TIGHTNESS OF A NEW AND ENHANCED SEMIDEFINITE RELAXATION FOR MIMO DETECTION

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Abstract. In this paper, we consider a fundamental problem in modern digital communications known as multi-input multi-output (MIMO) detection, which can be formulated as a complex quadratic programming problem subject to unit-modulus and discrete argument constraints. Various semidefinite relaxation (SDR) based algorithms have been proposed to solve the problem in the literature. In this paper, we first show that the conventional SDR is generically not tight for the problem. Then, we propose a new and enhanced SDR and show its tightness under an easily checkable condition, which essentially requires the level of the noise to be below a certain threshold. The above results have answered an open question posed by So in [22]. Numerical simulation results show that our proposed SDR significantly outperforms the conventional SDR in terms of the relaxation gap.

Key words. complex quadratic programming, semidefinite relaxation, MIMO detection, tight relaxation

AMS subject classifications. 90C22, 90C20, 90C46, 90C27

1. Introduction. Multiple-input multiple-output (MIMO) detection is a fundamental problem in modern digital communications [29]. Mathematically, the input-output relationship of the MIMO channel can be modeled as

\[(1.1) \quad r = Hx^* + v,\]

where
- \( r \in \mathbb{C}^m \) is the vector of received signals,
- \( H \in \mathbb{C}^{m \times n} \) is an \( m \times n \) complex channel matrix (for \( n \) inputs and \( m \) outputs with \( m \geq n \)),
- \( x^* \in \mathbb{C}^n \) is the vector of transmitted symbols, and
- \( v \in \mathbb{C}^m \) is an additive white Gaussian noise.

Assume that \( M \)-Phase-Shift Keying modulation scheme with \( M \geq 2 \) is adopted. Then each entry \( x^*_i \) of \( x^* \) belongs to a finite set of symbols, i.e.,

\[ x^*_i \in \{ e^{j \theta} \mid \theta = 2j\pi/M, \ j = 0, 1, \ldots, M - 1 \}, \ i = 1, 2, \ldots, n, \]

where \( j \) is the imaginary unit. The MIMO detection problem is to recover the vector of transmitted symbols \( x^* \) from the vector of received signals \( r \) based on the knowledge of the channel matrix \( H \). The mathematical formulation of the problem is

\[
\begin{align*}
\text{min} & \quad \|Hx - r\|_2^2 \\
\text{s.t.} & \quad |x_i|^2 = 1, \ \arg(x_i) \in A, \ i = 1, 2, \ldots, n,
\end{align*}
\]

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where \( \| \cdot \|_2 \) denotes the Euclidean norm, \( \arg ( \cdot ) \) denotes the argument of the complex number, and

\[
A = \{ 0, 2\pi/M, \ldots , 2(M - 1)\pi/M \}.
\]

It has been shown that minimizing the square error in the above MIMO detection problem (P) is equivalent to minimizing the probability of detection error (see, e.g., [27, Chapter 3]).

There are two related problems to MIMO detection problem (P). The first one is the following complex quadratic programming problem:

\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad x^\dagger Qx + 2 \text{Re}(c^\dagger x) \\
\text{s.t.} & \quad |x_i|^2 = 1, \ \arg(x_i) \in A, \ i = 1, 2, \ldots , n,
\end{align*}

where \( Q \in \mathbb{C}^{n \times n} \) is a Hermitian matrix, \( c \in \mathbb{C}^n \) is a complex column vector, \( (\cdot)^\dagger \) denotes the conjugate transpose, and \( \text{Re}(\cdot) \) denotes the element-wise real part of a complex matrix/vector/number. Problem (P) is a special case of problem (CQP) where \( Q = H^\dagger H \) and \( c = -H^\dagger r \). The second one is the following unit-modulus constrained quadratic programming problem:

\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad x^\dagger Qx + 2 \text{Re}(c^\dagger x) \\
\text{s.t.} & \quad |x_i|^2 = 1, \ i = 1, 2, \ldots , n,
\end{align*}

which can be seen as a continuous relaxation of problem (CQP) in the sense that the discrete argument constraints \( \arg (x_i) \in A \) for \( i = 1, \ldots , n \) in problem (CQP) are dropped.

Both problems (CQP) and (UQP) have been extensively studied due to their broad applications. For instance, the Max-Cut problem [6] and the Max-3-Cut problem [7] are special cases of problem (CQP) with homogeneous objective functions and with \( M \in \{ 2, 3 \} \); the classical binary quadratic programming problem [4] is also a special case of problem (CQP) with \( M = 2 \). Moreover, problems (CQP) and (UQP) have found wide applications in signal processing and wireless communications, including MIMO detection [9], angular synchronization [2, 20], phase retrieval [28], and radar signal processing [18, 19, 23]. However, it is known that both problems (CQP) and (UQP) are NP-hard [30], and the MIMO detection problem, as a special case of problem (CQP), is also NP-hard [26]. Therefore, there is no polynomial-time algorithms which can solve these problems to global optimality in general, unless \( P = NP \).

The semidefinite relaxation (SDR) based algorithms are perhaps the most popular ones to be used to solve problems (CQP) and (UQP), especially in the signal processing and wireless communication community [16]. For various SDR based algorithms for problems (CQP) and (UQP) under different signal processing and wireless communication scenarios, we refer the interested reader to [2, 9, 10, 12, 15, 17, 18, 19, 22, 25, 28, 30] and the references therein. The SDR based algorithms generally perform very well in some signal processing and wireless communication applications, as pointed out in [3]. Similar observations have also been made for MIMO detection [17], asynchronous multi-sensor registration [19], as well as angular synchronization [20]. For the above applications, the SDR based algorithms proved to be impressively effective if the so-called signal-to-noise ratio (SNR) is high.

Therefore, it has been a longstanding important question in the field as to understand why the performances of the SDR based algorithms are so remarkably good in
practice. One line of research is directed to analyze the approximation ratios of the SDR based algorithms. Along this direction, the approximation ratios of some SDR based algorithms have been analyzed in [21, 30] for problems (CQP) and (UQP) with homogeneous positive semidefinite objective functions. Another line of research is to identify conditions under which the SDRs are tight [2, 12, 22]. The phenomenon that some nonconvex problems are equivalent to their convex relaxations under certain conditions can also be regarded as a type of hidden convexity [24]. It is also worth remarking that, in some signal processing and wireless communication applications with high SNRs, even first-order algorithms are guaranteed to converge to the global solution of these nonconvex problems [5, 13, 14].

In this paper, we focus on the MIMO detection problem (P). For ease of presentation, we define the tightness of an SDR of the MIMO detection problem (P) as follows.

**Definition 1.1.** An SDR of problem (P) is called tight if the following two conditions hold:
- the gap between the SDR and problem (P) is zero; and
- the SDR recovers the true vector of transmitted signals.

For MIMO detection problem (P), the tightness of some SDRs has been studied in [8, 9, 11, 12, 22]. In particular, So proved in [22] that, for the case where $M = 2$, there exists a tight SDR (see (BSDP) further ahead) if the inputs $H$ and $v$ in (1.1) satisfy

$$\lambda_{\min}(\Re(H^\dagger H)) > \|\Re(H^\dagger v)\|_\infty,$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a given matrix and $\|\cdot\|_\infty$ denotes the $\infty$-norm. In [22], So also posed the following open question: Is the condition

$$\lambda_{\min}(H^\dagger H) > \|H^\dagger v\|_\infty$$

sufficient for the conventional (complex) SDR being tight for problem (P) with $M \geq 3$?

The main contributions of our paper are twofold. First, we show that the conventional SDR is generically not tight for (P) and thus answers an open question posed by So. Second, we propose an enhanced SDR for problem (P), which is much tighter than the conventional SDR. We prove that our proposed enhanced SDR is tight for the case where $M \geq 3$ if the following condition is satisfied:

$$\lambda_{\min}(H^\dagger H) \sin\left(\frac{\pi}{M}\right) > \|H^\dagger v\|_\infty.$$

To the best of our knowledge, our new enhanced SDR is the first one to have a theoretical guarantee of tightness if the SNR of the problem is sufficiently high. Numerical results show that the new enhanced SDR performs significantly better than the conventional SDR in terms of the relaxation gap as well as the ability to recover the vector of transmitted signals.

The rest of this paper is organized as follows. In Section 2, we review the conventional SDR for problem (P) and show that it is generically not tight. Then, we propose an enhanced SDR for problem (CQP) in Section 3 and prove it to be tight for problem (P), a special case of problem (CQP), if condition (1.5) holds in Section 4. In Section 5, we present some numerical results to show the effectiveness of the newly proposed SDR. Finally, we conclude the paper in Section 6.

