Electrically neutral Dirac particles in the presence of external fields: exact solutions

German V. Shishkin
Department of Theoretical Physics, Byelorussian State University
Minsk 220050, Byelorussia

Víctor M. Villalba
Centro de Física
Instituto Venezolano de Investigaciones Científicas, IVIC
Apdo 21827, Caracas 1020-A, Venezuela

Abstract

In the present article we present exact solutions of the Dirac equation for electric neutral particles with anomalous electric and magnetic moments. Using the algebraic method of separation of variables, the Dirac equation is separated in cartesian, cylindrical and spherical coordinates, and exact solutions are obtained in terms of special functions.
I. INTRODUCTION

The experimental data on the existence of anomalous magnetic moment and electric dipole moment for the Dirac particle require for their complete understanding an exact description of the corresponding single-particle states, i.e., exact solutions of the Dirac equation with non-minimal interactions. In some physical situations the interaction between the anomalous magnetic moment and electric dipole moments with the electromagnetic tensor can be solved exactly.

In fact, if for the electron and the muon (and apparently also for the $\tau$-lepton) the deviation of the magnetic moment from the Bohr’s magneton and the value of the electric dipole moment are extremely small, on the other hand, for the neutrino ($\nu_e, \nu_\mu, \nu_\tau$) despite the magnetic moment is very small it is one of their nonzero characteristic parameters. Also, we have that for the hadrons the values of the normal and anomalous magnetic moments are of the same order of magnitude. For all the hadrons with large mean life it has been established, when experiments make it possible, the presence of electric dipole moment. For electrically neutral hadrons (neutron, $\Lambda, \Sigma^0, \Xi^0$ hyperons) the interaction between the anomalous magnetic moment with the external field becomes significant.

Among the above mentioned particles, the neutron plays a privileged position because of their physical applications. In fact, one of the most crucial problems of the present century is the security of the nuclear energy. One of the most important ways for analyzing the kinematic parameters of a nuclear reactor is the injection of neutrons and the subsequent measurement of the decay on time of the net population inside the reactor. The mathematical models for this problem are still primitive: the momentum is replaced by a time dependent delta function, and in general, the space distribution of neutrons in impulse is unknown for the experimenter. Since neutron beams are controlled by means of magnetic and electric fields, in order to understand the space distribution of neutrons it would be of help to know the corresponding wave equations, i.e, the solutions of the Dirac equation for the neutron.
In view of all the above mentioned, we will examine in the present article exact solutions of the Dirac equation for electrically neutral particles with non minimal interaction with an external electromagnetic field, that is, we analyze the interaction between anomalous magnetic and electric dipole moments with the electromagnetic tensor.

The amount of articles devoted to the study of this problem is scarce, the reason is that, for solving the Dirac equation, as general rule, a complete separation of variables is required and, the inclusion of tensor field functions in the Dirac equation dramatically restricts the possibilities of separation of variables. In this case, the possibilities are even more limited that those obtained for the Dirac equation minimally coupled to the electromagnetic field, or in the presence of gravitational fields. This one is because the gravitational field goes into the Dirac equation in a geometric way, via the Lame’s coefficients, and no additional matrices in the equation if we choose to work in a diagonal tetrad gauge. Regarding the vector electromagnetic fields, the inclusion of them in the Dirac equation via the generalized momentum (minimal coupling), does not conduce to the apparition of new matrices, and the presence of tensor fields terms of the form $g\gamma^m\gamma^n T_{mn}$ introduces in the Dirac equation a new functional dependence. The difficulties that arise are those we find when we include non geometrized fields like scalar, pseudo-scalar or pseudo-vector fields.

Now, we proceed to mention the most relevant exact solutions of the Dirac equation with anomalous moment reported in the literature, among them we have the constant magnetic field problem solved by Strocchi and by Ternov et al., a generalization of the Volkov’s problem was found by Ternov et al., and more recently this problem has been revisited by Barut. The problem in spherical coordinates for the central field has been studied by Ternov and Barut, a good review on exact solutions of the Dirac equation with anomalous interaction can be found in.

In the present article we will discuss this problem from a unified point of view based on the algebraic method of separation of variables, which consists in reducing the original Dirac equation to a sum of two commuting differential operators as follows.
\[
\left\{ \hat{K}_1 + \hat{K}_2 \right\} \Phi = 0, \quad [\hat{K}_1 \hat{K}_2] = 0
\] (1.1)

where each operator depends on a different set of variables. Then, we reduce the problem of solving the original equation to obtain solutions for \( \hat{K}_1 \Phi = \lambda \Phi = -\hat{K}_2 \Phi \) where \( \lambda \) is a constant of separation. Now, applying the scheme (1.1) for \( \hat{K}_1 \Phi = \lambda \Phi \) and \( \hat{K}_2 \Phi + \lambda \Phi = 0 \) and iterating this process, we are able to reduce a system coupled partial differential equations to four systems of ordinary differential equations. Regrettably, not always a complete separation is possible because of the structure of the metric functions and the form of the external fields.

