Abstract. Let W be the germ of a smooth complex surface around an exceptional curve and let E be a rank 2 vector bundle on W. We study the cohomological properties of a finite sequence \{E_i\}_{1 \leq i \leq t} of rank 2 vector bundles canonically associated to E. We calculate numerical invariants of E in terms of the splitting types of E_i, 1 \leq i \leq t. If S is a compact complex smooth surface and E is a rank two bundle on the blow-up of S at a point, we show that all values of \(c_2(E) - c_2(\pi_*E^{**})\) that are numerically possible are actually attained.

2000 Mathematics Subject Classification: 14F05

0. Introduction

We consider exceptional curves in the following two cases. In the first case, let W be a smooth connected complex analytic surface which contains an exceptional divisor i.e. a smooth curve \(D \cong \mathbb{P}^1\) with \(O_D(-1)\) as normal bundle. Let U be a small tubular neighborhood of D in the Euclidean topology and let \(\pi: U \to Z\) be the contraction of D. In this case Z is the germ of a smooth surface around the point \(P:= \pi(D)\).

In the second case, let W be a smooth connected algebraic surface defined over an algebraically closed field \(K\) with arbitrary characteristic. We assume that W contains an exceptional curve D and denote by U the formal completion of W along D. Let \(\pi: U \to Z\) be the contraction of D. In this case Z is a formal smooth two-dimensional space supported at P.

In what follows we use the notation defined above to represent either case. Let \(I\) be the ideal sheaf of D in U and consider a rank 2 vector bundle E over U. Consider the pair of integers \((a,b)\) such that \(E|D \cong O_D(a) \oplus O_D(b)\). We will refer to the pair \((a,b)\) as the \textit{splitting type} of E. Since Z is a smooth surface the bidual \(\pi_*(E)^{**}\) is locally free and hence free because Z is 2-dimensional. There is a natural inclusion \(j: \pi_*(E) \to \pi_*(E)^{**}\) such that \(\text{coker}(j)\) has finite length. Set \(Q := \text{coker}(j)\).

We show that the pair \((z,w) := (h^0(Z,Q), h^0(Z,R^1\pi_*(E)))\) of numerical invariants of E is uniquely determined by a sequence of pairs of integers associated to E in \([B]\) using ele-
mentary transformations. We review the construction of the associated sequence and prove the following results.

**Theorem 0.1.** Let $E$ be a rank 2 vector bundle on $W$ with associated admissible sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$. Then we have the equalities

$$w := h^0(Z, R^1\pi_* (E)) = \sum_{1 \leq i \leq t} \max \{-b_i - 1, 0\} \quad \text{and} \quad z := h^0(Z, Q) = \sum_{1 \leq i \leq t} a_i - a_t^2 - \sum_{1 \leq i \leq t} \max \{-b_i - 1, 0\}.$$ 

Every admissible sequence is associated to a rank 2 vector bundle on $W$ (see [B] Th.0.2). For simplicity, we normalize our bundles to have splitting type $(j, -j + \varepsilon)$, with $\varepsilon = 0$ or $\varepsilon = -1$. We have the following existence theorem.

**Theorem 0.2.** For every pair of integers $(z, w)$ satisfying $j - 1 - \varepsilon \leq w \leq j(j-1)/2 - j\varepsilon$ and $1 \leq z \leq j(j+1)/2$ with $j \geq 0$ and $\varepsilon = 0$ or $-1$, there exists a rank 2 vector bundle $E$ on $W$ with splitting type $(j, -j + \varepsilon)$ having numerical invariants $h^0(Z, R^1\pi_* (E)) = w$ and $h^0(Z, Q) = z$.

**Remark 0.3.** It follows from theorem 0.2 that the strata defined in [BG] for spaces of bundles on the blow-up of $C^2$ are all non-empty.

We give also the following characterization of the split bundle.

**Proposition 0.4.** Let $E$ be a rank 2 vector bundle on $U$ with splitting type $(j, -j + \varepsilon)$ with $j > 0$ and $\varepsilon = 0$ or $-1$. The following conditions are equivalent:

(i) $E \cong O_U(-jD) \oplus O_U((j+\varepsilon)D)$

(ii) $c_2(E) - c_2(\pi_* (E)**) = j(j+\varepsilon)$

(iii) $h^0(Z, R^1\pi_* (E)) = j(j-1)/2$

(iv) $E$ has associated sequence $\{(a_i, b_i)\}_{1 \leq i \leq j-\varepsilon}$ with $b_i = -j-\varepsilon+i-1$ for every $i$.

