BRST Lagrangian construction for spin-$\frac{3}{2}$ field in Einstein space

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Abstract

We explore a hidden possibility of BRST approach to higher spin field theory to obtain a consistent Lagrangian for massive spin-$\frac{3}{2}$ field in Einstein space. Also, we prove that in the space under consideration the propagation of spin-$\frac{3}{2}$ field is hyperbolic and causal.

In this note we discuss the features of Lagrangian formulation for spin-$\frac{3}{2}$ field on a curved spacetime in framework of BRST approach [1–6]. It is well known that the Lagrangian formulation of the higher spin fields in arbitrary external background can be contradictory. The problem of consistent propagation of fields in different backgrounds and their Lagrangian description is one of the problems of higher spin field theory. Corresponding aspects of spin-$\frac{3}{2}$ field was studied in enormous number of papers (see e.g. [7–9] and the references therein). However, practically all consistent formulations for spin-$\frac{3}{2}$ were given or in flat or in AdS spaces.

BRST approach to higher spin field theories is a universal method for derivation of the Lagrangians for such fields beginning with on-shell relations, which define the higher spin fields (e.g. the relations defining irreducible representations of the Poincare or AdS groups). Following the general BRST-BFV construction we begin with a closed constraint algebra for the theory and built the Lagrangians. This approach yields consistent formulation for massless and massive, bosonic and fermionic arbitrary spin-$s$ fields in constant curvature space, however for consistency it demands the same space even for spin $s = 1, 2$ fields where the Lagrangian formulations exist in an arbitrary Riemann space and an Einstein space respectively. Such a puzzle was resolved in our paper [6] exploring some hidden possibility of the BRST approach. In this paper we demonstrate that the same hidden possibility exists for spin-$\frac{3}{2}$ field as well and allows us to get the consistent spin-$\frac{3}{2}$ Lagrangian formulation in arbitrary Einstein space.

We begin with a brief discussion of the main idea proposed in [6]. The Lagrangian construction in the BRST approach [1–5] was carried out for arbitrary spin fields. The basic notions of this approach are the Fock space vector $|\Psi_s\rangle$, corresponding to spin $s$ and the nilpotent BRST charge

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1 We do not discuss here the supergravity where consistency for massless spin-$\frac{1}{2}$ field is conditioned by supersymmetrical coupling to massless spin-2 field (see e.g. [10]).
where the new aspect of BRST approach elaborated in [6].

From (5) we see that if the traceless part of the Ricci tensor is zero, those equations which define the irreducible representations of the AdS group [11]. Thus the mass-shell equations are compatible with each other. Next from equations (2) are satisfied in the Einstein space. It is well known that spin-\(\frac{3}{2}\) field in such space using \(\tilde{R}_{\mu
u} = R_{\mu
u} - \frac{1}{d} g_{\mu\nu} R\). Extracting from \(\gamma_{\alpha\beta}\) totally antisymmetric part \(\gamma_{\nu\alpha\beta} = 1/6(\gamma_{\nu\alpha}\gamma_{\beta} + \gamma_{\nu\beta}\gamma_{\alpha} + \gamma_{\nu\gamma\alpha\beta})\) and substituting into (3) one gets

\[
\frac{i}{2} \tilde{R}_{\mu
u} \psi_{\mu} = 0.
\]

From (5) we see that if the traceless part of the Ricci tensor is zero \(R_{\mu
u} - \frac{1}{d} g_{\mu\nu} R = 0\) then equations (2) are compatible with each other. Next from \(R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R\) follows that \(R = const\), while the Weyl tensor \(C_{\mu\alpha\beta\nu}\) remains arbitrary. That is equations (2) are compatible with each other in the Einstein space. In the further part of the paper we construct Lagrangian for spin-\(\frac{3}{2}\) field in such space using the new aspect of BRST approach elaborated in [6].

