GEOGRAPHY OF SYMPLECTIC 4- AND 6-MANIFOLDS.

RAFAEL TORRES AND JONATHAN YAZINSKI

The geography of minimal symplectic 4-manifolds with arbitrary fundamental group and symplectic 6-manifolds with abelian fundamental group of small rank, and with arbitrary fundamental group are addressed.

1. Introduction

The geography problem for simply connected irreducible symplectic 4-manifolds with negative signature has been settled in its entirety \[22, 18, 40, 41, 1, 17, 8, 9, 3, 4\]. Much less is known about the nonsimply connected realm. It has been observed that the new construction techniques \[22, 36, 5\] prove to be rather effective when one wishes to study the symplectic geography problem for a variety of choices of fundamental groups \[22, 9, 3, 4, 49, 50\].

Concerning symplectic 4-manifolds with prescribed fundamental group, we obtained the following theorem.

**Theorem 1.** Let \(G\) be a group with a presentation consisting of \(g\) generators and \(r\) relations. Let \(e, \sigma\) be integers that satisfy \(2e + 3\sigma \geq 0, e + \sigma \equiv 0 \mod 4, e + \sigma \geq 8,\) and assume \(\sigma \leq -1\). Then there exists a minimal symplectic 4-manifold \(M(G)\) with fundamental group \(\pi_1(M(G)) \cong G\), characteristic numbers

\[(c_1^2(M(G)), \chi_h(M(G))) = (2e + 3\sigma + 4(g + r), \frac{1}{4}(e + \sigma) + (g + r)),\]

and with odd intersection form over \(\mathbb{Z}\). If \(G\) is finite, then \(M(G)\) admits exotic smooth structures.

Assume \(s \geq 1, n \in \mathbb{N}\). For each of the following pairs of integers

\[(c, \chi) = (8n - 8 + 8(g + r), 2s + n - 1 + (g + r)),\]

there exists a spin symplectic 4-manifold \(X(G)\) with \(\pi_1(X(G)) \cong G\), and characteristic numbers

\[(c_1^2(X(G)), \chi_h(X(G))) = (c, \chi).\]

Moreover, there is an infinite set \(\{X_m(G) : m \in \mathbb{N}\}\) of pairwise nondiffeomorphic symplectic and nonsymplectic manifolds homeomorphic to \(X(G)\).

If \(G\) is a residually finite group, the work of M. J. D. Hamilton and D. Kotschick \[20\] implies that the manifolds constructed are irreducible. Theorem 1 extends earlier results \[22, 7, 42, 9\]; we give a brief account of them in order to make clear

*Date: December 15, 2010. Revised: July 10, 2011.
2010 Mathematics Subject Classification. Primary 57R17; Secondary 57R55, 57M50.
Key words and phrases. Symplectic geography, symplectic sums, symplectic structures.
À toi: tu me manques encore, repose en paix.*
the contribution of the previous theorem. R. E. Gompf proved the well-known result that any finitely presented group can be the fundamental group of a symplectic 2n-manifold for \( n \geq 2 \) [22, Section 6]. S. Baldridge and P. Kirk took on the endeavor of minimizing the Euler characteristic with respect to the generators and relations of a presentation of the fundamental group in [7], using the elliptic surface \( E(1) \) and the manifold described in Section 2.2 as building blocks. They constructed a minimal symplectic manifold with Euler characteristic \( 12 + 12(g + r) \), and signature \(-8 - 8(g + r)\). J. Park systematically studied the geography in [42, Theorem 1] using the construction in [7] as a building block. Moreover, his result also addresses the existence of a myriad of exotic smooth structures by establishing a homeomorphism on the manifolds he constructed, using the results of S. Boyer [11] on homeomorphisms of simply connected 4-manifolds with a given boundary.

Using a minimal symplectic manifold homeomorphic to \( 3\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2 \) as a building block instead of \( E(1) \), Baldridge and Kirk constructed a smaller minimal symplectic example with Euler characteristic \( 10 + 6(g + r) \), and signature \(-2 - 2(g + r)\) in [9, Theorem 24] \((c_2^1, \chi_h) = (14 + 6(g + r), 2 + (g + r))\) under the coordinates used in the statement of Theorem 1. In [52], the second author of the present paper improved the known bounds by producing a minimal symplectic example with Euler characteristic \( 10 + 4(g + r) \), and signature \(-2 \) \((c_2^1, \chi_h) = (14 + 8(g + r), 2 + (g + r))\). His construction uses as a building block the symplectic manifold sharing the homology of \( S^2 \times S^2 \) built by R. Fintushel, B. D. Park, and R. J. Stern in [17, Section 4]; see Section 2.2. Theorem 1 offers an improvement and extends the results in [7, 42, 9, 52].

For known fundamental-group-dependent bounds on the geographical regions that can be populated by minimal symplectic manifolds with prescribed fundamental group and comparisons to fundamental-group-dependent results on the existence of smooth manifolds in dimension four, the reader is directed to the work of C. H. Taubes [47], of S. Baldridge and P. Kirk [7], of D. Kotschick [31], and of P. Kirk and C. Livingston [29, 30].

We now direct our attention to high dimensional manifolds, where existence results for symplectic 6-manifolds is somewhat uncharted territory. The geography of simply connected symplectic 6-manifolds was settled by M. Halic in [24] (see also [28, 39]), where he proved that any triple \((a, b, c) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\) satisfying the arithmetic conditions \(a \equiv c \equiv 0 \mod 2\), and \(b \equiv 0 \mod 24\) can be realized by a simply connected symplectic 6-manifold with Chern numbers \((c_3^1, c_1c_2, c_3) = (a, b, c)\). Although there is no established definition of minimality in dimension six yet (see [35, Definition 5.2]), the existence result of Halic relies heavily on blow ups along points and along surfaces of well chosen examples. In particular, the manifolds constructed in [24] are not minimal ([24, Lemma 4.1].) This scenario motivates interest for constructions of examples that do not involve blow ups. In this direction, we obtained the following two results.

First, by building greatly upon the aforementioned work on the geography of simply connected 4-manifolds, we have the following theorem.
Theorem 2. Let $G$ be a group among the choices $\{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}_p \oplus \mathbb{Z}, \mathbb{Z}, \pi_1(\Sigma_g), F_n\}$, where $\Sigma_g$ is a closed oriented surface of genus $g$, and $F_n$ is the free group of rank $n$. Let $e_i, \sigma_i$ $i = 1, 2$ be integers that satisfy $2e_i + 3\sigma_i \geq 0$, $e_i + \sigma_i \equiv 0 \bmod 4$, and $e_i + \sigma_i \geq 8$. There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$ with fundamental group $\pi_1(W_i(G)) \cong G$ for $i = 0, 1, 2$, and the following Chern numbers:

1. $c_1^3(W_0(G)) = 18 \cdot (\sigma_1 + \sigma_2) + 12 \cdot (e_1 + e_2)$
$c_1c_2(W_0(G)) = 6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2)$, and
$c_3(W_0(G)) = 2 \cdot (e_1 + e_2)$;

2. $c_1^3(W_1(G)) = 0$,
$c_1c_2(W_1(G)) = 0$, and
$c_3(W_1(G)) = 0$;

3. $c_1^3(W_2(G)) = -18 \cdot (\sigma_1 + \sigma_2) - 12 \cdot (e_1 + e_2) - 48$,
$c_1c_2(W_2(G)) = -6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) - 24$, and
$c_3(W_2(G)) = -2 \cdot (e_1 + e_2) - 8$.

Assume $s_1, s_2 \geq 1$, and $n_1, n_2 \in \mathbb{N}$. There exist spin symplectic 6-manifolds $Y_i(G)$ for $i = 0, 1, 2$ with fundamental group $\pi_1(Y_i(G)) \cong G$, and whose Chern numbers are

1. $c_1^3(Y_0(G)) = 48 \cdot (n_1 + n_2 - 2)$,
$c_1c_2(Y_0(G)) = 48 \cdot (s_1 + s_2) + 24 \cdot (n_1 + n_2 - 2)$, and
$c_3(Y_0(G)) = 48 \cdot (s_1 + s_2) + 8 \cdot (n_1 + n_2 - 2)$;

2. $c_1^3(Y_1(G)) = 0$,
$c_1c_2(Y_1(G)) = 0$, and
$c_3(Y_1(G)) = 0$;

3. $c_1^3(Y_2(G)) = -48 \cdot (n_1 + n_2 - 1)$,
$c_1c_2(Y_2(G)) = -48 \cdot (s_1 + s_2) - 24 \cdot (n_1 + n_2 - 1)$, and
$c_3(Y_2(G)) = -48 \cdot (s_1 + s_2) - 8(n_1 + n_2 - 1)$.

Next, the current understanding on the geography of minimal symplectic 4-manifolds with arbitrary fundamental group ([22], Theorem 1) is used to study the existence of symplectic 6-manifolds with prescribed fundamental group. Such enterprise had been pursued previously by R. E. Gompf in his seminal paper [22, Theorem 7.1]. The result obtained in this direction is the following theorem.

Theorem 3. Let $G$ be a group with a presentation that consists of $g$ generators and $r$ relations. Let $e_i, \sigma_i$ $i = 1, 2$ be integers that satisfy $2e_i + 3\sigma_i \geq 0$, $e_i + \sigma_i \equiv 0 \bmod 4$, and $e_i + \sigma_i \geq 8$. There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$ with fundamental group $\pi_1(W_i(G)) \cong G$ for $i = 0, 1, 2$, and the following Chern numbers:

1. $c_1^3(W_0(G)) = 18 \cdot (\sigma_1 + \sigma_2) + 12 \cdot (e_1 + e_2) + 48 \cdot (g + r)$,
$c_1c_2(W_0(G)) = 6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) + 24 \cdot (g + r)$, and
$c_3(W_0(G)) = 2 \cdot (e_1 + e_2) + 8 \cdot (g + r)$;
(2) \( c_1^3(W_1(G)) = 0, \)
\( c_1c_2(W_1(G)) = 0, \) and
\( c_3(W_1(G)) = 0; \)

(3) \( c_1^3(W_2(G)) = -18 \cdot (\sigma_1 + \sigma_2) - 12 \cdot (e_1 + e_2) - 48 \cdot (g + r) - 48, \)
\( c_1c_2(W_2(G)) = -6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) - 24 \cdot (g + r) - 24, \) and
\( c_3(W_2(G)) = -2 \cdot (e_1 + e_2) - 8 \cdot (g + r); \)

Assume \( s_1, s_2 \geq 1, \) and \( n_1, n_2 \in \mathbb{N}. \) There exist spin symplectic 6-manifolds \( Y_i(G) \) for \( i = 0, 1, 2 \) with fundamental group \( \pi_1(Y_i(G)) \cong G, \) and whose Chern numbers are

(1) \( c_1^3(Y_0(G)) = 48 \cdot (n_1 + n_2 - 2) + 48(g + r), \)
\( c_1c_2(Y_0(G)) = 24 \cdot (n_1 + n_2 - 2) + 24 \cdot (g + r), \) and
\( c_3(Y_0(G)) = 8 \cdot (n_1 + n_2 - 2) + 8 \cdot (g + r); \)

(2) \( c_1^3(Y_1(G)) = 0, \)
\( c_1c_2(Y_1(G)) = 0, \) and
\( c_3(Y_1(G)) = 0; \)

(3) \( c_1^3(Y_2(G)) = -48 \cdot (n_1 + n_2 - 2) - 48 \cdot (g + r) - 48, \)
\( c_1c_2(Y_2(G)) = -24 \cdot (n_1 + n_2 - 2) - 24 \cdot (s_1 + s_2) - 24 \cdot (g + r) - 24, \) and
\( c_3(Y_2(G)) = -8 \cdot (s_1 + s_2) - 8 \cdot (n_1 + n_2 - 2) - 8 \cdot (g + r) - 8. \)

A particular class of manifolds whose interest has sparkled the introduction of new techniques of symplectic constructions, are symplectic manifolds with canonical class \( c_1 = 0 \) that are known as Calabi-Yau manifolds. The special attention that these manifolds have received in recent years is of relevance for geographical studies. In dimension four, there is a fairly broad knowledge on the existence of these objects [33, Theorem 1.2], [33, Section 4]. The earlier known 6-dimensional examples were Kähler manifolds constructed from algebraic geometry, like the quintic hypersurface inside the complex projective 4-space \( \mathbb{CP}^4. \) Symplectic nilmanifolds appeared as the first examples of non-Kähler symplectic Calabi-Yau manifolds (see [48], [12]); these manifolds are quotients of nilpotent Lie groups with left-invariant symplectic forms.

