BETTI NUMBERS OF DETERMINANTAL IDEALS

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Abstract. Let \( R = k[x_1, \cdots, x_n] \) be a polynomial ring and let \( I \subset R \) be a graded ideal. In [10], Römer asked whether under the Cohen-Macaulay assumption the \( i \)-th Betti number \( \beta_i(R/I) \) can be bounded above by a function of the maximal shifts in the minimal graded free \( R \)-resolution of \( R/I \) as well as bounded below by a function of the minimal shifts. The goal of this paper is to establish such bounds for graded Cohen-Macaulay algebras \( k[x_1, \cdots, x_n]/I \) when \( I \) is a standard determinantal ideal of arbitrary codimension. We also discuss other examples as well as when these bounds are sharp.

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1. INTRODUCTION

Let \( R = k[x_1, \cdots, x_n] \) be a polynomial ring in \( n \) variables over a field \( k \), let \( \text{deg}(x_i) = 1 \) and let \( I \subset R \) be a graded ideal of arbitrary codimension. Consider the minimal graded free \( R \)-resolution of \( R/I \):

\[
0 \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0
\]

where we denote \( \beta_{i,j}(R/I) = \dim \text{Tor}_i^R(R/I, k)_j \) the \((i, j)\)-th graded Betti number of \( R/I \) and \( \beta_i(R/I) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(R/I) \) is the \( i \)-th total Betti number. Many important numerical invariants of \( I \) and the associated scheme can be read off from the minimal graded free \( R \)-resolution of \( R/I \). For instance, the Hilbert polynomial, and hence the multiplicity \( e(R/I) \) of \( I \), can be written down in terms of the shifts \( j \) such that \( \beta_{i,j}(R/I) \neq 0 \) for some \( i, 1 \leq i \leq p \).

Let \( c \) denote the codimension of \( R/I \). Then \( c \leq p \) and equality holds if and only if \( R/I \) is Cohen-Macaulay. We define

\[
m_i(I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}
\]

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to be the minimum degree shift at the \(i\)-th step and

\[ M_i(I) = \max\{ j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0 \} \]

to be the maximum degree shift at the \(i\)-th step. We will simply write \(m_i\) and \(M_i\) when there is no confusion. If \(R/I\) is Cohen-Macaulay and has a pure resolution, i.e. \(m_i = M_i\) for all \(1 \leq i \leq c\), then Herzog and Kühl [9] and Huneke and Miller [10] showed that

\[ e(R/I) = \frac{\prod_{i=1}^{c} m_i}{c!} \]

and

\[ \beta_i(R/I) = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{d_j - d_i} \quad \text{for} \quad i = 1, \ldots, c. \]

Since then there has been a considerably effort to bound the multiplicity of a homogeneous Cohen-Macaulay ideal \(I \subset R\) in terms of the shifts in its graded minimal \(R\)-free resolution; and Herzog, Huneke and Srinivasan have made the following conjecture (minimal conjecture): If \(I \subset R\) be a graded Cohen-Macaulay ideal of codimension \(c\), then

\[ \frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}. \]

There is a growing body of the literature proving special cases of the above conjecture. For example, it holds for complete intersections [8], powers of complete intersection ideals [7], perfect ideals with a pure resolution (i.e. \(m_i \geq M_{i-1}\)) [8], perfect ideals of codimension 2 [8], Gorenstein ideals of codimension 3 [13] and standard determinantal ideals of arbitrary codimension [14].

Another natural question which naturally arises in this context is whether under the Cohen-Macaulay assumption the \(i\)-th Betti number \(\beta_i(R/I)\) can be bounded above by a function of the maximal shifts in the minimal graded free \(R\)-resolution of \(R/I\) as well as bounded below by a function of the minimal shifts. In [16], Römer made a natural guess

\begin{equation}
\label{eq:romer}
\prod_{1 \leq j < i} \frac{m_j}{m_i - m_j} \prod_{i < j \leq c} \frac{m_j}{m_j - m_i} \leq \beta_i(R/I) \leq \prod_{1 \leq j < i} \frac{M_j}{M_i - M_j} \prod_{i < j \leq c} \frac{M_j}{M_j - M_i}
\end{equation}

for \(i = 1, \ldots, c\) and he showed that these bounds hold if \(I\) is a complete intersection or componentwise linear. Moreover, in these cases we have equality above or below if and only if \(R/I\) has a pure resolution. Unfortunately, these bounds are not always valid (see [16], Example 3.1). For Cohen-Macaulay algebras with strictly quasi-pure resolution (i.e., \(m_i > M_{i-1}\) for all \(i\)) he showed

\begin{equation}
\label{eq:romer2}
\prod_{1 \leq j < i} \frac{m_j}{M_i - M_j} \prod_{i < j \leq c} \frac{m_j}{m_j - M_i} \leq \beta_i(R/I) \leq \prod_{1 \leq j < i} \frac{M_j}{m_i - M_j} \prod_{i < j \leq c} \frac{M_j}{m_j - M_i}
\end{equation}

for \(i = 1, \ldots, c\) and again we have equalities if and only if \(R/I\) has a pure resolution. Notice that \(\prod_{1 \leq j < i} \frac{M_j}{m_i - M_j} \prod_{i < j \leq c} \frac{M_j}{m_j - M_i}\) may be negative and thus, in general, \(\prod_{1 \leq j < i} \frac{M_j}{m_i - M_j} \prod_{i < j \leq c} \frac{M_j}{m_j - M_i}\) is not a good candidate for being an upper bound. In [16],
Römer suggests as upper bound

\[ \beta_i(R/I) \leq \frac{1}{(i-1)!(c-i)!} \prod_{j \neq i} M_j \]

for \( i = 1, \cdots, c \) and he proved that the lower bound in \((1.2)\) and the upper bound in \((1.3)\) hold if \( R/I \) is Cohen-Macaulay of codimension 2 and Gorenstein of codimension 3.

