Decoherence Rates in Large Scale Quantum Computers and Macroscopic Quantum Systems

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Abstract. Markovian regime decoherence effects in quantum computers are studied in terms of the fidelity for the situation where the number of qubits \( N \) becomes large. A general expression giving the decoherence time scale in terms of Markovian relaxation elements and expectation values of products of system fluctuation operators is obtained, which could also be applied to study decoherence in other macroscopic systems such as Bose condensates and superconductors. A standard circuit model quantum computer involving three-state lambda system ionic qubits is considered, with qubits localised around well-separated positions via trapping potentials. The centre of mass vibrations of the qubits act as a reservoir. Coherent one and two qubit gating processes are controlled by time dependent localised classical electromagnetic fields that address specific qubits, the two qubit gating processes being facilitated by a cavity mode ancilla, which permits state interchange between qubits. With a suitable choice of parameters, it is found that the decoherence time can be made essentially independent of \( N \).

1 Introduction

The topic of quantum computation has developed enormously since the theoretical work of Feynman [1] and Deutsch [2] in the 1980s. Along with topics such as quantum teleportation, quantum cryptography and quantum measurement theory, quantum computation is one part of the expanding field of quantum information science [3]. Much of the current interest has been stimulated by potential applications of quantum computers in situations where they are expected to out-perform classical computers, such as in quantum algorithms for factoring large numbers [4] and searching large data bases [5], or in simulating the behavior of quantum systems [6]. The implementation of quantum computers has mainly focused on the standard quantum circuit model. Here quantum information is stored as entangled states of an array of two-state systems (qubits) which are initially prepared with all qubits in one state, the algorithm is then implemented as a sequence of unitary operations involving one or two qubits at a time, and the result for the computation is provided by a measurement on the final state of the quantum computer. Other approaches have also been proposed, including adiabatic quantum computation [6], continuous variable, topological...
or holonomic quantum computation [7], [8], [9] and quantum computation based on a sequence of projective measurements instead of unitary quantum gates [10], [11] - including those using cluster states [12]). A variety of physical realisations of the quantum circuit model have been proposed, differing in the nature of the qubit system (e.g., hyperfine ground states and metastable excited states in ions and neutral atoms, one photon states in optical systems with two polarisation modes, ...), the way the qubits are prepared in the initial state (e.g., optical pumping - ionic and neutral atom qubits), the method used for the gating process (e.g., two photon-resonant laser fields - one qubit gates in ions and atoms, one photon resonant laser fields combined with photons in high Q optical cavities - two qubit gates for atoms in cavity QED based systems, cold collisions between qubits - two qubit gates for neutral atoms, ..) and the technique used to detect qubit states (e.g., quantum jump techniques for ionic qubits). Small scale demonstrations of quantum computation have already been achieved, such as implementing the Deutsch-Josza algorithm [13] and factoring small numbers [14]. Quantum computer architectures, which specify the components and how they are integrated within a coherent plan, provide useful guides for the long term research program needed to develop practical quantum computers, and allow for developments of the basic architecture to overcome problems revealed as the research program is implemented. For example, ion trap quantum computers have significantly evolved from the original Cirac-Zoller single trap proposal [15] to an architecture involving qubits shuttling between a memory region and a processor region in order to build a larger scale system [16]. The implementation of quantum computers via quantum optical systems (atoms, ions and photons) is one of the more promising routes to follow [17], [18] and the present paper deals with such a system. A set of criteria for the successful realisation of circuit model quantum computers has been formulated by DiVincenzo [19], and comprehensive surveys of current knowledge of the subject are contained in recent reviews [20] and textbooks [3].

The idealised unitary evolution of the circuit model quantum computer does not occur in reality because the quantum computer interacts with the environment, both during gating processes and when no gating is taking place. The loss of unitarity is referred to as decoherence, and can be quantified in terms of the fidelity. This specifies how close the actual behavior of the qubit system density operator is to its idealised behavior when only coherent gating processes take place and system-reservoir interactions are switched off. The time for the fidelity to change significantly from unity defines the decoherence time scale. Other measures of decoherence time scales have also been proposed [21]. Decoherence is also of interest in other macroscopic systems such as Bose condensates and superconductors, and its general effect in macroscopic systems is important in quantum measurement theory and to understanding how classical behavior emerges [22], [23]. In the case of quantum computers, one of the DiVincenzo criteria for effective quantum computation is that decoherence time scales must be much larger than gating time scales and the time intervals when gating processes are absent, and hence a determination of the decoherence time scale is important.
in evaluating any specific quantum computer architecture. Decoherence is the enemy of quantum computation and a number of methods have been proposed for combating its effects. These include active (or error correcting) methods - quantum error correction [24], [25], [26], dynamical decoupling techniques [27], and passive (or error avoiding) methods - decoherence-free subspaces [28], [29], [30], [31], topological or holonomic quantum computing [7], [8], [9]. Combined methods, such as decoherence-free subspaces plus dynamical decoupling have also been proposed [32]. Quantum error correction implies a large overhead of redundant qubits and dynamical decoupling involves control pulses with time scales much shorter than reservoir correlation times, so in general terms it is desirable to implement error avoiding methods and use error correcting as a back up. It is well known that if the loss of fidelity in each gating process is kept below a certain threshold (estimated as being between $10^{-3}$ and $10^{-4}$ - see [33], [20]) then fault-tolerant quantum computing is possible using error correcting codes [34], [35]. Error avoiding methods will be the focus of the present paper, with the aim of keeping the fidelity loss below such a threshold. The scalability of a quantum computer architecture is another important DiVincenzo criterion. The large qubit case is important for implementing quantum computers in situations where they are expected to out-perform classical computers, such as factoring large numbers and searching large data bases [3]. In this paper, the primary aim will be to study decoherence effects in quantum computers for the situation where the number of qubits $N$ becomes large.

In general [36], the temporal behavior of a quantum system coupled to a zero temperature reservoir is of three distinct types, depending on the time regime: (a) Quadratic behavior at short times ($t \ll \tau_c$) (b) Exponential decay at intermediate times ($\tau_d \approx t \gg \tau_c$) and (c) Power law behavior at long times ($t \gg \tau_d$). Here $\tau_c$ is the reservoir correlation time, $\tau_d$ the system decay time. At non-zero temperature $T$, a thermal time scale $\tau_b = \hbar / k_B T$ is also involved. The short time regime is associated with the Quantum Zeno effect, the intermediate time regime involves Markovian decay and the long time regime behavior is due to a lower bound in the energy spectrum.

For quantum computers, the physical relevance of the short time regime is not clear, since (apart from architectures making use of dynamical decoupling methods for error correction purposes [27]) most feasible measurements and gating processes are likely to require time scales much longer than the reservoir correlation time. As Markovian theories for the intermediate time regime indicate that decoherence times decrease rapidly as the number of qubits $N$ increases, the decoherence time may finally become comparable to the reservoir correlation time. Apart from creating consistency problems for the theory, this could place a limit on quantum computer size. The relationship between the short and intermediate time regimes has been studied for a simple case of $N$ two-state systems, all initially in the lower state and coupled to the electromagnetic (EM) field in the vacuum state. A perturbation treatment correct to second order in the coupling constants [37] shows that the fidelity initially decreases quadratically for times less than $\tau_c$, then reaches a minimum and eventually
returns to unity. The Markoff theory fidelity equals one at all times. The effect is due to the non RWA terms in the system-reservoir coupling, and may merely reflect the artificial nature of the uncorrelated system-reservoir initial state. Alternative treatments of the short time regime based on methods preserving unitarity (38, 39) may overcome these difficulties for studying systems with very large $N$. A preliminary study [40] on a standard quantum computer model with two-state qubits obtained a quadratic behavior of the fidelity in the short time non-Markovian regime.

The present work deals with the intermediate time regime. The intermediate time regime is more physically relevant for reasons discussed above (the quantum computer model we study does not involve the use of dynamical decoupling), and here Markoff theory can be used. For internal consistency, the decoherence time scale $\tau_D$ must be long compared to $\tau_c$. Markovian expressions for the intermediate time behavior of the rate of change of fidelity are obtained for the general case where the qubit system (including any ancilla) are in a pure state and the reservoir (which may involve several components) is in a thermal state. The initial rate of change of the fidelity defines the decoherence rate and its inverse is the decoherence time scale $\tau_D$. This result gives the decoherence time scale in terms of Markovian relaxation elements and expectation values of products of fluctuation operators for the decohering quantum system. The expression is quite general and may have applications for decoherence in other macroscopic systems, such as Bose condensates or superconductors. For the quantum computer model studied, the characteristic decoherence time scale is evaluated at finite temperature for specific qubit states (such as Hadamard and GHZ states) in the situation of no gating processes occurring (memory decoherence). Decoherence effects on quantum computers due to one and two qubit gating processes are studied for zero temperature (gating decoherence). The zero temperature case should be most favourable for long decoherence times, and short decoherence times even at zero temperature during gating processes would be ominous for implementing large scale quantum computers.