In the paper [44], I. Smith, R. Thomas, and S. T. Yau produced several symplectic 6-manifolds with vanishing canonical class, which are potentially non-Kähler. The first example of a simply connected symplectic, yet non-Kähler Calabi-Yau 6-manifold was constructed by J. Fine and D. Panov in [15] (see [13] as well). Fine and Panov employ hyperbolic geometry to construct their examples. A new surgical procedure called coisotropic Luttinger surgery was introduced by S. Baldridge and P. Kirk to construct such manifolds in [10], where they also constructed several non-simply connected examples.

Several manifolds of Theorem 2 and Theorem 3 (Item (2)) satisfy \( c_1 = 0; \) this case is studied by A. Akhmedov in [2]. In particular, the first examples of manifolds \( Y_1((1)), Y_1(\mathbb{Z}), \) in item (2) of Theorem 3 with \( c_1 = 0 \) appeared in [2], Theorem 1, Theorem 2]. These manifolds have large third Betti number. In [16], Fine and
Panov constructed Calabi Yau manifolds with arbitrary fundamental group by resolving orbifolds. In particular, they constructed examples of such manifolds with $b_3 = 0$ \[16\], Theorem 3].

Finally, we recover and extend the main result in \[24\] to the geography of symplectic 6-manifolds for various choices of fundamental groups. The case of arbitrary fundamental group was studied previously by R. E. Gompf in \[22\], Section 7], and by F. Pasquotto in \[43\], Proposition 4.31]. The statement is the following.

**Corollary 4.** Let $X(G)$ be a symplectic 6-manifold. Then,

$$c_1^2(X(G)) \equiv c_3(X(G)) \equiv 0 \mod 2,$$

$$c_1c_2(X(G)) \equiv 0 \mod 24,$$

and $\pi_1(X(G)) \cong G$.

Conversely, any triple $(a,b,c)$ of integers satisfying $a \equiv c \equiv 0 \mod 2$ and $b \equiv 0 \mod 24$ occurs as a triple $(c_1^2,c_1c_2,c_3)$ of Chern numbers of a symplectic 6-manifold $X(G)$ with $\pi_1(X(G)) \cong G$, where $G$ is a finitely presented group.

Unlike the manifolds of Theorem 2 and Theorem 3, the construction of the manifolds of Corollary 4 involve blow ups along points and along surfaces. The fundamental tools of construction used to obtain all these results are R. E. Gompf’s symplectic sum \[22\] and Luttinger surgery \[36\].

The paper is organized as follows. The second section contains a description of our 4-dimensional building blocks. In Section 2.1, we present the building blocks that allows us to produce manifolds for several choices of fundamental groups. The building block used to construct manifolds with arbitrary fundamental group is described in Section 2.2. A useful explanations on the fundamental group computations of Theorem 1 is given in Section 2.3. The fundamental group computations are located in Section 3. Section 2.4 and Section 2.5 contain the building blocks we use to populate geographic regions. The geography of symplectic 4-manifolds with prescribed fundamental group is studied in Section 3. The proof of Theorem 1 is given in Section 3.1 and Section 3.2. We then use these results to study 6-manifolds in Section 4. The formulae needed to compute Chern numbers are given in Section 4.1. In Section 4.2, we build symplectic sums that are fundamental for our purposes. The tool that allow us to proof Corollary 4 is presented in Section 4.3. Where we work out an example to demonstrate the utility of these techniques for several fundamental groups. Section 5.1 provides a detailed description on our choices of gluing maps used in the 6-dimensional symplectic sums. The fundamental group computations involved in the statements of several of our results, including Theorem 2 and Theorem 3 are presented in Section 5.2. Finally, Section 5.3 contains the proofs of Theorem 4 and Corollary 4.

2. Building blocks

2.1. Symplectic 4-manifolds used to obtain several of our choices for $\pi_1$.

The following building block in our symplectic sums will allow us to vary our choices of fundamental groups without contributing anything to the characteristic numbers. Take the product of a torus and a genus two surface $T^2 \times \Sigma_2$. The characteristic
numbers are given by \( c_2(T^2 \times \Sigma_2) = 0 = \sigma(T^2 \times \Sigma_2) \). Endow \( T^2 \times \Sigma_2 \) with the symplectic product form. The torus \( T^2 \times \{x\} \) and the genus two surface \( \{x\} \times \Sigma_2 \) are symplectic submanifolds of \( T^2 \times \Sigma_2 \), which are geometrically dual to each other. Let \( F \) denote a parallel copy of the surface of genus two. Furthermore, \( T^2 \times \Sigma_2 \) contains 4 pairs of geometrically dual Lagrangian tori available to perform Luttinger surgery [30, 5] on them; see also [27, Section 2]. These tori are of the form \( \Sigma \times \Sigma \) and \( \Sigma \times \Sigma \times \Sigma \times \Sigma \).

Regarding the very relevant fundamental group computations, we have the following result. Let \( \{x, y\} \) be the generators of \( \pi_1(T^2) \), and let \( \{a_1, b_1, a_2, b_2\} \) be the standard set of generators of \( \pi_1(\Sigma_2) \).

**Proposition 5.** (Baldridge - Kirk [3] Proposition 7) The fundamental group 
\[
\pi_1(T^2 \times \Sigma_2 - (T_1 \cup T_2 \cup T_3 \cup T_4 \cup F))
\]
is generated by the loops \( x, y, a_1, b_1, a_2, b_2 \). Moreover, with respect to certain paths to the boundary of the tubular neighborhoods of the \( T_i \) and \( F \), the meridians and two Lagrangian push offs of the surfaces are given by

- \( T_1 : m_1 = x, l_1 = a_1, \mu_1 = [b^{-1}, y^{-1}] \),
- \( T_2 : m_2 = y, l_2 = b_1 a_1 b^{-1}, \mu_2 = [x^{-1}, b_1] \),
- \( T_3 : m_3 = x, l_3 = a_2, \mu_3 = [b_2^{-1}, y^{-1}] \),
- \( T_4 : m_4 = y, l_4 = b_2 a_2 b_2^{-1}, \mu_4 = [x^{-1}, b_2] \),
- \( \mu_F = [x, y] \).

The loops \( a_1, b_1, a_2, b_2 \) lie on the genus 2 surface and form a standard set of generators; the relation \([a_1, b_1][a_2, b_2] = 1 \) holds.

The reader is referred to the lovely paper [9] for details on constructions of symplectic 4-manifolds using Luttinger surgery. We finish this section by pointing out that the description above yields the building blocks needed to produce the manifolds of Theorem 1, Theorem 2, and Corollary 4 with surface fundamental groups \( \pi_1(\Sigma_g) \). Take the product symplectic manifold \( T^2 \times \Sigma_g \) of the 2-torus with a surface of genus \( g \geq 3 \). Endowed with the product symplectic form, the submanifold \( T^2 \times \{x\} \subset T^2 \times \Sigma_g \) is symplectic. Given that \( \{x\} \times \Sigma_g \) is geometrically dual to it, it is straightforward to see that the meridian \( \mu_{T^2} \) in \( T^2 \times \Sigma_g - T^2 \times \{x\} \) can be expressed in terms of the generators and relations of \( \pi_1(\{x\} \times \Sigma_g) \) see [8, 9] for details.

### 2.2. 4-dimensional building block with prescribed fundamental group.

**Theorem 6.** Let \( G \) be a group with a presentation with \( g \) generators and \( r \) relations. There exists a minimal symplectic 4-manifold \( X \) with fundamental group \( \pi_1(X(G)) \cong G \), and characteristic numbers

\[
c_1^2(X(G)) = 14 + 8(g + r), c_2(X(G)) = 10 + 4(g + r), \quad \text{and} \quad \sigma(X(G)) = -2
\]

\[
\chi_b(X(G)) = 2 + (g + r).
\]

The manifold \( X(G) \) contains a homologically essential Lagrangian torus \( T \), and a symplectic surface of genus two \( F \) with trivial normal bundle, and

\[
\pi_1(X(G) - T) \cong \pi_1(X(G) - F) \cong \pi_1(X(G)) \cong G.
\]

Let us describe the materials used to prove this result. The fundamental building block for these constructions is a symplectic manifold built in [7], which allows the
S. Balridge and P. Kirk take a 3-manifold $Y$ that fibers over the circle, and build the 4-manifold $N := Y \times S^1$. It admits a symplectic structure (p. 856 in [7]). Its Euler characteristic and its signature are both zero.

The fundamental group $\pi_1(Y \times S^1)$ has classes $s, t, \gamma_1, \ldots, \gamma_{g+r}$ so that

$$G \cong \pi_1(Y \times S^1)/N(s, t, \gamma_1, \ldots, \gamma_{g+r}),$$

where $N(s, t, \gamma_1, \ldots, \gamma_{g+r})$ is the normal subgroup generated by the aforementioned classes.

A result of W. Thurston [48] implies that the manifold $N$ is symplectic. There are $g+r+1$ symplectically imbedded tori $T_0, T_1, \ldots, T_{g+r} \subset N$, which have the following crucial traits for our purposes concerning fundamental group computations.

- The generators of $\pi_1(T_0)$ represent $s$ and $t$, and
- the generators of $\pi_1(T_i)$ represent $s$ and $\gamma_i$ for $i \geq 1$.

The curve $s$ has the form $\{y\} \times S^1 \subset Y \times S^1$, with $y \in Y$, and $\gamma_i$ has the form $\gamma_i \times \{x\} \subset Y \times \{x\}, x \in S^1$. We point out that the role of the curves $\gamma_i$ is to provide the $r$ relations in a presentation of $G$ ([7, Section 4]).

The details of the construction depend on the presentation of our group; we give a description of the construction with the purpose of making the paper a bit more self-contained. Let $G$ be a finitely presented group: $G = \langle x_1, \ldots, x_g | w_1, \ldots, w_r \rangle$ with $g$ generators $x_1, \ldots, x_g$ and $r$ relations $w_1, \ldots, w_r$. Let $n_i$ denote the length of the relation $w_i$, and set

$$n = 1 + \sum_{i=1}^r n_i.$$

Let $\Sigma_{gn}$ denote a surface of genus $gn$. We will define an automorphism $R$ on $\Sigma_{gn}$, and so we consider the following description of $\Sigma_{gn}$ to aid with specifying $R$. Consider a round 2-sphere $S^2 \subset \mathbb{R}^3$ centered at the origin, and let $\bar{R}$ denote a rotation of $S^2$ by an angle of $\frac{2\pi}{ng}$ through the axis passing through the points $(0,0,\pm 1)$. The orbit of $(1,0,0)$ under $\langle \bar{R} \rangle \subset SO(3)$ gives $ng$ distinct points on $S^2$, and performing a connected sum of a torus with $S^2$ at each point in $\langle \bar{R} \rangle \{ (1,0,0) \}$ in an equivariant way gives a surface $\Sigma_{gn}$ so that the action of $R$ on $S^2$ naturally gives an action $R$ of $\mathbb{Z}/(gn)$ on $\Sigma_{gn}$.

