OPTIMAL MAXIMALLY DECIMATED $M$-CHANNEL MIRRORED PARAUNITARY LINEAR PHASE FIR FILTER BANK DESIGN VIA NORM RELAXED SEQUENTIAL QUADRATIC PROGRAMMING

QING LIU, BINGO WING-KUEN LING*, QINGYUN DAI, QING MIAO AND CAIXIA LIU

Faculty of Information Engineering, Guangdong University of Technology
Guangzhou, 510006, China

(Communicated by Bin Li)

ABSTRACT. It is worth noting that the conventional maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response condition is defined in the frequency domain. As the frequency domain is a continuous set, it is expressed as a matrix functional (a continuous function of the frequency) equation. On the other hand, this paper expresses the condition as a finite number of discrete (a set of functions of the sampled frequencies) equations. Besides, this paper proposes to sample the magnitude responses of the filters with the total number of the sampled frequencies being more than the filter lengths. Hence, the frequency selectivities of the filters can be controlled more effectively. This filter bank design problem is formulated as an optimization problem in such a way that the total mirrored paraunitary linear phase error is minimized subject to the specifications on the magnitude responses of the filters at these sampling frequencies. However, this optimization problem is highly nonconvex. To address this difficulty, a norm relaxed sequential quadratic programming approach is applied for finding its local optimal solution. By iterating the above procedures using different initial conditions, a near global optimal solution is obtained. Computer numerical simulation results show that our proposed design outperforms the existing designs.

1. Introduction. There are many different types of filter banks such as the modulated filter bank and the paraunitary filter bank. Among them, the paraunitary filter bank is the most common type of filter banks used in the practical applications. This is because the paraunitary filter bank is the most common realization of the discrete time wavelet transform which finds many applications in the various engineering and science fields. Although there are many different types of the paraunitary filter banks such as the modulated paraunitary filter bank and the maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response filter bank [1, 2, 3, 4], the modulated paraunitary filter bank is less common. This is because the constraint imposed by the modulated structure of the modulated filter bank is too tight that the frequency selectivities of the filters are

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. M-channel mirrored paraunitary linear phase filter bank, optimal design, finite constraints, discrete equality constraints, frequency domain condition, norm relaxed sequential quadratic programming.

* Corresponding author: Bingo Wing-Kuen Ling.
poor. On the other hand, a maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response filter bank [1, 2, 3, 4], decomposes a signal into various components uniformly in the frequency domain. As these components are orthogonal to each others, the linear independency among the components does not exist. Also, as the filters are linear phase, the components do not suffer from the phase distortion. Moreover, since the filters are with the finite impulse responses, the stability of the filter bank is guaranteed. Hence, the filter bank can achieve a quite good performance [5, 6]. As a result, it is found in many applications such as in the image and video signal processing application [7, 8, 9, 10].

Although the design of the maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response filter bank has been studied for several decades, there are some unaddressed fundamental issues. In particular, an effective and an efficient design of this type of filter banks is challenging. To address this issue, the factorization approach [7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] is proposed for designing this type of filter banks. However, as the filter bank is realized as the cascade of a set of lattices and these lattices are characterized by their rotational angles, the frequency responses of the filters are expressed as a set of very high order polynomials of the trigonometric functions of the filter coefficients. Therefore, it is very difficult to find the gradient of the objective function of this optimization problem. Hence, the optimal filter bank design problem is not traceable. In general, it cannot guarantee to obtain a local optimal solution even though a computer aided design tool is used for finding a solution of this optimization problem. In this case, finding a near global optimal solution of this optimization problem is nearly impossible.

To address this difficulty, the paraunitary condition is expressed in the time domain. In particular, the vectors of the filter coefficients form a sub-matrix. By dividing the sub-matrix into a finite number of blocks with each block being a square matrix, circularly shifting these blocks, putting these shifted blocks into the sub-matrices and putting all these sub-matrices into a large square matrix, the paraunitary condition becomes the multiplication of this large square matrix and its conjugate transpose [22, 23] being equal to the identity matrix. However, as the condition is defined in the frequency domain, the frequency selectivities of the filters are not guaranteed.

The novelty of this paper is as follows. It is shown that the above paraunitary condition can be represented as a set of equations relating to the sum of the products of the frequency responses of the analysis filters. As these equality constraints are the quadratic equations of the filter coefficients, the problem formulation [22, 23] is simpler compared to the existing problem formulations defined in the frequency domain [7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Besides, this paper proposes to employ the norm relaxed sequential quadratic programming approach to find the solution of the optimization problem. The outline of this paper is as follows. The necessary and sufficient condition for the filter bank satisfying the maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response condition is derived in Section 2. To obtain the analysis filters with good frequency selectivities, the magnitude responses of the analysis filters are sampled with the total numbers of sampling frequencies being more than the filter lengths. The details are presented in Section 3. In Section 4, a method based on the norm relaxed sequential quadratic programming approach is employed for finding a local optimal solution of the optimization problem. Finally, the above procedures are
repeated with different initial conditions and a near global optimal solution is found. The discussions on the selection of the parameters of the algorithm are presented in Section 5. Computer numerical simulation results are presented in Section 6. It is shown that our proposed design method outperforms the conventional approaches. Finally, a conclusion is drawn in Section 7.

2. Necessary and sufficient condition for the filter bank satisfying the maximally decimated $M$-channel mirrored paraunitary linear phase finite impulsive response condition.

2.1. Paraunitary condition. Consider a maximally decimated $M$-channel paraunitary finite impulse response filter bank. Here, $M$ is the total number of the analysis filters. Assume that all the analysis filters have the same lengths. Let $N$ be the lengths of these analysis filters. Denote $h_m(k)$ for $m = 0, 1, \ldots, M-1$ and for $k = 0, \ldots, N-1$ as the impulse responses of these analysis filters. By putting the vectors of the filter coefficients into a sub-matrix, the dimension of the sub-matrix is $N \times M$. We assume that $N$ is a positive integer multiple of $M$. Define

$$L \equiv \frac{N}{M}. \quad (1)$$

Obviously, $L$ is a positive integer. By dividing the sub-matrix into a finite number of blocks with the size of each block being $M \times M$, we have $L$ blocks and the $l$th block is

$$H_l \equiv \begin{bmatrix} h_0 (lM) & \cdots & h_{M-1} (lM) \\ \vdots & \ddots & \vdots \\ h_0 ((lM + M - 1) & \cdots & h_{M-1} ((lM + M - 1) \end{bmatrix} \text{ for } l = 0, 1, \ldots, L - 1. \quad (2)$$

