Fractional Quadruple Laplace Transform and its Properties

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ABSTRACT
In this paper, we introduce definition for fractional quadruple Laplace transform of order \(0 < \alpha \leq 1\), for fractional differentiable functions. Some main properties and inversion theorem of fractional quadruple Laplace transform are established. Further, the connection between fractional quadruple Laplace transform and fractional Sumudu transform are presented.

KEYWORD: quadruple Laplace transform, Sumudu transform, fractional Difference.

INTRODUCTION
There are different integral transforms in mathematics which are used in astronomy, physics and also in engineering. The integral transforms were vastly applied to obtain the solution of differential equations: therefore there are different kinds of integral transforms like Mellin, Laplace, and Fourier and so on. Partial differential equations are considered one of the most significant topics in mathematics and others. There are no general methods for solve these equations. However, integral transform method is one of the most familiar methods in order to get the solution of partial differential equations [1, 2]. In [3, 9] quadruple Laplace transform and Sumudu transforms were used to solve wave and Poisson equations. Moreover the relation between them and their applications to differential equations have been determined and studied by [5, 6]. In this study we focus on quadruple integral determined and studied by [5, 6]. In this study we focus on quadruple integral transforms. First of all, we start to recall the definition of quadruple Laplace transform as follows.

Definition: Let \(f\) be a continuous function of four variables, then the quadruple Laplace transform of \(f(w, x, y, z)\) is defined by

\[
L_{wxyz} f(w, x, y, z) = F(p, q, r, s) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-pw}e^{-qx}e^{-ry}e^{-sz} f(w, x, y, z)dw dxdydz
\]

Where \(w, x, y, z > 0\) and \(p, q, r, s\) are Laplace variables, and

\[
f'(w, x, y, z) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\beta s} [\frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{\gamma s} [\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{r s} [\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} F(p, q, r, s)dz]dy]dx]dw
\]

is the inverse Laplace transform.

Fractional Derivative via Fractional Difference
Definition: let \(g(t)\) be a continuous function, but not necessarily differentiable function, then the forward operator \(FW(h)\) is defined as follows \(FW(h)g(t) = g(t + h)\), Where \(h > 0\) denote a constant discretization span.

Moreover, the fractional difference of \(g(t)\) is known as

\[
\Delta^\alpha g(t) = (FW - h)^\alpha g(t) = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} g(t + (\alpha - m)h)
\]
Where $0 < \alpha < 1$,
And the $\alpha$-derivative of $g(t)$ is known as
$$g^{(\alpha)}(t) = \lim_{h \to 0} \frac{\Delta^{\alpha} g(t)}{h^{\alpha}}$$
See the details in [9, 10].

**Modified Fractional Riemann-Liouville Derivative**
The author in [10] proposed an alternative definition of the Riemann-Liouville derivative

**Definition:** let $g(t)$ be a continuous function, but not necessarily differentiable function, then
Let us presume than $g(t) = K$, where $K$ is a constant, thus $\alpha$-derivative of the function $g(t)$
is $D^{\alpha}_t K = \begin{cases} K \Gamma^{-1}(1-\alpha) t^{-\alpha} & \alpha \leq 0, \\ 0 & \text{otherwise}. \end{cases}$
On the other hand, when $g(t) \neq K$ hence
$g(t) = g(0) + (g(t) - g(0))$, and fractional derivative of the function $g(t)$ will
become known as
$$g^{(\alpha)}(t) = D^{\alpha}_t g(0) + D^{\alpha}_t g(t) - g(0),$$
at any negative $\alpha$, ($\alpha < 0$) one has
$$D^{\alpha}_t g(t) - g(0) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{t} (t-\eta)^{-\alpha-1} g(\eta) d\eta, \alpha < 0.$$While for positive $\alpha$, we will put
$$D^{\alpha}_t g(t) - g(0) = D^{\alpha}_t g(t) = D^{\alpha}_t (g^{(\alpha-1)}).$$When $m < \alpha < m + 1$, we place
$$g^{(\alpha)}(t) = (g^{(\alpha-m)}(t))^m, m \leq \alpha < m + 1, m \geq 1.$$

