The existence of eigenvalues of Schrödinger operator on a lattice in the gap of the essential spectrum

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Abstract. We consider a three-particle discrete Schrödinger operator $H_{\mu,\gamma}(K)$, $K \in T^3$ associated to a system of three particles (two fermions and one different particle) interacting through zero range pairwise potential $\mu > 0$ on the three-dimensional lattice $\mathbb{Z}^3$. It is proved that the operator $H_{\mu,\gamma}(K)$, $\|K\| < \delta$, for $\gamma > \gamma_0$ has at least two eigenvalues in the gap of the essential spectrum for sufficiently large $\mu > 0$.

1. Introduction
In models of solid state physics [1], [2] and also in lattice quantum field theory [3], discrete lattice operators are considered which are lattice analogs of the three-particle Schrödinger operator in the continuum.

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions [4].

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms.

The dynamics of the ultracold atoms loaded in the lower or upper band is well described by the Bose-Hubbard Hamiltonian [4]; we give in section 3 the corresponding Schrödinger operator.

In the continuum case due to rotational invariance the Hamiltonian separates in a free Hamiltonian for the center of mass and in a Hamiltonian $H_{\text{rel}}$ for the relative motion. Bound states are eigenstates of $H_{\text{rel}}$.

The kinematics of the quantum particles on the lattice is rather exotic. The discrete laplacian is not rotationally invariant and therefore one cannot separate the motion of the center of mass.

The existence of eigenvalue of the three-particle Schrödinger operator $H_{\mu}(K) = H_0(K) - \mu V$ ($\mu$ is arbitrary) for dimensions $d = 1, 2$ is shown in the works [5], [6], the proofs of which are based on unboundedness of the norm of the Faddeev type operator $T(K, z)$ for $z$ spectral parameter close to the lower bound of the essential spectrum. If $d \geq 3$ then the operator $T(K, z)$ is bounded at the bottom of the essential spectrum, i.e., in this case the methods for $d = 1, 2$ are not applicable.

Efimov’s effect (i.e., infinity of the number of eigenvalues) of the three-particle Schrödinger operator $H_{\mu}(K)$ in a three-dimensional lattice for $K = 0$ and $\mu = \mu_0$, (see [7], [8], [9], [10]) and
finiteness of the number of the eigenvalues of the operator for non-trivial value of $K \in U_3(0)$ are given in [11]. In [12] is consider a system of three arbitrary quantum particles on a three-dimensional lattice, interacting with the help of pair contact potentials attraction. The condition for the appearance of a gap in the essential spectrum was found and the existence of an infinite number of eigenvalues in this gap was proved for the Hamiltonian of the corresponding system of three particles for $K = 0$. The existence and the number of the eigenvalues of the three particle discrete Schrödinger operator for $d \geq 3$ and sufficiently large $\mu$’s has not been investigated.

In [13] was investigated a model operator $H_\gamma$ associated with the three-particle discrete Schrödinger operator on the three-dimensional cubical lattice, with zero-range pair potentials. There were proved that there is a value $\gamma$ of the parameter such that only for $\gamma < \gamma^*$ the Efimov effect is absent for the sector of the Hilbert space which contains functions which are antisymmetric with respect to the two identical particles. Moreover, also were established an asymptotic for the number $N(\varepsilon)$ of eigenvalues below $\varepsilon < E_{\min}$ for $\gamma > \gamma^*$.

We remark that in the case of an antisymmetric wave function, the number $\gamma$ is a critical value for the mass ratio, where the Efimov effect is present or absent. Interestingly, the case of three fermions, two identical and different from the third one, was also considered from a more physical point of view by Petrov [14] and he also found a critical value for the mass ratio that allows or forbids the Efimov effect.

In the present work, in the system of three particles (two fermions and one different particle), there exists a critical value $\gamma_0$ of the mass ratio such that for $\gamma > \gamma_0$ the operator $H_{\mu,\gamma}(K)$ has at least two eigenvalues in the gap of the essential spectrum for sufficiently large $\mu$.

2. Representations of Hamiltonians associated to systems of two and three particles on a lattice. Statement of the main result

Let $Z^3$ be the three dimensional lattice. Let $\ell^2([Z^3]^m)$, ($m = 2, 3$) be a Hilbert space of square-summable functions $\varphi$, defined on the Cartesian product $([Z^3])^m$, and $\ell^2 antisym([Z^3]^m) \subset \ell^2([Z^3]^m)$ a subspace of antisymmetric functions with respect to the permutation of the first two coordinates.

We consider a Hamiltonian of a system of three quantum particles (two fermions and one different particle with masses $m_1 = m_2 = 1$ and $m_3$, respectively) interacting via zero-range attractive pair potentials on $Z^3$. Without loss of generality, we assume that the first and second particles are fermions, and the third is a particle of different nature. Since the Hamiltonian corresponding to system of fermions act in the subspaces of antisymmetric functions with respect to the permutation of two variables, there is no zero-range two-particle interaction of the first two particles in our system is absent (see, [6, 15]).

Energy operator of a system of two free arbitrary particles (fermion and different particle) on $Z^3$ in coordinate representation is associated with bounded self-adjoint operator $\hat{h}_{0,\gamma}$ in $\ell^2([Z^3]^2)$:

$$\hat{h}_{0,\gamma} = -\frac{1}{2}\Delta \otimes I - \frac{\gamma}{2} I \otimes \Delta,$$

where $\Delta$ is lattice Laplacian, $I$ is identity operator in $\ell^2([Z^3])$ and $\gamma = \frac{1}{m_3}$.

