Soft-Collinear Factorization and the Calculation of the 
$B \rightarrow X_s\gamma$ Rate

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Abstract

Using results on soft-collinear factorization for inclusive $B$-meson decay distributions, a 
systematic study of the partial $B \rightarrow X_s\gamma$ decay rate with a cut $E_\gamma \geq E_0$ on photon energy 
is performed. For values of $E_0 \leq 1.9$ GeV the rate can be calculated without reference to 
shape functions. The result depends on three large scales: $m_b$, $\sqrt{m_b\Delta}$, and $\Delta = m_b - 2E_0$. The 
sensitivity to the scale $\Delta \approx 1.1$ GeV (for $E_0 \approx 1.8$ GeV) introduces significant 
uncertainties, which have been ignored in the past. Our new prediction for the $B \rightarrow X_s\gamma$
branching ratio with $E_\gamma \geq 1.8$ GeV is $\text{Br}(B \rightarrow X_s\gamma) = (3.44 \pm 0.53 \pm 0.35) \times 10^{-4}$, 
where the errors refer to perturbative and parameter uncertainties, respectively. The 
implications of larger theory uncertainties for New Physics searches are explored with 
the example of the type-II two-Higgs-doublet model.

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1 Introduction

Given the prominent role of $B \rightarrow X_s \gamma$ decay in searching for physics beyond the Standard Model, it is of great importance to have a precise prediction for its inclusive rate and CP asymmetry in the Standard Model. The total inclusive $B \rightarrow X_s \gamma$ decay rate can be calculated using a conventional operator-product expansion (OPE) based on an expansion in logarithms and inverse powers of the $b$-quark mass. However, in practice experiments can only measure the high-energy part of the photon spectrum, $E_\gamma \geq E_0$, where typically $E_0 = 2 \text{ GeV}$ or slightly below (measured in the $B$-meson rest frame) [1, 2]. With $E_\gamma$ restricted to be close to the kinematic endpoint at $M_B/2$, the hadronic final state $X_s$ is constrained to have large energy $E_X \sim M_B$ but only moderate invariant mass $M_X \sim (M_B \Lambda_{\text{QCD}})^{1/2}$. In this kinematic region, an infinite number of leading-twist terms in the OPE need to be resummed into a non-perturbative shape function, which describes the momentum distribution of the $b$-quark inside the $B$ meson [3, 4].

Conventional wisdom based on phenomenological studies of shape-function effects says these effects are important near the endpoint of the photon spectrum, but they can be ignored as soon as the cutoff $E_0$ is lowered below about 1.9 GeV. In other words, there should be an instantaneous transition from the “shape-function region” of large non-perturbative corrections to the “OPE region”, in which hadronic corrections to the rate are suppressed by at least two powers of $\Lambda_{\text{QCD}}/m_b$. Below, we argue that this notion is based on a misconception. While it is correct that once the cutoff $E_0$ is chosen below 1.9 GeV the decay rate can be calculated using a local short-distance expansion, we show that this expansion involves three “large” scales. In addition to the hard scale $m_b$, an intermediate scale $\sqrt{m_b \Delta}$ corresponding to the typical invariant mass of the hadronic final state $X_s$, and a low scale $\Delta = m_b - 2E_0$ related to the width of the energy window over which the measurement is performed, become of crucial importance. The precision of the theoretical calculations is ultimately determined by the value of the lowest short-distance scale $\Delta$, which in practice is of order 1 GeV or only slightly larger. The theoretical accuracy that can be reached is therefore not as good as in the case of a conventional heavy-quark expansion applied to the $B$ system. More likely, it is similar to (if not worse than) the accuracy reached, say, in the description of the inclusive hadronic decay rate of the $\tau$ lepton.

While we are aware that this conclusion may come as a surprise to many practitioners in the field of flavor physics, we believe that it is an unavoidable consequence of our analysis. Not surprisingly, then, we find that the error estimates for the $B \rightarrow X_s \gamma$ branching ratio that can be found in the literature are, without exception, too optimistic. Since there are unknown $\alpha_s^2(\Delta)$ corrections at the low scale $\Delta \sim 1 \text{ GeV}$, we estimate the present perturbative uncertainty in the $B \rightarrow X_s \gamma$ branching ratio with $E_0$ in the range between 1.6 and 1.8 GeV to be of order 10–15%. In addition, there are uncertainties due to other sources, such as the $b$- and $c$-quark masses. The combined theoretical uncertainty is of order 15–20%, about twice as large as what has been claimed in the past. While this is a rather pessimistic conclusion, we stress that the uncertainty is limited by unknown, higher-order perturbative terms, not by non-perturbative effects, which we find to be under good control. Therefore, there is room for a reduction of the error by means of well-controlled perturbative calculations.
2 QCD factorization theorem

