SPECTRAL MULTIPlicITIES FOR INFINITE MEASURE
PRESERVING TRANSFORMATIONS

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ABSTRACT. Each subset $E \subset \mathbb{N}$ is realized as the set of essential values of the multiplicity function for the Koopman operator of an ergodic conservative infinite measure preserving transformation.

0. INTRODUCTION

Let $T$ be an ergodic conservative invertible measure preserving transformation of a $\sigma$-finite standard measure space $(X, \mathcal{B}, \mu)$. Consider an associated unitary (Koopman) operator $U_T$ in the Hilbert space $L^2(X, \mu)$:

$$U_T f := f \circ T.$$  

In the case of finite measure $\mu$, the operator $U_T$ is usually considered only in the orthocomplement to the subspace of constant functions. A general question of the spectral theory of dynamical systems is

(0-1) to find out which unitary operators can be realized as Koopman operators.

The crux of the problem is related to the multiplicative property $U_T(f_1 f_2) = U_T(f_1) U_T(f_2)$ which is specific for Koopman operators. For instance, in the case of finite $\mu$ this property implies that the discrete spectrum of any Koopman operator is a group. There are several important particular cases of (0-1): Banach problem on simple Lebesgue spectrum, Kolmogorov problem on group property of spectrum [KV], [St], Rokhlin problem on homogeneous spectrum and, more generally, spectral multiplicity problem. We state the last one as follows. Denote by $\mathcal{M}(T)$ the set of essential values for the spectral multiplicity function of $U_T$. Then

(0-2) which subsets of $\mathbb{N}$ are realizable as $\mathcal{M}(T)$ for an ergodic $T$?

Despite a significant progress achieved in works of many authors [Os], [Ro1], [Ro2], [G–L], [KwL], [Ka], [Ag1], [Ry1], [Ag2], [Da2], [Ry3], [Ag3], [KaL], [Da4], [Ry4], etc., this long-standing basic question of the spectral theory of finite measure preserving dynamical systems remains open.

In the present paper we consider (0-2) in the class of infinite measure preserving conservative ergodic (i.e. type $II_{\infty}$) transformations. It turns out that (0-2) can be solved completely in this class.
Main Theorem. Given any subset $E \subset \mathbb{N}$, there is an ergodic multiply recurrent infinite measure preserving transformation $T$ such that $\mathcal{M}(T) = E$.

To show this theorem we adapt the techniques developed in a recent paper [Da4] (which, in turn, absorbed the techniques from nearly all aforementioned papers) for probability preserving systems to the infinite setup.

We note that in the infinite measure preserving case we encounter some specific phenomena which are absent in the probability preserving case. For instance, ergodicity is not a spectral property. Hence availability of some weak limits of powers of the Koopman operator in Lemma 4.4 below does not guarantee that the underlying transformation is ergodic. Secondly, there is no a good definition of weak mixing. Indeed, for each $p > 0$, there is a transformation $T$ such that the $p$-fold Cartesian power $T^{\times p}$ is ergodic but $T^{\times (p+1)}$ is not [KPa]. Next, we recall that a transformation $T$ on $(X, \mu)$ is called multiply recurrent if for each subset $A \subset X$ of positive measure and each $p > 0$, there exists $k > 0$ such that $\mu(A \cap T^k A \cap \cdots \cap T^{kp} A) > 0$. This concept is a natural strengthening of conservativeness (that corresponds to $p = 1$). Furstenberg showed that if $\mu(X) < \infty$ then each $\mu$-preserving transformation is multiply recurrent [Fu]. However if $\mu(X) = \infty$ then there are ergodic transformations (even with all Cartesian powers ergodic) which are not multiply recurrent [AFS]. For more information on these and many other infinite counterexamples we refer to [Aa], [Da3], [DaS2] and references therein.

1. Preliminaries and notation

Let $G$ be a countable Abelian group, $H$ a subgroup of $G$ and $v : G \to G$ a group automorphism. We set

$$L(G, H, v) := \{\#(v^i(h) \mid i \in \mathbb{Z}) \cap H) \mid h \in H \setminus \{0\}\},$$

$$\mathcal{G} := \{a \in \hat{G} \mid \exists p > 0 \text{ with } \hat{v}^p(a) = a\},$$

$$l_g(a) := \frac{1}{p} \sum_{i=0}^{p-1} a(v^i(g)) \quad \text{for all } a \in \mathcal{G} \text{ with } \hat{v}^p(a) = a.$$

We now state without proof an algebraic lemma from [Da4]. It will play a crucial role in the proof of Main Theorem.

Lemma 1.1. Given any subset $E \subset \mathbb{N}$, there exist a countable Abelian group $G$, a subgroup $H \subset G$ and an automorphism $v : G \to G$ such that $E = L(G, H, v)$. Moreover, the following properties are satisfied:

(i) the subgroup $\mathcal{G}$ is countable and dense in $\hat{G}$,
(ii) $\#\{l_h(a) \mid a \in \mathcal{G}\} = \infty$ for each $h \in H \setminus \{0\}$,
(iii) if $g_1, g_2 \in \mathcal{G}$ and $v^i(g_1) \neq g_2$ for all $i \in \mathbb{Z}$ then there is $a \in \mathcal{G}$ such that $l_{g_1}(a) \neq l_{g_2}(a)$.

