You Shall Not Pass: Avoiding Spurious Paths in Shortest-Path Based Centralities in Multidimensional Complex Networks

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Abstract—The aggregation process may create spurious paths on the aggregated view of multidimensional (high order) complex networks. Consequently, these spurious paths may then cause shortest-path based centrality metrics to produce incorrect results, thus undermining the network centrality analysis. In this context, we propose a method built upon MultiAspect Graphs (MAGs) able to avoid taking into account spurious paths when computing centralities based on shortest paths in multidimensional complex networks. We show that, by using our proposed method, pitfalls usually associated with spurious paths resulting from aggregation in multidimensional networks can be avoided at the time of the aggregation process. As a result, shortest-path based centralities are assured to be computed correctly for multidimensional networks, without taking into account spurious paths that could otherwise lead to incorrect results. We also present case studies that show the impact of spurious paths in the computing of shortest paths and consequently of shortest-path based centralities, such as betweenness and closeness, thus illustrating the importance of this contribution.

Index Terms—Network centrality, time-varying networks, multilayer networks, high order networks.

I. INTRODUCTION

Centrality is a key concept for complex network analysis [1]. Typically seen as a ranking of the relative importance of vertices or edges in a complex network, many distinct centrality definitions have been proposed over the last decades for different purposes in several fields [2], [3]. More recently, many complex systems are being represented by time-varying, multilayer, and time-varying multilayer networks, i.e. multidimensional (or high order) networks. Traditionally, multidimensional networks were referred to as networks for which there are more than one edge connecting two vertices, i.e. a multigraph, or as networks where the dynamics of nodes are with multiple dimensions, such as having their location coordinates in 3D space. However, with the emergence of time-varying and multilayer networks, the definition of multidimensional networks evolved to support these newer types of higher order networks. Bergler et al. [4] state that dimensions in network data can be either explicit or implicit. In the first case the dimensions directly reflect the various interactions in reality; in the second case, the dimensions are defined by the analyst to reflect different interesting qualities of the interactions, which can be inferred from the available data, in structures as multilayer or multiplex networks. Therefore, inline with this interpretation of several related works [5]–[7], in this paper, we use the term multidimensional network to refer to higher-order complex networks that involve multiple aspects or features (i.e. dimensions), such as time instants, layers, and so on. More specifically, in this paper, we focus on shortest-path based centralities, such as betweenness and closeness, in multidimensional networks.

To assess node centralities, it is common to aggregate aspects of the multidimensional network in a dimensionality reduction process. However, it is well-known that the aggregation process may create spurious paths on the aggregated view of such multidimensional networks [5], [6]. Spurious paths emerge as additional paths that actually did not exist in the original network. As a consequence, this may generate shortest paths that also did not exist in the original networks. These artifacts on the aggregated network cause the shortest-path based centrality measures to include spurious shortest paths on their computation, leading to results that are not consistent with the original network and undermining the network centrality-based analysis of the multidimensional network. Fig. 1 presents a simple example of this issue: T1, T2, and T3 represent three instants of a time-varying graph (TVG). It can be seen that there is no path from vertex 1 at time T1 to vertex 4 at any time since the edge \{3, 4\} occurs at time T2, before edge \{2, 3\} at time T3. However, in the aggregated network

1. L.C. Costa had an LNCC fellowship during his involvement in this work.
2. We remark that all considerations and contributions of this paper about connectivity and avoidance of spurious paths remain valid if nodes may have multiple edges between them (a multigraph) or may have location coordinates in 3D space, for instance.

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There is a path from vertex 1 to vertex 4. Often such spurious paths are overlooked, thus potentially misleading the centrality analysis, or it becomes a cumbersome (and not scalable) process to disregard them in computing shortest paths at the aggregated view. This motivates the investigation of ways to avoid these spurious paths at the aggregation process.

In this paper, we propose a method\(^3\) able to avoid taking into account spurious paths when aggregating dimensions and, as a consequence, also when later computing centralities based on shortest paths in multidimensional networks. To the best of our knowledge, this is the first attempt to avoid the generation of spurious paths at the aggregation process in multidimensional networks. Further, we present a formalization of well-known centralities in the MAG environment (details in Section II), assuring that the subject of shortest-paths centralities in MAGs is well defined to provide the support proposed solution. The presence of aggregation artifacts on the aggregated network causes the centrality algorithms based on shortest paths to include these shortest spurious paths into the centrality calculation, leading to results that are not consistent with the original network. Then, as the main contribution of this work, we prove that, by using the proposed method, pitfalls usually associated with aggregation can be avoided in multidimensional networks. As a result, path-based centralities are assured to be computed correctly without taking into account spurious paths that could lead to incorrect results. Finally, we present case studies that show the impact of spurious paths in the computing of shortest paths and consequently of shortest-path based centralities, such as betweenness and closeness.

This paper is organized as follows. Section II presents a brief description of MAGs. Section III presents the proposed method that enables the avoidance of spurious paths in the sub-determination process. Section IV formalizes shortest-path based centralities and their sub-determined form on the MAG environment. Section V discusses the implementation of algorithms for evaluating shortest-path centralities on MAGs. Section VI analyzes a few case studies, while Section VII presents our final remarks.

II. BACKGROUND ON MULTIASPECT GRAPHS (MAGS)

We define a MAG as \( H = (A, E) \), where \( E \) is a set of edges and \( A \) is a finite list of sets, each of which is called an aspect. Each aspect \( \sigma \in A \) is a finite set, and the number of aspects \( p = |A| \) is called the order of \( H \). Each edge \( e \in E \) is a tuple with \( 2 \times p \) elements. All edges are constructed so that they are of the form \((a_1, \ldots, a_p, b_1, \ldots, b_p)\), where \( a_1, b_1 \) are elements of the first aspect of \( H \), \( a_2, b_2 \) are elements of the second aspect of \( H \), and so on, until \( a_p, b_p \) which are elements of the \( p \)-th aspect of \( H \).