We adopt the following somewhat standard notations in this paper. For a given complex vector $x$, we use $x_i$ (or $[x]_i$) to denote its $i$-th entry, $\|x\|_2$ to denote its
Euclidean norm, $\|x\|_1$ to denote its 1-norm, $\|x\|_\infty$ to denote its $\infty$-norm, and $\text{Diag}(x)$ to denote the diagonal matrix with the diagonal entries being $x$. For a given complex Hermitian matrix $A$, $A \succeq 0$ means $A$ is positive semidefinite, $\text{Trace}(A)$ denotes the trace of $A$, $A_{i,j}$ denotes the $(i,j)$-th entry of $A$, and $A^\dagger$ and $A^\top$ denotes the conjugate transpose and transpose of $A$, respectively. For two Hermitian matrices $A$ and $B$, $A \succeq B$ means $A - B \succeq 0$ and $A \bullet B$ means $\text{Re}(\text{Trace}(A^\dagger B))$. For any given matrix $C \in \mathbb{C}^{m \times n}$ (including the scalar case and the vector case), we use $\text{Re}(C)$ and $\text{Im}(C)$ to denote the component-wise real and imaginary parts of $C$, respectively. For a set $\mathcal{S}$, we use $\text{Conv}(\mathcal{S})$ to denote its convex envelope. Finally, we use $i$ to denote the imaginary unit which satisfies the equation $i^2 = -1$. In the remainder of this paper, we will focus on the MIMO detection problem. We denote

$$Q = H^\dagger H, \ c = -H^\dagger r, \ \text{and} \ \lambda_{\min} = \lambda_{\min}(Q)$$

unless otherwise specified.

2. Conventional semidefinite relaxations. By introducing $X = xx^\dagger$ and relaxing it to $X \succeq xx^\dagger$ and dropping the argument constraints $\text{arg}(x_i) \in A$ for all $i = 1, \ldots, n$, we get the following conventional SDR for (CQP):

$$\min_{x, X} \quad Q \bullet X + 2\text{Re}(c^\dagger x)$$

s.t. $\quad X_{i,i} = 1, \ i = 1, 2, \ldots, n,$

$$\quad X \succeq xx^\dagger,$$

where the variables $x \in \mathbb{C}^n$ and $X \in \mathbb{C}^{n \times n}$. (CSDP) is a complex semidefinite program, which has been widely used in the literature. Indeed, the approximation algorithms in [2, 9, 12, 17, 18, 21, 22, 28, 30] are all based on (CSDP).

In [22], So studied the tightness of an SDR for the MIMO detection problem. For the special case where $M = 2$, the argument constraints of (CQP) become the binary constraints $x \in \{-1, 1\}^n$. Therefore, (CSDP) can be posed as the following real SDR:

$$\min_{x, X} \quad \text{Re}(Q) \bullet X + 2\text{Re}(c^\top x)$$

s.t. $\quad X_{i,i} = 1, \ i = 1, 2, \ldots, n,$

$$\quad X \succeq xx^\dagger,$$

where the variables $x \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times n}$. Based on (BSDP), So proved the following theorem.

Theorem 2.1 ([22]). Suppose that $M = 2$. If the inputs $H$ and $v$ in (1.1) satisfy (1.3), then (BSDP) is tight for (P).

For the case where $M = 2$, Theorem 1 proposes a sufficient condition under which (BSDP) is tight for (P). In fact, for the same case, similar sufficient conditions for (BSDP) to be tight have been proposed in [8, 9, 11]. For instance, the condition proposed in [11, Theorem 1] is

$$\lambda_{\min}(\text{Re}(H^\dagger H)) > \|\text{Re}(H^\dagger v)\|_1$$

and the one proposed in [8, 9] is

$$\lambda_{\min}(\text{Re}(H^\dagger H)) > \|\text{Re}(H^\dagger v)\|_2.$$
For more general cases where $M \geq 3$, So posed an open question in [22]: Is condition (1.4) sufficient for (CSDP) to be tight for (P)? Next, we show that the answer to this question is negative. More specifically, we show that (CSDP) is not tight for almost all instances of (P) with nonzero random noise in the sense that the probability that there is no gap between (CSDP) and (P) is zero.

To tackle the problem, we first study the KKT condition of (UQP). Denote

$$
\hat{Q} = \begin{bmatrix}
\text{Re}(Q) & -\text{Im}(Q) \\
\text{Im}(Q) & \text{Re}(Q)
\end{bmatrix}, \quad \hat{c} = \begin{bmatrix}
\text{Re}(c) \\
\text{Im}(c)
\end{bmatrix}, \quad \text{and } y = \begin{bmatrix}
\text{Re}(x) \\
\text{Im}(x)
\end{bmatrix}
$$

and (UQP) in its real form can be written as

$$
\begin{align*}
\min_{y \in \mathbb{R}^{2n}} & \quad y^T \hat{Q} y + 2 \hat{c}^T y \\
\text{s.t.} & \quad y_i^2 + y_{n+i}^2 = 1, \quad i = 1, 2, \ldots, n.
\end{align*}
$$

The Lagrangian function of (2.2) is

$$
L(y; \lambda) = y^T \hat{Q} y + 2 \hat{c}^T y + \sum_{i=1}^{n} \lambda_i (y_i^2 + y_{n+i}^2 - 1),
$$

where $\lambda_i$ is the Lagrange multiplier corresponding to the constraint $y_i^2 + y_{n+i}^2 = 1$.

Let $x$ be an optimal solution of (UQP) and $y$ be the corresponding optimal solution of (2.2). Then there exist $\{\lambda_i \in \mathbb{R}\}_{i=1}^{n}$ such that

$$
\frac{\partial L(y; \lambda)}{\partial y_i} = 2 \left[ \hat{Q} y \right]_i + 2 \hat{c}_i + 2 \lambda_i y_i = 0, \quad i = 1, 2, \ldots, n
$$

and

$$
\frac{\partial L(y; \lambda)}{\partial y_{i+n}} = 2 \left[ \hat{Q} y \right]_{i+n} + 2 \hat{c}_{i+n} + 2 \lambda_i y_{i+n} = 0, \quad i = 1, 2, \ldots, n.
$$

Since

$$
2 \left[ \hat{Q} y \right]_i + 2 \hat{c}_i + 2 \lambda_i y_i = 2 \text{Re} \left( [Qx]_i + c_i \right) + 2 \lambda_i \text{Re} \left( x_i \right), \quad i = 1, 2, \ldots, n
$$

and

$$
2 \left[ \hat{Q} y \right]_{i+n} + 2 \hat{c}_{i+n} + 2 \lambda_i y_{i+n} = 2 \text{Im} \left( [Qx]_i + c_i \right) + 2 \lambda_i \text{Im} \left( x_i \right), \quad i = 1, 2, \ldots, n,
$$

the KKT condition of (UQP) in the complex form becomes

$$
[Qx]_i + c_i + \lambda_i x_i = 0, \quad i = 1, 2, \ldots, n.
$$

Now, we prove the next theorem.

**Theorem 2.2.** Suppose that $M \geq 2$. If (CSDP) is tight for (P), then there exist $\{\lambda^*_i \in \mathbb{R}\}_{i=1}^{n}$ such that

$$
[Q^*]_i = \lambda^*_i x^*_i, \quad i = 1, 2, \ldots, n,
$$

where $H$, $x^*$, and $v$ are given in (1.1).
Proof. If (CSDP) is tight for (P), then \( x^* \) and \( x^*(x^*)^\dagger \) are optimal to (CSDP). Since (CSDP) is an SDR of (UQP), \( x^* \) must also be an optimal solution of (UQP). Hence, \( x^* \) satisfies the KKT condition of (UQP), i.e., there exist \( \{\lambda_i^* \in \mathbb{R}\} \) such that

\[
[Qx^* + c]_i + \lambda_i^* x_i^* = 0, \quad i = 1, 2, \ldots, n.
\]

This, together with \( Qx^* + c = H^\dagger Hx^* - H^\dagger r = -H^\dagger v \), immediately implies the desired result (2.3). \( \square \)

The conditions in (2.3) mean that \( [H^\dagger v]_i \) and \( x_i^* \) have the same phase for all \( i = 1, 2, \ldots, n \). However, this is generically not true in real applications. In particular, the noise vector \( v \) is often assumed to follow the complex Gaussian distribution and therefore the probability that all conditions in (2.3) are simultaneously satisfied is zero. This immediately implies that (CSDP) is generically not tight for (P) (regardless of the condition in (1.4)). This answers the open question posed in [22]. It is also worth remarking that (CSDP) and (BSDP), even for the case where \( M = 2 \), are not equivalent to each other.