The following lemma was proved in [KaL] under slightly stricter conditions. We give here an alternative short proof of it.

Lemma 1.2. Let $V$ and $W$ be unitary operators with simple spectrum in Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ respectively. Assume moreover that for each $i = 1, \ldots, k$, there are two sequences $n^{(i)}_t \to \infty$, $m^{(i)}_t \to \infty$ and two complex numbers $\tilde{\kappa}_i \neq \kappa_i$ such that

(i) $V^{n^{(i)}_t} \to 0.5(\kappa_i I + V^*)$, $W^{m^{(i)}_t} \to 0.5(\kappa_i I + W^*)$ weakly,
Proof. Without loss of generality we can think that a cocycle and the conditions (i)–(iii) of Lemma 1.1 are satisfied. The subset of points in the $H$ orbit equivalence relation. A Borel map $V f$ and for all $f \in H$, $g \in \tilde{H}$, $z \in \mathbb{T}$. Then

$$\mathcal{H}^{\otimes k} \otimes \tilde{\mathcal{H}} = L^2_{\text{sym}}(\mathbb{T}^k, \sigma_V) \otimes L^2(\mathbb{T}, \sigma_W)$$

and

$$(V^{\otimes k} \otimes W)f(z_1, \ldots, z_k, z) = z_1 \cdots z_k z f(z_1, \ldots, z_k, z)$$

for all $f \in \mathcal{H}^{\otimes k} \otimes \tilde{\mathcal{H}}$. Let $Z$ stand for the $(V^{\otimes n} \otimes W)$-cyclic space generated by the constant function $1 \in \mathcal{H}^{\otimes k} \otimes \tilde{\mathcal{H}}$.

Denote by $\mathcal{A}$ the von Neumann algebra generated by $V^{\otimes n} \otimes W$. We consider elements of $\mathcal{A}$ as bounded functions on $\mathbb{T}^{k+1}$ which are invariant under any permutation of the $k$ first coordinates. From (i) and (ii) we deduce that the following two functions

$$(z_1, \ldots, z_k, z) \mapsto (\kappa_i + z) \cdot \prod_{l=1}^{k}(\kappa_i + z_l)$$

and

$$(z_1, \ldots, z_k, z) \mapsto (\tilde{\kappa}_i + z) \cdot \prod_{l=1}^{k}(\kappa_i + z_l)$$

are in $\mathcal{A}$ for each $i = 1, \ldots, k$. Since $\kappa_i \neq \tilde{\kappa}_i$, it follows that the function

$$(z_1, \ldots, z_k, z) \mapsto \prod_{l=1}^{k}(\kappa_i + z_l) = \sum_{l=0}^{k} \kappa_i^l P_l(z_1, \ldots, z_n)$$

is in $\mathcal{A}$ for each $i = 1, \ldots, k$. Hence $P_0, \ldots, P_k$ are all in $\mathcal{A}$ (by the property of Vandermonde determinant). It is easy to see that the polynomials $P_0, \ldots, P_k$ generate the entire algebra $P_{\text{sym}}(k)$ of symmetric polynomials in $k$ variables. Since $Z$ is invariant under $\mathcal{A}$, we then obtain that

$$Z \supset L^2_{\text{sym}}(\mathbb{T}^k, \sigma_V) \otimes 1.$$  

This yields $Z = L^2_{\text{sym}}(\mathbb{T}^k, \sigma_V) \otimes L^2(\mathbb{T}, \sigma_W)$. \hfill \Box

By Lemma 1.1, there exist a compact Polish Abelian group $K$, a closed subgroup $H$ of $K$ and a continuous automorphism $v$ of $K$ such that

$$E = L(\tilde{K}, \mathbb{R}/H, \tilde{v})$$

and the conditions (i)–(iii) of Lemma 1.1 are satisfied. The subset of $v$-periodic points in $K$ will be denoted by $\mathcal{K}$.

Let $T$ be an ergodic transformation of $(X, \mu)$. Denote by $\mathcal{R} \subset X \times X$ the $T$-orbit equivalence relation. A Borel map $\alpha$ from $\mathcal{R}$ to a compact group $K$ is called a cocycle of $\mathcal{R}$ if

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \text{ for all } (x, y), (y, z) \in \mathcal{R}.$$
Two cocycles \( \alpha, \beta : \mathcal{R} \to K \) are cohomologous if

\[
\alpha(x, y) = \phi(x)\beta(x, y)\phi(y)^{-1} \quad \text{at a.a. } (x, y) \in \mathcal{R}
\]

for a Borel map \( \phi : X \to K \). If a transformation \( S \) commutes with \( T \) (i.e. \( S \in C(T) \)) then a cocycle \( \alpha \circ S : \mathcal{R} \to K \) is well defined by \( \alpha \circ S(x, y) := \alpha(Sx, Sy) \).

