Yang-Mills sources in biconformal gravity

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October 27, 2020

Abstract

Biconformal gravity, based on gauging of the conformal group to $2n$ dimensions, reproduces $n$-dim scale-covariant general relativity on the co-tangent bundle in any dimension. We generalize this result to include Yang-Mills matter sources formulated as $SU(N)$ gauge theories with a twisted action on the full $2n$-dimensional biconformal space. We show that the coupling of the sources to gravity does not stop the reduction of effective dimension $2n \to n$ of the gravity theory, and instead forces the Yang-Mills source to reduce to ordinary $n$-dimensional Yang-Mills theory on the gravitating cotangent bundle, with the usual Yang-Mills energy tensor as gravitational source. The results apply as well to gravity with sources on Kähler manifolds and in double field theories.

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1 Introduction

In general relativity the coupling of matter sources to gravity is accomplished by making the matter action invariant under general coordinate transformations, then adding it to the Einstein-Hilbert action. In formulations of general relativity based on the conformal group, there may be additional conditions. Here we examine $SU(N)$ gauge theories as sources for gravity in a large class of spaces of doubled dimension.

Doubled dimension of spacetime arises in various contexts, frequently related to the idea of a relativistic phase space. Born reciprocity (M. Born, [1, 2]) was one early suggestion aimed at unifying relativity and quantum theory. The reciprocity involves the scaled symplectic exchange $x^a \rightarrow ap^a, p^\beta \rightarrow -bx^\beta$, thereby preserving Hamilton’s equations. Further developments include the study of Kähler manifolds, with mutually compatible metric, symplectic, and complex structures such that any two of the structures yield a specific third.

In a gauge theory approach to gravity, Ivanov and Niederle [3, 4] showed that general relativity can arise in a space of doubled dimension, now called biconformal space. Generalizing the 8-dimensional quotient of the conformal group of spacetime by its homogeneous Weyl subgroup led the authors to a class of curved geometries. From an action quadratic in the curvatures they found that suitable constraints reduced the field equations to the Einstein equation in 4-dimensions.

Another form of doubled dimension arises as a means of making the $O(d,d)$ symmetry of $T$-duality manifest. By introducing scalars to produce an additional $d$ dimensions, Duff [5] doubled the $X(\sigma,\tau)$ string variables to make this $O(d,d)$ symmetry manifest. Siegel brought the idea to full fruition by deriving results from superstring theory [6, 7, 8]. Allowing fields to depend on all $2d$ coordinates, Siegel introduced generalized Lie brackets, gauge transformations, covariant derivatives, and a section condition on the full doubled space, thereby introducing torsions and curvatures in addition to the manifest $T$-duality. By restricting half the coordinates—called imposing a section condition—one recovers the $d$-dimensional theory.

In [9] it was shown that biconformal spaces of any dimension $2n$ admit an action linear in the curvature with field equations reducing to the vacuum Einstein equation in $n$ dimensions. This reduction in both field count and the number of independent variables occurs by virtue of the field equations. Subsequently, results in [10, 11, 12] show that these spaces are generically Kähler. They also are found to share the properties of double field theories [13].

Because biconformal gravity is a gauge theory, it allows direct extension of the symmetry to include gravitational sources from $SU(N)$ gauge theories. However, such Yang-Mills type sources must be written in the space of doubled dimension. This means the introduction of far more potential source fields, $\frac{2n(2n-1)}{2} \times$
\( (N^2 - 1) \) instead of only \( \frac{n(n-1)}{2} \times (N^2 - 1) \), each depending on \( 2n \) independent variables instead of only \( n \).

There are two principal questions we address here. First is the question of the correct form of the Yang-Mills action in the doubled dimension. We find that the usual \( \int tr (\mathcal{F}^* \mathcal{F}) \) form must be augmented with a “twist” to reproduce familiar results, similar to the twist found in studies of double field theories [14, 15, 16]. The second question is whether the increase in fields and independent variables spoils the gravitational reduction, or at the other extreme, shares the reduction to \( n \) dimensions with gravity. We find that both gravity and sources reduce to \( n \)-dimensions. The reduction again occurs by virtue of the field equations, with no need for section conditions.

As noted above, though written in the language of biconformal gravitational gauge theory, our results apply to gravity theories on Kähler manifolds and to double field theories.

Before proceeding with our investigation of \( SU(N) \) sources in biconformal/double field theory/Kahler gravity, we look at sources in other gravity theories. We first exhibit general relativity with Yang-Mills sources as a gauge theory. This displays our general approach to gauging, as well as the reduced result we hope to achieve. In the remainder of this Section, we look at modifications of other conformally based gravity theories required in order to include sources. In general relativity, we must extend the symmetry of source actions to general coordinate from global Lorentz. By analogy, we expect that including matter in conformally based theories may require some additional conditions. We briefly discuss sources in scale-invariant gravity and Weyl gravity, before concluding the Section with a discussion of sources in doubled dimension theories.

### 1.1 Yang-Mills matter in general relativity

Defining a projection on the quotient of the \( \frac{n(n+1)}{2} \)-dimensional Poincaré group \( \mathcal{P} \) by its \( \frac{n(n-1)}{2} \)-dimensional Lorentz subgroup \( \mathcal{L} \) gives a principal fiber bundle with Lorentz fibers over an \( n \)-dimensional Minkowski spacetime. Generalizing the base space by changing the Maurer-Cartan connection forms of the Poincaré group and perhaps changing the manifold, the fiber structure is maintained by demanding horizontality of the curvature and torsion. The result is a Riemann-Cartan geometry characterized by curvature and torsion with local Lorentz symmetry.

Concretely, the generalization of the connection \( (\tilde{e}^i, \tilde{\omega}^a_{\ b}) \Rightarrow (e^i, \omega^a_{\ b}) \) takes the Maurer-Cartan equations of the Poincaré group,

\[
\text{d} \tilde{\omega}^a_{\ b} = \tilde{\omega}^c_{\ b} \wedge \tilde{\omega}^a_{\ c}
\]
\[
\text{d} \tilde{e}^a = \tilde{e}^b \wedge \tilde{\omega}^a_{\ b}
\]
to the Cartan equations,

\[
\begin{align*}
\text{d} \omega^a_{\ b} &= \omega^c_{\ b} \wedge \omega^a_{\ c} + R^a_{\ b} \\
\text{d} e^a &= e^b \wedge \omega^a_{\ b} + T^a
\end{align*}
\] (1) (2)

where horizontality of the curvature \( R^{ab} \) and the torsion \( T^a \) is captured by omitting any occurrence of the spin connection when writing them expressly as 2-forms

\[
\begin{align*}
R^a_{\ b} &= \frac{1}{2} R^a_{\ bcd} e^c \wedge e^d \\
T^a &= \frac{1}{2} T^a_{\ bc} e^b \wedge e^c
\end{align*}
\]

Horizontality insures the survival of the principal fiber bundle by guaranteeing that integrals of the curvatures over an area or of the connection forms over closed curves are independent of lifting.

Completing the description of the Riemann-Cartan geometry is the demand for integrability of the Cartan equations, which follows by exterior differentiation of Eqs. (1) and (2):

\[
\begin{align*}
\text{d} R^a_{\ b} &= 0 \\
\text{d} R^a_{\ b} &= e^b \wedge R^a_{\ b}
\end{align*}
\]

When torsion vanishes, this construction describes a general \( n \)-dimensional Riemannian spacetime with local Lorentz symmetry.

To include an additional \( SU(N) \) Yang-Mills symmetry in the fiber bundle we extend the \( \mathcal{P}/\mathcal{L} \) quotient to the quotient of the product \( \mathcal{P} \times SU(N) \) by the product \( \mathcal{L} \times SU(N) \):

\[
\mathcal{P} \times SU(N) \sslash \mathcal{L} \times SU(N)
\]

and carry out the same procedure. This still results in an \( n \)-dimensional spacetime but now the fibers of the principal bundle are isomorphic to \( \mathcal{L} \times SU(N) \). The Cartan Eqs. (1) and (2) are augmented by a third equation,

\[
\begin{align*}
\text{d} A^i &= -\frac{1}{2} c^i_{\ jk} A^j \wedge A^k + F^i
\end{align*}
\]
where indices beginning with $i$ have range $i, j, k, \ldots = 1, 2, \ldots, N^2 - 1$, and $\mathbf{F}^i$ is horizontal

$$\mathbf{F}^i = \frac{1}{2} F^i_{\ ab} e^a \wedge e^b$$

The integrability condition is $\mathbf{D} \mathbf{F}^i = 0$, where $\mathbf{D}$ is the $SU(N)$ covariant derivative. Here, indices from the first part of the alphabet have range $a, b, \ldots = 1, \ldots, n$.

To build a physical theory, we write an action functional using any of the tensor fields arising from the construction, $R^a_{\ b}, T^a, F^i, e^a, \eta_{ab}, e_{ab\cdots c}$, together with any representations of the original group.

In any dimension of spacetime, the action coupling the $SU(N)$ Yang-Mills field to general relativity is written as

$$S = S_{GR} + S_{YM} = \int R^{ab} \wedge e^c \wedge \ldots \wedge e^d \varepsilon_{abc\ldots d} - \frac{\kappa}{2} \int F^i \wedge \ast F_i$$

where $\ast F_i$ is the Hodge dual of the 2-form $F_i$. We vary the action with respect to the solder form, $e^a$, the spin connection, $\omega^a_{\ b}$, and the Yang-Mills connection $A^i$. Making the usual assumptions for the gravity theory to reduce to general relativity, this results in

$$R_{ab} - \frac{1}{2} \eta_{ab} R = \kappa \left( \eta^{cd} F^i_{\ ac} F_i \ bd - \frac{1}{4} \eta_{ab} F^i_{\ cd} F_i \ cd \right)$$

$$\tilde{D}^c F^i_{\ ac} = 0 \quad (3)$$

where $\tilde{D}^c$ is covariant with respect to both local Lorentz and local $SU(N)$ transformations. These methods generalize immediately to additional internal symmetries, such as the $SU(3) \times SU(2) \times U(1)$ of the standard model.

It is Eqs.(3) in $n$-dimensions which we hope to reproduce from theories formulated in $2n$-dimensions.

1.2 Sources for scale-invariant gravity

In addition to general covariance, scale-invariant gravity requires tracelessness of the energy tensor. This condition arises when local scale invariance is a symmetry of the action. Let $S_g$ be any locally scale invariant gravity action, with sources supplied by adding a matter action,

$$S_{\text{matter}} = \int \mathcal{L}_{\text{matter}} \sqrt{-g} d^4x$$
Then the source for gravity is the energy-momentum tensor given by the metric variation, 

\[ \delta_g S_{\text{matter}} = \int \frac{\delta (L_{\text{matter}} \sqrt{-g})}{\delta g^{\alpha \beta}} \delta g^{\alpha \beta} d^4x \]

\[ = \int T_{\alpha \beta} \delta g^{\alpha \beta} \sqrt{-g} d^4x \]

If this matter action is scale invariant, we may apply Noether’s theorem to the scaling symmetry transformation, \( \delta g^{\alpha \beta} = -2 \phi g^{\alpha \beta} \). This immediately gives

\[ 0 \equiv \delta_{\text{symmetry}} S_{\text{matter}} = -2 \int T_{\alpha \beta} \phi g^{\alpha \beta} \sqrt{-g} d^4x = -2 \int T^\alpha_{\alpha} \phi \sqrt{-g} d^4x \]

and since this must hold for any \( \phi \), the energy tensor must be traceless, \( T^\alpha_{\alpha} = 0 \).

### 1.3 Sources for Weyl gravity

More controversial is the case of Weyl gravity. Here the action is quadratic in the curvature, with metric variation leading to fourth order field equations. Some nonetheless argue [17, 18] for direct addition of a matter action with traceless energy tensor,

\[ S = \int C^{\alpha \beta \mu \nu} C_{\alpha \beta \mu \nu} \sqrt{-g} d^4x + S_{\text{matter}} \]  

(4)

It is easy to see that solutions to the vacuum Einstein equation are also solutions to the Bach equation,

\[ W^{\mu \nu} = D_{\alpha} D_{\beta} C^{\mu \alpha \nu \beta} - \frac{1}{2} C^{\mu \alpha \nu \beta} R_{\alpha \beta} = 0 \]  

(5)

since, up to boundary terms, the equation may be written entirely in terms of the Ricci tensor as

\[ 0 = R^{\mu \nu ; \beta ; \beta} - R_{\beta ; \beta} + R^{\mu \beta ; \nu ; \beta} - R^{\nu \beta ; \mu ; \beta} - 2 R^{\mu \beta} R^{\nu \beta} - \frac{1}{6} g^{\mu \nu} R_{\beta ; \beta} \]

\[ + \frac{1}{2} g^{\mu \nu} R^{\alpha \beta} R_{\alpha \beta} + \frac{2}{3} R^\mu_{\mu \nu} + 2 R R^\mu_{\mu \nu} - \frac{1}{6} g^{\mu \nu} R^2 \]  

(6)

Thus, it might be argued that evidence for vacuum general relativity is also evidence for vacuum Weyl gravity. However, the situation with sources is completely different. It cannot be claimed that solutions to general relativity with sources gives evidence for Weyl gravity with sources. In fact, it is not at all clear
what is meant by Weyl gravity with sources. While Bach tensor is symmetric with vanishing divergence,

\[ W^{\mu\nu} = W^{\nu\mu} \]

\[ W^{\mu\nu;\nu} = 0 \]

so that–presumably–we should add the actions as in Eq.(4) and vary the metric to find

\[ W^{\mu\nu} = \kappa T^{\mu\nu} \]

it is clear that the resulting solutions may differ substantially from the verified predictions of general relativity. Specifically, Flanagan has argued that the fourth order theory fails to agree with solar system values, and Yoon has criticized the Newtonian limit and the application to galactic rotation curves [for the critique and response see [19, 20, 21]]. On the other hand, Palatini-style variation has been shown for the vacuum case to reduce to scale covariant general relativity [22], and a novel coupling to matter might give better agreement with known results.

For Weyl gravity, whatever the form of \( S_{\text{matter}} \) in Eq.(4), Noether’s theorem takes the form,

\[ 0 \equiv \delta_{\text{symmetry}} S = \delta_{\text{symmetry}} \int C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \sqrt{-g} d^4 x + \delta_{\text{symmetry}} S_{\text{matter}} \]

\[ = \int C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \left( 2 g^{\mu\rho} g^{\nu\sigma} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g_{\lambda\delta} g_{\lambda\delta} \right) \sqrt{-g} d^4 x + \kappa \int T_{\alpha\beta} \delta g^{\alpha\beta} \sqrt{-g} d^4 x \]

\[ = \kappa \int T_{\alpha\beta} \delta g^{\alpha\beta} \sqrt{-g} d^4 x \]

where \( C^{\alpha\beta\nu} C_{\beta\alpha\nu} - \frac{1}{4} C^{\alpha\beta\mu\nu} C_{\alpha\beta\rho\sigma} g_{\lambda\theta} \equiv 0 \) in 4-dimensions. Since the Weyl curvature terms vanish identically, we must have

\[ T^{\alpha}_{\phantom{\alpha} \alpha} = 0 \]

This is also a direct consequence of the field equation, since \( W^\alpha_{\phantom{\alpha} \alpha} \equiv 0 \).
1.4 Sources for biconformal gravity, Kähler gravity, and double field theory

The situation may not be as complicated for biconformal gravity [4, 24, 9, 13], double field theory [6, 7, 8, 5, 25, 27], or gravity on a Kähler manifold [10, 13]. Each of these cases starts as a fully $2n$-dimensional theory but ultimately describes gravity on an $n$-dimensional submanifold. It is desirable to have a fully $2n$-dimensional form of the matter action which nonetheless also reduces to the expected $n$-dimensional source as a consequence of the field equations. It is this condition we address. We discuss the issue in biconformal space, since biconformal gravity is already a gauge theory and it generically includes the structures of both double field theory and Kähler manifolds [10, 13].

For matter fields we restrict our attention to Yang-Mills type sources. We find that although the usual form of $2n$-dimensional Yang-Mills action gives nonstandard coupling to gravity, including a “twist” matrix in the action corrects the problem.

Biconformal gravity arises as follows. The quotient of the conformal group $\mathcal{C}_{p,q} = \text{SO}(p+1, q+1)$ of an $\text{SO}(p,q)$-symmetric space $(p+q = n, \text{metric } \eta_{ab})$ by its homogeneous Weyl subgroup, $\mathcal{W}_{p,q} = \text{SO}(p,q) \times \text{SO}(1,1)$, leads to a $2n$-dimensional homogeneous space with local $\mathcal{W}_{p,q}$ symmetry. This homogeneous space, discussed in [4] and [24] and studied extensively in [10, 11, 12, 13], is found to have compatible symplectic, metric and complex structures, making it Kähler [10]. In addition, the restriction of the Killing form to the base manifold is nondegenerate and scale invariant, and the volume form of the base manifold is scale invariant. The homogeneous space and its curved generalizations are called biconformal spaces.