As mentioned previously, architectures involving ionic qubits are amongst the most promising for possible implementations of circuit model quantum computers [17], [18], [20]. However, directly scaling up the original Cirac-Zoller model [15] to large numbers of qubits is difficult not only because it is hard to trap large numbers of ions in a single trap, but also because using a collective vibrational mode as an ancilla for two qubit gates becomes impossible when the mode density becomes very large. An extensive study of decoherence effects as qubit numbers are increased in Cirac-Zoller type quantum computers where a vibrational mode is used for gating purposes, has been carried out by Plenio and Knight [41], [42] for both two state and three state lambda system qubits, and applied to real ions. Limitations on the size of quantum computers (and therefore on the size of numbers factorisable via the Shor algorithm [4]) based on this model were found. Even allowing for error correction this limit was quite small. A solution to the problem of scaling up qubit numbers is being developed using shuttled qubits [16], and a different approach involving a 2D array of trapped ions...
ions with a control qubit moved above its target target qubit to carry out a two qubit gate via a collisional process has also been suggested [43], [44]. The 2D arrays might be based on elliptical rather than linear ion traps [45]. Another approach not involving ion shuttling has also been proposed, with two qubit gating via Raman laser pulses applied to the pair of qubits [46]. Architectures involving high Q cavities are also attractive [17], [18], [20], and it is worthwhile to try to combine these with ionic qubits. Cavity QED systems involve using an optical (or microwave) cavity mode photon as an ancilla to enable two qubit gating to occur for atomic or ionic qubits, and were in fact amongst the earliest proposals for quantum computers [17], [18], [19]. High-finesse optical cavities have now been developed [50] which allow a large number of atomic qubits to be enclosed under strong coupling conditions, so that coupling of the cavity mode to the qubit system is faster than both cavity photon loss and qubit decay via spontaneous emission. In such circumstances, two qubit gating via cavity photons can occur much faster than decoherence due to the loss processes. However, it is difficult to create strong coupling conditions for ionic qubits, since the small optical cavity volume can interfere with the ion trap and the presence of electrodes can interfere with the cavity mode [51], though single ion trapping inside a cavity under weak coupling regime conditions has been realised [52], [53]. There have been several proposals for scalable quantum computers involving ionic qubits in optical cavities under weak coupling conditions [31], [54], [55], [56], involving probabilistic entanglement protocols or using dissipation to confine the evolution in decoherence-free subspaces. The optical cavity may be arranged with its axis perpendicular to the array of ions, so that two qubits at a time are in the cavity. Strong coupling regime proposals have also been formulated [57]. In the present paper, we consider a combined ionic qubit and high Q cavity quantum computer architecture with a large number of qubits in the cavity. Our model is somewhat similar to the two qubit case considered by Tregenna, Beige and Knight [56], but now we consider a large number of qubits and also allow for their vibrational motion rather than treat them as stationary. As the emphasis of the present work is to examine the effects of decoherence on the system considered by Tregenna et al [56] as qubit numbers are increased, the same parameters as in their work will be used, rather than those for real ions. The issue of developing a theory for real ions is discussed in the last section of the paper. Memory decoherence for the Cirac-Zoller model [15] due to the effects of vibrational motion has also been studied by Garg [58]. Note that in the present case (unlike in the work of Plenio and Knight [41], [42]), the vibrational modes act as a reservoir rather than as an ancilla to facilitate gating processes.

A standard model involving N ionic qubits is considered, the overall architecture being illustrated in figure 1. Each qubit is in a three-state lambda system [31], rather than a two-state system as previously treated in the short time regime [40]. The qubit states are the two lower states 0, 1. The quantum computer system also includes a high Q cavity mode, which is coupled to the qubits and acts as an ancilla. Lambda systems, as well as facilitating Raman
gating processes, should result in qubits that are less vulnerable to spontaneous emission (SE) based decoherence, the upper state 2 only being occupied during gating processes. Reducing SE by having qubits located in a high Q cavity is also desirable, and the cavity mode also facilitates two qubit processes. Our model involves $N$ ionic qubits localised around well-separated positions via trapping potentials, and the centre of mass vibrational motions of the qubits are now treated. Coherent one and two qubit gating processes are controlled by time dependent localised classical EM fields that address specific qubits. The one qubit gating process involves weak two-photon resonant Raman gating fields, well detuned from one photon resonance. The two qubit gating processes are facilitated by the cavity mode ancilla, which permits state interchange between qubits. For the two qubit gating process, resonant gating fields coupled to the 2-1 transition for the control qubit, and coupled to the 2-0 transition for the target qubit. The cavity mode is resonant with the 2-1 transition, but uncoupled to the well-detuned 2-0 transition (as in [56]). Two qubit gating processes take place in decoherence-free subspaces [31, 56]. In our model, the reservoir (or bath) has three constituents. The three-state qubits are coupled to a bath of spontaneous emission modes, and the cavity mode is coupled to a bath of cavity decay modes. For large $N$ the numerous vibrational modes of the ionic qubits also act as a reservoir, with Lamb-Dicke coupling to the internal qubit coordinates, cavity mode and gating fields. Non-RWA couplings are included. All qubit interaction terms (electric dipole, Rontgen, diamagnetic, ionic current) with these three baths are examined and the important contributions to the decoherence rate found. The qubit-bath coupling is amplitude coupling via $\sigma_{ia}^{\pm}$ optical coherence operators. In addition to these fundamental causes of decoherence, technical shortcomings in the implementation of the computer model can also cause decoherence. For example, the trapping potential producing the array of qubit trap sites could be subject to fluctuations. Such trapping potentials could be provided by off resonant near-classical optical fields or by magnetic fields, and these could fluctuate. Decoherence effects due to fluctuations in these fields could have significant effects and should be evaluated. However, we will concentrate in this paper on fundamental causes of decoherence, especially the effects of qubit vibrations, and technical causes will be left to a later time.

The plan of this paper is as follows. In section 2 we set out the Hamiltonian for the quantum computer model and derive our general expression for the decoherence time scale by applying Markovian evolution theory. In section 3 the decoherence time scales are evaluated for the no-gating, one qubit gating and two qubit gating cases. A summary of the main results of the paper is presented in section 4 along with a discussion of extensions of the theory for real ions.

2 Theory
2.1 Hamiltonian

The total Hamiltonian for the system can be written as

$$H = H_S + H_C + H_B + V_S + V_I$$  \hspace{1cm} (1)$$

where the Hamiltonian for the qubit system and cavity mode ancilla is

$$H_S = \sum_{ia} \hbar \omega_a \sigma_{aa}^i + \hbar \omega_b \sigma_{bb}$$  \hspace{1cm} (2)$$

the Hamiltonian for the collective vibrational motions of the qubit system is

$$H_C = \frac{1}{2m} \sum_{ia} p_{ia}^2 + \frac{1}{2} \sum_{ij \alpha \beta} \nu_{ij} \delta r_{i\alpha} \delta r_{j\beta}$$  \hspace{1cm} (3)$$

$$= \sum_K \hbar \nu_K A_K^\dagger A_K$$  \hspace{1cm} (4)$$

and the Hamiltonian for the bath of spontaneous emission and cavity decay modes is

$$H_B = \sum_k \hbar \omega_k a_k^\dagger a_k + \sum_k \hbar \xi_k b_k^\dagger b_k.$$  \hspace{1cm} (5)$$

The coherent coupling Hamiltonian for gating processes in the qubit system is

$$V_S = \sum_{i; a=0, 1} \hbar (\Omega_{ia} + \Omega_{ia}^*) (\sigma_+^{ia} + \sigma_-^{ia}) $$

$$+ \sum_{i; a=0, 1} \hbar (g_{ia} b + g_{ia}^* b^\dagger) (\sigma_+^{ia} + \sigma_-^{ia})$$  \hspace{1cm} (6)$$

