Corpus-compressed Streaming and the Spotify Problem

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1 Overview

In this work, we describe a problem which we refer to as the Spotify problem and explore a potential solution in the form of what we call corpus-compressed streaming schemes.

Spotify is a digital music streaming service that gives users access to music on demand. One of its most prominent features is the ‘playlist’ feature: Spotify (and other Spotify users) maintain and update lists of songs that other users may listen to at any time. The process of listening to these playlists is generally seamless on home networks due to the lack of significant bandwidth limitations, but this is not necessarily the case for some mobile users whose mobile broadband networks are less than reliable. The Spotify problem simply names the problem of improving the reliability of streaming playback for users who spend time in these constrained networks.

More generally, the Spotify problem applies in any number of practical domains where devices may be periodically expected to experience degraded communication or storage capacity. One obvious solution candidate which comes to mind immediately is standard compression. Though obviously applicable, standard compression does not in any way exploit all characteristics of the problem; in particular, standard compression is oblivious to the fact that a decoder has a period of virtually unrestrained communication. Towards applying compression in a manner which attempts to stretch the benefit of periods of higher communication capacity into periods of restricted capacity, we introduce as a solution the idea of a corpus-compressed streaming scheme.

This report begins with a formal definition of a corpus-compressed streaming scheme. Following a discussion of how such schemes apply to the Spotify problem, we then give a survey of specific corpus-compressed scheming schemes guided by an exploration of different measures of description complexity within the Chomsky hierarchy of languages.\(^1\)

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2 Definition of a Corpus-compressed Streaming Scheme

We define a **corpus-compressed streaming scheme** in a setting consisting of two parties, an encoder $A$ and a decoder $B$. $A$ holds a finite set of $n$ distinct binary strings $C = \{c_1, \ldots, c_n\}$, called a corpus set, in which $B$ is interested. Following a setup phase, $A$ has need to convey some stream of strings within this corpus set $c_1, c_2, \ldots$ to $B$ with the following constraints:

- During the setup phase, $A$ and $B$ may communicate freely.
- During the stream, communication between $A$ and $B$ is costly.
- During the entire exchange, memory is costly for $B$, meaning that $B$ desires not to store the entire corpus set.
- There is no characterization of the stream beyond the fact that all strings come from the corpus set.

We give a general definition of a corpus-compressed streaming scheme as well as a formal parameterization of the function of such a scheme:

**Definition 2.1. corpus-compressed streaming scheme** Let $C$ be a corpus set containing $n$ distinct strings. A corpus-compressed streaming scheme (CCSS), is a triplet of algorithms $(\text{Construct}(C), \text{Encode}(D, x), \text{Decode}(D, \hat{x}))$ respectively defined as follows:

1. $\text{Construct}(C)$ takes as input a corpus set and returns a schematic object $D$ along with (potentially empty) auxiliary output $A$. A schematic object is a data structure used to encode and decode elements of the corpus set.

2. $\text{Encode}(D, A, x)$ takes as input a valid schematic $D$ and associated auxiliary input $A$ and a string $x \in C$ and returns either an encoding of $x$ or $\bot$ if $x$ is invalid with respect to $D$.

3. $\text{Decode}(D, \hat{x})$ takes as input a valid schematic $D$ and an encoding $\hat{x}$ and returns some string $x \in C$ (where $D$ is a valid schematic for $C$).

**Definition 2.2. $\delta$-minimal $p(n, z) - \epsilon$-corpus-compressed streaming scheme** Let $C$ be a corpus set containing $n$ strings; let $z = \max_{c_i \in C}|c_i|$ be the length of the longest string in $C$. A $\delta$-minimal $p(n, z) - \epsilon$-corpus-compressed streaming scheme, shorthand $(\delta, p(n, z), \epsilon)$-CCSS, is a corpus-compressed streaming scheme $(\text{Construct}, \text{Encode}, \text{Decode})$ with the following properties:

1. Compression: For $D = \text{Construct}(C)$,

   $$\forall c_i \in C, |\text{Encode}(D, c_i)| \leq p(n, z)$$

   As defined, the parameter $p(n, z)$ gives the maximum length of the streaming code for any individual $c_i \in C$ as a function of the size of the corpus and the maximum element size.
2. **Correctness:** For $D = \text{Construct}(C)$,

$$\forall c_i \in C, \frac{\text{ham}(\text{Decode}(D, \text{Encode}(D, A, c_i)), c_i)}{|c_i|} \leq \epsilon$$

and where $\text{ham}(\cdot, \cdot)$ denotes Hamming distance. As defined, the parameter $\epsilon$ parameterizes the maximum reconstruction error of the scheme.

3. **Minimality:** For $D = \text{Construct}(C)$,

$$\frac{|D|}{|D^*|} \leq 1 + \delta$$

where $D^*$ is the minimum-length satisfactory schematic among some restriction of possible objects $D$. As defined, $\delta$ defines the factor by which the output of $\text{Construct}(D)$ is off from some definition of minimal. We will see in later sections that this notion of minimality has some interesting connections to concepts in algorithmic information theory.

Because this is a first presentation of corpus-compressed streaming, this work will explore only schemes guaranteeing exact reconstruction (in other words, we fix $\epsilon$ to be 0 in all explorations). We additionally provide the following trivial lower bound on the minimum achievable streaming code length of any CCSS which we will use in our exploration:

**Theorem 2.1.** Let $S$ be a valid $(\delta, p(n, z), 0)$ corpus-compressed scheming scheme. The maximum streaming code length for any corpus set must obey the inequality $p(n, z) \geq \log(n + 1) - 1$.

**Proof:** Assume not. Assume $p(n, z) < \log(n + 1) - 1$. Even if the scheme makes use of variable length streaming codes, the scheme may only encode

$$< \sum_{i=1}^{\log(n+1)-1} 2^i$$

$$< 2^{\log(n+1)-1+1} - 1$$

$$< n + 1 - 1$$

$$< n$$

distinct strings. Since $C$ includes $n$ distinct strings but can encode only less than that number, we conclude that $S$ cannot possibly be $\epsilon = 0$-correct.

### 2.1 Application to the Spotify Problem

We see immediately that the existence of a $(\delta, p(n, z), \epsilon)$-CCSS with reasonably sized schematics yields an effective solution to the Spotify problem. For the sake of illustration, say that we have a $(\delta, p(n, z), \epsilon)$-CCSS, $(\text{Construct}, \text{Encode}, \text{Decode})$. In the setting of the Spotify problem, we may apply the scheme in the following straight-forward manner:

1. Before leaving his or her home network, the user indicates to Spotify that he or she wishes to listen to playlist $P = \{s_1, ..., s_n\}$.

2. Spotify then treats the playlist of songs as a corpus set and sends $D = \text{Construct}(P)$ to the user’s device.

3. The user leaves his or her home network, entering a bandwidth-constrained mobile network.
4. In perpetuity, then, when the user requests song $i$, Spotify now sends $Encode(D, s_i)$ (of length less than or equal to $p(n, z)$) rather than the entirety of $s_i$.

