HOMOLOGICAL PROPERTIES OF REPRESENTATIONS OF P-ADIC GROUPS RELATED TO GEOMETRY OF THE GROUP AT INFINITY

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Abstract.
Let $G$ be a reductive group with compact center over a nonarchimedean local field, and let $X$ be its Bruhat-Tits building.

The central result of the first part of the work establishes a connection between the direct image of an equivariant sheaf on $X$ to the Borel-Serre compactification with Jacquet functor. More precisely, we prove that there exists an exact functor $L$ on a category of representations of the group, such that the following holds. Let $F$ be an equivariant simplicial sheaf on $X$; then the restriction of the direct image of $F$ to the boundary of the Borel-Serre compactification is canonically isomorphic to $L(\Gamma(F))$ (here $\Gamma$ stands for global sections). Furthermore, the restriction of $L$ to the category of admissible representations is described explicitly in terms of Jacquet functor.

As an application we obtain a canonical isomorphism between the homological duality on the derived category of smooth $G$-modules, and the composition of the Deligne-Lusztig functor with the Grothendieck-Serre duality (the latter comes from the action of the Bernstein center on every smooth $G$-module). (This isomorphism was formulated by J. Bernstein; another proof was recently obtained by P. Schneider).

In the second part I prove a conjecture by D. Kazhdan which expresses the Euler characteristic of Yoneda Ext’s between two admissible representations as the integral of the product of their characters over the set of elliptic conjugacy classes (another proof of this result was recently obtained by P. Schneider).
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1. Let $F$ be a non-archimedian local field, and $G = G(F)$ be the group of $F$-points of a reductive algebraic group $G$ defined over $F$. We assume for convenience that $G$ has compact center.

We fix an algebraically closed coefficient field $k$ of characteristic 0. $H$ will denote the Hecke algebra of $k$-valued locally constant compactly supported measures on $G$.

The main object of our study is the category $\mathcal{M}$ of finitely generated smooth representations of $G$ over $k$ [BZ].

2. In chapter 1 the category $\mathcal{M}$ is studied using equivariant sheaves on the Bruhat-Tits building $X$ of $G$.

For a $G$-space $Y$ we will denote by $Sh_G(Y)$ the category of $G$-equivariant sheaves on $X$. Let $Sh \subset Sh_G(X)$ be the category of simplicial $G$-equivariant sheaves of finite-dimensional vector-spaces on $X$. We have the derived functor of sections with compact support $R\Gamma_c : D^b(Sh) \to D^b(M)$, and Verdier dual functor $R\Gamma : D^b(Sh) \to D^b(M^{opp})$ (where $M^{opp}$ is the opposite category to $M$).

Consider the imbedding $j : X \to \overline{X}$ of $X$ into its Borel-Serre compactification $\overline{X}$ [BoSer], and the equivariant direct image functor $j^G_\ast : D^b(Sh) \to D^b(Sh_G(\overline{X}))$ (discussed below). Let $i$ stand for the imbedding $i : \overline{X} - X \hookrightarrow \overline{X}$. The main technical result of chapter 1 is existence of a functor $L : M^{opp} \to Sh_G(\overline{X} - X)$ such that:

i) We have $i^* \circ j^G_\ast = L \circ \Gamma$ canonically.

ii) $L$ is exact.

iii) Let $\mathcal{R} \subset M$ be the subcategory of admissible representations. Then $L|_{R^{opp}}$ is a well-known combinatorially defined functor: for $M \in \mathcal{R}$ and $y \in \overline{X} - X$ the stalk of $L(M)$ at $y$ is the Jacquet functor $r_{P_y}(M)$ where $P_y = Stab_G(y)$ is a parabolic subgroup in $G$. (The equivalence between $\mathcal{R}$ and $R^{opp}$ used above sends a representation $\rho \in \mathcal{R}$ to its contragradient $\rho^\vee$).

We remark that construction of $L$ and proof of i) and iii) are rather straightforward, while the proof of ii) required some effort.

To apply this result to representation theory one needs to be able to “localize” representations to sheaves on the building. An ideal result in this direction would be a one-sided inverse to the functor $R\Gamma_c$. Unfortunately such functor is unavailable at present. By the results of [Sch-St1], [Sch-St2] the full subcategory $D^b(\mathcal{R}) \subset D^b(M)$ lies in the image of $R\Gamma_c$. (And also $D^b(M)$ lies in the image of the functor $R\Gamma^c$ from $G$-equivariant simplicial sheaves of possibly infinite dimensional vector spaces to $D^b(\overline{M})$, where $\overline{M}$ is the larger category of all (not necessary finitely generated) smooth $G$-modules).

The result of [Sch-St2] is sufficient for our application. By a “general nonsense” argument we get another localization type claim sufficient for us (but much less explicit, and thus probably less useful than the Schneider-Stuhler’s result). It says the following: for any object $B \in D^b(M)$ and $i \in \mathbb{Z}$ one can find an object $A \in D^b(Sh)$ and a morphism $R\Gamma_c(A) \to B$ which induces isomorphism on $H^j$ for $j > i$.

Using the connection between the Verdier duality on sheaves with the homological duality on representations (also observed in Schneider’s work [Schn2]) one
derives the following statement (which belongs essentially to J. Bernstein and was the starting point for this part of the work): there exists a canonical isomorphism of functors:

\[ D_h \cong DL \circ D_{Gr} \]

where \( DL \) is the Deligne-Lusztig functor [DL]; \( D_h \) is the homological duality \( D_h : M \rightarrow RHom_G(M, C_\infty(G)) \); and \( D_{Gr}(M) \) is the Grothendieck dual of \( M \) as a module over the Bernstein center [BD] (for admissible \( M \) we have \( D_{Gr}(M) = M^\vee \) where \( M^\vee \) is the contragradient module). (See the main text for detailed definitions).

A proof of the latter isomorphism using (co)sheaves on the Borel-Serre compactification of the Bruhat-Tits building appears also in [Sch-St2].

3. The starting point for Chapter 2 was an attempt to find an “a’priori” proof for the following conjecture by Kazhdan (proved recently in [Sch-St2] by making use of an explicit resolution of a \( G \)-module constructed in [Sch-St2], [Sch-St1]).

Let \( Ell \) be the set of regular semisimple elliptic conjugacy classes in \( G \). Then \( Ell \) carries a canonical measure (an analogue of the Weyl integration measure for compact Lie groups, see section 4 of chapter 2 for a recollection), which we denote by \( d\mu \). For an admissible \( G \)-module \( \rho \) let \( \chi_\rho \) denote its character, and let us use the same notation for the corresponding locally constant function on \( Ell \). By [K] we have \( \chi_\rho \in L^2(Ell, d\mu) \) provided the base local field \( F \) has characteristic 0. Let now \( \rho_1, \rho_2 \) be two admissible representations. Then the conjecture (= Theorem 0.20 of chapter 2) says that

\[
\sum (-1)^i \dim Ext^i(\rho_1, \rho_2) = \int_{Ell} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g) d\mu(g) \tag{1}
\]

Recall the following construction (see e.g. [Br], §IX.2 or [Vig] §2 and references therein). Let \( A \) be a Noetherian associative algebra with 1, having finite homological dimension. Then to a pair \( (M, E) \) where \( M \) is a finitely generated \( A \)-module, and \( E \in \text{End}(M) \) one can associate an element of \( A/\mathfrak{A}, A \), denoted by \( Tr_{H-St}(M, E) \), and called the Hattori-Stallings trace of \( (M, E) \). This map enjoys the properties:

a) \( Tr_{H-St}(M, E_1 \circ E_2) = Tr_{H-St}(N, E_2 \circ E_1) \) for two modules \( M, N \) as above, and \( E_1 \in \text{Hom}(N, M), E_2 \in \text{Hom}(M, N) \);

b) \( Tr(A, R(a)) = a \mod [A, A] \) for \( a \in A \) where \( R(a) \) is the right multiplication by \( a \in A \) considered as an endomorphism of the free module;

c) additivity on short exact sequences.

It is characterized by these properties uniquely.

Applying this construction (with slight modifications necessary because \( \mathcal{H} \) is not unital, see 0.21 of chapter 2) to our situation we obtain a map \( Tr_{H-St} : \{ (\mathfrak{M}, E) \mid \mathfrak{M} \in \mathcal{M}, E \in \text{End}(\mathfrak{M}) \} \rightarrow \mathcal{H}/[\mathcal{H}, \mathcal{H}] \) satisfying above properties a), b), c). For \( \mathfrak{M} \in \mathcal{M} \) the element \( Tr_{H-St}(\mathfrak{M}, Id) \) is called a pseudo-coefficient or rank function of \( \mathfrak{M} \); we will denote it by \( \langle \mathfrak{M} \rangle \).

Recall that for \( h \in \mathcal{H} \) and a regular elliptic element \( g \in G \) the orbital integral \( O_g(h) \in k \) is defined by

\[
O_g(h) := \int_{y \in G} \frac{h}{\nu}(gy\nu^{-1}) d\nu(y),
\]
where \( \nu \) is a Haar measure on \( G \); the resulting expression does not depend on the choice of \( \nu \).

A formal argument shows (see section 4 of chapter 2) that (1) is equivalent to the equality

\[
O_g^{-1}(\langle \rho \rangle) = \chi_\rho(g)
\]

being true for any admissible representation \( \rho \) and regular elliptic element \( g \in G \).

Notice that the latter statement makes sense (and will be proved) without the restriction \( \text{char}(F) = 0 \) or \( k = \mathbb{C} \).

If \( \rho \) is cuspidal, and \( m_\rho \in \mathcal{H} \) is its matrix coefficient normalized by \( \int m_\rho \cdot \chi_\rho = 1 \), then \( \langle \rho \rangle = m_\rho \mod [\mathcal{H}, \mathcal{H}] \). Thus (2) generalizes a well-known equality going back to Harish-Chandra

\[
\chi_\rho(g) = O_g^{-1}(m_\rho).
\]

To prove (2) we extend the functional \( Tr(g, \rho) = \chi_\rho(g) \) from admissible to arbitrary smooth representations \( \rho \), and then show (Theorem 3.1) that for a projective \( \rho \cong \mathcal{H}^{\oplus n} E \), where \( E \in \text{Mat}_n(\mathcal{H}) \) is an idempotent we have

\[
Tr(g, \rho) = O_g^{-1}(\sum E_{ii}).
\]

The intuitive idea behind the definition of \( Tr(g, \mathfrak{M}) \) is that for a small open compact subgroup \( K \subset G \) one can find a finite dimensional \( g \)-invariant subspace \( \mathfrak{M}_0 \subset \mathfrak{M}^K \) such that the quotient \( \mathfrak{M}^K/\mathfrak{M}_0 \) “comes from the parabolic induction”; thus it is reasonable to set \( Tr(g, \mathfrak{M}) = \text{tr}(g, \mathfrak{M}_0) \). Realization of this idea exploits certain natural (multi)filtration on the Hecke algebra connected with the action of \( \mathcal{H} \) on the principal series (see section 2 of chapter 2).

To prove (3) we relate our filtration to the one defined “geometrically” in terms of support of a distribution \( h \in \mathcal{H} \). Here some geometry of \( G \) and its asymptotic cones \( (G/U \times G/U^-)/L \) at infinity is used.
1. Sheaves on the building and representations

Preliminaries and notations. In both chapters we will abuse the terminology by saying “a parabolic/Levi subgroup” instead of “the group of $F$-points of an $F$-rational parabolic subgroup/Levi subgroup of a proper parabolic subgroup”. By rank of a reductive group defined over $F$ we mean its split rank.

0.1. $k$ is a fixed coefficient field of characteristic 0; for a set $S$ we denote by $k[S]$ the space $\bigoplus_S k$. For a topological space $Z$ we write $C^\infty(Z)$ and $C_c^\infty(Z)$ for the space of locally constant functions on $Z$ and locally constant functions with compact support on $Z$. If $Z$ is an open subset of a group, we denote by $\mathcal{H}(Z)$ the space of locally constant measures on $Z$ with compact support; if $Z$ is biinvariant under the action of a compact open subgroup $K \subset G$, then $\mathcal{H}(Z,K) \subset \mathcal{H}(Z)$ is the subspace of $K$-biinvariant measures. If $Z$ is a semigroup, then $\mathcal{H}(Z)$ and $\mathcal{H}(Z,K)$ carry the structure of an associative algebra provided by convolution.

By $\underline{W}$ or $\underline{W}_Z$ we will denote the constant sheaf on $Z$ with stalk $W$; if $\iota : Z \to X$ is a closed embedding then we will also write $\underline{W}_Z$ instead of $\iota_*(\underline{W}_Z)$. By $\underline{Hom}$ we denote internal $\underline{Hom}$ in the category of ($G$-equivariant) sheaves.

For a $G$-space $X$ let $\mathcal{Sh}_G(X)$ denote the category of $G$-equivariant sheaves on $X$. Recall that for arbitrary topological group $G$ and a $G$-space $X$ a $G$-equivariant sheaf is the same as a sheaf $\mathcal{F}$ on $X$ equipped with a $G$-action such that each section of $\mathcal{F}$ is locally constant. Here a section $s \in \Gamma(U,\mathcal{F})$ is called locally constant if for $x \in U$ there exist a neighborhood $U'$ of $x$ and a neighborhood $V$ of identity $e \in G$ such that $g(s)|_{U' \cap g(U')} = s|_{U' \cap g(U')}$ for $g \in V$.

For a morphism of $G$-spaces $p : X \to Y$ we have the functor of inverse image $p^* : \mathcal{Sh}_G(Y) \to \mathcal{Sh}_G(X)$ and the adjoint functor $p_*^G : \mathcal{Sh}_G(X) \to \mathcal{Sh}_G(Y)$. Note that the inverse image commutes with the forgetful functor from equivariant sheaves to sheaves while the direct image does not: for $\mathcal{F} \in \mathcal{Sh}_G(X)$ one can describe $p_*^G(\mathcal{F})$ as the sheaf of locally constant sections of $p_*(\mathcal{F})$ (see [Sch1] for a detailed discussion).

From now on let $X$ denote the (semi-simple) building of $G$, unless stated otherwise. Let $\bar{X}$ be the Borel-Serre compactification of $X$ [BorSe]; put $Y = \bar{X} - X$, and denote the imbeddings by $j : X \hookrightarrow \bar{X}$ and $i : Y \hookrightarrow \bar{X}$. Recall that $Y$ is the spherical building of $G$. For a parabolic $P \subset G$ we let $\Delta_P \subset Y$ denote the corresponding simplex.

We will often say polysimplex meaning a polysimplex of the canonical polytriangulation of $X$ (facette in the terminology of [BT1]).

Let $\mathfrak{Sh}$ be the full subcategory of $\mathcal{Sh}_G(X)$ consisting of sheaves of finite dimensional vector spaces which are constant on every (poly)simplex.

0.2. By a representation we will mean a representation in a vector space over a field $k$; the field will be endowed with discrete topology.

Let $\tilde{\mathcal{M}}$ be the category of smooth $G$-modules; let $\mathcal{M}'$ be the dual category. It will be convenient to use an explicit realization of $\mathcal{M}'$. Namely $\mathcal{M}'$ is identified with the category of complete topological vector spaces $V$ having a basis of neighbourhoods of 0 consisting of vector subspaces of finite codimension, together with a continuous homomorphism $\pi : G \to \text{Aut}(V)$. (The group $\text{Aut}(V)$ is endowed with the topology of pointwise convergence.) The antiequivalence $* : \tilde{\mathcal{M}} \to \mathcal{M}'$ sends a $G$-module $M$ to its full linear dual equipped with the pointwise-convergence topology (= 1).
the inverse limit topology on $M^* = \varprojlim V^*$, where $V$ runs over the set of finite dimensional subspaces in $M$). The inverse antiequivalence (also denoted by $*$) sends a topological $G$-module $L$ to the space of continuous linear functionals $L \to k$.

Let $\mathcal{M} \subset \tilde{\mathcal{M}}$ (respectively $\mathcal{R} \subset \mathcal{M}$) be the full subcategory of finitely generated smooth (respectively admissible) $G$-modules. Let $\mathcal{M}' \subset \mathcal{M}'$ and $\mathcal{R}' \subset \mathcal{M}'$ be the dual categories.

We have an exact functor $Sm : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ sending a module $M \in \tilde{\mathcal{M}}$ to the module of smooth vectors in $M$. The functor $Sm$ induces an equivalence $\mathcal{R} \cong \mathcal{R}$. (Of course it does not send the category $\mathcal{M}'$ to $\mathcal{M}'$.) For $M \in \mathcal{M}$ we let $\mathcal{M}' = Sm(*(M)) \in \tilde{\mathcal{M}}$ be its contragradient.

We have left-exact functors $\Gamma_c : Sh \to \mathcal{M}$ and $\Gamma : Sh \to \mathcal{M}'$, where $\Gamma$ stands for global sections, and $\Gamma_c$ stands for sections with compact support. The linear topology on the space $\Gamma(X, F)$ is the inverse limit topology on $\Gamma(K, F) = \varprojlim \Gamma(K, F)$ where $K$ runs over all compact subsets of $X$.

As we will soon recall, $Sh$ has enough injectives (and projectives). Hence the right derived functors $R\Gamma : D^*(Sh) \to D^*(\mathcal{M})$ and $R\Gamma_c : D^*(Sh) \to D^*(\mathcal{M})$ where $* = b,+$ are defined.

1. Verdier duality for equivariant sheaves and homological duality. Here we recall some basic generalities on homological algebra of $Sh$, including Verdier duality and its connection with homological duality on representations. This section does not contain new results: Proposition 1.3a) follows from [Schn1], §3 and remark before [Schn2], Proposition 2; our 1.3b) is Proposition 4 in [Schn2]. We sketch the proof for completeness.

1.1. Let $Sh(X)$ be the category of sheaves of finite dimensional $k$-vector spaces on $X$ constant on any polysimplex of the canonical polytriangulation. Then $Sh(X)$ has enough injectives; the injective generators are the constant sheaves $\mathbb{K}^n$ on the closure of a simplex. From the fact that the stabilizer of any point of $X$ is compact, and hence its smooth representations form a semisimple category, it is easy to deduce that a sheaf $F \in Sh$ is injective iff it is injective as an object of $Sh(X)$. For a polysimplex $\Delta \subset X$ and a smooth finite dimensional representation $R$ of the stabilizer of $\Delta$ let $F_{\Delta,R} \in Sh$ be the $G$-equivariant sheaf $F_{\Delta,R} := \prod_{g \in G/Stab(\Delta)} k^{g\cdot \mathbb{K}^\Delta} \otimes C^\infty (g \cdot Stab(\Delta)) \otimes_{Stab(\Delta)} R$. Obviously $F_{\Delta,R}$ is injective.

Let $\mathcal{I}$ be the category of injective objects in $Sh(X)$, and $\mathcal{I}_G$ be the category of injective objects in $Sh$.

Let us call a module $M \in \mathcal{M}'$ (respectively $M \in \mathcal{M}$) standard injective (respectively projective) if it is a direct sum of modules induced (respectively compactly induced) from smooth finite dimensional representations of open compact subgroups.

Lemma 1.2. a) $Sh$ is of finite homological dimension and has enough injectives. Any injective object in $Sh$ is a direct sum of sheaves of the form $F_{\Delta,I}$. b) $\Gamma : Sh \to \mathcal{M}'$ sends injectives to injectives and $\Gamma_c : Sh \to \mathcal{M}$ sends injectives to projectives.

c) An injective (respectively projective) object of $\mathcal{M}'$ (respectively $\mathcal{M}$) is isomorphic to $\Gamma(F)$ (respectively $\Gamma_c(F)$) for $F \in \mathcal{I}_G$ if it is a standard injective (projective).
d) If \( F, \mathcal{G} \in \mathcal{J}_G \) then \( \underline{\text{Hom}}(F, \mathcal{G}) \) and \( F \otimes \mathcal{G} \) are also injective.

Proof Clear. \( \square \)

We see that any left-exact functor on \( \mathcal{S}h \) admits the (right) derived functor.

**Proposition 1.3.** a) There exists an object \( \underline{V} \in D^b(\mathcal{S}h) \) equipped with an isomorphism of functors on \( D^b(\mathcal{S}h) \)

\[
R\Gamma \circ R\underline{\text{Hom}}(\underline{V}, \underline{\_}) \cong \ast \circ R\Gamma_c
\]

Isomorphism (4) defines \( \underline{V} \) uniquely up to the unique isomorphism. \( \underline{V} \) is called the equivariant dualizing sheaf, and the functor \( \underline{V} := R\underline{\text{Hom}}(\underline{V}, \underline{\_}) \) is called the equivariant Verdier duality.

b) Let \( D_h : D^b(\mathcal{M}) \to D^b(\mathcal{M}) \) be the homological duality defined by \( D_h(M) = R\underline{\text{Hom}}(M, \mathcal{H}) \). We have a canonical isomorphism:

\[
R\Gamma_c \circ \underline{V} \cong D_h \circ R\Gamma_c
\]

Proof a) Obviously \( \underline{\text{Hom}}_{\mathcal{S}h}(F, \mathcal{G}) = (\underline{\Gamma}(\underline{\text{Hom}}(F, \mathcal{G})))^G \). Using Lemma 1.2b),d) we see that \( R\underline{\text{Hom}}(F, \mathcal{G}) = R\Gamma \circ R\underline{\text{Hom}}(R\underline{\text{Hom}}(F, \mathcal{G})) \) where \( I \) stands for invariants \( I : \mathcal{M} \to \text{Vect} \). So (4) yields an isomorphism of functors \( \underline{\text{Hom}}(\underline{V}, \underline{\_}) \cong H^0(R\Gamma \circ \ast \circ R\Gamma_c) \); uniqueness of \( \underline{V} \) follows.

To show existence we adapt a standard construction of the dualizing sheaf to our situation. For a polysimplex \( \iota : \Delta \to X \) let \( C_\Delta \) be the standard resolution of \( \iota(\underline{k}[^{\text{dim}}(\Delta)]) \) (so we have \( C_\Delta = \bigoplus_{\Delta \subset \Delta, \dim(\Delta') = -i} \underline{k}^{\Delta'} \)).

We set \( \underline{V}(\underline{k}) = C_\Delta \); this obviously extends to a contravariant functor \( \text{Kom}(\mathcal{J}) \to \text{Kom}(\mathcal{J}) \) and defines a contravariant functor \( \text{Kom}(\mathcal{J}_G) \to \text{Kom}(\mathcal{J}_G). \)

(We have \( \underline{V}(\underline{F}, \underline{\_}) = \prod_{g \in G/\text{Stab}(\Delta)} C_g \otimes C_\Delta^\infty (g \cdot \text{Stab}(\Delta)) \otimes R^* \)) The induced functor \( D^b(\mathcal{S}h) \to D^b(\mathcal{S}h) \) is also denoted by \( \underline{V} \). One can easily define isomorphism of complexes \( \underline{V}(\underline{F} \otimes \underline{\_}) \cong \underline{\text{Hom}}(\underline{F}, \underline{V}(\underline{\_})) \) for \( F, \mathcal{G} \in \mathcal{J}_G \); the isomorphism of bifunctors on \( D^b(\mathcal{S}h) \) follows. Hence \( \underline{V} \) is corepresented by \( \underline{V} = \underline{V}(\underline{k}) \).

For \( F \in \mathcal{J}_G \) we have a functorial map of complexes \( \underline{\Gamma}(\underline{V}(\underline{\_})) \to \ast (\underline{\Gamma}_c(\underline{\_})) \) which is a quasiisomorphism; here the pairing \( \langle \_ , \_ \rangle : \underline{\Gamma}_c(\underline{\_}) \times \underline{\Gamma}(\underline{V}(\underline{\_})) \to k \) is the standard Verdier pairing. This yields isomorphism (4).

b) The pairing \( \langle \_ , \_ \rangle \) gives a map \( \phi : \underline{\Gamma}(\underline{V}(\underline{\_})) \to \text{Hom}_G(\underline{\Gamma}_c(\underline{\_}), C_\infty^G(\underline{\_})) \); here \( (\phi(y))(x) = (x, g(y)) \). It is clear that \( \phi(\underline{\Gamma}_c(\underline{V}(\underline{\_}))) \subset \text{Hom}_G(\underline{\Gamma}_c(\underline{\_}), C_\infty^G(\underline{\_})) \subset \underline{\text{Hom}}_G(\underline{\Gamma}_c(\underline{\_}), C_\infty^G(\underline{\_})) = \underline{\Gamma}(\underline{V}(\underline{\_})) \); more precisely for \( F \in \mathcal{J}_G \) we get a functorial quasiisomorphism \( \underline{\Gamma}_c(\underline{V}(\underline{\_})) \to \underline{\text{Hom}}_G(\underline{\Gamma}_c(\underline{\_}), C_\infty^G(\underline{\_})) \) which yields (5). (In fact for \( F \in \mathcal{J}_G \) isomorphisms (4), (5) boil down to well-known equalities

\[
\ast (\text{Ind}_K^G(\underline{\_})) = \text{Ind}_K^G(\underline{\_})^*,
\]

\[
\text{Hom}_G(\text{Ind}_K^G(\underline{\_}), C_\infty^G(\underline{\_})) = \text{Ind}_K^G(\underline{\_})^*.\]

Proposition is proved. \( \square \)

Remark 1.4. In [Schn2] it is proved that \( \underline{V} \) is actually concentrated in one homological degree; we will not use this fact here.

2. A “localization” type theorem. Let us call a morphism in \( D^b(\mathcal{S}h) \) \( h \)-quasiisomorphism (respectively \( hc \)-quasiisomorphism) if it induces an isomorphism on \( R\Gamma \) (respectively \( R\Gamma_c \)).
Let $D^b(M)$, $D^0(M)$ be the full subcategories in $D^b(M)$, $D^b(M)$ respectively consisting of objects quasisiomorphic to a finite complex of standard injectives (respectively projectives). $LD^b(S\mathfrak{h})$, $L^0D^b(S\mathfrak{h})$ be the localizations of $D^0(S\mathfrak{h})$ respectively with respect to $h$-quasiisomorphism and $hc$-quasiisomorphisms.

**Proposition 2.1.** $R\Gamma$ provides an equivalence between $LD^b(S\mathfrak{h})$ and $D^0(M)$. $R\Gamma_c$ provides an equivalence between $L^0D^b(S\mathfrak{h})$ and $D^0(M)$.

For the proof of the Proposition we need the following

**Lemma 2.2.** Assume that $\mathcal{F}, \mathcal{G} \in S\mathfrak{h}$ and $\mathcal{F}$ is injective; let a morphism $\phi : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ be given. Then there exist a sheaf $\mathcal{F}' \in S\mathfrak{h}$ and morphisms $s : \mathcal{F}' \to \mathcal{F}$, $\phi' : \mathcal{F}' \to \mathcal{G}$ such that $s$ is an $h$-quasiisomorphism, and $\Gamma(\phi') = \phi \circ \Gamma(s)$.

**Proof:** First notice that $X \times X$ is a union of poly-simplicial subspaces $X_r$, such that $pr_i |_{X_r}$ is proper for $i = 1, 2$ and $pr_1 |_{X_r}$ has geodesically contractible fibers. (To see this let us choose an apartment $\mathfrak{X} \subset X$, an open simplex $\Delta \subset \mathfrak{X}$, and exhaust $\mathfrak{X}$ by simplicial convex neighborhoods $\mathfrak{C}_r$ of $\Delta$. Then take $X_r = G(\Delta \times \mathfrak{C}_r)$. The desired properties of $X_r$ are clear. We have $\bigcup X_r = X \times X$ because $G(\Delta) = X$ and any two points of $X$ are $G$-conjugate to two points lying in $\mathfrak{X}$.) Let $p_i = pr_i |_{X_r}$, $i = 1, 2$. Set $\mathcal{F}_r = p_{2*}p_1^* \mathcal{F}$.

We claim that for large enough $r$ we can take $\mathcal{F}' = \mathcal{F}_r$.

It is clear that if $\mathcal{F}$ is injective then $R\Gamma p_{2*}p_1^* \mathcal{F} = 0$ for $i \neq 0$. Hence $R\Gamma(\mathcal{F}_r) = R\Gamma(p_1^* \mathcal{F}) = R\Gamma(\mathcal{F})$ since $p_1$ has contractible fibers. Applying $p_{2*}$ to the morphism $p_1^* \mathcal{F} \to \delta_*(\mathcal{F})$ where $\delta : X \to X \times X$ is the diagonal imbedding we get a map $\mathcal{F}_r \to \mathcal{F}$ which induces isomorphism on cohomology.

It remains to construct $\phi' : \mathcal{F}_r \to \mathcal{G}$ for large $r$.

For any $x \in X$ consider the composition $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{G}) \to \mathcal{G}_x$ where $\mathcal{G}_x$ is the stalk of $\mathcal{G}$ at $x$. It is continuous and $\mathcal{G}_x$ is discrete; hence it factors through a map $\Gamma(\mathcal{F}) \to \Gamma(\mathfrak{C}, \mathcal{F}) \to \mathcal{G}_x$ for some compact $\mathfrak{C} = \mathfrak{C}(x) \subset X$. Then clearly the same is true for any $y$ lying in one polysimplex with $x$ with the same compact set $\mathfrak{C}(y) = \mathfrak{C}(x)$.

We can take a finite number of polysimplices $\Delta_i$ such that $G(\cup \Delta_i) = X$; and then find $r$ such that $X_r \supset \mathfrak{C}(x_i) \times \Delta_i$ for $x_i \in \Delta_i$. Then we see that the map $\Gamma(\mathcal{F}_r) = \Gamma(\mathcal{F}) \to \mathcal{G}_x$ factors through $\Gamma(\mathcal{F}_r) \to (\mathcal{F}_r)_x$ for all $x \in X$, i.e. the composition $k \otimes \Gamma(\mathcal{F}_r) \to k \otimes \Gamma(\mathcal{G}) \to \mathcal{G}$ factors through the canonical surjection $k \otimes \Gamma(\mathcal{F}_r) \to \mathcal{F}_r$. The Lemma is proved. □

**Proof** of the Proposition. First note that image of $R\Gamma$ is contained in $D^0(M)$ since an injective sheaf goes to a standard injective module.

