MEASURING THE WESS-ZUMINO
ANOMALY IN TAU DECAYS

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Abstract

We propose to measure the Wess-Zumino anomaly contribution by considering angular distributions in the decays $\tau \to \nu_\tau \eta \pi^- \pi^0$, $\tau \to \nu_\tau K^- \pi^- K^+$ and $\tau \to \nu_\tau K^- \pi^- \pi^+$. Radial excitations of the $K^*$, which cannot be seen in $e^+ e^-$, should be observed in the $K^- \pi^+ \pi^-$ decay channel.
With the experimental progress in $\tau$-decays an ideal tool for studying strong interaction physics has been developed. In this paper we show that several decays can be used to test the Wess-Zumino anomaly [1]. It appears that the anomaly violates the rule that the weak axialvector- and vector-currents produce an odd and an even number of pseudoscalars, respectively. From its structure it can be seen that the anomaly contributes possibly to $\tau$ decays into $\nu_\tau + n$ mesons (with $n \geq 3$). Of course the goldenplated decay is $\tau \to \nu_\tau \eta \pi^- \pi^0$ which has a vanishing contribution from the axialvector-current [2, 3, 4]. Therefore a detected $\eta \pi^- \pi^0$ final state implies a nonvanishing anomaly. A recent measurement of its width [5] confirms the CVC predictions [3, 4]. In this paper we will present the most general angular distribution of the $\eta \pi^- \pi^0$ system in terms of the vector current formfactor for this channel. Furthermore we demonstrate that also other decays modes into three pseudoscalars can be used to confirm (not only qualitatively) the presence of the Wess-Zumino anomaly. However, the most prominent decay channel into three pseudoscalars, i.e. $\pi^- \pi^- \pi^+$, cannot be used since $G$-parity forbids the anomaly to contribute. Beside the $\eta \pi^- \pi^0$ channel interesting candidates are the decay channels $K^- \pi^- K^+$ and $K^- \pi^- \pi^+$. Recently the corresponding branching ratios have been reconsidered [4] and it appears that the branching ratios alone are not sufficient to determine the presence of the anomaly. In the course of this paper we will show that a detailed study of angular distributions, as defined in [6, 7], is well suited to extract the contribution of the anomaly.

Our paper is organized as follows:

In section 2 we introduce the kinematical parameters which are adapted to the present experimental situation where the direction of flight of the tau lepton can not be reconstructed and only the hadrons are detected. Then we present, following [6, 7], the most general angular distribution of the three hadrons in terms of hadronic structure functions. The dependence of the $\tau$-polarization is included. By considering adequate moments in section 3 we show that all of the hadronic structure functions can be measured without reconstructing the $\tau$-restframe. Section 4 is devoted to the hadronic model [1] encoded in the structure functions. We present explicit parametrizations of the formfactors for the decays into $\eta \pi^- \pi^0$, $K^- \pi^- K^+$ and $K^- \pi^- \pi^+$. The different parameters of the model have been moved to appendix A.

Finally numerical results for the hadronic structure functions of the considered channels are presented in section 5, proving that an experimental determination of the anomaly is feasible. We anticipate our results and urge experimentalists to analyse the $K^- \pi^- \pi^+$-channel which could contain radial excitations of the $K^*$ which can not be obtained in $e^+e^-$ experiments.
2 Lepton Tensor and Angular Distributions

Let us consider the following \( \tau \)-decay

\[
\tau (l, s) \rightarrow \nu(l', s') + h_1(q_1, m_1) + h_2(q_2, m_2) + h_3(q_3, m_3)
\]

where \( h_i(q_i, m_i) \) are pseudoscalar mesons. The matrix element reads as

\[
\mathcal{M} = \frac{G}{\sqrt{2}} (\cos \theta_c \sin \theta_c) M_\mu J^\mu
\]

with \( G \) the Fermi-coupling constant. The cosine and the sine of the Cabbibo angle \( (\theta_C) \) in (2) have to be used for Cabibbo allowed \( \Delta S = 0 \) and Cabibbo suppressed \( |\Delta S| = 1 \) decays, respectively. The leptonic \( (M_\mu) \) and hadronic \( (J^\mu) \) currents are given by

\[
M_\mu = \bar{u}(l', s') \gamma_\mu (1 - \gamma_5) u(l, s)
\]

and

\[
J^\mu(q_1, q_2, q_3) = \langle h_1(q_1) h_2(q_2) h_3(q_3) | V^\mu(0) - A^\mu(0) | 0 \rangle
\]

\( V^\mu \) and \( A^\mu \) are the vector and axialvector quark currents, respectively. The most general ansatz for the matrix element of the quark current \( J^\mu \) in (5) is characterized by four formfactors [7, 4].

