Chiral fermions in two dimensions?

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Quenched studies of a global U(1) symmetric Wilson-Yukawa model in two dimensions show no evidence of a charged fermion in the vortex phase at strong Wilson-Yukawa coupling while there is strong indication of a massive neutral fermion. However, with the U(1) gauge field turned on, we use dimensional arguments to suggest that the neutral fermion appears to couple chirally to a massive vector boson state.

1. INTRODUCTION

The Wilson-Yukawa (W-Y) approach to chiral gauge theories on the lattice uses a gauge-invariant Wilson term, called the W-Y coupling term, which contains radially frozen scalar fields. The proposal fails for weak W-Y coupling $w$; only for strong $w$ the doubler fermions can be decoupled completely by making them heavier than the cut-off. Based on the results of our numerical studies and comparison with available analytic results some of us concluded that the approach still fails because the symmetric phase of the theory at strong $w$ contains only a massive neutral fermion whose couplings vanish as $a^2$.

The present work \cite{1} investigates the same approach in two dimensions. We find similar results as in four dimensions \cite{2}, namely that there does not seem to exist a charged fermion and there are very strong indications for a massive neutral fermion. In two dimensions, however, there is an interesting possibility suggested from naive power counting that the effective interaction of the neutral fermion might survive the continuum limit in the form of a chiral coupling to a vector boson state. We explore this issue and find positive evidence for an effective chiral coupling in two dimensions, although the theory emerges out to be very different from the original target addressed in the W-Y approach.

2. THE MODEL AND ITS PHASE DIAGRAM

The action of a W-Y model in two euclidean dimensions may be written as:

$$S = \sum_x \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_U + \mathcal{L}_\Phi + \mathcal{L}_\Psi,$$

$$\mathcal{L}_\Psi = \frac{1}{2} \sum_{\mu=1}^2 \bar{\Psi} \gamma_{\mu} \left( \left( D^+_{\mu} + D^-_{\mu} \right) P_L + \left( \partial^+_{\mu} + \partial^-_{\mu} \right) P_R \right) \Psi$$

$$+ \frac{w}{2} \left( \overline{\Phi} P_R + \Phi^* P_L \right) \Psi$$

$$- \frac{w}{2} \left( \overline{\Phi} P_R \right) \sum_{\mu=1}^2 \partial^+_{\mu} \partial^-_{\mu} \left( \Phi^* \Psi \right).$$

$\mathcal{L}_U$ gives the usual plaquette action with gauge coupling $g$ and $\mathcal{L}_\Phi$ is the usual lattice lagrangian for radially frozen scalar fields $\Phi_x$ with $\kappa$ as the scalar hopping parameter. $\mathcal{L}_\Psi$ includes the W-Y coupling $w$ and an usual Yukawa coupling $y$. Furthermore $D^+_{\mu} \Psi_x = U_{\mu x}^+ \Psi_{x+\hat{\mu}} - \Psi_x$, $D^-_{\mu} \Psi_x = \Psi_x - U_{\mu x}^- \Psi_{x-\hat{\mu}}$, $\partial^+_{\mu} = D^+_{\mu} |_{U=1}$ and $\partial^-_{\mu} = D^-_{\mu} |_{U=1}$. $P_{R,L} = \frac{1}{2} (\mathbb{1} \pm \gamma_5)$ with $\gamma_5 = -i \gamma_1 \gamma_2$. 

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The action (1) is invariant under the local gauge transformations: \( \Psi_{L,x} \rightarrow \Omega_{L,x} \Psi_{L,x} \)
\( \Psi_{L,x} \rightarrow \tilde{\Phi}_{L,x} \Omega_{L,x}^{*} \)
\( \Phi_{x} \rightarrow \Omega_{L,x} \Phi_{x} \)
\( U_{\mu x} \rightarrow \tilde{U}_{\mu x} \Omega_{L,x}^{*} \Omega_{L,x+\hat{\mu}} \)
with \( \Omega_{L,x} \in U(1)_L \).
It is furthermore invariant under the global transformations:
\( \tilde{\Phi}_{R,x} \rightarrow \Omega_{R} \tilde{\Phi}_{R,x} \)
\( \tilde{\Phi}_{R,x} \rightarrow \tilde{\Phi}_{R,x} \Omega_{R}^{*} \)
and \( \Phi_{x} \rightarrow \Phi_{x} \Omega_{R}^{*} \) with \( \Omega_{R} \in U(1)_R \).