3. An Enhanced Semidefinite Relaxation. In this section, we propose an enhanced SDR for (CQP). Recall that the (discrete) argument constraints \( \arg (x_i) \in \mathcal{A} \) for \( i = 1, 2, \ldots, n \) in (CQP) are ignored in its conventional relaxation (CSDP). The idea of designing the enhanced SDR for (CQP) is to better exploit the structure of the argument constraints and develop valid linear constraints for them to tighten (CSDP). Since these valid linear constraints are based on the real form of (CSDP), we first reformulate (CSDP) as the following real SDR:

\[
\begin{align*}
\min_{y, Y} & \quad \hat{Q} \bullet Y + 2\hat{c}^Ty \\
\text{s.t.} & \quad Y_{i,i} + Y_{n+i,n+i} = 1, \quad i = 1, 2, \ldots, n, \\
& \quad \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \succeq 0,
\end{align*}
\]

(RSDP)

where the variables \( y \in \mathbb{R}^{2n} \) and \( Y \in \mathbb{R}^{2n \times 2n} \) and \( \hat{Q}, \hat{c}, \) and \( y \) are defined in (2.1).

Remark that our real reformulation (RSDP) of complex (CSDP) is not the same as the ones in [7] and [28]. The dimension of the matrix variable in (RSDP) is \( 2n + 1 \) while the one of the matrix variable in [7] and [28] is \( 2n + 2 \). Hence, the equivalence between (CSDP) and (RSDP) cannot be shown by using the same argument in [7] and [28]. An equivalence proof of (CSDP) and (RSDP) is provided in Appendix A.

Next, we develop some valid linear constraints for the argument constraints based on (RSDP), which leads to an enhanced SDR for (CQP). First, for (RSDP), we define the following \( 3 \times 3 \) matrices:

\[
(3.1) \quad \mathcal{Y}(i) := \begin{bmatrix} 1 & y_i & y_{n+i} \\ y_i & Y_{i,i} & Y_{i,n+i} \\ y_{n+i} & Y_{n+i,i} & Y_{n+i,n+i} \end{bmatrix}, \quad i = 1, 2, \ldots, n.
\]

Each of \( \mathcal{Y}(i) \) contains the following 5 variables in (RSDP) (due to its symmetry):

\( y_i, \ y_{n+i}, \ Y_{i,i}, \ Y_{i,n+i}, \) and \( Y_{n+i,n+i} \).

Since \( y_i = \text{Re}(x_i), \ y_{n+i} = \text{Im}(x_i) \), and the constraints \( |x_i|^2 = 1 \) and \( \arg (x_i) \in \mathcal{A} \), it follows

\[
(y_i, y_{n+i}) \in \{ (\cos(\theta), \sin(\theta)) | \theta = 2j\pi/M, \ j = 0, 1, \ldots, M - 1 \}.
\]
Define the following $3 \times 3$ real symmetric matrices:

\begin{equation}
(3.2) \quad P_j = \begin{bmatrix}
1 & \cos \left( \frac{2j\pi}{M} \right) & \sin \left( \frac{2j\pi}{M} \right) \\
\cos \left( \frac{2j\pi}{M} \right) & 1 & \cos \left( \frac{2j\pi}{M} \right) \\
\sin \left( \frac{2j\pi}{M} \right) & \cos \left( \frac{2j\pi}{M} \right) & 1
\end{bmatrix}, \quad j = 0, 1, \ldots, M - 1.
\end{equation}

Then, each of $\mathcal{Y}(i)$ must equal one of matrices $P_j$ with $j = 0, 1, \ldots, M - 1$, i.e.,

$$
\mathcal{Y}(i) \in \{P_0, P_1, \ldots, P_{M-1}\}, \quad i = 1, 2, \ldots, n.
$$

The convex relaxation of the above constraints are

$$
\mathcal{Y}(i) \in \text{Conv} \{P_0, P_1, \ldots, P_{M-1}\}, \quad i = 1, 2, \ldots, n,
$$

which are equivalent to

$$
\mathcal{Y}(i) = \sum_{j=0}^{M-1} t_{i,j} P_j, \quad \sum_{j=0}^{M-1} t_{i,j} = 1, \quad t_{i,j} \geq 0, \quad j = 0, 1, \ldots, M - 1, \quad i = 1, 2, \ldots, n.
$$

Due to the symmetry of $\mathcal{Y}(i)$, the constraint $\mathcal{Y}(i) = \sum_{j=0}^{M-1} t_{i,j} P_j$ can be explicitly expressed as the following 5 linear constraints:

$$
Y_{i,i} = \sum_{j=0}^{M-1} t_{i,j} \cos \left( \frac{2j\pi}{M} \right), \quad Y_{n+i,n+i} = \sum_{j=0}^{M-1} t_{i,j} \sin \left( \frac{2j\pi}{M} \right),
$$

$$
Y_{i,i} = \sum_{j=0}^{M-1} t_{i,j} \sin \left( \frac{2j\pi}{M} \right), \quad Y_{n+i,n+i} = \sum_{j=0}^{M-1} t_{i,j} \cos \left( \frac{2j\pi}{M} \right),
$$

$$
Y_{i,n+i} = \sum_{j=0}^{M-1} t_{i,j} \cos \left( \frac{2j\pi}{M} \right) \sin \left( \frac{2j\pi}{M} \right), \quad Y_{n+i,i} = \sum_{j=0}^{M-1} t_{i,j} \sin \left( \frac{2j\pi}{M} \right) \cos \left( \frac{2j\pi}{M} \right).
$$

Obviously, the above equations and $\sum_{j=0}^{M-1} t_{i,j} = 1$ imply $Y_{i,i} + Y_{n+i,n+i} = 1$. By dropping redundant constraints, we get the following enhanced SDR for (CQP):

$$
\begin{align*}
\min_{y, Y, t} & \quad Q \bullet Y + 2\hat{c}^T y \\
\text{s.t.} & \quad \mathcal{Y}(i) = \sum_{j=0}^{M-1} t_{i,j} P_j, \quad \sum_{j=0}^{M-1} t_{i,j} = 1, \quad i = 1, 2, \ldots, n, \\
& \quad t_{i,j} \geq 0, \quad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, M - 1, \\
& \quad \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \succeq 0,
\end{align*}
$$

(ERSDP)

where the variables $y \in \mathbb{R}^{2n}$, $Y \in \mathbb{R}^{2n \times 2n}$, $t \in \mathbb{R}^{n \times M}$, $\hat{Q}$ and $\hat{c}$ are defined in (2.1), $\mathcal{Y}(i)$ is defined in (3.1), and $P_j$ is defined in (3.2).

We term the above real SDP “ERSDP”, since $\mathcal{Y}(i)$ for all $i = 1, 2, \ldots, n$ in (ERSDP) are constrained in the convex envelope with the Extreme points being $P_0, P_1, \ldots, P_{M-1}$, i.e., the variables $y_i, y_{n+i}, Y_{i,i}, Y_{i,n+i},$ and $Y_{n+i,n+i}$ must satisfy the linear constraints given in (3.3), which is the main difference between (ERSDP) and (RSDP). Hence, (ERSDP) is (strictly) tighter than (RSDP) (which is equivalent to (CSDP)).
4. Tightness of (ERSDP). In this section, we study the tightness of the newly proposed SDP relaxation (ERSDP).

Let us first look at a case where \( M = 2 \). In this case,

\[
P_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

and the linear constraints in (3.3) reduce to

\[
Y_{i,i} = 1, \quad y_i = t_{i,1} - t_{i,2}, \quad \text{and} \quad y_{n+i} = Y_{n+i,i} = Y_{n+i,n+i} = 0.
\]

By dropping the zero blocks in the matrix \( Y \), (ERSDP) reduces to (BSDP). Therefore, it follows from Theorem 2.1 that condition (1.3) is sufficient for (ERSDP) to be tight for problem (P) where \( M = 2 \).

In the remainder of this section, we study the tightness of (ERSDP) for (P) where \( M \geq 3 \). We prove that condition (1.5) is sufficient for (ERSDP) to be tight for (P). Our proof consists of two main steps: **Step I**, we derive a complex SDR called (CSDP2) based on (ERSDP) and show that (ERSDP) is tighter than (CSDP2); **Step II**, we show that (CSDP2) is tight for (P) where \( M \geq 3 \) under certain condition and hence (ERSDP) is also tight for (P) under the same condition.

