The role of images and priors in measuring $H_0$ from supernova Refsdal in galaxy cluster MACS J1149.5+2223

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ABSTRACT
Multiple image gravitational lensing systems with measured time delays provide a promising one-step method for determining $H_0$. MACS J1149, which lenses SN Refsdal into a quad S1-S4, and two other widely separated images, SX and SY, is a perfect candidate. If time delays are pinned down, the remaining uncertainty arises from the mass distribution in the lens. In MACS J1149, the mass in the relevant lens plane region can be constrained by many multiple images, the mass of the galaxy splitting S1-S4 (which, we show, is correlated with $H_0$), magnification of SX (also correlated with $H_0$), and prior assumptions on the mass distribution. To better understand what affects the measurement of $H_0$, we separate out the uncertainties associated with these constraints. Using images alone yields $\sim 1000\%$ uncertainty, despite the fact that the position of SX is recovered to within $\sim 0.04''$ (rms $\sim 0.26''$) by GRALE lens inversion. Fixing the mass of the galaxy that splits S1-S4 would reduce $1\sigma$ uncertainties to $20\%-33\%$, depending on the mass, while fixing the magnification of SX would reduce $1\sigma$ uncertainties to $35\%$. Smaller uncertainties, of order few percent, are a consequence of imposing assumptions on the shapes of the galaxy and cluster mass distributions, which may or may not apply in a highly non-equilibrium environment of a merging cluster. We propose that if a measurement of $H_0$ is to be considered reliable, it must be supported by a wide range of lens inversion methods.

Key words: gravitational lensing: strong – dark matter – galaxies: clusters: individual: MACS J1149.5+2223

1 INTRODUCTION
Multiple image gravitationally lensed systems can be used to measure $H_0$ in one step (Refsdal 1964), and completely independently of the distance ladder. The two inputs are the observed time delays between multiple images, and the mass distribution in the lens. In galaxy-size lenses, with quasars as sources, time delay measurements are currently uncertain at the few%–10% level (Bonvin et al. 2016, 2018; Courbin et al. 2018).

In galaxy clusters, with supernovae as sources, the uncertainty in time delay measurement can be reduced because clusters are large, and hence all scales, including time scales, get blown up, reducing the fractional errors in time delay measurements.

The first multiply imaged supernova (Kelly et al. 2015), nicknamed Refsdal, was discovered in Grism Lens Amplified Survey from Space (GLASS; PI T. Treu) as four quad images, S1-S4, surrounding an elliptical galaxy, in the Hubble Frontier Field (HFF; PI J. Lotz) galaxy cluster MACS J1149.5+2223 (hereafter MACS J1149). Refsdal was later confirmed to be a supernova, of a type similar to that of SN 1987A (Kelly et al. 2016b). The time delays between all the images of the quad were measured to be a few days to 3-4 weeks (Rodney et al. 2016). Viewed on cluster scale, images S1-S4 form in the second lowest minimum of the arrival time surface. Based on mass models (Sharon & Johnson 2015; Diego et al. 2016; Grillo et al. 2016; Oguri 2015; Kawanata et al. 2016), a cluster-scale saddle-point image, SX, was predicted (Treu et al. 2016), and then observed (Kelly et al. 2016a), at the time and position consistent with predictions. These successes prompted early estimates of $H_0$ from Refsdal (Vega-Ferrero et al. 2018; Grillo et al. 2018). The same cluster also hosts a transient, arising from a highly magnified, macro- and micro-lensed, massive high-redshift star (Kelly et al. 2018), whose images, Icarus (LS1/Lev16A) and Iapyx (Lev16B), separated by $\sim 0.3''$, straddle a nearby portion of the cluster critical line. The

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supernova and the transient source live in the same, triply imaged, face-on spiral galaxy at \( z = 1.489 \). The whole system of lensed images in MACS J1149 is spectacular and unique, and needs as much detailed investigation as possible.

This host galaxy cluster at \( z = 0.542 \) was first identified as one of 12 most distant X-ray luminous clusters detected at \( z > 0.5 \) by the Massive Cluster Survey, MACS (Ebeling et al. 2007). It is a double merger (Golovich et al. 2016), and one of the most complex merging clusters known (Ogrean et al. 2016). It was observed with the Advanced Camera for Surveys on HST, and analyzed and modeled by Zitrin & Broadhurst (2009), Smith et al. (2009) and Rau et al. (2014). They found three images of an extended, and surprisingly undistorted background spiral at \( z = 1.489 \), which is also the host of both SN Refsdal and the transient.

