Uniform Kazhdan Constant for some families of linear groups

Uzy Hadad

March 30, 2022

Abstract

Let \( R \) be a ring generated by \( l \) elements with stable range \( r \). Assume that the group \( EL_d(R) \) has Kazhdan constant \( \epsilon_0 > 0 \) for some \( d \geq r + 1 \). We prove that there exist \( \epsilon(\epsilon_0, l) > 0 \) and \( k \in \mathbb{N} \), s.t. for every \( n \geq d \), \( EL_n(R) \) has a generating set of order \( k \) and a Kazhdan constant larger than \( \epsilon \). As a consequence, we obtain for \( SL_n(\mathbb{Z}) \) where \( n \geq 3 \), a Kazhdan constant which is independent of \( n \) w.r.t generating set of a fixed size.

1 Introduction

Let \( k \in \mathbb{N} \) and \( 0 < \epsilon \in \mathbb{R} \). A group \( \Gamma \) is said to have Kazhdan property \( (T) \) with Kazhdan constant \( (k, \epsilon) \), if \( \Gamma \) has a set of generators \( S \), with \( |S| \leq k \) satisfying:

If \( (\rho, \mathcal{H}) \) is a unitary representation of \( \Gamma \), \( \rho : \Gamma \to \mathcal{U}(\mathcal{H}) \), with unit vector \( v \in \mathcal{H} \), such that for all \( s \in S \), \( \|\rho(s)v - v\| \leq \epsilon \), then \( \mathcal{H} \) contains a non-zero \( \Gamma \)-invariant vector.

In response to a question raised by Serre, Shalom [Sh1] and Kassabov [Kas2] showed that if one takes the elementary matrices as the set of generators, then one gets that \( (2(n^2 - n), \epsilon_n) \) with \( (42 \sqrt{n} + 860)^{-1} \leq \epsilon_n < 2n^{-\frac{1}{2}} \), is a Kazhdan constant for \( SL_n(\mathbb{Z}) \). Here we show that by taking a different set of generators, \( k \) and \( \epsilon \) can be made independent of \( n \):
Theorem 1.1. There exist $k \in \mathbb{N}$ and $0 < \epsilon \in \mathbb{R}$ s.t. for every $n \geq 3$, $SL_n(\mathbb{Z})$ has Kazhdan constant $(k, \epsilon)$.

Theorem 1.1 is deduced from a much more general result:

Theorem 1.2. Let $R$ be an associative ring generated by $a_1, \ldots, a_l$ with stable range $r$, and assume that for some $d \geq r + 1$, the group $EL_d(R)$ has Kazhdan constant $(k_0, \epsilon_0)$. Then there exist $\epsilon = \epsilon(\epsilon_0, l) > 0$ and $k = k(k_0, l) \in \mathbb{N}$, s.t. for every $n \geq d$, $EL_n(R)$ has Kazhdan constant $(k, \epsilon)$.

Although we require $d \geq r + 1$, the result is true also if $d < r + 1$ provided that $EL_m(R)$ is a bounded product of $EL_{m-1}(R)$ for every $r + 1 > m > d$.

For the definition of stable range and $EL_d(R)$ see Section 2. As $EL_3(\mathbb{Z}) = SL_3(\mathbb{Z})$ has property $(T)$ and the stable range of $\mathbb{Z}$ is 2, Theorem 1.1 follows from Theorem 1.2.

Shalom [Sh3, Sh4] proved that for $n > l + 2$, $SL_n(\mathbb{Z}[x_1, \ldots, x_l])$ has Kazhdan property $T$ (see Theorem 2.6 in this paper). Therefore we get:

Corollary 1.3. There exist $k = k(l)$ and $\epsilon = \epsilon(l)$ such that for every $n > l + 2$, $SL_n(\mathbb{Z}[x_1, \ldots, x_l])$ has Kazhdan constant $(k, \epsilon)$.

Since the stable range of any ring of integers $\mathcal{O}$ in a global field is 2, and it is known that $SL_3(\mathcal{O})$ has property $T$ (see [Sh1]), a similar result holds:

Corollary 1.4. For every ring of integers $\mathcal{O}$ in a global field $K$, there exist $k = k(\mathcal{O})$ and $\epsilon = \epsilon(\mathcal{O})$ such that for every $n \geq 3$, $SL_n(\mathcal{O})$ has Kazhdan constant $(k, \epsilon)$.