**Step I.** We derive a complex SDR to be called (CSDP2) from (ERSDP). Recall that, for any feasible solution \((y, Y)\) of (ERSDP), \( Y_i \) in (3.1) must lie in the convex envelope of \( \{P_0, \ldots, P_{M-1}\} \), which implies that

\[
(4.1) \quad (y_i, y_{i+n}) \in \text{Conv} \{ (\cos (\theta), \sin (\theta)) \mid \theta \in \mathcal{A} \},
\]

where \( \mathcal{A} \) is defined in (1.2). Note that \( y_i \) and \( y_{i+n} \) correspond to the real and imaginary parts of the complex variable \( x_i \). Then, one can show that (4.1) is equivalent to \( x_i \in \mathcal{F} \), where \( \mathcal{F} \) is the convex envelope of the set

\[
\mathcal{D} = \{w \mid |w| = 1, \quad \arg(w) \in \mathcal{A}\}.
\]

For the case where \( M \geq 3 \), the set \( \mathcal{F} \) can be represented as

\[
\left\{ w \in \mathbb{C} \mid \text{Re}(a_j^i w) \leq \cos \left( \frac{\pi}{M} \right), \quad j = 0, 1, \ldots, M - 1 \right\},
\]

where

\[
(4.2) \quad a_j = e^{i\theta_j} \quad \text{and} \quad \theta_j = \frac{(2j - 1)\pi}{M}.
\]

Furthermore, \( \text{Re}(a_j^i w) \leq \cos \left( \frac{\pi}{M} \right) \) is equivalent to

\[
\cos (\theta_j) \text{Re}(w) + \sin (\theta_j) \text{Im}(w) \leq \cos \left( \frac{\pi}{M} \right).
\]

The line associated with the above half plane connects the two points

\[
\cos \left( \frac{2(j - 1)\pi}{M} \right) + \sin \left( \frac{2(j - 1)\pi}{M} \right) i \quad \text{and} \quad \cos \left( \frac{2j\pi}{M} \right) + \sin \left( \frac{2j\pi}{M} \right) i
\]

in the complex domain. See Figure 1 for an illustration of the set \( \mathcal{F} \) for the case where \( M = 8 \).
Based on the above observations, we obtain the following complex SDR:

\[
\begin{align*}
\min_{x, X} & \quad Q \cdot X + 2 \text{Re}(c^\dagger x) \\
\text{s.t.} & \quad X_{ii} = 1, \ i = 1, 2, \ldots, n, \\
& \quad \text{Re}(a_j^i x_i) \leq \cos \left( \frac{\pi}{M} \right), \ i = 1, 2, \ldots, n, \ j = 0, 1, \ldots, M - 1,
\end{align*}
\]

where the variables \( x \in \mathbb{C}^n \) and \( X \in \mathbb{C}^{n \times n} \).

Two remarks on the comparison between \((\text{CSDP2})\) and \((\text{ERSDP})\) are in order. First, since the constraints \( \text{Re}(a_j^i x_i) \leq \cos \left( \frac{\pi}{M} \right) \) for all \( i = 1, 2, \ldots, n \) and \( j = 0, 1, \ldots, M - 1 \) in \((\text{CSDP2})\) are derived from \((\text{ERSDP})\), we know that \((\text{ERSDP})\) is at least as tight as \((\text{CSDP2})\). In fact, we have the following result.

**Theorem 4.1.** \((\text{ERSDP})\) is tighter than \((\text{CSDP2})\).

A rigorous proof of Theorem 4.1 can be found in Appendix B. Therefore, if \((\text{CSDP2})\) is tight for \((P)\), then \((\text{ERSDP})\) must also be tight. Second, \((\text{CSDP2})\) is tight enough for us to derive our main results in Theorems 4.2 and 4.3. Our analysis is based on a simpler reformulation of \((\text{CSDP2})\) and its dual.

**Step II.** We show the tightness of \((\text{CSDP2})\) in this part. Our basic idea is to construct a dual feasible solution such that \(\left(x^*, x^* (x^*)^\dagger\right)\) and the constructed dual solution jointly satisfy the KKT optimality conditions of \((\text{CSDP2})\) under some
conditions. The dual problem of (CSDP2) is

\[
\max_{\sigma, \lambda, \mu} \quad \sigma - \sum_{i=1}^{n} \lambda_i - \cos \left( \frac{\pi}{M} \right) \sum_{j=0}^{M-1} \mu_{i,j}
\]
\[
\text{s.t.} \quad \begin{bmatrix}
-\sigma & (c + g)\dagger \\
-c + g & Q + \text{Diag}(\lambda)
\end{bmatrix} \succeq 0,
\]
\[
\mu_{i,j} \geq 0, \quad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, M - 1,
\]

where the variables \( \sigma \in \mathbb{R}, \lambda \in \mathbb{R}^n, \mu \in \mathbb{R}^{n \times M} \) and \( g = [g_1, g_2, \ldots, g_n]^T \) with \( g_i = \sum_{j=0}^{M-1} \frac{\mu_{i,j}}{2} a_j, \quad i = 1, 2, \ldots, n \). For completeness, a detailed derivation of the above dual problem is provided in Appendix C. Since both problems (CSDP2) and its dual (4.3) are strictly feasible, it follows that the primal-dual Slater’s conditions are satisfied. Suppose that \((x, X)\) is an optimal solution of (CSDP2). Then there must exist feasible solution \((\lambda, \sigma, \mu)\) (to problem (4.3)) such that \((x, X)\) and \((\lambda, \sigma, \mu)\) jointly satisfy the following complementarity conditions:

\[
\mu_{i,j} \left( \cos \left( \frac{\pi}{M} \right) - \text{Re} \left( a_j^\dagger x_i \right) \right) = 0, \quad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, M - 1
\]

and

\[
\begin{bmatrix}
-\sigma & (c + g)\dagger \\
-c + g & Q + \text{Diag}(\lambda)
\end{bmatrix} \cdot \begin{bmatrix} 1 \\ x^\dagger \\ X \end{bmatrix} = 0.
\]

We are ready to present our main results in this paper.

**Theorem 4.2.** Suppose that \( M \geq 3 \). Let \( x^* \) with

\[
x_i^* = e^{2\pi i s_i/M}, \quad s_i \in \{0, 1, \ldots, M - 1\}, \quad i = 1, 2, \ldots, n
\]

be the vector of transmitted signals. Define

\[
t_i = \begin{cases} 
s_i + 1, & \text{if } s_i < M - 1; \\
0, & \text{if } s_i = M - 1
\end{cases}, \quad i = 1, 2, \ldots, n.
\]

If there exist

\[
\bar{\lambda}_i > -\lambda_{\text{min}}, \quad \bar{\mu}_{i,s_i} \geq 0, \quad \text{and} \quad \bar{\mu}_{i,t_i} \geq 0, \quad i = 1, 2, \ldots, n
\]

such that \( H \) and \( v \) in (1.1) satisfy

\[
[H^\dagger v]_i = \bar{\lambda}_i x_i^* + \frac{\bar{\mu}_{i,s_i}}{2} a_{s_i} + \frac{\bar{\mu}_{i,t_i}}{2} a_{t_i}, \quad i = 1, 2, \ldots, n,
\]

then \((x^*, x^*(x^*)\dagger)\) is the unique solution to (CSDP2).

**Proof.** Let \((\lambda^*, \sigma^*, \mu^*)\) be an optimal solution to problem (4.3). We consider the pair \((x^*, x^*(x^*)\dagger)\) and \((\lambda^*, \sigma^*, \mu^*)\) and show that they jointly satisfy the complementarity conditions (4.4) and (4.5), which then lead to the fact that \((x^*, x^*(x^*)\dagger)\) is an optimal solution of (CSDP2).

