Plane symmetric solutions in Hořava-Lifshitz theory

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Abstract

The purpose of this paper is to find and analyze plane symmetric, static (non static) solutions in Hořava-Lifshitz gravity. We discussed two versions of Horava gravity. First we showed that if the detailed balance principle have considered, there are both static and non-static solutions. We show that in static case there are two family of solvable models which either of them has a well defined EOS, in analogous to the perfect fluid solutions in GR. In non-static case we find a family of solutions. Some physical properties of these solutions was discussed. Secondly we investigated the plane symmetric solutions for a new modified version of Hořava gravity [18], which has the new terms inserted action in it.

1 Introduction

The Einstein’s equations for hypersurface-homogeneous space-time are reduced to a system of ordinary differential equations. At least for the spatially-homogeneous case, they form a well-posed Cauchy problem [1] and, although they have not been completely integrated, their qualitative properties have been discussed in many papers: (see Wainwright and Ellis [2] for an extensive survey of the results, and e.g. [3] Ryan and Shepley, MacCallum [4], Bogoyavlenskii [5], and Rosquist and Jantzen [6] for useful earlier reviews). Methods of dynamical systems theory which proved fruitful in elucidating these properties lead to ways of restricting the general case to more readily solvable subcases and thence to new exact solutions (see e.g. Uggl et al. [7] for a summary). Nearly all of these methods were initially developed for use in the spatially-homogeneous case (cosmologies, for brevity), and we shall therefore describe the methods in this context although they can be adapted to the $G_3$ on $T_3$ and $H_3$ on $V_3$ cases also (e.g. for an orthonormal tetrad method for G3 on T3 see Harness [8]). The number of freedom degrees , i.e. the number of essential arbitrary constants required in a general cosmology for each Bianchi type, has been studied by Siklos [9] and MacCallum [10], Wainwright and Ellis [11]). The residual set of ordinary

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differential equations which are going to be solved can be formulated in various ways [4]. Wainwright and Ellis [2]. One can use a time-independent basis as in and parameterize the components $g_{\alpha\beta}$ in suitable ways. This is called the metric approach. One may then choose spatial coordinations and a new time coordination $\tau$. 

The main alternative to the metric approach is the ortho normal tetrad approach using a tetrad basis. The two ways are closely related when variables are chosen as just described above [11], in both approaches a suitable choice of lapse (or, to include the $G_3$ on $T_3$ case, slicing gauge), or directly of independent variable, may decouple and simplify the equations; for details see e.g. Jantzen [12], Uggla et al. [7]. A power-law lapse, i.e. a product of powers of the dependent variables, an idea introduced in Bonanos [13], may be useful, as may an intrinsic slicing (e.g. making a product of invariantly-defined metric components the independent variable).

Recently, a power-counting renormalizable, ultra-violet (UV) complete theory of gravity was proposed by Hořava in [14, 15, 16, 17]. Although presenting an infrared (IR) fixed point, namely General Relativity, in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space of the form $x^i \rightarrow \ell x^i$, $t \rightarrow \ell^{\varepsilon} t$, where $\ell$, $\varepsilon$, $x^i$ and $t$ are the scaling factor, dynamical critical exponent, spatial coordination and temporal coordination, respectively. But recently Blas, et.al listed some inconsistencies of the Hořava-Lifshitz gravity as a complete description of Quantum gravity. They discussed the consistency of Hořava proposal for a theory of quantum gravity from the low-energy viewpoint. They uncover the additional scalar freedom degree arising from the explicit breaks the general covariance and its followed properties. Their analysis was done both in the original formulation of the theory and in the Stuckelberg frame. A unusual feature of the new mode is that it satisfies a first order(in time derivatives) equation of motion . In linear aproximation this extra freedom degree manifested only around non-static spatially inhomogeneous backgrounds. They found two serious problems associated with this mode. "First, the mode develops very fast exponential instabilities at short distances. Second, it becomes strongly coupled at an extremely low cutoff scale". They also discussed a version of Hořava gravity with projectable condition and stated that this version is a certain limit of the ghost condensate model. The theory is still problematic since the additional field generically forms caustics and, again, has a very low strong coupling scale. Also they clarify some subtleties that arise from the application of the Stuckelberg formalism to Hořava model due to its non-relativistic nature.