Using recent results on the factorization of inclusive $B$-meson decay distributions [3, 9], it is possible to derive a QCD factorization formula for the integrated $B \to X_s \gamma$ decay rate with a cut $E_\gamma \geq E_0$ on photon energy. In the region of large $E_0$, the leading contribution to the rate can be factorized in the form [7]

$$\Gamma_{B \to X_s \gamma}^{\text{leading}}(E_0) = \frac{G_F^2 \alpha}{32\pi^4} |V_{tb} V_{ts}^*|^2 \overline{m}_b(\mu_h) |H_\gamma(\mu_h)|^2 U_1(\mu_h, \mu_i)$$

$$\times \int_0^{\Delta_{E}} dP_+ (M_B - P_+)^3 \int_0^{P_+} d\hat{\omega} m_b J(m_b(P_+ - \hat{\omega}), \mu_i) \hat{S}(\hat{\omega}, \mu_i),$$

where $\Delta_E = M_B - 2E_0$ is twice the width of the window in photon energy over which the measurement of the decay rate is performed. The variable $P_+ = E_X - |\vec{P}_X|$ is the “plus component” of the 4-momentum of the hadronic final state $X_s$, which is related to the photon energy by $P_+ = M_B - 2E_\gamma$. The endpoint region of the photon spectrum is defined by the requirement that $P_+ \leq \Delta_E \ll M_B$, in which case $P_\mu$ is called a hard-collinear momentum [8].

In the factorization formula, $\mu_h \sim m_b$ is a hard scale, while $\mu_i \sim \sqrt{m_b\Lambda_{\text{QCD}}}$ is an intermediate hard-collinear scale of order the invariant mass of the hadronic final state. The precise values of these matching scales are irrelevant, since the rate is formally independent of $\mu_h$ and $\mu_i$. The hard corrections captured by the function $H_\gamma(\mu_h)$ result from the matching of the effective weak Hamiltonian of the Standard Model (or any of its extensions) onto a leading-order current operator of soft-collinear effective theory (SCET) [9]. At tree level, $H_\gamma(\mu_h) = C_{7\gamma}^{\text{eff}}(\mu_h)$ is equal to the “effective” coefficient $C_{7\gamma}^{\text{eff}} = C_{7\gamma} - \frac{1}{3} C_5 - C_6$. The expression valid at next-to-leading order can be found in Ref. [7]. The function $H_\gamma(\mu_h)$ is multiplied by the running $b$-quark mass $\overline{m}_b(\mu_h)$ defined in the MS scheme, which is part of the electromagnetic dipole operator $Q_{7\gamma}$.

The jet function $J(m_b(P_+ - \hat{\omega}), \mu_i)$ in (1) describes the physics of the final-state hadronic jet. An expression for this function valid at next-to-leading order in perturbation theory has been derived in Refs. [5, 6]. The perturbative expansion of the jet function can be trusted as long as $\mu_i^2 \sim m_b \Delta$ with $\Delta = m_b - 2E_0 \ll M_B$. Note that the “natural” choices $\mu_h \propto m_b$ and $\mu_i^2 \equiv m_b \mu_i$ with $\mu_i$ independent of $m_b$ remove all reference to the $b$-quark mass (other than in the arguments of running coupling constants) from the factorization formula.

The shape function $\hat{S}(\hat{\omega}, \mu_i)$ parameterizes our ignorance about the soft physics associated with bound-state effects inside the $B$ meson [3, 4]. Its naive interpretation is that of a parton distribution function, governing the distribution of the light-cone component $k_+$ of the residual momentum of the $b$ quark inside the heavy meson. Once radiative corrections are included, however, a probabilistic interpretation of the shape function breaks down [4]. For convenience, the shape function is renormalized in (1) at the intermediate hard-collinear scale $\mu_i$ rather than at a hadronic scale $\mu_{\text{had}}$. This removes any uncertainties related to the evolution from $\mu_i$ to $\mu_{\text{had}}$. Since the shape function is universal, all that matters is that it is renormalized at the same scale when comparing different processes.

