Aldous’ Spectral Gap Conjecture for Normal Sets

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Abstract

Let $G$ be a finite group and $\Sigma \subseteq G$ a symmetric subset. Every eigenvalue of the adjacency matrix of the Cayley graph $\text{Cay}(G, \Sigma)$ is naturally associated with some irreducible representation of $G$. Aldous’ spectral gap conjecture, proved by Caputo, Liggett and Richthammer [CLR10], states that if $\Sigma$ is a set of transpositions in the symmetric group $S_n$, then the second eigenvalue of $\text{Cay}(S_n, \Sigma)$ is always associated with the standard representation of $S_n$. Inspired by this seminal result, we study similar questions for other types of sets in $S_n$. Specifically, we consider normal sets: sets that are invariant under conjugation. Relying on character bounds due to Larsen and Shalev [LS08], we show that for large enough $n$, if $\Sigma \subset S_n$ is a full conjugacy class, then the largest non-trivial eigenvalue is always associated with one of eight low-dimensional representations. We further show that this type of result does not hold when $\Sigma$ is an arbitrary normal set, but a slightly weaker result does hold. We state a conjecture in the same spirit regarding an arbitrary symmetric set $\Sigma \subset S_n$.

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1 Introduction

Consider a finite group $G$ and a subset $\Sigma \subseteq G$ which is symmetric, namely, $g \in \Sigma \implies g^{-1} \in \Sigma$. The $|G| \times |G|$ adjacency matrix $A$ of the Cayley graph $\text{Cay}(G, \Sigma)$ is symmetric and equals

$$\sum_{g \in \Sigma} \rho_{\text{Reg}}(g)$$

where $\rho_{\text{Reg}}$ is the right regular representation of $G$, namely, $\rho_{\text{Reg}}(g)$ is the permutation matrix depicting the multiplication from the right by $g$. Recall that the regular representation of $G$ decomposes as a direct sum of all irreducible representations (irreps for short) of $G$, each

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appearing with multiplicity identical to its dimension\(^1\). An appropriate change of basis thus turns \(A\) into a block-diagonal matrix, with \(\dim(\rho)\) blocks of size \(\dim(\rho) \times \dim(\rho)\) for every irrep \(\rho\) of \(G\). The value of each of these \(\dim(\rho)\) blocks is \(\sum_{g \in \Sigma} \rho(g)\). This shows that the multiset of eigenvalues of \(A\) can be partitioned into sub-multisets, each of which is associated with some\(^2\) \(\rho \in \hat{G}\). For example, the largest, trivial eigenvalue of \(A\) is \(|\Sigma|\): this is the eigenvalue corresponding to the constant eigenfunction, and it is associated with the trivial representation of \(G\).

The current work focuses on the symmetric group \(S_n\). We consider the single eigenvalue associated with the sign representation to be trivial (in addition to the eigenvalue coming from the trivial representation), and denote by \(\lambda(S_n, \Sigma)\) the largest non-trivial eigenvalue of \(\lambda(S_n, \Sigma)\) a symmetric set \(\Sigma \subseteq S_n\). If we think of \(\Sigma\) as an element of \(\mathbb{R}[S_n]\), we can thus write

\[
\lambda(S_n, \Sigma) = \max_{\rho \in S_n \setminus \{\text{triv, sgn}\}} \lambda_1(\rho(\Sigma)),
\]

where \(\lambda_1(\rho(\Sigma))\) marks the largest eigenvalue of the matrix \(\rho(\Sigma)\) (which has only real eigenvalues as \(\Sigma\) is symmetric).

Many properties of a regular graph are related to the value of its second largest eigenvalue. Primarily, the spectral gap \(\lambda_1 - \lambda_2\) is a good measure for the extent to which the graph is “expanding” (see, e.g., the surveys [HLW06, Lub12]). Around 1992, David Aldous conjectured the following: whenever \(\Sigma \subset S_n\) is a set of transpositions, the largest non-trivial eigenvalue \(\lambda(S_n, \Sigma)\) is equal to the largest eigenvalue associated with the \((n-1)\)-dimensional irrep of \(S_n\) corresponding to the young diagram \((n-1, 1)\), often called the standard representation\(^3\). We denote this irrep \(\text{std}\). This conjecture was proved in 2009 by Caputo, Liggett and Richthammer. In fact, they proved a stronger version applying to weighted Cayley graphs as well:

**Theorem 1.1** (Aldous’ spectral gap conjecture, proved in [CLR10]). Let \(\Sigma \in \mathbb{R}[S_n]\) be supported on transpositions, with non-negative coefficients. Then the second eigenvalue of the weighted Cayley graph \(\text{Cay}(S_n, \Sigma)\) is equal to the largest eigenvalue of the standard representation. Namely,

\[
\lambda(S_n, \Sigma) = \lambda_1(\text{std}(\Sigma)).
\]

In other words, whenever \(\Sigma\) is supported on transpositions, the second eigenvalue of the Cayley graph \(\text{Cay}(S_n, \Sigma)\) is equal to the second eigenvalue of the Schreier graph \(\text{Sch}(S_n, \Sigma, [n])\) depicting the action of \(S_n\) on \([n] = \{1, \ldots, n\}\) with respect to the set \(\Sigma\).

Aldous’ conjecture cannot be extended as is to arbitrary symmetric sets \(\Sigma \subset S_n\). First, if \(\Sigma\) is contained in a proper transitive subgroup \(H \leq S_n\), \(H \neq A_n\), such as \((1 \, 2 \, \ldots \, n)\), then \(\text{Cay}(S_n, \Sigma)\) has more than two connected components and so \(\lambda(S_n, \Sigma) = \lambda_1(S_n, \Sigma) = |\Sigma|\), whereas the Schreier graph \(\text{Sch}(S_n, \Sigma, [n])\) is connected and so

\[
\lambda_1(\text{std}(\Sigma)) = \lambda_2(\text{Sch}(S_n, \Sigma, [n])) < \lambda_1(\text{Sch}(S_n, \Sigma, [n])) = |\Sigma|.
\]

In addition, there are known examples of generating sets of \(S_n\) for which the second eigenvalue of the Cayley graph is not associated with \(\text{std}\). For instance, if \(\Sigma = \{\text{id}, (1 \, 2), (1 \, 2 \, \ldots \, n)^{\pm 1}\}\),

\(^1\)Throughout this paper we use some standard facts from the theory of group representations and, more specifically, from the theory of representations of the symmetric groups \(S_n\). Good references are [FH91] for the general theory and [Ful97] for representations of \(S_n\). For the point of view of Group Representation when studying the spectrum of Cayley graphs, consult [Dia88].

\(^2\)We use the standard notation of \(\hat{G}\) for the set of isomorphism types of irreps of \(G\).

\(^3\)Occasionally, several different irreps give rise to an eigenvalue which is equal to \(\lambda(S_n, \Sigma)\). Aldous’ conjecture says that when \(\Sigma\) is a set of transpositions, the standard irrep is always one of these irreps.
it can be shown that $\lambda_1(\text{std}(\Sigma)) < \max(\lambda_1(\rho_1(\Sigma)), \lambda_1(\rho_2(\Sigma)))$ where $\rho_1$ is the irrep corresponding to the Young diagram $(n-2,2)$ and $\rho_2$ to $(n-2,1,1)$ – we elaborate in Example 4.2. Full conjugacy classes also occasionally have $\lambda(S_n, \Sigma)$ not associated with std, as described below.

However, the classification of multiply-transitive finite groups, which follows from the classification of finite simple groups, suggests a different possible extension of Aldous’ conjecture for arbitrary generating sets. According to this classification, a finite 4-transitive group is either $S_n$ ($n \geq 4$), $A_n$ ($n \geq 6$), or one of the four Mathieu groups $M_{11}, M_{12}, M_{23}$ and $M_{24}$ (where $M_k$ is a subgroup of $S_k$ and is 4-transitive in its action on $\{1, \ldots, k\}$ for $k = 11, 12, 23, 24$) [Cam99, Theorem 4.11]. It follows that for $n \geq 25$, if $\Sigma \subset S_n$ does not generate $A_n$ or $S_n$, it is not 4-transitive, and so the Schreier graph associated with the action of $S_n$ on 4-tuples of distinct elements in $[n]$ is not connected. The non-trivial irreps appearing in the decomposition of the action of $S_n$ on 4-tuples are precisely the 11 irreps associated with young diagrams.

