STRONG SUMMABILITY OF TWO-DIMENSIONAL VILENKIN–FOURIER SERIES

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We study the exponential uniform strong summability of two-dimensional Vilenkin–Fourier series. In particular, it is proved that the two-dimensional Vilenkin–Fourier series of a continuous function \( f \) is uniformly strongly summable to a function \( f \) exponentially in the power \( 1/2 \). Moreover, it is proved that this result is best possible.

1. Introduction

It is known that there exist continuous functions for which the trigonometric (Walsh) Fourier series do not converge. However, in 1905, Fejér proved [2] that the arithmetic means of the differences between the function and its Fourier partial sums uniformly converge to zero. The problem of strong summation was initiated by Hardy and Littlewood [16]. They generalized Fejér’s result by showing that the strong means also uniformly converge to zero for any continuous function. The investigation of the rate of convergence of strong means was originated by Alexits [1]. Numerous papers closely related to the problems of strong approximation and summability were published. Note that various significant results are due to Leindler [17–19], Totik [26–28], Gogoladze [9], and Goginava, Gogoladze, Karagulyan [13]; see also the monograph by Leindler [20].

The results on strong summation and approximation of trigonometric Fourier series have been extended to several other orthogonal systems. Thus, for the Walsh system, we refer the reader to [3–7, 11–13, 21–24] and, for the Ciselski system, to Weisz [29, 30]. The summability of multiple Walsh–Fourier series was investigated in [14, 15, 31].

Fridli and Schipp [5] proved that the following assertion is true:

**Theorem FS.** Let \( \Phi \) be the trigonometric (or Walsh system) and let \( \psi \) be a monotonically increasing function defined on \([0, \infty)\) for which \( \lim_{u \to 0^+} \psi(u) = 0 \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi((|S_k \Phi f(x) - f(x)|) = 0, \quad f \in C(G_2),
\]

if and only if there exists \( A > 0 \) such that \( \psi(t) \leq \exp(At), 0 \leq t < \infty \). Moreover, the convergence is uniform in \( x \), where \( S_k \Phi f \) denotes the \( k \)-th partial sum of the Fourier series of \( f \) in the orthonormal system \( \Phi \) and \( G_2 \) refers to the Vilenkin group \( G_m \) with \( m = (2, 2, \ldots) \).

In the present paper, we study the exponential uniform strong summability of two-dimensional Vilenkin–Fourier series. In particular, it is proved that the two-dimensional Vilenkin–Fourier series of a continuous function \( f \) is uniformly strongly summable to the function \( f \) exponentially in the power \( 1/2 \). Moreover, it is shown that this result is best possible.

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Let \( \mathbb{N}_+ \) denote the set of positive integers; \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \). Also let \( m := (m_0, m_1, \ldots) \) denote a sequence of positive integers not smaller than 2. By \( Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \) we denote an additive group of integers modulo \( m_k \). We define a group \( G_m \) as the complete direct product of the groups \( Z_{m_j} \) with the product of discrete topologies of \( Z_{m_j} \)'s. The direct product \( \mu \) of measures
\[
\mu_k(\{j\}) := \frac{1}{m_k}, \quad j \in Z_{m_k},
\]
is the Haar measure on \( G_m \) with \( \mu(G_m) = 1 \). If the sequence \( m \) is bounded, then \( G_m \) is called a bounded Vilenkin group. The elements of \( G_m \) can be represented in the form of sequences
\[
x := (x_0, x_1, \ldots, x_j, \ldots), \quad x_j \in Z_{m_j},
\]
The group operation \( + \) in \( G_m \) is defined as follows:
\[
x + y = (x_0 + y_0, x_1 + y_1, \ldots, x_k + y_k, \ldots),
\]
where \( x = (x_0, \ldots, x_k, \ldots) \) and \( y = (y_0, \ldots, y_k, \ldots) \in G_m \). The inverse of \( + \) is denoted by \( - \).