The last ingredient in the factorization formula is the function $U_1(\mu_h, \mu_i)$, which describes the renormalization-group (RG) evolution of the hard function $|H_\gamma|^2$ from the high matching scale $\mu_h$ down to the intermediate scale $\mu_i$, at which the jet and shape functions are
Figure 1: Dependence of the three scales $\mu_h = m_b$ (solid), $\mu_i = \sqrt{m_b \Delta}$ (dashed), and $\mu_0 = \Delta$ (dash-dotted) on the cutoff $E_0$, assuming $m_b = 4.7$ GeV. The gray area at the bottom shows the domain of non-perturbative physics. The light gray band in the center indicates the region where the MSOPE must be applied.

renormalized. The exact expression for this quantity and its perturbative expansion valid at next-to-next-to-leading logarithmic order can be found in Ref. [7].

As written in (1), the decay rate is sensitive to non-perturbative hadronic physics via its dependence on the shape function. This sensitivity is unavoidable as long as the scale $\Delta = m_b - 2E_0$ is a hadronic scale, corresponding to the endpoint region of the photon spectrum above, say, 2 GeV. Here we are interested in a situation where $E_0$ is lowered out of the shape-function region, such that $\Delta$ can be considered large compared with $\Lambda_{QCD}$. For orientation, we note that with $m_b = 4.7$ GeV and the cutoff $E_0 = 1.8$ GeV employed in a recent analysis by the Belle Collaboration [2] one gets $\Delta = 1.1$ GeV. The values of the three relevant physical scales as functions of the photon-energy cutoff $E_0$ are shown in Figure 1. This plot illustrates the fact that the transition from the shape-function region to the region where a conventional OPE can be applied is not abrupt but proceeds via an intermediate region, in which a short-distance analysis based on a multi-stage OPE (MSOPE) can be performed. The transition from the shape-function region into the MSOPE region occurs when the scale $\Delta$ becomes numerically (but not parametrically) large compared with $\Lambda_{QCD}$. Only for very low values of the cutoff ($E_0 < 1$ GeV or so) it is justified to treat $\Delta$ and $\sqrt{m_b \Delta}$ as scales of order $m_b$.

Separating the contributions associated with these scales requires a sophisticated multi-step procedure. The first step, the separation of the hard scale from the intermediate scale, has already been achieved in [11]. To proceed further, we use that integrals of smooth weight functions with the shape function $\hat{S}(\hat{\omega}, \mu)$ can be expanded in a series of forward $B$-meson matrix elements of local operators in heavy-quark effective theory (HQET) [10], provided that the integration domain is large compared with $\Lambda_{QCD}$ [5, 6]. The perturbative expansions of the associated Wilson coefficient functions can be trusted as long as $\mu \sim \Delta$. In order to complete the scale separation, it is therefore necessary to evolve the shape function in [11] from the intermediate scale $\mu_i \sim \sqrt{m_b \Delta}$ down to a scale $\mu_0 \sim \Delta$. This can be achieved using
the analytic solution to the integro-differential RG evolution equation for the shape function in momentum space [5, 11].

As a final comment, we stress that the main purpose of performing the scale separation using the MSOPE is not that this allows us to resum Sudakov logarithms. Indeed, the “large logarithm” \( \ln(m_b/\Delta) \approx 1.5 \) is only parametrically large, but not numerically. What is really important is to disentangle the physics at the low scale \( \mu_0 \sim \Delta \), which is “barely perturbative”, from the physics associated with higher scales, where a short-distance treatment is on much safer grounds. The MSOPE allows us to distinguish between the three coupling constants \( \alpha_s(m_b) \approx 0.22 \), \( \alpha_s(\sqrt{m_b \Delta}) \approx 0.29 \), and \( \alpha_s(\Delta) \approx 0.44 \) (for \( \Delta = 1.1 \text{ GeV} \)), which are rather different despite the fact that there are no numerically large logarithms in the problem. Given the values of these couplings, we expect that scale separation between \( \Delta \) and \( m_b \) is as important as that between \( m_b \) and the weak scale \( M_W \).

