Comments on “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]

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Abstract

We comment on the paper “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]. In particular, we give some counterexamples to the results presented there, and provide a correct result for degree two.

1 Introduction

For the description of the problem, its applications, used notation, and the theoretical background, see, e.g. [8, 2, 4, 7].

For some given positive numbers, $d$ (diameter) and $k$ (degree), the authors of [5] consider the following numbers:

- Let $m^*(d, k)$ the largest positive integer $m$ (number of vertices) such that there exists an $m$-element finite Abelian group $\Gamma$ and a $k$-element generating subset $A \subset \Gamma$ such that $\text{diam}(\text{Cay}(\Gamma, A)) \leq d$.
- Let $m(d, k)$ the largest positive integer $m$ such that there exists a cyclic group $\mathbb{Z}_m$ and a $k$-element generating subset $A \subset \mathbb{Z}_m$ such that $\text{diam}(\text{Cay}(\mathbb{Z}_m, A)) \leq d$.

Such authors claim that, for any integer $d \geq 2$, Jia and Hsu [4] proved that

$$m(d, 2) = \left\lfloor \frac{d(d + 4)}{3} \right\rfloor + 1, \quad (1)$$
but this was proved around ten years before by the author et al. [6, 7]. Thus, in [6], the following value can be found:

\[ m(d, 2) = \left\lceil \frac{(d + 2)^2}{3} \right\rceil - 1, \tag{2} \]

which is readily seen to be equivalent to (1). More generally, in Table I of [7] some other optimal values are shown (minimizing the diameter for some fixed number of vertices). A part of such a table is shown next with the corresponding gene rating sets \( \{a, b\} \) of the cyclic groups. (The values in boldface correspond to the ones given by (1) or (2).

| \( m(d, 2) \) | \( d \) | \( a \) | \( b \) (mod \( m \)) |
|---|---|---|---|
| \( 3x^2 \) | \( 3x - 1 \) | 1 | \( 3x - 1 \) |
| \( 3x^2 + x \) | \( 3x - 1 \) | 1 | \( 3x \) |
| \( 3x^2 + 2x \) | \( 3x - 1 \) | 1 | \( -3x \) |
| \( 3x^2 + 2x + 1 \) | \( 3x \) | 1 | \( 3x + 1 \) |
| \( 3x^2 + 3x + 1 \) | \( 3x \) | 1 | \( 3x + 2 \) |
| \( 3x^2 + 4x + 1 \) | \( 3x \) | 1 | \( -3x - 2 \) |
| \( 3x^2 + 4x + 2 \) | \( 3x + 1 \) | 1 | \( 3x + 3 \) |
| \( 3x^2 + 5x + 2 \) | \( 3x + 1 \) | 1 | \( 3x + 4 \) |
| \( 3x^2 + 6x + 2 \) | \( 3x + 1 \) | 1 | \( -3x + 4 \) |
| (= \( 3(x + 1)^2 - 1 \)) |

Also, as a main result, Mask, Schneider, and Jia [5, Th. 1.1] claimed that, for any \( d \) and \( k \),

\[ m_*(d, k) = m(d, k). \tag{3} \]

However, as shown by the counterexamples in the following section, such a result cannot be true even for degree \( k = 2 \). This is due to an error in the proof of such a theorem. Namely, the first \( r \) equalities in [5, Th. 1.1] should be understood modulo \( m_j \):

\[ x_j = \sum_{i=1}^{k} c_i a_{ij} \pmod{m_j} \quad \text{for } j = 1, 2, \ldots, r. \]

Thus, without this condition, the following equality in [5], which should be modulo \( m_{r-1}' = m_{r-1}m_r \), does not necessarily holds.

