On dominator coloring of degree splitting graph of some graphs.

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Abstract. A dominator coloring (DC) is a coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other class. In this paper, we obtain some results on DC in the context of degree splitting graph of Middle graph of any graph $G$ and Mycielskian graph of some graphs. Also we find DC of degree splitting graph (ds) of Star graph families such as $\chi_d(Ds(K_{1,n,n,n}))$, $\chi_d(Ds(K_{1,n,n}))$, $\chi_d(Ds(C(K_{1,n})))$ and $\chi_d(Ds(L(K_{1,n,n})))$ respectively.

Keywords: Dominator coloring, Degree splitting graph, Subdivision graph, Middle graph.

1. Introduction
All graphs considered here are finite, undirected, simple graphs. For graph theoretic terminology refer to D. B. West [11]. Let $G$ be a graph, with vertex set $V(gc{G})$ and edge set $E(G)$.

A set $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in $D$. An excellent details of domination is given in the book by Haynes et al., [3].

A DC a graph $G$ is a proper coloring of graph such that every vertex of $V$ dominates all vertices of at least one color class (possibly its own class). i.e., it is coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other class. DC, $\chi_d(G)$ is the minimum number of color classes in a DC of $G$, and this concept was introduced by Ralucca Michelle Gera in 2006 [1]. The DC was studied in [2]. The DC of Trees, Bipartite graph, Central, Middle graph of Path and Cycle graph were also studied in various papers [4, 5, 7, 8].

R. Ponaraj and S. Somasundaram have initiated a study of degree splitting graph $Ds(G)$ of a graph $G$ which is stated as: Let $G = (V,E)$ be a graph with $V = H_1 \cup H_2 \cup \ldots \cup H_t \cup T$ where each $H_k$ is a set of vertices having at least two vertices and having the same degree and $T = V - \cup H_k : 1 \leq k \leq t$. The degree splitting graph of $G$ is denoted by $Ds(G)$ is obtained from $G$ by adding vertices $w_1, w_2,\ldots , w_t$ and joining $w_k : 1 \leq k \leq t$ to each vertex of $S_k : 1 \leq k \leq t$.

The DC for Ds graph of various graph families have been investigated by Vaidya and Shukla [9].
2. Results and discussion

**Theorem 2.1** Let $n > 5$. For any graph $G$, $\chi_d(Ds(M(G)))$ is,

$$\chi_d(Ds(M(G))) \leq \begin{cases} |x_i| + \chi(Ds(M(G))), & \text{if } T = \emptyset \\ |x_i| + \chi(Ds(M(G))) - T, & \text{if } T \neq \emptyset. \end{cases}$$

**Proof.**

Let $M(G)$ be Middle graph of any graph $G$, $V(Ds(M(G))) = \{R_1, R_2, \ldots, R_t\} \cup T$ and $x_i : 1 \leq i \leq t$ be the set of vertices of all the corresponding sets of $R_i : 1 \leq i \leq t$ in $Ds(M(G))$. i.e.,

$$V(Ds(M(G))) = \{R_i : 1 \leq i \leq t\} \cup T \cup \{x_i : 1 \leq i \leq t\}.$$ 

**Case 1.** $T = \emptyset$.

If $T = \emptyset$ then the $\gamma(Ds(M(G)))$ is $|x_i|, 1 \leq i \leq t$, because each $x_i, 1 \leq i \leq t$ is an independent sets in $Ds(M(G))$ and also a maximal independent set of $Ds(M(G))$. We know that every maximal independent set is a minimal dominating set. Therefore $\gamma(Ds(M(G))) = |x_i|$.

A procedure to find $\chi_d(Ds(M(G)))$ as follows. Define a proper coloring $C$ for the graph $Ds(M(G))$ by assigning unique colors $c_1, c_2, c_3, \ldots, c_t$ to each vertex in $x_i, 1 \leq i \leq t$ respectively, which is a dominating set of graph $Ds(M(G))$ and all the remaining vertices in $Ds(M(G))$ i.e., $V(Ds(M(G)) - \cup x_i)$ are colored by number of required colors for producing a proper coloring of $Ds(M(G))$. Here every vertex in $Ds(M(G))$ dominates all vertices of at least one color class. Therefore it is easy to see that above coloring is a proper coloring and also a dominator coloring of $Ds(M(G))$. Hence an easy check shows that $\chi_d(Ds(M(G))) \leq |x_i| + \chi[Ds(M(G))]$.