We denote by \( T_{\alpha,H} \) the following transformation of the space \((X \times K/H, \lambda_{K/H})\):

\[
T_{\alpha,H}(x, k + H) := (Tx, \alpha(Tx, x) + k + H).
\]

It is called a skew product extension of \( T \). For brevity, \( T_{\alpha,(0)} \) will be denoted by \( T_\alpha \). As in the case of finite measure preserving transformations we have a decomposition of \( U_{T_{\alpha,H}} \) into orthogonal sum \( U_{T_{\alpha,H}} = \bigoplus_{\chi \in \widehat{K/H}} U_{T,\chi} \), where \( U_{T,\chi} \) is the following unitary in \( L^2(X, \mu) \):

\[
U_{T,\chi}f(x) = \chi(\alpha(Tx, x))f(Tx), \quad x \in X.
\]

It is straightforward to verify that if \( \alpha \circ S \) is cohomologous to \( v \circ \alpha \) for some \( S \in C(T) \) then \( U_{T,\chi} \) and \( U_{T,v^i(\chi)} \) are unitarily equivalent for each \( i \in \mathbb{Z} \).

2. Rokhlin problem on multiplicities for infinite measure preserving maps

Rokhlin problem on multiplicities can be stated as follows

— given \( n > 1 \), is there an ergodic transformation with homogeneous spectrum of multiplicity \( n \)?

This particular case of (0-1) plays an important role in the proof of Main Theorem. We note that in the finite measure preserving case Rokhlin problem was solved in [Ag1] and [Ry1] for \( n = 2 \) and in [Ag2] and explicitly in [Da2] for every \( n \). To solve Rokhlin problem in the infinite measure preserving case it is enough to consider natural factors of Cartesian powers (cf. [Ka], [Ag3]).

Lemma 2.1. Let \( \Gamma \) be a subgroup in \( \mathfrak{S}_k \). Let \( V \) be a unitary with a simple continuous spectrum such that \( V^{\otimes k} \) has a simple spectrum. Denote by \( V^{\otimes k}_\Gamma \) the restriction of \( V^{\otimes k} \) to the subspace of \( \Gamma \)-invariant tensors. Then \( V^{\otimes k}_\Gamma \) has a homogeneous spectrum of multiplicity \( k!/\#\Gamma \).

Proof. Let \( \sigma \) be a measure of maximal spectral type of \( V \). We note that the homomorphism \( \pi : \mathbb{T}^k \ni (z_1, \ldots, z_k) \mapsto z_1 \cdots z_k \in \mathbb{T} \) passes through the natural projections

\[
\mathbb{T}^k \to \mathbb{T}^k/\Gamma \to \mathbb{T}^k/\mathfrak{S}_k \to \mathbb{T}
\]

that correspond to the partitions into \( \Gamma \)- and \( \mathfrak{S}_k \)-orbits. Since \( \sigma \) is continuous, \( \mathfrak{S}_k \) acts freely in almost every fiber \( \pi^{-1}(z) \) furnished with the conditional measure \( \sigma^k|\pi^{-1}(z) \). If \( \mathbb{T} \ni z \mapsto l(z) \) is the spectral multiplicity function of \( V^{\otimes k} \) and \( \mathbb{T} \ni z \mapsto m(z) \) is the spectral multiplicity function of \( V^{\otimes k}_\Gamma \) then \( m(z) = (\#\mathfrak{S}_k/\#\Gamma) \cdot l(z) \). \( \Box \)
Corollary 2.2. Let $T$ be an ergodic conservative infinite measure preserving conservative map such that $U_T^{(k-1)}$ has a simple spectrum. Then $\mathcal{M}(T^{\otimes (k-1)} \times T) = \{k\}$.

Proof. Since $T$ is ergodic of type $II_\infty$, it follows that $U_T$ has a continuous spectrum. It remains to note that $U_{T^{\otimes (k-1)}} \times T = U_T^{\otimes (k-1)} \otimes U_T = (U_T)^{\otimes k}_{\otimes k-1}$ and apply Lemma 2.1. \qed

We note however that Corollary 2.2 is only a “quasi-solution” of the Rokhlin problem since it is unclear whether the transformation $T^{\otimes (k-1)} \times T$ is ergodic or not. A “complete” solution will appear in the proof of Main Theorem.

3. Construction

To prove Main Theorem we will use the $(C,F)$-construction (see [dJ], [Da1]–[Da3]). We now briefly outline its formalism. Let two sequences $(C_n)_{n \geq 0}$ and $(F_n)_{n \geq 0}$ of finite subsets in $\mathbb{Z}$ are given such that:

- $F_n = \{0, 1, \ldots, h_n - 1\}$, $h_0 = 1$, $\#C_n > 1$, $0 \in C_n$,
- $F_n + C_{n+1} \subset F_{n+1}$,
- $(F_n + c) \cap (F_n + c') = \emptyset$ if $c \neq c'$, $c, c' \in C_{n+1}$,
- $\lim_{n \to \infty} \frac{\#h_n}{\#C_1 \cdots \#C_n} = \infty$.

Let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$. Endow this set with the (compact Polish) product topology. The following map

$$(f_n, c_{n+1}, c_{n+2}) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots)$$

is a topological embedding of $X_n$ into $X_{n+1}$. We now set $X := \bigcup_{n \geq 0} X_n$ and endow it with the (locally compact Polish) inductive limit topology. Given $A \subset F_n$, we denote by $[A]_n$ the following cylinder: $\{x = (f, c_{n+1}, \ldots) \in X_n \mid f \in A\}$. Then $\{[A]_n \mid A \subset F_n, n > 0\}$ is the family of all compact open subsets in $X$. It forms a base of the topology on $X$.

