Three dimensional strings as collective coordinates of four dimensional non-perturbative quantum gravity

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ABSTRACT

A string theory in 3 euclidean spacetime dimensions is found to describe the semiclassical behavior of a certain exact physical state of quantum general relativity in 4 dimensions. Both the worldsheet and the three dimensional metric emerge as collective coordinates that describe a sector of the solution space of quantum general relativity. Additional collective coordinates exist which are interpreted as worldsheet degrees of freedom. The construction may be extended to the case in which the Kalb-Ramond field is included in the non-perturbative dynamics. It is possible that this mechanism is the inverse of the strong coupling limit by which some $D$ dimensional string theories are conjectured to give rise to $D+1$ dimensional field theories.

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1 Introduction

In the last years quantum gravity has progressed dramatically along two fronts. At the perturbative level string theory has been found to give a class of consistent descriptions of the interaction of gravitons with matter. Recent results strongly suggest that this description may even explain the thermodynamics of black holes. There are also many interesting results which suggest the existence of a single non-perturbative theory from which all these perturbative descriptions arise. However, the proper setting for this non-perturbative string theory is still unknown, despite a number of interesting proposals.

At the same time, a non-perturbative description has grown up for four dimensional quantum gravity based on the Ashtekar-Sen variables and their variants, which comes from considering spaces of states constructed from Wilson loops of the spacetime connection. Here there have been successes as well. There are robust predictions about discreteness of geometrical quantities at Planck scales. The spectra of the corresponding operators label a space of diffeomorphism invariant states built from spin networks. Moreover, recently it has been shown that these results may be couched in a completely rigorous formulation, giving non-perturbative quantum general relativity a mathematical status on the level of constructive quantum field theory.

However, there remain open problems, associated with the form of the Hamiltonian constraint. While the kinematical framework of the theory, at the level of spatially diffeomorphism invariant states, seems robust, there are many possible realizations of the Hamiltonian constraint as a quantum operator. These arise from different regularization procedures. Unfortunately, more than one of these define theories that lack massless gravitons and the long ranged order required to have a limit in which general relativity is recovered. A new principle is clearly needed to pick out a form of the Hamiltonian constraint that will naturally describe a critical point at which a good classical limit may emerge.

It is natural then to wonder whether some marriage of perturbative string theory and non-perturbative quantum gravity might resolve their common problems and lead to a good quantum theory of gravity. Spin networks and the associated observables (or some appropriate generalization of them)
might provide a language for a non-perturbative formulation of string theory, while string theory in turn might suggest new forms for the non-perturbative dynamics that escape the problems of quantum general relativity.

There is, indeed, a strong reason to suppose that non-perturbative quantum gravity must involve string theory if it is to succeed. Suppose that a form of non-perturbative quantum general relativity does exist which has a good classical limit. We should be able to use it to define a good, Poincare invariant, perturbative description of the interaction of gravitons on a background Minkowski spacetime. But after extensive search, the only such consistent descriptions that are known are string theories. So, unless the search for perturbative quantum gravity has missed something, the perturbation theory arising from the semiclassical limit of any non-perturbative quantum theory of gravity must be a string theory.

But there are no consistent perturbative string theories whose low energy limit describes only the graviton in four spacetime dimensions. Therefore, it seems a consistent perturbative description must have additional degrees of freedom. They might arise in three ways. It might be necessary to add supersymmetry and additional fields to the non-perturbative description for it to yield a good classical limit. Alternatively, it may be necessary to apply the technology of non-perturbative quantum gravity to 10 or 11 dimensional theories. Or it might be that additional degrees of freedom emerge directly from the non-perturbative physics.

We may note that it is not completely crazy that a string theory may emerge from a non-perturbative description of a theory based on Wilson loops. There are suggestions that non-critical strings may be associated with QCD. Further, there is the Klebanov-Susskind construction which suggests that critical string theory could emerge from a discrete theory based on Wilson loops in a certain limit. Non-perturbative quantum gravity gives precisely such a discrete starting point based on Wilson loops.

More generally, we may expect that to understand the emergence of classical behavior from the non-perturbative dynamics of spin networks, we will have to discover collective coordinates that define the large scale behavior of the physical states. To identify the collective coordinates we must study the systematics of generic spin network states that solve the Hamiltonian constraint. This is essentially what is done in this paper. The result is the identification of a set of collective coordinates that describe the solution space of a certain sector of four dimensional non-perturbative quantum general relativity. A subset of these collective coordinates label imbeddings of two dimensional surfaces in three dimensional metric manifolds.
Using this result we can then construct a particular physical state which has a semiclassical limit which describes the propagation of a string in a three dimensional classical spacetime background. The equations of motion of the string in this three dimensional background turn out to follow from the condition of stationary phase applied to this state. Furthermore, there arise other collective coordinates which describe additional degrees of freedom that propagate on the surfaces. The identification of these degrees of freedom, perhaps in terms of conformal fields on the surfaces, has not yet been carried out. Interestingly enough, the physics of these worldsheet degrees of freedom depends on the form of the Hamiltonian constraint.

The results described here thus represent a first step in the analysis of the collective coordinates of quantum general relativity. Even so, their form is interesting, as it is reminiscent of recent results that suggest that string theories in $D$ dimensions may have strong coupling limits which are described by $D + 1$ dimensional field theories\cite{4,26}. It is then possible that the mechanism described here implements the inverse process, by which a $D$ dimensional string theory is recovered from a weak coupling limit of a $D + 1$ dimensional nonperturbative quantum field theory.

In the next section I describe how two dimensional surfaces emerge as collective coordinates of quantum general relativity. Section 3 introduces an exact solution which gives rise to string theory in the semiclassical limit, which is the subject of sections 4 and 5. The Kalb-Ramond field is brought into the formalism in section 6. In the concluding section the question of whether these results can be extended to supergravity in four and higher dimensions is discussed. The implications of these results are discussed in the conclusions after which some technical issues associated with the choice of Hamiltonian constraint used here are discussed in the appendix. A new proposal for the form of the Hamiltonian constraint is described there as well.

2 Two surfaces as collective degrees of freedom of quantum general relativity

It is not difficult to see why two dimensional surfaces might naturally emerge from the solutions to the constraints of quantum general relativity. The key points depend only on the basic features of non-perturbative quantum general relativity. The quantum states of the theory have a basis which is diffeomorphism equivalence classes of spin networks\cite{30}. A spin network is a
graph, or more properly the diffeomorphism class of a graph\footnote{Unless otherwise specified all references to spin networks in this paper are to diffeomorphism classes. These are sometimes denoted \( \{ \Gamma \} \).}, whose edges are labeled by spins such that the laws of addition of angular momentum are satisfied at the vertices\footnote{If the node has more than three incident edges we decompose it into a product of trivalent nodes as described in \cite{24} such that there is an internal edge that connects the}. The dynamics is enforced by the Hamiltonian constraint, which is known to act only at the nodes of the spin networks\cite{11,27,28,34,35}. Given a certain definition of that operator, its action is intrinsically planar. This is because the action of the constraint on a spin network is to extend the network by creating new trivalent vertices in a particular way. This planar action of the constraint means that there are classes of solutions constructed from infinite superpositions of spin networks each of which span a given two dimensional surface.

