The multiset dimension of graphs

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Abstract

We introduce a variation of the metric dimension, called the multiset dimension.

The representation multiset of a vertex \( v \) with respect to \( W \) (which is a subset of the vertex set of a graph \( G \)), \( r_m(v|W) \), is defined as a multiset of distances between \( v \) and the vertices in \( W \) together with their multiplicities. If \( r_m(u|W) \neq r_m(v|W) \) for every pair of distinct vertices \( u \) and \( v \), then \( W \) is called a resolving set of \( G \). If \( G \) has a resolving set, then the cardinality of a smallest resolving set is called the multiset dimension of \( G \), denoted by \( md(G) \). If \( G \) does not contain a resolving set, we write \( md(G) = \infty \).

We present basic results on the multiset dimension. We also study graphs of given diameter and give some sufficient conditions for a graph to have an infinite multiset dimension.

Keywords: metric dimension, multiset dimension, distance.

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1. Introduction

In July 2013, at Graph Master 2013, which was held at the University of Lleida in Spain, the first author presented a survey on the metric dimension of graphs. The metric dimension was introduced separately by Slater \cite{3} and Harary and Melter \cite{2}.

Let \( G \) be a connected graph with the vertex set \( V(G) \). The distance \( d(u,v) \) between two vertices \( u,v \in V(G) \) is the number of edges in a shortest
A vertex \( w \) resolves a pair of vertices \( u, v \) if \( d(u, w) \neq d(v, w) \). For an ordered set of \( k \) vertices \( W = \{w_1, w_2, \ldots, w_k\} \), the representation of distances of a vertex \( v \) with respect to \( W \) is the ordered \( k \)-tuple

\[
r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).
\]

A set of vertices \( W \subset V(G) \) is a resolving set of \( G \) if every two vertices of \( G \) have distinct representations. A resolving set with minimum cardinality is called a metric basis and the number of vertices in a metric basis is called the metric dimension, denoted by \( \text{dim}(G) \).

After Graph Master 2013, Charles Delorme, who was in the audience, send a suggestion to the first author to look at the multiset of distances, instead of looking at the vector of distances to some given set of vertices.

Let us copy an excerpt of Charles’ email.

Instead of looking at the list of distances to some code in graph, we may look at the multiset of distances.

However, contrarily to the case of lists, where a sufficient large code allows the identification of all vertices, it may happen that no code provides identification. This is the case if the graph has too many symmetries.

For example, a complete graph with \( n \geq 3 \) vertices and a code with \( m \) vertices (with \( 1 \leq m < n \)) gives only two multisets, namely \( \{1^n\} \) for vertices out of the code and \( \{0, 1^{n-1}\} \) for vertices in the code.

Some graphs however have a code such that the multiset of distances identifies the vertices. It is the case for cycles with \( n \geq 6 \) vertices.

Other example; Petersen graph. We recall that all graphs with less than 7 vertices admit some non-trivial automorphism. Therefore, any code in Petersen graph with at most 6 vertices contains two vertices with the same multiset of distances. On the other hand, if the code has 7 or more vertices, some non-trivial automorphism of the whole graph preserves the complement of the code and the multisets in an orbit of this automorphism are the same.

Charles fell ill at the end of 2013 and he was not in a good health condition until his passing away in 2015. This is one of the reasons why we did not pursue his idea further. Four years later, we reencountered Charles’ email and felt that it is worthwhile to explore this new idea, since the notion is
interesting and it has different properties then the original notion of metric dimension.

We start by formally defining the multiset dimension.

**Definition 1.** Let $G$ be a simple and connected graph with vertex set $V(G)$. Suppose that $W$ is a subset of $V(G)$ and $v$ is a vertex of $G$. The representation multiset of $v$ with respect to $W$, $r_m(v|W)$, is defined as a multiset of distances between $v$ and the vertices in $W$ together with their multiplicities. If $r_m(u|W) \neq r_m(v|W)$ for every pair of distinct vertices $u$ and $v$, then $W$ is called a resolving set of $G$. A resolving set having minimum cardinality is called a multiset basis. If $G$ has a multiset basis, then its cardinality is called the multiset dimension of $G$, denoted by $md(G)$. If $G$ does not contain a multiset basis, we say that $G$ has an infinite multiset dimension and we write $md(G) = \infty$.

Note that the diameter of a graph $G$ is the distance between any two furthest vertices in $G$ and the Cartesian product of two graphs $G_1$ and $G_2$ will be denoted by $G_1 \square G_2$.

In this paper we study the multiset dimension of various graphs including graphs of given diameter. We also present some sufficient conditions for a graph to have an infinite multiset dimension.

