S-SHAPED AND BROKEN S-SHAPED BIFURCATION CURVES 
FOR A MULTIPARAMETER DIFFUSIVE LOGISTIC PROBLEM 
WITH HOLLING TYPE-III FUNCTIONAL RESPONSE

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Abstract. We study exact multiplicity and bifurcation curves of positive solutions for a multiparameter diffusive logistic problem with Holling type-III functional response
\[
\begin{cases}
  u''(x) + \lambda \left[ ru(1 - \frac{u}{q}) - \frac{u^p}{1+up} \right] = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\]
where \( u \) is the population density of the species, \( p > 1, q, r \) are two positive dimensionless parameters, and \( \lambda > 0 \) is a bifurcation parameter. For fixed \( p > 1 \), assume that \( q, r \) satisfy one of the following conditions: (i) \( r \leq \eta_{1,p}^* q \) and \((q, r)\) lies above the curve 
\[
\Gamma_1 = \left\{ (q, r) : a \left( \frac{2a^p - (p-2)}{a^p - (p-1)} \right), \quad r(a) = \frac{a^{p-1} [2a^p - (p-2)]}{(ap+1)^2} \right\}, \quad \forall p - 1 < a < C^*_p \},
\]
(ii) \( r \leq \eta_{2,p}^* q \) and \((q, r)\) lies on or below the curve \( \Gamma_1 \), where \( \eta_{1,p}^* \) and \( \eta_{2,p}^* \) are two positive constants, and \( C^*_p = \left( \frac{2^p + 3p - 4 + p\sqrt{p^2 + 6p - 7}}{(p+1)^{1/p}} \right)^{1/p} \). Then on the \((\lambda, \|u\|_\infty)\)-plane, we give a classification of three qualitatively different bifurcation curves: an S-shaped curve, a broken S-shaped curve, and a monotone increasing curve. Hence we are able to determine the exact multiplicity of positive solutions by the values of \( q, r \) and \( \lambda \).

1. Introduction. We study exact multiplicity and bifurcation curves of positive solutions for a multiparameter diffusive logistic problem with Holling type-III functional response
\[
\begin{cases}
  u''(x) + \lambda f(u) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\]
where \( u \) is the population density of the species, \( f(u) = ug(u) \) is the growth rate,
\[
g(u) = r \left( 1 - \frac{u}{q} \right) - \frac{u^{p-1}}{1+up},
\]

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is the growth rate per capita, $p > 1$, $q$, $r$ are two positive dimensionless parameters, and $\lambda > 0$ is a bifurcation parameter. For $p = 2$, problem (1) is a famous budworm problem in mathematical biology. For this budworm problem, roughly speaking, $r$ measures the foliage density while $q$ depends upon the properties of the budworm and the predators, but not upon forest conditions, see Ludwig et al. [6, p. 218]. Problem (1) with $p = 2$ has been extensively studied by many authors, see e.g. Ludwig et al. [6], Ludwig, et al. [7], Murray [9, 10], Wang and Yeh [16], Jiang and Shi [2], Shi and Shivaji [13], and Lee et al. [5].

Noy-Meir [11] studied a grazing system of herbivore-plant interaction, see also May [8]. He considered the differential equation
\[ \frac{dN}{dT} = G(N) - Hc(N), \]
where $N(T)$ is the vegetation biomass, $G(N)$ is the growth rate of vegetation in absence of grazing, $H$ is the herbivore population density, and $c(N)$ is the per capita consumption rate of vegetation by the herbivore. For problem (3), if $G(N)$ is given by the logistic function, and $c(N)$ is the Holling type III function, then (3) takes the form
\[ \frac{dN}{dT} = r_N N \left( 1 - \frac{N}{K_N} \right) - B \frac{N^p}{A^p + N^p}, \]
where $p > 1$ and $A$, $B$, $r_N$, $K_N > 0$, cf. [2, p. 37]. The Holling type III functional response was also considered in Sugie et al. [15] and Sugie and Katagama [14]. They studied the existence of stable limit cycle and global asymptotic stability for a predator-prey system
\[
\begin{align*}
\frac{dx}{dt} &= rx \left( 1 - \frac{x}{K} \right) - \frac{x^p y}{A^p + x^p}, \\
\frac{dy}{dt} &= y \left( \frac{\mu x^p}{A^p + x^p} - d \right).
\end{align*}
\]
In addition, the Holling type III functional response has also appeared in the dynamics of lake eutrophication
\[ \frac{dN}{dT} = a - bN + B \frac{N^p}{A^p + N^p}, \]
where $N(T)$ is the level of nutrients suspended in phytoplankton causing turbidity, $a$ is the nutrient loading, $b$ is the nutrient removal rate, and $B$ is the rate of internal nutrient recycling, see Carpenter et al. [1] and Scheffer et al. [12].

Adding a diffusion term to (4), we consider a reaction-diffusion model governed by the equation
\[ \frac{\partial N}{\partial T} = D \frac{\partial^2 N}{\partial X^2} + r_N N \left( 1 - \frac{N}{K_N} \right) - B \frac{N^p}{A^p + N^p}, \]
in spatial one dimension, where $D > 0$ is the diffusion coefficient. Under some assumptions and by using some transformations, we convert (5) into
\[ \begin{align*}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \lambda \left[ rv \left( 1 - \frac{v}{q} \right) - \frac{v^p}{1 + v^p} \right], \quad -1 < x < 1, \; t > 0, \\
v(-1, t) = v(1, t) = 0, \; t > 0,
\end{align*} \]
where $p > 1$ and $q$, $r$, $\lambda > 0$, see Wang and Yeh [16, p. 815] for $p = 2$. Let $u(x)$ denote a positive steady-state population density of (6). Then $u(x)$ satisfies problem (1).
For $p = 2$, problem (1) takes the form

$$\begin{cases}
u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^2}{1+u^2} \right] = 0, & -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases}$$

Applying the quadrature method (time-map method), Ludwig et al. [6] showed that the rough bifurcation curve goes from a monotone curve with a unique small steady state, to a broken S-shaped curve, to an S-shaped curve, and finally a monotone curve with a unique large steady state, when $r$ increases from $0^+$ to a large value. Note that the results of evolutionary bifurcation curves in Ludwig et al. [6] are not exact, and it was only shown that the equation has at least three positive solutions but not exactly three. Recently, Wang and Yeh [16, Theorems 2.1–2.3] gave a partial answer of this conjecture in Ludwig et al. [6]. Assume that either $r \leq \rho_1 q$ and $(q, r)$ lies above the curve

$$\Gamma = \left\{(q, r) : q(a) = \frac{2a^3}{a^2 - 1}, r(a) = \frac{2a^3}{(a^2 + 1)^2}, 1 < a < \sqrt{3} \right\},$$
or $r \leq \rho_2 q$ for some constants $\rho_1 \approx 0.0939$ and $\rho_2 \approx 0.0766$. Then on the $(\lambda, ||u||_\infty)$-plane, they gave a classification of three qualitatively different bifurcation curves: an S-shaped curve, a broken S-shaped curve, and a monotone increasing curve. Their results settled rigorously a long-standing open problem in Ludwig et al. [6].

Note that, for $p = 1$, the function $h(u) = u/(1 + u)$ is called a Holling type-II function. A $n$-dimensional Dirichlet problem of (1) with $p = 1$ was considered by Korman and Shi [3]. They obtained two qualitatively different bifurcation curves: a $\subset$-shaped curve and a monotone increasing curve, see [3, Theorem 3.1].

For problem (1) with fixed $p > 1$, in Fig. 1, we divide the first quadrant of $(q, r)$-parameter plane into the disjoint union of the three curves $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ and four regions $R_1$, $R_2$, $R_3$, $R_4$ defined as follows:

$$\Gamma_0 = \{(q, r) : r = m_p q > 0 \},$$

$$\Gamma_1 = \{(q, r) : q(a) = \frac{a[2ap - (p-2)]}{ap - (p-1)}, r(a) = \frac{a^{p-1}[2ap - (p-2)]}{(ap+1)^2}, \sqrt{p-1} < a < C_p^* \},$$

$$\Gamma_2 = \{(q, r) : q(a) = \frac{a[2ap - (p-2)]}{ap - (p-1)}, r(a) = \frac{a^{p-1}[2ap - (p-2)]}{(ap+1)^2}, C_p^* < a < \infty \},$$

and

$$R_1 = \{(q, r) : 0 < r < m_p q \text{ and } (q, r) \text{ lies above the curve } \Gamma_1 \},$$

$$R_2 = \{(q, r) : 0 < r < m_p q \},$$

$$R_3 = \{(q, r) : 0 < r < m_p q \text{ and } (q, r) \text{ lies between curves } \Gamma_1 \text{ and } \Gamma_2 \},$$

$$R_4 = \{(q, r) : r > m_p q > 0 \},$$

where

$$C_p^* = \left( \frac{p^2 + 3p - 4 + p\sqrt{p^2 + 6p - 7}}{4} \right)^{1/p} > \sqrt{p-1} > 0,$$

and

$$m_p = \frac{(C_p^*)^{p-2}[1 - p + (C_p^*)^p]}{[1 + (C_p^*)^p]^2} > 0.$$
Figure 1. Classified graphs of growth rate per capita \( g(u) = r(1 - \frac{u}{q}) - \frac{u^{p-1}}{1+u^p} \) on \((0, \infty)\) with fixed \( p > 1 \), drawn on the first quadrant of \((q, r)\)-parameter plane according to the monotonicity of \( g(u) \).

(Numerical simulations show that \( C_p^* \in (0, 2) \) and \( m_p \in (0, 1) \) for \( p > 1 \).) It is easy to show that curves \( \Gamma_1 \) and \( \Gamma_2 \) are continuous and strictly decreasing on the \((q, r)\)-plane, and the curve of the set \( \text{Cl}(\Gamma_1 \cup \Gamma_2) \) (the closure of \( \Gamma_1 \cup \Gamma_2 \)) has a cusp point \( P^* \in \text{Cl}(\Gamma_1 \cup \Gamma_2) \cap \Gamma_0 \) when setting \( a = C_p^* \) in (7) and (8). We thus write on curves \( \Gamma_1 \) and \( \Gamma_2 \) respectively, the functions \( r_1(q) \) and \( r_2(q) \) with \((q, r_i(q)) \in \Gamma_i, i = 1, 2 \) for \( q > q^*(p) \), which satisfy \( r_1(q) > r_2(q) \) for \( q > q^*(p) \), where

\[
q^*(p) = \frac{2 \left( p + 1 + \sqrt{p^2 + 6p - 7} \right)}{p - 1 + \sqrt{p^2 + 6p - 7}} \left( \frac{p^2 + 3p - 4 + p\sqrt{p^2 + 6p - 7}}{4} \right)^{1/p}.
\]

Note that, for \((q(a), r(a)) \in \Gamma_1\),

\[
\lim_{a \to (\frac{q^*}{p-1})^+} q(a) = \infty \quad \text{and} \quad \lim_{a \to (\frac{q^*}{p-1})^+} r(a) = \frac{1}{p} \left( p - 1 \right)^{\frac{p-1}{p-1}},
\]

and for \((q(a), r(a)) \in \Gamma_2\),

\[
\lim_{a \to \infty} q(a) = \infty \quad \text{and} \quad \lim_{a \to \infty} r(a) = 0.
\]

We define the bifurcation curve of (1)

\[
\tilde{S} = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1)} \}.
\]
In this paper we mainly study exact multiplicity of positive solutions and shapes of bifurcation curves $S$ of (1) for parameters $q, r > 0$ and $p > 1$. We first determine numbers of positive zeros of $g(u) = r(1 - \frac{u}{q}) - \frac{u^{p-1}}{1+u^p}$. We then give a classification of growth rate per capita $g(u)$ on the first quadrant of $(q, r)$-parameter plane according to the monotonicity of $g(u)$.