I will assume that the action of the Hamiltonian constraint is according to a set of simple rules. These may be derived from one particular regularization\cite{27,28,29} but this is not unique; there are other regularizations that will also lead to them. It is also possible to modify the rules I give slightly while preserving the most interesting results. So as not to bore non-experts, these and other technical issues associated with the different regularizations of the hamiltonian constraint are treated in the appendix.

\begin{itemize}
  \item \textbf{R1} The hamiltonian constraint \( \mathcal{C}(N) \) acts at each node of a spin network. The action at each node is a sum of terms, one for every pair of its non-collinear edges. In each term two new nodes are added, each on one of the two edges. These nodes are placed adjacent to the node at which the constraint acts. The two new nodes are then joined by a new edge with color 1. (We use as is traditional with spin networks a labeling in terms of twice the spin, so all labels are integer multiples of \( \hbar/2 \).) Thus the two nodes created by the action of the constraint are trivalent. The diffeomorphism class of the edge is chosen so that the triangle it forms with the two existing edges links no other edge of the graph. For each such term a linear sum of four new spin networks is then created, in each of which the colors of the edges joining the old node to the new nodes are raised or lowered by 1. The action of the Hamiltonian constraint is to produce these four new spin networks for every pair of non-collinear edges at every node of the graph, multiplied by coefficients, \( A_{\pm \pm}^\pm(i,j,k) \), where \( i \) and \( j \) are the spins of the two edges acted upon, and \( k \) is the third spin at the node.\end{itemize}
formula for the coefficient is given by eq. (25) of [28], but its exact form will not be relevant here. The two new vertices that are created are trivalent, and consist of the line of the new edge joining one of the old edges.

- **R2** The action of the Hamiltonian constraint on each node is multiplied by an arbitrary factor $N$ at each distinct node. This means that the solutions must be found independently for the action at each node. However, as we are giving the action on diffeomorphism invariant states we must give a rule that tells how distinct nodes are to be identified. This must respect diffeomorphism invariance, but reproduce the results of the standard action. The following rule, which makes use of the graph recognition problem [24], works

We multiply the action of the operator on each node $v$ by numbers $N(v)$, which are assumed to be assigned independently to all nodes of all networks, subject to the following restriction. When it is the case that a network $\Gamma$ may be identified as a sub-network of $\Gamma'$, such that a given vertex $v$ of $\Gamma$ is identified uniquely with a vertex $v'$ of $\Gamma'$ then $N(v)$ in the action on $\Gamma$ must be taken equal to $N(v')$ of $\Gamma'$.

- **R3** The Hamiltonian constraint acts thus at every pair of non-collinear tangents of every node, bivalent trivalent or higher, independent of whether the tangent vectors of the incident edges span the tangent space at the node.

We will shortly find reason to modify this last rule slightly, but it is useful to begin the discussion with this form.

### 2.1 How collective coordinates arise from the dynamics of spin networks

In order to extract the collective coordinates of a theory we must investigate the systematics of its solutions. The generic solution to the Hamiltonian constraint acts thus at every pair of non-collinear tangents of every node, bivalent trivalent or higher, independent of whether the tangent vectors of the incident edges span the tangent space at the node. However, as we are interested in the classical limit, and hence large complex graphs, this is sufficient.
Hamiltonian constraint of quantum general relativity is constructed from an infinite superposition of spin network states. To understand the properties of the solution, we may investigate the trajectories generated by the Hamiltonian constraint in the space of spin network states, $S$. To do this let us imagine that we begin with some initial network $|\Gamma\rangle$ and act on it with an infinite number of iterations of the Hamiltonian constraint, with arbitrary values of the lapses $N$, constrained by $R^2$. The result is a subspace $\mathcal{F}_\Gamma$. We would like to investigate some general properties of these subspaces, as solutions are going to be constructed from superpositions of states inside each of them.

The most important observation is that with the rules given above (and in most definitions of the Hamiltonian constraint in the literature\cite{8, 10, 27, 28, 29, 34, 35}) all trajectories converge on a particular subspace, $S^3$, which is spanned by spin networks with only trivalent nodes. This is because the Hamiltonian constraint in all these forms creates only trivalent vertices. Thus, $S^3$ is an attractor for the orbits of the Hamiltonian constraint.

For this reason all considerations of this paper will be restricted to either $S^3$ or $S^{2,3}$, the subspace consisting of networks with only trivalent and bivalent nodes. One may regard this as a simplifying assumption, but it may also be that the microscopic theory restricted to the trivalent sector may be sufficient to define a good quantum theory of gravity. Some reasons to expect this are described in the appendix\cite{5}.

A second key observation is that restricted to the trivalent sector the action of the Hamiltonian constraint defined by the above rules will produce graphs which are topologically planar. This means that for every two dimensional surface $S$ there is a collection of orbits of the Hamiltonian constraint such that each initial network $\Gamma$ and, every spin network in its orbit, is a skeletonization of $S$. This is how two dimensional surfaces arise as collective coordinates for solutions of the Hamiltonian constraint.

Once this is seen, one can observe also that the general orbit may be described in terms of a two dimensional surface imbedded in a link consisting of some number of non-intersecting loops (as these do not evolve\cite{8}). This will enable us to define a semiclassical limit in which the surface is embedded in a three dimensional metric, defined by the link.

We now proceed to describe how these results are obtained.

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4 Bivalent nodes are often called kinks

5 To the obvious objection that this sector is degenerate because all states have zero volume one may reply that once the theory is quantum deformed\cite{36}, this is no longer the case\cite{37}. This is discussed further below.
2.2 States associated with open surfaces

We begin with the simplest example of a graph that spans an open surface. Let us take for the initial spin network $\Gamma$ an unknotted triangle whose three edges each have spin $j$.

We will call the associated diffeomorphism invariant state $|\Delta, j>$. Let us describe the subspace $\mathcal{F}_{\Delta,j}$ generated by repeated application of the Hamiltonian constraint. The first time it acts, the Hamiltonian constraint will produce 12 planar graphs, each of which has one corner of the triangle bisected. ($12 = 3$ corners times four graphs made at each corner.) If we act again with the Hamiltonian constraint we now produce $28 = 7 \times 4$ new graphs, as there are now seven places where the Hamiltonian constraint will act. If we continue to iterate the action we will construct an infinite family of planar spin networks, whose boundaries are each the original triangle. These make a subspace of the diffeomorphism invariant state space associated to the original triangle, which we call $\mathcal{F}_{\Delta,j}$. This is a graded space, where each subspace $\mathcal{F}_{\Delta,j}^n$ consists of graphs produced by $n$ actions of the Hamiltonian constraint. We will label the elements in each graded subspace by an arbitrary index $\alpha$, so the states in $\mathcal{F}_{\Delta,j}$ will be called $|\Delta, j; n, \alpha>$. It is clear that we should find a set of solutions to the constraints inside of $\mathcal{F}_{\Delta,j}$. There are a set of matrices $\mathcal{M}_{\alpha\beta}^n$ which are defined by

$$C(N)|\Delta, j; n, \alpha>= \sum_{\hat{n} \in |\Delta,j;n,\alpha>} \sum_{\beta} N(\hat{n}) \mathcal{M}_{\alpha\beta}^n |\Delta, j; n+1, \beta>$$

where $\beta$ includes also the fact that we sum over the raisings and lowerings of the spins of the edges and the matrix elements contain the coefficients $A^{\pm \nu'}(j, k)$.

To find non-trivial solutions we will have to use a Hermitian ordered operator corresponding to the constraint. This means we are interested in the constraint

$$\mathcal{H}(N) = C(N) + C^\dagger(N)$$

where the hermitian conjugate is taken in the inner product defined for the Euclidean theory in [13, 17, 38]. In this inner product, the distinct diffeomorphism classes of spin networks comprise an orthogonal basis.

To define the Hermitian conjugate we also have to know what we mean by $N(n)$, where $n$ is the position of a node. We will assume that this is defined on diffeomorphism invariant states according to rule R2.

With this provision, the adjoint of the Hamiltonian constraint acts in the following way. It searches the graph for nodes which have adjacent to them
two vertices connected by an edge with spin 1, of the type created by the Hamiltonian constraint. This means that these vertices have two collinear edges which differ by ± one unit of spin. The operator than removes the edge, sets the spins on the two edges to agree with those outwards of the two vertices we have just removed and multiples by an associated coefficient $B^\pm \epsilon(j, k)$. It then multiples by $N(\hat{n})$ at that vertex.