We now consider a compact complex smooth surface $S$, so that we can calculate second Chern classes. If $E$ is a rank 2 bundle defined on the blow-up of $S$ at a point, then the difference of second Chern classes satisfies $j \leq c_2(E) - c_2(\pi_* (E)**) \leq j^2$ and is given by the sum $h^0(Z, R^1\pi_* (E)) + h^0(Z, Q)$ (see [FM]). Sharpness of these bounds was proven in [B] and in [G2] by different methods. We prove the following existence theorem.
Theorem 0.5. Let $S$ be the blow-up of a compact complex smooth surface $S$ at a point. Let $l$ denote the exceptional divisor and let $j$ be a non-negative integer. Then for every integer $k$ satisfying $j \leq k \leq j^2$ there exists a rank 2 vector bundle $E$ over $S$ with $E|_l \cong O_l(j) \oplus O_l(-j)$ satisfying $c_2(E) - c_2(\pi_0(E)**) = k$.

Note 0.6: In [G1, Thm. 3.5] it is shown that the number of moduli for the space of rank-2 bundles on the blow up of $\mathbb{C}^2$ at the origin with splitting type $j$ equals $2j-3$; and since such bundles are determined by their restriction to a formal neighborhood of the exceptional divisor it follows that we have the same number of moduli for bundles over the neighborhood $U$ of an exceptional curve on a surface $W$.

These results are proven in section 1, where we also review the construction of admissible sequences. On section 2 we consider briefly bundles of higher rank.

1. Rank 2 bundles

We briefly recall the construction of the associated sequences of pairs of bundles and splitting types given in the introduction of [B]. We first give the definitions of positive and negative elementary transformations.

Let $E$ be a rank 2 vector bundle on $W$ with splitting type $(a,b)$ with $a \geq b$. Fix a line bundle $R$ on $D$ and a surjection $r: E \to R$ induced by a surjection $\rho: E|_D \to R$. There exists such a surjection if and only if $\text{deg}(R) \geq b$. If $\text{deg}(R) = b < a$, then $\rho$ is unique, up to a multiplicative constant. Set $E' := \ker(r)$ and $R' = \ker(\rho)$. If $\text{deg}(R) = b < a$ the sheaf $E'$ is uniquely determined, up to isomorphism. Since $D$ is a Cartier divisor, $E'$ is a vector bundle on $U$. We will say that $E'$ is the bundle obtained from $E$ by making the negative elementary transformation induced by $r$. Note that $R'$ is a line bundle on $D$ with degree $\text{deg}(R') = a + b - \text{deg}(R)$. Since $\text{deg}(I/D) = 1$ it is easy to check that $\text{deg}(E'|_D) = a + b + 1$ and we have the exact sequence

$$0 \to O_D(1+\text{deg}(R)) \to E'|_D \to R' \to 0.$$  

Furthermore, using this exact sequence we obtain a surjection $t: E' \to R'$ such that $\ker(t) \cong E(-D)$. In particular $\ker(t)|_D \cong O_D(a+1) \oplus O_D(b+1)$. Thus, up to twisting by $O_D(-D)$, the negative elementary transformation induced by $r$ has an inverse operation and we will say that $E$ is obtained from $E'$ making a positive elementary transformation supported by $D$. The following diagram, called the display of the elementary transformation, summarizes the construction (see [M]).
Given two vector bundles $E_1$ and $E_2$ with splitting types $(a_1, b_1)$ and $(a_2, b_2)$ we say that $E_1$ is more balanced than $E_2$ if $a_1 - b_1 \leq a_2 - b_2$. Given a vector bundle $E$ with splitting type $(a, b)$ we say that $E$ is balanced if either $a = b$ (case $c_1$ even) or else $a = b + 1$ (case $c_1$ odd). Performing negative elementary transformations we will take the bundle $E$ into more balanced bundles. The sequence of elementary transformations finishes when we arrive at a balanced bundle. If $\deg(R) = b$, then $E'|D$ fits in the exact sequence

$$0 \rightarrow O_D(b+1) \rightarrow E'|D \rightarrow O_D(a) \rightarrow 0.$$  

If $b < a$ then $E'$ is more balanced than $E$. If $b \leq a - 3$, then (2) does not uniquely determine $E'|C$. If $b \leq a - 2$ and $E'$ is not balanced, we reiterate the construction starting from $E'$ taking $R'$ to be the factor of $E'|D$ of lowest degree and we take the unique surjection (up to a multiplicative constant) $\rho': E' \rightarrow R'$. In a finite number, say, $t-1$, of steps, we send $E$ into a bundle which, up to twisting by $O_U(sD)$, where $s = -(a+b+t-1) / 2$ has trivial restriction to $D$. The process ends with a bundle isomorphic to $O_U(sD)$ (see [B], Remark 0.1).