Is it possible that this set of irreps, or perhaps a larger finite set of irreps can not only capture the mere existence of a spectral gap but also the exact value of the spectral gap? The following potential generalization of Theorem 1.1 was raised during discussions between Gady Kozma and the second author.

**Definition 1.2.** A series of irreps $\{\rho_n \in \hat{S}_n\}_{n \geq n_0}$ is called a family of irreps if one of the following conditions holds:

1. Either the structure of the associated Young diagram outside the first row is constant, namely, for every $n \geq n_0$, $\rho_{n+1}$ is obtained from $\rho_n$ by adding a block to the first row, or

2. The structure of the associated Young diagram outside the first column is constant, namely, for every $n \geq n_0$, $\rho_{n+1}$ is obtained from $\rho_n$ by adding a block to the first column.

An element $\Sigma = \sum_{\sigma \in S_n} \alpha_\sigma \sigma \in \mathbb{R}[S_n]$ is called symmetric if $\alpha_\sigma = \alpha_{\sigma^{-1}}$ for every $\sigma \in S_n$ and non-negative if $\alpha_\sigma \geq 0$ for every $\sigma \in S_n$. We also denote $|\Sigma| \overset{\text{def}}{=} \sum_{\sigma \in S_n} \alpha_\sigma$.

**Question 1.3.** Is there a finite set of families of irreps $\rho^{(1)}, \ldots , \rho^{(m)}$ as in Definition 1.2 such that for every large enough $n$ and every symmetric non-negative $\Sigma \in \mathbb{R}[S_n]$, we have that

$$\lambda(S_n, \Sigma) = \max_{i=1}^{m} \lambda_1(\rho^{(i)}_n(\Sigma))?$$

This work studies a special case of this question, where $\Sigma$ is normal, in the sense that the coefficient $\alpha_\sigma$ are constant on every conjugacy class$^4$. In this case, the eigenvalues of the different irreps can be computed directly from character values (see Lemma 2.1 below). Our first result gives a positive answer to Question 1.3 when $\Sigma$ is a single conjugacy class. For every $n \geq 8$, consider the following set of eight irreps:

$$\text{EIGHT}_n = \left\{ (n-1,1), (n-2,2), (n-3,3), (n-3,2,1), (n-4,4), (n-1,1)^t, (n-2,2)^t, (n-2,1,1)^t \right\} \subset \hat{S}_n,$$

where $(n-1,1)^t$ is the irrep transpose to $(n-1,1)$, namely it is $(n-1,1) \otimes \text{sgn} = (2,1,1,\ldots,1)$, and so on.

\footnote{Note that a normal element $\Sigma \in S_n$ is symmetric, because $\sigma^{-1}$ is conjugate to $\sigma$ for every $\sigma \in S_n$.}
Theorem 1.4. There exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, if $\Sigma \subset S_n$ is a full, single conjugacy class, then $\lambda(S_n, \Sigma)$ is attained by one of the eight irreps in $EIGHT_n$:

$$\lambda(S_n, \Sigma) = \max_{\rho \in EIGHT_n} \lambda_1(\rho(\Sigma)).$$

In the proof of Theorem 1.4 we rely heavily on asymptotically sharp character bounds due to Larsen and Shalev [LS08] – see Section 2. The statement of Theorem 1.4 does not hold for $n = 16$: for $\Sigma = [(1 2 3 4 5) (6 7 8 9 10) (11 12 13 14 15)]^{S_{16}}$, the largest non-trivial eigenvalue $\lambda(S_{16}, \Sigma)$ is associated with the irreps $(11, 5)$ and its transpose. However, simulations suggest that this is the largest counter-example:

Conjecture 1.5. Theorem 1.4 holds with $N_0 = 17$.

When $\Sigma \in \mathbb{R}[S_n]$ is a general non-negative normal element, not only are the eight irreps from Theorem 1.4 insufficient, but no finite set of families of irreps suffices to capture $\lambda(S_n, \Sigma)$, and so the answer to Question 1.3 turns out to be negative:

Theorem 1.6. Let $\rho^{(1)}, \ldots, \rho^{(m)}$ be families of irreps of $\{S_n\}_{n \geq n_0}$ as in Definition 1.2, different from triv and sgn. Then for every large enough $n$, there is a non-negative normal element $\Sigma \in \mathbb{R}[S_n]$ such that

$$\lambda(S_n, \Sigma) > \max_{i=1}^m \lambda_1(\rho_n^{(i)}(\Sigma)).$$

However, our analysis for the case of a single conjugacy class does readily show that

Theorem 1.7. Let $\Sigma \in \mathbb{R}[S_n]$ be non-negative and normal. Then the spectral gap $|\Sigma| - \lambda(S_n, \Sigma)$ is bounded by the spectral gap of the standard representation multiplied by a decaying multiplicative factor:

$$|\Sigma| - \lambda(S_n, \Sigma) \geq [|\Sigma| - \lambda_1(\text{std}(\Sigma))] \cdot [1 - o_n(1)].$$

All the evidence we have so far supports the following generalization of Aldous’ spectral gap conjecture (Theorem 1.1), which was raised, as was Question 1.3 above, during discussions between Gady Kozma and the second author:

Conjecture 1.8 (Kozma-Puder). There is a finite set $\rho^{(1)}, \ldots, \rho^{(m)}$ of families of irreps of the groups $\{S_n\}$ and a universal constant $0 < c < 1$, so that for every large enough $n$ and for every symmetric non-negative element $\Sigma \in \mathbb{R}[S_n]$ we have

$$|\Sigma| - \lambda(S_n, \Sigma) \geq c \cdot \left[|\Sigma| - \max_{i=1}^m \lambda_1(\rho_n^{(i)}(\Sigma))\right].$$

This conjecture, if true, would yield that random pairs of permutations in $S_n$ give rise to a uniform family of expanders, which is a long standing open question (e.g., [Lub12, Problem 2.28]). It would also yield that for every generating set $\Sigma$ of $A_n$ of bounded size, the diameter of Cay($A_n$, $\Sigma$) is bounded by some $n^c$ where $c$ is a universal constant. This is very close to Babai’s conjecture [BS92, Conjecture 1.7] on the diameter of alternating groups. See Section 4 for more details.

The paper is organized as follows. In Section 2 we consider sets consisting of a single conjugacy class and prove Theorem 1.4. Section 3 deals with arbitrary normal sets and contains the proofs of Theorems 1.6 and 1.7. In Section 4 we further discuss Conjecture 1.8 and its consequences.
2 A Single Conjugacy Class

We start with the following standard lemma, which explains why all eigenvalues of $\text{Cay}(G, \Sigma)$ can be read off from the character table of $G$ when $\Sigma \in \mathbb{R}[G]$ is normal. The important quantity here is the normalized character of $\rho \in \hat{S}_n$ which we denote by $\hat{\chi}_\rho$:

$$\hat{\chi}_\rho(\sigma) \overset{\text{def}}{=} \frac{\chi_\rho(\sigma)}{\chi_\rho(1)},$$

where $\chi_\rho(\sigma)$ is the character of $\rho$ and $\sigma \in S_n$. Recall that the entire character table of $S_n$ consists of integers and that $|\chi_\rho(\sigma)| \leq \chi_\rho(1)$, so $\hat{\chi}_\rho(\sigma) \in \mathbb{Q} \cap [-1, 1]$ for every $\rho \in \hat{S}_n$ and every $\sigma \in S_n$.

**Lemma 2.1.** Let $G$ be a finite group and $\Sigma \in \mathbb{R}[G]$ be a normal element. Denote by $\alpha_C$ the coefficient in $\Sigma$ of every element $g$ in the conjugacy class $C$. Then for an irrep $\rho \in \hat{G}$, the matrix $\rho(\Sigma)$ is the scalar

$$\sum_C \alpha_C |C| \hat{\chi}_\rho(\sigma),$$

where the summation is over the conjugacy classes in $G$ and $\hat{\chi}_\rho(C)$ is the normalized character of each of the elements in $C$.

