LAMINAR CURRENTS AND BIRATIONAL DYNAMICS

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Abstract. We study the dynamics of a bimeromorphic map \( X \rightarrow X \), where \( X \) is a compact complex Kähler surface. Under a natural geometric hypothesis, we construct an invariant probability measure, which is mixing, hyperbolic and of maximal entropy. The proof relies heavily on the theory of laminar currents and is new even in the case of polynomial automorphisms of \( \mathbb{C}^2 \). This extends recent results by E. Bedford and J. Diller.

1. Introduction

Let \( X \) be a compact complex surface and \( f \) be a bimeromorphic self map on \( X \). Moreover assume \( X \) is Kähler. We are interested in the study of \((X, f)\) as a dynamical system. These mappings generalize polynomial automorphisms of \( \mathbb{C}^2 \) (viewed as birational on \( \mathbb{P}^2 \)), whose dynamics have turned out to be very rich. The general setting raises interesting problems, both in dynamics and in intersection theory of positive closed currents.

It is now classical to introduce the dynamical degree \( \lambda \), which is the asymptotic growth rate of the volumes of iterated submanifolds. This number is conjecturally related to the topological entropy of \( f \) by the equation \( h_{\text{top}}(f) = \log \lambda \) (see e.g. V. Guedj [Gu] for a general account). It is to be mentioned that in our context, this equality has been subject to intensive numerical study (N. Abarenkova et al. [Ab1-3]) motivated by questions in statistical physics.

An important contribution to the study of the dynamical system \((X, f)\) was made by J. Diller and C. Favre [DF]. They proved that the mappings with interesting dynamics are those with \( \lambda > 1 \). Under this hypothesis, they constructed positive closed currents \( T^\pm \) such that \((f^{\pm 1})^* T^\pm = \lambda T^\pm \). A classical additional observation is that if \( f \) is not birationally conjugate to an automorphism, then \( X \) is a rational surface.

For the purpose of extending the known results for polynomial automorphisms, a natural approach is to give a meaning to the intersection measure \( \mu = T^+ \wedge T^- \), which should have remarkable dynamical properties (see e.g. [FG, Ca, Di2]). In the most general context, this method, combining pluripotential theory for the definition of \( \mu \), and Pesin’s theory for its fine dynamical study, brings up several difficulties. The reason is the presence of indeterminacy points with possibly complicated dynamics. A recent breakthrough is the paper by E. Bedford and J. Diller [BeD] in which they construct the wedge product measure \( \mu \) and prove it to be mixing and hyperbolic (non zero Lyapounov exponents) under the hypothesis

\[
\sum_{n \geq 0} \frac{1}{\lambda^n} |\log \text{dist}(f^n(I(f^{-1})), I(f))| < \infty.
\]
Our approach differs crucially from the previous ones by the systematic use of the laminar structure of the currents $T^\pm$. This concept dates back to D. Ruelle and D. Sullivan \cite{RS} and was developed by E. Bedford, M. Lyubich and J. Smillie in their seminal paper \cite{BLST}.

Using the laminar structure allows us to define the measure $\mu$ without appealing to pluripotential theory, as the geometric intersection $\mu = T^+ \wedge T^-$ of the disks subordinate to $T^+$ and $T^-$. Next, we derive the dynamical properties of $\mu$ without use of Pesin’s theory, by using an argument in the style of M. Lyubich \cite{L} and J.Y. Briend and J. Duval \cite{BrD} along the laminar currents. The method we use is new even for complex Hénon mappings, and provides a new approach for the geometric analysis of the maximal entropy measure in \cite{BLST}, sections 4, 8, and 9. Since the Briend-Duval argument also works in higher dimensions this approach might open the way to the finer study of $\mu$ in higher dimension (cf. \cite{DS}).

We now state a precise result. The meaning of the “algebraically stable” assumption in the theorem will be made precise in the next section. This does not restrict the scope of the theorem, since any birational map is birationally conjugate to an algebraically stable map.

**Theorem 1.** Let $f$ be an algebraically stable birational map of a rational surface $X$ with dynamical degree $\lambda > 1$. Assume that the currents $T^+$ and $T^-$ have nontrivial geometric intersection $\mu = T^+ \wedge T^-$. Then

i. $\mu$ is an invariant measure which is mixing.

ii. For $\mu$-almost every $p$, there exist unit tangent vectors $e^u(p)$ and $e^s(p)$ at $p$, there exists $N' \subset N$ of density 1, such that

$$\lim \inf_{N' \ni n \to \infty} \frac{1}{n} \log |d f^n(e^u(p))| \geq \frac{\log \lambda}{2} \quad \text{and} \quad \lim \sup_{N' \ni n \to \infty} \frac{1}{n} \log |d f^n(e^s(p))| \leq -\frac{\log \lambda}{2}.$$ 

These bounds are sharp.

iii. $\mu$ has entropy $h_\mu(f) = \log \lambda$. In particular the topological entropy $h_{top}(f)$ is $\log \lambda$.

iv. $\mu$ has product structure with respect to local stable and unstable manifolds. In particular $(f, \mu)$ has the Bernoulli property.

In the case of a projective non rational surface $X$, $f$ is conjugate to an automorphism on a torus or $K3$ surface and the result is already known and due to S. Cantat \cite{Ca}. Nevertheless our proof can be adapted so as to apply in this setting as well.

From this theorem, it is natural to look for criteria ensuring that $T^+ \wedge T^- > 0$. Our second result is the following.

**Theorem 2.** Under the assumptions of Theorem 1 assume further that the Bedford-Diller condition (1) holds. Then $T^+ \wedge T^- = T^+ \wedge T^- > 0$, hence Theorem 1 applies. Moreover $\mu$ describes the asymptotic distribution of saddle orbits, and most saddle points lie inside $\text{Supp} \mu$.

Here “most” means the following: $f$ has approximately $\lambda^n$ periodic points of period $n$, and asymptotically (as $n \to \infty$) the number of saddle points inside $\text{Supp} \mu$ is equivalent to $\lambda^n$. The proof uses classical intersection theory of positive closed currents. It would be interesting in view of getting rid of hypothesis (1) to find a completely geometric argument ensuring that $T^+ \wedge T^- > 0$.

The structure of the paper is as follows. In \cite{DF} we recall some facts on birational dynamics, mainly from \cite{DF}, and some results on laminar currents from \cite{Du} that are crucial in the following. In \cite{Du3} we prove an equidistribution property for preimages of points along the
unstable current $T^-$, which is the analogue of the Lyubich-Briend-Duval lemma \cite{L} \cite{BrD} in our setting. Theorem 1 is proved in \cite{L} and theorem 2 in \cite{BrD}. Another approach to these results, allowing the use of Pesin’s theory, is outlined in the Appendix.

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2. Laminar structure

We first briefly introduce the dynamical setting we consider throughout the paper. For more details, the reader is referred to \cite{DF} \cite{BeD}.

Let $f : X \to X$ be a bimeromorphic map of a compact Kähler surface, with Kähler form $\omega$. We denote by $I(f)$ the indeterminacy set, which is a finite number of points, and by $C(f) = f^{-1}(I(f^{-1}))$ the critical set. We will often use the fact that $f(C(f)) = I(f^{-1})$ and $f(I(f)) = C(f^{-1})$.

We now review some results of \cite{DF}. First, up to a bimeromorphic change of surface, we may assume that $f$ is “algebraically stable”, which means that $\forall n, m \geq 0$, $I(f^n) \cap I(f^{-m}) = \emptyset$.

In this case the dynamical degree $\lambda$ is the spectral radius of the action of $f^*$ on $H^{1,1}(X)$.

From now on we will assume $\lambda > 1$. If $f$ is not birationally conjugate to an automorphism then $X$ is a rational surface (thus our bimeromorphic maps are rather birational). The case of automorphisms of projective non rational surfaces is treated in \cite{Ca}, so we will assume $X$ is a rational surface (see however Remark 2.6 below). There exist nef cohomology classes $\theta^+$ and $\theta^-$ in $H^{1,1}(X)$ such that

$$\frac{1}{\lambda} f^* \theta^+ = \theta^+ \text{ and } \frac{1}{\lambda} (f^{-1})^* \theta^- = \theta^-.$$  

Moreover there exist positive closed currents $T^+/\negthinspace-\negthinspace-$, respectively cohomologous to $\theta^+/\negthinspace-\negthinspace-$, so that for any smooth closed $(1,1)$ form $\alpha$ on $X$,

$$\lambda^{-k}(f^k)^* \alpha \to _{k \to \infty} \frac{\{\alpha\}, \theta^-}{(\theta^+, \theta^-)} T^+,$$

with a similar formula for $T^-$, where $\{\alpha\}$ is the cohomology class of $\alpha$ and $(\cdot, \cdot)$ is the intersection pairing in cohomology.

Here are some known properties of the currents $T^+/\negthinspace-\negthinspace-$:

- $f^*(T^+) = \lambda T^+$ and $(f^{-1})^* T^- = \lambda T^-$;
- $T^+/\negthinspace-\negthinspace-$ are extremal in the cone of positive closed currents;
- if the Lelong number $\nu(p, T^+) > 0$ then $p \in I(f^n)$ for some $n$ (similarly for $T^-$). In particular $T^+/\negthinspace-\negthinspace-$ give no mass to analytic curves. We call such currents diffuse.

The methods in the present article rely very much on some results on the laminar structure of the invariant currents, that we obtained in a series of papers \cite{Du1–3}. The motivation was precisely to extend the results of \cite{BLS} to the widest possible context. We will spend some time to recall some background on the topic.

The starting point is the following definition. The definition is local so here $\Omega$ denotes an open set in $\mathbb{C}^2$, and $T$ a $(1,1)$ positive current in $\Omega$. 


Definition 2.1 (BLS1).

- \( T \) is uniformly laminar if for every \( x \in \text{Supp}(T) \) there exists open sets \( V \supset U \ni x \), with \( V \) biholomorphic to the unit bidisk \( \mathbb{D}^2 \) so that in this coordinate chart \( T|_U \) is the direct integral of integration currents over a measured family of disjoint graphs in \( \mathbb{D}^2 \), i.e.:

  \[
  \text{there exists a measure } \lambda \text{ on } \{0\} \times \mathbb{D}, \text{ and a family } (f_a) \text{ of holomorphic functions } f_a : \mathbb{D} \to \mathbb{D} \text{ such that } f_a(0) = a, \text{ the graphs } \Gamma_{f_a} \text{ of two different } f_a \text{'s are disjoint, and}
  \]

  \[
  T|_U = \int_{\{0\} \times \mathbb{D}} \left[ \Gamma_{f_a} \cap U \right] d\lambda(a).
  \]

- \( T \) is laminar in \( \Omega \) if there exists a sequence of open subsets \( \Omega^i \subset \Omega \), such that \( \|T\| (\partial \Omega^i) = 0 \), together with an increasing sequence of currents \( (T^i)_{i \geq 0} \), \( T^i \) uniformly laminar in \( \Omega^i \), converging to \( T \).

