where $Z(i) = Z$ (the constant sheaf) for $i$ even and $Z(i)$ is the nontrivial local coefficient system determined by the covering map $(X \setminus Y) \to W$ for $i$ odd. By Poincaré duality, $H^p_\iota(W, Z) \cong H_{2-p}(W, Z)$, which is
\[ Z \text{ for } p = 2, Z^3 \text{ for } p = 1, 0 \text{ for } p = 0. \] The groups \( H^p_c(W, \mathbb{Z}(1)) \) are slightly harder to compute, but can be obtained, for example, from the exact sequence

\[ \cdots \rightarrow H^p_c(\mathbb{R} \times S^1, \mathbb{Z}(1)) \rightarrow H^p_c(W, \mathbb{Z}(1)) \rightarrow H^p_c(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}(1)) \rightarrow \cdots, \]

coming from the fact that deleting two line segments from \( W \), each one running between two of the branch points of the branched covering \( T^2 \rightarrow S^2 \), leaves an open subset diffeomorphic to \( \mathbb{R} \times S^1 \). Here \( H^p_c(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}(1)) \cong H^p_c(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}) \cong \mathbb{Z}^2 \) for \( p = 1 \), and 0 for other values of \( p \), since each component of \( \mathbb{R} \amalg \mathbb{R} \) is simply connected. The result is that \( H^p_c(W, \mathbb{Z}(1)) \) is isomorphic to \( \mathbb{Z}^2 \) for \( p = 1 \), \( \mathbb{Z}_2 \) for \( p = 2 \), and 0 for other values of \( p \). The spectral sequence is shown in Figure 3. Note that there is no room for any nontrivial differentials or for any nontrivial extensions, and the Proposition follows. \( \square \)

\[ q \backslash p \quad 0 \quad 1 \quad 2 \]

\[
\begin{array}{c|ccc}
q & H_c^0(\mathbb{R} \times S^1, \mathbb{Z}(1)) & H_c^1(W, \mathbb{Z}(1)) & H_c^2(\mathbb{R} \amalg \mathbb{R}, \mathbb{Z}(1)) \\
\hline
0 & 0 & \mathbb{Z}^3 & \mathbb{Z} \\
-1 & 0 & 0 & 0 \\
-2 & 0 & \mathbb{Z}^2 & \mathbb{Z}_2 \\
-3 & 0 & 0 & 0 \\
-4 & 0 & \mathbb{Z}^3 & \mathbb{Z} \\
\end{array}
\]

Figure 3. \( E_2 \) of the spectral sequence for computing \( KR_{\alpha}^\ast(X \setminus Y) \).

The sequence repeats with vertical period 4.

For the set of fixed points, \( Y \), the three options are

\[ KR_{\alpha}^{-i}(Y) = \begin{cases} 4KO^{-i}, & \alpha = (+, +, +, +) \\ 2KO^{-i} \oplus 2KSp^{-i}, & \alpha = (+, +, -, -) \\ 3KO^{-i} \oplus KSp^{-i}, & \alpha = (+, +, +, -). \end{cases} \] (27)

The case where \( \alpha = (+, +, +, +) \) just gives ordinary \( KR \)-theory, which was computed in (25) above. In the notation of [31] this is \( T^{1,2} \).

The relevant long exact sequence for \( \alpha = (+, +, +, -) \) is (via Proposition 2)

\[ \cdots \rightarrow KSC^{-i-1} \oplus 2K^{-i-1} \rightarrow KR^{-i}((+, +, +,-))(S^{1,1} \times S^{1,1}) \rightarrow 3KO^{-i} \oplus KSp^{-i} \rightarrow \cdots. \] (28)

This gives an extension problem in determining each of the \( KR \)-groups. Therefore, we need to look at some additional long exact sequences to determine \( KR_{\alpha}^\ast(S^{1,1} \times S^{1,1}) \).
Let $Y_+ = S^1_{(+)} \vee S^1_{(+)}$ be the wedge of 2 circles going through the three fixed points with sign choice +. In terms of Figure [2] this is the image of the dotted lines. Then we get a long exact sequence

\[ \cdots \to KR^{-i}(X \setminus Y_+) \to KR^{-i}_a(X) \to KR^{-i}_a(Y_+) \to \cdots. \]

Note that $X \setminus Y_+ \cong \mathbb{R}^{0,2}$, where the fixed point of $\mathbb{R}^{0,2}$ is given the sign choice −. Therefore

\[ KR^{-i}_a(X \setminus Y_+) \cong KR^{-i}_a(\mathbb{R}^{0,2}) \cong KSp^{-i+2}. \]

To determine $KR^{-i}_{(+,+)}(Y_+)$, first note that this reduces to ordinary $KR$-theory since the sign choices are all positive. Now consider the split long exact sequence

\[ \cdots \to KR^{-i}(Y_+ \setminus \{pt\}) \to KR^{-i}(Y_+) \to KR^{-i}(pt) \to \cdots, \]

where the basepoint is the joining point of the two circles (a fixed point with sign +). Therefore, $Y_+ \setminus \{pt\}$ is 2 copies of $\mathbb{R}^{0,1}$ and

\[ KR^{-i}(Y_+) \cong KO^{-i+1} \oplus KO^{-i+1} \oplus KO^{-i}. \]

Plugging $KR^{-i}_a(Y_+)$ into the exact sequence

\[ \cdots \to KR^{-i}_a(X \setminus Y_+) \cong KSp^{-i+2} \to KR^{-i}_a(X) \to KR^{-i}_a(Y_+) \to \cdots, \]

we find $KR^{-i}_a(S^{1,1} \times S^{1,1})$ is $\mathbb{Z}$ if $i = 4$ or 6, $\mathbb{Z}^2$ for $i = 5$, and $\mathbb{Z}^2$ for $i = 3$. There are extension problems for the other 4 indices mod 8.

