Infinitely Generated Hecke Algebras with Infinite Presentation

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Abstract
For a locally compact group $G$ and a compact subgroup $K$, the corresponding Hecke algebra consists of all continuous compactly supported complex functions on $G$ that are $K$–bi-invariant. There are many examples of totally disconnected locally compact groups whose Hecke algebras with respect to a maximal compact subgroups are not commutative. One of those is the universal group $U(F) +$, when $F$ is primitive but not 2–transitive. For this class of groups we prove the Hecke algebra with respect to a maximal compact subgroup $K$ is infinitely generated and infinitely presented. This may be relevant for constructing irreducible unitary representations of $U(F) +$ whose subspace of $K$–fixed vectors has dimension at least two. On the contrary, when $F$ is 2–transitive that Hecke algebra of $U(F) +$ is commutative, finitely generated admitting a single generator.

Keywords  Hecke algebras · Locally compact groups

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1 Introduction
The Hecke algebras are very useful tools to study the representation theory of locally compact groups. For example, in the particular case of a semi-simple algebraic group $G$ over a non-Archimedean local field there are two important Hecke algebras that can be associated with: the Hecke algebra of $G$ with respect to a good maximal compact subgroup, called the spherical Hecke algebra of $G$, and the Hecke algebra of $G$ with respect to a Iwahori subgroup, which is a smaller compact subgroup. The latter algebra is called the Iwahori–Hecke algebra of $G$ and plays a very important role in the representation theory of algebraic groups, especially in the Kazhdan–Lusztig theory, being an intense and rich field of research. The former one is used to study the spherical unitary dual of semi-simple and analogous groups. That Hecke algebra is moreover commutative and finitely generated with respect to the

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convolution product. The representation theory of both algebras is intimately related to the representation theory of $G$.

In this article we restrict our attention to Hecke algebras associated with specific totally disconnected locally compact groups and their maximal compact subgroups. First, let us recall the general definition of a Hecke algebra and some more specific motivation.

**Definition 1.1** Let $G$ be a locally compact group and $K \leq G$ be a compact subgroup. We denote by $C_c(G, K)$ the space of continuous, compactly supported complex-valued functions $\phi : G \to \mathbb{C}$ that are $K$-bi-invariant, i.e., functions that satisfy the equality $\phi(kgk') = \phi(g)$ for every $g \in G$ and all $k, k' \in K$. We view the $\mathbb{C}$-vector space $C_c(G, K)$ as an algebra whose multiplication is given by the convolution product

$$\phi * \psi : x \mapsto \int_G \phi(xg)\psi(g^{-1})d\mu(g)$$

where $\mu$ is the left Haar measure on $G$. Moreover, $C_c(G, K)$ is called the Hecke algebra corresponding to $K \leq G$. $(G, K)$ forms a Gelfand pair if the convolution algebra $C_c(G, K)$ is commutative.

For a locally compact group $G$ it is well known there is a natural one-to-one correspondence between (irreducible) unitary representations of $G$ and (irreducible) non-degenerate $\ast$-representations of $L^1(G)$ (see for example Dixmier [8, Chapter 13]). When restricted to a Hecke algebra of $G$ we have the following situation. With any (irreducible) unitary representation $(\pi, \mathcal{H})$ of $G$ admitting a non-zero $K$-invariant vector there is associated a canonical (irreducible) non-degenerate $\ast$-representation of the Hecke algebra $C_c(G, K)$ to $\text{End}(\mathcal{H}^K)$, where $\mathcal{H}^K$ is the space of $K$-invariant vectors with respect to $(\pi, \mathcal{H})$. In general the converse is not true, namely: the category of non-degenerate $\ast$-representation of the Hecke algebra $C_c(G, K)$ is not necessarily equivalent to the category of unitary representations of $G$ generated by $K$-fixed vectors (see for example the PhD thesis of Hall [10]).

The importance of Gelfand pairs in the theory of unitary representations of locally compact groups is given by the following well known two results. For the corresponding definitions one can consult van Dijk [13].

**Proposition 1.2** (See Proposition 6.3.1 in [13]) Let $G$ be a locally compact group and $K \leq G$ a compact subgroup. The pair $(G, K)$ is Gelfand if and only if for every irreducible unitary representation $(\pi, \mathcal{H})$ of $G$ the dimension of the subspace of $K$-fixed vectors is at most one.

**Corollary 1.3** (See Corollary 6.3.3 in [13]) Let $(G, K)$ be a Gelfand pair. The positive-definite spherical functions on $G$ correspond one-to-one to the equivalence classes of irreducible unitary representations of $G$ having a one-dimensional $K$-fixed vector space.

All known examples of connected, resp., totally disconnected, non-compact locally compact groups admitting Gelfand pairs are:

1. all semi-simple non-compact real Lie groups, with finite center, together with their maximal compact subgroups
2. locally compact groups $G$ that act continuously, properly and strongly transitively on a locally finite thick Euclidean building $\Delta$ together with the stabilizer in $G$ of a special vertex of $\Delta$.

In the latter case and assuming a type-preserving action, by the main theorem of [4] those are all Gelfand pairs that can appear. Recall, the second family of groups includes all semi-simple algebraic groups over non-Archimedean local fields and closed subgroups of $\text{Aut}(T)$.
that act 2–transitively on the boundary of $T$, where $T$ is a bi-regular tree with valence at least 3 at every vertex. Using the polar decomposition of those groups (see for example [5, Remark 4.6, Lemma 4.7]) and the Lemma of Bernstein [2], the commutative Hecke algebra $C_C(G, K)$, where $K$ is the stabilizer in $G$ of a special vertex of its corresponding Bruhat–Tits building, is finitely generated. Moreover, the main result of Bernstein [2] states that all reductive $p$-adic Lie groups are groups of type I (see Definition 5.4.2 in [8]). For the case of closed non-compact subgroups of $\text{Aut}(T_d)$ that act 2–transitively on the boundary of $T_d$, the type I property is still open in this generality see [6, 7]. After the online appearance of this paper, the author learned the recent result of [11, Theorems A,B] where for a special class of subgroups of the automorphisms of a locally finite tree the type I property holds if and only if the subgroup acts 2–transitively on the boundary of the tree. For the general case, if the group does not act 2–transitively of the boundary, then the group is not of type I.

Apart from the case of Gelfand pairs, the structure of Hecke algebras with respect to maximal compact subgroups is in general much less studied, even if those algebras can provide very useful information about the unitary representations of those locally compact groups. In this article we propose to study the structure of non-commutative Hecke algebras that are associated with a particular family of totally disconnected locally compact groups and their maximal compact subgroups. This family of groups is given by the universal groups $U(F)$ introduced by Burger–Mozes in [3, Section 3]. In his PhD thesis [1], Amann studies these groups from the point of view of their unitary representations.

First, let us recall the basic definitions.

**Definition 1.4** Denote by $\mathcal{T}$ the $d$-regular tree, with $d \geq 3$, and by $\text{Aut}(\mathcal{T})$ its full group of automorphisms, endowed with the compact-open topology. Let $\iota : E(\mathcal{T}) \to \{1, ..., d\}$, where $E(\mathcal{T})$ is the set of unoriented edges of the tree $\mathcal{T}$. The set $E(x) \subset E(\mathcal{T})$ consisting of all the edges containing the vertex $x \in \mathcal{T}$ is called the star of $x$. We say that $\iota$ is a **legal coloring** of the tree $\mathcal{T}$ if for every vertex $x \in \mathcal{T}$ the map $\iota|_{E(x)}$ is in bijection with $\{1, ..., d\}$.

**Definition 1.5** Let $F$ be a subgroup of permutations of the set $\{1, ..., d\}$ and let $\iota$ be a legal coloring of $\mathcal{T}$. We define the **universal group**, with respect to $F$ and $\iota$, to be

$$U(F) := \{ g \in \text{Aut}(\mathcal{T}) \mid \iota \circ g \circ (\iota|_{E(x)})^{-1} \in F, \text{ for every } x \in \mathcal{T} \}.$$  

When $F$ is the full permutation group $\text{Sym}(\{1, ..., d\})$, $U(F)$ equals $\text{Aut}(\mathcal{T})$. If $F = \text{id}$ then $U(F)$ is the group of all automorphisms preserving the coloring $\iota$ of $\mathcal{T}$.

We say an element $g \in \text{Aut}(\mathcal{T})$ is **edge-stabilizer** if there is an edge of $\mathcal{T}$ stabilized by $g$ pointwise. By $U(F)^+$ we denote the subgroup of $U(F)$ generated by the edge-stabilizers in $U(F)$. When $F = \text{Sym}(\{1, ..., d\})$, $U(F)^+ = \text{Aut}(\mathcal{T})^+$ is the group of type-preserving automorphisms of $\mathcal{T}$. When $F$ is transitive and generated by its point stabilizers, we have $U(F)^+ = U(F) \cap \text{Aut}(\mathcal{T})^+$.

By [1, Prop. 52] the groups $U(F)$, $U(F)^+$ are independent of the legal coloring $\iota$ of $\mathcal{T}$. From the definition $U(F)$ and $U(F)^+$ are closed subgroups of $\text{Aut}(\mathcal{T})$, and $U(F)^+$ is trivial or simple.

Another key property which is used in the sequel is the following.

**Definition 1.6** (See [12]) Let $T$ be a locally finite tree and let $G \leq \text{Aut}(T)$ be a closed subgroup. We say $G$ has **Tits’ independence property** if for every edge $e$ of $T$ we have the equality $G_e = G_{T_1}G_{T_2}$, where $T_i$ are the two infinite half sub-trees of $T$ emanating from the edge $e$ and $G_{T_i}$ is the pointwise stabilizer of the half-tree $T_i$. 

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Following [1, 3], $U(F)$ and $U(F)^+$ have Tits’ independence property.

