We address the two issues raised by Bayle, Vallisneri, Babak, and Petiteau (in their gr-qc document arxiv.org/abs/2106.03976) about our matrix formulation of Time-Delay Interferometry (TDI) (arxiv.org/abs/2105.02054) [1]. In so doing we explain and quantify our concerns about the results derived by Vallisneri, Bayle, Babak and Petiteau [2] by applying their data processing technique (named TDI-∞) to the two heterodyne measurements made by a two-arm space-based GW interferometer.

First we show that the solutions identified by the TDI-∞ algorithm derived by Vallisneri, Bayle, Babak and Petiteau [2] do depend on the boundary-conditions selected for the two-way Doppler data. We prove this by adopting the (non-physical) boundary conditions used by Vallisneri et al. and deriving the corresponding analytic expression for a laser-noise-canceling combination. We show it to be characterized by a number of Doppler measurement terms that grows with the observation time and works for any time-dependent time delays. We then prove that, for a constant-arm-length interferometer whose two-way light times are equal to twice and three-times the sampling time, the solutions identified by TDI-∞ are linear combinations of the TDI variable $X$.

In the second part of this document we address the concern expressed by Bayle et al. regarding our matrix formulation of TDI when the two-way light-times are constant but not equal to integer multiples of the sampling time. We mathematically prove the homomorphism between the delay operators and their matrix representation [1] holds in general. By sequentially applying two order-$m$ Fractional-Delay (FD) Lagrange filters of delays $l_1, l_2$ we find its result to be equal to applying an order-$m$ FD Lagrange filter of delay $l_1 + l_2$. On physical grounds this reflects the fact that sequentially applying two order-$m$ interpolators can’t result in a higher order $(3m - 2)$ interpolator!

We further show that the homomorphism holds in the general case of time-dependent arm-lengths for (i) the continuum limit and (ii) for fractional delays. In this case the operators do not commute. We further argue that the homomorphism extends to the general case of time-dependent arm-lengths.

## I. INTRODUCTION

Time-Delay Interferometry is the data processing technique for canceling the laser noise from the heterodyne measurements made by future space-based, unequal-arm gravitational wave (GW) interferometers. In its simplest-form implementation it entails properly time-shifting and linearly combining the two two-way Doppler data measured by an unequal-arm-length interferometer so as to cancel the laser phase fluctuations at any time $t$. TDI is a "local" operation in that, for a constant-arm-length interferometer for instance, it requires to properly combine four samples of the two Doppler data selected at times $t, t-l_1$ and $t-l_2$, with ($l_1, l_2$) being the times spent by the light to complete a round-trip within arm # 1 and # 2 respectively. In the case of delay-times changing linearly with time over a time-scale equal to the round-trip-light-time (RTLT) itself the so called “second-generation” TDI combination (TDI2) was then derived to account for the time evolution of the light-times and suppress the laser noise way below the secondary noises. In this case the laser noise suppression is achieved by combining eight samples of the two Doppler data. As pointed out in [3], TDI can be extended to any time-dependent time-delays by applying an iterative procedure in which the laser noise is effectively suppressed (but not exactly canceled) to a level many orders of magnitude lower than that defined by the remaining noise sources affecting the Doppler measurements.

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1 This is a dynamic configuration resulting from carefully selecting the spacecraft trajectories. It has been adopted by LISA and other currently proposed space-based GW interferometers to minimize the magnitude of the Doppler beat-notes between the received and receiving laser beams so that they fall within the photo receiver’s operational band-width.
In a recent publication by Vallisneri et al. [2] a new data processing algorithm for canceling the laser noise from the two two-way Doppler data measured by a two unequal-length arms space-based detector was proposed. In it the two sampled Doppler measurements are simultaneously processed by constructing an array containing the interleaved two Doppler data measurements. By then applying SVD decomposition to the rectangular matrix relating this array to the array associated with the laser noise, they can identify a number $n$ of combinations that are laser-noise-free (with $n$ being the number of data samples from each Doppler data). In our article [1] we pointed out that the "boundary conditions" (i.e. the relationship between the two Doppler measurements and the laser noise during the first RTLTs) adopted in their article were not physically correct. From there we argued that the solutions found by the TDI-$\infty$ algorithm necessarily had to reflect their implemented boundary conditions. Note we did not claim TDI-$\infty$ to be mathematically incorrect; we rather stated that the solutions discussed in their article reflected the boundary conditions assumed and therefore needed to be reanalyzed.