The time delay between Refsdal images S1 and SX can be determined to a 1-2\% precision (Kelly et al., in preparation). However, the mass distribution is an additional, and largest source of uncertainty. Several mass models reproduce the lensed images within observational uncertainties, but lensing degeneracies can prevent an accurate and precise determination of \( H_0 \). Models that quote low uncertainties usually rely on lens inversion methods that break many degeneracies through the use of parametric mass distribution assumptions.

The goal of this paper is to investigate how much precision in \( H_0 \) is possible based solely on the lensed image positions, and relations implied by lensing reconstructions, and how much precision is brought about through the use of priors, i.e., the assumptions about the density profiles of cluster galaxies and cluster dark matter. We do this by comparing the precision achieved by free-form GRALE reconstructions, or the assumptions about the density profiles of cluster galaxies and cluster dark matter. We perform well regarding fitness measure (i), but still predict unobserved additional images. To take this into account, an area that is larger than the region of the observed images is subdivided into a grid of triangles. By counting the number of backprojected triangles that overlap with the envelope of the backprojected image points, trial solutions can be penalized if they predicted additional images.

We used the following as input to our models: positions of all 93 multiple images of knots in the spiral source galaxy from Table 4 of Treu et al. (2016), and positions of the 4 Refsdal S1-S4 images. All these are at \( z = 1.489 \). Estimating source redshifts using lens inversion techniques can lead to biased estimates, because of lensing degeneracies (Williams et al. 2017). To avoid that we used 20 images that had spectroscopic redshifts (also from Treu et al. 2016). For sources at very high redshifts, the uncertainties in redshifts become less important; the fractional change in \( \Sigma_{\text{crit}} \) between \( z = 2 \) and 4 is 0.164, while between \( z = 4.5 \) and 6.5 it is 0.041, i.e., four times smaller. To supplement our image set, we included two sources (6 images) with photometric redshifts of 4.8 and 6.5 respectively. In all, we used 123 images from 40 sources. These image data presented in Treu et al. (2016) were based on spectroscopic redshifts from VLT-MUSE (Grillo et al. 2016), Keck DEIMOS and HST-WFC3 (Brammer et al. 2016; Wáng et al. 2017; Treu et al. 2015). The images we used are shown in Figure 1.

Treu et al. (2016) and Kelly et al. (2016a) present results of several lens inversion models, predicting, in advance of the event, the location and arrival time of Refsdal image at SX, which was first detected on December 11, 2015. In order to directly compare our results to the published ones, we did not use the observed position of SX as a constraint.
We also did not include the observed positions of the two transients, Icarus and Iapyx, as these had not yet appeared at the time of the published predictions. As all previous GRALE reconstructions, and in contrast to all but one other existing technique (Quinn Finney et al. 2018), our models do not use any information about galaxies in the lens or along the line of sight.

2.2 Inversion results

Each GRALE run consists of 9 sequential solutions, where the number of Plummer’s, and hence mass resolution, increases with each subsequent solution. We carry out a total of 75 of these runs\(^1\), each of which is started with its own random

\(^1\) These reconstructions are not the same as the v.4 set submitted to HST in early 2017. The v.4 reconstructions did not include S1-S4, and had a somewhat different, and smaller input image set.
Figure 2. **Left:** The points are all the images of Refsdal predicted by the BestFitness set of GRALE mass maps from 75 independent runs. To obtain these, each of the observed S1-S4 images was traced back to the source plane, and used individually as sources to generate predicted images. The total number of images is over 1000. The observed SX, S1-S4 are marked with black crosses. The images within the black dotted circle of radius 2′ around SX were used to calculate the average location and the rms of the predicted SX (displayed in the figure). The insets show the distribution of the ΔRA and ΔDec of predicted SX images. Vertical lines mark the observed values. The black dotted circles around S1-S4, of radii 2.5′ and 7′, are used to select reconstructions for the BFsubset; see Section 3 for details. **Right:** Same as the left panel, but for the HighResolution set of GRALE maps from the same 75 runs.

Figure 3. Of the 75 runs, we selected those that have no GRALE predicted images between the two black dotted circles centered on the S1-S4 quad, with radii of 2.5′ and 7′, and shown in Figure 2. This selection leaves 17 BestFitness (left) and 28 HighResolution (right) maps. For the BestFitness maps (left) the displacement between the observed and predicted SX, and rms of the scatter of predicted positions are 0.039′, and 0.200′, respectively. The corresponding values for the HighResolution (right) maps are, 0.023′, and 0.174′. The insets in both panels show the distribution of the ΔRA and ΔDec of predicted SX images. Vertical lines mark the observed values.
seed, so the runs are independent. Some runs slightly vary the number of Plummer's, as well as the center and size of the reconstruction grid.