## 2 Some counterexamples and a result

In [7] it was shown that for degree \( k = |A| = 2 \), the minimum diameter \( d \) of an Abelian group \( \Gamma \) with \( m \) vertices is \( d_{\min} = \left\lceil \sqrt{3m} \right\rceil - 2 \) (see [7, Eq. (9)]). That is,

\[ m_*(d, 2) \leq \left\lceil \frac{(d + 2)^2}{3} \right\rceil. \tag{4} \]
In fact the upper bound is attained when \( \Gamma = \mathbb{Z}_{3x} \times \mathbb{Z}_x \), with \( x \geq 1 \), and \( A = \{(1, 0), (-1, 1)\} \), leading to a (2-regular) Cayley digraph on \( m = 3x^2 \) vertices and diameter \( d = 3x - 2 \). However, it can be shown that, when \( x > 1 \), \( \text{rank } \Gamma = 2 \), so that \( \Gamma \) is not cyclic. In this case, the best result is obtained with the cyclic group \( \mathbb{Z}_m \) with \( m = \frac{1}{2}(d + 2)^2 - 1 \) and generating set \( A = \{a, b\} \), as shown in the following table.

| \( k \) | \( x \) | \( d = 3x - 2 \) | \( m_*(d, 2) = 3x^2 \) | \( A \subset \mathbb{Z}_{3x} \times \mathbb{Z}_x \) | \( m(d, 2) = 3x^2 - 1 \) | \( A \subset \mathbb{Z}_m \) |
| --- | --- | --- | --- | --- | --- | --- |
| 2 | 2 | 4 | 12 | \{(1, 0), (-1, 1)\} | 11 | \{1, 3\} |
| 2 | 3 | 7 | 27 | \{(1, 0), (-1, 1)\} | 26 | \{1, 8\} |
| 2 | 4 | 10 | 48 | \{(1, 0), (-1, 1)\} | 47 | \{1, 11\} |
| 2 | 5 | 13 | 75 | \{(1, 0), (-1, 1)\} | 74 | \{1, 14\} |
| 2 | 6 | 16 | 108 | \{(1, 0), (-1, 1)\} | 107 | \{1, 17\} |

For other values of \( m(d, 2) \), see [7, Table II] or the results in [3, 1]. In fact, from the results of these papers, and comparing the values of \( m(d, 2) \) in (2) with the upper bound for \( m_*(d, 2) \) in (4), one gets the following result for the case of degree \( k = 2 \):

**Proposition 2.1** For any diameter \( d \geq 2 \),

\[
m_*(d, 2) = \begin{cases} 
m(d, 2) + 1, & \text{if } d \equiv 1 \pmod{3}, \\
m(d, 2), & \text{otherwise}. 
\end{cases}
\]

In the case of the above digraph \( \text{Cay}(\mathbb{Z}_{3x} \times \mathbb{Z}_x, \{(1, 0), (-1, 1)\}) \), it can be shown that the two unique vertices at maximum distance \( d = 3x - 2 \) from the origin are \( (2x, x - 1) \) and \( (x, x - 1) \).

Similar counterexamples can be given to prove that the extremal Cayley digraphs with respect to their average distance are not necessarily attained for cyclic groups ([5, Th. 3.1]).

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**References**

[1] F. Aguiló, M.A. Fiol, An efficient algorithm to find optimal double loop networks, *Discrete Math.* 138 (1995) 15–29.
[2] J.-C. Bermond, F. Comellas, and D.F. Hsu, Distributed loop computer networks: a survey, *J. Parallel Distrib. Comput.* **24** (1995) 2–10.

[3] P. Esqué, F. Aguiló, M.A. Fiol, Double commutative-step digraphs with minimum diameters, *Discrete Math.* **114** (1993) 147–157.

[4] D.F. Hsu and X.D. Jia, Extremal problems in the construction of distributed loop networks, *SIAM J. Discrete Math.* **7** (1994) 57–71.

[5] A.G. Mask, J. Schneider, X. Jia, Extremal Cayley digraphs of finite Abelian groups, *J. Intercon. Networks* **12** (2011), no. 1-2, 125–135.

[6] P. Morillo, M.A. Fiol, and J. Fàbrega, The diameter of directed graphs associated to plane tessellations, *Ars Combin.* **20-A** (1985) 17–27.

[7] M.A. Fiol, J.L.A. Yebra, I. Alegre, M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.* **C-36** (1987) 702–713.

[8] C.K. Wong, D. Coppersmith, A combinatorial problem related to multinode memory organiztations, *J. Assoc. Comput. Machin.* **21** (1974) 392–402.