**Proof.** Since $\Sigma$ is in the center of $\mathbb{C}[G]$, $\rho(\Sigma)$ is an endomorphism of an irreducible representation, hence a scalar by Schur’s Lemma. The trace of $\rho(\Sigma)$ is $\sum_C \alpha_C |C| \chi_\rho(C)$. \hfill $\square$

This section deals with a single conjugacy class $\Sigma = 1_C$, which means that

$$\lambda(S_n, \Sigma) = |C| \cdot \max_{\rho \in \hat{S}_n \setminus \{\text{triv,sgn}\}} \hat{\chi}_\rho(C).$$

The statement of Theorem 1.4 is therefore equivalent to that for every $n \geq N_0$ and every permutation $\sigma \in S_n$ we have

$$\max_{\rho \in \hat{S}_n \setminus \{\text{triv,sgn}\}} \hat{\chi}_\rho(\sigma) = \max_{\rho \in EIGHT} \hat{\chi}_\rho(\sigma).$$

(2.3)

For a given $\sigma \in S_n$, if the maximum in (2.3) is obtained by some $\rho \in \hat{S}_n \setminus \{\text{triv,sgn}\}$, we say that “$\rho$ rules for $\sigma$ in $S_n$”.

Our main tool in analyzing the normalized characters $\hat{\chi}_\rho(\sigma)$ is the following asymptotically sharp character bounds established by Larsen and Shalev. We use the notation $c_\ell(\sigma)$ for the number of $\ell$-cycles in the permutation $\sigma \in S_n$. For example, $c_1(\sigma)$ is the number of fixed points.

**Theorem 2.2 ([LS08, Theorem 1.3]).** Let $\sigma \in S_n$ and let $f = \max(c_1(\sigma), 1)$. For every irrep $\rho \in \hat{S}_n$, its character $\chi_\rho$ satisfies

$$\hat{\chi}_\rho(\sigma) \leq |\chi_\rho(1)| \frac{\log(\omega f)}{\log n} + \varepsilon_n,$$

where $\varepsilon_n$ is a real number tending to 0 as $n \to \infty$.

(2.4)

Our strategy in proving Theorem 1.4 is as follows: using Theorem 2.2, we show that for large enough $n$, if $\sigma \in S_n$ has exactly two fixed points, then the standard representation std rules, namely, the maximal normalized character is $\hat{\chi}_{\text{std}}(\sigma) = \frac{1}{n!}$. Using a simple induction argument we then show that the same is true for every large enough $n$ when $c_1(\sigma) \geq 2$. Finally, we use Theorem 2.2 again to deal with the case $c_1(\sigma) \in \{0, 1\}$. Indeed, Theorem 1.4 follows immediately from the following two propositions, which we prove in the following two subsections.
Proposition 2.3. There is some $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$ and every $\sigma \in S_n$ with $c_1(\sigma) \geq 2$, the standard irrep std = $(n-1,1)$ rules.

Proposition 2.4. There is some $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$ and every $\sigma \in S_n$ with $c_1(\sigma) \leq 1$, one of the eight irreps in $\mathcal{EIGHT}_n$ rules.

2.1 The case $c_1(\sigma) \geq 2$

The following lemma is standard. We give its proof for completeness.

Lemma 2.5. Let $\left\{ \rho_n \in \hat{S}_n \right\}_{n \geq n_0}$ be a family of irreps as in Definition 1.2 with constant structure outside the first row, so that the first row of $\rho_n$ has exactly $n-k$ blocks. Then there is a polynomial $p \in \mathbb{Q}[c_1, \ldots, c_k]$ so that for $n \geq 2k$ and $\sigma \in S_n$,

$$\chi_{\rho_n}(\sigma) = p(c_1(\sigma), \ldots, c_k(\sigma)),$$

and, for the transpose family,

$$\chi_{\rho_n^t}(\sigma) = \text{sgn}(\sigma) \cdot p(c_1(\sigma), \ldots, c_k(\sigma)).$$

In particular, the dimension $\dim \rho_n = \chi_{\rho_n}(1)$ is given by a polynomial in $n$ equal to $p(n,0,\ldots,0)$.

The fact that $\chi_{\rho_n}(1) = \dim \rho_n$ is given by a polynomial in $n$ is also evident from the hook length formula (e.g., [FH91, Formula 4.12]). The polynomials associated with every family of irreps with at most four blocks outside the first row are listed in Table 1.

Proof. For a partition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \vdash n$, let $M^\lambda$ denote the reducible representation associated with $\lambda$: this is the permutation representation describing the action of $S_n$ on partitions of $\{1, \ldots, n\}$ with sizes of blocks given by $\lambda$ (see [Ful97, §7.2]). It is not hard to see that the statement of the lemma holds for families $\{M^{\lambda_n}\}_{n \geq n_0}$ when $\lambda_{n+1}$ is obtained from $\lambda_n$ by increasing $\lambda^{(1)}$ by one. For example, if $\lambda_n = (n-3,2,1)$, then $\chi_{M^{\lambda_n}}(\sigma)$, which equals the number of fixed points of $\sigma$ in this action of $S_n$, is $c_1 \cdot \left(\binom{c_1-1}{2} + c_1c_2\right)$.

For every $\lambda \vdash n$ as above, the character $\chi_{\rho^\lambda}$ of the irrep $\rho^\lambda$ corresponding to $\lambda$ is given by a linear combination with integer coefficients of the representations $\{M^\mu \mid \mu \triangleleft \lambda\}$, where “$\triangleleft$” marks the dominance relation (see [Ful97, §7.2]). Moreover, when $\lambda^{(1)} = n-k \geq \frac{3}{2}$, this linear combination is independent of $n$, namely, the coefficient of every $\mu \triangleleft \lambda$ depends only on the structure of $\mu$ outside the first row. For example,

$$\chi_{\rho^{(n-3,2,1)}} = \chi_{M^{(n-2,1,1)}} - \chi_{M^{(n-2,2)}} - \chi_{M^{(n-1,1)}} + \chi_{M^{(n)}}.$$

The statement of the lemma follows. \qed

Lemma 2.6. Let $n \geq 13$, and let $\rho \in \hat{S}_n$ be an irrep represented by a Young diagram with at least three blocks outside the first row and at least three blocks outside the first column. Then $\dim \rho \geq n^{2.05}$.

Proof. If $\rho$ has exactly three blocks outside the first row then $\rho$ is one of $(n-3,3)$, $(n-3,2,1)$ or $(n-3,1,1,1)$, in which case its dimension is $\frac{n(n-1)(n-5)}{6}$, $\frac{n(n-2)(n-4)}{3}$ or $\frac{(n-1)(n-2)(n-3)}{6}$, respectively. In each of these cases $\dim \rho \geq n^{2.05}$ for $n \geq 13$. The transpose case where $\rho$ has exactly three blocks outside the first column is identical.

For the case $\rho$ has at least four blocks outside the first row/column we use induction on $n$. It is easy to check directly (in the computer) that the statement is true for $n = 13, 14$: all
93 irreps of $S_{13}$ and 127 irreps of $S_{14}$ satisfying the assumption of the lemma have dimension $\geq n^{2.05}$. For $n \geq 15$, we assume the statement holds for $n - 1$ and for $n - 2$ and that $\rho \in \hat{S}_n$ has at least four blocks outside the first row and at least four blocks outside the first column. By the branching rule, $\dim \rho = \sum_{\rho' = \rho - \square} \dim \rho'$, the sum being over all $\rho' \in \hat{S}_{n-1}$ obtained from $\rho$ by removing one block. If the Young diagram corresponding to $\rho$ is not a rectangle, there are at least two such $\rho'$, each with at least three blocks outside the first row and outside the first column, and we are done as $2(n-1)^{2.05} \geq n^{2.05}$ for $n \geq 15$. Finally, if $\rho$ is a rectangle, it has at least $\frac{7}{8} > 4$ blocks outside the first row and outside the first column, and there are exactly two ways to remove two blocks from $\rho$. The branching rule now gives $\dim \rho = \dim \rho_1 + \dim \rho_2$ with $\rho_1, \rho_2 \in \hat{S}_{n-2}$ satisfying the assumption in the lemma. We are done as $2(n-2)^{2.05} \geq n^{2.05}$ for $n \geq 15$.

