Abstract

We consider the adversarial combinatorial multi-armed bandit (CMAB) problem, whose decision set can be exponentially large with respect to the number of given arms. To avoid dealing with such large decision sets directly, we propose an algorithm performed on a zero-suppressed binary decision diagram (ZDD), which is a compressed representation of the decision set. The proposed algorithm achieves either $O(T^{2/3})$ regret with high probability or $O(\sqrt{T})$ expected regret as the any-time guarantee, where $T$ is the number of past rounds. Typically, our algorithm works efficiently for CMAB problems defined on networks. Experimental results show that our algorithm is applicable to various large adversarial CMAB instances including adaptive routing problems on real-world networks.

1 Introduction

The multi-armed bandit (MAB) problem [29] has been extensively studied as a fundamental framework for online optimization problems with partial observations. In the MAB, a player chooses an arm (choice) from a set of possible arms. Then, the player incurs a cost and obtains feedback according to the selected arm. The aim of the player is to minimize the cumulative cost by exploring possible arms and exploiting those with low costs. There have been many studies on MAB applications, e.g., clinical trials [29], and recommendation systems [26].

In many real-world problems, each possible choice that the player can make is not expressed as a single arm but as a super arm, which is a set of arms that satisfies certain combinatorial constraints; a full set of super arms is called a decision set. This problem is called the combinatorial multi-armed bandit (CMAB) problem, and the CMAB is said to be adversarial if the cost of each arm is arbitrarily changed by an adversary. Examples of the adversarial CMAB include various important problems on networks such as the online shortest path (OSP) problem [4] [14], the dynamic Steiner tree (DST) problem [15], and the congestion game (CG) [30], although the original CG is a resource allocation problem over multiple players, it can be formulated as an adversarial CMAB if a player considers the other players to be adversaries. For instance, in the OSP on a traffic network, an arm corresponds to an edge (road) of a given network, a super arm is an $s$-$t$ path that connects the current point $s$ and the destination $t$, and the decision set is a set of all $s$-$t$ paths. Furthermore, in the OSP, the cost of an arm (road) represents the traveling time on the road, and it dynamically changes due to the time-varying amount of traffic or accidents (e.g., cyber attacks in the case of the OSP on communication networks).

In this paper, we focus on the adversarial CMAB.
The main difficulty with the adversarial CMAB is that the size of the decision set is generally exponential in the number of arms. To handle huge decision sets, existing methods for this problem assume that the decision set has certain properties. One such method is CombEXP [10], which can cope with the difficulty if the decision set consists of, for example, sets of arms satisfying a size constraint, or matchings on a given network. However, it has been hard to design practical algorithms for adversarial CMAB instances with complex decision sets defined on networks; for example, the OSP, DST, and CG on undirected networks.

In this paper, we develop a practical and theoretically guaranteed algorithm for the adversarial CMAB, which is particularly effective for network-based adversarial CMAB instances. We first propose CombWM (ComBAND [9] with Weight Modification), which is theoretically guaranteed to achieve either $O(T^{2/3})$ regret with high probability or $O(\sqrt{T})$ expected regret, where $T$ is the number of rounds. The above bounds are any-time guarantees [7], and we can choose which regret value CombWM actually achieves by setting its hyper parameter at an appropriate value. We then show that our CombWM can be performed on a compressed decision set; we assume that a decision set is given as a zero-suppressed decision diagram (ZDD) [27], which is a compact graph representation of a family of sets. The time and space complexities of CombWM with a ZDD are linear in the size of the ZDD, whereas those of the naive ComBAND is proportional to the size of a decision set. It is known that a ZDD tends to be small if it represents a set of subnetworks such as $s$-$t$ paths or Steiner trees [21]. Thus our algorithm is effective for network-based adversarial CMAB instances including the OSP, DST, and CG. Experimental results on OSP, DST, and CG instances show that our algorithm is more scalable than naive algorithms that directly deal with decision sets. To the best of our knowledge, this is the first work to implement algorithms for the adversarial CMAB and provide experimental results, thus revealing the practical usefulness of adversarial CMAB algorithms.

2 Related work

Many studies have considered the adversarial CMAB with specific decision sets, e.g., $m$-sets [20] and permutations [1]. In particular, the OSP, which is a CMAB problem on a network with an $s$-$t$ path constraint, has been extensively studied [4, 14] due to its practical importance. Whereas the previous studies have focused on the OSP on directed networks, our algorithm is also applicable to the OSP on undirected networks.

The adversarial CMAB with general decision sets has been also extensively studied in [3, 7, 5, 8, 9, 10, 31]. One of the best known algorithms for this problem is ComBAND [9], which has been proved to achieve $O(\sqrt{T})$ expected regret. Recently the algorithm has been also proved to achieve $O(T^{2/3})$ regret with high probability in [2]. More precisely, the regret of ComBAND is bounded by $O(t^{2/3})$ with high probability in any $t$-th round ($t = 1, \ldots, T$), which is called an any-time guarantee. Although ComBAND has the strong theoretical results, its time complexity generally depends on the size of decision sets, which can be prohibitively large in practice. To avoid such expensive computation, CombEXP [10] scales up ComBAND by employing a projection onto the convex hull of the decision set via KL-divergence. For some decision sets for which the projection can be done efficiently (e.g., $m$-sets or a set of matchings), CombEXP runs faster than ComBAND, achieving the same theoretical guarantees. However, it is difficult to perform the projection for other decision sets (e.g., $s$-$t$ paths or Steiner trees); actually it is NP-hard to do the projection in the case of the OSP and DST on undirected networks.

On the other hand, thanks to recent advances in constructing decision diagrams (DDs), optimization techniques using DDs have been attracting much attention [6, 11, 28]. Those techniques are advantageous in that DDs can efficiently store all solutions satisfying some complex constraints; for example, constraints that are hard to represent as a set of inequalities. The ZDD [27], which we use in our algorithm, is a

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1 The proof seems to include some mistakes. However, their techniques for the proof are still useful, and so we prove the $O(T^{2/3})$ high-probability regret bound of our algorithm by partially modifying their proof; the modified parts are the description of the algorithm and Lemma 2 in the supplementary materials.
kind of DD that is known to be suitable for storing specific network substructures (e.g., s-t paths or Steiner trees). Thus our algorithm with ZDDs runs fast in many CMAB instances defined on networks, including the OSP, DST, and CG.

3 Adversarial CMAB

We here define the adversarial CMAB, which is a sequential decision problem consisting of $T$ rounds. Let $[m] := \{1, \ldots, m\}$ for any $m \in \mathbb{N}$. We use $E = [d]$ to denote a set of arms and also use $S \subseteq 2^E$ to denote a decision set, where $X \in S$ is a super arm. At each $t$-th round ($t \in [T]$), an adversary secretly defines a loss vector $\ell_t := (\ell_{t,1}, \ldots, \ell_{t,d})^\top \in \mathbb{R}^d$ and a player chooses a super arm $X_t \in S$. Then, the player incurs and observes the cost $c_t = \ell_t^\top 1_{X_t}$, where $1_{X_t} \in \{0,1\}^d$ is an indicator vector such that its $i$-th element is 1 if $i \in X_t$ and 0 otherwise. Note that the player cannot observe $\ell_t$. The aim of the player is to minimize the regret $R_T$ defined as follows:

$$R_T := \sum_{t=1}^T \ell_t^\top 1_{X_t} - \min_{X \in S} \sum_{t=1}^T \ell_t^\top 1_X.$$ 

The first term is the cumulative cost and the second term is the total cost of the best single super arm selected with hindsight. Namely, $R_T$ expresses the extra cost that the player incurs against the best single super arm. As is customary, we assume $\max_{X \in S} |\ell_t^\top 1_X| \leq 1$.

