Creation of superposition of unknown quantum states

Michał Oszmaniec
Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warszawa and ICFO - Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain

Andrzej Grudka and Antoni Wójcik
Faculty of Physics, Adam Mickiewicz University, 61-614 Poznan, Poland

Michał Horodecki
Faculty of Mathematics, Physics and Informatics, Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdañsk, Poland

The superposition principle is one of the landmarks of quantum mechanics. The importance of quantum superpositions provokes questions about the limitations that quantum mechanics itself imposes on the possibility of their generation. In this work we systematically study the problem of creation of superpositions of unknown quantum states. First, we prove a no-go theorem that forbids the existence of a universal probabilistic quantum protocol producing a superposition of two unknown quantum states. Secondly, we provide an explicit probabilistic protocol generating a superposition of two unknown states, each having a fixed overlap with the known referential pure state. The protocol can be applied to generate coherent superposition of results of independent runs of subroutines in a quantum computer. Moreover, in the context of quantum optics it can be used to efficiently generate highly nonclassical states or non-gaussian states.

The existence of superpositions of pure quantum states is one of the most intriguing consequences of the postulates of quantum mechanics. Quantum superpositions are crucial for the path-integral formulation of quantum mechanics [1] and are responsible for numerous nonclassical phenomena that are considered to be the key features of quantum theory [2]. The prominent examples are: quantum interference [3–5] and quantum entanglement [6]. Coherent addition of wavefunctions is also responsible for quantum coherence, a feature of quantum states that recently received a lot of attention [7–9]. Quantum superpositions are not only important from the foundational point of view but also a feature of quantum mechanics that underpins the existence of ultra-fast quantum algorithms (such as Shor factoring algorithm [10] or Grover search algorithm [11]), quantum cryptography [12] and efficient quantum metrology [13].

The importance of quantum superpositions provokes questions about the restrictions that quantum mechanics itself imposes on the possibility of their generation. Studies of the limitations of the possible operations allowed by quantum mechanics have a long tradition and are important both from the fundamental perspective as well as for the applications in quantum information theory. On one hand quantum mechanics offers a number of protocols that either outperform all existing classical counterparts or even allow to perform tasks that are impossible in the classical theory (such as quantum teleportation [14]). On the other hand a number of no-go theorems [15–20] restrict a class of protocols that are possible to realise within quantum mechanics. Finally, such no-go theorems can be themselves useful for practical purposes. For instance a no-cloning theorem can be used to certify the security of quantum cryptographic protocols [12].

In this paper, we consider the scenario in which we are given two unknown pure quantum states and our task is to create, using the most general operations allowed by quantum mechanics, their superposition with some complex weights. Essentially the same question was posed in a parallel work of Alvarez-Rodriguez et al. [21], namely the authors asked about the existence of quantum adder - a machine, that would superpose two registers with the plus sign.

Here, we first prove a no-go theorem, showing that it is impossible to create superposition of two unknown states. We discuss the relation of our theorem with the no-go results of [21]. Subsequently, we provide a protocol that probabilistically creates superposition of two states having fixed nonzero overlaps with some referential state. We show that, by using appropriate encoding, the protocol can be used to generate superpositions of unknown vectors from the subspace perpendicular to the referential state, thus allowing for generation of coherent superpositions of the results of quantum subroutines of a given quantum algorithm. This actually shows how to circumvent our no-go theorem to some extent. We also discuss optical implementation of the protocol, with the referential state being the vacuum state. Finally, we discuss the differences between our results, and analogous results concerning cloning.

Introduction. Before we proceed we need to carefully analyse the concept of quantum superpositions. Recall first that the global phase of a wavefunction is not a physically accessible quantity. This redundancy can be removed when one interprets pure states as one dimensional orthogonal projectors acting on the relevant
Hilbert space. In what follows the pure state corresponding to a normalized vector $|\psi\rangle$ will be denoted by $|\psi\rangle_\psi$. Normalized vectors that rise to the same pure state $|\psi\rangle_\psi$ are called vector representatives of $|\psi\rangle_\psi$. They are defined up to a global phase i.e. $|\psi\rangle_\psi = |\psi\rangle_{\psi'}$ if and only if $|\psi\rangle = \exp (i\theta) |\psi\rangle$, for some phase $\theta$. Let now $\alpha, \beta$ be complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ and let $|\psi\rangle_\psi$, $|\phi\rangle_\phi$ be two pure states. By $P_{\alpha, \beta} (|\psi\rangle, |\phi\rangle)$ we denote the projector onto the superposition of $|\psi\rangle$ and $|\phi\rangle$.

$$P_{\alpha, \beta} (|\psi\rangle, |\phi\rangle) = |\psi\rangle_\psi \langle |\psi\rangle_\psi| + \alpha |\phi\rangle_\phi \langle |\phi\rangle_\phi| + \beta |\phi\rangle_\phi \langle |\phi\rangle_\phi| ,$$

where $N = \sqrt{1 + 2 \cdot \text{Re} (\alpha \beta^* \langle \psi | \phi \rangle)}$ is a normalization factor. The crucial observation is that $P_{\alpha, \beta} (|\psi\rangle, |\phi\rangle)$ is not a well-defined function of the states $|\psi\rangle_\psi$ and $|\phi\rangle_\phi$. This is because $P_{\alpha, \beta} (|\psi\rangle, |\phi\rangle)$ depends on vector representatives $|\psi\rangle$, $|\phi\rangle$, whose phases can be gauged independently. Consequently, we have the infinite family of pure states

$$P_{\alpha, \beta} (|\psi\rangle, \exp (i\theta) |\phi\rangle_\phi) , \quad \theta \in [0, 2\pi] ,$$

which can be legitimately called superpositions of $|\psi\rangle_\psi$ and $|\phi\rangle_\phi$. This phenomenon appears already in the simplest example of a qubit. For $P_{\psi} = \langle 0 \rangle_0 \langle 0 \rangle_0$, $P_{\phi} = \langle 1 \rangle_1 \langle 1 \rangle_1$, and $\alpha = \beta = \frac{1}{\sqrt{2}}$ the family given by (1) can be identified with the equator on the Bloch ball. The analogous analysis was conducted in [21] and it was argued there that the ambiguity of the relative phase forbids the existence of the universal quantum adding machine. In our approach we propose to relax the definition of superposing, so that it is not excluded from the very definition. However, we will still prove a no-go theorem.

We now settle the notation that we will used throughout the article. By $\text{Brm} (\mathcal{H})$, and $\text{D} (\mathcal{H})$ we denote respectively sets of hermitian operators and the set of density matrices on Hilbert space $\mathcal{H}$. By $\text{CP} (\mathcal{H}, \mathcal{K})$ we denote the set of completely positive (CP) maps $\Lambda : \text{Brm} (\mathcal{H}) \rightarrow \text{Brm} (\mathcal{K})$ ($\mathcal{K}$ is some arbitrary Hilbert space).