Remark 1.5. If $\mathcal{O}$ is generated by $l$ elements, then $SL_n(\mathcal{O})$ is a quotient of $SL_n(\mathbb{Z}[x_1, \ldots, x_l])$. Therefore for $n > l + 2$, $k(\mathcal{O})$ and $\epsilon(\mathcal{O})$ depend only on the number $g(\mathcal{O})$ of generators of $\mathcal{O}$ as a ring. Also it is known that $g(\mathcal{O})$ depends only on the discriminant of $\mathcal{O}$ (see [Pl]).

The idea of this work is inspired by the work of Shalom in [Sh1, Sh2, Sh3] who relates property $(T)$ to bounded generation and stable range, and also from the work of Kassabov in [Kas1] who extended Shalom’s results and proved that there exist $k \in \mathbb{N}$ and $\epsilon > 0$
s.t. for any finite commutative ring $R$, and any $n \geq 3$, the group $EL_3(M_n(R))$ has Kazhdan constant $(k, \epsilon)$ independent of $n$. Their proof is based on the fact that the group in question is boundedly generated by elementary matrices (see [CK] for the bounded elementary generation property of the group $SL_n(O)$ for $n \geq 3$). In our case we show that the groups in question are boundedly generated by elementary matrices and a common group with Kazhdan property $(T)$.

2 Background

2.1 Property (T)

Property (T) was introduced by Kazhdan in [Kaz]. Since then, it has found numerous applications in various areas of mathematics. Among them, for example, are constructions of expander graphs [Mar], the product replacement algorithm [LP] and bounds on mixing time of random walks on groups (see [Ch], [L1] for references and details).

Definition 2.1. Let $\Gamma$ be a discrete group, $S \subset \Gamma$ a subset, $\epsilon > 0$, and $(\rho, \mathcal{H})$ be a unitary representation of the group $\Gamma$. A vector $0 \neq v \in \mathcal{H}$ is called $(S, \epsilon)$-invariant, if $\|\rho(g)v - v\| \leq \epsilon \|v\| \ \forall g \in S$. A discrete group $\Gamma$ is said to have Kazhdan property (T), if there exist a finite set $S \subset \Gamma$ and $\epsilon > 0$, such that every unitary representation with $(S, \epsilon)$-invariant vector, contains a non-zero $\Gamma$-invariant vector. In that case $(|S|, \epsilon)$ is called a Kazhdan constant for $\Gamma$. We also sometimes say that $\Gamma$ has Kazhdan constant $\epsilon$ w.r.t the generating set $S$.

We will need the following known Lemma (see Lemma 2.2 in [Sh3] and also Lemma 1 in [KLN]).

Lemma 2.2. If $\Gamma$ is a product of subgroups $H_1, H_2, ..., H_k$, i.e. $\Gamma = H_1 \cdot H_2 \cdots H_k$, where each $H_i$ has Kazhdan constant $\epsilon_0$ w.r.t the generating set $S_i$ then $\Gamma$ has a Kazhdan constant $\epsilon = \epsilon(\epsilon_0, k)$ w.r.t their union $S = \bigcup S_i$.

For more information and introduction for the subject we refer the reader to [L1].
2.2 $EL_n(R)$ and Property $(T)$

**Notations:** Let $R$ be an associative ring with unit which is generated by the elements $a_1, \ldots, a_l$. Let $r \in R$ and let $i, j \in \mathbb{N}$ s.t. $1 \leq i \neq j \leq n$. Denote by $e_{ij}(r)$ the $n \times n$ matrix with 1 along the diagonal, $r$ in the $(i, j)$ position, and zero elsewhere. Note that $e_{ij}(-r)$ is the inverse of $e_{ij}(r)$ so that $e_{ij}(r) \in GL_n(R)$. These are the *elementary matrices*. The subgroup of $GL_n(R)$ which they generate is the elementary group $EL_d(R)$.