The qubits are coupled to both classical fields and the cavity mode. Each qubit is addressed by localised classical EM fields to facilitate 1 qubit and 2 qubit gating. For 1 qubit gating $\Omega_{ia}$ for $i$th (gated) qubit are two photon resonant Raman fields strongly detuned from 0-2 and 1-2 transitions. For 2 qubit gating $\Omega_{i1}$ is resonant with the 1-2 transition for the $i$th (control) qubit and $\Omega_{j0}$ is resonant with the 0-2 transition for the $j$th (target) qubit [56]. The cavity frequency $\omega_b$ is resonant with the 1-2 optical transition [56]. In the model of Tregenna et al [56] the cavity mode is only coupled to the 1-2 optical transition $g_{00} = 0$

Finally, the interaction of the qubit system and ancilla with the bath and centre of mass vibrations is given by

$$V_I = \sum_{i; a=0, 1} \hbar (\sigma_+^{ia} + \sigma_-^{ia}) \sum_k (g_k^{ia} a_k + g_k^{ia*} a_k^\dagger) $$

$$+ \hbar b \left[ \sum_k (v_k b_k + w_k b_k^\dagger) + \sum_{iK} t_{iK}^r (A_K^\dagger - A_K) \right] + HC$$
\[ + \sum_{i, a=0,1} \hbar (\sigma_{ia}^{+} + \sigma_{ia}^{-}) b \sum_{K} p_{K}^{i} (A_{K} + A_{K}^{\dagger}) + HC \]
\[ + \sum_{i, a=0,1} \hbar (\sigma_{ia}^{+} + \sigma_{ia}^{-}) \sum_{K} (\Theta_{K}^{i} + \Theta_{K}^{i*}) (A_{K} + A_{K}^{\dagger}) \] (7)

Terms include electric dipole coupling of qubits to SE modes, quasi-mode coupling of cavity mode to decay modes, Lamb-Dicke coupling of qubits, cavity mode, gating field to CM modes. Rontgen and diamagnetic terms are not included as their effects were shown to be small.

The centre of mass (CM) displacement of the qubits is related to the collective vibrational modes of the qubits via
\[ \delta r_{ia} = \sum_{K} S_{ia;K} \sqrt{\frac{\hbar}{2m\nu_{K}}} (A_{K} + A_{K}^{\dagger}) \] (8)
\[ \sum_{j\beta} V_{ij}^{\alpha\beta} S_{j\beta;K} = m\nu_{K}^{2} S_{ia;K} \] (9)

where the unitary real matrix \( S \) relates qubit CM displacements \( \delta r_{ia} (\alpha = x, y, z) \) to vibrational normal coordinates.

The qubit, cavity mode, centre of mass motion, bath modes operators are
\[ \sigma_{ia}^{+} = (|2\rangle\langle a|)_i \quad \sigma_{ia}^{-} = (|a\rangle\langle 2|)_i \quad a = 0,1 \]
\[ \sigma_{ia}^{\dagger} = (|a\rangle\langle b|)_i \quad a \neq b \quad a = 0,1 \]
\[ \sigma_{aa}^{\dagger} = (|a\rangle\langle a|)_i \quad a = 0,1,2 \]
\[ [b, b^{\dagger}] = 1 \quad [a_k, a_l^{\dagger}] = \delta_{kl} \]
\[ [b_k, b_l^{\dagger}] = \delta_{kl} \quad [A_K, A_L^{\dagger}] = \delta_{KL} \] (11)

These include qubit optical, Zeeman (or hyperfine) coherences and population operators, as well as bosonic annihilation, creation operators for the cavity, spontaneous emission, cavity decay and the CM vibrational modes.

The Hamiltonians involve certain coupling constants defined as follows:
\[ \Omega_{ia} = -i \sum_{c} \sqrt{\frac{\omega_c}{2\epsilon_0 h V}} (d_{2a} \cdot u_c) \alpha_c \exp(ik_c \cdot r_{i0} - \omega_c t) \] (12)
\[ g_{ia} = -i \sqrt{\frac{\omega b}{2\epsilon_0 h V_b}} (d_{2a} \cdot u_b) \exp(ik_b \cdot r_{i0}) \] (13)
\[ g_{ia}^{\dagger} = -i \sqrt{\frac{\omega_{i0}}{2\epsilon_0 h V_b}} (d_{2a} \cdot u_{i0}) \exp(ik \cdot r_{i0}) \] (14)
\[ p_{K}^{i} = \sqrt{\frac{\omega b}{2\epsilon_0 h V_b}} \sqrt{\frac{\hbar}{2m\nu_{K}}} (d_{2a} \cdot u_b) (k_b \cdot S_{ik}) \times \]
\[ \Theta_K^{\text{ia}} = \sum_c \sqrt{\omega_c} \sqrt{\frac{\hbar}{2m\nu_K}} (d_{2a} \cdot u_c)(k_c \cdot S_{iK}) \times \alpha_c \exp(i(k_c \cdot r_{i0} - \omega_c t)) \tag{15} \]

\[ t_K^i = ie_T \frac{1}{\sqrt{2e_0\omega_bV_b}} \sqrt{\frac{\nu_K}{2m}} (k_b \cdot S_{iK}) \exp(i k_b \cdot r_{i0}) \tag{16} \]

where \( a = 0, 1 \) unless stated otherwise. The cavity mode volume is \( V_b \), the SE mode volume \( V \). Each ion has charge \( e_T \).

2.2 Dynamics and Decoherence Time

The total density operator for the qubits, ancilla and reservoirs satisfies the Liouville-von Neumann equation

\[ i\hbar \frac{\partial W}{\partial t} = [H, W] \tag{18} \]

\[ H = H_S + V_S + H_R + V_I \tag{19} \]

The initial condition is given by

\[ W(0) = \rho_S(0)\rho_R(0) \tag{20} \]

and represents an uncorrelated state for qubits and reservoirs, the qubits initially being in a pure state \( |\psi_S\rangle \) and \( \rho_S(0) = |\psi_S\rangle\langle\psi_S| \), whilst the reservoirs are in thermal states.

The reduced density operator for qubits and ancilla is defined as

\[ \rho_S = Tr_R W \tag{21} \]

and its general evolution allows for both coherent coupling and reservoir interactions.

For coherent evolution due to \( V_S \) only, the reduced density operator for qubits and ancilla would satisfy

\[ i\hbar \frac{\partial \rho_{S0}}{\partial t} = [H_S + V_S, \rho_{S0}] \tag{22} \]

where the same initial condition \( \rho_{S0}(0) = \rho_S(0) \) can be chosen as for the general evolution.

The fidelity is defined by

\[ F = Tr_S \rho_{S0} \rho_S \tag{23} \]
and compares the actual and idealised evolution of the qubit system.

The decoherence timescale is defined as the time scale over which actual quantum computer evolution (non-unitary) differs significantly from idealised coherent evolution (unitary). The decoherence time scale will be defined via the time dependence of the fidelity. It is related to certain basic time scales due to qubit-environment coupling. These are: (a) the reservoir correlation time \( \tau_c \) - for EM field SE modes \( \tau_c \sim 10^{-17} \)s (b) Markovian relaxation times - \( T_1 \) for populations, \( T_2 \) for coherences - for EM field SE modes \( T_{1,2} \sim 10^{-8} \)s (c) the thermalisation time \( \tau_b = \hbar/k_BT \) \( \sim 10^{-5} \)s. The decoherence timescale will also depend on factors such as qubit system state, the numbers of qubits and the reservoirs involved.

Markovian evolution occurs for \( t \gg \tau_c \) and the reduced density operator satisfies a master equation

\[
\frac{i\hbar}{\partial t} \rho_S = [H_S + V_S, \rho_S] + L\rho_S
\]

\[
L\rho_S = -i \sum_{ab} \Delta_{ab} [S_a S_b^\dagger, \rho_S] + \sum_{ab} \Gamma_{ab} \{[S_a S_b^\dagger, \rho_S] + [S_b^\dagger S_a, \rho_S]\}.
\]

The master equation includes the Liouville superoperator term involving Markovian relaxation \( \Gamma_{ab} \) and frequency shift \( \Delta_{ab} \) matrices and system operators \( S_a \), where the system-reservoir interaction \( V_I \) is sum of products of system and reservoir operators

\[
V_I = \sum_a S_a R_a
\]

\[
[H_S, S_a] = \hbar \omega_a S_a.
\]

The Markovian evolution requires the reservoir correlation functions \( \langle \hat{R}_a(t) \hat{R}_b(t-\tau) \rangle \) approach zero for \( \tau \gg \tau_c \), and the Markovian relaxation \( \Gamma_{ab} = \Gamma_{ba}^* \) and frequency shift \( \Delta_{ab} = \Delta_{ba}^* \) matrices are given via reservoir correlation functions as:

\[
C_{ab} = \int_0^\infty d\tau \langle \hat{R}_a(t) \hat{R}_b(t-\tau) \rangle \exp(-i\omega_b + \epsilon) \tau
\]

\[
\Gamma_{ab} = (C_{ab} + C_{ba}^*)/2, \quad \Delta_{ab} = (C_{ab} - C_{ba}^*)/2i
\]