For each fixed $p > 1$, it is easy to see that $g(0) = r > 0$ and
\[
\begin{align*}
\lim_{u \to 0^+} g'(u) &= -\infty, \quad \text{if} \quad 1 < p < 2, \\
g'(0) &= -\frac{r}{q} - 1, \quad \text{if} \quad p = 2, \\
g'(0) &= -\frac{r}{q}, \quad \text{if} \quad p > 2,
\end{align*}
\]
and $\lim_{u \to \infty} g(u) = -\infty$. In addition $g(u)$ has at most three positive zeros and has at least one positive zero. According to Jiang and Shi [2, pp. 39–40], we classify all growth rate patterns according to the monotonicity of the growth rate per capita $g(u)$ on $[0, \infty)$ in (2). We have that:

(i) If $(q, r) \in R_4 \cup \Gamma_0$, then $g$ is of logistic type. We have that $g(0) = r > 0$ and $g'(u) < 0$ on $(0, \beta_1)$ except possibly at some value $\beta_0 \in (0, \beta_1)$ when $(q, r) \in \Gamma_0$, and $g(\beta_1) = 0$ and $g(u) < 0$ on $\beta_1$, see Fig. 1. Thus the bifurcation curve $S$ of (1) is a monotone increasing curve since $f(u) - uf'(u) = -u^2g'(u) > 0$ on $(0, \beta_1)$ except possibly at some value $\beta_0 \in (0, \beta_1)$ when $(q, r) \in \Gamma_0$; we omit the details of the proof.

(ii) If $(q, r) \in R_3 \cup \Gamma_2$, then $g$ is of weak hysteresis type. Notice that:

(a) In $R_3$, we have that $g(0) = r > 0$, $g'(u) < 0$ on $(0, \beta_1)$, $g(\beta_1) = 0$, and $g(u) < 0$ on $(\beta_1, \infty)$, see Fig. 1. Thus the bifurcation curve $S$ of (1) is a monotone increasing curve since $f(u) - uf'(u) = -u^2g'(u) > 0$ on $(0, \beta_1)$; we omit the details of the proof.

(b) In $\Gamma_2$, we have that $g(0) = r > 0$, $g'(u) < 0$ on $(0, \beta_1)$, $g(\beta_1) = g(\beta_2) = 0$, and $g(u) < 0$ on $(\beta_1, \beta_2) \cup (\beta_3, \infty)$, see Fig. 1. Thus the bifurcation curve $S$ of (1) is a monotone increasing curve since $f(u) - uf'(u) = -u^2g'(u) > 0$ on $(0, \beta_1)$; we omit the details of the proof.

(iii) If $(q, r) \in R_2 \cup \Gamma_1$, then $g$ is of hysteresis type. (In particular, $g$ is of strong hysteresis type if $(q, r) \in R_2$). Notice that:

(a) In $R_2$, we have that $g(0) = r > 0$, $g(\beta_1) = g(\beta_2) = g(\beta_3) = 0$, $g(u) > 0$ on $(0, \beta_1 \cup (\beta_2, \beta_3)$ and $g(u) < 0$ on $(\beta_1, \beta_2) \cup (\beta_3, \infty)$, see Fig. 1. In addition, for each fixed $q > q'(p)$, it is clear that $\beta_1$ and $\beta_3$ are differentiable.

Figure 2. (a) S-shaped bifurcation curve $S$ of (1). (b)–(c) Broken S-shaped bifurcation curves $\bar{S}$ of (1).
functions of \( r \). Hence we obtain that, for each fixed \( q > q^*(p) \),

\[
\frac{d}{dr} \int_{\beta_1}^{\beta_2} f(u)du = \frac{1}{6q} [(\beta_3)^3 - (\beta_1)^3] + \frac{1}{2r} \left[ \frac{(\beta_3)^{p+1}}{1 + (\beta_3)^p} - \frac{(\beta_1)^{p+1}}{1 + (\beta_1)^p} \right] > 0
\]

by \( f(\beta_1) = f(\beta_2) = 0 \). In addition,

\[
\int_{\beta_1}^{\beta_2} f(u)du < 0 \text{ for } r \text{ near } r_2(q)^+ \text{ and } \int_{\beta_1}^{\beta_2} f(u)du > 0 \text{ for } r \text{ near } r_1(q)^-,
\]

where the functions \( r_1(q) \) and \( r_2(q) \) with \( (q, r_i(q)) \in \Gamma_i, i = 1, 2 \) for \( q > q^*(p) \). It follows that, for each fixed \( q > q^*(p) \), there exists \( \bar{r}_2 = \bar{r}_2(q) \in (r_2(q), r_1(q)) \) such that

\[
\int_{\beta_1}^{\beta_2} f(u)du < 0 \text{ for } r_2(q) < r < \bar{r}_2(q),
\]

\[
\int_{\beta_1}^{\beta_2} f(u)du = 0 \text{ for } r = \bar{r}_2(q),
\]

\[
\int_{\beta_1}^{\beta_2} f(u)du > 0 \text{ for } \bar{r}_2(q) < r < r_1(q).
\]

(Note that \( \bar{r}_2(q) \) is a continuous function of \( q \) on \( (q^*(p), \infty) \).) Notice that \( \int_{\beta_1}^{\beta_2} f(u)du > 0 \) for \( \bar{r}_2(q) < r < r_1(q) \), then there exists a number \( \gamma \in (\bar{r}_2, \beta_3) \) such that \( \int_{\beta_1}^{\beta_2} f(u)du = 0 \) since \( f(u) < 0 \) on \((\beta_1, \beta_2)\) and \( f(u) > 0 \) on \((\beta_2, \beta_3)\). We thus define the curve

\[
\bar{\Gamma}_2 = \{(q, r) : q > q^*(p) \text{ and } r = \bar{r}_2(q)\}
\]

and regions

\[
\bar{R}_2 = \{(q, r) : 0 < r < m_p q, \text{ and } (q, r) \text{ lies between curves } \Gamma_1 \text{ and } \bar{\Gamma}_2 \},
\]

\[
\bar{R}_2 = \{(q, r) : 0 < r < m_p q, \text{ and } (q, r) \text{ lies on curve } \bar{\Gamma}_2 \text{ or between curves } \Gamma_2 \text{ and } \bar{\Gamma}_2 \}.
\]

(b) In \( \Gamma_1 \), we have that \( g(0) = r > 0, g(\beta_1) = g(\beta_3) = 0, g(u) > 0 \) on \((0, \beta_1) \cup (\beta_1, \beta_3)\) and \( g(u) < 0 \) on \((\beta_3, \infty)\), see Fig. 1. We define \( \gamma = \beta_1 \) in this subcase.

In Theorems 2.2 and 2.3 stated below we prove that the bifurcation curve \( \bar{S} \) of (1) is a broken S-shaped curve on the \((\lambda, ||u||_\infty)-\text{plane}\) when \((q, r) \in \bar{R}_2 \cup \Gamma_1 \) and \( r \leq \eta^*_2 \rho q \) for some positive constant \( \eta^*_2 \rho q \), see Fig. 2(b)–(c). In addition, it is easy to prove that \((q, r) \in \bar{R}_2 \), the bifurcation curve \( \bar{S} \) of (1) is a monotone increasing curve since \( f(u) - uf'(u) = -u^2g'(u) > 0 \) on \((0, \beta_1)\) and \( \int_{\beta_1}^{\beta_2} f(u)du \leq 0 \); we omit the details of the proof.

(iv) If \((q, r) \in R_1 \), then \( q \) is of weak hysteresis type. We have that \( g(0) = r > 0, g(u) \) is a decreasing-increasing-decreasing function on \([0, \beta_3)\), \( g(\beta_3) = 0, g(u) > 0 \) on \((0, \beta_3)\) and \( g(u) < 0 \) on \((\beta_3, \infty)\), see Fig. 1. In Theorem 2.1 stated below we prove that the bifurcation curve \( \bar{S} \) of (1) is an S-shaped curve on the \((\lambda, ||u||_\infty)-\text{plane}\) when \((q, r) \in R_1 \) and \( r \leq \eta^*_1 \rho q \) for some positive constant \( \eta^*_1 \rho q \), see Fig. 2(a).

The paper is organized as follows. Section 2 contains statements of the main results: Theorems 2.1–2.3. Section 3 contains lemmas needed to prove Theorems 2.1–2.3 and the proofs of main results.
2. **Main results.** The main results in this paper are Theorems 2.1–2.3. We first define two positive numbers \( \eta_{1,p} \) and \( \eta_{2,p} \) for \( p > 1 \) as follows:

(i) Let \( \eta_{1,p} \) be the unique positive intersection value of the two curves \( \eta = I(u) \) and \( \eta = K(u) \) for \( u > 0 \),

\[
I(u) = \frac{pu^{p-2}[-(p-1)+(p+1)u^p]}{2(1+u^p)^3} \tag{11}
\]

and

\[
K(u) = \frac{4}{u^4} \left[ \frac{u^2}{2} + \frac{u^2}{1+u^p} - 3 \int_0^u \frac{t}{1+t^p} \, dt \right]. \tag{12}
\]

See Figs. 3 and 5.

(ii) Let \( \eta_{2,p} \) be the unique positive intersection value of the two curves \( \eta = I(u) \) and \( \eta = M(u) \) for \( u > 0 \),

\[
M(u) = \frac{3}{u^3} \left[ u + \frac{u}{1+u^p} - 2 \int_0^u \frac{1}{1+t^p} \, dt \right]. \tag{14}
\]

See Figs. 3 and 6.