**Case 2.** $T \neq \emptyset$.

If $T \neq \emptyset$ then the $\gamma(Ds(M(G)))$ is $|x_i| + T$. In this case, at least any one of the vertex in $Ds[M(G)]$ is not existing in $R_i, 1 \leq i \leq t$, from this observation, $T \neq \emptyset$ and also $M(G)$ will be a subgraph of $Ds(M(G))$. Clearly an easy observation shows that $T$ and $x_i : 1 \leq i \leq t$ will be the dominating set of the $Ds(M(G))$. Hence $\gamma(Ds(M(G))) = |x_i| + T$.

Assign a proper coloring $c_t$ for $x_i, 1 \leq i \leq t$. Next $\chi[Ds(M(G))] - T$ number of colors will be assigned to the remaining vertices in $Ds(M(G))$. Since every vertex in $Ds(M(G))$ dominates all vertices of at least one color class. Hence it is easy to observe that this proper coloring pattern leads to be a dominator coloring of $Ds(M(G))$. Thus $\chi_d[Ds(M(G))] \leq |x_i| + \chi[Ds(M(G))] - T$.

**Theorem 2.2** For any graph $G$, $\chi_d(Ds(S(G)))$ is,

$$\chi_d(Ds(S(G))) = |x_i| + 2.$$

**Proof.** Let $V(G) = \{z_1, z_2, z_3, \ldots, z_n\}$. By the construction of subdivision graph, $V(S(G)) = \{z_1, z_2, z_3, \ldots, z_n\} \cup \{y_1, y_2, y_3, \ldots, y_n\} = R_1 \cup R_2 \cup R_3 \cup \ldots, \cup R_t \cup T$. For obtaining $Ds(S(G))$ from $S(G)$ add the vertices $x_1, x_2, x_3, \ldots, x_t$ corresponding to $R_1, R_2, R_3, \ldots, R_t$ respectively.

$$V(Ds(S(G))) = V(S(G)) \cup \{x_i : 1 \leq i \leq t\}.$$ 

Define $\chi_d$ to the vertex set $Ds(S(G))$ as follows.

For $1 \leq i \leq t$ assign the color $c_i$ to $x_i$. By the observation, the chromatic number of subdivision graph of any graph $G$ is two, so only two colors are enough to color the remaining vertices of $V(Ds(S(G)))$. Thus the remaining vertices of $Ds(S(G))$ are colored by using $c_{t+1}$ and $c_{t+2}$ colors.

By the observation, each vertex of $x_i$, dominates their own color class and all the remaining vertices dominate at least any one color class of $x_i, 1 \leq i \leq t$. Therefore $\chi_d(Ds(S(G))) \leq |x_i| + 2.$
To prove $\chi_d(Ds(S(G))) \geq |x_i| + 2$. Let us assume that $\chi_d(Ds(S(G))) < |x_i| + 2$, i.e., $\chi_d(Ds(S(G))) = |x_i| + 1$.

Suppose, if $|x_i| + 1$ colors are assigned to $Ds(S(G))$ then at least any one of the vertex will not dominate any color class (or the induced subgraph of at least any two vertices receives the same color).

Hence this contradicts the definition of DC. Therefore the DC with $|x_i| + 1$ color is not possible. Hence $\chi_d(Ds(S(G))) = |x_i| + 2$.

Theorem 2.3 Let $n \geq 5$, the dominator chromatic number of degree splitting graph of $\mu(C_p)$, $\chi_d(Ds(\mu(C_p)))$ is defined as follows:

$$\chi_d(Ds(\mu(C_p))) = \begin{cases} 5, & \text{when } p \text{ is even} \\ 6, & \text{when } p \text{ is odd} \end{cases}$$

Proof. Let $V(C_p) = \{y_k, 1 \leq k \leq p\}$ be the set of vertices of $C_p$. By the construction of Mycielski’s graph, $V(\mu(C_p)) = \{y_k, 1 \leq k \leq p\} \cup \{x_k, 1 \leq k \leq p\} \cup \{z\} = R_1 \cup R_2 \cup T$ where $R_1 = \{y_k, 1 \leq k \leq p\}, R_2 = \{x_k, 1 \leq k \leq p\}$, and $T = z$.