We now construct the admissible sequence associated to $E$. Step one: set $E_1 := E$, $a_1 := a$ and $b_1 := b$. If $a_1 = b_1$, set $t = 1$ and stop. Otherwise $a_1 > b_1$. Step two: in the case $a_1 > b_1$ set $E_2 := E'$ and let $(a_2, b_2)$ be the splitting type of $E'$. Note that $a_2 + b_2 = a_1 + b_1 + 1$ and $b_1 < b_2 \leq a_2 \leq a_1$. Hence $a_2 - b_2 < a_1 - b_1$ and $E_2$ is more balanced than $E_1$. If $a_2 = b_2$, set $t := 2$ and stop. If $a_2 > b_2$ reiterate the construction. Final step: in a finite number of steps (say $t-1$ steps) we arrive at a bundle $E_t$ with splitting type $(a_t, b_t)$ with $a_t = b_t$. Call $E_i$, $2 \leq i \leq t$, the bundle obtained after $i-1$ steps and let $(a_i, b_i)$ be the splitting type of $E_i$. The finite sequence of pairs $\{(a_i, b_i)\}$ $1 \leq i \leq t$ obtained in this way has the following properties:

i. $a_i \geq b_i \ \forall \ i > 0,$

ii. $a_i + b_i = a_{i-1} + b_{i-1} + i - 1 \ \forall i > 1,$

iii. $a_i \geq a_{i+1} \geq b_{i+1} > b_i \ \forall \ i \geq 1,$ and

iv. $a_t = b_t.$
We call **admissible** any such finite sequence of pairs of integers. We will say that a sequence \(\{(a_i,b_i)\}, 1 \leq i \leq t\) is the **admissible sequence associated to the bundle** E if this sequence is created by the algorithm just described. By [B] Th. 0.2, every admissible sequence is associated to a rank 2 vector bundle on \(W\).

**Examples:** Let us first set some notation. To represent the admissible sequence \(\{(a_i,b_i)\}, 1 \leq i \leq t\), we write \((a_1,b_1) \rightarrow (a_2,b_2) \rightarrow \cdots \rightarrow (a_t,b_t)\).

1. If the splitting type of E is \((b+2,b)\) then there is only one possibility for the admissible sequence associated to E, namely

\[(b+2,b) \rightarrow (b+2, b+1) \rightarrow (b+2, b+2).\]

2. If the splitting type of E is \((b+4,b)\) then there are 3 different possibilities for admissible sequences associated to E (which in particular will give rise to different values of the numerical invariants \((z,w)\)), these are:

   i. \((b+4,b) \rightarrow (b+4,b+1) \rightarrow (b+4, b+2) \rightarrow (b+4, b+3) \rightarrow (b+4, b+4)\)

   ii. \((b+4,b) \rightarrow (b+4,b+1) \rightarrow (b+3, b+3)\)

   iii. \((b+4,b) \rightarrow (b+3, b+2) \rightarrow (b+3, b+3)\)

We now calculate the numerical invariants of E in terms of admissible sequences. For every integer \(n \geq 0\) let \(D^{(n)}\) be the n-th infinitesimal neighborhood of \(D\) in \(U\). Hence \(D^{(n)}\) is the closed subscheme of \(U\) with \(I^{n+1}\) as ideal sheaf. In particular, \(D^{(0)} = D\) and \(D^{(n)}_{\text{red}} = D\) for every \(n \geq 0\). For each integer \(n \geq 0\) the following sequence is exact

\[0 \rightarrow I^n/I^{n+1} \rightarrow O_U/I^{n+1} \rightarrow O_U/I^n \rightarrow 0\]  

(3)

Suppose that E is a vector bundle normalized to have splitting type \((j,-j+\varepsilon)\) where \(j \geq 1\) and either \(\varepsilon = 0\) or \(\varepsilon = -1\). We denote by \(m\) be the maximal ideal of \(O_Z,p\). Consider the inclusion \(j: \pi_\ast(E) \rightarrow \pi_\ast(E)^{**}\) and let \(Q := \text{coker}(j)\), \(z := h^0(Z,Q)\), and \(w := h^0(Z,R^1\pi_\ast(E))\).