It remains open if these bounds hold for other interesting classes of ideals. The goal of this paper is to prove that the lower bound in \((1.2)\) and the upper bound in \((1.3)\) work in the following classes of ideals

- standard determinantal ideals of arbitrary codimension \( c \) (i.e., ideals generated by the maximal minors of a \( t \times (t+c-1) \) homogeneous polynomial matrix),
- symmetric determinantal ideals defined by the submaximal minors of a \( t \times t \) homogeneous symmetric matrix,
- determinantal ideals defined by the submaximal minors of a \( t \times t \) homogeneous matrix, and
- arithmetically Cohen-Macaulay divisors on a variety of minimal degree.

Determinantal ideals are a central topic in both commutative algebra and algebraic geometry. Due to their important role, their study has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results about determinantal ideals are due to Eagon and Hochster in \[3\], and to Eagon and Northcott in \[4\]. Eagon and Hochster proved that generic determinantal ideals are perfect. Eagon and Northcott constructed a finite graded free resolution for any determinantal ideal, and as a corollary, they showed that determinantal ideals are perfect. Since then many authors have made important contributions to the study of determinantal ideals, and the reader can look at \[2\], \[1\], \[12\] and \[5\] for background, history and a list of important papers.

Next we outline the structure of the paper. In section 2, we first recall the basic facts on standard determinantal ideals \( I \) of codimension \( c \) defined by the maximal minors of a \( t \times (t+c-1) \) homogeneous matrix \( A \) and the associated complexes needed later on. We determine the minimal and maximal shifts in the graded minimal free \( R \)-resolution of \( R/I \) in terms of the degree matrix \( U \) of \( A \) and we prove that the lower bound in \((1.2)\) and the upper bound in \((1.3)\) work for standard determinantal ideals of arbitrary codimension \( c \) (Theorem 2.4). We also discuss when these bounds are sharp. After some preliminaries, we devote section 3 to prove that the lower bound in \((1.2)\) and the upper bound in \((1.3)\) work for determinantal (resp. symmetric determinantal) ideals defined by the submaximal minors of a \( t \times t \) homogeneous (resp. symmetric) matrix (Theorems 3.3 and 3.6). We discuss other examples as well as when these bounds are sharp (Theorem 3.8).

2. Standard Determinantal ideals

In the first part of this section, we provide the background and basic results on determinantal ideals needed in the sequel, and we refer to \[2\] and \[5\] for more details.
Let $A$ be a homogeneous matrix, i.e., a matrix representing a degree 0 morphism $\phi : F \longrightarrow G$ of free graded $R$-modules. In this case, we denote by $I(A)$ the ideal of $R$ generated by the maximal minors of $A$ and by $I_j(A)$ the ideal generated by the $j \times j$ minors of $A$.

**Definition 2.1.** A homogeneous ideal $I \subset R$ of codimension $c$ is called a standard determinantal ideal if $I = I(A)$ for some $t \times (t + c - 1)$ homogeneous matrix $A$.

Let $I \subset R$ be a standard determinantal ideal of codimension $c$ generated by the maximal minors of a $t \times (t + c - 1)$ matrix $A = (f_{ij})_{i=1,...,t}^{j=1,...,t+c-1}$ where $f_{ij} \in k[x_1, ..., x_n]$ are homogeneous polynomials of degree $a_j - b_i$. The matrix $A$ defines a degree 0 map

$$F = \oplus_{i=1}^{t} R(b_i) \stackrel{A}{\longrightarrow} G = \oplus_{j=1}^{t+c-1} R(a_j)$$

$v \mapsto v \cdot A$

where $v = (f_1, \cdots, f_t) \in F$ and we assume without loss of generality that $A$ is minimal; i.e., $f_{ij} = 0$ for all $i, j$ with $b_i = a_j$. If we let $u_{ij} = a_j - b_i$ for all $j = 1, \ldots, t + c - 1$ and $i = 1, \ldots, t$, the matrix $U = (u_{ij})_{i=1,...,t,c}^{j=1,...,t+c-1}$ is called the degree matrix associated to $I$. By re-ordering degrees, if necessary, we may also assume that $b_1 \geq \ldots \geq b_t$ and $a_1 \leq a_2 \leq \ldots \leq a_{t+c-1}$. In particular, we have:

\begin{equation}
(2.1) \quad u_{ij} \leq u_{i+1,j} \quad \text{and} \quad u_{ij} \leq u_{i,j+1} \quad \text{for all} \quad i, j.
\end{equation}

Note that the degree matrix $U$ is completely determined by $u_{1,1}, u_{1,2}, \ldots, u_{1,c}, u_{2,1}, u_{2,2}, u_{2,3}, \ldots, u_{2,c+1}, \ldots, u_{t,t}, u_{t,t+1}, \ldots, u_{t,t+c-1}$ because of the identity $u_{ij} + u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} = 0$ for all $i, j$. Moreover, the graded Betti numbers in the minimal free $R$-resolution of $R/I(A)$ depend only upon the integers

$$\{u_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq c+i-1} \subset \{u_{ij}\}_{i=1,...,t}^{j=1,...,t+c-1}$$

as described below.