For obtaining $Ds(\mu(C_p))$ from $\mu(C_p)$ add the vertices $w_1, w_2$ corresponding to $R_1, R_2$ respectively. The vertex set of $Ds(\mu(C_p))$ is defined by,

$$V(Ds(\mu(C_p))) = V(\mu(C_p)) \cup \{w_1, w_2\}.$$ 

The dominator coloring $\varphi$ of $Ds(\mu(C_p))$ is defined as follows:

**Case 1.** $p$ is even

$$\varphi(x_k, y_k) = \begin{cases} c_1 \text{ for } k = 1, 3, 5, \ldots, p - 1 \\ c_2 \text{ for } k = 2, 4, 6, \ldots, p \end{cases}$$

and $\varphi(x_k, y_k) = c_3, \varphi(w_1) = c_4, \varphi(w_2) = c_5$.

**Case 2.** $p$ is odd

$$\varphi(x_k, y_k) = \begin{cases} c_1 \text{ for } k = 1, 3, 5, \ldots, p - 2 \\ c_2 \text{ for } k = 2, 4, 6, \ldots, p - 1 \end{cases}$$

and $\varphi(z) = c_4, \varphi(w_1) = c_5, \varphi(w_2) = c_6$ and $\varphi(y_n, x_n) = c_3$.

It is clear that the above assignment of colors will produce a dominator coloring of $Ds(\mu(C_p))$. The vertices $w_1, w_2$ and $z$ dominate their own color class and for $1 \leq i \leq n$, the vertices $x_i, y_i$, dominate at least any one color class of $w_1, w_2$. Hence an easy check shows that

$$\chi_d(Ds(\mu(C_p))) = \begin{cases} 5, & \text{when } p \text{ is even} \\ 6, & \text{when } p \text{ is odd} \end{cases}$$

Theorem 2.4 Let $n \geq 5$, $\chi_d(Ds(\mu(P_n))) = 6$.

Proof. Let $V(P_n) = \{y_i, 1 \leq i \leq n\}$ be the set of vertices of $P_n$. By the construction of Mycielski’s graph, $V(\mu(P_n)) = \{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n\} \cup \{z\} = R_1 \cup R_2 \cup R_3$ where $R_1 = \{x_i, y_i, x_n, y_n\}, R_2 = \{x_i, 2 \leq i \leq n\}, R_3 = \{y_i, 2 \leq i \leq n\}$ and $T = z$. For obtaining $Ds(\mu(P_n))$ from $\mu(P_n)$ add the vertex set $R_1, R_2$ and $R_3$ to corresponding vertices of $w_1, w_2$ and $w_3$ respectively. The vertex set of $Ds(\mu(P_n))$ is defined by,

$$V(Ds(\mu(P_n))) = V(\mu(P_n)) \cup \{w_1, w_2, w_3\}.$$
The dominator 6- coloring $\varphi$ of $Ds(\mu(P_n))$ is defined as follows:
$\varphi(w_1) = c_3, \varphi(w_2) = c_4, \varphi(w_3) = c_5$ and $\varphi(z) = c_6$. The remaining vertices are colored in the following cases.

**Case 1.** $n$ is odd
$$\varphi(x_i, y_i) = \begin{cases} c_1 & \text{for } i = 1, 3, 5, \ldots, n \\ c_2 & \text{for } i = 2, 4, 6, \ldots, n-1 \end{cases}$$

**Case 2.** $n$ is even
$$\varphi(x_i, y_i) = \begin{cases} c_1 & \text{for } i = 1, 3, 5, \ldots, n-1 \\ c_2 & \text{for } i = 2, 4, 6, \ldots, n \end{cases}$$

The vertex set $R_1$ dominate the color class of $w_1$, and the vertex set $R_2$ dominates the color class of $w_2$ and $R_3$ dominates the color class of $w_3$. Also the vertices $w_1, w_2, w_3$ and $z$ dominates their own color class. Hence $\chi_d(Ds(\mu(P_n))) \leq 6$.

