LARGE-TIME BEHAVIOR OF SOLUTIONS TO UNIPOLAR EULER-POISSON EQUATIONS WITH TIME-DEPENDENT DAMPING

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Abstract. This paper is concerned with the Cauchy problem of the 1-D unipolar hydrodynamic model for semiconductor device, a system of Euler-Poisson equations with time-dependent damping effect $-J(1 + t)^{-\lambda}$ for $-1 < \lambda < 1$, where $J$ denotes the current density, and the damping effect is asymptotically vanishing as $t \to \infty$ for $\lambda > 0$, and asymptotically enhancing to infinity as $t \to \infty$ for $\lambda < 0$. When the initial perturbation around the constant states are sufficiently small, by means of the time-weighted energy method, we prove that the smooth solutions to the Cauchy problem exist uniquely and globally. Particularly, we also obtain the large-time behavior of the solutions.

1. Introduction. Mathematical Models and equations. The hydrodynamic model of semiconductors is first proposed by Bløtekjær [1], that is usually employed to describe the dynamical phenomena of charged particles such as electrons and holes in semiconductor devices [1, 25], and positively and negatively charged ions in plasma [32]. The governing equations are Euler-Poisson equations with damping as follows:

\[
\begin{aligned}
  n_t + J_x &= 0, \\
  J_t + \left( \frac{J^2}{n} + p(n) \right)_x &= nE - \frac{J}{\tau}, \\
  E_x &= n - D(x).
\end{aligned}
\]  

(1.1)

Here $n(x,t), J(x,t)$ represent the electron density and the current density, respectively. $E(x,t)$ denotes electric field. $D(x) > 0$ is the doping profile standing for the density of impurities in the semiconductor device. $p(n)$ is the pressure function. $\tau > 0$ is the relaxation-time.

In system (1.1), the term $-\frac{J}{\tau}$ is called the damping effect, which is essential to the regularity of the solutions as well as the large-time behavior. In fact, as $\tau \to \infty$, the damping effect to system (1.1) will be vanishing, and the system becomes the pure Euler-Poisson system. In this case, the solutions will loss their regularity as we know. While, as $\tau \to 0^+$, the damping effect to the system will...
extremely enlarge. As $\tau \to \infty$ as well as $\tau \to 0^+$, it is full of interesting but also challenging to study the large-time behavior of the solutions to (1.1). In addressing this challenge, we mathematically take $\tau = (1 + t)^\lambda$ for some constant $\lambda$. In the case that $\lambda > 0$, $\tau = (1 + t)^\lambda \to \infty$ as $t \to \infty$, the damping effect in system (1.1) becomes time-gradually-degenerate. This case is called the under-damping. In the case that $\lambda < 0$, $\tau = (1 + t)^\lambda \to 0^+$ as $t \to \infty$, and the damping effect in system (1.1) becomes time-gradually-enhancing. This case is called the over damping. It is important and difficult to understand the structure of solutions in these cases from the mathematical point of view, and it is also the first attempt to the unipolar Euler-Poisson system as we know.

In this paper, we study the following 1-D isentropic unipolar Euler-Poisson system with time-dependent damping

\[
\begin{cases}
    n_t + J_x = 0, \\
    J_t + \left(\frac{J^2}{n} + p(n)\right)_x = nE - \frac{J}{(1 + t)^\lambda}, \\
    E_x = n - D(x),
\end{cases}
\]

with the initial-value conditions

\[
\begin{cases}
    (n, J)(x, 0) = (n_0, J_0)(x) \to (\bar{n}, \bar{J}_\pm) \quad \text{as} \quad x \to \pm \infty, \\
    E(-\infty, t) = E^-.
\end{cases}
\]

Here $\bar{n} > 0, \bar{J}_{\pm}, E^-$ are given constants. Technically throughout this paper we also assume

\[
p \in C^3(0, +\infty) \text{ with } p(s) \geq 0, p'(s) > 0 \text{ for } s > 0,
\]

\[-1 < \lambda < 1,
\]

\[D(x) \equiv \bar{n}.
\]

**Remark 1.**

1. The assumption (1.4) is physical. A typical example is polytropic gas which means $p(s) = \frac{1}{\gamma} s^\gamma$ for $\gamma \geq 1$, where $\gamma = 1$ is the isothermal case, and $\gamma > 1$ is the isentropic case.

2. The physical meaning of $D(x) \equiv \bar{n}$ is that the doping profile is flat (the flat doping profile means $|D'(x)| \ll 1$). The case for unipolar hydrodynamic model with $D(x) \equiv \bar{n}$ has not been studied yet, that is the first attempt. While, when $D(x) > 0$ is a general function, the expected asymptotic profiles of the equations are their corresponding stationary waves which is totally different from what we study here. This case is also interesting and more physical, but more complicated, and it will be expected in the coming work.

3. For the case $-1 < \lambda < 1$, the story comes from what follows. As clearly shown in [2, 21, 29, 30, 33] for the 1-D compressible Euler equations (the so-called $p$-system) with time-dependent damping

\[
\begin{cases}
    v_t - u_x = 0, \\
    u_t + p(v)_x = -\frac{u}{(1 + t)^\lambda},
\end{cases}
\]

in the case that $-1 < \lambda < 1$, the damping effect can prevent the singularity formation for the solutions like shocks, and the system can possess global-in-time solutions; while, in the case that $\lambda \geq 1$, the damping effect is too weak to stop the formation of shocks, and the system really behaves like the pure Euler system to possess shocks, namely, the solutions themselves are bounded but their derivatives
will blow up at finite time. Inspired by this, here we are mainly interested in the global existence of the solutions to unipolar Euler-Poisson equations with time-dependent damping effect \(-J_1/(1+t)^\lambda\) for \(-1 < \lambda < 1\).

**Background of relevant research.** In recent years, there are ever-increasing interests on hydrodynamic models for semiconductor devices, the theoretical theory and scientific computations on it has been one of hot spots in mathematical physics. For unipolar hydrodynamic model, the well-posedness of steady-state solutions has been studied in [4, 5, 7, 8], and stability of steady-state solutions has been studied in [9, 11, 15, 18, 23, 28, 34]. In [3, 20, 27, 31, 35], the authors gave the global existence of classical solutions and the entropy weak solutions, respectively. In [11, 12, 15, 18, 23, 34], the large-time behavior of solutions has been well studied. For the study on bipolar hydrodynamic model, great progress has been made, for example, see [6, 10, 13, 14, 16, 17, 22, 24] and the references therein.

For unipolar Euler-Poisson equations with regular damping (i.e. \(\lambda = 0\)), Huang-Mei-Wang-Yu [15] showed that the smooth solutions of (1.2)–(1.3) decay exponentially fast to steady-state solutions under the assumptions of

\[ J_+ \neq J_- \]

**Remark 2.** In physics, \(J_+ = J_-\) (or equivalently, \(E(\infty, t) - E(-\infty, t) = 0\)) stands for the switch-off case (no electric current). Correspondingly, the switch-on case means \(J_+ \neq J_-\) (or equivalently, \(E(\infty, t) - E(-\infty, t) \neq 0\)). It is more interesting but more challenging to study the switch-on case.

In the case that the damping is time-dependent, Chen-Li-Li-Mei-Zhang studied the following 1-D compressible Euler equations (the so-called \(p\)-system) in [2]

\[
\begin{aligned}
&v_t - u_x = 0, \\
&u_t = -p(v)_x - \frac{\mu}{(1+t)^\lambda}u,
\end{aligned}
\]

with the initial-value conditions

\[(v, u)(x, 0) = (v_0, u_0)(x) \to (v_\pm, u_\pm) \quad \text{as} \quad x \to \pm\infty, \quad (1.8)\]

and proved that, for \(0 < \lambda < 1\) and \(\mu > 0\) or \(\lambda = 1\) but \(\mu > 2\), when the derivatives of the initial date \((v_0, u_0)(x)\) are small, but the initial date themselves can be arbitrarily large, the solutions to (1.7)–(1.8) exist globally in time; while, for \(\lambda > 1\) or \(\lambda = 1\) but \(0 < \mu \leq 2\), the derivatives of the solutions will blow up even for small initial date.

For time-dependent damped bipolar Euler-Poisson equations

\[
\begin{aligned}
n_{1t} + J_{1x} &= 0, \\
J_{1t} + \left(\frac{J_1^2}{n_1} + p(n_1)\right)_x &= n_1E - \frac{J_1}{(1+t)^\lambda}, \\
n_{2t} + J_{2x} &= 0, \\
J_{2t} + \left(\frac{J_2^2}{n_2} + p(n_2)\right)_x &= -n_2E - \frac{J_2}{(1+t)^\lambda}, \\
E_x &= n_1 - n_2,
\end{aligned}
\]

with

\[(n_1, n_2, J_1, J_2) \big|_{t=0} = (n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_{1\pm}, J_{2\pm}) \quad \text{as} \quad x \to \pm\infty, \quad (1.10)\]
Li-Li-Mei-Zhang [22] proved that for $-1 < \lambda < 1$, the smooth solutions to (1.9)-(1.10) exist uniquely and globally, and time-asymptotically converges to the corresponding diffusion wave, when the initial perturbation around the diffusion wave is sufficiently small.

**Main results.** We denote the spatial asymptotic profile of the solutions as follows

$$f(\pm \infty, t) = f_{\pm}(t), \quad f \in \{n, J, E\},$$

and assume that

$$n_+(t) = n_-(t) = \bar{n}, \quad J_+(0) = \bar{J}_+, \quad J_-(0) = \bar{J}_-, \quad E_-(t) = E^- = 0, \quad (1.11)$$

$$E_+(t) = \int_{-\infty}^{+\infty} (n(x, t) - \bar{n})dx, \quad (1.13)$$

$$E^+ := E_+(0) = \int_{-\infty}^{+\infty} (n_0(x) - \bar{n})dx. \quad (1.15)$$

Define

$$\dot{n}(x, t) = E_+(t)m_0(x), \quad (1.16)$$

$$\dot{E}(x, t) = E_+(t)\int_{-\infty}^{\infty} m_0(y)dy, \quad (1.17)$$

$$\dot{J}(x, t) = J_-(t) + (J_+(t) - J_-(t))\int_{-\infty}^{\infty} m_0(y)dy, \quad (1.18)$$

where $m_0(x)$ is chosen as

$$m_0(x) \geq 0, \quad m_0(x) \in C_0^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} m_0(x)dx = 1.$$

We denote the initial perturbations as follows:

$$\phi_0(x) = \int_{-\infty}^{x} (n_0(y) - \bar{n} - \hat{n}(y, 0))dy, \quad \psi_0(x) = J_0(x) - \hat{J}(x, 0). \quad (1.19)$$

We are now able to state our main result in this paper.

**Theorem 1.1.** Assume that $-1 < \lambda < 1$, $(\phi_0, \psi_0)(x) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$, $\Phi_0 := ||\phi_0||_3 + ||\psi_0||_2$, and that $\delta := |E^+| + |J_+| + |J_-|$. There exists a suitably small constant $\delta_0 > 0$ such that, if $\delta + \Phi_0 \leq \delta_0$, the solutions $(n, J, E)(x, t)$ to IVP (1.2) - (1.3) uniquely and globally exist, and satisfy

$$\begin{cases}
    ||\partial_x^k (n - \bar{n} - \hat{n}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-1(k+2)(1+\lambda)}, & k = 0, 1, \\
    ||\partial_x^2 (n - \bar{n} - \hat{n}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-3-3\lambda}, \\
    ||\partial_x^k (J - \bar{J}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-1(k+1)(1+\lambda)-2}, & k = 0, 1, \\
    ||\partial_x^2 (J - \bar{J}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-4-2\lambda}, \\
    ||\partial_x^k (E - \bar{E}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-1(k+1)(1+\lambda)}, & k = 0, 1, 2, \\
    ||\partial_x^3 (E - \bar{E}) (t)||_{L^2(\mathbb{R})} \leq C\delta_0(1 + t)^{-3-3\lambda},
\end{cases}$$
and
\[
\begin{cases}
\|(n - \bar{n})(t)\|_{L^\infty(\mathbb{R})} = O(1)(1 + t)^{-\frac{2}{5}(1+\lambda)}, \\
\|J(t)\|_{L^\infty(\mathbb{R})} = O(1)(1 + t)^{-\frac{2}{5}(1+\lambda)}, \\
\|E(t)\|_{L^\infty(\mathbb{R})} = O(1)(1 + t)^{-\frac{2}{5}(1+\lambda)}.
\end{cases}
\]

The main difficulties in the proof of this theorem are as follows:

i) There are some \(L^\infty\)-gaps between the solutions and the constant states at far fields \(x = \pm \infty\) which makes the solutions not in \(L^2(\mathbb{R})\). To overcome this difficulty, we analyze what are the exact \(L^\infty\)-gaps and technically construct some correction functions to delete them.

ii) To obtain the decay estimate of the solutions, inspired by \([22]\), we use the time-weighted energy method with artfully choose the time-weight-functions.

