A DEGENERATE KAM THEOREM FOR PARTIAL DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, an infinite dimensional KAM theorem with double normal frequencies is established under qualitative non-degenerate conditions. This is an extension of the degenerate KAM theorem with simple normal frequencies in [3] by Bambusi, Berti and Magistrelli. As applications, for nonlinear wave equation and nonlinear Schrödinger equation with periodic boundary conditions, quasi-periodic solutions of small amplitude and quasi-periodic solutions around plane wave are obtained respectively.

1. Introduction. In [3], the authors studied infinite dimensional Hamiltonian systems with their frequencies being simple and analytically depending on one parameter. A degenerate KAM theorem for lower dimensional elliptic tori was proved. It was novel that the non-degeneracy conditions are qualitative instead of quantitative. Then the abstract KAM theorem was applied to one dimensional nonlinear wave equations with Dirichlet boundary conditions.

It is a natural question whether if the above KAM theorem could be extended to infinite dimensional Hamiltonian systems with multiple normal frequencies, and applied to one dimensional partial differential equations with periodic boundary conditions or higher dimensional partial differential equations.

In order to make clear the difficulty, we take one dimensional nonlinear wave equation

$$u_{tt} - u_{xx} + \xi u + f(x, u) = 0$$

(1)

for example, where the mass $\xi > 0$ belongs to some compact set $I \subset \mathbb{R}$. For (1) with Dirichlet boundary conditions, the eigenvalues

$$\lambda_j(\xi) = \sqrt{j^2 + \xi}, \quad j \in \mathbb{Z}^+: = \{1, 2, \cdots\},$$
which are different from each other. Choose an index set \( J = \{ j_1 < \cdots < j_n \} \subset \mathbb{Z}^+ \), and introduce tangential frequencies

\[
\omega_b(\xi) := \lambda_{j_b}(\xi), \quad 1 \leq b \leq n
\]

and normal frequencies

\[
\Omega_j(\xi) := \lambda_j(\xi), \quad j \in \mathbb{Z}^+ \setminus J.
\]

Then Bambusi, Berti and Magistrelli in [3] verified the following qualitative non-degeneracy conditions:

(i) for any \((c_1, \cdots, c_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}\), the function \( c_1 \omega_1(\xi) + \cdots + c_n \omega_n(\xi) + c_{n+1} \) is not identically zero on \( \mathcal{I} \),

(ii) for any \( l \in \mathbb{Z}^{n+1} \) with \(|l| := \sum_{j \in \mathbb{Z}^+ \setminus J} |l_j| \leq 2 \) and for any \((c_1, \cdots, c_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}\), the function \( c_1 \omega_1(\xi) + \cdots + c_n \omega_n(\xi) + c_{n+1} \sum_{j \in \mathbb{Z}^+ \setminus J} l_j \Omega_j(\xi) \) is not identically zero on \( \mathcal{I} \).

For (1) with periodic boundary conditions, the eigenvalues are

\[
\lambda_j(\xi) = \sqrt{j^2 + \xi}, \quad j \in \mathbb{Z}.
\] (2)

The multiplicity of \( \lambda_0(\xi) \) is 1, and the multiplicity of \( \lambda_j(\xi) \) for \( j \in \mathbb{Z} \setminus \{0\} \) is 2, i.e., \( \lambda_j = \lambda_{-j} \). By choosing the index set \( J = \{ j_1 < \cdots < j_n \} \subset \mathbb{Z} \) with \(|j_1|, \cdots, |j_n|\) different from each other, the condition (i) still holds true. However, the condition (ii) with \( \mathbb{Z} \) instead of \( \mathbb{Z}^+ \) is obviously not satisfied, for example, for any \( 1 \leq b \leq n \) with \(-j_b \in \mathbb{Z} \setminus J\),

\[
\omega_b(\xi) - \Omega_{-j_b}(\xi) = \lambda_{j_b}(\xi) - \lambda_{-j_b}(\xi) \equiv 0.
\] (3)

Fortunately, we observe that ‘real combinations of frequencies’ in the condition (ii) can be weakened as ‘integer combinations of frequencies’, that is,

(ii)’ for any \((k_1, \cdots, k_n) \in \mathbb{Z}^n\) and \( l \in \mathbb{Z}^{n+1} \setminus J\) with \(|l| \leq 2\), the function

\[
k_1 \omega_1(\xi) + \cdots + k_n \omega_n(\xi) + \sum_{j \in \mathbb{Z}^+ \setminus J} l_j \Omega_j(\xi)
\]

is not identically zero on \( \mathcal{I} \).

For the condition (ii)’ with \( \mathbb{Z} \) instead of \( \mathbb{Z}^+ \), generally (3) is still a counterexample. However, ‘integer combinations of frequencies’ provides a chance to avoid it by using some structure property of the equation. Now consider (1) with its nonlinearity not containing space variable \( x \) explicitly and periodic boundary conditions. The momentum is conserved. Passing to Fourier coefficients, the corresponding Hamiltonian takes the form

\[
e^{i(k_1 x_1 + \cdots + k_n x_n)} y_1^{m_1} \cdots y_n^{m_n} \prod_{j \in \mathbb{Z}^+ \setminus J} z_j^{\nu_j} \bar{z}_j^{\bar{\nu}_j}
\]

and (4) becomes

\[
\sum_{b=1}^{n} k_b j_b + \sum_{j \in \mathbb{Z}^+ \setminus J} l_j j = 0
\] (5)
with \( l_j := \alpha_j - \bar{\alpha}_j \). We can prove the conclusion of condition (ii)' for \( k, l \) additionally satisfying (5). In other words, such integer combinations as (3) are avoided by (5).

Consequently, we can establish an infinite dimensional KAM theorem with double normal frequencies under qualitative non-degenerate conditions (i) and (ii)' with (5).

Historically, KAM theory for partial differential equations was originated by Kuksin [18] and Wayne [29], where one dimensional nonlinear wave and Schrödinger equations with Dirichlet boundary conditions were studied. Then infinite dimensional KAM theory was deeply developed with applications to more partial differential equations, including both simple and multiple normal frequencies. For the case with simple normal frequencies, also see [3, 4, 17, 19, 20, 27, 28, 31] for example; for the case with multiple normal frequencies such as one dimensional nonlinear wave and Schrödinger equations with periodic boundary conditions and higher dimensional partial differential equations, see [6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 23, 26] for example; for partial differential equations with their nonlinearity containing spatial derivatives such as KdV, derivative nonlinear wave and Schrödinger equations, water wave equation, see [1, 2, 5, 21, 22, 24] for example. (We can not list all papers in this field.) Especially, for the introduction of degenerate KAM theory, see [3] and the references therein.

We now lay out an outline of the present paper:

In Subsection 2.1, we give a KAM iterative theorem (Theorem 2.1) and a measure estimate theorem (Theorem 2.2) which are copied from [3] with the following modifications: (1) the frequencies are double, and thus the index set is \( \mathbb{Z} \) instead of \( \mathbb{Z}^+ \); (2) the perturbation \( P \) is required to satisfy (9); (3) the assumption \( \delta < 1 \) is added in assumption (A). The proof of Theorem 2.1 is completely parallel to that in [3], and the proof of Theorem 2.2 is given in Section 3.

In Subsection 2.2, as the first application to the nonlinear wave equation with periodic boundary conditions, we state the existence of Cantor families of quasi-periodic solutions with small amplitude, seeing Theorem 2.3.