**Proposition 2.2.** Let $I \subset R$ be a standard determinantal ideal of codimension $c$ with degree matrix $U = (u_{ij})_{i=1,...,t}^{j=1,...,t+c-1}$ as above. Then we have:

1. $m_i = u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}$ for $1 \leq i \leq c$,
2. $M_i = u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t,t-c+i-1}$ for $1 \leq i \leq c$,
3. $\beta_i(R/I) = \binom{t+c-1}{i-1} \binom{t+i-2}{i-1}$ for $i = 1, \ldots, c$.

**Proof.** We denote by $\varphi : F \longrightarrow G$ the morphism of free graded $R$-modules of rank $t$ and $t + c - 1$, defined by the homogeneous matrix $A$ associated to $I$. The Eagon-Northcott complex $D_0(\varphi^*)$:

$\begin{align*}
0 \longrightarrow \wedge^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \wedge^t F & \longrightarrow \wedge^{t+c-2}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \ldots \longrightarrow \\
\wedge^t G^* \otimes S_0(F) \otimes \wedge^t F & \longrightarrow R \longrightarrow R/I \longrightarrow 0
\end{align*}$
gives us a graded minimal free $R$-resolution of $R/I$ (See, for instance [2]; Theorem 2.20 and [5]; Corollary A2.12 and Corollary A2.13). Now the result follows after a straightforward computation taking into account that

\[ ∧^t F = R(∑_{i=1}^{t} b_i), \]

\[ S_a(F) = ∐_{1≤i_1≤⋯≤i_a≤t} R(∑_{j=1}^{a} b_{ij}), \]

\[ G^* = ∐_{j=1}^{t+c-1} R(-a_j) \quad \text{and} \]

\[ ∧^b G^* = ∐_{1≤i_1<⋯<i_b≤t+c-1} R(-∑_{j=1}^{b} a_{ij}). \]

\[ \square \]

**Remark 2.3.** Let $I ⊂ R$ be a standard determinantal ideal. It is worthwhile to point out that the $i$-th total Betti number $β_i(R/I)$ in the minimal free $R$-resolution of $R/I$ depend only upon the size $t × (t + c − 1)$ of the homogeneous matrix $A$ associated to $I$.

We are now ready to state the main result of this short note.

**Theorem 2.4.** Let $I ⊂ R$ be a standard determinantal ideal of codimension $c$. Then, we have:

\[ \prod_{1≤j<i} \frac{m_j}{M_i-m_j} \prod_{i<j≤c} \frac{m_j}{M_j-m_i} ≤ β_i(R/I) ≤ \frac{1}{(i-1)!(c-i)!} \prod_{j≠i} M_j \]

for $i = 1, \cdots, c$. In addition, the bounds are reached for all $i$ if and only if $R/I$ has a pure resolution if and only if $u_{i,j} = u_{r,s}$ for all $1 ≤ i, r ≤ t$ and $1 ≤ j, s ≤ t + c − 1$.

**Proof.** We will first prove the result for $J$ being $J ⊂ R$ a standard determinantal ideal of codimension $c$ whose associated matrix $A$ is a $t × (t + c − 1)$ matrix with all its entries linear forms. In this case, for all $i = 1, \cdots, c$, we have (see Proposition 2.2):

\[ m_i(J) = M_i(J) = t + i - 1 \quad \text{and} \quad β_i(R/J) = \binom{t+c-1}{t+i-1} \binom{t+i-2}{i-1}. \]
Therefore, $R/J$ has a pure resolution and it follows from [9] and [10] that
\[
\prod_{1 \leq j < i} \frac{m_j(J)}{M_i(J) - m_j(J)} \prod_{i < j \leq c} \frac{m_j(J)}{M_j(J) - m_i(J)} = \prod_{1 \leq j < i} \frac{t + j - 1}{i - j} \prod_{i < j \leq c} \frac{t + j - 1}{j - i}
\]
\[
= \beta_i(R/J)
\]
\[
= \prod_{1 \leq j < i} \frac{t + j - 1}{i - j} \prod_{i < j \leq c} \frac{t + j - 1}{j - i}
\]
\[
= \frac{1}{(i - 1)!(c - i)!} \prod_{j \neq i} M_j(J)
\]

We will now prove the general case. Let $I$ be a standard determinantal ideal of codimension $c$ with associated degree matrix $\mathcal{U} = (u_{i,j})_{i=1,\ldots,c}$. Since, for all $i = 1, \ldots, c$, we have
\[
M_i(I) \geq m_i(I) \geq t + i - 1 = m_i(J) = M_i(J),
\]
it follows from Proposition 2.2 (3) and Remark 2.3 that
\[
\beta_i(R/I) = \beta_i(R/J)
\]
\[
= \prod_{1 \leq j < i} \frac{t + j - 1}{i - j} \prod_{i < j \leq c} \frac{t + j - 1}{j - i}
\]
\[
= \frac{1}{(i - 1)!(c - i)!} \prod_{j \neq i} M_j(J)
\]
\[
\leq \frac{1}{(i - 1)!(c - i)!} \prod_{j \neq i} M_j(I)
\]
and this completes the proof of the upper bound.