To avoid heavy notation, for the rest of the article we refer to the following convention.

**Convention 1.7** Fix $d \geq 3$. We denote by $\text{dist}_\mathcal{T}$ the usual distance on $\mathcal{T}$ and by $\partial \mathcal{T}$ the set of all ends of $\mathcal{T}$. Let $F$ be a primitive subgroup of $\text{Sym}(1, \ldots, d)$. Consider fixed a coloring $\iota$ of $\mathcal{T}$, a vertex $x$ of $\mathcal{T}$ and an edge $e$ in the star of $x$. For simplicity, set $\mathbb{G} := U(F)^+$, and $K := \mathbb{G}_x$, the stabilizer in $\mathbb{G}$ of the vertex $x$. As $\mathbb{G}$ is endowed with the compact-open topology, $K$ is an open and maximal compact subgroup of $\mathbb{G}$ (see [9, page 10, Thm. (5.2)]). Moreover, for a finite subset $A \subset \mathcal{T}$ we denote by $\mathbb{G}_A$ the pointwise stabilizer in $\mathbb{G}$ of the set $A$. For $\xi \in \partial \mathcal{T}$, $\mathbb{G}_\xi$ denotes the stabilizer in $\mathbb{G}$ of the ideal point $\xi$. Let $S(x, r) := \{ y \in \mathcal{T} \mid \text{dist}_\mathcal{T}(x, y) = r \}$, where $r \in \mathbb{N}^\ast$. Let $\mathcal{T}_{x,e}$ be the half-tree of $\mathcal{T}$ emanating from the vertex $x$ and containing the edge $e$. Set $V_{x,r} := S(x, r) \cap \mathcal{T}_{x,e}$, for every $r \in \mathbb{N}^\ast$. We have $|V_{x,r}| = (d - 1)^{r-1}$. For every two points $y, z \in \mathcal{T} \cup \partial \mathcal{T}$, we denote by $[y, z]$ the unique geodesic between $y$ and $z$ in $\mathcal{T} \cup \partial \mathcal{T}$. For a hyperbolic element $\gamma$ in $\mathbb{G}$ let $|\gamma| := \min_{x \in \mathcal{T}} \{ \text{dist}_\mathcal{T}(x, \gamma(x)) \}$, which is called the translation length of $\gamma$. Set $\text{Min}(\gamma) := \{ x \in \mathcal{T} \mid \text{dist}_\mathcal{T}(x, \gamma(x)) = |\gamma| \}$.

By [1, 3] the group $\mathbb{G}$ act 2–transitively on the boundary $\partial \mathcal{T}$ if and only if $F$ is 2–transitive. From this fact together with the main theorem in [4], applied to our case of $d$-regular trees, $(\mathbb{G}, K)$ is a Gelfand pair if and only if $F$ is 2–transitive. Therefore, when $F$ is primitive but not 2–transitive the Hecke algebra $C_c(\mathbb{G}, K)$ is not commutative. In particular, by Proposition 1.2 the group $\mathbb{G}$ admits an irreducible unitary representation whose subspace of $K$–fixed vectors has dimension at least two. Moreover, the theory of unitary representations of $\mathbb{G}$ is not at all developed when $F$ is primitive but not 2–transitive. Therefore, it is natural to study the structure of the Hecke algebra $C_c(\mathbb{G}, K)$ and its irreducible non-degenerate $*$–representations, when $F$ is primitive but not 2–transitive. Regarding the structure of this algebra we obtain the following theorem, which is the main result of this article.

**Theorem 1.8** Let $F$ be primitive. If $F$ is 2-transitive then the Hecke algebra $C_c(\mathbb{G}, K)$ is finitely generated admitting only one generator. If $F$ is not 2–transitive, then the Hecke algebra $C_c(\mathbb{G}, K)$ is infinitely generated with an infinite presentation.

Here by infinitely generated we mean there exists no finite set of generators, and by an infinite presentation we mean there are an infinite number of relations between the generators.

Understanding irreducible non-degenerate $*$–representations of $C_c(\mathbb{G}, K)$ when $F$ is primitive but not 2–transitive is left for a further study. It would be also interesting to see if Theorem 1.8 could be used to construct an explicit irreducible unitary representation of $\mathbb{G}$ whose subspace of $K$–fixed vectors has dimension at least two.

The article is structured as follows. In Section 2 we prove combinatorial formulas that are essential for the proof of our main theorem. In Section 3 we study the structure of the Hecke algebra $C_c(\mathbb{G}, K)$ when $F$ is primitive. This is used in Section 4 to prove Theorem 1.8.

## 2 Some Combinatorial Formulas

This section is meant to prove combinatorial formulas relating multinomial and binomial coefficients, as shown by Proposition 2.4 below. These formulas are proved using Newton’s...
General Binomial Theorem. As a consequence we obtain Lemma 2.5 below, which is one of the key ingredients involved in the main results of this article.

**Definition 2.1** Let $f, f' : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ be defined by

$$f(r, k) := k'(k-1)^{r-1}$$

for every $(r, k) \in \mathbb{N}^* \times \mathbb{N}^*$, where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and with the convention that $0^0 = 1$.

For $n \in \mathbb{N}$, set $\text{Sum}(n) := \{(k_1, \cdots , k_{n-1}) \mid k_i \in \mathbb{N} \text{ and } n = \sum_{i=1}^{n-1} k_i \cdot i\}$.

For $l \geq 2$ and $(k_1, \cdots , k_l) \in \mathbb{N}^l$ the corresponding **multinomial coefficient** is defined to be

$$\binom{k_1 + \cdots + k_l}{k_1, \cdots , k_l} := \frac{(k_1 + \cdots + k_l)!}{k_1! \cdot k_2! \cdots k_l!},$$

with the convention that $0! = 1$. The multinomial coefficient counts the total number of different words of length $k_1 + \cdots + k_l$ formed with $l$ distinct letters, where $k_i$ is the multiplicity of the $i^{\text{th}}$ letter. For $l = 2$ we obtain the binomial coefficient.

For every $r \in \mathbb{N}$ with $r \geq 2$ let

$$P_{1,r}(X) := X^{r-2} + \cdots + X + 1$$

where $X$ is the variable of the polynomial. Note $P_{1,r}(1) = r - 1$. Moreover

$$P_{1,r}(X) - (r - 1) = \sum_{j=1}^{r-2} (X^j - 1) = (X - 1) \left( \sum_{j=1}^{r-2} (X^{j-1} + \cdots + X + 1) \right).$$

For every $r \in \mathbb{N}$ with $r \geq 3$ define

$$P_{2,r}(X) := \frac{P_{1,r}(X) - (r - 1)}{X - 1} = \sum_{j=1}^{r-2} (X^{j-1} + \cdots + X + 1).$$

More generally, for every $i \in \mathbb{N}$ with $i \geq 2$ and every $r \in \mathbb{N}$ with $r \geq i + 1$ define

$$P_{i,r}(X) := \frac{P_{i-1,r}(X) - P_{i-1,r}(1)}{X - 1}.$$

**Lemma 2.2** For every $r \in \mathbb{N}$ with $r \geq 3$ and every $i \in \{2, \cdots , r - 1\}$

$$P_{i,r}(X) = \sum_{j=i}^{r-1} P_{i-1,j}(X). \quad (1)$$

**Proof** Given $r \geq 3$, we prove the lemma by induction on $i$. First we verify it for $i = 2$. Indeed, we have

$$P_{2,r}(X) = \frac{P_{1,r}(X) - P_{1,r}(1)}{X - 1} = \frac{X^{r-2} + X^{r-3} + \cdots + X + 1 - (r - 1)}{X - 1} = \frac{r-2}{X - 1} \sum_{j=1}^{r-2} X^{j-1}$$

$$= \sum_{j=2}^{r-1} (X^{j-2} + \cdots + 1) = \sum_{j=2}^{r-1} P_{1,j}(X).$$

Suppose the formula (1) is true for $i$ and we want to prove it for $i + 1$. We have

$$P_{i+1,r}(X) = \frac{P_{i,r}(X) - P_{i,r}(1)}{X - 1} = \sum_{j=i}^{r-1} \frac{P_{i-1,j}(X) - P_{i-1,j}(1)}{X - 1} = \sum_{j=i}^{r-1} P_{i,j}(X).$$

Therefore, the formula (1) holds for $i + 1$. By induction, the lemma is proven for all $i$.\[\square\]
as for \( j = i \), \( P_{i-1,i}(X) \) equals \( P_{i-1,i}(1) \) because the degree of the polynomial \( P_{i-1,i}(X) \) is zero.

We also need the following easy lemma.

**Lemma 2.3** For every \( j \in \mathbb{N} \) with \( j \geq 2 \) and every \( i \in \{1, \cdots, j-1\} \)

\[
P_{i,j}(1) = \binom{j-1}{i} = \frac{(j-1)!}{i!(j-i-1)!}. \tag{2}
\]

**Proof** First notice \( P_{1,j}(1) = j - 1 = \frac{(j-1)!}{1!(j-2)!} = \binom{j-1}{1} \).

Given \( j \geq 2 \), one can easily verify formula (2) by induction on \( i \) and by using formula (1) for \( X = 1 \) and the known equality

\[
\sum_{i=1}^{n} l(l+1) \cdots (l+k) = \frac{n(n+1) \cdots (n+k+1)}{k+2}.
\]

**Proposition 2.4** For every \( r \in \mathbb{N} \) with \( r \geq 2 \) and every \( i \in \{1, \cdots, r-1\} \)

\[
P_{i,r}(1) = \binom{r-1}{i} = \sum_{(k_1, \cdots, k_r-1) \in \text{Sum}(r) \atop k_1 + \cdots + k_r-1 = r-i} \binom{r-i}{k_1, \cdots, k_r-1}. \tag{3}
\]

**Proof** The key ingredient in order to prove the proposition is the following equality, known as Newton’s General Binomial Theorem. Let \( n \in \mathbb{N^n} \). Formally we have

\[
(1 + X + X^2 + \cdots)^n = \left( \sum_{k=0}^{\infty} X^k \right)^n = \left( \frac{1}{1-X} \right)^n = (1-X)^{-n}
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} X^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} X^k, \tag{4}
\]

where by definition \( \binom{-n}{k} := \frac{(-n)(-n-1) \cdots (-n-k+1)}{k!} \), thus \( (-1)^k \binom{-n}{k} = \frac{n(n+1) \cdots (n+k-1)}{k!} \).

