QUASI-INTEGRABLE MODULES,
A CLASS OF NON-HIGHEST WEIGHT MODULES OVER
TWISTED AFFINE LIE SUPERALGEBRAS

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Abstract. In this paper, we characterize quasi-integrable modules, of nonzero
level, over twisted affine Lie superalgebras. We show that quasi-integrable
modules are not necessarily highest weight modules. We prove that each quasi-
integrable module is parabolically induced from a cuspidal module, over a finite
dimensional Lie superalgebra having a Cartan subalgebra whose corresponding
root system just contain real roots; in particular, the classification of quasi-
integrable modules is reduced to the known classification of cuspidal modules
over such Lie superalgebras.

1. Introduction

Affine Lie superalgebras, which are the super version of affine Lie algebras, were
introduced and classified by Van de Leur in 1986. Affine Lie superalgebras are
divided into (1) nontwisted types \(X^{(1)}\), where \(X\) is the type of a finite dimensional
basic classical simple Lie superalgebra, and (2) twisted types \(A(2k - 1, 2\ell - 1)^{(2)}\)
\((k, \ell) \neq (1, 1)\), \(A(2k, 2\ell)^{(4)}\), \(A(2k, 2\ell - 1)^{(2)}\) and \(D(k, \ell)^{(2)}\) together with twisted affine
Lie algebras. Roughly speaking, a nontwisted affine Lie superalgebra is a certain
extension of a current Lie superalgebra \(k \otimes \mathbb{C}[t^{\pm 1}]\) in which \(k\) is a basic classical
simple Lie superalgebra but to construct a twisted affine Lie superalgebra, a finite
order automorphism gets involved as well.

Representation theory of affine Lie (super)algebras is one of the important topics
which has drawn considerable attention of mathematicians as well as physicists. In
this regard, after the study of finite dimensional modules, the first step is the study
of finite weight modules. In 1986 and 1988, V. Chari and A. Perssley [11 & 2] studied integrable finite weight modules over an affine Lie algebra and showed that the only irreducible integrable finite weight modules of nonzero level are irreducible integrable highest weight modules and their duals. Since then, there have been several attempts to study representation theory of affine Lie superalgebras; see [5], [7], [8], [11], [13] and the references therein.

For most affine Lie superalgebras with nonzero odd part, the even part contains
two affine Lie subalgebras in twisted case and two or three affine Lie subalgebras in
nontwisted case. In [7], the author shows that if \(M\) is an nonzero level irreducible
integrable finite weight module over a nontwisted affine Lie superalgebra, then \(M\)

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is a trivial module if the even part has at least two affine components and it is a highest weight module if the even part has only one affine component; see also [16].

The structure of an irreducible finite weight module $M$ over a twisted affine Lie superalgebra $\mathfrak{L}$ with root system $R$, strongly depends on the nature of the action of root vectors corresponding to nonzero real roots, i.e., roots which are not self-orthogonal. More precisely, each nonzero root vector corresponding to a nonzero real root $\alpha$, acts on $M$ either injectively or locally nilpotently. We denote by $R^{in}$ (resp. $R^{ln}$) the set of all nonzero real roots $\alpha$ whose corresponding nonzero root vectors act on $M$ injectively (resp. locally nilpotently). In [20], we showed that if $\{\delta\}$ generates the group generated by the set $R_{un}$ consisting of all roots which are orthogonal to all other roots, for each nonzero real root $\alpha$, one of the following occurs:

- $\alpha$ is full-locally nilpotent, i.e., $R \cap (\alpha + \mathbb{Z}\delta) \subseteq R^{ln}$,
- $\alpha$ is full-injective, i.e., $R \cap (\alpha + \mathbb{Z}\delta) \subseteq R^{in}$,
- $\pm \alpha$ are up-nilpotent (resp. down-nilpotent) hybrid, i.e., there is $m \in \mathbb{Z} > 0$ with $R \cap (\pm \alpha + \mathbb{Z}^{\pm m}\delta) \subseteq R^{un}$ (resp. $R^{ln}$) and $R \cap (\pm \alpha + \mathbb{Z}^{\leq -m}\delta) \subseteq R^{in}$ (resp. $R^{ln}$).

So, we can divide our study into two cases either all nonzero real roots are hybrid or some of nonzero real roots are not hybrid; we call $M$ respectively hybrid or tight.

In [20], we reduced the classification of hybrid irreducible finite weight modules over a twisted affine Lie superalgebra to the classification of cuspidal modules of finite dimensional Lie superalgebras; see [6]. In [21], we began the study of tight modules over a twisted affine Lie superalgebra $\mathfrak{L}$ whose even part has two affine components. The structure of such modules depends on whether non-hybrid roots occur for the roots of both affine components or not. We called the root systems of affine components of $\mathfrak{L}_0$, $R(1)$ and $R(2)$ and showed that if for a tight module with bounded weight multiplicity, $R(i) \cap R^{in}$ ($i = 1, 2$) is a proper nonempty subset of the set of real roots of $R(i)$, then $M$ is parabolically induced. In the present paper, we study the case that for a choice $i, j$ with $\{i, j\} = \{1, 2\}$, all nonzero real roots of $R(i)$ belong to $R^{ln}$ while all nonzero real roots of $R(j)$ are hybrid; we call such modules quasi-integrable modules. We also give an example of a module which is quasi-integrable while it is not a highest weight module.

This work is some part of a long-term project on representation theory of twisted affine Lie superalgebras. The main aim of this part is the classification of quasi-integrable modules over twisted affine Lie superalgebras. This in turn is divided into two subparts. In the first subpart, we give the structure of quasi-integrable modules and in the second part, we give the classification. In this paper, we focus on the first subpart. We prove that each quasi-integrable irreducible finite weight module is parabolically induced from a cuspidal module, over a direct sum of a reductive Lie algebra and finitely many basic classical simple Lie superalgebras whose root systems have just real roots. This is the most important step as the classification of cuspidal modules over such Lie superalgebras are known by [17], [14] and [15].

2. Some notations and definitions

In this section, to have a self-contained paper, we gather some information we need throughout the paper. All vector spaces and tensor products are considered over the field of complex numbers $\mathbb{C}$. The degree of a homogeneous element $x$ of a
Elements of a weight space are called weight vectors. A superspace $V = V_0 \oplus V_1$ is called a weight $\mathfrak{t}$-module (with respect to $\mathfrak{h}$) if $[x, y]v = x(yv) - (-1)^{ij} y(xv)$ $(i, j \in \{0, 1\})$, $v \in V$, $x, y \in \mathfrak{t}$, $\mathfrak{t}'V_j \subseteq V_{i+j}$ and $V = \oplus_{\lambda \in \mathfrak{h}^*} V^\lambda$ with $V^\lambda := \{v \in V \mid hv = \lambda(h)v \ (h \in \mathfrak{h})\}$ for each $\lambda \in \mathfrak{h}^*$. In this case, an element $\lambda$ of the support $\text{supp}(V) := \{\lambda \in \mathfrak{h}^* \mid V^\lambda \neq \{0\}\}$ of $V$, is called a weight of $V$ and the corresponding $V^\lambda$ is called a weight space. Elements of a weight space are called weight vectors. If all weight spaces are finite dimensional, the module $V$ is called a finite weight module (with respect to $\mathfrak{h}$).

If $\mathfrak{t}$ has a weight space decomposition with respect to $\mathfrak{h}$ via the adjoint representation, we say $\mathfrak{t}$ has a root space decomposition with respect to $\mathfrak{h}$; the set of weights of $\mathfrak{t}$ is called the root system and weights, weight vectors and weight spaces are called respectively roots, root vectors and root spaces.

If $\mathfrak{t}$ has a root space decomposition with respect to $\mathfrak{h}$ with root system $\Delta$ such that $\mathfrak{t}^0 = \mathfrak{h}$, the form on $\mathfrak{t}$ is restricted to a nondegenerate symmetric bilinear form on $\mathfrak{h}$ which in turn induces naturally a symmetric nondegenerate bilinear form, denoted again by $(\cdot, \cdot)$, on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. So, we can define

\begin{equation}
\Delta^\text{re} = \{\alpha \in \Delta \mid (\alpha, \alpha) \neq 0\}, \quad \Delta^\text{re}_c = \{\alpha \in \Delta \mid (\alpha, \alpha) = 0\}, \quad \Delta^\text{n} = \Delta \setminus \Delta^\text{re}, \quad \Delta^\text{n}_c = \{\alpha \in \Delta^\text{n} \mid (\alpha, \alpha) = 0\}, \quad \Delta^\text{ns} = \Delta^\text{n}_c \cup \{0\}.
\end{equation}

Elements of $\Delta^\text{im}$ (resp. $\Delta^\text{re}$ and $\Delta^\text{ns}$) are called imaginary roots (resp. real roots and nonsingular roots). We also have

\begin{equation}
\forall \alpha \in \mathfrak{h}^*, \text{ there is a unique } t_\alpha \in \mathfrak{h} \text{ with } \alpha(h) = (t_\alpha, h) \text{ for all } h \in \mathfrak{h}.
\end{equation}

In this case, if $V$ is a $\mathfrak{t}$-module and $S \subseteq \Delta$,

\begin{equation}
\text{we denote by } S^\text{in}(V) \text{ (resp. } S^\text{ns}(V)) \text{, the set of all roots } \alpha \in S \text{ with } (\alpha, \alpha) \neq 0 \text{ for which } 0 \neq x \in \mathfrak{t}^0 \text{ acts on } V \text{ locally nilpotently (resp. injectively).}
\end{equation}

If moreover, $V$ has a weight space decomposition $V = \oplus_{\lambda \in \mathfrak{h}^*} V^\lambda$ with respect to $\mathfrak{h}$ and $W$ is an $\mathfrak{h}$-submodule of $V$ then, we have $W = \oplus_{\lambda \in \mathfrak{h}^*} (W \cap V^\lambda)$; we next set

\begin{equation}
\begin{align*}
\mathfrak{S}_W :&= \{\alpha \in \text{span}_\mathbb{Z} \Delta \mid \# \{k \in \mathbb{Z}^>0 \mid \lambda + ka \in \text{supp}(W)\} < \infty (\forall \lambda \in \text{supp}(W))\}, \\
\mathfrak{C}_W :&= \{\alpha \in \text{span}_\mathbb{Z} \Delta \mid \alpha + \text{supp}(W) \subseteq \text{supp}(W)\}.
\end{align*}
\end{equation}