First of all we note that in the Einstein spaces the equations of motion (2) can be modified in such way that a parameter with dimension of length appears in the equations. In this case it is reasonable to construct the Lagrangian leading to mass-shell equations, formally coinciding with equations which define the irreducible representations of the AdS group [11]. Thus the mass-shell

\[Q.\] The equations of motion and gauge transformations are written in the form \(Q|\Psi_s\rangle = 0\) and \(\delta|\Psi_s\rangle = Q|\Lambda_s\rangle\) respectively, with the BRST operator \(Q\) being the same for all spins. Nilpotency of the BRST operator provides us the gauge transformations and fields \(|\Psi_s\rangle\) and \(|\Psi_s\rangle + Q|\Lambda_s\rangle\) are both physical. Since we consider all spins simultaneously, then from \(Q^2|\Lambda_s\rangle = 0\) follows \(Q^2 = 0\). But if we want to construct Lagrangian for the field with a given value \(s\) of spin, then it is sufficient to require a weaker condition that the BRST operator for given spin \(Q_s\) is not nilpotent in operator sense but will be nilpotent only on the specific Fock vector parameter \(|\Lambda_s\rangle\) corresponding to a given spin \(s\), \(Q^2_s|\Lambda_s\rangle = 0\) only and \(Q^2_s \neq 0\) on states of general form. Just this point allows us to construct Lagrangian for spin-\(\frac{3}{2}\) field in Einstein space.

We begin with pointing out that there exist the consistent equations of motion for spin-\(\frac{3}{2}\) field in a space-time different from AdS. It is well known that spin-\(\frac{3}{2}\) field \(\psi_{\mu}\) (the Dirac index is suppressed) will describe the irreducible massive representation of the Poincare group if the following conditions are satisfied

\[
(i\gamma^\nu \partial_\nu - m)\psi_{\mu} = 0, \quad \gamma^\mu \psi_{\mu} = 0, \quad \partial^\mu \psi_{\mu} = 0, \quad (1)
\]

with \(\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\). When we put these equations on an arbitrary curved spacetime we see that if we do not include the terms with the curvature and it unambiguously follows that in curved space equations (1) take the form

\[
(i\gamma^\nu \nabla_\nu - m)\psi_{\mu} = 0, \quad \gamma^\mu \psi_{\mu} = 0, \quad \nabla^\mu \psi_{\mu} = 0. \quad (2)
\]

Let us find what spaces do not give supplementary equations in addition to (2). For this purpose we take the divergence of the mass-shell equation and suppose that equations on \(\psi_{\mu}\) (2) are satisfied.

\[
0 = \nabla^\mu (i\gamma^\nu \nabla_\nu - m)\psi_{\mu} = i\gamma^\nu [\nabla^\mu, \nabla_\nu]\psi_{\mu} = i\tilde{R}^{\mu\nu}\gamma_\nu\gamma_{\alpha\beta}\psi_{\mu} - \frac{i}{4} R^{\alpha\beta\mu\nu}\gamma_\nu\gamma_{\alpha\beta}\psi_{\mu}, \quad (3)
\]

where \(\gamma_{\alpha\beta} = \gamma_{[\alpha\beta]} = 1/2(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)\) and \(\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R\). Extracting from \(\gamma_{\mu\nu\gamma_{\alpha\beta}}\) totally antisymmetric part \(\gamma_{\nu\alpha\beta} = 1/6(\gamma_{\nu\alpha}\gamma_{\beta} + \gamma_{\nu\beta}\gamma_{\alpha} + \gamma_{\nu\gamma\alpha\beta})\) and

\[
\gamma_{\nu\gamma_{\alpha\beta}} = \gamma_{\nu\alpha\beta} + \gamma_{\alpha} g_{\beta\nu} - \gamma_{\beta} g_{\alpha\nu} \quad (4)
\]

and substituting into (3) one gets

\[
\frac{i}{2} \tilde{R}^{\mu\nu}\gamma_\nu\psi_{\mu} = 0. \quad (5)
\]

\[2\text{An Einstein space is defined by the relation } R_{\mu\nu} = \text{const} \cdot g_{\mu\nu} \text{ with Weyl tensor be arbitrary (see e.g. [12]).}
\]

\[3\text{Our definition of the curvature tensor is } R^{\alpha\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\beta\nu} - \Gamma^\gamma_{\beta\mu} \Gamma^\alpha_{\gamma\nu}.
\]
equations which are expected to be resulted from the Lagrangian (up to gauge fixing) should have the form

\[
\left[ i\gamma^\sigma \nabla_\sigma - m - \frac{d-2}{2} r^\frac{1}{2} \right] \psi_\mu = 0, \quad \gamma^\mu \psi_\mu = 0, \quad \nabla^\mu \psi_\mu = 0, \quad (6)
\]

where \( r \) is defined from \( R_{\mu\nu\rho\sigma} = r(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}) + C_{\mu\nu\rho\sigma}, \) i.e. \( R = -rd(d-1). \)

