Second Order Photon Loops at Finite Temperature
Mahnaz Q. Haseeb\textsuperscript{a} and Samina S. Masood\textsuperscript{b}\textsuperscript{†}
\textsuperscript{a}Department of Physics, COMSATS Institute of Information Technology, Islamabad,
\textsuperscript{b}Department of Physics, University of Houston Clear Lake, Houston TX 77058
\textsuperscript{∗}mahnazhaseeb@comsats.edu.pk
\textsuperscript{†}masood@uhcl.edu

Abstract

We present two loop corrections to photon self energy at finite temperature in real
time formalism. An expression for renormalized coupling constant has been derived,
for the first time, in a form that is relevant for all temperature ranges of interest in
QED, specifically for temperatures $T \sim m$, where $m$ is electron mass. The latter
range of temperature is of specific interest from the point of view of the early universe
and stellar systems. Such calculations have acquired more significance recently due to
the possibility of producing $e^+e^-$ plasmas in laboratory. We use the calculations to
determine the dynamically generated mass of photon, Debye screening length, plasma
frequency up to order $\alpha^2$ as well as the electromagnetic properties of a medium at
high temperature.

PACS: 11.10.Wx, 12.20.-m, 11.10.Gh, 14.60.Cd

Keywords: renormalization, real time finite temperature field theory, photon self
energy, two-loop QED corrections

1 Introduction

Finite temperature field theory (FTFT) was formulated [1-4] a few decades
back, for various applications in condensed matter physics, astrophysics and
cosmology. Research work during the last couple of decades has led to an in-
creased development and application of methods of using the statistical distribu-
tion functions in non-relativistic situations [5] as well as relativistic regimes [6].
FTFT is relevant to address issues in phase transitions and particle-antiparticle
symmetry breaking after the Big Bang [7,8] and to trace the thermal history
of early universe. The temperatures equivalent to electron mass ($\sim 10^{10}K$) are
particularly significant from the point of view of primordial nucleosynthesis and
QED type corrections are highly important for calculating physically measure-
able values of QED parameters and their effect on abundance of elements in the
early universe.

Finite temperature effects are now extensively used not only in particle
physics but also to study relativistic plasmas. One of the possible extensions of
this work is to include the density effect at high temperatures. These extremely
involved calculations will provide a way to understand super dense electrodynami-
ics’ systems created in particle colliders and super dense hot stars. Such
effects are imperative in dense multi-particle systems in astrophysics and cosmology.
These calculations also help to move ahead in understanding phase transi-
tion in quantum electrodynamics of highly dense systems. More recently, possi-
bilities to create ultra-relativistic electron positron plasmas with high-intensity
lasers has generated renewed interest in finite temperature QED to study processes of pair-production [9-12] in QED plasmas. Lasers pulses hitting a thin gold foil can heat up the electrons in the foil up to several MeV’s leading to pair creation [13-19]. The latest laser facilities plan to attain colossal intensities of $10^{26}$ W/cm$^2$ which mark a substantial improvement over the ones highest recorded ($10^{22}$ W/cm$^2$), at Hercules facility in Michigan [20].

Quantum Field Theory (QFT) assumes particles to be analogous to excitations of a harmonically oscillating field which permeates space-time. Modern interpretation of vacuum describes it as an absence of particles, but not devoid of energy and fields. Vacuum can be treated as a hypothetical bath of virtual hot particles which can virtually mediate interactions between real particles for un-measurable short intervals of time. Virtual electron-positron pairs couple to photons through loops and give photon a temperature dependent finite size, charge and mass. Particles propagating in vacuum are assumed to be those with interactions switched off. The exact state of all the background particles is unknown when these particles propagate through a medium since they continually fluctuate between different configurations influencing particle dispersion. Thus properties of the system become different from a situation in which all particles are assumed to be completely independent of each other, behaving as freely propagating bare particles.

Finite temperature background is particularly interesting as it offers more realistic calculation and a possibility of measuring effects of the polarized vacuum by employing thermodynamic quantities. When dealing with sufficiently hot environments, the particles are considered to propagate in background heat bath at energies around thresholds for particle antiparticle pair production so that statistical effects due to temperature need to be appropriately taken into account. These effects arise due to continuous exchanges of mediating particles during physical interactions that take place in a heat bath containing hot particles and antiparticles. Thermal background effects are incorporated through radiative corrections and net statistical effects enter the theory through fermion and boson distributions. There is multitude of ways to describe physics at finite temperature. In real-time formalism, manifest covariance is restored through $u^\mu = (1, 0, 0, 0)$, the four component velocity of background heat bath. Finite temperature calculations also provide a guideline to estimate density corrections at higher order loops through chemical potential effects from background plasma.

Quantum electrodynamics (QED) is useful to study the conditions for existence of relativistic electron-positron plasmas. Self energies of particles in high temperature medium relevant in QED, acquire temperature corrections through virtual exchanges with real particles, in accordance with perturbative nature of the theory. Mass less photons gain an effective mass due to medium effects while propagating through a system comprising of a cloud of electrons and positrons at finite temperature, and vice versa. This gives rise to different terms in perturbation of states describing the system. Many of these terms can be reproduced by modifying particle propagators and from the poles of the propagators one obtains modified dispersion relation. Radiatively generated mass shift acts as
a kinematical cut-off in production rate of particles in the heat bath. Gauge bosons acquire a dynamically generated mass, at single and higher loop levels, due to plasma screening effect. One loop corrections have been studied in detail in real time formulation [21-24]. Medium properties such as electric permittivity and magnetic susceptibility or permeability also get modified by the thermal background effects.

