Derangement Polynomials and Excedances of Type $B$

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Abstract. Adopting the definition of excedances of type $B$ due to Brenti, we give a type $B$ analogue of the $q$-derangement polynomials. The connection between $q$-derangement polynomials and Eulerian polynomials naturally extends to the type $B$ case. Based on this relation, we derive some basic properties of the $q$-derangement polynomials of type $B$, including the generating function formula, the Sturm sequence property, and the asymptotic normal distribution. We also show that the $q$-derangement polynomials are almost symmetric in the sense that the coefficients possess the spiral property.

Keywords: signed permutation, $q$-derangement polynomial of type $B$, Eulerian polynomial of type $B$, spiral property, limiting distribution

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1 Introduction

In this paper, we define a type $B$ analogue of the $q$-derangement polynomials introduced by Brenti $^3$ by $q$-counting derangements with respect to the number of excedances of type $B$, also introduced by Brenti $^3$. We give some basic properties of these polynomials. It turns out that the connection between the $q$-derangement polynomials and the Eulerian polynomials naturally extends to the type $B$ case, where the type $B$ analogue of Eulerian polynomial has been given by Brenti $^3$, and has been further studied by Chow and Gessel in $^6$.

Let us now recall some definitions. Let $\mathcal{S}_n$ be the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For each $\sigma \in \mathcal{S}_n$, the descent set and the excedance set of $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ are defined respectively as follows,

$$\text{Des}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\},$$

$$\text{Exc}(\sigma) = \{i \in [n-1] : \sigma_i > i\}.$$

The descent number and excedance number are defined respectively by

$$\text{des}(\sigma) = |\text{Des}(\sigma)|, \quad \text{exc}(\sigma) = |\text{Exc}(\sigma)|.$$
The Eulerian polynomials \([11, 13]\) are defined by
\[
A_n(q) = \sum_{\sigma \in S_n} q^{\text{des}(\sigma)+1} = \sum_{\sigma \in S_n} q^{\text{exc}(\sigma)+1}, \quad n \geq 0.
\]

The Eulerian polynomials have the following generating function
\[
\sum_{n \geq 0} A_n(q) \frac{t^n}{n!} = (1 - q)e^{qt}e^{qt} - qe^{qt}.
\] (1.1)

A permutation \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n\) is a derangement if \(\sigma_i \neq i\) for any \(i \in [n]\). The set of derangements on \([n]\) is denoted by \(D_n\). Brenti [1] defined the \(q\)-derangement polynomials of type A by
\[
d_n(q) = \sum_{\sigma \in D_n} q^{\text{exc}(\sigma)},
\]
and proved that \(d_n(q)\) is symmetric and unimodal for \(n \geq 1\). The following formulas (1.2) and (1.3) are derived by Brenti [1].

**Theorem 1.1** For \(n \geq 0\),
\[
d_n(q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(q).
\] (1.2)

**Theorem 1.2** We have
\[
\sum_{n \geq 0} d_n(q) \frac{t^n}{n!} = \frac{1}{1 - \sum_{n \geq 2} (q + q^2 + \cdots + q^{n-1})t^n/n!}.
\] (1.3)

A combinatorial proof of the above identity is given by Kim and Zeng [8] based on a decomposition of derangements. Brenti further proposed the conjecture that \(d_n(q)\) have only real roots for \(n \geq 1\), which has been proved independently by Zhang [14], and Canfield as mentioned in [2].

**Theorem 1.3** The polynomials \(d_n(q)\) form a Sturm sequence. Precisely, \(d_n(q)\) has \(n\) distinct non-positive real roots, separated by the roots of \(d_{n-1}(q)\).

The following recurrence relation is given by Zhang [14], which has been used to prove Theorem 1.3.

**Theorem 1.4** For \(n \geq 2\), we have
\[
d_n(q) = (n-1)qd_{n-1}(q) + q(1-q)d'_{n-1}(q) + (n-1)qd_{n-2}(q).
\]
This paper is motivated by finding the right definition of a type $B$ analogue of the $q$-derangement polynomials of type $A$ so that we can get analogous properties to the above theorems for the type $A q$-derangement polynomials. We discover that the notion of excedances of type $B$ introduced by Brenti serves as the right choice for type $B$ derangement polynomials, although there are several possibilities to define type $B$ excedances, see \cite{3, 5, 12}. Nevertheless, it should be noted that the type $B$ derangement polynomials are not symmetric compared with the case of type $A$. On the other hand, we will be able to show that they are almost symmetric in the sense that their coefficients have the spiral property.

This paper is organized as follows. In Section 2, we recall Brenti’s definition of type $B$ excedances, and present the definition of $q$-derangement polynomials of type $B$, denoted by $d_n^B(q)$. In Section 3, we establish the connection between the derangement polynomials of type $B$ and the Eulerian polynomials of type $B$. This leads to a generating function formula for type $B$ derangement polynomials. We then extend the $U$-algorithm and $V$-algorithm given by Kim and Zeng \cite{8} to derangements of type $B$. This gives a combinatorial interpretation of the generating function formula. In Section 4, we prove that the polynomials $d_n^B(q)$ form a Sturm sequence. We also show that the coefficients of $d_n^B(q)$ possess the spiral property. In Section 5, by using Lyapunov’s theorem we deduce that the limiting distribution of the coefficients of $d_n^B(q)$ is normal.

## 2 The Excedances of Type $B$

In this section, we recall Brenti’s definition of type $B$ excedances, and give the definition of the $q$-derangement polynomials of type $B$. We adopt the notation and terminology on permutations of type $B$, or signed permutations, as given in \cite{5}. Let $B_n$ be the hyperoctahedral group on $[n]$. We can view the elements of $B_n$ as signed permutations of $[n]$, written as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, in which some elements are associated with a minus sign. We also express a negative element $-i$ in the form $\bar{i}$.