Consider the effect such an application would have on the music streaming process. In essence, the user ultimately utilizes his or her temporarily unconstrained bandwidth in the second step to receive a reasonably sized data structure that will allow him to stream music at a reduced cost in perpetuity; note that by choosing $p(n, z)$ as conservatively as $\frac{z}{r}$, where mobile bandwidth is $\frac{1}{r}$-fraction of home bandwidth (often, $r < 2$), we would negate any difference in playback he or she might otherwise observe, thus solving the problem.

3 Corpus-compressed Streaming Schemes: Concrete Constructions using Regular Output Automata

In this section, we attempt to design corpus-compressed streaming schemes by taking inspiration from elements of algorithmic information theory. Among these elements is the notion of the Kolmogorov complexity of an object, one of the most prevalent ideas in algorithmic information theory [4], defined as the length of the shortest Turing machine description which produces said object.

For our purposes, we both project this idea onto the notion of a corpus-compressed streaming scheme and direct our analysis to consider another segment of the Chomsky hierarchy of languages. In particular, where Kolmogorov complexity is interested in the length of the shortest Turing machine which outputs a string $x$, we would be interested in the shortest general output Turing machine $M$ which has the following property for a corpus set $C$:

$$\forall c_i \in C, \exists e_i, |c_i| \leq p(n, z), M(e_i) = c_i$$

We make two observations and derive one question which guides our exploration in this section. We first note that the decision variant of determining Kolmogorov complexity is undecidable, and so there is no algorithmic solution (in the way of Construct) capable of constructing $M$ given $C$. We also note that output Turing machines correspond to the most encompassing point of the Chomsky hierarchy of languages. These observations lead us to ask the following question: what if instead we restrict schematics to output automata corresponding to less encompassing points in the hierarchy?

This question guides the CCSS constructions we derive. This work begins to answer this question by considering corpus-compressed streaming schemes with schematics restricted to the least encompassing point in the Chomsky hierarchy: that of regular languages. A regular language may be defined as a language which may be recognized by a finite state machine; as we are interested in machines with output, in this work we will consider the schemes we may derive when we restrict schematics to the set of finite state machines with per-state output, known as Moore machines. More specifically, we first consider schemes with schematics restricted to Moore machines whose underlying acyclic graphs are acyclic and then use these results to make connections to the general (cyclic) case.

3.1 Formalizing our Restriction

In this section, we are interested in schemes $(Construct, Encode, Decode)$ where the output of $Construct(C)$ is a Moore machine defined by the tuple $(S, S_0, \Sigma, \Gamma, T, G)$ with the following properties:

1. $S_0$ is the unique start state of the machine.
2. $\Sigma$, the input alphabet, is the set $\{0, 1, \bot\}$. $\bot$ is a special end-of-input symbol that is read only at the end of every string.
3. $\Gamma = \Sigma$ is the output alphabet. In the case of the output alphabet, $\bot$ is a special blank symbol which only ever occurs at a starting or final state.
4. \( T : S \times \Sigma \Rightarrow S \) is a deterministic transition function mapping states to successor states given an input symbol. With respect to the transition function of a Moore machine, we use the convention of a unconditionaled transition, which is a transition which is taken regardless of whether or not the next input symbol is 0 or 1. Absent the presence of an explicit \( \perp \)-transition, this transition is taken even if there is no next input symbol.

5. \( G \) is an output function mapping states to their outputs.

The output of \( \text{Construct} \) must satisfy the further stipulation that a \((\delta, p(n, z), \epsilon)\)-CCSS maintain \( p(n, z) \)-compression and \( \epsilon \)-correctness for the following fixed, universal \( \text{Decode} \) procedure utilized by schemes under this restriction:

**Algorithm 1** Fixed Decoding Procedure

```
1: procedure Decode\((D = (S, S_0, \Sigma, T, G), x)\)
2:     Beginning at \( S_0 \), run \( D \) on input \( x \). If ever there is a next input symbol but no successor state, return \( \perp \).
3:     return the sequence of 0 and 1 outputs of \( D \).
4: end procedure
```

(Note: while this restriction does not necessarily stipulate a universal \( \text{Encode} \) procedure, the requirement remains that it must exist and be efficiently computable.)

Under this restriction of schematics, we define \( \delta \)-minimality with respect to the number of states in a machine. For a given corpus set \( C \), the minimal schematic is the Moore machine with the smallest number of states of any Moore machine satisfying the stated requirements.

In the remainder of this section, we refer to a CCSS under this restriction as a MM (Moore machine)-restricted CCSS.

### 3.2 Restricting Schematics to Acyclic Moore Machines

In [1], Bryant presented the binary decision diagram (BDD) data structure as a means of representing and manipulating Boolean functions. The core mechanism underlying applications of BDDs is their function as read-once branching programs: functions are represented as rooted, directed acyclic graphs consisting of decision junctions and terminal nodes. Each transition from a decision junction corresponds to a final assignment to exactly one variable, and terminal nodes correspond to function evaluations given assignments so far. The end result is a graph in which every distinct path corresponds to a distinct variable assignment \( \bar{x} \) ending with a terminal node having a label corresponding to whether it satisfies the formula.

Perhaps more importantly, BDDs are especially useful with respect to Boolean function representation because they are amenable to compression through the use of simple reduction rules. Given a formula \( \Phi \) in \( n \) variables and an ordering of those variables, there exists the notion of a reduced-ordered BDD (ROBDD) which is able to represent every assignment (and its image in \( \Phi \)) in a diagram having often far fewer than \( 2^n \) nodes for many practical instances.

Considering once more our interests in this work, there is at least one significant direct parallel between BDDs and the goals of corpus-compressed streaming schemes. In particular, consider a Moore machine schematic in which there are no cycles of states. If we view the graph created by the set of states \( S \) and the transition function \( T \), we see that we also have a rooted, directed acyclic graph with decision junctions at states having transition options for both input symbols 0 and 1 in which distinct paths lead to distinct outputs.

While BDDs and Moore machine schematics are obviously not exactly analogous with respect to goals and structure, these observable similarities naturally lead us to question whether similar reduction methods in corpus-compressed streaming might yield practical schemes. Inspired by this prospect, in this
section, we study corpus-compressed streaming in our chosen restriction with the additional requirement that the graphs underlying schematics be acyclic. We denote a valid CCSS under this restriction by the term ‘AMM (Acyclic Moore machine) -restricted CCSS’.

3.2.1 An Exact AMM-restricted \((0, \lfloor \log n \rfloor, 0)\)-Corpus-compressed Streaming Scheme

One of the most powerful reduction rules discussed in [1] is the merging of isomorphic subgraphs within non-reduced binary decision diagrams. We extend and apply this idea in order to derive an exact AMM-restricted \((0, \log n, 0)\)-CCSS. This section proceeds as follows: we (1) present our scheme, (2) provide a proof of this scheme’s validity, (3) note a fact about the streaming code length achieved, and (4) provide a worked example.

Scheme 1 We provide pseudocode for the Construct and Encode procedures of our AMM-restricted CCSS.