The total Hamiltonian $\hat{h}_{\mu,\gamma}$ of a system of two arbitrary particles with zero-range interaction acts in $\ell^2([Z^3]^2)$ and is bounded perturbation of the free Hamiltonian $\hat{h}_{0,\gamma}$:

$$\hat{h}_{\mu,\gamma} = \hat{h}_{0,\gamma} - \mu \hat{v}.$$

Here $\mu > 0$ is the energy of interaction of two particles (a fermion and another particle), and the operator $\hat{v}$ describes zero-range interaction of these particles:

$$(\hat{v}\hat{\psi})(x_\beta, x_\gamma) = \delta_{x_\beta x_\gamma} \hat{\psi}(x_\beta, x_\gamma),$$

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and $\delta_{\alpha\beta}$ is the Kronecker delta.

Similarly, the free Hamiltonian $\hat{H}_{0,\gamma}$ of a system of three particles (two fermions and one different particle) on $\mathbb{Z}^3$ is defined in $\ell^2_{as}[(\mathbb{Z}^3)^3]$ according to the formula:

$$\hat{H}_{0,\gamma} = -\frac{1}{2} \Delta \otimes I \otimes I - \frac{1}{2} I \otimes \Delta \otimes I - \frac{\gamma}{2} I \otimes I \otimes \Delta.$$  

The total Hamiltonian $\hat{H}_{\mu,\gamma}$ of a system of three-particles with the pairwise zero-range interaction is a bounded perturbation of the free Hamiltonian $\hat{H}_{0,\gamma}$:

$$\hat{H}_{\mu,\gamma} = \hat{H}_{0,\gamma} - \mu(\hat{V}_1 + \hat{V}_2),$$

where $\hat{V}_\alpha$, $\alpha = 1, 2$ multiplication operators are defined by

$$\hat{V}_1 \psi(x_1, x_2, x_3) = \delta_{x_2x_3} \psi(x_1, x_2, x_3),$$

and

$$\hat{V}_2 \psi(x_1, x_2, x_3) = \delta_{x_3x_1} \psi(x_1, x_2, x_3).$$

Let $T^3$ be a three-dimensional torus, $L^2_2([(T^3)^2] \subset L^2_2([(T^3)^2]$ be Hilbert space of square-integrable functions, defined on $(T^3)^2$ and antisymmetric with respect to the permutation of the coordinates. The study of the spectra of operators $\hat{h}_{\mu,\gamma}$ and $\hat{H}_{\mu,\gamma}$ is reduced to the study of the spectra of families of operators $h_{\mu,\gamma}(k), k \in T^3$ and $H_{\mu,\gamma}(K), K \in T^3$, respectively (see [6], [8]). On the torus $T^3$ is selected a unit measure $d\mathbf{p}$, i.e. $\int_{T^3} d\mathbf{p} = 1$.

Two-particle discrete Schrödinger operator

$$h_{\mu,\gamma}(k) = h_{0,\gamma}(k) - \mu v$$  \hspace{1cm} (1)

acts in $L_2(T^3)$, where

$$(h_{0,\gamma}(k)f)(\mathbf{p}) = \varepsilon_{k,\gamma}(\mathbf{p})f(\mathbf{p}), \quad \varepsilon_{k,\gamma}(\mathbf{p}) = \varepsilon(\mathbf{p}) + \gamma \varepsilon(\mathbf{k} - \mathbf{p}),$$

$$(vf)(\mathbf{p}) = \int_{T^3} f(\mathbf{s}) d\mathbf{s}.$$ 

The corresponding three-particle Schrödinger operator

$$H_{\mu,\gamma}(K) = H_{0,\gamma}(K) - \mu(V_1 + V_2)$$

acts in $L^2_2([(T^3)^2], where

$$(H_{0,\gamma}(K)f)(\mathbf{p}, \mathbf{q}) = E_{K,\gamma}(\mathbf{p}, \mathbf{q})f(\mathbf{p}, \mathbf{q}), \quad E_{K,\gamma}(\mathbf{p}, \mathbf{q}) = \varepsilon(\mathbf{p}) + \varepsilon(\mathbf{q}) + \gamma \varepsilon(\mathbf{K} - \mathbf{p} - \mathbf{q}),$$

$$(V_1 f)(\mathbf{p}, \mathbf{q}) = \int_{T^3} f(\mathbf{p}, \mathbf{s}) d\mathbf{s}, \quad (V_2 f)(\mathbf{p}, \mathbf{q}) = \int_{T^3} f(\mathbf{s}, \mathbf{q}) d\mathbf{s}.$$ 

Here $\varepsilon(\mathbf{p}) = 3 - \xi(\mathbf{p}), \xi(\mathbf{p}) = \sum_{i=1}^3 \cos p_i, p = (p_1, p_2, p_3) \in T^3, \mu > 0$ is the energy of interaction of two particles, $\gamma > 0$ - ratio of masses of the fermion and the different particle. 