The type $B$ descent set and the type $B$ ascent set of a signed permutation $\sigma$ are defined by

\[
\text{Des}_B(\sigma) = \{i \in [0, n - 1] : \sigma_i > \sigma_{i+1}\},
\]

\[
\text{Asc}_B(\sigma) = \{i \in [0, n - 1] : \sigma_i < \sigma_{i+1}\},
\]

where we set $\sigma_0 = 0$. The type $B$ descent and ascent numbers are given by

\[
\text{des}_B(\sigma) = |\text{Des}_B(\sigma)|, \quad \text{asc}_B(\sigma) = |\text{Asc}_B(\sigma)|.
\]
A derangement of type $B$ on $[n]$ is a signed permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ such that $\sigma_i \neq i$, for all $i \in [n]$. The fixed point is a position $i$ such that $\sigma_i = i$. The set of derangements in $B_n$ is denoted by $D_n^B$.

Let us recall the definitions of excedances and weak excedances of type $B$ introduced by Brenti [3]. For further information on statistics on signed permutations, see [3, 6, 9].

**Definition 2.1** Given $\sigma \in B_n$ and $i \in [n]$, we say that $i$ is a type $B$ excedance of $\sigma$ if either $\sigma_i = -i$ or $|\sigma|_{\sigma_i} > \sigma_i$. We denote by $\text{exc}_B(\sigma)$ the number of type $B$ excedances of $\sigma$. Similarly, we say that $i$ is a type $B$ weak excedance of $\sigma$ if either $\sigma_i = i$ or $|\sigma|_{\sigma_i} > \sigma_i$, and we denote by $\text{wexc}_B(\sigma)$ the number of type $B$ weak excedances of $\sigma$.

Based on the above definition of type $B$ excedances, we define the type $B$ analogue of the $q$-derangement polynomials.

**Definition 2.2** The type $B$ derangement polynomials $d_n^B(q)$ are defined by

$$d_n^B(q) = \sum_{\sigma \in D_n^B} q^{\text{exc}_B(\sigma)} = \sum_{k=1}^{n} d_{n,k} q^k, \quad n \geq 1, \quad (2.1)$$

where $d_{n,k}$ is the number of derangements in $D_n^B$ with exactly $k$ excedances of type $B$. For $n = 0$, we define $d_0^B(q) = 1$.

Below are the polynomials $d_n^B(q)$ for $n \leq 10$:

- $d_1^B(q) = q$,
- $d_2^B(q) = 4q + q^2$,
- $d_3^B(q) = 8q + 20q^2 + q^3$,
- $d_4^B(q) = 16q + 144q^2 + 72q^3 + q^4$,
- $d_5^B(q) = 32q + 752q^2 + 1312q^3 + 232q^4 + q^5$,
- $d_6^B(q) = 64q + 3456q^2 + 14576q^3 + 9136q^4 + 716q^5 + q^6$,
- $d_7^B(q) = 128q + 14912q^2 + 127584q^3 + 190864q^4 + 55624q^5 + 2172q^6 + q^7$,
- $d_8^B(q) = 256q + 62208q^2 + 977920q^3 + 2879232q^4 + 2020192q^5 + 314208q^6 + 6544q^7 + q^8$,
- $d_9^B(q) = 512q + 254720q^2 + 6914816q^3 + 35832320q^4 + 49168832q^5 + 18801824q^6 + 1697408q^7 + 19664q^8 + q^9$, 
- $d_{10}^B(q)$
\[ d_{10}^B(q) = 1024q + 1032192q^2 + 46429440q^3 + 394153728q^4 + 937670016q^5 \\
+ 704504832q^6 + 161032224q^7 + 8919456q^8 + 59028q^9 + q^{10}. \]

3 The Generating Function

The first result of this section is a formula expressing \( d_n^B(q) \) in terms of \( B_n(q) \), where \( B_n(q) \) are the Eulerian polynomials of type \( B \). This formula is analogous to that of Brenti [1] for the type \( A \) case, and it enables us to derive a formula for the generating function of \( d_n^B(q) \).

The Eulerian polynomials of type \( B \) are defined by Brenti [3] based on the number of descents of type \( B \):

\[ B_n(q) = \sum_{\sigma \in B_n} q^{\text{des}_B(\sigma)}. \quad (3.1) \]

Brenti [3] obtained the following formula for the generating function of the Eulerian polynomials of type \( B \), see, also, Chow and Gessel [6],

\[ \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!} = \frac{(1 - q)e^{t(1-q)}}{1 - qe^{2t(1-q)}}. \quad (3.2) \]

The following theorem is obtained by Brenti [3], which will be used to establish the formula for \( d_n^B(q) \).

**Theorem 3.1** There is a bijection \( \varphi : B_n \to B_n \) such that

\[ \text{asc}_B(\varphi(\sigma)) = \text{wexc}_B(\sigma), \]

for any \( \sigma \in B_n \).

The following formula indicates that the notion of excedances of type \( B \) introduced by Brenti is a right choice for type \( B \) derangement polynomials.

**Theorem 3.2** We have

\[ d_n^B(q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B_k(q). \quad (3.3) \]
Proof. From Theorem 3.1 we see that the number of excedances of type $B$ and the number of descents of type $B$ are equidistributed on $B_n$. So we deduce that

$$B_n(q) = \sum_{\sigma \in B_n} q^{\text{des}_B(\sigma)} = \sum_{\sigma \in B_n} q^{\text{exc}_B(\sigma)}. \quad (3.4)$$

We will establish the following relation

$$\sum_{\pi \in B_n} q^{\text{exc}_B(\pi)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in D_k^B} q^{\text{exc}_B(\sigma)}. \quad (3.5)$$

It suffices to construct a correspondence between $B_n$ and $\{(S, T, \sigma)| S \cup T = [n], S \cap T = \emptyset \text{ and } \sigma \in B_{|S|}\}$ such that $\text{exc}_B(\pi) = \text{exc}_B(\sigma)$ for any $\pi \in B_n$.