This document is organized as follows. In section II we derive a data combination corresponding to the non-physical boundary conditions. Such a combination is characterized by a number of data samples that grows with time and works for any time-dependent light-times. It is easy, however, to show that such a combination would not cancel the laser noise if applied to the actual measurements. To understand what the TDI-$\infty$ solutions would look like when the correct boundary conditions are implemented, we performed an analytic SVD decomposition (by using the program Mathematica) of the rectangular matrix relating the Doppler to the laser noise arrays. By assuming the two light-times to be constant and equal to integer multiples of the sampling time we found the resulting solutions to be all linear combinations of the TDI measurement $X$.

In section III we then turn to our matrix representation of TDI and mathematically prove that the homomorphism between the space of the delay operators and their corresponding matrices holds in general.

II. THE BOUNDARY CONDITIONS

TDI-$\infty$ establishes a linear relationship between the sampled Doppler measurements and the laser noise arrays. To understand its formulation let us consider again the simplified (and stationary) two-arm optical configuration. In it the laser noise, $C(t)$, folds into the two two-way Doppler data, $y_1(t)$, $y_2(t)$, in the following way (where the contributions from all other physical effects affecting the two-way Doppler data have been disregarded)

\[
y_1(t) = C(t - l_1(t)) - C(t),
\]

\[
y_2(t) = C(t - l_2(t)) - C(t),
\]

(2.1)

where $l_1$, $l_2$ are the two RTLTs, in general also functions of time $t$. Operationally, Eq. (2.1) says that each sample of the two-way Doppler data at time $t$ contains the difference between the laser noise $C$ generated at a RTLT earlier, $t - l_i(t)$, $i = 1, 2$ and that generated at time $t$. The important point to note here is what happens during the first $l_i$ seconds from the instant $t = 0$ when the laser is switched on. Since the $y_i$ measurements are the result of interfering the returned beam with the outgoing one, during the first $l_i$ seconds (i.e. from the moment the laser has been turned on) the $y_i$ measurements are identically equal to zero because no interference measurements can be performed during this time. In [2], however, only the first terms on the right-hand-sides of Eq. (2.1) were disregarded during these time intervals. Although this error has been minimized by Bayle et al. in their document, it is rather relevant to the matter discussed here. And it is even more relevant for correctly simulating the two-way Doppler data when assessing the performance of future GW interferometers. Even if we would consider start processing the Doppler data at any time $t$ after the first RTLT has past, we would still be confronted by the fact that the Doppler measurement $y_i$ at time $t$ contains laser noise generated at time $t$ and at time $t - l_i$. In other words, there exists a time-misalignment between the array of the Doppler measurement and that of the laser noise and physical boundary conditions have to be accounted for.

Although the TDI-$\infty$ technique is mathematically correct, by implementing with it the physical boundary conditions results in solutions that are quite different from those associated with the non-physical boundary conditions. By using the correct boundary conditions (in the stationary configuration with the RTLTs equal to twice and three times the sampling time) we have verified analytically that the $n - 6$ observables mentioned by Bayle et al. are actually equal to linear combinations of the TDI observable $X$ defined at each of the sampled times. By implementing the non-physical boundary conditions, on the other hand, one finds solutions equal to linear combinations of the TDI observable $X$ defined at each of the sampled times, plus some additional term that would not cancel the laser noise in the measured data. This additional term is a function of $y_1$ and $y_2$ defined at times $t < l_1, l_2$ and thus, is a manifestation of the non-physical boundary conditions.

