The space $\mathcal{B}'$ of distributions vanishing at infinity – duals of tensor products

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April 11, 2016

Abstract

Analogous to L. Schwartz’ study of the space $\mathcal{D}'(\mathcal{E})$ of semi-regular distributions we investigate the topological properties of the space $\mathcal{D}'(\mathcal{B})$ of semi-regular vanishing distributions and give representations of its dual and of the scalar product with this dual. In order to determine the dual of the space of semi-regular vanishing distributions we generalize and modify a result of A. Grothendieck on the duals of $E \hat{\otimes} F$ if $E$ and $F$ are quasi-complete and $F$ is not necessarily semi-reflexive.

Keywords: semi-regular vanishing distributions, duals of tensor products

MSC2010 Classification: 46A32, 46F05

1 Introduction

L. Schwartz investigated, in his theory of vector-valued distributions [24, 25], several subspaces of the space $\mathcal{D}'_{xy} = \mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ that are of the type $\mathcal{D}'_x(E_y)$ where $E_y$ is a distribution space.

Three prominent examples are:

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(i) $\mathcal{D}'(\mathcal{S}')$ – the space of semi-temperate distributions, for which the partial Fourier transform is defined [24, p. 123];

(ii) $\mathcal{D}'(\mathcal{D}'_{L^1,y})$ – the space of partially summable distributions, for which the convolution of kernels is defined [7, §2, p. 546], as a generalization of the convolvability condition for two distributions [24, pp. 131, 132];

(iii) $\mathcal{D}'(\mathcal{E}_y)$ – the space of semi-regular distributions (see [23] and [24, p. 99]).

In this paper we will be concerned with the space $\mathcal{D}'(\hat{\mathcal{B}})$ of “semi-regular vanishing” distributions.

**Notation and Preliminaries.** We will mostly build on notions from [26, 6, 24, 25]. $\mathcal{D}'(E)$ is defined as $\mathcal{D}' \in E$, which space coincides with $\mathcal{D}' \otimes E = \mathcal{D}' \otimes E =: \mathcal{D}' \otimes E$ in the 3 examples above. If $E, F$ are two locally convex spaces then $E \otimes E, E \otimes F$ and $E \otimes F$ denote the completion of their projective, injective, and inductive tensor product, respectively; writing $\otimes$ in place of $\hat{\otimes}$ means that we take the quasi-completion instead. The subscript $\beta$ in $E \otimes E$ [25, p. 12, $2^\star$] refers to the finest locally convex topology on $E \otimes E$ for which the canonical injection $E \times F \to E \otimes E$ is hypocontinuous with respect to bounded sets. Given a locally convex space $E$, $E'$ denotes its strong dual, $E'$ its weak dual and $E'$ its dual with the topology of uniform convergence on absolutely convex compact sets. In absence of any of these designations, $E'$ carries the strong dual topology. For the definition of $\mathcal{T}(E)$ see [24, p. 63] and [22, p. 94]. $\mathcal{B}'$ is the space of distributions vanishing at infinity, i.e., the closure of $\mathcal{E}'$ in $\mathcal{D}'_{L^\infty}$ [26, p. 200]. $\mathcal{N}(E, F)$ and $\mathcal{C}_{00}(E, F)$ denote the space of nuclear and compact linear operators $E \to F$, respectively. The normed space $\hat{E}_U$ for an absolutely convex zero-neighborhood $U$ in $E$ is introduced in [6, Chap. I, p. 7], with associated canonical mapping $\Phi_U: E \to \hat{E}_U$. $\mathcal{L}(E, F)$ is the space of continuous linear mappings $E \to F$. By $\mathcal{B}^c(E, F)$ and $\mathcal{B}(E, F)$ we denote the spaces of separately continuous and continuous bilinear forms $E \times F \to \mathbb{C}$, respectively, and $\mathcal{B}^h(E, F)$ is the space of separately hypocontinuous bilinear forms; for any of these spaces, the index $\epsilon$ denotes the topology of bi-equicontinuous convergence.

**Motivation.** In order to prove, e.g., the equivalence of $S(x - y)T(y) \in \mathcal{D}'(\mathcal{D}'_{L^1,y})$ for two distributions $S, T$ (which means, by definition, that $S, T \in$
\( \mathcal{D}'(\mathbb{R}^n) \) are convolvable) to the inequality

\[
\forall K \subset \mathbb{R}^n \text{ compact } \exists C > 0 \exists m \in \mathbb{N}_0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^{2n}) \text{ with } \supp \varphi \subseteq \{(x, y) \in \mathbb{R}^{2n} : x + y \in K\} : \\
|\langle \varphi(x, y), S(x)T(y) \rangle| \leq C \sup_{|\alpha| \leq m} \| \partial^\alpha \varphi \|_\infty ,
\]

it is advantageous to know of a “predual” of \( \mathcal{D}'(\mathcal{D}'_L) \), i.e., the space \( \mathcal{D}'(\mathcal{D}'_x) = \lim_{\rightarrow} X = D_x \mathcal{B} \) for which \( (\mathcal{D}'(\mathcal{D}'_x))_b = D_x \mathcal{D}'_L \) [7, Prop. 3, p. 541]. A “predual” of \( \mathcal{D}'(\mathcal{D}) \) and \( \mathcal{D}'(\mathcal{D}'_L) \) can easily be found by Corollary 3 in [24, p. 104], which states that \( (\mathcal{D}(\mathcal{D}'))_b = \mathcal{D}'(\mathcal{D})' \) if \( E \) is a Fréchet space. A “predual” of \( \mathcal{D}'(E) \) is the space \( D_x \mathcal{D}'(\mathcal{D}'_x) \) (see Propositions 1 and 2).

In the memoir [23] L. Schwartz investigated the space \( \mathcal{D}'(E) \) of semi-regular distributions. For reasons of comparison we present the main features thereof in Section 2, i.e., in Proposition 1 properties of \( \mathcal{D}'(E) \), in Proposition 2 the dual and a “predual” of \( \mathcal{D}'(E) \) and in Proposition 3 an explicit expression for the scalar product of \( K(x, y) \in \mathcal{D}'(E_x) \) with \( L(x, y) \in \mathcal{D}'(E_y) \). These propositions generalize the corresponding propositions in [23] and new proofs are given.

In [15] we found the condition

\[
\forall \varphi \in \mathcal{D} : (\varphi * S)T \in \mathcal{B}
\]

(1)

for two distributions \( S, T \), in order that \( (\partial_j S) * T = S * (\partial_j T) \) under the assumption that \( (\partial_j S, T) \) and \( (S, \partial_j T) \) are convolvable (see also [9, p. 559]). The equivalence of (1) and

\[
S(x - y)T(y) \in \mathcal{D}'_x(\mathcal{B}_y)
\]

(2)

is proven in [16]. Due to the regularization property

\[
S \in \mathcal{B} \iff S(x - y) \in \mathcal{D}'(\mathcal{B}_x)
\]

for a distribution \( S \) [26, remarque 3', p. 202] we are motivated to investigate the space \( \mathcal{D}'(\mathcal{B}) \) of “semi-regular vanishing distributions” analogously to \( \mathcal{D}'(E) \) in [23], i.e.,

- to state properties of \( \mathcal{D}'(\mathcal{B}) \) in Proposition 4,
- to determine the dual of \( \mathcal{D}'(\mathcal{B}) \) in Proposition 5,
- to express explicitly the scalar product in Proposition 6, and
• to determine the transpose of the regularization mapping $\hat{\mathcal{B}}' \rightarrow \mathcal{D}'(\hat{\mathcal{B}}_x)$, $S \mapsto S(x - y)$ in Proposition 7.

**Duals of tensor products.** Looking for $([\mathcal{D}'(\hat{\mathcal{B}})'])_b'$ we make use of the following duality result of A. Grothendieck which allows – in contrast to the corresponding propositions in [12, §45, 3.1, p. 301, §45, 3.5, p. 302, §45, 3.7, p. 304], [10, 16.1.7, p. 346], [20, IV, 9.9, p. 175; Corollary 1, p. 176; Ex. 32, pp. 198, 199] and [3, §4, Satz 1, p. 212] – to determine the duals of tensor products $E \otimes F$ in cases where $E$ and $F$ are of “different nature”:

$$[6, \text{Chap. II, § 4, n°1, Lemme 9, Corollaire, p. 90}]: \text{Let } E \text{ and } F \text{ be complete locally convex spaces, } E \text{ nuclear, } F \text{ semireflexive. If the strong dual } ([E \hat{\otimes} F])'_b' \text{ is complete then } ([E \hat{\otimes} F])'_b' = (E'_b \hat{\otimes} F'_b).$$

Note that for our example in the introduction the assumption of semireflexivity is not fulfilled. Nevertheless we reach the conclusion by observing that

$$([\mathcal{D}'(\hat{\mathcal{B}})])'_b = ([\mathcal{D}'(\mathcal{B}_c)])'_b,$$

$\mathcal{B}_c$ being the semireflexive space $\mathcal{D}_{L\infty}$ endowed with the topology of uniform convergence on compact subsets of $\mathcal{D}'_{L^1}$ [26, p. 203] which also can be described by seminorms [17, Prop. 1.3.1, p. 11]. Therefore, in Section 4, we prove a generalization of Grothendieck’s Corollary and a modification which applies to semi-reflexive locally convex Hausdorff spaces $F$ such that the completeness of $([E \hat{\otimes} F])'_b$ can be shown by the existence of a space $F_0$ such that $([E \hat{\otimes} F_0])'_b$ is complete and $([E \hat{\otimes} F])'_b = ([E \hat{\otimes} F_0])'_b'$.