It is easy to define a base for the neighborhoods of \( G_m \):
\[
I_0(x) := G_m, \\
I_n(x) := \{ y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1} \}
\]
for \( x \in G_m, \ n \in \mathbb{N} \). We also define \( I_n := I_n(0) \) for \( n \in \mathbb{N}_+ \) and set \( e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m \), where the \( n \)th coordinate is equal to 1 and the remaining coordinates to zeros \( (n \in \mathbb{N}) \).

If we define the so-called generalized number system based on \( m \) in the following way: \( M_0 := 1, \ M_{k+1} := m_k M_k, \ k \in \mathbb{N}, \) then every \( n \in \mathbb{N} \) can be uniquely expressed in the form
\[
n = \sum_{j=0}^{\infty} n_j M_j,
\]
where \( n_j \in Z_{m_j}, \ j \in \mathbb{N}_+, \) and only finitely many \( n_j \) differ from zero. We use the following notation: For \( n > 0 \), let \( |n| := \max\{k \in \mathbb{N} : n_k \neq 0\} \) (this means that \( M_{|n|} \leq n < M_{|n|+1} \)).

Further, on \( G_m \), we introduce an orthonormal system, which is called the Vilenkin system. First, we define complex-valued generalized Rademacher functions \( r_k(x) : G_m \to \mathbb{C} \) as follows:
\[
r_k(x) := \exp \frac{2\pi i x_k}{m_k}, \quad x^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.
\]
Thus, the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) can be defined as follows:
\[
\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.
\]
Specifically, for \( m \equiv 2 \), this system is called the Walsh–Paley system.
The Vilenkin system is orthonormal and complete in $L_1(G_m)$. It is well known that
\[ \psi_n(x)\psi_n(y) = \psi_n(x + y), \quad |\psi_n(x)| = 1, \quad n \in \mathbb{N}, \quad \psi_n(-x) = \overline{\psi_n(x)} \]
(see [25]).

We now introduce analogs of the ordinary definitions used in the Fourier analysis. If $f \in L_1(G_m)$, then we can get the following definitions in the ordinary way:

– the Fourier coefficients $\hat{f}(k) := \int_{G_m} f \overline{\psi_k} \, d\mu$, $k \in \mathbb{N}$;

– the partial sums $S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k$, $n \in \mathbb{N}_+$, $S_0 f := 0$;

and

– the Dirichlet kernels $D_n := \sum_{k=0}^{n-1} \psi_k$, $n \in \mathbb{N}_+$.

Recall that
\[
D_{M_n}(x) = \begin{cases} 
M_n & \text{if } x \in I_n, \\
0 & \text{if } x \in G_m \setminus I_n,
\end{cases} \quad (1)
\]
\[
D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{q=m_j-n_j}^{m_j-1} r_j^q(x), \quad f \in L_1(G_m), \quad n \in \mathbb{N}. \quad (2)
\]

It is well known that
\[
S_n f(x) = \int_{G_m} f(t) D_n(x - t) \, d\mu(t).
\]

Further, we introduce the notation from the theory of two-dimensional Vilenkin system. We fix $d \geq 1$, $d \in \mathbb{N}_+$. For the Vilenkin group $G_m$, let $G_m^d$ be the Cartesian product $G_m \times \ldots \times G_m$ of its $d$ copies. We denote the product measure $\mu \times \ldots \times \mu$ by $\mu$. The rectangular partial sums of the two-dimensional Vilenkin–Fourier series are defined as follows:
\[
S_{M,N}(f; x, y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i, j) \psi_i(x) \psi_j(y),
\]
where the number
\[
\hat{f}(i, j) = \int_{G_m \times G_m} f(x, y) \overline{\psi_i(x)} \overline{\psi_j(y)} \, d\mu(x, y)
\]
is said to be the $(i, j)$th Vilenkin–Fourier coefficient of the function $f$. 
Denote
\[ S_n^{(1)}(f; x, y) := \sum_{l=0}^{n-1} \hat{f}(l, y) \psi_l(x), \]
\[ S_m^{(2)}(f; x, y) := \sum_{r=0}^{m-1} \hat{f}(x, r) \psi_r(y), \]
where
\[ \hat{f}(l, y) = \int_{G_m} f(x, y) \psi_l(x) \, d\mu(x) \]
and
\[ \hat{f}(x, r) = \int_{G_m} f(x, y) \psi_r(y) \, d\mu(y). \]

