ON THE NÖRLUND LOGARITHMIC MEANS WITH RESPECT TO VILENKIN SYSTEM IN THE MARTINGALE HARDY SPACE $H_1$

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Abstract. In this paper we prove and discuss a new divergence result of Nörlund logarithmic means with respect to Vilenkin system in Hardy space $H_1$.

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1. Introduction

It is well-known that (see e.g. [1] and [5]) Vilenkin systems do not form bases in the space $L_1$. Moreover, there exists a function in the dyadic Hardy space $H_1$, such that the partial sums of $f$ are not bounded in $L_1$-norm.

In [10] (see also [13]) it was proved that the following is true:

**Theorem T1:** The maximal operator $\tilde{S}^*$ defined by

$$\tilde{S}^* := \sup_{n \in \mathbb{N}} \left| S_n \right| \log (n + 1)$$

is bounded from the Hardy space $H_1$ to the space $L_1$. Here $S_n$ denotes the $n$-th partial sum with respect to the Vilenkin system. Moreover, it was proved that the rate of the factor $\log(n + 1)$ is in a sense sharp.

Móricz and Siddiqi [6] investigate the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p$ functions in norm. Fridli, Manchanda and Siddiqi [3] improved and extended the results of Móricz and Siddiqi [6] to Martingale Hardy spaces. However, the case when $\{q_k = 1/k : k \in \mathbb{N}_+\}$ was excluded, since the methods are not applicable to Nörlund logarithmic means. In [1] Gát and Goginava proved some convergence and divergence properties of Walsh-Fourier series of the Nörlund logarithmic means of functions in the Lebesgue space $L_1$. In particular, they proved that there exists an function in the space $L_1$, such that

$$\sup_{n \in \mathbb{N}} \| L_n f \|_1 = \infty.$$

Analogical result for some unbounded Vilenkin systems was proved in [2].

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In [8] (see also [13]) it was proved that there exists a martingale \( f \in H_p, \quad (0 < p < 1) \) such that

\[
\sup_{n \in \mathbb{N}} \| L_n f \|_p = \infty,
\]

Here \( L_n \) is \( n \)-th Nörlund logarithmic means with respect to Vilenkin system.

In this paper we prove an analogous result for the bounded Vilenkin systems in the case when \( p = 1 \). Moreover, we discuss boundedness of weighted maximal operators on the Hardy space \( H_1 \).

2. Preliminaries

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \).

Let \( m := (m_0, m_1, \ldots) \) denote a sequence of the positive numbers not less than 2.

Denote by

\[
Z_{m_k} := \{0, 1, \ldots, m_k - 1\}
\]

the additive group of integers modulo \( m_k, k \in \mathbb{N} \).

Define the group \( G_m \) as the complete direct product of the group \( Z_{m_k} \) with the product of the discrete topologies of \( Z_{m_k} \)'s.

The direct product \( \mu \) of the measures

\[
\mu_k (\{j\}) := 1/m_k, (j \in Z_{m_k})
\]

is the Haar measure on \( G_m \), with \( \mu (G_m) = 1 \).

If \( \sup \limits_{n} m_n < \infty \), then we call \( G_m \) a bounded Vilenkin group. If the generating sequence \( m \) is not bounded, then \( G_m \) is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of \( G_m \) are represented by sequences

\[
x := (x_0, x_1, \ldots, x_k, \ldots), \quad (x_k \in Z_{m_k}).
\]

It is easy to give a base for the neighbourhood of \( G_m : \)

\[
I_0 (x) := G_m, \quad I_n (x) := \{ y \in G_m | y_0 = x_0, \ldots, y_{n-1} = x_{n-1}, (x \in G_m, n \in \mathbb{N}) \}.
\]

Denote \( I_n := I_n (0) \), for \( n \in \mathbb{N} \) and \( \bar{I}_n := G_m \setminus I_n \).

