Joining Spacetimes on Fractal Hypersurfaces

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Abstract

The theory of fractional calculus is attracting a lot of attention from mathematicians as well as physicists. The fractional generalisation of the well-known ordinary calculus is being used extensively in many fields, particularly in understanding stochastic process and fractal dynamics. In this paper, we apply the techniques of fractional calculus to study some specific modifications of the geometry of submanifolds. Our generalisation is applied to extend the Israel formalism which is used to glue together two spacetimes across a timelike, spacelike or a null hypersurface. In this context, we show that the fractional extrapolation leads to some striking new results. More precisely we demonstrate that, in contrast to the original Israel formalism, where many spacetimes can only be joined together through an intermediate thin hypersurface of matter satisfying some non-standard energy conditions, the fractional generalisation allows these spacetimes to be smoothly sewed together without any such requirements on the stress tensor of the matter fields. We discuss the ramifications of these results for spacetime structure and the possible implications for gravitational physics.

1 Introduction

The theory of fractional calculus has been considered a classical but obscure corner of mathematics [1, 2, 3]. It remained, until a few decades, a field by mathematicians, for mathematicians and of purely theoretical interest. Though it played a crucial role in the development of Abel’s theory of integral equations and many mathematicians like Liouville, Riemann, Heaviside and Hilbert took an active interest in it, fractional calculus found limited applications and was referred to only occasionally, to simplify complicated solutions. For example, this formalism has been used quite often to simplify the solutions of both the diffusion as well as the wave equation (for example, see [4], and [5]).

During the last few decades, however, this theory has found important applications for large number of practical real life situations. Indeed, fractional calculus is providing excellent tools to develop models of polymers and materials [6, 7]. In

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particular, it has been found that to understand properties of various materials which require long-range order to hold, fractional calculus provides a sound platform [8]. Fractional calculus have also been found to naturally incorporate some subtle effects in the dynamics of fluids, and these have found important applications in understanding mechanical, chemical and electrical properties of nano-fluids. However, possibly the most prominent application of these derivatives of non-integer order has been in the theory of fractals [9]. It has been found that for many stochastic processes, the phenomena progresses through increments which are not independent, but instead tend to retain some memory of previous increment, though not necessarily the immediately previous increment [9, 10, 11, 13]. In other words, these are random processes with long term memory. The theory of fractional Brownian motion, which provides a very natural explanation for these effects, incorporates these persistence effects (or anti-persistence effects) though the fractional modification of the usual Brownian formula relating displacement and time [14]. It has now been understood that statistically speaking, all the naturally occurring signals are of the Weierstrass type, i.e continuous but non-differentiable [9, 15] (here differentiation is in the sense of the usual calculus) and indeed, such Weierstrass-like functions arise even in many quantum mechanical situations. For example, it has been shown that many quantum mechanical problems involving discontinuous potentials possesses energy spectrum of the Weierstrass type [16]. Furthermore, the Feynman paths in the path integral formulation of quantum mechanics are also examples of these kind [17]. However, the most significant discovery has been that, though the naturally occurring functions are of the Weierstrass type and are endowed with a fractal dimension, they are fractionally differentiable and that the maximal order of differentiability is related to the box-dimension of the function [18, 19]. Thus, fractional calculus has been highly advantageous in modelling dynamical processes in self-similar systems and for analysing processes which generate chaotic signals and are apparently irregular.

In this paper, we apply the techniques of fractional calculus to general relativity. As is well known, the issue of final state of gravitational collapse is a long standing open question in general relativity. The appearance of spacetime singularity reveals the domain of failure of the classical theory of general relativity [20]. Quite naturally, it is assumed that general relativity must be corrected to eliminate these failures. Both the string theories and effective field theories necessitate that one must add terms involving higher order as well as higher derivatives in the Riemann tensor to incorporate the effects of physics at small scales. It is a general hope that these higher order corrections will certainly get rid of the singularities [21]. However, in absence of any comprehensive proof of these expectations, we propose to look at another alternative which may present itself at the small scales. As we shall see in the subsequent sections, fractional calculus, in any of it’s possible alternative forms, define differentiation through an integration. Hence, it naturally incorporates non-local spacetime correlations and long-range interactions, which are expected to be natural at high energy scales, into account. Thus, many subtle non-local effects may manifest itself if one replaces the ordinary differentiation by it’s fractional counterpart. One may immediately ask as to where should one look for such non-local terms to arise physically and envisage the regions of strong
gravity where the classical theory of general relativity is known to require modifications. An obvious candidate for the strong gravity regime is the black hole region since black holes are created due to gravitational collapse of matter fields in an intense gravitational field. There are two regions of the black hole spacetime which are ideally suited for the fractional effects to manifest itself. First is the horizon, which for small mass black holes are regions of intense gravitational field and second, near the singularity where the effects of strong gravity, though invisible to the asymptotic observer, are most spectacular. In either of these two situations, possibilities of long range order have quite interesting repercussions on modelling of spacetime.

Let us first discuss the region near the horizon. The black hole horizon, here taken to be an event horizon, is a null expansion- free hypersurface which lies in the region adjoining two spacetimes. So, the horizon may be thought of as a null hypersurface which glues the two spacetimes. Naturally, the joining of two spacetimes through the hypersurface requires that some conditions on the spacetime variables be satisfied on the hypersurface. The Israel- Darmois- Lanczos (IDL) junction condition demands that the metric on either side of the horizon, when pulled back to the hypersurface, must be continuous [25]. In contrast, the extrinsic curvature of the horizon is not required to be continuous. In fact, consistency requires that the Riemann tensor and hence the extrinsic curvature on the hypersurface admit delta function singularities. Using the Einstein equations, the Ricci part of this singularity is related to the stress tensor. Thus, the IDL condition only requires that the difference of the extrinsic curvatures of the hypersurface as embedded in these two spacetimes, must be proportional to the stress- energy tensor living on the horizon. In other words, due to the geometry itself, the hypersurface comes naturally equipped with a energy- momentum tensor. These junction conditions on the horizon has given rise to speculations of constructing a singularity free spacetimes, and in particular, non- singular black hole interiors, in the following way [22, 23]: Take the exterior of the Schwarzschild horizon as the future spacetime region and the interior to de- Sitter horizon as the past spacetime region. The boundary between these two regions, the common hypersurface to these two regions, will be a thin null hypersurface endowed with some specific energy- momentum tensor derived from the IDL matching conditions. Thus, one may have well defined matching conditions to create a singularity free universe, with the exterior a Schwarzschild spacetime while the interior being a de- Sitter spacetime. However, in most of the cases, the matching conditions leads to energy- momentum tensors which violate some of the well known energy conditions. In [23], there have been attempts at constructing a singularity free universe by adjoining the de- Sitter interior with the inner horizon of a Reissner- Nordstrom black hole with a particular values of charge and mass. In this particular case, the matching is smooth, with no requirement of any energy momentum tensor. In general situations, these matchings are not smooth and require energy condition violating energy- momentum tensors on the matching hypersurface.

\[1\] Most of these terms contribute non- local effects into the Green’s function. It should not be surprising if many of the effects of the fractional generalisation arise naturally in the string theories or any other quantum theory.
The second point is related to the another such attempt where, the de-Sitter spacetime is glued to the Schwarzschild interior though a spacelike hypersurface. This attempt was made by [24], in their famous proposal of limiting curvature. They devised a model in which the Schwarzschild metric inside the black hole region is matched to a de-Sitter one at some spacelike junction surface which represent a thin transition layer. As a requirement of their proposal, this layer is placed at a region very close to the singularity where the curvature reaches it’s limiting value. However again, for general singularity free matchings of the above kind, the junction layers admit energy momentum tensors which violate energy conditions. In particular, the effective stress-energy tensor of the model [24] violates the weak energy condition. In fact, in almost all similar attempts of creating singularity free models like that of [24], have energy condition violations. These kind of violations are actually characteristic of quantum effects which become important in strong gravitational field.