**The kernel identity for $\hat{\mathcal{B}}'$.** There is yet another condition equivalent to (1) and (2):

$$\delta(z - x - y)S(x)T(y) \in \mathcal{D}'_z \hat{\otimes} \mathcal{B}'_{xy}$$

which is nothing else than

$$\forall K \subset \mathbb{R}^n \text{ compact } \exists C > 0 \exists m \in \mathbb{N}_0 \forall \varphi \in \mathcal{D}(\mathbb{R}^{2n}) \text{ with } \text{supp } \varphi \subseteq \{(x, y) \in \mathbb{R}^{2n} : x + y \in K\} : \langle \varphi(x, y), S(x)T(y) \rangle \leq C \sup_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_1.$$

This equivalence can be shown by the use of the “kernel identity”

$$\mathcal{B}'_{xy} = \mathcal{B}'_x \hat{\otimes} \mathcal{B}'_y = \mathcal{B}'_x(\mathcal{B}'_y).$$
which we prove in Section 5 (Proposition 10). For similar identities see [24, Prop. 17, p. 59; Prop. 28, p. 98] and [6, Chap. I, Cor. 4, p. 61; Ex., p. 90].

A preliminary version of this paper was presented in two talks in Vienna and in Innsbruck by N. O.

## 2 Semi-regular distributions.

Whereas L. Schwartz stated that the space $\mathcal{D}'(\mathcal{E})$ is semireflexive [23, p. 110] we prove

**Proposition 1** (Properties of $\mathcal{D}'(\mathcal{E})$). The space of semi-regular distributions $\mathcal{D}'(\mathcal{E})$ is

(i) nuclear,

(ii) ultrabornological and

(iii) reflexive.

Proof. (i) The nuclearity follows by Grothendieck’s permanence result in [6, Chap. II, §2, n° 2, Théorème 9, 3°, p. 47] (see also [23, p. 110]).

(ii) M. Valdivia’s sequence space representation of $\mathcal{O}_M$ [28, Theorem 3, p. 478] and [6, Chap. I, §1, n°3, Proposition 6, 1°, p. 46] yield the isomorphisms

$$\mathcal{D}' \otimes \mathcal{E} \cong s' \hat{\otimes}s^N \cong (s' \hat{\otimes}s)^N \cong \mathcal{O}_M^N.$$ 

By [6, Chap. II, §4, n°4, Théorème 16, p. 131], $\mathcal{O}_M$ is ultrabornological and, hence, also $\mathcal{O}_M^N$, by [14, Folgerung 5.3.8, p. 106] (see also [10, 13.5.3, p. 281] or [11, §28, 8.(6), p. 392]).

(iii) The semi-reflexivity of $\mathcal{D}'(\mathcal{E})$ by [6, Chap. II, §3, n°2, Proposition 13, p. 76] (see also [23, p. 110]) and the barrelledness of $\mathcal{D}'(\mathcal{E})$ by (ii) yield the reflexivity of $\mathcal{D}'(\mathcal{E})$. 

In [23], the dual of $\mathcal{D}'(\mathcal{E})$ is described by the representation of its elements as finite sums of derivatives with respect to $y$ of functions $g(x, y) \in \mathcal{D}_x \hat{\otimes} \mathcal{D}^0_y$ [23, Proposition 1, p. 112]. Thus, $(\mathcal{D}'_x(\mathcal{E}_y))' = \lim_{\to m}(\mathcal{D}_x \hat{\otimes} \mathcal{E}_{c,y}^{m})$ algebraically.

Note that L. Schwarz asserts on the one hand $(\mathcal{D}'_x(\mathcal{E}_y))' = \lim_{\to m}(\mathcal{D}_x \hat{\otimes} \mathcal{E}_y^{m})$ in [23, Prop. 1, p. 112], whereas we find, on the other hand, $(\mathcal{D}'_x(\mathcal{E}_y))' = \lim_{\to m}(\mathcal{D}_x \hat{\otimes} \mathcal{E}_{c,y}^{m})$ algebraically.
\[ \lim_{m \to \infty} (D_x \hat{\otimes} E_{c,y}^m) \] in [23, Corollaire 1, p. 116]. It can be shown that the isomorphism is also a topological one. Other representations of the dual of \( D'(E) \) are given in:

**Proposition 2** (Dual of \( D'(E) \)). *We have linear topological isomorphisms*

\[ (D'(E))'_b \cong D \hat{\otimes} \varepsilon = D \hat{\otimes} \beta E' \]  

and linear isomorphisms

\[ (D'(E))'_b \cong \varepsilon'(D; \varepsilon) = \varepsilon'(D; \beta) \]  
\[ \cong N(D'; E') = \mathcal{C}_o(D'; E') \]  
\[ \cong N(E, D) = \mathcal{C}_o(E, D). \]

For the notation \( D(E'; \varepsilon) \) or \( D(E'; \beta) \) see [25, p. 54].

**Proof.** The first isomorphism of (4) results from the Corollary cited in the introduction: the 5 hypotheses are fulfilled due to Proposition 1 (ii). The equality \( E \hat{\otimes} \beta F = E \hat{\otimes} \varepsilon F \) is a consequence of the barrelledness of \( E \) and \( F \), in the case above: \( E = D, F = E' \).

The isomorphisms (5) and (6) follow by [25, Proposition 22, p. 103].

The coincidence of nuclear and compact linear operators is a consequence of [6, Chap. II, §2, n°1, Corollaire 4, 1°, p. 39] and the semi-Montel property of \( D \) and \( E' \).

**Proposition 3** (Existence and uniqueness of the scalar product). *There is one and only one scalar product*

\[ \langle \cdot, \cdot \rangle_x : (E_x' \hat{\otimes} \varepsilon D_y) \times (E_x \hat{\otimes} D_y') \to \mathbb{C} \]

which is partially continuous and coincides on \((E_x' \otimes D_y) \times (E_x \otimes D_y')\) with the product \( \varepsilon_x(\cdot, \cdot)_{E_x' \times E_x} \cdot \varepsilon_y(\cdot, \cdot)_{D_y' \times D_y} \), i.e., for decomposed elements \((S(x) \otimes \varphi(y), \psi(x) \otimes T(y))\) we have

\[ \langle \cdot, \cdot \rangle_x(S(x) \otimes \varphi(y), \psi(x) \otimes T(y)) = \varepsilon_x(\psi(x), S(x)) \varepsilon_y(\varphi(y), T(y)). \]
Proof. A first proof is given in [23, Proposition 1, p. 112], by means of the explicit representation of the elements of the strong dual $\mathcal{D}^\hat{} \otimes \mathcal{E}'$ of $\mathcal{D}'(\mathcal{E})$ hinted at before Proposition 2. The uniqueness can be presumed because L. Schwartz uses the word “le produit scalaire”.

A second proof consists in applying the “Théorèmes de croisement”, i.e., [25, Proposition 2, p. 18]: existence, uniqueness and partial continuity of the scalar product follow.

A third proof follows by composition of the vectorial scalar product

$$\langle \, , \rangle: \mathcal{E}_x(D'_y) \times (\mathcal{E}'_x \otimes \beta D_y) \rightarrow \mathcal{D}'_y \otimes \beta D_y$$

[25, Proposition 10, p. 57] with the scalar product

$$\langle \, , \rangle_y: \mathcal{D}'_y \otimes \beta D_y \rightarrow \mathbb{C}. \quad \square$$

3 “Semi-regular vanishing” distributions

Proposition 4 (Properties of $\mathcal{D}'(\mathcal{B})$ and $\mathcal{D}'(\mathcal{B}_c)$). The space of “semi-regular vanishing” distributions $\mathcal{D}'(\mathcal{B}) = \mathcal{D}' \otimes \mathcal{B}$ is ultrabornological but not semireflexive. $\mathcal{D}'(\mathcal{B}_c) = \mathcal{D}' \otimes \mathcal{B}_c$ is semireflexive but not bornological.

Proof. The non-semireflexivity and the semireflexivity, respectively, are consequences of Grothendieck’s permanence result [6, Chap. II, §3, n°2, Proposition 13 e., p. 76] due to the corresponding properties of $\mathcal{B}$ and $\mathcal{B}_c$, respectively.

The sequence space representations

$$\mathcal{B} \cong c_0 \hat{} \otimes s, \quad \mathcal{O}_M \cong s \hat{} \otimes s', \quad \mathcal{D}' \cong s^{\mathbb{N}}$$

(see [29, Theorem 3.2, p. 415; Theorem 5.3, p. 427] and [28, Theorem 3, p. 478]) show that

$$\mathcal{D}' \otimes \mathcal{B} \cong s^{\mathbb{N}} \otimes (s \hat{} \otimes c_0) \cong \mathcal{O}_M^{\mathbb{N}} \otimes c_0.$$

$\mathcal{O}_M^{\mathbb{N}}$ is ultrabornological (as seen in the proof of Proposition 1 (ii)), nuclear, and by [6, Chap. II, §2, n°2, Théorème 9, 2°, p. 47] also its dual $\mathcal{O}_M^{(\mathbb{N})}$ is nuclear, such that the bornologicity of $\mathcal{O}_M^{\mathbb{N}} \otimes c_0$ follows by [27, Proposition 2, p. 75].

