Remote State Estimation with Smart Sensors over Markov Fading Channels

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Abstract—We consider a fundamental remote state estimation problem of discrete-time linear time-invariant (LTI) systems. A smart sensor forwards its local state estimate to a remote estimator over a time-correlated multi-state Markov fading channel, where the packet drop probability is time-varying and depends on the current fading channel state. We establish a necessary and sufficient condition for mean-square stability of the remote estimation error covariance in terms of the state transition matrix of the LTI system, the packet drop probabilities in different channel states, and the transition probability matrix of the Markov channel states. To derive this result, we propose a novel estimation-cycle based approach, and provide new element-wise bounds of matrix powers. The stability condition is verified by numerical results, and is shown more effective than existing sufficient conditions in the literature. We observe that the stability region in terms of the packet drop probabilities in different channel states can either be convex or non-convex depending on the transition probability matrix of the Markov channel states. Our numerical results suggest that the stability conditions for remote estimation may coincide for setups with a smart sensor and with a conventional one (which sends raw measurements to the remote estimator), though the smart sensor setup achieves a better estimation performance.

Index Terms—Estimation, Kalman filtering, linear systems, stability, mean-square error, Markov fading channel

I. INTRODUCTION

A. Motivation

In the long-term evolution of wireless applications from conventional sensor networks (WSNs) to the Internet-of-Things (IoT) and the Industry 4.0, remote estimation is a key component [1]–[4]. Driven by Moore’s Law, the accelerated development and adoption of smart sensor technology enables low-cost sensors with high computational capability [5]. Thus, in a number of remote estimation applications, it is practical to use smart sensors (e.g., with Kalman filters) to pre-estimate the dynamic states, and then send the estimated states rather than the raw measurements to the remote estimator. In the presence of communication constraints, the smart sensors provide better estimation performance than the conventional sensors that purely send raw measurement data to the remote estimator [6].

Unlike wired communications, wireless communications are unreliable and the channel status varies with time due to multipath propagation and shadowing caused by obstacles affecting the wave propagation. The transition process of the fading channel states is usually modeled as a Markov process [7]–[9], and different channel states lead to different packet drop probabilities of transmissions. The presence of an unreliable wireless communication channel degrades the estimation performance, and in some cases even lead to instability. Whilst stability when using conventional sensors has been well investigated, see literature survey below, stability when using a smart sensor has been much less considered. In this paper, we tackle the fundamental problem: what are necessary and sufficient conditions on system parameters that ensure stochastic stability of a smart-sensor-based remote estimation system over a Markov fading channel?

B. Related Works

The existing work on remote estimation can be divided into two categories based on the sensor’s computational capability.

In the conventional sensor scenario, the sensor sends raw measurements to the remote estimator. When considering a static wireless channel, where neither the transceivers nor the wireless environment are moving, the packet drop probability during the remote estimation process remains fixed, so the packet arrival process is a Bernoulli process. It was proven in [10] that there exists a critical packet drop probability, such that the mean estimation error covariance is bounded for all initial conditions and diverges for some initial condition if the packet drop probability is less or greater than the critical probability, respectively. This result was further extended to a scenario with random packet delays in [6]. By modeling the packet arrival process as a Markovian binary switching process, sufficient conditions for stability in the sense of peak covariance were obtained in [11], [12]. For situations where the number of consecutive packet dropouts constitutes a bounded Markov process, peak covariance stability was investigated in [13]. By modeling the sequence of packet dropouts as a stationary finite-order Markov process, a necessary and sufficient stability condition was obtained in the sense of mean estimation error covariance in [14]. In contrast to [11]–[14], which directly model packet-dropouts as a Markovian process and abstract away the underlying wireless channel, Markovian fading channel states were explicitly considered in [15]. A sufficient condition for exponential stability was derived by using stochastic Lyapunov functions. With the same multi-state Markov channel model as [15], optimal transmit power allocation policy under different channel conditions was proposed in [16] to achieve the minimum remote estimation error.
More recently, closed-loop control systems over multi-state Markov channels were investigated in [17]–[19]. Under an ideal assumption of perfect sensor measurements, a necessary and sufficient stability condition was obtained in [17], where the sensor and the controller were co-located; a sufficient stability condition of a half-duplex control system was obtained in [18], where the controller applied a scheduling policy determining when to receive the sensor’s packet or to transmit a control packet to the actuator. In [19], sufficient stability conditions in terms of maximum allowable transmission interval of a nonlinear system was investigated. The work [20] focused on bit-rate limited error-free communication channels, where the number of bits to be transmitted in each time slot formed a Markov chain. By combining results from quantization theory with insights from Markov Jump Linear Systems, [20] examined how the quantization errors induced by finite-bit quantizers affect the control and estimation quality.

In the smart sensor scenario, an estimator (e.g., based on a Kalman filter) of the sensor side pre-processes the raw measurements, such that an estimate is transmitted to the remote estimator over the wireless channel. It has been rigorously proved in [6] that smart sensor scenario performs better than the conventional sensor scenario when taking into account the transmission delay and failures. However, unlike the conventional-sensor-based scenario, most of the theoretical research on smart-sensor-based remote estimation considered static channels and assumed independent and identically distributed (i.i.d.) packet dropouts [6], [21]–[28]. In [6], a necessary and sufficient condition for remote estimation stability was derived in the mean-square sense. In [21], an optimal sensor power scheduling policy under a sum power constraint was obtained. In [22], the optimal transmission scheduling policy of two sensors each measuring the state constraint was obtained. In [22], the optimal transmission policy was obtained by solving a Markov decision process problem. In [24], an optimal event-triggered transmission policy of a multi-sensor-multi-channel remote estimation system was proposed with a combined design target: the estimation error and the energy consumption of sensor transmissions.

In addition, optimal smart sensor transmission scheduling policies for single and multiple wireless channel scenarios were investigated under the presence of jamming attacks in [25] and [26], respectively; an optimal transmission scheduling policy under the presence of an eavesdropper was proposed in [27] to minimize the remote estimation error at the dedicated receiver while keeping the eavesdropper’s estimation error as large as possible.

More recently, a remote estimation system with retransmissions was proposed in [28], where the smart sensor can decide whether to retransmit the unsuccessfully transmitted local estimate (with a longer latency) or to send a new estimate (with a lower reliability). The obtained optimal retransmission scheduling policy found the optimal balance between the transmission latency and reliability on the remote estimation performance.

### C. Contributions

In this paper, we investigate mean-square stability of smart-sensor-based remote estimation over an error-prone multi-state time-homogeneous Markov channel. The $M$-state fading-channel model under consideration introduces an unbounded Markov chain in the analysis of the remote estimation system, which presents some non-trivial challenges. The main contributions are summarized as below.

