SOME COMPUTATIONS OF GENERALIZED HILBERT-KUNZ FUNCTION AND MULTIPLICITY

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Abstract. Let $R$ be a local ring of characteristic $p > 0$ which is $F$-finite and has perfect residue field. We compute the generalized Hilbert-Kunz invariant (studied in \cite{7, 8}) for certain modules over several classes of rings: hypersurfaces of finite representation type, toric rings, $F$-regular rings.

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1. Introduction

Let $R$ be a local ring of characteristic $p > 0$ which is $F$-finite and has perfect residue field. Let $M$ a finitely generated $R$-module. Let $F^n_R(M) = M \otimes_R F^n$ denote the $n$-fold iteration of the Frobenius functor given by base change along the Frobenius endomorphism. Let $\dim R = d$ and $q = p^n$. This paper constitutes a further study of the following:

$$f_{gHK}^M(n) := \ell(H^0_m(F^n(M)))$$

and

$$e_{gHK}(M) := \lim_{n \to \infty} \frac{f_{gHK}^M(n)}{p^{nd}},$$

which are called the generalized Hilbert-Kunz function and generalized Hilbert-Kunz multiplicity of $M$, respectively. These notions were first defined by Epstein-Yao in \cite{5} and were studied in details in \cite{7}. For instance, it is now known that $e_{gHK}(M)$ exists for all modules over a Cohen-Macaulay isolated singularity.

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It is a non-trivial and interesting problem to compute even the classical Hilbert-Kunz multiplicity. In this note we focus on computing $f_M^{gHK}$ and the limit $e_{gHK}(M)$ for certain modules in a number of cases: when $R$ is a normal domain of dimension $2$ (section 2), a hypersurface of finite representation type (section 3) and when $R$ is a toric ring (section 4). We also point out a connection between the generalized Hilbert-Kunz limits and tight closure theory in section 5. Namely, over $F$-regular rings, these limits detect depths of the module $M$ and all of its pull-back along iterations of Frobenius.

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2. Dimension two

In this section, we prove certain preliminary facts about behavior of $f_M^{gHK}$ when $R$ is normal and $M = R/I$ where $I$ is reflexive. We then apply them to give a formula for $e_{gHK}(R/I)$ when $I$ represents a torsion element in the class group of $R$.

Lemma 2.1. Let $R$ be a local normal domain of dimension at least 2 and $I$ a reflexive ideal that is locally free on the punctured spectrum. Then

$$\ell(H^0_m(R/I^{[q]})) = \ell(I^{[q]}/I^q) + \ell(I^q/I^{[q]})$$

Proof. Apply local cohomology functor to the sequence:

$$0 \to I^{[q]} \to R \to R/I \to 0.$$ 

Note that $H^0_m(R/I^{[q]}) = 0$ and $\ell(I^{[q]}/I^q), \ell(I^q/I^{[q]}) < \infty$ as the ideals coincide on the punctured spectrum. 

Proposition 2.2. Let $R$ be a local normal domain of dimension 2 and $I$ be a reflexive ideal. Then $e_{gHK}(R/I) = 0$ if and only if $I$ is principal.

Proof. Only one direction needs to be checked. Suppose $e_{gHK}(R/I) = 0$. Let $\mu()$ denote the minimal number of generators of an $R$-module. We have that $\ell(H^0_m(R/I[^q])) \geq \ell(H^0_m(R/I^q))$ by Lemma 2.1. It follows that $\limsup_{\mu} \frac{\ell(H^0_m(R/I^{[q]}))}{q^2} = 0$, so $I$ has analytic spread one by [10, Theorem 4.7], thus $[I]$ is principal.

Remark 2.3. The number $\limsup_{\mu} \frac{\ell(H^0_m(R/I^{[q]}))}{q^2} = 0$ is known as the epsilon multiplicity of $I$, $\epsilon(I)$. It has now been proved to exist as a limit under mild conditions, it see [4]. Lemma 2.1 says that $e_{gHK}(R/I) \geq \epsilon(I)$.