To show that $\mathcal{D}'(\mathcal{B}_c)$ is not bornological we have to find a linear form $K$ in $(\mathcal{D}'(\mathcal{B}_c))^*$, the algebraic dual of $\mathcal{D}' \otimes \mathcal{B}_c$, which is locally bounded (i.e.,
it maps bounded subsets of $D \hat{\otimes} B_C$ into bounded sets of complex numbers) but not continuous. Because $B_C$ is not bornological there exists $T \in (B_C)'$ which is locally bounded but such that $T \not\in (B_C)'$. Fixing any $\varphi_0 \in D$ with $\varphi_0(0) = 1$, we define $K$ by $K(u) := T(u(\varphi_0))$ for $u \in L(D, B_C)$. Then $K$ is locally bounded but not continuous. In fact, taking a net $(f_\nu)_\nu \to 0$ in $B_C$ such that $T(f_\nu)$ does not converge to zero, we define a net $(u_\nu)_\nu$ in $L_b(D, B_C)$ by $u_\nu(\varphi) := \varphi(0)f_\nu$. Then $u_\nu \to 0$, but $K(u) = T(u_\nu(\varphi_0)) = T(f_\nu)$ does not converge to zero.

Analogously to the explicit description of the elements in $(D'(E))'_b$, cited before Proposition 2, let us represent the elements of $(D'(\hat{B}))'$:

**Proposition 5' (Dual of $D'(\hat{B})$).** If $K(x, y) \in D'_{xy}$ we have the characterization

$$K(x, y) \in (D'_x(\hat{B}_y))' \iff \exists m \in \mathbb{N}_0 \ \exists g_\alpha(x, y) \in \mathcal{D}_x \hat{\otimes} L^1_y,$$

$$|\alpha| \leq m, \alpha \in \mathbb{N}_0^n,$$

such that

$$K(x, y) = \sum_{|\alpha| \leq m} \partial^\alpha g_\alpha(x, y),$$

i.e., $\langle D'(\hat{B}) \rangle' = \lim_{m \to \infty} (\mathcal{D}' \hat{\otimes} D^m_L)$ algebraically.

Furthermore, $(D'(\hat{B}))' = (D' \hat{\otimes} B_C)'$ algebraically.

$D^m_L$ is the strong dual of the Banach space $B^m_C$ [22, p. 99]. Note that $D \hat{\otimes} E = \mathcal{D} \hat{\otimes} E$ for a Banach space $E$ because separately continuous bilinear forms on $D \times E$ are continuous.

**Proof of Proposition 5'.** The following proof is a copy of the proof of [23, Proposition 1, p. 112].

“$\implies$”: If $K(x, y) \in (D'_x(\hat{B}_y))' = \mathcal{N}(D'_x, D'_L, y)$ (for the equality, see [25, Prop. 22, p. 103] and [6, Chap. II, §2, n°1, Corollaire 4 1., p. 39]) there exist a bounded sequence $(\varphi_\nu)_{\nu \in \mathbb{N}}$ in $D_x$, a bounded sequence $(T_\nu)_{\nu \in \mathbb{N}}$ in $D'_L, y$ and a sequence $(\lambda_\nu)_{\nu \in \mathbb{N}}$ in $\ell^1$ such that we have for $S \in D'_x$

$$\langle K(x, y), S(x) \rangle = \sum_{\nu} \lambda_\nu \langle \varphi_\nu(x), S(x) \rangle T_\nu(y).$$

The boundedness of $(T_\nu)_{\nu \in \mathbb{N}}$ implies

$$\exists m \in \mathbb{N}_0 \ \exists D > 0 \ \exists f_{\nu, \alpha} \in L^1, \nu \in \mathbb{N}, \alpha \in \mathbb{N}_0^n, |\alpha| \leq m,$$

with $\|f_{\nu, \alpha}\|_1 \leq D$ such that $T_\nu = \sum_{|\alpha| \leq m} \partial^\alpha f_{\nu, \alpha}$
[26, Remarque 2*, p. 202] and, hence,

\[
\langle K(x, y), S(x) \rangle = \sum_{|\alpha| \leq m} \partial^\alpha_y \sum_{\nu} \lambda_\nu \langle \varphi_\nu, S \rangle f_{\nu, \alpha}(y) \quad (7)
\]

and \( K = \sum_{|\alpha| \leq m} \partial^\alpha_y g_\alpha \) if we set \( g_\alpha(x, y) := \sum_{\nu} \lambda_\nu \varphi_\nu(x) \cdot f_{\nu, \alpha}(y) \). In order to see that \( g_\alpha(x, y) \in D_x \hat{\otimes} L^1_y \) it suffices to show that for \( S(x) \in D'_x \) the sequence of partial sums

\[
\sum_{\nu=1}^N \langle \varphi_\nu(x), S(x) \rangle f_{\nu, \alpha}(y), \quad N \in \mathbb{N},
\]

converges in \( L^1 \) and that

\[
D'_x \to L^1_y, \quad S(x) \mapsto \sum_{\nu=1}^\infty \lambda_\nu \langle \varphi_\nu(x), S(x) \rangle f_{\nu, \alpha}(y)
\]

maps bounded sets of \( D' \) into bounded sets of \( L^1 \). This follows from the boundedness of \( (\varphi_\nu) \) in \( D \).

\("\Leftarrow": \) If \( K(x, y) \in D'_{xy} \) has the representation

\[
K(x, y) = \sum_{|\alpha| \leq m} \partial^\alpha_y g_\alpha(x, y), \quad g_\alpha(x, y) \in D_x \hat{\otimes} L^1_y,
\]

then also \( g_\alpha(x, y) \in E_x \hat{\otimes} L^1_y \).

By [6, Chap. I, §2, n°1, Théorème 1, 1*, p. 51] there exists \( (\lambda_\nu, \nu) \in \ell^1 \), a bounded sequence \( (e_{\nu, \alpha})_\nu \) in \( E_x \) and a bounded sequence \( (f_{\nu, \alpha})_\nu \in L^1 \) such that

\[
g_\alpha(x, y) = \sum_{\nu} \lambda_\nu f_{\nu, \alpha}(y) e_{\nu, \alpha}(x).
\]

The compactness of the supports of \( g_\alpha \) with respect to \( x \) [24, p. 62] implies the existence of a function \( \phi \in D_x \) such that

\[
g_\alpha(x, y) = \phi(x) \cdot g_\alpha(x, y), \quad |\alpha| \leq m.
\]

Thus,

\[
g_\alpha(x, y) = \sum_{\nu} \lambda_\nu e_{\nu, \alpha}(x) \phi(x) \otimes f_{\nu, \alpha}(y)
\]

and

\[
K(x, y) = \sum_{\nu} \sum_{|\alpha| \leq m} \lambda_\nu e_{\nu, \alpha}(x) \phi(x) \otimes \partial^\alpha_y f_{\nu, \alpha}(y).
\]
Because $(e_{\nu,\alpha}(x)\phi(x))_{\nu,\alpha}$ is bounded (and hence equicontinuous) in $\mathcal{D}_x$ and
$(\partial^\alpha_{\nu} f_{\nu,\alpha}(y))_{|\alpha| \leq m, \nu}$ is an equicontinuous (i.e., bounded) subset of $\mathcal{D}'_{L^1,y}$ this proves that $K \in \mathcal{N}(\mathcal{D}'_x, \mathcal{D}'_{L^1,y})$.

In order to see that $(\mathcal{D}' \hat{\otimes} \mathcal{B})' \subseteq (\mathcal{D}' \hat{\otimes} \mathcal{B}_c)'$ (the converse inclusion is obvious) we return to equality (7). Then for each $f \alpha := \sum_\nu |\lambda_\nu| : f_{\nu,\alpha} | \in L^1$ there exists by [17, Prop. 1.2.1, p. 6] a function $g \alpha \in \mathcal{C}_0$, $g \alpha > 0$ such that $f \alpha / g \alpha \in L^1$.