We say the $\mathfrak{t}$-module $V$ to have shadow if

\begin{align*}
(\text{s1}) \Delta^\text{ns}_\mathfrak{S} &:= \{\alpha \in \Delta \mid (\alpha, \alpha) \neq 0\} = \Delta^\text{ns}(V) \cup \Delta^\text{in}(V), \\
(\text{s2}) \Delta^\text{in}(V) = \mathfrak{S}_V \cap \Delta^\text{ns}_\mathfrak{S} \text{ and } \Delta^\text{in}(V) = \mathfrak{C}_V \cap \Delta^\text{ns}_\mathfrak{S}.
\end{align*}

Suppose that $V$ is a $\mathfrak{t}$-module having shadow and $\mathfrak{G}$ is a subalgebra of $\mathfrak{t}$ containing $\mathfrak{h}$. Denote the root system of $\mathfrak{G}$ with respect to $\mathfrak{h}$ by $T$. If $W$ is a $\mathfrak{G}$-submodule of the $\mathfrak{G}$-module $V$, one can easily see that $W$ has also shadow and that $R^*(V) \cap T = T^*(W)$ for $* = \text{in}, \text{ln}$. In other words, that root vectors corresponding to real roots act locally nilpotently or injectively, depends only on $V$, so if $V$ is fixed, we may simply denote $R^*(V)$ and $T^*(W)$ by $R^*$ and $T^*$ respectively, for $* = \text{in}, \text{ln}$.

For a subset $S$ of $\Delta$ and a linear functional $f : \text{span}_{\mathbb{R}} S \longrightarrow \mathbb{R}$, the decomposition

\begin{equation}
S = S^+ \cup S^0 \cup S^-
\end{equation}

where

\begin{equation*}
S^\pm := \{\alpha \in S \mid f(\alpha) \gtrless 0\} \quad \text{and} \quad S^0 := \{\alpha \in S \mid f(\alpha) = 0\}
\end{equation*}

\footnote{We use $\#$ to indicate the cardinal number.}
is called a **triangular decomposition** for $S$. It is called **trivial** if $S = S_{2}^{r}$. A subset $P$ of the root system $\Delta$ of $\mathfrak{l}$ is called a **parabolic** subset of $\Delta$ if

$$\Delta = P \cup -P \quad \text{and} \quad \Delta \cap (P + P) \subseteq P.$$  

**Example 2.1.** Suppose that $\Delta = \Delta^{+} \cup \Delta^{0} \cup \Delta^{-}$ is a nontrivial triangular decomposition for $\Delta$ and $\Delta^{c} = \Delta^{0,+} \cup \Delta^{0,0} \cup \Delta^{0,-}$ is a triangular decomposition for $\Delta^{c}$. Then, $\Delta^{+} \cup \Delta^{0,+} \cup \Delta^{0,0}$ is a parabolic subset of $\Delta$.

Suppose that $P$ is a parabolic subset of $\Delta$ and set

$$\text{(2.6)} \quad \mathfrak{t}_{\mathfrak{p}}^{0} := \bigoplus_{a \in P \cap -P} \mathfrak{t}^{a}, \quad \mathfrak{t}_{\mathfrak{p}}^{+} := \bigoplus_{a \in P \setminus -P} \mathfrak{t}^{a}, \quad \mathfrak{t}_{\mathfrak{p}}^{-} := \bigoplus_{a \in -P \setminus P} \mathfrak{t}^{a} \quad \text{and} \quad \mathfrak{p} := \mathfrak{t}_{\mathfrak{p}}^{0} \oplus \mathfrak{t}_{\mathfrak{p}}^{+}.$$  

Assume $\Omega$ is a module over $\mathfrak{p}$. Consider $\Omega$ as a module over $\mathfrak{p}$ with trivial action of $\mathfrak{t}_{\mathfrak{p}}^{+}$. Denoting by $U(\cdot)$, the universal enveloping algebra of a Lie superalgebra, if the $\mathfrak{t}$-module $U(\mathfrak{t}) \otimes U(\mathfrak{p}) \Omega$ has a unique maximal submodule $Z$ intersecting $\Omega$ trivially, we set

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{t}}(\Omega) := \frac{U(\mathfrak{t}) \otimes U(\mathfrak{p}) \Omega}{Z}.$$  

An irreducible finite weight $\mathfrak{t}$-module $V$ is called **parabolically induced** if there is a parabolic subset $P$ of $\Delta$ and an irreducible module $\Omega$ over $\mathfrak{t}_{\mathfrak{p}}^{+}$ such that $V \simeq \text{Ind}_{\mathfrak{p}}^{\mathfrak{t}}(\Omega)$. An irreducible finite weight $\mathfrak{t}$-module $V$ is called **cuspidal** if it is not parabolically induced.

### 3. Twisted Affine Lie superalgebras

Suppose that $\mathfrak{g}$ is a finite dimensional basic classical simple Lie superalgebra of type $X = A(k, \ell)((k, \ell) \neq (1, 1)), D(k, \ell)$ with the standard Cartan subalgebra $H \subseteq \mathfrak{g}$; here $k$ is a nonnegative integer and $\ell$ is a positive integer. Suppose that $\kappa(\cdot, \cdot)$ is a nondegenerate supersymmetric invariant even bilinear form on $\mathfrak{g}$. In [15], the author introduces a certain automorphism $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

- $\sigma$ is of order $n = 4$ if $X = A(2k, 2\ell)$,
- $\sigma$ is of order $n = 2$ if $X = A(2k - 1, 2\ell - 1), A(2k, 2\ell - 1), D(k, \ell)$.

Suppose $\zeta$ is the $n$-th primitive root of unity. Then, we have

$$\mathfrak{g} = \bigoplus_{j=0}^{n-1} [j] \mathfrak{g} \quad \text{where} \quad [j] \mathfrak{g} := \{x \in \mathfrak{g} \mid \sigma(x) = \zeta^{j}x\} \quad (0 \leq j \leq n - 1).$$

For a two-dimensional vector space $\mathbb{C} c \oplus \mathbb{C} d$, set

$$\text{(3.1)} \quad \mathfrak{L} := \bigoplus_{j=0}^{n-1} ([j] \mathfrak{g} \otimes \mathfrak{t}^{0} \mathfrak{C}[\mathfrak{t}^{0}]) \oplus \mathbb{C} c \oplus \mathbb{C} d \quad \text{and} \quad \mathfrak{h} := (\{[0] \mathfrak{g} \cap H\} \otimes 1) \oplus \mathbb{C} c \oplus \mathbb{C} d.$$  

Then $\mathfrak{L}$, which is denoted by $X^{(\alpha)}\mathbb{C}$, together with

$$[x \otimes t^{p} + rc + sd, y \otimes t^{q} + r'c + s'd] := [x, y] \otimes t^{p+q} + pr(x, y)\delta_{p+q,0}c + sqy \otimes t^{d} - s'px \otimes t^{p}$$

(for $p, q \in \mathbb{Z}, x, y \in \mathfrak{g}$ and $r, r', s, s' \in \mathbb{C}$) in which “$\delta_{., .}$” indicates the Kronecker delta, is called the **twisted affine Lie superalgebra** of type $X^{(\alpha)}$. We refer to the central element $c$ as the **canonical central element** of $\mathfrak{L}$. For the details regarding affine Lie superalgebras see [18]; see also [20] Appendix] and [3].

The Lie superalgebra $\mathfrak{L}$ has a root space decomposition with respect to $\mathfrak{h}$. We denote the corresponding root system by $R$ and refer to $\mathfrak{h}$ as the **standard Cartan subalgebra** of $\mathfrak{L}$. We have $R = R_{0} \cup R_{1}$ where $R_{0}$ (resp. $R_{1}$) is the set of weights of $\mathfrak{L}_{0}$ (resp. $\mathfrak{L}_{1}$) with respect to $\mathfrak{h}$. 

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2It is the subalgebra of all diagonal matrices.
The form $\kappa(\cdot, \cdot)$ induces the following nondegenerate supersymmetric invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{L}$:

$$(x \otimes t^p + rc + sd, y \otimes t^q + r'c + s'd) := \kappa(x, y)\delta_{p+q,0} + rs' + sr'.$$

As the form is nondegenerate on $\mathfrak{h}$, one can transfer the form on $\mathfrak{h}$ to a form on $\mathfrak{h}^*$ denoted again by $(\cdot, \cdot)$. The root system $R$ of $\mathfrak{L}$ with respect to $\mathfrak{h}$ is as in the following table:

| $X^{(\alpha)}$ | $R$ |
|-----------------|------------------|
| $A(2k, 2\ell - 1)^{(2)}$ | $\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \epsilon_i \pm \delta_j | i \neq r, j \neq s\}$ $\cup (2\mathbb{Z} + 1)\delta \pm \{2\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. |
| $A(2k - 1, 2\ell - 1)^{(2)}$, $(k, \ell) \neq (1, 1)$ | $\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \epsilon_i \pm \delta_j | i \neq r, j \neq s\}$ $\cup (2\mathbb{Z} + 1)\delta \pm \{2\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. |
| $A(2k, 2\ell)^{(4)}$ | $\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j | 1 \leq i \leq k, 1 \leq j \leq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \epsilon_i \pm \delta_j | i \neq r, j \neq s\}$. |
| $D(k + 1, \ell)^{(2)}$ | $\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j | 1 \leq i \leq k, 1 \leq j \leq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j, \epsilon_i \pm \epsilon_r, \delta_j \pm \delta_s, \epsilon_i \pm \delta_j | i \neq r, j \neq s\}$. |

in which $\delta, \epsilon, \delta_p \in \text{span}_c R \ (1 \leq i \leq k, 1 \leq p \leq \ell)$ satisfy

$(\epsilon_i, \epsilon_j) = \delta_{i,j}$, $(\delta_p, \delta_q) = -\delta_{p,q}$, $(\epsilon_i, \delta_p) = 0$ and $(\delta, R) = \{0\}$.