The norm (or quasi-norm) of the spaces \( L_p (G_m) \) is defined by

\[
\| f \|_p := \left( \int_{G_m} | f |^p \, d\mu \right)^{1/p} \quad (0 < p < \infty).
\]

If we define the so-called generalized powers system based on \( m \) in the following way:

\[
M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
\]

\[
M_k := \sum_{j=0}^{k-1} M_j M_{k-j} \quad (1 \leq k \leq n).
\]
then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{k=0}^{\infty} n_k M_k \), where \( n_k \in \mathbb{Z}_{m_k} \) \((k \in \mathbb{N})\) and only a finite number of \( n_k \)'s differ from zero. Let \( |n| := \max\{k \in \mathbb{N} : n_k \neq 0\} \).

Next, we introduce on \( G_m \) an orthonormal system, which is called the Vilenkin system. First define the complex valued functions \( r_k (x) : G_m \rightarrow \mathbb{C} \), the generalized Rademacher functions, by
\[
r_k (x) := \exp \left( \frac{2\pi ix}{m} n_k \right), \quad (i^2 = -1, \ x \in G_m, k \in \mathbb{N})
\]
Now, define the Vilenkin systems \( \psi := (\psi_n) \) on \( G_m \) as:
\[
\psi_n (x) := \prod_{k=0}^{\infty} r_k^n (x), \quad (n \in \mathbb{N})
\]
Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \).

The Vilenkin systems are orthonormal and complete in \( L^2 (G_m) \) (see e.g. [1] [14]).

If \( f \in L^1 (G_m) \) we can establish Fourier coefficients, partial sums, Dirichlet kernels, with respect to Vilenkin systems in the usual manner:
\[
\hat{f} (n) := \int_{G_m} f \psi_n d\mu, \quad (k \in \mathbb{N}),
\]
\[
S_n f := \sum_{k=0}^{n-1} \hat{f} (k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (k \in \mathbb{N})
\]
Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers. The \( n \)-th Nörlund mean for the Fourier series of \( f \) is defined by
\[
t_n f = \frac{1}{l_n} \sum_{k=0}^{n-1} q_{n-k} S_k f.
\]
If \( q_k = 1/k \), then we get the Nörlund logarithmic means:
\[
L_n f := \frac{1}{l_n} \sum_{k=0}^{n} \frac{S_k f}{n-k}, \quad l_n := \sum_{k=1}^{n} \frac{1}{k},
\]
The kernel of the Nörlund logarithmic means is defined by
\[
F_n := \frac{1}{l_n} \sum_{k=0}^{n} D_k \frac{1}{n-k}.
\]
In the special case when \( \{q_k = 1 : k \in N\} \), we get the Fejér means
\[
\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f.
\]
We define \( n \)-th Fejér kernel
\[
K_n f := \frac{1}{n} \sum_{k=1}^{n} D_k.
\]
Let $f \in L_1(G_m)$. Then the maximal function is given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$ 

The Hardy martingale spaces $H_1$ consist of all martingales for which

$$\|f\|_{H_1} := \|f^*\|_1 < \infty.$$ 

It is well-known (see e.g. [9] and [16]) that if $f \in L_1$, then

$$\|f\|_{H_1} \sim \left\| \sup_{n \in \mathbb{N}} |S_{M_n} f| \right\|_1.$$ 

A bounded measurable function $a$ is a 1-atom if either $a = 1$ or

$$\int_I ad\mu = 0, \|a\|_\infty \leq \mu(I), \text{ supp}(a) \subset I.$$ 

3. Auxiliary propositions

The Hardy martingale space $H_1(G_m)$ has an atomic characterization (see [16], [17]):

**Lemma 1.** A function $f \in H_1$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of 1-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that

$$\sum_{k=0}^{\infty} \mu_k a_k = f, \text{ a.e.},$$

where

$$\sum_{k=0}^{\infty} |\mu_k| < \infty.$$ 

Moreover,

$$\|f\|_{H_1} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k| \right),$$

where the infimum is taken over all decomposition of $f$ of the form (2).