Ivanov and Niederle [4], wrote a gravity theory on an 8-dimensional biconformal space, using the curvature-quadratic action of Weyl gravity. By a suitable restriction of the coordinate transformations of the extra 4-dimensions, they showed that 4-dimensional general relativity describes the remaining subspace. Subsequently, Wehner and Wheeler [9] introduced a class of $\mathcal{W}$-invariant actions linear in the curvatures, defining biconformal gravity. Curvature-linear actions are possible because the $2n$-dimensional volume element is scale invariant. Unlike the theories above with actions quadratic in the curvature, the linear action functionals take the same form in any dimension. The doubled dimension is understood in terms of the symplectic structure, leading to a phase space interpretation for generic solutions. Lagrangian submanifolds represent the physical spacetime and have the original dimension. The class of torsion-free biconformal spaces has been shown to reduce to general relativity on the cotangent bundle of spacetime [13]. These reductions of the model work for any $(p,q)$. 
The most general action linear in the biconformal curvatures is given by

\[ S = \lambda \int e_{ac \cdots d}^{bc \cdots f} (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a \wedge f_b) \wedge e^c \wedge \cdots \wedge e^d \wedge e \wedge \cdots \wedge f \]  

(7)

where \( \Omega^a_b \) is the curvature of the spin connection and \( \Omega \) is the dilatational curvature. Here \( \lambda = (\frac{(-1)^n}{(n-1)(n+1)}) \) is a convenient constant, chosen to eliminate a combinatoric factor and to make our sign conventions agree with [13]. The cotangent bundle is spanned by the pair, \((e^a, f_b)\), called the solder form and the co-solder form, respectively. The variation is taken with respect to all \( (n+1)(n+2) \) gauge fields.

The reduction of a fully \( 2n \)-dimensional gravity theory to dependence only on the fields of \( n \)-dimensional gravity is a remarkable feature of biconformal gravity. While it has been shown to be a double field theory [3 4 6 7], double field theories require the assumption that fields depend on only half the coordinates. This artificial constraint is called a section condition. In sharp contrast, biconformal solutions do not require a section condition, reducing as a consequence of the field equations of torsion-free biconformal spaces. Thus, using the field equations with vanishing torsion, the components of the \( (n+1)(n+2) \) curvatures–initially dependent on \( 2n \) independent coordinates–reduce to the usual Riemannian curvature tensor in \( n \) dimensions. Correspondingly, the \( n \)-dim solder form determines all fields, up to coordinate and gauge transformations. Generic, torsion-free, vacuum solutions describe \( n \)-dimensional scale-covariant general relativity on the cotangent bundle.

With the exception of some general considerations and a scalar field example [23], studies of biconformal spaces [9 10 11 12 13 24 28 29 30 31] have considered pure gravity biconformal spaces, leading to vacuum general relativity.

Here we consider \( SU(N) \) Yang-Mills fields as gravitational sources. The central issue is to show that even with a completely general \( SU(N) \) gauge theory over a \( 2n \)-dimensional biconformal space, a full \( 2n \to n \) reduction occurs, both for gravity and the Yang-Mills field.

One issue that arises with \( SU(N) \) sources that is not present in the previous vacuum studies is the specification of the biconformal metric. The results of [13], for example, depend only on the orthonormal frame fields \( \tilde{e}^A = (e^a, f_b) \), but not significantly on their inner product, \( g^{AB} = \langle \tilde{e}^A, \tilde{e}^B \rangle \). However, the both the usual \( SU(N) \) action and the twisted action we introduce require this specification. Interestingly, there is more than one natural choice. Since biconformal spaces have both non-degenerate projected Killing form and Kähler structure, there are two candidate metrics. We show that only the Killing form produces gravity couplings consistent with the usual Yang-Mills stress-energy tensor.
As with the Riemann-Cartan construction of general relativity above, the development of biconformal spaces from group symmetry makes it straightforward to include the additional symmetry of sources. By extending the quotient to

$$M^{2n} = \mathcal{C}_{p,q} \times SU(N) / \mathcal{W}_{p,q} \times SU(N)$$

the local symmetry is enlarged by $SU(N)$ and we may add a Yang-Mills or similar action to Eq.(7). The central difficulty is then to show:

1. The number of field components in $2n$ dimensions reduces to the expected number on $n$ dimensional spacetime.
2. The functional dependence of the fields reduces from $2n$ to $n$ independent variables.
3. The gravitational source is the usual Yang-Mills stress-energy tensor.
4. The $SU(N)$ field equation is the usual $n$-dim Yang-Mills field equation.

To accomplish these goals we require two interdependent steps:

1. Find a form of the Yang-Mills action which gives the usual $n$-dimensional Yang-Mills source to the Einstein tensor.
2. Determine the most satisfactory variation (specifically, identify the metric).

We show that the standard Yang-Mills action

$$S_{YM}^0 = \int tr (F \wedge^* F)$$

cannot give the right couplings, and find a satisfactory modification

$$S_{YM} = \int tr (\tilde{F} \wedge^* F)$$

where the twisted field, $\tilde{F}$ is defined below. The twisted action, together with the restricted Killing form as metric, give the desired reduction.

The remainder of our presentation proceeds as follows. In the next Section, we introduce our notation and other conventions. In Section 3, we examine the usual form of Yang-Mills action, $S_{YM}^0$, with both possible
identifications of the metric, showing that neither choice can produce the usual coupling to gravity. The twisted form $\tilde{\mathbf{F}}$ is developed in Section 4, and the variation of $S_{YM}$ carried out. The Yang-Mills potentials are identified and varied in Section 5. The next Section contains the reduction of the gravitational field equations, as far as possible with the presence of sources. This reduction closely follows reference [13], and results in certain restrictions that must be applied to the matter sources.

The reduction of the number of fields and the number of independent variables is shown in Section 7 to follow from the full gravitational equations. We find that the reduction of fields that is necessary in the purely gravitational sector also forces reduction of the source fields. The Section concludes with the emergence of both the usual Yang-Mills gravitational source in $n$-dimensions, and the usual $n$-dimensional Yang-Mills equation. The final Section includes a brief review of the main results.

2 Notation and conventions

2.1 Conventions with biconformal tensors

2.1.1 Differential forms

The co-tangent space of biconformal manifolds are spanned by two sets of opposite conformal weight orthonormal frame fields, $\tilde{\mathbf{e}}^A = (\mathbf{e}^a, \mathbf{f}_b)$, with lowercase Latin indices $a, b, \ldots = 1, 2, \ldots, n$ indicating the use of these frames and upper case Latin $A, B, \ldots = 1, 2, \ldots, 2n$ to denote the pair. Coordinate indices are lower case Greek, $\mu, \nu, \ldots = 1, 2, \ldots, n$ so that we have, for example,

$$\mathbf{e}^a = e^{\mu a} dx^\mu + e^{\mu a} dy_\mu$$

A general 2-form may be written in the orthonormal basis as

$$\mathcal{F} = \frac{1}{2} F_{ab} \mathbf{e}^a \wedge \mathbf{e}^b + \frac{1}{2} F_{ab} \mathbf{f}^a \wedge \mathbf{f}^b + \frac{1}{2} F_{ab} \mathbf{f}_a \wedge \mathbf{f}_b$$

It is important to realize that $F_{ab}, F^a_b$ and $F_{ab}$ are distinct fields. Therefore, we cannot raise and lower indices in the usual way unless we choose different names for the separate independent components. As compensation for this, the raised or lowered position of an index reflects its conformal weight. Thus, $F_{ab}$ has
weight \(-2\) while \(F^{ab}\) has weight \(+2\). When practical these distinct fields will be given different names,

\[
\mathcal{F} = \frac{1}{2} F^{ab} e^a \wedge e^b + G^{ab} f_a \wedge e^b + H^{ab} f_a \wedge f_b
\]

but this can lead to an unnecessary profusion of field names.

When we need to explicitly refer to internal \(SU(N)\) indices, the field and its components will be given an additional index from the lower case Latin set \(\{i, j, k\}\). Other lower case Latin indices refer to the null orthonormal frame field, \((e^b, f_a)\). Thus \(F_{i,ab}\) represents the components of \(\frac{1}{2} F^{i,ab} G^{a}_{e}e^{a} \wedge e^{b}\) where \(G^{a}_{e}\) is a generator of \(SU(N)\) as \(i\) runs from 1 to \(N^2 - 1\). In most cases the internal index is suppressed.

Differential forms are written in boldface and always multiplied with the wedge product. For brevity in some longer expressions we omit the explicit wedge between forms. Thus, for example,

\[
f_a \wedge f_b \wedge f_c \wedge e^d \wedge e^e \iff f_{abc} e^{de}
\]

The bold font shows that these are differential forms, and therefore are to be wedged together.

As a compromise between keeping track of conformal weights, while being able to assess the symmetry or antisymmetry of components, we introduce a weight \(+1\) basis \(e^{A} \equiv (e^a, \eta^{ab} f_b)\), where \(\eta_{ab}\) is the \(n\)-dimensional metric of the original \(SO(p, q)\)-symmetric space, not the metric of the biconformal space. Thus, we may write

\[
\mathcal{F} = \frac{1}{2} F_{AB} e^{A} \wedge e^{B} \\
= \frac{1}{2} F_{ab} e^a \wedge e^b + G_{ab} \eta^{ac} f_c \wedge e^b + \frac{1}{2} \mathcal{H}_{ab} \eta^{cd} f_c \wedge f_d
\]

where we have defined

\[
G_{ab} \equiv \eta_{ac} G^{c}_{ b} \\
\mathcal{H}_{ab} \equiv \eta_{ac} \eta_{bd} H^{cd}
\]

The use of a different font is important because we are not using the biconformal metric, \(K_{AB}\), to change index positions. It is the original index positions and font, \(G^{a}_{ b}, H^{ab}\), that are the proper field components.
The matrix components $F_{AB}$ are then written as

$$
F_{AB} = \begin{pmatrix}
F_{ab} & F_{ac} \\
F_{ca} & F_{bc}
\end{pmatrix}
= \begin{pmatrix}
F_{ab} & -G_{ba} \\
G_{ab} & H_{ab}
\end{pmatrix}
$$

where the form of the upper right corner follows because

$$
-G_{ba} \epsilon^b \wedge \eta^{bc} \epsilon_c = G_{ba} \eta^{bc} \epsilon_c \wedge \epsilon^a = G^{c} \epsilon_a \wedge \epsilon^a
$$

With this convention, we can meaningfully define the transpose of matrices. Specifically, while the transpose of

$$
\begin{pmatrix}
F_{ab} & F_{ac} \\
F_{ca} & F_{bc}
\end{pmatrix}
$$

is ill-defined because of the mixed indices on the off-diagonal terms, the transpose of $F_{AB}$ in the weight $+1$ basis is

$$
\begin{pmatrix}
F_{ab} & -G_{ba} \\
G_{ab} & H_{ab}
\end{pmatrix}^t = \begin{pmatrix}
F_{ba} & G_{ba} \\
-G_{ab} & H_{ab}
\end{pmatrix}
$$

For $H_{ba} = -H_{ab}$ and $F_{ba} = -F_{ab}$ this is manifestly antisymmetric, as befits a 2-form. Notice that the effect of two transposes is the identity, so this operation provides an involutive automorphism even though $\eta_{ab}$ is not the biconformal metric.

From $[G_{ab}]^t = G_{ba}$ we have

$$
[\eta_{ac} G_{bc}^e]^t = [\eta_{be} G_{ac}^e]
$$

and therefore, $[G_{bd}]^t = \eta^{ad} \eta_{be} G_{ac}^e$.
2.1.2 Metric

Because all quadrants of the metric are used in the variation, it is desirable to retain both the name and index positions throughout. Since

\[
\begin{pmatrix}
K_{ab} & K^a_b \\
K^a_b & K^{ab}
\end{pmatrix}
\]

contains all index positions, the inverse metric and its components are written with an overbar,

\[
\begin{pmatrix}
\bar{K}^{ab} & \bar{K}^a_b \\
\bar{K}_a^b & \bar{K}_{ab}
\end{pmatrix}
\]

Thus \(K_{ab}\) is the first quadrant of the metric, while \(\bar{K}_{ab}\) is the final quadrant of the inverse metric. Here any changes of index position must be indicated with an explicit factor of \(\eta_{ab}\) or \(\eta^{ab}\).

2.1.3 Volume form

It is convenient to define a volume form as \(\Phi \equiv *1\) but in defining the Hodge dual operation a number of ambiguities need to be clarified. Because up and down indices have distinct conformal weight, we may partially order the indices on the Levi-Civita tensor. We establish the following conventions:

1. All factors of \(f_a\) are written first, followed by all of the \(e^b\).

2. The Levi-Civita tensor is written as \(e^{a\cdots c \cdots \cdot f}\), with all \(n\) up indices first. The partially ordered antisymmetric symbol is written as \(\varepsilon^{a\cdots c \cdots \cdot f}\).

3. When taking the dual of a \(p\)-form, we sum the components on the first \(p\) indices of the Levi-Civita tensor, then introduce factors of \((-1)\) to move indices to the default positions. For example, the dual of \(H = H^a_b f_a \wedge e^b\) is

\[
*H = \frac{1}{(n-1)! (n-1)!} H^a_b e_a^{bc \cdots d} \cdots f_{c \cdots d} \wedge e^c \wedge \cdots \wedge e^f
\]

\[
= \frac{(-1)^n}{(n-1)! (n-1)!} H^a_b e_a^{bc \cdots d} \cdots f_{c \cdots d} \wedge e^c \wedge \cdots \wedge e^f
\]

\[
= \frac{(-1)^n}{(n-1)! (n-1)!} H^a_b e_a^{bc \cdots d} \cdots f_{c \cdots d} e^c \cdots e^f
\]

Notice that in the final step, the number of wedged forms in \(f_{c \cdots d}\) may be inferred from the Levi-Civita tensor. Since the Levi-Civita tensor always has \(n\) up and \(n\) down indices, the number of basis forms of
each type is unambiguous. For example,

\[ e^{abc \ldots d} f_{e \ldots f} \wedge e^{e \ldots f} = e^{abc \ldots d} f_{e \ldots f} \frac{1}{n-2} \wedge e^{e \ldots f} \]

is a 2n - 2 form that includes the wedge product of n - 2 factors of \( f_a \) and n factors of \( e^a \).

4. An m-form is a polynomial with each term having different numbers of \( e \)'s and \( f \)'s, we write the terms in order of increasing number of \( f \)'s.

5. The partial ordering of indices on the Levi-Civita tensor reduces the normalization of the volume element from \( \frac{1}{(2n)!} \) to \( \frac{1}{n! \cdot n!} \).

It has been noted elsewhere [33] that there are alternative duals in biconformal space. For instance, we may use the symplectic form instead of the metric to connect indices. The difference resides in the relative signs between the \( e^a \), \( f_a \), and mixed terms. Here we use only the Hodge dual, taking care to keep the correct signs.

With these conventions in mind, we define

\[
\Phi = \ast 1
\]

\[
= \frac{1}{n! n!} e^{e \ldots f} f_{e \ldots f} \wedge e^{e \ldots f}
\]

\[
= \frac{1}{n! n!} \sqrt{K} e^{e \ldots f} f_{e \ldots f} \wedge e^{e \ldots f}
\]

and consequently,

\[
f_{e \ldots d} \wedge e^{e \ldots f} = \frac{1}{\sqrt{K}} e^{e \ldots f} e_{e \ldots f} \Phi
\]

\[
= \bar{e}^e_d f_{e \ldots f} \Phi
\]

(10)

where the overbar denotes the contravariant form of the Levi-Civita tensor. The contravariant form satisfies

\[
\varepsilon^{a \ldots b} e_{c \ldots d} \Phi = n! n!
\]

We also need the reduction formulas,

\[
e^{e \ldots f} m_{e \ldots f} h_{e \ldots f} c_{e \ldots f} = n!(n - 2)! (\delta^g_m \delta^h_n - \delta^g_n \delta^h_m)
\]

\[
e^{e \ldots f} m_{e \ldots f} h_{e \ldots f} g_{e \ldots f} = (n - 1)! (n - 1)! \delta^h_n \delta^m_g
\]

(12)
2.2 Conventions for invariant matrices

Possible actions can be constructed using curvatures naturally arising in a theory, together with any invariant
tensors consistent with the gauging. The biconformal gauging of the conformal group has a surprising number
of invariant objects. These invariant structures arise from internal symmetries of the conformal group are
induced into generic biconformal spaces \[10\].

The conformally invariant Killing form, restricted to the Kähler manifold:

\[
K_{AB} = \begin{pmatrix}
  0 & \delta_a^b \\
  \delta^a_b & 0
\end{pmatrix}
\]  

The symplectic form, underlying dilatations,

\[
\Omega_{AB} = \begin{pmatrix}
  0 & \delta_a^b \\
  -\delta^a_b & 0
\end{pmatrix}
\]  

Interestingly, this form manifests Born reciprocity \[1,2\]. The complex structure, arising from the sym-
metry between translations of the origin and translations of the point at infinity (i.e., special conformal
transformations),

\[
J^A_B := \begin{pmatrix}
  0 & -\eta^{ab} \\
  \eta_{ab} & 0
\end{pmatrix}
\]  

The Kähler metric, arising from the compatibility, \(g(u,v) = \Omega(u, Jv)\), of the symplectic and complex
structures,

\[
g_{AB} = \Omega_{AC}J^C_B
= \begin{pmatrix}
  0 & \delta_a^c \\
  -\delta^a_c & 0
\end{pmatrix}
\begin{pmatrix}
  0 & -\eta^{cb} \\
  \eta_{cb} & 0
\end{pmatrix}
= \begin{pmatrix}
  \eta_{ab} & 0 \\
  0 & \eta^{ab}
\end{pmatrix}
\]  

16
These three Kähler structures satisfy

\[ g(u, v) = \Omega(u, Jv) \]  

(17)

Notice that the Kähler metric is not invariant under the conformal structure.

