On a Spectral Theorem of Weyl

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Abstract
We give a geometric proof of a theorem of Weyl on the continuous part of the spectrum of Sturm-Liouville operators on the half-line with asymptotically constant coefficients. Earlier proofs due to Weyl and Kodaira depend on special features of Green’s functions for linear ordinary differential operators; ours might offer better prospects for generalization to higher dimensions, as required for example in non-commutative harmonic analysis.

1 Introduction
The purpose of this paper is to present a new approach to an old theorem of Hermann Weyl on the spectral theory of self-adjoint Sturm-Liouville operators on a half-line. Our aim is to invoke methods that are geometric in spirit, and more amenable to generalization, for instance to Plancherel formulas for spherical functions (this is an area that is closely related to Weyl’s work). We shall however stay fairly close to Weyl’s original context in this article.

Sturm-Liouville theory is of course concerned with the eigenvalues and eigenfunctions of linear differential operators

\[ D = -\frac{d}{dx} \cdot p(x) \cdot \frac{d}{dx} + q(x), \]

initially on a closed interval \([a, b]\). Assume for simplicity that \(p(x)\) and \(q(x)\) are smooth, real-valued functions on \([a, b]\), with \(p(x)\) positive everywhere. In examining the solutions of the eigenvalue problem

\[ Df_\lambda = \lambda f_\lambda, \]

it is conventional to impose boundary conditions, and the most obvious choice is

\[ f_\lambda(a) = 0 = f_\lambda(b). \]
The elements of Sturm-Liouville theory can then be summarized as follows \[^{1}\] using the $L^2$-inner product $\langle f, h \rangle$:

**1.1 Theorem.** The eigenvalues $\lambda$ for the above problem are real numbers, and each has multiplicity one. The set of all eigenvalues is a discrete subset of $\mathbb{R}$, bounded below, and if $h$ is any smooth function on $[a, b]$, then

$$h(x) = \sum_{\lambda} \frac{\langle f_\lambda, h \rangle}{\langle f_\lambda, f_\lambda \rangle} f_\lambda(x)$$

for $x \in (a, b)$.

In an influential paper \[^{Wey10}\] from early in his career, Weyl developed an analogous theory for Sturm-Liouville operators on $[0, \infty)$. Weyl’s paper addressed many issues, but our concern here is his treatment of the continuous spectrum of a Sturm-Liouville operator, and especially his version for the continuous spectrum of the expansion theorem above, which we shall now describe.

Weyl assumes that the coefficient functions $p(x)$ and $q(x)$ in (1.1) converge sufficiently rapidly to the constants 1 and 0, respectively, as $x$ tends to infinity. For the purposes of this introduction, let us assume even more, namely that

$$p(x) \equiv 1 \quad \text{and} \quad q(x) \equiv 0 \quad \text{if} \quad x \gg 0$$

(this assumption is too strong to be interesting in applications, but it allows us to quickly introduce Weyl’s ideas). For each $\lambda \in \mathbb{C}$ there is a one-dimensional space of eigenfunctions \[^{2}\] $F_\lambda$ for $D$ that satisfy the boundary condition

$$F_\lambda(0) = 0.$$  

If we focus on the case where $\lambda > 0$, and if we choose, as we may, $F_\lambda$ to be nonzero and real-valued, then our assumptions on the coefficient functions $p(x)$ and $q(x)$ imply that

$$F_\lambda(x) = c(\lambda)e^{i\sqrt{\lambda}x} + \overline{c(\lambda)}e^{-i\sqrt{\lambda}x} \quad \text{if} \quad x \gg 0,$$

for some nonzero $c(\lambda) \in \mathbb{C}$. We can now formulate Weyl’s result, at least in the simplified context of (1.4) that we are currently discussing.

\[^{1}\]For a precise formulation of the theorem and a thorough account of its proof, see for example \[^{DS88}\] Chapter XIII, especially Theorem 3 in Section 4.

\[^{2}\]We write $F_\lambda$ rather than $f_\lambda$ as a reminder that the eigenfunction need not be square-integrable in this context; in fact it is better to view it as a distribution.
1.2 Theorem. If \( h \) is a smooth, compactly supported function on \([0, \infty)\), then

\[
h(x) = \sum_{\lambda < 0} \frac{\langle F_\lambda, h \rangle}{\langle F_\lambda, F_\lambda \rangle} F_\lambda(x) + \frac{1}{4\pi} \int_0^\infty \frac{\langle F_\lambda, h \rangle}{|c(\lambda)|^2} F_\lambda(x) \frac{d\lambda}{\sqrt{\lambda}}
\]

for \( x \in (0, \infty) \). The first sum is over the square-integrable eigenfunctions associated to negative eigenvalues that satisfy the boundary condition (1.5), and there are finitely many of these.

We shall approach Weyl's theorem by comparing the Sturm-Liouville operator \( D \) to the simpler operator \( D_0 = -\frac{d^2}{dx^2} \) on \((-\infty, \infty)\). Our argument is roughly as follows. General theory guarantees an integral decomposition

\[
h(x) = \int \langle F_\lambda, h \rangle F_\lambda(x) d\mu(\lambda)
\]

for some measure on the spectrum of \( D \). If \( h \) is supported sufficiently far away from 0 \( \in [0, \infty) \), then, in view of (1.6), the integral in (1.7) can be viewed as a sort of approximate decomposition of \( h \) over the spectrum of \( D_0 \). It is not exact because (1.7) involves the negative spectrum of \( D \), which has no counterpart for \( D_0 \), and more crucially because it involves only some, but not all the eigenfunctions on \( D_0 \) (there are two linearly independent eigenfunctions of \( D_0 \) for every \( \lambda > 0 \), but only one specific linear combination contributes to the integral). However we can put together the integral formulas in (1.7) for all the forward translates of \( h \) along the line by averaging, and then invoke the translation-invariance of \( D_0 \) to obtain an exact formula for the spectral decomposition of \( D_0 \) in terms of the measure \( \mu \). Finally, we can invert this formula to describe the (positive part of the) spectral theory of \( D \) in terms of the spectral theory of \( D_0 \), which is of course known, and from this we shall recover Weyl's formula.

We shall give an abstract account of our approach in Section 3, where the main results are Theorems 3.3 and 3.6. Our reason for doing this is that other interesting instances of the abstract framework arise in harmonic

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\(^3\)We won't discuss in this introduction the nature of the convergence in the eigenfunction expansion, but see for example [Ban08] or [EK08], as well as the original source, of course, for more details. We shall make precise statements in the language of spectral theory later in the paper.
analysis, as we shall note in Section 3 (however we shall postpone until a future paper a detailed treatment of these examples). We shall apply our method to the continuous spectrum of Sturm-Liouville operators in Section 4, and we shall address the discrete part of the spectrum of $D$ in Section 5. But prior to all of that we shall quickly review the standard approach in Section 2 for the sake of comparison.

This work grew in part out of a project in noncommutative geometry [CCH16, CH16], which led us to try to understand more about the Plancherel formula, and hence Weyl’s theorem. Remarking on Weyl’s influence on representation theory, Borel [Bor01, p.38] writes that

“It was the reading of [Weyl’s 1910 paper] which suggested to Harish-Chandra that the measure should be the inverse of the square modulus of a function in $\lambda$ describing the asymptotic behaviour of the eigenfunctions ... and I remember well from seminar lectures and conversations that he never lost sight of that principle, which is confirmed by his results in the general case.”

We believe that the approach to Weyl’s theorem presented here offers good prospects for an alternative approach to some of Harish-Chandra’s results.

2 Review of Kodaira’s Approach

In this section we shall very briefly review the approach to Theorem 1.2 developed by Weyl [Wey10] and then substantially improved by Kodaira [Kod49]; see also [Wey50]. For brevity we shall continue to consider only the simplest possible case of a Sturm-Liouville operator $D$ on $[0, \infty)$ whose coefficients are eventually constant, as in (1.4).

We shall take it for granted that $D$ defines an essentially self-adjoint operator on $L^2(0, \infty)$, with initial domain the smooth, compactly supported functions on $[0, \infty)$ that vanish at 0, as in (1.5). We shall concentrate here on the positive part of the spectrum of $D$ and hence the integral expression in Theorem 1.2.

Of interest to us, therefore, are the spectral projections $P_{[\alpha, \beta]}$ for $D$ associated to closed intervals $[\alpha, \beta]$ in the positive real numbers. Following Kodaira we shall prove the following version of Weyl’s theorem:

2.1 Theorem. If $\beta > \alpha > 0$, then spectral projection $P_{[\alpha, \beta]}$ for $D$ is given by the
formula

\[(P_{\alpha,\beta} h)(x) = \int_0^\infty p_{\alpha,\beta}(x,y)h(y)\,dy,\]

where

\[p_{\alpha,\beta}(x,y) = \frac{1}{4\pi} \int_\alpha^\beta F_\lambda(x)F_\lambda(y) \frac{1}{|c(\lambda)|^2} \frac{d\lambda}{\sqrt{\lambda}},\]

To begin, assume that D is any self-adjoint Hilbert space operator. If \(\alpha\) and \(\beta\) are any real numbers (not necessarily positive, at this stage) that are not in the spectrum of D, then according to the Riesz functional calculus,

\[P_{\alpha,\beta} = \frac{1}{2\pi i} \int_\Gamma (\nu - D)^{-1} \,d\nu.\]

where the contour \(\Gamma\) is indicated in the figure.