Let

$$\gamma_0 = \left(\int_{T^3} \cos^2 s_1 - \cos s_1 \cos s_2 \frac{ds}{3 + \xi(s)} d\mathbf{s}\right)^{-1} \approx 5.39.$$

Our main results of the work are the following:

**Theorem 1.** Let $\gamma > \gamma_0$. Then there exist $\mu_\gamma > 0$ and $\delta > 0$ such that, for any $\mu > \mu_\gamma$ and $K$ satisfying $\|K\| < \delta$ the operator $H_{\mu,\gamma}(K)$ has at least two eigenvalues in the gap of the essential spectrum.
3. The spectrum of a two-particle operator \( h_{\mu,\gamma}(k) \)

According to Weyl's essential spectrum theorem [16], essential spectrum \( \sigma_{\text{ess}}(h_{\mu,\gamma}(k)) \) of the operator \( h_{\mu,\gamma}(k) \) coincides with the spectrum \( \sigma(h_{0,\gamma}(k)) \) unperturbed operator \( h_{0,\gamma}(k) \), i.e. with segment \([E_{\text{min,}\gamma}(k), E_{\text{max,}\gamma}(k)]\):

\[
\sigma_{\text{ess}}(h_{\mu,\gamma}(k)) = [E_{\text{min,}\gamma}(k), E_{\text{max,}\gamma}(k)],
\]

where

\[
E_{\text{min,}\gamma}(k) = \min_{q \in \mathbb{T}^3} E_{k,\gamma}(q) = 3(1 + \gamma) - \sum_{i=1}^{3} \sqrt{1 + 2\gamma \cos k_i + \gamma^2},
\]

\[
E_{\text{max,}\gamma}(k) = \max_{q \in \mathbb{T}^3} E_{k,\gamma}(q) = 3(1 + \gamma) + \sum_{i=1}^{3} \sqrt{1 + 2\gamma \cos k_i + \gamma^2}.
\]

Note that the functions \( E_{\text{min,}\gamma}(k) \) and \( E_{\text{max,}\gamma}(k) \) are symmetric with respect to permutation of any two variables \( k_i \) and \( k_j \), are even for each \( k_i \in [-\pi, \pi], i = 1, 2, 3 \). The function \( E_{\text{min,}\gamma}(k) \) is strongly monotone increasing, and \( E_{\text{max,}\gamma}(k) \) is strongly decreasing in each \( k_i \in [0, \pi], i = 1, 2, 3 \). Therefore,

\[
\min_{k \in \mathbb{T}^3} E_{\text{min,}\gamma}(k) = E_{\text{min,}\gamma}(0) = 0, \quad \max_{k \in \mathbb{T}^3} E_{\text{max,}\gamma}(k) = E_{\text{max,}\gamma}(0) = 6(1 + \gamma).
\]

Let \( \Delta_{\mu,\gamma}(k, z), z \in \mathbb{C} \setminus [E_{\text{min,}\gamma}(k), E_{\text{max,}\gamma}(k)] \), determinant of the Fredholm operator \( I - \mu vr_{0,\gamma}(k; z), r_{0,\gamma}(k; z) \) is a resolvent of the unperturbed operator \( h_{0,\gamma}(k) \), \( v \) is an integral operator with the kernel \( v(q, q') = 1 \). The function \( \Delta_{\mu,\gamma}(k; z) \) has the form:

\[
\Delta_{\mu,\gamma}(k, z) = 1 - \mu D_{\gamma}(k, z), \quad D_{\gamma}(k, z) = \int_{\mathbb{T}^3} \frac{dz}{E_{k,\gamma}(q) - z}. \tag{2}
\]

Lemma 1. A number \( z \in \mathbb{C} \setminus [E_{\text{min,}\gamma}(k), E_{\text{max,}\gamma}(k)] \) is an eigenvalue of the operator \( h_{\mu,\gamma}(k) \) if and only if \( \Delta_{\mu,\gamma}(k, z) = 0 \).

Lemma 1 is proved analogously as Lemma 2.1 work [7].

The function \( D_{\gamma}(k, \cdot) \) is symmetric with respect to permutation of any two variables \( k_i, k_j \), is even for every \( k_i \in [-\pi, \pi] \) and strongly increasing for each \( k_i \in [0, \pi], i = 1, 2, 3 \). For any fixed \( z \leq 0 \) the function \( D_{\gamma}(\cdot, z) \) is a strongly decreasing for \( k_i \in [0, \pi], i = 1, 2, 3 \). Moreover, the equalities hold:

\[
\min_{k \in \mathbb{T}^3} D_{\gamma}(k, E_{\text{min,}\gamma}(k)) = \max_{k \in \mathbb{T}^3} D_{\gamma}(k, 0) = D_{\gamma}(0, 0) = \frac{1}{\mu_0},
\]

where \( \mu_0 = \left( \int_{\mathbb{T}^3} \frac{dz}{E_{k,\gamma}(q) + \gamma^2} \right)^{-1} \) is harmonic mean value of the kinetic energy of the fermion and other particle. The proofs of these statements follow from the properties of the cosine and monotonicity of the Lebesgue integral.

Theorem 2. Let \( \mu > \mu_0 \). Then for every \( k \in \mathbb{T}^3 \) the operator \( h_{\mu,\gamma}(k) \) has unique simple eigenvalue \( z_{\mu,\gamma}(k) \), below the essential spectrum.

Proof. The function \( D_{\gamma}(k, \cdot) \) is continuous and strongly increasing in each half line \((\infty, E_{\text{min,}\gamma}(k)) \) and \((E_{\text{max,}\gamma}(k), \infty) \). The continuity of the function \( D_{\gamma}(k, \cdot) \) follows from the continuity of the integrand, and monotonicity follows from the inequality \( \frac{\partial D_{\gamma}(k, z)}{\partial z} > 0 \). Since for any \( k \in \mathbb{T}^3 \) the function \( D_{\gamma}(k, \cdot) \) is monotone in \((\infty, E_{\text{min,}\gamma}(k)) \), the following finite limit

\[
\lim_{z \to E_{\text{min,}\gamma}(k)} D_{\gamma}(k, z) = D_{\gamma}(k, E_{\text{min,}\gamma}(k))
\]
Lemma 1 follows that the operator \( h_{\mu, \gamma}(k) \) has a unique zero in the interval \((-\infty, \mathcal{E}_{\min, \gamma}(k))\). Further, from the Lemma 1 follows that the operator \( h_{\mu, \gamma}(k) \) has a unique simple eigenvalue \( \mu_{\mu, \gamma}(k) < \mathcal{E}_{\min, \gamma}(k) \).

