A CONCORDANCE ANALOGUE OF THE 4-DIMENSIONAL LIGHT BULB THEOREM

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Abstract. We prove a concordance analogue of Gabai’s 4-dimensional light bulb theorem. That is, we show that when $R$ and $R'$ are homotopically embedded 2-spheres in a 4-manifold $X^4$ where $\pi_1(X^4)$ has no 2-torsion and one of $R$ or $R'$ has a transverse sphere, then $R$ and $R'$ are concordant. When $\pi_1(X^4)$ has 2-torsion, we give a similar statement with extra hypotheses as in the 4-dimensional light bulb theorem. We also give similar statements when $R$ and $R'$ are orientable positive-genus surfaces.

1. Introduction

In this paper, we will study surfaces smoothly embedded in 4-manifolds. For ease of notation, we will always work in the smooth category – “embedding” or “immersion” should be taken to mean “smooth embedding,” or “smooth immersion,” respectively. Similarly, “isotopy” or “homotopy” should be taken to mean “smooth ambient isotopy,” or “smooth ambient homotopy,” respectively.

We will prove an analogue of Gabai’s recent 4-dimensional light bulb theorem \cite{G} in the setting of concordance. We discuss the 4-dimensional light bulb theorem in length in Section 3; for now it suffices to say that the 4-dimensional light bulb theorem regards when homotopic surfaces embedded in a 4-manifold are actually isotopic (given some other hypotheses).

Definition 1.1 (Concordance). Let $M^m$ and $N^m$ be $m$-dimensional submanifolds of $X^n$. We say that $M^m$ and $N^m$ are concordant if there exists an $(m+1)$-dimensional submanifold $H$ of $X^n \times I$ so that $H \cap (X^n \times 0) = M^m$, $H \cap (X^n \times 1) = N^m$, and $H \cong M^m \times I$. We call $H$ a concordance from $M^m$ to $N^m$.

The author of this paper previously wrote a note showing how the 4-dimensional light bulb theorem in $S^2 \times S^2$ can be used to construct concordances in $S^2 \times S^2$ \cite{M}. That is, how to construct a concordance from $R$ to $R'$ when $R, R'$ are genus-$g$ surfaces in the homology class $[S^2 \times \text{pt}]$ in $S^2 \times S^2$. In this setting, the existence of such a concordance is already known from the following theorem of Sunukjian \cite{Su}.

Theorem 1.2 (\cite{Su} Thm. 6.1). Let $X^4$ be a simply connected 4-manifold. Then surfaces $S, S'$ in $X^4$ are concordant if and only if they have the same genus and $[S] = [S']$ in $H_2(X^4)$.

In this paper, we extend the construction of \cite{M} to a more general 4-dimensional manifold, as in the 4-dimensional light bulb theorem. In particular, we do not
assume the ambient manifold is simply connected, so our first main theorem does not follow from Theorem 1.2 (unless $\pi_1(X^4) = 0$).

**Theorem 1.3.** Let $X^4$ be an orientable 4-manifold so that $\pi_1(X^4)$ has no 2-torsion. Let $R, R'$, and $G$ be embedded 2-spheres in $X^4$ so that the following conditions hold:

- $G$ has trivial normal bundle,
- $R \cap G = \text{pt}$, where the intersection is transverse.
- $R'$ is homotopic to $R$.

Then $R$ and $R'$ are concordant.

**Definition 1.4.** Let $R$ be a surface embedded in a 4-manifold $X^4$. We say $G$ is a transverse sphere to $R$ if $G$ has trivial normal bundle and intersects $R$ in one point (transversely).

Schwartz [Sc] has shown that the 2-torsion assumption of Theorem 1.3 is necessary. In [Sc], Schwartz constructs a pair of homotopic spheres $R, R'$ with a common transverse sphere $G$ in 4-manifold $X^4$ so that $R$ and $R'$ are not concordant. (The main purpose of this example is to show that a 2-torsion hypothesis is necessary in the 4-dimensional light bulb theorem to conclude that $R$ and $R'$ are isotopic, but this simultaneously obstructs the more general relation of concordance.) Schwartz has infinitely many distinct examples of such pairs (specifically, a finite number of examples in each of infinitely many ambient 4-manifolds). In these examples, the 4-dimensional light bulb theorem does not apply because $\pi_1(X^4)$ has 2-torsion.

In fact, the counterexamples of [Sc] also show that Theorem 6.2 of [Su] is not true. This theorem implies concordance of surfaces $S_0$ and $S_1$ in 4-manifold $X^4$ given three conditions:

- $\pi_1(S_i) \to \pi_1(X^4)$ is trivial for each $i$ (e.g. $S_i$ is a sphere),
- $[S_0] = [S_1]$ in $H_2(X^4; Z[\pi_1])$ (i.e. the lifts of $S_0, S_1$ to the universal cover of $X^4$ are componentwise homologous)
- There exists a third surface $S$ in $X^4$ so that $\pi_1(S) \to \pi_1(X)$ is trivial, $[S] = [S_0]$ in $H_2(X^4; Z[\pi_1])$, and the meridian of $S$ is nullhomotopic in $X - S$ (e.g. $S = S_0$ if $S_0$ has a transverse sphere).

However, Schwartz’s spheres $R$ and $R'$ (taking the place of $S_0$ and $S_1$) satisfy all three of these conditions but are not concordant. (The first condition is obviously satisfied and the third follows from $R$ and $R'$ having transverse spheres. The second condition appears during the construction of [Sc]; the lifts of $R$ and $R'$ to the universal cover of $X^4$ are in fact isotopic.)

Otherwise, Theorem 6.2 of [Su] would imply Theorem 1.3. In the long term, we hope that some modification in the presence of 2-torsion might correct this theorem. Theorem 1.3 (and Theorem 1.5, which has yet to be stated) cover the cases in which $S_0$ and $S_1$ are homotopic and $S_0$ has a transverse sphere.

Our second main theorem applies when $\pi_1(X^4)$ has 2-torsion.

**Theorem 1.5.** Let $X^4$ be an orientable 4-manifold. Let $R$ and $R'$ be 2-spheres embedded in $X^4$ so that $R$ has a transverse sphere $G$ and $R'$ is homotopic to $R$. 
Then up to an obstruction related to how a homotopy from \( R' \) to \( R \) interacts with 2-torsion elements in \( \pi_1(X^4) \), \( R' \) is concordant to \( R \).

Theorem 1.5 generalizes Theorem 1.3. We state Theorem 1.5 precisely in Section 3.3 after giving several necessary definitions.

Finally, applying an argument of Sunukjian [Su, Thm. 7.5], we obtain the following corollary.

Corollary 1.6. Let \( X^4 \) be a 4-manifold. Let \( R \) and \( R' \) be 2-spheres smoothly embedded in \( X^4 \) satisfying the hypotheses of Theorem 1.5.

Assume that \( \pi_1(X^4) \) is a good group (as in [FQ]) and that a meridian of \( R' \) is nullhomotopic in \( X^4 \setminus R' \).

There there is a homeomorphism from the pair \( (X^4, R') \) to \( (X^4, R) \).

This corollary is our only foray outside of the smooth category.