Given a signed permutation, we can decompose it into two parts separating the fixed points from the non-fixed points. Precisely, each $\pi \in B_n$ can be represented by $(S, T, \sigma)$, where $S$ is the set of non-fixed points of $\pi$, $T$ is the set of fixed points in $\pi$, and $\sigma$ is a reduced signed derangement of $\pi$. Keep in mind that $i$ is a fixed point of $\pi$ if $\pi_i = i$. Let $|S| = k$, then $\sigma$ is obtained from $\pi$ as a signed derangement on $[k]$ by deleting the fixed elements and reducing the resulting signed permutation to $[k]$. Formally speaking, let $\tau$ be the signed permutation obtained from $\pi$ by deleting the fixed points, then $\sigma$ is derived from $\tau$ by replacing the minimum element by 1 or $1$ depending on its sign, and replacing the second minimum element by 2 or $2$ depending on its sign, and so on. Note that the elements in $\tau$ are ordered regardless of their signs. For example, let $\pi = 4 6 3 7 5 1 2$. Then $S = \{1, 2, 4, 6, 7\}$, $T = \{3, 5\}$, $\tau = 4 6 7 1 2$ and $\sigma = 3 4 5 1 2$.

On the other hand, let $(S, T, \sigma)$ be a representation of a signed permutation on $[n]$ such that $S \cup T = [n]$, $S \cap T = \emptyset$, $|S| = k$ and $\sigma$ is a signed derangement on $[k]$. Then we can recover a unique signed permutation $\pi$ on $[n]$. Let $S = \{s_1, s_2, \ldots, s_k\}$ with $s_1 < s_2 < \cdots < s_k$ and $T = \{t_1, t_2, \ldots, t_{n-k}\}$ with $t_1 < t_2 < \cdots < t_{n-k}$. First $\tau$ can be obtained from $\sigma$ by replacing $i$ or $i$ in $\sigma$ by $s_i$ or $s_i$, then $\pi$ can be constructed from $\tau$ by inserting the fixed points.

We proceed to show that $\text{exc}_B(\pi) = \text{exc}_B(\sigma)$. Assume that $i$ is an excedance of $\pi$, that is, $\pi_i = -\pi$ or $\pi_{|\pi|} > \pi_i$. Clearly, the fixed points do not contribute to the number of excedances. Then there exists $j$ such that $\pi_i = \tau_j$. If $\pi_i = -i = \tau_j$, then there are $i - j$ fixed points in $\pi$ before $i$. This implies that $\tau_j = -i$ is the $j$-th minimum element in $\tau$ regardless of the signs. By the transformation from $\tau$ to $\sigma$ as given before, we see that $\sigma_j = -j$, that is, $j$ is an excedance of $\sigma$.

We now come to the case $\pi_{|\pi|} > \pi_i$. Clearly, $i$ is not a fixed point; otherwise, we would deduce that $\pi_{|\pi|} = \pi_i > \pi_i$. We have two cases depending
on the sign of $\pi_i$. First, we assume that $\pi_i$ is positive. Then $\pi_i$ is not a fixed point of $\pi$. For notational convenience, let $\pi_i = j$, so we have $\pi_j > j$. Since neither $i$ nor $j$ is a fixed point of $\pi$, both should appear in $\tau$. Thus there exist $j_1$ and $j_2$ such that $\pi_i = \tau_{j_1}$ and $\pi_j = \tau_{j_2}$. With the above notation, we see that $\tau_{j_2} > \tau_{j_1}$. This implies that $\sigma_{j_2} > \sigma_{j_1}$, since the transformation from $\tau$ to $\sigma$ is order preserving. In view of $\pi_i = j$ and $\pi_j = \tau_{j_2}$, we find that $\pi_i = j$ is the $j_2$-th minimum element in $\tau$ regardless of the signs, namely, $\sigma_{j_1} = j_2$. Hence we deduce that $\sigma_{\sigma_{j_1}} = \sigma_{j_2} > \sigma_{j_1}$, namely, $j_1$ is an excedance of $\sigma$ with $\sigma_{j_1}$ being positive. Conversely, given any excedance $j_1$ of $\sigma$ with $\sigma_{j_1}$ being positive, we can reverse the above procedure to generate an excedance $i$ of $\pi$ with $\pi_i$ being positive.

It remains to consider the case when $\pi_i = \tilde{j}$ is negative. It is clear that $i$ is not a fixed point of $\pi$. With this notation, we have $\pi_j > \tilde{j}$. Since $i$ is a not a fixed point of $\pi$ and $\pi_i$ is negative, both $i$ and $\tilde{j}$ will appear in $\tau$. Hence there exist $j_1$ and $j_2$ such that $\pi_i = \tau_{j_1}$ and $\pi_j = \tau_{j_2}$. Moreover, we see that $\tau_{j_2} > \tau_{j_1}$ and $\sigma_{j_2} > \sigma_{j_1}$. Using the same procedure as given before, we find that $\sigma_{j_1} = \tilde{j}_2$. It follows that $\sigma_{|\sigma_{j_1}|} = \sigma_{j_2} > \sigma_{j_1}$, i.e., $j_1$ is an excedance of $\sigma$ with $\sigma_{j_1}$ being negative. Conversely, given an excedance $j_1$ of $\sigma$ with $\sigma_{j_1}$ being negative, we can reverse the above procedure to generate an excedance $i$ of $\pi$ with $\pi_i$ being negative.

Combining the above cases, we arrive at the conclusion that $\text{exc}_B(\pi) = \text{exc}_B(\sigma)$, which implies (3.5). Hence we get the following relation

$$B_n(q) = \sum_{k=0}^{n} \binom{n}{k} d^B_n(q).$$

Using the binomial inversion, we arrive at (3.3). This completes the proof.