From the properties of the Fredholm determinant of the operator \( h_{\mu, \gamma}(k) \) the statements:

Lemma 2. The eigenvalue \( \mu_{\mu, \gamma}(k) = \mu_{\mu, \gamma}(k_1, k_2, k_3) \) is symmetric and even for each variable \( k_i \in [-\pi, \pi], i = 1, 2, 3 \), and strongly increasing in \( k_i \in [0, \pi], i = 1, 2, 3 \) and \( \mu_{\mu, \gamma}(0) = 0 \).

Lemma 3. For every \( \gamma > 0 \) and for every \( \mu > 3(1+\gamma) \) the following estimates hold true

\[
-\mu + 3(1+\gamma) - \frac{9(1+\gamma)^2}{\mu} \leq \mu_{\mu, \gamma}(p) \leq -\mu + 3(1+\gamma).
\]

4. Essential spectrum of a three-particle operator

In this section, we introduce the so-called "channel operators" through whose spectrum describes the essential spectrum of the Schrödinger operator \( H_{\mu, \gamma}(K) \).

Since in the three-particle system we are considering, two particles are the same, (i.e. the operators \( V_1 \) and \( V_2 \) are unitarily equivalent), therefore there is only one channel operator \( H_{\mu, \gamma}^ch(K) \), and in the momentum representation it is defined as is a self-adjoint operator acting in a Hilbert space \( L_2([T^3]^2) \) according to the formula

\[
H_{\mu, \gamma}^ch(K) = H_{0, \gamma}(K) - \mu V_1.
\]

Operator \( H_{\mu, \gamma}^ch(K) \) commutes with the group \( \{U_s, s \in \mathbb{Z}^3\} \) unitary operators

\[
(U_s f)(p, q) = \exp\{-i(s, p)\} f(p, q), \quad f \in L_2([T^3]^2),
\]

where

\[
(s, p) = s_1p_1 + s_2p_2 + s_3p_3, \quad s = (s_1, s_2, s_3) \in \mathbb{Z}^3, \quad p = (p_1, p_2, p_3) \in T^3.
\]

Therefore, ([16], Theorem XIII.84) operator \( H_{\mu, \gamma}^ch(K) \) decomposes into a direct operator integral

\[
H_{\mu, \gamma}^ch(K) = \int_{T^3} \oplus H_{\mu, \gamma}^ch(K, p) dp. \quad (3)
\]

From the uniqueness of the decomposition (3) follows (see [16], Theorem XIII.85), that the fiber operator \( H_{\mu, \gamma}^ch(K, p) \) has the form

\[
H_{\mu, \gamma}^ch(K, p) = h_{\mu, \gamma}(K - p) + \varepsilon(p) I, \quad (4)
\]

where \( I \) – identity operator and \( h_{\mu, \gamma}(k) \) is operator defined by (1).

From operator view \( H_{\mu, \gamma}^ch(K, p) \) for \( \mu > \mu_0 \) follows the equality

\[
\sigma(H_{\mu, \gamma}^ch(K, p)) = \{z_{\mu, \gamma}(K - p) + \varepsilon(p)\} \cup [\mathcal{E}_{\min, \gamma}(K - p), \mathcal{E}_{\max, \gamma}(K - p)].
\]

For every \( K \in T^3 \) put:

\[
E_{\min, \gamma}(K) = \min_{p, q \in T^3} E_{K, \gamma}(p, q), \quad E_{\max, \gamma}(K) = \max_{p, q \in T^3} E_{K, \gamma}(p, q).
\]

By the theorem (see for example [16]) on the spectrum of decomposable operators and the structure of the spectrum (5) operator \( H_{\mu, \gamma}^ch(K, p) \) we have

\[
\sigma(H_{\mu, \gamma}^ch(K)) = \bigcup_{p \in T^3} \{z_{\mu, \gamma}(K - p) + \varepsilon(p)\} \cup [E_{\min, \gamma}(K), E_{\max, \gamma}(K)].
\]
Let
\[
\tau_{\text{min},\gamma}(\mu, K) = \min_{p \in T^3} \{z_{\mu,\gamma}(K - p) + \varepsilon(p)\}, \quad \tau_{\text{max},\gamma}(\mu, K) = \max_{p \in T^3} \{z_{\mu,\gamma}(K - p) + \varepsilon(p)\}.
\]

**Lemma 4.** For the essential spectrum of the operator \(H_{\mu,\gamma}(K)\) has the equality holds true
\[
\sigma_{\text{ess}}(H_{\mu,\gamma}(K)) = \sigma(H_{\mu,\gamma}^{ch}(K)) = [\tau_{\text{min},\gamma}(\mu, K), \tau_{\text{max},\gamma}(\mu, K)] \cup [E_{\text{min},\gamma}(K), E_{\text{max},\gamma}(K)].
\]

**Proof.** A proof of a similar theorem is given in [15].

**Remark 1.** If \(\mu \geq 3(3 + \gamma)\), then a gap of the essential spectrum appears, that is, \(\tau_{\text{max},\gamma}(\mu, K) < E_{\text{min},\gamma}(K)\) for any \(K \in T^3\).