Now we introduce auxiliary Fock space generated by fermionic creation and annihilation operators \( a^+_\alpha, a_\alpha \) satisfying the anticommutation relations

\[
\{a^+_\alpha, a_\beta\} = \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \text{diag}(-, +, +, \cdots, +). \quad (7)
\]

Our further consideration is very close to [5], therefore we will omit some details of the calculations. As usual the tangent space indices and the curved space indices are converted one into another with the help of vielbein \( e^a_\mu \) which is assumed to satisfy the relation \( \nabla_\nu e^a_\mu = 0. \) Then in addition to the conventional gamma-matrices

\[
\{\gamma_\alpha, \gamma_\beta\} = -2\eta^{\alpha\beta}, \quad \eta^{\alpha\beta} = \text{diag}(-, +, +, \cdots, +). \quad (8)
\]

we introduce a set of \( d+1 \) Grassmann odd objects [3–5] which obey the following gamma-matrix-like conditions

\[
\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -2\eta^{ab}, \quad \{\tilde{\gamma}^a, \tilde{\gamma}\} = 0, \quad \tilde{\gamma}^2 = -1 \quad (9)
\]

and connected with the “true” (Grassmann even) gamma-matrices by the relation

\[
\gamma^a = \tilde{\gamma}^a\tilde{\gamma} = -\tilde{\gamma}\tilde{\gamma}^a. \quad (10)
\]

After this we define derivative operator

\[
D_\mu = \partial_\mu + \omega^a_\mu M_\alpha, \quad M_\alpha = \frac{1}{2}(a^+_\alpha a_\beta - a^+_\beta a_\alpha) - \frac{1}{8}(\tilde{\gamma}_a\tilde{\gamma}_b - \tilde{\gamma}_b\tilde{\gamma}_a), \quad (11)
\]

which acts on an arbitrary state vector in the Fock space

\[
|\psi\rangle = \sum_{n=0}^\infty |\psi_n\rangle, \quad |\psi_n\rangle = a^+_{\mu_1} \cdots a^+_{\mu_n} \psi_{\mu_1 \cdots \mu_n}(x)|0\rangle, \quad (12)
\]

as the covariant derivative\(^5\)

\[
D_\mu |\psi_n\rangle = a^+_{\mu_1} \cdots a^+_{\mu_n} (\nabla_\mu \psi_{\mu_1 \cdots \mu_n}) |0\rangle. \quad (13)
\]

As a result equations (6) can be realized in the operator form

\[
t_0 |\psi_1\rangle = 0, \quad t_1 |\psi_1\rangle = 0, \quad l_1 |\psi_1\rangle = 0, \quad (14)
\]

where

\[
t_0 = i\gamma^\mu D_\mu + \tilde{\gamma}(r^{\frac{1}{2}}g_0 - m), \quad g_0 = a^+_\mu a^\mu - \frac{d}{2}, \quad t_1 = \tilde{\gamma}^\mu a_\mu, \quad l_1 = -ia^\mu D_\mu. \quad (15)
\]

Lagrangian construction within the BRST approach [1–6] demands that we must have at hand a set of operators which is invariant under Hermitian conjugation and which forms an algebra [1–6].

\(^4\)Of course, in the case under consideration, we could also use bosonic creation and annihilation operators, but use of fermionic ones is simpler, cf. [5] and [4].

\(^5\)We assume that \( \partial_\mu a^+_\alpha = \partial_\mu a_\alpha = \partial_\mu |0\rangle = 0. \)
In order to determine the Hermitian conjugation properties of the constraints we define the following scalar product

$$\langle \Psi | \Phi \rangle = \int d^4x \sqrt{-g} \sum_{n, k=0} \langle 0 | \Psi_{\nu_1...\nu_k}^+(x) \gamma^0 a^{\nu_k} ... a^{\nu_1} a^{+\mu_1} ... a^{+\mu_n} \Phi_{\mu_1...\mu_n}(x) | 0 \rangle. \tag{16}$$

As a result we see that constraint $t_0$ is Hermitian and the two other are non-Hermitian.

