Abstract

For finite-dimensional bifurcation problems, it is well-known that it is possible to compute normal forms which possess nice symmetry properties. Oftentimes, these symmetries may allow for a partial decoupling of the normal form into a so-called “radial” part and an “angular” part. Analysis of the radial part usually gives an enormous amount of valuable information about the bifurcation and its unfoldings. In this paper, we are interested in the case where such bifurcations occur in retarded functional differential equations, and we revisit the realizability and restrictions problem for the class of radial equations by nonlinear delay-differential equations. Our analysis allows us to recover and considerably generalize recent results by Faria and Magalhães [11] and by Buono and Bélair [4].
1 Introduction

Delay-differential equations are used extensively in the modeling of a multitude of phenomena in the life sciences [3, 20, 22], physics [19, 27], atmospheric sciences [25], engineering [24], economics [28] and beyond. This has motivated a flurry of activity on the mathematical side to try to understand the behavior of this class of equations and to develop a theoretical framework suitable for their analysis.

It is now well-understood that retarded functional differential equations (RFDEs), a class which contains delay-differential equations, behave for the most part like infinite-dimensional ordinary differential equations. The upshot is that many of the techniques and theoretical results of finite-dimensional geometrical dynamical systems are portable to RFDEs. In particular, versions of the stable/unstable and center manifold theorems in neighborhoods of an equilibrium point exist for RFDEs [18]. For example, near a bifurcation point in a RFDE, the flow is essentially governed by a vector field on an invariant center manifold. This has allowed for the successful application of the vast machinery of bifurcation theory to many problems which are modeled by RFDEs, e.g. [2, 24]. Parallel to this, techniques for simplifying vector fields via normal form changes of coordinates have been adapted to RFDEs [8, 9], and has allowed for further insight into the qualitative behavior of RFDEs.

This paper is concerned with the bifurcation theory of RFDEs. In particular, we will be interested in the so-called realizability problem for normal forms of vector fields which arise via center manifold reduction of RFDEs.

Realizability problem:

Suppose $B$ is an arbitrary $m \times m$ matrix. For the sake of simplicity, suppose additionally that all eigenvalues of $B$ are simple. Let $C([-r, 0], \mathbb{R})$ designate the space of continuous functions from the interval $[-r, 0]$ into $\mathbb{R}$, and for any continuous function $z$, define $z_t \in C([-r, 0], \mathbb{R})$ as $z_t(\theta) = z(t+\theta), -r \leq \theta \leq 0$. It is then possible [10] to construct a bounded linear operator $L : C([-r, 0], \mathbb{R}) \to \mathbb{R}$ such that the infinitesimal generator $A_0$ for the flow associated with the functional differential equation

$$\dot{z}(t) = L z_t$$  \hspace{1cm} (1.1)

has a spectrum which contains the eigenvalues of $B$ as a subset. Thus, there exists an $m$-dimensional subspace $P$ of $C([-r, 0], \mathbb{R})$ which is invariant for the flow generated by $A_0$, and the flow on $P$ is given by the linear ordinary differential equation (ODE)

$$\dot{x} = Bx.$$ 

In our case, we will be especially interested in the case where the eigenvalues of $B$ all have zero real parts, and the spectrum of $A_0$ does not contain any elements with zero real part.
other than those which belong to the spectrum of $B$.

Now, suppose (1.1) is modified by the addition of a nonlinear delayed term

$$
\dot{z}(t) = Lz_t + az(t + \tau)^2,
$$

where $a \in \mathbb{R}$ is some coefficient and $\tau \in [-r, 0]$ is the delay time. Then the center manifold theorem for RFDEs[18] can be used to show that the flow for (1.2) admits an $m$-dimensional locally invariant center manifold on which the dynamics associated with (1.2) are given by a vector field which, to quadratic order, is of the form

$$
\dot{x} = Bx + ag(x),
$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is a fixed homogeneous quadratic polynomial which is completely determined by $L$ and $\tau$, and $a$ is the same coefficient which appears in (1.2). We immediately notice that for fixed $L$ and $\tau$, (1.3) has at most one degree of freedom in the quadratic term, corresponding to the one degree of freedom in the quadratic term in (1.2). However, whereas one degree of freedom is sufficient to describe the general scalar quadratic term involving one delay in (1.2), it is largely insufficient (if $m > 1$) to describe the general homogeneous quadratic polynomial $f : \mathbb{R}^m \to \mathbb{R}^m$. Therefore, there exist $m$-dimensional vector fields $\dot{x} = Bx + f(x)$ (where $f$ is homogeneous quadratic) which can not be realized by center manifold reduction (1.3) of any RFDE of the form (1.2). One quickly notices that the situation could be improved if we allow the nonlinear terms in (1.2) to depend on more than one delayed times, i.e.

$$
\dot{z}(t) = L(z_t) + \sum_{i_1 + \cdots + i_j = 2\atop i_1, \ldots, i_j = 0} a_{i_1i_2\ldots i_j} (z(t + \tau_1))^{i_1} \cdots (z(t + \tau_j))^{i_j},
$$

where the $a_{i_1i_2\ldots i_j}$ are real coefficients and $\tau_1, \ldots, \tau_j \in [-r, 0]$ are the delay times. The center manifold equations for (1.4) truncated to quadratic order are

$$
\dot{x} = Bx + \sum_{i_1 + \cdots + i_j = 2\atop i_1, \ldots, i_j = 0} a_{i_1i_2\ldots i_j} g_{i_1i_2\ldots i_j}(x),
$$

where $g_{i_1i_2\ldots i_j} : \mathbb{R}^m \to \mathbb{R}^m$ are fixed homogeneous quadratic polynomials which are completely determined by $L$ and $\tau_1, \ldots, \tau_j$. Thus, the subspace of $m$-dimensional vector fields $\dot{x} = Bx + f(x)$ (where $f$ is a homogeneous quadratic) of the form (1.5) is potentially larger than those of the form (1.3). Of course, there is nothing particularly special about the
quadratic order, and one could repeat the above discussion to include progressively higher order nonlinearities. Without loss of generality, we could also limit our attention to only those $f$ which are in normal form with respect to the matrix $B$. The particular version of the realizability problem which will interest us in this paper is the following:

Given:

- an $m \times m$ matrix $B$ whose spectrum consists solely of simple eigenvalues with zero real parts,
- a bounded linear operator $L : C([-r, 0], \mathbb{R}) \to \mathbb{R}$ such that the infinitesimal generator $A_0$ for the flow associated with the functional differential equation (1.1) has a spectrum which contains the eigenvalues $B$ as a subset, and no other part of its spectrum on the imaginary axis
- an integer $\ell \geq 2$
- a polynomial $f : \mathbb{R}^m \to \mathbb{R}^m$ of degree $\ell$ such that $f(0) = 0$ and $Df(0) = 0$, and $f$ is in normal form with respect to the matrix $B$

does there exist an RFDE of the form

$$\dot{z}(t) = L z_t + F(z(t + \tau_1), \ldots, z(t + \tau_j)), \quad (1.6)$$

such that the center manifold equations for (1.6), in normal form and truncated to order $\ell$, are $\dot{x} = Bx + f(x)$?

This question was answered in the affirmative for scalar RFDEs [16, 17] and in [10] for $n$-dimensional RFDEs in general. In the scalar case (which will be of interest to us), the result states that there is generically a solution to the realizability problem for general $f$ as stated above if $j$ (the number of distinct delays in (1.6)) is at least equal to $m$ (the dimension of the center subspace).

The main purpose of this paper is related to the optimality of the above sufficient number ($j = m$) of delays, in light of some recent results by Faria and Magalhães [11], and by Buono and Bélair [4], which we now describe.

Simple Hopf, $(0, \pm i\omega)$, and $(\pm i\omega_1, \pm i\omega_2)$ bifurcations:

In [11] and [4], the authors consider the optimality of the solution $j = m$ to the realizability problem for scalar RFDEs in some special cases. Consider one of the following three separate cases for the matrix $B$:
• $B$ is a $2 \times 2$ matrix whose eigenvalues are $\pm i\omega$, $\omega > 0$,
• $B$ is a $3 \times 3$ matrix whose eigenvalues are $0$ and $\pm i\omega$, $\omega > 0$,
• $B$ is a $4 \times 4$ matrix whose eigenvalues are $\pm i\omega_1$, $\pm i\omega_2$, where $\omega_1 > 0$ and $\omega_2 > 0$ are rationally incommensurate.

Let $L : C([−r,0], \mathbb{R}) \rightarrow \mathbb{R}$ be a bounded linear operator such that the infinitesimal generator $A_0$ for the flow associated with the functional differential equation (1.1) has a spectrum which contains the eigenvalues of $B$ as a subset, and has no other part of its spectrum on the imaginary axis. Therefore, a general RFDE of the form

$$\dot{z}(t) = Lz_t + N(z_t), \quad (1.7)$$

where $N(0) = 0$, $DN(0) = 0$ has an equilibrium solution $z = 0$ undergoing respectively a simple Hopf bifurcation, a steady-state/Hopf interaction, or a non-resonant double Hopf bifurcation. Normal forms and versal unfoldings for each of these bifurcations are well-known. In the first case, using normal form changes of coordinates and then converting to polar coordinates $x_1 = \rho \cos \theta$, $x_2 = \rho \sin \theta$ for the center manifold, the normal form (to cubic order) is

$$\dot{\rho} = a\rho^3, \quad \dot{\theta} = \omega + b\rho^2. \quad (1.8)$$

If the coefficient $a$ in (1.8) is non-zero, then the higher-order terms have no qualitative effects. In this case, the $\dot{\rho}$ equation completely determines the bifurcation. Now, from the above-mentioned solution $j = 2$ to the realizability problem [16, 17], we can conclude that if $N(z_t)$ is of the form $N(z_t) = F(z(t + \tau_1), z(t + \tau_2))$, then any value of $a$ and $b$ in (1.8) can be realized by means of center manifold reduction of (1.7). However, Faria and Magalhães show that, in fact, any value of $a$ in the determining $\dot{\rho}$ equation of (1.8) can be generically realized if $N(z_t)$ only involves one delay, i.e. $N(z_t) = F(z(t + \tau_1))$. Similarly, they show that the versal unfolding

$$\dot{\rho} = \lambda \rho + a\rho^3$$

of the $\dot{\rho}$ equation in (1.8) can be realized by a RFDE of the form $\dot{z}(t) = L(z_t) + \lambda z(t + \tau_1) + F(z(t + \tau_1))$.

For the $(0, \pm i\omega)$ case, the center manifold is three-dimensional. Normal form changes of coordinates and the use of center manifold coordinates $x_1$, $x_2 = \rho \cos \theta$, $x_3 = \rho \sin \theta$ yield the following equations (to quadratic order)

$$\begin{align*}
\dot{x}_1 &= b_1 x_1^2 + b_2 \rho^2 \\
\dot{\rho} &= a_4 x_1 \rho \\
\dot{\theta} &= \omega + O(|x_1, \rho|^2).
\end{align*}$$
If the coefficients $b_1$, $b_2$ and $a_1$ satisfy certain generic non-degeneracy conditions, then the higher-order terms have no qualitative effects. The bifurcation is thus characterized by the $\dot{x}_1, \dot{\rho}$ subsystem. In this case, Faria and Magalhães show that any value of $b_1$, $b_2$ and $a_1$ can be realized by center manifold reduction of an RFDE involving only 2 delays (which is less than the predicted value ($j = m = 3$) from the solution to the realizability problem [16, 17]).

Finally, a similar result was shown for the non-resonant double Hopf bifurcation in [4]: whereas the solution of the realizability problem [16, 17] predicts that $j = m = 4$ delays are sufficient, it is shown in [4] that 2 delays are sufficient to realize, to cubic order, the “radial part” $(\dot{\rho}_1, \dot{\rho}_2)$ of the center manifold equations

\[
\begin{align*}
\dot{\rho}_1 &= (\mu_1 + a_{11}\rho_1^2 + a_{12}\rho_2^2)\rho_1 \\
\dot{\rho}_2 &= (\mu_2 + a_{21}\rho_1^2 + a_{22}\rho_2^2)\rho_2 \\
\dot{\theta}_1 &= \omega_1 + O(|\mu_1, \mu_2|, |\rho_1, \rho_2|^2) \\
\dot{\theta}_2 &= \omega_2 + O(|\mu_1, \mu_2|, |\rho_1, \rho_2|^2).
\end{align*}
\]

In all three cases above, it is also shown that this smaller number of delays (1 in the simple Hopf case and 2 in both the $(0, \pm i\omega)$ and $(\pm i\omega_1, \pm i\omega_2)$ cases) is optimal, in the sense that anything less will lead to restrictions on realizability of the various coefficients which appear in these normal forms, and consequently to restrictions on the possible phase portraits in the classification of the versal unfolding of these respective singularities.

Overview:

The questions of realizability and restriction for normal forms and unfoldings of bifurcations in RFDEs are particularly important from a modeling point of view. Indeed, given a specific RFDE model (perhaps depending on many parameters) undergoing a local bifurcation, it is important to be able to characterize the range of possible dynamics accessible from within the model near the bifurcation point. From our discussion above, we see that knowledge of the abstract finite-dimensional bifurcation problem and its unfoldings is not sufficient in general to answer this question. Indeed, the specific form of the RFDE may restrict this range of possible dynamics. Depending on the functional form of the RFDE (e.g. how many distinct delays are involved), some phase diagrams which are possible in the unfolding of this given bifurcation may not be realizable in the RFDE. This could have important consequences in the interpretation of the model, especially as it pertains to the actual phenomenon being modeled.

The purpose of this paper is to further study these issues of realizability and restrictions, in light of the previously discussed results in [11] and [4]. We will develop a unified theoretical
framework for these results, which will consequently allow for considerable generalizations of these results.

Specifically, we will exploit and generalize the following common elements of the three specific cases studied in [11] and [4]: it is possible to make a canonical choice of normal form transformations on the center manifold equations which lead to a normal form with nice symmetry properties – it is equivariant with respect to an action of a torus group. This toroidal equivariance can be used to achieve a partial decoupling of the normal form into a “radial part” and an “angular part”. In many cases (in particular, in the three specific cases studied in [11] and [4]), the radial part characterizes the essential features of the bifurcation. It is then reasonable to

\begin{equation}
\text{investigate the realizability problem for the radial part of the normal form and its unfoldings},
\end{equation}

which is the goal we seek. Along with developing a theoretical framework to achieve this goal, we will in fact make the following generalizations to the results of [11] and [4]:

- we will assume that the spectrum of the matrix $B$ consists of simple eigenvalues, and has one of the two following forms

\begin{equation}
\text{spec}(B) = \{\pm i\omega_1, \ldots, \pm i\omega_p\} \quad \text{or} \quad \text{spec}(B) = \{0, \pm i\omega_1, \ldots, \pm i\omega_p\},
\end{equation}

(1.9)

where $\omega_1, \ldots, \omega_p > 0$ are independent over the rationals,

- we will investigate realizability of the radial part of the normal form to any order, and not just quadratic or cubic. This is important in cases where nonlinear degeneracies are present.

Similarly to [4, 11], we will limit our analysis to the case of scalar RFDEs. While studying the realizability problem in the context of general $n > 1$ dimensional systems of RFDEs is certainly very important, our computations indicate that there is an enormous increase in algebraic complexity involved. Consequently, a unified concise framework allowing for the simultaneous analysis of all cases (scalar and systems) appears at this point to be a difficult, albeit not impossible, goal to achieve.