Let us now formalise our scenario. We assume that we have access to two identical quantum registers (to each of them we associate a Hilbert space $\mathcal{H}$) and we know that the input state is a product of unknown pure states $|\psi\rangle_{\psi_{\text{in}}} \otimes |\phi\rangle_{\phi_{\text{in}}}$. Our aim is to generate from this input the superposition $P_{\alpha, \beta} (|\psi\rangle, |\phi\rangle)$ by the most general operations allowed by quantum mechanics. By such operations we understand the application of a quantum channel between $\mathcal{H}^\otimes 2$ and $\mathcal{H}$, followed by the postselection conditioned on the result of some generalized measurement [22]. This class of operations has a convenient mathematical characterization. It consists of CP maps $\Lambda \in \text{CP} (\mathcal{H}^\otimes 2, \mathcal{H})$ that do not increase the trace i.e. $\text{tr} [\Lambda (\rho)] \leq \text{tr} (\rho)$ for all $\rho \in \text{D} (\mathcal{H}^\otimes 2)$. For a given state $\rho$ the number $\text{tr} [\Lambda (\rho)]$ is the probability that the operation $\Lambda$ took place. If the operation takes place, the state $\rho$ undergoes the transformation $\rho \rightarrow \frac{\Lambda (\rho)}{\text{tr} [\Lambda (\rho)]}$.

**No-go theorem** We prove the no-go result in the strongest possible form. First, we impose the minimal assumptions on the generated superpositions, assuming only that vectors $|\psi\rangle$, $|\phi\rangle$ are vector representatives depending on the input states (in other words we are not interested in the relative phase $\theta$ of the superposition appearing in (1)). Secondly, we allow the probabilistic protocols, i.e. the superposition may be created with some probability.

**Theorem 1.** Let $\alpha, \beta$ be nonzero complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ and let $\dim \mathcal{H} \geq 2$. There exist no nonzero completely positive map $\Lambda \in \text{CP} (\mathcal{H}^\otimes 2, \mathcal{H})$ such that for all pure states $P_{\psi_{\text{in}}}, P_{\phi_{\text{in}}}$

$$\Lambda (P_{\psi_{\text{in}}} \otimes P_{\phi_{\text{in}}}) \propto |\psi\rangle \langle \psi| ,$$

where

$$|\psi\rangle = \alpha |\psi\rangle + \beta |\phi\rangle$$

and $|\psi\rangle |\phi\rangle = P_{\psi_{\text{in}}}, |\phi\rangle |\phi\rangle = P_{\phi_{\text{in}}}$ and the representants $|\psi\rangle, |\phi\rangle$ may in general depend on both $P_{\psi_{\text{in}}}$ and $P_{\phi_{\text{in}}}$.

**Remark.** In particular, for two pairs $(P_{\psi_{\text{in}}}, P_{\phi_{\text{in}}})$ and $(P_{\psi'_{\text{in}}}, P_{\phi'_{\text{in}}})$ the representant of $P_{\psi_{\text{in}}}$ can be different for each pair.

**Sketch of the proof.** Assume that there exist a nonzero CP map $\Lambda$ satisfying (3). Let the collection of operators $\{V_i\}_{i \in I}$, $V_i : \mathcal{H}^\otimes 2 \rightarrow \mathcal{H}$, form the Kraus decomposition [23] of $\Lambda$, $\Lambda (\rho) = \sum_{i \in I} V_i \rho V_i^\dagger$. Since operators $\lambda |\psi\rangle \langle \psi|$, $\lambda \geq 0$, belong to the extreme ray of the cone of nonnegative operators on $\mathcal{H}$ we must have

$$V_i P_{\psi_{\text{in}}} \otimes P_{\phi_{\text{in}}} V_i^\dagger \propto |\psi\rangle \langle \psi| , \quad \text{for all } i \in I .$$

Consequently, it is enough to consider only CP maps that have one operator in their Kraus decomposition. In such case (3) reduces to the investigation of a single linear operator. If (3) is satisfied then it necessary must hold for $P_{\psi_{\text{in}}}$, $P_{\phi_{\text{in}}}$ having support on two dimensional subspaces of $\mathcal{H}$. Therefore, it suffices to show that the qubit case only operators $V_i$ that satisfy condition (3) are the null operators. We present the proof of this in the Supplemental Material [24]. The main difficulty of the proof stems from the fact that the condition (3) is non-linear in the input state $P_{\psi_{\text{in}}} \otimes P_{\phi_{\text{in}}}$.

Theorem 1 shows that, even if we allow for postselection, there exist no quantum operation that produces superpositions of all unknown pure quantum states with some probability (we allowed this probability to be zero for some pairs of input states and in general it can be different for different inputs). We would like to stress that the creation of superpositions is still impossible even if we allow for the arbitrary dependence of the relative phase of the input states. Namely, in our formulation of
the problem we explicitly assumed that vector representatives |ψ⟩, |φ⟩ of states Pψ and Pφ are some functions of these states. As a matter of fact, otherwise one would not be able to formulate the problem of generation of superpositions in a consistent manner. We emphasize that in that respect the problem of creation of superpositions is different to quantum cloning [25]. Moreover, to our best knowledge, there is no immediate connection between the no-cloning theorem [15] [16] and its generalized variants (such as no-deleting theorem [18] or no-anticloning theorem [19]) to our result. This is a consequence of the fact that Λ must be non-invertible and therefore cannot be used to obtain a cloning map. Moreover, in the formulation of the theorem we allow for situations in which for some input states Pψ ⊗ Pφ the probability of success is zero.

Constructive protocol It is natural to study whether it is possible to create quantum superpositions if we have some knowledge about the input states. Except for specifying the class of input states for which a given protocol would work, it is also necessary to prescribe precisely which superpositions will be generated (see discussion before Eq. (2)). In what follows we present an explicit protocol that generates superpositions of unknown pure states Pψ, Pφ having fixed nonzero overlaps with some referential pure state Pχ (see Figure 1). Let us describe the superpositions that will be generated by our protocol. Let |χ⟩ be a vector representative of Pχ. For every pair of normalized vectors |ψ⟩, |φ⟩ satisfying ⟨χ|ψ⟩ ≠ 0, ⟨χ|φ⟩ ≠ 0 we define their superposition

\[ |Ψ⟩ = \alpha \frac{⟨χ|φ⟩}{|⟨χ|φ⟩|} |ψ⟩ + \beta \frac{⟨χ|ψ⟩}{|⟨χ|ψ⟩|} |φ⟩, \]

(6)
The norm of this vector is given by

\[ N_Ψ = \sqrt{1 + 2 \cdot \text{Re} \left( \frac{\text{tr} (P_χ |ψ⟩⟨ψ| P_ψ |φ⟩⟨φ|)}{|⟨χ|φ⟩| |⟨χ|ψ⟩|} \right)}. \]

(7)
The vector |Ψ⟩ changes only by a global phase once any of the vectors |ψ⟩, |φ⟩, |χ⟩ gets multiplied by a phase factor. Consequently, PΨ can be regarded as well-defined function of the states P|ψ⟩⟨ψ|, P|φ⟩⟨φ|, provided they have nonzero overlap with Pχ. This can be also seen from the explicit formula,

\[ |Ψ⟩⟨Ψ| = |α|^2 P_ψ + |β|^2 P_φ + \alpha \beta^* \frac{P_ψ P_χ P_φ}{\sqrt{\text{tr} (P_χ P_χ)} \sqrt{\text{tr} (P_φ P_φ)}} + h.c. \]

(8)

One could argue that the above choice of the superposition |Ψ⟩⟨Ψ| is somewhat arbitrary. However, the mapping (Pψ, Pφ) → |Ψ⟩⟨Ψ| is related to the so-called Pancharatnam connection and appears in studies concerning the superposition rules from the perspective of geometric approach to quantum mechanics [26] [27]. Moreover, it shown in [28] that Eq. (6) has a strong connection with the concept of the geometric phase. Finally, from the purely operational grounds, Eq. (6) constitute a rightful superposition of states Pψ, Pφ and as we vary coefficients α, β we can recover all possible superpositions of Pψ, Pφ.

