NOTE ON LINEARLY EQUIVALENT IDEAL TOPOLOGIES OVER NOETHERIAN MODULES

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Abstract. Let $R$ be a commutative Noetherian ring, and let $N$ be a non-zero finitely generated $R$-module. In this paper, the main result asserts that for any $N$-proper ideal $a$ of $R$, the $a$-symbolic topology on $N$ is linearly equivalent to the $a$-adic topology on $N$ if and only if, for every $p \in \text{Supp}(N)$, $\text{Ass}_{R_p} N_p$ consists of a single prime ideal and $\dim N \leq 1$.

1. Introduction

Let $R$ be a commutative Noetherian ring, $a$ an ideal of $R$ and let $N$ be a non-zero finitely generated $R$-module. For a non-negative integer $n$, the $n$th symbolic power of $a$ w.r.t. $N$, denoted by $(aN)^{(n)}$, is defined to be the intersection of those primary components of $a^nN$ which correspond to the minimal elements of $\text{Ass}_R N/aN$. Then the $a$-adic filtration $\{a^nN\}_{n \geq 0}$ and the $a$-symbolic filtration $\{(aN)^{(n)}\}_{n \geq 0}$ induce topologies on $N$ which are called the $a$-adic topology and $a$-symbolic topology, respectively. These two topologies are said to be linearly equivalent if, there is an integer $k \geq 0$ such that $(aN)^{(n+k)} \subseteq a^nN$ for all integers $n \geq 0$.

Our main point of the present paper concerns an invistigation of the linearly equivalent of the $a$-symbolic and the $a$-adic topology topologies on $N$. More precisely we shall show that:

Theorem 1.1. Let $R$ be a commutative Noetherian ring, and let $N$ be a non-zero finitely generated $R$-module. Then for any $N$-proper ideal $a$ of $R$, the $a$-symbolic topology on $N$ is linearly equivalent to the $a$-adic topology on $N$ if and only if, for every $p \in \text{Supp}(N)$, $\text{Ass}_{R_p} N_p$ consists of a single prime ideal and $\dim N \leq 1$.

The result in Theorem 1.1 is proved in Theorem 2.4. Our method is based on the theory of the asymptotic and essential primes of $a$ w.r.t. $N$ which were introduced by McAdam [6], and in [1]. Ahn extended these concepts to a finitely generated $R$-module $N$. One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper.

Proposition 1.2. Let $R$ be a commutative Noetherian ring and $a$ an ideal of $R$. Let $N$ be a non-zero finitely generated $R$-module such that $\dim N > 0$, and let $p \in \text{Supp}(N) \cap V(a)$. Then the following conditions are equivalent:

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A prime ideal $p$ of $R$ is called a quitesential (resp. quitasymptotic) prime ideal of $a$ w.r.t. $N$ precisely when there exists $q \in \text{Ass}_{R^p} N_p^*$ (resp. $q \in \text{mAss}_{R^p} N_p^*$) such that $\text{Rad}(aR_p^* + q) = pR_p^*$. The set of quitesential (resp. quitasymptotic) prime ideals of $a$ w.r.t. $N$ is denoted by $Q(a, N)$ (resp. $\tilde{Q}(a, N)$) which is a finite set.

We denote by $\mathcal{R}$ the graded Rees ring $R[u, at] := \oplus_{n \in \mathbb{Z}} a^u t^n$ of $R$ w.r.t. $a$, where $t$ is an indeterminate and $u = t^{-1}$. Also, the graded Rees module $N[u, at] := \oplus_{n \in \mathbb{Z}} a^n N$ over $\mathcal{R}$ is denoted by $\mathcal{N}$, which is a finitely generated graded $\mathcal{R}$-module. Then we say that a prime ideal $p$ of $R$ is an essential prime ideal of $a$ w.r.t. $N$, if $p = q \cap R$ for some $q \in Q(u\mathcal{R}, \mathcal{N})$. The set of essential prime ideals of $a$ w.r.t. $N$ will be denoted by $E(a, N)$.

Also, the asympotic prime ideals of $a$ w.r.t. $N$, denoted by $\tilde{A}^*(a, N)$, is defined to be the set $\{q \cap R \mid q \in Q^*(u\mathcal{R}, \mathcal{N})\}$.

In [14], Sharp et al. introduced the concept of integral closure of $a$ relative to $N$, and they showed that this concept have properties which reflect some of those of the usual concept of integral closure introduced by Northcott and Rees in [12]. The integral closure of $a$ relative to $N$ is denoted by $a^{-(N)}$. In [11], it is shown that the sequence $\{\text{Ass}_R R/(a^n)^{-(N)}\}_{n \geq 1}$, of associated prime ideals, is increasing and ultimately constant; we denote its ultimate constant value by $\tilde{A}^*(a, N)$. In the case $N = R$, $\tilde{A}^*(a, N)$ is the asymptotic primes $\tilde{A}^*(a)$ of $a$ introduced by Ratliff in [13]. Also, it is shown in [10] Proposition 3.2] that $\tilde{A}^*(a, N) = \tilde{A}^*(a, N)$.