In equations, there are matrices $N^n_{\alpha \beta}$ such that

$$C^\dagger(N)|\Delta, j; n, \alpha > = \sum_{\hat{n} \in |\Delta, j; n, \alpha >} \sum_{\beta} N(\hat{n}) N^n_{\alpha \beta} |\Delta, j; n - 1, \beta >$$

(3)

We may note that acting on any state in $F_{\Delta, j}$ the adjoint $C^\dagger(N)$ produces a state which is also in $F_{\Delta, j}$. Given the standard inner product in which the distinct spin networks are orthogonal this is true for any $F_{\Gamma}$ as long as

$$C^\dagger(N)|\Gamma > = 0$$

(4)

We may then construct solutions inside of $F_{\Delta, j}$ by expanding them as

$$|C > = \sum_n \sum_{\alpha} C(n, \alpha)|\Delta, j; n, \alpha >$$

(5)

We look for solutions such that

$$\mathcal{H}(N)|C >= \left(C(N) + C^\dagger(N)\right) |C >= 0$$

(6)

To find these we note that the matrices $M^n_{\alpha \beta}$ and $N^n_{\alpha \beta}$ may be described in more detail in terms of the nodes. Each nonzero element of $M^n_{\alpha \beta}$ comes from an $\alpha$ and a $\beta$ where $|\Delta, j; n + 1, \beta >$ is gotten from $|\Delta, j; n, \alpha >$ by dressing some node $\hat{n} \in |\Delta, j; n, \alpha >$. If we label arbitrarily the dressings by an index $d$ (which include the raising and lowerings of spins) then for each nonzero element there is a node $\hat{n} \in |\Delta, j; n, \alpha >$ and a dressing $d$ such that $\beta = f(\alpha, \hat{n}, d)$. Thus, we can write

$$C(N)|\Delta, j; n, \alpha > = \sum_{\hat{n} \in |\Delta, j; n, \alpha >} \sum_d N(\hat{n}) M^n_{\alpha \beta(\alpha, \hat{n}, d)} |\Delta, j; n + 1, \beta(\alpha, \hat{n}, d) >$$

(7)

Similarly, in $\mathcal{F}$ the sum over $\beta$ is nonvanishing only over those $\beta = g(\alpha, \hat{n})$ gotten from $\alpha$ by removing edges and pairs of nodes according to the procedure just described. Note that there is no dressing parameter as there is always a unique way to undress a node, because in a trivalent graph there
is always at most one unique edge whose two ends are adjacent to a node. Thus,

$$C^\dagger(N)|\Delta, j; n, \alpha> = \sum_{\hat{n}\in|\Delta, j; n, \alpha>} N(\hat{n})N_{\alpha_\beta(\alpha, \hat{n})}^n|\Delta, j; n - 1, \beta(\alpha, \hat{n})>$$

(8)

Hence, we have then to solve the set of equations,

$$0 = \sum_n \sum_\beta |\Delta, j; n, \beta> \left(\sum_\alpha \sum_{\hat{n}\in|\Delta, j; n, \alpha>} \sum_d C(n - 1, \alpha)N(\hat{n})M_{\alpha_\beta}^{n-1}\delta_{\beta f(\alpha, \hat{n}, d)}
+ \sum_\alpha \sum_{\hat{n}\in|\Delta, j; n, \alpha>} C(n + 1, \alpha)N(\hat{n})N_{\alpha_\beta}^{n+1}\delta_{\beta g(\alpha, \hat{n})}\right)$$

(9)

This gives us an independent equation to solve at every node of $\beta$. To see this note that in the first sum, $|\Delta, j; n - 1, \alpha>$ is necessarily a subgraph of $|\Delta, j; n, \beta>$ (to say this we allow graphs to have a spin zero edge). The sum over nodes can then be extended to a sum over nodes of $\beta$. We need only add the notion that the function $f(\alpha, \hat{n}, d)$ returns a trivial graph in the case that $\hat{n}$ is not a node of $\alpha$. In the second sum the situation is reversed, $|\Delta, j; n - 1, \beta>$ is a subgraph of $|\Delta, j; n, \alpha>$. But a node that is in $\alpha$ but not in $\beta$ would never contribute to the sum, because the node refers to one that is undressed, which means it remains. (The nodes removed are two that are adjacent to it.) Hence we can extract the sum over nodes, so we have

$$0 = \sum_n \sum_\beta |\Delta, j; n, \beta> \sum_{\hat{n}\in|\Delta, j; n, \beta>} N(\hat{n})\left(\sum_\alpha \sum_d C(n - 1, \alpha)M_{\alpha_\beta}^{n-1}\delta_{\beta f(\alpha, \hat{n}, d)}
+ \sum_\alpha C(n + 1, \alpha)N_{\alpha_\beta}^{n+1}\delta_{\beta g(\alpha, \hat{n})}\right)$$

(10)

Hence, for every $\beta$ and $\hat{n} \in |\Delta, j; n, \beta>$ we must solve,

$$0 = \left(\sum_\alpha \sum_d C(n - 1, \alpha)M_{\alpha_\beta}^{n-1}\delta_{\beta f(\alpha, \hat{n}, d)} + \sum_\alpha C(n + 1, \alpha)N_{\alpha_\beta}^{n+1}\delta_{\beta g(\alpha, \hat{n})}\right)$$

(11)

This gives us solutions, parameterized by the $C(n, \alpha)$. We see that taking the Hermitian part is necessary to get solutions, what is happening is that the amplitude of a graph created by the action of $C$ by adding to its subgraphs
must be balanced by the amplitude for $\mathcal{C}^\dagger$ to create the same graph by
removing an edge from a graph it is a subgraph of. Nothing is known presently about the space of solutions or its parameterization. I will therefor just parameterize them by a symbol $Z$, so the solutions to (given by (3) are called $|\Delta, j; Z \rangle$. Thus, we have a space of states associated to a planar
surface, bounded by a triangle, each of which presumably consists of an
infinite sum over spin network states that share the same triangle as its boundary. (Note however that in every state in this sum the spins on some part of the boundary will differ from the original assignment.) We shall call this the space of physical states $\mathcal{P}_{\Delta, j} \in \mathcal{F}_{\Delta, j}$.

One property of the solutions is clear by inspection. The linear equations that must be solved separate into three sets, each associated with the
dressings of one of the nodes of the triangle. This is an aspect of the problem of confined correlations, which is discussed at length in [24]. Thus, each
solution may be visualized in the following way. Consider three disks, labeled
by $\alpha = 1, 2, 3$, each of which is joined to the two others by an edge labeled
by $j$. Each of the disks stands for an infinite superposition of spin networks
which span the disk topologically, each with two external edges with spin $j$. Each disk may be labeled with a parameter $Z_\alpha$ that parameterizes the
solution in the neighborhood of the $\alpha$'th node of the initial triangle. These $Z_\alpha$ may be considered to be additional collective coordinates that describe
the solutions.

It is clear that this procedure can be immediately generalized to con-
struct the spaces $\mathcal{F}_\Gamma$ that are generated by acting on any initial trivalent
spin network $\Gamma$, as long as (4) is satisfied. We now proceed to discuss the
more interesting cases that are associated with closed surfaces.

### 2.3 States associated to closed surfaces

It is clear that a similar construction also lets us associate spaces of states
with closed surfaces. For example, we may construct planar states with the topology of $S^2$. To make the simplest example, consider first a simple tetrahedron, $\mathcal{T}$ whose edges are dressed with spins $j_i$, $i = 1, ..., 6$. We will
call the associated diffeomorphism invariant spin network state $|\mathcal{T}, j_i \rangle$. Associated to it we have the infinite dimensional space of states that we may
get by acting an arbitrary number of times with the Hamiltonian constraint,
which we call $\mathcal{F}_{\mathcal{T}, j_i}$. An element of which will be labeled $|\mathcal{T}, j_i, n, \alpha \rangle$ as before. These states have properties similar to the ones that we described
that dress triangles. In each state each triangle is dressed as before, but
along each edge one finds various patterns of edges that participate in the
dressings of one or the other of the edges it bounds.

