Manipulating motional states by selective vibronic interaction in two trapped ions

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(November 16, 2018)

We present a selective vibronic interaction for manipulating motional states in two trapped ions, acting resonantly on a previously chosen vibronic subspace and dispersively on all others. This is done respecting technical limitations on ionic laser individual addressing. We discuss the generation of Fock states and entanglement in the ionic collective motional degrees of freedom, among other applications.

PACS number(s): 42.50.Vk, 32.80.Pj, 42.50.Dv

Coherent manipulation of vibronic states of two or more trapped ions has become a subject of increasing interest in the last few years [1]. A number of experimental and theoretical advances has been done recently in this field, aiming to make applications possible and to test basic features of quantum mechanics. In this context, Bell states and multipartite entanglement of the ionic internal degrees of freedom have been studied [2] and partially realized [3]. Beside their fundamental importance, such states are essential ingredients for the implementation of several interesting applications, as quantum logic and quantum computing using trapped ions [1,7,8]. All these proposals are mostly concerned with the deterministic manipulation of internal states of the ions.

However, it is also important to find ways for manipulating several interesting applications, as quantum teleportation and entanglement swapping [9,10], acting resonantly on a previously chosen vibronic subspace and dispersively on all others. This is done respecting technical limitations on ionic laser individual addressing. We discuss the generation of Fock states and entanglement in the ionic collective motional degrees of freedom, among other applications.

In this article we present an effective interaction consisting of two dispersive Raman pulses simultaneously illuminating two trapped ions, which opens the possibility to manipulate coherently their vibrational motion. As the electronic Stark shifts, induced by the Raman pulses, depend on the motional state of the ions, the resulting dynamics can be described by a Jaynes-Cummings-like interaction acting distinctly on different subspaces of their vibronic Hilbert space. As we will show below, the frequencies of the two dispersive Raman pulses may be chosen in such a way that the effective interaction becomes resonant to a previously chosen vibronic subspace while remaining non-resonant to others. This enables us to excite selectively a desired subspace inside the motional Hilbert space of the ions. Vibrational state engineering of both center of mass and relative motion can then be done by means of this special property.

We consider two two-level ions of mass m, confined to move in the z direction in a Paul trap. They are cooled down to very low temperatures [11] and may perform small oscillations around their equilibrium positions, z_{10} = -d/2, z_{20} = d/2. We denote by \( \hat{Z} = (\hat{z}_1 + \hat{z}_2)/2 \) and \( \hat{z} = (\hat{z}_1 - \hat{z}_2)/2 \) the center of mass and relative position operators, respectively. Both ions are simultaneously illuminated by two classical homogeneous Raman effective pulses \( \hat{E}_I = \hat{E}_0 I e^{i(\hat{q}_1 \cdot \hat{r} - \omega_I t)} \) and \( \hat{E}_{II} = \hat{E}_0 II e^{i(\hat{q}_2 \cdot \hat{r} - \omega_{II} t)} \), with wave vectors \( \hat{q}_1 = \hat{q}_2 = \hat{q} \), parallel to the z direction and angular frequencies \( \omega_I \) and \( \omega_{II} \). The Raman pulses frequencies will be chosen to be quasi-resonant with a long-lasting electronic transition between two ionic hyperfine levels \( | \downarrow_j \rangle \) and \( | \uparrow_j \rangle \) (j=1,2), with energy \( \hbar \omega_0 \) and 0, respectively. The total Hamiltonian of the system may be written, in the optical RWA approximation, as

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},
\]

with

\[
\hat{H}_0 = \hbar \omega_o (\hat{S}_{+1} \hat{S}_{-1} + \hat{S}_{+2} \hat{S}_{-2}) + \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \nu_r \hat{b}^\dagger \hat{b},
\]

\[
\hat{H}_{\text{int}} = \hbar \Omega e^{i \hat{q} \hat{Z}} \left( \hat{S}_{+1} e^{i \nu z/2} + \hat{S}_{+2} e^{-i \nu z/2} \right) \times \left( e^{-i \nu_r t} + e^{-i \nu_r t} \right) + \text{H.c.}
\]

Here \( \hat{S}_{+j} = | \uparrow_j \rangle \langle \downarrow_j | \) is the flip operator associated with the electronic transition \( | \downarrow_j \rangle \rightarrow | \uparrow_j \rangle \) in the ion j. The operators \( \hat{a} \) and \( \hat{b} \) (\( \hat{a}^\dagger \) and \( \hat{b}^\dagger \)) are the annihilation (creation) operators associated with the center of mass mode of frequency \( \nu \) and the relative vibrational mode of frequency \( \nu_r \), respectively. For simplicity, we have assumed that...
the same Rabi frequency \( \Omega \) (taken as real) is associated with both lasers.