As a remedy to these energy condition violations, we argue in this paper that the notion of fractional derivatives offers a possibility of creating singularity free universe through smooth matching of spacetimes. More precisely, we demonstrate the following: First, that the IDL junction conditions both for timelike/spacelike as well as for null hypersurfaces are modified due to the fractional generalisation of the spacetime connection. This fractional generalisation of the IDL conditions will in turn modify the energy-momentum tensor on the hypersurface. Secondly, using specific examples, we show that this generalisation allows us to fix the conditions on the junction shell in such a way that the Schwarzschild or the Reissner-Nordstrom spacetimes can always be smoothly matched to the de-Sitter spacetime in the interior without any energy condition violating requirement on the energy-momentum tensors of the adjoining shell.

The paper is organised as follows: In the next section, section 2, we briefly discuss the mathematical formalism of fractional calculus and the relevant notations. In sections 3 and 4, we introduce the notations for timelike/spacelike and null hypersurfaces and discuss the generalisations of the IDL junction conditions for fractional exponents. We also argue that these generalised junction conditions leads to smooth joining of spacetimes, which otherwise are known to be joined only through a thin shell of matter. The implications of these are discussed in the Discussion section.

## 2 Mathematical preliminaries

Let us discuss some notations useful for the mathematical formulation of geometry of hypersurfaces. Let us consider a 4 dimensional spacetime \((\mathcal{M}, g)\) with signature \((- , +, +, +)\). Let a hypersurface \(\Delta\) be embedded in \(\mathcal{M}\) and is given by \(f : \Delta \to \mathcal{M}\). We shall assume that the embedding relation is such that the restriction of \(f\) to

2The fractional generalisation developed in this paper assumes that fractal like structures are present at very high energy scales, just as they are present at low energy scales. There is no experimental basis for such assumptions. However, this assumption leads to some interesting consequences as developed in this paper.
the image of $\Delta$ is $C^\infty$. Let, $\{x^\mu\}$ be a local coordinate chart on $\mathcal{M}$ and $\{y^a\}$ be a local coordinate chart on $\Delta$. The embedding relation implies $x^\mu = x^\mu(y^a)$. Let, $g_{\mu\nu}$ be the metric on the spacetime in terms of its local coordinates. The first fundamental form or the induced metric on $\Delta$ is the pull back of the metric $g$ under the map $f$. In the local coordinates this can be written as $h_{ab}$.

$$h_{ab} \equiv g(\partial_a, \partial_b) = \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} g(\partial_\mu, \partial_\nu) = e^\mu_a e^\nu_b g_{\mu\nu},$$  \hspace{1cm} (1)$$

where, $(\partial x^\mu/\partial y^a) = e^\mu_a$ and we have used that $e^\mu_a \partial_\mu$ is the push forward of the purely tangential vector field $\partial_a$ onto the full spacetime $\mathcal{M}$. One may define a linear connection and hence a derivative operator on the spacetime. Let, $T\mathcal{M}$ denote the tangent bundle on $\mathcal{M}$ and let, $X$ and $Y$ are two arbitrary vector fields on it. The covariant derivative is a linear map

$$\nabla : T\mathcal{M} \otimes T\mathcal{M} \to T\mathcal{M}$$

$$(X, Y) \to \nabla_X Y.$$

(2)

(3)

The Riemannian theory assumes the covariant derivative to be metric compatible, $\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$. On the tangent bundle one may also define a covariant derivative ($D$) on $\Delta$ using the Gauss decomposition formula:

$$D : T\Delta \otimes T\Delta \to T\Delta$$

$$\nabla_X Y = D_X Y + K(X,Y).$$  \hspace{1cm} (4)

(5)

$D_X Y$ is purely tangential and $K(X,Y)$ is an element of the normal bundle and refereed to as the extrinsic curvature. The Gauss equation also implies along with that metric compatibility of $\nabla$ with $g$ that the derivative operator $D_a$ is metric compatible with the metric on the hypersurface $h_{ab}$ (i.e. $D_a h_{bc} = 0$). In terms of the local coordinate charts, the Gauss equation gives the following expression for the derivative operator (for $X \equiv \partial_a$):

$$D_a Y_b = e^\mu_a e^\nu_b \nabla_\mu Y_\nu.$$  \hspace{1cm} (6)

The extrinsic curvature can also be defined for the hypersurface in terms of the local coordinates. The normal bundle for the hypersurface is one dimensional. Let, $n^\mu$ be the normal. The extrinsic curvature is

$$K_{ab} = e^\mu_a e^\nu_b \nabla_\mu n_\nu = (1/2) (\mathcal{L}_a g_{\mu\nu}) e^\mu_a e^\nu_b.$$  \hspace{1cm} (7)

For our later use, let us give the Gauss equation in terms of the local coordinates:

$$R^d_{\mu\nu\lambda\sigma} e^\mu_a e^\nu_b e^\lambda_c e^\sigma_d = R_{abcd} - (K_{ad} K_{bc} - K_{ac} K_{bd}).$$  \hspace{1cm} (8)

The Codazzi equation in local coordinates is given by:

$$R^d_{\mu\nu\lambda\sigma} n^\mu e^\nu_b e^\lambda_e e^\sigma_d = K_{ab|\sigma} - K_{ac|\sigma}.$$  \hspace{1cm} (9)
where $\nabla^\gamma$ denotes the covariant derivative with respect to the coordinates on the hypersurface.

Several of these spacetime functions have different values on either sides of a hypersurface. Then, it is required to express their continuity across the hypersurface. A useful and prominent example of this idea is that of the Israel- Darmois-Lanczos (IDL) junction condition \[25\]. Consider a hypersurface $\Delta$ which partitions the spacetime into two regions $(M^+, g^+)$ with coordinates $\{x^\mu^+\}$ and $(M^-, g^-)$ with coordinates $\{x^\mu^-\}$. The spacetime $M^+$ is assumed to be to the future of the spacetime $M^-$. Quite naturally, it is not generally true that the metrics on these two spacetimes could be continuously matched across the hypersurface $\Delta$ (The either side of the hypersurface $\Delta$ has been installed with coordinates $\{y^a\}$). The discontinuity in the metric would be reflected in the fact that Riemann tensor would have a delta- function singularity on the hypersurface. The Israel junction conditions provides a method to smoothly match these hypersurfaces by using the following trick: relate the Ricci part of the singular Riemann curvature tensor to the surface stress- tensor using the Einstein equations. For spacelike hypersurfaces, the Israel junction conditions for a smooth joining of hypersurfaces at $\Delta$ is given by

$$[h_{ab}] = 0 = [K_{ab}] \quad (10)$$

where $[A] \equiv A(M^+)_{|\Delta} - A(M^-)_{|\Delta}$. However, if the extrinsic curvature is not identical on both the sides on the hypersurface $\Delta$, the surface stress tensor $(S_{ab})$ on the hypersurface is

$$8\pi S_{ab} = [K_{ab}] - [K]h_{ab}. \quad (11)$$

However, on the null surface, the standard extrinsic curvature corresponding the normal of the hypersurface (which is also the tangent to null hypersurface) is always continuous and hence, one needs to define a transverse curvature \[25\]. The metric induced on the null hypersurface is again continuous but the discontinuities in the components of the transverse curvature is related to the energy- momentum tensor induced on the this hypersurface.