To solve the remaining extension problems, we can repeat the same process, but use the space $Y_-$ which is the one point union of 2 circles joined at the fixed point with sign − and going through 2 of the fixed points with sign choice +. This space is the image of the coordinate axes in Figure [2]. Note that $X \setminus Y_- \cong \mathbb{R}^{0,2}$ (with a + sign at the fixed point). If we remove one circle, which we can identify with $S^1_{(+)}$, from $Y_-$, then what remains is $\mathbb{R}^{0,1}$ (with a + sign), so we get an exact sequence

\[ \cdots \to KR^{-i}(\mathbb{R}^{0,1}) \to KR^{-i}_a(Y_-) \to KR^{-i}_{(+,-)}(S^{1,1}) \to \cdots, \]

or in other words,

\[ \cdots \to KO^{-i+1} \to KR^{-i}_a(Y_-) \to KSC^{-i+1} \to KO^{-i+2} \to \cdots. \]

In fact [31] splits, i.e., $KR^{-i}_a(Y_-) \cong KO^{-i+1} \oplus KSC^{-i+1}$, since the inclusion $S^{1,1}_{(+,-)} \hookrightarrow Y_-$ is split by the (sign-preserving) “fold map” sending both circles in $Y_-$ onto $S^{1,1}_{(+,-)}$. Putting our result for $Y_-$ into the exact sequence

\[ \cdots \to KR^{-i}(\mathbb{R}^{0,2}) \cong KO^{-i+2} \to KR^{-i}_a(X) \to KR^{-i}_a(Y_-) \to KO^{-i+3} \to \cdots \]

gives that $KR^{-i}_{(+,+,-)}(S^{1,1} \times S^{1,1})$ is $\mathbb{Z}$ for $i = 0$, $\mathbb{Z}^2$ for $i = 1$, $\mathbb{Z} \oplus (\mathbb{Z}_2)^2$ for $i = 2$ (for this case we must combine the information we get from [28], [30], and [31]),
and 0 for \(i = 7\). The results of the calculation are summarized in the last column of Table 4 in Section 5.1 below.

We could also use the spectral sequence in Theorem 3 for \(S^{1,1} \times S^{1,1}\) with \(\alpha = (+, +, +, -)\). To determine the \(E_2\) term, we need to look at the groups \(H^p_G(X;KR_{\alpha^q})\). These are most easily computed using the exact sequence

\[
\cdots \rightarrow H^p_{G,c}(X \times X';KR_{\alpha^q}) \rightarrow H^p_G(X;KR_{\alpha^q}) \rightarrow H^p_G(X';KR_{\alpha^q}) \rightarrow \cdots.
\]

Here \(H^p_{G,c}(X \times X';KR_{\alpha^q})\) is non-zero only for \(p = 0\), where it is \(3KO^q \oplus KS\), and \(H^p_{G,c}(X \times X';KR_{\alpha^q})\) was computed in the “Alternate proof” of Proposition 2. In (32) there is one potentially nonzero connecting map, \(\mathbb{Z}^4 \cong 3KO^q \oplus KS \rightarrow \mathbb{Z}^3\) when \(q \equiv 0 \pmod{4}\). This map can be computed by comparison with the corresponding sequences for the cases of \(S^{1,1}_{+,-}\) and \(S^{1,1}_{-,+}\), where the \(KR_{\alpha}\) groups were computed from equation (15) and the surrounding discussion. One finds that the connecting map has kernel \(\mathbb{Z}\) in all cases, is surjective for \(q \equiv 0 \pmod{8}\), and has a cokernel of \(\mathbb{Z}_2^3\) when \(q \equiv 4 \pmod{8}\). Thus the groups \(H^p_G(X';KR_{\alpha^q})\) are as in Figure 4. This calculation is consistent with our computation of \(KR_*(S^{1,1} \times S^{1,1})\), assuming that there are \(d_2\) differentials that kill off the \(\mathbb{Z}_2\)'s in positions (2, −2) and (2, −6).

The case \(\alpha = (+, +, +, -)\) can be obtained by the product of the type \(\tilde{IA}\) theory with itself or the type IA theory. The equivariant decomposition

\[
S^{1,1}_{(+,-)} \times S^{1,1} = (S^{1,1}_{(+,-)} \times \{pt\}) \amalg (S^{1,1}_{(+,-)} \times \mathbb{R}^{0,1})
\]

gives the calculation

\[
KR^i_{(+,+,-,-)}(S^{1,1} \times S^{1,1}) \cong KSC^i + KO^{i+1}.
\]

The same case can also be obtained by looking at

\[
S^{1,1}_{(+,-)} \times S^{1,1}_{(+,-)} = (S^{1,1}_{(+,-)} \times \{pt\}) \amalg (S^{1,1}_{(+,-)} \times \mathbb{R}^{0,1}).
\]

But crossing with \(\mathbb{R}^{0,1}\) has the same effect as crossing with \(\mathbb{R}^{4,1}\) or with \(\mathbb{R}^{3,0}\), and since \(KSC^*\) is 4-periodic, we get the same result as in (33).