The problem of renormalization in finite temperature theories is similar to that at zero temperature. The temperature acts as a regularization parameter for ultraviolet divergences. Infrared divergences introduced in finite temperature framework are also appropriately removable in particle decay processes, for example, via bremsstrahlung emission and absorption effects [22,23]. Renormalization of QED in this scheme has been already checked in detail at one loop level, using real time formulation, for all possible ranges of temperatures and chemical potentials [21-29]. The electron self-energy was calculated in detail, for all temperature ranges of relevance in QED with renormalized mass and wave function determined, at two-loop level [34-37] in this scheme of calculations. Method for re-summation over hard thermal loops (HTL) is commonly used to determine the higher loop corrections at finite temperature [30-32]. As far as renormalization is concerned, as discussed in Ref. [33], HTL corrections are not necessary. We aim here to calculate photon self energy in a framework valid for all temperatures relevant in QED.

We derive an expression for renormalized QED coupling constant in a background where electrons and positrons acquire temperature contribution from the heat bath along with photons. These calculations have been done in a manner that the limit of temperature $T \sim m$, around the threshold for creation of $e^+e^-$ pairs is inclusive. We now present calculations for photon self energy in a general temperature framework, for the first time, for renormalization of charge in this general framework of $T \sim m$ so that the calculations are valid for all the temperature ranges relevant in QED while the expressions in low temperature limit, $T << m$, presented in Ref. [38], are reproducible from these results.

Section 2 is based on calculations of second order in $\alpha$ self energy of photons that contribute in finite temperature background. Charge renormalization constant is evaluated in section 3. Expressions for electromagnetic properties in a thermal medium are obtained in section 4. Finally section 5 contains discussion of the results.

## 2 Two Loop Vacuum Polarization

In real time formalism, temperature dependent (hot) and temperature independent (cold) terms are simply combined, at one loop level, because the propagators include them separately as additive terms. At higher-loop level, the loop integrals involve a combination of temperature independent (cold) and temperature dependent (hot) terms which appear due to the overlapping propagator terms. Therefore, the calculations are much more involved.
Vacuum polarization tensor of photon at the two-loop level contributes to the second order hot corrections to charge renormalization constant of QED at finite temperature. This contribution basically comes from self mass in Fig. 1(a) and vertex type electron loop corrections inside the vacuum polarization at finite temperature. This contribution basically comes from self mass in Fig. 1(b). The expression for two loop self energy of photon from self energy type correction in Fig. 1(a) is

\[ \Pi_{\mu\nu}^{a}(p) = e^{4} \int \frac{d^{4}k}{(2\pi)^{4}} \int \frac{d^{4}q}{(2\pi)^{4}} \text{tr}[\gamma_{\mu}S_{\beta}(k)\gamma_{\nu}D_{\beta}^{a\sigma}(q-k)S_{\beta}(q)\gamma_{\sigma}S_{\beta}(k)\gamma_{\nu}S_{\beta}(p-k)]. \]

with \( k + l = q \). On substitution of the particle propagators at finite temperature, this gives

\[
\Pi_{\mu\nu}^{a}(p, T) = \Pi_{\mu\nu}^{a}(p, T = 0) + 2\pi e^{4} \int \frac{d^{4}k}{(2\pi)^{4}} \int \frac{d^{4}q}{(2\pi)^{4}} N_{\mu\nu}^{a}[\frac{\delta(k_{0} - E_{k})}{2} + \frac{\delta(k_{0} - E_{k})}{(k_{0} + E_{k})}] - \frac{\delta(q - m^{2})n_{F}(q)}{[k^{2} - m^{2}][l^{2}(p - k)^{2} - m^{2}][q^{2} - m^{2}]} \\
\times \frac{\delta'(k_{0} - E_{k})}{2} + \frac{\delta(k_{0} - E_{k})}{(k_{0} + E_{k})} - \frac{\delta(q^{2} - m^{2})n_{F}(q)}{(p - k)^{2}[k^{2} - m^{2}][q^{2} - m^{2}]} \delta((p - k)^{2} - m^{2})n_{F}(p - k) \\
- \frac{\delta(q^{2} - m^{2})n_{F}(q)\delta((p - k)^{2} - m^{2})n_{F}(p - k)}{[k^{2} - m^{2}][l^{2}(p - k)^{2} - m^{2}][q^{2} - m^{2}]} \times \left[ \frac{\delta'(k_{0} - E_{k})}{2} + \frac{\delta(k_{0} - E_{k})}{(k_{0} + E_{k})} - \frac{\delta(q^{2} - m^{2})n_{F}(q)}{(p - k)^{2}[k^{2} - m^{2}]} \right] \\
+ \frac{\delta((p - k)^{2} - m^{2})n_{F}(p - k)}{[k^{2} - m^{2}]} \delta(q^{2} - m^{2})n_{F}(q) \\
+ \frac{\delta((q - k)^{2} - m^{2})n_{B}((q - k))\delta((p - k)^{2} - m^{2})n_{F}(p - k)}{[q^{2} - m^{2}]} \\
+ \frac{\delta((q - k)^{2} - m^{2})n_{F}(q)\delta((p - k)^{2} - m^{2})n_{F}(p - k)}{[k^{2} - m^{2}][l^{2}(p - k)^{2} - m^{2}][q^{2} - m^{2}]},
\]