Thus set of operators $t_0, t_1, t_1^+, l_1, l_1^+$ is invariant under Hermitian conjugation. Then for constructing the BRST operators the underlying set of operators must form an algebra. Note that the nilpotency condition of the BRST operators is needed for existing of gauge symmetry. As it is known if we consider half-odd spin-s field and decompose the gauge parameter $|\Lambda_n\rangle$ ($s = n + 1/2$) in series of creation operators, then maximal tensorial rank of gauge parameters $|\lambda_k\rangle = a^{+\mu_1} ... a^{+\mu_k} \lambda_{\mu_1...\mu_k} | 0 \rangle$, entering in $|\Lambda_n\rangle$ is $k = n - 1$ (see e.g. [3]). Therefore if we want to construct Lagrangian for a particular half-odd spin-s field it is enough that this set of operators forms algebra only on states $|\lambda_k\rangle$ with $k < s - 1/2$.

Now in order to an algebra we add to the set of operators all the operators generated by the (anti)commutators of $t_0, t_1, l_1, t_1^+, l_1^+$, but unlike the case of arbitrary spin [3] the algebra must be closed on states $|\lambda_0\rangle = \lambda(x) | 0 \rangle$ only. Doing similar considerations as in [6] we arrive to the conclusion that there should be added the following three operators

$$l_0 = D^2 - m^2 + r \left( -g_0^2 + g_0 + t_1^+ t_1 + \frac{d(d+1)}{4} \right), \quad g_0 = a_\nu^\dagger a^\mu - \frac{d}{2}, \quad g_m = m, \tag{18}$$

where $D^2 = g^{\mu\nu}(D_\mu D_\nu - \Gamma^\sigma_{\mu\nu} D_\sigma)$. As a result set of operators $o_i = (t_0, l_0, t_1, l_1, t_1^+, l_1^+, g_0, g_m)$ is invariant under Hermitian conjugation and forms an algebra on states $|\lambda_0\rangle$ in the Einstein space. Note that the form of the algebra coincides with those obtained in [5], therefore we borrow from there all the results needed for construction of Lagrangian for spin-$\frac{3}{2}$ field in the space under consideration.

First, since the algebra contains operators $g_0$ and $g_m$ which are not constraints neither in the bra- nor in the ket-vector space then we should construct new expressions $o_i \rightarrow O_i$ for the operators of the algebra, so that the operators which are not constraints don’t give supplementary equations on the physical field and form an algebra. We borrow the result for these new expressions for the operators from [5]

$$T_0 = i\gamma^\mu D_\mu - \gamma m_0 - 2m_1 f^+ b + \frac{r}{2m_1} \left( b^+ b + 2h \right) b^+ f, \tag{20}$$

$$T_1^+ = a_\mu^\dagger \gamma^\mu + b^+, \quad L_1^+ = -i a^{+\mu} D_\mu + m_1 f^+ - \frac{r}{4m_1} b^{+2} f, \quad G_m = 0, \tag{21}$$

$$L_0 = D^2 - m_2^2 + r(\langle 0 | g_0 + t_1^+ t_1 + \frac{d(d+1)}{4} - r( b^+ b + 2h ) b^+ b - 2r( b^+ b + h + \frac{1}{2} ) f^+ f, \tag{22}$$

$$T_1 = \gamma^\mu a_\mu + \gamma^m m_0 f + (2f^+ f + b^+ b + 2h)b, \quad G_0 = g_0 + b^+ b + f^+ f + h, \tag{23}$$

$$L_1 = -i a^{+\mu} D_\mu + \gamma m_0 b + m_1 f^+ b^2 + \frac{m_0^2}{m_1} f - \frac{r}{m_1} ( b + \frac{1}{2} ) (b^+ b + h) f - \frac{r}{4m_1} b^{+2} b^2 f, \tag{24}$$

where in case of spin-$\frac{3}{2}$ field one should take $h = d/2 - 1$ and $m_0 = m + r\frac{1}{2}(d - 2)/2$. In eqs. (20)–(24) we have introduced one pair of fermionic $f^+, f$ and one pair of bosonic $b^+, b$ creation and annihilation operators with the standard (anti)commutation relations

$$\{ f^+, f \} = 1, \quad [ b^+, b ] = 1. \tag{25}$$

We assume that $(\gamma^\nu)^+ = \gamma^0 \gamma^\nu \gamma^0, \quad (\gamma^\nu)^+ = \gamma^0 \gamma^\nu \gamma^0 = -\gamma.$
Also expressions (20)–(24) contain arbitrary (nonzero) constant $m_1$ with dimension of mass. Its value remains arbitrary and it can be expressed from the other parameters of the theory $m_1 = f(m, r) \neq 0$. The arbitrariness of this parameter does not influence on the reproducing of the equations of motion for the physical field (6).