In just the same way we can solve the Hamiltonian constraints inside
$\mathcal{F}_{T,j_i}$, giving us a space of physical states $|T, j_i, Z>$ which each consists of
linear combinations of states that are topologically $S^2$. In this same way,
given any initial trivalent spin network $\Gamma$ that satisfies $\mathcal{F}_\Gamma$ we can construct a
space of physical states $\mathcal{P}_\Gamma \subset \mathcal{F}_\Gamma$.

As in the case of the triangle, the solutions are independent in the neighbor-
hoods of each node of $\Gamma$, because the hamiltonian constraint cannot is
block diagonal and does not mix labelings of the edges that it creates outside
of these neighborhoods. Thus the solutions can be pictured as a collection
of disks, each of which has two or three external edges, tied together with
the topology of the initial network $\Gamma$. Essentially what the Hamiltonian
constraint has done is to grow each node of $\Gamma$ into one of these disks. The
disks are connected by edges with the same spins as in $\Gamma$ and they are la-
beled by parameters $Z_\alpha$ which are collective coordinates that parameterize
the solutions.

We can then extend the disks, joining them in each $n$-gon of $\Gamma$ until they
form a continuous surface, $S$. This surface will in general be self-intersecting,
and it may be open or closed.

Inversely, we may begin with a surface $S$ and consider the state spaces
$\mathcal{F}_S = \oplus \mathcal{F}_\Gamma$ such that $\Gamma$ is a skeletonization of $S$ satisfying $\mathcal{F}_\Gamma$ and the corre-
spending spaces of physical states, $\mathcal{P}_S = \oplus \mathcal{P}_\Gamma$. Associated to each of these
there must as well be an algebra of physical observables $\mathcal{A}_S$, which is a sub-
algebra of observables of quantum gravity and which serve to distinguish
all the states in $\mathcal{P}_S$. This algebra describes a quantum field theory associ-
ated with the surface $S$, whose degrees of freedom include the collective
coordinates $Z$.

There is also an additional degree of freedom, associated with the choice
of the initial $\Gamma$ that spans $S$. Thus, the collective coordinates that describe
the solutions we have constructed are the triples $(S, \Gamma, Z)$.

Before turning to a study of the classical limit, it is appropriate to make
several comments.

First, with the rules given, the collective coordinates $Z_\alpha$ are non-propagating
as they are describe degrees of freedom associated to each of the disks got-
ten by solving the contraints in a neighborhood of each vertex of the initial
network $\Gamma$. This means they cannot be described by a massless field theory
on $S$. It would be much preferable to eliminate this problem so that the
collective coordinates $Z$ described degrees of freedom which could propagate
over the whole surface, as this means it might be possible to represent them in terms of a conformal field theory. A set of rules that accomplishes this is described in the appendix.

Second, it is easy to modify the theory so that there are no solutions associated to open surfaces. We need only modify assumption \textbf{R3} to read:

- \textbf{R3'} The Hamiltonian constraint acts thus at every pair of non-collinear tangents of every node, trivalent or higher, independent of whether the tangent vectors of the incident edges span the tangent space at the node. \textit{It does not act at bivalent vertices.}

How this modification may be accomplished is discussed in the appendix. I will assume from now on that a regularization of the Hamiltonian constraint satisfying \textbf{R3'} is made, so the surfaces are all closed. As a result, from now on we restrict attention to the space $S^3$ whose states have only trivalent vertices joining smooth edges.

Third, if we adopt the rule \textbf{R3'} it is straightforward to describe the general solution in the trivalent sector $S^3$. The reason is that any trivalent spin network $\rho$ can be decomposed into a set of connected trivalent graphs which we will call $\Gamma$ and a link, which consists of a set of non-intersecting loops, which we will call $\gamma$. In what follows, these symbols will stand for the pairs of the graphs and the spins that label their edges. When we need to refer to the spins on the components of $\gamma$ they will be called $j_i$. Note also that as we are working in $S^3$ there are no kinks.

Now, let us assume that we have an initial network $\rho = \{\Gamma, \gamma\}$ and let us consider the corresponding space $F_\rho$. The key point is that the links do not evolve, so that the general solution can be constructed from the solutions we have already described by decorating the initial trivalent component $\Gamma$ with non-intersecting loops.

Let us then consider a general element of $F_\rho$. It consists of states where the network $\Gamma$ has been dressed, while the link $\gamma$ remains unchanged. As a result, for every state $|\Gamma, n, \alpha \rangle \in F_\Gamma$ and every way to extend $\Gamma$ to $\rho$ by adding a link $\gamma$, there is a state $|\{\Gamma, \gamma\}, n, \alpha \rangle \in F_{\{\Gamma, \gamma\}}$. The correspondence extends to the physical states as well, so that for every solution in $P_\Gamma$ given by

\[
|\Gamma, Z \rangle = \sum_n \sum_\alpha C^Z[n, \alpha]|\Gamma, n, \alpha \rangle
\]

labeled by $Z$ and every way to add a link $\gamma$, there is a new solution

\[
|\{\Gamma, \gamma\}, Z \rangle = \sum_n \sum_\alpha C^Z[n, \alpha]|\{\Gamma, \gamma\}, n, \alpha \rangle.
\]
with the same coefficients \( C^Z[n, \alpha] \). The fact that the dressing procedure can be visualized as spreading each node out into a disk guarantees that in each term in the sum \( R \) the components of \( \gamma \) pass in between the disks, and not among the new edges. As a result, the diffeomorphism classes \( \{ \Gamma, \gamma \} \) are relevant also for the whole space of solutions \( \mathcal{P}_\rho \subset \mathcal{F}_\rho \).

A final comment is that the surface \( S \) may be self-intersecting. This is allowed in general, as one may begin with a trivalent graph which is not planar, in which case its decorations will describe self-intersecting surfaces. However, the same reasoning we have just given shows that because the dressing procedure consists of the spreading out of trivalent nodes into disks, there is always left a hole through which other elements of the network may pass. Thus, there is no obstruction to constructing the space of solutions \( \mathcal{F}_\Gamma \) associated to a trivalent network whose spanning surface is self-intersecting.

### 3 The string state

Now that we have identified a set of collective coordinates that involve surfaces and degrees of freedom defined on them we may see if we can use them to describe quantum states that might be interpreted in terms of strings or membranes.

To do this we may make use of a particularly natural observable which is defined on this space of states. This is the area of a non-self-intersecting surface \( S \) embedded in a link \( \gamma \) whose components \( \gamma_i \) are colored by spins \( j_i \). It is defined by \( [13, 19] \)

\[
\mathcal{A}[S, j_i] = l_{Planck}^2 \int \sum_i \text{Int}^+[S, \gamma_i] \sqrt{j_i (j_i + 1)} |\gamma_j, j_i >
\]

We may note that this definition makes use of the unoriented intersection number \( \text{Int}^+[S, \gamma_i] \), of the surface and the link, which is always positive, and the \( j_i \) in \( \sqrt{j_i (j_i + 1)} \) is the spin of the component of the link at the intersection.

We would like to make use of this definition to define an area associated with a generic physical quantum state \( |\{ \Gamma, \gamma \}, Z > \). This can be done in the case that the initial spin network \( \Gamma \) spans a surface \( S \) which is non-intersecting. In this case all the contributions to the area of \( S \) come from the links \( \gamma \), which are not part of the linear combinations of states evolved from \( \Gamma \) by the action of \( \mathcal{H}(N) \), out of which the solutions are built.

