Towards a Flow- and Path-Sensitive Information Flow Analysis: Technical Report

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Abstract

This paper investigates a flow- and path-sensitive static information flow analysis. Compared with security type systems with fixed labels, it has been shown that flow-sensitive type systems accept more secure programs. We show that an information flow analysis with fixed labels can be both flow- and path-sensitive. The novel analysis has two major components: 1) a general-purpose program transformation that removes false dataflow dependencies in a program that confuse a fixed-label type system, and 2) a fixed-label type system that allows security types to depend on path conditions. We formally prove that the proposed analysis enforces a rigorous security property: noninterference. Moreover, we show that the analysis is strictly more precise than a classic flow-sensitive type system, and it allows sound control of information flow in the presence of mutable variables without resorting to run-time mechanisms.

1 Introduction

Information-flow security is a promising approach to security enforcement, where the goal is to prevent disclosure of sensitive data by applications. Since Denning and Denning’s seminal paper [20], static program analysis has been widely adopted for information-flow control [37]. Among these program analyses, type systems (e.g., [32, 35, 40]) have enjoyed a great popularity due to their strong end-to-end security guarantee, and their inherently compositional nature to combine secure components forming a larger secure system as long as the type signatures agree.

Conventionally, we assume secrets are stored in variables, and security levels (e.g., P for public and S for secret) are associated with variables to describe the intended secrecy of the contents. The security problem is to verify that the final value of the public variables (outputs visible to the public) is not influenced by the initial value of the secret variables.

Many security type systems (e.g., [32, 35, 40]) assume fixed levels. That is, the security level for each variable remain unchanged throughout program execution. Though this fixed-level assumption simplifies the design of those type systems, one consequence is that they tend to be over-conservative (i.e., reject secure programs). For example, given that $s$ has a level $S$ (i.e., $s$
holds a secret value) and \( p \) has a level \( P \), a fixed-level type system rejects secure programs, such as 

\( (p := s; p := 0;) \), even though the publicly observable final value of \( p \) is always zero.

Previous work (e.g., [26]) observes that such inaccuracy roots from the flow-insensitive nature (i.e., the order of program execution is ignored) of fixed-level systems. From this perspective, the previous example is mistakenly considered insecure because the (impossible) execution order 

\( (p := 0; p := s;) \) is insecure.

Hunt and Sands [26] propose a classic flow-sensitive type system which allows a variable to have multiple security levels over the course of computation. For example, this floating-level type system correctly accepts the program 

\( (p := s; p := 0;) \) by assigning \( p \) with levels \( S \) and \( P \) after the first and second assignments respectively. However, this floating-level system is still path-insensitive, meaning that the predicates at conditional branches are ignored in the analysis. For example, it incorrectly rejects the following secure program since the (impossible) branch combination 

\( (y := s; p := y;) \) is insecure.

\[
\text{if} \ (x = 1) \ \text{then} \ y := 0 \ \text{else} \ y := s; \\
\text{if} \ (x = 1) \ \text{then} \ p := y
\]

This paper develops a flow- and path-sensitive information flow analysis that is precise enough to accept the aforementioned secure programs. The novel analysis is built on two key observations. First, flow-sensitivity can be gained via a general-purpose program transformation that eliminates false dataflow dependencies that confuse a flow-insensitive type system. Consider the example 

\( (p := s; p := 0;) \) again. The transformation removes the false dataflow dependency between \( s \) and \( p \) by introducing an extra copy of the variable \( p \) and keeps track of the final copy of each variable at the same time. So, the example is transformed to 

\( (p_1 := s; p_2 := 0;) \), where \( p_2 \) is marked as the final copy. Then, a fixed-level system can easily type-check this program by assigning levels \( S \) and \( P \) to \( p_1 \) and \( p_2 \) respectively.

Second, path-sensitivity can be gained via consolidating dependent type theory (e.g., [16, 38, 42]) into security labels. That is, a security label is, in general, a function from program states to security levels. Consider the second example above with branches. We can assign \( y \) a dependent security label: 

\( (x = 1? P : S) \), meaning that the level of \( y \) is \( P \) when \( x = 1 \), and \( S \) otherwise. Hence, the information flow from \( y \) to \( p \) can be judged as secure since it only occurs when \( x = 1 \) (hence, \( y \) has level \( P \)).

Based on the key observations, we propose a flow- and path-sensitive information flow analysis that consists of two major components: a general purpose program transformation that removes false dataflow dependencies that otherwise compromise the precision of a fixed-level system, as well as a fixed-label type system with dependent labels. Each component of our analysis targets one insensitive source of previous type systems. The modular design not only enables tunable precision of our analysis, but also sheds light on the design of security type systems: we show that a fixed-level system (e.g., [40]) plus the program transformation is as precise as [26] the classic flow-sensitive system in [26]; furthermore, a fixed-label dependent type system can soundly control information flow in the presence of mutable variables without resorting to run-time mechanisms (e.g., [23, 44]).

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1We note that in the information flow literature, different terms (such as “precision” and “permissiveness”) have been used to compare the amount of false positives of various mechanisms [15]. In this paper, we say a static analysis \( A \) is as precise as a static analysis \( B \) if \( A \) accepts every secure program that is accepted by \( B \). Moreover, we say \( A \) is (strictly) more precise than \( B \) if \( A \) is as precise as \( B \), and \( A \) accepts at least one secure program that is rejected by \( B \).
Figure 1: Examples: Imprecise Information Flow Analysis Rejects Secure Programs.

This paper makes the following key contributions:

1) We formalize a novel flow- and path-sensitive information flow analysis for a simple WHILE language. The analysis consists of a novel program transformation, which eliminates imprecision due to flow-insensitivity (Section 4), and a purely static type system using dependent security labels (Section 5).

2) We formally prove the soundness of our analysis (Section 6): the source program satisfies termination-insensitive noninterference whenever the transformed program type-checks. Novel proof techniques are required due to the extra variables introduced (for added precision) in the transformed program.

3) We show that our analysis is strictly more precise than a classic flow-sensitive type system [26] (Section 7). One interesting consequence is that the program transformation automatically makes a sound flow-insensitive type system (e.g., [40]) as precise as the classic flow-sensitive system [26].

4) We show that our dependent type system soundly controls information flow in the presence of mutable variables without resorting to dynamic mechanisms, such as the dynamic erasure mechanism in previous work [23, 44].

2 Background and Overview

2.1 Information Flow Analysis

We first review standard information flow terminology used in this paper. We assume all variables are associated with security levels. A security policy is specified as the ordering of the security levels, typically in the form of a security lattice. For data $d_1$ with security level $\ell_1$ and data $d_2$ with level $\ell_2$, the policy allows information flow from $d_1$ to $d_2$ if and only if $\ell_1 \sqsubseteq \ell_2$. In this paper, we use two distinguished security levels S (Secret) and P (Public) for simplicity, but keep in mind that the proposed theory is general enough to express richer security levels. The security policy on the levels P and S is defined as P $\sqsubseteq$ S, while S $\not\sqsubseteq$ P. That is, information flow from public data to secret variable is allowed, while the other direction is forbidden. Hereafter, we assume variable s is labeled as S, and variable p is labeled as P unless specified otherwise.
Explicit and Implicit Flows  An information flow analysis prohibits any explicit or implicit information flow that is inconsistent with the given policy. Explicit flows take place when confidential data are passed directly to public variables, such as the command \( p := s \), while implicit flows arise from the control structure of the program. For example, the following program has an implicit flow:

\[
\text{if } (s = 0) \text{ then } p := 0 \text{ else } p := 1
\]

Assume the secret variable \( s \) is either 0 or 1. This code is insecure since it is functionally equivalent to \( p := s \). That is, the confidential data \( s \) is copied to a public variable \( p \).

An information flow security system rules out all explicit and implicit flows; any violation of a given security policy results in an error. As in most information flow analyses, we do not consider timing, termination and other side channels in this paper; controlling side channel leakage (e.g., [1, 28, 43]) is largely an orthogonal issue.

### 2.2 Sources of Imprecision

Most information flow analyses provide soundness (i.e., if the analysis determines that a program is secure, then the program provably prevents disclosure of sensitive data). However, since the problem of checking information flow security is in general undecidable [37, 26], one key challenge of designing an information flow analysis is to maintain soundness, while improving precision (i.e., reject fewer secure programs).

In this section, we introduce the major sources of imprecision in existing type systems. In the next section (Section 2.3), we illustrate how does our novel information flow analysis alleviate those sources of imprecision.

Flow-Insensitivity  The first source of imprecision is flow-insensitivity, meaning that the order of execution is not taken into account in a program analysis [34, 26]. In the context of information flow analysis, the intuition is that an analysis is flow-insensitive if a program is analyzed as secure only when every subprogram is analyzed as secure [26].

Many security type systems, including [32, 35, 40], are flow-insensitive. Consider the program in Figure 1(a) (for now, ignore the brackets). This program is secure since the public variable \( p \) has a final value zero regardless of the secret variable \( s \). However, it is considered insecure by a flow-insensitive analysis because of the insecure subprogram \((x := s; p := x)\). Under the hood, the imprecision arises since the analysis requires fixed levels: the security level of a variable must remain the same throughout the program execution. But in this example, these is no fixed-level for the variable \( x \): when the level is \( S \), \( p := x \) is insecure; when the level is \( P \), \( x := s \) is insecure.

Path-Insensitivity  The second source of imprecision is path-insensitivity, meaning that the predicates at conditional branches are ignored in a program analysis [34, 26]. In the context of information flow analysis, the intuition is that an analysis is path-insensitive if a program is analyzed as secure only when every sequential program generated from one combination of branch outcomes is analyzed as secure.

For instance, the flow-sensitive type system in [26] is path-insensitive; consequently, it rejects the secure program shown in Figure 1(c) (due to Le Guernic and Jensen [29]). This example is secure since the value of the secret variable \( s \) never flows to the public variable \( p_2 \), since the
assignments \( y := s \) and \( x := y \) never execute together in the same program execution. However, the type system in \([26]\) rejects this program because it lacks the knowledge that the two if-statements cannot take the “then” branch in the same execution. Hence, it has to conservatively analyze the security of an impossible program execution: \( x := 0; y := 0; y := s; x := y; p := x \), which is insecure due to an explicit flow from \( s \) to \( p \).

Under the hood, we observe that the imprecision arises from the fact that a path-insensitive analysis (e.g., \([26]\)) requires that the security levels of a variable on two paths to be “merged” (as the least upper bound) after a branch. Consider the first branch in Figure 1(c). The “then” branch requires \( y \) to be \( S \) due to the flow from \( s \) to \( y \). So after that if-statement, the label of \( y \) must be \( S \) (i.e., which path is taken is unknown to the rest of the program). Similarly, \( x \) has label \( S \) after the second if-statement. Hence, \( p_2 := x \) is rejected due to an explicit flow from \( S \) to \( P \).

### 2.3 Overview

In order to alleviate analysis imprecision due to flow- and path-insensitivity, our novel information flow analysis has two major components: a program transformation that enables flow-sensitivity and a type system with dependent security labels, which enables path-sensitivity.

#### 2.3.1 Program Transformation

Consider the example in Figure 1(a) (for now, ignore the brackets). A fixed-level type system rejects this program since the levels of \( x \) at line 1 and 3 are inconsistent. We observe that there are indeed two copies of \( x \) in this program but only the final one (defined at line 2) is released. So without modifying a type system, we can explicitly transform the source program to a semantically equivalent one that explicitly marks different copies.