Theorem 3.3 We have

$$\sum_{n \geq 0} d^B_n(q) \frac{t^n}{n!} = \frac{(1 - q)e^{qt}}{e^{2qt} - qe^{2t}} = \frac{e^{qt}}{1 - \sum_{n \geq 2} 2^n(q + q^2 + \cdots + q^{n-1})t^n/n!}. \quad (3.7)$$

Proof. Using (3.2) and (3.6), we obtain

$$e^t \sum_{n \geq 0} d^B_n(q) \frac{t^n}{n!} = \sum_{n \geq 0} B_n(q) \frac{t^n}{n!} = \frac{(1 - q)e^{t(1-q)}}{1 - qe^{2t(1-q)}}. \quad (3.8)$$

The last equality of (3.7) is straightforward. This completes the proof.

Next, we give a combinatorial interpretation of the identity (3.7) based on a generalization of the decomposition of derangements given by Kim and Zeng [8] for their combinatorial proof of (1.3).
A Combinatorial Proof of Theorem 3.3. First, we give an outline of the proof of Kim and Zeng for ordinary derangements. We adopt the convention that a cycle $\sigma = s_1 s_2 \cdots s_k$ of length $k$ is written in such a way that $s_1$ is the minimum element and $\sigma_{s_i} = s_{i+1}$ with $s_{k+1} = s_1$. A cycle $\sigma$ (of length at least two) is called unimodal (resp. prime) if there exists $i$ ($2 \leq i \leq k$) such that $s_1 < \cdots < s_{i-1} < s_i > s_{i+1} > \cdots > s_k$ (resp. in addition, $s_{i-1} < s_k$). Let $(l_1, \ldots, l_m)$ be a composition of $n$, a sequence of prime cycles $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is called a $P$-decomposition of type $(l_1, \ldots, l_m)$ if $\tau_i$ is of length $l_i$ and the underlying sets of $\tau_1, \tau_2, \ldots, \tau_m$ form a partition of $[n]$. Define the excedance of $\tau$ as the sum of the excedances of its prime cycles, that is,

$$\text{exc}(\tau) = \text{exc}(\tau_1) + \cdots + \text{exc}(\tau_m),$$

and the weight of $\tau$ is defined by $q^{\text{exc}(\tau)}$. Note that one needs to express a cycle in the two row permutation form for the purpose of computing the excedances. The details are omitted here. In [8], Kim and Zeng obtained a bijection which maps the number of excedances of a derangement to the number of excedances of a $P$-decomposition with type $(l_1, \ldots, l_m)$, $l_i \geq 2$. Then the generating function of $d_n(q)$ follows from the generating function of $P$-decomposition with type $(l_1, \ldots, l_m)$, as given by

$$\left( \prod_{i=1}^{m} (q + \cdots + q^{l_i-1}) \right) \frac{p_1^{l_1} \cdots + p_m^{l_m}}{(l_1 + \cdots + l_m)!}.$$

Summing over $l_1, \ldots, l_m \geq 2$ and $m \geq 0$, we are led to the right hand side of the generating function formula (1.3).

We now proceed to extend the above construction to type $B$ derangements. We need the cycle decomposition of a signed permutation, which can be viewed as the cycle decomposition of an ordinary permutation with signs attached to some elements. There is one point that needs to be taken into account, that is, a signed permutation is a signed derangement if and only if the cycle decomposition does not have any one-cycle with a positive sign. More precisely, for any derangement $\pi$ of type $B$, we can decompose it into cycles

$$\pi = (C_1, C_2, \ldots, C_k),$$

where $C_1, C_2, \ldots, C_k$ are written in decreasing order of their minimum elements subject to the following order

$$\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < n.$$  \hspace{1cm} (3.9)

We first apply the $U$-algorithm [8] to decompose each cycle $\sigma = s_1 s_2 \cdots s_k$ of $\pi$ into a sequence of unimodal cycles, here we impose the order (3.9) in defining unimodal cycles. It should be noted that a cycle with only one
negative element is also considered as a unimodal and prime cycle. Then we define \( U(\pi) = (U(C_1), U(C_2), \ldots, U(C_k)) \).

**The U-algorithm**

1. If \( \sigma \) is unimodal, then set \( U(\sigma) = (\sigma) \).

2. Otherwise, let \( i \) be the largest integer such that \( s_{i-1} > s_i < s_{i+1} \) and
   \( j \) be the unique integer greater than \( i \) such that \( s_j > s_i > s_{j+1} \). Then set \( U(\sigma) = (U(\sigma_1), \sigma_2) \), where \( \sigma_1 = s_1 \cdots s_{i-1}s_{j+1} \cdots s_k \), and \( \sigma_2 = s_is_{i+1} \cdots s_j \) is unimodal.

We claim that the number of excedances of \( \pi \) is equal to total number of excedances of unimodal cycles in \( U(\pi) \). It suffices to verify that this statement is valid for each cycle \( \sigma = s_1s_2 \cdots s_k \) of \( \pi \). Clearly, it is true if \( \sigma \) is unimodal. Otherwise, it suffices to show \( \text{exc}_B(\sigma) = \text{exc}_B(\sigma_1) + \text{exc}_B(\sigma_2) \).

Assume \( |s_t| \) is an excedance of \( \sigma \), i.e., \( \sigma_{|\sigma_{|s_t|}|} > \sigma_{|s_t|} \). Then it is necessary to find an excedance in \( \sigma_1 \) or \( \sigma_2 \). By the cycle notation of \( \sigma \), we have \( \sigma_{|s_t|} = s_{t+1} \) and \( \sigma_{|\sigma_{|s_t|}|} = s_{t+2} \), then \( |s_t| \) is an excedance of \( \sigma \) implies \( s_{t+2} > s_{t+1} \). For \( t = i - 2 \), we have \( s_{t+2} = s_i \) and \( s_{t+1} = s_{i-1} \). On the other hand, we have \( s_i < s_{i-1} \) by the choice of \( i \), so it cannot be an excedance of \( \sigma \). Using the same argument, we see that when \( t = j - 1 \), \( |s_t| \) cannot be an excedance of \( \sigma \). Therefore, if \( 1 \leq t < i - 2 \) or \( j + 1 \leq t \leq k \), then \( |s_t| \) is an excedance of \( \sigma_1 \).