To define the area we must take into account the fact that with the rules as defined here the spin networks that make up the solutions do not
span the whole surface $S$, but only the disks we described in the previous section. This means that there is an ambiguity in how the linking among the spinnets in the diffeomorphism classes $\{(\Gamma, \gamma)\}$ may be considered to intersect the surfaces $S$. To resolve this we will define the intersections of the surface $S$ with the components of the link $\gamma$ in terms of the linking of $\gamma$ and the initial network $\Gamma$. To do this, let us divide the surface $S$ spanned by $\Gamma$ into polygons, $S_\alpha$ whose boundaries are the edges of $\Gamma$. We then define the unoriented intersection number (which is what is required for the area formula) as,

$$\text{Int}^+[S, \gamma] \equiv \sum_\alpha |L[\partial S_\alpha, \gamma]|$$

(15)

where $L[\partial S_\alpha, \gamma]$ is the standard Gauss linking number. This definition tells us that intersections of the link $\gamma$ with the surface $S$ are counted only if they are meaningful in terms of the diffeomorphism classes of the states that make up the solutions.

Given this, we may define an operator $\hat{A}$ which measures the area of physical states $|\{\Gamma, \gamma\}, Z>$. We define it such that

$$\hat{A}|\{\Gamma, \gamma\}, Z> = 0$$

(16)

when the surface $S$ spanned by $\Gamma$ is self-intersecting and, otherwise,

$$\hat{A}|\{\Gamma, \gamma\}, Z> = a[\{\Gamma, \gamma\}][\{\Gamma, \gamma\}, Z>$$

(17)

where, the area $a[\{\Gamma, \gamma\}]$ is defined by

$$a[\{\Gamma, \gamma\}] = \sum_\alpha \sum_i |L[\partial S_\alpha, \gamma_i]| t^2_{\text{Planck}} \sqrt{j_i(j_i + 1)}.$$

(18)

With these definitions, we may now define a special physical state, $\Psi_{\text{String}}[\rho]$ in the spin network basis. Because of the general form of solutions we found in the last section, any solution has the form

$$\Psi[\rho] = \Psi(\{\Gamma, \gamma\}, Z).$$

(19)

We choose to define $\Psi_{\text{String}}[\rho]$ so that

$$\Psi_{\text{String}}[\{\Gamma, \gamma\}, Z] = 0$$

(20)

This definition has the advantage that the subtleties associated with the presence of nodes in surfaces are not relevant.
whenever \( \Gamma \) describes a self-intersecting surface and, otherwise, 

\[
\Psi_{\text{String}}(\{\Gamma, \gamma\}, Z) = e^{ig^2a[\{\Gamma, \gamma\}]/l^2_{\text{Planck}}},
\]

(21)

Note that the state is parameterized by a dimensionless coupling constant \( g \), which introduces, as we will see, the string tension

\[
\alpha' = \frac{l^2_{\text{Planck}}}{g^2}
\]

(22)

4 The classical limit and the emergence of a string theory

We now consider what can be said about the classical limit of the state \( \Psi_{\text{String}} \) we have just defined. To do this we will make use of the dependence of the state on the non-intersecting link \( \gamma \). We may recall that there are special links \( \gamma \) that may be associated to classical slowly varying three metrics, using the weave construction [18].

It is easiest if we think of the state in the form \( \Psi_{\text{String}} \) as a function on spin networks, and not just on their diffeomorphism invariant classes. (We are free to do this as the state exists in both the kinematical state space and the space of diffeomorphism invariant states.)

We will use the definition of weaves given in [18] in which the correspondence to a classical spatial metric \( q_{ab} \) is given in terms of the areas of large surfaces. However, before going any further we must discuss two issues concerning the interpretation of weave states.

The first issue arises because the interpretation of non-intersecting weaves [18] is problematic, because these states have the special property that they are simultaneously in the kernel of the Hamiltonian constraint and the volume operator. This implies

\[
\hat{K}|\gamma > = [\hat{H}(N), \hat{V}]|\gamma > = 0
\]

(23)

where \( \mathcal{K} = \int d^3x \sqrt{q}K \), where \( K \) is the trace of the extrinsic curvature. Still, stranger, one can show using Thiemann’s methods [34] that \( \hat{T} \) implies

\[
\hat{T}(x)|\gamma > = 0
\]

(24)

\( ^7 \)This is true for all known forms of the volume operator, including the \( q \)-deformed operator.

\( ^8 \)This is where the restriction to states without kinks is used.
where $T(x)$ is the kinetic term in the Minkowskian signature Hamiltonian constraint which is quadratic in the extrinsic curvature $k_{ai}(x)$. This implies that the state is static, i.e. that if it corresponds to a classical spacetime it is one in which $k_{ai}(x) = 0$.

Thus, the background geometry defined by a weave without intersections is one in which there is no evolution so the time dimension plays no role. It is then natural to interpret the classical limit of such a state as describing a three dimensional Riemannian geometry. One can say that since no degrees of freedom can depend on the time dimension, what we have here is the quantum analogue of a static geometry. As far as dynamics is concerned it is natural to interpret this as saying that the quantum state describes a world that has spontaneously compactified, or better simply eliminated, the time direction, so that it describes a world of three rather than four dimensions.

The second issue concerns the possible ambiguities in the correspondence given in [18] between a slowly varying three metric $q_{ab}$ and a link $\gamma$. First, if there is one link $\gamma$ that corresponds to a given $q_{ab}$ there are many, so there is no unique map $\mathcal{W}: q_{ab} \rightarrow \gamma_q$.

For the following we will find it useful to resolve this ambiguity and fix a definition of a weave that results in a unique weave map

$$\mathcal{W}: q_{ab} \rightarrow \gamma$$

(25)

To do this we need to give a definite construction that results in a unique weave $\gamma$ given any slowly varying three metric. The prescription does not have to be optimal, any construction that gives a weave that satisfies the tests of agreements of areas of surfaces given in [18] will do. It is clear from the construction in [18] that this may be easily done, for example, the random selection of centers and angles prescribed there may be specified according to a particular random number generator. In this case there is a unique prescription given a three metric that produces a weave, such that measurements of areas of all surfaces of the two agree up to terms small in Planck units.

Given such a choice, we have, for all slowly varying metrics $q_{ab}$ a uniquely specified $\gamma[q_{ab}]$.

In essence, what we have done is identified a further set of collective coordinates associated with the dependence of the state on the non-intersecting

\footnote{according to the definition of [18]}
link $\gamma$ constain in a general trivalent spin network $\rho$. These collective coordinates are a slowly varying three dimensional metric. We must emphasize that, unlike those considered so far, these collective coordinates are not generally valid, as the weave correspondence is only sensible when the link has the property that it is the image of some slowly varying metric under the weave map. But it is sufficient to extract the physics of a sector of the theory, which is within the range of validity of the weave correspondence. The situation is perhaps analogous to how the physics of phonons may be extracted by constructing an effective field theory. The atoms in a system must be condensed into a solid for a description in terms of photons to be meaningful, so phonon degrees of freedom are collective coordinates only in a portion of the Hilbert space of the fundamental theory. Similarly, a description in terms of slowly varying geometry is only possible in a portion of the state space of quantum gravity.

Now assume that we have a surface $S'$ imbedded in three dimensional space. We would like to extend the weave map so that it associates to the pair $(q_{ab}, S')$ a spin network which consists of an imbedding of a trivalent spinnet $\Gamma$ that spans $S'$ in the link $\gamma$ that represents $q_{ab}$. This extended weave map will satisfy,

$$W : (q_{ab}, S') \rightarrow (\Gamma, \gamma)_{q,S'}$$

(26)

where we have picked the imbedding of $\Gamma$ in the link $\gamma$ such that

$$a[\Gamma, \gamma] = A^{\text{class}}[q_{ab}, S'] + O(l_{\text{Planck}}^2)$$

(27)

where $A^{\text{class}}[q_{ab}, S']$ is the classical formula for the area of $S'$ as a function of $q_{ab}$.

