Topological field theory and
the quantum double of $SU(2)$

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Abstract

We study the quantum mechanics of a system of topologically interacting particles in 2+1 dimensions, which is described by coupling the particles to a Chern–Simons gauge field of an inhomogeneous group. Analysis of the phase space shows that for the particular case of $ISO(3)$ Chern–Simons theory the underlying symmetry is that of the quantum double $\mathcal{D}(SU(2))$, based on the homogeneous part of the gauge group. This is in contrast to the usual $q$-deformed gauge group itself, which occurs in the case of a homogeneous gauge group. Subsequently, we describe the structure of the quantum double of a continuous group and the classification of its unitary irreducible representations. The comultiplication and the $R$-element of the quantum double allow for a natural description of the fusion properties and the nonabelian braid statistics of the particles. These typically manifest themselves in generalised Aharonov–Bohm scattering processes, for which we compute the differential cross sections. Finally, we briefly describe the structure of $\mathcal{D}(SO(2,1))$, the underlying quantum double symmetry of (2+1)-dimensional quantum gravity.

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1 Introduction

The study of topological quantum field theories has lead to important results in both mathematics and physics [1]. For example, the intrinsically three dimensional formulation of knot

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and link invariants is given by correlation functions in three dimensional Chern–Simons (CS) theory. Also, it is known that the Hilbert space of such Chern–Simons theories correspond to the chiral sectors of certain conformal field theories \cite{WZW}, and that the fusion and braid properties of these sectors can be described via so-called ‘hidden quantum symmetries’, see \cite{WZW} and references therein. More specifically, the wave functionals in the Wess–Zumino–Witten (WZW) model for which the chiral algebra is the Kac–Moody algebra $SU(2)_k$ obey the braiding and fusion relations of the quantum group $U_q(su(2))$, where the deformation parameter $q$ is related to the level $k$ of the WZW model, which in turn is the coupling constant of the corresponding CS theory. These models have found interesting applications in physical systems as for example the quantum Hall effect, Kondo-like impurity problems and high-$T_c$ superconductivity, and are also directly linked to the theory of integrable two-dimensional models in statistical mechanics.

In this paper we are concerned with the quantum description of particles which carry charges of the inhomogeneous group $ISO(3)$, and which interact topologically through their coupling to a CS gauge field. So one should think of particles which are characterised by some internal “Euclidean mass” $r$ and a spin $s$. To be precise, we actually consider the double cover of $ISO(3)$, which is the semi-direct product of $SU(2)$ with the group of translations in three dimensions, but for convenience we denote the gauge group by $ISO(3)$. It turns out that due to the inhomogeneous structure of the gauge group the physical phase space of the system has a natural cotangent bundle structure, which directly leads to the identification of the configuration space. We show that the internal degrees of freedom correspond to holonomies $g \in ISO(3)$ of noncontractable loops in physical space, which leads to a noncommutative multiplicative structure of the multi-particle configuration space. Upon quantisation of the single particle sectors one obtains that states in our system carry a ‘magnetic flux’ (topological charge) which is the holonomy $g \in SU(2)$, and an ‘electric charge’ $n$ which labels some irreducible representation of the centraliser subgroup $N_g \subset SU(2)$. These charges are not identified via Gauss’ constraint, in contrast to the case of ‘ordinary’ $SU(2)$ Chern–Simons theory, where the phase space does not have a natural cotangent bundle structure. The multi-particle sectors of the theory should exhibit two important features: firstly, the fusion properties have to be defined on the level of the multi-particle Hilbert space. Secondly, the topological interactions between the particles should be described by a suitable action of the braid group on this multi-particle Hilbert space. We show that for the model we are considering the correct mathematical structure, leading to the required braiding and fusion properties, is the so-called quantum double $D(G)$ of the homogeneous part of the gauge group, in this case $G = SU(2)$. This implies that the internal Hilbert space can be decomposed into irreducible unitary representations of this quantum double $D(G)$, and we will show that the labels of these representations precisely coincide with invariants associated to the ‘magnetic’ and ‘electric’ charges mentioned above, i.e. the internal mass and spin.

This $ISO(3)$ Chern–Simons theory may be referred to as Euclidean quantum gravity in 2+1 dimensions. To a certain extent we do indeed solve this problem, in the sense that we explicitly describe the Hilbert space and the underlying quantum symmetry. At the end of the paper we also do this for the realistic, Minkowskian version, i.e. the group $ISO(2,1)$. However, we do not want to over-emphasize the gravity aspect of our work, because in this paper we do not address the problem of identifying the internal “Chern–Simons” degrees of freedom with their external space time counterparts. We do return to this issue in a
forthcoming paper

At this point it may be interesting to note that quantum doubles have featured before in physics. For example, in \[6\] it has been shown that the sectors in certain rational conformal field theories (orbifold models) correspond to the irreducible representations of $\mathcal{D}(H)$, where $H$ denotes a finite group. The same quantum doubles have been used to describe the topological interactions of defects carrying discrete magnetic fluxes in gauge theories which are spontaneously broken to a finite residual symmetry group \[7\] \[8\]. It was observed that quantum states of such systems can be exactly described by elements of representation spaces of the quantum double of the residual (finite) group. As in our case, the use of this observation lies in the fact that the quantum double as underlying symmetry for these systems automatically comprises the correct notions of ‘fusion’, via the decomposition of tensor product representations, and ‘braiding’, via the action of the $R$-element of the quantum double. The identification of this quantum double symmetry allowed for explicit computation of differential cross sections in nonabelian generalisations of the Aharonov–Bohm effect. Finally, the associated $S$-matrix diagonalising the fusion rules generates an electric–magnetic duality transformation in these models. Both models just mentioned are related to the discrete topological field theories of Dijkgraaf and Witten \[9\]. In the present paper we consider a more intricate class of models which exhibit a manifest continuous quantum double symmetry.

The outline of the paper is as follows:

We start by describing the physical system of particles that carry quantum numbers of the group $ISO(3)$, and that have topological interactions mediated by an $ISO(3)$ Chern–Simons gauge field. In section 2 we consider the pure $ISO(3)$ Chern–Simons theory, without particles, and construct its physical phase space. Subsequently, in section 3 we include point particles, and arrive at an explicit parametrisation of the corresponding phase space. In section 4 we describe the quantisation of the single particle case. Next we discuss the nontrivial properties of the multi-particle quantum system, involving the aforementioned ‘braiding’ and ‘fusion’. Section 5 summarises the mathematical results of \[10\] and \[11\] on the quantum double of a (locally) compact group. In section 6 we turn to the quantum double of $SU(2)$ explicitly. We identify the labels of the irreducible unitary representations of $\mathcal{D}(SU(2))$ as the quantum numbers we discussed before, and show that elements in the corresponding representation spaces precisely coincide with the states we found by quantising the single particle sectors. Also, it is shown that the tensor products of such irreducible representations possess precisely the correct decomposition and braiding properties required for the quantised multi-particle sectors. In section 7 we give the detailed formulae for the fusion rules, which are the multiplicities of the Clebsch–Gordan series in $\mathcal{D}(SU(2))$, for all types of product representations. We also compute the Clebsch–Gordan coefficients. One important result is that we can calculate differential cross–sections for various generalised Aharonov–Bohm scattering processes, because we know the $R$-element and its action on tensor product states, this is discussed in section 8. In section 9 we show that there is a quantum double structure in 2+1 quantum gravity, corresponding to $\mathcal{D}(SO(2, 1))$, and in section 10 we present our conclusions.
We start by considering pure Chern–Simons theory in 2+1 dimensional spacetime $M$, because this will reveal much about the general structure of our classical phase space. It is convenient to include the particles later. For an introduction to CS theory we refer to [1] and references therein. The action is given by

$$S_{CS} = \frac{1}{2} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (2.1)$$

for $M$ an –in principle arbitrary– oriented three dimensional manifold, and a gauge field $A$ taking values in the Lie algebra $\mathfrak{g}$ of the symmetry group $G$. This defines a topological theory because the Lagrangian, which is a 3-form without metric dependence, is integrated over a three dimensional manifold, so the action is independent of the metric on $M$.

The case of compact $G$ has been well studied, and on the quantum level it is related to the $q$-deformation of $G$, or better, to the $q$-deformation of the universal enveloping algebra of $\mathfrak{g}$, see for instance [4]. In the present paper we consider a particular example of a noncompact, inhomogeneous group, being the semi-direct product of the groups of rotations and translations in three (internal) dimensions with Euclidean signature. The commutation relations read

$$[J_a, J_b] = \epsilon_{abc} J_c, \quad [J_a, P_b] = \epsilon_{abc} P_c, \quad [P_a, P_b] = 0,$$

with $P_a$ the generators of translations and $J_a$ the generators of $SO(3)$, and together they span the algebra of the (double cover of the) Euclidean group in three dimensions: $E(3) = ISO(3)$. The trace in Eq.(2.1) stands for the inner product on the algebra, which we take to be

$$\langle J_a, P_b \rangle = \delta_{ab}, \quad \langle P_a, P_b \rangle = \langle J_a, J_b \rangle = 0. \quad (2.3)$$

The fact that the inner product is not unique for this case was noted in [12].

The $ISO(3)$ gauge field is given by

$$A_\mu(x) = e^\mu_a(x) P_a + \omega^a_\mu(x) J_a, \quad x \in M, \ a = 0, 1, 2. \quad (2.4)$$

The components $e^\mu_a$ and $\omega^a_\mu$ of the gauge field are taken to be independent variables. The fact that we are dealing with an inhomogeneous gauge group has the important consequence that we can scale away any coupling constant in front of the action, simply by absorbing it in the inhomogeneous part of the gauge field. In the case of homogenous $G$ the coupling constant $k$ cannot be scaled away, and it will label physically distinct quantum theories via the deformation parameter $q$. For example, for the $SU(2)$ WZW-model at level $k$ we have that $q = e^{\pm i \pi k}$. The fact that in the $iso$ CS model we consider a the coupling constant can be scaled away, indicates that no ‘ordinary’ $q$-deformed quantum group will show up.

Observables in this theory must be topological invariants, so independent of the metric. As argued by Witten [2] these correspond to Wilson lines, defined to be the following functionals of the gauge field

$$W_R(C) := \text{tr}_R P \exp \oint_C A. \quad (2.5)$$

Here $C$ denotes a closed loop in $M$, and the trace is taken in some irreducible representation $R$ of the gauge group $G$. Later on we will orient the loop $C$ in the direction in $M$ that we
call ‘time’, and consider some region in $M$, where we may interpret these Wilson lines as the world lines of the particles.

Finally, we remark that this (pure) ISO(3) CS theory is completely equivalent to a (pure) SU(2) BF-theory on a three-dimensional closed manifold $M$. The latter are topological gauge theories, which for the case of nonabelian groups have the action

$$S_{BF} = \int_M \text{tr} \, B \, F.$$  \hfill (2.6)

Here $B$ is a Lie algebra valued 1-form, and $F$ is the curvature of some flat $G$-bundle over $M$. More details can be found in [1].

2.1 Phase space

The classical field equations from the Chern–Simons Lagrangian constrain the curvature $F$ of the connection $A$ to be zero. This means that the solutions are flat connections,

$$F_A = 0. \hfill (2.7)$$

In order to canonically quantise the theory we have to identify the phase space, which requires splitting of space and time coordinates in the action. As usual we take $M = \Sigma \times \mathbb{R}$, so two-dimensional Riemann surfaces $\Sigma$ as cross-sections of $M$ that move in (uniform) time. For later convenience we parametrise the surface $\Sigma$ by $z$. The action now reads

$$S = \int dt \int_\Sigma d^2 z \left( -2\varepsilon^{ij} e_i^a \partial_t \omega_{ja} + e_0^a R_a + \omega_0^a T_a \right) \hfill (2.8)$$

where we have defined the components of the curvature $F(z) = R_a(z) J^a + T_a(z) P^a$ as

$$R_a = \tfrac{1}{2} \varepsilon^{ij} \left( \partial_i \omega_{ja} - \partial_j \omega_{ia} + \epsilon_{abc} \omega_b^i \omega_c^j \right),$$

$$T_a = \tfrac{1}{2} \varepsilon^{ij} \left( \partial_i e_{ja} - \partial_j e_{ia} + \epsilon_{abc} (\omega_b^i e_c^j - \omega_c^i e_b^j) \right). \hfill (2.9)$$

The dynamical variables are $e_i^a$ and $\omega_i^a$, and their Poisson brackets which can be derived from the action read

$$\{ \omega_i^a(z_1), e_j^b(z_2) \} = \frac{i}{2} \delta_{ab} \varepsilon_{ij} \delta^{(2)}(z_1 - z_2),$$

$$\{ e_i^a(z_1), e_j^b(z_2) \} = \{ \omega_i^a(z_1), \omega_j^b(z_2) \} = 0. \hfill (2.10)$$

In the action $e_0^a$ and $\omega_0^a$ appear as Lagrange multipliers, variation with respect to them yields the constraints

$$R_a = 0, \quad T_a = 0, \hfill (2.11)$$

with Poisson brackets

$$\{ T_a(z_1), T_b(z_2) \} = \epsilon_{abc} T^c(z_1) \delta^{(2)}(z_1 - z_2),$$

$$\{ T_a(z_1), R_b(z_2) \} = \epsilon_{abc} R^c(z_1) \delta^{(2)}(z_1 - z_2),$$

$$\{ R_a(z_1), R_b(z_2) \} = 0. \hfill (2.12)$$

In general, constraints can either be imposed before, or after quantisation. If the unconstrained system is quantised, then the constraints have to be imposed as operators annihilating the physical states. However, for the current problem of quantisation of a CS gauge
field, it is more useful to impose the constraints on a classical level, because the resulting reduced phase space—called the physical phase space—turns out to be finite dimensional. The unreduced phase space is infinite dimensional, it consists of all gauge connections on \( \Sigma \). The physical phase space is the space of solutions to the equations of motion which satisfy the constraints, divided out by the gauge group \( \mathcal{G} \) generated by the constraints. So in our case it is the moduli space of flat ISO(3) connections on the Riemann surface \( \Sigma \):

\[
\mathcal{M}_{\text{phys}} = \{ \text{flat ISO}(3) \text{ connections on } \Sigma \} / \mathcal{G}.
\]  

(2.13)

This is not a very useful description if one wants to perform (canonical) quantisation of the system, and we recall the following argument (see also [13, 1]) in order to obtain a more explicit and practical description. A flat connection on a surface is, up to a gauge transformation, completely determined by its holonomies around noncontractable loops with a fixed base point on the surface. Such holonomies only depend on the homotopy class of the loops, not on the precise paths, so they are homomorphisms from the (based) fundamental group of the surface to the relevant symmetry group, here ISO(3). Gauge transformations only have an effect at the base point \( \ast \), where they act on the holonomies by conjugation with \( \mathcal{G}(\ast) \). Hence one arrives at

\[
\mathcal{M}_{\text{phys}} = \text{Hom}(\pi_1(\Sigma, \ast); ISO(3))/\sim
\]

(2.14)

where \( \sim \) denotes equivalence under conjugation by \( \mathcal{G}(\ast) \). This expression for the physical phase space shows that if there are no noncontractable loops in \( \Sigma \) the theory is trivial. In the next section we consider the case where noncontractable loops are present.

Finally, how is a classical state, being an element in \( \mathcal{M}_{\text{phys}} \), characterised? A point in the space of homomorphisms is described by the collection of all \( \ast \)-based noncontractable loops (more precisely, their homotopy classes) in \( \Sigma \), together with the values of their corresponding holonomies. It suffices to give the holonomies corresponding to a certain set of basis elements of \( \pi_1(\Sigma, \ast) \), because they determine the values of the holonomies corresponding to all other noncontractable loops. The gauge group acts via simultaneous conjugation with an element of \( \mathcal{G}(\ast) \) of all (basis) holonomies, and has to be divided out.

## 3 Including point particles on the classical level

From now on we assume that the spatial part \( \Sigma \) of our 2+1 dimensional space-time has no handles. To make the theory nontrivial we allow for punctures in \( \Sigma \), and at these points insert particles carrying ISO(3) charges. In \( M \) these punctures sweep out world lines, which will form a certain braid. Since the particles carry nonabelian charges, one may expect that on the quantum level the braid group will have a nontrivial realisation on the multi-particle Hilbert space, which may give rise to a (nonabelian) generalisation of the Aharonov–Bohm effect. We return to this issue in section 8.

Point particles can be introduced in various ways. In what is usually called a ‘dynamical’ way, the Riemann surface \( \Sigma \) is considered to have a boundary which consists of infinitesimally small circles with their interior removed, and a vanishing curvature everywhere on \( \Sigma \). This way the degrees of freedom are contained in the values of the gauge field on the boundary, and in the matching conditions of the simply connected patches with vanishing curvature.
that build up the non-simply connected surface $\Sigma$. This is the approach Witten suggested at the very end of [2]. By taking the three manifold $M$ the way we have described it above, the space of conformal blocks of a certain conformally invariant theory in 1+1 dimensions will be equivalent to the Hilbert space of a canonically quantised Chern–Simons model with charged point particles in 2+1 dimensions [3].

For multi-particle configurations this description via boundary conditions can get rather complicated, and in this paper we take a different route. We introduce particles ‘by hand’, by considering the removed points to actually be part of $\Sigma$, in such a way that the gauge field (and its curvature) can take values at these points, but loops cannot be pulled through them, i.e. these points still give rise to noncontractable loops. At the punctures we allow the gauge field to have curvature, so the constraints of Eq.(2.11) get modified in the presence of charged particles. They receive delta function contributions on the right hand side, and we will now explain how this can be done in a consistent way for a single particle.

3.1 The single particle case

We have seen that the observables in the theory come from Wilson lines as given in Eq.(2.5), and that they are directly related to irreducible representations of the gauge group. Following Witten [2], we consider a timelike Wilson line, i.e. the closed loop $C$ in Eq.(2.5) running along the time direction and closing at infinity, to be the world line of a particle. The charges of the particle correspond to the irreducible representation $R$ associated with the Wilson line.

Such a Wilson line will intersect a spacial slice $\Sigma$ at a certain time $t$ in the point $z_1 \in \Sigma$, where it is considered as a nonabelian charge acting as a source for the curvature.

As pointed out above Eq.(2.13) we want to arrive at the modified constraint equations (with sources) on a classical level. Therefore, we must find the classical equivalent of the charges of the particle, which on the quantum level reproduce the generators of the gauge symmetry at the position of the particle. To that aim we introduce a suitable classical mechanical system, which we associate to the Wilson line, such that upon quantisation we obtain the irreducible representation $R$. As is known from representation theory [14] an appropriate classical system is one which has a certain coadjoint orbit of the gauge group as its classical, reduced phase space. Subsequently, this classical system has to be coupled to the CS gauge field, effectively making the nonabelian charges interact topologically. This way one is indeed led to constraint equations where the symmetry generators of the classical system show up as the sources for the curvature. In the following subsections we will explain this procedure in more detail.

Coadjoint orbit method

For semi-simple Lie groups the coadjoint orbit method of Kirillov and Kostant allows one to construct the action for a classical system in such a way that the generators corresponding to the Noether symmetry under Poisson brackets satisfy the algebra isomorphic to the Lie algebra [14]. Quantisation will then automatically lead to an irreducible unitary representation of the corresponding Lie group. In fact, this method is not restricted to semi-simple Lie groups only, but also works for certain other groups, like some semi-direct product groups, amongst which $G = ISO(3)$, [13] [16]. We apply this method to construct the action for
a free particle carrying a Euclidean mass \( r \) and a spin \( s \), but we first recall the method in more detail for a general Lie group \( G \).