\[
J^\mu(q_1, q_2, q_3) = V_1^\mu F_1 + V_2^\mu F_2 + i V_3^\mu F_3 + V_4^\mu F_4
\]

with

\[
V_1^\mu = q_1^\mu - q_3^\mu - Q^\mu \frac{Q(q_1 - q_3)}{Q^2}
\]

\[
V_2^\mu = q_2^\mu - q_3^\mu - Q^\mu \frac{Q(q_2 - q_3)}{Q^2}
\]

\[
V_3^\mu = \epsilon^{\alpha \beta \gamma} q_1 \alpha q_2 \beta q_3 \gamma
\]

\[
V_4^\mu = q_1^\mu + q_2^\mu + q_3^\mu = Q^\mu
\]

The Wess-Zumino anomaly which is of main interest in the present paper give rise to the term proportional to \( F_3 \). The terms proportional to \( F_1 \) and \( F_2 \) originate from the axialvector-current. Together they correspond to a spin one hadronic final state while the \( F_4 \) term is due to the spin zero part of the axialvector-current. As it has been shown in [4] the spin zero contributions are extremely small and we neglect them in the rest of this paper, i.e. \( F_4 \) is set equal to zero.

The differential decay rate is obtained from

\[
d\Gamma(\tau \rightarrow \nu_\tau 3h) = \frac{1}{2m_\tau} \frac{G^2}{2} \left( \frac{\cos^2 \theta_c}{\sin^2 \theta_c} \right) \{ L_{\mu \nu} H^{\mu \nu} \} \ dPS^{(4)}
\]
where $L_{\mu\nu} = M_{\mu}(M_{\nu})^\dagger$ and $H^{\mu\nu} \equiv J^{\mu}(J^{\nu})^\dagger$.

Reaction (1) is most easily analyzed in the hadronic rest frame $\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$. The orientation of the hadronic system is characterized by three Euler angles ($\alpha, \beta$ and $\gamma$) introduced in [6, 7]. In current $e^+ + e^- \rightarrow \tau^+\tau^- (\rightarrow \nu, 3$ mesons) experiments two out of the three Euler angles are measurable. The measurable ones are defined by

$$\cos \beta = \hat{n}_L \cdot \hat{n}_\perp$$
$$\cos \gamma = -\frac{\hat{n}_L \cdot \hat{q}_3}{|\hat{n}_L \times \hat{n}_\perp|}$$

where ($\hat{a}$ denotes a unit three-vector)

- $\hat{n}_L = -\hat{n}_Q$, with $\hat{n}_Q$ the direction of the hadrons in the labframe,
- $\hat{n}_\perp = \hat{q}_1 \times \hat{q}_2$, the normal to the plane defined by the momenta of particles 1 and 2.

Note that the angle $\gamma$ defines a rotation around $\hat{n}_\perp$ and determines the orientation of the three hadrons within their production plane. The definition of the angles $\beta$ and $\gamma$ is shown in fig. 1.

Performing the integration over the unobservable neutrino and the unobservable Euler angle $\alpha$ we obtain the differential decay width for a polarized $\tau [6, 7]$: 

$$d\Gamma(\tau \rightarrow 3h) = \frac{G^2}{2m_\tau} \left( \frac{\cos^2 \theta_c}{\sin^2 \theta_c} \right) \left\{ \sum_X \bar{L}_X W_X \right\} \times$$

$$\frac{1}{(2\pi)^5} \frac{1}{64} \frac{(m_\tau^2 - Q^2)^2}{m_\tau^2} \frac{dQ^2}{Q^2} \frac{ds_1 ds_2}{2\pi} \frac{d\gamma}{\gamma} \frac{d\cos \beta}{\cos \beta} \frac{d\cos \theta}{\cos \theta}$$

In (9) we have defined the invariant masses in the Dalitz plot $s_i = (q_i + q_k)^2$ (where $i, j, k = 1, 2, 3; i \neq j \neq k$) and the square of the invariant mass of the hadron system $Q^2 \equiv (q_1 + q_2 + q_3)^2$. The angle $\theta$ is related to the hadronic energy in the labframe $E_h$ by [8, 6, 7]

$$\cos \theta = \frac{(2xm_\tau^2 - m_\tau^2 - Q^2)}{(m_\tau^2 - Q^2) \sqrt{1 - 4m_\tau^2/s}}$$

with

$$x = 2 \frac{E_h}{\sqrt{s}} \quad s = 4E^2_{beam}$$

Another quantity depending on this energy $E_h$ is

$$\cos \psi = \frac{x(m_\tau^2 + Q^2) - 2Q^2}{(m_\tau^2 - Q^2) \sqrt{x^2 - 4Q^2/s}}$$
which will be of some interest in the subsequent discussion. Finally in the case where the spin zero part of the hadronic current $J^\mu$ is zero ($F_4 = 0$ in (3)),

$$\sum_X L_X W_X$$

is given by a sum of nine terms $L_X W_X$ with $X \in \{A, B, C, D, E, F, G, H, I\}$ corresponding to nine density matrix elements of the hadronic system in a spin one state. One has