The weave map can then be further extended to give a map from pairs $(q_{ab}, S')$ and physical states, $|\{\Gamma \cup \gamma\}, Z, j_i >$ such that

$$W : (q_{ab}, S') \rightarrow |\{(\Gamma, \gamma)_{q,S'}\}, Z^* >$$

(28)

where $Z^*$ is some arbitrary value of the other collective coordinates.

We can now use the weave map to define an effective quantum state which is a function of the pair $(q_{ab}, S')$. This should be interpreted as any effective quantum theory: it describes a subset of the degrees of freedom of the physical states which are relevant for the long distance physics of a particular sector of the theory. It is well defined only when the pair $(q_{ab}, S')$ is slowly varying, in the sense defined in [18], so that the weave map is well defined.
Given a physical state $\Psi\{\{\Gamma, \gamma\}, j_i, Z\}$ the effective quantum state is defined by,

$$\tilde{\Psi}(q_{ab}, S') \equiv \Psi\{\{(\Gamma, \gamma)_{q, S'}\}, Z^*\}$$ (29)

As it is defined only for metrics and surfaces that are slowly varying on the Planck scale, the functional $\tilde{\Psi}(q_{ab}, S')$ provides a way to extract a semiclassical limit from a general state $\Psi[\rho]$.

The next question to ask is whether classical equations of motion might govern the approximate dependence of the effective state $\tilde{\Psi}(q_{ab}, S')$. If this is the case we will have defined a good classical limit for the physical state $\Psi\{\{\Gamma, \gamma\}, Z\}$. It is easy to see that this is the case for the string state. We have, for slowly varying surfaces and three metrics,

$$\tilde{\Psi}_{\text{string}}(q_{ab}, S') \equiv \Psi_{\text{string}}\{\{(\Gamma, \gamma)_{q, S'}\}, Z^*\} = e^{A_{\text{class}}[q_{ab}, S']/\alpha'} + O(l_{\text{Planck}}^2/A_{\text{class}}[q_{ab}, S'])$$ (30)

This state may be interpreted as giving a string theory, described by the motion of the surface $S'$ in the background three dimensional Euclidean spacetime described by $q_{ab}$. To see this note that the semiclassical limit is given by the principle of stationary phase. Applied to (30) we have, holding the background geometry fixed,

$$\frac{\delta A_{\text{class}}[q_{ab}, S']}{\delta X^a(\sigma, \tau)} = 0$$ (31)

where $X^a(\sigma, \tau)$ are the coordinates of the embedding of the surface $S'$. These are the equations of standard string theory in a background $q_{ab}$ (with the other background fields, the dilaton and antisymmetric tensor field vanishing.)

5 Discussion

There are a number of questions we may ask about the interpretation of the result we have just found.

The first is about the domain of validity of the approximation that gives rise to the interpretation of the state (21) as describing the propagation of strings in a static background. The weave interpretation requires a scale $R >> l_{\text{Planck}}$ on which the metric and all surfaces used in the construction are slowly varying [18]. We may note that there is naturally a second scale in the problem, which is $l_{\text{string}} = \sqrt{\alpha'}$. When $l_{\text{string}} >> l_{\text{Planck}}$ it may serve as
in this case the domain of validity of the stationary phase approximation may agree with that of the weave construction. This is also the case in which the degrees of freedom of string theory are cut off at a scale much larger than the Planck scale.

A second question we may ask is which string theory we have found. This depends on the dynamics of the other collective coordinates, which include the choice of Γ, the original skeletonization of S, and the Z’s. It is important to stress that the physics of these additional degrees of freedom on the surfaces will depend greatly on the form of the Hamiltonian constraint. For example, with the rules we have so far defined the Z’s cannot be massless on the worldsheet, as their correlations will be confined to regions of the worldsheet, reflecting the problem of the confinement of correlations discussed in [24]. However, with the definition given in the appendix this problem is avoided, so that these degrees of freedom may have long ranged correlations on the surface. In this case it would be very interesting to try to extract a conformal field theory to describe these additional collective coordinates.

Alternatively, it is possible to extend the construction by adding additional degrees of freedom to the non-perturbative theory as well. One case in which this can be done rather easily is that of the antisymmetric tensor gauge field (or Kalb-Ramond field). As fields of this kind play an important role in string theory, it is interesting to investigate them in this context, as is briefly done in the next section.

6 The Kalb-Ramond field and the surface representation

As described previously in [39], the Kalb-Ramond field turns out to fit rather neatly into non-perturbative quantum gravity. This is because there is a surface representation [10, 39] for antisymmetric tensor gauge fields $B_{ab}$, in which gauge invariant functionals are labeled by closed surfaces $S$ by means of

$$\tilde{\Psi}[S] = \int d\mu(B_{ab}) e^{\lambda \int_S B}.$$  \hspace{1cm} (32)

As in the loop representation, any Hamiltonian formulation involving gauge invariant functionals such as $H = dB$ may be rewritten in this surface formalism [10, 39]. Furthermore, the term of the Hamiltonian constraint for $B_{ab}$ is polynomial when written with density weight two, as is suitable for coupling to the Ashtekar form of general relativity [39].
When combined with the loop representation for quantum gravity one has a kinematical state space defined on diffeomorphism invariant functionals of spin networks and surfaces, whose states are of the form $\Psi[\{\Gamma, S\}]$. We see that by including $B_{ab}$ we have this form of the state space at the kinematical level, without the necessity to either restrict to the trivalent sector or solve the dynamics. We may then posit directly the “string state”,

$$\Psi_{\text{string}, B}[\{\Gamma, S\}] \equiv e^{2A[\{\Gamma, S\}]/l_{\text{Planck}}}$$

(33)

It is interesting to note that taking into account the phase factor in the semiclassical limit is proportional to the classical string action, $S_{\text{string}} = \int_{\Sigma} (\sqrt{g} + B)$. This has properties similar to (21), in particular, in the appropriate classical limit it describes extremal two dimensional surfaces imbedded in a background metric defined by the spin networks. However, it should be pointed out that $\Psi_{\text{string}, B}$ is not a solution of the Hamiltonian constraint including the terms for $B_{ab}$ (described in [39]). Thus, one cannot extend the full interpretation given to (21) to this case.

7 Conclusions

It is clear that the results described here are at most an indication of a possible connection between non-perturbative quantum gravity and string theory. To develop them further two kinds of implications may be explored. The first is to the possibility of a non-perturbative formulation of string theory. In this connection it is interesting to recall a suggestion of Witten that a four dimensional non-supersymmetric theory might be the strong coupling limit of a three dimensional supersymmetric string theory [26]. This is based on an analogy with the conjectured relationship between ten dimensional string theory and eleven dimensional supergravity [4]. While the nature of the four dimensional theory that follows from the strong coupling limit of 3D string theory is unclear, if the analogy holds there should be a field theory in four dimensions that plays the same role of eleven dimensional supergravity (That is it is at least the classical limit of the four dimensional theory). From the non-perturbative point of view, this theory must be diffeomorphism invariant, as it will be a theory of gravity. We also know from Witten’s argument that this theory will have zero cosmological constant, $\Lambda$ and no dilaton.

Moreover, if Witten’s conjecture is correct then the relationship might also work the other way as well, in which case this four dimensional field
theory should have a limit which is described by a three dimensional supersymmetric string theory.