In the Markovian intermediate time regime \( (\tau_d \approx t \gg \tau_c) \) the rate \( (\partial F / \partial t)_0 \) specifies the decoherence rate for the qubit system, and its inverse defines the decoherence time \( \tau_D \). The decoherence time can be expressed in terms of the Markovian relaxation matrix elements and qubit system averages of products of fluctuation
operators
\[ \frac{1}{\tau_D} \equiv -\left( \frac{\partial F}{\partial t} \right)_{t \gg \tau_c \to 0} = 2 \left( \sum_{ab} \Gamma_{ab} (\Delta S_a \Delta S_b^\dagger) S \right)_{t \gg \tau_c \to 0} \] (30)

Here the qubit and ancilla are in pure state \( |\chi_S\rangle \), which may differ from initial state \( |\psi_S\rangle \) due to gating processes. For \( t \gg \tau_c \to 0 \) we take \( \rho_{S0} = \rho_S = |\chi_S\rangle \langle \chi_S| \). In the above \( \Delta S_a = S_a - \langle S_a \rangle_S \) and \( \langle \Omega \rangle_S \equiv Tr_S(\Omega \rho_S) \). The expression for the decoherence time scale is quite general and may have applications for decoherence in other macroscopic systems, such as Bose condensates or superconductors or in the theory of quantum measurement. A special case of this result is given in Ref. [60], which deals with Bose condensates.

Expressions can be obtained for the loss of fidelity. During a time \( T \) much smaller than \( \tau_D \), the change in fidelity if no gating is occurring is
\[ \Delta F = -\left( \frac{1}{\tau_D} \right)_{\text{no-gating}} T \] (31)
and if gating is taking place during a time \( \Delta T \) the change in fidelity is
\[ \Delta F = -\left( \frac{1}{\tau_D} \right)_{\text{gating}} \Delta T \] (32)

For idealised quantum computation we require \( \Delta F \ll 1 \).

In the short-time regime \( t \ll \tau_c \) the time dependent fidelity may be written in a power series [40]
\[ F(t) = 1 - \left( \frac{t}{\tau_1} \right) - \left( \frac{t^2}{2\tau_2^2} \right) + .. \] (33)
\[ \frac{\hbar}{\tau_1} = 0 \] (34)
\[ \frac{\hbar^2}{2\tau_2^2} = Tr_{BC} (\langle V_I(0) \rangle^2) \rho_B(0) \rho_C(0) \]
\[ -Tr_{BC} \langle V_I(0) \rangle^2 \rho_B(0) \rho_C(0) \]
\[ \equiv Tr_{BC} (\Delta V_I(0)^2) \rho_B(0) \rho_C(0) \] (35)

The qubit and ancilla are initially in a pure state \( |\psi_S\rangle \) and \( \rho_S(0) = |\psi_S\rangle \langle \psi_S| \).
Here \( \Delta V_I(0) = V_I(0) - \langle V_I(0) \rangle_S \) where \( \langle \Omega \rangle_S \equiv Tr_S(\Omega \rho_S) \). The times \( \tau_1, \tau_2, .. \) specify characteristic decoherence times for the qubit and ancilla system, their inverses defining decoherence rates. For the uncorrelated initial state \( W(0) \), only \( \tau_2 \) is involved in specifying the short time decoherence. The decoherence time scale \( \tau_2 \) depends on qubit and reservoir averages of the fluctuation in the system-reservoir interaction operator squared. The quadratic time dependence of the fidelity is characteristic of the quantum Zeno effect. However, results obtained for \( \tau_2 \) are due to non RWA terms in the system-reservoir coupling, and may merely reflect the artificial nature of the uncorrelated system-reservoir initial state.
3 Results

3.1 Case of No Gating

For the situation where no gating processes are occurring, there is no upper state \( |2 \rangle \) amplitude, the cavity mode is in a no photon state \( |0 \rangle_A \) and the qubit-ancilla state \( |\chi_S \rangle \) is given by \( |\phi_Q \rangle |0 \rangle_A \), where qubit system state is \( |\phi_Q \rangle \). This situation corresponds to states produced after idealised coherent gating processes. To examine unfavourable scenarios, the reservoir temperature \( T \) is assumed non-zero and the qubits and the cavity is assumed to have a low \( Q \), so that spontaneous emission decay leads to a larger decoherence rate than would otherwise be the case. Spontaneous emission is the dominant decoherence process, and only its contribution is shown.

In the Markovian intermediate time regime \( (\tau_d \approx t \gg \tau_c) \) the decoherence time is given by

\[
\frac{1}{\tau_D} = \exp(-\frac{\hbar \omega_0}{k_B T}) \sum_{ab} \sqrt{\Gamma_a \Gamma_b} \cos \theta_{ab} \sum_i \langle \sigma_{i}^{a b} \rangle \tag{37}
\]

The expression involves Zeeman (or hyperfine) coherences for \( i \)th qubit \( \langle \sigma_{i}^{a b} \rangle (a \neq b) \) \( (a, b = 0, 1) \) and populations for \( i \)th qubit \( \langle \sigma_{i}^{a a} \rangle (a = 0, 1) \). The optical transition frequencies are \( \omega_{2a} \approx \omega_0 \) \( (a = 0, 1) \), the spontaneous emission decay rates for 2-a transition are \( \Gamma_a \) and the angle between dipole matrix elements \( d_{2a}, d_{2b} \) is \( \theta_{ab} \).

For the case of the Hadamard state (uncorrelated), the qubit state is \( |\phi_Q \rangle = \prod_i |\phi_Q \rangle_i \), where \( i \)th qubit state vector is \( |\phi_Q \rangle_i = (|0 \rangle_i + |1 \rangle_i)/\sqrt{2} \), we find that

\[
\frac{1}{\tau_D} = \frac{1}{2} N \exp(-\frac{\hbar \omega_0}{k_B T}) \sum_{ab} \sqrt{\Gamma_a \Gamma_b} \cos \theta_{ab} \tag{38}
\]

and note that the coherence time for Hadamard state can be infinite for the lambda qubit system if choose \( d_{20} + d_{21} = 0 \).

On the other hand, for the case of the GHZ state (correlated), where the qubit system state vector is \( |\phi_Q \rangle = (|00..0 \rangle + |11..1 \rangle)/\sqrt{2} \), we obtain

\[
\frac{1}{\tau_D} = \frac{1}{2} N \exp(-\frac{\hbar \omega_0}{k_B T}) \sum_a \Gamma_a \tag{39}
\]

Here the decoherence time scale is still very long, due to the upper state Boltzmann factor. With \( N \approx 10^4 \) qubits and \( \Gamma \approx 10^8 s^{-1} \), \( \omega_0 \approx 10^{15} s^{-1} \), \( T \approx 300 K \), we find \( \tau_D \approx 10^{19} s \).

Overall, spontaneous emission due to electric dipole coupling is dominant cause of decoherence. Other terms such as Lamb-Dicke coupling can be ignored, so the qubits can be treated as stationary. The decoherence time scales as \( 1/N \) for this intermediate time regime but it is very long. The present results for this case of memory decoherence are consistent with those of Garg [58], who also
found very low decoherence rates allowing for vibrational motion in Cirac-Zoller type quantum computers with large numbers of qubits.

3.2 Case of One Qubit Gating

For the one qubit gating process with two-photon resonant Raman gating fields detuned from one photon resonance, the upper state |2⟩ amplitude for ith (gated) qubit becomes slightly non-zero, though the cavity mode remains in the zero photon state |0⟩A |χS⟩ given by |φQ⟩|0⟩A, with the qubit state |φQ⟩ (aj = 0, 1) given as

\[ |φ_Q⟩ = \sum_{\{a\}} C_0(\{a\})|\{a\}\rangle + \sum_{\{a_i\}} C_0(\{a_i\}; 2_j)|\{a_i\}; 2_j⟩ \]  (40)

Here \( \{a\} \equiv \{a_1, a_2, .., a_N\} \), \( \{a_i\} \equiv \{a_1, a_i−1, a_i+1, ..\} \). However spontaneous emission decay is small in the high Q cavity. The reservoir temperature is assumed zero.