In each of the 75 runs we identify two solutions and the corresponding mass maps. (1) The first is the solution that has the best fitness of all the 9 sequential solutions. For MACS J1149 runs, these are usually the 4th, 5th or 6th solution in a specific run. We call these the BestFitness reconstructions, or maps. (2) The second solution is identified from the two last solutions—8th and 9th—as the one with the better fitness. We call these the HighResolution maps, because latter solutions have finer grids and more Plummer's making up the total mass map, which implies higher mass resolution. Higher resolution will be advantageous in MACS J1149 because it hosts many closely spaced images in the vicinity of Refsdal’s quad S1-S4. Selections (1) and (2) yield 75 maps each. (In 6 of the 75 runs, the BestFitness and the HighResolution maps are the same, so the total number of unique maps is 144, instead of 150. Because this is a small fraction of the total, we do not eliminate the redundant maps.)

In addition to the BestFitness and HighResolution sets of maps, we also analyze one subset from each of these: we call these BFsubset and HRsubset, respectively. These maps are selected based on how well they reproduce images S1-S4 (see Section 3 for details). In all, we have 4 sets of reconstructions, BestFitness, HighResolution, BFsubset and HRsubset.

### 3 IMAGE PLANE PREDICTIONS

The left and right panels of Figure 2 show the distribution of all GRALE predicted images of Refsdal for the BestFitness and HighResolution reconstructions, respectively. (Images corresponding to SY are outside the box shown, and are not considered in this paper.) These images were obtained as follows. For each of the GGRALE reconstructions, each of the four observed S1-S4 images was traced back to the source plane, and used as a source to generate lens plane images. Note that not all of these sources produced 4 images of the S1-S4 quad, but all produced at least one image very close to observed SX.

As expected, most of the images are associated with the S1-S4 quad, and the location of SX. The images of the S1-S4 quad age not reproduced very well, most likely because GGRALE does not have enough spatial resolution, even in HighResolution maps. The total number of images in each of the two panels is over one thousand. (In principle, each of the backprojected images of the quad should generate 5 images, and the SX image, for a total of 1800 images.) There are also a handful of images far from S1-S4 and SX; these are most likely spurious, i.e. have no observed counterpart. Also as expected, HighResolution has more spurious images than BestFitness.

The right panel of Figure 2 has a few predicted images in the upper right, scattered widely around (~8,3). These could be the fourth image of the cluster-wide Refsdal quad, with the fifth image hidden in the BCG. However, only a small fraction of our reconstructions have these images, and none of these remain in Figure 3. We conclude that it is most likely that the SY, S1-S4, and SX comprise a naked cusp, triple-image system split by the cluster. There are no additional images of Refsdal due to the cluster-wide potential.

The concentration of predicted images near the location of observed SX is very high, implying that the predicted position of SX is very well localized. The two insets in each of the two panels of Figure 2 show the histograms of ΔRA and ΔDec of predicted images around the observed SX, whose position is indicated by the vertical lines. To obtain the average coordinates of the predicted position, we include only the images within the $r = 2'$ radius from the observed position, shown as a black dotted circle. This eliminates most spurious images. Note that using the observed SX in this way does not bias the estimation of GGRALE predicted position, because the displacement between the mean predicted and observed SX position, $0.024''$ and $0.060''$ for the BestFitness and HighResolution reconstructions, and the rms, $0.310''$ and $0.355''$, are small by comparison. These values are displayed in the figure, as well as in Table 1.

The reconstructions that go into the BFsubset and HRsubset were selected based on how well they reproduce S1-S4 (whose positions were known to all groups in Kelly et al. 2016a), using two criteria. (i) Selected maps should not have any Refsdal images between 2.5'' and 7'' from the center of the S1-S4 quad. There are no observed Refsdal images in this annulus, so any model predicted images here are spurious. (ii) Selected maps should have at least one of the S2, S3, or S4 back-projected images predict the location of S1 to within 0.85'', which is half the distance between S1 and its nearest neighbor, S2. This criterion is somewhat loose, however, a stricter condition would eliminate most of our reconstructed maps. (GRALE’s limited spatial mass resolution prevents it from fitting these images better).

Figure 3 is similar to Figure 2, but for the BFsubset and HRsubset. Even though no cuts other than the two discussed above were imposed, BFsubset reconstructions have no spurious images, while the only remaining spurious images in the HRsubset are quite far from the observed SX.