Alternatively, a uniformly laminar current is the foliated cycle associated with an embedded Riemann surface lamination with an invariant transverse measure. More generally, we say that two disks are compatible if they have no isolated intersection points. A laminar current always has a laminar representation

\[
T = \int_A [\Delta_\alpha] d\mu(\alpha)
\]

as an integral over a family of compatible holomorphic disks, but with no lamination structure in general. This means that it is not possible in general to find open subsets \( U \) such that the components of \( \Delta_\alpha \cap U \) are closed in \( U \).

In a dynamical context, D. Ruelle and D. Sullivan [RS] constructed uniformly laminar currents subordinate to the stable and unstable laminations of a uniformly hyperbolic isolated compact set. Laminar currents were introduced in [BLS1] for the purpose of extending the Ruelle-Sullivan construction to the non-uniformly hyperbolic setting. The phenomenon of “folding”, which is apparent in the well known pictures of the Hénon attractor, is a manifestation of the non uniformity of the size of the disks in \( A \).

We proved in \([\text{Du1}]\) the following theorem, which gives a very rough indication of what the local geometry of the Julia sets \( J^\pm \) of a general birational map should look like. Notice that \( \text{Supp}(T^\pm) \subset J^\pm \) but whether equality holds is not known in general. See Diller [Di1] for definitions and results related to this question.

**Theorem 2.2 (Du1).** If \( f \) is an algebraically stable birational map on a rational surface \( X \) with \( \lambda > 1 \), then the Green currents \( T^+ \) and \( T^- \) are laminar.

The proof is not dynamical: we actually show that any limit of rational divisors \( \frac{1}{d_n} [C_n] \) in \( X \) with

\[
\text{genus}(C_n) + \sum_{p \in \text{Sing}(C_n)} n_p(C_n) = O(d_n),
\]

is a laminar current, and that the Green currents are of this form. By strongly approximable we mean a laminar current obtained in this way. A crucial point in the present paper is that these currents have additional properties, that were studied in \([\text{Du2}, \text{Du3}]\). We shall discuss many issues from \([\text{Du2} \in \{1, 1\}] \) and \([\text{Du3}] \) below so here we concentrate on \([\text{Du3}] \).

In the sequel, we will let the dynamics act on the laminar structure, so we need to know how it is organized. Notice first that the usual ordering on positive closed currents is compatible
with the laminar structure, in the following sense: if \( T_1 \) and \( T_2 \) are laminar currents with \( T_1 \leq T_2 \), then they admit representations

\[
T_i = \int_A [\Delta_{\alpha,i}] d\mu_i(\alpha)
\]

with \( \mu_1 \leq \mu_2 \). This allows us to identify disks and pieces of laminations subordinate to laminar currents.

By definition, a flow box is a closed lamination \( \mathcal{L} \) embedded in an open set \( U \simeq \mathbb{D}^2 \) such that in this coordinate chart \( \mathcal{L} \) is biholomorphic to a lamination by graphs over \( \mathbb{D} \) (\( \mathbb{D} \) denotes the unit disk). These graphs are called plaques. If \( \mathcal{L} \) is a flow box, we define the restriction \( T|_{\mathcal{L}} \) in terms of the representation (3) by integrating only over the disks in \( A \) lying inside one leaf if \( \mathcal{L} \).

Definition 2.3 (Du3).

- A holomorphic disk \( \Delta \) is subordinate to \( T \) if there exists a nonzero uniformly laminar current \( S \) with \( S \leq T \), and \( \Delta \) lies inside a leaf of the lamination associated to \( S \).
- A flow box subordinate to \( T \) is a flow box \( \mathcal{L} \) such that \( \text{Supp}(T|_{\mathcal{L}}) = \mathcal{L} \).
- The regular set \( \mathcal{R}(T) \) is the union of flow boxes, or equivalently the union of disks subordinate to \( T \).

The main result in [Du3] asserts that if \( T \) is strongly approximable, the flow boxes match correctly and for every flow box \( \mathcal{L} \), the restriction current \( T|_{\mathcal{L}} \) is uniformly laminar, i.e. \( T \) induces an invariant transverse measure on \( \mathcal{L} \). This will play the role of [BLS1, §4] in our context. More precisely, by weak lamination we mean a countable union of compatible flow boxes, where compatible here means the associated plaques do not meet at isolated points. A transversal is by definition a compact set in a flow box which meets each plaque at most once.

One feature of this definition is that the notions of leaf, holonomy, and transverse measure make sense in this setting.

Theorem 2.4 (Du3, Theorems 1.1 and 5.7). Let \( T \) be a diffuse strongly approximable current on the rational surface \( X \). The regular set \( \mathcal{R}(T) \) has the structure of a weak lamination in the preceding sense. Moreover \( T \) induces a holonomy invariant transverse measure on this weak lamination.

If \( T \) is extremal as a positive closed current, the transverse measure is ergodic, i.e. any measurable saturated set has zero or full measure.

The ergodicity of the weak lamination will be used in the paper through the following reformulation: for any pair of transversals \( \tau_1, \tau_2 \) of positive transverse measure, there exists a disk subordinate to \( T \) intersecting both \( \tau_1 \) and \( \tau_2 \). The theorem in [Du3] was stated for \( X = \mathbb{P}^2 \), nevertheless we explain how to adapt it to a general rational surface \( X \).

We first need to prove the fact that \( \mathcal{R}(T) \) being a weak lamination is invariant under birational conjugacy. It suffices to analyze the action of a birational map \( h : \mathbb{P}^2 \to X \) on a flow box \( \mathcal{L} \). Recall that \( h \) is the composition of finitely many point blow-ups and inverses of point blow-ups, so it suffices to understand the action of one single blow-up or blow-down on \( \mathcal{L} \).

Let \( U \) be an open set such that \( \mathcal{L} \) is embedded in \( U \). If \( \pi \) is the blow-up at some point \( p \in \mathcal{L} \), \( \pi : \pi^{-1}(U) \setminus \pi^{-1}(p) \to U \setminus \{p\} \) is a biholomorphism. Letting \( \mathcal{L}' \) denote \( \mathcal{L} \setminus L(p) \), where \( L(p) \) is the plaque through \( p \), it is easy to cover \( \pi^{-1}(\mathcal{L}') \) with at most countably many
flow boxes. The remaining leaf has zero transverse measure so holonomy invariance of the transverse measure is not affected.

Assume now $T$ is a diffuse strongly approximable laminar current in $\pi^{-1}(U)$. If $\pi_*T$ has non compatible flow boxes, the only possible point of non compatibility is $p$. But diffuse flow boxes cannot meet at a single point, so we get a contradiction.

Invariance by holonomy of the induced transverse measure as well as the statement concerning ergodicity are adapted in a similar fashion.

**Remark 2.5.** If $\mathcal{L}$ is a flow box crossing a component $V$ of the critical set $C(f)$, then the images of the plaques of $\mathcal{L}$ meet at the point $f(V)$. A geometric consequence is the following fact:

For every disk $\Delta$ subordinate to $T^-$ (resp. $T^+$), $\Delta \cap C(f^n)$ (resp. $\Delta \cap C(f^n)$) is a compatible intersection, that is, either $\Delta \subset C(f^n)$ or $\Delta \cap C(f^n) = \emptyset$.

The proof is a simple consequence of the invariance of the currents, together with Favre’s theorem [F] that for every $p \in I(f^n)$, the Lelong number $\nu(p,T^-)$ vanishes.

On the other hand, it is possible for a disk subordinate to $T^+$ to intersect a component $V$ of $C(f^n)$. This will yield a “pencil” of plaques through $f^n(V)$. This phenomenon is seemingly observed on computer pictures of stable and unstable manifolds of birational maps (see e.g. [BeD2]).

We will often need to estimate the transverse measure of a given set of plaques. If $T$ is strongly approximable, $\mathcal{L}$ is a flow box, and $\tau$ is a holomorphic disk transverse to $\mathcal{L}$, the induced transverse measure on $\tau$ is given by the wedge product $T|_\mathcal{L} \wedge \tau$. It is easily proved [Du3, Proposition 5.4] that if $\tau$ is the generic (in the measure theoretic sense) member of a smooth family of holomorphic transversals to $\mathcal{L}$, then the wedge product $T \wedge [\tau]$ is admissible (see below §4.1 for a formal definition) and

$$ T|_\mathcal{L} \wedge [\tau] = (T \wedge [\tau])|_{\mathcal{L} \cap \tau}. $$

Abusing notation, if $\tau$ is any transversal to the weak lamination (i.e. a closed set transverse to a flow box), we will denote the transverse measure induced by $T$ on $\tau$ by $T \wedge \tau$.

**Remark 2.6.** Theorem 2.4 is precisely where we use the rationality assumption on $X$. The dynamical analysis we perform in the next sections only rely on its conclusions. The invariant currents associated to automorphisms of projective $K3$ surfaces satisfy these conclusions (see the remarks in [Du3, §3]). In particular the discussion to come also makes sense in that setting, and provide a new approach to the results in [Ca].

### 3. Equidistribution of Preimages along the Unstable Current

In this section $f$ is an algebraically stable birational map on the rational $X$, with $\lambda > 1$. We denote by $M(\cdot)$ the mass of a current or measure and weak convergence of currents or measures is denoted by $\rightharpoonup$. We normalize invariant currents so that their mass is 1. Recall that a transversal is by definition a transversal in a flow box.

The main result in this section is the following equidistribution result. It asserts that generic points on the unstable current $T^-$ become close under backwards iteration. This approach is new even for complex Hénon mappings.

**Proposition 3.1.** If $\tau_1$ and $\tau_2$ are two transversals for the weak lamination associated to $T^-$, then

$$ (f^{-n})_* \left( \frac{T^-(\wedge \tau_1)}{M(T^- \wedge \tau_1)} \right) - (f^{-n})_* \left( \frac{T^-(\wedge \tau_2)}{M(T^- \wedge \tau_2)} \right) \rightharpoonup 0. $$
The proposition will be a consequence of the next lemma, which is the analogue of the Lyubich-Briend-Duval lemma in our context. Areas are computed with respect to the ambient Kähler form \( \omega \).