Lemma 2.7. There is some $N_3 \in \mathbb{N}$ such that std rules for every $n \geq N_3$ and $\sigma \in S_n$ with $c_1(\sigma) = 2$. Namely, for every $\rho \in \hat{S}_n \setminus \{\text{triv, sgn}\}$,

$$\hat{\chi}_\rho(\sigma) \leq \hat{\chi}_{\text{std}}(\sigma) = \frac{1}{n - 1}.$$  

Proof. Fix $N_3 \geq 13$ so that for every $n \geq N_3$, the term $\varepsilon_n$ in (2.4) satisfies

$$2.05 \cdot \left( - \frac{1}{2} \log 2 \log n + \varepsilon_n \right) \leq -1.$$  

Then, if $n \geq N_3$ and $\rho \in \hat{S}_n$ satisfies the assumptions in Lemma 2.6, it follows from Theorem 2.2 that

$$\hat{\chi}_\rho(\sigma) \leq (n^{2.05})^{-\frac{1}{2} + \frac{1.05}{2\log n} + \varepsilon_n} \leq \frac{1}{n} \leq \frac{1}{n - 1}.$$  

Finally, if $\rho$ is one of the five remaining irreps $(n - 1, 1)^t$, $(n - 2, 2)^t$, $(n - 2, 1, 1)^t$ and $(n - 2, 1, 1)^t$, we use the explicit expressions in Table 1 to show that $\hat{\chi}_\rho(\sigma) \leq \frac{1}{n - 1}$. For example, if $\rho = (n - 2, 1, 1)^t$ and $\sigma$ is odd, we get

$$\hat{\chi}_\rho(\sigma) = \frac{2c_2(\sigma)}{(n - 1)(n - 2)} \leq \frac{2 \cdot \frac{n - 2}{2}}{(n - 1)(n - 2)} = \frac{1}{n - 1},$$  

so at worst there is a tie between $(n - 1, 1)^t$ and $(n - 2, 1, 1)^t$ when $c_1(\sigma) = 2$. 

Table 1: Dimensions and characters of irreps with at most four blocks outside the first row.

| $\rho$ | $\dim(\rho) = \chi_\rho(1)$ | $\chi_\rho(\sigma)$ if $c_1(\sigma) = c_i$ |
|-------|-----------------|-----------------|
| $(n)$  | 1               | 1               |
| $(n - 1, 1)$ | $n - 1$       | $c_1 - 1$       |
| $(n - 2, 2)$ | $\frac{n(n - 3)}{2}$ | $c_1(c_1 - 3) + c_2$ |
| $(n - 2, 1, 1)$ | $(n - 1)(n - 2)$ | $(c_1 - 1)(c_1 - 2) - c_2$ |
| $(n - 3, 3)$ | $\frac{n(n - 1)(n - 5)}{6}$ | $c_1(c_1 - 1)(c_1 - 3) + c_2 + (c_1 - 1) c_2 + c_3$ |
| $(n - 3, 2, 1)$ | $\frac{n(n - 2)(n - 4)}{3}$ | $c_1(c_1 - 2)(c_1 - 4) - c_3$ |
| $(n - 3, 1, 1, 1)$ | $\frac{(n - 1)(n - 2)(n - 3)}{24}$ | $(c_1 c_1 - 2)(c_1 - 3) + (c_1 - 1) c_2 + c_3$ |
| $(n - 4, 4)$ | $\frac{n(n - 1)(n - 2)(n - 7)}{24}$ | $c_1(c_1 - 2)(c_1 - 6) + (c_1 - 1)(c_1 - 3)(c_1 - 6)$ + $c_2$ |
| $(n - 4, 3, 1)$ | $\frac{n(n - 1)(n - 3)(n - 6)}{8}$ | $c_1(c_1 - 2)(c_1 - 4)(c_1 - 5) + (c_2 - 2)c_2 - (c_1 - 1)c_3$ |
| $(n - 4, 2, 2)$ | $\frac{n(n - 1)(n - 4)(n - 5)}{12}$ | $c_1(c_1 - 2)(c_1 - 4)(c_1 - 5) + (c_1 - 2)c_2 - (c_1 - 1)c_3$ |
| $(n - 4, 2, 1, 1)$ | $\frac{n(n - 2)(n - 3)(n - 5)}{8}$ | $c_1(c_1 - 2)(c_1 - 3)(c_1 - 5)$ + $(c_1 - 2)c_2 - (c_1 - 1)c_3$ |
| $(n - 4, 1, 1, 1, 1)$ | $\frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24}$ | $(c_1 c_1 - 2)(c_1 - 3)(c_1 - 4)$ + $(c_1 - 2)(c_1 - 3)(c_1 - 4) + (c_1 - 2)c_2 - (c_1 - 1)c_3 - c_4$ |
The following easy but crucial lemma says that the normalized character of a permutation with at least one fixed point (which we can take without loss of generality to be \( n \)) is bounded from both sides by normalized characters with one block omitted.

**Lemma 2.8.** Let \( \sigma \in S_n \) be a permutation satisfying \( \sigma(n) = n \). Then \( \sigma \) can be considered also as an element of \( S_{n-1} \), and for every \( \rho \in \hat{S}_n \), the normalized character \( \tilde{\chi}_\rho(\sigma) \) is a weighted average of the normalized characters \( \{ \tilde{\chi}_{\rho'}(\sigma) \} \) for \( \rho' = \rho - \square \in \hat{S}_{n-1} \).

Here \( \rho' \) runs over all options to delete a block from \( \rho \) to obtain a Young diagram with \( n-1 \) blocks.

**Proof.** By the branching rule \( \chi_\rho(1) = \sum_{\rho' = \rho - \square} \chi_{\rho'}(1) \), and, similarly \( \chi_\rho(\sigma) = \sum_{\rho' = \rho - \square} \chi_{\rho'}(\sigma) \). For any real numbers \( x_1, \ldots, x_k \) and positive real numbers \( y_1, \ldots, y_k \), \( \frac{x_1 + \cdots + x_k}{y_1 + \cdots + y_k} \) is a convex combination of \( \frac{x_i}{y_i} \) for any \( 1 \leq i \leq k \).

**Corollary 2.9.** If \( \sigma \in S_n \) satisfies \( \sigma(n) = n \) and std rules for \( \sigma \) in \( S_{n-1} \), it also rules for \( \sigma \) in \( S_n \).

**Proof.** Denote the standard irrep in \( S_{n-1} \) by std\(^t\). Then

\[
\tilde{\chi}_{\text{std}}(\sigma) = \frac{c_1(\sigma) - 1}{n-1} \geq \frac{c_1(\sigma) - 2}{n-2} = \tilde{\chi}_{\text{std}}(\sigma).
\]

On the other hand, if \( \rho \in \hat{S}_n \setminus \{ \text{triv, sgn, std, std}^t \} \), then removing a block from \( \rho \) does not lead to the trivial or sign irreps of \( S_{n-1} \), and so by Lemma 2.8,

\[
\tilde{\chi}_{\rho}(\sigma) \leq \tilde{\chi}_{\text{std}}(\sigma).
\]

Finally, regarding std\(^t\), if \( \sigma \) is even then \( \tilde{\chi}_{\text{std}}(\sigma) = \tilde{\chi}_{\text{std}}(\sigma) \), and if \( \sigma \) is odd then as std rules for \( \sigma \) in \( S_{n-1} \), we have \( c_1(\sigma) \geq 2 \) and \( \tilde{\chi}_{\text{std}}(\sigma) = -\tilde{\chi}_{\text{std}}(\sigma) < \tilde{\chi}_{\text{std}}(\sigma) \).

The support of \( \sigma \in S_n \) is

\[
\text{supp}(\sigma) = \{ i \in \{1, \ldots, n\} | \sigma(i) \neq i \}.
\]

**Corollary 2.10.** Let \( N_3 \) be as in Lemma 2.7. Then std rules for every \( n \geq N_3 \) and \( \sigma \in S_n \) with \( c_1(\sigma) \geq 2 \) and \( |\text{supp}(\sigma)| \geq N_3 - 2 \).

**Proof.** Let \( k = |\text{supp}(\sigma)| + 2 \geq N_3 \). By omitting \( n-k \) fixed points from \( \sigma \), we may think of \( \sigma \) as representing a conjugacy class in \( S_k \) with exactly two fixed points. By Lemma 2.7, std rules for \( \sigma \) in \( S_k \). By applying Corollary 2.9 \( n-k \) times, we deduce that std also rules for \( \sigma \) in \( S_n \).

**Lemma 2.11.** Let \( \sigma \in S_r \) and consider it as a permutation in \( S_n \) for any \( n \geq r \) by appending \( n-r \) fixed points. For large enough \( n \), std rules for \( \sigma \) in \( S_n \).