The article is structured as follows, in Sec II, the Dirac equation with anomalous electric and magnetic moments is separated and some exact solutions are exhibited. In Sec. III, the Dirac equation is separated in cylindrical and spherical coordinates, also, some exact solutions in terms of special functions are presented. In Sec. IV, we present a particular configuration where Dirac equation is separable and exact solutions are obtained. Finally, in the conclusions, we summarize and discuss our results.

## II. CARTESIAN COORDINATES

In this section we proceed to separate variables and to find exact solutions of the Dirac equation with anomalous electric and magnetic moments when this interaction can be written in Cartesian coordinates. Then, for a neutral particle interacting with a magnetic field, Dirac equation in Cartesian coordinates reads

\[
\left\{ \gamma^0 \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z + m + g \epsilon_{ijk} \gamma^i \gamma^j H_k \right\} \Psi = 0
\] (2.1)

Applying the algebraic method of separation of variables\( ^6,^7 \), we can write Eq. (2.1) as follows

\[
\left( \hat{K}_{xy} + \hat{K}_{zt} \right) \Phi = 0, \quad \hat{K}_{xy} \Phi = \lambda \Phi = -\hat{K}_{zt} \Phi
\] (2.2)

with,

\[
\hat{K}_{xy} = \gamma^2 \partial_x - \gamma^1 \partial_y + \gamma^1 \gamma^2 m - g H_z(x, y),
\] (2.3)
\[
K_{zt} = \gamma^1\gamma^2\gamma^3\partial_z + \gamma^1\gamma^2\gamma^0\partial_0 - g\tilde{H}(z,t),
\] 

(2.4)

and

\[
\Phi = \gamma^1\gamma^2\Psi
\] 

(2.5)

where the magnetic vector \(H_z\) can be written as,

\[
H_z = H_z(x,y) + \tilde{H}_z(z,t)
\] 

(2.6)

applying the condition \(\nabla \cdot H = 0\), on (2.6) we find that \(\tilde{H}_z = \tilde{H}_k(t)\), and if we fix our attention to time independent fields this term can be omitted. In this way, \(\lambda\Phi = -\dot{K}_{zt}\Phi\) becomes in an algebraic equation that establishes the relationship between the different spinor components of \(\Phi\) and \(\Psi\), and the value of the constant of separation \(\lambda\), which satisfies the relation \(\lambda^2 = k_z^2 - E^2\).

Choosing to work in the following representation for the gamma matrices

\[
\begin{align*}
\gamma^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^0 &= \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},
\end{align*}
\] 

(2.7)

we have that,

\[
\left\{ \gamma^2\partial_1 - \gamma^1\partial_2 + m\gamma^1\gamma^2 - gH_z(x) - \lambda \right\} \Phi = 0
\] 

(2.8)

takes the form

\[
(\partial_x - gH_z(x) - \lambda)\Phi_{1,2} - (m + ik_y)\Phi_{4,3} = 0
\] 

(2.9)

\[
(\partial_x + gH_z(x) + \lambda)\Phi_{4,3} - (m - ik_y)\Phi_{1,2} = 0
\] 

(2.10)

on the other hand, from \(\lambda\Phi = -\dot{K}_{zt}\Phi\) we find,

\[
\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ (i\lambda - E)/k_z\Phi_1 \\ (i\lambda - E)/k_z\Phi_4 \\ \Phi_4 \end{pmatrix}
\] 

(2.11)
the coupled system of equations (2.9) and (2.10) can be solved in terms of special functions for some values of the magnetic field $H_z(x)$. Among them we have,

(a) $H_z = \beta$,  
(b) $H_z = \beta x$,  
(c) $H_z = \beta/x$,  
(d) $H_z = \beta \exp(\eta x)$,  

(2.12)

first, let us consider the simplest case of a constant magnetic field (a) $H_z = \beta$. Then, the solution of the system reads:

$$
\Phi_1 = c_1 \exp(\sqrt{m^2 + k_y^2 + (g\beta + \lambda)^2 x}) + c_2 \exp(-\sqrt{m^2 + k_y^2 + (g\beta + \lambda)^2 x})
$$  

$$
\Phi_2 = c_1 \frac{\sqrt{m^2 + k_y^2 + (gH - \lambda)^2 - gH + \lambda}}{m + ik_y} \exp(\sqrt{m^2 + k_y^2 + (gH - \lambda)^2 x}) - c_2 \frac{\sqrt{m^2 + k_y^2 + (gH - \lambda)^2 + gH + \lambda}}{m + ik_y} \exp(\sqrt{m^2 + k_y^2 + (gH - \lambda)^2 x})
$$  

(2.13)  

(2.14)

