POINTWISE METRIC COMPATIBLE CONNECTIONS
AND A CONJECTURE OF CHERN ON AFFINE
MANIFOLDS

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Let \( \xi \) be an oriented vector bundle over a manifold \( N \), and \( \nabla \) a
connection in \( \xi \). Let \( E \) be the total space of \( \xi \) and

\[
\pi : E \mapsto N
\]

be the canonical projection onto \( N \).

**Definition 0.1.** We say that \( \nabla \) is pointwise metric if for every \( p \in N \) there exist an open neighborhood \( U \subset N \) and a positive definite metric \( h \) defined on \( \pi^{-1}(U) \) such that \((\nabla h)(p) = 0\).

**Definition 0.2.** Let \( \xi \) be a vector bundle as in the previous definition. Let \( K \) denote the set of all locally metric connections in \( \xi \). A proper deformation in \( K \) is a smooth map \( \gamma : [0, 1] \to K \) which has the property that for every \( p \in N \) there exist an open neighborhood \( U \subset N \) such that every connection in \( \gamma([0, 1]) \) is pointwise metric on \( U \).

Our main result is:

**Theorem 0.3.** Let \( TM \) be the tangent bundle of a compact, even dimensional affine manifold \( M^{2m} \). Then the Euler class of \( TM \) is zero.

1. **The Euler form of a pointwise metric connection**

This section describes the construction of the Euler form of a pointwise metric connection. For technical details we will also refer the reader to [1]. In what follows the manifold \( M \) is a smooth, closed and even dimensional manifold of dimension \( n = 2m \). Let us briefly remember the construction of the Euler form associated to a Riemannian connection. Let \( M \) be an \( n \)-dimensional oriented manifold, \( g \) a Riemannian metric, and \( D \) its associated Levi-Civita connection. Let \( (e_i)_{i=1,\ldots,n} \) be a positive local orthonormal frame with respect to \( g \) and let \( (\theta_i)_{i=1,\ldots,n} \) be the connection forms with respect to the frame \( (e_i)_{i=1,\ldots,n} \). They are defined by the equations

\[
De_j = \theta_{ij}e_i.
\]
The matrix \((\theta_{ij})\) is skew-symmetric. The curvature forms are defined by Cartan’s second structural equation

\[
\Omega_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj}
\]

and the matrix \((\Omega_{ij})\) is skew symmetric as well. The matrix \((\Omega_{ij})\) globally defines an endomorphism of the tangent bundle, and therefore the trace is independent of the choice of the local frame \((e_i)\). Moreover, since the matrix \((\Omega_{ij})\) is skew-symmetric, its determinant is a ”perfect square”, hence the square root is also invariant under a change of the positive local frame. A heuristic definition of the Euler form of \(D\) is

\[
E(D) = \sqrt{\det \Omega}.
\]

From (3) we see that \(E\) is an \(n\)-form defined globally on \(M\), hence it defines a cohomology class.

In order to define the Euler form of a pointwise metric connection we need first some linear algebra and local considerations. Let \(V\) be a \(2n\)-dimensional vector space and let \(A\) be a skew-symmetric matrix with 2-forms as entries, that is

\[
A \in \Lambda^2(V, so(2n, \mathbb{R})).
\]

The Pfaffian \(Pf\) is map

\[
Pf : \Lambda^2(V, so(2n, \mathbb{R})) \mapsto \Lambda^{2m}(V).
\]

which, for a matrix

\[
A = \begin{bmatrix}
0 & a_{1,2} & \ldots & a_{1,2n} \\
-a_{1,2} & 0 & \ldots & a_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{2n,1} & -a_{2n,2} & \ldots & 0
\end{bmatrix},
\]

is defined as

\[
Pf(A) = \sum_{\alpha \in \Pi} sgn(\alpha) a_{\alpha}.
\]

Here \(a_{\alpha} = a_{i_1,j_1} \wedge a_{i_2,j_2} \wedge \ldots \wedge a_{i_n,j_n}\) and \(\Pi\) is the set of all partitions of the set \(\{1,2,3,\ldots,2n\}\) into pairs of elements. Since every element \(\alpha\) of \(\Pi\) can be represented as

\[
\alpha = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\},
\]

and since any permutation \(\pi\) associated to \(\alpha\) has the same signature as

\[
\pi = \begin{bmatrix}
1 & 2 & 3 & 4 & \ldots & 2n \\
i_1 & j_1 & i_2 & j_2 & \ldots & j_n
\end{bmatrix},
\]
the equality \([4]\) makes sense. The following lemma will allow us to define the Euler form of a pointwise metric connection.