Due to these features, there has been a large effort on examining and extending the properties of the theory itself [19, 20, 21, 22, 23, 27, 28, 29, 30, 31, 23, 26, 32]. Additionally, application of Hořava-Lifshitz gravity as a cosmological framework gives rise to Hořava-Lifshitz cosmology, which is going to lead from interesting behavior [33, 34, 35, 36, 37]. In particular, one can examine specific solution subclasses [38, 39, 40, 41] the perturbation spectrum [42, 43, 44, 45, 46, 47], the gravitational wave production [48, 49], the matter bounce [50, 51, 52], the black hole properties [53, 54, 55, 56, 57, 58, 59], the cosmic string solutions [60] the dark energy phenomenology [61, 62, 63, 64], the astrophysical phenomenology [65, 66] etc. However, despite this extended research, there are still many ambiguities if Hořava-Lifshitz gravity is reliable and capable from a successful description for the gravitational background of
our world, there still many ambiguities. In the present paper we would like to obtain plane-symmetric perfect fluid solutions in the framework of Hořava-Lifshitz gravity. At first we review such solutions in both static and non-static cases in the framework of general relativity. After a brief review of Hořava-Lifshitz gravity, we obtain and analyze plane symmetric, static Taub-like solution. We extend our study to the Purely time dependent solution. Also we discussed some simple solutions for a new version of modified Horava theory which was recently proposed by Blas, et al.

2 Plane-symmetric perfect fluid solutions in GR

2.1 Static solutions

The plane-symmetric static perfect fluids with a prescribed equation of state \( \mu = \mu(p) \) are given by \[ ds^2 = -e^{2\nu}dt^2 + z^2(dx^2 + dy^2) + zF(z)^{-1}dz^2 \] (1)

where

\[ 2zp' \frac{\mu(p)}{p} + p = 1 - \kappa_0 \frac{z^3}{F} = -2z\nu' \] (2)

\[ F' = -\kappa_0 \mu(p)z^2 \] (3)

For a given function \( \mu = \mu(p) \), the differential equations determine \( F(z) \) and \( p = p(z) \) and from \( p(z) \) then \( \nu = \nu(z) \). Equations (2,3) lead to the condition \[ p' \frac{z^2 + G'G + z^3}{z^3 - G} = -\frac{F(z)}{\kappa_0 \mu(p)} \]

So one may prescribe \( G(z) \) and then obtain \( p \) and \( \nu = \nu(z) \) as linear integrals. For an equation of state \( p = (\gamma - 1)\mu \) the function \( G \) was found by Collins. A solution for \( p = \mu/3 \) was obtained by Teixeira et al. The solution for \( \mu = \text{const} \) was given by Taub and by Horsky in terms of hypergeometric functions. Some other special solutions have been given by Davidson. Static plane-symmetric perfect fluids occur also as subcases of the static cylindrically-symmetric solutions.

2.2 Non-static solutions

Besides the classes of solutions which were described in a, several other classes have been found by making special assumptions for the metric or the equation of state. For \( p = \mu \), Tabensky and Taub reduced the field equation to a
A single linear differential equation,
\[ ds^2 = t^{-1/2}e^{\Omega}(dz^2 - dt^2) + t(dx^2 + dy^2), t > 0 \] (4)
\[ \Omega = 2 \int_t^{[\sigma_t^2 + \sigma_z^2]dt + 2\sigma_t\sigma_zdz} \] (5)
\[ \sigma_{tt} + t^{-1}\sigma_t - \sigma_{zz} = 0 \] (6)
\[ \kappa_0p = \kappa_0\mu = t^{1/2}e^{-\Omega}(\sigma_t^2 - \sigma_z^2), \] (7)

Tabensky and Taub [68] also gave the special solution
\[ \sigma = \alpha \log(t) + \beta \arccos(z) \] (8)
\[ \Omega = 2(\alpha^2 + \beta^2)\log(t) + 2\beta^2\log(1 - z^2t^2) + 4\alpha\beta \arccos(z) \] (9)

This solution represents the asymptotic Robertson-Walker Space-time. These functions are particular solutions which are inhomogeneous and anisotropic but toward to an homogeneous spacetime for large times. The solution depends on two parameters \( \alpha \) and \( \beta \). For \( \alpha^2 + \beta^2 = \frac{3}{2} \). We get Robertson-Walker space-time asymptotically. The variables \( z \) and \( t \) are restricted to \( z^2 \leq t^2 \).