In the second case for the spectrum of the matrix $B$ in (1.9), there is a technical subtlety which arises as it pertains to the possible unfoldings of this singularity. In fact, there are two algebraically different ways to construct an unfolding, depending on whether the “steady-state” mode (corresponding to the 0 eigenvalue) in the interaction is of saddle-node type or of transcritical type. It turns out that the transcritical case can be treated in the same framework as the first case for $\text{spec}(B)$ in (1.9), but it would be extremely cumbersome to attempt to treat the saddle-node case within this same framework. Therefore, we have chosen to treat the saddle-node case in a separate paper [5].
This paper is organized as follows. In Section 2, we will give a brief review of the theory of the center manifold reduction and normal form transformations of RFDEs as developed by Faria and Magalhães in [8, 9]. In Section 3, we will review how to make a canonical choice of normal form which possesses useful toroidal symmetry properties, and exploit this symmetry to achieve a partial decoupling of the normal form into a “radial part” and an “angular part”. The radial part possesses residual reflectional-type symmetry, which will be important for the subsequent analysis. In Section 4, we set a framework and establish an important surjectivity result which will be crucial to study the realizability problem for the radial part. Our main results on realizability and restrictions are given in Section 5. In this section, we also give some results which hint at the rudiments of a singularity theory for RFDEs. In Section 6 we show how the specific results of [11] and [4] are recovered by our main results. We end with some concluding remarks in Section 7. Some of the proofs in Section 3 are relegated to the appendices.

2 Functional Analytic Framework

In this section we will briefly recall some standard results and terminology in the bifurcation theory of RFDEs in order to establish the notation. For more details, see [8, 9, 18].

2.1 Infinite dimensional parameterized ODE

Suppose $r > 0$ is a given real number and $C = C([-r,0], \mathbb{R})$ is the Banach space of continuous functions from $[-r,0]$ into $\mathbb{R}$ with supremum norm. We define $z_t \in C$ as $z_t(\theta) = z(t+\theta), \quad -r \leq \theta \leq 0$. Let us consider the following parameterized family of nonlinear retarded functional differential equations

$$\dot{z}(t) = L(\mu)z_t + F(z_t, \mu),$$

(2.1)

where $L : C \times \mathbb{R}^s \to \mathbb{R}$ is a parameterized family of bounded linear operators from $C$ into $\mathbb{R}$ and $F$ is a smooth function from $C \times \mathbb{R}^s$ into $\mathbb{R}$. In this paper, we will assume the following hypothesis on $F$

Hypothesis 2.1 $F(0,0) = 0$ and $DF(0,0) = 0$.

A consequence of Hypothesis 2.1 is that in a Taylor expansion of (2.1), there are no terms which are $z_t$ independent and linear in $\mu$. While this is not a restriction in one of the cases we will be studying in this paper (multiple non-resonant Hopf bifurcation), it is a restriction in the other case (steady-state/multiple non-resonant Hopf interaction). Note however that
Hypothesis 2.1 includes as a special case the physically interesting case in which \( z = 0 \) is an equilibrium for (2.1) for all \( \mu \), which is the case of interaction between a transcritical bifurcation and multiple non-resonant Hopf bifurcation (see [21]). In a sequel to this paper, we will relax Hypothesis 2.1 to simply \( F(0,0) = 0 \) and \( D_1 F(0,0) = 0 \), which is the generic saddle-node case. This relaxation of Hypothesis 2.1 leads to some technical complications which would make a unified treatment of both cases simultaneously extremely cumbersome and lengthy. Therefore, for the sake of clarity, we have decided to treat these two cases separately (see [5]).

The bounded linear map \( L(\mu) \) can be represented in an integral form as
\[
L(\mu)\phi = \int_{-r}^{0} [d\eta_\mu (\theta)] \phi (\theta),
\]
where \( \eta_\mu (\theta) \) is a measurable function on \([-r, 0]\). Denote \( L_0 \equiv L(0) \), and rewrite (2.1) as
\[
\dot{z}(t) = L_0 z_t + (L(\mu) - L_0) z_t + F(z_t, \mu) = L_0 z_t + \tilde{F}(z_t, \mu),
\]
where \( \tilde{F}(z_t, \mu) = (L(\mu) - L_0) z_t + F(z_t, \mu) \).

Let \( A(\mu) \) be the infinitesimal generator of the flow for the linear system \( \dot{z} = L(\mu) z_t \), with spectrum \( \sigma(A(\mu)) \), and denote by \( \Lambda_\mu \) the set of eigenvalues of \( \sigma(A(\mu)) \) with zero real part.

The set \( \Lambda_0 \), which consists of the roots of the characteristic equation
\[
det \Delta(z) = 0, \quad \Delta(z) = z - \int_{-r}^{0} [d\eta_0 (\theta)] e^{z\theta},
\]
with zero real part will play an important role.

**Hypothesis 2.2** Throughout the rest of the paper, we assume the following hypotheses on \( \Lambda_\mu \) and \( \Lambda_0 \):

(a) \( \text{Card}(\Lambda_\mu) < \text{Card}(\Lambda_0) \) for \( \mu \) small,

(b) Each element of \( \Lambda_0 \) is a simple eigenvalue of \( A(0) \), and \( \Lambda_0 \) has one of the following two forms:

\[
\Lambda_0 = \{ \pm i\omega_1, \ldots, \pm i\omega_p \} \quad (\text{multiple non-resonant Hopf bifurcation}), \text{ or}
\]
\[
\Lambda_0 = \{ 0, \pm i\omega_1, \ldots, \pm i\omega_p \} \quad (\text{steady-state/multiple non-resonant Hopf interaction}),
\]
where \( \omega_1, \ldots, \omega_p \) are independent over the rationals, i.e. if \( r_1, \ldots, r_p \) are rational numbers such that \( \sum_{j=1}^{p} r_j \omega_j = 0 \), then \( r_1 = \cdots = r_p = 0 \).
Let $P$ be the invariant subspace for $A_0 \equiv A(0)$ associated with the eigenvalues in $\Lambda_0$, and let $\Phi = (\varphi_1 \ldots \varphi_m)$ be a matrix whose columns form a basis for $P$.

In a similar manner, we can define an invariant space, $P^*$, to be the generalized eigenspace of the transposed system, $A_0^T$ associated with $\Lambda_0$, having as basis the rows of the matrix $\Psi = \text{col}(\psi_1,\ldots,\psi_m)$. Note that the transposed system, $A_0^T$ is defined over a dual space $C^* = C([0,r], \mathbb{R})$, and each element of $\Psi$ is included in $C^*$. The bilinear form between $C^*$ and $C$ is defined as

$$(\psi, \phi) = \psi(0)\phi(0) - \int_0^0 \int_{-r}^0 \psi(\zeta - \theta) \left[ d\eta_0(\theta) \right] \phi(\zeta) \, d\zeta.$$  (2.4)

Note that $\Phi$ and $\Psi$ satisfy $\dot{\Phi} = B\Phi$, $\dot{\Psi} = -\Psi B$, where $B$ is an $m \times m$ matrix whose spectrum coincides with $\Lambda_0$.

We can normalize $\Psi$ such that $(\Psi, \Phi) = I$, and we can decompose the space $C$ using the splitting $C = P \oplus Q$, where the complimentary space $Q$ is also invariant for $A_0$.

Faria and Magalhães [8, 9] show that (2.1) can be written as an infinite dimensional ordinary differential equation on the Banach space $BC$ of functions from $[-r,0]$ into $\mathbb{R}$ which are uniformly continuous on $[-r,0)$ and with a jump discontinuity at 0, using a procedure that we will now outline. Define $X_0$ to be the function

$$X_0(\theta) = \begin{cases} 1 & \theta = 0 \\ 0 & -r \leq \theta < 0, \end{cases}$$

then the elements of $BC$ can be written as $\xi = \varphi + X_0\lambda$, with $\varphi \in C$ and $\lambda \in \mathbb{R}$, so that $BC$ is identified with $C \times \mathbb{R}$.

Let $\pi : BC \rightarrow P$ denote the projection

$$\pi(\varphi + X_0\lambda) = \Phi[(\Psi, \varphi) + \Psi(0)\lambda],$$

where $\varphi \in C$ and $\lambda \in \mathbb{R}$. We now decompose $z_t$ in (2.1) according to the splitting

$$BC = P \oplus \ker \pi,$$

with the property that $Q \subset \ker \pi$, and get the following infinite-dimensional ODE system which is equivalent to (2.1):

$$\dot{x} = Bx + \Psi(0) [(L(\mu) - L_0)(\Phi x + y) + F(\Phi x + y, \mu)]$$

$$\frac{d}{dt}y = A_{Q^*}y + (I - \pi)X_0 [(L(\mu) - L_0)(\Phi x + y) + F(\Phi x + y, \mu)],$$  (2.5)
where \( x \in \mathbb{R}^m \), \( y \in Q^1 \equiv Q \cap C^1 \), \((C^1)\) is the subset of \( C \) consisting of continuously differentiable functions), and \( A_{Q^1} \) is the operator from \( Q^1 \) into \( \ker \pi \) defined by

\[
A_{Q^1} \varphi = \dot{\varphi} + X_0 [L \varphi - \dot{\varphi}(0)].
\]

### 2.2 Faria and Magalhães normal form

Consider the formal Taylor expansion of the nonlinear terms \( \tilde{F} \) in (2.2)

\[
\tilde{F}(u, \mu) = \sum_{j \geq 2} \tilde{F}_j(u, \mu), \quad u \in C, \ \mu \in \mathbb{R}^s,
\]

where \( \tilde{F}_j(w) = H_j(w, \ldots, w) \), with \( H_j \) belonging to the space of continuous multilinear symmetric maps from \((C \times \mathbb{R}^s) \times \cdots \times (C \times \mathbb{R}^s) \) \((j \) times) to \( \mathbb{R} \). If we denote \( f_j = (f^1_j, f^2_j) \), where

\[
f^1_j(x, y, \mu) = \Psi(0) \tilde{F}_j(\Phi(x + y, \mu),
\]

\[
f^2_j(x, y, \mu) = (I - \pi) X_0 \tilde{F}_j(\Phi(x + y, \mu),
\]

then (2.5) can be written as

\[
\begin{align*}
\dot{x} &= Bx + \sum_{j \geq 2} f^1_j(x, y, \mu) \\
\frac{d}{dt} y &= A_{Q^1} y + \sum_{j \geq 2} f^2_j(x, y, \mu)
\end{align*}
\]  \( (2.6) \)

The spectral hypotheses we have specified in Hypothesis 2.2 are sufficient to conclude that the non-resonance condition of Faria and Magalhães [8, 9] holds. Consequently, using successively at each order \( j \) a near identity change of variables of the form

\[
(x, y) = (\hat{x}, \hat{y}) + U_j(\hat{x}, \mu) \equiv (\hat{x}, \hat{y}) + (U^1_j(\hat{x}, \mu), U^2_j(\hat{x}, \mu)),
\]  \( (2.7) \)

(where \( U^1, U^2 \) are homogeneous degree \( j \) polynomials in the indicated variables, with coefficients respectively in \( \mathbb{R}^m \) and \( Q^1 \)) system (2.6) can be put into formal normal form

\[
\begin{align*}
\dot{x} &= Bx + \sum_{j \geq 2} g^1_j(x, y, \mu) \\
\frac{d}{dt} y &= A_{Q^1} y + \sum_{j \geq 2} g^2_j(x, y, \mu)
\end{align*}
\]  \( (2.8) \)
such that the center manifold is locally given by $y = 0$ and the local flow of (2.1) on this center manifold is given by
\[ \dot{x} = Bx + \sum_{j \geq 2} g_j^1(x, 0, \mu). \] (2.9)

The nonlinear terms in (2.9) are in normal form in the classical sense with respect to the matrix $B$.

### 3 Bifurcations with Toroidal Normal Forms

With equation (2.9) in mind, in this section we will discuss normal form transformations of the general parameterized system
\[
\begin{align*}
\dot{x} &= Bx + f(x, \mu) \\
\dot{\mu} &= 0,
\end{align*}
\] (3.1)

where the spectrum $\Lambda_0$ of the matrix $B$ is as in Hypothesis 2.2(b). As much as possible, we will treat both cases of Hypothesis 2.2(b) (i.e. whether or not $\Lambda_0$ includes 0) simultaneously by adopting a notation which uses integers $\kappa$ and $d$, which should be interpreted as having the values $\kappa = 2p$ and $d = p$ in the case where $\Lambda_0 = \{\pm i\omega_1, \ldots, \pm i\omega_p\}$, and the values $\kappa = 2p + 1$ and $d = p + 1$ in the case where $\Lambda_0 = \{0, \pm i\omega_1, \ldots, \pm i\omega_p\}$.

It will be extremely useful to use complex coordinates for the last $2p$ components of the space $\mathbb{R}^\kappa$, so that we can identify
\[
\begin{align*}
\mathbb{R}^\kappa &= \begin{cases} 
\{ (x_1, \overline{x_1}, \ldots, x_p, \overline{x_p}) \mid x_j \in \mathbb{C}, j = 1, \ldots, p \} & \text{if } \kappa = 2p \\
\{ (x_0, x_1, \overline{x_1}, \ldots, x_p, \overline{x_p}) \mid x_0 \in \mathbb{R}, x_j \in \mathbb{C}, j = 1, \ldots, p \} & \text{if } \kappa = 2p + 1.
\end{cases}
\end{align*}
\]

Then, without loss of generality, we may assume that
\[
B = \text{diag}(i\omega_1, -i\omega_1, \ldots, i\omega_p, -i\omega_p) \quad \text{or} \quad B = \text{diag}(0, i\omega_1, i\omega_1, \ldots, i\omega_p, -i\omega_p) \tag{3.2}
\]
depending on which case of Hypothesis 2.2(b) is being considered.

At times, it will be convenient to write (3.1) as
\[
\dot{\tilde{x}} = \tilde{B} \tilde{x} + \tilde{f}(\tilde{x}), \tag{3.3}
\]
where $\tilde{x} = (x, \mu)$, $\tilde{f} = (f, 0)$ and
\[
\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}
\]
The section is divided into four subsections. In the first, we will give a brief review of results on symmetric normal forms with parameters. Most (if not all) of these results are largely well-known in the unparameterized case (see for example [7, 13]), and only minor modifications are required to obtain the parameterized versions we present herein.

In the second subsection, we will define an equivariant projection operator which will be useful in the computation of symmetric normal forms.

In the third subsection, we will specify how the symmetry of these normal forms can be exploited in order to achieve a partial decoupling of the normal form.

Finally, in the fourth subsection, we will introduce a splitting of our spaces of polynomials which naturally decomposes any vector field into a singular parameter independent part plus a perturbation.

3.1 Normal forms and toroidal symmetry

For $\tilde{B}$ as in (3.4), let $\tilde{B}^t$ denote the transpose of $\tilde{B}$ and let $\Gamma = \{ e^{s\tilde{B}^t} \mid s \in \mathbb{R} \}$ (where the closure is taken in the space of $(\kappa + s) \times (\kappa + s)$ matrices), and note that $\Gamma$ is an abelian connected Lie group isomorphic to $T_p$, where $T_p$ is the $p$-torus:

$$T_p = \left\{ \text{diag}(1, e^{i\theta_1}, \ldots, e^{i\theta_p}) \mid \theta_j \in S^1, j = 1, \ldots, p \right\} \text{ if } \kappa = 2p + 1$$

$$T_p = \left\{ \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_p}, 1, \ldots, 1) \mid \theta_j \in S^1, j = 1, \ldots, p \right\} \text{ if } \kappa = 2p$$

(3.5)

Definition 3.1 For a given integer $\ell \geq 2$, a given normed space $X$, and for $\kappa = 2p$ (respectively $\kappa = 2p + 1$), we denote by $H^{\kappa+s}_\ell(X)$ the linear space of homogeneous polynomials of degree $\ell$ in the $\kappa+s$ variables $x = (x_1, \overline{x_1}, \ldots, x_p, \overline{x_p})$ (respectively $x = (x_0, x_1, \overline{x_1}, \ldots, x_p, \overline{x_p})$) and $\mu = (\mu_1, \ldots, \mu_s)$ with coefficients in $X$. For $X = \mathbb{R}^{\kappa+s}$, define $H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}, \Gamma) \subset H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s})$ to be the subspace of $\Gamma$-equivariant maps, i.e.

$$\tilde{f} \in H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}, \Gamma) \iff \tilde{f} \in H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}) \text{ and } \gamma \tilde{f}(\gamma^{-1} \tilde{x}) = \tilde{f}(\tilde{x}), \forall \tilde{x} = (x, \mu) \in \mathbb{R}^{\kappa+s}, \forall \gamma \in \Gamma.$$

(3.6)

For the general class of near-identity changes of variables $\tilde{x} \mapsto \tilde{x} + h(\tilde{x})$ for (3.3), it is well-known that we can eliminate from (3.3) all nonlinear terms which are in the range of the homological operator

$$\mathcal{L}_{\tilde{B}} : H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}) \longrightarrow H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s})$$

$$\tilde{f} \mapsto (\mathcal{L}_{\tilde{B}} \tilde{f})(\tilde{x}) = D\tilde{f}(\tilde{x})\tilde{B}\tilde{x} - \tilde{B}\tilde{f}(\tilde{x}).$$

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Thus, we must define in $H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s})$ a complimentary space to range $\mathcal{L}_B$. Of course, such a space is not unique. However, there exists a nice canonical choice which will be extremely useful for our purposes (see for example [7, 13]).