Let $\mathcal{R}$ stand for the tail equivalence relation on $X$: two points $x, x' \in X$ are $\mathcal{R}$-equivalent if there is $n > 0$ such that $x = (f_n, c_{n+1}, \ldots)$, $x' = (f'_n, c'_{n+1}, \ldots) \in X_n$ and $c_m = c'_m$ for all $m > n$. There is only one non-atomic Borel infinite $\sigma$-finite measure $\mu$ on $X$ which is invariant (and ergodic) under $\mathcal{R}$ and such that $\mu(X_0) = 1$ for all $n$.

Now we define a transformation $T$ of $(X, \mu)$ by setting

$$T(f_n, c_{n+1}, \ldots) := (1 + f_n, c_{n+1}, \ldots) \text{ whenever } f_n < h_n - 1, \ n > 0.$$ 

This formula defines $T$ partly on $X_n$. When $n \to \infty$, $T$ extends to the entire $X$ (minus countably many points) as a $\mu$-preserving transformation. Moreover, the $T$-orbit equivalence relation coincides with $\mathcal{R}$ (on the subset where $T$ is defined). We call $T$ the $(C,F)$-transformation associated with $(C_{n+1}, F_n)_{n \geq 0}$.

We recall a concept of a $(C,F)$-cocycle (see [Da2]). From now on, the group $K$ is assumed Abelian. Given a sequence of maps $\alpha_n : C_n \to K$, $n = 1, 2, \ldots$, we first define a Borel cocycle $\alpha : \mathcal{R} \cap (X_0 \times X_0) \to K$ by setting

$$\alpha(x, x') := \sum_{n > 0} (\alpha_n(c_n) - \alpha_n(c'_n)),$$ 

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whenever \( x = (0, c_1, c_2, \ldots) \in X_0, x' = (0, c'_1, c'_2, \ldots) \in X_0 \text{ and } (x, x') \in \mathcal{R}. \) To extend \( \alpha \) to the entire \( \mathcal{R} \), we first define a map \( \pi : X \to X_0 \) as follows. Given \( x \in X \), let \( n \) be the least positive integer such that \( x \in X_n \). Then \( x = (f_n, c_{n+1}, \ldots) \in X_n. \) We set
\[
\pi(x) := (0, \ldots, 0, c_{n+1}, c_{n+2}, \ldots) \in X_0.
\]

Of course, \((x, \pi(x)) \in \mathcal{R}\). Now for each pair \((x, y) \in \mathcal{R}\), we let
\[
\alpha(x, y) := \alpha(\pi(x), \pi(y)).
\]

It is easy to verify that \( \alpha \) is a well defined cocycle of \( \mathcal{R} \) with values in \( K \). We call it the \((C, F)\)-cocycle associated with \((\alpha_n)_{n=1}^\infty\).

Let \( \bar{z} = (z_n)_{n=1}^\infty \) be a sequence of integers. The following statement is an infinite analogue of [Da2, Lemma 4.11].

**Lemma 3.1.** Suppose that \( \sum_{n=1}^\infty \frac{\#(C_n \triangle (C_n - z_n))}{\#C_n} < \infty \). We set
\[
X^\bar{z}_n := \{0, 1, \ldots, h_n - z_1 - \cdots - z_n\} \times \prod_{m>n} (C_m \cap (C_m - z_m)) \subset X_n.
\]

Then a transformation \( S^\bar{z}_n \) of \((X, \mu)\) is well defined by setting
\[
S^\bar{z}_n(x) := (z_1 + \cdots + z_n + f_n, z_{n+1} + c_{n+1}, z_{n+2} + c_{n+2}, \ldots)
\]
for all \( x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X^\bar{z}_n, n = 1, 2, \ldots. \) Moreover, \( S^\bar{z}_n \) commutes with \( T \) and \( U_{T^1 + \cdots + z_n} \to U_{S^\bar{z}_n} \) weakly as \( n \to \infty. \)

Now let \( C'_m := \{c \in C_m \cap (C_m - z_m) \mid \alpha_m(c + z_m) = v(\alpha_m(c))\}. \) If
\[
\sum_{n>0} (1 - \#C'_n/\#C_n) < \infty
\]
then the cocycle \( \alpha \circ S^\bar{z}_n \) is cohomologous to \( v \circ \alpha. \)

**Proof.** We need to verify that the subset \( D := \bigcup_{n>1} X^\bar{z}_n \) is of full measure in \( X. \) Fix \( n > 0. \) It follows from Borel-Cantelli lemma that
\[
X_n \setminus D \subset \bigcup_{m>n} \{\{h_m - z_1 - \cdots - z_m + 1, \ldots, h_m - 1\}\}_m.
\]

Therefore
\[
\mu(X_n \setminus D) \leq \sum_{m>n} \frac{z_1 + \cdots + z_m}{h_m} \mu(X_m) = \sum_{m>n} \frac{z_1 + \cdots + z_m}{\#C_1 \cdots \#C_m} < 2 \sum_{m>n} \frac{z_m}{\#C_m}.
\]

Since \( \sum_{m>1} z_m/\#C_m < \infty \), it follows that \( \mu(X_n \setminus D) \to 0 \) as \( n \to \infty. \)

The second claim can be shown by an obvious modification of the proof of Lemma 4.11 from [Da2]. \( \square \)

We will construct some special \((C, F)\)-transformation and its cocycle with values in \( K \). Fix a partition
\[
\mathbb{N} = \bigsqcup_{a \in \mathcal{K}} N_a \sqcup \bigsqcup_{b \in \mathcal{K}} M_b
\]
of \( \mathbb{N} \) into infinite subsets.