Now let us consider orientifolds of the 2-torus where the fixed set is 2 disjoint copies of \(S^1\). Topologically, this is \(S^{1,1} \times S^{2,0}\). There are 2 possible supersymmetric sign choices \((+, +)\) and \((+, -)\). As usual, the non-supersymmetric case \((-,-)\) can be obtained from \((+, +)\) by an index shift. When both fixed circles have sign choice +, \(KR_{\alpha}\) reduces to ordinary \(KR\)-theory,

\[
KR^{-i}(S^{1,1} \times S^{2,0}) \cong KR^{-i-1}(S^{1,1}) + KR^{-i}(S^{1,1})
\]

\[
\cong KO^{-i} \oplus KO^{-i} \oplus KO^{-i} \oplus KO^{-i+1}.
\]
The case $\alpha = (+, -)$ is just the product of the type $\tilde{IA}$ theory, $S^{1,1}_{(+,-)}$, with a fixed circle, $S^2_0$, so we find

$$KR^{-i}(S^{1,1} \times S^2_0) \cong KR^{-i-1}(S^{1,1}) \oplus KR^{-i}(S^{1,1}) \cong KSC^{-i} \oplus KSC^{-i+1}. \quad (35)$$

To conclude this section, we explain how to compute $KR$-theory for a 2-torus orientifold where the involution $\iota$ is orientation reversing and has a fixed set that is topologically $S^1$. Unlike the cases above, this orientifold does not split as a product of two circle orientifolds, so a somewhat more complicated calculation is required.

**Theorem 4.** Let $(X, \iota)$ be a Real space where $X = T^2$ and $\iota$ is smooth, orientation reversing, and has a fixed set that is topologically $S^1$. The quotient space $M = X/\iota$ is topologically a closed M"obius strip. (Such a space arises from taking $X$ to be the complex points of a smooth projective real curve of genus 1 when the real points have exactly one connected component, and taking $\iota$ to be the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$.) Then $KR^j(X, \iota) \cong (KO)^2 \oplus KU^{-j-1}$.  

**Proof.** Step 1. Since $X$ has a nonempty fixed set, $KR^{-j}(T^2, \iota) \cong K\overline{R}^{-j}(T^2, \iota) \oplus KO^{-j}$, and we only need to compute $K\overline{R}^{-j}(T^2, \iota)$. We begin by deducing two

| $q \backslash p$ | 0 | 1 | 2 |
|-----------------|---|---|---|
| 0               | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $-1$            | $(\mathbb{Z}_2)^3$ | 0 | 0 |
| $-2$            | $(\mathbb{Z}_2)^3$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2$ |
| $-3$            | 0 | 0 | 0 |
| $-4$            | $\mathbb{Z}$ | $(\mathbb{Z}_2)^2$ | $\mathbb{Z}$ |
| $-5$            | $\mathbb{Z}_2$ | 0 | 0 |
| $-6$            | $\mathbb{Z}_2$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2$ |
| $-7$            | 0 | 0 | 0 |

**Figure 4.** $E_2$ of the spectral sequence for computing $KR^\ast(S^{1,1} \times S^{1,1})$. The sequence repeats with vertical period 8.
useful exact sequences. The first comes from observing that $X \setminus X' \cong S^{0,2} \times \mathbb{R}^{0,1}$ (as a Real space). Thus $KR^{-j}(X \setminus X') \cong KR^{-j+1}(S^{0,2}) \cong KSC^{-j+1}$. Since $\widetilde{KR}^{-j}(X') \cong KR^{-j}(\mathbb{R}^{1,0}) \cong KO^{-j-1}$, we get the long exact sequence

$$\cdots \rightarrow KO^{-j-2} \rightarrow KSC^{-j+1} \rightarrow \widetilde{KR}^{-j}(T^2, i) \rightarrow KO^{-j-1} \rightarrow KSC^{-j+2} \rightarrow \cdots,$$

where the connecting map $\delta$ will be determined later. However, note for now that $\delta$ vanishes after inverting 2, since $KO^{-j-1}[\frac{1}{2}]$ is nonzero only for $j \equiv 3 \mod 4$ and $KSC^{-j+2} \cong \mathbb{Z}_2$ for these values of $j$. Thus the torsion-free part of $\widetilde{KR}^{-j}(X, i)$ is the same as for $KSC^{-j+1} \oplus KO^{-j-1}$ and is thus $\mathbb{Z}$ for $j \equiv 0 \mod 4$, and 0 for $j \equiv 2 \mod 4$.

To get the other exact sequence, choose an interval $I$ in $M = X/i$ transverse to the central circle and meeting the boundary in two points. The inverse image $\tilde{S}$ of this interval in $X$ is a copy of $S^{1,1}$, the unit circle in the complex plane with complex conjugation as the involution. Furthermore the complement of this copy of $S^{1,1}$ is isomorphic (as a Real space) to $(0, 1) \times S^{1,1}$. Since $S^{1,1}$ with one fixed point removed is isomorphic (as a Real space) to $\mathbb{R}^{0,1}$, $\widetilde{KR}^{-j}(S^{1,1}) \cong KR^j(\mathbb{R}^{0,1}) \cong KO^{j+1}$ via [3] Theorem 2.3 and $KR^j((0, 1) \times S^{1,1}) \cong KO^{j-1} \oplus KO^j$. So we get an exact sequence

$$\cdots \rightarrow KO^j \rightarrow KO^{j-1} \oplus KO^j \rightarrow \widetilde{KR}^j(X) \rightarrow KO^{j+1} \rightarrow KO^j \oplus KO^{j+1} \rightarrow \cdots.$$

**Step 2.** Observe next that the connecting maps $\delta$ and $\rho$ have to be compatible with cup products by the ground ring

$$KO^* \cong \mathbb{Z}[b^\pm, \xi, \eta]/(2\eta, \eta^2, \xi \eta, \xi^2 - 4b).$$

Here the torsion-free generators are $b$ in degree $-8$ and $\xi$ in degree $-4$, and the torsion generator $\eta$ is in degree $-1$. To prove this claim, simply replace $T^2$ by $T^2 \times \mathbb{R}^{p,q}$. Thus the connecting map $\delta : KO^{j-1} \rightarrow KSC^{j+2}$ has to be of the form $x \mapsto x \cdot y, x \in KO^*$ and $y$ some class in $KSC^3$, and the connecting map $\rho : KO^j \rightarrow KO^{j-1} \oplus KO^j$ has to be of the form $x \mapsto (ax, bx)$, where $a \in KO^{-1}$ and $b \in KO^0$.