Proof. We repeat the argument almost verbatim. In Theorem 1.5, we construct a concordance \( H \) from \( R' \) to \( R \). Lift \( H \subset X \times I \) to the universal cover \( \tilde{X} \times I \) of \( X \times I \) to find a cobordism \( \tilde{H} \) from \( \tilde{R}' \) to \( \tilde{R} \) (which are the lifts of \( R' \) and \( R \), respectively.) Each component of \( \tilde{R} \) has a transverse sphere in the lift \( \tilde{G} \) of \( G \), so every meridian of a component of \( \tilde{H} \) bounds a disk in \( \tilde{H}' := (\tilde{X} \times I) \setminus \tilde{H} \). Therefore, \( H' \) is the universal cover of \( H' := (X^4 \times I) \setminus H \). The Mayer-Vietoris sequence says that \( H' \) is an \( h \)-cobordism (here using the fact that \( X^4 \setminus \tilde{R}' \), \( \tilde{X}^4 \setminus \tilde{R} \), and \( H' \) are simply connected), so \( H' \) is also an \( h \)-cobordism. By additivity of Whitehead torsion (note \( H \) and \( H \cup H' = X^4 \times I \) are products), \( H' \) is actually an \( s \)-cobordism. Since \( \pi_1(X^4 \setminus R) \cong \pi_1(X^4) \) is good, \( H' \) is topologically a product. This product structure yields the desired homeomorphism. \( \square \)

Organization. We organize the paper as follows.

Section 2: We discuss the 3-dimensional light bulb theorem as lower-dimensional motivation. We give the proof of [Y] of a concordance analogue of the 3-dimensional light bulb theorem (also proved by [DNPR]).

Section 3: We discuss the 4-dimensional light bulb theorem and the statement of the main theorems.

Subsection 3.1: We state the 4-dimensional light bulb theorem and Theorem 1.3 for 4-manifolds with no 2-torsion in \( \pi_1 \).

Subsection 3.2: We remind the reader of important facts about homotopy and regular homotopy of surfaces in 4-manifolds.

Subsection 3.3: We state the general 4-dimensional light bulb theorem and Theorem 1.5. To do this, we give many definitions from [G].

Subsection 3.4: We recall the definition of a tubed surface from [G].

Section 4: We give the proofs of Theorems 1.3 and 1.5.

Acknowledgements. The author thanks her graduate advisor, David Gabai, for helpful conversations. Thanks to Danny Ruberman for useful discussion on regular homotopy and understanding Theorem 3.3 (by Hirsch [H]) and Nathan Sunukjian for interesting comments, including suggesting Corollary 1.6.
Figure 1. Left to right, top to bottom: the 3-dimensional light bulb trick. If \( K \) is a knot in \( S^1 \times S^2 \) intersecting \( pt \times S^2 \) in a single point, then we can effect crossing changes in \( K \) by sweeping a strand of \( K \) over the 2-sphere \( pt \times S^2 \).

The author is a fellow in the National Science Foundation Graduate Research Fellowship program, under Grant No. DGE-1656466.

2. A DIMENSION DOWN: THE LIGHT BULB THEOREM IN DIMENSION THREE

**Theorem 2.1** (3-dimensional light bulb theorem (folklore)). Let \( K \) be a circle embedded in \( S^1 \times S^2 \) so that \( K \) intersects \( pt \times S^2 \) geometrically once (and that intersection is transverse). Then \( K \) is isotopic to \( S^1 \times pt \).

**Proof.** Note that \( K \) is regularly homotopic to \( S^1 \times pt \) via a finite sequence of isotopies and crossing changes. The effect of each crossing change can be achieved via isotopy, by sweeping a strand of \( K \) parallel to \( pt \times S^2 \) (see Fig. 1; this is called the “light bulb trick”). Thus, \( K \) is in fact isotopic to \( S^1 \times pt \). \( \square \)

The statement of the 3-dimensional light bulb theorem requires that \( K \) have a transverse sphere. In this dimension, we mean that there must be a 2-sphere \( G \) (in this setting, \( pt \times S^2 \)) so that \( K \cap G = pt \). When we only know the algebraic intersection of \( K \) and \( G \), then the theorem does not necessarily hold (for example, the knot \( K \) in the leftmost frame of Fig. 2 is not isotopic to \( S^1 \times pt \), since \( \pi_1((S^1 \times S^2) \setminus K) \neq \mathbb{Z} \)). However, one can still construct a concordance from \( K \) to \( S^1 \times pt \).

**Theorem 2.2** (Concordance analogue of 3-dimensional light bulb theorem \[Y\] \[DNPR\]). Let \( K \) be a circle embedded in \( S^1 \times S^2 \) so that \( [K] = [S^1 \times pt] \) in \( H_1(S^1 \times S^2) \). Then \( K \) is concordant to \( S^1 \times pt \).

**Proof.** We illustrate this proof in Figure 2. This argument is due to Yildiz \[Y\].

As in the 3-dimensional light bulb theorem, \( K \) is regularly homotopic to \( S^1 \times pt \). This regular homotopy is a finite sequence of isotopies and crossing changes. We
build a concordance from $K$ (in this dimension, this means we build an annulus) in $(S^1 \times S^2) \times I$. Because visualizing submanifolds via movies is an important concept in paper, we attempt to give this easy example in detail.

Say $K$ is regularly homotopic to $S^1 \times pt$ via homotopy $f : S^1 \times I \to S^1 \times S^2 \times I$. Perturb $f$ so that all crossing changes happen at the same time, far from each other. Say there are $n$ crossing changes. Then take $f(S^1 \times t)$ to be smoothly embedded for $t \neq 1/2 + \epsilon$, and $f(S^1 \times (1/2 + \epsilon))$ to have $n$ self-intersections consisting of double-points (for some small $\epsilon > 0$).

Now we build an annulus $A$ in $(S^1 \times S^2) \times I$. Obtain $A'$ from $f(S^1 \times [0, 1/2])$ by attaching 2-dimensional 1-handles (bands) $b_1, \ldots, b_n$ to $f(S^1 \times 1/2)$. Specifically, attach one band for each crossing change of $f$. Each band lives in a neighborhood of the corresponding crossing change in $f(S^1 \times 1/2)$ and is embedded as in Figure 2 (second image).

Let $K' = (f(S^1 \times 1/2) \setminus (\cup_i b_i)) \cup ((\cup_i \partial b_i) \setminus f(S^1 \times 1/2))$. (In words, $K'$ is obtained from $f(S^1 \times 1/2)$ by deleting the intersection with $\partial b_i$ and adding in the rest of the boundary of $b_i$; this is normal band surgery.) View $K'$ as a subset of $S^1 \times S^2$ by identifying $S^1 \times S^2 \times 1/2$ with $S^1 \times S^2$. Note that $K'$ is a disjoint union of $n + 1$ circles. One of these components $C$ is isotopic to $S^1 \times pt$, while other $n$ components $U_1, \ldots, U_n$ are meridians of $C$. Each $U_i$ bounds a disk $D_i$ which does not intersect $K'$ in its interior (with $D_i \cap D_j = \emptyset$ for $i \neq j$).

Now let $A'' := A' \cup (C' \times [1/2, 3/4])$ in $S^1 \times S^2$. Attach the 2-dimensional 2-handle (disk) $D_i$ to $(U_i \times 3/4) \subset \partial A''$ for $i = 1, \ldots, n$; call the result $A'''$. (See Fig. 2, sixth image.) Finally, let $A = A''' \cup_{t \in [0, 1]} (g_t(S^1) \times (3 + t)/4)$, where $g : S^1 \times I \to S^1 \times S^2 \times I$ is an isotopy from $C$ to $S^1 \times pt$.