![FIG. 1. Graphical representation of the class of input states](image)

Theorem 2. Let Pχ be a fixed pure state on Hilbert space ℋ. There exist a CP map Λsup ∈ CP (C² ⊗ ℋ^⊗2, ℋ) such that for all pure states Pψ, Pφ on ℋ satisfying

\[ \text{tr} (P_χ P_ψ) = c_1, \text{tr} (P_χ P_φ) = c_2, \]

(9)

we have

\[ Λ_{sup} (P_ψ ⊗ P_ψ ⊗ P_φ) \propto |Ψ⟩⟨Ψ|, \]

(10)

where Pν, |ν⟩ = α|0⟩ + β|1⟩, is an unknown qbit state and the vector |Ψ⟩ is given by (6). Moreover, a CP map Λsup realizing (10) is unique up scaling.

Proof. We first present a protocol that realizes (10). Let us define and auxiliary normalized qbit vector |μ⟩ = C · (√c₁|0⟩ + √c₂|1⟩), where C is a normalization constant. We set Λsup = Λ₁ ⊙ Λ₃ ⊙ Λ₁, where

\[ Λ₁ (ρ) = V₁ρ V₁†, V₁ = |0⟩⟨0| ⊗ I ⊗ |1⟩⟨1| ⊗ S, \]

(11)
\[ Λ₂ (ρ) = V₂ρ V₂†, V₂ = I ⊗ Ω ⊗ |χ⟩⟨χ|, \]

(12)
\[ Λ₃ (ρ) = V₃ρ V₃†, V₃ = P_μ ⊗ I ⊗ Ω, \]

(13)
\[ Λ₄ (ρ) = \text{tr}_{13} (ρ). \]

(14)

In the above S denotes the unitary operator that swaps between two copies of ℋ and tr_{13} (·) is the partial trace over the first and the third factor in the tensor product C² ⊗ ℋ ⊗ ℋ. For a graphical presentation of the above protocol see Fig. 2. Operation Λsup is completely positive and trace non-increasing. Direct calculation shows that under the assumed conditions (10) indeed holds. We prove the uniqueness result in the Supplemental Material [24].

The probability that the above protocol will successfully create superpositions of states is given by

\[ P_{succe} = \text{tr} [Λ_{sup} (P_ψ ⊗ P_ψ ⊗ P_φ)] = \frac{c₁c₂}{c₁ + c₂} N_{Ψ}^2. \]

(15)
The map $\Lambda_{sup}$ cannot be rescaled to increase the probability of success. This follows from the (tight) operator inequality $(V_3V_2V_1)^\dagger(V_3V_2V_1) \leq I \otimes I \otimes I$. Taking into account the uniqueness (up to scaling) of $\Lambda$ we get that $P_{\text{succ}}$ from (15) is the maximal achievable probability of success (for inputs specified in the assumptions of Theorem (2)). However, for fixed coefficients $\alpha, \beta$ it is possible to design a CP map that can achieve higher probability of success $\Lambda$. Moreover, it is possible to generalize the protocol $\Lambda_{sup}$ to the situation when we have of $d$ input states (having nonzero overlap with $P_\chi$) and coefficients of superposition are encoded in an unknown state of a qdit $\Lambda$.

The existence of the map $\Lambda_{sup}$ shows that the problem of creating superpositions of quantum states differs greatly from the cloning problem. Probabilistic quantum cloning of pure states is possible if and only if we have a promise that the input states belong to the family of states whose vector representatives form a linearly independent set $[29]$. Consequently, the aforementioned family of states must be discrete. Our protocol shows that it is possible to probabilistically create superpositions from unknown quantum states belonging to uncountable families of quantum states.

**Applications** There exist deterministic circuits realizing classical arithmetic operations (like addition, multiplication, exponentiation etc.) on a quantum computer [30]. However, to our best knowledge there exist no protocols realizing addition on vectors belonging to the Hilbert space responsible for the computation. We now present a method to generate coherent superpositions of results of quantum computations. Assume that $\alpha = \beta = \sqrt{c_1} = \sqrt{c_2} = \frac{1}{\sqrt{2}}$. By setting the overlap of vector representatives of $P_\psi$ and $P_\phi$ with $|\chi\rangle$ to be positive we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\chi\rangle + \frac{1}{\sqrt{2}}|\psi^+\rangle, \quad |\phi\rangle = \frac{1}{\sqrt{2}}|\chi\rangle + \frac{1}{\sqrt{2}}|\phi^+\rangle,$$  \hspace{1cm} (16)$$

where unit vectors $|\psi^+\rangle, |\phi^+\rangle$ are perpendicular to $|\chi\rangle$. Input states $P_\psi, P_\phi$ are in one-to-one correspondence with the vectors $|\psi^+\rangle, |\phi^+\rangle$. By the application of $\Lambda_{sup}$ it is possible to obtain a state having the (non-normalized) vector representative

$$|\Psi\rangle = |\chi\rangle + \frac{1}{2}(|\psi^+\rangle + |\phi^+\rangle),$$  \hspace{1cm} (17)$$

with probability $P_{\text{succ}} = \frac{1}{4} \left(1 + \frac{1}{2} ||\psi^+\rangle + |\phi^+\rangle||^2\right) \geq \frac{1}{2}$.

We have obtained a state encoding the superposition of *unknown* vectors $|\psi^+\rangle, |\phi^+\rangle$ encoded in states $P_\psi$ and $P_\phi$ respectively. The method presented above effectively superposes the wavefunctions coherently, provided one has access to the auxiliary one dimensional subspace (spanned by $|\chi\rangle$). It is highly unexpected but by changing the perspective and by treating as "primary" objects the vectors perpendicular to $|\chi\rangle$ we have managed to effectively get around the no-go result from Theorem 1. To apply the above protocol, one has to run quantum computation in a the perpendicular space. In Supplemental Material [24], we present an exemplary scheme implementing such computation.

