Three-component soliton states in spinor $F = 1$ Bose-Einstein condensates

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Dilute-gas Bose-Einstein condensates are an exceptionally versatile testbed for the investigation of novel solitonic structures. While matter-wave solitons in one- and two-component systems have been the focus of intense research efforts, an extension to three components has never been attempted in experiments, to the best of our knowledge. Here, we experimentally demonstrate the existence of robust dark-bright-bright (DBB) and dark-dark-bright (DDB) solitons in a spinor $F = 1$ condensate. We observe lifetimes on the order of hundreds of milliseconds for these structures. Our theoretical analysis, based on a multiscale expansion method, shows that small-amplitude solitons of these types obey universal long-short wave resonant interaction models, namely Yajima-Oikawa systems. Our experimental and analytical findings are corroborated by direct numerical simulations highlighting the persistence of, e.g., the DBB states, as well as their robust oscillations in the trap.

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Solitons are localized waves propagating undistorted in nonlinear dispersive media. They play a key role in numerous physical contexts. Among the various systems that support solitons, dilute-gas Bose-Einstein condensates (BECs) provide a particularly versatile testbed for the investigation of solitonic structures. In single-component BECs, solitons have been observed either as robust localized pulses (bright solitons) or density dips in a background matter wave (dark solitons), typically in BECs with attractive or repulsive interatomic interactions, respectively. Extending such studies to two-component BECs has led to rich additional dynamics. Solitons have been observed in binary mixtures of different spin states of the same atomic species, so-called pseudo-spinor BECs. In particular, dark-bright (DB) and related SO(2) rotated states in the form of dark-dark solitons have experimentally been created in binary $^{87}$Rb BECs. Interestingly, although such BEC mixtures feature repulsive intra- and inter-component interactions, bright solitons do emerge due to an effective potential well created by the dark soliton through the inter-component interaction. Such mixed soliton states have been proposed for potential applications. Indeed, in the context of optics where these structures were pioneered, the dark soliton component was proposed to act as an adjustable waveguide for weak bright solitons. In multi-component BECs, compound solitons of the mixed type could also be used for all-matter-wave waveguiding, with the dark soliton building an effective conduit for the bright one, similar to all-optical waveguiding in optics. Apart from pseudo-spinor BECs, such mixed soliton states have also been predicted to occur in genuinely spinorial BECs, composed of different Zeeman sub-levels of the same hyperfine state. Indeed, pertinent works have studied the existence and dynamics of DB soliton complexes in spinor $F = 1$ BECs. However, experimental observation of such states has not been reported so far.

Here we report on the systematic experimental generation of three-component DB soliton complexes, of the dark-bright-bright (DBB) and dark-dark-bright (DDB) types, in a spinor $F = 1$ condensate of $^{87}$Rb atoms. While DB solitons normally consist of two atomic states (e.g., two $F = 1$ Zeeman sublevels or a combination of Zeeman sublevels of $F = 1$ and $F = 2$ states of $^{87}$Rb), here we use all three Zeeman $F = 1$ sublevels to generate three-component solitons in an elongated atomic cloud. In our theoretical analysis, we employ a multiscale expansion method to derive such vector soliton solutions of the pertinent Gross-Pitaevskii equations (GPEs). We thus show that DBB and DDB solitons can be approximated by solutions of Yajima-Oikawa systems. We thus provide a connection with universal long-short wave resonant interaction (LSRI) processes, which appear in a wide range of contexts, including plasmas, condensed matter, hydrodynamics, nonlinear optics, negative refractive index media, etc. Our experimental and analytical identification of these spinor solitonic structures is corroborated by direct numerical simulations.

To begin our discussion of the three-component solitonic structures, we first present examples for their realization in experiments. The three components are given by the three different Zeeman sub-levels of the $F = 1$ state of $^{87}$Rb, and are designated by their magnetic quantum numbers $|F,m_F⟩ = (1,−1), (1,0)$ and $(1,1)$. The experiments begin with a single-component BEC of approximately $0.8 \times 10^6$ atoms. The atoms are confined in an elongated harmonic trap with frequencies $\{\omega_x,\omega_y,\omega_z\} = 2\pi \times \{1.4,176,174\}$ Hz, where $z$ is the vertical direction. The trap is formed by a focused dipole laser beam and is independent of the atomic hyperfine state. A magnetic bias field of 45.5 G is applied along the weakly confining direction. This field leads to a Zeeman splitting of the energy levels. As a consequence, populations can be transferred between the three states by using radio frequency (RF) pulses or adiabatic radio frequency sweeps.
work, the wave functions of DBB solutions of Eqs. (1a)-(1b) reduce to the completely integrable Manakov system [52]. This model admits vector soliton solutions of the mixed type (i.e., DB soliton complexes – cf., e.g., Ref. [53] which may persist in the presence of the linear interatomic interactions, we employ (for \(q = 0\) and ignoring the spin-dependent terms of Eqs. (1a)-(1b) and the trap. Nevertheless, to better understand the role of the spin-dependent nonlinear interatomic interactions, we employ (for \(V(x) = 0\)) a multiscale expansion method to find approximate DBB soliton solutions of Eqs. (1a)-(1b). The lines of analysis are similar to those of Ref. [39], but with an important difference: in Ref. [39], a single-mode approximation (SMA) [50, 51] was used for the “symmetric” states \(|1, -1\rangle\) and \(|1, +1\rangle\); here, instead, we rely on the smallness of \(\lambda_s/|\lambda_a|\) to identify different types of solitons (dark and bright) for the states \(|1, -1\rangle\) and \(|1, +1\rangle\). Then, upon imposing nontrivial boundary conditions for the \(|1, -1\rangle\) component and trivial ones for the \(|1, 0\rangle\) and \(|1, +1\rangle\) components, we can derive DBB soliton solutions for sublevels \(m_F = -1, 0, +1\), respectively. More specifically, we can show that the wave functions of DBB solitons take the following form:

\[
\begin{align*}
\psi_{-1} &\approx \sqrt{n_0}e^{n_0/\mu}e^{-x/4\sqrt{C^2 + \mu_{-1}}}f(X, T) e^{-\mu_{-1}t + i\epsilon x/4\sqrt{C^2 + \mu_{-1}}}, \\
\psi_{0, +1} &\approx e^{i/4} q_{0, +1} (X, T) e^{i[C x - \mu_{0, +1} t/2]},
\end{align*}
\]  

(2)

where \(\epsilon\) is a formal small parameter, \(\mu_{-1} = (\lambda_s + \lambda_a) n_0\) and \(\mu_{0, +1} = (\lambda_s n_0)\) are the chemical potentials of the three components, while \(n_0\) and \(C^2 = \mu_{-1}\) denote, respectively, the steady-state density and the speed of sound of the \(|1, -1\rangle\) component. Finally, the unknown functions \(f(X, T)\) and \(q_{0, +1} (X, T)\), which depend on the stretched variables \(X = 0\). In our case, \(\lambda_s \approx 5.2 \times 10^{-3}\) and \(\lambda_a \approx -2.4 \times 10^{-5}\), i.e., \(\lambda_a/|\lambda_s|\) is a small parameter.

Based on this fact, it can readily be observed that – in the absence of the trap, and ignoring the spin-dependent interactions – the GPEs (1a)-(1b) reduce to the completely integrable Manakov system [52]. This model admits vector soliton solutions of the mixed type (i.e., DB soliton complexes – cf., e.g., Ref. [53] which may persist in the presence of the linear interatomic interactions, we employ (for \(V(x) = 0\)) a multiscale expansion method to find approximate DBB soliton solutions of Eqs. (1a)-(1b). The lines of analysis are similar to those of Ref. [39], but with an important difference: in Ref. [39], a single-mode approximation (SMA) [50, 51] was used for the “symmetric” states \(|1, -1\rangle\) and \(|1, +1\rangle\); here, instead, we rely on the smallness of \(\lambda_s/|\lambda_a|\) to identify different types of solitons (dark and bright) for the states \(|1, -1\rangle\) and \(|1, +1\rangle\). Then, upon imposing nontrivial boundary conditions for the \(|1, -1\rangle\) component and trivial ones for the \(|1, 0\rangle\) and \(|1, +1\rangle\) components, we can derive DBB soliton solutions for sublevels \(m_F = -1, 0, +1\), respectively. More specifically, we can show that the wave functions of DBB solitons take the following form:

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amplitudes. Second, apart from the traveling DBB solitons, we were also able to identify robust stationary such structures in the presence of the trap. A pertinent example is shown in the three bottom panels of Fig. 1 where a robust DBB soliton, persisting for a long time, is shown. This solution is constructed by identifying, at first, a stationary DBB soliton state of the form

$$\psi_-(x, 0) = \sqrt{\lambda} \tanh(\sqrt{\mu} x),$$

$$\psi_0(x, 0) = \text{Asech}(\sqrt{\mu} x)$$

and

$$\psi_{+1}(x, 0) = 0.$$  

Then, in line with our experimental protocol, switching on a Rabi coupling between components $|1, 0\rangle$ and $|1, +1\rangle$ for a finite time interval, atoms are transferred to $\psi_{+1}$ and a bright soliton is formed there too. After switching off the Rabi coupling between $\psi_0$ and $\psi_{+1}$, the percentage population of atoms in the three components is $97 : 2 : 1$.

We have also performed numerical simulations for the DBB solitons in the presence of a trap, in the case where the location of the soliton is displaced from the trap center. We have confirmed in such a case that the DBB solitons generically perform robust oscillations inside the trap. A typical example is shown in Fig. 2 illustrating that – despite the potential presence of sound waves inside the condensate – the oscillation persists for very long times of the order of many seconds.

