Large gap asymptotics for the Meijer-$G$ point process

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Abstract

We compute the constant term in the large gap asymptotic expansion of the Meijer-$G$ point process. This point process generalizes the Bessel point process and appears at the hard edge of Cauchy–Laguerre multi-matrix models and of certain product random matrix ensembles.

1 Introduction and main results

The Meijer-$G$ point process is a determinantal point process whose kernel is built out of Meijer $G$-functions. It appears in the study of the smallest squared singular values of certain product random matrices [17, 15] and in Cauchy multi-matrix models [8, 5] in the limit of large matrix dimension (see below for more details). Of particular interest is the distribution of the smallest particle, or equivalently, the probability of finding no particle in the interval $[0, s]$, $s > 0$. In some particular cases, this distribution is related to a system of partial differential equations [22, 21]. In this work, we are interested in the tail behavior of this distribution as $s \to +\infty$, known as the large gap asymptotics. The study of such asymptotics was initiated in [11], where the first two terms in the asymptotic expansion were found, and then pursued in [10], where the third term was obtained. The purpose of this paper is to obtain an explicit expression for the next term in the expansion, which is the constant term.

Meijer-$G$ point process. Let $r > q \geq 0$ be fixed integers and let

$$\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1.$$ 

The Meijer-$G$ point process is a determinantal point process on $\mathbb{R}^+ = (0, +\infty)$ whose kernel is given by

$$K(x, y) = \int_0^1 \binom{\mu_1, \ldots, -\mu_q}{0, -\nu_1, \ldots, -\nu_r} tx \binom{\mu_1, \ldots, \mu_q}{\nu_1, \ldots, \nu_r, 0} ty \, dt, \quad (1.1)$$

where $G$ is the Meijer $G$-function (see e.g. [19, Eq. 16.17.1] for a definition). Equivalently, the kernel (1.1) can be written as

$$K(x, y) = \int_{\gamma} \frac{du}{2\pi i} \int_{\tilde{\gamma}} \frac{dv}{2\pi i} F(u)x^{-u}y^{-1}, \quad (1.2)$$

where $F$ is given in terms of the Gamma function $\Gamma$ by

$$F(z) = \frac{\Gamma(z) \prod_{j=1}^q \Gamma(1 + \mu_j - z)}{\prod_{j=1}^r \Gamma(1 + \nu_j - z)}. \quad (1.3)$$
The contours $\gamma$ and $\tilde{\gamma}$ in (1.2) are disjoint, oriented upwards, and tend to infinity in sectors lying strictly in the left and right half-planes, respectively. Furthermore, they separate the poles of $\Gamma(z)$ from the poles of $\prod_{j=1}^{q} \Gamma(1 + \mu_j - z) \prod_{j=1}^{r} \Gamma(1 + \nu_j - z)$, see Figure 1.

The Meijer-G point process generalizes in a natural way the Bessel point process, which is the point process most commonly encountered at hard edges for random matrix ensembles. In fact, if $q = 0$, $r = 1$ and $\nu_1 = \nu > -1$, then the kernel (1.1) can be written as

$$K(x, y) = \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \int_0^1 J_\nu(2\sqrt{tx})J_\nu(2\sqrt{ty})dt = 4 \left(\frac{y}{x}\right)^{\frac{\nu}{2}} K_{\text{Be}}(4x, 4y),$$

where $K_{\text{Be}}$ is the kernel of the Bessel point process

$$K_{\text{Be}}(x, y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_\nu'(\sqrt{y}) - \sqrt{x}J_\nu'(\sqrt{x})J_\nu(\sqrt{y})}{2(x - y)},$$

and $J_\nu$ is the Bessel function of the first kind of order $\nu$.

### Large gap asymptotics.

We consider the probability to observe a gap on $[0, s]$ in the Meijer-G point process. It follows from the general theory of determinantal point processes [20] that this probability can be written as a Fredholm determinant:

$$\mathbb{P}(\text{gap on } [0, s]) = \det \left( 1 - K_{[0,s]} \right).$$

It is well-known [22] that the distribution of the smallest particle in the Bessel point process (which corresponds to $q = 0$ and $r = 1$) is naturally expressed in terms of the solution of a Painlevé V equation. For $q = 0$, $r \geq 2$ and integer values of $\nu_1, \ldots, \nu_r$, the Fredholm determinant (1.5) is instead related to a more involved system of partial differential equations [21]. The study of the tail behavior as $s \to +\infty$ of this distribution (i.e. the so-called large gap asymptotics) has been initiated by Claeys, Girotti and Stivigny in [11]. They proved that there exist real constants $\rho, a, b, c$ and $C$ such that

$$\det \left( 1 - K_{[0,s]} \right) = C \exp \left( -as^{2\rho} + bs^\rho + c \ln s + \mathcal{O}(s^{-\rho}) \right), \quad \text{as } s \to +\infty,$$
and derived the following explicit expressions for \(\rho, a\) and \(b\):

\[
\rho = \frac{1}{1 + r - q}, \quad a = \frac{(r - q)^2}{4}, \quad b = (1 + r - q)(r - q) - \sum_{j=1}^{r} \nu_j - \sum_{k=1}^{q} \mu_k.
\]

(1.7)

These asymptotics have been extended to all orders in \([10]\), where it was shown that, for any \(N \in \mathbb{N}\), there exist constants \(C_1, \ldots, C_N \in \mathbb{R}\) such that

\[
\det (1 - \mathbb{K}|_{[0,s]}) = C \exp \left( -as^{2\rho} + bs^{\rho} + c \ln s + \sum_{j=1}^{N} C_j s^{-j\rho} + \mathcal{O}(s^{-(N+1)\rho}) \right), \quad s \to +\infty.
\]

(1.9)

Furthermore, the following explicit expression for \(c\) was obtained (see [10, Remark 5.2]):

\[
c = \frac{r - q - 1}{12(r - q + 1)} - \frac{1}{2(r - q + 1)} \left( \sum_{j=1}^{r} \nu_j^2 - \sum_{k=1}^{q} \mu_k^2 \right).
\]

(1.10)

In this paper, we derive an explicit expression for the constant \(C\) in (1.9). The value of this constant was previously unknown except for some very particular choices of the parameters, see Section 1.2. Note that \(C\) is a multiplicative constant in (1.9), which means that there is no accurate description of the large gap asymptotics without its explicit expression. The following theorem is our main result.

**Theorem 1.1** (Explicit expression for the constant \(C\)). Let \(r > q \geq 0\) be fixed integers and let \(\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1\). The constant \(C\) that appears in the asymptotic formula (1.6) is given by

\[
C = \frac{\prod_{j=1}^{r} G(1 + \nu_j) \prod_{k=1}^{q} (2\pi)^{\frac{\mu_k}{2}}}{\prod_{k=1}^{q} G(1 + \mu_k) \prod_{j=1}^{r} (2\pi)^{\frac{\nu_j}{2}}} \exp \left\{ (1 - r + q) \zeta'(-1) \right\}
\]

\[
\times \exp \left\{ \left( \frac{1 - r + q - (r - q)^2}{2(1 + r - q)} \left[ \sum_{j=1}^{r} \nu_j^2 - \sum_{k=1}^{q} \mu_k^2 \right] + \frac{-2 + (r - q)^2(r - q - 1)}{24(1 + r - q)} \right) \right. \]

\[
\left. + \sum_{1 \leq j < k \leq r} \nu_j \nu_k + \sum_{1 \leq j < k \leq q} \mu_j \mu_k - \sum_{j=1}^{r} \nu_j \sum_{k=1}^{q} \mu_k + \sum_{j=1}^{r} \nu_j^2 \sum_{k=1}^{q} \mu_k^2 \right\} \ln(r - q) \}
\]

\[
\times \exp \left\{ \left( -\frac{2 - r + q}{2} \left[ \sum_{j=1}^{r} \nu_j^2 - \sum_{k=1}^{q} \mu_k^2 \right] - \frac{(r - q - 1)^2}{24} \right) \right. \]

\[
\left. - \sum_{1 \leq j < k \leq r} \nu_j \nu_k - \sum_{1 \leq j < k \leq q} \mu_j \mu_k + \sum_{j=1}^{r} \nu_j \sum_{k=1}^{q} \mu_k + \sum_{j=1}^{r} \nu_j^2 \sum_{k=1}^{q} \mu_k^2 \right\} \ln(1 + r - q) \}
\]

(1.11)

where \(G\) denotes Barnes’ \(G\)-function and \(\zeta'(-1)\) the derivative of Riemann’s zeta function evaluated at \(-1\).

The proof of Theorem 1.1 will be given in Section 6 after considerable preparations have been carried out in Sections 2-5.

As mentioned earlier, explicit expressions for the constant \(C\) have already appeared in the literature for some very particular choices of the parameters. We recall these expressions and show that Theorem 1.1 is consistent with them in Section 1.2.
1.1 Applications of Theorem 1.1 to hard edge scaling limits

The Meijer-$G$ point process appears at the hard edge scaling limit of several random matrix ensembles as the size of the matrices becomes large. In what follows, we explain how Theorem 1.1 can be used to obtain information on the large gap asymptotics at the hard edge for these models.

**The Cauchy-Laguerre two-matrix model.** The Cauchy two-matrix model has been introduced in [7] and is a model for two positive definite Hermitian matrices coupled in a chain. The probability density function is defined over all pairs $(M_1, M_2)$ of two $n \times n$ positive definite Hermitian matrices, and it takes the form

$$
\frac{1}{Z_n} \det(M_1)^{\alpha_1} \det(M_2)^{\alpha_2} e^{-\text{Tr}(V_1(M_1) + V_2(M_2))} \, dM_1 \, dM_2,
$$

(1.12)

where $Z_n$ is the normalization constant, the two scalar potentials $V_1(x)$, $V_2(x)$ grow sufficiently fast as $x \to +\infty$, and the parameters $\alpha_1$ and $\alpha_2$ satisfy $\alpha_1 > -1$, $\alpha_2 > -1$, and $\alpha_1 + \alpha_2 > -1$. The eigenvalues $x_1, \ldots, x_n$ of $M_1$ together with the eigenvalues $y_1, \ldots, y_n$ of $M_2$ form a two-level determinantal point process. The Meijer-$G$ kernel (1.1) was first discovered by Bertola, Gekhtman and Szmigielski in [8], where they considered the special case of $V_1(x) = V_2(x) = x$, known as the Cauchy-Laguerre two-matrix model. If we consider the point process involving only the eigenvalues of $M_1$, the associated limiting kernel as $n \to +\infty$ in the hard edge scaling limit, denoted by $G_{01}$ in [8], is given by $(\hat{\beta}^q)^{\alpha_1} K(x, y)$, where $K$ is given by (1.1) with $r = 2$, $q = 0$, $\nu_1 = \alpha_1 + \alpha_2$ and $\nu_2 = \alpha_1$. Letting $x_{\min} := \min\{x_1, \ldots, x_n\}$, we have (see [11, Appendix])

$$
\lim_{n \to +\infty} \mathbb{P}(x_{\min} > \hat{\beta}^q) = \det \left(1 - K[0, a]\right),
$$

(1.13)

and we can obtain the tail behavior as $s \to +\infty$ up to and including the constant for the right-hand-side of (1.13) by combining (1.9), (1.7), (1.8), (1.10) and (1.11) with $r = 2$, $q = 0$, $\nu_1 = \alpha_1 + \alpha_2$ and $\nu_2 = \alpha_1$.

**Cauchy-Laguerre multi-matrix models.** The Cauchy two-matrix model has been generalized to an arbitrary number $r$ of matrices in [5], and similar results have been obtained at the hard edge as in the case of two matrices. Large gap asymptotics up to and including the constant can also be obtained with the help of Theorem 1.1, with $q = 0$ but general values of $r$, $\nu_1, \ldots, \nu_r$. Note that in both works [8] and [5], the Meijer-$G$ kernel has only been obtained in the hard edge scaling limit for the special case when all potentials are linear (the Laguerre case). The same kernel is expected to appear for a large class of potentials (this is called universality in random matrix theory), but to prove this claim rigorously remains an open problem.

**Products of Ginibre matrices.** A complex Ginibre matrix is a random matrix whose entries are independent and identically distributed complex Gaussian variables. Let $G_1, \ldots, G_r$ be independent complex standard Ginibre matrices of size $(n + \nu_j) \times (n + \nu_{j-1})$, where $\nu_0 = 0$ and $\nu_1, \ldots, \nu_r$ are non-negative integers, and consider the product

$$
G_r \ldots G_1.
$$

(1.14)

If $r = 1$, the squared singular values are well-studied and follow the Laguerre Unitary Ensemble, which is a determinantal point process whose limiting kernel as $n \to +\infty$ in the hard edge scaling limit is given by the Bessel kernel (1.4) with $\nu = \nu_1$. For general $r \geq 2$, it is known from Akemann, Kieburg and Wei [3] that the squared singular values of (1.14) still form a determinantal point process, and from Kuijlaars and Zhang [17] that the limiting kernel in the hard edge scaling limit is
the Meijer-G kernel (1.1) with \( q = 0 \) (we refer to [2] for an excellent and more detailed overview of the existing literature on product random matrices). If \( x_{\text{min}} \) denotes the smallest squared singular value of (1.14), then the limit (1.13) holds, and we can obtain the tail behavior as \( s \to +\infty \) up to and including the constant for the right-hand-side of (1.13) by combining (1.9), (1.7), (1.8), (1.10) and (1.11), and setting \( q = 0 \).

**Products of truncated unitary matrices.** Let \( U_1, \ldots, U_r \) be \( r \) independent Haar distributed unitary matrices of size \( \ell_1 \times \ell_1, \ldots, \ell_r \times \ell_r \), respectively, and let \( T_j \) be the upper left \( (n+\nu_j) \times (n+\nu_{j-1}) \) truncation of \( U_j, j = 1, \ldots, r \). The parameters \( \ell_1, \ldots, \ell_r \) are positive integers, \( \nu_0 = 0 \) and \( \nu_1, \ldots, \nu_r \) are non-negative integers. Furthermore, assume that \( \ell_1 \geq 2n+1 \) and \( \ell_j \geq n+\nu_j+1 \) for \( j \geq 2 \). The squared singular values of the product

\[
T_r \ldots T_1
\]

form a determinantal point process [15]. Taking \( n \to +\infty \), we simultaneously have to let \( \ell_j \to +\infty \) for \( j = 1, \ldots, r \). We choose a subset \( J \) of indices

\[
J = \{ j_1, \ldots, j_q \} \subset \{2, \ldots, r\}, \quad \text{with } 0 \leq q = |J| < r
\]

and integers \( \mu_1, \ldots, \mu_q \) with \( \mu_k \geq \nu_k + 1 \), and assume that

\[
m_j - n \to +\infty, \quad \ell_j - n = \mu_k, \quad \text{for } j \in \{1, \ldots, r\} \setminus J, \quad \text{for } j_k \in J.
\]

Letting \( x_{\text{min}} \) denote the smallest squared singular value of (1.15), it follows from [15, Theorem 2.8] and [11, Appendix] that

\[
\lim_{n \to +\infty} \mathbb{P}(x_{\text{min}} > \frac{s}{c_n}) = \det (1 - K_{[0,n]}), \quad \text{with } c_n = n \prod_{j \in J} (m_j - n),
\]

and as in (1.13) we can obtain asymptotics as \( s \to +\infty \) for (1.16) up to and including the constant by combining (1.9), (1.7), (1.8), (1.10) and (1.11). Note that in the above two models of product random matrices, we only need to utilize Theorem 1.1 for integer values of the parameters.