Algorithm 2 Construct Routine, Scheme 1

1: procedure Construct\((C = \{c_1, \ldots, c_n\})\) 
2: STAGE 1: \(\triangleright\) Construct a MM with a directed tree topology 
3: Initialize \(D\) as a Moore machine with a single state \(S_0\).
4: for \(i = 1 \ldots n\) do 
5: Set \(s = S_0\).
6: for \(j = 1 \ldots \lfloor \log c_i \rfloor\) do 
7: Set \(b\) to be the \(j\)th bit of \(c_i\).
8: if \(s\) has no 0-, 1-, or unconditional transition edges then 
9: Add a new state \(s'\) to \(D\) which outputs \(b\).
10: Add an unconditional transition edge from \(s\) to \(s'\).
11: end if 
12: if \(s\) has an unconditional edge to a state \(s''\) with output \(1 - b\) then 
13: Modify the transition from \(s\) to \(s''\) to be a transition on input symbol \(1 - b\).
14: end if 
15: if \(s\) does not have a transition edge on input symbol \(b\) then 
16: Add a new state \(s'\) to \(D\) which outputs \(b\).
17: Add a transition on input symbol \(b\) from \(s\) to \(s'\).
18: end if 
19: Set \(s = s(b)\), where \(s(b)\) denotes the state reached following a transition on symbol \(b\).
20: end for 
21: Add a new node \(s'\) with the special \(\perp\) output symbol.
22: Add a transition from \(s\) to \(s'\) on end-of-input symbol \(\perp\).
23: Mark \(s'\) as a final state.
24: end for 
25: STAGE 2: \(\triangleright\) Enumerate states in depth order 
26: Initialize \(Q\) as an empty queue.
27: Initialize \(L\) as an empty list of length \(|S|\).
28: \(Q.enqueue(S_0)\)
29: Set \(i = 0\)
30: while \(|Q| > 0\) do 
31: Set \(s = Q.dequeue()\)
32: Set \(L[|S| - i] = s\)
33: Set \(i = i + 1\).
34: For all states \(s'\) such that there is a transition from \(s\) to \(s'\), \(Q.enqueue(s')\).
35: end while
Algorithm 2 Construct Routine (Continued)

36: STAGE 3:  \(\triangleright\) Apply reduction: merge isomorphic subgraphs
37: Initialize \(T\) as an empty associative array (dictionary).
38: for \(i = 1 \ldots |S|\) do
39: \(s = L[i]\). Let \(s.out\) denote the output symbol of state \(s\).
40: \(id = s.out\) if \(s(0)\) \& \(s(1)\) \& \(s(\bot)\).
41: if \(T[id] = null\) then
42: \(T[id] = s\).
43: else
44: Let \(r\) be the single node having a transition to \(s\). (Unless \(s\) is \(S_0\), in which case \textbf{break}).
45: Replace the transition from \(r\) to \(s\) with a transition on the same symbol to \(T[id]\).
46: Delete \(s\) from \(D\).
47: end if
48: end for
49: If \(S_0\) has only an unconditional transition to some state \(s'\), delete \(S_0\) and make \(s'\) the start state.
50: return \(D, A = \emptyset\).
51: end procedure

The \textit{Construct} routine of scheme 1 begins in stage 1 by constructing a Moore machine whose states and transitions take the form of a directed binary tree having \(n\) paths such that following each path produces a unique string in the corpus set. Starting from a non-output initial state, we iterate through all strings in the corpus set, bit by bit, branching where strings following the same path diverge. In stage 2, the procedure prepares for iteration through the states of this Moore machine in order of decreasing depth from the start state. In stage 3, the procedure merges isomorphic components in the machine using a light dynamic programming approach. We prove the correctness and properties of this procedure later in this section.

The encoding procedure of scheme 1 is given as

Algorithm 3 Encode Procedure, Scheme 1

1: procedure \textbf{Encode}(\(D = \text{Construct}(C), A, c_i\))
2: \hspace{1cm} If \(S_0.out == 1 - c_i[1]\), \textbf{return} error.
3: \hspace{1cm} Set \(x\) to be the empty string.
4: \hspace{1cm} Set \(s = S_0\).
5: for \(j = 1 \ldots |c_i|\) do
6: \hspace{2cm} If \(s\) has only an unconditional edge to a state which outputs \(1 - c_i[j]\), \textbf{return} error.
7: \hspace{2cm} If \(s\) has no transition to a state which outputs \(c_i[j]\), \textbf{return} error.
8: \hspace{2cm} If \(s\) has at least one non-unconditional transition, choose a symbol \(b\) which leads to a state which outputs \(c_i[j]\) and set \(x = x \parallel b\).
9: \hspace{2cm} Set \(s\) to be the successor state of \(s\) which outputs \(c_i[j]\).
10: end for
11: \hspace{1cm} If \(s\) has no \(\bot\)-transitions, \textbf{return} error.
12: \hspace{1cm} \textbf{return} \(x\).
13: end procedure

\textit{Encode} simply begins at the start state of the machine and records the transitions along a path in \(D\) which outputs the input string \(c_i\). We now move to prove that this scheme is indeed a \((0, \log n, 0) - \text{CCSS}\) as well as prove its runtime properties.

3.2.1.1 Proving \((0, \log n, 0)\)-CCSS Validity

We prove in this section that Scheme 1 is indeed a valid \((0, \log n, 0)\)-CCSS.
Lemma 3.1. At the end of stage 1 of Construct, $D$ has a tree topology in which the output along every path from the root ($S_0$) is unique.

Proof: That the graph underlying $D$ is a directed tree is immediate: when a new state is added, it is given a unique parent; likewise, when a new state is added, it is never given a transition to an existing state. Because $D$ is a directed tree, there is a unique path from the root to every internal state. Assume that there exist two of these unique paths $P_1$ and $P_2$ each starting with $S_0$ such that the output along these paths is equal. Because these paths cannot be the same, there must exist a first point of divergence along them. But at this point of divergence, there must be two transitions to two distinct states having the same output, which is impossible by lines 6-19. □

Lemma 3.2. At the end of stage 1 of Construct, there is a path from root to leaf in $D$ for each $c_i \in C$ along which the machine will output $c_i$.

Proof: Assume not; assume that there exists a $c_i$ such that there does not exist a path from $S_0$ to a leaf along which 0/1 outputs correspond to $c_i$ at the end of stage 1. There must exist a least index $k$ from 1 to $|c_i|$ such that there is a unique (by Lemma 3.1) path from $S_0$ which outputs $c_i[1], ..., c_i[k−1]$ (or the empty string if $k = 1$) but not $c_i[1], ..., c_i[k]$. During iteration $i$ of line 4, we see that the sequence of the first $k−1$ values of $s$ is precisely this path; since transitions are never deleted, this path must exist at the end of stage 1. But we see by lines 6-19 that a transition to a state with output $c_i[k]$, meaning that there does exist a path with output $c_i[1], ..., c_i[k]$, a contradiction. □

Lemma 3.3. At the end of stage 1, every path from root to leaf in $D$ outputs a string in $C$.

Proof: By construction, every leaf in $D$ has output symbol $\bot$. There are exactly $n$ such leaves added to $D$. Since paths in $D$ are unique ($D$ forms a tree), there are exactly $n$ paths from root to leaf in $D$. By Lemma 3.2 there must be a path from root to leaf in $D$ which outputs each $c_i \in C$. Since there are $n$ paths among $n$ strings, we conclude that there is a one-to-one correspondence between paths in $D$ and strings in $C$. Every path must therefore output a string in $C$. □

Lemma 3.4. Let $D$ be a Moore machine whose underlying graph is acyclic, rooted at $S_0$, in which $\bot$-output states are either $S_0$ or nodes without any outbound transitions (leaves). For any state $a$ in $D$, let $gen(a)$ be the set of strings which are generated by $D_a$, the machine whose underlying graph is the directed acyclic subgraph rooted at $a$ (i.e., the set of strings output by $D$ when starting at $a$ and following any path to a leaf).