In Subsection 2.3, as the second application, we consider the nonlinear Schrödinger equation with periodic boundary conditions

\[
iu_t = -u_{xx} + |u|^2u,
\]

where \( \iota \) is an positive integer. For \( m \in \mathbb{Z} \),

\[
u_m(x, t) = \rho e^{i(mx - \mu_m t)}
\]

are plane wave solutions to equation (6), where \( \rho > 0 \) is the amplitude and \( \mu_m := m^2 + \rho^2 \iota \) is the frequency. Note that the amplitude \( \rho \) need not be small. Here, we state the existence of Cantor families of quasi-periodic solutions around these plane wave solutions, seeing Theorem 2.4. Actually in [14] [30], Sobolev stability of plane wave solutions has been proved even for higher dimensional nonlinear Schrödinger equations. We will study the existence of quasi-periodic solutions around plane wave solutions of higher dimensional nonlinear Schrödinger equations in our future work.

In Section 3, Theorem 2.2 is proved. The key step is to use the analyticity of the frequencies to transform the qualitative non-degeneracy assumption (ND) into a quantitative non-resonance property, seeing Proposition 1. The authors in [3] discussed four cases, i.e., the Kolmogorov condition for \( l = 0 \), the first Melnikov condition for \( l = e_i \), the second Melnikov condition for \( l = e_i + e_j \), and the second Melnikov condition for \( l = e_i - e_j \). Then they proceeded by contradiction in every
case separately. We will also prove Proposition 1 by contradiction. Due to the double multiplicity of normal frequencies, the essential difference compared with [3] is to handle the case \( l = e_j - e_{-j} \) additionally. Nevertheless, we will not discuss case by case as in [3]. Alternatively, we will give a summarized proof for all \((k, l) \in \mathcal{X}\).

In Section 4, we give the proof of Theorem 2.3. We firstly write the equation (16) into the Hamiltonian form. Then we introduce standard angle-action coordinates and verify the conditions of Theorem 2.1 and Theorem 2.2. The verification of the first condition (I) of (ND) is completely the same as Lemma 6 in [3] for nonlinear wave equation with Dirichlet boundary conditions; in order to verify the second condition (II) of (ND), we prove that for every \((k, l)\) satisfying momentum condition (5) and \(0 < |l| \leq 2\), \(\langle k, \omega \rangle + \langle l, \Omega \rangle\) is a non-zero integer linear combination of \(\sqrt{j^2 + \xi}\), \(j \geq 0\), seeing Lemma 4.1.

In Section 5, we give the proof of Theorem 2.4. We firstly write the equation (18) in the Hamiltonian form. Then as in [14] [30], we eliminate the zero mode and normalize the quadratic Hamiltonian. Finally we introduce standard action-angle coordinates and verify the conditions of Theorem 2.1 and Theorem 2.2.

2. Main results. In this section, we give an abstract KAM theorem and its two applications. For convenience, we keep fidelity with the notation and terminology from [3].

2.1. An abstract KAM theorem. Define the index set \( J = \{ j_1 < \cdots < j_n \} \subset \mathbb{Z} \) and denote \( J^c = \mathbb{Z} \setminus J \). For \( a \geq 0, p > \frac{1}{2} \), define the Hilbert space \( \ell^{a, p} \) of all complex sequences \( z = (z_j)_{j \in J^c} \) such that

\[
||z||_{a, p}^2 := \sum_{j \in J^c} e^{2a|j|} (j)^{2p} |z_j|^2 < \infty,
\]

where \( (j) := \max\{1, |j|\} \) for \( j \in \mathbb{Z} \).

Consider the direct product

\[
\mathcal{P}^{a, p} := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a, p} \times \ell^{a, p}
\]

endowed with weighted norm

\[
||v||_{r, a, p} = |x| + \frac{|y|}{r^2} + ||z||_{a, p} + ||\bar{z}||_{a, p},
\]

where \( |\cdot| \) denotes the sup-norm for finite dimensional vectors. In the whole of this paper the parameter \( a \) is fixed. Denote \( \mathcal{P}^{a, p}_c \) as the complexification of \( \mathcal{P}^{a, p} \), and define the toroidal domain

\[
D(s, r) = D_{a, p}(s, r) = \{ (x, y, z, \bar{z}) : |\text{Im } x| < s, |y| < r^2, ||z||_{a, p} < r, ||\bar{z}||_{a, p} < r \}.
\]

We consider an infinite dimensional \( H = N + P \), where \( P \) is a small perturbation to the parameter dependent normal form

\[
N = \sum_{1 \leq b \leq n} \omega_b(\xi) y_b + \sum_{j \in J^c} \Omega_j(\xi) z_j \bar{z}_j
\]

defined on the phase space \( \mathcal{P}^{a, p} \) with the symplectic structure

\[
\sum_{1 \leq b \leq n} dy_b \wedge dx_b + i \sum_{j \in J^c} dz_j \wedge d\bar{z}_j.
\]
The tangential frequencies \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \) and the normal frequencies \( \Omega = (\Omega_j)_{j \in J^c} \in \mathbb{R}^{J^c} \) are real analytic in \( \xi \in \mathcal{I} \), a compact set in \( \mathbb{R} \). The perturbation term \( P \) is real analytic in the space coordinates and the parameter.

For the frequencies \( \omega, \Omega \) and the perturbation \( P \), we assume (A) and (R) respectively in the following.

(A) There exist \( d \geq 1, 0 < \eta < 1 \) fixed, and functions \( v_j : \mathcal{I} \to \mathbb{R} \) such that
\[
\Omega_j(\xi) = |j|^d + v_j(\xi), \quad j \in J^c,
\]
where each \( v_j(\xi) \) extends to an analytic function on the complex neighborhood of \( \mathcal{I} \)
\[
\mathcal{I}_\eta := \bigcup_{\xi \in \mathcal{I}} \{ \xi' \in \mathbb{C} : |\xi - \xi'| < \eta \} \subset \mathbb{C}.
\]
Also the function \( \omega : \mathcal{I} \to \mathbb{R}^n \) has an analytic extension on \( \mathcal{I}_\eta \). Moreover there exist \( \delta < \min\{1, d - 1\} \) and \( \Gamma \geq 1 \) such that
\[
\sup_{\mathcal{I}_\eta} \sup_{j \in J^c} |(j)^{-\delta} v_j(\xi)| \leq \Gamma, \quad \sup_{\mathcal{I}_\eta} |\omega(\xi)| \leq \Gamma.
\]