Then for any element $S$ of the polar $U := \{\varphi\}^\circ$, which is a $0$-neighborhood in $\mathcal{D}'$, and any $f \in \mathcal{B}$ we see that

\[
\langle \langle (K(x, y), S(x)), f(y) \rangle \rangle = \langle \sum_{|\alpha| \leq m} \partial^\alpha_y \sum_\nu \lambda_\nu \langle \varphi_\nu, S \rangle f_{\nu,\alpha}(y), f(y) \rangle
\]

\[
= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \langle \sum_\nu \lambda_\nu \langle \varphi_\nu, S \rangle f_{\nu,\alpha}(y), \partial^\alpha f(y) \rangle
\]

\[
= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int \sum_\nu \lambda_\nu \langle \varphi_\nu, S \rangle f_{\nu,\alpha}(y) \partial^\alpha f(y) \, dy
\]

and thus

\[
|\langle \langle (K(x, y), S(x)), f(y) \rangle \rangle| \leq \sum_{|\alpha| \leq m} \left| \int g_\alpha(y) \partial^\alpha f(y) \right| \frac{f_\alpha(y)}{g_\alpha(y)} \, dy
\]

\[
\leq \sum_{|\alpha| \leq m} \|g_\alpha \cdot \partial^\alpha f\|_\infty \int \frac{f_\alpha(y)}{g_\alpha(y)} \, dy.
\]

Hence, $\{\langle (K(x, y), S(x)) : S \in U \} \subseteq (\mathcal{B}_c)'$ is equicontinuous, which implies the claim. $\square$

**Proposition 5** (Dual of $\mathcal{D}' \hat{\otimes} \mathcal{B}$). We have linear topological isomorphisms

\[
(\mathcal{D}' \hat{\otimes} \mathcal{B}_c)_b' \cong \mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^1} = \mathcal{D}' \hat{\otimes} \mathcal{B}_c \mathcal{D}'_{L^1}
\]

and linear isomorphisms

\[
(\mathcal{D}' \hat{\otimes} \mathcal{B})' \cong \mathcal{D}(\mathcal{D}'_{L^1}; \varepsilon) = \mathcal{D}(\mathcal{D}'_{L^1}; \beta)
\]

\[
= \mathcal{N}(\mathcal{D}', \mathcal{D}'_{L^1})
\]

\[
\cong \mathcal{D}'_{L^1,\varepsilon}(\mathcal{D}; \varepsilon) = \mathcal{D}'_{L^1,\varepsilon}(\mathcal{D}; \beta).
\]

**Proof.** By Proposition 5’ we have $(\mathcal{D}' \hat{\otimes} \mathcal{B})' = (\mathcal{D}' \hat{\otimes} \mathcal{B}_c)'$ algebraically. Furthermore,

\[
(\mathcal{D}' \hat{\otimes} \mathcal{B})_b' \cong (\mathcal{D}' \hat{\otimes} \mathcal{B}_c)_b'.
\]
because $\mathcal{D}' \hat{\otimes} \mathcal{B}$ is distinguished: the bounded sets of $\mathcal{D}' \hat{\otimes} \mathcal{B}_c$ (and also of $\mathcal{D}' \otimes \mathcal{B}$) are contained in the weak closure of bounded sets in $\mathcal{D}' \otimes \mathcal{B}$. Hence,

$$(\mathcal{D}' \otimes \mathcal{B}_c)_b \cong (\mathcal{D}' \hat{\otimes} \mathcal{B})'_b.$$ 

Therefore, the first isomorphism in (8) results from Grothendieck’s duality Corollary cited in the introduction [6, Chap. II, §4, n°1, Lemme 9, Corollaire, p. 90]: $\mathcal{D}'$ is complete and nuclear, $\mathcal{B}_c$ is complete [22, p. 101] and semireflexive (see [17, Proposition 1.3.1, p. 11] and [24, p. 126]) and $(\mathcal{D}' \hat{\otimes} \mathcal{B})'_b$ is complete, due to Proposition 4. The equality of the $\nu$- and the $\beta$-topology in the first line follows because $\mathcal{D}$ and $\mathcal{D}'_{L^1}$ are barrelled spaces [25, p. 13]. □

**Remark.** We obtain, by Propositions 5 and 7, that $(\mathcal{D}'(\mathcal{B}))'_b = \mathcal{D}'(\mathcal{B})$ equals $\lim_{\tau \to \tau^m}(\mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^1})$ algebraically, which is a representation of the strong dual of $\mathcal{D}'(\mathcal{B})$ as a countable inductive limit.

In fact, for $K \in (\mathcal{D}'(\mathcal{B}))'$ we conclude by the implication “$\Rightarrow$” above that there exist $m \in \mathbb{N}_0$, $g_a(x, y) \in \mathcal{D}_x \hat{\otimes} L_{1y}$ with $K = \sum_{|\alpha| \leq m} \partial^\alpha g_a$. Hence, $K \in \mathcal{D}_x \hat{\otimes} D_{L^1,y}$. In order to see $\mathcal{D}_x \hat{\otimes} D_{L^1,y} \subset \mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y}$ it suffices, due to “$\Leftarrow$” above, to show the implication

$$K(x, y) \in \mathcal{D}_x \hat{\otimes} D_{L^1,y} \implies \exists g_a(x, y) \in \mathcal{D}_x \hat{\otimes} L_{1y} \text{ with } K = \sum_{|\alpha| \leq m} \partial^\alpha g_a.$$ 

However, this implication is a consequence of

$$\mathcal{D}_{L^1} = \sum_{|\alpha| \leq m} \partial^\alpha L^1 \quad \text{and} \quad \mathcal{D}_x \hat{\otimes} D_{L^1,y} = \sum_{|\alpha| \leq m} \partial^\alpha(D_x \hat{\otimes} L_{1y}).$$

**Proposition 6** (Existence and uniqueness of the scalar product). There is one and only one scalar product

$$\langle \cdot, \cdot \rangle_\mathcal{D}_x \otimes_\beta \mathcal{D}'_{L^1,y} \times (\mathcal{D}'_x \hat{\otimes} \mathcal{B}_y) \rightarrow \mathbb{C}$$

which is partially continuous and coincides on $(\mathcal{D}_x \otimes \mathcal{D}'_{L^1,y}) \times (\mathcal{D}'_x \otimes \mathcal{B}_y)$ with the product $\mathcal{D}_x \langle \cdot, \cdot \rangle_\mathcal{D}_x \cdot \mathcal{D}'_{L^1,y} \langle \cdot, \cdot \rangle_\mathcal{B}_y$.

**Proof.** A first proof follows from the “Théorèmes de croisement” [25, Proposition 2, p. 18].

A second proof consists in the composition of the vectorial scalar product given by [25, Proposition 10, p. 57], i.e.,

$$\langle \cdot, \cdot \rangle_\mathcal{D}_x \cdot \mathcal{D}'_{L^1,y} \rightarrow \mathcal{D}'_{L^1,y} \hat{\otimes} \mathcal{B}_y \rightarrow \mathcal{D}'_{L^1,y} \hat{\otimes} \mathcal{B}_y,$$

and the scalar product $\langle \cdot, \cdot \rangle: \mathcal{D}'_{L^1,y} \hat{\otimes} \mathcal{B}_y \rightarrow \mathbb{C}$. □
Remark. If $K(x, y) \in \mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y}$ has the representation

$$K(x, y) = \sum_{|\alpha| \leq m} \partial^\alpha_y g_\alpha(x, y)$$

with $g_\alpha(x, y) \in \mathcal{D}_x L^1_y$ and if $L(x, y) \in \mathcal{D}'_x(\hat{\mathcal{B}}_y)$ then

$$\langle \cdot, \cdot \rangle_x(K(x, y), L(x, y)) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} \mathcal{D}'_x(\partial^\alpha_y L(x, y), g_\alpha(x, y)) \mathcal{D}_x dy.$$ 

We find a third expression for the scalar product by means of vector-valued multiplication and integration:

**Proposition 6' (Scalar product).** If $K(x, y) \in \mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y}$ and $L(x, y) \in \mathcal{D}'_x(\hat{\mathcal{B}}_y)$ then the scalar product $\langle \cdot, \cdot \rangle_x$ (Proposition 6) can also be expressed as

$$\langle \cdot, \cdot \rangle_x(K(x, y), L(x, y)) = \mathcal{D}'_{L^1,xy} \langle (K(x, y))^* L(x, y), 1(x, y) \rangle_{\mathcal{B}_{c,xy}},$$

wherein $^*$ denotes the vectorial multiplicative product

$$(\mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y}) \times (\mathcal{D}'_x \hat{\otimes} \mathcal{B}_y) \rightarrow \mathcal{E}'_x \hat{\otimes} \varepsilon \mathcal{D}'_{L^1,y}.$$

**Proof.** The vectorial multiplicative product $^*$ exists uniquely as the composition of the canonical mapping defined by the “Théorèmes de croisement” [25, Proposition 2, p. 18],

$$(\mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y}) \times (\mathcal{D}'_x \hat{\otimes} \mathcal{B}_y) \rightarrow (\mathcal{D}_x \hat{\otimes} \mathcal{D}'_x) \in (\mathcal{D}'_{L^1,xy} \hat{\otimes} \mathcal{B}_y)$$

and the $\varepsilon$-product of the two multiplications

$$\mathcal{D}_x \hat{\otimes} \mathcal{D}'_x \rightarrow \mathcal{E}'_x \text{ and } \mathcal{D}'_{L^1,xy} \hat{\otimes} \mathcal{B}_y \rightarrow \mathcal{D}'_{L^1,xy},$$

namely

$$K(x, y)^* L(x, y) = [(\cdot, \cdot) \in (\cdot, \cdot)] \circ \varepsilon.$$

Note that this vectorial multiplication coincides with that defined in [25, Proposition 32, p. 127]. Due to the uniqueness of the scalar product and the continuity of the embedding $\mathcal{E}'_x \hat{\otimes} \mathcal{D}'_{L^1,y} \hookrightarrow \mathcal{D}'_{L^1,xy}$ the result follows. □
Proposition 7 (Existence of the regularization mapping and representation of its transpose). The regularization mapping

\[ \mathcal{B}' \to \mathcal{D}'_y \hat{\otimes} \mathcal{B}_x, \quad S \mapsto S(x - y) \]

is well-defined, linear, injective and continuous. Its transpose

\[ \mathcal{D}_y \hat{\otimes} \mathcal{D}'_{L^1,x} \to \mathcal{D}_{L^1,c} \]

is linear, continuous and given by

\[ K(x, y) \mapsto _{B_{c,x}} \langle 1(x), K(x - y, x) \rangle_{\mathcal{D}'_{L^1,x}(\mathcal{D}_{L^1,c,y})}, \]

Proof. 1. The well-definedness is L. Schwartz’ classical regularization result [26, Remarque 3°, p. 202].