3 Calculation of the shape-function integral

The scale dependence of the renormalized shape function is governed by an integro-differential RG evolution equation, whose exact solution in momentum space can be found using a technique developed in Ref. [11]. The result takes the remarkably simple form

\[
\hat{S}(\hat{\omega}, \mu_i) = U_2(\mu_i, \mu_0) \frac{e^{-\gamma E \eta}}{\Gamma(\eta)} \int_0^{\hat{\omega}} d\hat{\omega}' \frac{\hat{S}(\hat{\omega}', \mu_0)}{\mu_0^{n(\hat{\omega} - \hat{\omega}')^{1-\eta}}}. \tag{2}
\]

The exact expression for the evolution function \( U_2(\mu_i, \mu_0) \) can be found in Ref. [7], and

\[
\eta = \int_{\alpha_s(\mu_i)}^{\alpha_s(\mu_0)} d\alpha \frac{2\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} = \frac{\Gamma_0}{\beta_0} \ln \frac{\alpha_s(\mu_0)}{\alpha_s(\mu_i)} + \ldots \tag{3}
\]

is given in terms of the cusp anomalous dimension [12].

Relation (2) accomplishes the evolution of the shape function from the intermediate scale down to the low scale \( \mu_0 \sim \Delta \). The remaining task is to expand the integral over the shape function in (1) in a series of forward B-meson matrix elements of local HQET operators of increasing dimension, multiplied by perturbative coefficient functions. This can be done whenever \( \Delta = \Delta_E - \bar{\Lambda} = m_b - 2E_0 \) is large compared with \( \Lambda_{\text{QCD}} \) [5]. The result is

\[
\Gamma^{\text{leading}}_{B \to X_s \gamma}(E_0) = \frac{G_F^2 \alpha}{32\pi^4} |V_{tb} V_{ts}^*|^2 m_b^3 \bar{m}_b^2(\mu_h) |H_{\gamma}(\mu_h)|^2 U_1(\mu_h, \mu_i) U_2(\mu_i, \mu_0) \times
\]

\[
e^{-\gamma E \eta} \left( \frac{\Delta}{\mu_0} \right)^\eta \left[ 1 + \frac{C_F \alpha_s(\mu_i)}{4\pi} \mathcal{J}(\Delta) + \frac{C_F \alpha_s(\mu_0)}{4\pi} S(\Delta) \right] \times
\]

\[
\left[ 1 + \frac{\eta(\eta - 1)}{2} \left( \frac{-\lambda_1}{3\Delta^2} + \ldots \right) \right], \tag{4}
\]

where

\[
\mathcal{J}(\Delta) = 2 \ln^2 \frac{m_b \Delta}{\mu_i^2} - \left[ 4h(\eta) + 3 \right] \ln \frac{m_b \Delta}{\mu_i^2} + 2h^2(\eta) + 3h(\eta) - 2h'(\eta) + 7 - \frac{2\pi^2}{3},
\]
\[ S(\Delta) = -4 \ln^2 \frac{\Delta}{\mu_0} + 4 \left[ 2h(\eta) - 1 \right] \ln \frac{\Delta}{\mu_0} - 4h^2(\eta) + 4h(\eta) + 4h'(\eta) - \frac{5\pi^2}{6}, \]  

(5)

and \( h(\eta) = \psi(1 + \eta) + \gamma_E \) is the harmonic function generalized to non-integer argument. Even though it is parametrically larger than ordinary power corrections of order \( \left( \frac{\Lambda_{\text{QCD}}}{m_b} \right)^2 \), the “enhanced” \( \frac{\lambda_1}{\Delta^2} \) correction in (4) remains small in the region of “perturbative” values of \( \Delta \), where the MSOPE can be trusted. The net effect amounts to a reduction of the decay rate by less than 5%.

The rate in (4) is formally independent of the three matching scales, at which we switch from QCD to SCET (\( \mu_h \)), from SCET to HQET (\( \mu_i \)), and finally at which the shape-function integral is expanded in a series of local operators (\( \mu_0 \)). In practice, a residual scale dependence arises because we have truncated the perturbative expansion. Varying the three matching scales about their default values provides some information about unknown higher-order terms. In the limit where the intermediate and low matching scales \( \mu_i \) and \( \mu_0 \) are set equal to the hard matching scale \( \mu_h \), our result reduces to the conventional formula used in previous analyses of the \( B \rightarrow X_s \gamma \) decay rate. However, this choice cannot be justified on physical grounds.