Thus, we have constructed a surface $A$ in $S^1 \times S^2 \times I$. See Figure 2 for a clear schematic of the above construction. We remark that as described, $A$ is not smoothly embedded, but rather has corners (A is in horizontal-vertical position). We can standardly smooth these corners and take $A$ to be smoothly embedded. We won't remark on this distinction later in the paper.
By construction, \( A \cap (S^1 \times S^2 \times 0) = K \) and \( A \cap (S^1 \times S^2 \times 1) = S^1 \times \text{pt} \). Moreover, \( A \) is obtained from \( K \times [0, 1/2] \) by attaching \( n \) geometrically cancelling pairs of 1- and 2-handles, so \( A \) is an annulus. Therefore, \( A \) is a concordance from \( K \) to \( S^1 \times \text{pt} \).

3. THE LIGHT BULB THEOREM IN DIMENSION FOUR

Now we move up a dimension, to consider the 4-dimensional light bulb theorem.

3.1. When the ambient 4-manifold has no 2-torsion in its fundamental group.

**Theorem 3.1** (4-dimensional light bulb theorem, [G, Thm. 1.2]). Let \( X^4 \) be an orientable 4-manifold so that \( \pi_1(X^4) \) has no 2-torsion. Let \( R \) and \( R' \) be 2-spheres embedded in \( X^4 \) so that \( R \) and \( R' \) have a mutual transverse sphere \( G \) and \( R \) is homotopic to \( R' \). Then \( R \) and \( R' \) are isotopic.

In fact, the above theorem also states that the isotopy from \( R \) to \( R' \) can be taken to fix a neighborhood of \( G \) if \( R \) and \( R' \) coincide near \( G \).

To compare the 4-dimensional light bulb theorem with the 3-dimensional light bulb theorem, one should notice that \( G \) takes the role of \( \text{pt} \times S^2 \), \( R \) takes the role of \( S^1 \times \text{pt} \) and \( R' \) take the role of \( K \). The proof of the 4-dimensional light bulb theorem is considerably more involved than the proof of the 3-dimensional light bulb theorem.

In Section 2, we discussed a concordance analogue of the 3-dimensional light bulb theorem. In this paper, we give a concordance analogue of the 4-dimensional light bulb theorem.

**Theorem 1.3.** Let \( X^4 \) be an orientable 4-manifold so that \( \pi_1(X^4) \) has no 2-torsion. Let \( R \) and \( R' \) be 2-spheres embedded in \( X^4 \) so that \( R \) has a transverse sphere \( G \) and \( R \) is homotopic to \( R' \). Then \( R \) and \( R' \) are concordant.

Gabai [G] produces a more general version of the 4-dimensional light bulb theorem that may apply even when \( \pi_1(X^4) \) has 2-torsion. To state this theorem, we must first understand regular homotopy of surfaces.

3.2. Regular homotopy of surfaces in 4-manifolds. In the 4-dimensional light bulb theorem, one need not distinguish between homotopy and regular homotopy due to the following celebrated theorem of Smale [Sm].

**Theorem 3.2** ([Sm, Thm. D]). Two smooth embedded 2-spheres in an orientable 4-manifold \( X^4 \) are homotopic if and only if they are regularly homotopic.

The above theorem is actually stated more generally for immersions of 2-spheres in \( X^4 \) (with an extra restriction), but we need only concern ourselves with the statement for homotopy between embedded surfaces in this paper. A similar result of Hirsch [H] holds for homotopic positive-genus surfaces.
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Figure 3. Top row: a finger move along arc $\gamma$, as in Def. 3.4. Bottom row: a Whitney move along disk $W$, as in Def. 3.5. We may take $N_\partial$ to point out of the page.

Theorem 3.3 ([H, Thm. 8.3]). Two smooth embedded orientable genus-$g$ surfaces in an orientable 4-manifold $X^4$ are homotopic if and only if they are regularly homotopic.

The cited theorem is actually stated for immersed $k$-spheres in $2k$-manifolds, but the proof carries through in this setting as well – we first isotope the surfaces to agree outside of a disk, and then apply the arguments of [H] Lemma 8.1 and Theorem 8.3 to regularly homotope the remaining disk.

See [FQ] for more exposition on the following definitions and well-known proposition about regular homotopy.

Definition 3.4. Let $S$ be a surface smoothly immersed in 4-manifold $X^4$. Let $\gamma$ be an arc in $X^4$ with endpoints on $S$ so that $\hat{\gamma} \cap S = \emptyset$. Take $\partial \gamma$ to be far from self-intersections of $S$. A finger move along $\gamma$ is the regular homotopy of $S$ which homotopes one disk component of $\nu(\gamma) \cap S$ along $\gamma$ to create a new pair of oppositely-signed transverse intersections of $S$. See top of Figure 3.

We will usually name this finger move $f_i$, for some index $i$. Then we will write $\gamma_i$ to indicate the arc $\gamma$ along which the finger move takes place.

Definition 3.5. Let $S$ be a surface smoothly immersed in 4-manifold $X^4$. Let $x$ and $y$ be two distinct self-intersections of $S$, of opposite sign. Let $\alpha$ and $\beta$ be arcs embedded in $S$ from $x$ to $y$ so that $\hat{\alpha}$ and $\hat{\beta}$ do not meet self-intersections of $S$, and near $x$ and $y$ the arcs $\alpha, \beta$ live in different sheets of $S$. Assume there exists a disk...
A Whitney move along $W$ is the regular homotopy of $W$ which homotopes the sheet of $S$ containing $\beta$ along $W$ to remove the self-intersections $x$ and $y$. See bottom of Figure 3. (The extension of $N_\beta$ to the normal bundle of $W$ is a technical requirement to make this move possible.)

We will usually name this Whitney move $w_i$ for some index $i$. Then we will write $W_i$ to indicate the Whitney disk along which this Whitney move takes place.

**Remark 3.6.** The finger move and Whitney move are inverse operations. That is, let $f$ be a finger move, so $f$ is a regular homotopy from surface $S$ to surface $S'$, where $S$ has $n$ self-intersections and $S'$ has $n+2$ self intersections. Let $f'$ be the inverse of $f$, so $f'$ is a regular homotopy from $S'$ to $S$ which cancels the two self-intersections introduced by $f$. The homotopy $f'$ is a Whitney move, as implicitly illustrated in Figure 3.

Similarly, when $w$ is a Whitney move from $S$ to $S'$, then the inverse homotopy $\overline{w}$ is a finger move from $S'$ to $S$.

**Notation.** Usually, we will write a Whitney move from $S$ to $S'$ as $w_i$ for some index $i$. Then we will refer to the path along which the finger move $\overline{w_i}$ takes place as $\eta_i$. Recall that $\eta_i$ is an arc in $X^4$ with endpoints on $S'$ with $\eta_i \cap S' = \emptyset$.

**Proposition 3.7.** Let $S$ and $S'$ be smoothly embedded surfaces in the smooth 4-manifold $X^4$. Suppose $S$ is regularly homotopic to $S'$. Then up to isotopy, the regular homotopy can be obtained by a finite sequence of finger moves followed by a finite sequence of Whitney moves, with intermediate isotopy at each step.