2. Due to the (sequentially) closed graph of the regularization mapping the continuity is implied by [6, Chap. I, Théorème B, p. 17], if \( \mathcal{B}' \) and \( \mathcal{D}' \hat{\otimes} \mathcal{B} \) are ultrabornological. The sequence-space representation of \( \mathcal{B}' \cong c_0 \hat{\otimes} s' \) [2, Theorem 3, p. 13] shows that \( \mathcal{B}' \) is ultrabornological if [27, Proposition 2, p. 75] is applied. The space \( \mathcal{D}' \hat{\otimes} \mathcal{B} \) is ultrabornological by Proposition 4. Alternatively, for applying the closed graph theorem one can use that \( \mathcal{D}' \hat{\otimes} \mathcal{B} \) has a completing web [19, p. 736].

The transpose of the regularization mapping is continuous by [8, Corollary to Proposition 3.12.3, p. 256].

3. The representation in Proposition 6’ yields for \( K(x, y) \in \mathcal{D}_x \hat{\otimes} \mathcal{D}'_{L^1,y} \) and \( S(y - x) \in \mathcal{D}_y \hat{\otimes} \mathcal{B}_y \):

\[ \langle \cdot, \cdot \rangle_x (K(x, y), S(y - x)) = \mathcal{D}'_{L^1,x,y} \langle K(x, y)^\ast (y), S(y - x), 1(x, y) \rangle _{B_{c,x,y}}. \]

The linear change of variables

\[ x = v - u \quad u = y - x \]
\[ y = v \quad v = y \]

and the Theorem of Fubini [24, Corollary, pp. 136, 137] imply that the last expression equals

\[ _{B_{c,u,v}} \langle 1(u, v), K(v - u, v)^\ast (S(u) \otimes 1(v)) \rangle_{\mathcal{D}'_{L^1,u,v}} = \hat{\mathcal{B}}_u \langle S(u), _{B_{c,v}} \langle 1(v), K(v - u, v) \rangle_{\mathcal{D}'_{L^1,v}(\mathcal{D}_{L^1,c,u})} \rangle_{\mathcal{D}_{L^1,c,u}} \]

13
if we show that

\[ K(v, u) \in \mathcal{D}_v \hat{\otimes} \mathcal{D}'_{L^1, u} \implies K(v - u, v) \in \mathcal{D}'_{L^1, v}(\mathcal{D}_{L^1, c, u}). \] (9)

Then, the multiplicative product \( K(v - u, v) \cdot u \cdot v \) is defined as the image of \((K(v - u, v), S(u) \otimes 1(v))\) under the mapping

\[ \mathcal{D}'_{L^1, v}(\mathcal{D}_{L^1, c, u}) \times \mathcal{B}_{c,v}(\mathcal{B}'_u) \xrightarrow{\sim} \mathcal{D}'_{L^1, v}(\mathcal{D}'_{L^1, c, u}). \]

It remains to prove the implication (9): a vectorial regularization property similar to [1, Proposition 15] shows that

\[ K(v - w, u) \in \mathcal{D}'_{L^1, c, u} \implies K(v - u, v) \in \mathcal{D}'_{L^1, c, u} \]

because the convolution with \( \varphi \in \mathcal{D} \) maps the space \( \mathcal{D}'_{L^1, c} \) continuously into \( \mathcal{D}_{L^1, c} \). The vector-valued multiplication \( \mathcal{D}'_{L^1} \times \mathcal{B}_{c,v}(E) \to \mathcal{D}'_{L^1}(E) \) by [25, Proposition 25, p. 120] then yields

\[ K(v - u, v) \in \mathcal{D}_{L^1, c} \hat{\otimes} \mathcal{D}_{L^1, c, u} = \mathcal{D}'_{L^1, v}(\mathcal{D}_{L^1, c, u}). \]

\[ \square \]

4 On the duals of tensor products
- two complements

The goal of this section is the formulation of propositions which yield, as special cases, the strong duals of the spaces \( \mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^1} \) and \( \mathcal{D}' \hat{\otimes} \mathcal{B} \). These spaces are the “endpoints” in the scale of reflexive spaces \( \mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^p} \) and \( \mathcal{D}' \hat{\otimes} \mathcal{D}_{L^q} \), \( 1 < p, q < \infty \), the duals of which can be determined by the Corollaire [6, Chap. II, §4, n°1, Lemme 9, Corollaire, p. 90] cited in the introduction.

**Proposition 8** (Dual of a completed tensor product). Let \( \mathcal{H} = \varprojlim_k \mathcal{H}_k \) be the strict inductive limit of nuclear Fréchet spaces \( \mathcal{H}_k \) and \( F \) the strong dual of a distinguished Fréchet space. Then

\[ (\mathcal{H}'_b \hat{\otimes} F)' = \overline{\mathcal{H}(F'_b)} := \varinjlim_k (\mathcal{H}_k(F'_b)). \]

The space \( \overline{\mathcal{H}(F'_b)} \) is a complete, strict (LF)-space and \( \mathcal{H}'_b \hat{\otimes} F \) is distinguished. If \( F \) is reflexive, \( \mathcal{H}'_b \hat{\otimes} F \) is reflexive, too.
Proof. By [25, Prop. 22, p. 103] we have algebraically

\[(\mathcal{H}_b \hat{\otimes} F)' \cong F'_c(\mathcal{H}; \varepsilon) = F'_c(\mathcal{H}; \beta) = \lim_{k} (F'_c(\mathcal{H}_k; \beta))\]

due to the reflexivity of \(\mathcal{H}\) and due to the fact that a linear and continuous map \(T : F \to \mathcal{H}\) is bounded if and only if there exists \(k\) and a 0-neighborhood \(U\) in \(F\) such that \(T\) maps into \(\mathcal{H}_k\) and \(T(U) \subseteq \mathcal{H}_k\) is bounded. Because \(F\) is the strong dual of a distinguished Fréchet space,

\[F'_c(\mathcal{H}_k) = \text{L}_\beta((\mathcal{H}_k)'_c, F'_b) \cong L_c(F, \mathcal{H}_k) = F'_b \hat{\otimes} \mathcal{H}_k\]

by [25, p. 98, b)]. Hence,

\[F'_c(\mathcal{H}_k) = L_c((\mathcal{H}_k)'_c, F'_c) \cong L_c(F, \mathcal{H}_k) = F'_b \hat{\otimes} \mathcal{H}_k\]

(see [6, Chap. I, p. 75]). All together,

\[(\mathcal{H}_b \hat{\otimes} F)' = \lim_{k} \mathcal{H}_k(F'_b) = \mathcal{H}(F'_b).\]

The strong dual topology on \((\mathcal{H}_b(F))'\) is finer than the topology of uniform convergence on products of bounded subsets of \(\mathcal{H}_b'\) and \(F\) [25, Prop. 22, p. 103], i.e., the embedding \((\mathcal{H}_b(F))'_b \hookrightarrow \mathcal{H} \hat{\otimes} F'_b\) is continuous.

In order to show the continuity of the map

\[\text{Id} : \mathcal{H}(F'_b) \to (\mathcal{H}_b(F))'_b\]

we use an idea from the proof of [7, Prop. 3, p. 542]: it suffices that bounded sets in \(\mathcal{H}(F'_b)\) are bounded in \((\mathcal{H}_b(F))'_b\) because \(\mathcal{H}(F'_b)\) is bornological (note that \(F'_b\) is a Fréchet space by [5, p. 64], and that the inductive limit of the Fréchet spaces \(\mathcal{H}_b \hat{\otimes} F'_b = \mathcal{H}(F'_b)\) is bornological). If \(H \subseteq \mathcal{H}(F'_b)\) is bounded then by the regularity of the inductive limit there exists \(k\) such that \(H\) is bounded in \(\mathcal{H}_k(F'_b)\). By [6, Chap. II, §3, n°1, Prop. 12, p. 73] there are bounded subsets \(A \subseteq \mathcal{H}_k\) and \(B \subseteq F'_b\) such that \(H\) is contained in the closed absolutely convex hull of \(A \otimes B\). For each \(T \in \mathcal{H}_b \hat{\otimes} F \cong L_b(\mathcal{H}, F)\) the set \(T(H)\) is contained in the closed absolutely convex hull of \(T(A \otimes B) = (T(A), B)\), which is bounded because \(T(A) \subseteq F\) is bounded and \(B\) is equicontinuous. Hence, we see that \(H\) is weakly bounded and thus bounded in \((\mathcal{H}_b(F))'_b\) due to the completeness of \(\mathcal{H}_b(F)\).

\[
\square
\]

Remark. (1) The strong topology of \((\mathcal{H}_b(F))'_b\) coincides with the topology induced by \(L_b(\mathcal{H}_b', F'_b) = \mathcal{H} \hat{\otimes} F'_b\). By [25, Prop. 22, p. 103] this is equivalent with saying that bounded subsets of \(\mathcal{H}_b(F) = \mathcal{H}_b \hat{\otimes} F\) are \(\beta\)-\(\beta\)-decomposable.
(2) If the space $F$ is the strong dual of a reflexive Fréchet space then $\mathcal{H}_b' \hat{\otimes} F$ is reflexive too, i.e.,

$$ (\mathcal{H}_b' \hat{\otimes} F)'_b \cong \overline{\mathcal{H}}(F'_b) $$

and

$$ (\overline{\mathcal{H}}(F'_b))'_b \cong \mathcal{H}_b' \hat{\otimes} F $$

(for the second part use a suitable generalization of [25, Corollaire 3, p. 104]). This assertion generalizes the reasoning in [13, p. 314, 4.2, p. 315].