The group $EL_d(R)$, provided $d \geq 3$, is generated by the set $S_d(R)$, where $S_d(R)$ contains the set of $2(d^2 - d)$ elementary matrices with $\pm 1$ off the diagonal and the set of $4l(d - 1)$ elementary matrices $e_{ij}(\pm a_m)$ with $|i - j| = 1$ and $1 \leq m \leq l$.

**Definition 2.3.** The group $\Gamma = EL_d(R)$ is said to have the bounded elementary generation property if there is a number $N = BE_d(R)$ such that every element of $\Gamma$ can be written as a product of at most $N$ elementary matrices.

In [Kas1, Theorem 5] Kassabov generalized a result of Y. Shalom [Sh1] and proved the following:

**Theorem 2.4.** Suppose that $d \geq 3$ and $R$ is a finitely generated associative ring such that $EL_d(R)$ has the bounded elementary generation property. Then $EL_d(R)$ has property $T$ with an explicit lower bound for the Kazhdan constant of $EL_d(R)$ with respect to the generating set $S_d(R)$.

Moreover, Kassabov, in his proof of the above theorem, proved the following lemma (Lemma 1.1 in [Kas1] see also Corollary 3.5 in [Sh1]) which plays a crucial roll in this paper.

**Lemma 2.5.** There exists a constant $M(l) < 3\sqrt{2}(\sqrt{l} + 3)$ such that every f.g associative ring $R$, which is generated by $l$ elements and for every $k \geq 3$, the group $EL_k(R)$ satisfies the following property: Let $(\rho, \mathcal{H})$ be a unitary representation of the group $EL_k(R)$ and let $v \in \mathcal{H}$ be a unit vector s.t. $\|\rho(s)v - v\| < \epsilon$ for all $s \in S_k(R)$. Then $\|\rho(g)v - v\| \leq 2M(l)\epsilon$ for every elementary matrix $g$.

Recently Shalom in [Sh3, Sh4] gave a sufficient criterion for $EL_d(R)$ to have Kazhdan property $(T)$.
Theorem 2.6. Let $R$ be a f.g. associative ring with 1 and with stable range $r$. Then for all $d > \max\{2, r\}$, the group $ EL_d(R) $ has Kazhdan property ($T$).

It is clear that $M_{4n}(R)$ and $M_4(M_n(R))$ are isomorphic as rings. Now let us look at the multiplicative group $EL_4(M_n(R))$ contained in $M_4(M_n(R))$. It is easy to see that each elementary matrix in $EL_4(M_n(R))$ is a matrix in $EL_4n(R)$ and therefore $EL_4(M_n(R)) \subseteq EL_{4n}(R)$. Also it is clear that $EL_{2n}(R) \subseteq GL_2(M_n(R))$ and for $2n \geq 3$ we have $[EL_{2n}(R), EL_{2n}(R)] = EL_{2n}(R)$. Now, since $[GL_2(M_n(R)), GL_2(M_n(R))] \subseteq EL_4(M_n(R))$ (see Lemma 3.4 in this paper), we get that $EL_{2n}(R) \subseteq EL_4(M_n(R))$, sitting in the upper left block. The same is true for the case $EL_{2n}(R)$ sitting in the lower right block. This implies that the generating set $S_{2n}(R)$ for $EL_{2n}(R)$ is a subset of $EL_4(M_n(R))$, hence $EL_{4n}(R) = EL_4(M_n(R))$.

Let $R$ be a f.g. associative ring generated by $a_1 = 1, ..., a_l$. Let

$$A_i = \begin{pmatrix} a_i & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ (-1)^{n-1} & \ldots & \ldots & 0 \end{pmatrix}.$$

It is easy to see that the ring $M_n(R)$ is generated by the set $\{A_1, ..., A_l, B\}$, of size $l + 1$ which is independent of $n$.

For $n \geq 2$ and for arbitrary field $K$ it is obvious that $SL_n(K)$ is generated by the set of elementary matrices and thus $SL_n(K) = EL_n(K)$. Let $\mathcal{O}$ be a ring of integers in an algebraic number field. Bass, Milnor and Serre [BMS] have shown that every
matrix in $SL_n(\mathcal{O})$ for $n \geq 3$, may be written as a product of elementary matrices, therefore $EL_n(\mathcal{O}) = SL_n(\mathcal{O})$. Moreover, Suslin in [Sus] proved that $EL_n(\mathbb{Z}[x_1, ..., x_m]) = SL_n(\mathbb{Z}[x_1, ..., x_m])$, again for $n \geq 3$.