where
$$\begin{align*}
N_{\mu\nu}^a &= 8[(3m^2 - k^2 - 2k.q) \{k_\mu(p-k)_\nu + k_\nu(p-k)_\mu - g_{\mu\nu}k.(p-k)\} \\
&+ (k^2 - m^2) \{(q-k)_\mu(p-k)_\nu + (q-k)_\nu(p-k)_\mu \\
- g_{\mu\nu}(q-k).k.(p-k)\} + 2g_{\mu\nu}m^2(m^2 - k.q + k^2)].
\end{align*}$$

(3)

Vertex type correction to two loop self energy of photon in Fig. 1(b) can be written as

$$\begin{align*}
\Pi_{\mu\nu}^b(p) &= e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} tr[\gamma_\mu S_\beta(k)\gamma_\mu D^{\rho\sigma}_{\beta}(q-k)S_\beta(q)\gamma_\nu S_\beta(q-p)\gamma_\sigma S_\beta(k-p)],
\end{align*}$$

(4)

$$\begin{align*}
\Pi_{\mu\nu}^b(p, T) &= \Pi_{\mu\nu}^b(p, T = 0) + 2\pi e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} N_{\mu\nu}^b \\
&\times \left[ \frac{1}{(k-p)^2 - m^2} \frac{1}{(q-p)^2 - m^2} \frac{1}{q^2 - m^2} \right] \\
&\times \left[ \delta((k-p)^2 - m^2)n_F(k-p) - \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{q^2 - m^2} \right] \\
&\times \left[ \delta((q-k)^2)n_B(q-k) + \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-k)^2)n_B(q-k)\delta(q^2 - m^2)n_F(q)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} + \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-k)^2)n_B(q-k)\delta(q^2 - m^2)n_F(q)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-p)^2 - m^2)n_F(k-p)}{(k-p)^2 - m^2} + \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-k)^2)n_B(q-k)\delta(q^2 - m^2)n_F(q)}{(k-p)^2 - m^2} \right] \\
&\times \left[ \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} + \frac{\delta((q-p)^2 - m^2)n_F(q-p)}{(k-p)^2 - m^2} \right].
\end{align*}$$

(5)
where

\[
N^b_{\mu\nu} = -8[q.(k-p)\{k_\mu(q-p)_\nu - (q-p)_\mu k_\nu\} - q.(q-p)\{k_\mu(k-p)_\nu
\]

\[
+ (k-p)_\mu k_\nu\} + (q-p).(k-p)\{k_\mu q_\nu + q_\mu k_\nu\}
\]

\[
+k.q\{(k-p)_\mu(q-p)_\nu + (q-p)_\mu(k-p)_\nu\} - k.(k-p)\{q_\mu(q-p)_\nu
\]

\[
+ (q-p)_\mu q_\nu\} + k.(q-p)\{q_\mu(k-p)_\nu - (k-p)_\mu q_\nu\}
\]

\[
+ g_{\mu\nu}\{q.(k-p)k.(q-p) - k.q(k-p).(q-p) + k.(k-p)q.(q-p)\}
\]

\[
- m^2(2k_\mu(2q-p)_\nu - [k_\mu(k-p)_\nu + (k-p)_\mu k_\nu + g_{\mu\nu}k.(p-k)]
\]

\[
- [q_\mu(q-p)_\nu + (q-p)_\mu q_\nu - g_{\mu\nu}q.(q-p)]
\]

\[
+ 2(k-p)_\mu(2q-p)_\nu + m^4g_{\mu\nu},
\]

(6)

We find that most of the terms vanish on integrating over hot momenta before the cold ones. Therefore, we get respective nonzero contributions to these diagrams, only from the terms:

\[
\Pi^a_{\mu\nu}(p,T \neq 0) = -2\pi e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} N^a_{\mu\nu}
\]

\[
\frac{\delta(q-k)^2 n_B(q-k)}{[(p-k)^2 - m^2][q^2 - m^2]}
\]

\[
+ 4\pi^2\delta\{q^2 - m^2\} n_F(q)\delta\{(p-k)^2 - m^2\} n_F(p-k).
\]

(7)

\[
\Pi^b_{\mu\nu}(p,T \neq 0) = 8\pi^4 e^4 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} N^b_{\mu\nu} \delta(q-k)^2 n_B(q-k)
\]

\[
\times \frac{\delta(k^2 - m^2) n_F(k)\delta\{(q-p)^2 - m^2\} n_F(q-p)}{[(k-p)^2 - m^2][q^2 - m^2]},
\]

(8)