As a matter of notation, we say that \( A(H) \) is the aspect list of \( H \) and \( E(H) \) is the edge set of \( H \). Further, \( A(H)[n] \) is the \( n \)-th aspect in \( A(H) \), \( |A(H)|[n| = \tau_n \) is the number of elements in \( A(H)|[n], \) and \( p = |A(H)| \) is the order of \( H \). Further, we define the set

\[
\forall(H) = \times_{n=1}^{p} A(H)[n],
\]

which is the Cartesian product of all the \( p \) aspects of the MAG \( H \). We call the elements \( v \in \forall(H) \) composite vertices. Note that each composite vertex \( v \in \forall(H) \) has the form \((a_1, \ldots, a_p)\). Therefore, there is a close relation between MAG edges and pairs of composite vertices, since \((a_1, \ldots, a_p, b_1, \ldots, b_p) \sim (v, u) = ((a_1, \ldots, a_p), (b_1, \ldots, b_p))\), so that \( v = (a_1, \ldots, a_p) \) and \( u = (b_1, \ldots, b_p) \). From this relation between MAG edges and pairs of composite vertices, it is possible to build a directed graph of composite vertices, which is shown in [9] to be isomorphic to the MAG.

As a consequence of the isomorphism between a MAG and a directed graph, we then define the function

\[
g: (A(H), E(H)) \rightarrow (\forall(H), \forall(H) \times \forall(H))
\]

\[
H \rightarrow (\forall(H), \psi(E(H))),
\]

which maps every MAG \( H \) to its isomorphic directed graph \( g(H) \). Further, we define the set of functions

\[
\pi_i: \forall(H) \rightarrow A(H)[i]
\]

\[
(a_1, a_2, \ldots, a_p) \rightarrow a_i,
\]

to extract the \( n \)-th element of a composite vertex tuple.

Note that, by the definition of \( g(H) \) in Eq. 2, the vertices of the directed graph \( g(H) \) are tuples with \( p \) elements. It is also possible to have a more traditional graph representation where the vertices are simply points of a set with no additional meaning. In this case, the directed graph has to be complemented by a companion tuple, which allows the association of each vertex with its original MAG tuple. This companion tuple has \( p \) elements, where each element \( 0 < i < p \) is given as \( |A(H)|[i] \), the number of components of the \( i \)-th aspect. Further details regarding the construction and usage of the companion tuple of a MAG can be found in [10].

\(^3\)This is an extended version of the shorter paper [8] that showed only the theoretical part of the work.
Since every MAG $H$ is isomorphic to a directed graph, it follows that shortest paths between composite vertices are equivalent to shortest paths between vertices of a directed graph. Therefore, algorithms based on shortest paths between composite vertices can be constructed as traditional directed graph algorithms. In particular, algorithms for centralities based on shortest paths, like for instance, closeness and betweenness centralities, can be computed for composite vertices on a MAG with the same algorithms used for computing them in directed graphs.

In the case of sub-determined vertices on a MAG, however, the traditional algorithms for directed graphs may lead to problems, if applied directly. As shown in [9], [10], shortest paths and distances between sub-determined vertices do not have the same properties of shortest paths and distances on directed graphs.

In MAGs, similarly to traditional graphs, distances can be defined in different ways depending on the application. In this sense, one common way of defining distances between composite vertices in MAGs is the number of hops of a shortest path between these vertices. Edge attributes (weights) can be used to assign distances (e.g. metric distances) to an edge. Moreover, any MAG that has time instants as an aspect can be seen as a time-varying MAG. Hence, the time length of an edge is determined by the difference between the two time values on that edge. Therefore, time distances can be inferred directly from the edge composition. As a consequence, as with traditional directed graphs, algorithms based on shortest paths can be adapted for use with any other distance definition.

### III. SUB-DETERMINATION AND THE AVOIDANCE OF SPURIOUS PATHS

In this section, we define the sub-determination process and we illustrate how a spurious path may emerge from it. Then, we present a method that avoids spurious paths in the sub-determination process.

#### A. Sub-Determination

The sub-determination process can be seen as a partition of the original MAG, where composite vertices that are sub-determined to the same sub-determined vertex are on the same equivalence class. A MAG sub-determination is a generalization of the aggregation applied to multilayer or time-varying graphs, in which all layers or time instants can be aggregated, resulting in a traditional directed graph.

Since a MAG can have more than 2 aspects, a sub-determination can be performed in more ways than the aggregation. For instance, given an arbitrary MAG $H$, a sub-determination is constructed from a non-empty proper sublist of the aspects of $H$. Since this sublist excludes the empty sublist and the full aspect list of $H$, in a MAG with $p$ aspects there are $2^p - 2$ possible non-empty proper aspect sublists, each one of them originating a distinct sub-determination. Therefore, in a MAG $H$ with $p$ aspects we identify the possible sub-determination by a positive integer $\xi$, where $1 \leq \xi \leq 2^p - 2$.

A tuple with $p$ elements either 0 or 1 can be used as a set indicator to represent each possible sub-determination. An element with value 1 in the $i$-th position of the tuple indicates that the $i$-th aspect is present in the sub-determination while a 0 indicates that the $i$-th aspect is not present in the sub-determination. We remark that this tuple is equivalent to the binary representation of the integer $\xi$. As a consequence, we may use the symbol $\xi$ to represent the sub-determination both as an integer number or its equivalent tuple. Note that we adopt the notation $\zeta$ to denote any given sub-determination, whereas for a specific sub-determination we use a subscript to characterize the specific tuple associated with the sub-determination. For example, a sub-determination generated by the tuple $Y = [1, 0, 1]$ is represented as $\zeta_Y$.