In (4) we have accomplished a complete resummation of (parametrically) large logarithms at next-to-next-to-leading logarithmic order in RG-improved perturbation theory, which is necessary in order to calculate the decay rate with \( \mathcal{O}(\alpha_s) \) accuracy. Specifically, it means that terms of the form \( \alpha_s^n L^k \) with \( k = (n - 1), \ldots, 2n \) and \( L = \ln \left( \frac{m_b}{\Delta} \right) \) are correctly resummed to all orders in perturbation theory. To the best of our knowledge, a complete resummation at next-to-next-to-leading order has never been achieved before. Finally, we stress that the various next-to-leading order terms in the expression for the decay rate should be consistently expanded to order \( \alpha_s \) before applying our results to phenomenology.

Up to this point, the \( b \)-quark mass \( m_b \) entering the formula (4) for the decay rate is defined in the on-shell scheme. While this is most convenient for performing calculations using heavy-quark expansions, it is well known that HQET parameters defined in the pole scheme suffer from infra-red renormalon ambiguities. It is necessary to replace them in favor of some physical, short-distance parameters. For our purposes, the “shape-function scheme” provides a particularly suitable definition of the heavy-quark mass [5]. The idea is that a good estimate of a shape-function integral can be obtained using the mean-value theorem, replacing \( \hat{\omega} \) with

\[
\langle \hat{\omega} \rangle_\Delta = \frac{\int_0^{\Delta_E} d\hat{\omega} \; \hat{\omega} \; \hat{S}(\hat{\omega}, \mu_0)}{\int_0^{\Delta_E} d\hat{\omega} \; \hat{S}(\hat{\omega}, \mu_0)} \equiv \bar{\Lambda}(\Delta, \mu_0) = M_B - m_b(\Delta, \mu_0).
\]  

(6)

Here \( m_b(\Delta, \mu_0) \) is the running shape-function mass, which depends on a hard cutoff \( \Delta \) in addition to the renormalization scale \( \mu_0 \). The quantity \( \Delta \) in the shape-function scheme is defined by the implicit equation \( \Delta = \Delta_E - \bar{\Lambda}(\Delta, \mu_0) = m_b(\Delta, \mu_0) - 2E_0 \). The shape-function scheme provides a physical definition of \( m_b \), which can be related to any other short-distance definition using perturbation theory. Based on various sources of phenomenological information including \( \Upsilon \) spectroscopy and moments of inclusive \( B \)-meson decay spectra, the value of the shape-function mass at a reference scale \( \mu_s = 1.5 \text{ GeV} \) has been determined as \( m_b(\mu_s, \mu_s) = (4.65 \pm 0.07) \text{ GeV} \) [5].
The results discussed so far provide a complete description of the $B \to X_s \gamma$ decay rate at leading order in the $1/m_b$ expansion, where the two-step matching QCD $\to$ SCET $\to$ HQET is well understood. For practical applications, however, it is necessary to include corrections arising at higher orders in the heavy-quark expansion. Most important are “kinematic” power corrections of order $(\Delta/m_b)^n$, which are not associated with new hadronic parameters. Unlike the non-perturbative corrections, these effects appear already at first order in $\Delta/m_b$, and they are numerically dominant in the region where $\Delta \gg \Lambda_{\text{QCD}}$. Technically, the kinematic power corrections correspond to subleading jet functions arising in the matching of QCD onto higher-dimensional SCET operators, as well as subleading shape functions arising in the matching of SCET onto HQET operators. The corresponding terms are known in fixed-order perturbation theory, without scale separation and RG resummation [13, 14]. To perform a complete RG analysis of even the first-order terms in $\Delta/m_b$ is beyond the scope of our discussion. Since for typical values of $E_0$ the power corrections only account for about 15% of the $B \to X_s \gamma$ decay rate, an approximate treatment suffices at the present level of precision. Details of how these corrections are implemented can be found in Ref. [7].

4 Ratios of decay rates

The contributions from the three different short-distance scales entering the central result (4) and the associated theoretical uncertainties can be disentangled by taking ratios of decay rates. Some ratios probe truly short-distance physics (i.e., physics above the scale $\mu_h \sim m_b$) and so remain unaffected by the new theoretical results presented above. For some other ratios, the short-distance physics associated with the hard scale cancels to a large extent, so that one probes physics at the intermediate and low scales, irrespective of the short-distance structure of the theory.