The source language of our program analysis (Section 3) provides a tunable knob for improved precision: a \textit{bracketed assignment} in the form of \([x := e]\). Such an assignment is semantically identical to \( x := e \) but allows a programmer to request improved precision (the source language allows such flexibility since reduced precision might be preferred for reasons such as more efficient analysis on the program). In particular, for a bracketed assignment \([x := e]\), the program transformation (Section 4) generates a fresh copy for \( x \) and uses that copy in the rest of program until another new copy is generated. For example, given the bracketed assignment at line 2 of Figure 1(a), the transformed program is shown in Figure 1(b), where the second definition of \( x \) and its use at line 3 are replaced with \( x_1 \). The benefit is that the \textit{false dataflow dependency} from \( s \) to \( p \) in the source program is eliminated. Hence, the transformed program can be accepted by a fixed-level type system, by assigning \( x \) and \( x_1 \) to levels \( S \) and \( P \) respectively. In general, we prove that (when all assignments are bracketed) the transformation enables a fixed-level system to be at least as precise as a classic flow-sensitive type system (Section 7).

#### 2.3.2 Dependent Labels

Consider the example in Figure 1(c). A path-insensitive type system rejects this program since such a type system ignores the path conditions under which assignments occur. Consequently, the security level of \( y \) is conservatively estimated as \( S \) after line 2, though when \( p_1 \geq 0 \), variable \( y \) only carries public information.
In our system, path-sensitivity is gained via dependent security labels (i.e., security labels that depend on program states). Compared with a security level drawn directly from a lattice, a dependent security label precisely tracks all possible security levels from different branches; hence, path-sensitivity is gained. Since dependent security labels are orthogonal to bracketed assignments, extra precision can be gained in our system even in the absence of bracketed assignments. For example, while the program in Figure 1(c) can not be accepted using any simple security level for $y$, we can assign to $y$ a dependent label $(p_1 < 0?S : P)$, which specifies an invariant that the level of $y$ is $S$ when $p_1 < 0$ (i.e., the “then” branch is taken at line 2); the level is $P$ otherwise. Such an invariant can be maintained by the type system described in Section 5. For instance, to ensure that the explicit flow from $y$ to $x$ at line 3 is secure, the type system generates a proof obligation $(p_1 > 0 \Rightarrow (p_1 < 0?S : P) \sqsubseteq P)$, meaning that the information flow from $y$ to $x$ must be permissible under the path condition $p_1 < 0$. This proof obligation can easily be discharged by an external solver. The soundness of our type system (Section 6) guarantees that all security violations are detected at compile time.

3 Language Syntax and Semantics

In this paper, we consider a simple imperative WHILE language whose syntax and operational semantics are shown in Figures 2 and 3 respectively. The syntax and semantics are mostly standard: expressions $e$ consist of variables $x$, integers $n$, and composed expressions $e \ op e$, where $\ op$ is a binary arithmetic operation. Commands $c$ consist of standard imperative instructions, including $\text{skip}$, sequential composition $c_1; c_2$, assignments, conditional if branch and while loop. The semantics of expressions are given in the form of $\langle e, m \rangle \Downarrow n$ (big-step semantics), where memory $m$ maps variables to their values. The small-step semantics of commands has the form of $\langle c, m \rangle \rightarrow \langle c', m' \rangle$, where $\langle c, m \rangle$ is a configuration. We use $m\{x \mapsto n\}$ to denote the memory that is identical to $m$ except that variable $x$ is updated to the new value $n$.

The only interesting case is the bracketed assignment $\llbracket x := e \rrbracket$, which is semantically equivalent to normal assignment $x := e$ in the source language. These commands are tunable knobs for improved precision in our information flow analysis, as we show shortly.

4 Program Transformation

To alleviate the imprecision due to flow-insensitivity, one component of our analysis is a novel program transformation that introduces extra variable copies to the source program, so that false dataflow dependencies that otherwise may confuse flow-insensitive analyses are removed.
We propose a general and flexible design for the program transformation. In particular, the program transformation is triggered only for assignments that are marked with brackets. Such a design enables a tunable control of analysis precision for programmers or high-level program analysis built on our meta source language: when there is no bracketed assignment, the transformed program is simply identical to the source program; when all assignments have brackets, the transformation generates a fresh copy of $x$ for each bracketed assignment $Jx := eK$.

Due to the nature of the transformation, the transformed program follows the same syntax and semantics as the source language, except that all bracketed assignments are removed.

To avoid confusion, we use underlined notations for the transformed program: $\underline{e}$ for expressions, $\underline{c}$ for commands and $\underline{m}$ for memories, when both the original and the transformed programs are in the context; otherwise, we simply use $e$, $c$ and $m$ for the transformed programs as well.

### 4.2 Transformation Rules

The program transformation maintains one active copy for each variable in the source code. One invariant maintained by the transformation is that for each program point, there is exactly one active copy for each source-program variable. Intuitively, that unique active copy holds the most recent value of the corresponding source-program variable.

**Definition 1 (Active Set)** An active set $A : \text{Vars} \mapsto \text{Vars}$, is an injective function that maps a source variable to a unique variable in the transformed program.
\[ (n, A) \Rightarrow n \quad (x, A) \Rightarrow A(x) \]
\[ (e_1, A) \Rightarrow e_1' \quad (e_2, A) \Rightarrow e_2' \]
\[ (e_1 \text{ op } e_2, A) \Rightarrow e_1' \text{ op } e_2' \]

**TRSF-Skip**
\[ \langle \text{skip}, A \rangle \Rightarrow \langle \text{skip}, A \rangle \]

**TRSF-Assign**
\[ \langle e, A \rangle \Rightarrow e \quad \langle x := e, A \rangle \Rightarrow \langle x := e, A\{x \mapsto x\} \rangle \]

**TRSF-CREATE**
\[ \langle e, A \rangle \Rightarrow e \quad i \text{ is a fresh index for } x \]
\[ \langle [x := e], A \rangle \Rightarrow \langle x := e, A\{x \mapsto x_i\} \rangle \]

**TRSF-Assign-Create**
\[ \langle e, A \rangle \Rightarrow e \quad \langle c_1, A \rangle \Rightarrow \langle c_1, A_1 \rangle \quad \langle c_2, A \rangle \Rightarrow \langle c_2, A_2 \rangle \quad \Phi(A_1, A_2) \Rightarrow A_3 \]
\[ \langle \text{if } (e) \text{ then } c_1 \text{ else } c_2, A \rangle \Rightarrow \langle \text{if } (e) \text{ then } c_1; A_3 := A_1 \text{ else } c_2; A_3 := A_2, A_3 \rangle \]

**TRSF-While**
\[ \langle c, A \rangle \Rightarrow \langle c_1, A_1 \rangle \quad \Phi(A, A_1) \Rightarrow A_2 \quad \langle c, A_2 \rangle \Rightarrow \langle c, A_3 \rangle \quad \langle e, A_2 \rangle \Rightarrow e \]
\[ \langle \text{while } (e) c, A \rangle \Rightarrow \langle A_2 := A; \text{while } (e) c; A_2 := A_3, A_2 \rangle \]

**TRSF-If**
\[ \langle e, A \rangle \Rightarrow e \quad \langle c_1, A \rangle \Rightarrow \langle c_1, A_1 \rangle \quad \langle c_2, A \rangle \Rightarrow \langle c_2, A_2 \rangle \]
\[ \Phi(A_1, A_2) \Rightarrow A_3 \]

**TRSF-Seq**
\[ \langle c_1, c_2, A \rangle \Rightarrow \langle c_1; c_2, A \rangle \]
\[ \langle e, A \rangle \Rightarrow e \]

**TRSF-Call**
\[ \langle e_1 \text{ op } e_2, A \rangle \Rightarrow e_1' \text{ op } e_2' \]

**TRSF-Local**
\[ \langle v, A \rangle \Rightarrow v \]

\[ A_3 = \text{merge}(A_1, A_2) \]
\[ \Phi(A_1, A_2) \Rightarrow A_3 \]

**Figure 4:** Program Transformation. We use \( A := A' \) as a shorthand for \( \{ A(v) := A'(v) \mid v \in \text{Vars} \land A(v) \neq A'(v) \} \).

**Figure 5:** Merge Function.
For simplicity, we assume that the variables in the transformed program follow the naming convention of $x_i$ where $x \in \text{Vars}$ and $i$ is an index. Hence, for any variable $v$ in the range of $A$, we simply use $v \downarrow$ to denote its corresponding source variable (i.e., a variable without the index). Hence, $v = A(v \downarrow)$ always holds by definition. Moreover, since we frequently refer to the range of $A$, we abuse the notation of $A$ to denote active copies that $A$ may map to (i.e., the range of $A$). That is, we simply write $v \in A$ instead of $v \in \text{Ran}(A)$ in this paper. Moreover, we use $A\{x \mapsto x_i\}$ to denote an active set that is identical to $A$ except that $x$ is mapped to $x_i$.

The transformation rules are summarized in Figure 4. For an expression $e$, the transformation has the form of $\langle e, A \rangle \Rightarrow e$, where $e$ is the transformed expression. The transformation of an expression simply replaces the source variables with their active copies in $A$.

For a command $c$, the transformation has the form of $\langle c, A \rangle \Rightarrow \langle c', A' \rangle$, where $c$ is the source command and $c'$ is the transformed one. Since assignments may update the active set, $A'$ represents the active set after $c$.

Rule (TRSF-Assgin-Create) applies to a bracketed assignment $J x := e K$. It renames the assignee to a fresh variable. For example, line 1 of the transformed program in Figure 1(b) is exactly the same as the original program in 1(a); but the assignee of line 2 is renamed to $x_1$. Rule (TRSF-IF) uses a special $\Phi$ function, defined in Figure 5, to merge the active sets generated from the branches. In particular, $\Phi(A_1, A_2) \Rightarrow A_3$ generates an active set $A_3$ that maps $x$ to a fresh variable iff $A_1(x) \neq A_2(x)$. Transformation for the while loop is a little tricky since we need to compute an active set that is active both before and after each iteration. Rule (TRSF-WHILE) shows one feasible approach: the rule transforms the loop in a way that $A_1$ is a fixed-point: the active set is always $A_1$ before and after an iteration by the transformation.

We note that given an identity function as the initial active set $A$, a program without any bracketed assignment is transformed to itself with a final active set $A$. At the other extreme, the transformation generates one fresh active copy for each assignment when all assignments are bracketed.

4.3 Correctness of the Transformation

One important property of the proposed transformation is its correctness: a transformed program is semantically equivalent to the source program. To formalize this property, we need to build an equivalence relation on the memory for the source program ($m : \text{Vars} \rightarrow \mathbb{N}$) and the memory for the transformed program ($m' : \text{Vars} \rightarrow \mathbb{N}$). We note that the projection of $m$ on an active set $A$ defined as follows shares the same domain and range as $m$. Hence, it naturally specifies an equivalence relation on $m$ and $m$ w.r.t. $A$: $m$ can be directly compared with $m^A$.