Lemma 2.2 implies that for $\mathcal{F}, \mathcal{G} \in \mathfrak{I}_G$ the map $\text{Hom}_{LD^b(S\mathfrak{h})}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{M}(R\Gamma \mathcal{F}, R\Gamma \mathcal{G}) = \text{Hom}_{M}(\Gamma \mathcal{F}, \Gamma \mathcal{G})$ is surjective.

It is also injective for the following reason. Consider the commutative square

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{s} & \mathcal{G} \\
\uparrow & & \uparrow \\
\Gamma(\mathcal{F}) & \xrightarrow{\Gamma(s)} & \Gamma(\mathcal{G})
\end{array}
\]

where $\Gamma(s) : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ is the map on global sections induced by $s$, and $\Gamma(s)$ is the corresponding map of constant sheaves. The left vertical arrow in this diagram is surjective for injective $\mathcal{F}$, so we have $\Gamma(s) = 0$, $\mathcal{F} \in \mathfrak{I}_G \Rightarrow s = 0$. 
Thus $R\Gamma$ induces an isomorphism $\text{Hom}_{LD^b(Sh)}(F, G) \cong \text{Hom}_{\mathcal{M}^*}(R\Gamma F, R\Gamma G)$ for $F, G \in \mathcal{I}$, and hence for any $F, G \in D^b(Sh)$.

It remains to check that $R\Gamma : LD^b(Sh) \to D^0(\mathcal{M}^*)$ is surjective on isomorphism classes of objects. Let $C$ be a finite complex of standard injectives $0 \to I_a \to \ldots \to I_b \to 0$. We then have some injective sheaves $\mathcal{F}_i \in \mathcal{S}h$ with $\Gamma(\mathcal{F}_i) \cong I_i$. We now use inductively Lemma 2.2 to construct a complex of sheaves $\tilde{C} = 0 \to \mathcal{F}_a' \to \ldots \to \mathcal{F}_b' \to 0$, where $\mathcal{F}_i' = (\mathcal{F}_i)_r$, $H^j(F_i') = 0$ for $j \neq 0$ and $\Gamma(\tilde{C}) \cong C$.

The first statement is proved, and the second one follows by Verdier duality. □

Remark 2.3. I do not know whether in general $D^0(\mathcal{M}) = D^0(\mathcal{M})$, i.e. whether a localization type theorem stronger (but less explicit) than the one obtained by Schneider and Stuhler in [Sch-St1], [Sch-St2] holds. The positive answer to this question is equivalent to the claim that the Grothendieck group of $\mathcal{M}$ is generated by the classes of standard projectives.

Notice that the following unpublished result of J. Bernstein agrees with the positive answer to this question: the Grothendieck group of $\mathcal{M}$ tensored with $Q$ is generated by the classes of standard projectives.

Remark 2.4. Our method also does not answer the question whether $D^+(\mathcal{M}^*)$ or $D^+(\mathcal{M})$ is localization of $D^+(Sh)$. (Any module has infinite right resolution by standard injectives; our procedure of constructing a complex of sheaves from such a resolution was inductive “moving from right to the left”.)

3. Main theorem. We start the construction of the functor $L$ promised in the introduction.

Let $Z$ be a closed subset of $\overline{X} - X$.

Consider the subspace $\Gamma_c(\overline{X} - Z, j_*F) \subset \Gamma(X, F)$ and the subspace $\Gamma_c(\overline{X} - Z, j^G_*F) \subset \Gamma(\overline{X}, j^G_*F) = Sm(\Gamma(X, F))$. (The latter equality readily follows from compactness of $\overline{X}$.)

We now explain how to recover these subspaces from an object $\Gamma(X, F) \in \mathcal{M}$.

Fix a point $p \in X$. Let $a : G \times X \to X$ be the action; set $a_p(g) = a(g, p)$. Let $\mathfrak{S}_Z$ denote the filter of subsets in $G$ generated by the sets of the form $a_p^{-1}(U)$, where $U$ is a neighborhood of $Z$.

For a module $L \in \mathcal{M}$ define $L_Z = \{m \in L \mid \lim_{g} g^{-1}(m) = 0\}$, and $L_Z^{sm} = Sm(L) \cap L_Z$. (Here $\lim_{g}$ stands for the limit with respect to the filter $\mathfrak{S}$.)

Lemma 3.1. a) The filter $\mathfrak{S}$ is invariant under right translations on $G$. It does not depend on the choice of a point $p \in X$.

b) We have:
\[
\Gamma_c(\overline{X} - Z, j_*F) = (\Gamma(X, F))_Z
\]
\[
\Gamma_c(\overline{X} - Z, j^G_*F) = (\Gamma(X, F))_{Z}^{sm}
\]

Proof To prove a) we need the following well-known property of $\overline{X}$.

Fact 3.2. Assume that a sequence of points $x_n \in X$ has a limit $x \in \overline{X} - X$. Let $y_n \in X$ be another sequence, such that $d(x_n, y_n)$ is bounded. Then $y_n \to x$ as well.

Proof is included since no reference was found. We use Remark 5.5(2) in [BoSer]. It says the following. Fix an open polysimplex $\Delta_0 \subset X$, and for any apartment $A \subset X$ containing $\Delta_0$ let $\Xi : A \to X$ be the contraction of $X$ to $\Delta_0$ [BT1] 2.3.4, 7.4.19. Recall that an apartment $A$ is an affine space, and let $\overline{A}$ be its standard semisphere compactification (see below); denote by $j_{\overline{A}}$ the imbedding $A \hookrightarrow \overline{A}$. Then
$X$ is the closure of the image of the map $\prod (j_\mathfrak{A} \circ \Xi_\mathfrak{A}) : X \to \prod \overline{\mathfrak{A}}$. (The product is taken with the Tikhonov topology.) In particular two sequences $x_n, y_n$ have the same limit in $X$ if and only if $\lim \Xi_\mathfrak{A}(x_n) = \lim \Xi_\mathfrak{A}(y_n)$ for any $\mathfrak{A} \supset \Delta_0$. However, by [BT1] 7.4.20(ii) the map $\Xi_\mathfrak{A}$ does not increase distances. Hence the statement is reduced to the analogous statement about sequences in the compactified affine space $\overline{\mathfrak{A}}$, which is obvious. □

The statement a) follows from the Fact directly. □

3.3. For $\mathfrak{M} \in \mathcal{M}^r$ we now define a presheaf $\tilde{\mathcal{L}}(\mathfrak{M})$ on $Y$ by $\Gamma(U, \tilde{\mathcal{L}}(\mathfrak{M})) := \lim Sm(\mathfrak{M})/(\mathfrak{M}_Z \cap Sm(\mathfrak{M}))$ where $Z$ runs over all closed subsets $Z \subset U$. We define $\mathcal{L}(\mathfrak{M})$ to be the associated sheaf.

From the definition it is clear that the stalk of $\mathcal{L}(\mathfrak{M})$ at a point $z \in Y$ equals $Sm(\mathfrak{M})/(\mathfrak{M}_z \cap Sm(\mathfrak{M}))$.

The desired properties of the functor $\mathcal{L}$ follow from the next

Theorem 3.4. a) Let $z \in X - X$ be a point. Then the functor $\mathfrak{M} \to \mathfrak{M}^{pm}_{\{z\}}$ on the category $\mathcal{M}^r$ is exact.

b) Let $P$ denote the stabilizer of the point $z$, then $P$ is a parabolic subgroup in $G$. Let $r_P$ be the corresponding Jacquet functor (that is coinvariants with respect to the unipotent radical of $P$). For any $\mathfrak{M} \in \mathcal{M}^r$ the subspace $\mathfrak{M}^{pm}_{\{z\}} \subset Sm(\mathfrak{M})$ contains the kernel of the projection $Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M}))$. Furthermore, if $\mathfrak{M}$ lies in the category $\mathcal{R}$ then the sequence

$$0 \to \mathfrak{M}^{pm}_{\{z\}} \to Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M})) \to 0$$

is exact.

Proof occupies the rest of this section.

3.5. We abbreviate $\mathfrak{g} = \mathfrak{g}_{\{z\}}$. Our first aim is to get a more tractable description of $\mathfrak{g}$.

Fix a Levi decomposition $P = L \cdot N$, and a maximal split torus $T \subset L$. Let $A$ be the center of $L$, and $A^a$ be its maximal split subtorus.

We have vector spaces $a = \text{Hom}(\mathcal{F}^\times, A) \otimes \mathbb{R} = (A/A^c) \otimes \mathbb{R} = (L/L^c) \otimes \mathbb{R}$ where $A^c \subset A$ is the maximal compact subgroup and $L^c \subset L$ is the subgroup generated by all compact subgroups; and $a_T = \text{Hom}(\mathcal{F}^\times, T) \otimes \mathbb{R} = (T/T^c) \otimes \mathbb{R}$ where $T^c \subset T$ is the maximal compact subgroup. The imbedding $a \hookrightarrow a_T$ is obtained from the imbedding $\text{Hom}(\mathcal{F}^\times, A) = \text{Hom}(\mathcal{F}^\times, A_\mathfrak{g}) \hookrightarrow \text{Hom}(\mathcal{F}^\times, T)$.

(Here we use the notation $\mathcal{H}$ for an algebraic group over $F$, and $H$ for its group of $F$-points.) We identify $\text{Hom}(\mathcal{F}^\times, A)$ and $L/L^c$ with subgroups in $a$ and the groups $\text{Hom}(\mathcal{L}, \mathcal{F}^\times)$, $\text{Hom}(\mathcal{A}, \mathcal{F}^\times)$ with subgroups in the dual space $a^*$.

We fix a uniformizer $p$ in $\mathcal{F}^\times$, and get cocompact imbeddings $\text{Hom}(\mathcal{F}^\times, T) \hookrightarrow T$, $\text{Hom}(\mathcal{F}^\times, A) \hookrightarrow A$, given by $\psi \mapsto \psi(p)$. We denote the image of latter imbedding by $\Lambda$.

Let $\pi$ be the projection $L \to L/L^c$.

By $\Lambda^+$ we denote the set of $\lambda \in \Lambda$ such that $\pi(\lambda) \in a^+$ where $a^+$ is the positive chamber, $a^+ = \{a \in a | r(a) < 0 \text{ if } r \text{ is a root of } A \text{ in } N\}$.

We fix notations and recall some basic facts about the geometry of $X$ (see [BT1], [BT2], [L] for more details).

Recall that the set of maximal split tori is in bijection with apartments in $X$. Let $\mathfrak{A}$ be the apartment corresponding to $T$, and let $\overline{\mathfrak{A}} \subset \overline{X}$ be the closure of $\mathfrak{A}$.
Then $\mathfrak{A}$ is an affine space with underlying vector space $\mathfrak{a}_T$. The torus $T$ acts on $\mathfrak{A}$ through $T/T^c$ acting on the affine space by translations.

The closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ in $\overline{X}$ is the standard semispherical compactification of the affine space $\mathfrak{A}$. In particular $\overline{\mathfrak{A}} - \mathfrak{A}$ is the set of rays $(\mathfrak{a}_T - \{0\})/\mathbb{R}^*_+$. By our assumption the point $z$ lies in $\overline{\mathfrak{A}} - \mathfrak{A}$ and corresponds to a ray $\rho_z \in \mathfrak{a} \subset \mathfrak{a}_T$. We have $\rho_z \subset \mathfrak{a}^\perp$.

By a cone in a vector space we will mean a convex centered cone.

**Definition 3.6.** a) A z-good cone is an open cone $C \subset \mathfrak{A}$ such that $\rho_z \subset C$.

For a z-good cone set $\Lambda_C = \Lambda \cap \pi^{-1}(C \cap (L/L^c))$ and $\Lambda_C^+ = \Lambda \cap \pi^{-1}(C \cap (L/L^c))$.

b) A z-good semigroup is an open compactly generated semigroup $M \subset L$ such that

i) $M$ contains $\Lambda_C$ for a z-good cone $C$.

ii) We have $L = \Lambda \cdot M$.

It is easy to see that an intersection of two z-good semigroups is again a z-good semigroup.

**Proposition 3.7.** The filter $\mathfrak{F}$ is generated by the sets

$$K' \cdot M \cdot a \cdot K_p,$$

where $K'$ is an open compact subgroup in $G$, $M$ is a z-good semigroup, $a \in \Lambda$ and $p \in \mathfrak{A}$.

We need some more notations and basic facts about $\overline{X}$.

Recall that $d(\cdot, \cdot)$ denotes the canonical metric on $X$.

For $x \in X$ denote by $K_x$ the stabilizer of $x$. It is an open compact subgroup in $G$.

**Lemma 3.8.** a) For any points $c \in X$, $z \in \overline{X} - X$ there exists a unique geodesic $[c, z)$ connecting $c$ and $z$. (That is $[c, z)$ is the unique geodesic ray starting at $c$ and having $z$ as its limit point in $\overline{X} - X$.) For $c \in \mathfrak{A}$ the ray $[c, z)$ coincides with $\rho_z + c$.

b) Assume that $c_1, c_2 \in X$, $z \in \overline{X} - X$ and let $c_i^t$ be the unique point on $[c_i, z)$ at the distance $t$ from $c_i$ for $i = 1, 2$. Then we have

$$d(c_1^t, c_2^t) \leq d(c_1, c_2)$$

(6)

**Proof** a) follows from [Sch-St2], Proposition on p. 166. More precisely, in loc. cit. it is proven that there exists an apartment $\mathfrak{A}$ containing $c$ and $z$ in its closure, and that the ray $[c, z)_{\mathfrak{A}} \subset \mathfrak{A}$ of direction given by $z$ with vertex at $c$ does not depend on the choice of $\mathfrak{A}$.

This gives existence; to show uniqueness it remains only to check that any geodesic ray $\rho$ lies in some apartment. Let $\mathfrak{A}$ be any apartment containing the vertex $c$ of the ray $\rho$. By [BT1] 2.3.4 the set $\mathfrak{C}_t := \{g \in K_c | g(c')_t \in \mathfrak{A}\}$ is nonempty for all $t$. Each of the sets $\mathfrak{C}_t$ is closed (and open) since it is a finite union of cosets of an open compact subgroup $K_c \cap K'_c$; also $\mathfrak{C}_{t_1} \subset \mathfrak{C}_{t_2}$ if $t_1 > t_2$ because $\mathfrak{A}$ contains the geodesic connecting any two points of $\mathfrak{A}$, [BT1] 7.4.20(iii). Since $K_c$ is compact we have $\bigcap_t \mathfrak{C}_t \neq \emptyset$. Now we can take $\mathfrak{A}' := g^{-1}(\mathfrak{A})$ for any $g \in \bigcap_t \mathfrak{C}_t$.

Let us prove b). Let $x_1 = c_1, x_2, \ldots, x_n = c_2$ be a sequence of points on $[c_1, c_2]$ such that $[c_1, c_2) = \cup[x_i, x_{i+1})$, and $x_i, x_{i+1}$ lie in the closure of an open polysimplex $\Delta_i$.

By [BT1] 7.4.18(ii) there exists an apartment $\mathfrak{A}_i$ such that $\mathfrak{A}_i \supset \Delta_i$, and $\overline{\mathfrak{A}_i} \ni z$. Thus $[x_i, z)$, $[x_{i+1}, z)$ are parallel rays in the affine space $\mathfrak{A}_i$, so $d(x_i^t, x_{i+1}^t) = \ldots$
where neighborhoods of \( \mu_0 \) of \( \Delta_0 \) in \( A \times \overline{A} \).

Fix a point \( x \in \mathfrak{A} \) which lies inside an open polysimplex \( \Delta_0 \subset \mathfrak{A} \). Then \( K_x(z) \cap \overline{A} = \{ z \} \). Indeed, if, on the contrary, \( z \neq k(z) \) for some \( k \in K_x \), then \( k([x, z]) = [x, k(z')] \) is a ray in \( \mathfrak{A} \) different from \( [x, z] \); however, since \( k \) fixes the neighborhood \( \Delta_0 \) of \( x \) in \( \mathfrak{A} \), the intersection of the two rays with this neighborhood coincide, which is impossible.

Thus \( \mu^{-1}(z) = \text{Stab}_{K_x}(z) \times \{ z \} = (P \cap K_x) \times \{ z \} \).

Let \( P^- = L \cdot N^- \) be the parabolic opposite to \( P \). Since \( x \) lies in an open polysimplex of \( \mathfrak{A} \) we have the following triangular decomposition:

\[
K_x = K_x^- \cdot K_x^0 \cdot K_x^+ \tag{7}
\]

where \( K_x^- = K_x \cap N^- \), \( K_x^0 = K_x \cap L \), \( K_x^+ = K_x \cap N^+ \) (for \( P \) minimal this follows from [BT1] 7.1.4(2), and the general case is a corollary).

The fundamental system of neighborhoods of \( K_x \cap P = K_x^0 \cdot K_x^+ \) is formed by subgroups of the form \( O = O^- \cdot K_x^0 \cdot K_x^+ \) where \( O^- \) is a small open compact subgroup in \( N^- \).

The filter of neighborhoods of \( z \) in \( \overline{A} \) is generated by closures of the cones \( \overline{C} + t \) where \( \overline{C} \subset a\mathfrak{A} \) is an open cone containing \( \rho_z \), and \( t \in \mathfrak{A} \) is a point.

Since \( K_x \) and \( \overline{A} \) are compact it follows that the filter of neighborhoods of \( (K_x \cap P) \times \{ z \} \) is generated by the sets \( O \times (\overline{C} + t) \) in the above notations. Thus the sets \( O(\overline{C} + t) \) generate the filter of neighborhoods of \( z \) in \( \overline{X} \).

Further, it is enough to take only such \( \overline{C} \), \( t \) that \( \alpha |_{\overline{C}} \leq 0 \) if \( \alpha \) is a root of \( T \) in \( N \), and \( t \in x + \overline{C} \). Then we have \( \overline{C} + t \subset \overline{C} + x \subset \bigcup_{z', P' \subseteq P} \overline{[x, z']} \). Since \( K_x^+ \subset K_x \cap P' \) for any \( P' \subset P \) we see that \( K_x^+ \) fixes \( \overline{C} + t \) pointwise.

Thus the system of neighborhoods of \( z \) in \( \overline{X} \) is generated by \( O(\overline{C} + t) = O^- \cdot K_x^0 \cdot K_x^+(\overline{C} + t) = O^- \cdot K_x^0(\overline{C} + t) \) for \( O, \overline{C}, t \) as above; we will call \( O(\overline{C} + t) \) a standard neighborhood of \( z \).

3.10. Proof of Proposition 3.7. Fix a \( z \)-good semigroup \( M \), an element \( a \in \Lambda_C \) for a \( z \)-good cone \( C \), and an open subgroup \( K' \subset G \). First we want to find a neighborhood \( U \) of \( z \) in \( X \) such that \( a_p^{-1}(U) \subset K' \cdot M \cdot a \cdot K_p \), i.e. \( U \subset K' \cdot M \cdot a(p) \).

Claim Let \( \pi_T \) denote the projection from \( T \) to \( T/T^c \subset a_T \). Then the semigroup \( \pi_T(M \cap T) \) contains \( (T/T^c) \cap \overline{C} \) for some open cone \( \overline{C} \subset a_T \) containing \( \rho_z \).

Proof Notice that \( \pi_T(M \cap T) + A/A^c = T/T^c \). Moreover, \((\pi_T(M \cap T)) \cap a\) contains \( (T/T^c) \cap C = (A/A^c) \cap C \) for a \( z \)-good cone \( C \subset a \). This yields the claim by the following elementary argument.

Induction in \( \dim(a_T/a) \) reduces the statement to the situation when \( \dim(a_T/a) = 1 \); in this case one can argue as follows. By the second sentence of the proof \( C \) contains a basis \( v_1, \ldots, v_r \) of \( A/A^c \) such that \( \sum_i a_i v_i \in \rho \) for some \( a_i > 0 \). Let \( v \) be
an element of $\pi_T(M \cap T)$ which does not lie in $a$, and take $v' \in (−v + (A/A')) \cap \pi_T(M \cap T)$; denote the sublattice generated by $v, v_1, \ldots, v_r$ by $\Lambda'$.

The subset $\{a_0v + \sum a_iv_i \mid a_i ≥ 0\} \cup \{a_0v' + \sum a_iv_i \mid a_i ≥ 0\}$ obviously contains an open cone $C'$ such that $C' \supset \rho$. Any point of $\Lambda' \cap C'$ has nonnegative integral coordinates with respect to one of the bases $(v, v_1, \ldots, v_r)$ or $(v', v_1, \ldots, v_r)$, hence lies in $\pi_T(M \cap T)$.

On the other hand, $\Lambda'$ is a sublattice of finite index in $T/T^c$; let $x_1, \ldots, x_n \in T/T^c$ be a set of representatives for the $\Lambda'$ cosets. By the first sentence of the proof we can assume that $x_i \in \pi_T(M \cap T)$.

It is clear that for a narrow enough subcone $C \subset C'$, $C \supset \rho$, we have $y \in C \cap \pi_T(M \cap T), y \notin \rho \Rightarrow y \in C'$ for all $i$. For some $i$ we have $y - x_i \in \Lambda'$, so $y - x_i \in \Lambda' \cap C' \subset \pi_T(M \cap T)$, and hence $y = x_i + (y - x_i) \in \pi_T(M \cap T)$. If $y \in (T/T^c) \cap \rho$, then $y \in \pi_T(M \cap T)$ by the second sentence of the proof. □

We can assume that $|\alpha| \leq 0$ if $\alpha$ is a root of $T$ in $N$.

Also we can find $b \in \Lambda_C$ such that $b + a + p \in x + \tilde{C}$, so that $b + a + p + \tilde{C}$ is point-wise fixed by $K_\pm$. We can furthermore assume that $M \supset b \cdot K_\pm^0$. We now take $O = \tilde{O} - K_\pm^0 \cdot K_\pm^0$ as above such that $O \subset K'$. Then $K' \cdot M \cdot a(p) \supset \tilde{O} - K_\pm^0 \cdot ab \cdot (C \cap (T/T^c)) \cdot (p)$. So $U = O \cdot K_\pm^0 (a + b + C + p) = O(a + b + C + p)$ is the desired neighborhood.

It remains to show that for any neighborhood $U$ of $z$ in $\tilde{X}$ there exist $M, a, K'$ as above such that $K' \cdot M \cdot a \cdot K_\pm^0 \subset \tilde{a}^{-1}(U)$ i.e. $K' \cdot M \cdot a(p) \subset U$. We will do it with the help of another auxiliary geometric statement.

For $x \in X$, $r \in \mathbb{R}$ write $B(c, r) = \{y \in X \mid d(y, c) < r\}$ for the ball of radius $r$ centered at $c$. For a point $y \in \mathcal{A}$ and $\lambda \in \mathbb{R}^{≥ 0}$ we define a sector $S(y, \lambda)$ by $S(y, \lambda) = \cup_{t \in \mathbb{R}^{≥ 0}} B([y, \lambda], t)$ (notations of 3.8b)).

Lemma 3.11. a) For each neighborhood $U$ of $z$ in $X$ there exists a sector $S = S(y, \lambda)$ such that $S \subset U$.

b) Let $S \subset X$ be a sector. For any compact set $C \subset L$ and a $z$-good cone $C$ there exists a $a \in \Lambda_C$ such that $ag(S) \subset S$ for all $g \in C$.

Proof a) We can assume that $U = O(C + t)$ is a standard neighborhood.

It is obvious that $S(q, \lambda) \cap \mathcal{A} \subset U$ for $q \in U \cap \mathcal{A}$ and $\lambda$ small enough. It will be convenient to take $q$ inside of an open polysimplex $\Delta_q$ of the canonical polytriangulation.

By [BT1] 2.3.4, 7.4.19 any point $s \in X$ is $K_q$-conjugate to a unique point in $\mathcal{A}$. Moreover, by [BT1] 7.4.20 ii), for any $k \in K_q$ and $x_1, x_2 \in \mathcal{A}$ we have $d(k(x_1), x_2) ≥ d(x_1, x_2)$. Hence $S = S(t, \lambda) \subset K_q(S \cap \mathcal{A})$.

We can insert $q$ instead of $x$ in the triangular decomposition (7). Now for $\lambda$ small enough the cone we have $S(q, \lambda) \cap \mathcal{A} \subset a_T^\pm + q$, where $a_T^\pm = \{\alpha \in a_T \mid |\alpha| \leq 0\}$ if $\alpha$ is a root of $T$ in $N$, which implies that $S(q, \lambda) \cap \mathcal{A}$ is point-wise fixed by $K_\pm^0$ (see above). So $S(q, \lambda) \subset K_q(S \cap \mathcal{A}) = K_q^0 \cdot K_q^0 (S \cap \mathcal{A})$. It remains to notice that if we take e.g. $q = a(x)$ for appropriate $a \in \Lambda_C$ (a should be “deep enough” in $a_T^\pm$) then, besides $S(q, \lambda) \cap \mathcal{A} \subset U$, we will get $K_q^0 = a(K_q^0 a^{-1} \subset O^-, K_q^0 K_q^0 = K_q^0$. Then $S(q, \lambda) \subset K_q^- \cdot K_q^0 (S(q, \lambda) \cap \mathcal{A}) \subset O^- \cdot K_q^0 (S \cap \mathcal{A}) \subset U$. a) is proven.

b) Let $C \subset L$ be a compact subset. Then $d(x, g(x))$ is bounded by a constant $d_0$ for $g \in C$.

Set $d_1 = \sup_{t > 0} \inf_{a \in \Lambda_C} d(a(x), [x, z])$. It is clear that $d_1 < \infty$. 


Take $s \in S, g \in \mathcal{C}$; let $t_0 \in \mathbb{R}^+$, $a \in \Lambda_C$ be such that $d(s, [x, z]_{t_0}) < \lambda t_0$, $d([x, z]_{t_0}, a(x)) \leq d_1$. Then we have

$$d(\lambda t_0, a(x)) \leq d(\lambda t_0, a(x)) + d(\lambda t_0, a(x)) < \lambda t_0 + d(\lambda t_0, a(x)).$$

Here in the second step we used the inequality $d(\lambda t_0, a(x)) \leq d(\lambda t_0, a(x))$ which is just (6) applied to the case $c_1 = \lambda t_0, c_2 = [x, z]_t$. Note that $ag(z)^{z} = z$ because $a, g \in L$, hence $ag([x, z]) = [ag(x), z]$.

Thus we get

$$d(\lambda t_0, a(x)) \leq d(\lambda t_0, a(x)) + d(\lambda t_0, a(x)) \leq d_0 + d_1.$$

proved $\lambda t_0 > d_0 + d_1$. So $ag(S) \subset S$ and the Lemma is proven. □

3.12. We are now ready to finish the proof of Proposition 3.7. Assume a neighborhood $U$ of $z$ in $\bar{X}$ is given. We are looking for $M, a, K'$ as above such that $K' \cdot M \cdot a(p) \subset U$.

Let $S = S(y, \lambda)$ be a sector contained in $U$. There exists a $z$-good cone $C \subset a$ and $a \in \Lambda_C$ such that $a + C + p \subset S \cap \mathcal{A}$.

Let $\mathcal{C}$ be a compact set generating $L$ as a semigroup. By the previous Lemma we can find $b \in A$ such that $b \cdot \mathcal{C}$ preserves $S$.

We set $M$ to be generated by $\Lambda_C$, $K_y \cap L$ and $b \cdot \mathcal{C}$; the element $a$ is chosen so that $a + C + p \subset S \cap \mathcal{A}$.

We can assume $U \cap X = O(C + t)$ is a standard neighborhood, and take $K' = O$.

Let us check that $K' \cdot M \cdot a(p) \subset U$. By the construction $\Lambda_C \cdot a(p) \subset S \cap \mathcal{A}$. Since $L$ commutes with $\Lambda$, and $b \cdot \mathcal{C}$, $K_y \cap L$ preserve $S$ we have $M \cdot a(p) \subset S$. Hence $K' \cdot M \cdot a(p) \subset K'(S) = O(S) \subset U$.

The Proposition is proven. □

3.13. We return to the proof of the Theorem.

For an open compact subset $\mathcal{C} \subset G$ we denote by $e_{\mathcal{C}} \in H$ the constant measure on $\mathcal{C}$ of total volume 1. We write $\cdot$ for the convolution of compactly supported measures on a group.

We can fix a small open compact subgroup $K \subset K_p$ which is normal in $K_p$ and satisfies the triangular decomposition (7) (see 2.1 of part 2 below for a stronger statement).

Set: $L^+ = \{g \in L | g(K)g^{-1} \subset K^{-}\}$. It is readily seen that the semigroup $L^+$ is $z$-good.

We have a map $\phi: \mathcal{H}(L^+, K^0) \to \mathcal{H}(G, K)$ sending $h$ to $e_{K} \cdot i_*(h) \cdot e_{K}$ (here $i : L^+ \hookrightarrow G$ is the embedding, and $i_*$ denotes the direct image of a measure). We abbreviate $\mathcal{H}(L^+, K^0)$ to $\mathcal{H}_+^+$.

Lemma 3.14. $\phi$ is a homomorphism of algebras.

Proof see e.g. [Al], Theorem 1 (to apply loc. cit. as stated $L^+$ has to consist of semisimple elements, i.e. to be a minimal Levi, and $N, N^+$ must be maximal unipotent subgroups. However these more restrictive conditions are not used in the proof of Theorem 1). □

Let $\mathfrak{X}_\mathcal{H}$ be the filter of subspaces in $\mathcal{H}_+$ generated by $\mathcal{H}(a \cdot M, K^0)$ where $M \subset L^+$ is a $z$-good semigroup containing $K^0$ and $a \in \Lambda_C \subset M$. 
The Hecke algebra $\mathcal{H}(G)$ acts on every smooth $G$-module; this action is denoted by $h : m \mapsto h(m)$. For any $h \in \mathcal{H}(G)$ we denote by $h^*$ the image of $h$ under the involution $g \mapsto g^{-1}$.