Similarly, if \( i - 1 \leq t < j - 1 \), then \( |s_t| \) is an excedance of \( \sigma_2 \). If \( t = i - 1 \), i.e., \( |s_{i-1}| \) is an excedance of \( \sigma_1 \), then \( s_{i+1} > s_i \), which implies that \( |s_j| \) is an excedance of \( \sigma_2 \). If \( t = j \), i.e., \( |s_j| \) is an excedance of \( \sigma \), then \( s_{j+2} > s_{j+1} \), which implies that \( |s_{i-1}| \) is an excedance of \( \sigma_1 \). Conversely, given an excedance of \( \sigma_1 \) or \( \sigma_2 \), we can determine an excedance of \( \sigma \) by reversing the above procedure.

For example, let \( \pi = 35429687\bar{1} \). Then we have \( \text{exc}_B(\pi) = 5 \), and \( C_1 = 78, C_2 = \bar{6} \) and \( C_3 = 59\bar{1}342 \). Moreover, we find

\[
U(C_1) = (78), \quad U(C_2) = (\bar{6}), \quad U(C_3) = (59, \bar{1}342),
\]

and

\[
U(\pi) = (78, \bar{6}, 59, \bar{1}342).
\]

Note that \( \text{exc}_B(U(\pi)) = 5 \), in accordance with \( \text{exc}_B(\pi) = 5 \).

Next, we recall the V-algorithm given in [8], which transforms a sequence of unimodal cycles into a sequence of prime cycles. For signed derangements, we will use this algorithm by imposing the order relation [8,9].

**The V-algorithm**

1. If \( \sigma \) is prime, then set \( V(\sigma) = (\sigma) \).
2. Otherwise, let \( j \) be the smallest integer such that \( s_j > s_i > s_{j+1} > s_{i-1} \) for some integer \( i \) greater than 1. Then set \( V(\sigma) = (V(\sigma_1), \sigma_2) \), where \( \sigma_1 = s_1 \cdots s_{i-1} s_{j+1} \cdots s_k \), and \( \sigma_2 = s_is_{i+1} \cdots s_j \) is prime.

We claim that the total number of excedances of unimodal cycles in \( U(\pi) \) is equal to total number of excedances of prime cycles in \( V \circ U(\pi) \). It suffices to prove that the claim is valid for any unimodal cycle \( \sigma = s_1s_2 \cdots s_k \). Clearly, it is true if \( \sigma \) is prime. Otherwise, without loss of generality we may show that \( \text{exc}_B(\sigma) = \text{exc}_B(\sigma_1) + \text{exc}_B(\sigma_2) \). Assume \( |s_t| \) is an excedance of \( \sigma \), we have \( s_{t+2} > s_t \). As will be shown, we can find an excedance of \( \sigma_1 \) or \( \sigma_2 \). For \( t = j - 1 \), \( |s_t| \) cannot be an excedance of \( \sigma \), since \( s_{j+1} < s_j \) by the choice of \( j \). So, if \( 1 \leq t < i - 2 \) or \( j + 1 \leq t \leq k \), then \( |s_t| \) is an excedance of \( \sigma_1 \). If \( i \leq t < j - 1 \), we deduce that \( |s_t| \) is an excedance of \( \sigma_2 \). Since \( s_i > s_{i-1} \) by the choice of \( i \), when \( t = i - 2 \), \( |s_t| \) is an excedance of \( \sigma \). On the other hand, we have \( s_{j+1} > s_{i-1} \) by the choices of \( i \) and \( j \), which implies that \( |s_t| \) is an excedance of \( \sigma_1 \). For \( t = i - 1 \) or \( t = j \), we can use the same argument as in the \( U \)-algorithm.

Applying \( V \) to each cycle of \( U(\pi) \) in the above example, we obtain that

\[
V \circ U(\pi) = (78, 6\bar{5}9, \bar{1}2, 34),
\]

and \( \text{exc}_B(\pi) = 5 \).

Using the composition \( V \circ U \), we can transform a derangement in \( B_n \) to a \( P \)-decomposition of \([n]\). Moreover, it has been shown that this map is a bijection in \([8]\). In the type \( B \) case, we define the weight of each prime cycle \( \tau \) by \( q^{\text{exc}_B(\tau)} \), where \( \text{exc}_B(\tau) \) is the number of the excedances of the type \( B \) derangement with cycle decomposition \( \tau \). Note that in the cycle decomposition of a type \( B \) derangement, we allow cycles of length one with negative elements. Thus the corresponding \( P \)-decompositions have type \((1^k, l_1, \ldots, l_m), k \geq 0, l_i \geq 2 \). For a cycle containing only one negative element, the weight is \( q \). For a cycle of length \( l \geq 2 \), we have \( 2^l \) choices for the \( l \) elements in the prime cycle, so the weight of such a prime cycle on a \( l \)-set is \( 2^l(q + q^2 + \cdots + q^{l-1}) \). Hence the generating function of \( d_n^B(q) \) follows from the generating function of \( P \)-decompositions of type \((1^k, l_1, \ldots, l_m), k \geq 0, l_i \geq 2 \), as given by

\[
q^k t^k \left( \frac{l_1 + \cdots + l_m}{l_1, \ldots, l_m} \right) \prod_{i=1}^{m} 2^l(q + \cdots + q^{l-1}) \frac{t^{l_1 + \cdots + l_m}}{(l_1 + \cdots + l_m)!}.
\]

Summing over \( l_1, \ldots, l_m \geq 2 \) and \( k \geq 0, m \geq 0 \), we obtain the right hand side of \([3.7]\).
4 A Recurrence Relation

In this section, we will use the recurrence relation for Eulerian polynomials of type $B$ to derive a recurrence relation for the $q$-derangement polynomials $d^B_n(q)$. Applying a theorem of Zhang [13], we deduce that $d^B_n(q)$ form a Sturm sequence, that is, $d^B_n(q)$ has only real roots and separated by the roots of $d^B_{n-1}(q)$. Moreover, from the initial values, one sees that $d^B_n(q)$ has only non-positive real roots. Consequently, $d^B_n(q)$ is log-concave. Although the polynomials $d^B_n(q)$ are not symmetric, we show that they are almost symmetric in the sense that the coefficients have the spiral property.