Let $Y$ be a surface bundle over $S^1$ given by the monodromy $R^g$. That is, let

$$Y = \frac{F \times I}{(x,0) \sim (R^g(x), 1)}.$$

Let $p = (0,0,1)$, and let $\tau$ denote the loop in $Y$ that is the image in $Y$ of $\{p\} \times I$ under $\sim$. This is a loop since $R^g(p) = p$. Let $s, t \in \pi_1( Y \times S^1 )$ be given as follows: $t$ is represented by a parametrization of $\tau \times \{1\}$, and $s$ is a generator of $\{1\} \times \mathbb{Z} \subset \pi_1( Y \times \{1\} )$. We now use the presentation of $G$ given previously to get a new presentation of $G$. We add $g$ generators $y_1, \ldots, y_g$, we replace the relations $w_1, \ldots, w_r$ with new manipulation on the number of generators and relations on fundamental groups. We proceed to describe it.
relations $\tilde{w}_1, \ldots, \tilde{w}_r$, and we add $g$ more relations $x_1y_1, \ldots, x_gy_g$. Hence we have a new presentation of the same group $G$:

$$G = \langle x_1, \ldots, x_g, y_1, \ldots, y_g | \tilde{w}_1, \ldots, \tilde{w}_r, x_1y_1, \ldots, x_gy_g \rangle$$

The relation $\tilde{w}_i$ is obtained from the relation $w_i$ by replacing each occurrence of $x_k^{-1}$ with $y_k$. Hence each relation is expressed as a product of positive powers of the generators.

Each relation in the presentation of $G$ given in (7) gives an immersed curve $\gamma_i$ on the surface $\Sigma_{gn}$. Identify $\Sigma_{gn}$ with a fiber of $Y$, and then take the product of these curve with a circle to obtain tori $T_i := \gamma_i \times S^1 \subseteq Y \times S^1$. Figure 1 below is taken from [7], and here we see examples of how occurrences of $x_i$ and $y_i$ give us curves on this surface.

In order to make the construction work, the tori $T_i$ need to be disjoint but immersed, and the curves $\gamma_i$ need to be mapped to different parts of the surface. The details are provided in [7]; here we only briefly mention that by expressing the $r + g$ relations as only positive powers of the $2g$ generators, one ensures that we can perturb the tori $T_i$ so that they are symplectic.

A final torus that we consider is $T_0$, which is the product of the circle corresponding to $s, t \in \pi_1(Y \times S^1)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}
The presentation for \( \pi_1 (Y \times S^1) \) is
\[
(\pi_1 \Sigma_{gn}, t) R^g = t x t^{-1} \text{ for } x \in \pi_1 \Sigma_{gn},
\]
where \( \pi_1 \Sigma_{gn} \) has the presentation
\[
\pi_1 \Sigma_{gn} = \left\langle x_{1,1}, y_{1,1}, \ldots, x_{g,1}, y_{g,1}, x_{1,2}, y_{1,2}, \ldots, x_{g,2}, y_{g,2}, \ldots, x_{1,n}, y_{1,n}, \ldots, x_{g,n}, y_{g,n} \right| \prod_{l=1}^{g} [x_{k,l}, y_{k,l}] \right\rangle
\]

We symplectic sum along the tori \( T_0, \ldots, T_{r+g} \) with other symplectic manifolds to kill the pushoffs of the curves that generate \( \pi_1 (T_i) \) while not introducing additional generators to the fundamental group of our manifold. For example, if we symplectic sum in such a manner along \( T_0 \), then will kill the generators \( s \) and \( t \), and the presentation of \( \pi_1 \Sigma_{gn} \) reduces to
\[
(\pi_1 \Sigma_{gn}, R^g) = t x t^{-1} \text{ for } x \in \pi_1 \Sigma_{gn},
\]
Notice that in the notation of (6), \( R^g (x_{k,l}) = x_{k,l+1} \) and \( R^g (y_{k,l}) = y_{k,l+1} \), where the addition on the second subscript is taken modulo \( n \) (with \( n + 1 = 1 \)), and so setting \( x_k = x_{k,l} \) and \( y_k = y_{k,l} \) the presentation of (8) actually reduces further to,
\[
\left\langle x_{1,1}, y_{1,1}, \ldots, x_{g,1}, y_{g,1} \right| \prod_{l=1}^{g} [x_{k,l}, y_{k,l}] \right\rangle
\]
Summing along the tori \( T_{r+1}, \ldots, T_{r+g} \) introduced the relations \( x_k y_k = 1 \), and so the commutators \( [x_k, y_k] \) are trivial, so that the surface relation
\[
\prod_{l=1}^{g} [x_{k,l}, y_{k,l}]
\]
of (10) is trivial. Finally, performing a symplectic sum along the tori \( T_1, \ldots, T_r \) introduces the relations \( \tilde{w}_1, \ldots, \tilde{w}_r \), and the presentation of (10) is now
\[
\left\langle x_{1,1}, y_{1,1}, \ldots, x_{g,1}, y_{g,1} | x_{1,1} y_{1,1}, \ldots, x_{g,1} y_{g,1}, \tilde{w}_1, \ldots, \tilde{w}_r \right\rangle
\]
which is just the presentation of \( G \) in (7). Hence we have shown that fiber summing \( Y \times S^1 \) with the appropriate symplectic manifolds along the tori \( T_0, \ldots, T_{r+g} \) is a symplectic manifold with fundamental group \( G \).

For a proof of Theorem 6, the second author used a spin symplectic 4-manifold \( V \) with \( c_2 (V) = 4, \sigma = 0 \) that shares the cohomology ring of \( 2(S^2 \times S^2) \# S^3 \times S^1 \) and contains a symplectic torus \( T_V \) of self intersection zero carrying a generator of \( H_1 (V; \mathbb{Z}) \cong \mathbb{Z} \). This manifold was constructed in [17] Section 4; its construction consists of applying seven Luttinger surgeries [36, 5] to the product of two copies of a genus two surface \( \Sigma_2 \times \Sigma_2 \). There exists a symplectic surface of genus two and self intersection zero \( F \rightarrow V \).

A careful analysis of fundamental groups computations under cut and paste constructions was done by the second author in [52] Section 3, Section 4. The main ingredient that yields an improvement of [9, Theorem] is the symplectic sum [22, Theorem 1.3] of \( N \) and \( g + r \) copies of \( V \) along they symplectic tori \( T_i \) and \( T_V \) for \( 1 \leq i \leq g + r \); see [52] for details.
Denote this building block by $BK$. Its characteristic numbers are computed using the well known formulae in [22, p. 535] to be $c_2(BK) = 4(g + r)$, $\sigma(BK) = 0$.

2.3. Computations of $\pi_1$ Theorem 1. In order for the manifold $BK$ to be used successfully in the production of a manifold whose fundamental group has a presentation consisting of $g$ generators and $r$ relations, one builds a symplectic sum with a manifold $X$ that results in the resulting manifold having a fundamental group where the classes $s = 1 = t$ (where $s$ and $t$ are generators of $\pi_1(BK)$ as in Section 2.2). This was initially observed by R.E. Gompf [22], S. Baldridge and P. Kirk in [7], and used by the second author in the proof of Theorem 6.

Suppose there is a minimal symplectic 4-manifold $X$, which contains a symplectic torus $T \hookrightarrow X$ of self intersection zero such that $\pi_1(X - T) \cong \{1\}$ [52, Section 4]. Build the symplectic sum along tori $Z := BK_{T_0=TX}$.

In [52, Section 3, Section 4] it is proven that there exists an isomorphism $\pi_1(Z) \cong \langle g_1, \ldots, g_g | r_1, \ldots, r_r \rangle \cong G$.

The 4-manifold $X$ used in the proof of Theorem 6 is a minimal symplectic manifold homeomorphic to $3\mathbb{CP}^2 \# 5\mathbb{CP}^2$ constructed in [9, Theorem 18]. To prove Theorem 1 we will vary the choice of $X$ used in the construction described previously, where are choices come from from Proposition 11 and Proposition 15 in Section 2.4. The fundamental group computations follow verbatim from the work of the second author in [52].

2.4. 4-dimensional building blocks used to populate geographic regions

I. We proceed to describe the building blocks that have been successfully used by topologists to understand the geography and botany problems of symplectic 4-manifolds (see for example [22, 3, 9, 4, 49], and the references there).

**Proposition 11.** Let $b^- \in \{4, 5, \ldots, 17, 18, 19\}$. There exists a minimal symplectic simply connected 4-manifold $X_{3,b^-}$ with second Betti number given by $b_+^2(X_{3,b^-}) = 3$ and $b_1^-(X_{3,b^-}) = b^-$, which contains two homologically essential Lagrangian tori $T_1$ and $T_2$ such that $\pi_1(X_{3,b^-} - (T_1 \cup T_2)) = \{1\}$.

The characteristic numbers are $c_2(X_{3,b^-}) = 5 + b_2^-$, $\sigma = 3 - b^-$, and $c_1^2(X_{3,b^-}) = 10 + 2b^- + 9 - 3b_2^- = 19 - b^-$. 

Table 1 provides a blueprint on the construction of the manifolds in the statement of Proposition 11 and the sources of the building blocks. The notation used in Table 1 is similar to the one of Proposition 11 the manifolds $X_{1,n}$ are homeomorphic to $\mathbb{C}P^2 \# n\mathbb{C}P^2$. The elliptic surface used is denoted by $E(1) = \mathbb{C}P^2 \# 9\mathbb{C}P^2$. The symplectic sum of $X$ and $Y$ along $Z$ is denoted by $X \#_Z Y$.

The following building block was used in [3] to fill in vast regions of the geography.
Table 1. Minimal symplectic 4-manifolds with $b_2^+ = 3$.

| $b_2$ | $(c_1^2, \chi_h, \sigma)$ | Symplectic sum/manifold. | # of Luttinger surgeries. | Reference. |
|-------|-----------------|-----------------|-----------------|-----------------|
| 4     | $(15, 2, -1)$   | $X_{1,2}\#\Sigma_2 T^2 \times \Sigma_2$ | two             | [4] Section 9   |
| 5     | $(14, 2, -2)$   | $X_{3,5}$       | none            | [9] Theorem 18  |
| 6     | $(13, 2, -3)$   | $X_{1,3}\#\Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [4] Section 9   |
| 7     | $(12, 2, -4)$   | $X_{3,7}$       | none            | [9] Corollary 15|
| 8     | $(11, 2, -5)$   | $X_{1,5}\#\Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [4] Sections 3 and 4. |
| 9     | $(10, 2, -6)$   | $X_{1,5}\#\Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [9] Corollary 16. |
| 10    | $(9, 2, -7)$    | $X_{1,6}\#\Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [4] Lemma 15    |
| 11    | $(8, 2, -8)$    | $E(1)\# T^2 S$ | six             | [9] Lemma 16    |
| 12    | $(7, 2, -9)$    | $E(1)\# T^2 A$ | none            | Theorem 14      |
| 13    | $(6, 2, -10)$   | $E(1)\# T^2 B$ | none            | Theorem 14      |
| 14    | $(5, 2, -11)$   | $\mathbb{C}P^2 \# 12\mathbb{C}P^2 \# \Sigma_2 T^2 \times \Sigma_2$ | two          | [22] Building Block 5.6 |
| 15    | $(4, 2, -12)$   | $\mathbb{C}P^2 \# 13\mathbb{C}P^2 \# \Sigma_2 T^2 \times \Sigma_2$ | two          | [22] Building Block 5.6 |
| 16    | $(3, 2, -13)$   | $\mathbb{C}P^2 \# \Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [22] Building Blocks 5.6 and 5.6 |
| 17    | $(2, 2, -14)$   | $\mathbb{C}P^2 \# 13\mathbb{C}P^2 \# \Sigma_2 (T^4 \# 2\mathbb{C}P^2)$ | none | [22] Building Blocks 5.6 and 5.7 |
| 18    | $(1, 2, -15)$   | $S^1_1$         | none            | [22] Example 5.4|
| 19    | $(0, 2, -16)$   | $E(1)\# T^2 T^4 \# T^2 E(1)$ | none | [35]            |

**Definition 12.** [3 Definition 2]. An ordered triple $(X, T_1, T_2)$ consisting of a symplectic 4-manifold $X$ and two disjointly embedded Lagrangian tori $T_1$ and $T_2$ is called a *telescoping triple* if:

1. The tori $T_1$ and $T_2$ span a 2-dimensional subspace of $H_2(X; \mathbb{R})$.
2. The fundamental group is given by $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \cong \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2))$.
3. The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.
4. The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

A smooth 4-manifold is minimal if it does not contain any sphere of self intersection $-1$. If $X$ is a minimal manifold, then telescoping triple is called *minimal*. We emphasize the importance of the order of the tori in the definition. The meridians $\mu_{T_1}, \mu_{T_2}$ in $\pi_1(X - (T_1 \cup T_2))$ are trivial and the relevant fundamental groups are abelian. The push-off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of $T_i$ represents a well-defined element of $\pi_1(X - (T_1 \cup T_2))$ which is independent of the choices of framing and base-point.