**Proposition 3.15.** a) Fix $\mathfrak{M} \in \mathcal{M}$. Assume that $\ast(\mathfrak{M})$ is generated by its $K$-fixed vectors. Then for $f \in \mathfrak{M}^K$ the equality $\lim_{q} g^{-1}(f) = 0$ is equivalent to $\lim_{q} \phi(h)^*(f) = 0$.

b) The functor $H_\ast$ on the category of $\mathcal{H}_+$-modules sending a module $\mathfrak{M}$ to the space $\{f \in \mathfrak{M}^* \mid \lim_{q} \langle h(m), f \rangle = 0 \text{ for any } m \in \mathfrak{M} \}$ (where $\mathfrak{M}^*$ is the full linear dual, and the basic field is equipped with discrete topology) is exact.

**Proof** a) Assume that $\lim_{q} \phi(h)^*(f) = 0$. Then for any $m \in \ast(\mathfrak{M})^K$ we can find a $K$-invariant $z$-good semigroup $M \subset L^+$ and $a \in \Lambda_C$ such that

$$h \in \mathcal{H}(a \cdot M, K^0) \implies \langle m, \phi(h)^*(f) \rangle = 0 \quad (8)$$

Hence for any finite set $m_1, \ldots, m_l \in \ast(\mathfrak{M})^K$ there is some $M, a$ as above such that (8) is satisfied for $m \in \{m_1, \ldots, m_l\}$. Fix $m \in \ast(\mathfrak{M})^K$ and take the finite set to be $K_p(m)$ (recall that we assumed that $K_p$ normalizes $K$, hence preserves $\ast(\mathfrak{M})^K$). Then we get $M, a$ such that

$$h \in \mathcal{H}(a \cdot M) \implies \langle m, \phi(h)^*(f) \rangle = 0 \text{ for } k \in K_p.$$

This means that $\lim_{q} \phi(h)^*(f) = 0$ for $g \in K \cdot M : a \cdot K_p$ hence $\lim_{q} g^{-1}(f) = 0$.

It remains to notice that $\{m \in \ast(\mathfrak{M}) \mid \lim_{q} \langle g(m), f \rangle = 0 \}$ is a $G$-submodule in $\ast(\mathfrak{M})$ (because $f$ is right-translation invariant). Thus if $\ast(\mathfrak{M})$ is generated by its $K$-fixed vectors, then $\lim_{q} \langle m, g^{-1}(f) \rangle = 0$ for any $m \in \ast(\mathfrak{M})$.

Conversely, if $\lim_{q} g^{-1}(f) = 0$ then for any $m$ there is a $z$-good semigroup $M \subset L^+$ and $a \in \Lambda \cap L^+$ such that $\langle m, g^{-1}(f) \rangle = 0$ for $g \in M \cdot a$. We can assume that $a \in \Lambda_C$. Any $z$-good semigroup contains one normalized by $K^0$, so we can assume also that $M$ is normalized by $K^0$. Then $M \cdot K^0$ is again a $z$-good semigroup and $\langle m, \phi(h)^*(f) \rangle = 0$ for $h \in \mathcal{H}(a \cdot M \cdot K^0)$ provided $m$ is $K^0$-invariant. a) is proved.

16. To prove b) we need an auxiliary definition.

The homomorphism $\pi_{L^+} : L^+ \to L/L^c$ defines an $L/L^c$-grading on $\mathcal{H}_+$. 

**Definition 3.17.** A $z$-good subalgebra in $\mathcal{H}_+$ is a graded subalgebra $\mathcal{A} = \oplus_{\lambda \in L/L^c} \mathcal{A}_\lambda$ in $\mathcal{H}_+$ such that

i) For some $z$-good cone $C$ all elements of the form $e_{K^0, a}$ belong to $\mathcal{A}$ for $a \in \Lambda_C$.

ii) The map $pr_\ast : \mathcal{A} \to \mathcal{H}(L/L^c, K^0)$ is surjective.

**Sublemma 3.18.** Let $\mathcal{A}$ be a $z$-good semigroup, $C$ be a $z$-good cone such that $e_{\eta, K^0} \in \mathcal{A}$ for $\eta \in \Lambda_C$. Then for any $h \in \mathcal{H}(L, K^0)$ there exists $\nu \in \Lambda_C$ such that $\nu \cdot h \in \mathcal{A}$.

**Proof** We can assume that $h$ is supported in one $L^c$-coset $\lambda$. By condition ii) of the definition we have $pr_\ast(h) = \sum pr_\ast(h_i)$, where $h_i \in \mathcal{A}_{\lambda_i}$. Since $\mathcal{H}(L/L^c)$ is graded by the finite group $L/(L^c \cdot A)$, we can further assume that $\lambda_i = \lambda$ mod $\Lambda$. Then we get:

$$h = \sum (\lambda - \lambda_i)(h_i).$$

Clearly for “large enough” $\nu \in \Lambda_C$ we have $\nu + \lambda - \lambda_i \in \Lambda_C$, so $(\nu)h = \sum (\nu + \lambda - \lambda_i)(h_i) \in \mathcal{A}$. □

**Lemma 3.19.** a) The filter $\mathfrak{F}_{\mathcal{H}_+}$ is generated by the subspaces $a \cdot \mathcal{A}$, where $\mathcal{A}$ runs over the set of $z$-good subalgebras and $a$ runs over $\Lambda$.

b) There exist a set of $z$-good subalgebras $\mathcal{A}_i$ and elements $a_i \in \Lambda, e_{a_i, K^0} \in \mathcal{A}_i$ such that:
i) \( \mathfrak{F}_A^+ \) is generated by \( a_i \cdot A_i \).

ii) Each of the algebras \( A_i \) is finite over its center, which has finite type.

**Proof** a) It is obvious that if \( M \) is a \( z \)-good semigroup, then \( \mathcal{H}(M, K^0) \) is a \( z \)-good subalgebra.

On the other hand, let \( \mathcal{E} \) be a compact set generating \( L^c \) as a semigroup. Fix a \( z \)-good subalgebra \( A \). Since \( \mathcal{H}(\mathcal{E}, K^0) \) is finite dimensional Sublemma 3.18 gives a coweight \( \nu \in \Lambda_C \) such that \( (\nu) \cdot \mathcal{H}(\mathcal{E}, K^0) \subset A \).

Let \( M \) be the semigroup generated by \( K^0 \), \( \mathcal{E} \cdot \nu \) and \( \Lambda_C \). Obviously \( M \) is a \( z \)-good semigroup, and \( \mathcal{H}(M, K^0) \subset A \). This proves a).

The proof of b) consists in constructing the Noetherian algebras \( A_i \) explicitly.

It is known [BD] that \( \mathcal{H}(L, K^0) \) is finite over its center \( Z_L \); and \( Z_L \) is a finitely generated commutative algebra. Let \( z_1, \ldots, z_n \) be generators of \( Z_L \), and \( h_1, \ldots, h_l \) be the generators of \( \mathcal{H}(L, K^0) \) as \( Z_L \)-module. We have: \( h_i \cdot h_j = \sum z_{ij}^l h_l \) for \( z_{ij}^l \in Z_L \).

Fix a rational (with respect to the lattice \( L/L^c \subset a \) \( z \)-good cone \( C \subset a^+ \). Then the semigroup \( \Lambda_C \) is finitely generated.

We can find \( a \in \Lambda_C^+ \) such that \( a \cdot z_i, a \cdot z_{ij}^l \) and \( a \cdot h_l \) are supported inside \( L^+ \).

For such \( a, C \) define \( A_{C, a} \) to be the subalgebra generated by \( e_{\eta, K^0} \) for \( \eta \in \Lambda_C \) and \( a \cdot z_i, a \cdot z_{ij}^l, a \cdot h_l \).

Then the subalgebra \( Z_A \subset A_{C, a} \) generated by \( e_{\eta, K^0} \) for \( \eta \in \Lambda_C \) and by \( a \cdot z_i, a \cdot z_{ij}^l \) is central. Also \( A_{C, a} \) is finite over \( Z_A \); namely, it is clearly generated by \( a \cdot z_i, a \cdot z_{ij}^l, a \cdot h_l \) as a \( Z_A \)-module. Since \( Z_A \) is a finitely generated commutative algebra, it is Noetherian, so the center of \( A_{C, a} \) is a finite \( Z_A \)-module, hence itself is a finitely generated commutative algebra.

To finish the proof it is enough, in view of a), to check that every \( z \)-good subalgebra contains a subalgebra of the form \( A_{C, a} \). For this let \( A \) be a \( z \)-good algebra, and \( C \) be a \( z \)-good cone such that \( e_{\lambda, K^0} \in A \) for \( \lambda \in \Lambda_C \). Then Sublemma 3.18 guarantees existence of \( a \in \Lambda_C \) such that \( a \cdot z_i, a \cdot z_{ij}^l, a \cdot h_l \in A \), i.e. such that \( A_{C, a} \subset A \). \( \square \)

Let us finish the proof of the Proposition.

Fix \( A \in \{ A_i \} \). We can pick \( A_i, a_i \) in the Lemma so that \( A_i \subset A \).

The filter of subspaces in \( A \) generated by \( a_i A_i \) is invariant under both left and right multiplication by an element \( h \in A \), in the sense that for any \( i \) and \( h \in A \) there exists \( j \) such that \( h \cdot a_i A_i \subset a_j A_j \), \( a_i A_i \cdot h \subset a_j A_j \). Thus the space \( \lim_{\rightarrow} (A/a_i A_i)^* \) carries a natural \( A \)-module structure. Moreover, for any \( M \in \mathfrak{M} \) we have

\[
H_2(\mathfrak{M}) = \{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \lim_{\rightarrow} f_{n, h} (h(m), f) = 0 \} = \\
\{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \exists z \text{-good } A, a \in A \forall h \in aA : \langle h(m), f \rangle = 0 \} = \\
\{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \exists i \forall h \in a_i A_i : \langle h(m), f \rangle = 0 \} = \\
\Hom(\mathfrak{M}, \lim_{\rightarrow} (A/a_i A_i)^*) \quad (9)
\]

So to prove the Proposition we have only to show that \( \lim_{\rightarrow} (A/a_i A_i)^* \) is an injective \( A \)-module.

It is not hard to see that for fixed \( i \) the filter of subspaces in \( A \) generated by \( a_i^n A_i \), \( n \in \mathbb{Z}^+ \) is also invariant under left and right multiplication by \( h \in A \), thus we have an \( A \)-module \( A^i = \lim_{\rightarrow} (A/a_i^n A_i)^* \).

Notice that \( \lim_{\rightarrow} (A/a_i A_i)^* = \lim_{\rightarrow} (\lim_{\rightarrow} (A/a_i^n A_i)^*) = \lim_{\rightarrow} A^i \) (in fact we can assume that \( \forall i, n \exists j \mid A_i = A_j, a_i = a_j^n \).
Since $\mathcal{A}$ is Noetherian the direct limit of injective $\mathcal{A}$-modules is injective. Hence we have only to show that $\mathcal{A}_{i,j}$ is an injective $\mathcal{A}$-module. For any $\mathcal{A}$-module $M$ we have
\[
\text{Hom}(M, \mathcal{A}_{i,j}) = \{ f \in M^\ast \mid \forall m \in M \exists n \in \mathbb{Z}^\ast : \langle f, a^n_i \mathcal{A}(m) \rangle = 0 \} = \text{Hom}_\mathcal{A}(M, \varprojlim_n (\mathcal{A}_i/a^n_i \mathcal{A}_i)^\ast).
\]

So we need only to show that $\varprojlim_n (\mathcal{A}_i/a^n_i \mathcal{A}_i)^\ast$ is an injective $\mathcal{A}_i$-module.

Since $\mathcal{A}_i$ is Noetherian, this is equivalent to saying that the functor $\text{Hom}_\mathcal{A}_i(\underline{\_}, \varprojlim_n (\mathcal{A}_i/a^n_i \mathcal{A}_i)^\ast)$ is exact on the category of finitely generated $\mathcal{A}_i$-modules.

On this category the functor coincides with the functor $M \to \varprojlim_n (M/a^n)^\ast$, and its exactness is guaranteed by the standard Artin-Rees lemma because of 3.19b), property ii).

The Proposition is proved. $\Box$

3.20. We are ready at last to finish the proof of Theorem 3.4.

Proof of a): The statement is equivalent to saying that the functor $M \to (\ast(M))_{\{z\}}^\ast$ on the category $\mathcal{M}$ is exact. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{M}$. Let $K$ be as above (a compact open subgroup with triangular decomposition (7)), and assume that $M_i$ is generated by its $K$-fixed vectors for $i = 1, 2, 3$. Then by Proposition 3.15a):
\[
(\ast(M_i))_{\{z\}}^K = \{ f \in (M_i^K)^\ast \mid \lim_{\mathfrak{H}_+} \phi(h)^* (f) = 0 \}.
\]

Hence the sequence
\[
0 \to (\ast(M_3))_{\{z\}}^K \to (\ast(M_2))_{\{z\}}^K \to (\ast(M_1))_{\{z\}}^K \to 0
\]
is exact. The statement a) of 3.4 follows.

To prove b) it is convenient to use the following

**Lemma 3.21.** Let $M \in \mathcal{M}$ be a module, $K$ as above. For an element $m \in M^K$ the following are equivalent:

i) $m$ lies in the kernel of the projection $M \to r_P(M)$

ii) $e_{Ka^{-1}K}(m) = 0$ for some $a \in \Lambda^+$.

Proof is well-known; see e.g. [BZ], Lemma 3.22 for the case $G = GL(n)$, or [BR], Claim 15.4 for the (absolutely analogous) general case. $\Box$

Now for $\mathfrak{M} \in \mathcal{M}^\ast$ assume that $m \in (\mathfrak{M})^K$ lies in the kernel of the projection $Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M}))$. Then for some $a \in \Lambda^+$ we have:
\[
\phi(e_{aK_0})^* (m) = e_{Ka^{-1}K}(m) = 0 \implies \phi(e_{aK_0} \cdot \mathcal{H}_+)^* (m) = 0
\]
i.e. $m \in \mathfrak{M}_{\{z\}}$ by Proposition 3.15a).

Assume now that $\mathfrak{M} \in \mathcal{R}^\ast$, and $m \in \mathfrak{M}_K$. For $v \in (\ast(\mathfrak{M}))^K$ we have
\[
\lim_{\mathfrak{H}_+} \phi(h)^* (m, v) = 0 \quad \text{(by Proposition 3.15a)) hence there exists } a \in \Lambda^+ \quad \text{such that } (e_{Ka^{-1}K}(m), v) = (\phi(e_{aK_0})^* (m), v) = 0.
\]
Since $(\ast(\mathfrak{M}))^K$ is finite dimensional we can find one $a \in \Lambda^+$ such that $(e_{Ka^{-1}K}(m), v) = 0$ for all $v \in (\ast(\mathfrak{M}))^K$. But this means that $e_{Ka^{-1}K}(m) = 0$. By the last Lemma this finishes the proof of Theorem 3.4. $\Box$
Theorem 3.22. For \( M \in \mathcal{M} \) let \( L(M) \in \text{Sh}_G(Y) \) be the unique quotient of the constant sheaf \( \underline{M} \) such that for a parabolic \( P \) and \( p \in Y \) with \( \text{Stab}(p) = P \) (i.e. \( p \in \Delta_P \)) the stalk of \( L(M) \) at \( p \) is the Jacquet functor \( r_P(M) \).

The functor \( \mathcal{L} : \mathcal{M}^* \to \text{Sh}_G(Y) \) satisfies:

i) \( \mathcal{L} \circ \Gamma \cong i^* \circ j^*_G \)

ii) \( \mathcal{L} \) is exact

iii) We have a functorial surjection \( L(Sm(M)) \to \mathcal{L}(M) \); it is an isomorphism provided \( M \in \mathcal{R}^* \).

Proof. Property i) follows from the Lemma 3.1. Properties ii) and iii) follow from Theorem 3.4. \( \square \)

4. Application to representation theory: dualities. Recall the definition of Deligne-Lusztig duality (originally defined in the context of groups over finite field in [DL]). For two parabolic subgroups \( P_2 \subset P_1 \subset G \) we have the canonical morphism \( d_{P_1, P_2} : i_{P_1} \circ r_{P_1} \to i_{P_2} \circ r_{P_2} \). For any \( M \in \mathcal{M} \) define the complex \( DL(M) \) as follows. Set \((DL(M))^i = \bigoplus_{P_i \in S_i} i_P \circ r_P(M) \) where \( S_i \) is the set of standard parabolics of corank \( i \); the differential \( d_i := \sum_{P \in S_i, P \in S_{i+1}} \pm d_{P_1, P_2} \) (definition of signs is straightforward and can be found in [DL]). Obviously \( DL \) extends to an exact functor \( \text{Kom}(\mathcal{M}) \to \text{Kom}(\mathcal{M}) \) and defines a functor on the derived category.

We will need the following standard Lemma

Lemma 4.1. \( D^+(\mathcal{R}) \) (respectively \( D^b(\mathcal{R}) \)) is equivalent to the full subcategory of \( D^+(\mathcal{M}) \) (respectively \( D^b(\mathcal{M}) \)) consisting of complexes with admissible cohomology.

Proof. Let \( \mathcal{R} \subset \mathcal{M} \) be the full subcategory consisting of direct limits of admissible modules. By [BD] \( \mathcal{H} \) is finite over its center \( \mathcal{Z} \) which is locally of finite type. Let \( I \subset \mathcal{Z} \) be an ideal of finite codimension. Then the module \( \mathcal{H}/I^n \) is admissible for every \( n \), and by the Artin-Rees Lemma the functor \( M \to \text{Hom}(\mathcal{H}, \text{lim}_n(\mathcal{H}/I^n)) = \text{Hom}(\mathcal{G}, \text{lim}_n(\mathcal{G}/I^n)) \) is exact, i.e. the module \( \mathcal{H}_I := \text{lim}_n(\mathcal{H}/I^n) \) is injective.

It is clear that any module in \( \mathcal{R} \) injects into \( \mathcal{H}_I \) for some \( I \). Hence by [H], Proposition 1.4.8 the tautological functors \( D^b(\mathcal{R}) \to D^b(\mathcal{M}) \), \( D^+(\mathcal{R}) \to D^+(\mathcal{M}) \) induce equivalences with the full subcategories of objects, whose cohomology lie in \( \mathcal{R} \).

To finish the argument notice that for any complex \( C \in \text{Kom}^b(\mathcal{R}) \) (respectively \( C^* \in \text{Kom}^+(\mathcal{R}) \)) it is easy to represent \( C \) as a direct limit of its quasiisomorphic subcomplexes \( C_i \in \text{Kom}^b(\mathcal{R}) \) (respectively \( C_i^* \in \text{Kom}^+(\mathcal{R}) \)). It follows that \( D^{b,+}(\mathcal{R}) \) is equivalent to the full subcategory of \( D^{b,+}(\mathcal{R}) \) whose objects have admissible cohomology. \( \square \)

We are now in the position to prove

Theorem 4.2. a) There exists a canonical morphism of functors from \( D^b(\mathcal{M}) \) to \( D^b(\mathcal{M}) \)

\[ DL \circ \sim \to D_h. \quad (11) \]

b) Restriction of (11) to the full subcategory \( D^b(\mathcal{R}) \subset D^b(\mathcal{M}) \) gives an isomorphism of functors from \( D^b(\mathcal{R}) \) to itself

\[ D_h \cong DL \circ \sim. \quad (12) \]
Remark 4.3. Isomorphism (12), as well as its generalization (16), is due to J. Bernstein (unpublished). An attempt to interpret it through the geometry of $\overline{X}$ was the starting point for this part of the work.

Recall that another construction of (12) using compactified building appears in [Sch-St2].

Remark 4.4. Theorem 4.2 is the main step in the proof of Zelevinsky’s conjecture [Z], which says that the map on the Grothendieck group of the category of admissible representations induced by $DL$ sends the class of an irreducible representation to the class of an irreducible representation (up to a sign). The conjecture is proved in [B], [Sch-St2].

Proof of the Theorem. For $M \in \tilde{\mathcal{M}}$ denote by $\mathcal{DL}(M) \in Sh_G(\overline{X})$ the kernel of the canonical surjection $\mathcal{M}_\overline{X} \to L(M)$. Then $\mathcal{DL}$ is an exact functor from $\tilde{\mathcal{M}}$ to $Sh_G(\overline{X})$.

For an object $I \in \mathcal{G}$ we have the canonical surjections $pr_I : Sm(\Gamma(I)) \to j^G_I(I)$ and $L(\Gamma(I)) \to \mathcal{L}(\Gamma(I)) = i^*j^G_I(I)$ (3.22(iii)), which yield a morphism of functors $s : DL \circ Sm \circ \Gamma(I) \to Ker(j^G_I(I) \to i^*j^G_I(I)) = j_I(I)$.

Clearly $\mathcal{DL}$, $s$ extend to a functor (respectively, a morphism of functors) on bounded derived categories; we will denote these extensions by the same symbols.

According to [Schn1], Lemma 1 for any locally compact $G$-space $Z$ the category $Sh_G(Z)$ has enough injectives, so the derived functor of a left-exact functor on $Sh_G(Z)$ is defined. Suppose further that $Z$ is compact; then one checks immediately that for $F \in Sh_G(Z)$ the $G$-module $\Gamma(F)$ is smooth. Thus in this case we have the derived functor $R\Gamma : D^+(Sh_G(Z)) \to \tilde{\mathcal{M}}$, which commutes with the forgetful functors $For_{sh} : Sh_G(Z) \to Sh(Z)$, $For_{mod} : \tilde{\mathcal{M}} \to Vect_k$. By Corollary 3 in [Schn1] the forgetful functor $Sh_G(X) \to Sh(X)$ sends injective equivariant sheaves into $\omega$-soft sheaves; if $Z$ is compact they are adapted to $\Gamma$, hence we have $For_{mod} \circ R\Gamma \cong R\Gamma \circ For_{sh}$.

We will be interested in the case $Z = \overline{X}$; then all of the above properties are satisfied.

Lemma 4.5. $R\Gamma(\mathcal{DL}(M))$ is canonically isomorphic to $DL(R\Gamma(M))$ for any $M \in D^b(\mathcal{M})$.

Proof. Recall that $\overline{X} - X$ is the spherical building of $G$, that is the realization of the simplicial topological space, whose space of nondegenerate $n$-simplices is the disjoint union of partial flag varieties $G/P$, where $\text{corank}(P) = n + 1$; the structure maps come from the canonical projections $\pi_{P_1,P_2} : G/P_1 \to G/P_2$ for $P_1 \subset G/P_2$. The sheaf $L(M)$ is the realization of a simplicial sheaf on this space; such a sheaf has a canonical resolution by acyclic sheaves.

More precisely, for a set $s$ of simple roots (of the system of nonmultiplicable roots of $G$) let $\mathfrak{P}(s)$ denote the corresponding conjugacy class of parabolics (so $s$ is the set of simple roots which appear in the nilradical of $P \in \mathfrak{P}(s)$). Let $S(s) = \bigcup_{P \in \mathfrak{P}(s)} U(P)$ be the corresponding stratum of $\overline{X} - X$, and $U(s)$ be the union of the stars of all simplices $\Delta_P$, $P \in s$. Then we have $U(s_1 \cup s_2) = U(s_1) \cap U(s_2)$; thus $\{U_s \mid \#(s) = 1\}$ form an open covering of $\overline{X} - X$, and the corresponding Čech resolution of $L(M)$
has the form
\[ 0 \to \bigoplus_{|s|=1} j_{s}^* j_s (L(M)) \to \cdots \to \bigoplus_{|s|=\text{rank}(G)} j_{s}^* j_s (L(M)) \to 0, \quad (13) \]
where \( j_s \) denotes the imbedding \( U_s \hookrightarrow \mathcal{X} \). Hence for any \( M \in \mathcal{M} \) the complex
\[ 0 \to \mathcal{M} \to \bigoplus_{|s|=1} j_{s}^* j_s (L(M)) \to \cdots \to \bigoplus_{|s|=\text{rank}(G)} j_{s}^* j_s (L(M)) \to 0 \]
is a resolution for \( D\mathcal{L}(M) \). Moreover, there exists a canonical retraction of \( U_s \) on \( S(s) \), hence \( H^i(U_s, j_s^* (L(M))) = H^i(S(s), L(M)|_{S(s)}) = 0 \) for \( i > 0 \), and
\( H^0(U_s, j_s^* (L(M))) = H^0(S(s), L(M)|_{S(s)}) = i_{S(s)}^* o r_{S(s)}^*(M) \). Of course, \( H^i(\mathcal{M} \mathcal{R}) = 0 \) for \( i > 0 \) and \( H^0(\mathcal{M} \mathcal{R}) = M \) since \( \mathcal{X} \) is contractible. Thus the terms of the complex of global section of \( (14) \) are identified with that of \( D\mathcal{L}(M) \); it is easy to see that the differentials are also the same.

It is also evident how to extend the assignment \( M \mapsto (14) \) to the definition of an exact functor \( \mathcal{C}D\mathcal{L} \) from \( \text{Kom}(\mathcal{M}) \) to the complexes of \( \Gamma \)-adapted objects of \( \text{Sh}_{\mathcal{G}}(\mathcal{X}) \), together with a canonical quasiisomorphism \( \mathcal{L} \to \mathcal{C}D\mathcal{L} \), and a canonical isomorphism \( \Gamma \circ \mathcal{C}D\mathcal{L} \cong D\mathcal{L} \).

We can now construct the morphism (11). For \( F \in D^b(\mathcal{S}h) \) we have functorial morphisms
\[
\begin{align*}
D\mathcal{L}(R\Gamma_c(F)) &\cong D\mathcal{L}(S \circ R\Gamma(VF)) \overset{4.5}{\cong} R\Gamma(D\mathcal{L}(S \circ R\Gamma(VF))) \\
&\overset{R\Gamma(s)}{\longrightarrow} R\Gamma(j! (VF)) = R\Gamma_c(VF) \overset{1.3h}{\cong} D_h(R\Gamma_c(F)).
\end{align*}
\]

Hence by the second statement of Proposition 2.1 we get the desired morphism (11) on the full subcategory \( D^b(\mathcal{M}) \). a) of the Theorem now follows from the evident

**Claim 4.6.** a) Let \( \mathcal{A}, \mathcal{B} \) be additive categories, and \( \mathcal{C} \) be a full subcategory of \( \mathcal{A} \). Let \( F_1, F_2 : \mathcal{A} \to \mathcal{B} \) be additive functors, and \( s : F_2|_{\mathcal{C}} \to F_2|_{\mathcal{C}} \) be a transformation. Assume that every object of \( \mathcal{A} \) is isomorphic to a direct summand in an object of \( \mathcal{C} \). Then \( s \) extends in a unique way to a transformation of functors \( F_1 \to F_2 \).

b) Any object of \( D^b(\mathcal{M}) \) is a direct summand in an object of \( D^b(\mathcal{M}) \).

**Proof**. Take \( A, A' \in Ob(\mathcal{A}) \) such that \( A \oplus A' \in Ob(\mathcal{C}) \). By the definition of a morphism of functors we have \( s_{A \oplus A'} \circ (id \oplus 0) = id \oplus 0 \), \( s_{A \oplus A'} \circ (0 \oplus id) = 0 \oplus id \), hence \( s \) preserves the direct sum decomposition, i.e. \( s_{A \oplus A'} = s_A \oplus s_A' \). If \( A'' \in Ob(\mathcal{A}) \) is another object such that \( A \oplus A'' \in Ob(\mathcal{C}) \), then considering the morphism \( id \oplus 0 : A \oplus A' \to A \oplus A'' \) we see that \( s_A \) does not depend on the choice of \( A' \). This proves a).

To prove b) recall that \( \mathcal{M} \) has enough projectives and finite homological dimension ([BR], Theorem 38, see also e.g. [Vig], Proposition 37), so any object of \( D^b(\mathcal{M}) \) is represented by a finite complex of projective modules. Since standard projectives form a set of projective generators in \( \mathcal{M} \), we can add to such a complex a complex with zero differential to get a complex of standard projectives.

For part b) of the Theorem we need another

**Lemma 4.7.** For \( I \in \mathcal{I}_G \) the cohomology \( H^i (\mathcal{X}, j^*_s (I)) \) vanish for \( i > 0 \).
Sublemma 4.8. Let \( \Delta \subset X \) be a polysimplex of the canonical polytriangulation, \( x \in \Delta \) be a point, and \( R \) be a smooth finite dimensional representation of \( \text{Stab}(R) = K_x \). Let \( i_x \) be the imbedding \( i_x : G(x) \hookrightarrow X \), where \( G(x) \) is the \( G \)-orbit of \( x \), and set \( \mathcal{F}_{x,R} = i_{x*}^*\Delta (\mathcal{F}_{\Delta,R}) \) (notations of 1.1). Consider the canonical arrow \( \mathcal{F}_{\Delta,R} \to \mathcal{F}_{x,R} \), and the induced morphism \( f : j^G_x(\mathcal{F}_{\Delta,R}) \to j^G_x(\mathcal{F}_{x,R}) \). Then \( i^*(f) \) and \( R\Gamma(f) \) are isomorphisms.

Proof of the Sublemma. Let \( z \in \overline{X} - X \) be a point. Let \( K_i \subset G \) be compact open subgroups forming a fundamental system of neighborhoods of \( 1 \); then there exists a basis \( \{U_i \} \) of the filter of neighborhoods of \( z \) such that \( U_i \cap X \) is a \( K_i \)-invariant closed simplicial subset. Indeed, if \( \mathfrak{A} \) is an apartment containing \( z \) in its closure, then it is clear that one can find a basis \( C_i \) of the filter of neighborhoods of \( z \) in \( \mathfrak{A} \) such that \( C_i \cap \mathfrak{A} \) is closed and simplicial; then by 3.9 we get the desired basis of the filter setting \( U_i = K_i(C_i) \).