(R) The perturbation \( P \) is real analytic in \( \xi \in \mathcal{I}_\eta \). There exist \( s > 0, r > 0 \) such that, for each \( \xi \in \mathcal{I} \), the Hamiltonian vector field \( X_P := (\partial_y P, -\partial_z P, i\partial_z P, -i\partial_z P) \) defines in the neighborhood of \( \mathcal{T}_0 := \mathbb{T}^n \times \{y = 0\} \times \{z = 0\} \times \{\bar{z} = 0\} \) a real analytic map
\[
X_P : D(s, r) \to \mathcal{P}^n_{a, \bar{a}}, \quad \left\{ \begin{array}{l}
\bar{p} \geq p, \text{ for } d > 1 \\
p \geq p, \text{ for } d = 1
\end{array} \right.
\]
with \( p - \bar{p} \leq \delta \) and
\[
\|X_P\|_{r, a, \bar{a}, D(s, r) \times \mathcal{I}_\eta} := \sup_{D(s, r) \times \mathcal{I}_\eta} \|X_P\|_{r, a, \bar{a}} < +\infty.
\]
Moreover, the perturbation is taken from a special class of real analytic functions,
\[
\mathcal{A} = \{ P : P = \sum_{k \in \mathbb{Z}^n, \bar{a} \in \mathbb{N}^{J^c}} P_{k\bar{a}} e^{i(k, x) y + \bar{a} z} \},
\]
where \( \langle k, x \rangle = \sum_{b=1}^n k_b x_b \) and \( k, \alpha, \bar{a} \) has the following relation
\[
\sum_{1 \leq b \leq n} k_b x_b + \sum_{j \in J^c} (\alpha_j - \bar{\alpha}_j) j = 0.
\]
Define
\[
\langle l \rangle_d := \max\{1, \sum_j |j|^d l_j\}
\]
for \( l \in \mathbb{Z}^{J^c} \), and denote the set
\[
\mathcal{X} := \{(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^{J^c} \setminus (0, 0) : |l| \leq 2, \sum_{1 \leq b \leq n} k_b x_b + \sum_{j \in J^c} l_j j = 0\}.
\]
For the normal frequencies \( \Omega(\xi) \), define the norm \( |\Omega|_{-\delta} := \sup_{\xi \in \mathcal{X}} \sup_{j \in J^c} |(j)^{-\delta} \Omega_j| \) and the \( C^\mu_{\mathcal{X}} \)-norm, \( \mu \in \mathbb{N} \) as
\[
|\Omega|_{C^\mu_{\mathcal{X}}_{-\delta}} := \sum_{\mu=0}^{\mu} \left| \frac{d^\mu}{d\xi^\nu} \Omega(\xi) \right|_{-\delta}.
\]
The \( |\cdot|_{C^\mu} \)-norm of \( \omega : \mathcal{I} \to \mathbb{R}^n \) is defined analogously.
Theorem 2.1. (KAM iterative theorem) Consider the Hamiltonian system \( H = N + P \) satisfying (A) and (R). Let \( 9r^2 < \gamma < 1 \) and suppose
\[
\sum_{2i+j_1+j_2=4} \sup_{D(s,r) \times T} |\partial_y^i \partial_{\xi_1}^j \partial_{\xi_2}^l P| \leq \frac{\sqrt{7}}{3r}.
\] (11)
Then there is \( \epsilon_* > 0 \) such that, if the KAM-condition
\[
\epsilon := \gamma^{-1} ||X_P||_{r,a,p,D(s,r) \times T} \leq \epsilon_*
\] (12)
holds, then
1. there exist \( C^\infty \) maps \( \omega^* : \mathcal{I} \rightarrow \mathbb{R}^n, \Omega^* : \mathcal{I} \rightarrow \ell^{-d}_\infty \), satisfying for any \( \mu \in \mathbb{N} \),
\[
|\omega^* - \omega|^C_\mu \leq M(\mu) \epsilon_\gamma^{1-\mu}, \quad |\Omega^* - \Omega|^C_\delta \leq M(\mu) \epsilon_\gamma^{1-\mu}
\] (13)
for some constant \( M(\mu) > 0 \), where \( \ell^{-d}_\infty = \{ z = (z_j)_{j \in J} : \sup_j |z_j|^{-d} < \infty \} \);
2. there exists a smooth family of real analytic torus embedding
\[
\Phi : \mathbb{T}^n \times \mathcal{I}^* \rightarrow \mathbb{R}^{\mu,\bar{P}},
\]
where \( \mathcal{I}^* \) is the Cantor set
\[
\mathcal{I}^* := \left\{ \xi \in \mathcal{I} : |(k, \omega^*(\xi)) + (l, \Omega^*(\xi))| \geq \frac{2\gamma(l)d}{1 + |k|^d}, \forall (k,l) \in \mathcal{X} \right\},
\] (14)
such that, for each \( \xi \in \mathcal{I}^* \), the map \( \Phi \) restricted \( \mathbb{T}^n \times \{ \xi \} \) is an embedding of a rotational torus with frequencies \( \omega^*(\xi) \) for the Hamiltonian system \( H \), close to the trivial embedding \( \mathbb{T}^n \times \mathcal{I} \rightarrow \mathcal{T}_0 \).

Theorem 2.1 is a modification of KAM theorem in [3]. The proof is completely parallel to that in [3], which follows by Theorem 5.1 and Remark 5.1 in [4]. Therefore, the difference between the single and double eigenvalues rely only in proving the measure estimates.

Theorem 2.2. (Measure estimate). Assume that the frequency map \( (\omega, \Omega) \) fulfills condition (A) and non-degeneracy condition (ND):
(1) for any vector \( (c_1, \ldots, c_{n+1}) \in \mathbb{R}^{n+1} \setminus \{ 0 \} \), the function \( c_1 \omega_1 + \cdots + c_n \omega_n + c_{n+1} \) is not identically zero on \( \mathcal{I} \),
(2) for \( (k,l) \in \mathcal{X} \) with \( 0 < |l| \leq 2 \), the function \( k_1 \omega_1 + \cdots + k_n \omega_n + \sum_{j \in J^c} l_j \Omega_j \) is not identically zero on \( \mathcal{I} \).

Take
\[
M(\mu_0) \epsilon_\gamma^{1-\mu_0} \leq \beta/4, \quad M(\mu_0 + 1) \epsilon_\gamma^{-\mu_0} \leq 1,
\] (15)
where \( \mu_0, \beta > 0 \) are defined in (22) and \( M(\mu_0) \) in (13). Then there exist constants \( \tau, \gamma_* > 0, \mu_* \geq \mu_0 \), depending on \( d, \mu_0, \Gamma, \beta, \eta, \delta \) such that
\[
|\mathcal{I} \setminus \mathcal{I}^*| \leq (1 + |\mathcal{I}|) \left( \frac{\gamma}{\gamma_*} \right)^{1/\mu_*}
\]
for all \( 0 < \gamma \leq \gamma_* \).

Theorem 2.2 is a modification of Theorem 1 in [3], and the proof will be given in Section 3.

Remark 1. We do not require \( \Omega_j(\xi) \equiv \Omega_{-j}(\xi) \) in assumption (A). Actually, the case \( \Omega_j(\xi) \equiv \Omega_{-j}(\xi) \) is a special one of (8).
Remark 2. The condition (I) of (ND) is the non-degeneracy condition (i) in Section 1, while the condition (II) of (ND) is the non-degeneracy condition (ii)′ with \((k, l)\) additionally satisfying momentum condition, i.e., (5).

Remark 3. For systems with the zero mode being ruled off, i.e., the index set \(\mathbb{Z}\) being substituted with \(\bar{\mathbb{Z}} := \mathbb{Z} \setminus \{0\}\), Theorem 2.1 and Theorem 2.2 still holds true.

2.2. Application to nonlinear wave equation. The previous results apply to the nonlinear wave equation with periodic boundary conditions

\[
\begin{align*}
\left\{ \begin{array}{l}
u_{tt} - \nu_{xx} + \xi \nu + f(\nu) = 0 \\
u(t, x) = \nu(t, x + 2\pi),
\end{array} \right.
\end{align*}
\]

where the mass \(\xi\) is a real parameter on an interval \(\mathcal{I} := [\xi_1, \xi_2], 0 < \xi_1 < \xi_2\), the nonlinearity \(f(\nu)\) is a real analytic function near \(\nu = 0\) with \(f(0) = f'(0) = 0\).

We choose the index set \(J := \{j_1 < \cdots < j_n\}\) with

\[
|j_b| \neq |j_{b'}|, \quad b \neq b',
\]

such that the tangential frequencies will be different from each other. The linear equation \(\nu_{tt} - \nu_{xx} + \xi \nu = 0\) possesses quasi-periodic solutions

\[
u(t, x) = \sum_{b=1}^{n} B_b \cos(\lambda_j t + \theta_b) e^{ij_b x},
\]

where \(B_b, \theta_b \in \mathbb{R}\), and \(\lambda_j = \sqrt{j^2 + \xi}, \quad j \in \mathbb{Z}\) denote the eigenvectors of \(A := \sqrt{-\partial_{xx} + \xi}\) with periodic boundary conditions.