The contributions to the integral (2.2) from the vertical components of the contour \(\Gamma\) decrease to zero as the height of the contour decreases to zero, and so

\[P_{\alpha,\beta} = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \left( \int_{\alpha - i\varepsilon}^{\beta - i\varepsilon} (\nu - D)^{-1} \,d\nu - \int_{\alpha + i\varepsilon}^{\beta + i\varepsilon} (\nu - D)^{-1} \,d\nu \right),\]

or equivalently

\[P_{\alpha,\beta}(D) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_\alpha^\beta (D - \lambda - i\varepsilon)^{-1} - (D - \lambda + i\varepsilon)^{-1} \,d\lambda.\]

The integral on the right-hand side of (2.4) defines, for any \(\varepsilon > 0\), an operator of norm no more than 1. So, by approximating more general self-adjoint operators D by operators that do not contain \(\alpha\) or \(\beta\) in their spectrum, we find that
2.2 Lemma (Kodaira). The formula (2.4) holds for any self-adjoint operator \( D \) and any interval \([\alpha, \beta]\), as long as \( \alpha \) and \( \beta \) do not belong to the point spectrum of \( D \).

The value of the Kodaira formula\(^4\) is that when \( D \) is a Sturm-Liouville operator, the resolvent operators, particularly in the combination they appear in (2.4), may be computed quite explicitly. First of all, if \( \nu \in \mathbb{C} \setminus \mathbb{R} \), then we may write

\[
(D - \nu)^{-1} h = \int_0^{\infty} k_\nu(x, y) h(y) \, dy
\]

The kernel \( k_\nu(x, y) \) has the following properties:

(a) \( k_\nu \) is continuous on \([0, \infty) \times [0, \infty)\), smooth away from the diagonal, and converges to zero as \( x \to \infty \) or \( y \to \infty \).

(b) \( k_\nu(x, y) = 0 \) when \( x = 0 \) or \( y = 0 \).

(c) \( k_\nu(y, x) = k_\nu(x, y) \).

(d) \( (D - \nu)k_\nu(\cdot, y) = \delta_y \)

Now let \( F_\nu \) be a \( \nu \)-eigenfunction for \( D \) that vanishes at \( 0 \), and let \( G_\nu \) be a \( \nu \)-eigenfunction that vanishes at infinity. It follows rather easily from the above list of properties that

\[
k_\nu(x, y) = \frac{1}{w(\nu)} F_\nu(x) G_\nu(y) \quad \text{when } x < y,
\]

for some constant \( w(\lambda) \neq 0 \) (independent of \( x \) and \( y \)). Using (d) one computes that

\[
w(\nu) = -w_x(F_\nu, G_\nu) = -p(x) \left( F_\nu(x) G_\nu'(x) - F_\nu'(x) G_\nu(x) \right)
\]

(this is independent of \( x \)). This is an application of the relation

\[
\int_a^b (Dg)(y) h(y) \, dy - \int_a^b g(y)(Dh)(y) \, dy = w_a(g, h) - w_b(g, h)
\]

where

\[
w_x(g, h) = p(x) \left( g(x)h'(x) - g'(x)h(x) \right).
\]

\(^4\)The same formula was obtained in a slightly different context by Titchmarsh [Tit46], and the term “Kodaira-Titchmarsh formula” is therefore often used.
To return to Kodaira’s formula, we find that the integrand in (2.4) is represented by the integral kernel

\[ w(\lambda + i\epsilon)^{-1}F_{\lambda+i\epsilon}(x)G_{\lambda+i\epsilon}(y) - w(\lambda - i\epsilon)^{-1}F_{\lambda-i\epsilon}(x)G_{\lambda-i\epsilon}(y) \]

when \( x < y \). At this point we finally use the fundamental assumption (1.4) on the coefficients of \( D \). First, \( G_{\nu}(y) \) is a constant times \( e^{i\sqrt{\nu}y} \) where the square root with positive imaginary part is chosen to ensure \( G_{\nu} \) vanishes at infinity. Keeping this and (2.5) in mind, we compute the limit of (2.8) as \( \epsilon \searrow 0 \) to be

\[ \frac{-1}{2i\sqrt{\lambda|c(\lambda)|^2}}F_{\lambda}(x)F_{\lambda}(y). \]

It follows that

\[ P_{[\alpha,\beta]}(D) = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{-1}{2i\sqrt{\lambda|c(\lambda)|^2}}F_{\lambda}(x)F_{\lambda}(y) \, d\lambda \]

\[ = \frac{1}{4\pi} \int_{\alpha}^{\beta} \frac{F_{\lambda}(x)F_{\lambda}(y)}{|c(\lambda)|^2} \frac{d\lambda}{\sqrt{\lambda}}, \]

as required.

3 Asymptotically Related Representations

In this section we shall describe our alternative method of computing the continuous part of the spectral measure in Weyl’s theorem. Like Kodaira we shall make free use of techniques from the spectral theory of abstract self-adjoint operators, but we aim to combine spectral theory with some simple geometric ideas, rather than with information about Green’s functions. We shall formulate the method in general terms, adding various assumptions as we go along. We shall check these assumptions in the case of Sturm-Liouville operators in Section 4.

Let \( \mathbb{C} \) be a separable, commutative \( \mathbb{C}^* \)-algebra with Gelfand spectrum \( \Lambda \), so that of course

\[ \mathbb{C} \cong C_0(\Lambda). \]

We shall view elements of \( \mathbb{C} \) as continuous functions on \( \Lambda \) without further comment.

Let us suppose that we are given a non-degenerate representation of \( \mathbb{C} \) on a separable Hilbert space,

\[ \pi: \mathbb{C} \longrightarrow B(H). \]
Associated to $\pi$ there is a direct integral decomposition

\begin{equation}
H \cong \int_{\Lambda}^\oplus H_\lambda \, d\mu(\lambda).
\end{equation}

This means that there exists:

(i) a Borel-measurable field of Hilbert spaces, $\{H_\lambda\}_{\lambda \in \Lambda}$, as in[81, Part II, Chapter 1],

(ii) a measure $\mu$ on the Borel subsets of the second countable, locally compact and Hausdorff space $\Lambda$ that is finite on the compact subsets of $\Lambda$, and

(iii) a unitary isomorphism $H \ni g \mapsto [\lambda \mapsto g_\lambda] \in L^2(\Lambda, \{H_\lambda\}_{\lambda \in \Lambda}, d\mu)$, from $H$ to the Hilbert space of square-integrable sections of the measurable field, under which the representation $\pi$ corresponds to the representation of $C_0(\Lambda)$ on square-integrable sections by pointwise multiplication. Thus if $g \in H$ and $\varphi \in C$, then

$$(\pi(\varphi)g)_\lambda = \varphi(\lambda)g_\lambda$$

for almost all $\lambda \in \Lambda$ with respect to the measure $\mu$.

See [Dix81, Part II, Chapter 6, Theorem 2]. We shall make the following multiplicity one assumption:

(A) The representation $\pi$ has multiplicity one. That is, $\dim H_\lambda \leq 1$ for $\mu$-almost every $\lambda \in \Lambda$.

This isn’t strictly necessary (uniform finite-dimensionality would suffice), but it simplifies the statements of the results that follow, along with their proofs, and it is satisfied in the situations of interest to us.

Our aim is to compute the measure $\mu$ above, at least on an open subset of the locally compact space $\Lambda$, by comparing the representation $\pi$ to a second representation $\pi_0 : C \to B(H_0)$ that will in practice be easier to analyze. We shall assume the following:
(B) The Hilbert space $H_0$ carries a continuous, one-parameter group of unitary operators
$$U_t : H_0 \longrightarrow H_0$$
that commute with the operators in the representation $\pi_0$. There is a bounded operator
$$W : H_0 \longrightarrow H$$
with the property that if we define
$$W_t = W U_t : H_0 \longrightarrow H,$$
then
$$\lim_{t \to +\infty} \left[ \langle W_t g, \pi(\varphi) W_t h \rangle_H - \langle U_t g, \pi_0(\varphi) U_t h \rangle_{H_0} \right] = 0$$
for all $\varphi \in C$, and all $g, h \in H_0$.

3.1 Example. Our aim in this paper is to study Weyl’s Sturm-Liouville theorem, and for this purpose we shall take the operator
$$W : H_0 \longrightarrow H$$
to be the orthogonal projection from $L^2(-\infty, \infty)$ onto $L^2(0, \infty)$, while the $C^*$-algebra $C$ will be $C_0(\mathbb{R})$, acting on $H$ and $H_0$ via the functional calculus for the operators $D$ and $D_0$ discussed in the introduction. But other interesting examples arise in the context of Harish-Chandra’s Plancherel formula for spherical functions, as follows.

