SPECTRAL SHIFT FUNCTION FOR PERTURBED PERIODIC
SCHRÖDINGER OPERATORS.
THE LARGE-COUPLING CONSTANT LIMIT CASE.

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Abstract. In the large coupling constant limit, we obtain an asymptotic expansion in
powers of $\mu^{-\frac{1}{\delta}}$ of the derivative of the spectral shift function corresponding to the pair
$(P_\mu = P_0 + \mu W(x), P_0 = -\Delta + V(x))$, where $W(x)$ is positive, $W(x) \sim w_0(|x|^\delta$ near
infinity for some $\delta > n$ and $w_0 \in C^\infty(S^{n-1}; \mathbb{R}_+).$ Here $S^{n-1}$ is the unite sphere of the space
$\mathbb{R}^n$ and $\mu$ is a large parameter. The potential $V$ is real-valued, smooth and periodic with
respect to a lattice $\Gamma$ in $\mathbb{R}^n$.

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1
1. Introduction

Consider the perturbed periodic Schrödinger operator

\[ P_\mu = P_0 + \mu W(x), \quad \mu > 0, \]
\[ P_0 = -\Delta_x + V(x). \]

Here \( V \) is a real-valued, \( C^\infty \) function and periodic with respect to a lattice \( \Gamma \) of \( \mathbb{R}^n \). We assume that \( W \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) and satisfies the following estimate: for all \( \alpha \in \mathbb{N}^n \), there exists \( C_\alpha > 0 \) such that

\[ |\partial_x^\alpha W(x)| \leq C_\alpha (1 + |x|)^{-\delta - |\alpha|}, \quad \forall x \in \mathbb{R}^n, \quad \text{with } \delta > n. \]  

The operators \( P_0, P_\mu \) are self-adjoint on \( H^2(\mathbb{R}^n) \). Under the assumption (1.2) we show in Theorem 2.2 below that the operator \( [f(P_\mu) - f(P_0)] \) belongs to the trace class for all \( f \in C_0^\infty(\mathbb{R}) \). Following the general setup we define the spectral shift function, SSF, \( \xi_\mu(\lambda) := \xi(\lambda; P_\mu, P_0) \) related to the pair \( (P_\mu, P_0) \) by

\[ \text{tr}[f(P_\mu) - f(P_0)] = -\langle \xi_\mu(\cdot), f(\cdot) \rangle = \int_\mathbb{R} \xi_\mu(\lambda)f(\lambda)d\lambda, \quad \forall f \in C_0^\infty(\mathbb{R}). \]

By this formula \( \xi_\mu \) is defined modulo a constant but for the analysis of the derivative \( \xi'_\mu(\lambda) \) this is not important.

The notion of SSF was first singled out by the outstanding theoretical physicist I-M. Lifshits in his investigations in the solid state theory, in 1952, see [25]. It was brought into mathematical use in M-G. Krein’s famous paper [24], where the precise statement of the problem has given and explicit representation of the SSF in term of the perturbation determinant was obtained. The work of M-G. Krein’s on the SSF has been described in detail in [2]. Background information on the SSF theory can be found in [35] and [15, Chapter 8].

In the case where \( V = 0 \), the asymptotic behavior of the SSF of the Schrödinger operator has been intensively studied in different aspects (see [5, 20, 26, 27, 33, 34]) and the references given there).

In the semi-classical regime (i.e. \( H(h) = -h^2\Delta_x + W(x), (h \searrow 0) \)) a Weyl type asymptotics of \( \xi_h(\lambda) = \xi(\lambda; H(h), -h^2\Delta) \) with sharp remainder estimate has been obtained (see [33, 34, 36, 37]). On the other hand, if an energy \( \lambda > 0 \) is non-trapping for the classical hamiltonian \( p(x, \zeta) = |\zeta|^2 + W(x) \) (i.e. for all \( (x, \zeta) \in p^{-1}\{\lambda\}, |\exp(tH_p(x, \zeta))| \to \infty \) when \( t \to \infty \)) a complete asymptotic expansion in powers of \( h \) of \( \xi_h(\lambda) \) has been obtained (see [33, 34, 36, 37]). Similar results are well-known for the SSF at high energy (see [1, 6, 27, 28, 32]).

In the large coupling constant limit, the asymptotic behavior of \( \xi_\mu(\lambda) := \xi(\lambda; -\Delta + \mu W, -\Delta) \) depends both on the sign of the perturbation \( W \) and on its decay properties at infinity. For the case of non-negative perturbation \( W \geq 0 \) satisfying \( W(x) \sim w_0(\frac{x}{|x|})|x|^{-\delta} \) near infinity for some \( \delta > n \), it has been proved in [29] (see also [30]) that

\[ \xi_\mu(\lambda) = \mu^{\frac{\delta}{n}}(b_0 + o(1)), \quad \mu \to +\infty, \]

\[ b_0 = (2\pi)^{-n}\kappa_0 \int_{\mathbb{R}^n} \left( (\lambda)_+^{\frac{n}{\delta}} - (\lambda - w_0(\frac{x}{|x|})|x|^{-\delta})_+^{\frac{n}{\delta}} \right) dx, \]

where \( \kappa_0 = \text{vol}\{x \in \mathbb{R}^n; |x| < 1\} \) and \( (\lambda)_+ = \max(\lambda, 0) \).

Under the assumption that \( \omega_0 > 0 \) on \( S^{n-1} \) a complete asymptotic expansion in powers of \( \mu^{-\frac{1}{\delta}} \) is obtained in [12].
In the literature there are a lot of works concerning periodic Schrödinger operator with perturbations see \[1, 3, 5, 9, 10, 11, 14, 16, 18, 19, 21, 23, 43\] but there are only few ones dealing with the spectral shift function, see \[3, 11\] and also \[19\].

It should be mentioned that the tools in \[11\] are related to the asymptotic behavior of \(\xi(\lambda; P_0 + W(hx), P_0), \ h \searrow 0\).

To our best knowledge there are no works treating the large coupling constant limit in the case where \(V \neq 0\). The goal of this work is to generalize the results of \[12\] to the perturbed periodic Schrödinger operator \(P_\mu = P_0 + \mu W(x)\).

The paper is organized as follows: In the next section, we recall some well-known results concerning the spectra of a periodic Schrödinger operator (Subsection 2.1) and we state the assumptions and the results precisely (Subsection 2.2). We give an outline of the proofs in Subsection 2.3. Section 3 is devoted to the proofs. In Subsection 3.1 we built a semiclassical reference operator, denoted by \(Q := H(\mu^{-\frac{n-1}{2}})\), that we use in all the rest of the paper. The proof of the weak asymptotic expansion of \(\xi'_\mu\) is given in Subsection 3.2. At last, The pointwise asymptotic expansion of \(\xi''_\mu\) is proved in Subsection 3.3

2. Statements

2.1. Preliminaries.

Let \(\Gamma = \bigoplus \mathbb{Z} e_i\) be a lattice generated by some basis \((e_1, e_2, \cdots, e_n)\) of \(\mathbb{R}^n\). The dual lattice \(\Gamma^*\) is given by \(\Gamma^* := \{\gamma^* \in \mathbb{R}^n; \langle \gamma | \gamma^* \rangle \in 2\pi \mathbb{Z}, \forall \gamma \in \Gamma\}\). A fundamental domain of \(\Gamma\) (resp. \(\Gamma^*\)) is denoted by \(E\) (resp. \(E^*\)). If we identify opposite edges of \(E\) (resp. \(E^*\)) then it becomes a flat torus denoted by \(\mathbb{T} = \mathbb{R}^n/\Gamma\) (resp. \(\mathbb{T}^* = \mathbb{R}^n/\Gamma^*\)).