**Ratios insensitive to low-scale physics:** Physics beyond the Standard Model may affect the theoretical results for the $B \to X_s \gamma$ branching ratio and CP asymmetry only via the Wilson coefficients of the various operators in the effective weak Hamiltonian. As a result, the ratio of the $B \to X_s \gamma$ decay rate in a New-Physics model relative to that in the Standard Model remains largely unaffected by the resummation effects studied in the present work. From (4), we obtain

$$\frac{\Gamma_{\bar{B} \to X_s \gamma}|_{\text{NP}}}{\Gamma_{\bar{B} \to X_s \gamma}|_{\text{SM}}} = \frac{|H_\gamma(\mu_h)|^2_{\text{NP}}}{|H_\gamma(\mu_h)|^2_{\text{SM}}} + \text{power corrections.} \tag{7}$$

The power corrections would introduce some mild dependence on the intermediate and low scales $\mu_i$ and $\mu_0$, as well as on the cutoff $E_0$.

Another important example is the direct CP asymmetry in $B \to X_s \gamma$ decays, for which we obtain

$$A_{\text{CP}} = \frac{\Gamma_{\bar{B} \to X_s \gamma} - \Gamma_{B \to X_s \gamma}}{\Gamma_{\bar{B} \to X_s \gamma} + \Gamma_{B \to X_s \gamma}} = \frac{|H_\gamma(\mu_h)|^2 - |\overline{H}_\gamma(\mu_h)|^2}{|H_\gamma(\mu_h)|^2 + |\overline{H}_\gamma(\mu_h)|^2} + \text{power corrections,} \tag{8}$$
where $\overline{H}_\gamma$ is obtained from $H_\gamma$ by CP conjugation, which in the Standard Model amounts to replacing the CKM matrix elements by their complex conjugates. It follows that predictions for the CP asymmetry in the Standard Model and various New Physics scenarios \[15\] remain largely unaffected by our considerations.

**Ratios sensitive to low-scale physics:** The multi-scale effects studied in this work result from the fact that in practice the $B \to X_s \gamma$ decay rate is measured with a restrictive cut on the photon energy. These complications would be absent if it were possible to measure the fully inclusive rate. It is convenient to define a function $F(E_0)$ as the ratio of the $B \to X_s \gamma$ decay rate with a cut $E_0$ divided by the total rate,

$$F(E_0) = \frac{\Gamma_{B \to X_s \gamma}(E_0)}{\Gamma_{B \to X_s \gamma}(E^*)}.$$  \tag{9}

Because of a logarithmic soft-photon divergence for very low energy, it is conventional \[14\] to define the “total” inclusive rate as the rate with a very low cutoff $E^* = m_b/20$. The denominator in the expression for $F(E_0)$ can be evaluated using the standard OPE, which corresponds to setting all three matching scales equal to $\mu_h$. The numerator is given by our expression in (4), supplemented by power corrections.

Another important example of a ratio that is largely insensitive to the hard matching contributions is the average photon energy $\langle E_\gamma \rangle$, which has been proposed as a good way to measure the $b$-quark mass \[16\]. The impact of shape-function effects on the theoretical prediction for this quantity has been investigated and was found to be significant \[14\] \[17\]. Here we study the average photon energy in the MSOPE region, where a model-independent prediction can be obtained. It is structurally different from the one found using the conventional OPE in the sense that contributions associated with different scales are disentangled from each other. We stress that the hard scale $\mu_h \sim m_b$ affects the average photon energy only via second-order power corrections. This shows that it is not appropriate to compute the quantity $\langle E_\gamma \rangle$ using a simple heavy-quark expansion at the scale $m_b$, which is however done in the conventional approach \[16\]. This observation is important, because information about moments of the $B \to X_s \gamma$ photon spectrum is sometimes used in global fits to determine the CKM matrix element $|V_{cb}|$. Keeping only the leading power corrections, which is a very good approximation, we find that $\langle E_\gamma \rangle$ only depends on physics at the intermediate and low scales $\mu_i$ and $\mu_0$. For $E_0 = 1.8$ GeV, we obtain $\langle E_\gamma \rangle \approx [2.222 + 0.254\alpha_s(\sqrt{m_b\Delta}) + 0.009\alpha_s(\Delta)]$ GeV $\approx 2.30$ GeV.

### 5 Numerical results

We are now ready to present the phenomenological implications of our findings. A complete list of the relevant input parameters and their uncertainties is given in Ref. \[7\], where we also explain our strategy for estimating the perturbative uncertainty as well as the uncertainty due to parameter variations.