**Lemma 1.1.** Let \(A, B \in \Lambda^2(V, so(2n, \mathbb{R}))\) be two conjugate matrices, that is
\[
B = U^{-1}AU,
\]
for some nonsingular matrix \(U\) with positive determinant. Then
\[
Pf(A) = Pf(B).
\]

**Proof.** It is known that if
\[
B = Q^T AQ,
\]
for some orthogonal matrix \(Q\), then
\[
Pf(B) = \pm 1 Pf(A),
\]
where \(\pm 1 = \det Q\) (for a proof of this fact see [2], Appendix C, Lemma 9). From our assumption that both \(A\) and \(B\) are skew-symmetric and conjugate and if \(U\) is a diagonal matrix then it actually follows that \(A = B\) and hence
\[
Pf(A) = Pf(B).
\]

If \(U\) is an arbitrary matrix with positive determinant then by the Singular Decomposition Theorem for real matrices we have
\[
U = SDL,
\]
with \(S, L\) orthogonal with positive determinant and \(D\) diagonal with positive entries, so it follows that
\[
Pf(U^{-1}AU) = Pf(L^TD^{-1}S^TASDL) = Pf(A).
\]

\(\square\)

**Remark:** The fact that the two matrices in the hypothesis of Lemma 1.1 are skew-symmetric is essential. In general the conjugate of a skew-symmetric matrix is not skew-symmetric.

Lemma 1.1 will allow us to construct the Euler form of a pointwise metric connection. The construction is as follows.

Let \(\xi\) be an oriented vector bundle over a manifold \(N\), and let \(\nabla\) be a connection in \(\xi\). Let \(E\) be the total space of \(\xi\) and let
\[
\pi : E \mapsto N
\]
be the canonical projection onto \(N\).

Now let us consider \(p \in V \subset N\) where \(V\) is an open neighborhood and \(h\) a local metric compatible with \(\nabla\) at \(p\). If \(k \in 2\mathbb{Z}\) is the dimension of
the fiber of $\xi$ then let $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ be a positive oriented local orthonormal frame w.r.t $h$, that is
\[ h(\nu_i, \nu_j) = \pm 1. \]
The connection forms $\theta_{ij} \in \Omega^1(V)$ are defined by
\[ \nabla \nu_i = \theta_{ij} \nu_j, \]
and the curvature forms $\Omega_{ij} \in \Omega^2(V)$ are defined by
\[ \Omega_{ij} = d\theta_{ij} - \theta_{is} \wedge \theta_{sj}. \]
Because $(\nabla h)(p) = 0$, it follows that $\theta_{ij} = -\theta_{ji}$ and hence
\[ \Omega_{ij} = -\Omega_{ji}. \]
Now let us consider another metric compatible with $\nabla$ at $p$, that is a metric $g$ such that $(\nabla g)(p) = 0$.
Let $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$ be a positive local orthonormal frame w.r.t $g$. Doing the same thing as before for this metric we obtain a skew-symmetric curvature matrix $\tilde{\Omega}_{ij}$. We obviously have
\[ \tilde{\Omega} = A^{-1} \Omega A, \]
where $A$ is the matrix defined by
\[ \eta = A\nu, \]
and for which $\det A > 0$. Taking into account Lemma 1.1 we conclude that
\[ Pf(\Omega) = Pf(\tilde{\Omega}). \]
Therefore we define the Euler form of the locally metric connection $\nabla$ by
\[ \mathcal{E}_\nabla = Pf(\Omega). \]
By definition $\mathcal{E}_\nabla \in \Omega^k(N)$ is a globally defined form of degree $k$. We also have the following fundamental (local!) statement.

**Theorem 1.2.** The Euler form of a locally metric connection is a closed form. It therefore defines a cohomology class of $N$.