The root system $R_0$ of $\mathfrak{L}_0$ has a decomposition $R_0 = R(1) \cup R(2)$ with

| $X^{(\alpha)}$ | $R(1)$ | $R(2)$ if $k \neq 0$ |
|-----------------|------------|----------------|
| $A(2k, 2\ell - 1)^{(2)}$ | $(2\mathbb{Z}\epsilon_i + (1 - \delta_{i,1}))\mathbb{Z}\delta$ $\cup \mathbb{Z}\delta \pm \{\delta_j | 1 \leq j \neq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. | $\mathbb{Z}\delta \cup \mathbb{Z}\delta \pm \{\epsilon_i, \delta_j + \delta_s, \delta_j + \epsilon_i | i \neq r, j \neq s\}$ $\cup (2\mathbb{Z} + 1)\delta \pm \{2\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. |
| $A(2k - 1, 2\ell - 1)^{(2)}$, $(k, \ell) \neq (1, 1)$ | $(2\mathbb{Z}\epsilon_i + (1 - \delta_{i,1}))\mathbb{Z}\delta$ $\cup \mathbb{Z}\delta \pm \{\delta_j | 1 \leq j \neq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. | | $(2\mathbb{Z}\epsilon_i + (1 - \delta_{i,1}))\mathbb{Z}\delta$ $\cup \mathbb{Z}\delta \pm \{\epsilon_i \pm \epsilon_r, \delta_j | 1 \leq j \neq \ell\}$ $\cup (2\mathbb{Z} + 1)\delta \pm \{2\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. |
| $A(2k, 2\ell)^{(4)}$ | $2\mathbb{Z}\delta$ $\cup (2\mathbb{Z} + 1)\delta \pm \{\delta_j | 1 \leq j \leq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{\delta_j + \epsilon_i | 1 \leq j \neq \ell\}$ $\cup 4\mathbb{Z}\delta \pm \{2\delta_j | 1 \leq j \leq \ell\}$. | $2\mathbb{Z}\delta$ $\cup \mathbb{Z}\delta \pm \{\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{\epsilon_i \pm \epsilon_r, \delta_j | 1 \leq j \neq \ell\}$ $\cup (4\mathbb{Z} + 2)\delta \pm \{2\epsilon_i | 1 \leq i \leq k\}$. |
| $D(k + 1, \ell)^{(2)}$ | $2\mathbb{Z}\delta$ $\cup 2\mathbb{Z}\delta \pm \{\delta_j \pm \delta_s | 1 \leq j, s \leq \ell\}$ $\cup 2\mathbb{Z}\delta \pm \{\delta_j \pm \epsilon_i | 1 \leq j, s \leq \ell\}$. | $2\mathbb{Z}\delta$ $\cup \mathbb{Z}\delta \pm \{\epsilon_i | 1 \leq i \leq k\}$ $\cup 2\mathbb{Z}\delta \pm \{\epsilon_i \pm \epsilon_r, \delta_j | 1 \leq j \neq \ell\}$ $\cup (4\mathbb{Z} + 2)\delta \pm \{2\epsilon_i | 1 \leq i \neq k\}$. |

Also, we have $R_e \cup R_{im} = S(1) \cup S(2)$ where $R(2) = S(2) = \emptyset$ if $k = 0$,

$$(3.2) \quad S(1) = \mathbb{Z}\delta \cup R(1) \cup (R \cap \frac{1}{2} R(1)) \quad \text{and} \quad S(2) = \mathbb{Z}\delta \cup R(2) \cup (R \cap \frac{1}{2} R(2)) \quad \text{if} \quad k \neq 0.$$

Recall that a subset $T \subseteq R$ is called closed if $(T + T) \cap R \subseteq T$ and set

$$(3.3) \quad \mathcal{G}(i) := \bigoplus_{\alpha \in S(i)} \mathfrak{L}^\alpha \quad (i = 1, 2).$$

Since $S(i)$ is a closed subset of $R$, we get that $\mathcal{G}(i)$ is a sub-superalgebra of $\mathfrak{L}$.

Using (2.1), for each subset $T$ of $R$, we set

$$T_* := T \cap R_* \quad \text{and} \quad T_*^\times := T \cap R_*^\times \quad (*) = \text{re, im, ns},$$
Also, we set

\[ S_{\hat{\alpha}} := \{ \sigma \in \mathbb{Z}\delta \mid \hat{\alpha} + \sigma \in R \} \quad (\hat{\alpha} \in \hat{R}), \]

\[ \hat{T}_{\hat{\alpha}} := \{ \hat{\alpha} \in \hat{T} \mid \hat{\alpha} + S_{\hat{\alpha}} \subseteq \hat{R}_{\hat{\alpha}} \}, \quad \hat{T}_{\hat{\alpha}}^* := \hat{T}_{\hat{\alpha}} \setminus \{0\} \quad (\ast = \text{re, ns}), \]

\[ \hat{T}_{\hat{\alpha}}^r := \{ \hat{\alpha} \in \hat{T}_{\hat{\alpha}}^* \mid ((\hat{\beta}, \hat{\alpha}) \leq (\hat{\delta}, \hat{\delta}) \quad \forall \hat{\beta} \in \hat{T}_{\hat{\alpha}}^r \}, \]

\[ \hat{T}_{\text{ex}} := 2\hat{T}_{\hat{\alpha}} \cap T \quad \text{and} \quad \hat{T}_{\text{hy}} := \hat{T}_{\hat{\alpha}}^r \setminus (\hat{T}_{\hat{\alpha}} \cup \hat{T}_{\text{ex}}). \]

By \[20\] Table 4 and (3.9), we have:

(a) \( \forall \hat{\alpha} \in \hat{R}^\times \exists r_{\hat{\alpha}} \in \{1, 2, 4\} \) and \( 0 \leq k_{\hat{\alpha}} < r_{\hat{\alpha}} \gg S_{\hat{\alpha}} = (r_{\hat{\alpha}} \mathbb{Z} + k_{\hat{\alpha}})\delta \).

(b) \( \forall \hat{\alpha}, \hat{\beta} \in \hat{R}_{\hat{\alpha}}, \quad S_{\hat{\alpha}} = S_{\hat{\beta}} = \mathbb{Z}\delta \) and \( \forall \gamma, \eta \in \hat{R}_{\hat{\alpha}} \exists s \in \mathbb{Z}^{>0} \gg S_{\gamma} = S_{\delta} = s\mathbb{Z}\delta \).

(c) if \( T \) is a closed subset of \( R, \quad \mathbb{Z}\delta \subseteq T \) and \( \hat{\alpha} \in \hat{T} \), we have \( \hat{\alpha} + S_{\hat{\alpha}} \subseteq T \).

4. QUASI-INTEGRABLE MODULES

Definition 4.1. Using the same notations as in \[33\] we suppose that \( \mathfrak{L} \) is a twisted affine Lie superalgebra with \( R(2) \neq \emptyset \).

(i) For an irreducible finite weight module \( \mathcal{M} \) over \( \mathfrak{L} \), we say a subset \( S \) of \( R \) is tight if there is a nonzero real root \( \alpha \in S \) with \( (\alpha + \mathbb{Z}\delta) \cap S \subseteq S^{\text{fin}}(M) \) or \( (\alpha + \mathbb{Z}\delta) \cap S \subseteq S^{\text{fin}}(M) \); otherwise, we call it hybrid.

(ii) Recall \( \mathcal{G}(i) \) (\( i = 1, 2 \)) from \[33\]. Suppose \( M \) is an irreducible \( \mathfrak{L} \)-module. The \( \mathfrak{L} \)-module \( M \) is called integrable if \( R^h_{\alpha} = R^h \). If \( \{r, t\} = \{1, 2\} \), the irreducible \( \mathfrak{L} \)-module \( M \) is called \( t \)-quasi-integrable if \( S(r) \) is hybrid and \( M \) is integrable as a module over \( \mathcal{G}(t) \), that is \( S(t) \cap R^h_{\alpha} \subseteq R^h \).

Example 4.2. In this example, we present a module over \( \mathfrak{L} := A(2k - 1, 1)^{(2)} \) (\( k \geq 2 \)) which is \( 2 \)-quasi integrable but it is not a highest weight module. Suppose \( \mathfrak{h} \) is the standard Cartan subalgebra of \( \mathfrak{L} \) with corresponding root system \( R \).

We make a convention that in what follows for a subalgebra \( t \) of \( \mathfrak{L} \) containing \( \mathfrak{h} \) with corresponding root system \( s \), a parabolic subset \( P \) of \( s \) and an irreducible module \( \Omega \) over \( t^\times \), the module action on \( \text{Ind}_t^\mathfrak{L}(\Omega) \) is denoted, as usual, by juxtaposition while the module action on \( U(t) \otimes_{U(t^\times)} \Omega \) is denoted by \( \ast \ast \).

We recall that the root system \( R \) of \( A(2k - 1, 1)^{(2)} \) (\( k \geq 2 \)) is

\[ R = \mathbb{Z}\delta \cup \{ \pm 2\delta_1 + 2\mathbb{Z}\delta \} \cup \{ \pm \epsilon_i \pm \delta_1 \mid 1 \leq i \leq k \} + \mathbb{Z}\delta \]

\[ \cup \{ \pm 2 \epsilon_i \mid 1 \leq i \leq k \} + 2\mathbb{Z}\delta + \delta \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq k \} + \mathbb{Z}\delta \].

We define

\[ P_1 := \{0, \epsilon_k \pm \delta_1, \pm 2\delta_1\}, \]

\[ P_2 := \{0\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq k\} \cup \{\epsilon_i \pm \delta_1 \mid 1 \leq i \leq k - 1\} \cup \{\pm \epsilon_k \pm \delta_1\} \cup \{\pm 2\delta_1\}, \]

\[ P_3 := \{0\} \cup \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq k\} + \mathbb{Z}^{\geq 0}\delta \cup \{\pm \epsilon_i \mid 1 \leq i \leq k\} + \{\pm 2\delta_1\} + 2\mathbb{Z}^{\geq 0}\delta \]

\[ \cup \{\pm 2\delta_1\} + 2\mathbb{Z}^{\geq 0}\delta \cup \{\pm \epsilon_i \pm \delta_1 \mid 1 \leq i \leq k \} + \mathbb{Z}^{\geq 0}\delta \].