**Notation.** Let $h$ be a regular homotopy of a surface $R'$ which consists of finger moves $f_1, \ldots, f_n$ followed by Whitney moves $w_1, \ldots, w_n$ with intermediate isotopy. By slight abuse of notation, we will always refer to $f_i$ as a finger move of $R'$ for any $i$.

We will always denote the surface obtained from $R'$ by doing only the finger moves $f_1, \ldots, f_n$ by $S$. For $i = 2, \ldots, n$, we let $S_i$ denote the surface obtained from $S$ by performing Whitney moves $w_1, \ldots, w_{i-1}$. Then $w_i$ is a regular homotopy of $S_i$, so $\partial W_i \subset S_i$. Note by dimensionality that $W_i$ does not intersect $\eta_j$ for any $j < i$.

**3.3. When the ambient 4-manifold has nontrivial 2-torsion in its fundamental group.** We now move onto specific definitions required to parse the statement of the generalized 4-dimensional light bulb theorem. From now on, let $A$ abstractly be the 2-sphere. Given a sphere $Y$ embedded or immersed in $X^4$, let $\pi_Y : A \to X^4$ be the actual embedding or immersion.

**Remark 3.8.** Let $S$ be a 2-sphere embedded in $X^4$. Fix points $x$ and $y$ in $S$. Let $\gamma$ be an arc in $X^4$ from $x$ to $y$ with $\gamma \cap S = \emptyset$. Fix a point $z$ in $S$. Let $\gamma_{xz}, \gamma_{yz}$ be arcs in $S$ from $z$ to $x$ and $y$ to $z$, respectively. Then $\gamma$ uniquely determines the
element of $\pi_1(X^4, z)$ represented by $\gamma_{xz} \gamma \gamma_{yz}$. We write $[\gamma]$ to indicate this element of $\pi_1(X^4, z)$.

**Definition 3.9.** Let $R$ and $R'$ be regularly homotopic 2-spheres in $X^4$. Say that some regular homotopy $h$ from $R'$ to $R$ consists of the finger moves $f_1, \ldots, f_n$ followed by the Whitney moves $w_1, \ldots, w_n$ (with intermediate isotopies). Let $S$ be the surface obtained from $R'$ after performing the finger moves $f_1, \ldots, f_n$, so $S$ is a 2-sphere immersed in $X^4$ with $2n$ points of self-intersection.

Let $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ be pairs of distinct points in $A$ mapping to distinct self-intersections of $S$, so $\pi_S(x_i) = \pi_S(y_i)$. We take $\pi_S(x_i) = \pi_S(y_i)$ if $\pi_S(x_i) = \pi_S(y_i)$ is not cancelled by one of the Whitney moves $w_1, \ldots, w_{j-1}$. For each finger move $f_i$, there exist two distinct $i_1, i_2$ so that $\pi_S(x_{i_1}, x_{i_2}, y_{i_1}, y_{i_2})$ lie in the support of $f_i$ (i.e. are the two self-intersections introduced by $f_i$). Choose the labelings of each pair $(x_j, y_j)$ so that $x_{i_1}$ and $x_{i_2}$ lie in the same sheet of $S$ in this support, while $y_{i_1}$ and $y_{i_2}$ lie on the other.

We now refer to $L = (L_x, L_y)$ as a labeling of $h$, where $L_x = \{x_1, \ldots, x_n\}, L_y = \{y_1, \ldots, y_n\}$. There are $2^n$ distinct labelings of $h$.

**Definition 3.10 (\cite{G} Def. 5.15).** Let $R$ and $R'$ be regularly homotopic 2-spheres in $X^4$. Let $h$ be a regular homotopy from $R'$ to $R$ consisting of the finger moves $f_1, \ldots, f_n$ followed by the Whitney moves $w_1, \ldots, w_n$ (with intermediate isotopies). Let $L = (L_x, L_y)$ be a labeling of $h$, where $L_x = \{x_1, \ldots, x_n\}, L_y = \{y_1, \ldots, y_n\} \subset A$.

Let $W_i$ be the Whitney disk associated to Whitney move $w_i$, as in Definition 3.5 Then $\pi_S^{-1}(\partial W_i)$ consists of two arcs $\partial^1_i$ and $\partial^2_i$, whose four boundary points collectively consist of two points in $L_x$ and two points in $L_y$. If one of $\partial^1_i, \partial^2_i$ connects two points in $L_x$ while the other connects two points in $L_y$, then we say that $w_i$ is uncrossed with respect to $L$. If each of $\partial^1_i, \partial^2_i$ meet both $L_x$ and $L_y$, then we say that $w_i$ is crossed with respect to $L$.

Now we are able to state the general 4-dimensional light bulb theorem.

**Theorem 3.11 (4-dimensional light bulb theorem, \cite{G} Thm. 1.3).** Let $X^4$ be an orientable 4-manifold. Let $R$ and $R'$ be 2-spheres embedded in $X^4$ so that $R$ and $R'$ have a mutual transverse sphere $G$ and $R$ is homotopic to $R'$.

Let $h$ be a regular homotopy (via Thm. 3.2) from $R'$ to $R$ which consists of a sequence of finger moves $f_1, \ldots, f_n$ followed by Whitney moves $w_1, \ldots, w_n$ (with intermediate isotopies). Choose a labeling $L$ of $h$.

Let $\eta_i$ be the path along which the finger move $w_i$ takes place, as in Remark 3.6. By Remark 3.8, each $\eta_i$ represents an element $[\eta_i]$ of $\pi_1(X^4)$ with basepoint on $R$. Let $\mathcal{H}$ be the multiset $\{[\eta_i] \mid w_i$ is crossed with respect to $L\}$.

If each 2-torsion element of $\pi_1(X^4)$ appears an even number of times in $\mathcal{H}$, then $R'$ is isotopic to $R$.

From now on, when we refer to the “4-dimensional light bulb theorem,” we mean the statement of Theorem 3.11. When $\pi_1(X^4)$ has no 2-torsion, this restricts to the statement of Theorem 3.1.

The statement we give here of Theorem 3.11 does not exactly match the statement given in \cite{G}; we imagine a reader referring from this paper to \cite{G} for the first time.
might be confused by the difference. Gabai states Theorem 3.11 in terms of putting $R'$ into a normal form with respect to $R$ and then later gives the exact obstruction to this normally positioned surface being isotopic to $R$. Here, we have pulled back the whole result into one statement to make later discussion easier.

We extend Theorem 1.3 accordingly.

**Theorem 1.5.** Let $X^4$ be an orientable 4-manifold. Let $R$ and $R'$ be 2-spheres embedded in $X^4$ so that $R$ has a transverse sphere $G$ and $R$ is homotopic to $R'$.

Let $h$ be a regular homotopy (via Thm. 3.2 from $R'$ to $R$ which consists of a sequence of finger moves $f_1,\ldots,f_n$ followed by Whitney moves $w_1,\ldots,w_n$ (with intermediate isotopies). Choose a labeling $L$ of $h$.

Let $\eta_i$ be the path along which the finger move $\overline{w_i}$ takes place, as in Remark 3.6. By Remark 3.8 each $\eta_i$ represents an element of $\pi_1(X^4)$ with basepoint on $R$. Let $H$ be the multiset $\{[\eta_i] | w_i \text{ is crossed with respect to } L\}$.