(b) $H_z = \beta x$ For this linear magnetic field we have that the system of equations (2.9) (2.10) can be reduced to the following two second order differential equations

$$
[(\partial_x + g\beta x + \lambda)(\partial_x - g\beta x - \lambda) - (m^2 + k_y^2)]\Phi_1 = 0
$$  

$$
[(\partial_x - g\beta x - \lambda)(\partial_x + g\beta x + \lambda) - (m^2 + k_y^2)]\Phi_4 = 0
$$  

(2.15)  

(2.16)

after making the change of variables

$$
g\beta x + \lambda = \left(\frac{g\beta}{2}\right)^{1/2} y
$$  

(2.17)

we have that the equations (2.13) and (2.16) take the form:

$$
\left\{ \frac{d^2}{dy^2} - \frac{y^2}{4} - \frac{g\beta + m^2 + k_y^2}{2g\beta} \right\}\Phi_1 = 0
$$  

$$
\left\{ \frac{d^2}{dy^2} - \frac{y^2}{4} - \frac{-g\beta + m^2 + k_y^2}{2g\beta} \right\}\Phi_4 = 0
$$  

(2.18)  

(2.19)

the solution of the parabolic cylinder equation (2.18) can be written in terms of confluent hypergeometric functions $M(a, b, z)$.
\[
\Phi_1 = c_1 \exp(-y^2/4)M\left(\frac{1}{2} + \frac{m^2 + k_y^2}{4g\beta}, \frac{1}{2}, \frac{y^2}{2}\right) + c_2 \exp(-y^2/4)M\left(1 + \frac{m^2 + k_y^2}{4g\beta}, \frac{3}{2}, \frac{y^2}{2}\right)
\]

(2.20)

using (2.9) and the recurrence relation

\[
(b - 1)M(a - 1, b - 1, z) = (b - 1 - z)M(a, b, z) + zM'(a, b, z)
\]

(2.21)

we obtain

\[
\delta \Phi_4 = c_1 y \exp(-y^2/4)M\left(\frac{1}{2} + \frac{m^2 + k_y^2}{4g\beta}, \frac{3}{2}, \frac{y^2}{2}\right) + c_2 \exp(-y^2/4)M\left(\frac{m^2 + k_y^2}{4g\beta}, \frac{1}{2}, \frac{y^2}{2}\right)
\]

(2.22)

where the constant \(\delta\) reads,

\[
\delta = \frac{m + ik_y}{\sqrt{2}(g\beta)^{1/2}}
\]

(2.23)

The third case corresponds to the magnetic field \(c) H_z = \frac{\beta}{x}\). Substituting this expression into (2.9) and (2.10) we obtain

\[
\begin{align*}
\left\{ \frac{d^2}{dx^2} + \frac{2g\beta\lambda}{x} - \frac{g\beta(g\beta+1)}{x^2} - k_y^2 - \lambda^2 - m^2 \right\} \Phi_4 &= 0 \\
\left\{ \frac{d^2}{dx^2} - \frac{2g\beta\lambda}{x} - \frac{g\beta(g\beta+1)}{x^2} - k_y^2 - \lambda^2 - m^2 \right\} \Phi_1 &= 0
\end{align*}
\]

(2.24)

(2.25)

the solutions of (2.24) and (2.25) also can be expressed in terms of confluent hypergeometric functions as follows,

\[
\Phi_4 = \exp(-y^2/2) \left\{ a_0 y^{g\beta+1} M(1 + g\beta - \tilde{k}, 2 + 2g\beta, y) + a_1 y^{-g\beta} M(-g\beta - \tilde{k}, -2g\beta, y) \right\}
\]

(2.26)

\[
\Phi_1 = \exp(-y^2/2) \left\{ b_0 y^{g\beta} M(g\beta - \tilde{k}, 2g\beta, y) + b_1 y^{-g\beta} M(1 - g\beta - \tilde{k}, 2 - 2g\beta, y) \right\}
\]

(2.27)

where

\[
y = 2\sqrt{\lambda^2 + k_y^2 + m^2} x, \quad \tilde{k} = -\frac{g\beta\lambda}{\sqrt{\lambda^2 + k_y^2 + m^2}}
\]

(2.28)
and the coefficients $a_0$, $a_1$, $b_0$ and $b_1$ satisfy the following relations

$$\frac{a_1}{b_1} = \frac{2i}{k_y - im}(1 - 2g\beta)\sqrt{\lambda^2 + k_y^2 + m^2}$$

(2.29)

$$\frac{a_0}{b_0} = \frac{(ik_y - m)}{2(2g\beta + 1)\sqrt{\lambda^2 + k_y^2 + m^2}}$$