**Remark 1.** We know that for \( p > 1 \),

\[ \eta_{1,p} < J(C_p^*) = m_p \] and \( \eta_{2,p} < J(C_p^*) = m_p \)

by (43) and (56) stated below, where

\[
J(u) \equiv \frac{u^{p-2}(1-p+u^p)}{(1+u^p)^2}. \tag{15}
\]

See Fig. 3.
For \( p > 1 \) and \( 0 < \eta < m_p \), let \( B_{1,p}(\eta) \) be the smallest positive root of \( I(u) = \eta \) and \( C_{2,p}(\eta) \) be the largest positive root of \( J(u) = \eta \); see the proof of Lemma 3.2 in Section 3 below. We also define two positive numbers \( \eta_{1,p}^* \) and \( \eta_{2,p}^* \) for \( p > 1 \) as follows:

(i) Let

\[
\eta_{1,p}^* = \sup \{ \eta : 0 < \eta \leq \eta_{1,p} \text{ and } N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta)) > 0 \},
\]

where

\[
N_1(u) \equiv \frac{u^2[6 + (p - 2 - 5p + 12u^p)]}{2(1 + u^p)^2} - 6 \int_0^u \frac{t}{1 + t^p} dt,
\]

and

\[
N_2(u) \equiv \frac{u^2(6 + (p + 7u^p + 12u^p))}{2(1 + u^p)^2} - 6 \int_0^u \frac{t}{1 + t^p} dt,
\]

see Lemma 3.2.

(ii) Let

\[
\eta_{2,p}^* = \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \},
\]

where

\[
N_3(u) \equiv \frac{u[6 + (p - 2 - 5p + 12u^p)]}{3(1 + u^p)^3} + 2 \int_0^u \frac{t}{1 + t^p} dt,
\]

and

\[
N_4(u) \equiv \frac{u(6 + (p + 7u^p + 12u^p))}{3(1 + u^p)^2} - 2 \int_0^u \frac{t}{1 + t^p} dt,
\]

see Lemma 3.6.

Let \( u_\alpha \) be a positive solution of (1) with \( \alpha \equiv \|u_\alpha\|_\infty > 0 \).

**Theorem 2.1** (See Figs. 1 and 2(a)). Consider (1) with \( p > 1 \). If \( (q, r) \in R_1 \) and \( r \leq \eta_{1,p}^*q \), then

\[
\lim_{\alpha \to 0^+} \lambda(\alpha) = \hat{\lambda} \equiv \frac{\pi^2}{4p}, \quad \lim_{\alpha \to \hat{\lambda}^-} \lambda(\alpha) = \infty,
\]

and the bifurcation curve \( \bar{S} \) is an \( S \)-shaped curve on the \( (\lambda, \|u_\alpha\|_\infty) \)-plane. More precisely, \( \bar{S} \) consists of a continuous curve with exactly two turning points at some points \((\lambda_1, \|u_{\lambda_1}\|_\infty)\) and \((\lambda_* \equiv \lambda_{\hat{\lambda}}, \|u_{\lambda_*}\|_\infty)\) such that \( \hat{\lambda} \leq \lambda_* < \lambda^* < \infty \) and \( 0 < \|u_{\lambda_*}\|_\infty < \|u_{\lambda_*}\|_\infty < \beta_3 \). Problem (1) has:

(i) exactly three positive solutions \( w_\lambda, u_\lambda, v_\lambda \) with \( w_\lambda < u_\lambda < v_\lambda \) for \( \lambda < \lambda_* < \lambda^* \),

(ii) exactly two positive solutions \( w_\lambda, u_\lambda \) with \( w_\lambda < u_\lambda \) for \( \lambda = \lambda_* \) and exactly two positive solutions \( u_\lambda, v_\lambda \) with \( u_\lambda < v_\lambda \) for \( \lambda = \lambda^* \),

(iii) exactly one positive solution \( w_\lambda \) for \( \hat{\lambda} < \lambda < \lambda_* \) and exactly one positive solution \( v_\lambda \) for \( \lambda > \lambda_* \),

(iv) no positive solution for \( 0 < \lambda \leq \hat{\lambda} \).

Furthermore,

\[
\lim_{\lambda \to (\hat{\lambda})^+} \|w_\lambda\|_\infty = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta_3.
\]

In the next theorem for \( (q, r) \in \hat{R}_2 \) and \( r \leq \eta_{2,p}^*q \), we recall the number \( \gamma \in (\beta_2, \beta_3) \) satisfying \( \int_{\beta_1}^{\beta_3} f(u) du = 0 \).
Theorem 2.2 (See Figs. 1 and 2(c)). Consider (1) with \( p > 1 \). If \((q, r) \in \hat{R}_2 \) and \( r \leq \eta^*_2, q \), then

\[
\lim_{\alpha \to 0^+} \lambda(\alpha) = \lambda = \frac{\pi^2}{4r}, \quad \lim_{\alpha \to \beta_4^-} \lambda(\alpha) = \lim_{\alpha \to \gamma^+} \lambda(\alpha) = \lim_{\alpha \to \beta_4^+} \lambda(\alpha) = \infty,
\]

and the bifurcation curve \( S \) is a broken S-shaped curve on the \((\lambda, ||u||_\infty^\gamma)\)-plane. More precisely, \( S \) has two disjoint connected components such that the upper branch of \( S \) has exactly one turning point \((\lambda_*, ||u_{\lambda_*}||_\infty^\gamma)\), with \( \hat{\lambda} < \lambda_* < \infty \) and \( \gamma < ||u_{\lambda_*}||_\infty < \beta_3 \), where the curve turns to the right, and the lower branch of \( S \) is a monotone increasing curve starting at \((\hat{\lambda}, 0)\). Problem (1) has:

(i) exactly three positive solutions \( w_{\lambda}, u_\lambda, v_\lambda \) with \( w_\lambda < u_\lambda < v_\lambda \) for \( \lambda > \lambda_* \),
(ii) exactly two positive solutions \( w_\lambda, u_\lambda \) with \( w_\lambda < u_\lambda \) for \( \lambda = \lambda_* \),
(iii) exactly one positive solution \( w_\lambda \) for \( \hat{\lambda} < \lambda < \lambda_* \),
(iv) no positive solution for \( 0 < \lambda \leq \hat{\lambda} \).

Furthermore,

\[
\lim_{\lambda \to (\lambda_*)^+} ||w_\lambda||_\infty = 0, \quad \lim_{\lambda \to \infty} ||w_\lambda||_\infty = \beta_1, \quad \lim_{\lambda \to \infty} ||u_\lambda||_\infty = \gamma, \quad \text{and} \quad \lim_{\lambda \to \infty} ||v_\lambda||_\infty = \beta_3.
\]

Theorem 2.3 (See Figs. 1 and 2(b)). Consider (1) with \( p > 1 \). If \((q, r) \in \Gamma_1 \) and \( r \leq \eta^*_1, q \), then

\[
\lim_{\alpha \to 0^+} \lambda(\alpha) = \lambda = \frac{\pi^2}{4p}, \quad \lim_{\alpha \to \beta_4^-} \lambda(\alpha) = \lim_{\alpha \to \gamma^+} \lambda(\alpha) = \lim_{\alpha \to \beta_4^+} \lambda(\alpha) = \infty,
\]

and the bifurcation curve \( S \) is a broken S-shaped curve on the \((\lambda, ||u||_\infty^\gamma)\)-plane. More precisely, \( S \) has two disjoint connected components such that the upper branch of \( S \) has exactly one turning point \((\lambda_*, ||u_{\lambda_*}||_\infty^\gamma)\), with \( \hat{\lambda} < \lambda_* < \infty \) and \( \beta_1 < ||u_{\lambda_*}||_\infty < \beta_3 \), where the curve turns to the right, and the lower branch of \( S \) is a monotone increasing curve starting at \((\hat{\lambda}, 0)\). Problem (1) has:

(i) exactly three positive solutions \( w_{\lambda}, u_\lambda, v_\lambda \) with \( w_\lambda < u_\lambda < v_\lambda \) for \( \lambda > \lambda_* \),
(ii) exactly two positive solutions \( w_\lambda, u_\lambda \) with \( w_\lambda < u_\lambda \) for \( \lambda = \lambda_* \),
(iii) exactly one positive solution \( w_\lambda \) for \( \hat{\lambda} < \lambda < \lambda_* \),
(iv) no positive solution for \( 0 < \lambda \leq \hat{\lambda} \).

Furthermore,

\[
\lim_{\lambda \to (\lambda_*)^+} ||w_\lambda||_\infty = 0, \quad \lim_{\lambda \to \infty} ||w_\lambda||_\infty = \lim_{\lambda \to \infty} ||u_\lambda||_\infty = \beta_1, \quad \text{and} \quad \lim_{\lambda \to \infty} ||v_\lambda||_\infty = \beta_3.
\]

Remark 2. Numerical simulations show that, for \( p \in [1.01, 10] \),

\[ N_1(B_1_p(\eta_{1,p})) + N_2(C_2_p(\eta_{1,p})) > 0 \quad \text{and} \quad N_3(B_1_p(\eta_{2,p})) + N_4(C_2_p(\eta_{2,p})) > 0, \]

see Fig. 4. Hence we obtain that, for \( p \in [1.01, 10] \),

\[ \eta_{1,p}^* = \sup \{ \eta : 0 < \eta \leq \eta_{1,p} \quad \text{and} \quad N_1(B_1_p(\eta)) + N_2(C_2_p(\eta)) > 0 \} = \eta_{1,p} \]

and

\[ \eta_{2,p}^* = \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \quad \text{and} \quad N_3(B_1_p(\eta)) + N_4(C_2_p(\eta)) > 0 \} = \eta_{2,p}. \]

In particular, for \( p = 2 \), we obtain \( \eta_{1,p}^* = \eta_{1,p} \approx 0.0939 \) and \( \eta_{2,p}^* = \eta_{2,p} \approx 0.0766. \)
3. Lemmas and proofs of main results. To prove Theorem 2.1, we need the following Lemmas 3.1–3.4. For

\[ f(u) = ug(u) = ru \left(1 - \frac{u}{q}\right) - \frac{w^p}{1 + w^p} \]

with \( p > 1 \), by analyses for \( g(u) \) in Section 1, we obtain the following Lemma 3.1.

**Lemma 3.1.** Consider \( f(u) = ru(1 - \frac{u}{q}) - \frac{w^p}{1 + w^p} \) with \( p > 1 \).

(i) If \( (q, r) \in R_1 \), then there exists a positive number \( \beta_3 \) such that \( f(0) = f(\beta_3) = 0 \), \( f(u) > 0 \) on \( (0, \beta_3) \) and \( f(u) < 0 \) on \( (\beta_3, \infty) \).