Due to manifest covariance in the theory, physically measurable couplings can be evaluated through contraction of vacuum polarization tensor \(\Pi_{\mu\nu}\) with the metric in Minkowski space \(g_{\mu\nu}\) and the bath velocities \(u^\mu u^\nu\). Following Ref. [21], one can write

\[
\Pi_{\mu\nu}(p) = \Pi_L(p)P_{\mu\nu} + \Pi_T(p)Q_{\mu\nu},
\]

where

\[
\Pi_L(p) = -\frac{p^2}{|p|^2} u^\mu u^\nu \Pi_{\mu\nu}
\]

and

\[
\Pi_T(p) = -\frac{1}{2} \Pi_L(p) + \frac{1}{2} g^{\mu\nu} \Pi_{\mu\nu}.
\]

The coefficients \(P_{\mu\nu}\) and \(Q_{\mu\nu}\) are given by

\[
P_{\mu\nu} = (g_{\mu\nu} - u_\mu u_\nu) + \frac{1}{|p|^2} (p_\mu - \omega u_\mu)(p_\nu - \omega u_\nu)
\]

6
\[
Q_{\mu\nu} = \frac{1}{p^2|p|^2}\{(p^2 u_\mu + \omega(p_\mu - \omega u_\mu))\{p^2 u_\nu + \omega(p_\nu - \omega u_\nu)\}\}.
\]

If a preferred order of integration (integrating over temperature dependent variables before temperature independent ones) is not followed, the loop calculations become very cumbersome and the results become messed up. In Eqs. (7) and (8), integrating over \(d^4q\) (the temperature dependent momentum), using Feynman parameterization and then integrating over \(d^4k\), after a somewhat lengthy calculation, one gets

\[
u_{\mu} \Pi_{\nu}^{a+b} = 2\alpha^2 T^2 \left\{ \frac{1}{3} \left( \frac{1}{2\eta} + 1 - \frac{E^2}{6m^2} \right) + 4 \sum_{n,r,s=1}^{\infty} \frac{(-1)^{n+r}e^{-n\beta E}}{(n+s)(n-r)} e^{-m\beta(n-r)} \right\}
- \sum_{n,r,s=1}^{\infty} \frac{(-1)^{s+r}e^{-s\beta E}}{(n+s)} \left\{ \frac{2m^2 T}{m} \left[ e^{-m\beta(s+r)} \right] \right\}
- \frac{T}{(r+s)} \left\{ 4 + \frac{5}{v} \ln \left( 1 + \frac{v}{1-v} \right) - \frac{E}{v^2} \left[ \ln \left( 1 + \frac{v}{1-v} \right)^2 I_A \right] \right\},
\]

(9)

\[
g_{\mu\nu} \Pi_{\nu}^{a+b} = -\alpha^2 T^2 \left\{ \frac{5}{\eta} - 5 \ln(-m^2) - 1 + \frac{E^2}{mv^2} \right\}
+ 48 \sum_{n,r,s=1}^{\infty} \left\{ \frac{(-1)^{n+r} e^{-n\beta E}}{(n-r)} \left[ m + \frac{T}{(r-n)} \right] \right\}
\]
\[
+ e^{-s\beta E} \frac{T}{(n+s)} \left[ \frac{1}{(r+s)} \right]
\]
\[
- \frac{m^2}{2} \left\{ e^{-m\beta(s+r)} \right\} \right\},
\]

(10)

with \(v = \frac{|p|}{p_0}\) and \(\eta = \frac{1}{\varepsilon} - \gamma - \ln\left(\frac{4\pi n_e^2}{m^2}\right)\).

We need to remove the temperature enhanced ultraviolet divergences. The counter terms from Fig. (1c) provide the contribution necessary to cancel the divergences appearing in Fig. (1a) and Fig. (1b). Adding the results from all QED graphs in Figs. (1a) - (1c), it has been explicitly checked that all the hot and cold divergences cancel at the two loop level. Therefore, finite terms from Eqs. (9) and (10) are used to obtain longitudinal and transverse components of the two loop vacuum polarization tensor. Including the one loop corrections in Ref. [23] also, these components can be now written as.
\[
\Pi_L(p, T) = -\frac{p^2}{|p|^2} \varepsilon^\mu_\nu u^\nu \Pi_{\mu\nu}(p, T)
\]
\[
= \frac{16\alpha}{\pi} \left(1 - \frac{1}{v^2} \right) \left(1 - \frac{1}{2v} \ln \frac{1 + v}{1 - v} \right) \left(\frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2}\right)
\]
\[
+ \frac{b(m\beta)}{4} \left(2m^2 - \omega^2 + \frac{11v^2 + 37}{72} E^2\right)
\]
\[- \frac{2\alpha^2}{v^2} \left(1 + \frac{E^2}{2m^2}\right) + 4T^2 \sum_{n,r,s=1}^\infty e^{-n\beta E} \left(\frac{(-1)^{n+r} e^{-m\beta(n-r)}}{(n + s)(n - r)}\right)
\]
\[- 2m^2 \left(\frac{e^{-m\beta(s+r)}}{m} - (r + s)\beta Ei[-m\beta(r + s)]\right),
\]
(11)