In our numerical simulations, we have used the quenched approximation, and all the calculations involving fermions are performed in the global limit \( g = 0 \) and \( U_{\mu x} = 1 \). In this case the scalar part of the theory reduces to the XY model.

The XY model is known to have a phase transition at \( \kappa = \kappa_c \approx 0.56 \) which separates a vortex (VX) phase \( (\kappa < \kappa_c) \) with finite scalar correlation length from a spin-wave (SW) phase where the scalar correlation length is infinite. In practice, the SW and the VX phase behave on a finite lattice exactly as the broken (FM) and the symmetric (PM) phase of the four dimensional model.

In the quenched approximation \( \kappa_c \) is independent of \( y \) and \( w \). However, as in four dimensions, the fermionic sector has a crossover at \( y + dw \approx \sqrt{d}/2 \), i.e. in \( d = 2 \) at \( y + 2w \approx 1 \) separating weak coupling phases SW(W) and VX(W) from the strong coupling phases SW(S) and VX(S). The behavior of the fermion mass changes across the crossover and in the strong coupling phases, contrary to ‘expectations’, the fermion mass increases as \( \kappa \) decreases. Only in the strong coupling phases the doublers can be decoupled by making them heavier than the cutoff and we perform numerical simulations particularly in the VX(S) phase. (The VX(W) phase does not seem to be present in the full model with dynamical fermions and could be an artefact of the quenched approximation [3]. However, this does not affect our investigations in the VX(S) phase.)

3. FERMION MASSES AND HOPPING PARAMETER EXPANSION

In two dimensions, according to the well-known Mermin-Wagner-Coleman theorem, there cannot be a spontaneous breakdown of a continuous symmetry, the U(1)_L \( \otimes \) U(1)_R chiral symmetry in our case. However, two Dirac fields \( \Psi^{(n)} \) and \( \Psi^{(c)} \) may be constructed which transform vectorially under U(1)_R and U(1)_L respectively:

\[
\Psi^{(n)} = (\Phi^* P_L + P_R) \Psi, \quad \Psi^{(c)} = \Psi (\Phi P_R + P_L),
\]

\[
\Psi^{(c)} = (P_L + \Phi P_R) \Psi, \quad \Psi^{(c)} = \Psi (P_R + \Phi^* P_L).
\]

\( \Psi^{(n)} \) and \( \Psi^{(c)} \) will henceforth be called the neutral and the charged fermion field respectively because of their behavior under the U(1)_L, the group gauged in the action (1). Since these fields transform vectorially, mass terms can be written down for them.

Calculation of neutral and charged fermion propagators form the bulk of this work. First we present results for the propagators from the fermionic hopping parameter expansion (HPE).

To effect the HPE, the fermionic Lagrangian (4) is rewritten in terms of \( \Psi^{(n)} \):

\[
\mathcal{L}_{F} = \sum_{\mu=1}^{2} \left( |(\nabla^{(n)}_{L}) \Phi^{*} | (D_{\mu}^{+} + D_{\mu}^{-}) (\Phi \Psi^{(n)}_{L}) \right) + \mathcal{L}_{F}^{(n)} = \frac{1}{2} \Psi^{(n)} \Psi^{(n)} - \frac{w}{2} \Psi^{(n)} \Psi^{(n)} + \frac{w}{2} \Psi^{(n)} \Psi^{(n)}.
\]

Using the lagrangian (5) one finds, to lowest order in the hopping parameter \( \alpha = 1/(y + 2w) \), an expression for the neutral fermion propagator in momentum space, from which one can read off the fermion masses \( m^{(n)}_{F} \) and \( m^{(D)}_{F} \) for the physical fermion and the species doublers:

\[
m^{(n)}_{F} \approx yz^{-1}, \quad m^{(D)}_{F} \approx yz^{-1} + 2wlz^{-1}, \quad l = 1, 2,
\]

where \( l \) is the number of momentum components equal to \( \pi \) in the two dimensional Brillouin zone. The quantity \( z^2 = \langle \text{Re}(\Phi_{F}^{*} U_{\mu x} \Phi_{x+\hat{\mu}}) \rangle \) has a non-vanishing value in both VX and SW phases.