For a given MAG $H$ and a sub-determination $\xi$, we have a sublist $A^\xi(H) \subset A(H)$ such that $p^\xi = |A^\xi(H)|$ is the number of aspects in the sub-determination. From this, we define the set of composite sub-determined vertices of $H$:

$$V_\xi(H) = \times_{n=1}^{p^\xi} A^\xi(H)[n],$$

i.e., the Cartesian product of all the aspects in the sub-determination $\xi$. Further, we also define the function

$$S_\xi : V(H) \rightarrow V_\xi(H)$$

$$(a_1, a_2, \ldots, a_p) \mapsto (a_{\zeta_1}, a_{\zeta_2}, \ldots, a_{\zeta_{p^\xi}}),$$

which takes each composite vertex of $H$ to its sub-determined form.

Even though aggregation (or sub-determination for the case of MAGs) may cause additional paths to be present on the aggregated network, we show that it is possible to obtain a sub-determined Breadth-First Search (BFS) in which the additional paths potentially created by the sub-determination/aggregation process are not considered. The proof of this claim can be found in [10]. However, in order to make this work self-contained, we present a brief motivation of the purpose of this claim, providing a deeper understanding of how the issue of the spurious path can be avoided.

The construction of sub-determined algorithms relies on the use of functions to aggregate results according to the applied sub-determination. In some cases, this function can be as simple as summing up values obtained in composite vertices, which are reduced to the same sub-determined vertex. It follows that such summation can be done by matrix multiplication. Given a MAG $H$ and a sub-determination $\xi$, the sub-determination matrix $M_\xi(H) \in \mathbb{R}^{n_\xi \times n}$ is a rectangular matrix, where $n = |V(H)|$ is the number of composite vertices of $H$ and $n_\xi = |V_\xi(H)|$ is the number of composite vertices of the sub-determination $\xi$ applied to $H$. Since a sub-determination is a (proper) subset of the aspects of a MAG, it follows that $n_\xi \leq n$, i.e. the number of composite vertices of a MAG, is a multiple of the number of composite vertices in any of its sub-determinations. Further, $M_\xi(H)$ has the property of having exactly one non-zero entry in each column; and the position of this entry is determined by the numerical value of the sub-determined composite vertex.
Algorithm 1: Construction of $M_\zeta$.

```
input: $\tau(H)$ and $\zeta$
output: $M_\zeta(H)$
1 SubDetMatrix ($\tau(H)$, $\zeta$)
2 $T_\zeta = SubCompTuple(\tau(H), \zeta)$ \text{ // sub-determined companion tuple}
3 $n \leftarrow |V(H)|$
4 $n_\zeta \leftarrow |V_{\zeta}(H)|$
5 $M_\zeta(H) \leftarrow n_\zeta \times n$ \text{ // sparse matrix}
6 for $j \leftarrow 1$ to $n$
7 $u \leftarrow D^{-1}(j, \tau(H))$ \text{ // numeric tuple form of $j$}
8 $i \leftarrow D(u, T_\zeta)$ \text{ // sub-determined representation}
9 $M_\zeta(H)[i, j] \leftarrow 1$
10 end
11 return $M_\zeta(H)$
```

Algorithm 1 shows the construction of the sub-determination matrix $M_\zeta(H)$ for a given MAG $H$ and sub-determination $\zeta$. As Algorithm 1 is actually a classic BFS, it has time complexity $O(n + n_\zeta)$ and space complexity $O(n)$. The function $D$ takes a composite vertex to its numerical representation.

From Algorithm 1, it can be seen that the sub-determination matrix $M_{\tau\zeta}(R)$, used to do the sub-determination of the MAG $R$ shown in Fig. 2, has the form depicted in Eq. 6. Here, the sub-determination applied is given by the tuple $T = [1, 0]$, in this case sub-determining the time aspect on the MAG $R$.

$$
M_{\tau\zeta}(R) = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

B. An Example of the Emergence of a Spurious Path

Given the adjacency matrix $J(H)$ of an arbitrary MAG $H$ and a sub-determination matrix $M_\zeta(H)$ that represents the desired sub-determination, the adjacency matrix of the sub-determined MAG $H_\zeta$ is obtained by

$$
J_\zeta(H) = M_\zeta(H) J(H) M_\zeta(H)^T.
$$

The adjacency matrix $J(R)$ of the MAG $R$ in Fig. 2(a) is then given by

$$
J(R) = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Therefore, the adjacency matrix $J_{\tau\zeta}(R)$ of the sub-determined MAG $R$, shown in Fig. 2(b), is given by

$$
J_{\tau\zeta}(R) = M_{\tau\zeta}(R) J(R) M_{\tau\zeta}(R)^T.
$$

Due to the properties of matrix multiplication, it follows that the result of the algebraic sub-determination of a MAG results in a multi-graph with self-loops. Since we usually consider simple graphs, we may replace the main diagonal by a zero diagonal and (if necessary) substitute any non-zero entry by 1. In this way, the adjacency matrix $J_{\tau\zeta}(R)$ corresponding to Fig. 2(b) is given by

$$
J_{\tau\zeta}(R) = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$

Note that the spurious path from vertex 2 to vertex 3 in Fig. 2(b) is present in the adjacency matrix $J_{\tau\zeta}(R)$.