**Lemma 3.2.** Let \( \mathcal{L} = \{D_t, t \in \tau\} \) be a flow box subordinate to \( T^- \). For every \( \varepsilon > 0 \), there exists a positive constant \( C(\varepsilon) \) and a transversal \( \tau(\varepsilon) \subset \tau \), such that \( M(T^- \wedge \tau(\varepsilon)) \geq (1 - \varepsilon)M(T^- \wedge \tau) \) and

\[
\forall n \geq 1, \forall t \in \tau(\varepsilon), \quad \text{Area}(f^{-n}(D_t)) \leq \frac{C(\varepsilon)n^2}{\lambda^n}.
\]

**Proof:** we first analyze the action of the dynamics on the transverse measure. Assume \( \mathcal{L} \) is a flow box subordinate to \( T^- \), avoiding \( C(f) \cup I(f) \), and \( \tau \) is a transversal in \( \mathcal{L} \). Then \( f(\mathcal{L}) \) is a flow box for \( T^- \) and \( f(\tau) \) a transversal in \( f(\mathcal{L}) \), because \( f \) is a biholomorphism near \( \mathcal{L} \) and \( f_* T^- = \lambda T^- \).

We claim that \( T^- \wedge f(\tau) = \lambda^{-1}f_*(T^- \wedge \tau) \). By holonomy invariance it suffices to prove the result when \( \tau \) lies on a holomorphic disk \( \Delta \) satisfying \( L \). Then \( T^- \wedge \tau = (T^- \wedge \Delta)|_\tau \) is a genuine wedge product and

\[
(f_* T^- \wedge f(\tau) = (f_* T^-) \wedge f(\tau) = \lambda T^- \wedge f(\tau).
\]

In particular, since \( f_* \) acting on measures preserves masses, this implies \( M(T^- \wedge f(\tau)) = \lambda^{-1}M(T^- \wedge \tau) \).

We will now pull back transverse measures. If \( \mathcal{L} \) and \( \tau \) are as before, moving \( \tau \) if necessary we may assume that \( \tau \) being fixed, \( \tau \cap C(f^{-n}) \) is a finite set of points and \( \tau \cap I(f^{-n}) = \emptyset \). So up to a set of zero transverse measure, \( \tau \) is a disjoint union \( \tau = \bigcup \tau_j \), with \( \tau_j \cap C(f^{-n}) = \emptyset \). By the previous formula we get that \( M(T^- \wedge f^{-n}(\tau_j)) = \lambda^n M(T^- \wedge \tau_j) \), since \( f^{-n}(\tau_j) \) avoids \( C(f^n) \cup I(f^n) \).

On the other hand, if \( t_1 \neq t_2 \) in \( \bigcup \tau_j \), the disks \( D_{t_1} \) and \( D_{t_2} \) are disjoint and not contained in \( C(f^{-n}) \) so \( f^{-n}(D_{t_1}) \) and \( f^{-n}(D_{t_2}) \) have at most finitely many intersection points. The total \( \|T\| \)-mass of \( f^{-n}(\mathcal{L}) \) is

\[
\sum_j \int \text{Area}(f^{-n}(D_{f^n(s)}))(T^- \wedge f^{-n}(\tau_j))(s) \leq M(T) = 1.
\]

Since the total mass of \( \sum_j T^- \wedge f^{-n}(\tau_j) \) is \( \lambda^n M(T^- \wedge \tau) \), most disks \( f^{-n}(D_{f^n(s)}) \) have small area with respect to the transverse measure \( \sum_j (T^- \wedge f^{-n}(\tau_j)) \), more precisely

\[
\sum_j (T^- \wedge f^{-n}(\tau_j)) \left( \left\{ s, \text{Area}(f^{-n}(D_{f^n(s)})) \geq \frac{cn^2}{\lambda^n} \right\} \right) \leq \frac{\lambda^n}{cn^2}.
\]

Applying \( (f^n)_* \) yields

\[
(T^- \wedge \tau) \left( \left\{ t, \text{Area}(f^{-n}(D_t)) \geq \frac{cn^2}{\lambda^n} \right\} \right) \leq \frac{1}{cn^2}.
\]

We now get the conclusion of the lemma by considering all integers \( n \) and adjusting \( c = \frac{\pi^2}{6M(T^- \wedge \tau)} \). \( \square \)

From the lemma we deduce a first equidistribution result. Notice that since transversal measures do not charge points, all push forwards \( (f^{-n})_*(T^- \wedge \tau) \) are well defined.
Proposition 3.3. If $\tau_1$ and $\tau_2$ are two global transversals in a flow box $\mathcal{L}$, then
\[(f^{-n})_*(T^- \wedge \tau_1) - (f^{-n})_*(T^- \wedge \tau_2) \to 0.\]

Proof: recall from [BrD] the following basic area-diameter estimate:

Lemma 3.4. There exists a positive constant $c$, such that if $D \subset \tilde{D}$ are (possibly singular) disks in $X$, the following estimate holds
\[\text{Diam}(D)^2 \leq c \frac{\text{Area}(\tilde{D})}{\text{Modulus}(\tilde{D} \setminus D)}.\]

The estimate is only stated for smooth disks in $\mathbb{P}^k(\mathbb{C})$ in [BrD], however the proof depends only on the Lelong theorem, and the notion of extremal length, and it carries over for singular disks without modification.

If $\tau_1$ and $\tau_2$ are closed global transversals in $\mathcal{L}$, they have the same transverse mass by holonomy invariance. Fix a continuous function $\varphi$ on $X$. We must prove
\[\int \varphi \left[ (f^{-n})_*(T^- \wedge \tau_1) - (f^{-n})_*(T^- \wedge \tau_2) \right] \to 0.\]

First, by compactness, there exists a constant $m > 0$ such that for each plaque $\tilde{D}$ of $\mathcal{L}$, there exists a disk $D$, with $(\tau_1 \cap \tilde{D}) \subset D$ and $(\tau_2 \cap \tilde{D}) \subset D$ and $\text{Modulus}(\tilde{D} \setminus D) \geq m$. By the preceding lemmas, for most plaques $\tilde{D}$, points in $f^{-n}(D)$ get exponentially close under backwards iteration. Indeed for every $\varepsilon > 0$, there exists $\tau_i(\varepsilon)$, $i = 1, 2$, with transverse mass $M(T^- \wedge \tau_i(\varepsilon)) \geq (1 - \varepsilon)M(T^- \wedge \tau_i)$, such that for $t \in \tau_i(\varepsilon)$, $\text{Diam}(f^{-n}(D_t))^2 \leq \frac{c\varepsilon^2}{\text{Max}}$. Actually $\tau_1(\varepsilon)$ and $\tau_2(\varepsilon)$ correspond by holonomy since the property that $\text{Area}(f^{-n}(D_t))$ being small is independent of the transversal.

Thus the term in (6) writes as
\[\int \varphi \left[ (f^{-n})_*(T^- \wedge \tau_1|_{\tau_1(\varepsilon)}) - (f^{-n})_*(T^- \wedge \tau_2|_{\tau_2(\varepsilon)}) \right],\]

plus a remainder term not greater than $\varepsilon \|\varphi\| M(T^- \wedge \tau)$ because the mass $M(T^- \wedge \tau_1|_{\tau_1(\varepsilon)})$ is $M(T^- \wedge \tau)$ and $(f^{-n})_*$ preserves the mass of measures. The latter integral equals
\[\int_{\tau_1(\varepsilon)} [\varphi(f^{-n}(D_t \cap \tau_1)) - \varphi(f^{-n}(D_t \cap \tau_2))] d(T^- \wedge \tau_1)(t)\]

which is small because $\varphi$ is continuous and $\text{dist}(f^{-n}(D_t \cap \tau_1), f^{-n}(D_t \cap \tau_2))^2 \leq \frac{c\varepsilon^2}{\text{Max}}$. □

Proof of proposition 3.1: assume first $T^- \wedge \tau_1$ and $T^- \wedge \tau_2$ have the same (positive) mass. Since $T^-$ is extremal, almost every leaf through $\tau_1$ intersects $\tau_2$ (theorem 2.4). This means that for $T^- \wedge \tau_1$-almost every point $p$, there exists a disk through $p$, subordinate to $T^-$ and intersecting $\tau_2$. Such a disk is a neighborhood of a path joining $\tau_1$ and $\tau_2$ in the leaf through $p$. Fattening those disks in the weak lamination, it is standard to prove that for every $\varepsilon > 0$ there exist finitely many disjoint “long flow boxes” $\mathcal{L}_j$, such that $\tau_1 \cap \mathcal{L}_j$ and $\tau_2 \cap \mathcal{L}_j$ are global transversals in $\mathcal{L}_j$, and the transverse mass of $\bigcup_j \tau_1 \cap \mathcal{L}_j$ and $\bigcup_j \tau_2 \cap \mathcal{L}_j$ is greater than $(1 - \varepsilon)M(T^- \wedge \tau_1) = (1 - \varepsilon)M(T^- \wedge \tau_2)$. Now, the $(f^{-n})_*$ equidistribution of $T^- \wedge \tau_1$ and $T^- \wedge \tau_2$ follows as in the previous proposition.
In the general case choose a large integer $N$. For $i = 1, 2$, subdivide $\tau_i$ into $E(NM(T^- \wedge \tau_i))$ pieces $(\tau_{ij})_j$ of transverse mass $1/N$, plus a remainder piece of mass $< 1/N$, where $E(\cdot)$ denotes the integer part function. We may moreover assume the measure of $\overline{\tau_{ij}} \setminus \tau_{ij}$ is zero. By the first part of the proof, all pieces $T^- \wedge \tau_{ij}$ are $(f^{-n})_*$ equidistributed, i.e. for any two pairs $(i, j)$ and $(i', j')$,

$$\lim_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{ij}) - \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i'j'}) \right| = 0.$$ 

Thus for a continuous function $\varphi$ and every $i, j$,

$$\limsup_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_i) - E(NM(T^- \wedge \tau_i)) \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i,j}) \right| \leq \frac{\|\varphi\|}{N}.$$ 

This implies, for some constant $c$ depending only on $M(T^- \wedge \tau_i)$, that

$$\limsup_{n \to \infty} \left| \frac{1}{M(T^- \wedge \tau_1)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_1) - \frac{1}{M(T^- \wedge \tau_2)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_2) \right| \leq \frac{c\|\varphi\|}{N}.$$ 

Since $N$ is arbitrary the result follows.

\[\square\]

### 4. The geometric intersection measure

In order to convert the previous equidistribution statement into a convergence result, we need to find an invariant measure with some geometric structure. In this section we define the geometric intersection of strongly approximable laminar currents, and prove that, if nontrivial, the geometric intersection measure of $T^+$ and $T^-$ has interesting properties.