(ii) If \( (q, r) \in \bar{R}_2 \), then there exist positive numbers \( \beta_1 < \beta_2 < \beta_3 \) such that \( f(0) = f(\beta_1) = f(\beta_2) = f(\beta_3) = 0 \), \( f(u) > 0 \) on \( (0, \beta_1) \cup (\beta_2, \beta_3) \) and \( f(u) < 0 \) on \( (\beta_1, \beta_2) \cup (\beta_3, \infty) \). Also, there exists a positive number \( \gamma \in (\beta_2, \beta_3) \) satisfying \( \int_{\beta_3}^{\beta} f(u)du = 0 \).

(iii) If \( (q, r) \in \Gamma_1 \), then there exist positive numbers \( \beta_1 < \beta_3 \) such that \( f(0) = f(\beta_1) = f(\beta_3) = 0 \), \( f(u) > 0 \) on \( (0, \beta_1) \cup (\beta_1, \beta_3) \) and \( f(u) < 0 \) on \( (\beta_3, \infty) \).

Let \( F(u) \equiv \int_0^u f(t)dt \), and \( u_\alpha \) be a positive solution of (1) with \( \alpha \equiv \|u_\alpha\|_\infty > 0 \). The time map formula which we apply to study problem (1), takes the form as follows:

(i) if \( (q, r) \in R_1 \), then the time map

\[ T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{|F(\alpha) - F(u)|^{1/2}} du = \sqrt{\alpha} \] for \( \alpha \in (0, \beta_3) \); (22)

(ii) if \( (q, r) \in \bar{R}_2 \), then the time map

\[ T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{|F(\alpha) - F(u)|^{1/2}} du = \sqrt{\alpha} \] for \( \alpha \in (0, \beta_1) \cup (\gamma, \beta_3) \);

(iii) if \( (q, r) \in \Gamma_1 \), then the time map

\[ T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{|F(\alpha) - F(u)|^{1/2}} du = \sqrt{\alpha} \] for \( \alpha \in (0, \beta_1) \cup (\beta_1, \beta_3) \);

see Laetsch [4] for the derivation of the time map formula \( T(\alpha) \) for problem (1).

So positive solutions \( u_\alpha \) of (1) correspond to \( \|u_\alpha\|_\infty = \alpha \) and \( T(\alpha) = \sqrt{\alpha} \). Thus,
studying the exact number of positive solutions of (1) is equivalent to studying the number of roots of the equation \( T(\alpha) = \sqrt{\lambda} \).

We define
\[
\theta(u) = 2F(u) - uf(u) = \frac{r}{3q}u^3 - u - \frac{u}{1 + u^p} + 2 \int_0^u \frac{1}{1 + t^p} dt,
\] (23)
and
\[
H(u) = 3 \int_0^u tf(t) dt - u^2 f(u) = \frac{r}{4q}u^4 - \frac{1}{2}u^2 - \frac{u^2}{1 + u^p} + 3 \int_0^u \frac{t}{1 + t^p} dt.
\] (24)

We also compute
\[
\theta'(u) = f(u) - uf'(u) = u^2 \left( \frac{r}{q} - \frac{u^{p-2}(1 - p + u^p)}{(1 + u^p)^2} \right)
\] (25)
and
\[
\theta''(u) = -uf''(u) = 2u \left( \frac{r}{q} - \frac{pu^{p-2}(-(p - 1) + (p + 1)u^p)}{2(1 + u^p)^3} \right).
\] (26)

**Lemma 3.2.** Consider \( f(u) = ru(1 - \frac{u}{q}) - \frac{u^p}{1 + u^p} \) with \( p > 1 \). If \((q, r) \in R_1\) and \( r \leq \eta_{1,p}q\), then there exist positive numbers \( B_{1,p} < C_{1,p} < B_{2,p} < C_{2,p} < \beta_3 \) such that \( \theta'(C_{1,p}) = \theta'(C_{2,p}) = 0 \),

\[
\begin{align*}
\theta''(B_{1,p}) &= \theta''(B_{2,p}) = 0, \\
\theta''(u) &> 0 \quad \text{on } (0, B_{1,p}) \cup (B_{2,p}, \infty), \\
\theta''(u) &< 0 \quad \text{on } (B_{1,p}, B_{2,p}),
\end{align*}
\] (27)

and \( H(B_{2,p}) \leq 0 \).

**Figure 5.** Graphs of functions \( \eta = I(u), \eta = J(u), \eta = K(u) \) on \((0, \infty)\).
**Proof of Lemma 3.2.** By (11) and (26), it is clear that \( u = \hat{u} \) is a positive zero of \( \theta''(u) \) if and only if \( u = \hat{u} \) satisfies

\[
\frac{r}{q} = \frac{pu^{p-2}[-(p-1)+(p+1)u^p]}{2(1+u^p)^3} = I(u).
\]

Thus we obtain that

\[
\begin{cases}
\lim_{u \to 0^+} I(u) = -\infty, & \text{if } 1 < p < 2, \\
I(0) = -1, & \text{if } p = 2, \\
I(0) = 0, & \text{if } p > 2
\end{cases}
\]

and

\[
\begin{cases}
I\left(\left(\frac{p-1}{p+1}\right)^{1/p}\right) = 0, \\
I(u) < 0 & \text{on } (0, \left(\frac{p-1}{p+1}\right)^{1/p}), \\
I(u) > 0 & \text{on } \left(\left(\frac{p-1}{p+1}\right)^{1/p}, \infty\right).
\end{cases}
\]

We compute that

\[
I'(u) = \frac{-pu^{p-3}\left[(p^2 - 3p + 2) + 4(1-p^2)u^p + (p^2 + 3p + 2)u^{2p}\right]}{2(1+u^p)^4}.
\]

(28)

Thus we obtain that, if \( 1 < p \leq 2 \),

\[
\begin{cases}
I'(B^*_p) = 0, \\
I'(u) > 0 & \text{on } (0, B^*_p), \\
I'(u) < 0 & \text{on } (B^*_p, \infty),
\end{cases}
\]

and if \( p > 2 \),

\[
\begin{cases}
I'\left(\hat{B}_p\right) = I'(B^*_p) = 0, \\
I'(u) < 0 & \text{on } (0, \hat{B}_p), \\
I'(u) > 0 & \text{on } (\hat{B}_p, B^*_p), \\
I'(u) < 0 & \text{on } (B^*_p, \infty),
\end{cases}
\]

where

\[
\hat{B}_p = \left(\frac{2p^2 - 2 + \sqrt{3p^2(p^2 - 1)}}{p^2 + 3p + 2}\right)^{1/p}\left\{\begin{array}{ll}
< 0, & \text{if } 1 < p < 2, \\
= 0, & \text{if } p = 2, \\
\in (0, \left(\frac{p-1}{p+1}\right)^{1/p}), & \text{if } p > 2,
\end{array}\right.
\]

and

\[
B^*_p = \left(\frac{2p^2 - 2 + \sqrt{3p^2(p^2 - 1)}}{p^2 + 3p + 2}\right)^{1/p} > \left(\frac{p - 1}{p + 1}\right)^{1/p} > 0.
\]

In addition, it is clear that \( \lim_{u \to \infty} I(u) = 0 \), and hence

\[
\max_{u \in (0,\infty)} I(u) = I(B^*_p).
\]

By the above analysis for \( I(u) \), for \( 0 < r < I(B^*_p)q \), we obtain that the equation

\[ I(u) = \frac{r}{q} \]