Let $G = KAN$ be an Iwasawa decomposition of a real reductive group and let $M$ be the centralizer of $A$ in $K$. Let $H$ and $H_0$ be the $K$-fixed vectors within $L^2(G/K)$ and $L^2(G/MN)$, respectively. Choose an element $X \in a$ so that
$$\lim_{t \to +\infty} \exp(-tX)n\exp(tX) = e$$
for every $n \in N$, and define a one-parameter unitary group on $H_0$ using right translation by $\exp(tX)$ on $G/MN$ (the right translations are not measure-preserving, but they alter the measure by a scalar factor, so they are easily unitarized).

The $K$-invariant functions on $G/K$ and $G/MN$ identify with functions on $A^+$ and $A$ respectively, where $A^+$ is the dominant chamber in $A$, and a suitable operator $W$ may be defined by restriction of functions.\footnote{Actually we should restrict to a translation of $A^+$ by say $\exp(X)$, away from the walls of the chamber $A^+$, to ensure the operator $W$ is bounded.}
Finally, define $C$ to be the commutative $C^*$-subalgebra of the reduced $C^*$-algebra of $G$ generated by $K$-bi-invariant functions on $G$. It is a familiar idea in representation theory that the spaces $G/K$ and $G/MN$ are asymptotic to one another, and this ensures the crucial relation (3.2). Compare the diagram below, in which homogeneous spaces $G/K$ and $G/MN$ for $G = \text{SL}(2, \mathbb{R})$ are realized together in the coadjoint representation as coadjoint orbits.

The reader is referred to [Ban08] for an interesting and thorough discussion of the relation between Weyl’s theorem and harmonic analysis on symmetric spaces. It should also be noted that some of the spectral-theoretic methods from [SV12], which studies harmonic analysis on $p$-adic spherical varieties, are very closely related to the methods of this paper. See especially Section 8 of [SV12].

Let us return to our general argument. The unitary operators $U_t$ in (3.2) are actually superfluous since $\pi_0(\varphi)$ commutes with them, and we obtain from (B) the following key formula, which we shall use to compare the direct integral decomposition (3.1) for $\pi$ with a similar decomposition for $\pi_0$.

3.2 Lemma. If $\varphi \in C$ and if $t, g \in H$, then

$$\langle g, \pi_0(\varphi)h \rangle_{H_0} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \langle W_t g, \pi(\varphi)W_t h \rangle_H \, dt. \qquad \square$$

Next we shall make some assumptions concerning the direct integral decomposition of $\pi_0$. To avoid measure-theoretic complications we shall assume that it can be carried out in the following continuous fashion:

(C) There is an open subset $\Lambda_0 \subseteq \Lambda$

and there is a continuous field of Hilbert spaces $\{H_{0,\lambda}\}_{\lambda \in \Lambda_0}$ over $\Lambda_0$

\footnote{See for example [Dix77, Chapter 10] for the concept of continuous field.}
with constant and finite fiber dimension that decomposes the representation \( \pi_0 \) in the following sense. There is a dense subspace

\[ \mathcal{H}_0 \subseteq H_0 \]

and there are linear maps

\[ \varepsilon_{0,\lambda} : \mathcal{H}_0 \rightarrow H_{0,\lambda}, \]

defined for every \( \lambda \in \Lambda_0 \), that carry the elements of \( \mathcal{H}_0 \) to a total family\(^7\) of continuous sections of \( \{ H_{0,\lambda} \} \). Moreover there is a Borel measure \( \mu_0 \) on \( \Lambda_0 \) for which

\[ \langle h, \pi_0(\varphi) g \rangle_{H_0} = \int_{\Lambda_0} \langle \varepsilon_{0,\lambda}(h), \varepsilon_{0,\lambda}(g) \rangle_{H_{0,\lambda}} \varphi(\lambda) \, d\mu_0(\lambda) \]

for every \( \varphi \in \mathbb{C} \) and all \( h, g \in \mathcal{H}_0 \). Part of the assumption here is that the integrand, which is a continuous function on \( \Lambda_0 \), is in fact an integrable function.

We shall also assume compatibility between our continuous field and the one-parameter unitary group action on \( H_0 \):

\[ (D) \] The one-parameter unitary group \( \{ U_t \} \) on \( H_0 \) maps the subspace \( \mathcal{H}_0 \) into itself. Moreover the continuous field \( \{ H_{0,\lambda} \}_{\lambda \in \Lambda_0} \) carries a continuous, unitary action of \( \mathbb{R} \) such that

\[ \varepsilon_{0,\lambda}(U_t h) = U_t \varepsilon_{0,\lambda}(h) \]

for every \( h \in \mathcal{H}_0 \) and every \( \lambda \in \Lambda_0 \) (we shall use the same symbol \( U_t \) for the unitary action on the continuous field).

Underlying the continuous field \( \{ H_{0,\lambda} \}_{\lambda \in \Lambda_0} \) there is a measurable field of Hilbert spaces (for which a section of \( \{ H_{0,\lambda} \}_{\lambda \in \Lambda_0} \) is by definition measurable if its inner product with any continuous section is a measurable function), and the assumption \( (C) \) gives a direct integral decomposition

\[ (3.3) \quad H_0 \cong \int_{\Lambda_0} H_{0,\lambda} \, d\mu_0(\lambda) \]

of the representation \( \pi_0 \).

\(^7\)This means that the values of these sections at any point \( \lambda \) span \( H_{0,\lambda} \); see [Dix77, Definition 10.2.1].
We can now state our most important assumption, which relates the decompositions (3.1) and (3.3). We shall posit the existence of a measurable family of linear maps

$$A_\lambda: H_\lambda \to H_{0,\lambda}$$

with the following property: if $h \in H_0$ and $v_\lambda \in H_{\lambda}$, then

$$\lim_{t \to +\infty} \left| \langle A_\lambda v_\lambda, \varepsilon_{0,\lambda}(U_t h) \rangle_{H_{0,\lambda}} - \langle v_\lambda, (W_t h)_{\lambda} \rangle_{H_{\lambda}} \right| = 0.$$ 

As we shall see clearly in the next section, this means that $A_\lambda$ maps each $\lambda$-eigenfunction associated to the representation $\pi$ to a $\lambda$-eigenfunction for $\pi_0$ that is asymptotic to it.

Actually we shall need some uniformity in $\lambda$, and so we shall formulate our assumption precisely as follows:

(E) There is a measurable family of linear maps $A_\lambda$, as above, defined for $\mu$-almost every $\lambda \in \Lambda_0$, with the following property: if $K$ is a compact subset of $\Lambda_0$, if $h$ belongs to $H_0$, and if $\{v_\lambda\}_{\lambda \in K}$ is a measurable section of $\{H_\lambda\}_{\lambda \in K}$, then the difference

$$\left| \langle A_\lambda v_\lambda, \varepsilon_{0,\lambda}(U_t h) \rangle_{H_{0,\lambda}} - \langle v_\lambda, (W_t h)_{\lambda} \rangle_{H_{\lambda}} \right|$$

is bounded by $\|A_\lambda v_\lambda\|$ times a function of $t$ (independent of $\lambda \in K$) that converges to zero as $t \to +\infty$.

We can now state our main result on the comparison of $\pi$ with $\pi_0$:

3.3 Theorem. The measure $\mu_0$ is absolutely continuous with respect to $\mu$ on the open set $\Lambda_0$, with Radon-Nikodym derivative

$$\frac{d\mu_0}{d\mu}(\lambda) = \frac{\text{Trace}(A_\lambda^* A_\lambda)}{\dim(H_{0,\lambda})}.$$ 

The idea of the proof is simple. We shall obtain from the formula in Lemma 3.2 and the direct integral decomposition of $\pi$ in (3.1) a decomposition of $\pi_0$ that we can compare with (3.3). The theorem will then be a consequence of the following uniqueness result.

3.4 Lemma. Let $\{T_\lambda\}$ be a measurable field of positive operators on $\{H_{0,\lambda}\}_{\lambda \in \Lambda_0}$. Suppose that

$$\int_{\Lambda_0} \langle \varepsilon_{0,\lambda}(h), T_\lambda \varepsilon_{0,\lambda}(h) \rangle_{H_{0,\lambda}} \varphi(\lambda) \, d\mu(\lambda)$$

$$= \int_{\Lambda_0} \langle \varepsilon_{0,\lambda}(h), \varepsilon_{0,\lambda}(h) \rangle_{H_{0,\lambda}} \varphi(\lambda) \, d\mu_0(\lambda)$$

for every $h \in H_0$ and every continuous and compactly supported function $\varphi$. Then
(i) $\mu_0$ is absolutely continuous with respect to $\mu$.

(ii) $T_\lambda$ is a scalar multiple of the identity for $\mu$-almost all $\lambda$, and in fact

$$T_\lambda = \frac{d\mu_0}{d\mu}(\lambda) \cdot I_{H_0,\lambda}$$

$\mu$-almost everywhere.