In deriving the above relations, we have implicitly made two crucial assumptions: First, that the point functions are continuous and differentiable in the region under consideration. However, it may happen the scalar, vector or the tensor functions are only fractionally differentiable. In that case, the limiting values defined by our ordinary differential calculus become singular on the hypersurface. Thus, in addition to the IDL conditions, their fractional character must also be taken into account. Secondly, the spacetime connection is assumed to be a Levi- Civita connection. This arises since the spacetime is assumed to be a Riemannian spacetime and hence, the spacetime metric is compatible with the covariant derivative ($\nabla_{\gamma} g_{\alpha\beta} = 0$). The Gauss decomposition, eqn. (4), then implies that the connection on the hypersurface is also a Levi- Civita connection and that the extrinsic curvature is uniquely determined in terms of this connection. However, it may happen that in the strong gravity regime we are interested in, the spacetime is slightly modified from it’s Riemannian character and that the connection is not Levi- Civita connection derived from the metric. Quite naturally, in such a situation, the Gauss decomposition implies that the connection on the hypersurface will also be modified..
and the expression of the extrinsic curvature will also change.

In [26], a fractional generalisation of the Lie derivative has been proposed and utilised to generalise the definition of the extrinsic curvature for non-null hypersurfaces. In this fractional generalisation, which is based on Caputo’s modification of the Riemann-Liouville definition of fractional derivative (see the appendix), the usual definition of the extrinsic curvature $K_{ab} = (1/2)(L_n g_{\alpha\beta}) e^\alpha_a e^\beta_b$ is modified to give:

$$qK_{ab} = \frac{1}{2} (q L_n g_{\alpha\beta}) e^\alpha_a e^\beta_b$$

$$= \frac{1}{2} \left[ n^\gamma \mathcal{D}^q_{r-\Delta,\gamma} g_{\alpha\beta} + g_{\gamma\beta} \mathcal{D}^q_{r-\Delta,\gamma} n^\gamma + g_{\alpha\gamma} \mathcal{D}^q_{r-\Delta,\gamma} n^\gamma \right] e^\alpha_a e^\beta_b. \tag{12}$$

Here, the superscript $q$ denotes the fractional parameter, $0 < q \leq 1$ (see the appendix) and $\mathcal{D}^q_{r-\Delta,r}$ denotes the derivative:

$$\mathcal{D}^q_{r-\Delta,r}(g_{\alpha\beta}) = \frac{\Gamma(2-q)}{\Gamma(1-q)} \Delta^{1-q} \int_{r-\Delta}^r \frac{\partial g_{\alpha\beta}(w)}{\partial w} (r-w)^{-q} dw, \tag{13}$$

where the integration is carried out from a spacetime point $r-\Delta$ to $r$. In the context of matching of spacetimes across hypersurface, $\Delta$ is taken to be the thickness of the hypersurface. The junction conditions will be modified from eqn. (11) to

$$8\pi q S_{ab} = [qK_{ab}] - [qK] h_{ab}. \tag{14}$$

Naturally, because of the definition of the derivative, it has a non-local character imbedded into it. In the following sections, we shall utilize this generalisation of the definition of extrinsic curvature to modify the junction conditions for spacelike/timelike as well as null hypersurfaces. Additionally, we shall show that the junction conditions lead to a smooth matching of hypersurfaces.

### 3 Junction conditions for non-null hypersurfaces

Let us consider a non-null hypersurface $\Delta$. As discussed previously, the junction condition for the smooth joining of spacetimes along a timelike or spacelike hypersurface $\Delta$ is given by the following two conditions: $[h_{ab}] = 0$ and $[K_{ab}] = 0$. On the other hand, for joining spacetimes which contribute non-equal extrinsic curvatures on the hypersurface, a thin layer of matter is assumed to exist on the hypersurface with stress tensor $S_{ab} = (\epsilon/8\pi)([K_{ab}] - Kh_{ab})$. The quantity $\epsilon = n \cdot n$, distinguishes spacelike hypersurfaces ($\epsilon = -1$) from timelike ones ($\epsilon = -1$).

As described in the previous subsection, the junction conditions differ if the fractional derivatives are used. For the non-null hypersurface, we elaborate on this method through two explicit examples. In the first example, we give a detail step by step calculation showing the matching of a slowly rotating Kerr metric

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3In the appendix [6.3] we have developed a non-Levi-Civita connection based on a notion of fractional derivative and have shown to lead to modification of the tensor functions and the Einstein equations.
to a Minkowski metric on a timelike hypersurface. We show that depending on the width of the shell, the energy momentum tensor of the shell changes. We utilize this observation in the second example, which deals with matching of a Schwarzschild spacetime with a de- Sitter spacetime on a spacelike hypersurface. Again the energy- momentum tensor residing on the thin shell differs substantially from the standard results.

3.1 Joining Minkowski and slowly rotating Kerr metrics

Let us consider the metric of a Kerr spacetime in the slow- rotation approximation. We shall assume a shell of mass $M$ and angular momentum $J$ in the spacetime. The exterior spacetime $(M^+)$ has the following metric:

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega^2 - \frac{4Ma}{r} \sin^2 \theta dtd\phi,$$

where $f(r) = (1 - 2M/r)$. Let us assume that the shell is located at $r = R_0$. The induced metric on the shell becomes:

$$ds^2_{\Sigma} = -f(R_0) dt^2 + R_0^2 d\Omega^2 - \frac{4Ma}{R_0} \sin^2 \theta dtd\phi.$$

Using the definitions, $\psi = (\phi - \omega t)$ with $\omega = (2Ma/r^3)$, and keeping terms up to first order of $a$, we get the induced metric to be

$$h_{ab} dy^a dy^b = -f(r) dt^2 + R_0^2 (d\theta^2 + \sin^2 \theta d\psi^2).$$

Now let’s calculate the non vanishing components of extrinsic curvature. The definition of transverse component of the fractional generalisation is:

$$q K_{ab} = \frac{1}{2} (q \mathcal{L}_n g_{\alpha\beta}) e^\alpha_a e^\beta_b = \frac{1}{2} \left[ n^\gamma D^q_{r-\Delta,\gamma} g_{\alpha\beta} + g_{\gamma\beta} D^q_{r-\Delta,\gamma} n^n + g_{\alpha\gamma} D^q_{r-\Delta,\gamma} n^n \right] e^\alpha_a e^\beta_b,$$

where the projectors $e^\alpha_a$’s are $e^t_t \partial_t = (\partial_t + \omega \partial_\phi)$, $e^\theta_\theta \partial_\theta$, and $e^\phi_\phi \partial_\phi$. Let us first determine $q K_{tt}^+$, where + denotes that the variable is associated with the external spacetime. Note that since $e^t_t \partial_t = (\partial_t + \omega \partial_\phi)$, one get the only contribution from $q K_{tt}^+ = (1/2) q \mathcal{L}_n (g^{tt}_n)$. The other contribution to $q K_{tt}^+$ from $g^{tt}_n$ is neglected since $g^{tt}_n = -(2Ma \sin^2 \theta/r)$, is directly proportional to $a$ and further, together with $\omega = (2Ma/r^3)$ contributes an overall $a^2$ term. Note that due to the
form of the normal vector, and the metric, only the first term in expansion in \([19]\) contributes. Using the expression for \(D^{\nu}_{\tau-\Delta,\tau}(r^{-1})\) in the appendix, eqn. \([92]\), we get

\[
{^qK}^{+\pm}_{\nu\tau} = -\frac{M}{R_0^2} \left(1 - \frac{2M}{R_0}\right)^{1/2} \left[1 + 2\frac{1 - q\Delta}{2 - qR_0} + \ldots\right],
\]

and hence, using the metric, one easily determines that

\[
{^qK}^{+\pm}_{\nu\tau} = \frac{M}{R_0^2} \left(1 - \frac{2M}{R_0}\right)^{-1/2} \left[1 + 2\frac{1 - q\Delta}{2 - qR_0} + \ldots\right]. \tag{20}
\]