(3) The following list of distribution spaces illustrates possibilities of the applicability of Proposition 8:

$$\begin{align*}
\mathcal{H} : \mathcal{D}, \mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_+\Gamma \\
F : \mathcal{D}'_{L^1}, \mathcal{D}'_{L^1}, \ell^1, S^m, \mathcal{E}^m 
\end{align*}$$

(cf. [25, p. 154, p. 186]),

$m \in \mathbb{N}_0$).

One has to show that $S^m$ and $\mathcal{E}^m$ are distinguished.

(4) The dual of the space of partially summable distributions $\mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^1}$ was given first in [7, Prop. 3(2), p. 541], i.e., $(\mathcal{D}' \hat{\otimes} \mathcal{D}'_{L^1})'_b = \overline{\mathcal{D}(B)} = \mathcal{D}\hat{\otimes} B$.

By considering the proof of Proposition 5, i.e., $(\mathcal{D}'(B))'_b = (\mathcal{D}' \hat{\otimes} B_c)'_b = \mathcal{D} \hat{\otimes} \mathcal{D}'_{L^1}$, we were led to the following modification of A. Grothendieck’s Corollary [6, Chap. II, §4, n°1, Lemme 9, Corollaire, p. 90] cited in the introduction.

**Proposition 9** (Dual of a completed tensor product). Let $\mathcal{H}$ be a Hausdorff, quasicomplete, nuclear, locally convex space with the strict approximation property, $F$ a quasicomplete, semireflexive, locally convex space. Let $F_0$ be a locally convex space such that

$$ (\mathcal{H} \hat{\otimes} F)'_b = (\mathcal{H} \hat{\otimes} F_0)'_b $$

and $(\mathcal{H} \hat{\otimes} F_0)'_b$ is complete. Then,

$$ (\mathcal{H} \hat{\otimes} F)'_b \cong \mathcal{H}_b' \hat{\otimes} F'_b $$

and $(\mathcal{H} \hat{\otimes} F)'_b$ is semireflexive.

The proof is an immediate consequence of Proposition 10. The semireflexivity is a consequence of [6, Corollaire 2, p. 118].
Remark. (1) A. Grothendieck’s hypotheses “H complete, F complete and semireflexive” are weakened by the assumption of quasicompleteness at the expense of the additional hypothesis of the strict approximation property for H. The completeness of the strong dual $(H \hat{\otimes} F)'_b$ is implied by the existence of an additional space $F_0$ with the corresponding property.

(2) By checking the hypotheses of Proposition 9 we have shown in Proposition 5 that

$$(D' \hat{\otimes} B)'_b \simeq D \hat{\otimes} D_{L^1}.$$ 

Two other applications to concrete distribution spaces are:

$$(D' \hat{\otimes} B)'_b \simeq D \hat{\otimes} D_{L^1} = \overline{D(D_{L^1})}$$

with $F_0 = (D_{L^\infty}, \kappa(D_{L^\infty}, D_{L^1}))$, and

$$(D' \hat{\otimes} c_0)'_b \simeq D \hat{\otimes} \ell^1 = \overline{D(\ell^1)}$$

with $F_0 = (\ell^\infty, \kappa(\ell^\infty, \ell^1))$.

(3) As an application of Proposition 9 we see that spaces like $S' \hat{\otimes} B$ or $O_M \hat{\otimes} c_0$ are distinguished. This does not follow from [6, Chap. II, §3, n°2, Corollaire 2, p. 77]. In fact, $(S' \hat{\otimes} B)'_b \simeq S \hat{\otimes} D_{L^1}$ and $S \hat{\otimes} D_{L^1}$ is barrelled by [6, Chap. I, p. 78].

The proof of Proposition 9 rests on a generalization of Grothendieck’s Corollary on duality (cf. [6, Chap. II, §4, n°1, Lemme 9, Corollaire, p. 90]) which we prove now.

**Proposition 10 (Duals of tensor products).**

**Hypothesis 1:** Let $E$ be a nuclear, $F$ a locally convex space.

**Then:**

(i) Every element $u$ of the dual $B(E, F) = (E \hat{\otimes} F)' = (E \hat{\otimes}_\pi F)'$ is the image (under a canonical mapping) of an element $u_0$ of a space $E'_A \hat{\otimes}_\pi E'_B$, where $A$ and $B$ are absolutely convex, weakly closed, equicontinuous subsets of $E'$ and $F'$, respectively.

(ii) If the element $u$, of $E' \hat{\otimes} F' \subset E' \hat{\otimes}_\pi F'$ defined by $u_0$ is zero then $u$ is zero, from which we have a canonical injection of $B(E, F)$ into $E' \hat{\otimes} F' \subset E' \hat{\otimes}_\pi F'$.
(iii) $\mathcal{B}(E, F)$ is dense in $E' \hat{\otimes} F'$ and strictly dense in $E' \hat{\otimes} F'$.

If, in addition, we have

**Hypothesis 2:** $E$ is quasicomplete and has the strict approximation property and $F$ is quasicomplete,

then we obtain:

(iv) The topology $t_\iota$ induced from $E' \hat{\otimes} F'$ on $\mathcal{B}(E, F)$ is finer than the topology $t_b$ of $(E \hat{\otimes}_\pi F)'_b$, i.e., $t_\iota \geq t_b$.

If, in addition, we have

**Hypothesis 3:** $F$ is semireflexive,

then we obtain:

(v) The topology induced from $E' \hat{\otimes} F'$ on $\mathcal{B}(E, F)$ is identical with the topology of the strong dual $(E \hat{\otimes}_\pi F)'_b$, i.e., $t_\iota = t_b$. $E' \hat{\otimes} F'$ is the completion of $\mathcal{B}(E, F)$.

(vi) $(E \hat{\otimes}_\pi F)'_b \cong E' \hat{\otimes} F'$ if and only if $(E \hat{\otimes}_\pi F)'_b$ is complete.

$(E \hat{\otimes}_\pi F)'_b \cong E' \hat{\otimes} F'$ if and only if $(E \hat{\otimes}_\pi F)'_b$ is quasi-complete.

(vii) $E \hat{\otimes} F$ is semireflexive.

**Proof.** We shall modify the proof of Grothendieck and give more details.

(i) If $u \in \mathcal{B}(E, F)$ then the nuclearity of $E$ implies the existence of zero-neighborhoods $U$ in $E$ and $V$ in $F$ and of sequences $(e'_n)$ in $E'_U$ and $(f'_n)$ in $F'_V$ such that

\[
\sum_{n=1}^{\infty} \|e'_n\|_{U^\circ} \|f'_n\|_{V^\circ} < \infty
\]  

and

\[
u(e, f) = \sum_{n=1}^{\infty} \langle e, e'_n \rangle \langle f, f'_n \rangle \quad \forall (e, f) \in E \times F
\]

by [6, Chap. II, §2, n°1, Corollaire 4 to Théorème 6, p. 39] or [10, 21.3.5, p. 487]. Setting $A := U^\circ$, $B := V^\circ$ we then define $u_0 \in E'_A \hat{\otimes}_\pi E'_B$ by

\[
u_0 := \sum_{n=1}^{\infty} e'_n \otimes f'_n.
\]
The series in (11) converges because
\[
\sum_{n=1}^{\infty} |\langle e, e'_n \rangle \langle f, f'_n \rangle| \leq \left( \sum_{n=1}^{\infty} \| e_n' \|_{U^*} \| f'_n \|_{V^*} \right) \cdot \| e \|_U \cdot \| f \|_V
\]
due to Lemma 11 and inequality (10). Moreover, \( u_0 = \sum e'_n \otimes f'_n \) converges in the Banach space \( E'_A \widehat{\otimes}_\pi E'_B \):
\[
\| u_0 \| = \left\| \sum_{n=1}^{\infty} (e'_n \otimes f'_n) \right\| = \inf \left\{ \sum_{n=1}^{\infty} \| a_n \|_{U^*} \| b_n \|_{V^*} : u_0 = \sum_{n=1}^{\infty} a_n \otimes b_n \right\}
\leq \sum_{n=1}^{\infty} \| e'_n \|_{U^*} \| f'_n \|_{V^*}.
\]

Next let us describe in detail the canonical mapping \( E'_A \widehat{\otimes}_\pi E'_B \to \mathcal{B}(E,F) \) as the composition of the following three mappings
\[
E'_A \widehat{\otimes}_\pi F'_B \to (E_U)' \widehat{\otimes}_\pi (F_V)' \to (E_U \widehat{\otimes}_\pi F_V)' \to (E \widehat{\otimes}_\pi F)' = \mathcal{B}(E,F). \tag{12}
\]
For the first mapping in (12) we use that \( 't \Phi_U : (E_U)' \to E'_A \) is an isomorphism with inverse \( ( 't \Phi_U)^{-1} : E'_A \to (E_U)' \).

The second mapping in (12) is the continuous extension of the linear map on \( (E_U)' \otimes (E_V)' \) corresponding to the continuous bilinear map
\[
(E_U)' \times (E_V)' \to (E_U \widehat{\otimes}_\pi F_V)',
\]
\( (e', f') \mapsto e' \otimes f' \).