If the adversary and/or the player choose $\ell_t$ and $X_t$ in a stochastic manner, then $R_t$ is a random variable of a joint distribution $p(\ell_{1:t}, X_{1:t})$, where $X_{1:t} = \{X_1, \ldots, X_t\}$ and $\ell_{1:t} = \{\ell_1, \ldots, \ell_t\}$. In the adversarial CMAB, $p$ is assumed to satisfy the following conditional independence: $p(\ell_{1:T}, X_{1:T}) = \prod_{t \in [T]} p(X_{1:t} \mid \ell_{1:t-1}, X_{1:t-1})p(\ell_t \mid \ell_{1:t-1}, X_{1:t-1})$, where $X_{1:0} = \ell_{1:0} = \emptyset$. $p(\ell_t \mid X_{1:t-1}, \ell_{1:t-1})$ corresponds to the adversary’s strategy and $p(X_t \mid X_{1:t-1}, \ell_{1:t-1})$ corresponds to the player’s strategy. Since the player cannot directly observe $\ell_{1:t}$, the player’s strategy must satisfy $p(X_t \mid X_{1:t-1}, \ell_{1:t-1}) = p(X_t \mid X_{1:t-1}, C_{1:t-1})$. Using the joint distribution $p$, the expected regret $\overline{R}_T$ is defined as follows:

$$\overline{R}_T := \mathbb{E}_{\ell_{1:T}, X_{1:T} \sim p} [R_T].$$

The objective of the adversarial CMAB is to design the player’s strategy $p(X_t \mid X_{1:t-1}, C_{1:t-1})$ so that it minimizes $R_T$ or $\overline{R}_T$. In this paper, we use $p_t(X_t)$ as shorthand for $p(X_t \mid X_{1:t-1}, C_{1:t-1})$.

4 Proposed algorithm for adversarial CMAB

We here propose COMBW (COMBand with Weight Modification), which is an algorithm for designing the player’s strategy $p_t(X_t)$ with strong theoretical guarantees as described later. Algorithm\footnote{If the adversary behaves adaptively, the above interpretation of the regret is somewhat inappropriate; in such cases using the policy regret\footnote{Policy regret is considered to be more suitable. However, we here focus on the above regret and expected regret, leaving an analysis based on the policy regret for future work.} is considered to be more suitable. However, we here focus on the above regret and expected regret, leaving an analysis based on the policy regret for future work.} gives the details of COMBW. In what follows, we define $L := \max_{X \in S} \|1_X\|$ for any given $S \subseteq 2^E$, where $\|\cdot\|$ is the Euclidian norm. We also define $\lambda$ as the smallest non-zero eigenvalue of $\mathbb{E}_{X \sim u}(1_X 1_X^\top)$, where $u$ is the uniform distribution over $S$.

Given an arbitrary non-negative vector $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ and a decision set $S \subseteq 2^E$, we define the constrained distribution $p(X; w, S)$ over $S$ as follows:

$$p(X; w, S) := \frac{w(X)}{Z(w, S)}, \quad Z(w, S) := \sum_{X \in S} w(X), \quad w(X) := \prod_{i \in X} w_i.$$ 

Using the above, we define the player’s strategy $p_t(X_t)$, which appears in Step 4, as follows:

$$p_t(X_t) := (1 - \gamma_t)p(X_t; \overline{w_t}, S) + \gamma_t p_t(X_t; 1_E, S).$$
Algorithm 1 \textsc{CombWM}(\alpha, \mathcal{S})

1: $\tilde{w}_{1,i} \leftarrow 1$ (\text{if } i \in E)
2: for $t = 1, \ldots, T$ do
3: $\gamma_t \leftarrow \frac{\gamma_{t-1}}{2}$, $\eta_t \leftarrow \frac{\eta_{t-1}}{2L^2}$, $\eta_{t+1} \leftarrow \frac{\eta_{t+1}}{2L^2}$
4: $X_t \sim p_t$
5: $c_t \leftarrow \tilde{\ell}_i^\top 1_{X_i}$ ($\tilde{\ell}_i$ is unobservable)
6: $P_t(i,j) \leftarrow \sum_{X \in S : i,j \in X} p_t(X)$ (\text{if } i,j \in [d])
7: $\tilde{\ell}_i \leftarrow c_t P_t^{+} 1_{X_i}$
8: $\tilde{w}_{t+1,i} \leftarrow \frac{\tilde{w}_{t+1,i}}{\eta_{t+1}} \exp \left( - \eta_{t+1} \tilde{\ell}_{t+1,i} \right)$ (\text{if } i \in E)
9: end for
10: return $\{X_t \mid t \in [T]\}$

where $\tilde{w} = (\tilde{w}_{1,1}, \ldots, \tilde{w}_{t,d})^\top$ is the weight vector defined in Step 8, and $\gamma_t$ is the parameter defined in Step 3; we note that $p(X_t; 1_E, \mathcal{S})$ is the uniform distribution over $\mathcal{S}$. Thus $p_t$ is a mixture of two constrained distributions with the mixture rate $\gamma_t$.

Given a distribution $p$ over $\mathcal{S}$, a matrix $P$ is called a co-occurrence probability matrix (CPM) if its entry $P(i,j)$ is given by the co-occurrence probability $p(i \in X, j \in X) := \sum_{X \in S : i,j \in X} p(X)$. The matrix $P_t$ computed in Step 6 is the CPM of $p_t$, and $P_t^+$ used in Step 7 is the pseudo-inverse of $P_t$. From Eq. (2), the following equation holds:

$$P_t(i,j) = (1 - \gamma_t)p(i \in X, j \in X; \tilde{w}_i, \mathcal{S}) + \gamma_t p(i \in X, j \in X; 1_E, \mathcal{S}).$$

The above \textsc{CombWM} is based on \textsc{ComBand}[9]; if we replace Step 8 of \textsc{CombWM} with $\tilde{w}_{t+1,i} \leftarrow \tilde{w}_{t,i} \exp(-\eta_t \tilde{\ell}_{t,i})$, \textsc{CombWM} corresponds perfectly to the original \textsc{ComBand}. Hence the one and only one difference is the weight modification in Step 8. However, introducing this weight modification gives us the following theoretical guarantees (for proofs, see the supplementary materials):

\textbf{Theorem 1.} For any $\mathcal{S}$, \textsc{CombWM}(\alpha = 3, \mathcal{S}) achieves $R_T \leq O\left(\left(\frac{d}{\alpha^2} + \frac{L^2 \ln |\mathcal{S}| + 2}{\alpha} \right)T^{2/3}\right)$ with probability at least $1 - \delta$.

\textbf{Theorem 2.} For any $\mathcal{S}$, \textsc{CombWM}(\alpha = 2, \mathcal{S}) achieves $R_T \leq O\left(\left(\frac{d}{\alpha^2} + \frac{L^2 \ln |\mathcal{S}|}{\alpha} \right)\sqrt{T}\right)$.

In other words, \textsc{CombWM} achieves either $O(T^{2/3})$ regret with high probability or $O(\sqrt{T})$ expected regret as an any-time guarantee by choosing the hyper parameter $\alpha$ appropriately.

There are two difficulties when it comes to performing \textsc{CombWM}; the first is sampling from the player’s strategy $p_t(X_t)$ (Step 4), and the second is computing the CPM $P_t$ (Step 6). Naive methods for sampling from $p_t$ and computing $P_t$ require $O(|\mathcal{S}|)$ and $O(d^2 |\mathcal{S}|)$ computation times, respectively, where $|\mathcal{S}|$ is generally exponential in $d$, and so are the time complexities. In the following section, we propose efficient methods for sampling from any given constrained distribution $p(X; w, \mathcal{S})$ and for computing the CPM of $p(X; w, \mathcal{S})$. Because $p_t$ is a mixture of two constrained distributions, we can efficiently sample from $p_t$ and compute the CPM of $p_t$ using the proposed methods.