Similarly, one finds the contribution from \({^qK}_{t\psi}\) as follows:

\[
{^qK}_{t\psi} = (1/2) {^qL}_n(g_{ab}^+) e^a_{\xi} e^b_{\psi} = (1/2) {^qL}_n(g_{abl}^+) + (\omega/2) {^qL}_n(g_{\phi\omega}). \tag{21}
\]

Using \(D^{\eta}_{\tau-\Delta,\tau}(r^{-1})\) in the appendix, eqn. \([92]\), we get that

\[
{^qL}_n(g_{abl}^+) = n^\nu D^{\nu}_{\eta-\Delta,\nu}(\eta^2 \sin^2 \theta) = \frac{2Ma \sin^2 \theta}{R_0^2} \left(1 + 2\frac{1 - q\Delta}{2 - qR_0} + \ldots\right). \tag{22}
\]

Again, using \(D^{\eta}_{\tau-\Delta,\tau}(r^2)\) in the appendix, eqn. \([86]\), we get

\[
{^qL}_n(g_{\phi\omega}) = n^\nu D^{\nu}_{\eta-\Delta,\nu}(r^2 \sin^2 \theta)
\]

\[
= 2R_0 \sin^2 \theta \left(1 - \frac{2M}{R_0}\right)^{1/2} \left(1 - \frac{1 - q\Delta}{2 - qR_0} + \ldots\right). \tag{23}
\]

Putting \(\omega = (2Ma/R_0^2)\) in eqn. \([21]\), and using equations \([22]\) and \([23]\), we get

\[
{^qK}^{+\pm}_{t\psi} = \frac{3Ma \sin^2 \theta}{R_0^2} \left(1 - \frac{2M}{R_0}\right)^{1/2} \left[1 + \frac{1 - q}\left(\frac{\Delta}{R_0}\right)^2 + \ldots\right].
\]

The expression naturally leads to the following expressions for extrinsic curvatures:

\[
{^qK}^{+\pm}_{t\psi} = g^{\nu\tau} \left(\alpha{^qK}^{+\pm}_{\nu\tau}\right) = -\frac{3Ma \sin^2 \theta}{R_0^2} \left(1 - \frac{2M}{R_0}\right)^{-1/2} \left[1 + \frac{1 - q}{3 - q} \left(\frac{\Delta}{R_0}\right)^2 + \ldots\right]. \tag{24}
\]

\[
{^qK}^{+\pm}_{t\psi} = g^{\psi\psi} \left(\alpha{^qK}^{+\pm}_{t\psi}\right) = \frac{3Ma}{R_0^2} \left(1 - \frac{2M}{R_0}\right)^{1/2} \left[1 + \frac{1 - q}{3 - q} \left(\frac{\Delta}{R_0}\right)^2 + \ldots\right]. \tag{25}
\]

The angular components of the extrinsic curvatures are \({^qK}^{+\pm}_{\theta\theta}\) and \({^qK}^{+\pm}_{\psi\psi}\) and their expressions may be found in exactly the same method and we get:

\[
{^qK}^{+\pm}_{\theta\theta} = {^qK}^{+\pm}_{\psi\psi} = \frac{1}{R_0} \left(1 - \frac{2M}{R_0}\right)^{1/2} \left(1 - \frac{1 - q\Delta}{2 - qR_0} + \ldots\right). \tag{26}
\]

For interior spacetime, we take it to be the flat Minkowski spacetime. So, to the past of the hypersurface at \(r = R_0\), the spacetime \(M^-\) is given by the metric

\[
ds^2 = -\left(1 - \frac{2M}{R_0}\right)dt^2 + d\rho^2 + \rho^2 d\Omega^2 \tag{27}\]
where \( \rho \) is a radial coordinate. The intrinsic metric on the hypersurface from the interior matches with the induced metric from the exterior region. The normal to the hypersurface is \( n^\alpha = (\partial / \partial \rho)^\alpha \). The expressions for the extrinsic curvatures may be determined and the only non-vanishing components are \( qK^\theta_\theta \) and \( qK^\phi_\phi \):

\[
qK^\theta_\theta = qK^\psi_\psi = \frac{1}{R_0} \left( 1 - \frac{1 - q}{2 - q} \frac{\Delta}{R_0} + \ldots \right) \tag{28}
\]

Let us now determine the stress-energy tensor of the thin shell of matter forming the hypersurface joining the two spacetimes. The discontinuities in the extrinsic curvatures are related to the shell’s surface stress-energy tensor \( S^{ab} \).

\[
8\pi S^t_t = \left[ qK^\theta_\theta \right] + \left[ qK^\psi_\psi \right] \tag{29}
\]

\[
-8\pi S^\psi_t = \left[ qK^\theta_\psi \right] \tag{30}
\]

\[
-8\pi S^t_\psi = \left[ qK^\psi_\theta \right] \tag{31}
\]

\[
8\pi S^\theta_\theta = \left[ qK^t_t \right] + \left[ qK^\psi_\psi \right] \tag{32}
\]

The shell’s matter may be assumed to be made of perfect fluid, with density \( \sigma = -S^t_t \), pressure \( p = S^{\theta}_\theta \) and rotating with angular velocity \( \omega = -S^\psi_t / (-S^t_t + S^\psi_\psi) \). The expressions for these components of the energy momentum tensor are:

\[
S^t_\psi = \frac{3Ma \sin^2 \theta}{8\pi R_0^2} \left( 1 - \frac{2M}{r} \right)^{-1/2} \left[ 1 + \frac{1 - q}{3 - q} \left( \frac{\Delta}{R_0} \right)^2 + \ldots \right], \tag{33}
\]

\[
S^\psi_\psi = -\frac{3Ma}{8\pi R_0^4} \left( 1 - \frac{2M}{r} \right)^{1/2} \left[ 1 + \frac{1 - q}{3 - q} \left( \frac{\Delta}{R_0} \right)^2 + \ldots \right], \tag{34}
\]

\[
S^t_t = -\frac{1}{4\pi R_0} \left( 1 - \sqrt{1 - 2M/R_0} \right) \left[ 1 - \frac{1 - q}{2 - q} \Delta + \ldots \right], \tag{35}
\]

\[
S^\theta_\theta = \left[ \frac{(1 - 2M/R_0)^{-1/2}}{8\pi R_0^2} \right] \left[ 1 - M/R_0 - \sqrt{1 - 2M/R_0} \right] - \left[ \frac{(1 - 2M/R_0)^{-1/2}}{8\pi R_0} \right] \left[ 1 - 4M/R_0 + \sqrt{1 - 2M/R_0} \left( \frac{1 - q}{2 - q} \right) \left( \frac{\Delta}{R_0} \right) + \ldots \right]. \tag{36}
\]

\[
S^\theta_\psi = S^\psi_\psi. \tag{37}
\]

Quite naturally, all the expressions of the energy momentum tensor are modified due to the improved notion of fractional differential. The modification takes the thickness of the shell into account. One very interesting notion is the determination of the angular velocity of the shell. The angular velocity is obtained from \( \omega = S^t_\psi / (S^t_t - S^\psi_\psi) \). This gives for \( R_0 \gg 2M \),

\[
\omega_{\text{shell}} = \frac{3a}{2R_0^2} + \frac{3a}{2MR_0} \frac{1 - q}{2 - q} + \ldots. \tag{38}
\]

This expression given above for the angular velocity is different from that obtained in the usual case \[25\] but reduces to it in the limit \( \Delta / R_0 \to 0 \).
Figure 1: The Frolov, Markov and Mukhanov model in which the Schwarzschild black hole has a de Sitter world in the interior. The spacelike hypersurface $\Delta$ represents the matching hypersurface joining the two spacetimes. The $I^+$, $I^-$ and $i^0$ represent the future null, past null and the spatial infinities.