2. Best Approximation

We denote by \( E_{l,r}(f) \) the best approximation of a function \( f \in C(G^2_m) \) by Vilenkin polynomials of degree \( \leq l \) in the variable \( x \) and of degree \( \leq r \) in the variable \( y \). Also let \( E_{l,l}^{(1)}(f) \) be a partial best approximation of a function \( f \in C(G^2_m) \) by the Vilenkin polynomials of degree \( \leq l \) of a variable \( x \) whose coefficients are continuous functions of the remaining variable \( y \). Similarly, we can define \( E_{l,r}^{(2)}(f) \).

Let
\[ M_L \leq l < M_{L+1}, \quad M_R \leq r < M_{R+1}, \]
and
\[ E_{M_L,M_R}(f) := \| f - T_{M_L,M_R} \|_C, \]
where \( T_{M_L,M_R} \) is the Vilenkin polynomial of the best approximation for the function \( f \). Since [see (1)]
\[ \| S_{M_L,M_R}(f) \|_C \leq \| f \|_C, \]
we can write
\[ |S_{l,r}(f; x, y) - f(x, y)| \]
\[ \leq |S_{l,r}(f - S_{M_L,M_R}(f); x, y)| + \| S_{M_L,M_R}(f) - f \|_C \]
\[ \leq |S_{l,r}(f - S_{M_L,M_R}(f); x, y)| + \| S_{M_L,M_R}(f - T_{M_L,M_R}) \|_C + \| f - T_{M_L,M_R}f \|_C \]
\[ \leq |S_{l,r}(f - S_{M_L,M_R}(f); x, y)| + 2E_{M_L,M_R}(f). \] (3)
We can now show that the following inequality holds:

\[ E_{M_L,M_R}(f) \leq 2E^{(1)}_{M_L}(f) + 2E^{(2)}_{M_R}(f). \]  \hspace{1cm} (4)

Indeed, we have

\[ E_{M_L,M_R}(f) \leq \| f - S_{M_L,M_R}(f) \|_C = \| f - S^{(1)}_{M_L}(S^{(2)}_{M_R}(f)) \|_C \]
\[ \leq \| f - S^{(1)}_{M_L}(f) \|_C + \| S^{(1)}_{M_L}(S^{(2)}_{M_R}(f) - f) \|_C \]
\[ \leq \| f - S^{(1)}_{M_L}(f) \|_C + \| S^{(2)}_{M_R}(f) - f \|_C. \]

Let \( T^{(1)}_{M_L}(x, y) \) be a polynomial of the best approximation \( E^{(1)}_{M_L}(f) \). Then

\[ \| S^{(1)}_{M_L}(f) - f \|_C \leq \| f - T^{(1)}_{M_L} \|_C + \| S^{(1)}_{M_L}(f - T^{(1)}_{M_L}) \|_C \]
\[ \leq 2 \| f - T^{(1)}_{M_L} \|_C = 2E^{(1)}_{M_L}(f). \]

Similarly, we can show that

\[ \| S^{(2)}_{M_R}(f) - f \|_C \leq 2E^{(2)}_{M_R}(f). \]

Combining (5)–(7), we obtain (4).

It is easy to see that

\[ \| f - S_{M_L,M_R}(f) \|_C \leq 2E_{M_L,M_R}(f). \]

3. Main Results

**Theorem 1.** Let \( f \in C(G^2_{m}) \). Then the inequality

\[ \| \frac{1}{nm} \sum_{l=1}^{n} \sum_{r=1}^{m} (e^{A|S_{l,r}(f)|} - |S_{l,r}(f)| - 1) \|_C \]
\[ \leq c(f,A) \frac{1}{n} \sum_{l=1}^{n} \sqrt{E^{(1)}_{l}(f)} + c(f,A) \frac{1}{m} \sum_{r=1}^{m} \sqrt{E^{(2)}_{r}(f)} \]

is satisfied for any \( A > 0 \), where \( c(f,A) \) is a positive constant depending on \( A \) and \( f \).