**Corollary.** Let \(K = 0\). Then \(\sigma_{\text{ess}}(H_{\mu,\gamma}(0)) = [z_{\mu,\gamma}(0), z_{\mu,\gamma}(\pi) + 6] \cup [0, 6 + \frac{15}{\pi} \gamma]\), where \(\pi = (\pi, \pi, \pi) \in T^3\).

### 5. Discrete spectrum of a three-particle operator

Let \(\mu > 3(3 + \gamma)\). By Remark 1 in this case for any \(K \in T^3\) a gap of the essential spectrum appears. Now we find an equivalent equation for the eigenfunctions of the three-particle operator \(A\) if and only if 1 is eigenvalue of the operator \(L\).

**Lemma 5.** A number \(z \in (\tau_{\text{max},\gamma}(\mu, K), E_{\text{min},\gamma}(K))\) is an eigenvalue of the operator \(H_{\mu,\gamma}(K)\) if and only if 1 is eigenvalue of the operator \(A_{\mu,\gamma}(K, z)\).

**Proof.** Let the number \(z \in (\tau_{\text{max},\gamma}(\mu, K), E_{\text{min},\gamma}(K))\) is an eigenvalue of the operator \(H_{\mu,\gamma}(K)\) and \(f\) is the corresponding eigenfunction, i.e. equation
\[
H_{0,\gamma}(K)f - \mu \sum_{a=1}^{2} V_a f = zf
\]
has nonzero solution \(f \in L^2_{\text{loc}}(T^3)\). Introducing the notation
\[
\varphi(p) = (V_1 f)(p) = \int_{T^3} f(p, s)ds,
\]
from (6) for \(z \in \mathbb{C} \setminus [E_{\text{min},\gamma}(K), E_{\text{max},\gamma}(K)]\) we obtain
\[
f(p, q) = \mu \frac{\varphi(p) - \varphi(q)}{E_{K,\gamma}(p, q) - z}.
\]
Since the function $f$ is antisymmetric implies that the function $\varphi$, defined by the formulae (7) belongs in the space $L_2(T^3)$ and satisfies the condition

$$\int_{T^3} \varphi(p) \, dp = V_2 V_1 f = \int_{T^3} f(p, s) \, dp \, ds = 0.$$  

Substituting the expression (8) in (7) we get the equation

$$\varphi(p) \left(1 - \mu \int_{T^3} \frac{ds}{E_{K,\gamma}(p, s) - z}\right) = -\mu \int_{T^3} \frac{\varphi(s) \, ds}{E_{K,\gamma}(p, s) - z}$$

which has a nonzero solution. Using the notation (2) we have that $\varphi \in L_2(T^3)$ is a solution of the following equation

$$\varphi(p) = \frac{\mu}{-\Delta_{\mu,\gamma}(K - p, z - \varepsilon(p))} \int_{T^3} \frac{\varphi(s) \, ds}{E_{K,\gamma}(p, s) - z}.$$  

If we denote that $g(p) = \sqrt{-\Delta_{\mu,\gamma}(K - p, z - \varepsilon(p))} \varphi(p)$, then we have

$$g(p) = \frac{\mu}{\sqrt{-\Delta_{\mu,\gamma}(K - p, z - \varepsilon(p))}} \int_{T^3} \frac{g(s) \, ds}{(E_{K,\gamma}(p, s) - z)\sqrt{-\Delta_{\mu,\gamma}(K - s, z - \varepsilon(s))}},$$

i.e. 1 is an eigenvalue of the operator $A_{\mu,\gamma}(K, z)$ and

$$\int_{T^3} \frac{g(s) \, ds}{\sqrt{-\Delta_{\mu,\gamma}(K - s, z - \varepsilon(s))}} = 0.$$  

**Necessity.** Let for some $z \in (\tau_{\max,\gamma}(\mu), E_{\min,\gamma}(K))$ the number 1 is an eigenvalue of the operator $A_{\mu,\gamma}(K, z)$ and $g \in D(A_{\mu,\gamma}(K, z))$ corresponding to the eigenfunction. Then the function $f$, defined by the formulae (8) belongs to $L^2_{\alpha}(T^3)^2$ and satisfies the equation (6).

### 6. Eigenvalues of the operator $H_{\mu,\gamma}(0)$ for large $\mu$

In this section we discuss on the eigenvalues $z \in [2_{\mu,\gamma}(\bar{\pi}) + 6; 0]$ in the gap of the operator $H_{\mu,\gamma}(0)$ for sufficiently large $\mu$.

**Theorem 3.** Let $\gamma > \gamma_0$. Then there exists $\mu_\gamma > 0$ such that for every $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(0)$ has at least two eigenvalues in the gap $(2_{\mu,\gamma}(\bar{\pi}) + 6; 0)$ of the essential spectrum.