For any two distinct states $a$ and $b$, $gen(a) = gen(b)$ if and only if there exist states $x \in D_a$, $y \in D_b$ such that $D_x$ is isomorphic to $D_y$ but $x \neq y$.

Proof: ($\Rightarrow$). We show that $D_a \cong D_b$ implies $gen(a) = gen(b)$ by an inductive argument. Consider first the case where $a$ and $b$ are leaves. By our choice of $D$, $a$ and $b$ must be $\bot$-output leaves. Therefore $gen(a) = gen(b) = \{\bot\}$, the set containing the empty string. Assume now that $D_a \cong D_b \Rightarrow gen(a) = gen(b)$ holds for the children of some pair of states $x$ and $y$ in $D$ having $D_x \cong D_y$. Because $D_x \cong D_y$, the output bits of $x$ and $y$, $d_x$ and $d_y$, must be equal. Similarly, the subgraphs of the 0-,1-, and $\bot$-successors of $x$ must be respectively isomorphic to those of $y$, therefore having the same generated set of strings by our inductive assumption; call these $s_0, s_1, s_\bot$ respectively, and let $gen(s) = \emptyset$ if the given successor does not exist.

We may explicitly determine $gen(x)$ as $gen(x) = \cup_{s \in s_0, s_1, s_\bot} \{d_x \mid z, \forall z \in gen(s)\}$. Likewise, $gen(y) = \cup_{s \in s_0, s_1, s_\bot} \{d_y \mid z, \forall z \in gen(s)\}$. Because $d_x = d_y$, then, $gen(x) = gen(y)$. Since $x \neq y$, we conclude the proof in this direction.

($\Leftarrow$). Consider now any distinct states $a$ and $b$ with $gen(a) = gen(b) = g$. If $g$ contains the one-bit string $d_x$, then both $a$ and $b$ have $\bot$-successors $s_1^g$ and $s_1^g$, both leaves with the same set of generated strings. If $g$ contains strings which are 1 in the second bit, $a$ and $b$ must each have exactly one successor with output bit 1, $s_1^g$ and $s_1^g$, each generating the subset of $g$ which is 1 in the second bit. Likewise, if $g$
contains strings which are 0 in the second bit, \(a\) and \(b\) must each have a successor state with output bit \(0\), \(a^0\) and \(b^0\), each generating the subset of \(g\) which is 0 in the second bit. (By the same argument, if \(g\) does not contain strings which are 0 in the second bit, neither \(a\) nor \(b\) have 0-successors; the same holds for strings which are 1 in the second bit. Thus \(s^a \not\in \emptyset \iff s^b \not\in \emptyset\).

Now say that we wish to contradict the statement that there exist \(x \in D_a\), \(y \in D_b\) such that \(D_x\) is isomorphic to \(D_y\) but \(x \neq y\). Then it must be the case that \(D_x^1 \not\cong D_y^1 \lor s^1 = s^1\), \(D_x^0 \not\cong D_y^0 \lor s^0 = s^0\), and \(D_x^1 \not\cong D_y^0 \lor s^1 = s^0\). Indeed, equality cannot hold between all successors of \(a\) and \(b\), else it would be trivially true that \(a\) and \(b\) are isomorphic, and our original statement holds. There must then exist an \(A \in \{0, 1, \perp\}\) such that \(s^A_a \neq s^A_b\) (whose corresponding subgraphs re not isomorphic). But we have established that \(\text{gen}(s^A_a) = \text{gen}(s^A_b)\), and so we may repeat this argument starting from these two states. Following this pattern, we may continue until (a) all successors are equal, and we obtain an example, or (b) are are considering two distinct leaves \(x'\), \(D_a\), \(x'\) and \(y'\), \(D_b\) (we may not consider a leaf and a non-leaf, as it cannot be the case that a leaf and non-leaf generate the same string set). But, by choice of \(D\), \(x'\) and \(y'\) have the same output symbol, and so we simultaneously have \(x' \in D_a\), \(y' \in D_b\), \(x' \neq y'\), and \(D_{x'} \cong D_{y'}\).

**Lemma 3.5** (0-Minimality Condition). Fix a corpus set \(C = \{c_1, \ldots, c_n\}\). Let \(X\) be a Moore machine following the description given in Lemma \[3.4\] such that each has exactly \(n\) paths beginning with \(S_0\) and ending at a leaf and where each such path outputs a unique string in \(C_n\). Let \(Y\) be another Moore machine meeting the same requirements. Denote by \(|X|\) and \(|Y|\) respectively the number of states in \(X\) and \(Y\).

\(|X| < |Y|\) implies that there exist states \(u\) and \(v\) such that \(D_u \not\cong D_v\) or states \(w\) and \(x\) such that a path from \(S_0\) to \(w\) has the same output as one from \(S_0\) to \(x\). As a result, \(Y\) is minimal if there are no two distinct states \(u\) and \(v\) in \(Y\) such that \(D_u \cong D_v\) and no two distinct states \(w\) and \(x\) such that the paths from \(S_0\) to each have the same output.

**Proof:** We provide a proof by contrapositive. Assume (1) that for all pairs of distinct states \(u\) and \(v\) in \(Y\), \(D_u \not\cong D_v\). Then by Lemma \[3.4\], there do not exist any distinct states \(u\) and \(v\) in \(Y\) having \(\text{gen}(u) = \text{gen}(v)\). Further, (2) assume that there do not exist any distinct states \(w\) and \(x\) such that any path from \(S_0\) to either has the same output.

For the sake of contradiction, assume that \(X\) has \(|X| < |Y|\). By the pigeonhole principle, there necessarily exist strings \(c_i\) and \(c_j\) and indices \(k\) and \(l\) such that state \(s_k\) along the path \(P_i\) generating \(c_i\) in \(X\) is state \(s_l\) along the path \(P_j\) generating \(c_j\) in \(X\). Without loss of generality, say that \(s_k\) outputs the \(k\)th bit of \(c_i\) and that \(s_l\) outputs the \(l\)th bit of \(c_j\).

1. The prefix \(t\) of \(c_i\) generated by the sub-path of \(P_i\) from \(S_0\) to \(s_k\) in \(X\) is also a prefix of \(c_j\). But \(u_k\) and \(v_k\), respectively the states outputting bit \(k\) of \(c_i\) and \(c_j\) in \(Y\), must necessarily have a path from the root which outputs this prefix. This contradicts our choice of \(Y\).

2. The substring \(t_i\) of \(c_i\) generated by the sub-path of \(P_i\) from \(S_0\) to \(s_k\) in \(X\) is not a prefix of \(c_j\). Let \(t_j\) be the substring of \(c_j\) generated by the sub-path of \(P_j\) from \(S_0\) to \(s_l\) in \(X\). Since every path in \(X\) must correspond to a string in \(C\), we have that \(Q_i = \{t_i \parallel q, \forall q \in \text{gen}(s_k)\} \subseteq C\) and \(Q_j = \{t_j \parallel q, \forall q \in \text{gen}(s_l)\} \subseteq C\).