The third mapping in (12) is given as the transpose of \( \Phi_U \otimes \Phi_V : E \widehat{\otimes}_\pi F \to E_U \widehat{\otimes}_\pi F_V \).

The image of \( u_0 \) in \( \mathcal{B}(E,F) \) coincides with \( u \): denoting the image of \( u_0 \) in all spaces appearing in (12) by \( u_0 \) we obtain by going from right to left in the composition above:
\[
u_0(e, f) = u_0(e \otimes f) = u_0(\Phi_U(e) \otimes \Phi_V(f))
\]
\[
= \left( \sum_{n=1}^{\infty} \langle \Phi_U(e), \Phi_V(f) \rangle \Phi_U^{-1}_n(e'_n) \Phi_V^{-1}(f'_n) \right) \Phi_U(e) \otimes \Phi_V(f)
\]
\[
= \sum_{n=1}^{\infty} \langle \Phi_U(e), \Phi_V(f) \rangle \langle \Phi_U^{-1}(e'_n), \Phi_V^{-1}(f'_n) \rangle
\]
\[
= \sum_{n=1}^{\infty} \langle e, e'_n \rangle \langle f, f'_n \rangle = u(e, f).
\]
(ii) We also have a canonical mapping

$$(^t\Phi_U) \otimes (^t\Phi_V): E'_A \widehat{\otimes} F'_B = E'_A \otimes_i F'_B \to E' \widehat{\otimes} F'$$

which maps $u_0$ to an element $u \in E' \widehat{\otimes} F'$. Due to $$(^t\Phi_U) \otimes (^t\Phi_V) (u_0) = u_0 \circ (\Phi_U \otimes \Phi_V) \in B(E, F)$$
we conclude from $u_i = 0$ that $u_0$ vanishes on $\widehat{E}_U \times \widehat{F}_V$. By [6, Chap. I, §5, n°2, Corollaire 1, p. 181] or [10, Theorem 18.3.4, p. 406] the canonical mapping

$$E'_A \widehat{\otimes} F'_B \to B(\widehat{E}_U, \widehat{F}_V)$$

is injective because the zero-neighborhood $U$ can be chosen in such a manner that $\widehat{E}_U$ is a Hilbert space [6, Chap. II, §2, n°1, Lemme 3, p. 37], and, therefore, $\widehat{E}_U$ is reflexive and has the approximation property. Hence, the vanishing of $u_0$ on $\widehat{E}_U \times \widehat{F}_V$ implies $u_0 = 0$ and a fortiori $u = 0$.

(iii) follows from $E' \otimes F' \subset B(E, F) \subset E' \widehat{\otimes} F' \subset E' \widehat{\otimes} F'$.

(iv) If, in addition, Hypothesis 2 is fulfilled, we obtain by Lemma 12 below that $E \widehat{\otimes}_{\pi} F \cong B^h_c(E'_c, F'_c)$. Two topologies on $B(E, F)$ can be defined corresponding to the following two dual systems (cf. [18, Chap. III, §2, p. 46] or [8, Chap. 3, §2, p. 183]):

$$(B(E, F), E \widehat{\otimes}_{\pi} F \cong B^h_c(E'_c, F'_c))$$

$$j \quad m$$

$$(E' \widehat{\otimes} F', (E' \widehat{\otimes} F')' \cong B^h_c(E', F')).$$

The mapping $j$ is defined as the injection

$$B(E, F) \to E' \widehat{\otimes} F', \quad u \mapsto \sum_{n=1}^{\infty} e'_n \otimes f'_n$$

investigated in (i) and (ii).

The mapping $m$ is the injection of the space $B^h_c(E'_c, F'_c)$ of hypocontinuous bilinear forms on $E'_c \times F'_c$ into the space $B^s(E', F')$ of separately continuous bilinear forms on $E' \times F'$. Furthermore, we have canonical isomorphisms

$$k: E \widehat{\otimes}_{\pi} F \to B^h_c(E'_c, F'_c), \quad z \mapsto [(e', f') \mapsto \langle e' \otimes f', z \rangle],$$

$$\ell: B^s(E', F') \to (E' \widehat{\otimes}_s F'), \quad \ell(w)(e' \otimes f') = w(e', f').$$
On the one hand, we consider on $\mathcal{B}(E, F)$ the relative topology $t_\iota$ with respect to the embedding $j$. Because the topology of $E' \otimes \iota F'$ is the topology of uniform convergence on equicontinuous subsets of $(E' \otimes \iota F')' = \mathcal{B}^*(E', F')$ and equicontinuous subsets of this dual space are precisely the equicontinuous subsets of $\mathcal{B}^*(E', F')$ [6, Chap. I, §3, n’1, Proposition 13, p. 73] we see that a zero-neighborhood base of $t_\iota$ is given by the sets $j^{-1}(\ell(B')^\circ)$ with separately equicontinuous subsets $B'$ of $\mathcal{B}^*(E', F')$.

Let us now show that $t_\iota$ is finer than $t_b$; in particular, that for a given bounded set $B$ in $E \bar{\otimes}_\pi F$ the set $B' := m(k(B))$ is separately equicontinuous and that $j^{-1}(\ell(B')^\circ) = B^\circ$.

If $B \subset E \bar{\otimes}_\pi F \cong \mathcal{B}^b_\iota(E'_c, F'_\iota)$ is bounded the set $B(., f')$, for fixed $f' \in F'$, is bounded on all equicontinuous sets $U^\circ$, $U$ an absolutely convex, closed zero-neighborhood in $E$, i.e., $\forall U \exists \lambda_U > 0$ such that

$$\sup_{e' \in U^\circ} |B(e', f')| \leq \lambda_U,$$

so $B(., f') \subset \lambda_U U^\circ \cong \lambda_U U$ due to $(E'_c)' = E$. Hence, $B(., f')$ is bounded in $E$ and $B(., f')^\circ$ is a zero-neighborhood in $E'_b$, which means that $B(., f')$ is equicontinuous on $E'_b$. Similarly, $B(e', .)$ is equicontinuous on $F'_b$ for $e' \in E'$ fixed. Therefore, $B' := m(k(B))$ is a separately equicontinuous subset of $\mathcal{B}^*(E', F')$ which implies that $(\ell(B')^\circ)$ is a zero-neighborhood in $t_\iota$, which together with $j^{-1}(\ell(B')^\circ) = B^\circ$ will imply $t_b \leq t_\iota$.

In order to prove $j^{-1}(\ell(B')^\circ) = B^\circ$ for a given bounded subset $B \subset E \bar{\otimes}_\pi F$ with $B' = m(k(B))$ we write down the involved mappings explicitly. Let $u \in \mathcal{B}(E, F)$ with $j(u) = \sum_{i=1}^{\infty} e'_n \otimes f'_n$ and $z \in E \bar{\otimes}_\pi F$. Then,

$$\langle j(u), \ell(m(k(z))) \rangle \overset{\circ}{=} \ell(m(k(z))) \left( \sum_{n=1}^{\infty} e'_n \otimes f'_n \right) \overset{\circ}{=\circ} \sum_{n=1}^{\infty} \ell(m(k(z)))(e'_n \otimes f'_n) \overset{\circ}{=} \sum_{n=1}^{\infty} m(k(z))(e'_n, f'_n) \overset{\circ}{=} \sum_{n=1}^{\infty} k(z)(e'_n, f'_n) \overset{\circ}{=} \sum_{n=1}^{\infty} (e'_n \otimes f'_n)(z) \overset{\circ}{=} \langle u, z \rangle$$

$^1$ is the definition of $j$, $^2$ follows from the continuity of $\ell(m(k(z)))$. $^3$, $^4$ and $^5$ are the definitions of $\ell$, $m$ and $k$, respectively. The equality $^6$ is a consequence of the equality for $z$ in the strictly dense subspace $E \otimes F$ of
$E \otimes F$ and the continuity of

$$z \mapsto \sum_{n=1}^{\infty} (e'_n \otimes f'_n)(z)$$

which follows from the representation

$$\sum_{n=1}^{\infty} (e'_n \otimes f'_n)(z) = \sum_{n=1}^{\infty} \|e'_n\|_{U^o} \|f'_n\|_{V^o} \left( \frac{e'_n}{\|e'_n\|_{U^o}} \otimes \frac{f'_n}{\|f'_n\|_{V^o}} \right)(z)$$

and the inequality (using Lemma 11)

$$\left| \sum_{n=1}^{\infty} (e'_n \otimes f'_n)(z) \right| \leq \sum_{n=1}^{\infty} \|e'_n\|_{U^o} \|f'_n\|_{V^o} \left| \left( \frac{e'_n}{\|e'_n\|_{U^o}} \otimes \frac{f'_n}{\|f'_n\|_{V^o}} \right)(z) \right|$$

$$\leq \sum_{n=1}^{\infty} \|e'_n\|_{U^o} \|f'_n\|_{V^o} \left\| \frac{e'_n}{\|e'_n\|_{U^o}} \otimes \frac{f'_n}{\|f'_n\|_{V^o}} \right\|_{U^o \otimes V^o} \|z\|_{U \otimes V}$$

$$\leq \left( \sum_{n=1}^{\infty} \|e'_n\|_{U^o} \|f'_n\|_{V^o} \right) \|z\|_{U \otimes V}.$$ 

Note that the sets $\{e'_n/\|e'_n\|_{U^o}\}$ and $\{f'_n/\|f'_n\|_{V^o}\}$ are equicontinuous and hence also their tensor product is equicontinuous by [25, p. 14].