### 1.2 Consistency checks for Theorem 1.1

We provide three different consistency checks for Theorem 1.1; the first two verify consistency with known results in the literature for special choices of the parameters, while the third verifies consistency under the transformation \((r, q) \to (r + 1, q + 1)\) whenever \( \nu_{r+1} = \mu_{q+1} \). For ease of explanation, we sometimes indicate the dependence of \( \rho, a, b, c, C \) on the parameters \( r, \nu_1, \ldots, \nu_r, q \) and \( \mu_1, \ldots, \mu_q \) explicitly, e.g. for \( C \) we write

\[
C((r; \nu_1, \ldots, \nu_r), (q; \mu_1, \ldots, \mu_q)).
\]

**Consistency with known results for the Bessel point process.** For \( r = 1, q = 0 \) and \( \nu_1 = \nu \), the asymptotics for (1.5) are known from Deift, Krasovsky and Vasilevska [12]:

\[
\det \left(1 - K_{[0,n]}\right) = \frac{G(1 + \nu)}{(2\pi)^{\frac{n}{2}}} \exp \left(- s + 2\nu \sqrt{s} - \frac{\nu^2}{4} \ln(4s) + \mathcal{O}(s^{-1/2}) \right)
\]

as \( s \to +\infty \). We verify from (1.7), (1.8), (1.10) and (1.11) that

\[
\rho((1; \nu), (0; -)) = \frac{1}{2}, \quad a((1; \nu), (0; -)) = 1,
\]

\[
C((1; \nu), (0; -)) = 1.
\]
\[ b(1; \nu, (0; -)) = 2\nu, \quad c(1; \nu, (0; -)) = -\frac{\nu^2}{4}, \]

and

\[ C((1; \nu, (0; -)) = \frac{G(1 + \nu)}{(2\pi)^{1/2}} 2^{-\nu^2/2}, \]

which is consistent with (1.17).

**Consistency with known results for the Muttalib-Borodin ensembles.** The Muttalib–Borodin ensembles [18] are joint probability density functions of the form

\[
\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)(x_k^\theta - x_j^\theta) \prod_{j=1}^n x_j^\theta e^{-x_j} dx_j, \quad (1.18)
\]

where the \( n \) points \( x_1, \ldots, x_n \) belong to the interval \([0, +\infty)\), \( Z_n \) is a normalization constant and \( \theta > 0 \) and \( \alpha > -1 \) are two parameters of the model. This is a determinantal point process whose hard edge limiting kernel \( K_{MB} \) can be written in terms of Wright’s generalized Bessel functions [9].

The corresponding large gap asymptotics are of the form [11]

\[
\det \left( 1 - K_{MB}^{MB} \right) \sim C_{MB} \exp \left( -a_{MB} s^{2\rho_{MB}} + b_{MB} s^{\rho_{MB}} + c_{MB} \ln s + O(s^{-\rho_{MB}}) \right), \quad \text{as } s \rightarrow +\infty.
\]

The constants \( \rho_{MB}, a_{MB} \) and \( b_{MB} \) have been obtained in [11], and \( c_{MB} \) and \( C_{MB} \) in [10]. For \( q = 0 \) and certain particular choices of the parameters \( r, \nu_1, \ldots, \nu_r, \alpha, \) and \( \theta \), the kernels \( K \) and \( K_{MB} \) define the same point process (up to rescaling), see [16, Theorem 5.1]. More precisely, if \( r \geq 1 \) is an integer, \( \alpha > -1 \) and

\[
\theta = \frac{1}{r}, \quad \nu_j = \alpha + \frac{j - 1}{r}, \quad j = 1, \ldots, r, \quad (1.19)
\]

then the kernels \( K \) and \( K_{MB} \) are related by

\[
\left( \frac{x}{y} \right)^\alpha K(x, y) = r^r K_{MB}(r^r x, r^r y).
\]

Therefore, if the parameters satisfy (1.19), we obtain the following relations:

\[
\rho = \rho_{MB}, \quad a = a_{MB} r^{2\rho}, \quad b = b_{MB} r^{\rho}, \quad c = c_{MB}, \quad (1.20)
\]

\[
C = r^{r} C_{MB}. \quad (1.21)
\]

The relations (1.20) were already verified in [10, Remark 5.2], and we now verify that (1.21) holds. Let us explicitly write the dependence of \( C_{MB} \) on \( \theta \) and \( \alpha \). From [10, Theorem 1.1], we have

\[
C_{MB}(\theta, \alpha) = \frac{G(1 + \alpha)}{(2\pi)^{1/2}} \exp \left( d(1, \alpha) - d(\theta, \alpha) \right) \exp \left( \frac{24\alpha(\alpha + 2) + 15 + 3\theta + 4\theta^2}{24(1 + \theta)} \ln \theta \right)
\times \exp \left( \frac{6\alpha \theta - 6\alpha(1 + \alpha) - (\theta - 1)^2}{12\theta} \ln(1 + \theta) \right), \quad (1.22)
\]

where \( d(\theta, \alpha) \) is a regularized sum, see [10, Eq. (1.12)]. If \( \theta \) is rational, it follows from [10, Proposition 1.4] that \( d(\theta, \alpha) \) can be expressed in terms of \( \zeta'(-1) \) and Barnes’ \( G \)-function evaluated at certain points. By specializing [10, Proposition 1.4] for \( \theta = \frac{1}{r} \), we obtain

\[
d(\frac{1}{r}, \alpha) = r \zeta'(-1) + \frac{1 + (1 + 2\alpha)r}{4} \ln(2\pi) - \frac{1}{12} \left( 3 + \frac{1}{r} + r + 6\alpha(1 + r + \alpha r) \right) \ln r
\]
of the Fredholm determinant of an integrable kernel in terms of the solution

\begin{equation}
- \sum_{k=1}^{r} \ln G \left(1 + \alpha + \frac{k}{r} \right).
\end{equation}

Substituting (1.23) into (1.22) with \( \theta = \frac{1}{r} \), we obtain after a direct computation that

\begin{equation}
r^{rc}C_{MB}(\frac{1}{r}, \alpha) = \frac{G(1 + \alpha) \prod_{j=1}^{r} G (1 + \alpha + \frac{j}{r})}{G(2 + \alpha)(2\pi)^{\frac{r}{2}}} \exp\{\zeta(-1)r^{1/2}\} \times \exp \left\{ -\frac{1}{24} \frac{r(5 + 12\alpha) + 2r^2(1 + 6\alpha + 6\alpha^2)\ln r}{24(1 + r)} \right\} \times \exp \left\{ \frac{1 - 2r(1 + 3\alpha) + r^2(1 + 6\alpha + 6\alpha^2)\ln(1 + r)}{12r} \right\}.
\end{equation}

On the other hand, by substituting the particular values of \( \nu_j \) given by (1.19) into (1.11), another long but straightforward computation shows that

\begin{equation}
C((r; \alpha, \alpha + \frac{1}{r}, \ldots, \alpha + \frac{r-1}{r}), (0, -))
\end{equation}

is also given by the right-hand-side of (1.24), which proves (1.21).

**Poles-zeros cancellation.** If one increases simultaneously \( r \) and \( q \) by 1, with \( \nu_{r+1} \) and \( \mu_{q+1} \) such that \( \nu_{r+1} = \mu_{q+1} \), it is easy to see from (1.3) that \( F \) (and henceforth the kernel \( K \)) remains unchanged. We verify directly from (1.11) that

\begin{equation}
C((r + 1; \nu_1, \ldots, \nu_r, \nu_{r+1}), (q + 1; \mu_1, \ldots, \mu_q, \mu_{q+1})) = C((r; \nu_1, \ldots, \nu_r), (q; \mu_1, \ldots, \mu_q)),
\end{equation}

which is consistent with this observation.

### 1.3 Outline of the proof

The expressions (1.7), (1.8) and (1.10) for the coefficients \( \rho, a, b \) and \( c \), as well as the all-order expansion (1.9), were obtained in [11, 10] via a method that we briefly explain here. There is a standard procedure, named after Its, Izergin, Korepin and Slavnov (IIKS) [14], which expresses the logarithmic derivative\(^1\) of the Fredholm determinant of an integrable kernel in terms of the solution of a Riemann–Hilbert (RH) problem. The kernel \( K \) given in (1.2) is not integrable (in general), but it was shown in [11] that the IIKS procedure can still be applied (this fact is far from obvious—it is based on ideas from [4, 6] and uses the Mellin transform, see [11] for details). Using the IIKS procedure, the authors of [11] were able to express

\begin{equation}
\partial_s \ln \det \left(1 - K_{[0,s]} \right)
\end{equation}

in terms of the solution \( Y \) of a \( 2 \times 2 \) matrix Riemann–Hilbert (RH) problem. A Deift/Zhou nonlinear steepest descent analysis [13] of this RH problem yields an all-order expansion of \( Y \), and hence also of (1.25), as \( s \to +\infty \) [11, 10]. By substituting this expansion of (1.25) into the relation

\begin{equation}
\ln \det \left(1 - K_{[0,s]} \right) = \ln \det \left(1 - K_{[0,M]} \right) + \int_{M}^{s} \partial_{s'} \ln \det \left(1 - K_{[0,s']} \right) ds',
\end{equation}

where \( M \) is a sufficiently large but fixed constant, and then integrating with respect to \( s' \), we can deduce the existence of the all-order expansion (1.9) of the Fredholm determinant. The constants

\(^1\)The derivative can be taken with respect to any given parameter of the associated kernel, as long as the kernel depends smoothly on this parameter.
$C_1, C_2, \ldots$ appearing in (1.9) could also be computed in this way (with more efforts). However, with this method, the integration constant

$$\ln \det \left( 1 - K_{[0,M]} \right)$$

cannot be computed explicitly, and this is an essential obstacle for the evaluation of $C$.

Therefore, we need to use a different differential identity than (1.25) to evaluate $C$ (i.e. a differential identity with respect to another parameter than $s$). The large gap asymptotics for the Bessel point process is known up to and including the constant (see (1.17)), so the idea is to find a path in the set of parameters $r, q, \nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q$ which interpolates smoothly between the Bessel kernel and $K$. The existence of such a path is a priori not clear, since the parameters $r$ and $q$ are integers. The simple, but central idea of this paper is to first set the parameters $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q$ associated to $K$ equal to $\nu_{\text{min}}$, where

$$\nu_{\text{min}} = \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\},$$

and “smooth” the product of Gamma functions in (1.3) by considering the following kernel:

$$K_r(x, y) = \int \frac{du}{2\pi i} \int \frac{dv}{2\pi i} F_r(u) x^{-u} y^{v-1} F_r(v) \frac{x^{-u} y^{v-1}}{v-u},$$

where $r \geq 1^2$ and $\nu > -1$ are real-valued parameters and

$$F_r(z) = \frac{\Gamma(z)}{\Gamma(1+\nu-z)^r},$$

and where we choose the branch such that $F_r(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, 0]$. Starting with $r = 1$ (note that $K_r$ reduces to the Bessel kernel for $r = 1$), we first increase $r$ from 1 to $r - q$ and then successively move each of the remaining parameters from $\nu_{\text{min}}$ to its desired value. The process relies on the successive integration of appropriate differential identities for the following quantities:

$$\partial_r \ln \det \left( 1 - K_r[0,s] \right), \quad \partial_{\nu_\ell} \ln \det \left( 1 - K_r[0,s] \right), \quad \ell \in \{1, \ldots, r\}, \quad \ell \in \{1, \ldots, q\}.$$

More precisely, let us define $K^{(\ell)}$, $\ell \in \{0, 1, \ldots, r+1, \ldots, r+q\}$ by

$$K^{(\ell)} = \begin{cases} K_{[\nu_{\ell+1} = \ldots = \nu_r = \mu_1 = \ldots = \mu_q = \nu_{\text{min}}]}, & \text{if } \ell \in \{0, \ldots, r-1\}, \\ K_{[\mu_{r-\ell+1} = \ldots = \mu_q = \nu_{\text{min}}]}, & \text{if } \ell \in \{r, \ldots, r+q\}. \end{cases}$$

Note that $K^{(0)} = K_{r-q}$, where $K_{r-q}$ is defined by (1.27) (with $r$ replaced by $r - q$ and $\nu$ by $\nu_{\text{min}}$), and $K^{(r+q)} = K$. By integrating successively (1.29), (1.30) and (1.31), we obtain

$$\ln \det \left( 1 - K_{r-q}[0,s] \right) = \ln \det \left( 1 - K_{r=1}[0,s] \right) + \int_1^{r-q} \partial_r \ln \det \left( 1 - K_{r'}[0,s] \right) dr', \quad \ell = 1, \ldots, r,$$

$$\ln \det \left( 1 - K^{(\ell)}[0,s] \right) = \ln \det \left( 1 - K^{(\ell-1)}[0,s] \right) + \int_{\nu_{\text{min}}}^{\nu_{\ell}} \partial_{\nu_\ell} \ln \det \left( 1 - K^{(\ell)}[0,s] \right) d\nu_\ell, \quad \ell = 1, \ldots, r,$$

$$\ln \det \left( 1 - K^{(r+\ell)}[0,s] \right) = \ln \det \left( 1 - K^{(r+\ell-1)}[0,s] \right) + \int_{\nu_{\text{min}}}^{\nu_{\ell}} \partial_{\mu_\ell} \ln \det \left( 1 - K^{(r+\ell)}[0,s] \right) d\mu_\ell, \quad \ell = 1, \ldots, q.$$

\footnote{In fact, $K_r$ is well-defined for all $r > -1$ but we will use it only for $r \geq 1$.}
Since the large $s$ asymptotics of $\ln \det \left(1 - \mathbb{I}_{r=1}|_{(0,s)}\right)$ are known up to and including the constant term, see (1.17), this method allows us to obtain $C$ by keeping track of the term of order $1$ in the identities (1.33).

**Remark 1.2.** In this work, we focus on proving the expression (1.11) for $C$. We could also have computed the coefficients $\rho$, $a$, $b$ and $c$ with the same method (this would have provided an alternative proof and another consistency check for these constants), but in order to limit the complexity and length of the paper, we have decided to not pursue this direction.

### 1.4 Organization of the paper

By employing the IIKS procedure, we will express the quantities (1.29), (1.30) and (1.31) in terms of $Y$. Since we use the same RH problem as in [11, 10], we can recycle some of the analysis of these papers. We present the necessary material from [11, 10] in Section 2. In Section 3, we express the quantities (1.29), (1.30) and (1.31) in terms of $Y$. Section 4 is devoted to the first differential identity. More precisely, we compute the large $s$ asymptotics of (1.29) up to and including the constant term and then perform the integration with respect to $r'$ in (1.33). In Section 5, we proceed similarly with the differential identities with respect to $\nu_i$, $i = 1, ..., r$ and $\mu_j$, $j = 1, ..., q$. The proof of Theorem 1.1 is completed in Section 6.

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### 2 Preliminary results from [11, 10]

In this section we recall some results from [11, 10] that will be used throughout the paper. The notation adopted in this paper is the same as in [10], and is also almost identical to the notation used in [11]. The only difference with [11] is that the function $G$ is this paper and in [10] is instead denoted by $G$ in [11]. In this paper, as well as in [10], $G$ denotes Barnes’ $G$-function.$^3$

We start by stating the RH problem for $Y$.

**RH problem for $Y$**

(a) $Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where $\gamma$ and $\tilde{\gamma}$ are the contours shown in Figure 1.

(b) $Y$ has continuous boundary values $Y_+$ and $Y_-$ on $\gamma \cup \tilde{\gamma}$ from the left (+) side and right (−) side of $\gamma \cup \tilde{\gamma}$, respectively, and obeys the jump relations

\[
Y_+(z) = Y_-(z) \begin{pmatrix} 1 & -s^{-2}F(z) \\ 0 & 1 \end{pmatrix}, \quad \text{if } z \in \gamma, \quad (2.1)
\]

\[
Y_+(z) = Y_-(z) \begin{pmatrix} 1 & 0 \\ s^2 F(z)^{-1} & 1 \end{pmatrix}, \quad \text{if } z \in \tilde{\gamma}, \quad (2.2)
\]

where $F$ is defined as in (1.3).

---

$^3$If $G$ has subscripts and superscripts, such as $G_{p,q}^{m,n}$, then it denotes the Meijer $G$-function.
(c) $Y$ admits an expansion of the form

$$Y(z) = I + \frac{Y_1(s)}{z} + O(z^{-2}), \quad \text{as } z \to \infty, \quad (2.3)$$

where $Y_1$ is a $2 \times 2$ matrix that depends on $s$ but not on $z$.