We first construct a feasible solution \((\lambda^*, \sigma^*, \mu^*)\) for problem (4.3) as follows:

\[
\lambda_i^* = \bar{\lambda}_i, \quad i = 1, 2, \ldots, n
\]
and

\[ \mu_{i,j}^* = \begin{cases} \bar{\mu}_{i,j}, & \text{if } j \in \{s_i, t_i\}, \\ 0, & \text{if } j \notin \{s_i, t_i\}, \end{cases} \]

for all \( i = 1, 2, \ldots, n \), \( j = 0, 1, \ldots, M - 1 \),

where \( \bar{\lambda}_i \), \( \bar{\mu}_{i,s_i} \), and \( \bar{\mu}_{i,t_i} \) satisfy the conditions stipulated in (4.8). Set

\[ g_i^* = \sum_{j=0}^{M-1} \frac{\mu_{i,j}^*}{2} a_j, \quad i = 1, 2, \ldots, n \]

and denote \( \Lambda^* = \text{Diag}(\lambda^*) \). Since \( \lambda_i^* > -\lambda_{\min} \) for all \( i = 1, 2, \ldots, n \), it follows that \( Q + \Lambda^* > 0 \) (and thus \( Q + \Lambda^* \) is nonsingular). Furthermore, set

\[ \sigma^* = -(c + g^*)\dagger (Q + \Lambda^*)^{-1} (c + g^*) \]

Clearly

\[ \begin{bmatrix} -\sigma^* & (c + g^*)\dagger \\ c + g^* & Q + \Lambda^* \end{bmatrix} \succeq 0. \]

Therefore, the above constructed \((\lambda^*, \sigma^*, \mu^*)\) is feasible to problem (4.3).

Next we show that the pairs \((x^*, x^* (x^*)\dagger)\) and \((\lambda^*, \sigma^*, \mu^*)\) jointly satisfy the complementarity conditions (4.4) and (4.5). Note that \( \mu_{i,j}^* = 0 \) if \( j \notin \{s_i, t_i\} \) (cf. (4.11)) and \( \cos(\pi M) - \Re(a_j^* x_i^*) = 0 \) if \( j \in \{s_i, t_i\} \) (from (4.2) and (4.7)). Consequently, (4.4) is true. By simple calculation, we have

\[ \begin{bmatrix} -\sigma^* & (c + g^*)\dagger \\ c + g^* & Q + \Lambda^* \end{bmatrix} \begin{bmatrix} 1 \\ x^* \end{bmatrix} = (x^*)\dagger (Q + \Lambda^*) x^* + 2 \Re ((c + g^*)\dagger x^*) - \sigma^*. \]

By (4.9), (4.10), and (4.11), we obtain \( H^\dagger v = \Lambda^* x^* + g^* \), which, together with \( Q = H^\dagger H \) and \( c = -H^\dagger r = -H^\dagger H x^* - H^\dagger v \), further implies

\[ c + g^* = -H^\dagger H x^* - H^\dagger v + g^* = -H^\dagger H x^* - \Lambda^* x^* = -(Q + \Lambda^*) x^*. \]

From the above and the definition of \( \sigma^* \) (cf. (4.12)), we get

\[ \sigma^* = -(c + g^*)\dagger (Q + \Lambda^*)^{-1} (c + g^*) = -(x^*)\dagger (Q + \Lambda^*) x^*. \]

Substituting the above two equations into (4.13), we immediately obtain (4.5). Therefore, both of the complementarity conditions (4.4) and (4.5) hold.

From the above analysis, we can conclude that \((x^*, x^* (x^*)\dagger)\) is an optimal solution of (CSDP2). Since \( Q + \Lambda^* > 0 \), it follows from [1, Theorem 10] that \((x^*, x^* (x^*)\dagger)\) is the unique optimal solution of (CSDP2). The proof of Theorem 4.2 is completed.

As a direct consequence of Theorem 4.2, we have the following result.

**Theorem 4.3.** Suppose that \( M \geq 3 \). If the inputs \( H \) and \( v \) in (1.1) satisfy (1.5), then both (CSDP2) and (ERSDP) are tight for \((P)\).
Proof. Consider the set
\[ \mathcal{S}_i = \left\{ \lambda_i x_i^* + \frac{\mu_{i,s_i}}{2} a_{s_i} + \frac{\mu_{i,t_i}}{2} a_{t_i} \in \mathbb{C} \mid \lambda_i \geq -\lambda_{\min}, \mu_{i,s_i} \geq 0, \mu_{i,t_i} \geq 0 \right\}. \]
By (4.2) and (4.6), we have \( a_{s_i} = x_i^* e^{-\pi i/M} \) and \( a_{t_i} = x_i^* e^{\pi i/M} \). Hence, \( \mathcal{S}_i \) can be rewritten as \( \tilde{\mathcal{S}}_i = x_i^* \hat{\mathcal{S}}_i \), where
\[ \tilde{\mathcal{S}}_i = \left\{ \lambda_i + \frac{\mu_{i,s_i}}{2} e^{-\pi i/M} + \frac{\mu_{i,t_i}}{2} e^{\pi i/M} \in \mathbb{C} \mid \lambda_i \geq -\lambda_{\min}, \mu_{i,s_i} \geq 0, \mu_{i,t_i} \geq 0 \right\}. \]
Note that \( \hat{\mathcal{S}}_i \) is a polyhedron in the complex domain with the extreme point being \(-\lambda_{\min}\) and two extreme directions being \(e^{-\pi i/M}\) and \(e^{\pi i/M}\), that is, a polyhedron in the 2-dimensional real domain with the extreme point being \((-\lambda_{\min}, 0)\) and two extreme directions being \((\cos(\pi/M), -\sin(\pi/M))\) and \((\cos(\pi/M), \sin(\pi/M))\). One easily verifies that: (1) the origin lies in the set \( \hat{\mathcal{S}}_i \), and (2) the smallest distance from the origin to the boundary of the set \( \hat{\mathcal{S}}_i \) is \( \lambda_{\min} \sin \left( \frac{\pi}{M} \right) \). Therefore, if \( H \) and \( v \) in (1.1) satisfy the condition in (1.5), then \([H^* v]_j \) lies in the interior of \( \hat{\mathcal{S}}_i \) for all \( i = 1, 2, \ldots, n \), which further implies that the conditions in (4.9) are satisfied. Invoking Theorem 4.2, we conclude that (CSDP2) is tight for (P) if (1.5) is satisfied. Since (ERSDP) is even tighter than (CSDP2) (cf. Theorem 4.1), it follows that (ERSDP) is also tight for (P) if (1.5) is satisfied. The proof is completed. □

Theorem 4.3 shows that both (CSDP2) and (ERSDP) are tight for MIMO detection problem (P) under the condition (1.5) and thus answers the open question posed by So in [22]. Moreover, our result in Theorem 4.3 is for arbitrary \( M \geq 3 \), in contrast to all previous results in [8, 9, 11, 12, 22], which focus on the case \( M = 2 \).

5. Numerical experiments. In this section, we present some numerical experiment results to illustrate the tightness of our proposed enhanced (ERSDP) and the conventional (CSDP) for problem (P). We generate the instances of problem (P) as follows: we generate each entry of the channel matrix \( H \in \mathbb{C}^{m \times n} \) according to the complex standard Gaussian distribution (with zero mean and unit variance); for each \( i = 1, 2, \ldots, n \), we uniformly choose \( s_i \) from \( \{0, 1, \ldots, M-1\} \) and set each entry of the vector \( x^* \in \mathbb{C}^n \) to \( x_i^* = e^{i \pi s_i/M} \); we generate each entry of the noise vector \( v \in \mathbb{C}^m \) according to the complex Gaussian distribution with zero mean and with variance \( \sigma^2 \); and finally we compute \( r \) as in (1.1). In our numerical experiments, we set \( m = 15 \) and \( n = 10 \). In practical digital communications, \( M \) generally is \( 2, 4, 8, \ldots \). In our experiments, to study the tightness of (ERSDP) on various different settings, we consider all cases where \( M \in \{3, 4, 6, 8\} \).