**Definition 2 (Memory Projection on Active Set)** We use $m^A$ to denote the projection of $m$ on the active set $A$, defined as follows:

$$\forall x \in \text{Vars}. m^A(x) = m(A(x))$$

We formalize the correctness of our transformation as the following theorem. As stated in the theorem, the correctness is not restricted to any particular initial active set $A$. 
Theorem 1 (Correctness of Transformation) Any transformed program is semantically equivalent to its source:

\[ \forall c, c', m, m', A, A'. \]

\[ \langle c, A \rangle \Rightarrow \langle c, A' \rangle \land \langle c, m \rangle \rightarrow^* \langle \text{skip}, m' \rangle \]

\[ \land \langle c, m \rangle \rightarrow^* \langle \text{skip}, m' \rangle \land m = m^A \]

\[ \Rightarrow m' = (m')^{A'}. \]

**Proof sketch.** By induction on the transformation rules. The full proof is available in Appendix A.

4.4 Relation to Information Flow Analysis

Up to this point, it might be unclear why introducing extra variables can improve the precision of information flow analysis. We first note that transformed programs enable more precise reasoning for dataflows. Consider the program in Figure 1(a) and Figure 1(b). In the transformed program, it is clear that the value stored in \( x \) never flows to variable \( p \); but such information is not obvious in the source program. Moreover, Theorem 1 naturally enables a more precise analysis of the transformed program, since it implies that if any property holds on the final active set \( A' \) for the transformed program, then the property holds on the entire final memory for the original program. That is, in terms of information flow security, the original program leaks no information if the transformed program leaks no information in the subset \( A' \) of the final memory. Consider the example in Figure 1(b) again. Theorem 1 allows a program analysis to accept the (secure) program even though the variable \( x \), which is not in \( A' \), may leak the secret value.

In Section 7, we show that, in general, the program transformation automatically makes a flow-insensitive type system (e.g., the Volpano, Smith and Irvine’s system [40] and the system in Section 5) at least as precise as a classic flow-sensitive type system [26].

4.5 Relation to Single Static Assignment (SSA)

SSA [17] is used in the compilation chain to improve and simplify dataflow analysis. Viewed in this way, it is not surprising that our program transformation shares some similarity with the standard SSA-transformation. However, our transformation is different from the latter in major ways:

- Most importantly, our transformation does not involve the distinguishing \( \phi \)-functions of SSA. First of all, removing \( \phi \)-functions simplifies the soundness proof, since the resulting target language syntax and semantics are completely standard. Moreover, it greatly simplifies information flow analysis on the transformed programs. Intuitively, the reason is that in the standard SSA from, the \( \phi \)-function is added after a branch (i.e., in the form of \((\text{if } e \text{ then } c_1 \text{ else } c_2); x := \phi(x_1, x_2))\). However, without a nontrivial program analysis for the \( \phi \)-function, the path conditions under which \( x := x_1 \) and \( x := x_2 \) occur (needed for path-sensitivity) is lost in the transformed program. On the other hand, extra assignments are inserted under the corresponding branches in our transformation. The consequence is that the path information is immediately available for the analysis on the transformed program.
We defer a more detailed discussion on this topic to Section 5.6 after introducing our type system.

• As discussed in Section 4.4, the final active set $A'$ generated from the transformation is crucial for enabling a more precise program analysis on the transformed program (intuitively, an information flow analysis may safely ignore variables not in $A'$); however, such information is lost in the standard SSA form.

• Our general transformation offers a full spectrum of analysis precision: from adding no active copy to adding one copy for each assignment, but the standard SSA transformation only performs the latter.

5 Type System

The second component of the analysis is a sound type system with expressive dependent labels. The type system analyzes a transformed program along with the final active set; the type system ensures that the final values of the public variables in the final active set are not influenced by the initial values of secret variables.

5.1 Overview

We first introduce the nonstandard features in the type system: dependent security labels and program predicates.

Return to the example in Figure 1(c). We observe that this program is secure because: 1) $y$ holds a secret value only when $p_1 < 0$, and 2) the information flow from $y$ to $x$ at line 3 only occurs when $p_1 > 0$. Accordingly, to gain path-sensitivity, two pieces of information are needed in the type system: 1) expressive security labels that may depend on program states, and 2) an estimation of program states that may reach a program point.

We note that such information can be gained by introducing dependent security labels and program predicates to the type system. For the example in Figure 1(c), the relation between the level of $y$ and the value of $x$ can be described as a concise dependent label $(p_1 < 0?S:P)$, meaning that the security level of $x$ is $S$ when $p_1 < 0$; the level is $P$ otherwise. Moreover, for precision, explicit and implicit flows should only be checked under program states that may reach the program point. In general, a predicate overestimates such states. For the example in Figure 1(c), checking that the explicit flow from $y$ to $x$ is secure under any program state is too conservative, since it only occurs when $p_1 > 0$. With a program predicate that $p_1 > 0$ for the assignment $x := y$, the label of $y$ can be precisely estimated as $P$. Note that our analysis agrees with the definition of path-sensitivity: it understands that the two assignments $y := s$ and $x := y$; never execute together in one execution. The example in Figure 1(c) is accepted by our type system.

5.2 Challenge: Statically Checking Implicit Declassification

Though designing a dependent security type system may seem simple at the first glance, handling mutable variables can be challenging. The implicit declassification problem, as defined in [44], occurs whenever the level of a variable changes to a less restrictive one, but its value remains the
1. $x := 0; y := 0$

2. if ($p_1 < 0$) then
   3. $y := s$
   4. $[p_1 := 1]$;

5. if ($p_1 > 0$) then
   6. $x := y$
   7. $p_2 := x$

(a) Insecure Program.

(b) Transformed Program of 6(a)

Figure 6: Examples: Implicit Declassification.

The type systems in [23, 44] resort to a run-time mechanism to tackle the implicit declassification problem. However, that also means that the type system might change the semantics of the program being analyzed. In this paper, we aim for a purely static solution.

Program Transformation and Implicit Declassification Although the program transformation in Section 4 is mainly designed for flow-sensitivity, we observe that it also helps to detect implicit declassification. Consider the example in Figure 6(a) again, where the assignment at line 4 has brackets. The corresponding transformed program (Figure 6(b)) does not have an implicit declassification problem since updating $p_3$ at line 4 does not change $y$’s level, which depends on the value of $p_1$, rather than $p_3$. Moreover, the insecure program cannot be type-checked since both “then” branches might be executed together.

While adding extra variable copies helps in the previous example, it unfortunately does not eliminate the issue. The intuition is that even for a fully-bracketed program, variables modified in a loop might still be mutable (since the local variables defined in the loop might change in each iteration). Consider the program in 7(a). This program is insecure since it copies $s$ to $y$ in the first iteration, and copies $y$ to $p$ in the next iteration. When fully-bracketed, the loop body becomes

```
if ($x_2 \% 2 = 0$) then $y_1 := s$; $y_3 := y_1$ else ...;
```

$x_3 := x_2 + 1; x_2 := x_3; y_2 := y_3$;

where the labels of $y_1$ and $y_3$ depend on $x_2$. In this program, implicit declassification happens when $x_2$ is updated.

One naive solution is to disallow mutable variables in a program. However, dependence on mutable variables does not necessarily break security. Consider the program in Figure 7(b), which is identical to the previous example except that $y$ is updated at line 8. In this program, $y$’s level depends on the mutable variable $x$, but it is secure since the value of $s$ never flows to the next iteration.

Our Solution Our insight is that changing $y$’s level at line 7 in Figure 7(b) is secure since the value of $y$ is not used in the future (in terms of dataflow analysis, $y$ is dead after line 6).
observation motivates us to incorporate a customized liveness analysis (Section 5.4) into the type system: an update to a variable \( x \) is allowed if no labels of the live variables at that program point depend on \( x \).

5.3 Type Syntax and Typing Environment

In our type system, types are extended with security labels, whose syntax is shown in Figure 8. The simplest form of label \( \tau \) is a concrete security level \( \ell \) drawn from a security lattice \( \mathcal{L} \). Dependent labels, specifying levels that depend on run-time values, have the form of \( (e?\tau_1 : \tau_2) \), where \( e \) is an expression. Semantically, if \( e \) evaluates to a non-zero value, the dependent label evaluates to \( \tau_1 \), otherwise, \( \tau_2 \). A security label can also be the least upper bound, or the greatest lower bound of two labels.

We use \( \Gamma \) to denote a typing environment, a function from program variables to security labels. The integration of dependent labels puts constraints on the typing environment \( \Gamma \) to ensure soundness. In particular, we say \( \Gamma \) is well-formed, denoted as \( \vdash \Gamma \), if: 1) no variable depends on a more restrictive variable, preventing leakage from labels; 2) there is no chain of dependency. These restrictions are formalized as follows, where \( \text{FV}(\tau) \) denotes the free variables in \( \tau \):

**Definition 3 (Well-Formedness)** A typing environment \( \Gamma \) is well-formed, written \( \vdash \Gamma \), if and only if:

\[
\forall x \in \text{Vars}. \quad (\forall x' \in \text{FV}(\Gamma(x)). \quad \Gamma(x') \sqsubseteq \Gamma(x)) \\
\land (\forall x' \in \text{FV}(\Gamma(x)). \quad \text{FV}(\Gamma(x')) = \emptyset)
\]

We note that the definition rules out self-dependence, since if \( x \in \text{FV}(\Gamma(x)) \), we have \( \text{FV}(\Gamma(x)) = \emptyset \). Contradiction.
5.4 Predicates and Variable Liveness

Our type system is parameterized on two static program analyses: a predicate generator and a customized liveness analysis. Instead of embedding these analyses into our type system, we follow the modular design introduced in [44] to decouple program analyses from the type system. Consequently, the soundness of the type system is only based on the correctness of those analyses, regardless of the efficiency or the precision of those analyses.

Predicate Generator  We assume a predicate generator that generates a (conservative) program predicate for each assignment \( \eta \) in the transformed program, denoted as \( \mathcal{P}(\eta) \). A predicate generator is correct as long as each predicate is always true when the corresponding assignment is executed.

A variety of techniques, regarding the trade-offs between precision and complexity, can be used to generate predicates that describe the run-time state. For example, weakest preconditions [21] or the linear propagation [44] could be used. Our observation is that for path-sensitivity, only shallow knowledge containing branch conditions is good enough for our type system.

Liveness Analysis  Traditionally, a variable is defined as alive if its value will be read in the future. But in our type system, if a variable \( x \) is alive, then any free variable in the label of \( x \) should also be considered as alive, because the concrete level of \( x \) depends on those variables. Moreover, we assume at the end of a program, only the variables in the final active set are alive, due to Theorem [1].

The liveness analysis is defined in Figure 9 where \( s \) denotes a program command, and \( \text{final} \) refers to the last command of the program being analyzed. Here, \( \text{final} \) is the initial state for the backward dataflow analysis. \( \text{succ}[s] \) returns the successors (as a set) of the command \( s \). In the \( \mathcal{GEN} \) set of an assignment \( x := e \), both \( \mathcal{FV}(e) \), and \( \bigcup_{v \in \mathcal{FV}(e)} \mathcal{FV}(\Gamma(v)) \), the free variables inside their labels, are included. Since we are analyzing the transformed program, the state of the final active set is crucial for precision. Therefore, the analysis also enforces that, at the end of the program, all active copies in \( \mathcal{A} \) are alive. Other rules are standard for liveness analysis.

Interface to the Type System  We assume each assignment in the transformed language is associated with a unique identifier \( \eta \). We use \( \bullet \eta \) and \( \eta \bullet \) to denote the precise program points right
Γ ⊢ n : ⊥  
Γ ⊢ x : τ

Figure 10: Typing Rules: Expressions.

Γ, pc ⊢ skip
Γ, pc ⊢ c₁
Γ, pc ⊢ c₂

Γ ⊢ e : τ  Γ, τ ⊔ pc ⊢ c₁  Γ, τ ⊔ pc ⊢ c₂
Γ, pc ⊢ if (e) then c₁ else c₂

Γ ⊢ e : τ  Γ, τ ⊔ pc ⊢ c
Γ, pc ⊢ while (e) c

Figure 11: Typing Rules: Commands.

5.5 Typing Rules

The type system is formalized in Figure 10 and Figure 11. Typing rules for expressions have the form of \( \Gamma \vdash e : \tau \), where \( e \) is the expression being checked and \( \tau \) is the label of \( e \). The typing judgment of commands has the form of \( \Gamma, pc \vdash c \). Here, \( pc \) is the usual program-counter label \([37]\), used to control implicit flows.