2.3 Stable range

**Definition 2.7.** A sequence $\{a_1, ..., a_n\}$ in a ring $R$ is said to be left unimodular if $Ra_1 + ... + Ra_n = R$. In case $n \geq 2$, such a sequence is said to be reducible if there exist $r_1, ..., r_{n-1} \in R$ such that $R(a_1 + r_1a_n) + \cdots + R(a_{n-1} + r_{n-1}a_n) = R$.

This reduction notion leads directly to the definition of stable range.

**Definition 2.8.** A ring $R$ is said to have left stable range $\leq n$, if every left unimodular sequence of length $> n$ is reducible. The smallest such $n$ is said to be the left stable range of $R$.

Vaserstein has proved that for any ring $R$, the left stable range is equal to the right stable range [Vas2]. Thus, we write $sr(R)$ for this common value and call it simply the stable range of $R$.

The reader should be aware that there is an inconsistency of $\pm 1$ in the definition of the stable range in the literature.

In [Vas1] Vaserstein proves the following fundamental theorem:

**Theorem 2.9.** Let $R$ be a ring with stable range $r$, then the canonical mapping

$$S_n : GL_n(R)/EL_n(R) \to K_1(R)$$

is bijective for all $n \geq r + 1$.

As a consequence of this theorem it is easy to verify that for all $n \geq r + 2$,

$$EL_{n-1}(R) = EL_n(R) \cap GL_{n-1}(R).$$

We will need the following known fact which can be found in [HO,4.1.18].

**Theorem 2.10.** Let $R$ be a f.g. ring with stable range $k$, then $sr(M_n(R)) = 1 + \lceil \frac{k-1}{n} \rceil$ where $[x]$ is the greatest integer function.
3 Proof of Theorem 1.2:

We will show that every element in $EL_{4n}(R)$ is a bounded product of elementary matrices in $EL_d(M_n(R))$ and an element of $EL_d(R)$, where $4n > d$ and $d$ is a fixed number s.t $d$ is greater than the stable range of the ring $R$.

In [DV, Lemma 9] Dennis and Vaserstein proved the following lemma:

**Lemma 3.1.** Let $R$ be an associative ring with 1 with $sr(R) \leq r$. Then for any $n \geq r$ we have

$$GL_n(R) = ULUL \begin{pmatrix} GL_r(R) & 0 \\ 0 & I_{n-r} \end{pmatrix} = UL \begin{pmatrix} GL_r(R) & 0 \\ 0 & I_{n-r} \end{pmatrix} UL$$

where $U$ (resp., $L$) is the group of all upper (resp., lower) triangular matrices in $EL_n(R)$ with 1 along the main diagonal.

It is easy to see that for every $d$ with $n \geq d \geq r$ we also have:

$$GL_n(R) = ULUL \begin{pmatrix} GL_d(R) & 0 \\ 0 & I_{n-d} \end{pmatrix} = UL \begin{pmatrix} GL_d(R) & 0 \\ 0 & I_{n-d} \end{pmatrix} UL.$$  

Here is quantitative version which counts the number of elementary matrices for the case $n = 4$ and $sr(R) \leq 2$.

**Lemma 3.2.** Let $R$ be an associative ring with 1 with $sr(R) \leq 2$. Then every matrix in $GL_4(R)$ can be presented as a multiplication of 20 elementary matrices and a matrix of the form

$$\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(R).$$
Proof. Let $M$ be an arbitrary matrix in $GL_4(R)$, so $M$ is invertible, say

$$M = \begin{pmatrix}
* & * & * & a_1 \\
* & * & * & a_2 \\
* & * & * & a_3 \\
* & * & * & a_4 \\
\end{pmatrix}$$