**Step 3.** The ground ring for $KSC$ theory is

$$KSC^* \cong \mathbb{Z}[\beta^\pm, \eta']/((2\eta', \eta'^2),$$

where the periodicity element $\beta$ is in degree $-4$ and the torsion generator $\eta'$ is in degree $-1$. There is a canonical ring homomorphism $\varepsilon : KO^* \rightarrow KSC^*$ (the map on $KR$ induced by $S^{1,1} \rightarrow pt$). Then $\varepsilon(\eta) = \eta', \varepsilon(b) = \beta^2$, and $\varepsilon(\xi) = 2\beta$. These are standard facts which can be found in [3] §1, for example.

**Step 4.** We claim that $\delta$ is given by $x \mapsto \varepsilon(x) \cdot \beta^{-1} \eta'$ and that $\rho$ is given by $x \mapsto (x \cdot \eta, 0)$. We get this by playing off the sequences (36) and (37) against each other. Start with $\delta(x) = \varepsilon(x) \cdot y, y \in KSC^3$. If $y$ were 0, we’d have a short exact


sequence

\[ 0 \rightarrow KSC^{j+1} \rightarrow \overline{KR}^j(T^2, \iota) \rightarrow KO^{j-1} \rightarrow 0, \]

and this would imply for example that \( \overline{KR}^{-6}(T^2, \iota) \cong KSC^{-5} \cong \mathbb{Z}_2 \), which contradicts what we obtain from the other exact sequence \( [37] \) for \( j = -6 \). Thus \( y = \beta^{-1} \eta' \) (the generator of \( KSC^{3} \)) and the claim follows.

Recall that \( \rho \) is of the form \( x \mapsto (ax, bx) \) with \( a \in KO^{-1} \) and \( b \in KO^{0} \cong \mathbb{Z} \). The number \( b \) must be 0; otherwise the torsion-free part of \( \overline{KR}^{-1}(T^2, \iota) \) would contradict what we got in Step 1 from \( [36] \). And \( a \in KO^{-1} \) can’t vanish, because if it did, we’d have a short exact sequence

\[ 0 \rightarrow KO^{-2} \oplus KO^{-1} \rightarrow \overline{KR}^{-1}(T^2, \iota) \rightarrow KO^{0} \rightarrow 0, \]

giving \( \overline{KR}^{-1}(T^2, \iota) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z} \), while \( [36] \) gives that \( \overline{KR}^{-1}(T^2, \iota) \) is either \( \mathbb{Z} \) or \( \mathbb{Z} \oplus \mathbb{Z}_2 \). So this completes the calculation of the boundary maps \( \delta \) and \( \rho \).

**Step 5.** To conclude, we use a well-known fact in homotopy theory \([1, p. 206]\), which is that if \( KU \) and \( KO \) are the complex and real topological \( K \)-theory spectra, then there is a fiber/cofiber sequence of spectra

\[ \Sigma KO \cong KO \rightarrow KU. \]

This corresponds to a famous long exact sequence \([24, Theorem III.5.18]\) or \([8, Definition 1.13(2)]\):

\[ \cdots \rightarrow KO^{-n}(X) \rightarrow KO^{-n-1}(X) \rightarrow KU^{-n-1}(X) \rightarrow KO^{-n+1}(X) \rightarrow \cdots. \]

Here \( c \) is complexification, \( r \) is realification, and \( \beta_U \) is the complex Bott element.

Because of our calculation of the boundary map \( \rho \), \( KO^j \) splits off as a direct summand in \( \overline{KR}^{-j}(T^2, \iota) \), and the complement can be identified with the cofiber of \( \eta \) with a degree shift. So this completes the proof.

We conclude by noting that \([24, Theorem 4.8]\) says that if \( X \) is a smooth projective variety defined over \( \mathbb{R} \) (which in our case will be a curve of genus 1), identified with the Real space of its complex points with involution given by the action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), then the natural map \( K_j(X; \mathbb{Z}_2) \rightarrow KR^{-j}(X; \mathbb{Z}_2) \) sending algebraic to topological \( K \)-theory is an isomorphism for \( j \) sufficiently large (in our case \( j \geq 1 \) suffices). Here algebraic \( K \)-theory or \( KR \)-theory with \( \mathbb{Z}_2 \) coefficients is related to the integral theory by a universal coefficient sequence

\[ (38) \quad 0 \rightarrow KR^{-j}(X)/2 \rightarrow KR^{-j}(X; \mathbb{Z}_2) \rightarrow 2KR^{-j+1}(X) \rightarrow 0, \]

where \( 2KR^{-j+1}(X; \mathbb{Z}_2) \) denotes the 2-torsion in \( KR^{-j+1}(X) \), and similarly for \( K_j \). The torsion subgroup of \( K_j(X) \) was computed in \([32, Main Theorem 0.1]\) and agrees with our results under this isomorphism.

\[ ^4 \text{There is a small typo in the statement of } [32, \text{Main Theorem 0.1}]. K_2(X)_{\text{tors}} \text{ should contain } \nu + 1 \text{ copies of } \mathbb{Z}_2 \text{ (here } \nu \text{ is the species), not } \nu \text{ copies as written.} \]
5. More general twists and why they are needed for physics