The rest of this article is organized as follows. In Section 2, we first construct asymptotic profiles of the solutions, and technically construct the correction functions, then formulate a new problem to simplify the original IVP (1.2)−(1.3). In Section 3, the main effort is contributed to prove Theorem 1.1. Here the key step is to establish the a priori estimates for the solutions.

**Notations.** Throughout this article, \(C > 0\) denotes generic constant. \(L^2(\mathbb{R})\) is the space of square integrable real-valued functions defined on \(\mathbb{R}\) with the norm \(\| \cdot \| = \left( \int_{\mathbb{R}} |\cdot|^2 \, dx \right)^{\frac{1}{2}}\), and \(L^\infty(\mathbb{R})\) is the space of bounded measurable functions on \(\mathbb{R}\) with the norm \(\| \cdot \|_{L^\infty(\mathbb{R})} = \text{ess sup}_{x \in \mathbb{R}} |\cdot|\). For a nonnegative integer \(m\), \(H^m(\mathbb{R})\) denotes the Hilbert space with the norm \(\| \cdot \|_m\), especially \(\| \cdot \|_0 = \| \cdot \|\). We also denote
\[
\| (f_1, f_2, \cdots, f_k) \|^2 := \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_k\|^2.
\]
Let \(T > 0\), we denote by \(C([0, T]; H^m(\mathbb{R}))\) (resp. \(L^2(0, T; H^m(\mathbb{R}))\)) the space of continuous (resp.square integrable) functions on \([0, T]\) with values taken in \(H^m(\mathbb{R})\).

2. Preliminaries and reformulation of the problem.

2.1. Construction of the asymptotic profiles. First of all, we investigate the behavior of the solutions to (1.2)−(1.3) at far fields \(x = \pm \infty\). As \(x \to \pm \infty\), we further expect to reduce (1.2)−(1.3) to
\[
\frac{d}{dt} n_\pm(t) = 0, \quad (2.1)
\]
\[
\frac{d}{dt} J_\pm(t) = n_\pm(t) E_\pm(t) - \frac{J_\pm(t)}{(1 + t)\lambda}, \quad (2.2)
\]
\[
E(x, t) = E_-(t) + \int_{-\infty}^{x} E_-(x, t) \, dx = E_-(t) + \int_{-\infty}^{x} (n(x, t) - \bar{n}) \, dx. \quad (2.3)
\]
Since \(n_\pm(0) = n(\pm \infty, 0) = \bar{n}\), we have
\[
n_\pm(t) = n(\pm \infty, t) \equiv \bar{n}.
\]
Without loss of generality, we assume that
\[
E_-(t) = E(-\infty, t) = E^- = 0,
\]
then (2.2) shows that
\[
\begin{cases}
\frac{d}{dt} J_-(t) = - \frac{J_-(t)}{(1 + t)\lambda}, \\
J_-(0) = \bar{J}_-.
\end{cases}
\]
It follows immediately that
\[ J_-(t) = J_-e^{\frac{t}{1-E}}(1-(1+t)^{1-\lambda}). \]
Invoking \( E_-(t) = 0 \), one obtains from (2.3) that
\[ E(x,t) = \int_{-\infty}^x (n(x,t) - \bar{n})dx, \quad (2.4) \]
hence
\[ E^+ := E_+(0) = \int_{-\infty}^{+\infty} (n_0(x) - \bar{n})dx. \]
We denote
\[ \delta := |E^+| + |\bar{J}_+| + |\bar{J}_-|. \quad (2.5) \]
Differentiating (1.2)\textsuperscript{3} with respect to \( t \) and by (1.2)\textsubscript{1}, we get
\[ E_{xt}(x,t) = n_t(x,t) = -J_x(x,t). \quad (2.6) \]
Integrating (2.6) with respect to \( x \) over \( \mathbb{R} \) yields
\[ \frac{d}{dt}E_+(t) = E_+(\infty,t) = -\int_{-\infty}^{+\infty} J_+(x,t)dx = -\lambda(J_+(t) - J_-(t)), \quad (2.7) \]
then, we have
\[ E_+(t) = E(\infty,t) = \int_{-\infty}^{+\infty} (n_0(x) - \bar{n})dx + \int_0^t (J_+(s) - J_-(s))ds. \quad (2.8) \]
By (2.2), we get
\[ \frac{d}{dt}J_+(t) = \bar{n}E_+(t) - \frac{J_+(t)}{(1+t)^{\lambda}}. \quad (2.9) \]
Combining (2.7) with (2.9), we obtain
\[ \begin{cases} 
\frac{d^2}{dt^2}E_+(t) + \frac{1}{(1+t)^{\lambda}} \frac{d}{dt}E_+(t) + \bar{n}E_+(t) = 0, \\
\frac{d}{dt}E_+(0) = \bar{J}_+ - \bar{J}_-, \\
E_+(0) = E^+. 
\end{cases} \quad (2.10) \]
It is clear that (2.10) is well-posed. We next establish the decay estimates for \( E_+(t) \) when \(-1 < \lambda < 1\). To achieve this, we divide the estimates into two cases.

Case 1: \( 0 \leq \lambda < 1 \).
Multiplying (2.10)\textsubscript{1} by \( 2E_+'(t)(1 + t)^{-\lambda}E_+(t) \), where \( E_+'(t) = \frac{d}{dt}E_+(t) \), we arrive at
\[ \begin{align*}
&\left(E_+'(t) + (1 + t)^{-\lambda}E_+'(t) + \bar{n}E_+(t)\right) \left(2E_+'(t) + (1 + t)^{-\lambda}E_+(t)\right) \\
&= 2E_+'^2(t)E_+(t) + (1 + t)^{-\lambda}E_+'^2(t)E_+(t) + 2(1 + t)^{-\lambda}(E_+'(t))^2 \\
&\quad + (1 + t)^{-2\lambda}E_+'(t)E_+(t) + 2\bar{n}E_+(t)E_+'(t) + (1 + t)^{-\lambda}\bar{n}(E_+(t))^2 = 0,
\end{align*} \]
i.e.
\[ \frac{d}{dt} \left( (E_+'(t))^2 + (1 + t)^{-\lambda}E_+'(t)E_+(t) + \frac{\lambda}{2}(1 + t)^{-\lambda-1}(E_+(t))^2 \right) \\
+ \frac{1}{2} (1 + t)^{-\lambda}(E_+(t))^2 + \bar{n}(E_+(t))^2 \right) + (1 + t)^{-\lambda}(E_+'(t))^2 \\
+ \left( \frac{\lambda}{2}(\lambda + 1)(1 + t)^{-\lambda-2} + \lambda(1 + t)^{-2\lambda-1} + \bar{n}(1 + t)^{-\lambda} \right) (E_+(t))^2 = 0. \quad (2.11) \]
Set
\[ F(t) := (E_+^\prime(t))^2 + (1 + t)^{-\lambda} E_+^\prime(t) E_+(t) + \frac{\lambda}{2} (1 + t)^{-\lambda - 1} (E_+(t))^2 \] (2.12)
\[ + \frac{1}{2} (1 + t)^{-2\lambda} (E_+(t))^2 + \bar{n}(E_+(t))^2 \]
\[ = \frac{1}{2} (E_+^\prime(t))^2 + \frac{\lambda}{2} (1 + t)^{-\lambda - \lambda} (E_+(t))^2 + \bar{n}(E_+(t))^2 + \frac{1}{2} (E_+^\prime(t) + (1 + t)^{-\lambda} E_+(t))^2, \]
\[ \mu_1 \text{ is a positive constant, then (2.11) can be rewritten as} \]
\[ \frac{dF(t)}{dt} + \mu_1 (1 + t)^{-\lambda} F(t) + (1 - \frac{\mu_1}{2}) (1 + t)^{-\lambda} (E_+^\prime(t))^2 \]
\[ + \frac{\lambda}{2} (1 + t)^{-\lambda - 2} (E_+(t))^2 + (1 - \frac{\mu_1}{2}) \lambda (1 + t)^{-2\lambda - 1} (E_+(t))^2 \]
\[ + (1 - \mu_1) \bar{n}(1 + t)^{-\lambda} (E_+(t))^2 - \frac{\mu_1}{2} (1 + t)^{-\lambda} (E_+^\prime(t) + (1 + t)^{-\lambda} E_+(t))^2 = 0. \]
Noting that
\[ -\frac{\mu_1}{2} (1 + t)^{-\lambda} (E_+^\prime(t) + (1 + t)^{-\lambda} E_+(t))^2 \geq -\mu_1 (1 + t)^{-\lambda} (E_+^\prime(t))^2 - \mu_1 (1 + t)^{-3\lambda} (E_+(t))^2, \]
then we have
\[ \frac{dF(t)}{dt} + \mu_1 (1 + t)^{-\lambda} F(t) + (1 - \frac{\lambda}{2} \mu_1) (1 + t)^{-\lambda} (E_+^\prime(t))^2 \]
\[ + \frac{\lambda}{2} (1 + t)^{-\lambda - 2} (E_+(t))^2 + (1 - \frac{\mu_1}{2}) \lambda (1 + t)^{-2\lambda - 1} (E_+(t))^2 \]
\[ + (\bar{n}(1 - \mu_1)(1 + t)^{-\lambda} - \mu_1 (1 + t)^{-3\lambda}) (E_+(t))^2 \leq 0. \]
There exists some constant \( \mu_1 > 0 \) such that
\[ \frac{dF(t)}{dt} + \mu_1 (1 + t)^{-\lambda} F(t) \leq 0. \]
Thus, by Gronwall’s inequality, and using (2.5), we then have
\[ F(t) \leq F(0) e^{\frac{\mu_1}{2} t} (1 - (1 + t)^{-\lambda}) \leq C \delta e^{-C t^{1 - \lambda}}. \]
In this way, it follows from (2.12) that
\[ |E_+^\prime(t)| + |E_+(t)| \leq C \delta e^{-C t^{1 - \lambda}}. \] (2.14)
Since (2.7) shows that
\[ J_+(t) = -E_+^\prime(t) + J_-(t) = -E_+^\prime(t) + \bar{J}_- e^{\frac{\mu_1}{2} t} (1 - (1 + t)^{-\lambda}), \]
then, by (2.14), we have
\[ |J_+(t)| \leq C \delta e^{-C t^{1 - \lambda}}. \] (2.15)
Furthermore, (2.9) implies that
\[ |J_+^\prime(t)| \leq C \delta e^{-C t^{1 - \lambda}}. \] (2.16)
In summary, it holds that for \( 0 \leq \lambda < 1 \)
\[ \begin{cases} 
|n(\pm\infty, t) - \bar{n}| = 0, \\
|J(+\infty, t) - 0| = O(\delta) e^{-C t^{1 - \lambda}} \neq 0, \quad |J(-\infty, t) - 0| = O(\delta) e^{-C t^{1 - \lambda}} \neq 0, \\
|E(+\infty, t) - 0| = O(\delta) e^{-C t^{1 - \lambda}} \neq 0, \quad |E(-\infty, t) - 0| = 0.
\end{cases} \] (2.17)
Case 2: \(-1 < \lambda < 0\).
Multiplying (2.10) by \((2E_+(t) + (1 + t)\lambda E_+(t))\), we get
\[
(E_+''(t) + (1 + t)^{-\lambda}E_+'(t) + \bar{n}E_+(t)) (2E_+(t) + (1 + t)\lambda E_+(t)) = 2E_+''(t)E_+(t) + (1 + t)^{\lambda}E_+''(t)E_+(t) + 2(1 + t)^{-\lambda}(E_+'(t))^2 + E_+'(t)E_+(t) + 2\bar{n}E_+(t)E_+'(t) + (1 + t)^{\lambda}\bar{n}(E_+(t))^2 = 0,
\]
i.e.
\[
\frac{d}{dt} \left( (E_+(t))^2 + (1 + t)^{\lambda}E_+(t)E_+(t) - \frac{\lambda}{2} (1 + t)^{\lambda-1}(E_+(t))^2 \right) + \left( \frac{\lambda}{2} (\lambda - 1)(1 + t)^{\lambda-2} + \bar{n}(1 + t)^{\lambda} \right)(E_+(t))^2 = 0. \tag{2.18}
\]
Set \(G(t) := (E_+''(t))^2 + (1 + t)^{\lambda}E_+'(t)E_+(t) - \frac{\lambda}{2} (1 + t)^{\lambda-1}(E_+(t))^2 + \frac{1}{2}(E_+(t))^2 + \bar{n}(E_+(t))^2\), \(\mu_2\) is a positive constant, then (2.18) can be rewritten as
\[
\frac{dG(t)}{dt} + \mu_2(1 + t)^{\lambda}G(t) + \left( 2(1 + t)^{-\lambda} - (1 + \mu_2)(1 + t)^{\lambda} \right) (E_+'(t))^2 + \left( \frac{\lambda}{2} (\lambda - 1)(1 + t)^{\lambda-2} + \bar{n}(1 + t)^{\lambda} \right)(E_+(t))^2 = -\mu_2(1 + t)^{2\lambda}E_+(t)E_+(t) \
\]
we have
\[
\frac{dG(t)}{dt} + \mu_2(1 + t)^{\lambda}G(t) + \left( 2(1 + t)^{-\lambda} - (1 + \mu_2)(1 + t)^{\lambda} - \frac{\mu_2}{2} (1 + t)^{2\lambda} \right) (E_+'(t))^2 + \left( \bar{n} - \bar{n}\mu_2 - \frac{\mu_2}{2} \right)(1 + t)^{\lambda} + \frac{\lambda}{2} (\lambda - 1)(1 + t)^{\lambda-2} \]
\[
+ \frac{\lambda\mu_2}{2}(1 + t)^{2\lambda-1} - \frac{\mu_2}{2}(1 + t)^{2\lambda} \right)(E_+(t))^2 \leq 0.
\]
Then there exists some constant \(\mu_2\) such that
\[
\frac{dG(t)}{dt} + \mu_2(1 + t)^{\lambda}G(t) \leq 0.
\]
From Gronwall’s inequality and (2.5), it follows that
\[
G(t) \leq G(0)e^{\frac{\mu_2}{2}(1 - (1 + t)^{1+\lambda})} \leq C\delta e^{-C_1 t^{1+\lambda}}.
\]
Consequently, we have
\[
|E_+'(t)| + |E_+(t)| \leq C\delta e^{-C_1 t^{1+\lambda}}, \tag{2.19}
\]
\[
|J_+'(t)| + |J_+(t)| \leq C\delta e^{-C_1 t^{1+\lambda}}. \tag{2.20}
\]
In summary, it holds that for \(-1 < \lambda < 0\)
\[
\left\{ \begin{align*}
|n(\pm \infty, t) - \bar{n}| &= 0, \\
|J_+(\pm \infty, t) - 0| &= O(\delta)e^{-C_1 t^{1+\lambda}} \neq 0, \quad |J_-(\pm \infty, t) - 0| = O(\delta)e^{-C_1 t^{1+\lambda}} \neq 0, \\
|E(\pm \infty, t) - 0| &= O(\delta)e^{-C_1 t^{1+\lambda}} \neq 0, \quad |E(\pm \infty, t) - 0| = 0.
\end{align*} \right. \tag{2.21}
\]
Combining (2.17) with (2.21), we then obtain that for $-1 < \lambda < 1$