Let \(V\) be a real-valued potential, \(\mathcal{C}^\infty\) and \(\Gamma\)–periodic. For \(k \in \mathbb{R}^n\), we define the operator \(P(k)\) on \(L^2(\mathbb{T})\) by \(P(k) := (\partial_y + k)^2 + V(y)\). The operator \(P(k)\) is a semi-bounded self-adjoint with \(k\)-independent domain \(H^2(\mathbb{T})\). Since the resolvent of \(P(k)\) is compact, \(P(k)\) has a complete set of (normalized) eigenfunctions \(\Phi_n(\cdot, k) \in H^2(\mathbb{T})\), \(n \in \mathbb{N}\), called Bloch functions. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities, \(\lambda_1(k) \leq \lambda_2(k) \leq \cdots\). The operator \(P(k)\) satisfies the identity \(e^{-\text{i}y \gamma^*} P(k) e^{\text{i}y \gamma^*} = P(k + \gamma^*)\), \(\forall \gamma^* \in \Gamma^*\), then for every \(p \geq 1\), the function \(k \mapsto \lambda_p(k)\) is \(\Gamma^*\)-periodic.

Ordinary perturbation theory shows that \(\lambda_p(k)\) are continuous functions of \(k\) for any fixed \(p\), and \(\lambda_p(k)\) is even an analytic function of \(k\) near any point \(k_0 \in \mathbb{T}^*\) where \(\lambda_p(k_0)\) is a simple eigenvalue of \(P(k_0)\). The function \(\lambda_p(k)\) is called the band function and the closed intervals \(\Lambda_p := \lambda_p(\mathbb{T}^*)\) are called bands. See \[31, 11\] and also \[39, 40\].

Consider the self-adjoint operator on \(L^2(\mathbb{R}^n)\) with domain \(H^2(\mathbb{R}^n)\):

\[
(2.1) \quad P_0 = -\Delta_x + V(x), \quad \text{where} \quad \Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.
\]

The spectrum of \(P_0\) is absolutely continuous (see \[44\]) and consists of the bands \(\Lambda_p, \ p = 1, 2, \cdots\). Indeed, \(\sigma(P_0) = \sigma_{ac}(P_0) = \bigcup_{p \geq 1} \Lambda_p\). See also \[38\].

Definition 2.1. Let \(\lambda \in \mathbb{R}\) and \(F(\lambda) = \{k \in \mathbb{T}^*; \ \lambda \in \sigma(P(k))\}\) the corresponding Fermi-surface.
We will say that $\lambda \in \sigma(P_0)$ is a simple energy level if and only if $\lambda$ is a simple eigenvalue of $P(k)$, for every $k \in F(\lambda)$.

b) Assume that $\lambda$ is a simple energy level of $P_0$ and let $\lambda(k)$ be the unique eigenvalue defined on a neighborhood of $F(\lambda)$ such that $\lambda(k) = \lambda$, $\forall k \in F(\lambda)$. We say that $\lambda$ is a non-critical energy of $P_0$ if $d_k\lambda(k) \neq 0$ for all $k \in F(\lambda)$.

Note that in one dimension case $F(\lambda)$ is just a finite set of points.

Now, let us recall some well-known facts about the density of states associated with $P_0$, see [38]. The density of states measure $\rho$ is defined as follows:

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{p \geq 1} \int_{\{k \in E^*; \lambda_p(k) \leq \lambda\}} dk.$$  

Since the spectrum of $P_0$ is absolutely continuous, the measure $\rho$ is absolutely continuous with respect to the Lebesgue measure $d\lambda$. Therefore the density of states, $\frac{d\rho}{dE}(E)$, of $P_0$ is locally integrable.

2.2. Results.

Now, we introduce our perturbed periodic Schrödinger operator precisely:

$$P_\mu := P_0 + \mu W(x), \quad \mu > 0,$$

where $P_0$ is a periodic Schrödinger operator given in (2.1) and $W \in C^\infty(\mathbb{R}^n; \mathbb{R})$. Assume that:

(A1) $W$ is strictly positive

and satisfying the following condition:

(A2) There exists a sequence $(w_j)_{j \geq 0} \subset C^\infty(S^{n-1}; \mathbb{R})$ such that for all integer $N$, there exists $R_N(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ s.t.

$$W(x) = \sum_{j=0}^N w_j\left(\frac{x}{|x|}\right)|x|^{-\delta-j} + R_N(x), \quad \text{for all } x, \ |x| \geq 1,$$

where $[\delta > n]$ and for all $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that

$$|\partial_x^{\alpha}R_N(x)| \leq C_\alpha(1 + |x|)^{-\delta-N-1-|\alpha|}.$$

(A3) Assume also that $w_0 > 0$.

The operator $P_\mu$ is self-adjoint, semi-bounded on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$.

The assumption [A2] implies that $W$ goes to zero at infinity then by perturbation theory (Weyl theorem) yields:

$$\sigma_{ess}(P_\mu) = \sigma_{ess}(P_0) = \sigma(P_0) = \bigcup_{p \geq 1} \Lambda_p.$$  

Recall that $\sigma_{ess}(A)$, the essential spectrum of $A$, is defined by $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{disc}(A)$, where $\sigma_{disc}(A)$ is the set of isolated eigenvalues of $A$ with finite multiplicity. Here $A$ is an unbounded operator on a Hilbert space.

Our first theorem concerns the weak asymptotic of $\xi'_\mu(\lambda)$. 
Theorem 2.2 (Weak asymptotic). Let $I$ be a bounded open interval on $\mathbb{R}$. Assume that $W$ satisfies (A1), (A2) and (A3). For $f \in C^\infty_0(I)$, the operator $\left[f(P_\mu) - f(P_0)\right]$ is of trace class and

\begin{equation}
- \langle \xi_\mu, f \rangle := \text{tr}\left[f(P_\mu) - f(P_0)\right] \sim \mu^\frac{\pi}{\delta} \sum_{j=0}^{+\infty} a_j(f) \mu^{-\frac{j}{\delta}}, \quad \text{when } \mu \uparrow +\infty,
\end{equation}

with

\begin{equation}
a_0(f) = (2\pi)^{-n} \sum_{\rho \geq 1} \int_{\mathbb{R}^n} \int_{E^\ast} \left[f(\lambda_p(k) + w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}) - f(\lambda_p(k))\right] dk \, dx
\end{equation}

and

\begin{equation}
a_1(f) = (2\pi)^{-n} \sum_{\rho \geq 1} \int_{\mathbb{R}^n} \int_{E^\ast} f'(\lambda_p(k) + w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}) w_1\left(\frac{x}{|x|}\right)|x|^{-\delta-1} dk \, dx.
\end{equation}

The coefficients $a_j(f)$ are distributions on $f$.

