SETS IN $\mathbb{R}^d$ DETERMINING $k$ TAXICAB DISTANCES

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ABSTRACT. We address an analog of a problem introduced by Erdős and Fishburn, itself an inverse formulation of the famous Erdős distance problem, in which the usual Euclidean distance is replaced with the metric induced by the $\ell^1$-norm, commonly referred to as the taxicab metric. Specifically, we investigate the following question: given $d,k \in \mathbb{N}$, what is the maximum size of a subset of $\mathbb{R}^d$ that determines at most $k$ distinct taxicab distances, and can all such optimal arrangements be classified? We completely resolve the question in dimension $d = 2$, as well as the $k = 1$ case in dimension $d = 3$, and we also provide a full resolution in the general case under an additional hypothesis.

1. Introduction

In 1946, Erdős [2] asked a now famous question: given $n \in \mathbb{N}$, what is the minimum number of distinct distances determined by $n$ points in a plane? Denoting this minimum by $f(n)$, he proved via an elementary counting argument that $f(n) = \Omega(\sqrt{n})$, and he conjectured that the correct order of growth is $n/\sqrt{\log n}$, as attained by a $\sqrt{n} \times \sqrt{n}$ integer grid. After decades of incremental progress, this conjecture was effectively resolved in a celebrated result of Guth and Katz [5], who established that $f(n) = \Omega(n/\log n)$.

50 years after Erdős’s original paper, Erdős and Fishburn [3] addressed the same question from the inverse perspective, and aspired to precise results in fixed cases rather than general asymptotic results. Specifically, they investigated the following: given $k \in \mathbb{N}$, what is the maximum number of points in a plane that determine at most $k$ distinct distances, and can such optimal arrangements be classified? This question, which we refer to as the Erdős-Fishburn problem, was fully resolved by Erdős and Fishburn for $1 \leq k \leq 4$, then by Shinahara [8] for $k = 5$, and Wei [9] for $k = 6$, while it remains open for $k \geq 7$. By convention, in the quoted results and throughout this paper, 0 is not counted as a distance determined by a set of points.

These questions can also be adapted to higher dimensions, and to alternative notions of distance. Here we focus on a particular, well-known alternative metric.

Definition 1.1. For $d \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define the $\ell^1$-norm of $x$ by

$$\|x\|_1 = |x_1| + \cdots + |x_d|,$$

which in particular satisfies the triangle inequality $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$. Like every norm, the $\ell^1$-norm induces a metric on $\mathbb{R}^d$ by defining $\|x - y\|_1$ to be the $\ell^1$-distance between $x, y \in \mathbb{R}^d$.

The metric induced by the $\ell^1$-norm is commonly referred to as the taxicab metric, because it measures the length of the shortest path between two points in space, under the restriction that one can only travel in directions parallel to the coordinate axes, as if in a taxicab on a grid of city streets. For example, if two people at city intersections are separated by 3 blocks horizontally and 4 blocks vertically, then, as the crow flies, they are 5 blocks apart by the Pythagorean theorem. However, to actually make the journey without cutting through buildings, they must walk 7 blocks, which is the $\ell^1$-distance.

As noted in Chapters 0 and 1 of [4], one can show that the minimum number of $\ell^1$-distances determined by $n$ points in $\mathbb{R}^d$ is $\Omega(n^{1/d})$, and this order of growth is attained by $\{1, 2, 3, \ldots, \lceil n^{1/d} \rceil\}^d$. Therefore, in the case of the taxicab metric, the big picture asymptotic question is immediately resolved, which begs the question of whether this case can be analyzed more precisely. To begin this journey, we first consider the Erdős-Fishburn problem in the plane with $k = 1$. 

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We fix two points $U, V \in \mathbb{R}^2$, say $U = (-1, 0)$ and $V = (1, 0)$. With the usual notion of distance, if any additional points can be added without determining an additional distance, those points necessarily lie on the circles of radius 2 centered at $U$ and $V$, respectively. Those two circles intersect in only two points, and we find that they are $Q = (0, \sqrt{3})$ and $R = (0, -\sqrt{3})$. Since the distance between $Q$ and $R$ is greater than 2, only one of the two can be added to $\{U, V\}$ while maintaining only a single distance. In summary, a set $P \subseteq \mathbb{R}^2$ determining a single distance satisfies $|P| \leq 3$, and equality holds if and only if $P$ is the set of vertices of an equilateral triangle.

However, even in this simplest case, the taxicab metric case diverges from that of the usual distance. With the taxicab metric, the “circle” (which we refer to as an $l_1$-circle) of radius 2 centered at $U$ is in fact a square, rotated 45° from axis-parallel, with the four sides connecting the points $(-3, 0)$, $(-1, 2)$, $(1, 0)$, and $(1, -2)$. Similarly, the $l_1$-circle of radius 2 centered at $V$ is a square with sides connecting $(3, 0)$, $(1, -2)$, $(-1, 0)$, and $(1, 2)$. Like the usual distance case, these two circles intersect in exactly two points, this time $Q = (0, 1)$ and $R = (0, -1)$. The difference is that here $Q$ and $R$ are indeed separated by $l_1$-distance 2, and hence the four-point configuration $\{U, V, Q, R\}$ determines a single $l_1$-distance.

2. Main Definition and Results

Inspired by the four-point construction above, as well as additional trial and error, we define the following family of sets, which serve as our candidates for resolving the Erdős-Fishburn problem for the taxicab metric.

**Definition 2.1.** For integers $d > 0$ and $k \geq 0$, we define

$$\Lambda_d(k) = \{ n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d : \|n\|_1 \leq k, \ n_1 + \cdots + n_d \equiv k \pmod{2} \}.$$  

$\Lambda_d(k)$ is the union of the integer lattice points lying on the $l_1$-spheres (which in dimension $d$ are $2^d$-faced polytopes) centered at the origin of every other integer radius, starting with either 0 or 1 depending on the parity of $k$. The four-point configuration discussed in the introduction is $\Lambda_2(1)$, and some additional examples are pictured below.

![Figure 1](image)

(a) $\Lambda_2(2)$: 9 points in $\mathbb{R}^2$ determining two $l_1$-distances  
(b) $\Lambda_3(2)$: 19 points in $\mathbb{R}^3$ determining two $l_1$-distances

In Section 3 we establish the following properties of $\Lambda_d(k)$, including the crucial fact that it determines exactly $k$ distinct $l_1$-distances, the primary motivation for its definition.

**Theorem 2.2.** The following hold for all $d, k \in \mathbb{N}$:

(i) $\Lambda_d(k)$ determines exactly $k$ distinct $l_1$-distances, specifically $2, 4, \ldots, 2k$

(ii) $|\Lambda_1(k)| = k + 1$, $|\Lambda_d(1)| = 2d$

(iii) $|\Lambda_{d+1}(k)| = |\Lambda_d(k)| + 2 \sum_{j=0}^{k-1} |\Lambda_d(j)|$
Parts (ii) and (iii) of Theorem 2.2, combined with known formulas for sums of powers, allow one to determine explicit formulas for $|\Lambda_d(k)|$ for any fixed $d \in \mathbb{N}$. We include the first few examples in the following table:

| $d$  | $|\Lambda_d(k)|$                           |
|------|-------------------------------------------|
| 2    | $(k + 1)^2$                               |
| 3    | $\frac{2}{7}(k + 1)^3 + \frac{4}{7}(k + 1)$ |
| 4    | $\frac{1}{3}(k + 1)^4 + \frac{2}{3}(k + 1)^2$ |
| 5    | $\frac{2}{15}(k + 1)^5 + \frac{7}{5}(k + 1)^3 + \frac{1}{3}(k + 1)$ |
| 6    | $\frac{2}{35}(k + 1)^6 + \frac{4}{5}(k + 1)^4 + \frac{23}{35}(k + 1)^2$ |
| 7    | $\frac{4}{315}(k + 1)^7 + \frac{2}{5}(k + 1)^5 + \frac{28}{315}(k + 1)^3 + \frac{1}{3}(k + 1)$ |

Some of the patterns observed in Table 1 can be generalized using Faulhaber’s Formula for sums of powers, as seen in the following formulation, which we also prove in Section 3.

**Theorem 2.3.** For each $d \in \mathbb{N}$ and each integer $k \geq 0$, we have the formula

$$|\Lambda_d(k)| = \sum_{i=0}^{[d/2]-1} a_{d,i}(k + 1)^{d-2i},$$

where the coefficients $a_{d,i}$ satisfy the recursive formula

$$a_{d,i} = 2 \sum_{\ell=0}^{i} \frac{a_{d-\ell,i}}{d-2\ell} \left( \frac{d - 2\ell}{2(i - \ell)} \right) B_{2(i-\ell)},$$

where $B_i$ is the $i$-th Bernoulli number. In particular, we have the explicit formulas $a_{d,0} = 2^{d-1}/d!$ for all $d \in \mathbb{N}$ and $a_{d,1} = 2^{d-3}/(3(d-3)!)$ for all $d \geq 3$.