If each 2-torsion element of $\pi_1(X^4)$ appears an even number of times in $H$, then $R'$ is concordant to $R$.

To prove Theorem 1.5, we must understand some details of the proof of the 4-dimensional light bulb theorem.

### 3.4. Tubed surfaces

In [G], Gabai defines the class of tubed surfaces, which are defined by attaching tubes to embedded surfaces in a prescribed way.

**Definition 3.12** ([G Def. 5.4]). A framed embedded path is a smooth embedded path $\tau : I \to X^4$ with a framing $(\nu_1(t),\nu_2(t),\nu_3(t))$ of its normal bundle. Let $C(t)$ be a circle bounding a disk centered at $\tau(t)$ in the plane spanned by $\nu_1(t),\nu_2(t)$. Take each $C(t)$ to have small radius. We call the annulus $\bigcup_{t \in [0,1]} C(t)$ the cylinder from $C(0)$ to $C(1)$.

**Remark 3.13.** In the definition of a finger move $f$ along $\gamma$ of an immersed surface $S$, the path $\gamma$ is actually a framed embedded path. The framing on $\gamma$ is chosen so that $C(0) \subset S$ and $C(1)$ intersects $S$ in two points. The result $S'$ of the finger move can be obtained from $S$ by deleting the disk in $S$ bounded by $C(0)$ which contains $\gamma(0)$, then attaching the cylinder from $C(0)$ to $C(1)$ and a small disk $D$ bounded by $C(1)$ chosen so that $D \cap S = \emptyset$ and the resulting surface has transverse self-intersections.

**Definition 3.14** ([G Def. 5.5]). Let $S$ be an immersed surface in $X^4$. Fix a transverse sphere $G$ for $S$. Say $S$ has $n$ points of self-intersection, so there are $2n$ distinct points $x_1,\ldots,x_n,y_1,\ldots,y_n \in A$ with $\pi_S(x_i) = \pi_S(y_i)$ for each $i$. Let $z_0 = \pi_S^{-1}(z)$. A tubed surface $S_T$ on $S$ consists of the following data:

i) The immersion $\pi_S : A \to X^4$.

ii) For each $i = 1,\ldots,n$, an immersed path $\sigma_i \subset A$ from $x_i$ to $z_0$.

iii) Immersed paths $\alpha_1,\ldots,\alpha_r$ in $A$ with both endpoints at $z_0$ and for each $i = 1,\ldots,r$, pairs of points $(p_i,q_i)$ in $A$ with $p_i \in \alpha_i$ and a framed embedded path $\tau_i \subset X^4$ from $\pi_S(p_i)$ to $\pi_S(q_i)$ with $\bar{\tau}_i \cap (G \cup S) = \emptyset$.

iv) Pairs of immersed paths $(\beta_1,\gamma_1),\ldots,(\beta_s,\gamma_s)$ in $A$ where $\beta_i$ goes from $z_0$ to $b_i$ and $\gamma_i$ goes from $g_i$ to $z_0$ (for some $b_i, g_i \in A$) and framed embedded paths $\eta_i$ from $\pi_S(b_i)$ to $\pi_S(g_i)$ with $\bar{\eta}_i \cap (G \cup S) = \emptyset$.
The union of all arcs $\sigma_i, \alpha_j, \beta_k, \gamma_l, \tau_p, \eta_q$ is called the tube guide locus of $S_T$.

We require that the $\sigma_i, \alpha_j, \beta_k, \gamma_l$ curves be self-transverse and transverse to each other, and that their interiors not meet any points of the form $x_i, y_j, b_k, g_l$, and also be disjoint from $z_0$ and the $p_i$ points except as specified. At crossings of these curves, one sheet should be labeled as above or below the other sheet (as in a crossing in a standard knot diagram). The points of the form $x_i, y_j, p_k, q_l, b_m, g_n$ are all distinct (and distinct from $z_0$).

The curves $\tau_i$ and $\eta_j$ are pairwise disjoint. We require they be normal to $S$ near their boundaries. Recall that for each framed arc $\tau_i$ or $\eta_j$, we defined circles $C(0)$ and $C(1)$ near their boundaries in Definition 3.12. We restrict the allowed framings on $\tau_i$ and $\eta_j$, but do not state this condition until Construction 3.15.

We say that $S$ is the underlying surface of $S_T$.

From a tubed surface $S_T$ on $S$ we can construct an embedded surface.

**Construction 3.15** (Construction 5.7). Let $S_T$ be a tubed surface on $S$. From $S_T$, we construct an embedded surface $S_R$ called the realization of $S_T$ as follows (see Fig. 4 for an illustration that is likely more helpful than the ensuing wall of text):

i) For each $i$, remove from $S$ a disk $\pi_S(\nu(y_i))$. Attach to this new boundary component a disk $D(\sigma_i)$ consisting of a tube that follows $\pi_S(\sigma_i)$ and connects to a copy of $G \setminus \nu(z)$, pushed slightly off $G$.

ii) For each $\alpha_i$ arc, let $P(\alpha_i)$ be a 2-sphere obtained by attaching a copy of $G \setminus \nu(z)$ to each end of a tube following $\pi_S(\alpha_i)$, and pushing the copies of $G$ slightly off $G$ (and each other). The restriction on the framing of $\tau_i$ mentioned in Definition 3.14 is that if $C(0)$ and $C(1)$ are the circles near $\tau_i(0)$ and $\tau_i(1)$ as in Definition 3.12, then we require $C(0)$ to lie in $P(\alpha_i)$ and $C(1)$ to lie in $S$. Delete open disks in $P(\alpha_i)$ and $S$ bounded by $C(0)$ and $C(1)$ respectively and glue the resulting punctured surfaces together via the cylinder from $C(0)$ to $C(1)$ (as in Def. 3.12). This yields an embedded surface $\hat{S}$. We call the cylinders around $\tau_i$ a single tube.

iii) Now for each $\eta_i$ arc, construct disks $D(\beta_i)$ and $D(\gamma_i)$ consisting of copies of $G \setminus (\nu(z))$ (pushed slightly off $G$ and each other) with collars parallel to $\pi_S(\beta_i)$ and $\pi_S(\gamma_i)$ respectively, so the boundary of $D(\beta_i)$ lies in a disk normal to $S$ at $\pi_S(b_i)$ and the boundary of $D(\gamma_i)$ lies in a disk normal to $S$ at $\pi_S(g_i)$. Fix 4-balls $N_{b_i}$ and $N_{g_i}$ about $\pi_S(b_i)$ and $\pi_S(g_i)$ so that $\partial N_{b_i} \cap (S \cap D(\beta_i))$ is a Hopf link in the 3-sphere $\partial N_{b_i}$, and similarly $\partial N_{g_i} \cap (S \cap D(\gamma_i))$ is a Hopf link in the 3-sphere $\partial N_{g_i}$.