5.1. Twisted \( KO \)-theory. While twisted complex \( K \)-theory is by now well-known in both the mathematics literature (e.g., [15, 33, 4, 5, 25]) and the physics literature (e.g., [37, 9]), its cousin, twisted real \( K \)-theory, is defined similarly but is less familiar. One way to define it is by using the \( K \)-theory of real continuous-trace algebras of real type (see [33, §3]). In the separable case, after stabilization, such an algebra is the algebra of sections vanishing at infinity of a bundle whose fibers are the compact operators \( K_\mathbb{R} \) on an infinite-dimensional separable real Hilbert space \( \mathcal{H}_\mathbb{R} \). Since \( O(\mathcal{H}_\mathbb{R}) \) is contractible but the automorphism group of \( K_\mathbb{R} \) is the projective orthogonal group \( PO(\mathcal{H}_\mathbb{R}) = O(\mathcal{H}_\mathbb{R})/\mathbb{Z}_2 \), which is a \( K(\mathbb{Z}_2, 1) \) space, the relevant algebra bundles are classified by homotopy classes of maps from the space \( X \) to \( BPO(\mathcal{H}_\mathbb{R}) \), which is a \( K(\mathbb{Z}_2, 2) \) space. Thus they are classified by a single characteristic class \( \tilde{w}_2 \in H^2(X, \mathbb{Z}_2) \), which one can identify with the characteristic class for Witten’s type I string theory without vector structure in [36]. In other words, for each \( \tilde{w}_2 \in H^2(X, \mathbb{Z}_2) \), one gets an 8-periodic family of \( K \)-groups \( KO^*(X, \tilde{w}_2) \), reducing to \( KO^*(X) \) when \( \tilde{w}_2 = 0 \). This is analogous to twisting by \( H \)-flux for complex \( K \)-theory. Recall that the automorphism group of \( K \) (the compact operators on a complex infinite-dimensional separable Hilbert space \( \mathcal{H} \)) is the projective unitary group \( PU(\mathcal{H}) = U(\mathcal{H})/S^1 \). In this case the relevant algebra bundles are classified by homotopy classes of maps from \( X \) to \( BP\mathcal{U}(\mathcal{H}) \), which is a \( K(\mathbb{Z}, 3) \) space. Therefore they are classified by the \( H \)-flux \( H \in H^3(X; \mathbb{Z}) \).

Just as in the complex case, the twisted real \( K \)-theory groups can be computed using an Atiyah-Hirzebruch spectral sequence (AHSS)

\[
H^p_c(X, KO^q) \Rightarrow KO^{p+q}(X, \tilde{w}_2),
\]

where \( H^*_c \) is cohomology with compact supports and \( \tilde{w}_2 \) appears in the differentials. We will primarily be interested in the case \( X = T^2 \), in which case the “compact supports” modifier can be dropped and there is only room for one differential,

\[
d_2: H^0(T^2, KO^q) = KO^q \rightarrow KO^{q-1} \cong H^2(T^2, KO^{q-1}).
\]

This differential is cup product with \( \tilde{w}_2 \), viewed as an element of \( H^2(T^2, KO^{-1}) \cong \mathbb{Z}_2 \). So if \( \tilde{w}_2 \) is the nontrivial element of \( H^2(T^2, \mathbb{Z}_2) \), the \( E_2 \) term of the spectral sequence with the non-zero \( d_2 \) differentials indicated is shown in Figure 5 and the \( E_3 = E_\infty \) term is shown in Figure 6.

The groups \( KO^*(T^2, \tilde{w}_2) \) are thus determined up to extensions by summing along the diagonals (where \( p + q \) takes a constant value). We see that \( KO^0(T^2, \tilde{w}_2) \) is an extension of \( \mathbb{Z} \) by \( \mathbb{Z}^2_2 \), necessarily split, \( KO^{-2}(T^2, \tilde{w}_2) \) is an extension of \( \mathbb{Z}_2 \) by \( \mathbb{Z} \), and the remaining groups \( KO^j(T^2, \tilde{w}_2) \) are \( \mathbb{Z}_2 \) for \( j = -1 \), \( \mathbb{Z}_2^2 \) for \( j = -3 \), \( \mathbb{Z} \) for \( j = -4 \), 0 for \( j = -5 \), \( \mathbb{Z} \) for \( j = -6 \), \( \mathbb{Z}_2 \) for \( j = -7 \). The only case where we are left with an extension problem is \( j = -2 \). It turns out that \( KO^{-2}(T^2, \tilde{w}_2) \cong \mathbb{Z} \), which we can see as follows. A map of degree one \( T^2 \rightarrow S^2 \) collapsing the 1-skeleton \( S^1 \cup S^1 \) to a point induces a map of spectral sequences which is an isomorphism on the
| $q \backslash p$ | 0  | 1  | 2 |
|-------------------|----|----|----|
| 0                 | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| -1                | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ |
| -2                | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ |
| -3                | 0 | 0 | 0 |
| -4                | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| -5                | 0 | 0 | 0 |
| -6                | 0 | 0 | 0 |
| -7                | 0 | 0 | 0 |

**Figure 5.** $E_2$ of the spectral sequence for computing $KO^*(T^2, \tilde{w}_2)$. The sequence repeats with vertical period 8.

| $q \backslash p$ | 0  | 1  | 2 |
|-------------------|----|----|----|
| 0                 | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| -1                | 0 | $\mathbb{Z}_2^2$ | 0 |
| -2                | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | 0 |
| -3                | 0 | 0 | 0 |
| -4                | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| -5                | 0 | 0 | 0 |
| -6                | 0 | 0 | 0 |
| -7                | 0 | 0 | 0 |

**Figure 6.** $E_{\infty}$ of the spectral sequence for computing $KO^*(T^2, \tilde{w}_2)$. The sequence repeats with vertical period 8.
columns with \( p = 0 \) and \( p = 2 \), hence shows that \( KO^{-2}(T^2, \tilde{w}_2) \cong KO^{-2}(S^2, \tilde{w}_2) \) (with a non-trivial twist in \( H^2(S^2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \)), so we only need to compute this latter group and show that it is torsion-free. This will be done in Section 5.2 below.

