The quadrupole moment of slowly rotating stars

M Bradley and G Fodor

1 Department of Physics, Umeå University, SE-901 87 Umeå, Sweden
2 KFKI Research Institute for Particle and Nuclear Physics, H-1525, Budapest 114, P.O.B. 49, Hungary
E-mail: michael.bradley@physics.umu.se, gfodor@rmki.kfki.hu

Abstract. The second order field equations for the interior of slowly and rigidly rotating stars are solved numerically for different classes of possible equations of state and these solutions are then matched to the general asymptotically flat axisymmetric vacuum metric to second order in the rotational parameter. For these solutions we find that the quadrupole moment differs from that of the Kerr metric. Further we consider the post-Minkowskian limit analytically.

1. Introduction

In the present work we use the second order formalism for slowly and rigidly rotating stars, developed in [1], to study the quadrupole moment and its deviation from that of the Kerr metric for different classes of possible equations of state. For more details the reader is referred to a longer version of this paper, [2], and references therein. Some earlier numerical studies find quadrupole moments differing from that of the Kerr metric, see e.g. [3, 4, 5, 6]. In some recent papers [7, 8] it is shown that in the rigidly rotating case incompressible fluids and fluids with polytropic equation of state cannot be sources of the Kerr metric in the post-Minkowskian limit. These results are in accordance with the general expectation that the ellipsoidal shape of the rotating fluid produces an extra contribution to the quadrupole moment.

We also consider the post-Minkowskian limit analytically by expanding the field equations in the small parameter \( \lambda \equiv GM/c^2 \) and make a comparison with the results of [7, 8].

2. Preliminaries

To second order the metric of a slowly rotating axisymmetric object, both in the interior fluid region and the outside vacuum region, can be written as

\[
ds^2 = (1 + 2h)A^2 dt^2 - (1 + 2m) \frac{1}{B^2} dr^2 - (1 + 2k)r^2 \left[ d\theta^2 + \sin^2 \theta (d\phi - \omega dt)^2 \right],
\]

where \( \omega \) is first order and \( h, m \) are second order in the rotational parameter. The requirements of regularity at the centre and asymptotic flatness imply that the first order function \( \omega \) depends on \( r \) only. The second order functions can be given as \( h = h_0 + h_2 P_2(\cos \theta) \), \( m = m_0 + m_2 P_2(\cos \theta) \) and \( k = k_2 P_2(\cos \theta) \) where \( h_0, m_0 \) and \( h_2, m_2, k_2 \) are functions of \( r \) only and \( P_2 \) is the second order Legendre polynomial. For more details see [1].

The matter content of the interior is modelled by a perfect fluid \( T_{ab} = (p + \rho) u_a u_b - pg_{ab} \). The coordinate system used in (1) is assumed to be comoving with the fluid which also implies that the shear of the fluid is zero, so it rotates rigidly.
Einstein’s equations then give a subsystem for the functions $A$, $B$, $\omega$, $m_2$, $k_2$ and $h_2$ that decouples from $m_0$ and $h_0$. To close this system one more equation, e.g., an equation of state, $\rho = \rho(p)$, has to be specified. In this paper we will consider equations of state in the form:

$$\rho = d_1p + d_2 \left( \frac{p}{p_c} \right)^{1/\gamma} + d_3.$$  \hspace{1cm} (2)

Here $p_c$ is the central pressure. Equation (2) includes some of the more used approximations for the equation of state of dense stars, like Newtonian polytropes, relativistic polytropes with $d_1 = 1/(\gamma - 1)$, linear ones as well as the incompressible case.

Due to the requirement of a regular centre the solutions will only depend on three constants of integration, that are the central pressure to zeroth order, $p_{0c}$, the magnitude of the vorticity, $\omega_0$ and a second order quantity $h_1$.

In the exterior vacuum region a frame adapted to the asymptotically non-rotating observer is chosen. The solution to second order in the rotational parameter is characterized by the mass $M$, the first order rotation parameter $a$, and the second order constants $c_1, c_2$ and $q_1$. When $q_1$ takes the value zero the metric is the general asymptotically flat stationary and axisymmetric vacuum metric to second order. Using the method in [9], the Geroch-Hansen relativistic moments up to order two are found to be the mass to second order $M - c_2$, the angular momentum $J = Ma$ and the quadrupole moment $Q = -M \left( a^2 + \frac{16}{3} M^4 c_1 \right)$. We will be interested in the relative deviation of the quadrupole moment from that of the Kerr metric

$$\frac{\Delta Q}{Q} = \frac{Q - Q_{Kerr}}{Q_{Kerr}} = \frac{16M^4 c_1}{5a^2}. \hspace{1cm} (3)$$

When $q_1 \neq 0$ the metric cannot be asymptotically flat. It is important to keep in mind, however, that without the inclusion of this constant the matching conditions on the zero pressure surface are overdetermined in general.