In the Markovian intermediate time regime (\( τ_d \approx t \gg τ_c \)) the decoherence time is given by

\[ \frac{1}{τ_D} = 2\left( \sum_{k \neq i} \sum_{ab} \langle σ^{k}_{ab}⟩ Γ_{ka−C; kb−C} \right. \]
\[ + \sum_{ab} (\langle σ^{i}_{ab}⟩ − \langle σ^{i}_{+}\rangle \langle σ^{i}_{−}\rangle) Γ_{ia−; ib−} \]
\[ + \sum_{ab} (\langle σ^{i}_{+}\rangle δ_{ab} − \langle σ^{i}_{ab}\rangle) Γ_{ia+; ib+} \]
\[ + \sum_{ab} \langle σ^{i}_{ab}\rangle Γ_{ia−C; ib−C} + \sum_a \langle σ^{i}_{22}\rangle Γ_{ia+a; ia+a} \right) \]  (41)

The decoherence time involves Markovian relaxation elements and state dependent quantities for the qubit system. For the gated qubits terms, optical coherences \( ⟨σ^{i}_{ab}\rangle \), Zeeman (or hyperfine) coherences \( ⟨σ^{i}_{ab}\rangle \), upper state population \( ⟨σ^{i}_{22}\rangle \) and lower state populations \( ⟨σ^{i}_{aa}\rangle \) are involved. For the non-gated qubits (\( k \neq i \)), Zeeman (or hyperfine) coherences \( ⟨σ^{k}_{ab}\rangle \) and lower state populations \( ⟨σ^{k}_{aa}\rangle \) are present.

Expressions for the zero temperature relaxation matrix elements are as follows:

\[ Γ_{ka−C; kb−C} = \frac{i}{8}\hbar g_{ka}g_{kb}^{*} \frac{ω_{ab}}{ω_{0}} (k \neq i, k = i) \]  (42)
\[ Γ_{ia−; ib−} = \frac{i}{2}g^2(Ω_{ia}^{*}Ω_{ib}^{*} − Ω_{ia}Ω_{ib})/Δ_0 \]  (43)
\[ Γ_{ia+; ib+} = \frac{1}{2} \sqrt{Γ_a Γ_b} \cos θ_{ab} \]
In these formulae the Lamb-Dicke (LD) parameter is \( \eta \), the gating EM field Rabi frequencies are \( \Omega_{ia} (a = 0, 1) \), the one photon detunings for the Raman gating field are \( \Delta_{0i} \) and the one photon Rabi frequencies for the cavity mode are \( g_{ka} (a = 0, 1) \). The Zeeman (or hyperfine) transition frequencies are \( \omega_{ab} (a \neq b) \) \((a, b = 0, 1)\). Vibrational frequencies range from zero up to \( \nu_{\text{max}} \), and the vibrational modes have zero phonons at absolute zero. In the case studied the cavity mode is resonant with the 2-1 transition \( \omega_{b} = \omega_{21} \sim \omega_{0} \), but both one photon Rabi frequencies \( g_{ka} \) are assumed non-zero. Approximations based on \( \Delta_{0i} \gg \nu_{\text{max}}, \Gamma \) and \( \omega_{10} \gg \nu_{\text{max}} \) have been used.

The populations and coherences are found for large one photon detuning, assuming \( \Omega_{i0} = \Omega_{i1} \exp i\Delta\phi = \Omega(t) \), where the common amplitude \( \Omega \) has a maximum \( \Omega_{m} \) and a width \( \Delta T \). For a gated qubit initially in state \( |0\rangle \) (see Vitanov et al [61]) we find that

\[
\langle \sigma_{01} \rangle = \langle \sigma_{10}^\dagger \rangle = i \sin \theta \cos \theta \exp i\Delta \phi
\]

\[
\langle \sigma_{00} \rangle = \cos \theta, \quad \langle \sigma_{11} \rangle = \sin ^2 \theta
\]

\[
\langle \sigma_{-+} \rangle = \langle \sigma_{-0}^\dagger \rangle = -\frac{\Omega}{\Delta} \exp (-i\theta) \cos \theta
\]

\[
\langle \sigma_{++} \rangle = \langle \sigma_{11}^\dagger \rangle = -\frac{\Omega}{\Delta} \exp (-i\theta) i \sin \theta \exp i\Delta \phi
\]

\[
\langle \sigma_{22} \rangle = \frac{\Omega^2}{\Delta^2}
\]

In these expressions the quantity \( \theta(t) \) is defined by the integral of the two photon Rabi frequency \( \Omega_{R} = \Omega^2 / \Delta \)

\[
\theta(t) = \int_{-\infty}^{t} dt' \Omega(t')^2 / \Delta
\]

The gating time \( \Delta T \) is given by \( \theta = \pi / 2 \) for \( t \simeq \Delta T \), corresponding to the time the qubit takes to evolve into state \( |1\rangle \)

\[
\Delta T \simeq \frac{\pi \Delta}{2 \Omega_{m}^2}
\]

The parameters used in both the one and two qubit gating processes are set out in Table 1

**Table 1. Parameters used in gating processes**
\[ \omega_b \approx \omega_0 \approx 3 \times 10^{15} \text{s}^{-1} \]
\[ \omega_{10} = 6 \times 10^9 \text{s}^{-1} \]
\[ \Delta_{ci} = 3 \times 10^9 \text{s}^{-1} \]
\[ \nu_{\text{max}} = 8 \times 10^4 \text{s}^{-1} \]
\[ |\Omega_{ia}| = 3 \times 10^6 \text{s}^{-1} \]
\[ g_{ka} = 3 \times 10^8 \text{s}^{-1} \]
\[ \Gamma_a = 3 \times 10^9 \text{s}^{-1} \]
\[ \Gamma_{\text{cav}} = 3 \times 10^8 \text{s}^{-1} \]
\[ Q = 1.10^2 \]
\[ \eta = 6 \times 10^{-2} \]

These correspond to optical and hyperfine transitions with a high Q optical cavity in the medium coupling regime. As discussed previously, these parameters are chosen to be the same as in the work of Tregenna et al [56], so that their model can be compared to the present one where decoherence effects due to vibrational motion is allowed for as qubit numbers increase. The cavity coupling constant and cavity decay rate are made equal, and large compared to spontaneous emission (SE) rate, and the gating field one photon Rabi frequency is large compared to the SE decay rate. Two cases are studied, corresponding to the gating field one photon Rabi frequency being (i) small compared to (ii) the same as the cavity coupling constant and cavity decay rate. Case (i) applies in Tregenna et al [56]. The maximum vibration frequency and Lamb-Dicke parameter are calculated for a 3D Ca⁺ lattice with lattice spacing \( 3 \mu \). A standard approach to the theory of lattice vibrations is used [62], in which the vibrational potential energy is given by a quadratic form of the small displacements of the qubits from the lattice sites, as in Eq.3. The interaction between each pair of ionic qubits is obtained from electrostatics, from which the \( V_{ij}^{\alpha\beta} \) are obtained. For the cubic lattice case, the vibration frequencies are obtained from Eq.9, and are approximately proportional to the magnitude of the wave vector for each vibrational mode. Expressions for the quantities \( S_{\alpha;K} \) are obtained from Eq.9.

Using these parameters, the zero temperature relaxation elements can be calculated and the results are given in Table 2.

**Table 2. Relaxation elements in one qubit gating process**

| \( \Gamma_{ka-C; kb-C} \) | \( \Gamma_{ia-C; ib-C} \) | \( \Gamma_{ia-C; ib+} \) | \( \Gamma_{ia+; ib-C} \) | \( \Gamma_{ia+; ib+} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( 3 \times 10^{-3} \text{s}^{-1} \) | \( 3 \times 10^9 \text{s}^{-1} \) | \( 3 \times 10^8 \text{s}^{-1} \) | \( 1.10^4 \text{s}^{-1} \) | \( 1.10^4 \text{s}^{-1} \) |

The overall contributions to the decoherence rate can be obtained by combining the qubit factor (for \( \theta \approx \pi/4 \), midway through process) and the relaxation matrix elements for the various terms (1 - 5) involved in the expression for the decoherence rate \( 1/\tau_D \). The results are presented in Table 3. Here NG, G refer to non-gated and gated qubits respectively. The two gating field cases are: (i) \( \Omega = 3 \times 10^8 \text{s}^{-1} \) (ii) \( \Omega = 3 \times 10^6 \text{s}^{-1} \). The corresponding gating times are \( 10^{-2} \text{s} \), \( 10^{-6} \text{s} \) (see Eq.52), which are long compared to the correlation time for the vibrational modes reservoir of \( 10^{-7} \text{s} \) (given by \( 2\pi/\nu_{\text{max}} \)), and much longer than the correlations times associated with the SE or cavity decay modes reservoirs.