C. The Avoidance of Spurious Paths

We now show that given an arbitrary MAG $H$, its adjacency matrix $J(H)$, and a sub-determination $\zeta$, it is possible to obtain a BFS sub-determined by $\zeta$, in which no spurious path is considered. The BFS is closely related to matrix multiplication. This stems from the well-known property of the powers of the adjacency matrix, in which the $(i, j)$ entry of the $n$-th power of the adjacency matrix shows the number of existing walks of length $n$ from vertex $i$ to vertex $j$. From this, we could think that for a given MAG $H$, the series

$$
B = \sum_{i=0}^{\infty} J(H)^i = I + J(H) + J(H)^2 + J(H)^3 + J(H)^4 + \ldots
$$

would produce a matrix $B$, such that the entry $B_{i,j}$ indicates the number of walks of any length from vertex $i$ to vertex $j$. This is indeed the case when $H$ happens to be an acyclic MAG, making $J(H)$ a nilpotent matrix. The existence of cycles in $H$ makes that, for some vertices, there will exist walks of arbitrary length connecting them (namely, the
cycles), making the series of Eq. 12 divergent. However, since the objective is not to know the number of walks between each pair of vertices, but simply to know which vertices are reachable from each other (i.e. there is at least a path between them), this technical problem can be solved by multiplying the adjacency matrix $J(H)$ by a scalar $\rho_H$, i.e.

$$J_\rho(H) = \rho_H J(H),$$

such that

$$0 < \rho_H < \frac{1}{\rho(J(H))},$$

(14)

where $\rho(J(H))$ is the spectral radius of the matrix $J(H)$. Thus, the spectral radius of the matrix $J_\rho(H) < 1$. This also results that Eq. 12 constructed with the matrix $J_\rho(H)$ converges. Since the convergence of the series is assured, Eq. 12 can be re-expressed as

$$B = (I - J_\rho(H))^{-1}.$$  
(15)

The matrix $B$ has the property that, for any composite vertex $v \in V(H)$, the row $D(v)$ of $B$ has non-zero entries in every column that corresponds to a composite vertex $u \in V(H)$, such that $u$ is reachable from $v$. Hence, for a given composite vertex $v$, the row $D(v)$ corresponds to the result of a BFS started at that composite vertex.

Since we now have an algebraic formulation for the BFS, given by either Eq. 12 or Eq. 15, we can show that the order in which the sub-determination and the BFS are executed matters. Note that the equation

$$\sum_{i=0}^{\infty} \left(M_\xi(H) J_\rho(H) M_\xi(H)^T \right)^i$$

(16)

represents the case where the sub-determination is calculated first and the BFS is calculated afterward, while

$$M_\xi(H) \left( \sum_{i=0}^{\infty} J_\rho(H)^i \right) M_\xi(H)^T$$

(17)

represents the case where the BFS is calculated first and the sub-determination afterward. Note that in general the result obtained by Eq. 16 is different from the result obtained from Eq. 17. To see that this claim holds, note that an arbitrary power $k$ of the matrix $M_\xi(H) J_\rho(H) M_\xi(H)^T$ is given by

$$M_\xi(H) J_\rho(H) M_\xi(H)^T M_\xi(H) J_\rho(H) M_\xi(H)^T,$$

(18)

where $(M_\xi(H) J_\rho(H) M_\xi(H)^T)$ is multiplied $k$ times. Note, however, that

$$M_\xi(H)^T M_\xi(H) \neq I_n,$$

(19)

since $M_\xi(H) \in \mathbb{R}^{n \times n}$ is a rectangular matrix and $n_\xi < n$, so that the rank of the matrix $M_\xi(H)^T M_\xi(H)$ is less or equal to $n_\xi$, while the rank of the identity $I_n$ is $n > n_\xi$. Since Inequality 19 holds, our claim holds.

From the fact that Eq. 16 and Eq. 17 are not equivalent, we claim that results obtained from Eq. 17 do not consider spurious paths that can be created by sub-determination. Note that the term $\left(\sum_{i=0}^{\infty} J_\rho(H)^i \right)$ present in Eq. 17 is the BFS calculated for the original (i.e. not sub-determined) MAG $H$. Therefore, it follows that no spurious path is considered in this BFS since no sub-determination was done. As this BFS was calculated with no spurious paths, the result can now be sub-determined for calculating shortest-path centralities without considering unwanted artifacts that may lead to incorrect centrality results.

IV. CENTRALITY IN MAGS

A centrality can be understood as an indicator of the relative importance of the vertices or edges in the analyzed graph. Formally, for a given graph $G = (V, E)$, a vertex centrality $C_v$ can be seen as a function from the set of vertices to a set of nonnegative real numbers, i.e.

$$C_v : V(G) \to \mathbb{R}^+ \bigcup \{0\},$$

(20)

where $V(G)$ is the vertex set of the graph $G$. Note that Function 20 induces a linear order relation upon its domains, which can, in turn, be understood as a centrality notion. The adopted centrality function should then properly reflect the kind of vertex relative importance to be expressed by the centrality notion. Similarly, edge centralities may be defined to reflect the relative importance of an edge. For instance, the vertex betweenness centrality [11] is defined as

$$C_B(v) = \sum_{s,t \in V(G)} \frac{\sigma(s,t,v)}{\sigma(s,t)},$$

(21)

where $V(G)$ is the vertex set of the graph $G$, $s,t \in V(G)$ are vertices of $G$, $v \in V(G)$ is the vertex for which the centrality is calculated, $\sigma(s,t)$ is the number of shortest paths connecting vertices $s$ and $t$, and $\sigma(s,t,v)$ is the number of shortest paths from $s$ to $t$ which pass through vertex $v$. If calculated for each $v \in V(G)$, Eq. 21 can be seen as a possible implementation of Function 20.

A. Extending the Centrality Notion to MAGs

The evaluation of edge centralities in a MAG can be done in a straightforward way using the MAG’s composite vertex representation. Since this representation is a traditional directed graph and it carries all the edges of the MAG, thus preserving their topological properties (i.e., their adjacency properties), it follows that edge centralities can be computed using the same methods applied to traditional directed graphs.