Theorem 2.3. Under the above assumptions, there exists \(R_* > 0\) such that, for any \(B = (B_1, \cdots, B_n) \in \mathbb{R}^n\) with \(|B| := \sqrt{R} \leq R_*\), there is a Cantor set \(\mathcal{I}^* \subset \mathcal{I}\) with asymptotically full measure as \(R \to 0\), such that, for any \(\xi \in \mathcal{I}^*\), the nonlinear wave equation (16) has a quasi-periodic solution of the form

\[
u(t, x) = \sum_{b=1}^{n} B_b \cos(\lambda_{b} t + \theta_{b}) e^{ij_b x} + o(R),
\]

where \(o(R)\) is small in some analytic norm and \(\lambda_{b} \to \lambda_{j_b}\) as \(R \to 0\).

2.3. Application to nonlinear Schrödinger equation. The previous results also apply to the nonlinear Schrödinger equation with periodic boundary conditions

\[
\begin{align*}
i\nu_t &= -\nu_{xx} + |\nu|^2 \nu \\
u(t, x) &= \nu(t, x + 2\pi),
\end{align*}
\]

where \(i\) is a positive integer. For \(m \in \mathbb{Z}\),

\[
u_m(x, t) = \rho e^{i(mx - \mu_m t)}
\]

are plane wave solutions to equation (18), where \(\rho > 0\) is the amplitude and \(\mu_m := m^2 + \rho^2\) is the frequency. We prove that (18) admits many quasi-periodic solutions near \(\nu_m(x, t)\) corresponding to \(n\)-dimensional invariant tori of an associated infinite dimensional dynamical systems.
The same as in [14] [30], by the gauge invariance of (18), it suffices to assume 
\( m = 0 \). Then (19) becomes

\[ u_0(t) = \rho e^{-i\rho^2 t}. \]

Let \( \rho \) be a real parameter on an interval \( I := [\rho_1, \rho_2], \) \( 0 < \rho_1 < \rho_2 \). We choose the 
index set \( J = \{ j_1 < \cdots < j_n \} \subset \mathbb{Z} \) with

\[ |j_b| \neq |j_{b'}|, \quad b \neq b', \tag{20} \]

such that the tangential frequencies will be different from each other.

**Theorem 2.4.** Under the above assumptions, there exists \( R_* > 0 \) such that, for 
any \( B = (B_1, \cdots, B_n) \in \mathbb{R}^n \) with \( |B| := R \leq R_* \), there is a Cantor set \( \mathcal{I}^* \subset \mathcal{I} \) 
with asymptotically full measure as \( R \to 0 \), such that, for all \( \rho \in \mathcal{I}^* \), the nonlinear 
Schrödinger equation (18) has a quasi-periodic solution of the form

\[ u(x, t) = \tilde{\rho} e^{-i\tilde{\lambda}_0 t} + \sum_{b=1}^n B_b [a_b e^{-i(\tilde{\lambda}_b + \theta_b)} e^{i j_b x} + a'_b e^{i(\tilde{\lambda}_b + \theta_b)} e^{-i j_b x}] + o(R), \tag{21} \]

where \( a_b, a'_b \) are defined in (59), \( \theta_b \in \mathbb{R} \), \( o(R) \) is small in some analytic norm and

\[ \tilde{\rho} \to \rho, \quad \tilde{\lambda}_0 \to \rho^{2t}, \quad \tilde{\lambda}_b \to \sqrt{j_b^2 (j_b^2 + 2t \rho^{2t})} \]

as \( R \to 0 \).

3. **Proof of Theorem 2.2.** The first step is to use the analyticity of the frequencies to transform the non-degeneracy assumption (ND) into a quantitative 
non-resonance property.

**Proposition 1.** Let \( (\omega, \Omega) : \mathcal{I} \to \mathbb{R}^n \times \mathbb{R}^{l''} \) satisfy condition (A) and non-degeneracy 
condition (ND). Then there exist \( \mu_0 \in \mathbb{N} \) and \( \beta > 0 \) such that

\[ \max_{0 \leq k \leq \mu_0} \left| \frac{d^k}{d\xi^k} \left( (k, \omega(\xi)) + (l, \Omega(\xi)) \right) \right| \geq \beta |k| + 1 \tag{22} \]

for all \( \xi \in \mathcal{I}, (k, l) \in \mathcal{X} \).

**Proof.** The method of the proof comes from Proposition 3 in [3]. The authors in 
[3] discussed four cases, i.e., the Kolmogorov condition for \( l = 0 \), the first Melnikov 
condition for \( l = e_i \), the second Melnikov condition for \( l = e_i + e_j \), and the second 
Melnikov condition for \( l = e_i - e_j \). Then they proceeded by contradiction in every 
case separately. We will also prove (22) by contradiction. Due to the double multi-
plicity of normal frequencies, the essential difference compared with [3] is to handle 
the case \( l = e_j - e_{-j} \) additionally. Nevertheless, we will not discuss case by case 
as in [3]. Alternatively, we will give a summarized proof for all \( (k, l) \in \mathcal{X} \) in the 
following.

Proceed by contradiction and assume that for all \( \lambda \in \mathbb{N} \), \( \beta := 1/(\lambda + 1) \), there 
exist \( \xi_\lambda \in \mathcal{I}, (k_\lambda, l_\lambda) \in \mathcal{X} \) such that

\[ \max_{0 \leq \mu \leq \lambda} \left| \frac{d^\mu}{d\xi^\mu} \left( (k_\lambda, \omega(\xi)) + (l_\lambda, \Omega(\xi)) \right) \right| < \frac{|k_\lambda| + 1}{1 + \lambda}, \]

namely, for all \( \mu \geq 0 \), for any \( \lambda \geq \mu \), we have

\[ \left| \frac{d^\mu}{d\xi^\mu} \left( (k_\lambda, \omega(\xi)) + (l_\lambda, \Omega(\xi)) \right) \right| < \frac{|k_\lambda| + 1}{1 + \lambda}. \tag{23} \]

Note that for different \( \lambda \) in (23), \( l_\lambda \) may be in different cases, i.e., \( l_\lambda = 0, e_i, e_i + e_j, e_i - e_j \).
In view of \( \mu = 0 \) in (23), we have
\[
|\langle k_\lambda, \omega(\xi_\lambda) \rangle + \langle l_\lambda, \Omega(\xi_\lambda) \rangle| < \frac{|k_\lambda| + 1}{1 + \lambda}.
\] (24)
In view of \( \sup_{\mathcal{I}_\gamma} |\omega(\xi)| \leq \Gamma \) in assumption (A), we have
\[
|\langle k_\lambda, \omega(\xi_\lambda) \rangle| \leq \Gamma|k_\lambda|.
\] (25)
By (24) (25), we have
\[
|\langle l_\lambda, \Omega(\xi_\lambda) \rangle| < \frac{|k_\lambda| + 1}{1 + \lambda} + \Gamma|k_\lambda|.
\] (26)
In view of (8) and \( \sup_{\mathcal{I}_\gamma} \sup_{j \in \mathcal{J}} |(j)^{-\delta} v_j(\xi)| \leq \Gamma \) in assumption (A), we have
\[
\frac{|\langle l_\lambda, \Omega(\xi_\lambda) \rangle|}{\langle l_\lambda \rangle_d} \rightarrow 1 \quad \text{as} \quad \langle l_\lambda \rangle_d \rightarrow +\infty,
\]
and thus
\[
|\langle l_\lambda, \Omega(\xi_\lambda) \rangle| \geq \langle l_\lambda \rangle_d/2 \quad \text{for} \quad \langle l_\lambda \rangle_d \geq L_s,
\] (27)
where \( L_s \) is a constant bigger than 1. By (26) (27), we obtain
\[
\langle l_\lambda \rangle_d < \min \left\{ L_s, 2\left(\frac{|k_\lambda| + 1}{1 + \lambda} + \Gamma|k_\lambda|\right) \right\},
\] (28)
which means that, if \( k_\lambda \) are bounded, then \( \langle l_\lambda \rangle_d \) are bounded.