## 5 \textsc{CombWM} on compressed decision sets

As shown above, \textsc{CombWM} requires sampling from constrained distributions and computing CPMs as its building blocks, which generally require $O(|\mathcal{S}|)$ and $O(d^2 |\mathcal{S}|)$ computation times, respectively. Moreover, computing $L$ can also require $O(|\mathcal{S}|)$ time. Those computation costs can be prohibitively expensive since $|\mathcal{S}|$ is generally exponential in $d$. In this section, we present efficient algorithms for the building blocks that are based on dynamic programming (DP) on a ZDD, which is a compressed representation of $\mathcal{S}$. We first briefly describe ZDDs and then propose two DP methods for sampling and computing CPMs. $L$ can also be computed in a DP manner on a ZDD.
5.1 Zero-suppressed binary decision diagrams (ZDDs)

A ZDD [27] is a compact graph representation of a family of sets. Given $S \subseteq 2^E$, a ZDD for $S$ is a directed acyclic graph (DAG) denoted by $G_S = (V, A)$, where $V = \{0, 1, \ldots, |V| - 1\}$ is a set of vertices and $A \subseteq V \times V$ is a set of directed arcs. $G_S$ contains one root vertex $r \in V$ and two terminal vertices: 1-terminal and 0-terminal. Without loss of generality, we assume that $V = \{0, \ldots, |V| - 1\}$ is arranged in a topological order of $G_S$; $r = |V| - 1$ holds and the b-terminal ($b \in \{0, 1\}$) is denoted simply by $b \in V$. Each non-terminal vertex $v \in V \setminus \{0, 1\}$ is labeled by an integer in $E$ and has exactly two outgoing arcs: 1-arc and 0-arc. A vertex pointed by the b-arc of $v$ is called the b-child of $v$. We use $l_v$, $c_v^b$ and $c_v^0$ to denote $v$’s label, b-arc, and b-child, respectively. Consequently, $c_v^b = (v, c_v^0)$ holds. We use $R_{v,u} (v, u \in V)$ to denote a set of routes (directed paths) from $v$ to $u$ on $G_S$, where a route $R \in R_{v,u}$ is a set of directed arcs: $R \subseteq A$. Given $R \in R_{v,u}$, we define $X(R) \subseteq E$ as $X(R) := \{l_{v'} \mid (v', c_{v'}^b) \in R\}$. Then, $G_S$ satisfies

$$S = \{X(R) \mid R \in R_{r,1}\}.$$  

Therefore, $G_S$ represents the decision set $S$ as a set of all routes from its root $r$ to the 1-terminal. Note that once $G_S$ is obtained, $L = \max_{X \in S} \sqrt{|X|}$ is easily computed by a DP method to find $R \in R_{r,1}$ that maximizes $|X(R)|$.

In general, a ZDD is assumed to be ordered and reduced. $G_S$ is said to be ordered if $v > u \Rightarrow l_v < l_u$ holds for all $v, u \in V \setminus \{0, 1\}$. A non-terminal vertex $v$ is said to be redundant if $c_v^0 = 0$: its 1-arc directly points to the 0-terminal. A redundant vertex $v$ can be removed by replacing all $(u, v) \in A$ with $(u, c_v^0)$ without loss of the property [1]. A non-terminal vertex $v$ is said to be sharable if there exists another vertex $v'$ such that $l_v = l_{v'}$ and $c_v^0 = c_{v'}^b$ ($b \in \{0, 1\}$): $v$ and $v'$ have the same label and children. A sharable vertex $v$ can be removed by replacing $(u, v) \in A$ with $(u, v')$. $G_S$ is said to be reduced if no vertex is redundant or sharable. In this paper, we assume that $G_S$ is ordered and reduced. We show an example of a ZDD in Figure [1].

The ZDDs are known to store various families of sets compactly in many applications. In particular, if a decision set is a set of specific network substructures (e.g., a set of s-t paths or Steiner trees), the ZDD representing the decision set tends to be small. As we will see later, the time complexity of CombiWM with a ZDD $G_S = (V, A)$ is $O(d|V|)$, and so it runs fast if the ZDD is small. In theory, if $S$ is a set of specific network substructures, then $|V|$ is bounded by a value that is exponential in the pathwidth [17]. Thus, even if a network-based decision set is exponentially large in $d$, the time complexity of our algorithm in each round can be polynomial in $d$ if the pathwidth of the network is bounded by a small constant.

The frontier-based search [21], which is based on Knuth’s Simpath algorithm [24], has recently received much attention as a fast top-down construction algorithm for ZDDs that represent a family of subnetworks. In practice ZDDs are easily obtained via existing software [16] for various network-based constraints. In this paper, we omit the details of ZDD construction and assume that a decision set $S$ is represented by a ZDD $G_S$ rather than by the explicit enumeration of the components of $S$.

Figure 1: (a) An example network with an edge set $E = \{1, \ldots, 5\}$, and (b) a ZDD that stores all paths from the start to the goal; each non-terminal vertex $v$ is labeled $l_v \in E$, and 0-arcs and 1-arcs are indicated by dashed and solid lines, respectively. Note that we have $S = \{X(R) \mid R \in R_{r,1}\} = \{\{1, 4\}, \{2, 5\}, \{1, 3, 5\}, \{2, 3, 4\}\}$. 

5
5.2 Sampling from constrained distributions

We here propose an efficient algorithm for sampling from a constrained distribution \( p(X; w, \mathcal{S}) \). We first introduce the following forward weight (FW) \( F_v \) and backward weight (BW) \( B_v \) \((v \in V)\):

\[
(5) \quad F_v := \sum_{R \in R_v} w(R), \quad \quad B_v := \sum_{R \in R_{v,1}} w(R),
\]

where \( w(R) \) is an abbreviation of \( w(X(R)) = \prod_{i \in X(R)} w_i \). By combining Eq. (1), (4), and (5), we obtain \( Z(w, \mathcal{S}) = B_r = F_1, B := \{B_0, \ldots, B_r\} \text{ and } F := \{F_0, \ldots, F_r\} \) can be efficiently computed in a dynamic programming manner on \( G_S \) as shown in Algorithm Draw\((G_S, w)\) and BW\((G_S, w)\). Once we obtain B, we can draw a sample from \( p(X; w, \mathcal{S}) \) by top-down sampling on \( G_S \) without rejections as shown in Algorithm Draw\((G_S, w, B)\), where Ber\((\theta)\) is the Bernoulli distribution with the parameter \( \theta \in [0, 1] \). The space and time complexity when computing \( F \) and \( B \) is proportional to \(|V|\). This constrained sampling is based on the same idea as that used in logic-based probabilistic modeling [18, 19].

1: Algorithm FW\((G_S, w)\)
2: \( F_r \leftarrow 1 \)
3: \( F_v \leftarrow 0 \ (\forall v \in V \setminus \{r\}) \)
4: for \( v = r, \ldots, 2 \) do
5: \( F_v^{c,v} + F_v \)
6: \( F_v^{c,v} + w_v B_v^{c,v} \)
7: end for
8: \( F := \{F_0, \ldots, F_r\} \)
9: return \( F \)

1: Algorithm BW\((G_S, w)\)
2: \( B_1 \leftarrow 1 \)
3: \( B_v \leftarrow 0 \ (\forall v \in V \setminus \{1\}) \)
4: for \( v = 2, \ldots, r \) do
5: \( B_v \leftarrow B_v^{c,v} + w_v B_v \)
6: end for
7: \( B := \{B_0, \ldots, B_r\} \)
8: return \( B \)

1: Algorithm Draw\((G_S, w, B)\)
2: \( X \leftarrow \{\}, v \leftarrow r \)
3: while \( v > 1 \) do
4: \( \theta \leftarrow w_v B_v^{c,v}/B_v \)
5: \( b \sim \text{Ber}(\theta) \)
6: \( X \leftarrow X \cup \{b\} \) if \( b = 1 \)
7: \( v \leftarrow c_v \)
8: end while
9: return \( X \)

5.3 Computing co-occurrence probabilities

Given a constrained distribution \( p(X; w, \mathcal{S}) \), we define \( P_{i,j} := p(i \in X, j \in X; w, \mathcal{S}) \) as the co-occurrence probability of \( i \) and \( j \) \((i, j \in E)\). We here propose an efficient algorithm for computing \( P_{i,j} \) \((i \leq j)\), which suffices for obtaining \( P_{i,j} \) for all \( i, j \in [d] \) since \( P_{i,j} = P_{j,i} \). Using Eq. (1) and the notion of \( G_S \), \( P_{i,j} \) can be written as follows:

\[
(6) \quad P_{i,j} = \sum_{R \in R_{r,1}:i,j \in X(R)} \frac{w(R)}{Z(w, \mathcal{S})}.
\]

We first consider \( P_{i,i} \) as a special case of \( P_{i,j} \). By combining Eq. (5) and (6), we obtain

\[
P_{i,i} = \sum_{R \in R_{r,1}:i \in X(R)} \frac{w(R)}{Z(w, \mathcal{S})} = \sum_{v \in V: i = v} \sum_{R' \in R_{r,1}} \frac{w(R' \cup \{i\})}{B_r} = \sum_{v \in V: i = v} \frac{F_v w_v B_v^{c,v}}{B_r}.
\]