#### 4.1. Geometric intersection

Geometric intersection of laminar currents is discussed in [BLS1, Du2]. Nonuniqueness of the laminar representation [3] makes the general definition of the geometric intersection measure delicate. In the strongly approximable context, by using the notion of subordinate disks, we provide a nonambiguous definition.

Let $T_1 = dd^c u_1$ and $T_2 = dd^c u_2$ be two closed positive currents in $\Omega \subset \mathbb{C}^2$. We denote by $\|T\|$ the trace measure of the current $T$. We say that the wedge product $T_1 \wedge T_2$ is admissible if $u_1 \in L^1_{\text{loc}}(\|T_2\|)$. Notice that the condition is unambiguous since plurisubharmonic functions are defined pointwise. This condition is clearly independent of the choice of the potential $u_1$ (for convenience we drop the $\text{loc}$ subscript). Under this condition, the wedge product measure $T_1 \wedge T_2$ is defined by

$$T_1 \wedge T_2 = dd^c(u_1 T_2).$$

N. Sibony proved (see [Du1]) that the admissibility condition is symmetric in $T_1$ and $T_2$ and the wedge product operation is continuous under decreasing sequences of the potentials. A useful observation is the following: if $T_1 \wedge T_2$ is admissible and $S_k \leq T_k$, $k = 1, 2$, are positive closed currents, then $S_1 \wedge S_2$ is admissible and $S_1 \wedge S_2 \leq T_1 \wedge T_2$.

Following [BLS], §8 we now define the geometric wedge product of uniformly laminar currents.

**Definition 4.1.** Let $S_1$ and $S_2$ be diffuse uniformly laminar currents, endowed with representations

$$S_k = \int_{\tau_k} [D_{k,a}] d\mu_k(a), \ k = 1, 2$$

Since $\overline{\tau_{ij}} \setminus \tau_{ij}$ is zero. By the first part of the proof, all pieces $T^- \wedge \tau_{ij}$ are $(f^{-n})_*$ equidistributed, i.e. for any two pairs $(i, j)$ and $(i', j')$,

$$\lim_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{ij}) - \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i'j'}) \right| = 0.$$ 

Thus for a continuous function $\varphi$ and every $i, j$,

$$\limsup_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_i) - E(NM(T^- \wedge \tau_i)) \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i,j}) \right| \leq \frac{\|\varphi\|}{N}.$$ 

This implies, for some constant $c$ depending only on $M(T^- \wedge \tau_i)$, that

$$\limsup_{n \to \infty} \left| \frac{1}{M(T^- \wedge \tau_1)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_1) - \frac{1}{M(T^- \wedge \tau_2)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_2) \right| \leq \frac{c\|\varphi\|}{N}.$$ 

Since $N$ is arbitrary the result follows. \[\square\]

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Since $\overline{\tau_{ij}} \setminus \tau_{ij}$ is zero. By the first part of the proof, all pieces $T^- \wedge \tau_{ij}$ are $(f^{-n})_*$ equidistributed, i.e. for any two pairs $(i, j)$ and $(i', j')$,

$$\lim_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{ij}) - \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i'j'}) \right| = 0.$$ 

Thus for a continuous function $\varphi$ and every $i, j$,

$$\limsup_{n \to \infty} \left| \int \varphi(f^{-n})_{*}(T^- \wedge \tau_i) - E(NM(T^- \wedge \tau_i)) \int \varphi(f^{-n})_{*}(T^- \wedge \tau_{i,j}) \right| \leq \frac{\|\varphi\|}{N}.$$ 

This implies, for some constant $c$ depending only on $M(T^- \wedge \tau_i)$, that

$$\limsup_{n \to \infty} \left| \frac{1}{M(T^- \wedge \tau_1)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_1) - \frac{1}{M(T^- \wedge \tau_2)} \int \varphi(f^{-n})_{*}(T^- \wedge \tau_2) \right| \leq \frac{c\|\varphi\|}{N}.$$ 

Since $N$ is arbitrary the result follows. \[\square\]
as integrals of families of submanifolds. We define the product \( \hat{\wedge} \) by

\[
S_1 \hat{\wedge} S_2 = \int_{T_1 \times T_2} [D_{1,a} \cap D_{2,b}] d\mu_1(a) d\mu_2(b),
\]

where by convention the measure \([D_1 \cap D_2]\) is the sum of Dirac masses at the oints of intersection of the disks if they are isolated, zero if not.

Since the currents are diffuse, the set of non transverse intersections has zero measure by \cite{BLST} Lemma 6.4 so counting multiplicities does not affect the integral in (7). The next proposition asserts that when admissible, \( S_1 \hat{\wedge} S_2 \) is described as the geometric intersection of the disks constituting \( S_1 \) and \( S_2 \).

**Proposition 4.2** \cite{Du1}, §3. If the wedge product \( S_1 \hat{\wedge} S_2 \) is admissible, then \( S_1 \) and \( S_2 \) have geometric intersection, i.e. \( S_1 \hat{\wedge} S_2 = S_1 \wedge S_2 \).

Furthermore, if the leaves of the underlying laminations of \( S_1 \) and \( S_2 \) only intersect at isolated points, then \( S_1 \wedge S_2 \) is admissible.

We extend the definition of the geometric wedge product \( \hat{\wedge} \) to sums of uniformly laminar currents by summing the geometric intersections of all factors. We will repeatedly use the obvious fact that the product \( \hat{\wedge} \) is continuous under increasing sequences of the factors.

In the next proposition, we define a geometric wedge product for all strongly approximable currents.

**Proposition 4.3.** Let \( T_1 \) and \( T_2 \) be two diffuse strongly approximable currents on \( X \). There exists a measure \( T_1 \hat{\wedge} T_2 \) such that if \( S_1 \leq T_1 \) and \( S_2 \leq T_2 \) are uniformly laminar currents in \( \Omega \subset \Omega \), then \( S_1 \hat{\wedge} S_2 \leq T_1 \hat{\wedge} T_2 \). Furthermore \( T_1 \hat{\wedge} T_2 \) has finite mass and local product structure (i.e. is a countable sum of product measures).

If the wedge product \( T_1 \wedge T_2 \) is admissible, then \( T_1 \wedge T_2 \leq T_1 \hat{\wedge} T_2 \).

**Definition 4.4.** If \( T_1 \) and \( T_2 \) are two diffuse strongly approximable currents on \( X \), we say \( T_1 \) and \( T_2 \) have non trivial geometric intersection if \( M(T_1 \wedge T_2) > 0 \). The measure \( T_1 \wedge T_2 \) will be referred to as the geometric intersection measure of \( T_1 \) and \( T_2 \).

If moreover the wedge product \( T_1 \wedge T_2 \) is admissible and \( T_1 \hat{\wedge} T_2 = T_1 \wedge T_2 \), we say that \( T_1 \) and \( T_2 \) have (full) geometric intersection (or that the wedge product \( T_1 \wedge T_2 \) is geometric).

Observe that \( T_1 \) and \( T_2 \) have non trivial geometric intersection iff there exist disks \( D_k \), \( k = 1, 2 \), respectively subordinate to \( T_k \), with non trivial intersection.

Recall also that laminar currents were defined as increasing limits of currents uniformly laminar in \( \Omega \subset \Omega \). Hence if the wedge product \( T_1 \wedge T_2 \) is admissible, \( T_1 \) and \( T_2 \) have geometric intersection iff there are such increasing sequences \( T_k^i \uparrow T_k \), \( k = 1, 2 \), with \( T_k^i \hat{\wedge} T_2 \rightarrow T_1 \wedge T_2 \). This is obvious since \( T_k^i \wedge T_2 \leq T_1 \wedge T_2 \leq T_1 \hat{\wedge} T_2 \).

In order to prove proposition 4.3 we first give an a priori bound on masses of geometric intersections. We use hypotheses of global nature. This is actually needed only when the usual wedge product is not admissible, which is really the new case here.

**Lemma 4.5.** There exists a constant \( C \) depending only on \( X \) such that if \( S_i \leq T_i \), \( i = 1, 2 \), is an at most countable sum of uniformly laminar currents \( S_i = \sum_j S_{i,j} \), then \( M(S_1 \wedge S_2) \leq CM(T_1)M(T_2) \) (in case \( X = \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \), the right hand side can be replaced by the intersection pairing \( (\{T_1\}, \{T_2\}) \)).

Moreover if the wedge product \( T_1 \wedge T_2 \) is admissible then \( S_1 \hat{\wedge} S_2 \leq T_1 \wedge T_2 \).
Proof: assume first $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In this case we may regularize a positive closed current on $X$ by considering a family of shrinking neighborhoods $(N_\varepsilon)$ of $id$ in Aut$(X)$, that is, $T_\varepsilon = \frac{1}{\text{Vol}(N_\varepsilon)} \int_{N_\varepsilon} \Phi T d\Phi$ is a smooth positive closed current, and $T_\varepsilon \to T$ when $\varepsilon \to 0$. Observe that the approximation is linear in $T$.

First, by taking an increasing limit, we may assume that the sums $S_i = \sum_j S_{i,j}$ have only finitely many terms (here $i = 1$ or 2). The uniformly laminar current $S_{i,j}$ is closed in some open set $\Omega_{i,j}$. For any small $\alpha > 0$, define $\Omega_{i,j,\alpha}$ as the open subset

$$\Omega_{i,j,\alpha} = \{ p \in \Omega_{i,j}, \quad d(p, \partial \Omega_{i,j}) \geq \alpha \} .$$

As $\alpha \to 0$, $\Omega_{i,j,\alpha}$ increases to $\Omega_{i,j}$, so we have the following increasing limit of currents,

$$S_i = \sum_j S_{i,j}1_{\Omega_{i,j}} = \lim_{\alpha \to 0} \sum_j S_{i,j}1_{\Omega_{i,j,\alpha}} .$$

Fix now $\alpha > 0$. For $\varepsilon > 0$ small enough, we can define the regularization $S_{i,j,\varepsilon}$ in $\Omega_{i,j,\alpha}$, and $S_{i,j} = \lim_{\varepsilon \to 0} S_{i,j,\varepsilon}$ in $\Omega_{i,j,\alpha}$.

