Formula for the Projectively Invariant Quantization on Degree Three

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Abstract

We give an explicit formula for the projectively invariant quantization map between the space of symbols of degree three and the space of third-order linear differential operators, both viewed as modules over the group of diffeomorphisms and the Lie algebra of vector fields on a manifold.

1 Introduction

Let $M$ be a manifold of dimension $n$. Fix an affine connection $\nabla$ on $M$. Denote by $\mathcal{F}_\lambda(M)$ the space of $\lambda$-densities on $M$ (i.e. sections of the bundle $(\wedge^n T^* M)^{\otimes \lambda}$). This space admits naturally a structure of module over the group of diffeomorphisms $\text{Diff}(M)$ and the Lie algebra of vector fields $\text{Vect}(M)$. Consider $\mathcal{D}_{\lambda,\mu}(M)$ the space of linear differential operators acting from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. This space is a module over $\text{Diff}(M)$ and $\text{Vect}(M)$ (see [1, 2, 3, 4]). The action is given as follows: take $f \in \text{Diff}(M)$ and $A \in \mathcal{D}_{\lambda,\mu}(M)$ then

$$f^* A = f^*_{\mu} \circ A \circ f^*_{\lambda}^{-1}, \quad (1.1)$$

*Research supported by the Japan Society for the Promotion of Science.
where \( f^*_\alpha \) is the standard action of a diffeomorphism on \( \mathcal{F}_\lambda(M) \).

Differentiating the action of the flow of a vector field, one gets the corresponding action of \( \text{Vect}(M) \).

Denote by \( \mathcal{D}_{\lambda,\mu}^3(M) \) the space of third-order linear differential operators endowed with the structure of module (1.3). The module \( \mathcal{D}_{\lambda,\mu}^3(M) \) is viewed as a submodule of \( \mathcal{D}_{\lambda,\mu}(M) \).

Consider now \( \text{Pol}(T^*M) \) the space of functions on the cotangent bundle \( T^*M \), polynomials on the fibers. This space is naturally isomorphic to the space of symmetric contravariant tensor fields on \( M \). One can define a one-parameter family of \( \text{Diff}(M) \) modules by taking \( \text{Pol}_\delta(T^*M) := \text{Pol}(T^*M) \otimes \mathcal{F}_\delta(M) \). Let us give explicitly this action: take \( f \in \text{Diff}(M) \) and \( P \in \text{Pol}_\delta(T^*M) \) then

\[
f^*_\delta P = f^* P \cdot (J_f)^{-\delta},
\]

where \( f^* \) is the natural action of a diffeomorphism on contravariant tensor fields, and \( J_f \) is the Jacobian of \( f \).

Differentiating the action of the flow of a vector field, one gets the corresponding action of \( \text{Vect}(M) \).

Denote by \( \text{Pol}_0^3(T^*M) \) the space of symbols of degree three endowed with the module structure given by (1.2).

Suppose \( M := \mathbb{R}^n \) is endowed with a flat projective structure (coordinates change are projective transformations). In this case, Lecomte and Ovsienko in [4] construct a quantization map between the space \( \text{Pol}_\delta(T^*\mathbb{R}^n) \) and the space \( \mathcal{D}_{\lambda,\mu}(\mathbb{R}^n) \), equivariant with respect to the action of the Lie algebra \( \text{sl}_{n+1}(\mathbb{R}) \subset \text{Vect}(\mathbb{R}^n) \). Consider now any manifold \( M \) and fix an affine connection on it. It is interesting to ask if there exists a canonical quantization map associated to the given connection. On degree two, the author construct in [4] a quantization map depending only on the projective class of the connection (see also [3] for the conformal case). This approach generalizes Lecomte and Ovsienko’s approach for the flat case. On higher order, the problem of existence of the projectively invariant quantization map is open.

## 2 Main result

The main result of this note is

**Theorem 2.1** For \( n > 1 \), and for \( \delta \neq \frac{n+3}{n+1}, \frac{n+4}{n+1}, \frac{n+5}{n+1} \), there exits a quantization map \( Q : \text{Pol}_0^3(T^*M) \to \mathcal{D}_{\lambda,\mu}^3(M) \) given by

\[
P^{ijk} \mapsto P^{ijk} \nabla_i \nabla_j \nabla_k + \alpha \nabla_k P^{ijk} \nabla_i \nabla_j + \left( \beta_1 \nabla_i \nabla_j P^{ijk} + \beta_2 P^{ijk} R_{ij} \right) \nabla_k + \left( \eta_1 \nabla_i \nabla_j \nabla_k P^{ijk} + \eta_2 R_{ij} \nabla_k P^{ijk} + \eta_3 \nabla_i \nabla_j \nabla_k P^{ijk} \right) \tag{2.1}
\]

where \( R_{ij} \) are the components of the Ricci tensor of the connection \( \nabla \), the constants \( \alpha, \beta_1, \beta_2, \eta_1, \eta_2, \eta_3 \), are given by

\[
\alpha = \frac{6 + 3\lambda(1+n)}{4(1-\delta)(1+n)}, \quad \beta_1 = \frac{1 + \lambda(n+1)}{3 + (1-\delta)(1+n)} \alpha, \quad \beta_2 = \frac{2 + 3\lambda(1+n) - (4(1-\delta)(1+n)) \beta_1}{n-1}, \\
\eta_1 = \frac{\lambda(1+n) - \eta_1 (4(1-\delta)(1+n))}{n-1}, \quad \eta_2 = \frac{\lambda(1+n) \alpha - (10 + 3(1-\delta)(1+n)) \eta_1}{n-1}, \quad \eta_3 = \frac{\lambda(1+n)}{n-1},
\]
and have the following properties

(i) It depends only on the projective class of the connection $\nabla$ (see [3]).