The displacement between observed and predicted SX in our 4 sets ($<0.06''$; see Table 1) appear to be smaller than in most other existing models ($\sim 0.1 - 0.3$''). The smallest published offset is that for the Grillo et al. (2016) model, and is $\sim 0.045''$. Our rms values are in the range 0.17'' - 0.36'', while other models have rms values between 0.26'' and 0.9''.

Before we leave the image plane, let us look at the critical curve, for Refsdal’s $z_s$. Figure 4 shows the critical curves for BestFitness (thin pink), HighResolution (thin blue), BFsubset (thick pink), and HRsubset (thick blue) curves, and...
a zoom in the inset. Observed positions of Icarus and Iaplyx were not used as model inputs, and yet the Grale critical curves go between them. Near the transients, the largest separation between the 4 curves is $\sim 0.1''$, while the BF-subset and HR-subset curves are separated by $\sim 0.05''$. By comparison, the separation between the four models presented in Kelly et al. (2018) (namely, Jauzac et al. 2016; Kawamata et al. 2016; Zitrin et al. 2015; Keeton 2010) is $\sim 0.25''$. Furthermore, all Grale critical curves pass between the two transients, implying that Icarus and Iaplyx are counterimages of the same source. Two of the published models (Jauzac et al. 2016; Kawamata et al. 2016) make a similar prediction, while the other two models have both the transients to the North East of the critical curves.

4 GRALE TIME DELAY PREDICTIONS

In the previous section we saw that using only the image positions Grale is able to predict the position of SX and S1 and the locus of the critical curve in the vicinity of the transient, accurately and precisely. In this section we examine how well image positions can constrain $H_0$, assuming the observed time delay between Refsdal’s two images SX and S1 is known exactly.

4.1 Generalized mass sheet and monopole degeneracies

It is well known that the mass sheet degeneracy (MSD; Falco et al. 1985), also known as the steepness degeneracy (Saha 2000), is one of the lensing degeneracies that needs to be broken to measure $H_0$. Two mass maps, $\kappa_a(r)$ and $\kappa_b(r)$, are related by an MSD transformation if there exists a constant $\lambda$, such that $\kappa_b = \lambda \kappa_a + (1 - \lambda)$, or equivalently, $\Sigma_b = \lambda \Sigma_a + (1 - \lambda) \Sigma_{\text{crit}}$ over the entire lens plane, where $\Sigma$ is the surface mass density of the lens in physical units. Such a transformation would leave most lens observables—image positions, flux ratios, time delay ratios—unchanged, but will rescale all time delays by $\lambda$, thereby scaling the estimated $H_0$ by $\lambda$. Because $\kappa$ depends on the source redshift, the presence of multiply imaged sources at different redshifts breaks the degeneracy, as no single $\lambda$ works for all sources.

It was demonstrated in Liesenborgs et al. (2008a) and Liesenborgs & De Rijcke (2012) that a generalized version of the mass sheet degeneracy, gMSD, can be present despite sources at multiple redshifts. For gMSD, the added mass sheet is not simply $(1 - \lambda) \Sigma_{\text{crit}}$. Instead, it is a non-uniform disk, $\Sigma_{\text{gen}}$, centered on $\theta_c$, with its density profile arranged so that images at different redshifts and different locations see the same total enclosed mass as they would with the regular
MSD: $\Sigma_\delta(\theta) = \lambda \Sigma_{\epsilon}(\theta) + (1 - \lambda) \Sigma_{\text{gen}}(\theta - \theta_0 |)$.

In the regions of the lens plane that have images of different redshifts located close to each other, gMSD is suppressed, but not eliminated. Its approximate version, agMSD, that reproduces the image properties not exactly, but within observational uncertainty, presents an even wider range of possible mass models.

MSD and its kin, agMSD, are not the only degeneracies. In this paper we will also focus on the monopole degeneracy, MpD (Saha 2000; Liesenborgs et al. 2008b; Liesenborgs & De Rijcke 2012). In regions of the lens plane with no images, the projected mass distribution can be reshaped in any circularly symmetric fashion, as long as the net mass remains the same, and no extra images are produced. Since any number of such operations, with different centers, radii, and shapes can be performed and superimposed, the monopole degeneracy is very powerful.

An important difference between agMSD and MpD is that agMSD reshapes the lens mass distribution over the whole lens plane, while MpD redistributes mass between images only. The immediate consequence is that agMSD is not necessarily affected by the high density of images in the lens plane (as long as they are at approximately the same redshift), while MpD is suppressed in regions where the density of images is high. Other degeneracies can also be present, for example the source plane transformation (SPT; Schneider & Sluse 2014), which is related to MSD, and also reshapes the mass continuously across the lens plane. For brevity, we will talk about agMSD and MpD only, noting that other degeneracies are also possible.