As $M$ is invertible, we can write

$$M^{-1} = \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
b_1 & b_2 & b_3 & b_4
\end{pmatrix}$$

with $b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4 = 1$. In particular

$$Ra_1 + Ra_2 + Ra_3 + Ra_4 = R.$$ 

In this proof we consider only the most difficult case, where $a_i \neq 0$ for $i = 1, 2, 3, 4$. The other cases follow immediately. Since $sr(R) \leq 2$, we get that there are $t_1, t_2 \in R$ such that

$$R(a_1 + t_1(b_3a_3 + b_4a_4)) + R(a_2 + t_2(b_3a_3 + b_4a_4)) = R.$$ 

Thus, there exist $x_1, x_2 \in R$ such that

$$x_1(a_1 + t_1(b_3a_3 + b_4a_4)) + x_2(a_2 + t_2(b_3a_3 + b_4a_4)) = 1$$

For the elementary operation of adding a multiplication of a row $j$ by scalar $c$ to row $i$, we will use the following notation $R_i \leftarrow R_i + cR_j$. 

$$M = \begin{pmatrix}
* & * & * & a_1 \\
* & * & * & a_2 \\
* & * & * & a_3 \\
* & * & * & a_4 \\
\end{pmatrix} \quad R_1 \leftarrow R_1 + t_1b_4R_4 \quad \begin{pmatrix}
* & * & a_1 + t_1(b_3a_3 + b_4a_4) \\
* & * & a_2 \\
* & * & a_3 \\
* & * & a_4
\end{pmatrix}$$
This was done by 6 elementary operations.

Now since we have 1 in the lower right corner, we use 6 elementary matrices to annihilate all the rest of the last row and column. In a similar way we can use again 8 elementary matrices and bring $M$ to the form

$$
\begin{pmatrix}
* & * & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

\[\square\]

**Lemma 3.3.** Let $R$ be a f.g. associative ring with 1 and with $sr(R) \leq r$. If $m \geq d \geq r \geq 3$, then every element in $GL_m(R)$ is a product of at most 8 commutators, and an element of the form

$$
\begin{pmatrix}
S & 0 \\
0 & I_{m-d}
\end{pmatrix}
$$

where $S \in GL_d(R)$. 

9
Proof. From the remark after Lemma 3.1 we get that any element \( T \in GL_m(R) \) can be presented as
\[
T = L_1 U_1 L_2 U_2 \begin{pmatrix} S & 0 \\ 0 & I_{m-d} \end{pmatrix}
\]
where \( S \in GL_d(R) \). Van der Kallen [KW] asserted that every triangular matrix with entries in a ring \( R \) and 1 on the diagonal, can be expressed as a product of three commutators. This result was improved to two commutators by Dennis and Vaserstein [DV, Lemma 13].

Lemma 3.4. Let \( R \) be a f.g. associative ring with 1. Then every matrix in \( GL_4(R) \) of the form
\[
\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
where
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(R),
\]
is a commutator in \( GL_2(R) \), can be presented as a multiplication of 40 elementary matrices in \( EL_4(R) \).

Proof. We use the following identities:
\[
\begin{pmatrix} [h_1, h_2] & 0 \\ 0 & I_{2 \times 2} \end{pmatrix} = \begin{pmatrix} 0 & h_1 \\ -h_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -h_2^{-1} \\ h_2 & 0 \end{pmatrix} \begin{pmatrix} (h_2 h_1)^{-1} & 0 \\ 0 & h_2 h_1 \end{pmatrix},
\]
\[
\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -h^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & h^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & h \\ -h^{-1} & 0 \end{pmatrix} = \begin{pmatrix} I_{2 \times 2} & h \\ 0 & I_{2 \times 2} \end{pmatrix} \begin{pmatrix} I_{2 \times 2} & 0 \\ -h^{-1} & I_{2 \times 2} \end{pmatrix} \begin{pmatrix} I_{2 \times 2} & h \\ 0 & I_{2 \times 2} \end{pmatrix}
\]
Now since
\[
\begin{pmatrix}
1 & 0 & a & b \\
0 & 1 & c & d \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & c & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & d \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & a & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & b \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
and in the same way for the lower case, we get that every element of the form
\[
\begin{pmatrix}
I_{2\times2} & h \\
0 & I_{2\times2} \\
\end{pmatrix}
\text{or}
\begin{pmatrix}
I_{2\times2} & 0 \\
h & I_{2\times2} \\
\end{pmatrix}
\]
can be written as a products of at most 4 elementary matrices (and 2 for the case that \(h = \pm 1\)) in \(EL_4(R)\). The result now follows. \(\□\)

**Proposition 3.5.** For \(n \geq d > r\), every element in \(EL_{4n}(R)\) can be presented as product of at most 340 elementary matrices in \(EL_4(M_n(R))\) and an element in \(EL_d(R)\).