It is shown in [11] that the solution of the RH problem for $Y$ exists and is unique for all $s$. In the steepest descent analysis of the RH problem, the authors of [11] introduce a sequence of transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$, where $R$ is the solution of a small norm RH problem. We only recall here what is needed for the proof of Theorem 1.1, and refer to [11, 10] for more details. Let us choose the branch for $\ln F$ such that

$$\ln F(z) = \ln \Gamma(z) + \sum_{k=1}^{q} \ln \Gamma(1 + \mu_k - z) - \sum_{j=1}^{r} \ln \Gamma(1 + \nu_j - z), \quad (2.4)$$

and the branches on the right-hand-side of (2.4) are the principal ones. Then $z \mapsto \ln F(z)$ is analytic for $z \in \mathbb{C} \setminus ((-\infty,0) \cup [1 + \nu_{\min},+\infty))$. The first transformation $Y \mapsto U$ involves the change of variables

$$z(\zeta) = is^\rho \zeta + \frac{1 + \nu_{\min}}{2},$$

where we recall that $\nu_{\min} = \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$. The asymptotics of $\ln F(z(\zeta))$ as $s^\rho \zeta \to \infty$ are given by

$$\ln F\left(is^\rho \zeta + \frac{1 + \nu_{\min}}{2}\right) = is^\rho \zeta \ln(s) + is^\rho (c_1 \ln(i\zeta) + c_2 \ln(-i\zeta) + c_3 \zeta)$$

$$+ c_4 \ln(s) + c_5 \ln(i\zeta) + c_6 \ln(-i\zeta) + c_7 + \frac{c_8}{is^\rho \zeta} + O\left(\frac{1}{s^2 is^\rho \zeta}\right),$$

where the constants $c_1, \ldots, c_8$ are given by

$$c_1 = 1, \quad c_2 = r - q,$$
$$c_3 = -(r - q + 1), \quad c_4 = \frac{\nu_{\min}}{2} + \frac{1}{1 + r - q} \left(\sum_{k=1}^{q} \mu_k - \sum_{j=1}^{r} \nu_j\right),$$
$$c_5 = \frac{\nu_{\min}}{2}, \quad c_6 = (r - q) \frac{\nu_{\min}}{2} + \sum_{k=1}^{q} \mu_k - \sum_{j=1}^{r} \nu_j,$$
$$c_7 = \frac{1 + q - r}{2} \ln(2\pi), \quad c_8 = \frac{1 + r - q}{8} \left(\nu_{\min}^2 - \frac{1}{3}\right) - \frac{1}{2} \left(\sum_{k=1}^{q} \mu_k^2 - \sum_{j=1}^{r} \nu_j^2\right)$$
$$+ \frac{\nu_{\min}}{2} \left(\sum_{k=1}^{q} \mu_k - \sum_{j=1}^{r} \nu_j\right). \quad (2.5)$$

We define the function $\mathcal{G}$ by

$$\mathcal{G}(\zeta) = F\left(is^\rho \zeta + \frac{1 + \nu_{\min}}{2}\right) e^{-is^\rho (\zeta \ln s - h(\zeta))}, \quad (2.6)$$

where

$$h(\zeta) = -\zeta(c_1 \ln(i\zeta) + c_2 \ln(-i\zeta) + c_3).$$

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Note that the function $G(\zeta)$ also depends on the parameters $s, r, q, \nu_j, \mu_k$, although this is not indicated in the notation. The $T \mapsto S$ transformation utilizes a $g$-function $\zeta \mapsto g(\zeta)$ which is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_5$, where $\Sigma_5$ is the union of two segments

$$\Sigma_5 = [b_1, 0] \cup [0, b_2],$$

oriented from left to right, see Figure 2. The points $b_1$ and $b_2$ which characterize $\Sigma_5$ are defined by

$$b_2 = -\overline{b_1} = |b_2|e^{i\phi}, \quad \phi \in \left[0, \frac{\pi}{2}\right),$$

where

$$\text{Re} \ b_2 = -\text{Re} \ b_1 = 2\left(\frac{e_2}{e_1}\right)^{1+\frac{1}{1+\frac{1}{1+\frac{1}{e_1+e_2}}}} e^{-\frac{e_1+e_2+e_1}{e_1+e_2}} = 2(r-q)^{1+\frac{1}{1+\frac{1}{1+\frac{1}{e_1+e_2}}}},$$

$$\sin \phi = \frac{e_2 - e_1}{e_2 + e_1} = \frac{r-q-1}{r-q+1} \in [0, 1).$$

The $g$-function is defined via its second derivative given by

$$g''(\zeta) = -i\frac{e_1 + e_2}{2} \left(\frac{1}{\zeta} - \frac{1}{r(\zeta)} + \frac{i\text{Im} \ b_2}{\zeta r(\zeta)}\right),$$

where $\zeta \mapsto r(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma_5$ and is defined by

$$r(\zeta) = \sqrt{(\zeta - b_1)(\zeta - b_2)},$$

where the branch is fixed such that $r(\zeta) \sim \zeta$ as $\zeta \to \infty$. For $\zeta \in \Sigma_5$, one has $r_+(\zeta) + r_-(\zeta) = 0$. It can be shown that $g''(\zeta) = \mathcal{O}(\zeta^{-3})$ as $\zeta \to \infty$, and then $g'$ and $g$ are defined by

$$g'(\zeta) = \int_\infty^\zeta g''(\xi) d\xi, \quad \int_\infty^\zeta g'(\xi) d\xi,$$

where the path of integration lies in $\mathbb{C} \setminus \Sigma_5$. The RH problem for $S$ has exponentially decaying jumps outside $\Sigma_5$, and one needs to construct approximations to $S$ in different regions of the complex plane. Let $D_\delta(b_1)$ and $D_\delta(b_2)$ denote two disks of sufficiently small radius $\delta > 0$ centered at $b_1$ and $b_2$, respectively. For $\zeta \in \mathbb{C} \setminus \left(D_\delta(b_1) \cup D_\delta(b_2)\right)$, we approximate $S$ by a so-called global parametrix $P(\infty)$, while for $\zeta \in D_\delta(b_1) \cup D_\delta(b_2)$, $S$ is approximated by local parametrices $P$ that are defined in
The function $p$ and the following asymptotics as terms of Airy functions. We omit the exact definition of $P$ here, the interested reader can find it in [11]\textsuperscript{4}. The construction of $P^{(\infty)}$ is given in terms of a function $p$ that will be important for us and which is defined by

$$p(\zeta) = -\frac{r(\zeta)}{2\pi i} \int_{\Sigma_3} \frac{\ln \mathcal{G}(\xi)}{r_+(\xi)} \frac{d\xi}{\xi - \zeta},$$

where the branch of $\ln \mathcal{G}$ is such that

$$\ln \mathcal{G}(\zeta) = \ln F\left(i s^0 \zeta + \frac{1 + \nu_{\min}}{2}\right) - i s^0 (\zeta \ln s - h(\zeta)).$$

The function $p$ has the following jumps across $\Sigma_5$:

$$p_+(\zeta) + p_-(\zeta) = -\ln \mathcal{G}(\zeta), \quad \zeta \in \Sigma_5,$$

and the following asymptotics as $\zeta \to \infty$:

$$p(\zeta) = p_0 + \mathcal{O}(\zeta^{-1}), \quad \zeta \to \infty,$$

$$p_0 = \frac{1}{2\pi i} \int_{\Sigma_3} \frac{\ln \mathcal{G}(\xi)}{r_+(\xi)} d\xi.$$

The global parametrix $P^{\infty}(\zeta)$ is defined by [11, Eq. (3.51)]

$$P^{\infty}(\zeta) = e^{-p_0 \sigma_3} Q^{\infty}(\zeta) e^{p(\zeta) \sigma_3} \quad \text{with} \quad Q^{\infty}(\zeta) = \frac{1}{2} \left( \frac{1}{\gamma(\zeta) - \gamma(\zeta)^{-1}} \right),$$

where the branch of the function

$$\gamma(\zeta) = \left(\frac{\zeta - b_1}{\zeta - b_2}\right)^{\frac{i}{2}}$$

is chosen such that $\gamma(\zeta)$ is analytic on $\mathbb{C} \setminus \Sigma_5$ and $\gamma(\zeta) \sim 1$ as $\zeta \to \infty$. The solution of the RH problem for $R$ is then given by [10, Eq. (2.32)]

$$R(\zeta) = e^{p_0 \sigma_3} S(\zeta) \times \begin{cases} P(\zeta)^{-1} e^{-p_0 \sigma_3}, & \zeta \in \mathbb{D}_b(b_1) \cup \mathbb{D}_b(b_2), \\ P^{\infty}(\zeta)^{-1} e^{-p_0 \sigma_3}, & \zeta \in \mathbb{R} \setminus (\mathbb{D}_b(b_1) \cup \mathbb{D}_b(b_2)). \end{cases}$$

Let $\{\Sigma_j\}_{j=1}^4$ denote the contours

$$\Sigma_2 = -\Sigma_1 = b_2 + e^{i(\phi + \epsilon)} \mathbb{R}_{\geq 0}, \quad \Sigma_4 = -\Sigma_3 = b_2 + e^{-i\epsilon} \mathbb{R}_{\geq 0}$$

for some fixed $\epsilon \in (0, \frac{\pi}{10})$, with the orientation from left to right, and define

$$\tilde{\Sigma}_j = \Sigma_j \setminus (\partial \mathbb{D}_b(b_1) \cup \partial \mathbb{D}_b(b_2)), \quad j = 1, \ldots, 5. \quad (2.16)$$

The function $R$ defined in (2.15) is analytic for $\zeta \in \mathbb{C} \setminus \Gamma_R$, where

$$\Gamma_R = (\partial \mathbb{D}_b(b_1) \cup \partial \mathbb{D}_b(b_2)) \cup \bigcup_{j=1}^5 \tilde{\Sigma}_j,$$

and $\partial \mathbb{D}_b(b_1)$ and $\partial \mathbb{D}_b(b_2)$ are oriented clockwise, see Figure 3.

\textsuperscript{4}There are some minor typos in the construction of the local parametrices of [11], see [10, Section 2] for details.
Remark 2.1. The jumps for $Y$ depend on $F$. If $F$ is replaced by $F_r$ in (2.1)-(2.2), then the steepest descent analysis of $Y$ has not been carried out in [11]. However, it is not hard to see that the same analysis applies also in this case (with $\nu_{\min} = \nu$); the only difference is that the coefficients (2.5) are replaced by:

\[
\begin{align*}
    c_1 &= 1, & c_2 &= r, & c_3 &= -(r + 1), \\
    c_4 &= \nu \left( \frac{1}{2} - \frac{r}{1+r} \right), & c_5 &= \nu^2, & c_6 &= -\frac{r \nu^2}{2} , \\
    c_7 &= \frac{1-r}{2} \ln(2\pi), & c_8 &= \frac{1+r}{8} \left( \nu^2 - \frac{1}{3} \right).
\end{align*}
\]

(2.17)

3 Differential identities in $r$, $\nu_\ell$, and $\mu_\ell$

In this section, we express the logarithmic derivatives (1.29), (1.30) and (1.31) in terms of the RH problem for $Y$ via the IIKS procedure. By combining [11, Propositions 2.1 and 2.2] and [6, Theorem 2.1], we obtain

\[
\begin{align*}
    \partial_r \ln \det (1 - K_r|_{[0,s]}) &= \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z)Y'(z)\partial_r J(z)J^{-1}(z)] \frac{dz}{2\pi i}, \\
    \partial_{\nu_\ell} \ln \det (1 - K|_{[0,s]}) &= \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z)Y'(z)\partial_{\nu_\ell} J(z)J^{-1}(z)] \frac{dz}{2\pi i}, \quad \ell \in \{1, \ldots, r\}, \\
    \partial_{\mu_\ell} \ln \det (1 - K|_{[0,s]}) &= \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z)Y'(z)\partial_{\mu_\ell} J(z)J^{-1}(z)] \frac{dz}{2\pi i}, \quad \ell \in \{1, \ldots, q\},
\end{align*}
\]

(3.1)-(3.3)

where the RH solution $Y$ in (3.1) has the jumps (2.1)-(2.2) with $F$ replaced by $F_r$. The quantity $(\partial_r J)J^{-1}$ in (3.1) is given by

\[
\partial_r J(z)J^{-1}(z) = \ln \Gamma(1 + \nu - z)(J(z) - I)\sigma_3.
\]

(3.4)

In (3.2)-(3.3), $Y$ satisfies the jumps (2.1)-(2.2) with $F$ given by (1.3), and we have

\[
\partial_{\nu_\ell} J(z)J^{-1}(z) = \psi(1 + \nu_\ell - z)(J(z) - I)\sigma_3, \quad \ell \in \{1, \ldots, r\},
\]

(3.5)
\[ \partial_{\mu}J(z)J^{-1}(z) = -\psi(1 + \mu_\ell - z)(J(z) - I)\sigma_\ell, \quad \ell \in \{1, \ldots, q\}, \]  

(3.6)

where \( \psi = (\ln \Gamma)' \) denotes the di-gamma function. The same arguments as in the proof of [10, Lemma 6.1] apply here (so we do not provide details), and we obtain

\[ \partial_{\mu} \ln \det (1 - K_r|_{[0,s)}) = \frac{1}{2} \int_{\gamma \cup \tilde{\gamma}} \ln \Gamma(1 + s - z) \Tr[Y_+^{-1}(z)Y_+^\prime(z)\sigma_3 - Y_+^{-1}(z)Y_+^\prime(z)\sigma_3] \frac{dz}{2\pi i}, \]  

(3.7)

where \( \ell \in \{1, \ldots, r\} \) and \( \ell \in \{1, \ldots, q\} \) in (3.9).

Using the chain of transformations \( Y \mapsto U \mapsto T \mapsto S \mapsto R \) in the steepest descent analysis of [11], we rewrite the differential identities (3.7), (3.8), and (3.9) in a way that is more convenient for the asymptotic analysis as \( s \to +\infty \). Recall that the transformation \( Y \mapsto U \) involves the change of variables \( z = i\zeta s^\rho + (1 + \nu_{\min})/2 \) (and that \( \nu_{\min} = \nu \) in (3.7), see also Remark 2.1). The functions

\[ \ln \Gamma\left(\frac{1 + \nu}{2} - i\zeta s^\rho\right), \quad \psi\left(1 + 2\nu_{\min} - \nu - i\zeta s^\rho\right), \quad \psi\left(1 + \nu - \nu_{\min} - i\zeta s^\rho\right) \]  

(3.10)

appearing in (3.7), (3.8) and (3.9) have infinitely many poles on \( i\mathbb{R}^- \). These poles depend on \( s \) and approach 0 as \( s \to +\infty \). For example, the left-most function in (3.10) has simple poles at \( \{\zeta^j\}_{j=0}^{\infty} \subset i\mathbb{R}^- \), where

\[ \zeta_j = \frac{1}{is^\rho \left(\frac{1 + \nu}{2} + j\right)}, \quad j = 0, 1, \ldots \]

Following [10], we define for \( K > |b_2| \) the contour \( \sigma_K \) as given in Figure 4. The contour \( \sigma_K \) surrounds \( \Sigma_\delta \) in the positive direction in such a way that the poles of (3.10) lie in the region exterior to \( \sigma_K \). The circular part of \( \sigma_K \) has radius \( K \). We choose the contour \( \sigma_K \) to cross the imaginary axis at the point \( \zeta_0/2 \) and to have a horizontal part of constant length as \( s \) changes. Note that, since the poles of (3.10) approach 0 as \( s \to +\infty \), the contour \( \sigma_K \) depends on \( s \), even if \( K \) is independent of \( s \). We define \( \sigma = \sigma_{2|b_2|} \). Furthermore, we define the contour

\[ \tilde{\Sigma}_K = \bigcup_{j=1}^4 \tilde{\Sigma}_j \setminus \{|\zeta| \leq K\}, \]

where the contours \( \tilde{\Sigma}_j \) are defined by (2.16).