In Blahota, Gát [2] the following lemma (see Lemma 5) was proved for unbounded Vilenkin systems:

**Lemma 2.** Let $q_A = M_{2A} + M_{2A-2} + \ldots + M_0$. If $\log m_n = O(n^\beta)$, for some $0 < \delta < 1/2$, then

$$\|F_{q_A}\|_1 \geq c(\log q_A)^\beta,$$

for some $0 < \beta < 1 - \delta$.

For the proof of main result we also need the following new Lemma of independent interest:
Lemma 3. Let $G_m$ be a bounded Vilenkin system and $q_A = M_{2A} + M_{2A - 2} + \ldots + M_0$. Then
\[
\| F_{q_A} \|_1 \geq c \log q_A.
\]

Proof. During the proof we use some method of Blahota, Gát [2] (see Lemma 5) and Gát, Goginava [3] (see Lemma 2).

Set
\[
\theta_n := l_n F_n = \sum_{k=1}^{n-1} \frac{D_{n-k}}{k}.
\]

Then we have that
\[
\theta_{q_A}(x) = \sum_{k=1}^{M_{2A} + \ldots + M_0 - 1} \frac{1}{k} D_{M_{2A} + M_{2A - 2} + \ldots + M_0 - k}(x)
+ \sum_{k=M_{2A} - 2 + \ldots + M_0}^{M_{2A} + \ldots + M_0 - 1} \frac{1}{k} D_{M_{2A} + M_{2A - 2} + \ldots + M_0 - k}(x)
:= I + II.
\]

We first discuss $I$. Since $k < M_{2A} - 2 + \ldots + M_0$, then
\[
D_{M_{2A} + \ldots + M_0 - k}(x) = D_{M_{2A}}(x) + r_2 A D_{M_{2A - 2} + \ldots + M_0 - k}(x).
\]

This gives that
\[\tag{3} I = l_{q_A - 1} D_{M_{2A}}(x) + r_2 A G_{M_{2A - 2} + \ldots + M_0}(x).\]

Moreover, by using Abel transformation, we get that
\[
\begin{align*}
II & = \sum_{k=M_{2A} - 2 + \ldots + M_0}^{M_{2A} + \ldots + M_0 - 1} \frac{1}{k} D_{M_{2A} + M_{2A - 2} + \ldots + M_0 - k} \\
& = \frac{K_1}{M_{2A} + \ldots + M_0 - 1} - \frac{(M_{2A} + \ldots + M_0 - 2) K_{M_{2A} + \ldots + M_0 - 2}}{M_{2A} + \ldots + M_0 - 1} \\
& \quad + \sum_{k=M_{2A} - 2 + \ldots + M_0}^{M_{2A} + \ldots + M_0 - 2} \frac{M_{2A} + M_{2A - 2} + \ldots + M_0 - k}{k(k+1)} K_{M_{2A} + M_{2A - 2} + \ldots + M_0 - k}.
\end{align*}
\]

Since (for details see e.g. [3]) $\| K_n \|_1 \leq 2$, for all $n \in \mathbb{N}$, we obtain that
\[\tag{4} \| II \|_1 \leq \frac{2}{M_{2A} + \ldots + M_0 - 1} + \frac{2 (M_{2A} + \ldots + M_0 - 2)}{M_{2A} + \ldots + M_0 - 1} \\
+ 2 \sum_{k=M_{2A} - 2 + \ldots + M_0}^{M_{2A} + \ldots + M_0 - 2} \frac{M_{2A} + M_{2A - 2} + \ldots + M_0 - k}{k(k+1)} < c < \infty.
\]

Hence, by (3) and (4), we get that
\[
\| \theta_{q_A} \|_1 \geq \| l_{q_A - 1} D_{M_{2A}} + r_2 A \theta_{q_A - 1} \|_1 - c.
\]
We now discuss the right-hand side of this inequality, more exactly we give a lower bound for it. First, we consider the integral on the set $G_m \setminus I_{2A}$:

$$
\int_{G_m \setminus I_{2A}} |l_{qA-1} D_{M2A}(x) + r_{2A} \theta_{qA-1}(x)| \, d\mu(x)
= \int_{G_m \setminus I_{2A}} |\theta_{qA-1}(x)| \, d\mu(x) = \left\| \theta_{qA-1} \right\|_1 - \int_{I_{2A}} |\theta_{qA-1}(x)| \, d\mu(x)
= \left\| \theta_{qA-1} \right\|_1 - \frac{\theta_{qA-1}(0)}{M_{2A}}.
$$

