DESCRIPTIVE PROXIMAL HOMOTOPY.
PROPERTIES AND RELATIONS

J.F. PETERS AND T. VERGILI

Abstract. This paper introduces properties and relations for proximal homotopy as well as descriptive proximal homotopy. A number of results are given for the homotopy between Lodato-proximally continuous maps and for the homotopy between descriptive Lodato proximally continuous maps. This paper also introduces homotopic cycles in conjunction with paths in either proximity spaces in general or in descriptive proximity spaces. Three main results in this paper are (1) that the product of descriptive Lodato proximity (dlp) spaces is also a dlp space, (2) every descriptive dlp relation is an equivalence relation and (3) every homotopic cycle has a free group representation.

Contents
1. Introduction 1
2. Preliminaries 2
   2.1. Lodato Proximally Continuous Maps and Gluing Lemma 4
   2.2. Descriptive Lodato Proximity spaces 6
   2.3. Descriptive Lodato Proximally Continuous Maps and a dlpc Gluing Lemma 7
3. Proximal Homotopy 8
   3.1. Homotopy between L-proximally continuous maps 8
   3.2. Homotopy between descriptive Lodato proximally continuous maps 10
4. Paths in proximity and descriptive proximity spaces 11
   Appendix A. Free Group Representation of Homotopic Cycles 13
   Appendix B. Čech Proximity 14
   Appendix C. Čech-Lodato Proximity 14
   Appendix D. Descriptive Proximity 15
References 15

1. Introduction

This paper extends the homotopy of paths [18 §2.1,p.11] to paths in Čech proximity spaces [19 §2.5,p.439] and in descriptive proximity spaces [11]. A biproduct of this work is the introduction of proximal paths and constant proximal paths in Lodato proximity spaces [8] as well as descriptively close paths in descriptive

---

2010 Mathematics Subject Classification. 54E05 (proximity),55P57 (homotopy theory).
Key words and phrases. Cycle, Descriptive, Homotopy, Path, Proximally Continuous Map, Proximity.
Lodato proximity spaces \cite{1} with concomitant Lodato proximal (also, L-proximal) homotopy between paths and an extension of

**Theorem 1.** \cite{3} §1.1, p. 26] The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

and a main result in this paper, namely, Theorem \cite{13}

**Theorem 2.** Every descriptive L-proximal homotopy relation is an equivalence relation.

This paper also introduces homotopic cycles, leading to the result in Theorem \cite{15}

**Theorem 3.** Every homotopic cycle has free group representation.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{homotopy_classes.png}
\caption{Homotopy class $[h]$ \& homotopy class boundary and interior}
\end{figure}

2. Preliminaries

This section introduces notation and basic concepts for paths, homotopy of paths, homotopic paths, homotopy classes, proximal homotopy and descriptive proximal homotopy, starting with paths.

Let $I = [0, 1]$, the unit interval. A path in a space $X$ is a continuous map $h : I \to X$ with endpoints $h(0) = x_0$ and $h(1) = x_1$ \cite{18}, §2.1,p.11]. A homotopy of paths $h, h' : I \to X$ with fixed end points (denoted by $h \sim h'$), is a relation between $h$ and $h'$ defined by an associated continuous map $H : I \times I \to X$, where $H(s, t) = h_t(s)$ with $H(s, 0) = h(s)$ and $H(s, 1) = h'(s)$. In effect, in a homotopy of paths $h, h'$, path $h$ is continuously transformed into path $h'$. For $h \sim h'$, paths $h, h'$ are said to be homotopic paths.

**Theorem 4.** \cite{3} §1.1, p. 26] The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

*Proof.* For a detailed proof, see \cite{18} §0.6,p.3].
An equivalence class of a path \( h \) defined by the homotopy relation \( \sim \) (denoted by \([h]\)) is called the \textbf{homotopy class} of \( h \). In other words, a \textit{homotopy class} of \( h \) (i.e., paths equivalent to path \( h \)) is a collection \([h] = \{h_t : t \in [0,1] \text{ and } h \sim h_t\}\).

\textbf{Example 1.} A sample homotopy class \([h]\) of a path \( h \) is shown in Fig. 1.1.

\textbf{Remark 1.} The variable \( t \) can be viewed as time, i.e., the passage from \( t = 0 \) to \( t = 1 \) depends not only on the set of points in \( I \) that are passed through, but also on the clock time (also duration) in which each point is visited [11, VII.1, p. 175]. This observation becomes important, when we consider the descriptive proximity of paths in terms of the description of a path based on the rate of change of a path \( h \) with respect to time, denoted by \( \frac{\partial h}{\partial t} \), in the passage from \( h_0(s) \) to \( h_1(s) \) for \( t \) ranging from 0 to 1 over \( I \).

For a homotopy class \([h]\), the fixed points \( h(0), h(1) \) provide the \textit{class boundary} (denoted by \( \text{bdy}([h]) \)). The paths between the endpoints \( h(0), h(1) \) of \([h]\) constitute the \textit{class interior} (denoted by \( \text{int}([h]) \)).

\textbf{Example 2.} Sample homotopy class \([h]\) boundary and interior are shown in Fig. 1.2.

The fundamental parts of every homotopy class \([h]\) are gathered together in the closure of \([h]\), defined using the Hausdorff distance [4] (see, also, [5, §23, p. 128]) between all points in a space \( X \) and points in the closure of a homotopy class \([h]\) (denoted by \( \text{cl}([h]) \)).

\textbf{Definition 1.} Let \( h_t(s) \) be a point on a path in a homotopy class \([h] \subset X \) at time \( t \) and let \( p \) be a point in a space \( X \). The \textit{closure of a homotopy class} \( \text{cl}([h]) \) is defined by

\[
\text{D}(p, h_t(s)) = \inf \{ \|p - h_t(s)\| : h_t(s) \in \{\text{bdy}([h]) \cup \text{int}([h])\} \}. \\
\text{cl}([h]) = \{p \in X : \text{D}(p, h_t(s)) = 0\}.
\]

\textbf{Lemma 1.} Let \([h]\) be homotopy class on a space \( X \).