1) We derive a necessary and sufficient condition on the stability of a remote state estimation system in terms of the system matrix $A$, the packet drop probabilities in different channel states $\{d_1, \ldots, d_M\}$ and the matrix of the channel state transitions $M$. The remote state estimation is mean-square stable if and only if $\rho(A)\rho(M) < 1$, where $\rho(\cdot)$ denotes the spectral radius, and $D$ is the diagonal matrix generated by $\{d_1, \ldots, d_M\}$.

2) We derive asymptotic upper and lower bounds of the estimation error function in terms of the number of consecutive packet dropouts $i$, which are in the same order of $(\rho(A) + \epsilon)^i$ and $\rho(A)$, respectively, where $\epsilon$ is an arbitrarily small positive number.

To obtain these results, we propose a novel estimation-cycle based analytical approach. Moreover, we further develop the asymptotic theory of matrix power, which provides new element-wise bounds of matrix powers.

### D. Outline and Notations

The remainder of this paper is organized as follows: Section II presents the model of the remote estimation system using a Markov fading channel. Section III presents and discusses the main results of the paper. Section IV proposes a stochastic estimation-cycle based analysis approach and derives some element-wise bounds of matrix powers. They are used in Section V to prove the main results. Section VI numerically evaluates the performance of the remote estimation system, and verifies the theoretical results. Section VII draws conclusions.

**Notations:** Sets are denoted by calligraphic capital letters, e.g., $\mathcal{A}$. $\mathcal{A}\setminus\mathcal{B}$ denotes set subtraction. Matrices and vectors are denoted by capital and lowercase upright bold letters, e.g., $A$ and $a$, respectively. $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. $E[\cdot]$ is the expectation of the random variable $A$. $\cdot^\dagger$ is the matrix transpose operator. $\|v\|_1$ is the sum of the vector $v$’s elements. $|v| \triangleq \sqrt{v^\dagger v}$ is the Euclidean norm of a vector $v$. $\text{Tr}(\cdot)$ is the trace operator. $\text{diag}\{v_1, v_2, \ldots, v_K\}$ denotes the diagonal matrix with the diagonal elements $\{v_1, v_2, \ldots, v_K\}$. $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and non-negative integers, respectively. $\mathbb{R}^m$ denotes the $m$-dimensional Euclidean space. $\rho(A)$ is the spectral radius of $A$, i.e., the largest absolute value of its eigenvalues. $[u]_{B \times B}$ denotes the $B \times B$ matrix with identical elements $u$. $[A]_{j,k}$ denotes the element at the $j$th row and $k$th column of a matrix $A$. $\{v\}_{\mathbb{N}_0}$ denotes the semi-infinite sequence $\{v_0, v_1, \ldots\}$.
wireless link affected by random packet dropouts, as illustrated in Fig. 1.

A. Process Model and Smart Sensor

The discrete-time linear time-invariant (LTI) model is given as (see e.g., [6], [21], [29])

\[ x_{t+1} = Ax_t + w_t, \]
\[ y_t = Cx_t + v_t, \]

where \( x_t \in \mathbb{R}^n \) is the process state vector, \( A \in \mathbb{R}^{n \times n} \) is the state transition matrix, \( y_t \in \mathbb{R}^m \) is the measurement vector of the smart sensor attached to the process, \( C \in \mathbb{R}^{m \times n} \) is the measurement matrix, \( w_t \in \mathbb{R}^n \) and \( v_t \in \mathbb{R}^m \) are the process and measurement noise vectors, respectively. We assume \( w_t \) and \( v_t \) are independent and are identically distributed (i.i.d.) zero-mean Gaussian processes with corresponding covariance matrices \( W \) and \( V \), respectively. In this work, we focus on the stability condition of the remote estimation of the process \( x_t \) in the sense of average remote estimation mean-square error. Note that, if \( \rho^2(A) < 1 \), then the covariance of \( x_t \) is always bounded, and stability will trivially be satisfied. Thus, as commonly done in this context, in the sequel we focus on the more interesting case with \( \rho^2(A) \geq 1 \) indicating that the plant state grows exponentially fast.

Note that in practice, feedback control of open-loop unstable LTI systems with Gaussian noise always requires sensors with unbounded measurement range, and adaptive zooming-in/zooming-out measurement range have been widely adopted [30]. Beyond the idealized situation of LTI systems, system models with exponentially growing modes are also obtained through linearization at unstable equilibria, see for example the Pendubot system in [27] and references therein.

Since the sensor’s measurements are noisy, a smart sensor is used to estimate the state of the process, \( \hat{x}_t \). For that purpose a Kalman filter [21], [29] is used, which gives the minimum estimation MSE, based on the current and previous raw measurements:

\[ x^s_{t|t-1} = Ax^s_{t-1|t-1}, \]
\[ P^s_{t|t-1} = AP^s_{t-1|t-1}A^T + W, \]
\[ K_t = P^s_{t|t-1}C^T(CP^s_{t|t-1}C^T + V)^{-1}, \]
\[ x^s_{t|t} = x^s_{t|t-1} + K_t(y_t - Cx^s_{t|t-1}), \]
\[ P^s_{t|t} = (I - K_tC)x^s_{t|t-1}, \]

B. Wireless Channel

The main characteristic of the wireless fading channel is that the channel quality is a time-varying random process that changes over time in a correlated manner [7]–[9]. We consider a finite-state time-homogeneous Markov block-fading channel [7]. It is assumed that the channel power gain \( h_t \) remains constant during the \( t \)th time slot but may change slot by slot. We assume that the Markov channel has \( M \) states, i.e.,

\[ h_t \in \mathcal{B} \triangleq \{b_1, ..., b_M\}. \]

The transition probability from state \( i \) to state \( j \) is time-homogeneous and given by

\[ p_{i,j} \triangleq \text{Prob}[h_{t+1} = b_j | h_t = b_i], \forall i,j \in \mathcal{M}, t \in \mathbb{N}_0, \]

where \( \mathcal{M} \triangleq \{1, \cdots, M\} \). The matrix of channel state transition probability is given as

\[ M \triangleq \begin{bmatrix} p_{1,1} & \cdots & p_{1,M} \\ \vdots & \ddots & \vdots \\ p_{M,1} & \cdots & p_{M,M} \end{bmatrix}. \]
We assume that all the channel states are *aperiodic* and *positive recurrent*. Thus, the Markov chain induced by $\mathbf{M}$ is *ergodic*. 