### 2.3 Variation of the Kähler and Killing forms

Vacuum biconformal gravity depends only on the variation of the gauge fields and does not require introduction of a metric. Yang-Mills actions, however, make use of the Hodge dual and a metric is required. Given the presence of two symmetric forms in biconformal spaces, i.e., the scale invariant Killing form, Eq.(13), and the Kähler metric, Eq.(16), which is compatible with the symplectic and almost complex structures. The choice between these might seem arbitrary. We studied both cases, ultimately showing that the correct matter couplings arise only if we use the Killing form, \( K_{AB} \).

The gravitational field equations follow by variation of the connection forms, including the solder and co-solder forms and we must express the variation of the metric in terms of these. The relation between the solder/co-solder variation and metric variation follows using the equality of the metric and the inner product of the basis forms, so we have two alternative definitions of the inner product. We have either

\[ \langle \tilde{e}^A, \tilde{e}^B \rangle \equiv \tilde{K}^{AB} \]  

(18)

or

\[ \langle \tilde{e}^A, \tilde{e}^B \rangle \equiv \tilde{g}^{AB} \]  

(19)

where \( \tilde{e}^A = (e^a, f_b) \) and the overbar denotes the inverse metric. The variation of either metric candidate then follows.

Until the variation is complete, we need the general form of the inverse metric,

\[ \tilde{K}^{AB} = \begin{pmatrix} \tilde{K}^{ab} & \tilde{K}^a_b \\ \tilde{K}_b^a & \tilde{K}_{ab} \end{pmatrix} \]

Once the variations are expressed in terms of \( \delta\tilde{K}^{AB} \) or \( \delta\tilde{g}^{AB} \) any remaining components may be returned to the orthonormal form, Eq.(13) or Eq.(16).
Let the solder and co-solder variations be given by

\begin{align}
\delta e^a &= A^a e^c + B^{ac} f_c \\
\delta f_c &= C_{cd} e^d + D_c^d f_d
\end{align}

Then for variation of the components of the Kähler metric we expand Eq. (18)

\[ \delta \bar{K}^{ab} = \delta \langle e^a, e^b \rangle = \langle A^a e^c + B^{ac} f_c, e^b \rangle + \langle e^a, A^b e^c + B^{bc} f_c \rangle = A^a \bar{K}^{cb} + B^{ac} \bar{K}^{-b} + A^b \bar{K}^{ac} + B^{bc} \bar{K}^{a} 
\]

Since the coefficients \( A^a, B^{ac}, C_{cd}, D_c^d \) now represent the variation, we may return the remaining components of \( \bar{K}^{AB} \) to the null orthonormal form of Eq. (18),

\[ \delta \bar{K}^{ab} = B^{ab} + B^{ba} \]

Computing the remaining components in the same way, we arrive at the full set,

\begin{align}
\delta \bar{K}^{ab} &= B^{ab} + B^{ba} \\
\delta \bar{K}^{a} &= A^a + D_a^b \\
\delta \bar{K}^{b} &= D_b^a + A^b \\
\delta \bar{K}_{ab} &= C_{ab} + C_{ba}
\end{align}

The analogous calculation for the Kähler inner product, Eq. (19), gives

\begin{align}
\delta \bar{g}^{ab} &= A^a \eta^{cb} + A^b \eta^{ac} \\
\delta \bar{g}^{a} &= B^{ac} \eta_{bc} + C_{bc} \eta^{ac} \\
\delta \bar{g}^{b} &= C_{ac} \eta^{cb} + B^{bc} \eta_{ac} \\
\delta \bar{g}_{ab} &= D_a^c \eta_{cb} + D_b^c \eta_{ac}
\end{align}

For each proposed action functional below, we check each of the two inner products, until it becomes
clear that we must use the Killing form.

### 2.4 Further notation

The antisymmetric projection operator mapping for \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) tensors,

\[
\Delta_{ab}^{ac} = \frac{1}{2} (\delta^c_d \delta^a_b - \eta^{ac} \eta_{db}) = \frac{1}{2} \eta^{ce} \eta_{bf} \left( \delta^a_d \delta^f_e - \delta^a_e \delta^f_d \right)
\]

arises frequently.

### 3 Constructing the Yang-Mills action

In spacetime, the action for a Yang-Mills field may be written as

\[
S_{\text{YM}} = -\kappa \frac{1}{2} \int tr \mathcal{F} \wedge \ast \mathcal{F}
\]

and it is natural to consider the same form in biconformal space. However, as we show in this Section, this usual form leads to nonstandard coupling to gravity. We show in the next Section that a twisted action is required to give the usual coupling.

Expanded into independent components in the \( e^A \) basis,

\[
\mathcal{F}^i = \frac{1}{2} F^{ia} e^a \wedge e^b + \mathcal{F}^{ab} f_a \wedge e^b + \frac{1}{2} F^{ab} f_a \wedge f_b
\]

where \( i \) is an index of the internal Lie algebra. This index can be suppressed without loss of generality in the action and gravitational field equations. Also, for the Yang-Mills field it proves more transparent to give the three coefficients distinct names. Finally, since we must insure antisymmetry of the twisted field, it is most transparent to use the uniform weight basis, \( e^A = (e^a, \eta^{bc} f_c) = (e^a, f^b) \). We therefore write

\[
\mathcal{F} = \frac{1}{2} F_{AB} e^A \wedge e^B = \frac{1}{2} F_{ab} e^a \wedge e^b + G_{ab} f^a \wedge e^b + \frac{1}{2} H_{ab} f^a \wedge f^b
\]
In discussions of the Yang-Mills field equations (as opposed to the gravitational equations), the internal index becomes important and will be shown where necessary.

Note that $F^{ab}$ and $H_{ab}$ are antisymmetric. The cross-term may be written as

$$G^{ab} f_a \wedge e^b = \frac{1}{2} (G^{ab}) f_a \wedge e^b + \frac{1}{2} (-G^{ba}) e^a \wedge f_b$$

$$= G_{ab} f^a \wedge e^b$$

where $G_{ab}$ may be asymmetric. As a matrix, $F_{AB}$ takes the form given in Eq. (3).

### 3.1 The Hodge dual and standard Yang-Mills action

With these factors in mind, we find the Hodge dual of the Yang-Mills field. While we write the inverse Kähler metric $\bar{g}^{AB}$ throughout this Section, the replacement $\bar{g}^{AB} \rightarrow \bar{K}^{AB}$ in all expressions until the final variation immediately gives the Hodge dual using the Killing metric. In general terms the Hodge dual of a 2-form is given by

$$^*F = * \left( \frac{1}{2} F_{AB} \bar{e}^A \wedge \bar{e}^B \right)$$

$$= \frac{1}{(2n-2)!} \frac{1}{2} F_{AB} \bar{g}^{AC} \bar{g}^{BD} \varepsilon_{CDE...F} e^E \wedge \ldots \wedge e^F$$

However, we need to separate the distinct quadrants of each inverse metric. Expanding each upper case index $A, B, \ldots$, as a raised, weight +1 index and a lowered weight -1 index in turn, then collecting like terms leads to

$$^*F = \frac{1}{n! (n-2)!} \left( \frac{1}{2} F_{ab} \bar{g}^{am} \bar{g}^{bn} + G^{ab} \bar{g}_a \bar{g}_b m^n \right) \varepsilon_{c...d} e^c \ldots f e^{d...f}$$

$$+ \frac{(-1)^{n-1}}{(n-1)! (n-2)!} \left( \frac{1}{2} F_{ab} \bar{g}^{am} \bar{g}^{bn} + G^{ab} \bar{g}_a \bar{g}_b m^n \right) \varepsilon_{e...d} e^c \ldots f e^{d...f}$$

$$+ \frac{(-1)^n}{(n-1)! (n-3)!} \left( \frac{1}{2} F_{ab} \bar{g}^{am} \bar{g}^{bn} + G^{ab} \bar{g}_a \bar{g}_b m^n \right) \varepsilon_{e...d} e^c \ldots f e^{d...f}$$

$$+ \frac{1}{n! (n-2)!} \left( \frac{1}{2} F_{ab} \bar{g}^{am} \bar{g}^{bn} + G^{ab} \bar{g}_a \bar{g}_b m^n \right) \varepsilon_{e...d} e^c \ldots f e^{d...f}$$

where one readily sees the advantage of omitting the wedge between forms, $e^c \ldots \wedge e^f \leftrightarrow e^{c...f}$.

We form the usual Yang-Mills Lagrangian density as the wedge product, $F \wedge {}^*F$, eliminating the basis forms in favor of the volume form $\Phi$. After a bit of algebra, we find
\[
F \wedge \ast F = \left( \frac{1}{2} F_{mn} \bar{g}^{am} \bar{g}^{bn} + G^m_n \bar{g}^{an} m \bar{g}^{bn} + \frac{1}{2} H^m_n \bar{g}^a m \bar{g}^b n \right) F_{ab} \Phi \\
+ \left( F_{mn} \bar{g}_a ^m \bar{g}^{bn} + G^m_n \left( \bar{g}_{am} \bar{g}^{bn} - \bar{g}^a_n \bar{g}^b_m \right) + H^m_n \bar{g}_{am} \bar{g}^b n \right) G^a b \Phi \\
+ \left( \frac{1}{2} F_{mn} \bar{g}_a ^m \bar{g}^b n + G^m_n \bar{g}_{am} \bar{g}^b n + \frac{1}{2} H^m_n \bar{g}_{am} \bar{g}^b n \right) H^{ab} \Phi 
\]

(27)

and the matter action is given by Eq. (24). The full action is the combination of Eq. (7) and Eq. (24),

\[S = S_G + S_{YM}\]

The Lagrange density reduces, for the diagonal form of the Kähler metric, Eq. (16), to

\[
F \wedge \ast F = \left( \frac{1}{2} \eta^m \eta^n F_{ab} F_{mn} + \eta^m \eta^n G^m G^a b + \frac{1}{2} \eta^m \eta^n H^m H^{ab} \right) \Phi 
\]

(28)

while, replacing components of \(\bar{g}^{AB}\) in Eq. (27) with the corresponding components of \(\bar{K}^{AB}\) and substituting the null orthonormal form of the Killing metric, Eq. (13), the Lagrangian density becomes

\[
F \ast F = \left( H^{ab} F_{ab} - G^b_a G^a b \right) \Phi 
\]

(29)

We find that this Hodge dual form of the action is identical to the form of the action given in [31], despite the claim in [31] that the action is independent of the metric. We show below its equivalence to \(F \wedge \ast F\) using the Hodge dual. The presence of the metric is concealed in [31] because the Killing metric in this basis is comprised of Kronecker deltas.

Although varying the potentials with either Lagrange density yields the usual Yang-Mills equation, we show below that Eq. (28) and Eq. (29), give a nonstandard coupling to gravity. We define in the next Section a twisted action that gives the usual coupling to gravity, but nonetheless leads to the correct Yang-Mills equations.

### 3.2 The failure of the \(F \wedge \ast F\) action

Variation of the usual Lagrangian Eq. (27) gives

\[
\delta \int F \wedge \ast F = \int \left( F_{mn} \delta \bar{g}^m \bar{g}^{bn} + G^m_n \left( \delta \bar{g}^a m \bar{g}^{bn} + \bar{g}^a m \delta \bar{g}^b n \right) + H^m n \delta \bar{g}^a m \bar{g}^b n \right) F_{ab} \Phi 
\]
\[
+ (F_{mn} \delta g^m g^{bn} + G^m_n (\delta g_{am} g^{bn} - \delta g_a^m g^b_n) + H^{mn} \delta g_{am} g^b_n) G^n_b \Phi \\
+ (F_{mn} \delta g^m g^{bn} + G^m_n (\delta g_{am} g^{bn} - \delta g_a^m g^b_n) + H^{mn} \delta g_{am} g^b_n) \Phi \\
+ (F_{mn} \delta g_a^m g^b n + G^m_n (\delta g_{am} g^b n + \delta g_{am} g^b n) + H^{mn} \delta g_{am} g^b n) H^{ab} \Phi
\]

where we also consider replacement of the components of \(\delta g^{AB}\) with the corresponding components of \(\delta K^{AB}\).

Returning the remaining inverse metric components to the orthonormal form reduces this to

\[
\delta g \int \mathcal{F} \wedge ^* \mathcal{F} = \int (F_{mn} F_{ab} \delta g^{am} \eta^{bn}) \Phi \\
+ (G^m_n F_{ab} \eta^{bn} \delta g^a_m + F_{mn} G^a_n \eta^{bn} \delta g^a_m) \Phi \\
+ (G^m_n G^a_n \eta^{bn} \delta g_{am} + G^m_n G^a_n \eta^{bn} \delta g_{am}) \Phi \\
+ (H^{mn} G^a_n ^b \eta_{am} \delta g^b_n + G^m_n H^{ab} \eta_{am} \delta g^b_n) \Phi \\
+ (H^{mn} H^{ab} \eta_{bn} \delta g_{am}) \Phi
\]

where the Kähler variations are given by Eq. [23]. The resulting gravitational field equations are:

\[
\alpha (\Omega^b_m b_n - \Omega^b_n b_a \delta^n_m) + \beta (\Omega^a_m n_a \delta^b_m) + \alpha \delta^b_m = - (2\eta^{bc} F_{mc} F_{ab} + 2G^d_a G^e_m \eta_{de}) \eta^{mn} (30)
\]

\[
[\alpha (\Omega^a_m n_a - \Omega^b_n b_a \delta^m_a) + \beta (\Omega^a_m n_a \delta^m_a) + \alpha \delta^m_a] = - (2G^d_c G^m_b \eta_{dm} \eta^{bc} + 2\eta_{bc} \eta_{dm} H^{dc} H^{ab}) (31)
\]

\[
\alpha \Omega^a b_{mn} + \beta \Omega^b m = (2\eta^{ca} G^b_a F_{mc} + 2\eta_{bc} G^c_m H^{ab}) \eta_{bn} (32)
\]

\[
\alpha \Omega^b n b + \beta \Omega^m n = - (2\eta^{ca} G^m_a F_{bc} + 2\eta_{ac} G^c_b H^{am}) \eta^{nb} (33)
\]

The remaining four field equations involving the torsion and co-torsion are unchanged.

Notice that the sources on the right sides of Eqs. (32) and (33) differ only in the overall sign. This means that both the spacetime curvature, \(\Omega^a_{nam}\), and the momentum space curvature, \(\Omega^b_{n bm}\), are driven with equal strength. Thus, if spacetime curvature is nonzero, the momentum space must also be correspondingly curved. When we consider torsion-free solutions below, we find that the left hand side of Eq. (33) vanishes independently of the sources, implying a constraint on the source fields. Thus, for the \(\int \mathcal{F} \wedge ^* \mathcal{F} \) source and the Kähler case,

\[
\eta^{ca} G^m_a F_{bc} + \eta_{ac} G^c_b H^{am} = 0
\]

and this immediately shows that the source for the Einstein equation, Eq. (32), vanishes.
This suggestive possibility of corresponding spacetime and momentum curvatures could implement Born reciprocity, and will be explored elsewhere. However, momentum curvature also requires some part of the torsion to be nonvanishing, and this in turn requires a different gravitational solution than that known to reproduce general relativity. Thus, we cannot maintain the vanishing torsion without forcing the spacetime source to vanish.

In addition to the inescapability of torsion and momentum space curvature, the independence of the sources to Eqs. 30 and 31 also raise issues with the Kähler variation, because the method of solution employed in [13] makes use of the near identity of these two equations. At the very least, an entirely different form of reduction of the equations would be required, with no guarantee that the Einstein equation would emerge.

The same difficulties also arise from the twisted form of the action using the Kähler metric. Therefore, from here onward we use only the Killing form as metric.

These issues do not arise with the Killing variation, Eqs. 22—the source for the spacetime curvature and momentum space curvature are independent while remaining two variations are identical. The use of the Killing form as metric also makes good geometric sense, since it arises directly as a symmetric form in the Lie algebra and thus as metric of the co-tangent space. As such, it respects the conformal invariance of the full model. The Kähler structures, by contrast, reflect symmetries and dynamical properties within the conformal group and depend for their existence on the solution on the biconformal space, and the metric not conformally invariant.

Turning to the Killing form as inner product, we use Eqs. 22 instead. The variation now yields the alternative gravitational field equations,

\[
\begin{align*}
\alpha \Omega^a_{~b} \delta^m_b - \alpha \Omega^a_{~b} \delta^m_a + \beta \Omega^a_{~m} \delta^m_b - \beta \Omega^a_{~b} \delta^m_a + \Lambda \delta^m_b &= -\kappa \left( H^{bn} F_{bm} - G^b_{~m} G^m_{~b} + \frac{1}{2} \delta^m_b \left( F_{bc} H^{bc} - G^b_{~c} G^c_{~b} \right) \right) \\
\alpha \Omega^a_{~n} \delta^m_n - \alpha \Omega^a_{~n} \delta^m_a + \beta \Omega^a_{~n} \delta^m_n - \beta \Omega^a_{~n} \delta^m_a + \Lambda \delta^m_n &= -\kappa \left( H^{bn} F_{bn} - G^b_{~n} G^m_{~n} + \frac{1}{2} \delta^m_n \left( F_{bc} H^{bc} - G^b_{~c} G^c_{~b} \right) \right) \\
\alpha \Omega^a_{~nam} + \beta \Omega_{nm} &= \kappa \left( F_{an} G^a_{~n} + F_{an} G^m_{~a} \right) \\
\alpha \Omega^a_{~bm} + \beta \Omega_{nm} &= -\kappa \left( H^{nb} G^m_{~b} + H^{nb} G^m_{~b} \right)
\end{align*}
\]

Here the gravitational equations still require vanishing momentum curvature, so Eq. (37) implies

\[
H^{nb} G^m_{~b} + H^{nb} G^m_{~b} = 0
\]
The failure of this result is now not so immediate, but the reduction of Eqs. (34) and (35) gives a second constraint on $H^{ab}$ and $G^{a}_{\ b}$. The two constraints lead, at the most general, to vanishing $H^{ab}$ and symmetric $G^{a}_{\ b}$, leaving the source for the Einstein equation, Eq. (36) linear in $F_{ab}$ and therefore incompatible with the usual energy source for general relativity.