To compare the numerical performance of (ERSDP) and (CSDP), we generate a total of 40 instances with \( M = 3 \) and \( \sigma^2 \in \{0.01, 0.1, 1.0, 10\} \). Numerical results are summarized in Table 1, where “LBC” and “LBE” denote the lower bounds (i.e., the optimal objective values) returned by two SDP relaxations (CSDP) and (ERSDP), respectively; “UB” denotes the upper bound (i.e., the objective value at the feasible point) returned by the approximation algorithms in [21] and [30]; “GapC” (= UB−LBC) and “GapE” (= UB−LBE) denote the gaps between the upper bound and the corresponding lower bounds; “ClosedGap” denotes (LBE − LBC)/(UB − LBC), which measures how much of the gap in (CSDP) is closed by (ERSDP); and finally “Y/N” denotes whether (ERSDP) is tight or not and “Y” and “N” denote “Yes” and “No”, respectively. Obviously, the larger the value of ClosedGap is, the larger portion of the gap in (CSDP) is closed by (ERSDP) and the better the performance of (ERSDP) (compared to (CSDP)).
Table 1

Numerical results of (ERSDP) and (CSDP) on 40 randomly generated instances of problem (P) with different levels of noise and $M = 3$.

| ID | $\sigma^2$ | LBC   | LBE   | UB    | GapC | GapE  | ClosedGap | Y/N |
|----|------------|-------|-------|-------|------|-------|-----------|-----|
| 1  | 0.01       | -316.111 | -316.045 | -316.045 | 0.066 | 0.000 | 100.0%   | Y   |
| 2  | 0.01       | -299.784 | -299.696 | -299.696 | 0.089 | 0.000 | 100.0%   | Y   |
| 3  | 0.01       | -308.452 | -308.344 | -308.344 | 0.108 | 0.000 | 100.0%   | Y   |
| 4  | 0.01       | -255.438 | -255.368 | -255.368 | 0.070 | 0.000 | 100.0%   | Y   |
| 5  | 0.01       | -179.588 | -179.443 | -179.443 | 0.145 | 0.000 | 100.0%   | Y   |
| 6  | 0.01       | -270.658 | -270.618 | -270.618 | 0.041 | 0.000 | 100.0%   | Y   |
| 7  | 0.01       | -220.304 | -220.276 | -220.276 | 0.028 | 0.000 | 100.0%   | Y   |
| 8  | 0.01       | -223.241 | -223.097 | -223.097 | 0.144 | 0.000 | 100.0%   | Y   |
| 9  | 0.01       | -440.008 | -439.955 | -439.955 | 0.053 | 0.000 | 100.0%   | Y   |
| 10 | 0.01       | -282.097 | -282.001 | -282.001 | 0.095 | 0.000 | 100.0%   | Y   |
| 11 | 0.1        | -182.957 | -181.576 | -181.576 | 1.380 | 0.000 | 100.0%   | Y   |
| 12 | 0.1        | -221.830 | -220.881 | -220.881 | 0.949 | 0.000 | 100.0%   | Y   |
| 13 | 0.1        | -336.557 | -336.093 | -336.093 | 0.464 | 0.000 | 100.0%   | Y   |
| 14 | 0.1        | -321.355 | -319.680 | -319.680 | 1.674 | 0.000 | 100.0%   | Y   |
| 15 | 0.1        | -333.037 | -331.773 | -331.773 | 1.264 | 0.000 | 100.0%   | Y   |
| 16 | 0.1        | -287.849 | -286.448 | -286.448 | 1.402 | 0.000 | 100.0%   | Y   |
| 17 | 0.1        | -283.624 | -282.364 | -282.364 | 1.260 | 0.000 | 100.0%   | Y   |
| 18 | 0.1        | -264.686 | -263.655 | -263.655 | 1.031 | 0.000 | 100.0%   | Y   |
| 19 | 0.1        | -375.708 | -374.947 | -374.947 | 1.211 | 0.000 | 100.0%   | Y   |
| 20 | 0.1        | -385.069 | -383.395 | -383.356 | 1.714 | 0.039 | 97.7%    | N   |
| 21 | 1.0        | -267.937 | -260.953 | -258.935 | 9.002 | 2.018 | 77.6%    | N   |
| 22 | 1.0        | -310.189 | -290.710 | -289.668 | 11.521 | 1.042 | 91.0%    | N   |
| 23 | 1.0        | -462.995 | -455.912 | -452.758 | 10.237 | 3.153 | 69.2%    | N   |
| 24 | 1.0        | -399.650 | -393.103 | -393.103 | 6.547 | 0.000 | 100.0%   | Y   |
| 25 | 1.0        | -196.489 | -193.127 | -193.127 | 3.362 | 0.000 | 100.0%   | Y   |
| 26 | 1.0        | -252.685 | -241.514 | -240.779 | 11.906 | 0.734 | 93.8%    | N   |
| 27 | 1.0        | -341.429 | -326.867 | -325.963 | 15.466 | 0.904 | 94.2%    | N   |
| 28 | 1.0        | -382.724 | -370.711 | -369.352 | 13.373 | 1.359 | 89.8%    | N   |
| 29 | 1.0        | -443.267 | -432.466 | -432.372 | 10.895 | 0.094 | 99.1%    | N   |
| 30 | 1.0        | -322.118 | -315.922 | -314.113 | 8.005 | 1.809 | 77.4%    | N   |
| 31 | 1.0        | -257.180 | -218.040 | -194.286 | 62.894 | 23.754 | 62.2%    | N   |
| 32 | 1.0        | -437.213 | -416.125 | -407.382 | 29.831 | 8.744 | 70.7%    | N   |
| 33 | 1.0        | -301.418 | -271.875 | -242.414 | 59.004 | 29.464 | 50.1%    | N   |
| 34 | 1.0        | -656.062 | -607.805 | -591.345 | 64.718 | 16.460 | 74.6%    | N   |
| 35 | 1.0        | -146.455 | -131.433 | -118.948 | 27.507 | 12.485 | 54.6%    | N   |
| 36 | 1.0        | -214.375 | -200.038 | -167.521 | 46.854 | 32.517 | 30.6%    | N   |
| 37 | 1.0        | -442.110 | -404.595 | -395.709 | 46.401 | 8.887 | 80.8%    | N   |
| 38 | 1.0        | -638.142 | -588.624 | -584.318 | 53.824 | 4.306 | 92.0%    | N   |
| 39 | 1.0        | -306.110 | -270.383 | -246.080 | 60.030 | 24.303 | 59.5%    | N   |
| 40 | 1.0        | -250.783 | -223.968 | -206.346 | 44.437 | 17.622 | 60.3%    | N   |

We can observe from Table 1 that (ERSDP) is generally tight for (P) when the noise level $\sigma^2$ is low, i.e., when the SNR is high. More specifically, (ERSDP) is tight for 19 instances (out of 20) with $\sigma^2 \leq 0.1$ and for 2 instances with $\sigma^2 = 1$. In sharp
Table 2
Numerical results of (ERSDP) and (CSDP) on randomly generated instances of problem (P) with different M and different levels of noise.

| M | $\sigma^2$ | GapC | GapE | ClosedGap | TimeC | TimeE | ProbC | ProbE |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.01 | 0.096 | 0.000 | 100.0% | 0.05 | 0.08 | 0% | 100% |
| 6 | 0.01 | 0.106 | 0.000 | 100.0% | 0.05 | 0.08 | 0% | 100% |
| 8 | 0.1 | 0.100 | 0.000 | 99.9% | 0.05 | 0.09 | 0% | 99% |
| 4 | 0.1 | 0.971 | 0.015 | 98.9% | 0.05 | 0.08 | 0% | 100% |
| 6 | 0.1 | 0.871 | 0.015 | 98.9% | 0.05 | 0.08 | 0% | 80% |
| 8 | 0.1 | 0.979 | 0.056 | 96.2% | 0.05 | 0.09 | 0% | 59% |
| 4 | 1.0 | 9.399 | 0.934 | 91.0% | 0.05 | 0.07 | 0% | 32% |
| 6 | 1.0 | 9.423 | 2.646 | 75.7% | 0.05 | 0.07 | 0% | 7% |
| 8 | 1.0 | 9.162 | 3.739 | 60.8% | 0.05 | 0.07 | 0% | 1% |
| 4 | 10 | 54.251 | 11.328 | 78.6% | 0.05 | 0.07 | 0% | 0% |
| 6 | 10 | 26.827 | 8.085 | 68.2% | 0.05 | 0.07 | 0% | 0% |
| 8 | 10 | 14.803 | 4.617 | 68.0% | 0.05 | 0.07 | 0% | 0% |

contrast, (CSDP) is not tight for all instances. Furthermore, we can see from Table 1 that LBE is much larger than LBC and GapC is much larger than GapE for instances with high levels of noise, which show that (ERSDP) is much tighter than (CSDP) for instances with high levels of noise. For instance, when $\sigma^2 = 10$, GapE is generally much smaller than GapC and about 30% to 92% of the gap in (CSDP) is closed by (ERSDP).

To gain more insight towards the numerical performance of (ERSDP), we carry out numerical experiments on more problem instances with different $M \in \{4, 6, 8\}$ and different $\sigma^2 \in \{0.01, 0.1, 1.0, 10\}$. More specifically, we test the performance of (ERSDP) on 12 different settings and for each setting we randomly generate 100 instances. Table 2 summarizes the average results (over the 100 problem instances) and the statistical results (computed based on the 100 problem instances), where “TimeC” and “TimeE” denote the average CPU time for solving (CSDP) and (ERSDP), respectively; “ProbC” and “ProbE” denote the probability that (CSDP) and (ERSDP) are tight, respectively.