The restriction on the framing of $\eta_i$ mentioned in Definition 3.14 is that if $C(0)$ and $C(1)$ are the circles near $\eta_i(0)$ and $\eta_i(1)$ as in Definition 3.12, then we require $C(0)$ to lie in $\partial N_{b_i} \cap S$ and $C(1)$ to be $\partial D(\eta_i)$. Let $x(t) \in C(0)$ be the point in direction $\nu_1(t)$ in the framing of $\eta_i$. (see Def. 3.12)

Connect the specified Hopf links by two tubes parallel to $\eta_i$. One tube is the cylinder from $C(0)$ to $C(1)$ and connects $\partial N_{b_i} \cap S$ to $\partial D(\eta_i)$. The other tube is centered around $\cup_t x(t)$ and connects $\partial D(\beta_i)$ to $\partial N_{g_i} \cap S$. We call
these two tubes together a double tube. The resulting embedded surface is $S_R$. At each stage, whenever two tube segments correspond to arcs of the tube guide locus which intersect in $A$, take the tube corresponding to the “under” segment to have smaller radius and thus lie closer to $S$, to avoid self-intersections of $S_R$. This is a slight abuse of notation, as one arc of the tube guide locus in $A$ may cross itself – but simply take the piece of the tube corresponding to the “under” segment to be narrow.

We illustrate this construction in Figure 4.

A major part of the proof of the 4-dimensional light bulb theorem is the following proposition.

**Proposition 3.16 (G).** Let $S_T$ be a tubed surface on $S$, where $S$ is a 2-sphere embedded in $X^4$. Suppose that for each element $[\gamma]$ of 2-torsion in $\pi_1(X^4)$, $[\gamma]$ appears an even number of times in the list $[\eta_1],\ldots,[\eta_k]$ where $\eta_1,\ldots,\eta_k$ are as in Definition 3.14 (in words, $\eta_1,\ldots,\eta_k$ are the arcs yielding double tubes of $S_R$; recall by Remark 3.8 that $[\eta_i]$ is an element of $\pi_1(X^4)$ with basepoint in $S$). Then $S_R$ is isotopic to $S$.

4. Proof of Theorems 1.3 and 1.5

A very basic outline for the proof of Theorems 1.3 and 1.5 is as follows:

1. In $X^4 \times I$, build a cobordism from $R'$ to a positive-genus surface $S_+''$ by attaching 3-dimensional 1-handles to $R' \times I$.

2. Attach geometrically cancelling 3-dimensional 2-handles to the above cobordism to find a concordance from $R'$ to $R''$, where $R''$ is a sphere homotopic to $R'$ (and $R$) and $R'' \cap G = \emptyset$.

3. Argue that $R''$ is the realization of a tubed surface on $R$.

4. Apply Proposition 3.16 to conclude that $R''$ is isotopic to $R$ given the hypotheses of Theorem 1.5.

4.1. Construction of a concordance from $R'$. Let $z := R \cap G$. Recall that $h$ is a regular homotopy from $R'$ to $R$ consisting of finger moves $f_1,\ldots,f_n$ followed by Whitney moves $w_1,\ldots,w_n$ (with intermediate isotopies), and with labeling $L$.

Let $S$ be the surface obtained from $R'$ by performing only the finger moves $f_1,\ldots,f_n$, so $S$ is an immersed 2-sphere in $X^4$ with $2n$ points of self-intersection.

Recall that $\gamma_i$ is the path along which the finger move $f_i$ takes place, where $\gamma_i$ is an arc in $X^4$ with $\partial \gamma_i \subset R'$ and $\gamma_i \cap R' = \emptyset$ as in Definition 3.4. Recall also that $W_i$ is the Whitney disk associated to $w_i$, so that $W_i$ is a disk in $X^4$ with $\partial W_i \subset S_i$ and $W_i \cap S_i = \emptyset$ as in Definition 3.5. Again, $S_i$ denotes the surface obtained from $S$ after performing Whitney moves $w_1,\ldots,w_{i-1}$.

Let $L = (L_x,L_y)$ be a labeling of $h$ as in Definition 3.9 so $L_x = \{x_1,\ldots,x_n\}, L_y = \{y_1,\ldots,y_n\}$ are disjoint subsets of $A$ with $\pi_S(x_i) = \pi_S(y_i)$. The map $\pi_S^{-1}$ takes $S \cap (\text{support of } f_i)$ to two disjoint disks; in the definition of a labeling we require
that two points in $L_x$ be contained in one of these disks and two points in $L_y$ be in the other.

Let $S_+$ be the genus-$n$ embedded surface in $X^4$ obtained from $S$ by attaching a tube $T_i$ between the two self-intersections of $S$ created by $f_i$, as in Figure 3 (left two images). Specifically, if $\pi_S(x_j)$ and $\pi_S(x_k)$ are in the support of $\gamma_i$ (for $j \neq k$), fix an arc $\sigma_i$ in $A$ from $x_j$ to $x_k$. Take $\sigma_1, \ldots, \sigma_n$ to be disjoint. Then for $i = 1, \ldots, n,$
Although we have described Remark 4.1. Parallel to delete $\pi_S(\nu(y_i))$ from $S$. Attach $n$ tubes $T_1, \ldots, T_n$ to this bounded surface, with $T_i$ parallel to $\pi_S(\sigma_i)$. 

**Remark 4.1.** Although we have described $S_+$ as being obtained from $S$ by attaching tubes $T_i$ ($i = 1, \ldots, n$), we can alternatively obtain $S_+$ from $R'$ by attaching tubes $\bar{T}_i'$ ($i = 1, \ldots, n$). See Figure 3 (right two images). The tube $\bar{T}_i'$ lies in the support of $\gamma_i$. For each $i = 1, \ldots, n$, let $\bar{\gamma}_i : [0, 1 + \epsilon]$ be an extension of $\gamma_i$, so that $\bar{\gamma}_i|_{[0,1]} = \gamma_i$, $\bar{\gamma}_i|_{[1,1+\epsilon]} \cap R' = \emptyset$, $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$ for $i \neq j$, and $\bar{\gamma}_i$ intersects $R'$ transversely at $\bar{\gamma}_i(1)$.

Let $D_i := \bar{\gamma}_i \times I$ be contained in a small neighborhood of $\bar{\gamma}_i$, where the product direction is chosen so that $\bar{\gamma}_i(0) \times I \subset R'$ and $(\bar{\gamma}_i(0, 1 + \epsilon) \times I) \cap R' = \gamma_i(1)$. Let $\gamma_i' := \partial D_i \setminus (\bar{\gamma}_i(0) \times I)$ and frame $\gamma_i'$ so that $C(0)$ and $C(1)$ are both contained in $R'$. Then let $T_i'$ be the cylinder from $C(0)$ to $C(1)$. We obtain $S_+$ from $R'$ by deleting the interiors of small disks bounded by $C(0)$ and $C(1)$ and then attaching $T_i'$, for $i = 1, \ldots, n$.

For each $\gamma_i$, let $H_i$ be a narrow solid tube $\gamma_i' \times D^2$, where the product direction is taken so that $\partial H_i = (2$ disks in $R') \cup T_i'$. Let $M_1^3 \subset X^4 \times I$ be a cobordism from $R'$ to $S_+$ given by

$$M_1^3 = R' \times [0, 1/2] \cup \bigcup_{i=1}^n H_i \times 1/2 \cup S_+ \times [1/2, 1].$$

We fix the above handle description of $M_1^3$, so that “the 1-handles of $M_1^3$” will always refer to $H_1, \ldots, H_n$.