¿From the non-perturbative point of view, we may try to represent this theory in terms of spin networks, as they provide a very general language for describing the non-perturbative kinematics of any theory whose degrees of freedom may be defined in terms of a connection. Indeed, the simplest possibility would be that the conjectured theory is just quantum general relativity, perhaps coupled to some matter fields, quantized non-perturbatively and tuned to $\Lambda = 0$. This might seem quite implausible, but we have seen here that it is indeed the case that a sector of that theory has a semiclassical limit that describes a string theory in three dimensional spacetime\textsuperscript{[10]}. Further, the solutions we have been describing are for $\Lambda = 0$ and would not have existed for $\Lambda \neq 0$. We may also note that the theory we have studied has no dilaton and that the non-perturbative dimensional reduction mechanism that emerges here does not yield a compactification radius, both characteristics of the four dimensional theory Witten conjectures as the strong coupling limit of the three dimensional string theory.

Now, numerical simulations\textsuperscript{[48, 49, 50, 51]}, together with general renormalization group arguments suggest strongly that quantum general relativity at bare $\Lambda = 0$ cannot have a continuum limit that can be described in terms of a four dimensional field theory. This is consistent with what we have found so far, which is that there is a sector of that theory which has a continuum limit which may be described in terms of a three dimensional string theory. Of course, this is still a long way from realizing Witten’s picture, according to which the three dimensional string theory whose strong coupling limit gives a four dimensional field theory is supersymmetric. So far there is no evidence that the string theory found here is supersymmetric. However, it is not impossible that a form of the Hamiltonian constraint exists such that the additional collective coordinates describe fermionic conformal fields that, together with the embedding coordinates, realize worldsheet supersymmetry.

Whether or not this conjecture holds, it would be also interesting to see if the mechanism discovered here extends to supergravity in 11 dimensions. To investigate this, we should first study canonical quantization of supergravity in 10 and 11 dimensions. The required canonical formalisms do not, so

\textsuperscript{[10]} We may note that at the purely classical level there is a degenerate sector of general relativity, formulated in terms of Ashtekar variables, that describes a three dimensional theory\textsuperscript{[7]} It is not known if there is any relationship between this and the present results.
far as I know, so far exist, but they may be developed. This would also
be interesting as the study may reveal kinematical structures that could
underlie a non-perturbative string theory. Indeed, one might conjecture that
an extension of these results would show that $M$ theory was in fact nothing
more than a non-perturbative quantization of 11 dimensional supergravity.
But even if this is not the case it is possible that the phenomena described
here might be found to apply to that case and illuminate the physics of $M$
theory.

The second set of implications of these results are to non-perturbative
quantum gravity itself. This study shows that non-trivial structures may
emerge from the solution of some forms of the constraints, which can be
described in terms of collective degrees of freedom. Such structures may be
essential for understanding crucial questions of physical interest such as the
inner product and the continuum limit. For example, it may be that two
surfaces continue to play a key role in the parameterization of solutions, even
when we lift the restriction to trivalent states. One question of definite phys-
ical interest is the behavior of the new degrees of freedom associated with
surfaces, and their dependence on the choice of regularization procedures.

Still another direction to extend these results is from a canonical to a
path integral formulation. The new path integral formulation of Riesenberger\cite{11},
and Riesenberger and Rovelli\cite{12} suggests that fluctuations of surfaces may
play a role in a four dimensional covariant perturbation theory built around
semiclassical states.

However, to build a useful bridge between string theory and non-perturbative
quantum gravity the main obstacle to be overcome is the fact that each is
built from structures that are simplest in a particular dimension: ten in one
case, four in another. If there is to be such a bridge it will be likely based
on the existence of structures associated with 10 or 11 dimensions that play
roles analogous to those of spin networks and self-duality in four.

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APPENDIX

Here I discuss some details concerning the regularizations of the Hamiltonian constraint that satisfy the rules used.

A form of the Hamiltonian constraint that satisfies the first set of rules, R1-3 was described in [27, 28, 29]. One must amend the process described there in that the aim is to construct a Hamiltonian constraint, which is not diffeomorphism invariant, rather than a Hamiltonian, which must be. To do this one proceeds as follows. One defines the regulated operator acting on spin-network states as (compare eq. 5 of [27])

\[
\hat{C}^δ(N)|\Gamma> \equiv \int d^3x N(x) \hat{C}^δ(x)|\Gamma>
\]

\[
\equiv \int d^3x N(x) \int d^9 \int d3z f^δ(x, y) f^δ(x, z) \hat{θ}^{-1}_{ij}
\]

\[
\times \left( \hat{T}^{ab} \left[ γ_{x, \hat{a}, \hat{b}, e^2} \cdot γ_{xy} \cdot h(\hat{a})γ_{yx} \cdot γ_{xz} \cdot h(\hat{b})γ_{zx} \right] - \hat{T}^{-ab} \left[ γ_{x, \hat{a}, \hat{b}, e^2} \cdot γ_{xy} \cdot h(\hat{a})γ_{yx} \cdot γ_{xz} \cdot h(\hat{b})γ_{zx} \right] \right) |\Gamma>
\]

Here the notation is that of [13, 27, 28], and all smearing functions \( f^δ(x, y) \) depend on an arbitrary flat background metric \( h_{ab}^0 \). \( γ_{x, \hat{a}, \hat{b}, e^2} \) is a loop based at \( x \) in the \( \hat{a}\hat{b} \) plane with area \( e^2 \), \( γ_{xy} \) is a straight line (in the background metric) from \( x \) to \( y \) and \( h(\hat{a}) \) means that is where you make the insertion of the hand with index \( \hat{a} \). The only non-standard thing is that, as described in [27, 28, 29] there is an additional operator dependence which measures the angle \( θ_{ij} \) between the \( i \)'th and \( j \)'th tangent vectors incident at \( x \). This explicit operator dependence is necessary to cancel a factor of \( θ_{ij} \) that arises in the integrals over regulators. The use of such explicit additional operator dependence in the regulator is a cleaner way to describe what is sometimes called “state dependence of the regulator” and is described in more detail in [29].
After some steps that parallel those in \[27, 28, 29\] we arrive at
\[
\hat{C}(N)|\Gamma> = \sum_n \sum_{ij} \frac{N(n)}{\delta e^2} \frac{3}{10\pi} (16\pi l_p^2)^2 \gamma_{ij} \gamma_{ij} \delta \epsilon_{\epsilon^2} \gamma_{ij} |\Gamma> \tag{35}
\]
where the sums are over the nodes of $\Gamma$ labeled by $n$ and pairs of non-collinear edges $i$ and $j$ at $n$. We then define the renormalized operator by
\[
\hat{C}_{\text{ren}}(N)|\Gamma> \equiv \lim_{\epsilon \to 0} \lim_{\delta \to 0} \epsilon^2 \delta \epsilon \hat{C}(N)|\Gamma> \tag{36}
\]
Here, as described in more detail in \[27, 28, 29\] the first limit is taken in the kinematical state space while the second is taken in the space of diffeomorphism invariant states, with the loop chosen to lie in a triangle bordered by the edges $i$ and $j$ (more additional operator dependence built from $\dot{\gamma}(x)$.)

The result is an operator satisfying the rules $\text{R1-R3}$.

The next step is to modify this operator to eliminate the action on kinks, so that it satisfies rule $\text{R3}'$ instead of $\text{R3}$. As we learn from \[34, 35\] one way to change the class of nodes an operator acts on is to change its density weight. Given that we do a renormalization at the end, and so end up in any case with a non-diffeomorphism invariant operator, there is no objection to doing this. We then modify the Hamiltonian constraint to,
\[
C'(N) \equiv \int d^3 x N'(x) \hat{C}(x) q(x) \tag{37}
\]

Where $N'(x)$ is now a density of weight minus four. In order that the density $q$ not annihilate the trivalent vertices we are also going to have to $q$-deform the formalism, as in \[36, 37\]. Before discussing this we attend to the details of the regularization. We define the corresponding operator by keeping this order and regulating each piece separately
\[
\hat{C}_{\text{ren}}(N)|\Gamma> \equiv \int d^3 x N'(x) \hat{C}(x) q(x) L |\Gamma> \tag{38}
\]
where $\hat{C}(x)$ is defined in \[34\] and the regulated density $q(x) L$ is defined as in \[19, 37\]. We take a cube of linear size $L$ in the background metric centered at $x$ and define
\[
q^L \equiv \frac{1}{2^3 3! L^3} \hat{W}^{x,L} \tag{39}
\]
where $\hat{W}_{x,L}$ is the operator defined by eq. 10 of [14] for a the cube of size $L$ around $x$. We will then define the renormalized operator to be

$$\hat{C}_{\text{ren}}(N)|\Gamma> \equiv \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{L \to 0} \epsilon^2 \delta L^6 \hat{C}(N)|\Gamma>$$

(40)

with the limits taken in the order indicated.