We have

\[ s_1 := P_1 \cap -P_1 = \{0, \pm 2\delta_1\} \quad \text{(root system of } A_1), \]

\[ s_2 := P_2 \cap -P_2 = \{0, \pm \epsilon_k \pm \delta_1, \pm 2\delta_1\} \quad \text{(root system of } C(2)), \]

\[ s_3 := P_3 \cap -P_3 = \{0, \pm 2\delta_1, \pm \epsilon_i \pm \epsilon_j, \pm \epsilon_i \pm \delta_1 \mid 1 \leq i \neq j \leq k\} \quad \text{(root system of } D(k, 1)) \].
Note that $P_i$ is a parabolic subset of $\mathfrak{s}_{i+1}$ ($i = 1, 2$) and $P_3$ is a parabolic subset of $R$. Recall \(23\) as well as \(31\) and set

$$\mathcal{B}_i := \mathbb{C}c + Cd + \mathbb{C}t_{2\delta_1} + \bigoplus_{j=1}^{k} \mathbb{C}t_{\alpha_j} \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{g}_i} \mathfrak{g}^\alpha$$

and

$$\mathcal{B}_i^+ = \bigoplus_{0 \neq \alpha \in P_i \setminus -P_i} \mathfrak{g}^\alpha \quad (i = 2, 3).$$

Assume $\zeta \in \mathbb{Q} \setminus \mathbb{Z}$ and $\xi$ is an irrational number. Fix $e \in \mathcal{B}_{2\delta_1}$ and $f \in \mathcal{B}^{-2\delta_1}$ with $[e, f] = t_{2\delta_1}$; see \(23\) and \(19\) Lem. 3.1. Suppose $K_1$ is the vector space with basis \(\{v_\mu \mid \mu \in \zeta + 2\mathbb{Z}\}\). Then $K_1$ together with

$$\begin{align*}
    dv_\mu &= 0, & cv_\mu &= (2k + 2)v_\mu, & f v_\mu &= \frac{1}{2}(\xi - (\mu - 1)^2)v_{\mu-2}, & ev_\mu &= v_{\mu+2}, \\
    t_{2\delta_1}v_\mu &= 2\mu v_\mu, & t_{\alpha} v_\mu &= (k - i + 2)v_\mu & (1 \leq i \leq k, \mu \in \zeta + 2\mathbb{Z})
\end{align*}$$

is an irreducible finite weight module over the finite dimensional reductive Lie algebra

$$\mathfrak{b}_1 := \mathbb{C}c \oplus Cd \oplus \mathbb{C}t_{2\delta_1} \oplus \bigoplus_{j=1}^{k} \mathbb{C}t_{\alpha_j} \oplus \mathbb{C}e \oplus \mathbb{C}f$$

with respect to $\mathbb{C}c \oplus Cd \oplus \mathbb{C}t_{2\delta_1} \oplus \bigoplus_{j=1}^{k} \mathbb{C}t_{\alpha_j}$. We note that both $e$ and $f$ act injectively on $K_1$, so

$$\pm 2\delta_1 \in \mathcal{g}_{1}^\alpha(K_1).$$

Define

$$\varrho : \mathbb{C}c + Cd + \mathbb{C}t_{2\delta_1} + \bigoplus_{i=1}^{k} \mathbb{C}t_{\alpha_i} \longrightarrow \mathbb{C}$$

$$c \mapsto 2k + 2, \quad d \mapsto 0, \quad t_{2\delta_1} \mapsto -2\zeta, \quad t_{\alpha} \mapsto k - i + 2 \quad (1 \leq i \leq k).$$

Then

$$(4.1) \quad \text{supp}(K_1) = \varrho + 2\mathbb{Z}\delta_1.$$ 

We next set

$$K_i := \text{Ind}_{P_i}^{\mathcal{B}_i}(K_{i-1}) \quad (i = 2, 3, 4)$$

and note that

$$K_1 \overset{\text{as } \mathcal{B}_1-\text{submod}}{\longrightarrow} K_2 \overset{\text{as } \mathcal{B}_2-\text{submod}}{\longrightarrow} K_3 \overset{\text{as } \mathcal{B}_3-\text{submod}}{\longrightarrow} K_4.$$ 

Since $K_1$ is an irreducible finite weight module over $\mathcal{B}_1$, it is not hard to see that $K_i$ ($i = 2, 3$) is also an irreducible finite weight module over $\mathcal{B}_i$ and $M := K_4$ is an irreducible finite weight module over $\mathfrak{g}$; see \[5\] Pro. 1.8] and \[6\] Lem. 2.3. The aim of this example is showing that $M$ is 2-quasi-integrable while it is not a highest weight module.

For a nonzero nonsingular root $\alpha$, we have $2\alpha \notin R$ and $\dim(\mathfrak{g}^\alpha) = 1$. So, we get

$$(4.2) \quad \mathfrak{g}^\alpha \circ \mathfrak{g}^\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^\alpha]v = \{0\} \quad \text{for } \alpha \in R^+_\mathfrak{g} \cap \mathfrak{g}_i, \quad v \in K_i \quad (i = 1, 2, 3).$$

Also from the structure of $K_i$ ($i = 1, 2, 3$), we have

$$(4.3) \quad \mathfrak{g}^\alpha \ast K_i = \{0\} \quad \text{for } \alpha \in P_i \setminus -P_i \quad (i = 1, 2, 3).$$

**Step 1.** sup$(K_2) \subseteq \varrho + 2\mathbb{Z}\delta_1 \setminus \{0, 1, 2\} \epsilon_k$ : Since $K_2 = \text{Ind}_{P_i}^{\mathcal{B}_i}(K_1)$ is generated by $K_1$, we are done using \(11\), \(12\) and \(13\) together with the fact that $-P_i \setminus P_i = (-\epsilon_k \pm \delta_1) \subseteq (\mathfrak{s}_2)_n^\alpha$.

**Step 2.** $\{\pm (\epsilon_1 \pm \epsilon_2), \ldots, \pm (\epsilon_{k-1} \pm \epsilon_k)\} \subseteq \mathfrak{g}_{1, 2}^{\alpha}(K_3)$ : One knows that

$$B := \{\alpha_j := \epsilon_j - \epsilon_{j+1}, \alpha_k := \epsilon_k - \delta_1, \alpha_{k+1} := 2\delta_1 \mid 1 \leq j \leq k - 1\}$$

is a base of $\mathfrak{s}_i = D(k, 1)$. For each $1 \leq j \leq k + 1$, fix $e_{\alpha_j} \in \mathfrak{g}^{\alpha_j}$ and $f_{\alpha_j} \in \mathfrak{g}^{-\alpha_j}$ with $[e_{\alpha_j}, f_{\alpha_j}] = t_{\alpha_j}$. For $1 \leq j \leq k - 1$ and a weight vector $v \in K_i \subseteq K_2$ of weight $\lambda \in \varrho + 2\mathbb{Z}\delta_1$, as $B$ is a base, using PBW-Theorem, the $\mathfrak{g}_3$-submodule of $U(\mathfrak{g}_3) \otimes_U(\mathfrak{g}_2 \otimes \mathfrak{g}_3)$, $K_2$ generated by $f_{\alpha_j}^2 v$ is the linear span of the elements of the form