Next, to compute \( P_{i,j} \) \((i < j)\), we rewrite the right hand side of Eq. (6) using the backward weighted co-occurrence (BWC) \( C_{v,j} \ (j \geq l_v) \) as follows:

\[
P_{i,j} = \sum_{v \in V: i = v} \frac{F_v w_v C_v^{c,j}}{B_r}, \quad \quad C_{v,j} := \sum_{R \in R_{r,1}:i,j \in X(R)} w(R).
\]

Because \( C_{v,j} \) is a variant of \( B_v \), \( C := \{C_{v,j} \mid v \in V, j \geq l_v\} \) can be computed in a similar manner to \( B \) as shown in Algorithm BW\((G_S, w, B)\). To conclude, \( P_{i,j} \) can be computed by Algorithm CPM\((G_S, w, F, B, C)\). The total space and time complexity of computing \( P := \{P_{i,j} \mid i \leq j\} \) is \( O(d|V|) \).
We applied our \textsc{ComBW} with ZDDs to three network-based CMAB problems: the OSP, DST, and CG. In the OSP and DST, we used artificial networks to observe the scalability of our algorithm. In the CG, we used two real-world networks to show the practical utility of our algorithm. We implemented our algorithm in the C programming language and used Graphillion [16] to obtain the ZDDs. We note that constructing ZDDs with the software is not a drawback; in all of the following instances a ZDD was obtained within at most several seconds.

6 Experiments

We applied our \textsc{ComBW} with ZDDs to three network-based CMAB problems: the OSP, DST, and CG. In the OSP and DST, we used artificial networks to observe the scalability of our algorithm. In the CG, we used two real-world networks to show the practical utility of our algorithm. We implemented our algorithm in the C programming language and used Graphillion [16] to obtain the ZDDs. We note that constructing ZDDs with the software is not a drawback; in all of the following instances a ZDD was obtained within at most several seconds.

6.1 OSP and DST on artificial networks

**Experimental Setting**: We applied our \textsc{ComBW} with ZDDs to the OSP and DST instances on artificial networks, which are undirected grid networks with $3 \times m$ nodes ($m = 3, \ldots, 10$). In both problems, an arm corresponds to an edge of the given network. In the OSP, a decision set $S$ is a set of all $s$-$t$ paths from the starting node $s$ to the goal node $t$ that are placed on diagonal corners of the given grid. In the DST, $S$ is a set of all Steiner trees that contains the four corners of the grid. The aim of the player is to minimize the cumulative cost of the selected subnetworks over some time horizon. In this experiment, we define the loss vector $\ell_t$ as follows: We first uniformly sample $\bm{\mu}_0$ from $[0,1]^d$. In the $t$-th round, we set $\bm{\mu}_t = \bm{\mu}_{t-1}$ with probability 0.9 or draw a new $\bm{\mu}_t$ uniformly from $[0,1]^d$ with probability 0.1. Then, for each $i \in E$, we draw $h_i \sim \text{Ber}(\mu_{t,i})$ and set $\ell_{t,i} = 1/d$ if $h_i = 1$ otherwise $-1/d$. This setting is a stochastic CMAB with distributions $\text{Ber}(\mu_{t,i})$ in the short run, but the adversary secretly reset $\mu_i$ with probability 0.1 in each round to foil the player.

**Compression Power**: We first assess the compression power of ZDDs constructed for the decision sets of the OSP and DST instances. Table 1 shows the sizes of decision sets $S$ and those of the corresponding ZDDs $G_S$. In both problems, the ZDD size, $|V|$, grows much more slowly than $|S|$. In particular, with the DST on the $3 \times 10$ grid, we see that $|V|$ is five orders of magnitude smaller than $|S|$. In such cases, our \textsc{ComBW}, which only deals with a ZDD $G_S$, is much more scalable than the naive method that directly deals with $S$.

**Empirical Regret**: We next show that the empirical regrets of our \textsc{ComBW} and \textsc{ComBand} actually grow sublinearly, where \textsc{ComBand} is also performed on ZDDs. We applied these algorithms to the OSP and DST on a $3 \times 10$ grid and computed their empirical regrets over a time horizon. Figures 2 (a) and (b) summarize their regrets for the OSP and DST, respectively. We see that all of the algorithms achieved more or less the same sublinear regrets. It was confirmed that all of the regret values were lower than those of the theoretical bounds stated in Theorem 1 and Theorem 2; the precise values of the bounds are provided in the supplementary materials.
Table 1: The sizes of decision sets $S$ and the corresponding ZDDs $G_S$ for the OSP and DST (numbers with more than six digits are rounded to three significant digits).

| m  | OSP $|S|$ | OSP $|V|$ | DST $|S|$ | DST $|V|$ |
|----|--------|--------|--------|--------|
| 3  | 12     | 31     | 266    | 80     |
| 4  | 38     | 76     | 4,285  | 304    |
| 5  | 125    | 183    | 69,814 | 1,147  |
| 6  | 414    | 451    | $1.14 \times 10^6$ | 4,616  |
| 7  | 1,369  | 1,039  | $1.86 \times 10^7$ | 18,032 |
| 8  | 4,522  | 2,287  | $3.04 \times 10^8$ | 67,484 |
| 9  | 14,934 | 4,991  | $4.97 \times 10^9$ | 238,364|
| 10 | 49,322 | 11,071 | $8.12 \times 10^{10}$ | 933,394|

6.2 CG on real-world networks

**Experimental Setting:** We applied our COMBWM with ZDDs to the CG, which is a multi-player version of the OSP, on two real-world networks. The CG is described as follows: Given $m$ players and an undirected network with a starting node $s$ and a goal node $t$, the players concurrently send a message from $s$ to $t$. The aim of each player is to minimize the cumulative time needed to send $T$ messages. In this problem, an arm corresponds to an edge of a given network, and a super arm is an $s$-$t$ path. The loss value of an arm is the transmission time required when using the edge, and the cost of a super arm is the total transmission time needed to send a message along the selected $s$-$t$ path. In the experiments, we assume that the loss of each edge increases with the number of players who use the same edge at the same time; therefore, a player regards the other players as adversaries. We use $X^k_t \in S$ ($k \in \{m\}$) to denote the $k$-th player’s choice in the $t$-th round and use $X^k_{t,i} \in \{0, 1\}$ to denote the $i$-th element of $1_X^k_t$. We also use $\ell^k_{t,i}$ to denote the transmission time that the $k$-th player consumes when sending a message using the $i$-th edge at the $t$-th round. We here define $\ell^k_{t,i} := \beta_i \kappa N^{t,k}_{i} - k$ where $\beta_i \in \mathbb{R}$ is the length of the edge, $\kappa$ is an overhead constant, and $N^{t,k}_{i} := \sum_{k' \neq k} X^{k'}_{t,i}$ is the number of adversaries who also choose the $i$-th edge at the $t$-th round. Namely, we assume that the transmission time of each edge increases exponentially with the number of players using the same edge at the same time. Consequently, to reduce the total transmission time, the players should adaptively avoid contending with each other. Note that this setting violates the assumption $|c_t| < 1$; however, in practice, this violation barely matters. In the experiments, we set $m = 2$ and $\kappa = 10$.

We use two real-world communication networks in the Internet topology zoo [23]: the InternetMCI network (MCI) and the ATT North America network (ATT). Figure 2 (e) and (f) illustrate the topologies of the MCI and ATT, respectively. Both networks correspond to the U.S. map and we choose Los Angeles as the starting point $s$ and New York as the goal $t$. The statistics for each network are shown in Table 2.

Table 2: Statistics for two real-world communication networks.

| Network | # nodes | # edges | # s-t paths | $|S|$ | ZDD size | $|V|$ |
|---------|---------|---------|-------------|------|----------|------|
| MCI     | 19      | 33      | 1,444       | 756  | 12,397   | 756  |
| ATT     | 25      | 56      | 213,971     | 37,776| 238,364  | 37,776|

**Experimental Results:** Figures 2 (c) and (d) show the regret values of each player for the MCI and ATT, respectively. The figure shows that each player attained sublinear regrets. Figures 2 (e) and (f) show the top two most frequently selected paths for each player. We see that each player successfully avoided congestion. In the full information setting where the players can observe the costs of all s-t paths after choosing the current path, it is known that the Hedge algorithm [13] can achieve the Nash equilibria [22] on the CG. In this experiment, even though we employed the bandit setting where each player can only observe the cost of the selected path, the players successfully found almost optimal strategies on both networks. To conclude, the experimental results suggest that our algorithm is useful for adaptive routing problems on real-world networks.
Figure 2: (a) and (b) show the regret values for the OSP and DST, respectively. The regret values are averaged over 100 trials and the error bars indicate the standard deviations. (c) and (d) show the regret values of each player for the CG on the MCI and ATT, respectively. (e) and (f) are the topologies of the two networks; the triangles are the starting nodes and the squares are the goal nodes. The red (blue) paths indicate the top two paths most frequently chosen by player 1 (2).