Now we define a sequence \((C_n, h_n, z_n, \alpha_n)_{n=1}^{\infty}\) via an inductive procedure. Suppose we have already constructed this sequence up to index \(n\). Consider now two cases.

[\text{[I]}] Let \( n+1 \in \mathcal{N}_a \) for some \( a \in \mathcal{K} \). Now we set
\[
\begin{align*}
z_{n+1} & := 2ma nh_n, \quad r_n := n^3ma, \\
C_{n+1} & := 2h_n \cdot \{0, 1, \ldots, r_n - 1\}, \\
h_{n+1} & := 2r_nh_n.
\end{align*}
\]
Let \( \alpha_{n+1} : C_{n+1} \to K \) be any map satisfying the following conditions
\begin{align*}
(\text{A1}) \quad & \alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c) \quad \text{for all } c \in C_{n+1} \cap (C_{n+1} - z_{n+1}), \\
(\text{A2}) \quad & \text{for each } 0 \leq i < m \text{ there is a subset } C_{n+1,i} \subset C_{n+1} \text{ such that}
\end{align*}
\[
\begin{align*}
C_{n+1,i} - 2h_n & \subset C_{n+1}, \\
\alpha_{n+1}(c) = \alpha_{n+1}(c - 2h_n) + v^i(a) & \text{for all } c \in C_{n+1,i} \text{ and}
\end{align*}
\[
\left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - 1 \right| < \frac{2}{m}.
\]

[\text{[II]}] Let \( n+1 \in \mathcal{M}_b \) for some \( b \in \mathcal{K} \). We set
\[
\begin{align*}
z_{n+1} & := mb n(4h_n + 1), \quad r_n := 2n^3mb, \\
D_{n+1} & := 2h_n \cdot \{0, 1, \ldots, nbm - 1\} \cup (2h_n + 1) \cdot \{1, 2, \ldots, nbm\} + 2h_n(nbm - 1), \\
C_{n+1} & := D_{n+1} + z_{n+1} \cdot \{0, 1, \ldots, n^2 - 1\}, \\
h_{n+1} & := 2r_nh_n + r_n/2.
\end{align*}
\]
Let \( \alpha_{n+1} : C_{n+1} \to K \) be any map satisfying the following conditions
\begin{align*}
(\text{A3}) \quad & \alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c) \quad \text{for all } c \in C_{n+1} \cap (C_{n+1} - z_{n+1}), \\
(\text{A4}) \quad & \text{for each } 0 \leq i < m \text{ there is a subset } C_{n+1,i} \subset C_{n+1} \text{ such that}
\end{align*}
\[
\begin{align*}
C_{n+1,i} - 2h_n & \subset C_{n+1}, \\
\alpha_{n+1}(c) = \alpha_{n+1}(c - 2h_n) + v^i(a) & \text{for all } c \in C_{n+1,i} \text{ and}
\end{align*}
\[
\left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - \frac{1}{2m} \right| < \frac{2}{nm}.
\]
\begin{enumerate}
\item [(A5)] there is a subset \( C_{n+1,\Delta} \subset C_{n+1} \) such that
\[
\begin{align*}
C_{n+1,\Delta} - 2h_n - 1 & \subset C_{n+1}, \\
\alpha_{n+1}(c) = \alpha_{n+1}(c - 2h_n - 1) & \text{for all } c \in C_{n+1,\Delta} \text{ and}
\end{align*}
\[
\left| \frac{\#C_{n+1,\Delta}}{\#C_{n+1}} - \frac{1}{2} \right| < \frac{2}{n}.
\]
\end{enumerate}
Thus, \( C_{n+1}, h_{n+1}, z_{n+1}, \alpha_{n+1} \) are completely defined.

We now let \( F_n := \{0, 1, \ldots, h_n - 1\} \). Denote by \((X, \mu, T)\) the \((C, F)\)-transformation associated with the sequence \((C_{n+1}, F_n)_{n \geq 0}\). Since \( \mu(X_{n+1}) > 2\mu(X_n) \), it follows that \( \mu(X) = \infty \).

Let \( \mathcal{R} \) stand for the tail equivalence relation (or, equivalently, \( T \)-orbit equivalence relation) on \( X \). Denote by \( \alpha : \mathcal{R} \to K \) the cocycle of \( \mathcal{R} \) associated with the sequence \((\alpha_n)_{n>0}\).
4. Proof of the main result

Fix any number \( m > 0 \) such that \( m + 1 \in E \). Our purpose is to prove that \( \mathcal{M}(T_{\alpha,H}) = E \) if \( 1 \in E \) and \( \mathcal{M}(T^{\otimes m} \times T_{\alpha,H}) = E \) if \( 1 \notin E \).