Proof. For each point $\lambda_0 \in \Lambda_0$ there exists $h \in S_0$ such that the section $v_{0,\lambda} = \epsilon_{0,\lambda}(h)$ is nonzero at $\lambda_0$. It follows immediately from the uniqueness part of the Riesz representation theorem that $\mu_0$ is absolutely continuous with respect to $\mu$ near $\lambda_0$ with Radon-Nikodym derivative

$$\frac{d\mu_0}{d\mu}(\lambda) = \frac{\langle v_{0,\lambda}, T_\lambda v_{0,\lambda} \rangle_{H_0,\lambda}}{\langle v_{0,\lambda}, v_{0,\lambda} \rangle_{H_0,\lambda}}.$$

Since the derivative is independent of $\{v_{0,\lambda}\}$ this implies that

$$T_\lambda = \frac{d\mu_0}{d\mu}(\lambda) \cdot I_{H_0,\lambda},$$

almost everywhere, as required. $\square$

In the proof of Theorem 3.3 we shall use the following technical computations, which we shall deal with at the end of the section.

3.5 Lemma. Assume there is a measurable family of maps $A_\lambda$ as in (D) above. Let $K$ be a compact subset of $\Lambda_0$ and let $h \in S_0$.

(i) The quantity $\|A_\lambda\|$ is a $\mu$-square-integrable function of $\lambda \in K$.

(ii) The quantity $\|(W_t h)_{\lambda}\|$ is bounded independently of $t > 0$ by a $\mu$-square-integrable function of $\lambda \in K$.

Proof of Theorem 3.3 Let $h \in S_0$ and let $\varphi$ be a continuous and compactly supported function on $\Lambda_0$. According to Lemma 3.2

$$\langle h, \pi_0(\varphi) h \rangle_{H_0} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \langle W_t h, \pi(\varphi) W_t h \rangle_{H_0} dt.$$

Use the spectral decomposition of $\pi$ to write the integrand above as

$$\langle W_t h, \pi(\varphi) W_t h \rangle_{H_0} = \int_{\text{supp}(\varphi)} \langle (W_t h)_{\lambda}, (W_t h)_{\lambda} \rangle_{H_0,\lambda} \varphi(\lambda) d\mu(\lambda).$$

(3.4)
Since the spectral subspaces $H_\lambda$ are no more than one-dimensional, we can write

\[(3.5) \quad \langle (W_t h)_\lambda, (W_t h)_\lambda \rangle_{H_\lambda} = \langle (W_t h)_\lambda, v_\lambda \rangle_{H_\lambda} \cdot \langle v_\lambda, (W_t h)_\lambda \rangle_{H_\lambda},\]

where $\{v_\lambda\}$ is a measurable section that is a unit vector for almost all $\lambda$ for which $H_\lambda \neq 0$, and zero everywhere else. Thanks to Lemma 3.5, the expression (3.5) is bounded, uniformly in $t$, by a $\mu$-integrable function on $\text{supp}(\varphi)$.

Consider now the difference

\[
\langle \varepsilon_{0, \lambda}(U_t h), A_\lambda v_\lambda \rangle_{H_{0, \lambda}} \cdot \langle A_\lambda v_\lambda, \varepsilon_{0, \lambda}(U_t h) \rangle_{H_{0, \lambda}} - \langle (W_t h)_\lambda, v_\lambda \rangle_{H_\lambda} \cdot \langle v_\lambda, (W_t h)_\lambda \rangle_{H_\lambda},
\]

which we can write as

\[
\langle \varepsilon_{0, \lambda}(U_t h), A_\lambda v_\lambda \rangle_{H_{0, \lambda}} \left[ \langle A_\lambda v_\lambda, \varepsilon_{0, \lambda}(U_t h) \rangle_{H_{0, \lambda}} - \langle v_\lambda, (W_t h)_\lambda \rangle_{H_\lambda} \right]
+ \left[ \langle \varepsilon_{0, \lambda}(U_t h), A_\lambda v_\lambda \rangle_{H_{0, \lambda}} - \langle (W_t h)_\lambda, v_\lambda \rangle_{H_\lambda} \right] \langle v_\lambda, (W_t h)_\lambda \rangle_{H_\lambda}.
\]

According to Lemma 3.5 and assumption (E), the above expression is also bounded on $\text{supp}(\varphi)$ by a $\mu$-integrable function of $\lambda$, times a function of $t$ that converges to zero as $t \to +\infty$. Observing that

\[
\langle \varepsilon_{0, \lambda}(U_t h), A_\lambda v_\lambda \rangle_{H_{0, \lambda}} \langle A_\lambda v_\lambda, \varepsilon_{0, \lambda}(U_t h) \rangle_{H_{0, \lambda}}
= \langle U_t \varepsilon_{0, \lambda}(h), A_\lambda v_\lambda \rangle_{H_{0, \lambda}} \cdot \langle A_\lambda v_\lambda, U_t \varepsilon_{0, \lambda}(h) \rangle_{H_{0, \lambda}}
= \langle \varepsilon_{0, \lambda}(h), U_{-t} A_\lambda A_\lambda^* U_t \varepsilon_{0, \lambda}(h) \rangle_{H_{0, \lambda}},
\]

we find that the quantities (3.4) and

\[(3.6) \quad \int_{\text{supp}(\varphi)} \langle \varepsilon_{0, \lambda}(h), U_{-t} A_\lambda A_\lambda^* U_t \varepsilon_{0, \lambda}(h) \rangle_{H_{0, \lambda}} \varphi(\lambda) \, d\mu(\lambda) \]

are asymptotic to one another as $t \to +\infty$ (that is, the difference converges to zero). So we find that

\[
\langle h, \pi_0(\varphi) h \rangle_{H_0} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left( \int_{\text{supp}(\varphi)} \langle v_{0, \lambda}, U_{-t} A_\lambda A_\lambda^* U_t v_{0, \lambda} \rangle_{H_{0, \lambda}} \varphi(\lambda) \, d\mu(\lambda) \right) \, dt,
\]

where we have written $v_{0, \lambda} = \varepsilon_{0, \lambda}(h)$, and then, using Fubini’s theorem, that

\[
\langle h, \pi_0(\varphi) h \rangle_{H_0} = \lim_{T \to +\infty} \int_{\text{supp}(\varphi)} \left( \frac{1}{T} \int_0^T \langle v_{0, \lambda}, U_{-t} A_\lambda A_\lambda^* U_t v_{0, \lambda} \rangle_{H_{0, \lambda}} \, dt \right) \varphi(\lambda) \, d\mu(\lambda).
\]
The term in parentheses is bounded by an integrable function on supp(φ) that is independent of t. Therefore the dominated convergence theorem allows us to interchange the limit as $T \to +\infty$ and the integral over supp(φ) to obtain

$$\langle h, \pi_0(\varphi)h \rangle_{H_0} = \int_{\Lambda_0} \langle v_{0,\lambda}, \text{Av}[A_\lambda A_\lambda^*]v_{0,\lambda} \rangle_{H_{0,\lambda}} \varphi(\lambda) d\mu(\lambda),$$

where

$$\text{Av}[A_\lambda A_\lambda^*] = \lim_{T \to \infty} \frac{1}{T} \int_0^T U_{-t}A_\lambda A_\lambda^* U_t \, dt$$

(since we are dealing here with operators on the finite-dimensional space $H_{0,\lambda}$ the limit certainly exists).

We can now apply Lemma 3.4 which tells us that the operator $\text{Av}[A_\lambda A_\lambda^*]$ is a scalar multiple of the identity for $\mu$-almost-all $\lambda$. The computation

$$\text{Trace}(\text{Av}[A_\lambda A_\lambda^*]) = \text{Trace}(A_\lambda A_\lambda^*) = \text{Trace}(A_\lambda^* A_\lambda),$$

determines the multiple, and the theorem follows.

The theorem we have just proved gives a formula for the measure $\mu_0$ in terms of the measure $\mu$. But since our goal is to obtain information about the measure $\mu$, we should invert this formula, and for this purpose we shall make a final assumption:

(F) The operators $A_\lambda$ are nonzero for every $\lambda \in \Lambda_0$.

With this, the following is an immediate consequence of Theorem 3.3:

**3.6 Theorem.** The measure $\mu$ is absolutely continuous with respect to $\mu_0$ on $\Lambda_0$, and the Radon-Nikodym derivative of $\mu$ with respect to $\mu_0$ is

$$\frac{d\mu}{d\mu_0}(\lambda) = \frac{\dim(H_{0,\lambda})}{\text{Trace}(A_\lambda^* A_\lambda)}$$

on $\Lambda_0$.