Most rules are standard, thanks to the modular design of our type system. The only interesting one is rule (T-ASSIGN). For an assignment \( x :=_\eta e \), this rule checks that both the explicit and implicit flows are allowed in the security lattice: \( \tau \sqcup pc \sqsubseteq \Gamma(x) \). Note that since \( \tau \) might be a dependent label that involves free program variables, the \( \sqsubseteq \) relation is technically the lifted version of the relation on the security lattice. Hence, the constraint \( \tau \sqcup pc \sqsubseteq \Gamma(x) \) requires the label of \( x \) to be at least as restrictive as the label of current context \( pc \) and the label \( e \) under any program execution. For precision, the type system validates the partial ordering under the predicate \( P(\bullet \eta) \), the predicate that must hold for any execution that reaches the assignment.

Moreover, the assignment rule checks that for any variable in the liveness set after the assignment, its security label must not depend on \( x \); otherwise, its label might be inconsistent with its value. As discussed in Section 5.2, this check is required to rule out insecure implicit declassification.

At the top level, the type system collects proof obligations in the form of \( \models P \Rightarrow \tau₁ \sqsubseteq \tau₂ \), where \( \tau₁ \) and \( \tau₂ \) are security labels, and \( P \) is a predicate. Such proof obligations can easily be
discharged by theorem solvers, such as Z3 [19].

As an example, consider again the interesting examples in Figure 7. In both programs, we can assign \( y \) to the dependent label \((x \% 2 = 0? S : P)\), and assign \( x \) to the label \( P \). From the liveness analysis, we know that the live sets right after line 7 are \( \{x, y, s\} \) and \( \{x, p, s\} \) for Figure 7(a) and Figure 7(b) respectively. Hence, the type system correctly rejects the insecure program in Figure 7(a) since the check at line 7, \( \forall v \in L_A(\eta\cdot). x \notin FV(\Gamma(v)) \), fails. On the other hand, the check at line 7 succeeds for the program in Figure 7(b). For line 4 in Figure 7(b), the assignment rule generates one proof obligation

\[ \models (x \% 2 = 0) \Rightarrow P \sqcup S \subseteq (x \% 2 = 0? S : P) \]

which is clearly true for any value of \( x \). In fact, the secure program in Figure 7(b) is correctly accepted by the type system in Figure 10 and Figure 11.

5.6 Program Transformation and Information Flow Analysis

We now discuss the benefits of the program transformation in Section 4 for information flow analysis in details.

5.6.1 Simplifying Information Flow Analysis

As discussed in Section 4.5, our transformation does not involve the distinguishing \( \phi \)-functions of SSA. Doing so simplifies information flow analysis on the transformed programs. We illustrate this using the following example, where \( y \) is expected to have the label \((x = 1? P : S)\) afterwards.

\[
\text{if } (x = 1) \text{ then } y := 0 \quad \text{else } y := s
\]

Our transformation yields the following program, which can be verified with labels \( y_1 : P, y_2 : S, y_3 : (x = 1? P : S)\).

\[
\text{if } (x = 1) \text{ then } (y_1 := 0; y_3 := y_1) \\
\text{else } (y_2 := s; y_3 := y_2);
\]

In comparison, the standard SSA form is:

\[
(\text{if } (x = 1) \text{ then } y_1 := 0 \text{ else } y_2 := s) y_3 := \phi(y_1, y_2);
\]

To verify this program, a type system would need at least a nontrivial typing rule for \( \phi \), which somehow “remembers” that \( y_3 := y_2 \) occurs only when \( x = 1 \). Even with such knowledge, the type of \( y_2 \) cannot simply be \( S \), since otherwise, assigning \( y_2 \) to \( y_3 \) at \( \phi \) is insecure. In fact, the labels required for verification are \( y_1, y_2, y_3 : (x = 1? S : P) \).

Similar complexity is also involved for the \( \phi \)-functions inserted for loops: to precisely reason about information flow, the semantics and typing rules of \( \phi \) also need to track the number of iterations.

5.6.2 Improving Analysis Precision

Precision-Wise, bracketed assignments improve analysis precision in two ways. First, as discussed in Section 4.4, they improve flow-sensitivity by introducing new variable definitions. Second, they also improve path-sensitivity by enabling more accurate program predicates. Consider the following example.
\[ x := -1; \]
\[ \text{if } (x > 0) \text{ then } y := S \; \text{ else } y := 1; \]
\[ [x := -x]; \]
\[ \text{if } (x > 0) \text{ then } p := y; \]

This program is secure since \( p \) becomes \( 1 \) regardless of the value of \( s \). However, without the bracket shown, the type system rejects it since no such label \( \tau_y \) satisfies the constraints that \((x > 0) \Rightarrow (S \sqsubseteq \tau_y)\) (arising from the first if) and \((x > 0) \Rightarrow (\tau_y \sqsubseteq P)\) (arising from the second if).

However, with the bracket, the last two lines become
\[ x_1 := -x; \]
\[ \text{if } (x_1 > 0) \text{ then } p := y; \]

This program can be type-checked with \( y \)'s label as \((x > 0)?S : P)\) and a precise enough predicate generator, which generates \( x_1 = -x \) after the assignment \( x_1 := -x \), because constraints \((x > 0) \Rightarrow (S \sqsubseteq \tau_y)\) and \((x > 0 \land x_1 = -x) \Rightarrow (\tau_y \sqsubseteq P)\) can be solved with \( y \)'s label mentioned above.

## 6 Soundness

Central to our analysis is rigorous enforcement of a strong information security property. We formalize this property in this section and sketch a soundness proof. The complete proof is available in Appendix B.

### 6.1 Noninterference

Our formal definition of information flow security is based on noninterference [24]. Informally, a program satisfies noninterference if an attacker cannot observe any difference between two program executions that only differ in their confidential inputs. This intuition can be naturally expressed by semantics models of program executions.

Since a security label may contain program variables, its concrete level cannot be determined statically in general. But it can always be evaluated under a concrete memory:

**Definition 4** For a security label \( \tau \), we evaluate its concrete level under memory \( m \) as follows:

\[ V(\tau, m) = \ell, \text{ where } \langle \tau, m \rangle \Downarrow \ell \]

Moreover, to simplify notation, we use \( T_\Gamma(x, m) \) to denote the concrete level of \( x \) under \( m \) and \( \Gamma \) (i.e., \( T_\Gamma(x, m) = V(\Gamma(x), m) \)).

To formally define noninterference in the presence of dependent labels, we first introduce an equivalence relation on memories. Intuitively, two memories are \((\Gamma, \ell)\)-equivalent if all variables with a level below level \( \ell \) agree on both their concrete levels and values.

**Definition 5** \((\Gamma, \ell)\)-Equivalence Given any concrete level \( \ell \) and \( \Gamma \), we say two memories \( m_1 \) and \( m_2 \) are equivalent up to \( \ell \) under \( \Gamma \) (denoted by \( m_1 \approx^\ell_\Gamma m_2 \)) iff

\[ \forall x \in \text{Vars}. \quad (T_\Gamma(x, m_1) \sqsubseteq \ell \iff T_\Gamma(x, m_2) \sqsubseteq \ell) \land T_\Gamma(x, m_1) \sqsubseteq \ell \implies m_1(x) = m_2(x) \]
It is straightforward to check that \( \approx_\ell \) is an equivalence relation on memories. Note that we require type of \( x \) be bounded by \( \ell \) in \( m_2 \) whenever \( T_\ell(x, m_1) \subseteq \ell \). The reason is to avoid label channels, where confidential data is leaked via the security level of a variable \( [36,44] \).

Given initial labels \( \Gamma \) on variables and final labels \( \Gamma' \) on variables, we can formalize noninterference as follows:

**Definition 6 (Noninterference)** We say a program \( c \) satisfies noninterference w.r.t. \( \Gamma, \Gamma' \) if equivalent initial memories produce equivalent final memories:

\[
\forall m_1, m_2, \ell.
\]

\[
m_1 \approx_\ell m_2 \land \langle c, m_1 \rangle \rightarrow^* \langle \text{skip}, m'_1 \rangle \land
\]

\[
\langle c, m_2 \rangle \rightarrow^* \langle \text{skip}, m'_2 \rangle
\]

\[
\implies m'_1 \approx_\ell m'_2
\]

The main theorem of this paper is the soundness of our analysis: informally, if the transformed program type-checks, then the original program satisfies noninterference. Since the type system applies to the transformed program, we first need to connect the types in the original and the transformed programs. To do that, we define the projection of types for the transformed program in a way similar to Definition 2:

**Definition 7 (Projection of Types)** Given an active set \( \mathcal{A} \) and \( \Gamma \), types of variables in the transformed program, we use \( \Gamma^\mathcal{A} \) to denote a mapping from \( \text{Vars} \) to \( \tau \) as follows:

\[
\forall v \in \text{Vars}. \Gamma^\mathcal{A}(v) = \Gamma(\mathcal{A}(v))
\]

Formally, the soundness theorem states that if a program \( c \) under active set \( \mathcal{A} \) (e.g., an identity function) is transformed to \( \overline{c} \) and final active set \( \mathcal{A}' \), and \( \overline{c} \) is well-typed under the type system (parameterized on \( \mathcal{A}' \)), then \( c \) satisfies noninterference w.r.t. \( \Gamma^\mathcal{A} \) and \( \Gamma^\mathcal{A}' \):

**Theorem 2 (Soundness)**

\[
\forall c, \overline{c}, m_1, m_2, m'_1, m'_2, \ell, \Gamma, \mathcal{A}, \mathcal{A}' .
\]

\[
\langle c, \mathcal{A} \rangle \Rightarrow \langle \overline{c}, \mathcal{A}' \rangle \land \Gamma \land \Gamma \vdash \overline{c} \land m_1 \approx_{\mathcal{A}'} m_2 \land
\]

\[
\langle c, m_1 \rangle \rightarrow^* \langle \text{skip}, m'_1 \rangle \land \langle c, m_2 \rangle \rightarrow^* \langle \text{skip}, m'_2 \rangle
\]

\[
\implies m'_1 \approx_{\mathcal{A}'} m'_2
\]

To approach a formal proof, we notice that by the correctness of the program transformation (Theorem 1), it is sufficient to show that the transformed program leaks no information on the subset \( \mathcal{A}' \). Such connection is illustrated in Figure 12. We formalize the soundness for the transformed program w.r.t. initial and final active sets as follows:

**Theorem 3 (Soundness of Transformed Program)**

\[
\forall \overline{c}, m_1, m_2, m_3, m_4, \ell, \Gamma, \mathcal{A}, \mathcal{A}' .
\]

\[
\langle c, \mathcal{A} \rangle \Rightarrow \langle \overline{c}, \mathcal{A}' \rangle \land \Gamma \land \Gamma \vdash \overline{c} \land m^\mathcal{A}_1 \approx_{\mathcal{A}'} m^\mathcal{A}_2
\]

\[
\land \langle c, m_1 \rangle \rightarrow^* \langle \text{skip}, m_3 \rangle \land \langle c, m_2 \rangle \rightarrow^* \langle \text{skip}, m_4 \rangle
\]

\[
\implies m^\mathcal{A}_3 \approx_{\mathcal{A}'} m^\mathcal{A}_4
\]
Figure 12: Soundness of original and transformed programs.

\[
\text{erase}(m, x, \eta)(x') = \begin{cases} 
0, & x \in \text{FV}(x') \land x' \notin \mathcal{L}_{A'}(\eta) \\
\text{m}(x'), & \text{otherwise}
\end{cases}
\]

\[
\langle x, m \rangle \Downarrow n \quad m' = m\{x \mapsto n\} \quad \text{ST-ERASE}
\]

Figure 13: Erasure Semantics of Assignment.