**Remark 4.2.** Recall that $S_+$ is obtained from $S$ by attaching the tubes $T_1, \ldots, T_n$, where $T_i = \pi_S(\sigma_i) \times S^1$ for an arc $\sigma_i$ in $A$ from $x_j$ to $x_k$ (for some $j \neq k$). We refer to $\pi_S(\sigma_i(1/2)) \times S^1$ as the belt sphere of $T_i$. This belt intersects the belt sphere of the 1-handle $H_i$ in exactly one point, and does not intersect the belt sphere of $H_i$ ($l \neq i$) at all. Then attaching $n$ 3-dimensional 2-handles to $M_1^3$ along annular neighborhoods of the belt spheres of $T_1, \ldots, T_n$ would geometrically cancel the 1-handles of $M_1^3$. 

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**Figure 5.** Left: the immersed surface $S$ is obtained from $R'$ by doing finger moves $f_1, \ldots, f_n$. **Second:** We obtain $S_+$ from $S$ by surgery along tubes $T_1, \ldots, T_n$, where $T_i$ lies in the support of finger move $f_i$. **Third:** We obtain $S_+$ from $R'$ by surgery along tubes $T_i'$, where $T_i'$ lies in the support of finger move $f_i$. **Right:** The embedded surface $R'$. 

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[Image of the figure]
Recall that $w_i$ is a finger move of $R$, and $\eta_i$ is the path along which this finger move takes place. Then $\eta_i$ is an arc in $X^4$ with $\partial \eta_i \subset R$ and $\bar{\eta}_i \cap R = \emptyset$. Take $z = R \cap G$ to be far from $\partial \eta_i$ for each $i$.

Isotope and homotope $S_+ \cup S$ (respectively) near each Whitney disk $W_i$ as in Figure 6 to be contained in a small neighborhood of $R \cup \eta_1 \cup \cdots \cup \eta_n$. (Perform these moves in order of $i$. By dimensionality, $W_j$ is disjoint from tubes $\tilde{T}_k$ in the pictured support of $w_i$ for $i \leq j$. In this support, the tubes $\tilde{T}_k$ that are not parallel to $R$ are parallel to $\eta_i$.) Call the resulting surfaces $\bar{S}_+$ and $\bar{S}$, respectively. The isotopy of $S_+ \cup S$ to $\bar{S}_+ \cup \bar{S}$ takes tube $T_i$ to a tube $\tilde{T}_i$. Now $\bar{S}_+$ is obtained from $\bar{S}$ by attaching tubes $\tilde{T}_1, \ldots, \tilde{T}_n$. The belt sphere of $T_i$ is carried to a curve $B_i$ on $\tilde{T}_i$. We call $B_i$ the belt sphere of $T_i$. Up to reparametrization of $\sigma_i$, $B_i$ bounds a disk perpendicular to $\bar{S}_+$ which is centered at $b_i \subset \bar{S}_+ \cap R$. For $i \neq j$, take $b_i \neq b_j$.

For each $i = 1, \ldots, n$, let $\alpha_i$ be an arc embedded in $R$ from $b_i$ to $z$. Take the arcs $\alpha_i, \alpha_j$ to be disjoint when $i \neq j$, and take $\alpha_i$ to be far from the endpoints of $\eta_1, \ldots, \eta_n$. Also take $\alpha_i \cap b_j = \emptyset$ for all $i$ and $j$.

For $i = 1, \ldots, n$, we now find a disk $\bar{C}_i$ whose boundary is the belt sphere $B_i$ of $\tilde{T}_i$. See Figure 7 for an illustration of $\bar{C}_i$. 

Figure 6. **Top:** the neighborhood of a crossed or uncrossed Whitney disk $W_i$. We draw $S_+ \cap S$ in bold black, with the tubes $T_i$ in thin colored curves. In general, a tube $T_i$ may intersect this neighborhood many times. **Bottom:** We isotope $S_+$ and homotope $S$ in a neighborhood of each $W_i$ to obtain $\bar{S}_+$ and $\bar{S}$, respectively. Now $\bar{S}_+ \cap \bar{S}$ (in bold black) is contained in $R$. 


Figure 7. The disk $C_i$ has boundary the belt sphere $B_i$ of $\bar{T}_i$, then follows the path of $\alpha_i(t)$ before being capped off by a disk in $G$. We push $C_1, \ldots, C_n$ off of $G$ and each other to obtain $\bar{C}_1, \ldots, \bar{C}_n$. Here, we draw a movie of $\bar{C}_i$. At each time slice, we draw a 3-dimensional cross-section of $\nu(\alpha_i)$.

Let $T(\alpha_i)$ be a cylinder around $\alpha_i$, where $\alpha_i$ is framed so that $C(0) = B_i$ and $C(1) \subset G$. Let $\bar{C}_i$ be the disk obtained by capping off $T(\alpha_i)$ with a disk in $G$ which does not contain $z$. Take $T(\alpha_i)$ increasingly narrow so that $\bar{C}_i$ does not intersect itself, $\bar{T}_j$ for any $j$, or $T(\alpha_k)$ for any $k \neq i$. The disks $C_1, \ldots, C_n$ all mutually intersect in a disk in $G$. Since $G$ has trivial normal bundle, we can push the disks $C_1, \ldots, C_n$ slightly off of $G$ in different directions to obtain disjoint disks $\bar{C}_1, \ldots, \bar{C}_n$ where $\partial \bar{C}_i = \partial C_i = B_i$. Note that the interior of $\bar{C}_i$ does not intersect $S_+$.

Let $H'_i = \bar{C}_i \times I$, where the product direction is chosen so that $(\partial \bar{C}_i) \times I \subseteq \bar{T}_i$. Let $R''$ be the sphere obtained from $S_+$ by compressing along each $C_1, \ldots, C_n$, so $\partial (\bar{C}_i \times I) \subset S_+ \cup R''$.

Let $\phi_s : X^4 \to X^4|_{s \in [0,1]}$ be the ambient isotopy of $X^4$ taking $S_+$ to $\bar{S}_+$. Let $M_2^3$ be a cobordism from $S_+$ to $R''$ in $X^4 \times I$ given by

$$M_2^3 = \cup_{s \in [0,1]} (\phi_s(S_+) \times s/2) \cup_{i=1}^n (H'_i \times 1/2) \cup (R'' \times [1/2, 1]).$$

We illustrate $M_2^3$ in Figure 8.

Let $N^3$ be the cobordism from $R'$ to $R''$ in $X^4 \times I$ obtained by concatenating $M_1^3$ and $M_2^3$. In words, $N^3$ is obtained from $M_1^3$ by attaching the 2-handles $H'_1, \ldots, H'_n$. By Remark 4.2, these 2-handles geometrically cancel the 1-handles of $M_1^3$. Therefore, $N^3$ is a concordance from $R'$ to $R''$. 
The sphere $R''$ intersects $G$ in exactly the point $z$. Now we will prove that $R''$ is isotopic to $R$, using the 4-dimensional light bulb theorem.

4.2. Proof that the concordance goes from $R'$ to $R$. We will show that $R''$ is the realization of a tubed surface on $R$.

Recall that $S_+$ is obtained from $S$ by attaching tubes $T_1, \ldots, T_n$, where $T_i$ runs parallel to $\pi_S(\sigma_i)$ for an arc $\sigma_i$ between $x_j$ and $x_k$ (for some $j \neq k$), and that the isotopy $\phi_s$ from $S_+$ to $\tilde{S}_+$ takes $T_i$ to tube $\tilde{T}_i$. Let $B_1, \ldots, B_{2n}$ denote the components of $\tilde{T}_1, \ldots, \tilde{T}_n$ after compressing each $\tilde{T}_i$ along $C_i$. Each $B_i$ is a disk, and $R''$ is obtained from $S$ by deleting disks bounded by $\partial B_i$ and then attaching $\tilde{B}_i$ for each $i = 1, \ldots, 2n$. (The disks deleted from $S$ each meet a sheet of one of the $2n$ self-intersections of $\tilde{S}$; the sphere $R''$ is embedded.)