We will discuss the case \( d > 1 \) and the case \( d = 1 \) separately in the following.

**Case 1.** \( d > 1 \). Denote \( \lfloor l \rfloor := \max\{|j| : l_j \neq 0\} \) for \( 0 < |l| \leq 2 \) and \( \lfloor l \rfloor := 0 \) for \( l = 0 \). In view of (28), for \( (k_\lambda, l_\lambda) \) with \( l_\lambda \neq e_j - e_{-j} \), we have
\[
|l_\lambda|^{d-1} < C_1(|k_\lambda| + 1)
\] (29)
for some constant \( C_1 > 0 \). For \( (k_\lambda, l_\lambda) \) with \( l_\lambda = e_j - e_{-j} \), we have \( \sum_{1 \leq b \leq n} k_{\lambda,b} j_b + 2j = 0 \) and thus
\[
|l_\lambda| = |j| \leq \frac{1}{2} |k_\lambda| \sup_{1 \leq b \leq n} |j_b|.
\] (30)
By (29) (30), for all \( (k_\lambda, l_\lambda) \), we have
\[
|l_\lambda|^\min\{1,d-1\} < C_2(|k_\lambda| + 1)
\] (31)
for some constant \( C_2 > 0 \). Note that the exponent \( \min\{1, d-1\} > 0 \), and thus (31) trivially holds true for \( |l_\lambda| = 0 \).

By compactness there exist converging subsequences \( \xi_{k_\lambda} \rightarrow \xi \in \mathcal{I} \). We will consider the case \( k_{\lambda_n} \) bounded and the case \( k_{\lambda_n} \) unbounded separately in the following.

**Subcase 1.1.** \( k_{\lambda_n} \) bounded. By (31), \( |l_{\lambda_n}| \) is bounded, and thus the set \( \{l_{\lambda_n}\} \) is finite. Note that for different \( n \), \( l_{\lambda_n} \) may be in different cases, i.e., \( l_{\lambda_n} = 0, e_i, e_i + e_j, e_i - e_j \). By the finiteness of the sets \( \{k_{\lambda_n}\} \) and \( \{l_{\lambda_n}\} \), we can extract a constant subsequence \( \{k\} \) of the sequence \( \{k_{\lambda_n}\} \) and a constant subsequence \( \{l\} \) of the sequence \( \{l_{\lambda_n}\} \) with \( (k, l) \in \mathcal{X} \). Passing to the limit in (23), we get, for any \( \mu \geq 0 \),
\[
\frac{d\mu}{d\xi} \left( \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle \right) = 0.
\]
By analyticity, \( \langle \bar{k}; \omega(\xi) \rangle + \langle \bar{l}, \Omega(\xi) \rangle \) is identically zero on \( \mathcal{I} \). If \( \bar{l} = 0 \), this contradicts the assumption (I) in (ND); otherwise, this contradicts the assumption (II) in (ND).

**Subcase 1.2.** \( k_{\lambda_h} \) unbounded. The quantity \( \frac{k_{\lambda_h}}{1 + |k_{\lambda_h}|} \) converges, up to subsequence, to \( \bar{c} \in \mathbb{R}^n \), with \( 1/2 \leq |\bar{c}| \leq 1 \). For any \( \mu \geq 1 \), by the assumption (A) and Cauchy estimates, there exists \( c_\mu > 0 \) such that

\[
\left| \frac{d^\mu}{d\xi^\mu} \langle l_{\lambda_h}, \Omega(\xi_{\lambda_h}) \rangle \right| \leq c_\mu \sup_{l_{\lambda_h} \neq 0} \langle j \rangle^\delta,
\]

and thus by \( \delta < \min\{1, d-1\} \) and (31), we have

\[
\frac{1}{1 + |k_{\lambda_h}|} \left| \frac{d^\mu}{d\xi^\mu} \langle l_{\lambda_h}, \Omega(\xi_{\lambda_h}) \rangle \right| \to 0 \quad \text{as} \quad h \to \infty.
\]

Multiplying \( \frac{1}{1 + |k_{\lambda_h}|} \) in both sides of (23) and passing to the limit, we get

\[
\frac{d^\mu}{d\xi^\mu} \langle \bar{c}, \omega(\bar{\xi}) \rangle = 0,
\]

that is, all derivatives of \( \langle \bar{c}, \omega(\xi) \rangle \) vanish at \( \bar{\xi} \). Then by analyticity, the function \( \langle \bar{c}, \omega(\xi) \rangle \) is identically equal to some constant on \( \mathcal{I} \). This contradicts the assumption (I) in (ND).

**Case 2.** \( d = 1 \). In view of (28), there exists a constant \( C_3 > 0 \) such that, for \( (k_{\lambda}, l_{\lambda}) \) with \( l_{\lambda} \neq e_i - e_j \),

\[
[ l_{\lambda} ] < C_3 (|k_{\lambda}| + 1), \quad (32)
\]

and for \( (k_{\lambda}, l_{\lambda}) \) with \( l_{\lambda} = e_i - e_j \),

\[
\left| |i| - |j| \right| < C_4 (|k_{\lambda}| + 1). \quad (33)
\]

Note that for different \( \lambda \) in (32), \( l_{\lambda} \) may be in different cases, i.e., \( l_{\lambda} = 0, e_i, e_i + e_j, \) and \( [ l_{\lambda} ] \) is bounded if \( k_{\lambda} \) is bounded. Contrarily, for \( l_{\lambda} = e_i - e_j \), \( [ l_{\lambda} ] \) may be unbounded although \( k_{\lambda} \) is bounded by (33).

By compactness there exist converging subsequences \( \xi_{\lambda_h} \to \bar{\xi} \in \mathcal{I} \). We will discuss different cases according to the boundedness of \( k_{\lambda_h} \) and \( [ l_{\lambda_h} ] \).

**Subcase 2.1.** \( k_{\lambda_h} \) bounded while \( [ l_{\lambda_h} ] \) bounded. The proof is completely parallel to Subcase 1.1.

**Subcase 2.2.** \( k_{\lambda_h} \) bounded while \( [ l_{\lambda_h} ] \) unbounded. In view of (32) and (33), we obtain \( l_{\lambda_h} = e_i_{\lambda_h} - e_j_{\lambda_h} \) with \( |e_{i_{\lambda_h}}| - |j_{\lambda_h}| \) bounded. Then the quantity \( |i_{\lambda_h}| - |j_{\lambda_h}| \) converges, up to subsequence, to \( \bar{m} \in \mathbb{Z} \). Since \( [ l_{\lambda_h} ] \) are unbounded while \( |i_{\lambda_h}| - |j_{\lambda_h}| \) are bounded, we get both \( |i_{\lambda_h}| \) and \( |j_{\lambda_h}| \) are unbounded. In view of \( \delta < 0 \) here, we obtain

\[
\langle l_{\lambda_h}, \Omega_{\lambda_h}(\xi_{\lambda_h}) \rangle \to \bar{m}, \quad \text{as} \quad h \to \infty, \quad (34)
\]

and for any \( \mu \geq 1 \),

\[
\frac{d^\mu}{d\xi^\mu} \langle l_{\lambda_h}, \Omega_{\lambda_h}(\xi_{\lambda_h}) \rangle \to 0, \quad \text{as} \quad h \to \infty. \quad (35)
\]

By the boundedness of \( k_{\lambda_h} \), we extract constant subsequences \( k_{\lambda_h} \equiv \bar{k} \). Therefore passing to the limit in (23), we get, for any \( \mu \geq 0 \),

\[
\frac{d^\mu}{d\xi^\mu} \langle \bar{k}, \omega(\bar{\xi}) \rangle + \bar{m} = 0.
\]
Then by analyticity, the function $\langle k, \omega(\xi) \rangle + \bar{m}$ is identically equal to zero on $\mathcal{I}$. This contradicts the assumption (I) in (ND).