Lemma 2.4. Let $R$ be a local normal domain of dimension 2 and $I$ a reflexive ideal. Assume that $[I]$ is torsion in $\text{Cl}(R)$. Then $\ell(H^0_m(R/I^n))$ has quasi-polynomial behavior for $n$ large enough.

Proof. Let $r$ be some integer such that of $r[I] = 0$ in $\text{Cl}(R)$. Then $I^r = I_1 \cap I_2$, where $I_1$ is the determinant of $I^r$ and thus principal, and $I_2$ is $m$-primary. Let $I_1 = (x)$ we then have
$I^r = xJ$ where $J = I_2 : x$. Note that $J$ is $m$-primary. For any integer $n$, let $n = ar + b$. We have that $I^n = I^{ar+b} = x^aJ^b$. Then

$$H^0_m(R/I^n) \cong H^1_m(I^n) \cong H^1_m(J^bI^b) \cong H^0_m(R/J^aI^b)$$

To calculate the last term we use:

$$0 \to I^b/J^aI^b \to R/J^aI^b \to R/I^b \to 0$$

The leftmost term has finite length, thus what we want is equal to $\ell(I^b/J^aI^b) + \ell(H^0_m(R/I^b))$. Since $b$ is periodic and $a$ grows linearly with $n$, what we claimed follows. Note that the limit if $\ell(H^0_m(R/I^n))/n^2$ is equal to $e(J)/2r^2$.

\[\square\]

3. The finite representation type case

We now describe how to compute $e_{gHK}(M)$ when $M$ is a module of positive depth over a Gorenstein local ring of finite Cohen-Macaulay type. We first need some definitions.

**Definition 3.1.** Let $R$ be a Gorenstein complete local ring of finite Cohen-Macaulay type with perfect residue field (in particular, $R$ must be a hypersurface singularity, see [15]). Let $X_1, \ldots, X_n$ be all the indecomposable non-free Cohen-Macaulay modules.

We define the stable Cohen-Macaulay type of $M$ to be the vector $(u_1, \ldots, u_n)$ with $X = \oplus X_i^{u_i}$, here $X$ is a Cohen-Macaulay approximation $0 \to M \to N \to X \to 0$ where pd$_R N < \infty$. This is well-defined since $R$ is complete. As $R$ is also a hypersurface, by taking syzygy one can see that $X$ is stably equivalent to the e-syzygy of $M$ where $e = 2 \dim R$.

We also define $v_j = \lim_{n \to \infty} \frac{\#(nR, X_j)}{q^n}$, where $\#(nR, X_j)$ is the number of copies of $X_j$ in the decomposition of $nR$. This limit exists by [13, 14].

**Proposition 3.2.** Using the set up of Definition 3.1. Let $M$ be an $R$-module of positive depth. One has:

$$e_{gHK}(M) = \sum_{1 \leq i, j \leq n} u_i v_j \ell(Tor^R_1(X_i, X_j)) = \sum_{1 \leq i, j \leq n} u_i v_j \ell(Tor^R_2(X_i, X_j))$$

**Proof.** Take a MCM approximation $0 \to M \to N \to X \to 0$ and tensor with $nR$, we get

$$0 \to Tor^R_1(X, nR) \to M \otimes nR \to N \otimes nR$$

Note that depth$(N \otimes nR) = \text{depth } N = \text{depth } M > 0$ and Tor$_1^R(X, nR)$ has finite length as $R$ must have isolated singularity, we get that $\ell(H^0_m(M \otimes nR) = \ell(\text{Tor}^R_1(X, nR))$. The first equality is now obvious.

For the second equality we just need that $\ell(\text{Tor}^R_1(X, nR)) = \ell(\text{Tor}^R_2(X, nR))$ by [5].

\[\square\]

**Example 3.3.** Let $R = k[[x, y, z]]/(xy - z^r)$ where $k$ is a perfect field of characteristic $p > 0$. $R$ has finite type with $X_i = (x, z^i)$, $1 \leq i \leq r - 1$. It is not hard to check that $\ell(\text{Tor}^R_1(X_i, X_j)) = \min\{i, j, r - i, r - j\}$. Also, it is known that $v_j = 1/r$. So for a module $M$ with positive depth and stable CM type $(u_1, \ldots, u_n)$ one gets:

$$e_{gHK}(M) = \frac{1}{r} \sum_{1 \leq i, j \leq r - 1} u_i \min\{i, j, r - i, r - j\}$$
4. THE TORIC CASE

In this section, we show how to compute the generalized Hilbert-Kunz multiplicity of $R/I$, where $R$ is a normal toric ring and $I$ is an of $R$ generated by monomials. We fix the following notation.