Detailed analysis of $\Lambda_d(k)$ is perhaps of independent interest, but to make headway toward our goal, we need to address the important questions: does $\Lambda_d(k)$ have maximal size amongst subsets of $\mathbb{R}^d$ determining at most $k$ distinct $\ell^1$-distances? If so, is $\Lambda_d(k)$ the only such optimal arrangement? In anticipation of the latter question, we observe that any optimal arrangement can undergo any scaling, or any transformation that preserves the $\ell^1$-norm, and remain optimal, leading to the following definition.

**Definition 2.4.** For $d \in \mathbb{N}$, we say that two subsets of $\mathbb{R}^d$ are $\ell^1$-similar if one can be mapped to the other via a composition of translations, reflections about coordinate hyperplanes, dilations, and coordinate permutations, as these transformations either preserve or uniformly scale collections of $\ell^1$-distances.

We note that the list of transformations in Definition 2.4 does not include rotations, because, unlike the usual Euclidean metric, the taxicab metric is not invariant under rotation, unless the rotation can alternatively be obtained through reflection about coordinate hyperplanes and permutation of coordinates. This fact rears its head in our exploration of the taxicab metric in higher dimensions, and plays a key role in our discussions in Section 4. For now, though, the following result established in Section 4 completely resolves the taxicab analog of the Erdős-Fishburn problem in the plane.

**Theorem 2.5.** If $k \in \mathbb{N}$ and $P \subseteq \mathbb{R}^2$ determines at most $k$ distinct $\ell^1$-distances, then $|P| \leq (k + 1)^2$. Further, $|P| = (k + 1)^2$ if and only if $P$ is $\ell^1$-similar to $\Lambda_2(k)$.

As we discuss in Section 4, the $d = 2$ case is simplified by the fact that, for the purposes of analyzing distance sets, the $\ell^1$-norm in $\mathbb{R}^2$ is effectively the same as the $\ell^\infty$-norm defined by $\|(x, y)\|_\infty = \max\{|x|, |y|\}$.
However, this equivalence does not persist in dimension \( d \geq 3 \), and for this reason, our proof strategy does not immediately generalize to higher dimensions. (Although, for the interested reader, the proof does generalize to show that if \( P \subseteq \mathbb{R}^d \) determines at most \( k \) distinct \( \ell^\infty \)-distances, then \(|P| \leq (k+1)^d \), and equality holds if and only if \( P \) is \( \ell^1 \)-similar to \( \{0,1,2,\ldots,k\}^d \).)

With considerable additional effort, we successfully get our foot into the higher-dimensional door in Section 5, which assures us that the unique optimality of \( \Lambda_d(k) \) is not completely dependent on a connection to the \( \ell^\infty \)-norm.

**Theorem 2.6.** If \( P \subseteq \mathbb{R}^3 \) determines a single \( \ell^1 \)-distance, then \(|P| \leq 6 \). Further, \(|P| = 6\) if and only if \( P \) is \( \ell^1 \)-similar to \( \Lambda_3(1) \).

**Remark on previous work for** \( k = 1 \.\) After the initial posting of this paper to the arxiv server, we were alerted to previous work done in the \( k = 1 \) case (referred to as *equilateral sets*) in a variety of metric spaces, including \( \mathbb{R}^d \) with the taxicab metric (referred to as *rectilinear space*). Specifically, Theorem 2.6 above follows from Corollary 4.2 of [1], due to Bandelt, Chepoi, and Laurent, while Koolen, Laurent, and Schrijver [7] showed that if \( P \subseteq \mathbb{R}^d \) determines a single \( \ell^1 \)-distance, then \(|P| \leq 8 = |\Lambda_1(1)| \). This partially settles a question of Kusner (see Problem 0 in [10]), who asked if \(|P| \leq 2d = |\Lambda_1(d)| \) holds for subsets of \( \mathbb{R}^d \) determining a single \( \ell^1 \)-distance, and this remains open for \( d \geq 5 \). Conjecture 2.7 below can be thought of as a precise, multi-distance generalization of Kusner’s question. While the conclusion of Theorem 2.6 was known previously, we believe our alternative, elementary proof given in Section 5 remains of interest.

In Section 5 we explore the question of what additional hypotheses are required to prove the optimality of \( \Lambda_3(k) \) for all \( k \in \mathbb{N} \), or even \( \Lambda_d(k) \) in full generality. We find that the proof of Theorem 2.5 can be fully adapted with a seemingly mild additional assumption, leading us to make the following general conjecture.

**Conjecture 2.7.** If \( d, k \in \mathbb{N} \) and \( P \subseteq \mathbb{R}^d \) determines at most \( k \) distinct \( \ell^1 \)-distances, then \(|P| \leq |\Lambda_d(k)|\). Further, \(|P| = |\Lambda_d(k)| \) if and only if \( P \) is \( \ell^1 \)-similar to \( \Lambda_d(k) \).

3. Properties of \( \Lambda_d(k) \): Proof of Theorems 2.2 and 2.3

We begin this section by proving the essential properties of \( \Lambda_d(k) \) that make it a worthy candidate for resolving the Erdős-Fishburn problem for the taxicab metric.

3.1. **Proof of Theorem 2.2.** Fix \( k \in \mathbb{N} \). For [4], fix \( d \in \mathbb{N} \), note that by definition of \( \Lambda_d(k) \), we have \(|n|_1 \leq k \) for all \( n \in \Lambda_d(k) \). In particular, for any \( n, m \in \Lambda_d(k) \), we have by the triangle inequality that \(|n - m|_1 \leq |n|_1 + |m|_1 \leq k + k = 2k \).

Further, \(|n - m|_1 = |n_1 - m_1| + \cdots + |n_d - m_d| \) is certainly an integer, and by definition of \( \Lambda_d(k) \), and the fact that an integer is congruent to its absolute value modulo 2, we have

\[
|n_1 - m_1| + \cdots + |n_d - m_d| \equiv n_1 - m_1 + \cdots + n_d - m_d \\
\equiv (n_1 + \cdots + n_d) - (m_1 + \cdots + m_d) \\
\equiv k - k \\
\equiv 0 \pmod{2}.
\]

Therefore, the only possible values of \(|n - m|_1\) are \( 2,4,\ldots,2k \), and for each \( 1 \leq j \leq k \), the distance \( 2j \) is attained between the points \((j,0,\ldots,0)\) and \((-j,0,\ldots,0)\) if \( j \equiv k \pmod{2} \), or between \((j,1,\ldots,0)\) and \((-j,1,\ldots,0)\) if \( j \not\equiv k \pmod{2} \).

For [4], we first see that

\[
\Lambda_1(k) = \begin{cases} 
\{-k,-k+2,\ldots,-1,1,\ldots,k-2,k\} & \text{if } k \text{ is odd} \\
\{-k,-k+2,\ldots,-2,0,2,\ldots,k-2,k\} & \text{if } k \text{ is even} 
\end{cases}.
\]

In particular, \(|\Lambda_1(k)| = 2|k/2| = k+1\) if \( k \) is odd and \(|\Lambda_1(k)| = 2(k/2) + 1 = k + 1\) if \( k \) is even. Secondly, we see that \( \Lambda_2(1) \) is precisely \( \{\pm e_i : 1 \leq i \leq d\} \), where \( e_i \) is the standard basis for \( \mathbb{R}^d \).
For (iii), we see that the possible values of the final coordinate for elements of $\Lambda_{d+1}(k)$ are integers satisfying $-k \leq x_{d+1} \leq k$. Further, for a fixed value $x_{d+1} = c$, the intersection of this hyperplane with $\Lambda_{d+1}(k)$ is

$$\{(n_1, \ldots, n_d, c) \in \mathbb{Z}^{d+1} : |n_1| + \cdots + |n_d| \leq k - |c|, n_1 + \cdots + n_d \equiv k - c \equiv k - |c| \pmod{2}\},$$

which is in natural bijection with $\Lambda_d(k - |c|)$. Therefore,

$$|\Lambda_{d+1}(k)| = \sum_{c=-k}^{k} |\Lambda_d(k - |c|)| = |\Lambda_d(k)| + 2\sum_{j=0}^{k-1} |\Lambda_d(j)|.$$

We continue by establishing a detailed formula for $|\Lambda_d(k)|$, which in particular guarantees that it has the correct order of magnitude $\Omega(k^d)$.