$$\bar{L}_A = \frac{2}{3} K_1 + K_2 + \frac{1}{3} K_1 (3 \cos^2 \beta - 1)/2$$

$$\bar{L}_B = \frac{2}{3} K_1 + K_2 - \frac{2}{3} K_1 (3 \cos^2 \beta - 1)/2$$

$$\bar{L}_C = -\frac{1}{2} K_1 \sin^2 \beta \cos 2\gamma$$

$$\bar{L}_D = \frac{1}{2} K_1 \sin^2 \beta \sin 2\gamma$$

$$\bar{L}_E = K_3 \cos \beta$$

$$\bar{L}_F = \frac{1}{2} K_1 \sin 2\beta \cos \gamma$$

$$\bar{L}_G = - K_3 \sin \beta \sin \gamma$$

$$\bar{L}_H = -\frac{1}{2} K_1 \sin 2\beta \sin \gamma$$

$$\bar{L}_I = - K_3 \sin \beta \cos \gamma$$

where

$$K_1 = 1 - P \cos \theta - (m_\tau^2/Q^2) (1 + P \cos \theta)$$

$$K_2 = (m_\tau^2/Q^2) (1 + P \cos \theta)$$

$$K_3 = 1 - P \cos \theta$$

$$K_1 = K_1 (3 \cos^2 \psi - 1)/2 - 3/2 K_4 \sin 2\psi$$

$$K_2 = K_2 \cos \psi + K_4 \sin \psi$$

$$K_3 = K_3 \cos \psi - K_5 \sin \psi$$

$$K_4 = \sqrt{m_\tau^2/Q^2} P \sin \theta$$

$$K_5 = \sqrt{m_\tau^2/Q^2} P \sin \theta$$

In (14) $P$ denotes the polarization of the $\tau$ in the laboratory frame while $\theta$ and $\psi$ are defined in eqs. (10,12). For LEP-physics ($Z$-decay) $P$ is given by $P = \frac{-2 v_{\tau} a_{\tau}}{v_{\tau}^2 + a_{\tau}^2}$ with $v_{\tau} = -1 + 4 \sin^2 \theta_W$ and $a_{\tau} = -1$; while for lower energies $P$ vanishes. In this case (for ARGUS, CLEO) (14) simplifies to

$$K_1 = 1 - m_\tau^2/Q^2 = \frac{2 K_1}{(3 \cos^2 \psi - 1)}$$

$$K_2 = m_\tau^2/Q^2 = \frac{K_2}{\cos \psi}$$

(15)
Note that the full dependence on the $\tau$ polarization $P$, the hadron energy (through $\theta$ and $\psi$) and the angles $\beta$ and $\gamma$ is given in eqs. (13) to (15).

The hadronic functions $W_X$ contain the dynamics of the hadronic decay and depend in general on $s_1, s_2$ and $Q^2$. Let us recall that we are working in the hadronic rest frame with the $z-$ and $x-$ axis aligned along $\hat{n}_\perp$ and $\hat{q}_3$, respectively (see fig. 1). The hadronic tensor $H^{\mu\nu} = J^\mu (J^\nu)^\dagger$ (with $J^\mu$ given in (3)) is calculated in this frame and the hadronic structure functions $W_X$ are linear combinations of density matrix elements which are obtained from

$$H^{\sigma\sigma'} = \epsilon_\mu(\sigma) H^{\mu\nu} \epsilon_\nu^{*}(\sigma')$$  \hspace{1cm} (16)

where

$$\epsilon_\mu(\pm) = \frac{1}{\sqrt{2}} (0; \pm 1, -i, 0) \quad \epsilon_\mu(0) = (0; 0, 0, 1)$$  \hspace{1cm} (17)

are the polarization vectors for a hadronic system in a spin one state defined with respect to the normal on the three meson plane in the hadronic restframe. The pure spin-one structure functions are

\[
\begin{align*}
W_A &= H^{++} + H^{--} = H^{11} + H^{22} \\
W_B &= H^{00} = H^{33} \\
W_C &= -(H^{+-} + H^{-+}) = H^{11} - H^{22} \\
W_D &= i (H^{+-} - H^{-+}) = H^{12} + H^{21} \\
W_E &= H^{++} - H^{--} = -i (H^{12} - H^{21}) \\
W_F &= -(H^{+0} + H^{0+} - H^{-0} - H^{0-})/\sqrt{2} = H^{13} + H^{31} \\
W_G &= i (H^{+0} - H^{0+} - H^{-0} + H^{0-})/\sqrt{2} = -i (H^{13} - H^{31}) \\
W_H &= i (H^{+0} - H^{0+} + H^{-0} - H^{0-})/\sqrt{2} = H^{23} + H^{32} \\
W_I &= (H^{+0} + H^{0+} + H^{-0} + H^{0-})/\sqrt{2} = -i (H^{23} - H^{32})
\end{align*}
\]  \hspace{1cm} (18)