Take \( n := r - i \). Then for \( k = i \) the coefficient of \( X^i \) provided by formula (4) is

\[
\binom{r-i}{i} = \binom{r-1}{i}.
\]

Now compute the coefficient of \( X^i \) with respect to the product \((1 + X + X^2 + \cdots)^r-i\). By the definition of the multinomial coefficient this equals

\[
\sum_{k_1+2k_2+\cdots+i \cdot k_{i+1} = r-i \atop k_1+k_2+\cdots+k_{i+1} = r-i \atop k_1, \cdots, k_{i+1} \in \text{Sum}(r)} \binom{r-i}{k_1, \cdots, k_{i+1}}.
\]

Notice the following equality of sets:

\[
\{(k_1, k_2, \cdots, k_{i+1}) \mid k_2 + 2 \cdot k_3 + \cdots + i \cdot k_{i+1} = i \text{ and } k_1 + k_2 + \cdots + k_{i+1} = r - i\}
= \{(k_1, k_2, \cdots, k_{i+1}) \mid (k_1, k_2, \cdots, k_{i+1}, 0, \cdots, 0) \in \text{Sum}(r) \text{ and } k_1 + k_2 + \cdots + k_{i+1} = r - i\}
= \{(k_1, k_2, \cdots, k_r-1) \mid (k_1, k_2, \cdots, k_r-1) \in \text{Sum}(r) \text{ and } k_1 + k_2 + \cdots + k_{r-1} = r - i\},
\]

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as in the last set, \(k_i+2 = \cdots = k_{r-1}\) are always zero. Therefore

\[
\sum_{k_2+k_3+\cdots+k_{i+1}=i \atop k_1+k_2+\cdots+k_{i+1}=r-i} \binom{r-i}{k_1, \ldots, k_{i+1}} = \sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r) \atop k_1+\cdots+k_{r-1}=r-i} \binom{r-i}{k_1, \ldots, k_{r-1}}.
\]

We conclude \(\sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r) \atop k_1+\cdots+k_{r-1}=r-i} \binom{r-i}{k_1, \ldots, k_{r-1}} = \binom{r-1}{i} \). \(\square\)

**Lemma 2.5** Let \(r, k \in \mathbb{N}\) with \(r, k \geq 1\). Then

\[
k^{2r-1} - \sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r)} (k_1 + \cdots + k_{r-1}) f(1, k) k_1 f(2, k) k_2 \cdots f(r-1, k) k_{r-1} = f(r, k).
\]

In particular

\[
k^{r-1} - \sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r)} (k_1 + \cdots + k_{r-1}) f'(1, k) k_1 f'(2, k) k_2 \cdots f'(r-1, k) k_{r-1} = f'(r, k).
\]

**Proof** By Proposition 2.4

\[
k^{2r-1} - \sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r)} (k_1 + \cdots + k_{r-1}) f(1, k) k_1 f(2, k) k_2 \cdots f(r-1, k) k_{r-1}
\]

\[
= k^{2r-1} - \sum_{(k_1, \ldots, k_{r-1}) \in \text{Sum}(r)} (k_1 + \cdots + k_{r-1}) k^{r} k_{r+2} + \cdots + k_{r-1} k_{r-1}
\]

\[
= k^{r} \left( k^{r-1} - \sum_{k_1, \ldots, k_{r-1} \in \text{Sum}(r)} (k_1 + \cdots + k_{r-1} (k_1, \ldots, k_{r-1}) (k-1)^{r-i} \right)
\]

\[
= k^{r} \left( k^{r-1} - 1 - \sum_{i=2}^{r-1} \sum_{k_1, \ldots, k_{r-1} \in \text{Sum}(r) \atop k_1+\cdots+k_{r-1}=i} \binom{i}{k_1, \ldots, k_{r-1}} (k-1)^{r-i} \right)
\]

\[
= k^{r} \left( k^{r-1} - 1 - \sum_{i=2}^{r-1} P_{r-i,r}(1)(k-1)^{r-i} \right)
\]

\[
= k^{r} ((k-1)(k^{r-2} + \cdots + k + 1) - \sum_{i=2}^{r-1} P_{r-i,r}(1)(k-1)^{r-i-1})
\]

\[
= k^{r} (k-1) \left( P_{1,r}(k) - P_{1,r}(1) - \sum_{i=2}^{r-2} P_{r-i,r}(1)(k-1)^{r-i-1} \right)
\]

\[
= k^{r} (k-1)^2 \left( P_{2,r}(k) - P_{2,r}(1) - \sum_{i=2}^{r-3} P_{r-i,r}(1)(k-1)^{r-i-2} \right)
\]

\[
= \cdots
\]
\(= k^r (k - 1)^{r-3} \left( P_{r-3,r}(k) - P_{r-3,r}(1) - \sum_{i=2}^{r-1-(r-3)} P_{r-i,r}(1)(k-1)^{r-i-(r-3)} \right) \)

\(= k^r (k - 1)^{r-3} \left( (k-1)P_{r-2,r}(k) - \sum_{i=2}^{2} P_{r-i,r}(1)(k-1)^{r-i-(r-3)} \right) \)

\(= k^r (k - 1)^{r-3} \left( (k-1)P_{r-2,r}(k) - P_{r-2,r}(1)(k-1) \right) \)

\(= k^r (k - 1)^{r-2}(k-1)P_{r-1,r}(k) = k^r (k - 1)^{r-1}P_{r-1,r}(1) = k^r (k - 1)^{r-1} \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} \)

\(= k^r (k - 1)^{r-1} = f(r, k). \)

The last part of the lemma easily follows from the definition of \(f'(r, k). \)

\[ \square \]

3 Computation of the Hecke Algebra

As \(K\) is an open subgroup of \(G\), every function of the Hecke algebra \(C_c(G, K)\) is a finite linear combination of functions of the form \(1_{KgK}\) with \(g \in G\). To better understand the Hecke algebra \(C_c(G, K)\) one should be able to evaluate the convolution product of a finite number of functions \(1_{KgK}\) with \(g \in G\). The structure of the group \(G\) allows us to characterize its \(K\)–double cosets and to perform computations of the desired convolution products.

By \(F \leq \text{Sym}\{1, ..., d\}\) being primitive we mean primitive but not cyclic of prime order, implying \(F\) is transitive and generated by its point stabilizers.

When \(F\) is primitive and not generated by its point stabilizers, all point stabilizers of \(F\) are equal. This implies that all point stabilizers of \(F\) are just the identity, that \(F\) is cyclic of prime order and that the group \(U(F)^+\) is trivial, case that we want to avoid.

3.1 \(K\)–Double Cosets of \(G\)

The goal of this subsection is to describe and count \(K\)–double cosets of \(G\). Along the way we also give a description of some special hyperbolic elements of \(G\) which play an important role in that study.

When \(F\) is primitive but not 2–transitive, the group \(G\) still has some of the properties of closed non-compact subgroups of \( \text{Aut}(\mathcal{T})\) that act 2–transitively on the boundary \(\partial \mathcal{T}\).

Remark 3.1 As \(F\) is primitive, given an edge \(e'\) of \(\mathcal{T}\) at odd distance from \(e\), one can construct (using the definition of \(G\)) a hyperbolic element in \(G\) translating \(e\) to \(e'\). Moreover, every hyperbolic element in \(G\) has even translation length.

Lemma 3.2 (\(KA^+K\) decomposition) Let \(F\) be primitive. Then \(G\) admits a \(KA^+K\) decomposition, where

\[ A^+ := \{ \gamma \in G \mid \gamma \text{ is hyperbolic, } e \subset \text{Min}(\gamma), \gamma \text{ translates the edge } e \text{ inside } \mathcal{T}_{x,e} \} \cup \{ \text{id} \}. \]

Proof Let \(g \in G\). If \(g(x) = x\), then \(g \in K\). Suppose \(g(x) \neq x\). Consider the geodesic segment \([x, g(x)]\) in \(\mathcal{T}\) and denote by \(e_1\) the edge of the star of \(x\) that belongs to \([x, g(x)]\). Notice \([x, g(x)]\) has even length and there exists \(k \in K\) such that \(k(e_1) = e\); therefore, \(kg(x) \in \mathcal{T}_{x,e}\). By Remark 3.1 there is a hyperbolic element \(\gamma \in G\) of translation length
equal to the length of \([x, g(x)]\) that translates the edge \(e\) inside \(\mathcal{T}_{x,e}\) and such that \(\gamma(x) = kg(x)\); thus \(\gamma^{-1}kg \in K\). Notice the \(KA^+K\) decomposition of an element \(g \in G\) is not unique.

\[\square\]

**Remark 3.3** By Lemma 3.2 every \(K\)–double coset of \(G\) is of the form \(K\gamma K\) for some \(\gamma \in A^+\). When \(F\) is 2–transitive, \(A^+ = \langle a >^+\), where \(a\) is a hyperbolic element of \(G\), with \(|a| = 2\) and \(x \in \text{Min}(a)\). In this case we obtain the well-known polar decomposition of \(G\) and the \(K\)–double cosets of \(G\) are just \(\{Ka^nK\}_{n \geq 0}\).

In order to describe the hyperbolic elements in \(A^+\) we need the following easy but important remarks.