**Proof.** Without loss of generality, \( r \geq 5 \), so every \( \rho \in \hat{S}_r \setminus \{ \text{triv, sgn} \} \) is faithful, and \( c_1(\sigma) \geq 1 \) in \( S_r \), so if std\(^t\) rules for some \( n \geq r \), so does std. Assume that some \( \rho \in \hat{S}_r \setminus \{ \text{triv, sgn} \} \) rules for \( \sigma \) in \( S_r \). As \( \rho \) is faithful, \( B \overset{\text{def}}{=} \tilde{\chi}_\rho(\sigma) < 1 \). As \( n \) increases, as long as std does not rule, all normalized characters of irreps in \( \hat{S}_n \setminus \{ \text{triv, sgn} \} \) are bounded from above by \( B \); indeed, by induction this is true for all such irreps in \( \hat{S}_{n-1} \), and by Lemma 2.8 this is also true for every irrep in \( \hat{S}_n \setminus \{ \text{triv, sgn, std, std}^t \} \). For std and std\(^t\) this is true by the assumption that std does not rule. In contrast, the normalized character of std, which is \( \frac{n-|\text{supp}(\sigma)|-1}{n-1} \), tends to 1 as \( n \to \infty \). Therefore, for some \( n_0 \), std rules. Corollary 2.9 then implies that std rules for \( \sigma \) for every \( n \geq n_0 \).
Corollary 2.12. Let $M \in \mathbb{N}$ be a constant. There is some $N_4 = N_4(M) \in \mathbb{N}$ such that for every $n \geq N_4$ and every $\sigma \in S_n$ with $|\text{supp}(\sigma)| \leq M$, std rules for $\sigma$ in $S_n$.

Proof. Every such $\sigma$ belongs to a conjugacy class in $S_n$ which is obtained from blowing up (by appending fixed point) a fixed-point-free conjugacy class in $S_k$ for some $k \leq M$. This is a finite set of starting points, which means we need to apply Lemma 2.11 finitely many times.

Proof of Proposition 2.3. Let $N_3$ be the constant from Lemma 2.7 and Corollary 2.10, and $N_4 = N_4(M)$ with $M = N_3 - 3$ be the constant from Corollary 2.12. Set $N_1 = \max(N_3, N_4)$. Then std rules for every $n \geq N_1$ and for every $\sigma \in S_n$ with $c_1(\sigma) \geq 2$.

2.2 The case $c_1(\sigma) \leq 1$

The proof strategy of Proposition 2.4 is the same as in the case $c_1(\sigma) = 2$, albeit significantly more tedious. The difference is that the largest normalized character is not necessarily of order $\frac{1}{n}$ as in the $c_1(\sigma) = 2$ case, but can be of order as low as $\frac{1}{n^k}$. For example, this is the case when $\sigma$ is even, $c_1(\sigma) = c_3(\sigma) = 1$ and $c_2(\sigma) = 0$ (and see Table 2).

Lemma 2.13. Let $\{\rho_n\}_{n \geq n_0}$ be a family of irreps $\rho_n \in \widehat{S}_n$ as in Definition 1.2 with $k$ blocks outside the first row or outside the first column. Every monomial $\alpha_1^c \alpha_2^c \cdots \alpha_k^c$ in the associated polynomial $p$ from Lemma 2.5 satisfies $\sum_{i=1}^k i \cdot \alpha_i \leq k$. In addition, the polynomial $p(c_1, 0, \ldots, 0)$ giving the dimension of $\rho_n$ is of degree exactly $k$.

Proof. The first statement holds because it holds for the polynomials depicting the characters of the reducible representations $M^\lambda$ (see the proof of Lemma 2.5), and the character of $\rho_n$ is equal to a linear combination of $M^\lambda$’s with at most $k$ blocks outside the first row (namely, $\lambda^{(1)} \geq n - k$). The second statement is immediate from the hook length formula.

Lemma 2.14. For every large enough $n$ and every $\sigma \in S_n$

$$f(\sigma) \overset{\text{def}}{=} \max_{\rho \in \mathcal{LIHT}_n} \tilde{\chi}_\rho(\sigma) \geq \frac{3}{n(n-2)(n-4)}.$$

Proof. We deal with six different cases in the following table (the values of normalized characters can be read from Table 1):

| Assumptions on $\sigma$ | lower bound on $f(\sigma)$ |
|-------------------------|----------------------------|
| $c_1 \geq 2$            | $f(\sigma) \geq \tilde{\chi}_{\text{std}}(\sigma) \geq \frac{1}{n-1}$ |
| $c_1 = 1, c_3 \geq 1$   | $f(\sigma) \geq \tilde{\chi}_{(n-3,3)}(\sigma) = \frac{4(n-2)}{n(n-1)(n-5)} \geq \frac{6}{n(n-1)(n-5)}$ |
| $c_1 = 1, c_3 = 0$      | $f(\sigma) \geq \tilde{\chi}_{(n-3,2,1)}(\sigma) = \frac{2(n-1)}{n(n-2)(n-4)}$ |
| $c_1 = 0, c_2 \geq 1$   | $f(\sigma) \geq \tilde{\chi}_{(n-2,2)}(\sigma) = \frac{2(n-3)}{n(n-1)(n-3)} \geq \frac{2}{n(n-3)}$ |
| $c_1 = 0, c_2 = 0, \sigma$ is even | $f(\sigma) \geq \tilde{\chi}_{(n-2,1,1)}(\sigma) = \frac{2(n-4)}{n(n-1)(n-3)}$ |
| $c_1 = 0, c_2 = 0, \sigma$ is odd   | $f(\sigma) \geq \tilde{\chi}_{\text{std}}(\sigma) = \frac{1}{n-1}$ |

Lemma 2.15. Let $\{\rho_n\}_{n \geq n_0}$ be a family of irreps $\rho_n \in \widehat{S}_n$ as in Definition 1.2 with $k \geq 5$ blocks outside the first row or outside the first column. Then for large enough $n$ and every $\sigma \in S_n$ with $c_1(\sigma) \leq 1$,

$$\tilde{\chi}_{\rho_n}(\sigma) \leq f(\sigma) \overset{\text{def}}{=} \max_{\rho \in \mathcal{LIHT}_n} \tilde{\chi}_\rho(\sigma).$$
Proof. If $c_2 (\sigma) \geq 2$ then

$$f (\sigma) \geq \tilde{x}_{(n-2,2)} (\sigma) \geq \frac{2 (c_2 (\sigma) - 1)}{n (n-3)} \geq \frac{2}{n (n-3)}.$$  

In contrast, by Lemma 2.13, as $c_2 (\sigma), \ldots, c_k (\sigma) \leq n$, we have $\chi_{\rho_n} (\sigma) \leq O \left( \frac{n^{\frac{k}{2}}}{n-3} \right)$ and $\chi_{\rho_n} (1)$ is a polynomial in $n$ of degree $k$. We deduce that the normalized character $\tilde{\chi}_{\rho_n} (\sigma) \leq O \left( \frac{n^{\frac{k}{2}}}{n-3} \right)$ as $k \geq 5.$

If $c_2 (\sigma) \leq 1$ then by Lemma 2.14 $f (\sigma) \geq \frac{3}{n (n-2) (n-4)}$ whereas, by Lemma 2.13, $\chi_{\rho_n} (\sigma) \leq O \left( \frac{n^{\frac{k}{2}}}{n-3} \right)$ and so the normalized character $\tilde{\chi}_{\rho_n} (\sigma) \leq O \left( \frac{n^{\frac{k}{2}}}{n-3} \right) = O \left( \frac{n}{n-3} \right)$ as $k \geq 5.$

Of course, Lemma 2.15 is good enough to deal with every family of irreps separately, but not for all irreps uniformly. For this, we need to use Larsen-Shalev’s Theorem 2.2. First, as above, we need a uniform lower bound on the dimension of almost all irreps:

Lemma 2.16. Let $n \geq 39$, and let $\rho \in \hat{S}_n$ be an irrep represented by a Young diagram with at least $14$ blocks outside the first row and at least $14$ blocks outside the first column. Then $\dim \rho \geq n^{6.05}.$

Proof. We verified the statement numerically for $n = 39, \ldots, 48$. For the general case, assume that $\rho$ has exactly $14$ blocks outside the first row (the transpose case is identical). Consider the hook lengths at the first block of the Young diagram associated with $\rho$. The hook length at the $15$th block is $n - 28$, since the second row is of length $\leq 14$. The hook length at the $16$th block is $n - 29$ as so on. For $1 \leq i \leq 14$, the first $(i-1)$ columns contain at least $2$ $(i-1)$ blocks, so the hook length of the $i$th block is at most $n - 2 (i-1)$. The product of hook lengths of all blocks outside the first row is at most $14!$ (by the hook length formula for Young diagrams with $14$ blocks). Thus, by the hook length formula, we obtain that

$$\dim \rho = \frac{n!}{\text{product of hook lengths}} \geq \frac{n!}{n (n-2) (n-4) \cdots (n-26) \cdot (n-28)! \cdot 14!} = \frac{(n-1) (n-3) (n-5) \cdots (n-27)}{14!},$$

which is greater than $n^{6.05}$ for $n \geq 47$. For $n \geq 49$ and $\rho \in \hat{S}_n$ with at least $15$ blocks outside the first row or outside the first column, we proceed by induction exactly as in the proof of Lemma 2.6.