The r.h.s of eqs. (18) refers to the cartesian components of $H^{\mu\nu}$. The structure functions can thus be expressed in terms of the formfactors $F_i$ as follows [7]

\[
\begin{align*}
W_A &= (x_1^2 + x_3^2) |F_1|^2 + (x_2^2 + x_3^2) |F_2|^2 + 2(x_1 x_2 - x_3^2) \text{Re}(F_1 F_2^*) \\
W_B &= x_3^2 |F_3|^2 \\
W_C &= (x_1^2 - x_3^2) |F_1|^2 + (x_2^2 - x_3^2) |F_2|^2 + 2(x_1 x_2 + x_3^2) \text{Re}(F_1 F_2^*)
\end{align*}
\]
\[ W_D \ = \ 2 \left[ x_1 x_3 |F_1|^2 - x_2 x_3 |F_2|^2 + x_3 (x_2 - x_1) \text{Re} \ (F_1 F_2^*) \right] \]

\[ W_E \ = \ -2x_3 (x_1 + x_2) \text{Im} \ (F_1 F_2^*) \]

\[ W_F \ = \ 2x_4 \left[ x_1 \text{Im} \ (F_1 F_3^*) + x_2 \text{Im} \ (F_2 F_3^*) \right] \]

\[ W_G \ = \ -2x_4 \left[ x_1 \text{Re} \ (F_1 F_3^*) + x_2 \text{Re} \ (F_2 F_3^*) \right] \]

\[ W_H \ = \ 2x_3 x_4 \left[ \text{Im} \ (F_1 F_3^*) - \text{Im} \ (F_2 F_3^*) \right] \]

\[ W_I \ = \ -2x_3 x_4 \left[ \text{Re} \ (F_1 F_3^*) - \text{Re} \ (F_2 F_3^*) \right] \] (19)

where \( x_i \) are functions of the Dalitz plot variables and \( Q^2 \). One has

\[ x_1 = q_1^x - q_3^x \]

\[ x_2 = q_2^x - q_3^x \]

\[ x_3 = q_1^y = -q_2^y \]

\[ x_4 = \sqrt{Q^2 x_3 q_3^x} \] (21)

Here \( E_i \) and \( q_i \) refers to the components of the hadron momenta in the hadronic rest frame with

\[ E_i = \frac{Q^2 - s_i + m_i^2}{2\sqrt{Q^2}} \]

\[ q_3^x = \sqrt{E_3^2 - m_3^2} \]

\[ q_2^x = \frac{(2E_2 E_3 - s_1 + m_2^2 + m_3^2)/(2q_3^x)}{2} \]

\[ q_1^x = \frac{(2E_1 E_3 - s_2 + m_2^2 + m_3^2)/(2q_3^x)}{2} \]

\[ q_2^y = -\sqrt{E_2^2 - (q_3^y)^2 - m_2^2} \]

\[ q_1^y = \sqrt{E_1^2 - (q_3^y)^2 - m_1^2} = -q_2^y \]

Note that the structure functions \( W_{B,F,G,H,I} \) are related to the anomaly formfactor \( F_3 \). In the following we will consider these structure functions in more detail.

### 3 Definition of Moments

Equation (19) provides the full description of the angular distribution of the decay products from a single polarized \( \tau \). They reveal that the measurement of the structure functions \( W_i \) and therefore the measurement of the anomaly formfactor...
$$F_3$$ is possible in currently ongoing high statistics experiments. In the following we will concentrate on the $$s_1, s_2$$ integrated structure functions

$$w_i(Q^2) \equiv \int ds_1 ds_2 W_i(Q^2, s_1, s_2) \quad (22)$$

A possible strategy to isolate the various structure functions in (9) is to take suitable moments on the differential decay distribution \([7]\). Let us define

$$\langle f(\beta, \gamma) \rangle = \int \frac{8\pi \sqrt{Q^2} d\Gamma(\tau \rightarrow \nu_{\tau} 3h)}{dQ^2 d\cos \theta d\gamma d\cos \beta} f(\beta, \gamma) \frac{d\cos \beta d\gamma}{2} (23)$$

which yields

$$\langle 1 \rangle = R_s(Q^2) \left( 2K_1 + 3K_2 \right) (w_A + w_B)$$

$$\langle (3 \cos^2 \beta - 1)/2 \rangle = R_s(Q^2) \frac{1}{5} K_1 (w_A - 2w_B)$$

$$\langle \cos 2\gamma \rangle = -R_s(Q^2) \frac{1}{2} K_1 w_C$$

$$\langle \sin 2\gamma \rangle = R_s(Q^2) \frac{1}{2} K_1 w_D$$

$$\langle \cos \beta \rangle = R_s(Q^2) K_3 w_E$$

$$\langle \sin 2\beta \cos \gamma \rangle = R_s \frac{2}{3} K_1 w_F$$

$$\langle \sin \beta \sin \gamma \rangle = -R_s(Q^2) K_3 w_G$$

$$\langle \sin 2\beta \sin \gamma \rangle = -R_s(Q^2) \frac{2}{5} K_1 w_H$$

$$\langle \sin \beta \cos \gamma \rangle = -R_s(Q^2) K_3 w_I$$

where the function $$R_s(Q^2)$$ has been defined by

$$R_s(Q^2) = \frac{G^2}{12m_{\tau}^3} \left( \cos^2 \theta_c \right) \frac{(m_{\tau}^2 - Q^2)^2}{(4\pi)^5 \sqrt{Q^2}} \quad (25)$$

Some comment are in order here:

- First note that after integration over the angles $$\beta$$ and $$\gamma$$ the preceding expressions are still dependent on $$P$$ and $$E_h$$ (through $$\theta$$ and $$\psi$$) while the hadronic structure functions $$w_X$$ are functions of $$Q^2$$.