By circularly shifting these blocks, putting these shifted blocks into the sub-matrices and putting all these sub-matrices into a matrix, the dimension of this matrix is $N \times N$. Denote this matrix as

$$H \equiv \begin{bmatrix} H_0 & H_{L-1} & \cdots & H_1 \\ \vdots & \ddots & \ddots & \vdots \\ H_{L-1} & \cdots & H_0 \\ \vdots & \ddots & \ddots & \vdots \\ \end{bmatrix}. \quad (3)$$

Let $U$ be the $N \times N$ discrete Fourier transform matrix. That is, for $p = 0, \ldots, N-1$ and for $q = 0, \ldots, N-1$, the element in the $(p + 1)^{th}$ row and the $(q + 1)^{th}$ column of $U$ is $\frac{1}{\sqrt{N}} e^{-j2\pi pq/N}$. Here, $j$ is the pure imaginary number. That is, $j \equiv \sqrt{-1}$. Also, define $I_{N \times N}$ as the $N \times N$ identity matrix and

$$Z \equiv UHU^+, \quad (4)$$

where the superscript “$+$” denotes the conjugate transposition operator. It is well known that $Z$ is a diagonal matrix if $M = 1$. That means, there exists a filter with its impulse response denoted as $\hat{h}(n)$ for $n = 0, \ldots, N-1$ and the discrete Fourier transform of this impulse response denoted as $\hat{H}(k)$ for $k = 0, \ldots, N-1$ such that the diagonal elements of $Z$ are $\hat{H}(0), \ldots, \hat{H}(N-1)$. Here, $\hat{H}(k)$ for $k = 0, \ldots, N-1$ are defined as:

$$\hat{H}(k) = \sum_{n=0}^{N-1} \hat{h}(n) e^{-j2\pi nk/N} \text{ for } k = 0, 1, \ldots, N-1. \quad (5)$$
Also, if
\[ HH^+ = I_{N \times N}, \]
then
\[ |\hat{H}(k)| = 1 \quad \text{for} \quad k = 0, 1, \ldots, N - 1, \]
where $|z|$ denotes the modulus of the complex number $z$. That means, $\hat{H}(k)$ has the allpass characteristic at the sampling frequencies $\omega_k = \frac{2\pi k}{N}$ for $k = 0, 1, \ldots, N - 1$. However, we have $M > 1$ for the $M$-channel filter bank. Therefore, it is interesting to study the corresponding characteristic for the $M$-channel filter bank. Define the discrete Fourier transform of $h_m(n)$ for $n = 0, 1, \ldots, N - 1$ as $H_m(k)$ for $k = 0, 1, \ldots, N - 1$ and for $m = 0, 1, \ldots, M - 1$. That is,
\[ H_m(k) = \sum_{n=0}^{N-1} h_m(n) e^{-j\frac{2\pi nk}{N}} \]
for $k = 0, 1, \ldots, N - 1$ and for $m = 0, 1, \ldots, M - 1$.

Let the element in the $(p+1)$th row and the $(q+1)$th column of $Z$ be $z_{p,q}$ for $p = 0, \ldots, N - 1$ and for $q = 0, \ldots, N - 1$. Denote $Z$ as the set of integers. Then, we have the following lemma:

**Lemma 2.1.** For $p = 0, \ldots, N - 1$ and $q = 0, \ldots, N - 1$, we have
\[ z_{p,q} = \begin{cases} \frac{1}{M} \sum_{m=0}^{M-1} H_m(p) e^{j\frac{2\pi q m}{N}} & \frac{q-L}{L} \in \mathbb{Z} \\ 0 & \frac{q-L}{L} \notin \mathbb{Z} \end{cases} \]
Define
\[ Z = \begin{bmatrix} Z_{0,0} & \cdots & Z_{0,M-1} \\ \vdots & \ddots & \vdots \\ Z_{M-1,0} & \cdots & Z_{M-1,M-1} \end{bmatrix}. \]
Lemma 2.1 implies that $Z_{s,t}$ for $s = 0, \ldots, M - 1$ and for $t = 0, \ldots, M - 1$ are $L \times L$ diagonal matrices. The diagonal elements are the average values of the modulated discrete Fourier transform coefficients of the filters. Here, the averaging operator is with respect to the filter index $m$ and the modulating frequencies depend on the filter index $m$ and the position index of the diagonal elements $q$. This is the generalization of the existing result when $M = 1$ and $L = N$. Also, $Z$ reduces to a diagonal matrix and the diagonal elements of $Z$ reduce to $\hat{H}(k)$ for $k = 0, 1, \ldots, N - 1$.

Denote the superscript $^\ast$ as the conjugate operator. Then, we have the following paraunitary condition:

**Theorem 2.2.** If
\[ HH^+ = I_{N \times N}, \]
then
\[ \frac{1}{M} \sum_{m=0}^{M-1} H_m(Ls + a) H_m^\ast(Lt + a) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases} \]
for $s = 0, \ldots, M - 1$, for $s = 0, \ldots, M - 1$, and for $a = 0, \ldots, L - 1$.

This theorem reveals that the paraunitary condition defined in the frequency domain can be characterized by a set of discrete equations instead of a set of functional equations. More precisely, all the sampling frequencies are grouped into $M$ blocks with each block containing $L$ sampling frequencies. Then, we pick up two
discrete Fourier transform coefficients sampled at the \((a + 1)\)th element of both the \((s + 1)\)th block and the \((t + 1)\)th block of the same filter with the filter index \(m\). Next, one discrete Fourier transform coefficient is multiplied by the conjugate of another discrete Fourier transform coefficient. After that, we evaluate the average values of these products over the filter index \(m\). This theorem reveals that these averaged values are equal to the delta function defined according to these two block indices.

2.2. Real valued mirrored linear phase condition. Now, consider the real valued maximally decimated \(M\)-channel mirrored paraunitary linear phase finite impulse response filter bank. Since the analysis filters are mirrored to each others, \(M\) is a positive integer multiple of 2. As \(N\) is a positive integer multiple of \(M\), \(N\) is also a positive integer multiple of 2.