Using the equality

$$\frac{1}{1 - x} = 1 + x + \frac{x^2}{1 - x}, \quad (x \neq 1)$$

we have

$$\frac{1}{E_{0,\gamma}(p, q) - z} = \frac{1}{6 + 3\gamma - z} \left(1 + \frac{(\xi(p) + \xi(q) + \gamma \xi(p + q))}{6 + 3\gamma - z} + \frac{\zeta(\gamma; p, q)}{6 + 3\gamma - z}\right), \quad (9)$$

where

$$\zeta(\gamma; p, q) = \frac{(\xi(p) + \xi(q) + \gamma \xi(p + q))^2}{E_{0,\gamma}(p, q) - z}.$$
Let $g \in D(A_{\mu,\gamma}(0, z))$. Taking into account equality (9) we obtain

$$(A_{\mu,\gamma}(0, z)g)(p) = (A_{\mu,\gamma}^{(0)}(0, z)g)(p) + (A_{\mu,\gamma}^{(1)}(0, z)g)(p),$$

where

$$(A_{\mu,\gamma}^{(0)}(0, z)g)(p) = \frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{-\Delta_{\mu,\gamma}(p, z - \varepsilon(p))}} \int_{T^3} \frac{\xi(s) + \gamma \xi(p + s)g(s)ds}{\sqrt{-\Delta_{\mu,\gamma}(s, z - \varepsilon(s))}},$$

and

$$(A_{\mu,\gamma}^{(1)}(0, z)g)(p) = \frac{\mu(6+3\gamma-z)^{-2}}{\sqrt{-\Delta_{\mu,\gamma}(p, z - \varepsilon(p))}} \int_{T^3} \zeta(\gamma; p, s)g(s)ds.$$
has a nonzero solution $\psi \in L^2_e(T^3)$. Denoting through
\[
C_i = \int_{T^3} \frac{\cos s_i \psi(s) ds}{\sqrt{-\Delta_{\mu,\gamma}(s, z - \varepsilon(s))}}, \quad i = 1, 2, 3, \tag{11}
\]
equation (10) rewrite as
\[
\lambda \psi(p) = \frac{\mu(6 + 3\gamma - z)^{-2}}{\sqrt{-\Delta_{\mu,\gamma}(p, z - \varepsilon(p))}} \left[ \sum_{i=1}^{3} C_i (1 + \gamma \cos p_i) \right].
\]
Substituting $\psi$ to the right side of equality (11) we obtain a system of equations for $C_j, j = 1, 2, 3$:
\[
\lambda C_j = \frac{\mu}{(6 + 3\gamma - z)^2} \int_{T^3} \frac{\cos s_j \cos s_i ds}{-\Delta_{\mu,\gamma}(s, z - \varepsilon(s))} \left[ \sum_{i=1}^{3} C_i (1 + \gamma \cos s_i) \right] ds.
\]
The Fredholm determinant of a homogeneous system of equations has the form
\[
\Lambda_{\mu,\gamma}(\lambda; z) = [\lambda + a_{\mu}^{(1)}(z) + b_{\mu,\gamma}^{(11)}(z)]^3 + 2[a_{\mu}^{(1)}(z) + b_{\mu,\gamma}^{(12)}(z)]^3 - 3[a_{\mu}^{(1)}(z) + b_{\mu,\gamma}^{(11)}(z)][\lambda + a_{\mu}^{(1)}(z) + b_{\mu,\gamma}^{(11)}(z)],
\]
where
\[
a_{\mu}^{(1)}(z) = \frac{\mu}{(6 + 3\gamma - z)^2} \int_{T^3} \frac{\cos s_i ds}{\Delta_{\mu,\gamma}(s, z - \varepsilon(s))},
\]
\[
b_{\mu,\gamma}^{(11)}(z) = \frac{\mu \gamma}{(6 + 3\gamma - z)^2} \int_{T^3} \frac{\cos^2 s_1 ds}{\Delta_{\mu,\gamma}(s, z - \varepsilon(s))},
\]
\[
b_{\mu,\gamma}^{(12)}(z) = \frac{\mu \gamma}{(6 + 3\gamma - z)^2} \int_{T^3} \frac{\cos s_1 \cos s_2 ds}{\Delta_{\mu,\gamma}(s, z - \varepsilon(s))}.
\]
Solving the equation $\Lambda_{\mu,\gamma}(\lambda; z) = 0$ with respect to $\lambda$ we find that $\lambda_{1,2}(\mu, \gamma; z) = b_{\mu,\gamma}^{(12)}(z) - b_{\mu,\gamma}^{(11)}(z)$ is a double zero function $\Lambda_{\mu,\gamma}(\lambda; z)$.
Solution to equation (10) at $\lambda = \lambda_{1,2}(\mu, \gamma; z)$ has the form
\[
\psi(p) = \frac{C_1 \cos p_1 + C_2 \cos p_2 - (C_1 + C_2) \cos p_3}{\sqrt{-\Delta_{\mu,\gamma}(p, z - \varepsilon(p))}}.
\]
The following statement can be easily proved:

**Lemma 7.** For any $z \in (\tau_{\max,\gamma}(\mu, K), E_{\min,\gamma}(K))$ the operator $A_{\mu,\gamma}^{(0-)}(0, z)$, has a unique threefold negative eigenvalue
\[
\lambda_3(\mu, \gamma; z) = \frac{\mu \gamma}{(6 + 3\gamma - z)^2} \int_{T^3} \frac{\sin^2 s_1 ds}{\Delta_{\mu,\gamma}(s, z - \varepsilon(s))}.
\]
Lemma 8. Let $\gamma > 0$ and $\mu > 3(1 + \gamma)$. Then the following equality
\[
\frac{1}{\Delta_{\mu,\gamma}(p, z - \varepsilon(p))} = \frac{(3 + 3\gamma - z_{\mu,\gamma}(p))(6 + 3\gamma - z)}{\mu(z_{\mu,\gamma}(p) + \varepsilon(p) - z)} \frac{1}{1 + Q(\mu, \gamma, z; p)}
\]
holds for all $p \in T^3$, where
\[
Q(\mu, \gamma, z; p) = \int_{T^3} \left[ \frac{\xi(s) + \gamma\xi(p + s)}{\varepsilon(s) + \gamma\varepsilon(p + s) - z_{\mu,\gamma}(p)} + \frac{\xi(s) + \gamma\xi(p + s) + \xi(p)}{\varepsilon(s) + \gamma\varepsilon(p + s) + \varepsilon(p) - z} \right] \, ds.
\]
After simple calculations, one can get the following estimates:

Remark 2. There exists $\mu_\gamma > 0$ such that, for all $\mu > \mu_\gamma$ the following estimates
\[
\frac{\xi(p)}{\mu} - \frac{C}{\mu^2} \leq Q(\mu, \gamma, z; p) \leq \frac{\xi(p)}{\mu} + \frac{C}{\mu^2}
\]
for all $p \in T^3$ (12) hold, where $C$ is a positive constant depending only on $\gamma$.