Moreover, if $\lambda$ is a non-critical energy of $P_0$ for all $\lambda \in I$, then $a_j(f) = -\langle \gamma_j(\cdot), f \rangle$, for all $f \in C^\infty_0(I)$. Here $\gamma_j(\lambda)$ are smooth functions of $\lambda \in I$. In particular,

\begin{equation}
\gamma_0(\lambda) = \frac{d}{d\lambda} \left[ \int_{\mathbb{R}^n} \left\{ \rho(\lambda) - \rho\left(\lambda - w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}\right) \right\} dx \right]
\end{equation}

and

\begin{equation}
\gamma_1(\lambda) = \frac{d^2}{d\lambda^2} \left[ \int_{\mathbb{R}^n} \rho\left(\lambda - w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}\right) w_1\left(\frac{x}{|x|}\right)|x|^{-\delta-1} dx \right].
\end{equation}

The proof of this theorem is contained in Subsection 3.2.

Our main result concerning the derivative of the spectral shift function near the bottom of the spectrum is the following.

Let $\lambda_0 = \inf \sigma(P_0) \in \mathbb{R}$. Let $\lambda_1(k)$ be the first Floquet eigenvalue. It is well-known (see [31, 39, 40]) that there exists a bounded interval $[a, b] \subset \lambda_1(T^\ast) \subset \sigma(P_0)$ near $\lambda_0$ such that for all $\lambda = \lambda_1(k) \in [a, b]$, $\lambda$ is a non-critical energy of $P_0$ and satisfies:

\begin{equation}
\Delta_k \lambda_1(k) > 0, \quad \text{for all } k \in F(\lambda).
\end{equation}

Recall that $F(\lambda)$ is a Fermi-surface associated to $\lambda$.

Theorem 2.3 (Pointwise asymptotic). Fix $[a, b]$ as above (satisfying (*)). Assume (A1), (A2) and (A3). Then the following asymptotic expansion holds:

\begin{equation}
\xi_\mu^\ast(\lambda) \sim \mu^n \sum_{j \geq 0} \gamma_j(\lambda) \mu^{-\frac{j}{\delta}}, \quad \text{as } \mu \uparrow +\infty,
\end{equation}

uniformly for $\lambda \in [a, b]$. The coefficients $\left\{\gamma_j(\lambda)\right\}_{j \geq 0}$ are given in Theorem 2.2. Furthermore, this expansion has derivate in $\lambda$ to any order.
2.3. Outline of the proofs.

The purpose of this section is to provide a broad outline of the proofs. As we have noticed in the introduction the asymptotics like (2.5) and (2.10) are well-known for $\xi'(\lambda; P_0 + W(hx), P_0)$ where $h = \mu^{-\frac{1}{3}} \downarrow 0$ and $W$ is a regular potential, bounded with all its derivatives (see [14, 15]). Our strategy in this work will be to show that:

$$\xi'(\lambda) = \xi'(\lambda; Q, P_0) + O(\mu^{-\infty}),$$  \hspace{1cm}  (2.11)

where $Q := H(\mu^{-\frac{1}{3}}) = P_0 + \varphi(\mu^{-\frac{1}{3}}x, \mu^{-\frac{1}{3}})$ and

$$\varphi(x, h) = \phi_0(x) + h\phi_1(x) + \cdots,$$  \hspace{1cm}  (2.12)

has a full asymptotic expansion in powers of $h := \mu^{-\frac{1}{3}}$, with coefficients are uniformly bounded on $x$ together with their derivatives. This makes it possible to apply the results of [14] and [15] (see also [11]).

The main idea of the proof of (2.11) are the two following facts.

1. Since $W > 0$ by assumptions [A1] and [A2] it follows that for all $C > 0$ there exists $\mu_C > 1$ such that for $\mu > \mu_C$ the set

$$\{(x, \zeta) \in \mathbb{R}^{2n}; |x| < C \text{ and } \zeta^2 + V(x) + \mu W(x) \in [a, b]\} = \emptyset.$$  

Here $[a, b]$ is any bounded interval of $\mathbb{R}$. Thus, on the symbol level, $\text{tr}(f(P_\mu) - f(P_0))$ for $f \in \mathcal{C}_b^\infty(\mathbb{R})$, only depends on the asymptotic behavior of $W(x)$ at infinity.

2. On the other hand, by assumption [A2], $\mu W(x) = \varphi(hx, h)$, where $\varphi(x, h)$ satisfies (2.12) with $h = \mu^{-\frac{1}{3}}$ and $\phi_j(x) = w_j(x)|x|^{-\delta_j}$, $j = 1, 2, \cdots$ for $|x| > 1$.

Armed with the above remarks, we will construct in the next subsection a semiclassical reference operator $Q := H(\mu^{-\frac{1}{3}}) = P_0 + \varphi(hx; h)$ with $\varphi(x, h)$ satisfies (2.12) (see Lemma 3.1) and satisfying the following crucial identity:

$$\xi'(\lambda) - \xi'(\lambda; Q, P_0) = \text{tr}(A_1(z))A_2(z)(z - Q)^{-1} + (z - P_0)^{-1}A_3(z)(z - Q)^{-1},$$  \hspace{1cm}  (2.13)

where $z \to A_i(z)$, $(i = 1, 2, 3)$ is analytic in a complex neighborhood $\mathcal{U}$ of $[a, b]$. Moreover, $A_2(z)$ and $A_3(z)$ are of trace class and for all $s, s' \in \mathbb{R}$, we have

$$\left\| (hx)^s A_j(z)(hx)^{s'} \right\|_{L^2} = O(\mu^{-\infty}), \hspace{1cm} j = 2, 3$$  \hspace{1cm}  (2.14)

uniformly on $z \in \mathcal{U}$ (see Lemma 3.4 and Proposition 3.5). We recall that $h = \mu^{-\frac{1}{3}}$.

Using (2.13) and the Helffer-Sjöstrand formula we get (2.11) in the sense of distributions for all $f \in \mathcal{C}_b^\infty([a, b])$. Thus, Theorem 2.2 follows from Theorem 1.3 in [13].

On the other hand, since $z \to A_i(z)$, $(i = 1, 2, 3)$ are analytic, it follows from (2.13) and the stone’s formula (see also Proposition 3.6) that

$$\xi'(\lambda) - \xi'(\lambda; Q, P_0) = \text{tr}(A_2(z))\left[(\lambda + i0 - P_\mu)^{-1} - (\lambda - i0 - P_\mu)^{-1}\right] + \text{tr}(\left[(\lambda + i0 - P_\mu)^{-1} - (\lambda - i0 - P_\mu)^{-1}\right]A_3(z)\left[(\lambda + i0 - Q)^{-1} - (\lambda - i0 - Q)^{-1}\right])$$  \hspace{1cm}  (2.15)

$$= (A) + (B).$$

Using the cyclicity of the trace, we obtain

$$\text{tr}(\left[(hx)^s A_2(z)(hx)^{s'}\right]A_3(z)(hx)^{-s}) = (A) + (B).$$  \hspace{1cm}  (2.16)
Using (*), we will prove that: for \( s > \frac{1}{2} \)

\[
(2.17) \quad \|\langle h x \rangle^{-s} (\lambda \pm i 0 - Q)^{-1} \langle h x \rangle^{-s} \| = O(h^{-1}), \quad \|\langle h x \rangle^{-s} (\lambda \pm i 0 - P_\mu)^{-1} \langle h x \rangle^{-s} \| = O(h^{-1}),
\]

uniformly on \( \lambda \in [a, b] \). Recall that \( h = \mu^{-\frac{1}{2}} \). Combining (2.14) and (2.17) we get \((A) = O(\mu^{\infty})\) uniformly on \( \lambda \in [a, b] \). The same arguments give \((B) = O(\mu^{-\infty})\). Thus, it follows from (2.15) that (2.11) holds uniformly on \( \lambda \in [a, b] \) and therefore Theorem 2.3 follows from [15] (see also [11, Theorem 3]).