Writing the action first in terms of the charged fermion and then doing HPE for the charged fermion propagator, similar formulas can be obtained for the masses of the charged fermion (assuming it exists for the moment) and its species doublers:

\[
m^{(c)}_{F} \approx (y + 4w)z^{-1} - 4wz, \quad m^{(c)}_{D} \approx m^{(c)}_{F} + 2wlz.
\]
4. COMPARISON OF THE NUMERICAL RESULTS WITH THE HPE

We compute numerically the neutral and the charged fermion propagators on equilibrated scalar field configurations and analyze them in terms of the free Wilson fermion ansatz to extract the rest energies $E_F^{(n)}$ and $E_F^{(c)}$ of the neutral and the charged fermion and the corresponding rest energies $E_D^{(n)}$ and $E_D^{(c)}$ for the lowest lying doubler fermions. In fig. 1 we have displayed the numerical values for $E_F^{(n)}$, $E_D^{(n)}$, $E_F^{(c)}$ and $E_D^{(c)}$ as a function of $y$ for $\kappa = 0.4$ and $w = 2.0$, which is well inside the VXS phase. The dashed, solid, dash-dotted and dotted lines correspond respectively to the HPE results for the rest energies $E_F^{(n)}$, $E_D^{(n)}$, $E_F^{(c)}$ and $E_D^{(c)}$, as obtained from formulas (6) and (7) and then using lattices dispersion relations. The figure shows that the agreement between the numerical result and the analytic prediction is quite impressive for the rest energies $E_F^{(n)}$ and $E_D^{(n)}$ while the HPE curves for $E_F^{(c)}$ and $E_D^{(c)}$ exhibit a strong deviation from the numerical results. In the case of $E_F^{(c)}$ the deviation is larger than a factor two. The figure shows furthermore that $E_F^{(n)}$ appears to vanish in the limit $y \to 0$, in agreement with the GP shift symmetry mentioned before, whereas $E_D^{(n)}$ stays above 1 for all values of $y$ which implies the decoupling of the species doublers of the neutral fermion in the continuum limit. We refer the reader to ref. [1] for a full account of the numerical results.

5. EFFECTIVE THEORY AT STRONG W-Y COUPLING $w$

The excellent agreement of the numerical data with the HPE predictions for the neutral fermions suggests that the physics in the strong coupling region is well described by the lagrangian (1) in terms of the neutral fermion fields. The charged fermion fields $\psi^{(c)} = \Phi \psi^{(n)}$ and $\bar{\psi}^{(c)} = \bar{\psi}^{(n)} \Phi^*$ can then be regarded as composite fields and the charged fermion, provided it exists at all in the particle spectrum, is to be considered as a bound state with binding energy $\epsilon_B$ defined as

$$E_F^{(c)} = E_F^{(n)} + E_{\Phi} + \epsilon_B ,$$

where $E_{\Phi}$ is the rest energy of the scalar.

Table 1

Various rest energies and $\epsilon_B$ in dependence of $L$ on lattices $L \times 64$ at the point $(\kappa, y, w) = (0.45, 0.3, 2.0)$. The errors of $E_F^{(n)}$ are negligible.

| $L$   | $E_{\Phi}$ | $E_F^{(n)}$ | $E_F^{(c)}$ | $\epsilon_B$ |
|-------|------------|-------------|-------------|--------------|
| 16    | 0.126(3)   | 0.275       | 0.368(13)   | -0.033(16)   |
| 32    | 0.115(3)   | 0.277       | 0.374(9)    | -0.018(12)   |
| 48    | 0.120(5)   | 0.276       | 0.379(12)   | -0.017(17)   |
| 64    | 0.119(5)   | 0.277       | 0.389(11)   | -0.007(16)   |

Table 1 shows the numerically obtained values of the various rest energies and the binding energy at a point in parameter space inside the VXS(S) phase for various $L$ on $L \times 64$ lattices. The binding energy is clearly very small and almost compatible with zero even on the smaller lattices. Furthermore there is a systematic trend of $|\epsilon_B|$ to decrease when $L$ increases. This along with many other similar pieces of evidence in ref. [1] indicate that the signal detected in the charged fermion propagators can very well be just a two particle
state of the neutral fermion and the scalar particle.