By condition (2) on \(Y\), there must exist a unique state \(u_i\) in \(Y\) which generates the substring \(t_i\) through all \(Q_i\) are generated, and there must exist a unique state \(v_j\) in \(Y\) which generates the substring \(t_j\) through all \(Q_j\) are generated. This implies that \(\text{gen}(s_k) \subseteq \text{gen}(u_i)\) and also that \(\text{gen}(s_l) \subseteq \text{gen}(v_j)\). Without loss of generality, say that condition (2) holds also for \(X\) (given an \(X\) for which this is not true, we may simply construct one by merging identical prefix paths). If there exists a string \(r \in \text{gen}(v_j)\) not in \(\text{gen}(s_k)\), then the string \(t_i \parallel r\) is in \(C\) but cannot be generated by \(X\) by condition (2); the same argument holds for strings in \(\text{gen}(u_i)\). Indeed, then, it is the case that \(\text{gen}(v_j) = \text{gen}(u_i)\), therefore that there exist \(a\) and \(b\) such that \(D_a \cong D_b\) by Lemma \[3.4\] a contradiction.

Q.E.D.
Theorem 3.1 (Validity of Scheme 1). Scheme 1 is a valid \((0, \log n, 0)\)-CCSS.

Proof:

\textbf{0-correctness} We demonstrate that scheme 1 is 0-correct. By lemmas 3.1, 3.2, and 3.3 at the end of stage 1 of Construct, \(D\) has a directed tree topology and has exactly one path from root to leaf which outputs one unique string in \(C_i\). In stage 2 of construct, we order this tree in decreasing order of the depth of states in \(D\). For each state from root to leaf, we then in stage 3 merge isomorphic components (which therefore generate the same sets of strings). Because only states generating the same sets of strings are merged, \(D\) preserves both the number of paths and the set of strings generated from the root (the corpus set).

So long as the string \(c_i\) given to \(Encode\) is a string in \(C\), there exists a path in \(D\) returned by \(Encode\) which outputs \(c_i\). Further, by the construction of \(D\), there is exactly one path through \(D\) for every prefix of any string, and so the procedure given in lines 5-10 of \(Encode\) will generate \(x\) as the sequence of non-unconditional inputs needed to generate \(c_i\). It follows directly, then, that running \(D\) on \(x\) (and reading \(\perp\) at the end of \(x\)) will yield \(c_i\) without any reconstruction error. It thus holds that

\[
\forall c_i \in C, \frac{\text{ham}(\text{Decode}(D, \text{Encode}(D, A, c_i)), c_i)}{|c_i|} \leq \epsilon
\]

\textbf{[log n]-compression} By lemmas 3.1, 3.2, and 3.3 and as we have shown in our demonstration of 0-correctness, the schematic returned by Construct has exactly \(n\) paths. There can therefore be at most \([\log n]\) junctions at which there is more than one transition. At any junction, one of the following must be true:

1. There is an unconditional transition and a \(\perp\) transition. Then all that is needed is a single unary signal indicating to continue. (Include an additional 1 in the input to continue; else end the input where it is.)

2. There is a 0-transition and a 1-transition. Then all that is needed is a single binary signal indicating which transition to take.

Since in both cases each junction requires only a one-bit indicator, we conclude that

\[
\forall c_i \in C, |\text{Encode}(D, c_i)| \leq \log n
\]

\textbf{0-minimality} Note in the specification of Construct that, for every unique prefix among strings in \(C\), \(D\) has a unique path from \(S_0\) to some node \(x\) which generates that prefix at the end of stage 1. Thus, for \(D\) (at the end of stage 1), there do not exist \(w\) and \(x\) such that a path from \(S_0\) to \(w\) has the same output as one from \(S_0\) to \(x\). Because stage 3 does not add new states or cause any state to generate a new set of strings (by the argument given in our discussion of 0-correctness), we conclude that the final schematic \(D\) returned maintains this property.

We also claim that the final schematic \(D\) returned does not contain any distinct states \(a\) and \(b\) such that \(\text{gen}(a) = \text{gen}(b)\). We show this by induction on the depth of states. Base case: consider any two distinct states \(a\) and \(b\) with depth greater than or equal to \(m\), the maximum depth of any state in \(D\). \(a\) and \(b\) are necessarily leaves, thus also necessarily \(\perp\)-output states. They will both thus have \(id = 1\) || \(\emptyset\) || \(\emptyset\) || \(\emptyset\) in stage 3, line 39, and would have been merged. As an inductive hypothesis, assume that there are no distinct pairs of nodes \(x\) and \(y\) at depth greater than or equal to \(k\) such that \(\text{gen}(x) = \text{gen}(y)\). Say that there exists a pair \(a\) and \(b\) at or below depth greater than or equal to \(k - 1\) such that \(\text{gen}(a) = \text{gen}(b)\). Then \(\text{gen}(a(0)) = \text{gen}(b(0))\), \(\text{gen}(a(1)) = \text{gen}(b(1))\), and \(\text{gen}(a(\perp)) = \text{gen}(b(\perp))\). This implies that the successors of \(a\) respectively generate the same set of strings as the successors of \(b\). But since all of these successors are at depth greater than \(\geq k\), they cannot be distinct by the inductive hypothesis. Therefore, where \(o\) is the output symbol of \(a\) and \(b\), \(id = o || s(0) || s(1) || s(\perp)\) will be equal for these nodes, and so they would therefore be merged in stage 3 of Construct. We note that the procedure given in stage 3 enforces the invariant of this inductive argument by processing states in decreasing depth order.
We've seen thus far that Scheme 1 is a (0, [log \(n\)], 0)-CCSS. From Theorem 2.1, we know that the minimum possible code length for any corpus set and any CCSS is is bounded from below by \(\log(n + 1) - 1\). By giving Scheme 1 with \(p(n, z) = [\log n]\), we have therefore provided a CCSS admitting a streaming code length only additive factor of \([\log n] - (\log(n + 1) - 1) \leq \log\frac{n}{n+1} + 2\) from optimal. Noting that this factor tends towards 2 for large \(n\) and is strictly less than 2 for all other \(n\), we also see that this scheme is nearly optimal with respect to streaming code length in a very strong sense.

The remainder of this section provides and illustrates a worked example of Scheme 1 in order to motivate questions surrounding how to extend the scheme. Consider the corpus set consisting of the vowels in the English alphabet (excluding y):

\[C = \{a, e, i, o, u\}\]
For the sake of exposition, say that each of these vowels are given using a naive alphabetical encoding where any letter is represented by its ordinal position in the English alphabet. We may view our corpus set now as

\[
C = \{ a = 1 = 000001, \\
e = 5 = 000111, \\
i = 9 = 001011, \\
o = 15 = 001111, \\
u = 21 = 010101 \}
\]

Let us say now that we run \textit{Construct}(C) (for the \textit{Construct} procedure of Scheme 1) according to the given pseudo-code. We depict below the Moore machine schematic that would be obtained via this procedure:

![Figure 1: Schematic for C, Output of Scheme 1 Construct](image)

As can be seen in Figure 1 above, the schematic for \( C \) in Scheme 1 is an acyclic Moore machine in 11 states. As promised by the validity of scheme 1 as a \((0, \lceil \log n \rceil, 0)\)-CCSS, we have that we may now convey vowels in less than or equal to \( \lceil \log n \rceil = 3 \) bits according to the following stream encoding:

\[
\begin{align*}
a : & \ 000 \\
e : & \ 001 \\
i : & \ 010 \\
o : & \ 011 \\
u : & \ 1
\end{align*}
\]

As we see in the case of this example, scheme 1 portrays the properties of a CCSS that we desired in order to address the Spotify problem: we see a 40\% reduction in the bandwidth required to express vowels under the given naive encoding without needing to explicitly store the encodings for all vowels.