On the other hand, if the vertex $w_1$ or $w_2$ is colored by $c_5$, then the vertices in $R_1$ or $R_2$ does not dominate any color class, which implies $\chi_d(Ds(\mu(P_n))) \neq 6$. Hence $\chi_d(Ds(\mu(P_n))) = 6$.

**Theorem 2.5** Let $n \geq 2$, $\chi_d(Ds(K_{1,n,n}))$ is,
$$\chi_d(Ds(K_{1,n,n})) = 4.$$ 

**Proof.** Let $V(K_{1,n,n}) = R_1 \cup R_2 \cup T$ where $R_1 = \{v_i : 1 \leq i \leq n\}, R_2 = \{u_i : 1 \leq i \leq n\}$ and $T = \{v\}$. By the definition of degree splitting graph, $Ds(K_{1,n,n})$ is obtained by adding a vertex $x_1$ to $R_1$ and $x_2$ to $R_2$. The vertex set is defined by
$$V(Ds(K_{1,n,n})) = V(K_{1,n,n}) \cup \{x_1\} \cup \{x_2\}.$$ 

The dominator 4- coloring $\varphi$ of $Ds(K_{1,n,n})$ is defined as follows:
$$\varphi(v, u_i) = c_1, \text{ if } i = 1, 2, 3, \ldots, n$$
$$\varphi(v_i) = c_2 \text{ for } 1 \leq i \leq n.$$ 
$$\varphi(x_1) = c_3$$
$$\varphi(x_2) = c_4.$$ 

Here the above coloring pattern satisfies the condition to be a DC of $Ds(K_{1,n,n})$ as the vertex $v, v_i, 1 \leq i \leq n$ dominate the color class $c_3$. Next the vertices $u_i : 1 \leq i \leq n$ dominate the color class of $c_2$ and the vertices $x_1, x_2$ dominates itself. Hence an easy check shows that $\chi_d(Ds(K_{1,n,n})) = 4$.

**Theorem 2.6** Let $s \geq 2$, $\chi_d(Ds(K_{1,s,s,s}))$ is,
$$\chi_d(Ds(K_{1,s,s,s})) = 5.$$ 

**Proof.** Let $V(K_{1,s,s,s}) = R_1 \cup R_2 \cup T$ where $R_1 = \{v_k, u_k : 1 \leq k \leq s\}, R_2 = \{w_k : 1 \leq k \leq t\}$ and $T = \{v\}$. By the definition of degree splitting graph, $Ds(K_{1,s,s,s})$ is obtained by adding a vertex $x_1$ to $R_1$ and $x_2$ to $R_2$. The vertex set of $Ds(K_{1,s,s,s})$ is defined by
$$V(Ds(K_{1,s,s,s})) = V(K_{1,s,s,s}) \cup \{x_1\} \cup \{x_2\}.$$ 

The following procedure gives the dominator chromatic number of $Ds(K_{1,s,s,s})$. Consider the color class $C = \{c_1, c_2, c_3, c_4\}$. 

- Assign the colors $c_1, c_2, c_3$ to the vertices in $\gamma(Ds(K_{1,s,s,s}))$.
- For $1 \leq k \leq s$, assign color $c_4$ to $v_k$ and $w_k$.
- Assign color $c_5$ to $u_k : 1 \leq k \leq s$. 


Clearly the above defined proper coloring pattern gives a DC for the respective graph. An easy observation shows that $\gamma$ set in $Ds(K_{1,s,s})$ dominates itself. For $1 \leq k \leq s$, the vertex $w_k$ dominate the color class $x_2$ and the vertices $u_k, v_k$ dominate the color class of $x_1$. Hence $\chi_d(Ds(K_{1,s,s})) = 5$.

**Theorem 2.7** If $n \geq 2$, then $\chi_d(Ds(C(K_{1,n}))) = n + 2$.

**Proof.** By the definition of central graph, subdividing each edge of $K_{1,n}$ exactly once and then joining each pair of vertices of $K_{1,n}$ which were non adjacent. Let $V(C(K_{1,n})) = R_1 \cup R_2 \cup T$, where $R_1 = v, v_i : 1 \leq i \leq n, R_2 = u_i : 1 \leq i \leq n$. By the definition of degree splitting graph, we have $V(Ds(C(K_{1,n}))) = R_1 \cup R_2 \cup T$ and $x_1, x_2$ be the vertices of all the corresponding sets of $R_1, R_2$ in $Ds(C(K_{1,n}))$.

$$V(Ds(C(K_{1,n}))) = V(C(K_{1,n})) \cup \{x_1\} \cup \{x_2\}.$$ Clearly, in $Ds(C(K_{1,n}))$ the vertices $x_1, v_i, 1 \leq i \leq n$ forms a clique of order $n + 1$.