2. Basic results

From the definition of the multiset dimension and the metric dimension it is clear that the multiset dimension of any graph $G$ is at least the metric dimension of $G$.

**Lemma 1.** $md(G) \geq dim(G)$.

This bound is tight, since it is well-known that the metric dimension of a graph $G$ is one if and only if $G$ is a path and we obtain an equivalent result for the multiset dimension.

**Theorem 1.** The multiset dimension of a graph $G$ is one if and only if $G$ is a path.

**Proof.** The set containing a pendant vertex of a path is a resolving set, thus $md(P_n) = 1$. Now we show that if $md(G) = 1$, then $G$ is a path. Let $W = \{w\}$ be a multiset basis of a graph $G$. Then $d(u, w) \neq d(v, w)$ for any
two vertices $u, v \in V(G)$, which means that there exists a vertex $x$ such that $d(x, w) = n - 1$ (where $n$ is the order of $G$). This implies that the diameter of $G$ is $n - 1$, hence $G$ is the path $P_n$. \(\Box\)

Let us prove that no graph has the multiset dimension 2.

**Lemma 2.** No graph has the multiset dimension 2.

**Proof.** Assume that $md(G) = 2$ for some graph $G$. Let $W = \{w_1, w_2\}$ be a resolving set of $G$. Then $r_m(w_1|W) = \{0, d(w_1, w_2)\} = r_m(w_2|W)$, a contradiction. \(\Box\)

The following theorem is a corollary of Lemma 2.

**Theorem 2.** Let $G$ be a graph other than a path. Then $md(G) \geq 3$.

Another bound relates the multiset dimension of a graph with its diameter. Theorem 3 gives a better lower bound than the one presented in Theorem 2. For positive integers $n$ and $d$, we define $f(n, d)$ to be the least positive integer $k$ for which $\frac{(k+d-1)!}{k!(d-1)!} + k \geq n$.

**Theorem 3.** If $G$ is a graph of order $n \geq 3$ and diameter $d$, then $md(G) \geq f(n, d)$.

**Proof.** Let $W$ be a multiset basis of $G$ having $k$ vertices. If $x$ is a vertex not in $W$, then $r_m(x|W) = \{1^{m_1}, 2^{m_2}, \ldots, d^{m_d}\}$, where $m_1 + m_2 + \cdots + m_d = k$ and $0 \leq m_i \leq k$ for each $i = 1, 2, \ldots, k$. Then there are $C((k + d - 1, d - 1)$ different possibilities for representation of $x$. Since we have $n - k$ vertices not in $W$, $\frac{(k+d-1)!}{k!(d-1)!}$ must be at least $n - k$. Hence $\frac{(k+d-1)!}{k!(d-1)!} + k \geq n$. \(\Box\)

### 3. Graphs with infinite multiset dimension

In this section we give some sufficient conditions for a graph to have an infinite multiset dimension.

**Lemma 3.** Let $G$ be a graph and let $W'$ be a set of vertices, where $|W'| \geq 2$. If every pair of vertices in $W'$ is of distance at most 2, then $W'$ is not a resolving set of $G$. 


Proof. We prove the result by contradiction. Assume that every pair of vertices in $W$ is of distance at most 2 and $W$ is a resolving set of $G$. We denote the vertices in $W$ by $w_1, w_2, \ldots, w_p$, where $p \geq 2$. For $i = 1, 2, \ldots, p$, we have $r_m(w_i|W) = \{0, 1^{m_1}, 2^{m_2}\}$, where $m_1 + m_2 = p - 1$. Since we have $p$ vertices in $W$ and they have different representations, their representations must be $\{0, 1^{p-1}\}, \{0, 1^{p-2}, 2\}, \ldots, \{0, 1, 2^{p-2}\}, \{0, 2^{p-1}\}$. Without loss of generality we can assume that the vertex having the representation $\{0, 1^{p-1}\}$ is $w_1$ and the vertex having the representation $\{0, 2^{p-1}\}$ is $w_p$. Since $r_m(w_1|W) = \{0, 1^{p-1}\}$, it follows that $d(w_1, w_p) = 1$, a contradiction to $r_m(w_p|W) = \{0, 2^{p-1}\}$. Hence, $W$ is not a resolving set of $G$.

In a graph of diameter at most 2, the distance between any two vertices is at most 2, thus Theorem 4 is a corollary of Lemma 3.

Theorem 4. If $G$ is a graph of diameter at most 2 other than a path, then $md(G) = \infty$.