\[
\begin{cases}
|n(\pm \infty, t) - \bar{n}| = 0, \\
|J(\pm \infty, t) - 0| = O(\delta)e^{-Ct}\sigma \neq 0, \\
|J(-\infty, t) - 0| = O(\delta)e^{-Ct}\sigma \neq 0, \\
|E(\pm \infty, t) - 0| = O(\delta)e^{-Ct}\sigma \neq 0, \\
|E(-\infty, t) - 0| = 0
\end{cases}
\] (2.22)

for some constant $\sigma > 0$.

2.2. Correction functions. One can observe from (2.22) that there are some gaps between $J(\pm \infty, t)$ and 0, and $E(\pm \infty, t)$ and 0, which implies

\[J(x, t) \notin L^2(\mathbb{R}), \ E(x, t) \notin L^2(\mathbb{R}).\]

To delete these gaps, we need to construct some correction functions. Inspired by [16], we choose the correction functions $(\hat{n}, \hat{J}, \hat{E})(x, t)$ such that

\[
\begin{cases}
\hat{n}_t + \hat{J}_x = 0, \\
\hat{J}_t = \hat{n}\hat{E} - \frac{\hat{J}}{1 + t}x, \\
\hat{E}_x = \hat{n}, \\
\hat{n}(x, t) \to 0 \quad \text{as} \quad x \to \pm \infty, \\
\hat{J}(x, t) \to J_\pm(t) \quad \text{as} \quad x \to \pm \infty, \\
\hat{E}(x, t) \to 0 \quad \text{as} \quad x \to -\infty, \\
\hat{E}(x, t) \to E^+(t) \quad \text{as} \quad x \to +\infty,
\end{cases}
\]

where $\hat{n} = \hat{n}(x), \hat{J}(x, 0)$ and $\hat{E}(x, 0)$ are selected such that

\[
\hat{n}(x) \to \bar{n}, \ \hat{J}(x, 0) \to \bar{J}_\pm, \ \hat{E}(x, 0) \to E^\pm \quad \text{as} \quad x \to \pm \infty.
\]

In fact, we choose

\[
\begin{cases}
\hat{n}(x) = \bar{n}, \\
\hat{n}(x, 0) = m_0(x) \int_{-\infty}^{\infty} (n_0(x) - \bar{n}) dx, \\
\hat{J}(x, 0) = \bar{J}_- + (\bar{J}_+ - \bar{J}_-) \int_{-\infty}^{x} m_0(y) dy, \\
\hat{E}(x, 0) = E^+ \int_{-\infty}^{x} m_0(y) dy,
\end{cases}
\]

where $m_0(x)$ is chosen as

\[m_0(x) \geq 0, \ m_0(x) \in C_0^\infty(\mathbb{R}), \ \int_{\mathbb{R}} m_0(x) dx = 1.\]

Here, from (2.23), the following compatibility condition

\[
\hat{E}_x(x, 0) = \hat{n}(x, 0)
\]

holds. In fact, from (2.24), we obtain

\[
\hat{E}_x(x, 0) = \left(\hat{E}(x, 0)\right)_x = \left(E^+ \int_{-\infty}^{x} m_0(y) dy\right)_x = E^+ m_0(x)
\]

\[= m_0(x) \int_{-\infty}^{+\infty} (n_0(x) - \bar{n}) dx = \hat{n}(x, 0).\]
According to the idea of separating variables, set \( \hat{n}(x,t) = f(x)g(t) \). Integrating (2.23)\(_1\) with respect to \( x \) over \( \mathbb{R} \) yields
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \hat{n}(x,t)dx = - \int_{-\infty}^{+\infty} \dot{J}_x(x,t)dx = - (J_+(t) - J_-(t)).
\]
Integrating this equation over \((0,t)\) obtain
\[
\int_{-\infty}^{+\infty} \hat{n}(x,t)dx = \int_{-\infty}^{+\infty} \hat{n}(x,0)dx + \int_{0}^{t} (J_-(s) - J_+(s))ds,
\]
then
\[
\hat{n}(x,t) = \left( \int_{-\infty}^{+\infty} \hat{n}(x,0)dx + \int_{0}^{t} (J_-(s) - J_+(s))ds \right) f(x).
\] (2.25)
Hence
\[
\hat{n}(x,0) = f(x) \int_{-\infty}^{+\infty} \hat{n}(x,0)dx = f(x) \int_{-\infty}^{+\infty} (n_0(x) - \bar{n})dx.
\]
Then (2.24)\(_1\) implies that
\[
f(x) = m_0(x).
\] (2.26)
Thus, we get from (2.25), (2.26) and (2.8) that
\[
\hat{n}(x,t) = E_+(t)m_0(x).
\] (2.27)
Then \( \hat{E}_x(x,t) = \hat{n}(x,t) = E_+(t)m_0(x) \), which implies
\[
\hat{E}(x,t) = E_+(t) \int_{-\infty}^{x} m_0(y)dy.
\] (2.28)
Integrating (2.23)\(_1\) with respect to \( x \) over \((-\infty,x)\) and using (2.27), we then have
\[
\hat{J}(x,t) = J_-(t) + (J_+(t) - J_-(t)) \int_{-\infty}^{x} m_0(y)dy.
\] (2.29)
It is easy to verify that \( (\hat{n}, \hat{J}, \hat{E})(x,t) \) satisfies (2.23). Noting the decay estimates for \( E_+(t) \) and \( J_+(t) \) (see (2.14)-(2.16) and (2.19)-(2.20)) and \( m_0(x) \in C_0^{\infty}(\mathbb{R}) \), then we have the following estimates
\[
|\partial_x^k \partial_t^l (\hat{n}, \hat{J}, \hat{E})(x,t)| \leq C \delta e^{-Ct^\sigma}
\] (2.30)
for some constant \( \sigma > 0 \).

On the other hand, noting (1.2)\(_1\) and (2.23)\(_1\), i.e.
\[
(n - \bar{n} - \hat{n})_t = -(J - \hat{J})_x,
\]
after integrating it with respect to \( x \) over \( \mathbb{R} \), we have
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} (n(x,t) - \bar{n} - \hat{n}(x,t))dx = \int_{-\infty}^{+\infty} (-J_x(x,t) + \hat{J}_x(x,t))dx
\]
\[
= \left( -J(+\infty,t) + \hat{J}(+\infty,t) \right) - \left( -J(-\infty,t) + \hat{J}(-\infty,t) \right) = 0,
\]
which implies
\[
\int_{-\infty}^{+\infty} (n(x,t) - \bar{n} - \hat{n}(x,t))dx = \int_{-\infty}^{+\infty} (n_0(x) - \bar{n} - \hat{n}(x,0))dx = 0.
\] (2.31)
Summarizing (2.27)-(2.31), we have the following lemma.
Lemma 2.1. There holds
\[ \| \partial^k_x \partial^l_t (\hat{n}, \hat{J}, \hat{E})(t) \|_{L^\infty(\mathbb{R})} \leq C \delta e^{-Ct^\sigma} \] (2.32)
for \( k \geq 0, \ l = 0, 1, \) and some constant \( \sigma > 0, \) and
\[ J(\pm \infty, t) - \hat{J}(\pm \infty, t) = 0, \] (2.33)
\[ E(\pm \infty, t) - \hat{E}(\pm \infty, t) = 0, \] (2.34)
\[ \int_{-\infty}^{+\infty} [n(x, t) - \bar{n} - \bar{n}(x, t)] dx = 0, \] (2.35)
supp \( \bar{n} \subset \text{supp} \ m_0. \) (2.36)