Call \(O_D(x)\) the degree x line bundle on \(D\). Twisting the exact sequence (3) by \(E\) and using the fact that \(I^n/I^{n+1}\) has degree \(n\), we obtain the exact sequence

\[0 \rightarrow O_D(j+n) \otimes O_D(-j+\varepsilon+n) \rightarrow E|_D^{(n)} \rightarrow E|_D^{(n-1)} \rightarrow 0\]  

(4)

**Lemma 1.1.** The integers \(z\) and \(w\) satisfy the inequalities:

\[1 \leq z \leq j(j+1)/2 \quad \text{and} \quad j-1-\varepsilon \leq w \leq j (j-1)/2 - \varepsilon j.\]
Proof. By the Theorem on Formal Functions we have the bounds for \( z \) and we have that
\[
w \leq \sum_{n \geq 0} h^1(D, O_D(-j+\epsilon+n)) = j(j-1)/2 - \epsilon j.
\]
The upper bound for \( w+z \) was stated in [FM] Remark 2.8, and proven for bundles with arbitrary rank in [Bu] Prop.2.8. Consequently we have an alternative proof of the upper bound for \( z \). The lower bound for \( w \) will be proven in Remark 1.4. For the case of rank two and \( \epsilon = 0 \) [G2] shows that these bounds are sharp.

Since \( Q \) is a quotient of \( O_{U,P}^{\oplus 2} \) the dimension of the fiber of \( Q \) at \( P \) is either 1 or 2. The sheaf \( Q \) is isomorphic to the structure sheaf of a subscheme of \( Z \) supported by \( P \) and with length \( z \) if and only if the dimension of this fiber is 1. We will check that this is always true (see Proposition 1.3). We first check the split case.

**Lemma 1.2.** Suppose that \( E \cong O_U(-jD) \oplus O_U((-j+\epsilon)D) \) then \( z = j(j+1)/2, \ w = j(j-1)/2 - \epsilon j \) and \( Q \) is isomorphic to the structure sheaf of a subscheme of \( Z \) supported by \( P \) and with \( m^j \) as ideal sheaf.

Proof. Since \( D \) is an exceptional divisor, we have \( \pi_* (O_U((-j+\epsilon)D)) = \pi_* (O_U) = O_Z \) for every \( j \geq \epsilon \) and \( \pi_* (O_U(-jD)) = m^j \) if \( j > 0 \).

**Proposition 1.3.** Let \( E \) be a rank 2 vector bundle on \( W \) with splitting type \( (j,-j+\epsilon) \) with \( j > 0 \). Then \( Q \) is isomorphic to the structure sheaf of a length \( z \) subscheme \( Q \) of \( Z \) with \( Q_{\text{red}} = P \) and \( Q \subseteq P(j-1) \).

Proof of 1.3. The first assertion is well-known and follows from the proof of Lemma 1.2. Since \( Q \) is a quotient of \( O_Z^{\oplus 2} \), in order to prove the second assertion it is sufficient to check that its fiber at \( P \) is a 1-dimensional vector space. Since \( E \) has splitting type \( (j,-j+\epsilon) \), we have an extension
\[
0 \to O_U((-j+\epsilon)D) \to E \to O_U(jD) \to 0 \tag{5}
\]
([BG] Lemma 1.2, or in [G1] Thm. 2.1 in the case \( \epsilon = 0 \)). Call \( e \) the extension (5) giving \( E \). For each \( t \in K \setminus \{0\} \) consider the extension of \( O_U(jD) \) by \( O_U((-j+\epsilon)D) \) given by extension class \( te \), this extension has as middle term a vector bundle isomorphic to \( E \). Using the extension \( e \) for \( t = 0 \), we construct a family \( \{\lambda e\}_{\lambda \in K} \) of extensions. We call \( E_\lambda \) the corresponding middle term and \( Q_\lambda \) the corresponding sheaf. Since \( E_\lambda \cong E \) for \( \lambda \neq 0 \), we have \( Q_\lambda = Q \) for \( \lambda \neq 0 \), and because \( E_0 \cong O_W(jD) \oplus O_W((-j+\epsilon)D) \), we have that \( Q_0 = P(j-1) \), and the result follows by semi-continuity of the fiber dimension at \( P \).