For completeness, let us explain the proof of (2.17). Following [18, 21, 24, 7] and the references given there, the spectral study of \((z - Q)\) for \( z \) in a small complex neighborhood of \( \inf(\sigma(P_0)) \) is reduced to the spectral study of an \( h\)-pseudodifferential operator (called effective hamiltonian)

\[
E_{zz}(z, h) = z - \lambda_1(hD_x) - \phi_0(x) + O(h),
\]

via the following well-known identity \((z - Q)^{-1} = E(z) + E_{zz}(z)E_{zz}(z, h)^{-1}E_{zz}(z)\) where \( E(z), E_{zz}(z) \) and \( E_{zz}(z) \) are holomorphic for \( z \) in a small complex neighborhood of \( \inf(\sigma(P_0))\) (see [32], identity (3.15)). Here \( \phi_0 \) is given in Lemma 3.1.

By construction of \( \phi_0 \) we have \( \lambda_1(k) + \phi_0(x) \in [a, b] \) implies that \( \phi_0(x) = \omega_0(\frac{x}{|x|})|x|^{-\delta} \).

Combining this with (*) we deduce that the interval \([a, b]\) is a non-trapping region for the classical hamiltonian \( p(k, x) = \lambda_1(k) + \phi_0(x) \). Consequently, (2.17) follows from a standard limiting absorption principle (see [37], Lemma 3.5) and also [17]).

3. PROOFS

3.1. Construction of semiclassical reference operator \( Q := H(\mu^{-\frac{1}{2}}) \).

In this subsection, we reduce the study of SSF \( \zeta'_\mu(\cdot) \) to a semiclassical problem via the following formula: For \( f \in C_0^\infty(\mathbb{R}) \),

\[
\text{tr}[f(P_\mu) - f(P_0)] = \text{tr}[f(Q) - f(P_0)] + O(\mu^{-\infty}), \quad \mu \gg 1,
\]

where \( Q := H(\mu^{-\frac{1}{2}}) = P_0 + \varphi(\mu^{-\frac{1}{2}} x, \mu^{-\frac{1}{2}}) \) and \( \varphi(x, \mu^{-\frac{1}{2}}) \) has a full asymptotic expansion on \( \mu^{-\frac{1}{2}} \), with coefficients are uniformly bounded on \( x \) together with their derivative. Moreover \( \varphi(\mu^{-\frac{1}{2}} x, \mu^{-\frac{1}{2}}) \) coincide with \( \mu W(x) \) outside \( \Omega_M(\mu^{-\frac{1}{2}}) = \{ x \in \mathbb{R}^n; \mu W(x) \geq M \} \) for \( M \) sufficiently large.

Set \( h = \mu^{-\frac{1}{2}} \). For \( M > 0 \), put \( \Omega_M(h) = \{ x \in \mathbb{R}^n; h^{-\delta} W(x) > M \} \). Let \( r_1, r_2 \) two real satisfying \( 0 < r_1 < (\min w_0)^{\frac{1}{\delta}} \leq (\max w_0)^{\frac{1}{\delta}} < r_2 \). The two real \( r_1, r_2 \) exist since \( w_0 > 0 \) and continuous on the unit sphere.

According to (A2) and (A3) there exists \( h_0 > 0 \) such that

\[
(3.1) \quad B(0, r_1 M^{-\frac{1}{2}}) \subset \Omega_M(h) \subset B(0, r_2 M^{-\frac{1}{2}}), \quad \text{for all } 0 < h \leq h_0.
\]

Here \( B(0, r) \) denotes the ball of center 0 and radius \( r \).

Let \( \chi \in C_0^\infty \left(B(0, r_1 M^{-\frac{1}{2}}); [0, 1]\right) \equiv 1 \) near zero. We introduce the following quantities:

(i) \[ \varphi(x, h) = \left[ 1 - \chi(x) \right] h^{-\delta} W \left( \frac{x}{h} \right) + M \chi(x), \]

(ii) \[ \bar{W}(x) = h^{-\delta} W(x) - \varphi(h x, h) = \chi(h x) \left[ h^{-\delta} W(x) - M \right]. \]
By construction of $\varphi$ and $\tilde{W}$, we have:

**Lemma 3.1.** The two functions $\varphi$ and $\tilde{W}$ are $C^\infty$, and have the following properties:

$$\text{supp}(\tilde{W}) \subset \text{supp}(\chi(h)) \subset \Omega_M(h),$$

(3.2)

$$\varphi(hx, h) > \frac{M}{2} \quad \text{for all } x \in \Omega_M(h),$$

(3.3)

$$|\partial_x^n \varphi(x, h)| \leq C_\alpha \quad \text{uniformly for } h \in [0, h_0],$$

and satisfies for all integer $N$, there exist $\phi_0, \phi_1, \cdots, \phi_N, K_N(h, x) \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to $h \in [0, h_0]$ together with their derivatives such that

$$\varphi(x; h) = \sum_{j=0}^N \phi_j(x) h^j + h^{N+1} K_N(h, x).$$

Moreover, the function $\phi_0$ is given by:

$$\phi_0(x) = (1 - \chi(x)) w_0 \left( \frac{x}{|x|^4} \right) |x|^{-\delta} + M \chi(x).$$

**Lemma 3.2.** If $\phi_0(x) < M$ then $\phi_0(x) = w_0 \left( \frac{x}{|x|^4} \right) |x|^{-\delta}$.

In fact, if $x \in \text{supp} \chi$ then $w_0 \left( \frac{x}{|x|^4} \right) |x|^{-\delta} > r_1^4 |x|^{-\delta} > M$, which implies that

$$\phi_0(x) = (1 - \chi(x)) w_0 \left( \frac{x}{|x|^4} \right) |x|^{-\delta} + M \chi(x) > M \quad \text{for all } x \in \text{supp} \chi.$$

**Choice of the constant $M$:** Let $I = [a, b]$ be a bounded interval of $\mathbb{R}$. The first condition on $M$, we choose $M > |a| + |b|$ such that $\lambda_0(k) + \varphi(x; h) \in I$ therefore $\phi_0(x) < M$ for all $0 < h \leq h_0$.

Let $\Theta \in C^\infty(\mathbb{R}; \left\lbrack \frac{M}{2}, +\infty \right\rbrack)$ satisfying $\Theta(t) = t$, for all $t \geq \frac{M}{2}$. We define

$$F_1(x; h) = \Theta \left( \varphi(hx; h) \right) \quad \text{and} \quad F_2(x; h) = \Theta \left( h^{-\delta} W(x) \right),$$

for all $x \in \mathbb{R}^n$.