Then from Lemma 8 and Remark 2 we obtain:

Lemma 9. Let $\gamma > \gamma_0$ and $z \in [z_{\mu,\gamma}(\bar{s}) + 6, z_{\mu,\gamma}(\bar{s}) + 6 + 3\gamma)$. Then there exists $\mu_\gamma > 0$ such that for every $\mu > \mu_\gamma$ the following estimate
\[
\|A^{(1)}_{\mu,\gamma}(0, z)\| \leq \frac{C}{\mu}
\]
holds, where $C$ is a positive constant depending only on $\gamma$.

Lemma 10. Let $\gamma > \gamma_0$. Then there exists $\mu_\gamma > 0$ such that, for every $\mu > \mu_\gamma$ the eigenvalue of the operator $A^{(1)}_{\mu,\gamma}(0, z_{\mu,\gamma}(\bar{s}) + 6)$ satisfies the following inequality
\[
\lambda_{1,2}(\mu, \gamma; z_{\mu,\gamma}(\bar{s}) + 6) \geq \frac{\gamma}{\gamma_0} - \frac{C}{\mu},
\]
where $C$ is a positive constant depending only on $\gamma$.

Proof. Using Lemma 3, Lemma 8 and inequality (12) we get the following estimate
\[
\int_{T^3} \cos^2 s_1 - \cos s_1 \cos s_2 \, ds = \frac{1}{\Delta_{\mu,\gamma}(s, z_{\mu,\gamma}(\bar{s}) + 6 - \varepsilon(s))}
= \frac{1}{\mu} \int_{T^3} (3 + 3\gamma - z_{\mu,\gamma}(p))(3\gamma - z_{\mu,\gamma}(\bar{s})) \, ds
\geq \frac{(3 + 3\gamma - z_{\mu,\gamma}(\bar{s}))}{\mu} \int_{T^3} (3 + \xi(s))(1 + Q(\mu, \gamma, z_{\mu,\gamma}(\bar{s}) + 6; s)) \, ds
\geq (3\gamma - z_{\mu,\gamma}(\bar{s})) \left[ \int_{T^3} \cos^2 s_1 - \cos s_1 \cos s_2 \, ds - \frac{C}{\mu} \right].
\]
From this and Lemma 3 we have
\[
\lambda_{1,2}(\mu, \gamma; z_{\mu,\gamma}(\bar{s}) + 6) = \frac{\mu\gamma}{(3\gamma - z_{\mu,\gamma}(\bar{s}))^2} \int_{T^3} \Delta_{\mu,\gamma}(s, z_{\mu,\gamma}(\bar{s}) + 6 - \varepsilon(s)) \, ds
\geq \frac{\mu\gamma}{3\gamma - z_{\mu,\gamma}(\bar{s})} \left[ \int_{T^3} \frac{(\cos^2 s_1 - \cos s_1 \cos s_2)}{3 + \xi(s)} \, ds - \frac{C}{\mu} \right]
\geq \frac{\mu\gamma}{\mu + 3 + \frac{9(1 + \gamma)^2}{\mu}} \left[ \frac{1}{\gamma_0} - \frac{C}{\mu} \right] \geq \frac{\gamma}{\gamma_0} - \frac{C}{\mu}.
\]

**Lemma 11.** Let $\gamma > \gamma_0$. Then there exists $\mu_\gamma > 0$ such that, for every $\mu > \mu_\gamma$ the eigenvalue of the operator $A^{(0+)}_{\mu, \gamma}(0, z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma)$ satisfies the inequality

$$\lambda_{1,2}(\mu, \gamma; z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma) < 1.$$  

**Proof.** Using the representation for the eigenvalue (see Lemma 6), taking into account Lemma 3, Lemma 8, we have

$$\lambda_{1,2}(\mu, \gamma; z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma) =$$

$$= -\frac{\mu\gamma}{(6 + 3\gamma - (z_{\mu, \gamma}(\vec{s}))^2)\int_{T^3} \Delta_{\mu, \gamma}(s, z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma - \varepsilon(s)) ds}$$

$$\leq \frac{\gamma}{-z_{\mu, \gamma}(\vec{p}) \int_{T^3} (z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma - z_{\mu, \gamma}(s) - \varepsilon(s)) (1 + Q(\mu, \gamma, z_{\mu, \gamma}(\vec{p}) + 6; s)) ds}$$

$$\leq \frac{2(1 + 1 + \gamma^2)}{\mu - 3(1 + \gamma)} \int_{T^3} 3\gamma (1 + Q(\mu, \gamma, z_{\mu, \gamma}(\vec{p}) + 6; s))$$