Proof of 0.1. Given the admissible sequence of splitting types \( \{(a_i,b_i)\}_{1 \leq i \leq t} \) associated to \( E \) we want to show that
\[ w := h^0(Z, R^1 \pi_\#(E)) = \Sigma_{1 \leq i \leq t} \max \{-b_i-1, 0\} \quad \text{and} \]
\[ z := h^0(Z, Q) = \Sigma_{1 \leq i \leq t} a_i - a_i^2 - \Sigma_{1 \leq i \leq t} \max \{-b_i-1, 0\}. \]

We use induction on \( t \), the case \( t = 1 \) arising if and only if \( a_1 = b_1 \), equivalently, when \( E \equiv O_w(-a_1D)^{\oplus 2} \) (this follows immediately from the definition of admissible sequence).

Since
\[ R^1 \pi_\#(O_w(xD)) = 0 \quad \forall \ x \leq 1 \quad \text{and} \]
\[ R^1 \pi_\#(O_w(yD)) = y(y-1)/2 \quad \forall \ y > 0, \]

we have the equality for \( w \) in the split case. Assume \( t \geq 2 \). By the definition of the sequence \( \{E_i\}_{1 \leq i \leq t} \) associated to \( E \) we have that \( E_1 = E \) and that there is an exact sequence
\[ 0 \to E_2 \to E_1 \to O_D(-b_1D) \to 0. \quad (6) \]

First assume \( b_1 < 0 \), in which case we have that \( h^0(Z, R^1 \pi_\#(O_D(-b_1D))) = 0 \) and \( h^0(Z, R^1 \pi_\#(O_w(-b_1D))) = -b_1-1. \) Hence \( w := h^0(Z, R^1 \pi_\#(E)) = h^0(Z, R^1 \pi_\#(E_2)) -b_1 + 1 \) and since \( E_2 \) has \( \{(a_{i+1}, b_{i+1})\}_{1 \leq i < t} \) as admissible sequence, the claim follows.

Now assume \( b_1 \geq 0 \), from the exact sequence (6) it follows that \( h^0(Z, R^1 \pi_\#(E)) \leq h^0(Z, R^1 \pi_\#(E_2)). \) Since \( b_i > b_1 \) for every \( i > 1 \), we have \( h^0(Z, R^1 \pi_\#(E_2)) = 0. \) Hence, by the inductive assumption on the length of the admissible sequence, it follows that \( h^0(Z, R^1 \pi_\#(E)) = 0, \) proving the first assertion. The value of \( z := h^0(Z, Q) \) comes from the equalities \( c_2(E) - c_2(\pi_\#(E)^{**}) = \Sigma_{1 \leq i < t} a_i - a_i^2 \) and \( c_2(E) - c_2(\pi_\#(E)^{**}) = h^0(Z, Q) + h^0(Z, R^1 \pi_\#(E)) \) proved in [B, Th. 0.3] and in [FM] respectively. Here, of course, we assume that \( E \) is extended to a compactification, however these integers do not depend upon the choices of compactification and of extension of \( E \).

**Proof of 0.2.** By [B] Th. 0.2 every admissible sequence \( (a_i, b_i) \) is associated to a rank two bundle \( E \) on \( W \), moreover, the intermediate steps of the construction of \( E \) give bundles \( E_i \) with splitting types \( (a_i, b_i) \) for each \( i \). Now use Th. 0.1 to calculate \( z \) and \( w \).

**Remark 1.4.** If we assume that \( E \) has splitting type \( (j, -j+\varepsilon) \) with \( j \geq 1+\varepsilon \), then because \( b_1 = -j+\varepsilon \), we obtain \( w \geq j-1-\varepsilon \).

**Proof of 0.4.** By [B] Th. 0.5 we know that (i) and (ii) are equivalent. By Lemma 1.2 (i) implies (iv). Since \( b_1 = -j + \varepsilon, b_i > b_{i-1} \forall i > 1 \) holds, and \( a_1 = j \) and \( a_i+b_i = \varepsilon+i-1 \forall i \), it follows from Theorem 0.1 that (iv) implies (ii).
Proof of 0.5. Given bundles G on S and F on W with \(c_1(G)=0=c_1(F)\) there exists a bundle E on S satisfying \(E \mid S - l = \pi^* E \mid S - \{p\}\) and \(E \mid W = F\) (see [G3] Cor. 3.4). It then follows that \(c_2(E) - c_2(\pi_\#(E)^{**}) = R^1\pi_\#(F) + l(Q)\) and the result follows from Th. 0.2.