**Remark 3.4** Let \(\gamma, \gamma' \in A^+\) be two hyperbolic elements. We claim \(\gamma\gamma'\) is again a hyperbolic element of \(A^+\) and \(|\gamma| + |\gamma'| = |\gamma\gamma'|\). Indeed, the edge \(e\) endowed with the orientation pointing towards the boundary \(\partial \mathcal{T}_{x,e}\) is sent by any \(\eta \in A^+\) to an edge of \(\mathcal{T}_{x,e}\) with induced orientation that points towards the boundary \(\partial \mathcal{T}_{x,e}\). This observation proves the claim.

**Remark 3.5** For \(i \in \{1, \ldots, d\}\) let \(F_i\) be the stabilizer in \(F\) of \(i\). As \(F\) is transitive on the set \(\{1, \ldots, d\}\) the number of orbits of \(F_i\) acting on the set \(\{1, \ldots, d\} \setminus \{i\}\) is independent of the choice of the color \(i\). Therefore, we denote by \(k\) the total number of all such orbits. In addition, independently of the choice of a color \(i \in \{1, \ldots, d\}\) we are allowed to denote by \(n_j\), where \(j \in \{1, \ldots, k\}\), the number of elements of each \(F_i\)–orbit in \(\{1, \ldots, d\} \setminus \{i\}\). Notice \(n_1 + \cdots + n_k = d - 1\). If \(F\) is 2–transitive \(k = 1\). If \(F\) is not 2–transitive, then \(k \geq 2\).

Let us now describe the elements in \(A^+\) of translation length 2. By Remarks 3.1 and 3.5 it is easy to see there are exactly \(k\) distinct \(K\)–double cosets \(K\gamma K\), with \(\gamma \in A^+\) and \(|\gamma| = 2\). Let \(K\gamma K\) be such a \(K\)–double coset and let \(\gamma_1, \gamma_2 \in A^+ \cap K\gamma K\). Then \(|\gamma_1| = 2\), for every \(i \in \{1, 2\}\). We notice there are two cases: either there is \(k \in G_e\) such that \(k\gamma_1(e) = \gamma_2(e)\), or \(k\gamma_1(e) \neq \gamma_2(e)\) for every \(k \in K\). Moreover, the latter mentioned case occurs if and only if \(F\) is not 2–transitive.

**Definition 3.6** Let \(F\) be primitive and let \(\gamma_1, \gamma_2 \in A^+\) be such that \(|\gamma_1| = |\gamma_2| = 2\).

We say the ordered pair \((\gamma_1, \gamma_2)\) is a \(K\)–**primitive pair** if there exists \(k \in K\) such that \(k\gamma_1(e) = \gamma_2(e)\). If this is the case, then \(k \in G_e\) and \(K\gamma_1 K = K\gamma_2 K\). In particular, for every \(k' \in G_{[x, \gamma_2(e)]}kG_{[x, \gamma_1(e)]}\) we have \(k'\gamma_1(e) = \gamma_2(e)\).

Note, if \((\gamma_1, \gamma_2)\) is a \(K\)–primitive pair then \((\gamma_2, \gamma_1)\) is such as \(\gamma_1(e) = k^{-1}\gamma_2(e)\). Also, two elements in disjoint \(K\)–double cosets of two elements of \(A^+\) with translation length 2 do not form a \(K\)–primitive pair.

By Remark 3.5 we deduce:

**Lemma 3.7** Let \(F\) be primitive. Then every \(K\)–double coset \(K\gamma K\), with \(\gamma \in A^+\) of translation length 2, admits exactly \(k\) \(K\)–primitive pairs up to \(K\)–left-right multiplication. In particular there are exactly \(k^2\) \(K\)–primitive pairs of \(A^+\), up to \(K\)–left-right multiplication.

For a further use and following Lemma 3.7 let us make the following notation.

**Notation 3.8** For every \(\gamma \in A^+\), with \(|\gamma| = 2\), we denote by \(\{\gamma(j)\}_{j \in \{1, \ldots, k\}} \subset A^+\) a set of different elements of translation length 2 such that for every \(\eta \in K\gamma K \cap A^+\) there exists a unique \(j \in \{1, \ldots, k\}\) with the property that \((\eta, \gamma(j))\) is a \(K\)–primitive pair.
Lemma 3.9 Let $F$ be primitive. Let $\gamma, \gamma' \in A^+ \setminus \{id\}$ be such that there exists $k \in K$ with $k\gamma(e) = \gamma'(e)$. Let $(\gamma_1, \gamma'_1)$ be a $K$–primitive pair (resp., let $\gamma_2, \gamma'_2 \in A^+$ with $|\gamma_2| = |\gamma'_2|$ and such that $K\gamma_2 K = K\gamma'_2 K$). Then there exists $k' \in K$ with the property that $k'\gamma\gamma_1(e) = \gamma'\gamma'_1(e)$ (resp., $k'\gamma\gamma_2(x) = \gamma'\gamma'_2(x)$).

Proof We have to find $k' \in K$ that sends the geodesic segment $[x, \gamma\gamma_1(e)]$ into the geodesic segment $[x, \gamma'\gamma'_1(e)]$.

As $\gamma, \gamma' \in A^+$ are hyperbolic the following inclusions of geodesic segments hold: $[x, \gamma(e)] \subset [x, \gamma\gamma_1(e)]$ and $[x, \gamma'(e)] \subset [x, \gamma'\gamma'_1(e)]$. Also, because $k\gamma(e) = \gamma'(e)$ then $k \in \mathbb{G}_e$, and so $k$ sends the geodesic segment $[x, \gamma(e)]$ into the geodesic segment $[x, \gamma'(e)]$.

We just need to find $k' \in \mathbb{G}_{[x, \gamma'(e)]}$ that sends $[\gamma'(e), k\gamma\gamma_1(e)]$ into $[\gamma'(e), \gamma'\gamma'_1(e)]$. Suppose the contrary that there is no such $k' \in \mathbb{G}_{[x, \gamma'(e)]}$. Then, by Tits’ independence property of $\mathbb{G}$ (see [1]), there is no $k' \in \mathbb{G}_{[x, \gamma'(e)]}$ that sends the segment $[\gamma'(e), \gamma'\gamma'_1(e)]$ into the segment $[\gamma'(e), k\gamma\gamma_1(e)]$. So the segment $[\gamma'(e), \gamma'\gamma'_1(e)]$ is not contained in the $\mathbb{G}_{\gamma'(e)}$-orbit of the segment $[\gamma'(e), k\gamma\gamma_1(e)]$. Apply now $(\gamma')^{-1}$. The latter assumption becomes: the segment $[e, \gamma'_1(e)]$ is not contained in the $\mathbb{G}_e$-orbit of the segment $[e, (\gamma')^{-1}k\gamma\gamma_1(e)]$. But $(\gamma')^{-1}k\gamma \in \mathbb{G}_e < K$ and recall $(\gamma_1, \gamma'_1)$ is a $K$–primitive pair. We obtained thus a contradiction and the conclusion follows.

If we just consider $K\gamma_2 K = K\gamma'_2 K$, one proceeds in the same way as above by replacing the edge $e$ with the vertex $x$ where is needed.

The importance of $K$–primitive pairs is expressed by the next proposition where two $K$–double cosets of $G$ are compared. Its proof relies on the fact that every element in $A^+$ can be written as a product of elements of $A^+$ that are of translation length 2; the aim of describing the elements of $A^+$ being thus achieved.

Proposition 3.10 Let $F$ be primitive and let $\gamma \in A^+$. Then $\gamma = \gamma_1\gamma_2 \cdots \gamma_t$, where $\gamma_i \in A^+$ is such that $|\gamma_i| = 2$, for every $i \in \{1, \ldots, t\}$. Moreover, for any $\gamma_1, \ldots, \gamma_t, \gamma'_1, \ldots, \gamma'_t \in A^+$, with $|\gamma_1| = |\gamma'_1| = 2$ for every $i \in \{1, \ldots, t\}$, the following equivalence is true: $K\gamma_1\gamma_2 \cdots \gamma_t K = K\gamma'_1\gamma'_2 \cdots \gamma'_t K$ if and only if $(\gamma_1, \gamma'_1)$ are $K$–primitive for every $i \in \{1, \ldots, t-1\}$ and $K\gamma_t K = K\gamma'_t K$.

Proof Let us prove the first part of the proposition. Notice, if $|\gamma| = 2$ the conclusion follows. Assume $|\gamma| > 2$ and denote the vertices of the geodesic $[x, \gamma(x)] \subset \mathcal{T}_{x,e}$ by $x = x_0, x_1, x_2, \ldots, x_2$. Then by Remark 3.1 there exists $\gamma_1 \in A^+$ with $|\gamma_1| = 2$ such that $\gamma_1(e) = [x_2, x_3]$. Notice $\gamma_1^{-1}\gamma$ is still an element of $A^+$, with $|\gamma_1^{-1}\gamma| = |\gamma| - 2$. Indeed, this is because the orientation of the edge $\gamma_1^{-1}\gamma(e)$ induced from the orientation of $e$ that points towards the boundary of the half-tree $\mathcal{T}_{x,e}$ is preserved. Apply now the above procedure to $\gamma_1^{-1}\gamma$. There exists thus $\gamma_2 \in A^+$ with $|\gamma_2| = 2$ and such that $\gamma_2^{-1}\gamma_1^{-1}\gamma$ is in $A^+$ with $|\gamma_2^{-1}\gamma_1^{-1}\gamma| = |\gamma| - 4$. By induction we obtain the elements $\gamma_1, \gamma_2, \ldots, \gamma_t$ with the property that $\gamma = \gamma_1\gamma_2 \cdots \gamma_t$. The conclusion follows.

Let us prove the second part of the proposition.