**Subcase 2.3.** $k_{\lambda_h}$ unbounded. The proof is completely parallel to Subcase 1.2. \(\square\)

As a corollary of Proposition 1 and by (13), also the final frequencies $(\omega^*, \Omega^*)$ satisfy a non-resonance property similar to (22).

**Lemma 3.1.** Assume $M(\mu_0)\epsilon \gamma^{1-\mu_0} \leq \beta/4$, where $\mu_0$ and $\beta$ are defined in Proposition 1 and $M(\mu_0)$ is the constant in (13). Then

$$\max_{0 \leq \mu \leq \mu_0} \left| \frac{d\mu}{d\xi} \right| (\langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle) \geq \frac{\beta}{2}(|k| + 1)$$

for all $\xi \in \mathcal{I}, (k, l) \in \mathcal{X}$.

We now proceed with the proof of Theorem 2.2. By (14) we have

$$\mathcal{I} \setminus \mathcal{I}^* \subset \bigcup_{(k, l) \in \mathcal{X}} \mathcal{R}_{kl}(\gamma)$$

with resonant regions

$$\mathcal{R}_{kl}(\gamma) := \left\{ \xi \in \mathcal{I} : \frac{|\langle k, \omega^*(\xi) \rangle + \langle l, \Omega^*(\xi) \rangle|}{1 + |k|} \leq \frac{2\gamma(l)d}{1 + |k|^{\gamma+1}} \right\}.$$  

Assume

$$\begin{cases} \tau > \mu_0(n + \frac{2}{d-1}), & d > 1 \\
\tau > \mu_0(n + 1)(1 - \frac{\mu_0}{\delta}), & d = 1 \end{cases}$$

and let

$$\begin{cases} \mu_* = \mu_0, & d > 1 \\
\mu_* = -\frac{\mu_0(\mu_0 - \delta)}{\delta}, & d = 1 \end{cases}$$

**Lemma 3.2.** There exists a positive constant $\gamma_*$ depending on $d, n, \mu_0, \Gamma, \beta, \eta, \delta$ such that

$$\left| \bigcup_{(k, l) \in \mathcal{X}} \mathcal{R}_{kl}(\gamma) \right| \leq (1 + |\mathcal{I}|)(\frac{\gamma}{\gamma_*})^{\frac{\mu_*}{\mu_0}}.$$  

**Proof.** The proof is parallel to that of Lemma 4 and Lemma 5 in [3]. By Lemma 3.1, we only need to count the number of $\mathcal{R}_{kl}$. As usual, for any fixed $k$, we discuss the number of nonempty $\mathcal{R}_{kl}$ with $(k, l) \in \mathcal{X}$. For $l = 0, e_j, e_i + e_j$ and $e_i - e_j$ with $i \neq \pm j$, the proof is parallel to that in [3]. The key difference lies in the case $l = e_j - e_{-j}$. In this case, we rewrite $\mathcal{R}_{kl}$ as $\mathcal{R}_{kj}$. Then for every $k, j$, the measure of $\mathcal{R}_{kj}$ can be estimated as usual. To count the number of $\mathcal{R}_{kj}$, we use the condition $\sum_{1 \leq b \leq n} k_{b} b + 2j = 0$, which implies that for any fixed $k \in \mathcal{Z}^n$, the number of $\mathcal{R}_{kj}$ is at most 1. Also see the proof of Lemma 4.6 in [25] for details. \(\square\)

4. **Proof of Theorem 2.3.** Introducing $q = \frac{1}{\sqrt{2}}(A^{\frac{1}{2}}u - iA^{-\frac{1}{2}}u_t)$, the equation (16) can be rewritten into the Hamiltonian form

$$q_t = i\frac{\partial H}{\partial q},$$  

$$H = \frac{1}{2} \int_\mathcal{T} (Aq)q dx + \int_\mathcal{T} g \left( A^{-\frac{1}{2}} \left( \frac{q + \bar{q}}{\sqrt{2}} \right) \right) dx,$$

where $g$ is a primitive of $f$. 

Letting
\[ q(x) = \sum_{j \in \mathbb{Z}} q_j \sqrt{\frac{1}{2\pi}} e^{ijx}, \]
then the system (36) is equivalent to the lattice Hamiltonian equations
\[ \dot{q}_j = \frac{\partial H}{\partial \bar{q}_j}, \quad H = \sum_{j \in \mathbb{Z}} \lambda_j q_j \bar{q}_j + G(q, \bar{q}), \]
where
\[ G(q, \bar{q}) = \int g \left( \sum_{j \in \mathbb{Z}} q_j e^{ijx} + \bar{q}_j e^{-ijx} \right) \right) dx = \sum_{\alpha, \bar{\alpha} \in \mathbb{N}^\mathbb{Z}} G_{\alpha \bar{\alpha}} q^\alpha \bar{q}^{\bar{\alpha}} \]
satisfying
\[ G_{\alpha \bar{\alpha}} = 0, \quad \text{if} \quad \sum_{j \in \mathbb{Z}} (\alpha_j - \bar{\alpha}_j) j \neq 0. \]

Next we introduce standard angle-action coordinates
\[(x, y, z, \bar{z}) \in T^n \times \mathbb{R}^n \times \ell^{a,p} \times \ell^{a,p}\]
by letting
\[ q_j = \sqrt{I_b + y_b e^{ix}}, \quad \bar{q}_j = \sqrt{I_b + y_b e^{-ix}}, \quad b = 1, \ldots, n, \]
\[ q_j = z_j, \quad \bar{q}_j = \bar{z}_j, \quad j \in J^c \]
with
\[ I_b \in \left( \frac{r^2 \theta}{2}, r^2 \theta \right], \quad \theta \in (0, 1). \]
Then the Hamiltonian in (39) is given by
\[ H = \langle \omega(\xi), y \rangle + \sum_{j \in J^c} \Omega_j(\xi) z_j \bar{z}_j + P(x, y, z, \bar{z}, \xi) \]
with the symplectic structure \( \sum_{1 \leq b \leq n} dy_b \wedge dx_b + i \sum_{j \in J^c} dz_j \wedge d\bar{z}_j, \) where
\[ \omega_b(\xi) = \lambda_j(\xi), \quad \Omega_j(\xi) = \lambda_j(\xi), \]
and \( P \) is just \( G \) expressed in terms of \( (x, y, z, \bar{z}) \). More precisely, by (40), \( P \) is of the form
\[ \sum_{k \in \mathbb{Z}^n, j \in \mathbb{N}^\mathbb{Z}, \alpha, \bar{\alpha} \in \mathbb{N}^\mathbb{Z}} P_{k \alpha \bar{\alpha}} e^{i(k \cdot x)} y^\alpha z^{\bar{\alpha}} \]
satisfying
\[ P_{k \alpha \bar{\alpha}} = 0 \quad \text{if} \quad \sum_{b=1}^n k_b b + \sum_{j \in J^c} (\alpha_j - \bar{\alpha}_j) j \neq 0, \]
i.e., \( P \in \mathcal{A} \).