$^1$ Note that these moments differ from the moments defined in \([8]\) by the factor $$R_s$$ defined in (25).
The sum \( w_A + w_B \) is closely related to the spin one part of the spectral function:

\[
\rho_1(Q^2) = \frac{1}{6} \frac{1}{(4\pi)^4} \frac{1}{Q^4} (w_A + w_B) \tag{26}
\]

and we obtain the standard form for the total width

\[
\Gamma(\tau \to \nu \tau^3 h) = \frac{G^2}{8\pi m_\tau} (\cos^2 \theta_c) \int dQ^2 (m_\tau^2 - Q^2)^2 \left( 1 + \frac{2Q^2}{m_\tau^2} \right) \rho_1(Q^2) \tag{27}
\]

\[
= \int \frac{dQ^2}{\sqrt{Q^2}} R_3(Q^2)\frac{m_\tau^2 + 2Q^2}{Q^2}(w_A + w_B)
\]

In our figures in section (5) we will present the functions \( R_3(Q^2)w_X(Q^2) \) as well as numerical results for the hadronic structure functions itself.

In the next section we present an explicit parametrization of the formfactors which are used in our numerical simulations in order to test whether the anomaly can be measured experimentally.

4 Formfactors

In a recent paper [4] we have given an explicit parametrization of the form factors and compared successfully with measured widths. The physical idea behind the model for the formfactors can be resumed to:

- In the chiral limit the formfactors are normalized to the \( U(3)_L \times U(3)_R \) chiral model.
- Meson-vertices are independent of momentum.
- The full momentum dependence is given by Breit-Wigner propagators of the resonances occurring in the different channels. Resonances occur either in \( Q^2 \) which are then three body resonances or in \( s_i \) which are two body resonances.

Now we present our parametrization for the formfactors \( F_i \) defined in (5) which fulfill all these requirements.

First we present the formfactors induced by the anomaly

\[
F_3^{(\eta\pi^0)}(s_1, Q^2) = \frac{1}{2\sqrt{6}f_\pi^2} T_\rho^{(2)}[Q^2] T_\rho^{(1)}[s_1] \tag{28}
\]

\[
F_3^{(K^-\pi^-K^+)}(s_1, s_2, Q^2) = \frac{-1}{2\sqrt{2}f_\pi^2} T_\rho^{(2)}[Q^2] T_{\rho K^*}(s_2, s_1, \alpha) \tag{29}
\]

\[
F_3^{(K^-\pi^+)}(s_1, s_2, Q^2) = \frac{1}{2\sqrt{2}f_\pi^2} T_{\rho K^*}^{(1,2)}[Q^2] T_{\rho K^*}(s_1, s_2, \alpha) \tag{30}
\]
The parametrization of the $\eta \pi^0$ channel is obtained from $e^+ + e^-$-data via CVC [3, 4]. It is given as a product of two functions describing the resonances in $Q^2$ and $s_i$. The same $Q^2$ dependence ($T^{(2)}_\rho(Q^2)$) can be used for the $K^-\pi^-K^+$-channel. Of course the two-body channels have to be modified since they involve a $\rho$ and a $K^*$ as well, we have included these contributions in $T_{\rho K}(s_1, s, \alpha)[4]$. The same function $T_{\rho K}$ also enters in the $K^-\pi^-\pi^+$ channel. Unfortunately nothing is known experimentally on the three-body resonances in this channel. In [4] only the $K^*$ resonance ($T^{(1)}_{K^*}$) has been included. In our numerical results we will also use a different parametrization including more $\Delta S = 1$ vector resonances.

Second the axialvector current induces two formfactors $F_1$ and $F_2$ for the $K^-\pi^-K^+$ and $K^-\pi^-\pi^+$ channels with the following parametrization

$$F_1^{(K^-\pi^-K^+)}(s_2, Q^2) = -\frac{\sqrt{2}}{3f_\pi} BW_{A_1}[Q^2] T^{(1)}_\rho[s_2] \tag{31}$$

$$F_2^{(K^-\pi^-K^+)}(s_1, Q^2) = -\frac{\sqrt{2}}{3f_\pi} BW_{A_1}[Q^2] T^{(1)}_K[s_1] \tag{32}$$

and

$$F_1^{(K^-\pi^-\pi^+)}(s_2, Q^2) = -\frac{\sqrt{2}}{3f_\pi} \hat{BW}_{K_1}[Q^2] T^{(1)}_{K^*}[s_2] \tag{33}$$

$$F_2^{(K^-\pi^-\pi^+)}(s_1, Q^2) = -\frac{\sqrt{2}}{3f_\pi} \hat{BW}_{K_1}[Q^2] T^{(1)}_\rho[s_1] \tag{34}$$

Note that $G$-parity forbids the axialvector current to contribute to the $\eta \pi^0$ channel. In the axialvector channel we assume the dominance of a resonance in each channel, i.e. the $A_1$ and the $K_1$ in the $\Delta S = 0$ and $\Delta S = 1$ channel, respectively. The two-body channels are again parametrized by the functions $T^{(1)}_\rho$ and $T^{(1)}_{K^*}$. We have moved explicit expressions of these functions and all numerical parameters (taken from [4]) to appendix A.