**Remark 3.6.** The constant 340 related to bounded generation can be improved.

**Proof.** We take an arbitrary element \(T\) in \(EL_{4n}(R)\). By Theorem 2.10 \(sr(M_n(R)) \leq 2\), hence by the argument after Theorem 2.6 and Lemma 3.2 we get that \(T\) can be presented as a multiplication of 20 elementary matrices of \(EL_4(M_n(R))\) and a matrix of the form
\[
\begin{pmatrix}
A & B & 0 & 0 \\
C & D & 0 & 0 \\
0 & 0 & Id & 0 \\
0 & 0 & 0 & Id \\
\end{pmatrix}
\]
where
\[
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix} \in GL_2(M_n(R)) \subseteq GL_{2n}(R).
\]
As \(2n \geq sr(R) + 2\), by the argument after Theorem 2.6 it follows that
\[
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix} \in EL_{2n}(R).
\]
Now by Lemma 3.3 (for \( m = 2n \)) we get that
\[
\begin{pmatrix}
A & B & 0 & 0 \\
C & D & 0 & 0 \\
0 & 0 & Id & 0 \\
0 & 0 & 0 & Id
\end{pmatrix}
\]
can be written as a product of 8 commutators and an element \( S \in EL_d(R) \). By Lemma 3.4, applied to \( EL_4(M_n(R)) \), the result follows.

We will need the following known result which is proved in many papers (see Ch. 3, Cor. 11 in [HV] and Lemma 2.5 in [Sh1]):

**Lemma 3.7.** Let \((\rho, H)\) be a unitary representation of a group \( \Gamma \). Suppose that for some unit vector \( v \in H \), one has for all \( g \in \Gamma \): \( \|\rho(g)v - v\| < \sqrt{2} \). Then there exist a non-zero \( \Gamma \)-invariant vector in \( H \).

Before we get to the proof of the Theorem 1.2 we give a general property of a Kazhdan group.

**Lemma 3.8.** Let \( \Gamma \) be a group generated by a set \( S \) with Kazhdan constant \( \epsilon > 0 \), and let \( \epsilon > \delta > 0 \) be given. Let \((\rho, H)\) be a unitary representation of \( \Gamma \) and assume that there exists \( v \in H \) with \( \|v\| = 1 \) s.t for all \( s \in S \), \( \|\rho(s)v - v\| \leq \delta \). Then \( v \) is \((\Gamma, 2\cdot \frac{\delta}{\epsilon})\)-invariant.

**Proof.** Decompose \( \rho \) into the trivial component \( \sigma_0 \) and the non-trivial component \( \sigma_1 \), \( \rho = \sigma_0 + \sigma_1 \), and accordingly decompose \( v = v_0 + v_1 \).

For all \( s \in S \),
\[
delta \geq \|\rho(s)v - v\| = \|\sigma_0(s)v_0 - v_0 + \sigma_1(s)v_1 - v_1\| = \|\sigma_1(s)v_1 - v_1\|
\]
Taking the maximum over \( S \), we get
\[
delta \geq \max_{s \in S} \|\sigma_1(s)v_1 - v_1\| > \epsilon \cdot \|v_1\|. \tag{1}
\]
This implies that \( \|v_1\| < \frac{\delta}{\epsilon} \), hence for all \( g \in \Gamma \), we get \( \|\rho(g)v - v\| = \|\rho(g)v_1 - v_1\| < 2\cdot \frac{\delta}{\epsilon} \) as required.

\( \square \)
Now we are ready to prove Theorem 1.2. Let $R$ be an associative ring generated by $l$ elements s.t. $sr(R) \leq r$. Let $d > r$ and assume that $EL_d(R)$ is generated by the set $F$, where $k_0 = |F|$, and with Kazhdan constant $\epsilon_0$ w.r.t $F$. Notice that by Theorem 2.6 we know that $EL_d(R)$ has Kazhdan property $(T)$.