Now we verify the conditions of Theorem 2.1 and Theorem 2.2. The assumption (A) holds with \( d = 1, \delta = -1 \) and \( \eta = \xi_1/2 \). Also the assumption (R) holds with \( \bar{p} = p + 1 \). Fix
\[ \theta \in \left( \frac{2}{3}, 1 \right), \quad \gamma := r^\sigma, \quad 0 < \sigma < \frac{3\theta - 2}{\mu_0 + 1} \]
with \( r > 0 \) small enough. Then \( 9r^2 < \gamma < 1 \) and thus

\[
\sum_{2i+j_1+j_2=4}^{D(s,r) \times I_n} \left| \frac{\partial^i \partial^{j_1}_y \partial^{j_2}_{\bar{z}} P}{\sqrt{\gamma}} \right| = O(1) \leq \frac{\sqrt{\gamma}}{3r},
\]

which verifies the condition (11). In view of

\[
\epsilon := \gamma^{-1}||X_P||_{r,p+1, D(s,r) \times I_n} = O\left(\gamma^{-1},3\theta^{-2}\right),
\]

the condition (12) and (15) hold true.

It remains to verify the non-degeneracy condition (ND). By the choice of the index set \( J \) with (17), we have

\[ \omega_b \neq \omega_{b'}, \quad b \neq b'. \]

Then the verification of the first condition (I) of (ND) is completely the same as Lemma 6 in [3] for nonlinear wave equation with Dirichlet boundary conditions. In order to verify the second condition (II) of (ND), we give the following lemma.

**Lemma 4.1.** For every \((k,l) \in \mathcal{X}\) with \( 0 < |l| \leq 2, \ (k, \omega) + (l, \Omega) \) is a non-zero integer linear combination of \( \sqrt{j^2 + \xi}, \ j \geq 0. \)

**Proof.** In view of the definition of \( \omega \) and \( \Omega \) in (41) and the fact \( \lambda_j(\xi) \equiv \lambda_{-j}(\xi) \), it is equivalent to prove that the following combinations of \( (k, \omega) + (l, \Omega) \) do not exist:

\[
(42) \quad \pm (\omega_b - \Omega_{-j_b}), \quad 1 \leq b \leq n, -j_b \in J^c,
\]

\[
(43) \quad \pm (\omega_b + \omega_{b'} - \Omega_{-j_b} - \Omega_{-j_{b'}}), \quad 1 \leq b, b' \leq n, -j_b, -j_{b'} \in J^c,
\]

\[
(44) \quad \pm (\omega_b - \omega_{b'} - \Omega_{-j_b} + \Omega_{-j_{b'}}), \quad 1 \leq b, b' \leq n, -j_b, -j_{b'} \in J^c.
\]

Calculating directly, the momentum \( \sum_{b=1}^{n} k_b j_b + \sum_{j \in J^c} l_j j \) for (42), (43) and (44) equal to \( \pm 2j_b, \pm 2(j_b + j_{b'}) \) and \( \pm 2(j_b - j_{b'}) \) respectively, which are not zero. This contradicts \((k,l) \in \mathcal{X}\).

Finally, the second condition (II) of (ND) follows from Lemma 6 in [3].

5. **Proof of Theorem 2.4.** As in [21], the equation (18) can be rewritten in the Hamiltonian form

\[
H = \int_T |u_x|^2 dx + \frac{1}{t+1} \int_T |u|^{2(t+1)} dx.
\]

Letting

\[
u = \sum_{j \in \mathbb{Z}} u_j(t) \sqrt{\frac{1}{2\pi}} e^{ijx},
\]

the system (45) is equivalent to the lattice Hamiltonian equations

\[
\dot{u}_j = -i \frac{\partial H}{\partial u_j},
\]

\[
H = \sum_{j \in \mathbb{Z}} j^2 u_j \bar{u}_j + \frac{1}{(t+1)(2\pi)^t} \sum_{\sum_{i=1}^{t+1} j_i = \sum_{i=1}^{t+1} k_i} u_{j_1} \cdots u_{j_{t+1}} \bar{u}_{k_1} \cdots \bar{u}_{k_{t+1}}.
\]
Denote $\rho := ||u(0)||_{L^2}$ and assume that the initial datum is concentrated the zero mode. In order to eliminate the zero mode, as in [14] [30], define the symplectic reduction of $u_0$:

$$\{ u_j, \tilde{u}_j \}_{j \in \mathbb{Z}} \rightarrow (\rho^2, \theta_0, \{ v_j, \tilde{v}_j \}_{j \in \mathbb{Z}}),$$

$$u_0 = e^{i\theta_0} \sqrt{\rho^2 - \sum_{j \in \mathbb{Z}} |v_j|^2}, \quad u_j = v_j e^{i\theta_0}, \quad \forall j \in \mathbb{Z},$$  \hspace{1cm} (48)

and then, up to a constant,

$$H = N + P,$$  \hspace{1cm} (49)

where

$$N = \sum_{j \in \mathbb{Z}} \left( j^2 + \frac{j^2 \rho^2}{(2\pi)^2} \right) |v_j|^2 + \frac{j^2 \rho^2}{(2\pi)^2} (v_j v_{-j} + \tilde{v}_j \tilde{v}_{-j})$$

and $P$ is at least three order of $v, \tilde{v}$ with its monomial $v_{k_1} \cdots v_{k_r} \tilde{v}_{l_1} \cdots \tilde{v}_{l_s}$ satisfying $k_1 + \cdots + k_r = l_1 + \cdots + l_s$.

Still as in [14] [30], for $j \in \mathbb{Z}$, denote

$$\lambda_j = \sqrt{j^2 (j^2 + \frac{2j^2 \rho^2}{(2\pi)^2})},$$  \hspace{1cm} (50)

$$S_j = \frac{j^2 \rho^2}{\sqrt{(\lambda_j + j^2) (\lambda_j + j^2 + \frac{2j^2 \rho^2}{(2\pi)^2})}} \left( \lambda_j + j^2 + \frac{j^2 \rho^2}{(2\pi)^2} \right) \left( \lambda_j + j^2 + \frac{j^2 \rho^2}{(2\pi)^2} \right),$$

and set

$$\left( \frac{q_j}{q_{-j}} \right) = S_j \left( \frac{v_j}{v_{-j}} \right).$$  \hspace{1cm} (51)

Then

$$N = \sum_{j \in \mathbb{Z}} \lambda_j |q_j|^2,$$

and $P$ is at least three order of $q, \tilde{q}$ with its monomial $q_{k_1} \cdots q_{k_r} \tilde{q}_{l_1} \cdots \tilde{q}_{l_s}$ satisfying $k_1 + \cdots + k_r = l_1 + \cdots + l_s$.  \hspace{1cm} (52)

Denote $J^c = \mathbb{Z} \setminus J$. Next we introduce standard action-angle coordinates

$$(x, y, z, \bar{z}) \in \mathbb{T}^n \times \mathbb{R}^n \times \ell^a - \ell^p \times \ell^a - \ell^p$$

by letting

$$q_{jb} = \sqrt{I_b + y_b e^{i x_b}}, \quad q_{jb} = \sqrt{I_b + y_b e^{-i x_b}}, \quad b = 1, \ldots, n,$$

$$q_j = z_j, \quad \bar{q}_j = \bar{z}_j, \quad j \in J^c$$  \hspace{1cm} (53)

with

$$I_b \in \left( \frac{\ell^2}{2}, \ell^2 \right], \quad \theta \in (0, 1).$$

Then the Hamiltonian in (49) is given by

$$H = \langle \omega(\xi), y \rangle + \sum_{j \in J^c} \Omega_j(\xi) z_j \bar{z}_j + P(x, y, z, \bar{z}, \xi)$$

with the symplectic structure

$$\sum_{1 \leq b \leq n} dy_b \wedge dx_b + i \sum_{j \in J^c} dz_j \wedge d\bar{z}_j,$$

where

$$\omega_b(\xi) = \sqrt{j^2 b^4 + j^2 b^2 \xi}, \quad \Omega_j(\xi) = \sqrt{j^4 + j^2 \xi}.$$
with
\[ \xi = \frac{2\rho^2}{(2\pi)^2}. \] (55)