Since the impulse responses of the analysis filters are real valued, we have

\[
H_m \left( \frac{N}{2} + p \right) = H_m^* \left( \frac{N}{2} - p \right) \quad \text{for } m = 0, \ldots, M - 1, \text{ and for } p = 0, \ldots, \frac{N}{2} - 1.
\]

Also, as the analysis filters are linear phase, we have

\[
\text{Imag} \left( e^{i \frac{2\pi p(N-1)}{2N}} H_m(p) \right) = 0 \quad \text{for } m \text{ being an even number and for } p = 0, \ldots, N - 1,
\]

\[
\text{Real} \left( e^{i \frac{2\pi p(N-1)}{2N}} H_m(p) \right) = 0 \quad \text{for } m \text{ being an odd number and for } p = 0, \ldots, N - 1.
\]

Here, \(\text{Real}(z)\) and \(\text{Imag}(z)\) are denoted as the real part and the imaginary part of the complex number \(z\), respectively. Let \(x_m(p)\) for \(m = 0, \ldots, M - 1\) and for \(p = 0, \ldots, \frac{N}{2}\) be the real valued sequences such that

\[
x_m(p) = e^{i \frac{2\pi p(N-1)}{2N}} H_m(p) \quad \text{for } m \text{ being an even number and for } p = 0, \ldots, \frac{N}{2},
\]

as well as

\[
x_m(p) = \frac{1}{j} e^{i \frac{2\pi p(N-1)}{2N}} H_m(p) \quad \text{for } m \text{ being an odd number and for } p = 0, \ldots, \frac{N}{2}.
\]

As the analysis filters are mirrored to each others, we have

\[
H_m(p) = -H_{M-1-m}^* \left( \frac{N}{2} - p \right) \quad \text{for } p = 0, \ldots, \frac{N}{2} \text{ and for } m = 0, \ldots, \frac{M}{2} - 1.
\]

Denote the superscript "\(^T\)" as the transposition operator. Define

\[
x_m \equiv \left[ x_m(0) \cdots x_m \left( \frac{N}{2} \right) \right]^T \quad \text{for } m = 0, \ldots, \frac{M}{2} - 1,
\]

and

\[
x \equiv \left[ x_0^T \cdots x_{\frac{M}{2}-1}^T \right]^T.
\]

Then, we have the following condition for the filter bank satisfying the \(M\)-channel mirrored paraunitary linear phase finite impulse response condition.

**Theorem 2.3.** A filter bank satisfies the \(M\)-channel mirrored paraunitary linear phase finite impulse response condition if there exist \(Q_{s,t,a}\) and \(Q_{s,t,a}^*\) for \(s = 0, \ldots, M - 1\), for \(t = 0, \ldots, M - 1\) and for \(s = a, \ldots, L - 1\) such that

\[
x^T \text{Real} (Q_{s,t,a}) x \times \delta(s - t) = 0
\]

(13)
and

\[ x^T \text{Imag} (Q_{s,t,a}) x = 0 \]  

for \( s = 0, \ldots, M - 1 \), for \( t = 0, \ldots, M - 1 \) and for \( a = 0, \ldots, L - 1 \).

It is worth noting that our derived condition is different from the conventional condition defined in the frequency domain. As the dimension of the polyphase matrix is \( M \times M \), there are \( M \times M \) equations in the conventional condition. For each equation, it is a function of frequency. As the frequency domain is a continuous set which consists of an infinite number of frequencies, the conventional condition is expressed as \( M \times M \) functional constraints with each functional constraint consisting of an infinite number of constraints. In general, it is very challenging to guarantee that all these infinite numbers of constraints are satisfied. Besides, the paraunitary filter bank is lossless. This implies that the polyphase matrix is unitary. Nevertheless, for any two different unitary matrices, a convex combination of these two unitary matrices is not unitary. Therefore, the feasible set satisfying the paraunitary condition is nonconvex. As the dimension of the polyphase matrix is \( M \times M \), the feasible set of the optimization problem is the intersection of these \( M \times M \) nonconvex sets. In general, it is very difficult to find the near global optimal solutions of such kind of nonconvex optimization problems. Although a numerical optimization computer aided design tool can be employed for finding a local optimal solution, the required computational power is too high to be implemented in the practical situations. To address this difficulty, the paraunitary condition is defined in the time domain [24]. In particular, the filter coefficient vectors are put into the columns of a matrix. The filter coefficient matrix is divided into the submatrices. By circularly shifting these submatrices in the filter coefficient matrix and computing the products of the transpose of the filter coefficient matrix and the shifted filter coefficient matrices, the paraunitary condition is expressed as the matrix equality constraints. More precisely, the products of the transpose of the filter coefficient matrix and the shifted filter coefficient matrices are equal to the identity matrices multiplying to the delta function defined according to the shift indices. However, as this paraunitary condition is defined in the time domain, in general it is very difficult to guarantee that the frequency selectivities of the analysis filters are satisfied. To address the above difficulty, this paper reformulates the paraunitary condition in the frequency domain via another approach. In particular, we derive the paraunitary condition in the time domain first and then apply the discrete Fourier transform to convert the condition in the time domain to the condition in the frequency domain. As the paraunitary condition defined in the time domain is expressed as a finite number of quadratic equations and the discrete Fourier transform is invertible, there is only a finite number of quadratic constraints expressed in the frequency domain. Here, it is worth noting that the obtained condition is not derived from sampling on these infinite number of constraints in the frequency domain. Therefore, no error is introduced neither by the sampling scheme nor by the transformation between the time domain and the frequency domain. From the above, it can be seen that our derived condition is much simpler than the conventional condition defined in the frequency domain. On the other hand, for the conventional condition defined in the time domain, there are \( M \times M \) quadratic equations with each quadratic equation involving the sum of \( N \) terms. Whereas, there are \( M \times M \times L \) quadratic equations in our proposed method with each quadratic equation involving the sum of \( N \) terms. Since \( N = M \times L \), the required computational powers for these two methods are the same. However, as our proposed method represents the paraunitary condition in
the frequency domain, our proposed method is much easier to capture the frequency selectivities of the filters into the consideration.

3. **Sampling the magnitude responses of the analysis filters.** Now, consider the frequency selectivities of the analysis filters. Here, $H_m(k)$ for $m = 0, \ldots, M/2 - 1$ are the frequency responses of the analysis filters evaluated at $2\pi k/N$ for $k = 0, \ldots, N - 1$. However, there are only $N$ sampling frequencies. Even though the specifications on the frequency responses of the analysis filters are satisfied at these $N$ sampling frequencies, it cannot be guaranteed that the specifications are also satisfied at the frequencies between any two consecutive sampling frequencies. In this paper, we propose to sample the magnitude responses of the analysis filters with the total number of the sampling frequencies more than the filter lengths so that we can have more sampling frequencies for formulating the specifications on the frequency responses of the analysis filters. Define the magnitude responses of the analysis filters as

$$
\bar{H}_m(\omega) \equiv \begin{cases} 
\sum_{n=0}^{N/2 - 1} 2 \cos \left( \left( \frac{N-1}{2} - n \right) \omega \right) h_m(n) & m \text{ is an even number,} \\
\sum_{n=0}^{N/2 - 1} 2 \sin \left( \left( \frac{N-1}{2} - n \right) \omega \right) h_m(n) & m \text{ is an odd number,}
\end{cases}
$$