We then work in the $q$-deformed formalism [36, 37] based on $q$-deformed spin networks [43]. In that formalism all trivalent nodes are eigenstates of the operators $\hat{W}$ defined on every box small enough that it encloses only one of them. Let us call the eigenvalue $w(i,j,k)$, where $i,j$ and $k$ are the three spins incident on it; it is computed in [37]. These eigenvalues are non-vanishing, without regard for any linear dependences among the tangent vectors of the incident edges. In this respect (as well as in the $q$-deformation) it differs from the operator defined in [20]. At the same time $\hat{W}$ for any box annihilates kinks. The action of the modified renormalized operator is then to ignore the kinks and, when acting on trivalent nodes, multiply the coefficients $A_{\pm}(j,k)$ by $w(i,j,k)$.

The result is an operator which satisfies the modified rule $R3'$ instead of $R3$.

There is one more form of the Hamiltonian constraint that gives that I want to discuss. This is a regularization defined directly in the diffeomorphism invariant spin-network language. As discussed in more detail in [24] the idea is that the form of the regulated operator that represents the Hamiltonian constraint need not be derived from a regularization procedure. It is sufficient that it agree, when applied to a non-diffeomorphism invariant state in the connection representation and evaluated for connections that are slowly varying (in the topology of the graph) with a form of the constraint that is derived from a point-split regularization. Such a prescription can be given directly by a set of rules. One such operator is given in [24]. I give here an alternate form that has the property that its action closes on the trivalent spin networks\textsuperscript{11}. It is called $C'_{\text{new}}$.

- $\mathbf{N1} C'_{\text{new}}(N)$ acts on an element $\Gamma$ of the spin network basis at each pair of non-collinear edges $e_1$ and $e_2$ of every trivalent node $v$ in the following way. It finds the first nodes adjacent to $v$ along $e_1$ and $e_2$, which will be called $v_1$ and $v_2$.

\textsuperscript{11}For those familiar with [24] the only modifications are in the restriction to action on trivalent nodes and in the step where new edges are created.
Suppose there is an edge joining $v_1$ and $v_2$, which will be called $e_{12}$. The action of $C'_{\text{new}}$ produces a sum of six terms in which the colors along $e_1$, $e_2$ and $e_{12}$ which we call $i$, $j$ and $k$ respectively are updated by $\pm 1$. Each is multiplied by an amplitude $A_{\pm, \pm', \pm''}(i, j, k; r, s, t)$ which I give below. Here we assume that each of the nodes is written in the form in which the two edges in the problem are joined to a third edge at a trivalent vertex with an edge with definite color. The colors associated with these edges for $v, v_1$ and $v_2$, respectively, are $r, s$ and $t$. $\pm, \pm'$ and $\pm''$ refer respectively to the updating of $i, j$ and $k$. The amplitude is then,

$$A_{\pm, \pm', \pm''}(i, j, k; r, s, t) = \pm'' ij \{iii \pm 1; 112\} \{jjj \pm' 1; 112\} \{i \pm 1ir; j \pm' 1j1\} \times \{i \pm 1is; k \pm'' 1k1\} \{j \pm' 1jt; kk \pm'' 11\} \times \Theta(i, j, r) \Theta(j, k, t) \Theta(i, k, s) [r + 1][s + 1][t + 1]$$

Here \{iii \pm 1; 112\} are the $6 - j$ symbols, and $\Theta(i, j, r)$ is the theta function defined in [28]. The formula is written in a way that is good for either the ordinary or $q$-deformed case, so $[n]$ is the quantum integer [28], which is equal to $n$ in the ordinary case.

There is also the case in which there is in $\Gamma$ no edge joining $v_1$ and $v_2$. In this case the operator adds a new edge with color 1. Here the definition must differ from that given in [28] so that no vertices are produced with more than 3 incident edges. To do this we break the edge $e_1$ joining $v$ to $v_1$ and and insert a new node $v'_1$. We do the same thing to $e_2$ creating a new node $v'_2$ between $v$ and $v_2$. The two halves of $e_1$ and $e_2$ are each colored by the same spins that colored the edges originally. Then we join these two new vertices with a new edge with color 1, which we call $e_{12}$. The topology of the edge is chosen so the loop it forms with the segments of $e_1$ and $e_2$ links or intersects no other edge of the network. One then applies the above formula with $r = i$, $s = j$, $k = 0$ and $\pm'' = +$, producing in this case four terms.

What this combinatorial formula corresponds to is adding a loop as usual to represent the $F_{ab}$ in the plane of the tangent vectors of the two edges. The combinatorics are as in [28], except that the new loop is taken to go around the triangle $e_1, e_2, e_{12}$.

To complete the definition of the operator we must divide by the area of the triangle $e_1, e_2, e_{12}$. We may note that, as determined by
the area operator, this will often vanish, but it may instead be defined using Thiemann’s length operator\cite{22} as follows. If we call $\hat{L}_1, \hat{L}_2$ and $\hat{L}_3$ the length operators of the edges of a triangle $\Delta$ of a spin-network, we may define an operator that measures its area as,

\[
\hat{A}_\Delta = \frac{1}{4} \left( \frac{\hat{L}_1^2 \hat{L}_2^2}{2} + \frac{\hat{L}_2^2 \hat{L}_3^2}{2} + \frac{\hat{L}_3^2 \hat{L}_1^2}{2} - \frac{\hat{L}_1^4}{4} - \frac{\hat{L}_2^4}{4} - \frac{\hat{L}_3^4}{4} \right)
\] (42)

where we have used the standard formula from Euclidean geometry for the area. (If the operators fail to commute we take symmetric ordering.) That this will often yield a different answer than a direct measurement of the area is an inevitable consequence that we are working with a quantum field theory, in which functional relationships between classical observables may not be preserved.

We may then define this step as follows: If there is a term with no triangle corresponding to the three original edges we do nothing. If there is we multiply the state gotten by the first two steps by the operator $\hat{A}_\varepsilon^{-1}$. We need to define the inverse so it is well defined in the case that the area is zero. We do so by $\hat{A}^{-1} \equiv \hat{A}^{-2} \hat{A}$, where $\hat{A}^{-2}$ is defined on the space orthogonal to the kernel of $\hat{A}$, so that terms that might contribute zero area are projected out.

As discussed in more detail in\cite{23} this operator may resolve the problem of bounded correlations. The solutions it generates will still be characterized by two dimensional surfaces, but there will in general be no single graph in the superposition of states which plays a special role as the ancestor of the other graphs. Instead, it is possible that this rule will lead to a flow that is free on the space of all spin networks $\Gamma$ that span a given two dimensional surface, $S$. If true this will mean that the parameters $Z$ that distinguish the different solutions on $S$ will not be restricted to particular regions, but will depend on how the spin networks propagate over the whole surface. If so they must be be described in terms of conformal fields on each $S$. Investigation of this form of the theory is in progress.

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\footnote{Note also that the formula for $\hat{L}$ of an edge given by Thiemann in\cite{22} is ultimately expressed in terms of $6-j$ symbols, which means that the $q$ deformation can be immediately written down by using the $q$ deformed $6-j$ symbols defined in\cite{43}.}
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