4.2 Illustrating agMSD and MpD in grale reconstructions

We will demonstrate the presence and effect of agMSD and MpD using our grale maps. Figure 5 shows the cross-section of the $\kappa$ density maps taken along the line connecting Gralle predicted positions of SX and S1. The line is parametrized by the $\Delta \alpha$ coordinate, with respect to the center of the cluster. The pink curves show the BFsubset maps, while the blue lines show the HRsubset. As expected, all the Gralle curves follow the same general trend, for example, all show the presence of the galaxy at $\Delta \alpha \approx -7''$, which creates the S1-S4 quad. But because HighResolution maps have Plummer spheres with smaller widths (i.e., higher spatial resolution), they have a sharper density peak associated with that galaxy.

The five black curves in Figure 5 show publicly available reconstructions: parametric models by Sharon et al., Keeton et al., Plavic team, CATS team, and free-form hybrid model by Diego et al., along a line connecting observed SX and S1 Refsdal images. These models show less dispersion than Gralle because they assume specific functional forms for the mass distributions of the galaxy and cluster components, which eliminate many degenerate mass models. Gralle reconstructions do not have such assumptions, and so Gralle curves show a fuller range of density distributions allowed by the multiple images. The realistic range of mass distributions probably lies somewhere between these two types of models: on one hand, some Gralle models are likely to be astrophysically implausible, but on the other, parametric models are too restrictive, and likely do not account for the complexity of mass distribution in lenses, especially in merging, far-from-relaxed galaxy clusters, such as MACS J1149.

In Figure 6 the pink curves (left panel) show the Best-Fitness set of all 75 Gralle maps, while the blue curves (right panel) show the HighResolution set of the same 75 Gralle maps. As an illustration of agMSD, we highlight two Gralle maps that are closely related by it. In the left panel of Figure 6, these are $\kappa_1$ (thick solid black curve), and $\kappa_2$ (thin solid black curve). A third curve (thick dashed black) shows $\lambda \kappa_1 + (1 - \lambda)$, with $\lambda = 0.73$, i.e., an MSD-transformed $\kappa_1$ cross-section. The cross-sections of these two maps (thin solid and thick dashed) are very similar, but not the same, implying that other degeneracies, such as MpD, are contributing in addition to agMSD. In the right panel, the two agMSD related mass maps are $\kappa_3$ (thick solid black curve), and $\kappa_4$ (thin solid black curve), and the corresponding factor is $\lambda = 0.55$.

To demonstrate the presence of MpD we first approximately take out the standard MSD, as follows. For every one of the $\kappa$ cross-sections presented in Figure 6, we find $\lambda$ such that the curve $\lambda \kappa + (1 - \lambda)$ attains $\kappa = 0$ at some location within the $\Delta \alpha$ range shown in the figure. The resulting $\kappa$ cross-sections are shown in Figure 7. Though the maps attain $\kappa = 0$ at different locations, for the vast majority it happens within a narrow interval $0.5'' < \Delta \alpha < 0.3''$. The bar code at the bottom indicates the locations of multiple images within $\pm 6''$ of the SX-S1 line, mapped on to the $\Delta \alpha$ axis. The $\pm 6''$range includes all but 4 of the knots of the spiral galaxy at $z = 1.489$: more distant multiple images have lesser influence on the mass distribution near SX and the Refsdal quad. Because the bulk of the images in this region are knots of the same spiral galaxy, most of the agMSD is just the standard MSD and its approximate versions, arising due to small uncertainties in the image positions.

If the maps were related by agMSD transformations, all the curves would be nearly identical (in the case of pure MSD they would be identical). To judge the importance of agMSD one must compare the corresponding panels of Figures 6 and 7. The curves on the right side of both panels of Figure 7, at $\Delta \alpha \lesssim -2''$, show less dispersion compared to the corresponding panels in Figure 6, implying that agMSD is present in Gralle maps, and was taken out, and that the small residual differences are due to other degeneracies, mostly MpD. At $\Delta \alpha \lesssim -2''$ multiple images are abundant, so MpD does not have much room to play out, but is still present.

On the left side of the panels, at $\Delta \alpha \gtrsim -2''$, where the images are sparse, MpD becomes more important, diminishing the visibility of MSD, and making the density cross-sections have different shapes. In other words, it is the redistribution of mass between images due to MpD that accounts for most of the scatter in the curves at $\Delta \alpha \gtrsim -2''$ in both panels of Figures 6 and 7.

4.3 $H_0$ from lensed images only

We conclude that both types of degeneracies, agMSD and MpD, contribute to mass maps, and will therefore contribute to the model time delay, $\tau_{\text{model}}$, and estimation of $H_0$.