Now we see that \( \Gamma(U_i, \mathcal{F}_{\Delta,R}) \cong \prod_{\Delta' \subseteq G(\Delta), \Delta' \subset U_i} \Gamma(U_i, \mathcal{F}_{\Delta,R}) \cong \prod_{G(x) \cap U_i} \Gamma(U_i, \mathcal{F}_{\Delta,R}) \).

Hence \( j^G_x(\mathcal{F}_{\Delta,R})|_z = \lim_{\rightarrow} \Gamma(U_i \cap X, \mathcal{F}_{\Delta,R}) |_{\Delta'} \to \lim_{\rightarrow} \Gamma(U_i \cap X, \mathcal{F}_{\Delta,R}) |_{\Delta'} |_z \) (here we write \( ?|_z \) for a stalk of a sheaf \( ? \) at a point \( z \)). This proves the first statement.

It implies that \( f \) is a surjection of sheaves, and the kernel of \( f \) is (noncanonically) isomorphic to \( R\Gamma(\mathfrak{A} - x) \) extended by 0 to \( \overline{X} \). Since \( H_x^i(\mathfrak{A} - x) = 0 \) for all \( i \) the latter sheaf is acyclic, so we get the second statement.

Proof of the Lemma. \( \mathcal{F}_{x,R} \) is an injective object in \( \text{Sh}_G(X) \); hence \( j^G_x(\mathcal{F}_{x,R}) \) is an injective object of \( \text{Sh}_G(\overline{X}) \). Hence by [Schn1], Corollary 3 it is a \( c \)-soft sheaf; since \( \overline{X} \) is compact this implies \( H^{>0}(\overline{X}, j^G_x(\mathcal{F}_{x,R})) = 0 \). Thus Lemma follows from the Sublemma.

4.9. Proof of Theorem 4.2b). To prove b) we must check the following. Suppose that \( \mathcal{F} \in D^b(\text{Sh}) \) is such that \( R\Gamma_c(\mathcal{F}) \cong M_1 \oplus M_2 \) where \( M_1 \in D^b(\mathbb{R}) \subset D^b(\mathcal{M}) \), \( M_2 \in D^b(\mathcal{M}) \). Then we have \( R\Gamma(s) = S_1 \oplus S_2 \), where \( S_i : DL(M_i) \to D_h(M_i) \) for \( i = 1, 2 \), and \( S_1 \) is an isomorphism. We have \( R\Gamma(s) = S_1 \oplus S_2 \) for some \( S_1, S_2 \) by Claim 4.6. Now recall that for \( I \in \mathcal{G} \) the morphism \( s : DL(I) \to j^G_x(\mathcal{I}) \) includes in a functorial morphism of exact sequences

\[
0 \longrightarrow \mathcal{D} \mathcal{L} \longrightarrow \text{Sm}(\Gamma(I)) \longrightarrow L(\text{Sm}(\Gamma(I))) \longrightarrow 0
\]

Thus we have a functorial morphism of distinguished triangles in \( D^b(\mathcal{M}) \)

\[
DL(\text{Sm} \circ R\Gamma(\mathcal{F})) \longrightarrow \text{Sm} \circ R\Gamma(\mathcal{F}) \longrightarrow R\Gamma(L(\text{Sm} \circ R\Gamma(\mathcal{F})))
\]

The second vertical map is an isomorphism by Lemma 4.7.

Also it is clear that decomposition \( R\Gamma_c(\mathcal{F}) \cong M_1 \oplus M_2 \) splits the whole diagram into the direct sum of two. The condition \( M_1 \in D^b(\mathbb{R}) \) implies that the right vertical arrow in the first summand is an isomorphism by Theorem 3.22 ii), iii).
Hence $S_1$ is also an isomorphism. By Lemma 4.1 the quasiisomorphism (12) in the category $D^b(\widetilde{M})$ yields the corresponding quasiisomorphism in $D^b(\mathcal{R})$.

Remark 4.10. We could have used the Schneider-Stuhler “localization” Theorem [Sch-St2] instead of our Proposition 2.1 in order to deduce the isomorphism (12) from Theorem 3.22.

4.11. We finish the chapter with a generalization of the last Theorem which connects homological and Deligne-Lusztig dualities on $D^b(\mathcal{M})$ (not only on complexes with admissible cohomology).

Recall that the Bernstein center $Z$ is by the definition the algebra of endomorphisms of identity functor on $\mathcal{M}$; so it is a commutative algebra acting on every module $M \in \mathcal{M}$ commuting with the $G$-action. It is known [BD] that $Z$ is an infinite product of finitely generated commutative algebras; in fact each factor in this decomposition is isomorphic to algebra of functions on an algebraic torus of dimension $\leq \text{rank}(G)$ invariant under the action of a finite group.

We also recall that the strong admissibility Theorem [BD] asserts that for any $M \in \mathcal{M}$ and any open compact subgroup $K \subset G$ the space of $K$-invariants $M^K$ is a finite $Z$-module supported on a finite number of components of $\text{Spec}(Z)$. If $M$ is a $G$-module, and $V$ is a $Z$-module then the space $\text{Hom}_Z(M, V)$ is naturally a $G$-module; moreover, from the strong admissibility Theorem it follows that $\text{Sm}(\text{Hom}_Z(M, V)) \in \mathcal{M}$ if $M \in \mathcal{M}$ and $V$ is locally finitely generated. Since $M$ has enough projectives we can derive the bifunctor $(M, V) \mapsto \text{Sm}(\text{Hom}_Z(M, V)) \in \mathcal{M}$ if $M \in \mathcal{M}$ and $V$ is locally finitely generated. Since $M$ has enough projectives we can derive the bifunctor $\mathcal{D}^-(\mathcal{M}) \times \mathcal{D}^+(\mathcal{R}) \to \mathcal{D}^+(\mathcal{M})$, where $\mathcal{R} = \mathcal{Z} \otimes B$ is the category of locally finitely generated $Z$-modules. Let $D \in \mathcal{D}^+(\mathcal{R})$ be the Grothendieck dualizing complex.

We denote by $D_{Gr}$ the Grothendieck-Serre duality functor $R\text{Hom}(\mathcal{M}, D)$; we have $D_{Gr} : D^b(\mathcal{M}) \to D^+(\mathcal{M})$.

Note that for a finite dimensional module $V$ over a commutative algebra we have $D_{Gr}(V) \cong V^*$ canonically. It follows that $D_{Gr}|_{D^b(\mathcal{R})} \cong \mathcal{R}$.

Remark 4.12. The quotient of a smooth variety by an action of a finite group is known to be Cohen-Macaulay, hence the dualizing complex $D$ is concentrated in one homological dimension. Also, for such a variety the Grothendieck-Serre duality functor sends the bounded derived category $D^b(\text{Coh})$ into itself, where $\text{Coh}$ stands for coherent sheaves. It follows that $D_{Gr}$ sends $D^b(\mathcal{M})$ to itself. We will not use these facts below; the latter one follows also from the next Theorem.

Theorem 4.13. We have a canonical isomorphism of functors on $D^b(\mathcal{M})$

$$D_h \cong DL \circ D_{Gr} \quad (16)$$

In order to prove the Theorem we need to extend the statement 4.2 to the following set-up. Let $B$ be a commutative unital $k$-algebra of finite type. Let $\widetilde{\mathcal{M}}_B$ be the category of smooth $G \times B$ modules, i.e. $B$-modules equipped with a smooth $G$-action; in other words $\widetilde{\mathcal{M}}_B$ is a category of modules over the algebra $\mathcal{H}_B := \mathcal{H} \otimes B$. The category $\mathcal{M}_B \subset \widetilde{\mathcal{M}}_B$ consists of all finitely generated modules. We also define $\widetilde{\mathcal{M}}_B^c$ to be the category of complete topological $G \times B$ modules having a basis of neighbourhoods of 0 consisting of $B$-submodules such that the quotient is a finite $B$-module; the subcategory $\mathcal{M}_B^c \subset \widetilde{\mathcal{M}}_B$ consists of modules which are topologically finitely generated.
We call a module $M \in \mathcal{M}_B$ or $M \in \mathcal{M}'_B$ admissible if the space of $K$-invariants in $M$ is a finite $B$-module for any open compact subgroup $K \subset G$. We denote by $\mathcal{R}_B \subset \mathcal{M}_B$ and $\mathcal{R}'_B \subset \mathcal{M}'_B$ the (full) subcategories of admissible modules.

**Lemma 4.14.** The category $D^+(\mathcal{R}_B)$ (respectively $D^b(\mathcal{R}_B)$) is equivalent to the full subcategory of $D^+(\mathcal{M}_B)$ (respectively $D^b(\mathcal{M}_B)$) consisting of complexes with admissible cohomology.

**Proof** is parallel to that of 4.1. □

For $V \in B - \text{mod}$ we have a contravariant functor $\text{Hom}_B(\underline{\cdot}, V) : \mathcal{M}_B \to \mathcal{M}_B$. We can derive it in both arguments to get a bifunctor $D^-(\mathcal{M}_B) \times D^+(B - \text{mod}) \to D^+(\mathcal{M}_B)$.

We have Deligne-Lusztig functors $DL : D^{b, \pm}(\mathcal{M}_B) \to D^{b, \pm}(\mathcal{M}_B)$; the definition remains the same.

**Proposition 4.15.** For any $V \in D^b(B - \text{mod})$ there exists a canonical isomorphism of functors $D^b(\mathcal{R}_B) \to D^+(\mathcal{R}_B)$

\[ R\text{Hom}_{G \times B}(\underline{\cdot}, \mathcal{H} \otimes V)|_{\mathcal{R}_b} \cong DL \circ R\text{Hom}_B(\underline{\cdot}, V) \]

**Proof** is parallel to the proof of 4.2. The only difference is that one should consider $G$-equivariant sheaves of $B$-modules instead of $G$-equivariant sheaves of $k$-vector spaces, and replace the Verdier dualizing sheaf by the tensor product of Verdier dualizing sheaf (with coefficients in $k$) and $V \in D^+(B - \text{mod})$. □

4.16. By the definition we have imbedding of a full subcategory $I : \mathcal{M} \hookrightarrow \mathcal{M}_Z$. By the strong admissibility Theorem its image actually lies in $\mathcal{R}_Z$. So to finish the proof of the Theorem it is enough to prove the following standard fact.

**Proposition 4.17.** We have a functorial isomorphism

\[ R\text{Hom}_{G \times Z}(I \otimes \text{Id}(M), N \otimes D) \cong I(R\text{Hom}_G(M, N)) \]

where $M \in D^-(\mathcal{M})$; $N$ is a bounded below complex of finitely generated smooth $G$-bimodules; $D \in D^b(Z - \text{mod})$ is the dualizing complex. Here both sides lie in $D^+(\mathcal{M}_Z)$.

**Proof** We have an isomorphism

\[ \text{Hom}_{G \times Z}(I(M), ?) = \text{Hom}_G(M, \text{Hom}_{G \times Z}(I(\mathcal{H}), ?)), \]

where the action of $G$ on $\text{Hom}_{G \times Z}(I(\mathcal{H}), ?)$ comes from the right $G$-action on $\mathcal{H}$; and also:

\[ \text{Hom}_{G \times Z}(I(\mathcal{H}), ?) = \text{Hom}_{Z \otimes Z}(Z, ?) \]

where the first of the two $Z$-actions on ? comes from the $G$-action on ?, and $G$ acts on $\text{Hom}_{Z \otimes Z}(Z, ?)$ via its action on ?. Since $\text{Hom}_{Z \otimes Z}(Z, ?)$ is injective provided ? is injective, we get the corresponding isomorphisms of derived functors.

We have the standard quasiisomorphism in the derived category of $Z$-modules

\[ R\text{Hom}_{Z \otimes Z}(Z, N \otimes D) = i^! \circ pr_1^!(N) \cong (pr_1 \circ i)^! N \cong N \]

where $i$ is the diagonal embedding $i : \text{Spec}(Z) \to \text{Spec}(Z \otimes Z)$, $pr_1 : \text{Spec}(Z \otimes Z) \to \text{Spec}(Z)$ is the projection, and we do not distinguish between a module over a commutative algebra and the corresponding quasicoherent sheaf over its spectrum.

It is easy to lift this quasiisomorphism to a quasiisomorphism in the category of $G$-bimodules. More precisely, if $N$ lies in the heart of the derived category
(i.e. \( H^i(N) = 0 \) for \( i \neq 0 \); the only case we need \( N = \mathcal{H} \) satisfies this assumption) then \( RHom_{Z \otimes Z}(Z, N \otimes D) \) has only non-trivial cohomology in degree 0 isomorphic to \( N \), hence in this case we have a canonical quasiisomorphism \( N \leftarrow \tau_{\leq 0}(RHom_{Z \otimes Z}(Z, N \otimes D)) \rightarrow RHom_{Z \otimes Z}(Z, N \otimes D) \). If \( N' \) is any complex (of injective modules) and \( D' \) is any complex of injective \( Z \)-modules quasiisomorphic to the dualizing sheaf, then \( RHom_{Z \otimes Z}(Z, N \otimes D) \) is represented by the bicomplex \( \text{Hom}_{Z \otimes Z}(Z, N' \otimes D') \); we can apply the above canonical quasiisomorphism in each column of this bicomplex to get the desired quasiisomorphism for any \( N \).

4.18. **Proof** of Theorem 4.13. Since \( I(M) \in \mathcal{R}_Z \) for \( M \in \mathcal{M} \) we see using 4.17 that

\[
D_h(M) = RHom_G(M, \mathcal{H}) \cong RHom_{G \times Z}(I(M), \mathcal{H} \otimes D)
\]

(more precisely we apply the forgetful functor \( G \times \mathcal{Z} \rightarrow \mathcal{G} \otimes \mathcal{D} \) to the isomorphism 4.17 to get the last isomorphism). By 4.15 we have:

\[
RHom_{G \times Z}(I(M), \mathcal{H} \otimes D) \cong DL(RHom_Z(M, D))
\]

The Theorem is proved. \( \square \)
2. Elliptic pairing and filtrations on the Hecke algebra

Preliminaries and notations. In this chapter we prove

Theorem 0.19. The equality (2):

$$O_{g^{-1}}(\rho) = \chi_{\rho}(g)$$

holds for any $\rho \in \mathcal{R}$, and any elliptic element $g \in G$.

and deduce from it

Theorem 0.20. Assume $\text{char}(F) = 0$ and $k = \mathbb{C}$. The equality (1):

$$\sum (-1)^{i} \dim \text{Ext}^{i}(\rho_{1}, \rho_{2}) = \int_{\mathcal{E}} \chi_{\rho_{1}}(g^{-1})\chi_{\rho_{2}}(g)d\mu(g)$$

is valid for all $\rho_{1}, \rho_{2} \in \mathcal{R}$.

It will be convenient to allow $G$ to be an arbitrary reductive group (not requiring that $G$ has compact center). Notice however that a group with non-compact center does not contain elliptic elements (in the terminology we adopt), so the RHS of (1) vanishes; it is well-known (see Claim 4.3 below) that so does the LHS. The statement of Theorem 0.19 is vacuous for a group with noncompact center.

0.21. Hattori-Stallings trace for the Hecke algebra. Since the Hecke algebra $\mathcal{H}$ is not unital, a comment on the definition of the Hattori-Stallings trace appearing in the formulation of Theorem 0.19 is required.

Recall from [BD] that for a nice open compact subgroup $K \subset G$ (see [BD] 2.1b) for the definition) the full subcategory $\mathcal{M}^{K} \subset \mathcal{M}$ consisting of modules generated by $K$-fixed vectors is a direct summand in $\mathcal{M}$. Moreover, the functors $\mathfrak{M} \to \mathfrak{M}^{K}$ and $M \to M \otimes_{\mathcal{H}(G,K)} \mathcal{H}$ are mutually inverse equivalences between $\mathcal{M}^{K}$ and the category of $\mathcal{H}(G,K)$-modules (here $\mathcal{H}(G,K)$ is the subalgebra of $K$-biinvariant measures in $\mathcal{H}$).

Hence $\mathcal{H}(G,K)$ is a Noetherian unital algebra of finite homological dimension, so the usual construction of the Hattori-Stallings trace applies to it. Now for any $\mathfrak{M} \in \mathcal{M}^{K}$, $E \in \text{End}_{G}(\mathfrak{M})$ and $K' \subset K$ we have $\mathfrak{M}^{K'} = \mathfrak{M}^{K} \otimes_{\mathcal{H}(G,K)} \mathcal{H}(G,K')$, hence the natural map $\mathcal{H}(G,K)/[.] \to \mathcal{H}(G,K')/[.]$ induced by the imbedding $\mathcal{H}(G,K) \to \mathcal{H}(G,K')$ sends $Tr_{H-S_{1}}(\mathfrak{M}^{K}, E_{\mathfrak{M}^{K}})$ to $Tr_{H-S_{1}}(\mathfrak{M}^{K'}, E_{\mathfrak{M}^{K'}})$. Thus for small enough $K'$ the image of $Tr_{H-S_{1}}(\mathfrak{M}^{K'}, E_{\mathfrak{M}^{K}})$ in $\mathcal{H}/[\mathcal{H}, \mathcal{H}]$ does not depend on $K'$; this is by the definition the Hattori-Stallings trace of $(\mathfrak{M}, E)$.

0.22. A regular elliptic element $g \in G$ is fixed throughout the argument.

Since $g$ is elliptic it is also compact; we fix a maximal compact subgroup $K_{0}$ containing $g$.

Let $K \subset K_{0}$ be a normal subgroup; since $g$ normalizes $K$ it acts on the set of double cosets $K \backslash G / H$ for any subgroup $H \subset G$. We say that $g$ is $K$-elliptic if it acts on $K \backslash G / P$ without fixed points for any proper parabolic $P \subset G$. Obviously $g$ is $K$-elliptic for small enough normal open $K \subset K_{0}$.

We say that $K$ is nice if it is nice with respect to some minimal Levi subgroup $L_{0}$, which is in good position with $K_{0}$ (see 0.24, 2.1 below).

We fix a normal open subgroup $K \subset K_{0}$, such that $g$ is $K$-elliptic.

From now on we will denote by $\mathcal{H}$ the subalgebra $\mathcal{H}(G,K) \subset \mathcal{H}(G)$ of $K$-biinvariant distributions.

We will in fact work with the following form of Theorem 0.19
Theorem 0.23. One can define for any \( g \in \mathcal{E}_L \), and any finitely generated \( \mathcal{H} \)-module \( M \) a number \( \text{Tr}(g, M) \in k \) so that

i) \( \text{Tr}(g, M) \) is additive on short exact sequences in \( M \).

Fix \( \mathfrak{M} \in \mathcal{M} \).

ii) If \( \mathfrak{M} \) is admissible, and its character \( \chi_{\mathfrak{M}} \) is constant on the coset \( K \cdot g \), then \( \text{Tr}(g, \mathfrak{M}^K) = \chi_{\mathfrak{M}}(g) \) is the character value.

iii) Let \( M \) be a projective \( \mathcal{H} \)-module. Then there exists an idempotent \( E \in \text{Mat}_m(\mathcal{H}) \) such that \( M \) is isomorphic to the image of \( E \) acting on \( \mathcal{H}^{\otimes m} \) on the right.

Suppose that \( K \) is nice, and the locally constant invariant function \( x \to O_{x-1}((\mathfrak{M})) \) defined on the regular elliptic set is (defined and) constant on the coset \( K \cdot g \). Then we have

\[
\text{Tr}(g, M) = O_{g^{-1}}(\sum E_{ii}).
\]

It is immediate to deduce 0.19 from 0.23 (see 3.8).

The next three sections are devoted to the construction of \( \text{Tr}(g, M) \) and verification of the properties i)–iii).

0.24. Notations. Let us choose a minimal parabolic with Levi decomposition \( P_0 = L_0 \cdot U_0 \subset G \) so that \( L_0 \) is in good relative position with \( K_0 \), i.e. the point \( p \) of the Bruhat-Tits building fixed by \( K_0 \) lies in the apartment normalized by \( L_0 \).

A standard parabolic is the one containing \( P_0 \); a standard Levi is the Levi subgroup of a standard parabolic containing \( L_0 \).

The letters \( P, L, U \) with indices will be reserved for a standard parabolic, its standard Levi, and its unipotent radical respectively, unless stated otherwise.

Let \( W = Nm(L_0)/L_0, W_L = (Nm(L_0) \cap L)/L_0 \) and \( W_{aff} = Nm(L_0)/L_0 \) be respectively the Weyl group, the Weyl group of \( L \) and the affine Weyl group.

For an \( L \)-module \( \rho \) we will sometimes write \( i_G^L(\rho) \) instead of \( i_G^L(\rho) \) for the parabolically induced module (of course there is no ambiguity here, because \( P \) and \( L \) determine each other uniquely).

Let \( A_L \) be the center of \( L \), \( X_L \) be the lattice of coweights of \( A_L \). Thus \( X := X_{L_0} \) is the lattice of abstract coweights of \( G \). Set \( a = X \otimes \mathbb{R} \); let \( a_+ \subset a \) (respectively \( a^+ \)) be the dominant Weyl chamber (respectively its closure). For a subset \( \mathfrak{a} \subset a \) we denote the intersection \( \mathfrak{a} \cap a^+ \) by \( \mathfrak{a}^+ \).

For a standard Levi \( L \) let \( a_L \subset a \) denote \( X_L \otimes \mathbb{R} \).

Fixing a uniformizer \( p \in F^\times \) we obtain a canonical imbedding \( \iota_L : X_L \hookrightarrow A_L \), \( \chi \mapsto \chi(p) \), inducing an imbedding of finite index \( X_L \hookrightarrow L/L^c \).

For a topological group \( H \) we denote by \( H^c \) the subgroup generated by all compact subgroups.

Consider the composition \( L/L^c \hookrightarrow (L/L^c) \otimes \mathbb{R} = a_L \hookrightarrow a \). We identify \( L/L^c \) with its image under this map.

By \( \text{Def}_L \) we denote the orthogonal projection \( a \to a_L \).

By a root we will mean a nonmultiplicable root, and by a coroot the corresponding coweight of \( A_{L_0} \). Thus coroots (respectively, roots) form a reduced root system in \( a \) (respectively \( a^+ \)) (see [BoT], Theorem 7.2). The choice of a minimal parabolic \( P_0 \) determines the basis of simple (co)roots, and the dual basis of fundamental (co)weights.


Let $\Sigma^+$ be the set of positive coroots; for a Levi $L$ let $\Sigma^+_L \subset \Sigma^+$ be the set of positive coroots of the derived group $L^\ad$.

If $\alpha$ is a simple coroot, $\omega_\alpha$ is the dual fundamental weight, and $\lambda_1, \lambda_2 \in X$ then we will write $\lambda_1 \leq_{\alpha} \lambda_2$ meaning $(\lambda_1, \omega_\alpha) \leq (\lambda_2, \omega_\alpha)$. We also write $\lambda \leq n$ meaning $(\lambda, \omega_\alpha) \leq n$.

The standard partial order $\leq$ on $\alpha$ is defined by $\lambda_1 \leq \lambda_2$ if $\lambda_1 \leq \lambda_2$ for all $\alpha$; as usual we write $\lambda_1 \prec \lambda_2$ instead of $\lambda_1 \leq \lambda_2$ & $\lambda_1 \neq \lambda_2$ etc.

Recall the notation $k[S] := \bigoplus_{s \in S} k$. When $S$ is a (semi)group this is the (semi)group algebra; for $s \in S$ let $[s] \in k[S]$ be the delta-function of $s$.

If $(\Lambda, \blacktriangleleft)$ is a (partially) ordered semigroup, we use the notation $\Lambda_{\blacktriangleleft \blacktriangleright} := \{ \mu \in X \mid \mu \blacktriangleleft \lambda \}$ for $\lambda \in \Lambda$.

We have a $(\Lambda, \blacktriangleleft)$-filtration on the semigroup algebra $k[\Lambda]$, defined by $k[\Lambda]_{\blacktriangleleft \blacktriangleright} := k[\Lambda_{\blacktriangleleft \blacktriangleright}]$.

Analogous notations apply when $\blacktriangleleft$ is a filtration on $\Lambda$ indexed by another (partially) ordered semigroup.

We use the same symbol for a partial order on $\alpha$ and its restriction to a sublattice.

We will say that $\lambda \in X_L$ is dominant (or large) enough if $(\lambda, r) \gg 0$ when $r$ is a root of $U$. (The latter condition will be abbreviated as $(\lambda, \Sigma^+ - \Sigma^+_L) \gg 0$).

1. **“Spectral” filtration: definition of the trace functional.**

1.1. Let $\rho$ be an admissible representation of $L$. We can form a (non-admissible) $G$-module $\Pi_{\rho} := i^G_L(\rho \otimes k[L/L^c]) = ind_{L^c,L}(\rho).$ Then $\Pi_{\rho}$ has also a $k[L/L^c]$-action commuting with the $G$-action, which comes from the action of $k[L/L^c]$ on the second multiple of $\rho \otimes k[L/L^c]$:

$$l(f)(g) = (Id \otimes l)(f(g))$$  \hspace{1cm} (17)

for $l \in L/L^c$, $f : G \to \rho \otimes k[L/L^c]$, $f \in \Pi_{\rho}$. For $\lambda \in L/L^c$ let $[\lambda]_\rho \in \text{End}(\Pi_{\rho})$ denote action (17).

For an unramified character $\psi : L/L^c \to k^\times$ we have a canonical isomorphism of $G$-modules (but not of $L/L^c$-modules) $I_{\psi} : \Pi_{\rho} \xrightarrow{\psi} \Pi_{\rho \otimes \psi}$, satisfying

$$I_{\psi}(l(m)) = \psi(l)(I_{\psi}(m))$$ \hspace{1cm} (18)

for $l \in L/L^c$, $m \in \Pi_{\rho}$.

From (18) it follows that $E \in \text{End}(\Pi_{\rho})$ commutes with $\lambda_\rho$ for all $\lambda \in L/L^c$ iff $I_{\psi} \circ E \circ I_{\psi}^{-1}$ commutes with all $[\lambda]_\rho \otimes \psi$, thus we have a canonical isomorphism

$$\text{End}_{L/L^c}(\Pi_{\rho}) \cong \text{End}_{L/L^c}(\Pi_{\rho \otimes \psi}).$$ \hspace{1cm} (19)

For any standard Levi $L$ let us choose a set $\text{Cusp}_L$ of cuspidal irreducible representations of $L$, in such a way that any cuspidal irreducible representation of $L$ is isomorphic to a unique representation $\rho \otimes \psi$, where $\rho \in \text{Cusp}_L$, and $\psi : L/L^c \to k$ is an unramified character of $L$. We can (and will) make this choice in such a way that $\rho|_{L(X_L)}$ is trivial for $\rho \in \text{Cusp}_L$.

Let $\text{Cusp}_L^K \subset \text{Cusp}_L$ be the set of such $\rho \in \text{Cusp}_L$ that $i^G_L(\rho)^K \neq 0$; set $\text{Cusp} := \bigcup_L \text{Cusp}_L$ and $\text{Cusp}^K := \bigcup_L \text{Cusp}_L^K$.

Set $\mathcal{H} := \bigoplus_{\rho \in \text{Cusp}^K} \text{End}_{k[L/L^c]}(\Pi_{\rho}^K)$. 

1.2. We are going to define an $a$-multifiltration on $\hat{\mathcal{H}}$. To clarify the idea we first carry out the construction under an additional simplifying assumption that the maximal compact subgroup $K_0$ containing $g$ is good, i.e. $G = K_0 \cdot P_0$ (the general case will be treated in the next subsection).

If $K_0$ is good, then the $k[L/L^c]$-module $\Pi_{\rho}$ can be canonically trivialized in the following fashion.

By the definition the space of the induced representation $\Pi_{\rho}$ is the space of functions on $G$ with values in $\rho \otimes k[L/L^c]$ which transform accordingly under the right action of $P$. Let us say that $f \in \Pi_{\rho}$ is constant if $f|_{K_0}$ takes values in $\rho \otimes 1 \subset \rho \otimes k[L/L^c]$. Then the space of constant elements is a $K_0$-submodule, and generates $\Pi_{\rho}$ freely as a $k[L/L^c]$-module. We denote the space of constant elements in $\Pi_{\rho}$ by $(\Pi_{\rho})_0$. For an unramified character $\psi$ of $L$ the isomorphism $I_\psi$ sends $(\Pi_{\rho})_0$ to $(\Pi_{\rho \otimes \psi})_0$, and we have the canonical isomorphism $(\Pi_{\rho \otimes \psi})_0 \overset{\sim}{\longrightarrow} I_\psi(\psi \otimes \rho) = \Pi_{\rho \otimes k[L/L^c]} \psi$.

We abbreviate $(\Pi_{\rho})_0^K = i_\rho(\rho)$. So we have $\hat{\mathcal{H}} = \bigoplus_{\rho \in \text{Cusp}} \text{End}(\Pi_{\rho})_0^k \otimes k[L/L^c] = \bigoplus_{\rho \in \text{Cusp}} \text{End}(i_\rho(\rho)) \otimes k[L/L^c]$.

For a (partial) order $\blacktriangleleft$ on $a$ we have an $(a, \blacktriangleleft)$-filtration on $\hat{\mathcal{H}}$ given by $\hat{\mathcal{H}}_{\blacktriangleleft, \lambda} := \bigoplus_{\rho \in \text{Cusp}} \text{End}(i_\rho(\rho)) \otimes k[L/L^c]_{\blacktriangleleft, \lambda}$.