We say that a function \( \psi \) belongs to the class \( \Psi \) if it increases on \([0, +\infty)\) and

\[ \lim_{u \to 0} \psi(u) = \psi(0) = 0. \]
Theorem 2.

(a) Let \( \varphi \in \Psi \) and let the inequality

\[
\limsup_{u \to \infty} \frac{\varphi(u)}{\sqrt{u}} < \infty
\]  

be true. Then, for any function \( f \in C(G_m^2) \), the equality

\[
\lim_{n,m \to \infty} \left\| \frac{1}{nm} \sum_{l=1}^{n} \sum_{r=1}^{m} \left( \varphi(\Delta_{l,r}(f)) - 1 \right) \right\|_C = 0
\]

is satisfied.

(b) For any function \( \varphi \in \Psi \) satisfying the condition

\[
\limsup_{u \to \infty} \frac{\varphi(u)}{\sqrt{u}} = \infty
\]

there exists a function \( F \in C(G_m^2) \) such that

\[
\limsup_{u \to \infty} \frac{1}{m^2} \sum_{l=1}^{m} \sum_{r=1}^{m} \varphi(|\Delta_{l,r}(F;0,0) - F(0,0)|) = +\infty.
\]

4. Auxiliary Results

Lemma 1 [8]. Let \( p \in \mathbb{N}_+ \). Then

\[
\sup_n \left( \int_{G_m^p} \frac{1}{M_n} \left| \sum_{l=M_n}^{M_{n+1} - 1} \prod_{k=1}^{p} D_l(s_k) \right| d\mu(s_1, \ldots, s_p) \right)^{1/p} \leq cp,
\]

where \( c \) is a positive constant.

Lemma 2 [9]. Let \( \varphi, \psi \in \Psi \) and let the equality

\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{l=1}^{n} \sum_{r=1}^{m} \psi(|\Delta_{l,r}(f; x, y) - f(x, y)|) = 0
\]

be satisfied at the point \((x_0, y_0)\) or uniformly on the set \( E \subset I^2 \). If

\[
\limsup_{u \to \infty} \frac{\varphi(u)}{\psi(u)} < \infty,
\]
then the equality
\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{l=1}^{m} \sum_{r=1}^{n} \varphi \left( |S_{l,r}(f; x, y) - f(x, y)| \right) = 0
\]
is satisfied at the point \((x_0, y_0)\) or uniformly on the set \(E \subset I^2\).

**Lemma 3.** Let \(p > 0, A, B \in \mathbb{N}\). Then

\[
\left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f; x, y)|^p \right\}^{1/p} \leq c \|f\|_C (p + 1)^2. \tag{11}
\]

**Proof.** Since

\[
\left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f; x, y)|^p \right\}^{1/p} \leq \left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f; x, y)|^{p+1} \right\}^{1/(p+1)},
\]

without loss of generality we can assume that \(p = 2^m, m \in \mathbb{N}_+\). Thus, we can write

\[
|S_{n,l}(f; x, y)|^2 = S_{n,l}(f; x, y) \overline{S}_{n,l}(f; x, y)
\]

\[
= \int_{G_n^2} f(x - s_1, y - t_1) D_n(s_1)D_l(t_1) \, d\mu(s_1, t_1)
\]

\[
\times \int_{G_n^2} \overline{f}(x - s_2, y - t_2) \overline{D}_n(s_2)\overline{D}_l(t_2) \, d\mu(s_2, t_2)
\]

\[
= \int_{G_n^2} f(x - s_1, y - t_1) D_n(s_1)D_l(t_1) \, d\mu(s_1, t_1)
\]