We start by taking the frequencies \( \omega_I \) and \( \omega_{II} \) as:

\[
\omega_I = \omega_0 + k\nu + k_r\nu - \delta \\
\omega_{II} = \omega_0 + \delta.
\]

with \( \delta \ll \nu, \nu_r \). The frequencies are chosen so that \( \omega_I + \omega_{II} = k\nu + k_r\nu + 2\omega_0 \).

Following the usual treatment for one single ion interacting with a laser field, we make the rotating-wave-approximation (RWA) with respect to the vibrational frequencies and select the terms that oscillate with minimum frequency. In the interaction picture we obtain, for the interaction Hamiltonian, the following expression:

\[
\hat{H}_{\text{int}} = \hbar \Omega \left( \hat{S}_r \hat{a}^\dagger \hat{b}^\dagger \hat{S}_k, \hat{k} \hat{r}, \hat{c}_0 e^{i\delta t} + \hat{S}_r^\dagger \hat{H}_{0,0} e^{-i\delta t} \right) + \text{H.c.},
\]

where

\[
\hat{S}_r = \hat{S}_{r+1} e^{i\phi/2} + (-)^{k_r} \hat{S}_{r+2} e^{-i\phi/2}, \\
\hat{S}_r^\dagger = \hat{S}_{r+1} e^{i\phi/2} + \hat{S}_{r+2} e^{-i\phi/2}.
\]

Here \( \phi = 2\pi d \) is the phase difference due to the equilibrium distance between the two ions, which, in this paper, may be taken as 2\( \pi/4 \)m. In the interaction picture we obtain, following the usual treatment for one single ion in resonance with the Stark shifted levels associated with both lasers. The relative modes can be excited. It can be written as such that

\[
\hat{a} \quad \text{are the eigenvectors of the number operators } \hat{a}^\dagger \hat{a}, \text{ respectively.}
\]

If \( \delta \) is large enough so that \( \delta \gg \Omega \) and for times \( t \) such that \( \delta t \gg 1 \), it is possible to derive a two-photon effective time-independent Hamiltonian, where both CM and relative modes can be excited. It can be written as a sum of three terms:

\[
\hat{H}_{\text{eff}} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3,
\]

with

\[
\hat{H}_1 = \hbar \Omega_0 \hat{c}_k, \hat{S}_{r+1} \hat{S}_{r+2} \left[ \hat{a}^\dagger \hat{b}^\dagger \hat{k} \hat{r} \hat{S}_k, \hat{k} \hat{r}, \hat{c}_{0,0} \right] + \text{H.c.},
\]

\[
\hat{H}_2 = \hbar \Omega_0 \hat{c}_k, \hat{S}_{r+1} \hat{S}_{r+2} \left[ \hat{a}^\dagger \hat{b}^\dagger \hat{k} \hat{r} \hat{S}_k, \hat{k} \hat{r}, \hat{c}_{0,0} \right] + \text{H.c.},
\]

\[
\hat{H}_3 = \hbar \Omega_0 \hat{c}_k, \hat{S}_{r+1} \hat{S}_{r+2} \left[ \hat{a}^\dagger \hat{b}^\dagger \hat{k} \hat{r} \hat{S}_k, \hat{k} \hat{r}, \hat{c}_{0,0} \right] + \text{H.c.},
\]