We can conclude from Table 2 that: 1) the probability that (ERSDP) is tight for (P) is very high if the noise level is low; 2) if the noise level is high, (ERSDP) is much tighter than (CSDP) and (ERSDP) narrows down over 60% of the gap due to (CSDP); and 3) the CPU time of solving (ERSDP) is not much longer than the one of solving (CSDP). Finally, (ERSDP) performs better in terms of the tightness on instances with smaller $M$ (if the noise level is fixed). This is consistent with the sufficient condition in (1.5), which shows that a lower level of the noise is required to guarantee the tightness of (ERSDP) for problem (P) with a larger $M$.

6. Conclusion. In this paper, we considered the MIMO detection problem (P), an important class of quadratic programming problems with unit-modulus and discrete argument constraints. We showed that the conventional (CSDP) is generically not tight for (P). Moreover, we proposed a new enhanced SDR called (ERSDP) and showed that (ERSDP) is tight for problem (P) under the condition in (1.5). To the best of our knowledge, our proposed (ERSDP) is the first SDR that is theoretically guaranteed to be tight for general cases of problem (P). Our above results answered an open question posed by So in [22]. In addition to enjoying strong theoretical guar-
Tightness of a New and Enhanced SDR for MIMO Detection

Anteas, our proposed (ERSDP) also performs very well numerically. Our experiment results show that (ERSDP) can return the true vector of transmitted signals with high probability if the level of the noise is low and it can narrow down more than 60% of the gap due to (CSDP) on average if the level of the noise is high. We believe that our proposed (ERSDP) and related techniques will find further applications such as in the development of quantized precoding for massive multi-user MIMO communications [10].

Appendix A: Equivalence between (CSDP) and (RSDP). We first show that, for any feasible point \((x, X)\) of (CSDP), we can construct a feasible point \((y, Y)\) of (RSDP) that satisfies \(Q \cdot Y + 2c^T y = Q \cdot X + 2\text{Re}(c^T x)\). Since

\[
\begin{bmatrix}
1 \\
x
\end{bmatrix} \succeq 0,
\]

we assume, without loss of generality, that it has the following decomposition:

\[
(6.1) \quad \begin{bmatrix}
1 \\
x
\end{bmatrix} = \sum_{j=1}^{n+1} t_j \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix}
\]

where \(t_j \in \mathbb{R}_+\) and \(v_j \in \mathbb{C}^n\) for all \(j = 1, 2, \ldots, n + 1\). Let

\[
(6.2) \quad y = \sum_{j=1}^{n+1} t_j \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix} \quad \text{and} \quad Y = \sum_{j=1}^{n+1} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix}^T.
\]

It is simple to see that

\[
\begin{bmatrix}
1 \\
y
\end{bmatrix} = \sum_{j=1}^{n+1} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix}^T \succeq 0.
\]

It follows from (6.1) and (6.2) that, for any \(i = 1, 2, \ldots, n\), we have

\[
Y_{i,i} + Y_{n+i,n+i} = \sum_{j=1}^{n+1} \left( |\text{Re}(v_j)|^2 + |\text{Im}(v_j)|^2 \right) = \sum_{j=1}^{n+1} |v_j|^2 = X_{i,i} = 1.
\]

Hence, \((y, Y)\) in (6.2) is feasible to (RSDP). Moreover, we have, from (6.1) and (6.2), that

\[
\hat{Q} \cdot Y + 2c^T y = \hat{Q} \cdot \left( \sum_{j=1}^{n+1} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix} \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix}^T \right) + 2c^T \left( \sum_{j=1}^{n+1} t_j \begin{bmatrix}
\text{Re}(v_j) \\
\text{Im}(v_j)
\end{bmatrix} \right)
\]

\[
= Q \cdot \left( \sum_{j=1}^{n+1} v_j v_j^\dagger \right) + 2\text{Re} \left( c^\dagger \left( \sum_{j=1}^{n+1} t_j v_j \right) \right)
\]

\[
= Q \cdot X + 2\text{Re}(c^\dagger x).
\]

We can conclude from the above that (CSDP) is at least as tight as (RSDP).
Conversely, for any given feasible point \((y, Y)\) of (RSDP), we can construct a feasible point \((x, X)\) of (CSDP) that gives the same objective value, i.e., \(Q \cdot X + 2\text{Re}(c^\dagger x) = \hat{Q} \cdot Y + 2\text{c}^T y\). Suppose that \(y = \begin{bmatrix} a \\ b \end{bmatrix}\) and \(Y = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\), where \(a, b \in \mathbb{R}^n\), \(A, B, C \in \mathbb{R}^{n \times n}\), \(A = A^T\), and \(C = C^T\). Let
\[(6.3) \quad x = a + \bar{b}i \quad \text{and} \quad X = (A + C) + (B^T - B)i.\]
Next, we first show
\[(6.4) \quad \hat{X} := \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0.\]
Define
\[M_1 = \begin{bmatrix} 1 & a^T & 0 & b^T \\ a & A & \zeta & B \\ 0 & \zeta^T & 0 & \zeta^T \\ b & B^T & \zeta & C \end{bmatrix}\quad \text{and} \quad M_2 = \begin{bmatrix} 0 & \zeta^T & 0 & \zeta^T \\ \zeta & C & -b & -B^T \\ 0 & -b^T & 1 & a^T \\ \zeta & -B & a & A \end{bmatrix},\]
where \(\zeta\) is the all-zero column vector of dimension \(n\). Then, it follows from
\[\begin{bmatrix} 1 \\ y \end{bmatrix} \begin{bmatrix} y^T \\ Y \end{bmatrix} \succeq 0,\]
that
\[M_1 \succeq 0 \quad \text{and} \quad M_2 = \begin{bmatrix} I_{n+1} & I_{n+1} \\ -I_{n+1} & -I_{n+1} \end{bmatrix} M_1 \begin{bmatrix} I_{n+1} \\ -I_{n+1} \end{bmatrix} \succeq 0,\]
which further implies
\[(6.5) \quad M := M_1 + M_2 = \begin{bmatrix} 1 & a^T & 0 & b^T \\ a & A + C & -b & B - B^T \\ 0 & -b^T & 1 & a^T \\ b & B^T - B & a & A + C \end{bmatrix} \succeq 0.\]
Recall that \(v \in \mathbb{C}^{n+1}\) is an eigenvector of \(\hat{X}\) in (6.4) if and only if
\[\begin{bmatrix} \text{Re}(v) \\ \text{Im}(v) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\text{Im}(v) \\ \text{Re}(v) \end{bmatrix}\]
are two eigenvectors of \(M\) in (6.5) corresponding to the same eigenvalue [7, Page 448]. Then, we have \(\hat{X} \succeq 0\). The positive semidefiniteness of \(\hat{X}\) further implies that all of its diagonal entries are nonnegative. This, together with (6.3), immediately shows that \(X_{i,i} = 1\) for all \(i = 1, 2, \ldots, n\). Moreover, it is simple to check that \(x\) and \(X\) in (6.3) satisfy \(Q \cdot X + 2\text{Re}(c^\dagger x) = \hat{Q} \cdot Y + 2\text{c}^T y\). This shows that (RSDP) is at least as tight as (CSDP).

From the above analysis, we conclude that (CSDP) are (RSDP) are equivalent to each other.
Appendix B: Proof of Theorem 4.1. We first show that, for any feasible point \((y, Y)\) of (ERSDP), we can construct a feasible point \((x, X)\) of (CSDP2) that satisfies \(Q \bullet X + 2\text{Re}(c^\dagger x) = \hat{Q} \bullet Y + 2e^\dagger y\). Similar to the proof in Appendix A, we can construct the same \(x\) and \(X\) in (6.3) and show that \(\hat{X}\) in (6.4) satisfies \(\hat{X} \succeq 0\) and \(X_{i,i} = 1\) for all \(i = 1, 2, \ldots, n\), and \(Q \bullet X + 2\text{Re}(c^\dagger x) = \hat{Q} \bullet Y + 2e^\dagger y\). For each \(i = 1, 2, \ldots, n\), since \(\mathcal{Y}(i) \in \text{Conv}\{P_0, P_1, \ldots, P_{M-1}\}\) in (ERSDP), it follows that
\[
(y_i, y_{i+n}) \in \text{Conv}\{(\cos \theta, \sin \theta) | \theta \in \mathcal{A}\},
\]
which is equivalent to
\[
\text{Re}(a_j^\dagger x_i) \leq \cos \left( \frac{\pi}{M} \right), \quad j = 0, 1, \ldots, M - 1,
\]
where \(x_i = y_i + y_{i+n}i\) and \(a_j\) is given in (4.2). Hence, \((x, X)\) is feasible to (CSDP2). Based on the above analysis, we conclude that (ERSDP) is at least as tight as (CSDP2).