(2.30)

finally, we have the exponential depending magnetic field given by $(d) H_z = \beta \exp(\eta x)$. Here we have that, after making the change of variables $\mu = \exp(\eta x)$, the system of equations (2.9) (2.10) takes the form

$$(ik_y + m)\Phi_4 - (\eta \mu \partial_\mu - (\lambda + \beta \mu)\Phi_1 = 0$$

(2.31)

$$(ik_y - m)\Phi_1 + (\eta \mu \partial_\mu + (\lambda + \beta \mu)\Phi_4 = 0$$

(2.32)

the system of equations (2.31) (2.32) is similar to the one obtained when we solve the Dirac equation for an electron in a magnetic field like $d$. Then we have that,

$$\Phi_4 = \exp(-\frac{\beta \mu}{\eta}a_0\mu i\frac{2i}{k_y - im}\frac{\lambda}{\eta} + \frac{i k_x}{\eta} + \frac{2 i k_x}{\eta} + 1, \frac{2 \beta \mu}{\eta})$$

+ \exp(-\frac{\beta \mu}{\eta}a_1\mu^{-i\frac{2i}{\eta}M}(\frac{\lambda}{\eta} - \frac{i k_x}{\eta} + \frac{2 i k_x}{\eta} + 1, \frac{2 \beta \mu}{\eta})$$

(2.33)

$$\Phi_1 = ia_0\frac{i k_x + \lambda}{k_y + im}\exp(-\frac{\beta \mu}{\eta}a_0\mu i\frac{2i}{k_y + im}\frac{\lambda}{\eta} + \frac{i k_x}{\eta} + \frac{2 i k_x}{\eta} + 1, \frac{2 \beta \mu}{\eta})$$

- ia_1\frac{i k_x - \lambda}{k_y + im}\exp(-\frac{\beta \mu}{\eta}a_1\mu^{-i\frac{2i}{\eta}M}(\frac{\lambda}{\eta} - \frac{i k_x}{\eta} + \frac{2 i k_x}{\eta} + 1, \frac{2 \beta \mu}{\eta})$$

(2.34)

### III. EXACT SOLUTIONS IN CYLINDRICAL AND SPHERICAL COORDINATES

In this section we are going to find some exact solutions of the Dirac equation with anomalous moment in cylindrical and spherical coordinates. The Dirac equation in spherical coordinates for a chargeless particle with anomalous electric moment, expressed in the diagonal tetrad gauge, reads,
\[
\left\{ \gamma^0 \partial_0 + \gamma^1 \partial_x + \frac{\gamma^2}{r} \partial_\vartheta + \frac{\gamma^3}{r \sin \vartheta} + m + g \gamma^1 \gamma^0 E \right\} \Psi = 0 \quad (3.1)
\]

Substituting into equation (3.1) the electric field given \( E \) by the expression \( E = \epsilon/r \), and separating the angular variables from the radius and time, we have

\[
\begin{align*}
\left[ \gamma^0 \partial_r + \gamma^1 \partial_t + m \gamma^1 \gamma^0 + \frac{g \epsilon}{r} + \frac{k}{r} \right] \Phi &= 0 \quad (3.2) \\
\left[ \gamma^2 \partial_\vartheta + \frac{\gamma^3}{\sin \vartheta} \partial_\varphi \right] \gamma^1 \gamma^0 \Phi &= k \Phi \quad (3.3)
\end{align*}
\]

where \( k \) is a constant of separation associated with the angular dependence. It is worth noting that the operator (3.3) correspond to the one obtained in the separation of variables for the Dirac equation in the Schwarzschild metric when we work in the diagonal tetrad gauge. The spinor \( \Phi \) is related to \( \Psi \) by means of the expression, \( \Phi = \gamma^1 \gamma^0 \Psi \).

Working in the standard representation of the gamma matrices,

\[
\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3 \quad (3.4)
\]

and substituting (3.4) into (3.2) we obtain,

\[
\begin{align*}
\left[ \frac{d}{dr} + \frac{i(g \epsilon + k)}{r} \right] \Phi_1 - (m - E) \Phi_4 &= 0 \quad (3.5) \\
\left[ \frac{d}{dr} - \frac{i(g \epsilon + k)}{r} \right] \Phi_4 + (m + E) \Phi_1 &= 0 \quad (3.6)
\end{align*}
\]

where the constant \( E \) is the eigenvalue of the operator \( i \partial_t \). After substituting (3.3) into (3.6) we obtain a second order differential equation given by the expression

\[
\left[ \frac{d^2}{dr^2} + \frac{(g \epsilon + \kappa)^2 - i(g \epsilon + \kappa)}{r^2} + m^2 - E^2 \right] \Phi_1 = 0 \quad (3.7)
\]

whose solution can be expressed in terms of \( Z_\nu(z) = \alpha J_\nu(z) + \beta N_\nu(z) \) where \( J_\nu(z) \) and \( N_\nu(z) \) are the Bessel and Neumann functions respectively, and \( \alpha \) and \( \beta \) are arbitrary constants,
\( \Phi_1 = a_0 r^{1/2} Z_{1/2 - i(g \epsilon + k)}(\sqrt{m^2 - E^2} x) \) (3.8)

then, substituting (3.8) into (3.5) we obtain,

\( \Phi_4 = \frac{\sqrt{m^2 - E^2}}{m - E} a_0 r^{1/2} Z_{1/2 - i(g \epsilon + k)}(\sqrt{m^2 - E^2} x) \) (3.9)

here, it is worth noting that, given the matrix structure of eq. (3.2), the spinor components \( \Phi_1 \) and \( \Phi_4 \) are proportional to \( \Phi_2 \) and \( \Phi_3 \) respectively.