For a proof of this fundamental theorem see [1] (Lemma 18.1).

The proof of Theorem 0.3 now follows.

**Proof.**
Let $g$ be a global Riemannian metric on $M$ that has $D$ as its Levi Civita connection. First we will prove that $\nabla$ can be deformed into the global metric connection $D$ through pointwise metric connections. Second we will prove that $E$, the Euler form of $\nabla$, and $E'$ the Euler form of $D$, represent the same cohomology class.

We begin by constructing a one parameter family of pointwise metric connections on $TM$ denoted $\nabla^t$ for $t \in [0, 1]$. Take $p \in M$. Let $U$ be a contractible affine neighborhood of $p$. Since the restricted holonomy group of $\nabla$ with respect to $p$ is trivial, then there exist a UNIQUE Riemannian metric $h$ on $U$ such that $\nabla h = 0$ and $h(p) = g(p)$.

Consider the metric on $U$ defined by $h^t = (1 - t)h + tg$ and let $D^t$ be its Levi Civita connection. Let $X$ be a tangent vector field on $U$ and $v \in T_p M$ and we define the covariant derivative $(\nabla^t)(p)$ as

$$\nabla^t_v X = D^t_v X.$$ 

From its construction it is obvious that $(\nabla^t)((1 - t)h + tg)(p) = 0$ and that $\nabla^0 = \nabla$ and $\nabla^1 = D$.

Let $\pi : M \times [0, 1] \to M$ be defined as $\pi(p, t) = p$.

First we need to prove that the deformation $\nabla^t$ of $\nabla$ into $D$ defines a pointwise metric connection on $\tau = \pi^*(TM)$. We set $\pi^*(\nabla^t) = D^t$, and note that $g^t = \pi^*(h^t)$ are metrics compatible with $D^t$ on $\pi^{-1}(p)$.

We define the connection $\mathbb{D}$ on $\tau$ by defining its action on a smooth section $\sigma$ of $\tau$

$$(9) \quad (\mathbb{D}\sigma)(x, t) = (D^t\sigma)(x, t).$$

On $\pi^{-1}(p) = \{p\} \times \mathbb{R}$ we define a metric $g$ by
(10) \[ g(v_1, v_2)(p, t) = g_t(v_1, v_2)(p, t) \]
for \( v_1, v_2 \in T_{(p,t)}M \times [0,1] \). If \( X \in T(M \times [0,1]) \) is an arbitrary vector
field, taking the derivative of the metric \( g \) in the direction defined by \( X \),
\[
(\mathbb{D}_X g)(\sigma_1, \sigma_2) = X(g(\sigma_1, \sigma_2)) - g(\mathbb{D}_X \sigma_1, \sigma_2) - g(\sigma_1, \mathbb{D}_X \sigma_2),
\]
and now evaluating it at \((p, t) \in M \times [0,1]\), we obtain
\[
(\mathbb{D}_X g)(p, t) = ((D_t)_{Xg})(\sigma_1, \sigma_2)(p, t).
\]
Therefore
\[
(\mathbb{D}_X g)(p, t) = 0.
\]
It follows that the connection defined by (9) is pointwise metric and its Euler form \( \mathcal{A} \) is well defined. Thus according to Theorem 1.2 we have that
\[ d\mathcal{A} = 0. \]
We define a family of maps
\[ i_t : M \to M \times [0,1] \]
by
\[ i_t(x) = (x, t). \]
Since the Euler form behaves nicely with respect to pullbacks (see [1] the proof of Lemma 18.2), we have
\[ i_0^* \mathcal{A} = \mathcal{E} \]
and
\[ i_1^* \mathcal{A} = \mathcal{E}'. \]
Because the two maps \( i_0 \) and \( i_1 \) are homotopic, they induce the same map in cohomology and it follows that
\[ \mathcal{E} - \mathcal{E}' \]
is exact on \( M \), and the conclusion of the theorem follows. □

References

[1] Madsen M. and Tornehave J. , From calculus to cohomology: de Rham cohomology and characteristic classes, Cambridge University Press, Cambridge 1997.

[2] Milnor J. and Stasheff J., Characteristic Classes, Annals of Mathematics Studies, Princeton University Press 1974.