We assume that the channel state information is available at both the sensor and the remote estimator, which can be achieved by standard channel estimation and feedback techniques, see e.g. [33] and the references therein. Let $\gamma_t = 1$ and $\gamma_t = 0$ denote the successful and failed packet detection of the remote estimator during time slot $t$, respectively. The packet drop probability in channel state $b_t$ is

$$d_t \triangleq \text{Prob}[\gamma_t = 0|h_t = b_t], \forall i \in \mathcal{M}, t \in \mathbb{N}_0.$$  

(5)

Note that the transmission is always perfect if $\gamma_n$ techniques, see e.g. [33] and the references therein. Let

$$\hat{x}_t = \begin{cases} A\hat{x}_{t-1} + \gamma_{t-1} = 0, \\ A\hat{x}_{t-1}^\top, \quad \gamma_{t-1} = 1. \end{cases}$$  

(8)

Assuming a packet was successfully received at time $t'$, and the following transmission consequently failed for $\delta \geq 1$ times before the current time $t$, i.e., $t = t' + \delta + 1$, from (8), it can be obtained that

$$\hat{x}_{t'} + 1 = A^\delta \hat{x}_{t'}^\top,$$

(9)

where $\delta_t \in \mathbb{N}_0$ is the number of consecutive packet dropouts before time slot $t$. In other words, $(\delta_t + 1)$ can be treated as the age-of-information (AoI) of the remote estimator in time slot $t$. 

Then, the estimation error covariance is given as

$$P_t \triangleq \mathbb{E} \left[ (\hat{x}_t - x_t)(\hat{x}_t - x_t)^\top \right]$$

(10)

$$= v(\delta_t + 1)(P_0)$$

(11)

where (11) is obtained by substituting (9) and (1) into (10) and with:

$$v(X) \triangleq AXX^\top + \mathbf{W}$$

(12)

Thus, the quality of the remote estimation error in time slot $t$ can be quantified via $\text{Tr}(P_t)$. We introduce the following function

$$c(i) \triangleq \text{Tr}(v^i(P_0)), \forall i \in \mathbb{N}.$$  

(13)

From (11), we can write

$$\text{Tr}(P_t) \triangleq c(\delta_t + 1) + 1.$$  

(14)

Since $P_t$ is a countable stochastic process taking value from a countable infinity set

$$\{v^i(P_0), v^{2i}(P_0), \ldots \}$$

it will grow during periods of consecutive packet dropouts when $\rho(A) \geq 1$. Since periods of consecutive packet dropouts have unbounded support, at best one can hope for some type of stochastic stability. In the present work, our focus is on the mean-square stability.

**Definition 1** (Mean-Square Stability). The remote estimation system is mean-square stable if and only if the average estimation MSE $J$ is bounded, where

$$J \triangleq \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \text{Tr}(P_t) \right],$$  

(15)

and $\limsup_{T \to \infty}$ is the limit superior operator.

Note that establishing necessary and sufficient stability conditions is non-trivial as we consider correlated fading-channel model in the remote estimation system which induces a countable (and unbounded) Markov chain in the analysis. Some of the existing works adopt stochastic Lyapunov functions to elucidate such situations (see e.g., [13]). These however, merely lead to sufficient conditions.

**III. MAIN RESULTS**

In this section, we present and discuss the main results of the paper, which will be proved in Section V.
A. The Necessary and Sufficient Stability Condition

Theorem 1. Let Assumption 1 hold. The remote estimation system described by (1), (2) and (9) is mean-square stable over the Markov channel defined by (4) and (6) if and only if the following condition holds:
\[ \rho^2(A) \rho(DM) < 1. \]  
(16)

Theorem 1 shows that the stability condition depends on the system matrix A, the packet drop probability matrix D and the matrix of the channel state transitions M. It is important to note that the necessary and sufficient condition is determined by both the spectral radiuses of A and the product of two matrices D and M. Since \( \rho(A) \) measures how fast the dynamic process varies, \( \rho(DM) \) can be treated as an effective measurement of the Markov channel quality.

Remark 1. In Corollary 1, a sufficient condition in terms of exponential stability of a conventional-sensor-based remote estimation system over Markov channel is obtained as
\[ \hat{\rho}^2(A) \max_{i \in M} \left\{ \sum_{j=1}^{M} p_{ij}d_j \right\} < 1, \]  
(17)
where \( \hat{\rho}(A) \) is the largest singular value of A. Using Perron–Frobenius theorem \([37]\), we have \( \max_{i \in M} \left\{ \sum_{j=1}^{M} p_{ij}d_j \right\} \geq \rho(DM) = \rho(DM) \). In addition, due to the fact that the largest singular value is no smaller than the spectral radius, i.e., \( \hat{\rho}(A) \geq \rho(A) \), it can be proved that the sufficient condition \((17)\) is more restrictive than \((16)\).

Corollary 1 (Special Case I). Consider the same assumption and system model in Theorem 1. For the special case of i.i.d. packet dropout channel with packet dropout probability \( d \), the remote estimation system is mean-square stable if and only if the following condition holds:
\[ \rho^2(A)d < 1. \]  
(18)

Remark 2. For the special case of i.i.d. packet dropouts, our stability condition obtained from Theorem 1 is identical to the conventional result in \([9]\).

Corollary 2 (Special Case II). Consider the same assumption and system model in Theorem 1. For the special case of Markovian packet dropout channel with packet dropout probability \( d_1 = 0 \) and \( d_2 = 1 \) and the channel state transition matrix
\[ M = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \]
the remote estimation system is mean-square stable if and only if the following condition holds:
\[ \rho^2(A)p_{22} < 1. \]  
(19)

Remark 3. For the Markovian on-off channel in Corollary 2, it is interesting to see that the stability condition only depends on one element of the 2-by-2 matrix M, which is the state transition probability from the bad state to the bad state.

We would like to compare our result with the one obtained in \([17]\), which considered a conventional sensor scenario.

In \([17]\) Theorem 2, a necessary stability condition is obtained as
\[ \rho^2(A) \min\{p_{22}, (1 - p_{12})\} < 1, \]  
(20)
which is less restrictive than our current result \([19]\).

B. Upper and Lower Bounds of the Estimation Error Function

A pair of asymptotic upper and lower bounds of the estimation error function are given below.

Proposition 1 (Asymptotic upper bound of the estimation-error function). For any \( \epsilon > 0 \), there exists \( N > 0 \) and \( \kappa > 0 \) such that
\[ c(i) < \kappa (\rho^2(A) + \epsilon)^i, \forall i > N. \]

Proposition 2 (Asymptotic lower bound of the estimation-error function). There exists a constant \( N > 0 \) and \( \eta > 0 \) such that \( c(i) \geq \eta (\rho(A))^{2i}, \forall i > N. \)

Propositions 1 and 2 show that when a large number of consecutive packet dropouts occur, i.e., \( i \gg 1 \), the remote estimation error is upper and lower bounded by exponential functions in terms of \( i \).

Remark 4. It can be observed that the estimation-error function \( c(i) \) grows as exponentially fast as \( \rho^2(A) \).

IV. ANALYSIS OF THE AVERAGE ESTIMATION MSE

In this section, we first investigate an estimation-cycle based performance analysis approach of the remote state estimation, and then develop new element-wise bounds of matrix powers. The results and technical lemmas obtained in this section will be used for the proofs of the main results of the paper.