**Proposition 3.2**

$$H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}) = H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}, \Gamma) \oplus \text{range } \mathcal{L}_B$$

The usefulness of Proposition 3.2 is that it is straightforward to compute the general element of $H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}, \Gamma)$.

**Lemma 3.3** Let $\tilde{B}$ be as in (3.3). Then a smooth vector field $\tilde{f} : \mathbb{R}^{\kappa+s} \rightarrow \mathbb{R}^{\kappa+s}$ is $\Gamma^p$-equivariant if and only if $\tilde{f}$ has one of the following forms

$$\tilde{f}(x, \mu) = \begin{cases} 
\left( \frac{a_1(x_1x_1, \ldots, x_px_p, \mu) x_1}{a_1(x_1x_1, \ldots, x_px_p, \mu) x_1} \\
\vdots \\
\frac{a_p(x_1x_1, \ldots, x_px_p, \mu) x_p}{a_p(x_1x_1, \ldots, x_px_p, \mu) x_p} \\
b_1(x_1x_1, \ldots, x_px_p, \mu) \\
\vdots \\
b_s(x_1x_1, \ldots, x_px_p, \mu),
\end{cases}$$

or

$$\begin{cases} 
\left( \frac{a_0(x_0, x_1x_1, \ldots, x_px_p, \mu)}{a_1(x_0, x_1x_1, \ldots, x_px_p, \mu) x_1} \\
\vdots \\
\frac{a_p(x_0, x_1x_1, \ldots, x_px_p, \mu) x_p}{a_p(x_0, x_1x_1, \ldots, x_px_p, \mu) x_p} \\
b_1(x_0, x_1x_1, \ldots, x_px_p, \mu) \\
\vdots \\
b_s(x_0, x_1x_1, \ldots, x_px_p, \mu),
\end{cases}$$

respectively if $\kappa = 2p$ or $\kappa = 2p + 1$, where $a_1, \ldots, a_p$ are smooth and complex-valued, and $a_0, b_1, \ldots b_s$ are smooth and real-valued.

**Proof** This is a standard result which is a consequence of Schwarz lemma [23]. See also [13].

Proposition 3.2 and Lemma 3.3 are not exactly in a form suitable for our purposes, since the vector field $\tilde{f}$ in (3.3) has the special form $\tilde{f} = (f, 0)$ which we require our normal form
changes of variables to preserve. Since we are only interested in the first $\kappa$ components of (3.3), we would like to obtain a splitting of $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa})$ akin to the splitting of $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa+s})$ in Proposition 3.2. For this purpose, we will need the following

Definition 3.4

(a) We define $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}, T_p)$ to be the subset of $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa})$ consisting of mappings $f : \mathbb{R}^{\kappa+s} \rightarrow \mathbb{R}^{\kappa}$ whose components are of the form of the first $\kappa$ components of (3.7). Note that $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}, T_p)$ consists precisely of the $T_p$-equivariant elements of $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa})$; that is,

$$f \in H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}, T_p) \iff f \in H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}) \text{ and } f(\gamma_0 x, \mu) = \gamma_0 f(x, \mu), \forall \gamma_0 \in \Gamma_0, \forall (x, \mu) \in \mathbb{R}^{\kappa+s},$$

where $\Gamma_0$ is the group of $\kappa \times \kappa$ matrices which is isomorphic to $T_p$, and is parameterized as

$$\Gamma_0 = \left\{ \begin{array}{ll}
\{ \text{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}) & | \theta_j \in \mathbb{S}^1, j = 1, \ldots, p \} & \text{if } \kappa = 2p \\
\{ \text{diag}(1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}) & | \theta_j \in \mathbb{S}^1, j = 1, \ldots, p \} & \text{if } \kappa = 2p + 1,
\end{array} \right. \quad (3.8)$$

(b) We define the following operator

$$\mathcal{L}_B : H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}) \rightarrow H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa})$$

$$f \mapsto (\mathcal{L}_B(f))(x, \mu) = D_x f(x, \mu) B x - B f(x, \mu). \quad (3.9)$$

Note that $\mathcal{L}_B(f, 0) = (\mathcal{L}_B f, 0)$.

Proposition 3.5

$$H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}) = H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa}, T_p) \oplus \text{range } \mathcal{L}_B.$$

Proof The proof is given in the appendix.

3.2 Equivariant projection

In this section, we will construct an appropriate linear projection associated with the splitting of $H_{\ell}^{\kappa+s}(\mathbb{R}^{\kappa})$ given in Proposition 3.5. This projection has very nice algebraic properties, and will be useful when we prove our main results later.
Definition 3.6  Let \( \int_{\Gamma_0} d\gamma \) denote the normalized Haar integral on \( \Gamma_0 \cong \mathbb{T}^p \) (see (3.8)). We define the linear operator

\[
A : H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \rightarrow H^{\kappa+s}_\ell(\mathbb{R}^\kappa)
\]

\[
f \mapsto (Af)(x,\mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x,\mu) d\gamma
\]

Proposition 3.7  \( A \) is a projection. Furthermore,

\[
\text{range } A = H^{\kappa+s}_\ell(\mathbb{R}^\kappa, \mathbb{T}^p) \quad (3.10)
\]

and

\[
\ker A = \text{range } \mathcal{L}_B \quad (3.11)
\]

Proof  The proof is given in the appendix.

Since \( A \) is a projection, then \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) = \ker A \oplus \text{range } A \), and Proposition 3.7 shows that this decomposition is precisely the decomposition of \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \) given in Proposition 3.5. Thus, for any \( f \in H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \), write

\[
f = Af + (I - A)f,
\]

and note that \( Af \) is \( \mathbb{T}^p \)-equivariant and that \( (I - A)f \in \ker A \). From Proposition 3.7, there exists \( h \in H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \) such that \( \mathcal{L}_B h = (I - A)f \).

3.3 Phase decoupling

The following example serves as an illustration of a trivial (well-known) case in which normal form toroidal symmetry leads to a decoupling of the equations in the normal form.

Example 3.8  In the case where \( B = \text{diag}(i\omega, -i\omega) \), the normal form has the rotational symmetry of a one-dimensional torus: \( (x, \overline{x}) \rightarrow (e^{i\theta} x, e^{-i\theta} \overline{x}) \), \( \theta \in \mathbb{T}^1 \), and it is easy to verify that the most general \( \mathbb{T}^1 \)-equivariant differential equation has the form

\[
\dot{x} = f(x, \overline{x}) x, \quad (x \in \mathbb{C}) \quad (3.12)
\]

and its complex conjugate where \( f \) is complex-valued, and \( f(0) = i\omega \). Writing \( x = re^{i\theta} \) leads to the equations

\[
\dot{r} = \text{Re}(f(r^2)) r, \quad \dot{\theta} = \text{Im}(f(r^2)). \quad (3.13)
\]
We note that $\theta$ does not appear in the $\dot{r}$ equation, and that the $\dot{r}$ equation has a reflectional symmetry $r \to -r$. The analysis of the normal form (3.12) then essentially reduces to a one-dimensional problem (the $\dot{r}$ equation in (3.13)) which possesses some residual (reflectional) symmetry.

In fact, this example is a special case of a more general result which holds when the spectrum of $B$ satisfies Hypothesis 2.2(b), and which we now outline.

From Lemma 3.3, we get the following

**Corollary 3.9** Suppose the spectrum of $B$ satisfies Hypothesis 2.2(b). Then a smooth vector field $f : \mathbb{R}^{\kappa+s} \to \mathbb{R}^{\kappa}$ with $f(0, 0) = 0$, $Df(0, 0) = 0$ is $\Gamma_0 \simeq T^p$-equivariant if and only if $f$ has the form

$$f(x, \mu) = \begin{cases} 
\begin{pmatrix} a_1(x_1, \ldots, x_p, \mu) x_1 \\ a_1(x_1, \ldots, x_p, \mu) x_1 \\ \vdots \\ a_p(x_1, \ldots, x_p, \mu) x_p \\ a_p(x_1, \ldots, x_p, \mu) x_p \end{pmatrix} & \text{if } \kappa = 2p \\
\begin{pmatrix} a_0(x_0, x_1, \ldots, x_p, \mu) \\ a_1(x_0, x_1, \ldots, x_p, \mu) x_1 \\ a_1(x_0, x_1, \ldots, x_p, \mu) x_1 \\ \vdots \\ a_p(x_0, x_1, \ldots, x_p, \mu) x_p \\ a_p(x_0, x_1, \ldots, x_p, \mu) x_p \end{pmatrix} & \text{if } \kappa = 2p + 1,
\end{cases}
$$

(3.14)

where $a_1, \ldots, a_p$ are smooth and complex-valued, and $a_0$ is smooth and real-valued.

**Proposition 3.10** Consider a differential equation $\dot{x} = Bx + f(x, \mu)$, where $f$ is as in (3.14). Then under the under the change of variables $x_0 = \rho_0$, $x_j = \rho_j e^{i\theta}$, $j = 1, \ldots, p$, this
The differential equation transforms into

\[ \dot{\rho}_j = \text{Re}(a_j(\rho_1^2, \ldots, \rho_p^2, \mu)) \rho_j, \quad j = 1, \ldots, p \] if \( \kappa = 2p \)

\[ \begin{align*}
\dot{\rho}_0 &= a_0(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \\
\dot{\rho}_j &= \text{Re}(a_j(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)) \rho_j, \quad j = 1, \ldots, p
\end{align*} \] if \( \kappa = 2p + 1 \),

(3.15)

and

\[ \begin{align*}
\dot{\theta}_j &= \text{Im}(a_j(\rho_1^2, \ldots, \rho_p^2, \mu)), \quad j = 1, \ldots, p \quad \text{if} \quad \kappa = 2p \\
\dot{\theta}_j &= \text{Im}(a_j(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)), \quad j = 1, \ldots, p \quad \text{if} \quad \kappa = 2p + 1.
\end{align*} \] (3.16)

**Proof**  This is a simple computation.

We will call the subsystem (3.15) the *uncoupled radial part* of the normal form (3.14). For many practical purposes of interest, it is sufficient to consider only the uncoupled radial part (3.15) in the analysis of (3.14). For example, small-amplitude equilibria of (3.15) correspond to periodic solutions or invariant tori of the full normal form (3.14). Oftentimes, given some normal hyperbolicity conditions, these invariant objects for (3.14) persist as invariant objects in the original system (3.1). In fact, in the case of non-resonant double Hopf bifurcation (\( \Lambda_0 = \{ \pm i\omega_1, \pm i\omega_2 \} \)) and in the case of saddle-node/Hopf interaction (\( \Lambda_0 = \{ 0, \pm i\omega_1 \} \)), it is well-known [26] that given some generic non-degeneracy conditions on the coefficients of the lower-order nonlinearities, the radial equations (3.15) (suitably truncated) completely determine the dynamics in the full system (3.1) up to topological equivalence. So, it is reasonable to investigate the realizability of the uncoupled radial part (3.15) by center manifold reduction (2.9) of the RFDE (2.1).

We now introduce an integer \( d \) which should be interpreted such that \( d = p \) in the case where \( \Lambda_0 = \{ \pm i\omega_1, \ldots, \pm i\omega_p \} \), and \( d = p + 1 \) in the case where \( \Lambda_0 = \{ 0, \pm i\omega_1, \ldots, \pm i\omega_p \} \). Denote by \( \mathbb{Z}_{2,p} \) the group whose action on \( \mathbb{R}^d \) is given by

\[ (\rho_1, \ldots, \rho_p) \rightarrow (\lambda_1 \rho_1, \ldots, \lambda_p \rho_p) \quad \text{if} \quad d = p \]

\[ (\rho_0, \rho_1, \ldots, \rho_p) \rightarrow (\rho_0, \lambda_1 \rho_1, \ldots, \lambda_p \rho_p) \quad \text{if} \quad d = p + 1, \]

(3.17)

where \( \lambda_j \in \{ 1, -1 \}, \quad j = 1, \ldots, p. \)

**Definition 3.11**  For a given integer \( \ell \geq 2 \), a given normed space \( X \), and for \( d = p \) (respectively \( d = p + 1 \)), we denote by \( \mathcal{H}^{d+s}_\ell(X) \) the linear space of homogeneous polynomials of degree \( \ell \) in the \( d + s \) variables \( \rho = (\rho_1, \ldots, \rho_p) \) (respectively \( \rho = (\rho_0, \rho_1, \ldots, \rho_p) \)) and \( \mu = (\mu_1, \ldots, \mu_s) \) with coefficients in \( X \). Denote by \( \mathcal{H}^{d+s}_\ell(\mathbb{R}^d, \mathbb{Z}_{2,p}) \subset \mathcal{H}^{d+s}_\ell(\mathbb{R}^d) \) the subspace of \( \mathcal{H}^{d+s}_\ell(\mathbb{R}^d) \) consisting of \( \mathbb{Z}_{2,p} \)-equivariant polynomials.
It is easy to show (see [13]) that the most general element of $H_{d+s}^\ell(\mathbb{R}^d, \mathbb{Z}_{2,p})$ has the form

\[
\begin{pmatrix}
  h_1(\rho_1^2, \ldots, \rho_p^2, \mu) \rho_1 \\
  \vdots \\
  h_p(\rho_1^2, \ldots, \rho_p^2, \mu) \rho_p,
\end{pmatrix}
\quad \text{if } d = p
\]

\[
\begin{pmatrix}
  h_0(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \\
  h_1(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \rho_1 \\
  \vdots \\
  h_p(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \rho_p,
\end{pmatrix}
\quad \text{if } d = p + 1
\]

and one immediately notices the similarity with (3.15). It then becomes useful to define the following surjective linear mapping

\[
\Pi : H_{d+s}^\ell(\mathbb{R}^\kappa, \mathbb{T}_p^\ell) \longrightarrow H_{d+s}^\ell(\mathbb{R}^d, \mathbb{Z}_{2,p})
\]

which is defined by sending the general element (3.14) of $H_{d+s}^\ell(\mathbb{R}^\kappa, \mathbb{T}_p^\ell)$ to the following element of $H_{d+s}^\ell(\mathbb{R}^d, \mathbb{Z}_{2,p})$:

\[
\begin{pmatrix}
  \text{Re}(a_1(\rho_1^2, \ldots, \rho_p^2, \mu)) \rho_1 \\
  \vdots \\
  \text{Re}(a_p(\rho_1^2, \ldots, \rho_p^2, \mu)) \rho_p
\end{pmatrix}
\quad \text{if } d = p
\]

\[
\begin{pmatrix}
  a_0(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \\
  \text{Re}(a_1(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)) \rho_1 \\
  \vdots \\
  \text{Re}(a_p(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)) \rho_p
\end{pmatrix}
\quad \text{if } d = p + 1.
\]

The following characterization of the mapping $\Pi$ will be very useful later for computational purposes: if $G$ is an element of $H_{d+s}^\ell(\mathbb{R}^\kappa, \mathbb{T}_p)$, then

\[
(\Pi G)(\rho, \mu) = C \cdot \gamma \cdot G(\gamma^{-1} \cdot R, \mu),
\]

(3.20)
where $\gamma$ is any fixed element of $\Gamma_0$, $R = (p_1, p_1, \ldots, p_p, p_p)$ if $d = p$ and $R = (p_0, p_1, \ldots, p_p, p_p)$ if $d = p + 1$, and where $C$ is the following $d \times \kappa$ matrix

$$
C = \begin{cases}
\begin{pmatrix}
1/2 & 1/2 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & \cdots & \cdots & 0 & 0 \\
& \vdots & & & & & & \\
0 & 0 & 0 & 0 & \cdots & \cdots & 1/2 & 1/2
\end{pmatrix} & \text{if } d = p, \kappa = 2p \\
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 1/2 & 1/2 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & \cdots & \cdots & 0 & 0 \\
& \vdots & & & & & & \\
0 & 0 & 0 & 0 & \cdots & \cdots & 1/2 & 1/2
\end{pmatrix} & \text{if } d = p + 1, \kappa = 2p + 1.
\end{cases}
$$

(3.21)

### 3.4 Parameter splitting

There is a canonical direct sum decomposition of $H^{\kappa^+s}_\ell(X)$ which will turn out to be quite useful for our purposes. Note that $H^{\kappa^+s}_\ell(X)$ contains $H^\kappa_\ell(X)$ (the $\mu$-independent polynomials) as a subspace, and consequently we can write

$$
H^{\kappa^+s}_\ell(X) = H^\kappa_\ell(X) \oplus P^{\kappa^+s}_\ell(X),
$$

(3.22)

where $q \in P^{\kappa^+s}_\ell(X)$ if and only if $q \in H^{\kappa^+s}_\ell(X)$ and $q(x, 0) = 0$.