2. Bundles of higher rank

In this section we consider vector bundles with rank \(r \geq 3\). Fix a rank \(r\) vector bundle \(E\) on \(U\). We use the notation of [BG] \(\beta 3\) for the admissible sequence \(\{E_i\}, 1 \leq i \leq t\) of vector bundles associated to \(E\). In particular we denote by \((a(i,1),...,a(i,r))\) the splitting type of \(E_i\) where for \(a(i,1) \geq ... \geq a(i,r)\). We make the strong assumption that \(a(i,r-1) \geq -1\) for every \(i\) and compute \(h^0(Z,R^1\pi_\#(E))\).

**Proposition 2.1.** Let \(E\) be a rank \(r\) vector bundle on \(W\) whose associated sequence of vector bundles \(\{E_i\}\) has splitting type \((a(i,1),...,a(i,r))\) with \(1 \leq i \leq t\) and \(a(i,r-1) \geq -1\), for all \(i\). Then we have \(h^0(Z,R^1\pi_\#(E)) = \sum_{1 \leq i \leq \min{-a(r,1)-1,0}}\).

**Proof.** We first observe that the proof of the corresponding inequality for rank 2 bundles works verbatim (both cases \(t = 1\) and \(t > 1\)), because for each \(i\) with \(1 \leq i \leq t\) at most one of the integers \(a(i,j)\) is not at least \(-1\) and \(h^1(P^1,L) = 0\) for every line bundle \(L\) on \(P^1\) with \(\deg(L) \geq -1\).

In the case \(r \geq 3\), the sequence of elementary transformations made to balance the bundle is not, a priori, uniquely determined, and hence the sequence of associated bundles is not uniquely determined by \(E\). The condition \(a(1,r-1) \geq -1\) implies that there is an associated sequence in which we make always an elementary transformation with respect to \(O_D(a(r,i))\) to pass from \(E_i\) to \(E_{i+1}\) for some \(a(r,i) \leq -1\) (which gives that \(h^0(Z,R^1\pi_\#(E_i)) = h^0(Z,R^1\pi_\#(E_{i+1})) - a(r,i) + 1\)). We continue to perform elementary transformations until we arrive at an integer \(m \leq t\) such that \(a(m,j) \geq -1\) for every \(i\). It is then quite easy to check that \(h^0(Z,R^1\pi_\#(E_m)) = 0\) and the result follows.

In the general case the same method gives the following partial result.

**Proposition 2.2.** Let \(E\) be a rank \(r\) vector bundle on \(W\) whose associated sequence \(\{E_i\}, 1 \leq i \leq t\) of vector bundles has splitting type \((a(i,1),...,a(i,r))\) with \(1 \leq i \leq t\). Then we have \(h^0(Z,R^1\pi_\#(E)) \leq \sum_{1 \leq i \leq t, 1 \leq j \leq \min{-a(j,i)-1,0}}\).

References

[B] E. Ballico, *Rank 2 vector bundles in a neighborhood of an exceptional curve of a smooth surface*, Rocky Mountain J. Math., 29 (1999), 1185-1193.
[BG1] E. Ballico, E. Gasparim, *Vector bundles on a formal neighborhood of a curve in a surface*, Rocky Mountains J. Math. 30, n. 3 (2000), 1-20.

[BG2] E. Ballico, E. Gasparim, *Numerical invariants for bundles on blow-ups*, Proc. of the AMS, to appear.

[Bu] N. P. Buchdahl, *Blow-ups and gauge fields*, alg-geom/9505006

[FM] R. Friedman and J. Morgan, *On the diffeomorphism type of certain algebraic surfaces, II*, J. Diff. Geometry 27 (1988), 371-398.

[G1] E. Gasparim, *Holomorphic vector bundles on the blow-up of $C^2$*, J. Algebra 199 (1998), 581-590.

[G2] E. Gasparim, *Chern classes of bundles on blown-up surfaces*, Comm. Algebra, Vol. 28, n.10,(2000) 4912-4926.

[M] M. Maruyama, *On a generalization of elementary transformations of algebraic vector bundles*, Rend. Sem. Mat. Univ. Politec. Torino (1987), 1-13.

Edoardo Ballico  
Dept. of Mathematics, University of Trento  
38050 Povo (TN) - Italy  
fa0: 39-0461881624  
e-mail: ballico@science.unitn.it

Elizabeth Gasparim  
Department of Mathematics  
University of Texas at Austin  
Austin TX 78712  
e-mail: gasparim@math.utexas.edu