Next, we give an example to illustrate that (ERSDP) is indeed (strictly) tighter than (CSDP2). Let \(m = n = 2\) and \(M = 3\) (and hence the set \(\mathcal{A}\) in (1.2) is \(\{0, 2\pi/3, 4\pi/3\}\) in this case); let
\[
x^* = \begin{pmatrix} -1 + \sqrt{3} & \sqrt{3} \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 8 - 6i & 8 + 6i \\ 3 + 4i & -4 - 3i \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} 5 - 6i \\ 4 - 4i \end{pmatrix};
\]
and finally set \(r = Hx^* + v\) as in (1.1). Then, we numerically solve (CSDP), (CSDP2), and (ERSDP), and their optimal objective values are
\[-76.3176, \quad -45.1273, \quad \text{and} \quad -25.4763,\]
respectively. Hence, (CSDP2) is tighter (CSDP) and (ERSDP) is further tighter than (CSDP2).

Appendix C: Derivation of the Dual Problem of (CSDP2). The canonical form of (CSDP2) is
\[
\begin{array}{ll}
\min & \hat{C} \bullet \hat{X} \\
\text{s.t.} & E_i \bullet \hat{X} = 1, \quad i = 1, 2, \ldots, n + 1, \\
& \text{Re}(a_j^\dagger \hat{X}_{i+1,1}) + y_{i,j} = \cos \left( \frac{\pi}{M} \right), \quad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, M - 1, \\
& y \succeq 0, \quad \hat{X} \succeq 0,
\end{array}
\]
where the variables \(y \in \mathbb{R}^{M \times n}\) and \(\hat{X} \in \mathbb{C}^{(n+1) \times (n+1)}\), \(\hat{C} = \begin{pmatrix} 0 & c^\dagger \\ c & Q \end{pmatrix}\), and \(E_i \in \mathbb{R}^{(n+1) \times (n+1)}\) is the all-zero matrix except its \(i\)-th diagonal entry being 1. Let \(\sigma\) be the Lagrange multiplier corresponding to the constraint \(E_i \bullet \hat{X} = 1\), \(-\lambda_i\) be the Lagrange multiplier corresponding to the constraint \(E_{i+1} \bullet \hat{X} = 1\), and \(-\mu_{i,j}\) be the Lagrange multiplier corresponding to the constraint \(\text{Re}(a_j^\dagger \hat{X}_{i+1,1}) + y_{i,j} = \cos \left( \frac{\pi}{M} \right)\). Notice that both the nonnegative orthant cone and the semidefinite cone are self-dual. Thus, (4.3) is the dual problem of (CSDP2).

REFERENCES
1. F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton, Complementarity and nondegeneracy in semidefinite programming, Math. Program., 77 (1997), pp. 111–128.
2. A. S. Bandeira, N. Boumal, and A. Singer, Tightness of the maximum likelihood semidefinite relaxation for angular synchronization, Math. Program., 163 (2017), pp. 145–167.
3. A. S. Bandeira, Y. Khoo, and A. Singer, Open problem: Tightness of maximum likelihood semidefinite relaxations, J. Mach. Learn. Res.: Workshop and Conference Proceedings, 35 (2014), pp. 1265–1267.
4. A. Beck and M. Teboulle, Global optimality conditions for quadratic optimization problems with binary constraints, SIAM J. Optim., 11 (2000), pp. 179–188.
5. N. Boumal, Nonconvex phase synchronization, SIAM J. Optim., 26 (2016), pp. 2355–2377.
6. M. Goemans and D. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, Journal of the ACM, 42 (1995), pp. 1115–1145.
7. M. Goemans and D. Williamson, Approximation algorithms for Max-3-Cut and other problems via complex semidefinite programming, J. Comput. Syst. Sci., 68 (2004), pp. 442–470.
8. J. Jaldén, Detection for multiple input multiple output channels, Ph.D. Thesis, School of Electrical Engineering, KTH, Stockholm, Sweden, 2006.
9. J. Jaldén, C. Martin, and B. Ottersten, Semidefinite programming for detection in linear systems – Optimality conditions and space-time decoding, in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP’03), Hong Kong, 2003, pp. 9–12.
10. S. Jacobsson, G. Durisi, M. Goldstein, and C. Studer, Quantized precoding for massive MU-MIMO, IEEE Trans. Commun., to appear, DOI: 10.1109/TCOMM.2017.2730090.
11. M. Kisialiou and Z.-Q. Luo, Efficient implementation of a quas-maximum-likelihood detector based on semi-definite relaxation, in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP’07), Honolulu, 2007, pp. 1329–1332.
12. M. Kisialiou and Z.-Q. Luo, Probabilistic analysis of semidefinite relaxation for binary quadratic minimization, SIAM J. Optim., 20 (2010), pp. 1906–1922.
13. H. Liu, M.-C. Yue, and A. M.-C. So, On the statistical performance of the generalized power method for angular synchronization, preprint, arXiv:1603.00211, 2016.
14. H. Liu, M.-C. Yue, and A. M.-C. So, A discrete first-order method for large-scale MIMO detection with provable guarantees, in Proceedings of the 18th IEEE Workshop on Signal Processing Advances in Wireless Communications (SPAWC’17), Sapporo, 2017, pp. 669–673.
15. Y.-F. Liu, M. Hong, and Y.-H. Dai, Max-min fairness linear transceiver design problem for a multi-user SIMO interference channel is polynomial time solvable, IEEE Signal Process. Lett., 20 (2013), pp. 27–30.
16. Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, Semidefinite relaxation of quadratic optimization problems, IEEE Signal Process. Mag., 27 (2010), pp. 20–34.
17. W.-K. Ma, P.-C. Ching, and Z. Ding, Semidefinite relaxation based multiuser detection for M-ary FSK multiuser systems, IEEE Trans. Signal Process., 52 (2004), pp. 2862–2872.
18. A. D. Maio, S. D. Nicola, Y. Huang, Z.-Q. Luo, and S. Zhang, Design of phase codes for radar performance optimization with a similarity constraint, IEEE Trans. Signal Process., 57 (2009), pp. 610–621.
19. W. Pu, Y.-F. Liu, J. Yan, S. Zhou, H. Liu, and Z.-Q. Luo, A two-stage optimization approach to the asynchronous multi-sensor registration problem, in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP’17), New Orleans, 2017, pp. 3271–3275.
20. A. Singer, Angular synchronization by eigenvectors and semidefinite programming, Appl. Comput. Harmon. Anal., 30 (2011), pp. 20–36.
21. A. M.-C. So, J. Zhang, and Y. Ye, On approximating complex quadratic optimization problems via semidefinite programs, Math. Program., 110 (2007), pp. 93–110.
22. A. M.-C. So, Probabilistic analysis of the semidefinite relaxation detector in digital communications, in Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’10), Austin, 2010, pp. 698–711.
23. M. Soltanalian and P. Stoica, Designing unimodular codes via quadratic optimization, IEEE Trans. Signal Process., 62 (2014), pp. 1221–1234.
24. J. Sun, Q. Qu, and J. Wright, When are nonconvex problems not scary, preprint, arXiv:1510.06096, 2016.
25. P. H. Tan and L. K. Rasmussen, The application of semidefinite programming for detection in CDMA, IEEE J. Sel. Areas Commun., 19 (2001), pp. 1442–1449.
26. S. Verdú, Computational complexity of optimum multiuser detection, Algorithmica, 4 (1989), pp. 303–312.
[27] S. Verdú, *Multiuser Detection*, Cambridge Univ. Press, New York, 1998.
[28] I. Waldspurger, A. Aspremont, and S. Mallat, *Phase recovery, MaxCut and complex semidefinite programming*, Math. Program., 149 (2015), pp. 47–81.
[29] S. Yang and L. Hanzo, *Fifty years of MIMO detection: The road to large-scale MIMOs*, IEEE Commun. Surveys Tuts., 17 (2015), pp. 1941–1988.
[30] S. Zhang and Y. Huang, *Complex quadratic optimization and semidefinite programming*, SIAM J. Optim., 16 (2006), pp. 871–890.