The homological operator $L_B$ (see (3.9)) preserves the decomposition (3.22):

$$
L_B(H^\kappa_\ell(\mathbb{R}^\kappa)) \subset H^\kappa_\ell(\mathbb{R}^\kappa) \quad \text{and} \quad L_B(P^{\kappa^+s}_\ell(\mathbb{R}^\kappa)) \subset P^{\kappa^+s}_\ell(\mathbb{R}^\kappa).
$$

Moreover,

$$
H^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p) = H^\kappa_\ell(\mathbb{R}^\kappa, T^p) \oplus P^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p),
$$

(3.23)

where $H^\kappa_\ell(\mathbb{R}^\kappa, T^p) = H^\kappa_\ell(\mathbb{R}^\kappa) \cap H^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p)$ and $P^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p) = P^{\kappa^+s}_\ell(\mathbb{R}^\kappa) \cap H^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p)$.

If $L_B|_1$ and $L_B|_2$ represent respectively the restrictions of $L_B$ on $H^\kappa_\ell(\mathbb{R}^\kappa)$ and on $P^{\kappa^+s}_\ell(\mathbb{R}^\kappa)$, then we have the following refinement of Proposition 3.5:

**Proposition 3.12**

$$
H^\kappa_\ell(\mathbb{R}^\kappa) = H^\kappa_\ell(\mathbb{R}^\kappa, T^p) \oplus \text{range } L_B|_1,
$$

$$
P^{\kappa^+s}_\ell(\mathbb{R}^\kappa) = P^{\kappa^+s}_\ell(\mathbb{R}^\kappa, T^p) \oplus \text{range } L_B|_2.
$$
The equivariant projection operator $A$ defined in Definition 3.6 also preserves the decomposition (3.22), and we get the following refinement of Proposition 3.7

**Proposition 3.13**

\[
A(\mathcal{H}_\kappa^\ell(\mathbb{R}^\kappa)) = \mathcal{H}_\kappa^\ell(\mathbb{R}^\kappa), \quad A(\mathcal{P}_\kappa^{\ell+s}(\mathbb{R}^\kappa)) = \mathcal{P}_\kappa^{\ell+s}(\mathbb{R}^\kappa). \tag{3.24}
\]

If $A|_1$ and $A|_2$ represent respectively the restrictions of $A$ on $\mathcal{H}_\kappa^\ell(\mathbb{R}^\kappa)$ and on $\mathcal{P}_\kappa^{\ell+s}(\mathbb{R}^\kappa)$, then

\[
\ker A|_1 = \text{range } \mathcal{L}_B|_1
\]
\[
\ker A|_2 = \text{range } \mathcal{L}_B|_2 \tag{3.25}
\]

**Remark 3.14** We note that there exist similar direct sum decompositions of $\mathcal{H}_\kappa^{d+s}(\mathbb{R}^d)$ and of $\mathcal{H}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2)$ using the subspace $\mathcal{H}_\kappa^d(\mathbb{R}^d) \subset \mathcal{H}_\kappa^{d+s}(\mathbb{R}^d)$ of $\mu$-independent polynomials

\[
\mathcal{H}_\kappa^{d+s}(\mathbb{R}^d) = \mathcal{H}_\kappa^d(\mathbb{R}^d) \oplus \mathcal{P}_\kappa^{d+s}(\mathbb{R}^d)
\]
\[
\mathcal{H}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2) = \mathcal{H}_\kappa^d(\mathbb{R}^d, \mathbb{Z}_2) \oplus \mathcal{P}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2) \tag{3.26}
\]

where $q \in \mathcal{P}_\kappa^{d+s}(\mathbb{R}^d)$ if and only if $q \in \mathcal{H}_\kappa^{d+s}(\mathbb{R}^d)$ and $q(x,0) = 0$, and where $\mathcal{H}_\kappa^d(\mathbb{R}^d, \mathbb{Z}_2) = \mathcal{H}_\kappa^d(\mathbb{R}^d) \cap \mathcal{H}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2)$ and $\mathcal{P}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2) = \mathcal{P}_\kappa^{d+s}(\mathbb{R}^d) \cap \mathcal{H}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2)$.

Note that the mapping $\Pi$ defined in (3.18)-(3.20) preserves (3.23) and (3.26), i.e.

\[
\Pi(\mathcal{H}_\kappa^\ell(\mathbb{R}^\kappa, \mathbb{T}^p)) = \mathcal{H}_\kappa^d(\mathbb{R}^d, \mathbb{Z}_2) \quad \text{and} \quad \Pi(\mathcal{P}_\kappa^{\ell+s}(\mathbb{R}^\kappa, \mathbb{T}^p)) = \mathcal{P}_\kappa^{d+s}(\mathbb{R}^d, \mathbb{Z}_2).
\]

Combining the results of this section with the Faria and Magalhães normal form procedure described in section 2, we get the following version of Theorem 5.8 of [8] and Theorem 2.16 of [9] which is adapted for our purposes

**Theorem 3.15** Consider the system (2.6)

\[
\dot{x} = Bx + \sum_{j \geq 2} f^1_j(x, y, \mu)
\]
\[
\frac{d}{dt} y = A_Q^1 + \sum_{j \geq 2} f^2_j(x, y, \mu). \tag{3.27}
\]

Write

\[
f^1_j(x, 0, \mu) = h_j(x) + q_j(x, \mu), \tag{3.28}
\]
where \( h_j \in H^\kappa_j(\mathbb{R}^\kappa) \) and \( q_j \in P^{\kappa+s}_j(\mathbb{R}^\kappa) \). Then there is a formal near-identity change of variables

\[
(x,y) \mapsto (\hat{x}, \hat{y}) + (U^1(\hat{x}), U^2(\hat{x})) + (W^1(\hat{x}, \mu), W^2(\hat{x}, \mu))
\]

(where \( W^1(\hat{x}, 0) = 0, W^2(\hat{x}, 0) = 0 \)) which transforms (3.27) into system (2.8) (upon dropping the hats), and the flow on the invariant local center manifold \( y = 0 \) is given by

\[
\dot{x} = Bx + \sum_{j \geq 2} ((A|_1(h_j + Y_j))(x) + (A|_2(q_j + Z_j))(x,\mu))
\]

(3.29)

where \( Y_2 = 0, Z_2 = 0 \), and for \( j \geq 3 \), \( Y_j(x) \) and \( Z_j(x,\mu) \) are the extra contributions to the terms of order \( j \) coming from the transformation of the lower order \(( < j)\) terms, and \( Z_j(x, 0) = 0 \).

### 4 Realizability: linear analysis

In this section, we present the first of our main results on the realizability of the radial part (3.19) of toroidal normal forms (3.14) to any order for RFDEs (2.6) via the center-manifold normal form equations (3.29).

We will define a linear operator between suitable spaces of polynomials, which arises in the context of the normal form transformations of (2.6). Our main result in this section will be to establish the surjectivity of this operator. Surjectivity will be the main ingredient in the proof of our main realizability results which will be presented in the next section.

Again, we will try as much as possible to use concise notation which will allow for the simultaneous treatment of both cases for \( \Lambda_0 \) in Hypothesis 2.2(b).

#### 4.1 Preliminaries

For given integers \( p \geq 1 \) and \( \ell \geq 2 \), and for \( \kappa = 2p \) (respectively \( 2p+1 \)) and \( d = p \) (respectively \( p + 1 \)), recall that \( H^{d+s}_\ell(\mathbb{R}^d, \mathbb{Z}_{2p}) \) is the linear space of homogeneous polynomials of degree \( \ell \) in the \( d+s \) variables \( \rho = (\rho_1, \ldots, \rho_p) \) (respectively \( \rho = (\rho_0, \rho_1, \ldots, \rho_p) \equiv (\rho_0, \tilde{\rho}) \)) and \( \mu = (\mu_1, \ldots, \mu_s) \) with coefficients in \( \mathbb{R}^d \), and which are equivariant with respect to the \( \mathbb{Z}_{2p} \) action (3.17) on \( \mathbb{R}^d \). Recall also that \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \) is the space of homogeneous polynomials of degree \( \ell \) in the \( \kappa+s \) variables \( x = (x_1, x_1, \ldots, x_p, \tilde{x}_p) \) (respectively \( x = (x_0, x_1, \tilde{x}_1, \ldots, x_p, \tilde{x}_p) \)) and \( \mu = (\mu_1, \ldots, \mu_s) \) with coefficients in \( \mathbb{R}^\kappa \), and \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa, \mathbb{T}^p) \) is the subset of \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \) consisting of \( \mathbb{T}^p \)-equivariant mappings.

**Definition 4.1** Denote by \( V^{d+s}_\ell(\mathbb{R}) \subset H^{d+s}_\ell(\mathbb{R}) \) the subspace of homogeneous degree \( \ell \) polynomials in the \( d+s \) variables \( v = (v_1, \ldots, v_p) \) (respectively \( v = (v_0, v_1, \ldots, v_p) \equiv (v_0, \tilde{v}) \)).
\[ \mu = (\mu_1, \ldots, \mu_s) \] with real coefficients, spanned by the basis

\[
\begin{align*}
\{ \mu^d v_{2k}^c | c \in \{1, \ldots, p\}, \ (k, q) \in \mathbb{N}_0^{d+s}, |q| + 2|k| + 1 = \ell \} & \quad \text{if } d = p \\
\{ \mu^d \tilde{v}_0^{k_0} \tilde{v}_{2k}^c | c \in \{1, \ldots, p\}, \ ((k_0, k), q) \in \mathbb{N}_0^{d+s}, |q| + k_0 + 2|k| + 1 = \ell \} \cup \\
\{ \mu^d \tilde{v}_0^{k_0} \tilde{v}_{2k}^c | ((k_0, k), q) \in \mathbb{N}_0^{d+s}, |q| + k_0 + 2|k| = \ell \} & \quad \text{if } d = p + 1, 
\end{align*}
\]

where it is understood that if \((k, q) = (k_1, \ldots, k_p, q_1, \ldots, q_s) \in \mathbb{N}_0^{p+s}, \) then \(\mu^q = (\mu_1)^{q_1} \cdots (\mu_s)^{q_s}, \) \(v_{2k} = \tilde{v}_{2k} = (v_1)^{2k_1} \cdots (v_p)^{2k_p}, \ |q| = \sum_j q_j \) and \(|k| = \sum_j k_j.\)

Note that, \(V_{d+s}(\mathbb{R})\) is isomorphic to the vector space \(H_{d+s}(\mathbb{R}, \mathbb{Z}_{2,p}),\) since this latter space has the following basis

\[
\begin{align*}
\{ \mu^d \rho_{2k}^c e_c | c \in \{1, \ldots, p\}, \ (k, q) \in \mathbb{N}_0^{d+s}, |q| + 2|k| + 1 = \ell \} & \quad \text{if } d = p \\
\{ \mu^d \rho_0^{k_0} \rho_{2k}^c e_{c+1} | c \in \{1, \ldots, p\}, \ ((k_0, k), q) \in \mathbb{N}_0^{d+s}, |q| + k_0 + 2|k| + 1 = \ell \} \cup \\
\{ \mu^d \rho_0^{k_0} \rho_{2k}^c e_1 | ((k_0, k), q) \in \mathbb{N}_0^{d+s}, |q| + k_0 + 2|k| = \ell \} & \quad \text{if } d = p + 1, 
\end{align*}
\]

where \(e_j\) is a column vector with zeros on each row except the \(j^{th}\) row, which is 1. Therefore, \(\dim H_{d+s}(\mathbb{R}, \mathbb{Z}_{2,p}) = \dim V_{d+s}(\mathbb{R}).\)

Since \(B\) is as in (3.2), then this corresponds to the following choice of basis for the center subspace \(P:\)

\[
\Phi(t) = \begin{cases} 
(e^{i\omega_1 t} & e^{-i\omega_1 t} & \ldots & e^{i\omega_p t} & e^{-i\omega_p t} ) & \text{if } d = p, \ \kappa = 2p \\
(1 & e^{i\omega_1 t} & e^{-i\omega_1 t} & \ldots & e^{i\omega_p t} & e^{-i\omega_p t} ) & \text{if } d = p + 1, \ \kappa = 2p + 1. 
\end{cases}
\]

It follows that \(\Psi(0)\) in (2.5) is a \(\kappa \times 1\) matrix

\[
\Psi(0) = \begin{cases} 
\col(u_1, u_1^*, \ldots, u_p, u_p^*) & \text{if } \kappa = 2p \\
\col(u_0, u_1, u_1^*, \ldots, u_p, u_p^*) & \text{if } \kappa = 2p + 1, 
\end{cases}
\]

where \(u_0 \neq 0\) is real and \(u_j \neq 0\) are complex, \(j = 1, \ldots, p.\)
4.2 Linear analysis

Definition 4.2 Let $S$ denote the normed real linear space of $d \times \kappa$ matrices of the form

$$M = \begin{cases} 
\left( \begin{array}{cccc}
\alpha_{1,1} & \alpha_{1,1} & \cdots & \alpha_{1,p} \\
\alpha_{2,1} & \alpha_{2,1} & \cdots & \alpha_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p,1} & \alpha_{p,1} & \cdots & \alpha_{p,p}
\end{array} \right) & \text{if } d = p, \kappa = 2p \\
\left( \begin{array}{cccc}
\alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,p} \\
\alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,p} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p,0} & \alpha_{p,1} & \cdots & \alpha_{p,p}
\end{array} \right) & \text{if } d = p + 1, \kappa = 2p + 1,
\end{cases}$$

(4.4)

(where the $\alpha_{i,0}$ are real and the $\alpha_{i,j}$, $j \geq 1$ are complex numbers) equipped with norm $||M|| = \max(|\alpha_{i,j}|)$. For any given $M \in S$, we define the $\ell$-mapping associated to $M$

$$\mathcal{J}_M^{\ell} : H^{d+s}_\ell(\mathbb{R}) \longrightarrow H^{k+s}_\ell(\mathbb{R}^\kappa)$$

by

$$(\mathcal{J}_M^{\ell}(h))(x, \mu) = \Psi(0) h \left( x (M)^T, \mu \right),$$

where $x = (x_1, x_2, \ldots, x_p, x_p)$ (respectively $x = (x_0, x_1, x_1, \ldots, x_p, x_p)$).