The first condition assures us that the Lagrangian tori $T_1$ and $T_2$ are linearly independent in $H_2(X; \mathbb{R})$. This allows for the symplectic form on $X$ to be slightly perturbed so that one of the $T_i$ remains Lagrangian while the other becomes symplectic [22 Lemma 1.6]. Four our purposes, we will require for $T_1$ to be symplectic.
Proposition 13. (cf. [3, Proposition 3]). Let \((X,T_1,T_2)\) and \((X',T_1',T_2')\) be two telescoping triples. Then for an appropriate gluing map the triple

\((X \# T_2, T_1', T_1, T_2')\)

is again a telescoping triple. If \(X\) and \(X'\) are minimal symplectic 4-manifolds, then the resulting telescoping triple is minimal. The Euler characteristic and the signature of \(X \# T_2, T_1' X'\) are given by \(c_2(X) + c_2(X')\) and \(\sigma(X) + \sigma(X')\).

The proof of Proposition 13 follows from [22, 3, 4], and the claim about minimality follows from Usher’s theorem [51]. The results minding the existence of telescoping triples are gathered in the following theorem, which was proven in [3, Section 5], [50, Lemma 6, Lemma 7].

Theorem 14. Existence of telescoping triples.

- There exists a minimal telescoping triple \((A,T_1,T_2)\) satisfying \(c_2(A) = 5\), \(\sigma(A) = -1\).
- For each \(g \geq 0\), there exists a minimal telescoping triple \((B_g,T_1,T_2)\) satisfying \(c_2(B_g) = 6 + 4g\), \(\sigma(B_g) = -2\).
- There exists a minimal telescoping triple \((C,T_1,T_2)\) satisfying \(c_2(C) = 7\), \(\sigma(C) = -3\).
- There exists a minimal telescoping triple \((D,T_1,T_2)\) satisfying \(c_2(D) = 8\), \(\sigma(D) = -4\).
- There exists a minimal telescoping triple \((F,T_1,T_2)\) satisfying \(c_2(F) = 10\), \(\sigma(F) = -6\).

2.5. 4-dimensional building blocks used to populate geographic regions II. The building blocks of the previous sections are used to produce our guiding building blocks we will use in the proofs of our main results by using R.E. Gompf’s symplectic sum [22, Theorem 1.3]. A precise statement is the following.

Proposition 15. (cf. [3, Theorem A], [4, Theorem 1]). Let \(e, \sigma\) be integers that satisfy \(2e + 3\sigma \geq 0\), \(e + \sigma \equiv 0 \mod 4\), \(e + \sigma \geq 8\), and assume \(\sigma \leq -1\). Then there exist simply connected minimal symplectic 4-manifolds \(Z_{1,1}\) and \(Z_{1,2}\), both with Euler characteristic \(e\), signature \(\sigma\), and with odd intersection form over \(\mathbb{Z}\). Moreover, the 4-manifold \(Z_{1,1}\) contains two homologically essential Lagrangian tori \(T_1\) and \(T_2\) such that

\[\pi_1(Z_{1,1} - (T_1 \cup T_2)) \cong \{1\}.\]

The 4-manifold \(Z_{1,2}\) contains a homologically essential Lagrangian torus \(T_1\), and a symplectic surface of genus two \(F\) such that

\[\pi_1(Z_{1,2} - (T_1 \cup F)) \cong \{1\}.\]

Remark 16. At this point we emphasize that Proposition 17 and Proposition 15 are fruits of the efforts of several topologists (see [8, 9, 3, 4, 19] and the references provided there). The observation regarding the existence of the submanifolds is crucial for our purposes, and it has not appeared in the literature previously.
Proof. The construction of the 4-manifold \( Z_{1,1} \) is obtained by an iterated use of the telescoping triples of Theorem 14, Proposition 11, Proposition 5 (which cover most of the geographic points), and the results of [9, 3, 4] and [17, Section 4]. The details are left to the reader (see [49] for details).

The construction of the 4-manifold \( Z_{1,2} \) goes as follows. Build the symplectic sum \( Z := Z_{1,1} \#_{T_1 = T_2} T^2 \times \Sigma_2 \). Apply four Luttinger surgeries to obtain a simply connected symplectic 4-manifold \( Z_{1,2} \); see Proposition 5 for the fundamental group computations. The torus \( T_2 \) is contained in \( Z_{1,2} \), and so is a parallel copy of the genus two surface \( \{x\} \times \Sigma_2 \subset T^2 \times \Sigma_2 \) that we now call \( F \) [22, Corollary 1.7]. The meridian \( \mu_F \) is trivial in the complement by Proposition 5.

\[ \square \]

3. Geography of symplectic 4-manifolds with prescribed fundamental group.

In this Section we give a proof of Theorem 1. The proof is split in two parts.

3.1. Manifolds with odd intersection form and negative signature. The first part of Theorem 1 is proven in this section. We build greatly upon the progress on the geography problem for nonspin simply connected symplectic 4-manifolds with negative signature [22, 42, 8, 9, 17, 3, 4], and the study of minimal symplectic 4-manifolds with arbitrary fundamental group [7, 52].

**Theorem 17.** Let \( G \) be a group with a presentation consisting of \( g \) generators and \( r \) relations. Let \( e, \sigma \) be integers that satisfy \( 2e + 3\sigma \geq 0, e + \sigma \equiv 0 \pmod{4}, e + \sigma \geq 8, \) and assume \( \sigma \leq -1 \). Then there exists a minimal symplectic 4-manifold \( M(G) \) with fundamental group \( \pi_1(M(G)) \cong G \), and characteristic numbers

\[ c_2(M(G)) = e + 4(g + r), \quad \sigma(M(G)) = \sigma. \]

The intersection form over \( \mathbb{Z} \) of \( M(G) \) is odd. If the group \( G \) is residually finite, then \( M(G) \) is irreducible. If the group \( G \) is finite, then \( M(G) \) admits exotic smooth structures.

**Proof.** The proof of the existence claim is verbatim to the proof of [52, Theorem 1]. It consists of building the symplectic sum [22, Theorem 1.3] \( M(G) := Z_{1,1} \#_{T_1 = T_2} BK \) along a symplectic torus of the manifold \( BK \) described in Section 2.2 and a manifold \( Z_{1,1} \) from Proposition 15.

If the group \( G \) is residually finite, then \( M(G) \) is irreducible by [26].

The claim minding the existence of exotic smooth structures on 4-manifolds follows from a pigeonhole argument (cf [52, Remark 1.2]). An infinite family of pairwise nondiffeomorphic manifolds sharing the same fundamental group and characteristic numbers is constructed using the techniques in [18, 19, 17], and keeping track of the Seiberg-Witten invariants using [46, 38]; see [9, Remark on p. 343]. It is proven in [23, Corollary 1.5] that there are only finitely many homeomorphism types of closed oriented 4-manifolds with finite fundamental group, and a given Euler characteristic. Thus, there exist exotic smooth structures as it was claimed.

\[ \square \]

3.2. Spin manifolds with negative signature. In this Section, we prove the second part of Theorem 1. B. D. Park and Z. Szabó settled the geography of simply connected spin symplectic 4-manifolds in [40] (see [41] as well). The task at
hand is to extend their results to the geography problem for spin manifolds with finitely presented fundamental group using the building block described in Section 1.2. We then follow J. Park’s exposition in [42]. Let $V$ be a compact connected simply connected 4-manifold with a given boundary. Results on homeomorphisms of simply connected 4-manifolds with a given boundary are obtained using S. Boyer’s work [11] on homeomorphisms of simply connected 4-manifolds with boundary and R. Fintushel and R. Stern’s Knot surgery [18] to construct exotic smooth structures on the 4-manifolds constructed by using these techniques.

In the following, we then follow J. Park’s work [42] to build exotic smooth structures on the 4-manifolds $V$ with finitely presented fundamental group using the building block described in Section 1.2. Particularly useful is the following corollary, which appears in J. Park’s work.

**Corollary 19.** [cf. [12], Corollary 2.3] Assume that $\partial V_1 \cong \partial V_2$ is a homology 3-sphere. For any morphism $(f, \Lambda) : V_1 \to V_2$ between simply connected smooth 4-manifolds $V_1, V_2$, there is a homeomorphism $F : V_1 \to V_2$ such that $(f, \Lambda) = (F|_{\partial V_1}, F_*)$.

Before we begin proving Theorem 1, we take a moment to recall S. Boyer’s [11] results on homeomorphisms of simply connected 4-manifolds with a given boundary. We follow J. Park’s exposition in [42]. Let $V_1$ and $V_2$ be compact connected simply connected 4-manifolds with connected boundary $\partial V_1 \cong \partial V_2$, and let $f : \partial V_1 \to \partial V_2$ be an orientation-preserving homeomorphism. According to Boyer, there are two obstructions that need be overcome in order to extend $f$ to a homeomorphism $F : V_1 \to V_2$. First, an isometry $\Lambda : (H_2(V_1; \mathbb{Z}), Q_{V_1}) \to (H_2(V_2; \mathbb{Z}), Q_{V_2})$ for which the following diagram commutes

$$
\begin{array}{cccccc}
0 & \longrightarrow & H_2(\partial V_1) & \longrightarrow & H_2(V_1) & \longrightarrow & H_2(V_1, \partial V_1) & \longrightarrow & H_1(\partial V_1) & \longrightarrow & 0 \\
\text{\text{f}_*} & & \text{\Lambda} & & \text{\Lambda}^* & & \text{\text{f}_*} & & & \\
0 & \longrightarrow & H_2(\partial V_2) & \longrightarrow & H_2(V_2) & \longrightarrow & H_2(V_2, \partial V_2) & \longrightarrow & H_1(\partial V_2) & \longrightarrow & 0
\end{array}
$$

must exist. In this context, an isometry is an isomorphism $\Lambda : H_2(V_1; \mathbb{Z}) \to H_2(V_2; \mathbb{Z})$ that preserves intersection forms $Q_{V_1}$. The morphism $\Lambda^*$ in the diagram is the adjoint of $\Lambda$ with respect to the identification of $H_2(V_1, \partial V_1)$ with $\text{Hom}(H_2(V_1); \mathbb{Z})$ that arises from Lefschetz duality. This information is encoded in the pair $(f, \Lambda)$, which is called a **morphism** and it is denoted symbolically by $(f, \Lambda) : V_1 \to V_2$.