**Notation 4.1.** Let $k$ be a field of characteristic $p$ and $M \cong \mathbb{Z}^d$ be a lattice and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let $\sigma \subset M_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and $R = k[\sigma \cap M] = k[\{X^m \mid m \in \sigma \cap M\}]$ be a normal toric ring. Let $I = (X^{m_1}, \ldots, X^{m_s})$ be a monomial ideal of $R$. We put $\Gamma_I$ the convex hull of $\bigcup_{i=1}^s [m_i + \sigma]$ and $W_I = \bigcup_{i=1}^s [m_i + \sigma]$. We define a subset $\text{LC}_I$ of $M_{\mathbb{R}}$ by

$$m \in \text{LC}_I \iff \lfloor m + \sigma \rfloor \cap \lfloor m + \sigma \rfloor \cap W_I \text{ has finite volume}$$

**Proposition 4.2.** With the notation above:

1. For $m \in M$, $x^m \in I$ : $J^\infty$ (where $J$ is the maximal ideal) if and only if $m \in \text{LC}_I$.
2. $\text{LC}_{I^{(q)}} = q\text{LC}_I$ and $W_{I^{(q)}} = qW_I$.
3. $\text{LC}_I \setminus W_I$ is a bounded region in $M_{\mathbb{R}}$.

**Proof.** It is clear from the definition that $m \in \text{LC}_I$ iff $x^m J^t \subseteq I$ for $t \gg 0$. (2) is also clear. For (3), note that the region in question is defined by finitely many half planes. Thus, if it has infinite volume, there will be $q$ big enough such that $\text{LC}_I \setminus W_I$ contains infinitely many points in $\frac{1}{q} \mathbb{Z}^d$. In other words, there are infinitely many integral points in $\text{LC}_{I^{(q)}} \setminus W_{I^{(q)}}$. But the integral points in that region simply correspond to the monomials in $H^0_m(R/I^{(q)})$, a contradiction. □

**Remark 4.3.** In the picture below, $\text{LC}_I \setminus W_I$ can be seen as the combination of the red and green regions.

**Theorem 4.4.** Let $R$ and $I$ be as above. Then $e_{gHK}(R/I) = \text{vol}(\text{LC}_I \setminus W_I)$. In particular, $e_{gHK}(R/I) \in \mathbb{Q}$.

**Proof.** The previous Proposition tells us that $m \in \text{LC}_{I^{(q)}}$ iff $m/q \in \text{LC}_I$, from which the result follows. □

We demonstrate the ideas of the last Theorem with two concrete examples.
Proposition 4.5. Let \( R = k[[x^r, x^{r-1}y, \ldots, y^r]] \) be isomorphic to the \( r \)-Veronese of \( k[[x, y]] \) and \( I_m = (x^r, x^{r-1}y, \ldots, x^{r-m}y^m) \subset R \) be the one of the reflexive ideals of \( R \) (note that \( I \) corresponds to the element \( m \in \mathbb{Z}/(r) \equiv \text{Cl}(R) \)). Then
\[
e_{\text{gHK}}(R/I_m) = \frac{m(m+1)}{2r}
\]

Proof. Let \( I = I_m \). Note that \( I^r = (x^r, x^{r-1}y, x^{r-m}y^m) = x^{(r-m)m} \). We use Lemmas 2.1 and 2.4 to calculate the relevant lengths. It follows that \( \lim \ell(R^m(R/I^q))/(q^d) = e(m^m)/2r^2 = m^2/2r \).