Proof of Lemma 3.5 Since the continuous field $\{H_{0,\lambda}\}_{\lambda \in K}$ has finite and constant fiber dimension, and since the sections associated to elements of $H_0$ constitute a total set, there is a finite set of elements $\{h_j\}$ in $H_0$ such that

$$\|w_\lambda\|_{H_{0,\lambda}} \leq \sum_j \left| \langle w_\lambda, \varepsilon_{0,\lambda}(h_j) \rangle_{H_{0,\lambda}} \right|$$
for all $\lambda \in K$ and all $w_\lambda \in H_{0,\lambda}$. Applying this inequality to $w_\lambda$ of the form $U_t A_\lambda v_\lambda$ we get

$$\|A_\lambda v_\lambda\|_{H_{0,\lambda}} \leq \sum_j \left| \langle A_\lambda v_\lambda, \epsilon_{0,\lambda}(U_t h_j) \rangle_{H_{0,\lambda}} \right|$$

for all $t > 0$, all $\lambda \in K$ and all $v_\lambda \in H_{0,\lambda}$. Now it follows from assumption $[E]$ that there is some function $f(t)$ that converges to zero as $t \to +\infty$ such that

$$\sum_j \left| \langle A_\lambda v_\lambda, \epsilon_{0,\lambda}(U_t h_j) \rangle_{H_{0,\lambda}} \right| \leq \sum_j \left| \langle v_\lambda, (W_t h_j)_{\lambda} \rangle_{H_{\lambda}} \right| + \|A_\lambda v_\lambda\|_{H_{0,\lambda}} \cdot f(t)$$

for all $t > 0$, all $\lambda \in K$ and all $v_\lambda \in H_{\lambda}$. Rearrange this as

\begin{equation}
(3.7) \quad \|A_\lambda v_\lambda\|_{H_{0,\lambda}} (1 - f(t)) \leq \sum_j \left| \langle v_\lambda, (W_t h_j)_{\lambda} \rangle_{H_{\lambda}} \right|
\end{equation}

and take $v_\lambda$ to be a measurable unit vector field in the nonzero fibers of $\{H_{\lambda}\}$. Fix $t > 0$ large enough so that $1 - f(t) > 0$. The right-hand side of (3.7) is a square-integrable function of $\lambda$, and the left hand side is a fixed multiple of $\|A_\lambda\|$, so part (i) of the lemma is proved.

As for part (ii), it follows from assumption $[E]$ that

$$\left| \langle v_\lambda, (W_t h)_{\lambda} \rangle_{H_{\lambda}} \right| \leq \left| \langle A_\lambda v_\lambda, \epsilon_{0,\lambda}(U_t h) \rangle_{H_{0,\lambda}} \right| + \|A_\lambda v_\lambda\|_{H_{0,\lambda}} \cdot g(t)$$

for all $t > 0$, all $\lambda \in K$ and all $v_\lambda \in H_{\lambda}$, for some $g(t)$ that converges to zero as $t \to +\infty$. Once again, take $v_\lambda$ to be a measurable unit vector field in the nonzero fibers of $\{H_{\lambda}\}$ to conclude from Cauchy-Schwarz that

$$\left| \langle v_\lambda, (W_t h)_{\lambda} \rangle_{H_{\lambda}} \right| \leq \|A_\lambda\|_{H_{0,\lambda}} \cdot \max_{\lambda \in K} \|\epsilon_{0,\lambda}(h)\| + \|A_\lambda\|_{H_{0,\lambda}} \cdot g(t).$$

So part (ii) follows from part (i). 

\section*{4 Sturm-Liouville Operators}

In this section we shall apply the approach of Section 3 to Sturm-Liouville operators on the half-line. So let $D$ be a linear differential operator of the form $[\mathcal{D}]$, where the coefficient functions $p(x)$ and $q(x)$ are smooth and real-valued on $[0, \infty)$, and where $p(x)$ is everywhere positive. We shall begin by assuming in addition that

\begin{equation}
(4.1) \quad \lim_{x \to \infty} p(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} q(x) = 0,
\end{equation}
and also that
\[(4.2) \lim_{x \to \infty} p'(x) = 0.\]
Later on in the section we shall make stronger assumptions about the rates of convergence in the limits above.
We shall take for granted the following result, which from a modern perspective is straightforward:

**4.1 Theorem.** The operator $D$ is essentially self-adjoint on the domain of smooth, compactly supported functions $h: [0, \infty) \to \mathbb{C}$ with $h(0) = 0$.

**4.2 Remark.** We have made a simple and explicit choice of boundary conditions in the theorem, but nothing in what follows depends on the boundary conditions, as long as they determine an essentially self-adjoint operator.

Associated to the unbounded self-adjoint operator $D$ on the Hilbert space $H = L^2[0, \infty)$ is the functional calculus morphism
\[
\pi: C_0(\mathbb{R}) \to B(H)
\]
\[
\pi: \phi \mapsto \phi(D).
\]
We shall apply the considerations of the previous section to this representation.

Fix a direct integral decomposition as in (3.1). As we already noted, one is guaranteed to exist by abstract theory, and we shall use the following standard technique to extract some concrete information about it. Suppose that a topological vector space $S$ is included in $H$ via a continuous map that factors through a Hilbert-Schmidt operator. Thus suppose we have a commuting diagram

$\begin{tikzcd}
S \arrow{r}{\text{inclusion}} \arrow{dr}{\text{continuous}} & H \arrow{d}{\text{Hilbert-Schmidt}} \\
& K
\end{tikzcd}$

where $K$ is a Hilbert space. Then for almost all $\lambda$ (with respect to the measure $\mu$) there are continuous operators
\[(4.3) \epsilon_\lambda: S \to H_\lambda\]
with dense range such that if $h \in S$, then $h_\lambda = \epsilon_\lambda(h)$ for almost every $\lambda \in \Lambda$. See [Ber88, Section 1] for a succinct account that is well aligned with
the outlook of this paper. Or see the Fundamental Theorem in [Mau67, Chapter VII, Section 1].

In our case we can take $S = C^\infty_c[0,\infty)$ (compare [Ber88, Section 1, Lemma 2.3]). Since the maps (4.3) have dense range for almost every $\lambda$, the adjoint maps

$$(4.4) \quad \varepsilon^*_\lambda : H^*_\lambda \rightarrow S^*$$

are almost always injective. This leads to a description of $H_\lambda$ as a space of eigenfunctions, based on the fact that

$$H^*_\lambda \cong \Pi_\lambda.$$

In fact, if $h$ is smooth and compactly supported in the open half-line $(0, \infty)$, while $g \in S$, then

$$\langle Dh, g \rangle = \langle h, Dg \rangle,$$

and as a result

$$D\varepsilon^*_\lambda(g) = \lambda\varepsilon^*_\lambda(g)$$

in the sense of distributions on $(0, \infty)$, for almost all $\lambda \in \Lambda$. So for almost all $\lambda$, the map $\varepsilon^*_\lambda$ embeds $H^*_\lambda$ into the space $S^*_\lambda$ of $\lambda$-eigendistributions for $D$. By linear ODE theory $S^*_\lambda$ consists of smooth functions on $[0, \infty)$ and is two-dimensional.

**4.3 Lemma.** For almost every $\lambda \in \Lambda$ the adjoint maps (4.4) embed $H^*_\lambda$ into the space of smooth function solutions in $S^*$ of the differential equation $DF_\lambda = \lambda F_\lambda$ that satisfy the boundary condition $F_\lambda(0) = 0$.

**Proof.** If $h \in S$, and if $g$ is in the domain of $D$ (by which we mean, both here and subsequently, the domain of the self-adjoint closure of $D$), then

$$(4.5) \quad \langle \varepsilon_\lambda(Dh), g_\lambda \rangle_{H_\lambda} - \langle \varepsilon_\lambda(h), (Dg)_\lambda \rangle_{H_\lambda}$$

$$= \int_0^\infty (Dh)(x)G_\lambda(x) \, dx - \int_0^\infty h(x)(DG_\lambda)(x) \, dx$$

$$= p(0) \left( \overline{h'}(0)G_\lambda(0) - \overline{h}(0)G'_\lambda(0) \right)$$

(c.f. (2.6)). Here $G_\lambda = \varepsilon^*_\lambda g_\lambda$. The top expression in (4.5) is an integrable function of $\lambda$, and therefore so is the bottom.

If we choose $h \in S$ so that $h(0) = 0$, then $h \in \text{dom}(D)$. In this case the top expression in (4.5) integrates to zero for $\mu$-almost all $\lambda$. So if we choose $h$ so that in addition $h'(0) \neq 0$, then we find that $G_\lambda(0)$ is an integrable
function of \( \lambda \), and it integrates to zero. Replacing \( g \in \text{dom}(D) \) with \( \varphi(D)g \), where \( \varphi \) is any bounded Borel function, we get
\[
\int_{\Lambda_0} G_\lambda(0) \varphi(\lambda) \, d\mu(\lambda) = 0,
\]
and therefore \( G_\lambda(0) = 0 \) for almost every \( \lambda \). It follows that for almost every \( \lambda \) the image of the adjoint map \((4.4)\) is contained in the one-dimensional space of smooth \( \lambda \)-eigenfunctions for \( D \) that satisfy the boundary condition \( G_\lambda(0) = 0 \), as required.