Let us now prove the lower bound. Recall that if $r \geq m$ and $s \geq n$ then $u_{r,s} \geq u_{m,n}$. Therefore, for $1 \leq j < i$, we get, using Proposition 2.2 that
\[
m_j = u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1}
\]
\[
\leq (t + j - 1)u_{t,t+j-1},
\]
\[
M_i - m_j = u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t,t+c-i} + u_{t,t+c-i+1} + u_{t,t+c-i+2} + \cdots + u_{t,t+c-1}
\]
\[
- (u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1})
\]
\[
\geq u_{t,t+c-i+j} + u_{t,t+c-i+j+1} + \cdots + u_{t,t+c-1}
\]
\[
\geq (i - j)u_{t,t+c-i+j},
\]
and
\[
\frac{m_j}{M_i - m_j} \leq \frac{(t + j - 1)u_{t,t+j-1}}{(i - j)u_{t,t+c-i+j}}
\]
\[
\leq \frac{t + j - 1}{i - j}.
\]
For \( i < j \leq c \), we have

\[
m_j = u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1}
\]

\[
= \begin{cases} 
  u_{1,1} + u_{1,2} + \cdots + u_{1,t+i-1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,t+j-1}, & \text{if } t + i \leq j, \\
  u_{1,1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,i-j+1} + \cdots + u_{t,t+j-1}, & \text{if } t + i > j;
\end{cases}
\]

\[
M_j - m_i = u_{1,c-j+1} + u_{2,c-j+2} + \cdots + u_{t,t+c-j} + u_{t,t+c-j+1} + u_{t,t+c-j+2} + \cdots + u_{t,t+c-1}
\]

\[
- (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1})
\]

\[
\geq u_{t,t+c+i-j} + u_{t,t+c+i-j+1} + \cdots + u_{t,t+c-1}
\]

\[
\geq \begin{cases} 
  u_{1,t+i} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,t+j-1}, & \text{if } t + i \leq j, \\
  u_{t+i-j+1,t+i} + \cdots + u_{t,t+j-1}, & \text{if } t + i > j.
\end{cases}
\]

Therefore, if \( i < j \leq c \) and \( t + i \leq j \), we get

\[
\frac{m_j}{M_j - m_i} \leq \frac{u_{1,1} + u_{1,2} + \cdots + u_{1,t+i-1} + u_{t,t+i} + \cdots + u_{t,t+j-1}}{(j - i)u_{1,t+i}} + 1
\]

\[
\leq \frac{(t + i - 1)u_{1,t+i-1}}{(j - i)u_{1,t+i}} + 1
\]

\[
\leq \frac{t + i - 1}{j - i} + 1
\]

\[
= \frac{t + j - 1}{j - i},
\]

and if \( i < j \leq c \) and \( t + i > j \), we get

\[
\frac{m_j}{M_j - m_i} \leq \frac{u_{1,1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t+t+i-j+i-1}}{u_{t+i-j+1,t+i} + \cdots + u_{t,t+j-1}} + 1
\]

\[
\leq \frac{(t + i - 1)u_{t+i-j+i-1}}{(j - i)u_{t+t+i-j+i-1}} + 1
\]

\[
\leq \frac{t + i - 1}{j - i} + 1
\]

\[
= \frac{t + j - 1}{j - i}.
\]

Hence, putting altogether, we obtain

\[
\prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} \leq \prod_{1 \leq j < i} \frac{t + j - 1}{i - j} \prod_{i < j \leq c} \frac{t + j - 1}{j - i}
\]

\[
= \beta_i(R/J)
\]

\[
= \beta_i(R/I)
\]

and this completes the proof of the lower bound. Checking the inequalities we easily see that we have equality above and below for all \( 1 \leq i \leq c \) if and only if \( R/I \) has a pure resolution. This concludes the proof of the Theorem. \(\square\)

**Remark 2.5.** Since a complete intersection ideal \( I \) of arbitrary codimension and Cohen-Macaulay ideals of codimension 2 are examples of standard determinantal ideals, we recover \([16]\); Theorem 2.1 and Corollary 4.2.
Since the power $I^s$ of a complete intersection ideal $I \subset R$ is an example of standard determinantal ideal, as a corollary of Theorem 2.4 we have

**Corollary 2.6.** Let $I \subset R$ be a complete intersection ideal of codimension $c$ and let $s$ be any positive integer. Then, it holds

\[
\prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} \leq \beta_i(R/I^s) \leq \frac{1}{(i-1)!(c-i)!} \prod_{j \neq i} M_j
\]

for $i = 1, \ldots, c$.