3.2 Matching the Schwarzschild and the de-Sitter spacetimes

The joining of exterior spacetime of the Schwarzschild black hole (taken as the exterior spacetime) with the de-Sitter spacetime has been the subject of many investigations, which were particularly directed to create singularity free models of black hole interior. One particularly interesting application was considered by Frolov, Markov and Mukhanov \[24\] to exemplify their limiting curvature hypothesis. They suggested that inside the Schwarzschild black hole, very close to the singularity, when the Planck scale is reached, there would be corrections to the Einstein theory of gravity. These corrections would not allow the curvature of the spacetime to dynamically grow to infinite values. Instead, the effective curvature of the spacetime would be bounded from below by $\ell_p^2$, where $\ell_p$ is the Planck length. Naturally, this hypothesis implies that there will be no curvature singularity. Instead, the model in \[24\] proposes that very close to the spacetime singularity, where the curvature reaches the $\ell_p^2$, the spacetime makes a transition from the Schwarzschild to the de-Sitter spacetime by passing through a very thin transition layer. The spacetime passes through a deflation stage and instead of singularity, reaches a new inflating universe free of singularity. The matching of these two spacetimes require stress-energy tensors on the joining shell which violate energy conditions.

In the following, we recalculate the stress-energy tensor on the matching shell using the fractional calculus and show that the stress-tensor is modified. The modified stress tensor will be shown to lead to smooth matching of the spacetimes. Let us match the de-Sitter spacetime with the interior Schwarzschild spacetime. The metric for the two spacetimes may be written in a combined form as:

$$ds^2 = f(r) dv^2 + 2dvdr + r^2 d\Omega^2,$$ \hspace{1cm} (39)
where \( f(r) = (2M/r - 1) \) for the Schwarzschild metric and \( f(r) = [(r/l)^2 - 1] \) for the de-Sitter metric. For simplification, let us define a new set of coordinates: \( v = \lambda/\sqrt{J} \). The induced metric on the spacelike surface becomes \( ds^2 = d\lambda^2 + r^2d\Omega^2 \). The normal to this surface is given by:

\[
\mathbf{n}_\alpha = \left[ 0, -\frac{1}{\sqrt{J}}, 0, 0 \right],
\]

(40)

and \( n^\alpha = (1/\sqrt{J}, \sqrt{J}, 0, 0) \). The coordinates in the spacetime is taken to be \( x^\alpha = (v, r, \theta, \phi) \) and that of the hypersurface to be \( y^\alpha = (\lambda, \theta, \phi) \). This implies that \( e^\alpha_q 0_\partial = (1/\sqrt{J})(\partial/\partial v)^\alpha, e^\alpha_q 0_\partial = (\partial/\partial \theta)^\alpha \) and \( e^\alpha_q 0_\partial = (\partial/\partial \phi)^\alpha \).

Let us evaluate the extrinsic curvatures. The general expressions for these quantities for either of these spacetimes are given by the following:

\[
_q^K_{qq} = \frac{1}{2\sqrt{J}} \mathcal{D}_r^{q\lambda\partial}(g_{ve}),
\]

(41)

\[
_q^K_{q\theta} = (\sqrt{J}/2) \mathcal{D}_r^{q\lambda\partial}(g_{\theta\theta}),
\]

(42)

\[
_q^K_{q\phi} = (\sqrt{J}/2) \mathcal{D}_r^{q\lambda\partial}(g_{\phi\phi}).
\]

(43)

For the Schwarschild Metric, which is taken to be the interior spacetime, these expressions are obtained using the equations (92) and (86) are:

\[
\_q^K_{qq} = -\frac{M}{R_0^2} \left[ 2M^2 - 1 \right]^{-1/2} \left[ 1 + 2\frac{1 - q \Delta}{2 - q R_0} + \cdots \right],
\]

(44)

\[
\_q^K_{q\theta} = \frac{1}{R_0} \left[ 2M^2 - 1 \right]^{1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right].
\]

(45)

For the de-Sitter metric, taken to be the external or the future spacetime, the same expressions are given as, using (86) are:

\[
\_q^K_{qq} = \frac{R_0}{l^2} \left[ (R_0/l)^2 - 1 \right]^{-1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right],
\]

(46)

\[
\_q^K_{q\theta} = \frac{1}{R_0} \left[ (R_0/l)^2 - 1 \right]^{1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right].
\]

(47)

The jump in the components of the extrinsic curvatures are given by:

\[
\kappa = [^q^K_{qq}] = \frac{R_0}{l^2} \left[ (R_0/l)^2 - 1 \right]^{-1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]
\]

\[
+ \frac{M}{R_0^2} \left[ 2M^2 - 1 \right]^{-1/2} \left[ 1 + 2\frac{1 - q \Delta}{2 - q R_0} + \cdots \right],
\]

(48)

\[
\lambda = [^q^K_{q\theta}] = -\frac{1}{R_0^2} \left[ 2M^2 - 1 \right]^{1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]
\]

\[
+ \frac{1}{R_0} \left[ (R_0/l)^2 - 1 \right]^{1/2} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right].
\]

(49)
The components of the stress-energy tensor is given by \( S^{\eta}_{\eta} = \lambda/4\pi \) and \( S^{\phi}_{\phi} = (\kappa + \lambda)/8\pi \). Quite noticeably, the values of the energy momentum tensors are markedly different from those obtained in [24]. The values differ by quantities which are proportional to the ratio \((\Delta/R_0)\), and hence by choosing the value of this ratio judiciously, it can be easily seen that the energy momentum tensor can be made to vanish. Hence, one may match the two spacetimes smoothly across a spacelike hypersurface.

4 Junction conditions for null hypersurfaces

Let us consider a null hypersurface that partitions the 4-dimensional spacetime into two regions \([\mathcal{M}^+, g^{\mu\nu}(x^+)\)] and \([\mathcal{M}^-, g^\mu\nu(x^-)\)], which we shall conveniently call as the future and the past respectively. Let us denote the coordinates of the spacetime as \(x^\alpha\), \(\alpha = 0, 1, 2, 3\), whereas the coordinates on either side of the hypersurface will be denoted by \(y^a\), \(a = 1, 2, 3\), which will mean the collective coordinates \((\lambda, \theta^A)\), where \(\theta^A, A = (2, 3)\) denotes the variables on the two-dimensional cross-sections of the hypersurface. On each side of the hypersurface, one may construct the tangents to the generators of the null hypersurface \((\ell^a)\) and the transverse spacelike vectors \((e^a_A)\), which are tangents to the cross-sections (taken to be compact) of the hypersurface. These vectors shall be denoted by:

\[
\ell^a = e^a_{\lambda} = \left(\frac{\partial x^a}{\partial \lambda}\right)_{\theta^A}; \quad e^a_A = \left(\frac{\partial x^a}{\partial \theta^A}\right)_{\lambda},
\]

(50)

with the following properties: \(\ell^a \ell_\alpha = 0\), \(\ell_\alpha e^a_A = 0\). These vectors may be constructed for both sides of the null hypersurface. Further, on each side, the basis needs four vectors and the fourth vector, will be taken to be a null vector. It will be denoted by \(n^a\) with the following properties: \(\ell^an_\alpha = -1\), \(n^an_\alpha = 0\), \(n_\alpha e^a_A = 0\).