Note that the new expressions for the operators do not obey the usual properties

$$(T_0)^+ \neq T_0, \quad (L_0)^+ \neq L_0, \quad (T_1)^+ \neq T_1^+, \quad (L_1)^+ \neq L_1^+ \quad (26)$$

if one uses the standard rules of Hermitian conjugation for the new creation and annihilation operators

$$(b)^+ = b^+, \quad (f)^+ = f^+ \quad (27)$$

To restore the proper Hermitian conjugation properties for the additional parts we change the scalar product in the Fock space generated by the new creation and annihilation operators as follows:

$$\langle \tilde{\Psi}_1 | \Psi_2 \rangle_{\text{new}} = \langle \tilde{\Psi}_1 | K | \Psi_2 \rangle, \quad (28)$$

for any vectors $|\Psi_1\rangle, |\Psi_2\rangle$ with some operator $K$. Since the problem with the proper Hermitian conjugation of the operators is only in $(b^+, f^+)$-sector of the Fock space, the modification of the scalar product takes place only in this sector. Therefore operator $K$ acts as a unit operator in the entire Fock space, but for the $(b^+, f^+)$-sector where the operator has the form

$$K = \sum_{k=0}^{\infty} \frac{C_h(k)}{k!} \left[ |0, k\rangle \frac{1}{2h+k}(0, k) + |1, k\rangle \frac{2m_0^2 - rh(2h+k+1)}{4h m_1^2} (1, k) \right. 
\left. + |1, k\rangle \frac{\tilde{m}_0 k}{2h m_1} (0, k+1) + |0, k+1\rangle \frac{\tilde{m}_0 k}{2h m_1} (1, k) \right], \quad (29)$$

where

$$C_h(k) = 2h(2h+1) \ldots \cdot (2h+k-2)(2h+k-1)(2h+k), \quad (30)$$
$$|0, k\rangle = (b^+]^k|0\rangle, \quad |1, k\rangle = f^+(b^+]^k|0\rangle. \quad (31)$$

Next step is constructing the BRST operator on the base of algebra generated by (20)–(24). Its explicit form can be found in [5]. Then using the found BRST operator one constructs (up to an overall factor) Lagrangian and gauge transformation for spin-$\frac{3}{2}$ field in the Einstein space (the details can be found in e.g. [4,5])

$$\mathcal{L} = \langle \bar{\chi}^0 | K \tilde{T}_0 | \chi^0 \rangle + \frac{1}{2} \langle \bar{\chi}^1 | K \{ \tilde{T}_0, q_i^+ q_i \} | \chi^1 \rangle + \langle \bar{\chi}^0 | K \Delta Q | \chi^1 \rangle + \langle \bar{\chi}^1 | K \Delta Q | \chi^0 \rangle, \quad (32)$$

$$\delta |\chi^0\rangle = \Delta Q |\Lambda\rangle, \quad \delta |\chi^1\rangle = \tilde{T}_0 |\Lambda\rangle, \quad (33)$$

where

$$\tilde{T}_0 = T_0 + 2i(\eta_1^+ p_1 - \eta_1 p_1^+) + \frac{r}{2}(2g_0 - G_0) (q_1 P_1^+ + q_i^+ P_i) \quad (34)$$
$$\Delta Q = \eta_1^+ T_1 + \eta_1 T_1^+ + q_i^+ L_1 + q_1 L_1^+ - \frac{r}{4} \left[ q_1(2t_1^+ - T_1^+) + q_1^+(2t_1 - T_1) \right] (q_1 P_1^+ + q_i^+ P_i) \quad (35)$$