The following recurrence formula (4.1) for $B_n(q)$ is a special case of Theorem 3.4 in Brenti [3], see also Chow and Gessel [6], which will play a key role in establishing a recurrence relation for $d^B_n(q)$.

**Theorem 4.1** We have

$$B_n(q) = ((2n-1)q + 1)B_{n-1}(q) + 2q(1-q)B'_{n-1}(q), \quad n \geq 1, \quad (4.1)$$

where $B_0(q) = 1$.

**Theorem 4.2** For $n \geq 2$, we have

$$d^B_n(q) = (2n-1)qd^B_{n-1}(q) + 2q(1-q)d^B'_{n-1}(q) + 2(n-1)qd^B_{n-2}(q). \quad (4.2)$$

**Proof.** By (3.3) and (4.1), we obtain

$$d^B_n(q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B_k(q)$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \left( \binom{n-1}{k-1} + \binom{n-1}{k} \right) B_k(q)$$

$$= -d^B_{n-1}(q) + \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} ((2k-1)q + 1)B_{k-1}(q) + 2q(1-q)B'_{k-1}(q))$$

$$= -qd^B_{n-1}(q) + 2q \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} - \binom{n-1}{k} \right) kB_{k-1}(q) + 2q(1-q)d^B'_{n-1}(q)$$

$$= -d^B_{n-1}(q) + 2nq \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} B_{k-1}(q)$$

$$+ 2q(n-1) \sum_{k=1}^{n} (-1)^{n-k-1} \binom{n-2}{k-1} B_{k-1}(q) + 2q(1-q)d^B'_{n-1}(q)$$

$$= (2n-1)qd^B_{n-1}(q) + 2(n-1)qd^B_{n-2}(q) + 2q(1-q)d^B'_{n-1}(q),$$

as desired.

Equating coefficients on both sides of (4.2), we are led to the following recurrence relation for the numbers $d_{n,k}$.
Corollary 4.3 For \( n \geq 2 \) and \( k \geq 1 \), we have

\[
d_{n,k} = 2kd_{n-1,k} + (2n - 2k + 1)d_{n-1,k-1} + 2(n - 1)d_{n-2,k-1}.
\]

(4.3)

From the above relation (4.3), it follows that \( d_{n,1} = 2^n \) for \( n > 1 \). The recurrence relation (4.2) on \( d_n^B(q) \) enables us to show that the polynomials \( d_n^B(q) \) form a Sturm sequence. The proof turns out to be an application of the following theorem of Zhang [15].

Theorem 4.4 Let \( f_n(q) \) be a polynomial of degree \( n \) with nonnegative real coefficients satisfying the following conditions:

1. For \( n \geq 2 \), \( f_n(q) = a_nf_{n-1}(q) + b_nf_{n-1}(1 + cq)f'_{n-1}(q) + d_nf_{n-2}(q) \), where \( a_n > 0, b_n > 0, c_n \leq 0, d_n \geq 0 \);
2. For \( n \geq 1 \), zero is a simple root of \( f_n(q) \);
3. \( f_0(q) = e, f_1(q) = e_1q \) and \( f_2(q) \) has two real roots, where \( e \geq 0 \) and \( e_1 \geq 0 \).

Then the polynomial \( f_n(q) \) has \( n \) distinct real roots, separated by the roots of \( f_{n-1}(q), n \geq 2 \).

It can be easily verified that the recurrence relation (4.2) satisfies the conditions in the above theorem. Thus we reach the following assertion.

Theorem 4.5 The polynomials \( d_n^B(q) \) form a Sturm sequence, that is, \( d_n^B(q) \) has \( n \) distinct non-positive real roots, separated by the roots of \( d_{n-1}^B(q) \).

As a direct consequence of the above theorem, we see that the coefficients of \( d_n^B(q) \) are log-concave. Although the coefficients are not symmetric as in the type \( A \) case, we will show that they are almost symmetric in the sense that they satisfy the spiral property. The spiral property was first observed by Zhang [16] in his proof of a conjecture of Chen and Rota [4].

Theorem 4.6 The polynomials \( d_n^B(q) \) have the spiral property. Precisely, for \( n \geq 2 \), if \( n \) is even, then

\[
d_{n,n} < d_{n,1} < d_{n,n-1} < d_{n,2} < d_{n,n-2} < \cdots < d_{n,\frac{n}{2}+2} < d_{n,\frac{n}{2}-1} < d_{n,\frac{n}{2}+1} < d_{n,\frac{n}{2}}.
\]

and if \( n \) is odd, then

\[
d_{n,n} < d_{n,1} < d_{n,n-1} < d_{n,2} < d_{n,n-2} < \cdots < d_{n,\frac{n+1}{2}} < d_{n,\frac{n-1}{2}} < d_{n,\frac{n+3}{2}}.
\]
Proof. Let

\[ f(n) = \begin{cases} 
\frac{n}{2} - 1, & \text{if } n \text{ is even}, \\
\frac{n-1}{2}, & \text{if } n \text{ is odd}.
\end{cases} \]

In this notation, the spiral property can be described by the following inequalities

\[ d_{n,n+1-k} < d_{n,k} < d_{n,n-k} \] \hspace{1cm} (4.4)

for any \( 1 \leq k \leq f(n) \), and in addition, the inequality

\[ d_{n,\frac{n}{2}+1} < d_{n,\frac{n}{2}} \] \hspace{1cm} (4.5)

when \( n \) is even.