$$f_{\alpha_1} \cdots f_{\alpha_p} e_{\alpha_{j_1}} \cdots e_{\alpha_{j_t}} f_{\alpha_j}^2 v, \quad f_{\alpha_1} \cdots f_{\alpha_p} f_{\alpha_{j_1}}^2 v, \quad e_{\alpha_1} \cdots e_{\alpha_{j_t}} f_{\alpha_j}^2 v, \quad f_{\alpha_j}^2 v, \quad f_{\alpha_j}^2 v$$
for $1 \leq j_1, \ldots, j_t, p_1, \ldots, p_s \leq k + 1$. Since $j \in \{1, \ldots, k - 1\}$ and for $p \neq k + 1$, by (3.3), $e_p \ast v = 0$, we have for $1 \leq p \leq k + 1$ that
\[
eq_{\alpha_p} f_{\alpha_j}^2 \ast v = \begin{cases} f_{\alpha_j}^2 e_p \ast v = 0 & p \neq j, k + 1, \\ (2g - \alpha_j)(t_{\alpha_j}) f_{\alpha_j} \ast v = 0 & p = j, \\ f_{\alpha_j}^2 e_{\delta_1} \ast v \in f_{\alpha_j}^2 \ast K_1^{\lambda + 2\delta_1} & p = k + 1, \end{cases}
\]
that is
\[
eq_{\alpha_p} f_{\alpha_j}^2 \ast K_1^{\lambda} = \neq_{\alpha_p} f_{\alpha_j}^2 \ast K_1^{\lambda} \subseteq f_{\alpha_j}^2 \ast K_1^{\lambda + 2\delta_1}.
\]
So, we get
\[
\neq_{\alpha_{j_1}} \cdots \neq_{\alpha_{j_t}} f_{\alpha_j}^2 \ast K_1 \subseteq f_{\alpha_j}^2 \ast K_1.
\]
This implies that the support of the $B_3$-module generated by $f_{\alpha_j}^2 \ast v (1 \leq j \leq k - 1)$ lies in $g + Z\delta_1 - 2(\epsilon_j - \epsilon_{j+1}) + \sum_{t=1}^{k} Z^{\leq 0} \alpha_t$. This, in particular, together with Step 1 implies that the $B_3$-submodule of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g} \oplus \mathfrak{h}_+)} K_2$ generated by $f_{\alpha_j}^2 \ast v$ intersects $K_2$ trivially and so $f_{\alpha_j}^2 v = 0 (1 \leq j \leq k - 1)$; in other words,
\[
(4.4) - \alpha_j = -(\epsilon_j - \epsilon_{j+1}) \in a_0^\ast(K_3) \quad (1 \leq j \leq k - 1).
\]
Next, note that
\[
B := \{ \beta_i := \epsilon_i - \epsilon_{i+1}, \beta_{k-1} := \epsilon_{k-1} + \epsilon_k, \beta_k := -\epsilon_k - \delta_1, \beta_{k+1} := 2\delta_1 | 1 \leq i \leq k - 2\}
\]
is also a base of $\mathfrak{g}_3$. Contemplating Step 1 and suppose
\[
r := \max \{ j \mid \exists n \in Z \ni g - j\epsilon_k + n\delta_1 \in \text{supp}(K_2) \} \quad \text{and} \quad \mu := g - r\epsilon_k + m\delta_1 \in \text{supp}(K_2)
\]
for some $m \in Z$. Fix $0 \neq v \in K_2^\ast = K_2^{\mu - r\epsilon_k + m\delta_1}$. For $1 \leq i \leq k - 1$, fix $e_{\beta_i} \in \mathfrak{g}_i^{\delta_1}$ and $f_{\beta_i} \in \mathfrak{g}^{\delta_1}$ with $[e_{\beta_i}, f_{\beta_i}] = t_{\beta_i}$. Setting
\[
s := \mu(t_{\beta_{k-1}}) = (g - r\epsilon_k + m\delta_1)(t_{\beta_{k-1}}) = g(t_{\epsilon_{k-1}}) + g(t_{\epsilon_k}) - r = 5 - r \geq 0,
\]
as $e_{\beta_k} \ast v \in K_2^{\mu - r\epsilon_k - \delta_1} = K_2^{\delta_1 + (r+1)\epsilon_k + (m-1)\delta_1} = \{0\}$, we have using (3.3) that
\[
eq_{\beta_i} f_{\beta_{k-1}}^t \ast v = \begin{cases} f_{\beta_{k-1}}^t e_{\beta_i} \ast v = 0 & t = k, 1 \leq t < k - 2, \\ (s + 1)(\mu(t_{\beta_{k-1}}) - s)f_{\beta_{k-1}}^t v = 0 & t = k - 1, \\ f_{\beta_{k-1}}^t e_{2\delta_1} \ast v & t = k + 1. \end{cases}
\]
So for $1 \leq j_1, \ldots, j_t \leq k$, we get
\[
eq_{\beta_{j_1}} \cdots \neq_{\beta_{j_t}} f_{\beta_{k-1}}^t \ast v \in f_{\beta_{k-1}}^t \ast \sum_{j \in Z} K_2^{\mu + 2\delta_1 - r\epsilon_k}.
\]
This implies that the support of the $B_3$-submodule of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g} \oplus \mathfrak{h}_+)} K_2$ generated by $f_{\beta_{k-1}}^t \ast v$ lies in $g + Z\delta_1 - r\epsilon_k - (s + 1)\beta_{k-1} + \sum_{t=1}^{k} Z^{\leq 0} \beta_t$. This together with Step 1 implies that the $B_3$-submodule generated by $f_{\beta_{k-1}}^t \ast v$ intersects $K_2$ trivially and so $f_{\beta_{k-1}}^t v = 0$. This means that $-\beta_{k-1} = -\epsilon_{k-1} - \epsilon_k \in a_0^\ast(K_3)$. Since $\{ \epsilon_i - \epsilon_{i+1}, \epsilon_{k-1} + \epsilon_k | 1 \leq i \leq k - 1\}$ is a base of $D(k)$, this together with (4.1), (4.2) and (20) Thm. 4.7 completes the proof of this step.

**Step 3.** $3 \leq \ell \leq (\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_{i+1}, 2\epsilon_k - \delta \mid 1 \leq i \leq k - 1 \} \subseteq R^{\text{lin}}(K_4)$: We first mention that $\Delta := \{ \gamma_1 := -2\delta_1, \gamma_2 := \delta_1 + \epsilon_k, \gamma_{i+1} := -\epsilon_i + \epsilon_{i+1}, \gamma_{k+2} := -\epsilon_{k-1}, \gamma_{k+1} := 2\delta_1 - 2\epsilon_1 | 2 \leq i \leq k \}$ is a linearly independent subset of $R$ with $R \subseteq \text{span}_{\mathbb{Z} \geq 0} \Delta \cup \text{span}_{\mathbb{Z} \leq 0} \Delta$. Suppose that $0 \neq v \in K_1 \subseteq K_2 \subseteq K_3$ is a weight vector of weight $\lambda \in g + 2Z\delta_1$. We note that as
\[
\Delta \setminus \{-2\delta_1\} \subseteq (P_3 \setminus P_2) \cup (P_2 \setminus P_2) \cup (P_1 \setminus P_1),
\]
we have using (4.3) that $e_{\gamma_i} \ast v = 0 (2 \leq i \leq k + 2)$. So, for $1 \leq p \leq k + 1$, we have
\[
eq_{\gamma_p} f_{\gamma_{k+2}} \ast v = \begin{cases} f_{\gamma_{k+2}} e_{\gamma_p} \ast v = 0 & 2 \leq p \leq k + 1, \\ \lambda(t_{\gamma_{k+2}}) v = 0 & p = k + 2, \\ f_{\gamma_{k+2}} e_{-2\delta_1} \ast v & p = 1. \end{cases}
\]
If the following proposition:

Proposition 4.5.

Remark

Since by Step 3, \( k, \ell \leq r \) and \( \supp(M) \subseteq \mathbb{Z}^+ \), there is \( \lambda, \alpha \subseteq \mathbb{Z}^0 \), such that \( \lambda + \alpha \in \supp(W) \) and \( 2(\lambda, \alpha) \in \mathbb{Z}^+ \). Moreover, if \( \lambda \in \supp(W) \) and \( 2(\lambda, \alpha) \in \mathbb{Z}^+ \), then \( \lambda - \alpha \in \supp(W) \).

Proposition 4.4. Suppose that \( T \) is a closed subset of \( R \). Set \( \mathcal{G} := \oplus_{\alpha \in \mathcal{T}} \Sigma^\alpha \). Assume \( W \) is a \( \mathcal{G} \)-submodule of \( R \) and \( -\alpha \subseteq T \cap R^{\alpha} \). Then we have the following:

(i) If \( \lambda, \alpha + \lambda \in \supp(W) \), then \( \Sigma^\alpha W^\lambda \neq \{0\} \).

(ii) For \( \lambda \in \supp(W) \), \( \frac{2(\lambda, \alpha)}{\alpha, \alpha} \in \mathbb{Z} \). Moreover, if \( \lambda \in \supp(W) \) and \( 2(\lambda, \alpha) \in \mathbb{Z}^+ \) (resp. \( \alpha \in \mathbb{Z}^0 \)), then \( \lambda - \alpha \in \supp(W) \) (resp. \( \lambda + \alpha \in \supp(W) \)).

Proposition 4.5. Let \( T \) be a closed subset of \( R \) and \( \mathcal{G} := \oplus_{\alpha \in \mathcal{T}} \Sigma^\alpha \) (in particular, \( h = \mathbb{Z} \in \mathcal{G} \)). Suppose \( W \) is a \( \mathcal{G} \)-submodule of \( M \).

(i) Assume \( \mathcal{A} \subseteq \supp(W) \) is nonempty and \( S \subseteq T \) is a nonempty finite subset with \( S \cap \mathcal{R}_M = \emptyset, S \subseteq \mathcal{B}_W \), \( -S \subseteq \mathcal{C}_W \) and \( (\mathcal{A} + S) \cap \supp(W) \subseteq \mathcal{A} \).

Then, there is \( \lambda \in \mathcal{A} \) such that \( (\lambda + \supp(Z^\alpha S)) \cap \supp(W) = \{\lambda\} \).
(ii) Suppose that $f$ is a functional on $\text{span}_R T$ with $f(\delta) \neq 0$ and corresponding triangular decomposition $T = T^+ \cup T^0 \cup T^-$. Assume for $K := T_{re} \setminus \frac{1}{2} T_{re}$, $J := K \cup (\mathcal{Z} \delta \cap \text{span}_R K)$ is an affine root system. If

$$T_{re}^+ := T^+ \cap R_{re} \subseteq \mathcal{B}_W \quad \text{and} \quad T_{re}^- := T^- \cap R_{re} \subseteq \mathcal{C}_W,$$

we have the following:

(a) Let $r \in \{\pm 1\}$ with $f(r \delta) > 0$, then there are $\lambda \in \text{supp}(W)$ and a positive integer $p$ with $(\lambda + Z^{>p}(r \delta)) \cap \text{supp}(W) = \emptyset$.

(b) Recall (3.3) and suppose $\lambda, \delta$.

Proof. (i) See [20] Pro. 3.6.

(ii) (a) follows from (i) together with the same argument as in [20] Lem. 5.1.

(ii) (b) follows from the same argument as in [20] Pro. 3.7 by using part (ii)(a) and changing the role of $\delta$ with $-\delta$ if $f(\delta) < 0$. \hfill $\square$

Proposition 4.6. Recall (3.3) and suppose $M$ is of nonzero level. Suppose $W \subseteq M$ and \{1, 2\}.

(i) Let $f$ be a linear functional on $\text{span}_R S(i)$ with corresponding triangular decomposition $S(i) = S(i)^+ \cup S(i)^0 \cup S(i)^-$. If $W$ is a $\mathcal{G}(i)$-submodule of $M$ and $f(\delta) \neq 0$, $S(i)^+ \cap R_{re} \subseteq R_{in}$, $S(i)^- \cap R_{re} \subseteq R_{in}$,

then there is $\mu \in \text{supp}(W)$ such that $\mu + \alpha \notin \text{supp}(W)$ for all $\alpha \in S(i)^+$.

(ii) Assume $W$ is an integrable $\mathcal{G}(j)$-submodule of $M$ and $\lambda \in \text{supp}(W)$ satisfies $\text{supp}(W) \cap (\lambda + Z^{>p} r \delta) = \emptyset$ for some $r \in \{\pm 1\}$. Then for each $\hat{\alpha} \in \tilde{S}(j)$, see [3.7], there is $m_{\alpha} \in Z^{>0}$ such that $\lambda + \hat{\alpha} + nr \delta \notin \text{supp}(W)$ for all $n \geq m_{\alpha}$.