Now we are going to solve Eq. (3.1) when a constant electric field \( E = \epsilon \) along the radial direction is present. In this case, the set of separated equations reads:

\[
\begin{align*}
\left[ \frac{d}{dr} + ig \epsilon + \frac{ik}{r} \right] \Phi_1 - (m - \epsilon) \Phi_4 &= 0 \quad (3.10) \\
\left[ \frac{d}{dr} - ig \epsilon - \frac{ik}{r} \right] \Phi_4 + (m + \epsilon) \Phi_1 &= 0 \quad (3.11)
\end{align*}
\]

the solution of the coupled system of equations (3.10)-(3.11) can be written by the help of the Bessel functions,

\[
\Phi_1 = cr^{1/2} Z_{1/2 - i \omega}(\sqrt{m^2 - E^2} r) \quad (3.12)
\]

\[
\frac{m - \epsilon}{\sqrt{m^2 - E^2}} \Phi_4 = cr^{1/2} Z_{1/2 - i \omega}(\sqrt{m^2 - E^2} r) \quad (3.13)
\]

The Dirac equation in cylindrical coordinates expressed in the diagonal tetrad gauge takes the form,

\[
\left\{ \gamma^0 \partial_t + \gamma^1 \partial_r + \frac{\gamma^2}{r} \partial_\theta + \gamma^3 \partial_z + m + g \gamma^1 \gamma^2 H \right\} \Psi = 0 \quad (3.14)
\]

where we have included a term associated with the anomalous magnetic moment. Eq. (3.14) can be written as a sum of two commuting differential operators:

\[
\left( \gamma^2 \partial_r - \frac{\gamma^1}{r} \partial_\theta + m \gamma^1 \gamma^2 - g H + \lambda \right) \Phi = 0 \quad (3.15)
\]

\[
\left( \gamma^1 \gamma^2 \gamma^3 \partial_z + \gamma^1 \gamma^2 \gamma^0 \partial_\theta \right) \Phi = \lambda \Phi, \quad \Psi = \gamma^1 \gamma^2 \Phi \quad (3.16)
\]
here, it is convenient to work in the following representation of the Dirac matrices,

\[
\gamma^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\] (3.17)

then, after substituting (3.17) into (3.15) we arrive at,

\[
(\sigma^2 \partial_r - \frac{\sigma^1}{r} \partial_\theta + im\sigma^3 - gH + \lambda)\Theta = 0 \quad (3.18)
\]

\[
(-\sigma^2 \partial_r + \frac{\sigma^1}{r} \partial_\theta + im\sigma^3 - gH + \lambda)\chi = 0 \quad (3.19)
\]

where, taking into account (3.16) we have

\[
\Phi = \begin{pmatrix} \Theta \\ \chi \end{pmatrix} = \begin{pmatrix} \Theta \\ \Theta_1 \\ iE\sigma^3 \Theta \end{pmatrix}, \Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}
\] (3.20)

where \( \lambda^2 = k_z^2 - E^2 \). Then, substituting the explicit representation of the Pauli matrices into (3.18) we obtain that the equations governing the radial dependence of \( \Theta \) are:

\[
(d_r - \frac{k_\theta}{r})\Theta_1 - (m + i(\lambda - gH))\Theta_2 = 0 \quad (3.21)
\]

\[
(d_r + \frac{k_\theta}{r})\Theta_2 - (m - i(\lambda - gH))\Theta_1 = 0 \quad (3.22)
\]

Here, there are two simple cases for which the system (3.21)-(3.22) can be solved exactly in terms of special functions. They are: a) the constant magnetic field \( H \), and b) \( H = \alpha/r \).