The protocol $\Lambda_{sup}$ can be also used to generate non-classical states in the context of quantum optics. Let the states $P_\psi, P_\phi$ describe quantum fields in two different optical modes. Hilbert spaces associated each of the modes is isomorphic and can be identified with the single-mode bosonic Fock space. Moreover, let the auxiliary qbit be encoded in a polarization of a single photon in different optical mode or in another two level physical system. In such a setting the natural choice of the state $P_\chi$ is the Fock vacuum $|0_F\rangle|0_F\rangle$ describing the state of the field with no photons. As an input we can put coherent or pure Gaussian states $[31]$ that have fixed overlaps with the vacuum. Then, the protocol $\Lambda_{sup}$ generically creates highly nonclassical or respectively non-gaussian states. Operations $\Lambda_2, \Lambda_3, \Lambda_4$ are relatively easy to realize in this setting. The most demanding operation is the conditional swap $\Lambda_1$. However, conditional swap can be realized in the optical setting via implementation of phase flip operation and standard beam splitters $[32]$. The phase flip operation on the other hand can in principle $[33]$ be obtained in the optical setting by coupling light to atoms inside the cavity, trapped ions, or by the usage of cross-Kerr nonlinearities in materials with electromagnetically induced transparency. Despite the possible difficulties with the implementation the map $\Lambda_{sup}$ is worth realizing as it gives the maximal probability of success. Moreover, the protocol $\Lambda_{sup}$ is universal and can be used in different physical scenarios.  

**Discussion** There is a number of open questions we did not adress here. First of all, the relation of our no-go theorem to other no-go results in quantum mechanics is not clear and requires further investigation. The constructive protocol presented by us suggest a connection with the recent works concerning the problem of controlling an unknown unitary operation $[34,37]$ (the referential pure states $P_\psi$ can be regarded as an analogue of the known eigenvector of the “unknown” operation $U$ which allows for its control). It would be also natural to study the problem of approximate generation of quantum superpositions in (in analogy to the problem of approximate cloning $[38]$). Another possible line of research is to
investigate the probabilistic protocols designed especially to generate superpositions of states naturally appearing in the experimental context (like pure coherent or Gaussian states).

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∗michal.oszmaniec@icfo.es

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SUPPLEMENTAL MATERIAL

Part A: No-go theorem

Final step of the proof of Theorem 1. We will show that there exist no nonzero linear mapping \( V : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \) such that for all pairs of input pure qubit states \( \mathbb{P}_\psi, \mathbb{P}_\phi \) we have

\[
V\mathbb{P}_\psi \otimes \mathbb{P}_\phi V^\dagger \propto |\Psi\rangle \langle \Psi | ,
\]

(S.1)

where

\[
|\Psi\rangle = \alpha |\psi\rangle + \beta |\phi\rangle ,
\]

(S.2)

where \( \alpha, \beta \) are fixed nonzero complex numbers satisfying \( |\alpha|^2 + |\beta|^2 = 1 \), and vector representatives \( |\psi\rangle, |\phi\rangle \) are given by

\[
|\psi\rangle = \mathcal{F}(\mathbb{P}_\psi, \mathbb{P}_\phi) , \quad |\phi\rangle = \mathcal{G}(\mathbb{P}_\psi, \mathbb{P}_\phi) ,
\]

for some fixed functions \( \mathcal{F}, \mathcal{G} \). Let us fix the standard product basis of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) and let us order it in a lexicographic order.

\[
|v_1\rangle = |0\rangle |0\rangle , \quad |v_2\rangle = |0\rangle |1\rangle , \quad |v_3\rangle = |1\rangle |0\rangle , \quad |v_4\rangle = |1\rangle |1\rangle .
\]

(S.4)

Likewise, we introduce the standard basis in the output Hilbert space \( \mathbb{C}^2 \), \( |f_1\rangle = |0\rangle \), \( |f_2\rangle = |1\rangle \). For such a choice of the basis the operator \( V \) can be described as 2 \times 4 matrix

\[
V = \begin{pmatrix}
a & b & c & d \\
e & f & g & h
\end{pmatrix} .
\]

(S.5)

The condition [S.1] can be written in the form

\[
V|\psi\rangle |\phi\rangle = \mathcal{C}(|\psi\rangle , |\phi\rangle) (\alpha |\psi\rangle + \beta |\phi\rangle) ,
\]

(S.6)

where \( |\psi\rangle, |\phi\rangle \in \mathbb{C}^2 \) are unit vectors, \( \mathcal{C}(|\psi\rangle , |\phi\rangle) \) is a complex-valued function and \( Z(|\psi\rangle , |\phi\rangle) \) is a function taking values in the unit circle and satisfying \( Z(|\psi\rangle , |\phi\rangle) = Z(\exp(i\theta) |\psi\rangle , \exp(i\theta) |\phi\rangle) \) for all \( \theta \in \mathbb{R} \). Taking \( |\psi\rangle = |\phi\rangle \) in [S.6] we obtain

\[
V|\psi\rangle |\psi\rangle \perp |\psi^\perp\rangle ,
\]

(S.7)

where \( |\psi^\perp\rangle \in \mathbb{C}^2 \) is an arbitrary vector perpendicular to \( |\psi\rangle \). Using [S.7] for

\[
|\psi\rangle \propto |0\rangle + \alpha |1\rangle , \quad |\psi^\perp\rangle \propto -\bar{\alpha} |0\rangle + |1\rangle , \quad \alpha \in \mathbb{C}
\]

(S.8)

we obtain

\[
V = \begin{pmatrix}
f + g & b & c & 0 \\
0 & f & g & b + c
\end{pmatrix} .
\]

(S.9)

Using the above we obtain that for every pair of vectors \( |\psi\rangle, |\phi\rangle \in \mathbb{C}^2 \),

\[
V|\psi\rangle |\phi\rangle = \langle \chi_1 | \phi \rangle |\psi\rangle + \langle \chi_2 | \psi \rangle |\phi\rangle ,
\]

(S.10)

where

\[
|\chi_1\rangle = g |0\rangle + \bar{b} |1\rangle ,
\]

(S.11)

\[
|\chi_2\rangle = f |0\rangle + \bar{c} |1\rangle .
\]

(S.12)

From [S.10] and [S.6] we obtain that for all unit vectors \( |\psi\rangle, |\phi\rangle \in \mathbb{C}^2 \)

\[
(\alpha \cdot \mathcal{C}(|\psi\rangle , |\phi\rangle) - \langle \chi_1 | \phi \rangle |\psi\rangle + \beta \cdot \bar{\mathcal{C}}(|\psi\rangle , |\phi\rangle) - \langle \chi_2 | \psi \rangle \rangle |\phi\rangle = 0 ,
\]

(S.13)

where \( \mathcal{C}(|\psi\rangle , |\phi\rangle) = \mathcal{C}(|\psi\rangle , |\phi\rangle) \mathcal{Z}(|\psi\rangle , |\phi\rangle) \). Setting in the above \( |\psi\rangle = |\chi_2^\perp\rangle \) \( (|\chi_2^\perp\rangle \) is some unit vector perpendicular to \( |\chi_2\rangle \) we obtain

\[
(\alpha \cdot \mathcal{C}(|\chi_2^\perp\rangle , |\phi\rangle) - \langle \chi_1 | \phi \rangle |\chi_2^\perp\rangle + \beta \cdot \bar{\mathcal{C}}(|\chi_2^\perp\rangle , |\phi\rangle) |\phi\rangle = 0 .
\]