Proof of Proposition 2.4. As in the proof of Lemma 2.7, we deduce from Theorem 2.2 and Lemma 2.16 that for large enough $n$, for all $\sigma \in S_n$ with $c_1 (\sigma) \leq 1$ and for all $\rho \in S_n$ with at least $14$ blocks outside the first row or outside the first column, we have

$$\tilde{\chi}_\rho (\sigma) \leq \frac{1}{n^3},$$

which, by Lemma 2.14, is less than the maximal normalized character of $\sigma$ among the irreps in $\mathcal{EIGHT}_n$. For the finite number of families of irreps with $5 \leq k \leq 13$, where $k$ is the number of blocks outside the first row or column, we use Lemma 2.15. Finally, the normalized characters of all irreps with $k \leq 4$ appear in Table 1. Comparing them for large $n$ yields that for every $\sigma \in S_n$ with $c_1 (\sigma) \leq 1$, some $\rho \in \mathcal{EIGHT}_n$ rules. We omit the technical details of this comparison, but see Tables 2 and 3 for the ruling irrep in each case.
| $\rho$                      | $\rho$ rules for even $\sigma$ when: |
|-----------------------------|------------------------------------|
| $(n - 1, 1)$                | $c_1 \geq 2$                      |
| $(n - 1, 1)^t$              |                                    |
| $(n - 2, 2)$                | $c_1 = 1, c_2 \geq 2$             |
| $(n - 2, 2)^t$              | $c_1 = 0, c_2 \geq 1$             |
| $(n - 2, 1, 1)$             | $c_1 = 0, c_2 = 0, c_3 \leq \frac{n-4}{3}$ |
| $(n - 2, 1, 1)^t$           |                                    |
| $(n - 3, 3)$                | $c_1 = 1, c_2 = 1, c_3 \geq 1$    |
| $(n - 3, 3)^t$              | $c_1 = 1, c_2 = 0, c_3 \geq 2$    |
| $(n - 3, 2, 1)$             | $c_1 = 1, c_2 = 0, c_3 = 1, c_4 \leq \frac{n-5}{4}$ |
| $(n - 3, 2, 1)^t$           | $c_1 = 1, c_2 = 0, c_3 = \frac{n}{3}$ |
| $(n - 4, 4)$                | $c_1 = 1, c_2 = 1, c_3 = 0, c_4 \geq \frac{n+3}{8}$ |
| $(n - 4, 4)^t$              | $c_1 = 1, c_2 = 0, c_3 = 1, c_4 = \frac{n-4}{8}$ |
| $(n - 3, 3)$                | $c_1 = 1, c_2 = 1, c_3 = \frac{n-4}{3}$ |

Table 2: For large enough $n$, this table shows which $\rho \in EIGHT_n$ rules for every even $\sigma \in S_n$. Note that at least one of every pair of irreps in the left column belongs to $EIGHT_n$.

| $\rho$                      | $\rho$ rules for odd $\sigma$ when: |
|-----------------------------|------------------------------------|
| $(n - 1, 1)$                | $c_1 \geq 2$                      |
| $(n - 1, 1)^t$              | $c_1 = 0, c_2 \leq \frac{n-3}{4}$ |
| $(n - 2, 2)$                | $c_1 = 0, c_2 = \frac{3}{4}$      |
| $(n - 2, 2)^t$              | $c_1 = 1, c_2 = 0$                |
| $(n - 2, 1, 1)^t$           | $c_1 = 2, c_2 = \frac{n-2}{2}$    |
| $(n - 2, 1, 1)$             | $c_1 = 1, c_2 \geq 2$             |
| $(n - 2, 1, 1)^t$           | $c_1 = 1, c_2 = 1, c_3 \leq \frac{n-4}{3}$ |
| $(n - 3, 3)$                | $c_1 = 1, c_2 = 1, c_3 = \frac{n-4}{3}$ |

Table 3: For large enough $n$, this table shows which $\rho \in EIGHT_n$ rules for every odd $\sigma \in S_n$. 
3 Arbitrary Normal Sets

In this section we prove the negative result, Theorem 1.6, stating that no finite set of families of irreps is enough to capture \( \lambda (S_n, \Sigma) \) for an arbitrary symmetric non-negative element \( \Sigma \in \mathbb{R} [S_n] \), nor even for a normal non-negative element \( \Sigma \in \mathbb{R} [S_n] \). We also prove the positive result, Theorem 1.7, which says that for every normal non-negative \( \Sigma \in \mathbb{R} [S_n] \), std alone captures the spectral gap \( |\Sigma| - \lambda (S_n, \Sigma) \) up to a decaying multiplicative factor.

3.1 A negative result

Before proving Theorem 1.6, we remark that the irreducible characters of \( S_n \) constitute a linear basis for the space of class functions on \( S_n \), hence there are normal elements \( \Sigma \in \mathbb{R} [S_n] \) giving any prescribed values for the eigenvalues of every ir reps as in (2.1). One could still hope that as we restrict to non-negative normal elements, the answer to Question 1.3 would still be affirmative in the case of normal sets. However, this is not the case, as we now prove.

Proof of Theorem 1.6. Let \( \rho^{(1)}, \ldots, \rho^{(m)} \) be arbitrary families of non-trivial irreps of \( S_n \) as in Definition 1.2. Assume each of these families has at most \( k \) blocks outside the first row/column. In particular, for \( n \geq 2k \), the evaluation of the characters \( \rho^{(1)}, \ldots, \rho^{(m)} \) on \( \sigma \in S_n \) depends only on the numbers \( c_1 (\sigma), \ldots, c_k (\sigma) \) of short cycles – see Lemma 2.5. Moreover, as the characters are given by polynomials in \( c_1, \ldots, c_k \), their expected values for some \( \Sigma \in \mathbb{R} [S_n] \) depend only on the joint distribution of \( \{(c_1 (\sigma), \ldots, c_k (\sigma))\}_{\sigma \in \Sigma} \), and even more particularly on the distribution in \( \Sigma \) of the monomials \( c_1^{\alpha_1} \cdots c_k^{\alpha_k} \) with \( \sum i \cdot \alpha_i \leq k \) (see Lemma 2.13).