(15)

for $m = 0, \ldots, M/2 - 1$ and for $\omega \in [-\pi, \pi]$. Suppose that we have $K$ sampling frequencies for each filter. Here, $K > N$. Let $\tilde{\omega}_k$ for $k = 0, \ldots, K - 1$ be these sampling frequencies. Denote $\text{diag}(z_0, \ldots, z_{d-1})$ as the diagonal matrix with their diagonal elements being equal to $(z_0, \ldots, z_{d-1})$. Similarly, denote $\text{skewdiag}(z_0, \ldots, z_{d-1})$ as the skew diagonal matrix with their skew diagonal elements being equal to $(z_0, \ldots, z_{d-1})$. Denote

$$
\Gamma = \begin{bmatrix} I_{N/2 \times N/2} & 0_{N/2 \times N/2} \\
0_{(N-1) \times 1} & \text{skewdiag} \left( e^{-j2\pi \frac{0(N-1)}{2N}}, \ldots, e^{-j2\pi \frac{(N-1)(N-1)}{2N}} \right) \end{bmatrix}
$$

Define

$$
G_m = \begin{cases} 
\begin{bmatrix} \cos \left( \frac{\tilde{\omega}_0(N-1)}{2} \right) & \cdots & \cos \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) \\
\vdots & \ddots & \vdots \\
\cos \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) & \cdots & \cos \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) 
\end{bmatrix} & m \text{ is an even number,} \\
\begin{bmatrix} \sin \left( \frac{\tilde{\omega}_0(N-1)}{2} \right) & \cdots & \sin \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) \\
\vdots & \ddots & \vdots \\
\sin \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) & \cdots & \sin \left( \frac{\tilde{\omega}_K(N-1)}{2} \right) 
\end{bmatrix} & m \text{ is an odd number,} 
\end{cases}
$$

(16)
for $M = 0, \ldots, M/2 - 1$. Then, we have
\[
\begin{bmatrix}
\tilde{H}_m(\tilde{\omega}_0) \\
\vdots \\
\tilde{H}_m(\tilde{\omega}_{K-1})
\end{bmatrix} = G_m
\begin{bmatrix}
x_m(0) \\
\vdots \\
x_m(N/2)
\end{bmatrix}
\text{ for } m = 0, \ldots, M/2 - 1. \tag{17}
\]

To formulate the specifications on the frequency responses of the analysis filters as the constraints of the optimization problem, define $D_m$ for $m = 0, \ldots, M/2 - 1$ as the vectors containing the desirable magnitude responses of the analysis filters evaluated at $\tilde{\omega}_k$ for $k = 0, \ldots, K - 1$,
\[
D \equiv \begin{bmatrix} D_0^T & \cdots & D_{M/2-1}^T \end{bmatrix}.
\tag{19}
\]

Define $\varepsilon$ as the vector containing the maximum allowable errors on the real parts of the ripple responses of the analysis filters. Then, the specifications on the frequency responses of the analysis filters can be formulated as the following constraints:
\[
|\text{Real}(Gx - D)| \leq \varepsilon \tag{20}
\]
and
\[
|\text{Imag}(Gx)| = 0_{K\times 1} \tag{21}
\]

4. Methods. It is worth noting that the matrices in the quadratic constraints in our proposed approach are not positive definite. Therefore, the optimization problem is nonconvex. The required computational power for finding a near global optimal solution of the nonconvex optimization problem is still very high. To address this issue, the optimization problem is reformulated. First, it is worth noting that the filter bank is guaranteed to achieve the exact perfect reconstruction condition if the paraunitary condition is satisfied. (Here, the exact perfect reconstruction condition refers to the condition that the output of the filter bank is equal to a pure delay gain of its input.) However, the exact perfect reconstruction condition is a tight condition that could result to a poor performance of the filter responses. Therefore, this paper relaxes the exact perfect reconstruction condition to a near perfect reconstruction condition as well as subject to the frequency selectivities of the filters with more sampling points. (Here, the near perfect reconstruction condition refers to the condition that there is a small but an acceptable error between the output and the pure delay gain of the input of the filter bank.) In particular, this paper proposes to formulate the filter bank design problem as an optimization problem in such a way that the total mirrored paraunitary linear phase error is minimized subject to the specifications on the magnitude responses of the filters at more sampling frequencies. Since the total mirrored paraunitary linear phase error is minimized, the near perfect reconstruction error of the filter bank will be low. To find a local optimal solution of the optimization problem, a norm relaxed sequential quadratic programming approach [25, 26, 27, 28, 29] is employed. As each optimization problem in the sequence of the quadratic programming problems is a standard quadratic programming problem and the required computational power for finding the global optimal solutions of these standard quadratic programming problems is very low, the required computational power for finding a local optimal solution of the original optimization problem is very low. Finally, these procedures
The following inequality constraints are imposed instead:

\[ \text{Imag} (x) \]

The exact equality constraint \( \text{Imag} (x) \) is very difficult to be satisfied. Hence, instead of imposing \( \text{Imag} (x) \) to zero. Define

\[ \text{Problem (P)} \]

\[
\min_J (x) = \sum_{s=0}^{M-1} \sum_{t=0}^{L-1} \left| (x^T \text{Real} (Q_{s,t,a}) x - \delta (s-t)) \right| + \sum_{s=0}^{M-1} \sum_{t=0}^{L-1} \left| x^T \text{Imag} (Q_{s,t,a}) x \right|,
\]

s.t. \( |\text{Real} (G) x - D| \leq \varepsilon \),

and \( \text{Imag} (G) x = 0_{KM} \times 1 \).

Due to the numerical error generated by the computer aided design tool, the exact equality constraints are very difficult to be satisfied. Hence, instead of imposing the exact equality constraint \( \text{Imag} (G) x = 0_{KM/2 \times 1} \) to the optimization problem, the following inequality constraints are imposed instead: \( \text{Imag} (G) x \leq \bar{\varepsilon} \) and \(-\text{Imag} (G) x \leq \bar{\varepsilon} \). Here, \( \bar{\varepsilon} \) is a vector with its elements being positive but very close to zero. Define

\[
q_1 (x) = \text{Real} (G) x - D - \delta,
\]

\[
q_2 (x) = -\text{Real} (G) x + D - \delta,
\]

\[
q_3 (x) = \text{Imag} (G) x - \bar{\varepsilon},
\]

and

\[
q_4 (x) = -\text{Imag} (G) x - \bar{\varepsilon}.
\]

For \( i = 1, \ldots, KM/2 \), let the \( i \)-th element of \( q_1 (x), q_2 (x), q_3 (x) \) and \( q_4 (x) \) be \( q_{1,i} (x) \), \( q_{2,i} (x) \), \( q_{3,i} (x) \) and \( q_{4,i} (x) \), respectively. Then, Problem (P) can be approximated by the following optimization problem:

\[ \text{Problem (P)} \]

\[
\min_J (x) = \sum_{s=0}^{M-1} \sum_{t=0}^{L-1} \left| (x^T \text{Real} (Q_{s,t,a}) x - \delta (s-t)) \right| + \sum_{s=0}^{M-1} \sum_{t=0}^{L-1} \left| x^T \text{Imag} (Q_{s,t,a}) x \right|,
\]

s.t. \( q_{1,i} (x) \leq 0 \) for \( i = 1, 2, \ldots, KM/2 \),

\[
q_{2,i} (x) \leq 0 \text{ for } i = 1, 2, \ldots, KM/2 ;
\]

\[
q_{3,i} (x) \leq 0 \text{ for } i = 1, 2, \ldots, KM/2 ;
\]

and \( q_{4,i} (x) \leq 0 \) for \( i = 1, 2, \ldots, KM/2 \).