### 3. Ideals Defined by Submaximal Minors

The first goal of this section is to prove that the lower bound in (1.2) and the upper bound in (1.3) work for $k$-algebras $k[x_1, \ldots, x_n]/I$ being $I$ a perfect ideal generated by the submaximal minors of a $t \times t$ homogeneous symmetric matrix. A classical homogeneous ideal that can be generated by the submaximal minors of a $t \times t$ homogeneous symmetric matrix is the ideal of the Veronese surface $X \subset \mathbb{P}^5$. Indeed, the ideal of the Veronese surface $X \subset \mathbb{P}^5 = \text{Proj}(k[x_1, \ldots, x_6])$ can be generated by the $2 \times 2$ minors of the symmetric matrix

\[
\begin{pmatrix}
x_6 & x_1 & x_2 \\
x_1 & x_3 & x_4 \\
x_2 & x_4 & x_5
\end{pmatrix}
\]

Let us now fix some notation. Let $I \subset S = k[x_1, \ldots, x_n]$ be a codimension 3, perfect ideal generated by the submaximal minors of a $t \times t$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\ldots,t}$ where $f_{ji} \in k[x_1, \ldots, x_n]$ are homogeneous polynomials of degree $a_i + a_j$, i.e., $I = I_{t-1}(\mathcal{A})$. We denote by

\[
\mathcal{U} = \begin{pmatrix}
2a_1 & a_1 + a_2 & \cdots & a_1 + a_t \\
a_1 + a_2 & 2a_2 & \cdots & a_2 + a_t \\
\vdots & \vdots & \ddots & \vdots \\
a_1 + a_t & a_2 + a_t & \cdots & 2a_t
\end{pmatrix}
\]

the degree matrix of $\mathcal{A}$. We assume that $a_1 \leq a_2 \leq \cdots \leq a_t$. The determinant of $\mathcal{A}$ is a homogeneous polynomial of degree $\ell := 2(a_1 + a_2 + \cdots + a_t)$. Note that $a_i + a_j$ is an integer for all $1 \leq i \leq j \leq t$ while $a_i$ does not necessarily need to be an integer.

Note that the degree matrix $\mathcal{U}$ is completely determined by $a_1, \ldots, a_t$. Moreover, the graded Betti numbers in the minimal free $S$-resolution of $S/I_{t-1}(\mathcal{A})$ depend only upon the integers $a_1, \ldots, a_t$ as we will describe now. To this end, we recall Jozefiak’s result about the resolution of ideals generated by minors of a symmetric matrix.

Let $R$ be a commutative ring with identity and let $X = (x_{ij})$ be a symmetric $t \times t$ matrix with entries in $R$. Write $Y = (y_{ij})$ for the matrix of cofactors of $X$, i.e., $y_{ij} = (-1)^{i+j}X^i_j$ where $X^i_j$ stands for the minor of $X$ obtained by deleting the $i$-th column and the $j$-th row of $X$. The matrix $Y$ is also a symmetric matrix. Let $M_t(R)$ be the free $R$-module of all $t \times t$ matrices over $R$ and let $A_t(R)$ be the free $R$-submodule of $M_t(R)$ consisting of
all alternating matrices. Denote by \( tr : M_t(R) \rightarrow R \) the trace map. By [I], Theorem 3.1, the free complex of length 3 associated to \( X \):

\[
0 \rightarrow A_t(R) \xrightarrow{d_3} Kcr(M_t(R) \xrightarrow{tr} R) \xrightarrow{d_2} M_t(R)/A_t(R) \xrightarrow{d_1} R
\]

where the corresponding differentials are defined as follows:

- \( d_3(A) = AX \),
- \( d_2(N) = XN \mod A_t(R) \), and
- \( d_1(M \mod A_t(R)) = tr(YM) \)

is acyclic and gives a free resolution of \( R/I_{t-1}(X) \). So, we obtain

**Proposition 3.1.** Let \( I \subset S = k[x_1, \ldots, x_n] \) be a perfect ideal of codimension 3 generated by the submaximal minors of a symmetric matrix \( A \). Let

\[
U = \begin{pmatrix}
2a_1 & a_1 + a_2 & \cdots & a_1 + a_t \\
-1 & 2a_2 & \cdots & a_2 + a_t \\
\vdots & \vdots & & \vdots \\
a_1 + a_t & a_2 + a_t & \cdots & 2a_t
\end{pmatrix}
\]

be the degree matrix and \( \ell := 2(a_1 + a_2 + \cdots + a_t) \). Then, we have:

1. \( m_1 = \ell - 2at \) and \( M_1 = \ell - 2a_1 \),
2. \( m_2 = \ell - a_1 + a_t \) and \( M_2 = \ell - a_1 + a_t \),
3. \( m_3 = \ell + a_1 + a_2 \) and \( M_3 = \ell + a_{t-1} + a_t \), and
4. \( \beta_1(R/I) = \binom{\ell + 1}{2} \), \( \beta_2(R/I) = t^2 - 1 \), and \( \beta_3(R/I) = \binom{t}{2} \).

**Proof.** By [I]; Theorem 3.1, \( I \) has a minimal free \( S \)-resolution of the following type:

\[
(3.1) \quad 0 \rightarrow \oplus_{1 \leq i < j \leq t} S(-a_i - a_j - \ell) \rightarrow (\oplus_{1 \leq i, j \leq t} S(-\ell - a_i + a_j))/S(-\ell) \rightarrow \oplus_{1 \leq i < j \leq t} S(a_i + a_t - \ell) \rightarrow I \rightarrow 0.
\]

So, the maximum and minimum degree shifts at the \( i \)-th step are

1. \( m_1 = \ell - 2at \) and \( M_1 = \ell - 2a_1 \),
2. \( m_2 = \ell - a_1 + a_t \) and \( M_2 = \ell - a_1 + a_t \),
3. \( m_3 = \ell + a_1 + a_2 \) and \( M_3 = \ell + a_{t-1} + a_t \),

and the \( i \)-the total Betti numbers are:

4. \( \beta_1(R/I) = \binom{\ell + 1}{2} \), \( \beta_2(R/I) = t^2 - 1 \), and \( \beta_3(R/I) = \binom{t}{2} \)

which proves what we want. \( \square \)

**Remark 3.2.** Let \( I \subset R \) be a perfect ideal of codimension 3 generated by the submaximal minors of a symmetric matrix. It is worthwhile to point out that the \( i \)-th total Betti number \( \beta_i(R/I) \) in the minimal free \( R \)-resolution of \( R/I \) depend only upon the size \( t \times t \) of the homogeneous symmetric matrix \( A \) associated to \( I \).
We are now ready to bound the $i$-the total Betti number of codimension 3, perfect ideals generated by the submaximal minors of a symmetric matrix in terms of the shifts in its minimal free $R$-resolution.