and

\[
\Pi_T(p, T) = -\frac{1}{2} \Pi_L(p, T) - g^\mu_\nu \Pi_{\mu\nu}(p, T)
\]
\[
= \frac{8\alpha}{\pi} \left(1 - \frac{1}{v^2} \right) \left(1 - \frac{1}{2v} \ln \frac{1 + v}{1 - v} \right) \left(\frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2}\right)
\]
\[
+ \frac{1}{8} \left(2m^2 + E^2[1 - \frac{107 - 131v^2}{72}]\right) b(m\beta)
\]
\[
+ \frac{\alpha^2}{3} T^2 \left[1 - \frac{1}{v^2}(1 + \frac{E^2}{2m^2})\right]
\]
\[
+ \sum_{n,r,s=1}^\infty (-1)^{s+r} e^{-s\beta E} \left(\frac{T}{(n + s)} \right) \left(\frac{T}{(r + s)}\right) \left(4 + \frac{5}{v^2} \ln \frac{1 + v}{1 - v}\right)
\]
\[- \frac{2m^2}{4} \left(\frac{e^{-m\beta(s+r)}}{m} - (r + s)\beta Ei[-m\beta(r + s)]\right)\].
(12)

where

\[
a(m\beta) = \ln(1 + e^{-m\beta}),
\]
(13)

\[
b(m\beta) = \sum_{n=1}^\infty (-1)^n Ei(-nm\beta),
\]
(14)

\[
c(m\beta) = \sum_{n=1}^\infty (-1)^n \frac{e^{-nm\beta}}{n^2},
\]
(15)
$I_A$ is infrared divergence at finite temperature and $E_i$ is error integral given by

$$E_i(-x) = -\int_x^\infty \frac{dt}{t} e^{-t}.$$  

(16)

These expressions are in a form such that all the relevant limits of temperature including $T \sim m$ are valid.

## 3 Charge Renormalization

Virtual photons are emitted and reabsorbed by electrons and tend to smear out charge distribution and correct the electron mass. The virtual fermion loops get polarized in electromagnetic fields, which act to damp the field. Renormalized electron mass and renormalized wavefunction have been already calculated in detail [36]. Vacuum polarization in a medium gives modification to electric charge and QED coupling constant. The electric charge couples with the medium through vacuum polarization get screened, and picks up thermal corrections accordingly. We derive an expression for the renormalized charge following Ref. [21] and using Eq. (11). The electron charge renormalization constant up to the order $\alpha^2$ can therefore be expressed as:

$$Z_3 = 1 - \frac{\alpha}{3\pi\varepsilon} + \frac{8\alpha}{\pi m^2} \left( \frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2} + \frac{b(m\beta)}{4} (m^2 + \frac{1}{3} \omega^2) \right)$$

$$+ \frac{\alpha^2}{m^2} \frac{T^2}{6} + \sum_{n,r,s=1}^{\infty} (-1)^{s+r} e^{-s\beta E} \frac{T}{(n+s)} \left( \frac{24T}{(r+s)} \right)$$

$$- 12m^2 \left[ \frac{e^{-m\beta(s+r)}}{m} - (r+s)\beta E_1\{-m\beta(r+s)\} \right].$$  

(17)

Using $Z_3$, a corresponding value of QED coupling constant is obtained to be:

$$\alpha_R = \alpha(T=0) \left[ 1 + \frac{8\alpha}{\pi m^2} \frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2} + \frac{b(m\beta)}{4} (m^2 + \frac{1}{3} \omega^2) \right]$$

$$+ \frac{\alpha^2}{m^2} \frac{T^2}{6} + \sum_{n,r,s=1}^{\infty} (-1)^{s+r} e^{-s\beta E} \frac{T}{(n+s)} \left( \frac{24T}{(r+s)} \right)$$

$$- 12m^2 \left[ \frac{e^{-m\beta(s+r)}}{m} - (r+s)\beta E_1\{-m\beta(r+s)\} \right].$$  

(18)

It can be seen from the above equations that electron charge and hence the QED coupling constant gets smaller with an increased order of loops which clearly assures the renormalization of electron charge. This modification in the coupling constant can be used to determine changes in electromagnetic properties of medium.
4 Electromagnetic Properties of the Medium

It is a fact that the electromagnetic properties of media depend on their statistical properties. Higher order calculation of thermal corrections to self energy of photons is much more complicated. However, it is interesting to determine the corresponding changes in electromagnetic properties of hot media which arise due to radiative emission and absorption of hot particles in the heat bath. It can be checked using Ref. [22] that electric permittivity of a medium at the two loop level modifies to

\[
\varepsilon(p, T) = 1 - |p|^2 \Pi_L(p, T)
\]

= \[1 - \frac{16\alpha m^2}{\pi v^2 E^2}\left((1 - \frac{1}{2v}\ln\frac{1 + v}{1 - v})\left(\frac{m\alpha(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2}\right)\right.
\]

+ \[\frac{b(m\beta)}{4}\left(2m^2 - E^2[1 - \frac{11v^2 + 37}{72}]ight)\]

+ \[2\alpha^2 p^2\left(\frac{T^2}{3}(1 + \frac{E^2}{2m^2}) - 4T^2\sum_{n,r,s=1}^{\infty} e^{-n\beta E}\left(\frac{(-1)^{n+r}e^{-m\beta(n-r)}}{(n + s)(r - n)}\right)\right.
\]