**Table 3. Contributions to decoherence rate in one qubit gating process**

| \( \Omega = 3 \times 10^8 \text{s}^{-1} \) | \( \Omega = 3 \times 10^6 \text{s}^{-1} \) |
|-----------------|-----------------|
| \( 1.10^8 \text{s}^{-1} \) | \( 1.10^8 \text{s}^{-1} \) |
The scaling of the decoherence rate with qubit number for the case of one qubit Raman gating gives an overall decoherence time which is essentially independent of $N$. This is because the terms for gated qubit $i$ do not scale with $N$, whilst the non-gated qubit contributions (which scale with $N$) are negligible in comparison, even for $N \sim 10^4$ qubits. These features are clear from the results in Table 3. For both the small and larger gating field cases, the largest overall contribution to the decoherence rate arises from the terms associated with Lamb-Dicke coupling of the gated qubit with the gating field $\sum_{ab} (\langle \sigma_{ab}^i \rangle - \langle \sigma_{ia}^i \rangle \langle \sigma_{ib}^i \rangle) \Gamma_{ia+; ib-}$. The next largest overall contribution arises from the terms associated with Lamb-Dicke coupling of the gated qubit with the cavity mode $\sum_{a} \langle \sigma_{i2}^i \rangle \Gamma_{ia+C; ia+C}$. The larger relaxation rate $\Gamma_{ia+C; ia+C}$ for the latter is balanced by the smaller qubit factor $\langle \sigma_{i2}^i \rangle$ associated with the upper state population.

The overall fidelity loss can then be obtained. Since terms associated with Lamb-Dicke coupling with the gating field are dominant, these terms combined with the gating time gives for the overall fidelity loss

$$\Delta F \simeq -\frac{1}{2} \frac{\eta^2 \Omega_m^2}{\Delta} \frac{\pi}{2} \frac{\Delta}{\Omega_m^2}$$

$$\simeq -\frac{\pi}{4} \frac{\Delta}{\eta^2}$$

We note that the fidelity loss only depends on Lamb-Dicke parameter. For $\eta = 6.10^{-2}$, the fidelity loss is $2.10^{-3}$. This is reasonably small though still somewhat large for fault-tolerant quantum computation - see [33], [20].

### 3.3 Case of Two Qubit Gating

For the two qubit gating process, with resonant gating fields coupled to the 2-1 transition for $i$th (control) qubit, and coupled to the 2-0 transition for the $j$th (target) qubit, and the cavity mode resonant with 2-1 transition, but uncoupled to the 2-0 transition, the upper state $|2\rangle$ amplitude for the $i$th (control) and $j$th (target) gated qubits could be non-zero. The cavity mode is in zero photon
state $|0\rangle_A$ when one or both gated qubits $i,j$ are in the upper state, but could be in one photon state $|1\rangle_A$ when both are in lower states. Here $|\chi_S\rangle$ given by $|\phi_Q^0\rangle_0 + |\phi_Q^1\rangle_1$, with qubit states $|\phi_Q^i\rangle$

$$|\phi_Q^0\rangle = \sum_{\{a_{ij}\}} C_0(\{a_{ij}\}; 2, i, j)|\{a_{ij}\}; 2, i, j\rangle + \sum_{\{a_i\}} C_0(\{a_i\}; 2, 2)|\{a_i\}; 2, 2\rangle$$

$$|\phi_Q^1\rangle = \sum_{\{a\}} C_1(\{a\})|\{a\}\rangle.$$

Here $\{a\} \equiv \{a_1, a_2, \ldots a_N\}$, $\{a_i\} \equiv \{a_i, a_{i-1}, a_{i+1}\}$, $\{a_j\} \equiv \{a_j, a_{j-1}, a_{j+1}\}$, and $(a_i, a_j = 0, 1)$. However spontaneous emission decay is small in the high Q cavity. The reservoir temperature is assumed zero.

In the Markovian intermediate time regime ($\tau_d \approx t \gg \tau_c$) the decoherence time is given as the sum of non-gated and gated qubit contributions as

$$\left(\frac{1}{\tau_D}\right) = \left(\frac{1}{\tau_D}\right)_{NG} + \left(\frac{1}{\tau_D}\right)_G. \tag{57}$$

The non-gated (NG) qubits contribution involves 7 terms

$$\left(\frac{1}{\tau_D}\right)_{NG} = 2\{ \sum_{k \neq i,j} \sum_{ab} (\sigma_{ab}^k) \Gamma_{ka--; kb--}$$

$$+ \sum_{k \neq i,j} \sum_{ab} (\sigma_{ab}^k \sigma_{ba}^k) \Gamma_{ka--; kb--}$$

$$+ \sum_{k \neq i,j} \sum_{ab} (\sigma_{ab}^k \sigma_{ba}^k) \Gamma_{ka--; kb--}$$

$$+ \sum_{k \neq i,j} \sum_{ab} (\sigma_{ab}^k \sigma_{ba}^k) \Gamma_{ka--; kb--}$$

The gated qubits contribution consist of 120 terms. Some of the terms are

$$\left(\frac{1}{\tau_D}\right)_G^{(23)} = 2\sum_{ab} (\langle \sigma_{ab}^i \rangle - \langle \sigma_{ab}^i \rangle \langle \sigma_{ab}^i \rangle) \Gamma_{ia--; ib--} \tag{59}$$

$$\left(\frac{1}{\tau_D}\right)_G^{(24)} = 2\sum_{ab} (\langle \sigma_{ab}^j \rangle - \langle \sigma_{ab}^j \rangle \langle \sigma_{ab}^j \rangle) \Gamma_{ia--; ib--} \tag{60}$$

$$\left(\frac{1}{\tau_D}\right)_G^{(34)} = 2\sum_{ab} (\langle \sigma_{ab}^j \rangle - \langle \sigma_{ab}^j \rangle \langle \sigma_{ab}^j \rangle) \Gamma_{ia--; ib--} \tag{61}$$
\[
\left( \frac{1}{\tau_D} \right)_G^{(35)} = 2 \sum_{ab} \langle \sigma_{ab} \rangle - \langle \sigma_{ab}^\dagger \rangle \langle \sigma_{ab}^\dagger \rangle \Gamma_{ja-\rightarrow jb-} 
\]
\[
\left( \frac{1}{\tau_D} \right)_G^{(120)} = 2 \langle b|b \rangle - \langle b\rangle \langle b \rangle \Gamma_{C+;C+} 
\]
Some of these are equivalent to those for one qubit gating, others involve different expressions, since the gating fields are now resonant rather than having a large detuning. In addition, there are many new terms only present for two qubit gating. Again, the decoherence time involves Markovian relaxation elements and state dependent quantities for qubit system. The terms for non-gated qubits \((k \neq i, j)\) involve Zeeman (or hyperfine) coherences \(\langle \sigma_{ab}^g \rangle\) and lower state populations \(\langle \sigma_{aa}^g \rangle\), as well as quantities also involving the cavity mode operators \(\langle \sigma_{ab}^h \rangle, \langle \sigma_{ab}^h \rangle, \langle \sigma_{ab}^h \rangle, \langle \sigma_{ab}^h \rangle, \langle \sigma_{ab}^h \rangle\) and \(\langle \sigma_{ab}^h \rangle\). For the terms involving gated qubits, optical coherences \(\langle \sigma_{ab}^g \rangle, \langle \sigma_{ab}^g \rangle\), Zeeman (or hyperfine) coherences \(\langle \sigma_{ab}^g \rangle\), upper state populations \(\langle \sigma_{ab}^g \rangle\), lower state populations \(\langle \sigma_{ab}^g \rangle\) \((g \text{ refers to } i, j)\), as well as two qubit state transitions \(\langle \sigma_{ab}^g \rangle, \langle \sigma_{ab}^g \rangle, \langle \sigma_{ab}^g \rangle\) \(\text{ are all involved. The two qubit transitions are } \sigma_{ab;cd}^g = \langle \langle a|c \rangle \rangle \langle \langle b|d \rangle \rangle_h, \text{ where } g \neq h \text{ refers to } i \neq j\) \(\text{ In addition, there are quantities involving cavity mode operators also, similar to those for the non-gated qubits.}\)