The problem therefore does not lie solely in the choice of inner product, but rather in the use of the usual action. In the context of other double field theories, a twist allows dimensional reduction to preserve gauging of supersymmetries [14, 15, 16]. Here we find that including a twist insures consistency under dimensional reduction not only for supersymmetry, but also gives the correct coupling of Yang-Mills sources to gravity.

For the remainder of our discussion, we take the inner product of the basis forms $(e^{a}, f_{b})$ to be given by the Killing form as in Eq. (18), and additionally include a twist in the action.

### 4 Metric variation of the twisted Yang-Mills Action

Instead of the usual spacetime action, Eq. (24), we consider a biconformal Yang-Mills theory with an action functional of the form

$$S_{TYM} = -\kappa \int tr \tilde{\mathcal{F}} \wedge * \mathcal{F}$$

where $*$ is the usual Hodge dual, $\mathcal{F}$ is a curvature 2-form, $\tilde{\mathcal{F}}$ is a twisted conjugate curvature, and the trace is over the $SU(N)$ generators. The twist matrix is formed using both the Killing metric and the Kähler form, $K^{A}_{\ B} \equiv \tilde{K}^{AC}g_{CB}$. While the twist matrix is similar to that used to preserve supersymmetry in other double field theories, [14, 15, 16], we define the twisted Yang-Mills field by

$$\tilde{\mathcal{F}}_{AB} = \frac{1}{2} (K^{C}_{A}\mathcal{F}_{CB} + \mathcal{F}_{AC}K^{C}_{B})$$

We find that this form is necessary to preserve the antisymmetry of the field while giving the required interchange of source fields.
4.1 Details of the twist

The twist is accomplished using $K^A_B \equiv \tilde{K}^{AC} g_{CB}$ where $K^A_B \equiv \tilde{K}^{AC} g_{CB} = g_{BC} \tilde{K}^{CA} = K_B^A$, since both $g_{AB}$ and $K_{AB}$ are symmetric. In the null-orthonormal form and the $e^A$ basis, this matrix is simply

$$K^A_B = K_B^A = \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix}$$

and the required form of the field is

$$\tilde{F}_{AB} = \frac{1}{2} \left( K_A^C F_{CB} + F_{AC} K_B^C \right) = \frac{1}{2} \begin{pmatrix} 0 & \delta_a^c \\ \delta_b^c & 0 \end{pmatrix} \begin{pmatrix} F_{cb} & -G_{bc} \\ G_{cb} & H_{cb} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} F_{ac} & -G_{ca} \\ G_{ac} & H_{ac} \end{pmatrix} \begin{pmatrix} 0 & \delta_b^c \\ \delta_b^c & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{G}_{[ab]} & \frac{1}{2} (F_{ab} + \mathcal{H}_{ab}) \\ \frac{1}{2} (F_{ab} + \mathcal{H}_{ab}) & \mathcal{G}_{[ab]} \end{pmatrix}$$

(39)

Note that this transformation maintains the antisymmetry while interchanging the diagonal and anti-diagonal elements.

However, Eq.(39) is insufficient. Until we complete the metric variation we must use the generic form of the metric in computing the twist matrix,

$$K^A_B = \begin{pmatrix} K^{ac} & K^a \epsilon^e \eta^{ec} \\ \eta^{ae} K_e^c & \eta^{ae} \eta^{ef} \bar{K}_e^f \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & \eta_{bc} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{K}^{ac} \eta_{cb} & \tilde{K}_b^a \\ \tilde{K}_a^b & \eta^{ae} \tilde{K}_e^b \end{pmatrix}$$

$$K_A^B = \begin{pmatrix} \eta_{ad} \tilde{K}_{db} & \tilde{K}_a^b \\ \tilde{K}_a^b & \eta_{ae} \tilde{K}_{eb} \end{pmatrix}$$

with symmetry given by $\tilde{K}_b^a = \eta^{ae} \tilde{K}_e^c \eta_{cb}$ (see Appendix A for details of the symmetry). Then $\tilde{F}_{AB}$ becomes
\[ \mathcal{F}_{AB} = \frac{1}{2} (K_A^C F_{CB} + F_{AC} K_C^B) \]

\[ = \frac{1}{2} \left( \begin{array}{cc}
\eta_{ad} \bar{K}_{dc}^a & \bar{K}^c_a \\
\eta_{ad} \bar{K}^d e^e e^c & \bar{K}_{ac} \eta^e e^c
\end{array} \right) \left( \begin{array}{c}
F_{eb} - G_{eb} \\
G_{eb} - H_{eb}
\end{array} \right) + \frac{1}{2} \left( \begin{array}{cc}
F_{ac} - G_{ca} \\
G_{ac} - H_{ac}
\end{array} \right) \left( \begin{array}{c}
\bar{K}^{ce} \eta_{eb} \\
\eta^d \bar{K}^d e^e \eta_{eb} - \eta^e \bar{K}^e e^e \eta_{eb}
\end{array} \right) \]

\[ = \frac{1}{2} \left( \begin{array}{cc}
\eta_{ad} \bar{K}_{dc}^a F_{eb} + \bar{K}^c_a G_{eb} \\
\eta_{ad} \bar{K}^d e^e e^c F_{eb} + \bar{K}_{ac} e^e G_{eb}
\end{array} \right) + \frac{1}{2} \left( \begin{array}{c}
- \eta_{ad} \bar{K}_{dc}^a G_{eb} + \bar{K}^c_a H_{eb} \\
- \eta_{ad} \bar{K}^d e^e e^c G_{eb} + \bar{K}_{ac} e^e H_{eb}
\end{array} \right) + \frac{1}{2} \left( \begin{array}{c}
F_{ac} \bar{K}^{ce} \eta_{eb} - G_{ca} \eta^d \bar{K}^d e^e \eta_{eb} \\
G_{ac} \bar{K}^{ce} \eta_{eb} + H_{ac} \eta^d \bar{K}^d e^e \eta_{eb}
\end{array} \right) \]

(40)

and we check that \[44\] agrees with Eq. \[39\] when we restore the null orthonormal frame for the metric. The twisted field is simpler when written as a 2-form,

\[ \mathcal{F} = \frac{1}{2} \mathcal{F}_{AB} e^A \wedge e^B \]

\[ = \frac{1}{2} \left( F_{ac} \bar{K}^{ce} \eta_{eb} + \bar{K}^c_a G_{eb} \right) e^a \wedge e^b \]

\[ + \frac{1}{2} \left( F_{ac} \bar{K}^{ce} \eta_{eb} + \bar{K}^c_a G_{eb} \right) e^a \wedge e^b \]

\[ + \frac{1}{2} \left( H_{ac} \eta^d \bar{K}^d e^e \eta_{eb} + G_{ac} \bar{K}^{ce} \eta_{eb} \right) e^a \wedge e^b \]

(41)

We may interchange \( \bar{K}^c_a \) and \( \bar{K}^b_a \), when convenient since these have identical variations and both restrict to \( \delta^a_b \) in the null orthonormal basis.

### 4.2 The action

The dual field, replacing \( \bar{g}^{AB} \) with \( K^{AB} \) in Eq. \[20\], is

\[ *\mathcal{F} = \frac{1}{n!(n-2)!} \left( \frac{1}{2} F_{ab} \bar{K}^{am} \bar{K}^{bn} + G_{gb} \eta^g a \bar{K}^a m \bar{K}^b n + \frac{1}{2} H_{gh} \eta^g a \bar{K}^a m \eta^{hb} \bar{K} b n \right) \epsilon^{c...d} m_{n-2} e_{c...d} e^{e...f} \]

\[ + \frac{(-1)^{n-1}}{(n-1)! (n-1)!} \left( \frac{1}{2} F_{ab} \bar{K}^{am} \bar{K}^{bn} + G_{gb} \eta^g a \bar{K}^a m \bar{K}^b n + \frac{1}{2} H_{gh} \eta^g a \eta^{hb} \bar{K} a m \bar{K} b n \right) \epsilon^{mc...d} n_{e...f} e_{c...d} e^{e...f} \]

\[ + \frac{(-1)^n}{(n-1)! (n-1)!} \left( \frac{1}{2} F_{ab} \bar{K}^{am} \bar{K}^{bn} + G_{gb} \eta^g a \bar{K}^a m \bar{K}^b n + \frac{1}{2} H_{gh} \eta^g a \eta^{hb} \bar{K} a m \bar{K} b n \right) \epsilon^{mc...d} n_{e...f} e_{c...d} e^{e...f} \]

\[ + \frac{1}{n! (n-2)!} \left( \frac{1}{2} F_{ab} \bar{K}^{a} m \bar{K}^{b} n + G_{gb} \eta^g a \bar{K}^a m \bar{K}^b n + \frac{1}{2} H_{gh} \eta^g a \eta^{hb} \bar{K} a m \bar{K} b n \right) \epsilon^{mc...d} e_{c...d} e^{e...f} \]

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We rewrite the twisted field as

\[ \widetilde{\mathcal{F}} = \frac{1}{2} \left( F_{rq} \tilde{R}^{qr} \eta_{ts} + \tilde{R}_r{}^q \tilde{G}_{qs} \right) e^r \wedge e^s + \frac{1}{2} \left( F_{rc} \tilde{R}^{rc} \eta_{qs} + \tilde{R}_r{}^c \tilde{H}_{cs} \right) \eta^{sw} e^r \wedge f_w + \frac{1}{2} \left( -G_{sq} \tilde{R}^{sq} \eta_{tr} - G_{qr} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} e^r \wedge f_w + \frac{1}{2} \left( G_{rs} \tilde{R}^{rs} + H_{rs} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} e^r \wedge f_w + \frac{1}{2} \left( -G_{sq} \tilde{R}^{sq} \eta_{tr} - G_{qr} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} e^r \wedge f_w + \frac{1}{2} \left( G_{rs} \tilde{R}^{rs} + H_{rs} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} e^r \wedge f_w \]

where we have renamed the indices in \( \widetilde{\mathcal{F}} \) to avoid duplication when we wedge the two expressions together. Each term of the wedge product is proportional to the volume form, using \( e^c \wedge \cdots \wedge e^f = \tilde{e}^c \cdots \tilde{e}^f \Phi \). Then, replacing the double Levi-Civita tensors with Kronecker deltas according to Eqs. (12), we fully distribute the lengthy expression. We summarize the essential features here, but details including the full wedge product \( \widetilde{\mathcal{F}} \wedge * \mathcal{F} \) and its reduction to \( \widetilde{\mathcal{F}} \wedge * \mathcal{F} |_{\text{contributing}} \) below are given in Appendix B.

Since we are interested only in the variation, and will return the metric to the null orthonormal form of Eq. (13) after variation, certain terms clearly do not contribute to the field equations. For example, in the product

\[ (-G_{sq} \tilde{R}^{sq} \eta_{tr} - G_{qr} \eta^{qt} \tilde{K}_{ts}) \eta^{sw} \left( \frac{1}{2} F_{ab} \tilde{K}_a{}^m \tilde{K}_b{}^n + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} \tilde{K}_a{}^m \tilde{K}_b{}^n \right) \]

none of the four terms after distribution will contribute to the field equation because the variation of any one factor of metric components always leaves an unvaried \( \tilde{K}_{ts} \) or \( \tilde{K}_b{}^n \),

\[ \delta \left( \tilde{K}_a{}^m \tilde{K}_ts \tilde{K}_b{}^n \right) = \delta \tilde{K}_a{}^m \tilde{K}_ts \tilde{K}_b{}^n + \tilde{K}_a{}^m \delta \tilde{K}_ts \tilde{K}_b{}^n + \tilde{K}_a{}^m \tilde{K}_ts \delta \tilde{K}_b{}^n \]

Dropping such terms, we are only required to vary terms linear in \( \tilde{K}_{ab} \) or \( \tilde{K}^{ab} \), or cubic in the off diagonal components \( \tilde{K}_a{}^b \). Collecting these and using the symmetries of the fields, finally yields,

\[ \mathcal{F} \wedge * \mathcal{F} |_{\text{contributing}} = \left( \frac{1}{2} F_{bc} F_{da} \eta^{ac} + F_{bc} \tilde{H}^{ac} \eta_{ad} + \frac{1}{2} (G_{ad} - 2G_{da}) \eta_{bc} \tilde{G}^{ac} \right) \tilde{K}^{bd} \Phi + \left( G_{ba} \tilde{H}^{dc} + F_{ab} G^{cd} \right) \tilde{K}^{b}{}_{e} \tilde{K}^{c}{}_{d} \tilde{K}_a{}^e \Phi + \left( \frac{1}{2} \tilde{H}_{ac} \tilde{H}^{ac} \eta_{cd} + F_{ac} \tilde{H}^{ac} \eta_{cd} + \frac{1}{2} (G_{ba} - 2G_{ab}) \eta_{cd} \tilde{G}_{ca} \right) \tilde{K}_{bd} \Phi \]

(42)

We may now vary the metric.
To carry out the variation of the Yang-Mills potentials we may write the action in the null orthonormal frame. This form is still contained in the expression above, since it depends only on the purely off-diagonal terms, cubic in $\bar{K}^{ab}$. This form of the action follows immediately as

$$S_{TYM} = \kappa \int tr (\mathcal{F} \wedge * \mathcal{F}) = \kappa \int tr (G^{ab} (H_{ab} + F_{ab})) \Phi$$  \hspace{1cm} (43)

4.3 Variation

Using the variation of the Killing metric given in Eq.(22) and the variation of the volume form given by

$$\delta \Phi = -\frac{1}{2} K_{AB} \delta \bar{K}^{AB} \Phi$$

$$= -\frac{1}{2} [K_{ab} (B^{ab} + B^{ba}) + K^a_b (D_a^b + A_a^b) + K_b^a (A_b^a + D_a^b) + K^{ab} (C_{ab} + C_{ba})] \Phi$$

$$= -\delta^a_b (A^b_a + D_a^b) \Phi$$

the variation of Eq.(42) yields

$$\delta (\mathcal{F} \wedge * \mathcal{F}) = \left( \frac{1}{2} F_{ca} F_{da} \eta^{ac} + F_{ca} \mathcal{H}^{ac} \eta_{ad} + \frac{1}{2} (G_{ad} - 2G_{da}) \eta_{bc} G^{ac} \right) 2B_{bd} \Phi$$

$$+ \left( (2G_{ad} - G_{da}) F_{ab} - (2G_{da} - G_{ad}) \mathcal{H}_{ab} - (F_{ac} + \mathcal{H}_{ac}) G^{ac} \eta_{cd} \right) (A_{cd} + D_{cd}) \Phi$$

$$+ \left( \frac{1}{2} \mathcal{H}_{ac} \mathcal{H}^{ab} \eta^{cd} + F_{ac} \mathcal{H}^{ab} \eta^{cd} + \frac{1}{2} (G_{ab} - 2G_{ba}) \eta^{cd} \mathcal{G}_{cd} \right) 2C_{bd} \Phi$$  \hspace{1cm} (44)

This variation couples to the $(e^a, f_a)$ variation of the gravity action, Eq.(7). The resulting field equations and their reduction to the gravity theory are given in Section 6 below.

5 Yang-Mills potentials and field equations

We also need to vary the potential to find the Yang-Mills field equations. We start with the action of Eq.(43) in the null orthonormal basis,

$$S_{TYM} = \kappa \int tr \mathcal{F}^i \wedge * \mathcal{F}^i = \kappa \int G^{iab} (H_{i ab} + F_{i ab}) \Phi$$  \hspace{1cm} (45)

In this Section, we make the internal symmetry explicit, varying $S_{YM}$ with respect to the $SU(N)$ potentials. It is most convenient to work in the $e^i = (e^a, f_i)$ basis. Internal indices are labeled with letters $i, j, k$, while frame indices are chosen from the beginning of the alphabet, $a, b, c, \ldots$. 

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5.1 The field components and potentials

The $SU(N)$ field is given in the $\bar{e}^A$ basis by Eq. (46),

\[
\mathcal{F}^i = \frac{1}{2} F_{i}^{\ ab} e^a \wedge e^b + G_{i}^{\ ac} f_a \wedge e^b + \frac{1}{2} \mathcal{H}_{i}^{\ abc} f_c \wedge f_c
\]

where as a matrix,

\[
\mathcal{F}^i_{\ AB} = \begin{pmatrix}
F_{i}^{\ ab} & G_{i}^{\ a} b \\
G_{i}^{\ a} b & \mathcal{H}_{i}^{\ ab}
\end{pmatrix}
\]

with $G_{i}^{\ ab} = - G_{i}^{\ a} b$.