As it is clear that Theorem 1 holds for the special cases with \( D = 0 \) or \( I \), in the following we only focus on the cases with \( D \neq 0 \) nor \( I \).

A. Stochastic Estimation-Cycle Based Analysis

Before analyzing the long-term average MSE of the remote estimation system and derive the stability condition, we need to introduce and analyze estimation cycle. To be specific, the \( k \)th estimation cycle starts after the \( k \)th successful transmission and ends at the \( (k+1) \)th successful transmission as illustrated in Fig. 2. In other words, the estimation process is divided by the estimation cycles.

The channel state at the beginning of estimation cycle \( k \), i.e., a post-success channel state, is denoted by \( S_k \in B' \subset B \), where
\[ B' \triangleq \{ b_j : \max_{i \in M} (1 - d_i)p_{i,j} > 0, \forall j \in M \} \]  
(21)
is the set of post-success channel states and the cardinality of \( B' \) is \( M' \leq M \). Without loss of generality, we assume that \( B' \) contains the first \( M' \) elements of \( B \). In other words, none of the last \( (M - M') \) elements of \( B \) can be a post-success channel state, while the others can.

Example 2. Consider a two-state Markov channel with
\[ M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]
\[d_1 = 1 \text{ and } d_2 = 0.\] It is clear that the channel states deterministically switches between the two states, and the transmission can be successful only in channel state 2. Thus, channel state 1 is the only post-success state, i.e., \(B' = \{b_1\} \subset B = \{b_1, b_2\}.\)

Then, we have the following property of \(S_k\).

**Lemma 1.** \(\{S\}_{N_0}\) is a time-homogeneous ergodic Markov chain with \(M'\)-irreducible states of \(B'\). The state transition matrix of \(\{S\}_{N_0}\) is \(G'\), which is the \(M'\)-by-\(M'\) matrix taken from the top-left corner of \(G\), where

\[
G = \sum_{j=0}^{\infty}(DM)^j(I - D)M,
\]

and the last \((M - M')\) columns of \(G\) are all zeros. The stationary distribution of \(\{S\}_{N_0}\) is \(\beta \triangleq [\beta_1, \cdots, \beta_{M'}]^T\), which is the unique null-space vector of \((I - G')^T\) and \(\beta_i > 0, \forall i \in M'\), where \(M' \triangleq \{1, 2, \cdots, M'\}\).

**Proof.** See Appendix A. \(\square\)

**Remark 5.** Our analysis investigates the sequence of successful reception instances. This has also been considered in [38], where the instances of successful reception are return times of a Markov chain. Different to [38], we focus on the channel states right after these instances, which form an ergodic Markov chain. Our approach will shed lights on the future work of the analysis of closed-loop control systems over Markov channels.

Let \(T_k\) denote the sum number of transmissions in the \(k\)th estimation cycle. The sum MSE in the \(k\)th estimation cycle, say \(C_k\), is given as

\[
C_k = g(T_k) \triangleq \sum_{j=1}^{T_k} c(j).
\]

From (15) it directly follows that the average estimation MSE can be rewritten as

\[
J = \lim_{K \to \infty} \frac{\sum_{i=1}^{K} C_i + C_2 + \cdots + C_K}{T_1 + T_2 + \cdots + T_K}.
\]

Since \(C_k\) is determined by \(T_k\), and the distribution of \(T_k\) depends on \(S_k\) and the distribution of \(S_k\) is time-invariant, the unconditional distributions of \(T_k\) and of \(C_k\) are also time-invariant. We thus drop the time indexes of \(T_k\), \(C_k\) and \(S_k\). Then, the time average of \(\{\cdots, T_k, T_{k+1}, \cdots\}\) and \(\{\cdots, C_k, C_{k+1}, \cdots\}\) can be translated to the following ensemble averages as

\[
E[T] = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} T_k = \sum_{m=1}^{M} \beta_m E[T|S = b_m],
\]

and

\[
E[C] = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} C_k = \sum_{m=1}^{M} \beta_m E[C|S = b_m],
\]

where \(\beta_m\) is defined in Lemma 1 for \(m \in \{1, \cdots, M'\} \) and \(\beta_m = 0\) when \(m > M'\).

From the definition of estimation cycle and the property of channel state transition, the conditional probability of the length of an estimation cycle is obtained as

\[
\text{Prob}[T = i|S = b_m] = \sum_{k=1}^{M} (DM)^{i-1}(I - D)M_{m,k}.
\]

If we now replace (27) into (25) and into (26), then after some algebraic manipulations, one can obtain:

\[
E[T] = \sum_{i=1}^{M} \sum_{k=1}^{M} \sum_{l=1}^{M} i \mathbb{E}(i)_{j,k},
\]

\[
E[C] = \sum_{i=1}^{M} \sum_{k=1}^{M} \sum_{l=1}^{M} g(i) \mathbb{E}(i)_{j,k},
\]

where

\[
\mathbb{E}(i) = \text{diag}\{\beta_1, \cdots, \beta_{M'}\}(DM)^{i-1}(I - D)M.
\]

Taking (28) and (29) into (30), we have

\[
J = \lim_{K \to \infty} \frac{1}{K} \left( \frac{C_1 + C_2 + \cdots + C_K}{T_1 + T_2 + \cdots + T_K} \right) = \frac{E[C]}{E[T]}.
\]

Therefore, it turns out that the average estimation MSE \(J\) depends on the estimation error function \(c(i)\) and the function \((DM)^i(I - D)M\), both of which involve matrix powers. In what follows, we will introduce and prove some technical lemmas about the element-wise upper and lower bounds of matrix powers, which are the key steps for analyzing the sufficient and necessary stability conditions of the remote estimation system.
B. Element-Wise Bounds of Matrix Powers

We give an element-wise upper bound of matrix powers as below.

Lemma 2 (Element-wise upper bound of matrix power). Consider a $z$-by-$z$ matrix $Z$ with $A$ different eigenvalues \{\lambda_1, \lambda_2, \ldots, \lambda_A\}$, where $1 \leq A \leq z$, and define $\mathcal{Z} \triangleq \{1, \ldots, z\}$. Then, for any $\epsilon > 0$, there exist $N > 0$ and $\kappa > 0$ such that

$$||Z||_{j,k}^2 < \kappa (\rho(Z) + \epsilon)^{2l}, \forall j, k \in \mathcal{Z}, \forall i > N.$$  

Proof. See Appendix B.