+ \[\sum_{n,r,s=1}^{\infty} (-1)^{s+r}e^{-s\beta E}\left(\frac{T}{(n + s)}\frac{T}{(r + s)}\left(4 + \frac{5}{v}\ln\frac{1 + v}{1 - v}\right)\right.
\]

- \[2m^2\left(e^{-m\beta(s+r)} \right) (r + s)\beta \text{Ei}[-m\beta(r + s)]\],

whereas magnetic permeability becomes

\[
\frac{1}{\mu(p, T)} = 1 + \frac{1}{p^2}\left[\Pi_T(p, T) - \frac{1}{v^2}\Pi_L(p, T)\right]
\]

= \[1 + \frac{8\alpha}{\pi v^2 m^2}\left(\frac{1}{8}(E^2[1 + \frac{109v^2 - 129}{72}] - 6m^2)b(m\beta)\right.
\]

+ \[\frac{(1 - \frac{1}{v^2})(1 - \frac{1 - v^2}{2v}\ln\frac{1 + v}{1 - v})\left(\frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2}\right)\}
\]

+ \[\frac{\alpha^2 T^2}{3}\left\{\frac{1}{2m^2} - \frac{1}{E^2v^2}(1 + \frac{E^2}{2m^2})(1 + \frac{2}{v^2})\right\}
\]

+ \[\sum_{n,r,s=1}^{\infty} (-1)^{s+r}e^{-s\beta E}\left(\frac{T}{(n + s)}\frac{4T}{(s - r)}e^{-m\beta(s-r)}\left(1 + \frac{1}{v^2}\right)\right.
\]

- \[\frac{T}{(r + s)}\left(4 + \frac{5}{v}\ln\frac{1 + v}{1 - v}\right) - 2m^2\left(6 + \frac{1}{v^2}\right)
\]

\[
\times\left\{e^{-m\beta(s+r)} \right) (r + s)\beta \text{Ei}[-m\beta(r + s)]\}\].

Equations (19) and (20) show that the radiative emissions and absorptions of real hot photons from the heat bath up to order \(\alpha^2\) lead to deviations from unity.
for the values of dielectric constant and magnetic permeability. There are two possible limits for $\Pi_L$ and $\Pi_T$ [22] as $p^2 \to 0$ and their physical interpretations depend on the order of taking the limits $p_0 = 0$ and $|p| \to 0$. In Eqs. (11) and (12), if we take $p_0 = |p|$ in the rest frame of the heat bath and then limit $|p| \to 0$ is taken, we get

$$
\kappa_L^2 \sim \lim_{|p| \to 0} \Pi_L(0, |p|, T)
\approx \frac{16\alpha}{\pi} \left\{ \frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2} + \frac{m^2b(m\beta)}{2} \right\}
+ 2\alpha^2 \left\{ \frac{T^2}{3} - 24 \sum_{n,r,s=1}^{\infty} \frac{(-1)^{n+r}}{(n+s)(r-n)} e^{-m\beta(n+r)} \right\}
\sum_{n,r,s=1}^{\infty} \frac{4T^2}{(n+s)(r+s)}
- 2m^2 \left\{ \frac{e^{-m\beta(s+r)}}{m} - (r+s)\beta Ei[-m\beta(r+s)] \right\},

(21)
$$

$$
\kappa_T^2 \sim \lim_{|p| \to 0} \Pi_T(0, |p|, T)
\approx \frac{\alpha}{\pi} m^2b(m\beta) - \alpha^2 \left[ \sum_{n,r,s=1}^{\infty} \frac{T}{(n+s)} \left\{ \frac{4T}{s-r} e^{-m\beta(s-r)} \right\} \right]
- \frac{24T}{(r+s)} + 6 \left( m + \frac{T}{r-s} \right) - 10m^2 \left\{ \frac{e^{-m\beta(s+r)}}{m} \right\}
-(r+s)\beta Ei[ -m\beta(r+s)] \right\} + \frac{T^2}{3},

(22)
$$

On the other hand if we set $p_0 = 0$ with $|p| \to 0$ then this result in:

$$
\omega_L^2 \sim \lim_{|p| \to 0} \Pi_L(|p|, |p|, T) = 0,

(23)
$$

$$
\omega_T^2 \sim \lim_{|p| \to 0} \Pi_T(|p|, |p|, T)
\approx \frac{8\alpha}{\pi} \left\{ \frac{ma(m\beta)}{\beta} - \frac{c(m\beta)}{\beta^2} + \frac{m^2b(m\beta)}{4} \right\}
+ \alpha^2 \left\{ \frac{T^2}{6} + \sum_{n,r,s=1}^{\infty} \frac{T}{(n+s)} \left\{ \frac{24T}{(r+s)} e^{-m\beta(s-r)} \right\} \right\}
+ 24T \frac{e^{-m\beta(s-r)}}{(s-r)} \left( m + \frac{T}{r-s} \right) - 12m^2 \left\{ \frac{e^{-m\beta(s+r)}}{m} \right\}
-(r+s)\beta Ei\{-m\beta(r+s)\}],

(24)
$$
The vacuum polarization tensor in $p^2 \to 0$ limit provides the dynamically generated mass of photon.