3.2. Proof of Theorem 2.3. We first note that by Theorem 2.2(ii), we have $|\Lambda_1(k)| = k + 1$ for all $k \geq 0$. We now fix $d \geq 2$, let $h = \lceil d/2 \rceil - 1$, and make the inductive hypothesis that

$$|\Lambda_{d-1}(k)| = a_{d-1,0}(k + 1)^{d-1} + a_{d-1,1}(k + 1)^{d-3} + \cdots + a_{d-1,h}(k + 1)^{d-1-2h}$$

for all $k \geq 0$. Faulhaber’s formula gives

$$F_p(n) = \sum_{j=1}^{n} j^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{p+1} \sum_{i=0}^{p-1} \binom{p+1}{i} B_{p+1-i} n^i,$$

for all $n, p \in \mathbb{N}$, where $B_i$ is the $i$-th Bernoulli number. By Theorem 2.2(iii), we have

$$|\Lambda_d(k)| = 2 \sum_{j=0}^{k} |\Lambda_{d-1}(j)| + |\Lambda_{d-1}(k)| = 2 \sum_{j=0}^{k} |\Lambda_{d-1}(j)| - |\Lambda_{d-1}(k)|,$$

which combines with (1) to yield

$$|\Lambda_d(k)| = 2 \left( a_{d-1,0} \sum_{j=0}^{k} (j + 1)^{d-1} + \cdots + a_{d-1,h} \sum_{j=0}^{k} (j + 1)^{d-1-2h} \right) - |\Lambda_{d-1}(k)|$$

$$= 2 \left( a_{d-1,0} \sum_{j=1}^{k-1} j^{d-1} + \cdots + a_{d-1,h} \sum_{j=1}^{k-1} j^{d-1-2h} \right) - |\Lambda_{d-1}(k)|$$

$$= 2 \left( a_{d-1,0} F_d(k + 1) + \cdots + a_{d-1,h} F_{d-1-2h}(k + 1) \right) - |\Lambda_{d-1}(k)|.$$

This tells us that we can indeed write $|\Lambda_d(k)|$ as a polynomial in $k + 1$, but we wish to establish the claimed explicit and recursive formulas for the coefficients, as well as the fact that every other coefficient is zero.

First we consider the $(k + 1)^d$ coefficient, which only arises from the term $2a_{d-1,0} F_{d-1}(k + 1)$. Since the $n^{p+1}$ coefficient of $F_p(n)$ is $1/(p+1)$, we have $a_{d,0} = 2a_{d-1,0}/d!$. Using the base case $a_{1,0} = 1$, we have by induction that $a_{d,0} = 2^{d-1}/d!$, as claimed.

Next we consider the $(k + 1)^{d-1}$ coefficient, which arises from two sources: the $(k + 1)^{d-1}$ coefficients of $2a_{d-1,0} F_{d-1}(k + 1)$ and $-|\Lambda_{d-1}(k)|$, respectively. The former is $2a_{d-1,0}(1/2) = a_{d-1,0}$, while the latter is $-a_{d-1,0}$, which means that the $(k + 1)^{d-1}$ coefficient of $|\Lambda_d(k)|$ is indeed 0. More generally, for other coefficients corresponding to terms of the form $(k + 1)^{d-1-2i}$, we use the following three facts: the $(k + 1)^{d-1-2i}$ coefficient on $2a_{d-1,i} F_{d-1-2i}(k + 1) = a_{d-1,i}$ by the same logic as above, the $(k + 1)^{d-1-2i}$ coefficient of $-|\Lambda_{d-1}(k)|$ is $-a_{d-1,i}$, and the $(k + 1)^{d-1-2i}$ coefficient of $F_{d-1-2i}(k + 1)$ is 0 for all $\ell < i$, because $B_n = 0$ for all odd $n \geq 3$. Therefore, all $(k + 1)^{d-1-2i}$ coefficients of $|\Lambda_d(k)|$ are 0.

For the $(k + 1)^{d-2}$ coefficient, we begin by noting that a direct calculation using (2) and (3) yields $|\Lambda_3(k)| = \frac{2}{3}(k + 1)^3 + \frac{1}{4}(k + 1)$, hence $a_{3,1} = 1/3$, which serves as the base case for another induction. Fixing $d \geq 4$ and assuming the claimed formula $a_{d-1,1} = 2^{d-4}/(3(d - 4)!)$ holds, the $(k + 1)^{d-2}$ coefficient of $|\Lambda_d(k)|$ is formed by two contributions, from $2a_{d-1,0} F_{d-1}(k + 1)$ and $2a_{d-1,1} F_{d-3}(k + 1)$, respectively.
The former is given by

\[ 2a_{d-1,0} \left( \frac{1}{d} \right) \left( \frac{d}{d-2} \right) B_2 = 2 \cdot \frac{2^{d-2}}{(d-1)!} \cdot \frac{1}{d} \cdot \frac{d(d-1)}{2} \cdot \frac{1}{6} = \frac{2^{d-3}}{3(d-2)!}, \]

while the latter is given by

\[ 2a_{d-1,1} \cdot \frac{1}{d-2} = 2 \cdot \frac{2^{d-4}}{3(d-4)!} \cdot \frac{1}{d-2} = \frac{2^{d-3}}{3(d-4)!(d-2)}. \]

Therefore, we have

\[ a_{d,1} = \frac{2^{d-3}}{3(d-2)!} + \frac{2^{d-3}}{3(d-4)!(d-2)} = \frac{2^{d-3}}{3(d-2)!} = \frac{2^{d-3}}{3(d-3)!}, \]

as claimed.

More generally, by (2) and (3), we see that the \((k + 1)^{d-2i}\) coefficient of \(|\Lambda_d(k)|\) receives a contribution from \(2a_{d-1,\ell} F_{d-1-2\ell}(k + 1)\) for each \(0 \leq \ell \leq i\). Specifically, that contribution is

\[ 2a_{d-1,\ell} \cdot \frac{1}{d-2\ell} \cdot \left( \frac{d-2\ell}{d-2i} \right) B_{2i-2\ell} = 2a_{d-1,\ell} \left( \frac{d-2\ell}{d-2i} \right) B_{2(i-\ell)}, \]

and the recursive formula for \(a_{d,\ell}\) follows.

\[ \square \]

4. Optimality in Two Dimensions: Proof of Theorem 2.5

In this section, we prove the unique optimality of \(\Lambda_2(k)\), in that it is the unique subset of \(\mathbb{R}^2\), up to \(\ell^1\)-similarity, of maximal size amongst sets determining at most \(k\) distinct \(\ell^1\)-distances. As referenced in Section 2, the proof is in part enabled by an equivalence between the \(\ell^1\)-norm and the \(\ell^\infty\)-norm on \(\mathbb{R}^2\). For the sake of exposition, we frame our discussion entirely in the context of the \(\ell^1\)-norm, but the connection is implicit in our proof, particularly the following lemma.

**Lemma 4.1.** Let \(v_1 = (1, 1)\) and \(v_2 = (-1, 1)\). If \(x \in \mathbb{R}^2\) with \(x = c_1 v_1 + c_2 v_2\), then

\[ \|x\|_1 = 2 \max\{|c_1|, |c_2|\}. \]

**Proof.** Let \(v_1 = (1, 1), v_2 = (-1, 1)\), fix \(x \in \mathbb{R}^2\), and write \(x\) uniquely as \(x = c_1 v_1 + c_2 v_2 = (c_1 - c_2, c_1 + c_2)\). By potentially reflecting over the diagonal \(x_1 = x_2\) and/or replacing \(x\) by \(-x\), both of which preserve the \(\ell_1\)-norm, we can assume without loss of generality that \(|c_1| \geq |c_2|\) and \(c_1 \geq 0\). In this case,

\[ \|x\|_1 = |c_1 - c_2| + |c_1 + c_2| = c_1 - c_2 + c_1 + c_2 = 2c_1 = 2 \max\{|c_1|, |c_2|\}. \]

\[ \square \]

Our main strategy for proving Theorem 2.5 is inspired by Erdős and Fishburn [3]. Specifically, we suppose that \(P \subseteq \mathbb{R}^2\) determines at most \(k\) distinct \(\ell^1\)-distances, and we seek an upper bound on the number of points we must remove from \(P\) in order to eliminate the largest \(\ell^1\)-distance, hence reducing to the case of \(k - 1\) distinct \(\ell^1\)-distances and allowing us to invoke an inductive hypothesis. The following sequence of lemmas formalizes this strategy. Here we define an \(\ell^1\)-ball in the expected way, as the region bounded by an \(\ell^1\)-sphere, which for \(d = 2\) is an \(\ell^1\)-circle.

**Lemma 4.2.** Suppose \(P \subseteq \mathbb{R}^2\) is finite. If \(D\) is the largest \(\ell^1\)-distance determined by \(P\), then \(P\) is contained in a closed \(\ell^1\)-ball of diameter \(D\).

**Proof.** Suppose \(P \subseteq \mathbb{R}^2\) is finite. Let \(v_1 = (1, 1)\) and \(v_2 = (-1, 1)\). Since \(\{v_1, v_2\}\) forms a basis for \(\mathbb{R}^2\), every \(x \in P\) can be written uniquely as \(x = c_1 v_1 + c_2 v_2\). Choose \(x_1, x_2, x_3, x_4 \in P\) such that \(x_1\) maximizes \(c_1\), \(x_2\) minimizes \(c_1\), \(x_3\) maximizes \(c_2\), and \(x_4\) minimizes \(c_2\). Call these values \(c_{1,\text{max}}, c_{1,\text{min}}, c_{2,\text{max}}, c_{2,\text{min}}\), respectively. These choices contain \(P\) inside of a rectangle \(R\), rotated 45° from axis parallel, determined by the inequalities \(c_{1,\text{min}} \leq c_1 \leq c_{1,\text{max}}\) and \(c_{2,\text{min}} \leq c_2 \leq c_{2,\text{max}}\).
Let \( w_1 = c_{1,\text{max}} - c_{1,\text{min}} \) and \( w_2 = c_{2,\text{max}} - c_{2,\text{min}} \), and assume without loss of generality that \( w_1 \geq w_2 \). By Lemma 4.4, we have that \( \|x_1 - x_2\|_1 = 2w_1 \) and \( \|p_1 - p_2\|_1 \leq 2w_1 \) for all \( p_1, p_2 \in \mathbb{R} \), so \( D = 2w_1 \) is the largest \( \ell_1 \)-distance determined by \( P \). Let \( c_{2,\text{new}} = c_{2,\text{max}} - w_1 \leq c_{2,\text{min}} \), and let \( B \supseteq R \supseteq P \) be defined by the inequalities \( c_{1,\text{min}} \leq c_1 \leq c_{1,\text{max}} \) and \( c_{2,\text{new}} \leq c_2 \leq c_{2,\text{max}} \). \( B \) is a square rotated 45° from axis parallel, or in other words a closed \( \ell_1 \)-ball, of diameter \( D \), as required.