The field is given in terms of its $U(1)$ or $SU(N)$ potential by the Cartan equation,

\[
\mathcal{F}^i = d A^i - \frac{1}{2} \varepsilon^{i j k} A^j \wedge A^k
\]

where the potentials are biconformal 1-forms,

\[
A^i = A^i_a e^a + B^i_a f_a
\]

In terms of $A^i_a, B^i_a$, the field becomes

\[
\mathcal{F}^i = D A^i_a \wedge e^a - \frac{1}{2} A_{i}^{\ k} \wedge A_{k}^{\ b} e^b \\
+ D B^i_a \wedge f_a + B^i_a S_a - \frac{1}{2} A_i^{\ k} \wedge B^k_b f_b
\]

where $A_{i}^{\ k}$ is the $SU(N)$ connection in the adjoint representation,

\[
A_{i}^{\ k} \equiv c^{i j k} A^j
\]

and the Weyl-covariant derivatives of the potentials are given by

\[
D A^i_a = d A^i_a - \omega^a_c A_i^a_c + A_i^a \omega
\]

\[
D B^i_a = d B^i_a - B^i_c \omega^c_a - B^i_a \omega
\]
Note that the covariant derivative of $\eta^{ab}$ does not necessarily vanish, $D\eta^{ab} = d\eta^{ab} - 2\omega^{ab}$ where $d\eta^{ab}$ takes into account the conformal equivalence class of $\eta_{ab} \in \{ e^{2\varphi}\eta_{ab}^0|\varphi = \varphi(x,y)\}$.

Now we separate Eq.(47) into its independent $e^a \wedge e^b$, $f_a \wedge e^b$ and $f_a \wedge f_b$ parts. We begin by expanding the exterior derivatives in the orthonormal basis

\[
d A^i{}_a = A^i{}_{a,b}e^b + A^i{}_{a}e^bf_b
\]
\[
d B^i{}^a = B^i{}_{c,b}e^b + B^i{}_{c}e^bf_b
\]

and similarly the covariant exterior derivatives

\[
D A^i{}_a = A^i{}_{a;b}e^b + A^i{}_{a}e^bf_b
\]
\[
D B^i{}^a = B^i{}_{a;c}e^b + B^i{}_{a}e^bf_b
\]

We also expand the $SU(N)$ connection, $\mathcal{A}^iG_j$, in the adjoint representation:

\[
\mathcal{A}^i{}^k = c^i{}_{jk}A^j{}^a e^a + c^i{}_{jk}B^j{}^a f_a
\]
\[
= \alpha^i{}_{k} + \beta^i{}_{k}
\]

Finally, writing the general form of the co-torsion

\[
S_a = \frac{1}{2}S_{abc}e^b \wedge e^c + S_a{}^b e^b \wedge f_c + \frac{1}{2}S_a{}^{bc}f_b \wedge f_c
\]

we arrive the component fields in terms of the potentials:

\[
F^i{}_{ab} = A^i{}_{b;a} - A^i{}_{a;b} - c^i{}_{jk}A^j{}_{a}A^k{}_{b} + B^i{}_{c}S_{cab}
\]
\[
G^i{}_{ab} = A^i{}_{b} - B^i{}_{a} - c^i{}_{jk}B^j{}_{a}A^k{}_{b} + B^i{}_{c}S_{a}{}^{c}{}_{b}
\]
\[
H^i{}_{ab} = B^i{}_{b;a} - B^i{}_{a;b} - c^i{}_{jk}B^j{}_{a}B^k{}_{b} + B^i{}_{c}S_{c}{}^{ab}
\]

These expressions are covariant with respect to both $SU(N)$ and Weyl group transformations.
5.2 The field equations for the potentials

Using Eqs. (48)-(50) in the action, Eq. (45), the variation of \(A^i_a\) gives

\[
\delta A S_{YM} = \kappa \int \left( \delta A G^{iab} (H_{i ab} + F_{i ab}) \right) \Phi \\
+ \kappa \int G^{iab} \left( \delta A H_{i ab} + \delta A F_{i ab} \right) \Phi \\
= \kappa \int \left( \delta A^k b \eta^{bc} \left( (H_{k ac} + F_{k ac})^{ia} - \epsilon^i j k B^j a (H_{i ac} + F_{i ac}) \right) \right) \Phi \\
+ \kappa \int \delta A^k b \left( (G_{k ab} - G_{k ba})^{ia} - \epsilon^i j k A^j a (G_{i ba} - G_{i ab}) \right) \Phi \\

\]

Therefore, using the adjoint form of the connection, the field equation becomes

\[
0 = \eta^{bc} (H_{k ac} + F_{k ac})^{ia} + \eta^{bc} (H_{i ac} + F_{i ac}) \beta^i_k \quad a \\
+ (G_{k ab} - G_{k ba})^{ia} - (G_{i ab} - G_{i ba}) \alpha^i b \\
\]

For the \(B^i a\) variation,

\[
\delta B S_{YM} = \kappa \int \left( \delta B G^{iab} (H_{i ab} + F_{i ab}) \right) \Phi \\
+ \kappa \int \left( \delta B H_{i ab} + \delta B F_{i ab} \right) \Phi \\
= \kappa \int \delta B^i a \left( \eta^{bc} (H_{j ac} + F_{j ac})^{ia} - \epsilon^i j k A^k b \eta^{bc} (H_{i ac} + F_{i ac}) + S_a^e \right) \Phi \\
+ \kappa \int \delta B^i a \left( (G_{j ba} - G_{j ab})^{ia} - (G_{i ab} - G_{i ba}) \epsilon^i j k B^k b + \frac{1}{2} (G_{j cb} - G_{j bc}) S_a^{cb} \right) \Phi \\
+ \kappa \int \delta B^i a \left( G_{j cb} - G_{j bc} \right) \frac{1}{2} S_{acb} \Phi \\
\]

Extracting the field equation, recalling that \(i, j\), are internal indices and \(a, b, c, e\) orthonormal indices,

\[
0 = \eta^{bc} \left( (H_{j ac} + F_{j ac})^{ia} + \alpha^i j b (H_{i ac} + F_{i ac}) + S_a^e \right) \Phi \\
\]

Notice that there is no dynamical equation for the symmetric part of \(G^{k cd}\). Moreover, the action depends only on the antisymmetric part of \(G^{k cd}\). Therefore, from here on we assume that like \(F^{i ab}\) and \(H^{i ab}\), \(G^{k cd}\)
is antisymmetric, \( G_{ab} = G_{[ab]} \). Also notice that there is no separate Yang-Mills field equation for \( F_{ab} \) and \( H_{ab} \). Both field equations contain only their sum, \( H^k_{ab} + F^k_{ab} \), although \( F_{ab} \) and \( H_{ab} \) enter the gravitational equations separately. We therefore define a new field,

\[
K^k_{ab} \equiv \frac{1}{2} (H^k_{ab} + F^k_{ab})
\]

In terms of these, the field equations become

\[
0 = \eta^{bc} K_{a c i} ^{a} + \eta^{bc} K_{i a c} ^{j} k ^{a} + G_{k a} ^{ab} \alpha + G_{i a k} ^{ab} \alpha ^{i}
\]

\[
0 = \eta^{bc} (K_{i a} ^{ac} b + K_{i a} ^{ac} \alpha ^{i} j b + K_{e c} ^{e} S_{a} ^{b c})
+ G_{j a} ^{ab} \alpha ^{i} j ^{b} + \frac{1}{2} G_{j e} ^{bc} S_{a} ^{b c} + \frac{1}{2} G_{j e} ^{bc} S_{abc}
\]

These have the expected form of a divergence of the field strength.

Eqs. (51) and (52) represent the first of three main goals of this paper. They are the field equations of an SU(\(N\)) gauge theory on a 2\(n\)-dimensional space. Our second goal is to understand how the corresponding SU(\(N\)) gravitational sources, Eq. (44), affect the gravitational solutions. Once we have reduced the gravitational field equations to more simply describe the underlying geometry, we will return to Eqs. (51) and (52). Our final result is to show that the reduction of the underlying geometry to the co-tangent bundle of general relativity simultaneously forces the reduction of the source equations to the usual Yang-Mills sources for general relativity.

Since the reduction of the gravitational field equations has been presented in detail elsewhere [13], we only give a brief description in the next Section. Subsequently, in Sec. (7), particular attention will be given to the resulting twofold reduction of the SU(\(N\)) fields: (1) from \((N^2 - 1) \times 2^{2(n-1)}\) field components to \((N^2 - 1) \times \frac{n(n-1)}{2}\) components, and (2) the restriction of the number of effective independent variables, \(F(x, y) \rightarrow F(x)\).
6 Reduction of the gravitational field equations

6.1 Field equations for the twisted action

The full action is

\[ S = S_G + S_{YM} \]

\[ = \lambda \int e_{a c . . . d}^{b e . . . f} \left( a \Omega^a_{ b} + \beta \delta^a_{ b} \Omega + \gamma e^a \wedge f_b \right) \wedge e^c \wedge \cdots \wedge e^d \wedge f_e \wedge \cdots \wedge f_f \]

\[ - \frac{\kappa}{2} \int \text{tr} F^\wedge F \quad (53) \]

We choose the combinatoric factor \( \lambda \) so the final coupling is \( \kappa \).

6.1.1 Curvatures, Bianchi identities, and gravity variation

The gravitational field equations are given by varying \( S_G \) in Eq.(53) with respect to all connection 1-forms, then combining with the Yang-Mills sources found in Eq.(44). The curvature components are given in terms of the connection by the Cartan structure equations,

\[ \Omega^a_{ b} = d\omega^a_{ b} - \omega^c \wedge \omega^a_{ c} - 2\Delta^a_{ db} f_c e^d \quad (54) \]

\[ T^a = de^a - e^c \omega^a_{ c} - \omega e^a \quad (55) \]

\[ S_a = df_a - \omega^c \wedge f_c - f_a \omega \quad (56) \]

\[ \Omega = d\omega - e^c f_c \quad (57) \]

We also have the integrability conditions—generalized Bianchi identities—which follow by exterior differentiation of the Cartan equations,

\[ D\Omega^a_{ b} = 2\Delta^a_{ db} f_c T^d - 2\Delta^a_{ db} S_c e^d \quad (58) \]

\[ DT^a = e^c \Omega^a_{ c} - \Omega e^a \quad (59) \]

\[ DS_a = -\Omega^c f_c + f_a \Omega \quad (60) \]

\[ D\Omega = -T^c f_c + e^c S_e \quad (61) \]

The variation is discussed in detail in [13], and involves only variation of the connection forms, so we
simply state the result. The variation of the spin connection \( \omega^a{}_b \) and Weyl vector \( \omega \) give

\[
T^{ae} - T^{ea} - S_e^{ae} = 0 \quad (62)
\]

\[
T^a + S^a_c - S^a_e = 0 \quad (63)
\]

\[
\alpha \Delta^{arb} (T^m - \delta^m_a S_e^c) = 0 \quad (64)
\]

\[
\alpha \Delta^{arb} (\delta^b_d T_{ad} + S_c^b - \delta^b_e S_d^d) = 0 \quad (65)
\]

and these acquire no sources since the Yang-Mills action is independent of these connection forms, as noted in [9]. The variation of the solder and co-solder forms lead to

\[
\begin{align*}
[\alpha (\Omega^n_b - \Omega^a_b \delta^n_a) + \beta (\Omega^n_m - \Omega^a_m \delta^n_m) + \Lambda \delta^n_a] A^m_n \\
[\alpha (\Omega^m_a - \Omega^a_b \delta^m_b) + \beta (\Omega^m_n - \Omega^a_n \delta^m_n) + \Lambda \delta^m_n] D^m_n \\
- [\alpha \Omega^a_{nam} + \beta \Omega^a_{nm}] B^{mn} \\
[\alpha \Omega^n_b - \beta \Omega^a_{nm}] C^{mn} 
\end{align*}
\]

(66)

with the arbitrary variations \( A^m_n \), \( B^{mn} \), \( C_{mn} \), and \( D^m_n \) defined in Eqs. (20) and (21). In [13] the expressions above are equated to zero, but they now acquire sources.

### 6.1.2 Combined equations

Equating corresponding parts of Eqs. (12) and Eqs. (60) and symmetrizing appropriately,

\[
\begin{align*}
[\alpha (\Omega^a_c \delta^d_b - \Omega^c_d \delta^a_b) + \beta (\Omega^a b - \Omega^c d \delta^a_d) + \Lambda \delta^a_d] &= \kappa W^a_b \\
[\alpha (\Omega^a_c \delta^d_b - \Omega^c_d \delta^a_b) + \beta (\Omega^a b - \Omega^c d \delta^a_d) + \Lambda \delta^a_d] &= \kappa W^a_b \quad (67)
\end{align*}
\]

\[
\begin{align*}
- [\alpha \Omega^a b + \beta \Omega^a d] &= \kappa T_{ab} \\
[\alpha \Omega^a b + \beta \Omega^a d] &= \kappa S^{ab} \\
[\alpha \Omega^a b + \beta \Omega^a d] &= \kappa S^{ab} \quad (68)
\end{align*}
\]

where, recalling the antisymmetry of \( G^{ab} \),

\[
T_{ab} = tr \left( F_{ac} F_{bd} \eta^{cd} + H_{ac} F_{bd} \eta^{cd} + H_{bc} F_{ad} \eta^{cd} + 3 \eta^{cd} G_{ac} G_{bd} \right) \quad (71)
\]

\[
S^{ab} = \eta^{bd} \eta^{ac} tr \left( \eta^{ef} H_{ec} H_{fd} + \eta^{ef} H_{ec} F_{fd} + \eta^{ef} H_{ed} F_{fc} + 3 \eta^{ef} G_{ce} G_{fd} \right) \quad (72)
\]

\[
W^a b = tr \left( 3 G^{ca} F_{cb} + 3 G^{ca} H_{cb} - (F_{cd} + H_{cd}) G^{cd} \delta^a_b \right) \quad (73)
\]

34
In addition, we have the field equations for the $U(1)$ field,

$$
0 = \eta^{bc} K^i_k a_{ac} + \eta^{bc} K_i a_{\beta \alpha} i_k a + G_{k a \alpha} a + G_{i \alpha} a \beta a_{ka} \\
0 = \eta^{bc} (K_j a_{ac} + K_i a_{\alpha \beta} j_b + K_j e c S_a e_b) \\
+ G_j a b + G_i a b \beta a_{j b} + \frac{1}{2} G_j a b c S_a b c + \frac{1}{2} G_j a b c S_{abc}.
$$

(74)

(75)

Our gravitational solution now follows many of the steps presented in detail in [13].

6.2 Solving the field equations for the twisted action

For the remainder of the gravitational solution, the particular forms of $T_{ab}, S_{ab}$ and $W_{ab}$ make little difference; indeed, the solution of this Section holds for the metric variation of any sources at all. While the form of these source tensors varies with the fields and the matter action, the positions in which they occur in the field equations and their symmetries follow knowing only the variation of $\bar{K}^{AB}$. For the present, this is all we need.

We first turn to the consequences of vanishing torsion, $T^a = 0$.

6.2.1 Vanishing torsion

Similarly to general relativity, with vanishing torsion the torsion Bianchi identity, Eq.(59), simplifies to an algebraic relation,

$$
0 = e^c \Omega^a c_e - \Omega e_a
$$

The algebraic condition $e^c \Omega^a c_e = \Omega e_a$ expands to three independent component equations,

$$
\Omega^a_{[bcd]} = \delta^a_{[b} \Omega_{cd]} \\
\Omega^a b c_d - \Omega^a d c_b = \delta^a b \Omega^c d - \delta^a d \Omega^c b \\
\Omega^a b c d = \delta^a b \Omega^c d
$$

(76)

(77)

(78)
Since $\eta_{ea}\Omega^a_{\ b} \ cd = -\eta_{ba}\Omega^a_{\ e} \ cd$, the $ab$ trace of Eq.(78) leaves $\Omega^{cd} = 0$. Therefore each term vanishes separately,

$$\Omega^a_{\ b} \ cd = 0 \tag{79}$$

$$\Omega^{cd} = 0 \tag{80}$$

The $ad$ contraction of Eq.(77) gives

$$\Omega^a_{\ b} \ c_a = -(n-1)\Omega^c_{\ b} \tag{81}$$

Combining Eqs.(79) and (80) with Eq.(70) we immediately find that the gravitational fields force a constraint on the source fields. This is our first source constraint:

$$S^{ab} = 0 \tag{82}$$

We next look at the field equations for the curvature.