5 Results and Discussion

Thermal corrections at the two loop level are not as simple as the one loop. Temperature dependence to vacuum polarization tensor at the two loop level enter as an overlap of photon and fermions hot loop with momenta due to effectively mutual interactions while propagating through the medium. An unusual behavior of hot integrals appears due to the overlap of hot and cold terms. The order of integration between the cold and hot loop not only affect the length of calculations but also the right order of integration is required to show the renormalizability of the theory. With the integration over temperature dependent variables before temperature independent ones, the loop calculations become lesser cumbersome and also the cancellation of singularities can be clearly seen. In case of wavefunction, the terms with the reverse order of integrations have to be included to fully establish the renormalization. The standard regularization techniques of vacuum theory like dimensional regularization would only be valid in a covariant framework of a Lorentz invariant system. Once the hot momenta are integrated out, usual field theoretical techniques of Feynman parameterization and dimensional regularization can be applied to remove singularities arising from cold loops. The results reduce to finite values after order by order cancellation of singularities on addition of counter terms of the same order.

At the higher loop level, temperature corrections in hot medium imply convergence of the perturbative series. The renormalized self energies are used to obtain an estimation of dielectric constant and magnetic permeability of a hot medium. Both of these quantities deviate from unity even for $T << m$ whereas they do not deviate from the vacuum value with the first order loop corrections [23]. This difference of loop correction can be interpreted as at the one loop level self-interaction of the photon loops does not contribute whereas at the two loop level the hot photon loop develops self-interaction due to the dynamically generated mass. This type of effect has been earlier observed for self-mass of gluon even at the one-loop level [39] due to self-coupling of gluons. With an increase in temperature, photon mass sufficiently affects the behavior of this coupling, especially when $T \gtrsim m$. This is due to the self mass of photon at finite temperature which may lead to nonzero self interaction.

Further we obtain respective propagation vectors and frequencies when we take the limiting values for longitudinal and transverse component of vacuum polarization tensor in Eqs. (7) and (8). In particular eqn. (22) $\kappa^{-1}_L$ gives Debye screening length of electric force in QED plasma whereas the transverse component of vacuum polarization tensor in Eq. (12) corresponds to the dynamically generated mass of photon. QED effective coupling is determined in Eq. (18).

The framework of calculations adopted here has a form such that the temperatures $T \sim m$ (by taking $m\beta \sim 1$) are inclusive while the limits of temperature
classified as, high temperature \( (T > m, \text{ with } e^{-mβ} \text{ ignorable as compared to } \frac{T^2}{m^2}) \) and low temperature \( T << m \) (with fermions contribution negligible) can be retrieved from the results obtained in previous sections.

It is worth-mentioning that all hot corrections give a similar dominant \( T^2 \) dependence, as in case of low temperature [38]. The analysis here is useful to evaluate decay rates and scattering cross sections of particles and can be generalized for nonzero density that can be used as an effective tool to study quark gluon plasma, cores of neutron stars, supernovae, etc. The calculations presented here can be extended to the study of electroweak decay rates [40-44] at higher loop level. In particular such processes will be interesting to study, for a better understanding about the medium effects in physical interactions associated with black holes and primordial nucleosynthesis. Two loop calculations with imaginary time formulation were done earlier for hard lepton pair (dilepton) production in the context of quark gluon plasma [45-50].

References

1. T. Matsubara, Prog. Theor. Phys. 14 (1955) 351; D. A. Kirzhnits, *Field Theoretical Methods in Many Body Systems*, Peramgon, Oxford (1967); E. M. Lifshitz and L. P. Pitaevskii, *Course on Theoretical Physics- Physical Kinetics*, Pergamon Press, New York (1981).

2. J. Schwinger, J. Math. Phys. 2 (1961) 407; L. V. Keldysh, Sov. Phys. 20 (1964) 1018.

3. D. A. Kirzhnits, *Field Theoretical Methods in Many Body Systems*, Peramgon, Oxford (1967); H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States*, North Holland, Amsterdam (1982).

4. R. Mills, *Propagators for Many Particle Systems*, Gordon and Breach, New York (1969).

5. L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, W. A. Benjamin, Reading (1978); A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, Dover, New York (1975); A. Fetter and J.D. Walecka, *Quantum Theory of Many Particle Systems*, Mc Graw Hill, New York (1971).

6. J. I. Kapusta and C. Gale, *Finite Temperature Field Theory*, Cambridge University Press, New York (2006); B. Muller, *Physics of the Quark-Gluon Plasma*, Lecture Notes in Physics 225, Springer, Berlin (1985); C.-Y. Wong, *Introduction to High-Energy Heavy-Ion Collisions*, World Scientific Pub Co, (1994); J. Bartke, *Relativistic Heavy Ion Physics*, World Scientific, Singapore (2003); M. LeBellac, *Thermal Field Theory*, Cambridge University Press (1996); A. Das, *Finite Temperature Field Theory*, World Scientific, Singapore (1997).

7. S. Weinberg, *Gravitation and Cosmology* Wiley, New York (1972).

8. V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press (2005).
9. W. Dittrich and H. Gies, *Probing the quantum vacuum*, Springer-Verlag, Berlin, Germany (2000).