More precisely, by (52), \( P \) is of the form
\[ \sum_{k \in \mathbb{Z}^n, l \in \mathbb{N}^n, \alpha, \bar{\alpha} \in \mathbb{N}_c} P_{k\alpha \bar{\alpha}} e^{i\langle k, x \rangle y l \alpha \bar{\alpha}} \]
satisfying
\[ P_{k\alpha \bar{\alpha}} = 0 \quad \text{if} \quad \sum_{b=1}^n k_b j_b + \sum_{j \in J_c} (\alpha_j - \bar{\alpha}_j) j \neq 0. \]

Now we verify the conditions of Theorem 2.1 and Theorem 2.2 with \( \bar{Z} \) instead of \( Z \), seeing Remark 3 in Section 2. The assumption (A) holds with \( d = 2, \delta = 0 \) and \( \eta = \frac{1}{2} \). Also the assumption (R) holds with \( \bar{p} = p \). Fix \( \theta \in \left( \frac{2}{3}, 1 \right), \gamma := r^\sigma, \quad 0 < \sigma < \frac{3\theta - 2}{\mu_0 + 1} \) with \( r > 0 \) small enough. Then \( 9r^2 < \gamma < 1 \) and thus
\[ \sum_{2i+j_1+j_2=4} \sup_{D(s,r) \times I_\eta} |\partial^i_y \partial^{j_1}_\xi \partial^{j_2}_\xi P| = O(1) \leq \frac{\sqrt{\gamma}}{3r} \]
which verifies the condition (11). In view of
\[ \epsilon := \gamma^{-1} |X_P|_{r,p,D(s,r) \times I_\eta} = O(\gamma^{-1} r^{3\theta - 2}), \]
the condition (12) and (15) hold true.

Note that the zero mode here is ruled off, i.e., the index set \( Z \) is substituted with \( \bar{Z} \). By Remark 3, in order to use Theorem 2.1 and Theorem 2.2, it only remains to verify the non-degeneracy condition (ND) with \( \bar{Z} \) instead of \( Z \). Obviously, Lemma 4.1 still holds true with \( \bar{Z} \) instead of \( Z \). Therefore, it is sufficient to prove that, for any \( 0 < j_1 < \cdots < j_L \), for any \((c_0, c_1, \cdots, c_L) \in \mathbb{R}^{L+1} \setminus \{0\}\), the function \( c_0 + c_1 \lambda_{j_1} + \cdots + c_L \lambda_{j_L} \) is not identically zero on \( \bar{Z} \), which is implied by the following lemma.

**Lemma 5.1.** Define
\[
D = \begin{pmatrix}
\frac{d\lambda_{j_1}}{d\xi} & \frac{d\lambda_{j_2}}{d\xi} & \cdots & \frac{d\lambda_{j_L}}{d\xi} \\
\frac{d^2\lambda_{j_1}}{d\xi^2} & \frac{d^2\lambda_{j_2}}{d\xi^2} & \cdots & \frac{d^2\lambda_{j_L}}{d\xi^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^L\lambda_{j_1}}{d\xi^L} & \frac{d^L\lambda_{j_2}}{d\xi^L} & \cdots & \frac{d^L\lambda_{j_L}}{d\xi^L}
\end{pmatrix}.
\]

Then
\[ |\det D| \geq \left( \prod_{r=1}^L \frac{(2r - 3)!!}{2^r} \right) \left( \prod_{i=1}^L j_i^{2L} (j_i^4 + 2j_i^2 \rho_2^2)^{-L+\frac{1}{2}} \right). \] (56)

**Proof.** In view of (50),(55) and \( \rho \in [\rho_1, \rho_2] \) with \( 0 < \rho_1 < \rho_2 \), we get
\[ \lambda_j = \sqrt{j^4 + j^2 \xi}, \quad \xi = \frac{2\rho^2}{(2\pi)^2} \in \left[ \frac{2\rho_1^2}{(2\pi)^2}, \frac{2\rho_2^2}{(2\pi)^2} \right]. \]
By explicit computation one has
\[
\frac{d^r \lambda_j}{d \xi^r} = j_i^{2r} \frac{(2r-3)!!}{2^r} \frac{(-1)^{r+1}}{(j_i^4 + j_i^2 \xi)^{r+\frac{1}{2}}},
\]
where \((-1)!! = 1\). Substituting (57) into the left hands of (56) and using the linearity of the determinant, we get
\[
\det D = \left( \prod_{r=1}^{L} (-1)^{r+1} \frac{(2r-3)!!}{2^r} \right) \left( \prod_{i=1}^{L} j_i^2 (j_i^4 + j_i^2 \xi)^{-\frac{1}{2}} \right)
\]
\[
\times \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\chi_1 & \chi_2 & \cdots & \chi_L \\
\vdots & \vdots & \ddots & \vdots \\
\chi_1^{L-1} & \chi_2^{L-1} & \cdots & \chi_L^{L-1}
\end{vmatrix},
\]
where \(\chi_i = j_i^2 (j_i^4 + j_i^2 \xi)^{-1}\). The last is a Vandermond determinant whose value is given by
\[
\prod_{i\neq j \leq L} (\chi_i - \chi_j).
\]
Now one has
\[
|\chi_i - \chi_j| = |j_i^2 (j_i^4 + j_i^2 \xi)^{-1} - j_j^2 (j_j^4 + j_j^2 \xi)^{-1}|
\geq j_i^2 (j_i^4 + j_i^2 \xi)^{-1} j_j^2 (j_j^4 + j_j^2 \xi)^{-1}
= \chi_i \chi_j,
\]
then (58) is estimated by
\[
\prod_{i=2}^{L} \chi_i = \prod_{i=1}^{L} \chi_i = \prod_{i=1}^{L} \chi_i^{L-1} = \prod_{i=1}^{L} j_i^{2(L-1)} (j_i^4 + j_i^2 \xi)^{-(L-1)},
\]
from which, (56) immediately follows.

By the conclusions of Theorem 2.1, we get quasi-periodic solutions in \(D(s, r)\) with frequencies \(\omega^*\) and \(\Omega^*\). Returning to the coordinates \(q, \bar{q}\) in (53) and (54), we have
\[
q_{j_b} = O(r^b), \quad 1 \leq b \leq n,
q_j = O(r), \quad j \in \mathbb{Z}_*.
\]
Back to the coordinates \(v, \bar{v}\) in (51), for \(1 \leq b \leq n\),
\[
\begin{pmatrix} v_{j_b} \\ \bar{v}_{-j_b} \end{pmatrix} = S_{j_b}^{-1} \begin{pmatrix} q_{j_b} \\ \bar{q}_{-j_b} \end{pmatrix} = \begin{pmatrix} a_b \\ a'_b \end{pmatrix} q_{j_b} + O(r),
\]
where
\[
a_b := \frac{i \rho^{2q} (\lambda_{j_b} + j_b^2 + \frac{i \rho^{2q}}{(2\pi)^q})}{(2\pi)^q \sqrt{(\lambda_{j_b} + j_b^2)(\lambda_{j_b} + j_b^2 + \frac{2 \rho^{2q}}{(2\pi)^q})}},
\]
\[
a'_b := - \frac{i^2 \rho^{4q} j_b}{(2\pi)^q \sqrt{(\lambda_{j_b} + j_b^2)(\lambda_{j_b} + j_b^2 + \frac{2 \rho^{2q}}{(2\pi)^q})}},
\]
satisfy \(a_b^2 - (a'_b)^2 = 1\); for \(j \neq \pm j_b\),
\[
v_j = O(r).
\]
Finally, returning to the coordinates \(u\) in (48), we get (21) in Theorem 2.4.
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