Next, we note that on the set $I_{2A}$ we have that

$$
\int_{I_{2A}} |l_{qA-1} D_{M2A}(x) + r_{2A} \theta_{qA-1}(x)| \, d\mu(x) = l_{qA-1} - \frac{\theta_{qA-1}(0)}{M_{2A}}.
$$

It follows that

$$
(5) \quad \left\| \theta_{qA} \right\|_1 \geq \left\| \theta_{qA-1} \right\|_1 + l_{qA-1} - \frac{2\theta_{qA-1}(0)}{M_{2A}} - c.
$$

Moreover, according to simple estimation,

$$
\theta_n(0) = \sum_{k=1}^{n} \frac{n-k}{k} = n \sum_{k=1}^{n-1} \frac{1}{k} = n + 1 = nl_n + o(n).
$$

Then, since

$$
q_{A-1} \leq M_{2A-2} (1 + \frac{1}{4} + \frac{1}{16} + ...) = \frac{4}{3} M_{2A-2} \leq \frac{1}{3} M_{2A}
$$

we obtain that

$$
(6) \quad \frac{2}{l_{qA} M_{2A}} \theta_{qA-1}(0) = \frac{2 q_{A-1} l_{qA-1}}{M_{2A} l_{qA}} + o(1) \leq \frac{2 l_{qA-1}}{3 l_{qA}} + o(1).
$$

Finally, by using (5) and (6), we conclude that

$$
\|F_{qA}\|_1 = \frac{1}{l_{qA}} \left\| \theta_{qA} \right\|_1
\geq \frac{1}{l_{qA}} \left\| \theta_{qA-1} \right\|_1 + \frac{l_{qA-1}}{l_{qA}} - \frac{2 l_{qA-1}}{3 l_{qA}} - o(1)
\geq \frac{1}{l_{qA}} \left\| \theta_{qA-1} \right\|_1 + \frac{l_{qA-1}}{3 l_{qA}} - o(1)
\geq \frac{1}{l_{qA}} \left\| \theta_{qA-1} \right\|_1 + \frac{1}{6} - o(1)
\geq \frac{l_{qA-1}}{l_{qA}} \|F_{qA-1}\|_1 + \frac{1}{6} - o(1)
\geq \frac{l_{qA-1}}{l_{qA}} \|F_{qA-1}\|_1 + \frac{1}{8}
$$
By iterating this estimate we get

\[ \|F_{qA}\|_1 \geq \frac{1}{8} \sum_{k=1}^{A-1} \frac{l_{qA}}{k} \geq \frac{c}{8A} \sum_{k=1}^{A-1} k \geq cA \geq c \log qA. \]

The proof is complete. □

4. FORMULATION OF THE MAIN RESULTS

Our main results read:

**Theorem 1.** Let \( G_m \) be a bounded Vilenkin system. Then there exist a martingale \( f \in H_1 \) such that

\[ \sup_{n \in \mathbb{N}} \|L_nf\|_1 = +\infty. \]  

**Remark 1.** In one point of view Theorem 1 of Blahota and Gát [2] is better than our result (see Theorem 1) because in their result boundedness of the group \( G_m \) is not necessary and in the other point of view theorem of Blahota and Gát is a slightly weaker since they construct function in the Lebesgue space \( L_1 \).