Assume $K\gamma_1\gamma_2 \cdots \gamma_t K = K\gamma'_1\gamma'_2 \cdots \gamma'_t K$. This means there exist $k, k' \in K$ such that $k\gamma_1\gamma_2 \cdots \gamma_t k' = \gamma'_1\gamma'_2 \cdots \gamma'_t$; this is equivalent to saying there is $k \in K$ such that the geodesic segment $[x, \gamma_1\gamma_2 \cdots \gamma_t(x)]$ is sent by $k$ into the geodesic segment $[x, \gamma'_1\gamma'_2 \cdots \gamma'_t(x)]$. From here we deduce that in fact $k\gamma_1(e) = \gamma'_1(e) \subset [x, \gamma'_1\gamma'_2 \cdots \gamma'_t(x)]$. 

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Therefore, \((y'_t)^{-1}k\gamma_1 =: k_1 \in \mathbb{G}_e < K\) and \(k_1\gamma_2 \cdots \gamma_t k' = y'_2 \cdots y'_t\). We apply again the above procedure and by induction, \((\gamma_t, y'_t)\) are \(K\)-primitive for every \(i \in \{1, \ldots, t-1\}\) and \(K\gamma_i = K\gamma'_i K\). The implication follows.

Assume now that \((\gamma_i, y'_i)\) are \(K\)-primitive for every \(i \in \{1, \ldots, t-1\}\) and \(K\gamma_i = K\gamma'_i K\). We need to find \(k \in K\) such that \(k[\{x, \gamma_1\gamma_2 \cdots \gamma_t(x)\}] = [x, y'_1 y'_2 \cdots y'_t(x)]\). We prove this by induction on \(i \in \{1, \ldots, t\}\) by applying Lemma 3.9. Notice, for \(i = 1\) we just use \((\gamma_1, y'_1)\) is a \(K\)-primitive pair. Suppose the induction step is true for \(i\) and we want to prove it for \(i + 1\). This means there is \(k_i \in K\) that sends the geodesic segment \([x, \gamma_1\gamma_2 \cdots \gamma_t(x)]\) into the geodesic segment \([x, y'_1y'_2 \cdots y'_t(x)]\) and we want to find \(k_{i+1} \in K\) that sends the geodesic segment \([x, \gamma_1\gamma_2 \cdots \gamma_t\gamma_{i+1}(e)]\) into the geodesic segment \([x, y'_1y'_2 \cdots y'_i\gamma_{i+1}(e)]\). This follows by applying Lemma 3.9, even for the last induction step when \(i + 1 = t\).

The following lemma counts the \(K\)-double cosets of \(G\). Together with Proposition 3.10 the goal of this subsection is achieved.

**Lemma 3.11** Let \(F\) be primitive. Using Convention 1.7, for every \(r \geq 1\) the total number of orbits of the group \(G_e\) acting on \(V_{x, 2r}\) is exactly \(k^{2r - 1}\).

For any two distinct \(G_e\)-orbits \([x_1]\), \([x_2]\) of the \(2r\)-level \(V_{x, 2r}\), there exist hyperbolic elements \(\gamma_1, \gamma_2 \in A^+\) such that \(\gamma_i(x) = x_i\), for \(i \in \{1, 2\}\). In addition, for any such choice of \(\gamma_1, \gamma_2\) having the above properties, \(K\gamma_1 K \neq K\gamma_2 K\).

Therefore, for every \(r \geq 1\) there are exactly \(k^{2r - 1}\) disjoint \(K\)-double cosets \(K\gamma K\), where \(\gamma \in A^+\) and \(|\gamma| = 2r\).

**Proof** We prove the first assertion by induction on \(r\). For \(r = 1\), this is Remark 3.5.

Now suppose the induction hypothesis is true for \(r \geq 1\) and we want to prove it for \(r + 1\). Fix any vertex \(y\) of the \(2r\)-level \(V_{x, 2r}\) and denote by \([z, y]\) the corresponding last edge of the geodesic segment \([x, y]\). We claim there are exactly \(k^2\) orbits of \(G_{[z, y]}\) acting on \(V_{z, 3}\). Indeed, this follows from the fact that \((d - 1)^2 = (n_1 + \cdots + n_k)^2 = \sum_{i \neq j} n_i n_j\), where we have \(k^2\) terms as the pairs \((i, j)\) and \((j, i)\) give disjoint orbits of the group \(G_{[z, y]}\) acting on \(V_{z, 3}\). Applying then Tits’ independence property of \(G\) for the subgroup \(G_{[z, y]}\), we obtain there are \(k^2\) orbits of \(G_{[x, y]}\) acting on \(V_{z, 3}\). From the above claim the induction step follows easily.

The first part of the second assertion of the lemma is Remark 3.1. Notice for a given \(G_e\)-orbit \([x_1]\), the hyperbolic element \(\gamma_1\) does not depend, up to \(G_e\)-conjugation, on the chosen representative \(x_1\).

It remains to prove for any two such hyperbolic elements \(\gamma_1, \gamma_2\), \(K\gamma_1 K \neq K\gamma_2 K\). Indeed, consider two disjoint \(G_e\)-orbits \([x_1]\), \([x_2]\) of the \(2r\)-level \(V_{x, 2r}\) and two hyperbolic elements \(\gamma_1, \gamma_2 \in G\) such that \(\gamma_i(x) = x_i\), for \(i \in \{1, 2\}\). By contraposition, suppose \((K\gamma_1 K) \cap (K\gamma_2 K) \neq \emptyset\). Then \(\gamma_i = k_1 \gamma_2 k_2\), for some \(k_1, k_2 \in K\). As \(\gamma_i(x) = x_i\) and \(x_1, x_2 \in V_{x, 2r}\), \(x_1 = k_1(x_2)\) and \(k_1 \in \mathbb{G}_e\). This is a contradiction with \([x_1] \neq [x_2]\) as \(G_e\)-orbits.

### 3.2 Convolution Products of \(K\)-Double Cosets

Recall, our main goal is to prove the Hecke algebra \(C_c(G, K)\) is infinitely generated when \(F\) is primitive but not 2-transitive. Therefore, we need to understand the elements \(f \in C_c(G, K)\) and convolution products of those. As the support of any function \(f \in C_c(G, K)\)
is compact and $K$–bi-invariant, using Lemma 3.2 the support of $f$ is covered with a finite number of compact-open subsets of the form $K\eta K$, with $\eta \in A^+$. Thus

$$f = \sum_{i \in I} a_i \mathbb{1}_{K\eta_i K},$$

(5)

where $I$ is a finite set, $a_i \in \mathbb{C}$ and $\eta_i \in A^+$ for every $i \in I$. This implies the set of all functions $\mathbb{1}_{K\gamma K}$, with $\gamma \in A^+$, forms an infinite (countable) base for the $\mathbb{C}$–vector space $C_c(\mathbb{G}, K)$ endowed with the addition of functions.

To study convolution products of elements in $C_c(\mathbb{G}, K)$ it is enough to compute convolution products of the form $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K} \ast \cdots \ast \mathbb{1}_{K\gamma_n K}$, where $\gamma_i \in A^+$ for every $i \in \{1, \cdots, n\}$. Firstly, given two elements $\gamma_1, \gamma_2 \in A^+$ not necessarily of the same translation length, we want to evaluate the $K$–double cosets appearing in the decomposition (5) of the convolution product $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$.

Recall, by definition

$$\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}(x) := \int_G \mathbb{1}_{K\gamma_1 K}(gx) \mathbb{1}_{K\gamma_2 K}(g^{-1}) d\mu(g).$$

For $x$ to be in the support of $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$, we need $xg \in K\gamma_1 K$ and $g^{-1} \in K\gamma_2 K$. This gives $x \in K\gamma_1 K\gamma_2 K$, implying the support of the function $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$ is contained in $K\gamma_1 K\gamma_2 K$. It is easy to check the support of $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$ equals $K\gamma_1 K\gamma_2 K$.

**Definition 3.12** Let $X$ be a compact $K$–bi-invariant subset of $\mathbb{G}$. Notice, $X$ is covered with a finite number of (open) $K$–double cosets $K\eta K$, where $\eta \in A^+$. We say $K\eta K$ is a **maximal $K$–double coset** of $X$, where $\eta \in A^+$, if the translation length $|\eta|$ is maximal among all $K$–double cosets that appear in the above decomposition of $X$.

Let $\gamma_1, \gamma_2 \in A^+$. Let $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K} = \sum_{i \in I} a_i \mathbb{1}_{K\eta_i K}$ as in equality (5). We say $a_i$ is a **maximal coefficient** of $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$ if the corresponding $K$–double coset $K\eta_i K$ of the support of $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K}$ is maximal.

**Proposition 3.13** Let $F$ be primitive and let $\gamma_1, \gamma_2 \in A^+$. By Proposition 3.10, let $\gamma_1 = \gamma_1' \gamma_1'' \cdots \gamma_1'^{t_1}$ and $\gamma_2 = \gamma_2' \gamma_2'' \cdots \gamma_2'^{t_2}$, where $\gamma_1', \gamma_1'' \in A^+$ are such that $|\gamma_1'| = |\gamma_2'| = 2$, for every $i \in \{1, \cdots, t_1\}$ and every $j \in \{1, \cdots, t_2\}$.

Then the maximal $K$–double cosets that appear in $K\gamma_1 K\gamma_2 K$ are of the form

$$K\gamma_1' \gamma_2' \cdots \gamma_1'^{t_1} \gamma_1'' \gamma_2'' \cdots \gamma_2'^{t_2} K,$$

where $\gamma$ can be any element of $A^+ \cap K\gamma_1' K$. In particular, we obtain exactly $k$ such maximal $K$–double cosets.