Let $\tau = (\tau_1, \ldots, \tau_d)$ be a vector (as of yet unspecified) in $\mathbb{R}^d$. Define

$$E_\tau = \begin{pmatrix}
\Phi(\tau_1) \\
\vdots \\
\Phi(\tau_d)
\end{pmatrix} = \begin{cases} 
\left( \begin{array}{cccc}
e^{i\omega_1 \tau_1} & e^{-i\omega_1 \tau_1} & \cdots & e^{i\omega_p \tau_1} \\
e^{i\omega_1 \tau_2} & e^{-i\omega_1 \tau_2} & \cdots & e^{i\omega_p \tau_2} \\
\vdots & \vdots & \ddots & \vdots \\
e^{i\omega_1 \tau_p} & e^{-i\omega_1 \tau_p} & \cdots & e^{i\omega_p \tau_p}
\end{array} \right) & \text{if } d = p \\
\left( \begin{array}{cccc}
e^{i\omega_1 \tau_1} & e^{-i\omega_1 \tau_1} & \cdots & e^{i\omega_p \tau_1} \\
e^{i\omega_1 \tau_2} & e^{-i\omega_1 \tau_2} & \cdots & e^{i\omega_p \tau_2} \\
\vdots & \vdots & \ddots & \vdots \\
e^{i\omega_1 \tau_{p+1}} & e^{-i\omega_1 \tau_{p+1}} & \cdots & e^{i\omega_p \tau_{p+1}}
\end{array} \right) & \text{if } d = p + 1,
\end{cases}$$

(4.5)

and note that $E_\tau$ belongs to the space $S$ of Definition 4.2. Define the linear mapping

$$\mathcal{E}_\tau^{\ell} : H^{d+s}_\ell(\mathbb{R}) \longrightarrow H^{k+s}_\ell(\mathbb{R}^\kappa)$$
by
\[(E_{\ell}(h))(x, \mu) \equiv (J'_{E_{\tau}})(h)(x, \mu)\] (4.6)
where \(J'_{E_{\tau}}\) is the \(\ell\)-mapping associated to \(E_{\tau}\).

Now, let \(\Pi : H^{\kappa+s}_{\ell}(\mathbb{R}^\kappa, \mathbb{T}^p) \rightarrow H^{d+s}_{\ell}(\mathbb{R}^d, \mathbb{Z}_{2,p})\) be the mapping defined in (3.18)-(3.20), and let \(A : H^{\kappa+s}_{\ell}(\mathbb{R}^\kappa) \rightarrow H^{\kappa+s}_{\ell}(\mathbb{R}^\kappa, \mathbb{T}^p)\) be the group averaging operator defined in Definition 3.6. Our main result in this section is the following:

**Proposition 4.3** For an open and dense set \(U \subset \mathbb{R}^d\), the following linear mapping is surjective for all \(\tau \in U\):

\[
\Pi \circ A \circ E_{\tau} : H^{d+s}_{\ell}(\mathbb{R}) \rightarrow H^{d+s}_{\ell}(\mathbb{R}^d, \mathbb{Z}_{2,p}).
\]

**Proof** Let \(K\) be the \(d \times d\) matrix such that \(K_{j,k}\) is equal to \(-1\) if \(j + k > d + 1\) and is equal to \(1\) otherwise. It is easy to row reduce \(K\) to the identity matrix, so \(K\) is invertible. Therefore, \(K\) induces an automorphism of the space \(H^{d+s}_{\ell}(\mathbb{R})\):

\[
\mathcal{K} : H^{d+s}_{\ell}(\mathbb{R}) \rightarrow H^{d+s}_{\ell}(\mathbb{R})
\]
\[
(Kh)(v, \mu) = h(v (K)^T, \mu).
\]

(4.7)

Define \(\hat{V}^{d+s}_{\ell}(\mathbb{R}) \equiv \mathcal{K}^{-1}(V^{d+s}_{\ell}(\mathbb{R}))\), where \(V^{d+s}_{\ell}(\mathbb{R})\) is as in Definition 4.1, and let

\[
\mathcal{N}_{\tau}^d : \hat{V}^{d+s}_{\ell}(\mathbb{R}) \rightarrow H^{d+s}_{\ell}(\mathbb{R}^d, \mathbb{Z}_{2,p})
\]

be the restriction of \(\Pi \circ A \circ E_{\tau}\) to \(\hat{V}^{d+s}_{\ell}(\mathbb{R})\). Our approach to proving Proposition 4.3 will be to prove that there exists an open and dense set of points \(U \subset \mathbb{R}^d\) such that \(\mathcal{N}_{\tau}^d\) is invertible for all \(\tau \in U\).

If \((\mathcal{N}_{\tau}^d)\) is any matrix representation of \(\mathcal{N}_{\tau}^d\), then \(\det((\mathcal{N}_{\tau}^d))\) is a real-analytic function of \(\tau_1, \ldots, \tau_d\) (in fact, it is a polynomial in \(\cos \omega_k \tau_q\) and \(\sin \omega_k \tau_q\), \(k \in \{1, \ldots, p\}, q \in \{1, \ldots, d\}\)). Therefore, if we can show that \(\det((\mathcal{N}_{\tau}^d))\) is not identically zero, the conclusion is a trivial consequence of this analyticity. This amounts to showing that there exists at least one point \(\tau^* \in \mathbb{R}^d\) such that with \(E_{\tau^*}\) as in (4.5), the mapping \(\mathcal{N}_{\tau^*}^d\) is invertible. We will prove this last claim with a sequence of five lemmas.

**Lemma 4.4** Let \(S\) be as in Definition 4.2. If \(M_\ast \in S\) is such that the restriction of the map \(\Pi \circ A \circ J_{M_\ast}^\ell\) to \(\hat{V}^{d+s}_{\ell}(\mathbb{R})\):

\[
\Pi \circ A \circ J_{M_\ast}^\ell : \hat{V}^{d+s}_{\ell}(\mathbb{R}) \rightarrow H^{d+s}_{\ell}(\mathbb{R}^d, \mathbb{Z}_{2,p})
\]

is invertible, then there is a \(\delta = \delta(M_\ast) > 0\) such that for all \(M\) in the \(\delta\)-ball centered on \(M_\ast\), the restriction \(\Pi \circ A \circ J_{M}^\ell : \hat{V}^{d+s}_{\ell}(\mathbb{R}) \rightarrow H^{d+s}_{\ell}(\mathbb{R}^d, \mathbb{Z}_{2,p})\) is invertible.
**Proof**  This follows from the fact that the determinant of the map \( \Pi \circ A \circ J^\ell_M : \hat{V}^d_s(\mathbb{R}) \to H^d_s(\mathbb{R}^d, \mathbb{Z}_{2,p}) \) is continuous in the entries of \( M \).

As mentioned above, \( \Psi(0) \) in (4.3) is such that each of its components is non-zero. Therefore:

**Lemma 4.5** There exists \( \sigma_1, \ldots, \sigma_p \) such that

\[
\text{Re}(e^{i\sigma_j u_j}) \neq 0, \quad j = 1, \ldots, p.
\]

**Lemma 4.6** Let \( S \) be as defined in Definition 4.2 and \( \sigma_1, \ldots, \sigma_p \) be as in Lemma 4.5. Consider the following element \( I \in S \) of the form (4.4) where \( \alpha_{0,0} = 1 \) in the case \( d = p + 1 \), and:

\[
\alpha_{j,k} = \begin{cases} 
  e^{i\sigma_j} & \text{if } j = k \\
  0 & \text{if } j \neq k
\end{cases}
\]

If \( J^\ell_I : H^d_s(\mathbb{R}) \to H^{\kappa + s}(\mathbb{R}^\kappa) \) is the \( \ell \)-mapping associated to \( I \), then the restriction to \( V^d_s(\mathbb{R}) \):

\[
\Pi \circ A \circ J^\ell_I : \hat{V}^d_s(\mathbb{R}) \to H^d_s(\mathbb{R}^d, \mathbb{Z}_{2,p})
\]

is invertible.

**Proof**  We give only the proof in the case \( d = p, \kappa = 2p \). The other case \( (d = p + 1, \kappa = 2p + 1) \) is treated in a completely similar manner.

Consider the basis element \( \mu^q v^{2k} v_c \) of \( V^p_s(\mathbb{R}) \) (see (4.1)). Then after an appropriate translation of the integration variables, we have

\[
(\Pi \circ A \circ J^\ell_I)(\mu^q v^{2k} v_c) = \\
\frac{\mu^q}{(2\pi)^p} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{G}(\rho_1(e^{-i\theta_1} + e^{i\theta_1}))^{2k_1} \cdots (\rho_p(e^{-i\theta_p} + e^{i\theta_p}))^{2k_p} \rho_c(e^{-i\theta_c} + e^{i\theta_c}) d\theta_1 \cdots d\theta_p,
\]

where \( \mathcal{G} = C \cdot \text{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}) \cdot \text{diag}(e^{i\sigma_1}, e^{-i\sigma_1}, \ldots, e^{i\sigma_p}, e^{-i\sigma_p}) \cdot \Psi(0), \) (\( C \) as in (3.21) and \( \Psi(0) \) as in (4.3)). A simple computation then shows that

\[
(\Pi \circ A \circ J^\ell_I)(\mu^q v^{2k} v_c) = \\
\text{Re}(e^{i\sigma_c u_c}) \frac{2k_c + 1}{k_c + 1} \left[ \frac{(2k_1)!}{(k_1!)^2} \frac{(2k_2)!}{(k_2!)^2} \cdots \frac{(2k_p)!}{(k_p!)^2} \right] \mu^q \rho^{2k} \rho_c e_c,
\]

26
where we remind the reader that $e_σ$ is a $p$-dimensional column vector with zeros on each row except the $c^{th}$ row, which is 1. Taking into account (4.1) and (4.2), under a suitable choice of bases for the spaces $V_{t+}^p(\mathbb{R})$ and $H_{t+}^p(\mathbb{R}^p, \mathbb{Z}_{2p})$, the matrix representation of the restriction of $\Pi \circ A \circ \mathcal{J}_E^{|}$ to $V_{t+}^p(\mathbb{R})$ is diagonal with non-zero diagonal entries.

**Lemma 4.7** Let $S$ be as in Definition 4.2, $σ_1, \ldots, σ_p$ as in Lemma 4.5 and $I$ as in Lemma 4.6. Consider the element $E_σ ≡ KI ∈ S$, where the $d \times d$ matrix $K$ is such that $K_{j,k}$ is equal to $-1$ if $j + k > d + 1$ and is equal to 1 otherwise. If $J_{E_σ}^{|} : H_{t+}^d(\mathbb{R}) → H_{t+}^\kappa(\mathbb{R}^\kappa)$ is the $\ell$-mapping associated to $E_σ$, then the restriction $\Pi \circ A \circ J_{E_σ}^{|} : \hat{V}_{t+}^d(\mathbb{R}) → H_{t+}^d(\mathbb{R}^d, \mathbb{Z}_{2p})$ is invertible.

**Proof** Let $I$ and $J_{E_σ}^{|}$ be as in Lemma 4.6. Since $E_σ = KI$, it follows that $J_{E_σ}^{|} = J_{E_σ}^{|} \circ K$, where $K$ is the automorphism defined in (4.7). So $Π \circ A \circ J_{E_σ}^{|} = (Π \circ A \circ J_{E_σ}^{|}) \circ K$. Consequently, the restriction of $Π \circ A \circ J_{E_σ}^{|}$ to $\hat{V}_{t+}^d(\mathbb{R}) ≡ K^{-1}(V_{t+}^d(\mathbb{R}))$ is invertible.

**Lemma 4.8** Let $E_σ$ be as in Lemma 4.7 and let $δ = δ(E_σ) > 0$ be as in Lemma 4.4. There exists a $τ^* ∈ \mathbb{R}^p$ such that $E_{τ^*}$ in (4.5) satisfies $||E_{τ^*} - E_σ|| < δ$, and consequently if $E_{τ^*} ≡ J_{E_σ}^{|} : H_{t+}^d(\mathbb{R}) → H_{t+}^\kappa(\mathbb{R}^\kappa)$ is the $\ell$-mapping associated to $E_{τ^*}$, then the restriction $\mathcal{N}_{τ^*} ≡ Π \circ A \circ E_{τ^*}^{|} : \hat{V}_{t+}^d(\mathbb{R}) → H_{t+}^d(\mathbb{R}^d, \mathbb{Z}_{2p})$ is invertible (from Lemma 4.4).

**Proof** Since the $ω_1, \ldots, ω_p$ are independent over the rationals, it follows that the set
\[
\{(e^{iω_1 t}, e^{iω_2 t}, \ldots, e^{iω_p t}) \mid t ∈ \mathbb{R}\}
\]
is dense on the $p$-torus $\mathbb{T}^p$. Consequently, it is possible to choose a $τ^* ∈ \mathbb{R}^d$ such that each row of $E_{τ^*}$ is as close (in any given norm) as we wish to the corresponding row of $E_σ$.

The proof of Proposition 4.3 follows immediately from Lemmas 4.4-4.8.

### 4.3 Refinement

In the next section, we will need a finer version of Proposition 4.3. If $H_{t+}^d(\mathbb{R})$ denotes the subspace of $\mu$-independent elements of $H_{t+}^d(\mathbb{R})$, then we have
\[
H_{t+}^d(\mathbb{R}) = H_{t+}^d(\mathbb{R}) + P_{t+}^d(\mathbb{R}),
\]
where $q ∈ P_{t+}^d(\mathbb{R})$ if and only if $q ∈ H_{t+}^d(\mathbb{R})$ and $q(ν, 0) = 0$. Define $V_{t+}^d(\mathbb{R}) = V_{t+}^d(\mathbb{R}) ∩ H_{t+}^d(\mathbb{R})$ and $W_{t+}^d(\mathbb{R}) = V_{t+}^d(\mathbb{R}) ∩ P_{t+}^d(\mathbb{R})$ and note that
\[
V_{t+}^d(\mathbb{R}) = V_{t+}^d(\mathbb{R}) ∩ W_{t+}^d(\mathbb{R})
\]
is precisely the decomposition of $V^{d+s}(\mathbb{R})$ into the direct sum of $\mu$-independent elements of $V^{d+s}(\mathbb{R})$ and elements of $V^{d+s}(\mathbb{R})$ which vanish at $\mu = 0$. The automorphism $K$ of $H^{d+s}(\mathbb{R})$ defined in (4.7) preserves these decompositions, and we have

$$
\hat{V}^{d+s}(\mathbb{R}) = K^{-1}(V^{d+s}(\mathbb{R})) = K^{-1}(V^d(\mathbb{R})) \oplus K^{-1}(W^{d+s}(\mathbb{R}))
$$

(4.10)

which is the decomposition of $\hat{V}^{d+s}(\mathbb{R})$ into the direct sum of $\mu$-independent elements of $\hat{V}^d(\mathbb{R})$ and elements of $\hat{V}^{d+s}(\mathbb{R})$ which vanish at $\mu = 0$.