The reader may perhaps have noticed at this point that there are yet states in \( \mathbb{1} \) that may be merged to obtain a smaller schematic whilst maintaining the acyclicity property of the restriction. We illustrate such a merge below:
Figure 2: Modified Schematic

Figure 2 shows that we may merge two states to obtain a smaller schematic (9 states). Note, however, that this reduction in size increases the number of paths through the machine, meaning that stream encodings must convey more information. In particular, notice that the above schematic contains a path encoding $42 = 0101001$, not a vowel (not even a letter in our naive encoding); indeed, this schematic no longer satisfies $\log n$-compression, as the longest encoding required for any vowel increases from 3 to 5:

$$
\begin{align*}
 a & : 000 \\
 e & : 00101 \\
 i & : 010 \\
 o & : 011 \\
 u & : 1101
\end{align*}
$$

While it is certainly true that this schematic no longer satisfies the definition of a $(0, \log n, 0)$-CCSS, this modification shows the benefit of increasing maximum streaming code length from the near-optimal point of $\log n$: by doing so, we may reduce the size of the schematic. This phenomenon motivates our study of the case where $p(n, z) > \log n$.

### 3.2.2 On the Hardness of Maintaining Minimality for a $p(n, z) > \log n$ AMM-restricted CCSS

The previous section portrays very clearly the advantage of increasing maximum streaming code length from the near-optimal point of $\log n$: by doing so, we may reduce the size of the schematic. Though this is an attractive prospect, we will show that this is in fact NP-hard to do while maintaining strict schematic minimality in virtually all interesting cases. As in the remainder of this work, we are considering strictly 0-correct CCSS constructions.

Specifically, we will show that (i) it is NP-hard to give an AMM-restricted CCSS maintaining 0-minimality for unbounded maximum code length $p(n, z)$ and the stronger result that (ii) it is NP-hard to give an AMM-restricted CCSS maintaining 0-minimality for maximum code length $p(n, z) = \frac{z}{\beta}$ for any fixed $\beta \geq 1$. We will then extend our discussion to consider the maintenance of $\delta$-minimality for general $\delta > 0$, formulating a manner in which to relax CCSS constraints to allow us to give CCSS constructions which still perform well in practice.

**Theorem 3.5** (AMM-restriction Hardness for Unbounded Streaming Codes). It is NP-hard to give a 0-minimal, 0-correct AMM-restricted CCSS with unbounded streaming code length ($p(n, z) = \infty$).
Proof: We show the hardness of this problem via reduction from the well-known NP-hard shortest common supersequence (SCS) problem for binary alphabets (shown to be NP-complete in [5]). The SCS problem is as follows: given a set of $n$ strings composed of letters from a fixed (binary, in our case) alphabet, determine the shortest possible string $s$ such that each string in the input set is a subsequence of $s$.

The reduction from SCS to AMM-restricted $(0,\infty,0)$-CCSS is direct. Say that we have a $(0,\infty,0)$-CCSS $S = (\text{Construct}, \text{Encode}, \text{Decode})$. Consider now an instance $I$ of the SCS problem: $I = \{c_1, ..., c_n\}$. As this notation implies, take now $I$ as our corpus set $C$ and run $\text{Construct}(C = I)$ to obtain an AMM schematic $D$.

By the definition of an AMM-restricted CCSS, $D$ is a Moore machine whose underlying directed graph is acyclic. Since this graph is acyclic, we may enumerate the states of $D$ in topological order $O$. Take $s$ as the string formed by taking the output symbols of each 1- or 0-state in the same order as $O$. We claim now that $s$ is a shortest common supersequence of the original instance $I$, and we show this in two parts:

1. $s$ is a supersequence of all strings in $I$. Take any string $c_i \in I$. Since $D$ is an AMM schematic for $C = I$, there exists a path of states through which $D$ outputs $c_i$. The $j$th state along this path outputs the $j$th bit of $c_i$, and the edges along the path which outputs $c_i$ must obey the topological ordering, meaning there is a subsequence of states in the topological ordering $O$ which outputs $c_i$. Taking $s$ as defined, there is therefore a subsequence of bits in $s$ which is equal to $c_i$; $s$ is therefore a supersequence of $c_i$.

2. $s$ is as short as any other supersequence of the strings in $I$. Assume not. Then there exists a string $s'$ shorter than $s$. We will use this string $s'$ to construct a new Moore machine $D'$: for each bit in $s'$, introduce a new state which outputs the same bit. Next, for each string $c_i \in I$, create a path through these states by adding transitions between states $E_j$ and $E_{j+1}$, respectively corresponding to the $j$th and the $j+1$th bits of $c_i$, for all $j = 1, ..., |c_i| - 1$. By adding start and end states with the special symbol $\perp$ according to the convention we’ve seen thus far, we obtain a valid Moore machine $D'$ which is a valid schematic for $I$. But $D'$ has as many states as are bits in $s'$, therefore fewer states than bits in $s$ and thus fewer states than $D$, contradicting the 0-minimality of our scheme.

\[ \square \]

Theorem 3.6 (AMM-restriction Hardness for $\frac{z}{\beta}$-bounded Streaming Codes). It is NP-hard to give a 0-minimal, 0-correct AMM-restricted CCSS with streaming code length bounded by $p(n, z) = \frac{z}{\beta}$ for any fixed $\beta \geq 1$.

Proof: Our proof relies on a result by Jiang and Li in [3] which states that it is NP-hard to approximate the SCS problem with a constant approximation ratio. In particular, we show that the existence of a $(0, \frac{z}{\beta}, 0)$ AMM-restricted CSS yields a polynomial-time $(\beta + 1)$-approximation algorithm for the SCS problem for any $\beta \geq 1$; the truth of this theorem will then follow from Jiang and Li’s result.

Say that we have a $(0, \frac{z}{\beta}, 0)$ AMM-restricted CSS $S$ for some $\beta \geq 1$. Consider an instance $I = \{c_1, ..., c_n\}$ of the SCS problem (for binary alphabets). We use $S$ to approximate the shortest common supersequence of $I$ using the following polynomial-time method:

1. Add to $I$ an all-0 string of polynomial length $\beta(z \log_2 \sum_{i=1}^{n}|c_i| + 1)$, where $z$ is (as before) the length of the longest string in $I$, obtaining a new set of strings $I'$.

2. Run $\text{Construct}(I')$ to obtain an AMM schematic $B$.

3. Obtain a supersequence $s$ of the strings in $I$ by using the topological ordering method from Theorem 3.5 except this time traversing only the states which are along the output path of some string in the original set $I$. Return $s$. 