It is worth noting that the objective function of Problem (P) is nonconvex because many eigenvalues of \( \text{Real} (Q_{s,t,a}) \) and \( \text{Imag} (Q_{s,t,a}) \) for \( s = 0, \ldots, M-1 \), for \( t = 0, \ldots, M-1 \) and for \( a = 0, \ldots, L-1 \) are zero. The required computational power for finding a local optimal solution of this nonconvex problem is very high. In order to address this issue, a norm relaxed sequential quadratic programming approach is employed for finding a local optimal solution of Problem (P). However, the objective function of Problem (P) is not differentiable. Hence, the norm relaxed sequential quadratic programming approach cannot be directly applied. To address this difficulty, the absolute operator in the objective function of Problem (P) is smoothen by the following operator:

\[
g_{\delta} (z) = \begin{cases} 
|z| & |z| \geq \delta \\
\frac{z^4}{25} + \frac{3z^2}{28} & \text{otherwise}
\end{cases}
\]

\[ (22) \]

Now, Problem (P) can be approximated by the following problem:

\[ \text{Problem (P}_\delta) \]
the feasible set of Problem $(\bar{P})$ of the original optimization problem. However, the conditions of the sequential quadratic programming will result to a more accurate optimization problem as its near global optimal solution. Obviously, more initial local optimal solutions of the original optimization problem are found by generating $x^\ast$ as the initial conditions of the sequential quadratic programming. Different local optimal solutions of the original optimization problem are found if the total number of the initial conditions of the sequential quadratic programming cover all the local optimal solutions of the optimization problem. Therefore, it is difficult to guarantee that these local optimal solutions obtained by using different initial conditions of the sequential quadratic programming are large enough (The local optimal solutions of the original optimization problem obtained by using different initial conditions of the sequential quadratic programming cover all the local optimal solutions of the optimization problem), then the global optimal solution of the original optimization problem can be found. However, in the practical situation, only a finite number of the initial conditions are generated in the sequential quadratic programming. Thus, it is difficult to guarantee that these local optimal solutions obtained by using different initial conditions of the sequential quadratic programming cover all the local optimal solutions of the original optimization problem. As a result, only a near global optimal solution of the original optimization problem can be found. To find a near global optimal solution of the original optimization problem, a finite number of vectors is generated as the initial conditions of the sequential quadratic programming. Different local optimal solutions of the original optimization problem are found by using different initial conditions of the sequential quadratic programming. Take the local optimal solution with the minimum objective functional value of the original optimization problem as its near global optimal solution. Obviously, more initial conditions of the sequential quadratic programming will result to a more accurate near global optimal solution of the original optimization problem. However, the required computational power will increase. Hence, there is a tradeoff between the required computational power and the accuracy of the near global optimal solution of the original optimization problem.

5. Discussion. In this paper, an example of $N = 8$ and $M = 4$ is considered. For the determination of $K$, it is worth noting that the maximum absolute changes of
the magnitude responses of the filters between two consecutive sampling frequencies are directly proportional to the maximum absolute values of the gradients of the magnitude responses of the filters between two consecutive sampling frequencies and inversely proportional to the total numbers of the sampling frequencies. Here, the gradients of the magnitude responses of the filters are dependent on the filter lengths and their filter coefficients, in which the filter coefficients are designed according to the frequency bands of the filters where they are indirectly related to $M$. However, the relationship on $N, M$ and $K$ for the critical satisfaction of these infinite constraints is very complicated. In general, the maximum absolute changes of the magnitude responses of the filters between two consecutive sampling frequencies is small if $N$ is large, $M$ is small and $K$ is a large. Hence, this paper chooses $K = 100$ for $M = 4$ and $M = 8$. It can be checked that $K/N = 12.5$ for $M = 4$. This ratio is larger than those ratios used in the existing designs [14, 30]. Hence, this selected value of $K$ should be large enough to guarantee the satisfaction of these infinite constraints.

To characterize the response of a desirable filter, it is required to define its passband and its stopband and well as its transition band bandwidth. Denote the magnitude responses of the desirable analysis filters as $D_m(\omega)$ for $m = 0, \ldots, M - 1$. Let the passbands and stopbands of $D_m(\omega)$ be $B^p_m$ and $B^s_m$ for $m = 0, \ldots, M - 1$, respectively. Let the transition band bandwidths of $D_m(\omega)$ be $\Delta_m$ for $m = 0, \ldots, M - 1$. Here, $D_m(\omega)$ for $m = 0, \ldots, M - 1$ are chosen as the ideal rectangle magnitude responses. That is,

$$|D_m(\omega)| = \begin{cases} \sqrt{M} & \omega \in B^p_m \\ 0 & \omega \in B^s_m \end{cases} \text{ for } m = 0, \ldots, M - 1. \quad (23)$$

The ideal rectangle magnitude responses are chosen because they correspond to the responses with the best frequency selectivities. For the transition band bandwidths of $D_m(\omega)$ for $m = 0, \ldots, M - 1$, their values should be small in order to have a set of filters with good frequency selectivities. However, too small values of the transition band bandwidths would result to the infeasibility of the optimization problems. That is, the optimization problems do not have the solutions. In general, the smallest values of the transition band bandwidths that result to the existence of the solutions of the optimization problems are dependent on both the values of $M$ and $N$. However, the relationship on these parameters is too complicated to be analytically represented by a mathematical formulae. Hence, for the simplicity reason, they are chosen as $\Delta_m = \Delta = \frac{\pi}{2M}$ for $m = 0, \ldots, M - 1$. For the passbands and the stopbands of $D_m(\omega)$ for $m = 0, \ldots, M - 1$, they are mainly dependent on the values of $M$. In particular, they are chosen as follows:

$$B^p_0 = \left[0, \frac{\pi}{M} - \Delta \right], \quad (24)$$

$$B^s_0 = \left[\frac{\pi}{M} + \Delta, \pi \right], \quad (25)$$

$$B^p_m = \left[\frac{m\pi}{M} + \Delta, \frac{(m+1)\pi}{M} - \Delta \right] \text{ for } m = 1, \ldots, M - 2, \quad (26)$$

$$B^s_m = \left[0, \frac{m\pi}{M} - \Delta \right] \cup \left[\frac{(m+1)\pi}{M} + \Delta, \pi \right] \text{ for } m = 1, \ldots, M - 2, \quad (27)$$