Apart from DBB solitons, in our experiments we have also observed the emergence of DDB ones, again with lifetimes on the order of hundreds of milliseconds. To generate DDB solitons, a procedure similar to that of the DBB soliton generation is followed. We begin with all atoms in the $|1, 0\rangle$ state. A small fraction of atoms is then transferred from the $|1, 0\rangle$ state to the $|1, +1\rangle$ state. Subsequently, a weak magnetic gradient is again applied and leads to the formation of DBB solitons. In this experiment, the dark solitons reside in the $|1, 0\rangle$ state, while the bright solitons are formed by atoms in the $|1, +1\rangle$ state. After the DBB solitons are formed, the magnetic gradient is removed, which is necessary to ensure long lifetimes of the solitonic structures. To convert the DBB solitons into DDB ones, an RF sweep is used to transfer an adjustable fraction of the atoms from the $|1, 0\rangle$ state to the $|1, -1\rangle$ state. This completes the formation of a DDB soliton.

In our experiments, we have found that the DDB solitons (and also the DBB solitons discussed above) have lifetimes on the order of hundreds of milliseconds.

The existence of these features appears to be fairly insensitive to the exact population ratio of the three Zeeman states.

For example, we have experimentally verified the existence of DDB structures for different percentage population of atoms in the three states including $71 : 21 : 8$, $33 : 66 : 1$, and $97 : 2 : 1$. These results highlight the generic robustness of the DDB structures. A pertinent example is shown in Fig. 3.

The formation of DDB solitons can also be predicted in the framework of the multiscale expansion method. In this case, assuming approximately equal chemical potential for all spin components, $\mu \approx [\lambda_0 (1+r^2) - \lambda_0 (1-r^2)] n_0$ (where $n_0$ is the pertinent steady-state density and $r = |\psi_0|/|\psi_{-1}|$), we can show that DDB solitons do exist, and assume the following
The above equations constitute the single component Yajima-Oikawa (YO) system [41]. The YO system is completely integrable and possesses soliton solutions of the form
\[ \psi \approx \sqrt{n_0 + \epsilon \rho(X,T)} e^{-i\mu t + i\frac{\epsilon}{2}\sqrt{\pi} \int \rho(X,T) dX}, \]
where \( \epsilon \) is again a small parameter, \( X = e^{1/2}(x - \sqrt{\mu} t) \) and \( T = ct \) are stretched variables, and the functions \( \rho(X,T) \) and \( q(X,T) \) are governed by the equations:
\[ \partial_T \rho = -\sqrt{\pi}(\lambda_a + \lambda_b) \partial_X (|q|^2), \]
\[ i\partial_X q + \frac{1}{2} \partial^2_X q - (\mu/n_0) pq = 0. \]

The above equations constitute the single component Yajima-Oikawa (YO) system [41]. The YO system is completely integrable and possesses soliton solutions of the form \( \psi \approx -\text{sech}^2(k_s X - \omega_s T) \) and \( q \approx \text{sech}(k_s X - \omega_s T) \), where \( k_s, \omega_s \) are constants. These expressions, when substituted into Eqs. [4], give rise to approximate DDB solitons, for the \( n_0 = -1, 0, +1 \) spin components, respectively. Note that we have found (results not shown here) that such DDB solitons are also long-lived in our direct numerical simulations.

In conclusion, we have demonstrated the creation of dark-bright-bright (DBB) and dark-dark-bright (DDB) solitons in a spinor \( F = 1^{87}\text{Rb} \) condensate. It was found that these structures are quite robust, featuring lifetimes on the order of several hundreds of milliseconds, and can be formed for different relative populations of atoms in the three Zeeman states. We have employed a perturbative approach to show that these mixed solitons can be approximated by solutions of the multi- and single-component Yajima-Oikawa systems. This connection also underscores the breadth of relevance of these patterns and supports their robustness. Direct numerical simulations corroborate our results indicating that these solitons can persist but also that they can robustly oscillate inside the condensates.

The experimental, theoretical and numerical manifestation of such states paves the way for a number of interesting studies in the future. For instance, it will be particularly relevant to explore more systematically the oscillations of these solitary waves in the trap and to identify their oscillation frequency as a function of both the trap frequency and the atomic fractions of the different components, similarly to the cases of one- and two-components [57, 58]. Another possibility is to explore the generalizations of such spinorial states in higher dimensions constructing spinorial analogues of vortex-bright (or baby-skyrmion, or filled-core vortex) states [4] and understand their dynamics and interactions. Soliton interaction dynamics and stability over parametric variations (e.g., of the spin-dependent part of the Hamiltonian) would also be particularly relevant to consider even in the one-dimensional case. More broadly, spinor BECs open an avenue to proceed beyond 2-component soliton dynamics that we expect will yield exciting developments in the near future.

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For imaging, the trap is suddenly switched off and a vertical magnetic gradient is applied. Due to this gradient, the three Zeeman states are separated along the vertical z-direction after a brief time of flight (13 ms). An image is then taken along the y-direction, showing the position of the three atom clouds.

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