The evaluation of vertex centralities in MAGs is more complex than in a traditional graph. The first difference is that a MAG can be sub-determined and at the limit any aspect element can become a vertex. Nevertheless, the composite vertices representation of a MAG, which is shown to be isomorphic to the MAG [9], is a directed graph. Hence, vertex
centrals in a MAG can be defined in terms of the composite vertices present on the MAG’s composite vertices representation, as well as, in terms of any sub-determination of the composite vertices.

B. Composite Vertex Centralities

Formally, a composite vertex centrality can be seen as a function that goes from the set of composite vertices to the set of nonnegative real numbers, i.e.

$$C_v : \mathcal{V}(H) \rightarrow \mathbb{R}^+ \cup \{0\},$$

(22)

where \(\mathcal{V}(H)\) is the set of all composite vertices in \(H\).

As the MAG composite vertex representation is a directed graph in which the vertices are the MAG’s composite vertices, it follows that any centrality form known for directed graphs can be directly applied to calculate MAG composite vertex centralities. For instance, the composite vertex betweenness centrality can be defined as

$$C^B_v : \mathcal{V}(H) \rightarrow \mathbb{R}^+ \cup \{0\},$$

(23)

where \(v, s, t \in \mathcal{V}(H)\) are composite vertices of \(H\), \(\sigma(s, t)\) is the number of shortest paths from \(s\) to \(t\), and \(\sigma(s, t/v)\) is the number of shortest paths from \(s\) to \(t\) passing in \(v\).

C. Sub-Determined Vertex Centralities

The centrality definition for sub-determined vertices is similar to the centrality definition for composite vertices:

$$C_{\zeta} : \mathcal{V}_\zeta(H) \rightarrow \mathbb{R}^+ \cup \{0\},$$

(24)

where \(\mathcal{V}_\zeta(H)\) is the set of sub-determined vertices of the MAG \(H\). However, the computation of such sub-determined centrality may require distinct algorithms as the computation of composite vertices centralities. Consider the sub-determined betweenness centrality, i.e.

$$C^B_{\zeta} : \mathcal{V}_\zeta(H) \rightarrow \mathbb{R}^+ \cup \{0\},$$

(25)

where \(v_\zeta, s_\zeta, t_\zeta \in \mathcal{V}_\zeta(H)\) are sub-determined composite vertices of the MAG \(H\) by a given sub-determination \(\zeta\), \(\sigma(s_\zeta, t_\zeta)\) is the number of shortest paths from \(s_\zeta\) to \(t_\zeta\), and \(\sigma(s_\zeta, t_\zeta/v_\zeta)\) is the number of shortest paths from \(s_\zeta\) to \(t_\zeta\) which pass through \(v_\zeta\).

The shortest paths between a given pair of sub-determined vertices \(s_\zeta, t_\zeta \in \mathcal{V}_\zeta(H)\) cannot, however, be computed with the traditional algorithm used for composite vertices. Actually, the computation of \(\sigma(s_\zeta, t_\zeta/v_\zeta)\) and, therefore, all centralities that are defined in terms of shortest paths have to be computed with algorithms constructed based upon the sub-determined shortest paths algorithms discussed in Section II.

D. Single Aspect Centralities

Formally, for a given MAG \(H = (A, E)\), a single aspect centrality can be seen as a function that takes each element from a given aspect to a nonnegative real number, i.e.

$$C_{\alpha} : A(H)[j] \rightarrow \mathbb{R}^+ \cup \{0\},$$

(26)

where \(A(H)[j]\) is the \(j\)-th aspect of the MAG \(H\).

Note that a single aspect centrality is a particular case of sub-determined centralities, since the set \(A(H)[j]\) can be obtained by a sub-determination \(\zeta_j\) in which the \(j\)-th entry of the sub-determination tuple is 1 and all other entries are 0. Therefore, a single aspect centrality is a particular case of sub-determined centrality, and can then be computed using the same algorithms used for sub-determined centralities. Hence, given any sub-determination \(\zeta_j\) that preserves a single aspect, the sub-determined betweenness centrality \(C^B_{\zeta_j}\), as defined in 25, is a single aspect betweenness centrality. Another example of a single aspect centrality is degree centrality. In this case, the centrality function simply takes each aspect element to the number of edges incident to it. In particular, for any MAG that has an aspect designated as time, a time centrality is simply an aspect centrality for this particular aspect [12].

V. ALGORITHMS

In this section, we show with a betweenness centrality example how centrality algorithms for MAGs can be derived from well-known algorithms for directed graphs.

We start by presenting Algorithm 2, which evaluates betweenness centrality for all composite vertices of a given MAG, as defined by Expression 23. We remark that this is just an implementation of the Brandes algorithm [13], which for unweighted sparse networks has as time complexity \(O(nm)\) and as space complexity \(O(nm)\), where \(n\) is the number of composite vertices and \(m\) the number of edges. Note that, for practical purposes, multidimensional complex networks are in general sparse. The original Brandes algorithm can be used directly to evaluate the centralities of composite vertices of a MAG since the composite vertices of a MAG form a directed graph. Therefore, all of the known properties of the Brandes algorithm for directed graphs apply for MAGs when evaluating betweenness centralities for composite vertices. Since this algorithm is well-known, we have included its complete description in Algorithm 2, mainly to serve as a basis for comparison with the sub-determined betweenness centrality algorithm we present in Algorithm 3.

When considering the sub-determined form of betweenness centrality, defined by Expression 25, we first have to take into account that calculating betweenness centrality of a sub-determined MAG is not the same as calculating the sub-determined betweenness centrality of a MAG. This is a consequence of the fact that composite vertices that are not connected in a MAG may be connected on a sub-determined form of this MAG, meaning that the sub-determination process may create paths on the sub-determined MAG that have no real correspondence to paths existing on the original MAG. As an
The sub-determined betweenness centrality, presented in Algorithm 3, is based upon the original Brandes algorithm. Algorithm 3 evaluates sub-determined betweenness centrality (i.e. betweenness centrality of sub-determined vertices) without considering the spurious paths potentially generated by MAG sub-determination.