For $g > 0$ the VX phase is expected to turn into a confinement phase where scalar particles confine into massive bosonic particles. Writing the effective gauge field combination $U'_{\mu x} = \Phi^*_x U_{\mu x} \Phi_{x+\bar{\mu}}$ in (3) in the standard fashion: $U'_{\mu x} = z^2 + H_{\mu x} + iW_{\mu x}$, where $H_x$ and $W_{\mu x}$ are interpolating fields for the scalar and vector bosonic bound states in the confinement phase and $m_H$ is some mass scale (introduced for dimensional reason) and after a trivial rescaling of the fields $\Psi_L^{(n)}$ and $\overline{\Psi}_L^{(n)}$ we obtain for $\mathcal{L}_F$ the form

\[
\mathcal{L}_F = \frac{1}{2} \sum_{\mu=1}^2 \left[ \overline{\Psi}_L^{(n)} \gamma_{x+\bar{\mu}} \gamma_{x+\bar{\mu}} \Psi_L^{(n)} - \overline{\Psi}_L^{(n)} \gamma_{x+\bar{\mu}} \gamma_{x+\bar{\mu}} \Psi_L^{(n)} \right]
+ \frac{g}{z} \overline{\Psi}_L^{(n)} \psi^{(n)} - \frac{w}{2z} \overline{\Psi}_L^{(n)} \sum_{\mu=1}^2 \partial_\mu \partial_{\bar{\mu}} \psi^{(n)}
+ \frac{m_H}{z^2} \frac{1}{2} \sum_{\mu=1}^2 H_x \overline{\Psi}_L^{(n)} \gamma_{x+\bar{\mu}} \Psi_L^{(n)}
- \frac{\overline{\Psi}_L^{(n)} \gamma_{x+\bar{\mu}} \gamma_{x+\bar{\mu}} \Psi_L^{(n)}}{z^2}
+ \frac{1}{z^2} \sum_{\mu=1}^2 iW_{\mu x} \overline{\Psi}_L^{(n)} \gamma_\mu \Psi_L^{(n)}
+ \frac{\overline{\Psi}_L^{(n)} \gamma_{x+\bar{\mu}} \gamma_{x+\bar{\mu}} \Psi_L^{(n)}}{z^2}
\]

In the above (3) describes a free neutral fermion with mass $m_f^{(n)} = \frac{y}{z}$, (10) suggests that the coupling of the neutral fermion to the Higgs-like bound state vanishes like $a$. However, in (11) the neutral fermion couples chirally to the vector boson field $W_{\mu x}$ if its dimension is one, as suggested by the naive dimensional analysis.

In order to find out whether the fields $W_{\mu x}$ and $H_{\mu x}$ are indeed dimension one operators we have computed numerically the scale dependence of the corresponding wave-function renormalization constants $Z_H$ and $Z_W$ in the gauge-Higgs system. Fig. 2 collects our results and displays strong support for the naive dimensionalities.

The fermion couplings in the VXS phase may then be summarized qualitatively by the following effective lagrangian

\[
\mathcal{L}_{F}^{eff} = \overline{\Psi}_L^{(n)} \gamma_{\mu} \psi^{(n)} + m_f^{(n)} \overline{\Psi}_L^{(n)} \psi^{(n)}
\]

Figure 2. The wave-function renormalization constants $Z_W$ (squares) and $Z_H$ (circles) are plotted respectively as a function of $m^2 = a^2 m^2_{W,phys}$ and $m^2 = a^2 m^2_{H,phys}$ for the fixed ratio $m_H/m_W = 1.14$. The dotted lines are to guide the eye.

where $g_R = \sqrt{Z_W/z^2}$, $\psi^{(n)}_L$ etc. are continuum fermionic fields and $W_{\mu}^{(c)}$ is the vector field in the continuum with standard normalization.

Although the original target of the W-Y approach is not achieved, we end by noting that in two dimensions the neutral fermion exhibits indeed a non-vanishing chiral coupling to the massive vector boson; however, in the global limit there does not seem to be a strong enough $\Psi^{(n)}_{\mu}$-Φ interaction to produce a charged fermion.

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