7 Conclusion

We proposed CombWM with ZDDs, which is a practical and theoretically guaranteed algorithm for the adversarial CMAB. We also showed that our algorithm is effective for network-based adversarial CMAB instances, which include various important problems such as the OSP, DST, and CG. The efficiency of our algorithm is thanks to the compression of the decision sets via ZDDs, and its time and space complexities are linear in the size of ZDDs; more precisely, they are $O(d|V|)$. We showed experimentally that the ZDDs for the OSP, DST, and CG are much smaller in size than original decision sets. Our algorithm is also theoretically guaranteed to achieve either $O(T^{2/3})$ regret with high probability or $O(\sqrt{T})$ expected regret as an any-time guarantee; we experimentally confirmed that our algorithm attained sublinear regrets. The results on CG showed that our algorithm is useful for adaptive routing problems on real-world networks.
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Supplementary material

In what follows we prove Theorem 1 and Theorem 2. Section S1 presents two concentration inequalities that are important in the proofs. In Section S2 we provide some preliminaries for the proofs. Section S3 and Section S4 provide the proofs of Theorem 1 and Theorem 2 respectively.

S1 Concentration inequalities

The following concentration inequalities play crucial roles in the subsequent discussion.

**Theorem 3** (Azuma-Hoeffding inequality). If a martingale difference sequence \( \{Z_t\}_{t=1}^T \) satisfies \( a_t \leq Z_t \leq b_t \) almost surely with some constants \( a_t, b_t \) for \( t = 1, \ldots, T \), then the following inequality holds with probability at least \( 1 - \delta \):

\[
\sum_{t=1}^T Z_t \leq \sqrt{\frac{\ln(1/\delta)}{2} \sum_{t=1}^T (b_t - a_t)^2}.
\]

**Theorem 4** (Bennett’s inequality [12]). If a supermartingale difference sequence \( \{Z_t\}_{t=1}^T \) with respect to a filtration \( \{\mathcal{F}_t\}_{t=0}^T \) satisfies \( Z_t \leq b \) with some constant \( b > 0 \) for \( t = 1, \ldots, T \), then, for any \( v \geq 0 \), we have the following with probability at least \( 1 - \delta \):

\[
\sum_{t=1}^T \text{Var}[Z_t | \mathcal{F}_{t-1}] \geq v \quad \text{or} \quad \sum_{t=1}^T Z_t \leq \frac{b}{3} \ln \frac{1}{\delta} + \sqrt{2v \ln \frac{1}{\delta}}.
\]

S2 Preliminaries for the proofs

**Algorithm 2** COMBWM(\( \alpha, S \))

1: \( \tilde{w}_{1,i} \leftarrow 1 \) and \( w_{1,i} \leftarrow 1 \) (\( i \in E \))
2: for \( t = 1, \ldots, T \) do
3: \( \gamma_t \leftarrow \frac{t-1/\alpha}{2}, \eta_t \leftarrow \frac{t-1/\alpha}{2L}, \eta_{t+1} \leftarrow \frac{t+1-1/\alpha}{2L^2} \)
4: \( X_t \sim \mathcal{P}_t \)
5: \( c_t \leftarrow \ell_t \cdot 1_{X_t} \) (\( \ell_t \) is unobservable)
6: \( P_t \leftarrow (1 - \gamma_t)Q_t + \gamma_t U \)
7: \( \hat{\ell}_t \leftarrow c_t P_t^{\dagger} 1_{X_t} \)
8: \( w_{t+1,i} \leftarrow \tilde{w}_{t,i} \exp \left( -\eta_t \hat{\ell}_{t,i} \right) \) (\( i \in E \))
9: \( \tilde{w}_{t+1,i} \leftarrow u_{t+1,i}^{n+1/2} \) (\( i \in E \))
10: end for
11: return \( \{X_t \mid t \in [T]\} \)

We here rewrite Algorithm 1 equivalently as in Algorithm 2 which will be helpful in terms of understanding the subsequent discussion. In what follows, we let \( K := |S| \) and \( \mu := 1/K \). We also define \( E_t[\cdot] := \mathbb{E}[\cdot \mid X_{1:t-1}, \ell_{1:t}] \) as the conditional expectation in the \( t \)-th round given all the history of rounds 1, \ldots, \( t-1 \) and the loss vector in round \( t \). Similarly, we define the conditional variance in round \( t \) as \( \text{Var}_t[\cdot] := \text{Var}[\cdot \mid X_{1:t-1}, \ell_{1:t}] \). For any vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) and \( p > 0 \), we define the \( p \)-norm of \( x \) as \( \|x\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \), and we often use \( \|x\|_2 \) to express \( \|x\|_2 \). For any matrix \( P \in \mathbb{R}^{n \times n} \), we denote its \( i,j \) entry as \( P(i,j) \). We define the trace of \( P \) as \( \text{Tr}(P) := \sum_{i=1}^n P(i,i) \) and denote the spectral norm of \( P \) as \( \|P\| \), i.e., \( \|P\| \) is the largest singular value of \( P \). For any symmetric matrices \( P, Q \in \mathbb{R}^{n \times n} \), we use \( P \succeq Q \) to express the fact that the smallest eigenvalue of \( P - Q \) is non-negative.
For all \( t \in [T] \), we define distributions \( u \) and \( q_t \) over \( \mathcal{S} \), and \( d \times d \) matrices \( U \) and \( Q_t \) as follows:

\[
\begin{align*}
    u(X) & := p(X; 1_E, \mathcal{S}) = \mu, \quad q_t(X) := p(X; \vec{w}_t, \mathcal{S}) = \frac{\vec{w}_t(X)}{\sum_{X' \in \mathcal{S}} \vec{w}_t(X')}, \\
    U & := \mathbb{E}_{X \sim u} [1_X 1_X^\top] = \sum_{X \in \mathcal{S}} \mu 1_X 1_X^\top, \quad Q_t := \mathbb{E}_{X \sim q_t} [1_X 1_X^\top] = \sum_{X \in \mathcal{S}} q_t(X) 1_X 1_X^\top,
\end{align*}
\]

where \( \vec{w}_t(X) \) is an abbreviation of \( \prod_{i \in E} \vec{w}_{t,i} \). Note that we have the following for any \( X \in \mathcal{S} \) and \( t \in [T] \):

\[
    p_t(X) = (1 - \gamma_t)q_t(X) + \gamma_t u(X), \\
    P_t = \mathbb{E}_{X \sim p_t} [1_X 1_X^\top] = \sum_{X \in \mathcal{S}} p_t(X) 1_X 1_X^\top = (1 - \gamma_t)Q_t + \gamma_t U,
\]

where \( p_t(X) \) and \( P_t \) are those defined in Eq. (2) and (3), respectively. We note that the weight values \( \vec{w}_{t,i} (i \in E) \) defined in Step 8 of Algorithm 2 satisfy the following for any \( X \in \mathcal{S} \) and \( t \geq 2 \):

\[
\begin{align*}
    (S1) \quad w_t(X) & = \exp \left( - \eta_{t-1} \sum_{t'=1}^{t-1} \vec{\ell}_{t'} 1_X \right), \\
    (S2) \quad q_t(X) & = \frac{w_t(X)}{\sum_{X' \in \mathcal{S}} w_t(X')}. 
\end{align*}
\]

For convenience, we let \( \eta_0 := \eta_1 \) in what follows, which makes Eq. (S2) hold for \( t = 1 \) since we have \( w_{1,i} = \vec{w}_{1,i} = 1 \) for all \( i \in E \).