The following lemma is a particular case of [Da1, Lemma 2.4] or [DaS1, Lemma 5.2]. We state it here without proof.

**Lemma 4.1.** Fix \( p > 0 \) and a map \( \delta : \mathbb{Z}^p \to \mathbb{R}_+ \) such that \( \sum_{g \in \mathbb{Z}^p} \delta(g) < \frac{1}{2} \). If for each \( n > 0 \) and \( f_1, \ldots, f_p, f'_1, \ldots, f'_p \in F_n \), there are a subset \( A \subset [f_1]_n \times \cdots \times [f_p]_n \) and \( s \in \mathbb{Z} \) such that

\[
\mu(A) > \delta(f_1 - f'_1, \ldots, f_p - f'_p) \mu([f_1]_n \times \cdots \times [f_p]_n)
\]

and \( T^{x_p}A \subset [f'_1]_n \times \cdots \times [f'_p]_n \) then \( T^{x_p} \) is ergodic.

As we noted in Section 0, ergodicity is not a spectral property for infinite measure preserving maps. Hence instead of deducing it from Lemma 4.4 below (as in the finite measure preserving case) we have to establish ergodicity of \( T^{\otimes m} \times T_{\alpha,H} \) in a separate claim.

**Proposition 4.2.** The transformation \( T^{x_p} \times T_\alpha \) is ergodic for each \( p > 0 \).

**Proof.** We first show that \( T^{x_p} \) is ergodic. For simplicity, we will consider only the case when \( p = 2 \). (The general case is considered in a similar way.) We note that for each \( n \in \mathcal{M}_b - 1 \), there are subsets \( C'_{n+1} \) and \( C''_{n+1} \) in \( C_{n+1} \) such that

(a) if \( c \in C'_{n+1} \) then \( 2h_n + c \in C_{n+1} \),
(b) if \( c \in C''_{n+1} \) then \( 2h_n + 1 + c \in C_{n+1} \),
(c) \( \#C'_{n+1} \geq 1/3 \cdot \#C_{n+1} \) and \( \#C''_{n+1} \geq 1/3 \cdot \#C_{n+1} \).

Suppose we are given an arbitrary \( n > 0 \) and \( f, f', d, d' \in F_n \). Assume for definiteness that \( f \geq f' \) and \( d \geq d' \). Let \( s := \max(f - f', d - d') \). We now find \( k > n \) such that the intersection \( \{n+1, \ldots, k\} \cap \bigcup_{b \in \mathcal{K}} (\mathcal{M}_b - 1) \) consists of \( s \) points, say \( l_1, \ldots, l_s \). We now set \( A := f + \sum_{i=1}^{k} A_i \) and \( B := d + \sum_{i=1}^{k} B_i \), where

\[
A_i := \begin{cases} C_i & \text{if } i \notin \{l_1, \ldots, l_s\} \\ C'_{i} & \text{if } i \in \{l_{f-f'+1}, \ldots, l_s\} \\ C''_{i} & \text{if } i \in \{l_{f-f'+1}, \ldots, l_s\} \end{cases} \quad \text{and} \quad B_i := \begin{cases} C_i & \text{if } i \notin \{l_1, \ldots, l_s\} \\ C'_{i} & \text{if } i \in \{l_{d-d'+1}, \ldots, l_s\} \\ C''_{i} & \text{if } i \in \{l_{d-d'+1}, \ldots, l_s\} \end{cases}
\]

It is easy to deduce from (a)–(c) that

(o) \( \mu([A]_k) \subset [f]_n, [B]_k \subset [d]_n \)

(o) \( \mu([A]_k) \geq \frac{1}{3} \mu([f]_n), \mu([B]_k) \geq \frac{1}{3} \mu([d]_n) \)

(o) \( T^{2(h_{l_1}+\cdots+h_{l_s})}[A]_k \subset [f]_n, T^{2(h_{l_1}+\cdots+h_{l_s}-1)}[B]_k \subset [d]_n \).

It remains to apply Lemma 4.1.

Now we verify that \( T^{x_p} \times T_\alpha \) is ergodic. It is convenient to consider this transformation as a skew product \( (T^{(p+1)})_{\otimes a} \). Given \( n > 0, f_1, \ldots, f_{p+1} \in F_n \) and \( a \in \mathcal{K} \), we can find \( k > n \) such that \( k + 1 \in \mathcal{N}_a \). We set

\[
A_i := f_i + C_{n+1} + \cdots + C_k + C_{k+1,0}
\]

and \( A := [A]_{k+1} \times \cdots \times [A_{p}]_{k+1} \subset [f_1]_n \times \cdots \times [f_p]_n \). It follows from (A2) that

\[
(T^{x_p})^{-2h_k}[A]_{k+1} \times \cdots \times [A_{p}]_{k+1} \subset [f_1]_n \times \cdots \times [f_p]_n,
\]

\[
\frac{\mu(A)}{\mu([f_1]_n \times \cdots \times [f_p]_n)} > \left( \frac{1}{2m_a} \right)^p
\]

and \( 1 \otimes \alpha(x, (T^{x_p})^{-2h_k}) = \alpha(x_{p+1}, T^{-2h_k} x_{p+1}) = a \).
for all \( x = (x_1, \ldots, x_{p+1}) \in A \). Since \( K \) is dense in \( K \), we deduce from the standard ergodicity criterium for cocycles [Sc] that \( 1 \otimes \alpha \) is ergodic, i.e. \( T^{\times p} \times T_\alpha \) is ergodic. \( \Box \)