The typical situation with a null surface is that the usual extrinsic curvature, \(K_{ab} = (1/2) (\mathcal{L}_{\ell} g_{\alpha\beta}) e^a_{\alpha} e^b_{\beta}\), corresponding to the normal to the hypersurface is continuous, since the normal is also the tangent \(\ell^a\). So, one usually defines the transverse component of the extrinsic curvature corresponding to the null vector field normal to the transverse cross-sections of the hypersurface. This vector is \(n^a\), such that \(\ell.n = -1\). The transverse extrinsic curvature may be defined as \(C_{ab} = (1/2) (\mathcal{L}_n g_{\alpha\beta}) e^a_{\alpha} e^b_{\beta}\). The stress-energy tensor of the shell is given by:

\[
S^{\alpha\beta} = \mu \ell^a \ell^b + p \sigma^{AB} e^a_A e^b_B, \quad \text{where} \quad \mu = (-1/8\pi)\sigma^{AB}[C_{AB}] \text{ is the shell’s surface density and } p = (-1/8\pi)[C_{\lambda\lambda}] \text{ is the surface pressure.}
\]

4.1 Null Charged Shell Collapsing on a Charged Black Hole

Let us consider a spherically symmetric charged black hole of mass \(M\) and charge \(Q\) on which a null charged shell of mass \(E\) and charge \(q\) collapses. Outside the shell, the spacetime outside the total configuration may be viewed as a spherically symmetric Reissner-Nordstrom type geometry with mass \((M + E)\) and charge \((Q + q)\). As before, the spacetimes outside and inside the shell shall be denoted by
\[ ds^2 = -f_\pm(r) + dv^2 + 2dvdr + r^2d\Omega^2 \] (51)

\[ f_+(r) = 1 - \frac{2(M + E)}{r} + \frac{(Q + q)^2}{r^2} \] (52)

\[ f_-(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \] (53)

The coordinates of the spacetime is given by \( x^\alpha = (v, r, \theta, \phi) \). The surface of the shell is given by \( v = v_0 \) with coordinates of the shell being \( y^\alpha = (r, \theta, \phi) \). The vector fields are given by \( e^\alpha_\alpha \partial_\alpha = \ell^\alpha = -(\partial/\partial r)^\alpha, e^\alpha_\theta \partial_\alpha = (\partial/\partial \theta)^\alpha, e^\alpha_\phi \partial_\alpha = (\partial/\partial \phi)^\alpha \). Note that the vector \( \ell^\alpha \) is the generator of the null surface. The transverse null vector required to complete the basis is \( n^\alpha = \{f_\pm(r)/2\} (\partial/\partial r)^\alpha \).

The metric is continuous across the shell. Let us check that the extrinsic curvature of the null surface corresponding to the null normal \( \ell^\alpha \) is also continuous on either side of the surface. This is always true for the non-fractional case and precisely for this reason, the concept of the transverse curvatures have been introduced. We show that for the fractional case too, the extrinsic curvatures corresponding to the null normal of the surface is continuous. For the interior solution, we get that \( \ell^- = (0, 1, 0, 0) \) and hence, the components of the fractional extrinsic curvature on the cross-sections are:

\[ qK^-_{\theta\theta} = (1/2)q \mathcal{L}_\ell g_{\theta\theta}. \] Using the formulae from the appendix, equation (86), we get

\[ qK^-_{\phi\phi} = R_0 \sin^2 \theta \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \]

These two equations may be combined to the following form:

\[ qK^-_{AB} = R_0^2 \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]\sigma_{AB} \] (54)

For the exterior solution too, the null normal is given by \( \ell^+ = (0, 1, 0, 0) \) and the extrinsic curvature corresponding to this null normal is \( qK^+_{AB} = \frac{1}{2}q \mathcal{L}_\ell g_{AB} \) is given by:

\[ qK^+_{AB} = R_0^2 \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]\sigma_{AB} \] (55)

This implies that the extrinsic curvatures are also continuous \( qK^+_{AB} = qK^-_{AB} \).

The transverse extrinsic curvature is not continuous for this metric. The expression for \( qC^+_{\theta\theta} \) is given by

\[ qC^+_{\theta\theta} = (1/2) \left[ n^\rho \mathcal{D}^g_{\rho - \Delta r}(g_{\theta\theta}) \right] \]

\[ = f_+(r) R_0 \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \] (56)

Similarly for \( qC^+_{\phi\phi} \), we get:

\[ qC^+_{\phi\phi} = f_+(r) R_0 \sin^2 \theta \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]. \] (57)
So these two expressions may be combined to give:

\[ qC^+_{AB} = \frac{f_+(r)}{R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \sigma_{AB}, \]

where \( \sigma_{AB} = R_0^2 + R_0^2 \sin^2 \theta \). Similarly, for the interior spacetime, the transverse component of the extrinsic curvature is given by:

\[ qC^-_{AB} = \frac{f_-(r)}{R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \sigma_{AB} \]

These equations immediately imply that the shell’s surface pressure is zero for this case. The shell’s surface density is

\[ \mu = -\frac{1}{4\pi R} \left( \frac{2Qq + q^2}{R^2} - \frac{2E}{R} \right) \left[ 1 - \frac{1 - q \Delta}{2 - q R} + \cdots \right]. \]

This relation clearly implies that to satisfy the weak energy condition, we must have

\[ 2E \geq \frac{2Qq + q^2}{M + \sqrt{M^2 - Q^2}}. \]

As a simple application, let us study if the charged black hole may be overcharged, so that the total charge \((Q + q)\) exceed the total mass \((M + E)\). It is a simple matter to check that the condition for overcharging violates the weak energy condition. So, even in the fractional modification, a charged black hole cannot be overcharged.

### 4.2 Matching Schwarschild and de-Sitter spacetimes across horizons

Let’s start with a general form of the metric and then we shall specialize to the individual cases. The general form for a spherical symmetric metric in the advanced Eddington -Finkelstein coordinates is given by:

\[ ds^2 = -f(r) \, dv^2 + 2dv \, dr + r^2 \, d\Omega^2. \]

The coordinates of the spacetime is given by \( x^\alpha = (v, r, \theta, \phi) \). Let us assume a null hypersurface (a shell) given by \( r = r_0 \) with coordinates of the shell being \( y^\alpha = (v, \theta, \phi) \). The null surface is foliated by compact surface \( S^2 \). The vector fields tangent to the sphere are given by, \( e_\theta^\alpha = (\partial/\partial \theta)^\alpha \) \( e_\phi^\alpha = (\partial/\partial \phi)^\alpha \). Let us now determine the set of null vectors tangent to the null surface which is given by the relation \( f(r_0) = 0 \). The generator of the null surface is \( \ell^\alpha = (\partial/\partial v)^\alpha \) and the transverse null vector is \( n^\alpha = -(\partial/\partial r)^\alpha \).

Let us consider the interior metric \((\mathcal{M}^-)\) to be the de- Sitter spacetime:

\[ ds^-_\infty = -\left[ 1 - \left( \frac{r}{l} \right)^2 \right] dv^2 + 2dv \, dr + r^2 \, d\Omega^2. \]
As usual, the standard extrinsic curvatures associated to the null normals of the
are continuous and hence let us calculate the transverse extrinsic curvatures $q C^-_{\theta \theta}$
and $q C^-_{\phi \phi}$:

$$q C^-_{\theta \theta} = R_0 \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right],$$

and similarly for $q C^-_{\phi \phi}$, we get

$$q C^-_{\phi \phi} = R_0 \sin^2 \theta \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right].$$

So combining them together, we get:

$$q C^-_{AB} = \frac{1}{R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \sigma_{AB}, \hspace{1cm} (64)$$

where $\sigma_{AB} = R_0^2 + R_0^2 \sin^2 \theta$. The quantity $q C^-_{vv}$ gives:

$$q C^-_{vv} = \frac{1}{a^2 R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]. \hspace{1cm} (65)$$

The exterior spacetime is taken to be the Schwarzschild spacetime $(M^+)$, with
the metric

$$ds^2 = -\left(1 - 2m/r\right)dv^2 + 2dvdr + r^2 d\Omega^2 \hspace{1cm} (66)$$

Again, the coordinate on null shell are $(v, \theta, \phi)$. Let us calculate transverse curva-
tures. Just as in the previous case, the result is

$$q C^+_AB = \frac{1}{R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \sigma_{AB}. \hspace{1cm} (67)$$