Second, one must find a homeomorphism $F : V_1 \to V_2$ such that the morphism $(f, \Lambda)$ is realized geometrically, if this condition is satisfied. In [11], S. Boyer studied conditions under which a given morphism can be geometrically realized.

**Theorem 18.** (Boyer [11], Theorem 0.7, Proposition 0.8) If $(f, \Lambda) : V_1 \to V_2$ is a morphism between two simply connected compact smooth 4-manifolds $V_1, V_2$ with boundary $\partial V_1 \cong \partial V_2$, there is an obstruction $\theta(f, \Lambda) = f^! (\partial V_1)$ such that $(f, \Lambda)$ is realized geometrically if and only if $\theta(f, \Lambda) = 0$. Moreover, if $H_1(\partial V_1; \mathbb{Q}) = 0$, then $\theta(f, \Lambda) = 0$.

Particularly useful is the following corollary, which appears in J. Park’s work.

**Corollary 19.** (cf. [12], Corollary 2.3) Assume that $\partial V_1 \cong \partial V_2$ is a homology 3-sphere. For any morphism $(f, \Lambda) : V_1 \to V_2$ between simply connected smooth 4-manifolds $V_1, V_2$, there is a homeomorphism $F : V_1 \to V_2$ such that $(f, \Lambda) = (F|_{\partial V_1}, F_*)$.

We are now ready to prove our first result. The precise statement of Theorem 1 reads as follows.

**Theorem 20.** Let $s \geq 1$, $n \in \mathbb{N}$, and let $G$ be a finitely presented group. For each of the following pairs of integers

$$
(1, s) \quad (s, 1) \quad (s, s)
$$

...
there exists a symplectic spin 4-manifold $X$ with
\[
\pi_1(X(G)) = G \text{ and } (c_1(X(G)), \chi_h(X(G))) = (c, \chi).
\]
If the group $G$ is residually finite, then $X(G)$ is irreducible. Moreover, there is an infinite set \( \{X_m(G) : m \in \mathbb{N} \} \) of pairwise nondiffeomorphic symplectic and nonsymplectic manifolds homeomorphic to $X(G)$ for every $m \in \mathbb{N}$.

The characteristic numbers of the examples of Theorem 20 can be stated as
\[
c_2(X(G)) = 4(n - 4 + 6s + (g + r)), \text{ and } \sigma(X(G)) = -16s.
\]

The following proof is a modification to J. Park’s work done in [42].

**Proof.** Let $Z$ be an irreducible simply connected symplectic spin 4-manifold of [30] (see also [41]). B. D. Park and Z. Szabó constructed $Z$ using simply connected minimal elliptic surfaces without multiple fibers $E(n)$, which are known to contain Gompf nuclei $C_n \subset E(n)$ [21]. In particular, there exists a symplectic torus $T \hookrightarrow Z$ of self intersection zero, with $\pi_1(Z - T) \cong \{1\}$. Build the symplectic sum [22, Theorem 1.3]
\[
X(G) := Z \#_TBK,
\]
where $BK$ is the manifold described in Section 2.2. By [22, Proposition 1.2], the manifolds $X(G)$ are spin. The computations of the characteristic numbers are straight-forward, since $c_1(T^2) = 0$, $c_2(BK) = 4(g + r)$, and $\sigma(BK) = 0$. The claim $\pi_1(X(G)) \cong G$ follows from the computations in Section 2.3. If the group $G$ is residually finite, then $M(G)$ is irreducible by [26]. The existence claim follows from a variation of the choice of symplectic manifold $Z$ one takes from [40] in the construction of the symplectic sum $X(G)$.

From the existence of Gompf nuclei inside the building block $Z$, one concludes that there exists a symplectic torus $T' \hookrightarrow X(G)$ such that the inclusion induces the trivial map $\pi_1(T') \rightarrow \pi_1(X(G) - T') \cong G$. The infinite set of pairwise nondiffeomorphic symplectic and nonsymplectic 4-manifolds \( \{X_m(G) : m \in \mathbb{N} \} \) is constructed using R. Fintushel - R. Stern’s knot surgery [18], varying the choice of knot.

To prove the claim on the homeomorphism type of the manifolds constructed, we use an argument due to J. Park [42, Proposition 2.2]. Let the 4-manifold with arbitrary fundamental group $X_i(G) := Z_{K_i} \#TBK \in \{X_m(G) : m \in \mathbb{N} \}$ be the result of applying knot surgery with knot $K_i$. We claim that there exists a homeomorphism between the manifolds $X_i(G) = Z_{K_i} \#TBK$ and $X_j(G) = Z_{K_j} \#TBK$ for any $i, j \in \mathbb{N}$. In this notation, we are specifying that the surgery occurs in the $Z$ block of the construction, and calling $Z_{K_i}$ the homeomorphic simply connected manifolds produced using knot surgery with knot $K_i$. The spin manifold $Z_{K_i}$ contains a Gompf nucleus as a codimensional zero submanifold, independent of the knot $K_i$. Let $C_n$ be the Gompf nucleus, let $\Sigma$ be a homology 3-sphere, and set $Z_{K_i} := Z_{K_i} - C_n$. The symplectic sum $Z_{K_i} \#TBK$ can be decomposed as
\[
Z_{K_i} \cup_{\Sigma} (BK \#_TC_n).
\]

The endeavor at hand is to construct a morphism $(f, \Lambda)$, and then use Boyer’s results to show that it is geometrically realized. The choice of homeomorphism $f$ is
canonical by setting \( g := \text{id} : BK \#_T C_n \to BK \#_T C_n \), so that its restriction yields \( g|_{\partial(BK \#_T C_n)} = f : \Sigma \to \Sigma \). Take an isomorphism \( \Lambda : H_2(Z^n_K; \mathbb{Z}) \to H_2(Z^n_K; \mathbb{Z}) \). Minding the intersection forms, \( Q_{Z^n_K} \) and \( Q_{Z^n_K} \) are unimodular, indefinite forms over \( \mathbb{Z} \) that have the same rank, same signature, and same type. By the classification of unimodular indefinite integral forms, \( Q_{Z^n_K} \equiv Q_{Z^n_K} \). We have that \( \Lambda : (H_2(Z^n_K; \mathbb{Z}), Q_{Z^n_K}) \to (H_2(Z^n_K; \mathbb{Z}), Q_{Z^n_K}) \) is an isomorphism that preserves intersection form, i.e., an isometry. Therefore, we have a required morphism \( (f, \Lambda) : Z^n_K \to Z^n_K \).

Since \( H_1(\partial(Z^n_K); \mathbb{Q}) = H_1(\partial(Z^n_K); \mathbb{Q}) = H_1(\Sigma; \mathbb{Q}) = 0 \), then \( \theta(f, \Lambda) = 0 \) by Theorem 18. It follows from Corollary 19 that there exists a homeomorphism \( F : Z^n_K \to Z^n_K \) such that \( (f, \Lambda) = (F|_\Sigma, F|_\Sigma) \). Thus, for \( i, j \in \mathbb{N} \) any two manifolds \( X_i(G) = Z^n_K \#_T BK \) and \( X_j(G) = Z^n_K \#_T BK \) are homeomorphic.

**Remark 21.** On the geography for other choices of \( \pi_1 \) and other regions. The techniques used in this paper yield similar results to Theorem 7 for several choices of fundamental groups. For example, surface groups, 3-manifolds groups, knot group of \((p, q)\)-torus knots, the 3-dimensional Heisenberg group, the infinite dihedral group, \( PSL(2, \mathbb{Z}) \), free groups of rank \( n \), amongst others cf. \([7]\), and Example 27 below. Moreover, other regions can be populated using similar methods cf. \([8]\).

### 4. Geography of symplectic 6-manifolds.

#### 4.1. Computing Chern numbers.

Minding the 6-manifolds arising as products of 4- and 2-manifolds, we have the following lemma.

**Lemma 22.** Let \( Y \) be a symplectic 4-manifold, and \( \Sigma_g \) a symplectic surface of genus \( g \). Let \( e := e(Y) \) be the Euler characteristic and \( \sigma := \sigma(Y) \) the signature. The Chern numbers of the product symplectic 6-manifold \( X := Y \times \Sigma_g \) are given as follows.

- \( c_1^2(X) = (2e + 3\sigma) \cdot (6 - 6g) \),
- \( c_1 c_2(X) = (e + \sigma) \cdot (6 - 6g) \), and
- \( c_3(X) = e \cdot (2 - 2g) \).

**Proof.** We need to identify the Chern classes. For such a purpose, consider the projections

\[
\pi_1 : X = Y \times \Sigma_g \to Y, \quad \pi_2 : X = Y \times \Sigma_g \to \Sigma_g.
\]

The total Chern class can be written as \( c(X) = \pi_1^*(e(Y)) \cdot \pi_2^*(e(\Sigma_g)) \). The lemma follows by integrating over \( X = Y \times \Sigma_g \).

The change on the Chern numbers of 6-manifolds after blow ups at points and along surfaces are given by the following formulas.

**Lemma 23.** Let \( X \) be a symplectic 6-manifold. Let \( X' \) be the symplectic 6-manifold obtained by blowing up a point in \( X \). The Chern numbers of \( X' \) are as follows:

- \( c_1^2(X') = c_1^2(X) - 8 \),
- \( c_1 c_2(X') = c_1 c_2(X) \),
- \( c_3(X') = c_3(X) + 2 \).
Let $\Sigma_g \hookrightarrow X$ be a symplectic surface of genus $g$, and denote by $N(\Sigma_g)$ its normal bundle. Let $X'_g$ be the symplectic 6-manifold obtained by blowing up $X$ along $\Sigma_g$. The Chern numbers of $X'_g$ are as follows:

- $c_1^3(X'_g) = c_1^3(X) + 6(g - 1) - 2 \langle c_1(N(\Sigma_g)), [\Sigma_g] \rangle$,
- $c_1c_2(X'_g) = c_1c_2(X)$,
- $c_3(X'_g) = c_3(X) - 2(g - 1)$.

See [24] Lemma 2.1 for a proof.

Regarding the 6-manifolds obtained as symplectic sums, we have the following lemma.

**Lemma 24.** Let $X_1, X_2$ be symplectic 6-manifolds that contain 4-dimensional symplectic submanifolds $Y_i \subset X_i$ for $i = 1, 2$ with trivial normal bundle such that $Y_1 \cong Y_2$. Set $Y := Y_1$. The Chern numbers of the symplectic sum $X := X_1 # Y X_2$ are

- $c_1^3(Y) = c_1^3(X_1) + c_1^3(X_2) - 6c_1^2(Y)$,
- $c_1c_2(Y) = c_1c_2(X_1) + c_1c_2(X_2) - 2(c_1^2(Y) + c_2(Y))$, and
- $c_3(Y) = c_3(X_1) + c_3(X_2) - 2c_2(Y)$.

See [24] Proposition 1.3 for a proof (cf. [32] Lemma 3.23).

The existence of a symplectic structure induces an almost-complex structure on the underlying manifold. The possible restrictions on the Chern numbers of almost-complex 6-manifolds are computed in [39] Proposition 9], [20], and see [32] for interesting 6-dimensional phenomena.

### 4.2. 6-dimensional symplectic sums

We use our 4-dimensional building blocks to construct symplectic 6-manifolds using Gompf’s symplectic sum construction [22] Theorem 1.3 (see also [57]) as our main manufacture tool. In our notation, $X #_Z Y$ denotes the symplectic sum of $X$ and $Y$ along the submanifold $Z$.