(iii) Suppose that $S(i)$ is up-nilpotent (resp. down-nilpotent) hybrid. If $W$ is an invariant under the action of both $\mathcal{G}(i)$ and $\mathcal{G}(j)$ such that it is integrable as a module over $\mathcal{G}(j)$, then there are $\mu \in \text{supp}(W)$ and a functional $f$ on $\text{span}_R W$ with $f(\delta) > 0$ (resp. $f(\delta) < 0$) and $f(\tilde{S}(j)) = \{0\}$ such that

- $\mu + \alpha \notin \text{supp}(W)$ for all $\alpha \in S(i)$ with $f(\alpha) > 0$,
- for each $\hat{\alpha} \in \tilde{S}(j)$, there is $m_{\alpha} \in Z^{>0}$ such that $\mu + \hat{\alpha} + n \delta \notin \text{supp}(W)$ (resp. $\mu + \hat{\alpha} - n \delta \notin \text{supp}(W)$) for all $n \geq m_{\alpha}$.

Proof. (i) We note that $S(i)^{\infty} \subseteq \mathcal{C}_W$. Since $M$ has shadow, $R_{in} \subseteq \mathcal{B}_M \subseteq \mathcal{B}_W$. Therefore, the result follows from [4.3] and Proposition 4.4(ii)(b).

(ii) We first assume $r = 1$. Suppose $\lambda$ is as in the statement and $\hat{\alpha} \in \tilde{S}(i)^{\infty}$. Recalling (3.6), for each $n \in Z$ we have

$$\hat{\alpha} + (k_{\alpha} + r_{\alpha} n) \delta \in R_{re}^c.$$

This in particular together with Proposition 4.4(iii) implies that

$$2(\hat{\alpha} + k_{\alpha} \delta, \lambda)/(\hat{\alpha}, \hat{\alpha}), 2(\hat{\alpha} + (k_{\alpha} + r_{\alpha}) \delta, \lambda)/(\hat{\alpha}, \hat{\alpha}) \in Z$$

which in turn implies that $2(r_{\alpha} \delta, \lambda)/(\hat{\alpha}, \hat{\alpha}) \in Z$. Since the canonical central element $c$ acts on $M$ as a nonzero scalar, we get

$$(\lambda, \delta)M^\lambda = \lambda(c)M^\lambda = cM^\lambda \neq \{0\}.$$

Therefore, $(\lambda, \delta) \neq 0$ and so $2(r_{\alpha} \delta, \lambda)/(\hat{\alpha}, \hat{\alpha}) \in Z \setminus \{0\}$.

---

3In Proposition 3.7 of [20], we are working with $\mathcal{C}$ or $\mathcal{Z}$ instead of $\mathcal{G}$ introduced here, but our assumptions on $T$ make the same situation as in Proposition 3.7 of [20].
Case 1. \( 2(\lambda, r_\alpha \delta)/(\check{\alpha}, \check{\alpha}) \in \mathbb{Z}_{>0} \) : For each \( \check{\alpha} \in \check{S}(i)^{\times} \), considering (1.6), we choose \( N \in \mathbb{Z}_{>0} \) such that \( 2(\lambda, \check{\alpha} + (k_\alpha + r_\alpha N)\delta)/(\check{\alpha}, \check{\alpha}) \) is positive. We claim that for \( n > k_\alpha + r_\alpha N \), \( \lambda + \check{\alpha} + n\delta \notin \text{supp}(W) \). In fact, if \( n > k_\alpha + r_\alpha N \), then \( \lambda + \check{\alpha} + n\delta \notin \text{supp}(W) \), since

\[
2(\lambda + \check{\alpha} + n\delta, \check{\alpha} + (k_\alpha + r_\alpha N)\delta)/(\check{\alpha}, \check{\alpha}) = (2(\lambda, \check{\alpha} + (k_\alpha + r_\alpha N)\delta)/(\check{\alpha}, \check{\alpha})) + 2 > 0,
\]

we get from Proposition 4.4(iii) that \( \lambda + (n - k_\alpha - r_\alpha N)\delta \notin \text{supp}(W) \) which is a contradiction.

Case 2. \( 2(\lambda, r_\alpha \delta)/(\check{\alpha}, \check{\alpha}) \in \mathbb{Z}_{<0} \): We claim that \( \lambda + \check{\alpha} + n\delta \notin \text{supp}(W) \) for all \( \check{\alpha} \in \check{S}(i)^{\times} \) and all positive integers \( n \). To the contrary, let \( \lambda + \check{\alpha} + n\delta \notin \text{supp}(W) \) for some \( \check{\alpha} \in \check{S}(i)^{\times} \) and some positive integer \( n \). Consider Proposition 4.4(iii) and choose \( p \in \mathbb{Z}_{>0} \) such that \( pr_\alpha > k_\alpha \) and \( 2(\lambda, \check{\alpha} + r_\alpha (-p)\delta + k_\alpha \delta)/(\check{\alpha}, \check{\alpha}) > 0 \). This gives that

\[
pr_\alpha - k_\alpha > 0 \quad \text{and} \quad 2(\lambda + \check{\alpha} + n\delta, \check{\alpha} + r_\alpha (-p)\delta + k_\alpha \delta)/(\check{\alpha}, \check{\alpha}) > 0,
\]

and so Proposition 4.4(iii) implies that

\[
\lambda + (n + pr_\alpha - k_\alpha)\delta = \lambda + \check{\alpha} + n\delta - (\check{\alpha} + r_\alpha (-p)\delta + k_\alpha \delta) \notin \text{supp}(W)
\]

which is a contradiction.

A mild modification of what we did for \( r = 1 \) gives the result in case \( r = -1 \).

(iii) Let \( P := S(i)^{n} \cup -S(i)^{n} \cup \mathbb{Z}\delta \) if \( S(i) \) is up-nilpotent hybrid and \( P := S(i)^{n} \cup -S(i)^{n} \cup \mathbb{Z}\delta \) if \( S(i) \) is down-nilpotent hybrid. Then \( P \) is a parabolic subset of \( S(i) \); see the proof of [20] Lem. 5.4. Since \( R(i) = (S(i)^{n} \setminus \frac{1}{2} S(i)^{n}) \cup (\mathbb{Z}\delta \cap \text{span}_\mathbb{Z} S(i)^{n}) \) is the root system of an affine Lie algebra and \( R(i) \cap P \) is a parabolic subset of \( R(i) \), by [4] Theorem 2.10(ii), there is a functional \( f \) on \( \text{span}_\mathbb{R} R(i) \) such that

\[
R(i) \cap P = \{ \alpha \in R(i) \mid f(\alpha) \geq 0 \}.
\]

This in turn implies that \( P = \{ \alpha \in S(i) \mid f(\alpha) \geq 0 \} \). Extend \( f \) on \( \text{span}_\mathbb{R} R \) with \( f(S(j)) = 0 \), we get the result using parts (i),(ii).

Proposition 4.7. Suppose that \( \{ i, j \} = \{ 1, 2 \} \) and assume \( M \) has nonzero level. Assume \( S(i) \) is up-nilpotent (resp. down-nilpotent) hybrid and \( M \) is integrable as a module over \( \mathcal{O}(j) \). Then, there are a nonzero weight vector \( v \in M \) and a functional \( f \) on \( \text{span}_\mathbb{R} R \) with corresponding triangular decomposition \( R = R^{+} \cup R^{-} \cup R^{0} \) such that

- \( f(\delta) > 0 \) (resp. \( f(\delta) < 0 \)) and \( f(S(j)) = 0 \),
- \( \forall \check{\alpha} \in R^{+} \cap S(i), \mathcal{L}^n \check{\alpha} v = 0 \),
- \( \forall \check{\alpha} \in R^{-} \setminus S(i), \exists N \in \mathbb{Z}_{>0} \ni \mathcal{L}^{\check{\alpha} + n\delta} v = 0 \) (resp. \( \mathcal{L}^{\check{\alpha} - n\delta} v = 0 \)).

Proof. As the proof of the case that \( S(i) \) is down-nilpotent hybrid is similar to the proof of the case that \( S(i) \) is up-nilpotent hybrid, we just give the proof of the case that \( S(i) \) is up-nilpotent hybrid. This is a modified version of what we give to prove (5.10) of [20]; but as the proof is so technical, for the convenience of readers, we give the proof.

- \( \mathcal{L} \neq \mathcal{A}(2k - 1, 2\ell - 1)^{(2)} \): By Proposition 4.1(iii), there is a functional \( f \) on \( \text{span}_\mathbb{R} R \) such that \( f(\delta) > 0 \) and

\[
F := \{ \lambda \in \text{supp}(M) \mid (\lambda + (S(i) \cap R^{+})) \cap \text{supp}(M) = \emptyset \}
\]

is a nonempty subset of \( \text{supp}(M) \). Set

\[
A := \{ v \neq 0 \in M \mid \mathcal{L}^n v = 0 \ (\alpha \in S(i) \cap R^{+}) \}
\]

is a nonempty subset of \( \text{supp}(M) \). Set

\[
A := \{ v \neq 0 \in M \mid \mathcal{L}^n v = 0 \ (\alpha \in S(i) \cap R^{+}) \}
\]

Pick \( \lambda \in F \) and fix \( 0 \neq v \in M^\lambda \). Then, \( v \in A \). So to complete the proof, we just need to show that for each \( \check{\alpha} \in R_{ns}^{\times} \), there is a positive integer \( N \) such that for each \( n \geq N \),

\[
\mathcal{L}^{\check{\alpha} + n\delta} v = 0.
\]
Suppose that \( \alpha \in \check{R}_{ns}^{\infty} \). Then by Table 1 we have \( \check{\alpha} = \check{\beta} + \check{\gamma} \), for some \( \check{\beta}, \check{\gamma} \in \check{R}_{sh} \), and by (3.8b), we have

\[
S_{\check{\beta}} = S_{\check{\gamma}} = \mathbb{Z} \delta \text{ as well as } S_{\check{\alpha}} = s \mathbb{Z} \delta \quad \text{ (for some } s \in \mathbb{Z}^{>0}).
\]