Define $S_{i,\varepsilon}$ as $\sum_j S_{i,j,\varepsilon}1_{\Omega_{i,j,\alpha}}$. We get that

$$S_{1,\varepsilon} \wedge S_{2,\varepsilon} = \left( \sum_j S_{1,j,\varepsilon}1_{\Omega_{i,j,\alpha}} \right) \wedge \left( \sum_k S_{2,k,\varepsilon}1_{\Omega_{2,k,\alpha}} \right) \leq T_{1,\varepsilon} \wedge T_{2,\varepsilon}$$

because of the linearity of the approximation. On the other hand the measure on the right hand side has mass ($(\{T_1\}, \{T_2\})$. If the wedge products $S_{1,j} \wedge S_{2,k}$ are locally admissible, for fixed $\alpha$, the convergence $S_{1,j,\varepsilon} \wedge S_{2,k,\varepsilon} \to S_{1,j} \wedge S_{2,k}$ holds in $\Omega_{1,j,\alpha} \cap \Omega_{2,k,\alpha}$, when $\varepsilon \to 0$. This may be seen for instance as a consequence of geometric intersection of uniformly laminar currents. Hence

$$M\left( S_{1} \wedge S_{2} |_{\bigcup_{j,k} \Omega_{1,j,\alpha} \cap \Omega_{2,k,\alpha}} \right) \leq (\{T_1\}, \{T_2\}) .$$

We conclude that $M(S_1 \wedge S_2) \leq (\{T_1\}, \{T_2\})$ by letting $\alpha$ tend to zero. The second assertion of the lemma is obvious.

In the general (non admissible wedge product) case just remark that $S_{i,j}$ can be written as $S_{i,j} = R_{i,j} + Q_{i,j}$, where $R_{i,j}$ is made up of the disks not subordinate to $S_{2,k}$ and having non trivial intersection with $S_{2,k}$. Hence by definition of the geometric wedge product, $S_{1,j} \wedge S_{2,k}$ equals $R_{1,j} \wedge S_{2,k}$. Now the wedge product $R_{1,j} \wedge S_{2,k}$ is admissible by proposition 4.2 so $R_{1,j} \wedge S_{2,k} = S_{1,j} \wedge S_{2,k}$ and we conclude as before, replacing $S_{1,j}$ by $R_{1,j}$.

For an arbitrary rational surface $X$, consider a rational map $h : X \to \mathbb{P}^2$. Since $T_1$ and $T_2$ are diffuse, $S_1 \wedge S_2$ charge neither points nor curves, so

$$M(S_1 \wedge S_2) = M(S_1 \wedge S_2|_{X \setminus (C(h)) \cup \overline{I(h))}}) = M((h_*S_1 \wedge h_*S_2)|_{\mathbb{P}^2 \setminus (C(h^{-1}) \cup \overline{I(h^{-1}))}})$$

where the last equality follows from the fact that $h|_{X \setminus (C(h)) \cup \overline{I(h))}}$ is a biholomorphism.

Let $N$ be any norm on $H^2(X, \mathbb{C})$. Observe that $(h_*\{T_1\}, h_*\{T_2\}) \leq CN(T_1)N(T_2)$, because $h_*$ is linear and $H^2(X, \mathbb{C})$ is finite dimensional. It is an easy exercise in Kähler geometry to prove that $N(T_i)$ can be replaced by the mass $M(T_i)$ in this inequality. We now conclude using the previously discussed case $X = \mathbb{P}^2$, because $h_*(S_i) \leq h_*T_i$, $i = 1, 2$, the $h_*T_i$ are strongly approximable currents on $\mathbb{P}^2$. □
Proof of proposition 4.3: Fix a neighborhood basis \((\omega_j)\) of \(X\). We assume all \(\omega_j\) are biholomorphic to bidisks. For every \(\omega_j \simeq \mathbb{D}^2\), we consider the two sub-bidisks \(\omega_j'\) and \(\omega_j''\) corresponding to \(\mathbb{B}' = \mathbb{D} \times D(0, \frac{1}{4})\) and \(\mathbb{B}'' = D(0, \frac{1}{4}) \times \mathbb{D}\), where \(\mathbb{D}\) denotes the unit disk. It is a very basic observation that for every line \(L\) in \(\mathbb{D}^2\) intersecting \(\mathbb{B}' \cap \mathbb{B}''\), \(L\) is either a graph over the second projection in \(\mathbb{B}'\) or a graph over the first projection in \(\mathbb{B}''\). Rename as \((\Omega_j)\) the family \((\omega_j') \cup (\omega_j'')\).

Let \(D\) be any disk subordinate to \(T_1\), and \(p \in D\). Because at small scales, \(D\) is close to its tangent space at \(p\), by the preceding observation, there exists \(\Omega_j \ni p\) such that \(D \cap \Omega_j\) is a graph for one of the two natural projections. Thus \(D = \bigcup_{j \in J_D} D \cap \Omega_j\), where \(J_D\) is the set of indices such that \(D \cap \Omega_j\) is a graph in the bidisk \(\Omega_j\). The open set \(\Omega_j\) being fixed, the set of such graphs in \(\Omega_j\) subordinate to \(T_1\) form a lamination \(\mathcal{L}_{1,j}\) in \(\Omega_j\). We let \(T_{1,j} = T_1|_{\mathcal{L}_{1,j}}\).

Doing the same construction with \(T_2\), for every \(j\) we form the geometric intersection measure \(\mu_j = T_{1,j} \cap T_{2,j}\). Now the family \(\sup(\mu_1, \ldots, \mu_j)\) is increasing and we define \(T_1 \hat{\wedge} T_2\) to be its increasing limit, which has finite mass by the preceding lemma.

Let \(S_i \leq T_i\) be uniformly laminar currents. Take \(p \in \text{Supp}(S_1 \hat{\wedge} S_2)\), and let \(D_1\) be a disk subordinate to \(S_1\) through \(p\). There exists a bidisk \(\Omega^1 \ni p\) from the neighborhood basis, such that \(D_1\) is a graph over some direction. This also holds for the corresponding leaves close enough to \(p\). We do the same for \(S_2\). Since the disks subordinate to a strongly approximable current are compatible, there is at most one disk subordinate to \(T_i\) through \(p\) so with the preceding notation \(S_i \leq T_{i,j}\), and near \(p\) in \(\Omega^1 \cap \Omega^2\), \(S_1 \hat{\wedge} S_2 \leq T_1 \hat{\wedge} T_2\).

It only remains to check the product structure. If \(\mathcal{L}_i\) is a flow box subordinate to \(T_i\), \(i = 1, 2\), then, if non trivial, the measure \(T_1|_{\mathcal{L}_1} \hat{\wedge} T_2|_{\mathcal{L}_2}\) has product structure. Moreover if \(p \in \mathcal{L}_i\), there is exactly one disk through \(p\) subordinate to \(T_i\). Hence

\[
(T_1 \hat{\wedge} T_2)|_{\mathcal{L}_1 \cap \mathcal{L}_2} = T_1|_{\mathcal{L}_1} \hat{\wedge} T_2|_{\mathcal{L}_2}.
\]

We may now pick a countable collection of disjoint product sets \(\mathcal{L}_1 \cap \mathcal{L}_2\), of full measure, and \(T_1 \hat{\wedge} T_2\) has product structure on each of them. \(\square\)

4.2. Invariant measure. We now study the dynamical properties of the geometric intersection measure, provided it is non zero. Recall the statement of our first main theorem.

Theorem 4.6. Let \(f\) be an algebraically stable birational map on a rational surface \(X\), satisfying \(\lambda > 1\). Assume further \(T^+\) and \(T^-\) have nontrivial geometric intersection \(\mu = T^+ \hat{\wedge} T^-\). Then

i. \(\mu\) is an invariant measure which is mixing.

ii. For \(\mu\)-almost every \(p\), there exist unit tangent vectors \(e^u(p)\) and \(e^s(p)\) at \(p\), depending measurably on \(p\), there exists \(N \subset \mathbb{N}\) of density 1, such that

\[
\liminf_{N \ni n \to \infty} \frac{1}{n} \log |df^n(e^u(p))| \geq \frac{\log \lambda}{2} \quad \text{and} \quad \limsup_{N \ni n \to \infty} \frac{1}{n} \log |df^n(e^s(p))| \leq -\frac{\log \lambda}{2}.
\]

iii. \(\mu\) has entropy \(h_{\mu}(f) = \log \lambda\). In particular the topological entropy \(h_{\text{top}}(f)\) is \(\log \lambda\).

iv. \(\mu\) has product structure with respect to local stable and unstable manifolds.

Item ii. requires a few comments. Lyapounov exponents are only defined when \(\log |df| \in L^1(\mu)\). We do not know whether this hypothesis is true in our context, while (8) always make sense. Of course when \(\log |df| \in L^1(\mu)\) then ii. is a statement about Lyapounov exponents.
The bound $\frac{\log \lambda}{2}$ on the Lyapounov exponents is sharp. This is clear if not only birational maps on rational surfaces are allowed, but also holomorphic diffeomorphisms on torii: consider for instance the map induced by the linear map $\frac{2}{1}$ on $\mathbb{C}^2/\mathbb{Z}[\sqrt{-1}]^2$. Its Lyapounov exponents –relative to Lebesgue measure– are $\chi^u = \log \frac{1 + \sqrt{5}}{2} > 0 > \chi^s = \log \frac{1 - \sqrt{5}}{2}$. The topological entropy is $2 \max(-\chi^s, \chi^u) = -2\chi^s$. On the other hand it was observed by S. Cantat and C. Favre [CF, Example 3.2] that such a map gives rise to an automorphism of a rational surface, obtained as the desingularization of the quotient of the torus $\mathbb{C}^2/\mathbb{Z}[\sqrt{-1}]^2$, by the multiplication by $\sqrt{-1}$.

As shown in [OW, p.86], i. and iv. imply $(f, \mu)$ has the Bernoulli property, i.e. is measurably conjugate to a Bernoulli shift.

**Proof:** we will prove the items separately. Note that iv. follows from proposition 4.3 as soon as the disks subordinate to $T^+$ and $T^-$ are respectively identified as being stable and unstable disks, which will be a consequence of the proof of ii. The measure $\mu$ has finite mass, so by normalization we assume $\mu$ is a probability measure.