has exactly two positive roots, say, \( B_{1,p}, B_{2,p} \) with \( \left(\frac{p-1}{p+1}\right)^{1/p} < B_{1,p} < B^*_p < B_{2,p} \), see Fig. 5. Thus, for \( 0 < r < I(B^*_p)q \), \( \theta''(u) = 2u[\left(r/q\right) - I(u)] \) has two positive zeros \( B_{1,p}, B_{2,p} \) such that (27) holds.
By (15) and (25), it is clear that \( u = \hat{u} \) is a positive zero of \( \theta'(u) \) if and only if \( u = \hat{u} \) satisfies
\[
\frac{r}{q} = \frac{u^{p-2}(1 - p + u^p)}{(1 + u^p)^2} = J(u).
\]
Thus we obtain that
\[
\left\{\begin{array}{l}
\lim_{u \to 0^+} J(u) = -\infty, \quad \text{if } 1 < p < 2, \\
J(0) = -1, \quad \text{if } p = 2, \\
J(0) = 0, \quad \text{if } p > 2
\end{array}\right.
\]
and
\[
\left\{\begin{array}{l}
J(\sqrt[p]{p - 1}) = 0, \\
J(u) < 0 \quad \text{on } (0, \sqrt[p]{p - 1}), \\
J(u) > 0 \quad \text{on } (\sqrt[p]{p - 1}, \infty).
\end{array}\right. \quad (29)
\]
We compute that
\[
J'(u) = \frac{u^{p-3}[-(p^2 - 3p + 2) + (p^2 + 3p - 4)u^p - 2u^{2p}]}{(1 + u^p)^3}.
\]
Thus we obtain that, if \( 1 < p \leq 2 \),
\[
\left\{\begin{array}{l}
J'(C_p^*) = 0, \\
J'(u) > 0 \quad \text{on } (0, C_p^*), \\
J'(u) < 0 \quad \text{on } (C_p^*, \infty),
\end{array}\right.
\]
and if \( p > 2 \),
\[
\left\{\begin{array}{l}
J'(\tilde{C}_p) = J'(C_p^*) = 0, \\
J'(u) < 0 \quad \text{on } (0, \tilde{C}_p), \\
J'(u) > 0 \quad \text{on } (\tilde{C}_p, C_p^*), \\
J'(u) < 0 \quad \text{on } (C_p^*, \infty),
\end{array}\right.
\]
where
\[
\tilde{C}_p = \left(\frac{p^2 + 3p - 4 - p\sqrt{p^2 + 6p - 7}}{4}\right)^{1/p} \left\{\begin{array}{l}
< 0, \quad \text{if } 1 < p < 2, \\
= 0, \quad \text{if } p = 2, \\
\in (0, \sqrt[p]{p - 1}), \quad \text{if } p > 2,
\end{array}\right.
\]
and
\[
C_p^* = \left(\frac{p^2 + 3p - 4 + p\sqrt{p^2 + 6p - 7}}{4}\right)^{1/p} > \sqrt[p]{p - 1} > 0.
\]
In addition, it is clear that \( \lim_{u \to \infty} J(u) = 0 \), and hence
\[
\max_{u \in (0, \infty)} J(u) = J(C_p^*).
\]
By the above analysis for \( J(u) \), for \( 0 < r < J(C_p^*)q \), we obtain that the equation \( J(u) = r/q \) has exactly two positive roots, say, \( C_{1,p}, C_{2,p} \) with \( \sqrt[p]{p - 1} < C_{1,p} < C_p^* < C_{2,p} \), see Fig. 5. Thus, for \( 0 < r < J(C_p^*)q \),
\[
\theta'(u) = u^2 \left[\frac{r}{q} - J(u)\right]
\]
has two positive zeros \( C_{1,p}, C_{2,p} \) such that
\[
\left\{\begin{array}{l}
\theta'(C_{1,p}) = \theta'(C_{2,p}) = 0, \\
\theta'(u) > 0 \quad \text{on } (0, C_{1,p}) \cup (C_{2,p}, \infty), \\
\theta'(u) < 0 \quad \text{on } (C_{1,p}, C_{2,p}).
\end{array}\right. \quad (32)
\]
By (12) and (24), it is clear that \( u = \tilde{u} \) is a positive zero of \( H(u) \) if and only if \( u = \tilde{u} \) satisfies
\[
\frac{r}{q} = \frac{4}{u^2} \left[ \frac{u^2}{2} + \frac{u^2}{1 + u^p} - 3 \int_0^u \frac{t}{1 + t^p} dt \right] = K(u).
\]
We compute that
\[
K'(u) = \frac{-4}{u^3 (1 + u^p)^2} \left[ 6u^2 + (p + 7)u^{p+2} + u^{2p+2} - 12(1 + u^p)^2 \int_0^u \frac{t}{1 + t^p} dt \right].
\]
Let
\[
\tilde{K}(u) \equiv \frac{u^5}{48} K'(u) = -\frac{u^2 [6 + (p + 7)u^p + u^{2p}] + 12(1 + u^p)^2 \int_0^u \frac{t}{1 + t^p} dt}{12(1 + u^p)^3},
\]
and hence
\[
\tilde{K}'(u) = \frac{u^{p+1} [-(p^2 - 3p + 2) + (p^2 + 3p - 4)u^p - 2u^{2p}]}{12(1 + u^p)^3}.
\]
Thus we obtain that, if \( 1 < p \leq 2 \),
\[
\begin{cases}
K'(C_p^*) = 0, \\
\tilde{K}'(u) > 0 \text{ on } (0, C_p^*), \\
\tilde{K}'(u) < 0 \text{ on } (C_p^*, \infty),
\end{cases}
\]
and if \( p > 2 \),
\[
\begin{cases}
\tilde{K}'(C_p^*) = \tilde{K}'(C_p^*) = 0, \\
\tilde{K}'(u) < 0 \text{ on } (0, C_p^*), \\
\tilde{K}'(u) > 0 \text{ on } (C_p^*, C_p^*), \\
\tilde{K}'(u) < 0 \text{ on } (C_p^*, \infty).
\end{cases}
\]
Since \( \tilde{K}(0) = 0 \) and \( \lim_{u \to \infty} \tilde{K}(u) = -\infty \), we obtain that there exist two positive numbers \( D_p^* \in (C_p^*, C_p^*) \) (if \( p > 2 \)) and \( D_p^* \in (C_p^*, \infty) \) such that, if \( 1 < p \leq 2 \),
\[
\begin{cases}
\tilde{K}(D_p^*) = 0, \\
\tilde{K}(u) > 0 \text{ on } (0, D_p^*), \\
\tilde{K}(u) < 0 \text{ on } (D_p^*, \infty),
\end{cases}
\]
and if \( p > 2 \), either
\[
\begin{cases}
\tilde{K}(D_p^*) = \tilde{K}(D_p^*) = 0, \\
\tilde{K}(u) < 0 \text{ on } (0, D_p^*), \\
\tilde{K}(u) > 0 \text{ on } (D_p^*, D_p^*), \\
\tilde{K}(u) < 0 \text{ on } (D_p^*, \infty)
\end{cases}
\]
or
\[
\begin{cases}
\tilde{K}(C_p^*) \leq 0, \\
\tilde{K}(u) < 0 \text{ on } (0, C_p^*) \cup (C_p^*, \infty).
\end{cases}
\]
By (33), we obtain that, if \( 1 < p \leq 2 \),
\[
\begin{cases}
K'(D_p^*) = 0, \\
K'(u) > 0 \text{ on } (0, D_p^*), \\
K'(u) < 0 \text{ on } (D_p^*, \infty),
\end{cases}
\]
and if \( p > 2 \), either
\[
\begin{cases}
K'(D_p^*) = K'(D_p^*) = 0, \\
K'(u) < 0 \text{ on } (0, D_p^*), \\
K'(u) > 0 \text{ on } (D_p^*, D_p^*), \\
K'(u) < 0 \text{ on } (D_p^*, \infty)
\end{cases}
\]
or
\[
\begin{cases}
K'(C_p^* ) \leq 0, \\
K'(u) < 0 \text{ on } (0, C_p^* ) \cup (C_p^* , \infty). 
\end{cases}
\tag{34}
\]

In addition, it is clear that
\[
\lim_{u \to 0^+} K(u) = \begin{cases}
-\infty, & \text{if } 1 < p < 2, \\
-1, & \text{if } p = 2, \\
0, & \text{if } p > 2,
\end{cases}
\]

and \(\lim_{u \to \infty} K(u) = 0\), and hence there exists a positive number
\[
\bar{D}_p \in \begin{cases}
(0, D_p^* ), & \text{if } 1 < p \leq 2, \\
(\bar{D}_p, D_p^* ), & \text{if } p > 2
\end{cases}
\]
satisfying
\[
\begin{cases}
K(\bar{D}_p) = 0, \\
K(u) < 0 \text{ on } (0, \bar{D}_p), \\
K(u) > 0 \text{ on } (\bar{D}_p, \infty), 
\end{cases}
\]

and
\[
\max_{u \in (0, \infty)} K(u) = K(D_p^* ).
\]

(Note that the case of (34) contradicts to \(\lim_{u \to 0^+} K(u) = \lim_{u \to \infty} K(u) = 0\).) By the above analysis for \(K(u)\), for \(0 < r < K(D_p^* )q\), we obtain that the equation
\[
K(u) = r/q
\]
has exactly two positive roots, say, \(D_{1,p}, D_{2,p}\) with \(\bar{D}_p < D_{1,p} < D_p^* < D_{2,p}\), see Fig. 5. Thus, for \(0 < r < K(D_p^* )q\),
\[
H(u) = \frac{u^4}{4} \left[ \frac{r}{q} - K(u) \right]
\tag{35}
\]
has two positive zeros \(D_{1,p}, D_{2,p}\) such that
\[
\begin{cases}
H(D_{1,p}) = H(D_{2,p}) = 0, \\
H(u) > 0 \text{ on } (0, D_{1,p}) \cup (D_{2,p}, \infty), \\
H(u) < 0 \text{ on } (D_{1,p}, D_{2,p}). 
\end{cases}
\tag{36}
\]

We compute that
\[
I(u) - J(u) = \frac{u^{p-2} \left[ - (p^2 - 3p + 2) + (p^2 + 3p - 4)u^p - 2u^{2p} \right]}{2(1 + u^p)^3} = \frac{u}{2} f'(u)
\tag{37}
\]
and
\[
J(u) - K(u) = \frac{\left[ 6 + (p + 7)u^p + u^{2p} \right]}{u^2(1 + u^p)^2} + 12 \int_0^u \frac{t}{1 + t^p} dt = \frac{u}{4} K'(u)
\tag{38}
\]
by (30) and (33). Thus, if \(1 < p \leq 2\),
\[
\begin{cases}
I(C_p^* ) - J(C_p^* ) = 0, \\
I(u) - J(u) > 0 \text{ on } (0, C_p^* ), \\
I(u) - J(u) < 0 \text{ on } (C_p^* , \infty), 
\end{cases}
\tag{39}
\]

\[
\begin{cases}
J(D_p^* ) - K(D_p^* ) = 0, \\
J(u) - K(u) > 0 \text{ on } (0, D_p^* ), \\
J(u) - K(u) < 0 \text{ on } (D_p^* , \infty), 
\end{cases}
\tag{40}
\]
and if \( p > 2 \),
\[
\begin{align*}
I(\tilde{C}_p) - J(\tilde{C}_p) &= I(C^*_p) - J(C^*_p) = 0, \\
I(u) - J(u) < 0 &\quad \text{on } (0, \tilde{C}_p), \\
I(u) - J(u) > 0 &\quad \text{on } (C^*_p, C^*_p), \\
I(u) - J(u) < 0 &\quad \text{on } (C^*_p, \infty),
\end{align*}
\]
(41)
\[
\begin{align*}
J(\tilde{D}_p) - K(\tilde{D}_p) &= J(D^*_p) - K(D^*_p) = 0, \\
J(u) - K(u) < 0 &\quad \text{on } (0, \tilde{D}_p), \\
J(u) - K(u) > 0 &\quad \text{on } (D^*_p, D^*_p), \\
J(u) - K(u) < 0 &\quad \text{on } (D^*_p, \infty).
\end{align*}
\]
(42)

It is easy to check that \( B^*_p < C^*_p \) and hence
\[
I(B^*_p) = \max_{u \in (0, \infty)} I(u) > I(C^*_p) = J(C^*_p)
\]
by (39) and (41). In addition, we know that \( C^*_p < D^*_p \) and hence
\[
J(C^*_p) = \max_{u \in (0, \infty)} J(u) > J(D^*_p) = K(D^*_p)
\]
by (40) and (42). So we obtain

1. \( B^*_p < C^*_p < D^*_p \) and \( I(B^*_p) > J(C^*_p) > K(D^*_p) \).

In addition, we also obtain

2. \( \tilde{D}_p > \frac{p}{p-1} > \left( \frac{p-1}{p+1} \right)^{1/p} \) since
\[
\tilde{D}_p \in \begin{cases} 
(0, D^*_p), & \text{if } 1 < p \leq 2, \\
(\tilde{D}_p, D^*_p), & \text{if } p > 2,
\end{cases}
\]

and by (40), (42), (29) and \( K(\tilde{D}_p) = 0 \).

3. \( \lim_{u \to \infty} I(u) = \lim_{u \to \infty} J(u) = \lim_{u \to \infty} K(u) = 0 \).

4. \( I(u) < 0 \) on \( (0, \left( \frac{p-1}{p+1} \right)^{1/p}) \) and \( I(u) \) has exactly one critical point, a local maximum, at \( B^*_p \), on \( \left( \frac{p-1}{p+1} \right)^{1/p}, \infty \). \( J(u) < 0 \) on \( (0, \frac{p}{p-1}) \) and \( J(u) \) has exactly one critical point, a local maximum, at \( C^*_p \), on \( \left( \frac{p}{p-1}, \infty \right) \). \( K(u) < 0 \) on \( (0, \tilde{D}_p) \) and \( K(u) \) has exactly one critical point, a local maximum, at \( D^*_p \), on \( (\tilde{D}_p, \infty) \).

