Smaller Extended Formulations for the Spanning Tree Polytope of Bounded-Genus Graphs

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Abstract We give an $O\left(\frac{g^{1/2}n^{3/2} + g^{3/2}n^{1/2}}{2}\right)$-size extended formulation for the spanning tree polytope of an $n$-vertex graph embedded in a surface of genus $g$, improving on the known $O(n^2 + gn)$-size extended formulations following from Wong (Proceedings of 1980 IEEE International Conference on Circuits and Computers, pp 149–152, 1980) and Martin (Oper Res Lett 10:119–128, 1991).

Keywords Spanning tree polytope · Extended formulation · Genus

1 Introduction

An extended formulation of a (convex) polytope $P \subseteq \mathbb{R}^d$ is a linear system $Ax + By \leq b$, $Cx + Dy = c$ in variables $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$ that provides a description of $P$ in the sense that...
\[ P = \{ x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^k : Ax + By \leq b, \ Cx + Dy = c \}. \]

The size of an extended formulation is defined as its number of inequalities. The extension complexity \( xc(P) \) is the minimum size of an extended formulation of \( P \). Notice that equalities are not accounted for in the size of an extended formulation. In fact, we may equivalently define the extension complexity of \( P \) as the minimum number of facets of a polytope that affinely projects to \( P \).

Let \( G = (V, E) \) be a connected (simple, finite, undirected) graph. The spanning tree polytope of \( G \) is the convex hull of the 0/1-vectors in \( \mathbb{R}^E \) that are the characteristic vector of some spanning tree of \( G \). We denote this polytope as \( \text{P}_{\text{sp.\,trees}}(G) \), and use the notation

\[
\text{P}_{\text{sp.\,trees}}(G) = \text{conv}\{ \chi^T \in \{0, 1\}^E \mid T \subseteq E, \ T \text{ spanning tree of } G \}. 
\]

The following result gives the best known upper bound on the extension complexity of the spanning tree polytope for general graphs, and is due to Wong [10] and Martin [7].

**Theorem 1** For every connected graph \( G = (V, E) \), \( xc(\text{P}_{\text{sp.\,trees}}(G)) = O(|V| \cdot |E|) \).

For planar graphs, a linear bound was proved by Williams [9].

**Theorem 2** For every connected planar graph \( G = (V, E) \), \( xc(\text{P}_{\text{sp.\,trees}}(G)) = O(|V|) \).

Let \( S \) be a surface. By the classification theorem for surfaces, \( S \) is homeomorphic to a sphere with \( g \) handles, or a sphere with \( g \) crosscaps, for some \( g \). We call \( g \) the genus of \( S \). Our main result is an improvement of Theorem 1 for graphs embedded in a surface of genus \( g \).

**Theorem 3** For every connected graph \( G = (V, E) \) embedded in a surface of genus \( g \), \( xc(\text{P}_{\text{sp.\,trees}}(G)) = O(g^{1/2} |V|^{3/2} + g^{3/2} |V|^{1/2}) \). In particular, \( xc(\text{P}_{\text{sp.\,trees}}(G)) = O(|V|^{3/2}) \) if \( g \) is fixed.

This gives an improvement over Theorem 1 for all fixed \( g \). For instance, for toroidal graphs we obtain a \( O(|V|^{3/2}) \)-size extended formulation, while the previously known extended formulations are of size \( \Omega(|V|^2) \).

For other polytopes, smaller extended formulations have also been obtained when restricting to graphs of bounded genus. For example, Gerards [4] proved that the perfect matching polytope has a polynomial-size extended formulation for graphs embedded in a fixed genus surface. This is in stark contrast to the situation for general graphs: Rothvoß [8] showed that the perfect matching polytopes of complete graphs have exponential extension complexity.

Going back to the spanning tree polytope, we conjecture that the bound in Theorem 3 can be improved to match the corresponding bound for planar graphs.

**Conjecture 1** If \( G = (V, E) \) is a connected graph embedded in a fixed surface, then \( xc(\text{P}_{\text{sp.\,trees}}(G)) = O(|V|) \).
Indeed, the same bound may even hold more generally for proper minor-closed families of graphs.

**Conjecture 2** If $\mathcal{C}$ is a proper minor-closed family of graphs and $G = (V, E)$ is a connected graph in $\mathcal{C}$, then $\text{xc}(P_{\text{sp.trees}}(G)) = O(|V|)$.

We remark that this conjecture is known to hold if the graphs in $\mathcal{C}$ have bounded treewidth [6]. To provide some additional support for the conjecture, we observe that it is also true when the graphs in $\mathcal{C}$ are $k$-apex for some fixed $k$. Recall that a graph $G = (V, E)$ is $k$-apex if there is a set $X \subseteq V$ with $|X| \leq k$ such that $G - X$ is planar. It is easily checked that the set of $k$-apex graphs is a proper minor-closed family of graphs.

**Theorem 4** Let $G = (V, E)$ be a connected $k$-apex graph. Then $\text{xc}(P_{\text{sp.trees}}(G)) = O(k \cdot |E|) = O(k^2 \cdot |V|)$.

### 2 The Proofs

In this section we prove Theorems 3 and 4. We first gather the necessary ingredients.