The $q C^+_vv$ is given by:

$$q C^+_vv = -m \frac{1}{R_0^2} \left[ 1 + 2 \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]. \hspace{1cm} (68)$$

Let us calculate the quantities associated with the shell. The surface density $\mu = 0$.
The pressure is given by:

$$p = -\frac{1}{8\pi} \left[ \left( \frac{R_0}{l^2} + \frac{m}{R_0^2} \right) - \left( \frac{R_0}{l^2} - \frac{2m}{R_0^2} \right) \frac{1 - q \Delta}{2 - q R_0} + \cdots \right]. \hspace{1cm} (69)$$

So, if the matching surface is the horizon, $R_0 = 2m = l$ and hence the pressure
must by non vanishing. However, in the fractional modification, we may choose
the $\Delta/R_0$ judiciously to get a smooth matching of the two spacetimes.
Figure 2: The Barrabes- Israel model in which the inner horizon of the non-extremal Reissner-Nordstrom black hole is joined to a de Sitter world in the interior. The null surface \( \Delta \) represents the matching hypersurface joining the two spacetimes. The \( I^+ \), \( I^- \) and \( i^0 \) represent the future null, past null and the spatial infinities.

4.3 Matching the Reissner-Nordstrom and the de-Sitter spacetimes

Let us determine the criteria for matching the Reissner-Nordstrom spacetime and the de-Sitter spacetimes on the inner horizon of the non-extremal charged black hole. Interestingly the matching is to be carried out on the inner horizon as was first proposed in [23]. The Penrose diagram is given in fig 2.

The external spacetime is the Reissner-Nordstrom spacetime \((M^+)\) with the following metric:

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dv^2 + 2dvd\rho + r^2d\Omega^2.
\end{align*}
\]

The co-ordinates on null shell are \((v, \theta, \phi)\) and \( R_0 = m - \sqrt{m^2 - Q^2} \). The transverse extrinsic curvatures \( qC_{\theta\theta}^+ \) and \( qC_{\phi\phi}^+ \) may be written as

\[
qC_{AB}^+ = \frac{1}{R_0} \left[ 1 - \frac{1 - q \Delta}{2 - q R_0} + \cdots \right] \sigma_{AB}.
\]

\[
qC_{\theta v}^+ = \left( \frac{Q^2}{R_0^2} - \frac{m}{R_0^2} \right) - \frac{1 - q \Delta}{2 - q R_0} \left( \frac{2m}{R_0^2} - \frac{3Q^2}{R_0^3} \right) + \cdots.
\]

The interior spacetime is the de-Sitter spacetime with matching at \( R_0 = l \). The curvatures have already been found out in the previous subsection. The properties of the shell may be immediately obtained. The surface density \( \mu = 0 \) but the pressure is

\[
p = -\frac{1}{8\pi} \left[ \left( \frac{R_0}{l^2} + \frac{m}{R_0^2} - \frac{Q^2}{R_0^3} \right) - \left( \frac{R_0}{l^2} - \frac{2m}{R_0^2} + \frac{3Q^2}{R_0^3} \right) \frac{1 - q \Delta}{2 - q R_0} + \cdots \right].
\]
So, again, in the standard case, when $\left(\Delta/R_0\right) = 0$ the spacetimes matching requires a shell which shall hold this pressure and hence the matching is not smooth. Incidentally, in [23], the authors noted that for special case like $3l^2 = Q^2$, there is a smooth matching of the two spacetimes. This matching is a special case. The fractional generalisation however, shows that it is possible to adjust the parameter $\left(\Delta/R_0\right)$ to get a vanishing pressure and hence, a smooth matching of the spacetimes on the hypersurface.

5 Discussions

In this paper, we have developed the fractional generalisation of the Israel- Darmois-Lanczos junction conditions for spacelike/timelike as well as for null hypersurfaces. We have observed that there is a significant modifications due to the fractional generalisation. First, due to the definition of the fractional differentiation through an integral, it automatically incorporates the non-local spacetime correlations into itself. As a manifestation of this, the thickness of the shell gets incorporated into to the values of the shell’s properties like the energy and pressure. We have taken several examples and have demonstrated that by choosing this thickness parameter $\Delta/R_0$ judiciously, it is possible to join many spacetimes smoothly across spacelike timelike or null hypersurfaces.

A point of crucial importance is that must be mentioned here is that the dimension of the spacetime has been taken to be integral. Fractal dimensions may also be possibly included. In fact, general relativity may also be suitably adapted for fractal spacetimes, which would also require revising our notions of coordinate transformations and covariance. However, we have not attempted this path, of altering the theory of general relativity to recast it for all spacetime dimensions, integral or non-integral. Instead, we have looked for alternate avenues by generalising the notion of Lie-derivative which is intrinsically attached to the differentiable structure of the spacetime [24]. Unlike the usual Levi-Civita connection, there is no requirement of the metric and hence the Lie derivative is much primitive and turns out to be most useful. This generalisation has been used to construct the extrinsic curvatures and hence the surface properties of the shell. In the appendix, we have developed the reasons as to why we should expect that there should be some modification in the dynamics as well. We show that the Einstein equations modify significantly. The ramifications of these issues shall be dealt with in future papers.

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6 Appendix

6.1 Fractional derivative

The Riemann-Liouville definition of fractional calculus is usually given in the form of an integral transform of a specialised type, as given below [1, 2, 3]:

\[ D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - y)^{\nu-1} f(y) \, dy, \]  

(74)

where \( \nu > 0 \). This definition is the foundation of the theory of fractional differentiation (and integration), but breaks down at the integral points, \( \nu = 0, -1, -2, \ldots \). At those points the integration may however be replaced by the ordinary integration formula.

The Caputo derivative is a modification of the Riemann-Liouville derivative where suitable modification have been applied so that it satisfies all the rules of a derivative. The Caputo derivative is defined as follows [1, 2, 3]:

\[ D_x^q f(x) = \frac{1}{\Gamma(1 - q)} \int_a^x (x - y)^{-q} \frac{\partial f(y)}{\partial y} \, dy, \]  

(75)

where the superscript \( q \) denotes the fractional parameter, \( 0 < q \leq 1 \). To take into account of the tensor indices, a further modification is added in [26] as follows:

\[ D_{x,k}^q (x'^i) = \frac{\Gamma(2 - q)}{\Gamma(1 - q)(\Delta)^{1-q}} \int_x^{x'} \frac{\partial g_{ij}}{\partial y^k} (x' - y)^{-q} \, dy. \]  

(76)

For example, if the integration of the metric variable \( g_{ij}(r) \) is to be carried out from one end of the shell (of thickness \( \Delta \)) to the other, the above definition gives:

\[ D_{r-\Delta}^q (g_{ij}) = \frac{\Gamma(2 - q)}{\Gamma(1 - q)(\Delta)^{1-q}} \int_{r-\Delta}^r \frac{\partial g_{ij}(w)}{\partial w} (r - w)^{-q} \, dw, \]  

(77)

where the integration limits have been chosen appropriately. This definition has been utilised in this paper.