**Proof** Let $(\xi_i, \xi_i^{+})$ be the translation axis of $\gamma_i$, for $i \in \{1, 2\}$. As $K\gamma_1 K\gamma_2 K$ is evidently a maximal $K$–double coset of $K\gamma_1 K\gamma_2 K$, it remains to find all other $K$–double cosets $K\eta K$ such that $\eta \in A^+$ and $|\eta| = |\gamma_1| + |\gamma_2|$.

To compute the maximal $K$–double cosets that appear in the decomposition of $K\gamma_1 K\gamma_2 K$, it is enough to study elements of the form $\gamma_k K\gamma_2 K$ for $k \in K$ with the property that $k[x, \xi_2] \cap [\xi_1, \xi_1^{+}] \subseteq [x, \xi_1^{+}]$. Indeed, if $k[x, \xi_2] \cap [\xi_1, \xi_1^{+}]$ was not just equal to $x$ and not a subset of $[x, \xi_1^{+}]$, one would see $K\gamma_1 k\gamma_2 K$ is not a maximal $K$–double coset of $K\gamma_1 K\gamma_2 K$.

Consider first the case when $k \in \mathbb{G}_e$. For such $k$ we claim $K\gamma_1 k\gamma_2 K = K\gamma_1 K\gamma_2 K$. Indeed, we apply Lemma 3.9, as we need to find $k_1 \in \mathbb{G}_e$ such that $k_1 \gamma_1 k\gamma_2(x) = \gamma_1 \gamma_2(x)$.
The claim follows. In addition, notice, such $k_1$ fixes point-wise the geodesic segment $[x, \gamma_1(e)]$.

Consider now the case when $k \in K$ is such that $k(e) \notin (\xi_{1-}, \xi_{1+})$. Notice, for every edge in the star of $x$, which is not on the bi-infinite geodesic $(\xi_{1-}, \xi_{1+})$, there exists some $k \in K$ sending $e$ to it. For such $k \in K$ we need to decompose the element $\gamma_1 k \gamma_2$, which is in $A^+$, using elements $\gamma \in A^+$ with $|\gamma| = 2$. By the hypothesis on $k$, $|\gamma_1 k \gamma_2| = |\gamma_1| + |\gamma_2| = t_1 + t_2$. There exist thus $\gamma', \gamma'' \in A^+$ such that $\gamma_1 k \gamma_2 = \gamma' \gamma''$, with $|\gamma'| = |\gamma_1|$ and $|\gamma''| = |\gamma_2|$. Because of the choice of $k$, $\gamma_1(x) = \gamma'(x)$; thus $(\gamma')^{-1} \gamma_1 \in K$ and $\gamma'(e)$ belongs to the star of $\gamma_1(x)$, being different from $\gamma_1(e)$ and the edge belonging to the geodesic segment $[x, \gamma_1(x)]$. Therefore, the first $t_1 - 1$ terms of the decomposition of $\gamma_1 k \gamma_2$ are $\gamma_1' \gamma_1'' \cdots \gamma_{t_1-1}$ and the $t_1$–term appearing in the decomposition of $\gamma_1 k \gamma_2$ is an element $\gamma \in A^+ \cap K \gamma'_1 K$. Notice, this $t_1$–term depends strictly on the element $k$ and up to $K$–left-right multiplication, every element of $A^+ \cap K \gamma'_1 K$ appears as a $t_1$–term of $\gamma_1 k \gamma_2$. From the equality $\gamma_1 k \gamma_2 = \gamma' \gamma''$ and the above properties $K \gamma_2 K = K \gamma'' K$. Following Lemma 3.9, we conclude there exists $k' \in K$ such that $k' \gamma' \gamma''(x) = \gamma_1 k \gamma_2(x)$.

Combining the two cases studied above and using Proposition 3.10, $K \gamma_1 k \gamma_2 K = K \gamma_1' \gamma_2' \cdots \gamma_{t_1-1}' \gamma_1'' \gamma_2'' \cdots \gamma_{t_2}'' K$, where $\gamma \in A^+ \cap K \gamma'_1 K$. Moreover, up to $K$–primitivity, there are $k$ such $\gamma$ elements in $A^+ \cap K \gamma'_1 K$.

The next proposition computes the maximal coefficients appearing in $\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K}$.

**Proposition 3.14** Let $F$ be primitive and let $\gamma_1, \gamma_2 \in A^+$. By Proposition 3.10, let $\gamma_1 = \gamma_1' \gamma_1'' \cdots \gamma_{t_1}'$ and $\gamma_2 = \gamma_2' \gamma_2'' \cdots \gamma_{t_2}'$, where $\gamma_i', \gamma_i'' \in A^+$ are such that $|\gamma_i'| = |\gamma_i''| = 2$, for every $i \in \{1, \ldots, t_1\}$ and every $j \in \{1, \ldots, t_2\}$.

Then every maximal coefficient appearing in $\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K}$ equals $\mu(K)$, where $\mu$ is the left Haar measure on $G$. In particular, by normalizing $\mu(K) = 1$, all maximal coefficients equal one.

**Proof** As in equality (5), $\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K} = \sum_{i \in I} a_i \mathbb{1}_{K \eta_i K}$, where $I$ is a finite set, $a_i \in \mathbb{C}$ and $\eta_i \in A^+$ for every $i \in I$. Let $K \eta_i K$ be a maximal $K$–double coset and let $h \in K \eta_i K$.

By Proposition 3.13

$$\eta_i = \gamma_1' \gamma_2' \cdots \gamma_{t_1-1}' \gamma \gamma_1'' \gamma_2'' \cdots \gamma_{t_2}'' = \gamma_1(\gamma_1')^{-1} \gamma \gamma_2$$

where $\gamma$ is an element of $A^+ \cap K \gamma'_1 K$. In particular, using Proposition 3.10, we can suppose, without loss of generality, $\gamma(x) = \gamma_1'(x)$.

We want to compute $\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K}(h)$ which equals $a_i$. As all functions involved are $K$–bi-invariant, we can suppose $h = \eta_i$. It remains to evaluate

$$\int_G \mathbb{1}_{K \gamma_1 K}(\eta_i g) \mathbb{1}_{K \gamma_2 K}(g^{-1}) d\mu(g).$$

This reduces to find all $g \in K \gamma_2^{-1} K$ such that $\eta_i g \in K \gamma_1 K$; this is equivalent to evaluate the intersection $\eta_i^{-1} K \gamma_1 K \cap K \gamma_2^{-1} K$. We would have

$$\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K}(\eta_i) = a_i = \mu(\eta_i^{-1} K \gamma_1 K \cap K \gamma_2^{-1} K) = \mu(\gamma_2^{-1}(\gamma_1(\gamma_1')^{-1}\gamma)^{-1} K \gamma_1 \gamma_2^{-1} K) = \mu(\gamma^{-1}(\gamma_1' \gamma_{t_1-1}' \cdots)^{-1} K \gamma_1 \gamma_2 \gamma_2^{-1} K).$$
Notice the following. For every $g \in \gamma_2 K \gamma_2^{-1} K$, we have
\[
\text{dist}_T (g(x), \gamma_2(x)) = \text{dist}_T (\gamma_2 k_1 \gamma_2^{-1} k_2(x), \gamma_2(x)) = |\gamma_2|.
\]
For $g \in \gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1} K \gamma_1 K$ we have
\[
\text{dist}_T (g(x), \gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1}(x)) = \text{dist}_T (k_3 \gamma_1(x), x) = |\gamma_1|.
\]
As $\gamma_2(x) \in T_{x,e}$ and $\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1}(x) \in T \setminus T_{x,e}$, and because
\[
|\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1}| = |\gamma_1|^{-1}
\]
we obtain
\[
\overline{B}(\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1}(x), |\gamma_1|) \cap \overline{B}(\gamma_2(x), |\gamma_2|) = \{x\}
\]
where $\overline{B}(y, r) \subset T$ denotes the closed ball centered at the vertex $y$ and of radius $r$. This implies for every $g \in \gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1} K \gamma_1 K \cap \gamma_2 K \gamma_2^{-1} K$ we necessarily have $g(x) = x$; therefore, $\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1} K \gamma_1 K \cap \gamma_2 K \gamma_2^{-1} K \subset K$. In fact, we claim
\[
\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1} K \gamma_1 K \cap \gamma_2 K \gamma_2^{-1} K = K.
\]
Indeed, it is immediate $K \subset \gamma_2 K \gamma_2^{-1} K$. It remains to show $K \subset \gamma^{-1}(\gamma_1' \gamma_1' - 1) K \gamma_1 K$. As we have supposed $\gamma(x) = \gamma_1'(x)$, $\gamma^{-1}(\gamma_1' - 1) \subset K$, so $\gamma^{-1}(\gamma_1' - 1) \gamma_1 K = K$. This proves the claim and
\[
a_i = \mu(\gamma^{-1}(\gamma_1' \gamma_2' \cdots \gamma_{i-1}')^{-1} K \gamma_1 K \cap \gamma_2 K \gamma_2^{-1} K) = \mu(K).
\]

4 The Hecke Algebra is Infinitely Generated

The goal of this section is to prove the Hecke algebra $C_c(G, K)$ endowed with the convolution product is infinitely generated.

By contraposition suppose $C_c(G, K)$ would be a finitely generated algebra with respect to the convolution product. Let $S = \{f_1, \ldots, f_m\}$ be a finite set of generators for $C_c(G, K)$. Then every $f \in C_c(G, K)$ would be written as a finite linear combination of $\{f_1, \ldots, f_m\}$ and finite convolution products of those. By Eq. 5 we decompose every $f_i \in S$ as $f_i = \sum_{j=1}^{n_i} a_{ij} \mathbb{1}_{K \gamma_j K}$, where $\gamma_j \in A^+$ and $a_{ij} \in \mathbb{C}$. We notice the finite set $S' := \{\mathbb{1}_{K \gamma_j K} \}_{\gamma_j \in C_c(G, K)}$ also generates $C_c(G, K)$. In addition, by letting $N := \max_{\gamma_j} (|\gamma_j|) < \infty$, the finite set $S'' := \{\mathbb{1}_{K \gamma K} | \gamma \in A^+ \text{ with } |\gamma| \leq N\}$ generates $C_c(G, K)$ too. Therefore, if $C_c(G, K)$ was finitely generated, we could assume, without loss of generality, there exists a finite set $S'' = \{\mathbb{1}_{K \gamma K} | \gamma \in A^+ \text{ with } |\gamma| \leq N\}$ that generates $C_c(G, K)$.