Of course there is a precisely one-dimensional space of eigenfunctions \( F_\lambda \) for which \( F_\lambda(0) = 0 \), and so it follows immediately from the lemma that \( \dim H_\lambda \leq 1 \) for \( \mu \)-almost every \( \lambda \in \Lambda \), as required in assumption (A) from the previous section.

Next we define \( H_0 = L^2(-\infty, \infty) \) which obviously contains \( H \) as a closed subspace; we shall denote by
\[
W: H_0 \longrightarrow H
\]
the orthogonal projection. Define
\[
D_0 = -\frac{d^2}{dx^2},
\]
which we shall treat as an essentially self-adjoint operator on the Hilbert space \( H_0 \) with domain the smooth, compactly supported functions on the line, and define
\[
\pi_0: C_0(\mathbb{R}) \longrightarrow B(H_0)
\]
by \( \pi_0(\varphi) = \varphi(D_0) \). Define \( U_t: H_0 \rightarrow H_0 \) to be the translation operator
\[
(U_t h)(x) = h(x-t).
\]
Obviously each \( \varphi(D_0) \) commutes with each \( U_t \). The following computation checks assumption \( (B) \) from the previous section.

\textbf{4.4 Lemma.} If \( \varphi \in C_0(\mathbb{R}) \) and if \( g, h \in L^2(0, \infty) \), then
\[
\lim_{t \rightarrow +\infty} \left[ \langle U_t g, \varphi(D)U_t h \rangle_{L^2(0, \infty)} - \langle U_t g, \varphi(D_0)U_t h \rangle_{L^2(-\infty, \infty)} \right] = 0.
\]

\textit{Proof.} Much more is true, namely that
\[
\lim_{t \rightarrow +\infty} \| \varphi(D)U_t - \varphi(D_0)U_t \|_{B(H_0)} = 0
\]
for every $\varphi \in C_0(\mathbb{R})$. To prove this, observe first that the set of all $\varphi \in C_0(\mathbb{R})$ satisfying (4.6) is a norm-closed subalgebra of $C_0(\mathbb{R})$, so it suffices to show that the resolvent functions $\varphi(\lambda) = (\lambda \pm i)^{-1}$ belong to it.

Let $\psi$ be a smooth function on $\mathbb{R}$ that is identically zero in a neighborhood of $(-\infty, 0]$ and identically one in a neighborhood of $[1, \infty)$, and for $t > 0$ let $M_t$ be the bounded operator of pointwise multiplication by $\lambda \mapsto \psi(t^{-1}\lambda)$ (it is an operator on $H$ whose range lies in $H$). Then

$$U_t = M_t U_t : H \rightarrow H$$

and by standard Sobolev space estimates (the basic elliptic estimate, applied to $D_0$)

$$\lim_{t \rightarrow +\infty} \|M_t \varphi(D_0) - \varphi(D_0)M_t\| = 0$$

for $\varphi(\lambda) = (\lambda \pm i)^{-1}$, or indeed for every $\varphi \in C_0(\mathbb{R})$. Now

$$\varphi(D)U_t - \varphi(D_0)U_t = \varphi(D)M_t U_t - \varphi(D_0)M_t U_t$$

as operators from $H$ to $H$, and the right-hand side is asymptotic in norm to

$$\varphi(D)M_t U_t - M_t \varphi(D_0)U_t.$$

For the particular case where $\varphi(\lambda) = (\lambda \pm i)^{-1}$, the above may be expressed as

$$\varphi(D) [M_t D_0 - D M_t] \varphi(D_0)U_t.$$  

The expression in the middle is, for each $t > 0$, a second order differential operator

$$a_t(x) \frac{d^2}{dx^2} + b_t(x) \frac{d}{dx} + c_t(x)$$

on the line and as $t \rightarrow +\infty$ the coefficient functions converge uniformly to zero. So by the basic estimates (4.7) is a bounded operator whose norm converges uniformly to zero as $t$ tends to infinity.

The spectral theory of the operator $D_0$ is of course easily obtained from Fourier theory. Let

$$\Lambda_0 = (0, \infty),$$

and for $\lambda \in \Lambda_0$ define $H_{0,\lambda}$ to be the two-dimensional vector space of functions on the line spanned by $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$. Equip $H_{0,\lambda}$ with the inner product that makes these two functions an orthonormal basis. The family
\{H_{0,\lambda}\}_{\lambda>0} \text{ obviously forms a continuous field of Hilbert spaces over } \Lambda_0 \text{ with constant and finite fiber dimension. Now let } \mathcal{H}_0 \text{ be space of smooth and compactly supported functions in } H_0. \text{ The Fourier transform }
\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x) e^{-i \xi x} \, dx
\text{ associates to each } h \in \mathcal{H}_0 \text{ a continuous section } \{\varepsilon_{0,\lambda}(h)\} \text{ of the continuous field, namely }
\varepsilon_{0,\lambda}(h) = \hat{h}(\sqrt{\lambda}) e^{i \sqrt{\lambda} x} + \hat{h}(-\sqrt{\lambda}) e^{-i \sqrt{\lambda} x}.
\text{ We obtain a total family of sections, and it follows from Plancherel’s formula that }
\langle h, \varphi(D_0)g \rangle_{L^2(\mathbb{R})} = \int_{\Lambda_0} \langle \varepsilon_{0,\lambda}(h), \varepsilon_{0,\lambda}(g) \rangle_{H_{0,\lambda}} \varphi(\lambda) \, d\mu_0(\lambda),
\text{ where }
(4.8) \quad d\mu_0(\lambda) = \frac{1}{4\pi \sqrt{\lambda}} \frac{d\lambda}{\sqrt{\lambda}}.
\text{ This takes care of assumption (C), and moreover assumption (D) is a simple consequence Fourier theory, too.}

Finally, we need to analyze the asymptotics of the } \lambda \text{-eigenfunctions of } D. \text{ Our method is essentially the same as Weyl’s [Wey10], and it is in any case standard (moreover it is perhaps worth noting that in the simple case where the coefficients of } D \text{ are eventually constant, nothing from here up to the formulation of Theorem 4.8 is needed at all).}

4.5 Lemma. \text{ Suppose that a smooth function } u: [0, \infty) \to \mathbb{C}^n \text{ is a solution of the differential equation }
u'(x) = Cu(x) + Q(x)u(x)
\text{ where } C \text{ is a constant } n \times n \text{ matrix and } Q \text{ is a smooth } n \times n \text{ matrix-valued function. Let }
(4.9) \quad k(x) = \int_x^{\infty} \| \exp(-xC)Q(x) \exp(xC) \| \, dx
\text{ and assume that } k(0) < \infty. \text{ If }
u(x) = \exp(-xC)u(x)
\text{ then the limit }
\nu(\infty) = \lim_{x \to +\infty} \nu(x)
exists. Moreover,

\[(4.10) \quad \|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|s(\infty)\|,\]

where the constant can be chosen to be a continuous function of \(k(0)\). In particular, if the limit \(s(\infty)\) is zero, then \(s(x)\) is identically zero.

**4.6 Remark.** Since all norms on finite-dimensional spaces are equivalent, we can choose any norm in (4.9) and below. We shall choose the Hilbert-Schmidt norm

\[\|T\|^2 = \text{Trace}(T^*T).\]

This is a submultiplicative norm, and it has the advantage over, for example, the operator norm, of being a smooth function on the space of nonzero matrices.

**Proof of the Lemma.** Consider the linear differential equation

\[U'(x) = CU(x) + Q(x)U(x)\]

in which \(U(x)\) is a smooth, \(n \times n\) matrix-valued function. There exists a unique solution for the initial condition

\[U(0) = I,\]

and it is defined for all \(x\). Moreover each \(U(x)\) is invertible: the inverse matrices can be obtained by solving the linear differential equation

\[V'(x) = -V(x)C - U(x)Q(x)\]

with initial condition \(V(0) = I\). If we write

\[S(x) = \exp(-xC)U(x),\]

then

\[S'(x) = \exp(-xC)Q(x) \exp(xC)S(x),\]

and so of course

\[(4.11) \quad \|S'(x)\| \leq \|\exp(-xC)Q(x) \exp(xC)\| \cdot \|S(x)\|.\]

This, together with the simple inequality

\[\left| \frac{d}{dx} \|S(x)\| \right| \leq \|S'(x)\|\]
gives us the estimate

\[
(4.12) \quad \left| \frac{d}{dx} \log \|S(x)\| \right| \leq \| \exp(-xC)Q(x) \exp(xC) \|.
\]

The integrability hypothesis of the lemma now implies that

\[
\sup_{x \in [0, \infty)} \log \|S(x)\| < \infty,
\]

and therefore

\[
(4.13) \quad \sup_{x \in [0, \infty)} \|S(x)\| < \infty.
\]

Both suprema are bounded by a continuous function of \(k(0)\). Returning to (4.11), it follows from (4.13) that

\[
(4.14) \quad \|S'(x)\| \leq \text{constant} \cdot \| \exp(-xC)Q(x) \exp(xC) \|,
\]

where the constant can be chosen to be a continuous function of \(k(0)\). So by applying the integrability hypothesis a second time we find that \(S(x)\) converges to a limit \(S(\infty)\) as \(x\) tends to infinity.