For the non-gated qubits, all relaxation elements \(\Gamma_{ka-\rightarrow kb-}, \Gamma_{ka-\rightarrow kb-C}, \Gamma_{ka-\rightarrow kb-C+}, \Gamma_{ka-C;kb-C} \text{ and } \Gamma_{ka+C;kb+C} \text{ are zero for the specific gating process [56] involved. In particular, the last two are zero because the cavity mode is resonantly coupled to the 2-1 transition and uncoupled to the 2-0 transition, as may be seen from the following expressions:}\)

\[
\Gamma_{ka-C;kb-C} = \frac{i}{8} \eta^2 g_{ka} g_{kb}^* \frac{\omega_{ab}}{\omega_0^*} \Gamma_{ka-C;kb-C} 
\]
\[
\Gamma_{ka+C;kb+C} = i \eta^2 g_{ka} g_{kb}^* \frac{(1 - \delta_{ab})(-1)^n}{\nu_{\text{max}}} \Gamma_{ka+C;kb+C} 
\]

\[
\Gamma_{ia-;ib-} = i \delta_{ai} \delta_{bi} \eta^2 (\Omega_{i1} \Omega_{i1} - \Omega_{i1}^* \Omega_{i1}^*) / \nu_{\text{max}} \Gamma_{ia-;ib-} \]
\[
\Gamma_{ia-;jb-} = i \delta_{ai} \delta_{ji} \eta^2 (\tilde{k}_{ci} \cdot \tilde{k}_{cj}) (\Omega_{i1} \Omega_{j0} - \Omega_{i1}^* \Omega_{j0}^*) / \nu_{\text{max}} \times \frac{x}{x_{ij}} \int_0^{x_{ij}} \sin x dx / x \Gamma_{ia-;jb-} \]
\[
\Gamma_{ja-;ib-} = (\Gamma_{ia-;jb-})^* \Gamma_{ja-;ib-} \]
\[
\Gamma_{ja-;jb-} = i \delta_{ai} \delta_{ji} \eta^2 (\Omega_{j0} \Omega_{j0} - \Omega_{j0}^* \Omega_{j0}^*) / \nu_{\text{max}} \Gamma_{ja-;jb-} \]
\[
\Gamma_{C+;C+} = \frac{1}{2} \Gamma_{\text{cav.}} \Gamma_{C+;C+} \]

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The quantity $x_{ij} = \sqrt{3}a / |r_{i0} - r_{j0}|$ relates the qubit separation to the lattice size. As stated above, the cavity mode is resonant with the 2-1 transition ($\omega_b = \omega_{21} \sim \omega_0$), the cavity mode is uncoupled to the 2-0 transition ($g_{k0} = 0$), the control gating field is coupled to the 2-1 transition, the target gating field is coupled to the 2-0 transition and both gating fields are on resonance. Approximations based on $\omega_0 \gg \omega_{10} \gg \nu_{\text{max}}$ are used in the derivations. The vibrational modes have zero phonons at absolute zero.

Expressions for qubit populations, coherences and two qubit transitions could be obtained based on the work of Tregenna et al [56]. Gating is based on the use of decoherence-free subspaces. States for the gated qubits ($|0_i0_j\rangle$, $|0_i1_j\rangle$, $|1_i0_j\rangle$, $|1_i1_j\rangle$, $|A\rangle = (|1_i2_j\rangle - |2_i1_j\rangle)/\sqrt{2}$), with the cavity mode in zero photon state $|0\rangle_A$ are not directly coupled to one photon cavity states (see [56]) via the qubit-cavity interaction. A CNOT gate can thus be performed with negligible cavity mode excitation, thereby avoiding decoherence due to cavity mode decay. Their treatment assumes $|\Omega_i1\rangle = |\Omega_j0\rangle = \Omega(t)$, where $\Omega$ has a maximum $\Omega_m$ and a width $\Delta T$.

The gating time is given by

$$\Delta T \simeq \frac{2\pi}{\Omega_m} \quad (71)$$

The parameters used for the two qubit gating case are the same as for one qubit gating, except in accordance with Tregenna et al, the gating field is weak, and only $\Omega_m = 3.10^6 \text{s}^{-1}$ is used (see Table 1). The corresponding gating time is $2.10^{-6} \text{s}$ (see Eq (71)), which is long compared to the correlation time for the vibrational modes reservoir of $10^{-7} \text{s}$ (given by $2\pi/\nu_{\text{max}}$), and much longer than the correlations times associated with the SE or cavity decay modes reservoirs. Some relaxation elements obtained are given in Table 4.

### Table 4. Relaxation elements in two qubit gating process

| Element | Value |
|---------|-------|
| $\Gamma_{i0\rightarrow ib^-}$ | $4.10^8 \text{s}^{-1}$ |
| $\Gamma_{i0\rightarrow ib^-}$ | $4.10^8 \text{s}^{-1}$ |
| $\Gamma_{j0\rightarrow jh^-}$ | $4.10^8 \text{s}^{-1}$ |
| $\Gamma_{C^+\rightarrow C^-}$ | $1.10^8 \text{s}^{-1}$ |
| $\Gamma_{k0\rightarrow C^+}$ | $1.10^8 \text{s}^{-1}$ |
| $\Gamma_{k0\rightarrow C^+}$ | $1.10^8 \text{s}^{-1}$ |

The scaling of the decoherence rate with qubit number for the case of the two qubit gating process treated in Tregenna et al [56] gives an overall decoherence time which is independent of $N$. This is because the terms for gated qubits $i, j$ do not scale with $N$, whilst the non-gated qubit contributions (which scale with $N$) are zero for the present case where the 2-1 transition resonantly coupled to cavity mode and the 2-0 transition is uncoupled. However, other parameter choices, such as having both optical transitions coupled to the cavity mode, could lead to large non-gated contributions, due to $\Gamma_{k0\rightarrow C^+, kb\rightarrow C^+} \sim 1.10^8 \text{s}^{-1}$ associated with LD coupling of the qubits with the cavity and vibrational modes. If one photon is present, so that $\langle \sigma^k_{a0} b^d b \rangle \sim 1$, then even modest size qubit numbers
$N \sim 10^3$ would lead to non-gated qubit contributions exceeding those from other contributions, such as $\langle b^\dagger b \rangle \Gamma_{C^+;C^+}$, where $\Gamma_{C^+;C^+} \sim 1.1 \times 10^8 \text{s}^{-1}$.

As for one qubit gating, the contributions from the various terms to $(1/\tau_d)_{\text{gating}}$ (and hence to the change in fidelity during the two qubit gating period) are significantly different in size. The relaxation element factor may be more important than the qubit state factor, and vice versa. We would need to investigate all 120 terms to determine which contribution is dominant. For example, for the terms numbered 23, 24, 34 and 35, the relaxation element is $\sim 4.1 \times 10^2 \text{s}^{-1}$ and the qubit factors are $\sim 1$, giving a product $\sim 4.1 \times 10^2 \text{s}^{-1}$. For term number 120, the relaxation element is $\sim 1.1 \times 10^8 \text{s}^{-1}$ and the qubit factor gives the probability $\langle b^\dagger b \rangle$ of finding one photon in the cavity mode. If this probability is greater than $\sim 4.1 \times 10^{-6}$ (and it could be as high as unity if decoherence free subspaces were not utilised during the gating process), the term 120 would be more important than the terms 23, 24, 34 and 35. Term 120 is due to cavity decay. If the cavity decay term was the most important, the reduction in fidelity would be given by

$$
\Delta F = -\langle b^\dagger b \rangle - \langle b^\dagger \rangle \langle b \rangle \Gamma_{\text{cavity}} \frac{2\pi}{\Omega_m} \tag{72}
$$

which is $\sim 6.1 \times 10^2 \langle b^\dagger b \rangle$. The probability of finding a photon in the cavity mode must be less than $10^{-5}$ if the fidelity loss is to be reasonably small.

4 Discussion

The scaling of decoherence effects in circuit model quantum computers have been studied for the situation where the number of qubits $N$ becomes large. Decoherence effects were specified via the fidelity, with its initial rate of change defining the decoherence time scale. Expressions for the decoherence time scale were obtained for the intermediate time regime via Markovian theory. The general case was treated where the qubit system was in any pure state, the reservoirs being in thermal states. The decoherence time scale was expressed in terms of Markovian relaxation elements and expectation values of products of fluctuation operators for the decohering quantum system. The expression given in Eq. 30 for the decoherence time scale is quite general and may have applications for treating decoherence in other macroscopic systems, such as Bose condensates or superconductors or in quantum measurement theory.