Now consider the uniform distribution on \( S_{2k} \). By orthogonality of irreducible characters, for every triv \( \neq \rho \in \widehat{S_{2k}}^t \), the expected value of \( \chi_\rho (\sigma) \) is zero. Now, for every \( n \geq 2k \), if \( \Sigma \in \mathbb{R} [S_n] \) is a normal non-negative element with the same joint distribution of \( c_1, \ldots, c_k \) as the uniform distribution in \( S_{2k} \), we get that the eigenvalues of Cay \( (S_n, \Sigma) \) associated with \( \rho_n^{(1)}, \ldots, \rho_n^{(m)} \) all vanish. For large enough \( n \), one can construct such \( \Sigma \) with large values of, say, \( c_{k+1} \), and supported on odd conjugacy classes. This would assure that there is some irrep \( \rho \) with \( k + 1 \) blocks outside the first row or outside the first column with large positive average value of the character \( \chi_\rho \). Hence \( \rho^{(1)}, \ldots, \rho^{(m)} \) do not rule for such \( \Sigma \). \( \Box \)

Example 3.1. Let us illustrate the proof for the families \( \rho^{(1)} = (n - 1, 1) \), \( \rho^{(2)} = (n - 2, 2) \) and \( \rho^{(3)} = (n - 2, 1, 1)^t \) and \( n = 100 \). So we can take \( k = 2 \) and consider the joint distribution of \( (c_1, c_2) \) in \( S_4 \):

| \( c_1 \) | \( c_2 \) | probability |
|-----|-----|---------|
| 4   | 0   | \( \frac{1}{21} \) |
| 2   | 1   | \( \frac{1}{3} \) |
| 1   | 0   | \( \frac{1}{3} \) |
| 0   | 2   | \( \frac{1}{8} \) |
| 0   | 0   | \( \frac{1}{4} \) |

Consider the following permutations in \( S_{100} \):

\[
\begin{align*}
\sigma_1 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots 88 \ 89 \ 90 \ 91 \ldots 96) \\
\sigma_2 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots 94 \ 95 \ 96 \ 97 \ 98) \\
\sigma_3 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots 91 \ 92 \ 93 \ 94 \ldots 99) \\
\sigma_4 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots 88 \ 89 \ 90 \ 91 \ldots 96 \ 97 \ 98 \ 99 \ 100) \\
\sigma_5 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ldots 94 \ 95 \ 96 \ 97 \ldots 100)
\end{align*}
\]
and notice that they all have many 3-cycles and are all odd. Define \( \Sigma \in \mathbb{R}[S_{100}] \) by
\[
\Sigma = \sum_{i=1}^{5} \alpha_i \cdot \sigma_i^{S_{100}}
\]
so that \( \alpha_i \cdot |\sigma_i^{S_{100}}| \) is equal to the probability in the \( i \)th line in the table. For example, \( \alpha_1 \cdot |\sigma_1^{S_{100}}| = \frac{1}{2} \). This choice of \( \Sigma \) assures that for \( i = 1, \ldots, m \), \( \rho^{(i)}(\Sigma) \) is the zero matrix, but there is an irrep with three blocks outside the first row or column with \( \rho(\Sigma) \) being a positive scalar matrix (in our case, this is true for the irreps \((97, 3), (97, 2, 1)^t \) and \((97, 1, 1, 1)\)).

**Remark 3.2.** In fact, it seems the proof of Theorem 1.6 can work also with the joint distribution of \((c_1, \ldots, c_k)\) induced from the uniform distribution on \( S_k \), rather than on \( S_{2k} \). For example, when \( k = 2 \), the joint distribution in \( S_2 \) of \((c_1, c_2)\) is \((2, 0)\) and \((0, 1)\) each with probability \( \frac{1}{2} \).

For any normal \( \Sigma \in \mathbb{R}[S_n] \) with such joint distribution on \((c_1, c_2)\), the matrix \( \rho(\Sigma) \) is zero for \( \rho \neq \text{triv}, \text{sgn} \) with at most 2 blocks outside the first row or the first column.

**Remark 3.3.** Theorem 1.6 is true also if one restricts attention to normal sets in \( S_n \), namely, to normal elements \( \Sigma \in \mathbb{R}[S_n] \) with \( 0 - 1 \) coefficients. The point is that there is a lot of flexibility in construction \( \Sigma \) in the proof, so as \( n \to \infty \) one can construct normal sets with joint distribution of \((c_1, \ldots, c_k)\) tending to the one in \( S_{2k} \) (or in \( S_k \)), while keeping the average value of \( c_{k+1} \) at least, say, \( \frac{n}{2k+1} \), and all conjugacy classes odd. This assures that some irrep with \( k + 1 \) blocks outside the first row/column eventually beats every irrep with at most \( k \) blocks outside the first row/column.

### 3.2 A positive result

We now prove Theorem 1.7, based on the following lemma:

**Lemma 3.4.** Let \( N_0 \) be the constant from Theorem 1.4. For every \( n \geq N_0 \) there is a constant \( \delta_n > 0 \) tending to zero as \( n \to \infty \), so that for every \( \sigma \in S_n \) and \( \rho \in \hat{S}_n \backslash \{\text{triv}, \text{sgn}\} \) we have
\[
1 - \bar{\chi}_\rho(\sigma) \geq [1 - \bar{\chi}_{\text{std}}(\sigma)] \cdot (1 - \delta_n).
\] (3.1)

**Proof.** By Proposition 2.3 (and assuming \( N_0 \geq N_1 \)), if \( c_1(\sigma) \geq 2 \) then std rules and (3.1) holds even with \( \delta_n = 0 \). If \( c_1(\sigma) \leq 1 \) then one of \( \rho \in E\GammaGH_n \) rules, so it is enough to check (3.1) for each of the seven irreps in \( E\GammaGH_n \backslash \{\text{std}\} \). And, indeed, (3.1) holds in this cases. The worst case is when \( \sigma \) has \( c_1(\sigma) = 0 \) and \( c_2(\sigma) = \frac{n}{2} \), where \( (n - 2, 2) \) rules, and \( 1 - \bar{\chi}_{(n-2, 2)}(\sigma) = 1 - \frac{1}{n-3} \), whereas \( 1 - \bar{\chi}_{\text{std}}(\sigma) = 1 + \frac{1}{n-1} \). Hence one can take \( \delta_n = \frac{2(n-2)}{n(n-3)} \). \( \square \)

**Proof of Theorem 1.7.** By Lemma 2.1, for any non-negative normal element \( \Sigma \in \mathbb{R}[S_n] \), the eigenvalues associated with \( \rho \in \hat{S}_n \) are all equal to \( \sum_C \alpha_C |C| \bar{\chi}_\rho(C) \). The gap between this eigenvalue and \( |\Sigma| \) is
\[
|\Sigma| - \sum_C \alpha_C |C| \bar{\chi}_\rho(C) = \sum_C \alpha_C |C| (1 - \bar{\chi}_\rho(C)).
\]
By Lemma 3.4, if \( n \geq N_0 \), this is at least
\[
\sum_C \alpha_C |C| [1 - \bar{\chi}_{\text{std}}(\sigma)] \cdot (1 - \delta_n) = [||\Sigma| - \lambda_1(\text{std}, \Sigma)| \cdot (1 - \delta_n) \]. \( \square \)
4 A Conjecture for Arbitrary Symmetric Sets and Its Consequences

This section gives more justification, motivation and background for Conjecture 1.8.

Background and Examples

Recall that Conjecture 1.8 says that there is a finite set of families of irreps of $S_n$, as in Definition 1.2, which are nearly dominant spectrally. This means that (for every large enough $n$ and for every symmetric non-negative $\Sigma \in \mathbb{R}[S_n]$), this set of irreps determines the spectral gap $|\Sigma| - \lambda(S_n,\Sigma)$ up to a universal multiplicative factor. In fact, even the following stronger version of Conjecture 1.8 is conceivable:

**Question 4.1.** Let $\mathcal{R}_n \subseteq \hat{\mathcal{S}}_n$ be the set of 11 irreps with one to four blocks outside the first row of the groups $\{S_n\}_n$. Is it true that for every symmetric non-negative $\Sigma \in \mathbb{R}[S_n]$, $$|\Sigma| - \lambda(S_n,\Sigma) \geq \left| \Sigma \right| - \max_{\rho \in \mathcal{R}_n \cup \mathcal{R}_n^t} \lambda_1(\rho(\Sigma)) \cdot [1 - o_n(1)],$$ where $\mathcal{R}_n^t = \{\rho^t \mid \rho \in \mathcal{R}_n\}$? Perhaps even $\mathcal{R}_n$ suffices (without $\mathcal{R}_n^t$)?