For further solutions see Collins and Lang [74], Goode [75], Carot and Sintes [76], Bray [77], Tariq and Tupper [78], Shikin [79], Gupta and Sharma [80]. We want to obtain the analogous of the metric solution (3) in Horava theory. We use from this gauge, since firstly it has a well known simple GR analogous and secondly since the field equations has only one gauge function \( \Omega \) and also this solution is spatially Ricci flat, i.e \( R = 0 \) and several terms in Horava action will be vanished, especially the Cotton tensor \( C_{ij} = 0 \).

3 Review of Hořava-Lifshitz gravity with detailed condition

Following from the ADM decomposition of the metric [81], and the Einstein equations, the fundamental objects of interest are the fields \( N(t, x), N_i(t, x), g_{ij}(t, x) \) corresponding to the lapse, shift and spatial metric of the ADM decomposition. In the \((3+1)\)-dimensional ADM formalism, where the metric can be written as
\[ ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \] (10)
and for a spacelike hypersurface with a fixed time, its extrinsic curvature \( K_{ij} \) is
\[ K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_iN_j - \nabla_jN_i) \]
where ”dot” denotes a derivative from ”t” and covariant derivatives defined with respect to the spatial metric \( g_{ij} \), the action of Hořava-Lifshitz theory for \( z = 3 \) is
\[ S = \int_M dt^3x\sqrt{g}(\mathcal{L}_K - \mathcal{L}_V) \] (11)
we define the space-covariant derivative on a covector $v_i$ as $\nabla_i v_j \equiv \partial_i v_j - \Gamma^l_{ij} v_l$ where $\Gamma^l_{ij}$ is the spatial Christoffel symbol, $g$ is the determinant of the 3-metric and $N = N(t)$ is a dimensionless homogeneous gauge field. The kinetic term is

$$\mathcal{L}_K = \frac{2}{\kappa^2} \mathcal{O}_K = \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2)$$

Here $N_i$ is a gauge field with scaling dimension $[N_i] = z - 1$.

The potential term $\mathcal{L}_V$ of the (3 + 1)-dimensional theory is determined by the principle of detailed balance \[3\], requiring $\mathcal{L}_V$ to follow, in a precise way, from the gradient flow generated by a 3-dimensional action $W_g$. This principle was applied to gravity with the result that the number of possible terms in $\mathcal{L}_V$ are drastically reduced with respect to the broad choice available in an potential is

$$\mathcal{L}_V = \alpha_0 C_{ij} C^{ij} - \alpha_5 \epsilon_{ij}^l R_{im} \nabla_j R^{ml} + \alpha_4 R_{ij} R^{ij} - \frac{4\lambda - 1}{4(3\lambda - 1)} R^2 + \alpha_2 (R - 3\Lambda_W)$$ \[12\]

The coupling constants $\alpha_i$ define by

$$\begin{align*}
\alpha_2 &= \frac{\alpha_4 \Lambda_w}{3\lambda - 1} \\
\alpha_4 &= \frac{\kappa^2 \mu^2}{8} \\
\alpha_6 &= \frac{\kappa^2}{2\nu^2} \\
\alpha_5 &= \frac{\kappa^2 \mu}{2\nu^2}
\end{align*}$$

Where in it $C_{ij}$ is the Cotton tensor \[3\] which is defined as,

$$C^{ij} = \epsilon^{kli} (\nabla_k R^l_{ij})$$

Following \[38\] we can write the action as

$$S = \int dt dx^3 (\mathcal{L}_0 + \mathcal{L}_1) \quad \text{(13)}$$

$$\begin{align*}
\mathcal{L}_0 &= \sqrt{g} N (\kappa^2 \mu^2 (K_{ij} K^{ij} - \lambda K^2) + \kappa^2 \mu^2 (\Lambda_w R - 3\Lambda^2_w)) \\
\mathcal{L}_1 &= \sqrt{g} N (\kappa^2 \mu^2 (1 - 4\lambda) R^2 - \frac{\kappa^2}{2\nu^2} (C^{ij} - \frac{\mu w^2}{2} R^{ij})(C_{ij} - \frac{\mu w^2}{2} R_{ij})) \quad \text{(14)}
\end{align*}$$