### 5 Numerical results

In this section we will present numerical results for $R_s(Q^2) \cdot w_X(Q^2)$ (defined in (22,25)) as well as for the hadronic structure functions $w_X(Q^2)$ separately for the different decay channels. We prefer to present both $R_s \cdot w_X$ and $w_X$ in order to show the effect of the phase space (included in the function $R_s$) while the hadronic physics is more visible in the plots for $w_X$ alone. Although by integrating over $s_1$ and $s_2$ we have lost information on the resonances in the two body decays we observe still interesting structures.
Let us start with the Cabibbo allowed decay \( \tau \rightarrow \nu_\tau + \eta \pi^- \pi^0 \). As mentioned before this channel has a vanishing contribution from the axialvector current (\(G\)-parity) which implies that only \( w_B \) is different from zero. A comparison of the data and our prediction for \( R_c \cdot w_B \) and \( w_B \) in fig. (2a,b) would be highly interesting, especially a confirmation of higher lying \( \rho \) resonances in \( T_\rho^{(2)}(Q^2) \) (the shoulder at \( Q^2 = 3\text{GeV}^2 \) in fig. (2a,b)).

Next process is the Cabibbo allowed decay \( \tau \rightarrow \nu_\tau K^- \pi^- K^+ \) which has contributions from both the axialvector and vector currents. Therefore all nine structure functions are different from zero. In fig. 3a we present the structure function combinations obtained from the \( \langle 1 \rangle \) and \( \langle (3 \cos^2 \beta - 1)/2 \rangle \) moment. Note that a measurement of the differential decay width (proportional to \( \langle 1 \rangle \)) is not enough to separate \( w_A \) and \( w_B \). We observe a sizeable effect of \( w_B \) which makes a determination of \( F_3 \) possible. Note this sizeable effect is due to heavy \( \rho' \) resonances in the anomaly channel, which existence is predicted from the description of \( e^+ e^- \rightarrow \eta \pi \pi \). In order to get a feeling of the effects of the phase space in this channel we present the combinations \( w_A + w_B \) and \( w_A - 2w_B \) as well as the structure functions \( w_A \) and \( w_B \) in fig. 3b. The moments which measure the interference of the axialvector and vector-currents are presented in fig. 3c. The size of these moments is comparable to those in fig. 3a and the very peculiar shape would make them measurable too. For completeness we present the remaining moments in fig. 3d.

Finally we discuss the Cabibbo suppressed decay \( \tau \rightarrow \nu_\tau K^- \pi^- \pi^+ \) in figs. 4a-d for the parametrization with \( T_K^{(1)} \) in (30). Note that although this decay is Cabibbo-suppressed the moments are comparable in size to the \( K^- \pi^+ K^+ \) case (suppression due to the mass of the supplementary kaon in the phase space is comparable to the Cabibbo-suppression). All moments of figs. 4a-d have a shape which shows the strong presence of the \( K_1 \) resonance in the axial channel. A measurement of the structure functions related to the anomaly seems very hard since \( F_3 \) is very small in this parametrization. Of course this unfavorable result could have been deduced since the contribution of the anomaly to the rate, as computed in [4], was of the order of 1%. However, we should note that our parametrization of the anomaly form factor [4] includes only a \( K^* \), which can never be on mass shell, and therefore produces no strong enhancement. On the other hand in this channel we have no CVC prediction which could tell us if heavier resonances are present in this channel. In order to get a feeling for possible effects of heavier \( K^* \) resonances we propose the following parametrization which is \( \rho \)-channel inspired (see eq[41]).

\[
T_K^{(2)}[s] = \frac{1}{1 + \beta + \delta} \left\{ BW_{K^*(1680)}[s] + \beta BW_{K^*(1410)}[s] + \delta BW_{K^*}[s] \right\}
\]  

(35)
\[ \delta = -26 \quad m_{K^*} = 0.892 \text{ GeV} \quad \Gamma_{K^*} = 0.050 \text{ GeV} \]
\[ \beta = 6.5 \quad m_{K^*(1410)} = 1.412 \text{ GeV} \quad \Gamma_{K^*(1410)} = 0.227 \text{ GeV} \]
\[ m_{K^*(1680)} = 1.714 \text{ GeV} \quad \Gamma_{K^*(1680)} = 0.323 \text{ GeV} \]

With this parametrization we obtain the results in figs. 5a-c. Note that fig. 4d is not changed.