a) For this case we obtain the following second order differential equations

\[
\left(\frac{d^2}{dr^2} - \frac{k_\theta(k_\theta + 1)}{r^2} - m^2 - (\lambda - gH)^2\right)\Phi_2 = 0 \quad (3.23)
\]

\[
\left(\frac{d^2}{dr^2} - \frac{k_\theta(k_\theta - 1)}{r^2} - m^2 - (\lambda - gH)^2\right)\Phi_1 = 0 \quad (3.24)
\]

whose solutions take the form\[4\]

\[
\Phi_1 = c_1 r^{1/2} Z_{k_\theta - 1/2}(i\sqrt{m^2 + (\lambda - gH)^2}r) \quad (3.25)
\]
\[ \Phi_2 = c_2 r^{1/2} Z_{k_\phi + 1/2}(i \sqrt{m^2 + (\lambda - gH)^2}r) \]  \hspace{1cm} (3.26)

where \( Z_\alpha(z) \) is the general solution of the Bessel equation, and the constants \( c_1 \) and \( c_2 \) are related as follows,

\[ c_1 = i c_2 \frac{m + i(\lambda - gH)}{\sqrt{m^2 + (\lambda - gH)^2}} \]  \hspace{1cm} (3.27)

b) Substituting \( H = \alpha/r \) into (3.21)-(3.22) we get,

\[ (d_r - \frac{k_\theta}{r}) \Theta_1 - (m + i\lambda - ig\alpha \sqrt{m^2 + \lambda^2}) \Theta_2 = 0 \]  \hspace{1cm} (3.28)

\[ (d_r + \frac{k_\theta}{r}) \Theta_2 - (m - i\lambda + ig\alpha \sqrt{m^2 + \lambda^2}) \Theta_1 = 0 \]  \hspace{1cm} (3.29)

the system of equations (3.28)-(3.29) resembles the one obtained in solving the Dirac equation with a Coulomb potential. Therefore it is possible to express the solution of (3.28)-(3.29) by means of hypergeometric functions as follows:

\[ \Theta_1 = \sqrt{1 - i \frac{\lambda}{m}} e^{-\frac{\gamma}{2}} (F_1(\rho) + F_2(\rho)) \]  \hspace{1cm} (3.30)

\[ \Theta_1 = \sqrt{1 + i \frac{\lambda}{m}} e^{-\frac{\gamma}{2}} (F_1(\rho) - F_2(\rho)) \]  \hspace{1cm} (3.31)

where \( \rho = 2\sqrt{m^2 + \lambda^2}r \), and \( F_1 \) and \( F_2 \) read:

\[ F_2(\rho) = c_0 \rho^\gamma M(\gamma - \frac{g\alpha \lambda}{\sqrt{m^2 + \lambda^2}}, 2\gamma + 1, \rho) \]  \hspace{1cm} (3.32)

\[ F_1(\rho) = -c_0 \gamma \frac{\lambda}{k_\theta - i \frac{g\alpha \gamma}{\sqrt{m^2 + \lambda^2}}} \rho^\gamma M(\gamma - \frac{g\alpha \lambda}{\sqrt{m^2 + \lambda^2}} + 1, 2\gamma + 1, \rho) \]  \hspace{1cm} (3.33)

where \( c_0 \) is a constant arbitrary. and \( \gamma = \sqrt{k_\theta^2 + g^2\alpha^2} \)

Making use of the expression for the asymptotic behavior of the confluent hypergeometric functions for \( \rho \to \infty \):

\[ M(a, b, x) \to e^{-ix\alpha} \frac{\Gamma(b)}{\Gamma(b-a)} \rho^{-a} + \frac{\Gamma(b)}{\Gamma(a)} \rho^{a-b} e^\rho \]  \hspace{1cm} (3.34)

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we have that the solutions of (3.28)-(3.29) are regular at infinity only if
\[
\frac{1}{\Gamma(\gamma - \frac{ga\lambda}{\sqrt{m^2 + \lambda}})} = 0 \tag{3.35}
\]
or equivalently,
\[
E^2 = k_z^2 - \frac{m^2}{(\gamma + n)^2 - 1} \tag{3.36}
\]
where \(n = 0, 1, 2, \ldots\), and \(E\) is the energy.

the expression (3.36) takes a very simple form when \(n = 0\),
\[
E^2 = k_z^2 + m^2 + \left(\frac{g\alpha m}{k_\theta}\right)^2 \tag{3.37}
\]

Now, we are going to solve the Dirac equation for a neutral particle with anomalous electric interaction. For an electric field along the \(z\) axis we have that the Dirac equation takes the form
\[
\left(\gamma^0 \partial_t + \frac{\gamma^2}{r} \partial_\theta + \gamma^3 \partial_z + m + g \gamma^3 \gamma^0 E\right) \Psi = 0 \tag{3.38}
\]