(v) The space $E$ is nuclear and quasicomplete and, hence, $E$ is semireflexive (see [21, Prop. 3, exp. 17, p. 5], [6, Chap. II, Cor. 1, p. 38] and [8, Cor. to Prop. 3.15.4, p. 277 and Prop. 3.9.1, p. 231]). Thus, with Hypothesis 3 both spaces $E$ and $F$ are semireflexive. Moreover, by [6, Chap. I, p. 11]:

$$B^s(E'_b, F'_b) = L(E'_b, (F'_b)'_\sigma) = L(E'_b, (F'_\sigma)'_b) = B^s(E'_b, F'_b) = \ldots = B^s(E'_\sigma, F'_\sigma),$$

$$B^s(E'_c, F'_c) = B^s(E'_c, F'_c)$$

and

$$B^h(E'_b, F'_b) = B^s(E'_b, F'_b),$$

because $E'_b$ and $F'_b$ are barrelled [8, Prop. 3.8.4, p. 228]. Hence,

$$B^h_b(E'_c, F'_c) = B^h_b(E'_b, F'_b) = B^h_b(E'_\sigma, F'_\sigma).$$

Let us show that $t_b$ is finer than $t_c$, $t_b \geq t_c$: if $B \subset B^s(E', F')$ is separately equicontinuous, $B$ is pointwise bounded such that Mackey’s theorem [4, IV.1, Prop. 1 (ii)] implies its boundedness in $B^h_b(E'_b, F'_b)$, i.e., $B$ is bounded in $E \overline{\otimes}_F F$. In virtue of (iii), $E' \overline{\otimes}_F F'$ is the completion of $B(E, F)$ and $E' \overline{\otimes}_F F'$ its quasi-completion.
(vi) follows from \((E \bar{\otimes}_\pi F)' = \mathcal{B}(E, F)\).

(vii) is a consequence of [6, Chap. I, §4, n°2, Corollaire of Prop. 24, p. 118].

**Lemma 11.** Let \(U\) be an absolutely convex zero-neighborhood in \(E\), \(e' \in E'_{U^\circ}\) and \(e \in E\). Then \(\langle e, e' \rangle \leq \|e\|_U \|e'\|_{U^\circ}\).

**Proof.** The inequality follows either by restricting the scalar product on \(E \times E'\) to the Banach spaces \(E_U \times E'_{U^\circ}\) (taking into account that \((E_U)' = E'_{U^\circ}\)) or, more elementary:

\[
\|e'\|_{U^\circ} = \sup_{e \in U} |\langle e, e' \rangle| \implies \forall e \in U : |\langle e, e' \rangle| \leq \|e'\|_{U^\circ}
\]

\[
\implies \forall e \in E, \forall \varepsilon > 0 : \left| \frac{e}{\|e\|_U + \varepsilon}, e' \right| \leq \|e'\|_{U^\circ},
\]

i.e., \(\langle e, e' \rangle \leq \|e\|_U \|e'\|_{U^\circ}\).

**Lemma 12.** By assuming the hypotheses 1 and 2 on \(E\) and \(F\) we have \(E \bar{\otimes}_\pi F \cong \mathcal{B}_h^b(E'_c, F'_c)\).

**Proof.** The nuclearity of \(E\) implies that \(E \bar{\otimes}_\pi F = E \bar{\otimes}_\varepsilon F\). By the quasi-completeness of \(E\) and \(F\) and the strict approximation property of \(E\), we see that \(E \bar{\otimes}_\varepsilon F = E \varepsilon F = \mathcal{B}_h^b(E'_c, F'_c)\) [24, Corollaire 1, p. 47].

## 5 Decomposition of the space \(\dot{\mathcal{B}}'_{xy}\)

As mentioned in the introduction the decomposition

\[
\dot{\mathcal{B}}_{xy} = \dot{\mathcal{B}} \bar{\otimes}_\varepsilon \dot{\mathcal{B}}_y
\]

is proven in [24, Proposition 17, p. 59]. It is an analogue of A. Grothendieck’s example

\[
\mathcal{C}_0(M \times N) = \mathcal{C}_0(M) \bar{\otimes}_\varepsilon \mathcal{C}_0(N)
\]

for locally compact topological spaces \(M\) and \(N\) [6, Chap. I, p. 90]. Thus, it seems to us of its own interest to state an analogue decomposition of \(\dot{\mathcal{B}}'_{xy}\) besides its applicability in proving the equivalence \((1) \iff (3)\) hinted at in the introduction.
Proposition 13 (Decomposition of $\hat{B}'_{xy}$).

\[
\hat{B}'_{xy} = \hat{B}'_x \otimes \hat{B}'_y = \hat{B}'_z \neq \hat{B}'(\hat{B}'_y).
\]

Proof. First, we see that the isomorphism $\hat{B}'_{xy} \cong \hat{B}'_x \otimes \hat{B}'_y$ is a consequence of the sequence-space representation $\hat{B}' \cong c_0 \otimes s'$ [2, Theorem 3, p. 13], A. Grothendieck’s example above and the commutativity of the $\varepsilon$-product:

\[
\hat{B}'_{xy} \cong c_{0,jm} \otimes s'_{kl} \cong (c_{0,j} \otimes c_{0,m}) \otimes (s'_{k} \otimes s'_{l}) \cong (c_{0,j} \otimes s'_{k}) \otimes (c_{0,m} \otimes s'_{l}) \cong \hat{B}'_x \otimes \hat{B}'_y.
\]

Alternatively, for the algebraic equality $\hat{B}'_{xy} = \hat{B}'_x \otimes \hat{B}'_y$, the characterization of $\hat{B}'$ by regularization yields for $K(x, y) \in \mathcal{D}'_{xy}$:

\[
K(x, y) \in \hat{B}'_{xy} \iff K(x - z, y - w) \in \mathcal{D}'_{zw} \otimes \hat{B}_{xy}
\]

and, hence, by [24, Prop. 17, p. 59 and Prop. 28, p. 98]

\[
\iff K(x - z, y - w) \in (\mathcal{D}'_{z} \otimes \mathcal{D}'_{w}) \otimes (\hat{B}_x \otimes \hat{B}_y).
\]

By the commutativity of the $\varepsilon$-product we obtain

\[
\iff K(x - z, y - w) \in \mathcal{D}'_{z} (\hat{B}_x (\mathcal{D}'_{w} \otimes \hat{B}_y)),
\]

and by the vectorial regularization property [1, Proposition 15, p. 11]

\[
\iff K(x, y - w) \in \hat{B}'_x (\mathcal{D}'_{w} \otimes \hat{B}_y) = (\mathcal{D}'_{w} \otimes \hat{B}_y)(\hat{B}'_x)
\]

\[
\iff K(x, y) \in \hat{B}'_x \otimes \hat{B}_y = \hat{B}'_{xy} = \hat{B}'_y(\hat{B}'_x).
\]

For the topological equality we show that any 0-neighborhood in $\hat{B}'_x \otimes \hat{B}'_y = \mathcal{L}_x(\mathcal{D}_{L^1,c,x}, \hat{B}'_y)$ is a neighborhood in $\hat{B}'_{xy}$, i.e., the topology of $\hat{B}'_{xy}$ is finer than the topology on $\hat{B}'_{xy}$ induced by $\hat{B}'_x \otimes \hat{B}'_y$, i.e., Id: $\hat{B}'_{xy} \to \hat{B}'_x \otimes \hat{B}'_y$ is continuous: a base of 0-neighborhoods in $\mathcal{L}_x(\mathcal{D}_{L^1,c,x}, \hat{B}'_y)$ is given by means of bounded subsets $B_1 \subset \mathcal{D}_{L^1,c,x}, B_2 \subset \mathcal{D}_{L^1,y}$ in the form of sets

\[
W(B_1, B_2) = \{ K(x, y) \in \hat{B}'_x \otimes \hat{B}'_y : K(B_1) \subset B_2 \}
\]

\[
= \{ K \in \hat{B}'_{xy} : |K(B_1, B_2)| \leq 1 \}
\]

But the set $W(B_1, B_2)$ is a 0-neighborhood in $\hat{B}_{xy}$ because $B_1 \otimes B_2$ is a bounded set in $\mathcal{D}_{L^1,xy}$. Hereby we made use of $(\hat{B}')' \cong \mathcal{D}_{L^1}$. 

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Due to the isomorphism
\[ \hat{\mathcal{B}}_x' \hat{\otimes}_\varepsilon \hat{\mathcal{B}}_y' \cong c_{0,jm} \hat{\otimes} s_{kl} \]
the space \( \hat{\mathcal{B}}_x' \hat{\otimes}_\varepsilon \hat{\mathcal{B}}_y' \) is ultrabornological (apply [27, Proposition 2, p. 75]). As seen in the proof (2.) of Proposition 7, the space \( \hat{\mathcal{B}}_{xy}' \) is also ultrabornological. Hence, [6, Chap. I, Théorème B, p. 17] implies that \( \text{Id} \) is an isomorphism. □

**Acknowledgments.** E. A. Nigsch was supported by the Austrian Science Fund (FWF) grants P23714 and P26859.

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