Definition 2 (Asymptotically and periodically lower bounded). A function $r(k)$ is asymptotically and periodically lower bounded by $\rho(k)$ with a period $l \in \mathbb{N}$ if there exists $N \in \mathbb{N}$ such that

$$\max\{r(i), r(i + 1), \ldots, r(i + l - 1)\} \geq \rho(i), \forall i \geq N.$$  

Thus, if $r(k)$ is asymptotically and periodically lower bounded by $\rho(k)$ with a period $l$, then the sum of the function $r(k)$ with a consecutive of $l$ samples is lower bounded by $\rho(k)$. When a direct lower bound of the function $r(k)$ is intractable or very loose, we can resort to finding a periodical lower bound $\rho(k)$, which might introduce a lower bound of the average of $r(k)$ per $l$ samples, i.e., $\rho(k)/l$. It is clear that the periodical lower bound is tighter if the period $l$ is smaller. In Lemma 3, we will show how to determine the period of a specific problem in details. Definition 2 will be used to capture the lower bound of the average sum MSE in (23) for analyzing the necessary stability condition.

Definition 3 (Asymptotically lower bounded). A function $r(k)$ is asymptotically lower bounded by $\rho(k)$ if it is asymptotically and periodically lower bounded by $\rho(k)$ with period 1.

Given the preceding definitions, we can obtain an element-wise lower bound of matrix powers as below.

Lemma 3 (Element-wise lower bound of matrix power).

(i) Consider a $z$-by-$z$ matrix $Z$. Then there exist $\eta > 0$ and $j, k \in \mathcal{Z}$ such that $||Z||_{j,k}^2$ is asymptotically and periodically lower bounded by $\eta(\rho(Z))^{21}$. The period is a positive integer no larger than the number of eigenvalues of $Z$ with the same maximum magnitude.

(ii) Consider a pair of $z$-by-$z$ matrices $Z$ and $Q$ with the assumptions that $Q$ is symmetric positive semidefinite and $(Z, \sqrt{Q})$ is controllable. Then there exist $\eta > 0$ and $j, k \in \mathcal{Z}$ such that $||Z||_{j,k}^2$ is asymptotically and periodically lower bounded by $\eta(\rho(Z))^{2i}$. The period has the same property as in (i).

Proof. See Appendix B.

V. PROOF OF THE MAIN RESULTS

In this section, we prove Propositions 1 and 2 and Theorem 1.

A. Proof of Proposition 1

Taking (12) into (13), we have

$$c(i) = \text{Tr} (A^i \sqrt{F_0}(A^i \sqrt{F_0})^\top) + \sum_{m=0}^{i-1} \text{Tr} (A^m \sqrt{W}(A^m \sqrt{W})^\top).$$  

(31)

From (31) and Lemma 2, for any $\epsilon > 0$, there exists $\kappa, \kappa', N > 0$ such that for all $i > N$ we have

$$c(i) \leq n^2 \left( \max_{j,k \in \mathcal{N}} \left( A^i \sqrt{F_0}_{j,k} \right)^2 + \sum_{m=0}^{i-1} \max_{j,k \in \mathcal{N}} \left( A^m \sqrt{W}_{j,k} \right)^2 \right) \leq n^2 \left( \kappa (\rho(A) + \epsilon)^{2l} + \sum_{m=N+1}^{i} \kappa' (\rho(A) + \epsilon)^{2m} \right) \leq n^2 \left( (i - N + 1) \max\{\kappa, \kappa'\} (\rho(A) + \epsilon)^{2l} + \sum_{m=0}^{N} \max_{j,k \in \mathcal{N}} \left( A^m \sqrt{W}_{j,k} \right)^2 \right),$$  

(32)

where $\mathcal{N} \triangleq \{1, \ldots, n\}$. Recall that $A$ is an $n$-by-$n$ matrix. Thus, for any $\epsilon' > \epsilon$, we can find $N' > N$ and $\kappa'' > 0$ such that $c(i) \leq \kappa'' (\rho(A) + \epsilon')^{2l}, \forall i > N'$. This completes the proof of Proposition 1.

B. Proof of Proposition 2

From Lemma 3(ii), we note that there exists $\eta > 0$ and $j, k \in \mathcal{N}$ such that $||A^iW||_{j,k}^2$ is asymptotically and periodically lower bounded by $\eta(\rho(A))^{2i}$ with the period $l$, which is no larger than the dimension of the matrix $A$. Then, from (31), when $i$ is sufficiently large, we have

$$c(i) \geq \sum_{m=1-l}^{i-l} \text{Tr} (A^m \sqrt{W}(A^m \sqrt{W})^\top) \geq \sum_{m=1-l}^{i-1} \left( ||A^mW||_{j,k}^2 \right) \geq \eta(\rho(A))^{2(i-l)} = \eta (\rho(A))^{-2l} (\rho(A))^{2i}. $$  

(33)

This completes the proof of Proposition 2.

C. Proof of Theorem 1

Besides the upper and lower bounds of the estimation error function, to obtain the necessary and sufficient stability condition, we need some additional properties of $(DM)^i$ and $(DM)^i((I - D)M)$.

Lemma 4 (Property of matrix $(DM)^i$). Consider the stochastic matrix $M$ and the diagonal matrix $D$ defined in (4) and (5), respectively. Let $\mathcal{J}_0 \triangleq \{j | d_j = 0, j \in \mathcal{M}\}$ and $\mathcal{J}_0 \triangleq \mathcal{M}\setminus\mathcal{J}_0 = \{j | d_j \neq 0, j \in \mathcal{M}\} \neq \emptyset$.

(i) If $\mathcal{J}_0 = \emptyset$, there exists $\eta > 0$ such that $||\text{tr}(\sqrt{DM})||_{j,k}^2$ is asymptotically and periodically lower bounded by $\eta\rho'(DM)$, $\forall j, k \in \mathcal{M}$.
(ii) If \( \mathcal{J}_0 \neq \emptyset \), there exists \( \eta > 0 \), \( j \in \mathcal{J}_0 \) and \( k \in \mathcal{J}_0 \) such that \( [(DM)^j](i,k) \) is asymptotically and periodically lower bounded by \( \eta \rho^j(DM) \).

(iii) \( \rho(DM) < 1 \).

Proof. See Appendix C.

\[ \]

\textbf{Lemma 5} (Property of matrix \((DM)^j(I - D)M\)). Given the stochastic matrix \( M \) and the diagonal matrix \( D \) defined in (4) and (5), respectively, there exist \( \eta > 0 \) and \( j, k \in \mathcal{M} \) such that \( [(DM)^j(I - D)M](i,j) \) is asymptotically and periodically lower bounded by \( \eta \rho^j(DM) \).

Proof. See Appendix C.

By using the properties in Lemmas 4 and 5, Theorem 1 can be proved as in Appendix D.

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VI. NUMERICAL RESULTS

In this section, we illustrate and compare the stability regions of the remote estimation system obtained using Theorem 1 of the current paper and based on Corollary 1 of our previous work [15]. We also present simulated results of the average estimation MSE in (24) based on the average of \( 10^5 \) time steps.