### 6.2.2 Curvature equations

We now combine the vanishing torsion simplifications with the curvature and dilatation field equations, Eqs.(67) and (68). The reduction of these equations begins by noting that the difference between Eq.(67) and Eq.(68) immediately gives equality of the traces,

$$\Omega^c_{\ b} \ a_c = \Omega^a_{\ c} \ b_c \tag{83}$$

Next, formally lowering an index in Eq.(77)

$$\eta_{ea}\Omega^a_{\ b} \ c_d - \eta_{ea}\Omega^a_{\ d} \ c_b = \eta_{eb}\Omega^c_{\ a} \ c_d - \eta_{ed}\Omega^c_{\ a} \ b_c$$

we cycle $ebd$, then add the first two and subtract the third. Using the the antisymmetry of the curvature on the first two indices, $\eta_{ea}\Omega^a_{\ d} \ c_b = -\eta_{da}\Omega^a_{\ c} \ b$ we find

$$\Omega^a_{\ c} \ d_e = -2\Delta^{ab}_{\ de}\Omega^c_{\ b} \tag{84}$$

Substituting Eq.(84) into the trace symmetry, Eq.(83) to the two contractions of Eq.(83) constrains the
cross-dilatation,

\[-\delta_d^a \Omega^e_c + \eta^{ae} \eta_{cd} \Omega^c_e = -(n-1) \Omega^a_d\]

Contracting with $\eta_{ba}$, we see that the antisymmetric part vanishes,

\[(n-2) (\eta_{bc} \Omega^e_d - \eta_{cd} \Omega^e_b) = 0\]

in dimensions greater than 2, while an explicit check confirms the vanishing antisymmetry in 2-dimensions as well. Therefore, the symmetric part, $\eta_{bc} \Omega^e_d + \eta_{dc} \Omega^e_b = \frac{2}{n} \eta_{bd} \Omega^e_c$, becomes a solution for the full cross-dilatation in terms of its trace,

$$\Omega^e_d = \frac{1}{n} \delta_b^e \Omega^c_c$$  

This, in turn, combines with Eq.(84) to give the cross-curvature in terms of the trace of the dilatation,

$$\Omega^a_c b d = -\frac{2}{n} \Delta_{bd}^a \Omega^e_e$$  

We have one remaining cross-curvature field equation, Eq.(67), which couples the cross-dilatation trace, $\Omega^a_a$, to the Yang-Mills source fields. Using Eqs.(85) and (86) to replace the cross-curvature and the cross-dilatation in Eq.(67), and simplifying,

$$\frac{1}{n} (n-1) \left[\left((n-1) \alpha - \beta\right) \Omega^e_c + n\Lambda\right] \delta_b^a = W^a_b$$

so that $W^a_b = f \delta_b^a$ for some function $f$. The constant $\Lambda$ is given by $\Lambda = (n-1) \alpha - \beta + n^2 \gamma$. Contracting then substituting back, the gravitational field equations force a second source constraint:

$$W^a_b = -\frac{1}{n} W^c_c \delta_b^a$$  

where

$$W^c_c = (n-1) \left[\left((n-1) \alpha - \beta\right) \Omega^e_c + n\Lambda\right]$$

The traced source tensor on the right, $W^c_c$, therefore drives the entire cross-curvature and cross-dilatation.
It is striking that the only source dependence for these components is the Yang-Mills Lagrangian density,

\[ W^a_a = 3 G^{i\, ac} (\mathcal{H}_{i\, ac} + F_{i\, ac}) = 3 \mathcal{L} \]

### 6.2.3 Spacetime terms

Finally, we combine the remaining field equation, Eq.(69),

\[ \alpha \Omega^c_{bca} + \beta \Omega^a_{ba} = -\kappa T_{ab} \]

and the corresponding part of the vanishing torsion Bianchi, Eq.(76), which expanded becomes

\[ \Omega^a_{bcd} + \Omega^a_{cdb} + \Omega^a_{dbc} = \delta_b^c \Omega_{cd} + \delta_c^a \Omega_{db} + \delta_d^a \Omega_{bc} \]

The \( ac \) trace reduces this to

\[ \Omega^c_{bcd} - \Omega^c_{dcb} = -(n - 2) \Omega_{bd} \]

Combining this with the antisymmetric part of the field equation, \( \alpha (\Omega^a_{nam} - \Omega^a_{man}) = -2 \beta \Omega_{nm} \) shows that

\[ ((n - 2) \alpha - 2 \beta) \Omega_{ab} = 0 \]

so that generically (i.e., unless \( (n - 2) \alpha = 2 \beta \)), the spacetime dilatation vanishes. Note that this is true for any symmetric source tensor, so spacetime dilatation is never driven by ordinary matter. As a result,

\[ \Omega^c_{acb} = \Omega^c_{(ac)(b)} = -\frac{\kappa}{\alpha} T_{ab} \quad (88) \]

\[ \Omega_{nm} = 0 \quad (89) \]

### 6.2.4 Dilatation

Having reduced the dilatational curvature to a single function,

\[ \Omega = \chi e^a \wedge f_a \quad (90) \]
where $\chi \equiv \frac{1}{n} \Omega^a_{\phantom{a}a}$, we can now use the dilatational integrability condition, Eq. (61), to press further. Substituting Eq. (90) into the Bianchi identity, Eq. (61),

$$0 = d\Omega - e^b \wedge S_b$$

where we have used $d(e^a \wedge f_a) = D(e^a \wedge f_a) = T^a \wedge f_a - e^a \wedge S_a$. Setting $d\chi = \chi f_c + \chi^c f_c$, expanding the co-torsion into components, and combining like forms yields three independent equations,

$$(1 + \chi) S_{[acd]} = 0$$

$$(1 + \chi) (S_{c^\phantom{c}a}^\phantom{c}d - S_{d^\phantom{c}a}^\phantom{c}c) = \chi d^a_c - \chi c^a_d$$

$$(1 + \chi) S_{a}^{\phantom{a}cd} = \chi^d c^a_d - \chi^c d^a_c$$

We may now use the co-torsion field equations to gain insight into $\chi$.

With vanishing torsion, the co-torsion field equations Eqs. (62)-(65) reduce to

$$S_{ae}^e = 0$$

$$S_c^a_{\phantom{a}a} - S_a^a_{\phantom{a}c} = 0$$

$$\alpha \Delta_{ab} \left( S_{c^\phantom{c}b}^\phantom{c}a - \delta^b_c S_{d^\phantom{c}a}^\phantom{c}d \right) = 0$$

Using the field equation Eq. (95) to replace the co-torsion terms in the trace of the Bianchi identity, Eq. (92), gives

$$(1 + \chi) (S_c^a_{\phantom{a}a} - S_a^a_{\phantom{a}c}) = -(n-1) \chi_c$$

$$\chi_c = 0$$

Then, combining Eq. (94) with the ad trace of Eq. (93),

$$(1 + \chi) S_{a}^{\phantom{a}ca} = -(n-1) \chi^c$$

$$\chi^c = 0$$
and therefore,
\[ d\chi = 0 \]
(97)

The dilatation therefore takes the form
\[ \Omega = \chi e^a \wedge f_a \]
with \( \chi \) constant. The remainder of the development of the solution continues as in the homogeneous case but with a different constant value,
\[ \chi = \frac{1}{(n-1)\alpha - \beta} \left( \frac{1}{n-1} \Lambda - \frac{\kappa}{n(n-1)} W^c \right) \]
for the magnitude of the dilatation cross-term.

Importantly, the constancy of \( \chi \) implies the constancy of \( W^c \), and via the second source constraint, Eq. (87), the (mostly zero) constancy of all of \( W^a_b \).

6.2.5 The Frobenius theorem and the final reduction

With vanishing torsion, \[ \text{Eq. (55)} \] shows that the solder form becomes involute, and we may write
\[ e^a = e^a_\alpha dx^\alpha \]
where \( x^\alpha \) comprise \( n \) of the \( 2n \) coordinates. Holding \( x^\alpha \) constant, \( e^a = 0 \), and the residual field equations describe a submanifold. Here the discussion exactly parallels that of [13]:

1. Solve for the connection on the \( e^a = 0 \) submanifold. Here, because the curvature and dilatation vanish (see Eqs. (79) and (80)), we may gauge the restricted components of the spin connection and Weyl vector to zero. A careful coordinate choice puts the submanifold basis in the form \( h_a = e_\mu^a dy_\mu \).

2. Now let \( x^\alpha \) vary, extending the solution back to the full biconformal space. This allows all connection forms to acquire an additional \( dx^\alpha \) or \( e^a \) term,
\[ \omega^a_b = \omega^a_{bc} e^c, \quad e^a = e_\alpha^a dx^\alpha, \quad f_a = e_\mu^a dy_\mu + c_{ab} e^b, \quad \omega = W_a e^a \]
(98)
3. Note that Eq.(55) is now purely quadratic in $e^a$, and therefore requires the coefficients to depend only on the $x$-coordinates, $e^a_{\alpha} = e^a(x)$. Solving for the connection separates it into a compatible piece and a Weyl vector piece,

$$\omega^a_b = \alpha^a_b - 2\Delta^{ac}_{db} W_c e^d$$

where $de^a = e^b \alpha^a_b$.

4. Substitute these reduced forms of $e^a, f_b$ into the dilatation, Eq.(57) and solve for the Weyl vector. This yields

$$\omega = -(1 + \chi) y_a e^a$$

where $y_a = e_{\mu}^a y_{\mu}$.

These steps give the final expressions for the connection forms, except for the form of $c_{ab} = c_{ba}$ in the expansion of the co-solder form $f_a$.

We note that the submanifolds found by setting either $e^a = 0$ by holding $x^\mu$ constant, or $h_a = 0$ by holding $y_\mu$ constant are Lagrangian submanifolds.

6.2.6 The curvature

The final steps in the gravitational reduction are to substitute the partial solution for the connection forms Eq.(98) into Eqs.(54) and (56) to impose the final field equations.

To express the remaining undetermined component of the curvature, $\Omega^a_{bcd}$, we define the Schouten tensor

$$\mathcal{R}_a = \frac{1}{n - 2} \left( R_{ab} - \frac{1}{2(n - 1)} \eta_{ab} R \right) e^b$$

where $R_{ab} = (n - 2) \mathcal{R}_{ab} + \eta_{ab} R$ is the Ricci tensor. The generalization of the Schouten tensor to an integrable Weyl geometry is then (see [34])

$$\mathcal{R}_a = \mathcal{R}_a + D_{(\alpha, x)} W_a + W_a \omega - \frac{1}{2} \eta_{ab} W^2 e^b$$

In terms of the Schouten tensor, the decomposition of the Riemann curvature 2-form into the Weyl curvature 2-form and trace parts is

$$R^a_{\ b} = C^a_{\ b} - 2\Delta^{ac}_{db} R_c \wedge e^d$$

Because of the manifest involution of $h_a = e_{\mu}^a (x) dy_\mu$, the subspace spanned by the solder form, $e^a$, is
a submanifold. Because $\Omega_{ab} = 0$ the submanifold geometry is always an integrable Weyl geometry, so the Weyl vector may be removed from the spacetime submanifold by a gauge transformation. The spacetime submanifold is simply a Riemannian geometry with local scale invariance.

Now, introducing the reduced form of the connection into Eq. (54) and imposing the corresponding field equation, Eq. (69) shows that

$$\frac{1}{1 + \chi} R_{ab} + c_{ab} = -\frac{\kappa}{\alpha} T_{ab}$$

(99)

with the full spacetime component of the biconformal curvature given by the Weyl curvature, $\Omega^a_{\ bcd} = C^a_{\ bcd}$.

### 6.2.7 The co-torsion

A similar introduction of the reduced connection into Eq. (56) for the co-torsion shows that the momentum and cross-terms vanish, while (following the somewhat intricate calculation of [13]) the remaining component is given by

$$S_a = d(x) c_a + c_b \wedge \omega^b_a + \omega \wedge c_a$$

(100)

where $c_a$, in turn, is determined by Eq. (99).

Expanding $c_a$ fully to separate the Weyl vector parts,

$$c_a = -\frac{1}{1 + \chi} \left( R_a + D_{(\alpha,x)} W_a + W_a \omega - \frac{1}{2} \eta_{ab} W^2 e^b \right) - \frac{\kappa}{\alpha} T_a$$

$$= b_a - \frac{1}{1 + \chi} \left( D_{(\alpha,x)} W_a + W_a \omega - \frac{1}{2} \eta_{ab} W^2 e^b \right)$$

where $T_a = T_{ab} e^b$ is the remaining source field and $b_a \equiv -\frac{1}{1 + \chi} R_a - \frac{\kappa}{\alpha} T_a$. Then substituting into Eq. (100), after multiple cancellations the co-torsion becomes

$$S_a = \frac{1}{1 + \chi} W_b R^b_a - D_{(\alpha,x)} \left( \frac{1}{1 + \chi} R_a + \frac{\kappa}{\alpha} T_a \right)$$

$$+ 2 \Delta c_{a} d W_b \left( \frac{1}{1 + \chi} R_b + \frac{\kappa}{\alpha} T_b \right) \wedge e^c$$

(101)

with the cross term and momentum term of the co-torsion vanishing.

This result is quite similar to an integrability condition. It is shown in [34] that the condition for the
existence of a conformal gauge in which the Einstein equation, $G_{ab} = \kappa T_{ab}$, holds is

$$0 = \phi_{,b} R^b_a - D_{(a,x)} (R_a - \kappa T_a) + 2\Delta_{ca} \phi_{,d} (R_b - \kappa T_b) \wedge e^c$$

(102)

where

$$T_a = \frac{1}{n-2} \left( T_{ab} - \frac{1}{n-1} T \eta_{ab} \right)$$

When $T_a = 0$, Eq. (102) reduces to the well-known condition, $D_{(a,x)} R_a - \varphi_{,b} C^b_a = 0$, for the existence of a Ricci flat conformal gauge.

There are two differences between Eq.(101) and Eq.(102). First, the co-torsion on the left hand side of Eq. (101) obstructs the integrability condition, Eq.(102), and we cannot set $S_a = 0$ because the Triviality Theorem shows that when both torsion and co-torsion vanish, biconformal space must be trivial. The second difference is that the Weyl vector on the right is not integrable on the full biconformal space.

These issues have a common solution. The part of structure equation for the co-solder form involving $h_a$ is

$$dh_a = \omega^c \wedge h_c + h_a \wedge \omega$$

(103)

Therefore, as briefly noted above, $h_a = e_a \mu dy^\mu$ is in involution. Holding $y_\mu = y_\mu^0$ constant shows that $e^a$ spans a submanifold. On that submanifold, the Weyl vector becomes exact,

$$\omega = W_a e^a = d (y_\mu^0 x^\mu)$$

This means that on the $y_\mu = y_\mu^0$ spacetime submanifold, the right side takes the form of the integrability condition.

At the same time, we may use the form of $S_a$ as the covariant derivative of $e_a$, Eq.(100) with a suitable choice of $c_{ab}$. In [13] it is shown that $c_{ab}$ is symmetric and divergence free. While the interpretation given in [13] of a phenomenological energy tensor is consistent, it is at odds with the more fundamental interpretation of sources given here. Instead, we identify $c_{ab}$ as proportional to the Minkowski metric—the only invariant, symmetric tensor available. It is also divergence free with respect to the compatible connection, since $D_{(a,x)} \eta_{ab} = 0$. However, as noted in the previous Section the fully biconformal-covariant derivative of $\eta_{ab}$ does not necessarily vanish. Since by Eq.(100) the co-torsion is given by the full biconformal derivative of
c_{ab}, the identification $c_{ab} = \Lambda_0 \eta_{ab}$, implies

$$S_a = 2 \left(1 + \chi \right) \Delta_{ca}^{bc} \Lambda_0 \eta_{cd} e^d \wedge e^e$$

thereby avoiding the Triviality Theorem. This residual form of the co-torsion may now be combined into the right hand side of Eq. (101).

Combining these observations, on the h_{a} = 0 spacetime submanifold with $c_{ab} = \Lambda_0 \eta_{ab}$, setting $\phi, a = y^0_a$, and using $D_{(\alpha,x)} (\Lambda_0 \eta_{ab}) = 0$, it follows that

$$0 = \frac{1}{1 + \chi} \phi, b R^b_a - D_{(\alpha,x)} \left( \frac{1}{1 + \chi} \mathcal{R}_a + \frac{\kappa}{\alpha} T_a + \Lambda_0 \eta_{ab} e^b \right) + 2 \Delta_{cd}^{ab} \phi, d \left( \frac{1}{1 + \chi} \mathcal{R}_b + \frac{\kappa}{\alpha} T_b + \Lambda_0 \eta_{be} e^e \right) \wedge e^e$$

(104)

This is now the condition for the existence of a conformal transformation such that

$$\frac{1}{1 + \chi} \mathcal{R}_b + \Lambda_0 \eta_{be} e^e = - \frac{\kappa}{\alpha} T_b$$

Expressed in terms of the Einstein tensor,

$$G_{ab} + \Lambda_C \eta_{ab} = -(n - 2) \frac{\kappa (1 + \chi)}{\alpha} (T_{ab} - \eta_{ab} T)$$

(105)

where the net effect of $c_{ab}$ is a cosmological constant, $\Lambda_C = -(n - 1) (n - 2) (1 + \chi) \Lambda_0$.

If we make the conformal transformation that produces Eq. (105), the co-torsion equation (104) reduces to $\phi, b R^b_a = 0$. In generic spacetimes this requires $W_\mu = y^0_\mu = 0$.

7 The source for gravity

7.1 The reduction of sources forced by coupling to gravity

The necessary source constraints from the gravitational couplings, Eqs. (82) and (87), may be written as

$$0 = \mathcal{H}_{ac} \mathcal{H}_{i bd} \eta^{cd} + (\mathcal{H}_{ac} F_{i bd} + \mathcal{H}_{bc} F_{i ad}) \eta^{cd} + 3 \mathcal{G}_{ac} \mathcal{G}_{i bd} \eta^{cd}$$

(106)

$$0 = \mathcal{G}^i ca (F^i_{cb} + \mathcal{H}^i_{cb}) - \frac{1}{n} \delta^a_b G^{i cd} (F^i_{cd} + \mathcal{H}^i_{cd})$$

(107)
where the full contraction of the second, $G^{i\,cd} (F_{\,cd}^i + H_{\,cd}^i)$, is constant.

These conditions must continue to hold for small physical variations of the independent potentials, $A^i_a$ and $B^{ia}$. We may imagine two nearby solutions differing only in one or both of the potentials and look at their difference. The change in $F_{\,ab}^i$ as we change $A^i_a$ is given by

$$
\delta F_{\,ab}^i = \delta A^i_{\,b,a} - \delta A^i_{\,a,b} - c^j \delta A_{\,j}^i a A_k^b - c^j \delta A_{\,j}^i a A_k^b \delta A^k_{\,b}
$$

$$
= \left( \delta A_{\,b,a}^i - \alpha^i_{\,k,b} \delta A_k^b \right) - \left( \delta A_{\,a,b}^i - \alpha^i_{\,j,b} \delta A_j^b \right)
$$

$$
= D_a (\delta A_{\,b}^i) - D_b (\delta A_{\,a}^i)
$$

where $D_a$ is covariant with respect to local Lorentz, dilatational and $SU(N)$ transformations. Similarly we find for $G^{i\,a}_b$ and $H^{i\,ab}$,

$$
\delta A G_{\,a}^{i\,b} = D_a (\delta A_{\,b}^i)
$$

$$
\delta A H_{\,a}^{i\,b} = 0
$$

Of course, under changes of gauge, these fields are invariant.