**Theorem 3.3.** Let $I \subset S$ be a perfect ideal of codimension 3 generated by the submaximal minors of a $t \times t$ symmetric matrix. Then, we have:

\[
\prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq 3} \frac{m_j}{M_j - m_i} \leq \beta_i(R/I) \leq \frac{1}{(i - 1)!(c - i)!} \prod_{j \neq i} M_j
\]

for $1 \leq i \leq 3$. In addition, the bounds are reached for all $i$ if and only if $R/I$ has a pure resolution if and only if $a_1 = \cdots = a_t$.

**Proof.** We will first prove the result for $J$ being $J \subset R$ a codimension 3 perfect ideal generated by the submaximal minors of a $t \times t$ symmetric matrix $A$ with linear entries. In this case, for all $1 \leq i \leq 3$, we have (see Proposition 3.1)

\[
m_i(J) = M_i(J) = t + i - 2,
\]

\[
\beta_1(R/J) = \binom{t+1}{2}, \quad \beta_2(R/J) = t^2 - 1 \quad \text{and} \quad \beta_3(R/J) = \binom{t}{2}.
\]

Therefore, $R/J$ has a pure resolution and it follows from [9] and [10] that

\[
\prod_{1 \leq j < i} \frac{m_j(J)}{M_i(J) - m_j(J)} \prod_{i < j \leq 3} \frac{m_j(J)}{M_j(J) - m_i(J)} = \prod_{1 \leq j < i} \frac{t + j - 2}{i - j} \prod_{i < j \leq 3} \frac{t + j - 2}{j - i} = \beta_i(R/J) = \prod_{1 \leq j < i} \frac{t + j - 2}{i - j} \prod_{i < j \leq 3} \frac{t + j - 2}{j - i} = \prod_{1 \leq j < i} \frac{M_j(J)}{m_i(J) - M_j(J)} \prod_{i < j \leq 3} \frac{M_j(J)}{m_j(J) - M_i(J)} = \frac{1}{(i - 1)!(3 - i)!} \prod_{j \neq i} M_j(J).
\]

We will now prove the general case. Let $I$ be a perfect ideal of codimension 3 generated by the submaximal minors of a $t \times t$ symmetric matrix, let

\[
U = \begin{pmatrix}
2a_1 & a_1 + a_2 & \cdots & a_1 + a_t \\
a_1 + a_2 & 2a_2 & \cdots & a_2 + a_t \\
\vdots & \vdots & & \vdots \\
a_1 + a_t & a_2 + a_t & \cdots & 2a_t
\end{pmatrix}
\]

be its degree matrix and $\ell := 2(a_1 + a_2 + \cdots + a_t)$. Since, for all $1 \leq i \leq 3$, we have

\[
M_i(I) \geq m_i(I) \geq t + i - 2 = m_i(J) = M_i(J),
\]
it follows from Proposition 3.1 (3) and Remark 3.2 that
\[ \beta_i(R/I) = \beta_i(R/J) \]
\[ = \prod_{1 \leq j < i} \left( \frac{t + j - 2}{i - j} \right) \prod_{i < j \leq 3} \left( \frac{t + j - 2}{j - i} \right) \]
\[ = \frac{1}{(i - 1)!(3 - i)!} \prod_{j \neq i} M_j(J) \]
\[ \leq \frac{1}{(i - 1)!(3 - i)!} \prod_{j \neq i} M_j(I) \]
and this completes the proof of the upper bound.

Let us now prove the lower bound. Using again Proposition 3.1 and Remark 3.2, we have
\[ \frac{m_1}{M_3 - m_1} \frac{m_2}{M_3 - m_2} = \frac{\ell - 2a_t}{3a_t + a_{t-1} a_{t-1} + 2a_t - a_1} \]
\[ \leq \frac{\ell - 2a_t \ell - a_t + a_1}{4a_{t-1} 2a_t} \]
\[ \leq \frac{2(t - 1)a_{t-1} 2ta_t}{4a_{t-1} 2a_t} \]
\[ = \frac{(t - 1)t}{2} \]
\[ = \beta_3(R/J) \]
\[ = \beta_3(R/I). \]
Analogously, we obtain
\[ \frac{m_1}{M_2 - m_1} \frac{m_3}{M_3 - m_1} \leq (t - 1)(t + 1) = \beta_2(R/J) = \beta_2(R/I) \]
and
\[ \frac{m_2}{M_2 - m_1} \frac{m_3}{M_3 - m_1} \leq \frac{t(t + 1)}{2} = \beta_1(R/J) = \beta_1(R/I) \]
and this completes the proof of the lower bound. It is easy to see that the inequality is an equality for all \( i \) if and only if \( a_1 = a_2 \cdots = a_t \) if and only if \( R/I \) has a pure resolution. \( \square \)

Now, we will focus our attention on determinantal ideals \( I \) generated by the submaximal minors of a \( t \times t \) quadratic homogeneous matrix. By [6]; Théorème 2, \( I \) is a Gorenstein ideal of codimension 4 and we will prove the lower bound in (L.2) and the upper bound in (L.3) for such kind of perfect ideals.