A coadjoint orbit is obtained via the action of the group on a certain coadjoint vector \( Y \in \mathfrak{g}^* \), where \( \mathfrak{g}^* \) denotes the dual of the Lie algebra. It has a natural symplectic structure \( \omega \), and can be quantised under Weil’s integrality condition, which means that “\( \omega \) must be an integral element of the second cohomology group”, see for instance [17]. So a coadjoint orbit can be regarded as the phase space of some (classical) system, and geometrical quantisation leads to a Hilbert space which forms a particular irreducible unitary representation of the Lie group we started off with.

To find the action for the classical system, we consider the canonical one-form \( \theta_Y \) on the coadjoint orbit. Note that if \( \omega \) corresponds to a nontrivial element of the second cohomology group it cannot be exact, so the relation \( \omega = -d\theta_Y \) does not hold globally. This implies that a globally valid expression for \( \theta_Y \) cannot be obtained from the symplectic form \( \omega_Y \) on the orbit, which follows directly from the Poisson–Lie structure on \( \mathfrak{g}^* \).

As is well known the coadjoint orbit can be obtained as the reduced phase space of a particle moving on the group \( G \). We denote basis elements of \( \mathfrak{g} \) by \( T^a, a = 1, \ldots, n \), with \( n = \text{dim}(G) \), and the relations \( [T^a, T^b] = f^{ab}_c T^c \) with \( f^{ab}_c \) the structure constants. The dual basis on \( \mathfrak{g}^* \) is denoted by \( \{t_a\} \), with \( t_a(T^b) = \delta_a^b \). On the full (unreduced) phase space \( T^*G \simeq G \times \mathfrak{g}^* \) there is a standard canonical one-form, defined via its contraction with a vector field on \( T^*G \)

\[
\theta(x, Y) \cdot (X(x), Z) = Y(X^{(e)}) = \langle \tilde{Y}, X^{(e)} \rangle 
\]

Here \( (X^{(x)}, Z) \) is an element of the tangent space to \( T^*G \) in the point \( (x, Y) \in T^*G \), which is isomorphic to \( T_xG \times \mathfrak{g}^* \). The element \( \tilde{Y} \) lying in \( \mathfrak{g} \) is by definition dual to \( Y \in \mathfrak{g}^* \). For semi-simple groups it is simply the element itself, because the inner product \( \langle \cdot, \cdot \rangle \) is given by the trace in the fundamental representation, \( \langle T^a, T^b \rangle = \text{tr}(T^aT^b) = -\frac{1}{2} \delta^{ab} \), identifying the algebra with its dual. Note that for \( G = \text{ISO}(3) \), with the inner product on the algebra as given in Eq.(2.3), this is not the case.

The (left) action of \( G \) on itself induces a symplectic action on \( T^*G \), i.e. it leaves \( \omega \) invariant, and it has the moment map \( J : T^*G \to \mathfrak{g}^* \), given by

\[
\langle J((x, Y, x)), \xi \rangle = Y(\xi), \quad Y \in \mathfrak{g}^*, \xi \in \mathfrak{g}, x \in G, 
\]

with \( x, Y \) denoting the coadjoint action of \( x \) on \( Y \). Moment maps are directly related to the symmetries of the system, and it can be shown [18] that the reduced phase space, which is by construction a symplectic manifold, is isomorphic to the coadjoint orbit

\[
J^{-1}(Y)/N_Y \simeq G/N_Y \simeq \mathcal{O}_Y, \quad Y \in \mathfrak{g}^* 
\]

where \( N_Y = \{x \in G \mid x.Y = Y\} \) leaves the coadjoint vector \( Y \) invariant. Clearly, the particular choice of \( Y \in \mathcal{O}_Y \subset \mathfrak{g}^* \) is irrelevant. The orbit can therefore be labeled by the set of invariants of \( Y \) under the action of \( G \), which we will denote by \( R \). We use the canonical one-form on \( T^*G \) to find a local expression for the one-form on the reduced phase space \( J^{-1}(Y)/N_Y \). For the moment map as given in Eq.(3.16) the pre-image of a certain coadjoint vector \( Y \) is the trivial bundle over \( G \), with in each element \( x \in G \) the vector \( Y \) as the fiber. This bundle is isomorphic to the group \( G \). The one-form on the reduced phase space can be taken to be equal to the one-form on \( T^*G \), restricted to the pre-image of a certain fixed
$Y \in \mathfrak{g}^*$, and it is obviously invariant under the action of $N_Y$. We conclude that locally on the orbit $\mathcal{O}_R$ the canonical one-form is given by

$$\theta_Y(x,Y) = \langle \tilde{Y}, x^{-1}dx \rangle, \quad \text{fixed } Y \in \mathfrak{g}^*. \quad (3.18)$$

In general, for a phase space with coordinates $\{\zeta\}$ (denoting ‘positions’ and ‘momenta’) we may construct a first order Lagrangian \[9\]

$$L dt = \theta(\zeta) - H(\zeta)dt, \quad (3.19)$$

with $H(\zeta)$ some Hamiltonian, and $\theta$ the canonical one-form. For the simple example where the phase space is given by $T^*Q$, and $Q$ is $n$-dimensional, the Lagrangian is given by $L = p_i \dot{q}^i - H(p,q)$, $i = 1, \ldots, n$.

For the case of a coadjoint orbit we may take the canonical Hamiltonian equal to zero, because at this point we are only interested in the construction of the corresponding Hilbert space, and not in some possible dynamics of the particle on the coadjoint orbit. The action is now given by

$$S = \int L dt = \int \left( \theta_Y(Y,x), \frac{d}{dt} \right) dt, \quad (3.20)$$

with the large brackets $\langle , \rangle$ denoting the contraction between the one-forms and the vector fields on $\mathfrak{g}^*$.

An element $\xi \in \mathfrak{g}$ is a linear function on the dual vector space, and thus corresponds to a vector field $X_\xi$ on $\mathfrak{g}^*$. The symplectic two-form $\omega_Y$ can be shown to be given by \[18\]

$$\omega_Y(X_\xi, X_\eta) = Y([\xi, \eta]). \quad (3.21)$$

As expected, this corresponds precisely to the Poisson–Lie structure on $\mathfrak{g}^*$:

$$\{F,G\}(Y) = Y([dF_Y, dG_Y]). \quad (3.22)$$

The differential $dF_Y$ of the function $F$ on $\mathfrak{g}^*$, being a linear map from the tangent space to $\mathfrak{g}^*$ in the point $Y$, is an element of $\mathfrak{g}^{**} \simeq \mathfrak{g}$, so $[ , ]$ is the Lie bracket, see for instance \[16\]. It can be shown that the two-form $\omega_Y$ is nondegenerate on the coadjoint orbit, corresponding to the well-known fact that the coadjoint orbits are the symplectic leaves of the dual of the Lie algebra.

Consider the coordinate functions on $\mathfrak{g}^*$

$$Q^a = \langle Q, T^a \rangle, \quad Q = Q^a t_a \in \mathfrak{g}^*. \quad (3.23)$$

They can be shown to generate the infinitesimal transformations $\delta Y = T^a(Y) = Y(T^a)$, their Poisson brackets being given by

$$\{Q^a, Q^b\} = f^{ab} c Q^c \quad (3.24)$$

In particular this holds for the coordinate functions on the orbit of some fixed $Y$, where $Q = x.Y$, so they are the Noether charges corresponding to the global symmetries on the coadjoint orbit.
Coadjoint orbits of ISO(3)

For the case at hand, \( G = ISO(3) \cong SU(2) \ltimes \mathbb{R}^3 \cong SU(2) \ltimes su(2) \). In computations we use the \( 4 \times 4 \) representation of \( ISO(3) \), writing an element as

\[
U = \begin{pmatrix}
\Lambda & \vec{a} \\
0 & 1
\end{pmatrix}
\]

with \( \Lambda \in SU(2) \) and \( \vec{a} \) an element of the translation group. Denoting \( U \) as \( (\Lambda, \vec{a}) \), the multiplication is given by

\[
(\Lambda_1, \vec{a}_1)(\Lambda_2, \vec{a}_2) = (\Lambda_1 \Lambda_2, \vec{a}_1 + \Lambda_1 \vec{a}_2),
\]
and the inverse by \( (\Lambda^{-1}, -\Lambda^{-1} \vec{a}) \). The one-form \( dU \) is given by

\[
d(\Lambda, \vec{a}) = \begin{pmatrix}
d\Lambda & d\vec{a} \\
0 & 0
\end{pmatrix}.
\]

The Lie algebra is \( isu(2) = su(2) \oplus \mathbb{R}^3 \), its dual is \( isu(2)^* = su(2)^* \oplus (\mathbb{R}^3)^* \simeq \mathbb{R}^3 \oplus \mathbb{R}^3 \). The coadjoint action of \( ISO(3) \) on \( isu(2)^* \) is denoted by “dot” and reads

\[
(\Lambda, \vec{a}).(\vec{j}, \vec{p}) = (\Lambda \vec{j} + \vec{a} \wedge \Lambda \vec{p}, \Lambda \vec{p}), \quad \vec{j} \in su(2)^*, \vec{p} \in \mathbb{R}^3,
\]

where it is understood that \( \Lambda \) acts in the coadjoint representation on \( \vec{j} \). The invariants of the orbit of \( (\vec{j}, \vec{p}) \) are the norm \( |\vec{p}| \), and the inner product of \( \vec{p} \) and \( \vec{j} \), both via the ordinary inner product on \( \mathbb{R}^3 \). We call these invariants \( \vec{p}^2 = r^2, r \geq 0, \) and \( \vec{p} \cdot \vec{j} = rs \). As a manifold the orbit \( (r, s), r > 0, \) is equivalent to the cotangent bundle \( T^* S^2 \), since the orbit of \( \vec{p} \) is a 2-sphere, where in each point \( \vec{p} \) the \( \vec{j} \) that satisfy \( \vec{p} \cdot \vec{j} = rs \) span an \( \mathbb{R}^2 \). For \( r = 0 \) a coadjoint orbit is given by an \( SU(2) \) orbit of \( \vec{j} \), so also an \( S^2 \). For a proof of the fact that coadjoint orbits of semi-direct product groups indeed have a nondegenerate symplectic structure we refer to [20] and [16]. As can be concluded from the general derivation by Rawnsley [15] the coadjoint orbits of \( ISO(3) \) can be quantised for all \( r \geq 0 \). Then \( s \in \frac{r}{2} \mathbb{Z} \) for \( r \neq 0 \), and \( s \in \frac{1}{2} \mathbb{Z} \) with the restriction that the spin \( s \geq 0 \) in case the Euclidean mass \( r = 0 \), which indeed yields all irreducible representations of \( ISO(3) \).

To obtain the irreducible representation \( (r, s) \) of \( ISO(3) \) we have to find a classical system which has the coadjoint orbit \( O_{r,s} \) as its physical phase space. The coadjoint element from which we construct the orbit is \( (\vec{j}_0, \vec{p}_0) = ((s, 0, 0), (r, 0, 0)) \), corresponding to \( \hat{Y} = r \mathcal{J}_0 + s \mathcal{P}_0 \), due to the form of the inner product on \( isu(2) \) as given in Eq.(2.3). From the general discussion above it follows that the canonical one-form is given by

\[
\theta_{r,s}(\vec{j}_0, \vec{p}_0) = \langle r \mathcal{J}_0 + s \mathcal{P}_0, U^{-1} dU \rangle.
\]

The second entry takes values in the Lie algebra of \( ISO(3) \), so it can be written as

\[
U^{-1} dU = \langle \Lambda^{-1} d\vec{a}, \mathcal{P}^b \rangle \mathcal{P}^b + \langle \Lambda^{-1} d\Lambda, \mathcal{J}^b \rangle \mathcal{J}^b
\]

where it is obvious in which algebra the inner products have to be taken. Together with Eq.(3.20) this leads to the free particle action

\[
S_p = \int dt \left( \vec{p} \cdot \partial_t \vec{a} - \frac{s}{2} \text{tr} \left( \Lambda^{-1} \partial_t \Lambda \mathcal{J}_0 \right) \right).
\]
The canonically conjugate momentum to $\vec{a}$ is defined by $\vec{p}$ with components $p_b := r\Lambda_b^0$, as will be explained shortly. We remark that the same action has been used by De Sousa Gerbert in [21] for the case of $(2+1)$-dimensional gravity, where the symmetry group is the Poincaré group in three dimensions, $G = ISO(2,1)$.

Let us take a closer look at Eq. (3.31). The first term simply corresponds to a particle with position vector $\vec{a}$, momentum $\vec{p}$, and (Euclidean) mass $|\vec{p}| = r$. The second term corresponds to a pure spin $s$; it is nothing but the action of a system which has the coadjoint orbit $O_s$ of $SU(2)$ as its phase space (a coadjoint orbit $O_j$ of $SU(2)$ is a two-sphere of radius $j$). The two terms are related, however, due to the fact that $\Lambda$ is contained in the definition of $\vec{p}$. This is reflected in the Poisson brackets of the symmetry generators, as can be understood as follows. From the general coadjoint orbit construction we know that the symmetries of this is reflected in the Poisson brackets of the symmetry generators, as can be understood as follows. From the general coadjoint orbit construction we know that the symmetries of the action in Eq. (3.31) are generated by the coordinate functions defined in Eq. (3.23). The coadjoint element $Q = x.Y$ now corresponds to

$$\left(\vec{j},\vec{p}\right) = U(\vec{j}_0,\vec{p}_0) = (\Lambda\vec{j}_0 + \vec{a} \wedge \Lambda\vec{p}_0, \Lambda\vec{p}_0),$$  

leading to the coordinate functions

$$P_b = \langle \vec{j}_0, \vec{p}_0 \rangle, \quad J_b = \langle \vec{j}_0, \vec{p}_0 \rangle.$$  

From the Poisson brackets of $\vec{P}$ with $\vec{a}$ with respect to the variable $\vec{a}$ and its canonically conjugate momentum $\vec{p}$, it follows that $\vec{P}$ generates translations in $\vec{a}$, so it is simply equal to $\vec{p}$. The second line of Eq. (3.33) can now be identified as the angular momentum $\vec{j} = \vec{l} + \vec{s}$, where $\vec{l} = \vec{a} \wedge \vec{p}$ is the orbital part, and $\vec{s}$ is the intrinsic part. The latter is in fact the generator of the $SU(2)$ symmetry of the second term in the action. So we can say that $\vec{P} = \vec{p}$ and $\vec{J} = \vec{j}$, with components $j^b = \epsilon^{bcd}a_c p_d + \frac{s}{r} p^b$. By construction they satisfy the Poisson brackets

$$\{p_a, p_b\} = 0, \quad \{j_a, p_b\} = \epsilon_{abc} p^c, \quad \{j_a, j_b\} = \epsilon_{abc} j^c,$$

and the relations $p^a p_a = r^2$ and $p^a j_a = rs$.

A derivation of the above Poisson relations via Dirac’s theory of constrained Hamiltonian systems is discussed in [21].

For $r \neq 0$ the coadjoint orbit $O_{r,s}$ is four dimensional. One can either consider the $(\vec{j},\vec{p})$ with the constraints $|\vec{p}| = r$ and $\vec{p} \cdot \vec{j} = rs$ as the degrees of freedom, or the $U$ of Eq. (3.32), which are determined up to right multiplication with elements $V$ in the two-dimensional subgroup of $ISO(3)$ that leave $(\vec{j}_0,\vec{p}_0)$ invariant. This two-dimensional centraliser consists of the abelian semi-direct product group $U(1) \ltimes T^0$ of rotations around $\vec{p}_0$ with translations in the ‘time’ direction $T^0$. For $r = 0$ the coadjoint orbit $O_{0,j}$ is the two-dimensional coadjoint orbit $O_j$ of $SU(2)$, corresponding to the $U \in ISO(3)$ up to right multiplication with elements of the four-dimensional centraliser $SU(2) \ltimes T$.

**Coupling the particle to the gauge field**

Above we have derived the action for a free, spinning particle of Euclidean mass $r$ and spin $s$. To introduce nonabelian interactions for such particles they have to be coupled to the
ISO(3) CS gauge field. This is simply done via minimal substitution in the vector field $\frac{d}{dt}$ in Eq.(3.20), using the component of the gauge field in the direction of the Wilson line, which is the world line of the particle

$$\frac{d}{dt} \rightarrow D_t = \frac{d}{dt} + e^a_0 P_a + \omega^a_0 J_a.$$  

(3.35)

The total action for the coupled particle thus becomes

$$S_p + S_{\text{int}} = \int dt \left( p_b (\partial_t a^b + e^b_0 + \epsilon^{bcd} \omega_{cd} a_d) - \frac{s}{2} \text{tr} \left( \Lambda^{-1} \left( \partial_t + \omega^b_0 J_b \right) \Lambda J_0 \right) \right),$$  

(3.36)

from which we derive the following interaction term in the Lagrangian

$$L_{\text{int}} = e^b_0 p_b + \omega^b_0 j_b.$$  

(3.37)

Adding the total action of Eq.(3.36) to the gauge field action of Eq.(2.8) now leads to the desired constraint relations

$$R_a(z) = p_a \delta^{(2)}(z - z_1), \quad T_a(z) = j_a \delta^{(2)}(z - z_1),$$  

(3.38)

instead of Eq.(2.11). This shows that indeed there is nonvanishing curvature at the position $z = z_1$ of the particle. In particular, note that the Poisson brackets of the sources for the curvature, as given in Eq.(3.34), are consistent with the Poisson brackets of the constraints as in Eq.(2.12).

We now turn to the gauge symmetry of the model. It is well known that CS theories are not invariant under gauge transformations at their boundaries, and since punctures are part of the boundary of $\Sigma$, we may not allow for gauge transformations at their positions. This means that effectively we only take gauge equivalence under the group of small gauge transformations, i.e. in Eq.(2.13) we only divide out by these elements of the gauge group that take the value $e$ (unit in $G$) at the punctures. Without this restriction the internal degrees of freedom of a particle, corresponding to the $U$ (or $(\vec{j}, \vec{p})$) of Eq.(3.32), could have been completely gauged away, reducing the internal phase space to a single point, namely the unit element of ISO(3).

The total phase space in the presence of a particle

As our goal is to quantise the system of topologically interacting particles, we should study the total phase space which includes both the degrees of freedom of the gauge field, as well as the degrees of freedom of the particles coupled to them. At first sight this seems to lead to quite an elaborate analysis – this phase space being the tensor product of $M_{\text{phys}}$ with the tensor product of the phase spaces of each of the particles. The latter, in principle, for each particle consists of the coordinates and momenta in space time, together with the internal coadjoint orbit that leads to its nonabelian charges. However, this total phase space can be largely simplified, as follows from the fact that we have to impose the modified constraint equations from Eq.(3.38). We will now show that this basically allows one to eliminate the internal degrees of freedom of the particles in favour of the holonomy degrees of freedom of the gauge field.
By introducing the puncture we have effectively created a noncontractable loop in $\Sigma$, which contains nonvanishing curvature as given by Eq.(3.38). Before dividing out the gauge symmetry as in Eq.(2.13), or Eq.(2.14), the phase space of the gauge field in the presence of the particle consists of the holonomy

$$g_1 := w(C_1) = P \exp(\oint_{C_1} A_j dx^j) = \exp(\int_{\sigma} F_{ij} dx^i dx^j) = \exp(p^a J_a + j^a P_a) \in ISO(3).$$

(3.39)

Here $C_1$ is a closed loop around the particle which encloses the surface $\sigma$. It follows that if we –for the moment– ignore the gauge symmetry, i.e. we don’t take conjugation equivalence under $G(\ast)$ into account, we see that the classical state of the gauge field is given by the $ISO(3)$ group element $g_1 = w(C_1)$. Thus, the degrees of freedom of the gauge field, as given by the holonomy in Eq.(3.39), are in direct correspondence with the internal degrees of freedom $(j^a, p^a)$ of the particle, for a given Euclidean mass $r$ and spin $s$.