Next a procedure to find $\chi_d(Ds(C(K_{1,n})))$ as follows.

In detail the dominator $n + 2$- coloring $\varphi$ of $Ds(C(K_{1,n}))$ in the following way:

$$\varphi(v_i) = c_i, \text{ if } i = 1, 2, 3, \ldots, n.$$ $$\varphi(x_1) = c_{n+1}$$ $$\varphi(u_i) = c_{n+1}, 1 \leq i \leq n.$$ $$\varphi(v) = c_1$$ $$\varphi(x_2) = c_{n+2}.$$ Thus the above assignment of colors will produce a DC of $Ds(C(K_{1,n}))$. The vertex $x_2$ dominates itself, for $1 \leq i \leq n$, the vertex $v_i$ dominate any one color class $c_i$ and $u_i$ dominate the color class $c_{n+1}$. Next the vertex $v$ dominates the color class $c_{n+2}$.

On the other hand, if the color $c_{n+1}$ is assigned to $x_1$ and $x_2$ then the vertex $x_2$ does not dominate any color class. Therefore assigning $n + 1$ colors to $Ds(C(K_{1,n}))$ is not possible. Hence an easy check shows that $\chi_d(Ds(C(K_{1,n}))) = n + 2$.

**Theorem 2.8** If $s \geq 2$, then $\chi_d(Ds(L(K_{1,s,s}))) = s + 2$.

**Proof.**

Let $V(L(K_{1,s,s})) = R_1 \cup R_2$ where $R_1 = v_k : 1 \leq k \leq s, R_2 = u_k : 1 \leq k \leq s$. By the construction of $Ds$, let $V(Ds(L(K_{1,s,s})))$ is $R_1 \cup R_2$ and $x_1, x_2$ be the vertices of the corresponding sets of $R_1, R_2$ in $Ds(L(K_{1,s,s})))$. i.e.,

$$V(Ds(L(K_{1,s,s}))) = R_1 \cup R_2 \cup \{x_1, x_2\}.$$ Next a procedure to obtain $\chi_d(Ds(L(K_{1,s,s})))$ as follows.

Clearly, in $Ds(L(K_{1,s,s}))$ the vertices $x_1, v_k, 1 \leq k \leq n$ forms a clique of order $s + 1$.

Thus $\chi_d(Ds(L(K_{1,s,s}))) \geq s + 1$.

Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_s, c_{s+1}, c_{s+2}\}$. Assign the color $c_k$ for $v_k : 1 \leq k \leq s$. Next consign the color $c_{s+1}$ to $x_1$ and $u_k : 1 \leq k \leq s$. At last assign the color $c_{s+2}$ to $x_2$. For $1 \leq k \leq s$, the vertex $v_k$ dominate any one color class $c_k$ and $u_k$ dominate the color class $c_{s+2}$. Also the vertex $x_1$ dominate the color class $c_k$ and $x_2$ dominates itself. Hence an easy observation shows that $\chi_d(Ds(L(K_{1,s,s}))) \leq s + 2$.

To prove $\chi_d(Ds(L(K_{1,s,s}))) \geq s + 2$. Let $\chi_d(Ds(L(K_{1,s,s}))) < s + 2$. Suppose, if any preused colors $c_k$ or $c_{s+1}$ are assigned to the vertices $x_2, u_k, 1 \leq k \leq s$ then the vertices $x_2, u_k$ does not dominate any color class. Therefore dominator coloring with less than $s + 2$ color is not possible. Hence $\chi_d(Ds(L(K_{1,s,s}))) = s + 2$. 

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3. Conclusion
In this paper, we obtained some results on DC in the context of DS of Middle graph of any graph \( G \) and Mycielskian graph of some graphs. Also we found \( \chi_d(D_s) \) of some star graph families. This paper can be further extended by identifying graph families of graphs for which these chromatic numbers are equal to other kinds of chromatic numbers.

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