and

$$|\chi^0\rangle = [-ia^+ \psi_\mu(x) + f^+ \varphi(x) + b^+ \bar{\psi}(x)] |0\rangle, \quad |\chi^1\rangle = [P_1^+ \chi_1(x) - ip_1^+ \bar{\chi}(x)] |0\rangle, \quad (36)$$
$$|\Lambda\rangle = [P_1^+ \bar{\gamma} \lambda_1(x) - ip_1^+ \bar{\lambda}(x)] |0\rangle. \quad (37)$$
Thus we have obtained the Lagrangian for massive spin-$\frac{3}{2}$ field with the Lagrangian multiplier $\chi$ obeying the (anti)commutation relations

$$
\{\eta, \mathcal{P}^+\} = \{\eta^+_i, \mathcal{P}_1\} = 1, \quad [q_1, p_1^+] = [q_1^+, p_1] = i
$$

and possess the standard ghost number distribution $gh(C^i) = -gh(\mathcal{P}_i) = 1$ and act on the vacuum state as follows

$$
(q_1, p_1, \eta, \mathcal{P}_1)|0\rangle = 0.
$$

Substituting (36), (37) into (33), we find the action (up to an overall factor) for a spin-$\frac{3}{2}$ field in the Einstein spacetime in the component form

$$
S = \int d^dx \sqrt{-g} \left\{ i\gamma^\nu \nabla_\nu \psi - m_0 \psi - \nabla_\mu \chi - i\gamma_\mu \lambda_1 \right\} - \left[ (d - 2) \tilde{\psi} + \frac{m_0}{m_1} \tilde{\varphi} \right] \left\{ i\gamma^\mu \nabla_\mu m_0 \varphi + \frac{r(d - 1)}{2m_1} \varphi + \tilde{\chi} \right\}
$$

where $M^2 = m_0^2 - \frac{1}{r} r(d - 1)(d - 2)$ and $m_0 = m + r^\frac{1}{2}(d - 2)/2$. Substituting (36), (37) into (33), we find the gauge transformations in the component form

$$
\delta \psi = i\gamma^\mu \nabla_\mu \chi, \quad \delta \varphi = m_1 \lambda,
$$

$$
\delta \chi = \left[ i\gamma^\mu \nabla_\mu - m_0 \right] \lambda + 2\lambda_1, \quad \delta \lambda_1 = - \left[ i\gamma^\mu \nabla_\mu + m_0 \right] \lambda_1 - \frac{r}{2}(d - 1) \lambda.
$$

Beginning with the Lagrangian (40) we can obtain the other Lagrangians for spin-$\frac{3}{2}$ field containing less number of involved fields. Let us present the action in terms of one basic field $\psi_\mu$. To this end, we get rid of the fields $\varphi, \psi$, by using their gauge transformations and the gauge parameters $\lambda, \lambda_1$, respectively. Having expressed the field $\chi$, using the equation of motion $\chi = i\gamma^\mu \psi_\mu$, we see that the terms with the Lagrangian multiplier $\chi$ disappear. As a result, we obtain

$$
\mathcal{L} = \tilde{\psi}^\mu (i\gamma^\sigma \nabla_\sigma - m_0) \psi_\mu - i\tilde{\psi}^\mu (\gamma_\nu \nabla_\mu + \gamma_\mu \nabla_\nu) \psi_\nu - \tilde{\psi}^\mu \gamma_\nu (i\gamma^\sigma \nabla_\sigma + m_0) \gamma^\mu \psi_\mu.
$$

Thus we have obtained the Lagrangian for massive spin-$\frac{3}{2}$ field in $d$-dimensional Einstein space only in term of the basic field. In the massless case $m = 0$ ($m_0 = r^\frac{1}{2}(d - 2)/2$) and Lagrangian (43) becomes invariant under gauge transformation

$$
\delta \psi_\mu = \nabla_\mu \lambda - \frac{ir^\frac{1}{2}}{2} \gamma_\mu \lambda.
$$

Our next aim is discussing the problem of causality for massive spin-$\frac{3}{2}$ field with Lagrangian (43) in Einstein space. Consideration is based on the Velo and Zwanziger method and reformulated for the theories in curved spacetime.