Recall that \( \lambda \) is the smallest non-zero eigenvalue of \( U = \mathbb{E}_{X \sim u} [1_X 1_X^\top] \), and that \( \|c_t\| \leq 1 \) holds because of the loss value assumption. The following basic results will be used repetitively in what follows.

**Lemma 1 (Basic results).** For any \( X \in \mathcal{S} \) and \( t \in [T] \), we have

\[
\begin{align*}
    (S3) \quad \|P_t^+\| & \leq \frac{1}{\gamma_t \lambda} \quad \text{and} \quad |\vec{\ell}_t 1_X| \leq \frac{L^2}{\gamma_t \lambda}, \\
    (S4) \quad \mathbb{E}_t [1_{X_t} P_t 1_{X_t}] & \leq d, \\
    (S5) \quad P_t P_t^+ 1_X = 1_X, \\
    (S6) \quad \mathbb{E}_t [\vec{\ell}_t 1_X] = \vec{\ell}_t 1_X.
\end{align*}
\]

**Proof.** The first inequality of Eq. (S3) comes from \( P_t \geq \gamma_t U \), and the second one is obtained from \( |c_t| \leq 1 \) as follows:

\[
|\vec{\ell}_t 1_X| = |c_t 1_{X_t} P_t 1_X| \leq \|1_{X_t}\| \|P_t^+\| \|1_X\| \leq \frac{L^2}{\gamma_t \lambda}.
\]

Eq. (S4) can be obtained as follows:

\[
\mathbb{E}_t [1_{X_t} P_t^+ 1_{X_t}] = \mathbb{E}_t [\text{Tr}(P_t^+ 1_{X_t} 1_{X_t})] = \text{Tr}(P_t^+ P_t) \leq d.
\]

The proof of Eq. (S5) is presented in [9] Lemma 14. Finally, Eq. (S6) is obtained with Eq. (S5) as follows:

\[
\mathbb{E}_t [\vec{\ell}_t 1_X] = \mathbb{E}_t [\vec{\ell}_t 1_{X_t} 1_{X_t} P_t^+ 1_{X_t}] = \vec{\ell}_t^\top P_t P_t^+ 1_X = \vec{\ell}_t^\top 1_X.
\]

\( \square \)
S3 Proof for the high-probability regret bound

We show the complete proof of Theorem 1. Below is a detailed statement of the theorem.

**Theorem 5.** The sequence of super arms \( \{X_t\}_{t \in [T]} \) obtained by \textit{COMBWM} (\( \alpha = 3, \mathcal{S} \)) satisfies the following inequality for any \( X \in \mathcal{S} \) with probability at least \( 1 - \delta \):

\[
\sum_{t=1}^{T} (\ell_t^T 1_{X_t} - \ell_t^T 1_X) \leq \left( 3d(e-2)\lambda \frac{4L^2}{\lambda} + \frac{3}{2} + L \sqrt{\frac{7}{\lambda} \ln \frac{K+2}{\delta}} \right) T^{2/3} + o(T^{2/3}).
\]

Let \( \bar{x}_t := \sum_{X \in \mathcal{S}} q_t(X) 1_X \). As in [7], the proof is obtained by bounding each term on the right hand side of the following equation:

\[
\sum_{t=1}^{T} (\ell_t^T 1_{X_t} - \ell_t^T 1_X) = \sum_{t=1}^{T} (\ell_t^T 1_{X_t} - \ell_t^T \bar{x}_t) + \sum_{t=1}^{T} (\ell_t^T \bar{x}_t - \ell_t^T 1_X) + \sum_{t=1}^{T} (\ell_t^T 1_X - \ell_t^T 1_X),
\]

where \( X \in \mathcal{S} \) is an arbitrary super arm. To bound them, we prove the following three lemmas.

**Lemma 2.** For any \( X \in \mathcal{S} \), we have

\[
\sum_{t=1}^{T} (\ell_t^T \bar{x}_t - \ell_t^T 1_X) \leq \frac{\ln K}{\eta_t} + (e-2) \left( \frac{d}{2} \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t} + L^2 \left[ \frac{1}{2} - \ln \frac{1}{\delta} \sum_{t=1}^{T} \frac{\eta_t^2}{1 - 2 \eta_t} \right] \right)
\]

with probability at least \( 1 - \delta \).

**Proof.** With the weight values \( w_{t,i} (i \in E) \) used in Algorithm 2, we define \( w_t(X) := \prod_{i \in X} w_{t,i} \) and \( W_t := \sum_{X \in \mathcal{S}} w_t(X) \); we measure the progress of the algorithm in each round via \( \ln(W_t^{\eta_t^{-1}}/W_t^{\eta_t^{-1}-1}) \). By Hölder’s inequality, \( \|x\|_s \geq K^{1/s - 1} \|x\|_r \) holds for any \( x \in \mathbb{R}^K \) and \( 0 < r \leq s \). Thus, letting \( s = \eta_t^{-1} / \eta_t \) and \( r = 1 \), we obtain

\[
W_t^{\eta_t^{-1}} = \left( \sum_{X \in \mathcal{S}} w_t(X)^{\eta_t^{-1}/\eta_t} \right)^{\eta_t^{-1}} \geq K^{1/\eta_t} \left( \sum_{X \in \mathcal{S}} w_t(X)^{\eta_t^{-1}} \right)^{\eta_t^{-1} - 1}. \]

Hence we have

\[
\ln \frac{W_t^{\eta_t^{-1}}}{W_t^{\eta_t^{-1}-1}} - \ln K^{1/\eta_t} \leq \frac{1}{\eta_t} \ln \frac{W_t^{\eta_t^{-1}}}{\sum_{X \in \mathcal{S}} w_t(X)^{\eta_t^{-1}}} \leq \frac{1}{\eta_t} \ln \frac{\sum_{X \in \mathcal{S}} w_t(X)^{\eta_t^{-1}} \exp(-\eta_t \ell_t^T 1_X)}{\sum_{X \in \mathcal{S}} w_t(X)^{\eta_t^{-1}}} \leq \frac{1}{\eta_t} \ln \sum_{X \in \mathcal{S}} q_t(X) \left( 1 - \eta_t \ell_t^T 1_X + (e-2) \eta_t^2 (\ell_t^T 1_X)^2 \right) \leq \frac{1}{\eta_t} \ln \left( 1 - \eta_t \ell_t^T \bar{x}_t + (e-2) \eta_t^2 \sum_{X \in \mathcal{S}} q_t(X)(\ell_t^T 1_X)^2 \right) \leq -\ell_t^T \bar{x}_t + (e-2) \eta_t \sum_{X \in \mathcal{S}} q_t(X)(\ell_t^T 1_X)^2,
\]

3
where the second inequality comes from $e^{-x} \leq 1 - x + (e - 2)x^2$ for any $|x| \leq 1$; note that $\eta_t$ is defined to satisfy $\eta_t \| \hat{\ell}^T_t \mathbf{1}_X \| \leq \eta_t L^2 / (\gamma_t \lambda) = 1$. The third inequality is obtained by $\ln(1 + x) \leq x$ for any $x \geq -1$.

The second term on the right hand side can be bounded from above by using the Azuma–Hoeffding inequality as follows:

$$
\sum_{X \in S} q_t(X) (\hat{\ell}^T_t \mathbf{1}_X)^2 \leq \sum_{X \in S} p_t(X) (\ell^T_t \mathbf{1}_X)^2 \leq \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t}.
$$

Therefore, we have

$$
\frac{1}{\eta_t} \ln W_{t+1} - \frac{1}{\eta_{t-1}} \ln W_t \leq \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \ln K - \hat{\ell}^T_t \mathbf{x}_t + (e-2) \eta_t \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t}.
$$

Summing up both sides of the above for $t = 1, \ldots, T$, we obtain the following inequality from $W_1 = K$:

$$
\frac{1}{\eta_T} \ln W_{T+1} \leq \frac{1}{\eta_T} \ln K - \sum_{t=1}^T \hat{\ell}^T_t \mathbf{x}_t + (e-2) \sum_{t=1}^T \eta_t \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t}.
$$

On the other hand, we have $w_{t+1} = \exp \left( - \eta_T \sum_{t=1}^T \hat{\ell}_{t,i} \right)$ by Eq. (S1). Thus the following holds for any $X \in S$:

$$
\frac{1}{\eta_T} \ln W_{T+1} \geq \frac{1}{\eta_T} \ln w_{T+1}(X) = - \sum_{t=1}^T \hat{\ell}^T_t \mathbf{1}_X.
$$