The second part involves \( \ell(I^q/I^{[d]}q) \). The monomials that are in \( I^q \) but not in \( I^{[d]}q \) are contained in the right triangles whose hypotenuses are the intervals \((iq, (r-i)q), ((i-1)q, (r-i+1)q)\) with \( i = r, \ldots, r-m+1 \). It is clear that the number of such monomials, which is the length we want, is of order \( mq^2/2r \). So the second term contribute \( m/2r \) to the limit. We conclude that:
\[
e_{\text{gHK}}(R/I) = m^2/2r + m/2r = m(m+1)/2r
\]

\( \square \)

Proposition 4.6. Let \( R = k[[x, y, z]]/(xy - z^r) \) and \( I_m = (x, z^m) \subset R (m < r) \). Then
\[
e_{\text{gHK}}(R/I_m) = \frac{m(r-m)}{r}
\]

Proof. Let \( I = I_m \). Note that \( I^r = x^m(x^{r-m}, x^{r-m-1}z^m, \ldots, z^{r-m}y^{m-1}, y^m) = x^mJ \). We again use Lemmas 2.1 and 2.4.

If we assign point \( x \to (r, -1), y \to (0, 1), z \to (1, 0) \), the points corresponding to \((x^{r-m}, x^{r-m-1}z^m, \ldots, z^{r-m}y^{m-1}, y^m)\) are \((r(r-m), -(r-m)), \ldots, (r-m, m-1), (0, m)\), lying on a line of slope \(1/(r-m)\). This line and the cone defined by \( x \geq 0 \) and \( y \geq -x/r \) form a triangle of area \( r(r-m)/2 \), this means the multiplicity of the ideal \( J \) is \( rm(r-m) \). Hence, \( \lim \ell(R^m(R/I^q))/(q^d) = m(r-m)/2r \).

On the other hand, \( \ell(I^q/I^{[d]}q) \) corresponds to the triangle whose vertices are \((qm, 0), (qr, -q)\) and \((qr, -q(r-m)/r)\). The area is \( m(r-m)q^2/r \). Summing up we have \( e_{\text{gHK}}(R/I) = m(r-m)/r \).

\( \square \)

5. The F-regular case

Lastly, we study a connection between generalized Hilbert-Kunz multiplicity and tight closure theory. We first recall the following criterion for tight closure due to Hochster-Huneke.

Lemma 5.1. Let \( R \) be equidimensional and either complete or essentially of finite type over a field and \( N \subseteq L \subseteq G \) be finitely generated \( R \)-modules such that \( L/N \) has finite length. Then \( e_{\text{gHK}}(G/N) \geq e_{\text{gHK}}(G/L) \), and equality occurs if and only if \( L \subseteq N^*_G \).

We now want to show:

Proposition 5.2. Let \( R \) be F-regular (i.e, all ideals are tightly closed) and \( M \) be a finitely generated \( R \)-module. The following are equivalent:

1. \( e_{\text{gHK}}(M) = 0 \).
(2) depth $F^n(M) > 0$ for all $n \geq 0$.

Proof. We only need to show (1) implies (2). It is harmless to complete $R$ and $M$ (see Exercise 4.1 in [9]). Suppose there exists $n \geq 0$ such that depth $F^n(M) = 0$, we need to prove that $e_{gHK}(M) > 0$. Replacing $M$ by $F^n(M)$ if necessary, we may assume depth $M = 0$. Now take a short exact sequence $0 \to N \to G \to M \to 0$ where $G$ is free. Let $x \in G$ represent an element in the socle of $M$, we know that $L = (N,x) \not\subseteq N = N_G^*$, thus $e_{gHK}(M) > e_{gHK}(G/L) \geq 0$ by Lemma 5.1. □

Remark 5.3. When $R$ is strongly $F$-regular, one can prove the above Proposition as follows. The assumption means that we have decompositions of $R$-modules $nR = R^{a_i} \oplus M_q$ and $c = \lim_{n \to \infty} \frac{a_n}{q^n} > 0$. Then it is clear that $e_{gHK}(M) \geq c\ell(H^0_m(M))$, so the non-trivial direction (1) implies (2) is now easy to see.

Corollary 5.4. Let $R$ be $F$-regular of dimension at least 2 and $I$ be a reflexive ideal that is locally free on the punctured spectrum. Then $e_{gHK}(R/I) = 0$ if and only if $I$ is principal.