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By the proof of Theorem 3.5, we know that $s$ is at least a supersequence of all the strings in $I$. All that remains to show is the optimality gap between $s$ and the true shortest common supersequence of $I$. Note that the states of the AMM schematic $B$ may be partitioned into a group of states which (a) are along the output path for some $c_i \in I$ but not the path for the string added to obtain $I'$ or (b) are along the output path for the long string added to $I$. Let the former be $D$, and let the latter be $Z$. Trivially, we have $|B| = |\hat{D}| + |Z|$.

Note now that the length of $s$ corresponds to the number of states enumerated in the topological ordering step: this is equal to all states along an output path for some $c_i \in I$, including those also along the output path for the added string, $D$: $|\hat{D}| = |D| + |D \cap Z|$. The optimal SCS $s^*$ corresponds (by our proof of Theorem 3.5) to the number of states in the would-be schematic of a AMM-restricted $(0, \infty, 0)$-CCSS, $D^*$ given by $\text{Construct}(I)$.

Because our CCSS is 0-minimal, we know that $|B|$ is minimized; additionally, since $|Z|$ is fixed, we know that $|\hat{D}|$ is minimized. We note now that $D^*$ may be converted to an AMM schematic for $B'$ by simply adding one state per bit of the added string. This is true because each state in $D^*$ has at most $|D^*| \leq \sum_{i=1}^{n} |c_i|$ transitions, meaning that the streaming code gives at most $\log_2 \sum_{i=1}^{n} |c_i|$ bits of information per transition, therefore that, for the minimum $(0, 0, 0)$ schematic, at most $\log_2 \sum_{i=1}^{n} |c_i|$ bits are needed to represent any string: since our effective $z$ in $I'$ is $\beta(\log_2 \sum_{i=1}^{n} |c_i| + 1)$, $z/\beta$ (our maximum encoding length) is strictly larger than this quantity. Moreover, adding 0-states as mentioned creates exactly one additional choice at the start node (hence the +1 term), and the resulting schematic $B'$ for $I'$ intersects with precisely no states along the output path for strings in $I$ in $D^*$.

To continue, given that $D^*$ may be used to obtain an AMM schematic for $I'$ in which the states along the output paths of strings in $I$ intersect with no states along the output path corresponding to the added string, we know that $|\hat{D}| \leq |D^*|$ by the 0-minimality of our CCSS, thus that $|\hat{D}| + |\hat{D} \cap Z| \leq |D^*| + |D \cap Z|$. Furthermore, because $|Z| = z\beta \leq |D^*|$, we can state $|\hat{D}| + |\hat{D} \cap Z| \leq |D^*|(\beta + 1)$. Applying this, we see then that the approximation ratio of this algorithm, given by

$$\frac{|\hat{D}| + |\hat{D} \cap Z|}{|D^*|}$$

is in fact

$$\leq \frac{|D^*| (\beta + 1)}{|D^*|} = \beta + 1$$

$\square$.

3.2.2.1 Considering $\delta > 0$-Minimality

We saw in Theorems 3.5 and 3.6 that it is NP-hard to maintain 0-minimality in an AMM-restricted CCSS when the streaming code length is either unbounded or bounded as a constant fraction of the length of the longest string in a corpus set. A natural question to ask now is this: does the problem remain hard when we relax the $\delta$ parameter? I.e., is it hard to maintain $\delta$-minimality when varying code length bound?

We give a result in this section which indicates that the answer to this question is yes. We specifically rely on yet another result from [3] to show that there exists a natural floor to the maximum value of $\delta$ below which unexpected complexity-theoretic results are implied.

**Theorem 3.7** (Minimality Floor of AMM-restricted Schemes). For every $\beta \geq 1$, there exists an $\alpha$ such that the existence of a $(\log^\alpha n, \frac{\delta}{2}, 0)$-CCSS implies that $NP \not\subseteq \text{DTIME}(2^{\log^\alpha n})$.

**Proof:** Our proof relies on the following result of Jiang and Li [3]: there exists a constant $\gamma > 0$ such that, if SCS has a polynomial-time approximation algorithm with ratio $O(\log^\gamma n)$, then $NP \not\subseteq \text{DTIME}(2^{\log^{\log(n)}})$. We then need only to show that, for generic $\beta$, the existence of a $(f(n), \beta, 0)$-CCSS implies the existence of an $O(f(n) + \beta)$-ratio SCS approximation algorithm. The theorem then follows directly from the result of Jiang and Li.
That a \((f(n), \beta, 0)\)-CCSS yields an \(O(f(n) + \beta)\)-ratio SCS approximation algorithm follows almost exactly according to the argument given in the proof of Theorem 3.6 with the following exception: where \(\bar{D}\) is the set of states along the output paths of the strings of \(I\) and \(D^*\) is the set of states in the minimal AMM schematic, we use the fact that \(|\bar{D}| \leq |D^*|\) to instead obtain that \(f(n)|\bar{D}| \leq f(n)|D^*|\). Introducing \(D'\) as the corresponding set of states for the schematic returned by the \(f(n)\)-minimal scheme, we can (using the same strategy in the proof of Theorem 3.6) show that \(|\bar{D}'| + |\hat{D}' \cap Z| \leq f(n)|D^*| + \beta|D^*|\). The approximation ratio is immediate.

3.2.2.2 Relaxing the Problem: from \(f(\cdot)\)-Minimality to \(E[f(\cdot)]\)-Minimality

This section overall has shown that analysis of AMM-restricted CCSS constructions through a strict interpretation of minimality reveals very unyielding hardness results. Theorems 3.5 and 3.6 show us that it is hard to give CCSS constructions bounding code length as a fraction of maximum string length; indeed, this hardness will present itself once more if we instead consider code-length bounds \(p(n,z)\) as a function of \(n\) on instances where \(n << z\), and so the prospect of exploring this possibility seems equally grim. Theorem ?? shows us that there is a poly-logarithmic minimality floor under which the prospect of giving efficient schemes seems unlikely; this is especially daunting when we consider that, in practice, we would like to see \(\delta\)-minimality for small, constant values of \(\delta\), e.g. 1.25 or 1.5.

More positively, however, the structure of the problem of giving AMM-restricted CCSS constructions and how closely related it seems to the SCS problem does show some promise for relaxations which still allow resulting constructions to be useful in practice. In particular, [3] notes that simple greedy algorithms for the SCS problem yield solutions which are bad in the worst case but are nearly optimal in the expected case. Toward the end of exploiting this similarity, we suggest the exploration of a new class of AMM-restricted CCSS constructions: constructions in which minimality is viewed in the expected case. Where we previously were married to the idea of giving an AMM-restricted \((f(\cdot), \cdot, \cdot)\)-CCSS which is strictly \(f(\cdot)\)-minimal in all cases, we suggest considering AMM-restricted \((E[f(\cdot)], \cdot, \cdot)\) schemes which is \(f(\cdot)\)-minimal in the expected case. If the structural similarities with SCS hold to this point, it may be possible that greedy approaches perform well in the expected case.