Following the pairwise scheme of separation, we have that Eq. (3.38) can be reduced to a sum of two commuting first order differential operators
\[
\left(\gamma^0 \gamma^1 \gamma^3 \partial_r + \frac{\gamma^0 \gamma^2 \gamma^3}{r} \partial_\theta + gE - \lambda\right) \Phi = 0 \tag{3.39}
\]
\[
\left(\gamma^3 \partial_0 + \gamma^0 \partial_z + m \gamma^3 \gamma^0 + \lambda\right) \Phi = 0 \tag{3.40}
\]
with \(\Phi = \gamma^3 \gamma^0 \Psi\). Choosing to work in the Dirac matrices representation given by the expression (3.4) we have that Eq. (3.39) can be written as:
\[
\left(\sigma^2 \partial_r - \frac{\sigma^1}{r} \partial_\theta - (gE - \lambda)\right) \Theta = 0 \tag{3.41}
\]
\[
\left(\sigma^2 \partial_r - \frac{\sigma^1}{r} \partial_\theta + (gE - \lambda)\right) \chi = 0 \tag{3.42}
\]
with
\[ \Phi = \begin{pmatrix} \Theta \\ \chi \end{pmatrix} = \begin{pmatrix} \Theta \\ \frac{i(m+E)}{\chi - k_z} \sigma^3 \Theta \end{pmatrix}, \quad \Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \] (3.43)

where the constant of separation \( \lambda \) satisfies the relation: \( \lambda^2 = k_z^2 + m^2 - E^2 \). From (3.41) we have that the components of \( \Theta \) satisfy the relation:

\[
\left( \partial_r - \frac{k_\theta}{r} \right) \Theta_1 = i(\lambda - gE)\Theta_2 \] (3.44)

\[
\left( \partial_r + \frac{k_\theta}{r} \right) \Theta_2 = -i(\lambda - gE)\Theta_1 \] (3.45)

the solution of this system reads

\[
\Theta_1 = c r^{\frac{k_\theta}{2}} Z_{k_\theta - 1/2}(i(\lambda - gE)r) \] (3.46)

\[
\Theta_2 = -c r^{\frac{k_\theta}{2}} Z_{k_\theta + 1/2}(i(\lambda - gE)r) \] (3.47)

\[ \text{IV. A SOLVABLE EXAMPLE} \]

The following example could be regarded as surprising because of the big amount of functions appearing in the Dirac equation. This example seems to be contradictory since, following the remarks made at the introduction, the presence of additional matrices in the Dirac equation dramatically reduces the possibilities of separation of variables. But such a contradiction does not exist in the case to be presented. All the functions considered in the example, depend only on the radial variable \( r \) namely in the form \( 1/r \), therefore their structure repeat the Lamé function arising when we use spherical coordinates. This one explains the restriction in the election of the functions allowing complete separation of variables.

Then, let us consider a Dirac particle in a Coulomb field with anomalous electric dipole interaction and also scalar and pseudoscalar interactions. At a first glance this problem seems artificial, because we do not relate the Coulomb term with the dipole one, but the
problem includes the possibility of a unified approach of a series of physical situations like the Hydrogen atom (Coulomb field), the non minimal electric dipole interaction, and confinement model (scalar potential) and the quarkonium theory.\textsuperscript{22–25}

The Dirac equation expressed in the diagonal tetrad gauge takes the form
\[
\left\{ \gamma^1 \partial_r + \frac{1}{r} (\gamma^2 \partial_\vartheta + \frac{\gamma^3}{\sin \vartheta} \partial_\varphi) + \gamma^0 (\partial_t - \frac{Ze^2}{r}) + q \gamma^1 \gamma^0 \frac{\epsilon}{r} + \zeta \frac{S}{r} + \xi \frac{P(\vartheta, \varphi)}{r} \right\} \Psi = 0 \quad (4.1)
\]

here \(Ze^2/r\) is the Coulomb potential, \(q\epsilon/r\) is the electric dipole interaction, \(S\) is the scalar potential, \(P\) is the pseudoscalar potential, and \(\zeta\) and \(\xi\) are constants. After separating the angular dependence, (4.1) takes the form
\[
\left\{ \gamma^0 \partial_r + \frac{k - q \epsilon}{r} - \gamma^1 (\partial_t - \frac{Ze^2}{r}) + \gamma^1 \gamma^0 m + \zeta \gamma^1 \gamma^0 \frac{S}{r} \right\} \Phi = 0 \quad (4.2)
\]
\[
\left( \gamma^2 \gamma^1 \gamma^0 \partial_\vartheta + \frac{\gamma^3 \gamma^1 \gamma^0}{\sin \vartheta} \partial_\varphi - \gamma^2 \gamma^3 \xi P(\vartheta, \varphi) \right) \Phi = k \Phi \quad (4.3)
\]

where \(\Psi = \gamma^1 \gamma^0 \Phi\)

We will not solve the equation (4.3) governing the angular dependence of the problem because this one is not possible without knowing the \(P(\vartheta, \varphi)\) function. Obviously, the solution for the particular case \(P = 0\) will be expressed in terms of spherical harmonics like the Coulomb case.