We consider an example involving the Pendubot, a two-link planar robot [39]. A linearized continuous time model for balancing the Pendubot in the upright position can be found in [40]. With a sampling time of 15 ms, we can then obtain the following discrete time model [27]:

\[
A = \begin{bmatrix}
1.0058 & 0.0150 & -0.0016 & 0.0000 \\
0.7808 & 1.0058 & -0.2105 & -0.0016 \\
-0.0060 & 0.0000 & 1.0077 & 0.0150 \\
-0.7962 & -0.0060 & 1.0294 & 1.0077
\end{bmatrix}, \quad (36a)
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad (36b)
\]

\[
W = uu^T, \quad u = \begin{bmatrix} 0.003 \\
1.0000 \end{bmatrix}, \quad -0.005 \\
-2.150 \end{bmatrix}^T, \quad (36c)
\]

\[
V = 0.001 \times I. \quad (36d)
\]

Thus, \( \rho(A) = 1.15 \) and \( \bar{\rho}(A) = 2 \). Unless otherwise stated, we consider a two-state Markov channel model characterised by the transition matrix \( M = \begin{bmatrix} 0.1 & 0.9 \\
0.5 & 0.5
\end{bmatrix} \) and conditional dropout probabilities \( d_1 = 0.8 \), and \( d_2 = 0.1 \).

Fig. 3 shows the stability regions (in the dropout probability plane) for different \( A \) and \( M \). In this figure, the solid and dashed line bounded regions are obtained from Theorem 1 and [15, Corollary 1], respectively. Specifically, we set \( A = A_1 = \begin{bmatrix} 1.129 \\
0.7808 \end{bmatrix} \) and \( \bar{\rho}(A_1) = 2 \), and set \( \text{M} = M_1 = \begin{bmatrix} 0.1 & 0.9 \\
0.9 & 0.1
\end{bmatrix} \) in case (b), where \( \rho(A_1) = 1.2 \) and \( \bar{\rho}(A_1) = 2 \), and set \( M = M_1 = \begin{bmatrix} 0.1 & 0.9 \\
0.9 & 0.1
\end{bmatrix} \) in cases (c) and (d), respectively. From cases (a)-(d), it is clear that our current necessary and sufficient stability region is much larger than the sufficient stability region established in [15]. Also, it can be observed that the necessary and sufficient stability region is convex in case (d) and is non-convex in cases (a)-(c). Note that the convexity of a stability region is important in practice. For example, if one has tested a set of communications parameter vectors that can stabilize the remote estimation system and the stability region is proved to be convex, any parameter vector that belong to the convex hull of the set can stabilize the system as well. Comparing (b) with (a), it is clear to see a smaller stability region as the system in (b) is more unstable than (a). Comparing (c) with (d), it is interesting to see that if the Markov channel has a longer memory, i.e., it has a higher chance to stay in a poor channel condition, then the remote estimation system has a smaller stability region.

Fig. 4 shows the original unstable process \( x_1 \) of the system [36], and the remote estimation \( \hat{x}_1 \). We see that the estimator tracks the unstable process well, and the relative estimation error, i.e., \( |x_1 - \hat{x}_1|/|x_1| \) decreases with time.

Fig. 5 shows the simulated average estimation MSEs of the smart-sensor-based and a conventional-sensor-based remote estimator [6] with different packet drop probabilities. Under the stability condition (illustrated as the gray area in Fig. 3(a)), we see that although the local estimator guarantees a better performance than the remote estimator, the performance gap is non-negligible only when the packet dropout probabilities are large at all channel states. It is interesting to note that the two cases actually have the same stability condition when packet dropout are i.i.d., see [6]. This motivates the hypothesis that the smart-sensor-based and the conventional-sensor-based remote estimation systems have the same stability condition under the Markov channel in terms of the LTI system transition matrix, the packet drop probabilities and the channel state transition matrix.

In Figs. 6 and 7 we give contour plots of the average estimation MSE in the smart-sensor-based and the conventional-sensor-based cases, respectively. It can be observed that the average estimation MSEs grow up dramatically outside the
developing the asymptotic theory of matrices which provides new (periodic) element-wise bounds of matrix powers. Our numerical results have verified the correctness of the stability condition and have shown that it is much more effective than existing sufficient conditions in the literature. It has been observed that the stability region in terms of the packet drop probabilities in different channel states can either be convex or non-convex depending on the transition probability matrix. Our simulation results have suggested that the stability conditions may coincide for schemes with a smart sensor and with a conventional sensor. This inspires our future work on analyzing the stability condition of the case without a smart sensor. Furthermore, the derived stability conditions can be used to design the optimal policy of transmission power control (e.g., via off-line design of $D$) as well as multi-sensor scheduling policies. In addition to stability, we will look into the performance of the smart sensor based remote estimation
The hitting time from \( S \) where the state transition probability is \( G \), the solution of

given any state \( S \) and the ergodicity of \( G \) are finite with a positive probability. This completes the proof of (22). From the definition of \( B' \) in (21), it is clear that the last \((M-M')\) columns of \((I-D)M\) are all zeros, completing the proof of Lemma [1]

**APPENDIX B: PROOFS OF LEMMAS 2 AND 3**

### A. Preliminaries

Assume a \( z \)-by-\( z \) matrix \( Z \) has \( A \) different eigenvalues \( \{\lambda_1, \lambda_2, \ldots, \lambda_A\} \). Represent \( Z \) in its Jordan normal form of \( Z = UJU^{-1} \), where \( U \) is an invertible matrix and

\[
J = \begin{bmatrix} J_1 & \cdots & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \cdots & J_A \end{bmatrix},
\]

\[
J_m = \begin{bmatrix} \lambda_m & 1 \\ & \ddots & \ddots \\ & & 1 \end{bmatrix}, \forall m = 1, \ldots, A.
\]

Let \( u_m \) denote the size of Jordan block \( J_m \). Then, \( U \) and \( U^{-1} \) can be represented as

\[
U = [F_1|F_2|\cdots|F_A]
\]

\[
U^{-1} = [G_1|G_2|\cdots|G_A]^{\top}
\]

where \( F_m \) and \( G_m \) are \( z \)-by-\( u_m \) matrices. Since \( U \) is of full rank, \( F_m \) and \( G_m \) have full column rank of \( u_m, \forall m = 1, \ldots, A \).