One such a combination can be derived in the following way. Let us consider the two-way Doppler measurement $y_i(t)$ during the first RTLT $l_i$, i.e. $t_0 \leq t < t_0 + l_i$ ($t_0$ being the time when the laser is switched on). Its expression,
as given in [2], is equal to
\[ y_i(t) = -C(t), \quad t_0 \leq t < t_0 + l_i. \] (2.2)
During the second RTLT we then have
\[ y_i(t) = C(t - l_i) - C(t), \quad t_0 + l_i \leq t < t_0 + 2l_i, \] (2.3)
which can then be rewritten in the following form after including the expression for the laser noise given by Eq. (2.2)
\[ y_i(t) + y_i(t - l_i) = -C(t), \quad t_0 \leq t < t_0 + 2l_i. \] (2.4)
It is then easy to derive the following expression for the laser noise at an arbitrary time \( t < t_0 + Nl_i \)
\[ \rho_i \equiv \sum_{k=0}^N y_i(t - kl_i) = -C(t). \] (2.5)
If we now take the difference \( \rho_1 - \rho_2 \) we end up canceling the laser noise at any time \( t \) and achieve sensitivity to GWs.
Note that the above derivation can easily be extended to the case in which the RTLTs are arbitrary functions of time since each Doppler data is delayed by its own RTLT. Also note the number of Doppler measurements entering in the combination \( \rho_1 - \rho_2 \) increases with \( t \). As a final observation, Eq.(2.5) would allow us to measure the noise of a laser at any time \( t \) and therefore have a noiseless laser!

In summary, our findings lead us to the conclusion that the results quoted in [2] depend on the boundary conditions selected by the authors. This is not to say that TDI-∞ is mathematically incorrect. Rather, it makes it difficult to understand the nature of the solutions it can find in the general case of unequal and arbitrarily time-changing arms. It was the difficulty we experienced to understand what was happening “under the hood” of TDI-∞ that resulted in our matrix formulation of TDI. As stated by Vallisneri et al. [2] and by us [1] a matrix formulation of the technique for canceling the laser noise may provide computational advantages to the data analysis tasks of space-based GW missions.

III. THE HOMOMORPHISM: DELAY OPERATORS IN MATRIX AVATAR

The homomorphism concept is fundamental and should hold in every situation of time delays; whether they are integer multiples of the sampling interval, or fractional or time dependent. Here we argue that this is indeed so. We have already shown that this is valid for the case of integer multiples of sampling interval [1]. As a matter of principle, one may argue that if the Doppler data could be sampled at a rate as high as required by TDI (corresponding to a sampling time of about \( 10 \) m/c sec), then the issue raised on constant fractional delays would not exist and the equality \( \phi(D_1 D_2) = \phi(D_1) \phi(D_2) \) would hold. In fact in the continuum limit of the sampling interval \( \Delta t \rightarrow 0 \), the matrix representation of a delay operator \( D_1 \) with delay \( l_1(t) \) tends to a delta function \( \delta[t' - (t - l_1(t))] \equiv D_1(t, t') \). Here the matrix \( D_1(t, t') \) acts on the continuous data stream \( y(t) \) as follows:
\[ D_1 y(t) = \int dt'' D_1(t, t') y(t') = \int dt'' \delta[t'' - (t - l_1(t))] y(t') = y(t - l_1(t)), \] (3.1)
which is consistent with the usual definition. Here the homomorphism \( \phi \) is \( \phi(D_1) = D_1(t, t') \). If one takes two such operators even with time-dependent delays \( l_1(t) \) and \( l_2(t) \), and applies the two operators successively then the result is again a delta function with a delay \( l_1(t) + l_2(t - l_1(t)) \) as shown below:
\[ \phi(D_1) * \phi(D_2) = (D_1 * D_2)(t, t'') \]
\[ = \int dt' D_1(t, t') D_2(t', t'') \]
\[ = \int dt' \delta[t' - (t - l_1(t))] \delta[t'' - (t' - l_2(t'))], \] (3.2)
\[ = \delta[t'' - \{t - l_1(t) - l_2(t - l_1(t))\}] \equiv \phi(D_1 D_2). \] (3.3)
This proves that the matrix representation in the continuum case is also a homomorphism although,
\[ \phi(D_1) * \phi(D_2) = D_1 * D_2 \neq D_2 * D_1 = \phi(D_2) * \phi(D_1). \]
The operators do not commute in general when the arm lengths are time dependent. The operators then form a non-commutative polynomial ring, sometimes also called a free algebra (more about this later). When the delays are constants, the operators $D_1$ and $D_2$ commute and the operators form a commutative polynomial ring. Thus we have shown that the homomorphism holds in the continuum limit in addition to the case of delays being integer multiples of the sampling interval (constant time-delays) - the opposite end, so to speak.