Then

1° \( \text{cl}([h]) = \text{bdy}([h]) \cup \text{int}([h]) \).

2° \( \text{bdy}([h]) = \text{cl}([h]) \setminus \text{int}([h]) \).

3° \( \text{int}([h]) = \text{cl}([h]) \setminus \text{bdy}([h]) \).

\textbf{Proof.} 1°: Immediate from Def. [1]

2°: From 1°, we obtain the boundary points of the closure of a homotopy class \( \text{cl}([h]) \) by subtracting all points in the interior \( \text{int}([h]) \) from \( \text{cl}([h]) \), i.e., \( \text{bdy}([h]) = \text{cl}([h]) \setminus \text{int}([h]) \).

3°: 1° and 2° imply 3°.

From the Čech proximity \( \delta \) in App. [3] we can consider the closeness of homotopy classes in a proximity space \((X, \delta)\).

\textbf{Theorem 5.} Let \([h], [k] \) be homotopy classes in Čech proximity space \((X, \delta)\). Then

1° \( \text{cl}([h]) \cap \text{cl}([k]) \neq \emptyset \Rightarrow \text{cl}([h]) \delta \text{cl}([k]) \).
Figure 2. Overlapping homotopy classes $[h], [k]$.

2° $\text{bdy}([h]) \cap \text{bdy}([h]) \neq \emptyset \Rightarrow \text{bdy}([h]) \delta \text{bdy}([h])$.

3° $\text{int}([h]) \cap \text{int}([h]) \neq \emptyset \Rightarrow \text{int}([h]) \delta \text{int}([h])$.

Proof. 1°: Replace $A, B$ with $\text{cl}([h]), \text{cl}([k])$ in the Axioms in App. [13] It is a straightforward task to show that each of the Čech axioms is satisfied. In particular, from Axiom (P.2), we obtain $\text{cl}([h]) \cap \text{cl}([k]) \neq \emptyset$ implies $\text{cl}([h]) \delta \text{cl}([k])$, i.e., $\text{cl}([h])$ is close to $\text{cl}([k])$.

2°,3°: From Lemma [1] we obtain the definitions of $\text{bdy}([h]), \text{int}([h])$ and $\text{bdy}([k]), \text{int}([k])$.

The proofs of 2°,3° are then symmetric with the proof of 1° after substituting the definitions of the boundaries and interiors of the homotopy classes $[h], [k]$ in 1°. □

Example 3. The homotopy classes $[h], [k]$ in Fig. 2 be in Čech proximity space $(X, \delta)$. It is easy to verify that pairs of nonvoid homotopy classes in this proximity space satisfy the Čech axioms in App. [13] In that case, closures $\text{cl}([h]), \text{cl}([k])$ with nonempty intersection imply $\text{cl}([h]) \delta \text{cl}([k])$ (from Axiom P.2), i.e.,

$\text{cl}([h]) \cap \text{cl}([k]) \neq \emptyset \Rightarrow \text{cl}([h]) \delta \text{cl}([k])$ from Thm 5.1°.

Remark 2. The closures of the homotopy classes $\text{cl}([h]), \text{cl}([k])$ are strongly close, provided $\text{int}([h]) \cap \text{int}([k]) \neq \emptyset$ (this occurs in space $X$ in Fig. 2 where the interiors of the homotopy classes overlap). In that case, we have the strong proximity of the homotopy classes, denoted by $\text{cl}([h]) \overset{\delta}{\sim} \text{cl}([k])$. In a topological space $X$ equipped with the strong proximity relation $\delta$ and with $\text{cl}([h]), \text{cl}([k]) \subset X$, the overlapping interiors of the homotopy classes in Fig. 2 satisfy Axiom N4 in [15, §2, pp. 1-2], namely,

(N4): $\text{int}([h]) \cap \text{int}([k]) \neq \emptyset \Rightarrow \text{cl}([h]) \overset{\delta}{\sim} \text{cl}([k])$.  □

2.1. Lodato Proximally Continuous Maps and Gluing Lemma.

This section introduces gluing lemma for proximity spaces, defined via proximally continuous maps over a pair of Čech-Lodato proximity spaces defined in terms of the proximity $\delta_L$ (see Appendix C). Therefore, throughout the paper we assume that all proximity spaces are Čech-Lodato proximity spaces, or Lodato proximity...
spaces, for short, unless otherwise stated.

**Definition 2.** Proximally continuous map \([16, p. 5], [2]\).

A map \(f : (X, \delta_1) \rightarrow (Y, \delta_2)\) between two proximity spaces is proximally continuous, provided \(f\) preserves proximity, i.e., \(A \delta_1 B\) implies \(f(A) \delta_2 f(B)\) for \(A, B \in 2^X\).

**Remark 3.** Proximally continuous maps were introduced by V.A. Efremović [2] and Yu. M. Smirnov [16, 17] in 1952 and elaborated by S.A. Naimpally and B.D. Warrack [10] in 1970. To distinguish between proximally continuous maps defined in terms of an Efremović proximity space and a proximally continuous map \(f\) on a Lodato proximity space \((X, \delta)\), we write L-proximally continuous.

The following theorem shows that the composition of two L-proximally continuous maps is L-proximally continuous but it is also true for any types of proximally continuous maps.

**Lemma 2.**
Composition of two L-proximally continuous maps is L-proximally continuous.

**Proof.** Let \(f : (X, \delta_1) \rightarrow (Y, \delta_2)\) and \(g : (Y, \delta_2) \rightarrow (Z, \delta_3)\) be L-proximally continuous maps and \(A \delta_1 B\) in \(X\). Then \(f(A) \delta_2 f(B)\) since \(f\) is L-proximally continuous and \(g \circ f(A) \delta_3 g \circ f(B)\), since \(g\) is L-proximally continuous. \(\square\)

**Theorem 6.** Composition of two proximally continuous maps is proximally continuous.

**Proof.** In terms of proximally continuous maps, the proof is symmetric with the proof of Lemma 2. \(\square\)

The following theorem is necessary for the proof of Theorem 11.