There are many ways of seeing that this sort of \( \tilde{w}_2 \) twisting of \( KO \) is needed for D-brane classification in the “no vector structure” theory of [39]. But the key feature is that Chan-Paton bundles are given not by \( O(n) \) bundles but by \( PO(n) \) bundles [39, §2.1], which is precisely how our twisting was defined.

The physics literature suggests that there should be a T-duality between the “type I with no vector structure” theory on \( T^2 \) and the type IIA orientifold on an elliptic curve with antiholomorphic involution of species 1 (i.e., a fixed set which is topologically just a single circle) [27]. The D-brane charges in this theory are described by the groups \( KR^i(T^2, \iota) \), where \( \iota \) is an involution on \( T^2 \) with fixed set \( S^1 \). These groups were computed above in Theorem 4. Table 4 shows \( KO^j(T^2, \tilde{w}_2) \) for \( \tilde{w}_2 \neq 0 \), \( KR^j(T^2, \iota) \) for the species 1 antiholomorphic involution \( \iota \), and \( KR^j_{(+,+,-)}(S^{1,1} \times S^{1,1}) \) from Section 4. The second column agrees precisely with the first column shifted down by 1, and the third column agrees with the second column shifted down by 1, as is predicted by T-duality. Note that the data of the \( B \)-field for the type IIB theory on \( S^{1,1} \times S^{1,1} \) with \( \alpha = (+, +, -) \) is encoded in the non-triviality of the \( d_2 \) differential for the spectral sequence in Figure 4. In [16] we will describe how the \( B \)-field is described by a sign choice under T-duality.

| \( j \mod 8 \) | \( KO^j(T^2, \tilde{w}_2) \) | \( KR^j(T^2, \iota) \) | \( KR^j_{(+,+,-)}(S^{1,1} \times S^{1,1}) \) |
|---|---|---|---|
| 0 | \( \mathbb{Z} \oplus \mathbb{Z}_2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |
| -1 | \( \mathbb{Z}_2 \) | \( \mathbb{Z} \oplus \mathbb{Z}_2 \) | \( \mathbb{Z} \) |
| -2 | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z} \oplus \mathbb{Z}_2 \) |
| -3 | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \) |
| -4 | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |
| -5 | 0 | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \) |
| -6 | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) |
| -7 | \( \mathbb{Z} \) | \( \mathbb{Z} \) | 0 |

Table 4. \( KO^j(T^2, \tilde{w}_2) \), \( KR^j(T^2, \iota) \), and \( KR^j_{(+,+,-)}(S^{1,1} \times S^{1,1}) \)

5.2. Twisted \( KO \)-theory with an \( H^1 \) twist. Twisting of \( KO^*(X) \) by \( H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \) was already defined by Donovan and Karoubi in [15]. (The group of twists \( HO(X) \) is actually a non-split abelian extension of \( H^2(X, \mathbb{Z}_2) \) by \( H^1(X, \mathbb{Z}_2) \).) For \( X \) compact and \( \mathcal{A} \) a bundle over \( X \) whose fibers are \( \mathbb{Z}_2 \)-graded simple \( \mathbb{R} \)-algebras, with \( w(\mathcal{A}) = \alpha \in HO(X) \), \( KO^\alpha(X) \) is the Grothendieck group of graded real vector bundles \( X \) which are finitely generated projective modules for \( \mathcal{A} \). Here \( w(\mathcal{A}) = (w_1(\mathcal{A}), w_2(\mathcal{A})) \), where vanishing of \( w_2(\mathcal{A}) \in H^2(X, \mathbb{Z}_2) \) is
the condition for $\mathcal{A}$ to be the endomorphism bundle of a $\mathbb{Z}_2$-graded vector bundle, and $w_1(\mathcal{A}) = w_1(V)$ if $\mathcal{A}$ is the Clifford algebra bundle of a real vector bundle $V$ for a nontrivial real line bundle over $\mathbb{R}P^2$, (39) when $w_1 = 0$, we get back the twisted $KO$-groups of Section 5.1. The basic composition rule in $HO(X)$ is that
\[
w_1(\mathcal{A} \otimes \mathcal{B}) = w_1(\mathcal{A}) + w_1(\mathcal{B}), \quad w_2(\mathcal{A} \otimes \mathcal{B}) = w_2(\mathcal{A}) + w_2(\mathcal{B}) + w_1(\mathcal{A}) \cdot w_1(\mathcal{B}).
\]

For general $X$, $HO(X)$ can have elements of order 4, but this won’t happen if (as for $S^1$ or $T^2$) every element of $H^1(X, \mathbb{Z}_2)$ has square 0. Thus (assuming this condition) every element of $HO(X)$ is its own inverse, and by the Thom Isomorphism Theorem of $[15, \S 6]$, if $V$ is a real vector bundle over $X$,
\[
KO^j(V) \cong KO^{j-\dim V}(X, w_1(V), w_2(V)).
\]