(S.14)

Since \( \alpha, \beta \neq 0 \) and \( |\phi\rangle \) can be chosen in arbitrary manner we obtain

\[
\alpha \cdot \mathcal{C}(|\chi_2^\perp\rangle , |\phi\rangle) - \langle \chi_1 | \phi \rangle = 0 ,
\]

(S.14)

\[
\bar{\mathcal{C}}(|\chi_2^\perp\rangle , |\phi\rangle) = 0 ,
\]

(S.15)

whenever \( |\chi_2^\perp\rangle \) not linearly dependent with \( |\phi\rangle \). Consequently we obtain that \( \mathcal{C}(|\chi_2^\perp\rangle , |\phi\rangle) = \bar{\mathcal{C}}(|\chi_2^\perp\rangle , |\phi\rangle) = 0 \) and consequently \( \langle \chi_1 | \phi \rangle = 0 \) for all \( |\phi\rangle \) not parallel to \( |\chi_2^\perp\rangle \). Consequently we obtain \( \langle \chi_1 | = 0 \). An analogous argument shows that \( \langle \chi_2 | = 0 \). \qed

Part B: Constructive protocols for superposition of two states

In this part we complete the proof of Theorem 2 and derive the formulas for the maximal probability of success for the generation of superposition \( \mathbb{P}_\psi \) via the usage of CP maps \( \Lambda \in \mathcal{CP}(\mathbb{C}^2 \otimes \mathcal{H}^\otimes_2, \mathcal{H}) \).

Proof of the uniqueness result from Theorem 2. Let \( |\nu\rangle = \alpha |0\rangle + \beta |1\rangle \) and let \( \mathbb{P}_\psi, \mathbb{P}_\chi \) be states on \( \mathcal{H} \) satisfying \( \text{tr}(\mathbb{P}_\psi \mathbb{P}_\chi) = c_1 \), \( \text{tr}(\mathbb{P}_\phi \mathbb{P}_\chi) = c_2 \). Let now \( \Lambda \in \mathcal{CP}(\mathbb{C}^2 \otimes \mathcal{H}^\otimes_2, \mathcal{H}) \) be the CP map satisfying

\[
\Lambda(\mathbb{P}_\nu \otimes \mathbb{P}_\psi \otimes \mathbb{P}_\phi) \propto |\Psi\rangle \langle \Psi | ,
\]

(S.16)

where

\[
|\Psi\rangle = \alpha \frac{|\chi\rangle \langle \phi|}{|\chi\rangle \langle \phi|} |\psi\rangle + \beta \frac{|\chi\rangle \langle \psi|}{|\chi\rangle \langle \psi|} |\phi\rangle
\]

(S.17)
is the superposition of states we want to generate. Let \( \{ V_i \}_{i \in I}, V_i : \mathbb{C}^2 \otimes \mathcal{H}^{\otimes 2} \to \mathcal{H} \), form the Kraus decomposition \([23]\) of \( \Lambda \). Using the analogous argumentation to the one presented in the proof of Theorem 1 we get that

\[
V_i \mathcal{P}_\nu \otimes \mathcal{P}_\psi \otimes \mathcal{P}_\phi V_i^\dagger \propto \ket{\Psi} \bra{\Psi}, \text{ for all } i \in I. \tag{S.18}
\]

Let us focus on a single Kraus operator \( V_i \). In what follows we will drop the index \( i \) for simplicity. From \([S.18]\) we get

\[
V_\nu \ket{\psi} \bra{\phi} = \mathcal{A}(\alpha, \beta, |\psi\rangle, |\phi\rangle) \cdot \left( \frac{\alpha}{\sqrt{|\langle \chi | \phi \rangle|}} \ket{\psi} + \beta \frac{\bra{\chi}}{\sqrt{|\langle \psi | \chi \rangle|}} \bra{\phi} \right), \tag{S.19}
\]

for all vectors \( |\psi\rangle, |\phi\rangle \in \mathcal{H} \) satisfying

\[
|\langle \chi | \psi \rangle| = \sqrt{c_1}, \quad |\langle \chi | \phi \rangle| = \sqrt{c_2}, \tag{S.20}
\]

arbitrary \( |\nu\rangle = \alpha|0\rangle + \beta|1\rangle \) and for \( \mathcal{A}(\alpha, \beta, |\psi\rangle, |\phi\rangle) \) being some unknown function. We will now show that condition \([S.19]\) defines \( V \) uniquely up to a multiplicative constant. Having this result we will be able infer the uniqueness of \( \Lambda \) (up to scaling). Using the linearity of the left hand side of \([S.19]\) in \( |\nu\rangle \) we get

\[
\mathcal{A}(\alpha, \beta, |\psi\rangle, |\phi\rangle) = \mathcal{A}(|\psi\rangle, |\phi\rangle). \tag{S.21}
\]

Moreover, from the linearity of \( V \) and the condition \([S.19]\) it follows that

\[
\mathcal{A}(|\psi\rangle, |\phi\rangle) = \mathcal{A}(\exp(i\theta_1) |\psi\rangle, \exp(i\theta_2) |\phi\rangle), \tag{S.22}
\]

where \( \theta_1, \theta_2 \) are arbitrary phases. Because of this property it suffices to check the condition \([S.19]\) for vectors of the form

\[
|\psi\rangle = \sqrt{c_1} |\chi\rangle + q_1 |\psi^+\rangle, \tag{S.23}
\]

\[
|\phi\rangle = \sqrt{c_2} |\chi\rangle + q_2 |\phi^+\rangle, \tag{S.24}
\]

where \( |\chi\rangle \) is some fixed vector representative of \( \mathcal{P}_{\chi} \), \( q_1 = \sqrt{1 - c_1} \), and vectors \( |\psi^+\rangle, |\phi^+\rangle \) are normalized and belong to \( \mathcal{H}_\perp^{\perp} \), the orthogonal complement of \( |\chi\rangle \) in \( \mathcal{H} \).