It remains to investigate the subsets $\{\gamma_1, \ldots, \gamma_n\} \subset A^+$ such that the finite set $\{\mathbb{1}_{K \gamma_1 K}, \ldots, \mathbb{1}_{K \gamma_n K}\}$ would generate $C_c(G, K)$ with respect to the convolution product. For this we need to solve in $C_c(G, K)$ systems of equations of convolution products and to introduce a notion of linear independence for such systems of equations.

4.1 Sub-Bases and Weakly Linearly Independent Equations of Degree $n$

In Section 3.2 we have computed the convolution product of two functions $\mathbb{1}_{K \gamma_1 K}, \mathbb{1}_{K \gamma_2 K} \in C_c(G, K)$, where $\gamma_1, \gamma_2 \in A^+$. We have obtained the equation
\[
\mathbb{1}_{K \gamma_1 K} \ast \mathbb{1}_{K \gamma_2 K} = \sum_{|\eta| = |\gamma_1| + |\gamma_2|} a_i \mathbb{1}_{K \eta_i K} + \sum_{|\eta| = |\gamma_1| + |\gamma_2|} \mu(K) \mathbb{1}_{K \eta_j K},
\]
where $I$ is a finite set, $a_i \in \mathbb{C}$, $|J| = k$, $\eta_i, \eta_j \in A^+$ and $\eta_j$, with $j \in J$, are as in Proposition 3.13.
From now on we assume the left Haar measure $\mu$ of $\mathbb{G}$ is normalized with $\mu(K) = 1$.

By considering $\gamma_1, \gamma_2, \ldots, \gamma_m \in A^+$ we obtain a more general equation

$$\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K} \ast \cdots \ast \mathbb{1}_{K\gamma_m K} = \sum_{|\eta_1| < |\gamma_1| + \cdots + |\gamma_m|, \ i \in I} a_i \mathbb{1}_{K\eta_i K} + \sum_{|\eta_j| = |\gamma_1| + \cdots + |\gamma_m|, \ j \in J} \mathbb{1}_{K\eta_j K},$$

where $I$ is a finite set, $a_i \in \mathbb{C}$, $|J| = k^{m-1}$, $\eta_i, \eta_j \in A^+$ and $\eta_j$ are as in Proposition 3.13.

**Definition 4.1** Let $f \in C_c(\mathbb{G}, K)$. Recall, by equality (5)

$$f = \sum_{i \in I} a_i \mathbb{1}_{K\eta_i K},$$

where $I$ is a finite set, $a_i \in \mathbb{C}$ and $\eta_i \in A^+$ for every $i \in I$. We say (the equation) $f$ is of degree $n$ if for every $i \in I$, $|\eta_i| \leq n$ and there exists $i \in I$ such that $|\eta_i| = n$. In particular, Eq. 7 is of degree $n$ if $|\gamma_1| + \cdots + |\gamma_m| = n$.

**Definition 4.2** Let $I$ be finite. We say $\{E_i\}_{i \in I}$ is a system of equations of degree $n$ if:

1. for every $i \in I$, $E_i = \mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K} \ast \cdots \ast \mathbb{1}_{K\gamma_m K}$ for some $\gamma_i \in A^+$, where $m_i$ is finite and $j \in \{1, \ldots, m_i\}$
2. $E_i$ is of degree $n$ for every $i \in I$.

**Notation 4.3** To simplify the notation, from now on we reserve the letter $E$ to represent a convolution product of the form $\mathbb{1}_{K\gamma_1 K} \ast \mathbb{1}_{K\gamma_2 K} \ast \cdots \ast \mathbb{1}_{K\gamma_m K}$, where $\gamma_i \in A^+$ for every $i \in \{1, \ldots, m\}$.

**Definition 4.4** A system of equations $\{E_1, \ldots, E_r\}$ of degree $n$ is weakly linearly dependent if there exist $b_1, \ldots, b_r \in \mathbb{C}$, not all zero, such that $b_1 E_1 + \cdots + b_r E_r$ is of degree strictly less than $n$. We say a system of equations $\{E_1, \ldots, E_r\}$ of degree $n$ is weakly linearly independent if it is not weakly linearly dependent.

**Definition 4.5** We say the set $\{\mathbb{1}_{K\gamma_i K}\}_{i \in I}$, where $I$ is finite and $\{\gamma_i\}_{i \in I} \subset A^+$, forms a sub-base of $C_c(\mathbb{G}, K)$ of degree $\leq n$ if the following conditions are satisfied:

1. $|\gamma_i| \leq n$, for every $i \in I$
2. every function $\mathbb{1}_{K\eta K}$, with $\eta \in A^+$ and $0 < |\eta| \leq n$, can be written as a sum of a finite linear combination of $\{\mathbb{1}_{K\gamma_i K}\}_{i \in I}$ and a finite linear combination of convolution products of $\{\mathbb{1}_{K\gamma_i K}\}_{i \in I}$ having degree $\leq n$
3. $|I|$ is minimal with the above properties.

**Remark 4.6** Let $F$ be primitive. It is immediate that a sub-base of $C_c(\mathbb{G}, K)$ of degree 2 exists and it is unique. Its cardinality is $k$.

Before considering the general case, let us warm up proving the following lemma.

**Lemma 4.7** Let $F$ be primitive. Then $C_c(\mathbb{G}, K)$ admits a sub-base of degree $\leq 4$. Its cardinality is $k + k^2(k - 1)$. A sub-base of $C_c(\mathbb{G}, K)$ of degree $\leq 4$ is not unique.

**Proof** For the proof recall Definition 2.1 and Notation 3.8. Let also $\{\mathbb{1}_{K\gamma_j K}\}_{j \in \{1, \ldots, k\}}$ be the sub-base of $C_c(\mathbb{G}, K)$ degree 2.
Let \( \eta_1, \eta_2 \in \{ \gamma_1, \cdots, \gamma_k \} \). By Proposition 3.13, the only way to obtain functions of the from \( I_{K\eta_1}^{(i)} \cap I_{K\eta_2}^{(j)} \), with \( i \in \{1, \cdots, k\} \), as maximal \( K \)-double coset using the sub-base of degree 2 is by convoluting \( I_{K\eta_1} \) and \( I_{K\eta_2} \):

\[
I_{K\eta_1} \ast I_{K\eta_2} = \sum_{|\eta_1|+|\eta_2|} a_i \sum_{i=1}^{k} I_{K\eta_1}^{(i)} \cap I_{K\eta_2}^{(j)}.
\]

Following Proposition 3.10, in Eq. 8 all functions \( I_{K\eta_1}^{(i)} \cap I_{K\eta_2}^{(j)} \), with \( i \in \{1, \cdots, k\} \), appear. As disjoint \( K \)-double cosets give rise to different \( K \)-primitive pairs, the above equation is the only one where those functions can appear.

In order to find a sub-base of \( C_c(\mathbb{G}, K) \) of degree \( \leq 4 \) we have to choose from Eq. 8 \( f'(2, k) = k - 1 \) different elements of \( A^+ \) of translation length 4. This number is independent of the choices made for the elements \( \{ \gamma_{ji}^{(i)} \}_{i=1}^{k} \). By doing this, one of the terms of degree 4 appearing in Eq. 8 will be written in terms of the chosen elements. Without loss of generality, we can suppose we have chosen \( \{ I_{K\eta_1}^{(i)} \cap I_{K\eta_2}^{(j)} \}_{i=1}^{2} \) to form a sub-base of degree \( \leq 4 \). Moreover, by the same reasoning and making a choice, we choose the following set \( \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i, l \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \cup \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \cup \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \cup \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \) as a sub-base of degree \( \leq 4 \). No fewer elements than above can be chosen to form such a sub-base. The cardinality of the chosen sub-base of degree \( \leq 4 \) is \( f(2, k) + f(1, k) = k^2 - k \). This number is independent of the choices made. Note, the sub-base \( \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i, l \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \cup \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \) is not unique.

**Proposition 4.8** Let \( F \) be primitive and let \( \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid j \in \{1, \cdots, k\} \} \) be the sub-base of \( C_c(\mathbb{G}, K) \) degree 2. For every \( r \geq 3 \) there exists a sub-base of \( C_c(\mathbb{G}, K) \) of degree \( \leq 2r \). Its cardinality is \( f(1, k) + f(2, k) + \cdots + f(r, k) \). In particular, if \( k = 1 \) then \( f(1, k) + f(2, k) + \cdots + f(r, k) = 1 \), for every \( r \geq 1 \).

**Proof** We prove the proposition by induction on \( r \). First recall Definition 2.1 and Notation 3.8. By Lemma 4.7 we take \( \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid i, l \in \{1, \cdots, k\} \text{ and } j \in \{2, \cdots, k\} \} \cup \{ I_{K\gamma_1}^{(i)} \cap I_{K\gamma_2}^{(j)} \mid j \in \{1, \cdots, k\} \} \) as a sub-base of \( C_c(\mathbb{G}, K) \) of degree \( \leq 4 \).

Fix a sequence \( \{ \eta_i \}_{i=1}^{\ell} \subset \{ \gamma_1^{(j_1)}, \cdots, \gamma_k^{(j_k)} \} \) for \( i_1, \cdots, i_k \in \{1, \cdots, k\} \).