We proceed to prove the relations (4.4) and (4.5) by induction on \( n \). It is easily seen that (4.4) and (4.5) hold for \( n = 2 \) and \( n = 3 \). We now assume that they hold for all integers up to \( n \). We now aim to show that

\[ d_{n+1,n+2-k} < d_{n+1,k} < d_{n+1,n+1-k} \] \hspace{1cm} (4.6)

for any \( 1 \leq k \leq f(n+1) \), and it is also necessary to show that when \( n+1 \) is even,

\[ d_{n+1,\frac{n+1}{2}} < d_{n+1,\frac{n+1}{2}}. \] \hspace{1cm} (4.7)

For \( k = 1 \), we have \( d_{n+1,n+1}-d_{n+1,1} = 1-2^{n+1} < 0 \). For \( 2 \leq k \leq f(n+1) \), by the recurrence relation (4.3) for \( d_{n,k} \), we have

\[ d_{n+1,n+2-k} = 2(n+2-k)d_{n,n+2-k} + (2k-1)d_{n,n+1-k} + 2nd_{n-1,n+1-k}, \] \hspace{1cm} (4.8)

\[ d_{n+1,k} = 2kd_{n,k} + (2n-2k+3)d_{n,k-1} + 2nd_{n-1,k-1}, \] \hspace{1cm} (4.9)

\[ d_{n+1,n+1-k} = 2(n+1-k)d_{n,n+1-k} + (2k+1)d_{n,n-k} + 2nd_{n-1,n-k}. \] \hspace{1cm} (4.10)

It follows from (4.8) and (4.9) that

\[ d_{n+1,n+2-k} - d_{n+1,k} = (2n-2k+3)(d_{n,n+2-k} - d_{n,k-1}) + 2k(d_{n,n+1-k} - d_{n,k}) + 2n(d_{n-1,n+1-k} - d_{n-1,k-1}) + (d_{n,n+2-k} - d_{n,n+1-k}). \]

By the inductive hypothesis, we see that the difference in every parenthesis in the above expression is negative. This implies that for \( 2 \leq k \leq f(n+1) \)

\[ d_{n+1,n+2-k} - d_{n+1,k} < 0. \] \hspace{1cm} (4.11)

Similarly, for \( 2 \leq k \leq f(n+1) \), in view of (4.9) and (4.10) we find

\[ d_{n+1,k} - d_{n+1,n+1-k} = (2k+1)(d_{n,k} - d_{n,n-k}) + 2n(d_{n-1,k-1} - d_{n-1,n-k}) \]

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\[ + (2n + 3 - 2k)(d_{n,k-1} - d_{n,n+1-k}) + (d_{n,n+1-k} - d_{n,k}). \]

Again, by the inductive hypothesis, we deduce that for \(2 \leq k \leq f(n+1),\)
\[ d_{n+1,k} - d_{n+1,n+1-k} < 0. \tag{4.12} \]

Combining (4.11) and (4.12) gives (4.6) for \(1 \leq k \leq f(n+1).\)

It remains to verify (4.7) when \(n+1\) is even. By the recurrence relation (4.3) of \(d_{n,k},\) we have
\[
\begin{align*}
    d_{n+1, \frac{n+3}{2}} &= (n+3) d_{n, \frac{n+3}{2}} + n d_{n, \frac{n+1}{2}} + 2n d_{n-1, \frac{n+1}{2}}, \\
    d_{n+1, \frac{n+1}{2}} &= (n+1) d_{n, \frac{n+1}{2}} + (n+2) d_{n, \frac{n-1}{2}} + 2n d_{n-1, \frac{n-1}{2}}.
\end{align*}
\]

This yields
\[
\begin{align*}
    d_{n+1, \frac{n+3}{2}} - d_{n+1, \frac{n+1}{2}} &= (n+2) (d_{n, \frac{n+3}{2}} - d_{n, \frac{n-1}{2}}) + (d_{n, \frac{n+3}{2}} - d_{n, \frac{n+1}{2}}) \\
    &+ 2n (d_{n-1, \frac{n+3}{2}} - d_{n-1, \frac{n-1}{2}}).
\end{align*}
\]

Again, by the inductive hypothesis, we immediately obtain (4.7). This completes the proof.

5 The Limiting Distribution

In this section, we show that the limiting distribution of the coefficients of \(d_n^{B}(q)\) is normal. The type \(A\) case has been studied by Clark [7]. It has been shown that the limiting distribution of the coefficients of \(d_n(q)\) is normal. Let \(\xi_n\) be the number of type \(B\) excedances in a random type \(B\) derangement on \([n]\). We first compute the expectation and the variance of \(\xi_n\). Then we use Lyapunov’s theorem to deduce that \(\xi_n\) is asymptotically normal.

**Theorem 5.1** We have
\[
\begin{align*}
    \mathbb{E}\xi_n &= \frac{n}{2} + \frac{1}{4} + o(1), \tag{5.1} \\
    \text{Var}\xi_n &= \frac{n}{12} - \frac{1}{16} + o(1). \tag{5.2}
\end{align*}
\]

**Proof.** By the recurrence relation (4.11) for \(B_n(x),\) we have for \(n \geq 1,\)
\[ B_n'(x) = (2n - 1)B_{n-1}(x) + (2nx - 5x + 3)B_{n-1}'(x) + 2x(1-x)B_{n-1}''(x). \tag{5.3} \]
Since $B_n(1) = 2^n n!$ for $n \geq 0$, setting $x = 1$ in (5.3) gives the following recurrence relation for $B'_n(1)$:

$$B'_n(1) = (2n - 1)(n - 1)! 2^{n-1} + (2n - 2)B'_{n-1}(1).$$

It can be verified that for $n \geq 1$,

$$B'_n(1) = \frac{n2^n n!}{2}.$$  \hspace{1cm} (5.4)

Moreover, by (5.3) we get

$$B''_n(x) = (4n - 6)B'_{n-1}(x) + (2nx - 9x + 5)B''_{n-1}(x) + 2x(1 - x)B''_{n-1}(x).$$  \hspace{1cm} (5.5)

Setting $x = 1$ in (5.5) and using (5.4), we obtain

$$B''_n(1) = (2n - 3)(n - 1)2^{n-1}(n - 1)! + (2n - 4)B''_{n-1}(1).$$  \hspace{1cm} (5.6)

Then for $n \geq 2$,

$$B''_n(1) = \frac{(3n^2 - 5n + 1)2^n n!}{12}.$$  \hspace{1cm} (5.7)

Since $B_n(1) = 2^n n!$, in view of the formula (3.3), we see that

$$d^n_B(1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B_k(1).$$  \hspace{1cm} (5.8)

Let

$$s_n = \sum_{k=0}^{n} (-1)^{k} \frac{1}{2^k \cdot k!}.$$  

Then $d^n_B(1)$ can be written as $2^n n! s_n$.