Then, taking into account (3.26), we have

**Proposition 4.9**

$$
\dim \hat{V}^{d}(\mathbb{R}) = \dim H^{d}(\mathbb{R}^d, \mathbb{Z}_2, p) \quad \text{and} \quad \dim \hat{W}^{d+s}(\mathbb{R}) = \dim P^{d+s}(\mathbb{R}^d, \mathbb{Z}_2, p).
$$

(4.11)

Furthermore, if $\tau$ is as in Proposition 4.3, then

$$(\Pi \circ A \circ \mathcal{E}^\ell)(\hat{V}^{d}(\mathbb{R})) = H^{d}(\mathbb{R}^d, \mathbb{Z}_2, p) \quad \text{and} \quad (\Pi \circ A \circ \mathcal{E}^\ell)(\hat{W}^{d+s}(\mathbb{R})) = P^{d+s}(\mathbb{R}^d, \mathbb{Z}_2, p).$$

Proof. We give the proof in the case $d = p$, the other case being treated in a similar manner.

We note that

$$V^p(\mathbb{R}) = \text{span} \{ v^{2k} v_c | c \in \{1, \ldots, p\}, \, k \in \mathbb{N}_0, 2k + 1 = \ell \},$$

and

$$H^p(\mathbb{R}^d, \mathbb{Z}_2, p) = \text{span} \{ \rho^{2k} \rho_c e_c | c \in \{1, \ldots, p\}, \, k \in \mathbb{N}_0, 2k + 1 = \ell \}.$$ 

Equation (4.11) follows from (4.9), (4.10) and the fact that $\hat{V}^p(\mathbb{R}) = K^{-1}(V^p(\mathbb{R}))$. It now follows from the theory presented in section 3 that

$$(\Pi \circ A \circ \mathcal{E}^\ell)(\hat{V}^p(\mathbb{R})) \subset H^p(\mathbb{R}^d, \mathbb{Z}_2, p) \quad \text{and} \quad (\Pi \circ A \circ \mathcal{E}^\ell)(\hat{W}^{d+s}(\mathbb{R})) \subset P^{d+s}(\mathbb{R}^d, \mathbb{Z}_2, p).$$

The reverse inclusions then follow from the invertibility of $\mathcal{N}^\ell_\tau$ and from (3.26), (4.8) and (4.11).
5 Main Results

We are now ready to state and prove our main realizability results for both cases of Hypothesis 2.2(b), with the convention that respectively \((d, \kappa) = (p, 2)\) and \((d, \kappa) = (p+1, 2p+1)\). It will be convenient to define the following linear spaces of (non-homogeneous) polynomials

**Definition 5.1** For an integer \(\ell \geq 2\), define

\[
\hat{V}_{d}^{\ell+s}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} \hat{V}_{d}^{j+s}(\mathbb{R}),
\]

\[
\hat{W}_{d}^{\ell+s}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} \hat{W}_{d}^{j+s}(\mathbb{R}),
\]

\[
\mathcal{H}_{d}^{\ell+1}(\mathbb{R}; \mathbb{Z}_{2,p}) \equiv \bigoplus_{j=2}^{\ell} H_{d}^{j+1}(\mathbb{R}; \mathbb{Z}_{2,p}),
\]

\[
\mathcal{H}_{d}^{\ell+1}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} H_{d}^{j+1}(\mathbb{R}),
\]

Our first result addresses the issue of realizability of singularities and unfoldings within the class of scalar delay-differential equations with \(d\) delays.

**Theorem 5.2** Consider the RFDE (2.1), and let \(\Lambda_{0}\) denote the set of solutions of (2.3) with zero real part. Suppose that Hypothesis 2.2 is satisfied. Let \(\ell \geq 2\) be a given integer. For each \(h \in \mathcal{H}_{d}^{\ell}(\mathbb{R}^{d}; \mathbb{Z}_{2,p})\):

\[
h(\rho) = \sum_{j=2}^{\ell} h_{j}(\rho),
\]

\((h_{j} \in H_{d}^{j}(\mathbb{R}^{d}; \mathbb{Z}_{2,p}), j = 2, \ldots, \ell)\) and each \(q \in \mathcal{P}_{d+s}^{\ell}(\mathbb{R}^{d}; \mathbb{Z}_{2,p})\):

\[
q(\rho, \mu) = \sum_{j=2}^{\ell} q_{j}(\rho, \mu),
\]

\((q_{j} \in P_{d+s}^{j}(\mathbb{R}^{d}; \mathbb{Z}_{2,p}), j = 2, \ldots, \ell)\), there are \(d\) distinct points \(\tau_{1}, \ldots, \tau_{d} \in [-r, 0]\), an \(\eta \in \hat{V}_{d}^{\ell}(\mathbb{R})\):

\[
\eta(v) = \sum_{j=2}^{\ell} \eta_{j}(v), \quad (5.1)
\]

\((\eta_{j} \in \hat{V}_{d}^{j}(\mathbb{R}), j = 2, \ldots, \ell)\), and a \(\xi \in \hat{W}_{d}^{\ell+s}(\mathbb{R})\):

\[
\xi(v, \mu) = \sum_{j=2}^{\ell} \xi_{j}(v, \mu), \quad (5.2)
\]
\( \eta \in \hat{W}_j^{d+s}(\mathbb{R}) \), \( j = 2, \ldots, \ell \), such that if

\[
\hat{F}(z_t, \mu) = \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu)
\]

in (2.2), then in polar coordinates, the radial part of the center manifold equations (3.29) in \( \mathbb{T}^p \)-equivariant normal form up to degree \( \ell \) reduces to \( \dot{\rho} = h(\rho) + q(\rho, \mu) \), where \( \rho \equiv (\rho_1, \ldots, \rho_p) \) or \( \rho = (\rho_0, \rho_1, \ldots, \rho_p) \). In fact, \( \tau \) can be chosen in an open and dense set of \([-r, 0]^d\), independently of the particular \( h \) and \( q \) to be realized (i.e. only \( \eta \) and \( \xi \) must be changed in order to account for different jets to be realized).

**Proof** Choose a point \( \tau \in [-r, 0]^d \) such that the previously defined linear mappings

\[
\mathcal{N}_\tau^j : \hat{V}_j^{d+s}(\mathbb{R}) \rightarrow H_j^{d+s}(\mathbb{R}^d, Z_{2p})
\]

are invertible for all \( j = 2, \ldots, \ell \) (from Proposition 4.3, this is possible for an open and dense set of points in \([-r, 0]^d\)). Suppose \( \eta \) is an arbitrary polynomial of the form (5.1), \( \xi \) is and arbitrary polynomial of the form (5.2), and suppose \( \hat{F} \) in (2.2) is such that \( \hat{F}(z_t, \mu) = \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu) \). Using Theorem 3.15, it is possible to define successively at each order near identity changes of variables of the form

\[
(x, y) = (\hat{x}, \hat{y}) + (U_1^j(\hat{x}) + W_1^j(\hat{x}, \mu), U_2^j(\hat{x}) + W_2^j(\hat{x}, \mu)), \quad (5.3)
\]

where \( W_i^j(\hat{x}, 0) = 0, i = 1, 2 \), which transform (2.6) into (2.8), and the center manifold equations are as in (2.9), with

\[
\begin{align*}
g_2^j(x, 0, \mu) &= A(E_2^j \eta_2)(x) + A(E_2^j \xi_2)(x, \mu) \\
g_3^j(x, 0, \mu) &= A(E_3^j \eta_3 + Y_3)(x) + A(E_3^j \xi_3 + Z_3)(x, \mu) \\
&\vdots \\
g_j^1(x, 0, \mu) &= A(E_j^j \eta_j + Y_j)(x) + A(E_j^j \xi_j + Z_j)(x, \mu) \\
&\vdots
\end{align*}
\]

In (5.4), \( E_j^j \) are as in (4.6), \( A \) is the \( \mathbb{T}^p \) averaging operator (3.6), and \( Y_j(x) \) and \( Z_j(x, \mu) \) are the extra contributions to the terms of order \( j \) coming from the lower order \( (< j) \) changes of variables, and \( Z_j(x, 0) = 0 \). Hence, the terms \( Y_j \) and \( Z_j \) are completely determined once the normalizing procedure arrives at order \( j \). More precisely, \( Y_j \) is determined explicitly in terms of \( \eta_2, \ldots, \eta_{j-1}, U_2^i, \ldots, U_{j-1}^i, i = 1, 2 \) and \( Z_j \) is determined explicitly in terms of \( \eta_2, \ldots, \eta_{j-1}, \xi_2, \ldots, \xi_{j-1}, U_2^i, \ldots, U_{j-1}^i, W_2, \ldots, W_{j-1}^i, i = 1, 2 \). Taking into account (5.4) and using the convention \( Y_2 = 0, Z_2 = 0 \), the center manifold equations (2.9) are \( \mathbb{T}^p \)-equivariant, and in polar coordinates, the uncoupled radial part (truncated at order \( \ell \)) is of the form

\[
\dot{\rho} = \sum_{j=2}^{\ell} \left[ (\mathcal{N}_\tau^j \eta_j + (\Pi \circ A)(Y_j))(\rho) + (\mathcal{N}_\tau^j \xi_j + (\Pi \circ A)(Z_j))(\rho, \mu) \right].
\]

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Thus, using Proposition 4.9, we get the desired result if we set
\[ \eta_j = (\mathcal{N}_\tau^{-1})(h_j - (\Pi \circ A)(Y_j)), \quad \xi_j = (\mathcal{N}_\tau^{-1})(q_j - (\Pi \circ A)(Z_j)), \quad j = 2, \ldots, \ell. \] (5.5)

Theorem 5.2 has an important interpretation in terms of the singularity and unfolding theory of scalar delay-differential equations. Suppose (2.1) satisfies the hypotheses of Theorem 5.2. Let \( h(\rho) \) be any given (parameter independent) element of \( \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2p}) \), \( \ell \geq 2 \). Then Theorem 5.2 implies that (under generic conditions on \( \tau_1, \ldots, \tau_d \)) there exists an un-parameterized nonlinear polynomial delay-differential equation
\[ \dot{z}(t) = L_0 z_t + \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) \] (5.6)
whose dynamics on a center manifold up to order \( \ell \) have as uncoupled radial equations \( \dot{\rho} = h(\rho) \). Therefore, generically, any finitely-determined singularity within the space of \( \mathbb{Z}_{2p} \)-equivariant radial equations can be realized by an appropriate choice of \( \eta \) in (5.6).

Now, suppose that \( \tilde{h} \in \mathcal{H}_\ell^{d+s}(\mathbb{R}^d, \mathbb{Z}_{2p}) \) is an equivariant unfolding of the finitely-determined singularity \( h \) above, i.e. \( \tilde{h}(\rho, \mu) \) is such that \( \tilde{h}(\rho, 0) = h(\rho) \). Then \( q(\rho, \mu) \equiv \tilde{h}(\rho, \mu) - h(\rho) \) is an element of \( \mathcal{P}_\ell^{d+s}(\mathbb{R}^d, \mathbb{Z}_{2p}) \). Theorem 5.2 implies that there exists a parameterized nonlinear polynomial delay-differential equation of the form
\[ \dot{z}_t = L_0 z_t + \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu), \] (5.7)
with \( \xi(z(t + \tau_1), \ldots, z(t + \tau_d), 0) = 0 \), whose dynamics on a center manifold up to order \( \ell \) have as uncoupled radial equations \( \dot{\rho} = \tilde{h}(\rho, \mu) = h(\rho) + q(\rho, \mu) \). Therefore, the unfolding \( \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu) \) of \( \eta \) realizes the unfolding \( \tilde{h}(\cdot, \mu) \) of the singularity \( h \) on the center manifold.

In the theory of classification of singularities of equivariant vector fields [12, 13], one often defines a suitable equivalence relation on a given space of vector fields (by requiring preservation of certain local qualitative features of the flow associated to the vector field), and then classifies the equivalence classes in terms of a (hopefully finite) set of conditions of the Taylor coefficients of the vector field. One then wishes to characterize the “likelihood” of a given singularity, \( f \), by computing its codimension, which roughly speaking, is the codimension of the equivalence orbit through \( f \). Finally, one then uses this idea of codimension to construct a versal unfolding of the singularity (perturbing in transversal directions to the equivalence orbit).

Suppose \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a smooth vector field vanishing at the origin and equivariant with respect to the group \( \mathbb{Z}_{2p} \) previously defined. Typically, the computation of codimension of the singularity \( f \) is done by first identifying a polynomial \( h \in \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2p}) \) (for some
suitable \ell) which is equivalent to \( f \) regardless of the Taylor coefficients of \( f \) of order greater than \( \ell \). Then, one constructs the tangent space within \( \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) \) to the equivalence orbit of \( h \) through \( h \), \( T_h \subset \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) \). Finally one finds a complementary subspace \( C_h \subset \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) \) such that \( \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) = T_h \oplus C_h \). The codimension of \( f \) is then identified with the dimension of \( C_h \) (i.e. \( \text{codim} \ f \equiv \text{codim} \ T_h = \text{dim} \ C_h \)). We say that \( f \) is generic with respect to the equivalence relation if the codimension of \( f \) is zero. The following theorem addresses this issue within the context of realizability.

**Theorem 5.3** Consider the RFDE (2.1) in the unparametrized \((s = 0)\) case, and let \( \Lambda_0 \) denote the set of solutions of (2.3) with zero real part. Suppose that Hypothesis 2.2 is satisfied. Suppose that the nonlinear term \( F(z) \) is of the general form

\[
F(z) = \eta(z(t + \tau_1), \ldots, z(t + \tau_d)),
\]

where \( \eta \) is smooth. Then the local dynamics of (2.1) near the origin on an invariant center manifold can be described by a system of ordinary differential equations on \( \mathbb{R}^\kappa \). Moreover, this ODE system can be brought into \( \mathbb{T}^p\)-equivariant normal form to any desired order \( \ell \), and the resulting (truncated at order \( \ell \)) normal form can be uncoupled into two sub-systems

\[
\dot{\rho} = h(\rho; \eta, \tau) \quad \quad \quad \quad (5.8)
\]

\[
\dot{\theta} = k(\rho; \eta, \tau), \quad \quad \quad \quad (5.9)
\]

where \( \tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \), \( h(\cdot; \eta, \tau) \in \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) \) and \( k(\cdot; \eta, \tau): \mathbb{R}^p \rightarrow \mathbb{R}^p \). For given \( \tau \in \mathbb{R}^d \), consider the following mapping:

\[
\mathcal{F}_\tau : \mathcal{H}_\ell^d(\mathbb{R}) \rightarrow \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p})
\]

\[
\eta \mapsto \mathcal{F}_\tau(\eta) = h(\cdot; \eta, \tau),
\]

where \( h \) is as in (5.8). Then there is an open and dense set \( \mathcal{U} \subset \mathbb{R}^d \), such that for all \( \tau \in \mathcal{U} \), \( \mathcal{F}_\tau \) is a submersion. Consequently, if \( \mathcal{M} \subset \mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p}) \) is a smooth manifold, then for all \( \tau \in \mathcal{U} \), \( \mathcal{F}_\tau^{-1}(\mathcal{M}) \) is a smooth submanifold of \( \mathcal{H}_\ell^d(\mathbb{R}) \), and

\[
\text{codim} \mathcal{F}_\tau^{-1}(\mathcal{M}) = \text{codim} \mathcal{M} \quad \quad \quad \quad (5.10)
\]

**Proof** The fact that the center manifold equations are given by (5.8) and (5.9) has already been proved.
The mapping $F_\tau$ is computable similarly to (5.5): if $\eta = \sum_{j=2}^\ell \eta_j$, with $\eta_j \in H^d_j(\mathbb{R})$, then

$$F_\tau(\eta) = h(\cdot; \eta, \tau) = \sum_{j=2}^\ell \left( (\Pi \circ A \circ E_j^\tau)(\eta_j) + (\Pi \circ A)(Y_j) \right),$$

where $Y_2 = 0$ and $Y_j$ is a smooth function of $\eta_2, \ldots, \eta_{j-1}$ for $j > 2$. Thus, if $\zeta = \sum_{j=2}^\ell \zeta_j$, with $\zeta_j \in H^d_j(\mathbb{R})$, then

$$DF_\tau(\eta) \cdot \zeta = \sum_{j=2}^\ell \left( \left( \Pi \circ A \circ E_j^\tau \right)(\zeta_j) + (\Pi \circ A)\left( \sum_{i=2}^\ell Y_{ji}(\eta) \zeta_i \right) \right),$$

where $Y_{ji} = 0$ if $i \geq j$. From Proposition 4.3, there is an open and dense set $U \subset \mathbb{R}^d$ such that for all $\tau \in U$, $DF_\tau(\eta)$ is onto $H^d_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)$, and consequently $F_\tau$ is a submersion. Equation (5.10) follows from the transversal mapping theorem [1].