2.3. The reformulated problem. By (1.2) and (2.23), we have
\[
\begin{cases}
(n - \bar{n} - \bar{n})_t + (J - \hat{J})_x = 0, \\
(J - \hat{J})_t + \left( \frac{J^2}{n} + p(n) \right)_x = nE - \bar{n}E - \frac{J - \hat{J}}{(1 + t)\lambda}, \\
(E - \hat{E})_x = n - \bar{n} - \bar{n}.
\end{cases}
\] (2.37)

Let
\[
\begin{cases}
\phi(x, t) = \int_{-\infty}^{x} (n(y, t) - \bar{n} - \bar{n}(y, t)) dy, \\
\psi(x, t) = J(x, t) - \hat{J}(x, t), \\
\phi_0(x) = \phi(x, 0) = \int_{-\infty}^{x} (n_0(y) - \bar{n} - \bar{n}(y, 0)) dy, \\
\psi_0(x) = \psi(x, 0) = J_0(x) - \hat{J}(x, 0).
\end{cases}
\] (2.38)

Then (2.37) is reduced to
\[
\begin{cases}
\phi_t + \psi = 0, \\
\psi_t + \frac{\psi}{(1 + t)\lambda} + p'(\bar{n})\phi_{xx} - \bar{n}\phi = a_1 - a_2, \\
E - \hat{E} = \phi, \\
(\phi, \psi)|_{t = 0} = (\phi_0, \psi_0)(x).
\end{cases}
\] (2.39)

Here
\[
\begin{align*}
\phi_1 & := (n\phi - \bar{n}\phi) + (\phi_x + \bar{n})\hat{E}, \\
\phi_2 & := \left( \frac{\psi + J}{\phi_x + \bar{n}} \right) + p(\phi_x + \bar{n} + \bar{n}) - p(\bar{n}) - p'(\bar{n})\phi_x.
\end{align*}
\]

Theorem 2.2. Assume that \( (\phi_0, \psi_0)(x) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}), \) \( \Phi_0 = \| \phi_0 \|_3 + \| \psi_0 \|_2. \) There exists a suitably small constant \( \delta_0 > 0 \) such that, if \( \delta + \Phi_0 \leq \delta_0, \) then the solutions \( (\phi, \psi)(x, t) \) to the IVP (2.39) uniquely and globally exist, and satisfy
\[
\sum_{k=0}^{2} (1 + t)^{(k+1)(1 + \lambda)} \| \partial^k_x \phi(t) \|^2 + (1 + t)^{3 + 3\lambda} \| \phi_{xx}(t) \|^2
\] 
\[
+ \sum_{k=0}^{1} (1 + t)^{(k+1)(1 + \lambda) + 2} \| \partial^k_x \phi(t) \|^2 + (1 + t)^{4 + 2\lambda} \| \phi_{xx}(t) \|^2
\] 
\[
+ \sum_{k=0}^{1} (1 + t)^{(k+1)(1 + \lambda) + 3 + \lambda} \| \partial^k_x \phi_{tt}(t) \|^2 \leq C(\delta + \Phi_0^2).
\]
Using the Sobolev inequality
\[ \|f\|_{L^\infty(\mathbb{R})} \leq C \|f\|^{1/2} \|f_x\|^{1/2}, \]  
(2.40)
and noting Lemma 2.1, one can further derive the following estimates.

**Corollary 2.1.** Under the assumptions of Theorem 2.2, we have
\[
\begin{aligned}
&\| (n - \bar{n}) (t) \|_{L^\infty(\mathbb{R})} \leq C (1 + t)^{-\frac{7+\lambda}{4}}, \\
&\| J(t) \|_{L^\infty(\mathbb{R})} \leq C (1 + t)^{-\frac{3+3\lambda}{4}}, \\
&\| E(t) \|_{L^\infty(\mathbb{R})} \leq C (1 + t)^{-\frac{3+3\lambda}{4}},
\end{aligned}
\]  
(2.41)

Theorem 1.1 is a direct consequence of Theorem 2.2 and Corollary 2.1.

### 3. A priori estimates

In this section, we will prove global existence of smooth solutions to (2.39). Substituting (2.39) into (2.39) we get
\[
\begin{cases}
\phi_{tt} + \frac{1}{1 + t} \phi_t - p'(\bar{n}) \phi_{xx} + \bar{n} \phi = f_1 - f_2 - f_3 - f_4, \\
\phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = -\psi_0(x).
\end{cases}
\]  
(3.1)
Here
\[
\begin{align*}
&f_1 := \frac{(-\phi_t + J)^2}{\phi_x + \bar{n} + \bar{n}}, \\
&f_2 := p(\phi_x + \bar{n} + \bar{n}) - p(\bar{n}) - p'(\bar{n}) \phi_x, \\
&f_3 := (\phi_x + \bar{n}) \phi, \\
&f_4 := (\phi_x + \bar{n}) \hat{E}.
\end{align*}
\]
Equivalently, we only need to study the global existence of smooth solutions to (3.1), it can be proved by the classical energy method with the continuation argument based on the local existence and the a priori estimates. As in [26], the local existence of solutions can be obtained by the standard iteration method together with the energy estimates. Thus, the key step is to establish the a priori estimates for the solutions by technical time-weighted energy method, which is the main target in the rest of this article.

For \( T > 0 \), we define the solution space as follows
\[ X(T) = \left\{ \phi(x, t) \mid \partial_t^l \phi \in C([0, T]; H^{3-l}(\mathbb{R})), l = 0, 1, 2 \right\} \]
equipped with the norm
\[
N(T)^2 := \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{2} (1 + t)^{(k+1)(1 + \lambda)} \| \partial_x^k \phi(t) \|^2 + (1 + t)^{3+3\lambda} \| \phi_{xx}(t) \|^2 \\
+ \sum_{k=0}^{1} (1 + t)^{(k+1)(1 + \lambda)} \| \partial_x^k \phi_t(t) \|^2 + (1 + t)^{4+2\lambda} \| \phi_{xxx}(t) \|^2 \\
+ \sum_{k=0}^{1} (1 + t)^{(k+1)(1 + \lambda) + 3+\lambda} \| \partial_x^k \phi_{tt}(t) \|^2 \right\}.
\]

Assume that \( N(T) + \delta \ll 1 \), then by Sobolev inequality (2.40) and simple computations, we have
\[
0 < \frac{\bar{n}}{2} \leq n = \phi_x + \bar{n} + \bar{n} \leq \frac{3}{2} \bar{n}.
\]
Lemma 3.1. There holds
\[
(1 + t)^{1 + \lambda} \| (\phi_t, \phi_x, \phi)(t) \|^2 + \int_0^t (1 + s)^{\lambda} \| (\phi_x, \phi)(s) \|^2 ds + \int_0^t (1 + s) \| \phi_t(s) \|^2 ds \\
\leq C(\delta + \Phi_0^2)
\] (3.2)
provided with \( N(T) + \delta \ll 1 \).

Proof. Multiplying (3.1) by \((2(\alpha + t)^\kappa \phi_t + \beta(\alpha + t)^{\kappa - 1} \phi)\), where \(\alpha \geq 1, \beta > 0, \kappa \geq 0\) are some constants, and integrating the resultant equation over \(\mathbb{R}\), we get
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^\kappa \phi_t^2 + p'(\bar{n})(\alpha + t)^\kappa \phi_x^2 + \bar{n}(\alpha + t)^\kappa \phi^2 + \beta(\alpha + t)^{\kappa - 1} \phi \phi_t \right) dx \\
\leq \frac{3}{2} (\kappa - 1)(\alpha + t)^{\kappa - 2} \phi^2 + \frac{3}{2} (\alpha + t)^{\kappa - 1} \phi^2 \right) dx \\
+ \int_{\mathbb{R}} (\beta - \kappa) p'(\bar{n})(\alpha + t)^{\kappa - 1} \phi_x^2 dx \\
+ \int_{\mathbb{R}} \frac{2(\alpha + t)^\kappa}{(1 + t)^{\lambda}} - (\beta + \kappa)(\alpha + t)^{\kappa - 1} \phi_t^2 dx \\
+ \int_{\mathbb{R}} \frac{3}{2}(\kappa - 1)(\kappa - 2)(\alpha + t)^{\kappa - 3} \\
- \left( \frac{3}{2}(\alpha + t)^{\kappa - 1} \right) + (\beta - \kappa) \bar{n}(\alpha + t)^{\kappa - 1} \phi^2 dx \\
= \int_{\mathbb{R}} \left( 2(\alpha + t)^\kappa \phi_t + \beta(\alpha + t)^{\kappa - 1} \phi \right) (f_1 + f_2 - f_3 + f_4) dx.
\] (3.3)

The right hand side of (3.3) will be estimated as follows. Applying Young’s inequality, and using the Sobolev inequality
\[
\| \phi_t(x, t) \|_{L^\infty(\mathbb{R})} \leq C \| \phi_t(x, \cdot) \|_{L^2(\mathbb{R})} \| \phi_{xx}(x, t) \|_{L^1(\mathbb{R})} \leq C N(t)(1 + t)^{-1},
\] (4.4)
\[
\| \phi_{xx}(x, t) \|_{L^\infty(\mathbb{R})} \leq C \| \phi_{xx}(x, \cdot) \|_{L^2(\mathbb{R})} \| \phi_{xxx}(x, t) \|_{L^1(\mathbb{R})} \leq C N(t)(1 + t)^{-\frac{3}{2}(1 + \lambda)},
\] (5.5)
and
\[
\| \phi(x, t) \|_{L^\infty(\mathbb{R})} \leq C \| \phi(x, \cdot) \|_{L^2(\mathbb{R})} \| \phi_{xx}(x, t) \|_{L^1(\mathbb{R})} \leq C N(t)(1 + t)^{-\frac{3}{2}(1 + \lambda)}^{-1},
\] (6.6)
and the decay estimates of \( \tilde{n}_x, \tilde{J}_x \) shown in Lemma 2.1, one obtains
\[
\int_{\mathbb{R}} 2(\alpha + t)^\kappa \phi_t f_{1x} dx \\
= \int_{\mathbb{R}} 2(\alpha + t)^\kappa \phi_t \left( \frac{J^2}{n} \right) x dx = \int_{\mathbb{R}} 2(\alpha + t)^\kappa \phi_t \left( \frac{2JJ_x}{n} - \frac{J^2n_x}{n^2} \right) dx \\
= \int_{\mathbb{R}} \frac{4}{n} (\alpha + t)^\kappa (-\phi_t + \tilde{J})(-\phi_{tx} + \tilde{J}_x) \phi_t dx + \int_{\mathbb{R}} \frac{2}{n^2}(\alpha + t)^\kappa (-\phi_t + \tilde{J})^2 (\phi_{xx} + \tilde{n}_x) \phi_t dx \\
\leq C \int_{\mathbb{R}} (\alpha + t)^\kappa |\phi_{tx}| \phi_t^2 dx + C \int_{\mathbb{R}} (\alpha + t)^\kappa |\tilde{J}_x| \phi_t^2 dx \\
+ C \int_{\mathbb{R}} (\alpha + t)^\kappa |\tilde{J}_t| \phi_t^2 dx + C \int_{\mathbb{R}} (\alpha + t)^\kappa |\tilde{n}_x| \phi_t^2 dx \\
+ C \int_{\mathbb{R}} (\alpha + t)^\kappa \tilde{J}^2 |\phi_{xx}| dx + C \int_{\mathbb{R}} (\alpha + t)^\kappa |\tilde{n}_x| \phi_t^2 dx.
By Taylor’s formula and Lemma 2.1, we have

\[ \begin{align*}
\leq & C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - 1} \phi_t^2 dx + C \int_R (\alpha + t)^{\kappa} |\dot{J}|(\phi_t^2 + \phi_{tx}^2) dx \\
+ & C \int_R (\alpha + t)^{\kappa} |\dot{f}|(\dot{J}_x^2 + \phi_t^2) dx + C \int_R (\alpha + t)^{\kappa} \dot{J}^2 (\phi_{xx}^2 + \dot{\phi}_t^2) dx \\
+ & C \int_R (\alpha + t)^{\kappa} \dot{J}^2 (\dot{n}_x^2 + \dot{\phi}_t^2) dx \\
\leq & C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - 1} \phi_t^2 dx + C \delta e^{-Ct^*} \int_R (\phi_{tx}^2 + \dot{J}_x^2 + \phi_t^2 + \dot{n}_x^2) dx \\
\leq & C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - \lambda} \phi_t^2 dx + C \delta e^{-Ct^*}.
\end{align*} \] (3.7)