**Proof sketch.** One challenge in the formal proof is that the equivalence relation \(\approx^\ell_{\Gamma}\) only holds on the active copies and it may break temporarily during the program execution. Consider the example in Figure 7(b). During the first iteration of the loop body, \(y\) holds a secret value but its level is \(P\) right after line 8. Hence, the relation \(\approx^\ell_{\Gamma}\) may break at that point in the small-step evaluation starting from two memories that only differ in secrets. To tolerate such temporary violation of the \(\approx^\ell_{\Gamma}\) relation, we prove the soundness with a new semantics which enforces that the relation \(\approx^\ell_{\Gamma}\) holds for all variables, and the final values of variables in \(A'\) agree with those in the standard semantics. The new semantics, called the *erasure semantics* is shown in Figure 13. The semantics is parameterized on the final active set \(A'\). The only difference from the standard one is for assignments: the new assignment rule (ST-ERASE) sets variables that are not alive and whose types depend on \(x\) to be zero. It is easy to check that the erasure semantics agrees on the final value of the variables in \(A'\). Also, it removes the temporary violation of the equivalence relation by forcing value of \(y\) to be zero after line 7 of Figure 7(b). The complete proof is available in Appendix B.

7 Enabling Flow-Sensitivity with Program Transformation

Recall that the dependent security type system (without program transformation) is flow-insensitive; yet, our program analysis is flow-sensitive with the novel program transformation in Section 4. In this section, we show that this is not a coincidence: the program transformation automatically makes a flow-insensitive type system (e.g., the Volpano, Smith and Irvine’s system [40] and the system in Section 5) flow-sensitive.
7.1 The Hunt and Sands System

In [26], Hunt and Sands define a classic flow-sensitive type system where the security level of a program variable may “float” in the program. In particular, Hunt and Sands (HS) judgments have the form of $pc \vdash_{HS} \Gamma \{ \text{c} \} \Gamma'$, where $\Gamma$ and $\Gamma'$ are intuitively the typing environments before and after executing $c$ respectively.

Consider the program in Figure 1(a). While a flow-insensitive type system rejects it, the HS system accepts it with the following typing environments:

$$
\Gamma \{ x := s \} \Gamma \{ x := 0 \} \Gamma' \{ p := x \} \Gamma'
$$

where $\Gamma = \{ x \mapsto S, p \mapsto P \}$ and $\Gamma' = \{ x \mapsto P, p \mapsto P \}$.

The HS typing rules for commands are summarized in Figure 14. We use $\vdash_{HS}$ to distinguish those judgments from the ones in our system. The interesting rules are rule (HS-I) and rule (HS-WHILE): the former computes the type for each variable as the least upper bound of its labels in the two branches; the latter computes the least fixed-point of a monotone function (the while loop) on a finite lattice.

7.2 Program Transformation and Flow-Sensitivity

We show that the program transformation in Section 4 along with a flow-insensitive type system subsumes the HS system: for any program $c$ that can be type-checked in the HS system, the transformed program of $\mathcal{J}c$ (i.e., a fully-bracketed program) can be type-checked in a flow-insensitive type system. This result has at least two interesting consequences:

1. The program transformation removes the source of “flow-insensitivity”; a flow-insensitivity type system can be automatically upgraded to a flow-sensitive one.

2. The flow- and path-sensitive system in this paper strictly subsumes the HS system: any secure program accepted by the latter is accepted by the former, but not vice versa (e.g., the program in Figure 1(c)).
C-SKIP \[ (pc,\Gamma,A)\{\text{skip} \Rightarrow \text{skip}\}(\Gamma,A) \rightarrow \Gamma_A \]

C-SEQ \[ (pc,\Gamma,A)\{c_1 \Rightarrow c_1\}(\Gamma''',A'') \rightarrow \Gamma_i (pc,\Gamma,A'')\{c_2 \Rightarrow c_2\}(\Gamma',A') \rightarrow \Gamma_j \]

C-ASSIGN \[ pc \vdash_{\text{HS}} \Gamma\{x := e\} \Gamma (\{x := e\},A) \Rightarrow \Gamma_i (pc,\Gamma,A)\{x := e\} \Gamma (\{x := e\},A) \rightarrow \Gamma_A \]

C-IF \[ (pc,\Gamma,A)\{\text{if}\ (e) \text{ then } c_1 \text{ else } c_2 \Rightarrow \text{if}\ (e) \text{ then } c_1 \text{ else } c_2\} (\Gamma',A') \rightarrow \Gamma_0 \]

C-WHILE \[ pc \vdash_{\text{HS}} \Gamma\{\text{while}\ (e)\ c\} \Gamma' (c,e) \Rightarrow (c,e) (\Gamma',A') \rightarrow \Gamma_0 \]

Figure 15: Type Construction in Transformed Program.

To construct types in the transformed program, we first introduce a few notations. Given a typing environment \( \Gamma : \text{Vars} \rightarrow \tau \) for the original program and an active set \( \mathcal{A} \), we can straightforwardly construct a (minimal) typing environment, written \( \Gamma_{\mathcal{A}} \), whose projection on \( \mathcal{A} \) is \( \Gamma \):

\[ \forall v \in \mathcal{A}, \Gamma_{\mathcal{A}}(v) \triangleq \Gamma(v) \]

Easy to check that \( (\Gamma_{\mathcal{A}})^{\mathcal{A}} = \Gamma \).

Moreover, given a sequence of tying environments for the transformed program, say \( \Gamma_1, \Gamma_2, \ldots \), we define a merge function, denoted as \( \cup_{\mathcal{A}} \), that returns the union of \( \Gamma_1, \Gamma_2, \ldots \) so that conflicts in the environments are resolved in the order of \( \Gamma_1, \Gamma_2, \ldots \). For example, \( \cup_{\mathcal{A}}\{x_1 \mapsto S, y_2 \mapsto P\}, \{x_1 \mapsto P, y_2 \mapsto P\} = \{x_1 \mapsto S, y_2 \mapsto P\} \).

For a fully bracketed program \([c]\), we can inductively define the construction of \( \Gamma \) as inference rules in the form of

\[ (pc,\Gamma,A)\{[c] \Rightarrow c\}(\Gamma',A') \rightarrow \Gamma \]

where \( pc,\Gamma,c,\Gamma' \) are consistent with the HS typing rules in the form of \( pc \vdash_{\text{HS}} \Gamma\{c\} \Gamma' ; A, [c],A' \), \( c \) are consistent with the program transformation rules in the form of \( (\{c\},A) \Rightarrow (c,A') \). \( \Gamma \) is the constructed typing environment that, as we show shortly in Theorem 4, satisfies \( \Gamma, pc \vdash c \). The construction algorithm is formalized in Figure 15.

Most parts of the rules are straightforward; they are simply constructed to be consistent with the HS typing rules and the transformation rules in Figure 4. The following lemma makes such connections explicit.

Lemma 1

\[ \forall pc,\Gamma,\Gamma',A,c,c', pc \vdash_{\text{HS}} \Gamma\{c\} \Gamma' \land (\{c\},A) \Rightarrow (c,A') \]

\[ \Rightarrow \exists \Gamma_{\mathcal{A}} (pc,\Gamma,A)\{[c] \Rightarrow c\}(\Gamma',A') \rightarrow \Gamma \]

Proof. By induction on the structure of \( c \). \qed
To construct types for the transformed program: for skip, we use $\Gamma, A$ (the typing environment before this command); for assignment, since $x_i$ must be fresh, we can simply augment $\Gamma, A$ with $\{x_i \mapsto \tau\}$. Other rules simply merge constructed types from subexpressions in a conflict-solving manner, using $\sqcup$. An eagle-eyed reader may find the construction is intuitively correct if there is no conflict at all in the merge operations.

We show that there is no conflict during construction by two observations. First, if a variable has the same active copy before and after transforming a fully-bracketed command $[c]$, then its type must remain the same (before and after $c$ does not conflict at all in the merge operations).

Theorem 4 is at least as precise as the HS system: $A$ and $\Gamma, A'$.

Lemma 2

$$pc \vdash_{HS} \Gamma\{c\} \Gamma' \land \langle [c], A \rangle \Rightarrow \langle \xi, A' \rangle \Rightarrow \forall v \in \text{Vars}. (A(v) = A'(v)) \Rightarrow (\Gamma(v) = \Gamma'(v))$$

Proof sketch. By induction on the structure of $c$. The most interesting cases are for branch and loop.

- if $(e)$ then $c_1$ else $c_2$: by the HS typing rule, $pc \vdash_{HS} \Gamma\{c_1\} \Gamma_1 \land pc \vdash_{HS} \Gamma\{c_2\} \Gamma_2 \land \Gamma' = \Gamma_1 \sqcup \Gamma_2$. By the transformation rules, $\langle [c_i], A \rangle \Rightarrow \langle \xi_i, A_i \rangle, i \in \{1, 2\}$. Suppose $A(v) \neq A_1(v), A_1(v)$ must be a fresh variable generated in $\xi_1$, and hence, cannot be in $A_2$. By the definition of $\Phi, A_2(v)$ must be fresh. This contradicts the assumption $A(v) = A'(v)$.

- while $(c)$: By rule (TRSF-WHILE), we have $\langle [c], A \rangle \Rightarrow \langle \xi_1, A_1 \rangle$, where $A'$ is $A_1$ in this case. Hence, by the assumption, we have $A(v) = A_1(v)$. By rule (HS-WHILE), there is a sequence of environments $\Gamma_i, \Gamma'' = \Gamma_i \sqcup \Gamma''$. By the induction hypothesis, $\Gamma''(v) = \Gamma_i'(v)$. Since $\Gamma'_0 = \Gamma$ and $\Gamma'_{i+1} = \Gamma \sqcup \Gamma''_i$ in rule (HS-WHILE), we can further infer that $\Gamma'_{i+1}(v) = \Gamma''_i(v)$. Hence, we have $\Gamma'(v) = \Gamma_n(v) = \Gamma_0(v) = \Gamma(v)$.

Second, the constructed environment is minimal, meaning that it just specifies types for the variables in $A$ and the freshly generated variables in $\xi$ (denoted as $F\text{Vars}(\xi)$).

For technical reasons, we formalize this property along with the main correctness theorem of the construction, stating that the transformed program $\xi$ type-checks under the constructed environment $\Gamma$. Note that given any $A$, a fully bracketed command $[c]$ always transforms to some $\xi$ and $A'$. Hence, by Lemma 1, the following theorem is sufficient to show that our program analysis is at least as precise as the HS system:

Theorem 4

$$\forall c, \xi, pc, A, A', \Gamma, \Gamma', \xi \vdash_{HS} (pc, \Gamma, A)\{[c]\} \Rightarrow \xi \Rightarrow (\Gamma', A') \Rightarrow \Gamma \Rightarrow \text{Dom}(\Gamma) \subseteq A \cup F\text{Vars}(\xi) \land \xi, pc \vdash c$$

Proof. Complete proof is available in Appendix C. □
An interesting corollary of Theorem 4 is that the transformed program can be type-checked under the classic fixed-level system in [40] as well.

**Corollary 1** Theorem 4 also applies to the type system in Figure 10 and Figure 11 with the restriction that all labels are security levels (i.e., non-dependent labels), which is identical to the system in [40].