In practice however, one has nonzero sampling intervals $\Delta t > 0$ (we assume uniform sampling) and also to avoid oversampling the data one could apply an appropriate fractional delay filter to the Doppler measurement and achieve digitally the oversampling mentioned above. To compute fractional delays if one uses the Lagrange interpolation say on $m$ points, then one can envisage a $m \times m$ matrix of Lagrange polynomials $D(\alpha)$, where $\alpha$ is the delay, acting on the data $y$ (we write the delays as $\alpha, \beta, ...$ in order to not confuse with the Lagrange polynomials which are also denoted by $l_j$). If one considers another delay $\beta$ we have the matrix $D(\beta)$. If one uses the same sample points also for all delays including the total delay, then one can show that $D(\alpha + \beta) = D(\alpha)D(\beta)$. This follows from the properties of Lagrange polynomials. We give below the proof:

For concreteness sake, consider just $m = 3$ points at $t = 0, 1, 2$ and let $l_j(t)$, $j = 0, 1, 2$ be the Lagrange polynomials. We do not need them explicitly. Let $p(t)$ be the interpolating polynomial which is required to pass through the points $y_0, y_1, y_2$ at $t = 0, 1, 2$ respectively. Then we have,

$$p(t) = l_0(t)y_0 + l_1(t)y_1 + l_2(t)y_2.$$  \hspace*{1cm} (3.4)

We just need to use the property of Lagrange polynomials:

$$l_j(t = k) = \delta_{jk}$$ \hspace*{1cm} (3.5)

From this we have $p(k) = y_k$ and so:

$$p(t) = l_0(t)p(0) + l_1(t)p(1) + l_2(t)p(2).$$ \hspace*{1cm} (3.6)

Consider the first term of the product matrix, $l_0(\alpha + \beta) \equiv p(\alpha)$ where $\beta$ is held constant. Then at each value of $\alpha = 0, 1, 2$ we have $p(k) = l_0(\beta + k)$. Thus we get:

$$l_0(\alpha + \beta) = \sum_{k=0}^{2} l_k(\alpha)l_0(\beta + k).$$ \hspace*{1cm} (3.7)

In general we have:

$$l_n(\alpha + \beta) = \sum_{k} l_k(\alpha)l_n(\beta + k).$$ \hspace*{1cm} (3.8)

This is in fact the addition theorem for Lagrange polynomials for integer valued nodes, say at $k = 0, 1, ..., m$. The matrix $D(\alpha)$ for $m = 2$ is:

$$D(\alpha) = \left| \begin{array}{ccc} l_0(\alpha) & l_1(\alpha) & l_2(\alpha) \\ l_0(\alpha + 1) & l_1(\alpha + 1) & l_2(\alpha + 1) \\ l_0(\alpha + 2) & l_1(\alpha + 2) & l_2(\alpha + 2) \end{array} \right| \equiv D_\alpha,$$

\hspace*{1cm} (3.9)

where $D_\alpha = l_k(\alpha + j)$. Taking two such matrices corresponding to $\alpha$ and $\beta$ and multiplying them together, we have,

$$\sum_k D_\alpha D_\beta = \sum_k l_k(\alpha + j)l_n(\beta + k) \equiv l_n(\alpha + \beta + j) = D_{\alpha n}(\alpha + \beta),$$ \hspace*{1cm} (3.10)

where we have used the addition theorem in Eq. (3.8). Although we just used 3 time stamps the results are true for $m$ points. Also one might think, that since the product of Lagrange polynomials appears as entries in the product of the matrices, it might lead to polynomials of degree $2m - 2$ (we do not understand $3m - 2$). But this does not happen, as the addition theorem shows; the terms of degree greater than $m - 1$ cancel out, leaving behind a $m - 1$ degree polynomial.