\[
\times \int_{G_n^2} \overline{f}(x + s_2, y + t_2) D_n(s_2)D_l(t_2) \, d\mu(s_2, t_2)
\]

\[
= \int_{G_n^2} f(x - s_1, y - t_1) \overline{f}(x + s_2, y + t_2)
\]

\[
\times D_n(s_1)D_n(s_2)D_l(t_1)D_l(t_2) \, d\mu(s_1, t_1, s_2, t_2).
\]
Hence, we get

\[ |S_{n,l}(f; x, y)|^p = \left( |S_{n,l}(f; x, y)|^2 \right)^{p/2} \]

\[ = \left( \int_{G_n^p} f(x - s_1, y - t_1) f(x + s_2, y + t_2) \right)^{p/2} \]

\[ \times D_n(s_1) D_n(s_2) D_l(t_1) D_l(t_2) d\mu(s_1, t_1, s_2, t_2) \]

\[ = \int_{G_n^p} \prod_{k=1}^{p/2} f(x - s_{2k} - 1, y - t_{2k} - 1) \]

\[ \times \prod_{r=1}^{p/2} f(x + s_{2r}, y + t_{2r}) \prod_{i=1}^p D_n(s_i) \prod_{j=1}^p D_l(t_j) d\mu(s_1, t_1, \ldots, s_p, t_p), \]

\[ \left\{ \frac{1}{MA MB} \sum_{n=M_A}^{MA+1-1} \sum_{l=M_B}^{MB+1-1} |S_{n,l}(f; x, y)|^p \right\}^{1/p} \]

\[ \leq \left( \int_{G_n^p} \prod_{k=1}^{p/2} \left| f(x - s_{2k} - 1, y - t_{2k} - 1) \right| \prod_{r=1}^{p/2} \left| f(x + s_{2r}, y + t_{2r}) \right| \right)^{p/2} \]

\[ \times \frac{1}{MA MB} \left| \sum_{n=M_A}^{MA+1-1} \sum_{l=M_B}^{MB+1-1} \prod_{i=1}^p D_n(s_i) \prod_{j=1}^p D_l(t_j) \right| d\mu(s_1, t_1, \ldots, s_p, t_p) \]

\[ \leq \|f\|_C \left( \int_{G_n^p} \frac{1}{MA} \left| \sum_{n=M_A}^{MA+1-1} \prod_{i=1}^p D_n(s_i) \right| d\mu(s_1, \ldots, s_p) \right)^{1/p} \]

\[ \times \left( \int_{G_n^p} \frac{1}{MB} \left| \sum_{l=M_B}^{MB+1-1} \prod_{j=1}^p D_l(t_j) \right| d\mu(t_1, \ldots, t_p) \right)^{1/p} \]

\[ \leq cp^2 \|f\|_C. \]

Lemma 3 is proved.
Lemma 4. Let $f \in C(G_{m}^{2})$ and $p > 0$. Then

$$\frac{1}{nk} \sum_{l=1}^{n} \sum_{r=1}^{k} |S_{l,r}(f; x, y) - f(x, y)|^p$$

$$\leq c^p(p + 1)^{2p} \left\{ \frac{1}{nM} \sum_{l=1}^{n} (E^{(1)}_l(f))^p + \frac{1}{k} \sum_{r=1}^{k} (E^{(2)}_r(f))^p \right\}. \quad (12)$$