For $EL_{4n}(R)$ where $n \geq d$ we will show that the group $EL_{4n}(R)$ has Kazhdan constant $(k_1, \epsilon_1)$ with respect to the generating set $\tilde{S} = F \cup S_4(M_n(R))$, where $k_1 = k_0 + |S_4(M_n(R))|$ and $\epsilon_1 = \epsilon_1(\epsilon_0, l)$ ($S_4(M_n(R))$ was defined in Subsection 2.2 and its order is independent of $n$).

Let $(\rho, \mathcal{H})$ be a unitary representation of $EL_{4n}(R)$ and suppose that $v \in \mathcal{H}$ is a unit vector which is $(\tilde{S}, \epsilon_1)$-invariant (the constant $\epsilon_1$ will be determined later).

Since $EL_4(M_n(R)) = EL_{4n}(R)$, we get that $v$ is $(S_4(M_n(R)), \epsilon_1)$-invariant, hence by Lemma 2.5 any elementary matrix $g \in EL_4(M_n(R))$ satisfies:

$$\|\rho(g)v - v\| < 2M(l + 1)\epsilon_1$$

provided $\epsilon_1 < \epsilon_0$.

Now, restricting $\rho$ to the subgroup $EL_d(R)$, we see that $v$ is $(F, \epsilon_1)$-invariant and hence by Lemma 3.8 we get for any $g \in EL_d(R)$:

$$\|\rho(g)v - v\| < 2 \cdot \frac{\epsilon_1}{\epsilon_0}.$$ 

Let $g$ be an arbitrary element in $EL_{4n}(R)$. By Proposition 3.5, $g$ is expressible as a product of at most 340 elementary matrices of $EL_4(M_n(R))$ and an element $g_c \in EL_d(R)$:

$$g = g_1 \cdot \ldots \cdot g_{340} \cdot g_c.$$  

Therefore

$$\|\rho(g)v - v\| \leq 340 \sum_{i=1}^{340} \|\rho(g_i)v - v\| + \|\rho(g_c)v - v\| < 340 \cdot 2M(l + 1)\epsilon_1 + 2 \cdot \frac{\epsilon_1}{\epsilon_0}.$$ 

If we choose $\epsilon_1 < \frac{\epsilon_0 \sqrt{7}}{880M(l+1)+2}$ we obtain for all $g \in EL_{4n}(R)$
\[ \|\rho(g)v - v\| < \sqrt{2}. \]

Therefore \( v \) is \( (EL_{4n}(R), \sqrt{2}) \) invariant and hence by Lemma 3.7 there exists \( 0 \neq v_0 \in \mathcal{H} \) which is \( EL_{4n}(R) \)-invariant.

To complete the proof, the only cases left are \( EL_m(R) \) where \( 4n < m < 4(n+1) \) or \( d < m < 4d \). Let \( m = 4n + 1 \) (resp. \( m = d + 1 \)) and \( T \in EL_m(R) \). Since \( m > sr(R) \), by using elementary operations (in the same way we did in the proof of Lemma 3.2) we can reduce \( T \) to a matrix which has 1 in the lower right corner. The elementary matrices we use can be decomposed to two sets:

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & * \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \ldots & 1 & * \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 & \ldots & 0 & * \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The left set can be seen as part of the group \( EL_{m-1}(R) \) which sits in the lower right part of \( EL_m(R) \). For the right set, since all the elementary subgroups are conjugate, this set is part of some conjugate of \( EL_{m-1}(R) \) which sits (say) in the lower right part of \( EL_m(R) \). This implies that we can bring \( T \) to a matrix which has 1 in the lower right corner by using a bounded product of elements from groups which are isomorphic to \( EL_{m-1}(R) \), independent of \( m \). Use the 1 in the lower right corner to annihilate all the rest of the last column and row (this can be done in a similar way as a bounded product of groups isomorphic to \( EL_{m-1}(R) \)). Now from the argument after Theorem 2.9 it is easy to verify that \( EL_m(R) \) is a bounded product of groups isomorphic to \( EL_{m-1}(R) \), independent of \( m \). The same procedure we do for \( 4n < m < 4(n+1) \) (resp. \( d < m < 4d \)), and we get that \( EL_m(R) \) is a bounded product of groups isomorphic to \( EL_n(R) \) where \( n = 4 \cdot \lceil \frac{m}{4} \rceil \) (resp. groups isomorphic to \( EL_d(R) \)), and by Lemma 2.2 we get uniform Kazhdan constant for \( EL_n(R) \) where \( n \geq d \).

\[ \square \]
4 Acknowledgments.