and

$$B^p_{M-1} = \left[\frac{(M-1)\pi}{M} + \Delta, \pi \right], \quad (28)$$

$$B^s_{M-1} = \left[0, \frac{(M-1)\pi}{M} - \Delta \right]. \quad (29)$$
For the specifications on the maximum ripple magnitudes of the analysis filters, it is worth noting that there is a tradeoff between the mirrored paraunitary linear phase error and the maximum allowable errors on the real parts of the ripple responses of the filters. This is because the larger the maximum allowable errors on the real parts of the ripple responses of the filters will result to the larger feasible sets. As a result, the objective functional values of the optimization problems will be lower. However, too large maximum allowable errors on the real parts of the ripple responses of the filters will result to the filters with the unacceptable responses. On the other hand, too small specification values may result to the infeasibilities of the optimization problem. In this paper, the specification value is chosen as 0.4918 for $N = 8$. This value is smaller than those values used in the existing designs [14, 30]. Hence, this value should be small enough to obtain the acceptable frequency selectivities of the filters. Similarly, $\tilde{\epsilon} = \epsilon$ are chosen for the same reason. That is, $\epsilon = \tilde{\epsilon} = 0.4918$ is chosen for $N = 8$, where $\mathbf{a} \times 1$ denotes the $a \times 1$ column vector with all its elements being equal to one.

For the smoothing operator, a small value of $\delta$ will achieve a good approximation between $g_\delta(z)$ and the corresponding ideal nonsmooth function. However, a too small value of $\delta$ may suffer from a numerical computational problem. Hence, $\delta = 10^{-6}$ is chosen.

To find a local optimal solution of the optimization problem via the norm relax sequential quadratic programming approach, the parameters in the algorithm are selected the same as those employed in [25, 26, 27, 28, 29]. That is $\delta_1 = 1, \delta_2 = 1, \sigma = 0.5, \alpha = 0.45, \hat{G} = 0.5, \beta_{-1} = 0$ and $Q_0 = I$. Here, $I_{N \times N}$ denotes the identity matrix. However, the optimization problem is nonconvex. In [30], an initial matrix is generated to compute a local optimal maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response filter bank and 100 different initial matrices are generated to obtain a near global optimal solution. Here, those 100 initial matrices generated in [30] are also used as the initial conditions of the norm relax sequential programming algorithm and a near global optimal solution is obtained.

6. Results. The maximally decimated $M$-channel mirrored paraunitary linear phase finite impulse response filter bank designed by the method discussed in [14] is the most common one used in various engineering applications. On the other hand, the filter bank designed by the method discussed in [30] is the most recent one found in the literature. Therefore, these two optimal filter bank design methods [14, 30] are compared in this paper. For $N = 8$ and $M = 4$, the normalized magnitude responses of the analysis filters in decibels designed by various methods as well as that designed by the methods discussed in both [14] and [30] are shown in Figure 1. Here, the normalized magnitude responses refer to the magnitude responses divided by the DC gains of the corresponding ideal lowpass filters. In fact, these DC gains are equal to $\sqrt{M}$. That is, $|D_0(0)| = \sqrt{M}$. In other words, Figure 1 plots $20\log_{10} \left( \frac{|H_m(\omega)|}{\sqrt{M}} \right)$ for $m = 0, 1, 2, 3$ of the analysis filters designed by various methods. Table 1 lists the maximum ripple magnitudes of the analysis filters in decibels designed by various methods. That is, $\max_{\omega \in \mathcal{B}_m \cup \bar{B}_m} 20\log_{10} \left( ||H_m(\omega)|| - |D_m(\omega)|| \right)$ for $m = 0, 1, 2, 3$.

It can be seen from both Figure 1 and Table 1 that the maximum ripple magnitudes of both the first and the fourth analysis filters designed by the methods
discussed in both [14] and [30] are much larger than those designed by our proposed method. Also, the specifications on these filters designed by the methods discussed in both [14] and [30] are not satisfied, while those designed by our proposed method satisfy the required specifications. Although the maximum ripple magnitudes of both the second and the third analysis filters designed by the methods discussed in both [14] and [30] are smaller than those designed by our proposed method, these filters designed by our proposed method still satisfy the required specifications. This is because the corresponding constraints are imposed in our proposed method. Besides, the maximum ripple magnitude among all these four filters designed by the methods discussed in both [14] and [30] is much larger than that designed by our proposed method. On the other hand, the maximum ripple magnitude among all these four filters designed by our proposed method is almost the same.

Table 1. \[
\max_{\omega \in B^p_m \cup B^s_m} 20 \log_{10} \left( |H_m(\omega)| - |D_m(\omega)| \right) \text{ for } m = 0, \ldots, 3 \text{ of the analysis filters in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30].}
\]

| Method discussed in [14] | The maximum ripple magnitude of the first analysis filter in decibel | The maximum ripple magnitude of the second analysis filter in decibel | The maximum ripple magnitude of the third analysis filter in decibel | The maximum ripple magnitude of the fourth analysis filter in decibel |
|---------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| -0.3251dB | -11.3525dB | -11.3525dB | -0.3251dB |
| Method discussed in [30] | -0.7366dB | -13.0137dB | -12.9932dB | -0.7372dB |
| Our proposed method | -6.1628dB | -6.1930dB | -6.1930dB | -6.1628dB |

It is worth noting that the paraunitary condition is defined as \( HH^+ = I_{N \times N} \). As \( H \) is a block shift matrix consisting of \( L \times L \) blocks with each block having the size \( M \times M \), the paraunitary error can be represented by
Figure 1. $20 \log_{10} \left( \frac{|H_m(\omega)|}{\sqrt{M}} \right)$ for $m = 0, \ldots, 3$ of the analysis filters in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30].

Table 2 lists $\log_{10}(err_{para}(l))$ for $l = 1, \ldots, L - 1$ of the filter banks in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30]. Figure 2(a) and Figure 2(b) show the magnitude distortions and the total aliasing distortions of the filter banks in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30]. Here, the magnitude distortions and the total aliasing distortions are defined as $10 \log_{10}(|| \frac{1}{M} \sum_{m=0}^{M-1} H_m(\omega) \tilde{H}_m(\omega) - 1 ||)$ and


Figure 2. (a) $10\log_{10}\left(|\frac{1}{M} \sum_{m=0}^{M-1} H_m(\omega) \tilde{H}_m(\omega)| - 1\right)$ and (b) $10\log_{10}\left(|\frac{1}{M} \sum_{k=1}^{M-1} \sum_{m=0}^{M-1} H_m(\omega - \frac{2\pi k}{M}) \tilde{H}_m(\omega)|\right)$ of the filter banks in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30].