In practice, choosing the same set of sample points may not be feasible for delays much greater than the sampling interval and so different sets of sample points must be chosen for different delays but then the matrices may appear different. But then care must be taken to translate the matrices to a common reference in order to compare them. Then the closure property of the polynomials can be explicitly seen to hold. Since here we are concerned about matters of principle, we may choose $m$ sufficiently large to cover all delays.
We would like to emphasize that Eq. (3.10) is valid for time dependent delays also. Both $\alpha$ and $\beta$ become now functions of time. If one applies the delay $\beta$ first and then $\alpha$, the combined delay is $\alpha + \beta(\alpha) \equiv \alpha \oplus \beta$ and in the reverse case it is $\beta + \alpha(\beta) = \beta \oplus \alpha$ which are in general unequal. Then we have the situation:

$$D(\alpha \oplus \beta) = D[\alpha(\beta)] = D(\alpha)D[\beta(\alpha)] \neq D(\beta)D[\alpha(\beta)] = D(\beta \oplus (\alpha(\beta))].$$ (3.11)

Eq. (3.11) shows that the homomorphism also holds for time-dependent fractional delays with non-commuting operators.

It is possible that some other interpolation method may be employed for obtaining the fractional delay filter. In that case, the fractional delay filter may only produce an approximation to the true matrix representation of the operator albeit very accurate. This approximation may result in producing $D(\alpha)D(\beta)$ different from $D(\alpha + \beta)$ making it appear that it is not a homomorphism. But both $D(\alpha)D(\beta)$ and $D(\alpha + \beta)$ are two different approximations to the same exact matrix, which exists in principle and also perhaps realized in practice with a higher sampling rate. Therefore, while applying approximation methods, their limitations must be kept in mind.

IV. TIME DEPENDENT ARM LENGTHS AND NON-COMMUTATIVE POLYNOMIAL RINGS

Although in reference [1] we have mostly dealt with constant arm length delays, in section IV A we have described the general ring structure of the operators with time dependent arm-lengths. We then have a non-commutative ring of operators. When discussing homomorphisms, our presentation in [1] might have been misconstrued that we were only referring to the situation of 3 constant arm-lengths which was first addressed by Dhurandhar, Nayak and Vinet (2002) [4]. As correctly pointed out by Bayle et al. the TDI observables defined in [4] cannot deal with time-dependent arm-lengths, because the operators commute. But we would like to point out that since then more work has been done. Ref.[5] obtains the first module of syzygies for 6 constant arm-lengths with 6 commuting operators, where one has a rigidly rotating LISA triangle and the Sagnac effect makes the up and down links between spacecraft unequal. Here too the ring is commutative. This is the so-called TDI 1.5. The problem of TDI with time-dependent arm-lengths has been addressed in [6] and [7]. In this case we have a non-commutative polynomial ring which again leads to the first module of syzygies. Here the operators do not commute and any element of the ring is a word or a string of operators, and the order of the operators is important. For example, if $x$ and $y$ are the operators, then there are 4 distinct monomials $x^2, xy, yx, y^2$ of second degree, instead of 3 in the commutative case. At third degree there are 8 distinct monomials and so on. In this general case, obtaining the TDI variables is a very difficult problem. It is not even clear whether the Gröbner basis is finite. Nevertheless, partial approximate solutions for the special case of only two arms of LISA functioning and with arm-lengths slowly varying, have been obtained. The solutions are of the Michelson type [7] and are TDI-2. All the above literature has been reviewed comprehensively in [3].

In summary, the matrix representation of the delay operators is correct and it captures all the well known properties of the TDI space and its generators.

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