**Lemma 3.1** (Differential identities). Let \( K \) be such that \( K > 2|b_2| \). Then the following statements hold:

(a) Let \( r \geq 1 \) and \( \nu = \nu_{\min} > -1 \). Then

\[ \partial_r \ln \det (1 - K_r|_{[0,s)}) = I_{1,r} + I_{2,r} + I_{3,r}(K) + I_{4,r}(K), \]  

(3.11)

where

\[ I_{1,r} = -s^\rho \int_{\sigma} \ln \Gamma\left(\frac{1 + \nu}{2} - i\zeta s^\rho\right) q'(\zeta) \frac{d\zeta}{2\pi i}, \]  

(3.12)

\[ I_{4,r} = -s^\rho \int_{\sigma} \ln \Gamma\left(\frac{1 + \nu}{2} - i\zeta s^\rho\right) g'(\zeta) \frac{d\zeta}{2\pi i}, \]  

(3.13)
Figure 4: The contour $\sigma_K$ surrounds $\Sigma_5$ but does not enclose any of the poles of the functions in (3.10).

\[ I_{2,r} = -\frac{1}{2} \int_{\sigma} \ln \Gamma \left( \frac{1+\nu}{2} - is^\theta \zeta \right) \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1} P^{(\infty)}(\zeta)'^{\sigma} \right] \frac{d\zeta}{2\pi i}, \]  
(3.13)

\[ I_{3,r}(K) = -\frac{1}{2} \int_{\sigma_K} \ln \Gamma \left( \frac{1+\nu}{2} - is^\theta \zeta \right) \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1} e^{-p_0^{\sigma_3} R^{-1}(\zeta) R'(\zeta) e^{p_0^{\sigma_3}} P^{(\infty)}(\zeta)'^{\sigma_3}} \right] \frac{d\zeta}{2\pi i}, \]  
(3.14)

\[ I_{4,r}(K) = \frac{1}{2} \int_{\Sigma_K} \ln \Gamma \left( \frac{1+\nu}{2} - is^\theta \zeta \right) \times \] 
\[ \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1} e^{-p_0^{\sigma_3} (R_+^{-1}(\zeta) R'_+(\zeta) - R_-^{-1}(\zeta) R'_-(\zeta)) e^{p_0^{\sigma_3}} P^{(\infty)}(\zeta)'^{\sigma_3}} \right] \frac{d\zeta}{2\pi i}. \]  
(3.15)

(b) Let $r > q \geq 0$ be integers. Let $\ell \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, q\}$. Then

\[ \partial_{\nu\ell} \ln \det (1 - K|_{[0,s]}) = I_{1,\nu\ell} + I_{2,\nu\ell} + I_{3,\nu\ell}(K) + I_{4,\nu\ell}(K), \]  
(3.16)

\[ \partial_{\mu k} \ln \det (1 - K|_{[0,s]}) = -I_{1,\mu k} - I_{2,\mu k} - I_{3,\mu k}(K) - I_{4,\mu k}(K), \]  
(3.17)

where, for $\alpha \in \{\nu_\ell, \mu_k\}$,

\[ I_{1,\alpha} = -s^\rho \int_{\sigma} \psi \left( \frac{1+2\alpha - \nu_{\min}}{2} - is^\theta \zeta \right) g'(\zeta) \frac{d\zeta}{2\pi i}, \]  
(3.18)

\[ I_{2,\alpha} = -\frac{1}{2} \int_{\sigma} \psi \left( \frac{1+2\alpha - \nu_{\min}}{2} - is^\theta \zeta \right) \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1} P^{(\infty)}(\zeta)'^{\sigma_3} \right] \frac{d\zeta}{2\pi i}, \]  
(3.19)

\[ I_{3,\alpha}(K) = -\frac{1}{2} \int_{\sigma_K} \psi \left( \frac{1+2\alpha - \nu_{\min}}{2} - is^\theta \zeta \right) \times \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1} e^{-p_0^{\sigma_3} R^{-1}(\zeta) R'(\zeta) e^{p_0^{\sigma_3}} P^{(\infty)}(\zeta)'^{\sigma_3}} \right] \frac{d\zeta}{2\pi i}, \]  
(3.20)
\[ I_{4,\alpha}(K) = \frac{1}{2} \int_{\Sigma_K} \psi \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - is^\alpha \zeta \right) \text{Tr} \left[ P(\infty)(\zeta)^{-1} e^{-p_0\sigma_3} \right] \times \left( R_r^{-1}(\zeta) R_r'(\zeta) - R_r^{-1}(\zeta) R_r'(\zeta) \right) e^{ip_0\sigma_3} P(\infty)(\zeta)^{\sigma_3} \frac{d\zeta}{2\pi i}. \] (3.21)

**Proof.** The proof is analogous to the proof of [10, Lemma 6.2] and consists of implementing the chain of transformations \( Y \mapsto U \mapsto T \mapsto S \mapsto R \) in (3.7), (3.8), and (3.9), and performing a contour deformation. \( \square \)

The remainder of the paper is devoted to the computation of the constant terms in the large \( s \) asymptotics of the right-hand sides of (3.11) (see Section 4) and (3.16)-(3.17) (see Section 5) and then integrating these identities (see (1.33) in the outline). In Section 6, we prove Theorem 1.1 by combining the computations from Section 4 and 5.

### 4 Asymptotics of the differential identity in \( r \)

In this section, we compute the large \( s \) asymptotics of the four quantities \( I_{1,r}, I_{2,r}, I_{3,r}(K), \) and \( I_{4,r}(K) \) appearing on the right-hand side of the differential identity (3.11) in \( r \). By integrating the resulting asymptotics with respect to \( r \), we obtain the constant term in the large \( s \) asymptotics of \( \ln \text{det} \left( 1 - K_r \right) \).

Throughout this section, we assume that \( r \geq 1 \) and \( \nu > -1 \). The quantities \( c_1, \ldots, c_8 \) and \( b_1, b_2 \) are defined by (2.17) and (2.8), respectively. As mentioned in Remark 1.2, we focus in this work on proving the expression (1.11) for \( C \) and we do not attempt to obtain the coefficients \( \rho, a, b \) and \( c \). Therefore, to avoid unnecessary computations, we introduce the notation \( \Omega \).

**Notation.** Let \( t \in \mathbb{R} \) and \( f, g : (t, \infty) \to \mathbb{C} \). The notation

\[
  f(s) = \Omega(g(s)), \quad \text{as } s \to +\infty,
\]

means that either \( f \equiv 0 \) or that there exist \( c > 0 \) and \( s_0 > 0 \) independent of \( s \) such that

\[
  |f(s)| \geq cg(s), \quad \text{for all } s \geq s_0.
\]

#### 4.1 Asymptotics of \( I_{1,r} \)

**Proposition 4.1.** Let \( \nu > -1 \) and let \( I_{1,r} \) be the function defined by (3.11). Then

\[
  I_{1,r} = \Omega(\ln(s^\rho)) + I_{1,r}^{(c)} + O \left( \frac{\ln(s^\rho)}{s^\rho} \right) \] (4.1)

as \( s \to +\infty \) uniformly for \( r \) in compact subsets of \([1, +\infty)\), where

\[
  I_{1,r}^{(c)} = \frac{c_1 + c_2}{2} \left\{ 1 - 3\nu^2 \right\} + \zeta'(-1) - \ln G \left( \frac{\nu + 1}{2} \right) + \frac{\nu - 1}{2} \ln \Gamma \left( \frac{\nu + 1}{2} \right) \right\} \left( 1 - \frac{\text{Im} b_2}{|b_2|} \right) 

  + \frac{c_1 + c_2}{48} (1 - 3\nu^2) \left( 1 + \frac{\text{Im} b_2}{|b_2|} \right) \ln \left( \left| b_2 \right| + \text{Im} b_2 \right) - 2 \frac{\text{Im} b_2}{|b_2|} \ln |b_2| (4.2)

**Proof.** We define the function

\[
  \Psi(\zeta) = s^\rho \int_{0}^{\zeta} \ln \Gamma \left( \frac{1 + \nu}{2} - is^\alpha \xi \right) d\xi. \] (4.3)
Then $\Psi(\zeta)$ is analytic on $\sigma$ and an integration by parts and (2.9) yield

$$I_{1, r} = \int_{\sigma} \Psi(\zeta) g''(\zeta) \frac{d\zeta}{2\pi i}$$

$$= i \left( c_1 + c_2 \right) \int_{\sigma} \Psi(\zeta) \left( 1 - i \frac{\text{Im} b_2}{\zeta} \right) \frac{1}{r(\zeta)} \frac{d\zeta}{2\pi i},$$

where we have used the fact that $\frac{1}{\zeta} \Psi(\zeta)$ has no pole at $\zeta = 0$. We first collapse the contour $\sigma$ onto $\Sigma_5$. Second, recalling that $r_+ (\zeta) + r_- (\zeta) = 0$ for $\zeta \in \Sigma_5$, we rewrite the resulting integral only in terms of $r_+$. Finally, we deform the contour on the $+$ side of $\Sigma_5$ into another contour $\gamma_{b_2 b_1}$, which is the part of the counterclockwise oriented circle with radius $|b_2|$ centered at the origin going from $b_2$ to $b_1$. We note again that, since $\frac{1}{\zeta} \Psi(\zeta)$ is analytic at $\zeta = 0$, there is no residue at $\zeta = 0$ during this contour deformation. This gives

$$I_{1, r} = i \left( c_1 + c_2 \right) \int_{\gamma_{b_2 b_1}} \Psi(\zeta) \left( 1 - i \frac{\text{Im} b_2}{\zeta} \right) \frac{1}{r(\zeta)} \frac{d\zeta}{2\pi i}. \quad (4.4)$$

It remains to compute the large $s$ asymptotics of $\Psi(\zeta)$ uniformly for $\zeta \in \gamma_{b_2 b_1}$. The following formula is useful for us (cf. [19, Eq. 5.17.4])

$$\int_{1}^{\infty} \ln \Gamma(z') dz' = \frac{z - 1}{2} \ln(2\pi) - \frac{(z - 1)z}{2} + (z - 1) \ln \Gamma(z) - \ln G(z), \quad (4.5)$$

where $G$ is Barnes’ $G$-function. By applying (4.5) twice with

$$z = \frac{1 + \nu}{2} - is^\rho \zeta \quad \text{and} \quad z = \frac{1 + \nu}{2},$$

in (4.3), we obtain

$$\Psi(\zeta) = i \int_{\frac{1 + \nu}{2}}^{\frac{1 + \nu}{2} - is^\rho \zeta} \ln \Gamma(z') dz'$$

$$= \frac{i}{2} \left[ 2s^\rho \zeta^2 + is^\rho \zeta (\nu - \ln(2\pi)) - 2 \ln \frac{G(\frac{1 + \nu}{2} - is^\rho \zeta)}{G(\frac{1 + \nu}{2})} - (1 - \nu) \ln \frac{\Gamma(\frac{1 + \nu}{2} - is^\rho \zeta)}{\Gamma(\frac{1 + \nu}{2})} - 2is^\rho \zeta \ln \Gamma(1 + \frac{\nu}{2} - is^\rho \zeta) \right]. \quad (4.6)$$

The large $z$ asymptotics of $\ln \Gamma(z)$ and $\ln G(z)$ are given by (cf. [19, Eqs. 5.11.1 and 5.17.5])

$$\ln G(z + 1) = \frac{z^2}{4} + z \ln \Gamma(z + 1) - \left( \frac{z(z + 1)}{2} + \frac{1}{12} \right) \ln z - \frac{1}{12} + \zeta'(-1) + O(z^{-2}),$$

$$\ln \Gamma(z) = (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} + O(z^{-3}) \quad (4.7)$$

as $z \to \infty$ uniformly for $|\text{arg} z| < \pi - \epsilon$ for some $\epsilon > 0$. This implies

$$\Psi(\zeta) = \Omega \left( \ln(s^\rho) \right) - \frac{i}{24} \left[ 1 - 3\nu^2 - \frac{i\pi}{2} + 24\zeta'(-1) + \frac{3i\pi\nu^2}{2} - 24 \ln G(\frac{\nu + 1}{2}) \right. + 12(\nu - 1) \ln \Gamma(\frac{\nu + 1}{2}) + (1 - 3\nu^2) \ln \zeta \left. \right] + O(s^{-\rho}) \quad (4.8)$$
as \( s \to +\infty \) uniformly for \( \zeta \in \gamma_{b_2b_1} \) and \( r \) in compact subsets of \((0, +\infty)\). Substituting (4.8) into (4.4) gives (4.1) and, in particular,

\[
I_{1,r}^{(c)} = (c_1 + c_2) \left\{ \frac{1 - 3\nu^2}{24} \left( 1 - \frac{\pi i}{2} \right) + \zeta'(-1) - \ln G \left( \frac{\nu + 1}{2} \right) + \frac{\nu - 1}{2} \ln \Gamma \left( \frac{\nu + 1}{2} \right) \right\} 
\times \int_{\gamma_{b_2b_1}} \left( - \frac{i\text{Im}b_2}{\zeta} \right) \frac{1}{r(\zeta)} \frac{d\zeta}{2\pi i} + c_1 + c_2 \frac{1}{24} (1 - 3\nu^2) \int_{\gamma_{b_2b_1}} \ln(\zeta) \left( 1 - \frac{i\text{Im}b_2}{\zeta} \right) \frac{1}{r(\zeta)} \frac{d\zeta}{2\pi i}.
\]

It was shown in [10, Lemma 7.2] that

\[
2 \int_{\gamma_{b_2b_1}} \frac{1}{r(\zeta)} \frac{d\zeta}{2\pi i} = 1, \quad 2 \int_{\gamma_{b_2b_1}} \frac{\ln(\zeta)}{r(\zeta)} \frac{d\zeta}{2\pi i} = \ln i(b_2 + \text{Im}b_2) - \ln 2,
\]

\[
2 \int_{\gamma_{b_2b_1}} \frac{1}{\zeta r(\zeta)} \frac{d\zeta}{2\pi i} = -\frac{i}{|b_2|}, \quad 2 \int_{\gamma_{b_2b_1}} \frac{\ln(\zeta)}{\zeta r(\zeta)} \frac{d\zeta}{2\pi i} = \ln \left( \frac{2|b_2|^2}{|b_2 + \text{Im}b_2|} \right) = \ln \pi i,
\]

which proves (4.2) and the proposition.

\[\square\]

### 4.2 Asymptotics of \( I_{2,r} \)

In this subsection we compute large \( s \) asymptotics for \( I_{2,r} \), which we recall is given by

\[
I_{2,r} = -\frac{1}{2} \int_{\sigma} \ln \Gamma \left( 1 + \frac{\nu}{2} - is^p\zeta \right) \text{Tr} \left[ P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta')\sigma_3 \right] \frac{d\zeta}{2\pi i}.
\]

Recalling the definition (2.14) of the global parametrix \( P^{(\infty)}(\zeta) \), a straightforward calculation yields

\[
\text{Tr} \left[ P^{(\infty)}(\zeta)^{-1}P^{(\infty)}(\zeta')\sigma_3 \right] = \text{Tr} \left[ Q^{(\infty)}(\zeta)^{-1}Q^{(\infty)}(\zeta')\sigma_3 \right] + \text{Tr} \left[ p'(\zeta)I \right] = 2p'(\zeta).
\]

Therefore, integrating by parts, using the jump condition (2.13) of \( p(\zeta) \), and then collapsing the contour \( \sigma \) onto \( \Sigma_5 \), we obtain

\[
I_{2,r} = -is^p \int_{\sigma} \psi \left( 1 + \frac{\nu}{2} - is^p\zeta \right) p(\zeta) \frac{d\zeta}{2\pi i} = is^p \int_{\Sigma_5} \psi \left( 1 + \frac{\nu}{2} - is^p\zeta \right) \left( p_+(\zeta) - p_-(\zeta) \right) \frac{d\zeta}{2\pi i} = Z_r + X_r,
\]

where

\[
Z_r = -2is^p \int_{\gamma_{b_2b_1}} \psi \left( 1 + \frac{\nu}{2} - is^p\zeta \right) p(\zeta) \frac{d\zeta}{2\pi i},
\]

\[
X_r = is^p \int_{\Sigma_5} \psi \left( 1 + \frac{\nu}{2} - is^p\zeta \right) \ln G(\zeta) \frac{d\zeta}{2\pi i}.
\]

It remains to find the large \( s \) asymptotics of \( Z_r \) and \( X_r \).