Using \([S.19]\) we obtain the following condition

\[
V_\nu \otimes (\sqrt{c_1 c_2} |\chi\rangle + \sqrt{c_1 q_2} |\chi\rangle |\phi^+\rangle + \sqrt{c_2 q_1} |\psi^+\rangle |\chi\rangle + q_1 q_2 |\psi^+\rangle |\phi^+\rangle) = \mathcal{A}(|\psi^+\rangle, |\phi^+\rangle) \cdot (\alpha \sqrt{c_1} |\chi\rangle + q_1 |\psi^+\rangle + q_2 |\phi^+\rangle), \tag{S.25}
\]

where \( \mathcal{A}(|\psi^+\rangle, |\phi^+\rangle) = \mathcal{A}(|\psi\rangle, |\phi\rangle) \) for \( |\psi\rangle, |\phi\rangle \) given by \([S.23]\) and \([S.24]\). For the fixed normalized vectors \( |\psi^+_0\rangle, |\phi^+_0\rangle \in \mathcal{H}_\perp^{\perp} \) the function

\[
(\theta_1, \theta_2) \to \mathcal{A}(\exp(i\theta_1) |\psi^+_0\rangle, \exp(i\theta_2) |\phi^+_0\rangle), \tag{S.26}
\]

is a smooth function on a torus \( S_1 \times S_1 \). It follows from the expression

\[
\mathcal{A}(\exp(i\theta_1) |\psi^+_0\rangle, \exp(i\theta_2) |\phi^+_0\rangle) = \frac{f(\theta_1, \theta_2)}{g(\theta_1, \theta_2)}, \tag{S.27}
\]

where \( f \) and \( g \) are smooth and \( g \neq 0 \). Equation \([S.27]\) follows from the definition of \( \mathcal{A} \) and equations \([S.18]\), \([S.21]\) and \([S.22]\). Now, by inserting \( |\psi^+\rangle = \exp(i\theta_1) |\psi^+_0\rangle \) and \( |\phi^+\rangle = \exp(i\theta_2) |\phi^+_0\rangle \) into \([S.25]\), we can view expressions appearing on both sides of equality \([S.25]\) as integrable vector-valued functions of the pair of angles \( (\theta_1, \theta_2) \). Using the linearity of \( V \) and comparing Fourier coefficients on both sides of \([S.25]\) we obtain that for all \( \theta_1, \theta_2 \in [0, 2\pi) \)

\[
\mathcal{A}(\exp(i\theta_1) |\psi^+_0\rangle, \exp(i\theta_2) |\phi^+_0\rangle) = \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle), \tag{S.28}
\]

Moreover, we get

\[
V_\nu \ket{\chi} \bra{\chi} = \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle) \frac{\alpha \sqrt{c_1} + \beta \sqrt{c_2}}{\sqrt{c_1 c_2}} |\chi\rangle, \tag{S.29}
\]

\[
V_\nu \ket{\psi^+} \bra{\phi^+} = 0, \tag{S.30}
\]

\[
V_\nu \ket{\psi^+_0} \bra{\chi} = \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle) \frac{\alpha}{\sqrt{c_2}} |\psi^+_0\rangle, \tag{S.31}
\]

\[
V_\nu \ket{\psi^+_0} \bra{\phi^+_0} = \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle) \frac{\beta}{\sqrt{c_1}} |\phi^+_0\rangle. \tag{S.32}
\]

Using \([S.28]\) and the fact that \([S.31]\) and \([S.32]\) must hold for all normalized \( |\psi^+_0\rangle, |\phi^+_0\rangle \in \mathcal{H}_\perp^{\perp} \), we get that \( \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle) = \mathcal{A} = const \). We complete the proof by noticing that for constant \( \mathcal{A}(|\psi^+_0\rangle, |\phi^+_0\rangle) \) the above conditions uniquely specify the action of a linear map \( V \) on every vector \( |\Phi\rangle \in \mathbb{C}^2 \otimes \mathcal{H} \).

A careful analysis of the above proof shows that the protocol also works for all input states satisfying \( \text{tr}(\mathcal{P}_\chi \mathcal{P}_\psi) = \lambda c_1, \text{tr}(\mathcal{P}_\chi \mathcal{P}_\phi) = \lambda c_2, \) where \( \lambda \in \left(0, \frac{1}{\max(c_1, c_2)}\right] \). The appropriate superpositions are then generated with probability \( P_{succ} = \lambda P_{succ} \). The set of possible inputs for which a given \( \Lambda_{succ} \) works is characterized by the condition \( \text{tr}(\mathcal{P}_\chi \mathcal{P}_\psi) = \text{tr}(\mathcal{P}_\chi \mathcal{P}_\phi) = c \). Combining this
with uniqueness result we get that it is not possible to probabilistically generate superpositions \(\text{(S.17)}\) for all input states having nonzero overlap with \(\mathbb{P}_\chi\).

We now present an explicit protocol that generates the superposition \(\text{(S.17)}\) with the higher probability of success than the one given in the proof of Theorem 2 but works for the fixed coefficients \(\alpha, \beta\) satisfying \(|\alpha|^2 + |\beta|^2 = 1\). Let \(\Lambda_{\text{sup}}(\rho) = W\rho W^\dagger\), for a linear mapping \(W: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}\) defined by \(W = W_2W_1\), where

\[
W_1 = \frac{\alpha}{\sqrt{c_1}} I \otimes I + \frac{\beta}{\sqrt{c_2}} S, \quad (\text{S.33})
\]

\[
W_2 = I \otimes \langle \chi |. \quad (\text{S.34})
\]

In the above \(S\) denotes the unitary operator that swaps between two copies of \(\mathcal{H}\), and \(|\chi\rangle\) is a vector representative of \(\mathbb{P}_\chi\). The action of \(V_2\) on simple tensors is given by

\[
I \otimes \langle \chi | (|x\rangle y) = |x\rangle \langle x| \chi , \quad (\text{S.35})
\]

for all \(|x\rangle, |y\rangle \in \mathcal{H}\). Explicit computation shows that for vectors \(|\psi\rangle, |\phi\rangle\) which are vector representatives of the input states \(\mathbb{P}_\psi, \mathbb{P}_\phi\) we have

\[
W|\psi\rangle|\phi\rangle = \alpha \frac{\langle \chi |\phi \rangle}{\langle \chi |\psi \rangle} |\psi\rangle + \beta \frac{\langle \chi |\psi \rangle}{\langle \chi |\psi \rangle} |\phi\rangle . \quad (\text{S.36})
\]

which shows that \(W\mathbb{P}_\psi \otimes \mathbb{P}_\phi W^\dagger \propto |\psi\rangle|\psi\rangle\). The map \(\Lambda_{\text{sup}}\) is not normalized i.e. it might happen that it increases the trace. Since \(\Lambda_{\text{sup}}\) can be expressed via a single Krauss operator, it is trace non-increasing if and only if \(\mathbb{P}_\phi\) operator \(W\) satisfies \(W^\dagger W \leq I \otimes I\). We have

\[
W^\dagger W = \frac{|\alpha|^2}{c_1} I \otimes |\chi\rangle\langle\chi| + \frac{|\beta|^2}{c_2} |\chi\rangle\langle\chi| \otimes I + \frac{\alpha\beta}{\sqrt{c_1c_2}} I \otimes |\chi\rangle\langle\chi| S + S \frac{\alpha\beta}{\sqrt{c_1c_2}} I \otimes |\chi\rangle\langle\chi| . \quad (\text{S.37})
\]