The key differences between Algorithm 3 and the traditional Brandes algorithm is that in Algorithm 3 both the sub-determined and the full (non sub-determined) MAG are taken into account. Actually, the Breadth-First Search (BFS) part of the algorithm is done upon the full MAG, while the paths are built upon the sub-determined vertices. By doing this, we have that only the paths that exist on the full MAG will be taken into account for calculating the centrality, whose result is expressed in terms of sub-determined vertices.

By comparing the traditional and the sub-determined algorithms, it can be seen that the main differences are at lines 3, 8, 16 to 20, and 30 to 33. In Algorithm 3, for each sub-determined vertex of the MAG, a BFS is conducted using the original MAG and shortest paths are computed for each distinct sub-determined vertex found by the BFS. Further, note that given a sub-determined vertex \( s \), instead of starting from a single vertex, the BFS starts from all vertices whose sub-determined form is equal to \( s \). This is the equivalent of starting a BFS on a sub-determined MAG. However, since the BFS is conducted upon the sub-determined MAG, it follows that the BFS respects the reachability found on the full MAG and, therefore, it disregards the spurious paths created by sub-determining the MAG. Since the functional difference between Algorithm 3 and the traditional betweenness centrality algorithm is that Algorithm 3 uses the sub-determined BFS algorithm proposed in [10], it follows that Algorithm 3 has the same properties of the traditional Brandes algorithm. In particular, since the BFS problem is done over the full MAG, and if we consider that \( n_z \) (the number of \( z \) sub-determined vertices) is of the same order of \( n \) (the number of composite vertices), it follows that Algorithm 3 can be computed in \( O(nm) \) time and requires \( O(n + m) \) space, given that this algorithm is for unweighted MAGs (note that \( m \) is the number of edges). It is worth noting that this adaptation can be implemented for any other shortest path based algorithm, such as closeness, stress and other centralities of this kind.

For example, remark that the betweenness centrality for the sub-determined MAG \( R \) shown in Fig. 2(b) is \([0, 1, 0]\) while the sub-determined betweenness centrality for the MAG \( R \) shown in Fig. 2(a) is \([0, 0, 0]\). It can be seen that the betweenness centrality (Algorithm 2) considers the presence of a path from vertex 1 to 3 in Fig. 2(b), making the centrality of vertex 2 to have value 1. In the case of sub-determined betweenness centrality (Algorithm 3) of the MAG \( R \) shown in Fig. 2(a), the result is \([0, 0, 0]\), from which it can be seen that no path from vertex 1 to 3 is considered in this case.

VI. ILLUSTRATIVE CASE STUDIES

In this section, we present a set of case studies designed to show, in an experimental approach, the differences between
Algorithm 3: Sub-determined betweenness for MAGs.

\textbf{Input:} \( H, \tau(H), \zeta \) // \text{MAG, companion tuple, and sub-determination}
\textbf{Output:} \( C^B_{\xi} \) // Betweenness centrality vector

1. \text{SubBetweennessCentrality}(H, \tau(H), \zeta)
2. \( n \leftarrow |V(H)|// \text{Number of composite vertices of the MAG} \)
3. \( n_{\xi} \leftarrow |V_{\xi}(H)| // \text{Number of composite vertices of the sub-determined MAG} \)
4. \( C^B_{i_{\xi}} \) // vector of \( n_{\xi} \) integers, all 0
5. \text{for } s_i \in V_{\xi}(H) // \text{Single Source shortest path (SSSP) problem} \text{ do}
6. \( \text{color} \) // vector of \( n_{\xi} \) integers, all 0
7. \( \text{color}_{i_{\xi}} \) // vector of \( n_{\xi} \) integers, all 0
8. \( S \leftarrow \text{empty stack} \)
9. \( P \leftarrow \text{a vector of } n_{\xi} \text{ empty lists} \)
10. \( \sigma \leftarrow \text{vector of } n_{\xi} \text{ integers for counting shortest paths} \)
11. \( \sigma[s_i] \leftarrow 1 // \text{Each vertex has 1 shortest path to itself} \)
12. \( d \leftarrow \text{vector of } n_{\xi} \text{ values} = -1 \text{ for distances} \)
13. \( d[s_i] = 0 // \text{Distance from } s \text{ to itself is 0} \)
14. \( Q \leftarrow \text{empty queue} \)
15. \( \text{for } i \in V_{\xi}(H), \text{ where } s_i \equiv i_{\xi} // \text{For all vertices with same sub-determination as } s \)
16. \text{do}
17. \( \text{color}[i] \leftarrow \text{BLACK} \)
18. \( \text{enqueue } i \rightarrow Q // \text{All vertices equivalent to } s \)
19. \text{end}
20. \( \text{while } Q \text{ is not empty} // \text{Sub-determined vertices BFS} \text{ do}
21. \text{dequeue } v \leftarrow Q \)
22. \( v_{\xi} \leftarrow v \text{ sub-determined by } \xi \)
23. \( \text{if } \text{color}_{i_{\xi}}[v_{\xi}] \text{ is WHITE then} \)
24. \( \text{color}_{i_{\xi}}[v_{\xi}] \leftarrow \text{BLACK} \)
25. \( \text{push } v_{\xi} \rightarrow S \)
26. \text{end}
27. \text{for } \text{each composite vertex } w \text{ such that } (v, w) \in E(H) \text{ do}
28. \text{if } \text{color}[w] \text{ is WHITE then} \)
29. \( \text{color}[w] \leftarrow \text{BLACK} \)
30. \( \text{enqueue } w \rightarrow Q \)
31. \text{end}
32. \( \text{w}_{\xi} \leftarrow w \text{ sub-determined by } \xi \)
33. \( \text{if } d[w_{\xi}] = -1 // \text{if } w_{\xi} \text{ found for the first time} \)
34. \text{then}
35. \( \text{d}[w_{\xi}] \leftarrow d[v_{\xi}] + 1 // \text{Distance } s_i \rightarrow w_{\xi} \)
36. \text{end}
37. \( \text{if } d[w_{\xi}] = d[v_{\xi}] + 1 // \text{If there is a shortest path to } w_{\xi} \text{ via } v_{\xi} \)
38. \text{then}
39. \( \sigma[w_{\xi}] \leftarrow \sigma[w_{\xi}] + \sigma[v_{\xi}] // \text{Counts shortest paths} \)
40. \( \text{append } v_{\xi} \rightarrow P[w_{\xi}] // v_{\xi} \text{ is predecessor of } w_{\xi} \)
41. \text{end}
42. \text{end}
43. \( \delta \leftarrow \text{vector of } n_{\xi} \text{ integers, all 0} \)
44. \( \text{while } S \text{ is not empty} \)
45. \( \text{pop } w_{\xi} \leftarrow S \)
46. \( \text{for } v_{\xi} \text{ in } P[w_{\xi}] // \text{For all predecessors of } w_{\xi} \)
47. \text{do}
48. \( \delta[v_{\xi}] \leftarrow \delta[v_{\xi}] + (\sigma[v_{\xi}] / \sigma[w_{\xi}]) \times (1 + \delta[w_{\xi}]) \)
49. \text{end}
50. \( \text{if } w_{\xi} = s_i \text{ then} \)
51. \( C^B_{i_{\xi}}[w_{\xi}] \leftarrow C^B_{i_{\xi}}[w_{\xi}] + \delta[w_{\xi}] \)
52. \text{end}
53. \text{end}
54. \text{return } C^B_{i_{\xi}}