We begin by presenting predictions for the CP-averaged $B \to X_s \gamma$ branching fraction with a cutoff $E_\gamma \geq E_0$ applied on the photon energy measured in the $B$-meson rest frame. Lowering
$E_0$ below 2 GeV is challenging experimentally. The first measurement with $E_0 = 1.8$ GeV has recently been reported by the Belle Collaboration \[2\]. It yields\(^2\)

\[
\text{Br}(B \to X_s \gamma) \bigg|_{E_\gamma > 1.8 \text{ GeV}} = (3.38 \pm 0.30 \pm 0.29) \cdot 10^{-4},
\]

\[
\langle E_\gamma \rangle \bigg|_{E_0 = 1.8 \text{ GeV}} = (2.292 \pm 0.026 \pm 0.034) \text{ GeV}.
\] (10)

For $E_0 = 1.8$ GeV we have $\Delta \approx 1.1$ GeV, which is sufficiently large to apply the formalism developed in the present work. (For comparison, the value $E_0 = 2.0$ GeV adopted in the CLEO analysis \[1\] implies $\Delta \approx 0.7$ GeV, which we believe is too low for a short-distance treatment.) We find

\[
\text{Br}(B \to X_s \gamma) \bigg|_{E_0 = 1.8 \text{ GeV}} = (3.44 \pm 0.53 \text{ [pert.]} \pm 0.35 \text{ [pars.]}) \cdot 10^{-4},
\] (11)

where the first error refers to the perturbative uncertainty and the second one to parameter variations. The largest parameter uncertainties are due to the $b$- and $c$-quark masses. Our result is in excellent agreement with the experimental value shown in (10). Comparing the two results, and naively assuming Gaussian errors, we conclude that

\[
\text{Br}(B \to X_s \gamma)_{\text{exp}} - \text{Br}(B \to X_s \gamma)_{\text{SM}} < 1.4 \cdot 10^{-4} \quad (95\% \text{ CL}).
\] (12)

Mainly as a result of the enlarged theoretical uncertainty, this bound is much weaker than the one derived in Ref. \[18\], where this difference was found to be less than $0.5 \cdot 10^{-4}$. Hence, we obtain a much weaker constraint on New Physics parameters. For instance, for the case of the type-II two-Higgs-doublet model, we may use the analysis of Ref. \[19\] to deduce

\[
m_{H^+} > \text{(slightly below) 200 GeV} \quad (95\% \text{ CL}),
\] (13)

which is significantly weaker than the constraints $m_{H^+} > 500$ GeV (at 95% CL) and $m_{H^+} > 350$ GeV (at 99% CL) found in Ref. \[18\].

The function $F(E_0)$ provides us with an alternative way to discuss the effects of imposing the cutoff on the photon energy. In contrast to the branching ratio, it is independent of several input parameters (e.g., $\overline{m}_b(\overline{m}_b)$, $|V_{ts}^* V_{tb}|$, $\tau_B$, $\lambda_{1,2}$), and it shows a very weak sensitivity to variations of the remaining parameters. We obtain

\[
F(1.8 \text{ GeV}) = (92^{+7}_{-10} \text{ [pert.]} \pm 1 \text{ [pars.]})\%.
\] (14)

This is the first time that this fraction has been computed in a model independent way. The result may be compared with the values $(95.8^{+1.3}_{-1.2})\%$ and $(95 \pm 1)\%$ obtained from two studies of shape-function models \[14\, 17\], in which perturbative uncertainties have been ignored. We obtain a significantly smaller central value with a much larger uncertainty.

The last quantity we wish to explore is the average photon energy. As mentioned above, this observable is very sensitive to the interplay of physics at the intermediate and low scales. The study of uncertainties due to parameter variations exhibits that the prime sensitivity is

\(^2\)To obtain the first result we had to undo a theoretical correction accounting for the effects of the cut $E_\gamma > 1.8$ GeV, which had been applied to the experimental data.
to the $b$-quark mass, which is expected, since $\langle E_\gamma \rangle = m_b/2 + \ldots$ to leading order. The next-important contribution to the error comes from the HQET parameter $\lambda_1$. To a very good approximation, we have

$$\langle E_\gamma \rangle \bigg|_{E_\gamma=1.8\text{ GeV}} = (2.27^{+0.05}_{-0.07}) \text{ GeV} + \frac{\delta m_b}{2} - \frac{\delta \lambda_1}{4m_b},$$