**Integral with Respect to $(dt)^{\alpha}$**
The next lemma show the solution of fractional differential equation

$$dy = g(t)(dt)^{\alpha}, t \geq 0, y(0) = 0$$
By integration with respect to $(dt)^{\alpha}$

**Lemma:** If $g(t)$ is a continuous function, so the solution of (2) is defined as the following
$$y(t) = \int_{0}^{1} g(\eta) (d\eta)^{\alpha}, y(0) = 0$$
$$= \alpha \int_{0}^{1} (t-\eta)^{\alpha-1} g(\eta) d\eta, 0 < \alpha < 1$$
For more results and varies views on fractional calculus, see for example [15, 16, 17, 18, 19, 20, 21]

**Fractional quadruple Sumudu Transform**
Recently, in [13] the author defined quadruple sumudu transform of the function depended on two variables. Analogously, fractional quadruple sumudu transform was defined and some properties were given as the following

**Definition:** [14] The fractional quadruple sumudu transform of function $f(x, y, z, t)$ is known as
$$S^4_{q}(f(x, y, z, t)) = G^4_{q}(p, q, r, s),$$
$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} E_{\alpha}(-x + y + z + t)^{\alpha}(pxqy + rzst)(dx)^{\alpha}(dy)^{\alpha}(dz)^{\alpha}(dt)^{\alpha}$$
Where $q, r, s \in \mathbb{C}, x, y, z, t > 0$ and
$$E_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^{m}}{\Gamma(am+1)}$$
is the Mittag-Leffler function.

**Some Properties of Fractional Quadruple Sumudu Transform**
We recall some properties of Fractional Quadruple Sumudu Transform

**Definition 6:** If $f(x, y, z, t)$ is a function where $x, y, z, t > 0$, then Quadruple Laplace Transform of
Fractional order of $f(x, y, z, t)$ is defined as
$$L^4_{\alpha}(f(x, y, z, t)) = F^4_{\alpha}(p, q, r, s)$$
$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} E_{\alpha}(-(px + qy + rz + st)^{\alpha} f(x, y, z, t)(dx)^{\alpha}(dy)^{\alpha}(dz)^{\alpha}(dt)^{\alpha}$$
Where $p, q, r, s \in \mathbb{C}$ and $E_{\alpha}(x)$ is the Mittag-Leffler function.

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Corollary 10.1 By using the Mittag-Leffler property then we can rewrite the formula (5) as the following:

\[ L_a^4 \{ f(x, y, z, t) \} = F_a^4(p, q, r, s) \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_a\left(-\left(p + \frac{q}{a} + \frac{r}{b} + \frac{s}{c} \right)\right) f(x, y, z, t)(dx)^a(dy)^a(dz)^a(dt)^a \]

Remark 8: In particular case, fractional quadruple Laplace transform (5) turns to quadruple Laplace transform (1) when \( a = 1 \).

Some properties of fractional quadruple Laplace transform

In this section, various properties of fractional quadruple Laplace transform are discussed and proved such as linearity property, change of scale property and so on.

1. Linearity property

Let \( f_1(x, y, z, t) \) and \( f_2(x, y, z, t) \) be functions of the variables \( x \) and \( t \), then

\[ L_a^4 \{ a_1 f_1(x, y, z, t) + a_2 f_2(x, y, z, t) \} = a_1 L_a^4 \{ f_1(x, y, z, t) \} + a_2 L_a^4 \{ f_2(x, y, z, t) \} \]

Where \( a_1 \) and \( a_2 \) are constants.

2. Changing of scale property

If

\[ L_a^4 \{ f(x, y, z, t) \} = F_a^4(p, q, r, s) \]

\[ L_a^4 \{ f(ax, by, cz, dt) \} = \frac{1}{a^a b^a c^a d^a} F_a^4(p, q, r, s) \]

Whenever \( a \) and \( b \) are constants.