In the fluid region the matching surface $S$ is the surface of vanishing pressure. For slow rotation $S$ is given by $r = r_1 + \xi \equiv r_1 + \xi_0 + \xi_2 P_2(\cos \theta)$ where $r = r_1$ is the zeroth order radius and the constants $\xi_0$ and $\xi_2$ are determined from the pressure. Similarly, in the vacuum exterior region suitable surfaces for matching are determined by $r = r_1 + \chi \equiv r_1 + \chi_0 + \chi_2 P_2(\cos \theta)$, where $\chi_0$ and $\chi_2$ are constants to be determined from the matching conditions.

The two regions are matched using the Darmois-Israel procedure, where the respective induced metrics $ds^2_{(i)}$, $ds^2_{(v)}$ (for vacuum) and induced extrinsic curvatures $K_{(i)}$, $K$ are equated with each other as

$$ds^2_{(i)}|S = ds^2|S = K_{(i)}|S = K|S. \hspace{1cm} (4)$$

Here $K \equiv K_{ab} dx^a dx^b \equiv h_{\xi} h_{\chi} n_{(\xi, \chi)} dx^a dx^b$ in terms of the unit normal $n_a$ of the matching surface $S$ and the projection operator $h_{\xi} = n_a n^a + \delta_{\xi}^b$. To adjust the different coordinate systems with each other we apply a rigid rotation in the fluid region by setting $\varphi \rightarrow \varphi + \Omega t$ where $\Omega$ is a constant. (The $t$-coordinates in the two regions are also adjusted by rescaling one of them.) The constants $M$, $r_1$, $a$, $\Omega$, $c_1$, $c_2$, $q_1$, $\chi_0$ and $\chi_2$ can now be solved for consistently in terms of the interior solution. Note that we did not impose the equation of state (2) when calculating the matching conditions. Hence the vacuum metric above is general enough for describing the exterior of any axisymmetric rigidly rotating perfect fluid ball up to second order.

3. Numerical solutions

In this section we consider solutions with equations of state (2), with special attention to the quadrupole moment and its deviation, $\Delta Q/Q$, from that of the Kerr metric. In scanning the parameter space $\omega_0$ was fixed to 0.1 due to a scaling invariance and the requirement of
asymptotically flatness indirectly fixes the value of $h_1$. Hence, for a given equation of state, we only need to vary the central pressure, $p_{0c}$. When considering different equations of state we use another scale invariance to fix one of the constants $d_2$ or $d_3$. When performing the corresponding rescaling $r \rightarrow \alpha r$ the quantity $\Delta Q/Q$ is invariant.

3.1. Incompressible fluids (interior Schwarzschild)
When the central pressure approaches infinity and the radius the Buchdahl limit $9M/4$, then $\Delta Q/Q$ approaches 0.0212. This value is independent of the density. This case was also considered in [4].

3.2. Linear equations of state $\rho = d_1 p + d_3$
First consider the case $d_1 < -1$. For all these $\rho_{0c} = -p_{0c}$, corresponding to the anti-de Sitter spacetime, when $p_{0c} = d_3/(-d_1 - 1)$. Since the zero-pressure surface is further and further pushed outward when approaching this limit, all these configurations also become infinite in extension. The ratio $r_1/2M$ tends towards one and the quadrupole moment approaches that for Kerr. Even if these spacetimes are quite unphysical, a study of them still is helpful in understanding which conditions are needed for a successful matching to the Kerr metric.

In the numerical runs, due to scaling invariance, we put $d_3 = 1$. Negative values of $d_3$ are excluded if the surface density should be larger than zero, and configurations with $d_3 = 0$ are not finite in size. The quantity $\Delta Q/Q$ for a sequence of values for $d_1$ are given in Figure 1. As seen, for more realistic configurations with $d_1 \geq 1$ it differs significantly from zero.

![Figure 1](image1.png)

**Figure 1.** The quantity $\Delta Q/Q$ as a function of central pressure for a sequence of fluid balls with linear equation of state $\rho = d_1 p + 1$.