**Lemma 4.3.** If \( P \subseteq \mathbb{R}^2 \) is contained in a closed \( \ell^1 \)-ball \( B \) of diameter \( D \), then the \( \ell^1 \)-distance \( D \) can be eliminated from \( P \) by removing the points of \( P \) contained in any two adjacent sides of the boundary of \( B \).

**Proof.** Suppose \( P \subseteq \mathbb{R}^2 \) is contained in a closed \( \ell^1 \)-ball \( B \) of diameter \( D \).

Let \( a_1, a_2 \) be the left and right vertices of \( B \), respectively, so in particular \( \|a_1 - a_2\|_1 = D \). Let \( U \) denote the closed (including \( a_1, a_2 \)) upper \( \ell^1 \)-semicircle connecting \( a_1 \) and \( a_2 \), and let \( L \) denote the open (not including \( a_1, a_2 \)) lower \( \ell^1 \)-semicircle connecting \( a_1 \) and \( a_2 \). Since the \( \ell^1 \)-norm is invariant under 90° rotation, it suffices to establish the conclusion of the lemma for removing the points of \( P \) lying in \( U \). Suppose \( x_1, x_2 \in P \setminus U \).

**Case 1:** At least one of \( x_1, x_2 \) lies in \( B \setminus (U \cup L) \), which is an open \( \ell^1 \)-ball of radius \( D/2 \).

Assume without loss of generality that \( x_1 \in B \setminus (U \cup L) \), and let \( c \) be the center of \( B \). Therefore, \( \|x_1 - c\|_1 < D/2 \) and \( \|x_2 - c\|_1 \leq D/2 \). By the triangle inequality, \( \|x_1 - x_2\|_1 \leq \|x_1 - c\|_1 + \|c - x_2\|_1 < D/2 + D/2 = D \).

**Case 2:** \( x_1, x_2 \in L \). After possibly reflecting, assume without loss of generality that \( x_1 \) is to the left of \( x_2 \) and \( \|x_1 - a_1\|_1 \leq \|x_2 - a_2\|_1 \), so \( x_1 \) is positioned at least as high as \( x_2 \). By replacing \( x_1 \) with \( a_1 \), we move up and to the left, so both the horizontal and vertical components of the \( \ell^1 \)-distance to \( x_2 \) get larger, hence

\[
\|x_1 - x_2\|_1 < \|a_1 - x_2\|_1 = D.
\]

In both cases, all distances amongst points in \( P \setminus U \) are strictly less than \( D \), and the lemma follows.

**Lemma 4.4.** If \( P \subseteq \mathbb{R}^d \) is contained in a line and determines at most \( k \) distinct \( \ell^1 \)-distances, then \( |P| \leq k + 1 \).

Further, if \( |P| = k + 1 \), then \( P \) is an arithmetic progression, meaning the \( \ell^1 \)-distances are \( \lambda, 2\lambda, \ldots, k\lambda \) for some \( \lambda > 0 \).

**Proof.** Since \( \ell^1 \)-distance along a straight line in \( \mathbb{R}^d \) is just a constant multiple, depending on the direction of the line, times the standard Euclidean distance, it suffices to establish the lemma with \( d = 1 \), for which we induct on \( k \).

The base case \( k = 1 \) is trivial, as three points \( x_1 < x_2 < x_3 \) in \( \mathbb{R} \) automatically determine two distances \( x_2 - x_1 < x_3 - x_1 \), and any two points form an arithmetic progression.

Now, fix \( k \geq 2 \), and assume that if \( Q \subseteq \mathbb{R} \) determines at most \( k - 1 \) distances, then \( |Q| \leq k \), and further, if \( |Q| = k \), then \( Q \) is an arithmetic progression. Now suppose \( P \subseteq \mathbb{R} \) determines at most \( k \) distances.

Let \( P = \{x_1 < x_2 < \cdots < x_n\} \). The \( n - 1 \) distances \( x_2 - x_1 < x_3 - x_1 < \cdots < x_n - x_1 \) are all distinct, hence \( n - 1 \leq k \), or in other words \( n \leq k + 1 \). Further, suppose \( n = k + 1 \). By removing \( x_{k+1} \), we also remove the longest distance \( x_{k+1} - x_1 \), so the set \( Q = \{x_1, \ldots, x_k\} \) determines \( k - 1 \) distances. By our inductive hypothesis, \( Q \) must be an arithmetic progression, in other words \( Q = \{x_1, x_1 + \lambda, x_1 + 2\lambda, \ldots, x_1 + (k-1)\lambda\} \).

If \( x_{k+1} < x_1 + k\lambda \), then both \( x_{k+1} - x_1 > (k-1)\lambda \) and \( x_{k+1} - x_k < \lambda \) are new distances not determined by \( Q \). If \( x_{k+1} > x_1 + k\lambda \), then both \( x_{k+1} - x_k > k\lambda \) and \( x_{k+1} - x_2 > (k-1)\lambda \) are new distances not determined by \( Q \). In either case, \( P \) determines at least \( k + 1 \) distinct distances, contradicting the assumption that it determines at most \( k \) distances. Therefore, \( x_{k+1} \) must be \( x_1 + k\lambda \), and the lemma follows.

\[ \Box \]
Lemma 4.5. If \( S \subseteq \mathbb{R}^2 \) is contained in two adjacent sides of an \( \ell^1 \)-circle and determines at most \( k \) distinct \( \ell^1 \)-distances, then \( |S| \leq 2k + 1 \). Further, if \( |S| = 2k + 1 \), then the points of \( S \) on each side form an arithmetic progression containing the shared vertex.

Proof. Suppose \( S \subseteq \mathbb{R}^2 \) is contained in two adjacent sides of an \( \ell^1 \)-circle and determines at most \( k \) distinct \( \ell^1 \)-distances. Assume without loss of generality that the two adjacent sides are the closed upper semicircle. We know from Lemma 4.4 that there are at most \( k + 1 \) points on each of the two sides.

Further, if \( |S| \geq 2k + 1 \), then there are exactly \( k + 1 \) points on one side, assume the left, and at least \( k \) points on the right side. We note that the \( \ell^1 \)-distance from the leftmost point to any point on the right side is at least the \( \ell^1 \)-distance from the leftmost point to the shared vertex. In particular, if the shared vertex is not included amongst the \( k + 1 \) points on the left side, then at least \( k + 1 \) distinct \( \ell^1 \)-distances occur from the leftmost point, contradicting our assumption.

Therefore, if \( |S| \geq 2k + 1 \), it must be the case that there are exactly \( k + 1 \) points on both the left and right sides, including the shared vertex, meaning in fact \( |S| = 2k + 1 \). Finally, by Lemma 4.4 we know that the \( k + 1 \) points on each side must form an arithmetic progression. \( \square \)

We are now fully armed to show the unique optimality of \( \Lambda_2(k) \).

Proof of Theorem 2.5. We induct on \( k \). For our base case, consider \( k = 0 \). In order for a set to determine 0 \( \ell^1 \)-distances (as always, not including 0), it can contain at most 1 = \((0 + 1)^2 \) point, and if it contains a point, then it is trivially a translation of \( \Lambda_2(0) = \{(0,0)\} \).

Now, fix \( k \in \mathbb{N} \), assume the conclusion of the theorem holds for \( k - 1 \), and suppose \( P \subseteq \mathbb{R}^2 \) determines at most \( k \) distinct \( \ell^1 \)-distances. By Lemma 4.2, \( P \) is contained in a closed \( \ell^1 \)-ball \( B \) of diameter \( D \), where \( D \) is the largest \( \ell^1 \)-distance determined by \( P \). By Lemma 4.3 we can remove the distance \( D \) by removing the points of \( P \) that lie on the closed upper \( \ell^1 \)-semicircle \( U \) on the boundary of \( B \). Since \( D \) has been removed as an \( \ell^1 \)-distance, we know that \( T = P \setminus U \) determines at most \( k - 1 \) distinct \( \ell^1 \)-distances. By our inductive hypothesis, \( |T| \leq k^2 \), and if \( |T| = k^2 \), then \( T \) is \( \ell^1 \)-similar to \( \Lambda_2(k - 1) \).