$10\log_{10}\left(|\frac{1}{M} \sum_{k=1}^{M} \sum_{m=0}^{M-1} H_m(\omega - \frac{2\pi k}{M}) \tilde{H}_m(\omega)|\right)$, respectively. Table 3 lists the corresponding maximum values. That is, $\max_\omega 10\log_{10}(\left|\frac{1}{M} \sum_{m=0}^{M-1} H_m(\omega) \tilde{H}_m(\omega)| - 1\right|$ and $\max_\omega 10\log_{10}(\left|\frac{1}{M} \sum_{k=1}^{M-1} \sum_{m=0}^{M-1} H_m(\omega - \frac{2\pi k}{M}) \tilde{H}_m(\omega)|\right)$.

It can be seen from Table 2 that the paraunitary errors of the filter banks designed by the methods discussed in both [14] and [30] are very small because the exact paraunitary condition
Table 2. $\log_{10}(\text{err}_{\text{para}}(l))$ for $l = 0, \ldots, L - 1$ of the filter banks in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30].

| Method discussed in [14] | Method discussed in [30] | Our proposed method |
|--------------------------|--------------------------|---------------------|
| $\log_{10}(\text{err}_{\text{para}}(0))$ | -66.7193dB | -141.8743dB |
| $\log_{10}(\text{err}_{\text{para}}(l))$ | -141.4721dB | -140.8206dB | 1.1415dB |

Table 3. $\max_{\omega}10\log_{10}\left(\left|\left|\sum_{m=0}^{M-1} H_m(\omega) \tilde{H}_m(\omega)\right|-1\right|\right)$ and $\max_{\omega}10\log_{10}\left(\left|\sum_{k=1}^{M-1} \sum_{m=0}^{M-1} H_m(\omega - 2\pi k/M) \tilde{H}_m(\omega)\right|\right)$ of the filter banks in decibels designed by our proposed method as well as those designed by the methods discussed in both [14] and [30].

| Method discussed in [14] | Method discussed in [30] | Our proposed method |
|--------------------------|--------------------------|---------------------|
| $\max_{\omega}10\log_{10}(\left|\sum_{m=0}^{M-1} H_m(\omega) \tilde{H}_m(\omega)\right|-1)$ | -72.7399dB | -142.8249dB | -2.9505dB |
| $\max_{\omega}10\log_{10}(\left|\sum_{k=1}^{M-1} \sum_{m=0}^{M-1} H_m(\omega - 2\pi k/M) \tilde{H}_m(\omega)\right|)$ | -146.7204dB | -145.1184dB | -6.0642dB |

is imposed to these design methods. However, very small errors still exist. This is because the filter coefficients are obtained by the numerical optimization computer aided design tool where the calculations are processed under the finite word length operations. On the other hand, the paraunitary errors of the filter bank designed by our proposed method are slightly larger than those designed by the methods discussed in both [14] and [30]. This is because the exact paraunitary condition is relaxed. Similarly, it can be seen from both Figure 2 and Table 3 that both the maximum magnitude distortions and the maximum total aliasing distortions of the filter banks designed by the methods discussed in both [14] and [30] are very small because the exact perfect reconstruction condition is imposed to these design methods. Similarly, both the maximum magnitude distortion and the maximum total aliasing distortion of the filter bank designed by our proposed method are slightly larger than those designed by the methods discussed in both [14] and [30]. This is because the exact perfect reconstruction condition is relaxed. As discussed in the above that the larger the maximum allowable errors on the real parts of the ripple responses of the analysis filters (The constraint bounds of the optimization problem) will result to a lower mirrored paraunitary linear phase error of the filter bank (The objective functional value of the optimization problem), our proposed method relaxes both the exact paraunitary condition and the exact perfect reconstruction error so as to achieve the analysis filters with better frequency selectivities. Actually, this is an advantage because it provides a flexible tradeoff among both the paraunitary error and the perfect reconstruction error of the filter bank as well as the maximum ripple magnitudes of the analysis filters.

It is worth noting that the required computational power depends on the total numbers of the filter coefficients required to be designed. This is because this is
related to the dimension of the decision vector of the optimization problem. To compare the required computational powers of our proposed method to those of the methods discussed in both [14] and [30], all computer numerical simulations are conducted by the same computer. Here, an Intel(R) Xeon(R) E3-1225 V2 CPU operating at 3.2GHz with a 16GB memory is employed for running the computer numerical simulations. The algorithm is executed using the Matlab Version 7.11.0.584 (R2010b) operating under the 64 bit Microsoft Windows 7 Version 6.1 with Service Pack 1 and Java 1.6.0_17-b04. It is found that our proposed method only takes 2.3 seconds to obtain a near global optimal solution based on 100 initial conditions. On the other hand, the methods discussed in [14] and [30] take 1 minute and 27 seconds and 15.6 seconds to obtain the near global optimal solutions based on the same 100 initial conditions, respectively. Obviously, the required computational power based on our proposed method is much lower than those based on the methods discussed in both [14] and [30].

7. Conclusions. This paper proposes a norm relax sequential quadratic programming approach for designing the maximally decimated M-channel mirrored paraunitary linear phase finite impulse response filter bank. First, the necessary and sufficient condition for the filter bank satisfying the maximally decimated M-channel mirrored paraunitary linear phase finite impulse response condition is derived. Then, the filter bank design problem is formulated as an optimization problem in such a way that the mirrored paraunitary linear phase error is minimized subject to the specifications on the sampled magnitude responses of the analysis filters. A near global optimal solution of this optimization problem is found via a norm relax sequential quadratic programming algorithm. Computer numerical simulation results show that our proposed method could achieve a better overall frequency selectivity performance and require a lower computational power compared to the existing design methods. Although the required computational power of our proposed method executing in the Intel(R) Xeon(R) E3-1225 V2 CPU operating at 3.2GHz with a 16GB memory is not fast enough for the applications required the paraunitary filter bank to be designed in the real time mode, most of the applications require the paraunitary filter bank to be designed in the offline mode such as the denoising operations for the consumer electronic applications. This is because the hardware cost for designing the real time paraunitary filter bank is too high to be afforded by the consumers. Instead, most of the coefficients of the paraunitary filter bank are pre-designed in the offline mode and these filter coefficients are preloaded in the consumer electronic products. In the case where it is required the paraunitary filter bank to be designed in the real time mode, GPU can be used for the design.

Acknowledgments. This paper was supported partly by the National Nature Science Foundation of China (no.1701266, no.61372173 and no.61671163), the Team Project of the Education Ministry of the Guangdong Province (2017KCXTD011), the Guangdong Higher Education Engineering Technology Research Center for Big Data on Manufacturing Knowledge Patent (no.501130144), and Hong Kong Innovation and Technology Commission, Enterprise Support Scheme (no.S/E/070/17).
REFERENCES

[1] Y.-J. Chen, S. Oraintara and K. S. Amaratunga, Dyadic-based factorizations for regular paraunitary filterbanks and M-band orthogonal wavelets with structural vanishing moments, *IEEE Transactions on Signal Processing*, 53 (2005), 193–207.