where \( \Omega_0 = \Omega^2/\delta \). The first term, \( \hat{H}_1 \), gives rise to an anti Jaynes-Cummings dynamics, leading to a simultaneous excitation (or de-excitation) of the electronic states of the two ions, accompanied by the creation (or annihilation) of \( k \) vibrational quanta in the CM mode and \( k_r \) vibrational quanta in the relative mode. The factor \( \epsilon_r = (1 + (-)^{k_r}) \) prevents excitations of odd number of quanta in the relative mode, so that the symmetry by exchange of the two ions is maintained. The second term, \( \hat{H}_2 \), generates a dynamics where, simultaneously, one ion undergoes a transition from the ground to the excited electronic state and the other ion makes a transition in the inverse direction. This process is not accompanied by any excitation of the vibrational modes. The third term, \( \hat{H}_3 \), generates motional dependent dynamical energy shifts in the electronic levels. Due to this dependence, this term turns the processes induced by \( \hat{H}_1 \) and \( \hat{H}_2 \) more or less resonant, depending on the particular level of excitation of the vibrational modes. Note that the sensitivity of the energy shifts to the vibrational state of the ions increases with increasing values of the Lamb-Dicke parameters. For not too small values of the theses parameters, it is possible to make the interaction \( \hat{H}_1 \) completely resonant inside a previously chosen subspace \( \{|↓↓, N, N_r \rangle, |↑↑, ↑, N + k, N_r + k_r \rangle \} \), whereas remaining largely non resonant inside other subspaces. The same applies for \( \hat{H}_2 \) in a given subspace \( \{|↓↓, ↑, M, M_r \rangle, |↑↑, ↑, M, M_r \rangle \} \). This may occur, for example, when the Raman lasers frequencies, originally given by Eq. [3] are modified and correctly tuned to take into account the motional dependent energy shifts.

To describe these effects in more detail, we turn our attention to specific cases. We first consider excitations to the first blue side band of the center of mass mode \( (k = 1, k_r = 0) \). If we start from the electronic ground state, only \( \hat{H}_1 \) and \( \hat{H}_3 \) will be effective, and we may write:

\[
\begin{align*}
\hat{H}_{\text{eff}} &= \hbar \Omega_0 \hat{c}_k, \hat{S}_{r+1} \hat{S}_{r+2} \left[ \hat{a}^\dagger \hat{F}_1, \hat{F}_0 \right] + \\
&\quad + \frac{1}{2} \left( \hat{S}_{r+1} \hat{S}_{r+2} - \hat{S}_{r+2} \hat{S}_{r+1} \right) \left[ \eta^2 \hat{a}^\dagger \hat{F}_1^2 - \hat{F}_0^2 \right] \\
&\quad + \frac{1}{2} \left( \hat{S}_{r+1} \hat{S}_{r+2} - \hat{S}_{r+2} \hat{S}_{r+1} \right) \left[ \eta^2 \hat{a}^\dagger \hat{F}_1^2 - \hat{F}_0^2 \right] + \text{H.c.}. \quad (11)
\end{align*}
\]

As can be easily seen from Eq. [11], the energy shifts of levels \( |↓↓, n, n_r \rangle \) and \( |↑↑, ↑, n + 1, n_r \rangle \) are given by:

\[
\Delta_{↓↓, n,n_r}^{n,n_r} = 2\hbar \Omega_0 g_n \left[ \eta^2 \right] \left( \eta^2 \right) \left( n^2 \right) \left( n + 1 \right) \left( f_0^2 \right) \left( n + 1 \right) \left( n \right) \left( f_0^2 \right) \left( n \right) \quad (12)
\]

respectively. In Eq. [13]

\[
\begin{align*}
\eta_1(n) &= e^{-\eta^2/2} \frac{m!}{(m+k)!} L_m(\eta^2) \\
\eta_2(n) &= e^{-\eta^2/2} \frac{m!}{(m+k)!} L_m(\eta^2).
\end{align*}
\]