This paper is part of the author’s PhD thesis. The author wishes to thanks Moshe Jarden and Ehud de Shalit for their comments regarding the number of generators of ring of integers. Thanks to Noa Edelstein and Yehuda Shalom for useful discussions. Thanks are also due to E. Bagno for his careful reading and several helpful comments and to the anonymous referee for his report. The author is grateful to his advisor Alex Lubotzky for introducing him the problem and for fruitful conversations.

References

[BMS] H. Bass, J. Milnor and J. P. Serre. Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$, Publ. Math. I.H.E.S. 33(1967), 59–137.

[CK] D. Carter and G. Keller, Bounded elementary generation of SLn(O), Amer. J. Math., 105 (1983), 673-687.

[Ch] F.R.K. Chung, Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92), American Mathematical Society, Providence, RI, 1994.

[DV] R.K. Dennis and L. N. Vaserstein. On a question of M. Newman on the number of commutators. J. Algebra, 118, (1988), 150–161.

[HO] A.J. Hahn and O.T. O’Meara. The classical groups and $K$-theory. Springer-Verlag, Berlin, 1989.

[HV] P. de la Harpe, A. Valette. La propriété $(T)$ de Kazhdan pour les groupes localement compacts, Astérisque No. 175, Société Math, de France (1989).

[Kas1] M. Kassalov. Universal lattices and unbounded rank expanders. Invent. Math, to appear.

[Kas2] M. Kassabov. Kazhdan constants for $SL_n(\mathbb{Z})$. Internat. J. Algebra Comput. 15 (2005), no. 5-6, 971–995.
[Kaz] D.A. Kazhdan. On the connection of the dual space of a group with the structure of its closed subgroups. Funk. Anal. Pril., 1:71–74, 1967.

[KLN] M. Kassabov, A. Lubotzky, N. Nikolov. Finite simple groups as expanders. Proc. Natl. Acad. Sci. USA 103 (2006), no. 16, 6116–6119.

[KW] W. van der Kallen. $SL_3(C[x])$ does not have bounded length. Proc. Algebraic K-theory, Part I (Oberwolfach, 1980), pp. 357–361, Lecture Notes in Math., 966, Springer, Berlin-New York, 1982.

[L1] A. Lubotzky. Discrete Groups, Expanding graphs and Invariant Measures, Birkhäuser, 1994.

[LP] A. Lubotzky, I. Pak, The product replacement algorithm and Kazhdans property ($T$), J. Amer. Math. Soc. 14 (2001), no. 2, 347-363.

[Mar] G. A. Margulis, Explicit constructions of expanders, Problemy Peredachi Informacii 9 (1973), no. 4, 71-80.

[Pl] P. A. B. Pleasants. The number of generators of the integers of a number field. Mathematika 21 (1974), 160–167.

[Sh1] Y. Shalom. Bounded generation and Kazhdan property($T$), Publ. Math. IHES, 90 (1999), 145–168.

[Sh2] Y. Shalom. Explicit Kazhdan constants for representations of semisimple and arithmetic groups. Ann. Inst. Fourier (Grenoble) 50 (2000), no. 3, 833–863.

[Sh3] Y. Shalom. The algebrization of Kazhdan’s property ($T$), Proc. of ICM 2006.

[Sh4] Y. Shalom. Elementary linear groups and Kazhdans property ($T$). In preparation.

[Sus] A. A. Suslin. The structure of the special linear group over rings of polynomials. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 235–252, 477.

[Vas1] L. N. Vaseršteǐn. On the stabilization of the general linear group over a ring. Mat. Sb. (N.S.) 79 (121) 405–424 (Russian); translated as Math. USSR-Sb. 8 1969 383–400.
[Vas2] L. N. Vaseršteǐn. The stable range of rings and the dimension of topological spaces.
Funkcional. Anal. i Priložen. 5 1971 no. 2, 17–27.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem
91904, Israel

E-mail address: ouzy@math.huji.ac.il