This together with the fact that \( f(\delta) > 0 \) and \( \lambda \in F \), guarantees the existence of a large enough \( n \) such that \( \lambda + \check{\beta} + s n' \delta, \lambda + \check{\gamma} + s n' \delta \notin \text{ supp}(M) \) for all \( n' \geq n \). So, for each nonnegative integer \( t \), we have

\[
\mathcal{L}^{\alpha + s(2n+t)\delta} v = [\mathcal{L}^{\check{\beta} + (s+1)\delta} \check{\beta} + \check{\gamma} + s n' \delta, \mathcal{L}^{\check{\gamma} + s n' \delta} \check{\gamma} + s n' \delta] M^{\lambda + \check{\beta} + s(n+1)\delta} + \mathcal{L}^{\check{\beta} + (s+1)\delta} \check{\beta} + \check{\gamma} + s n' \delta} M^{\lambda + \check{\gamma} + s(n+1)\delta} = \{0\}.
\]

Therefore, for each \( n' \geq 2n \), we have \( \mathcal{L}^{\check{\alpha} + s n' \delta} v = \{0\} \) and so we are done as \( S_{\check{\alpha}} = s \mathbb{Z} \delta \).

\[ \bullet \mathcal{L} = A(2k - 1, 2l - 1)^{2} : \text{ We have } R_{v e} \subset R_{0}. \text{ Recalling } [20 \text{ Rem. 3.1}], \text{ if } \alpha \in R_{v e} \cup R_{v m} \text{ and } \epsilon \in R_{ns}^{\infty} \text{ with } \epsilon + \alpha \in R_{v e}, \text{ then, we have } \epsilon + \alpha \in R_{ns}^{\infty}. \text{ Set}
\]

\[
W := \sum_{\lambda \in \text{ supp}(M)} \sum_{\epsilon \in R_{ns}^{\infty}} \mathcal{L}^{\epsilon} M^{\lambda}.
\]

For \( \alpha \in S(i) \) \((i = 1, 2)\), we have

\[
\mathcal{L}^{\alpha} W = \mathcal{L}^{\alpha} \sum_{\lambda \in \text{ supp}(M) \epsilon \in R_{ns}^{\infty}} \mathcal{L}^{\epsilon} M^{\lambda} = \sum_{\lambda \in \text{ supp}(M) \epsilon \in R_{ns}^{\infty}} \mathcal{L}^{\alpha} \mathcal{L}^{\epsilon} M^{\lambda}
\]

\[
\subseteq \sum_{\lambda \in \text{ supp}(M) \epsilon \in R_{ns}^{\infty}} \sum_{\gamma \in \text{ supp}(M)} \gamma M^{\lambda} + \sum_{\lambda \in \text{ supp}(M) \epsilon \in R_{ns}^{\infty}} \mathcal{L}^{\epsilon} \mathcal{L}^{\gamma} M^{\lambda} \subseteq W; \quad \text{in other words, } W \text{ is a } \mathcal{G}(i)-\text{submodule of } M. \text{ Using Proposition 3.4(iii), there are a functional } f \text{ on span}_{R} R \text{ with corresponding triangular decomposition } R = R^{+} \cup R^{s} \cup R^{-} \text{ such that } f(\delta) > 0 \text{ and a weight } \mu \text{ of } W \text{ such that } \mu + \alpha \text{ is not a weight of } W \text{ if } \alpha \in S(i) \cap R^{+} \text{ and moreover, for each } \check{\alpha} \in \hat{S}(j), \text{ there is a positive integer } N \text{ such that } \mu + \check{\alpha} + n \delta \notin \text{ supp}(W) \text{ for all } n \geq N. \text{ Since } \mu \text{ is a weight of } W, \text{ there is a nonzero nonsingular root } \epsilon_{s} \text{ and } \lambda \in \text{ supp}(M) \text{ such that } \mathcal{L}^{\epsilon_{s}} M^{\lambda} \neq \{0\} \text{ and } \mu = \epsilon_{s} + \lambda. \text{ For } 0 \neq v \in \mathcal{L}^{\epsilon_{s}} M^{\lambda}, \text{ we have}
\]

\[
\mathcal{L}^{\alpha} v \subseteq W^{\alpha + \mu} \{0\} \quad (\alpha \in R^{+} \cap S(i));
\]

\[
\text{and}
\]

\[
\forall \check{\alpha} \in \hat{S}(j) \exists N \in \mathbb{Z}^{\geq 0} \text{ such that } \mathcal{L}^{\check{\alpha} + n \delta} v \subseteq W^{\check{\alpha} + n \delta + \mu} \{0\} \quad (n \geq N).
\]

To complete the proof, we need to show (3.9) holds for all \( \check{\alpha} \in \check{R} \setminus \hat{S}(i)^{\infty} = \check{R}_{sh}^{\infty} \cup \hat{S}(j)^{\infty} \). We first note that \( \dim(\mathcal{L}^{\epsilon_{s}}) = 1 \) and that two times of a nonzero nonsingular root is not a root (in particular, \( \mathcal{L}^{2\epsilon_{s}} = \{0\} \)) so

\[
\mathcal{L}^{2\epsilon_{s}} v \subseteq \mathcal{L}^{2\epsilon_{s}} M^{\lambda} \subseteq [\mathcal{L}^{\epsilon_{s}}, \mathcal{L}^{\epsilon_{s}}] M^{\lambda} = \{0\}.
\]

Suppose

\[
\epsilon_{s} = \check{\epsilon}_{s} + s \delta \quad \text{for some } \check{\epsilon}_{s} \in \check{R}_{ns}^{\infty} \text{ and } s \in \mathbb{Z}.
\]

For each \( \check{\alpha} \in \check{R}_{ns}^{\infty} \), by (20 Rem. 3.1), one of the following happens:

- \( \exists \check{\beta}_{1} \in \check{R}_{sh} \ni \check{\alpha} = \check{\epsilon}_{s} + \check{\beta}_{1} \),
- \( \exists \check{\beta}_{1} \in \check{R}_{sh}, \check{\beta}_{2} \in \check{R}_{ns} \ni \check{\epsilon}_{s} + \check{\beta}_{1} \in \check{R}_{ns}^{\infty}, \check{\alpha} = \check{\epsilon}_{s} + \check{\beta}_{1} + \check{\beta}_{2} \),
- \( \exists \check{\beta}_{1} \in \check{R}_{sh}, \check{\beta}_{2}, \check{\beta}_{3} \in \check{R}_{ns}^{\infty} \ni \check{\epsilon}_{s} + \check{\beta}_{1} \in \check{R}_{ns}^{\infty}, \check{\alpha} = \check{\epsilon}_{s} + \check{\beta}_{1} + \check{\beta}_{2} + \check{\beta}_{3} \).

- In the first case, using (4.7) together with (3.8b)) and the fact that \( f(\delta) > 0 \), we choose \( t_{1} \in \mathbb{Z}^{>0} \) such that

\[
(\lambda + \check{\beta}_{1} + 2 t_{1} \delta) \cap \text{ supp}(M) = \emptyset.
\]

So, for \( t > t_{1} + s \), we have

\[
\mathcal{L}^{\check{\beta}_{1} + (t-s)\delta} v = [\mathcal{L}^{\check{\beta}_{1} + (t-s)\delta}, \mathcal{L}^{\epsilon_{s}}] v \subseteq \mathcal{L}^{\check{\beta}_{1} + (t-s)\delta} \mathcal{L}^{\epsilon_{s}} v + \mathcal{L}^{\check{\beta}_{1} + (t-s)\delta} \mathcal{L}^{\epsilon_{s}} v \subseteq \{0\}.
\]
• In the second case, contemplating (3.6(b)), we use (4.7) and the fact that \( f(\delta) > 0 \) to choose \( t_1, t_2 \in \mathbb{Z}^* \) with

\[
(\lambda + \beta_1 + Z^{t_1} \delta) \cap \text{supp}(M) = \emptyset, \quad \beta_2 + t_2 \delta \in R \quad \text{and} \quad \lambda + \beta_2 + t_2 \delta \notin \text{supp}(M).
\]

This implies that for \( t \geq t_1 + t_2 + s \), we have \( \mathcal{L}^{\beta_1 + (t-t_2-s) \delta} v = \{0\} \) and \( \mathcal{L}^{\beta_2 + t_2 \delta} v = \{0\} \). So (4.10) implies that

\[
\mathcal{L}^{\alpha + t \delta} v = [\mathcal{L}^{\beta_2 + t_2 \delta}, [\mathcal{L}^{\beta_1 + (t-t_2-s) \delta}, \mathcal{L}^{\epsilon_*}] v = \{0\}.
\]

• In the third case, we choose \( t_1, t_2, t_3 \in \mathbb{Z}^* \) with

\[
(\lambda + \beta_1 + Z^{t_1} \delta) \cap \text{supp}(M) = \emptyset, \quad \beta_1 + t_3 \delta \in R \quad \text{and} \quad \lambda + \beta_1 + t_3 \delta \notin \text{supp}(M) \quad (i = 2, 3).
\]

Then for \( t \geq t_1 + t_2 + t_3 + s \), we have

\[
\mathcal{L}^{\alpha + t \delta} v = [\mathcal{L}^{\beta_1 + t_3 \delta}, [\mathcal{L}^{\beta_2 + t_2 \delta}, [\mathcal{L}^{\beta_1 + (t-t_2-t_3-s) \delta}, \mathcal{L}^{\epsilon_*}] v = \{0\}.
\]

This completes the proof. \( \square \)

**Theorem 4.8.** Suppose that \( \{r, t\} = \{1, 2\} \) and \( M \) is a \( t \)-quasi-integrable \( \mathcal{S} \)-module of nonzero level. Then there are triangular decompositions \( R = R^+ \cup R^0 \cup R^- \) and \( R^0 = R_{\alpha^0}^+ \cup R_{\beta^0}^0 \cup R_{\gamma^0}^- \), with \( \delta \notin \mathbb{R}^* \) as well as \( R_{\alpha^0}^+ \subseteq S(r) \), and a cuspidal module \( \mathcal{Q} \) over \( \mathcal{S}_{p^0}^\circ \), for \( P := R^+ \cup R_{\alpha^0}^0 \cup R_{\gamma^0}^- \), such that \( M \cong \text{Ind}_{p}(\mathcal{Q}) \). We also have that \( \mathcal{S}_{p^0}^\circ \) is a direct sum of a reducible finite dimensional Lie algebra and finitely many basic classical simple Lie superalgebras of types \( B(0, p) \) \( (p \in \mathbb{Z}^*) \). In particular, the classification of quasi-integrable irreducible finite weight \( \mathcal{S} \)-modules is reduced to the classification of cuspidal modules over \( \mathcal{S}_{p^0}^\circ \), see \([11, 13, 15]\).