3.3. Polytropes
3.3.1. Newtonian polytropes
In a recent paper [8] it was shown that slowly and rigidly rotating polytropes cannot be sources of the Kerr metric in the post-Minkowskian limit. Here we numerically find that slowly and rigidly rotating polytropes with arbitrary strength of the gravitational field cannot be matched to Kerr either.

We use the equation of state $\rho = d_2 \left( \frac{p}{p_{0c}} \right)^{1/\gamma}$ and use scaling invariance to put $d_2 = 1$. In Figure 2 $\Delta Q/Q$ is shown as a function of the central pressure for a sequence of values of $\gamma$, including the physically interesting cases $\gamma = 4/3$ and $\gamma = 5/3$. As seen $\Delta Q/Q$ never seems...
to approach zero. Furthermore, for physically reasonable configurations with $p_c \leq \rho_c = 1$, the quantity differs significantly from zero.

3.3.2. Relativistic polytropes For a fluid with one type of constituent particles, a relativistic polytrope is given by $p = C n^{\gamma}$, where $C$ is a constant, in terms of the particle density $n$ and the polytropic index $\gamma$. This equation is suitable, e.g., to describe an ideal degenerate neutron gas. Using the energy conservation equation for a perfect fluid it is then easy to show that the equation of state becomes $p = \rho/\gamma - 1 + dp/(\gamma^2)$ in terms of the pressure $p$. As seen the equation of state approaches a linear equation of state for large pressures and a Newtonian polytrope for low pressures. The results of the numerical runs are similar to the previous ones.

4. The post-Minkowskian limit

Here the weak gravity limit is considered by making an expansion in the small parameter $\lambda = M/r_1$, or in SI units $GM/r_1^2$, where $M$ is the mass of the fluid ball and $r_1$ is its radius. A similar study, using global harmonic coordinates and the Lichnerowicz matching conditions, was performed in [7] and [8].

Using the condition that $q_1$ should vanish (to first order in $\lambda$) for an asymptotically flat solution one obtains

$$c_1 = -\frac{5\omega_1^2/2}{48\lambda} \frac{d}{dr} \log \left( \frac{\left(h_{21} + \omega_1^2 r^2/3\right)}{\left(r^4 dA/\lambda\right)} \right)_{r=r_1},$$

that should be zero for Kerr. Here $h_{21}$ is $h_2$ to order $\lambda$ and $\omega_1^2$ constant is $\omega$ to order $\sqrt{\lambda}$. Note that generically $c_1$ diverges as $1/\lambda^3$, whereas $\Delta Q/Q$ diverges as $1/\lambda$.

For the incompressible case ($\rho = \rho_1$ constant) the equations can be integrated completely and the relative difference to the Kerr quadrupole moment in terms of the central pressure is $\Delta Q/Q = 25\rho_1/(16p_c)$.

For Newtonian polytropes, $p = p_c \left( \frac{\rho}{\rho_c} \right)^\gamma$, where $p_c$ and $p_c$ are the central density and pressure respectively, we can use a theorem in [8] to show that $c_1 \neq 0$, in agreement with their result. Furthermore, in [10] we show that the rotating fluid ball is oblate in shape if $h_{21r=r_1} = 2 \omega_1^2 r_1 < 0$. To first order in $\lambda$ this is equivalent to $(h_{21} + \omega_1^2 r^2/3)_{r=r_1} > 0$. For Newtonian polytropes this implies that $c_1$ as given by (5) is positive (see [2] for details). Hence $\Delta Q/Q = 16M^4c_1/5a^2 > 0$ for oblate Newtonian polytropes.

References

[1] Hartle J B 1967 Astrophys. J. 150 1005
[2] Bradley M and Fodor G 2009 Phys. Rev. D 79 044018
[3] Hartle J B and Thorne K S 1968 Astrophys. J. 153 807
[4] Chandrasekhar S and Miller J C 1974 Mon. Not. R. Astr. Soc. 167 63
[5] Berti E, White F, Maniopoulou A and Bruni M 2005 Mon. Not. R. Astr. Soc. 358 923
[6] Lassarros W G and Poisson E 1999 Astrophys. J. 512 282
[7] Cabezas J A, Martín J, Molina A and Ruiz E 2007 Gen. Rel. Grav. 39 707
[8] Martín J, Molina A and Ruiz E 2008 Class. Quantum Grav 25 105019
[9] Fodor G, Hoenselaers C A and Perjes Z 1989 J. Math. Phys. 30 2252
[10] Bradley M, Eriksson D, Fodor G and Rácz I 2007 Phys. Rev. D 75 024013