4 Exact solutions

We shall restrict ourselves to situations where the space time has plane symmetry. the problem is invariant under transformations of the form

$$\begin{align*}
x &\to x + a \\
y &\to y + b
\end{align*}$$

$$\begin{align*}
x &\to x \cos(\theta) + y \sin(\theta) \\
y &\to y \cos(\theta) - x \sin(\theta)
\end{align*}$$
The metric of a spacetime which admits plane symmetry may be written as [51]

$$ds^2 = -e^{2F}dt^2 + \frac{1}{e^2}(e^{2H}(dx^2 + dy^2) + e^{2G}dz^2)$$  \hspace{1cm} (16)$$

Where $F, G$ and $H$ are just functions of $z$ and $t$ alone. By substituting (16) in (14,15) finally we obtain the following form of action

$$S = \int dt \int d^3x \frac{1}{c^3} e^{2H+G+F} [\frac{2}{\kappa^2} e^{-2F} [(1-\lambda)G'^2 + 2(1-2\lambda)\dot{H}^2 - 4\lambda \dot{G}\dot{H}] + \frac{\kappa^2 \mu^2}{8(1-3\lambda)}(\Lambda_w R - 3\Lambda_w^2)] (17)$$

Where in it $f' \equiv df/\partial z, \dot{f} \equiv df/\partial t$ and the spatial Ricci Scalar for metric (15) reads as,

$$R = 2c^2 e^{-2G}(3H'^2 + 2H'' - 2H'G')$$

By varying $G, H$ and $F$, we obtain the equations of motion. Considering the case where $3\lambda - 1 > 0$, one can see, from the modified Friedmann equations in the context of the Horava theory, a negative $\Lambda$ is required. In Ref. [38], the authors point out that a positive $\Lambda$ can be obtained by making an analytical continuation for parameters. In this work we emphasized on the $3\lambda - 1 \leq 0$.

Choosing $\lambda$ like above avoids us from a negative cosmological constant term which is so harmful for the cosmological evidences.

4.1 Static Taub-like solution

In this section we want to obtain a Taub like solution (1) as metric functions (2,3) with action (17). Comparing two metrics (16) and (1) we observe that we must take the metric functions as $H = \log(cz), G = \log(\sqrt{F(z)})$ and $F = \nu(z)$.

In this case the scalar Ricci is $R = \frac{2F''}{F}$ and the full action (17) convert to the next functional:

$$S = \int dt \int d^3x \Phi(\nu, F(z), F'(z)) (18)$$

$$\Phi(\nu, F(z), F'(z)) = -\frac{1}{32} e^\nu \kappa^2 \mu^2 z^{-3.5} F(z)^{-0.5}(4\lambda - 1)^{-1}(3\lambda - 1)^{-1}\sum_{n=0} f_n z^n (19)$$

The $\nu$ equation of motion is:

$$\sum_{n=0} f_n z^n = 0$$  \hspace{1cm} (20)$$

Where the Functional coefficients $f_n$ were defined by:

$$f_0 = (24\lambda^2 - 11\lambda + 1) F^2$$  \hspace{1cm} (21)$$

$$f_1 = (-144\lambda^2 + 90\lambda - 14) FF'$$  \hspace{1cm} (22)$$

$$f_2 = (24\lambda^2 - 11\lambda + 1) F'^2$$  \hspace{1cm} (23)$$

$$f_3 = 8\Lambda_w(4\lambda - 1) F$$  \hspace{1cm} (24)$$

$$f_4 = 8\Lambda_w(4\lambda - 1) F'$$  \hspace{1cm} (25)$$

$$f_6 = 12\Lambda_w^2(1 - 4\lambda)$$  \hspace{1cm} (26)$$
Just $F(z)$ field equation is more complicated, we do not write it here. We mentioned here that the general solution for metric function $F(z)$ and the energy density function $\rho$ for action (19) is:

$$F(z) = -e^{-1} I(a, b; z)[12\lambda(4\lambda - 1)I(c, d; z) + (1 - 3\lambda)c_1]$$

$$h = 8\lambda(z - 3) - z + 7$$

$$I(\alpha, \beta; z) = \int h^\alpha z^\beta dz$$

$$\kappa_0 \rho = e^{-1} z^{-2}[h^a z^b(12\lambda(4\lambda - 1)I(c, d; z) + (1 - 3\lambda)c_1) + 12\Lambda(4\lambda - 1)h^c z^d I(a, b; z)]$$