By properly adjusting the laser frequencies we may put them in resonance with the Stark shifted levels associated...
with a previously chosen vibronic subspace \( \{ | \downarrow, \downarrow, N, N_e \rangle, | \uparrow, \uparrow, N + 1, N_e \rangle \} \), while preventing resonant transitions in other subspaces with \( n \neq N \) and \( n_e \neq N_e \). This can be done by modifying the laser frequencies to

\[
\omega_I = \tilde{\omega}_0 + \nu - \delta, \quad \omega_{II} = \tilde{\omega}_0 + \delta,
\]

where \( 2\tilde{\omega}_0 = 2\omega_0 + \Delta^{N+1,N_e} - \Delta^{N,N_e} \) is the renormalized splitting of the levels. Notice that, for very small values of the Lamb-Dicke parameters, the motional dependence of the dynamical Stark shift disappears. For this reason it is important to work beyond the Lamb-Dicke regime in order to effectively select a chosen subspace out of the whole vibronic Hilbert space. It is noteworthy to mention that for special values of the Lamb-Dicke parameter \( \eta \), it may happen that energy shifts \( \Delta^{\pi+1,N_e} \) and \( \Delta^{\pi,N_e} \) are equal for certain values of \( n \), irrespectively of the state of the relative vibrational mode. For example, for \( \eta \approx 0.51, 0.42, 0.24 \), transitions inside the subspaces \( \{ | \downarrow, \downarrow, N, N_e \rangle, | \uparrow, \uparrow, N + 1, N_e \rangle \} \), become resonant for \( N = 1, 2, 8 \), respectively. Clearly, in this case it is not necessary to correct the laser frequencies.

In order to check this model, numerical simulations were done using the time dependent Hamiltonian given in Eq. (4). Starting from the state \( | \downarrow, \downarrow, 0, 0 \rangle \) and selecting the laser frequencies as in Eq. (3), we were able to observe complete Rabi oscillations between the states \( | \downarrow, \downarrow, 0, 0 \rangle \) and \( | \uparrow, \uparrow, 1, 0 \rangle \), in agreement with the model discussed above (see Fig. 1). In particular, for a \( \pi \) pulse, the state \( | \uparrow, \uparrow, 1, 0 \rangle \) is generated with \( \approx 100\% \) efficiency. For the same laser frequencies, we plot the Rabi oscillations of the population corresponding to the initial state \( | \downarrow, \downarrow, 1, 0 \rangle \). We can see that, indeed, the Rabi oscillations for this case have a very small amplitude.

Similar results may be obtained also for transitions leading to even excitations of the relative vibrational mode only. This is done by choosing \( k = 0 \) in Eq. (3) and correctly tuning the lasers to take into account the self energy terms. We checked numerically that it is possible, in this case, to drive resonantly Rabi oscillations inside selected subspaces \( \{ | \downarrow, \downarrow, N, N_e \rangle, | \uparrow, \uparrow, N, N_e + k_e \rangle \} \).

The Hamiltonian (5) could be used to generate a large set of motional states. For example, any Fock state associated with the center of mass motion and with even excitations of the relative vibrational mode could be obtained from the initial state \( | \downarrow, \downarrow, 0, 0 \rangle \) by successively applying \( \pi \) pulses with different frequencies. However, if one is interested in generating a highly excited Fock state, this process could take an unsatisfactory long time. A more efficient, non unitary, way of producing such states, as well as engineering other vibrational states, is to start from a product of the electronic ground state and any motional state \( | \psi_{\text{ vib}} \rangle \). By selecting the laser frequencies, we excite only a chosen vibronic transition \( \{ k, k_e \} \) with a \( \pi \) pulse. Ideally, we would end up with a superposition of the two states \( | \uparrow, \uparrow, N + k, N_e + k_e \rangle \) and \( | \downarrow, \downarrow \rangle \otimes \langle | N_e, N_e | \psi_{\text{ vib}} \rangle \). Measurement of the electronic levels projects out either the Fock state \( | N + k, N_e + k_e \rangle \) or the original state with a “hole” in the \( N, N_e \) component.