Due to the scope of this report and in the interest of being able to touch a broader range of issues related to MM-restricted CCSS constructions in general, we leave both the formulation of \(E[f(\cdot)]\)-minimality and the presentation of algorithms fitting its definition open.

3.3 Restricting Schematics to General Moore Machines

The reader may recall from section 3.2.1.3 that we saw a method of reducing AMM schematic size by merging states and introducing new paths. In the rest of this report, we do away with the acyclicity requirement. Now that we have the ability to introduce cycles into the graph underlying our schematics, we may in fact reduce schematic size even further. Anecdotally, we illustrate how the addition of a single self-loop reduces the size of the schematic depicted in Figure 2 from 9 to 7:
In truth, the ability to include cycles in our schematics does more than give us the ability to reduce schematic size: not even the same hardness properties no longer hold. For example, Theorem 3.5 showed that schematic minimization is NP-hard for unbounded code lengths in the AMM-restricted case; the relaxed MM-restriction contradicts this directly, as invariably something similar to one of the following schematics will be optimal:

![Figure 4: Cyclic Schematic for Degenerate Case, Unbounded Code Length](image)

Above, we see that an NP-hard instance of CCSS construction problem in the AMM restriction becomes the degenerate case for the general MM restriction. Owing to potentially significant savings in schematic size and this significant gap in hardness, we in this section begin to explore the prospect of MM-restricted CCSS constructions. In the interest of adhering to the expected scope of this report, this exploration is limited to an informal presentation of an equivalent formulation meant to motivate future study of the problem.

### 3.4 An Equivalent Formulation of the Construction Problem

We denote by the *construction problem* the task of designing the *Construct* algorithm of a CCSS. In this section, we informally present an equivalent formulation of the MM-restricted construction problem as a bicriteria partitioning problem in directed graphs. We then use this formulation as a basis of discussing directions for continued work.

As stated, we equivalently pose the MM-restricted construction problem as a partitioning problem in directed graphs. The input for such a problem is a directed acyclic graph $F = (V, E)$, an associated labeling function $L(u)$ which maps each node in $V$ to a color in $\{0, 1\}$, a natural number parameter $k$, a natural number parameter $\tau$, and a set of *trace-paths* $P_1, ..., P_m$ through $F$. The output desired by this problem is a disjoint monochromatic (as in each partition is monochromatic) $k$-partitioning $G_1, ..., G_k$ of $F$ which satisfies $\tau$ in the following specific sense:
• For each trace path $P_i$, define the partition-path of $P_i$, $p(P_i)$, to be the sequence of partitions reached along the path $P_i$.

• For each partition $G_i$, define the degree of $G_i$, $\text{deg}(G_i)$ to be the number of distinct partitions $G_j$ such that $G_j$ is the successor of $G_i$ along some partition path.

• We say that a partition $G_1, \ldots, G_k$ satisfies $\tau$ if and only if $\max_{P_i} \sum_{j=1}^{\lfloor \log \text{deg}(p(P_i)[j]) \rfloor} \leq \tau$.

We omit a formal proof of the correspondence between this formulation and the construction problem for the sake of maintaining the scope of this report, but we summarize it informally. In the above formulation, $k$ corresponds to the number of states in a schematic, and the parameter $\tau$ corresponds to the maximum streaming code length. The quantity which must satisfy $\tau$ corresponds precisely to the length of the longest streaming code. Thus, to give an MM-restricted $(0, p(n, z), 0)\text{-CCSS}$, we would solve the given problem for the minimum $k$ yielding a partition satisfying $p(n, z)$. Again, we note that we leave the formal proof of this correspondence open.

This formulation is useful because it gives us a natural framework in which to develop schematic minimization algorithms, but it is also useful for the sake of analyzing hardness as we did for the AMM-restricted schemes. We can see this, for example, even in an ability to view hardness of the AMM-restricted case through the lens of this formulation: if we impose a partial acyclicity constraint, namely dictating that there must not exist edges with start- and end-points in the same partition, the above formulation reduces to the problem of determining the chromatic number of an arbitrary undirected graph (when $\tau$ is unbounded).

Focusing once more on the MM-restricted case, our hope is that this formulation makes it possible to analyze the limits of MM-restricted schemes in a natural way. We leave the problem of this analysis open, but we note that other similar partitioning problems [2] are both hard to solve exactly and approximately. We additionally hope that this formulation simplifies the task of developing specific MM-restricted CCSS constructions, but we leave also this problem open.

4 Future Work

We have in this work explored the surface of corpus-compressed streaming as a solution to the Spotify problem, substantiating a particular strategy which utilizes regular function automata as schematics. Even with respect to this strategy, our concrete results are limited to the development of a single AMM-restricted $(0, \lceil \log n \rceil, 0)\text{-CCSS}$ and a few hardness results for this specific restriction. While our discussion has touched on extensions of this strategy, our work nonetheless leaves open quite a few questions which seem worthwhile to explore in the future. We outline these areas for future work below.

• **Lossy Reconstruction** All of our schemes, hardness results, and discussion thus far have been restricted to consideration of lossless schemes having the $\epsilon$ parameter fixed at 0. Future work could consider developing new lossless CCSS constructions, extending the AMM-restricted scheme we’ve developed to make strategic use of loss, or even seeking to extend or refine our hardness results in the lossless case.

• **Expected Minimality** In our presentation of results for AMM-restricted schemes, we noted the possible utility of developing a notion of expected-case minimality rather than strict minimality of a CCSS. Future work could look at giving a precise formulation of this notion and, moreover, using it to construct practically useful schemes with $p(n, z)$ varied away from $\lceil \log n \rceil$.

• **Taking Other Parameters in Expectation** Related to the previous point, it seems natural to also consider the benefit of also taking the maximum streaming code length and loss parameters in expectation.

[2] Indeed, the problem which corresponds exactly requires a slight modification in that the input graph must support having a constant number of unlabeled nodes, but we leave this out for ease of exposition.
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• **MM-restricted Schemes**  The last section gave an overview of the benefits of studying MM-restricted constructions. Future work in this area could look at formally proving properties of these constructions (potentially using the equivalent formulation we have provided) or, perhaps more importantly, developing MM-restricted schemes.

• **Schemes from Other Function Automata**  This work has focused exclusively on constructing schematics from regular function automata. While the hardness results we found at this level of the hierarchy may be a deterrent from any attempt at operating at a higher level, studying what happens when we do could nonetheless be worthwhile.

• **Other Restriction Constraints**  Related to the previous, it would of course be worthwhile to consider other means of restricting schematics independent of language considerations. One particular area of interest could be exploring restrictions of schematics to auto-encoders with specific properties.

• **Practical Evaluation of Schemes**  Outside of the realm of theory, one major area of future work concerns the implementation and evaluation of CCSS constructions on real-world data and in real-world environments.

5 Conclusion

In this work, we have defined the **Spotify problem** and explored a potential solution in the form of a new algorithmic goal which we refer to as **corpus-compressed streaming schemes**. After formally substantiating the notion of a corpus-compressed streaming scheme, we explored a specific strategy of constructing them based upon a Kolmogorov-like use of regular function automata as an ‘almost self-extracting’ archive. Following a presentation of results including a concrete, nearly optimal scheme (under a specific restriction) and hardness properties, we further motivate and outline opportunities for future work in this area.

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