**Remark 4.3.** We also note that \( T^{\times p} \) is multiply recurrent for each \( p > 0 \). This follows from [DaS1, Remark 2.4(i)]. Now [In] yields that \( T^{\times p} \times T_\alpha \) is also multiply recurrent.

**Lemma 4.4.** Let \( a, b \in \mathcal{K} \). Then for each \( \chi \in \widehat{K} \),

\[
\begin{align*}
(i) & \quad U_{T,\chi}^{2h_n} l_\chi(a) \cdot I \text{ as } N_\alpha - 1 \ni n \to \infty \text{ and } \\
(ii) & \quad U_{T,\chi}^{2h_n} \to 0.5(l_\chi(b) \cdot I + U_{T,\chi}^*) \text{ as } \mathcal{M}b - 1 \ni n \to \infty.
\end{align*}
\]

**Proof.** We show only (ii) since (i) is proved in a similar way but a bit simpler. Let \( n + 1 \in \mathcal{M}_b \).

Take any subset \( A \subset F_n \). We note that \([A]_n = [A + C_{n+1}]_{n+1}\). Therefore it follows from (A4) that for each \( x \in T[F_n]_n \),

\[
U_{T,\chi}^{2h_n} 1_{[A]_n}(x) = \sum_{i=0}^{m_b-1} \chi(\alpha(T^{2h_n}x, x))1_{[A+C_{n+1}, i]}_{n+1}(T^{2h_n}x) + \omega(1)
\]

\[
= \sum_{i=0}^{m_b-1} \chi(v^i(b))1_{[A+C_{n+1}, i-2h_n]}_{n+1}(x) + \omega(1)
\]

\[
= \sum_{i=0}^{m_b-1} \chi(v^i(b))1_{[A+C_{n+1}, i-2h_n]}_{n+1}(x) + U_{T,\chi}^{*} 1_{[A+C_{n+1}, i-2h_n]}_{n+1}(x) + \omega(1).
\]

Therefore

\[
U_{T,\chi}^{2h_n} - \sum_{i=0}^{m_b-1} \chi(v^i(b))1_{[C_{n+1}, i-2h_n]}_{n+1} - U_{T,\chi}^* 1_{[C_{n+1}, i-2h_n]}_{n+1} \to 0
\]

as \( \mathcal{M}b - 1 \ni n \to \infty \). Here the functions \( 1_{[C_{n+1}, i-2h_n]}_{n+1} \) and \( 1_{[C_{n+1}, i-2h_n]}_{n+1} \) are considered as multiplication operators (orthogonal projectors) in \( L^2(X, \mu) \). It remains to use the inequalities from (A4) and (A5) and a standard fact that for any sequence \( C^* \subset C \) such that \( \#C^* / \#C \to \delta \), we have

\[
1_{[C^*]_n} \to \delta I \text{ weakly as } n \to \infty.
\]

\( \Box \)

**Proof of Main Theorem.** It follows from Proposition 4.2 that \( T^{\otimes m} \times T_{\alpha,H} \) is ergodic.

Assume first that \( 1 \not\in E \). There is a natural decomposition of \( U_{T^{\otimes m} \times T_{\alpha,H}} \) into an orthogonal direct sum

\[
U_{T^{\otimes m} \times T_{\alpha,H}} = \bigoplus_{\chi \in \widehat{K}/H} (U_T^{\otimes m} \otimes U_{T,\chi}).
\]
Therefore it remains to show the following claims:

(i) \( U_T \otimes U \) has a homogeneous spectrum \( m + 1 \),

(ii) \( U_T \otimes U_{T, \chi} \) has a simple spectrum if \( \hat{K}/\hat{H} \ni \chi \neq 0 \),

(iii) \( U_T \otimes U_{T, \chi} \) and \( U_T \otimes U_{T, \xi} \) are unitarily equivalent if \( \chi \) and \( \xi \) belong to the same \( \hat{v} \)-orbit,

(iv) the measures of maximal spectral type of \( U_T \otimes U_{T, \chi} \) and \( U_T \otimes U_{T, \xi} \) are mutually singular if \( \chi \) and \( \xi \) do not belong to the same \( \hat{v} \)-orbit.

Since \( \mathcal{M}(U_T \otimes U) = \mathcal{M}((U_T)_{S^m}^{(m+1)}) \), Lemma 2.1 implies (i).

Since \( T \) is of rank one and the map \( [f] \ni x \mapsto \alpha(T x, x) \in K \) is constant for each \( f \in F_n \setminus \{h_n - 1\}, n \in \mathbb{N} \), it follows that the operator \( U_{T, \chi} \) has a simple spectrum.