**Theorem 25.** (cf. Theorem 2). Let $G \in \{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \pi_1(\Sigma_g), \pi_n\}$, where $\Sigma_g$ is a closed oriented surface and $\pi_n$ is the free group of rank $n$. Let $e_i, \sigma_i$ for $i = 1, 2$ be integers that satisfy $2e_i + 3\sigma_i \geq 0, e_i + \sigma_i \equiv 0 \mod 4$, and $e_i + \sigma_i \geq 8$.

There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$, with fundamental group $\pi_1(W_i(G)) \cong G$ for $i = 0, 1, 2$, and the following Chern numbers:

1. $c_1^3(W_0(G)) = 6 \cdot (3\sigma_1 + 2e_1 + 3\sigma_2 + 2e_2)$, $c_1c_2(W_0(G)) = 6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2)$, and $c_3(W_0) = 2 \cdot (e_1 + e_2)$;

2. $c_1^3(W_1(G)) = 0$, $c_1c_2(W_1(G)) = 0$, and $c_3(W_1(G)) = 0$;

3. $c_1^3(W_2(G)) = -6 \cdot (3\sigma_1 + 2e_1 + 3\sigma_2 + 2e_2) - 48$, $c_1c_2(W_2(G)) = -6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) - 24$, and $c_3(W_2(G)) = -2 \cdot (e_1 + e_2) - 8$.

Assume $s_1, s_2 \geq 1$, and $n_1, n_2 \in \mathbb{N}$. There exist spin symplectic 6-manifolds $Y_i(G)$ for $i = 0, 1, 2$ with fundamental group $\pi_1(Y_i(G)) \cong G$, and whose Chern
numbers are

\[ c_1^3(Y_0(G)) = 48 \cdot (n_1 + n_2 - 2), \]
\[ c_1 c_2(Y_0(G)) = 48 \cdot (s_1 + s_2) + 24 \cdot (n_1 + n_2 - 2), \text{ and} \]
\[ c_3(Y_0(G)) = 48 \cdot (s_1 + s_2) + 8 \cdot (n_1 + n_2 - 2); \]

(2) \( c_1^3(Y_1(G)) = 0, \)
\[ c_1 c_2(Y_1(G)) = 0, \text{ and} \]
\[ c_3(Y_1(G)) = 0; \]

(3) \( c_1^3(Y_2(G)) = -48 \cdot (n_1 + n_2 - 1), \)
\[ c_1 c_2(Y_2(G)) = -48 \cdot (s_1 + s_2) - 24 \cdot (n_1 + n_2 - 1), \text{ and} \]
\[ c_3(Y_2(G)) = -48 \cdot (s_1 + s_2) - 8(n_1 + n_2 - 1). \]

**Proof.** Let us construct the manifolds \( W_i(G) \) for \( i = 0, 1, 2; \) the method of construction of the manifolds \( Y_i(G) \) is similar. Let \( X_1, X_2 \) be minimal symplectic 4-manifolds of Proposition 15. Moreover, using the procedure described in Section 5.2 on \( X_1 \), we can produce a minimal symplectic 4-manifold \( X_1(G) \) such that \( \pi_1(X_1(G)) \cong G \). The manifold \( X_1(G) \) contains a homologically essential Lagrangian torus \( T := T_1 \) such that \( \pi_1(X_1(G) - T) \cong G \). Perturb the symplectic sum on \( X_1(G) \) so that \( T \) becomes symplectic [22, Lemma 1.6]. We make use of the following prototypes of symplectic sums according to the item in the statement of the theorem.

1. \( W_0(G) := X_1(G) \times S^2 \# T^2 \times S^2 \times X_2 \times S^2 \)
2. \( W_1(G) := X_1(G) \times T^2 \# T^2 \times T^2 \times X_2 \times T^2 \)
3. \( W_2(G) := X_1(G) \times \Sigma_2 \# \Sigma_2 \times X_2 \times \Sigma_2. \)

The reader should notice that the manifolds \( X_1, X_2 \) are to be chosen from the manifolds \( Z_{1,1}, Z_{1,2} \) of Proposition 15 appropriately, so that they contain the necessary symplectic submanifolds for the symplectic sums performed.

A description of the choices of gluing maps for the symplectic sums is given in Section 5.1. The computations of Chern numbers are straight-forward, and follow from Lemma 22 and Lemma 24. Minding the claim about fundamental groups, we refer the reader to Section 6. The manifolds \( Y_i(G) \) are constructed in a similar way by taking the manifolds \( X_1, X_2 \) in the procedure described above from 10. The details are left to the reader.

\[ \square \]

### 4.3. Realization of all possible combinations of Chern numbers.

The geography of non-minimal simply connected symplectic 6-manifolds was settled by M. Halic in [24]. The examples given by him rely heavily on the blow-ups of some basic building blocks. His main tool to realize all possible combinations of Chern numbers with a symplectic 6-manifold is stated in the following result.

**Lemma 26.** M. Halic [24, Lemma 4.1]. Let \( X \) be a symplectic 6-manifold whose Chern numbers are given by \( (c_1^2, c_1 c_2, c_3) = (2a, 24b, 2c) \), and which has the following properties. We denote by \( N(E_X) \) the normal bundle of a projective line \( E \subset X \).
The manifold $X$ contains a symplectically embedded product $U \times D$, where $U$ is an open subset of a symplectic 4-manifold and $D$ is a disc. Assume that $U$ contains a projective line $E$ having the property that

$$-\alpha := \langle c_1(N(E_X)), [E] \rangle \leq -1.$$ 

There is a symplectic genus two surface $\Sigma_2 \subset X$, which is disjoint from the exceptional sphere $E$, and $[\Sigma_2]^3 = 0$.

Then, by blowing up $p$ points, $r$ distinct copies of $E$, and $z$ distinct copies of $\Sigma_2$, one obtain all triples of Chern numbers of the form $(2a', 24b, 2c')$, where $a', c'$ are arbitrary integers.

Theorem 25 allows us to recover the main result of [24], and extend it to manifolds with other choices of fundamental groups as exemplified in the following statement (cf. Remark 21 and Corollary 4).

**Example 27.** Let $X(G)$ be a symplectic 6-manifold. Then,

$$c_1^3(X(G)) \equiv c_3(X(G)) \equiv 0 \mod 2,$$

and

$$c_1c_2(X(G)) \equiv 0 \mod 24.$$

Conversely, let $G \in \{ \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \pi_1(\Sigma_g), F_3 \}$. Any triple $(a, b, c)$ of integers satisfying $a \equiv c \equiv 0 \mod 2$ and $b \equiv 0 \mod 24$ occurs as a triple $(c_1^3, c_1c_2, c_3)$ of Chern numbers of a symplectic 6-manifold $X(G)$ with $\pi_1(X(G)) \cong G$.

**Proof.** Let $X$ be a symplectic 6-manifold. Then $X$ admits an almost-complex structure, and Proposition 9 implies that the Chern numbers of $X$ satisfy $c_1^3(X) = c_3(X) \equiv 0 \mod 2$ and $c_1c_2(X) \equiv 0 \mod 24$.

Now, let $X_1(G)$ be the manifold as in the proof of Theorem 25. There are two things to be checked in order to establish the converse. First, one needs to see that the result of blowing up at a point the manifolds constructed in Theorem 25 satisfy the hypothesis of Lemma 26. As a consequence, we are then able to conclude that all the possible values of $c_1^3$ and $c_3$ are realized by a symplectic 6-manifold with the given fundamental group. Second, it needs to be seen that such manifolds fulfill all the possible values of $c_1c_2$ (values of $b$ in the statement of Lemma 26). Notice that by Lemma 23, the value $c_1c_2$ is invariant under blow ups of points and under blow ups along surfaces.

By Proposition 15, there exists a symplectic surface of genus two $F$ inside the submanifold $X_1(G) \times \{ x \} \subset X_1(G) \times \Sigma_g$ for every 6-manifold obtained as a symplectic sum in Theorem 25. By [22], every manifold of Theorem 25 contains a symplectic surface of genus two. The manifold obtained by blowing up a point that is disjoint from the surface of genus two satisfies the hypothesis of Lemma 26.

We now consider the possible values of $c_1c_2$. The claim is that all possible values are obtained by the manifolds constructed in Theorem 25. The case $c_1c_2 = 0$ is clear. The claim is equivalent to finding $e_i, \sigma_i, e'_i, \sigma'_i$ for $i = 1, 2$ that produce integers $k_+$ and $k_-$ that satisfy

$$6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) = 24k_+$$

and

$$-6 \cdot (e'_1 + \sigma'_1 + e'_2 + \sigma'_2) = 24k_-.$$ 

The numbers $e_i, \sigma_i, e'_i, \sigma'_i$ for $i = 1, 2$ are as in the statement of Theorem 25 and they correspond to the Euler characteristic and signatures of minimal 4-manifolds of Proposition 15 where $X_1(G)$ is constructed from. The corresponding manifolds to these integer numbers can be chosen so that the equalities $(e_1 + \sigma_1 + e_2 + \sigma_2) = 4k_+$ and $(e'_1 + \sigma'_1 + e'_2 + \sigma'_2) = 4k_-$ are satisfied. Thus, all possible values of $c_1c_2$ are...
realized by manifolds from Theorem 25. The corollary now follows from Lemma 26. □

5. Fundamental group computations

5.1. Choices of gluing maps for symplectic sums in Section 4.2. An important piece of data in the 6-dimensional symplectic sum constructions of Section 4.2 are the gluing maps. Here we proceed to describe them in detail in order to conclude the fundamental group computations claimed in the statements of Theorem 24 and Theorem 23. The setting is as follows. Let \( X \) and \( Z \) be symplectic 4-manifolds containing symplectic tori \( T \subset X \), and \( T' \subset Z \) or symplectic surfaces of genus two \( F \subset X \) and \( F' \subset Z \) (see Proposition 15); notice that according to Proposition 15 or Theorem 6 one might need to perturb the symplectic form of the ambient 4-manifold using [22, Lemma 1.6] for the homologically essential Lagrangian tori to be assumed symplectic.

Regarding the homotopy computations, we assume
\[
\pi_1(X - T) \cong \pi_1(X) \quad \text{and} \quad \pi_1(Z - T') \cong \pi_1(Z) \cong \{1\},
\]
\[
\pi_1(X - F) \cong \pi_1(X) \quad \text{and} \quad \pi_1(Z - F') \cong \pi_1(Z) \cong \{1\}.
\]

Take a symplectic surface of genus \( g \), and build the 6-manifolds \( X \times \Sigma_g \) and \( Z \times \Sigma_g \) equipped with the product symplectic form. For \( g = 0, 1 \), these symplectic 6-manifolds contain a symplectic submanifold \( T^2 \times \Sigma_g \): \( T \times \Sigma_g \subset X \times \Sigma_g \) and \( T' \times \Sigma_g \subset Z \times \Sigma_g \). Similarly, we have the symplectic submanifolds (case \( g = 2 \)) \( F \times \Sigma_2 \subset X \times \Sigma_2 \) and \( F' \times \Sigma_2 \subset Z \times \Sigma_2 \). In all our constructions, \( \Sigma_g \) is either a 2-sphere \( S^2 \) (\( g = 0 \)), a 2-torus \( T^2 \) (\( g = 1 \)), or a surface of genus two \( \Sigma_2 \) (\( g = 2 \)); surfaces of higher genus can be used with a small modification to our arguments.