**Proof.** We note that the construction in Figure 15 only uses the non-dependent part of our type system. Given all labels are security levels, it is straightforward to check that our type system degenerates to the system in [40]. □

Theorem 4 has a strong prerequisite that all assignments in the original program are bracketed. We note that the result remains true when such prerequisite is relaxed. Intuitively, a bracket is unnecessary when the old and new definitions have the same security label. Otherwise, a bracket is needed for flow-sensitivity. For example, to gain flow-sensitivity, only the second assignment in Figure 1(a) needs a bracket. The strong prerequisite is used in Theorem 4 to make the result general (i.e., type-agnostic).

According to Corollary 1, the result that any secure program accepted by the HS system is accepted by our analysis is true even if all dependent security labels degenerate to simple security levels. On the other hand, introducing dependent security labels makes our analysis strictly more precise than the HS system. For example, the program in Figure 1(c) cannot be verified without dependent security labels, but it can be type-checked with a label $y : (p_1 < 0?S: P)$.

### 7.3 Comparison with the transformation in [26]

Hunt and Sands show that if a program can be type-checked in the HS system, then there is an equivalent program which can be type-checked by a fixed level system [26]. However, their construction of the equivalent program is type guided, meaning that the program transformation assumes that security labels have already been obtained in the HS system, while our program transformation (Figure 4) is general and syntax-directed. An interesting application of our transformation is to test the typeability of the HS system without obtaining the types needed in the HS system in the first place.

It is noteworthy that our transformation is arguably simpler than the HS transformation since our rule for loop has no fixed-point construction while the latter has one. The reason is that compared with the HS transformation, the goal of our transformation is easier to achieve: our transformation improves analysis precision, while the HS transformation infers the type for each variable in a program. For example, consider a loop with only one assignment $x := x + 1$, and $x$ is initially $P$. In the HS system, the transformed program is $x_P := x_P + 1$, where $x_P$ is the public version of variable $x$. On the other hand, our transformation generates $x_1 := x_2; x_2 := x_1 + 1$. From the perspective of inferring labels, introducing $x_1$ and $x_2$ might seem unnecessary since they must have the same label according to the type system. However, doing so might improve analysis precision (e.g., the type system can specify the dependencies on $x_1$ and $x_2$ separately with two copies of $x$).
8 Related Work

We refer to [37] for a comprehensive survey of static information flow analysis. Here, we focus on the most relevant ones.

Dependent Labels and Information Flow Security Dependent types have been widely studied and have been applied to practical programming languages (e.g., [7, 16, 31, 32, 41, 42]). New challenges emerge for information flow analysis, such as precise, sound handling of information channels arising from label changes.

For security type systems, the most related works are SecVerilog [23, 44], Lourenço and Caires [30] and Murray et al. [31]. SecVerilog is a Verilog-like language with dependent security labels for verifying timing-sensitive non-interference in hardware designs. The type systems in [23, 44] are not purely static: they remove implicit declassification by a run-time enforcement that modifies program semantics. A recent extension to SecVerilog [22] alleviates such limitation by hardware-specific static reasoning. However, those type systems do not handle loops (absent in hardware description languages), which gives rise to new challenges for soundness. Moreover, they are not flow-sensitive. The recent work [30] also allows the security type to depend on runtime values. However, the system is flow-insensitive, and it does not have a modular design that allows tunable precision. Moreover, the language has limited expressiveness: it has no support for recursion, and it disallows dependence on mutable variables. Exploring dependent labels to their full extent exposes new challenges that we tackle in this work, such as implicit declassification. Murray et al. [31] present a flow-sensitive dependent security type system for shared-memory programs. The type system enforces a stronger security property: timing-sensitive non-interference for concurrent programs. However, even when the extra complexity due to concurrency and timing sensitivity are factored out, extra precision in their system is achieved via a floating type system that tracks the typing environments and program states throughout the program. In comparison, our analysis achieves flow-sensitivity via a separate program transformation, which results in an arguably simpler type system. Moreover, for dependency on mutable variables, their system only allows a variable’s security level to upgrade to a higher one, while our system allows a downgrade to a lower level when doing so is secure.

Some prior type systems for information flow also support limited forms of dependent labels [25, 27, 32, 38, 39, 45]. The dependence on run-time program state, though, is absent in most of these, and most of them are flow- and path-insensitive.

Flow-Sensitive Information Flow Analysis Flow-sensitive information flow control [8, 26, 36] allows security labels to change over the course of computation. Those systems rely on a floating type system or a run-time monitor to track the security labels at each program point. On the other hand, the program transformation in our paper eliminates analysis imprecision due to flow-insensitivity. Moreover, the bracketed assignments in a source program provide tunable control for needed analysis precision. These features offer better flexibility and make it possible to turn a flow-insensitive analysis to be flow-sensitive.

Semantic-Based Information Flow Analysis Another direction of information flow security is to verify the semantic definition of noninterference based on program logics. The first work that
used a Hoare-style semantics to reason about information flow is by Andrews and Reitman [5].

Independence analysis based on customized logics [2–4] was proposed to check whether two variables are independent or not. Self-Composition [9, 18] composes a program with a copy of itself, where all variables are renamed. The insight is that noninterference of a program \( P \) can be reduced to a safety property for the self-composition form of \( P \).

Relational Hoare Logic [13] was first introduced for a core imperative program to reason about the relation of two program executions. It was later extended to verify security proofs of cryptographic constructions [11] and differential privacy of randomized algorithms [10, 12]. In the context of information flow security, Relational Hoare Type Theory [33] extends Hoare Type Theory and has been used to reason about advanced information flow policies.

Though some semantic-based information flow analyses are flow- and path-sensitive, most mechanisms incur heavy annotation burden and steep learning curve on programmers. We believe our approach shows that it is not necessary to resort to those heavyweight methods to achieve both flow- and path-sensitivity.

9 Conclusions and Future Work

This paper presents a sound yet flow- and path-sensitive information flow analysis. The proposed analysis consists of a novel program transformation as well as a dependent security type system that rigorously controls information flow. We show that our analysis is both flow- and path-sensitive. Compared with existing work, we show that our analysis is strictly more precise than a classic flow-sensitive type system, and it tackles the tricky implicit declassification issue completely at the compile time. Moreover, the novel design of our analysis allows a user to control the analysis precision as desired. We believe our analysis offers a lightweight approach to static information flow analysis along with improved precision.

The proposed analysis alleviates analysis imprecision due to data- and path-sensitivity, but it still may suffer from other sources of imprecision, such as the presence of insecure dead code and false control-flow dependency. For example, consider the secure program in Figure 16 (simplified from an example in [14]) with security labels \( s : S, p : P \). In this example, although \( x \) is updated under a confidential branch condition, both branches result in the same state where \( x = 1 \); thus, the outcome of \( p \) is independent of the value of \( s \). However, our analysis rejects this program since rule (T-ASSIGN) conservatively assumes that any public variable modified in a confidential branch would leak information. Motivated by the type system in [14], a promising direction that we plan to investigate is to incorporate sophisticated static program analyses so that the implicit flows can be ignored for the variables whose values are independent of branch outcomes. Additionally, hybrid information flow monitors (e.g., [6, 14, 36]) are shown to be more precise than static flow-sensitive type systems. We plan to compare the analysis precision with those systems in our future work.

Figure 16: False Control-Flow Dependency.
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A Correctness of the Transformation

We first show a few lemmas needed to prove the correctness of the program transformation.

**Lemma 3 (Equal Expression)** Any transformed expression e evaluates to the same value as in the original program, and the transformation does not introduce fresh variables.

\[ \forall e, A, m, m', m'' = m^A \land \langle e, A \rangle \Rightarrow e \land \langle e, m \rangle \downarrow n \land \langle e, m \rangle \downarrow n' \Rightarrow n = n' \land \text{Vars}(e) \subseteq A. \]

**Proof.** By induction on the structure of the expression e:

- Case \( e = n \): trivial since \( e = n \) and \( \text{Vars}(e) = \emptyset \).

- Case \( e = x \): trivial since we have \( e = A(x) \) by the transformation rule. By the assumption \( m = m^A \), we have \( m(x) = m^A(x) \). Moreover, \( \text{Vars}(e) = A(x) \in A \).

- Case \( e = e_1 \text{ op } e_2 \): from the transformation, we know that \( e \) has the form of \( e_1 \text{ op } e_2 \). By the induction hypothesis, we have \( \langle e_1, m \rangle \downarrow n_1, \langle e_2, m \rangle \downarrow n_2, \langle e_1, m \rangle \downarrow n'_1, \langle e_2, m \rangle \downarrow n'_2 \), and \( n_1 = n'_1 \land n_2 = n'_2 \). Thus, we have \( e \) and \( e \) evaluate to the same value. Moreover, \( \text{Vars}(e) \subseteq A \) by the induction hypothesis.

□

**Lemma 4 (Set-Assignment)**

\[ \forall A_1, A_2, m_1, m_2. \]

\[ \langle A_2 := A_1, m_1 \rangle \rightarrow \langle \text{skip}, m_2 \rangle \Rightarrow m_1^{A_1} = m_2^{A_2} \]

**Proof.** Recall that \( A_2 := A_1 \) is just a shorthand for assigning \( v_j \in A_1 \) to \( v_i \in A_2 \) for each \( v \in \text{Vars} \) when \( i \neq j \). Hence, for all \( x \in \text{Vars} \) such that \( A_1(x) \neq A_2(x) \), we have \( m_2^{A_2}(x) = m_1^{A_1}(x) \). For any other variable \( x \) (such that \( A_1(x) = A_2(x) \)), we know its value is not updated. Thus, we have \( m_1^{A_1} = m_2^{A_2} \).

□

**Proof of Theorem 1** Any transformed program is semantically equivalent to its source:

\[ \forall c, c', m, m', m'', A, A'. \]

\[ \langle c, A \rangle \Rightarrow \langle c, A' \rangle \land \langle c, m \rangle \Rightarrow \langle \text{skip}, m' \rangle \land \langle c, m \rangle \Rightarrow \langle \text{skip}, m' \rangle \land m = m^A \Rightarrow m' = (m')^{A'}. \]

**Proof.** By induction on the transformation rules in the form of \( \langle c, A \rangle \Rightarrow \langle c, A' \rangle \).

- Case \( \langle \text{skip}, A \rangle \Rightarrow \langle \text{skip}, A \rangle \): trivial since no change is made to the memory \( m \) and the active set \( A \).
• Case \( \langle x := e, \mathcal{A} \rangle \wedge \langle x \mapsto x \rangle \): From Lemma 3, we know that given \((e, m) \downarrow n\) and \((e, m) \downarrow n'\), then \(n = n'\). We know from the semantics rule S-Assign that \(m'(x) = n\) and \((m')^{\mathcal{A}}(x) = m(x) = n'\). Thus, we have \(m'(x) = (m')^{\mathcal{A}}(x)\). For all variables other than \(x\), they remain unchanged. Thus, we have \(m' = (m')^{\mathcal{A}}\).

• Case \( \langle \lfloor x := e \rfloor, \mathcal{A} \rangle \wedge \langle x \mapsto x \rangle \): Similar to the previous case.