Therefore, we obtain

$$
\sum_{t=1}^T (\hat{\ell}^T_t \mathbf{x}_t - \hat{\ell}^T_t \mathbf{1}_X) \leq \frac{\ln K}{\eta_T} + (e-2) \sum_{t=1}^T \eta_t \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t}.
$$

The second term on the right hand side can be bounded from above by using the Azuma–Hoeffding inequality (Theorem 3) for the martingale difference sequence $\frac{\eta_t}{1-\gamma_t} (1_{X_t} P_t^+ 1_{X_t} - E_t[1_{X_t} P_t^+ 1_{X_t}])$ as follows. First, note that we have

$$
\mathbb{E}_t[1_{X_t} P_t^+ 1_{X_t}] \leq d \quad \text{and} \quad 0 \leq \eta_t \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t} \leq \frac{\eta_t L^2}{(1-\gamma_t) \gamma_t \lambda}
$$

by Lemma 3. Thus, by the Azuma-Hoeffding inequality, the following holds with probability at least 1 $- \delta$:

$$
\sum_{t=1}^T \eta_t \frac{1}{1-\gamma_t} 1_{X_t} P_t^+ 1_{X_t} \leq d \sum_{t=1}^T \frac{\eta_t}{1-\gamma_t} + \frac{2 \ln(1/\delta)}{\lambda} \sqrt{\frac{\ln(1/\delta)}{2} \sum_{t=1}^T \frac{\eta_t^2}{\gamma_t^2 (1-\gamma_t)^2}}.
$$

Hence we obtain

$$
\sum_{t=1}^T (\hat{\ell}^T_t \mathbf{x}_t - \hat{\ell}^T_t \mathbf{1}_X) \leq \frac{\ln K}{\eta_T} + (e-2) \left( d \sum_{t=1}^T \frac{\eta_t}{1-\gamma_t} + \frac{L^2}{\lambda} \sqrt{\frac{\ln(1/\delta)}{2} \sum_{t=1}^T \frac{\eta_t^2}{\gamma_t^2 (1-\gamma_t)^2}} \right).
$$

$$
\square
$$

**Lemma 3.** The following inequality holds with probability at least 1 $- \delta$:

$$
\sum_{t=1}^T (\ell^T_t \mathbf{1}_{X_t} - \hat{\ell}^T_t \mathbf{x}_t) \leq 2 \sum_{t=1}^T \gamma_t + \frac{1}{3} \left( 2 + \frac{L}{\sqrt{N} \gamma_T (1-\gamma_T)} \right) \ln \frac{1}{\delta} + \sqrt{2 \left( T + \frac{3L^2}{\lambda} \sum_{t=1}^T \frac{\gamma_t}{(1-\gamma_t)^2} \right) \ln \frac{1}{\delta}}.
$$
Proof. Let $z := \sum_{X \in S} \mu_1 X$ and $\bar{x}_t := \mathbb{E}_t[1_X] = (1 - \gamma_t)\bar{x}_t + \gamma_t z$. We obtain the proof by using Bennett’s inequality (Theorem 3) for the martingale difference sequence

$$
Y_t := \ell_t^\top 1_X - \ell_t^\top \bar{x}_t - \mathbb{E}_t[\ell_t^\top 1_X, - \ell_t^\top \bar{x}_t]
= \ell_t^\top 1_X - \ell_t^\top \bar{x}_t - \ell_t^\top \bar{x}_t + \ell_t^\top \bar{x}_t
= \ell_t^\top 1_X - \ell_t^\top \bar{x}_t + \gamma_t \ell_t^\top (\bar{x}_t - z).
$$

We first bound the values of $|Y_t|$ and $\text{Var}_t[Y_t]$. By $Q_t \leq \frac{1}{1 - \gamma_t} P_t$ and Jensen’s inequality $\bar{x}_t \bar{x}_t^\top \leq Q_t$, we have

$$
(\ell_t^\top \bar{x}_t)^2 \leq c_1^2 1_i X_t P_t^+ Q_t 1_t X_t \leq \frac{1_t X_t P_t^+ 1_t X_t}{1 - \gamma_t} \leq \frac{L^2}{\lambda \gamma_t (1 - \gamma_t)},
$$

and hence

$$
|Y_t| \leq 1 + \frac{L}{\sqrt{\lambda \gamma_T (1 - \gamma_T)}} + 2\gamma_T \leq 2 + \frac{L}{\sqrt{\lambda \gamma_T (1 - \gamma_T)}}.
$$

The variance of $Y_t$ is bounded as follows:

$$
\text{Var}_t[Y_t] \leq \mathbb{E}_t[(\ell_t^\top 1_X - \ell_t^\top \bar{x}_t)^2] = \mathbb{E}_t[c_t^2 (1 - 1_X) P_t^+ \bar{x}_t]^2 \leq \mathbb{E}_t[(1 - 1_X) P_t^+ \bar{x}_t]^2
= 1 - 2\bar{x}_t^\top P_t^+ \bar{x}_t + \bar{x}_t^\top P_t^+ 1_X 1_X^\top P_t^+ \bar{x}_t
= 1 - \frac{2}{1 - \gamma_t} \bar{x}_t^\top P_t^+ (\bar{x}_t - \gamma_t z) + \frac{1}{(1 - \gamma_t)^2} (\bar{x}_t - \gamma_t z)^\top P_t^+ (\bar{x}_t - \gamma_t z)
= 1 - \frac{1 - 2\gamma_t}{(1 - \gamma_t)^2} \bar{x}_t^\top P_t^+ \bar{x}_t + \frac{\gamma_t^2}{(1 - \gamma_t)^2} (z - 2\bar{x}_t)^\top P_t^+ z
\leq 1 + \frac{\gamma_t^2}{(1 - \gamma_t)^2} (z - 2\bar{x}_t)^\top P_t^+ z
\leq 1 + \frac{3\gamma_t L^2}{(1 - \gamma_t)^2}.
$$

where the third inequality comes from $1 - 2\gamma_t \geq 0$, and the last inequality is obtained by Lemma 1 with $\|z\| \leq L$ and $\|\bar{x}_t\| \leq L$. Therefore, by using Bennett’s inequality, we obtain

$$
\sum_{t=1}^T (\ell_t^\top 1_X - \ell_t^\top \bar{x}_t + \gamma_t \ell_t^\top (\bar{x}_t - z)) \leq \frac{1}{3} \left(2 + \frac{L}{\sqrt{\lambda \gamma_T (1 - \gamma_T)}}\right) \ln \frac{1}{\delta} + \sqrt{2 \left(T + \frac{3L^2}{\lambda} \sum_{t=1}^T \frac{\gamma_t}{(1 - \gamma_t)^2}\right) \ln \frac{1}{\delta}}.
$$

The proof is completed by $|\ell_t^\top (\bar{x}_t - z)| \leq 2$. \qed

Lemma 4. The following inequality holds for all $X \in S$ simultaneously with probability $1 - \delta$:

$$
\sum_{t=1}^T (\ell_t^\top 1_X - \ell_t^\top 1_X) \leq \frac{1}{3} \left(1 + \frac{L^2}{\gamma_T \lambda}\right) \ln \frac{K}{\delta} + \sqrt{\frac{2L^2}{\lambda} \ln \frac{K}{\delta} \sum_{t=1}^T \frac{1}{\gamma_t}}.
$$

Proof. We fix $X \in S$ arbitrarily. The proof is obtained by using Bennett’s inequality for the martingale difference sequence $\ell_t^\top 1_X - \ell_t^\top 1_X$; note that $\mathbb{E}_t[\ell_t^\top 1_X - \ell_t^\top 1_X] = 0$ holds by Lemma 1. First, the
absolute value and variance of $\ell_t^\top 1_X - \ell_t^\top 1_X$ are bounded as follows:

$$|\ell_t^\top 1_X - \ell_t^\top 1_X| \leq 1 + \frac{L^2}{\gamma_t \lambda},$$

$$\text{Var}[\ell_t^\top 1_X - \ell_t^\top 1_X] \leq E_t[(\ell_t^\top 1_X)^2] \leq E_t[1^T_P^+ 1_X, 1^T_X, P^+_t 1_X] \leq 1^T P^+_t 1_X \leq \frac{L^2}{\gamma_t \lambda}.$$ 

Hence, by Bennett’s inequality, we have

$$\sum_{t=1}^T (\ell_t^\top 1_X - \ell_t^\top 1_X) \leq \frac{1}{3} \left( 1 + \frac{L^2}{\gamma_T \lambda} \right) \ln \frac{K}{\delta} + \sqrt{\frac{2L^2}{\lambda} \ln \frac{K}{\delta} \sum_{t=1}^T \frac{1}{\gamma_t}}$$

with probability at least $1 - \delta/K$. Taking the union bound over all super arms $X \in \mathcal{S}$, we obtain the claim. 