If \( \hat{K}/\hat{H} \ni \chi \neq 0 \) then by Lemma 1.1(ii), there are \( a_1, \ldots, a_{m+1} \in K \) such that the numbers \( l_\chi(a_i), i = 1, \ldots, m + 1 \), are pairwise different and they are different from 1. Lemma 4.4(ii) yields

\[
U_{T, \chi}^{2h_n} \to 0.5(l_\chi(a_i) \cdot I + U_{T, \chi}^*) \quad \text{and} \quad U_{T, \xi}^{2h_n} \to 0.5(I + U_{T, \xi}^*)
\]
as \( \mathcal{M}_{a_i} - 1 \ni n \to \infty \). Now (ii) follows from Lemma 1.2.

Since

\[
\sum_{n>0} \frac{\#(C_n \triangle (C_n - z_n))}{\#C_n} = \sum_{n>0} \frac{2}{n^2},
\]
it follows from Lemma 3.1 that a transformation \( S_z \) of \((X, \mu)\) is well defined by the formula (3-1) and \( S_z \in C(T) \). It follows from (A1) and (A3) that (3-2) is satisfied. Hence by Lemma 3.1,

\[\alpha \circ S_z \quad \text{is cohomologous to} \quad v \circ \alpha\]

and (iii) follows.

Let characters \( \chi, \xi \in \hat{K} \) do not belong to the same \( \hat{v} \)-orbit. By Lemma 1.1(iii), there is \( a \in K \) with \( l_\chi(a) \neq l_\xi(a) \). We now deduce from Lemma 4.4(i)

\[
(U_T \otimes U_{T, \chi})^{2h_n} \to l_\chi(a) I \quad \text{and} \quad (U_T \otimes U_{T, \xi})^{2h_n} \to l_\xi(a) I
\]
as \( \mathcal{N}_{a_i} - 1 \ni n \to \infty \). This yields (iv).

If \( 1 \in E \) then we use the orthogonal decomposition

\[U_T = \bigoplus_{\chi \in \hat{K}/\hat{H}} U_{T, \chi}\]

and the following properties:

(i') \( U_{T, \chi} \) has a simple spectrum if \( \chi \neq 0 \),

(ii') \( U_{T, \chi} \) and \( U_{T, \xi} \) are unitarily equivalent if \( \chi \) and \( \xi \) belong to the same \( \hat{v} \)-orbit,

(iii') the measures of maximal spectral type of \( U_{T, \chi} \) and \( U_{T, \xi} \) are mutually singular if \( \chi \) and \( \xi \) do not belong to the same \( \hat{v} \)-orbit.

\[\square\]
5. Concluding remarks

There are several ways in which the main results of this paper can be further strengthened and generalized.

The first one is to consider mixing realizations of spectral multiplicities, i.e. assume, in addition, that $T$ in (0-2) is mixing. We recall that in the finite measure preserving case mixing realizations in (0-2) were under study in [Ro2], [Ry2], [Ry3], [Ag3], [Da5]. An infinite measure preserving transformation $T$ is called mixing or, equivalently, of zero type if $U^n_T \to 0$ weakly as $n \to \infty$ (see [DaS2] and references therein). We note that mixing does not imply ergodicity nor conservativeness in the infinite measure preserving case. We announce the following result.

**Theorem 5.1.**

(i) Given any subset $E \subset \mathbb{N}$, there is a mixing ergodic conservative infinite measure preserving transformation $T$ such that $\mathcal{M}(T) = E$.

(ii) There is a mixing ergodic conservative infinite measure preserving transformation $T$ such that the operator $\exp(U_T)$ has a simple spectrum.

It follows from (ii) that the Poisson suspension of $T$, which is a probability preserving transformation, is mixing of all orders [Roy1] with a simple spectrum [Ne]. For the definition of Poisson suspensions we refer to [CFS] and [Ne]. Non-mixing Poisson suspensions (of some transformations constructed in [Ag1]) with a simple spectrum were considered in [Roy2]. A proof of Theorem 5.1 will appear in a subsequent paper of the authors.

It is also worthy to note that the results of this paper can be generalized to actions of more general groups such as $\mathbb{Z}^d$ or even arbitrary countable discrete Abelian groups. For instance, Section 2 extends to arbitrary Abelian group actions verbally, since the specific structure of $\mathbb{Z}$ is used nowhere in that section. As for the $(C,F)$-construction, it was developed mainly to work with general groups and to bypass limitations of the geometrical intuition which is inherent to the classical cutting-and-stacking for $\mathbb{Z}^d$-actions [dJ], [Da1], [Da3]. It looks plausible that the construction in Section 3 extends to the Abelian group actions, provided that the weak closure of the operators of the associated Koopman representation contains a right list of weak limits.

We also note that that the results of the paper extend to non-singular transformations which do not admit an equivalent invariant $\sigma$-finite measure. Such transformations are called of type III. We conjecture that Main Theorem is true for the subclass of those type III transformations whose associated flow is AT, i.e. approximately transitive. We refer to the survey [DaS2] devoted to type III systems, where definitions of the associated flow and the AT-property can be found.

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