(ii) If $\mathcal{M} = \mathbb{R}^n$ is endowed with a flat projective structure the map (2.1) is the unique map that preserves the principal symbols, equivariant with respect to the action of the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{R}) \subset \mathfrak{Vect}(\mathbb{R}^n)$.

**Proof.** Let us give an idea of the proof. Let $\hat{\nabla}$ be another connection projectively equivalent to $\nabla$. Denote by $Q_{\hat{\nabla}}$ the quantization map (2.1) written with the connection $\hat{\nabla}$. We have to prove that $Q_{\hat{\nabla}} = Q_{\nabla}$.

We need some formulæ.

Since $\hat{\nabla}$ is projectively equivalent to $\nabla$ there exists a 1-form $\omega$ on $\mathcal{M}$ such that the Christoffel symbols of the connections $\nabla$ and $\hat{\nabla}$ are related by

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k \omega_j + \delta_j^k \omega_i, \quad (2.2)$$

(see [3]). It follows that, for any $\phi \in \mathcal{F}_{\lambda}$, one has $\nabla_k \phi = \hat{\nabla}_k \phi + \lambda(1 + n)\omega_k$. In the same manner we can express the tensors $\nabla_i \nabla_j \phi, \nabla_i \nabla_j \nabla_k \phi$, with the tensors $\hat{\nabla}_i \hat{\nabla}_j \phi, \hat{\nabla}_i \hat{\nabla}_j \hat{\nabla}_k \phi$, respectively.

Using also formula (2.2) one has $\nabla_i P^{ijk} = \hat{\nabla}_i P^{ijk} + ((1 + n)\delta - (n + 5))\omega_i P^{ijk}$. In the same manner we can express the tensors $\nabla_j \nabla_i P^{ijk}, \nabla_k \nabla_j \nabla_i P^{ijk}, R_{ij}, \nabla_k R_{ij}$, with the tensors $\hat{\nabla}_j \hat{\nabla}_i P^{ijk}, \hat{\nabla}_k \hat{\nabla}_j \hat{\nabla}_i P^{ijk}, \hat{R}_{ij}, \hat{\nabla}_k \hat{R}_{ij}$. Replacing now these formulæ into (2.1), we see that $Q_{\hat{\nabla}} = Q_{\nabla}$ if and only if the constants $\alpha, \beta_1, \beta_2, \eta_1, \eta_2, \eta_3$ are given as above.

To prove part (ii), recall that the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{R})$ can be identified with the Lie subalgebra of $\mathfrak{Vect}(\mathbb{R}^n)$ generated by the vector fields $\partial_i, x^i\partial_j, x^ix^j\partial_j$, where $(x^i)$ is the coordinates of the projective structure. The proof now is a simple computation (see [3]).

For the particular values of $\delta$ :

**Proposition 2.2** If $\delta = \frac{n+3}{n+1}, \frac{n+4}{n+1}, \frac{n+5}{n+1}$, there exists a quantization map given by (2.1) with particular values of $\lambda, \mu$, given in the following table

| $\delta$ | $\lambda$ | $\mu$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\eta_1$ | $\eta_2$ | $\eta_3$ |
|----------|------------|-------|-----------|------------|------------|-----------|-----------|-----------|
| $n+5$    | $-2$       | $n+3$ | $t$       | $t$        | $4$        | $1$       | $4$        | $t$       |
| $n+4$    | $-2$       | $n+2$ | $0$       | $t$        | $4+2$      | $2$       | $2$        | $6+2t$    |
| $n+3$    | $-2$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+3$ | $t$       | $1$        | $1$        | $9+t$     | $3$        | $3-3n$    |
| $n+3$    | $-1$       | $n+2$ | $3$       | $1$        | $1$        | $4$       | $2$        | $1+2t$    |
| $n+3$    | $-1$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+1$ | $2$       | $0$        | $1$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
| $n+3$    | $-1$       | $n+1$ | $0$       | $0$        | $4$        | $1$       | $4$        | $2+1+t$   |
Here $t$ is a parameter.

**Remark 2.3** (i) For the particular values of $\delta$, the quantization map (2.1) is not unique (it is given by the parameter $t$).

(ii) In the one dimensional case, the quantization map was given in [2, 4].

(iii) Another approach to the quantization map equivariant with respect to the action of the conformal group in a Riemannian manifold was given in [3, 7].

**Acknowledgments.** I am grateful to Ch. Duval and V. Ovsienko for the statement of the problem. I am also grateful to H. Gargoubi and S. E. Loubon Djounga for fruitful discussions, and Y. Maeda and Keio University for their hospitality.

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