**Theorem 7** (Gluing Lemma for proximity spaces).

Suppose \((X, \delta_1)\) and \((Y, \delta_2)\) are Lodato proximity spaces and \(A\) and \(B\) are closed subsets of \(X\) such that \(A \cup B = X\). If \(f : (A, \delta_1) \rightarrow (Y, \delta_2)\) and \(g : (B, \delta_1) \rightarrow (Y, \delta_2)\) are L-proximally continuous maps such that \(f(x) = g(x)\) for all \(x \in A \cap B\), then the map \(h : (X, \delta_1) \rightarrow (Y, \delta_2)\) defined by

\[
h(x) = \begin{cases} f(x), & x \in A, \\ g(x), & x \in B \end{cases}
\]

is also L-proximally continuous.

**Proof.** Let \(C, D\) be subsets of \(X\) such that \(C \delta_1 D\) so that these two sets are near. That is, there exist \(c \in C\) and \(d \in D\) that are either equal \(c = d\) or near to each other \(\{c\} \delta_1 \{d\}\). If \(c = d\), then we are done.

Assume \(\{c\} \delta_1 \{d\}\). Note that \(c \in A (\in B)\) implies \(d \in A (\in B)\), since \(A (B)\) is closed. Therefore we have the following three cases.

**Case 1:** \(c, d \in A\).
In that case, we have \(h(\{c\}) = f(\{c\}) \delta_2 h(\{d\}) = f(\{d\})\) so that \(h(C) \delta_2 h(D)\).
Case 2: $c, d \in B$. 
In that case, we have $h(\{c\}) = g(\{c\}) \delta \_2 h(\{d\}) = g(\{d\})$ so that $h(C) \delta \_2 h(D)$.

Case 3: $c, d \in A \cap B$. 
In that case, we have $h(\{c\}) = f(\{c\}) = g(\{c\}) \delta \_2 h(\{d\}) = f(\{d\}) = g(\{d\})$ so that $h(C) \delta \_2 h(D)$.

In all cases, $h$ satisfies the L-proximal continuity property. □

![Figure 3. Descriptively close homotopy classes, where $[h] \sim f([h])$](image)

2.2. Descriptive Lodato Proximity spaces.

Let $(X, \delta_\Phi)$ be a descriptive Lodato proximity space. Then the descriptive closure of $A \subset X$ (denoted by $\text{cl}_\Phi A$) is the set of all points in $X$ descriptively near to $A$, i.e.,

$$\text{cl}_\Phi A = \{x \in X : x \delta_\Phi A\} = \{x \in X : \Phi(x) \in \Phi(A)\}.$$ 

Note that $A$ is descriptively closed, provided $\text{cl}_\Phi A = A$.

The following corollary is straightforward.

**Corollary 1.** Suppose $A$ is a descriptively closed subset of a descriptive Lodato proximity space $(X, \delta_\Phi)$. Then

$$x \in A \iff \Phi(x) \in \Phi(A).$$
Definition 3 (Descriptive intersection). [1]
The descriptive intersection $A \cap_{\Phi} B$ of two nonempty subsets $A$ and $B$ of a descriptive Lodato proximity space $(X, \delta_{\Phi})$, is the set of all points in $A \cup B$ such that $\Phi(A)$ and $\Phi(B)$ have common descriptions, i.e.
$$A \cap_{\Phi} B = \{ x \in A \cup B : \Phi(x) \in \Phi(A) \cap \Phi(B) \}.$$ 

It’s obvious that $A \cap B \subseteq A \cap_{\Phi} B$. However the converse may not be true in general.

Lemma 3. If $A$ and $B$ are descriptively closed subsets of a descriptive Lodato proximity space, then $A \cap_{\Phi} B = A \cap B$.

Proof. It’s enough to show that $A \cap_{\Phi} B \subseteq A \cap B$. Let $x \in A \cap_{\Phi} B$ so that $\Phi(x) \in \Phi(A)$ and $\Phi(x) \in \Phi(B)$, which implies $x \in \text{cl}_{\Phi} A = A$ and $x \in \text{cl}_{\Phi} B = B$. □

2.3. Descriptive Lodato Proximally Continuous Maps and a dlpc Gluing Lemma.

This section introduces an advanced form of the relation between descriptive Lodato proximity spaces introduced in 2016 in [12, §4.15.2, p. 155]. An important result given here is an extension of the gluing lemma (Theorem 7) given in Theorem 9 in terms of a pair of dlpc maps over a pair of descriptive Lodato spaces. Therefore, throughout this paper we assume that all descriptive proximity spaces are descriptive Lodato proximity spaces, unless otherwise stated.

Definition 4 (Descriptive proximally continuous maps).
A map $f : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is descriptive Lodato proximally continuous (dlpc), provided $A \delta_{\Phi_1} B$ implies $f(A) \delta_{\Phi_2} f(B)$ for $A, B \subset X$. ■

Example 4. Let the homotopy class $[h]$ be represented in Fig. 3.1 and let the paths in this class be described in terms of the rate of change $h \in [h]$ with respect to time $t$ (denoted by $\frac{\partial h}{\partial t}$). Let the map $f : (X, \delta_{\Phi}) \to (X, \delta_{\Phi})$ be dlpc with the result $f([h])$ of the mapping as shown in Fig. 3.2. Then we have
$$\Phi(h \in [h]) = \frac{\partial h}{\partial t}, \text{ and}$$
$$\Phi(k \in f([h])) = \frac{\partial k}{\partial t}, \text{ and}$$
$$\Phi(h \in [h]) = \Phi(k \in f([h])) \text{ for each blue path } k \in f([h]). \text{ Hence,}$$
$$[h] \cap_{\Phi} f([h]) \neq \emptyset \Rightarrow [h] \\text{ or } [h] = f([h]) \text{ from Def. 4}.$$ ■

Definition 5. Each path $k$ in a homotopy class $[h]$ has a rate of change with respect to time (denoted by $\frac{\partial k}{\partial t}$). This rate of change of a path $k$ in $[h]$ is a temporal signature of the path at each instant in time.