As an example of (39), we can compute $KO^j(S^1, w_1)$ for the nontrivial element $w_1 \in H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Indeed, we have
\[
KO^j(S^1, w_1) \cong KO^{j+1}(V) \cong KO^{j+1}(\mathbb{R}P^2),
\]
where $V$ is the nontrivial real line bundle over $S^1$, that is, the Möbius strip. And the $KO$-groups of $\mathbb{RP}^2$ were computed in $[18$, Theorem 1$]$. The result is given in Table 5. Here the surprise is the existence of 4-torsion in $KO^0(\mathbb{R}P^2) \cong KO^{-1}(S^1, w_1)$.

| $j$  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|------|----|----|----|----|----|----|----|----|
| $KO^j(S^1, w_1)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0  | 0  | 0  | $\mathbb{Z}_2$ |

Table 5. $KO^j(S^1, w_1)$ for the nontrivial twist

The groups in Table 5 can once again be explained by a twisted Atiyah-Hirzebruch spectral sequence with starting point $H^p(S^1, KO^q)$, but this time the only differential is $d_1$, which is multiplication by 2 in the places indicated by the arrows in Figure 7.

The calculation of $KO^*(S^1, w_1)$ also enables us to compute $KO^*(T^2, w_1)$ for any choice of a twisting $w_1 \in H^1(T^2, \mathbb{Z}_2)$. The reason is that for any such $w_1 \neq 0$, we can choose a topological splitting $T^2 = S^1 \times S^1$ with respect to which $w_1$ lives only on the first factor, so that $(T^2, w_1) \cong (S^1, w_1) \times (S^1, 0)$. It follows that $KO^j(T^2, w_1)$ splits as $KO^j(S^1, w_1) \oplus KO^{j-1}(S^1, w_1)$.

The calculation of $KO^*(S^1, w_1)$ also enables us to compute $KO^*(T^2)$ in the sense of Donovan-Karoubi for a twist $\alpha$ with both $w_1(\alpha)$ and $w_2(\alpha)$ nonzero. Indeed, let $V$ again be the nontrivial real line bundle over $S^1$, that is, the Möbius strip. Then $V \times V$ (the Cartesian product) is a rank-two real vector bundle over $S^1 \times S^1 = T^2$. If $a$ and $b$ are the elements of $H^1(T^2, \mathbb{Z}_2)$ dual to the two circles in the decomposition $T^2 = S^1 \times S^1$, then $V \times V$ can be identified with the Whitney sum $L_a \oplus L_b$, since the fiber of $V \times V$ over $(x, y) \in S^1 \times S^1$ is $L_a(x, y) \times L_b(x, y) = L_a(x, y) \oplus L_b(x, y)$. 
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\[
\begin{array}{c|cc}
q \setminus p & 0 & 1 \\
\hline
0 & \mathbb{Z} & \mathbb{Z} \\
-1 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
-2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
-3 & 0 & 0 \\
-4 & \mathbb{Z} & \mathbb{Z} \\
-5 & 0 & 0 \\
-6 & 0 & 0 \\
-7 & 0 & 0 \\
\end{array}
\]

Figure 7. \(E_1\) of the spectral sequence for computing \(KO^*(S^1, w_1)\).

The sequence repeats with vertical period 8. Arrows represent multiplication by 2, so in \(E_2 = E_\infty\), each \(\mathbb{Z}\) in the \(p = 1\) column is replaced by a \(\mathbb{Z}_2\), and each \(\mathbb{Z}\) in the \(p = 0\) column dies.

Note that \(w_1(L_a \oplus L_b) = a + b\) and \(w_2(L_a \oplus L_b) = ab\), a generator of \(H^2(T^2, \mathbb{Z}_2)\). So \(KO^j(T^2, a + b, ab) = KO^{j+2}(V \times V)\). The same holds for \(KO^j(T^2, w_1, w_2)\) for any nonzero \(w_1, w_2\) since there is a self-homeomorphism of \(T^2\) sending \(w_1\) to \(a + b\).

Finally we can compute \(KO^j(T^2, w_1, w_2)\) is a 2-primary torsion group for all \(j\).

Finally, we mention still another application of the Thom isomorphism (39), namely the completion of the calculation of \(KO_*^*(T^2, \tilde{w}_2)\) when the twist is nonzero. Observe that the nonzero element \(\tilde{w}_2 \in H^2(T^2, \mathbb{Z}_2)\) is pulled back from the generator of \(H^2(S^2, \mathbb{Z}_2)\) under a map \(T^2 \to S^2\) of degree one, so to compute \(KO_*^*(T^2, \tilde{w}_2)\), we can begin by computing \(KO_*^*(S^2, \tilde{w}_2)\). The generator of \(H^2(S^2, \mathbb{Z}_2)\) is \(\tilde{w}_2\) for the underlying real 2-plane bundle of the Hopf (complex) line bundle over \(S^2 \cong \mathbb{C}P^1\), for which the total space is \(\mathbb{C}P^2 \setminus \{\text{pt}\}\). So by (39), \(KO^{-j}(S^2, \tilde{w}_2) \cong \widetilde{KO}^{-j+2}(\mathbb{C}P^2)\), which is computed in \([18, \text{Theorem 2}]\). (The degree 0 part was computed earlier in \([34, \S 3.6]\).) Rather surprisingly, \(\widetilde{KO}^* (\mathbb{C}P^2)\) is entirely torsion-free, with
copies of $\mathbb{Z}$ in all even degrees and nothing in odd degrees. Thus
$KO^{-2}(S^2, \tilde{\omega}_2) \cong KO^{-2}(T^2, \tilde{\omega}_2) \cong \tilde{KO}^0(\mathbb{C}P^2) \cong \mathbb{Z}$, not $\mathbb{Z} \oplus \mathbb{Z}_2$.