Where

$$a = \frac{704\lambda^2 - 400\lambda + 55}{(8\lambda - 1)(24\lambda - 7)}$$

$$b = -\frac{4\lambda - 1}{24\lambda - 7}$$

$$c = \frac{(4\lambda - 1)(8\lambda - 3)}{(8\lambda - 1)(24\lambda - 7)}$$

$$d = \frac{40\lambda - 11}{24\lambda - 7}$$

$$e = (-1 + 3\lambda)^{-1}$$

the integral can be calculated by using the Hypergeometric functions as

$$I(c, d; z) = \int h^c z^d dz = \frac{z^{1+d}}{1+d}(7 - 24\lambda)^c F[1 + d, -c, 2 + d, z(-8\lambda + 1)]$$

Without overloading the paper with lengthy formulae, the pressure $p(z)$ and metric function $\nu(z)$ can be calculated from (2,3). It was argued by Horava that in the infrared limit the higher order terms become negligible and so one may expect to recover GR if the parameter $\lambda$ flows to 1 in that limit. But Blas et al [18] argued that this expectation is incorrect: the explicit breaking of general covariance leads to the appearance of an extra degree of freedom in the infrared which becomes strongly coupled when $\lambda$ approaches 1. More precisely, the additional mode is weakly coupled only in a narrow window at low energies. This window depends both on $\lambda$ and the parameters of the background geometry; it shrinks to zero when $\lambda \to 1$ or when the background curvature vanishes. Thus the solution has no familiar GR limit. Also as stated by Blas et al [18], the only consistent solution for this theory must be time dependent and inhomogeneous. Another reason for considering the plane symmetric metrics goes to the Mukohyama work [82]. He considered the version without detailed balance condition with the projectability condition and addressed one aspect of the theory: avoidance of caustics for constant time hypersurfaces. He showed that there is no caustic with plane symmetry in the absence of matter source if $\lambda \neq 1$. If $\lambda = 1$ is a stable IR fixed point of the renormalization group flow then $\lambda$ is expected to be deviated from 1 near would-be caustics, where the extrinsic

$$2F[a, b; c; x] = \sum_{k=0} \frac{(a)_k b^k x^k}{c^k k!}$$

$$(a)_k = \frac{(a+k-1)!}{(a-1)!}$$

\[7\]
curvature increases and high energy corrections become important. Therefore, the absence of caustics with $\lambda \neq 1$ implies that caustics cannot form with this symmetry when the source is absent. He argued that inclusion of matter source will not change the conclusion. He also argued that caustics with codimension higher than one will not form because of repulsive gravity generated by nonlinear higher curvature terms. These arguments support Mukohyama conjecture that there is no caustic for constant time hypersurfaces. Our exact plane symmetric solution endorsement this conjecture directly.

### 4.2 Other special solutions

**Case $\lambda = 1/3$:**

As the observation in \[16\] the value $\lambda = 1/3$ corresponds to the action being invariant under an anisotropic conformal (Weyl) symmetry. The case $\lambda = 1/3$, corresponds to an anisotropic Weyl-invariant theory, is another relevant value for which the equations of motion simplify and hence admit quite explicit solutions. Recently \[84\] derived general static spherically symmetric solutions in the Hořava theory of gravity with nonzero shift field. These represent hedgehog versions of black holes with radial hair arising from the shift field. For the case of the standard de Witt kinetic term ($\lambda = 1$) there is an infinity of solutions that exhibit a deformed version of re parametrization invariance away from the general relativistic limit. Special solutions also arise in the anisotropic conformal point $\lambda = 1/3$. This is a simple reason that why we investigated solution with $\lambda = 1/3$.

The field equation is:

$$\Lambda w z^3 = \frac{2}{3} (zF)'$$

The general solution for this differential equation is:

$$F(z) = \frac{3\Lambda w}{8} z^3 + c_1/z$$

To find the unknown metric function $\nu$ we must use (2,3). By solving the simple differential equations for $\nu(z), p(z)$ and by substituting $\mu(z)$ from (3), after a simple integration it leads to the

$$\mu(z) = 1/8(-\frac{9\Lambda w}{\kappa_0} + 8\frac{c_1}{\kappa_0 z^3})$$

$$p(z) = -0.3735\xi \frac{\theta}{\eta}$$

$$\theta = \sqrt{c_1}\xi P(-0.25, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}}) - 3.1768 c_1 P(0.75, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}})$$

$$+ c_2 \sqrt{c_1} Q(-0.25, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}}) - 3.1768 c_1 c_2 Q(0.75, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}})$$

$$\eta = \sqrt{c_1}\kappa_0(-\xi^2 + 8c_1)(c_2 Q(-0.25, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}}) + P(-0.25, 1.0308, 0.3535 \frac{\xi}{\sqrt{c_1}}))$$