#### 4.2.1 Asymptotics of \( X_r \)

The large \( s \) asymptotics of \( X_r \) are described in terms of the Hurwitz zeta function \( \zeta(z,u) \) which is defined for \( \text{Re } z > 1 \) and \( u \neq 0, -1, -2, \ldots \) by

\[
\zeta(z,u) = \sum_{n=0}^{\infty} \frac{1}{(n + u)^z}.
\]
Proposition 4.2. Let $\nu \geq 0$ and $X_r$ be defined by (4.11). Then

$$X_r = \Omega(\ln(s^\rho)) + X_{1,r}^{(c)} + X_{2,r}^{(c)} + X_{3,r}^{(c)} + O\left(\frac{\ln(s^\rho)}{s^\rho}\right)$$  (4.12)

as $s \to +\infty$ uniformly for $\nu$ in compact subsets of $[1, +\infty)$, where

$$X_{1,r}^{(c)} = \frac{i}{4\pi} \left[ 6\nu^2 \left( \ln^2(-ib_1) - \ln^2(-ib_2) \right) + 6\nu \ln(2\pi) \left( \ln(ib_2) - \ln(ib_1) \right) ight]$$

$$+ \left[ \ln(-ib_2) - \ln(-ib_1) \right] \left( (1+r)(1-3\nu^2) + 6\nu(1-2r)\ln(2\pi) \right)$$

$$+ 6\nu^2 \left( \ln(-ib_2) \ln(ib_2) - \ln(-ib_1) \ln(ib_1) \right) \right],$$

$$X_{2,r}^{(c)} = \ln G(1+\nu) - \zeta'(-1) - \frac{\nu \ln(2\pi)}{4} + \frac{3\nu^2 - 1}{24} - \zeta'\left( -1, \frac{1+\nu}{2} + 1 \right) + \frac{\nu + 1}{2} \ln \left( \frac{\nu + 1}{2} \right),$$

$$X_{3,r}^{(c)} = \frac{1+r+3(r-3)\nu^2 \ln(b_1/b_2) \ln(-b_1b_2)}{24} - \frac{(1-3\nu^2)(1+r) - 6(r-1)\nu \ln(2\pi) \ln(b_1/b_2)}{24 \pi i}. \quad (4.13)$$

where $\zeta'(z,u) = \partial_z \zeta(z,u)$ denotes the derivative in the $z$-variable of the Hurwitz zeta function.

Proof. We first rewrite the integral $X_r$ in a convenient way. Performing an integration by parts yields

$$X_r = -\frac{\ln \Gamma \left( \frac{1+\nu}{2} - is^\rho \zeta \right) \ln G(\zeta)}{2\pi i} \bigg|_{z=b_1}^{b_2} + \int_{\Sigma_b} \ln \Gamma \left( \frac{1+\nu}{2} - is^\rho \zeta \right) G'(\zeta) \frac{d\zeta}{2\pi i}. \quad (4.14)$$

By the definition (2.6) of $G(\zeta)$ (with $F$ replaced by $F_r$) and the identity $\rho^{-1} = c_1 + c_2$, it holds that

$$\frac{G'(\zeta)}{G(\zeta)} = is^\rho \left\{ \psi \left( \frac{1+\nu}{2} + is^\rho \right) - \ln(is^\rho \zeta) + r \psi \left( \frac{1+\nu}{2} - is^\rho \right) - r \ln(-is^\rho \zeta) \right\}. \quad (4.15)$$

By substituting (4.17) into (4.14) and using the change of variables $w = is^\rho \zeta$, we split $X_r$ as

$$X_r = X_{1,r} + X_{2,r} + X_{3,r}, \quad (4.16)$$

where $X_{1,r}$, $X_{2,r}$ and $X_{3,r}$ are given by

$$X_{1,r} = -\frac{\ln \Gamma \left( \frac{1+\nu}{2} - is^\rho \zeta \right) \ln G(\zeta)}{2\pi i} \bigg|_{z=b_1}^{b_2},$$

$$X_{2,r} = \int_{is^\rho \Sigma_b} \ln \Gamma \left( \frac{1+\nu}{2} - w \right) \left\{ \psi \left( \frac{1+\nu}{2} + w \right) - \ln(w) - f(w) \right\} \frac{dw}{2\pi i},$$

$$X_{3,r} = \int_{is^\rho \Sigma_b} \ln \Gamma \left( \frac{1+\nu}{2} - w \right) \left\{ r \psi \left( \frac{1+\nu}{2} - w \right) - r \ln(-w) + f(w) \right\} \frac{dw}{2\pi i},$$

with

$$f(w) = \frac{\nu}{2} \left( \frac{1}{w-m} - \frac{m}{(w-m)^2} \right) + \frac{1-3\nu^2}{24(w-m)^2} \quad \text{and} \quad m = \frac{1+\nu}{2}. \quad (4.18)$$
We have added and subtracted the term \( f(w) \) in order to make the integrand of \( X_{2,r} \) vanish as \( w^{-2} \ln w \) as \( w \to \infty \). Indeed, from (4.7) and the asymptotic formula [19, Eq. 5.11.2] of the di-gamma function given by

\[
\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + O\left(\frac{1}{z^4}\right), \quad z \to \infty, \tag{4.20}
\]

for \( |\arg z| < \pi - \delta \) with some fixed \( \delta > 0 \), we have

\[
\psi\left(\frac{1 + \nu}{2} + w\right) = \ln w + \frac{\nu}{2w} + \frac{1 - 3\nu^2}{24w^2} + O\left(\frac{1}{w^3}\right), \quad w \to \infty, \tag{4.21}
\]

\[
= \ln w + f(w) + O\left(\frac{1}{w^5}\right), \quad w \to \infty, \tag{4.22}
\]

where \( |\arg w| < \pi - \delta \) with some fixed \( \delta > 0 \). Note that the integral \( X_{2,r} \) is convergent as long as \( m \notin is^\rho \Sigma_5 \); the choice \( m = \frac{\nu + m}{\nu} \) is made because it makes the upcoming computations easier. The remainder of the proof consists of computing the large \( s \) asymptotics of \( X_{1,r}, X_{2,r} \) and \( X_{3,r} \).

**Asymptotics of \( X_{1,r} \).** From [11, Eq. (3.15)], we have

\[
\ln G(\zeta) = c_4 \ln s + c_5 \ln(i\zeta) + c_6 \ln(-i\zeta) + c_7 + \frac{c_8}{i s^\rho \zeta} + O\left(\frac{1}{s^{2\rho} \zeta^2}\right), \quad s^\rho \zeta \to \infty. \tag{4.23}
\]

The asymptotic formula (4.23) is in particular valid for \( \zeta = b_1 \) and \( \zeta = b_2 \). By combining these asymptotics together with (4.7) and (2.17), we obtain

\[
X_{1,r} = \Omega(\ln(s^\rho)) + X_{1,r}^{(c)} + O\left(\frac{\ln(s^\rho)}{s^\rho}\right) \quad \text{as} \quad s \to +\infty, \tag{4.24}
\]

where \( X_{1,r}^{(c)} \) is given by (4.13).

**Asymptotics of \( X_{2,r} \).** Recall that \( X_{2,r} \) is given by

\[
X_{2,r} = \int_{is^\rho \Sigma_5} \ln \Gamma\left(\frac{1 + \nu}{2} - w\right) \left\{ \psi\left(\frac{1 + \nu}{2} + w\right) - \ln(w) - f(w) \right\} \frac{dw}{2\pi i},
\]

and that the integrand is \( O(w^{-2} \ln w) \) as \( w \to \infty \). Thus we have

\[
X_{2,r} = \int_{\gamma_\infty} \ln \Gamma\left(\frac{1 + \nu}{2} - w\right) \left\{ \psi\left(\frac{1 + \nu}{2} + w\right) - \ln(w) - f(w) \right\} \frac{dw}{2\pi i} + O\left(\frac{\ln(s^\rho)}{s^\rho}\right)
\]

as \( s \to +\infty \), where \( f(w) \) is defined by (4.19) and where the contour \( \gamma_\infty \) is a line oriented upwards and approaching infinity which crosses the real line between the origin and \( m = \frac{\nu + m}{\nu} \) (see Figure 5). We will compute \( X_{2,r} \) by integration and then contour deformation. However, we first need to add and subtract the term \( \frac{\nu}{2w} \) in the integrand and to split \( X_{2,r} \) into two parts as follows:

\[
X_{2,r}^{(c)} = \int_{\gamma_\infty} \ln \Gamma\left(\frac{1 + \nu}{2} - w\right) \left\{ \psi\left(\frac{1 + \nu}{2} + w\right) - \ln(w) - \frac{\nu}{2w} - \frac{1 - 3\nu^2}{24(w - m)^2} \right\} \frac{dw}{2\pi i}
\]

\[20\]
\[
+ \int_{\gamma_{\infty}} \ln \Gamma \left( \frac{1 + \nu}{2} - w \right) \left\{ \frac{\nu}{2w} - \frac{\nu}{2} \left( \frac{1}{w - m} - \frac{m}{(w - m)^2} \right) \right\} \frac{dw}{2\pi i}.
\]

Now, we integrate by parts the first integral, while the second integral can be evaluated explicitly by deforming the contour to infinity on the left half-plane (there is only a residue at \( w = 0 \)). This gives

\[
X_{2,r}^{(c)} = \int_{\gamma_{\infty}} \psi \left( \frac{1 + \nu}{2} - w \right) \left\{ \ln \Gamma \left( \frac{1 + \nu}{2} + w \right) + w \left( 1 - \ln(w) \right) - \frac{\nu}{2} \ln(w) + \frac{1 - 3\nu^2}{12(2w - \nu - 1)} \ln(2\pi) \right\} \frac{dw}{2\pi i} + \frac{\nu}{2} \ln \Gamma \left( \frac{1 + \nu}{2} \right).
\]

For the remaining integral, we deform the contour to infinity in the right half-plane. Note that this would not have been possible without adding and subtracting the term \( \frac{\nu}{2w} \). Since \( \psi \left( \frac{1 + \nu}{2} - w \right) \) has simple poles with residue 1 at \( w = m + n, \ n = 0, 1, 2, \ldots \), we pick up the following residue contributions

\[
- \left\{ \ln \Gamma(1 + \nu + n) + (n + m)(1 - \ln(n + m)) - \frac{\nu}{2} \ln(n + m) + \frac{1 - 3\nu^2}{24n} - \frac{\ln(2\pi)}{2} \right\}
\]

at the points \( n + m \) for \( n = 1, 2, \ldots \), and

\[
- \left\{ \ln \Gamma(1 + \nu) + m(1 - \ln(m)) - \frac{\nu}{2} \ln(m) - \frac{\ln(2\pi)}{2} \right\} + \frac{\gamma_E(1 - 3\nu^2)}{24}
\]

at \( w = m \), where \( \gamma_E \) is Euler’s gamma constant. This yields

\[
X_{2,r}^{(c)} = -\sum_{n=0}^{\infty} \left\{ \ln \Gamma(1 + \nu + n) + (n + m)(1 - \ln(n + m)) - \frac{\nu}{2} \ln(n + m) + \frac{1 - 3\nu^2}{24(n + 1)} - \frac{\ln(2\pi)}{2} \right\}
\]

\[
+ \frac{\gamma_E(1 - 3\nu^2)}{24} + \frac{\nu}{2} \ln \left( \frac{1 + \nu}{2} \right).
\]
The series is convergent since it arises from a convergent integral. We rewrite Euler’s gamma constant (see [19, Eq. 5.2.3]) as
\[
\gamma_E = \sum_{n=1}^{\infty} \frac{1}{n} - \ln \left( 1 + \frac{1}{n} \right),
\]  
(4.25)
which implies
\[
X_{2,v}^{(c)} = -\sum_{n=0}^{\infty} \left\{ \ln \Gamma(1 + \nu + n) + (n + m)(1 - \ln(n + m)) + \frac{1 - 3\nu^2}{24} \ln \left( 1 + \frac{1}{n + 1} \right) - \frac{\nu}{2} \ln(n + m) - \frac{\ln(2\pi)}{2} \right\}
\]  
(4.26)
This series can be computed explicitly. From the formula (cf. [19, Eq. 5.17.1])
\[
G(z + 1) = \Gamma(z)G(z),
\]
we deduce
\[
-\sum_{n=0}^{N} \ln \Gamma(1 + \nu + n) = -\ln G(2 + \nu + N) + \ln G(1 + \nu).
\]
The asymptotic formula (4.7) then implies that
\[
-\sum_{n=0}^{N} \ln \Gamma(1 + \nu + n) = \Omega(\ln N) - \zeta'(-1) - \frac{\ln(2\pi)}{2}(1 + \nu) + \ln G(1 + \nu) + O(N^{-1})
\]  
(4.27)
as \( N \to +\infty \), where \( \zeta'(-1) \) denotes the derivative of the Riemann zeta function evaluated at \( -1 \). Furthermore, from [10, Eq. (10.11)] with \( \theta = 1 \), we have
\[
\sum_{n=0}^{N} \left( \frac{1 + \nu}{2} + n \right) \ln \left( \frac{1 + \nu}{2} + n \right) = \frac{1 + \nu}{2} \ln \left( \frac{1 + \nu}{2} \right) + \Omega(\ln N)
\]  
\[+ \frac{3(1 + \nu^2) + 8 + 12\nu}{24} - \zeta'(-1, 1 + \nu + 1) + O(N^{-1}), \quad N \to +\infty,
\]  
(4.28)
where we recall that \( \zeta(z,u) \) is the Hurwitz zeta function and \( \zeta'(-1, m + 1) = \partial_z \zeta(z, m + 1)|_{z=-1} \). Also, it is easy to verify from \( \Gamma(z + 1) = z\Gamma(z) \) that
\[
\sum_{n=0}^{N} \ln \left( \frac{1 + \nu}{2} + n \right) = \ln \Gamma \left( \frac{1 + \nu}{2} + N + 1 \right) - \ln \Gamma \left( \frac{1 + \nu}{2} \right),
\]
and thus, by (4.7),
\[
\frac{\nu}{2} \sum_{n=0}^{N} \ln \left( \frac{1 + \nu}{2} + n \right) = \Omega(\ln N) + \frac{\nu}{4} \ln(2\pi) - \frac{\nu}{2} \ln \Gamma \left( \frac{1 + \nu}{2} \right) + O(N^{-1}), \quad N \to +\infty.
\]  
(4.29)
Another straightforward calculation shows that
\[
\sum_{n=0}^{N} \ln \left( 1 + \frac{1}{n + 1} \right) = \ln(N + 2) = \ln N + O(N^{-1}), \quad N \to +\infty.
\]  
(4.30)
Substituting (4.27), (4.28), (4.29), and (4.30) into (4.26) gives

$$X_{2,r} = X_{2,r}^{(c)} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right), \quad s \to +\infty,$$  (4.31)

where $X_{2,r}^{(c)}$ is given by (4.14).