Proof. Since

$$(a + b)^\beta \leq 2^\beta (a^\beta + b^\beta), \quad \beta > 0,$$

by using (3), (4), (8), and Lemma 3 we conclude that

$$\frac{1}{nM} \sum_{n=M_A}^{M_{A+1}} \sum_{l=M_B}^{M_{B+1}} |S_{n,l}(f; x, y) - f(x, y)|^p$$

$$\leq \frac{2^p}{M_A M_B} \sum_{n=M_A}^{M_{A+1}} \sum_{l=M_B}^{M_{B+1}} |S_{n,l}(f - S_{M_A M_B}(f); x, y)|^p$$

$$+ \frac{2^p}{M_A M_B} (M_{A+1} - M_A)(M_{B+1} - M_B) E_{MA MB}^p(f)$$

$$\leq c^p(p + 1)^{2p} \| f - S_{M_A M_B}(f) \|_{C}^p + c^p \left( (E^{(1)}_{MA}(f))^p + (E^{(2)}_{MB}(f))^p \right)$$

$$\leq c^p(p + 1)^{2p} \left( (E^{(1)}_{MA}(f))^p + (E^{(2)}_{MB}(f))^p \right). \quad (13)$$

Let $M_L \leq n < M_{L+1}$ and $M_R \leq k < M_{R+1}$. Thus, it follows from (13) that

$$\frac{1}{nk} \sum_{l=1}^{n} \sum_{r=1}^{k} |S_{l,r}(f; x, y) - f(x, y)|^p$$

$$\leq \frac{1}{nk} \sum_{l=1}^{n} \sum_{r=1}^{k} |S_{l,r}(f; x, y) - f(x, y)|^p$$

$$= \frac{1}{nk} \sum_{A=0}^{L} \sum_{B=0}^{R} \sum_{l=M_A}^{M_{A+1}} \sum_{r=M_B}^{M_{B+1}} |S_{l,r}(f; x, y) - f(x, y)|^p$$

$$\leq \frac{c^p(p + 1)^{2p}}{nk} M_A M_B \sum_{A=0}^{L} \sum_{B=0}^{R} \left( (E^{(1)}_{MA}(f))^p + (E^{(2)}_{MB}(f))^p \right).$$
\[
\leq \frac{c^p(p + 1)^{2p}}{nk} \sum_{A=0}^{L} \sum_{B=0}^{R} \sum_{l=M_{A-1}}^{M_{A-1}} \sum_{r=M_{B-1}}^{M_{B-1}} ((E_{MA}^{(1)}(f))^p + (E_{MB}^{(2)}(f))^p)
\]
\[
\leq \frac{c^p(p + 1)^{2p}}{nk} \sum_{A=0}^{L} \sum_{B=0}^{R} \sum_{l=M_{A-1}}^{M_{A-1}} \sum_{r=M_{B-1}}^{M_{B-1}} ((E_{l}^{(1)}(f))^p + (E_{r}^{(2)}(f))^p)
\]
\[
\leq \frac{c^p(p + 1)^{2p}}{nk} \sum_{l=1}^{n} \sum_{r=1}^{k} ((E_{l}^{(1)}(f))^p + (E_{r}^{(2)}(f))^p)
\]
\[
\leq c^p(p + 1)^{2p} \left\{ \frac{1}{n} \sum_{l=1}^{n} (E_{l}^{(1)}(f))^p + \frac{1}{k} \sum_{r=1}^{k} (E_{r}^{(2)}(f))^p \right\}.
\]

Lemma 4 is proved.

5. Proofs of the Main Results

The Walsh–Paley version of Theorem 1 was proved in [12]. In view of inequality (12), the same construction works in Vilenkin’s case. Therefore, the proof of Theorem 1 is omitted.

**Proof of Theorem 2.** (a) It is easy to see that if \( \varphi \in \Psi \), then \( e^{\varphi} - 1 \in \Psi \). In addition, (9) implies the existence of a number \( A \) such that

\[
\limsup_{u \to \infty} \frac{e^{\varphi(u)} - 1}{e^{A u^{1/2}} - 1} < \infty.
\]

Hence, in view of Lemma 2, in order to prove Theorem 2 it is sufficient to show that

\[
\lim_{n,m \to \infty} \left\| \frac{1}{nm} \sum_{l=1}^{m} \sum_{r=1}^{m} \left( e^{A|S_{l,r}(f)-f|^{1/2}} - 1 \right) \right\|_C = 0.
\] (14)

The validity of equality of (14) immediately follows from Theorem 1.