There is one subtlety, however, which concerns the following. It is clear that due to the exponentiation in Eq.(3.39) one cannot distinguish between states of the gauge field that are related to particles with Euclidean masses $r$, and states that are related to particles with masses $r + 2\pi m$, with $m \in \mathbb{N}$. So upon imposing the constraints the “space of possible total phase spaces” gets compactified in the $r$-direction, as opposed to the infinite range $r \geq 0$ for the coadjoint orbits, which correspond to the internal phase spaces. In particular, states with Euclidean masses $2\pi m$ cannot be distinguished from massless states.

Concerning the spatial degrees of freedom we can say that for the case of a single particle they completely decouple from the rest of phase space. In fact, the base point $\ast$ can be chosen such that the particle is static at a certain point $z_1 \in \Sigma$.

Let us now consider the question of the gauge symmetry. In fact, the physical phase space of the gauge field as given in Eq.(2.13) is invariant under (small) gauge transformations. This means that we still have to divide the classical single particle state given in Eq.(3.39) by conjugation with $G(\ast)$. Therefore, in the presence of just a single particle the phase space $M_1$ consists of only one point, with only the labels of the conjugacy class of $g_1$ as nontrivial data. In other words, all dynamical degrees of freedom are gauged away, leaving only the gauge invariant quantities. For a general Lie group $G$ conjugation is in fact the definition of the adjoint action in the Lie algebra,

$$\exp(\text{Ad}_g \xi) := g(\exp \xi)g^{-1}, \quad g \in G, \xi \in \mathfrak{g}.$$  

(3.40)

This in turn determines the coadjoint action on the coordinates

$$\exp(\text{Ad}_g (\xi^a T_a)) = \exp(\xi^a \text{Ad}_g T_a) = \exp((C_{\xi^a} T_a).$$

(3.41)

At least for $SU(2)$ this makes clear that we can regard a conjugacy class of the group as the exponentiation of a coadjoint orbit, where again we see that the space of orbits becomes compactified. For $ISO(3)$ it means that the coadjoint orbit with invariants $(r, s)$ corresponds to the conjugacy class with invariants $(r \text{ mod } 2\pi, s)$.

### 3.2 The multi-particle case

The situation becomes considerably more interesting if there is more than a single particle present. As before, we impose the constraints from Eq.(3.38), effectively eliminating the
internal degrees of freedom of the particles in favour of the degrees of freedom of the gauge field. The latter are purely contained in the choice of basis of the (based) fundamental group \( \pi_1(\Sigma, \ast) \) of \( \Sigma \), and in the holonomies of the flat gauge field around the basic loops.

From the identification of the phase space \( \mathcal{M} \) in Eq.(2.14) we can see that it has a definite multiplicative structure under the composition of \( \ast \)-based noncontractible loops in \( \Sigma \). To give a well defined description of the phase space as given in Eq.(2.14) we must first define the generators of \( \pi_1(\Sigma, \ast) \). Choosing another set of generators will give a different characterisation of the phase space, but it doesn’t change the physical contents, of course. We choose to order the punctures according to their angle \( \phi_k \) (increasing in the clockwise direction) relative to the base point, and by their distance to the base point if they have the same angle. The generators of \( \pi_1(\Sigma, \ast) \) are chosen to be the homotopy classes of the noncontractable loops around single punctures, which we denote by \( C_1, C_2, \ldots, C_N \), see figure 1. To each \( C_i \) we assign a holonomy in the usual way, \( g_i := w(C_i) \), as in Eq.(3.39), where by definition \( C_i \) has to be traversed in the counterclockwise direction. Traversing \( C_i \) in the clockwise direction yields the inverse holonomy. From now on we refer to the \( C_i \) as the ‘basic loops’, instead of the ‘homotopy classes of the basic loops’.

Take \( \Sigma \) with \( N \) punctures with nonvanishing curvature. Before dividing out the gauge symmetry the internal classical state will be given by \((p_{a_1}^1, j_{a_1}^1, \ldots, p_{a_N}^N, j_{a_N}^N)\). In the total phase space this corresponds to holonomies \((g_1, \ldots, g_N)\) with

\[
g_k = e^{p_k^a J_a + j_k^a P_a}, \quad k = 1, \ldots, N. \tag{3.42}
\]

In the following we will in fact identify the individual holonomies with the (classical states of the) individual particles. That is, we will often refer to \( g_k \) as the ‘\( k \)-th particle’.

We will now discuss two characteristic features of the classical multi-particle phase space, which we expect –at least partially– to remain valid on the quantum level. They are ‘fusion’ and ‘braiding’.

**Classical fusion**

Firstly, consider the case where there are two particles present. In a given ordering one can simply determine the over-all holonomy, because by traversing \( C_{1:2} := C_1 \circ C_2 \) one determines the properties of the effective particle within the loop, which consists of the two individual punctures (in the given ordering). So the total holonomy is \( g_{1:2} = w(C_{1:2}) = g_1 g_2 \), which corresponds to *fusion* of particle 1 and 2. This is the definite—-in principle nonabelian—multiplicative structure of phase space mentioned above.
We call \( g_3 := g_{1,2} \). So the classical state of the total (effective) particle, \( g_3 \), is uniquely determined by the classical states of the particles associated with the basic loops. In particular, the invariants \( r_3 \) and \( s_3 \) that label the conjugacy class of \( g_3 \) are uniquely determined by the group multiplication.

In case there are \( N \) punctures, \( \Sigma \) is noncompact, and we take the set \( \{ C_i \}_{i=1,...,N} \) defined above as the basis for \( \pi_1(\Sigma, \ast) \), we will find the over-all holonomy

\[
g_{\text{tot}} = \prod_{i=1}^{N} g_i. \tag{3.43}
\]

Note that gauge transformations are not allowed at infinity (they are not a symmetry of the CS-action), so we can shrink infinity to a point and consider \((N + 1)\) punctures on the compactified \( \Sigma \), with the condition that \( \prod_{i=1}^{N+1} g_i = e \), where \( e \) denotes the unit element in the group.

**Classical braiding**

A second essential feature of topologically interacting particles consists of the kinematical relations which their internal degrees of freedom have to satisfy, if one considers the trajectories of the particles in space time. In two spatial dimensions these relations can be nontrivial, due to the fact that the configuration space of distinguishable particles moving in the plane is non-simply connected.

Consider the case of two particles. At a certain initial time \( t \), for a given ordering of the basic loops \( C_1(t) \) and \( C_2(t) \), the classical state is given by \( (g_1, g_2) \), as depicted in figure [2]. Suppose that we carry particle 1 in the \( z_x \)-direction (one of the two independent directions on \( \Sigma \), which we parametrise by \( z \)), relatively to particle 2. At some final time \( t' > t \) the configuration of the particles will be as in Figure [3]. According to the chosen ordering of the basic loops \( C_1(t') \) will now encircle particle 2, and loop \( C_2(t') \) will go around particle 1. To relate the holonomies of the final configuration to the holonomies of the initial configuration we must express \( C_1(t') \) and \( C_2(t') \) in \( C_1(t) \) and \( C_2(t) \). It is clear that \( C_2(t') = C_1(t) \). The loop \( C_1(t') \) can be decomposed in the loops \( C_1(t) \) and \( C_2(t) \)

\[
C_1(t') = C_1(t) \circ C_2(t) \circ C_1^{-1}(t), \tag{3.44}
\]

from which it follows that the corresponding holonomy is given by \( g_1 g_2 g_1^{-1} \). Thus a counter clockwise interchange of the particles can be described by

\[
\mathcal{R}(g_1, g_2) = (g_1 g_2 g_1^{-1}, g_1). \tag{3.45}
\]
Figure 3: Basic loops at time $t'$ and their decomposition in loops at time $t$.

If particle 1 had passed particle 2 above, instead of underneath, the interchange would have been clockwise, and the first holonomy would have been conjugated by the inverse of the second one, i.e. we would have ended up with the fluxes $(g_2, g_2^{-1} g_1 g_2)$. Note that in the initial as well as the final state fusion leads to the same result, since $g_1 g_2 = (g_1 g_2 g_1^{-1}) g_1 = g_2 (g_2^{-1} g_1 g_2)$, so the total holonomy remains constant. This concludes our description of braiding and fusion on the classical level.

To take the gauge symmetry into account we must divide out the simultaneous conjugation of all holonomies. From now on we take $\Sigma$ to be compact. Let us, for simplicity, consider again the case of three punctures, where one of them represents spatial infinity. By a gauge transformation we can put one of the holonomies in the simple form of a diagonal rotation matrix and a pure time translation, we call this form ‘diagonal’ as well. However, the ‘relative differences’ between holonomies are invariant, by which we mean the following. Like on a coadjoint orbit, any element of a conjugacy class can be written as a transformation of a representative element under the group action. In Eq.(3.32) an arbitrary element $(\vec{j}, \vec{p})$ has been written as the transformation of the element $(\vec{j}_0, \vec{p}_0)$ by the group element $U \in ISO(3)$. For any $(\vec{j}, \vec{p})$ and a fixed $(\vec{j}_0, \vec{p}_0)$ the element $U$ is determined up to right multiplication by an element $V \in ISO(3)$ which leaves $(\vec{j}_0, \vec{p}_0)$ invariant. Similarly, any element $g$ in a conjugacy class $C_{r,s}$ can be written as the conjugation of a representative element in that class. We choose this representative element to be the diagonal element $g_{r,s} = \exp(r \mathcal{J}_0 + s \mathcal{P}_0)$, so we write $g = U g_{r,s} U^{-1}$. As explained in the previous subsection for a single particle this $U$ can always be gauged away, and it does not contain any dynamical information. However, in the multi-particle case a gauge transformation acts simultaneously on all holonomies, which must satisfy the ‘fusion constraint’ we described above. We write the fusion as

$$U_1 g_{r_1,s_1} U_1^{-1} U_2 g_{r_2,s_2} U_2^{-1} U_3 g_{r_3,s_3} U_3^{-1} := U_3 g_{r_3,s_3} U_3^{-1}.$$  (3.46)

From this it is clear that simultaneous conjugation of all holonomies will leave the relative differences $U_1^{-1} U_2$, $U_2^{-1} U_3$ and $U_3^{-1} U_1$ invariant, and that they can contain dynamical degrees of freedom.

Finally, to specify a certain classical state in the case of $N$ punctures on a noncompact space $\Sigma$, we must define the ordering of the punctures in space, and for each of them give the corresponding holonomy. Per puncture the holonomy gives four degrees of freedom, since we consider the $r$ and $s$ to be fixed. Also, we must divide out the gauge symmetry. This leaves us with a $4N - 4$ dimensional phase space for $N \geq 1$, plus the choice of the ordering of the
basic loops. As argued before, for $N = 1$ the phase space is a single point.

4 The quantum theory

In the previous section we have determined the classical physical phase space in the presence of particles. Now we turn to the problem of quantisation. It is important to observe the natural cotangent bundle structure of the phase space

$$M_{\text{phys}} = T^*N$$ (4.47)

$$N = \text{Hom}(\pi_1(\Sigma, \ast); SU(2))/\sim$$ (4.48)

where $\sim$ now denotes equivalence under conjugation by $SU(2)$ elements. This makes a choice of polarisation (i.e. choice of ‘momenta’ and ‘coordinates’ inside phase space) straightforward. $N$ is the configuration space, where we emphasise that it has the unusual property that its elements do not commute. Upon quantisation, the Hilbert space consists of square integrable functions on this configuration space.

Parametrisations of $SU(2)$

To discuss the quantisation in more detail we first fix some convenient parametrisations of $SU(2)$. In the Euler–angle parametrisation each $g \in SU(2)$ can be written as

$$g = g_\phi a_\theta g_\psi$$ (4.49)

with

$$g_\phi = \begin{pmatrix} e^{\frac{1}{2}i\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\phi} \end{pmatrix}, \quad a_\theta = \begin{pmatrix} \cos \frac{1}{2}\theta & -\sin \frac{1}{2}\theta \\ \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{pmatrix}$$ (4.50)

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad -2\pi \leq \psi \leq 2\pi.$$ (4.51)

Of course, all elements $g_\phi$ form the diagonal $U(1)$ subgroup.

Conjugacy classes of $SU(2)$ consist of all elements with the same angle of rotation $r$, in arbitrary directions, and thus can be thought of as 2-spheres with radius $r$. We take the diagonal element $g_r$ to be the representative of the conjugacy class

$$C_r := \{x g_r x^{-1} | x \in SU(2)\}, \quad 0 \leq r \leq 2\pi.$$ (4.52)

Let $0 < r < 2\pi$, then $C_r$ clearly consists of the elements

$$g(r, \theta, \phi) := g_\phi a_\theta g_r a_\theta^{-1} g_\phi^{-1}.$$ (4.53)

We call the ‘centraliser’ of an element the subgroup which leaves that element invariant. Elements in the same conjugacy class have isomorphic centralisers, so we can speak about the centraliser $N_r$ of the conjugacy class $C_r$, which we define as the centraliser of the representative of the conjugacy class. This means that for $0 < r < 2\pi$ we have $N_r = U(1) = \{g_\phi | -2\pi \leq \phi \leq 2\pi\}$, and its irreps are labeled by integers and half-integers. For $r = 0, 2\pi$ the centraliser is $SU(2)$ itself, with the irreps labeled by $j = 0, \frac{1}{2}, 1, \ldots$

Comparison of Eq. (4.53) with Eq. (4.52) shows that $x$ actually denotes the direction within
the conjugacy class $C_r$, so on the 2-sphere, and that it corresponds to the coset $g_\phi a_\theta N_r$. Take $\vec{r}$ to be the vector of generators of $SU(2)$ in the fundamental representation, with $\tau_3$ diagonal. Then $g(r, \theta, \phi)$ can also be written as

$$g(r, \theta, \phi) = \exp(i \frac{r}{2} \hat{n}(\theta, \phi) \cdot \vec{r}) = \mathds{1} \cos \frac{r}{2} + i \hat{n} \cdot \vec{r} \sin \frac{r}{2}$$  

(4.54)

so the representative of the aforementioned coset $x N_r$ is in 1–1 correspondence with the unit vector $\hat{n}(\theta, \phi)$, and we will choose it to be $x = g_\phi a_\theta$. The unit and minus unit element correspond to $r = 0$ and $r = 2\pi$ respectively.

### 4.1 Quantising the single particle system

In section 3 we introduced particles, and showed that this leads to the constraints of Eq.(3.38). They imply that for the case of a single particle the phase space as given in Eq.(4.47) is the conjugacy class $C_{r,s}$, instead of the coadjoint orbit $O_{r,s}$ for the internal phase space of the particle. In principle we should quantise the conjugacy class $C_{r,s}$ directly, and the resulting physical states should reflect the difference between $C_{r,s}$ and $O_{r,s}$. However, we know the mapping between the two spaces (the exponential mapping), and it is clear that the group action is the same on each of them, as can be seen in Eqs.(3.40) and (3.41). Therefore we will now first quantise the coadjoint orbit, and simply accomodate for the difference by making the appropriate identifications in the space of labels of irreps. Then we identify the resulting Hilbert space as the representation space of a 'new' kind of underlying symmetry, the true benefit of which will become fully clear if we study the multi-particle case in the next section.

The coadjoint orbit $O_{r,s}$ can be geometrically quantised for $r > 0$ iff $s \in \frac{1}{2} \mathbb{Z}$, and for $r = 0$ iff $s \in \frac{1}{2} \mathbb{Z}$ with $s > 0$. In particular, its Poisson structure is given by

$$\{F,G\}(\vec{j},\vec{p}) = \left< \vec{j}, \left[ \frac{\partial F}{\partial \vec{j}}, \frac{\partial G}{\partial \vec{j}} \right] \right> + \left< \vec{p}, \rho' \left( \frac{\partial F}{\partial \vec{j}} \right) \frac{\partial G}{\partial \vec{p}} \right> - \left< \vec{p}, \rho' \left( \frac{\partial G}{\partial \vec{j}} \right) \frac{\partial F}{\partial \vec{p}} \right>$$  

(4.55)

see Marsden et al in [20], and also [16]. In Eq.(4.55) the functional derivative $\frac{\partial F}{\partial \vec{j}}$ is regarded as an element of $su(2)$, and $\frac{\partial F}{\partial \vec{p}} \in \mathbb{R}^3$. Also, $\langle , , \rangle$ denotes both the pairing between $su(2)$ and $su(2)^*$, and the inner product on $\mathbb{R}^3$. The $\rho'$ is the induced Lie algebra representation.

By construction quantisation of this orbit leads to the irreducible unitary representation $(r,s)$ of $ISO(3)$. The carrier space (Hilbert space) of the corresponding representation is given by

$$\mathcal{H}^{r,s} = L^2(O_r) \otimes \mathcal{H}_s,$$  

(4.56)

where $O_r$ is the orbit of $\vec{p}_0$ under the coadjoint action of $SU(2)$, so the 2-sphere of radius $|\vec{p}_0| = r$. The Hilbert space $\mathcal{H}_s$ is the carrier space of the irreducible unitary representation $\Pi_s$ of the centraliser $N_r$.

The action of the group in this representation is usually described as follows. Let $\sigma$ be a Borel section for $SU(2)/U(1) \simeq O_r$, meaning a (Borel) mapping $\sigma : O_r \rightarrow SU(2)$ such that $\sigma(\vec{p})\vec{p}_0 = \vec{p}$ (again, ‘dot’ denotes the (coadjoint) action of $\sigma(\vec{p}) \in SU(2)$ on the element $\vec{p}_0 \in \mathbb{R}^3$, we will omit it from now on). In other words, the section $\sigma$ assigns to a vector $\vec{p} \in \mathbb{R}^3$ an element $x \in SU(2)$ which rotates the representative element $\vec{p}_0$ to $\vec{p}$. If one
chooses $\vec{p}_0$ along the $z$-axis, then $x$ corresponds to the direction $(\theta_p, \phi_p)$ of $\vec{p}$. The action of an element $(\Lambda, \vec{a})$ on an element $\psi$ of the space of Eq.(4.56) is given by

\begin{equation}
((\Lambda, \vec{a})\psi_{r,s}) (\vec{p}) = e^{i\vec{p} \cdot \vec{a}} \Pi_s (\sigma(\vec{p})^{-1} \Lambda \sigma(\Lambda^{-1} \vec{p})) \psi_{r,s}(\Lambda^{-1} \vec{p})
\end{equation}

where $\vec{p}' = \Lambda^{-1} \vec{p}$, $x' = \sigma(\vec{p}')$. For later applications we note that the space in Eq.(4.56) is equivalent to the following space

\begin{equation}
L^2_s(SU(2), H_s) := \left\{ \phi : SU(2) \to H_s \mid \phi(xh) = \Pi_s(h^{-1})\phi(x), \forall h \in N_r \right\},
\end{equation}

which equals the space of sections of an $H_s$-bundle over the orbit $O_r$. In this case the action of $ISO(3)$ simplifies to

\begin{equation}
((\Lambda, \vec{a})\phi^{r,s}) (x) = e^{i\vec{p} \cdot \vec{a}} \phi^{r,s}(\Lambda^{-1} x), \quad \vec{p} = x \vec{p}_0.
\end{equation}

We apologise for the slightly misleading notation, where $x$ denotes the direction of $\vec{p}$, and has nothing to do with a possible canonically conjugate variable to $\vec{p}$.