Figure 8 shows the cross-section of the arrival time surface along the same straight lines that were used in Figures 6 and 7. On the vertical axis the curves were shifted
Figure 6. Similar to Figure 5. Left: The pink curves show the BestFitness set of all 75 GraLE maps. The thick solid black curve highlights one of these maps (which we call $\kappa_1$), while thin solid black curve highlights another map ($\kappa_2$). These two are related by approximate, generalized MSD, because $\lambda \kappa_1 + (1-\lambda)$ curve, shown as the thick dashed black curve, is very similar to $\kappa_2$. Right: Same, but for the 75 HighResolution maps. Note that the extent of the vertical axes are different in the two panels.

Figure 7. Left: Similar to Figure 6, but here we have taken out MSD, so that other degeneracies become visible. To do that, for every one of the 75 BestFitness $\kappa$ cross-sections presented in the left panel of Figure 6, we find $\lambda$ such that the curve $\lambda \kappa + (1-\lambda)$ attains $\kappa = 0$ at some location within the $\Delta$RA range shown in the figure. The bar code at the bottom shows the locations of multiple images within $\pm 6'$ from the SX-S1 line, mapped on to the $\Delta$RA axis. Right: Same as the left panel, but for HighResolution. The four cross-sections highlighted in black are the reconstructions labeled A, B (solid lines), and C, D (dot-dashed lines) in Figures 9 and 10.
to go through zero at the location of predicted SX, to make it easier to read off the model time delay between SX and S1, $\Delta \tau_{\text{model}}$. The dashed curves in both panels show all 75 BestFitness and 75 HighResolution models, while the solid thick curves represent the BFsubset and HRsubset.

The model time delay, $\Delta \tau_{\text{model}}$, and the observed time delay, $\tau_{\text{obs}}$, are connected by $\Delta t_{\text{obs}} = (H_{0,\text{fid}}/H_{0}) \Delta \tau_{\text{model}}$, where $H_{0,\text{fid}}$ is the fiducial value of the Hubble constant assumed in GRALE runs. If the observed time delay is known, measuring $\tau_{\text{model}}$ is equivalent to measuring $H_{0}$. Because of the lensing degeneracies, there is a range of predicted $\Delta \tau_{\text{model}}$ values. Hence, using just the lensed images for mass reconstruction, as GRALE does, leaves one with many degenerate models. $\Delta \tau_{\text{model}}$ values span a factor of 10, from ~0.2 to ~2 years. Therefore the 123 lensed images that we use, with no additional constraints, can constrain $H_{0}$ only to within a factor of 10.

To obtain a competitive measure of $H_{0}$ given $\Delta \tau_{\text{obs}}$, one needs to isolate a narrow range of mass models. Though lensing by itself cannot break the degeneracies, it can help us determine what additional information would be helpful. Specifically, lensing mass reconstructions can tell us what observables correlate with $\Delta \tau_{\text{model}}$.

### 4.4 $H_{0}$ from lensed images and galaxy masses

Figure 9 shows that the (projected) mass of the galaxy interior to S1-S4 is correlated with model time delay, $\Delta \tau_{\text{model}}$. The mass is proportional to the difference between the average $\kappa$ densities within $r \leq 1.5''$ and $r = 2.0'' \rightarrow 2.5''$, where the latter is assumed to be representative of the projected density of the cluster in the vicinity of the Refsdal quad. We call this quantity $\Delta \kappa_{\text{gal}}$. Because the radii quoted above are the same throughout the analysis, $\Delta \kappa_{\text{gal}}$ is proportional to the mass of the galaxy. The image radius is about 1.5'', so we are measuring the mass of the galaxy interior to the images.

Note that $\Delta \kappa_{\text{gal}}$ is equivalent to the slope of the density profile of the total (dark matter and galaxy) mass distribution. MSD, which is sometimes called the steepness transformation, changes the slope of the total mass responsible for lensing. The image separation of the S1-S4 quad fixes the total mass enclosed by the four images, but does not say how that mass is partitioned between the galaxy and the cluster’s smooth dark matter at that location. Changing that partitioning affects the total density slope. In the rest of the paper we will refer to $\Delta \kappa_{\text{gal}}$ as the galaxy mass, but one must keep in mind that it is analogous to steepness.

We have tried radial ranges other than the ones specified above, but these give the strongest correlation with $\Delta \tau_{\text{model}}$, i.e., the most optimistic scenario. The small symbols in Figure 9 represent the full BestFitness (pink circles) and HighResolution (blue triangles) sets, while the large symbols represent the corresponding BFsubset and HRsubset. All but a small handful of models have SX arriving after S1, as one would expect for the ordering of the saddle and the neighboring minimum, in a naked cusp configuration.