To this end, let \( A = (f_{ji})_{i,j=1,...,t} \) be a homogeneous quadratic matrix with entries homogeneous polynomials \( f_{ji} \in k[x_1, ..., x_n] \) of degree \( a_j - b_i \). We assume without loss of generality that \( A \) is minimal; i.e., \( f_{ji} = 0 \) for all \( i, j \) with \( b_i = a_j \). If we let \( u_{ji} = a_j - b_i \) for all \( 1 \leq i, j \leq t \), the matrix \( U = (u_{ji})_{i,j=1,...,t} \) is called the degree matrix associated to \( I \). By re-ordering degrees, if necessary, we may also assume that \( b_1 \leq ... \leq b_t \) and \( a_1 \leq ... \leq a_t \).
The determinant of $\mathcal{A}$ is a homogeneous polynomial of degree
\[ s := \deg(\det(A)) = \sum_{j=1}^{t} a_j - \sum_{i=1}^{t} b_i. \]

Let us now compute the $i$-th total Betti numbers in the minimal free $R$-resolution of $R/I_{t-1}(\mathcal{A})$ in terms of the degree matrix $\{u_{j,i}\}_{i,j=1,\ldots,t}$ of $\mathcal{A}$.

**Proposition 3.4.** Let $I \subset R$ be a determinantal ideal of codimension 4 generated by the submaximal minors of a quadratic homogeneous matrix $\mathcal{A}$ with degree matrix $\mathcal{U} = (u_{j,i})_{i,j=1,\ldots,t}$ with $u_{j,i} = a_j - b_i$. Set $s := \sum_{j=1}^{t} a_j - \sum_{i=1}^{t} b_i$. Then we have:

1. $m_1(I) = s + b_1 - a_t$, $M_1(I) = s + b_t - a_1$,
2. $m_2(I) = \min(s + b_1 - b_t, s + a_1 - a_t)$, $M_2(I) = \max(s + b_t - b_1, s + a_t - a_1)$,
3. $m_3(I) = s - b_t + a_1$, $M_3(I) = s + a_t - b_1$,
4. $m_4(I) = M_4(I) = 2s$,
5. $\beta_1(R/I) = t^2$, $\beta_2(R/I) = t^2 - 2$, $\beta_3(R/I) = t^2$ and $\beta_4(R/I) = 1$.

**Proof.** We denote by
\[ F := \oplus_{i=1}^{t} R(b_i) \xrightarrow{A} G := \oplus_{j=1}^{t} R(a_j) \]
the morphism of free graded $R$-modules of rank $t$, defined by the homogeneous matrix $\mathcal{A}$ associated to $I$. By [6]: Théorème 2, $R/I$ has a minimal graded free $R$-resolution of the following type:

\[
0 \longrightarrow R(-2s) \longrightarrow \bigoplus_{1 \leq i,j \leq t} R(b_j - a_i - s) \longrightarrow \\
( \bigoplus_{i \neq j} R(a_i - a_j - s)) \oplus ( \bigoplus_{i \neq j} R(b_i - b_j - s)) \oplus R(-s)^{2t-2} \longrightarrow \bigoplus_{1 \leq i,j \leq t} R(b_i - b_j - s) \longrightarrow I \longrightarrow 0
\]

and, hence, the maximum and minimum degree shifts at the $i$-th step are

1. $m_1 = s + b_1 - a_t$, $M_1 = s + b_t - a_1$,
2. $m_2 = \min(s + b_1 - b_t, s + a_1 - a_t)$, $M_1 = \max(s + b_t - b_1, s + a_t - a_1)$,
3. $m_3 = s - b_t + a_1$, $M_3 = s + a_t - b_1$,
4. $m_4 = M_4 = 2s$.

and the $i$-the total Betti numbers are:

1. $\beta_1(R/I) = t^2$, $\beta_2(R/I) = t^2 - 2$, $\beta_3(R/I) = t^2$ and $\beta_4(R/I) = 1$.

which proves what we want. \hfill \Box

**Remark 3.5.** Let $I \subset R$ be a perfect ideal of codimension 4 generated by the submaximal minors of a homogeneous $t \times t$ matrix. It is worthwhile to point out that the $i$-th total Betti number $\beta_i(R/I)$ in the minimal free $R$-resolution of $R/I$ depend only on $t$.

Arguing as in Theorems 2.3 and 3.3 we can bound the $i$-th total Betti number of a codimension 4, Gorenstein ideal generated by the submaximal minors of a quadratic matrix in terms of the shifts in its minimal free $R$-resolution and we get:
Theorem 3.6. Let $I \subset R$ be a determinantal ideal of codimension 4 generated by the submaximal minors of $t \times t$ quadratic homogeneous matrix $A$. Then, we have:

$$(3.4) \quad \prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq 4} \frac{m_j}{M_j - m_i} \leq \beta_i(R/I) \leq \frac{1}{(i-1)!(4-i)!} \prod_{j \neq i} M_j$$

for $1 \leq i \leq 4$. In addition, the bounds are reached for all $i$ if and only if $R/I$ has a pure resolution if and only if $u_{i,j} = u_{r,s}$ for all $1 \leq i, r, j, s \leq t$.