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$^a P(a, b, x), Q(a, b, x)$ are the associated Legendre functions of the first and second kind, new coordinate $\xi$ is defined as $\xi = (3\Lambda w z^4 + 8c_1)^{1/2}$
The expression of the metric function $\nu$ is volumeless and we do not write it here. We must mention here that this spacetime represents a new exact solution for GR with a complicated implicit EOS $p = p(\mu)$ which could be derived by eliminating the $z$ coordinate between the both pressure and energy density expressions given by the above functions.

**Case $\lambda = 1/4$:**

In this case the differential equation (20) is:

$$\frac{1}{2}z^2(F')^2 + zFF' + \frac{1}{2}F^2 = 0$$

(36)

This equation has a simple solution

$$F(z) = \frac{c_1}{z}$$

(37)

this is the special case of solution (28) with the $\Lambda_w = 0$. We can write the following expressions for $\mu(z), p(z)$:

$$\mu(z) = \frac{c_1}{\kappa_0 z^4}$$

(38)

$$p(z) = \mu(z)(4 - \sqrt{17} \tanh \frac{\sqrt{17}}{2} (c_2 - \log z))$$

(39)

by eliminating the $z$ between (35,36) we can write the EOS as:

$$p(\mu) = \mu(4 - \sqrt{17} \tanh \frac{\sqrt{17}}{2} (c_2 + 1/4 \log \mu))$$

(40)

Where $c'_2 = c_2 - 1/4 \log(c_1/\kappa_0)$.

The metric function $\mu(z)$ in this case has simple expression as:

$$\nu(z) = \log(\frac{z^{3/2}}{\sqrt{\frac{17}{2}}} \cosh \left(\frac{\sqrt{17}}{2} (\log z - c'_2)\right))$$

(41)

Thus the metric (16) transforms to the following form:

$$ds^2 = z^2[-z^{-1/2} \cosh \left(\frac{\sqrt{17}}{2} (\log z - c'_2)\right) dt^2 + dx^2 + dy^2 + \frac{1}{c_1} dz^2]$$

(42)

### 4.3 Purely time dependent solution

If we consider the non-static, plane symmetric solutions with the metric anstaz (4). In synchronous time $t$, the Plane ADM metric has $N_i = 0$, where $\Omega$ is the usual Tabensky and Taub metric function. we assumed that the metric function $\Omega = \Omega(t)$. Since the metric (3) is Spatial Ricci flat, and further the Spatial Ricci tensor and consequently the result Cotton tensor are vanished, then the full curvature dependence part of the action (13), i.e the $\mathcal{L}_1 = 0$. The easiest way to obtain the solution for the full Lagrangian is to substitute the metric anstaz (3) into the action, and then vary the function $\Omega$. This is a valid procedure in analogous with cylindrically and spherical kinds. The result reduces Lagrangian to an overall scaling constant, is given by

$$\mathcal{L} = \sqrt{\epsilon^\Omega} \left( \frac{2}{\kappa^2} [t^{-3/2} e^{-\Omega}] \frac{1}{2} + \frac{1}{4} \sqrt{\frac{17}{2}} (1 - 5/4 + t^{-1/4} \Omega)^2 - \frac{\lambda}{4} \frac{1}{z} + t \Omega^2) \right] - \frac{3}{2} \frac{\kappa^2}{1 - 3\lambda} \Lambda_w^2$$

(43)
Perhaps surprising power $5/2$ can be understood by observing that this form of $\mathcal{L}$ is invariant under the reflection. we assumed that the metric function $\Omega = \Omega(t)$. The remaining equation is obtained by variation of Lagrangian (15). The field equation is:

$$\frac{d}{dt} \left( t \frac{d\Omega}{dt} \right) = b \sqrt{7} e^\Omega$$ \hspace{1cm} (44)

where

$$b = \frac{3\kappa^4 \mu^2 \Lambda_w^2}{(3\lambda - 1)(\lambda - 1)}$$ \hspace{1cm} (45)