5. \[
\begin{align*}
I(C^*_p) &= J(C^*_p), \\
I(u) > J(u) &\quad \text{on } \left( \frac{p}{p-1}, C^*_p \right), \\
I(u) < J(u) &\quad \text{on } (C^*_p, \infty)
\end{align*}
\]

and
\[
\begin{align*}
J(D^*_p) &= K(D^*_p), \\
J(u) > K(u) &\quad \text{on } (\tilde{D}_p, D^*_p), \\
J(u) < K(u) &\quad \text{on } (D^*_p, \infty)
\end{align*}
\]
by (39), (40), (41), (42) and \( \tilde{C}_p < \frac{p}{p-1}, \tilde{D}_p < \tilde{D}_p \) if \( p > 2 \).

6. Functions \( I(u) \), \( J(u) \) and \( K(u) \) are all strictly decreasing on \( (D^*_p, \infty) \).

7. The largest positive zeros of functions \( I(u) - J(u) \), \( J(u) - K(u) \) on \( (0, \infty) \) are \( C^*_p \) and \( D^*_p \) respectively. We define the largest positive zero of function \( I(u) - K(u) \) as \( \xi_p \). Thus we obtain \( C^*_p < \xi_p < D^*_p \) by \( C^*_p < D^*_p \) and property (5). Hence
\[
\eta_{1,p} = I(\xi_p) = K(\xi_p) < K(D^*_p) < J(C^*_p) < I(B^*_p)
\]
(43)
Thus $N = \eta - \eta^+$, which exists. In addition, if $q = r/q$, then $N(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta)) > 0$.

Lemma 3.3. Consider $f(u) = ru(1-\frac{u}{q}) - \frac{u^p}{1+u^p}$ with $p > 1$, $q, r > 0$, and let $r/q = \eta$. Then

$$N_1(u) = \frac{u^2}{4(1+u^p)^3} \left[ -12 + (p^2 - 5p - 24)u^p + (-p^2 - 5p - 12)u^{2p} \right] + 6 \int_0^u \frac{t}{1 + tv^p} dt,$$

and

$$N_2(u) = \frac{u^2}{2(1+u^p)^2} \left[ 6 + (p + 7)u^p + u^{2p} \right] - 6 \int_0^u \frac{t}{1 + tv^p} dt.$$

For $0 < \eta < \eta^+$, it is easy to prove that $B_{1,p}(\eta) \to \left( \frac{p-1}{p+1} \right)^1/\eta$ and $C_{2,p}(\eta) \to \infty$ as $\eta \to 0^+$, and

$$\lim_{u \to \infty} \frac{\int_0^u \frac{t}{1 + tv^p} dt}{u^2} = 0.$$
by (37) and (38). In addition, by $\eta = r/q = J(C_{2,p})$, (35) and (38), we obtain that

$$-2H(C_{2,p}) = -\frac{C_{2,p}}{2} \left[ J(C_{2,p}) - K(C_{2,p}) \right]$$

$$= \frac{C_{2,p}}{2} [6 + (p + 7)C_{2,p}^p + C_{2,p}^{2p}] - 6 \int_{0}^{C_{2,p}} \frac{t}{1 + tp} dt.$$ 

Hence $2H(B_{1,p}^2) - B_{1,p}^2 \theta'(B_{1,p}) - 2H(C_{2,p}) = N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta)).$

We compute that

$$N_1'(u) = \frac{pmu^{p+1}[(p^2 - 3p + 2) + 4(1 - p^2)u^p + (p^2 + 3p + 2)u^{2p}]}{4(1 + u^p)^4} = -\frac{u^4}{2} I'(u)$$

by (28). Thus we obtain that, if $1 < p \leq 2$,

$$\begin{cases}
N_1'(B_p^*) = 0, \\
N_1'(u) < 0 \text{ on } (0, B_p^*), \\
N_1'(u) > 0 \text{ on } (B_p^*, \infty),
\end{cases} \quad (44)$$

and if $p > 2$,

$$\begin{cases}
N_1'(B_p^*) = N_1'(u) = 0, \\
N_1'(u) > 0 \text{ on } (0, B_p^*), \\
N_1'(u) < 0 \text{ on } (B_p^*, B_p^*), \\
N_1'(u) > 0 \text{ on } (B_p^*, \infty).
\end{cases} \quad (45)$$

By (44), (45) and $\bar{B}_p < \left(\frac{p-1}{p+1}\right)^{1/p} < B_p^*$, we obtain that $N_1(u)$ is strictly decreasing on $\left(\left(\frac{p-1}{p+1}\right)^{1/p}, B_p^*\right)$. In addition, we know that, for $0 < \eta \leq \eta_{1,p}^*, B_{1,p}(\eta) \in \left(\left(\frac{p-1}{p+1}\right)^{1/p}, B_p^*\right)$ and $B_{1,p}(\eta)$ is strictly increasing on $[0, \eta_{1,p}^*]$. Thus $N_1(B_{1,p}(\eta))$ is strictly decreasing on $[0, \eta_{1,p}^*]$. On the other hand, we also compute that

$$N_2'(u) = \frac{-u^{p+1}[-(p^2 - 3p + 2) + (p^2 + 3p - 4)u^p - 2u^{2p}]}{2(1 + u^p)^3} = -\frac{u^4}{2} J'(u)$$

by (30). Thus we obtain that, if $1 < p \leq 2$,

$$\begin{cases}
N_2'(C_p^*) = 0, \\
N_2'(u) < 0 \text{ on } (0, C_p^*), \\
N_2'(u) > 0 \text{ on } (C_p^*, \infty),
\end{cases} \quad (46)$$

and if $p > 2$,

$$\begin{cases}
N_2'(C_p^*) = N_2'(u) = 0, \\
N_2'(u) > 0 \text{ on } (0, C_p^*), \\
N_2'(u) < 0 \text{ on } (C_p^*, C_p^*), \\
N_2'(u) > 0 \text{ on } (C_p^*, \infty).
\end{cases} \quad (47)$$

By (46) and (47), we obtain that $N_2(u)$ is strictly increasing on $(C_p^*, \infty)$. In addition, we know that, for $0 < \eta \leq \eta_{1,p}^*, C_{2,p}(\eta) \in (C_p^*, \infty)$ and $C_{2,p}(\eta)$ is strictly decreasing on $(0, \eta_{1,p}^*)$. Thus $N_2(C_{2,p}(\eta))$ is strictly decreasing on $(0, \eta_{1,p}^*)$. So $N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta))$ is strictly decreasing on $(0, \eta_{1,p}^*)$ and hence for $0 < \eta \leq \eta_{1,p}^*$,

$$2H(B_{1,p}^2) - B_{1,p}^2 \theta'(B_{1,p}) - 2H(C_{2,p}) = N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta))$$

$$\geq N_1(B_{1,p}(\eta_{1,p}^*)) + N_2(C_{2,p}(\eta_{1,p}^*)) \geq 0.$$ 

The proof of Lemma 3.3 is complete.
By Lemmas 3.1(i), 3.2–3.3, and [16, Lemma 3.1], we obtain the following Lemma 3.4.

**Lemma 3.4.** Consider (1) with $p > 1$. If $(q, r) \in R_1$ and $r \leq \eta_{1,p}^*$, then

$$\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2\sqrt{r}} \text{ and } \lim_{\alpha \to \beta_3^*} T(\alpha) = \infty.$$  

In addition, $T(\alpha)$ has exactly two positive critical points, at some $\alpha^* < \alpha_*$, on $(0, \beta_3)$, such that $T(\alpha^*)$ is a local maximum on $(0, \beta_3)$ and $T(\alpha_*)$ is a local minimum on $(0, \beta_3)$.

**Proof of Theorem 2.1.** By Lemma 3.4 and (22), the results in (20) hold and the bifurcation curve $S$ is an $S$-shaped curve on the $(\lambda, ||u||_{\infty})$-plane. More precisely, $S$ consists of a continuous curve with exactly two turning points at some points $(\lambda_*, ||u_{\lambda_*}||_{\infty})$ and $(\lambda_*, ||u_{\lambda_*}||_{\infty})$ such that $0 < \lambda_* < \lambda^* < \infty$ and $0 < ||u_{\lambda_*}||_{\infty} < ||u_{\lambda_*}||_{\infty} < \beta_3$. In addition, since

$$f(u) = ru\left(1 - \frac{u}{q}\right) - \frac{u^p}{1 + up^p} < ru\left(1 - \frac{u}{q}\right) \equiv f_0(u) \text{ for } u \in (0, \beta_3),$$

by a comparison theorem of Laetsch [4, Theorem 2.3], we obtain that, for $\alpha \in (0, \beta_3)$

$$T(\alpha) > T_{f_0}(\alpha) \equiv \frac{1}{\sqrt{2}} \int_{0}^{\alpha} \frac{1}{\left[F_0(\alpha) - F_0(u)\right]^{1/2}} du,$$

where $F_0(u) = \int_{0}^{u} f_0(t) dt$. We compute

$$T'_{f_0}(\alpha) = \frac{1}{2\sqrt{2\alpha}} \int_{0}^{\alpha} \frac{\theta f_0(\alpha) - \theta f_0(u)}{\left[F_0(\alpha) - F_0(u)\right]^{3/2}} du,$$

where $\theta f_0(u) = 2F_0(u) - uf_0(u)$. By (48) and $\theta f_0(u) = \frac{\pi}{q} u^2 > 0$ for $u \in (0, \beta_3)$, we obtain that $T_{f_0}(\alpha)$ is strictly increasing on $(0, \beta_3)$. Thus

$$T(\alpha) > T_{f_0}(\alpha) > \lim_{\alpha \to 0^+} T_{f_0}(\alpha) = \frac{\pi}{2\sqrt{r}} \text{ for } \alpha \in (0, \beta_3)$$

by [4, Theorem 2.10], and hence $\lambda_* > \frac{\pi^2}{4} = \hat{\lambda}$ by (22). Thus $\hat{\lambda} < \lambda_* < \lambda^* < \infty$. So we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(iv). The proofs of results in (21) are easy but tedious; we omit them.

The proof of Theorem 2.1 is now complete.

To prove Theorems 2.2–2.3, we need Lemma 3.1 and the following Lemmas 3.5–3.8.