Further, by Lemma 4.5 we know that \( S = P \cap U \) satisfies \( |S| \leq 2k + 1 \), and if \( |S| = 2k + 1 \), then \( S \) consists of two \((k + 1)\)-term arithmetic progressions, one on each side of \( U \), which meet at the shared vertex. Therefore, \( |P| \leq |T| + |S| \leq k^2 + 2k + 1 = (k + 1)^2 \), and \( |P| = (k + 1)^2 \) if and only if \( T \) is \( \ell^1 \)-similar to \( \Lambda_2(k - 1) \) and \( S \) is a union of two arithmetic progressions meeting at the shared vertex. Finally, the only way these two sets can be combined without creating additional \( \ell^1 \)-distances is for \( S \cup T \) to be \( \ell^1 \)-similar to \( \Lambda_2(k) \). \( \square \)

5. Single \( \ell^1 \)-distance in three dimensions: Proof of Theorem 2.6

Without analogs of Lemmas 4.1 and 4.2 in dimension \( d \geq 3 \), our strategy for proving Theorem 2.5 does not naturally generalize to higher dimensions. However, in the case of \( k = 1 \), we make the observation that if \( P \subseteq \mathbb{R}^d \) determines a single \( \ell^1 \)-distance, then all but the “southernmost” point (the point minimizing the last coordinate) of \( P \) lie on a single closed upper \( \ell^1 \)-hemisphere. The following sequence of lemmas provide a detailed investigation into how \( \ell^1 \)-distance behaves between points on a single upper \( \ell^1 \)-hemisphere in \( \mathbb{R}^3 \), which consists of four flat faces, one for each quadrant determined by the first two coordinates, intersecting at a single northernmost point. The three lemmas correspond to the cases where the points lie on the same face, opposite faces, or neighboring faces, respectively.

Lemma 5.1. Suppose \( V, W \in \mathbb{R}^3 \) with \( V = (x_1, y_1, z_1) \) and \( W = (x_2, y_2, z_2) \). If \( \|V\|_1 = \|W\|_1 \) and \( x_1 x_2, y_1 y_2, z_1 z_2 \geq 0 \), then
\[
\|V - W\|_1 = 2 \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}.
\]
Proof. Suppose $V, W \in \mathbb{R}^3$, $V = (x_1, y_1, z_1)$, $W = (x_2, y_2, z_2)$, $\|V\|_1 = \|W\|_1 = \lambda$, and $x_1 x_2, y_1 y_2, z_1 z_2 \geq 0$. After reflections about coordinate planes, coordinate permutations, and relabeling $V$ and $W$ (which all preserve both sides of the equation in the conclusion of the lemma), we can assume without loss of generality that all coordinates are nonnegative and $x_1 - x_2 \geq |y_1 - y_2| \geq |z_1 - z_2|$. Since 

$$\|V\|_1 = x_1 + y_1 + z_1 = \|W\|_1 = x_2 + y_2 + z_2 = \lambda,$$

we have in particular that $(x_1 - x_2) + (y_1 - y_2) + (z_1 - z_2) = 0$. Since the largest coordinate distance is in the $x$-direction, and $x_1 \geq x_2$, we must have $y_1 \leq y_2$ and $z_1 \leq z_2$. Therefore

$$\|V - W\|_1 = (x_1 - x_2) + (y_2 - y_1) + (z_2 - z_1) = x_1 - x_2 + y_2 - y_1 + (\lambda - x_2 - y_2) - (\lambda - x_1 - y_1) = 2(x_1 - x_2),$$

and the lemma follows. \hfill \Box

Lemma 5.2. Suppose $V, W \in \mathbb{R}^3$ with $V = (x_1, y_1, z_1)$ and $W = (x_2, y_2, z_2)$. If $\|V\|_1 = \|W\|_1 = \lambda$, $x_1 x_2 \leq 0$, $y_1 y_2 \leq 0$, and $z_1, z_2 \geq 0$, then

$$\|V - W\|_1 = 2(\lambda - \min\{z_1, z_2\}).$$

Proof. Suppose $V, W \in \mathbb{R}^3$ with $V = (x_1, y_1, z_1)$, $W = (x_2, y_2, z_2)$, $\|V\|_1 = \|W\|_1 = \lambda$, $x_1 x_2 \leq 0$, $y_1 y_2 \leq 0$, and $z_1, z_2 \geq 0$. After reflections about coordinate planes and relabeling $V$ and $W$, we can assume without loss of generality that $x_1, y_1 \geq 0$, $x_2, y_2 \leq 0$, and $z_1 \leq z_2$. Therefore, $x_1 + y_1 = \lambda - z_1$ while $-x_2 - y_2 = \lambda - z_2$, hence

$$\|V - W\|_1 = (x_1 - x_2) + (y_1 - y_2) + (z_2 - z_1) = \lambda - z_1 - z_2 + z_2 - z_1 = 2(\lambda - z_1),$$

and the lemma follows. \hfill \Box

Lemma 5.3. Suppose $V, W \in \mathbb{R}^3$ with $V = (x_1, y_1, z_1)$, $W = (-x_2, y_2, z_2)$, $\|V\|_1 = \|W\|_1 = \lambda$, and $x_1 x_2, y_1 y_2, z_1 z_2 \geq 0$. If $\|V - W\|_1 = \lambda$, then $|x_1| \leq \lambda/2$.

Proof. Suppose $V, W \in \mathbb{R}^3$ with $V = (x_1, y_1, z_1)$, $W = (-x_2, y_2, z_2)$, $\|V\|_1 = \|W\|_1 = \lambda$, $x_1 x_2, y_1 y_2, z_1 z_2 \geq 0$. After reflecting about coordinate planes and scaling, we can assume $x_1, x_2, y_1, y_2, z_1, z_2 \geq 0$, and $\lambda = 2$. If $\|V - W\|_1 = 2$, then the largest possible value of $y_2 + z_2$ is $y_1 + z_1 + 2 - (x_1 + x_2)$. However, since $\|W\|_1 = 2$, we must have $y_2 + z_2 = 2 - x_2$, hence $2 - x_2 \leq y_1 + z_1 + 2 - (x_1 + x_2)$, which rearranges to $x_1 \leq y_1 + z_2 = 2 - x_1$, hence $x_1 \leq 1$, as required. \hfill \Box

We now establish the unique optimality of $A_3(1)$ by conducting a case analysis based on the concentration of the points of $P$, apart from the southernmost point, on the four faces of a single closed upper $\ell^1$-hemisphere.

Proof of Theorem 2.6. Suppose $P \subseteq \mathbb{R}^3$ determines a single $\ell^1$-distance $\lambda$, and choose a point $c \in P$ that minimizes the $z$-coordinate. By translating and dilating, we can assume without loss of generality that $c = (0, 0, 0)$ and $\lambda = 2$, and hence the remaining elements of $P$ are all contained in the closed upper $\ell^1$-hemisphere $H$ of radius 2 centered at $(0, 0, 0)$. We note that the southernmost point of $A_3(1)$ is $(0, 0, -1)$, so our end goal in this proof is to show that $|P| < 6$ unless $P$ is $A_3(1)$ shifted up by 1.
Case 1: \( V = (0, 0, 2) \in P \).

For \( Q = (x, y, z) \in H \), we have by Lemma \( 2.5 \) that \( \|Q - V\|_1 = 2(2 - z) = 2 \), hence \( z = 1 \). In particular, the elements of \( P \) other than \((0, 0, 0)\) and \((0, 0, 2)\) take the form \((x, y, 1)\) with \(|x| + |y| = 1\), and all pairs are separated by \( \ell^1 \)-distance 2. By Theorem \( 2.5 \) there can be at most four such elements, and the only choice of four that works is \((1, 0, 1), (0, 1, 1), (0, 1, 1)\), and \((0, -1, 1)\). The resulting arrangement is \( \Lambda_3(1) \) translated up by 1, which establishes Theorem \( 2.6 \) in this case.

For the remainder of the proof, we may assume that \((0, 0, 2) \not\in P\), and we consider the different ways that \( P \) can be concentrated on the faces of \( H \). To this end, we define \( H_{++} = \{(x, y, z) \in H : x, y \geq 0\} \) and \( H_{+-} = \{(x, y, z) \in H : x \geq 0, y \leq 0\} \), with analogous definitions for \( H_{-+} \) and \( H_{--} \). We refer to the pair \( H_{++}, H_{--} \) as opposite faces, and likewise for \( H_{+-}, H_{-+} \).

Case 2: \( V = (0, 0, 2) \not\in P\), and \( P \) contains three points \( U, V, W \) such that \( U \) and \( V \) lie on the same face, and \( W \) lies on the opposite face.