Lemma 4. Every path in a homotopy class has its own temporal signature.

Proof. Let $k$ be a path in a homotopy class $[h]$. From Def. 5 each $k$ has its own temporal signature, namely, $\frac{\partial k}{\partial t}$. □
Theorem 8. Composition of two dlpc maps is dlpc.

Proof. Let \( f : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2}) \) and \( g : (Y, \delta_{\Phi_2}) \to (Z, \delta_{\Phi_3}) \) be dlpc maps and \( A \delta_{\Phi_1} B \) in \( X \). Then \( f(A) \delta_{\Phi_2} f(B) \) since \( f \) dlpc and \( g \circ f(A) \delta_{\Phi_3} g \circ f(B) \) since \( g \) is dlpc. \( \square \)

We adapt the gluing lemma (Theorem 7) for descriptive Lodato proximally continuous maps.

Theorem 9. [Descriptive Gluing Lemma]

Let \( (X, \delta_{\Phi_1}) \) and \( (Y, \delta_{\Phi_2}) \) be two descriptive Lodato proximity spaces and let \( A \) and \( B \) be two descriptively closed subsets of \( X \) with \( A \cup B = X \). If \( f : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2}) \) and \( g : (B, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2}) \) are dlpc maps such that \( f(x) = g(x) \) for all \( x \in A \cap B \), then the map \( h : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2}) \) is defined by

\[
h(x) = \begin{cases} f(x), & \Phi_1(x) \in \Phi_1(A) \quad (\equiv x \in A \text{ by Corollary } \mathbb{I}), \\ g(x), & \Phi_1(x) \in \Phi_1(B) \quad (\equiv x \in B \text{ by Corollary } \mathbb{I}) \end{cases}
\]

is also dlpc.

Proof. Let \( C, D \) be subsets of \( X \) such that \( C \delta_{\Phi_1} D \) (so, these two sets are descriptively near). That is, there exist \( c \in C \) and \( d \in D \) that are either equal \( c = d \) or descriptively near to each other \( \{c\} \delta_{\Phi_1} \{d\} \). If \( c = d \), then we are done. Assume \( \{c\} \delta_{\Phi_1} \{d\} \). Note that \( c \in A \ (\subseteq B) \) implies \( d \in A \ (\subseteq B) \) since \( A \ (B) \) is descriptively closed. Therefore we have the following three cases.

Case 1: \( c, d \in A \).
In that case, we have \( h(\{c\}) = f(\{c\}) \delta_{\Phi_2} h(\{d\}) = f(\{d\}) \) so that \( h(C) \delta_{\Phi_2} h(D) \).

Case 2: \( c, d \in B \).
In that case, we have \( h(\{c\}) = g(\{c\}) \delta_{\Phi_2} h(\{d\}) = g(\{d\}) \) so that \( h(C) \delta_{\Phi_2} h(D) \).

Case 3: \( c, d \in A \cap B \).
In that case, we have \( h(\{c\}) = f(\{c\}) = g(\{c\}) \delta_{\Phi_2} h(\{d\}) = g(\{d\}) = g(\{c\}) = f(\{c\}) \delta_{\Phi_2} h(\{d\}) \) so that \( h(C) \delta_{\Phi_2} h(D) \).

In all cases, \( h \) satisfies the descriptive Lodato proximal continuity property. \( \square \)

3. Proximal Homotopy

3.1. Homotopy between L-proximally continuous maps.

The following theorem is also valid for all types of proximity spaces. For our purpose, we only consider the case of Lodato proximity spaces.

Theorem 10. The product of Lodato proximity spaces is a Lodato proximity space.

For two Lodato proximity spaces \( (X, \delta_1) \) and \( (Y, \delta_2) \), let \( X \times Y \) denote their product. Then the subsets \( A \times B \) and \( C \times D \) of \( X \times Y \) are near, provided \( A \delta_1 C \) and \( B \delta_2 D \).
**Definition 6 (L-proximal Homotopy).** Let \((X, \delta_1)\) and \((Y, \delta_2)\) be Lodato proximity spaces and \(f, g : (X, \delta_1) \to (Y, \delta_2)\) L-proximally continuous maps. Then we say \(f\) and \(g\) are L-proximally homotopic, provided there exists an L-proximally continuous map \(H : X \times [0, 1] \to Y\) such that \(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\). Such a map \(H\) is called an L-proximal homotopy between \(f\) and \(g\). In keeping with Hilton’s notation [6], we write \(f \sim g\), provided there is a proximal homotopy between them.

**Theorem 11.** Every L-proximal homotopy relation is an equivalence relation.

*Proof.* It’s easy to check that \(\sim\) is reflexive and symmetric.

Now let \(F\) and \(G\) be L-proximal homotopies between \(f\) and \(g\) and between \(g\) and \(h\), respectively. Then the function \(H : X \times [0, 1] \to Y\) defined by

\[
H(x) = \begin{cases} 
F(x, 2t), & t \in [0, \frac{1}{2}] \\
G(x, 2t - 1), & t \in [\frac{1}{2}, 1]
\end{cases}
\]

is L-proximally continuous by Theorem 7, so that this defines an L-proximal homotopy between \(f\) and \(h\).