**Lemma 3.5.** Consider $f(u) = ru(1 - \frac{u}{q}) - \frac{u^p}{1 + up^p}$ with $p > 1$. If $(q, r) \in \hat{R}_2 \cup \Gamma_1$ and $r \leq \eta_{2,p} q$, then there exist positive numbers $B_{1,p} < C_{1,p} < E_{1,p} < B_{2,p} < C_{2,p} < E_{2,p} < \beta_3$ such that $\theta(E_{1,p}) = \theta(E_{2, p}) = \theta'(C_{1,p}) = \theta'(C_{2,p}) = 0$ and (27) holds.

**Proof of Lemma 3.5.** In the proof of Lemma 3.2, we know that:

(A) $\theta''(u)$ has two positive zeros $B_{1,p} < B_{2,p}$ satisfying (27) for $0 < r < I(B_{1,p}^*)q$.

(B) $\theta'(u)$ has two positive zeros $C_{1,p} < C_{2,p}$ satisfying (32) for $0 < r < J(C_{2,p}^*)q$.

By (14) and (23), it is clear that $u = \tilde{u}$ is a positive zero of $\theta(u)$ if and only if $u = \tilde{u}$ satisfies

$$\frac{r}{q} = \frac{3}{u^3} \left[u + \frac{u}{1 + up} - 2 \int_{0}^{u} \frac{1}{1 + tp} dt\right] = M(u).$$
We compute that
\[
M'(u) = \frac{-3}{u^4(1+u^p)^2} \left[ u(6 + (p + 8)u^p + 2u^{2p}) - 6(1+u^p)^2 \int_0^u \frac{1}{1+t^p} dt \right]
\]
\[
\equiv \frac{18}{u^2} \tilde{M}(u), \tag{49}
\]
where
\[
\tilde{M}(u) = -\frac{u(6 + (p + 8)u^p + 2u^{2p})}{6(1+u^p)^2} + \int_0^u \frac{1}{1+t^p} dt.
\]
For \(\tilde{M}(u)\), we compute that
\[
\tilde{M}'(u) = \frac{u^p[-(p^2 - 3p + 2) + (p^2 + 3p - 4)u^p - 2u^{2p}]}{6(1+u^p)^3}.
\]
Thus we obtain that, if \(1 < p \leq 2\),
\[
\begin{cases}
\tilde{M}'(C_p^*) = 0, \\
\tilde{M}'(u) > 0 \text{ on } (0, C_p^*), \\
\tilde{M}'(u) < 0 \text{ on } (C_p^*, \infty),
\end{cases}
\]
and if \(p > 2\),
\[
\begin{cases}
\tilde{M}'(\tilde{C}_p) = \tilde{M}'(C_p^*) = 0, \\
\tilde{M}'(u) < 0 \text{ on } (0, \tilde{C}_p), \\
\tilde{M}'(u) > 0 \text{ on } (\tilde{C}_p, C_p^*), \\
\tilde{M}'(u) < 0 \text{ on } (C_p^*, \infty).
\end{cases}
\]
Since $\dot{M}(0) = 0$ and $\lim_{u \to \infty} \dot{M}(u) = -\infty$, we obtain that there exist two positive numbers $\bar{E}_p \in (\bar{C}_p, \bar{C}_p^*)$ (if $p > 2$) and $E_p^* \in (C_p^*, \infty)$ such that, if $1 < p \leq 2$, 
\[
\begin{cases}
\dot{M}(E_p^*) = 0, \\
\dot{M}(u) > 0 \text{ on } (0, E_p^*), \\
\dot{M}(u) < 0 \text{ on } (E_p^*, \infty),
\end{cases}
\]
and if $p > 2$, either 
\[
\begin{cases}
\dot{M}(\bar{E}_p) = \dot{M}(E_p^*) = 0, \\
\dot{M}(u) < 0 \text{ on } (0, \bar{E}_p), \\
\dot{M}(u) > 0 \text{ on } (\bar{E}_p, E_p^*), \\
\dot{M}(u) < 0 \text{ on } (E_p^*, \infty),
\end{cases}
\]
or 
\[
\begin{cases}
\dot{M}(C_p^*) \leq 0, \\
\dot{M}(u) < 0 \text{ on } (0, C_p^*) \cup (C_p^*, \infty).
\end{cases}
\]
By (49), we obtain that, if $1 < p \leq 2$, 
\[
\begin{cases}
M'(E_p^*) = 0, \\
M'(u) > 0 \text{ on } (0, E_p^*), \\
M'(u) < 0 \text{ on } (E_p^*, \infty),
\end{cases}
\]
and if $p > 2$, either 
\[
\begin{cases}
M'(\bar{E}_p) = M'(E_p^*) = 0, \\
M'(u) < 0 \text{ on } (0, \bar{E}_p), \\
M'(u) > 0 \text{ on } (\bar{E}_p, E_p^*), \\
M'(u) < 0 \text{ on } (E_p^*, \infty),
\end{cases}
\]
or 
\[
\begin{cases}
M'(C_p^*) \leq 0, \\
M'(u) < 0 \text{ on } (0, C_p^*) \cup (C_p^*, \infty).
\end{cases}
\]
In addition, it is clear that 
\[
\lim_{u \to 0^+} M(u) = \begin{cases}
-\infty, & \text{if } 1 < p < 2, \\
-1, & \text{if } p = 2, \\
0, & \text{if } p > 2,
\end{cases}
\]
and $\lim_{u \to \infty} M(u) = 0$, and hence there exists a positive number 
\[
E_P \in \begin{cases}
(0, E_p^*), & \text{if } 1 < p \leq 2, \\
(\bar{E}_p, E_p^*), & \text{if } p > 2
\end{cases}
\]
satisfying 
\[
\begin{cases}
M(\bar{E}_p) = 0, \\
M(u) < 0 \text{ on } (0, \bar{E}_p), \\
M(u) > 0 \text{ on } (\bar{E}_p, \infty),
\end{cases}
\]
and 
\[
\max_{u \in (0, \infty)} M(u) = M(E_p^*).
\]
(Note that the case of (50) contradicts to $\lim_{u \to 0^+} M(u) = \lim_{u \to \infty} M(u) = 0$.) By the above analysis for $M(u)$, for $0 < r < M(E_p^*)q$, we obtain that the equation 
$M(u) = r/q$ has exactly two positive roots, say $E_{1,p}, E_{2,p}$ with $\bar{E}_p < E_{1,p} < E_p < E_{2,p}$, see Fig. 6. Thus, for $0 < r < M(E_p^*)q$, 
\[
\theta(u) = \frac{u^3}{3} \left[ \frac{r}{q} - M(u) \right]
\]
has two positive zeros $E_{1,p}, E_{2,p}$ such that
\[
\begin{cases}
\theta(E_{1,p}) = \theta(E_{2,p}) = 0, \\
\theta(u) > 0 \text{ on } (0, E_{1,p}) \cup (E_{2,p}, \infty), \\
\theta(u) < 0 \text{ on } (E_{1,p}, E_{2,p}).
\end{cases}
\] (52)

We compute that
\[
J(u) - M(u) = -\frac{(6 + (p + 8)u^p + 2u^{2p})}{u^2(1 + u^p)^2} + \frac{6}{u^3} \int_0^u \frac{1}{1 + t^p} dt = \frac{u}{3} M'(u)
\] by (49). Thus, if $1 < p \leq 2$,
\[
\begin{cases}
J(E_{1,p}) - M(E_{1,p}) = 0, \\
J(u) - M(u) > 0 \text{ on } (0, E_{1,p}), \\
J(u) - M(u) < 0 \text{ on } (E_{1,p}, \infty),
\end{cases}
\] (54)

and if $p > 2$,
\[
\begin{cases}
J(E_{1,p}) - M(E_{1,p}) = J(E_{1,p}) - M(E_{1,p}) = 0, \\
J(u) - M(u) < 0 \text{ on } (0, E_{1,p}), \\
J(u) - M(u) > 0 \text{ on } (E_{1,p}, E_{2,p}), \\
J(u) - M(u) < 0 \text{ on } (E_{2,p}, \infty).
\end{cases}
\] (55)

It is easy to check that $B_p^* < C_p^*$ and $I(B_p^*) > J(C_p^*)$. In addition, we know that $C_p^* < E_p^*$ and hence
\[
J(C_p^*) = \max_{u \in (0, \infty)} J(u) > J(E_p^*) = M(E_p^*)
\] by (54) and (55). So we obtain

(1) $B_p^* < C_p^* < E_p^*$ and $I(B_p^*) > J(C_p^*) > M(E_p^*)$.

In addition, we also obtain

(2) $E_p > \sqrt[p]{p-1} > \left(\frac{p-1}{p+1}\right)^{1/p}$ since
\[
\begin{cases}
E_p \in \left\{ \frac{m}{p+1}, \quad \text{if } 1 < p \leq 2, \\
E_p, E_p^*, \quad \text{if } p > 2, 
\end{cases}
\]

and by (54), (55), (29) and $M(E_p) = 0$.

(3) $\lim_{u \to \infty} I(u) = \lim_{u \to \infty} J(u) = \lim_{u \to \infty} M(u) = 0$.

(4) $I(u) < 0$ on $\left(0, \frac{m}{p+1}\right)^{1/p}$ and $I(u)$ has exactly one critical point, a local maximum, at $B_p^*$, on $\left((\frac{m}{p+1})^{1/p}, \infty\right)$. $J(u) < 0$ on $\left(0, \sqrt[p]{p-1}\right)$ and $J(u)$ has exactly one critical point, a local maximum, at $C_p^*$, on $\left(\sqrt[p]{p-1}, \infty\right)$. $M(u) < 0$ on $\left(0, E_p\right)$ and $M(u)$ has exactly one critical point, a local maximum, at $E_p^*$, on $\left(E_p, \infty\right)$.

(5)
\[
\begin{cases}
I(C_p^*) = J(C_p^*), \\
I(u) > J(u) \text{ on } \left(\sqrt[p]{p-1}, C_p^*\right), \\
I(u) < J(u) \text{ on } \left(C_p^*, \infty\right)
\end{cases}
\]
and
\[
\begin{cases}
J(E_p^*) = M(E_p^*), \\
J(u) > M(u) \text{ on } \left(E_p, E_p^*\right), \\
J(u) < M(u) \text{ on } \left(E_p^*, \infty\right)
\end{cases}
\]

by (39), (41), (54), (55) and $C_p^* < \sqrt[p]{p-1}$, $E_p < E_p^* \text{ if } p > 2$. 

(6) Functions $I(u)$, $J(u)$ and $M(u)$ are all strictly decreasing on $(E_p^*, \infty)$.
(7) The largest positive zeros of functions $I(u) - J(u)$, $J(u) - M(u)$ on $(0, \infty)$
are $C_p$ and $E_p^*$ respectively. We define the largest positive zero of function
$I(u) - M(u)$ is $\zeta_p$. Thus we obtain $C_p^* < \zeta_p < E_p^*$ by $C_p^* < E_p^*$ and property
(5). Hence
\[
\eta_{2,p} = I(\zeta_p) = N(E_p^*) < M(E_p^*) < J(C_p^*) < I(B_p^*)
\]
by (13) and property (1). See Fig. 6 for graphs of functions $I(u)$, $J(u)$, and
$M(u)$ on $(0, \infty)$.