1.3. Consider now the case when $K_0$ is an arbitrary maximal compact subgroup.

Recall that $p$ is the point of the Bruhat-Tits building $X$ fixed by $K_0$; let $\mathfrak{A} \ni p$ be the apartment corresponding to $L_0$. Let $W_p = K_0 \cap \text{Nm}(L_0)/L_0 \subset W_{aff}$ be the stabilizer of $p$, and $W_{aff} \subset W_{aff}$ be the subgroup $(L^c \cap \text{Nm}(L_0))/L_0$. The groups $\text{Nm}(L_0)$, $L^c \cap \text{Nm}(L_0)$ act on $\mathfrak{A}$ respectively through $W_{aff}$, $W_{aff}.^L$.

Notice that $P^c = L^c \cdot U$.

**Lemma 1.4.** The map $w \mapsto K_0 w P^c$ provides a bijection

$$W_p \backslash W_{aff}/W_{aff}^L \overset{\sim}{\longrightarrow} K_0 \backslash G/P^c.$$

**Proof** First notice that since $K_0$, $L_0$ are in good relative position we have $K_0 \supset L_0^c$, and obviously $L_0^c \subset P^c$, so the expression $K_0 w P^c$ is meaningful.

Let $W_{aff}(L)$ denote $\text{Nm}_L(L_0)/L_0^c$. On the LHS of (21) the group $W_{aff}(L)/W_{aff}^L$ acts from the right. From the Bruhat decomposition $L = (P_0 \cap L) \cdot \text{Nm}_L(L_0) \cdot (P_0 \cap L)$ we see, using the inclusion $P_0 \cap L = L_0 \cdot (U \cap L) \subset L_0 \cdot L^c$, that the map $W_{aff}(L)/W_{aff}^L = \text{Nm}_L(L_0)/\text{Nm}_L(L_0) \rightarrow L/L^c$ is an isomorphism.

On the RHS of (21) the group $L/L^c = P/P^c$ acts from the right. The map (21) obviously agrees with the action of $L/L^c$. Moreover, we claim that the action on both sides is free. Indeed, the right action of $L/L^c = P/P^c$ on $G/P^c$ is obviously
free; since the map $G/P^c \to K_0 \backslash G/P^c$ is proper, the stabilizer of any point in $K_0 \backslash G/P^c$ is compact. However $L/L^c$ is a (discrete) free abelian group, so it does not contain compact subgroups. The same argument applies to the LHS (with “finite-to-one” replacing “proper”).

Thus it is enough to prove that the map $W_p \backslash W_{aff}/W_{aff}(L) \to K_0 \backslash G/P$ is an isomorphism. This can be reformulated via the Bruhat-Tits building $X$ as follows.

Any points $x \in X$, $z \in \mathfrak{X} X$ lie in the closure of some apartment $\mathfrak{A}$. Moreover, if $x' \in \mathfrak{A}$, $z' \in \mathfrak{X} - \mathfrak{A}$ is another pair of points, such that $x' = g(x)$, $z' = g(z)$, then also $x' = w(x)$, $z' = w(z)$ for some element $w \in W_{\mathfrak{A}}$, where $W_{\mathfrak{A}} = Nm(\mathfrak{A})/Stab(\mathfrak{A}) \cong W_{aff}$.

The fact that $x$ and $z$ lie in one apartment follows from [BT1] 7.4.18(ii). To check the second statement recall that by Lemma 3.8 of part 1 there exists a unique geodesic ray connecting a point in $X$ and a point in $\mathfrak{X} - X$. In particular the rays $[x, z]$, $g([x, z]) = [x', z']$ lie in $\mathfrak{A}$ since there endpoints lie in $\mathfrak{X}$. Hence by [BT1] 7.4.8 there exists $w \in W_{\mathfrak{A}}$ such that $g|[x, z] = w|[x, z]$. By continuity of the action of $g$, $w$ we get $g(z) = w(z)$.

1.5. For $x \in W_p \backslash W_{aff}/W_{aff}^L$ let $(G/U)_x$ be the corresponding orbit of $K_0 \times P^c$.

Let the subgroup $W_p^f \subset Aut(\mathfrak{A})$ be generated by reflections in all hyperplanes parallel to root hyperplanes and passing through $p$; let $W_{aff}^f$ be generated by $W_{aff}$ and $W_p^f$. Let $\Lambda$ be the intersection of $W_{aff}^f$ with the group of translations; it is obvious that $\Lambda$ is a lattice in $a = X \otimes \mathbb{R}$ containing $X$, and $W_{aff}^f = W_p^f \ltimes \Lambda$.

Consider the projection $\pi_L : W_p \backslash W_{aff}/W_{aff}^L \to W_p^f \backslash W_{aff}^f/W_{aff} = \Lambda/W_{aff}^L \rightarrow \pi_L(a_L) \subset a$; denote its image by $\Lambda_L$. For $\lambda \in \Lambda_L$ set $(G/U)_{\lambda} = \bigcup_{\pi_L(x) = \lambda} (G/U)_x$.

Now we get a $\Lambda_L$-grading on $\Pi_\rho$ for $\rho \in Cusp_{\Lambda_L}$ as follows

$$\Pi_\rho = C_c(G/U)^K \otimes_{\mathcal{H}(L^c)} \rho = \bigoplus_{\lambda \in \Lambda_L} C_c(G/U)_\lambda^K \otimes_{\mathcal{H}(L^c)} \rho. \quad (22)$$

For $\lambda \in a$ we now set

$$\mathcal{H}_\lambda = \{ h \in \mathcal{H} \mid h(\Pi^K_\rho)_\nu \subset (\Pi^K_\rho)_{\nu + \lambda} \}, \quad (23)$$

$$\mathcal{H}_{\lambda, \mu} = \bigoplus_{\mu \in \lambda} \mathcal{H}_\mu. \quad (24)$$

Comparison of the definitions shows that the latter filtration coincides with the one introduced in 1.1 when $K_0$ is good.

1.6. For a (partial) order $\bullet$ on $a$ we have defined an $(a, \bullet)$-filtration on $\mathcal{H}$. Let $\mathcal{H}_{\lambda, \bullet}$ be the induced filtration on $\mathcal{H} \subset \mathcal{H}$.

The next Proposition is the key technical result needed for our definition of the trace functional.

For a standard Levi $L_1$, $\rho \in Cusp_{L_1}$, the space of $\Pi_\rho$ is the space of $\rho \otimes k[L_1/L_1^c]$-valued functions on $G$ transforming accordingly under the right action of $P_1$; for $L \supset L_1$ and $x \in K/G/P$ let $m_p^\rho \in End(\Pi^K_\rho)$ be multiplication by the $\delta$-function of the corresponding right $K \times P$ double coset.

Set also $m_x = \sum_{L \subset L_1, \rho \in Cusp_{L_1}} m_p^\rho$. 


Proposition 1.7. Fix a standard Levi $L \supset L_1$, $\rho \in \text{Cusp}^\text{w}_{L_1}$, $x \in K \setminus G/P$ and $a \in \mathbb{R}_{>0}$. Then for dominant enough $\lambda \in X^+_L$ there exists an element $h^\rho_{x,\lambda} \in \mathcal{H}$ such that $I(h^\rho_{x,\lambda}) = \sum_{L_2 = w(L_1) \subseteq L} |\lambda|_{w(\rho)} \cdot m_{w(\rho)}^x + h'$, where $h' = \sum_{r \in \Sigma^+ - \Sigma^+_L} \mathcal{H}_{r < \lambda - a \cdot r}$.

Remark 1.8. The construction of the trace functional will use only the fact $I(h^\rho_{x,\lambda}) - \sum_{L_2 = w(L_1) \subseteq L} |\lambda|_{w(\rho)} \cdot m_{w(\rho)}^x \in \mathcal{H}_{\leq \lambda}$. The stronger condition stated in the Proposition will play a role only in the proof that the constructed functional satisfies property i) of 0.23.

We will deduce the Proposition from the matrix Payley-Wiener Theorem which describes the image of the embedding $I$ (see [BR], §21). Let us first recall its contents.

If $M_1, M_2$ are two modules over a free abelian group $\mathbb{Z}^n$ then by a rational morphism from $M_1$ to $M_2$ we will mean an element of $\text{Hom}_R(M_1 \otimes k[\mathbb{Z}^n], \text{Frac}(k[\mathbb{Z}^n]))$, $M_2 \otimes k[\mathbb{Z}^n] \text{Frac}(k[\mathbb{Z}^n])$ where $\text{Frac}(\cdot)$ is the field of fractions of a commutative ring $\cdot$.

We recall the necessary part of the theory of intertwining operators. Suppose that standard Levi $L_1$, $L_2$ are conjugate, so we have $L_2 = w(L_1)$ for some $w \in W$.

Then there exists a canonical rational isomorphism of $G$-modules $I_w^\rho : \Pi_\rho \rightarrow \Pi_{w(\rho)}$, such that $I_w \circ |\lambda|_\rho = |w(\lambda)|_{w(\rho)} \circ I_w$ for $\lambda \in \mathbb{L}/\mathbb{L}^e$ satisfying:

i) Transitivity: if $L_2 = w_1(L_1)$, $L_3 = w_2(L_2)$ are standard Levi then $I_{w_2w_1}(m) = I_{w_1} \circ I_{w_1} \circ I_{w_2}(f_{w_1,w_2}m)$ for some rational function $I_{w_1,w_2}(\cdot) \in \text{Frac}(k[L_1/L_1^e])$.

ii) Compatibility with induction: if $L \supset L_1, L_2$, $L_2 = w(L_1)$ then for a representation $\rho \in \text{Cusp}^\text{w}_{L_1}$ we have $I_w^\rho = i_{L_1}^L(I_w^\rho L)$, where $i_{L_1}^L : i_{L_1}^L(\rho \otimes k[L_1/L_1^e]) \rightarrow i_{L_2}^L(\rho \otimes k[L_2/L_2^e])$ is the intertwining operator between the $L$-modules.

We define the rational morphisms $R_w^\rho : \text{End}_{L_1/L_1^e}(\Pi_\rho) \rightarrow \text{End}_{L_2/L_2^e}(\Pi_{w(\rho)})$ by $R_w^\rho(E) = I_w \circ E \circ I_w^{-1}$. Notice that the operators $R_w^\rho$ satisfy the stronger transitivity condition:

$$R_{w_1w_2}^\rho = R_{w_1} \circ R_{w_2}.$$

Let $L_1$, $L_2$ be standard Levi subgroups such that $L_2 = w(L_1)$ and $\rho_1 \in \text{Cusp}^\text{w}_{L_1}$, $\rho_2 \in \text{Cusp}^\text{w}_{L_2}$ be such that $\rho_2 = w(\rho_1) \otimes \psi$ for an unramified character $\psi$. Identify $\text{End}_{L_2/L_2^e}(\Pi_{w(\rho_1)})$ with $\text{End}_{L_1/L_1^e}(\Pi_{w(\rho_2)})$ by means of (19); then $R_w^\rho$ provides a rational morphism $\text{End}_{L_2/L_2^e}(\Pi_{w(\rho_1)}) \rightarrow \text{End}_{L_2/L_2^e}(\Pi_{w(\rho_2)})$. We denote it again by $R_w$ or $R_w^\rho$, this abuse of notation hopefully will not lead to a confusion.

For future reference we record the following

Lemma 1.9. For standard Levi subgroups $L_2 = w(L_1)$ and $\rho_1 \in \text{Cusp}^\text{w}_{L_1}$, $\rho_2 \in \text{Cusp}^\text{w}_{L_2}$ such that $\rho_2 = w(\rho_1) \otimes \psi$ as above we have $R_w(|\lambda|_{\rho_1}) = |w(\lambda)|_{\rho_2}$ provided $\lambda$ lies in $X_{L_1} \subseteq L_1/L_1^e$.

Proof Since $R_w^\rho(|\lambda|_{\rho_1}) = |w(\lambda)|_{w(\rho_1)}$ we have only to check that $|w(\lambda)|_{w(\rho_1)} = |w(\lambda)|_{\rho_2}$ (more precisely, that the two sides agree under (19)). By (18) this reduces to $\psi(\mu) = 1$ for $\mu \in X_{L_2}$. But this is clear because $X_{L_2}$ acts trivially both on $w(\rho_1)$ and on $\rho_2 \in \text{Cusp}^\text{w}_{L_2}$.

There exists also another type of operators acting on $\bigoplus \Pi_{w(\rho)}$. Namely, for a given irreducible representation $\rho$ of $L$ there might exist a finite number of unramified characters $\psi$ such that $\psi \otimes \rho \cong \rho$. For such $\psi$ the canonical isomorphism...
$I_\psi : \Pi_\rho \rightarrow \Pi_{\rho \otimes \psi}$ induces an automorphism of $\Pi_\rho$. This automorphism is defined
uniquely up to a constant (we have to choose an isomorphism $J : \rho \otimes \psi \rightarrow \rho$, and then get $i^C_L(J) \circ I_\psi : \Pi_\rho \rightarrow \Pi_\psi$); thus we have a uniquely defined automorphism $T_\psi$
of $\text{End}_{L^\prime/(L^\prime \cap \rho)}$. $T_\psi : E \mapsto (i^C_L(J) \circ I_\psi) \circ E \circ (i^C_L(J) \circ I_\psi)^{-1}$.

The matrix Payley-Wiener Theorem ([BR], Theorem 33) asserts that

An element $E = \sum_{Cusp^K} E_\rho \in \mathcal{H}$ lies in the image of $I$ iff

i) For any conjugate standard Levi subgroups $L_2 = w(L_1)$ and $\rho_1 \in Cusp_{L_1}$, $\rho_2 \in Cusp_{L_2}$ such that $\rho_2 \cong w(\rho_1 \otimes \psi)$ for an unramified character $\psi$ we have $R^\psi_w(E_{\rho_1}) = E_{\rho_2}$.

ii) $T_\psi(E_\rho) = E_\rho$ whenever $\psi \otimes \rho \cong \rho$.

1.10. We return to the proof of 1.7.

For standard Levi subgroups $L = w(L)$ and $\rho \in Cusp^K_L$ we will not distinguish between $\Pi_w(\rho)$ and $\Pi_{w(\rho) \otimes \psi_\rho}$ when this is not likely to lead to a confusion. We will also abuse notations writing $R_w$ instead of $T_{\psi_\rho} \circ R_w$. Thus $R_w : \text{End}(\Pi_\rho) \rightarrow \text{End}(\Pi_{w(\rho)})$ is a rational endomorphism of $\mathcal{H}$.

The plan is to start with $m^\rho_x \cdot [\lambda]_\rho$ and then average over the set of intertwining operators; we have to take care of the poles of intertwining operators, and also to ensure that we really get an element with the desired highest term. It will be not hard to see that each summand in the resulting sum is $T_\psi$-invariant.

**Lemma 1.11.** If $L_2 = w(L_1)$, $L \supset L_1$, $L_2$ are standard Levi, $x \in K \setminus G/P$, $\lambda \in X_L \subset X_{L_1}, X_{L_2}$ then we have $R_w(m^\rho_x \cdot [\lambda]_\rho) = m^\rho_x \cdot [\lambda]_\rho$ for any $\rho$.

**Proof.** By 1.9 the action by intertwining operators induces the standard geometric action on coweights, thus $R_w([\lambda]_\rho) = [\lambda]_w(\rho)$ for $\lambda \in X_L$ and $w \in W_L$.

If we think of $\Pi_w = i^C_L(\rho \otimes k[L_1/L_L]) = i^C_L(i^C_L(\rho \otimes k[L_1/L_L]))$ as sections of the corresponding sheaf on $G/P$ then the action of $m^\rho_x$ comes from an action on the sheaf; moreover the corresponding endomorphism of a stalk is either identity or $0$.

From $R_w(Id) = Id$ we conclude by compatibility that it sends $m^\rho_x$ to $m^w(\rho)$. $\square$

For $h \in \mathcal{H}$, $h = \sum_{L, \lambda \in \Lambda_L} h_\lambda$, where $h_\lambda \in \mathcal{H}_\lambda$, let the support of $h$ be

$$\text{supp}(h) := \{ \lambda \in \mathfrak{a} \mid h_\lambda \neq 0 \}. \quad (25)$$

1.12. Consider the following situation. Let $L_1 \neq L_2$ be two conjugate standard Levi lying in a standard Levi $L$ with rank($L$) = rank($L_1$) + 1 = rank($L_2$) + 1. Then necessary $L_2 = w(L_1)$, where $w_0 \in W_L$ is the longest element. Let $\alpha$ be a nonzero element of the group $X_{L_1} \cap (a_L)^\perp$. Notice that $\alpha$ is proportional to $\Psi_{L_1}(r)$ where $r$ is the only simple coroot of $L$ which does not lie in $\Sigma_{L_1}$.

**Lemma 1.13.** a) Components of the divisor of poles of $R^L_{w_0}$ have the form $[\alpha]_{w_0} = \text{const}$; in other words for some $f \in k[t]$ for any $h \in \text{End}_{L_1/L_2} \Pi^K_{w_0}$ we have $R^L_{w_0}(f([\alpha]_{w_0})h) \in \text{End}_{L_1/L_2} \Pi^K_{w_0} \subset \text{End}_{L_1/L_2} \Pi^K_{w_0} \otimes \text{Frac}(k[L_2/L_2])$.

b) There exists $n_0 \in \mathbb{Z}$ such that the following holds. Suppose that $h \in \text{End}_{L_1/L_2} \Pi^K_{w_0}$ is such that $R^L_{w_0}(h)$ is regular; i.e. belongs to $\text{End}_{L_1/L_2} \Pi^K_{w_0}$ rather than to $\text{End}_{L_1/L_2} \Pi^K_{w_0} \otimes \text{Frac}(k[L_2/L_2])$. Then $\nu \in \text{supp}(R^L_{w_0}(h)) \Rightarrow (w_0)^{-1}(\nu) + n\alpha \in \text{supp}(h)$ for some $n$ with $|n| \leq n_0$. 
Proof. a) We can decompose $L$ as an almost direct product of its center $A_L$ and derived group $L^{[i]}$, i.e., we have a homomorphism $\pi : A_L \times L^{[i]} \to L$ with finite kernel and cokernel. Obviously for $j = 1, 2$ the pull-back of the representation $i_{L_j}^L(k[L_j/L_j^\rho] \otimes \rho)$ under $\pi$ decomposes as the tensor product of the free module $k[L^\rho/A_L^\rho]$ with the parabolically induced module $\left( \bigoplus_{L_j/L_j^\rho \cdot A_L(L^{[i]} \cap L_j)} k[(L_j \cap L^{[i]})/L_j \cap L^{[i]}] \otimes \rho |_{L_j \cap L^{[i]}} \right)$. The intertwining operator $I_{K,L}^{[i]}$ also decomposes as $Id \otimes I_{K,L}^{[i]} |_{L_j \cap L^{[i]}}$, hence so does $R_{K,L}^{[i]}$. Thus the image of the divisor of poles of $R_{K,L}^{[i]}$ under the finite covering $Spec(k[L/L^\rho]) \to Spec(k[A_L/A_L^\rho]) \times Spec(k[L^{[i]}/L_j])$ is the product of $Spec(k[A_L/A_L^\rho])$ with a finite subscheme of $Spec(k[[L^{[i]} \cap L_j]/(L^{[i]} \cap L_j)^\rho])$ (notice that $(L^{[i]} \cap L_j)/(L^{[i]} \cap L_j)^\rho$ is a free cyclic group, so $Spec(k[[L^{[i]} \cap L_j]/(L^{[i]} \cap L_j)^\rho])$ is the punctured affine line). This finite subscheme is necessary killed by the function $\prod_{j=1}^n (([\alpha]_\rho - q_j)$ for some finite set of elements $q_1, \ldots, q_n \in k^\times$, hence the same holds for the divisor of poles of $R_{K,L}^{[i]}$.

Let us prove b). For $j = 1, 2$ consider the projection $\pi_j : K \backslash G/P^c \to K \backslash G/P^c$. Let $y \in K \backslash G/P^c$ denote $(\Pi^K_{\rho})_y := \bigoplus_{\pi_j(x) = y} (\Pi^K_{\rho})_{\pi_j(x), y} : \bigoplus_{\pi_j(x) = y} (\Pi^K_{\rho})_{\pi_j(x)}$. Notice that $(\Pi^K_{\rho})_y$ is a $[\mathbb{Z}\alpha]$-submodule for any $y \in K \backslash G/P^c$.

By a) the intertwining operator $I_{\rho}^{[i]}$ is a well-defined morphism $\Pi^K_{\rho} \otimes_{[\mathbb{Z}\alpha]} k([\alpha]_\rho) \to \Pi^K_{\rho} \otimes_{[\mathbb{Z}\alpha]} k([\mu]_{\rho})$. From compatibility with induction it follows that $I_{\rho}^{[i]}(\Pi^K_{\rho})_y \otimes k([\alpha]_\rho) \subset (\Pi^K_{\rho})_y \otimes k([\mu]_{\rho})$.

Now from the definition it is easy to see that for $j = 1, 2$ the following diagram is commutative:

\[
\begin{array}{ccc}
K \backslash G/P^c_{\rho} & \xrightarrow{\pi_j} & K \backslash G/P^c \\
\pi_{L_j} & \downarrow & \pi_L \\
\Lambda_{L_j} & \xrightarrow{\Psi_{L_j}} & \Lambda_L
\end{array}
\]

Hence for $\mu \in \Lambda_L$ we have $k[\mathbb{Z}\alpha]$-submodule $(\Pi^K_{\rho})_{\mu} := \bigoplus_{\lambda \in \Psi_{L_j}^{-1}(\mu)} (\Pi^K_{\rho})_{\lambda}$, and $k([\mu]_{\rho})$-submodule $(\Pi^K_{\rho})_{\mu} := \bigoplus_{\lambda \in \Psi_{L_j}^{-1}(\mu)} (\Pi^K_{\rho})_{\lambda}$, so that $I_{\rho}^{[i]}((\Pi^K_{\rho})_{\mu} \otimes k([\alpha]_\rho)) \subset (\Pi^K_{\rho})_{\mu} \otimes k([\mu]_{\rho})$.

It follows that $R_{\rho}^{[i]}((End_{L_j/L_j^\rho} \Pi^K_{\rho})) \subset (End_{L_j/L_j^\rho} \Pi^K_{\rho})_{\mu} \otimes k([\mu]_{\rho})$ where the $\mu$ (respectively $[\mu]_{\rho}$) invariant $\Lambda_L$-grading on $End_{L_j/L_j^\rho} \Pi^K_{\rho}$ (respectively on $End_{L_j/L_j^\rho} \Pi^K_{\rho}$) is induced by the grading on $\Pi^K_{\rho}$, $\Pi^K_{\rho}$ as in (23).

Recall the following elementary fact. For a Laurent polynomial $P = \sum a_i t^i \in k[t, t^{-1}]$ set $\text{supp}(P) = \{ n \in \mathbb{Z} : a_n \neq 0 \}$. Let $f(t) \in k(t)$ be a rational function, and set $n_f = \max(\text{ord}_0(f), -\text{ord}_\infty(f))$. Then for $P \in k[t, t^{-1}]$ we have $fP \in k[t, t^{-1}] \Rightarrow \text{supp}(fP) \subset [\text{inf}(\text{supp}(P)) - n_f, \text{sup}(\text{supp}(P)) + n_f]$. 

Now notice that for $\mu \in \Lambda_L$: \( \Psi_{L_1}^{-1}(\mu) = w_0^L(\Psi_{L_1}^{-1}(\mu)) \), and recall that by Lemma 1.9 \( R_{w_0^L}((\alpha)\rho) = [w_0^L(\alpha)]_{w_0^L}(\rho)R_{w_0^L}(h) \).

It is clear that \((\text{End}_{L_1/L_1}^{\varphi})_{\mu} \) is a finite graded \(k[\rho]\)-module for \(\mu \in \Lambda_L\). Hence the above elementary fact implies that at least for fixed \(\mu\) there exists such \(n_0\) that the condition b) of the Lemma holds for \(h \in (\text{End}_{L_1/L_1}^{\varphi})_{\mu} \). But if it holds for some \(h\) it also holds (with the same \(n_0\)) for \(h' = l(h)\) where \(l\) is any element of \(A_L/A^+_L\). Since a finite number of components of the \(\Lambda_L\)-grading generate \(\text{End}_{L_1/L_1}^{\varphi} \) as an \(A_L/A^+_L\)-module, b) is proved. \(\Box\)

### 1.14 The next statement implies Proposition 1.7.

Let us choose such \(m \in \mathbb{Z}_{>0}\) that \(\Psi_L (mr) \in X_L\) for any \(L\) and any coroot \(r\). In the notations of 1.13 choose \(\alpha\) to be \(\Psi_{L_1}(mr)\).

Let \(q_1, \ldots, q_n \in k^\times\) be a set of elements such that \(R^L_{w_0^L}((\prod_i (\alpha)\rho - q_i)h)\) is regular for any \(\rho\), \(L_1, L_2, L\), \(w_0^L\) as in 1.13, \(\alpha\) as above and \(h \in \text{End}_{L_1/L_1}^{\varphi}(\Pi_{L}(\rho))\) (such a set exists by 1.13a). Let \(\lambda \in L_1/L_1^0\) denote

\[
M^0_{x,\lambda} = m^0_x \cdot [\lambda]_\rho \prod_{r \in \Sigma^+ - \Sigma^+_L} \prod_{i=1}^n (\Psi_L (mr))_\rho - q_i.
\]

### Proposition 1.15. Let \(L_2 = w(L_1) \neq L_1, L \supset L_1\) be standard Levi subgroups, \(\rho \in \text{Cusp}_{L_1}\), and \(x \in K \backslash G/P\).

a) For any \(\lambda \in L_1/L_1^0\) the element \(R_w(M^0_{x,\lambda})\) is \(\Psi_L(mr)\)-regular (i.e. lies in \(\text{End}(\Pi_{w(\rho)}(\rho))\) rather than \(\text{End}(\Pi_{w(\rho)}) \otimes \text{Frac}(k[L_2/L_2^0])\)).

b) If \(L_2 \subset L\) then \(R_w(M^0_{x,\lambda}) = M^w_{x,w(\lambda)}\). Moreover, if the integer \(m\) used in the definition of \(M^0_{x,\lambda}\) is large enough, then

\[
M^w_{x,w(\lambda)} - m_x [\lambda]_w(\rho) \in \sum_{r \in \Sigma^+ - \Sigma^+_L} \tilde{H}_{\langle \lambda + 2n\delta - m \rangle r}.
\]

c) If \(\lambda \in X_L\) is dominant enough and \(L_2 \not\subset L\), then \(R_w(M^0_{x,\lambda}) \in \sum_{r \in \Sigma^+ - \Sigma^+_L} \tilde{H}_{\langle \lambda + 2n\delta - m \rangle r} \), where \(\delta = \frac{1}{2} \sum_{r \in \Sigma^+ - \Sigma^+_L} r\).

**Proof.** b) The first statement follows directly from Lemma 1.11 and 1.9. To see the second notice that

\[
\text{supp}(R_w(M^0_{x,\lambda} - m_x [\lambda]_w(\rho))) = \{ \lambda + \sum_{r \in \Psi_{L_1}(\Sigma^+ - \Sigma^+_L^+)} i_r mr \mid 0 \leq i_r \leq n \} -
\]
\[
\{ \lambda + \sum_{r \in \Psi_{L_1}(\Sigma^+ - \Sigma^+_L^+)} nm r \} \subset \bigcup_{r \in \Psi_{L_1}(\Sigma^+ - \Sigma^+_L^+)} a_{\langle \lambda + 2n\delta - m \rangle r}.
\]

(Here we used that \(\lambda, 2\delta \in X_L \subset X_{L_1}\), so \(\Psi_{L_1}(\lambda + 2n\delta) = \lambda + 2n\delta L\). It is well-known that \(\Psi_{L_1}\) preserves \(\leq\) (see e.g. [BW] Lemma 6.4 on p. 139, statement 2), so the statement is clear.

Let us prove a). Recall that the set of standard Levi conjugate to \(L_1\) is in bijection with chambers into which the root hyperplanes divide \(a_{L_1}\): the bijection sends a chamber \(C\) to \(L_C = w_C(L_1)\), where \(w_{C} \in W\) is an element such that \(w(C) \in a^+\) (such an element is defined uniquely up to right multiplication by an element of \(W_L\); we also have \(w_C(C) = a^+_L\)). Let \(C_+ = a^+_L\), \(C\) be the chambers
corresponding respectively to $L_1$, $L_2$. On the set of chambers we have the length function $\ell(C) = \#\{r \in \Psi_{L_i}(\Sigma^+) \mid \langle r, \alpha \rangle < 0 \text{ for } \alpha \in C\}$ (the latter condition will be abbreviated as $\langle r, C \rangle < 0$). We can choose a sequence of chambers $C_0 = C_1, C_2, \ldots, C_n = C$ so that $C_i$ and $C_{i+1}$ have a common face of codimension 1, and $\ell(C_i) = \ell(C_{i-1}) + 1$. Let $r_i \in \Psi_{L_i}(\Sigma^+)$ be the (only) element such that $\langle r_i, C_{i-1} \rangle > 0$, $\langle r_i, C_i \rangle < 0$, and $L_i = L_{C_i}$. The sequence $C_1, \ldots, C_n$ can be chosen so as to satisfy the following additional property:

$$r_i \in \Psi_{L_i}(\Sigma^+_L) \implies r_j \in \Psi_{L_j}(\Sigma^+_L) \& L_j \subset L \text{ for all } j < i. \tag{27}$$

Indeed, such a sequence can be constructed inductively as follows. Suppose that $C_1, \ldots, C_i$ are already chosen, and that $L_1, \ldots, L_i \subset L, r_1, \ldots, r_i \in \Psi_{L_i}(\Sigma^+_L)$. Assume first that there exists a coroot $r \in \Sigma_L$ separating $C_i$ from $C$ (i.e. $\langle r, \alpha \rangle < 0$ for $\alpha \in C_1, \beta \in C$). Then there exists also a coroot $r \in \Sigma^+_L$ which is orthogonal to a codimension 1 face of $C_i$, and separates $C_i$ from $C$, for otherwise all simple roots of $L$ are nonnegative on the cone $w_C(C)$, while some positive root of $L$ is negative there, which is impossible. Any (co)root separating $C_i$ from $C$ separates also $C_1$ from $C$; since $r \in \Sigma^+_L$ we get $\langle r, C_1 \rangle > 0$, $\langle r, C_i \rangle > 0$. We now take $r_{i+1} = \Psi_{L_i}(r) \in \Psi_{L_i}(\Sigma^+_L)$, and $C_{i+1}$ to be the chamber neighboring with $C_i$ and separated from $C_i$ by $r'$. It is clear that $L_{i+1} \subset L$.