The next result can be found in [13] and [8], respectively:

**Corollary 1.** Let \( G_m \) be a bounded Vilenkin system. Then there exists a martingale \( f \in H_1 \) such that

\[ \|L^*f\|_1 = +\infty. \]

**Corollary 2.** Let \( 0 < p < 1 \) and \( G_m \) be a bounded Vilenkin system. Then there exists a martingale \( f \in H_p \) such that

\[ \|L^*f\|_p = +\infty. \]

**Theorem 2.** a) Let \( G_m \) be a bounded Vilenkin system. Then the maximal operator

\[ \tilde{L}^* := \sup_{n \in \mathbb{N}} \frac{|L_n f|}{\log (n+1)} \]

is bounded from the Hardy space \( H_1(G_m) \) to the space \( L_1(G_m) \).

b) Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a nondecreasing function satisfying the condition

\[ \lim_{n \to \infty} \frac{\log (n+1)}{\varphi(n)} = +\infty. \]

Then there exists a martingale \( f \in H_1(G_m) \), such that the maximal operator

\[ \sup_{n \in \mathbb{N}} \frac{|L_n f|}{\varphi(n)} \]

is not bounded from the Hardy space \( H_1(G_m) \) to the Lebesgue space \( L_1(G_m) \).

**Remark 2.** Note that b) shows that the statement in part a) of Theorem 2 is in the sense sharp with respect to the logarithmic factor in [2].
5. Proofs of the Main Results

Proof of Theorem 1. Let \( \{ \alpha_k : k \in \mathbb{N} \} \) be an increasing sequence of positive integers such that

\[
\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_{\eta}} < \frac{M_{2\alpha_k}}{\alpha_k^{1/2}},
\]

(9)

\[
\frac{M_{2\alpha_{k-1}}}{\alpha_{k-1}^{1/2}} < \alpha_k.
\]

(10)

Since \( \alpha_k \to \infty \),
\[
\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_{\eta}} \to \infty, \quad \frac{M_{2\alpha_k}}{\alpha_k^{1/2}} \to \infty, \quad \text{as} \quad k \to \infty
\]

and all of three sequences are strictly increasing, we can construct lacunar increasing sequence \( \{ \alpha_k : k \in \mathbb{N} \} \) of natural number \( \mathbb{N} \), for which

\[
\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_{\eta}} < \frac{M_{2\alpha_k}}{\alpha_k^{1/2}} < \alpha_{k+1}.
\]

We also note that under condition (10) we get that

\[
\frac{M_{2\alpha_{k-1}}}{\alpha_{k-1}^{1/2}} < \alpha_k \alpha_k^{1/2} \leq \alpha_k^{3/2}
\]

and

\[
\sum_{k=0}^{\infty} \alpha_k^{1/2} \leq \alpha_0 + \sum_{k=0}^{\infty} \frac{1}{M_{2\alpha_{k-1}}} < c < \infty.
\]

(11)

Similar constructions of martingales can be found in [10]-[13] (see also [7]).

Let

\[
f(x) = \sum_{k=0}^{\infty} \lambda_k a_k,
\]

where

\[
\lambda_k = \frac{m_{2\alpha_k}}{\alpha_k}, \quad a_k = \frac{1}{m_{2\alpha_k}} \left( D_{M_{2\alpha_k+1}} - D_{M_{2\alpha_k}} \right).
\]

From (11) and Lemma 1 we can conclude that \( f \in H_1 (G_m) \). It is easy to show that

\[
\hat{f}(j) = \begin{cases} \frac{1}{\alpha_k^{1/2}}, & \text{if} \quad j \in \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}, \quad k = 0, 1, 2, \ldots, \\
0, & \text{if} \quad j \notin \bigcup_{k=1}^{\infty} \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}.
\end{cases}
\]

(12)
We can write that

\( L_{q_{\alpha k}} f = \frac{1}{l_{q_{\alpha k}}} \sum_{j=1}^{q_{\alpha k}} S_j f \)

\[ = \frac{1}{l_{q_{\alpha k}}} \sum_{j=1}^{M_{2\alpha k} - 1} \frac{S_j f}{q_{\alpha k} - j} + \frac{1}{q_{\alpha k}} \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} S_j f \]

\[ = \frac{1}{l_{q_{\alpha k}}} \sum_{j=1}^{M_{2\alpha k} - 1} \frac{S_j f}{q_{\alpha k} - j} + \frac{1}{q_{\alpha k}} \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} S_j f \]

\[ = I + II. \]