By properties (1)–(7), we obtain that for $0 < r \leq \eta_{2,p}q$, $B_{1,p} < C_{1,p} < E_{1,p} \leq
B_{2,p} < C_{2,p} < E_{2,p}$, see Fig. 6. In addition, we know that, for $(q, r) \in \hat{R}_2 \cup \Gamma_1$ and
$r \leq \eta_{2,p}q$, $g(u)$ changes from decreasing to increasing then to decreasing on $[0, \beta_3]$.
Thus, by $\theta'(u) = -u^2g'(u)$ and (32), we obtain that $C_{2,p} < \beta_3$. By (9), we know
that, for $(q, r) \in \hat{R}_2 \cup \Gamma_1$ and $r \leq \eta_{2,p}q$, $f(u)du > 0$. Then $\theta(\beta_3) = 2F(\beta_3) > 2F(\beta_1) > 0$ by Lemma 3.1(ii)(iii). Thus $E_{2,p} < \beta_3$ by (52) and $\beta_3 > C_{2,p} > E_{1,p}$.
So we obtain that, for $(q, r) \in \hat{R}_2 \cup \Gamma_1$ and $r \leq \eta_{2,p}q$, $B_{1,p} < C_{1,p} < E_{1,p} \leq B_{2,p} < C_{2,p} < E_{2,p} < C_{2,p}$.

The proof of Lemma 3.5 is complete.

**Lemma 3.6.** Consider $f(u) = ru(1 - \frac{u}{q}) - \frac{u^p}{1+u^p}$ with $p > 1$, $q, r > 0$, and let $r/q = \eta$. Then
\[
\eta_{2,p} = \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \}
\]
exists. In addition, if $0 < \eta < \eta_{2,p}$, then $\theta(B_{1,p}) - B_{1,p}\theta'(B_{1,p}) - \theta(C_{2,p}) > 0$.

**Proof of Lemma 3.6.** Recall (18) and (19),
\[
N_3(u) = -\frac{[6u - (p^2 - 4p - 12)u^{p+1} + (p^2 + 4p + 6)u^{2p+1}]}{3(1 + u^p)^3} + 2 \int_0^u \frac{1}{1 + t^p} dt
\]
and
\[
N_4(u) = \frac{u(6 + (p + 8)u^p + 2u^{2p})}{3(1 + u^p)^2} - 2 \int_0^u \frac{1}{1 + t^p} dt.
\]
For $0 < \eta < \eta_{2,p}$, it is easy to prove that $B_{1,p}(\eta) \to \left(\frac{p-1}{p+1}\right)^{1/p}$ and $C_{2,p}(\eta) \to \infty$ as
$\eta \to 0^+$, and
\[
\lim_{u \to \infty} \int_0^u \frac{1}{1 + t^p} dt = 0.
\]
Thus $N_3(B_{1,p}(\eta)) \to N_3\left(\left(\frac{p-1}{p+1}\right)^{1/p}\right)$ and $N_4(C_{2,p}(\eta)) \to \infty$ as $\eta \to 0^+$. Hence
$N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) \to \infty$ as $\eta \to 0^+$. It follows that
\[
\{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \} \neq \emptyset,
\]
and it is bounded above by $\eta_{2,p}$. By the Completeness Axiom, we obtain
\[
\eta_{2,p} = \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \}
\]
exists and hence $N_3(B_{1,p}(\eta_{2,p})) + N_4(C_{2,p}(\eta_{2,p})) \geq 0$.

Next, we want to prove that, for $0 < \eta \leq \eta_{2,p}$, $\theta(B_{1,p}) - B_{1,p}\theta'(B_{1,p}) - \theta(C_{2,p}) \geq 0$,
which we prove as follows:
By \( \eta = \frac{r}{q} = I(B_{1,p}) \), (31) and (51), we obtain that
\[
\theta(B_{1,p}) - B_{1,p}\theta'(B_{1,p}) = \frac{B_{1,p}^3}{3}[I(B_{1,p}) - M(B_{1,p})] - B_{1,p}^3[I(B_{1,p}) - J(B_{1,p})] = -\left[\frac{6B_{1,p} - (p^2 - 4p - 12)B_{1,p}' + (p^2 + 4p + 6)B_{1,p}'}{3(1 + B_{1,p}'^2)}\right] + 2\int_{0}^{B_{1,p}} \frac{1}{1 + u^p} dt.
\]

by (37) and (53). In addition, by \( \eta = \frac{r}{q} = J(C_{2,p}) \), (51) and (53), we obtain that
\[
-\theta(C_{2,p}) = -\frac{C_{2,p}^3}{3}[J(C_{2,p}) - M(C_{2,p})] = \frac{C_{2,p}}{3}\int_{0}^{C_{2,p}} \frac{1}{1 + u^p} dt.
\]

Hence \( \theta(B_{1,p}) - B_{1,p}\theta'(B_{1,p}) - \theta(C_{2,p}) = N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) \).

We compute that
\[
N_3'(u) = \frac{pu^n((p^2 - 3p + 2) + 4(1 - p^2)u^n + (p^2 + 3p + 2)u^{2n})}{3(1 + u^n)^3} = -\frac{2}{3}u^3J'(u)
\]
by (28). Thus we obtain that, if \( 1 < p \leq 2 \),
\[
\begin{cases}
N_3'(B_p^*) = 0, \\
N_3'(u) < 0 \text{ on } (0, B_p^*), \\
N_3'(u) > 0 \text{ on } (B_p^*, \infty),
\end{cases}
\]
and if \( p > 2 \),
\[
\begin{cases}
N_3'(\tilde{B}_p) = N_3'(B_p^*) = 0, \\
N_3'(u) > 0 \text{ on } (0, \tilde{B}_p), \\
N_3'(u) < 0 \text{ on } (\tilde{B}_p, B_p^*), \\
N_3'(u) > 0 \text{ on } (B_p^*, \infty).
\end{cases}
\]

By (57), (58) and \( \tilde{B}_p < \left(\frac{p-1}{p+1}\right)^{1/p} < B_p^* \), we obtain that \( N_3(u) \) is strictly decreasing on \( \left(\frac{p-1}{p+1}\right)^{1/p}, B_p^* \). In addition, we know that, for \( 0 < \eta \leq \eta_{2,p}^* \), \( B_{1,p}(\eta) \in (\left(\frac{p-1}{p+1}\right)^{1/p}, B_p^*) \) and \( B_{1,p}(\eta) \) is strictly increasing on \( (0, \eta_{2,p}^*]. \) Thus \( N_3(B_{1,p}(\eta)) \) is strictly decreasing on \( (0, \eta_{2,p}^*]. \) On the other hand, we also compute that
\[
N_3'(u) = \frac{-u^n[(p^2 - 3p + 2) + (p^2 + 3p - 4)u^n - 2u^{2n}]}{3(1 + u^n)^3} = -\frac{u^3}{3}J'(u)
\]
by (30). Thus we obtain that, if \( 1 < p \leq 2 \),
\[
\begin{cases}
N_4'(C_p^*) = 0, \\
N_4'(u) < 0 \text{ on } (0, C_p^*), \\
N_4'(u) > 0 \text{ on } (C_p^*, \infty),
\end{cases}
\]
and if \( p > 2 \),
\[
\begin{cases}
N_4'(\tilde{C}_p) = N_4'(C_p^*) = 0, \\
N_4'(u) > 0 \text{ on } (0, \tilde{C}_p), \\
N_4'(u) < 0 \text{ on } (\tilde{C}_p, C_p^*), \\
N_4'(u) > 0 \text{ on } (C_p^*, \infty).
\end{cases}
\]
By (59) and (60), we obtain that \(N_4(u)\) is strictly increasing on \((C^*_p, \infty)\). In addition, we know that, for \(0 < \eta \leq \eta^*_p\), \(C_{2,p}(\eta) \in (C^*_p, \infty)\) and \(C_{2,p}(\eta)\) is strictly decreasing on \((0, \eta^*_p]\). Thus \(N_4(C_{2,p}(\eta))\) is strictly decreasing on \((0, \eta^*_p]\). So \(N_3(B_1,p(\eta)) + N_4(C_{2,p}(\eta))\) is strictly decreasing on \((0, \eta^*_p]\) and hence for \(0 < \eta \leq \eta^*_p\),

\[
\theta(B_{1,p}) - B_{1,p} \theta'(B_{1,p}) - \theta(C_{2,p}) = -N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) \\
\geq N_3(B_{1,p}(\eta^*_p)) + N_4(C_{2,p}(\eta^*_p)) \geq 0.
\]

The proof of Lemma 3.6 is complete.

By Lemmas 3.1(ii)(iii), 3.5–3.6, and [16, Lemmas 3.2–3.3], we obtain the following Lemmas 3.7–3.8.

**Lemma 3.7.** Consider (1) with \(p > 1\). If \((q, r) \in \tilde{R}_2\) and \(r \leq \eta^*_p, q\), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2\sqrt{r}} \quad \text{and} \quad \lim_{\alpha \to \beta^-_1} T(\alpha) = \lim_{\alpha \to \gamma^+} T(\alpha) = \lim_{\alpha \to \beta^-_3} T(\alpha) = \infty.
\]

In addition, \(T(\alpha)\) is strictly increasing on \((0, \beta_1)\) and \(T(\alpha)\) has exactly one positive critical point at some \(\alpha_*\) on \((\gamma, \beta_3)\), such that \(T(\alpha_*)\) is a local minimum on \((\gamma, \beta_3)\).

**Lemma 3.8.** Consider (1) with \(p > 1\). If \((q, r) \in \Gamma_1\) and \(r \leq \eta^*_p, q\), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2\sqrt{r}} \quad \text{and} \quad \lim_{\alpha \to \beta^-_1} T(\alpha) = \lim_{\alpha \to \beta^-_3} T(\alpha) = \lim_{\alpha \to \beta^-_4} T(\alpha) = \infty.
\]

In addition, \(T(\alpha)\) is strictly increasing on \((0, \beta_1)\) and \(T(\alpha)\) has exactly one positive critical point at some \(\alpha_*\) on \((\beta_1, \beta_3)\), such that \(T(\alpha_*)\) is a local minimum on \((\beta_1, \beta_3)\).

The proofs of Theorems 2.2–2.3 follows by Lemmas 3.7–3.8 and the fact that \(\lambda < \lambda_\ast\), which see the proof of Theorem 2.1, and hence we omit it.

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