We begin with a brief outline of the method. If one has a system of the first order differential equations for a set of fields $\varphi^B$

$$
G^A_B \partial_\mu \varphi^B + \ldots = 0, \quad \mu, \nu = 0, \ldots, d - 1
$$

6
then to verify that the system (45) describes hyperbolic propagation one should check that all solutions \( n_0(n_i), (i = 1, \ldots, d - 1) \) of the algebraic equation

\[
\det(G_B^A \mu \gamma \mu) = 0
\]  

(46)

are real for any given real set of \( n_i \). The hyperbolic system is called causal if there are no timelike vectors among solutions \( n_\mu \) of (46).

In many physical cases (including spin-3/2 field) equation (46) fulfills identically. In this case one should replace the initial system of equations by another equivalent system of equations supplemented by constraints at a given initial time. Then the above analysis should be applied to this new system.

Let us turn to our spin-3/2 field described by Lagrangian (43). The equations of motion are

\[
E_\mu = (i \gamma^\sigma \nabla_\sigma - m_0)\psi_\mu - i(\gamma_\nu \nabla_\mu + \gamma_\mu \nabla_\nu)\psi^\nu - \gamma_\mu (i \gamma^\sigma \nabla_\sigma + m_0)\gamma^\nu \psi_\nu = 0.
\]  

(47)

If we consider equation (46) for equations (47) then we find that it fulfills identically. Therefore one should replace equations (47) by another equivalent system of equations with constraints on initial data. It can be done by the same method as in [8] and we will not repeat all the steps and proofs. The system of equations equivalent to (47) is

\[
E_\mu + i \gamma_\mu C + i \nabla_\mu D + m_0 \gamma_\mu D = (i \gamma^\sigma \nabla_\sigma - m_0)\psi_\mu = 0,
\]  

(48)

where

\[
C = \frac{ir(d - 2) \gamma^\mu E_\mu - 4m_0 \nabla^\mu E_\mu}{4m_0^2 - r(d - 2)^2} = \nabla^\mu \psi_\mu + \gamma^\sigma \nabla_\sigma \gamma^\nu \psi_\nu,
\]  

(49)

\[
D = \frac{4m_0 \gamma^\mu E_\mu + i(d - 2) \nabla^\mu E_\mu}{4m_0^2 - r(d - 2)^2} = \gamma^\nu \psi_\nu
\]  

(50)

and it is supplemented by constraints at an initial time (say \( t = 0 \))

\[
E_0|_{t=0} = 0, \quad \gamma^\mu \psi_\mu|_{t=0} = 0.
\]  

(51)

Thus, like in the flat case, the equations for spin-3/2 field (47), following from Lagrangian (43), are hyperbolic and causal for the background under consideration.

To conclude, we have shown that the BRST approach to higher spin field theories yields consistent Lagrangian construction for spin-3/2 field in general Einstein space. As usual in the BRST approach, massive spin-3/2 Lagrangian is obtained in gauge invariant form with suitable St"uckelberg auxiliary fields. Analysis has been based on some hidden aspect of the BRST-BFV construction which was explored in context of higher spin field theory in [6]. We saw that the BRST approach perfectly works if one requires that the BRST operator \( Q_s \) is nilpotent in weak sense, i.e. \( Q_s^2 |\Lambda_s\rangle = 0 \) for some spin \( s \), where \( |\Lambda_s\rangle \) is a Fock space valued gauge parameter. One can show that above condition is realized only for exceptional spins \( s = 1, \frac{3}{2}, 2 \). The cases \( s = 1, 2 \) have been considered in [6]. Here we have studied the last exceptional case \( s = \frac{3}{2} \). Also we have proved that in Einstein space the spin-3/2 field propagation is hyperbolic and causal. As we know such proof was known before only for the constant curvature space.

Acknowledgements. The authors are grateful to P.M. Lavrov for discussions on BRST-BFV construction. The work is partially supported by the RFBR grant, project No. 09-02-00078, the RFBR-Ukraine grant, project No. 10-02-90446, grant for LRSS, project No. 3558.2010.2. The work of I.L.B. is also partially supported by the RFBR-DFG grant, project No. 09-02-91349 and the DFG grant, project No. 436 RUS 113/669/0-4.