• Case \( \langle c_1; c_2, \mathcal{A} \rangle \wedge \langle c_1; c_2, \mathcal{A}_2 \rangle \): By the induction hypothesis, given \(\langle c_1, \mathcal{A} \rangle \wedge \langle c_1, \mathcal{A}_1 \rangle\), \(\langle c_1, m \rangle \rightarrow^* \langle \text{skip}, m_1 \rangle \) and \(\langle c_1, m \rangle \rightarrow^* \langle \text{skip}, m_1 \rangle\), we have \(m_1 = m_1^{\mathcal{A}_1}\). By the semantics, \(\langle \text{skip}; c_2, m_1 \rangle \rightarrow \langle c_2, m_1 \rangle \rightarrow^* \langle \text{skip}, m_2 \rangle\) and \(\langle \text{skip}; c_2, m_1 \rangle \rightarrow \langle c_2, m_1 \rangle \rightarrow^* \langle \text{skip}, m_2 \rangle\). By the induction hypothesis on \(\langle c_2, \mathcal{A}_1 \rangle \wedge \langle c_2, \mathcal{A}_2 \rangle\), we have \(m_2 = m_2^{\mathcal{A}_2}\). Thus, we have \(m' = (m')^{\mathcal{A}}\).

• Case \( \langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, \mathcal{A} \rangle \wedge \langle \text{if} (e) \text{ then } c_1; \mathcal{A}_3 := \mathcal{A}_1 \text{ else } c_2; \mathcal{A}_3 = \mathcal{A}_2, \mathcal{A}_3 \rangle\): From Lemma 3, we know \(e\) and \(c\) must evaluate to some value \(n\). Thus, both evaluation must take the same branch. Without losing generality, we consider the case when \(n \neq 0\).

By the induction hypothesis on the transformation \(\langle c_1, \mathcal{A} \rangle \wedge \langle c_1, \mathcal{A}_1 \rangle\), we know that given \(\langle c_1, m \rangle \rightarrow^* \langle \text{skip}, m_1 \rangle\) and \(\langle c_1, m \rangle \rightarrow^* \langle \text{skip}, m_1 \rangle\), we have \(m_1 = m_1^{\mathcal{A}_1}\). For the rest of the evaluation \(\langle \text{skip}; \mathcal{A}_3 := \mathcal{A}_1, m_1 \rangle \rightarrow \langle \mathcal{A}_3 := \mathcal{A}_1, m_1 \rangle \rightarrow^* \langle \text{skip}, m_3 \rangle\), by Lemma 4 we have \(m_3^{\mathcal{A}_3} = m_1^{\mathcal{A}_1} = m_1\). Thus, we have \(m' = m_1 = m_3^{\mathcal{A}_3} = (m')^{\mathcal{A}}\).

• Case \( \langle \text{while} (e) c, \mathcal{A} \rangle \wedge \langle \mathcal{A}_1 := \mathcal{A}_1 \text{ while } (e) \langle c; \mathcal{A}_1 := \mathcal{A}_2, \mathcal{A}_1 \rangle \rangle\): The evaluation looks like:

\[
\langle \mathcal{A}_1 := \mathcal{A}_1 \text{ while } (e) \langle c; \mathcal{A}_1 := \mathcal{A}_2, \mathcal{A}_1 \rangle \rangle
\rightarrow
\langle \text{if} (e) \langle c; \mathcal{A}_1 := \mathcal{A}_2, \mathcal{A}_1 \rangle \rangle
\rightarrow
\langle c; \text{while} (e) c, m \rangle
\rightarrow
\langle \text{skip; while} (e) c, m \rangle
\rightarrow
\langle \text{while} (e) c, m \rangle
\rightarrow
\ldots
\]

By Lemma 4, we have \(m_1^{\mathcal{A}_1} = m^{\mathcal{A}} = m\). We proceed by induction on the number of iterations being executed in the evaluation:

- Base case: 0 iteration is executed in the original program. It must be true that \((e, m) \downarrow 0\) and \(m' = m\). Since the transformation requires that \(\langle e, \mathcal{A}_1 \rangle \wedge \langle e, m_1 \rangle \downarrow 0\) by Lemma 3. Hence, 0 iteration is executed in the transformed program, and \(m' = m_1\).

- Induction case for \(N\) iterations \((N \geq 1)\): The evaluation of original while looks like:

\[
\langle \text{while} (e) c, m \rangle
\rightarrow
\langle \text{if} (e) \text{ then } (c; \text{while} (e) c) \text{ else skip}, m \rangle
\rightarrow
\langle c; \text{while} (e) c, m \rangle
\rightarrow
\langle \text{skip; while} (e) c, m \rangle
\rightarrow
\langle \text{while} (e) c, m \rangle
\rightarrow
\ldots
\]
The evaluation of the transformed while looks like:

\[
\langle \text{while} (e) (c; A_1 := A_2), m_1 \rangle \\
\to \langle \text{if} (e) \text{ then } (c; A_1 := A_2; \text{while} (e) (c; A_1 := A_2)) \text{ else } \text{skip}, m_1 \rangle \\
\to \langle c; A_1 := A_2; \text{while} (e) (c; A_1 := A_2), m_1 \rangle \\
\to^* \langle \text{skip}; A_1 := A_2; \text{while} (e) (c; A_1 := A_2), m_2 \rangle \\
\to \langle A_1 := A_2; \text{while} (e) (c; A_1 := A_2), m_2 \rangle \\
\to \langle \text{while} (e) (c; A_1 := A_2), m_3 \rangle \to ...
\]

Here, we know that the transformed program must take the “if” branch due to the same argument as in the base case. Since \( m_1^{A_1} = m \), by (structural) induction hypothesis on \( c \) and the transformation rule which requires \( \langle c, A_1 \rangle \Rightarrow \langle c, A_2 \rangle \), we have \( m_1 = m_2^{A_2} \).

By Lemma 4, we have \( m_3^{A_1} = m_2^{A_2} = m_1 \). By the induction hypothesis when the original program runs for \( N - 1 \) iterations, we have \( m_1^{A_1} = m' \) as desired.

\[\square\]
\[ e ::= \ldots \mid [n] \]
\[ c ::= \ldots \mid [c] \]
\[ n \sim n \quad [n_1] \sim [n_2] \quad m_1 \sim m_2 \implies \forall x.m_1(x) \sim m_2(x) \]

\[ c \sim c \quad c_1 \sim c_3 \quad c_2 \sim c_4 \quad [c_1] \sim [c_2] \]

Figure 17: Extended Syntax

Figure 18: Equivalence on Memories and Commands

B Soundness

We need more definitions before showing the soundness of our analysis. We first use a distinguished label \( L \) ("low") to define what is observable to the low observer. Since the lemmas and theorems are valid regardless of what level \( L \) is, the propositions proved hold for any label \( \ell \) in the security lattice.

As discussed in Section 6, we prove the soundness based on the erasure semantics in Figure 13. The connection between the erasure and the standard semantics is established by the following lemma:

Lemma 5

\[ \forall c, m, m_1, m_2, A. \langle m, c \rangle \rightarrow^* \langle m_1, \text{skip} \rangle \land \langle m, c \rangle \rightarrow^*_{ER(A)} \langle m_2, \text{skip} \rangle \implies \forall v \in A. m_1(v) = m_2(v) \]

Proof. We note that the erasure semantics only changes the values of dead variables. Hence, the result is trivial given that the live variable analysis is correct. \[ \square \]

Next, we prove that any well-typed target program under the erasure semantics satisfies the noninterference property. To simplify notation, we will use \( \rightarrow \) instead of \( \rightarrow_{ER(A)} \) hereafter. To prove soundness, we extend the language syntax and semantics with explicitly marked high values and commands. Memory is extended to track high values as well. The extension is useful since the low equivalence relation we defined earlier corresponds to the equivalence relation on the marked memories.

By showing the completeness of the extended language, all interesting proof are then conducted on the extended language. To do that, we first prove several useful lemmas, and then show the type system enforces noninterference.

B.1 Extended Language

Extended syntax The extended syntax is shown in Figure 17. We augment memories to map high variables to bracketed results. In a similar way, syntax is augmented to include bracketed results, and bracketed commands. Intuitively, bracketed results represent values from high memory, and bracketed commands represent commands executed in a high pc context (such as in a branch with a high guard).
\[
\langle [n], m \rangle \Downarrow [n] \quad \langle [e_1, m] \Downarrow [n_1] \quad \langle [e_2, m] \Downarrow n_2 \quad n = n_1 \text{ op } n_2
\]
\[
\langle [e_1 \text{ op } e_2, m] \Downarrow [n] \rangle
\]
\[
\langle [e_1, m] \Downarrow n_1 \quad \langle [e_2, m] \Downarrow [n_2] \quad n = n_1 \text{ op } n_2
\]
\[
\langle [e_1 \text{ op } e_2, m] \Downarrow [n] \rangle
\]
\[
\langle [e_1, m] \Downarrow [n_1] \quad \langle [e_2, m] \Downarrow [n_2] \quad n = n_1 \text{ op } n_2
\]
\[
\langle [e_1 \text{ op } e_2, m] \Downarrow [n] \rangle
\]

**Figure 19: Extended Semantics: Expressions**

\[
\text{erase}(m, x, \eta)(x') = \begin{cases} 
0, & x \in \text{FV}(x') \land x' \not\in \mathcal{L}_A(\eta) \land \mathcal{T}(x', m) \sqsubseteq L \\
[0], & x \in \text{FV}(x') \land x' \not\in \mathcal{L}_A(\eta) \land \mathcal{T}(x', m) \nsubseteq L \\
m(x'), & \text{otherwise}
\end{cases}
\]

**S-Skip1**

\[
\langle [\text{skip}], m \rangle \rightarrow \langle \text{skip}, m \rangle
\]

**S-Bracket**

\[
\langle [c], m \rangle \rightarrow \langle \text{[c]}, m' \rangle
\]

**S-Asgn1**

\[
\langle [e, m] \Downarrow n \quad \mathcal{T}(x, m) \sqsubseteq L \quad m' = m\{x \mapsto n\}
\]
\[
\langle x := \eta e, m \rangle \rightarrow \langle \text{skip, erase}(m', x, \eta) \rangle
\]

**S-Asgn2**

\[
\langle [e, m] \Downarrow n \quad \mathcal{T}(x, m) \nsubseteq L \quad m' = m\{x \mapsto [n]\}
\]
\[
\langle x := \eta e, m \rangle \rightarrow \langle \text{skip, erase}(m', x, \eta) \rangle
\]

**S-Asgn3**

\[
\langle [e, m] \Downarrow [n] \quad m' = m\{x \mapsto [n]\}
\]
\[
\langle x := \eta e, m \rangle \rightarrow \langle \text{skip, erase}(m', x, \eta) \rangle
\]

**S-If3**

\[
\langle [e, m] \Downarrow [n] \quad n \neq 0
\]
\[
\langle \text{if } (e) \text{ then } c_1 \text{ else } c_2, m \rangle \rightarrow \langle [c_1], m \rangle
\]

**S-If4**

\[
\langle [e, m] \Downarrow [n] \quad n = 0
\]
\[
\langle \text{if } (e) \text{ then } c_1 \text{ else } c_2, m \rangle \rightarrow \langle [c_2], m \rangle
\]

**Figure 20: Extended Semantics: Commands**
Figure 21: Extended Typing Rules: Expressions

\[ \frac{\ell \not\subseteq L}{\Gamma \vdash [n] : \ell} \quad \text{T-BracketExp} \]

T-BracketCmd

\[ \frac{\Gamma, \tau \vdash c \quad pc \subseteq \tau \quad \tau \not\subseteq L}{\Gamma, pc \vdash [c]} \]

Figure 22: Extended Typing Rules: Commands

Extended Semantics

The operational semantics is augmented to propagate brackets, as shown in Figure[19, 20] All rules are extensions to the original grammar except that (S-ASGN) is split into three rules: (S-ASGN1), (S-ASGN2), (S-ASGN3). Moreover, the erasure semantics for the extended language also adds brackets when needed. All rules with brackets work the same way as the normal rules from computational perspective. Brackets are just syntactic markers.