The conditions (106) and (107) must continue to hold throughout such small changes. Substituting these variations into the first constraint,

$$
0 = \delta F_{\,ed}^i (H_{\,ac}^{i\,b} + H_{\,bc}^{i\,a} + c^j H_{\,j}^{i\,e} c^d G_{\,b}^{i\,d} G_{\,a}^{i\,d} + G_{\,a}^{i\,d} \eta_{\,bd} G_{\,i}^{d\,e} c^d \eta^{cd})
$$

(108)

The first term of Eq. (106) has dropped out because $H_{\,i\,ab}$ is independent of $A^i_a$.

Now we expand Eq. (108) in terms of the variation $\delta A_{\,a}^i$ and its derivatives,

$$
0 = \delta A_{\,d,e}^i (H_{\,ac}^{i\,b} c^d + H_{\,bc}^{i\,a} c^d - H_{\,ac}^{i\,b} c^d - H_{\,bc}^{i\,a} c^d)
$$

$$
+ \delta A_{\,c}^i c^e 3 (\eta_{\,ae} G_{\,i}^{bd} + G_{\,i}^{ad} \eta_{\,b} e) \eta^{cd}
$$

$$
+ \delta A_{\,a}^f \left( -\omega_{\,b}^{\,i\,f\,e} + W_{\,d}^{\,i\,f\,e} + \alpha_{\,k}^{\,i\,f\,e} \right) (H_{\,k}^{\,b} c^d \eta^{cc} + H_{\,k}^{\,b} c^d \eta^{cc} - H_{\,k}^{\,b} c^d \eta^{cc} - H_{\,k}^{\,b} c^d \eta^{cc})
$$

$$
+ \delta A_{\,i \,f \,e}^k (3 c^j k \beta^j - c^j k \beta^j) (\eta_{\,ae} G_{\,i}^{bd} \eta^{cd} + G_{\,i}^{ad} \eta_{\,b} e \eta^{cd})
$$

where we collect terms proportional to $\delta A_{\,f}^k$ and $\delta A_{\,f\,e}^k$ separately, noting that the gravitational solution reduces the $y$-covariant derivative to a partial, $\delta A_{\,b}^i = \delta A_{\,b}^i \cdot a$. 

45
While the field equations determine the second derivatives of the potentials, the potential itself and its first derivative are arbitrary initial conditions on any Cauchy surface. Therefore, the three variations \( \delta A^k_f, \delta A^k_{f,e} \) and \( \delta A^k_{f,e} \) are independent, and the coefficient of each must vanish separately:

\[
0 = \mathcal{H}_{i\ ac}\delta^c_b\eta^{cd} + \mathcal{H}_{i\ bc}\delta^e_a\eta^{ce} - \mathcal{H}_{i\ ac}\delta^d_b\eta^{ce} - \mathcal{H}_{i\ bc}\delta^d_a\eta^{ce} \tag{109}
\]

\[
0 = 3 (\eta_{ac}\mathcal{G}_{i\ bd} + \mathcal{G}_{i\ ad}\eta_{be}) \eta^{cd} \tag{110}
\]

\[
0 = -\omega_{de}^f + W_c\delta^i_d + \alpha^i_{kd}\delta^f_e \left( \mathcal{H}_{k\ ac}\delta^e_b\eta^{cd} + \mathcal{H}_{k\ bc}\delta^e_a\eta^{cd} - \mathcal{H}_{k\ ac}\delta^d_b\eta^{ce} - \mathcal{H}_{k\ bc}\delta^d_a\eta^{ce} \right) \\
+ (3\epsilon^i_{kj}B^j) \left( \eta_{ae}\mathcal{G}_{i\ bd}\eta^{fd} + \mathcal{G}_{i\ ad}\eta_{be}\eta^{fd} \right) \tag{111}
\]

For the \( x \)-derivative part of the constraint, Eq.\( \ref{109} \), we contract \( eb \) and lower the \( d \) index to show that \( \mathcal{H}_{i\ ac} \) must vanish,

\[
0 = n\mathcal{H}_{i\ ac}
\]

Similarly, contracting \( ac \) in Eq.\( \ref{110} \) expressing the independence of the \( y \)-derivative, shows that \( G_{i\ be} \) must also vanish.

\[
0 = 3G_{i\ be}
\]

With these two conditions, the final equation Eq.\( \ref{111} \) is identically satisfied.

These conditions satisfy both gravitational conditions on the sources, Eqs.\( \ref{82} \) and \( \ref{87} \).

### 7.2 The source for gravity

We have shown that

\[
\frac{1}{1 + \chi} \mathcal{R}_b + \Lambda_0\eta_{be}e^x = -\frac{\kappa}{\alpha}T_b
\]

Expressed in terms of the Einstein tensor this is

\[
G_{ab} + \Lambda G_{ab} = -(n - 2)(1 + \chi)\frac{\kappa}{\alpha} (T_{ab} - \eta_{ab}T)
\]
where

\[ T_{ab} = F^i_{ca} F_i db \eta^{cd} \]

The trace of this is well-known to be gauge dependent, and the conformal symmetry requires the energy tensor to be trace free. Therefore, we are justified in adjusting the SU\( (N) \) gauge to give

\[ G_{ab} + \Lambda C \eta_{ab} = -\lambda \left( F^i_{ca} F_i db \eta^{cd} - \frac{1}{4} \eta_{ab} \left( \eta^{ce} \eta^{df} F^i_{cd} F_i ef \right) \right) \]

where

\[ \lambda = (n - 2) (1 + \chi) \frac{\kappa}{\alpha} \]

### 7.3 The Yang-Mills equation

With \( H^k_{\; ab} = G^i_{\; a b} = 0 \), Eqs.\( 51 \) and \( 52 \) reduce to

\[ 0 = \eta^{bc} F^i_{k ac} - \eta^{bc} F^i_{ac} \beta^i_k^a \]
\[ 0 = \eta^{bc} (F^i_{ac} + F^i_{a c} \alpha^i_{jk}) \]

where

\[ \mathcal{A}^i_k = c^i_{jk} A^j_a e^a + c^i_{jk} B^j_a f^a_n \]
\[ \mathcal{A}^i_k = \alpha^i_k + \beta^i_k \]

We may use these results and the form of the co-torsion,

\[ S_a = 2 (1 + \chi) \Delta_{ef}^{bc} \eta_{ab} \Lambda \eta_{cd} e^d \land e^e \]

to solve for the potentials,

\[ F^i_{ab} = A^i_{b,ca} - A^i_{a;cb} - c^i_{jk} A^j_a A^k_b + (1 + \chi) \Lambda B^i_{ac} \left( \Delta_{ef}^{gf} y_g \eta_{fa} - \Delta_{ac}^{gf} y_g \eta_{fb} \right) \]
\[ 0 = A^i_{b,a} - A^i_{a;b} - c^i_{jk} B^j_a A^k_b \]
\[ 0 = B^i_{b,a} - B^i_{a;b} - c^i_{jk} B^j_a B^k_b \]  
\( \text{(112)} \)
The third equation is the vanishing of the Yang-Mills field strength on the $y$-submanifold,

$$d(y)B^i = -\frac{1}{2} c^i_{jk} B^j \wedge B^k$$

so that $B^k$ is a pure-gauge connection for any fixed $x^\alpha$. Therefore, for each $x^\alpha_0$ we may choose an $SU(N)$ gauge $\Lambda(x^\alpha_0, y_\beta)$ such that $B^k = 0$. But this makes the value of $B^k$ independent of $x^\alpha$ as well, so $B^k = 0$ everywhere. As a result, the fields in terms of the potentials reduce to

$$F_{i ab} = A_{i b a} - A_{i a b} - c^i_{jk} A^j a A^k b$$

$$A_{i b : a} = 0$$

Now, when we write the field equations in terms of the potentials and set $B^{k a} = 0$, we have

$$0 = F_{k ab} : a$$

$$0 = \eta^{bc} (F_{j ac b} + F_{i ac c} c^i_{jk} A^j a)$$

The first shows that $F_{k ab}$ is independent of $y_a$ and the second shows it to be covariantly divergence free.

8 Conclusions

Gravitational field theories in doubled dimensions include biconformal gravity [4, 24, 9, 13], double field theory [6, 7, 8, 5, 25, 27], and gravity on a Kähler manifold [10, 13]. Each of these cases starts as a fully $2n$-dimensional theory but ultimately is intended to describe gravity on an $n$-dimensional submanifold. We have found a satisfactory $2n$-dimensional form of Yang-Mills matter sources and shown that they also reduce to the expected $n$-dimensional sources as a consequence of the field equations. Our gravitational reduction and the consequent reduction of the Yang-Mills fields and field equations does not require a section condition.

While we discussed the issue in biconformal space, our results hold in the related forms of double field theory and Kähler manifolds [10, 13].

For matter fields we restrict our attention to gauged $SU(N)$ sources (Yang-Mills type). While we find that the usual form of $2n$-dimensional Yang-Mills action gives incorrect coupling to gravity, we find that including a “twist” matrix in the action corrects the problem.

For the gravitational fields we use the most general action linear in the biconformal curvatures. The
variation is taken with respect to all \( \frac{(n+1)(n+2)}{2} \) conformal gauge fields. In the absence of sources, the use of the gravitational field equations to reduce fully 2n-dimensional gravity theory to dependence only on the fields of n-dimensional gravity is well established. The field equations of torsion-free biconformal space restrict the \( \frac{1}{2} (n+1)(n+2) \) curvature components, each initially dependent on 2n independent coordinates, to the usual locally scale covariant Riemannian curvature tensor in n dimensions. Ultimately, the n-dim solder form determines all fields, up to coordinate and gauge transformations. Generic, torsion-free, vacuum solutions describe n-dimensional scale-covariant general relativity on the co-tangent bundle.

Here we have shown that the same reduction occurs when gauged SU(\( N \)) field strengths are included as matter sources. The result goes well beyond any previous work. With the exception of some general considerations and a scalar field example \[23\], studies of biconformal spaces \[24, 25, 26, 27, 28, 29, 30, 12, 10, 11, 13\] have considered the pure gravity biconformal spaces, leading to vacuum general relativity. With SU(\( N \)) Yang-Mills fields as gravitational sources, the central issue is to show that a completely general SU(\( N \)) gauge theory over a 2n-dimensional biconformal space does not disrupt the gravitational reduction to general relativity, but rather itself reduces to a suitable n-dim gravitational source.

As with the Riemann-Cartan construction of general relativity above, the development of biconformal spaces from group symmetry made it straightforward to include the additional symmetry of sources. By extending the quotient to

\[ M^{2n} = \left[ SO(p+1, q+1) \times SU(N) \right] / \left[ SO(p, q) \times SO(1, 1) \times SU(N) \right] \]

the local symmetry is enlarged by SU(\( N \)). We considered the effects of adding an SU(\( N \)) action to the gravitational action Eq.(7). As central results we successfully showed:

1. The number of fields \( \frac{2n(2n-1)}{2} \times (N^2 - 1) \) field components in 2n dimensions) reduces to the expected number \( \frac{n(n-1)}{2} (N^2 - 1) \) on n-dimensional spacetime.
2. The functional dependence of the fields reduces from 2n to 2 independent variables.
3. The usual form of Yang-Mills stress-energy tensor provides the source for the scale-covariant Einstein equation on n-dimensional spacetime.
4. The usual Yang-Mills field equation holds on the spacetime submanifold.

To accomplish these goals we required two interdependent intermediate steps:
1. We considered alternate forms of 2n-dimensional Yang-Mills action, finding that the usual action,

\[ S_{YM}^0 = \int tr (F \wedge^* F) \]

gives nonstandard coupling to gravity. Instead, including a “twist” matrix in the action

\[ S_{YM} = \int tr (\tilde{F} \wedge^* F) \]

with twisted form

\[ \tilde{F}_{AB} = \frac{1}{2} (K^C_A F_{CB} + F_{AC} K_B^C) \]

leads to both the usual n-dimensional Yang-Mills source to the Einstein tensor and the usual Yang-Mills equation for the SU(N) fields. A similar twist has been found in other double field theory studies in order to enforce supersymmetry. Here, the twist is required for the bosonic fields alone. Interestingly, the twist matrix \( K^A_B = \tilde{K}^{AC} g_{CB} \) makes use of both the Kähler and Killing forms, \( g_{AB} \) and \( K_{AB} \), respectively.

2. We considered two naturally occurring inner products for the orthonormal frame fields: the Kähler metric and the restriction to the base manifold of the Killing form. We showed the Kähler form cannot lead to the usual field equations while the variation of the Killing form in the twisted action gives usual Yang-Mills equations and usual coupling to gravity. Previous results in biconformal gravity did not require the inner product.

9 Appendices

Appendix A: Symmetry of the metric

Now the generic case, using explicit \( \eta \)s:

\[
K^A_B = \begin{pmatrix}
\bar{K}^{ac} & \bar{K}^a_c \eta^{ec} \\
\eta^{ac} \bar{K}^c_e & \eta^{ae} \eta^{ef} \bar{K}_{ef}
\end{pmatrix}
\begin{pmatrix}
\eta_{cb} & 0 \\
0 & \eta_{cm} \eta_{bn} \eta^{mn}
\end{pmatrix}
\]

\[
K_A^B = \begin{pmatrix}
\eta_{ad} \bar{K}^{db} & \bar{K}^d_a \\
\eta_{ad} \bar{K}^d_e \eta^{eb} & \bar{K}_{ae} \eta^{eb}
\end{pmatrix}
\]
Symmetry of $K_A^B$ is just the symmetry of $g_{AB}$:

\[
g_{AB} = g_{BA} \\
K_A^B = K^A_C g_{CB} = K^A_C g_{BC} \\
= g_{BC} K^C_A = K_B^A
\]

and

\[
K_B^A = g_{BC} K^C_A = \begin{pmatrix}
\eta_{bc} & 0 \\
0 & \eta_{bm} \eta_{cn} \eta^{mn}
\end{pmatrix}
\begin{pmatrix}
K^c_a & K^c_e \eta^{ea} \\
\eta^{ce} K^e_a & \eta^{ce} \eta^{af} K^e_f
\end{pmatrix}
\]

Check symmetry,

\[
\begin{pmatrix}
K_A^B \\
K^A_B
\end{pmatrix} = \begin{pmatrix}
K^a_{bc} & \eta^{ae} \tilde{K}^e_a \\
\eta^{ce} \tilde{K}^e_a & \eta^{ce} \eta^{af} \tilde{K}^e_f
\end{pmatrix} = \begin{pmatrix}
\eta_{bc} K^c_a & \eta_{bc} \tilde{K}^c_e \eta^{ea} \\
\tilde{K}^a_{bc} & \tilde{K}^a_{bc} \eta^{ca}
\end{pmatrix}
\]

Therefore,

\[
\begin{align*}
K^a_{bc} &= \eta_{bc} K^c_a \\
\tilde{K}^a_{bc} &= \eta_{bc} \tilde{K}^c_e \eta^{ea} \\
\eta^{ae} \tilde{K}^e_a \eta_{bc} &= \tilde{K}^a_{bc} \\
\eta^{ae} \tilde{K}^e_a \eta_{cd} &= \eta^{db} \tilde{K}^a_{bc} \\
\eta^{ae} \tilde{K}^e_{ab} &= \tilde{K}^a_{bc} \eta^{ca}
\end{align*}
\]

The first and last follow from the symmetry of $\eta_{ab}$, $K^a_{ab}$ and $K_{ab}$. But we also have symmetry of $K_{AB}$ itself:

\[
K^A_C = \begin{pmatrix}
\tilde{K}^{ac} & \tilde{K}^a_e \eta^{ec} \\
\eta^{ae} \tilde{K}^e_c & \eta^{ae} \eta^{af} \tilde{K}^e_f
\end{pmatrix}
\]
\[ [\bar{K} t]^{CA} = \begin{pmatrix} \bar{K}^{ca} & \eta^{ce} \bar{K}_e^a \\ \bar{K}_c^a \eta^{ca} & \eta^{ae} \eta^{cf} \bar{K}_e^f \end{pmatrix} \]

This shows that

\[ \bar{K}_a^a \eta^{ce} = \eta^{ce} \bar{K}_e^a \]

\[ \bar{K}_a^a \eta^{ce} \eta_{bc} = \eta_{bc} \eta^{ce} \bar{K}_e^a \]

\[ \bar{K}_{ab} = \bar{K}_b^a \]

Combine this with

\[ \eta^{ae} \bar{K}_e^c \eta_{cb} = \bar{K}_b^a \]

and we get

\[ \eta^{ae} \bar{K}_e^c \eta_{cb} = \bar{K}_{ab} \]

so that all forms are equivalent,

\[ \eta^{ae} \bar{K}_e^c \eta_{cb} = \bar{K}_b^a = K_a^a = \eta_{bc} \bar{K}_c^e \eta^{ea} \]

With this, we may write

\[ K^A_B = \begin{pmatrix} \bar{K}^{ac} \eta_{cb} & \bar{K}_b^a \\ \bar{K}_{ab} & \eta^{ae} \bar{K}_e^b \end{pmatrix} \]

\[ K_B^A = \begin{pmatrix} \eta_{bd} \bar{K}^{da} & \bar{K}_b^a \\ \bar{K}_b^a & \bar{K}_b^{ea} \end{pmatrix} = K_B^A \]