If none of the roots separating $C_i$ from $C$ lies in $\Sigma_L$, we choose the remaining $C_{i+1}, \ldots, C_n = C$ arbitrary (keeping only the requirements $\dim(C_i \cap C_{i+1}) = \dim(C_i) - 1, \ell(C_{i+1}) = \ell(C_i) + 1$). Since $r_i$ separates $C_j$ from $C$ for $j < i$ we see that the resulting sequence of chambers satisfies (27).

Fix $(C_1, \ldots, C_n)$ satisfying (27); let $l$ be the smallest integer for which $r_j \not\in \Sigma^+_L$. We can write:

$$R_w(M^\rho_{x,\lambda}) = (R_{w_{C_n} \cdot w_{C_{n-1}}^{-1}}) \circ \ldots \circ (R_{w_{C_{i+1}} \cdot w_{C_i}^{-1}}) \circ R_{w_{C_i} (M^\rho_{x,\lambda})} =$$

$$= (R_{w_{C_n} \cdot w_{C_{n-1}}^{-1}}) \circ \ldots \circ (R_{w_{C_{i+1}} \cdot w_{C_i}^{-1}}) \circ \ldots$$

$$= \left( R_{w_{C_n} \cdot w_{C_{n-1}}^{-1}} \prod [(w_{C_{n-1}}(mr_n))w_{C_{n-1}}(\rho) - q_j] \right) \circ \ldots$$

$$= \left( w_{C_1}(\lambda)w_{C_1}(\rho) \prod_{r \in \Psi_{L_1}(\Sigma^+), r \neq r_j} \prod_{j=1}^n [(mr)w_{C_1}(\rho) - q_j] \right). \tag{28}$$

Notice that $L_{i-1}, L_i$ generate a standard Levi $L^i$ of rank equal to $\dim(L_i) + 1$, $w_{C_i}w_{C_{i-1}}^{-1} \in W_{L_{i-1}}$. $w_{L_{i-1}}(r_i)$ is the only simple root of $L^i$ which is not a root of $L_{i-1}$. Hence, by the choice of $q_1, \ldots, q_n$, each of the operators $R_{w_{C_i} \cdot w_{C_{i-1}}^{-1}} \prod [(w_{C_{i-1}}(mr_i))w_{C_{i-1}}(\rho) - q_j]$ is regular, and thus the whole expression (28) is regular.

Let us prove c). We show that for $C_1, \ldots, C_n$, $l$ as above and $i > l$

$$\mu \in \supp(w_{C_1}(M^\rho_{x,\lambda})) \Rightarrow \exists \nu \in \supp(w_{C_{i-1}}(M^\rho_{x,\lambda})), r \in \Sigma^+ - \Sigma^+_L : \mu \preceq \nu - ar.$$
Recall that \( w_{C_{i-1}}^{-1} w_{C_i}^{-1} \in w_{C_l}^{-1} \cdot W_{L_l}; \) also the vector \( \alpha := w_{C_{i-1}}(r_i) \in X_L \) is orthogonal to \( X_{L_l}. \) Thus by Lemma 1.13b there exists \( \nu \in \text{supp}(w_{C_{i-1}}(M_{x,\lambda}^{\rho})) \) such that \( \nu = w_{L_l}^L(\mu) + N\alpha, \) where \( |N| < n_0. \) Decompose \( \nu \) as \( \nu = \mathcal{P}_L(\nu) + \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \)

Let \( \tilde{r}_i \in \Sigma^+ \) be a coroot such that \( \mathcal{P}_{L_{i-1}}(\tilde{r}_i) = r_i. \) Then \( w_{C_{i-1}}(\tilde{r}_i) \succ 0 \succ w_L^L(\alpha) = (w_{L_l}^L)^{-1}(\alpha)\mathcal{P}_{L_{i-1}}(w_{C_i}(\tilde{r}_i)). \) Moreover, since \( \alpha \) belongs to a fixed finite subset of \( \mathfrak{a}^+_L, \) there exists \( b > 0 \) and \( r \in \Sigma^+ - \Sigma^+_L \) such that \( \alpha \succ b r. \)

To finish the proof it is enough to show that \( \langle \nu, \alpha \rangle \gg 0 \) if \( \lambda \in X_L \) is very dominant; in fact it is enough to ensure that

\[
\langle \nu, \alpha \rangle \gg \left( \frac{a}{b} + n_0 \right) \langle \alpha, \alpha \rangle, \tag{29}
\]

for then we get \( \mu = (w_{L_l}^L)^{-1}(\nu - N\alpha) = \mathcal{P}_L(\nu) + \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \mu - \mathcal{P}_L(\nu) \gg \mathcal{P}_L(\nu) + \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \gg \nu - ar. \)

To check (29) notice that from 1.13b) it follows by induction on \( i \) that for \( w = w_{C_i} \) there exists a finite set \( S \) such that \( \text{supp}(R_w(M_{x,\lambda}^\rho)) \) is contained in the convex hull of \( w(\text{supp}(M_{x,\lambda}^\rho)) \cup S \) for any \( \lambda, \rho \) (where we used the notation \( A + B = \{ a + b | a \in A, b \in B \} \)). Hence

\[
\min_{\nu \in \text{supp}(R_{w_{C_{i-1}}}(M_{x,\lambda}^\rho))} \langle \nu, \alpha \rangle \leq \min_{\nu \in \text{supp}(M_{x,\lambda}^\rho)} \langle \nu, \alpha \rangle + \text{const.}
\]

Since \( \alpha = w_{C_{i-1}}(r_i) \) we have

\[
\min_{\nu \in \text{supp}(M_{x,\lambda}^\rho)} \langle \nu, r_i \rangle \gg 0 \quad \text{if} \quad \nu \in X_L \quad \text{is very dominant, because our assumption} \quad i > l \quad \text{implies that} \quad r_i \in \Sigma^+ - \Sigma^+_L.
\]

This proves (29). \( \Box \)

1.16. To deduce Proposition 1.7 from 1.15 we take

\[
h_{x,\lambda}^\rho := \sum_{L_2 = w(L_1)} R_w(M_{x,\lambda}^\rho). \tag{30}
\]

From Proposition 1.15 it follows that \( h_{x,\lambda}^\rho \in \mathcal{H} \) and that \( h_{x,\lambda}^\rho - \sum_{L_2 = w(L_1) \subset L} [\lambda]_{w(\rho)} w_{x}^{w(\rho)} \in \sum_{r \in \Sigma^+ - \Sigma^+_L} \mathcal{H}_{\xi^\rho, \lambda + 2n \delta^L - ar}. \) It is obvious that \( h_{x,\lambda}^\rho \) is invariant under all the intertwining operators \( R_w \) (more precisely \( R_w(b\rho') = h_{w(b\rho')} \)) where \( h_{x,\lambda}^\rho = \sum_{C_{\text{sup}} \rho, \rho'} h_{x,\lambda}^\rho, \) \( h_{x,\lambda}^\rho \in \text{End}(\Pi(x')) \). Also it is immediate to check that \( T_\psi(m_{x}^{\rho}) = m_{x}^{\rho}, R_w \circ T_\psi \circ R_w^{-1} = T_{w(\psi)} \) when \( T_\psi \) is defined. If \( \rho \otimes \psi \cong \psi \) for some irreducible representation \( \rho \) of a standard Levi \( L, \) then \( \psi|_{L_L}(x_L) = 1, \) hence for \( \lambda \in X_L \) we have \( T_\psi([\lambda]_{\rho}) = [\lambda]_{\rho \otimes \psi}. \)

This implies that each summand in (30) is invariant under \( T_\psi \) when the latter is defined, provided \( \lambda \in X_L. \) By the matrix Payley-Wiener Theorem we conclude that \( h_{x,\lambda}^\rho \in I(\mathcal{H}). \) Since \( 2\delta^L \in X_L \) the coweight \( \lambda + 2n\delta^L \) runs over the set of very dominant elements of \( X_L \) when \( \lambda \) does. Proposition 1.7 is proved. \( \Box \)
Corollary 1.17. Set

\[ h_x^\mu := \sum_{L_1 \in L, \rho \in Cusp_{L_1}} \frac{1}{\#\{L' \subset L | L' \in \mathbb{W}_L(L_1)\}} h_{x, \lambda}^\rho. \]

Then we have

\[ I(h_x^\mu) - [\lambda]m_x \in \sum_{r \in \Sigma^+ - \Sigma_0^+} \mathcal{H}_{\leq \lambda - a - r} \]

provided \( \lambda \in X_L^+ \) is dominant enough, and the integer \( m \) used in (26) is large. \( \square \)

1.18. We next want to consider associated graded objects, so we have to pass to a complete order.

Let \( \preceq \) be any complete order on \( X \) strengthening \( \leq \). It yields filtrations on \( \mathcal{H}, \mathcal{H} \) as in 1.1–1.6.

We write \( gr\mathcal{H} \) for the associated graded algebra of \( \mathcal{H}, \leq \), and use the letter \( I \) for the induced imbedding \( gr\mathcal{H} \hookrightarrow gr\mathcal{H} \cong \mathcal{H} \).

We have an obvious map \( K_0 \to gr_0\mathcal{H} \).

Theorem 1.19. a) \( gr\mathcal{H} \) is Noetherian.

b) Let \( M \) be any finitely generated graded module over \( gr\mathcal{H} \). Then for all but finitely many \( \lambda \in X \) we have \( Tr(g, M_\lambda) = 0 \).

Proof Let \( Z \) be the center of \( \mathcal{H} \).

The explicit description of \( Z \) (Theorem 2.13 of [BD]) implies that the following elements form a basis in \( I(gr Z) \subset gr\mathcal{H} \cong \mathcal{H} \). For a standard Levi \( L, \rho \in Cusp^L \), \( \lambda \in (L/L^\rho)^+ \) such that \( \psi(\lambda) = 1 \) whenever \( \psi \otimes \rho \cong \rho \) denote \( z_{\lambda}^\rho := \sum_{w(\lambda) = \lambda} [\lambda] w(\rho) \otimes \psi_w(\rho) \),

where \( w \) runs over the set of such \( w \in W/W_L \) for which \( w(\lambda) \) is a standard parabolic and \( w(\lambda) = \lambda \).

It follows that for \( \lambda \in X^+ \) we have \( [\lambda] \in gr Z \).

It is clear that \( gr Z \) is central in \( gr\mathcal{H} \), and (as we will soon see) it is of finite type.

Thus to prove a) it suffices to show that \( gr\mathcal{H} \) is a finite \( gr Z \) module. The main step is the next:

Lemma 1.20. There exists \( \lambda_0 \in a \) such that \( gr\mathcal{H}_\lambda \neq 0 \Rightarrow \lambda \in \lambda_0 + a^+ \).

Proof of the Lemma. Assume the contrary. Then for any \( N > 0 \) we can find \( \lambda \in X \) such that \( (\lambda, r) < -N \) for a simple coroot \( r \), and \( gr\mathcal{H}_\lambda \neq 0 \). Let \( \lambda \) be like that, then there exists \( h \in \mathcal{H} \) such that \( \lambda \in \text{supp}(h) \subset X_{\leq \lambda} \). Decompose \( h = \sum h_{\rho}, h_{\rho} \in \text{End}(H_{\rho}) \), and fix \( \rho \in Cusp^L \) for which \( \text{supp}(h_{\rho}) \ni \lambda \). Since \( (r, \lambda) \neq 0 \) we have \( r \not\in \Sigma_L \). Let \( L' \) be the standard Levi such that \( L' \supset L, \text{rank}(L') = \text{rank}(L) + 1 \) and \( r \in \Sigma_{L'} \). Then writing \( \lambda = \mathfrak{L}_L(L) + \frac{(\lambda, r)}{[\mathfrak{L}_L(L), \mathfrak{L}_L(r)]} \mathfrak{L}_L(r) \) we get by 1.13b) an element \( \nu \in \text{supp}(R_{w_0(L)}(h)) = \text{supp}(h) \) such that \( \nu = w_0(L) (\lambda) + N' w_0(L) (\mathfrak{L}_L(r)) = \mathfrak{L}_L(L) + \frac{(\lambda, r)}{[\mathfrak{L}_L(L), \mathfrak{L}_L(r)]} + N' \mathfrak{L}_L(r) \), where \( |N'| < n_0 \). We have \( \mathfrak{L}_L(L) > 0, \mathfrak{L}_L(r) = w_0(L) (\mathfrak{L}_L(r)) < 0 \), hence taking \( N > n_0 \cdot [\mathfrak{L}_L(L), \mathfrak{L}_L(r)] \) we obtain \( \nu > \mathfrak{L}_L(L) \Rightarrow \nu > 0 \). This contradicts the condition \( \text{supp}(h) \subset a_{\leq \lambda} \). \( \square \)

Lemma 1.21. For any \( L \) fix \( \lambda_L \in X_L^+ \). The subalgebra \( gr\mathcal{Z}_L \subset gr Z \) generated by \( [\lambda] \) for \( \lambda : \lambda_L + X_L^+ \), \( L' \subset L \) is of finite type. The algebra \( \mathcal{Z}_L := \)
\[ \bigoplus_{L', \rho \in \text{Cusp}_L^*} \bigoplus_{\lambda \in (L'/L) \cap a_L^+} k \cdot [\lambda]_\rho \subset \tilde{H}, \text{ where } L' \text{ runs over the set of standard Levi,} \]

is a finite \( grZ_L \)-module.

**Proof** follows from the next easy fact. Fix \( n \in \mathbb{Z}_{\geq 0} \) and consider the subsemigroup in \( \mathbb{Z}_{\geq 0}^m \) generated by \( \mathbb{Z}_{\geq 2n}^I \), where \( I \) runs over the subsets of \([1, m]\). This semigroup is finitely generated, and \( \mathbb{Z}_{\geq 0}^m \) is the union of a finite number of cosets of this semigroup. \( \square \)

It is clear that \( \bigoplus_{\lambda \in \lambda_0 + a_*} \text{gr} H_\lambda \) is a finite module over \( Z_G \). Hence by 1.21 it is also finite over \( gr Z_G \). By 1.20 we have \( gr \mathcal{H} \subset \bigoplus_{\lambda \in \lambda_0 + a_*} \text{gr} H_\lambda \) for some \( \lambda_0 \) and by 1.21 \( gr Z_G \) is Noetherian, hence \( gr (H) \) is finite over \( gr Z_G \). a) is proved. Notice that along the way we have shown also that \( gr \mathcal{H} \) is finite over \( gr Z \supset gr Z_G \).

Proof of b). Consider a subalgebra \( S_L \subset \mathcal{H} \) generated by the highest terms of the elements \( h_\lambda^\mu \) (see 1.17), with large \( \lambda \in X_L^+ \), i.e. by the elements \( [\lambda] m_x \).

Notice that \( S_L \supset gr Z_L \) (for appropriate choice of coweights \( \lambda_L \) in 1.21), because \( \sum_{x \in K(G/P)} [\lambda] m_x = [\lambda] \). Hence \( gr \mathcal{H} \) is finite over \( S_L \), and thus any finite \( gr \mathcal{H}_L \)-module is also finite over \( S_L \).

We prove by induction on corank of \( L \), that for any graded finitely generated \( S \)-module \( M \) we have \( Tr(g, M_\lambda) = 0 \) for all but finitely many \( \lambda \in X_L \). The step of induction is as follows.

Let \( M \) be an \( X_L \)-graded finitely generated \( S_L \)-module.

By the definition \( S_L \) contains for some \( \lambda \in X_L^+ \) the subalgebra \( A_L[\lambda_0 + X_L^+] \), where

\[ A_L := \bigoplus_{x \in K \setminus G/P} k \cdot m_x. \tag{31} \]

Hence the proof of the Hilbert Basis Theorem implies that for any graded finitely generated \( S_L \)-module \( M \) there exists \( \mu_0 \) and a finite \( A_L \)-module \( M_0 \) such that \( M_\mu \cong M_0 \) for \( \mu \in \mu_0 + X_L^+ \), this isomorphism being compatible with the action. In other words we have an isomorphism of graded \( A_L[\lambda_0 + X_L^+] \)-modules \( \bigoplus_{\mu \in \mu_0 + X_L^+} M_\mu \cong M_0 [\mu_0 + X_L^+] \).

Further, for \( \lambda \in \lambda_0 + X_L^+ \), \( M_\lambda \cong M_0 \) is the direct sum of the images of idempotents \( m_x \in A_L \), and we have \( g(\text{Im}(m_x)) = \text{Im}(m_{g(x)}) \). We have assumed that \( g \) acts on \( K \setminus G/P \) without fixed points, hence \( tr(g, M_\lambda) = 0 \).

It is obvious that \( \Lambda - \cup (\lambda_0 + X_L^+) \) is a finite union of cosets \( \mu_i + X_L^+, L_i \supset L \). Then \( \bigoplus_{\lambda \in \mu_i + X_L^+} M_\Lambda \) is a finitely generated graded module over \( S_L \), because it is contained in \( \bigoplus_{\lambda \in \mu_i + a_L^+} M_\Lambda \), which is finite over \( gr Z \). Hence the statement for \( M \) follows from the induction hypothesis. The Theorem is proved. \( \square \)

Now for any finitely generated graded module \( N \) over \( gr \mathcal{H} \) we can define \( Tr(g, N) := \sum_{\lambda} Tr(g, N_\lambda) \) the sum being finite by the last Theorem.

Let \( M \) be a finitely generated \( \mathcal{H} \)-module.
1.22. We can choose an $(\alpha, \preceq)$-filtration $F$ on $M$ compatible with $\preceq$ so that associated graded module $gr_F M$ is a finite $gr H$-module (such a filtration is called a good filtration).

1.23. From now on we assume that the complete order $\preceq$ is induced by the lexicographic order under the isomorphism $a \cong \mathbb{R}^r, \lambda \mapsto ((e_1, \lambda), \ldots, (e_r, \lambda))$, where $(e_1, \ldots, e_r)$ is a basis in $(X \otimes \mathbb{Q})^*$, such that $(e_i, r) \geq 0$ for $r \in \Sigma^+$.

**Proposition 1.24.** a) $\text{Tr}(g, gr_F M)$ does not depend on $F$.

We set $\text{Tr}(g, M) := \text{Tr}(g, gr_F M)$. Then $\text{Tr}(g, M)$ satisfies:

b) 0.23i), and
c) 0.23ii).

**Proof** Recall the following:

**Fact 1.25.** Let $H$ be an algebra with an increasing $\mathbb{Z}$-filtration bounded below (i.e. $F_{\leq n} = 0$ for $n \ll 0$), such that the associated graded algebra is Noetherian; let $M$ be a finite $H$-module.

Let $gr(H)$-Mod be the category whose objects are finitely generated graded $gr(H)$-modules and morphisms are homomorphisms of modules preserving the degree up to a shift (i.e. $\text{Mor}(M, N) = \sum \text{Mor}(M, N)_\nu$ where $\text{Mor}(M, N)_\nu = \{ \phi : M \rightarrow N \mid \phi(M_\lambda) \subset N_{\lambda + \nu} \}$). Then the class of $gr M$ in the Grothendieck group $K^0(gr(H)$-Mod) does not depend on the choice of a good filtration on $M$.

Thus $[M] \rightarrow [gr M]$ yields a well-defined homomorphism from $K^0(H\text{-Mod})$ to $K^0(gr H\text{-Mod})$.

**Proof**: see e.g. [CG] Corollary 2.3.19 on p. 79. (The authors of loc. cit. consider the Grothendieck group of the category of finitely generated $gr H$-modules without grading; however their argument actually proves the above statement as well.)

We apply 1.25 inductively.

Let the sublattice $\Lambda_0 \subseteq \alpha$ be generated by $\Lambda_L$ for all $L$. It is obvious that the set of indices $\lambda$ for which $gr H_\lambda \neq 0$ is contained in $\Lambda_0$.

For $i \in \{1, \ldots, n\}$ let $\Lambda^{(i)}$ be the image of $\Lambda_0$ in $\mathbb{Q}^i$ under $pr_i : \lambda \mapsto ((\lambda, e_1), \ldots, (\lambda, e_i))$; it is convenient to choose $e_i$ so that $\Lambda^{(i)} \subset \mathbb{Z}^i$. Equip $\Lambda^{(i)}$ with the lexicographic order $\preceq$ induced from $\mathbb{Z}^i$. We have a $\Lambda^{(i)}$-filtration $F^{(i)}(H) = \bigcup_{n \geq \lambda, \text{pr}_i(\lambda)} H_{\lambda}$ on $H$; notice that $F^{(i)}$ is induced by the corresponding $\Lambda^{(i)}$-filtration on $\tilde{H}$.

Furthermore, consider the one parameter filtration $\tilde{H}$ given by $\tilde{H}_{i+1} = \bigoplus_{(\lambda, e_i) \leq n} \tilde{H}_\lambda$, and the induced filtration on $H$ denoted by the same symbol.

It yields a filtration on $gr_{F^{(i)}} H$ given by the rule: $gr_{F^{(i)}} H \leq n = \bigoplus_{X \in X^{(i)}} [gr_{F^{(i)}} H]_{X^{(i)}} \leq n$ where $[gr_{F^{(i)}} H]_{X^{(i)}} \leq n = F^{(i)}_{\lambda}(H) \cap H_{i+1} / F^{(i)}_{\lambda}(H) \cap H_{i+1} .

Notice also that this filtration on $gr_{F^{(i)}} H$ is induced by the $\preceq$-filtration on $\tilde{H} \cong gr_{F^{(i)}} (\tilde{H})$ under the imbedding $gr_{F^{(i)}} (I)$.

We see that $gr_{F^{(i+1)}} H = gr_{i+1} [gr_{F^{(i)}} H]$. 

If now a good $X$-filtration $F$ on an $H$-module $M$ is given, then again we can form $\Lambda^{(i)}$-filtrations $F^{(i)}(M)$, where $F^{(i)}(M) = \bigcup_{\lambda, pr_3(\lambda) = \lambda} M_{\leq \lambda}$. Consider the associated graded $gr_{F^{(i)}} M$; on $gr_{F^{(i)}} M$ we have a one parameter filtration denoted by $\leq_i$, and given by $[gr_{F^{(i)}} M]_{\leq i} n = \bigcup_{pr_3(\lambda) = \lambda, (e_{i+1}, \lambda) \leq n} M_{\leq \lambda} = \bigcup_{pr_3(\lambda) = \lambda, (e_{i+1}, \lambda) \leq n} M_{\leq \lambda}$.

By inverse induction in $i$ we see that $\leq_i$ is good; hence using 1.25 and induction in $i$ we get the following statement:

The class of $gr^{(i)} F$ in the Grothendieck group of the category of $\Lambda^{(i)}$-graded $gr^{(i)} H$-modules, with morphisms preserving grading up to a shift, is independent on $F$.

(Actually to make a step of induction we need a variant of 1.25 applicable in the situation when $H$ and $M$ have some multi-grading respected by the filtration while $gr H$ and $gr M$ are considered as multi-graded objects with one additional index. One may say that what we need is exactly the statement 1.25 but with algebras/modules being algebras/modules in the category of multigraded vector spaces rather than in the category of vector spaces).

a), b) follow from the last italicized statement directly. c) is obvious: if $\mathfrak{M}$ is an admissible representation, and $M = \mathfrak{M}^K$ then for large $\lambda$ we have:

$$Tr(g, M) = Tr(g, F_{\leq \lambda}(M)) = tr(g, M) = tr(\delta_k \cdot g, \mathfrak{M}) = 
\int_{g' \in K} \chi\mathfrak{M}(g'g) dg' = \chi\mathfrak{M}(g)$$

provided the integrand is constant. $\square$

2. Geometric filtration and asymptotic cones. In 1.24 we have defined a functional $Tr(g, M)$ satisfying property i), ii) of Theorem 0.23; it remains to prove that it satisfies also property iii). This will be accomplished in the next two sections by passing from the “spectral” filtration constructed above to a “geometric” one defined in terms of support of a distribution $h \in H$; the latter filtration is directly related to orbital integrals.

The main results of this section used in further arguments are Theorem 2.7, Proposition 2.12, and Theorem 2.15.

2.1. We can assume that $K$ is nice with respect to $L_0$ (nice in the terminology of [BD] 2.1b), i.e. that for any pair of opposite parabolics $P = L \cdot U, P^- = L \cdot U^-$, with $L \supset L_0$ standard we have $K = K^+ \cdot K^0 \cdot K^-$ where as usual $K^+ = U \cap K$, $K^- = U^- \cap K$, $K^0 = L \cap K$. (More precisely, the condition that $K_0, L_0$ are in good relative position implies that there exists a $K_0$-stable lattice $g \subset \text{Lie} G$ such that $g = (g \cap \text{Lie} U^-) \oplus (g \cap \text{Lie} L) \oplus (g \cap \text{Lie} U)$; we can take $g$ to be the Lie algebra of the $O$-algebraic group attached to the $K_0$-fixed point $p$ of the building by the Bruhat-Tits theory [BT2]. We then obtain $K$ from $p^N g$ for large $N$ as in [BD][2.1b]).

Fix opposite parabolics $P = L \cdot U, P^- = L \cdot U^-$ with $P, L$ standard, and set $C_p = (G/U \times G/(U^-))/L$. By $H(C_p)$ we denote the space of $K$-biinvariant compactly supported measures on $C_p$.

Consider the imbedding $\psi: L \hookrightarrow C_p$ given by $\psi: l \mapsto (l \cdot U, 1 \cdot U^-)$ mod $L$. 
Let $T \subseteq L_0$ be the (unique) maximal split torus contained in $L_0$. For $\lambda \in X$ let $T_\lambda \in T/T^c$ be the coset of $\lambda$. (Recall that we have an imbedding $X \rightarrow T$, $\chi \mapsto \chi(p)$).

**Proposition 2.2.** Let $C \subset G$ be a compact $K$-biinvariant set. For $\lambda \in X^+$ such that $\left< \lambda, \Sigma^+ - \Sigma^+_L \right> \gg 0$, and $t \in T_\lambda$, there exists a unique bijection $\Psi_{\lambda,t} : K \cdot C \cdot t \cdot C/K \rightarrow K \cdot C \cdot \psi(t) \cdot C/K$ which satisfies $\Psi_{\lambda,t}(K \cdot c_1tc_2 \cdot K) = K \cdot c_1\psi(t)c_2 \cdot K$ for $c_1, c_2 \in C$.

**Proof.** We have only to check that

$$K \cdot g_1tg_2 \cdot K = K \cdot g_1'g_2' \cdot K \iff K \cdot g_1(\psi(t))g_2 \cdot K = K \cdot g_1'(\psi(t))g_2' \cdot K$$

for $g_1, g_2, g_1', g_2' \in C$.

Set $C' = \{g_1g_2^{-1}|g_1, g_2 \in C\} \cup \{g_2g_1^{-1}|g_1, g_2 \in C\}$, and $K' := \bigcap \{gK \cdot K \}$

for $g \in C$. It is clearly enough to verify that

$$g_1(\psi(t))g_2 = \psi(t), \ g_1, g_2 \in C' \implies g_1tg_2 \in K' \cdot t \cdot K' \quad \quad (32)$$

$$g_1tg_2 = t, \ g_1, g_2 \in C' \implies \Psi(\psi(t))g_2 \in K' \cdot \psi(t) \cdot K' \quad \quad (33)$$

If equality in the LHS of (32) holds then we have: $g_1 = tu$, $g_2 = t^{-1}Lu$ where $u \in U$, $u^- \in U^-$, $l \in L$. Since $g_1, g_2$ lie in the compact set $C'$, the elements $l, t^{-1}lt, u, u^-$ are in some bounded subsets of $L, U, U^-$. Hence the condition $\left< \lambda, \Sigma^+ - \Sigma^+_L \right> \gg 0$ implies that $(t(t^-)u(t^-)) \in K$, $(lt)u^-(lt)^{-1} \in K$, which yields $g_1tg_2^{-1} = [(lt)u^-(lt)^{-1}]^{-1}t^{-1}ut \in K\cdot t\cdot K'$.

If equality in the LHS of (33) holds, then $g_1 = t(g_2^{-1}t^{-1} \in C' \cap tCt^{-1}$. It is well-known that for $t \in T_\mu$, $\left< \mu, \Sigma^+ - \Sigma^+_L \right> \gg 0$ we have $C' \cap tCt^{-1} \subset C\cdot \cdot C\cdot C\cdot C\cdot C\cdot C\cdot C\cdot$, where $C^+, C^L$ are bounded (independently of $t$) subsets of respectively $U$ and $L$, and $C^-$ is an arbitrary small subset of $U^-$; in particular we can assume $C^- \subset (K')^-$. Thus we have $g_1 = u^- \cdot l \cdot u$ with $u^- \in K^-, u \in C^L$. Then $g_2 = t^{-1}g_1$, and we can assume $t^{-1}ut \in (K')^+$. So finally we get $(g_1, g_2)(\psi(t)) = (u^-, t^{-1}ut)(\psi(t)) \in K' \times (K')^\prime$.