Let \( r = 3 \). Using the chosen sub-base of \( C_c(\mathbb{G}, K) \) of degree \( \leq 4 \), we form all equations \( E \) of degree 2 or 6 where functions of the form \( I_{K\eta_1}^{(i)} \cap I_{K\eta_2}^{(j)} \), with \( i_1, i_2 \in \{1, \cdots, k\} \), can appear (the total number of these functions is \( k^2 \)). By a combinatorial argument, Lemmas 2.5, 4.7 and equality (7) the total number of these equations is exactly

\[
\sum_{(k_1,k_2) \in \text{Sum}(3)} \left( \begin{array}{c} k_1 + k_2 \\ k_1, k_2 \end{array} \right) f'(1, k)^{k_1} f'(2, k)^{k_2} = k^2 - f'(3, k).
\]

The above combinatorial argument is the following. The number \( k_1 \) (resp., \( k_2 \)), which can be 1 or 3 (resp., 1 or 0), represents how many degree–1 (resp., degree–2) \( K \)-double cosets we use for the convolution to form all equations \( E \). The terms \( f'(1, k)^{k_1}, f'(2, k)^{k_2} \) represent the number of choices for those \( K \)-double cosets including their multiplicities \( k_1, k_2 \). And by its definition, the multinomial coefficient \( \left( \begin{array}{c} k_1 + k_2 \\ k_1, k_2 \end{array} \right) \) gives exactly what we want.

\( \square \)
We claim all the above \( k^2 - f'(3, k) \) equations form a system of weakly linearly independent equations of degree 6. Indeed, note all functions \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^j} \) with \( i_1, i_2 \in \{1, \ldots, k\} \), appear at least once in the above system of equations. In particular, these functions appear from the convolution product \( \mathbb{1}_{K\eta_1} \ast \mathbb{1}_{K\eta_2} \ast \mathbb{1}_{K\eta_3} \). Moreover, the function \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^j} \) appears only once, hence, only from the convolution product \( \mathbb{1}_{K\eta_1} \ast \mathbb{1}_{K\eta_2} \ast \mathbb{1}_{K\eta_3} \). This is because the sub-base of degree \( \leq 4 \) is chosen to be \( \{\mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^j} \mid i, l \in \{1, \ldots, k\} \text{ and } j \in \{2, \ldots, k\}\cup\{\mathbb{1}_{K\eta_j^2} \mid j \in \{1, \ldots, k\}\} \). Moreover, functions of the form \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^j} \ast \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^j} \), for every \( i \in \{2, \ldots, k\} \) appear twice: once from the convolution product \( \mathbb{1}_{K\eta_1} \ast \mathbb{1}_{K\eta_2} \ast \mathbb{1}_{K\eta_3} \) and once, respectively, from the convolution product \( \mathbb{1}_{K\eta_1} \ast \mathbb{1}_{K\eta_2} \ast \mathbb{1}_{K\eta_3} \), where \( j \in \{2, \ldots, k\} \). These remarks imply the system of equations we are interested in are indeed weakly linearly independent. This proves our claim.

By the theory of linear algebra, in order to form a sub-base of degree \( \leq 6 \), given the sub-base of degree \( \leq 4 \), we have to choose (from the above system of equations of degree 6) \( f'(3, k) \) characteristic functions corresponding to \( K \)-double cosets of degree 6. This argument is independent and valid for every of the \( k^3 \) systems of equations.

By the proof of the above claim and above facts, to obtain a sub-base of degree \( \leq 6 \) we make a choice and select the set \( \{\mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^{i_3}} \mid i_1, i_2 \in \{2, \ldots, k\} \text{ and } j_1, j_2, j_3 \in \{1, \ldots, k\}\} \). This set together with the sub-base of \( C_r(G, K) \) of degree \( \leq 4 \) is minimal and its cardinality is indeed \( f(1, k) + f(2, k) + f(3, k) \). In particular, all the functions of the form \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^{i_3}} \ast \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^{i_3}} \ast \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\eta_3^{i_3}} \), with \( i_1, i_2, j_1, j_2, j_3 \in \{1, \ldots, k\}\), are written using the chosen sub-base of degree \( \leq 6 \); they do not appear as elements of that sub-base.

Let us now suppose the conclusion of the proposition is true for all \( r \leq r \) and the sub-base of \( C_r(G, K) \) of degree \( \leq 2r \) does not contain any function of the form

\[
\mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\cdots \eta_{l-1}^{(i_{l-1})}} K
\]

where \( l \leq r \) and at least one of \( \{i_1, \ldots, i_{l-1}\} \) is 1. We have to prove this is also true for \( r + 1 \). Indeed, using only the sub-base of degree \( \leq 2r \), constructed in the previous induction steps, we form all equations \( E \) of degree 2\( (r + 1) \) where functions of the form \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\cdots \eta_r^{(i_r)} \eta_{r+1}^2 K}, \) with \( i_1, i_2, \ldots, i_r \in \{1, \ldots, k\}, \) can appear (the total number of these functions is \( k^r \)). By a combinatorial argument, Lemmas 2.5, 4.7 and equality (7) the total number of these equations is exactly

\[
\sum_{(k_1, \ldots, k_r) \in \text{Sum}(r+1)} \binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r} f'(1, k)^{k_1} f'(2, k)^{k_2} \cdots f'(r, k)^{k_r} = k^r - f'(r + 1, k).
\]

We claim all the above \( k^r - f'(r + 1, k) \) equations form a system of weakly linearly independent equations of degree 2\( (r + 1) \). Indeed, every function \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\cdots \eta_r^{(i_r)} \eta_{r+1}^2 K} \) with \( i_1, \ldots, i_r \in \{1, \ldots, k\} \) appears at least once in the above system of equations, specifically, from the convolution product \( \mathbb{1}_{K\eta_1} \ast \mathbb{1}_{K\eta_2} \ast \cdots \ast \mathbb{1}_{K\eta_{r+1}} \). In addition, the function \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\cdots \eta_r^{(i_r)} \eta_{r+1}^2 K} \) appears exactly once in that system of equations. Moreover, every equation of the above system determines in a unique way a function \( \mathbb{1}_{K\eta_1^{(i_1)}\eta_2^{(i_2)}\cdots \eta_r^{(i_r)} \eta_{r+1}^2 K} \) where the number of appearances of 1 among the coefficients \( i_1, \ldots, i_r \) is maximal. This function uniquely depends on the convolution product involved in that equation. In addition, there exists at least one different equation of that system of equations such that the latter mention
function appears, but it is not anymore ‘maximal’. The only exception is the function
\[ 1_{K_{\eta_1}^{(1)} \eta_2^{(2)} \cdots \eta_r^{(r)} K}. \]

If the above system of equations \((E_1, \ldots, E_n)\), where \(n = k' - f'(r + 1, k)\), was not weakly linearly independent there would exist coefficients \(c_1, \ldots, c_n \in \mathbb{C}\), not all zero, such that \(c_1 E_1 + \cdots + c_n E_n\) is of degree strictly less than \(2(r+1)\). By the remarks made above this would imply that we have reduced all functions
\[ 1_{K_{\eta_1}^{(1)} \eta_2^{(2)} \cdots \eta_r^{(r)} K_{\eta_{r+1}} K} \]
where at least one of \(\{i_1, \ldots, i_r\}\) is 1. This cannot be possible as every equation determines in a unique way a ‘maximal’ such function. Therefore, the weakly linearly independence follows. This argument is independent and valid for every of the other \(k^{r+1}\) systems of equations.

To complete the sub-base of degree \(\leq 2r\) to a sub-base of degree \(\leq 2(r + 1)\) we choose the functions
\[ 1_{K_{\gamma_1}^{(1)} \gamma_2^{(2)} \cdots \gamma_r^{(r)} \gamma_{r+1} K}, \]
where \(i_1, \ldots, i_r \in \{2, \ldots, k\}\) and \(j_1, \ldots, j_{r+1} \in \{1, \ldots, k\}\). This concludes the induction step.

\[ \square \]

4.2 The Proof of the Main Theorem

\textbf{Theorem 4.9} Let \(F\) be primitive. If \(k = 1\) then the Hecke algebra \(C_c(G, K)\) is finitely generated admitting only one generator. If \(k > 1\) then the Hecke algebra \(C_c(G, K)\) is infinitely generated with an infinite presentation.

\textbf{Proof} Let \(k = 1\). Then \(F\) is 2–transitive. By Remark 3.3, \(A^+\) equals \(< a >^+\) where \(a\) is a hyperbolic element of \(G\), with \(|a| = 2\) and \(x \in \text{Min}(a)\). The \(K\)–double cosets of \(G\) are just \((Ka^n K)_{n \geq 0}\). It is then easy to see the function \(1_{K_a K}\) alone generates the Hecke algebra \(C_c(G, K)\) (one can also apply the general result of Proposition 4.8).

Consider now the case \(k > 1\) and suppose \(C_c(G, K)\) is a finitely generated algebra with respect to the convolution product. Arguing as in the introduction to Section 4 we can assume, without loss of generality, \(C_c(G, K)\) is generated by the finite set \(S'' = \{1_{K_{\gamma} K} | \gamma \in A^+ \text{ with } |\gamma| \leq N\}\), for some \(N \in \mathbb{N}^*\). By Proposition 4.8, there exists a sub-base of \(C_c(G, K)\) of degree \(\leq N\) and by our assumption this finite sub-base of \(C_c(G, K)\) of degree \(\leq N\) generates \(C_c(G, K)\). This is in contradiction with Proposition 4.8 as the sub-base of \(C_c(G, K)\) of degree \(\leq N + 1\) strictly contains the one of degree \(\leq N\). Therefore, \(C_c(G, K)\) is infinitely generated. Its infinitely presentation is the one coming from the convolution products of the generators.

\[ \square \]

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