(b) First, we prove that if \( \psi \in \Psi \) and

\[
\limsup_{u \to \infty} \frac{\psi(u)}{u} = \infty,
\]

then there exists a function \( f \in C(G_m) \) and a sequence of positive integers \( \{A_k : k \geq 1\} \) such that

\[
\psi(\left| S_{N_{A_k}}(f, 0) \right|) > 5(A_k - 1) \ln a,
\] (15)

where

\[
a := \sup_j m_j
\]
and
\[ N_{A_j} := \sum_{k=A_{j-1}}^{A_j-1} \left[ \frac{m_{2k}}{2} \right] M_{2k}. \]

Let \( \{B_k : k \geq 1\} \) be an increasing sequence of positive integers such that
\[ B_1 > c', \]
\[ B_j > 2B_{j-1}, \]
\[ \frac{\psi(B_j)}{B_j} > \frac{5j \ln a}{c'}, \]
where the constant \( c' \) is determined below.

We set
\[ A_k := \left\lfloor \frac{kB_k}{c'} \right\rfloor + 1, \]
\[ f_j(x) := \frac{1}{j+1} \sum_{s=A_{j-1}}^{A_j-1} \sum_{x_{2s+1}=0}^{m_{2s+1}-1} \cdots \sum_{x_{2A_{j-1}}=0}^{m_{2A_j}-1} \exp \left( -i \arg \left( D_{N_{A_j}}(x) \right) \right) \]
\[ \times \mathbb{I}_{I_{2A_j}(0,\ldots,0,x_{2s}=m_{2s}-1,x_{2s+1},\ldots,x_{2A_j-1})}(x), \]
\[ f(x) := \sum_{j=1}^{\infty} f_j(x), \]
\[ f(0) = 0, \]
where \( \mathbb{I}_E \) is the characteristic function of the set \( E \subset G_m \).

Since
\[ I_{2A_j}(0,\ldots,0,x_{2s}=m_{2s}-1,x_{2s+1},\ldots,x_{2A_j-1}) \]
\[ \cap I_{2A_j}(0,\ldots,0,x_{2l}=m_{2l}-1,x_{2l+1},\ldots,x_{2A_j-1}) = \emptyset, \quad l \neq s, \]
and
\[ 1/(j+1) \to 0 \quad \text{as} \quad j \to \infty, \]
we conclude that \( f \in C(G_m) \).

We can write
\[ \left| S_{N_{A_k}}(f;0) - f(0) \right| = \left| S_{N_{A_k}}(f;0) \right| = \left| \int_{G_m} f(t) \overline{D_{N_{A_k}}(t)} d\mu(t) \right|. \]
It follows from the definition of the function $f$ that

$$J_1 = \frac{1}{k+1} \left| \sum_{s=A_{k-1}}^{A_k-1} \sum_{t_{2s+1}=0}^{m_{2s+1}-1} \int_{I_{2s+1}^{2s}} \exp(-i \arg(\overline{D}_{N_{Ak}}(t))) \overline{D}_{N_{Ak}}(t) d\mu(t) \right|$$

$$= \frac{1}{k+1} \sum_{s=A_{k-1}}^{A_k-1} \sum_{t_{2s+1}=0}^{m_{2s+1}-1} \int_{I_{2s+1}^{2s}} |D_{N_{Ak}}(t)| d\mu(t).$$

Since (see [10])

$$|D_{N_{Ak}}(t)| \geq c_{M_{2s+1}} \text{ for } t \in I_{2s+1}^{2s}(0, \ldots, 0, t_{2s} = m_{2s} - 1), \ s = A_{k-1}, \ldots, A_k - 1,$$

in view of (17), we can write

$$J_1 \geq \frac{c}{k+1} \sum_{s=A_{k-1}}^{A_k-1} M_{2s+1} \sum_{t_{2s+1}=0}^{m_{2s+1}-1} \int_{I_{2s+1}^{2s}} |D_{N_{Ak}}(t)| d\mu(t)$$