Let $U$ be a small complex neighborhood of $I$. From now on, we choose $M$ large enough so that

$$\sup_{x \in \mathbb{R}^n} \left[ V(x) + F_i(x; h) - \Re(z) \right] \geq \frac{M}{4} \quad \text{uniformly for } z \in U, \quad i = 1, 2.$$

This choice of $M$ implies that the function defined by $z \mapsto (z - P_{F_i})^{-1}$ is holomorphic from $U$ to $L(L^2(\mathbb{R}^n))$, where $P_{F_i} = P_0 + F_i$, $i = 1, 2$. Moreover, it follows from (3.3) that $\partial_x^\alpha F_i(x, h) = O_\alpha(h^{-\delta})$.

Finally (3.2) shows that

$$\text{dist} \left( \text{supp}(\tilde{W}), \text{supp}[\varphi(hx, h) - F_1(x, h)] \right) \geq \frac{a_1(M)}{h},$$

(3.4)

$$\text{dist} \left( \text{supp}(\tilde{W}), \text{supp}[h^{-\delta} W(x) - F_2(x, h)] \right) \geq \frac{a_2(M)}{h},$$

with $a_1(M), a_2(M) > 0$ independent of $h$.

Let $Q := H(\mu^{-\frac{1}{2}}) = P_0 + \varphi(hx; h)$. The operator $Q$ with domain $H^2(\mathbb{R}^n)$ is self-adjoint.
Proposition 3.3. For all \( z \in \mathcal{U} \setminus \left[ \sigma(P_\mu) \cup \sigma(Q) \right] \), we have
\[
(z - P_\mu)^{-1} - (z - Q)^{-1} = (z - P_{F_2})^{-1} \tilde{W}(z - P_{F_1})^{-1} + (z - P_{F_2})^{-1} \tilde{W}(z - P_{F_1})^{-1} G_1(z - Q)^{-1}
\]
\[
+ (z - P_{F_2})^{-1} \tilde{W} G_2(z - P_{F_2})^{-1},
\]
where \( G_1(\cdot; h) = \Phi\left( \varphi(h\cdot; h) \right) \) and \( G_2(\cdot; h) = \Phi\left( h^{-\delta} W(\cdot) \right) \), with \( \Phi(t) = t - \Theta(t) \), \( t \in \mathbb{R} \).

Proof. From the resolvent equation, we have:
\[
(z - P_\mu)^{-1} - (z - Q)^{-1} = (z - P_\mu)^{-1} \tilde{W}(z - Q)^{-1},
\]
\[
(z - P_\mu)^{-1} = (z - P_{F_2})^{-1} + (z - P_\mu)^{-1} \left[ h^{-\delta} W(x) - F_2(x; h) \right] (z - P_{F_2})^{-1}
\]
\[
= (z - P_{F_2})^{-1} + (z - P_\mu)^{-1} G_2(z - P_{F_2})^{-1},
\]
and
\[
(z - Q)^{-1} = (z - P_{F_1})^{-1} + (z - Q)^{-1} \left[ \varphi(hx, h) - F_1(x; h) \right] (z - P_{F_1})^{-1}
\]
\[
= (z - P_{F_1})^{-1} + (z - P_\mu)^{-1} G_1(z - Q)^{-1}.
\]

By inserting (3.7) in the right-hand side of (3.6), we obtain
\[
(z - P_\mu)^{-1} - (z - Q)^{-1} = (z - P_{F_2})^{-1} \tilde{W}(z - Q)^{-1}
\]
\[
+ (z - P_{F_2})^{-1} G_2(z - P_{F_2})^{-1} \tilde{W}(z - Q)^{-1}.
\]

Now, we replace \( (z - Q)^{-1} \) in the first term of the right-hand side of (3.9) by (3.8), we get (3.5). \( \square \)

Note that the supports of \( G_1(\cdot; h) \) and \( G_2(\cdot; h) \) are contained in the complementary of the set \( \Omega_{\frac{\alpha}{2}}(h) \).

Lemma 3.4. Let \( \chi_1(\cdot; h), \chi_2(\cdot; h) \) be two \( h \)-dependent functions and \( C^\infty_b \) w.r.t \( x \) in \( \mathbb{R}^n \). We assume that:
(i) the function \( \chi_1 \) is compactly supported,
(ii) there exists \( C > 0 \) independent on \( h \) such that \( \text{dist}(\text{supp}\chi_1, \text{supp}\chi_2) \geq C > 0 \),
(iii) there exist \( m_j \) such that for all \( \alpha \in \mathbb{N}^n \), \( \partial_x^\alpha \chi_j(x, h) = O_h(h^{-m_j}) \), \( (j = 1, 2) \).

Then, for all two reals \( s, s' \), the operator \( K_i := \langle hx \rangle^s \left[ \chi_1(hx, h)(z - P_{F_1})^{-1}\chi_2(hx, h) \right] \langle hx \rangle^{s'} \) is of trace class and its trace norm is \( O(h^{\infty}) \) uniformly on \( h \). Here \( i = 1, 2 \).

Recall that \( a(x) = O(\langle x \rangle^{-\infty}) \) means that for all integer \( N \) there exist \( C_N > 0 \) such that \( |a(x)| \leq C_N (\langle x \rangle^{-N}) \) and \( C^\infty_b \) is the space of \( C^\infty \) functions on \( \mathbb{R}^n \) that are uniformly bounded with respect to \( h \in [0, h_0] \) together with all their derivatives.

Proof. Following Proposition 3.3 in [8] and remembering (i), (ii), the integral kernel \( k_i(x, y, z; h) \) of \( K_i := \langle hx \rangle^s \left[ \chi_1(hx, h)(z - P_{F_1})^{-1}\chi_2(hx, h) \right] \langle hx \rangle^{s'} \), \( (i = 1, 2) \), satisfies: for all \( \alpha, \beta \in \mathbb{N}^n \),
\[
\partial^\alpha_x \partial_y^\beta [k_i(x, y, z; h)] = O_{\alpha, \beta} \left( h^{\infty} \exp \left[ -\frac{1}{C} (d(x, \text{supp}\chi_1(h\cdot, h)) + d(y, \text{supp}\chi_2(h\cdot, h))) \right] \right),
\]
where \( C \) and \( O_{\alpha, \beta} \) are independent of \( (z, h) \in \mathcal{U} \times [0, h_0] \) for some \( h_0 > 0 \) small.
Now, Lemma 3.4 follows from the classical results on trace class operators (see for instance [13, Chapter 9]) and the trace norm of $K_i$ can be estimated by
\[
\|K_i\|_{tr} \leq c_n \sum_{|\alpha|+|\beta| \leq 2n+1} \|\partial_\alpha \partial_\beta k_i(x,y,z;h)\|_{L^1(\mathbb{R}^{2n})},
\]
with a constant $c_n$ depending only on $n$. \hfill \Box

For $z \in \mathcal{U}$, $\Im(z) \neq 0$, put
\[
(3.10) \quad Q(z) = (z - P_\mu)^{-1} - (z - Q)^{-1} - (z - P_{F_2})^{-1}(z - P_{F_1})^{-1}.
\]

**Proposition 3.5.** The operator $Q(z)$ is of trace class and satisfies the following estimate:
\[
(3.11) \quad \|Q(z)\|_{tr} = \mathcal{O}(h^\infty|\Im(z)|^{-2}),
\]
uniformly on $z \in \mathcal{U}$, $\Im(z) \neq 0$.