This means that cycles with at most 5 vertices, complete graphs, stars, the Petersen graph and strongly regular graphs have multiset dimension infinity.

Lemma 4. If $G$ contains a vertex which is adjacent to (at least) three pendant vertices, then $md(G) = \infty$.

Proof. Let $v_1, v_2, v_3$ be three pendant vertices adjacent to some vertex in $G$. Let $W$ be any resolving set of $G$. Then either at least two of them (say $v_1$ and $v_2$) are in $W$, or at least two of them (say $v_1$ and $v_2$) are not in $W$. In both cases these vertices cannot be resolved, because $d(v_1, v) = d(v_2, v)$ for any other vertex $v \in V(G)$.

4. Multiset dimension of trees

The sufficient conditions in Theorem 4 and Lemma 4 are, unfortunately, not necessary for a graph to have infinite multiset dimension. We shall give an example of a tree of diameter 4, having no vertex adjacent to at least 3 pendant vertices, that has infinite multiset dimension. Let $T$ be the rooted tree of height 2 with the root vertex having 3 neighbours, each of them adjacent to 2 pendant vertices. Let $W$ be any resolving set of $T$. Since there exist 3 pairs of pendant vertices of distance 2, then to avoid the same representations, exactly one pendant vertex from each pair must be in $W$. Now, consider the 3 vertices in the first level. Either at least two of them
are in $W$ or at least two of them are not in $W$. In both cases, the two vertices have the same representations with respect to $W$, regardless what the members of $W$ are.

In this section, however, we shall present a family of tree where the conditions in Theorem 4 and Lemma 4 are both necessary and sufficient for the tree to have infinite multiset dimension.

**Theorem 5.** The multiset dimension of a complete $k$-ary tree is finite if and only if $k = 1$ or $2$. Moreover, if $T$ is a complete binary tree of height $h$, then $md(T) = 2^h - 1$.

**Proof.** Let $T$ be a complete $k$-ary tree of height $h \geq 1$. If $k \geq 3$, then by Lemma 4, $md(T) = \infty$. If $k = 1$, then $T$ is a path and $md(T) = 1$ (by Theorem 1).

Let $k = 2$. If $h = 1$, then $T$ is a path having two edges and $md(T) = 1$. So, let us study binary trees of height $h \geq 2$. Let $W$ be any resolving set of $T$. Consider the last level $h$ of $T$ containing $2^{h-1}$ pairs of pendant vertices of distance 2. Note that exactly one pendant vertex of each pair must be in $W$ (otherwise their representations would be equal). Now consider level $h - 1$ of $T$ having $2^{h-2}$ pairs of vertices of distance 2. Vertices of each pair have the same representations with respect to the vertices in $W$ which are in level $h$ and they have the same distance to any other vertex of $T$. So the pair cannot be resolved by other vertices, which means that exactly one of the vertices of each pair is in $W$. We can repeat similar arguments for next levels (levels $h - 2$, $h - 3$, $\ldots$, 1) to obtain $2^{h-1} + 2^{h-2} + \cdots + 1 = 2^h - 1$ vertices that must be in $W$, thus $md(T) \geq 2^h - 1$.

In each level $i$ of $T$, where $1 \leq i \leq h$, there are exactly $2^{i-1}$ pairs of vertices of distance 2. Let $W$ contains exactly one vertex of each such pair. So $|W| = \sum_{i=1}^{h} 2^{i-1} = 2^h - 1$. We prove that $W$ is a resolving set. Let us show that any two vertices $u, v$ of $T$ are resolved by $W$. We consider two cases.

(1) $u$ and $v$ are in different levels, say $i$ and $j$, where $0 \leq i < j \leq h$:

The distance between $v$ and $2^{h-2}$ vertices of $W$ which are in level $h$ is $j + h$. On the other hand, there is no vertex in $W$ of distance $j + h$ from $u$. Thus $u$ and $v$ have different representations.

(2) $u$ and $v$ are in the same level $i$, where $1 \leq i \leq h$: 


If exactly one of them is in \( W \), clearly they have different representations. If none of them is in \( W \) or both of them are in \( W \), then let us denote by \( x \) the central vertex of the path connecting \( u \) and \( v \). This path has even number of edges, say \( 2s \), and then \( x \) is in level \( i - s \). We know that \( x \) is adjacent to two vertices, say \( x_1, x_2 \), in level \( i - s + 1 \). Both, \( x_1 \) and \( x_2 \), belong to the path and exactly one of them, say \( x_1 \), is in \( W \). Clearly \( d(u, x_1) \neq d(v, x_1) \) and it can be checked that \( u \) and \( v \) have the same representations with respect to \( W \setminus \{x_1\} \). Hence \( W \) is a resolving set and \( \text{md}(T) \leq 2^h - 1 \). The proof is complete.

\[ \square \]

5. Graphs with multiset dimension 3

From Theorem 2, we know that the multiset dimension of any graph other than a path is at least 3. We present two families of graphs having the multiset dimension 3. The proof for the first result was given by Charles Delorme.