If $t \in T_\mu$ where $\left< \mu, \alpha \right>$ is so large for $\alpha \in \Sigma^+ - \Sigma^+_L$ that $\Psi_{\mu,t}$ is defined for $t \in T_\mu$ and $C' := \{g_1g_2^{-1}|g_1, g_2, g_3 \in C\}$, then obviously $\Psi_{\lambda,t}$ and $\Psi_{\mu,t'}$ agree on the intersection $\psi(t) \cdot C' \cap \psi(t') \cdot C'$ whenever this intersection is nonempty. Thus we have a bijection $\Psi : K \cdot \{ \sum \cdot T_\lambda \cdot \cdot C/K \rightarrow K \cdot \{ \sum \cdot \psi(T_\lambda) \cdot C/K \cdot (\lambda, \Sigma^+ - \Sigma^+_L) \gg 0 \}$.

2.3. The map $w \mapsto K_0 \cdot w \cdot K_0$ yields a canonical bijection

$$W_p \cdot W_a \cdot W_p \rightarrow K_0 \cdot W_a \cdot K_0$$

see [BT1] 7.4.15 (notice that since $K_0$ and $L_0$ are in good relative position we have $K_0 \subseteq L_0^c$, so the expression $K_0 \cdot w \cdot K_0$ is meaningful).

Consider the projection $pr_L : W_p \cdot W_a \cdot W_p \rightarrow W_p \cdot W_a \cdot W_p = \Lambda / \Lambda \cdot W_0 = \Lambda^+$. For $x \in W_p \cdot W_a \cdot W_p$ we write $G_x$ for the corresponding double coset in $G_0$, $H_x$ for the space of elements in $H$ supported in $G_x$, and we set $G_\lambda := \bigcup_{pr_L(x) = \lambda} G_x = \bigcup_{pr_L(x) = \lambda} H_x$.

(Support of an element $h \in H$ appearing here should not be confused with support of the corresponding element $I(h) \in H$ considered before (see (25)): the
first one is an open compact subset of $G$ while the second is a finite subset of $X$. To rule out the very possibility of such confusion we will sometimes refer to the first notion as to the geometric support, and to the second one as to the spectral support and write respectively $\text{supp}^{\text{spec}}$ and $\text{supp}^{\text{geom}}$.

Let us choose $\mathcal{C}$ to be large enough; more precisely, we require the following. We can choose a finite number of elements $w_i \in W_{\text{aff}}$, so that for any $w \in W_{\text{aff}}$ there exist $i_1, i_2$ such that $w_i, w w_{i_2} \in X^+$. Let us take $\mathcal{C}$ to be $\bigcup \{ (K_0 \cdot w_i) \cup \bigcup_i (w_i K_0) \}$. Then we have

$$\bigcup_{i \in \mathcal{C}} \mathcal{C} \cdot T_\lambda \cdot \mathcal{C} \supset \bigcup_{\mu \in \mathfrak{a}^+, (\mu, \Sigma^+ - \Sigma_L^+)} G_\mu.$$

Denote $(C_P)_\mu := \bigcup_{g \in G_\mu} \Psi(K g K), \mathcal{H}(C_P)_\mu = \{ f \in \mathcal{H}(C_P) \mid \text{supp } f \subset (C_P)_\mu \}.

Thus we have a bijection

$$\bigcup_{\mu \in \mathfrak{a}^+, (\mu, \Sigma^+ - \Sigma_L^+)} K \setminus G_\mu / K \overset{\sim}{\rightarrow} \bigcup_{\mu \in \mathfrak{a}^+, (\mu, \Sigma^+ - \Sigma_L^+)} K \setminus C_\mu / K.$$

Let us set

$$\mathcal{H}_N^P = \bigoplus_{(\mu, \alpha) \geq N, \alpha \in \Sigma^+ - \Sigma_L^+} \mathcal{H}_\mu, \mathcal{H}(C_P)_N = \bigoplus_{(\mu, \alpha) \geq N, \alpha \in \Sigma^+ - \Sigma_L^+} \mathcal{H}(C_P)_\mu,$$

and use the notation $\Psi$ for the isomorphism $\mathcal{H}_N^P \overset{\sim}{\rightarrow} \mathcal{H}(C_P)_N$ sending the $\delta$-function of a double $K$-coset $x$ on $G$ to the $\delta$-function of $\Psi(x)$. (Here $N \gg 0$).

**Lemma 2.4.** Let $\mathcal{C} \subset G$ be compact. Let $N$ be so large that $\Psi_{\mathcal{C}, \mathcal{C}, t}$ is defined for $(\mu, \Sigma^+ - \Sigma_L^+) \geq N$, $t \in T_\mu$ where $\mathcal{C} \cdot \mathcal{C} = \{ c_1 c_2 \mid c_1, c_2 \in \mathcal{C} \}$. Then we have $\text{supp}(h) \subset \mathcal{C}$, $h' \in \mathcal{H}_N^P \Rightarrow \Psi(h \ast h')$, $\Psi(h' \ast h)$ are defined and

$$\Psi(h \ast h') = h \ast \Psi(h'),$$

$$\Psi(h' \ast h) = \Psi(h') \ast h$$

*Proof* follows directly from 2.2. □

**Remark 2.5.** Asymptotic semigroups. Before going on with the argument let us explain the geometric picture standing behind it.

Recall from [V1] that for a semisimple algebraic group $G$ there exists a canonical way to degenerate $G$ into one of its asymptotic cones $\overline{C}_P$ isomorphic to the affine closure of the algebraic variety $(G/U \times G/U^-) / L$.

More precisely, in [V1] Vinberg constructs the so called universal semigroup $\tilde{G}$ enjoying the following properties. (He works over an algebraically closed field of characteristic $0$; his construction carries over immediately to the case of a split group over a field of characteristic $0$).

Let $R$ be the set of simple roots of $G$; let $T \cong (\mathbb{A}^1)^R$ be the canonical partial compactification of the abstract Cartan group of the adjoint group $G / A$ (where $A$ is the finite center of $G$). Notice that $T$ has a natural “coordinate cross” stratification, with the set of strata being in bijection with subsets of $R$, i.e. with conjugacy classes of parabolics. For a parabolic $P$ let $T_P \subset T$ be the corresponding stratum, and let $e_P \in T_P$ be the point all of whose coordinates are either $0$ or $1$ (so the coordinate of $e_P$ corresponding to a (co)root $r \in \Sigma_L$ is $0$, and the one corresponding to a (co)root in $\Sigma - \Sigma_L$ is $1$).
\( \tilde{G} \) is an algebraic semigroup equipped with a flat morphism \( \varphi : \tilde{G} \to T \), such that the preimage of the open stratum is isomorphic to \( G \times A \ T \). Further, for \( x \in T \) the preimage \( \varphi^{-1}(x) \) is isomorphic to the algebraic variety \( (G/U \times G/U^-)/L \).

Now the canonical bijection \( \Psi \) can be characterized as follows.

For any parabolic \( P \) there exists an \( A_L \)-invariant open neighborhood \( \mathcal{U}_P \) of \( \varphi^{-1}(e_P) \), and a continuous \( A_L \)-equivariant projection \( \pi_P : \mathcal{U}_P \to \varphi^{-1}(e_P) \), such that \( \pi_P|_{\varphi^{-1}(e_P)} = \text{id} \), and the following holds.

Whenever \( KgK \subset \mathcal{U}_P \) and \( KxK \subset \mathcal{U}_P \) for \( g \in G \) and \( x \in C_P \), where \( P' \supset P \), we have \( KxK = \Psi(KgK) \iff \pi_P(KxK) = \pi_P(KgK) \). In other words \( KxK \) and \( KgK \) are close in the topology of \( G \).

Notice also that the domain of definition of \( \Psi : K\backslash G/K \to K\backslash(C_p)/K \) is a neighborhood of the stratum corresponding to \( P \) in the DeConcini-Procesi compactification of \( G \) [DCPr].

Probably it is possible to work directly with this geometric definition; we however found it more economical (though less transparent) to use the techniques employed here.

**Lemma 2.6.** Suppose that \( l \in L_\mu \), where \( \langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0 \). Then \( \Psi(KlK) = K\psi(l)K \).

**Proof.** Obvious. \( \Box \)

For a partial order \( \ll \) on \( a \) we write \( G_{<a} = \bigcup_{\mu \ll a} G_\mu \), \( \text{Geom}_{<a}(\mathcal{H}) = \bigoplus_{\mu \ll a} \mathcal{H}_\mu \).

We are now ready for the proof of the next

**Theorem 2.7.** For dominant enough \( \lambda \) we have \( \mathcal{H}_{<\lambda} \cdot \text{Geom}_{<\lambda}(\mathcal{H}) \subset \text{Geom}_{<\lambda+\mu}(\mathcal{H}) \).

**Lemma 2.8.** For \( h \in \mathcal{H} \) and \( \lambda \in \mathfrak{a}^+ \) the following are equivalent:

1. \( h \in \mathcal{H}_{<\lambda} \).
2. For any parabolic, and any function \( f \in C_c^\infty(G/U)^K \) we have \( \text{supp}(f) \subset (G/U)_\mu \Rightarrow \text{supp}(h \ast f) \subset \bigcup_{\nu \ll \lambda+\mu} (G/U)_\nu \).

**Proof.** Assume that i) holds. Recall that for any \( \rho \in \text{Cusp}^K \) we have \( \Pi^K_\mu = C_c^\infty(G/U)^K \otimes_{\mathcal{H}(L_\gamma)} R_p \), the grading on \( \Pi_\mu \) comes from the grading on the first multiple in the RHS, given by \( C_c^\infty(G/U)^K = \bigoplus_{\mu} C_c^\infty(G/U)^K_\mu \). i) \( \Rightarrow \) i) is now obvious.

Conversely, suppose that ii) does not hold, i.e. for some \( \nu \neq \lambda \) there exists \( f \in C_c^\infty(G/U)^K \) such that \( \text{supp}(h \ast f) \cap (G/U)_{\lambda+\nu} \neq \emptyset \). Thus the morphism of free \( \mathcal{H}(L_\gamma) \)-modules \( C_c^\infty(G/U)^K_{\mu} \to C_c^\infty(G/U)^K_{\mu+\nu} \), \( f \mapsto h \ast f|_{(G/U)_{\mu+\nu}} \), is nonzero. Hence (recall that \( \mathcal{H}(L_\gamma) \) is finite over its center) there exists an irreducible representation \( \rho \) of \( L_\gamma \) such that the tensoring the latter morphism over \( \mathcal{H}(L) \) with \( \rho \) we still obtain a nonzero morphism. Further, \( \rho \) is a subquotient in a representation of the form \( i_{L_1}(\rho_1) \mid_{L_\gamma} \), where \( \rho_1 \) is a cuspidal representation of a standard Levi \( L_1 \subset L \). Then there exists \( \nu_1 \in \rho_1 \), with \( \mathcal{P}_L(\nu_1) = \nu \) such that the morphism \( C_c^\infty(G/U)^K_{\mu} \to C_c^\infty(G/U)^K_{\mu+\nu_1} \), \( f \mapsto h \ast f|_{(G/U)_{\mu+\nu_1}} \), is nonzero. This means that \( \text{supp}^{\text{pec}}(I_{\rho_1}(h)) \nsubseteq \nu_1 \), where \( \rho_1' \in \text{Cusp} \) differs from \( \rho \) by twisting with an unramified character; so \( \text{supp}^{\text{pec}}(h) \nsubseteq \nu_1 \).
If i) holds then $\nu_1 \preceq \lambda$. Since $\lambda \in a^+$ we have $P(\lambda) \preceq \lambda$: since $P_L$ preserves $\preceq$ (by [BW], Lemma 6.4 on p.139, statement 2) we get $\lambda \preceq P_L(\lambda) \preceq \nu$. We arrived at a contradiction, which proves i) $\Rightarrow$ ii). □

**Lemma 2.9.** Let $C \subset G$ be a compact set. Suppose that $x_1 \in (C_P)_{\mu_1}, x_2 \in (C_P)_{\mu_2}, \langle \mu_1, \Sigma^+ - \Sigma^+_L \rangle \gg 0$, and $x_2 = cx_1$ for $c \in C$. Identify the left $G$-orbit on $C_P$ passing through $x_1, x_2$ with $G/U$, and suppose that $x_1 \in (G/U)_{\nu_1}, x_2 \in (G/U)_{\nu_2}$. Then $\nu_2 - \nu_1 = P_L(\mu_2 - \mu_1)$.

**Proof.** We have a map
$$W_{aff} \times W_{aff}(L) W_{aff} \to K_0 \setminus C_P/ K_0$$
which sends $(w_1, w_2)$ to $K_0 \cdot (w_1, w_2) \cdot K_0$ (it is immediate to check that it is actually well-defined). For $x \in W_{aff} \times W_{aff}(L) W_{aff}$ denote the corresponding double coset by $(C_P)_{x}$. As in 2.3 we pick a finite set $\{w_i\} \subset W_{aff}$ so that for any $w \in W_{aff}$ we have $w_i w w_j \in X^+$ for some $i, j$. For $(\mu, \Sigma^+ - \Sigma^+_L) \gg 0$ Proposition 2.2 implies that
$$(C_P)_{(w_i, w_j)} = \bigcup_{g \in G_{w_i w_j}} \Psi(KgK).$$
In particular, $x_i \in (C_P)_{w_i, \lambda w_i}$ for $i = 1, 2$, where $\lambda_i$ is such that $(\lambda_i, \Sigma^+ - \Sigma^+_L) \gg 0$, and $w_i \lambda_i w'_i \in W_{f} \cdot \mu_i \cdot W_{f}'$. Since $x_1, x_2$ lie in one left $G$-orbit we have $(\lambda_2 w'_2)^0 \subset P^- (\lambda_1 w'_1)^0$, which is possible only if $w'_2 \in W_{aff}(L) w_2$. Thus we can (and will) assume that $w'_1 = w'_2 = \nu'$. Pick $x \in (C_P)_{w_0} \cap G(x_1)$, and identify $G/U$ with the left orbit $G(x_1) = G(x_2)$ by means of the map $g \mapsto g(x)$. Then we get $x_i \in (K_0 \cdot (w_i, \lambda_i) \cdot (K_0 \cap L) U) / U \subset (G/U)_{(w_i, \lambda_i)} = \bigcup_{g \in G_{w_i, \lambda_i}} \Psi(KgK)$. In particular $P_L(\mu_i) = P_L(\nu_i + \mu')$. The Lemma follows. □

**Proof of Theorem 2.7.** Consider the “Rees algebra” $Rees(\mathcal{H}) := \bigoplus_{\lambda \in X^+} \mathcal{H}_{\prec \lambda}$. The explicit description of the center $Z \subset \mathcal{H}$ ([BD], Theorem 2.13) readily implies that the central subalgebra $Rees(Z) \subset Rees(\mathcal{H})$ is of finite type. Further, from Lemma 1.13b) it is not hard to deduce that there exists $\lambda_0 \in X^+$ such that for $\lambda \in a_+$, $\rho \in Cusp^L_\lambda$ we have $h \in \mathcal{H}_{\prec \lambda}, \nu \in \supp^{pc}(I_{\rho}(h)) \Rightarrow w(\nu) \prec \lambda + \lambda_0$ whenever $w(L)$ is a standard Levi.

Thus we have $Rees(\mathcal{H}) \hookrightarrow \bigoplus_{L \rho Cusp^L_\lambda \in a_+} \bigoplus_{w} \bigcap \End(\Pi_{\rho})_{w(\alpha_{\lambda + \lambda_0})}$. The RHS is a finite $Rees(Z)$-module. Hence the LHS is, thus $Rees(\mathcal{H})$ is a Noetherian graded algebra, and in particular is finitely generated.

Let $h_i \in \mathcal{H}_{\prec \lambda_i}, i = (1, \ldots, n)$ be the generators. Then it is enough to ensure that $f \in Geom_{\prec \mu}(\mathcal{H}) \Rightarrow h_i f \in Geom_{\prec \lambda_i + \mu}(\mathcal{H})$.

Pick a number $N$ which satisfies the condition of 2.4 with $C = \cup \supp(h_i)$ for all $P$.

Also let $N'$ be such that $h_i \in Geom_{\prec N'}(\mathcal{H})$ for all $i, \alpha$. 


Suppose that $\lambda \in a^+$ is large enough, $\mu \not\leq \lambda$, and $h \in \mathcal{H}_\mu$.

Pick a simple root $\alpha$. We consider two cases.

I. $\langle \mu, \alpha \rangle > N$. Then let $P$ be the maximal proper parabolic corresponding to $\alpha$.

Applying 2.4 we see that $h_i \ast h = \Psi^{-1}(h_i \ast \Psi(h))$.

We can decompose $C_P = \bigcup_{x \in P \backslash G/K} (C_P)_x$, where $(C_P)_x$ is the fiber of the projection $C_P \rightarrow P \backslash G/K$, $(g_1 U, g_2 U) L \mapsto P \dashv g_2 K$.

Let $\Phi$ now write $h$ as the sum of $h_\alpha$, where $\supp(\Phi(h_\alpha)) \in (C_P)_x$.

We have an isomorphism of $G$-modules $C_\infty((C_P)_x)^K \cong C_\infty(G/U)^K \cap P$. (Here the invariants are taken with respect to the right action).

Hence applying to each $h_\alpha$ 2.8 and 2.9 we obtain:

$$h_i \ast h \in \text{Geom}_\alpha^{\langle \mu, \omega_\alpha \rangle}(\mathcal{H}) \subset \text{Geom}_\alpha^{\langle \lambda, \omega_\alpha \rangle}(\mathcal{H}).$$

II. If on the other hand $\langle \mu, \alpha \rangle \leq N$ then

$$\langle \lambda - \mu, \omega_\alpha \rangle \geq \frac{(\lambda, \alpha) - N}{(\alpha, \alpha)},$$

because the RHS is the distance from $\lambda$ to the hyperplane $\{ \eta \mid \langle \eta, \omega_\alpha \rangle = N \}$, while $\mu$ is separated from $\lambda$ by this hyperplane.

If $\lambda$ is large then the RHS of (34) is greater than $N'$, so in that case

$$h_i \ast h \in \text{Geom}_\alpha^{\langle \mu, \omega_\alpha \rangle + N'}(\mathcal{H}) \subset \text{Geom}_\alpha^{\langle \lambda, \omega_\alpha \rangle}(\mathcal{H}).$$

Thus in any case $h_i \ast \text{Geom}_\alpha^{\langle \lambda, \omega_\alpha \rangle}(\mathcal{H}) \subset \text{Geom}_\alpha^{\langle \lambda, \omega_\alpha \rangle + N'}(\mathcal{H})$ provided $\lambda$ is large.

Since $h_i$ generate $\text{Rees}(\mathcal{H})$ this finishes the proof. $\square$

To derive an important corollary we need some information on compatibility of our filtrations with convolution.

**Lemma 2.10.** We have

a) $\text{Geom}_\alpha^{\langle \mu \rangle}(\mathcal{H}) \cdot \text{Geom}_\alpha^{\langle \mu \rangle}(\mathcal{H}) \subseteq \text{Geom}_\alpha^{\langle \mu + \mu \rangle}(\mathcal{H})$.

b) $G_\lambda((G/U)_\mu) \subset \bigcup_{\nu \leq \lambda + \mu} ((G/U)_{\nu}).$

**Proof** Recall that we have fixed a minimal Levi subgroup in good relative position with $K_0$. This choice determines an apartment $\mathfrak{A}$ of the Bruhat-Tits building $X$ containing the $K_0$-fixed point $p$. Let $\Xi : X \rightarrow \mathfrak{A}$ be the contraction centered at an open polysimplex, which contains $p$ in its closure. Then an element $g \in G$ lies in $G_{\nu}$ iff the coweight $\Xi(g(p)) - p \in \alpha$ is $W$-conjugate to $\nu$. Now the proof is parallel to the usual proof of the fact that $\Xi$ does not increase distances (see [BT1] 7.4.20(ii)).

Take now $g_1 \in G_{\nu_1}$, $g_2 \in G_{\nu_2}$; we must check that $g_1 g_2 \in G_{\nu}$ where $\nu$ satisfies $\nu \leq \nu_1 + \nu_2$.

Let us break the segment $[g_1(p), g_1 g_2(p)]$ into the union $[z_0 = g_1(p), z_1], [z_1, z_2], \ldots [z_{n-1}, g_1 g_2(p) = z_n]$, where $z_i$ and $z_{i+1}$ lie in the closure of one open polysimplex (loc. cit. 7.4.21).

According to [BT1] 7.4.19, 7.4.18(i) the map $\Xi$ restricted to the closure of any polysimplex coincides with the action of some $g \in G$, hence we have $\Xi(z_i) = x_i(z_i)$, $\Xi(z_{i+1}) = x_i(z_{i+1})$ for some $x_i \in G$.

On the other hand $\Xi([p, g_2(p)]) = g([p, g_2(p)])$ for some $g \in G$ is a line segment, and by the above remark $w(\Xi(g_2(p)) - p) = \nu_2$ for some $w \in W$. 

Then obviously $\nu_2 = \sum_{i=0}^{n-1} \nu^i$, where $\nu^i = w(\Xi(g_2(z_i)) - \Xi(g_2(z_{i-1}))) \in a^+$.

By [BT1] 7.4.8 for points $p_1, p_2, p'_1, p'_2 \in X$ the elements $p_1 - p_2$ and $p'_1 - p'_2 \in a$ are $W$-conjugate provided $p_i = g(p'_i)$ for some $g \in G$.

Thus we see that $\Xi(z_i) - \Xi(z_{i-1})$ is $W$-conjugate to $\Xi(g_2(z_i)) - \Xi(g_2(z_{i-1})) = g_1x^{-1}_i(\Xi(z_i)) - g_1x^{-1}_i(\Xi(z_{i-1}))$, and hence also to $\nu^i$. Let $w_i \in W$ be such that $\Xi(z_i) - \Xi(z_{i-1}) = w_i(\nu^i)$. Suppose that $g_1g_2 \in G_{\nu}$, i.e. $w'(\Xi(g_1g_2(p) - p)) = \nu$ for some $w' \in W$. Then we have

$$\nu = w'(\Xi(g_1g_2(p) - p)) = w'(\nu_1 + \sum w_i(\nu^i)) = w'(\nu_1) + \sum w'_iw_i(\nu^i) \preceq \nu + \sum \nu^i = \nu_1 + \nu_2.$$ 

This proves a).

To prove b) we reinterpret the condition $gU \in (G/U)_\mu$ as follows. Let $\Xi' : X \rightarrow X$ be the contraction centered at a “vector chamber” corresponding to $P_0$. Then $gU \in (G/U)_\mu$ if and only if $\Xi'(p - \Xi'(g^{-1}(p))) = \nu$.

From the definition ([BT1] 7.4.25) it is clear that $\Xi'(x) = g_\Delta$ for any polysimplex for some $g_\Delta \in G$. Hence repeating the arguments used in a) we can find $\lambda^i \in a^+$ and $w_i \in W$, such that $\Xi(g^{-1}_2(p)) - \Xi(g^{-1}_2g^{-1}_1) = \sum w_i(\lambda^i)$, while $\lambda = \sum \lambda^i$. Since by [BW], Lemma 6.4 on p.139 $\Psi_L$ preserves $\leq$ we see that if $g_1 \in G_{\lambda}, g_2U \in (G/U)_\mu$, $g_1g_2 \in (G/U)_{\mu'}$ then

$$\mu' = \Psi_L(p - \Xi'(g^{-1}_2(p))) + \Psi_L(\Xi'(g^{-1}_2(p)) - \Xi'(g^{-1}_2g^{-1}_1)) = \Psi_L(\mu) + \Psi_L(\sum w_i(\lambda^i)) \preceq \mu + \sum \lambda_i = \mu + \lambda.$$ 

This proves the Lemma. □

**Corollary 2.11.** For any $\lambda \in X^+$ we have $H_{<\lambda} \supset Geom_{<\lambda}(H)$.

**Proof** Compare Lemma 2.8 with Lemma 2.10b). □

The latter statement together with Theorem 2.7 yield the following

**Proposition 2.12.** Fix $\lambda_0$ in $X^+$, and define filtration $F_{<\lambda}(H) := \sum_{\mu \in a_{<\lambda}} Geom_{<\lambda}(H)$. Then $F_{<\lambda}$ is a good filtration on the free $H$-module, provided $\lambda_0$ is large enough.

**Proof** From Theorem 2.7 we see that the filtrations on the algebra and on the module are compatible if $\lambda_0$ is large.

It remains to show that for all but finitely many $\nu$ we have

$$F_{<\lambda} \subseteq \sum_{\lambda_2 \in a^+_{<\lambda}} \sum_{\lambda_1 \in a_{<\lambda} - \lambda_2} H_{<\lambda_1} \cdot F_{<\lambda_2}. \quad (35)$$

We will in fact show that

$$F_{<\lambda} \subseteq \sum_{\lambda_2 \in a^+_{<\lambda}} \sum_{\lambda_1 \in a_{<\lambda} - \lambda_2} Geom_{<\lambda_1}(H) \cdot F_{<\lambda_2} \quad (36)$$

for almost all $\lambda$; (36) is stronger than (35) by 2.11.

Pick some $\lambda$ and $\nu$ such that $H_{<\nu} \subseteq F_{<\lambda}(H)$. We want to check that $H_{<\nu}$ lies in the RHS of (36).
By the definition $\nu \leq \lambda' + \lambda_0$ for some $\lambda' \in a_+^\perp$. If $\lambda' < \lambda$ then $H_\nu$ does lie in the RHS of (36), for it lies in the summand corresponding to $\lambda_2 = \lambda'$, $\lambda_1 = 0$. So assume that $\lambda' = \lambda \geq \nu$.

For any $w \in W_p^f$ consider the intersection $w^{-1}W_{aff}w \cap \Lambda$. It is a sublattice of finite index in $\Lambda$; in particular there exists some $n \in \mathbb{Z} > 0$ such that $n\omega_\alpha \in w^{-1}W_{aff}w \cap \Lambda$ for any $w \in W_p^f$ and any simple coroot $\alpha$.

For all but finitely many $\lambda$ there exists a simple coroot $\alpha$ such that $(\lambda, \omega_\alpha) \geq n+1$; thus we can suppose that $\lambda, \alpha$ are like that. If $\nu \leq n$ then $H_\nu \subset F_{\leq \lambda - \omega_\alpha}$, so we can take $\lambda_1 = 0$, $\lambda_2 = \lambda - \omega_\alpha$.

It remains to consider the case $(\nu, \omega_\alpha) > n$.

We have $H_\nu = \bigoplus H_x$, where $x$ runs over double cosets $W_p^f w W_p \subset W_p \backslash W_{aff} / W_p = K_0 \backslash G / K_0$ such that $w \in W_{aff} \cap (W_p^f \nu W_p^f)$.

Fix one such $w$; let us write it down as $w = w_1 \nu w_2$, $w_i \in W_p^f$, and consider the element $y = W_p \cdot (w_1(n\omega_\alpha)w_2^{-1}) \cdot W_p \subset W_p \backslash W_{aff} / W_p$.

It is obvious that for any triple double cosets $z_1$, $z_2$, $z_3$ where $z_1 = W_p \cdot w' W_p$, $z_2 = W_p \cdot w'' w W_p$, $z_3 = W_p \cdot w'' w W_p \subset W_p \backslash W_{aff} / W_p$ we have $H_{z_1} \cdot H_{z_2} \supset H_{z_3}$.

Hence we have $H_x \subset H_y \cdot H_z$, where $z = W_p^f w_1 (\nu - n\omega_\alpha) w_2$.

Since $H_x \subset H_{\nu - n \omega_\alpha} \subset H_{(\lambda - n \omega_\alpha)} \subset F_{\leq \lambda - n \omega_\alpha}$ we see that in this case $H_\nu$ lies in the summand of the RHS of (36) corresponding to $\lambda_1 = n \omega_\alpha$, $\lambda_2 = \lambda - n \omega_\alpha$.

The Proposition is proved. □

**Remark 2.13.** We will also use the following equivalent description of $F_{\leq \lambda}$.

For a coweight $\lambda \in a$ let $\varphi(\lambda)$ denote the point of $a_+$ closest to $\lambda$. Then

$$
\varphi(\lambda) = \sum_{\alpha} a_\alpha \omega_\alpha
\varphi(\lambda) - \lambda = \sum_{\beta} b_\beta \beta;
$$

where $\alpha$ and $\beta$ run over nonintersecting sets of simple roots, and $a_\alpha > 0$, $b_\beta \geq 0$; moreover for any coweight $\lambda$ there exists a unique such decomposition (the latter statement is the so-called Langlands combinatorial Lemma, see e.g. [BW], Lemma 6.11 on p.143).

It is not hard to see that $\varphi(\mu)$ is the minimal element in the set $\{ \lambda \in a_+ | \lambda \geq \mu \}$. It follows that

$$
F_{\leq \lambda}(H) = \bigoplus_{\varphi(\mu - \lambda_0) \leq \lambda} H_\mu
$$

2.14. Another result we will need is the following

**Theorem 2.15.** Fix $\lambda_1 \in a^+$. Make the following assumptions:

(e1, r_j) > 0 for all $r_j \in \Sigma^+$;

$h_2^\omega$ (notation of 1.17) satisfies the condition of 1.17 with large enough $a$ (the bound on $a$ depends on $\lambda_1$);

$\mu \in a_+$ is very dominant.

Then we have:

- a) If $\nu \in a_+$ is such that $(\nu, \Sigma^+ - \Sigma_L) \gg 0$ then

$$
\varphi(\nu - \lambda_0) \leq \mu + \lambda_1, h \in H_\nu \implies \Psi(h_2^\omega h) = \iota(\lambda) m_2 \Psi(h) + \Psi(h'),
$$

where $h' \in F_{\leq \lambda + \mu}(H)$.

- b) Otherwise $h_2^\omega h \in F_{\leq \lambda' + \mu}(H)$.
Lemma 2.16. For $L' \subset L$ and $f \in C_c(G/U')^K$ we have $h_\lambda^L f = \iota(\lambda)m_x(f) + f'$, where $f' \in \sum_{r \in \Sigma^+ - \Sigma_L^+} C_c(G/U')_{\xi_\mu + \lambda - a_\Psi L'}(r)$.