**Invariance and mixing.** By hypothesis, $\mu$ is the geometric intersection measure of diffuse laminar currents, so $\mu$ gives no mass to subvarieties. In particular we may check the invariance of $\mu$ in $X \setminus (I(f^\pm) \cup C(f^\pm))$. On any open set $\Omega$ where $f$ is a biholomorphism, $f_*(T^+ \wedge T^-) = f_*T^+ \wedge f_*T^- = T^+ \wedge T^-$ so it follows that $\mu$ is invariant.

The proof that $\mu$ is mixing is slightly reminiscent of the celebrated *Hopf argument* for the ergodicity of the geodesic flow (see [KH, p. 217]). By construction, $\mu$ is an integral of measures of the form $T^- \wedge [D]$, where $D$ is a disk subordinate to $T^+$. Moreover $T^- \wedge [D]$ decomposes as an at most countable sum of induced transverse measures $T^- \wedge \tau$ on transversals to $T^-$: indeed this is the case for every $T^-|_\mathcal{L} \wedge [D]$, where $\mathcal{L}$ is a flow box for $T^-$.

So by proposition 5.1 $\mu$ itself is equidistributed with measures of the form $\frac{T^- \wedge \tau}{M(T^- \wedge \tau)}$, i.e. for every transversal $\tau$,

\[(f^{-n})_*\mu - (f^{-n})_* \left( \frac{T^- \wedge \tau}{M(T^- \wedge \tau)} \right) = \mu - (f^{-n})_* \left( \frac{T^- \wedge \tau}{M(T^- \wedge \tau)} \right) \to 0.\]

If $\varphi$ is a piecewise constant function on a given flow box $\mathcal{L}$, we get similarly

\[\mu - (f^{-n})_* \left( \frac{\varphi \mu}{\int \varphi \mu} \right) \to 0.\]

These functions are uniformly dense among continuous functions on $\mathcal{L}$. Hence (9) holds for continuous $\varphi$ on $\mathcal{L}$. For global $\varphi$, just write $\varphi = \sum 1_{\mathcal{L}_i} \varphi$, where $(\mathcal{L}_i)$ is a collection of disjoint flow boxes of full $\mu$-measure. To conclude, we remark that (9) is a reformulation of mixing.

**Lyapounov exponents.** We show that there exists a measurable unit vector field $e^u$, such that for fixed $\varepsilon > 0$, for $\mu$-a.e. $p$, there exists $N_\varepsilon$ of density $\geq 1 - \varepsilon$ such that

\[\liminf_{N_\varepsilon \to \infty} \frac{1}{n} \log |df^n(e^u(p))| \geq \frac{\log \lambda}{2}.\]

It will then suffice to put $N' = \bigcup_{\varepsilon > 0} N_\varepsilon$. 
Fix $\varepsilon > 0$ and consider a collection $A = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_N$ of disjoint flow boxes for $T^-$, such that $\mu(\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_N) \geq 1 - \frac{2\varepsilon}{\lambda}$. If $p \in A$, we denote by $D_p$ the plaque of $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_N$ through $p$, and $e^n(p)$ the unit tangent vector to $D_p$ at $p$.

Removing a set of plaques of small $T^-$-transverse measure, hence of small $\mu$-measure, we get by lemma 3.2 a set still denoted by $A$, with measure $\geq 1 - \frac{2\varepsilon}{\lambda}$, such that if $p \in A$, and for every $n$, $\text{Area}(f^{-n}(D_p)) \leq Cn^\frac{2\varepsilon}{\lambda}$. Making a further reduction we end up with a set $A$ with $\mu(A) \geq 1 - \varepsilon$, such that for each plaque $D$ of $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_N$, $A \cap D$ is relatively compact in $D$, with a uniform bound on $\text{dist}(A, \partial D)$.

By the Birkhoff Ergodic Theorem, for a.e. $p$, the set
\[
N_\varepsilon = \{ n \in \mathbb{N}, \ f^n(p) \in A \}
\]
has density $\geq 1 - \varepsilon$. If $p \in A$ and $n \in N_\varepsilon$ is large enough, $f^{-n}D_{f^n(p)}$ has small area, and reducing $D_{f^n(p)}$ slightly if necessary, small diameter, so $f^{-n}D_{f^n(p)} \subset D_p$. Since $D_{f^n(p)}$ lies on a finite set of flow boxes, by Cauchy estimates (or Koebe distortion), the derivative $df_{f^n(p)}^{-n}(e^u(f^n(p)))$ has norm
\[
\left| df_{f^n(p)}^{-n}(e^u(f^n(p))) \right| = \left| (df^n_p)^{-1}(e^u(f^n(p))) \right| \leq \frac{Cn}{\lambda^2}.
\]
This gives (10) if $p \in A$.

For $\mu$-generic $p$, $p$ does not belong to $\bigcup_n C(f^n)$ and for some $n_0(p)$, $f^{n_0}(p) \in A$—more precisely $f^{n_0}(p)$ belongs to the full measure subset $A' \subset A$ of points satisfying (10). Since $p \notin \bigcup_n C(f^n)$, the differential $df^{n_0}$ is invertible and (10) holds at $p$ by pulling back by $f^{n_0}$.

We also proved that points in the plaques $D_p$ become exponentially close under backwards iteration, so $D_p$ is the local unstable manifold of $p$.

**Entropy.** Defining topological entropy requires some care because $f$ has indeterminacy points. The definition of topological entropy we use is Bowen’s definition via $(n, \varepsilon)$ separated sets on $X \setminus \bigcup I(f^n)$ with respect to the ambient Riemannian metric. The Gromov inequality [Gr] asserts that $h_{\text{top}}(f) \leq \log \lambda$. The variational principle need not hold in this context, but the inequality $h_\nu(f) \leq h_{\text{top}}(f)$ persists for any invariant probability measure $\nu$. This can be seen for instance by restricting to ergodic measures, considering partitions by balls of radius $\varepsilon$ in the definition of metric entropy and applying the Shannon-McMillan-Breiman theorem.

Let us recall some material from entropy theory. We shall use the formalism of measurable partitions and conditional measures (see e.g. [BLS1] for a presentation adapted to our context). If $\xi$ is a measurable partition, a probability measure $\nu$ may be disintegrated with respect to $\xi$, giving rise to a probability measure $\nu(\cdot|\xi(x))$ on each atom of $\xi$. We have the following disintegration formula: for every continuous function $\phi$,
\[
\int \left( \int \phi(y) d\nu(y|\xi(x)) \right) d\nu(x) = \int \phi d\nu.
\]
The partition $\xi$ is said to be $f^{-1}$-invariant if $f^{-1}\xi$ is a refinement of $\xi$, i.e. for every $x$, $f^{-1}(\xi(f(x))) \subset \xi(x)$. Given partitions $\xi_i$, we denote by $\bigvee \xi_i$ the joint partition, i.e. $(\bigvee \xi_i)(x) = \bigcap (\xi_i(x))$. A partition is called a generator if $\bigvee_{n \in \mathbb{Z}} f^n \xi$ is the partition into points.

Given a partition $\xi$, we consider the $f^{-1}$-invariant partition $\xi^u = \bigvee_{n \in \mathbb{N}} f^n \xi$. 
Proposition 4.7 (Rokhlin). If $\xi$ is a generator with finite entropy, then

$$h_\mu(f) = h_\mu(f, \xi^u) = -\int \log \mu(f^{-1}(\xi^u(x)))\xi^u(x) d\mu(x) = \int \log J^u(x) d\mu(x),$$

where $J^u(x) := \left(\mu\left(f^{-1}(\xi^u(f(x)))|\xi^u(x)\right)\right)^{-1}$ is the unstable Jacobian.

We do not define the entropy $h_\mu(f, \xi^u)$ here but we stress that the entropy finiteness hypothesis is satisfied because

$$h_\mu(f, \xi^u) \leq h_\mu(f) \leq h_{\text{top}}(f) \leq \log \lambda.$$

Proposition 4.8 (Pesin, see [LS]). There exists a measurable $f^{-1}$-invariant generator $\xi^u$, whose atoms are open subsets of local unstable manifolds, and such that $h_\mu(f) = h_\mu(f, \xi^u)$.

Proof: the proposition is stated in the context of Pesin theory in [LS] Proposition 3.1, but it holds in our context. More precisely what is exactly needed in [LS] is a family of local unstable manifolds $V_{\text{loc}}$ satisfying the conclusions of [LS] Proposition 3.3: items (3.3.1) to (3.3.6), except (3.3.5), assert that the family of manifolds $V_{\text{loc}}$ has controlled geometry on a set of large $\mu$ measure, and (3.3.5) means that points in the same local leaf become exponentially close under backwards iteration, uniformly on sets of large measure. The reader will easily check these properties are true for the unstable disks constructed above, that is, the set of disks subordinate to $T^-$.

We are now ready to compute $h_\mu(f)$. Consider the unstable partition provided by the previous proposition. Since $\mu$ has product structure relative to $T^+$ and $T^-$, for $\mu$-a.e. $x T^+ \land [\xi^u(x)]$ has positive mass. As an obvious consequence of the product structure of $\mu$ and the definition of geometric wedge product $\land$, the conditional measures $\mu(\cdot|\xi^u(x))$ are induced by $T^+$, more specifically

$$\mu(\cdot|\xi^u(x)) = \frac{T^+ \land [\xi^u(x)]}{M(T^+ \land [\xi^u(x)])}.$$

From the invariance relation $f^*T^+ = \lambda T^+$ (see equation (5)) we deduce that

$$T^+ \land [f^{-1}(\xi^u(f(x)))] = (T^+ \land [\xi^u(x)]) | f^{-1}(\xi^u(f(x))) = \frac{1}{\lambda}f^*(T^+ \land [\xi^u(f(x))]),$$

hence the unstable Jacobian $J^u_\mu$ satisfies the multiplicative cohomological equation

$$J^u_\mu(x) = \lambda \frac{\rho(x)}{\rho(f(x))} \text{ a.e.}, \text{ where } \rho(x) = M(T^+ \land [\xi^u(x)]).$$

Using the invariance of both $\mu$ and the partition, the Birkhoff Ergodic Theorem implies

$$h_\mu(f) = \int \log J^u_\mu d\mu = \log \lambda,$$

see [BLS1] Proposition 3.2] for more details. This concludes the proof of theorem 4.6

5. The Bedford-Diller setting

The aim of this section is to prove that the class of maps considered in [BeD] satisfy the hypotheses of theorem 4.6. The currents $T^+$ and $T^-$ actually have full geometric intersection in this setting.
5.1. Geometric intersection. We prove that under some potential theoretic conditions, the wedge product of two strongly approximable currents is geometric. The results here generalize those of [Du2] and we refer the reader to this paper for more details.