6.2 Beta Function and relation to the Hypergeometric functions

In the paper, we have frequently made use of the Beta function, defined as:

\[ B_x(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} \, dt, \]  

(78)

where \( a > 0, b > 0 \). In general, we may also write it as

\[ B_x(a, b) = x^a \left( \frac{1}{a} + \frac{1 - b}{1 + a} x + \cdots \right), \]  

(79)
and hence the above equation implies naturally that:

\[ B_{\frac{\Delta}{R_0}}(1-q,2) = \left( \frac{\Delta}{R_0} \right)^{1-q} \frac{1}{1-q} \left[ 1 - \frac{1-q}{2-q} \frac{\Delta}{R_0} + \cdots \right]. \]  

(80)

If, \((\Delta/R) \neq 1\), we may also use the relation between Beta function and Hypergeometric function:

\[ B_x(a,b) = \frac{x^a}{a} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{x^a}{a} \sum_{n=0}^{\infty} \frac{(a)_n x^n}{n!}, \]

(81)

where

\[ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = 1 + \frac{ab}{c} \frac{c}{(c+1)} + \cdots. \]  

(82)

This gives the following two useful forms:

\[ B_{\frac{\Delta}{R_0}}(1-q,-1) = \left( \frac{\Delta}{R_0} \right)^{1-q} \frac{1}{1-q} \left[ 1 + 2 \frac{1-q \Delta}{2-q R_0} + \frac{1-q}{3-q} \left( \frac{\Delta}{R_0} \right)^2 + \cdots \right] \]  

(83)

\[ B_{\frac{\Delta}{R_0}}(1-q,-2) = \left( \frac{\Delta}{R_0} \right)^{1-q} \frac{1}{1-q} \left[ 1 + 3 \frac{1-q \Delta}{2-q R_0} + 12 \frac{1-q}{3-q} \left( \frac{\Delta}{R_0} \right)^2 + \cdots \right]. \]  

(84)

These equations have been used below to derive the results used in the main text. Let us first evaluate \( D_{r-\Delta,r}(r^2) \). Using eqn. (77), we get:

\[ D_{r-\Delta,r}(r^2) = 2 \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int_{r-\Delta}^{r} (r-w)^{-q} dw, \]

(85)

where the integration limit is chosen to take the thickness of the hypersurface into account. Using the change of variables, \( r-w = t \), the limit also changes from \( r-\Delta \) to \( \Delta \) and \( r \) to 0. This gives us:

\[ D_{r-\Delta,r}(r^2) = 2 \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int_{0}^{\Delta} (r-t)^{-q} dt. \]  

(86)

Again, make a change of variables \( t/r = y \) and also put \( r = R_0 \) as we match on the hypersurface placed at \( r = R_0 \).

\[ D_{r-\Delta,r}(r^2) = 2 \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} \int_{0}^{\Delta} (1-y)^{-q} dy \int_{0}^{\Delta} (1-y)^{-q} \]  

(87)

\[ = 2 \frac{\Gamma(2-q)}{\Gamma(1-q)(\Delta)^{1-q}} R_0^{2-q} B_{\frac{\Delta}{R_0}}(1-q,2). \]  

(88)

Using the form of eqn. (80) and property of Gamma function \((1-q)\Gamma(1-q) = \Gamma(2-q)\), we get,

\[ D_{r-\Delta,r}(r^2) = 2R_0 \left[ 1 - \frac{1-q \Delta}{2-q R_0} + \cdots \right]. \]  

(89)
For $\mathcal{D}_{r-\Delta,r}^q(r^{-1})$, a similar calculation yields the following result:

$$\mathcal{D}_{r-\Delta,r}^q(r^{-1}) = \frac{\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-1-q} \int_0^{\Delta R_0} (1-y)^{-q} \, dy, \quad (90)$$

$$= -\frac{\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-1-q} B \frac{\Delta}{R_0} (1-q,-1). \quad (91)$$

Using eqn. (83) and property of Gamma function i.e $(1-q)\Gamma(1-q) = \Gamma(2-q)$ we get:

$$\mathcal{D}_{r-\Delta,r}^q(r^{-1}) = -\frac{1}{R_0^q} \left[ 1 + 2\frac{1-q}{2-q} \Delta + \ldots \right]. \quad (92)$$

The computation for $\mathcal{D}_{r-\Delta,r}^q(r^{-2})$ proceeds along similar lines and gives:

$$\mathcal{D}_{r-\Delta,r}^q(r^{-2}) = -\frac{2\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-2-q} \int_0^{\Delta R_0} (1-y)^{-3} \, dy \quad (93)$$

$$= -\frac{2\Gamma(2-q)}{\Gamma(1-q)\Delta^{1-q}} R_0^{-2-q} B \frac{\Delta}{R_0} (1-q,-2), \quad (94)$$

which using the equation (84) and $(1-q)\Gamma(1-q) = \Gamma(2-q)$ gives us:

$$\mathcal{D}_{r-\Delta,r}^q(r^{-2}) = -\frac{2}{R_0^q} \left[ 1 + 3\frac{1-q}{2-q} \Delta R_0 + \ldots \right]. \quad (95)$$

### 6.3 Modification of the Einstein equations

The fractional derivative leads to a modification of the partial derivative. From the previous sections, we note that the Caputo derivative modifies the derivative through a factor $(1-q)\Delta/R_0$. Let us use this form to write for any function $g$, a modification of the derivative operator as:

$$\mathcal{D}g = \partial g \left[ 1 \pm \beta (1-q) \frac{\Delta}{R_0} \pm \ldots \right] \quad (96)$$

here $\beta$ is some constant, and $q$ denotes the fractional parameter. Using this definition of the derivative, the relation between new Christoffel symbol (for non-Levi-Civita connection) and old Christoffel symbol (for Levi-Civita connection) becomes:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} \pm \alpha (1-q) \frac{\Delta}{R_0} + \ldots \quad (97)$$

where $\alpha$ is some constant. This gives a relation between old Riemann tensor and new Riemann tensor. The usual definition

$$R^{\rho}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\rho}_{\beta\delta} - \partial_\delta \Gamma^{\rho}_{\beta\gamma} + \Gamma^{\nu}_{\beta\delta} \Gamma^{\rho}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma} \Gamma^{\rho}_{\nu\delta} \quad (98)$$

is modified to a new definition

$$\tilde{R}^{\rho}_{\beta\gamma\delta} = \mathcal{D}_\gamma \tilde{\Gamma}^{\rho}_{\beta\delta} - \mathcal{D}_\delta \tilde{\Gamma}^{\rho}_{\beta\gamma} + \tilde{\Gamma}^{\nu}_{\beta\delta} \tilde{\Gamma}^{\rho}_{\nu\gamma} - \tilde{\Gamma}^{\nu}_{\beta\gamma} \tilde{\Gamma}^{\rho}_{\nu\delta}. \quad (99)$$
The relation between the two Riemann tensors is given by:

\[
\tilde{R}^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} \pm (1 - q) \frac{\Delta}{R_0} [\mu] \pm \cdots
\]  

(100)

where \( \mu = \left[ \beta \partial_\gamma \Gamma^\alpha_{\beta\delta} + \partial_\gamma \tilde{\alpha} - \beta \partial_\delta \Gamma_\gamma^\alpha - \partial_\delta \tilde{\alpha} \pm \tilde{\alpha} (\Gamma^\nu_{\beta\gamma} \pm \Gamma^\nu_{\beta\gamma} \mp \Gamma^\nu_{\alpha\delta}) \right] \). The Ricci tensor is

\[
\tilde{R}_{\alpha\beta} = R_{\alpha\beta} \pm (1 - q) \frac{\Delta}{R_0} [\eta] \pm \cdots
\]  

(101)

Similarly Ricci scaler is :

\[
\tilde{R} = R \pm (1 - q) \frac{\Delta}{R_0} [\eta] \pm \cdots
\]  

(102)

where \( \eta \) and \( \tau \) are some constants. The Einstein field equations get modified as well:

\[
\tilde{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \tilde{R} = 8\pi G T_{\alpha\beta} \pm (1 - q) \frac{\Delta}{R_0} \left[ \eta - \tau \frac{1}{2} g_{\alpha\beta} \right] \pm \cdots
\]  

(103)

So, the dynamics of the gravitational fields get modified for the fractional generalisation.

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