If $(R, m)$ is local, then $R^*$ (resp. $N^*$) denotes the completion of $R$ (resp. $N$) w.r.t. the $m$-adic topology. In particular, for every prime ideal $p$ of $R$, we denote $R_p^*$ and $N_p^*$ the $pR_p$-adic completion of $R_p$ and $N_p$, respectively. For any ideal $b$ of $R$, the radical of $b$, denoted by $\text{Rad}(b)$, is defined to be the set $\{x \in R : x^n \in b \text{ for some } n \in \mathbb{N}\}$. Finally, for each $R$-module $L$, we denote by $\text{mAss}_R L$ the set of minimal prime ideals of $\text{Ass}_R L$.

Recall that an ideal $b$ of $R$ is called $N$-proper if $N/bN \neq 0$, and, when this the case, we define the $N$-height of $b$ (written $\text{height}_N b$) to be $\inf\{\text{height}_N p : p \in \text{Supp } N \cap V(b)\}$, where $\text{height}_N p$ is defined to be the supremum of lengths of chains of prime ideals of $\text{Supp}(N)$ terminating with $p$. Also, we say that an element $x$ of $R$ is an $N$-proper element if $N/xN \neq 0$. For any unexplained notation and terminology we refer the reader to [3] or [7].

2. THE MAIN RESULT

Let $R$ be a commutative Noetherian ring, and let $N$ be a non-zero finitely generated $R$-module. The purpose of the present paper is to give an investigation of the linearly equivalent of the $a$-symbolic and the $a$-adic topology topologies on $N$. The main goal of this section is Theorem 2.4. The following proposition plays a key role in the proof of the main theorem.
Proposition 2.1. Let $a$ be an ideal of $R$ and let $N$ be a non-zero finitely generated $R$-module with $\dim N > 0$. Let $p \in \text{Supp}(N) \cap V(a)$. Then the following conditions are equivalent:

(i) $p \in \tilde{A}^*(a, N)$.
(ii) $p \in \tilde{A}^*(ab, N)$, for any $N$-proper ideal $b$ of $R$ with $\text{height}_N b > 0$.
(iii) $p \in \tilde{A}^*(xa, N)$, for any $N$-proper element $x$ of $R$ with $x \notin \bigcup_{p \in \text{Ass}_R N} p$.
(iv) $p \in \tilde{A}^*(xa, N)$, for some $N$-proper element $x$ of $R$ with $x \notin \bigcup_{p \in \text{Ass}_R N} p$.

Proof. (i)$\implies$(ii): Let $p \in \tilde{A}^*(a, N)$ and let $b$ be an $N$-proper ideal of $R$ such that $\text{height}_N b > 0$. Then, in view of [10, Remark 2.4],

$$p/\text{Ann}_R N \in \tilde{A}^*(a + \text{Ann}_R N/\text{Ann}_R N).$$

Hence, as by [8, Theorem 2.1],

$$\text{height}_N b = \text{height}(b + \text{Ann}_R N/\text{Ann}_R N) > 0,$$

it follows from [5, Proposition 3.26] that

$$p/\text{Ann}_R N \in \tilde{A}^*(ab + \text{Ann}_R N/\text{Ann}_R N).$$

Therefore by using [10, Remark 2.4], we obtain that $p \in \tilde{A}^*(ab, N)$, as required.

(ii)$\implies$(iii): Let (ii) hold and let $x$ be an $N$-proper element of $R$ such that $x \notin \bigcup_{p \in \text{Ass}_R N} p$. Then it is easy to see that $\text{height}_N xR > 0$, and so according to the assumption (ii), we have $p \in \tilde{A}^*(xa, N)$.

(iii)$\implies$(iv): Since $\dim N > 0$, there exists $q \in \text{Supp} N$ such that $\text{height}_N q > 0$. Hence $q \notin \bigcup_{p \in \text{Ass}_R N} p$, and so there is $x \in q$ such that $x \notin \bigcup_{p \in \text{Ass}_R N} p$. Consequently, it follows from the hypothesis (iii) that $p \in \tilde{A}^*(xa, N)$.

(iv)$\implies$(i): Let $x$ be an $N$-proper element of $R$ such that $x \notin \bigcup_{p \in \text{Ass}_R N} p$ and let $p \in \tilde{A}^*(xa, N)$. Then

$$p/\text{Ann}_R N \in \tilde{A}^*(xa + \text{Ann}_R N/\text{Ann}_R N),$$

by [10, Remark 2.4]. Now, since $x \notin \bigcup_{p \in \text{Ass}_R N} p$, it is easy to see that $x + \text{Ann}_R N$ is not in any minimal prime $R/\text{Ann}_R N$. Therefore, it follows from [5, Proposition 3.26] that

$$p/\text{Ann}_R N \in \tilde{A}^*(a + \text{Ann}_R N/\text{Ann}_R N).$$

Consequently, in view of [10, Remark 2.4], $p \in \tilde{A}^*(a, N)$, and this completes the proof. \qed

Before we state Theorem 2.4 which is our main result, we give a couple of lemmas that will be used in the proof of Theorem 2.4.