The fact that two laminar currents on $X$ intersect geometrically is a local property near every point in $X$. So throughout this paragraph, $\Omega$ denotes an open set in $\mathbb{C}^2$. Moreover, reducing $\Omega$ slightly if necessary, we may replace all the $L^p_{loc}$ conditions by $L^p$ conditions in $\Omega$.

We first state a local property of strongly approximable currents, which is proved in [Du2, Prop. 4.4].

**Proposition 5.1.** Let $T$ be a strongly approximable laminar current, and $\Omega \subset \mathbb{C}^2$ as above.

Let $\pi_1$ and $\pi_2$ be generic linear projections. Then for subdivisions $S_1, S_2$ of the respective projection bases into squares of size $r$, if

$$Q = \{\pi_1^{-1}(s_1) \cap \pi_2^{-1}(s_2), (s_1, s_2) \in S_1 \times S_2\}$$

denotes the associated subdivision of $\Omega$ into affine cubes of size $r$, there exists a current $T_Q \leq T$ in $\Omega$, uniformly laminar in each $Q \in Q$, and satisfying the estimate

$$M(T - T_Q) \leq Cr^2,$$

with $C$ independent of $r$.

We say that a laminar current satisfying the conclusions of the preceding proposition is strongly approximable in $\Omega$.

**Theorem 5.2.** Let $T_1 = d\omega u_1$ and $T_2 = d\omega u_2$ be two strongly approximable currents in $\Omega \subset \mathbb{C}^2$. Assume $u_1 \in L^1(\|T_2\|)$, $u_2$ has derivatives in $L^2(T_1)$ and $u_1$ has derivatives in $L^2(T_2)$. Then the wedge product $T_1 \wedge T_2$ is geometric.

$L^2$ spaces on positive currents are considered in [BeD, BS]. Let $\Omega \subset \mathbb{C}^2$, and $T$ be a positive current in $\Omega$. Let $u, v$ be smooth functions. Following [BeD] we define the pairing

$$E(u, v) = \int du \wedge d^c v \wedge T,$$

and denote by $|\cdot|_T$ the associated seminorm, $|u|_T = (\int du \wedge d^c u \wedge T)^{\frac{1}{2}}$. If $u$ is a p.s.h. function in $\Omega$ we say that $u$ has derivatives in $L^2(T)$ if for every regularizing sequence $u_j \downarrow u$, $(u^j)$ is a Cauchy sequence for $|\cdot|_T$. If $u$ has derivatives in $L^2(T)$, then $u$ has derivatives in $L^2(S)$ for every $S \leq T$.

**Proof of theorem 5.2** we follow the approach of [Du2] closely, only differing in the final estimate. The letter $C$ denotes a constant that may change from line to line, remaining independent of $r$. The currents $T_1$ and $T_2$ being strongly approximable, by proposition 5.1 there exist for each $r > 0$, a subdivision $Q$, which we may assume is the same for $T_1$ and $T_2$, and for each $Q \in Q$ a uniformly laminar current $T_{k,Q}$, $k = 1, 2$, such that

$$M(T_k - T_{k,Q}) = M(T_k - \sum_{Q \in Q} T_{k,Q}) \leq Cr^2, \quad k = 1, 2.$$

We have to estimate the mass of

$$T_1 \wedge T_2 - \sum_{Q \in Q} T_{1,Q} \wedge T_{2,Q},$$

where the second term is a geometric wedge product because of uniform laminarity.
The first step is to choose an adapted subdivision so that \( T_1 \wedge T_2 \) is not too concentrated near the boundary of the cubes. More specifically, for \( \lambda < 1 \) close to 1 and \( Q \in \mathcal{Q} \), let \( Q^\lambda \) be the homothetic cube of \( Q \) with respect to its center, with factor \( \lambda \). As in [Du2] lemma 4.5, up to a translation of \( Q \), we may choose \( \lambda \) independent of \( r \) so that the mass of \( T_1 \wedge T_2 \) in the union of \( Q \setminus Q^\lambda \) is small (i.e. smaller than \( 2(1 - \lambda^4) \)).

To handle the remaining part of [13], \( Q \), and \( \lambda \) being fixed by now, let \( \chi \) be a nonnegative \( C^\infty \) function, with \( \chi = 1 \) near every \( Q^\lambda \), vanishing near the boundary of every \( Q \in \mathcal{Q} \), and with derivatives bounded by \( C/r \) in uniform norm. The problem reduces to bounding

\[
\int \chi (T_1 \wedge T_2 - T_1,Q \wedge T_2,Q) = \sum_{Q \in \mathcal{Q}} \int \chi ((T_1 \wedge T_2)|_Q - T_1,Q \wedge T_2,Q).
\]

Moreover in each cube \( Q \),

\[
T_1 \wedge T_2 - T_1,Q \wedge T_2,Q = T_1 \wedge (T_2 - T_2,Q) + T_2,Q \wedge (T_1 - T_1,Q) \leq T_1 \wedge (T_2 - T_2,Q) + T_2 \wedge (T_1 - T_1,Q),
\]

so by taking the union over all cubes \( Q \in \mathcal{Q} \), we infer that

\[
T_1 \wedge T_2 - T_1,Q \wedge T_2,Q \leq T_1 \wedge (T_2 - T_2,Q) + T_2 \wedge (T_1 - T_1,Q).
\]

Of course we need only consider the first term because the hypotheses are symmetric. Using the Schwarz inequality and (12), we infer

\[
\int \chi d\bar{\chi} \, u_1 \wedge (T_2 - T_2,Q) = - \int d\chi \wedge d\bar{\chi} \, u_1 \wedge (T_2 - T_2,Q)
\]

\[
\leq \left( \int du_1 \wedge d\bar{\chi} \, u_1 \wedge (T_2 - T_2,Q) \right)^{1/2} \left( \int d\chi \wedge d\bar{\chi} \wedge (T_2 - T_2,Q) \right)^{1/2}
\]

\[
\leq \frac{C}{r} \mathcal{M}(T_2 - T_2,Q)^{1/2} \left( \int du_1 \wedge d\bar{\chi} \, u_1 \wedge (T_2 - T_2,Q) \right)^{1/2}
\]

\[
\leq C \left( \int du_1 \wedge d\bar{\chi} \, u_1 \wedge (T_2 - T_2,Q) \right)^{1/2} = C |u_1|_{T_2 - T_2,Q}
\]

where the Stokes theorem is valid because \( \chi \) has compact support and \( T_2 - T_2,Q \) is closed in every \( Q \in \mathcal{Q} \).

Let \( u_1^\varepsilon \) be a regularizing family. We write \( u_1 = u_1^\varepsilon + (u_1 - u_1^\varepsilon) \) and use the triangle inequality

\[
|u_1|_{T_2 - T_2,Q} \leq |u_1^\varepsilon|_{T_2 - T_2,Q} + |u_1 - u_1^\varepsilon|_{T_2 - T_2,Q} \leq |u_1^\varepsilon|_{T_2 - T_2,Q} + |u_1 - u_1^\varepsilon|_{T_2}
\]

Since \( u_1 \) has derivatives in \( L^2(T_2) \), we may fix \( \varepsilon \), independent of \( r \), so that \( |u_1 - u_1^\varepsilon|_{T_2} \) is small. For fixed \( \varepsilon > 0 \), the function \( u_1^\varepsilon \) is smooth so by weak convergence, \( |u_1^\varepsilon|_{T_2 - T_2,Q} \) tends to zero when \( \mathcal{M}(T_2 - T_2,Q) \) does, i.e. when \( r \to 0 \).

\[ \square \]

Remark 5.3. Using the same argument together with proposition [A.2] allows to prove the following:

**Theorem.** Let \( T_1 = d\bar{\chi} \, u_1 \) and \( T_2 = d\bar{\chi} \, u_2 \) be two strongly approximable currents in \( \Omega \subset \mathbb{C}^2 \). Assume \( u_1 \in L^1(||T_2||) \), \( u_2 \) has derivatives in \( L^2(T_1) \) and \( T_1 \) gives no mass to pluripolar sets. Then the wedge product \( T_1 \wedge T_2 \) is geometric.
Proof: following step by step the proof of the previous theorem only allows to prove that
\( T_2 \land (T_1 - T_1, Q) \) tends to zero as \( r \to 0 \), i.e. that \( T_2 \land T_1 \) is approximated by the “semi geometric” wedge products \( T_2 \land T_1, Q \). We claim that these wedge products are geometric.

Indeed, since \( T_1 \) does not charge pluripolar sets, neither does \( T_1, Q \), and proposition \( \text{A.2} \) asserts that in each cube \( Q, T_1, Q \) is the increasing limit of a sequence of uniformly laminar currents \( S_j \) with continuous potential. Moreover, the potentials of \( S_j \) may be chosen to form a decreasing sequence: just write \( S_{j+1} = S_j + R_j \) and choose a nonpositive potential for \( R_j \). We thus infer that \( T_2 \land S_j \to T_2 \land T_1, Q \).

On the other hand \( T_2 \land S_j \) is a geometric wedge product because of theorem \( \text{B.2} \) indeed a continuous plurisubharmonic function has derivatives in \( L^2(T) \) for any positive closed current \( T \); this is a corollary of the polarization identity

\[
2 du \land d^c u \land T = dd^c(u^2) \land T - 2udd^c u \land T.
\]

The theorem is proved. \( \square \)

Another consequence of proposition \( \text{A.2} \) is that if \( T_1 \) and \( T_2 \) do not charge pluripolar sets, then neither does \( T_1 \land T_2 \). This is the case under assumption \( \text{(13)} \) below.

5.2. Dynamics. We turn back to the dynamical context, and give the proof of theorem \( \text{2} \). Due to the possibly complicated dynamics of indeterminacy points, it is not known whether the wedge product \( T^+ \land T^- \) is admissible in general. E. Bedford and J. Diller \( \text{[BeD]} \) managed to construct the wedge product measure \( \mu = T^+ \land T^- \) and study some of its dynamical properties under the condition

\[
\left( \sum_{n \geq 0} \frac{1}{\lambda^n} |\log \text{dist}(f^n(I(f^{-1})), I(f))| \right) < \infty
\]

(where \( \text{dist} \) is the ambient Riemannian distance function) which is satisfied for many birational maps, and is symmetric with respect to \( f \) and \( f^{-1} \) \( \text{[Di1, Theorem 5.2]} \). Under this hypothesis, they proved the following: if \( \omega^+/^- \) are smooth forms representing the cohomology classes \( \theta^+/^- \), then \( T^+/^- = \omega^+/^- + dd^c g^+/^- \), where \( g^+ \) is a quasi-p.s.h. function with derivatives in \( L^2(\omega + T^-) \), and similarly \( g^- \) has derivatives in \( L^2(\omega + T^+) \).