**Asymptotics of $X_{3,r}$.** The integrand of $X_{3,r}$ is analytic on the left of $is^\rho\Sigma$. By deforming the contour $is^\rho\Sigma$ to $is^\rho\gamma_{b_2}$, where $\gamma_{b_2}$ is defined as in (4.4), we rewrite $X_{3,r}$ as follows:

$$X_{3,r} = -\int_{is^\rho\gamma_{b_2}} \ln\Gamma\left(\frac{1 + \nu}{2} - w\right)\left\{\psi\left(\frac{1 + \nu}{2} - w\right) - r \ln(-w) + f(w)\right\} \frac{dw}{2\pi i}$$

$$= -\int_{is^\rho\gamma_{b_2}} \ln\Gamma\left(\frac{1 + \nu}{2} - w\right)\left\{\psi\left(\frac{1 + \nu}{2} - w\right) - r \ln(-w) + \frac{\nu}{2w} + \frac{1 - 3\nu^2}{24w^2}\right\} \frac{dw}{2\pi i} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right)$$

as $s \to +\infty$. Using the asymptotic formulas (4.7) and (4.20) for the integrand of $X_{3,r}$, we find

$$X_{3,r} = - \int_{is^\rho\gamma_{b_2}} \left\{\frac{\nu}{2}(r - 1)(\ln(-w) - 1)
+ 1 + r - 3\nu^2(1 + r) - 6(r - 1)\nu \ln(2\pi) - (1 + r + 3(r - 3)\nu^2) \ln(-w)
+ \mathcal{O}\left(w^{-2}\ln(-w)\right)\right\} \frac{dw}{2\pi i} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right), \quad s \to +\infty.$$

After the change of variables $w = is^\rho\zeta$, we obtain

$$X_{3,r} = \Omega\left(\ln(s^\rho)\right) + \frac{1 + r + 3(r - 3)\nu^2}{24} \int_{\gamma_{b_2}} \frac{\ln(-i\zeta)}{2\pi i} \frac{d\zeta}{\zeta}$$

$$- \frac{1 + r - 3\nu^2(1 + r) - 6(r - 1)\nu \ln(2\pi)}{24} \int_{\gamma_{b_2}} \frac{1}{\zeta} \frac{d\zeta}{2\pi i} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right)$$  (4.32)

as $s \to +\infty$. Since

$$\int_{\gamma_{b_2}} \frac{1}{2\pi i} \frac{d\zeta}{\zeta} = \frac{\ln(b_1/b_2)}{2\pi i}, \quad \int_{\gamma_{b_2}} \frac{\ln(-i\zeta)}{\zeta} \frac{d\zeta}{2\pi i} = \frac{\ln(b_1/b_2) \ln(-b_1b_2)}{4\pi i},$$  (4.33)

the asymptotics (4.32) can be rewritten as

$$X_{3,r} = \Omega\left(\ln(s^\rho)\right) + X_{3,r}^{(c)} + \mathcal{O}\left(\frac{\ln(s^\rho)}{s^\rho}\right), \quad s \to +\infty,$$  (4.34)

where $X_{3,r}^{(c)}$ is given by (4.15).

**Asymptotics of $X_r$.** By substituting (4.24), (4.31), and (4.34) into (4.18), we obtain (4.12). This completes the proof of Proposition 4.2.
4.2.2 Asymptotics of $Z_r$

In this subsection, we compute the asymptotics of

$$Z_r = -2is^\rho \int_{\gamma_{b_1}} \psi\left(\frac{1+\nu}{2} - is^\rho \zeta\right)p(\zeta)\frac{d\zeta}{2\pi i},$$

(4.35)

where $p(\zeta)$ is defined by (2.12). Some of the following computations are similar to those performed in [10, Section 8]. In particular, the quantity $f_1$ in [10, Eq (3.15)] is in our case equal to

$$f_1 = -\frac{3\nu^2 - 1}{24}$$

and appears naturally in the asymptotics of $Z_r$.

**Proposition 4.3.** Let $\nu > -1$ and $Z_r$ be defined by (4.35). Then

$$Z_r = \Omega\left(\ln(s^\rho)\right) + Z_r^{(c)} + O\left(\ln s^\rho\right)$$

(4.36)

as $s \to +\infty$ uniformly for $r$ in compact subsets of $[1, +\infty)$, where

$$Z_r^{(c)} = -2i\left\{ \frac{1}{4} \left( \frac{c_8}{2\rho} \frac{c_4}{2} + \frac{\pi c_8}{2\rho} \right) \left( 1 - \frac{\ln b_2}{2} \right) + \frac{1}{2\pi i} \left( \frac{ic_8}{4} \left( \ln b_1 - \ln b_2 \right)^2 \right) 
+ \frac{\pi(c_8 - 2i\hat{f}_1)}{2|b_2|} \left[ \frac{|b_2| - \ln b_2}{2} \ln \left( \frac{2i|b_2|^2}{|b_2| + \ln b_2} \right) + \ln b_2 \ln \left( \frac{i|b_2| + \ln b_2}{2} \right) \right] 
+ \frac{\nu}{10\pi} \left( -4i\pi \ln|b_2| - 4i\pi c_0 \ln \left( \frac{2}{|b_2| + \ln b_2} \right) + 3b_1 \ln(2c_7 + i\pi(3c_5 + c_6)) 
- (c_5 + c_6) \ln^2(b_1) + \ln(b_2)(2c_7 + i\pi(c_5 + c_6)) + (c_5 + c_6) \ln^2(b_2) + 2\pi^2c_5 \right) \right\}. \quad (4.37)$$

**Proof.** By (4.20), we see that

$$\psi\left(\frac{1+\nu}{2} - is^\rho \zeta\right) = \ln(s^\rho) + \ln(-i\zeta) - \frac{\nu}{2is^\rho \zeta} + O(s^{-2\rho}), \quad \text{as } s \to +\infty, \quad (4.38)$$

uniformly for $\zeta \in \gamma_{b_1}$. From [10, beginning of Section 8.3 and Eq. (3.38)] with the coefficients $c_1, \ldots, c_8$ given by (2.17), we have

$$p(\zeta) = \frac{c_4}{2\rho} \ln(s^\rho) + \frac{B(\zeta)}{2} + \frac{A(\zeta)}{s^\rho} + O(s^{-2\rho}) \quad (4.39)$$

as $s \to +\infty$ uniformly for $\zeta \in \gamma_{b_1}$ and $r$ in compact subsets of $[1, +\infty)$, where

$$B(\zeta) = -c_5r(\zeta) \int_0^{\infty} \frac{d\xi}{r(\xi)(\xi - \zeta)} - c_6r(\zeta) \int_{-\infty}^{-i\infty} \frac{d\xi}{r(\xi)(\xi - \zeta)} - c_5 \ln(i\zeta) - c_6 \ln(-i\zeta) - c_7, \quad (4.40)$$

$$A(\zeta) = \frac{ic_8}{2\zeta} + r(\zeta) \frac{c_8 - \frac{2\pi^2 - 1}{2|b_2|}}{2\zeta|b_2|}. \quad (4.41)$$

After substituting (4.38) and (4.39) into the definition (4.35) of $Z_r$, we write

$$Z_r = \Omega\left(\ln(s^\rho)\right) - 2i\left( \int_{\gamma_{b_1}} A(\zeta) \left( \ln(\zeta) - \frac{\nu}{2i} \right) \frac{d\zeta}{2\pi i} - \frac{\nu}{4i} \int_{\gamma_{b_1}} \frac{B(\zeta)}{\zeta} \frac{d\zeta}{2\pi i} \right) + O\left(\frac{\ln s^\rho}{s^\rho}\right) \quad (4.42)$$

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as $s \to +\infty$. It was shown in [10, Lemma 8.4] that

\[
\int_{\gamma_{b_1 b_2}} \mathcal{A}(\zeta) d\zeta = -\frac{c_8 (\arg b_1 - \arg b_2)}{2} - \frac{\pi (c_8 - \frac{3\nu^2 - 1}{12})}{2} \left( 1 - \frac{\Im b_2}{|b_2|} \right), \quad (4.43)
\]

\[
\int_{\gamma_{b_1 b_2}} \ln(\zeta) \mathcal{A}(\zeta) d\zeta = \frac{i c_8}{4} (\ln b_1)^2 - (\ln b_2)^2 + \frac{\pi (c_8 - \frac{3\nu^2 - 1}{12})}{2|b_2|} \left\{ |b_2| - \Im b_2 - |b_2| \ln \left( \frac{2|b_2|^2}{|b_2| + \Im b_2} \right) \right\}, \quad (4.44)
\]

\[
\int_{\gamma_{b_1 b_2}} \frac{\mathcal{B}(\zeta)}{\zeta} d\zeta = \frac{1}{2} \left\{ -4\pi \ln(|b_2|)(c_5 - c_6) + 4i\pi c_6 \ln \left( \frac{2}{|b_2| + \Im b_2} \right) + \ln(b_1)(-2c_7 + i\pi(3c_5 + c_6)) - (c_5 + c_6)(\ln b_1)^2 + \ln(b_2)(2c_7 + i\pi(c_5 - c_6)) + (c_5 + c_6)(\ln b_2)^2 + 2\pi^2 c_5 \right\} \quad (4.45)
\]

By substituting (4.43), (4.44), and (4.45) into (4.42), we obtain (4.36) after a long computation. This completes the proof of the proposition.

### 4.3 Asymptotics of $I_{3,r}(K)$ and $I_{4,r}(K)$

In this section we compute the large $s$ asymptotics of $I_{3,r}(K)$ and $I_{4,r}(K)$ defined in (3.14) and (3.15).

**Proposition 4.4.** Let $\nu > -1$ and $K = s^\rho$. Then

\[
I_{3,r}(K) = \Omega \left( \ln(s^\rho) \right) + I^{(c)}_{3,r} + \mathcal{O} \left( \frac{\ln(s^\rho)}{s^\rho} \right) \quad (4.46)
\]

as $s \to +\infty$ uniformly for $r$ in compact subsets of $[1, +\infty)$, where

\[
I^{(c)}_{3,r} = \frac{-2(1 + r) + 3r(r^2 - 1)\nu^2 + 2r(1 + 3\nu^2)\ln(r)}{24r(1 + r)^2}.
\]

**Proof.** Proceeding as in the proof of [10, Proposition 9.1], one obtains

\[
I_{3,r}(K) = -\frac{1}{2s^\rho} \int \ln I \left( \frac{1 + \nu}{2} - is^\rho \zeta \right) W(\zeta) \frac{d\zeta}{2\pi i} + \mathcal{O} \left( \frac{\ln(s^\rho)}{s^\rho} \right), \quad s \to +\infty,
\]

where $\bar{\sigma}$ surrounds the horizontal segment $[b_1, b_2]$ but does not surround the origin, and is oriented counterclockwise. The function $W(\zeta)$ is defined by

\[
W(\zeta) = \frac{1}{\bar{\sigma}} \Re \left( \left[ \left( -\frac{A}{(\zeta - b_1)^2} - \frac{2B}{(\zeta - b_1)^2} + \frac{A}{(\zeta - b_2)^2} - \frac{2B}{(\zeta - b_2)^2} \right) \right] \zeta - i\Re b_2 \right),
\]

where $\bar{\sigma} = \sqrt{(\zeta - b_1)(\zeta - b_2)}$ has a branch cut on $[b_1, b_2]$, such that $\bar{\sigma}(\zeta) \sim \zeta$ as $\zeta \to \infty$. The matrices $A$ and $B$ denote the coefficients appearing in the large $s$ asymptotics of $R$ (cf. [10, Proposition 4.1]) and are given by

\[
A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} 5b_1 \\ 48(c_1 + c_2) \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \quad (4.48)
\]
with

\[ A_{1,1} = \frac{3i\text{Im}b_2 + 2i\text{Re}b_2 - 12|b_2|(c_5 - c_6)(c_5 + c_6) + (c_5^2 + c_6^2)\text{Im}b_2 + 2ic_5c_6\text{Re}b_2}{48(c_1 + c_2)\text{Re}b_2}, \]

\[ A_{1,2} = \frac{4i(3|b_2|(c_5 - c_6)(1 + c_5 + c_6) + \text{Im}b_2 + 3(c_5 + c_6^2 + c_6 + c_6^2)\text{Im}b_2)}{48(c_1 + c_2)\text{Re}b_2} - \frac{(5 + 12c_6 + 12c_5(1 + 2c_6))}{48(c_1 + c_2)} , \]

\[ A_{2,1} = \frac{12i|b_2|(c_5 - c_6)(-1 + c_5 + c_6) + 4i(1 + 3(c_5 - 1)c_5 + 3(c_6 - 1)c_6)\text{Im}b_2}{48(c_1 + c_2)\text{Re}b_2} + \frac{-5 + 12(c_5 + c_6 - 2c_5c_6)}{48(c_1 + c_2)} , \]

\[ A_{2,2} = -A_{1,1}. \]

Using the asymptotics (4.7) of \( \ln \Gamma \), we obtain

\[ I_{3,r}(K) = -\frac{i}{2} \int_{\tilde{\sigma}} (1 - \ln(-is^p\zeta))\zeta W(\zeta) \frac{d\zeta}{2\pi i} + O\left(\frac{\ln(s^p)}{s^p}\right), \quad s \to +\infty. \] (4.49)

From (4.47), we see that

\[ W(\zeta) = -\frac{\text{Tr}[\{A - \tilde{A}\} \sigma_3]}{\zeta^2} + O(\zeta^3) = -\frac{i}{6(c_1 + c_2)}\zeta^2 + O(\zeta^3) \quad \text{as } \zeta \to \infty. \]

Thus, after splitting the leading term in (4.49) into two parts, we obtain

\[ I_{3,r}(K) = -\frac{i}{2} \frac{1 - 12c_5c_6}{6(c_1 + c_2)} + \frac{i}{2} \int_{\tilde{\sigma}} \ln(-i\zeta)\zeta W(\zeta) \frac{d\zeta}{2\pi i} + O\left(\frac{\ln(s^p)}{s^p}\right) \] (4.50)

as \( s \to +\infty \), where we have deformed \( \tilde{\sigma} \) to infinity for the first part. The last integral in (4.50) can be evaluated as follows:

\[ \frac{i}{2} \int_{\tilde{\sigma}} \ln(-i\zeta)W(\zeta) \frac{d\zeta}{2\pi i} = \lim_{R \to \infty} \left\{ \frac{i}{2} \int_{C_R} \ln(-i\zeta)W(\zeta) \frac{d\zeta}{2\pi i} + \frac{i}{2} \int_{-iR}^{0} \zeta W(\zeta) d\zeta \right\} = \lim_{R \to \infty} \left\{ -\frac{i}{2} \text{Tr}[\{A - \tilde{A}\} \sigma_3] \int_{C_R} \frac{\ln(-i\zeta)}{\zeta} \frac{d\zeta}{2\pi i} + \frac{i}{2} \int_{-R}^{0} i\zeta W(it) dt \right\} , \]

where \( C_R \) is the circle centered at the origin of radius \( R \) oriented positively. Since

\[ \int_{C_R} \frac{\ln(-i\zeta)}{\zeta} \frac{d\zeta}{2\pi i} = \ln(R), \]

\[ \tilde{r}(it) = -i\sqrt{(t - \text{Im} b_2)^2 + (\text{Re} b_2)^2}, \quad \text{for } t < 0, \]

we compute the integral \( \int_{-iR}^{0} \zeta W(\zeta) d\zeta \) by a rather long primitive calculation, which uses the definition (4.47) of \( W(\zeta) \). Then, after substituting the expressions (2.5) and (2.8), we obtain

\[ \frac{i}{2} \int_{\tilde{\sigma}} \ln(-i\zeta)W(\zeta) \frac{d\zeta}{2\pi i} = \frac{1 + r}{24r(1 + r)^2} \left(1 + r\left(-2 + r\left(2 + (9r - 3)\nu^2\right)\right) + 2r(1 + 3r\nu^2)\ln(r)\right). \] (4.51)

Substituting (4.51) into (4.50) and using again (2.17), we obtain (4.46), which finishes the proof.  \( \Box \)
Proposition 4.5. Let $\nu > -1$ and $K = s^\rho$. Then, for any integer $N \geq 1$,
\[ I_{4,r}(K) = \mathcal{O}(s^{-N\rho}) \tag{4.52} \]
as $s \to +\infty$ uniformly for $r$ in compact subsets of $[1, +\infty)$.

Proof. The proof is analogous to the proof of [10, Proposition 9.2] and relies on the large $s$ asymptotics of $R$.