Choosing to work in the representation
\[
\gamma^0 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.4)
\]
equation (4.2) takes the form,
\[
\left[ i\sigma^3 \partial_r + \frac{k - q \epsilon}{r} + \sigma^1 (\partial_t - i\frac{Ze^2}{r}) + \sigma^2 (\frac{S}{r} + m) \right] \Phi_1 = 0 \quad (4.5)
\]
substituting (4.4) into (4.3) we arrive at,
\[
\left\{ \frac{d}{dr} - i \frac{k - q \epsilon}{r} \right\} \Theta_1 + \left( -(E + m) - \frac{1}{r} (Ze^2 + \zeta S) \right) \Theta_2 = 0 \quad (4.6)
\]
\[
\left\{ \frac{d}{dr} + i \frac{k - q \epsilon}{r} \right\} \Theta_2 + \left( (E - m) + \frac{1}{r} (Ze^2 - \zeta S) \right) \Theta_1 = 0 \quad (4.7)
\]
where $\Theta_1$ and $\Theta_2$ are the components of the spinor $\Phi_1$

$$\Phi_1 = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \quad (4.8)$$

Introducing the notation $i(k-\epsilon q) = M$, $m+E = A$, $m-E = B$, $\zeta S + Ze^2 = \alpha$, $Ze^2 - \zeta S = \beta$, and $rD = \rho$ we have that eq. (4.6), (4.7) take the form

$$\left\{ \frac{d}{d\rho} - \frac{M}{\rho} \right\} \Theta_1 + \left( -\frac{A}{D} - \frac{\alpha}{\rho} \right) \Theta_2 = 0 \quad (4.9)$$

$$\left\{ \frac{d}{d\rho} + \frac{M}{\rho} \right\} \Theta_2 + \left( -\frac{B}{D} + \frac{\beta}{\rho} \right) \Theta_1 = 0 \quad (4.10)$$

we shall look for solutions of the system (4.9), (4.10) in the form of power series:

$$\Theta_1 = e^{-\rho} \sum_{\upsilon=0}^{\infty} \rho^{s+\upsilon} a_\upsilon, \quad \Theta_2 = e^{-\rho} \sum_{\upsilon=0}^{\infty} \rho^{s+\upsilon} b_\upsilon \quad (4.11)$$

Substituting (4.11) into (4.9), (4.10) and putting the coefficients of $\rho^{s+\upsilon+1}$ equal to zero we find

$$(s + M)a_0 + \beta b_0 = 0 \quad (4.12)$$

$$\alpha a_0 - (s - M)b_0 = 0 \quad (4.13)$$

$$-a_{\upsilon-1} + (s + \upsilon + M)a_\upsilon - \frac{B}{D} b_{\upsilon-1} + \beta b_\upsilon = 0 \quad (4.14)$$

$$b_{\upsilon-1} - (s + \upsilon - M)b_\upsilon + \frac{A}{D} a_{\upsilon-1} + \alpha b_\upsilon = 0 \quad (4.15)$$

from (4.12) and (4.13) it follows that the parameter $s$ takes the value

$$s = (M^2 - \alpha \beta)^{1/2} \quad (4.16)$$

where we have considered a positive root in order to avoid a divergent solution at the origin.

From (4.14) and (4.15) we find the following relation between the coefficients $a_\upsilon$ and $b_\upsilon$
\{D(s + v + M) + B\alpha\} a_\nu = \{B(s + v - M) - D\beta\} b_\nu \quad \text{(4.17)}

The series (4.11) will have a good behavior at infinity if they terminate for a finite value \(N\). Putting \(a_{N+1} = b_{N+1} = 0\) in (4.14) and (4.15) with \(a_N \neq 0\) and \(b_N \neq 0\) we arrive at

\[
\frac{a_N}{b_N} = -\frac{B}{D} \quad \text{(4.18)}
\]

then, from (4.17), (4.18) and the definitions of \(B\) and \(D\) we arrive at:

\[
EZe^2 = (s + N)\sqrt{m^2 - E^2} + m\zeta S \quad \text{(4.19)}
\]

The expression (4.19) is the condition of quantization of the energy, and its structure is very similar to the one obtained for the hydrogen atom. From (4.19) can obtain some particular cases: a) \(\epsilon = 0, S = 0\) Hydrogen atom. b) \(\epsilon = 0, Z = 0\) Confinement of quarks by a scalar potential c) \(Z = 0, S = 0\). No minimal electric dipole interaction.

Obviously, exact solutions of the Dirac equation containing such set of terms like (4.1) is not only of physical interest, but also is interesting from a mathematical point of view. Regretfully the amount of examples of this kind are relatively scarce and therefore not exact solutions are known from most of the problems.

V. CONCLUDING REMARKS

In this article we have obtained some exact solutions of the Dirac equation for electric neutral Dirac particle with anomalous magnetic moment. The results presented in this paper show the capabilities of the algebraic method of separation of variables provided a suitable representation for the gamma matrices. The separation of variables for other curvilinear system of coordinates was not possible because of the matrix character of the anomalous interaction in the Dirac equation. This peculiarity dramatically restricts the possibilities of finding exact solution in other geometric and physical configurations. Also, it would be of interest to find exact solutions when other interactions are present.
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