$$= \frac{c}{k+1} \sum_{s=A_{k-1}}^{A_k-1} \frac{M_{2s+1} m_{2s+1} \cdots m_{2A_k-1}}{M_{2A_k}} = \frac{c}{k+1} (A_k - A_{k-1}).$$
Further, since [see (16)]

\[ A_k - A_{k-1} = \left[ \frac{kB_k}{c'} \right] - \left[ \frac{(k-1)B_{k-1}}{c'} \right] \]

\[ \geq \frac{kB_k}{c'} - \frac{(k-1)B_{k-1}}{c'} - 1 = \frac{k(B_k - B_{k-1})}{c'} + \frac{B_{k-1}}{c'} - 1 \]

\[ > \frac{k(B_k - B_{k-1})}{2c'} \geq \frac{A_k}{2} - \frac{1}{2} > \frac{A_k}{4}, \]

for \( J_1 \) we find

\[ J_1 \geq \frac{c}{k+1} A_k^4. \]

For \( J_2 \), we get

\[ J_2 \leq \sum_{j=k+1}^{\infty} \frac{1}{j + 1} \sum_{s=A_{j-1}}^{A_j-1} \frac{1}{M_{2s}} N_{A_k} \leq \frac{1}{k} \sum_{s=A_k}^{\infty} \frac{1}{M_{2s}} N_{A_k} \leq \frac{c}{k}. \]  

(20)

By (2), in view of the construction of the function \( f_j \), we can write

\[ \text{supp}(f_j) \cap \text{supp}(D_{N_{A_k}}) = \emptyset, \quad j = 1, 2, \ldots, k - 1. \]

Consequently,

\[ J_3 = 0. \]  

(21)

Combining (18)–(21), we conclude that

\[ |S_{N_{A_k}}(f; 0)| = |S_{N_{A_k}}(f; 0) - f(0)| \geq \frac{c'A_k}{k} \geq B_k, \]

\[ \psi\left(|S_{N_{A_k}}(f; 0)|\right) \geq \psi(B_k) \geq \frac{5k\ln a}{c'} B_k \geq 5(A_k - 1) \ln a. \]

Hence, inequality (15) is proved.

We now write \( \varphi(u) = \lambda(u)\sqrt{u} \) and define \( \psi(u) := \lambda\left(u^2\right)u \). This yields

\[ \limsup_{u \to \infty} \frac{\psi(u)}{u} = +\infty. \]

Therefore, there exist a function \( f \in C(G_m) \) and a sequence of positive integers \( \{A_k : k \geq 1\} \) such that

\[ \psi\left(|S_{N_{A_k}}(f, 0)|\right) > 5(A_k - 1) \ln a. \]  

(22)
We set

\[ F(x, y) := f(x) f(y). \]

It is easy to see that

\[
\varphi(\left| S_{N_{Ak}, N_{Ak}}(F; 0, 0) \right|) = \varphi(\left| S_{N_{Ak}}(f; 0) \right|^2) = \lambda(\left| S_{N_{Ak}}(f; 0) \right|^2) |S_{N_{Ak}}(f; 0)|
\]

\[ = \psi(\left| S_{N_{Ak}}(f; 0) \right|). \]

Since \( N_{Ak} \leq a^{2A_k} \), it follows from (22) that

\[
\frac{1}{N_{Ak}^2} \sum_{i=1}^{N_{Ak}} \sum_{j=1}^{N_{Ak}} e^{\varphi(\left| S_{i,j}(F; 0, 0) \right|)} \geq \frac{1}{N_{Ak}^2} e^{\varphi(\left| S_{N_{Ak}, N_{Ak}}(F; 0, 0) \right|)} = \frac{1}{N_{Ak}^2} e^{\psi(\left| S_{N_{Ak}}(f; 0) \right|)}
\]

\[
\geq \frac{e^{5(A_k-1) \ln a}}{a^{4A_k}} \to \infty \quad \text{as} \quad k \to \infty.
\]

Theorem 2 is proved.

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