4.4 Integration of the differential identity in $r$

In this subsection, we compute the constant term $C_r$ in the large gap asymptotics for the point process defined by $\mathbb{K}_r$. With minor adjustments of [11] and [10], these asymptotics are of the form
\[ \det (1 - \mathbb{K}_r|_{[0,s]}) = C_r \exp \left( -a_r s^{2\rho} + b_r s^\rho + c_r \ln s + \mathcal{O}(s^{-\rho}) \right), \tag{4.53} \]
with the constants $\rho, a_r, b_r$ and $c_r$ given by
\[
\rho = \frac{1}{1+r}, \quad a = \frac{(r+1)^2}{4(r+1)}, \quad b = (1+r)\nu, \quad c = \frac{r-1}{12(r+1)} - \frac{r\nu^2}{2(r+1)}. \]

Proposition 4.6. Let $\nu > -1$ and $r \geq 1$. Then
\[ \det (1 - \mathbb{K}_r|_{[0,s]}) = C_r \exp \left( -a_r s^{2\rho} + b_r s^\rho + c_r \ln s + \mathcal{O}(s^{-\rho}) \right), \]
where
\[
C_r = \frac{G(1+\nu)^r}{(2\pi)^{\nu + 1}} \exp \left\{ - (r-1)\zeta'(-1) \right\} \exp \left\{ - \frac{2 + r^2(r-1+12\nu^2)}{24(r+1)} \ln(r) \right\} \times \exp \left\{ - \frac{(r-1)^2 + 12r \nu^2}{24} \ln(1+r) \right\}. \]

Proof. It follows from the analysis of [11, 10] that the error term in (4.53) can be differentiated with respect to $r$ and that its $r$-derivative is of order $\mathcal{O}(s^{-\rho} \ln(s^\rho))$ uniformly for $r$ in compact subsets of $[1, +\infty)$. Therefore we have
\[
\partial_r \ln \det (1 - \mathbb{K}_r|_{[0,s]}) = -2\partial_r (\rho) a_r s^{2\rho} \ln s - \partial_r a_r s^{2\rho} + \partial_r (\rho) b_r s^\rho \ln s + \partial_r (b_r) s^\rho + \partial_r (c_r) \ln(s) + \partial_r (\ln C_r) + \mathcal{O}(s^{-\rho} \ln(s^\rho))
\]
as $s \to +\infty$, uniformly for $r$ in compact subsets of $[1, +\infty)$. Thus, from Lemma 3.1 and Propositions 4.1, 4.2, 4.3, 4.4, and 4.5, we infer that
\[
\partial_r (\ln C_r) = I_{1,r}^{(c)} + X_{1,r}^{(c)} + X_{2,r}^{(c)} + X_{3,r}^{(c)} + Z_{r}^{(c)} + I_{3,r}^{(c)}
\]
\[
= -\frac{2 - 2r + 3r \nu^2}{24r} - \frac{\nu \ln(2\pi)}{4} + \frac{1 + r(r - 1 + r^2 + 6(2 + r)\nu^2)}{12(r + 1)^2} \ln(r)
\]
\[
- \frac{1}{12} \frac{(r-1+6\nu^2)\ln(1+r)}{\ln(r)} + \frac{\nu - 1}{2} \ln \left( \frac{1 + \nu}{2} \right) - \ln \left( \frac{1 + \nu}{2} \right) - \frac{\nu \ln(2\pi)}{4}
\]
\[
+ \ln \left( \frac{3\nu^2 - 1}{24} \right) - \zeta' \left( -1; \frac{1 + \nu}{2}, 1 \right) + \nu + 1 \ln \left( \frac{\nu + 1}{2} \right). \]
Applying the identity [1, Eq (18)]

$$\ln G(z + 1) = \zeta'(-1) - \zeta'(-1, z + 1) + z \ln \Gamma(z + 1)$$  \hspace{1cm} (4.54)

with \(z = \frac{1 + \nu}{2}\), we get

$$\partial_r(\ln C_r) = \frac{r - 2}{24r} - \frac{\nu \ln(2\pi)}{2} + \frac{1 + r(r - 1 + r^2 + 6(2 + r)^2 \nu^2)}{12(r + 1)^2} \ln(r)$$

$$- \frac{1}{12} (r - 1 + 6\nu^2) \ln(1 + r) - \zeta'(-1) + \ln G(1 + \nu).$$

Integrating this identity with respect to \(r\) from \(r = 1\) to a fixed \(r \geq 1\), we obtain

$$\int_1^r \partial_r(\ln C_r) dr' = \frac{\nu^2}{2} \ln(2) - \frac{\nu(r - 1) \ln(2\pi)}{2} + (r - 1) \left( \ln G(1 + \nu) - \zeta'(-1) \right)$$

$$+ \frac{-2 + r^2(r - 1 + 12\nu^2)}{24(r + 1)} \ln(r) - \frac{(r - 1)^2 + 12r\nu^2}{24} \ln(1 + r).$$

Since \(\ln C_1 = \ln G(1 + \nu) - \frac{\nu}{2} \ln(2\pi) - \frac{\nu^2}{2} \ln 2\) by (1.17), we arrive at

$$\ln C_r = \nu \ln G(1 + \nu) - \frac{\nu \ln(2\pi)}{2} - (r - 1) \zeta'(-1)$$

$$+ \frac{-2 + r^2(r - 1 + 12\nu^2)}{24(r + 1)} \ln(r) - \frac{(r - 1)^2 + 12r\nu^2}{24} \ln(1 + r)$$

and the proposition follows by exponentiating both sides.

\[\square\]

5 Asymptotics of the differential identities in \(\nu_\ell\) and \(\mu_\ell\)

In this section, we compute the large \(s\) asymptotics of the differential identities (3.16) and (3.17). For \(\alpha \in \{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}\), these identities express \(\partial_\alpha \ln \det(1 - K_{[0, s]}(K))\) in terms of the quantities \(I_{1,\alpha}, I_{2,\alpha}, I_{\delta,\alpha}(K)\), and \(I_{1,\alpha}(K)\) defined in Lemma 3.1. By computing the asymptotics of these quantities and then integrating with respect to \(\alpha\), we can deduce the large \(s\) asymptotics of \(\ln \det (1 - K_{[0, s]}(K))\), see also the outline in Section 1.3; in particular (1.30)-(1.31).

In the remainder of the paper, we let \(r > q \geq 0\) be integers and let \(\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1\) be the parameters associated to the kernel \(K\) defined in (1.1). We set \(\nu_{\min} := \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}\). The constants \(c_1, c_2, \ldots, c_8\) and \(b_1, b_2\) are defined in (2.5) and (2.8), respectively, and we will often use the notation \(\Omega\) introduced at the beginning of Section 4.

5.1 Asymptotics of \(I_{1,\alpha}\)

Proposition 5.1. Let \(I_{1,\alpha}\) be the function defined in (3.18). Then

$$I_{1,\alpha} = \Omega \left( \ln(s^\rho) \right) + I_{1,\alpha}^{(0)} + O \left( \frac{\ln(s^\rho)}{s^\rho} \right)$$  \hspace{1cm} (5.1)

as \(s \to +\infty\) uniformly for \(\alpha\) in compact subsets of \((-1, +\infty)\), where

$$I_{1,\alpha}^{(0)} = -\frac{(c_1 + c_2)}{2} \left( \frac{\ln(2\pi)}{2} - \ln(1 + 2\alpha - \nu_{\min}) \right) \left( 1 - \frac{\ln b_2}{|b_2|} \right)$$
\[ -(c_1 + c_2) \frac{2\alpha - \nu_{\text{min}}}{2} \left( \ln(|b_2| + \text{Im} b_2) - \ln(2) - \frac{\text{Im} b_2}{2|b_2|} \ln \left( \frac{2|b_2|^2}{|b_2| + \text{Im} b_2} \right) \right). \] (5.2)

**Proof.** Integrating \( I_{1,\alpha} \) by parts as in the proof of Proposition 4.1, we find

\[ I_{1,\alpha} = -i(c_1 + c_2) \int_{\gamma_{b_1}} \tilde{\Psi}(\zeta) \left( 1 - \frac{i \text{Im} b_2}{\zeta} \right) \frac{1}{r(\zeta) 2\pi i} \, d\zeta, \] (5.3)

where \( \gamma_{b_2} \) is defined as in (4.4) and

\[ \tilde{\Psi}(\zeta) = -s^\rho \int_{0}^{\zeta} \psi \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - is^\rho \xi \right) d\xi = -i \ln \Gamma \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - is^\rho \right) + i \ln \Gamma \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} \right). \]

A direct computation using the asymptotics (4.7) yields

\[ \tilde{\Psi}(\zeta) = \Omega \left( \ln(s^\rho) \right) - i \frac{2\alpha - \nu_{\text{min}}}{2} \ln(-i\zeta) - i \frac{\ln(2\pi)}{2} + i \ln \Gamma \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} \right) + \mathcal{O} \left( \frac{\ln(s^\rho)}{s^\rho} \right) \] (5.4)

as \( s \to +\infty \) uniformly for \( \alpha \) in compact subsets of \((-1, +\infty)\), and uniformly for \( \zeta \in \gamma_{b_2} \). We obtain (5.1)-(5.2) after substituting (5.4) into (5.3), using (4.9) and then simplifying.

### 5.2 Asymptotics of \( I_{2,\alpha} \)

Let \( I_{2,\alpha} \) be the function defined in (3.19). Using (4.10), we can write

\[ I_{2,\alpha} = \int_{\sigma} \mathcal{F}(\zeta) p(\zeta) \frac{d\zeta}{2\pi i}, \]

where \( p(\zeta) \) is defined by (2.12) and \( \mathcal{F}(\zeta) \) is defined by

\[ \mathcal{F}(\zeta) = -\psi \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - is^\rho \zeta \right). \] (5.5)

We integrate by parts and then deform the contour and use the jumps for \( p \) as in the beginning of Section 4.2. This yields

\[ I_{2,\alpha} = Z_\alpha + X_\alpha, \]

where

\[ Z_\alpha = -2 \int_{\gamma_{b_1}} \mathcal{F}(\zeta) p(\zeta) \frac{d\zeta}{2\pi i}, \quad X_\alpha = \int_{\Sigma_\alpha} \mathcal{F}(\zeta) \ln \mathcal{G}(\zeta) \frac{d\zeta}{2\pi i}. \] (5.6)

**Proposition 5.2.** Let \( X_\alpha \) be defined by (5.6). Then

\[ X_\alpha = \Omega \left( \ln(s^\rho) \right) + X_{1,\alpha}^{(c)} + X_{2,\alpha}^{(c)} + X_{3,\alpha}^{(c)} + \mathcal{O} \left( \frac{\ln(s^\rho)}{s^\rho} \right) \quad \text{as } s \to +\infty, \] (5.7)

uniformly for \( \alpha \) in compact subsets of \((-1, +\infty)\), where

\[ X_{1,\alpha}^{(c)} = \frac{1}{2\pi i} \left\{ c_5 \left( \ln(-ib_1) \ln(ib_1) - \ln(-ib_2) \ln(ib_2) \right) \right\}. \]
\[ + c_0 \left( \ln^2(-ib_1) - \ln^2(-ib_2) \right) + c_7 \left( \ln(-ib_1) - \ln(-ib_2) \right) \}, \quad (5.8) \]

\[ X_{2,\alpha}^{(c)} = \sum_{k=0}^{\infty} \left\{ \ln \left( k + \frac{1}{2} \alpha - \frac{\nu_{\text{min}}}{2} \right) - \psi(k+1+\alpha) + \frac{\nu_{\text{min}}}{k+1} \right\}, \quad (5.9) \]

\[ X_{3,\alpha}^{(c)} = \left\{ - \frac{\nu_{\text{min}}}{2} + \sum_{j=1}^{r} \frac{2\nu_j - \nu_{\text{min}}}{2} - \sum_{k=1}^{q} \frac{2\mu_k - \nu_{\text{min}}}{2} \right\} \ln(b_1/b_2) \ln(|b_2|). \quad (5.10) \]

**Proof.** Recalling the definition (2.6) of \( G(\zeta) \) and using the identity \( \rho^{-1} = c_1 + c_2 \), we see that

\[
\ln G(\zeta) = \ln \Gamma \left( \frac{1 + \nu_{\text{min}}}{2} + is^\theta \zeta \right) + \sum_{k=1}^{\infty} \ln \Gamma \left( \frac{1 + 2\mu_k - \nu_{\text{min}}}{2} - is^\theta \zeta \right) - \sum_{j=1}^{r} \ln \Gamma \left( \frac{1 + 2\nu_j - \nu_{\text{min}}}{2} - is^\theta \zeta \right)
- is^\theta \zeta (c_1 \ln(is^\theta \zeta) + c_2 \ln(-is^\theta \zeta) + c_3).
\]

Hence, recalling the values (2.5) of \( c_1, c_2 \) and \( c_3 \),

\[
\frac{G'(\zeta)}{G(\zeta)} = is^\theta \left( \psi \left( \frac{1 + \nu_{\text{min}}}{2} + is^\theta \zeta \right) - \sum_{k=1}^{q} \psi \left( \frac{1 + 2\mu_k - \nu_{\text{min}}}{2} - is^\theta \zeta \right) + \sum_{j=1}^{r} \psi \left( \frac{1 + 2\nu_j - \nu_{\text{min}}}{2} - is^\theta \zeta \right)
- \ln(is^\theta \zeta) - (r - q) \ln(-is^\theta \zeta) \right).
\]

Therefore, after integrating by parts, we can write

\[ X_\alpha = X_{1,\alpha} + X_{2,\alpha} + X_{3,\alpha}, \]

where

\[ X_{1,\alpha} = \frac{\mathcal{F}(\zeta) \ln G(\zeta) \bigg|_{\zeta=b_2}}{2\pi i}, \]

\[ X_{2,\alpha} = \int_{is^\theta \Sigma_5} \psi \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - w \right) \left\{ \psi \left( \frac{1 + \nu_{\text{min}}}{2} + w \right) - \ln(w) - \frac{\nu_{\text{min}}}{2(w-m)} \right\} \frac{dw}{2\pi i}, \]

\[ X_{3,\alpha} = \int_{is^\theta \Sigma_5} \psi \left( \frac{1 + 2\alpha - \nu_{\text{min}}}{2} - w \right) - \sum_{j=1}^{r} \psi \left( \frac{1 + 2\nu_j - \nu_{\text{min}}}{2} - w \right) + \sum_{k=1}^{q} \psi \left( \frac{1 + 2\mu_k - \nu_{\text{min}}}{2} - w \right)
+ \sum_{j=1}^{r} \psi \left( \frac{1 + 2\nu_j - \nu_{\text{min}}}{2} - w \right) - (r - q) \ln(-w) + \frac{\nu_{\text{min}}}{2(w-m)} \frac{dw}{2\pi i}. \]

Here we have used the change of variables \( is^\theta \zeta = w \) in the expressions for \( X_{2,\alpha} \) and \( X_{3,\alpha} \), and \( m \) is an arbitrary constant which lies in \( \mathbb{C} \setminus is^\theta \Sigma_5 \); it will be convenient to henceforth choose \( m = \frac{1 + 2\alpha - \nu_{\text{min}}}{2} \).

As in the proof of Proposition 4.2, we have added and substracted one term in order to ensure that \( X_{2,\alpha} \) has a limit as \( s \to +\infty \). It remains to compute the large \( s \) asymptotics of \( X_{1,\alpha}, X_{2,\alpha}, \) and \( X_{3,\alpha} \).