This concludes the quantisation of the internal space for the single particle case. The Hilbert space is given in Eq.(4.56), or equivalently in Eq.(4.58). It has an $ISO(3)$ symmetry, and the action of the group is given in Eq.(4.57), or Eq.(4.59).

So far we have considered the quantisation of the coadjoint orbits of $ISO(3)$, whereas, as we pointed out before, we are actually interested in quantising the conjugacy classes. Recall that the group action is the same in both cases. We will now argue that quantisation of the conjugacy classes leads to the same set of functions that form the Hilbert space of Eq.(4.58), where two important remarks have to be made. The first one is that the label $r$ has to be restricted to the interval $0 \leq r \leq 2\pi$, reflecting the physical fact that the Euclidean masses are only defined modulo $2\pi$. An important result of this paper is that this compactification of $r$, with the fact that the group action is still valid, does have a representation-theoretical foundation. This leads to the second remark, which is that the functions of Eq.(4.58) may be interpreted differently, namely as representations of a new algebraic object, being the quantum double of $SU(2)$, rather than as representations of $ISO(3)$. This quantum double of $SU(2)$ is denoted by $D(SU(2))$, and we will show that the set

\begin{equation}
\{\text{conjugacy classes of } SU(2)\} \times \{\text{centraliser representations}\}
\end{equation}

corresponds exactly to the complete set of irreducible unitary representations of $D(SU(2))$. Also, the $ISO(3)$-action is covered by the action of $D(SU(2))$, as will be explicitly shown in section 6.

For the single particle case this observation is not of crucial importance, since the $ISO(3)$ representation theory with compactification of the $p^a$-space is sufficient to describe the physical Hilbert space. However, for the multi-particle case this new symmetry is explicitly needed, because the algebraic structures of the quantum double exactly cover the physical properties of multi-particle systems with topological interactions, as will be shown in the following sections. The fact that our physical Hilbert space forms an irreducible unitary representation of the quantum double of $SU(2)$ provides a direct answer to the problem of
quantisation of a conjugacy class in ISO(3), but a direct mathematical derivation will not be given here.

Let us briefly consider a possible characterisation of states in the Hilbert space we have found. From the description of the Hilbert space as in Eq.(4.58) we see that for the single particle case a quantum state of the gauge field can be written as a function $\phi^{r,s}$ on the group $SU(2)$, with $N_r$-covariance under right multiplication of its argument. This is effectively the same as a function on a conjugacy class $r$. In section 6 we will give an explicit basis for the Hilbert space in terms of Wigner functions. At this point it suffices to remark that one may choose an alternative basis of ‘delta-sections’ on the group, i.e. delta functions which by definition have the covariance property

$$\delta^s_x(y) = \delta^s_{e^{x^{-1}y}}(y), \quad \delta^s_{xy}(y) = \delta^s_{x^{-1}y}(y) = \Pi_s(h^{-1})\delta^s_x(y). \quad (4.61)$$

In bra-ket notation we can denote the (quantum) state corresponding to this wave function by $|x\rangle_{r,s}$. This state is completely localised on the homogeneous part of the group, and corresponds to the classical holonomy

$$g = e^{paJ^a} = e^{Ad_x(rJ_0)} = xg_rx^{-1}, \quad x \in SU(2). \quad (4.62)$$

Alternatively, the state may be denoted by $|g\rangle_s$. In analogy with the Aharonov–Bohm case we call $g \in SU(2)$ the flux of the particle, and $s$ its charge. As already remarked in the previous section, due to the gauge equivalence in Eq.(4.48) the $x$ will not be a dynamical degree of freedom in the single particle case. A gauge transformation can always diagonalise a single holonomy, leaving only $r$ and $s$ as (gauge) invariants, which therefore constitute the physical observables.

### 4.2 Quantising the interacting multi-particle system

In section 3 we pointed out that most of the interesting features of the model only become manifest in the multi-particle context, and we are faced with the problem of implementing them on the quantum level. The conventional approach, where one uses the tensor product space of the free particles as a starting point for a perturbative scheme, is not suitable for the current situation, because topological interactions are nonperturbative in nature. Also, a canonical analysis constructing the symplectic structure of the multi-particle phase space of Eq.(2.14) is far from straightforward. We will now study how braiding and fusion of the particles, which have nontrivial effects on the classical states, are reflected on the quantum level, and how this in turn fixes the appropriate (mathematical) framework.

#### Quantum fusion

Given two particles on $\Sigma$, with fixed Euclidean masses $r_1, r_2 \neq 0, 2\pi$ and spins $s_1, s_2$. We take them each to be in the flux eigenstate $|g_i\rangle_{s_i} = |x_ig_ix^{-1}\rangle_{s_i}, \ i = 1, 2, \ \text{with} \ x_i \in SU(2)$, and $g_{r_i}$ the representative, diagonal element in conjugacy class $C_{r_i}$ as in Eq.(4.50). So $g_i$ is

1Technically, it would be interesting to study the connection between deformation quantisation of iso(3) as a Lie bi-algebra, and the quantum double $\mathcal{D}(SU(2))$. In particular the role of the deformation parameter is not clear in this case, since it is not manifest in $\mathcal{D}(SU(2))$.
the homogeneous part of the classical state of particle \(i\). To the total holonomy, whose loop encloses both particles, we should assign a quantum state, which must also correspond to the homogeneous part of its classical counterpart. From the multiplication in \(ISO(3)\) we see that it is given by \(|g_3\rangle = |g_1g_2\rangle\). On the quantum level fusion can be written as

\[
|g_1\rangle \otimes |g_2\rangle = |g_3\rangle.
\]

What do we know of the state \(|g_3\rangle\)? Firstly, its Euclidean mass \(r_3\) depends on \(r_1\) and \(r_2\), and on the relative difference \(x_1^{-1}x_2\), as follows from the constraint

\[
x_1g_1x_1^{-1}x_2g_2x_2^{-1} = x_3g_3x_3^{-1}.
\]

Secondly, the \(x_3\) corresponding to \(g_3\) is determined by the group multiplication in \(SU(2)\). Recall that we can also write \(g_k = \exp(\vec{p}_k \cdot \vec{J}) = x_kg_kx_k^{-1}\). It follows that in general, for noncommuting \(g_1\) and \(g_2\), this implies that \(\vec{p}_3\) is not the vector sum of \(\vec{p}_1\) and \(\vec{p}_2\). We return to this explicit example in subsection [7.3], but for the moment we note the following.

Despite the fact that states of individual particles correspond to a subset of irreps of \(ISO(3)\), the two points above clearly show that we are certainly not dealing with ordinary \(ISO(3)\) tensor product decomposition, as described for instance by Rno in [22], where he treats a.o. the Clebsch–Gordan coefficients for \(ISO(3)\) state in an irrep \((\Sigma, r, s)\). In the case of \(E(3)\) a state in an irrep \((r, s)\) is given by \(|\vec{p}\rangle\) with \(|\vec{p}| = r\), and the tensor product state \(|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle \in (r_1, s_1) \otimes (r_2, s_2)\) simply corresponds to the vector sum \(|\vec{p}_1 + \vec{p}_2\rangle\). Also, for \(E(3)\) the resulting representation label \(r_3\) lies in the interval \([|r_1 - r_2|, r_1 + r_2]\), whereas in our case, due to the fact that the tensor product state has to obey the group multiplication rule, we have that \(r_3\) is restricted to a smaller set.

\section*{Quantum braiding}

Counterclockwise interchange of the two particles must affect the quantum state of the particles in a similar way as it affects the classical state. This transformation is due to the fact that for \(n\) distinguishable particles moving in the plane the configuration space \(C_n = (\Sigma^n - D)\), with \(D\) the singular configurations in which two or more particles coincide, is non-simply connected. This means that there is no unique way to quantise such multi-particle systems, but that it depends on the kinematics of the particles, i.e. the braid formed by their world lines. More precisely, there is a quantisation associated to each unitary irreducible representation of the fundamental group of the configuration space, \(\pi_1(C_n)\), which for two spatial dimensions can be shown to be equal to the pure braid group, which we denote by \(B_n(\Sigma)\). Thus the quantum Hilbert space must (also) form an irreducible unitary representation of \(B_n(\Sigma)\). This group is generated by the monodromy operators \(\gamma_{ij}\), which by definition transport particle \(i\) in a counterclockwise direction around particle \(j\), without encircling other particles. For more details we refer to [3] and references therein.

The action of the monodromy operator \(\gamma_{12}\) is equal to twice performing a counterclockwise interchange of two particles, which is denoted by \(\mathcal{R}^2\). A single counterclockwise interchange affects the classical state as described in Eq.(3.43), restricting it to the homogeneous parts we find that

\[
\mathcal{R} : |g_1\rangle |g_2\rangle \mapsto |g_1g_2g_1^{-1}\rangle |g_1\rangle, \quad g_i \in SU(2), i = 1, 2.
\]

(4.65)
This means that in general, for noncommuting \( g_1 \) and \( g_2 \), upon interchange of the particles the wave function transforms in a nontrivial way. This will certainly affect the results of scattering processes, and gives rise to a nonabelian generalisation of the well known Aharonov–Bohm effect \([23]\). For the case of topological interacting defects in broken gauge theories, where the residual symmetry group is finite, this has been studied in detail in \([8]\). One of our objectives is to compute a two-particle scattering amplitude for particles carrying charges of the continuous, nonabelian group \( ISO(3) \), we come back to this in more detail in section \([8]\).

Summarising, we see that the states in our Hilbert space must satisfy highly nontrivial combination rules, and transform under the action of the braid group. These properties are to be interpreted as the manifestation of a new type of underlying symmetry which acts on the Hilbert space, and which we briefly discussed towards the end of section \([4.1]\). As was pointed out there, for the case at hand this new symmetry corresponds to the quantum double of \( SU(2) \), and we will study it in detail in the next section.

5  The quantum double

A quantum double group –or simply ‘quantum double’– is an example of a quasitriangular Hopf algebra. We will briefly explain its most important properties, and refer to the appendix for a summary of the basic ingredients and properties.

In general we denote a Hopf algebra by \( \mathcal{A} \). It has a multiplication \( m \), which is a mapping

\[
m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.
\]  (5.1)

It also has a comultiplication \( \Delta \), which is a mapping

\[
\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}.
\]  (5.2)

The fact that the quantum double is a quasitriangular Hopf algebra means that it contains an invertible element \( R \in \mathcal{A} \otimes \mathcal{A} \) satisfying a set of relations, amongst which the quantum Yang–Baxter equation on the space \( V_1 \otimes V_2 \otimes V_3 \), which reads

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]  (5.3)

The \( V_i \) are vector spaces on which \( \mathcal{A} \) acts, and \( R_{ij} \) denotes the representation of \( R \) on the vectorspace \( V_i \otimes V_j \), with \( R_{ij} : V_i \otimes V_j \to V_j \otimes V_i \).

In the appendix we summarise the construction of the quantum double \( \mathcal{D}(G) \) for a locally compact group \( G \). Despite certain formal mathematical objections \([14]\), we may stick to a straightforward generalisation of the construction for finite \( G \) and take

\[
\mathcal{D}(G) := C(G) \otimes \mathbb{C}[G],
\]  (5.4)

where \( G \) now denotes a (locally) compact Lie group. This means that we consider \( \mathcal{D}(G) \) as the tensor product of the algebra \( C(G) \) of continuous functions on \( G \), with the group algebra \( \mathbb{C}[G] \). A general element of \( \mathcal{D}(G) \) is denoted by \( f \otimes g \), with \( f \) a (continuous) function on \( G \), and \( g \) in the group algebra \( \mathbb{C}[G] \). For our applications we don’t run into difficulties with ill-defined expressions, products of Dirac delta functions, etc, which is the reason we can use the above definition for \( \mathcal{D}(G) \), with the advantage that it makes a clear distinction between the actions of \( C(G) \) and \( \mathbb{C}[G] \) within the quantum double action.
5.1 The irreducible unitary representations of \( D(G) \)

The derivation and classification of the unitary irreducible representations (irreps) of \( D(G) \) for locally compact \( G \) has been given in [10] in the mathematically precise formulation with \( D(G) = C(G \times G) \). We recall a main result in the formulation with \( D(G) = C(G) \otimes \mathbb{C}[G] \).

Let us first define the following space of functions

\[
L^2_\alpha(G, V_\alpha) := \{ \phi : G \to V_\alpha \mid \phi(xh) = \alpha(h^{-1})\phi(x), \ \forall h \in N, \ \text{and} \ \|\phi\|^2 := \int_G \|\phi(x)\|^2_{V_\alpha} \, dx < \infty \} \tag{5.5}
\]

Here, \( N \) is a subgroup of \( G \) acting on a Hilbert space \( V_\alpha \) via an irreducible unitary representation \( \alpha \in \hat{N} \), and \( dx \) is the normalised Haar measure on \( G \). By \( \text{Conj}(G) \) we denote the collection of conjugacy classes of the compact group \( G \). For each conjugacy class \( C_A \) we now choose some (representative) element \( g_A \in C_A \), with \( N_A \) its centraliser in \( G \). Also, we choose a unitary irrep \( \alpha \in \hat{N}_A \) of \( N_A \) on some Hilbert space \( V_\alpha \). The following theorem from [10] classifies the irreps of \( D(G) \).

**Theorem 5.1** For \( A \in \text{Conj}(G) \) and \( \alpha \in \hat{N}_A \) an irreducible unitary representation \( (A, \alpha) \) of \( D(G) \) on \( L^2_\alpha(G, V_\alpha) \) is given by

\[
((f \otimes g)A\phi_{\alpha})(x) := f(xg_Ax^{-1})A\phi_{\alpha}(g^{-1}x), \tag{5.6}
\]

and every irrep of \( D(G) \) is equivalent to some \( (A, \alpha) \).

In Eq.\((5.6)\) it is understood that \( f \otimes g \) acts in the representation \( (A, \alpha) \). We have used the more physical convention of labeling the element of the representation space, \( A\phi_{\alpha} \), instead of the element of the quantum double, \( f \otimes g \).

What is the physical content of Eq.\((5.6)\)? We will soon work it out in detail for our case of \( ISO(3) \) CS theory, but for now we remark the following about the function \( A\phi_{\alpha} \), and the action of the quantum double element \( f \otimes g \). Consider a particle inserted at a puncture in \( \Sigma \), which carries quantum numbers \( (A, \alpha) \). These quantum numbers correspond to the invariant \( A \) of the holonomy \( g \) around the puncture (its conjugacy class), and the irrep \( \alpha \) of the group that leaves \( g \) invariant. The element \( x \in G \), which is related to the holonomy via \( g = xg_Ax^{-1} \), contains all internal configuration space degrees of freedom for the gauge field in the presence of the particle at the puncture. The \( A\phi_{\alpha}(x) \) corresponds to the wavefunction of the total, interacting system. We may describe the functions of Eq.\((5.5)\) in the more physical–bra-ket notation,

\[
A\phi_{\alpha,a}(x) := \langle x \mid (A, \alpha) a \rangle, \tag{5.7}
\]

where \( a \) denotes an element of the set of labels of the basis functions on the conjugacy class. In Eq.\((5.6)\) we give an explicit example for the case of irreps of \( D(SU(2)) \). This notation will turn be useful in the tensor product decomposition, but to describe the action of the quantum double the functions \( A\phi_{\alpha}(x) \) are more suitable than the kets \( |(A, \alpha) a \rangle \).

Concerning the element \( f \otimes g \in D(G) \), we can distinguish two types of actions in Eq.\((5.6)\). Firstly, the group-part of the quantum double, i.e. \( g \) in \( f \otimes g \), acts as an ordinary representation of \( G \) on the linear space \( L^2(G) \) of square integrable functions on \( G \). This is extended to an action of the group algebra in the obvious way. Secondly, the function-part of the quantum double, so \( f \) in \( f \otimes g \), picks up the holonomy \( g = xg_Ax^{-1} \). It is precisely this last
feature which will enable us to give a first quantised description of the nontrivial properties of multi-particle systems, like the fusion and braiding we have discussed in the previous sections.

6 The quantum double of SU(2)

In this section we explicitely construct the unitary irreducible representations (irreps) of \( \mathcal{D}(SU(2)) \), and point out that their representation spaces correspond precisely to the total quantum Hilbert spaces that we described in section 4.1.

From the parametrisation of \( SU(2) \) as given in Eq.(4.53) and the classification of the irreps of \( \mathcal{D}(G) \) for general compact \( G \) we see that for \( \mathcal{D}(SU(2)) \) we can distinguish two classes of irreps, as described in [10]. They differ by the subset of \( \text{Conj}(SU(2)) \) where \( r \) belongs to, and correspond to two types of particles:

(i) generic representations
For \( 0 < r < 2\pi \) the centraliser \( N_r \) equals \( U(1) \), and its irreps are labeled by \( n \in \frac{1}{2}\mathbb{Z} \). Their representation spaces are equal to \( V_n = \mathbb{C} \). (We denote these irreps by \( n \in \frac{1}{2}\mathbb{Z} \) to distinguish them from the irreps \( (r,s) \) of \( \text{ISO}(3) \).) The Hilbert space of the irreps \( (r,n) \) of \( \mathcal{D}(SU(2)) \) is given by

\[
V^r_n := L^2_n(SU(2)) = \{ \phi : SU(2) \to \mathbb{C} \mid \phi(xh) = n(h^{-1})\phi(x), \ \forall h \in U(1) \}. \tag{6.1}
\]

Note that \( n(h) \) here means ‘the element \( h \in U(1) \) in the \( n \)-th irreducible unitary representation of \( U(1) \)’.

Physically these representations corresponds to particles with nonzero Euclidean mass \( r \) (up to \( 2\pi \)), and spin \( n \), which can take any integer or half integer value. We emphasise that the variable \( x \) denotes the direction of the ‘Euclidean momentum’ \( \vec{p} \), not a Euclidean space time coordinate.

We now show that \( \text{ISO}(3) \) in a sense is contained in \( \mathcal{D}(SU(2)) \), by identifying which particular elements of \( \mathcal{D}(SU(2)) \) correspond to particular elements of \( \text{ISO}(3) \), in a single irreducible unitary representation. Consider a state \( ^r\phi_n \) in a fixed irrep \( (r,n) \) of \( \mathcal{D}(SU(2)) \). The action of \( (f \otimes g) \in \mathcal{D}(SU(2)) \) on this state reads

\[
((f \otimes g)^r\phi_n)(x) = f(xg,x^{-1})^r\phi(g^{-1}x). \tag{6.2}
\]

Now take for \( f \) the particular function

\[
f_{\vec{a}}(\tilde{g}) := e^{i\vec{p} \cdot \vec{a}}, \quad \tilde{g} = xg,x^{-1} \in SU(2), \quad \vec{p} = x\vec{p}_0, \ |\vec{p}_0| = r, \tag{6.3}
\]

then the transformation of \( ^r\phi_n \) is given by

\[
((f_{\vec{a}} \otimes g)^r\phi_n)(x) = e^{i\vec{p} \cdot \vec{a}} ^r\phi_n(g^{-1}x). \tag{6.4}
\]

Compare this to Eq.(4.59), which describes the transformation of the same function \( ^r\phi_n \) under the action of \( \text{ISO}(3) \) in the irrep \( (r,n) \), then it follows that

\[
f_{\vec{a}} \otimes g \in \mathcal{D}(SU(2)) \iff (g,\vec{a}) \in \text{ISO}(3), \quad g \in SU(2), \vec{a} \in \mathbb{R}^3. \tag{6.5}
\]
Thus we see that under a single irrep, i.e. on the one-particle level, \(ISO(3)\) is contained in \(D(SU(2))\).