Appendix B: Details of the metric variation of the action

We have the dual field,

\[ {}^* \mathcal{F} = \frac{1}{n! (n-2)!} \left( \frac{1}{2} F_{ab} \bar{K}^{am} \bar{K}^{bn} + \mathcal{G}_{gb} \eta^{ga} \bar{K}_a^m \bar{K}^{bn} + \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \bar{K}_a^m \eta^{hb} \bar{K}_b^n \right) \varepsilon^{c\cdots d} m_{ne\cdots f} \mathcal{F}_{c\cdots d} \varepsilon^{e\cdots f} + \frac{(-1)^{n-1}}{(n-1)! (n-1)!} \left( \frac{1}{2} F_{ab} \bar{K}^{am} \bar{K}^{bn} + \mathcal{G}_{gb} \eta^{ga} \bar{K}_a^m \bar{K}^{bn} + \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \bar{K}_a^m \bar{K}_b^n \right) \varepsilon^{mc\cdots d} m_{ne\cdots f} \mathcal{F}_{c\cdots d} \varepsilon^{e\cdots f} \]

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We must wedge these together, then vary the metric.

\[
+ \frac{(-1)^n}{(n-1)!} \left( \frac{1}{2} F_{ab} K^{am} K^{b}_n + G_{gh} \eta^{ga} K^{m} K^{b}_n + \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} K^{m} K^{b}_n \right) \varepsilon^{nc\cdots d} m_{e\cdots f} e^{e\cdots f} + m_{e\cdots f} e^{e\cdots f} \\
+ \frac{1}{n! (n-2)!} \left( \frac{1}{2} F_{ab} \tilde{K}^{am} \tilde{K}^{b}_n + G_{gh} \eta^{ga} \tilde{K}^{m} \tilde{K}^{b}_n + \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \tilde{K}^{m} \tilde{K}^{b}_n \right) \varepsilon^{mnc\cdots d} e_{e\cdots f} e^{e\cdots f}
\]

and, changing indices to avoid duplication, the barred field,

\[
\tilde{F} = \frac{1}{2} \left( F_{rq} \tilde{K}^{q} \eta_{ts} + \tilde{K}_{r} \tilde{q} \mathcal{G}_{qs} \right) e^{e} \wedge e^{s} + \frac{1}{2} \left( F_{rc} \tilde{K}^{c} + \tilde{K}_{r} c \mathcal{H}_{cs} \right) \eta^{sw} e^{e} \wedge f_{w} + \frac{1}{2} \left( -G_{sq} \tilde{K}^{q} \eta_{tr} - G_{qr} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} e^{e} \wedge f_{w} + \frac{1}{2} \left( G_{rq} \tilde{K}^{q} + \mathcal{H}_{rq} \eta^{qt} \tilde{K}_{ts} \right) \eta^{sw} f_{w} \wedge f_{x}
\]

We must wedge these together, then vary the metric.

Wedgeing,

\[
\tilde{F} \wedge * \tilde{F} = \frac{1}{2} \frac{1}{n! (n-2)!} \left( F_{rq} \tilde{K}^{q} \eta_{ts} + \tilde{K}_{r} \tilde{q} \mathcal{G}_{qs} \right) \\
\times \left( G_{gh} \eta^{ga} \tilde{K}^{m} \tilde{K}^{b}_n + \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \tilde{K}^{m} \tilde{K}^{b}_n \right) \varepsilon^{nec\cdots d} m_{e\cdots f} e^{e\cdots f} \wedge f_{w} + \frac{1}{2} \left( F_{rc} \tilde{K}^{c} + \tilde{K}_{r} c \mathcal{H}_{cs} \right) \eta^{sw}.
\]

Now use the relation between the basis forms and the volume element, and the Kronecker reduction of pairs.
of Levi-Civita tensors,

\[
\varepsilon_{c \ldots d} \wedge \varepsilon^{c \ldots f} = \tilde{\varepsilon}_{c \ldots d} \varepsilon^{c \ldots f} \Phi
\]

\[
\varepsilon^{mnc \ldots d}_{\varepsilon \ldots fc_{pqc \ldots d}} \varepsilon^{c \ldots f} = n! (n-2)! \left( \delta^{m}_{p} \delta^{n}_{q} - \delta^{m}_{p} \delta^{n}_{q} \right)
\]

\[
\varepsilon^{mce \ldots d}_{\varepsilon \ldots fc_{pce \ldots d}} \varepsilon^{c \ldots f} \eta^{m} = (n-1)! (n-1)! \delta^{m}_{p} \eta^{n}
\]

to reduce the Lagrange density to

\[
\mathcal{F} \wedge * \mathcal{F} = \frac{1}{2} \left( F_{rq} K^{qt} \eta_{ts} + K_{r} q G_{qs} \right) \left( G_{gh} \eta^{ga} K_{a} m K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} K_{a} m \eta^{hb} K_{b} n \right) \left( \delta^{m}_{r} \delta^{n}_{s} - \delta^{n}_{s} \delta^{m}_{r} \right)
\]

\[
- \frac{1}{2} \left( F_{rc} K^{c s} + K_{r} c H_{cs} \right) \eta^{sw} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \delta^{m}_{w} \delta^{n}_{t} \Phi
\]

\[
+ \frac{1}{2} \left( F_{rc} K^{c s} + K_{r} c H_{cs} \right) \eta^{sw} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \delta^{m}_{w} \delta^{n}_{t} \Phi
\]

\[
- \frac{1}{2} \left( -G_{sq} K^{qt} \eta_{tr} - G_{qr} \eta^{qt} K_{ts} \right) \eta^{sw} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \delta^{m}_{w} \delta^{n}_{t} \Phi
\]

Then, absorbing the \( \eta_{ab} \) and \( \delta_{b}^{a} \) factors,

\[
\mathcal{F} \wedge * \mathcal{F} = \frac{1}{2} \left( F_{mq} K^{qt} \eta_{tm} - F_{nm} K^{qt} \eta_{tn} - K_{m} q G_{qn} - K_{n} q G_{qm} \right) \left( G_{gh} \eta^{ga} K_{a} m K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} K_{a} m \eta^{hb} K_{b} n \right) \Phi
\]

\[
- \frac{1}{2} \left( F_{mc} K^{c s} + K_{m} c H_{cs} \right) \eta^{sm} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \Phi
\]

\[
+ \frac{1}{2} \left( F_{mc} K^{c s} + K_{m} c H_{cs} \right) \eta^{sm} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \Phi
\]

\[
- \frac{1}{2} \left( -G_{sq} K^{qt} \eta_{tm} - G_{qm} \eta^{qt} K_{ts} \right) \eta^{sm} \left( \frac{1}{2} F_{ab} K^{a m} K^{bn} + \frac{1}{2} H_{gh} \eta^{ga} \eta^{hb} K_{am} K_{b n} \right) \Phi
\]

\[
+ \frac{1}{2} \left( -G_{rq} K^{q s} \eta^{m} \eta^{n} - G_{rq} K^{q s} \eta^{m} \eta^{n} + H_{rq} \eta^{q t} K_{ts} \eta^{m} \eta^{n} - H_{rq} \eta^{q t} K_{ts} \eta^{m} \eta^{n} \right)
\]

After varying the metric, the null-orthonormal form of the metric is restored, so we may anticipate this and drop terms in the product such as \( G_{gh} \eta^{ga} K_{a} m K^{bn} G_{sq} K^{qt} \eta_{tr} \) which will ultimately vanish. Terms with two or more factors of \( K^{ab} \) and/or \( K_{ab} \) will vanish, so we have only terms with one of these and two off diagonal
factors such as $\bar{K}_a^b$, or terms with three off-diagonal factors. In all cases where there is one factor of $\bar{K}^{ab}$ or $\bar{K}_{ab}$, it is this factor that must be varied so the off-diagonal factors may be replaced. For example, once the null orthonormal basis is restored, the only surviving term of the variation of

$$F_{mq} \eta_{ln} \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \bar{K}^{q\ell} \bar{K}_a^m \bar{K}_b^n$$

will be

$$F_{mq} \eta_{ln} \frac{1}{2} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \delta \bar{K}^{q\ell} \delta_a^m \delta_b^n = \frac{1}{2} \left( F_{aq} \eta_{lb} \mathcal{H}_{gh} \eta^{ga} \eta^{hb} \right) \delta \bar{K}^{q\ell}$$

Terms with three off-diagonal components of the metric must be retained until after variation.

Distributing fully, then making these reductions where possible, we collect terms

$$\mathcal{F} \wedge \star \mathcal{F} = \frac{1}{2} \left( \frac{1}{2} F_{ec} \eta^{cb} F_{db} - \frac{1}{2} F_{ec} \eta^{ca} F_{ad} + \frac{1}{2} F_{ad} \eta_{eb} H^{ab} + \frac{1}{2} H_{ec} \eta^{cb} F_{db} - \frac{1}{2} F_{bd} \eta_{ea} H^{ab} - \frac{1}{2} H_{ec} \eta^{ca} F_{ad} \right) \bar{K}^{de} \Phi + \frac{1}{2} \left( -G_{ed} G_{ca} \eta^{ca} + G_{ac} G_{ed} \eta^{ca} - \alpha_{ed} \alpha_{ca} \eta^{cb} G_{gb} \eta^{ga} \right) \bar{K}^{de} \Phi$$

$$+ \frac{1}{2} \left( -\frac{1}{2} F_{bc} \eta^{ce} H^{db} - H_{bc} \eta^{ce} \frac{1}{2} H^{db} + \frac{1}{2} \alpha_{ac} \eta^{ce} H^{ad} + \frac{1}{2} H_{ac} \eta^{ce} H^{ad} + \frac{1}{2} H^{ad} \eta^{cb} F_{ab} - \frac{1}{2} H^{bd} \eta^{ca} F_{ab} \right) \bar{K}_{de} \Phi$$

$$+ \frac{1}{2} \left( G_{rs} \eta^{re} \eta^{sb} G_{gb} \eta^{rd} - G_{rs} \eta^{rb} \eta^{sc} G_{rb} \eta^{gd} - G_{qs} \eta^{qd} \eta^{rb} G_{gb} \eta^{ga} \right) \bar{K}_{de} \Phi$$

$$+ \frac{1}{2} \left( \mathcal{H}_{qn} H^{ab} \delta_m^c - \frac{1}{2} \mathcal{H}_{qm} H^{ab} \delta_n^c + F_{mq} \eta^{cb} G_{gn} \eta^{ga} \right) \bar{K}^{q\ell} \bar{K}_a^m \bar{K}_b^n \Phi$$

$$+ \frac{1}{2} \left( \mathcal{H}_{qs} \eta^{sh} G_{gn} \eta^{ga} \delta_m^c + \frac{1}{2} \mathcal{H}_{rq} \eta^{ra} \eta^{cb} F_{mn} - \frac{1}{2} \mathcal{H}_{rq} \eta^{rb} \eta^{ca} F_{mn} \right) \bar{K}_a^m \bar{K}_b^n \bar{K}^{q\ell} \Phi$$

and consolidate using symmetries,

$$\mathcal{F} \wedge \star \mathcal{F} = \frac{1}{2} \left( F_{dc} \eta^{cb} F_{eb} + 2 F_{ad} H^{ab} \eta_{be} \right) \bar{K}^{de} \Phi$$

$$+ \frac{1}{2} \left( G_{ac} G_{de} \eta^{ca} - 2 G_{da} G_{cb} \eta^{ca} \right) \bar{K}^{de} \Phi$$

$$+ \frac{1}{2} \left( \mathcal{H}_{ab} H^{ad} \eta^{be} + 2 F_{ac} H^{ad} \eta^{ce} \right) \bar{K}_{de} \Phi$$

$$+ \frac{1}{2} \left( G_{ca} G_{db} \eta^{ab} - 2 G_{db} G_{cb} \eta^{ab} \right) \eta^{de} \eta^{ef} \bar{K}_{ef} \Phi$$

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\[
\frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K_a m K_b n K^d e \Phi \\
+ \frac{1}{2} \left( H_{ds} \eta^{ab} g_{gn} \eta^{ga} \delta_m + \frac{1}{2} g_{rd} \eta^a \eta^{cb} F_{mn} - \frac{1}{2} g_{rd} \eta^b \eta^{ca} F_{mn} \right) K^m a K^c b K^d e \Phi 
\]

Checking the limit in the orthonormal basis, we find the correct form, \( \tilde{\mathcal{F}} \wedge * \mathcal{F} = (g_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} g_{cd}) \Phi \).

Proceeding, we vary the metric

\[
\delta (\tilde{\mathcal{F}} \wedge * \mathcal{F}) = \frac{1}{2} \left( F_{dc} \eta^b F_{cb} + 2 F_{ad} H^{ab} \eta^{be} + (g_{ae} - 2 g_{ea}) g_{cd} \eta^{ca} \right) \delta K^{dc} \Phi \\
+ \frac{1}{2} K_b n K^d e \left( \frac{1}{2} g_{db} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_b + F_{md} \eta^b g_{gb} \eta^{ga} \right) \delta K_a m \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K_a m K_b n K^d e \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K_a m K_b n K^d e \Phi \\
+ \frac{1}{2} \left( H_{ds} \eta^{ab} g_{gn} \eta^{ga} \delta_m + \frac{1}{2} g_{rd} \eta^a \eta^{cb} F_{mn} - \frac{1}{2} g_{rd} \eta^b \eta^{ca} F_{mn} \right) K^m a \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( H_{ds} \eta^{ab} g_{gn} \eta^{ga} \delta_m + \frac{1}{2} g_{rd} \eta^a \eta^{cb} F_{mn} - \frac{1}{2} g_{rd} \eta^b \eta^{ca} F_{mn} \right) K^m a \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( H_{ad} H^{ad} \eta^{be} + 2 F_{ae} \eta^{ad} \eta^{be} \right) \delta K_{de} \Phi + \frac{1}{2} g_{ca} \eta^{ab} (g_{db} - 2 g_{bd}) \eta^{de} \eta^{cf} K_{ef} \Phi \\
+ (g_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} g_{cd}) \delta \Phi 
\]

At this point we may replace remaining unvaried metric components, leaving

\[
\delta (\tilde{\mathcal{F}} \wedge * \mathcal{F}) = \frac{1}{2} \left( F_{dc} \eta^b F_{cb} + 2 F_{ad} H^{ab} \eta^{be} + (g_{ae} - 2 g_{ea}) g_{cd} \eta^{ca} \right) \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{db} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_b + F_{md} \eta^b g_{gb} \eta^{ga} \right) \delta K_a m \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K_a m \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K_a m \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( H_{ds} \eta^{ab} g_{gn} \eta^{ga} \delta_m + \frac{1}{2} g_{rd} \eta^a \eta^{cb} F_{mn} - \frac{1}{2} g_{rd} \eta^b \eta^{ca} F_{mn} \right) K^m a \delta K_{de} \Phi \\
+ \frac{1}{2} \left( H_{ds} \eta^{ab} g_{gn} \eta^{ga} \delta_m + \frac{1}{2} g_{rd} \eta^a \eta^{cb} F_{mn} - \frac{1}{2} g_{rd} \eta^b \eta^{ca} F_{mn} \right) K^m a \delta K_{de} \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{db} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_b + F_{md} \eta^b g_{gb} \eta^{ga} \right) K^m a \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( \frac{1}{2} g_{dn} H^{ab} \delta_m - \frac{1}{2} g_{dm} H^{ab} \delta_n + F_{md} \eta^b g_{gn} \eta^{ga} \right) K^m a \delta K^{dc} \Phi \\
+ \frac{1}{2} \left( H_{ad} H^{ad} \eta^{be} + 2 F_{ae} \eta^{ad} \eta^{be} \right) \delta K_{de} \Phi + \frac{1}{2} g_{ca} \eta^{ab} (g_{db} - 2 g_{bd}) \eta^{de} \eta^{cf} K_{ef} \Phi \\
+ (g_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} g_{cd}) \delta \Phi 
\]

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Finally, we collect all terms by type of variation,

\[
\delta (\mathcal{F} \wedge \ast \mathcal{F}) = \frac{1}{2} (F_{de} \eta^{eb} F_{eb} + 2 F_{ad} H^{ab} \eta_{be} + (G_{ae} - 2 G_{ea}) G_{ca} \eta^{ca}) \delta K^{de} \Phi \\
+ \frac{1}{2} (H^{ac} \eta^{af} + 2 F_{ac} H^{ac} \eta^{cf}) + G_{ca} \eta^{ab} (G_{db} - 2 G_{bd}) \eta^{de} \eta^{cf} \bar{K}_{ef} \Phi \\
+ (G_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} G_{cd}) \delta \Phi
\]

This allows further simplifications, then including the variation of the volume form we have the final result,

\[
\delta (\mathcal{F} \wedge \ast \mathcal{F}) = \frac{1}{2} (F_{de} \eta^{eb} F_{eb} + 2 F_{ad} H^{ab} \eta_{be} + (G_{ae} - 2 G_{ea}) G_{ca} \eta^{ca}) \delta K^{de} \Phi \\
+ H^{ab} (2 G_{ma} - G_{am}) \delta K^{m} \Phi \\
+ F_{ma} (G^{na} - 2 G^{an}) \delta K^{m} \Phi \\
+ \frac{1}{2} (H^{ac} \eta^{af} + 2 F_{ac} H^{ac} \eta^{cf}) + G_{ca} \eta^{ab} (G_{db} - 2 G_{bd}) \eta^{de} \eta^{cf} \bar{K}_{ef} \Phi \\
+ (G_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} G_{cd}) \delta \Phi
\]

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