**Asymptotics of \( X_{1,\alpha} \).** By combining (4.23) and (4.38), we obtain directly that

\[ X_{1,\alpha} = \Omega(\ln(s^\theta)) + X_{1,\alpha}^{(c)} + \mathcal{O} \left( \frac{\ln(s^\theta)}{s^\theta} \right), \quad s \to +\infty, \]

where \( X_{1,\alpha}^{(c)} \) is given by (5.8).
Asymptotics of $X_{2,\alpha}$. The expansion (4.21) implies that
\[
X_{2,\alpha} = \int_{-\infty}^{\infty} \psi \left( \frac{1 + 2\alpha - \nu_{\min}}{2} - u \right) \left\{ \psi \left( \frac{1 + \nu_{\min}}{2} + w \right) - \ln(w) - \frac{\nu_{\min}}{2w - m} \right\} \frac{dw}{2\pi i} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right)
\]
as $s \to +\infty$. We deform the contour of integration to infinity in the right half-plane and pick up infinitely many residue contributions at the points $m + k$, $k = 1, 2, \ldots$, of the form
\[
- \left\{ \psi(k + 1 + \alpha) - \ln(k + \frac{1 + 2\alpha - \nu_{\min}}{2}) - \frac{\nu_{\min}}{2k} \right\}
\]
and one residue contribution $-\psi(1 + \alpha) + \ln(m) - \frac{\gamma \nu_{\min}}{2}$ at the point $m$. Using the identity (4.25), it follows that
\[
X_{2,\alpha} = - \sum_{k=1}^{\infty} \left\{ \psi(k + 1 + \alpha) - \ln(k + \frac{1 + 2\alpha - \nu_{\min}}{2}) - \frac{\nu_{\min}}{2k} \right\}
- \psi(1 + \alpha) + \ln(1 + 2\alpha - \nu_{\min}) - \gamma \nu_{\min} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right)
\]
\[
= X_{2,\alpha}^{(c)} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right), \quad s \to +\infty.
\]
Asymptotics of $X_{3,\alpha}$. For $X_{3,\alpha}$, we first deform the contour $is^\rho \Sigma_5$ to $is^\rho \gamma_{b_2 b_1}$, then we apply the change of variables $w = is^\rho \zeta$ and the large $s$ asymptotics (4.38) of $\psi$. This gives
\[
X_{3,\alpha} = \Omega\left( \ln(s^\rho) \right) + \left\{ \frac{\nu_{\min}}{2} + \sum_{j=1}^{r} 2\nu_j - \nu_{\min} - \sum_{k=1}^{q} 2\mu_k - \nu_{\min} \right\} \int_{\gamma_{b_2 b_1}} \frac{\ln(-i\zeta)}{2\pi i} \frac{d\zeta}{\zeta} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right)
\]
as $s \to +\infty$. Using the second integral in (4.33), we obtain
\[
X_{3,\alpha} = \Omega\left( \ln(s^\rho) \right) + X_{3,\alpha}^{(c)} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right), \quad s \to +\infty.
\]

Proposition 5.3. Let $Z_{\alpha}$ be defined by (5.6). Then
\[
Z_{\alpha} = \Omega\left( \ln(s^\rho) \right) + Z_{\alpha}^{(c)} + \mathcal{O}\left( \frac{\ln(s^\rho)}{s^\rho} \right)
\]
as $s \to +\infty$ uniformly for $\alpha$ in compact subsets of $[0, +\infty)$, where
\[
Z_{\alpha}^{(c)} = \frac{1}{4\pi i} \left\{ -4\pi i \ln(|b_2|)(c_5 - c_6) + 4i\pi c_6 \ln \left( \frac{2}{|b_2| + \text{Im } b_2} \right)
\right.
\frac{\ln(b_1)(-2c_7 + i\pi(3c_5 + c_6)) - (c_5 + c_6)(\ln b_1)^2}{\ln b_2(2c_7 + i\pi(c_5 - c_6)) + (c_5 + c_6)(\ln b_2)^2 + 2\pi^2 c_5}. \quad (5.12)
\]

Proof. A formula for the large $s$ asymptotics of the function $F(\zeta)$ defined in (5.5) can be deduced from (4.38) (with $\nu$ replaced by $2\alpha - \nu_{\min}$). This formula is uniform for $\zeta \in \gamma_{b_2 b_1}$, and can be differentiated with respect to $\zeta$. Hence
\[
F'(\zeta) = -\frac{1}{\zeta} + \mathcal{O}(s^{-\rho}) \quad \text{as } \ z \to +\infty, \quad (5.13)
\]
uniformly for $\zeta \in \gamma_{h_2}$. Substituting (5.13) and the asymptotic formula (4.39) for $\rho(\zeta)$ into the definition (5.6) of $Z_\alpha$, we infer that

$$Z_\alpha = -\frac{c_4}{\rho} \ln(s^\rho) \int_{\gamma_{h_2}} \frac{d\zeta}{\zeta 2\pi i} + \int_{\gamma_{h_2}} B(\zeta) \frac{d\zeta}{\zeta 2\pi i} + O\left(\frac{\ln s^\rho}{s^\rho}\right)$$

as $s \to +\infty$ uniformly for $\alpha$ in compact subsets of $(-1, +\infty)$, where $B(\zeta)$ is defined by (4.40). Recalling (4.45), the proposition follows.

5.3 Asymptotics of $I_{3,\alpha}(K)$ and $I_{4,\alpha}(K)$

We next show that the quantities $I_{3,\alpha}(K)$ and $I_{4,\alpha}(K)$ defined in (3.20) and (3.21) vanish as $s$ tends to $+\infty$ for $K = s^\rho$.

**Proposition 5.4.** Let $\nu > -1$, $K = s^\rho$, and let $N \geq 1$ be an integer. Then

$$I_{3,\alpha}(K) = O\left(\frac{\ln(s^\rho)}{s^\rho}\right) \quad \text{and} \quad I_{4,\alpha}(K) = O(s^{-N\rho}) \quad (5.14)$$

as $s \to +\infty$ uniformly for $\alpha$ in compact subsets of $(-1, +\infty)$.

**Proof.** The proof for $I_{3,\alpha}(K)$ is similar to (but easier than) the proof of Proposition 4.4. In fact, it follows from (4.38) that

$$\psi\left(\frac{1 + 2\nu - \nu_{\min}}{2} - is^\rho \zeta\right) = O(\ln(s^\rho))$$

as $s \to +\infty$ uniformly for $\zeta \in \tilde{\sigma}$ (see the proof of Proposition 4.4 for the definition of $\tilde{\sigma}$), from which we immediately deduce that

$$I_{3,\alpha}(K) = O\left(\frac{\ln(s^\rho)}{s^\rho}\right), \quad \text{as} \ s \to +\infty.$$  

The proof for $I_{4,\alpha}(K)$ is analogous to the proof of Proposition 4.5 and we omit the details.  

5.4 Integration of the differential identities in $\nu_\ell$ and $\mu_\ell$

By using the results of the previous subsections, we can compute the large $s$ asymptotics of $\partial_\alpha \ln C$ for any $\alpha \in \{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$. Integration with respect to $\alpha$ then yields the following proposition.

**Proposition 5.5.** Let $r > q \geq 0$ be integers and suppose that $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q > -1$. Let $\nu_{\min} = \min\{\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q\}$. If $\ell \in \{1, \ldots, r\}$, then

$$\int_{\nu_{\min}}^{\nu_\ell} \partial_{\nu_\ell}(\ln C) d\nu_\ell = (\nu_\ell - \nu_{\min}) \left(\sum_{j=1}^{r} \nu_j - \sum_{k=1}^{q} \mu_k\right) \left(\ln(r - q) - \ln(1 + r - q)\right)$$

$$+ \nu_\ell^2 - \nu_{\min}^2 \frac{1 + r - q - (r - q)^2}{1 + r - q} \ln(r - q) - \frac{1}{2} \frac{\nu_\ell^2 - \nu_{\min}^2}{\nu_\ell^2} \ln(1 + r - q)$$

$$+ \ln G(1 + \nu_\ell) - \ln G(1 + \nu_{\min}) + (\nu_{\min} - \nu_\ell) \frac{\ln(2\pi)}{2}. \quad (5.15)$$
If $\ell \in \{1, \ldots, q\}$, then
\[
\int_{\nu_{\min}}^{\mu_{\ell}} \partial_{\nu_{\ell}}(\ln C) d\nu_{\ell} = (\nu_{\min} - \mu_{\ell}) \left( \sum_{j=1}^{r} \nu_j - \sum_{k=1}^{q} \mu_k \right) \left( \ln(r - q) - \ln(1 + r - q) \right) + \frac{\mu_{\ell}^2 - \nu_{\min}^2}{2} \frac{1 + r - q + (r - q)^2}{1 + r - q} \ln(r - q) - (r - q) \mu_{\ell}^2 - \nu_{\min}^2 \ln(1 + r - q) \]
\[- \ln G(1 + \mu_{\ell}) + \ln G(1 + \nu_{\min}) - (\nu_{\min} - \mu_{\ell}) \frac{\ln(2\pi)}{2}. \tag{5.16}
\]

**Proof.** We only consider the case $\alpha = \nu_{\ell}, \ell \in \{1, \ldots, r\}$. The case $\alpha = \mu_{\ell}, \ell \in \{1, \ldots, q\}$, is analogous. We start by integrating the term $X_{2,\nu_{\ell}}^{(c)}$ defined in (5.9). By Fubini’s theorem, we can interchange the order of integration and summation, which implies
\[
\int_{\nu_{\min}}^{\nu_{\ell}} X_{2,\nu_{\ell}}^{(c)} d\nu_{\ell} = \sum_{k=0}^{\infty} \{- \ln \Gamma \left( k + 1 + \nu_{\ell} \right) - \nu_{\ell} + \left( k + \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) \ln \left( k + \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) \}
\]
\[
\frac{\nu_{\min}}{2}(\nu_{\ell} - \nu_{\min}) \ln \left( 1 + \frac{1}{k + 1} \right) + \ln \Gamma \left( k + 1 + \nu_{\min} \right) + \nu_{\min}
\]
\[- \left( k + \frac{1 + \nu_{\min}}{2} \right) \ln \left( k + \frac{1 + \nu_{\min}}{2} \right) \}
\]
Simplification gives (see (4.27), (4.28), and (4.30))
\[
\int_{\nu_{\min}}^{\nu_{\ell}} X_{2,\nu_{\ell}}^{(c)} d\nu_{\ell} = \frac{\ln(2\pi)}{2}(\nu_{\min} - \nu_{\ell}) + \ln G(1 + \nu_{\ell}) - \ln G(1 + \nu_{\min}) \]
\[
+ \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \ln \left( 1 + \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) - \frac{1 + \nu_{\min}}{2} \ln \left( 1 + \frac{\nu_{\min}}{2} \right) \]
\[- \zeta' \left( -1, \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) + \zeta' \left( -1, \frac{1 + \nu_{\min}}{2} \right) + \frac{\nu_{\ell}}{2}(\nu_{\ell} - \nu_{\min}). \tag{5.17}
\]
On the other hand, part (b) of Lemma 3.1 and Propositions 5.1, 5.2, 5.3, and 5.4 imply that
\[
\partial_{\nu_{\ell}}(\ln C) - X_{2,\nu_{\ell}}^{(c)} = \ln \Gamma \left( \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) - \frac{\ln(2\pi)}{2} \]
\[
+ \left( \sum_{j=1}^{r} \nu_j - \sum_{k=1}^{q} \mu_k \right) \ln(r - q) + \nu_{\ell} \frac{1 + r - q - (r - q)^2}{1 + r - q} \ln(r - q) \]
\[- \left( \sum_{j=1}^{r} \nu_j - \sum_{k=1}^{q} \mu_k \right) \ln(1 + r - q) - (2 + q - r)\nu_{\ell} \ln(1 + r - q). \tag{5.18}
\]
Using the identity (4.5), we can integrate the term in (5.18) involving the Gamma function:
\[
\int_{\nu_{\min}}^{\nu_{\ell}} \ln \Gamma \left( \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) d\nu_{\ell} = \frac{\ln(2\pi)}{2}(\nu_{\ell} - \nu_{\min}) - \frac{\nu_{\ell}}{2}(\nu_{\ell} - \nu_{\min}) \]
\[- \frac{1 - 2\nu_{\ell} + \nu_{\min}}{2} \ln \Gamma \left( \frac{1 + 2\nu_{\ell} - \nu_{\min}}{2} \right) + \frac{1 - \nu_{\min}}{2} \ln \Gamma \left( \frac{1 + \nu_{\min}}{2} \right) \]
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to prove Theorem 5.17 shows that, for $\nu = \frac{1}{2}$, the function $\nu^2 - \ln(1 + \nu^2)$ is analogous.

In view of (4.54) and the identities $\Gamma(z + 1) = z\Gamma(z)$ and $G(z + 1) = \Gamma(z)G(z)$, this expression simplifies to (5.15). The proof of (5.16) is analogous.

6 Proof of Theorem 1.1

We use the strategy described in Section 1.3 to prove Theorem 1.1. Let $C^{(\ell)}$ be the constant arising in the large gap asymptotics for the point process induced by $K^{(\ell)}$, $\ell \in \{0, \ldots, r + q\}$, where $K^{(\ell)}$ is given by (1.32). The final constant is given by

$$\ln C = \ln C_{r-q} + \sum_{\ell=1}^{r} \int_{\nu_{\min}}^{\nu_{\ell}} \partial_{\nu_{\ell}}(\ln C^{(\ell)})d\nu_{\ell} + \sum_{\ell=1}^{r} \int_{\nu_{\min}}^{\nu_{\ell}} \partial_{\nu_{\ell}}(\ln C^{(r+\ell)})d\mu_{\ell},$$

where $C_{r-q}$ is the constant in (4.53) associated with $K_{r-q}$. Proposition 4.6 with $r$ replaced by $r - q$ gives

$$\ln C_{r-q} = (r - q)\ln G(1 + \nu_{\min}) - \frac{\ln(2\pi)}{2} \nu_{\min}^2 - (r - q - 1)\zeta'(1 - 1) + 2(r - q - 1)\nu_{\min}^2$$

$$+ \frac{2 + (r - q)^2(1 + 2\nu_{\min}^2)}{24(r - q + 1)} \ln(r - q) - \frac{(r - q - 1)^2 + 12(r - q)\nu_{\min}^2}{24} \ln(1 + r - q).$$

On the other hand, Proposition 5.5 shows that, for $\ell = 1, \ldots, r$,

$$\int_{\nu_{\min}}^{\nu_{\ell}} \partial_{\nu_{\ell}}(\ln C^{(\ell)})d\nu_{\ell} = (\nu_{\ell} - \nu_{\min}) \left( \sum_{j=1}^{\ell-1} \nu_j + (r - q - \ell)\nu_{\min} \right)(\ln(r - q) - \ln(1 + r - q))$$

$$- \ln G\left(\frac{1 + 2\nu_{\ell} - \nu_{\min}}{2}\right) + \ln G\left(\frac{1 + \nu_{\min}}{2}\right).$$
\[\frac{\nu^2 - \nu_{\text{min}}^2}{2} \frac{1 + r - q - (r - q)^2}{1 + r - q} \ln(r - q) - (2 + q - r) \frac{\nu^2 - \nu_{\text{min}}^2}{2} \ln(1 + r - q)\]
\[+ \ln G(1 + \nu) - \ln G(1 + \nu_{\text{min}}) + (\nu_{\text{min}} - \nu) \frac{\ln(2\pi)}{2}\]  
(6.3)

and, for \(\ell = 1, \ldots, q\),
\[\int_{\nu_{\text{min}}}^{\mu} \partial_{\nu}^{\ell} \ln G^{(r+\ell)} d\nu = (\nu_{\text{min}} - \mu) \left( \sum_{j=1}^{r} \nu_j - \sum_{k=1}^{\ell-1} \mu_k - (q - \ell) \nu_{\text{min}} \right) \left( \ln(r - q) - \ln(1 + r - q) \right)\]
\[+ \frac{\mu^2 - \nu_{\text{min}}^2}{2} \frac{1 + r - q + (r - q)^2}{1 + r - q} \ln(r - q) - (r - q) \frac{\mu^2 - \nu_{\text{min}}^2}{2} \ln(1 + r - q)\]
\[- \ln G(1 + \mu) + \ln G(1 + \nu_{\text{min}}) - (\nu_{\text{min}} - \mu) \frac{\ln(2\pi)}{2}\]  
(6.4)

By substituting (6.2), (6.3), and (6.4) into (6.1), we find the expression (1.11). This completes the proof of Theorem 1.1. \(\square\)

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