After reflecting about coordinate planes, we can assume that \( U, V \in H_{++} \) and \( W \in H_{--} \). The condition of Case 2 precisely means that the pair \((U, V)\) satisfies the conditions of Lemma \( 5.1 \) while the pairs \((U, W)\) and \((V, W)\) satisfy the conditions of Lemma \( 5.2 \). Therefore, if \( U = (x_0, y_0, z_0), V = (x_1, y_1, z_1) \), and \( W = (x_2, y_2, z_2) \), we have

\[
2 = \|U - V\|_1 = 2 \max\{|x_0 - x_1|, |y_0 - y_1|, |z_0 - z_1|\} = 2(2 - \min\{z_0, z_1\}) = 2(2 - \min\{z_1, z_2\}),
\]

and hence

\[
\max\{|x_0 - x_1|, |y_0 - y_1|, |z_0 - z_1|\} = \min\{z_0, z_2\} = \min\{z_1, z_2\} = 1.
\]

In particular, all three \( z \) coordinates are at least 1, and since \((0, 0, 2) \not\in P\), we have \(|z_0 - z_1| < 1\). Therefore, we simultaneously know that \( 0 \leq x_0, x_1, y_0, y_1 \leq 1 \) and \( \max\{|x_0 - y_0|, |x_1 - y_1|\} = 1 \).

This implies that (after potentially relabeling) either \( U = (1, 0, 1) \) and \( V = (0, y, 2 - y) \) for \( 0 < y \leq 1 \) or \( U = (x, 0, 2 - x) \) for \( 0 < x \leq 1 \) and \( V = (0, 1, 1) \). In either case, \( U \in H_{++} \cap H_{+-} \) and \( V \in H_{++} \cap H_{-+} \), so \( P \) contains at least one element on every face of \( H \). Therefore, by Lemma \( 5.2 \), all points of \( P \) lying on \( H \) have \( z \)-coordinate at least 1. Further, by the same reasoning as above, there are at most two points of \( P \) on each face, and the only way two points of \( P \) can lie on the same face is if they lie on opposite sides of the boundary, as with \( U \) and \( V \). In particular, at most four points of \( P \) lie on \( H \), and hence \( P \) contains at most five points in total.

We note that if \((0, 0, 2) \not\in P\), \(|P \cap H| \geq 5\), and all four faces of \( H \) contain at most two points of \( P \), then \( P \) necessarily falls into Case 2, which means the proof of Theorem \( 2.6 \) now reduces to the following final case.

Case 3: \((0, 0, 2) \not\in P\), and there exists a face of \( H \) containing at least three elements of \( P \).

After reflecting about coordinate planes, we can assume \( P \) contains three points \( U, V, W \in H_{++} \). Let \( x_0, y_0, \) and \( z_0 \) be the minimum \( x, y, \) and \( z \)-coordinates, respectively, attained by \( U, V, \) and \( W \). In what follows, we repeatedly utilize Lemma \( 5.1 \), which tells us that for every pair out of points in \( \{U, V, W\} \), the maximum coordinate distance is exactly 1.

We know that the maximum \( x, y, \) and \( z \)-coordinates attained by \( U, V, \) and \( W \) are at most \( x_0 + 1, y_0 + 1, \) and \( z_0 + 1 \). Suppose that this inequality is strict in at least one coordinate. By permuting coordinates and relabeling points we may assume that \( U = (x_0, y_1, z_1) \), and neither of \( V \) and \( W \) has \( x \)-coordinate \( x_0 + 1 \). In this case (after possibly relabeling), we know that \( V \) is either \( (x_0, y_1 + 1, z_1 - 1) \) or \( (x_0 + \alpha, y_1 - 1, z_1 + (1 - \alpha)) \) for \( 0 \leq \alpha < 1 \), while \( W \) is either \( (x_0, y_1 - 1, z_1 + 1) \) or \( (x_0 + \beta, y_1 + (1 - \beta), z_1 - 1) \) for \( 0 \leq \beta < 1 \). However, no combination of choices for \( V \) and \( W \) have a maximum coordinate distance of 1 from each other, so this arrangement is impossible. Therefore, all the maxima \( x_0 + 1, y_0 + 1, \) and \( z_0 + 1 \) are indeed achieved.

If two of these maximum coordinates appear simultaneously in a single point, then since all the points have \( \ell^1 \)-norm 2, it must be the case that \( x_0 = y_0 = z_0 = 0 \), in which case the three points are \((1, 1, 0), (0, 1, 1), \) and \((1, 0, 1)\), and no fourth point can be added.
If two of the minimum coordinates appear simultaneously in a single point, say \( U = (x_0, y_0, z_1) \), then since one of the coordinates must change by 1, the total net change must be 0, and \( x_0 \) and \( y_0 \) cannot be decreased, all of the decrease must happen in the \( z \)-coordinate, hence \( z_1 = z_0 + 1 \) and \( x_0 + y_0 + z_0 = 1 \). In this case, \( V = (x_0 + \alpha, y_0 + (1 - \alpha), z_0) \) and \( W = (x_0 + \beta, y_0 + (1 - \beta), z_0) \) for some \( 0 \leq \alpha, \beta \leq 1 \). Further, the maximum coordinate distance between \( V \) and \( W \) can only be 1 if \( \alpha = 0 \) and \( \beta = 1 \) or vice versa. In summary, this case yields the arrangement \((x_0, y_0, z_0 + 1), (x_0, y_0 + 1, z_0), (x_0 + 1, y_0, z_0)\) for some \( x_0, y_0, z_0 \geq 0 \) with \( x_0 + y_0 + z_0 = 1 \), and no fourth point can be added.

The only remaining case is that each point contains exactly one minimum coordinate and one maximum coordinate, so after permuting coordinates and relabeling points we assume \( U = (x_0, y_0 + 1, z_0 + \alpha) \) for some \( 0 < \alpha < 1 \). In this case, in order to meet the above conditions and preserve the \( \ell^1 \)-norm, \( V \) and \( W \) must be \((x_0 + 1, y_0 + \alpha, z_0)\) and \((x_0 + \alpha, y_0, z_0 + 1)\), in some order, where \( x_0, y_0, z_0 \geq 0 \) and \( x_0 + y_0 + z_0 = 1 - \alpha \), and no fourth point can be added. In fact, extending the range of \( \alpha \) to include 0 and 1 allows us to capture all three subcases in this single form.

We now note that if \( P \) contains any points in \( H_{-1} \), then we fall back into Case 2, so we can assume that all remaining points of \( P \cap H \) lie on \((H_{++} \cup H_{--}) \setminus H_{++} \). By Lemma 5.3, in order for a point \( Q \in P \cap (H_{--} \setminus H_{++}) \) to be \( \ell^1 \)-distance 2 away from \( V = (x_0 + 1, y_0 + \alpha, z_0) \), we must have \( x_0 = 0 \).

If there is a second point \( R \in P \cap (H_{--} \setminus H_{++}) \), then by considering the three points \( U, Q, R \), we fall back into Case 2.

If instead there is a point \( R \in P \cap (H_{++} \setminus H_{++}) \) that is \( \ell^1 \)-distance 2 from \( U = (x_0, y_0 + 1, z_0 + \alpha) \), then by Lemma 5.3, we again have \( y_0 = 0 \). In this case, our arrangement looks like \( U = (0, 1, 1), V = (1, \alpha, 1 - \alpha), W = (\alpha, 0, 2 - \alpha) \), so by considering the three points \( U, Q, W \), we again fall back into Case 2.

Therefore, every configuration in Case 3 allows for at most four points of \( P \) to lie on \( H \), hence \( P \) contains at most five points in total, and the proof is complete.

6. Conditional Results in Higher Dimensions

In the remainder of our discussion, we use the terms \( \ell^1 \)-sphere and \( \ell^1 \)-ball as before, defined analogously to regular spheres and balls in \( \mathbb{R}^d \), with the usual distance replaced by \( \ell^1 \)-distance. As noted at the beginning of Section 5, our proof of Theorem 2.5 does not naturally generalize to higher dimensions, because in dimension \( d \geq 3 \), it is not necessarily the case that if the largest \( \ell^1 \)-distance determined by a finite set \( P \subseteq \mathbb{R}^d \) is \( \lambda \), then \( P \) is contained in a closed \( \ell^1 \)-ball of diameter \( \lambda \).

However, consider the case where the largest \( \ell^1 \)-distance determined by \( P \) is \( \lambda \) and \((0, 0, \ldots, -\lambda/2), (0, 0, \ldots, \lambda/2) \) \( \in P \). In this case, \( P \) is contained in the intersection of the closed \( \ell^1 \)-balls of radius \( \lambda \) centered at \((0, 0, \ldots, -\lambda/2)\) and \((0, 0, \ldots, \lambda/2) \) \( \in P \), respectively, which happens to be the closed \( \ell^1 \)-ball of radius \( \lambda/2 \), hence diameter \( \lambda \), centered at the origin. Using our intuition from the usual Euclidean metric, it is tempting to say that we can reduce to this case by translating and rotating our original set \( P \), but we must contend with the fact that the taxicab metric is not invariant under rotation. The following definition codifies when we can reduce to this model case.

**Definition 6.1.** Given \( P \subseteq \mathbb{R}^d \) and an \( \ell^1 \)-distance \( \lambda > 0 \), we say that \( \lambda \) occurs in an axis parallel direction if there exist \( x \in P \) and \( 1 \leq i \leq d \) such that \( x + \lambda e_i \in P \), where \( e_i \) is the \( i \)-th standard basis vector.

The following definition introduces notation intended to streamline the induction arguments that follow.