The points in Figure 9 show a well defined trend, with scatter. The main trend is primarily due to agMSD, while the scatter is due to MpD, as well as the approximate nature...
Thus, under the assumptions inherent in parametric modeling, the plot equivalent to our Figure 9 would look like a single curve, and so most, if not all degeneracies are broken by measuring the mass of the galaxy interior to S1-S4. (This is usually done using a proxy, like the central velocity dispersion.)

Since \( \Delta \text{model} \) correlates well with the mass of the galaxy interior to S1-S4, one might ask if a similarly good correlation exists between \( \Delta \text{model} \) and the mass of the galaxy, or galaxies, in the close vicinity of SX. Unfortunately, no. This is related to the fact that near S1-S4, agMSD is the dominant degeneracy, and mass distributions connected by it are simple scalings of each other. On the other hand, the main degeneracy near image SX is MpD, and there is no simple relation between the mass maps connected by MpD. So the prevalence of MpD near SX means that nearby galaxy masses will not provide a useful constraint.

Therefore the only galaxy mass that correlates with \( \Delta \text{model} \), and hence is useful for constraining \( H_0 \), is the mass of the galaxy enclosed by S1-S4. If the mass of that galaxy is estimated accurately, and no parametric assumptions are used, \( H_0 \) can be estimated to within 20% – 33%, depending on the mass of the galaxy.

### 4.5 \( H_0 \) from lensed images, galaxy masses, and parametric assumptions

The predicted time delay between S1 and SX obtained by parametric models (Jauzac et al., Sharon et al. Oguri et al., Grillo et al., Zitrin et al.) and free-form hybrid models (Diego et al.) are presented in Figure 3 of Kelly et al. (2016a). Most models have 1\( \sigma \) ~ 25 days, or ~ 7% uncertainty in \( \Delta \text{model} \). The transition from 20% – 33% precision on \( H_0 \) to ~ 7% is accomplished by introducing parametric forms describing the distribution of mass associated with the cluster dark matter, and the individual galaxies in the cluster. Because these priors differ between various lensing inversion methods, some models—even those using the same LENSTOOL software (Kneib et al. 1993; Jullo et al. 2007)—Sharon et al. and Jauzac et al.—differ by 2\( \sigma \) – 3\( \sigma \). This is an indication that the parametric assumptions are somewhat too restrictive; in other words, not all of these assumptions can be correct.

An eye-ball estimate of the combination of these models gives an uncertainty of ~ 18%, similar to that obtained through Bayesian analysis from these data, where one finds \( H_0 = 64^{+3}_{-15} \text{km s}^{-1} \text{Mpc}^{-1} \) (Vega-Ferrero et al. 2018).

### 4.6 \( H_0 \) from lensed images and absolute magnification of SX

Because the absolute fluxes of supernovae Type Ia are standardizable, they allow for precise determination of the magnification at the location of their images, and can in principle be used to break MSD in cluster lenses (Rodney et al. 2015). Though Refsdal is not Type Ia, one can still ask if knowing the magnification will break some existing degeneracies.

Magnification, \( \mu \), is related to normalized surface mass density \( \kappa \), and shear \( \gamma \), through \( \mu = \frac{1}{(1 - \kappa - \gamma)(1 - \kappa + \gamma)}^{-1} \). Because of MSD, model time delay is related to \( \kappa \), and to \( \mu \). Therefore one should expect that \( \Delta \text{model} \) between SX and S1...
is related to $\mu$ at SX; this is shown in Figure 11. We will now account for the main trend, through the MSD parameter $\lambda$, used in Section 4.2 and Figure 7, where all GRALE models have been MSD transformed to have zero mass sheet in the lens plane region between SX and the S1-S4 quad. The GRALE reconstructed density (Figure 6) and the density in Figure 7 are related by, $\kappa_{\text{Grale}} = \lambda \kappa_{\text{Fig.6}} + (1 - \lambda)$. Typical $\kappa$ at SX in Figure 7 is around 1, with scatter. The black thick filled circle is the center of the cluster elliptical, which splits images S1-S4. The fractional difference in the predicted $H_0$ is $\sim 22\%$. Right: Same as the left panel, but for the two models labeled C and D. The fractional difference in the predicted $H_0$ is $\sim 80\%$. The offset between the center of the observed galaxy and the center of the galaxy peak in the two panels ranges from 0.07″ to 0.24″. Note that the galaxy interior to S1-S4 was not used in the reconstruction.