3. Shifting property

Let

\[ L_a^4 \{ f(x, y, z, t) \} = F_a^4(p, q, r, s) \]

then

\[ L_a^4 \{ E_a\left(-(p + \frac{q}{a} + \frac{r}{b} + \frac{s}{c} \right)\right) f(x, y, z, t) \] \[ = F_a^4(p + a, q + b, r + c, s + d) \]

Proof:

\[ L_a^4 \{ E_a\left(-(a + p)x + (b + q)y + (c + r)(d + s)z \right\} f(x, y, z, t) \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_a\left(-(a + p)x + (b + q)y + (c + r)(d + s)z \right\} f(x, y, z, t)(dx)^a(dy)^a(dz)^a(dt)^a \]

4. Multiplication by \( x^a t^a \)

If

\[ L_a^4 \{ f(x, y, z, t) \} = F_a^4(p, q, r, s) \]

\[ L_a^4 \{ x^a y^a z^a f(x, y, z, t) \} = \frac{\partial^2}{\partial p^a \partial q^a \partial r^a \partial s^a} \]

\[ L_a^4 \{ f(x, y, z, t) \} \]

Proof:

\[ L_a^4 \{ x^a y^a z^a t^a f(x, y, z, t) \} \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^a E_a\left(-(p + \frac{q}{a} + \frac{r}{b} + \frac{s}{c} \right)\right) f(x, y, z, t)(dx)^a(dy)^a(dz)^a(dt)^a \]

By using the fact \( D_s^a (E_a\left(-s^a x^a \right) = -x^a E_a\left(-s^a x^a \right) \), then

\[ L_a^4 \{ x^a y^a z^a t^a f(x, y, z, t) \} \]
The Convolution Theorem of the Fractional Double Laplace Transform

**Theorem:** The double convolution of order α of functions $f(x, y, z, t)$ and $g(x, y, z, t)$ can be defined as the expression

$$
(f \ast \ast \ast_a g)(x, y, z, t)
$$

where

$$
= \int_0^\infty \int_0^\infty \int_0^\infty f(x - \eta, y - \theta, z - \gamma, t) dt d\eta d\theta d\gamma
$$

and

$$
g(x, y, z, t) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\partial^\alpha}{\partial \eta^\alpha} \frac{\partial^\alpha}{\partial \theta^\alpha} \frac{\partial^\alpha}{\partial \gamma^\alpha} E_a\left(-\eta^\alpha\right) E_a\left(-\theta^\alpha\right) E_a\left(-\gamma^\alpha\right)
$$

Proof: We obtain the definition of fractional quadruple Laplace transform and fractional quadruple convolution above, then we obtain

$$
\left(\frac{\partial^\alpha}{\partial x^\alpha} \right)\left(\frac{\partial^\alpha}{\partial y^\alpha} \right)\left(\frac{\partial^\alpha}{\partial z^\alpha} \right)\left(\frac{\partial^\alpha}{\partial t^\alpha} \right) E_a\left(-x^\alpha\right) E_a\left(-y^\alpha\right) E_a\left(-z^\alpha\right) E_a\left(-t^\alpha\right)
$$

**Remark:** All results above are suitable for quadruple Laplace transform when $\alpha = 1$.
In special case, we have
\[
\int \int \int g(x, y, z, t) \delta_a(x, y, z, t)(dx)^a(dy)^a(dz)^a(dt)^a
= a^4 g(0,0,0,0)
\]

**Example:** we can obtain fractional quadruple Laplace transform of function \( \delta_a(x - a, y - b, z - c, t - d) \) as follows
\[
I^4_a[\delta_a(x - a, y - b, z - c, t - d)]
= \int_0^\infty \int_0^\infty \int_0^\infty E_a(-px + qy + rz + st)^a(dz)^a(dt)^a
= a^4 E_a(-pa + qy + rz + st)^a(10)
\]
In particular, we have \( I^4_a[\delta_a(x, y, z, t)] = a^2 \)