**Definition 7 (Relative L-proximal Homotopy).**

Let \((X, \delta_1)\) and \((Y, \delta_2)\) be Lodato proximity spaces and \(A \subset X\). Then two L-proximally continuous maps \(f, g : (X, \delta_1) \to (Y, \delta_2)\) are said to be L-proximally homotopic relative to \(A\), provided there exists an L-proximal homotopy \(H\) between \(f\) and \(g\) such that \(H(a, t) = f(a) = g(a)\) for all \(a \in A\) and \(t \in [0, 1]\). We write \(f \sim g\) (rel \(A\)), provided there is an L-proximal homotopy relative to \(A\). ■

**Proposition 1.** Suppose \(f, g : (X, \delta_1) \to (Y, \delta_2)\) are L-proximally homotopic. If \(h : (Y, \delta_2) \to (Z, \delta_3)\) is L-proximally continuous, then the maps \(h \circ f\) and \(h \circ g\) are also L-proximally homotopic.

*Proof.* Let \(F : X \times [0, 1] \to Y\) be the L-proximal homotopy between \(f\) and \(g\) so that \(F(x, 0) = f(x)\) and \(F(x, 1) = g(x)\). Note that \(h \circ f\) and \(h \circ g\) are L-proximally continuous by Lemma 2 and the map \(H : X \times [0, 1] \to Y\) defined by \(H(x, t) = h \circ F(x, t)\) is the desired L-proximal homotopy between them. ■

**Proposition 2.** Suppose \(f, g : (X, \delta_1) \to (Y, \delta_2)\) are L-proximally homotopic. If \(k : (W, \delta_0) \to (X, \delta_1)\) is L-proximally continuous, then the maps \(f \circ k\) and \(g \circ k\) are also L-proximally homotopic.

*Proof.* Let \(F : X \times [0, 1] \to Y\) be the L-proximal homotopy between \(f\) and \(g\) so that \(F(x, 0) = f(x)\) and \(F(x, 1) = g(x)\). Note that \(f \circ k\) and \(g \circ k\) are L-proximally continuous by Lemma 2 and the map \(K : Z \times [0, 1] \to Y\) defined by \(K(z, t) = F(k(z), t)\) is the desired L-proximal homotopy between them. ■

**Definition 8.** An L-proximally continuous map is L-proximally nullhomotopic, provided it is L-proximally homotopic to a constant map. ■
**Definition 9.** An L-proximity space is L-proximally contractible, provided the identity map on it is L-proximally homotopic to a constant map. □

**Definition 10.** Two Lodato proximity spaces \((X, \delta_1)\) and \((Y, \delta_2)\) are L-proximally homotopy equivalent, provided there exist L-proximally continuous maps \(f : (X, \delta_1) \to (Y, \delta_2)\) and \(g : (Y, \delta_2) \to (X, \delta_1)\) such that \(g \circ f\) and \(f \circ g\) are L-proximally homotopic to the identity maps on \(X\) and \(Y\), respectively.

### 3.2. Homotopy between descriptive Lodato proximally continuous maps.

This section extends Lodato proximity space (lps) result for the product of a pair of lps's in terms of the product of descriptive Lodato proximity spaces (dlp) which is also a dlp space (see Theorem 12), which can also be extended to all types of descriptive proximity spaces.

**Theorem 12.** The product of dlp spaces is a dlp space.

**Proof.** Let \(\{(X_i, \delta_{\Phi_i})\}_{i \in J}\) be a family of dlp spaces, where \(J\) is an index set. Then we can define a descriptive nearness relation \(\delta_{\Phi}\) on the product space \(X := \prod_{i \in J} X_i\) with the probe function \(\Phi := \prod_{i \in J} \Phi_i\) by declaring that two subsets \(A, B\) of \(X\) are descriptively near, provided \(A \delta_{\Phi} B\) if and only if \(\text{pr}_i(A) \delta_{\Phi_i} \text{pr}_i(B)\) for all \(i \in J\), where \(\text{pr}_i\) is the \(i\)th projection map of \(X\) onto \(X_i\). □

**Remark 5.** To define the descriptive homotopy between dlpc maps, we impose a descriptive nearness relation on the closed interval \([0, 1]\) in the following manner. Two subsets \(A\) and \(B\) of \([0, 1]\) are descriptively near, provided \(D(A, B) = 0\) (that is, the descriptive proximity relation and the (metric) proximity relation coincide).

The descriptive nearness relation introduced in Remark 5 leads to descriptive homotopic maps.

**Definition 11 (Descriptive L-proximal Homotopy).**

Let \((X, \delta_{\Phi_1})\) and \((Y, \delta_{\Phi_2})\) be dlp spaces and \(f, g : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})\) dlpc maps. Then we say \(f\) and \(g\) are descriptive L-proximally homotopic, provided there exists a dlpc map \(H : X \times [0, 1] \to Y\) such that \(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\). Such a map \(H\) is called a descriptive L-proximal homotopy between \(f\) and \(g\). We denote \(f \sim_{\Phi} g\), provided there exists a descriptive L-proximal homotopy between them. □

**Theorem 13.** Every descriptive L-proximal homotopy relation is an equivalence relation.

**Proof.** It’s easy to check that \(\sim\) is reflexive and symmetric.

Let \(F\) and \(G\) are the descriptive L-proximal homotopies between \(f\) and \(g\) and between \(g\) and \(h\), respectively. Then the function \(H : X \times [0, 1] \to Y\) defined by

\[
H(x) = \begin{cases} 
F(x, 2t), & t \in [0, \frac{1}{2}], \\
G(x, 2t - 1), & t \in [\frac{1}{2}, 1]
\end{cases}
\]
DESCRIPTIVE PROXIMAL HOMOTOPY

is dlpc by Theorem [9] so that this defines a descriptive L-proximal homotopy between \( f \) and \( h \).

\[ \square \]

**Definition 12** (Descriptive L-proximal Relative Homotopy).

Let \((X, \delta_{\Phi_1})\) and \((Y, \delta_{\Phi_2})\) be dlpc spaces and \( A \subset X \). Then two dlpc maps \( f, g : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2}) \) are said to be descriptive L-proximally homotopic relative to \( A \), provided there exists a descriptive L-proximal homotopy \( H \) between \( f \) and \( g \) such that \( H(a, t) = f(a) = g(a) \) for all \( a \in A \) and \( t \in [0, 1] \). We write \( f \sim_{\Phi} g \) (rel \( A \)), provided there is a descriptive L-proximal homotopy relative to \( A \).

\[ \square \]

4. Paths in proximity and descriptive proximity spaces

By Remark [5], we know that the descriptive nearness relation also induces a descriptive proximity relation on \([0, 1]\). This leads to the introduction of (descriptive) proximal paths in a (descriptive) proximity space.