\footnote{In curved spacetime the question of causality should be considered locally at some arbitrary point \( x_0 \) choosing around \( x_0 \) the Riemann normal coordinates. In this case \( E_0 \) appears as a constraint.}
References

[1] I.L. Buchbinder, A. Pashnev, M. Tsulaia, Phys. Lett. B523 (2001) 338; I.L. Buchbinder, A. Pashnev, M. Tsulaia, Massless higher spin fields in the AdS background and BRST construction for nonlinear algebras, arXiv:hep-th/020626; X. Bekaert, I.L. Buchbinder, A. Pashnev, M. Tsulaia, Class. Quant. Grav. 21 (2004) S1457; I.L. Buchbinder, V.A. Krykhtin, A. Pashnev, Nucl. Phys. B711 (2005) 367; I.L. Buchbinder, V.A. Krykhtin, Nucl. Phys. B727 (2005) 537; “BRST approach to higher spin field theories,” arXiv:hep-th/0511276; “Progress in Gauge Invariant Lagrangian Construction for Massive Higher Spin Fields,” arXiv:0710.5715 [hep-th]; I.L. Buchbinder, A. Fotopoulos, A.C. Petkou, M. Tsulaia, Phys. Rev. D74 (2006); I.L. Buchbinder, V.A. Krykhtin and P. M. Lavrov, Nucl. Phys. B 762 (2007) 344; I.L. Buchbinder, A.V. Galajinsky, V.A. Krykhtin, Nucl. Phys. B779 (2007) 155; I.L. Buchbinder, V.A. Krykhtin, H. Takata, Phys. Lett. B656 (2007) 253; I.L. Buchbinder, V.A. Krykhtin, L.L. Ryskina, Mod. Phys. Lett. A24 (2009) 401;

[2] A. Pashnev and M. Tsulaia, Mod. Phys. Lett. A 13 (1998) 1853; C. Burdik, A. Pashnev, M. Tsulaia, Nucl. Phys. B (Proc. Suppl.) 102 (2001) 285; C. Burdik, A. Pashnev, M. Tsulaia, Mod. Phys. Lett. A16 (2001) 731; A. Sagnotti, M. Tsulaia, Nucl. Phys. B682 (2004) 83; A. Fotopoulos, K.L. Panigrahi, M. Tsulaia, Phys. Rev. D74 (2006) 085029; A. Fotopoulos, M. Tsulaia, Phys. Rev. D76 (2007) 025014; A. Fotopoulos, N. Irges, A.C. Petkou, M. Tsulaia, JHEP 0710 (2007) 021; P.Yu. Moshin, A.A. Reshetnyak, JHEP 0710 (2007) 040; A. Fotopoulos, M. Tsulaia, Int. J. Mod. Phys. A24 (2009) 1.

[3] I. L. Buchbinder, V. A. Krykhtin, L. L. Ryskina and H. Takata, Phys. Lett. B 641 (2006) 386.

[4] I. L. Buchbinder, V. A. Krykhtin and A. A. Reshetnyak, Nucl. Phys. B 787 (2007) 211.

[5] I. L. Buchbinder, V. A. Krykhtin and L. L. Ryskina, Nucl. Phys. B 819 (2009) 453.

[6] I. L. Buchbinder, V. A. Krykhtin and P. M. Lavrov, Phys. Lett. B 685 (2010) 208.

[7] K. Johnson and E. C. G. Sudarshan, Annals Phys. 13 (1961) 126; C. R. Hagen, Phys. Rev. D 4 (1971) 2204; M. Hortacsu, Phys. Rev. D 9 (1974) 928; A. Z. Capri and R. L. Kobes, Phys. Rev. D 22 (1980) 1967;

[8] G. Velo and D. Zwanziger, Phys. Rev. 186 (1969) 1337; D. Zwanziger, Lecture Notes in Physics, v73 (1978) 143.

[9] S. Deser, V. Pascalutsa and A. Waldron, Phys. Rev. D 62 (2000) 105031; S. Deser and A. Waldron, Phys. Lett. B 501 (2001) 134; Nucl. Phys. B 631 (2002) 369; M. Porrati and R. Rahman, Phys. Rev. D 80 (2009) 025009.

[10] I. L. Buchbinder, S. M. Kuzenko, Ideas and Methods of Supergravity and Supergravity, IOP Publishing, Bristol and Philadelphia, 1998.

[11] R. R. Metsaev, Class. Quant. Grav. 11 (1994) L141; Class. Quant. Grav. 14 (1997) L115; Phys. Lett. B 419 (1998) 49.

[12] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact solutions of Einstein’s field equations, Cambridge University Press, 2003