Let \( j < M_{2\alpha k} - 1 \). By combining (9) and (12) we get that

\[ |S_j f| \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha k}}^{M_{2\alpha k}+1-1} \left| \tilde{f}(v) \right| \]

\[ \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha k}}^{M_{2\alpha k}+1-1} \frac{1}{\sqrt{\alpha \eta}} \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha k}}{\sqrt{\alpha \eta}} \]

\[ \leq c \frac{2 M_{2\alpha k}}{\sqrt{\alpha k-1}}. \]

Since

\[ \sum_{j=0}^{M_{2\alpha k} - 1} \frac{1}{q_{\alpha k} - j} \leq \sum_{j=M_{2\alpha k}}^{q_{\alpha k}} \frac{1}{j} \leq \log M_{2\alpha k} - \log M_{2\alpha k-1} \]

\[ \leq \log \frac{M_{2\alpha k}}{M_{2\alpha k-1}} < c < \infty \]

according to (10) and (14) we can conclude that

\[ |I| \leq \frac{2c}{\alpha_k} \sum_{j=0}^{M_{2\alpha k} - 1} \frac{1}{q_{\alpha k} - j} \frac{M_{2\alpha k}}{\sqrt{\alpha k-1}} \]

\[ \leq \frac{c}{\alpha_k} \frac{M_{2\alpha k}}{\sqrt{\alpha k-1}} < c < \infty. \]

Hence,

\[ \|I\|_1 < c < \infty. \]
Let \( M_{2\alpha_k} \leq j \leq q_{\alpha_k} \). Then we have that
\[
S_j f = \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_k}}^{j-1} \tilde{f}(v)\psi_v + \sum_{v=M_{2\alpha_k}}^{j-1} \tilde{f}(v)\psi_v
\]
\[
= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_\eta}} \left( D_{M_{2\alpha_k}+1} - D_{M_{2\alpha_k}} \right)
\]
\[
+ \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left( D_j - D_{M_{2\alpha_k}} \right).
\]
This gives that
\[
II = \frac{1}{l_{q_{\alpha_k}} \qquad q_{\alpha_k}} \left[ \frac{1}{q_{\alpha_k} - j} \sum_{\eta=0}^{k-1} \frac{1}{\sqrt{\alpha_\eta}} \left( D_{M_{2\alpha_k}+1} - D_{M_{2\alpha_k}} \right) \right]
\]
\[
+ \frac{1}{l_{q_{\alpha_k}} \sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} \frac{1}{q_{\alpha_k} - j} \sum_{\eta=0}^{\infty} \frac{1}{\sqrt{\alpha_\eta}} = II_1 + II_2.
\]
Applying (9) in \( II_1 \) we have that
\[
\|II_1\|_1 \leq \frac{1}{l_{q_{\alpha_k}} \qquad q_{\alpha_k}} \left[ \frac{1}{q_{\alpha_k} - j} \sum_{\eta=0}^{k-1} \frac{1}{\sqrt{\alpha_\eta}} \right] \left( D_{M_{2\alpha_k}+1} - D_{M_{2\alpha_k}} \right) \|_1
\]
\[
\leq \frac{1}{l_{q_{\alpha_k}} \qquad q_{\alpha_k}} \left[ \frac{1}{q_{\alpha_k} - j} \sum_{\eta=0}^{\infty} \frac{1}{\sqrt{\alpha_\eta}} \right] < c < \infty.
\]
Since
\[
D_{j+M_n} = D_{M_n} + \psi_{M_n} D_j, \quad \text{when} \ j < M_n,
\]
for \( II_2 \) we obtain that
\[
II_2 = \frac{1}{l_{q_{\alpha_k}} \sqrt{\alpha_k}} \sum_{j=0}^{q_{\alpha_k}-1} \frac{D_j}{q_{\alpha_k} - j}
\]
\[
= \frac{l_{q_{\alpha_k}-1}}{l_{q_{\alpha_k}} \sqrt{\alpha_k}} \psi_{M_{2\alpha_k}} F_{q_{\alpha_k}-1}.
\]
Hence, if we apply Lemma 2 in the case when $\delta = 1/8$ and $\beta = 3/4$ for the sufficiently large $k$ we can estimate as follows