Using the Hölder inequality and the Sobolev inequality

\[ \|\phi(\cdot, t)\|_{L^\infty(R)} \leq C N(t)(1 + t)^{-\frac{\lambda}{2}(1 + \lambda)} \leq C N(t), \]

we have

\[ \begin{align*}
\int_R 2(\alpha + t)^{\kappa} \phi_t (f_3 + f_4) dx \\
= \int_R 2(\alpha + t)^{\kappa} \phi_t (\phi + E) dx \\
\leq C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - \lambda} \phi_t^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - 1} \phi_t^2 dx + C \delta e^{-Ct^*}.
\end{align*} \] (3.10)

In the same manner we see that

\[\begin{align*}
\int_R \beta(\alpha + t)^{\kappa - 1} \phi(f_{1x} + f_{2x} - f_3 - f_4) dx \\
\leq C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - \lambda} \phi_t^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - 1} \phi_t^2 dx \\
+ C(N(t) + \delta) \int_R (\alpha + t)^{\kappa - 1} \phi_t^2 dx + C \delta e^{-Ct^*}.
\end{align*}\] (3.11)
Combining (3.3), (3.7), (3.9), (3.10) and (3.11), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^{\kappa} \phi_t^2 + p'(\bar{n})(\alpha + t)^{\kappa} \phi_x^2 + \bar{n}(\alpha + t)^{\kappa} \phi^2 + \beta(\alpha + t)^{\kappa-1} \phi \phi_t \right)
- \frac{\beta}{2} (\kappa - 1)(\alpha + t)^{\kappa-2} \phi^2 + \frac{\beta(\alpha + t)^{\kappa-1}}{2(1 + t)^{\lambda}} \phi^2 \right) dx
+ \int_{\mathbb{R}} (\beta - \kappa) p'(\bar{n})(\alpha + t)^{\kappa-1} \phi_t^2 dx
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{\kappa}}{(1 + t)^{\lambda}} - (\beta + \kappa)(\alpha + t)^{\kappa-1} \right) \phi_t^2 dx
+ \int_{\mathbb{R}} \left( \frac{\beta}{2} (\kappa - 1)(\alpha + t)^{\kappa-3} \right) \phi_t^2 dx
- \left( \frac{\beta(\alpha + t)^{\kappa-1}}{2(1 + t)^{\lambda}} \right) \phi_t^2 dx
\leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\kappa-\lambda} \phi_t^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\kappa-1} \phi_x^2 dx
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\kappa-1} \phi^2 dx + C\delta e^{-C\tau}. \tag{3.12}
\]
Taking \( \kappa = 1 + \lambda \), \( \beta = 2 \) in (3.12), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^{1+\lambda} \phi_t^2 + p'(\bar{n})(\alpha + t)^{1+\lambda} \phi_x^2 + \bar{n}(\alpha + t)^{1+\lambda} \phi^2 + 2(\alpha + t)^{\lambda} \phi \phi_t \right)
- \lambda(\alpha + t)^{\lambda-1} \phi^2 + \frac{(\alpha + t)^{\lambda}}{(1 + t)^{\lambda}} \phi^2 \right) dx
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{1+\lambda}}{(1 + t)^{\lambda}} - (3 + \lambda)(\alpha + t)^{\lambda} \right) \phi_t^2 dx
+ \int_{\mathbb{R}} \left( \lambda(\alpha - 1)(\alpha + t)^{\lambda-2} - \lambda(\alpha + t)^{\lambda-1} \right) \phi_t^2 dx
+ \int_{\mathbb{R}} \left( (\alpha + t)^{1+\lambda} \phi_x^2 \right) dx
\leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\lambda} \phi_t^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\lambda} \phi_x^2 dx
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\lambda} \phi^2 dx + C\delta e^{-C\tau}. \tag{3.13}
\]
Since
\[
2(\alpha + t)^{\lambda} \phi \phi_t = 2 \left( \frac{\sqrt{2}}{2} (\alpha + t)^{\frac{\lambda+1}{2}} \phi_t \right) \sqrt{2(\alpha + t)^{\frac{\lambda+1}{2}}} \phi \geq \frac{1}{2} (\alpha + t)^{\lambda+1} \phi_t^2 - 2(\alpha + t)^{\lambda-1} \phi^2,
\]
if \( \alpha \) is sufficiently large, it holds
\[
\bar{n}(\alpha + t)^{1+\lambda} \phi^2 - 2(\alpha + t)^{\lambda-1} \phi^2 - \lambda(\alpha + t)^{\lambda-1} \phi^2 + \frac{(\alpha + t)^{\lambda}}{(1 + t)^{\lambda}} \phi^2 \geq C(\alpha + t)^{1+\lambda} \phi^2,
\]
then, we have
\[
(\alpha + t)^{1+\lambda} \phi_t^2 + \bar{n}(\alpha + t)^{1+\lambda} \phi^2 + 2(\alpha + t)^{\lambda} \phi \phi_t - \lambda(\alpha + t)^{\lambda-1} \phi^2 + \frac{(\alpha + t)^{\lambda}}{(1 + t)^{\lambda}} \phi^2 \geq C(\alpha + t)^{1+\lambda}(\phi_t^2 + \phi^2). \tag{3.14}
\]
Noticing that, if \( \alpha \) is sufficiently large, it holds
\[
\frac{2(\alpha + t)^{1+\lambda}}{(1 + t)^\lambda} - (3 + \lambda)(\alpha + t)^\lambda
\]
\[
= (\alpha + t)^{1+\lambda} \left( \frac{2}{(1 + t)^\lambda} - \frac{3 + \lambda}{\alpha + t} \right) \geq (\alpha + t)^{1+\lambda} \cdot \frac{1}{(1 + t)^\lambda} \geq \alpha + t \quad (3.15)
\]
and
\[
\lambda(\lambda - 1)(\alpha + t)^{\lambda-2} - \lambda(\alpha + t)^{\lambda-1} \left( \frac{(\alpha + t)^\lambda}{(1 + t)^\lambda} + (1 - \lambda)\bar{n}(\alpha + t)^\lambda \right) \geq C(\alpha + t)^\lambda. \quad (3.16)
\]
Then integrating (3.13) over \((0, t)\), and by (3.14)-(3.16), we can prove that
\[
(1 + t)^{1+\lambda} \| (\phi_t, \phi_x, \phi)(t) \|^2 + \int_0^t (1 + s)^\lambda \| (\phi_x, \phi)(s) \|^2 ds + \int_0^t (1 + s) \| \phi_t(s) \|^2 ds 
\]
\[
\leq C(\delta + \Phi_0^2)
\]
provided with \( N(T) + \delta \ll 1 \). \( \square \)

**Lemma 3.2.** There holds
\[
(1 + t)^{2+2\lambda} \| (\phi_{xt}, \phi_{xx}, \phi_x)(t) \|^2 + \int_0^t (1 + s)^{2+\lambda} \| (\phi_{xt}(s) \|^2 ds 
\]
\[
+ \int_0^t (1 + s)^{1+2\lambda} \| (\phi_{xx}, \phi_x)(s) \|^2 ds \leq C(\delta + \Phi_0^2) \quad (3.17)
\]
provided with \( N(T) + \delta \ll 1 \).

**Proof.** Differentiating (3.1) in \( x \) yields
\[
\phi_{xt} + \frac{1}{(1 + t)\lambda} \phi_{xt} - p'(\bar{n})\phi_{xxx} + \bar{n}\phi_x = f_{1xx} + f_{2xx} - f_{3x} - f_{4xx}. \quad (3.18)
\]

Multiplying (3.18) by \((2(\alpha + t)^{2+2\lambda}\phi_{xt} + 4(\alpha + t)^{1+2\lambda}\phi_x)\), where \( \alpha \geq 1 \) is some constant, and integrating the resultant equation over \( \mathbb{R} \) gives
\[
\frac{d}{dt} \int_\mathbb{R} \left( (\alpha + t)^{2+2\lambda} \phi_x^2 + p'(\bar{n})(\alpha + t)^{2+2\lambda} \phi_{xx}^2 + \bar{n}(\alpha + t)^{2+2\lambda} \phi_x^2 
\]
\[
+ 4(\alpha + t)^{1+2\lambda} \phi_{xt} \phi_x - 2(1 + 2\lambda)(\alpha + t)^{2\lambda} \phi_x^2 \right) + \frac{2(\alpha + t)^{1+2\lambda}}{(1 + t)^\lambda} \phi_x^2 dx 
\]
\[
+ \int_\mathbb{R} \left( \frac{2(\alpha + t)^{2+2\lambda}}{(1 + t)^\lambda} - (6 + 2\lambda)(\alpha + t)^{1+2\lambda} \right) \phi_{xt}^2 dx 
\]
\[
+ \int_\mathbb{R} (2 - 2\lambda)p'(~\bar{n}~)(\alpha + t)^{1+2\lambda} \phi_x^2 dx 
\]
\[
+ \int_\mathbb{R} \left( (2 - 2\lambda)\bar{n}(\alpha + t)^{1+2\lambda} + 4\lambda(1 + 2\lambda)(\alpha + t)^{2\lambda-1} 
\]
\[
+ 2\lambda(\alpha + t)^{1+2\lambda} \right) \left( \phi_{xx}^2 \right) - \frac{2(1 + 2\lambda)(\alpha + t)^{2\lambda}}{(1 + t)^\lambda} \phi_x^2 dx 
\]
\[
= \int_\mathbb{R} \left( 2(\alpha + t)^{2+2\lambda} \phi_{xt} + 4(\alpha + t)^{1+2\lambda} \phi_x \right) (f_{1xx} + f_{2xx} - f_{3x} - f_{4xx}) dx. \quad (3.19)
\]
The right hand side of (3.19) will be estimated as follows. we first calculate
\[
\int_\mathbb{R} 2(\alpha + t)^{2+2\lambda} \phi_{xt} f_{1xx} dx
\[
\begin{align*}
\int_R \left(2\frac{\alpha + t}{n} + 2\frac{J^2}{n^2} - \frac{4JJ_x}{n^2} + \frac{2J^2n_x}{n^3} - \frac{J^2n_{xx}}{n^4}\right) dx &=: \int_R I_1 dx + \int_R I_2 dx + \int_R I_3 dx, \\
&= \int_R I_1 dx + \int R I_2 dx + \int R I_3 dx,
\end{align*}
\]
where
\[
\begin{align*}
I_1 &= 2(\alpha + t) + 2\phi_{xt}\left(\frac{2J^2}{n} - \frac{4JJ_x n_x}{n^2} + \frac{2J^2n_x}{n^3}\right), \\
I_2 &= 2(\alpha + t) + 2\phi_{xt}\left(\frac{2JJ_x}{n}\right), \\
I_3 &= 2(\alpha + t) + 2\phi_{xt}\left(-\frac{J^2n_{xx}}{n^2}\right).
\end{align*}
\]
Using Young’s inequality, (3.4), (3.5) and (2.32), one gets
\[
\begin{align*}
\int_R I_1 dx \leq C \int_R (\alpha + t)^{2+2\lambda} |\phi_{xt}|(\phi_x^2 + \phi_{xx}^2 + J_x^2 + \hat{n}_x^2) dx \\
&\quad + C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} |\phi_{xt}(\phi_{xx}^2 + \hat{n}_x)| dx \\
&\leq C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} |\phi_{xt}| dx \\
&\quad + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma}. \\
\end{align*}
\]
Applying the Sobolev inequality, we obtain
\[
|J| = |\phi_t + \hat{J}| \leq C(N(t) + \delta).
\]
It then follows that
\[
\begin{align*}
\int_R I_2 dx &= \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt}\left(\frac{2JJ_x}{n}\right) dx \\
&= \int_R \left(\frac{4\phi_t}{n} - \frac{4\hat{J}}{n}\right)(\alpha + t)^{2+2\lambda} \phi_{xx} dx \\
&\quad + C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma} \\
&\leq \int_R \left(\frac{2\phi_t}{n} - \frac{2\hat{J}}{n}\right)(\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx \\
&\quad + C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma} \\
&\leq C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma} \\
\end{align*}
\]
and
\[
\begin{align*}
\int_R I_3 dx &= \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt}\left(-\frac{J^2n_{xx}}{n^2}\right) dx \\
&= \int_R \left(\frac{2\phi_t}{n} - \frac{2\hat{J}}{n}\right)(\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx \\
&\quad + C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma} \\
&\leq C(N(t) + \delta) \int_R (\alpha + t)^{2+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ce^\sigma} \\
\end{align*}
\]
Substituting (3.21)-(3.23) into (3.20), one gets