For completeness we present the results of the total decay width \( \Gamma_{\eta\pi^-\pi^0} \), \( \Gamma_{K^-\pi^-K^+} \) and \( \Gamma_{K^-\pi^-\pi^+} \) normalized to the electronic width \( \Gamma_e \) (\( \Gamma_e/\Gamma_{\text{tot}} \approx 18\% \)). One has[4]

| Channel \((abc)\) | \( \frac{\Gamma_{(abc)}}{\Gamma_e} \) | Contribution from \( F_3 \) in % |
|-----------------|--------------------------|------------------|
| \( \eta\pi^-\pi^0 \) | 0.0108 | 100 % |
| \( K^-\pi^-K^+ \) | 0.0061 | 39.3 % |
| \( K^-\pi^-\pi^+ \) | 0.0316 | with \( T_{K^*}^{(1)} \) |
| \( K^-\pi^-\pi^+ \) | 0.0325 | 4.2 % | with \( T_{K^*}^{(2)} \) |

In view of this result we urge our experimental colleagues to study carefully this Cabibbo suppressed channel.

6 Conclusions

In this paper we have proposed to measure moments (eq. (23)) which allow to determine quantitatively the contribution of the Wess-Zumino-anomaly to \( \tau \) decays into three mesons. We have considered the channels \( \eta\pi^-\pi^0 \), \( K^-\pi^-K^+ \) and \( K^-\pi^-\pi^+ \). We have shown that measuring the unique moment of \( \eta\pi^-\pi^0 \) channel allows to verify the CVC prediction, especially the presence of heavy \( \rho \) excitations observed in \( e^+e^- \) data [10]. In the \( K^-\pi^-K^+ \) channel we can define much more moments because of the interference of the anomaly with the axial vector contributions. In our prediction Fig. (2) the effect of the heavier \( \rho \) is again clearly seen.

The interest of the analysis of the \( K^-\pi^-\pi^+ \) is twofold: we learn something about resonances, first in the axialvector channel and second in the vector channel. We noted that in contradistinction to the Cabibbo allowed decays the vector channel for Cabibbo suppressed cannot be predicted through CVC from \( e^+e^- \) data.
A Parameters used in the formfactors

As stated in section 4 the formfactors are dominated by resonances. The effects of these resonances are described by functions $BW_p(Q^2)$, which are normalized ($BW(0) = 1$) Breit-Wigner propagators. We will use two kind of Breit-Wigners:

- energy-dependent width

$$BW_R[s] \equiv \frac{-M_R^2}{[s - M_R^2 + i\sqrt{s}\Gamma_R(s)]} \quad (36)$$

The energy-dependent width ($\Gamma_R(s)$) is computed from its usual definition.

$$\Gamma_R(s) = \frac{1}{2\sqrt{s}} |M_{R\rightarrow fi}|^2 d\Phi(\mathcal{Q} - \sum p_i) \quad \Gamma_R(s)_{s=M_R^2} = \Gamma_R \quad (37)$$

- constant width

$$\hat{BW}_R[s] \equiv \frac{-M_R^2 + iM_R\Gamma_R}{[s - M_R^2 + iM_R\Gamma_R]} \quad (38)$$

First we define the parameters which arise in the axialvector three-body channel: In the Breit-Wigner of the $A_1$ we use

$$m_{A_1} = 1.251 \text{ GeV} \quad \Gamma_{A_1} = 0.599 \text{ GeV}$$

$$\sqrt{s}\Gamma_{A_1}(s) = m_{A_1}\Gamma_{A_1}\frac{g(s)}{g(m_{A_1}^2)} \quad (39)$$

where the function $g(s)$ has been calculated in [3]

$$g(s) = \begin{cases} 4.1(s - 9m_\rho^2)^3(1 - 3.3(s - 9m_\rho^2) + 5.8(s - 9m_\rho^2)^2) & \text{if } s < (m_\rho + m_\pi)^2 \\ s(1.623 + \frac{10.38}{s} - \frac{9.32}{s^2} + \frac{0.65}{s^3}) & \text{else} \end{cases} \quad (40)$$

In this equation all masses and $\sqrt{s}$ are expressed in GeV.

In case of the $K_1$ resonance we use a constant width Breit-Wigner $\hat{B}_{K_1}(s)$ with

$$m_{K_1} = 1.402 \text{ GeV} \quad \Gamma_{K_1} = 0.174 \text{ GeV}$$

This is because the decay of the $K_1$ is experimentally not well known.

The Cabibbo allowed vector formfactor is obtained from CVC and yields [10]

$$T^{(2)}_\rho[s] = \frac{1}{1 + \beta + \delta} \left\{ BW_{\rho'}[s] + \beta BW_{\rho'}[s] + \delta BW_{\rho}[s] \right\} \quad (41)$$
\[ \delta = -26 \quad m_\rho = 0.773 \text{ GeV} \quad \Gamma_\rho = 0.145 \text{ GeV} \]
\[ \beta = 6.5 \quad m_{\rho'} = 1.500 \text{ GeV} \quad \Gamma_{\rho'} = 0.220 \text{ GeV} \]
\[ m_{\rho''} = 1.750 \text{ GeV} \quad \Gamma_{\rho''} = 0.120 \text{ GeV} \]

For its \( \Delta S = 1 \) we propose (no experimental data) either
\[
T_{K^*}[s] \equiv BW_{K^*}[s]
\] (42)
\[ m_{K^*} = 0.892 \text{ GeV} \quad \Gamma_{K^*} = 0.051 \text{ GeV} \]
or the function \( T_{K^*}[s] \) defined in (35).