**Definition 13** (Proximal Path).

Let \((X, \delta)\) be a Lodato proximity space and \( x_0, x_1 \in X \). Then a **proximal path** between \( x_0 \) and \( x_1 \) is an L-proximally continuous map \( \alpha : [0, 1] \to X \) such that \( \alpha(0) = x_0 \) and \( \alpha(1) = x_1 \), i.e., for two subsets of \( A, B \) in \([0, 1]\), \( D(A, B) = 0 \) implies \( \alpha(A) \delta \alpha(B) \).

In this section, we introduce constant proximal paths and their descriptive forms.

**Definition 14** (Constant Proximal Path).

For a Lodato proximity space \((X, \delta)\), the constant proximal path \( c : [0, 1] \to X \) at \( x_0 \in X \) is the proximal path such that \( c(t) = x_0 \) for every \( t \in [0, 1] \).

**Definition 15.** [Descriptive proximal path]

Let \((X, \delta_{\Phi})\) be a Lodato descriptive proximity space and \( x_0, x_1 \in X \). Then a descriptive proximal path between \( x_0 \) and \( x_1 \) is a dlpc map \( \alpha : [0, 1] \to X \) such that \( \alpha(0) = x_0 \) and \( \alpha(1) = x_1 \), i.e., for two subsets of \( A, B \) in \([0, 1]\), \( D(A, B) = 0 \) implies \( \alpha(A) \delta_{\Phi} \alpha(B) \) or equivalently, \( \alpha(A) \cap \alpha(B) \neq \emptyset \) (from Axiom (DP.2) and Lemma [2] in App. [7]).

Descriptive Lodato proximally continuous maps were informally introduced in [13], defined here in terms of path descriptions, utilizing the descriptive proximity relation \( \delta_{\Phi} \) (see App. [7]).

**Definition 16** (Path description).

Let \( h, k \) be L-proximally homotopic paths in a Lodato proximity space \( X \).

**the set of feature vectors that describe path** \( h \)

\[
\Phi(h) = \left\{ \Phi(h(s)) : s \in [0, 1] \right\} \subseteq \mathbb{R}^n
\]

**descriptively close paths**

\[
\Phi(h) = \Phi(k) \iff h \delta_{\Phi} k.
\]
Similarly, for descriptively close homotopy classes \([h], [k]\), we write

**descriptively families of paths**

\[ \Phi([h]) = \Phi([k]) \iff [h] \delta_{\Phi} [k]. \]

In other words, the closeness of descriptions of paths (and families of paths) is expressed using the descriptive proximity relation \(\delta_{\Phi}\).

**Definition 17.** Let \([h], [k]\) be nonempty families of paths in a Lodato proximity space \(X\). A map \(f: (2^X \times I, \delta_\Phi) \to (2^X \times I, \delta_\Phi)\) is descriptive proximally continuous (dpc), provided

\[ [h] \delta_{\Phi} [k] \text{ implies } f([h]) \delta_{\Phi} f([k]). \]

Unlike the constant proximal path, descriptive proximal paths (from Def.15) fall into two niches, namely, **(ordinary descriptive) constant paths** and **degenerate descriptive constant paths**, introduced in this section. These proximal paths lead to the construction of (descriptive) proximal free groups of a proximity space.

**Definition 18.** A finite group \(G\) is free, provided every element \(x \in G\) is a linear combination of its basis elements (called generators). We write \(\mathcal{B}\) to denote a nonempty set of generators \(\{g_1, \ldots, g_{|\mathcal{B}|}\}\) and \(G(\mathcal{B}, +)\) to denote the free group with binary operation \(+\).

**Example 5.** A homotopic cycle \(E\) (denoted by cyc\(E\)) is defined to be \(\{h_i\}_{i=1}^n\), a set of \(n\) paths in a proximity space \(X\), where \(h_i(0) = h_n(1)\) and the initial point of \(h_{i+1}\) is the terminal point of \(h_i\) for \(2 \leq i \leq n-1\), i.e., \(h_i(0) = h_{i-1}(1)\). In cyc\(E\), there is a starting point \(h_1(0)\), which is also the ending point of the cycle. Then \(h_1(0)\) plays the role of a generator of a free group \(G(\{g\}, +)\) in which there is a sequence of \(k\) maps (denoted by \(kg\)) from vertex \(h_1(0)\) to any other cycle vertex \(h_{k+1 \mod n}(0)\).

A zero move serves as the identity element in \(G\). Every path \(h\) can be written as a summation of \(k\) moves \(+\) from \(g\) to \(h(1)\). Zero moves from \(g\) is denoted by \(0g\).

Every \(kg\) in \(G\) has an inverse, namely, \(-kg = -kh(0)\) so that \(kg - kg = 0g\). A zero move serves as the identity element in \(G\). A reverse path \(h(t) = h(t - 1)\) serves as an inverse in \(G\), so that \(h(t) + h(t - 1) = h(t) - h(t) = 0\). In effect, every homotopic cycle cyc\(E\) has free group representation. For the details, see Appendix [A].

**Definition 19. Degenerate Descriptive Constant Map.**

Let \((X, \delta_{\Phi_1})\) and \((Y, \delta_{\Phi_2})\) be descriptive proximity spaces. Then a map \(d: X \to Y\) is said to be a _degenerate descriptive constant_, provided \(\Phi_2(d(x_0)) = \Phi_2(d(x_1))\) for all \(x_0, x_1 \in X\).