**Proof.** Identity (3.5) gives
\[
Q(z) = (z - P_{F_2})^{-1}\overline{W}(z - P_{F_1})^{-1}G_1(z - Q)^{-1} + (z - P_\mu)^{-1}G_2(z - P_{F_2})^{-1}\overline{W}(z - Q)^{-1}.
\]

Applying the previous lemma and the fact that $\|(z - Q)^{-1}\|, \|(z - P_\mu)^{-1}\| = \mathcal{O}(|\Im(z)|^{-1})$, we get the proposition 3.5. \hfill \Box

Now we are ready to state the main result of this subsection. Let $J$ be an open interval of $\mathbb{R}$ such that $I \subset J \subset \mathcal{U} \cap \mathbb{R}$.

**Proposition 3.6.** The operator $f(P_\mu) - f(Q)$ is of trace class with $\mathcal{O}(h^\infty)$ as trace norm, uniformly on $f \in \mathcal{C}_0^\infty(J;\mathbb{R})$; $\equiv 1$ near of $\overline{T}$. Moreover,
\[
(3.12) \quad \text{tr}[f(P_\mu) - f(Q)] = \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{\mathbb{R}} f(\lambda) \left[\text{tr}(Q(\lambda + i\epsilon)) - \text{tr}(Q(\lambda - i\epsilon))\right] d\lambda,
\]
where the limit is taken in the sense of distributions.

**Proof.** Let $f \in \mathcal{C}_0^\infty(J;\mathbb{R})$; $\equiv 1$ near of $\overline{T}$. Let $\tilde{f} \in \mathcal{C}_0^\infty(\mathcal{U})$ be an almost analytic extension of $f$, i.e. $\tilde{f}|_{\mathbb{R}} = f$ and $\overline{\partial} \tilde{f}(z) = O_N(|\Im(z)|^N)$ for all $N \in \mathbb{N}$, here $\overline{\partial} = \frac{\partial}{\partial z}$. The functional calculus due to Helffer-Sjöstrand (see for instance [13, Chapter 8]) yields
\[
[f(P_\mu) - f(Q)] = -\frac{1}{\pi} \int \overline{\partial} \tilde{f}(z) \left((z - P_\mu)^{-1} - (z - Q)^{-1}\right) L(dz),
\]
where $L(dz) = dx dy$, $z = x + iy$, $(x,y) \in \mathbb{R}^2$. Remembering the definition of $Q(z)$ with (3.5), we obtain
\[
[f(P_\mu) - f(Q)] = -\frac{1}{\pi} \int \overline{\partial} \tilde{f}(z)(z - P_{F_2})^{-1}\overline{W}(z - P_{F_1})^{-1} L(dz) - \frac{1}{\pi} \int \overline{\partial} \tilde{f}(z)Q(z) L(dz).
\]
Since $(z - P_{F_j})^{-1}, j = 1, 2$, is holomorphic in a neighborhood of $\text{supp}(\tilde{f})$, the first term in the r.h.s. of the previous identity vanishes. Consequently,
\[
[f(P_\mu) - f(Q)] = -\frac{1}{\pi} \int \overline{\partial} \tilde{f}(z)Q(z) L(dz).
\]
Combining this with Proposition 3.5 and using (3.11) and that $\overline{\partial f}(z) = O(|\Im(z)|^2)$, we get $\|f(P_\mu) - f(Q)\|_{tr} = O(h^\infty)$ and

$$
\text{tr} \left[ f(P_\mu) - f(Q) \right] = -\frac{1}{\pi} \int \overline{\partial f}(z) \text{tr} [Q(z)] L(dz)
$$

$$
= \lim_{\epsilon \to 0} \left[-\frac{1}{\pi} \int_{\{\Im(z) > 0\}} \overline{\partial f}(z) \text{tr} [Q(z + i\epsilon)] L(dz) - \frac{1}{\pi} \int_{\{\Im(z) < 0\}} \overline{\partial f}(z) \text{tr} [Q(z - i\epsilon)] L(dz) \right].
$$

Using that $\text{tr} [Q(z + i\epsilon)]$ (resp. $\text{tr} [Q(z - i\epsilon)]$) is holomorphic on $\{z \in \mathcal{U}, \Im(z) > 0\}$ (resp. $\{z \in \mathcal{U}, \Im(z) < 0\}$) and applying Green formula, we have (3.12). \qed

3.2. Proof of the weak asymptotic expansion of $\xi_\mu(\cdot)$.

Let $I$ be an interval of $\mathbb{R}$. Let $f \in C_0^\infty(I)$, we have

$$
f(P_\mu) - f(P_0) = \left[ f(Q) - f(P_0) \right] + \left[ f(P_\mu) - f(Q) \right].
$$

Proposition 3.6 imply that $\left[ f(P_\mu) - f(Q) \right]$ is of trace class and $\| [f(P_\mu) - f(Q)] \|_{tr} = O(h^\infty)$. On the other hand, applying Theorem 1.3 of [14] to the operator $Q$, with $h = \mu^{-\frac{1}{2}} \downarrow 0$ we obtain that $\left[ f(Q) - f(P_0) \right]$ is of trace class and

$$
\text{tr} \left[ f(Q) - f(P_0) \right] \sim h^{-n} \sum_{j=0}^{+\infty} a_j(f) h^j, \quad \text{when } h \downarrow 0,
$$

with

$$
a_0(f) = (2\pi)^{-n} \sum_{\mu \geq 1} \int_{\mathbb{R}^n} \int_{E^*} \left[ f(\lambda_\mu(k) + \phi_0(x)) - f(\lambda_\mu(k)) \right] dk dx
$$

and

$$
a_1(f) = (2\pi)^{-n} \sum_{\mu \geq 1} \int_{\mathbb{R}^n} \int_{E^*} f'(\lambda_\mu(k) + \phi_0(x)) \phi_1(x) dk dx.
$$

Note that, the sum in these equalities is finite, since $\lim_{\mu \to +\infty} \lambda_\mu(k) = +\infty$ and $\phi_0$ is bounded.

The coefficient $a_j(f)$ is a finite sum of term of the form $\int \int c_l(x,k) f^{(l)}(b(x,k)) \, dx dk$, where $c_k$ depends on $\phi$, and their derivatives and $b(x,k) \in \{\lambda_\mu(k), \lambda_\mu(k) + \phi_0(x)\}$, see [13] Chapter 8, Identity (8.16). Then

$$
\text{tr} \left( f(P_\mu) - f(P_0) \right) = \text{tr} \left( f(Q) - f(P_0) \right) + O(h^\infty)
$$

and recall that $h = \mu^{-\frac{1}{2}} \downarrow 0$, we get

$$
\text{tr} \left[ f(P_\mu) - f(P_0) \right] \sim \mu^\frac{n}{2} \sum_{j=0}^{+\infty} a_j(f) \mu^{-\frac{j}{2}}, \quad \text{when } \mu \uparrow +\infty,
$$

with

$$
a_0(f) = (2\pi)^{-n} \sum_{\mu \geq 1} \int_{\mathbb{R}^n} \int_{E^*} \left[ f(\lambda_\mu(k) + w_0(\frac{x}{|x|}|x|^{-\delta})) - f(\lambda_\mu(k)) \right] dk dx
$$
and
\[ a_1(f) = (2\pi)^{-n} \sum_{p \geq 1} \int_{\mathbb{R}^n} \int_{E^*} f'\left(\lambda_p(k) + w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}\right) w_1\left(\frac{x}{|x|}\right)|x|^{-\delta-1} \, dk \, dx, \]
here we have used Lemma 3.2 for the expression of \( \phi_0(x) \).