For surfaces of genus \( g = 0, 1 \), the 6-manifolds are symplectic sums of the form
\[
W_g := X \times \Sigma_g \#_{T \times \Sigma_g = T' \times \Sigma_g} Z \times \Sigma_g,
\]
where the gluing map is a diffeomorphism
\[
\phi_g : \partial(T \times \Sigma_g \times D^2) \to \partial(T' \times \Sigma_g \times D^2) \cong T^2 \times \Sigma_g \times S^1.
\]

Similarly, for surfaces of genus two we built
\[
W_2 := X \times \Sigma_2 \#_{F \times \Sigma_2 = F' \times \Sigma_2} Z \times \Sigma_2,
\]
where the gluing map given by a diffeomorphism
\[
\phi_2 : \partial(F \times \Sigma_2 \times D^2) \to \partial(F' \times \Sigma_2 \times D^2) \cong \Sigma_2 \times S^2 \times S^1.
\]

Lemma 28. \( \pi_1(W_g) \cong \pi_1(X) \) for \( g = 0, 1, 2 \).

Proof. We use Seifert-van Kampen Theorem to compute the fundamental group of \( W_g \) for \( g = 0, 1, 2 \). In order to obtain \( \pi_1(W_g) \cong \pi_1(X) \), we choose a diffeomorphism \( \phi_g \) that maps the generators of the fundamental groups involved as follows; remember we can assume \( \pi_1(Z - T') \cong \pi_1(Z) \cong \{1\} \) (see Proposition 15). The case \( g = 0 \) is clear, since there are isomorphisms \( \pi_1(X \times S^2 - T \times S^2) \cong \pi_1(X) \) and \( \pi_1(Z \times S^2 - T' \times S^2) \cong \{1\} \).

We work out the case \( g = 1 \); the case \( g = 2 \) follows verbatim from our argument. In this case, the symplectic 4-manifold along which the gluing in the symplectic sum [22] is performed is the 4-torus.
\[ T \times T^2 \cong T^4 \cong T' \times T^2. \]

Let the groups \( \pi_1(X \times T^2 - T \times T^2) \cong \pi_1(X) \oplus \mathbb{Z} \oplus \mathbb{Z} \) and \( \pi_1(Z \times T^2 - T' \times T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \) be generated by \( \{x_1, x_2, \ldots, x_n, x = 1, y = 1, w, z\} \) and \( \{x' = 1, y' = 1, w', z'\} \) respectively, where \( \{x_1, x_2, \ldots, x_n\} \) are generators of \( \pi_1(X) \), \( \{x, y\} \) are push offs of generators of \( \pi_1(T) \), and each element \( \{w', z'\} \) generates a \( \mathbb{Z} \). Let \( \mu \) and \( \mu' \) be the respective (based) meridians of the submanifolds. Notice that in both cases the meridians \( \mu = 1 = \mu' \) are trivial; we assume \( \mu \) gets identified with \( \mu^{-1} \) during the gluing employed to construct \( W_1 \). Choose \( \phi_1 \) to be the diffeomorphism that identifies the generators of the fundamental groups as follows:

\[ x \mapsto w', y \mapsto z', w \mapsto x', z \mapsto y'. \]

This choice of diffeomorphism allows us to conclude that the extra generators \( w, z, w', z' \) are killed during the gluing. An application of the Seifert-van Kampen Theorem implies \( \pi_1(W_1) \cong \pi_1(X) \), as desired. □

5.2. Computations of \( \pi_1 \) Theorem 2, Theorem 3, Corollary 4 and Theorem 25

Minding the geography of symplectic 6-manifolds, the desired fundamental groups are obtained as follows. A minimal 4-manifold \( Z_{1,1} \) of Proposition 15 contains two homologically essential Lagrangian tori with trivial meridians; in particular, there is an isomorphism \( \pi_1(Z_{1,1} - (T_1 \cup T_2)) \rightarrow \pi_1(Z_{1,1}) \cong \{1\} \) (similar argument works for \( Z_{1,2} \)).

The idea is to build the symplectic sum of \( Z_{1,1} \) along \( T_1 \) and a chosen manifold, in order to obtain a manifold \( P \) with the desired fundamental group \( G \). The homologically essential Lagrangian torus \( T_2 \subset P \) satisfies that the homomorphism induced by inclusion \( T_2 \hookrightarrow P \) is an isomorphism \( \pi_1(P - T_2) \rightarrow \pi_1(P) \cong G \). This is then used in our 6-dimensional symplectic sums in Section 4.2 to obtain the prescribed fundamental group. We proceed to prove the fundamental group claims on the four-dimensional building blocks employed.

Proof.

- Case abelian group of small rank. To be precise, we construct manifolds with fundamental group among \( \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}\} \).

Consider the 4-torus \( T^2 \times T^2 \) endowed with the product symplectic form. The 2-torus \( \{x\} \times T^2 \subset T^2 \times T^2 \) is a symplectic submanifold of self intersection zero. Start by building the symplectic sum

\[ Z(\mathbb{Z} \oplus \mathbb{Z}) := Z_{1,1} T_1 = T_2 T^2 \times T^2. \]

Since \( \pi_1(Z_{1,1}) \cong \pi_1(Z_{1,1} - T_1) \cong \{1\} \), it follows from Seifert-van Kampen Theorem that \( \pi_1(Z(\mathbb{Z} \oplus \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z} \). There are two pairs of geometrically dual Lagrangian tori inside \( Z(\mathbb{Z} \oplus \mathbb{Z}) \), on which one can perform Luttinger surgeries [36] [5]. These tori come from the Lagrangian tori inside the 4-torus used in the symplectic sum construction. Applying the appropriate surgery yields the desired fundamental groups: see [8 Section 2], [9 Theorem 1] for the fundamental group computations, and [40] for details.

- Case surface groups. Manifolds with fundamental group \( \{1\} \) and \( \mathbb{Z} \oplus \mathbb{Z} \) were constructed in the previous item. To construct a manifold \( S \) with \( \pi_1(S) \cong \pi_1(\Sigma_g) \) for \( g \geq 2 \), we proceed as follows. Consider the product \( T^2 \times \Sigma_g \) of a 2-torus and a genus \( g \geq 2 \) surface endowed with the product
symplectic form. The submanifold $T^2 \times \{x\} \subset T^2 \times \Sigma_g$ is symplectic and its self intersection number is zero. One builds the symplectic sum

$$S := Z_{1,1-T_1=T^2} \times \Sigma_g.$$

Since $\pi_1(\Sigma_{1,1}) \cong \pi_1(\Sigma_{1,1-T_1}) \cong \{1\}$, it follows from Seifert-van Kampen Theorem that $\pi_1(S) \cong \pi_1(\Sigma_g)$ as desired.

- Case free groups of rank $n \in \mathbb{N}$, $F_n$. The case $n = 1$ was proven previously. Let $\Sigma_n$ be a surface of genus $n$, and denote the standard generators of $\pi_1(\Sigma_n)$ by $x_1, y_1, \ldots, x_n, y_n$. Let $\phi: \Sigma_n \to \Sigma_n$ be a diffeomorphism consisting of the composition of $n - 1$ Dehn twists along the loops $x_i$; denote by $Y$ the mapping torus of $\phi$ that fibers over the circle.

Build the product $Y \times S^1$, which admits a symplectic structure [48]. This manifold contains a symplectic torus $T'$, given by $T' = S \times S^1$, where $S$ is a section of $Y \to S^1$. The fundamental group $\pi_1(Y \times S^1)$ is calculated by expressing $\pi_1(Y)$ as an HNN extension, which yields

$$\pi_1(Y \times S^1) = \langle x_i, y_i, t : tx_t t^{-1} = x_i, ty_t y^{-1} = y_i x_i \rangle \cong \mathbb{Z}_s.$$

Build the symplectic sum

$$F(n) := Z_{1,1-T_1=T^2} Y \times S^1.$$

Since $\pi_1(\Sigma_{1,1}) \cong \pi_1(\Sigma_{1,1-T_1}) \cong \{1\}$, it follows from Seifert-van Kampen Theorem that the generators $t$ and $s$ are killed during the gluing. The relations $t = 1 = s$ imply that $x_i = 1$ for all $i$. The Seifert-van Kampen Theorem implies that the group $\pi_1(F(n))$ is generated by the $y_i$’s with no relations among them in the group presentation. That is, $\pi_1(F(n))$ is a free group on $n$ generators.

\[\square\]

**Remark 29.** On the minimality of 6-manifolds. The notion of minimality for symplectic 6-manifolds (see [55] Definition 5.2) in generality is yet to be established (compare with the notion in dimension four [29]); this is an interesting problem (cf. [55]).

### 5.3. Proofs of Theorem [3] and Corollary [4]

The paper finishes with the proofs of the results promised in the Introduction [1]. The precise statements are the following.

**Theorem 30.** (cf. Theorem [3]). Let $G$ be a group with a presentation that consists of $g$ generators and $r$ relations. Let $e_i, \sigma_i, i = 1, 2$ be integers that satisfy $2e_i + 3\sigma_i \geq 0, e_i + \sigma_i \equiv 0 \pmod{4}$. There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$ with the following Chern numbers:

1. $c_1^3(W_0(G)) = 6 \cdot (3\sigma_1 + 2(e_1 + 4(g + r)) + 3\sigma_2 + 2e_2),$
   $c_1c_2(W_0(G)) = 6 \cdot ((e_1 + 4(g + r)) + \sigma_1 + e_2 + \sigma_2),$ and
   $c_3(W_0) = 2 \cdot ((e_1 + 4(g + r)) + e_2);$

2. $c_1^3(W_1(G)) = 0,$
   $c_1c_2(W_1(G)) = 0,$ and
   $c_3(W_1(G)) = 0;$
There is an isomorphism \( \pi_1(W_i) \cong G \) for \( i = 0, 1, 2 \).

Assume \( s_1, s_2 \geq 1 \), and \( n_1, n_2 \in \mathbb{N} \). There exist spin symplectic 6-manifolds \( Y_i(G) \) for \( i = 0, 1, 2 \) with \( \pi_1(Y(G)) \cong G \), and whose Chern numbers are

\[
\begin{align*}
(1) & \quad c_2^3(Y_0(G)) = 6 \cdot (8u_1 - 8 + 8(g + r) + 8n_2 - 8), \\
& \quad c_1 c_2(Y_0(G)) = 6 \cdot (4n_1 - 4 + 4(g + r) + 8s_1 + 4n_2 - 4 + 8s_2), \\
& \quad c_3(Y_0(G)) = 2 \cdot (4n_1 - 4 + 4(g + r) + 24s_1 + 4n_2 - 4 + 24s_2); \\
(2) & \quad c_2^3(Y_1(G)) = 0, \\
& \quad c_1 c_2(Y_1(G)) = 0, \\
& \quad c_3(Y_1(G)) = 0; \\
(3) & \quad c_2^3(Y_2(G)) = -6 \cdot (8u_1 - 8 + 8(g + r) + 8n_2 - 8) - 48, \\
& \quad c_1 c_2(Y_2(G)) = -6 \cdot (4n_1 - 4 + 4(g + r) + 8s_1 + 4n_2 - 4 + 8s_2) - 24, \\
& \quad c_3(Y_2(G)) = -2 \cdot (4n_1 - 4 + 4(g + r) + 24s_1 + 4n_2 - 4 + 24s_2) - 8.
\end{align*}
\]

**Proof.** The argument is verbatim to the one employed in proof of Theorem 25 modulo one of the building blocks. This new piece is going to be taken from the construction of Theorem 5. The changes go as follows. Take a minimal symplectic 4-manifold \( X_i \) from Proposition 15 and perturb the symplectic form so that the torus \( T_i \) becomes symplectic 22 Lemma 1.6). Build the 4-dimensional symplectic sum 22

\[ X_1(G) := BK \#_{T_i} X_i \]

where \( BK \) is the minimal symplectic 4-manifold with Euler characteristic given by \( c_2(BK) = 4(g + r) \), signature \( \sigma(BK) = 0 \), and fundamental group \( \pi_1(BK) \cong \langle g, \ldots, g, r \rangle \) described in Section 2.2. A straightforward computation yields \( c_2(X_1(G)) = c_2(X_i) + 4(g + r), \sigma(X_1(G)) = \sigma(X_i) \). The discussion in Section 2.3 implies \( \pi_1(X_1(G)) \cong G \), which explains our choice of notation.