the outcomes of centrality measures when using classical centrality algorithms on previously sub-determined MAGs or when using sub-determined algorithms applied on (non-sub-determined) MAGs.

A. Real-World and Synthetic Networks

We consider two real-world multidimensional networks as well as synthetic ER (Erdős-Rényi) random MAGs:

- MOSAR network – MOSAR (Mastering hOSpital Anti-microbial Resistance and its spread) is a scientific collaboration project [14] that comprises several medical, biochemistry, and computing research institutions. The MOSAR project focuses on antimicrobial-resistant bacteria (AMRB) transmission dynamics in high-risk environments, such as intensive care units. The adopted dataset consists of the records of in-person contacts (from physicians, nurses, staff members, and patients) in a certain medical ward for a period of two weeks (between July 25 and August 08, 2009). Each one of the 160 volunteers who participated in the study was equipped with a RFID device that detected the presence of another device within a small distance (one meter). Device identification was always associated with the same person. Every 30 s during the two-weeks period, each device registered the list of all devices (nodes) that were within its coverage area in order to establish the arrangement of the contacts among them (edges). We have then modeled the MOSAR dataset as an order 2 MAG representing a TVG with 160 participants (elements in the first aspect) and 40320 time instants (elements in the second aspect). Contacts are represented by non-oriented edges. With spurious paths, the characteristic path length and diameter of the MOSAR network are 2.39 and 5, respectively. Disregarding spurious paths, the characteristic path length and diameter of the MOSAR network are 2.64 and 9, respectively.

- Brazilian domestic air transportation network – This dataset refers to the air transportation network from Brazil, as published by the Brazilian National Civic Aviation Agency (ANAC) in May, 12th 2017. We model this dataset as an order 3 MAG to represent the multidimensional network equivalent to a multilayer time-varying network. The first aspect has 110 elements corresponding to the airports. The 2nd aspect consists of the multilayer part in which each layer involves the air flight network of each airline company. In this 2nd aspect, there are 14 elements as 2 layers are used for each of the 7 airline companies present in the dataset. The 3rd aspect is composed of 7057 elements (i.e., time instants). This arrangement results in a MAG with 48339 composite vertices and 66195 edges. The weights in the edges represent the flights duration, boardings, landings, and the waiting time.

The outcomes of these studies show the effectiveness of the proposed algorithms in handling spurious paths and improving the accuracy of the centrality measures.
between flights. Each airline company uses 2 layers: one layer of the flights and the other for the waiting times between flights. The layer with waiting times is where we put the edge to make the weekly cycle. With spurious paths, the characteristic path length and diameter of this network are 2.39 and 5, respectively. Disregarding spurious paths, the characteristic path length and diameter are 2.47 and 7, respectively.

- Synthetic ER random MAGs – We have generated 2000 synthetic ER random MAGs, where half are order 2 MAGs and half are order 3 MAGs. The order 2 MAGs represent time-varying graphs (TVGs) while the order 3 MAGs represent higher order networks, such as multilayer time-varying networks. Each sample for both kinds of synthetic ER random MAGs has 10000 composite vertices connected through 42586 edges. The difference is that these composite vertices are divided in the order 2 MAG as 1000 elements in the first aspect and 10 elements in the second aspect; while in the order 3 MAG as 1000 elements in the first aspect, 2 elements in the second aspect, and 5 elements in the third aspect. The characteristic path length in ER random networks is \( O(\log n) \) and the diameter \( l \approx \log n / \log k \), where \( k = 2m/n \) is the average degree. This results in a characteristic path length in the order of 4 hops and an expected value for the diameter of about 4.30 hops synthetic ER random MAGs.