5.3. **Twisted $KR_\alpha$-theory with an $H^1$ and/or $H^2$ twist.** Finally, without going into great detail, we point out that the work of [13] and [14] suggests that for full generality in applications to physics of orientifolds, one should allow $H^1$ and $H^2$ twisting, not just of $KO$ theory, but of $KR_\alpha$ theory as in section 4. Again this can take many different forms. A first possibility is to modify Definition 2 to allow nontrivial Dixmier-Douady invariants. A more subtle kind of twisting, which will be needed in [16], is analogous to that in section 5.2 above. For these twisted theories, there will again be a spectral sequence analogous to (12), but what was previously the $E_2$ may now only be $E_1$, and there can be two additions to the differentials: product with a class in $H^1_G$, and product with a class in $H^2_G$. (Note that these reduce to what we had in section 5.2 when $\iota$ is trivial.)

As an example, for $S^{1,1} \times S^{1,1}$ with $\alpha = (+, +, +, -)$ the $E_2$ term pictured in Figure 4 should be considered as the $E_1$ term. An $H^1$ twist would correspond to a non-trivial $d_1$ differential in the $q \equiv -4 \pmod{8}$ rows.

6. Conclusion

$KR$-theory with a sign choice (Definition 2) allows us to give a mathematical description of $D$-brane charges for all orientifolds including ones with both $O^+$- and $O^-$-planes. The additional data of a sign choice is required to distinguish between topologically equivalent spaces with different $O$-plane content. As we saw, $KR$-theory with a sign choice gives a purely mathematical description of the $D$-branes in the type $I\tilde{A}$ theory. This calculation provides further evidence for $T$-duality rather than requiring its assumption to determine the brane charges.

In addition to providing new tests of $T$-duality, $KR$-theory with a sign choice predicts the $D$-brane content in theories that could not be computed previously (which in turn can aid in the discovery of unknown dualities). We are not aware of the $D$-brane content for the type I theory without vector structure or either of its $T$-dual theories appearing anywhere in the literature. This extends the usefulness of $K$-theory as a first check for $D$-brane content to orientifold theories. As noted previously, the $K$-theoretic description cannot determine the sources for the $D$-brane charge, only that there is a stable charge. Determining the stable charges using $KR$-theory with a sign choice can greatly constrain what sources need to be tested for stability at different points in the moduli using other methods (such as considering the boundary state description). Since boundary state descriptions can be quite difficult for orientifolds, any constraints are very useful, and as we will show in [16] most of the sources can often be determined from the $KR$-theory using what we know about $O^{\pm}$-planes.

As noted in the introduction one of our original motivations for a detailed analysis of $T$-duality via orientifold plane charges in $KR$-theory was the special case
of \( c = 3 \) Gepner models as studied in \cite{6}. The authors of that paper used simple current techniques in CFT to construct the charges and tensions of Calabi-Yau orientifold planes. Using twisted KR-theory with a sign choice to classify the brane charges does not depend on the specific structure of \( c = 3 \) Gepner models, nor even on a rational conformal field theoretic description. In \cite{17} a twisted equivariant \( K \)-theory description of the \( D \)-brane charge content for WZW models is provided. Current work in progress attempts to generalize this work by establishing an isomorphism between a suitable (real) variant of twisted equivariant \( K \)-theory, sufficient to capture orientifold charge content, and our KR-theory with sign choices for Gepner models. Such an isomorphism would allow the computation of twisted KR-theory with a sign choice for complicated Calabi-Yau manifolds through a simpler computation at the Gepner point.

KR-theory with a sign choice provides a universal \( K \)-theory for classifying \( D \)-brane charges. In addition to being able to describe new orientifold cases it reduces to all other known classifications on smooth manifolds when using the correct involution. This unifies the \( K \)-theoretic classification of \( D \)-brane charges by not requiring one to change \( K \)-theories for different string theories. While its definition was motivated by a problem in physics, the last point exemplifies why KR-theory with a sign choice is also interesting mathematically.

KR-theory with a sign choice provides a framework for studying the underlying structure of \( K \)-theory. It was very surprising to see that \( KSC \)-theory (the KR-theory of \( S^{0,2} \)) can be described as a twisting of the KR-theory of \( S^{1,1} \). While we have explicitly shown that twisted KR-theory with a sign choice satisfies all possible \( T \)-duality relationships for spaces where the compact dimensions are a circle or a 2-torus, in this paper we did not look at why there are isomorphisms between the twisted KR-theories of \( T \)-dual theories. The purpose of this paper was simply to set up the necessary topology to correctly classify brane charges. In \cite{16} we will explore why \( T \)-duality gives isomorphisms of twisted KR-theory with a sign choice. We will see that the extra data that we needed to include is contained in the geometry of \( T \)-dual theories.

We have already seen how considering the geometry is important. The complex structure constrains what involutions are possible on a 2-torus. Since the physical theory depends on the involution, the geometry of the torus constrains the allowable string theories. Another well known example that played a role in our analysis is the \( B \)-field, which is determined by the Kähler modulus. We were also compelled to explore more exotic twists in order to account for the \( T \)-duality of the type I theory without vector structure. Without the physical motivation we might not have considered looking at such additional mathematical structures. We have shown how such twistings must behave via an Atiyah-Hirzebruch spectral sequence. In \cite{16} we will give a more geometric reason for why such twistings must be included.
By exploring the underlying topology and geometry we were able to gain physical
information and new evidence for hypothesized dualities. Additionally, this work
shows how we can go in the opposite direction and use the additional structure of
physics to gain insight into the underlying geometry and topology. This gives us
a greater understanding of the interplay between the three structures: topology,
geometry, and physics.

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