As before, let $G = (V, E)$ be a connected graph. The *subgraph polytope* of $G$ is defined as $P_{\text{sub}}(G) = \text{conv}\{(\chi^S, \chi^F) \in \{0, 1\}^V \times \{0, 1\}^E \mid S \subseteq V, \ F \subseteq E(S)\}$, where $E(S)$ denotes the set of edges of $G$ with both endpoints in $S$. It is easy to verify using total unimodularity that

$$ P_{\text{sub}}(G) = \{(x, y) \in \mathbb{R}^V \times \mathbb{R}^E \mid \forall v, w \in V \text{ with } vw \in E : 0 \leq y_{vw} \leq x_v \leq 1\}. $$

Hence, the subgraph polytope has at most $3|E| + |V|$ facets, and in particular $\text{xc}(P_{\text{sub}}(G)) = O(|E|)$.

We will mostly be interested in the variant of the subgraph polytope known as the *non-empty subgraph polytope*, defined as

$$ P_{\text{sub}}^*(G) = \text{conv}\{(\chi^S, \chi^F) \in \{0, 1\}^V \times \{0, 1\}^E \mid \emptyset \subseteq S \subseteq V, \ F \subseteq E(S)\}. $$

Notice that $P_{\text{sub}}^*(G)$ is nothing else than the convex hull of the vertices of $P_{\text{sub}}(G)$ distinct from the origin $(0^V, 0^E)$.

The non-empty subgraph polytope turns out to be tightly connected to the spanning tree polytope: Conforti, Kaibel, Walter and Weltge [2] proved that the extension complexities of the two polytopes are essentially equal.

**Theorem 5** For every connected graph $G = (V, E)$, $\text{xc}(P_{\text{sp.trees}}(G)) = \text{xc}(P_{\text{sub}}^*(G)) + \Theta(|E|)$.

In particular, it follows from this and Theorem 2 that $\text{xc}(P_{\text{sub}}^*(G)) = O(|V|)$ for every connected planar graph $G = (V, E)$.

Balas’ union of polytopes [1] is a basic tool to construct extended formulations. It provides an upper bound on the extension complexity of the convex hull of a union of polytopes.
Theorem 6  Let $P_1, \ldots, P_k$ be non-empty polytopes in $\mathbb{R}^d$, and let $P = \text{conv}(\bigcup_{i=1}^k P_i)$. Then $xc(P) \leq \sum_{i=1}^k \max\{1, xc(P_i)\}$.

The next observation follows easily from Balas’ union of polytopes.

Lemma 7  Let $G = (V, E)$ be a connected graph and $X \subseteq V$ be a set of vertices. Then

$$P^\star_{\text{sub}}(G) = \text{conv} \left( P^\star_{\text{sub}}(G - X) \cup \bigcup_{v \in X} \{ P_{\text{sub}}(G) \cap \{(x, y) \in \mathbb{R}^V \times \mathbb{R}^E \mid x_v = 1\} \} \right),$$

thus $xc(P^\star_{\text{sub}}(G)) \leq xc(P^\star_{\text{sub}}(G - X)) + O(|X| \cdot |E|)$.

Proof  We may assume that $X$ is a proper, non-empty subset of $V$, since otherwise the result holds. From Theorem 6,

$$xc(P^\star_{\text{sub}}(G)) \leq xc(P^\star_{\text{sub}}(G - X)) + \sum_{v \in X} xc(P_{\text{sub}}(G)) = xc(P^\star_{\text{sub}}(G - X)) + O(|X| \cdot |E|).$$

(Remark: If $|X| = |V| - 1$ then $P^\star_{\text{sub}}(G - X)$ is empty and is thus not part of the list of polytopes we apply Theorem 6 on, as expected.)

We are now ready to prove Theorem 4.

Proof of Theorem 4  Let $G = (V, E)$ be a connected $k$-apex graph, and let $X \subseteq V$ be any set of at most $k \geq 1$ vertices whose deletion from $G$ gives a planar graph. By Theorem 5, Lemma 7 and Theorem 2,

$$xc(P_{\text{sp.trees}}(G)) \leq xc(P^\star_{\text{sub}}(G)) + O(|E|) \leq xc(P^\star_{\text{sub}}(G - X)) + O(|X| \cdot |E|) = O(k \cdot |E|).$$

Notice that $|E| \leq k \cdot (|V| - 1) + 3(|V| - k) - 6 = O(k \cdot |V|)$, thus $O(k \cdot |E|) = O(k^2 \cdot |V|)$.

For Theorem 3, we need one additional result of Djidjev and Venkatesan [3]. The same result for orientable surfaces was obtained earlier by Hutchinson and Miller [5].

Theorem 8  For every graph $G = (V, E)$ embedded in a surface of genus $g$, there exists a set $X$ of $O(\sqrt{g|V|})$ vertices such that $G - X$ is planar.

Finally, we prove our main result, Theorem 3.
Proof of Theorem 3 Let \( G = (V, E) \) be a connected graph embedded in a surface of genus \( g \). The result follows by combining Theorem 5, Lemma 7, Theorem 2, Theorem 8, and the upper bound \( |E| = O(|V| + g) \) (by Euler’s formula).

More explicitly, letting \( X \subseteq V \) be as in Theorem 8,

\[
xc(P_{\text{sp.trees}}(G)) \leq xc(P_{\text{sub}}^*(G)) + O(|E|) \\
\leq xc(P_{\text{sub}}^*(G - X)) + O(|X| \cdot |E|) \\
= O(|V|) + O(g^{1/2}|V|^{1/2} \cdot (|V| + g)) \\
= O(g^{1/2}|V|^{3/2} + g^{3/2}|V|^{1/2}).
\]

\( \Box \)

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