where the error accounts for the perturbative uncertainty. The quantities $\delta m_b$ and $\delta \lambda_1$ parameterize possible deviations of the relevant input parameters from their central values $m_b = 4.65\text{ GeV}$ and $\lambda_1 = -0.25\text{ GeV}^2$. Our prediction is in excellent agreement with the Belle result in (10). This finding provides support to the value of the $b$-quark mass in the shape-function scheme extracted in Ref. [5]. We stress, however, that the large perturbative uncertainties in the formula for $\langle E_\gamma \rangle$ impose significant limitations on the precision with which $m_b$ can be extracted from a measurement of the average photon energy. Our estimate above implies a perturbative uncertainty of $\delta m_b[\text{pert.}] = 140^{+100}_{-100} \text{ MeV}$. This is in addition to twice the experimental error in the measurement of $\langle E_\gamma \rangle$, which at present yields $\delta m_b[\text{exp.}] = 86 \text{ MeV}$. 

6 Conclusions and outlook

We have performed the first systematic analysis of the inclusive decay $B \to X_s \gamma$ in the presence of a photon-energy cut $E_\gamma \geq E_0$, where $E_0$ is such that $\Delta = m_b - 2E_0$ can be considered large compared to $\Lambda_{\text{QCD}}$, while still $\Delta \ll m_b$. This is the region of interest to experiments at the $B$ factories. The first condition ($\Delta \gg \Lambda_{\text{QCD}}$) ensures that a theoretical treatment without shape functions can be applied. However, the second condition ($\Delta \ll m_b$) means that this treatment is not a conventional heavy-quark expansion in powers of $\alpha_s(m_b)$ and $\Lambda_{\text{QCD}}/m_b$. Instead, we have shown that three distinct short-distance scales are relevant in this region. They are the hard scale $m_b$, the hard-collinear (or jet) scale $\sqrt{m_b \Delta}$, and the low scale $\Delta$. To separate the contributions associated with these scales requires a multi-scale operator product expansion (MSOPE).

Our approach allows us to study analytically the transition from the shape-function region, where $\Delta \sim \Lambda_{\text{QCD}}$, into the MSOPE region, where $\Lambda_{\text{QCD}} \ll \Delta \ll m_b$, into the region $\Delta = \mathcal{O}(m_b)$, where a conventional heavy-quark expansion applies. This is a significant improvement over previous work. For instance, it has sometimes been argued that exactly where the transition to a conventional heavy-quark expansion occurs is an empirical question, which cannot be answered theoretically. Our formalism provides a precise, quantitative answer to this question. In particular, for $B \to X_s \gamma$ with a realistic cut on the photon energy, one is not in a situation where a short-distance expansion at the scale $m_b$ can be justified. The analysis makes it evident that the precision that can be achieved in the prediction of the $B \to X_s \gamma$ branching ratio is, ultimately, determined by how well perturbative and non-perturbative corrections can be controlled at the lowest relevant scale $\Delta$, which in practice is of order 1 GeV. Consequently, we estimate much larger theoretical uncertainties than previous authors. These uncertainties are dominated by yet unknown higher-order perturbative effects. Non-perturbative, hadronic effects at the scale $\Delta$ appear to be small and under control.

This is not the first time in the history of $B \to X_s \gamma$ calculations that issues of scale setting have changed the prediction and error estimate for the branching ratio (see, e.g., the discussion
in Ref. [18]. In our case, however, the change in perspective about the theory of $B \to X_s\gamma$ decay is more profound, as it imposes limitations on the very validity of a short-distance treatment. If the short-distance expansion at the scale $\Delta$ fails, then the rate cannot be calculated without recourse to non-perturbative shape functions, which introduces an irreducible amount of model dependence. In practice, while $\Delta \approx 1.1 \text{ GeV}$ (for $E_0 \approx 1.8 \text{ GeV}$) is probably sufficiently large to trust a short-distance analysis, it would be unreasonable to expect that yet unknown higher-order effects should be less important than in the case of other low-scale applications of QCD.

Obtaining a precise prediction for the $B \to X_s\gamma$ decay rate in the Standard Model is an important target of heavy-flavor theory. The present work shows that the ongoing effort to calculate the dominant parts of the next-to-next-to-leading corrections in the conventional heavy-quark expansion is only part of what is needed to achieve this goal. Equally important will be to compute the dominant higher-order corrections proportional to $\alpha_s^2(\Delta)$ and $\alpha_s^2(\sqrt{m_b\Delta})$, and to perform a renormalization-group analysis of the leading kinematic power corrections of order $\Delta/m_b$. In fact, our error analysis suggests that these effects are potentially more important that the hard matching corrections at the scale $m_b$.

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