\[ \int_R \left( -\frac{2f^2}{n^2} \right) (\alpha + t)^{2+2\lambda} \phi_{xt} \phi_{xxx} dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xt}^2 dx + C\delta e^{-Ct^\sigma} \]
\[ = \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xx} \left( \phi_{xtt} \left( \frac{f^2}{n^2} \right) + \phi_{xt} \left( \frac{f^2}{n^2} \right) \right) dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xt}^2 dx + C\delta e^{-Ct^\sigma} \]
\[ \leq \frac{d}{dt} \int_R (\alpha + t)^{2+2\lambda} \left( \frac{f^2}{n^2} \right) \phi_{xx}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xt}^2 dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct^\sigma}. \quad (3.23) \]

Taking integration by parts, using Taylor’s formula (3.8) and Lemma 2.1, we obtain

\[ \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt} f_{xx} dx \]
\[ = \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt} \left( p''(n)(\phi_{xx} + \hat{n}_x)^2 + p'(n)(\phi_{xxx} + \hat{n}_{xx}) - p'(\hat{n})\phi_{xx} \right) dx \]
\[ \leq \int_R 2(\alpha + t)^{2+2\lambda} \left( p'(n) - p'(\hat{n}) \right) \phi_{xt} \phi_{xxx} dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xt}^2 dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct^\sigma} \]
\[ \leq -\frac{d}{dt} \int_R (\alpha + t)^{2+2\lambda} \left( p'(n) - p'(\hat{n}) \right) \phi_{xx}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xx}^2 dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct^\sigma}. \quad (3.25) \]

It follows from the Young inequality that

\[ - \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt} (f_{3x} + f_{4x}) dx \]
\[ = \int_R 2(\alpha + t)^{2+2\lambda} \phi_{xt} \left( (\phi_{xx} + \hat{n}_x)(\phi + \hat{E}) + (\phi_x + \hat{n})(\phi_{x} + \hat{E}_x) \right) dx \]
\[ \leq C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xt}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx \]
\[ + C(N(t) + \delta) \int_R (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct^\sigma}. \quad (3.26) \]
In the same manner we see that
\[
\int_{\mathbb{R}} 4(\alpha + t)^{1+2\lambda} \phi_x (f_1x + f_2x - f_3x - f_4x) dx
\]
\[
\leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{2+\lambda} \phi_x^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_x^2 dx
\]
\[
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_x^2 dx + C\delta e^{-Ct}\sigma. \quad (3.27)
\]
Summarizing (3.19) and (3.24)-(3.27), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^{2+2\lambda} \phi_x^2 + (p'(n) + O(1)(N(t) + \delta)) (\alpha + t)^{2+2\lambda} \phi_x^2 \right) dx
\]
\[
+ \bar{n}(\alpha + t)^{2+2\lambda} \phi_x^2 + 4(\alpha + t)^{1+2\lambda} \phi_x \phi_{xt} - 2(1 + 2\lambda)(\alpha + t)^{2\lambda} \phi_x^2 + \frac{2(\alpha + t)^{1+2\lambda}}{(1 + t)^{\lambda}} \phi_x^2 dx
\]
\[
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{2+2\lambda}}{(1 + t)^{\lambda}} - (6 + 2\lambda)(\alpha + t)^{1+2\lambda} \right) \phi_x^2 dx
\]
\[
+ \int_{\mathbb{R}} (2 - 2\lambda)p'(\bar{n})(\alpha + t)^{1+2\lambda} \phi_x^2 dx
\]
\[
+ \int_{\mathbb{R}} \left( (2 - 2\lambda)\bar{n}(\alpha + t)^{1+2\lambda} + 4\lambda(1 + 2\lambda)(\alpha + t)^{2\lambda - 1} \phi_x^2 \right) dx
\]
\[
\leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{2+\lambda} \phi_x^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_x^2 dx
\]
\[
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_x^2 dx + C\delta e^{-Ct}\sigma. \quad (3.28)
\]
Since
\[
4(\alpha + t)^{1+2\lambda} \phi_x \phi_{xt}
\]
\[
= 2 \left( \sqrt{2} \frac{(\alpha + t)^{1+\lambda} \phi_{xt}}{2} \right) \left( 2\sqrt{2}(\alpha + t)^{\lambda} \phi_x \right) \geq - \frac{1}{2} (\alpha + t)^{1+\lambda} \phi_x^2 - 8(\alpha + t)^{2\lambda} \phi_x^2,
\]
if \(\alpha\) is sufficiently large, it holds
\[
-8(\alpha + t)^{2\lambda} \phi_x^2 - 2(1 + 2\lambda)(\alpha + t)^{2\lambda} \phi_x^2 + \frac{2(\alpha + t)^{1+2\lambda}}{(1 + t)^{\lambda}} \phi_x^2 \geq C(\alpha + t)^{1+\lambda} \phi_x^2,
\]
then, we have
\[
(\alpha + t)^{2+2\lambda} \phi_x^2 + 4(\alpha + t)^{1+2\lambda} \phi_x \phi_{xt} - 2(1 + 2\lambda)(\alpha + t)^{2\lambda} \phi_x^2 + \frac{2(\alpha + t)^{1+2\lambda}}{(1 + t)^{\lambda}} \phi_x^2
\]
\[
\geq C(\alpha + t)^{2+2\lambda} \phi_x^2 + C(\alpha + t)^{1+\lambda} \phi_x^2. \quad (3.29)
\]
Noticing that, if \(\alpha\) is sufficiently large, it holds
\[
\frac{2(\alpha + t)^{2+2\lambda}}{(1 + t)^{\lambda}} - (6 + 2\lambda)(\alpha + t)^{1+2\lambda}
\]
\[
= (\alpha + t)^{2+2\lambda} \left( \frac{2}{1 + t} - \frac{6 + 2\lambda}{\alpha + t} \right) \geq (\alpha + t)^{2+2\lambda} \cdot \frac{1}{(1 + t)^{\lambda}} \geq (\alpha + t)^{2+\lambda} \quad (3.30)
\]
and
\[ (2 - 2\lambda)\tilde{u}(\alpha + t)^{1+2\lambda} + 4\lambda(1 + 2\lambda)(\alpha + t)^{2\lambda - 1} + \frac{2\lambda(\alpha + t)^{1+2\lambda}}{(1 + t)^{1+\lambda}} - \frac{2(1 + 2\lambda)(\alpha + t)^{2\lambda}}{(1 + t)^{\lambda}} \geq C(\alpha + t)^{1+2\lambda}. \] (3.31)

Then integrating (3.28) over \((0, t)\), and by (3.29)-(3.31), we can prove that
\[ (1 + t)^{2+2\lambda}\|\phi_{xt}, \phi_{xx}, \phi_x(t)\|^2 + \int_0^t (1 + s)^{2+\lambda}\|\phi_{xt}(s)\|^2 ds \]
\[ + \int_0^t (1 + s)^{1+2\lambda}\|\phi_{xx}, \phi_x(s)\|^2 ds \leq C(\delta + \Phi_0^2) \]
provided with \(N(T) + \delta \ll 1\).

**Lemma 3.3.** There holds
\begin{align*}
(1 + t)^{3+3\lambda}\|\phi_{xx}, \phi_{xxx}, \phi_x(t)\|^2 &+ \int_0^t (1 + s)^{3+2\lambda}\|\phi_{xt}(s)\|^2 ds \\
+ \int_0^t (1 + s)^{2+3\lambda}\|\phi_{xxx}, \phi_x(s)\|^2 ds &\leq C(\delta + \Phi_0^2) \tag{3.32}
\end{align*}
provided with \(N(T) + \delta \ll 1\).

**Proof.** Differentiating (3.18) in \(x\) yields
\[ \phi_{xxtt} + \frac{1}{(1 + t)^{\lambda}} \phi_{xxt} - p'(\tilde{n})\phi_{xxx} + \tilde{n}\phi_{xx} = f_{1xxx} + f_{2xxx} - f_{3xx} - f_{4xx}. \] (3.33)

Multiplying (3.33) by \((2(\alpha + t)^{3+3\lambda}\phi_{xxt} + 6(\alpha + t)^{2+3\lambda}\phi_{xx})\), where \(\alpha \geq 1\) is some constant, and integrating the resultant equation over \(\mathbb{R}\) gives
\begin{align*}
\frac{d}{dt} \int_\mathbb{R} &\left( (\alpha + t)^{3+3\lambda}\phi_{xxt}^2 + p'(\tilde{n})(\alpha + t)^{3+3\lambda}\phi_{xxx}^2 + \tilde{n}(\alpha + t)^{3+3\lambda}\phi_{xx}^2 \right) dx \\
&\quad + 6(\alpha + t)^{2+3\lambda}\phi_{xx}\phi_{xxt} - 3(2 + 3\lambda)(\alpha + t)^{1+3\lambda}\phi_{xx}^2 + \frac{3(\alpha + t)^{2+3\lambda}}{(1 + t)^{\lambda}}\phi_{xx}^2
\end{align*}
(3.34)
\begin{align*}
&\quad + \int_\mathbb{R} \left( \frac{2(\alpha + t)^{3+3\lambda}}{(1 + t)^{\lambda}} - (9 + 3\lambda)(\alpha + t)^{2+3\lambda} \right) \phi_{xxt}^2 dx \\
&\quad + \int_\mathbb{R} (3 - 3\lambda)p'(\tilde{n})(\alpha + t)^{2+3\lambda}\phi_{xxx}^2 dx \\
&\quad + \int_\mathbb{R} \left( (3 - 3\lambda)\tilde{n}(\alpha + t)^{2+3\lambda} + 3(2 + 3\lambda)(1 + 3\lambda)(\alpha + t)^{3\lambda} + \frac{3\lambda(\alpha + t)^{2+3\lambda}}{(1 + t)^{1+\lambda}} - \frac{3(2 + 3\lambda)(\alpha + t)^{1+3\lambda}}{(1 + t)^{\lambda}} \right) \phi_{xx}^2 dx \\
&\quad = \int_\mathbb{R} \left( (\alpha + t)^{2+3\lambda}\phi_{xxt} + 6(\alpha + t)^{2+3\lambda}\phi_{xx} \right) (f_{1xxx} + f_{2xxx} - f_{3xx} - f_{4xx}) dx.
\end{align*}

Analysis similar to (3.20)-(3.24) shows that
\begin{align*}
\frac{d}{dt} \int_\mathbb{R} (\alpha + t)^{3+3\lambda}\phi_{xxt} f_{1xxx} dx &\leq \frac{d}{dt} \int_\mathbb{R} (\alpha + t)^{3+3\lambda} \left( \frac{J^2}{n^2} \right) \phi_{xxx}^2 dx + C(N(t) + \delta) \int_\mathbb{R} (\alpha + t)^{3+2\lambda}\phi_{xxt}^2 dx
\end{align*}
As in (3.25), we obtain
\[ \int_R (\alpha + t)^{2+\lambda} \phi_{xx}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \varphi_{x}^2 dx + C\delta e^{-Ct}. \] (3.35)