The general solution for this ODE is

$$\Omega(t) = \log \left( \frac{(8c_1 b - 9) \sec^2 \left( \frac{1}{4} \sqrt{8c_1 b - 9(c_2 - \log(t))} \right)}{8b t^{3/2}} \right)$$ \hspace{1cm} (46)

Thus the general solution for metric (3) with Lagrangian (15) in context of Horava theory is:

$$ds^2 = \frac{(8c_1 b - 9) \sec^2 \left( \frac{1}{4} \sqrt{8c_1 b - 9(c_2 - \log(t))} \right)}{8b t^{3/2}} \left( dz^2 - dt^2 \right) + t(dx^2 + dy^2)$$ \hspace{1cm} (47)

The interpretation is as follows: we choose $d$ as the proper distance from $z = 0$ along $x, y, t$ constant. The density vanishes at $z = \infty$ and at $z = 0$. The plane $z = 0$ behaves like a hard wall. It can be shown that all time-like geodesics oscillate between two $z$ planes determined by initial conditions. No time-like geodesic ever touches the wall. We can, imagine the space-time for $z < 0$ to be the vacuum solution. Strictly speaking that one doesn’t know the matching condition for singular hypersurfaces but the type of divergence from both sides of $z = 0$ is exactly the same. We conclude that the only way to have a static solution is to introduce an additional boundary condition.We shall now show that in asymptotic limit, space-time is homogeneous, and for some of the parameter values, Robertson-Walker\footnote{Asymptotic Robertson-Walker Space-time}. All we have to do is to carry out the inversion for comoving coordinations for large $t$’s. The calculation is straight-forward but rather long, so we omit it.Therefore as similar to the GR, we get that for any group of fluid world-lines, the space-time is homogeneous.If the four-velocity is irrotational, it can be written in terms of a scalar function $\sigma(z, t)$ as $u_\alpha = (-\sigma^\alpha \sigma_\alpha)$, and Modified Einstein field equations for a stiff fluid read $p = \rho$.

5 Plane symmetric solution for modified Horava-Lifshitz gravity

In this section we attempt to find analytic solutions for the modified Horava-Lifshitz gravity proposed recently by Blas, et. al\footnote{Asymptotic Robertson-Walker Space-time}. For this we choose the action to be

$$S = \int d^3x dt N \sqrt{g} (\alpha (K_{ij} K^{ij} - \lambda K^2) + \beta C_{ij} C^{ij} + \xi R + a_1(a_i a^i)) \hspace{1cm} (48)$$

$$a_i = \frac{\partial_i N}{N}$$
The same analysis was performed by this action and by imposing static gauges for Lapse and the metric functions (with a vanishes shift function) by Kiritsis is Spherical symmetry [83], and we are going to look for plane symmetric solutions with zero shift of the form

\[ ds^2 = -N(z)^2 dt^2 + dz^2 + u(z)^2 (dx^2 + dy^2) \] (49)

For such static-gauge the kinetic terms do not contributes the equations of motion. We insert this metric ansatz in the action we have:

\[ S = \int d^3 x dt L \]
\[ L = 2\beta Nu'^2\left(\frac{u''}{u} - \frac{u'^2}{u^2}\right)^2 + 2\xi(u'^2 + 2uu'') + a_1 u^2 \frac{N'^2}{N} \]

5.1 Solution in the absence of the Cotton tensor

There is only one simple solution in case \( \beta = 0 \). In this case the contribution from Cotton tensor disappeared. This solution is written as:

\[ N(z) = \sqrt{\frac{\xi}{2a_1}} \coth^{-1}(c_1 \sqrt{\frac{a_1}{2\xi}})z) \] (50)
\[ u(z) = c_2 N(z)^{1/4} \sqrt{N'(z)} \] (51)