It is clear from the definition of the degenerate descriptive constant map that the degenerate descriptive constant map need not map every element to a fixed element, but instead it fixes the description. That is \(|\text{im } d| \geq 1\) but \(|\Phi_2(d(X))| = 1\). We say that \(d\) is ordinary descriptive constant map, provided \(|\text{im } d| = 1\).

**Theorem 14.** \(d\) is dlp\(c\).

**Proof.** For two subsets \(A\) and \(B\) of \(X\), suppose that \(A \delta_{\Phi_1} B\). By the definition of a degenerate descriptive constant map, we have \(\Phi_2(d(A)) = \Phi_2(d(B))\) so that \(d(A) \delta_{\Phi_2} d(B)\) which completes the proof. \(\square\)
Conjecture 1. We can always find paths in distinct homotopy classes that are descriptively proximal.

Remark 6. From Lemma 4, each path has its own temporal signature. Let \( \partial \frac{h}{t} \) and \( \partial \frac{h'}{t} \) be temporal signatures in the descriptions \( \Phi(k), \Phi(k') \) for paths \( k, k' \) in distinct homotopy classes \( [k], [k'] \), respectively. Assuming that \( k \cap k' \neq \emptyset \), we can then conclude \( k \delta_k k' \). This does not prove Conj. 1, since we have not guaranteed that we can find distinct paths with the same temporal signatures.

![Figure 4. Homotopic cycle \( cycE = \{h_i\}_{i=1}^n \).](image)

APPENDIX A. FREE GROUP REPRESENTATION OF HOMOTOPIE CYCLES

A homotopic cycle \( E \) (denoted by \( cycE \)) is defined to be \( \{h_i\}_{i=1}^n \), a set of \( n \) paths in a proximity space \( X \), where \( h_i(0) = h_n(1) \) and the initial point of \( h_{i+1} \) is the terminal point of \( h_i \) for \( 2 \leq i \leq n-1 \), i.e., \( h_i(0) = h_{i(1)}(1) \) as shown in Fig. 4.

Each path is a mapping \( h : [0, 1] \rightarrow X \) and \( h_i(0) \) is a vertex in a finite set of cycle vertices. A reverse path \( \bar{h}(t) := h(t-1) \) gives us an inverse map, so that

\[
 h_i(0) - \bar{h}_i(1) = h_i(0) - h_i(0) = 0.
\]

In cycle \( cycE \), every vertex \( v_i \) is reachable by \( k \) maps from a distinguished vertex \( h_1(0) = v_0 \), i.e.,

\[
 kv_0 := h_1(0) + \cdots + h_{k+1}(0)
\]

**i.e., \( k \) maps to reach \( h_{k+1}(0) \) from \( h_1(0) \)**

\[
 := h_1 \rightarrow \cdots \rightarrow h_{k+1}.
\]

Here, + represents a move from one vertex to another one in the cycle, which translates to a homotopic path between vertices.

A 0 move in a homotopic cycle serves as an identity element in a free group representation of the cycle. We write

\[
 0v_i(0) = 0 + v_i(0) = v_i(0).
\]

**Definition 20.** A free group \( G (\{g_1, \ldots\}, +) \) is a collection of paths with a binary operation + such that every path \( h_i \) is a linear combination of the generators \( kg \),
which are distinguished paths in $G$, i.e., starting at a vertex $h_1(0) = v_0$, a vertex $h_i(0)$ can be reached by a sequence of maps from $h_1(0)$ to $h_i(0)$.

\textbf{Theorem 15.} Every homotopic cycle has free group representation.

\textit{Proof.} Immediate from the definition of a homotopic cycle. \hfill \square

\textbf{Appendix B. Čech Proximity}

A nonempty set $X$ equipped with the relation $\delta$ is a Čech proximity space (denoted by $(X, \delta)$) [19 §2.5, p 439], provided the following axioms are satisfied.

\textbf{Čech Axioms}

(P.0): All nonempty subsets in $X$ are far from the empty set, i.e., $A \delta \emptyset$ for all $A \subseteq X$.

(P.1): $A \delta B \Rightarrow B \delta A$.

(P.2): $A \cap B \neq \emptyset \Rightarrow A \delta B$.

(P.3): $A \delta (B \cup C) \Rightarrow A \delta B$ or $A \delta C$.

\textbf{Appendix C. Čech-Lodato Proximity}

A Čech proximity space $(X, \delta_L)$ is a Čech-Lodato proximity space, provided $\delta$ satisfies the Lodato proximity axiom [8].

(P.5): $A \delta_L B$ and $\{b\} \delta_L C$ for each $b \in B \Rightarrow A \delta_L C$.

The closure of a subset $A$, denoted by $\text{cl}A$, of the proximity space $X$ is the set of all points in $X$ which are near $A$:

$$\text{cl}A = \{x \in X : x \delta_L A\}.$$ 

Note that $A$ is closed, provided $\text{cl}A = A$.

\textbf{Lemma 5.} [16, p. 9] The closure of any nonempty set $E$ in a proximity space $X$ is the set of all points which are close to $E$.

We define a nearness relation on $\mathbb{R}$ as follows [9 §1.7, p. 48]. Two nonempty subsets $A$ and $B$ of $\mathbb{R}$ are near if and only if the Hausdorff distance $D(A, B) = 0$, where

$$D(A, B) = \begin{cases} \inf\{|a - b| : a \in A \text{ and } b \in B\}, & \text{if } A, B \neq \emptyset, \\ \infty, & \text{if } A = \emptyset \text{ or } B = \emptyset. \end{cases}$$

Note that $\mathbb{R}$ is symmetric (or weakly regular), since $\mathbb{R}$ satisfies the following condition [9 §3.1, p. 71].

\((*)\) $x$ is near $\{y\} \Rightarrow y$ is near $\{x\}$.