Let $X_i = \phi_1(\text{support of } w_i)$. See Figure 8 (top three rows) for illustrations of $S$, $S_+$, and $R''$ in $X_i$.

For each $i = 1, \ldots, n$, say $\pi^{-1}_S(W_i) = \partial_i^1 \cup \partial_i^2$ where $\partial_i^1$ and $\partial_i^2$ are arcs in $A$. If $w_i$ is uncrossed, take $\partial_i^2$ to have both endpoints in $L_y$. We perform the following operation to $R''$, illustrated in Figure 9 (bottom). For $i = 1, \ldots, n$:

- If $w_i$ is crossed, then take $R'' \cap X_i$ as in Figure 9 (third row, left).
- If $w_i$ is uncrossed, suppose $\sigma_l$ crosses $\partial_i^2$ for some $l$. (See Fig. 9 second row, rightmost.) Then some segment of $\tilde{B}_r \cap X_i$ runs parallel to $\eta_i$ as in Figure 9 (third row, third picture). For some $m \neq s$, $\tilde{B}_m$ and $\tilde{B}_s$ both have ends in $X_i$. Assume $r \neq s$ (by perhaps allowing $r = m$) and slide this segment of $\tilde{B}_r$ over the disk $\tilde{B}_s$ and out of $X_i$, as in Figure 9 (third row, rightmost). Repeat for each intersection of a $\sigma_l$ curve (for any $l$) with $\partial_i^2$.

Now we see that $R''$ is the realization of a tubed surface on $R$. The tube guide locus curves for $R''$ in $A$ are all of the form $\alpha_i, \beta_i$, and $\gamma_i$. Every $\tilde{B}_j$ lies in a small neighborhood of $R$. Near the boundary of the disk $\tilde{B}_j$, we find one of the two following situations:
• The ends of two $\tilde{B}_j$’s join at a single tube parallel to $\eta_k$ where $w_k$ is uncrossed (recall from Remark 3.13 that $\eta_k$ is a framed path).

• The ends of two $\tilde{B}_j$’s meet opposite ends of a double tube parallel to $\eta_k$ where $w_k$ is crossed.

Thus, the curves of the form $\tau_i$ for $R''$ are exactly $\{\eta_k \mid w_k \text{ uncrossed}\}$ while the curves of the form $\eta_i$ for $R'$ are exactly $\{\eta_k \mid w_k \text{ crossed}\}$.

Assume $L$ is as in the hypothesis of Theorem 1.5. That is, each 2-torsion element of $\pi_1(X^4, z)$ appears an even number of times in the multiset $\{[\eta_k] \mid w_k \text{ crossed}\}$. Then by Proposition 3.16, $R''$ is isotopic to $R$. Thus, $R'$ is concordant to $R$. This completes the proof of Theorem 1.5 (and hence also Theorem 1.3).

5. Concordance of surfaces of positive genus

When $R$ and $R'$ are positive-genus surfaces rather than spheres, Gabai [G] proves the following extension of the light bulb theorem.

**Theorem 5.1 (G, Thm. 9.7).** Let $X^4$ be an orientable 4-manifold so that $\pi_1(X^4)$ has no 2-torsion. Let $R$ and $R' \subset X^4$ be orientable genus-$g$ surfaces embedded in $X^4$ so that $R$ and $R'$ have a mutual transverse sphere and $R'$ is homotopic to $R$. Moreover, assume the maps $\pi_1(R \setminus G) \to \pi_1(X^4 \setminus G)$ and $\pi_1(R' \setminus G) \to \pi_1(X^4 \setminus G)$ induced by inclusion are both trivial.

Then $R$ and $R'$ are isotopic.

Note that when $g = 0$, Theorem 5.1 specializes to Theorem 3.1. The analogous extension of Theorem 1.3 is thus as follows.

**Theorem 5.2.** Let $X^4$ be an orientable 4-manifold so that $\pi_1(X^4)$ has no 2-torsion. Let $R$ and $R' \subset X^4$ be orientable genus-$g$ surfaces embedded in $X^4$ so that $R$ has a mutual transverse sphere and $R'$ is homotopic to $R$. Moreover, assume the map $\pi_1(R \setminus G) \to \pi_1(X^4 \setminus G)$ induced by inclusion is trivial.

Then $R$ and $R'$ are concordant.

**Proof.** By Theorem 3.3, $R$ and $R'$ are regularly homotopic. We repeat the argument of Theorem 1.3 to construct an embedded surface $R''$ which is the realization of a tubed surface on $R$ so that $R''$ is concordant to $R'$ and $R \cap G = \text{pt}$. (In exactly the same fashion as in Theorem 1.3 we build a concordance from $R$ to $R''$ by attaching a 1-handle to $R \times I$ for each finger move in the regular homotopy, and then attach cancelling 2-handles using the transverse sphere $G$.)

Note that $R''$ is built from $R'$ by surgery along immersed 3-balls (1- and 2-handle pairs) which meet $R'$ in a disk and can thus be homotoped to be trivial (see Figure 10 for an illustration). Therefore, $R''$ is homotopic to $R'$ and hence $R$. Moreover, every loop in $R''$ can be isotoped off the tubes attached to $R$ and into $R$ itself, so the map $\pi_1(R'' \setminus G) \to \pi_1(X^4 \setminus G)$ induced by inclusion is trivial. Then by Theorem 5.1 $R''$ is isotopic to $R$, so $R'$ is concordant to $R$. □
Figure 9. **Top row:** The Whitney disk $W_i$ with boundary in $S$. **Second row:** we attach tubes $T_1, \ldots, T_n$ to $S$ to obtain $S_+$. In the leftmost picture, the Whitney move $w_i$ is crossed. In the middle and left pictures, $w_i$ is uncrossed. In the rightmost picture, some $\sigma_l$ intersects $\partial_2$. (Here, the green tube is parallel to $\pi_S(\sigma_l)$). **Third row:** We isotope $S_+$ to $\tilde{S}_+$ and then compress along disks $C_i$ (not visible in this diagram) to obtain $R''$. In the third picture, some $\tilde{B}_r$ corresponds to the previously pictured segment of $T_l$. There are two $\tilde{B}_m, \tilde{B}_s$ with ends pictured, with $m \neq s$. So without loss of generality, take $r \neq s$ and slide $\tilde{B}_r$ over $\tilde{B}_s$. **Bottom row:** We find that $R''$ is the realization of a tubed surface on $R$. We give schematics for the tube guide locus arcs contained in $A$. 

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Figure 10. Left to right: Part of a homotopy from $R''$ to $R$. Here we draw $R'$ and a schematic of a 1-handle and the (collar of a) core of the 2-handle which geometrically cancels it, projected to one $X^4 \times t$. The cancelling 2-handle and other 2-handle may intersect the 1-handle. The 1-handle and cancelling 2-handle together form an immersed 3-ball, which we shrink over time during the homotopy. To obtain $R'$, we repeat for each 1-handle of $M^3_0$.

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