10. P. H. Cox, W. S. Hellman, and A. Yildiz, Ann. Phys. 154 (1984) 211.

11. M. Loewe and J. C. Rojas, Phys. Rev. D 46 (1992) 2689.

12. P. Elmfors and B-S. Skagerstam, Phys. Lett. B 348 (1995) 141.

13. N. B. Narozhny et al., Phys. Lett. A 330 (2004) 1.

14. A. R. Bell and John G. Kirk, Phys. Rev. Lett. 101 (2008) 200403.

15. J. G. Kirk, A. R. Bell, and I. Arka, Plasma Phys. Control. Fusion 51 (2009) 085008.

16. E.P. Liang, S.C. Wilks, and M. Tabak, Phys. Rev. Lett. 81 (1998) 4887.

17. B. Shen, Phys. Rev. E 65 (2001) 016405.

18. M.H. Thoma, Eur. Phys. J. D 55 (2009) 271; M.H. Thoma, Rev. Mod. Phys. 81 (2009) 959.

19. Ben King, Antonino Di Piazza, and Christoph H. Keitel, Nature Photonics 4 (2010) 92.

20. V. Yanovsky et al., Opt. Express 16 (2008) 2109.

21. H.A. Weldon, Phys. Rev. D 26 (1982) 1394.

22. J.F. Donoghue, B.R. Holstein, and R.W. Robinett, Ann. Phys. (N.Y.) 164 (1985) 233.

23. K. Ahmed and Samina S. Masood, Ann. Phys. (N.Y.) 207 (1991) 460.

24. J.F. Donoghue and B.R. Holstein, Phys. Rev. D 28 (1983) 340; ibid 29 (1983) 3004(E).

25. Samina S. Masood, Phys. Rev. D 44 (1991) 3943.

26. Samina S. Masood, Phys. Rev. D 47 (1993) 648.

27. W. Keil and R. Kobes, Physica A 158 (1989) 47.

28. K. Ahmed and Samina Saleem, Phys. Rev. D 35 (1987) 1861.

29. K. Ahmed and Samina Saleem, Phys. Rev. D 35 (1987) 4020.

30. E. Braaten and R.D. Pisarski, Nucl. Phys. B 337 (1990) 569.

31. E. Braaten and R.D. Pisarski, Phys. Rev. D 42 (1990) 2156.

32. R.D. Pisarski, Phys. Rev. Lett. 63 (1989) 1129.
33. E. Mottola and Z. Szép, Phys. Rev. D 81 (2010) 025014.
34. Mahnaz Qader, Samina S. Masood, and K. Ahmed, Phys. Rev. D 44 (1991) 3322.
35. Mahnaz Qader, Samina S. Masood, and K. Ahmed, Phys. Rev. D 46 (1992) 5633.
36. Mahnaz Q. Haseeb and Samina S. Masood, Chin. Phys. C 35 (2011) 608.
37. Mahnaz Q. Haseeb and Samina S. Masood, Phys. Lett. B 704 (2011) 66.
38. Samina S. Masood and Mahnaz Q. Haseeb, Int. J. of Mod. Phys. A 23 (2008) 4709.
39. Samina S. Masood and Mahnaz Q. Haseeb, Astropart. Phys. 3 (1995) 405.
40. Samina S. Masood and Mahnaz Qader, Phys. Rev. D 46 (1992) 5110; Samina S. Masood and Mahnaz Qader, Finite temperature and density corrections to electroweak decays, in Proc. 4th Regional Conference on Mathematical Physics, Sharif Univ. of Technology, Tehran, Iran, 12–17 May 1990, eds. F. Ardalan, H. Arafae and S. Rouhani (Sharif University of Technology Press, 1990), p. 334.
41. D. A. Dicus, E. W. Kolb, A. M. Gleeson, E. C. G. Sudarshan, V. L. Teplitz, and M. S. Turner, Phys. Rev. D 26 (1982) 2694.
42. J.-L. Cambier, J.R. Primack, and M. Sher, Nucl. Phys. B 209 (1982) 372.
43. A.E. Johansson, G. Peresutti, and B.S. Skagerstam, Nucl. Phys. B 278 (1986) 324.
44. R. Baier, E. Pilon, B. Pire, and D. Schif, Nucl. Phys. B 336 (1990) 157.
45. M.E. Carrington, A. Gynther, P. Aurenche, Phys. Rev. D 77 (2008) 045035.
46. P. Aurenche, M.E. Carrington, N. Marchal, Phys. Rev. D 68 (2003) 056001.
47. P. Aurenche, F. Gelis, G.D. Moore, and H. Zaraket, J. High Energy Phys. 0212 (2002) 006.
48. P. Aurenche, F. Gelis, and H. Zaraket, J. High Energy Phys 0207 (2002) 063.
49. P. Aurenche, F. Gelis, R. Kobes, and H. Zaraket, Phys. Rev. D 60 (1999) 076002.
50. M.E. Carrington, A. Gynther, and D. Pickering, Phys. Rev. D 78 (2008) 045018.

Acknowledgement One of the authors (MQH) thanks Higher Education Commission, Pakistan for partial funding under a research grant no. 1925 during this work.

Figure caption

Fig. 1. Vacuum polarization diagrams at the second order in $\alpha$. 