$$\leq \frac{2}{3}(1 + \frac{C}{\mu}) < 1.$$  

**Proof of Theorem 2.** According to the perturbation theory and from Lemma 9 in a neighborhood of a point $\lambda_{i,2}^{(0+)}(\mu, \gamma; z)$, operator

$$A_{\mu, \gamma}(0, z) = A^{(0)}_{\mu, \gamma}(0, z) + A^{(1)}_{\mu, \gamma}(0, z)$$

has exactly two eigenvalues $\lambda_i(\mu, \gamma; z), i = 1, 2$. Let’s choose $\mu_\gamma$ so that the inequalities $\lambda_i(\mu, \gamma; z_{\mu, \gamma}(\vec{p}) + 6) > 1 > \lambda_i(\mu, \gamma; z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma)$. From the continuity of the function $\lambda_i(\mu, \gamma; \cdot), i = 1, 2$ by $z \in [z_{\mu, \gamma}(\vec{p}) + 6, z_{\mu, \gamma}(\vec{p}) + 6 + 3\gamma]$ there are numbers $z_i(\mu, \gamma), i = 1, 2$, such that $\lambda_i(\mu, \gamma; z_i(\mu, \gamma)) = 1, i = 1, 2$. According to Lemma 5 the numbers $z_i(\mu, \gamma), i = 1, 2$ are the eigenvalues of the operator $H_{\mu, \gamma}(0)$.

**Proof of Theorem 1.** Let rewrite the operator $H_{\mu, \gamma}(K)$ in the form:

$$H_{\mu, \gamma}(K) = H_{\mu, \gamma}(0) + \gamma \tilde{H}_0(K),$$

where $\tilde{H}_0(K)$ is the operator multiplication by the function

$$\tilde{E}_K(p, q) = \sum_{i=1}^{3} [(1 - \cos K_i) \cos(p_i + q_i) + \sin K_i \sin(p_i + q_i)].$$

It is known that the norm of the operator of multiplication by a function is equal to the maximum modulus of this function, i.e.

$$\|\tilde{H}_0(K)\| = \max_{p, q} \|\tilde{E}_K(p, q)\| = \sqrt{\sum_{i=1}^{3} 4 \sin^2 \frac{K_i}{2}} \leq \|K\| = \sqrt{K_1^2 + K_2^2 + K_3^2}.$$  

According to the perturbation theory, it follows, there exist $\mu_\gamma > 0$ and $\delta > 0$ such that, for any $\mu > \mu_\gamma$ and $K$ satisfying $\|K\| < \delta$ the operator $H_{\mu, \gamma}(K)$ has at least two eigenvalues in the gap $(\tau_{\text{max}, \gamma}(\mu, K), E_{\text{min}, \gamma}(K))$ of the essential spectrum.
References

[1] Mattis D C 1986 The few-body problem on lattice Rev. Mod. Phys. 58 361–379.
[2] Mogilner A I 1991 Hamiltonians of solid state physics at few-particle discrete Schrödinger operators: problems and results Adv. Sov. Math. 5 139–94
[3] Malishev V A and Minlos R A 1995. Linear Infinite-particle Operators (Translations of Mathematical Monographs vol 143) (Providence, RI: American Mathematical Society)
[4] Winkler K, Thalhammer G, Frimm R, Denschlag J H, Daley A J, Kantian A, Büchler H P and Zoller P 2006 Repulsively bound atom pairs in an optical lattice. Letters. 441 -15.
[5] Lakaev S N, Dell’Antonio G F and Khalkhuzaev A M 2016 Existence of an isolated band of a system of three particles in an optical lattice J. Phys. A: Math. Theor. 49 145204–15
[6] Lakaev S N, Lakaev Sh S 2017 The existence of bound states in a system of three particles in an optical lattice J. Phys. A: Math. Theor. 50 335202–17
[7] Lakaev S N 1993 The Efimov Effect of a system of three identical quantum lattice particles. Funct. Anal. Appl. 27 166–175.
[8] Abbeverio S, Lakaev S N, Muminov Z I 2004 Schrödinger Operators on Lattices. The Efimov Effect and Discrete Spectrum Asymptotics. Ann. Henri Poincaré 5 743–72 25.
[9] Abbeverio S, Khalkhujaev A M, Lakaev S N 2012. Number of Eigenvalues of the Three-Particle Schrodinger Operators on Lattices Markov Processes and Related Fields 3 -33
[10] Abdullaev J, Lakaev S N 2003 Asymptotics of the Discrete Spectrum of the Three-Particle Schrödinger Difference Operator on a Lattice Theor. and Math. Physics 136(2)-14
[11] Abdullaev Zh 2007 Finiteness of the discrete spectrum for non-trivial values of the full quasi-momentum in the system of three bosons on a lattice Russian Math. Surveys, 62(1) -5
[12] Muminov M. I. 2009 The infiniteness of the number of eigenvalues in the gap in the essential spectrum for the three-particle Schrodinger operator on a lattice Theoret. and Math. Phys. 159(2) 667–683
[13] Dell Antonio G F, Muminov Z I and Shermatova Y M 2011 On the number of eigenvalues of a model operator related to a system of three particles on lattices J. Phys. A: Math. Theor. 44 315302-27
[14] Petrov D S 2003 Three-body problem in Fermi gases with short-range interparticle interaction Phys. Rev. A 67 010703
[15] Khalkhuzaev A M 2017 The essential spectrum of the three-particle discrete operator corresponding to a system of three fermions on a lattice, Russian Mathematics 61 -12
[16] Reed M and Simon B 1979 Methods of Modern Mathematical Physics: VI. Analysis of Operators (New York: Academic)