The next result states that the number of delays, $d$, shown above to be sufficient to realize any arbitrary element of $H^d_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)$, is optimal for that purpose.

**Theorem 5.4** Consider the RFDE (2.1) in the unparameterized ($s = 0$) case, and let $\Lambda_0$ denote the set of solutions of (2.3) with zero real part. Suppose that Hypothesis 2.2 is satisfied. Suppose that the nonlinear term $F(z_t)$ is of the general form

$$F(z_t) = \eta(z(t + \tau_1), \ldots, z(t + \tau_{d-1})),
$$

where $\eta$ is smooth. Then the local dynamics of (2.1) near the origin on an invariant center manifold can be described by a system of ordinary differential equations on $\mathbb{R}^\kappa$. Moreover, this ODE system can be brought into $\mathbb{T}^p$-equivariant normal form to any desired order $\ell$, and the resulting (truncated at order $\ell$) normal form can be uncoupled into two sub-systems

$$\dot{\rho} = h(\rho; \eta, \tau)$$

and

$$\dot{\theta} = k(\rho; \eta, \tau),$$

where $\tau = (\tau_1, \ldots, \tau_{d-1}) \in \mathbb{R}^{d-1}$, $h(\cdot; \eta, \tau) \in H^d_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)$ and $k(\cdot; \eta, \tau) : \mathbb{R}^p \rightarrow \mathbb{R}^p$. For given $\tau \in \mathbb{R}^{d-1}$, consider the following mapping:

$$F_\tau : H^d_{\ell-1}(\mathbb{R}) \rightarrow H^d_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)$$

$$\eta \mapsto F_\tau(\eta) = h(\cdot; \eta, \tau),$$

where $h$ is as in (5.11). Then there is an integer $\ell_0 \geq 2$ such that $F_\tau$ is not surjective if $\ell \geq \ell_0$. 

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Proof. It will be sufficient to show that for fixed $d$,

$$\frac{\dim \mathcal{H}^{d-1}_\ell(\mathbb{R})}{\dim \mathcal{H}^{d}_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)} = O(\ell^{-1})$$

(5.14)

as $\ell \to \infty$. First, note that it is well-known [14] that for given integers $m \geq 1$ and $\ell \geq 1$, the number of solutions in non-negative integers for the equation

$$k_1 + \cdots + k_m = \ell$$

is

$$\binom{m+\ell-1}{m-1}.$$

Thus,

$$\dim \mathcal{H}^{d-1}_\ell(\mathbb{R}) = \sum_{j=2}^{\ell} \dim \mathcal{H}^{d-1}_j(\mathbb{R}) = \sum_{j=2}^{\ell} \binom{d+j-2}{d-2}$$

(5.15)

$$= \binom{d-1+\ell}{d-1} - d = O(\ell^{d-1}) \text{ as } \ell \to \infty.$$

Using a similar (but slightly lengthier) computation, we can show that

$$\dim \mathcal{H}^{d}_\ell(\mathbb{R}^d, \mathbb{Z}_2, p) = O(\ell^d) \text{ as } \ell \to \infty,$$

which establishes (5.14) and concludes the proof of this theorem.

Theorem 5.4 is important in the problem of establishing whether or not there are restrictions on the possible phase portraits for an unfolding of a given singularity $h \in \mathcal{H}^{d}_\ell(\mathbb{R}^d, \mathbb{Z}_2, p)$ when such an unfolding arises from center manifold reduction and phase/amplitude decoupling of a nonlinear delay-differential equation (2.1). This theorem allows one to conclude that, at least for $\ell$ large enough, such restrictions are likely to occur if the number of delays in the nonlinearity $F$ in (2.2) is less than $d$. For example, this question was addressed in [4] in the context of the non-resonant double Hopf bifurcation. In this case, they show that if the nonlinear part of (2.1) contains 2 delays, then generically any cubic order radial equation (5.8) can be realized by appropriate choice of the nonlinear coefficients in (2.1) (note that our Theorem 5.2 recovers and generalizes that result). However, in [4], it is also shown that if the nonlinear part of (2.1) depends on only one delay, then not all equivalence classes of phase portraits in the versal unfolding of the radial equations (5.11) can be attained by variation of the nonlinear coefficients in (2.1), for fixed values of $\omega_1$, $\omega_2$ and $\tau$. The next example treats this specific case in the context of Theorem 5.4.
Example 5.5 In the case where \( \Lambda_0 = \{ \pm i\omega_1, \pm i\omega_2 \} \), we get from (5.15) that

\[
\dim \mathcal{H}_1^\ell(\mathbb{R}) = \ell - 1.
\]

It is also easy to show that if \( \ell = 2L + j \), where \( L \geq 0 \) is an integer and \( j \in \{0, 1\} \), then

\[
\dim \mathcal{H}_2^\ell(\mathbb{R}^2, \mathbb{Z}_{2,2}) = L(L + 3).
\]

Thus, from \( \ell = 3 \) onward, we have \( \dim \mathcal{H}_1^\ell(\mathbb{R}) < \dim \mathcal{H}_2^\ell(\mathbb{R}^2, \mathbb{Z}_{2,2}) \). In particular, \( \dim \mathcal{H}_3^1(\mathbb{R}) = 2 \) and \( \dim \mathcal{H}_3^2(\mathbb{R}^2, \mathbb{Z}_{2,2}) = 4 \). Therefore, at cubic order, the mapping \( \mathcal{F}_\tau \) in (5.13)

\[
\mathcal{F}_\tau : \mathcal{H}_3^1(\mathbb{R}) \rightarrow \mathcal{H}_3^2(\mathbb{R}^2, \mathbb{Z}_{2,2})
\]

\[\eta \mapsto \mathcal{F}_\tau(\eta) = h(\cdot; \eta, \tau),\]

is not surjective, and so there are elements of \( \mathcal{H}_3^2(\mathbb{R}^2, \mathbb{Z}_{2,2}) \) which can not be realized by any element of \( \mathcal{H}_3^1(\mathbb{R}) \). In fact, \( \mathcal{F}_\tau(\mathcal{H}_3^1(\mathbb{R})) \) is a two-dimensional smooth surface in the 4-dimensional space \( \mathcal{H}_3^2(\mathbb{R}^2, \mathbb{Z}_{2,2}) \), such that \( \mathcal{F}_\tau(0) = 0 \). Specifically, if we write the general element of \( \mathcal{H}_3^1(\mathbb{R}) \) as

\[
b_2 v^2 + b_3 v^3
\]

and the general element of \( \mathcal{H}_3^2(\mathbb{R}^2, \mathbb{Z}_{2,2}) \) as

\[
(a_{11}\rho_1^2 + a_{12}\rho_2^2)\rho_1
\]

\[
(a_{21}\rho_1^2 + a_{22}\rho_2^2)\rho_2.
\]

then the mapping \( \mathcal{F}_\tau \) can be represented by the following mapping from \( \mathbb{R}^2 \) into \( \mathbb{R}^4 \):

\[
a_{ij}(b_2, b_3) = \alpha_{ij} b_2^2 + \beta_{ij} b_3, \quad i, j = 1, 2
\]

(5.16)

where the real coefficients \( \alpha_{ij} \) and \( \beta_{ij} \) are determined from \( \tau, \omega_1 \) and \( \omega_2 \).

Note however that the problem of determining whether or not there are restrictions is somewhat more subtle than one of surjectivity, since the topological types of the possible phase diagrams in the unfolding space for the double Hopf bifurcation are determined by the sign of the cubic coefficients in the radial equations (and not their actual values). In the \((b_2, b_3)\) plane, the zero level sets of the \( a_{ij} \) in (5.16) are (at most) four distinct curves (parabolae generically) which intersect only at the origin. Consequently, there are at most four distinct open regions in the \((b_2, b_3)\) plane in which the signs of the coefficients \( a_{ij} \) are constant and non-zero. It is then easy to see that it is impossible to realize the twelve possible sign combinations (see [15]) which characterize the complete unfolding space of the double Hopf bifurcation, and so there will be restrictions on the phase portraits when the nonlinear terms in (2.1) contain only one delay.
For general RFDEs (i.e. not necessarily delay-differential equations), we have the following result on realization of unfoldings:

**Theorem 5.6** Consider the general nonlinear RFDE

\[
\dot{z}(t) = L_0 z_t + N(z_t) \tag{5.17}
\]

where \(L_0 : C \to \mathbb{R}\) is a bounded linear operator from \(C \equiv C \left(\left[-r, 0\right], \mathbb{R}\right)\) into \(\mathbb{R}\), and \(N\) is a smooth function from \(C\) into \(\mathbb{R}\), with \(N(0) = 0\), \(DN(0) = 0\). Let \(\Lambda_0\) denote the set of solutions of (2.3) with zero real part and suppose that Hypothesis 2.2 is satisfied. Then the local dynamics of (5.17) near the origin on an invariant center manifold can be described by a system of ordinary differential equations on \(\mathbb{R}^\kappa\). Moreover, this ODE system can be brought into \(\mathbb{T}^p\)-equivariant normal form to any desired order \(\ell\), and the resulting (truncated at order \(\ell\)) normal form can be uncoupled into an uncoupled \(d\)-dimensional system and a \(p\)-dimensional system

\[
\dot{\rho} = h(\rho; N) \tag{5.18}
\]

\[
\dot{\theta} = k(\rho; N), \tag{5.19}
\]

where for given \(N\), \(h(\cdot; N)\) is some element of \(\mathcal{H}_\ell^d(\mathbb{R}^d, \mathbb{Z}_{2,p})\), and \(k(\cdot; N) : \mathbb{R}^p \to \mathbb{R}^p\). Let \(\tilde{h}(\rho, \mu)\) be an \(s\)-parameter equivariant unfolding of \(h\) of degree at most \(\ell\), i.e. \(\tilde{h} \in \mathcal{H}_\ell^{d+s}(\mathbb{R}^d, \mathbb{Z}_{2,p})\) and \(\tilde{h}(\cdot, 0) = h(\cdot; N)\). Then there exists an \(s\)-parameter unfolding of (5.17) of the form

\[
\dot{z}(t) = L_0(z_t) + N(z_t) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu) \tag{5.20}
\]

(where \(\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d\), and \(\xi \in \hat{W}_{\ell}^{d+s}(\mathbb{R})\) vanishes at \(\mu = 0\)) which realizes the unfolded radial equations

\[
\dot{\rho} = \tilde{h}(\rho, \mu)
\]

on an invariant center manifold for (5.20).

**Proof** Choosing \(W_j^1 = 0, W_j^2 = 0\) and choosing \(U_j^1\) and \(U_j^2\) appropriately in (5.3), the center manifold equations for (5.17) truncated at order \(\ell\) are equivalent to (5.18) and (5.19).

Using Theorem 3.15, for arbitrary \(\xi = \sum_{j=2}^\ell \xi_j \in \hat{W}_{\ell}^{d+s}(\mathbb{R})\), there is a sequence of near-identity changes of variables (5.3) (with \(U_j^1\) and \(U_j^2\) as above) for which the uncoupled radial part of the center manifold equations for (5.20) truncated at order \(\ell\) are

\[
\dot{\rho} = h(\rho; N) + \sum_{j=2}^\ell (N_j^1 \xi_j + (\Pi \circ A)(Z_j))
\]
where \((\Pi \circ A)(Z_j)\) is some known element of \(P^d_{j+s}(\mathbb{R}^d, Z_{2^p})\). The conclusion follows from setting
\[
\xi_j = (\mathcal{I}_{p_j}^{-1}(\tilde{h} - h - (\Pi \circ A)(Z_j)).
\]

6 The \(\pm i\omega, (\pm i\omega_1, \pm i\omega_2)\) and \((0, \pm i\omega)\) Singularities

Our results in Theorems 5.2, 5.3 and 5.6 allow us to recover some previous results on realizability and (lack of) restrictions for Hopf bifurcation, non-resonant double Hopf bifurcation, and the \((0, \pm i\omega)\) singularity in scalar RFDEs [11, 4].

**Corollary 6.1 (Theorem 1 of [11])** Consider the RFDE (2.1) in the unparameterized case
\[
\dot{z}(t) = L_0 z_t + F(z_t), \tag{6.1}
\]
such that the characteristic equation (2.3) has simple purely imaginary roots \(\pm i\omega \neq 0\) and no other roots on the imaginary axis (simple Hopf bifurcation). If
\[
F(z_t) = A_2(z(t + \tau))^2 + A_3(z(t + \tau))^3, ~ \tau \in [-r, 0] \tag{6.2}
\]
then the uncoupled radial part of the center manifold equations to cubic order are
\[
\dot{\rho} = a \rho^3, \tag{6.3}
\]
where \(a = a(A_2, A_3; \tau, \omega)\). Generically, the non-degeneracy condition \(a \neq 0\) is satisfied. In fact, for any \(a \in \mathbb{R}\), (6.3) can be realized with \(A_2 = 0\) for an appropriate choice of \(A_3\) in (6.2). Furthermore, in the case \(a \neq 0\), the versal unfolding
\[
\dot{\rho} = \mu \rho + a \rho^3
\]
of (6.3) is generically realized (modulo a rescaling of the parameter) by the following unfolding of (6.1)
\[
\dot{z}(t) = L_0 z_t + F(z_t) + \mu z(t + \tau).
\]

**Corollary 6.2 (Theorem 3.1(1) of [4])** Consider the RFDE (2.1) in the unparameterized case
\[
\dot{z}(t) = L_0 z_t + F(z_t), \tag{6.4}
\]
such that the characteristic equation (2.3) has simple non-resonant purely imaginary roots ±iω1, ±iω2, and no other roots on the imaginary axis (non-resonant double Hopf bifurcation).