Lemma 2.2. Let $(R, m)$ be a local ring and let $N$ be a non-zero finitely generated $R$-module such that $\dim N > 0$ and that $\text{Ass}_R N$ has at least two elements. Then there is an ideal $a$ of $R$ such that $m \in Q(a, N) \setminus \text{mAss}_R N/aN$.

Proof. See [2, Proposition 4.2]. \qed

Lemma 2.3. Let $N$ be a non-zero finitely generated $R$-module and let $a$ be an $N$-proper ideal of $R$. Then $E(a, N) = \text{mAss}_R N/aN$ if and only if the $a$-symbolic topology is linearly equivalent to the $a$-adic topology.
Proof. The assertion follows easily from [9, Theorem 4.1].

We are now ready to state and prove the main theorem of this paper which is a characterization of the certain modules in terms of the linear equivalence of certain topologies induced by families of submodules of a finitely generated module $N$ over a commutative Noetherian ring $R$. We denote by $Z_R(N)$ the set of zero divisors on $N$, i.e., $Z_R(N) := \{ r \in R \mid rx = 0 \text{ for some } x(\neq 0) \in N \}$. 

**Theorem 2.4.** Let $N$ be a non-zero finitely generated $R$-module. Then the following conditions are equivalent:

(i) For every $N$-proper ideal $b$ of $R$, the $b$-symbolic topology is linearly equivalent to the $b$-adic topology.

(ii) $\dim N \leq 1$ and $\text{Ass}_{R_p} N_p$ consists of a single prime ideal, for all $p \in \text{Supp}(N)$.

*Proof.* Suppose that (i) holds. Firstly, we show that $\dim N \leq 1$. To achieve this, suppose the contrary is true. That is $\dim N > 1$. Then there exists $p \in \text{Supp}(N)$ such that $\text{height}_N p > 1$. Hence $p \not\subseteq \bigcup_{q \in \text{Ass}_{R_p} N} q$, and so there exists $x \in p$ such that $x \not\in \bigcup_{q \in \text{Ass}_{R_p} N} q$. Now, since $p \in A^+(pN)$ and $xN \neq N$, it follows from Proposition 2.1 that $p \in A^+(xpN)$. Therefore, in view of [1, Theorem 3.17] we have $p \in E(xpN)$.

On the other hand, since $x \not\in \bigcup_{q \in \text{Ass}_{R_p} N} q$, it is easily seen that $p \not\in m_{R_p}/xNp$, and so by the assumption (i) and Lemma 2.3 we have $p \not\in E(xpN)$, which is a contradiction. Hence, $\dim N \leq 1$. Now, we show that $\text{Ass}_{R_p} N_p$ consists of a single prime ideal, for all $p \in \text{Supp}(N)$. To do this, if $\dim N = 0$, then $\dim N_p = 0$. Hence $\text{Ass}_{R_p} N_p = \{ q_{R_p} \}$, as required. Consequently, we have $\dim N_p = 1$. Now, if $\text{Ass}_{R_p} N_p$ has at least two elements, then in view of Lemma 2.2 there exists an ideal $a_{R_p}$ of $R_p$ such that $p_{R_p} \in Q(a_{R_p}, N_p)$ but $p_{R_p} \not\in m_{R_p}/a_{R_p}$. Therefore, in view of [1, Lemma 3.2 and Theorem 3.17], $p \in E(aN) \setminus m_{Ann N} N$, which is a contradiction.

In order to show the implication (ii)$\implies$(i), in view of Lemma 2.3 it is enough for us to show that $E(b, N) = m_{Ass} N/bN$. To this end, let $p \in E(b, N)$. By virtue of [1, Lemma 3.2], we may assume that $(R, p)$ is local.

Firstly, suppose $\dim N = 0$. Then it readily follows that $p \in m_{Ass} N/bN$, as required. So we may assume that $\dim N = 1$. There are two cases to consider:

**Case 1.** $b \not\subseteq Z_R(N)$. Then $\text{grade}(b, N) > 0$. Since $\dim N = 1$, it follows that $\text{height}_N b = 1$, and so $b + \text{Ann}_R N$ is $p$-primary. Hence $p \in m_{Ass} N/bN$, as required.

**Case 2.** Now, suppose that $b \subseteq Z_R(N)$. Then there exists $z \in \text{Ass}_N$ such that $b \subseteq z$. Since $\text{Ass}_R N$ consists of a single prime ideal, so $\text{Ass}_R N = \{ z \}$. Hence in view of [1, Proposition 3.6], $p/z \in E(b + z/N, R/z)$. Since $b \subseteq z$, it follows from [1, Remark 2.3] that $p = z$, which is a contradiction, because $\dim N = 1$. Consequently, $b \not\subseteq Z_R(N)$ and the claim holds. 

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