We would like now to state a nice conjecture which naturally arises in this context. Indeed, the results in [16] together with Theorems 2.4, 3.3 and 3.6 and Corollary 2.6 suggest -and prove in many cases- the following conjecture

Conjecture 3.7. Let $I \subset R$ be a graded Cohen-Macaulay ideal of codimension $c$. Then, we have:

$$(3.5) \quad \prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} \leq \beta_i(R/I) \leq \frac{1}{(i-1)!(c-i)!} \prod_{j \neq i} M_j$$

for $i = 1, \cdots, c$.

We will end this paper with some other examples which give support to the above conjecture.

Theorem 3.8. Let $X \subset \mathbb{P}^n = \text{Proj}(S) = \text{Proj}(k[x_1, \cdots, x_{n+1}])$ be a reduced arithmetically Cohen-Macaulay subscheme of degree $d > c = \text{codim}X$. Suppose $X$ is a divisor on a variety of minimal degree. Then, we have:

$$(3.6) \quad \prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} \leq \beta_i(S/I(X)) \leq \frac{1}{(i-1)!(c-i)!} \prod_{j \neq i} M_j$$

for $i = 1, \cdots, c$.

Proof. First of all recall that $\text{deg}(X) = e(S/I(X))$. By [15], Theorem 1.3, the minimal graded free $S$-resolution of $I(X)$ has the following shape

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I(X) \to 0$$

where

$$F_i = S(-1-i)^{\alpha_i} \oplus S(-t+1-i)^{\beta_i} \oplus S(-t-i)^{\gamma_i}, \quad 1 \leq i \leq c,$$

$$d = tc + 1 - p \quad \text{with} \quad 1 \leq p \leq c$$

and

$$\alpha_i = i \left( \begin{matrix} c \\ i + 1 \end{matrix} \right), \quad \text{for} \quad 1 \leq i \leq c,$$

$$\beta_i = \begin{cases} p \left( \begin{matrix} c \\ i-1 \end{matrix} \right) - c \left( \begin{matrix} c-1 \\ i-2 \end{matrix} \right) & \text{if} \quad 1 \leq i \leq p \\ 0 & \text{if} \quad p < i \leq c, \end{cases}$$

$$\gamma_i = \begin{cases} 0 & \text{if} \quad 1 \leq i \leq p \\ c \left( \begin{matrix} c-1 \\ i-1 \end{matrix} \right) - p \left( \begin{matrix} c \\ i \end{matrix} \right) & \text{if} \quad p < i \leq c. \end{cases}$$

Therefore, we have

$$m_i = 1 + i \quad \text{for} \quad 1 \leq i \leq c, \text{ and}$$
\[ M_i = \begin{cases} 
 t + i - 1 & \text{if } 1 \leq i \leq p \\
 t + i & \text{if } p < i \leq c 
\end{cases} \]
and the result follows after a straightforward computation taking into account that

\[ \beta_i(S/I(X)) = \alpha_i + \beta_i + \gamma_i \]
for all all \( 1 \leq i \leq c \). \hfill \square

REFERENCES

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[2] W. Bruns and U. Vetter, *Determinantal rings*, Springer-Verlag, Lectures Notes in Mathematics 1327, New York/Berlin, 1988.

[3] J.A. Eagon and M. Hochster, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. 93 (1971), 1020-1058.

[4] J.A. Eagon and D.G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. London 269 (1962), 188-204.

[5] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Springer-Verlag, Graduate Texts in Mathematics 150 (1995).

[6] M. T. Gulliksen and O.G. Negård, *Un complexe résolvant pour certains idéaux déterminantielis*, C.R. Acad. Sc. Paris 274 (1972), 16-18.

[7] E. Guardo and A. Van Tuyl, *Powers of complete intersections: graded Betti numbers and applications*, Illinois Journal of Math., to appear.

[8] J. Herzog and H. Srinivasan, *Bounds for multiplicities*, Trans AMS 350 (1998), 2879-2902.

[9] J. Herzog and M. Kühfl, *On the Betti numbers of finite pure and linear resolutions*, Comm. Alg. 12 (1984) 1627-1646.

[10] C. Huneke and M. Miller *A note on the multiplicity of Cohen-Macaulay algebras with pure resolutions*, Can. J. Math. 37 (1985), 1149-1162

[11] T. Józefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helvetici 53 (1978), 595-607.

[12] J. Kleppe and R.M. Miró-Roig, *Dimension of families of determinantal schemes*, Trans A.M.S 357 (2005), 2871-2907.

[13] J. Migliore, U. Nagel and T. Romer, *The multiplicity conjecture in low codimension*, Math. Res. Lett. 12 (2005), 731-747.

[14] R.M. Miró-Roig, *A note on the multiplicity of determinantal ideals*, J. Alg. 299 (2006) 714-724.

[15] U. Nagel, *On arithmetically Buchsbaum subschemes and liaison*, Preprint, Paderborn, 1996.

[16] T. Römer, *Betti numbers and shifts in minimal graded free resolutions*, arXiv AC/070119.

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