### B. Methodology for the Experimental Study

For all the analyses we conduct in the presented networks, we sub-determine the MAG representing the multidimensional network into a single aspect and then we evaluate the betweenness and the closeness centralities. For instance, for each random MAG, the results correspond to the centrality of the first aspect elements. In its turn, the centrality we analyze in the sub-determined MOSAR network refers to each person of the original face-to-face network (shown as node IDs in Section VI-C). Finally, for the Brazilian air transportation network, the single aspect puts forward the centrality of the airports (shown in airport codes in Section VI-D).

Moreover, we compare the results from centrality metrics computed by the classical approach and by our approach, developed to evaluate sub-determined centralities on MAG, thus avoiding spurious paths. We actually compare the centrality rankings generated by both cases and we analyze the position changes when the two rankings are compared. More specifically, to compare the rankings, we use the Ranking-Biased Overlap (RBO) [15], which is a ranking similarity measure. In contrast to the classic Kendall coefficient to compare rankings, RBO allows us to evaluate the similarity between rankings by applying weights (i.e., distinct importance) to the positions of the considered rankings. For example, the top positions of the rank (and their value) may have higher importance than positions far from the beginning of the rank. Note that RBO allows us to control how to weight rank positions. For example, we are able to fix \( x \) percent of the weight to the first \( n \) elements of the rank. Also note that RBO measures similarity, not distance (i.e., RBO is not a metric). However, RBO can trivially be turned into a distance measure, denominated rank-biased distance (RBD), where \( RBD = 1 - RBO \), as proved in [15]. In particular, for all experiments presented in this section, RBO and RBD will be set to assign 85% of the total weight to the top 10% positions of the rankings.

In short, as we previously stated, our goal is to show that, in the case of shortest path based centralities, the classic centrality algorithms may fail due to the issues resulting from the presence of spurious paths caused by the aggregation (and sub-determination).

### C. Evaluating MOSAR – A Human Contact Network

In this experiment, we evaluate the difference obtained when applying the proper algorithms for sub-determined results and classical sub-determined algorithms on the MOSAR network. Figs. 3 and 4 show the top 10 vertices for the betweenness and the closeness centralities, for both approaches (i.e., the classical sub-determined algorithms and the use of proper algorithms for sub-determined results). In this sense, we compare the centrality rankings considering the presence of spurious paths (or not). In both figs., for both network centralities, we clearly note a difference between the position of the top 10 vertices. In fact, according to Fig. 3, which shows the results for betweenness centrality, the two rankings differ from the third position. The rankings for closeness centrality (Fig. 4) diverge from the fourth position on the ranking. Moreover, in Fig. 3, we also note the presence of elements that do not figure on both top 10% rankings.

When comparing the top 10% of central nodes, the rankings present a significant RBD distance. In this case, the RBD
distance between the top 10% of the ranking is 0.188, while the RBD distance for the closeness centrality is slightly smaller (0.119). In short, spurious paths incur a considerable misleading assessment of node centrality.

D. Evaluating the Brazilian Air Transportation Network

We evaluate the betweenness and closeness centralities of the Brazilian domestic air transportation network. Figs. 5 and 6 graphically highlight the ranking differences. According to Fig. 5, the difference between the two rankings is important. For the betweenness, we note pair exchanges in the ranking positions, starting from the 2nd. position of the ranking. As shown in Fig. 6, the differences between both closeness centrality rankings is smaller. We only note one pair exchanging their relative position and, a pair of distinct airports in the 10th ranking position. This similarity between both rankings is expected once the closeness centrality algorithm calculates a mean of the distances from each vertex, therefore reducing the influence of spurious paths.

In this evaluation, using RBO, we have assigned 85% of the ranking weight to the top 10% of the ranking (i.e., 11 vertices). In this case, we observe an RBO correlation to the betweenness centrality rankings of 0.787. The RBO correlation to the pair of closeness centrality rankings is 0.956. In other words, the distance between betweenness centrality rankings is 0.213, while the RBD distance for the closeness centrality rankings is considerably smaller (i.e. 0.044).

E. Evaluating Synthetic ER Random MAGs

We present the correlation and distances obtained for betweenness and closeness centralities, considering the algorithms suppressing spurious paths and the classic algorithms that do not suppress spurious paths. We first evaluate the 1000 synthetic ER random order 2 MAGs (equivalent to time-varying graphs – TVGs) and then we evaluate the 1000 synthetic ER random order 3 MAGs. Unless indicated otherwise, we consider that 85% of the RBO and RBD weight is assigned for the first 10% of the rankings.

Tables I and II summarize the RBO and RBD values for the centralities when comparing rankings of TVG (order 2 MAG) networks with/without spurious paths. First, spurious paths impose a relevant effect on the centrality metrics we analyze. For betweenness (Table I), the mean RBD between both rankings is 0.386. Moreover, the lowest difference observed is of 0.175. The RBD values for closeness (Table II) are as high as those for betweenness. In this case, the mean RBD is superior to 0.6, while the minimum RBD is about 0.37. Then, analyzing path-related centralities using the classical approach may lead to misconceptions about the network centrality assessment.

We also observe high RBD values for the random order 3 MAGs for both betweenness and closeness centralities (Tables III and IV). For example, the mean RBD for betweenness and closeness centralities is superior to 0.39 and 0.6,
VII. CONCLUSION

To the best of our knowledge, we propose the first method to avoid taking into account spurious paths at the aggregation process in multidimensional complex networks. This assures the aggregated view of the higher order network to be free of spurious paths, thus allowing an accurate shortest-path based centrality computation in multidimensional complex networks. We have evaluated both, the traditional approach to calculate centrality metrics in multidimensional networks disregarding the potential existence of spurious paths and our proposal that avoids the generation of spurious paths at the aggregation process. This latter case is interesting because it shows the impact of properly avoiding spurious paths on the network centrality computation. The python implementation of the algorithms are publicly available.4

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4 http://github.com/wehmuthklaus/MAG_Algorithms