**Definition 6.2.** For a fixed \( d, k \in \mathbb{N} \), we let \( \text{Opt}(d, k) \) denote the proposition that \( \Lambda_d(k) \) is the unique (up to \( \ell^1 \)-similarity) maximal subset of \( \mathbb{R}^d \) determining at most \( k \) distinct \( \ell^1 \)-distances. For example, we know that \( \text{Opt}(3, 1) \) and \( \text{Opt}(2, k) \) for all \( k \in \mathbb{N} \) hold by Theorems 2.6 and 2.5, respectively, while Conjecture 2.7 says precisely that \( \text{Opt}(d, k) \) holds for all \( d, k \in \mathbb{N} \).
6.1. Conditional induction in three dimensions. Since we have fully resolved the Erdős-Fishburn problem for the taxicab metric in dimension \( d = 2 \), we can bootstrap this result to dimension \( d = 3 \), provided we can reduce to the model case discussed above.

**Theorem 6.3.** Suppose \( k \geq 2 \), \( \text{Opt}(3, k - 1) \) holds, and \( P \subseteq \mathbb{R}^3 \) determines at most \( k \) distinct \( \ell_1 \)-distances. If the largest \( \ell_1 \)-distance determined by \( P \) occurs in an axis parallel direction, then \( |P| \leq |\Lambda_3(k)| \). Further, if \( |P| = |\Lambda_3(k)| \), then \( P \) is \( \ell_1 \)-similar to \( \Lambda_3(k) \).

**Proof.** Suppose \( k \geq 2 \), \( \text{Opt}(3, k - 1) \) holds, and \( P \subseteq \mathbb{R}^3 \) determines at most \( k \) distinct \( \ell_1 \)-distances. Since \( \text{Opt}(3, k - 1) \), we know that \( P \) is not of maximal size if it determines fewer than \( k \) distinct \( \ell_1 \)-distances, so we can assume it determines exactly \( k \) distinct \( \ell_1 \)-distances \( \lambda_1 < \lambda_2 < \cdots < \lambda_k \). If \( \lambda_k \) occurs in an axis parallel direction, then by scaling, translating and permuting coordinates, we can assume that \( k = 2k \) and \((0, 0, k), (0, 0, k) \in P \).

In this case, \( P \) is contained in the closed \( \ell_1 \)-balls of radius \( 2k \) centered at \((0, 0, -k)\) and \((0, 0, k)\), respectively. Conveniently, the intersection of these two \( \ell_1 \)-balls is itself an \( \ell_1 \)-ball, with radius \( k \) and center \((0, 0, 0)\). We now proceed as in the proof of Theorem 2.5 by noting that the distance \( 2k \) can be eliminated from \( P \) by removing the points of \( P \) that lie on the closed upper \( \ell_1 \)-hemisphere \( H \) of radius \( k \), centered at \((0, 0, 0)\). For a point \( U = (x, y, z) \in P \cap H \), we have \( \|U - (0, 0, k)\|_1 = |x| + |y| + k - z = 2(k - z) \), so the points of \((P \cap H) \setminus (0, 0, k) \) all have \( z \)-coordinates in the list \( z_1 > z_2 > \cdots > z_k \), where \( z_k = -\lambda_1/2 \).

For each \( z_i \), the points of \( H \) with \( z \)-coordinate equal to \( z_i \) take the form \((x, y, z_i) \) where \( |x| + |y| = k - z_i \), and we refer to the set of such points as \( S_i \). With regard to \( \ell_1 \)-distances, \( S_i \) is equivalent to an \( \ell_1 \)-circle in \( \mathbb{R}^2 \), centered at \((0, 0)\) with radius \( k - z_i \). All distances determined by \( P \cap S_i \) are at most \( 2(k - z_i) = \lambda_i \), so \( P \cap S_i \) determines at most \( i \) distinct \( \ell_1 \)-distances. By Lemma 4.5 the upper \((y \geq 0)\) and lower \((y \leq 0)\) semicircles of \( S_i \) can contain at most \( 2i + 1 \) points of \( P \), respectively.

Further, neither semicircle can exceed \( 2i - 1 \) points without including the boundary points \((k - z_i, 0, z_i), (z_i - k, z_i)\), and the maximum number of \( 4i \) total points can only occur if the points of \( P \) on each side of \( S_i \) form an arithmetic progression containing both endpoints, and in particular the first \( i \) \( \ell_1 \)-distances form an arithmetic progression. Therefore, \( |P \cap S_i| \leq 4i \), and equality holds for all \( 1 \leq i \leq k \) if and only if \( P \cap S_i = \{(x, y, z_i) : x, y \in \mathbb{Z}, |x| + |y| = i\} \).

Combining these facts for all \( i \), we have that
\[
|P \cap H| \leq 1 + \sum_{i=1}^{k} 4i = 2k^2 + 2k + 1 = |\Lambda_3(k)| - |\Lambda_3(k - 1)|,
\]
and equality holds if and only if \( P \cap H = \{(x, y, z) : x, y, z \in \mathbb{Z}, z \geq 0, |x| + |y| + z = k\} \). Since \( P \setminus H \) does not determine \( 2k \) as an \( \ell_1 \)-distance, \( \text{Opt}(3, k - 1) \) tells us that \( |P \setminus H| \leq |\Lambda_3(k - 1)| \), and therefore \( |P| = |P \cap H| + |P \setminus H| \leq |\Lambda_3(k)| \). Further, equality only holds if the \( \ell_1 \)-distances are \( 2, 4, \ldots, 2k \) and \( P \setminus H \) is \( \ell_1 \)-similar to \( \Lambda_3(k - 1) \), which forces \( P \setminus H \), given that \((0, 0, -k)\) is its southernmost point, to be \( \Lambda_3(k) \) shifted down by \( 1 \). In this case, \( P = (P \cap H) \cup (P \setminus H) \) is precisely \( \Lambda_3(k) \), and the induction is complete. \( \square \)

6.2. Generalized conditional induction. Finally, we generalize the observation that reducing to the case of the largest \( \ell_1 \)-distance occurring in an axis parallel direction allows us to bootstrap the unique optimality of our sets up a dimension. As seen in the proof of Theorem 6.3, the proof in dimension \( d \) also requires classification of optimal subsets of \( \ell_1 \)-spheres in dimension \( d - 1 \). This yields the following conjecture, which serves as our attempt to keep track of the minimal assumptions we currently require to fully resolve the Erdős-Fishburn problem for the taxicab metric.

**Conjecture 6.4.** Suppose \( d, k \in \mathbb{N} \). If \( P \subseteq \mathbb{R}^d \) is of maximal size amongst sets in \( \mathbb{R}^d \) determining at most \( k \) distinct \( \ell_1 \)-distances, then the largest \( \ell_1 \)-distance determined by \( P \) occurs in an axis parallel direction. The same holds within the class of sets contained in an \( \ell_1 \)-sphere in \( \mathbb{R}^d \).

As noted above, an adaptation of the proof of Theorem 6.3 requires knowledge about optimal subsets of \( \ell_1 \)-spheres, which we conditionally establish before proceeding to the main conjecture.
Theorem 6.5. If the $\ell^1$-sphere component of Conjecture 6.4 holds, then $\Lambda_d(k) \setminus \Lambda_d(k-2)$ is the unique (up to $\ell^1$-similarity) set of maximal size amongst sets contained in an $\ell^1$-sphere in $\mathbb{R}^d$ determining at most $k$ distinct $\ell^1$-distances for all $d, k \in \mathbb{N}$ (taking $\Lambda_d(-1)$ to be empty).

Proof. Suppose the $\ell^1$-sphere component of Conjecture 6.4 holds. As seen in the proof of Theorem 6.3, Lemma 4.5 implies that the conclusion of the theorem holds for $d = 2$ and $k \geq 2$, since $\Lambda_2(k) \setminus \Lambda_2(k-2)$ is a set of $4k$ equally spaced points around an $\ell^1$-circle, including all four vertices. We now fix $d \geq 3$ and $k \geq 2$, and we make the inductive hypothesis that the conclusion of the theorem holds for (dimension, number of $\ell^1$-distances) pairs $(d, k-1)$ and $(d-1, k)$.

Suppose $P \subseteq \mathbb{R}^d$ is contained in the $\ell^1$-sphere $S$ of radius $k$ centered at the origin, and that $P$ has maximal size amongst all such sets determining at most $k$-distinct $\ell^1$-distances. To establish Theorem 6.5, we must show that $P = \Lambda_d(k) \setminus \Lambda_d(k-2)$. We proceed in a manner analogous to our investigation of $P \cap H$ in the proof of Theorem 6.3. Again, thanks to our inductive hypothesis, we can assume $P$ determines exactly $k$ distinct $\ell^1$-distances, not fewer, and we denote those $\ell^1$-distances by $\lambda_1 < \cdots < \lambda_k$. By the $\ell^1$-sphere component of Conjecture 6.4, we know that $\lambda_k$ occurs in an axis-parallel direction. By permuting coordinates, we assume that $\lambda_k$ occurs in the last coordinate direction, in other words $(x_1, \ldots, x_d), (x_1, \ldots, x_d + \lambda_k) \in P$ for some $x_1, \ldots, x_d \in \mathbb{R}$ with $|x_1| + \cdots + |x_d| = |x_1| + \cdots + |x_d + \lambda_k| = k$, which in particular forces $x_d = -\lambda_k/2$ and $|x_1| + \cdots + |x_d - 1| = k - \lambda_k/2$. This transformation is allowable because our end goal, $\Lambda_d(k) \setminus \Lambda_d(k-2)$, is invariant under coordinate permutation.