**Proof.** We assume \( S(r) \) is up-nilpotent hybrid and carry out the proof; the proof in case \( S(r) \) is down-nilpotent hybrid, is similarly done.

By Proposition \([17]\) there is a functional \( f \) on \( \text{span}_R v \) with \( f(\delta) = 0 \) and \( f(\alpha) > 0 \) such that the set \( A \) consisting of all nonzero weight vectors \( v \) satisfying

\[
\begin{align*}
&\mathcal{L}^v \alpha = \{0\} \text{ for all } \alpha \in S(r) \text{ with } f(\alpha) > 0, \\
&\forall \alpha \in R \setminus \delta S(r) \ni m_\alpha \in \mathbb{Z}^* \text{ with } \mathcal{L}^{v \alpha + \delta \delta} v = \{0\} \text{ for all } k > m_\alpha
\end{align*}
\]

is nonempty. For \( v \in A \), set

\[
C_v := \{ \alpha \in R \setminus \delta S(r) \mid \mathcal{L}^v \alpha \neq \{0\}, \quad f(\alpha) > 0 \}.
\]

We mention that as \( f(\delta) > 0 \), \( C_v \) is a finite set and claim that there is \( w_0 \in A \) with \( C_{w_0} = \emptyset \). Pick \( v \in A \) such that \( C_v \) is of minimal cardinality. If \( C_v = \emptyset \), we take \( w_0 := v \) and we are done. Otherwise, we pick \( \alpha_*(v) \in C_v \) with

\[
f(\alpha_*(v)) = \max \{f(\alpha) \mid \alpha \in C_v\}.
\]

Suppose \( 0 \neq w_1 \in \mathcal{L}^{\alpha_*(v)} v \). We claim that

\[
w_1 \in A \quad \text{and} \quad C_{w_1} = C_v,
\]

• \( w_1 \in A \): We show it in the following two steps:

**Step 1.** For \( \alpha \in S(r) \) with \( f(\alpha) > 0 \), since \( v \in A \), we have

\[
\mathcal{L}^{\alpha} w_1 \subseteq \mathcal{L}^{\alpha} \mathcal{L}^{\alpha_*(v)} v \subseteq [\mathcal{L}^{\alpha}, \mathcal{L}^{\alpha_*(v)}] v + \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha} v \subseteq \mathcal{L}^{\alpha + \alpha_*(v)} v.
\]

We have

\[
f(\alpha + \alpha_*(v)) > f(\alpha_*(v)) > 0.
\]

If \( \alpha + \alpha_*(v) \notin R \) or \( \alpha + \alpha_*(v) \notin S(r) \), we get \( \mathcal{L}^{\alpha + \alpha_*(v)} v = \{0\} \) as \( v \in A \) and so \( \mathcal{L}^{\alpha} w_1 \subseteq \mathcal{L}^{\alpha + \alpha_*(v)} v = \{0\} \). Also if \( \alpha + \alpha_*(v) \in R \setminus S(r) \), due to the choice of \( \alpha_*(v) \), \( \alpha + \alpha_*(v) \notin C_v \) and so \( \mathcal{L}^{\alpha + \alpha_*(v)} v = \{0\} \). Therefore, again we have \( \mathcal{L}^{\alpha} w_1 \subseteq \mathcal{L}^{\alpha + \alpha_*(v)} v = \{0\} \).

**Step 2.** Since \( v \in A \), we choose \( N \) such that

\[
(4.12) \quad f(\hat{\alpha} + n\delta) > 0 \quad \text{and} \quad \mathcal{L}^{\hat{\alpha} + n\delta} v = \{0\} \quad (\hat{\alpha} \in \hat{R}, \ n > N).
\]
Suppose $\dot{\alpha} \in \dot{R} \setminus \dot{S}(r)$ and $n > N$. If $\dot{\alpha} + n\delta + \alpha_*(v) \in S(r)$, since $v \in A$ and $f(\dot{\alpha} + n\delta + \alpha_*(v)) > f(\alpha_*(v)) > 0$, we get $\mathcal{L}^{\dot{\alpha} + n\delta + \alpha_*(v)} \subseteq \{0\}$. Also, if $\dot{\alpha} + n\delta + \alpha_*(v) \in R \setminus S(r)$, then as $f(\dot{\alpha} + n\delta + \alpha_*(v)) > f(\alpha_*(v))$, we get $\dot{\alpha} + n\delta + \alpha_*(v) \notin C_v$ and so again $\mathcal{L}^{\dot{\alpha} + n\delta + \alpha_*(v)} = \{0\}$. These altogether imply that for all $n > N$, we have

$$\mathcal{L}^{\dot{\alpha} + n\delta} W_1 \subseteq \mathcal{L}^{\dot{\alpha} + n\delta} \mathcal{L}^{\alpha_*(v)} \subseteq \left[\mathcal{L}^{\dot{\alpha} + n\delta}, \mathcal{L}^{\alpha_*(v)}\right] v + \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\dot{\alpha} + n\delta} v \subseteq \mathcal{L}^{\dot{\alpha} + n\delta + \alpha_*(v)} v = \{0\}$$

as we desired.

- $C_{w_1} = C_v$: Suppose that $\alpha \in C_{w_1}$. Since $f(\alpha + \alpha_*(v)) > f(\alpha_*(v)) > 0$, we get $\mathcal{L}^{\alpha + \alpha_*(v)} v = \{0\}$. So, we have

$$\{0\} \neq \mathcal{L}^{\alpha} W_1 \subseteq \mathcal{L}^{\alpha + \alpha_*(v)} \subseteq \left[\mathcal{L}^{\alpha}, \mathcal{L}^{\alpha_*(v)}\right] v + \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha} v \subseteq \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha} v$$

which in turn implies that $\mathcal{L}^{\alpha} v \neq \{0\}$, that is, $\alpha \in C_v$. So $C_{w_1} \subseteq C_v$. Since $C_v$ is of minimal cardinality, we get that $C_v = C_{w_1}$, as we desired.

Since $C_{w_1} = C_v$, we have in particular that

$$\alpha_*(v) \in C_{w_1} \quad \text{and} \quad f(\alpha_*(v)) = \max \{f(\alpha) \mid \alpha \in C_{w_1}\}.$$ 

As $\alpha_*(v) \in C_{w_1}$ and $\mathcal{L}^{\alpha_*(v)}$ is 1-dimensional, we get $\{0\} \neq \mathcal{L}^{\alpha_*(v)} W_1 = \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha_*(v)} v$ and $\alpha_*(v) \in R_{\alpha_*(v)}$, since $R_{\alpha_*(v)} \subseteq R_1$ and two times of a nonzero nonsingular root is not a root, we get $\{0\} = [\mathcal{L}^{\alpha_*(v)}, \mathcal{L}^{\alpha_*(v)}] v + \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha_*(v)} v$ which is a contradiction. So

$$\alpha_*(v) \in S(t)_{\alpha_*(v)}.$$ 

Repeating the above process for $w_1$ instead of $v$, we get $0 \neq w_2 \in \mathcal{L}^{\alpha_*(v)} W_1 \subseteq \mathcal{L}^{\alpha_*(v)} \mathcal{L}^{\alpha_*(v)} W_1$ with

$$C_v = C_{w_2} = C_{w_1}, \quad \alpha_*(v) \in C_{w_1} = C_{w_2}$$ 

and $f(\alpha_*(v)) = \max \{f(\alpha) \mid \alpha \in C_{w_1} = C_{w_2}\}$.

Continuing this process, for each $n \in \mathbb{Z}^+$,

$$\{0\} \neq \mathcal{L}^{\alpha_*(v)} \cdots \mathcal{L}^{\alpha_*(v)} v \subseteq \mathcal{L}^{\alpha + n\alpha_*(v)}$$

which is a contradiction as by our assumption and (4.13), $\alpha_*(v) \in R^{\alpha} \subseteq B_M$. So, $C_v$ cannot be nonempty. This means that $\mathcal{L}^+ v = \{0\}$. So by [20] Prop. 3.3], $N := \{w \in \mathcal{L}^+ w = \{0\}\}$ is an irreducible module over $\oplus_{\alpha \in R^+} \mathcal{L}^\alpha$ and for $P' := R^+ \cup R^\alpha$, $M \simeq \text{Ind}_{P'}(N)$. Since $\delta \notin R^\alpha$, $R^\alpha$ is finite and $\oplus_{\alpha \in R^\alpha} \mathcal{L}^\alpha$ is a finite dimensional Lie superalgebra. Since

$$\text{inj} N := R^\alpha \cap R^\alpha \cap R_0 \subseteq R(r),$$

using [3] Thm. 3.6, one has a triangular decomposition $R^\alpha = R^{\alpha,+} \cup R^{\alpha,0} \cup R^{\alpha,-}$ for $R^\alpha$ with $R^{\alpha,0} \subseteq S(r)$ and a cuspidal module $\Omega$ over $\bigoplus_{\alpha \in R^{\alpha,0}} \mathcal{L}^\alpha$ such that for $P'' := R^{\alpha,+} \cup R^{\alpha,0}$, $N \simeq \text{Ind}_{P''}(\Omega)$. Therefore,

$$M \simeq \text{Ind}_{P'}(N) \simeq \text{Ind}_{P'}(\text{Ind}_{P''}(\Omega)) \simeq \text{Ind}_P(\Omega),$$

where $P = R^+ \cup R^{\alpha,+} \cup R^{\alpha,0}$. As $R^{\alpha,0} \subseteq S(r)$ just contains real roots, it is a direct sum of a finite root system and finitely many root systems of types $B(0,p)$ ($p \in \mathbb{Z}^+$) and so we are done.

\[\square\]

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