(ii) special representations
When \(r = 0, 2\pi\) the corresponding conjugacy class consists of only one element (namely the unit and minus the unit, respectively), and the centraliser is \(SU(2)\). By Remark 3.12 in [10] the corresponding irrep of \(D(SU(2))\) can act directly on \(V^0_l = V^{2\pi}_l = \mathbb{C}^{2l+1}\). However, the definition of the representation spaces in general, Eq.(5.3), tells us that for the special representations we should consider the space \(L^2_l(SU(2), \mathbb{C}^{2l+1})\). There is the following relation:

\[
\mathbb{C}^{2l+1} \iff L^2_l(SU(2), \mathbb{C}^{2l+1}) \quad (6.6)
\]

\[
v \iff \phi : x \mapsto \Gamma_l(x^{-1})v \quad (6.7)
\]

and thus \(v = \phi(e)\), and we should also label such a \(\phi\) with the basis vector \(v\) it is related to.

To avoid confusion with functions on the group we denote the \(l\)-th representation of \(SU(2)\) by \(\Gamma_l\). Eq.(6.7) follows from the defining property of the elements of the representation space as given in Eq.(5.5):

\[
\phi(xg^{-1}) = \Gamma_l(g)\phi(x), \quad x, g \in SU(2)
\]

\[
take g = x \Rightarrow \phi(e) = \Gamma_l(x)\phi(x) \quad (6.8)
\]

The special representations correspond to sources that have zero Euclidean mass (again up to \(2\pi\)). They carry \(SU(2)\) angular momentum, in contrast to the \(U(1)\) angular momentum of the generic representations.

There is a very convenient choice of basis of the Hilbert spaces for all irreps of \(D(SU(2))\), corresponding to the Wigner functions \(D^j_{mn}\), with restrictions on the labels for various distinct representations. Indeed, the Wigner functions span the algebra of functions on \(SU(2)\), and later on their properties will allow us to perform explicit calculations for the decomposition of tensor product representations of \(D(SU(2))\). Wigner functions are well studied, we follow the notation of [24], and have made extensive use of [25]. For \(x \in SU(2)\) parametrised by the Euler angles as in Eq.(4.49) the Wigner function \(D^j_{mn}\) corresponding to the \(m, n\)-th matrix element in the \(j\)-th irreducible representation takes the value

\[
D^j_{mn}(x) = e^{-im\phi} P^j_{mn}(\cos \theta) e^{-in\psi}, \quad (6.9)
\]

where \(P^j_{mn}\) are a subset of the Jacobi polynomials. For more details we refer to section 3.3 in [25].

(i) Let us first consider again the generic representations. Recall that any \(g \in SU(2)\) in a given conjugacy class \(g_r\) can be written as \(g = xg_rx^{-1} = g_\phi a_\theta g_r g_\phi^{-1} a_{\theta^{-1}}\). The representation space from Eq.(5.1) will be denoted by \(V^r_n\), and the covariance property reads

\[
\phi(xh) = e^{-in\zeta} \phi(x), \quad \forall h = e^{i\zeta} \in U(1). \quad (6.10)
\]

For clarity we denote the element of the centraliser by \(h\) instead of \(e^{i\zeta}\). The conjugacy class representative will still be denoted by \(g_r\), although they are in the same centraliser \(U(1)\).

From Eq.(6.3) we see that

\[
D^j_{mn}(xh) = e^{-in\zeta} D^j_{mn}(x), \quad \forall h = e^{i\zeta} \in U(1). \quad (6.11)
\]
This shows that the set \( \{ D_{mn}^j \mid n \text{ fixed, } j \geq n, -j \leq m \leq j \} \) has exactly the right covariance property, in other words, they are indeed sections of \( V_n \)-bundles over \( C_r \cong SU(2)/U(1) \).

As the Wigner functions form a complete set on \( SU(2) \), the aforementioned set forms a basis for a Hilbert space corresponding to an irrep of \( D_n \) irreps with the same property, in other words, they are indeed sections of \( V \). This shows that the set \( D_{mn}^j \) should also be given a label \( r \) which denotes the behaviour under the action of \( D(SU(2)) \). We write \( ^rD_{mn}^j \in V_r^n \) with

\[
\left( (f \otimes g)^rD_{mn}^j \right) (x) = f(xg,x^{-1})^rD_{mn}^j(g^{-1}x).
\] (6.12)

An arbitrary element of the representation space can be written as

\[
^r\phi_n(x) = \sum_{j \geq n} \sum_{-j \leq m \leq j} c_{jm}^r D_{mn}^j(x), \quad x \in G.
\] (6.13)

Note that the sum over \( j \) is infinite, so the generic irreducible unitary representations of \( D(SU(2)) \) are infinite dimensional. In more physical terms we say that the wavefunction for the system in the presence of a particle with nonzero (up to \( 2\pi \)) Euclidean mass and spin \( n \) can be expanded on the set of Wigner functions with a fixed label \( n \) and \( j \geq n \).

The bra-ket notation of Eq. (5.7) for this case reads

\[
^rD_{mn}^j(x) = \langle x \mid (r, n) jm \rangle,
\] (6.14)

so the label \( a \) of Eq. (5.7) here corresponds to the labels \( j, m \) of the basis functions on any conjugacy class \( r \neq 0, 2\pi \). These are nothing but the labels of the spherical harmonics, since the generic conjugacy classes are 2-spheres.

(ii) Next we consider the special representations. The representation space is given in Eq. (6.6). Let \( v_n = v_n^{(l)} \) be the \( n \)-th basis vector in \( \mathbb{C}^{2l+1} \). Then the \( m \)-th component of the \((2l+1)\)-dimensional basis function (or section of a fibre bundle over the group) \( ^0\phi_{l,n} \) is given by

\[
\left( ^0\phi_{l,n} \right)_m (x) = D_{mn}^j(x^{-1}).
\] (6.15)

For clarity we will omit the labels \((0, l)\) when it is clear which representation we are working in. The label \( n \) in fact has to be fixed, as a matter of choice of basis in the fibres over the group. This makes it more obvious that we are dealing with ordinary \( SU(2) \) representations, where \( l \) denotes the irrep and \( m \) the state. The action of \( D(SU(2)) \) on this basis reads

\[
(f \otimes g) (\phi_n)_m (x) = \left( (f \otimes g)D_{mn}^j \right) (x^{-1}) = f(e)D_{mn}^j((g^{-1}x)^{-1}) = f(e) D_{mn}^j(x^{-1}g).
\] (6.16)

The action of \( SU(2) \subset D(SU(2)) \) indeed is the ordinary group action on the \( l \)-th irreducible unitary representation, as can be seen from the following arguments:

\[
\psi_n(x) := (1 \otimes g)\phi_n(x) = \phi_n(g^{-1}x), \quad g \text{ fixed}
\]
\[
= \Gamma_l(x^{-1})\Gamma_l(g)v_n
\]
\[
\psi_n \leftrightarrow \Gamma_l(g)v_n.
\] (6.17)
This means that after the action of $(1 \otimes g)$ on $\phi_n$ the resulting function $\psi_n$ has to be expanded on the basis $\Gamma_i(g)v_i$, instead of the basis $v_i$. In terms of Wigner functions this is explicitly given by
\[
\sum_m (1 \otimes g)_{pm} D^l_{mn}(x^{-1}) = \sum_{m'} D^l_{m'}(x^{-1}) D^l_{m'n}(g),
\]
and it shows that it is in fact a right action. The labels $(p, m)$ denote the matrix element in the $(0, l)$-th irreducible representation of $D(SU(2))$. This description of the Hilbert spaces of the special irreps may at first sight appear to be unnecessarily complicated, especially in view of the equivalence in Eqs. (6.6)–(6.7). However, in calculating the decomposition of tensor products of irreps it will turn out to be convenient.

Finally, we remark that this Hilbert space can also be considered to be spanned by the kets $\{|(0, l) m\rangle\}, -l \leq m \leq l$, reflecting the fact that the conjugacy class is just the unit, or minus unit element, and the Hilbert space forms an irrep of the centraliser $SU(2)$.

7 Tensor product representations

In this section we will explain that the quantum double has precisely the correct (Hopf) algebraic structures to describe topologically interacting multi-particle systems. We start with a general remark on tensor product actions.

Suppose that we have two particles, each corresponding to an irrep of some underlying symmetry group $G$. To describe the transformation properties of the combined system we must give the action of $G$ on the tensor product space of the two representations. This is defined via the so-called comultiplication $\Delta : G \otimes G$, which for ordinary groups is given by $\Delta(g) = g \otimes g$, for any $g \in G$. For more complicated algebraic structures which may arise as a symmetry of physical systems, like the quantum double $\mathcal{D}(G)$, the comultiplication can be much less trivial. For example, for the algebra of functions on a group $C(G)$ the comultiplication is given in the Appendix in Eq. (A.3). In general, it is the comultiplication which determines the decomposition of tensor products into irreducible components, and we now turn to the case of $\mathcal{D}(G)$.

In the appendix the explicit form of the comultiplication of $\mathcal{D}(G)$ is given in Eq. (A.11). Formally this action has been derived in [11], we will rephrase it here in the (physically) more convenient description with $\mathcal{D}(G) = C(G) \otimes \mathbb{C}[G]$.

Let $(A, \alpha)$ and $(B, \beta)$ be irreducible unitary representations, then for the representation space of the tensor product we take the following space of vector valued functions on $G \times G$ (for simplicity we here omit the argument of the $L^2$-space)
\[
L^2_{\alpha\beta} := \{ \Phi : G \times G \to V_A \otimes V_B | \Phi(x_1 h_1, x_2 h_2) = \alpha(h_1^{-1}) \otimes \beta(h_2^{-1}) \Phi(x_1, x_2), \forall h_1 \in N_A, h_2 \in N_B \}
\]
and it is understood that the functions are square integrable. The action of $\mathcal{D}(G)$ in the representation $(A, \alpha) \otimes (B, \beta)$ is given by
\[
(\Delta(f \otimes g)^{A,B}_{\alpha,\beta})(x_1, x_2) = f(x_1 g_A x_1^{-1} x_2 g_B x_2^{-1}) A^{A,B}_{\alpha,\beta}(g^{-1} x_1, g^{-1} x_2),
\]
where we have used the comultiplication of $f \otimes g$ as given in Eq. (A.11). It is understood that $f \otimes g$ acts in the representation $(A, \alpha) \otimes (B, \beta)$ via the explicit labeling of the function
The functions of the form

\[ A^B \Phi_{\alpha,\beta}(x_1, x_2) = A^\alpha \Phi(x_1) \otimes B^\beta \Phi(x_2) \in V_\alpha \otimes V_\beta \]  

(7.3)

with \( A^\alpha \Phi(x_1) \) and \( B^\beta \Phi(x_2) \) basis functions of the representation spaces for \( (A, \alpha) \) and \( (B, \beta) \) respectively, span a dense subspace of \( L^2_{\alpha,\beta}(G \times G, V_\alpha \otimes V_\beta) \), and the positive–definite inner product reads

\[ \langle \Phi_1, \Phi_2 \rangle := \int_G \int_G \langle A^\alpha \Phi_1(x_1), A^\alpha \Phi_2(x_1) \rangle_{V_\alpha} \langle B^\beta \Phi_2(x_2), B^\beta \Phi_2(x_2) \rangle_{V_\beta} dx_2 dx_1. \]  

(7.4)

We have omitted the labels \((A, \alpha, B, \beta)\) on the functions \( \Phi \), and will always do so when it is obvious which representation we are working in.

### 7.1 ISO(3) versus \( \mathcal{D}(SU(2)) \)

In Eq.(6.3) we showed that the inhomogeneous group ISO(3) is contained in the quantum double \( \mathcal{D}(SU(2)) \), in the sense that within a single irrep a subset of elements of \( \mathcal{D}(SU(2)) \) can be identified with ISO(3). In this section, however, we will show explicitly that there exists a major difference between ISO(3) and \( \mathcal{D}(SU(2)) \), which is their action on tensor product representations.

To that aim, first consider the representation \((r_1, s_1) \otimes (r_2, s_2)\) of ISO(3), with \( r_1, r_2 \neq 0 \). The action of an element \((g, \bar{a})\) is defined via its comultiplication

\[ \Delta(g, \bar{a}) = (g, \bar{a}) \otimes (g, \bar{a}) \]  

(7.5)

and it reads

\[ (\Delta(g, \bar{a})\Phi)(x_1, x_2) = e^{i\bar{p}_1 + \bar{p}_2} \Phi(g^{-1}x_1, g^{-1}x_2), \]  

(7.6)

where it is understood that \( \Phi \in \mathcal{H}_r^{x_1, s_1} \otimes \mathcal{H}_r^{x_2, s_2} \). The vectors \( \bar{p}_1 \) and \( \bar{p}_2 \) are defined via the action of \( x_1 \) and \( x_2 \) respectively, as in Eq.(4.39).

Next, consider the representation \((r_1, n_1) \otimes (r_2, n_2)\) of \( \mathcal{D}(SU(2)) \), for \( r_1, r_2 \neq 0, 2\pi \), and the quantum double element \( f_{\bar{a}} \otimes g \), which is related to \((g, \bar{a}) \in ISO(3)\). With the tensor product action for the quantum double as given in Eq.(7.2) we find that

\[ (\Delta(f_{\bar{a}} \otimes g)\Phi)(x_1, x_2) = f_{\bar{a}}(x_1 g_{r_1} x_1^{-1} x_2 g_{r_2} x_2^{-1}) \Phi(g^{-1}x_1, g^{-1}x_2) \]
\[ = e^{i\bar{p}_3} \Phi(g^{-1}x_1, g^{-1}x_2), \]  

(7.7)

with \( x_3 \) defined by the constraint given in Eq.(4.64), and the vector \( \bar{p}_3 = x_3(\bar{p}_3)_{0} \), with \( |\bar{p}_3| = r_3 \).

Comparison of Eqs.\((7.6)\) and \((7.7)\) explicitly shows the difference between ISO(3) and \( \mathcal{D}(SU(2)) \) under tensor product representations. It also makes clear that the quantum double takes the fusion process described in section 4 properly into account, whereas the inhomogeneous group does not. This becomes even more obvious if one studies the decomposition of tensor products into single irreps, which we now turn to.
7.2 Fusion rules and Clebsch–Gordan coefficients

In general the decomposition of the tensor products of two irreps into single irreps is described by the Clebsch–Gordan series, which can be denoted by

\[(A, \alpha) \otimes (B, \beta) \simeq \bigoplus \gamma \int N_{\alpha \beta C}^{AB \gamma} (C, \gamma) \ d\mu(C). \tag{7.8}\]

Here \(\mu\) denotes in fact an equivalence class of measures on the set of conjugacy classes, we refer to [11] for more explanation. The coefficients \(N_{\alpha \beta C}^{AB \gamma}\) are called the ‘multiplicities’, or ‘fusion rules’.

When dealing with ordinary groups one may calculate the multiplicities via the characters

\[n_{ab}^c = \int_{h \in H} \chi^c(h)\chi^a(h)\chi^b(h) \, dh. \tag{7.9}\]

However, for technical reasons this cannot be generalised straightforwardly for the (infinite dimensional) irreps of the quantum double. To obtain an expression for the multiplicities \(N_{\alpha \beta C}^{AB \gamma}\) we construct the projection of a tensor product state onto the direct integral/sum of Hilbert spaces in the decomposition of Eq.(7.8). We subsequently compare the squared norms of the ingoing state with the direct integral/sum of squared norms in the irreducible Hilbert spaces, and thus deduce an implicit expression for the multiplicities. The construction of this projection has been derived in [11], here we recall the result.

We fix the irreducible unitary representations \((A, \alpha)\), \((B, \beta)\), and \((C, \gamma)\), and take \(\xi \in N_A \backslash G/N_B\). The representation spaces \(V_\alpha, V_\beta\) and \(V_\gamma\) have dimensions \(d_\alpha, d_\beta\) and \(d_\gamma\) respectively. For \(k = 1, ..., d_\alpha\) and \(l = 1, ..., d_\beta\) and \(i, j = 1, ..., d_\gamma\) the projection of a tensor product state onto a state in a single irrep is given by the following

**Definition 7.1** An intertwining mapping between the representations \((A, \alpha) \otimes (B, \beta)\) and \((C, \gamma)\),

\[\rho^{C}_{\gamma,k,l,j} : L^2_{\alpha \beta}(G \times G, V_\alpha \otimes V_\beta) \to L^2_{\gamma}(G, V_\gamma) \tag{7.10}\]

is given by

\[(\rho^{C(\xi)}_{\gamma,k,l,j} \Phi)_{i}(x) := \int_{N_C} \gamma_{ij}(h) \Phi_{kl}(xhw(\xi)^{-1}, xhw(\xi)^{-1}y(\xi)) \, dh, \tag{7.11}\]

where \(C(\xi), w(\xi)\) and \(y(\xi)\) are by definition related by

\[g_A y(\xi) g_B y(\xi)^{-1} = w(\xi) g_C(\xi) w(\xi)^{-1}. \tag{7.12}\]

Let us examine the occurring variables more closely: \(\xi\) denotes an element of the double coset \(N_A \backslash G/N_B\), which parametrises the possible relative difference \(y(\xi) = x_1^{-1}x_2\) between the entries \((x_1, x_2)\) of \(\Phi\). Effectively, we have constructed a new function \((\rho \Phi)\) on \(G\) out of a function \(\Phi\) on \(G \times G\). The new function depends on the degrees of freedom of the aforementioned relative difference, in the sense that the conjugacy class \(C\) which labels the irrep the new function belongs to, is uniquely determined by \(\xi\) via Eq.(7.12). The variable \(y(\xi)\) thus assures that we include the comultiplication correctly. We assume there exists a continuous mapping

\[\xi \mapsto y(\xi) : N_A \backslash G/N_B \to G, \tag{7.13}\]
which via Eq. (7.12) determines \( w(\xi) \in G \) up to right multiplication by an element \( h' \in N_{C(\xi)} \). This assumption does not seem to be restrictive \([11]\), and in fact only comes down to fixing the \( N_A \otimes N_B \) ‘phase’ factor of the ingoing state \( A, \bar{\alpha} \). By this we mean that we choose for which \( (x_1, x_2) \in G \otimes G \) the \( N_A \otimes N_B \)-factor that we can pull out by the covariance condition in Eq. (7.1) is equal to \( \alpha(\varepsilon) \otimes \beta(\varepsilon) = \mathbb{I} \otimes \mathbb{I} \). The choice of the \( N_{C(\xi)} \)-factor \( h' \) on the right of \( w(\xi) \) (in other words: the choice of the representative of the coset \( w(\xi)N_{C(\xi)} \) is nothing but the choice of the \( N_C \) phase factor of the outgoing function \( \langle \rho \Phi \rangle \).

If we write \( \Phi = \langle (A, \alpha) a | (B, \beta) b \rangle = \langle (A, \alpha) a | (B, \beta) b \rangle \), then Eq. (7.11) can be rewritten in the more familiar form

\[
\rho_C^\gamma \left( \langle (A, \alpha) a | (B, \beta) b \rangle \right) = \{ \langle (C, \gamma) c | (A, \alpha) a, (B, \beta) b \rangle \} \langle (C, \gamma) c \rangle. \tag{7.14}
\]

If the kets span orthogonal bases in the corresponding representation spaces, we may call the coefficients on the right hand side the Clebsch–Gordan coefficients of \( D \). The explicit example of \( D(SU(2)) \) will be treated in the next section, Clebsch–Gordan coefficients are given in Eq. (7.32).

As a last remark on the general formula in Eq. (7.11) we note that the mapping \( \rho_C^\gamma \) will not exist for all \( \gamma \in \tilde{N}_C \), in other words, there are selection rules. In \([11]\) it was shown that only those \( \gamma \) are allowed for which

\[
\gamma(z) = \alpha(z) \otimes \beta(z), \quad z \in Z(G), \tag{7.15}
\]

where \( Z(G) \) denotes the centre of \( G \).