**Theorem 6.** [1] Let \( n \geq 6 \). Then \( \text{md}(C_n) = 3 \).

**Proof.** The description of the representations of vertices depends on the parity of \( n \); in both cases, we check that the set \( W = \{v_0, v_1, v_3\} \) with a usual labelling of the cycle is convenient.

If \( n = 2t \) with \( t \geq 3 \), the representations of vertices are the following:

\[
\begin{align*}
v_0 & \ {\{0, 1, 3\}} & v_1 & \ {\{0, 1, 2\}} & v_2 & \ {\{1, 1, 2\}} & v_3 & \ {\{0, 2, 3\}} \\
v_i \ (3 < i < t) & & v_t & & v_{t+1} & & v_{t+2} \\
\{i - 3, i - 1, i\} & & \{t - 3, t - 1, t\} & & \{t - 2, t - 1, t\} & & \{t - 2, t - 1, t - 1\} \\
v_t+3 & & v_{i+t} \ (3 < i < t) & & \{t - 3, t - 2, t\} & & \{t - i, t - i + 1, t - i + 3\}
\end{align*}
\]

If \( n = 2t + 1 \) with \( t \geq 3 \), the representations of vertices are given by:

\[
\begin{align*}
v_0 & \ {\{0, 1, 3\}} & v_1 & \ {\{0, 1, 2\}} & v_2 & \ {\{1, 1, 2\}} & v_3 & \ {\{0, 2, 3\}} \\
v_i \ (3 < i < t) & & v_t & & v_{t+1} & & v_{t+2} \\
\{i - 3, i - 1, i\} & & \{t - 3, t - 1, t\} & & \{t - 2, t, t\} & & \{t - 1, t - 1, t\} \\
v_{t+3} & & v_{i+t} & & v_{i+t+1} \ (3 < i < t) & & \{t - 2, t - 1, t\} & & \{t - i, t - i + 1, t - i + 3\}
\end{align*}
\]

\[ \square \]
Theorem 7. Let $m \geq 3$ and $n \geq 2$. Then $md(P_m \Box P_n) = 3$.

Proof. We can write $V(P_m \Box P_n) = \{v_{i,j} \mid i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n\}$. Then $E(P_m \Box P_n) = \{v_{i,j}v_{i,j+1} \mid i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n-1\} \cup \{v_{i,j}v_{i+1,j} \mid i = 1, 2, \ldots, m-1, \ j = 1, 2, \ldots, n\}$.

Let us show that $W = \{v_{1,1}, v_{1,2}, v_{3,1}\}$ is a resolving set of the graph $P_m \Box P_n$. We present representations of vertices with respect to $W$ as follows

\[
\begin{align*}
    r_m(v_{1,1}|W) &= \{0, 1, 2\}, \\
    r_m(v_{2,1}|W) &= \{1, 1, 2\}, \\
    r_m(v_{i,1}|W) &= \{i-3, i-1, i\} \text{ for } 3 \leq i \leq m, \\
    r_m(v_{1,j}|W) &= \{j-2, j-1, j+1\} \text{ for } 2 \leq j \leq n, \\
    r_m(v_{2,j}|W) &= \{j-1, j, j\} \text{ for } 2 \leq j \leq n, \\
    r_m(v_{i,j}|W) &= \{i+j-4, i+j-3, i+j-2\} \text{ for } 3 \leq i \leq m, \ 2 \leq j \leq n.
\end{align*}
\]

Since no two vertices have the same representations, $W$ is a resolving set and hence $md(P_m \Box P_n) = 3$. \hfill \Box

6. Open problems

Since the multiset dimension is a new invariant, there is a very large space for research in this area. Let us state a few interesting problems.

Problem 1. Let $G$ be a graph having $n$ vertices and $m$ edges. If $md(G)$ is finite, find lower and upper bounds on the multiset dimension of $G$ with respect to $n$ and $m$.

Problem 2. Find bounds on the multiset dimension with respect to minimum degree of a graph.

Problem 3. Characterize all trees having finite multiset dimension. Give exact values of the multiset dimension of trees if it is finite.

Problem 4. Study the multiset dimension for Cayley graphs of cyclic, Abelian, and non-Abelian groups.

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