$$
\left\| L_{q_{0k}} f \right\|_1 \geq \| II_2 \|_1 - \| II_1 \|_1 - \| I \|_1 \\
\geq \frac{1}{2} \| II_2 \|_1 \geq \frac{c \| F_{q_{0k}-1} \|_1}{\sqrt{\alpha_k}} \\
\geq \frac{c \alpha_k^{3/4}}{\sqrt{\alpha_k}} \geq c \alpha_k^{1/4} \to \infty, \text{ as } k \to \infty.
$$

The proof is complete. \hfill \square

**Proof of Theorem 2.** It is obvious that

$$
\sup_{n \in \mathbb{N}} \frac{|L_nf|}{\log (n+1)} \leq \frac{1}{l_n} \sum_{k=0}^{n} \frac{1}{(n-k)} \sup_{k \in \mathbb{N}} \frac{|S_k f|}{\log (k+1)} \leq \sup_{n \in \mathbb{N}} \frac{|S_n f|}{\log (n+1)}.
$$

By using Theorem T1 we can conclude that

$$
\left\| \sup_{n \in \mathbb{N}} \frac{|L_nf|}{\log (n+1)} \right\|_1 \leq \left\| \sup_{n \in \mathbb{N}} \frac{|S_n f|}{\log (n+1)} \right\|_1 \leq c \| f \|_{H_1}.
$$

The proof of part a) is complete.

Under condition (8), there exists a positive integers \( \{n_k; k \in \mathbb{N}_+\} \subset \{\lambda_k; k \in \mathbb{N}_+\} \) such that

$$
\lim_{k \to \infty} \frac{n_k}{\varphi(q_{n_k})} = \infty.
$$

Set

$$
f_{n_k} = D_{M_{2n_k+1}} - D_{M_{2n_k}}.
$$

It is evident

$$
\widehat{f}_{n_k}(i) = \begin{cases} 
1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\
0, & \text{otherwise}.
\end{cases}
$$

Then we have that

$$
S_i f_{n_k} = \begin{cases} 
D_i - D_{M_{2n_k}}, & \text{if } i = M_{2n_k} + 1, \ldots, M_{2n_k+1} - 1, \\
f_{n_k}, & \text{if } i \geq M_{2n_k+1}, \\
0, & \text{otherwise}.
\end{cases}
$$

From (11) and (16) we get that

$$
\| f_{n_k} \|_{H_1} < c < \infty.
$$
It is easy to show that

\[ L_{q_k} f_{n_k} = \frac{1}{l_{q_k}} \sum_{j=1}^{q_n} \frac{S_j f_{n_k}}{q_{n_k} - j} \]

\[ = \frac{1}{l_{q_k}} \sum_{j=M_{2n_k}+1}^{q_n} \frac{D_j - D_{M_{2n_k}}}{q_{n_k} - j} \]

\[ = \frac{1}{l_{q_k}} \sum_{j=1}^{q_n-1} \frac{D_{j+M_{2n_k}} - D_{M_{2n_k}}}{q_{n_k-1} - j} \]

By applying (15) we find that

\[ \left| L_{q_k} f_{n_k} \right| = \frac{1}{l_{q_k}} \sum_{j=1}^{q_n-1} \frac{D_j}{q_{n_k} - j} \to c_{n_k} \rightarrow \infty, \text{ as } k \to \infty. \]

By now using Lemma 3 we can conclude that

\[ \int_{G_m} \left| L_{q_k} f_{n_k} \right| d\mu \geq c_{n_k} \to \infty, \text{ as } k \to \infty. \]

The proof is complete.

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