In this case, by \( \text{[BeD] §3]} \), \( \mu = T^+ \land T^- \) is a well defined wedge product, and \( \mu \) has positive mass for cohomological reasons. As said before, the wedge product \( T^+ \land T^- \) being geometric is a local property near every point in \( X \), so by theorem \( \text{5.2} \) \( \mu = T^+ \land T^- \). Hence theorem \( \text{1.6} \) applies to give the dynamical properties of \( \mu \)–some of which (mixing and non zero exponents) were already given in \( \text{[BeD]} \).

As an example, Diller shows in \( \text{[Di1 §7]} \) that a polynomial birational map in \( \mathbb{C}^2 \), which is algebraically stable in \( \mathbb{P}^2 \) satisfies condition \( \text{(13)} \).

Another result in \( \text{[BeD]} \) is that \( C^\infty \) functions with logarithmic poles at points of \( I(f) \) are \( \mu \)-integrable. This is the case in particular for \( - \log \text{dist}(x, I) \), near \( I \in I(f) \), as well as \( \log^+ \|df\| \) and \( \log^+ \|d^2 f\| \). This allows us to use the construction of Lyapounov charts and Pesin’s theory (see e.g. the appendix of \( \text{[Kl]} \)).

A consequence is the equidistribution of saddle orbits, following \( \text{BLS2} \).

**Theorem 5.4.** Assume that \( f \) and \( X \) are as in theorem \( \text{4.4} \) and that \( \log^+ \|df\| \) and \( \log^+ \|d^2 f\| \) are \( \mu \)-integrable.
Then saddle points are equidistributed towards $\mu$, that is, if $\text{PER}_n$ denotes the set of saddle periodic points of period $n$,
\[
\frac{1}{\lambda^n} \sum_{p \in \text{PER}_n} \delta_p \to \mu.
\]
Moreover for every $n$ there exists a set $\mathcal{P}_n$ of saddle points with $\#\mathcal{P}_n/\lambda^n \to 1$, such that every $p \in \mathcal{P}_n$ lies in the support of $\mu$.

Similarly, Lyapunov exponents can be evaluated by averaging on saddle orbits, that is,
\[
\frac{1}{\lambda^n} \sum_{p \in \text{PER}_n} \chi^u(p) \to \chi^u(\mu),
\]
where $\chi^u(p)$ (resp. $\chi^u(\mu)$) denotes the positive Lyapunov exponent of $p$ (resp. $\mu$).

We use the formalism of Pesin boxes from [BLS1]. Pesin boxes are sets $Q$ of positive $\mu$-measure, together with neighborhoods $N(Q)$ so that for $x \in Q$, $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$ are transverse connected boundaryless submanifolds in $N(Q)$. Moreover $Q$ may be chosen so that the angle between intersecting stable and unstable manifolds in $N(Q)$ is uniformly bounded from below and the resulting $Q$ has product structure. In our setting, the measure $\mu$ has product structure in Pesin boxes. We denote by $\mathcal{L}^s(Q)$ and $\mathcal{L}^u(Q)$ the stable and unstable laminations in $N(Q)$.

**Proof:** following [BLS2], the equidistribution statements (15) and (16) are formal consequences of mixing, product structure and the upper bound $\lambda^n + C$ on the number of periodic points of period $n$ [DF Theorem 0.6].

It remains to prove that the saddle points constructed with the method of [BLS2] lie in $\text{Supp}(\mu)$; let $\mathcal{P}_n$ be this set of saddle points. We adapt the argument of [BLS1 §9].

Points in $\mathcal{P}_n$ arise as intersection points of stable-like and unstable-like disks in open neighborhoods $N(Q)$ of Pesin boxes, biholomorphic to bidisks. The important fact is that for any $p \in \mathcal{P}_n$, there exists a Pesin box $Q$ so that $W^s_{\mathcal{P}_n}(Q) \subset N(Q)$ is a global transversal of $\mathcal{L}^s(Q)$ in $N(Q)$, and similarly for $W^u_{\mathcal{P}_n}(Q)$. Here the subscript $N(Q)$ means: connected component of $p$ in $N(Q)$ of the manifold under consideration. Without loss of generality, we may assume $p$ is a fixed point. Consider the restriction currents $T^+|\mathcal{L}^s(Q)$ and $T^-|\mathcal{L}^u(Q)$. These currents have positive mass because $\mu(Q) > 0$ and the leaves of $\mathcal{L}^s(Q)$ (resp. $\mathcal{L}^u(Q)$) are subordinate to $T^+$ (resp. $T^-$). Furthermore $\mu|Q = T^+|\mathcal{L}^s(Q) \wedge T^-|\mathcal{L}^u(Q)$. By the hyperbolic Lambda lemma (the Inclination lemma), for every leaf $\mathcal{L}^s$ of the stable lamination $\mathcal{L}^s(Q)$, the sequence of cut-off iterates $(f^{-n}(\mathcal{L}^s))|_{N(Q)}$ converges in the $C^1$ topology to $W^s_{\mathcal{P}_n}(Q)$. Hence
\[
\frac{1}{\lambda^n} \left( f^\mu_{N(Q)} \right)^* (T^+|\mathcal{L}^s(Q)) \leq T^+
\]
is a uniformly laminar current, with leaves arbitrarily $C^1$-close to $W^s_{\mathcal{P}_n}(Q)$, where the notation $f_{N(Q)}$ means all iterates are successively restricted to $N(Q)$—this is the Graph Transform operator for currents. There is an analogous result in the unstable direction. In particular if we let
\[
\mu_n = \frac{1}{\lambda^n} \left( f^\mu_{N(Q)} \right)^* (T^+|\mathcal{L}^s(Q)) \wedge (f^\mu_{N(Q)})^* (T^-|\mathcal{L}^u(Q))
\]
then $0 < \mu_n \leq \mu$ and the measure $\mu_n$ has support arbitrarily close to $p$. \qed
APPENDIX A. AN ALTERNATE APPROACH TO THEOREM 4.6

The discussion in section 3 and 4 was designed to avoid the use of Pesin’s theory. We sketch here how to recover theorem 4.6 by allowing Pesin’s theory. The point is to relate the laminar structure of the currents and the stable and unstable manifolds. This provides yet another approach to the results in [BLS1], §4 and 8.

The setting is the following: we adopt the hypotheses of [5.2] that is f is a birational map on X satisfying [14]. We assume Q is a Pesin box and 0 < μ₁ = S⁺ ∩ S⁻ ≤ μ is a measure supported by Q, where S⁺⁻ ≤ T⁺⁻ are uniformly laminar currents in N(Q), and the leaves of the underlying laminations are disks. For x ∈ Q, we let S⁺⁻(x) be the disk of the corresponding current S⁺⁻ through x.

Proposition A.1. With notations as above, for μ₁ a.e. x ∈ Q, S⁺(x) ⊂ W⁺loc(x) (resp. S⁻(x) ⊂ W⁻loc(x)).

Before proving the result, we make two observations. The first is that a current with continuous potential. From the previous discussion and geometric

Proposition A.2. Let S be a uniformly laminar current, integral of holomorphic graphs in the bidisk, S = ∫[Γα]dμ(α). Assume S gives no mass to pluripolar sets. Then S can be written as a countable sum S = ∑ Sj, where the Sj = ∫[Γα]dμj(α) have continuous potential and disjoint support.

Proof of proposition A.1. we prove the result for S⁺. One may assume from the previous observations that S⁺ and S⁻ have continuous potentials.

Suppose the result is false, that is, there exists R ⊂ Q of positive μ₁-mass such that for x ∈ R, W⁺loc(x) ≠ S⁺(x). Slightly moving x if necessary, makes the intersection between W⁺loc(x) and S⁺(x) transverse. Indeed, μ₁ has product structure with respect to S⁺ and S⁻, and R has positive measure, so we may assume the ([S⁺(x)] ∩ S⁻)-mass of R inside S⁺(x) is positive. For y ∈ R ∩ S⁺(x) near x, W⁻loc(y) is transverse to S⁺(y) = S⁺(x) since the local stable manifolds are disjoint (see [BLS1] Lemma 6.4). Without loss of generality we write x for y. Reducing N(Q) once again, we assume S⁺(x) is a global transversal to the family of stable manifolds. This does not affect the fact that μ₁(R) > 0.

A corollary of transversality is that S⁺ ∩ [W⁺loc(x)] > 0 because a set of positive transverse measure of disks intersect W⁺loc(x) transversally –the existence of the wedge product is ensured since S⁺ has continuous potential.

On the other hand, the current S⁻ induces the measure S⁻ ∩ [S⁺(x)] on the disk S⁺(x), which is a measure with continuous potential on S⁺(x). Up to a normalizing factor this measure coincides with the conditional measure μ₁(·|S⁺(x)). Now

μ₁(·|S⁺(x))|R∩S⁺(x) ≤ μ₁(·|S⁺(x))

so the restriction ν = μ₁(·|S⁺(x))|R has continuous potential also.

Let C = ∫[W⁺loc(x)]dμ(y) be the uniformly laminar current constructed from the transversal S⁺(x) to the family of stable manifolds and the conditional measure ν. One proves (see [Du3], Proposition 6.2) that C has continuous potential. From the previous discussion and geometric intersection it follows that C ∩ S⁺ > 0.
Let $\psi$ be a nonnegative test function in $N(Q)$, $\psi = 1$ near $x$. Then
\[ 0 < \int \psi C \wedge S^+ \leq \int \psi C \wedge T^+. \]
Consider now the sequence of currents $\lambda^{-n}(f^n)_*(\psi C)$, where the action $f_*$ has to be understood here as a proper transform near indeterminacy points. A result of J. Diller [Di1], built on classical arguments, asserts that the cluster points of this sequence of currents are positive closed currents of mass $\int \psi C \wedge T^+ > 0$. However, $C$ is an integral of local stable manifolds so $M(\lambda^{-n}(f^n)_*(\psi C)) \to 0$. We have reached a contradiction. □

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