If \( \lambda \) is a non-critical energy of \( P_0 \) for all \( \lambda \in I \). Then \( d(\lambda_p(k)) \neq 0 \) and \( d(\lambda_p(k) + \phi_0(x)) \neq 0 \) for all \( k \in F(\lambda) \). We recall that \( F(\lambda) \) is the Fermi surface. Therefore, \( a_j(f) = -\langle \gamma_j(\cdot), f \rangle \), for all \( f \in C_0^\infty(I) \) and \( \gamma_j(\lambda) \) are smooth functions of \( \lambda \in I \), in particular,
\[ \gamma_0(\lambda) = \frac{d}{d\lambda} \left[ \int_{\mathbb{R}^n} \{ \rho(\lambda) - \rho\left(\lambda - w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}\right) \} \, dx \right] \]
and
\[ \gamma_1(\lambda) = \frac{d^2}{d\lambda^2} \left[ \int_{\mathbb{R}^n} \rho\left(\lambda - w_0\left(\frac{x}{|x|}\right)|x|^{-\delta}\right) w_1\left(\frac{x}{|x|}\right)|x|^{-\delta-1} \, dx \right], \]
which complete the proof of the theorem 2.2.

### 3.3. Proof of the pointwise asymptotic expansion of \( \xi_\mu(\cdot) \).

Let \( \lambda_0 = \inf \sigma(P_0) \in \mathbb{R} \). Let \( \lambda_1(k) \) be the first Floquet eigenvalue. As noticed in Subsection 2.2 there exists a bounded interval \( \{a, b\} \subset \lambda_1(\mathbb{T}^*) \subset \sigma(P_0) \) near \( \lambda_0 \) such that for all \( \lambda = \lambda_1(k) \in [a, b] \), \( \lambda \) is a non-critical energy of \( P_0 \) and satisfies:
\[ \Delta_k \lambda_1(k) > 0, \quad \text{for all } k \in F(\lambda). \]

From now on, we assume that \( \{a, b\} \) satisfies the above properties. Let \( \mathcal{U} \) be a small compact neighborhood of \( [a, b] \). Put \( \mathcal{U}_\pm = \mathcal{U} \cap \{ z \in \mathbb{C}; \pm \Im(z) > 0 \} \). Recall that \( h = \mu^{-\frac{1}{2}} \).

**Proposition 3.7** (limiting absorption principle). Assume \((A1), (A2)\) and \((A3)\). For all \( \alpha > l - \frac{1}{2} \) with \( l \in \mathbb{N}^* \), we have
\[ \| \langle hx \rangle^{-\alpha}(z - P_\mu)^{-l}\langle hx \rangle^{-\alpha} \| = \mathcal{O}(h^{-l}), \]
uniformly on \( z \in \mathcal{U}_\pm \).

**Proof.** First assume that the operator \( Q \) satisfies the limiting absorption principle, i.e. for all \( \alpha > l - \frac{1}{2} \) with \( l \in \mathbb{N}^* \),
\[ \| \langle hx \rangle^{-\alpha}(z - Q)^{-l}\langle hx \rangle^{-\alpha} \| = \mathcal{O}(h^{-l}), \]
uniformly on \( z \in \mathcal{U}_\pm \). Then we prove the estimation (3.13). In fact, it follows from identity (3.9) that for all \( z \in \mathcal{U} \setminus \sigma(Q) \cup \sigma(P_\mu) \),
\[ (z - P_\mu)^{-1}[\text{Id} - \mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W}(z - Q)^{-1}] = [\text{Id} + (z - P_{F_2})^{-1}\tilde{W}] (z - Q)^{-1}. \]
Then, the weighted resolvent \( (z - P_\mu)^{-1} \) satisfies the identity:
\[ (z - P_\mu)^{-1}\left[\text{Id} - \mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W}\right]_{-\alpha}(z - Q)^{-1} = \]
\[ [\text{Id} + \langle hx \rangle^{-\alpha}(z - P_{F_2})^{-1}\tilde{W}\langle hx \rangle^{-\alpha}](z - Q)^{-1}. \]
Here, we used the notation \( A_\gamma := <hx, A <hx>^{-\gamma} \).

Using the fact that \( z \mapsto (z - P_{F_2})^{-1} \) is holomorphic, \( \tilde{W} \) is compactly supported and the estimation (3.14) for \( (z - Q)^{-1} \) we conclude that for all \( \alpha > \frac{1}{2} \) the right hand side of (3.15)
is $O(h^{-1})$ uniformly on $z \in \mathcal{U}_\pm$ as a bounded operator from $L^2(\mathbb{R}^n)$ on itself. Note that by classical result (see [31]) the integral kernel $r_2(x,y,z;h)$ of $(z - P_{F_2})^{-1}$ satisfies: for all $\beta, \tilde{\beta} \in \mathbb{N}^n$,
\[
\partial^\beta_x \partial^\tilde{\beta}_y r_2(x,y,z;h) = O_{\beta, \tilde{\beta}}(h^{-\delta}) e^{-c|x-y|}, \quad |x-y| > 1,
\]
where $c$ and $O_{\beta, \tilde{\beta}}$ are independent of $(z, h) \in \mathcal{U} \times ]0, h_0].$

On the other hand, Lemma 3.3 yields that $(\mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W})_\alpha$ is $O(h^\infty)$ as a bounded operator from $L^2(\mathbb{R}^n)$ on itself uniformly on $z \in \mathcal{U}$. Then with (3.14) the operator $\left[\text{Id} - (\mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W})_\alpha(z - Q)^{-1}\right]$ is invertible as a bounded operator from $L^2(\mathbb{R}^n)$ on itself uniformly on $z \in \mathcal{U}_\pm$ for $h \in ]0, h_0], h_0 > 0$ small and his inverse is $O(1)$ as a bounded operator from $L^2(\mathbb{R}^n)$ on itself uniformly on $z \in \mathcal{U}_\pm$ for $h \in ]0, h_0], h_0 > 0$.

Hence, for all $\alpha > \frac{1}{2} (z - P_{F_2})^{-1} = O(h^{-1})$ as a bounded operator from $L^2(\mathbb{R}^n)$ on itself uniformly on $z \in \mathcal{U}_\pm$ for $h \in ]0, h_0], h_0 > 0$. The same argument work for $l \geq 2$.

It is well-known that by the method of effective hamiltonian spectral problems of $Q = H(h)$ can be reduced to similar problem of systems of $h$- pseudodifferential operators, (see [18, 17, 14]). In our situation the principal symbol $p$ of the $h$-pseudodifferential operators associated to $Q$ is given by $p(k,x) := \lambda_1(k) + w_0(\frac{x}{|x|})|x|^{-\delta}$ near $p^{-1}([a,b])$. By observing that $x \cdot \nabla_x (w_0(\frac{x}{|x|})) = 0$, we get
\[
\{\nabla \lambda_1(k) \cdot x, p(k,x)\} = |\nabla \lambda_1(k)|^2 + \delta w_0(\frac{x}{|x|})|x|^{-\delta} \Delta \lambda_1(k) \geq c_0 > 0.
\]
in $p^{-1}([a,b])$, here $\{u,v\} := \frac{\partial u}{\partial x} \frac{\partial v}{\partial k} - \frac{\partial u}{\partial k} \frac{\partial v}{\partial x}$ is the poisson bracket between $u$ and $v$ and $c_0$ is some positive constant. This shows that every energy $\lambda \in [a,b]$ is non-trapping for the classical Hamiltonian $p(k,x)$ (see for instance [22, Proposition 21.3]). Now, by Lemma 3.5 in [37] we get the estimation (3.14). For more details see [15].