Applying the formula (3.3) again and using the evaluation (5.4) for $B'_n(1)$, we find that for $n \geq 1$

$$d^n_{B'}(1) = \sum_{m=1}^{n} (-1)^{n-m} \binom{n}{m} \cdot 2^{m-1} m \cdot m!$$

$$= 2^n n! \sum_{m=0}^{n-1} (-1)^m \frac{n - m}{m! 2^{m+1}}$$

$$= 2^n n! \left( \frac{n}{2} \sum_{m=0}^{n-1} (-1)^m \frac{1}{m! 2^m} + \frac{1}{4} \sum_{m=0}^{n-1} (-1)^{m-1} \frac{1}{2^{m-1} (m-1)!} \right)$$

$$= \frac{2^n n!}{2} \left( ns_{n-1} + \frac{1}{2} s_{n-2} \right).$$  \hspace{1cm} (5.9)
Differentiating (3.3) twice and invoking (5.7), we deduce that for $n \geq 2$

\[
d_B''(1) = \sum_{m=2}^{n} (-1)^{n-m} \binom{n}{m} 2^m \frac{3m^2 - 5m + 1}{12}
\]

\[
= \frac{2^n n!}{12} \sum_{m=2}^{n} (-1)^{n-m} \frac{3m^2 - 5m + 1}{(n-m)!2^{n-m}}
\]

\[
= \frac{2^n n!}{12} \sum_{m=0}^{n-2} (-1)^m \frac{3(n-m)^2 - 5(n-m) + 1}{m!2^m}
\]

\[
= \frac{2^n n!}{12} \left( \sum_{m=0}^{n-2} (-1)^m \frac{3n^2 - 5n + 1}{m!2^m} + \sum_{m=0}^{n-2} (-1)^m \frac{-6n + 5}{(m-1)!2^m} \right)
\]

\[
= \frac{2^n n!}{12} \left( (3n^2 - 5n + 1)s_{n-2} + \frac{1}{2}(6n - 5)s_{n-3} + \frac{3}{4}s_{n-4} - \frac{3}{2}s_{n-3} \right)
\]

\[
= \frac{2^n n!}{12} \left( (3n^2 - 5n + 1)s_{n-2} + (3n - 4)s_{n-3} + \frac{3}{4}s_{n-4} \right); \quad (5.10)
\]

It is easy to see that $s_{n-r}/s_n = 1 + o(1)$ for $r = 1, 2, 3, 4$. From (5.8), (5.9) and (5.10), we conclude that

\[
E\xi_n = \frac{d_B''(1)}{d_B(1)} =\frac{n}{2} + \frac{1}{4} + o(1), \quad (5.11)
\]

\[
Var\xi_n = \frac{d_B''(1)}{d_B(1)} + E\xi_n - (E\xi_n)^2 = \frac{n}{12} - \frac{1}{16} + o(1), \quad (5.12)
\]
as desired.

Given the formulas for the expectation and variance of $\xi_n$, we will use Lyapunov’s theorem [10, Section 1.2] to derive that the limiting distribution of $\xi_n$ is normal. Recall that a triangular array of independent random variables $\xi_{nk}, k = 1, 2, \ldots, n, n = 1, 2, \ldots$, is called a Poisson sequence if

\[
P\{\xi_{nk} = 1\} = p_k, \quad P\{\xi_{nk} = 0\} = q_k,
\]

where $p_k = p_k(n), q_k = q_k(n)$ and $p_k + q_k = 1$. Then Lyapunov’s theorem can be used to derive asymptotically normal distributions [10, Section 1.2].

**Theorem 5.2 (Lyapunov)** Let

\[
V_n^2 = \sum_{k=1}^{n} p_k q_k, \quad \eta_n = V_n^{-1} \sum_{k=1}^{n} (\xi_{nk} - p_k).
\]

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If \( V_n \to \infty \) as \( n \to \infty \), then the sequence \( \{\eta_n\} \) is asymptotically standard normal.

The above theorem enables us to derive the asymptotic distribution of the random variable \( \eta_n \).

**Theorem 5.3** The distribution of the random variable

\[
\eta_n = \frac{\xi_n - E\xi_n}{\sqrt{\text{Var}\xi_n}}
\]

converges to the standard normal distribution as \( n \to \infty \).

**Proof.** Since the polynomials \( d_n^B(q) \) have distinct, real and non-positive roots, we may express \( d_n^B(q) \) as

\[
d_n^B(q) = q(q + \alpha_1)(q + \alpha_2) \cdots (q + \alpha_{n-1}).
\]

Hence the random variable \( \xi_n \), namely, the number of type \( B \) excedances in a random type \( B \) derangement on \([n]\), can be represented as the sum of independent random variables

\[
\xi_n = \xi_{n1} + \xi_{n2} + \cdots + \xi_{n,n-1} + \xi_{n,n},
\]

where \( \xi_{n1}, \xi_{n2}, \ldots, \xi_{n,n-1}, \xi_{n,n} \) form a Poisson sequence with \( p_k = \frac{1}{1+q_k} \), for \( k = 1, 2, \ldots, n-1 \) and \( p_n = 1 \). On the other hand, the formula (5.12) implies that \( \text{Var}\xi_n \to \infty \) as \( n \to \infty \). Therefore, from Theorem 5.2 we deduce that \( \eta_n \) is asymptotically standard normal.

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