5.2 Constant (spatially) curvature solutions

We know that constant curvature solutions form a rich family of investigations both in GR and f(R) metric gravity. If we accept that there is a Cosmological constant term in the universe which dominated in epoch, thus all the models of the universe imposed that the field equations govern the spacetime manifolds must be asymptotically de Sitter. This means that the spacetime posses a constant Ricci scalar. Also in the context of the string theory the existence of the constant negative curvature solution (AdS) is fundamental. Far away from any of this reasons as an attempt we can suppose that there is a constant spatial Ricci scalar solution for our field equations in this modified form of the Horava theory. This family of exact solutions can be obtained by supposing that for this solution the Ricci scalar \( R \) is constant. Solving the field equation for \( u(z) \) we obtain:

\[ u(z) = \sqrt{\frac{2A}{R}} \cosh(\theta_0 + \sqrt{\frac{R}{2A}}z) \] (52)

for finding the lapse function \( N(z) \) one must solve the following differential equation

\[ \psi' + \psi (\frac{1}{2} \psi + 2(\log(u(z)))') = g(z) \] (53)

\[ \psi = \frac{N'}{N} \]
\[ g(z) = \frac{f(z)}{2a_1 u(z)^2} \]
\[ f(z) = 2\beta u'(z)(\frac{R}{4} - 2\left(\frac{u'(z)}{u(z)}\right)^2) \]
This is a standard form of a Riccati equation. There is not found a general solution for it, yet for non-vanishing values of $g(z)$. But if we set $\beta = 0$ then this differential equation solves by simple methods. The final result is:

$$N(z) = c_2 [2A \tanh^2(\sqrt{R/2A}z) + 4\sqrt{2ARc_1} \tanh(\sqrt{R/2A}z) + 4Rc_1^2]$$  \hspace{1cm} (54)

6 Conclusion

The Horava-Lifshitz gravity is a power-counting renormalizable theory, which has anisotropic scale between space and time in the UV limit, and thus breaks the Lorentz invariance. By applying this theory to plane symmetric spacetimes, one finds a set of simple differential equations. This system could be solved analytically and two family of static Taub like solutions are obtained. In both families the cosmological constant term is positive. We mentioned here that we do not have any analytical continuation on the parameters of the model. We observe that in contraction of the cosmological models based on the Horava theory, there are no trouble problematic features of model in the special case $\lambda = 1/3, 1/4$. So the negative cosmological constant problem can be solved successfully in the plane symmetry. In this case, the system has both a state and a non static one. The former corresponds to the two families corresponds to the $\lambda = 1/3, 1/4$. The first solution $\lambda = 1/3$ is a generalization of the Taub fluid solution in GR. But the EOS of the fluid in this case is so complicated and contains Legendre functions. Once the equation of state $w = p/\rho$ of a perfect fluid reaches to infinity. If we rewrite the field equations in terms of a first order system of autonomous equations, this stable critical point coincides with the saddle one. Thus the stable state is broken. It may be expected that the equation of state may change with the energy scale. Actually, if $\lambda = 1/3$, this solution represents the role of an infinite charge in the Kehagias-Sfetsos (KS) black hole[85]. We notice that if $g_{\mu\nu}$ is a solution then $\psi g_{\mu\nu}$ is also a solution, whenever $\psi$ is constant. From now on it shall be understood that all line elements can be multiplied by a constant conformal factor. The non static solution recovers the GR solution in IR regime. It is homogenous and asymptotically Robertson-Walker Space-time in comoving coordinations. In spite of the GR stiff matter solutions, unfortunately there are not many non-static solutions with only one Killing vector. We may can state a theorem as:

**Theorem:** If the metric (4) satisfies the vacuum field equations for a Horava type of gravity, then the perfect fluid field equations for a stiff fluid are satisfied by the metric (4) which differs from (3) by the same simple (but not so trivial as the GR) in metric function $\Omega(t, z)$ respectively.

We lead proof of this theorem to another work. The technique described by the above theorem can be applied to many solutions: the class of vacuum metrics (4) belongs to those metrics with an orthogonally transitive Abelian groups. Moreover, since the metric function $g_{xx} = g_{yy} = t^2$ does not change under these generation techniques, the same functions $\sigma$ and $\Omega$ can be used for all vacuum solutions obtained from the same vacuum seed. A different way of looking at the class of solutions covered by this Theorem is for starting from a stiff fluid solution, performing a transformation to a vacuum solution, applying a soliton-generation technique and then going back to the stiff fluid: one then may speak of a solution describing solitons travelling in the background of a (particular)
stiff fluid of higher symmetry. Of course, since σ and Ω need not be changed, it suffices to immediately transform only when the vacuum part of the metric. Also we derived two family of exact solution for the modified version of Horava theory which recently proposed by Blas et al. It was shown that there is one solution which it’s spatial curvature is constant. Another solution is obtained in the absence of the Cotton tensor.

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