The limit \(S(\infty)\) is an invertible matrix. Indeed we can apply the argument we’ve just given to the matrix-valued function

\[
T(x) = S(x)^{-1} = U(x)^{-1} \exp(xC),
\]

in place of \(S(x)\). The function \(T(x)\) is a solution of the differential equation

\[
T'(x) = -T(x) \exp(-xC)Q(x) \exp(xC),
\]

and the argument above shows that \(T(x)\) converges to a limit as \(x\) tends to infinity, and is bounded by a continuous function of \(k(0)\). And of course

\[
\lim_{x \to \infty} T(x) \cdot \lim_{x \to \infty} S(x) = \lim_{x \to \infty} T(x)S(x) = I.
\]

To complete the proof, it follows from the uniqueness of solutions property for ODE’s that

\[
u(x) = U(x)u(0),
\]

so that

\[s(x) = \exp(-xC)u(x) = \exp(-xC)U(x)u(0) = S(x)u(0).\]

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So the limit $s(\infty)$ exists. As for (4.10), we can write
\[
\|s(\infty) - s(x)\| \leq \|S(\infty)\| \cdot \|I - T(\infty)S(x)\| \cdot \|u(0)\|,
\]
and then estimate the middle norm on the right-hand side by
\[
\int_x^{\infty} \|T(\infty)\|\|S'(x)\| \, dx.
\]
From (4.14) we obtain
\[
\|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|u(0)\|.
\]
Since $u(0) = T(\infty)s(\infty)$, we obtain
\[
\|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|s(\infty)\|,
\]
for a constant that is a continuous function of $k(0)$, as required.

Let us apply this to the Sturm-Liouville eigenvalue equation. If $\lambda \in \mathbb{C}$ and if $F_\lambda$ is any $\lambda$-eigenfunction for $D$, then the vector-valued function
\[
(4.15) \quad u_\lambda(x) = \begin{bmatrix} p(x)F'_\lambda(x) \\ F_\lambda(x) \end{bmatrix}
\]
is a solution of the differential equation
\[
(4.16) \quad u'(x) = C_\lambda u(x) + Q(x)u(x),
\]
where
\[
(4.17) \quad C_\lambda = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \quad \text{and} \quad Q(x) = \begin{bmatrix} 0 \\ 1 - p(x)^{-1} \end{bmatrix} q(x).
\]

4.7 Proposition. Suppose that
\[
\int_{x_0}^{\infty} |1 - p(x)^{-1}| \, dx < \infty \quad \text{and} \quad \int_{x_0}^{\infty} |q(x)| \, dx < \infty.
\]
Let $K$ be a compact subset of $(0, \infty)$. There is a positive function $j(x)$ on $[0, \infty)$ for which
\[
\lim_{x \to \infty} j(x) = 0
\]
with the following property: if $\lambda \in K$, and if $F_\lambda$ is any $\lambda$-eigenfunction of $D$, then there is a unique $\lambda$-eigenfunction $F_{0,\lambda}$ of $D_0$ such that
\[
(4.18) \quad |F_\lambda(x) - F_{0,\lambda}(x)| \leq j(x) \cdot \|F_{0,\lambda}\|_{H_{0,\lambda}}, \quad \forall x > 0.
\]
Proof. If \( \lambda > 0 \), then the matrices \( C_\lambda \) in (4.17) are skew-adjoint for inner products on \( \mathbb{C}^2 \) that depend continuously on \( \lambda \) and so (using the fixed norm on matrices that we chose earlier)

\[
(4.19) \quad \sup_{\lambda \in K, x \in \mathbb{R}} \| \exp(xC_\lambda) \| < \infty.
\]

As a result, the hypotheses on \( p(x) \) and \( q(x) \) in the proposition imply that if \( C = C_\lambda \) and \( Q \) are as in (4.17), and if \( k(x) = k_\lambda(x) \) is as in Lemma 4.5 then

\[
\sup_{\lambda \in K} k_\lambda(0) < \infty.
\]

Let us now apply Lemma 4.5 to the vector-valued function \( u_\lambda(x) \) in (4.15). Define the function \( F_{0,\lambda}(x) \) to be the bottom entry of the vector-valued function

\[
F_{0,\lambda}(x) = \exp(xC_\lambda)s_\lambda(\infty)
\]

with \( s_\lambda(\infty) \) as in the lemma. This is a linear combination of \( \exp(\pm i\sqrt{\lambda}x) \), and hence an element of \( H_{0,\lambda} \). In fact if \( s_\lambda(\infty) = \begin{bmatrix} a_\lambda \\ b_\lambda \end{bmatrix} \), and if we write

\[
\begin{bmatrix} a_\lambda \\ b_\lambda \end{bmatrix} = \left( a_\lambda/2 + b_\lambda/2i\sqrt{\lambda} \right) \begin{bmatrix} 1 \\ i\sqrt{\lambda} \end{bmatrix} + \left( a_\lambda/2 - b_\lambda/2i\sqrt{\lambda} \right) \begin{bmatrix} 1 \\ -i\sqrt{\lambda} \end{bmatrix},
\]

and note that the vectors on the right are eigenvectors for \( C_\lambda \) with eigenvalues \( \pm i\sqrt{\lambda} \), then we find that

\[
(4.20) \quad F_{0,\lambda}(x) = \frac{1}{2}(b_\lambda + a_\lambda i\sqrt{\lambda}) \exp(i\sqrt{\lambda}x) + \frac{1}{2}(b_\lambda - a_\lambda i\sqrt{\lambda}) \exp(-i\sqrt{\lambda}x).
\]

This formula also shows that the norm of \( s_\lambda(\infty) \) is uniformly bounded by a multiple of the norm of \( F_{0,\lambda} \) as \( \lambda \) varies over \( K \). The required estimate follows from this fact together with the conclusion of Lemma 4.5 and another application of (4.19).

Now let us define operators \( A_\lambda : H_\lambda \rightarrow H_{0,\lambda} \) for \( \mu \)-almost all \( \lambda > 0 \) by

\[
H_\lambda \ni F_\lambda \mapsto A_\lambda F_\lambda = F_{0,\lambda} \in H_{0,\lambda}
\]

where \( F_\lambda \) and \( F_{0,\lambda} \) are as in Proposition 4.7. If \( h \) is a smooth, compactly supported function on \( \mathbb{R} \), and if \( v_\lambda = F_\lambda \), then

\[
\langle A_\lambda v_\lambda, \epsilon_{0,\lambda}(U_t h) \rangle_{H_{0,\lambda}} - \langle v_\lambda, (W_t h)_\lambda \rangle_{H_\lambda} = \int_0^\infty (\overline{F_{0,\lambda}(x)} - F_{0,\lambda}(x)) h(x-t) \, dx
\]

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(this formula holds as long as \( t \) is large enough that \( h(x-t) \) is supported on the positive x-axis). Proposition 4.7 implies that if \( \lambda \) is confined to a compact set in \((0, \infty)\), then the integral is bounded by the norm of the vector
\[
A_\lambda v_\lambda = F_{0,\lambda}
\]
in \( H_{0,\lambda} \) times a function in \( t \) that converges to zero as \( t \) converges to infinity. So we have checked assumption (E) from the previous section. It also follows from Proposition 4.7 that the operators \( A_\lambda \) are injective, so assumption (F) is satisfied, too.

Let us summarize. Let \( D \) be a Sturm-Liouville operator on \([0, \infty)\) with smooth, real-valued coefficient functions \( p \) and \( q \) that satisfy \( p > 0 \) as well as
\[
\lim_{x \to \infty} q(x) = \lim_{x \to \infty} (1 - p(x)) = \lim_{x \to \infty} p'(x) = 0
\]
and
\[
\int_1^\infty |1 - p(x)^{-1}| \, dx < \infty \quad \int_1^\infty |q(x)| \, dx < \infty.
\]

View \( D \) as an essentially self-adjoint operator with domain the smooth, compactly supported functions on \([0, \infty)\) that vanish at 0. We obtain from the above the following version of Weyl’s theorem:

**4.8 Theorem.** Assume that the coefficient functions of the Sturm-Liouville operator \( D \) satisfy (4.21) and (4.22) above. Let \( g \) and \( h \) be smooth and compactly supported functions on \([0, \infty)\). If \( \beta > \alpha > 0 \), and if \( P_{[\alpha,\beta]} \) is the spectral projection for \( D \) associated to the interval \([\alpha, \beta]\), then

\[
\langle g, P_{[\alpha,\beta]} h \rangle = \frac{1}{4\pi} \int_\alpha^\beta \langle g, F_\lambda \rangle \langle F_\lambda, h \rangle \frac{1}{|c(\lambda)|^2} \frac{d\lambda}{\sqrt{\lambda}}
\]

where \( F_\lambda \) is the unique \( \lambda \)-eigenfunction with \( F_\lambda(0) = 0 \) and \( F'_\lambda(0) = 1 \), and \( c(\lambda) \) is characterized by

\[
\lim_{x \to +\infty} \left( F_\lambda(x) - c(\lambda)e^{i\sqrt{\lambda}x} - \bar{c}(\lambda)e^{-i\sqrt{\lambda}x} \right) = 0.
\]