As shown before, we have
\[ \frac{d}{dt} \int_R (\alpha + t)^{3+\lambda} \phi_{xxt} f_{2xxx} dx \]
\[ \leq - \frac{d}{dt} \int_R (\alpha + t)^{3+\lambda} (p'(n) - p'(\bar{n})) \phi_{xxx}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xx}^2 dx 
+ C(N(t) + \delta) \int_R (\alpha + t)^{2+\lambda} \phi_{xxx}^2 dx + C\delta e^{-Ct}. \] (3.36)

Using the fact
\[ \| \phi_x(\cdot, t) \|_{L^\infty(\mathbb{R})} \leq C N(t)(1 + t)^{-\frac{\lambda}{2}(1+\lambda)}, \]
we have
\[ - \int_R 2(\alpha + t)^{3+\lambda} \phi_{xxt}(f_{3xx} + f_{4xx}) dx \]
\[ \leq C(N(t) + \delta) \int_R (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx 
+ C(N(t) + \delta) \int_R (\alpha + t)^{2+3\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct}. \] (3.37)

As shown before, we have
\[ \int_R 6(\alpha + t)^{2+3\lambda} \phi_{xxx}(f_{1xxx} + f_{2xxx} - f_{3xx} - f_{4xx}) dx \]
\[ \leq C(N(t) + \delta) \int_R (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_R (\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx 
+ C(N(t) + \delta) \int_R (\alpha + t)^{2+3\lambda} \phi_{xx}^2 dx + C\delta e^{-Ct}. \] (3.38)

Thus, by (3.34)-(3.38), we have
\[ \frac{d}{dt} \int_R \left( (\alpha + t)^{3+\lambda} \phi_{xxt}^2 + (p'(n) + O(1)(N(t) + \delta)) (\alpha + t)^{3+3\lambda} \phi_{xxx}^2 \right. 
+ \bar{n}(\alpha + t)^{3+3\lambda} \phi_{xx}^2 + 6(\alpha + t)^{2+3\lambda} \phi_{xxx} \phi_{xxt} 
- 3(2 + 3\lambda)(\alpha + t)^{1+3\lambda} \phi_{xx}^2 + \frac{3(\alpha + t)^{2+3\lambda}}{(1 + t)^\lambda} \phi_{xx}^2 dx 
\left. + \int_R \left( \left( 2(\alpha + t)^{3+3\lambda} \frac{1}{(1 + t)^\lambda} - (9 + 3\lambda)(\alpha + t)^{2+3\lambda} \right) \phi_{xxt}^2 dx 
+ \int_R (3 - 3\lambda)p'(\bar{n})(\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx 
+ \int_R \left( (3 - 3\lambda)\bar{n}(\alpha + t)^{2+3\lambda} + 3(2 + 3\lambda)(1 + 3\lambda)(\alpha + t)^{3\lambda} \right. 
+ \frac{3\lambda(\alpha + t)^{2+3\lambda}}{(1 + t)^{1+\lambda}} - \frac{3(2 + 3\lambda)(\alpha + t)^{1+3\lambda}}{(1 + t)^\lambda} \right) \phi_{xx}^2 dx \right] dx. \]
Lemma 3.4. There holds

\[ (1 + t)^{3+1/2} \|\phi_{xxt}(t)\|^2 + \int_0^t (1 + s)^{3+2/3} \|\phi_{xx}(s)\|^2 ds \leq C(\delta + \Phi_0^2) \]

provided with \( N(T) + \delta \ll 1 \). \( \square \)
Proof. Differentiating (3.1) in $t$ yields
\[ \phi_{tt} - \frac{\lambda}{(1 + t)^{\lambda+1}} \phi_t + \frac{1}{(1 + t)^{\lambda}} \phi_{tt} - p'((\bar{n}) \phi_{xt}) + \bar{n} \phi_t = f_{1xt} + f_{2xt} - f_{3t} - f_{4t}. \] (3.44)

Multiplying (3.44) by $(2(a + t)\theta \phi_{tt} + 6(a + t)^{\theta-1} \phi_t)$, where $\alpha \geq 1, \theta \geq 0$ are some constants, and integrating the resultant equation over $\mathbb{R}$, we obtain
\[
\frac{d}{dt} \int_\mathbb{R} (a + t)^{\theta} \phi_{tt}^2 - \frac{\lambda(a + t)^{\theta}}{(1 + t)^{\lambda+1}} \phi_t^2 + p'(\bar{n})(a + t)\theta \phi_{xt}^2 + \bar{n}(a + t)^{\theta} \phi_t^2 + 6(a + t)^{\theta-1} \phi_t \phi_{tt} - 3(\theta - 1)(a + t)^{\theta-2} \phi_t^2 + \frac{3(a + t)^{\theta-1}}{(1 + t)^{\lambda}} \phi_t^2 \) \] 
\[
\int_\mathbb{R} (6 - \theta) \bar{n}(a + t)^{\theta-1} + 3(\theta - 1)(\theta - 2)(a + t)^{\theta-3} + \frac{(\theta - 3)\lambda(a + t)^{\theta-1}}{(1 + t)^{1+\lambda}} \phi_t^2 dx \] 
\[
= \int_\mathbb{R} (2(a + t)^{\theta} \phi_{tt} + 6(a + t)^{\theta-1} \phi_t) (f_{1xt} + f_{2xt} - f_{3t} - f_{4t}) dx. \] (3.45)

Analysis similar to that in the proof of Lemma 3.2 shows that
\[
\int_\mathbb{R} (2(a + t)^{\theta} \phi_{tt} (f_{1xt} + f_{2xt} - f_{3t} - f_{4t}) dx \] (3.46)
\[
\leq - \frac{d}{dt} \int_\mathbb{R} (a + t)^{\theta} \left( p'(n) - p'((\bar{n}) - \frac{J^2}{n^{2\lambda}}) \right) \phi_{xt}^2 dx + C(N(t) + \delta) \int_\mathbb{R} (a + t)^{\theta-1} \phi_{tt}^2 dx + C(N(t) + \delta) \int_\mathbb{R} (a + t)^{\theta-1} \phi_t^2 dx + C\delta e^{-Ct^\sigma} \] 
and
\[
\int_\mathbb{R} 6(a + t)^{\theta-1} \phi_t (f_{1xt} + f_{2xt} - f_{3t} - f_{4t}) dx \] 
\[
\leq C(N(t) + \delta) \int_\mathbb{R} (a + t)^{\theta-1} \phi_{xt}^2 dx + C(N(t) + \delta) \int_\mathbb{R} (a + t)^{\theta} \phi_{tt}^2 dx + C(N(t) + \delta) \int_\mathbb{R} (a + t)^{\theta} \phi_t^2 dx + C\delta e^{-Ct^\sigma}. \] (3.47)

Thus,
\[
\frac{d}{dt} \int_\mathbb{R} (a + t)^{\theta} \phi_{tt}^2 - \frac{\lambda(a + t)^{\theta}}{(1 + t)^{\lambda+1}} \phi_t^2 + (p'(n) + O(1)(N(t) + \delta))(a + t)^{\theta} \phi_{xt}^2 + \bar{n}(a + t)^{\theta} \phi_t^2 + 6(a + t)^{\theta-1} \phi_t \phi_{tt} - 3(\theta - 1)(a + t)^{\theta-2} \phi_t^2 + \frac{3(a + t)^{\theta-1}}{(1 + t)^{\lambda}} \phi_t^2 \) \] 
\[
\int_\mathbb{R} (2(a + t)^{\theta} \phi_{tt} - (6 - \theta)(\theta + 1)(\theta - 2)(a + t)^{\theta-3} + \frac{(\theta - 3)\lambda(a + t)^{\theta-1}}{(1 + t)^{1+\lambda}} \phi_t^2 dx \] 
\[
- \frac{\lambda(a + t)^{\theta}}{(1 + t)^{\lambda+2}} - \frac{3(\theta - 1)(a + t)^{\theta-2}}{(1 + t)^{\lambda}} \phi_t^2 dx \]
\[ \leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\theta - \lambda} \phi_t^2 \, dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\theta - 1} \phi_x^2 \, dx \]
\[ + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{\theta - 1} \phi^2 \, dx + C\delta e^{-Ct}. \] (3.48)

Taking \( \kappa = 3 + \lambda \) in (3.48), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^{3+\lambda} \phi_t^2 - \frac{\lambda(\alpha + t)^{3+\lambda}}{(1 + t)^{1+\lambda}} \phi^2_t + (p'(n) + O(1)) (N(t) + \delta) (\alpha + t)^{3+\lambda} \phi_t^2 \\
+ \frac{3(\alpha + t)^{2+\lambda}}{(1 + t)^{1+\lambda}} \phi_t^2 \right) \, dx \\
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{3+\lambda}}{(1 + t)^{1+\lambda}} - (9 + \lambda)(\alpha + t)^{2+\lambda} \right) \phi_t^2 \, dx \\
+ \int_{\mathbb{R}} \left( (3 - \lambda) p'(\bar{n})(\alpha + t)^{2+\lambda} \phi^2_t \right. \, dx \\
+ \int_{\mathbb{R}} \left( (3 - \lambda) \bar{n}(\alpha + t)^{2+\lambda} + 3(2 + \lambda)(1 + \lambda)(\alpha + t)^{\lambda} + \frac{\lambda^2(\alpha + t)^{2+\lambda}}{(1 + t)^{1+\lambda}} \\
- \frac{\lambda(\lambda + 1)(\alpha + t)^{3+\lambda}}{(1 + t)^{2+\lambda}} - \frac{3(2 + \lambda)(\alpha + t)^{1+\lambda}}{(1 + t)^{\lambda}} \right) \phi_t^2 \, dx \\
\leq C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+\lambda} \phi_t^2 \, dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{2+\lambda} \phi_x^2 \, dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{2+\lambda} \phi^2 \, dx + C\delta e^{-Ct}. \] (3.49)

Since
\[ 6(\alpha + t)^{2+\lambda} \phi_t \phi_x = 2 \left( \frac{\sqrt{3}}{2} (\alpha + t)^{\frac{\lambda}{2} + \frac{3}{2}} \phi_x \right) \left( 2\sqrt{3} (\alpha + t)^{\frac{\lambda}{2} + \frac{3}{2}} \phi_t \right) \]
\[ \geq - \frac{3}{4} (\alpha + t)^{\lambda + 3} \phi^2_t - 12(\alpha + t)^{\lambda + 1} \phi_t^2, \]
if \( \alpha \) is sufficiently large, it holds
\[ \bar{n}(\alpha + t)^{3+\lambda} \phi_t^2 - \frac{\lambda(\alpha + t)^{3+\lambda}}{(1 + t)^{1+\lambda}} \phi^2_t - 3(2 + \lambda)(\alpha + t)^{1+\lambda} \phi_t^2 \\
+ \frac{3(\alpha + t)^{2+\lambda}}{(1 + t)^{1+\lambda}} \phi_t^2 - 12(\alpha + t)^{\lambda + 1} \phi_t^2 \geq C(\alpha + t)^{3+\lambda} \phi_t^2, \]
then, we have
\[ (\alpha + t)^{3+\lambda} \phi_t^2 - \frac{\lambda(\alpha + t)^{3+\lambda}}{(1 + t)^{1+\lambda}} \phi^2_t + \bar{n}(\alpha + t)^{3+\lambda} \phi_t^2 + 6(\alpha + t)^{2+\lambda} \phi_t \phi_x \\
- 3(2 + \lambda)(\alpha + t)^{1+\lambda} \phi_t^2 + \frac{3(\alpha + t)^{2+\lambda}}{(1 + t)^{1+\lambda}} \phi_t^2 \\
\geq C(\alpha + t)^{3+\lambda} \phi_t^2 + C(\alpha + t)^{3+\lambda} \phi_t^2. \] (3.50)