A standard model involving $N$ ionic qubits, each a three-state lambda system, was studied, with localised, well-separated qubits undergoing vibrational motion in a lattice of trapping potentials. Coherent one and two qubit gating processes were controlled by time dependent localised classical EM fields, the two qubit gating processes being facilitated by a high Q cavity mode. The qubits were coupled to reservoir of spontaneous emission (SE) modes, the cavity mode was coupled to a bath of cavity decay modes. For ionic qubits, the numerous collective vibrational qubit modes also acted as a reservoir, with Lamb-Dicke
coupling to the internal qubit system. A key objective of the work was to investigate decoherence effects due to the qubit vibrational motion. Parameters similar to those in the model treated by Tregenna et al. were chosen, with comparable cavity decay and cavity Rabi frequencies, both much larger than the spontaneous emission decay rate and the Rabi frequencies of the two qubit gating fields. One optical transition was resonant with the cavity mode. For two qubit gating, the other transition was also assumed not coupled to the cavity mode. Our primary aim was to evaluate fundamental rather than technical causes of decoherence in standard qubit based quantum computers.

For the standard model we investigated, cavity decay, spontaneous emission and Lamb-Dicke coupling to the vibrational modes were the most important fundamental causes of decoherence. Effects due to Rontgen and diamagnetic interactions were found to be negligible. Technical causes of decoherence, such as fluctuations in the trapping fields, though needed to relate decoherence times to current experiments were not studied here.

Characteristic decoherence time scales were evaluated for specific qubit states (Hadamard, GHZ) at finite temperature in the situation with no gating processes occurring. The decoherence time scaled as $1/N$. The decoherence time scale for the uncorrelated Hadamard state could be made infinite by choosing two optical dipole matrix elements that added to zero. The decoherence time scale for the GHZ state was very long, due to the Boltzmann factor $\tau_D$ being about $10^{19}$s for $N \approx 10^4$ qubits, even if the free SE decay rate of $10^8s^{-1}$ applied.

For the case of one qubit gating processes due to weak two photon resonant Raman fields with a large one photon detuning, the decoherence time scale was evaluated, but at zero temperature. Decoherence was mainly due to Lamb-Dicke coupling of the gated qubit with the Raman fields, but the loss of fidelity during the gating process was small, being proportional to the square of the Lamb-Dicke parameter. Scaling effects were associated with non-gated qubits and were small. For both optical transitions coupled to the cavity mode, decoherence was associated with Lamb-Dicke coupling of non-gated qubits with the cavity mode, no photon being present. However, the effects were negligible even for $N \approx 10^4$ qubits. Scaling effects were absent for only one coupled transition.

For the case of two qubit gating processes due to one photon resonant gating fields, as in the work of Tregenna et al., the decoherence time scale was evaluated, also at zero temperature. Scaling effects were absent for the parameters chosen, so overall the decoherence time is independent of qubit numbers. However, other parameter choices, such as having both optical transitions coupled to the cavity mode would lead to a significant contribution associated with Lamb-Dicke coupling with the cavity mode, one photon being present. In this case, modest qubit numbers $N \approx 10^3$ qubits would result in non-gated contributions that exceed those for the gated qubits.

In our model the parameters used have been the same as for the theoretical model studied by Tregenna et al., rather than those where real ions are
involved. This was done in order to compare for quantum computer models of this type, the effects of including (or not including) the scaling up of qubit numbers and allowing for decoherence due to vibrational motion. A treatment for real ions based on three state lambda systems and involving only electric dipole transitions is generally too simplified. The presence of other states (such as additional magnetic substates, or states associated with other hyperfine levels) may need to be taken into account, the actual transitions involved may be of electric quadrupole or magnetic dipole character, and magnetic fields may need to be present in order that only the 0-2 and 1-2 transitions are resonant with the two-qubit gating laser fields. There are several possibilities which involve storing the qubit in states 0, 1 and utilising an excited state 2 in the gating processes, so that although these key states form a lambda system, other states or non electric dipole transitions may be involved. Consider the case where states 0, 1 are associated with two hyperfine sublevels of a ground electronic level and state 2 is an optical excited state. A simple system of this type involves a $^2S_{1/2}$ ground level and a $^2P_{1/2}$ excited state, but with a non-zero nuclear spin $I = 1/2$. With $^2S_{1/2}$ ($F = 0, M_F = 0$) as state 0, $^2S_{1/2}$ ($F = 1, M_F = +1$) as state 1 and with $^2P_{1/2}$ ($F = 1, M_F = +1$) as state 2, suitable polarisations for the gating laser fields can be chosen to only cause transitions between these states. However, spontaneous emission causes transitions into the $^2S_{1/2}$ ($F = 1, M_F = 0$) state, so there is no longer a three state lambda system. A second example is where states 0, 1 are associated with two Zeeman substates of a ground electronic level and state 2 is an optical excited state. A simple system of this type exists in $^{40}$Ca$^+$ where states 0, 1 are the ground level $^2S_{1/2}$ ($M_J = -1/2$) and ($M_J = +1/2$) states and state 2 is say the metastable $^2D_{5/2}$ ($M_J = +1/2$) state. Here the nuclear spin is $I = 0$, so no hyperfine structure is involved. With suitable polarisations the required one and two qubit gating processes that do not involve other states can be performed, the presence of a non-zero magnetic field detuning the transition between $^2S_{1/2}$ ($M_J = -1/2$) and the additional $^2D_{3/2}$ ($M_J = -1/2$) state. Spontaneous emission from state 2 only causes transitions to states 0, 1, so here a genuine lambda system is involved. However, an electric quadrupole transition connects state 2 with 0 and 1 rather than an electric dipole transition. If the state 2 was chosen as say the lowest $^2P_{3/2}$ ($M_J = +1/2$) state so that electric dipole transitions are involved, then spontaneous emission processes to $^2D_{5/2}$ and $^2D_{3/2}$ states occur and more than three states would be involved. A final example involves storing the qubit in states 0, 1 where 0 is a ground state and state 1 is a metastable excited state, so that $\omega_{10}$ is an optical rather than a Zeeman or hyperfine frequency. A such case exists in $^{40}$Ca$^+$ with the states 0, 1 and 2 being magnetic substates of the lowest $^2S_{1/2}$, $^2D_{5/2}$ and $^2P_{3/2}$ energy levels (with respective substates $M_J = 1/2, 5/2, 7/2$ for example). However, even with suitable laser field polarisations for the one and two qubit gating fields, the additional $M_J = 1/2, 3/2$ substates of the $^2D_{5/2}$ level become involved due to spontaneous emission from the $^2P_{3/2}$ ($M_J = 3/2$) state. Thus, the theory would need to be extended to allow for the actual states and radiative transitions involved for a particular ion of interest. This choice of ion would be
made to minimise the numbers of states needed - suggesting avoidance of cases where there are many lower energy (fine structure, hyperfine structure) levels, together with as small an upper state spontaneous decay rate as possible - suggesting avoidance of electric dipole downward transitions in favour of electric quadrupole or magnetic dipole processes. The 0-2, 1-2 transitions also need to be in the optical frequency range in order to couple these transitions to a high Q cavity. Also, cases where there are other levels between 0,1 and 2 are also unfavourable, as other such states may be populated via spontaneous emission.

In conclusion, lambda systems localised in a high Q cavity, which can both facilitate two qubit gating processes and reduce decoherence caused by spontaneous emission and cavity decay, are a useful system for research on scalable quantum computers. However, for real ions the model needs to be expanded to include the presence of all magnetic substates and to treat the case of electric quadrupole and magnetic dipole transitions. The case of neutral qubits, where the vibrational modes are independent and do not constitute a reservoir, is also of significant interest and a treatment via the present Markovian theory would be worthwhile.

5 Acknowledgements

The author is grateful for helpful discussions with A. Beige, S. Barnett, J. Compagno, H. Carmichael, I. Deutsch, J. Eberly, J. Eschner, F. Haake, E. Hinds, D. Jaksch, P. Knight, J. Pachos, D. Pegg, W. Phillips, M. Plenio, S. Scheel, F. Schmidt-Kaler, B. Varcoe and H. Wiseman on various aspects of this work. Helpful comments from a referee are also acknowledged.

6 Figure caption

Figure 1. Model of an \( N \) qubit quantum computer. Three-state lambda system qubits are localised around well-separated positions via trapping potentials, and undergo collective centre of mass (CM) vibrational motions. Coherent one and two qubit gating processes are controlled by time dependent localised classical electromagnetic (EM) fields that address specific qubits. Two qubit gating processes are facilitated by a cavity mode ancilla, which permits state interchange between qubits. The lambda system qubits are coupled to a bath of EM field spontaneous emission (SE) modes, and the cavity mode is coupled to a bath of cavity decay modes. For large \( N \) the numerous collective vibrational modes of the qubits also act as a reservoir, coupled to the qubits, the cavity mode and the SE modes.
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