If

\[ F(z_t) = A_{20}(z(t + \tau_1))^2 + A_{11}z(t + \tau_1)z(t + \tau_2) + A_{02}(z(t + \tau_2))^2 + A_{30}(z(t + \tau_1))^3 + A_{21}(z(t + \tau_1))^2z(t + \tau_2) + A_{12}z(t + \tau_1)(z(t + \tau_2))^2 + A_{03}(z(t + \tau_2))^3, \]  

(6.5)

where \( \tau_1, \tau_2 \in [-r, 0] \), then the uncoupled radial part of the center manifold equations to cubic order are

\[
\begin{align*}
\dot{\rho}_1 &= (a_{11}\rho_1^2 + a_{12}\rho_2^2)\rho_1 \\
\dot{\rho}_2 &= (a_{21}\rho_1^2 + a_{22}\rho_2^2)\rho_2,
\end{align*}
\]

(6.6)

where \( a_{ij} = a_{ij}(A_{20}, A_{11}, A_{02}, A_{30}, A_{21}, A_{12}, A_{03}; \tau_1, \tau_2, \omega_1, \omega_2) \). Generically, the non-degeneracy condition \( a_{11}a_{22} - a_{21}a_{12} \neq 0 \) is satisfied. In fact, for any \( a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \), (6.6) can be realized with \( A_{20} = A_{11} = A_{02} = 0 \) for an appropriate choice of \( A_{30}, A_{21}, A_{12}, A_{03} \) in (6.5). Furthermore, in the case \( a_{11}a_{22} - a_{21}a_{12} \neq 0 \), the versal unfolding

\[
\begin{align*}
\dot{\rho}_1 &= (\mu_1 + a_{11}\rho_1^2 + a_{12}\rho_2^2)\rho_1 \\
\dot{\rho}_2 &= (\mu_2 + a_{21}\rho_1^2 + a_{22}\rho_2^2)\rho_2
\end{align*}
\]

(6.7)

of (6.6) is generically realized (modulo a linear change of parameters) by the following unfolding of (6.4)

\[
\dot{z}(t) = L_0z_t + F(z_t) + \mu_1 z(t + \tau_1) + \mu_2 z(t + \tau_2).
\]

Remark 6.3 We would like to clarify the statement “modulo a linear change of parameters” in the preceding Corollary. According to the notation we have established in this paper, we have

\[
V_2^{2+2}(\mathbb{R}) = \hat{V}_2^{2+2}(\mathbb{R}) = W_2^{2+2}(\mathbb{R}) = \hat{W}_2^{2+2}(\mathbb{R}) = \text{span} \{ \mu_1 v_1, \mu_1 v_2, \mu_2 v_1, \mu_2 v_2 \}
\]

and

\[
H_2^{2+2}(\mathbb{R}^2, \mathbb{Z}_{2,2}) = P_2^{2+2}(\mathbb{R}^2, \mathbb{Z}_{2,2}) = \text{span} \left\{ \mu_1 \left( \begin{array}{c} \rho_1 \\ 0 \end{array} \right), \mu_1 \left( \begin{array}{c} 0 \\ \rho_2 \end{array} \right), \mu_2 \left( \begin{array}{c} \rho_1 \\ 0 \end{array} \right), \mu_2 \left( \begin{array}{c} 0 \\ \rho_2 \end{array} \right) \right\}.
\]

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From Proposition 4.3, the mapping
\[ N_2^\tau : V_2^{+2}(\mathbb{R}) \rightarrow H_2^{+2}(\mathbb{R}^2, \mathbb{Z}_{2,2}) \]
is generically invertible. Since the mapping \( N_2^\tau \) does not have any effect on the parameters \( \mu_1 \) and \( \mu_2 \), generically we have
\[
(N_2^\tau)^{-1} \begin{pmatrix} \mu_j \\ \rho_j \end{pmatrix} = \begin{pmatrix} m_{11} v_1 + m_{12} v_2, \\
 m_{21} v_1 + m_{22} v_2, 
\end{pmatrix}, \quad j = 1, 2,
\]
and it follows that the \( 2 \times 2 \) matrix \( M = (m_{ij}) \) is invertible. Consequently, the unfolding (6.7) of (6.6) is realized by the following unfolding of (6.4)
\[
\dot{z}(t) = L_0 z_t + F(z_t) + \mu_1 (m_{11} z(t + \tau_1) + m_{12} z(t + \tau_2)) + \mu_2 (m_{21} z(t + \tau_1) + m_{22} z(t + \tau_2)).
\]
We get the conclusion of Corollary 6.2 by performing the linear change of parameters
\[
\tilde{\mu}_1 = m_{11} \mu_1 + m_{21} \mu_2, \quad \tilde{\mu}_2 = m_{12} \mu_1 + m_{22} \mu_2
\]
and dropping the tildes.

**Corollary 6.4 (Theorem 2 of [11])** Consider the RFDE (2.1) in the unparameterized case
\[
\dot{z}(t) = L_0 z_t + F(z_t), \tag{6.8}
\]
such that the characteristic equation (2.3) has simple purely imaginary roots \( \pm i \omega \neq 0 \), a simple root at 0, and no other roots on the imaginary axis (interaction of a simple bifurcation and a Hopf bifurcation). If
\[
F(z_t) = A_{20}(z(t + \tau_1))^2 + A_{11} z(t + \tau_1) z(t + \tau_2) + A_{02}(z(t + \tau_2))^2 \tag{6.9}
\]
where \( \tau_1, \tau_2 \in [-r, 0] \), then the uncoupled radial part of the center manifold equations to quadratic order are
\[
\dot{\rho}_0 = b_1 \rho_0^2 + b_2 \rho_1^2, \quad \dot{\rho}_1 = a_1 \rho_0 \rho_1, \tag{6.10}
\]
where the coefficients \( a_1, b_1 \) and \( b_2 \) are functions of \( (A_{20}, A_{11}, A_{02}; \tau_1, \tau_2, \omega) \). Generically, the non-degeneracy conditions \( a_1 \neq 0, b_1 \neq 0, b_2 \neq 0 \) and \( a_1 \neq b_2 \) are satisfied.
Corollary 6.5 Consider the singularity (6.10) in the non-degenerate case $a_1 \neq 0$, $b_1 \neq 0$, $b_2 \neq 0$ and $a_1 \neq b_2$. Then the following Langford unfolding [21] of (6.10) in the transcritical case
\[
\dot{\rho}_0 = \mu_1 \rho_0 + b_1 \rho_0^2 + b_2 \rho_1^2 \\
\dot{\rho}_1 = \mu_2 \rho_1 + a_1 \rho_0 \rho_1,
\]
(6.11)
is generically realized (modulo a linear change of parameters) by the following unfolding of (6.8)
\[
\dot{z}(t) = L_0 z_t + F(z_t) + \mu_1 z(t + \tau_1) + \mu_2 z(t + \tau_2).
\]

7 Conclusions

We have established a framework for the realizability problem for scalar RFDEs which exploits fully the toroidal equivariance of normal forms of bifurcations associated with purely imaginary eigenvalues. This has allowed us to recover and significantly generalize recent results of Faria and Magalhães [11] and of Buono and Bélair [4]. As mentioned in the Introduction, it is important for modelers using RFDEs to be able to accurately assess the range of possible dynamics accessible within their models. For this purpose, this paper gives a thorough analysis of this question in the case where the model is a nonlinear delay-differential equation undergoing non-resonant multiple Hopf bifurcation or transcritical/non-resonant multiple Hopf interaction. Specifically, we split the dynamics of the normal form into components which are normal to the orbits of a torus group, and components which are tangent to these group orbits. Sharp estimates on the number of delays are then given for the realizability of the normal “radial” part of the normal form by nonlinear delay-differential equations. The case of saddle-node/non-resonant multiple Hopf interaction will be treated using similar techniques in a subsequent paper [5].

The generalizations we have achieved in our paper are twofold. First, we can treat within a unified framework the general case of $p$ non-resonant Hopf eigenvalues and the interaction between simple steady-state bifurcation and $p$ non-resonant Hopf bifurcation. Second, in contrast to [11] and [4] where only the generic (non-degenerate) cases are treated, we can treat the general finitely-determined case (whether degenerate or not) and its unfoldings, also within a unified framework. Note that in parameterized families of vector fields with sufficiently many parameters, it becomes possible to violate any specified non-degeneracy condition which is expressed in terms of the Taylor coefficients of the vector field up to some finite order. Therefore, it becomes desirable to have a framework in which these degenerate cases and their unfoldings can be systematically treated. Our results provide such a framework.
Open problems of interest related to this analysis and worthy of further investigation are

- relaxing the restriction to scalar RFDEs in order to consider \( n > 1 \) dimensional systems of RFDEs
- incorporating resonances in the purely imaginary eigenvalues and repeated eigenvalues with Jordan blocks.

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A Proof of Proposition 3.5

Let \( f \) be a given element of \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) \), and consider \( \tilde{f} = (f, 0) \in H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}) \). From Proposition 3.2, there exists \( \tilde{h} = (h_1, h_2) \in H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}) \) and a unique \( \tilde{g} = (g_1, g_2) \in H^{\kappa+s}_\ell(\mathbb{R}^{\kappa+s}, \Gamma) \) such that

\[
(f, 0) = \mathcal{L}_B(h_1, h_2) + (g_1, g_2). \tag{A.1}
\]

Now, \( \mathcal{L}_B(h_1, h_2) = (\mathcal{L}_B h_1, D_x h_2 B x) \), so it follows that \( g_2 = -D_x h_2 B x \). Consequently, (A.1) can be rewritten as

\[
(f, 0) = (\mathcal{L}_B h_1, 0) + (g_1, 0),
\]

and thus

\[
f = \mathcal{L}_B h_1 + g_1, \tag{A.2}
\]

where \( g_1 \in H^{\kappa+s}_\ell(\mathbb{R}^\kappa, T^p) \). So \( H^{\kappa+s}_\ell(\mathbb{R}^\kappa) = H^{\kappa+s}_\ell(\mathbb{R}^\kappa, T^p) + \text{range } \mathcal{L}_B \). Suppose \( f = 0 \) in (A.2), then it is easy to see that \( \mathcal{L}_B(h_1, 0) + (g_1, 0) = (0, 0) \), and from Proposition 3.2, it follows that \( g_1 = 0 \) and \( \mathcal{L}_B h_1 = 0 \). Therefore,

\[
H^{\kappa+s}_\ell(\mathbb{R}^\kappa) = H^{\kappa+s}_\ell(\mathbb{R}^\kappa, T^p) \oplus \text{range } \mathcal{L}_B.
\]

\[ \blacksquare \]
B Proof of Proposition 3.7

For a given $f \in H_{\ell}^{\kappa+s}(\mathbb{R}_\kappa)$, let $g = Af$; then

$$\begin{align*}
(Ag)(x, \mu) &= \int_{\Gamma_0} \tilde{\gamma} g(\tilde{\gamma}^{-1} x, \mu) d\tilde{\gamma} = \int_{\Gamma_0} \tilde{\gamma} \left( \int_{\Gamma_0} \gamma f(\gamma^{-1} \tilde{\gamma}^{-1} x, \mu) d\gamma \right) d\tilde{\gamma} \\
&= \int_{\Gamma_0} \left( \int_{\Gamma_0} \tilde{\gamma} \gamma f(\tilde{\gamma}^{-1} x, \mu) d\gamma \right) d\tilde{\gamma} \\
&= \int_{\Gamma_0} \left( \int_{\Gamma_0} \gamma f(\gamma^{-1} x, \mu) d\gamma \right) d\tilde{\gamma} = \int_{\Gamma_0} \gamma f(\gamma^{-1} x, \mu) d\gamma \\
&= (Af)(x, \mu),
\end{align*}$$

where the second to last line holds because of the translation invariance and the normalization of the Haar integral. So $A$ is a projection.

Now, let $f \in \text{range } A$, then $Af = f$, i.e.

$$f(x, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1} x, \mu) d\gamma.$$

So, for any $\sigma \in \Gamma_0$, we have

$$\sigma f(\sigma^{-1} x, \mu) = \sigma \int_{\Gamma_0} \gamma f(\gamma^{-1} \sigma^{-1} x, \mu) d\gamma = \int_{\Gamma_0} \sigma \gamma f((\sigma \gamma)^{-1} x, \mu) d\gamma$$

$$= \int_{\Gamma_0} \gamma f(\gamma^{-1} x, \mu) d\gamma = f(x).$$

Therefore, $f \in H_{\ell}^{\kappa+s}(\mathbb{R}_\kappa, \mathbb{T}_\rho)$. On the other hand, if $f \in H_{\ell}^{\kappa+s}(\mathbb{R}_\kappa, \mathbb{T}_\rho)$, then

$$(Af)(x, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1} x, \mu) d\gamma = \int_{\Gamma_0} f(x, \mu) d\gamma = f(x, \mu),$$

so $f \in \text{range } A$. This establishes (3.10). We now establish (3.11). Since $A$ is a projection, then

$$H_{\ell}^{\kappa+s}(\mathbb{R}_\kappa) = \text{range } A \oplus \ker A.$$

From Proposition 3.5, we conclude that $\dim \ker A = \dim \text{range } \mathcal{L}_B$. Thus, we need only show that $\text{range } \mathcal{L}_B \subset \ker A$. In order to show this, we will need the following

**Lemma B.1** Let $g : \Gamma_0 \to \mathbb{R}_\kappa$ be a continuous function, then

$$\int_{\Gamma_0} g(\gamma) d\gamma = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} g(e^{Rs}) ds.$$
Proof. For a given $q \in \{1, 2, \ldots, p\}$, consider the rotation matrix

$$R_q(\theta) = \text{diag}(e^{i\omega_q \theta}, e^{-i\omega_q \theta})$$

which is $T_q \equiv 2\pi/\omega_q$-periodic in $\theta$. Then $\tilde{R}_q(\theta) \equiv R_q(\theta T_q/(2\pi))$ is $2\pi$-periodic in $\theta$. By hypothesis, the set

$$\left\{ \frac{2\pi}{T_1}, \ldots, \frac{2\pi}{T_p} \right\}$$

is algebraically independent. Let $T_p \equiv [0, 2\pi]^p$, then we can parameterize $\Gamma_0$ as follows:

$$h : T^p \rightarrow \Gamma_0$$

$$h(\theta_1, \ldots, \theta_p) = \begin{cases} \text{diag}(\tilde{R}_1(\theta_1), \ldots, \tilde{R}_p(\theta_p)) & \text{if } \kappa = 2p \\ \text{diag}(1, \tilde{R}_1(\theta_1), \ldots, \tilde{R}_p(\theta_p)) & \text{if } \kappa = 2p + 1. \end{cases}$$

Define $\tilde{g} : T^p \rightarrow \mathbb{R}^\kappa$ by $\tilde{g}(\theta_1, \ldots, \theta_p) = (g \circ h)(\theta_1, \ldots, \theta_p)$. Obviously, $\tilde{g}$ is $2\pi$-periodic in each of its entries, and

$$\frac{1}{(2\pi)^p} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{g}(\theta_1, \ldots, \theta_p) d\theta_1 \cdots d\theta_p = \int_{\Gamma_0} g(\gamma) d\gamma.$$

Noting that $h\left(\frac{2\pi s}{T_1}, \ldots, \frac{2\pi s}{T_p}\right) = e^{Bs}$ and using Lemma 4.1, P.430 of [6], we get that

$$\frac{1}{(2\pi)^p} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{g}(\theta_1, \ldots, \theta_p) d\theta_1 \cdots d\theta_p = \lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{g} \left(\frac{2\pi s}{T_1}, \ldots, \frac{2\pi s}{T_p}\right) ds$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T g(e^{Bs}) ds,$$

which yields the desired result. \hfill \blacksquare

Now, let $f \in \text{range } \mathcal{L}_B$; then there exists $g \in H^{\kappa+s}(\mathbb{R}^\kappa)$ such that

$$D_x g(x, \mu) Bx - Bg(x, \mu) = f(x, \mu), \quad \forall (x, \mu) \in \mathbb{R}^{\kappa+s}.$$
Therefore, using Lemma B.1, we get

\[(Af)(x, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x, \mu) \, d\gamma = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{Bs} f(e^{-Bs}x, \mu) \, ds\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{Bs} \left(D_x g(e^{-Bs}x, \mu)Be^{-Bs}x - Bg(e^{-Bs}x, \mu)\right) \, ds
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d}{ds} \left(e^{Bs} g(e^{-Bs}x, \mu)\right) \, ds
\]

\[
= \lim_{T \to \infty} \frac{e^{BT} g(e^{-BT}x, \mu) - g(x, \mu)}{T}
\]

and this last limit is equal to 0, since the numerator is bounded in T for any given \((x, \mu) \in \mathbb{R}^{\kappa + s}\). So we conclude that \(f \in \ker A\), and thus that \(\ker A = \text{range } \mathcal{L}_B\). This establishes (3.11), and concludes the proof of Proposition 3.7.

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