Note that this question differs from Conjecture 1.8 both by giving a specific set of irreps, and by suggesting that the multiplicative factor tends to 1 as $n \to \infty$. The evidence we gathered so far supports Conjecture 1.8 and even its stronger version – Question 4.1. The conjecture is certainly true, and with std alone, for normal elements (Theorem 1.7) and for elements supported on transpositions (Theorem 1.1). The example of non-generating sets, where the spectral gap is zero (see Section 1), shows that std alone is not sufficient. The latter is also demonstrated by the following example relating to the generating set consisting of an $n$-cycle and a sole transposition:

**Example 4.2.** Let $$\Sigma = \frac{1}{4} \left[ \text{id} + (1 \ 2) + (1 \ 2 \ldots n) + (1 \ 2 \ldots n)^{-1} \right] \in \mathbb{R}[S_n].$$

In this case, one of $(n-2,2)$ or $(n-2,1,1)$ rules, at least up to a multiplicative constant factor on the spectral gap, whereas $1 - \lambda_1(\text{std}(\Sigma))$ is of a different order. More concretely, it is known [DSC93, Example 1, Page 2139] that $$1 - \lambda(S_n,\Sigma) \geq \frac{1}{18n^3},$$ and that $\frac{1}{n^3}$ is the right order of the spectral gap. The proof of the upper bound to the spectral gap in [DSC93] can be adapted as follows: Consider the Schreier graph depicting the action of $S_n$ on ordered pairs $\{(x,y) \in [n]^2 \mid x \neq y\}$ with respect to $\Sigma$, and let $f$ be a function on its vertices given by $f((x,y)) = (x - y \mod n) - \frac{n}{2}$. As $f$ is orthogonal to the constant function, its Rayleigh quotient $\langle (f,A)f, f \rangle$, which roughly equals $\frac{6}{n^3}$, gives a lower bound on the spectral gap of this Schreier graph, where $A$ is the adjacency operator of the graph. The adjacency operator of this graph decomposes to the irreps $(n-2,1,1)$, $(n-2,2)$, std = $(n-1,1)$ and triv = $(n)$, so the second eigenvalue comes from one of $(n-2,1,1)$, $(n-2,2)$ or $(n-1,1)$. However, $(n-1,1)$ is not possible because $1 - \lambda_1(\text{std}(\Sigma))$ is of order $\frac{1}{n^2}$: this is the spectral gap of the random walk on the connected 4-regular $n$-vertex Schreier graph $\text{Sch}(S_n,\Sigma,[n])$, and it follows from Cheeger inequality that $1 - \lambda_1(\Sigma) \geq \frac{1}{8n^2}$. (It can be shown that,
in fact, that $n^2 \cdot (1 - \lambda_1 (\text{std} (\Sigma))) \xrightarrow{n \to \infty} 1$. Simulations for small values of $n$ suggest that $(n - 2, 1, 1)$ is associated with $\lambda (S_n, \Sigma)$ for this $\Sigma$, but we do not know whether this is true for all $n$.

Let us also mention another attempt to generalize Theorem 1.1 (Aldous’ conjecture) which is attributed to Caputo in [Ces16, Page 301] and is named there “the $c$-shuffles conjecture”. For $A \subseteq [n] = \{1, \ldots, n\}$, let $J_A = \sum_{\sigma \in S_n} \text{s.t. supp(}\sigma\text{)\subseteq} A \sigma \in \mathbb{R}[S_n]$. The conjecture states that for every linear combination with positive coefficients of the $J_A$’s

$$\Sigma = \sum_{A \subseteq [n]} \alpha_A \cdot J_A, \quad \alpha_A \geq 0$$

the standard representation rules, namely, $\lambda (S_n, \Sigma) = \lambda_1 (\text{std} (\Sigma))$. Some special cases of this conjecture were proven in work in progress of Gil Alon, Gady Kozma and the second author.

**Consequence 1: Random pairs of permutations expand**

It is well-known that if $g, h \in S_n$ are chosen uniformly and independently at random then they generate $A_n$ or $S_n$ with probability tending to 1 as $n \to \infty$ [Dix69]. So with high probability $\lambda (S_n, \frac{1}{4} (g + g^{-1} + h + h^{-1})) < 1$. But do random pairs of $S_n$ also generate a family of expander Cayley graphs? Namely,

**Question 4.3.** Is there some $\varepsilon > 0$ so that for uniformly random $g, h \in S_n$

$$\text{Prob} \left[ \lambda \left( S_n, \frac{1}{4} (g + g^{-1} + h + h^{-1}) \right) < 1 - \varepsilon \right] \xrightarrow{n \to \infty} 1? \quad (4.1)$$

This question is asked for all finite non abelian simple groups in [Lub12, Problem 2.28] (with uniform $\varepsilon$ for all), and is proven in [BGGT15] for finite simple group of Lie type and bounded Lie rank. However, the case of $A_n$ remains wide open. The best known bound is given in [HSZ15, Theorem 1.2], where (4.1) is proven with $\varepsilon$ replaced with $\frac{c_1}{n^2 (\log n)^2}$ for some absolute constants $c_1, c_2$. In fact, it was a major challenge to show that $A_n$ (or $S_n$) can even be turned into an expanding family – this was done in [Kas07], and it is still not known whether two generators suffice for this goal.

Conjecture 1.8, if true, would yield a positive answer to Question 4.3. This is based on the fact, proven in [FJR+98], that for every fixed $\ell$ and $r$, the Schreier graphs depicting the action of $S_n$ on $\ell$-tuples of different elements in $[n]$ with respect to $r$ random permutations form an expander family with high probability, namely, there is some $\varepsilon_\ell > 0$ so that

$$\text{Prob} [\lambda_1 - \lambda_2 \geq \varepsilon_\ell] \xrightarrow{n \to \infty} 1.$$ 

The decomposition of this action of $S_n$ on $\ell$-tuples contains exactly all the irreps with at most $\ell$ blocks outside the first row, so for each given family of irreps as in Definition 1.2(1) we get a uniform gap between $|\Sigma|$ and $\lambda_1$ with high probability. A small adaptation of the argument in [FJR+98] allows one to prove an analog statement for families of irreps of the second type (Definition 1.2(2)): introducing a random sign to the action on $\ell$-tuples only adds more randomness and thus can only increase the expected spectral gap. This explains how Conjecture 1.8, if true, yields a positive answer to Question 4.3.

**Remark 4.4.** A weaker version of Conjecture 1.8 states that in order to approximate the spectral gap $|\Sigma| - \lambda (S_n, \Sigma)$, it is enough to consider, for every $n$, a set $\mathcal{L}_n \subseteq \hat{S}_n$ of irreps with at most $\ell_n$ blocks outside the first row/column, where $\ell_n$ grows slowly with $n$ (say, $\ell_n \sim c \cdot \log n$). This may still be enough to yield a positive answer to Question 4.3. We
remark that the analog result for the mere existence of a spectral gap, namely, that $S_n$ has no $3\log n$-transitive subgroups but itself and $A_n$, was proved by Jordan in 1895 [Jor95]. Unlike the fact mentioned in Section 1 that there are no 4-transitive subgroups of $S_n$ other than $A_n$ and $S_n$ (for $n \geq 25$), this weaker result of Jordan has elementary proofs, which, in particular, do not depend on the classification of finite simple groups – see [BS87] and the references therein.

Consequence 2: Bounds on the diameter of Cayley graphs of $S_n$

In [BS92, Conjecture 1.7], Babai conjectures that for every finite simple group $G$ and every generating set $\Sigma \subset G$,

$$\text{diam} \left( \text{Cay} \left( G, \Sigma \right) \right) \leq \left( \log |G| \right)^{O(1)},$$

where the implied constant is absolute. The special case of $G = A_n$ is referred to as a “folklore” conjecture. In this case, the conjecture translates to that $\text{diam} \left( \text{Cay} \left( A_n, \Sigma \right) \right) \leq n^{O(1)}$. The best upper bound to date is quasi-polynomial and is due to Helfgott and Seress [HS14].

Consider the Schreier graph of $A_n$ on $\ell$-tuples with respect to $g, h \in A_n$. If $g$ and $h$ are generating, this graph is connected, and the spectral gap of the simple random walk on this graph is bounded from below by $\frac{1}{n^2}$ for some constants $c_1, c_2 > 0$. For example, using the Cheeger inequality, one gets

$$\frac{1}{n^2} \leq h \left( \text{Sch} \right) \leq \sqrt{32 \left( 1 - \lambda_2 \left( \text{Sch} \right) \right)},$$

which gives $1 - \lambda_2 \left( \text{Sch} \right) \geq \frac{1}{32n^2}$.

If conjecture 1.8 holds, this gives a lower bound of the same order on the spectral gap of $\text{Cay} \left( S_n, \frac{1}{2} \left( g + g^{-1} + h + h^{-1} \right) \right)$. Finally, a lower bound of this kind on the spectral gap yields a polynomial upper bound on the diameter of the Cayley graph: for instance, [SC04, Equation (6.6)] says that if $1 - \lambda_2$ is the spectral gap of the random walk on a Cayley graph of a finite group $G$, then the diameter of this graph is at most

$$\frac{3 \log |G|}{\sqrt{1 - \lambda_2}}.$$
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