**Decomposition of \( D(SU(2)) \) tensor products**

For the case \( G = SU(2) \) the definitions of the previous paragraph can be made explicit, and the formulas simplify considerably. For convenience we recall the definition of the Clebsch–Gordan coefficients of \( SU(2) \), using the notation of \([24]\)

\[
D_{m_1 n_1}^{j_1}(x)D_{m_2 n_2}^{j_2}(x) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,n=-j}^j C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 n_1 j_2 n_2}^{j n} D_j^{ij}(x), \quad x \in G. \tag{7.16}
\]

The Clebsch–Gordan coefficients are the elements of the unitary matrix which performs the direct and inverse transformations between tensor product state vectors \( |j_1 m_1 j_2 m_2 \rangle = |j_1 m_1 \rangle \otimes |j_2 m_2 \rangle \) and \( |j_1 j_2 j m \rangle \). The latter is a state in the irreducible representation \( j \), coming from the irreducible representations \( j_1 \) and \( j_2 \). Since \( C_{j_1 m_1 j_2 m_2}^{j m} = 0 \) if \( m \neq m_1 + m_2 \) we can work out the summations over \( m \) and \( n \), so for fixed \( m_1, m_2, n_1, n_2 \):

\[
D_{m_1 n_1}^{j_1}(x)D_{m_2 n_2}^{j_2}(x) = \sum_{j} C_{j_1 m_1 j_2 m_2}^{j m(n_1 + n_2)} C_{j_1 n_1 j_2 n_2}^{j m(n_1 + n_2)} D_j^{ij}(x). \tag{7.17}
\]

The primed summation runs from \( \max(|j_1 - j_2|, |m_1 + m_2|, |n_1 + n_2|) \) to \( (j_1 + j_2) \). In the following we will work out the summations as far as possible, and leave out the range of summation for \( j \)-type labels.

Let us identify the ingredients for the mapping in Eq. (7.11). From section 6 we know that the conjugacy classes \( A, B, C \) for \( G = SU(2) \) are denoted by \( r_1, r_2, r_3 \). The centraliser
representations are labeled by \( n_i \) for \( r_i \neq 0, 2\pi \) and by \( j_i \) otherwise, with dimensions 1 and \((2j_i + 1)\) respectively. We now consider various fusion processes, by combining different sectors:

(I) The case \( r_1, r_2 \neq 0, 2\pi \).

Consider the tensor product representation \((r_1, n_1) \otimes (r_2, n_2)\). Take as a basis function

\[
r_{1,2} \Phi_{n_1,n_2} = r_1 D_{m_1 n_1}^{j_1} \otimes r_2 D_{m_2 n_2}^{j_2} : (x_1, x_2) \mapsto r_1 D_{m_1 n_1}^{j_1} (x_1) r_2 D_{m_2 n_2}^{j_2} (x_2),
\]

\( j_i \geq n_i, -j_i \leq m_i \leq j_i, i = 1, 2. \)

We emphasise that \( r_1 \) and \( r_2 \) have been fixed. In the parametrisation of \( SU(2) \) as given in Eq.\((4.53)\) relation \((7.12)\) reads

\[
g_1 (r_1, 0, 0) g_2 (r_2, \theta, 0) = g_3 (r_3, \theta, \phi_3).
\]

This means that for \( r_{1,2} \Phi_{n_1,n_2} \) we have fixed the \( U(1) \otimes U(1) \) phase factor to be 1. The double coset \( U(1) \backslash SU(2) / U(1) \) is parametrised by the angle \( \theta \), and the mapping \((7.13)\) is given by

\[
\theta \mapsto y(\theta) = \begin{pmatrix} \cos \frac{1}{2} \theta & -\sin \frac{1}{2} \theta \\ \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{pmatrix}.
\]

Taking the trace on both sides of Eq.\((7.19)\) tells us how the outgoing conjugacy class label \( r_3 \) depends on \( \theta \):

\[
\cos \frac{r_3 (\theta)}{2} = \cos \frac{r_1}{2} \cos \frac{r_2}{2} - \cos \theta \sin \frac{r_1}{2} \sin \frac{r_2}{2}.
\]

From this we see that

\[
\cos(\frac{r_1 + r_2}{2}) \leq \cos \frac{r_3}{2} \leq \cos(\frac{r_1 - r_2}{2}),
\]

and thus

\[
|r_1 - r_2| \leq r_3 \leq \min (r_1 + r_2, 4\pi - (r_1 + r_2)),
\]

which restricts the possible outgoing irreducible representations, i.e. the ones in the direct sum/integral in Eq.\((7.3)\). In principle, for all incoming generic representations, one always has that \( 0 \leq r_3 \leq 2\pi \), however, in [11] it has been shown that \( r_3 = 0 \) and \( r_3 = 2\pi \) have measure zero in the image of the mapping from \( U(1) \backslash SU(2) / U(1) \) to \( \text{Conj}(SU(2)) \). This means that on a formal level special representations do occur in the tensor product decomposition of two generic representation (i.e. when \( r_1 = r_2 \), but that they do not contribute to the squared norm of a generic tensor product state. So in practice, given two generic representations \((r_1, n_1), (r_2, n_2)\), one doesn’t have to compute the mapping \( \rho_{r_3}^{r_1,r_2} \) for \( r_3 = 0, 2\pi \).

As follows from the selection rules not all \( n_3 \in \tilde{N}_{r_3} \) will occur, but only

\[
n_3 \in \left( \tilde{N}_{r_3} \right) \epsilon = \{ n \in \tilde{N}_{r_3} | n|_Z = \epsilon \mathbb{1} \}
\]

with

\[
n_1(z) \otimes n_2(z) = \epsilon (z) \mathbb{1}_{v_1 \otimes v_2}, \quad z \in \{ e, -e \} \subset SU(2),
\]

which implies that \( n_3 \) can only be integer, or half-integer, when \((n_1 + n_2)\) is integer, or half-integer, respectively. Thus the Clebsch–Gordan series reads

\[
(r_1, n_1) \otimes (r_2, n_2) \simeq \bigoplus_{n_3 \in (n_1+n_2) \mod \mathbb{Z}} \int_{I_{r_1,r_2}} (r_3, n_3) dr_3,
\]

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where \( I_{r_1, r_2} = [r_1 - r_2, \min(r_1 + r_2, 4\pi - r_1 - r_2)] \).

The variables \( w(\theta) \) and \( y(\theta) \) are related via
\[
g_{r_1} y(\theta) g_{r_2} y(\theta)^{-1} = w(\theta) g_{r_3} w(\theta)^{-1}. \tag{7.27}
\]

From Eq.\((7.19)\) we can also derive that
\[
\sin \frac{r_3}{2} \hat{n}_3 = \sin \frac{r_1}{2} \cos \frac{r_2}{2} \hat{n}_1 + \cos \frac{r_1}{2} \sin \frac{r_2}{2} \hat{n}_2 - \sin \frac{r_1}{2} \sin \frac{r_2}{2} (\hat{n}_1 \wedge \hat{n}_2) \tag{7.28}
\]
where \( \hat{n}_1 \wedge \hat{n}_2 \) denotes the exterior product, and \( \hat{n}_3 \) corresponds to \( w(\theta) \), see Eqs.\((4.53)\) and \((4.54)\). The choices made in Eq.\((7.19)\) mean that
\[
\hat{n}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix} \tag{7.29}
\]
and thus we can calculate
\[
\hat{n}_3 = \hat{n}_{w(\theta)} = \begin{pmatrix} \sin \theta w \cos \phi w \\ \sin \theta w \sin \phi w \\ \cos \theta w \end{pmatrix} = \frac{1}{\sin \frac{r_3}{2}} \begin{pmatrix} \sin \theta \cos \frac{r_1}{2} \sin \frac{r_2}{2} \\ \sin \theta \sin \frac{r_1}{2} \sin \frac{r_2}{2} \\ \sin \frac{r_1}{2} \cos \frac{r_2}{2} + \cos \theta \sin \frac{r_1}{2} \sin \frac{r_2}{2} \end{pmatrix}. \tag{7.30}
\]

The irreducible unitary representations \( n_1, n_2, n_3 \) of \( N_{r_1} = N_{r_2} = N_{r_3} = U(1) \) are 1-dimensional, so the indices \( i, j, k, l \) in Eq.\((7.11)\) won’t occur. The mapping \( \rho_{n_3} \Phi_{n_1, n_2} \) now reads:
\[
\left( \rho_{n_3}^{r_3(\theta)} \Phi \right)(x) = \int_{U(1)} e^{i n_3 \zeta} \left( \rho_{n_1, n_2}^{r_1, r_2} \Phi \right) \left( x h w(\theta)^{-1}, x h w(\theta)^{-1} y(\theta) \right) dh
\]
\[
= \sum_{j=|j_1 - j_2|}^{j_1 + j_2} \sum_{m=p=|j_2|}^{j_2} \sum_{p_2=-j_2}^{j_2} C_{j_1 m_1 j_2 m_2}^{j p} C_{j_1 n_1 j_2 p_2}^{j} D_{j_2}^{m_2} (y(\theta)) \times
\]
\[
\int_{U(1)} e^{i n_3 \zeta} \left( D_{j_1}^{r_1} (r_1, n_1) \right) \left( D_{l_2}^{r_2} (r_2, n_2) \right) D_{n_3}^{l_3} (w(\theta)^{-1}) dh \tag{7.31}
\]
\[
= \sum_{j} \langle (r_3, n_3) j m | (r_1, n_1) j_1 m_1; (r_2, n_2) j_2 m_2 \rangle D_{n_3}^{j} (x), \tag{7.32}
\]
with \( h = e^{i \zeta} \). The primed summation over \( j \) has been explained under Eq.\((7.14)\), the summation over \( p_2 \) runs from \( \max((-j - n_1), -j_2) \) to \( \min((j - n_1), j_2) \). From this it follows that the Clebsch–Gordan coefficients of the quantum double group of \( SU(2) \) are given by
\[
\langle (r_3, n_3) j m | (r_1, n_1) j_1 m_1; (r_2, n_2) j_2 m_2 \rangle = \sum_{p_2} C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 n_1 j_2 p_2}^{j (n_1 + p_2)} D_{p_2 m_2}^{j_2} (y(\theta)) D_{n_3}^{l_3} (w(\theta)). \tag{7.32}
\]
Clearly they depend on the representation labels \((r_i, n_i), i = 1, 2, 3\) and on the specific states the functions represent, which are labeled by the \( j_1, m_1 \), etc., just as would be expected. **(II)** The case \( r_1 = r_2 = 0 \)
In case both incoming ‘magnetic fluxes’ are trivial (the conjugacy classes of the corresponding irreducible representations are the unit element in the group) the ‘electric charges’ (centraliser representations) correspond to representations of the full group \( SU(2) \), we will denote
them by \( j_1 \) and \( j_2 \) respectively. The constraining relation (7.27) shows that now only \( r_3 = 0 \) can occur, and that \( y(\theta) \) and \( w(\theta) \) never affect this constraining relation. Any \( \theta \) (and thus \( y(\theta) \) and \( w(\theta) \)) will satisfy Eq.(7.27), so in the definition of \( \rho \) we may simply take them to be unity. This means that we take the same basis relative to which we choose the basis in the incoming \( SU(2) \)-fibres. So, for these special representations we can in fact consider the tensor product states as sections in the same point.

The \((m_1, m_2)\)-component of the incoming two particle wavefunction reads

\[
(\Phi_{n_1, n_2})_{m_1, m_2} (x_1, x_2) = (D^{j_1}_{m_1 n_1} \otimes D^{j_2}_{m_2 n_2})(x_1^{-1}, x_2^{-1}).
\]

(7.33)
The \( n_1 \) and \( n_2 \) occur for the same reason as for the single irreducible representation functions, as explained under Eq.(6.4), i.e. they are related to the basis vectors \( v_{n_1} \otimes v_{n_2} \) in \( V_{j_1} \otimes V_{j_2} \). The mapping \( \rho \) reads

\[
\left( \rho_{j_3, m_1, m_2, n_3}^0 \Phi \right)_{m_3} (x) = \int_{SU(2)} D^{j_3}_{m_3 n_3} (g) D^{j_1}_{m_1 n_1} (g^{-1} x^{-1}) D^{j_2}_{m_2 n_2} (g^{-1} x^{-1}) \, dg =
\]

\[
= \sum_{p_1 = -j_1}^{j_1} \sum_{p_2 = -j_2}^{j_2} \left( \int D^{j_3}_{m_3 n_3} (g^{-1}) D^{j_1}_{m_1 p_1} (g^{-1}) D^{j_2}_{m_2 p_2} (g^{-1}) \, dg \right) D^{j_1}_{p_1 n_1} (x^{-1}) D^{j_2}_{p_2 n_2} (x^{-1})
\]

\[
= \frac{1}{2 j_3 + 1} C^{j_3 n_3}_{j_1 m_1 j_2 m_2} C^{j_3 (n_1 + n_2)}_{j_1 n_1 j_2 n_2} D^{j_3}_{m_3 (n_1 + n_2)} (x^{-1}).
\]

(7.34)

From the properties of the \( SU(2) \) Clebsch–Gordan coefficients we see that this mapping only differs from zero if \( n_3 = m_1 + m_2 \). Eq.(7.34) gives the \( m_3 \)-rd component of a section which is related to \( v_{n_1 + n_2} \in V_{j_3} = \mathbb{C}^{2j_3+1} \) as indicated under Eq.(6.7).

We have combined two pure electric states corresponding to irreducible unitary \( SU(2) \)-representations, and decomposed it in single pure electric states. This should be completely equivalent to the case of ordinary \( SU(2) \)-representation theory, so we would like to make explicit connection with that. To that aim recall our discussion at the end of section 3 where we remarked that in order to obtain a normal \( SU(2) \)-representation one should fix the label \( n \) of the functions \( D^{j}_{n m} \). The same holds for tensor products, that is, one should fix \( n_1 \) and \( n_2 \), which is nothing more than choosing a specific basis in the tensor product representation space. This implies that the basis in the irreducible components has been fixed to \((n_1 + n_2)\). The second \( SU(2) \)-Clebsch–Gordan coefficient in Eq.(7.34) now is only a fixed number, depending on the basis choice. Indeed only one \( SU(2) \)-Clebsch–Gordan coefficient remains, and we have recovered the ordinary \( SU(2) \) representation theory.

The case \( r_1 \neq 0, 2\pi, r_2 = 0 \)

We consider the tensor product of a generic representation with a special representation corresponding to a flux \( e \). By construction we can never have that \( r_3 = 0, 2\pi \). The incoming wavefunction takes values in \( \mathbb{C} \otimes V_{j_2} \simeq V_{j_2} \), and we take the \( m_2 \)-component to be given by \((\Phi(z_1, z_2))_{m_2} = D^{j_1}_{m_1 n_1} (z_1) D^{j_2}_{m_2 n_2} (z_2^{-1}) \). Relation (7.27) tells us that \( r_3 = r_1 \) and that \( w(\theta) \in U_{j_3} = U(1) \). As under (II) the variable \( y(\theta) \) may take any value. From Eq.(7.27) it follows that \( w(\theta) = e \). Thus we get

\[
\left( \rho_{n_3, n_2}^{j_3} \Phi \right) (x) = \int_{U(1)} e^{i n_3 \zeta} D^{j_1}_{m_1 n_1} (x h w(\theta)^{-1}) D^{j_2}_{m_2 n_2} (x h w(\theta)^{-1})^{-1} dh =
\]

\[
= \int_{U(1)} e^{i n_3 \zeta} D^{j_1}_{m_1 n_1} (x) D^{j_2}_{m_2 n_2} (x^{-1}) e^{-i n_1 \zeta} e^{i m_2 \zeta} d\zeta =
\]

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As a first check we see that this function has the correct behaviour under right multiplication of its argument by an arbitrary element of \( U \). As expected, the outgoing electric charge \( n_3 \) depends on the \( SU(2) \)-state of the second incoming particle. This \( m_2 \) can vary between \(-j_2 \) and \( j_2 \), so the outgoing electric label \( n_3 \) can vary between \((n_1 - j_2)\) and \((n_1 + j_2)\), if we don’t specify the electric state of the second ingoing particle. The corresponding Clebsch–Gordan coefficient reads
\[
\langle (r_3, n_3) jm \mid (r_1, n_1) j_1 m_1; (0, j_2) m_2 \rangle = (-1)^{m_1 - m_2 - m} C_{j_1 m_1 j_2 m_2}^{jm} C_{j_1 m_1 j_2 m_2}^{j(m_1 - m_2)} \delta_{n_3, n_1 - m_2},
\]
as follows from Eq.\((7.35)\).

\((\text{IVi})\) \(r_1 \neq 0, 2\pi\) and \(r_2 = 2\pi\). Same result as Eq.\((7.35)\), only with \(r_3 = r_1 + 2\pi\), due to the fact that Eq.\((7.27)\) now reads \(g_{r_1}(-e) = -g_{r_1} = g_{r_1 + 2\pi} = wg_{r_3}w^{-1}\), where again we may choose \(w = e\).

\((\text{IVii})\) \(r_1 = r_2 = 2\pi\). Same as Eq.\((7.34)\), with \(r_3 = 0\).