**Relationship between Two Variables Delta Function of Order \( \alpha \) and Mittag-Leffler Function**

The relationship between \( E_a(x + y + z + t)^a \) and \( \delta_a(x, y, z, t) \) is clarified by the following theorem

**Theorem:** The following formula holds
\[
\frac{a^4}{(M_a)^4\alpha} \int \int \int \int E_a(i(-hx)^aE_a(i(-uy)^aE_a(i(-vz)^a
E_a(i(-wt)^a(du)^a(dv)^a(dw)^a
= \delta_a(x, y, z, t)(11)
\]
Where \( M_a \) satisfy the equivalence \( E_a(i((M_a)^a) = 1 \), and it is called period of the Mittag-Leffler function.

**Proof:** We test that (11) agreement with
\[
a^2 = \int \int \int \int E_a(i(hx)^aE_a(i(uy)^aE_a(i(vz)^a
E_a(i(wt)^a(dx)^a(dy)^a(dz)^a(dt)^a
\]
We replace \( \delta_a(x, y, z, t) \) in above equality by (11) to get
\[
a^2 = \int \int \int \int (dx)^a(dy)^a(dz)^a(dt)^a \frac{\alpha^4}{(M_a)^4\alpha}
\]
\[
E_a(i(hx)^aE_a(i(uy)^aE_a(i(vz)^a
E_a(i(wt)^aE_a(i(-px)^aE_a(i(-qy)^aE_a(i(-rz)^a
E_a(i(-st)^a(dp)^a(dq)^a(dr)^a(ds)^a
\]
\[
= \int \int \int \int (dx)^a(dy)^a(dz)^a(dt)^a \frac{\alpha^4}{(M_a)^4\alpha}
\]

**Note that one has as well**
\[
\frac{a^4}{(M_a)^4\alpha} \int \int \int \int E_a(i(-hx)^aE_a(i(-uy)^aE_a(i(-vz)^a
E_a(i(-wt)^a(du)^a(dv)^a(dw)^a = \delta_a(x, y, z, t)
\]

**Inversion Theorem of Quadruple Fractional Laplace Transform**

**Theorem:** Here we recall the fractional quadruple Laplace transform (5) for convenience
\[
I^4_a[f(x, y, z, t)] = F^4_a(p, q, r, s)(12)
\]
And its inverse formula define as
\[
f(x, y, z, t) = \frac{1}{(M_a)^4\alpha} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E_a(px + qy + rz + st)^aF^4_a(p, q, r, s)(dp)^a(dq)^a(dr)^a(ds)^a
\]

**Proof:** Substituting (12) into (13) and using the formula (11), (9) respectively, we obtain in turn
\[
f(x, y, z, t) = \frac{1}{(M_a)^4\alpha} \int_0^\infty \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_a(px)^a
E_a(qy)^aE_a(rz)^aE_a(st)^a(dp)^a(dq)^a(dr)^a(ds)^a
\]
\[
= \int_0^\infty \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_a(-(pj + qk + rl + sm)^a
f(j, k, l, m)(dp)^a(dq)^a(dr)^a(dm)^a
\]
\[
\frac{1}{(M_\alpha)^{4\alpha}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} f(\beta, \gamma, \psi, \theta) (d\beta)^\alpha (d\gamma)^\alpha (d\psi)^\alpha (d\theta)^\alpha \\
(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{(M_\alpha)^{4\alpha}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(\beta, \gamma, \psi, \theta) \delta_\alpha(j-x, y-k, l-z, m-s) (d\beta)^\alpha (d\gamma)^\alpha (d\psi)^\alpha (d\theta)^\alpha \\
= f(x, y, z, t)
\]

**Conclusion**

In this present work, fractional quadruple Laplace transform and its inverse are defined, and several properties of fractional quadruple transform have been discussed which are consistent with quadruple Laplace transform when \( \alpha = 1 \). More over convolution theorem is presented.

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