We proceed to build the 6-dimensional examples. Let \( X_2 \) be a minimal symplectic 4-manifold of Proposition 15. We make use of the following prototypes of symplectic sums according to the item in the statement of the theorem.

\[
\begin{align*}
(1) & \quad W_0(G) := X_1(G) \times S^2 \#_{T_2 \times S^2} X_2 \times S^2 \\
(2) & \quad W_1(G) := X_1(G) \times T^2 \#_{T_2 \times T^2} X_2 \times T^2 \\
(3) & \quad W_2(G) := X_1(G) \times \Sigma_2 \# \Sigma_2 \times \Sigma_2 X_2 \times \Sigma_2.
\end{align*}
\]

A detailed description of the gluing maps chosen for the symplectic sums is given in Section 5.1. Varying our choices of manifolds \( X_1 \) and \( X_2 \) results in the realization of the Chern numbers claimed; under our notation \( c_2(X_i) = e_i, \sigma(X_i) = \sigma_i \) for \( i = 1, 2 \). The computations of the Chern numbers are straightforward, and follow from Lemma 22 and Lemma 24.

The construction of the manifolds \( Y_i(G) \) for \( i = 0, 1, 2 \) follows the same mechanism described above, using as building blocks the manifolds of Theorem 20 and the reference [40]. \( \square \)
We prove Corollary 4 (cf. Proof of Example 27).

**Corollary 31.** (cf. [43, Proposition 4.31].) Let \((a, b, c)\) be any triple of integers, and \(G\) a finitely presented group. There exists a symplectic 6-manifold \(X(G)\) with \(\pi_1(X(G)) \cong G\), and whose Chern numbers are 
\[
(c_1(X(G)), c_1c_2(X(G)), c_3(X(G))) = (2a, 24b, 2c).
\]

**Proof.** Let \(X(G) := X_1(G)\) be as in the proof of Theorem 30. There are two things to be checked. First, one needs to see that the result of blowing up at a point the manifolds constructed in Theorem 25 satisfy the hypothesis of Lemma 26. As a consequence, we are then able to conclude that all the possible values of \(c_1\) and \(c_3\) are realized by a symplectic 6-manifold with the given fundamental group. Second, it is needed to be seen that such manifolds fulfill all the possible values of \(c_1c_2\) (values of \(b\) in the statement of Lemma 26). Notice that by Lemma 23, the value \(c_1c_2\) is invariant under blow ups of points and under blow ups along surfaces. By Proposition 15 there exists a symplectic surface of genus two \(F\) inside the submanifold \(X(G) \times \{x\} \subset X(G) \times \Sigma_g\) for every 6-manifold obtained as a symplectic sum in Theorem 25. By [22], every manifold of Theorem 25 contains a symplectic surface of genus two. The manifold obtained by blowing up a point that is disjoint from the surface of genus two satisfies the hypothesis of Lemma 26.

We now mind the possible values of \(c_1c_2\). The claim is that all possible values are obtained by the manifolds constructed in Theorem 25. The case \(c_1c_2 = 0\) is clear. The claim is equivalent to finding \(e_i, \sigma_i, e'_i, \sigma'_i\) \(i = 1, 2\) that produce integers \(k_+\) and \(k_-\) that satisfy
\[
6 \cdot ((e_1 + 4(g + r)) + \sigma_1 + e_2 + \sigma_2) = 24k_+ \quad \text{and} \quad -6 \cdot ((e'_1 + 4(g + r)) + \sigma'_1 + e'_2 + \sigma'_2) = 24k_-.
\]

The numbers \(e_i, \sigma_i, e'_i, \sigma'_i\) \(i = 1, 2\) are as in the statement of Theorem 25 and they correspond to the Euler characteristic and signatures of minimal 4-manifolds of Proposition 15 used to construct \(X(G)\). Such manifolds can be done so that their characteristic numbers satisfy the equalities \((e_1 + \sigma_1 + e_2 + \sigma_2) = 4k_+ + 4(g + r)\) and \((e'_1 + \sigma'_1 + e'_2 + \sigma'_2) = 4k_- + 4(g + r)\). Thus, all possible values of \(c_1c_2\) are realized by manifolds from Theorem 25 and Theorem 30 The corollary now follows from Lemma 26.

\[\square\]

6. ACKNOWLEDGEMENTS

R. T. thanks Ronald J. Stern for suggesting useful references that motivated this paper, and Tian-Jun Li for sending him a copy of his preprint [27]. The hospitality of the math department at the University of Toronto during the production of part of this manuscript is much appreciated. R. T. gratefully acknowledges support from the Simons Foundation. J. Y. thanks the math department at the University of Hawaii at Manoa for its hospitality under which part of this paper was written. The authors wish to thank Robert E. Gompf for pointing out the preprint [2] to us, and John Etnyre and Joel Fine for useful observations that helped us improve the paper.

**References**

[1] A. Akhmedov, *Small Exotic 4-Manifolds*, Algebraic & Geometric Topology 8 (2008), no.3, 1781-1794.
[2] A. Akhmedov, Simply connected symplectic Calabi-Yau 6-manifolds, arXiv: 1107.2623v1 (2011).
[3] A. Akhmedov, S. Baldridge, R. I. Baykur, P. Kirk and B. D. Park, Simply connected minimal symplectic 4-manifolds with signature less than -1, J. Eur. Math. Soc. 12 (2010) 133-161.
[4] A. Akhmedov and B.D. Park, Exotic smooth structures on small 4-manifolds with odd signatures, Invent. Math. 181 (2010), 577-603.
[5] D. Auroux, S. K. Donaldson and L. Katzarkov, Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves, Math. Ann. 326 (2003) 185 - 203 MR1981618.
[6] S. Baldridge and P. Kirk, Symplectic 4-manifolds with arbitrary fundamental group near the Bogomolov-Miyaoka-Yau line, J. Symp. Geo. vol 4 - 4 (2006) 63 - 70.
[7] S. Baldridge and P. Kirk, On symplectic 4-manifolds with prescribed fundamental group, Comment. Math. Helv. 82 (2007), no. 4, 845 - 875, MR 2341842.
[8] S. Baldridge and P. Kirk, A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$, Geom. Topol. 12 (2008), 919-940.
[9] S. Baldridge and P. Kirk, Constructions of Small Symplectic 4-Manifolds using Luttinger Surgery, Journal of Differential Geometry, 82, No. 2 (2009), 155-205.
[10] S. Baldridge and P. Kirk, On symplectic 4-manifolds with prescribed fundamental group, Comment. Math. Helv. 82 (2007), no. 4, 845 - 875, MR 2341842.
[11] S. Baldridge and P. Kirk, Building symplectic manifolds using hyperbolic geometry, Proceedings of the 16th Gökova Geometry-Topology Conference (2009), 124-136.
[12] J. Fine and D. Panov, Hyperbolic geometry and non-Kähler manifolds with trivial canonical bundle, Geom. Topol. 14, no. 3 (2010), 1731-1763.
[13] R. Fintushel and R. Stern, K notions of almost complex manifolds, Proceedings of the AMS Vol. 129 No 12 (2001), 3749-3752.
[14] R. E. Gompf, Nuclei of elliptic surfaces, Topology 30 (1991), 479-511.
[15] R. E. Gompf, A new construction of symplectic manifolds, Ann. of Math. (2) 142 (1995), no. 3, 527-595.
[16] R. E. Gompf and A. I. Stipsicz, 4-Manifolds and Kirby Calculus. Graduate Studies in Mathematics, 20. AMS, Providence, RI, 1999.
[17] C. I. Ho and T. J. Li, Luttinger surgery and Kodaira dimension, to appear Asian J. Math. arXiv:1108.0479.
[31] D. Kotschick, *Minimizing Euler characteristics of symplectic four-manifolds*, Proc. Amer. Math. Soc. 134 (2006), no. 10, 3081 - 3083, MR 2231635.

[32] C. LeBrun, *Topology versus Chern numbers for complex 3-folds*, Pacific J. Math., 191 (1999), 123-131.

[33] T. J. Li, *The Kodaira dimension of symplectic 4-manifolds*, Clay Mathematics Institute Proceddings 4, Floer Homology, Gauge Theory and Low Dimensional Topology, CMI/AMS Book Series, (2006), 249 - 261.

[34] T. J. Li, *Symplectic 4-manifolds with Kodaira dimension zero*, J. Diff. Geom. 74, no 2, (2006), 321-352.

[35] T. J. Li, *Symplectic birational geometry*, New perspectives and challenges in symplectic field theory, 307326, CRM Proc. Lecture Notes, 49, arXiv: 0906.3265v1 (2009).

[36] K. M. Luttinger, *Lagrangian Tori in $\mathbb{R}^4$*, J. Diff. Geom. 42 (1995) 220-228.

[37] J. McCarthy and J. Wolfson, *Symplectic normal connect sum*, Topology 33 (1994), 729 - 764.

[38] J. W. Morgan, T. S. Mrowka, and Z. Szabó, *Product formulas along $T^3$ for Seiberg-Witten invariants*, Math. Res. Lett. 4 (1997) 915929 MR1492130.

[39] C. Okonek and A. Van de Ven, *Cubic forms and complex 3-folds*, L’Enseignement mathématique, 41 (1995), 297-333.

[40] B. D. Park and Z. Szabó, *The geography problem for irreducible spin four-manifolds*, Trans. Amer. Math. Soc. 352 (2000), 3639-3650.

[41] J. Park, *The geography of spin symplectic 4-manifolds*, Math. Z. 240 (2002), 405 - 421.

[42] J. Park, *The geography of symplectic 4-manifolds with an arbitrary fundamental group*, Proc. Amer. Math. Soc. 135 (2007), No. 7, 2301-2307.

[43] F. Pasquotto, *On the geography of symplectic manifolds*, PhD Thesis, Thomas Stieltjes Institute for Mathematics, Leiden University (2004).

[44] I. Smith, R. Thomas and S. T. Yau, *Symplectic conifold transitions*, J. Diff. Geom. 62 (2002), 209-242.

[45] Z. Szabó, *Simply-connected irreducible 4-manifolds with no symplectic structures*, Invent. Math. 132, (1998), 457-466.

[46] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1(6) (1994), 809 - 822. MR1306023.

[47] C. H. Taubes, *SW $\Rightarrow$ Gr*, *From the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996), 845-918.

[48] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (2) (1976), 467-468.

[49] R. Torres, *Geography and botany of irreducible symplectic 4-manifolds with abelian fundamental group*, preprint (2009).

[50] R. Torres, *Irreducible 4-manifolds with abelian non-cyclic fundamental group of small rank*, Top. Appl. 57 (2010), 831 - 838.

[51] M. Usher, *Minimality and symplectic sums*, Int. Math. Res. Not. (2006) article ID 49857.

[52] J. Yazinski, *A new bound on the size of symplectic 4-manifolds with prescribed fundamental group*, to appear J. of Symplectic Geometry (2010).

Mathematical Institute - University of Oxford, 24-29 St Giles’, Oxford, OX1 3LB, England
E-mail address: torres@maths.ox.ac.uk

McMaster University - Dept. Mathematics and Statistics, 1280 Main Street West, L8S 4K1, Hamilton, ON.
E-mail address: yazinski@math.mcmaster.ca