As a numerical example we have taken the initial state to be the product of the electronic ground state with a coherent state of the CM motion and the vacuum of the relative motion. We start from a coherent state \( | \tilde{n} \rangle \) with \( \tilde{n} = 4.0 \) and excite the center of mass transition from \( n = 4 \) to \( n = 5 \). A dark event in a fluorescent measurement should leave us with an state that is close to the Fock state with \( n = 5 \). In Fig. 2 we show the results for the phonon distribution. As expected, it is possible to make a “hole” in the vibrational quanta distribution (Fig. 2a) while creating a quasi Fock state (Fig. 2b) using a \( \pi \) pulse. The state with a “hole” will be associated with the ground electronic state, while the approximate Fock state of Fig. 2b will be associated with the excited electronic state. As shown in Fig. 2b, small contamination occurs around the target Fock state because transitions to levels other than \( n = 5 \) are not totally suppressed. For the case studied, the probability of finding the approximated Fock state after fluorescence is about 30%. A Fock state with \( n = 6 \), for example, can now be obtained if we apply subsequently to the ions another \( \pi \) pulse resonant to the transition \( | \uparrow, \uparrow, 5, 0 \rangle \leftrightarrow | \downarrow, \downarrow, 6, 0 \rangle \). If by measuring the electronic states we find \( | \downarrow, \downarrow \rangle \) (\( \text{a priori} \) probability of \( \approx 87\%) \), we are left with a state very close to the Fock state \( | 6, 0 \rangle \). In this case the fidelity is \( \approx 99\% \) (See Fig. 2c).

Depopulation of a region around a certain value of \( N \), can also be achieved by proper choice of the Lamb Dicke parameter and the detuning \( \delta \), taking advantage of the quasi resonance character of the interaction for \( n' \) very close to \( N \).

Another interesting application is the generation of maximally entangled states inside the subspace \( | 1, 0 \rangle, | 0, 2 \rangle, | 1, 2 \rangle, | 0, 0 \rangle \) of the vibrational modes. Starting from the electronic and vibrational ground state, a \( \pi/2 \) pulse with a given frequency would generate the state \( \frac{1}{\sqrt{2}} \{ | 1, 0 \rangle + | 0, 2 \rangle \} \). A subsequent \( \pi \) pulse with another frequency would then lead to the state \( \frac{1}{\sqrt{2}} \{ | \uparrow, \uparrow \rangle, | 1, 0 \rangle + | 0, 2 \rangle \} \) which is a maximally entangled state. Similar procedures could also lead to any maximally entangled state.

Entanglement transfer may also be achieved between the internal and motional degrees of freedom [5]. For example, assume that we start with the state \( \frac{1}{\sqrt{2}} \{ | \downarrow, \downarrow \rangle + | \uparrow, \uparrow \rangle \} \otimes | 0, 0 \rangle \). If we apply to the ions a \( \pi \) pulse, connecting the states \( | \downarrow, \downarrow, 0, 0 \rangle \rightarrow | \uparrow, \uparrow, 1, 2 \rangle \), we obtain \( | \uparrow, \uparrow \rangle \otimes (| 0, 0 \rangle + | 1, 2 \rangle) \). In this case the efficiency is very high since the state \( | \uparrow, \uparrow \rangle \otimes | 0, 0 \rangle \) does not couple to any state in an anti Jaynes-Cummings transition. Of course, the inverse process, where entanglement is transferred from the motional degrees of freedom to electronic one, is also possible.

In summary, we have engineered an interaction that, respecting limitations of ionic individual addressing, enhances our possibilities of manipulating and generating diverse vibronic states of two trapped ions. This interac-
tion acts selectively in a previously chosen vibronic subspace, \{ | ↓, ↓, N, N_r⟩, | ↑, ↑, N + k, N_r + k_r⟩ \}, permitting, in principle, a complete transfer of populations inside this subspace. The applications mentioned above are only a few relevant examples of what may be done by selectively addressing the initial vibronic states.

This work was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), the Programa de Apoio a Núcleos de Excelência (PRONEX) and the Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ).

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FIG. 1. Center of mass first side band blue excitation. The larger oscillations correspond to the population of the state | ↓, ↓, 0, 0⟩. The smaller oscillations correspond to the population of the state | ↓, ↓, 1, 0⟩. \eta = 0.5, \delta = 40\eta\Omega.

FIG. 2. Vibrational distributions \(P(n)\) for the center of mass motion after a \(\pi\) pulse resonant to the transition from \(n = 4\) to \(n = 5\). The initial state is the product of a coherent state with \(\bar{n} = 4.0\) and | ↓, ↓⟩. (a): Distribution correlated to | ↓, ↓⟩. (b): Distribution correlated to | ↑, ↑⟩. (c) Distribution correlated to | ↓, ↓⟩ after the second pulse. \(\eta = 0.3\) and \(\delta = 40.0\eta\Omega\).