Explicit computation shows that the maximal eigenvalue of \(W^\dagger W\) is given by

\[
\lambda_{\text{max}}(W^\dagger W) = \max \left\{ \left| \frac{\alpha}{\sqrt{c_1}} + \frac{\beta}{\sqrt{c_2}} \right|^2, \frac{|\alpha|^2}{c_1} + \frac{|\beta|^2}{c_2} \right\} . \quad (\text{S.38})
\]

The largest possible \(s \in \mathbb{R}_+\) such that \(s:\Lambda_{\text{sup}}\) is trace non-increasing is \(s_{\text{max}} = \left[\lambda_{\text{max}}(W^\dagger W)\right]^{-1}\). The probability that \(s_{\text{max}}:\Lambda_{\text{sup}}\) will produce the superposition \(\mathbb{P}_\psi\) is given by

\[
\tilde{P}_{\text{succ}} = s_{\text{max}} \cdot \text{tr} (\Lambda_{\text{sup}}(\mathbb{P}_\psi \otimes \mathbb{P}_\phi)) = \frac{\lambda_{\text{max}}^2}{\lambda_{\text{max}}(W^\dagger W)} , \quad (\text{S.39})
\]

Comparing \(\text{(S.39)}\) with \(\text{(15)}\) and using \(\text{(S.38)}\) we see that \(\tilde{P}_{\text{succ}} \geq P_{\text{succ}}\) if an only if

\[
\frac{1}{c_1} + \frac{1}{c_2} \geq \max \left\{ \frac{|\alpha|^2 + |\beta|^2}{c_2} \right\} , \quad (\text{S.40})
\]

for \(c_1, c_2 \in (0, 1]\) and \(|\alpha|^2 + |\beta|^2 = 1\). Inequality \(\text{(S.40)}\) can be easily checked via elementary means. Just like in the case of \(\Lambda_{\text{sup}}\) it is possible to show that \(\Lambda_{\text{sup}}\) is defined uniquely up to scaling. Consequently, the probability of success \(\tilde{P}_{\text{succ}}\) given in \(\text{(S.39)}\) is the highest possible one, provided the coefficients \(\alpha, \beta\) are fixed.

**Part C: Constructive protocol for superposition of multiple states**

In this part we generalize the protocol presented in the proof of Theorem 2 to the case of multiple superpositions.

First, we generalize the formula \(\text{(6)}\). Assume that we are given a known pure state \(\mathbb{P}_\chi\) and \(d\) unknown pure states \(\mathbb{P}_{\psi_1}, \ldots, \mathbb{P}_{\psi_d}\) satisfying

\[
\text{tr} (\mathbb{P}_\chi \mathbb{P}_{\psi_i}) \neq 0 , \quad i = 1, \ldots, d , \quad (\text{S.41})
\]

We now introduce the mapping \((\mathbb{P}_{\psi_1}, \ldots, \mathbb{P}_{\psi_d}) \rightarrow |\psi\rangle|\psi\rangle\) that would associate to any sequence of such pure states their superposition. Let \(|\chi\rangle\) be a vector representative of \(\mathbb{P}_\chi\) and let \(|\psi_i\rangle\), be vector representatives of states \(\mathbb{P}_{\psi_i}\), \(i = 1, \ldots, d\). For a given sequence of complex coefficients \(\alpha_1, \ldots, \alpha_d\) (satisfying \(\sum_{i=1}^d |\alpha_i|^2 = 1\)) we set

\[
|\Psi_d\rangle = \sum_{i=1}^d \alpha_i \prod_{k \neq i} (|\psi_k\rangle|\psi_k\rangle) |\psi_i\rangle . \quad (\text{S.42})
\]

The above formula is a direct generalization of \(\text{(6)}\) and analogous arguments show that \(|\psi\rangle|\psi\rangle\) is a well-defined function of states \(\mathbb{P}_{\psi_1}, \ldots, \mathbb{P}_{\psi_d}\). This can be also seen from the explicit formula

\[
|\Psi_d\rangle = \sum_{i=1}^d |\alpha_i|^2 \mathbb{P}_{\psi_i} \prod_{k \neq i} \text{tr} (\mathbb{P}_\chi \mathbb{P}_{\psi_k}) \sqrt{\text{tr} (\mathbb{P}_\chi \mathbb{P}_{\psi_i})} , \quad (\text{S.43})
\]

where

\[
M_{ij} = \mathbb{P}_\chi \prod_{k \neq i, k \neq j} \mathbb{P}_{\psi_k} , \quad (\text{S.44})
\]

and it is understood that the product of operators indexed by the empty set is the identity operator.
Remark. In the context of geometric approaches to quantum mechanics, there appears the following superposition rule,

$$|\Psi_d\rangle = \sum_{i=1}^{d} \alpha_i \langle\psi_i|\rangle |\psi_i\rangle. \quad (S.45)$$

For \(d > 2\) the mappings

$$(\mathbb{F}_{\psi_1}, \ldots, \mathbb{F}_{\psi_d}) \rightarrow |\Psi_d\rangle, \quad (\mathbb{F}_{\psi_1}, \ldots, \mathbb{F}_{\psi_d}) \rightarrow |\Psi_d\rangle|\Psi_d\rangle.$$  

(S.46)

differ from each other. It would be interesting to explore if states of the geometric significance of the superposition rule used differ from each other. It would be interesting to explore if states of the form \(|\chi\rangle\) can be generated by processes allowed by quantum mechanics.

**Theorem 3.** Let \(\mathbb{F}_X\) be a fixed pure state on Hilbert space \(\mathcal{H}\) and let \(d > 1\) be a natural number. There exist a CP map \(\Lambda^{d}_{\text{sup}} \in \mathcal{CP} (\mathcal{C}^d \otimes \mathcal{H}^{\otimes d}, \mathcal{H})\) such that for all pure states \(\mathbb{F}_{\psi_i}, i = 1, \ldots, d,\) on \(\mathcal{H}\) satisfying

$$\text{tr} (\mathbb{F}_{\psi_i} \mathbb{F}_{\psi_i}) = c_i, \quad i = 1, \ldots, d,$$

we have

$$\Lambda^{d}_{\text{sup}} \left( \mathbb{F}_{\nu} \otimes \bigotimes_{i=1}^{d} \mathbb{F}_{\psi_i} \right) \propto |\Psi_d\rangle|\Psi_d\rangle, \quad \text{(S.48)}$$

where

$$\mathbb{F}_{\nu}, \nu = \sum_{i=1}^{d} \alpha_i |i\rangle,$$  

(S.49)

is an unknown qdit state and the vector \(|\Psi_d\rangle\) is given by \((S.42)\). Moreover, a CP map \(\Lambda^{d}_{\text{sup}}\) realising \((S.48)\) is unique up scaling.