Proof: Similar to the proof of Lemma 2.8. It is enough to prove that for any irreducible representation $\rho_L$ of $L$ the action of $h_\lambda^L$ on $\Pi_{\rho_L}^K = C_c(G/U')^K \otimes H(L')$ is very dominant for $\rho_L$. The irreducible representation $\rho_L$ is subquotient in the induced representation $\Pi_{L'}^K(\rho_L')$, where $\rho_L'$ is a cuspidal representation of a standard Levi. Then $\Pi_{\rho_L} = \Pi_{\rho_L'}$, and the statement reduces to $I_{\rho_L'}(h_\lambda^L) - [\lambda]_{\rho_L'} \in \sum_{a \in \Sigma^+ - \Sigma_L^+} \rho_L(\rho_L')_{\xi_\lambda - a_\Psi}$, which follows from Proposition 1.7.

□

Lemma 2.17. For a weight $\nu \in a_\lambda$ such that $(\nu, \Sigma^+ - \Sigma_L^+) \gg 0$ we have $\varphi(\nu + \lambda_1 - a_\Psi r - \lambda_0) \leq \varphi(\nu - \lambda_0)$ for $r \in \Sigma^+ - \Sigma_L^+$, provided $\lambda_1 > a_\Psi (\lambda_1, \lambda_1)$. For any simple root $\alpha \in \Sigma^+ - \Sigma_L^+$.

Proof: Let $\alpha \in \Sigma^+ - \Sigma_L^+$ be a simple root such that $\lambda_1 \leq \lambda_0 - a_\Psi r$. If $\nu$ satisfies $(\nu, \Sigma^+ - \Sigma_L^+) \gg 0$ then also $\varphi(\nu - \lambda_0) + \lambda_1 - a_\Psi \lambda_0, \Sigma^+ - \Sigma_L^+) \gg 0$. Since $(\alpha, \beta) = 0$ for $\beta \in \Sigma_L^+$ it follows that $\varphi(\nu - \lambda_0) + \lambda_1 - a_\Psi \lambda_0, \Sigma^+ - \Sigma_L^+) \gg 0$. From the condition of Lemma we get $e_\nu, \varphi(\nu - \lambda_0) + \lambda_1 - a_\Psi \lambda_0, \Sigma^+ - \Sigma_L^+) \ll (e_\nu, \varphi(\nu - \lambda_0), \Sigma^+ - \Sigma_L^+)$. Hence $\varphi(\nu - \lambda_0) + \lambda_1 - a_\Psi \lambda_0, \Sigma^+ - \Sigma_L^+) \ll \varphi(\nu - \lambda_0)$. The Lemma now follows from (38).

Proof of Theorem 2.15 follows the same scheme as the proof of 2.7.

Let $L'$ be minimal among the standard Levi subgroups for which $\Psi_{L'}|G'$ is defined. Under the conditions of a) we have $L' \subset L$.

In the notations of the proof of 2.7, we can assume that $\Psi(h) \in H(C_{P'})_{\lambda_0}$ for some $x \in P^- \backslash G/K$. Then using Lemma 2.16 together with 2.8, 2.9 we get

$$\Psi(h_\lambda^L h) - m_\lambda'(\lambda)\Psi(h) \in \sum_{r \in \Sigma^+ - \Sigma_L^+} \sum_{\eta \in \Psi_{L'}(\eta)_{\xi_\mu + \lambda - a_\Psi}} H(C_{P'})_{\eta},$$

hence

$$\text{supp}^\text{geom}(h_\lambda^L h - \Psi^{-1}(m_\lambda'(\lambda)\Psi(h))) \subset \bigcup_{r \in \Sigma^+ - \Sigma_L^+} \bigcup_{\eta \in \Psi_{L'}(\eta)_{\xi_\mu + \lambda - a_\Psi}} G_{\eta.}$$

Let, first, $\eta$ be such that $G_{\eta} \cap \text{supp}^\text{geom}(h_\lambda^L h - \Psi^{-1}(m_\lambda'(\lambda)\Psi(h))) = \emptyset$. As we have just seen there exists $r \in \Sigma^+ - \Sigma_L^+$ such that $\eta \leq \nu + \lambda - a_\Psi$ for any simple coroot $\alpha \in \Sigma^+ - \Sigma_L^+$. By Lemma 2.17 $\nu + \lambda - a_\Psi \leq \lambda_0 + \lambda'$, where $\lambda' \leq \mu + \lambda$.

Thus for a simple coroot $\alpha \in \Sigma^+ - \Sigma_L^+$ we have $\eta \leq \nu + \lambda - a_\Psi \leq \lambda_0 + \lambda'$.

On the other hand, if $\alpha$ is a simple coroot of $L'$ then $\langle \omega_\alpha, \nu \rangle$ is bounded. Since $\lambda_0$ is very dominant we have $\eta \leq \lambda_0 - a_\Psi \lambda_0 + \lambda'$.

So $\eta \leq \lambda_0 + \lambda'$ for any simple coroot $\alpha$, i.e. $\eta \leq \lambda_0 + \lambda'$, which implies $h_x^L h - \Psi^{-1}(m_\lambda'(\lambda)\Psi(h)) \in F_{\xi_\mu + \lambda}(H)$. a) is proved.

Assume now we are in the situation of b). Then for some simple root $\alpha \in \Sigma_L^+$ the number $\langle \nu, \omega_\alpha \rangle$ is bounded.

The condition $(e_1, \alpha) > 0$ implies that we can find $a \in \mathbb{R}_{>0}$ such that $\lambda_1 - a_\Psi \alpha \leq 0$; we can then assume $\lambda \geq a_\Psi \alpha$. Also we can assume (after possibly increasing $\lambda_0$) that $H_{c_\lambda} \subset \text{Geom}_{c_\mu}(H) \subset \text{Geom}_{c_\lambda + \mu}$ holds already for $\mu \geq \lambda_0 - a_\Psi \alpha$. By the
Precisely, let \( \phi \) be \( (\lambda_0,\lambda_1) \) for some \( \zeta \leq \mu + \lambda_1 \). But since \( (\omega_\alpha,\nu) \) is bounded we also have \( \nu \leq \lambda_0 + \zeta - a \alpha \). Then we have \( h_\phi h \in Geom_{\lambda_0+\zeta-a \alpha+\lambda} \subseteq F_{\lambda-a \alpha+\zeta} = F_{\lambda+\mu+(\lambda_1-a \alpha)} \). \( \square \)

3. End of the proof: orbital integrals.

**Theorem 3.1.** The functional \( Tr(g,M) \) defined above satisfies 0.23iii).

We start with recalling the following standard computation.

We will say that a subset \( \mathcal{S} \subseteq \Lambda^+ \) is large if it contains \( (\Lambda^+)_{\leq \lambda} \) for large \( \lambda \).

For any \( \mathcal{S} \subseteq \Lambda^+ \) denote \( G_\mathcal{S} = \bigcup_{\lambda \in \mathcal{S}} G_\lambda \), \( \mathcal{H}_\mathcal{S} = \bigoplus_{\lambda \in \mathcal{S}} \mathcal{H}_\lambda \) and let \( pr_\mathcal{S} : \mathcal{H}^m \rightarrow \mathcal{H}^m_{\mathcal{S}} \subset \mathcal{H}^m_{\mathcal{S}^+} \) be the projector along \( \mathcal{H}^m_{\mathcal{S}^+} \).

**Lemma 3.2.** Keep the assumptions of 3.1. Then for any large enough finite set \( \mathcal{S} \subseteq \Lambda^+ \) we have

\[
O_{g^{-1}}(\sum E_{ii}) = tr(g \circ E \circ pr_\mathcal{S})
\]

(the RHS is the well-defined trace of a finite-rank endomorphism of \( \mathcal{H}^m \)).

**Proof of the Lemma.** We have \( tr(g \circ E \circ pr_\mathcal{S} \circ k[G/K]^m) = tr(\delta_g K \circ E \circ pr_\mathcal{S} \circ k[G/K]^m) = \int_{g' \in K} tr(g' \circ E \circ pr_\mathcal{S} \circ k[G/K]^m) \), where we reused the symbol \( pr_\mathcal{S} \) to denote also the projector \( k[G/K]^m \rightarrow k[G/K]^m \subset k[G/K]^m \) along \( k[G+\mathcal{S}/K]^m \).

In \( k[G/K] \) we have the standard basis consisting of delta-functions; it gives also a basis in \( k[G/K]^m \). We use this basis to “compute” the trace of \( g' \circ E \circ pr_\mathcal{S} \).

Precisely, let \( dg \) be the Haar measure on \( G \) such that \( \int_K dg = 1 \); if \( x \in G/K \), and \( x_{(i)} = (0,\ldots,0,\delta_x,0,\ldots,0) \) is a basis element of \( \mathcal{H}^m \), then the corresponding diagonal entry of \( g' \circ E \circ pr_\mathcal{S} \) is equal to

\[
\begin{cases}
0 & \text{if } x \notin G_\mathcal{S}/K \\
E_{ii} \delta_g (x^{-1}(g')^{-1}x) & \text{if } x \in G_\mathcal{S}/K
\end{cases}
\]  

(39)

If \( K \cdot g \ni g' \) is contained in the regular elliptic set, then for \( \mathcal{S} \) large enough we have \( E_{ii} \delta_g (x^{-1}(g')^{-1}x) \neq 0 \Rightarrow x \in G_\mathcal{S}/K \). For such \( \mathcal{S} \) summation of (39) over \( x \) gives \( O_{(g')^{-1}}(E_{ii}) \).

Altogether we get

\[
tr(g \circ E \circ pr_\mathcal{S}, \mathcal{H}^m) = tr(\int_{g' \in K \cdot g} g' \circ E \circ pr_\mathcal{S} \circ k[G/K]^m) dg' =
\]

\[
= \int_{g' \in K \cdot g} O_{(g')^{-1}}(\sum E_{ii}) = O_{g^{-1}}(\sum E_{ii})
\]

where the last equality follows from the assumption that \( O_{(g')^{-1}}(\sum E_{ii}) \) is constant for \( g' \in g \cdot K \). \( \square \)

We proceed now to the proof of the Theorem.

Fix very dominant \( \lambda_0 \), and set \( \mathcal{S}_\lambda = \bigcup_{\mu \leq \lambda} X_{\leq \mu + \lambda_0}^+ \). Then \( \mathcal{H}^m_{\mathcal{S}_\lambda} = F_{\mathcal{S}_\lambda}(\mathcal{H}^m) \) is a good filtration on the free module of rank \( m \) by 2.12.

Thus the image of \( F_{\mathcal{S}_\lambda}(\mathcal{H}^m) \) under \( E \) is a good filtration on \( M \), hence we have

\[
Tr(g,M) = tr(g,E(F_{\mathcal{S}_\lambda}(\mathcal{H}^m))) \text{ for large } \lambda.
\]
So to prove the Theorem we need only to show that
\[ \text{tr}(g, E(H_{\Theta, \lambda}^{\oplus m})) = \text{tr}(g \circ E \circ pr_{\Theta, \lambda}, H_{\Theta, \lambda}^{\oplus m}). \] (40)

It is convenient to consider the vector space \( E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m} \) which is invariant under the action of \( E, pr_{\Theta, \lambda}, g \). Obviously
\[
\text{tr}(g, E(H_{\Theta, \lambda}^{\oplus m})) = \text{tr}(g \circ E, E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}).
\]
\[
\text{tr}(g \circ E \circ pr_{\Theta, \lambda}, E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}).
\]

Thus the difference of the two sides of (40) coincides with
\[
\text{tr}(g \circ E \circ (Id - pr_{\Theta, \lambda}), E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) = \text{tr}(g \circ E \circ pr_{X, \Theta, \lambda}, E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) =
\]
\[
\text{tr}(g \circ pr_{X, \Theta, \lambda} \circ E, (E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) \cap H_{X, \Theta, \lambda}^{\oplus m}),
\]
so the Theorem is reduced to the following assertion.

**Proposition 3.3.** The endomorphism \( g \circ pr_{X, \Theta, \lambda} \circ E \) of the vector space \( (E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) \cap H_{X, \Theta, \lambda}^{\oplus m} \) is traceless for large \( \lambda \).

Denote
\[
(E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) \cap H_{X, \Theta, \lambda}^{\oplus m} = pr_{X, \Theta, \lambda} (E(H_{\Theta, \lambda}^{\oplus m})) \longrightarrow (E(H_{\Theta, \lambda}^{\oplus m}) + H_{\Theta, \lambda}^{\oplus m}) / H_{\Theta, \lambda}^{\oplus m}
\]
by \( W_{\lambda} \). For \( \nu \leq \lambda \) let us denote by \( (W_{\lambda})_{\leq \nu} \subset W_{\lambda} \) the subspace \( (E(F_{\leq \nu}(H_{\Theta, \lambda}^{\oplus m}))) / (F_{\leq \lambda}(H_{\Theta, \lambda}^{\oplus m})) = (E(F_{\leq \nu}(H_{\Theta, \lambda}^{\oplus m}))) / (F_{\leq \lambda}(H_{\Theta, \lambda}^{\oplus m})) \).

For a number \( N \) let \( \lambda_N \) be the maximal element in the (finite) set \( X_N := \{ \lambda \in X^+ \mid (\lambda, e_1) \leq N \} \). Then \( X^+_{\leq \lambda_N} = X_N \); since \( X_N \) is arbitrary large for \( N \gg 0 \) it is enough (in view of 3.2) to prove the Proposition for \( \lambda = \lambda_N \). We write \( W_N, \Theta_N \) instead of \( W_{\lambda_N}, \Theta_{\lambda_N} \).

Consider the associated graded space \( \text{gr}W := \bigoplus_{N} \bigoplus_{\lambda \leq X_N} (W_{\lambda})_{\leq \nu} / (W_{\lambda})_{< \nu} \) which is induced by multiplication in \( \mathcal{H} \).

**Lemma 3.4.** \( \text{gr}W \) is a finitely generated \( \text{gr}\mathcal{H} \)-module.

**Proof** It is obvious that \( \text{gr}(W_N)_{\lambda} \neq 0 \Rightarrow N - n \leq (\lambda, e_1) \leq N \) where \( n \) is some constant (depending on \( \supp\rho_{\text{even}}(E_{ij}) \)). Thus \( \text{gr}W \) is a finite sum of modules \( \text{gr}W^{(i)} := \bigoplus_{N - (\lambda, e_1 = i)} \text{gr}(W_N)_{\lambda} \), each one of which is a quotient of \( \text{gr}_F(H_{\Theta, \lambda}^{\oplus m}) \), so is finitely generated. \( \square \)

Now by the proof of 1.19b) we can break \( \Lambda_0^+ \) into a finite union of cosets
\[
\Lambda_0^+ = \bigcup_{\mu_i} + X_{L_i}^+
\]
so that \( \lambda \in \mu_i + X_{L_i}^+ \Rightarrow \text{gr}W_{\lambda} = M_i \) where \( M_i \) is a finite \( \mathcal{A}_{L_i} \)-module (notations of (31)).

Notice that for \( \lambda \in a_+ \) the set \( \Theta_{\lambda} - \Theta_{< \lambda} \) looks as follows. Suppose that \( \lambda \) lies strictly inside \( a_+^\ast \). Then \( \Theta_{\lambda} - \Theta_{< \lambda} = (\lambda_0 + \mathcal{g}^{-1}(\lambda_0)) \cap \Lambda^+ = \Lambda^+ \cap \{ \lambda - \sum a_i r_i \mid 0 \leq a_i \leq \langle \omega_i, \lambda_0 \rangle \} \). Let \( \lambda_0 \) runs over the set of simple roots of \( L \).

In particular we have \( (\lambda, \Sigma^+ - \Sigma_0^+) \gg 0 \), \( \nu \in \Theta_{\lambda} - \Theta_{< \lambda} \Rightarrow \langle \nu, \Sigma^+ - \Sigma_0^+ \rangle \gg 0 \).

Proceeding by induction in \( \dim(a_i \lambda) \) we can choose decomposition (41) so that \( \langle \nu, \Sigma^+ - \Sigma_0^+ \rangle \) is arbitrary large if \( \nu \in \Theta_{\lambda} - \Theta_{< \lambda} \) where \( \lambda \in \mu_i + X_{L_i}^+ \). In particular
we can guarantee that \( G_{\delta_n - \delta_{<\eta}} \) lies in the domain of \( \Psi : K \backslash G / K \to K \backslash (C_P) / K \) for \( \eta \in \mu, \lambda \). We shall make this assumption from now on.

Then we get a direct sum decomposition:

\[
\mathcal{H}_{\delta_n - \delta_{<\eta}}^{\oplus m} = \bigoplus_{x \in K \backslash G / P} \Psi^{-1} C^\infty ((C_P^{\delta_n - \delta_{<\eta}})_x)^{\oplus m}
\]  

(42)

So take \( \eta \in \mu, \lambda \), and consider the surjection \( \pi : \mathcal{H}_{\delta_n - \delta_{<\eta}}^{\oplus m} \to gr(W_N)_{\eta} \approx E(H_{\delta_n - \delta_{<\eta}}^{\oplus m}) / (E(H_{\delta_n - \delta_{<\eta}}^{\oplus m}) \cap (E(H_{\delta_n - \delta_{<\eta}}^{\oplus m}) + H_{\delta_n - \delta_{<\eta}}^{\oplus m})).

Since its target is an \( \mathcal{A}_L \)-module, it also decomposes as a direct sum:

\[
gr(W_N)_{\eta} = \bigoplus_{x \in K \backslash G / P} gr(W_N)_{\eta}^x.
\]

It is not hard to see that \( \pi \) commutes with the \( \mathcal{A}_L \)-action, and hence sends respective summands to respective summands (i.e. \( \pi \Psi^{-1} (C^\infty (C_P^{\delta_n - \delta_{<\eta}})_x) \subset gr(W_N)_{\eta}^x \)).

Hence for each \( N, \eta \) we can choose a subspace \( (V_N)_\eta \subset \mathcal{H}_{\delta_n - \delta_{<\eta}}^{\oplus m} \) in such a way that:

a) \( \pi \) maps \( (V_N)_\eta \) to \( gr(W_N)_{\eta} \) isomorphically.

b) \( (V_N)_{\eta} = \bigoplus_{x \in K \backslash G / P} \left( \Psi^{-1} C^\infty ((C_P^{\delta_n - \delta_{<\eta}})_x)^{\oplus m} \right) \cap (V_N)_\eta \).

c) \( (V_N)_{\eta} \) is \( K_0 \) invariant.

d) For \( \mu \in X_L^+ \), the isomorphism \( \Psi^{-1}_{\eta} \circ \iota(\mu) \circ \Psi_{\eta} : \mathcal{H}_{\delta_n - \delta_{<\eta}}^{\oplus m} \approx \mathcal{H}_{\delta_n - \delta_{<\eta + \nu}}^{\oplus m} \) sends \( (V_N)_{\eta} \) to \( (V_{(\nu, \mu, \lambda + \nu)} + (V_{(\nu, \mu, \lambda)})^{+} \).

Set now \( (W_N)_{\eta} = E((V_N)_{\eta}) \).

By property a) we have a direct sum decomposition \( W_N = \bigoplus (W_N)_{\nu} \).

Let \( E^\nu_N \in End ((W_N)_{\nu}) \) denote the corresponding diagonal block of \( pr_{X^+ - \delta \nu} \circ E \).

To finish the proof we need the following (key)

**Lemma 3.5.** For \( N \) large and any \( x \in P \backslash G / K \) we have:

a) If \( (\nu, \Sigma^+ - \Sigma^+_L) \gg 0 \) then \( h_{x, \lambda} \cdot (W_N)_{\nu} \subset (W_N + (e_1, \lambda))_{\nu + \lambda} + \mathcal{H}_{\delta_n + (e_1, \lambda)} \).

b) \( h_{x, \lambda} \cdot (W_N)_{\nu} \subset \mathcal{H}_{\delta_n + (e_1, \lambda)} \) otherwise.

**Proof** By 2.4 property d) it implies that \( \Psi_{\nu}(W_N)_{\nu + \lambda} = \iota(\lambda) \Psi_{\nu}(W_N)_{\nu} \) if \( (\nu, \Sigma^+ - \Sigma^+_L) \gg 0 \). Also from the definition it is clear that under the same condition on \( \nu \) the space \( \Psi_{\nu}(W_N)_{\nu} \) is invariant under the action of \( m_x \). So a) follows from Theorem 2.15a) (take \( \lambda_1 \) to be any coweight satisfying \( E_{ij} \in \mathcal{H}_{\leq \lambda_i} \)).

Statement b) follows Theorem 2.15b).

**Corollary 3.6.** \( E^\nu_N \) commutes with the action of \( \mathcal{A}_L \) on \( (W_N)_{\nu} \approx gr(W_N)_{\nu} \) provided \( (\nu, \Sigma^+ - \Sigma^+_L) \gg 0 \).

**Proof** Fix \( x \in K \backslash G / P \) and consider the element \( h_\lambda^x \in \mathcal{H} \) (see 1.17), where \( \lambda \in X_L^+ \) is dominant enough as in 1.17.

We have \( h' \in (W_N)_{\nu} \Rightarrow h' \cdot E = E^\nu_N(h') + \sum_{\nu' \neq \nu} h_{\nu'} + h_0 \) where \( h_{\nu'} \in (W_N)_{\nu'} \) and \( h_0 \in \mathcal{H}_{\delta_n} \). Multiplying the last equality by \( h_\lambda^x \) on the left and applying 3.5 we get the statement. \( \Box \)

**Proof** of 3.3. From 3.6 it follows that for large \( N \) and any \( \nu \) the endomorphism \( E^\nu_N \) preserves the direct sum decomposition \( (W_N)_{\nu} = \bigoplus_{x \in K \backslash G / P} ((W_N)_{\nu})_x \). Since
g permutes these summands without fixed points, the endomorphism $g \circ E_\nu^N$ is traceless. Hence $tr(g \circ pr_{X+ \xi(N)} = \sum \nu tr(g \circ E_\nu^N) = 0$.

Proposition 3.3, and thus also Theorem 3.1 is proved. □

We can now summarize our results in the following

**Theorem 3.7.** Let $\mathfrak{M}$ be a projective $G$-module, and let $K \subset K_0$ be a normal open subgroup, so small that:

a) $g$ is $K$-elliptic, so $Tr(g, \mathfrak{M})$ is defined (see 0.22, 0.23).

b) $\mathfrak{M}$ is generated by its $K$-fixed vectors.

c) Conditions of 0.23iii) are satisfied for $M = \mathfrak{M}^K$.

Then

$$Tr(g, \mathfrak{M}^K) = O_{g^{-1}}((\mathfrak{M}))$$

(43)

**Proof** There exists an idempotent $E \in \text{Mat}_m(H)$, such that $\mathfrak{M}$ is isomorphic to the image of $E$ acting on the right on $k[H/K]^{\oplus m}$. Since $K$ is nice by the assumption (see 0.23iii), we have $\langle \mathfrak{M} \rangle = \sum E_{ii} \mod [H(G), H(G)]$, so the statement follows from Theorem 3.1. □

3.8. **Proof of Theorem 0.19.** Let $0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \rho \rightarrow 0$ be a projective resolution of $\rho$. Pick a nice open compact $K$, such that $g$ is $K$-elliptic, properties $\beta)$ and $\gamma)$ of 3.7 hold for $M = P_i$. Then by additivity of $Tr(g, \rho^K)$ (0.23(i)=1.24(b)) and 3.7 we get $Tr(g, \rho^K) = O_{g^{-1}}((\rho))$. The proof is finished. □

4. **Elliptic pairing.** It remains to deduce Theorem 0.20 from 0.19. This argument can also be found in [Sch-St2] and is included here for the sake of completeness.

We set $k = \mathbb{C}$, char($F$) = 0 till the end of the argument.

Recall that the measure $d\mu$ on the set $Ell$ of regular semisimple elliptic conjugacy classes is characterized by the equality

$$\int_{Ell} O_g(h) d\mu(g) = \int_G h$$

(44)

being true for all $h \in H$ supported inside the set of regular elliptic elements.

**Lemma 4.1.** For any $\mathfrak{M} \in \mathcal{M}$ and $\rho \in \mathcal{R}$ we have

$$\sum (-1)^i \dim Ext^i(\mathfrak{M}, \rho) = \int_G \chi_\rho \cdot \langle \mathfrak{M} \rangle.$$

**Proof** Since both sides are additive on short exact sequences in $\mathfrak{M}$, it is enough to consider the case of projective $\mathfrak{M}$. As usual we write $\mathfrak{M}$ as the image of the right action of an idempotent $E \in \text{Mat}_m(H)$ on $H^{\oplus m}$. We have

$$\text{Hom}_G(H, \rho) \cong \hat{\rho},$$

(45)

where $\hat{\rho} = \varprojlim_K(\rho_K)$. (For $\rho \in \mathcal{R}$ we have also $\hat{\rho} = * (\rho)$). The right action of $\text{Mat}_n(H)$ on $H^{\oplus m}$ induces an action of $\text{Mat}_n(H)$ on $\text{Hom}_G(H^{\oplus m}, \rho)$, $m(f)(x) :=$
f(m(x)). Evidently, this action agrees under (45) with the action of Mat_n(\mathcal{H}) on \hat{\rho}^{\oplus n} inherited from the action of \mathcal{H} on \rho. Thus we have
\[
\dim(\text{Hom}_G(\mathfrak{M}, \rho)) = \dim(\text{Hom}_G(\text{Im}(E, \mathcal{H}^{\oplus n}), \rho)) = \dim(\text{Im}(E, \hat{\rho}^{\oplus n})) = \dim(\text{Im}(E, \rho^{\oplus n})) = \text{tr}(E, \rho^{\oplus n}) = \int_G \chi_\rho \cdot \sum_i E_{ii}. \tag{46}
\]
The Lemma is proved.

**Lemma 4.2.** Let \Phi be an invariant generalized function on G supported on the set of regular non-elliptic elements. Then for any admissible representation \rho we have:
\[
\int_G \langle \rho \rangle \cdot \Phi = 0.
\]

**Proof.** By Theorem 0 in [K] it is enough to show that
\[
\int_G \langle \rho \rangle \cdot \chi_{i_G}(\rho_L) = 0
\]
for any admissible representation \rho_L of a proper standard Levi subgroup L. By Lemma 4.1 it we have to show that
\[
\sum_i (-1)^i \dim \text{Ext}_G^i(\rho, i_G(\rho_L)) = 0.
\]
By Frobenius adjointness we have \text{Ext}_G^i(\rho, i_G(\rho_L)) \cong \text{Ext}_L^i(\rho_L, \rho_L). The Lemma now follows from the next

**Claim 4.3.** [B1] If the center of G is not compact, then for any admissible representations \rho_1, \rho_2 \in \mathcal{R}(G) we have
\[
\sum_i (-1)^i \dim \text{Ext}_G^i(\rho_1, \rho_2) = 0.
\]

**Proof.** For any \mathfrak{M} \in \text{Sm}(G) the space \text{Ext}_G^i(\mathfrak{M}, \rho_2) is finite dimensional. Thus the map \mathfrak{M} \rightarrow \sum_i (-1)^i \dim \text{Ext}_G^i(\mathfrak{M}, \rho_2) is a well-defined homomorphism \text{K}^0(\mathcal{M}(G)) \rightarrow \mathbb{Z}.

So we will be done if we show that the class of \rho_1, \rho_1 \in \text{K}^0(\mathcal{M}(G)) vanishes.

G/G^c is a free abelian group of rank equal to the split rank of the center of G. Hence, if the center of G is not finite, G/G^c is nontrivial, and there exists a nontrivial homomorphism G \rightarrow \mathbb{Z} with open kernel. Let us endow \mathbb{Z}[\mathbb{Z}] with the structure of a smooth G-module by means of this homomorphism. Consider the short exact sequence of G-modules:
\[
0 \rightarrow \mathbb{Z}[\mathbb{Z}] \xrightarrow{n \mapsto n+1} \mathbb{Z}[\mathbb{Z}] \rightarrow k \rightarrow 0.
\]

Tensoring it with \rho_1 we get
\[
0 \rightarrow \rho_1 \otimes k[\mathbb{Z}] \rightarrow \rho_1 \otimes k[\mathbb{Z}] \rightarrow \rho_1 \rightarrow 0. \tag{47}
\]

It is easy to see that \rho_1 \otimes k[\mathbb{Z}] is finitely generated provided \rho_1 is admissible. Thus (47) implies that \|\rho_1\| = 0. \qed
4.4. Proof of 0.20. By Lemma 4.1
\[ \sum (-1)^i \dim \text{Ext}^i(\rho_1, \rho_2) = \int_G \chi_{\rho_2} \cdot \langle \rho_1 \rangle. \]

By Harish-Chandra’s Theorem on integrability of characters we can rewrite
\[ \int_G \chi_{\rho_2} \cdot \langle \rho_1 \rangle = \lim_U \int_U \chi_{\rho_2} \cdot \langle \rho_1 \rangle, \]
where \( U \) runs over an exhausting family of conjugation-invariant subsets of regular elements. Each \( U \) in this family can be written as a disjoint union \( U = U^{\text{ell}} \cup U^{\text{nonell}} \) of a subset of the elliptic (respectively, non-elliptic) set. We have \( \int_{U^{\text{nonell}}} \chi_{\rho_2} \cdot \langle \rho_1 \rangle = 0 \), for all \( U \) by 4.2, while
\[ \int_{U^{\text{ell}}} \chi_{\rho_2} \cdot \langle \rho_1 \rangle = \int_{U^{\text{ell}}/\text{Ad}} \chi_{\rho_2}(g) O_\rho((\rho_1)) d\mu = \int_{U^{\text{ell}}/\text{Ad}} \chi_{\rho_2}(g) \chi_{\rho_1}(g^{-1}) d\mu, \]
where the first equality is (44), and the second follows from 0.19. As \( U^{\text{ell}} \) increases, the latter expression tends to \( \int_{\text{Ell}} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g) d\mu(g) \). The proof is finished.
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