Proposition 3.8. One has $\text{tr}\left[Q(z)\right] = O(h^\infty)$, uniformly for $z \in \mathcal{U}_\pm$.

Proof. Recall the expression of $Q(z)$:
\[
Q(z) = (z - P_{\mu})^{-1} - (z - Q)^{-1} - (z - P_{F_2})^{-1}\tilde{W}(z - P_{F_1})^{-1}.
\]
Following the proposition 3.3 we have
\[
Q(z) = (z - P_{F_2})^{-1}\tilde{W}(z - P_{F_1})^{-1}\mathcal{G}_1(z - Q)^{-1} + (z - P_{\mu})^{-1}\mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W}(z - Q)^{-1}.
\]
Therefore $\text{tr}\left[Q(z)\right] = \text{tr}(I_1(z)) + \text{tr}(I_2(z))$, where
\[
(3.16) \quad I_1(z) = (z - P_{F_2})^{-1}\tilde{W}(z - P_{F_1})^{-1}\mathcal{G}_1(z - Q)^{-1}
\]
and
\[
I_2(z) = (z - P_{\mu})^{-1}\mathcal{G}_2(z - P_{F_2})^{-1}\tilde{W}(z - Q)^{-1}.
\]
Using the cyclicity of the trace, we show that:
\[
\text{tr}(I_1(z)) = \text{tr} \left[ \langle hx \rangle^{-1} \langle hx \rangle (z - P_{F_2})^{-1} \tilde{W}(z - P_{F_1})^{-1} G_1(z - Q)^{-1} \right] \\
= \text{tr} \left[ \langle hx \rangle (z - P_{F_2})^{-1} \langle hx \rangle^{-1} \cdot \langle hx \rangle \tilde{W}(z - P_{F_1})^{-1} G_1 \right] \langle hx \rangle \cdot \langle hx \rangle^{-1} (z - Q)^{-1} \langle hx \rangle^{-1}.
\]
which implies that
\[
(3.17) \quad \left| \text{tr}(I_1(z)) \right| \leq \left\| \langle hx \rangle (z - P_{F_2})^{-1} \langle hx \rangle^{-1} \right\| \times \left\| \langle hx \rangle \tilde{W}(z - P_{F_1})^{-1} G_1 \right\|_{\text{tr}} \times \left\| \langle hx \rangle^{-1} (z - Q)^{-1} \langle hx \rangle^{-1} \right\|.
\]
Since dist(\(\tilde{W}, G_1\)) \(\geq \frac{C(M)}{h}\). Then, uniformly for \(z \in \mathcal{U}_\pm\),
\[
\left\| \langle hx \rangle \tilde{W}(z - P_{F_1})^{-1} G_1 \right\|_{\text{tr}} = O(h^\infty),
\]
we use here the lemma 3.4. On the other hand, the proposition 3.7 gives that the third term of (3.17) is \(O(h^{-1})\) uniformly on \(z \in \mathcal{U}_\pm\) and by pseudodifferential calculus we get
\[
\left\| \langle hx \rangle^{-1} (z - Q)^{-1} \langle hx \rangle^{-1} \right\| = O(1).
\]
Then we obtain \(\text{tr}(I_1(z)) = O(h^\infty)\) uniformly for \(z \in \mathcal{U}_\pm\).

For \(\text{tr}(I_2(z))\), we use again the resolvent identity:
\[
(z - P_\mu)^{-1} = (z - Q)^{-1} + (z - Q)^{-1} \tilde{W}(z - P_\mu)^{-1}.
\]
Therefore \(\text{tr}(I_2(z)) = \text{tr}(I_{21}(z)) + \text{tr}(I_{22}(z))\), where
\[
I_{21}(z) = (z - Q)^{-1} G_2(z - P_{F_2})^{-1} \tilde{W}(z - Q)^{-1}
\]
and
\[
I_{22}(z) = (z - Q)^{-1} \tilde{W}(z - P_{F_\mu})^{-1} G_2(z - P_{F_2})^{-1} \tilde{W}(z - Q)^{-1}.
\]
The two terms \(\text{tr}(I_{21}(z)), \text{tr}(I_{22}(z))\) are in some sense analogous then we discuss only the term \(\text{tr}(I_{22}(z))\). For this, we use again the cyclicity of the trace and the fact that the multiplication operator by \(\langle hx \rangle^{-2}\) is bounded, we obtain:
\[
\text{tr}(I_{22}(z)) = \text{tr} \left[ \langle hx \rangle^2 \tilde{W} \langle hx \rangle \cdot \langle hx \rangle^{-1} (z - P_{\mu})^{-1} \langle hx \rangle^{-1} \right. \\
\left. \cdot \left( \langle hx \rangle G_2(z - P_{F_2})^{-1} \tilde{W} \langle hx \rangle^2 \right) \cdot \langle hx \rangle^{-2} (z - Q)^{-2} \langle hx \rangle^{-2} \right].
\]
Since dist(\(G_2, \tilde{W}\)) \(\geq \frac{C(M)}{h}\). Then, uniformly for \(z \in \mathcal{U}_\pm\),
\[
\left\| \langle hx \rangle G_2(z - P_{F_2})^{-1} \tilde{W} \right\|_{\text{tr}} = O(h^\infty).
\]
See Lemma 3.4. The proposition 3.7 gives the following estimations:
\[
\left\| \langle hx \rangle^{-1} (z - P_{\mu})^{-1} \langle hx \rangle^{-1} \right\| = O(h^{-1}), \quad \left\| \langle hx \rangle^{-2} (z - Q)^{-2} \langle hx \rangle^{-2} \right\| = O(h^{-2}).
\]
Using the definition of \(\tilde{W}\), more precisely, the fact that his support is in \(B(0, r_1 M^{-\frac{5}{2}})\), we get \(\|\langle hx \rangle^3 \tilde{W}\|_{\infty} = O_M(1)\). Therefore \(\text{tr}(I_{22}(z))\), (and in the same way \(\text{tr}(I_{21}(z))\)), is \(O(h^\infty)\), uniformly for \(z \in \mathcal{U}_\pm\), which finish the proof of Proposition 3.3. \(\square\)
End of the proof of Theorem 2.3. Let $\xi_\mu(\lambda)$ (resp. $\xi_h(\lambda)$) be the spectral shift function associated to the pair $(P_\mu, P_0)$ (resp. $(Q, P_0)$). An immediate consequence of the previous proposition and Proposition 3.6, identity (3.12) is that:

$$
\xi'_\mu(\lambda) = \xi'_h(\lambda) + O(h^\infty),
$$
uniformly for $\lambda \in [a, b]$,

which together with Theorem 2.3 in [15] imply (2.10).

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