We argue (informally for the moment) that the only reasonable choice is $\lambda_k = 2k$ and $x_1 = \cdots = x_d-1 = 0$, meaning $ke_d, -ke_d \in P$. This is because, since $\lambda_k$ is the largest $\ell^1$ distance, $P$ is contained in the intersection of the closed $\ell^1$-balls of radius $\lambda_k$ centered at $(x_1, \ldots, x_d-1, -\lambda_k/2)$ and $(x_1, \ldots, x_d-1, \lambda_k/2)$, respectively, and this intersection is the $\ell^1$-ball of radius $\lambda_k/2$ centered at $(x_1, \ldots, x_d-1, 0)$. However, $P$ is also contained in $S$, so if it is not the case that $\lambda_k = 2k$, then $P$ would in fact be contained in the intersection of an $\ell^1$-sphere with a closed $\ell^1$-ball of a smaller radius, which is at most a closed $\ell^1$-hemisphere. The idea that a maximal subset of an $\ell^1$-sphere determining at most $k$ distinct $\ell^1$-distances could actually be contained in a closed $\ell^1$-hemisphere is intuitively suspect, and we return to this issue near the end of the proof. For now, we assume $-ke_d, ke_d \in P$.

As in the proof of Theorem 6.3, we consider the closed upper $\ell^1$-hemisphere of $S$, which we denote by $H$. For a point $U = (x_1, \ldots, x_d) \in P \cap H$, we have $\|U - ke_d\|_1 = |x_1| + \cdots + |x_d-1| + k - x_d = 2(k-x_d)$, so the points of $(P \cap H) \setminus ke_d$ all have $x_d$-coordinates in the list $z_1 > z_2 > \cdots > z_k$, where $z_i = k - \lambda_i/2$.

For each $z_i$, the points of $H$ with $x_d$-coordinate equal to $z_i$ take the form $(x_1, \ldots, x_{d-1}, z_i)$ where $|x_1| + \cdots + |x_{d-1}| = k - z_i$, and we refer to the set of such points as $S_i$. With regard to $\ell^1$-distances, $S_i$ is equivalent to an $\ell^1$-sphere in $\mathbb{R}^{d-1}$, centered at the origin with radius $k - z_i$. All distances determined by $P \cap S_i$ are at most $2(k - z_i) = \lambda_i$, so $P \cap S_i$ determines at most $i$ distinct $\ell^1$-distances. By our inductive hypothesis, $|P \cap S_i| \leq |\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i-2)|$, with equality holding if and only if the projection $P \cap S_i$ onto the first $d-1$ coordinates is $\ell^1$-similar to $\Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i-2)$, so in particular $\lambda_1, \ldots, \lambda_i$ form an arithmetic progression. Since $k - z_i = \lambda_i/2$ and $\lambda_k = 2k$, this equality holds for all $1 \leq i \leq k$ if and only if $P \cap S_i = \{(x, k-i) : x \in \Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i-2)\}$ for all $1 \leq i \leq k$. Here we note that if it were not the case that $ke_d, -ke_d \in P$ as previously assumed, then $P \cap S_i$ would be, at most, equivalent to an $\ell^1$-hemisphere in $\mathbb{R}^{d-1}$, in which case our inductive hypothesis would prohibit it from having $|\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i-2)|$ elements.

In summary,

$$|P \cap H| \leq \sum_{i=0}^k |\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i-2)|,$$

taking $\Lambda_{d-1}(-1)$ and $\Lambda_{d-2}(-2)$ to be empty, and equality holds if and only if

$$P \cap H = \bigcup_{i=0}^k \{(x, k-i) : x \in \Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i-2)\}.$$

Further, $P \cap H$ can contain at most $\sum_{i=0}^{k-1} |\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i-2)|$ points with $x_d > 0$. 

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Letting $H'$ denote the closed lower $\ell^1$-hemisphere of $S$, we employ the identical reasoning as above to yield the same upper bound \[4\) on $|P \cap H'|$, with equality holding if and only if

$$P \cap H' = \bigcup_{i=0}^{k} \{(x, i - k) : x \in \Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i - 2)\}.$$  

Further, $P \cap H'$ can contain at most $\sum_{i=0}^{k-1} |\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i - 2)|$ points with $x_d < 0$. Putting all this together, we have

$$|P| = |P \cap H| + |P \cap H'| - |P \cap H \cap H'|$$

$$\leq 2 \left( \sum_{i=0}^{k} |\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i - 2)| \right) - (|\Lambda_{d-1}(k)| - |\Lambda_{d-1}(k - 2)|)$$

$$= |\Lambda_d(k)| - |\Lambda_d(k - 2)|,$$

where the last step comes from Theorem 2.2(iii), and equality holds if and only if

$$P = \bigcup_{i=-k}^{k-1} \{(x, i) : x \in \Lambda_{d-1}(k - |i|) \setminus \Lambda_{d-1}(k - |i| - 2)\} = \Lambda_d(k) \setminus \Lambda_d(k - 2),$$

and the induction is complete. $\square$

We conclude our discussion with the conditional proof of the unique optimality of $\Lambda_d(k)$ for all $d, k \in \mathbb{N}$.

**Theorem 6.6.** Conjecture 6.4 implies Conjecture 2.4.

**Proof.** We assume that Conjecture 6.4 holds, so in particular the conclusion of Theorem 6.5 holds. Theorem 6.5 allows us to adapt the proof of Theorem 6.3 to the higher-dimensional setting, and the higher-dimensional details are very similar to the proof of Theorem 6.5, so we streamline them a bit here. By Lemma 4.4 and Theorem 6.6, respectively, we know that $\text{Opt}(1, k)$ and $\text{Opt}(2, k)$ hold for all $k \in \mathbb{N}$. Also, we can extend the definition to include $\text{Opt}(d, 0)$, which trivially holds for all $d \in \mathbb{N}$, as maximal subsets determining $0$ $\ell^1$-distances, as well as $\Lambda_d(0)$, are single points.

We now fix $d \geq 3$ and $k \in \mathbb{N}$, and we make the inductive hypothesis that $\text{Opt}(d-1, k)$ and $\text{Opt}(d-1, k - 1)$ each hold. Suppose $P \subseteq \mathbb{R}^d$ determines at most $k$ distinct $\ell^1$-distances, and has maximal size amongst subsets of $\mathbb{R}^d$ with this property. Since $\text{Opt}(d-1, k)$ holds, we can assume that $P$ determines exactly $k$ distinct $\ell^1$-distances $\lambda_1 < \cdots < \lambda_k$. By Conjecture 6.4, we know that $\lambda_k$ occurs in an axis parallel direction, so by translating, scaling, and permuting coordinates, we can assume $\lambda_k = 2k$ and $ke_d, -ke_d \in P$.

Once again, we know that $P$ is contained in the intersection of the closed $\ell^1$-balls of radius $2k$ centered at $ke_d$ and $-ke_d$, respectively, which is the closed $\ell^1$-ball of radius $k$ centered at the origin. As in the proof of Theorem 6.3, we note that the $\ell^1$-distance $2k$ can be removed from $P$ by removing all points that lie on the closed upper $\ell^1$-hemisphere $H$ of radius $k$ centered at the origin. By Theorem 6.5, as seen in the induction step of that proof,

$$|\Lambda_{d-1}(i)| - |\Lambda_{d-1}(i - 2)| = |\Lambda_{d-1}(k)| + |\Lambda_{d-1}(k - 1)| = |\Lambda_d(k)| - |\Lambda_d(k - 1)|,$$

where the last step follows from Theorem 2.2(iii), and equality holds if and only if

$$P \setminus H = \bigcup_{i=0}^{k} \{(x, i - k) : x \in \Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i - 2)\}.$$  

Since $P \setminus H$ determines at most $k - 1$ distinct $\ell^1$-distances, and $\text{Opt}(d, k - 1)$ holds, we know that $|P| \leq |\Lambda_d(k - 1)|$, and equality holds if and only if $P$ is $\ell^1$-similar to $\Lambda_d(k - 1)$. 

$$\square$$
In this case, since $-ke_d \in P \setminus H$ and the largest remaining $\ell^1$-distance is $2(k - 1)$, $P \setminus H$ is in fact $\Lambda_d(k - 1)$ shifted by $-e_d$. Combining this with (5) and (6), we have that

$$|P| \leq |\Lambda_d(k - 1)| + |\Lambda_d(k)| - |\Lambda_d(k - 1)| = |\Lambda_d(k)|$$

and equality holds if and only if

$$P = \{x - e_d : x \in \Lambda_d(k - 1)\} \cup \bigcup_{i=0}^{k} \{(x, k - i) : x \in \Lambda_{d-1}(i) \setminus \Lambda_{d-1}(i - 2)\} = \Lambda_d(k),$$

so Opt($d, k$) holds and the theorem follows. □

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