Then, we have

\[
Z = UJU^{-1} = \sum_{m=1}^{A} F_m J_m G_m^{\top},
\]

where

\[
J_m^i = \begin{bmatrix} \lambda_m^i & (i)\lambda_m^{i-1} & \cdots & (i)_{u_m-1}\lambda_m^{i-u_m+1} \\ \cdots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (i)_{u_m-1}\lambda_m^{i-u_m+1} & \cdots & (i)\lambda_m^{i-1} & \lambda_m^i \end{bmatrix}
\]

Note that \( J_m^i \) has a full rank of \( u_m \) for all \( m \in \mathbb{N} \) if \( \lambda_m \neq 0 \). From (43) and (44), the element at the \( j \)th row and \( k \)th column of \( F_m J_m^i G_m^{\top} \) denoted by \( [F_m J_m^i G_m^{\top}]_{j,k} \), can be rewritten as a polynomial in terms of \( m \) as

\[
[F_m J_m^i G_m^{\top}]_{j,k} = \lambda_m [(i)\lambda_m^{i-u_m-1}, (i)\lambda_m^{i-u_m-2}, \ldots, (i)_{u_m-1}\lambda_m^i] A_{m,(j,k)},
\]

where \( A_{m,(j,k)} \) is a column vector determined by \( F_m \) and \( G_m \) and is independent with \( i \).
B. Proof of Lemma 2
From (43) and (45), we have
\[
||Z||_{j,k} = \left| \sum_{m=1}^{\infty} F_m J_m^i G_m^\top \right| \leq \kappa \epsilon \sum_{m=1}^{\infty} |\lambda_m|^i \leq A \kappa \epsilon \rho_i(Z),
\]
where \(\kappa\) is a positive constant.

The result follows upon noting that \(\lim_{i \to \infty} i^\epsilon \rho_i(Z)/(\rho(Z) + \epsilon)^i = 0, \forall \epsilon > 0\).

C. Proof of Lemma 3

Before proceeding to prove the element-wise lower bound of matrix powers, we need the following technical lemma.

Lemma 6. Consider a \(z\)-by-\(z\) matrix \(Z\) with \(A\) different eigenvalues \(\{\lambda_1, \lambda_2, \cdots, \lambda_A\}\), where \(A \leq z\), and a \(z\)-by-\(z\) symmetric and positive semidefinite matrix \(Q\). If \((Z, \sqrt{Q})\) is controllable and \(\lambda_m \neq 0\) for some \(m \in \{1, \cdots, A\}\), then \(F_m J_m^i G_m^\top \sqrt{Q} \neq 0, \forall i \in N_0\).

Proof. Assume that there exists \(i \in N_0\) such that \(F_m J_m^i G_m^\top \sqrt{Q} = 0\). Then, we have
\[
0 = \text{Rank} \left( F_m J_m^i G_m^\top \sqrt{Q} \right) 
\geq \text{Rank} (F_m) + \text{Rank} \left( J_m^i G_m^\top \sqrt{Q} \right) - u_m 
= \text{Rank} (J_m^i G_m^\top \sqrt{Q}) 
= \text{Rank} (G_m^\top \sqrt{Q}),
\]
where (48) is due to Sylvester’s rank inequality [41]. Thus, \(G_m^\top \sqrt{Q} = 0\).

From the definitions of \(F_m\) and \(G_m\), it is clear that
\[
\sum_{k=1}^{\infty} F_k G_k^\top = \mathbf{I}.
\]
By multiplying \(\sqrt{Q}\) on the both sides of (51), we have
\[
\sqrt{Q} = \sum_{k \in \{1, \cdots, A\}\setminus m} F_k G_k^\top \sqrt{Q}.
\]
Applying \(G_m^\top \sqrt{Q} = 0\) on (53), it can be obtained that
\[
Z^i \sqrt{Q} = \sum_{k \in \{1, \cdots, A\}\setminus m} F_k J_k^i G_k^\top \sqrt{Q}, \forall i \in N_0.
\]
Jointly using (52) and (53), it is easy to see that each column of the matrix concatenation \(\sqrt{Q}, Z\sqrt{Q}, \cdots, Z^{z-1}\sqrt{Q}\) is in the span of the columns of \(\{F_k | k \in \{1, \cdots, A\}\setminus m\}\). Therefore,
\[
\text{Rank} \left( \begin{bmatrix} \sqrt{Q}, Z\sqrt{Q}, \cdots, Z^{z-1}\sqrt{Q} \end{bmatrix} \right) 
\leq \sum_{k \in \{1, \cdots, A\}\setminus m} \text{Rank} (F_k) 
= z - \text{Rank} (F_m) 
< z,
\]
which, however, contradicts with the assumption that \(\sqrt{Q}, Z\sqrt{Q}, \cdots, Z^{z-1}\sqrt{Q}\) is of full rank. This completes the proof.

Proof of Lemma 2. If \(\lambda_m \neq 0\), \(J_m\) is a full-rank square matrix and hence \(J_m^i G_m^\top\) has a full row rank of \(u_m\). Since \(F_m\) has a full column rank of \(u_m\), by using Sylvester’s rank inequality, we have
\[
\text{Rank} (F_m J_m^i G_m^\top) \geq \text{Rank} (F_m) + \text{Rank} (J_m^i G_m^\top) - u_m
= u_m > 0.
\]
Therefore, \(F_m J_m^i G_m^\top \neq 0, \forall i \in N_0\). From (45), we can find a pair of \(j, k \in Z\) such that \(A_{m,(j,k)} \neq 0\). Thus, without loss of generality, we assume that the dominant term of the polynomial \(\{F_m J_m^i G_m^\top\}_{j,k} = \lambda_m^{i} u_m^{m-1}, i \geq m-2, \cdots, i, 1\} \Lambda_{m,(j,k)}\) is \(\Lambda_{m,(j,k)} \Lambda_{m}^{i} u_m^{m-1}\) when \(i \to \infty\), where \(\Lambda_{m,(j,k)} \neq 0\) and \(u_m, (j,k) \in \{0, \cdots, u_m - 1\}\).

If \(|\lambda_m| = \rho(Z)\) and \(\lambda_m\) is the unique eigenvalue that has the maximum magnitude, it is clear that the dominant term of \(Z^i j, k = \sum_{m=1}^{\infty} \{F_m J_m^i G_m^\top\}_{j,k}\) is \(\Lambda_{m,(j,k)} \Lambda_{m}^{i} u_m^{m-1}\). Thus, one can find \(\eta > 0\) such that \(||Z||_{j,k}^i\) is asymptotically lower bounded by \(\eta \rho(Z)^i\).

If there are multiple eigenvalues having the same maximum magnitude, i.e., \(Z^i \triangleq \{\lambda_i = \rho(Z), \forall i \in Z\\} \) and \(|Z|^i > 1\), where \(Z \triangleq \{1, 2, \cdots, A\}\), we consider the following two complementary cases:

Case 1): There exists \(m \in Z^i\) such that \(\Lambda_{m,(j,k)} \neq 0\) and \(u_m > \Lambda_{m,(j,k)} \neq 0, \forall (m', j, k) \in Z^i \setminus \{m\}\). In this case, the dominant term of \(\{Z^i\}_{j,k} = \sum_{m=1}^{\infty} \{F_m J_m^i G_m^\top\}_{j,k}\) is still \(\Lambda_{m,(j,k)} \Lambda_{m}^{i} u_m^{m-1}\). Thus, one can find \(\eta > 0\) such that \(||Z||_{j,k}^i\) is asymptotically lower bounded by \(\eta \rho(Z)^i\).