## 8 The action of the R-element

The existence of the \( R \)-element in \( D(SU(2)) \) enables us to compute the action of the braid group on irreducible unitary representations. The action of the universal \( R \)-element reads
\[
(((A, \alpha) \otimes (B, \beta)) R) \Phi(x, y) = \Phi(x, x g_A^{-1} y^{-1}) x^{-1} y).
\]

The braid operator \( R \) is an intertwining mapping between \((A, \alpha) \otimes (B, \beta)\) on \(V_{\alpha} \otimes V_{\beta}\) and \((B, \beta) \otimes (A, \alpha)\) on \(V_{\beta} \otimes V_{\alpha}\) given by
\[
R_{\alpha}^{AB} \Phi := (\sigma_L \circ ((A, \alpha) \otimes (B, \beta)) (R)) \Phi
\]
where
\[
(\sigma_L \Phi)(x, y) = \sigma(\Phi(y, x)), \quad \sigma(v \otimes w) := w \otimes v, \quad v \in V_{\alpha}, w \in V_{\beta},
\]
interchanging the representations \((A, \alpha)\) and \((B, \beta)\). Hence
\[
(R_{\alpha}^{AB} \Phi)(x, y) = \sigma_L (((A, \alpha) \otimes (B, \beta)) (R) \Phi(x, y)) = \sigma (\Phi(y, y g_A^{-1} y^{-1} x)).
\]

At first sight, comparing to Eq.\((4.62)\), this is not quite what one would expect. However, a state \(|g\rangle\) corresponds to the delta function of Eq.\((4.61)\), and it can be checked that by taking \(\Phi = \delta_{x_1} \otimes \delta_{x_2}\) Eq.\((8.4)\) leads to the correct interchange transformation. This has been done for \(R^2\) in Eq.\((8.19)\)

With the previous equation it is not difficult to show that
\[
(R^2 \Phi)(y_1, y_2) = \Phi(y_2 g_B^{-1} y_2^{-1} y_1, y_2 g_B^{-1} y_2^{-1} y_1 g_A^{-1} y_1^{-1} y_2 g_B y_2^{-1} y_2),
\]

\[34\]
where $R^2 = R_{\beta \alpha} R_{\alpha \beta}$. For our $SU(2)$ example we can simply choose a basis function for $\Phi$. The monodromy operation on such a function, i.e. winding one of the particles counterclockwise around the other, is the action of $R^2$ in the representation $(r_1, n_1) \otimes (r_2, n_2)$, and it reads

$$R^2 (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (y_1, y_2) = (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (y_2g_{r_1}^{-1}y_2^{-1}y_1, y_2g_{r_1}^{-1}y_2^{-1}y_1g_{r_1}^{-1}y_1^{-1}y_2g_{r_2}).$$

(8.6)

As alluded to before, our Hilbert space must also form a unitary irreducible representation of the pure braid group. For two particles this group $B_2(\Sigma)$ is abelian, so it has one-dimensional irreps. The eigenvalue of the monodromy operator must satisfy the generalised spin–statistics connection

$$K^{ABC}_{\alpha \beta \gamma} R^2 = e^{2\pi i (s(C, \gamma) - s(A, \alpha) - s(B, \beta))} K^{ABC}_{\alpha \beta \gamma}$$

(8.7)

where $K^{ABC}_{\alpha \beta \gamma}$ denotes the projection on the irreducible component $(C, \gamma)$ of $(A, \alpha) \otimes (B, \beta)$, given by our mapping $\rho^C_\Phi$ in Eq.(7.11). The (generalised) spin $s_{(A, \alpha)}$ of the sector $(A, \alpha)$ is given by the action of the central element of the quantum double on a state in the $(A, \alpha)$-representation:

$$\left( \int d\phi g \delta g \otimes g \right) A \phi_\alpha (y) = A \phi_\alpha (y g_A^{-1}) = \alpha(g_A) A \phi_\alpha (y) = e^{2\pi i s_{(A, \alpha)}} A \phi_\alpha (y).$$

(8.8)

For the case $G = SU(2)$ this means that

$$e^{2\pi i s_{(r, n)}} = e^{inr} \quad \longleftrightarrow \quad s_{(r, n)} = \frac{nr}{2\pi}.$$  

(8.9)

We also verify Eq.(8.7) for this case

$$\left( \rho^r_{n_3} R^2 (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) \right) (x) = \int_{U(1)} \frac{d h n_3 (h) D^{ij}_{m_1n_1} (x h g_{r_1}^{-1} w^{-1} g_{r_1}, x h g_{r_1}^{-1} w^{-1} y g_{r_2})}{d h n_3 (h g_{r_3}) (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (x h w^{-1}, x h w^{-1} y)} = e^{-in_1 r_1} e^{-in_2 r_2} \int_{U(1)} \frac{d h n_3 (h g_{r_3}) (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (x h w^{-1}, x h w^{-1} y)}{d h n_3 (h g_{r_3}) (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (x h w^{-1}, x h w^{-1} y)} = e^{-in_1 r_1} e^{-in_2 r_2} e^{in_3 r_3} \left( \rho^r_{n_3} \right) (D^{ij}_{m_1n_1} \otimes D^{j2}_{m_2n_2}) (x),$$

(8.10)

where we have used that $g_{r_1} y g_{r_2} y^{-1} = w g_{r_2} w^{-1}$.

### 8.1 Scattering

The fact that the internal quantum numbers of encircling point particles with purely topological interactions affect the scattering cross-section is well known from the Aharonov–Bohm effect [23], which deals with charged point particles with fluxes exponentiating to elements of an abelian group. We recall the differential scattering cross-section for an external charge $q$ (corresponding to an irreducible representation of $U(1)$) scattering off a particle with only magnetic flux $\phi$ (corresponding to the $U(1)$-element $e^{i\phi}$):

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi} \frac{\sin^2 \frac{q \phi}{2}}{\sin^2 \frac{\theta}{2}}.$$  

(8.11)
where $p$ denotes the momentum of the incoming particle, and $\theta$ is the scattering angle. The nonabelian generalisation of this formula involves the monodromy $\mathcal{R}^2$, which now typically will be nondiagonal, unless we have chosen a particular incoming two-particle state being an eigenstate of the monodromy operator. It has been derived by Verlinde \cite{26}

\[
\frac{d\sigma}{d\theta} = \frac{1}{4\pi p \sin^2 \frac{\theta}{2}} (1 - \text{Re}(\langle \text{in}\vert \mathcal{R}^2 \vert \text{in}\rangle)).
\]  

(8.12)

As explained in \cite{8}, this formula was obtained by understanding that the braid group for the two-particle case is abelian, and therefore the monodromy matrix $\mathcal{R}^2$ can be diagonalised, reducing the nonabelian problem to a solved abelian problem. It is also important to remark that this is an inclusive cross-section, which means that the detector measuring the scattered particles does not distinguish between their internal electric states, in other words, scattering may go via all possible fusion channels. This is made more clear by Eq.(8.16).

In general, a two-particle Hilbert space can be decomposed into representation spaces of quantum double irreps, as shown in the former section. This means that a complete basis for this two-particle Hilbert space can be given by the direct sum of bases of all possible irreps of the quantum double. If we denote this basis by $\{\langle (A, \alpha) a \rangle\}$, $A \in \text{Conj}(G)$, $\alpha \in \tilde{N}_A$, and $a$ labeling the basis functions on $C_A$ as in Eq.(5.7), we can write

\[
\langle \text{in} \vert \mathcal{R}^2 \vert \text{in} \rangle = \int_{\text{Conj}(G)} d\mu(A) \sum_{\alpha, a} \langle \text{in} \vert (A, \alpha) a \rangle \langle (A, \alpha) a \vert \mathcal{R}^2 \vert \text{in} \rangle.
\]  

(8.13)

To be more specific, for $G = SU(2)$ the complete basis $\{\langle (A, \alpha) a \rangle\}$ is given by

\[
\{\vert (r_3, n_3)jm \rangle, \text{ with } 0 < r_3 < 2\pi, n_3 \in \frac{1}{2} \mathbb{Z}, j \geq n_3, j \in \frac{1}{2} \mathbb{N}, -j \leq m \leq j \} \bigcup \{\vert (0, l)m' \rangle, \vert (2\pi, l)m' \rangle, \text{ with } l \in 4 \mathbb{N}, -l \leq m' \leq l \}.
\]  

(8.14)

We will now calculate the differential cross-sections for the scattering of various types of particles.

(i) Consider two generic representations, so the incoming state is an element of $V_{n_1}^{r_1} \otimes V_{n_2}^{r_2}$. Firstly, take the incoming state $\vert \text{in} \rangle$ as given in Eq.(7.18). Then the first factor in Eq.(8.13) is simply the Clebsch–Gordan coefficient from Eq.(7.32). We have seen that these Clebsch–Gordan coefficients are only nonzero for a subset of all possible fusion channels, so the sum/integral in Eq.(8.13) is only over the irreps that occur in the Clebsch–Gordan series, see Eq.(7.20). For the second factor of Eq.(8.13) we use the generalised spin–statistics connection of Eq.(8.7)

\[
\langle (r_3, n_3)jm \vert \mathcal{R}^2 \vert \text{in} \rangle = e^{i(n_3r_3 - n_1r_1 - n_2r_2)} \langle (r_3, n_3)jm \vert \text{in} \rangle.
\]  

(8.15)

We arrive at

\[
\langle \text{in} \vert \mathcal{R}^2 \vert \text{in} \rangle = e^{-i(n_1r_1 + n_2r_2)} \sum_{n_3 \in (n_1 + n_2)\mod \mathbb{Z}} \sum_{j,m} \int_{I_{r_1, r_2}} e^{in_3r_3} \vert \langle (r_1, n_1)jm_1, (r_2, n_2)jm_2 \vert (r_3, n_3)jm \rangle \vert^2 \, dr_3.
\]  

(8.16)

Unfortunately, it cannot easily be seen that this expression is finite, which we expect to be the case. An alternative way is to compute the left hand side of Eq.(8.13) directly. We use the
explicit action of $\mathcal{R}^2$ as given in Eq. (8.6), and take the inner product according to Eq. (7.4). Working it all out leaves us with the following unattractive, but finite (!), expression:

$$\langle \text{in} | \mathcal{R}^2 | \text{in} \rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{k=|k_1-k_2|}^{j_1+j_2} \sum_{l=|l_1-l_2|}^{k_1+k_2} \sum_{m=|m_1-m_2|}^{l_1+l_2} \sum_{n=|n_1-n_2|}^{m_1+m_2} \sum_{r=|r_1-r_2|}^{n_1+n_2} \sum_{s=|s_1-s_2|}^{r_1+r_2} C_{jm}^{j_1m_1m_2} C_{jm'}^{j_1m_1m_2} C_{jk}^{j_1m_1m_2} C_{jk'}^{j_1m_1m_2} C_{kl}^{j_1m_1m_2} C_{kl'}^{j_1m_1m_2} C_{kr}^{j_1m_1m_2} C_{kr'}^{j_1m_1m_2}$$

(8.17)

Inserting this in Eq. (8.12) yields a finite differential scattering cross-section for the case where the incoming wave function is given by $r^1 \mathcal{D}_{m_1n_1}^{j_1} \otimes r^2 \mathcal{D}_{m_2n_2}^{j_2}$.

Secondly, as a different example of an incoming state in the same representation, we consider a wave function corresponding to two particles with sharply determined fluxes $g_1 = x_1 g_{r_1} x_1^{-1}$ and $g_2 = x_2 g_{r_2} x_2^{-1}$. This means that $|\text{in}\rangle$ is given by $r^1 \delta_{x_1,n_1} \otimes r^2 \delta_{x_2,n_2}$, a tensor product of 'delta-sections' for which

$$\delta_{xh^{-1},n} = e^{-i\zeta \delta_{x,n}}, \quad \forall h = e^{i\zeta} \in U(1),$$

(8.18)

where from now on we omit the $r$-labels, as before. Formally, such delta-sections are not elements of the Hilbert spaces of irreps of $\mathcal{D}(SU(2))$, because they are not square integrable (normalisable), as was required in Eq. (7.5). However, they correspond to the more classical picture of incoming particles of which we know the fluxes precisely, therefore we would like to know the scattering cross-section for such wave functions (distributional sections) as well. From the model for topologically interacting excitations in broken gauge theories [3], which corresponds to the quantum double of a finite group, we know that the action of the monodromy operator comes down to conjugation of each flux by the total flux, and that there can be nontrivial actions in the centraliser representation spaces. It is easy to check that this corresponds to the action of $\mathcal{R}^2$ as given in Eq. (8.3), which now reads

$$\left( \mathcal{R}^2 \delta_{x_1} \otimes \delta_{x_2} \right) (x, y) = e^{-i\zeta_{1r_1}} \left( \delta_{g_1 g_{2x_1}} \otimes \delta_{g_1 x_2} \right) (x, y).$$

(8.19)

Consider the inner product between two delta-sections in a generic irrep $(r, n)$:

$$\langle \delta_x | \delta_{x'} \rangle = \int \langle \delta_{x,n}(y) | \delta_{x',n}(y) \rangle_{V_n} dy = \delta_{x',n}(x).$$

(8.20)

We decide that we extract all non-equal $U(1)$ phase factors from the right. This means the following: in general any $x \in G$ can be written as $xh$, with $x \in G$ a chosen representative of the coset $xH$, and $h \in H$. To compare differences in centraliser states between elements of the Hilbert space $V_n^r$ we choose the same representative in each coset. For the $SU(2)$ case in the Euler–angle parametrisation this comes down to choosing the same angle $\psi$, see Eq. (4.49). So $x = xh$ and $x' = x'h'$, with now $x, x' \in SU(2)$, and $h, h' \in U(1)$. This yields

$$\delta_{x',n}(x) = n(x' x^{-1}) \delta_{x',n}(\bar{x})$$

(8.21)

where the delta on the right side is an ordinary delta function on the group. In calculating $\langle \text{in} | \mathcal{R}^2 | \text{in} \rangle$ we find

$$\langle \text{in} | \mathcal{R}^2 | \text{in} \rangle = e^{-i\zeta_{1r_1}} \delta_{g_1 g_{2x_1}, n_1}(x_1) \delta_{g_1 x_2, n_2}(x_2).$$

(8.22)
This will always be zero, unless \( g_1 \) and \( g_2 \) commute. They do so iff \( \vec{x}_1 = \vec{x}_2 \), or \( \vec{x}_1 = \vec{x}_2 a_\pi \). In these cases we can 'pull' the factors \( g_1 g_2 \) and \( g_1 \) through \( x_1 \) and \( x_2 \) respectively. Since we have chosen the representatives of the conjugacy classes to lie in (the same) \( U(1) \) we obtain
\[
\delta_{g_1 g_2 x_1, n_1}(x_1) = e^{im_1 (r_1 + r_2)} \delta_{x_1}(x_1),
\]
(8.23)
and a similar factor for \( \delta_{x_2, n_2} \), and similar term for the case \( \vec{x}_1 = \vec{x}_2 a_\pi \). Such a fulfilled delta function of course gives rise to infinities, which are to be expected, since we started off with non-normalisable states. We find that
\[
\langle \text{in} | \mathcal{R}^2 | \text{in} \rangle = e^{im_1 r_1} e^{im_2 r_1} \delta_{x_1}(x_2) + e^{-im_1 r_1} e^{-im_2 r_1} \delta_{x_1}(x_2 a_\pi),
\]
(8.24)
and upon substituting this in Eq. (8.12) we see that the differential cross section for the scattering of two particles with magnetic fluxes \( g_1 \) and \( g_2 \) (and equal electric phase factor) becomes
\[
\frac{d\sigma}{d\theta} = \frac{1}{4\pi p} \frac{1}{\sin^2 \frac{\theta}{2}} (1 - (\delta_{x_1}(x_2) + \delta_{x_1}(x_2 a_\pi)) \cos(n_2 r_1 + n_1 r_2)).
\]
(8.25)
Note that the delta functions live on the set of coset representatives.

From this we conclude that the scattering of two precisely determined fluxes that do not commute gives
\[
\frac{d\sigma}{d\theta} = \frac{1}{4\pi p} \frac{1}{\sin^2 \frac{\theta}{2}}.
\]
(8.26)
If the fluxes do commute we in a sense retrieve the abelian case, were it not for the fulfilled delta functions in front of the cosine. If these are somehow regularised (by approximating the continuous set of coset representatives by a finite set, and taking Kronecker delta’s instead of Dirac delta functions) we find that
\[
\frac{d\sigma}{d\theta} = \sin^2 \frac{\theta}{4} \left( \frac{n_2 r_1 + n_1 r_2}{2\pi p} \right) \frac{1}{\sin^2 \frac{\theta}{2}}.
\]
(8.27)
(ii) In this case we consider the incoming state to correspond to a generic and a special representation, \(|\text{in}\rangle \in V_{n_1}^{r_1} \otimes V_{j_2}^0 \). In particular, we take
\[
|\text{in}\rangle = r_1 D_{m_1 n_1}^{j_1} \otimes 0 D_{m_2 n_2}^{j_2},
\]
(8.28)
then it can be shown that
\[
\frac{d\sigma}{d\theta} = \frac{1}{4\pi p} \frac{1}{\sin^2 \frac{\theta}{2}} \left( 1 - \cos(r_1 m_2) \sum_j ' \left( C_{j_1 m_1 j_2 - n_2}^{j_1 n_1 - m_2} C_{j_1 n_1 j_2 - m_2}^{j_1 n_1 - m_2} \right)^2 \right),
\]
(8.29)
where we have used Eq. (8.33).

(iii) Consider again a generic and a special representation, now with \(|\text{in}\rangle = r_1 D_{m_1 n_1}^{j_1} \otimes 2\pi D_{m_2 n_2}^{j_2}.\) This will give
\[
\frac{d\sigma}{d\theta} = \frac{1}{4\pi p} \frac{1}{\sin^2 \frac{\theta}{2}} \left( 1 + \cos((n_1 + m_2) r_1 + 2\pi m_2) \sum_j ' \left( C_{j_1 m_1 j_2 - n_2}^{j_1 n_1 - m_2} C_{j_1 n_1 j_2 - m_2}^{j_1 n_1 - m_2} \right)^2 \right).
\]
(8.30)
(iv) For pure charges, so \(|\text{in}\rangle \in V_{j_1}^0 \otimes V_{j_2}^0 \) it is easy to see that the monodromy matrix leaves the incoming state invariant, and thus \( \frac{d\sigma}{d\theta} = 0. \)
9 Gravity in 2+1 dimensions, and $\mathcal{D}(SO(2, 1))$.

In this section we show that, under the assumption of zero cosmological constant, for gravitating particles in two spatial and one time dimension there also is the structure of a quantum double symmetry, now corresponding to the locally compact group $SO(2, 1)$. We briefly recall the special features of 2+1 gravity, and the classical point particle solutions. Then we summarise Witten’s gauge theoretical formulation of 2+1 gravity, and show that it is almost equal to the CS theory discussed in section 3, but now for the gauge group $ISO(2, 1)$ (or its universal covering, but for now we only consider $ISO(2, 1)$).

We start with a general remark about gravity in three dimensions, which shows the topological nature of the theory. The Einstein–Hilbert action is given by

$$S_{EH} = \int d^3x \sqrt{-g} R,$$

(9.1)

with $g$ the determinant of the full three-dimensional metric of the spacetime, and $R$ the associated scalar curvature. In general, that is in arbitrary dimensions, the equations of motion are the Einstein equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0,$$

(9.2)

with $R_{\mu\nu}$ the Ricci tensor and $G_{\mu\nu}$ the Einstein tensor. When matter terms are added to the action the field equations become

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $\kappa = 8\pi G$ the coupling constant, and $G$ Newton’s constant. The Riemann four-index tensor $R_{\alpha\mu\beta\nu}$ carries all the information about the curvature of space time. What makes three dimensional space time special is that in that case the curvature tensor $R_{\alpha\mu\beta\nu}$ and Ricci tensor $R_{\mu\nu}$ have the same number of degrees of freedom. This allows one to express the curvature tensor in terms of the Einstein tensor

$$R_{\alpha\beta\mu\nu} = -\epsilon_{\alpha\beta\mu\nu} \epsilon_{\gamma\delta\epsilon} G^{\gamma\delta}.$$

(9.3)

Together with Eq. (9.2) we immediately see that in empty regions the curvature tensor vanishes, so the geometry is flat. We will only consider point sources, so the curvature is concentrated along the worldlines of the particles.

Next, we briefly consider classical particle solutions in this theory [27]. From now on we put $8\pi G = 1$. For a single, static, particle of mass $\mu$ located at the origin the energy-momentum tensor reads

$$T^{00}(z) = \mu \delta^{(2)}(z), \quad T^{0i} = T^{ij} = 0.$$

(9.4)

Solving the Einstein equations in the ADM formulation of general relativity, and making an appropriate coordinate transformation, yields the following expression for the spatial line element

$$dl^2 = dr^2 + r^2 d\theta^2,$$

(9.5)

which is manifestly flat. However, the range of the angular variable $\theta$ is restricted

$$0 \leq \theta \leq 2\pi \alpha, \quad \alpha := 1 - \frac{\mu}{2\pi}.$$

(9.6)

For $\mu < 2\pi$ this corresponds to a cone with deficit angle $\mu$. These boundary conditions are equivalent to the holonomy one obtains upon performing a closed loop around the particle in
the original, not manifestly flat coordinates. (This can for instance be measured by carrying a gyroscope around the particle, because the holonomy corresponds to a rotation over \( \mu \).) Only values \( 0 \leq \mu < 2\pi \) are physically admissable, because for masses equal or larger than \( 2\pi \) the space vanishes. In case the particle also carries intrinsic angular momentum, i.e. it has spin \( s \), it can be shown that upon performing a closed loop the holonomy also contains a jump in time of magnitude \( s \), in our units. Here we will not discuss the possible consequences of the metric described above, but restrict ourselves to configurations whose total mass is positive, but smaller than \( 2\pi \), and allow for translations in time.