Using the above three lemmas, we prove Theorem 3 as follows.

**Proof of Theorem 3** Note that we have $\gamma_t = \frac{t^{1/3}}{L}$ and $\eta_t = \frac{\lambda}{L} \gamma_t = \frac{\lambda t^{1/3}}{2L^2}$. By using Lemma 2, we have the following with probability at least $1 - \delta/(K + 2)$:

$$\sum_{t=1}^T (\ell_t^\top x_t - \hat{\ell}_t^\top 1_X) \leq \frac{\ln K}{\eta_T} + (e - 2) \left( d \sum_{t=1}^T \frac{\eta_t}{1 - \gamma_t} + \frac{L^2}{\lambda} \sqrt{\frac{1}{2} \ln \frac{K + 2}{\delta} \sum_{t=1}^T \frac{\eta_t^2}{\gamma_t (1 - \gamma_t)^2}} \right)$$

$$\leq \frac{2L^2}{\lambda} \ln K T^{1/3} + (e - 2) \left( \frac{3d}{2L^2} T^{1/3} + \frac{3}{2} \sqrt{\frac{2L^2}{\lambda} T^{1/3}} \right) \ln \frac{K + 2}{\delta}. $$

We also obtain the following inequality with probability at least $1 - \delta/(K + 2)$ by using Lemma 3:

$$\sum_{t=1}^T (\ell_t^\top 1_X - \ell_t^\top \check{x}_t)$$

$$\leq 2 \sum_{t=1}^T \gamma_t + \frac{1}{3} \frac{L}{\lambda \sqrt{T (1 - \gamma_T)}} \ln \frac{K + 2}{\delta} + \sqrt{2 \left( T + \frac{3L^2}{\lambda} \sum_{t=1}^T \frac{\gamma_t}{(1 - \gamma_t)^2} \right) \ln \frac{K + 2}{\delta}}$$

$$\leq \frac{3}{2} \frac{L^2}{\lambda} T^{1/3} + \frac{1}{3} \left( \frac{\sqrt{2} L}{\lambda} \sqrt{T (1 - \gamma_T)} \right) \ln \frac{K + 2}{\delta} + \sqrt{2 \left( T + \frac{9L^2}{\lambda} T^{2/3} \right) \ln \frac{K + 2}{\delta}}.$$ 

Furthermore, we have the following inequality with probability at least $1 - K \delta/(K + 2)$ by using Lemma 4:

$$\sum_{t=1}^T (\ell_t^\top 1_X - \ell_t^\top 1_X) \leq \frac{1}{3} \left( 1 + \frac{L^2}{\gamma_T \lambda} \right) \ln \frac{K + 2}{\delta} + \sqrt{\frac{2L^2}{\lambda} \ln \frac{K + 2}{\delta} \sum_{t=1}^T \frac{1}{\gamma_t}}$$

$$\leq \frac{1}{3} \left( 1 + \frac{2L^2}{\lambda} T^{1/3} \right) \ln \frac{K + 2}{\delta} + \sqrt{\frac{2L^2}{\lambda} \ln \frac{K + 2}{\delta} \sum_{t=1}^T \frac{1}{\gamma_t}}$$

Summing up both sides of the three inequalities and taking the union bound, we obtain the theorem. □
We then show the proof of Theorem 2; the detailed statement is as follows.

**Theorem 6.** The sequence of super arms \( \{X_t\}_{t \in [T]} \) obtained by COMBWM(\( \alpha = 2, S \)) satisfies the following inequality for any \( X \in S \):

\[
E \left[ \sum_{t=1}^{T} (\ell_t^\top 1_{X_t} - \ell_t^\top 1_X) \right] \leq \left( \frac{2L^2 \ln K}{\lambda} + \frac{(e - 2)d}{L^2} + 2 \right) \sqrt{T} + o(\sqrt{T}).
\]

Let \( \bar{x}_t := \sum_{X \in S} q_t(X)1_X \). The proof is obtained by bounding each term on the right hand side of the following equation for any \( X \in S \):

\[
(S8) \quad E \left[ \sum_{t=1}^{T} (\ell_t^\top 1_{X_t} - \ell_t^\top 1_X) \right] = E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top 1_{X_t} - \ell_t^\top 1_X] \right] = E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top 1_{X_t} - \ell_t^\top \bar{x}_t] \right] + E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top \bar{x}_t - \ell_t^\top 1_X] \right],
\]

where the second equality comes from Lemma 4. To bound these terms, we prove the following two lemmas.

**Lemma 5.** For any \( X \in S \), we have

\[
E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top \bar{x}_t - \ell_t^\top 1_X] \right] \leq \frac{\ln K}{\eta T} + (e - 2)d \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t}.
\]

*Proof.* As in the proof of Lemma 2, we have Eq. (S7):

\[
\sum_{t=1}^{T} (\ell_t^\top \bar{x}_t - \ell_t^\top 1_X) \leq \frac{\ln K}{\eta T} + (e - 2) \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t}.
\]

Taking the expectation of both sides, we obtain

\[
E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top \bar{x}_t - \ell_t^\top 1_X] \right] \leq \frac{\ln K}{\eta T} + (e - 2) E \left[ \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t} E_t[1_{X_t}^\top P_t^+ 1_{X_t}] \right]
\]

\[
\leq \frac{\ln K}{\eta T} + (e - 2) d \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t},
\]

where the second inequality is obtained by Lemma 4. \( \square \)

**Lemma 6.** The following inequality holds:

\[
E \left[ \sum_{t=1}^{T} E_t[\ell_t^\top 1_{X_t} - \ell_t^\top \bar{x}_t] \right] \leq 2 \sum_{t=1}^{T} \gamma_t.
\]

*Proof.* Since \( |\ell_t^\top 1_X| \leq 1 \) holds for any \( X \in S \), we have

\[
E_t[\ell_t^\top 1_{X_t} - \ell_t^\top \bar{x}_t] = E_t[\ell_t^\top 1_{X_t}] - \ell_t^\top \bar{x}_t = \sum_{X \in S} p_t(X)\ell_t^\top 1_X - \sum_{X \in S} q_t(X)\ell_t^\top 1_X
\]

\[
= \sum_{X \in S} \gamma_t \mu_{t, X} 1_X - \sum_{X \in S} \gamma_t q_t(X)\ell_t^\top 1_X \leq 2 \gamma_t.
\]

Summing up both sides for \( t = 1, \ldots, T \) and taking the expectation, we obtain the claim. \( \square \)

We now prove Theorem 6 as follows.
Proof of Theorem 6. Recall that we have \( \gamma_t = \frac{t^{1/2}}{2} \) and \( \eta_t = \frac{\lambda}{L^2} \gamma_t = \frac{\lambda^{t-1/2}}{2L^2} \). The proof is readily obtained by Eq. (SS), Lemma 5 and Lemma 6 as follows:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} (\ell_t^T 1_X - \hat{\ell}_t^T 1_X) \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t [\ell_t^T 1_X - \hat{\ell}_t^T \tilde{x}_t] \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}_t [\hat{\ell}_t^T \tilde{x}_t - \hat{\ell}_t^T 1_X] \right] \\
\leq \frac{\ln K}{\eta T} + (e-2)\frac{d}{T} \sum_{t=1}^{T} \frac{\eta_t}{1 - \gamma_t} + 2 \sum_{t=1}^{T} \gamma_t \\
\leq \frac{2L^2 \ln K}{\lambda} \sqrt{T} + \frac{(e-2)\lambda d}{L^2} \left( \sqrt{T} + \frac{1}{2} \ln(2\sqrt{T} - 1) \right) + 2\sqrt{T} - 1.
\]

\( \square \)