Proof. By Proposition 5.2 we only need to show that depth $R/I^{[q]} = 0$ for some $q$. But suppose it is not the case, then Lemma 2.1 implies that $I^q = I^{[q]}$ for all $q$, thus the analytic spread is one. □

Before moving on we recall the following limits studied in [7]. Let $i \geq 0$ be an integer. Let

$$e_{gHK}^i(M) := \lim_{n \to \infty} \frac{\ell(H^i_m(F^n(M)))}{p^{nd}}$$

Let IPD($M$) denote the set of prime ideals $p$ such that pd$_{R_p} M_p = \infty$.

Lemma 5.5. Let $R$ be of depth $d$. Let $N$ be an $R$-module such that IPD($N$) $\subseteq \{m\}$. Let $M$ be a $t$-syzygy of $N$. Then $H^i_m(F^n(M)) \cong H^i_m(F^n(N))$ for $0 \leq i \leq d - t - 1$.

Proof. We begin with tensoring the exact sequence $0 \to \text{syz} N \to F \to N \to 0$ with $nR$ to get

$$0 \to \text{Tor}^R_1(N, nR) \to F^n(\text{syz} N) \to F^n(N) \to 0$$

which we break into:

$$0 \to \text{Tor}^R_1(N, nR) \to F^n(\text{syz} N) \to C \to 0$$

and

$$0 \to C \to F^n(F) \to F^n(N) \to 0$$

Note that Tor$_1^n(N, nR)$ has finite length, so the long sequence of local cohomology for the first sequence gives $H^i_m(F^n(\text{syz} N)) \cong H^i_m(C)$ for $i > 0$. For the second sequence, we have that $H^i_m(F^n(N)) \cong H^i_{m+1}(C)$ for $0 \leq i \leq d - 2$. Thus

$$H^i_m(F^n(N)) \cong H^{i+1}_m(F^n(\text{syz} N))$$

for $0 \leq i \leq d - 2$. A simple induction finishes the proof. □
Theorem 5.6. Let $R$ be $F$-regular of dimension $d \geq 2$ and $0 \leq a \leq b \leq d - 1$ be integers. Let $M$ be an $R$-module that is locally free on the punctured spectrum and depth $M \geq a$. The following are equivalent:

1. $e_{gHK}^i(M) = 0$ for $a \leq i \leq b$.
2. $H_m^n(F^a(M)) = 0$ for all $a \leq i \leq b$ and all $n \geq 0$.

Proof. We use induction on $b - a$. It is enough to prove the case $b = a$, since the conclusion implies that depth $M \geq a + 1$, and we can replace $a$ by $a + 1$. As depth $M \geq a$, we can pushforward $a$ times and write $M$ as syz$^a N$ for some module $N$. Proposition 5.2 and Lemma 5.5 finish the proof.

Corollary 5.7. Let $R$ be $F$-regular and $I$ be a reflexive ideal that is locally free on the punctured spectrum. If $[I]$ is torsion in the class group of $R$ then $I$ is Cohen-Macaulay.

Proof. We can assume $R$ has dimension is at least 3. We just note that the double dual of $F^n(I)$, $F^n(I)^{**}$, is isomorphic to $I^{(q)}$, which corresponds to the element $q[I]$ in $\text{Cl}(R)$. The natural map $F^n(I) \to F^n(I)^{**}$ has kernel and cokernel of finite length. It follows that $H_m^n(F^n(I)) \cong H_m^n(F^n(I)^{**}) \cong H_m^n(I^{(q)})$ for $i \geq 2$. But the isomorphism classes of $I^{(q)}$ will be periodic as $[I]$ is torsion. Thus $e_{gHK}^i(I) = 0$ for $2 \leq i \leq d - 1$, and by Theorem 5.6 $H_m^n(I) = 0$ for $2 \leq i \leq d - 1$, which is all we need to prove.

Remark 5.8. If the order of $[I]$ is prime to the characteristic of $R$, the result was first proved, without condition that $I$ is locally free on the punctured spectrum, for strongly $F$-regular rings in [16]. The condition on the order of $[I]$ was removed in [12, Corollary 3.3]. All of these results will be extended in [6], with a more direct approach.

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