What about magnification of Refsdal image S1? Since GRALE does not accurately predict the location of the quad images, using GRALE predicted magnification—which depends on the second derivative of the lensing potential—is ill advised.

5 SUMMARY AND DISCUSSION

The value of $H_0$ affects almost every aspect of cosmology, from distance scale to fundamental physics. The goal of this paper was to understand how various constraints affect the uncertainties on $H_0$ measurement based on cluster lensing. To separate out the constraints provided by lensed images alone, we used free-form GRALE, a lens inversion method that relies solely on image positions. No information about any cluster galaxy was used. GRALE predicts very accurately $\Delta = 0.17^\circ - 0.36^\circ$ and precisely (rms = 0.023°–0.060°) the position of the cluster-wide saddle point, SX, and the locus of the cluster critical curve ($< 0.1^\circ$), which goes between the transients.

While GRALE predictions in the image plane are excellent, the predicted time delays span a very large range, implying that these lensed images alone do not constrain $H_0$ significantly. This is due to the presence of unbroken degeneracies, mostly the approximate generalized mass sheet degeneracy (agMSD), and the monopole degeneracy (MpD). We showed that their regions of dominance vary across the lens plane. In regions where multiple images at the same redshift are plentiful, agMSD dominates, but MpD is nearly broken. This is well exemplified by the region around Refs-
Figure 11. GRALE time delay vs. magnification at the predicted location of SX. (Five points from each of HighResolution and BestFitness sets are outside the limits of the plot.) The black thick curve was obtained using the MSD parameter, $t$ (Section 4.2 and Figure 7), as discussed in Section 4.6. The thin dashed lines contain approximately 68% of GRALE models. If magnification at SX is known exactly, then the model time delay, and hence $H_0$ are constrained to ±35%, at 1σ.

We conclude that using the currently available data, the multiple images by themselves limit the range of allowable $\Delta \tau_{\text{model}}$ to 0.2–2 years, or ~1000% uncertainty in $H_0$. Obviously, additional constraints are needed. We showed that the mass of the galaxy splitting S1-S4 quad correlates well with GRALE model time delay, which is proportional to $H_0$. Including an accurate estimate of the galaxy mass would cut down on the range of mass models, and reduce uncertainties to 20%–33% (at 1σ), or potentially better, if image positions are reproduced very well. Absolute magnification of image SX also correlates with model time delay; its measurement would yield 35% uncertainties at 1σ. However, combining galaxy mass and magnification information is unlikely to improve precision significantly because the main remaining degeneracy is MpD, whose contribution to the mass distribution in the lens plane is 'stochastic' and cannot be described by a scaling relation.

Uncertainties smaller than 20% – 35%, of order a few percent, are a consequence of imposing assumptions on the shapes of the galaxy and cluster mass distributions. An interesting future exercise with GRALE would be to determine what density of images would be required to suppress MpD sufficiently, to achieve a measurement of $H_0$ with a few percent precision.

We propose that a reliable estimate of $H_0$ fromRefsdal can be obtained by exploiting the advantages of two types of lensing inversion techniques: parametric, and free-form methods, like GRALE, that are capable of exploring a wide range of mass models. The advantage of parametric methods is that their models incorporate our knowledge about average galaxy and cluster properties, while the advantage of GRALE and similar methods, is that they recognize that averages might not be applicable, especially in a highly non-equilibrium environment of a merging cluster like MACS J1149.

Carrying out a reliable measurement of $H_0$ would also require some additional data. It is essential to obtain an accurate estimate of the mass of the galaxy enclosed by S1-S4. Measuring spectroscopic redshifts for all known images will help improve the accuracy of the mass model (Johnson & Sharon 2016; Williams et al. 2017). Any additional images, beyond the current set, will be also welcome.

Given these data, free-form methods would be able to narrow down $H_0$ to 20%–33% or somewhat better, and, more importantly, provide a conservative estimate of the uncertainties (Priewe et al. 2017). Individual parametric models will yield uncertainties of <10%, or smaller, but these need not agree with each other. One way to combine the parametric models is by using the Bayesian formalism outlined in Vega-Ferrero et al. (2018), but to weight the models (using $q_i$ parameter in eq. 7) by the lens plane rms before including them in the Bayesian analysis. A low lens plane rms is a necessary, but not sufficient measure of how well a mass model approximates the true mass distribution. However, it remains the best available measure.

An $H_0$ measurement can be deemed reliable if GRALE, or similar method, and Bayesian-combined parametric models, yield the same value to less than ~2σ. The combined parametric models would provide the optimistic estimate of uncertainties, while GRALE, or similar, would provide the conservative uncertainties.

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