Noticing that, if \( \alpha \) is sufficiently large, it holds
\[ \frac{2(\alpha + t)^{3+\lambda}}{(1 + t)^{1+\lambda}} - (9 + \lambda)(\alpha + t)^{2+\lambda} \]
\[
(\alpha + t)^{3+\lambda} \left( \frac{2}{1+t} - \frac{9+\lambda}{\alpha + t} \right) \geq (\alpha + t)^{3+\lambda} \cdot \frac{1}{(1+t)^{\lambda}} \geq (\alpha + t)^{3} \quad (3.51)
\]

and

\[
(3 - \lambda)\bar{n}(\alpha + t)^{2+\lambda} + 3(2 + \lambda)(1 + \lambda)(\alpha + t)^{\lambda} + \frac{\lambda^2(\alpha + t)^{2+\lambda}}{(1 + t)^{2+\lambda}} - \frac{\lambda(\lambda + 1)(\alpha + t)^{3+\lambda}}{(1 + t)^{2+\lambda}} - 3(2 + \lambda)(\alpha + t)^{1+\lambda} \geq C(\alpha + t)^{2+\lambda}. \quad (3.52)
\]

Then integrating (3.49) over \((0, t)\), and by (3.50)-(3.52), we prove that

\[
(1 + t)^{3+\lambda}||\varphi_{xx}(t)||^2 + \int_{0}^{t} (1 + s)^{3+\lambda}||\varphi_{xx}(s)||^2ds
\]

\[
+ \int_{0}^{t} (1 + s)^{2+\lambda}||\varphi_{x}(s)||^2ds \leq C(\delta + \Phi_0^2)
\]

provided with \(N(T) + \delta \ll 1\). \qed

**Lemma 3.5.** There holds

\[
(1 + t)^{4+2\lambda}||\varphi_{xtt}(t)||^2 + \int_{0}^{t} (1 + s)^{4+\lambda}||\varphi_{xtt}(s)||^2ds
\]

\[
+ \int_{0}^{t} (1 + s)^{3+2\lambda}||\varphi_{x}(s)||^2ds \leq C(\delta + \Phi_0^2)
\]

provided with \(N(T) + \delta \ll 1\).

**Proof.** Differentiating (3.44) in \(x\) yields

\[
\varphi_{xtt} - \frac{\lambda}{(1+t)^{\lambda+1}}\varphi_{xt} + \frac{1}{(1+t)^{\lambda}}\varphi_{xtt} - p'(\bar{n})\varphi_{xxxt} + \bar{n}\varphi_{xt} = f_{1xt} + f_{2xt} - f_{3xt} - f_{4xt}. \quad (3.54)
\]

Multiplying (3.54) by \((2(\alpha + t)^{4+2\lambda}\varphi_{xtt} + 6(\alpha + t)^{3+2\lambda}\varphi_{xt})\), where \(\alpha \geq 1\) is some constant, and integrating it over \(\mathbb{R}\) yields

\[
\frac{d}{dt} \int_{\mathbb{R}} ((\alpha + t)^{4+2\lambda}\varphi_{xtt} - \frac{\lambda(\alpha + t)^{4+2\lambda}}{(1 + t)^{1+\lambda}}\varphi_{xt}^2 - p'(\bar{n})(\alpha + t)^{4+2\lambda}\varphi_{xt}^2
\]

\[
+ \bar{n}(\alpha + t)^{4+2\lambda}\varphi_{xt}^2 + 6(\alpha + t)^{3+2\lambda}\varphi_{xt}\varphi_{xtt} - 3(3 + 2\lambda)(\alpha + t)^{3+2\lambda}\varphi_{xt}^2
\]

\[
+ \frac{3(\alpha + t)^{3+\lambda}}{(1 + t)^{\lambda}}\varphi_{xt}^2)dx
\]

\[
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{4+2\lambda}}{(1+t)^{\lambda}} - (10 + 2\lambda)(\alpha + t)^{3+2\lambda} \right) \varphi_{xtt}^2dx
\]

\[
+ \int_{\mathbb{R}} (2 - 2\lambda)p'(\bar{n})(\alpha + t)^{3+2\lambda}\varphi_{xtt}^2dx
\]

\[
+ \int_{\mathbb{R}} \left( (2 - 2\lambda)\bar{n}(\alpha + t)^{3+2\lambda} - \frac{3(3 + 2\lambda)(\alpha + t)^{2+2\lambda}}{(1 + t)^{1+\lambda}} - \frac{\lambda(7 + 2\lambda)(\alpha + t)^{3+2\lambda}}{(1 + t)^{1+\lambda}}
\]

\[
- \frac{\lambda(\lambda + 1)(\alpha + t)^{4+2\lambda}}{(1 + t)^{2+\lambda}} + 3(3 + 2\lambda)(2 + 2\lambda)(\alpha + t)^{1+2\lambda} \right) \varphi_{xtt}^2dx
\]

\[
= \int_{\mathbb{R}} (2(\alpha + t)^{4+2\lambda}\varphi_{xtt} + 6(\alpha + t)^{3+2\lambda}\varphi_{xt}) (f_{1xt} + f_{2xt} - f_{3xt} - f_{4xt})dx. \quad (3.55)
\]
Substituting (3.56) and (3.57) into (3.55), we get
\begin{align*}
\int_{\mathbb{R}} 2(\alpha + t)^{4+2\lambda} \phi_{xxtx} (f_{xxt} + f_{xxt} - f_{xxt} - f_{xxt}) dx \\
\leq - \frac{d}{dt} \int_{\mathbb{R}} (\alpha + t)^{4+2\lambda} \left( p'(n) - p'(\bar{n}) - \frac{f^2}{n^2} \right) \phi_{xxt}^2 dx + C \int_{\mathbb{R}} (\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx \\
+ C \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{4+\lambda} \phi_{xxt}^2 dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C\delta e^{-Ct^\sigma}
\end{align*}

and
\begin{align*}
\int_{\mathbb{R}} 6(\alpha + t)^{3+2\lambda} \phi_{xxtx} (f_{xxt} + f_{xxt} - f_{xxt} - f_{xxt}) dx \\
\leq C \int_{\mathbb{R}} (\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx + C \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{4+\lambda} \phi_{xxt}^2 dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C\delta e^{-Ct^\sigma},
\end{align*}

where we have used the fact that
\[ \|\phi(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C N(t)(1 + t)^{-\frac{7+3\lambda}{4}} \]

and
\[ \|\phi(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C N(t)(1 + t)^{-\frac{7+3\lambda}{4}}. \]

Substituting (3.56) and (3.57) into (3.55), we get
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}} \left( (\alpha + t)^{4+2\lambda} \phi_{xxt}^2 - \frac{\lambda(\alpha + t)^{4+2\lambda}}{(1 + t)^{1+\lambda}} \phi_{xx}^2 \\
+ (p'(n) + O(1)(N(t) + \delta))(\alpha + t)^{4+2\lambda} \phi_{xxt}^2 + \bar{n}(\alpha + t)^{4+2\lambda} \phi_{xxt}^2 \\
+ 6(\alpha + t)^{3+2\lambda} \phi_{xxt} \phi_{xt} - 3(3 + 2\lambda)(\alpha + t)^{2+2\lambda} \phi_{xxt}^2 + \frac{3(\alpha + t)^{3+2\lambda}}{(1 + t)\lambda} \phi_{xxt} \right) dx \\
+ \int_{\mathbb{R}} \left( \frac{2(\alpha + t)^{4+2\lambda}}{(1 + t)^{1+\lambda}} - (10 + 2\lambda)(\alpha + t)^{3+2\lambda} \phi_{xxt}^2 \\
+ \int_{\mathbb{R}} (2 - 2\lambda)p'(\bar{n})(\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx \\
+ \int_{\mathbb{R}} \left( (2 - 2\lambda)\bar{n}(\alpha + t)^{3+2\lambda} - \frac{3(3 + 2\lambda)(\alpha + t)^{2+2\lambda}}{(1 + t)\lambda} - \frac{\lambda(\alpha + t)^{4+2\lambda}}{(1 + t)^{1+\lambda}} \\
- \frac{\lambda(\lambda + 1)(\alpha + t)^{4+2\lambda} + 3(3 + 2\lambda)(2 + 2\lambda)(\alpha + t)^{1+2\lambda}}{(1 + t)^{1+\lambda}} \right) \phi_{xx}^2 dx \\
\leq C \int_{\mathbb{R}} (\alpha + t)^{2+3\lambda} \phi_{xxx}^2 dx + C \int_{\mathbb{R}} (\alpha + t)^{1+2\lambda} \phi_{xx}^2 dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{4+\lambda} \phi_{xxt}^2 dx + C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx \\
+ C(N(t) + \delta) \int_{\mathbb{R}} (\alpha + t)^{3+2\lambda} \phi_{xxt}^2 dx + C\delta e^{-Ct^\sigma}.
\end{align*}

(3.58)
Since
\[ 6(\alpha + t)^{3+2\lambda} \phi_{xt} \phi_{xtt} = 2 \left( \frac{\sqrt{3}}{2} (\alpha + t)^{2+\lambda} \phi_{xtt} \right) \left( 2\sqrt{3}(\alpha + t)^{1+\lambda} \phi_{xt} \right) \]
\[ \geq - \frac{3}{4}(\alpha + t)^{4+2\lambda} \phi_{xtt}^2 - 12(\alpha + t)^{2+2\lambda} \phi_{xt}^2, \]
if \( \alpha \) is sufficiently large, it holds
\[ \bar{n}(\alpha + t)^{4+2\lambda} \phi_{xt}^2 - \frac{\lambda(\alpha + t)^{4+2\lambda}}{(1+t)^{1+\lambda}} \phi_{xt}^2 \]
\[ + 3(\alpha + t)^{3+2\lambda} \phi_{xt}^2 - 12(\alpha + t)^{2+2\lambda} \phi_{xt}^2 \geq C(\alpha + t)^{4+2\lambda} \phi_{xt}^2, \] (3.59)

Noticing that, if \( \alpha \) is sufficiently large, it holds
\[ \frac{2(\alpha + t)^{4+2\lambda}}{(1+t)^{2+\lambda}} - (10 + 2\lambda)(\alpha + t)^{3+2\lambda} \]
\[ = (\alpha + t)^{4+2\lambda} \left( \frac{2}{(1+t)^{\lambda}} - \frac{10 + 2\lambda}{\alpha + t} \right) \geq (\alpha + t)^{4+2\lambda}, \quad \frac{1}{(1+t)^{\lambda}} \geq (\alpha + t)^{4+\lambda} \quad (3.60) \]

and
\[ (2 - 2\lambda) \bar{n}(\alpha + t)^{3+2\lambda} - \frac{3(3 + 2\lambda)(\alpha + t)^{2+2\lambda}}{(1+t)^{3+2\lambda}} - \frac{\lambda(7 + 2\lambda)(\alpha + t)^{3+2\lambda}}{(1+t)^{1+\lambda}} \]
\[ - \frac{\lambda(\lambda + 1)(\alpha + t)^{4+2\lambda}}{(1+t)^{2+\lambda}} + 3(3 + 2\lambda)(2 + 2\lambda)(\alpha + t)^{1+2\lambda} \geq C(\alpha + t)^{3+2\lambda}. \quad (3.61) \]

Then integrating (3.58) over \((0, t)\), by Lemma 3.2.3.3 and (3.59)-(3.61), we prove that
\[ (1 + t)^{4+2\lambda} ||(\phi_{xtt}, \phi_{xxxt}, \phi_{xt}) (t)||^2 + \int_0^t (1 + s)^{4+\lambda} ||\phi_{xtt}(s)||^2 ds \]
\[ + \int_0^t (1 + s)^{3+2\lambda} ||(\phi_{xxxt}, \phi_{xt}) (s)||^2 ds \leq C(\delta + \phi_0^2) \]
provided with \( N(T) + \delta \ll 1 \).

\[ \square \]

Theorem 2.2 is a direct consequence of Lemma 3.1-3.5.

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