[2] M. T. de Gouvêa and D. Odloak, A new treatment of inconsistent quadratic programs in a sqp-based algorithm, *Computers & Chemical Engineering*, 22 (1998), 1623–1651.

[3] Y.-T. Fong and C.-W. Kok, Correction to “Iterative least squares design of DC-leakage free paraunitary cosine modulated filter banks”, *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, 50 (2003), 238–243.

[4] X. Q. Gao, T. Q. Nguyen and G. Strang, Theory and lattice structure of complex paraunitary filterbanks with filters of (hermitian-) symmetry/antisymmetry properties, *IEEE Transactions on Signal Processing*, 49 (2001), 1028–1043.

[5] X. Q. Gao, T. Q. Nguyen and G. Strang, On factorization of M-channel paraunitary filterbanks, *IEEE Transactions on Signal Processing*, 49 (2001), 1433–1446.

[6] L. Gan and K.-K. Ma, A simplified lattice factorization for linear-phase paraunitary filter banks with pairwise mirror image frequency responses, *IEEE Transactions on Circuits and Systems II: Express Briefs*, 51 (2004), 3–7.

[7] N. Holighaus, Z. Průša and P. L. Søndergaard, Reassignment and synchrosqueezing for general time-frequency filter banks, subsampling and processing, *Signal Processing*, 125 (2016), 1–8.

[8] M. Ikehara, T. Nagai and T. Q. Nguyen, Time-domain design and lattice structure of FIR paraunitary filter banks with linear phase *Signal Processing*, 80 (2000), 333–342.

[9] M. Ikehara and T. Q. Nguyen, Time-domain design of linear-phase pr filter banks, 1997 IEEE International Conference on Acoustics, Speech, and Signal Processing, 3 (1997), 2077–2080.

[10] J.-B. Jian, Q.-J. Xu and D.-L. Han, A strongly convergent norm-relaxed method of strongly sub-feasible direction for optimization with nonlinear equality and inequality constraints, *Applied Mathematics and Computation*, 182 (2006), 854–870.

[11] J.-B. Jian, X.-Y. Ke and W.-X. Cheng, A superlinearly convergent norm-relaxed sqp method of strongly sub-feasible directions for constrained optimization without strict complementarity, *Applied Mathematics and Computation*, 214 (2009), 632–644.

[12] C. W. Kok, T. Nagai, M. Ikehara and T. Q. Nguyen, Lattice structures parameterization of linear phase paraunitary matrices with pairwise mirror-image symmetry in the frequency domain with an odd number of rows, *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, 48 (2001), 633–636.

[13] Y.-P. Lin and P. Vaidyanathan, Linear phase cosine modulated maximally decimated filter banks with perfect reconstruction, *IEEE Transactions on Signal Processing*, 43 (1995), 2525–2539.

[14] C. Liu, B. W. Ling, C. Y. Ho and Q. Dai, Finite number of necessary and sufficient discrete condition in frequency domain for maximally decimated m-channel mirrored paraunitary linear phase fir filter bank, 2014 IEEE International Conference on Consumer Electronics-China, (2014), 1–5.

[15] B. W.-K. Ling, N. Tian, C. Y.-F. Ho, W.-C. Siu, K.-L. Teo and Q. Y. Dai, Maximally decimated paraunitary linear phase FIR filter bank design via iterative SVD approach, *IEEE Transactions on Signal Processing*, 63 (2015), 466–481.

[16] T. Q. Nguyen, A quadratic-constrained least-squares approach to the design of digital filter banks, 1992 IEEE International Symposium on Circuits and Systems, 3 (1992), 1344–1347.

[17] T. Q. Nguyen, A. K. Soman and P. Vaidyanathan, A quadratic-constrained least-squares approach to linear phase orthonormal filter bank design, 1993 IEEE International Symposium on Circuits and Systems, (1993), 383–386.

[18] S. Oraintara, T. D. Tran, P. N. Heller and T. Q. Nguyen, Lattice structure for regular paraunitary linear-phase filterbanks and M-band orthogonal symmetric wavelets, *IEEE Transactions on Signal Processing*, 49 (2001), 2659–2672.

[19] S. Patel, R. Dhuli and B. Lall, Design and analysis of matrix wiener synthesis filter for multirate filter bank, *Signal Processing*, 102 (2014), 256–264.

[20] M. Sangnier, J. Gauthier and A. Rakotomamonjy, Filter bank learning for signal classification, *Signal Processing*, 113 (2015), 124–137.
[22] A. K. Soman, P. P. Vaidyanathan and T. Q. Nguyen, Linear phase paraunitary filter banks: Theory, factorizations and designs, IEEE Transactions on Signal Processing, 41 (1993), 3480–3496.

[23] A. K. Soman and P. P. Vaidyanathan, A complete factorization of paraunitary matrices with pairwise mirror-image symmetry in the frequency domain, IEEE Transactions on Signal Processing, 43 (1995), 1002–1004.

[24] C. G. Shen, W. J. Xue and X. D. Chen, Global convergence of a robust filter SQP algorithm, European Journal of Operational Research, 206 (2010), 34–45.

[25] T. D. Tran and T. Q. Nguyen, On m-channel linear phase fir filter banks and application in image compression, IEEE Transactions on Signal Processing, 45 (1997), 2175–2187.

[26] T. D. Tran, M. Ikehara and T. Q. Nguyen, Linear phase paraunitary filter bank with filters of different lengths and its application in image compression, IEEE Transactions on Signal Processing, 47 (1999), 2730–2744.

[27] T. D. Tran, R. L. De Queiroz and T. Q. Nguyen, Linear-phase perfect reconstruction filter bank: Lattice structure, design, and application in image coding, IEEE Transactions on Signal Processing, 48 (2000), 133–147.

[28] P. G. Vouras and T. D. Tran, Factorization of paraunitary polyphase matrices using subspace projections, 2008 42nd Asilomar Conference on Signals, Systems and Computers, (2008), 602–605.

[29] Z. M. Xu and A. Makur, On the arbitrary-length M-channel linear phase perfect reconstruction filter banks, IEEE Transactions on Signal Processing, 57 (2009), 4118–4123.

[30] W. J. Xue, C. G. Shen and D. G. Pu, A penalty-function-free line search sqp method for nonlinear programming, Journal of Computational and Applied Mathematics, 228 (2009), 313–325.

Received August 2019; revised November 2019.

E-mail address: 17661556@qq.com
E-mail address: yongquanling@gdut.edu.cn
E-mail address: daiqy@gdut.edu.cn
E-mail address: miaoqing598@163.com
E-mail address: cailiver@qq.com