**Proof.** We present an explicit protocol that realizes \((S.48)\), which is analogous to the one given in the proof of Theorem 2. Let us first define and auxiliary normalized qdit vector

$$|\mu_d\rangle = C_d \sum_{i=1}^{d} \frac{1}{\prod_{k \neq i} \sqrt{C_i}} |i\rangle,$$  

(S.50)

where \(C_d\) is a normalization constant. We set \(\Lambda^{d}_{\text{sup}} = \Lambda_4 \circ \Lambda_3 \circ \Lambda_2 \circ \Lambda_1\), where

$$\Lambda_1 (\rho) = V_1 \rho V_1^\dagger, \quad V_1 = \sum_{i=1}^{d} |i\rangle \otimes I_{1,i},$$

(S.51)

$$\Lambda_2 (\rho) = V_2 \rho V_2^\dagger, \quad V_2 = I_d \otimes I \otimes \left( |\chi\rangle \langle \chi| \right)^{\otimes d-1},$$

(S.52)

$$\Lambda_3 (\rho) = V_3 \rho V_3^\dagger, \quad V_3 = \mathbb{F}_{\mu_d} \otimes I^{\otimes d},$$

(S.53)

$$\Lambda_4 (\rho) = \text{tr}_{1,3,\ldots,d+1} (\rho).$$

(S.54)

In the above \(S_{1,i} : \mathcal{H}^{\otimes d} \rightarrow \mathcal{H}^{\otimes d}\) denotes the unitary operator that swaps between first and \(i\)th copy of the Hilbert space \(\mathcal{H}^{\otimes d}\), \(I_d\) and \(I\) are identity operators on \(\mathcal{C}^d\) and \(\mathcal{H}\) respectively, \(\text{tr}_{1,3,\ldots,d+1}(\rho)\) is the partial trace over all except for the second factor in the tensor product \(\mathcal{C}^d \otimes \mathcal{H} \otimes \mathcal{H}^{\otimes d-1}\). Operation \(\Lambda^{d}_{\text{sup}}\) is manifestly completely positive and trace non-increasing. Direct calculation shows that under the assumed conditions \((S.48)\) indeed holds. The proof of uniqueness of the map \(\Lambda^{d}_{\text{sup}}\) is analogous to the one given in the case \(d = 2\) (covered by Theorem 2).

**Part D: Creation of superpositions of results of subroutines of quantum computations run in parallel.**

Here we show how to implement the protocol coherently superposing results of subroutines of quantum computation in the standard quantum circuit formalism. Assume we want to superpose states \(\mathbb{F}_{\psi_1}, \mathbb{F}_{\psi_2}\) that correspond to application of some quantum circuits on \(N\) qubits (the Hilbert space of the system \(\mathcal{H} = (\mathbb{C}^2)^{\otimes N}\)),

$$\mathbb{F}_{\psi} = U|x\rangle|y\rangle U^\dagger, \quad \mathbb{F}_{\phi} = V|y\rangle|y\rangle V^\dagger,$$

(S.55)

where \(|x\rangle|y\rangle\) are classical states encoding the input to the computation,

$$|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_N\rangle, \quad |y\rangle = |y_1\rangle \otimes \cdots \otimes |y_N\rangle,$$

(S.55)

and \(U, V\) are unitary operators on \((\mathbb{C}^2)^{\otimes N}\). We introduce the auxiliary qubit that will allow us to encode the state \(\mathbb{F}_{\psi}\) without altering the computation (from now on we will consider the Hilbert space \(\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}\)). We set \(|\chi\rangle = |0\rangle^{\otimes N+1}\) and we introduce new initial states as projectors onto vectors

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}} (|\chi\rangle + |1\rangle|x\rangle),$$

(S.56)

$$|\tilde{y}\rangle = \frac{1}{\sqrt{2}} (|\chi\rangle + |1\rangle|y\rangle).$$

(S.56)

By applying the controlled versions of the unitaries \(U, V\),

$$\mathcal{C}(U) = |0\rangle|0\rangle \otimes I + |1\rangle|1\rangle \otimes U, \quad \mathcal{C}(V) = |1\rangle|1\rangle \otimes I + |1\rangle|1\rangle \otimes V,$$

we obtain the states represented by the vectors

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}} (|\chi\rangle + |1\rangle U|x\rangle),$$

$$|\tilde{\phi}\rangle = \frac{1}{\sqrt{2}} (|\chi\rangle + |1\rangle V|y\rangle).$$

(S.58)

These vectors are exactly of the form given by \((16)\) and thus we can apply to them protocol from the main text of the manuscript (keep in mind that we set \(|\chi\rangle = |0\rangle^{\otimes N+1}\) that with the probability

$$P_{\text{succ}} = \frac{1}{4} \left( 1 + \frac{1}{4} \|[U|x\rangle + V|y\rangle]\|^2 \right).$$

(S.59)
will produce a state having a vector representative

\[ |\Psi\rangle = \frac{1}{\sqrt{P_{\text{succ}}}} (|0\rangle^{\otimes N+1} + \frac{1}{2} |1\rangle \langle U|x\rangle + V|y\rangle) \quad (S.60) \]

Note that from the state (S.60) it is possible to extract (by postselecting with respect to obtaining the result "1" in the auxiliary qbit) the state encoding the superposition \( U|x\rangle + V|y\rangle \) in the computational register.

Let us discuss the method presented above. First of all, in order to be able to create the desired superpositions we need to encode input states in the extended space (see (S.56)) and use the controlled versions of the gates \( U, V \) (see (S.57)). The controlled versions of the unitary gate are defined up to a phase standing next to the unitary which is controlled \( U \) and therefore we can always add an additional phase in front of the second terms in the sums (S.58). However, this is not a problem as we can always decode (probabilistically) the original computation from states of the form (S.58). Another possible problem may come from the necessity of implementing the controlled versions of gates \( U, V \). This can be always done if one knows the classical description of these gates. In particular, assume that the \( N \) qbit gate \( U \) can be decomposed as a sequence of basic gates \( U_1, U_2, \ldots, U_k \).

\[ U = U_k \circ \ldots U_2 \circ U_1 \quad (S.61) \]

Then, a controlled version of this gate can be obtained by composing controlled versions of the basic gates,

\[ \mathcal{C}(U) = \mathcal{C}(U_k) \circ \ldots \mathcal{C}(U_2) \circ \mathcal{C}(U_1) \quad (S.62) \]

For the graphical illustration of the preparation of the state \( P_{\text{S}} \) see Figure 3.

To sum up, the protocol presented above creates superpositions of results of subroutines of quantum computations run in parallel with probability \( P_{\text{succ}} \geq \frac{1}{4} \). In order to implement the method one has to know what quantum subroutines are implemented. However, the inputs of the computations can be arbitrary (there are no constrains on the classical input states \( |x\rangle\langle x| \) and \( |y\rangle\langle y| \)).
FIG. 3. Computing in the subspace perpendicular to $|\chi\rangle$. (a) original circuit, including preparation of classical input $|x\rangle$ and the quantum algorithm producing $U|x\rangle$. (b) the new circuit uses previous gates controlled by ancillary qubit. The blue part of the circuit prepares $|\tilde{x}\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle^{\otimes N} + |1\rangle|x\rangle)$. The green part runs computing on $|x\rangle$ resulting in $|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle^{\otimes N} + |1\rangle U|x\rangle)$. The reference vector is $\chi = |0\rangle^{\otimes N+1}$. 