Case 2): There exists a set \(Z'' \subseteq Z^i\) with cardinality \(z'' \geq 2\) such that \(u_m, (j,k) = u_m, (j', k) \neq 0, \forall (m'', j', k') \in Z'' \setminus \{m''\}\) and \(u_m > \Lambda_{m,(j,k)} \neq 0, \forall (m'', j', k') \in Z'' \setminus \{m''\}\). In this case, \(||Z||_{j,k}^i\) may not be asymptotically lower bounded by \(\eta \rho(Z)^i\) due to the multiple eigenvalues with identical magnitude but different phases.

In the following, we will show that \(\sum_{m \in Z''} \Lambda_{m,(j,k)} \Lambda_{m}^{i} u_m^{m-1}\) is asymptotically and periodically bounded by \(\eta \rho(Z)^i\). Let \(m_i\) denote the index of the \(l\)th eigenvalue in \(Z''\), where \(l \in \{1, \cdots, z''\}\). Thus, \(\lambda_{m_i} \triangleq \rho(Z) e^{j\phi_{m_i}}, \phi_{m_i} \in [0, 2\pi]\). We have the following matrix
\[
\Pi \triangleq \begin{bmatrix}
\lambda_{m_1}^{i} & \lambda_{m_2}^{i} & \cdots & \lambda_{m_{z''}}^{i} \\
\lambda_{m_1}^{i+1} & \lambda_{m_2}^{i+1} & \cdots & \lambda_{m_{z''}}^{i+1} \\
\vdots & \vdots & & \vdots \\
\lambda_{m_1}^{i+z''-1} & \lambda_{m_2}^{i+z''-1} & \cdots & \lambda_{m_{z''}}^{i+z''-1}
\end{bmatrix}
= \text{diag}\left\{\rho^{i}(Z) e^{j\phi_{m_1}}, \rho^{i+1}(Z) e^{j(i+1)\phi_{m_1}}, \cdots, \rho^{i+z''-1}(Z) e^{j(i+z''-1)\phi_{m_1}}\right\} \Pi',
\]
where
\[
\begin{bmatrix}
1 & e^{\lambda_1(t+1)}(\phi_{m_1^2}) & \cdots & e^{\lambda_{m_1}(t+1)}(\phi_{m_1^m}) \\
1 & e^{\lambda_1(t+1)}(\phi_{m_1^2}) & \cdots & e^{\lambda_{m_1}(t+1)}(\phi_{m_1^m}) \\
\vdots & \vdots & \ddots & \vdots \\
1 & e^{\lambda_1(t+z''-1)}(\phi_{m_1^2}) & \cdots & e^{\lambda_{m_1}(t+z''-1)}(\phi_{m_1^m})
\end{bmatrix} = \Pi' \Phi
\]
denotes a Vandermonde matrix, which is invertible due to the fact that \(\lambda_{m_i} \neq \lambda_{m_i'}\) for \(i \neq i'\), see [42]. Let
\[
b = [\Lambda_{m_1,(j,k)}, \Lambda_{m_2,(j,k)}, \ldots, \Lambda_{m_r,(j,k)}] \neq 0.
\]
Since \(\Pi''\) is invertible, using the inequality of matrix-vector product [43], [44], we have
\[
|\Pi''b| = |\Pi''\Phi b| \geq |(\Pi'')^{-1} |-^1| b| > 0,
\]
where \(|(\Pi'')^{-1} |-^1\neq 0\) is the minimum magnitude of the eigenvalues of \(\Pi''\). Since the largest magnitude of the elements of \(\Pi''b\) is no smaller than \(|\Pi''b|/\sqrt{z''}\), we have
\[
\max_{l=0, \ldots, z''t} \sum_{i=1}^{z''} \phi_{m_1^1}(t+l') \leq \frac{1}{\sqrt{z''}} |(\Pi'')^{-1} |-^1 |b| > 0,
\]
and hence
\[
\max_{l=0, \ldots, z''t} \sum_{i=1}^{z''} \phi_{m_1^1}(t+l') \leq \frac{1}{\sqrt{z''}} |(\Pi'')^{-1} |-^1 |b| > 0.
\]
Therefore, \(|Z|\leq \text{asymptotically and periodically lower bounded by } \eta p^2(\gamma)|\) with period \(z''\), where \(\eta\) is a positive constant.

**Proof of Lemma 4 (ii).** From (45), if \(\lambda_{m_i} \neq 0\), it is easy to see that
\[
[F_m J_m G_m \sqrt{Q}]_{j,k} = \lambda_{m}^{i} \gamma_{m_{i+1}} \gamma_{m_{i+2}} \cdots \gamma_{m_{i+1}}(Z)_{j,k}.
\]

**Appendix C: Proofs of Lemmas 4 and 5**

**D. Proof of Lemma 4**

For part (i), since \(M\) is an irreducible non-negative matrix and \(d_j > 0, \forall j \in M\), the \(M\)-by-\(M\) matrix \(M\) is also irreducible and non-negative, i.e., one can associate with the matrix a certain directed graph \(G\), which has exactly \(M\) vertices, and there is an edge from vertex \(j\) to vertex \(k\) when \([M]_{j,k} > 0\), and \(G\) is strongly connected. By using Lemma 5, there exists \(j, k \in M\) such that \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(M)\), with the period no larger than the number of eigenvalues of \(DM\) with the same maximum magnitude. Then using the non-negative and irreducible property of \(DM\), we get \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\), which completes the proof of part (i).

For part (ii), we construct a diagonal matrix \(D' = \text{diag}(d_j, \ldots, d_M)\), where \(d_j = d_j, \forall j \in M\), and \(D' = \text{diag}(d_j, \ldots, d_M)\). Since \(DM\) is irreducible and non-negative, we can find \(l \in \mathbb{N}^+\) such that \(d_{k'}^l\) is reachable from \(k\) in \(l\) steps, i.e., \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\). This completes the proof of part (ii).

For part (iii), since \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\), \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\). The proof is similar to part (ii).

**E. Proof of Lemma 5**

For the case that \(J_0 = 0\), since \(D \neq I\) and \(M\) is a stochastic matrix, there exists \(k', k \in M\) such that \([I - D]M^{k',k} = \eta > 0\). Using Lemma 4 (ii), for any \(j, k' \in M\), \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\). Thus, there exists \(j, k' \in M\) such that
\[
[D_t]_{j,k'} \geq \eta p^2(DM),
\]
where \(\eta\) is a positive constant. For the case that \(J_0 \neq 0\), given any \(k' \in J_0\), we can find \(k \in M\) such that \([I - D]M^{k',k} = \eta > 0\). Using Lemma 4 (ii), there exists \(j \in J_0, k' \in J_0\) such that \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\). Thus, there exists \(j, k' \in J_0\) and \(k \in M\) such that, also in this case, \([D_t]\) is asymptotically and periodically lower bounded by \(\eta p^2(DM)\).
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