To a mass \( \mu \) spin \( s \) particle in the origin we thus associate the holonomy corresponding to the element \((\Lambda_\mu, a_s)\) of the Poincaré group \( \text{ISO}(2,1) = \text{SO}(2,1) \ltimes T(3) \) in three dimensions, which is a pure rotation over \( \mu \) and a pure time translation over \( s \). (From now on we omit vector arrows for three-vectors, to avoid confusion with spatial two-vectors.) In case the particle is not static but moving in an arbitrary Lorentz frame, an observer should first boost (with an \( \text{SO}(2,1) \) element \( L \)) and translate (over a three-vector \( t \)) to a frame where the particle is at rest in the origin, perform the closed loop to pick up the rotation and translation due to its mass and spin, and then transform back to his/her original restframe. This means that the holonomy \( U \) corresponding to such a particle is given by a conjugation of \((\Lambda_\mu, a_s)\) with the Poincaré element \((L,t)\).

\[
U = (\Lambda, a) = (L, t) (\Lambda_\mu, a_s) (L, t)^{-1} = \exp \left( \text{Ad}_{(L,t)} (\mu \mathcal{J}_0 + s \mathcal{P}_0) \right) .
\]

(9.7)

It is not hard to see that in the presence of more than one particle a closed loop around several particles will give the product of the encircled holonomies, in which we recognise the fusion property as described in subsection 3.2.

Let us return to the case of pure gravity, no particles present. An important observation is that the Einstein–Hilbert action in three dimensions can be rewritten as the action of a Chern–Simons theory for an \( \text{ISO}(2,1) \) gauge field \([29][12]\). To that aim one uses the vielbein formalism for curved manifolds, so one introduces the dreibein \( e^a_\mu \) and the associated spin connection \( \omega^a_{\mu \nu} \). They obey the well known relation

\[
g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab},
\]

(9.8)

and the zero torsion condition

\[
T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\nu b} e^b_\mu - \omega^a_{\mu b} e^b_\nu = 0.
\]

(9.9)

With the definition \( \omega^a_{\mu \nu} = \frac{1}{2} \epsilon^{a bc} \omega^b_{\mu \nu} \) the Einstein–Hilbert action of Eq.(9.1) can be written as

\[
S = \int d^3 x \epsilon^{\mu\nu\rho} e_{ap} \left( \partial_\mu \omega^a_{\nu \rho} - \partial_\nu \omega^a_{\mu \rho} + \epsilon^{abc} \left( \omega^b_{\rho \mu} \omega^c_{\nu} - \omega^b_{\nu} \omega^c_{\mu} \right) \right) .
\]

(9.10)

Consider the \( \text{ISO}(2,1) \) gauge field

\[
A_\mu = e^a_\mu \mathcal{P}_a + \omega^a_{\mu} \mathcal{J}_a ,
\]

(9.11)

with \( \mathcal{J}_a \) and \( \mathcal{P}_a \) the generators of the three-dimensional Poincaré group \( \text{ISO}(2,1) \),

\[
[\mathcal{J}_a, \mathcal{J}_b] = \epsilon_{abc} \mathcal{J}^c, \quad [\mathcal{J}_a, \mathcal{P}_b] = \epsilon_{abc} \mathcal{P}^c, \quad [\mathcal{P}_a, \mathcal{P}_b] = 0,
\]

(9.12)
and the following inner product on the algebra

$$\langle J_a, P_b \rangle = \delta_{ab}, \quad \langle P_a, P_b \rangle = \langle J_a, J_b \rangle = 0. \quad (9.13)$$

Then Eq.(9.10) is nothing but the Chern–Simons action as given in Eq.(2.1) for the gauge field of Eq.(9.11). We have obtained the same theory as in section 2, however, now for the inhomogeneous group ISO(2,1), which is reflected in the fact that the contraction of the flat indices $a,b,c,...$ goes via the Minkowski metric $\eta_{ab}$.

Inclusion of particles in the theory can be done in a similar fashion as in section 3, that is, by allowing for curvature and torsion at the positions of the particles, just as in Eq.(3.38). The resulting holonomy degrees of freedom are exactly the same as given in Eq.(3.39), but now take values in the group ISO(2,1). This coincides with the holonomy of Eq.(9.7), since

$$\exp(\text{Ad}(L,t)(\mu J_a + s P_a)) = e^{i\mu J_a + j^a P_a}, \quad (9.14)$$

with $p^a = \Lambda^a_0 p^0$ and $j^a = \Lambda^a_0 j^0 + \epsilon_{abc}^i (\Lambda^i p)^c$, compare Eq.(3.28). In ordinary CS theory these $p^a$ and $j^a$ are internal degrees of freedom, taking values in the dual of the Lie algebra. For 2+1 gravity, however, they denote the real, external (space time) momentum and angular momentum of a particle in the Minkowski frame at the position of the particle. We return to this crucial point at the end of this section.

For a single particle the total phase space consists of the conjugacy class labeled by $\mu$ and $s$ in $\mathcal{M} = ISO(2,1)$, which has a natural cotangent bundle structure, just as in Eqs.(4.47) and (4.48). This allows for a straightforward choice of polarisation of the total phase space, and since the homogeneous part of an ISO(2,1) holonomy corresponds to the three-momentum $p^a$ of the particle, it follows directly that upon quantisation the Hilbert space consists of wave functions of the momentum. For a given fixed mass $\mu$ this momentum $p^a$ lies on the hyperboloid $(p^0)^2 - (p^i)^2 = \mu^2$, with $0 < \mu < 2\pi$ as it should, and where $p^i$ denotes the spatial momentum. Note that, as in the case of ISO(3), the elements of the configuration space, which are holonomies depending on the momenta, are noncommuting. Also the generators $j^a$ of ‘translations’ in configuration space are noncommuting, compare Eq.(3.34).

Mathematically spoken we can proceed with quantisation just as we did in section 4. This means that for the case of 2+1 gravity the Hilbert space decomposes into irreducible unitary representation of $\mathcal{D}(SO(2,1))$, the quantum double of $SO(2,1)$. This is in contrast to a decomposition into irreps of ISO(2,1), although the important difference mainly shows up in the interacting multi-particle systems. The sectors of the theory are labeled by conjugacy classes in $SO(2,1)$ and irreps of their centralisers. Due to the locally compact structure of $SO(2,1)$ there are more than just two different types of sectors, as was the case for $\mathcal{D}(SU(2))$ where we had the generic and the special representations corresponding to $0 < r < 2\pi$ and $r = 0, 2\pi$ respectively. Recall that $SO(2,1)$ is isomorphic to $SL(2,\mathbb{R})$ and to $SU(1,1)$. The classification of the irreps of $\mathcal{D}(SL(2,\mathbb{R}))$ has been given in [10]. Of the ten types of conjugacy classes described there, we associate the first type to the massive particles in 2+1 gravity. This conjugacy class is given by $C_\mu = \{xg_\mu x^{-1} \mid x \in SO(2,1)\}$, with $g_\mu$ corresponding to the diagonal element as defined in Eq.(4.50), for $0 < \mu < 2\pi$. Note that this representation-theoretical restriction for the values of $\mu$ agrees with the physically allowed values for the mass. The centraliser of such conjugacy classes in SO(2,1) has irreps labeled by $s \in \frac{1}{2}\mathbb{Z}$. 
The corresponding Hilbert space $V_s^\mu$ is given by
\[ V_s^\mu := L^2_s(SO(2,1)) = \{ \phi : SO(2,1) \to \mathbb{C} \mid \phi(xh) = s(h^{-1})\phi(x), \ \forall h \in U(1) \}, \tag{9.15} \]
which equals the space of $V_s$ sections over the orbit $C_\mu \simeq SO(2,1)/U(1)$, which as a space equals the two dimensional hyperboloid $H^2$. We emphasise again that the $x$ here denote the direction of the momentum $p^\mu$, and have nothing to do with spatial coordinates. From harmonic analysis on $H^2$ \[25\] it is known that the Plancherel decomposition of $L^2(SL(2,\mathbb{R})/SO(2))$ only contains the continuous irreps of $SL(2,\mathbb{R})$, also called the first and second fundamental series \[25\]. More precisely, a function on $SL(2,\mathbb{R})/SO(2) \simeq SO(2,1)/SO(2) \simeq SU(1,1)/U(1)$ can be decomposed in a subset of the matrix elements of the continuous irreducible unitary representations of $SO(2,1)$. For the case of a $V_s$ section over $SO(2,1)/SO(2)$ this subset consists of matrix elements of either the first or the second fundamental series, depending whether the fixed right hand side label $s$ is integer or half-integer, respectively. We introduce $s = s \mod \mathbb{Z}$, which can either be 0 or $\frac{1}{2}$. An arbitrary element of $V_s^\mu$ can be written as
\[ \hbar^2 \phi_s(x) = \sum_{\epsilon=0,\frac{1}{2}} \delta_{\epsilon,s} \int_0^\infty \sum_{m'=m+\epsilon} c^\epsilon_m(\lambda) \lambda \tanh(\lambda + i\epsilon) \mathfrak{D}_{m's}^{-\frac{1}{2}+i\lambda}(x) d\lambda, \quad x \in SU(1,1) \tag{9.16} \]
which should be compared to the case of $\mathcal{D}(SU(2))$ in Eq.(6.13). The integration in Eq.(9.16) uses the Plancherel measure $\rho_s(\lambda)$, which normalises the functions $T_{ms}^\lambda := \mathfrak{D}_{ms}^{-\frac{1}{2}+i\lambda}$ as follows
\[ \langle T_{ms}^\lambda | T_{ms'}^{\lambda'} \rangle = \frac{1}{\rho_s(\lambda)} \delta(\lambda - \lambda') = (\pi \lambda \tanh(\lambda + i\tilde{s}))^{-1} \delta(\lambda - \lambda'). \tag{9.17} \]
So the set
\[ \{ \mathfrak{D}_{ms}^{-\frac{1}{2}+i\lambda} \mid s \text{ fixed}, \ \lambda \in \mathbb{R}^+, \ m \in (\mathbb{Z} + \tilde{s}) \} \tag{9.18} \]
forms a basis for $V_s^\mu$. Just as the Wigner functions $D^j_{mn}$ are related to the Jacobi polynomials $P^j_{mn}$ as given in Eq.(9.19), the functions $\mathfrak{D}_{ms}^{-\frac{1}{2}+i\lambda}(x)$ are related to the so-called Jacobi functions $\mathfrak{D}^j_{mn}$. For more details we refer to \[25\].

Summarising, due to the fact that 2+1 gravity with point particles can be written as the CS theory of an inhomogeneous group, namely $ISO(2,1)$, we find that its quantum theory is governed by the quantum double of the homogeneous part of that group, $\mathcal{D}(SO(2,1))$. For a single particle with fixed mass $0 < \mu < 2\pi$ and spin $s \in \frac{1}{2}\mathbb{Z}$ we have found the total Hilbert space $V_s^\mu$ of Eq.(9.13), i.e. the space of wave functions for the metric and particle together, under the constraints that the curvature and torsion of the gauge field are equal to the momentum and angular momentum, respectively. The wave functions depend on the energy and momentum, and transform covariantly under right action with $N_\mu = U(1)$. Due to the properties of the comultiplication of the quantum double the fusion properties of multi-particle systems are automatically reflected on the quantum level, analogously to the case of $\mathcal{D}(SU(2))$ CS theory, which is described in detail in section 7 of the current article. Thus, a consistent construction of a multi-particle Hilbert space in 2+1 gravity is (at least partially) solved by identifying the underlying quantum double structure. This is supported by the fact that the quantum double provides us with another important tool, namely the $R$-element, which assures that the Hilbert space also forms a representation of
the braid group $B_n(\Sigma)$ for $n$ distinguishable particles. In principle this allows us to compute differential cross-sections of scattering processes with massive, spinning particles in 2+1 gravity. However, here one faces the interesting problem of reconstructing the physical space time from the Hilbert space which arises upon canonical quantisation of the multi-particle phase space $\mathcal{M} = \text{Hom}(\pi_1(\Sigma,*) ; ISO(2,1))/\sim$. This problem has been recognised and addressed by other authors as well [30] [31], but they impose different matching conditions on the space time coordinates. We will return to these issues in a forthcoming paper [5].

10 Conclusions

We have shown that the representation theory of $\mathcal{D}(SU(2))$, the quantum double of $SU(2)$, can be used to give a quantised description of purely topological interacting point particles coupled to an $ISO(3)$ CS gauge field in 2+1 dimensions. We discussed the classical properties of the system, and used the explicit model of the $ISO(3)$ CS theory to get an explicit parametrisation of the physical phase space. Due to the fact that (classical) observables are only measured via noncontractable loops the phase space has a multiplicative structure, which must be maintained on a quantum level. This meant that we had to quantise the particles plus their interactions in one go, instead of quantising single, free particles and describing their interaction via an interaction term in the Hamiltonian. Indeed, due to the topological nature of the interactions, it turned out to be possible to directly incorporate the interactions on the level of the Hilbert space. In section 4 we discussed the properties of the quantum system for single and multi-particle cases, and the two types of interactions, ‘fusion’ and ‘braiding’. In section 5 we made a side step to pure mathematics, and described the quantum double $\mathcal{D}(G)$ of a (compact) group $G$. Subsequently we gave its irreducible unitary representations, and worked them out for the case $G = SU(2)$. Simply by comparing the labels of the irreps, their degrees of freedom, and their properties under fusion and braiding, with the quantum numbers, configuration space, and interaction properties of the wave functions of the quantised system, we showed that the representation theory of $\mathcal{D}(SU(2))$ precisely covers the quantum theory of our system.

The nontrivial comultiplication, which is part of the Hopf algebra structure of the quantum double, implies that the action on tensor product states is different from the tensor product action of groups, and therefore that the decomposition in irreducible unitary components (the Clebsch–Gordan series) becomes different and much more involved than for the case of ordinary groups. In section 7 we worked it out in detail for the case $G = SU(2)$. Although it is a rather technical aspect, it is interesting to note that the derivation of the Clebsch–Gordan coefficients for $\mathcal{D}(SU(2))$ requires solution of the problem of combining Wigner functions in different points of $SU(2)$. Next we studied the braiding of two particles. First we described how the $R$-element of the quantum double describes the effect of braiding, then we used the nonabelian generalisation of the Aharonov–Bohm formula, Eq. (8.12), to calculate the differential cross-section for various scattering processes. We concluded by pointing out the quantum double structure $\mathcal{D}(SO(2,1))$ underlying the quantum theory of 2+1 gravity, as a preview to [3].

Finally, the connection with the ‘ordinary’ $q$-deformed structure for CS theories with homogeneous, compact gauge group deserves further analysis. Related to this for the inhomogeneous case is the identification of the ‘algebra of observables’, corresponding to the func-
tions on classical phase space, and leading to the quantum double structure (or $q$-deformed structure for compact CS theories) of the quantum theory.

A The quantum double

We summarise the construction of the quantum double for a locally compact group as given in [10]. More generally the quantum double of a Hopf algebra has been introduced by Drinfel’d in [32]. Quantum doubles are important examples of quasitriangular (quasi) Hopf algebras, and are well-studied in mathematics, see also the introduction in [11]. Readers who are not familiar with Hopf algebras we refer to [33] and [34]. A much more compact introduction to quantum groups is given in [35].

**Definition A.1** Let $A$ be a Hopf algebra and $A^0$ the dual Hopf algebra to $A$ with the opposite comultiplication. Then $D(A)$ is the unique quasitriangular Hopf algebra with universal $R$-matrix $R \in D(A) \otimes D(A)$ such that

1. As a vector space, $D(A) = A \otimes A^0$.
2. $A = A \otimes 1$ and $A^0 = 1 \otimes A^0$ are Hopf subalgebras of $D(A)$.
3. The mapping $x \otimes \xi \mapsto x\xi : A \otimes A^0 \rightarrow D(A)$ is an isomorphism of vector spaces. Here $x\xi$ is short notation for the product $(x \otimes 1)(1 \otimes \xi)$.
4. Let $(e_i)_{i \in I}$ be a basis of $A$ and $(e^i)_{i \in I}$ the dual basis of $A^0$. Then

\[ R = \sum_{i \in I} (e_i \otimes 1) \otimes (1 \otimes e^i), \] (A.1)

independent of the choice of the basis.

For a finite group $G$ we take $A := C(G)$, the space of all complex valued functions on $G$, which becomes a Hopf algebra under pointwise multiplication:

\[ (f_1 \cdot f_2)(x) = f_1(x)f_2(x), \quad x \in G \] (A.2)

with comultiplication

\[ (\Delta f)(x, y) := f(xy), \quad x, y \in G \] (A.3)

and with antipode

\[ (Sf)(x) := f(x^{-1}), \quad x \in G. \] (A.4)

The dual of $C(G)$ is the group algebra $\mathbb{C}[G]$ with as multiplication the ordinary group multiplication, and with comultiplication

\[ \Delta(x) = x \otimes x, \quad x \in G. \] (A.5)

The antipode is the inverse in the group. The pairing between $C(G)$ and $\mathbb{C}[G]$ is given by

\[ \langle f, x \rangle = f(x), \quad f \in C(G), x \in G. \] (A.6)
For the quantum double of $C(G)$ we now write
\[ D(G) := D(C(G)) = C(G) \otimes \mathbb{C}[G] \simeq C(G, \mathbb{C}[G]). \]  
(A.7)
Thus $f \otimes g \in C(G) \otimes \mathbb{C}[G]$ can be considered as a mapping from $G \to \mathbb{C}[G]$ via
\[ (f \otimes g) : z \mapsto f(z)g. \]  
(A.8)

Also $D(G) \otimes D(G) \simeq C(G \times G, \mathbb{C}[G] \otimes \mathbb{C}[G])$. The full quasi triangular Hopf algebra structure of $D(G)$ is given by

- Multiplication
  \[ (f_1 \otimes g_1)(f_2 \otimes g_2) = f_1(.)f_2(g_1^{-1}.g_1) \otimes g_1g_2 : z \mapsto f_1(z)f_2(g_1^{-1}zg_1)g_1g_2. \]  
(A.9)

Especially
\[ (1 \otimes g)(f \otimes e) = f(g^{-1}.g) \otimes g, \quad (e \text{ unit in } G). \]  
(A.10)

- Comultiplication
  \[ \Delta(f \otimes g)(y, z) = f(yz)g \otimes g, \quad (g, y, z \in G, f \in C(G)) \]  
(A.11)

- Unit
  \[ 1 \otimes e : z \mapsto e. \]  
(A.12)

- Co-unit
  \[ \epsilon(f \otimes x) = f(e). \]  
(A.13)

- Antipode
  \[ S(f \otimes g) = f(g(.).g^{-1}) \otimes g^{-1} : z \mapsto f(gz^{-1}g^{-1})g^{-1}. \]  
(A.14)

- $R$-element
  \[ R = \sum_{g \in G} (\delta_g \otimes e) \otimes (1 \otimes g) : (x, y) \mapsto e \otimes x, \quad (x, y) \in G \times G \]  
(A.15)

where $\delta_g$ is the Kronecker delta on $g \in G$, and thus a (basis) element of $C(G)$.

As explained in [10] this construction cannot be uniquely generalised to the case of continuous $G$, even not for compact $G$. To avoid the difficulties the construction can be reformulated with aid of the following linear bijection:
\[ D(G) = C(G) \otimes \mathbb{C}[G] \leftrightarrow C(G \times G), \]  
(A.16)
for which we can write
\[ f \otimes g \leftrightarrow ((x, y) \mapsto f(x)\delta_g(y)) \]
\[ \sum_{z \in G} F(., z) \otimes z \leftrightarrow F \]  
(A.17)
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