Finally we define the functions describing the resonances in the two-body channel [9]
\[
T_{\rho K^*}(s_1, s_2, \alpha) = \frac{T_{\rho}^{(1)}[s_1] + \alpha T_{K^*}[s_2]}{1 + \alpha}
\] (44)

where \( T_{\rho}^{(1)} \) is given in (43) and \( \alpha = -0.2 \) [4].
References

[1] J. Wess and B. Zumino, Phys. Lett. B37B 95 (1971)

[2] G. Kramer, W. F. Palmer and S. Pinsky, Phys. Rev. D30 89 (1984)
   G. Kramer, W. F. Palmer, Z. Phys. C25 195 (1984)
   G. Kramer, W. F. Palmer, Z. Phys. C39 423 (1988)

[3] A. Pich, Phys. Lett. B196 561 (1987)

[4] R. Decker, E. Mirkes, R. Sauer and Z. Was, Karlsruhe preprint TTP92-25,
   to be published in Z. Phys. C, in press.

[5] CLEO preprint, Measurement of $\tau$ Decays Involving $\eta$ Mesons,
   CLEO 92-7.

[6] J.H. Kühn and E. Mirkes, Phys. Lett. B 286 (1992) 381

[7] J.H. Kühn and E. Mirkes, Karlsruhe preprint TTP92-20 to appear in Z. Phys. C, in press.

[8] J.H. Kühn and F. Wagner, Nucl. Phys. B236 (1984) 16.

[9] J. H. Kühn and A. Santamaria, Z. Phys. C48 445 (1990)

[10] DM2 Collaboration, A. Antonelli et al., Phys. Lett. B212 133 (1988)
    J. J. Gomez-Cadenas, M. C. Gonzales-Garcia and A. Pich,
    Phys. Rev. D42 3093 (1990)
Figure captions

Fig. 1 Definition of the polar angle \( \beta \) and the azimuthal angle \( \gamma \). \( \beta \) denotes the angle between \( \vec{n}_\perp \) and \( \vec{n}_L \). \( \gamma \) denotes the angle between the \((\vec{n}_L, \vec{n}_\perp)\) plane and the \((\vec{n}_\perp, \hat{q}_3)\)-plane.

Fig. 2 a) \( Q^2 \) dependence of \( w_B \cdot R_C \) for the decay channel \( \eta \pi^- \pi^0 \)
b) \( Q^2 \) dependence of \( w_B \) for the decay channel \( \eta \pi^- \pi^0 \)

Fig. 3 a) \( Q^2 \) dependence of \((w_A + w_B) \cdot R_C, (w_A - 2w_B) \cdot R_C, w_A \cdot R_C \) and \( w_B \cdot R_C \) for the decay channel \( K^- \pi^- K^+ \)
b) \( Q^2 \) dependence of \((w_A + w_B), (w_A - 2w_B), w_A \) and \( w_B \) for the decay channel \( K^- \pi^- K^+ \)
c) \( Q^2 \) dependence of \( w_F \cdot R_C, w_G \cdot R_C, w_H \cdot R_C \) and \( w_I \cdot R_C \) for the decay channel \( K^- \pi^- K^+ \)
d) \( Q^2 \) dependence of \( w_C \cdot R_C, w_D \cdot R_C \) and \( w_E \cdot R_C \) for the decay channel \( K^- \pi^- K^+ \)

Fig. 4 a) \( Q^2 \) dependence of \((w_A + w_B) \cdot R_S, (w_A - 2w_B) \cdot R_S \) and \( w_B \cdot R_S \) for the decay channel \( K^- \pi^- \pi^+ \) with the parametrization \( T_{K^*}^{(1)} \) in (12)
b) \( Q^2 \) dependence of \((w_A + w_B), (w_A - 2w_B) \) and \( w_B \) for the decay channel \( K^- \pi^- \pi^+ \) with the parametrization \( T_{K^*}^{(1)} \) in (12)
c) \( Q^2 \) dependence of \( w_F \cdot R_S, w_G \cdot R_S, w_H \cdot R_S \) and \( w_I \cdot R_S \) for the decay channel \( K^- \pi^- \pi^+ \) with the parametrization \( T_{K^*}^{(1)} \) in (12)
d) \( Q^2 \) dependence of \( w_S \cdot R_S, w_D \cdot R_S \) and \( w_E \cdot R_S \) for the decay channel \( K^- \pi^- \pi^+ \) with the parametrization \( T_{K^*}^{(1)} \) in (12)

Fig. 5 a-c) same as Fig. 4 with the parametrization \( T_{K^*}^{(2)} \) in (13).