In that case, this nearness relation defines a Lodato proximity $\delta_L$ on $\mathbb{R}$ by [9 §3, Theorem 3.1] $A \delta_L B \Leftrightarrow \text{cl}A \cap \text{cl}B \neq \emptyset$, where $\text{cl}E = \{x \in \mathbb{R} : D(x, E) = 0\}$. 
The topological space $X$ satisfying (\(\star\)) becomes a Čech-Lodato proximity space $(X, \delta_L)$ where $\delta_L$ is defined by

\[
A \delta_L B \iff \text{cl}A \cap \text{cl}B \neq \emptyset,
\]

and $\text{cl}E$ is the closure of $E \subset X$ with respect to the topology on $X$.

We assume that the proximity on the closed interval $[0, 1]$ is the subspace proximity \[\text{[9, \S3.1, p. 74]}\] induced by the (metric) proximity on $\mathbb{R}$.

**Appendix D. Descriptive Proximity**

This section gives the axioms for a descriptive proximity space $(X, \delta_{\Phi})$ in which $\delta_{\Phi}$ is a descriptive proximity relation on a nonempty set $X$. Nonempty sets $A, B \subset X$ with overlapping descriptions are descriptively proximal (denoted by $A \delta_{\Phi} B$).

The descriptive intersection $\Phi$ of nonempty subsets in $A \cup B$ (denoted by $A \cap_{\Phi} B$) is defined by

\[
i.e.,\ \text{Descriptions } \Phi(A) \& \Phi(B) \text{ overlap}
\]

\[
A \cap_{\Phi} B = \{ x \in A \cup B : \Phi(x) \in \Phi(A) \cap \Phi(B) \}.
\]

Let $2^X$ denote the collection of all subsets in a nonvoid set $X$. A nonempty set $X$ equipped with the relation $\delta_{\Phi}$ with non-void subsets $A, B, C \in 2^X$ is a descriptive proximity space, provided the following descriptive forms of the Čech axioms are satisfied.

**Descriptive Lodato Proximity Axioms** \[1\ pp. 97-98\].

- **(dP.0)**: All nonempty subsets in $2^X$ are descriptively far from the empty set, i.e., $A \delta_{\Phi} \emptyset$ for all $A \in 2^X$.
- **(dP.1)**: $A \delta_{\Phi} B \Rightarrow B \delta_{\Phi} A$.
- **(dP.2)**: $A \cap_{\Phi} B \neq \emptyset \Rightarrow A \delta_{\Phi} B$.
- **(dP.3)**: $A \delta_{\Phi} (B \cup C) \Rightarrow A \delta_{\Phi} B$ or $A \delta_{\Phi} C$.
- **(dP.4)**: $A \delta_{\Phi} B$ and $\{ b \} \delta_{\Phi} C \forall b \in B \Rightarrow A \delta_{\Phi} C$.

**Lemma 6.** \[14\] Let $X$ be equipped with the relation $\delta_{\Phi}$, $A, B \in 2^X$. Then $A \delta_{\Phi} B$ implies $A \cap_{\Phi} B \neq \emptyset$.

**References**

1. A. Di Concilio, C. Guadagni, J.F. Peters, and S. Ramanna, *Descriptive proximities, properties and interplay between classical proximities and overlap*, Math. Comput. Sci. 12 (2018), no. 1, 91–106, MR3767897, Zbl 06972895.
2. V.A. Efremovič, *The geometry of proximity I (in Russian)*, Mat. Sb. (N.S.) 31(73) (1952), no. 1, 189–200.
3. A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, UK, 2002, xii+544 pp. ISBN: 0-521-79160-X, MR1867354.
4. F. Hausdorff, *Grundzüge der mengenlehre*, Veit and Company, Leipzig, 1914, viii + 476 pp.
5. , *Set theory, trans. by j.r. aumann*, AMS Chelsea Publishing, Providence, RI, 1957, 352 pp.
6. P.J. Hilton, *An introduction to homotopy theory*, Cambridge tracts in mathematics and mathematical physics, no. 43, Cambridge University Press, Cambridge, U.K., 1953, viii+142 pp., MR0056289.
7. S. Leader, *On products of proximity spaces*, Mathematische Annalen 154 (1964), 185–194, MR0162221.
8. M.W. Lodato, *On topologically induced generalized proximity relations*, Ph.D. thesis, Rutgers University, Department of Mathematics, 1962, supervisor: S. Leader.

9. S.A. Naimpally and J.F. Peters, *Topology with applications. Topological spaces via near and far*, World Scientific, Singapore, 2013, xv + 277 pp, Amer. Math. Soc. MR3075111.

10. S.A. Naimpally and B.D. Warrack, *Proximity spaces*, Cambridge Tract in Mathematics No. 59, Cambridge University Press, Cambridge, UK, 1970, x+128 pp., Paperback (2008),MR0278261.

11. M.H.A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, Cambridge, U.K., 1953, 2nd ed., vii+214 pp., MR0044820, reviewer A.D. Wallace.

12. J.F. Peters, *Topology of digital images. visual pattern discovery in proximity spaces*, Intelligent Systems Reference Library, vol. 63, Springer, 2014, xv + 411 pp, Zentralblatt MATH Zbl 1295 68010.

13. J.F. Peters and C. Guadagni, *Strong proximities on smooth manifolds and Voronoi diagrams*, Adv. Math., Sci. J. 4 (2015), no. 2, 91–107, Zbl 1339.54020.

14. E. Čech, *Topological spaces*, John Wiley & Sons Ltd., London, 1966, fr seminar, Brno, 1936-1939; rev. ed. Z. Frolik, M. Katětov.

Computational Intelligence Laboratory, University of Manitoba, WPG, MB, R3T 5V6, Canada and Department of Mathematics, Faculty of Arts and Sciences, Adiyaman University, 02040 Adiyaman, Turkey,

Email address: james.peters3@umanitoba.ca

Department of Mathematics, Karadeniz Technical University, Trabzon, Turkey,

Email address: tane.vergili@ktu.edu.tr