**Proof.** We shall compute \( \|P_{[\alpha,\beta]} h\|^2 \) (the formula in the statement of the theorem will follow by polarization). First, according to the definition of a direct integral decomposition,

\[
\|P_{[\alpha,\beta]} h\|^2 = \int_\alpha^\beta \| h_\lambda \|^2_{H_\lambda} \, d\mu(\lambda).
\]
Now let \( \{v_\lambda\} \) be the section of \( \{H_\lambda\} \) for which \( \epsilon(\lambda) = \frac{F_\lambda}{\langle v_\lambda, v_\lambda \rangle_{H_\lambda}} \), with \( F_\lambda \) as in the statement of the theorem. Then

\[
\int_{\alpha}^{\beta} \|h_\lambda\|_{H_\lambda}^2 \, d\mu(\lambda) = \int_{\alpha}^{\beta} \frac{|\langle F_\lambda, h_\lambda \rangle_{H_\lambda}|^2}{\langle v_\lambda, v_\lambda \rangle_{H_\lambda}} \, d\mu(\lambda),
\]

and applying Theorem 3.6 we get

\[
\int_{\alpha}^{\beta} \|h_\lambda\|_{H_\lambda}^2 \, d\mu(\lambda) = \int_{\alpha}^{\beta} \frac{|\langle F_\lambda, h_\lambda \rangle_{H_\lambda}|^2}{\langle v_\lambda, v_\lambda \rangle_{H_\lambda}} \, d\mu(\lambda) = 2 \int_{\alpha}^{\beta} \frac{|\langle F_\lambda, h_\lambda \rangle_{H_\lambda}|^2}{\langle \lambda_\lambda v_\lambda, \lambda_\lambda v_\lambda \rangle_{H_0,\lambda}} \, d\mu_0(\lambda).
\]

It follows from our definition of \( \Lambda_\lambda \) that this is

\[
\int_{\alpha}^{\beta} \frac{|\langle F_\lambda, h_\lambda \rangle_{H_\lambda}|^2}{|c(\lambda)|^2} \, d\mu_0(\lambda),
\]

and the theorem follows from the explicit formula for \( \mu_0 \) in (4.8).

\[\square\]

5 Non-Positive Spectrum

In this concluding section we shall briefly examine the non-positive part of the spectrum of a Sturm-Liouville operator \( D \) of the type considered in the previous section, with coefficient functions satisfying (4.21) and (4.22).

The value \( \lambda = 0 \) belongs to the spectrum of \( D \), of course, because the spectrum is closed. But for the purposes of determining the measure \( \mu \) we need to determine whether or not \( 0 \) is an eigenvalue.

The answer is that \( \lambda = 0 \) is not an eigenvalue, at least if we assume a bit more about the rate of convergence of the coefficients \( p(x) \) and \( q(x) \) to their asymptotic values.

5.1 Lemma. Suppose that

\[
\int_1^{\infty} x^2 |1 - p(x)^{-1}| \, dx < \infty \quad \text{and} \quad \int_1^{\infty} x^2 |q(x)| \, dx < \infty.
\]

Then \( D \) has no non-zero square-integrable 0-eigenfunctions.

Proof. This is a consequence of Lemma 4.5. When \( \lambda = 0 \) the integral in the statement of Lemma 4.5 is finite for the matrices \( C \) and \( Q \) in (4.17). So any 0-eigenfunction for \( D \) is asymptotic to a 0-eigenfunction for \( D_0 \). But the latter are the functions \( ax + b \), and we find that no 0-eigenfunction for \( D \) can be square-integrable. \( \square \)
Let us consider now the negative part of the spectrum of $D$. The argument below is not optimal\(^8\) but it uses the same ideas we have already developed to handle the continuous spectrum. Moreover it is adequate to handle the operators that arise in harmonic analysis.

**5.2 Lemma.** If we assume that
\[
\int_1^\infty e^{\alpha x} |1 - p(x)^{-1}| \, dx < \infty \quad \text{and} \quad \int_1^\infty e^{\alpha x} |q(x)| \, dx < \infty.
\]
for some $\alpha > 0$, then $\lambda = 0$ is not a limit point of the set of eigenvalues of the self-adjoint Hilbert space operator $D$.

*Proof.* For every $\lambda \in \mathbb{C}$ the matrix $C_\lambda$ in (4.17) satisfies $C_\lambda^2 = -\lambda I$, and therefore
\[
\exp(xC_\lambda) = \cosh(x\sqrt{\lambda})I + \frac{\sinh(x\sqrt{\lambda})}{\sqrt{\lambda}} C_\lambda
\]
for any square root of $\lambda$. It follows from this that if $\varepsilon > \delta > 0$, then
\[
|\lambda| \leq \delta \quad \Rightarrow \quad \| \exp(xC_\lambda) \| \leq \text{constant} \cdot e^{\varepsilon x}
\]
for all $x$ and some constant independent of $\lambda$ and $x$. Now choose $\varepsilon = \alpha/4$. The estimate (5.2) and Lemma 4.5 imply that for $u_\lambda(x)$ as in (4.15) the limit
\[
w_\lambda := \lim_{x \to \infty} \exp(-xC_\lambda) u_\lambda(x)
\]
exists whenever $|\lambda| \leq \delta$, and moreover
\[
|\lambda| \leq \delta \quad \Rightarrow \quad \| \exp(-xC_\lambda) u_\lambda(x) - w_\lambda \| \leq \text{constant} \cdot e^{-2\varepsilon x}
\]
for some constant that is again independent of $\lambda$ and $x$. If we write
\[
u_\lambda(x) = \exp(xC_\lambda) w_\lambda + \exp(xC_\lambda) \left[ \exp(-xC_\lambda) u_\lambda(x) - w_\lambda \right]
\]
then we find from (5.2) and (5.3) that
\[
u_\lambda \in L^2 \quad \Leftrightarrow \quad \exp(xC_\lambda) w_\lambda \in L^2,
\]
and so $f_\lambda$ is square-integrable if and only if the second entry of the vector-valued function $\exp(xC_\lambda) w_\lambda$ is a square-integrable function.

\(^8\)See Weyl’s paper [Wey10] or [DS88, Chapter XII, Section 7] for sharper results.
Now if \( z > 0 \) and if \( \lambda = -z^2 \), and if we write \( w_\lambda = \begin{bmatrix} a_\lambda \\ b_\lambda \end{bmatrix} \), then

\[
\exp(xC_\lambda)w_\lambda = \left( a_\lambda/2 + b_\lambda/2z \right) e^{xz} \begin{bmatrix} 1 \\ z \end{bmatrix} + \left( a_\lambda/2 - b_\lambda/2z \right) e^{-xz} \begin{bmatrix} 1 \\ -z \end{bmatrix},
\]

as we noted in (4.20). The second term on the right is always square-integrable. So we find that the second entry of \( \exp(xC_\lambda)w_\lambda \) is a square-integrable function if and only if the first term on the right-hand side is zero, or in other words

\[
f_\lambda \in L^2 \iff a_\lambda z + b_\lambda = 0
\]

(as long as \( z > 0 \)). But now \( w_\lambda \), and therefore the quantity \( a_\lambda z + b_\lambda \), is holomorphic in a sufficiently small neighborhood of \( 0 \in \mathbb{C} \). It is not identically zero because for example if \( z \) is nonzero and purely imaginary (so that \( \lambda = -z^2 \) is positive), then \( a_\lambda \) and \( b_\lambda \) are real and at least one is nonzero. So there are at most finitely many \( L^2 \)-eigenvalues in a sufficiently small neighborhood of \( 0 \in \mathbb{C} \). □

We can say more using perturbation theory. The operator \( D \) is a semi-bounded and relatively compact perturbation of the positive operator

\[
-d/dx \cdot p(x) \cdot d/dx.
\]

So the negative part of its spectrum consists of at most countable set of eigenvalues accumulating only at 0. Compare [Kat76, Chapter IV, Theorem 5.35]. But Lemma 5.2 rules out this latter possibility. Hence:

5.3 Theorem. If we assume that

\[
\int_1^\infty e^{\alpha x} |1 - p(x)^{-1}| \, dx < \infty \quad \text{and} \quad \int_1^\infty e^{\alpha x} |q(x)| \, dx < \infty.
\]

for some \( \alpha > 0 \), then the operator \( D \) has at most finitely many \( L^2 \)-eigenfunctions satisfying the boundary condition \( f_\lambda(0) = 0 \), all associated to negative eigenvalues. □

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