The extended language requires extra typing rules, shown in Figure[21] Rule (T-BracketExp) treats bracketed expression as high. Note that this rule requires a security level \( \ell \), which by definition cannot depend on any program state. Rule (T-BracketCmd) in Figure[22] is given to support the soundness proof. Bracketed command should be type-checked under a pc label that is not bounded by \( L \) in the type system.

Equivalence on Memories and Commands

We define the equivalence of memories and commands up to label \( L \) as in Fig.[18] Intuitively, bracketed memory and commands are indistinguishable. An equivalence relation \( \sim \) is defined on memories such that \( m_1 \sim m_2 \) if and only if they agree on all low variables, and the rest of variables are all high (with brackets).

The type system enforces an important invariant on the memory: a variable holds bracketed value if and only if the security level of that variable is high. This is formalized as follows:

Definition 8 (Well-Formedness) A variable \( x \) is well-formed under memory \( m \), denoted as \( \vdash_m x \) if the following condition holds:

\[ T(x, m) \not\subseteq L \iff \exists n. \langle x, m \rangle \downarrow [n] \]

A memory \( m \) is well-formed, denoted as \( \vdash m \) if all variables are well formed under \( m \):

\[ \vdash m \iff (\forall x \in m. \vdash_m x) \]

Completeness of the Extended Language

It is then clear that for any two low-equivalent standard memories \( m_1, m_2 \), there are augmented memories, simply by putting brackets for high variables, that agree with standard memories on all low variables; and vise versa. Hence, the completeness result (Lemma[6]) justifies the noninterference result in the unextended language, by showing that starting from any \( m_1 \sim m_2 \) that are both well-formed, the resulting memories are still equivalent in the augmented language.

Completeness means that every step in the new semantics can be performed in the unextended semantics (maybe with removal of brackets) and vice versa. More formally, given that \( c \) is a command in the extended language, let us use the notation of \( \lfloor c \rfloor \) to denote removal of all brackets from \( c \) in the obvious way, yielding a command from the original language. Similarly, we define \( \lfloor m \rfloor \) to convert memory. Completeness can be expressed as the following lemma.

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Lemma 6 (Completeness of the Extended Language)

\[ \Gamma, pc \vdash c \land \{\lfloor c \rfloor, \lfloor m \rfloor\} \rightarrow^* \langle \text{skip}, m' \rangle \implies \exists m''. \langle c, m \rangle \rightarrow^* \langle \text{skip}, m'' \rangle \land m' = \lfloor m'' \rfloor \]

Proof. By rule induction on each evaluation step. \qed

B.2 Soundness Proof

Next, we prove the soundness of the type system on the extended language language with the erasure semantics. We first introduce a couple of useful lemmas.

Lemma 7 The type comparison is conservative:

\[ \forall m, \tau_1, \tau_2. \tau_1 \sqsubseteq \tau_2 \implies V(\tau_1, m) \sqsubseteq V(\tau_2, m) \]

Proof. Clear from the lifted definition of \( \sqsubseteq \) on (dependent) labels. \qed

Lemma 8 Low expressions always evaluate to ordinary integers (without brackets) under well-formed memory:

\[ \vdash m \land \Gamma \vdash e : \tau \land V(\tau, m) \sqsubseteq L \implies \exists n. \langle e, m \rangle \Downarrow n \]

Proof. By induction on the structure of the expression \( e \):

- Case \( e = n \): trivial.
- Case \( e = [n] \): contradiction to the typing rule (T-BRACKETEXP).
- Case \( e = x \): \( x \) could be either low or high.
  - Case \( x \) is low (\( T(x, m) \sqsubseteq L \)): clear from the definition of \( \vdash m \).
  - Case \( x \) is high (\( T(x, m) \not\sqsubseteq L \)): contradiction to the assumption \( V(\tau, m) \sqsubseteq L \).
- Case \( e = e_1 \oplus e_2 \): From \( \Gamma \vdash e : \tau \), we can infer that \( \Gamma \vdash e_1 : \tau_1 \), \( \Gamma \vdash e_2 : \tau_2 \) and \( \tau = \tau_1 \sqcup \tau_2 \). By Lemma 7, \( V(\tau_1, m) \sqsubseteq L \) and \( V(\tau_2, m) \sqsubseteq L \). Hence by induction hypothesis, \( \exists n_1, n_2. \langle e_1, m \rangle \Downarrow n_1, \langle e_2, m \rangle \Downarrow n_2 \). Thus, \( \langle e, m \rangle \Downarrow n \), where \( n = n_1 \oplus n_2 \) by the semantics. \qed

Lemma 9 (PC Subsumption)

\[ \Gamma, pc \vdash c \land pc' \sqsubseteq pc \implies \Gamma, pc' \vdash c \]

Proof. By rule induction on the typing derivation for \( c \):

- Case \text{skip} : From typing rule (T-SKIP), we know that \( \Gamma, pc' \vdash \text{skip} \) for any \( pc' \).
- Case \( x := \eta e \): From typing rule (T-ASSIGN), we know that \( \Gamma \vdash e : \tau \) and \( \vdash \mathcal{P}(\bullet \eta) \Rightarrow \tau \sqcup pc \sqsubseteq \Gamma(x) \) and \( \forall v \in L(A(\bullet \eta), x \not\in \text{FV}(\Gamma(v))) \). Since \( pc' \sqsubseteq pc \), we have \( \tau \sqcup pc' \sqsubseteq \tau \sqcup pc \sqsubseteq \Gamma(x) \). Thus, we have \( \vdash \mathcal{P}(\bullet \eta) \Rightarrow \tau \sqcup pc' \sqsubseteq \Gamma(x) \) and there is no change to the other conditions. So we can derive \( \Gamma, pc' \vdash x := \eta e \).
• Case $c_1; c_2$: From typing rule (T-SEQ), we know that $\Gamma, pc \vdash c_1$ and $\Gamma, pc \vdash c_2$. By the induction hypothesis, we have $\Gamma, pc' \vdash c_1$ and $\Gamma, pc' \vdash c_2$. Thus, we can derive $\Gamma, pc' \vdash c_1; c_2$.

• Case if $(e)$ then $c_1$ else $c_2$: From typing rule (T-IF), we know that $\Gamma \vdash e : \tau$, $\Gamma, pc \sqcup \tau \vdash c_1$ and $\Gamma, pc \sqcup \tau \vdash c_2$. Since $pc' \subseteq pc$, we have $\tau \sqcup pc' \subseteq \tau \sqcup pc$. So by the induction hypothesis, we have $\Gamma, pc' \sqcup \tau \vdash c_1$ and $\Gamma, pc' \sqcup \tau \vdash c_2$. Hence, $\Gamma, pc' \vdash$ if $(e)$ then $c_1$ else $c_2$.

• Case while $(e) \ c$: From typing rule (T-WHILE), we know that $\Gamma \vdash e : \tau$, $\Gamma, pc \sqcup \tau \vdash c$. Since $pc' \subseteq pc$, we have $\tau \sqcup pc' \subseteq \tau \sqcup pc$. So by the induction hypothesis, we have $\Gamma, pc' \sqcup \tau \vdash c$. Hence, $\Gamma, pc' \vdash$ while $(e) \ c$.

\[\square\]

**Lemma 10 (Preservation)**

\[\Gamma, pc \vdash c \land \langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle \land \vdash m_1 \Rightarrow \vdash m_2 \land \Gamma, pc \vdash c'\]

**Proof.** By rule induction on the evaluation rules $\langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle$:

- Case $\langle [\text{skip}], m_1 \rangle \rightarrow \langle \text{skip}, m_1 \rangle$: trivial.

- Case $\langle \text{skip}; c, m_1 \rangle \rightarrow \langle c, m_1 \rangle$: Trivial since $m$ does not change and $\Gamma, pc \vdash c$ is required in rule (T-SEQ).

- Case $\langle c_1; c_2, m_1 \rangle \rightarrow \langle c'_1; c_2, m_2 \rangle$: From the assumption, we have $\langle c_1, m_1 \rangle \rightarrow \langle c'_1, m_2 \rangle$. From typing rule (T-SEQ), we have $\Gamma, pc \vdash c_i, i \in \{1, 2\}$. So by the induction hypothesis, we have $\Gamma, pc \vdash c'_1$ and $\vdash m_2$. Hence we can derive $\Gamma, pc \vdash c'_1; c_2$.

- Case $\langle [c], m_1 \rangle \rightarrow \langle [c'], m_2 \rangle$: From typing rule (T-BRACKETCMD), we know that there exists some label $\tau$ such that $\tau \notin L \land pc \subseteq \tau$ and $\Gamma, \tau \vdash c$. From the assumption, we know that $\langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle$. By induction hypothesis, we have $\Gamma, \tau \vdash c'$ and $\vdash m_2$. Thus, we can derive $\Gamma, pc \vdash [c']$.

- Case $\langle x :=_{\eta} e, m_1 \rangle \rightarrow \langle \text{skip}, \text{erase}(m', x, \eta) \rangle$: We have $\Gamma, pc \vdash \text{skip}$ trivially. Next, we prove that $\vdash m_2$ (in this case, $m_2$ is $\text{erase}(m', x, \eta)$) by showing that every variable $x'$ is well-formed under $m_2$. To do so, we first note that $\mathcal{T}(x', m_2) = \mathcal{T}(x', m')$ since there is no chain of dependence or self dependency.

  - Case $x \notin \text{FV}(\Gamma(x'))$: We first show that the level of $x'$ does not change after the assignment. Since $x'$ does not depend on $x$, we have $\mathcal{T}(x', m_1) = \mathcal{T}(x', m')$. Further, we can infer that $\mathcal{T}(x', m_2) = \mathcal{T}(x', m') = \mathcal{T}(x', m_1)$.

    By the definition of erase, we have $m_2(x') = m'(x')$. So if $x' \neq x$, we have $\vdash m_2 x'$ trivially since neither its value (since $m'(x') = m(x')$) nor its type is changed after the assignment; otherwise, if $x' = x$ we have three cases:

      * Case S-Asgn1: We have $m_2(x) = m'(x) = m_1(x \mapsto n)(x) = n$ and $\mathcal{T}(x, m_2) = \mathcal{T}(x, m_1) \subseteq L$. Thus, $\vdash m_2 x$.  

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* Case S-Asgn2: We have $m_2(x) = m'(x) = m_1\{x \mapsto \lceil n \rceil\}(x) = \lceil n \rceil$ and $\mathcal{T}(x, m_2) = \mathcal{T}(x, m_1) \not\subseteq L$. Thus, we can derive $\vdash_{m_2} x$ since $x$ is high and is given a bracketed value.

* Case S-Asgn3: We have $m_2(x) = m'(x) = m_1\{x \mapsto \lceil n \rceil\}(x) = \lceil n \rceil$. We also know from the typing rule (T-ASSGN) that $\Gamma \vdash e : \tau$, $\mid \mathcal{P} \cdot \eta \rangle \Rightarrow \tau \cup pc \subseteq \Gamma(x)$. By assumption, $\langle e, m_1 \rangle \Downarrow \lceil n \rceil$. By Lemma 8, $\mathcal{V}(\tau, m_1) \not\subseteq L$. Hence, $\mathcal{T}(x, m_1) \not\subseteq L$ due to Lemma 7. Thus, we have $\mathcal{T}(x, m_2) = \mathcal{T}(x, m_1) \not\subseteq L$. So $\vdash_{m_2} x$ since $x$ is high and is given a bracketed value.

Thus, in all three cases, we have $\vdash_{m_2} x'$.

– Case $x \in \text{FV}(x')$: by the definition of erase, $x'$ is erased to 0 or [0] according to $\mathcal{T}(x', m')$. We already showed $\mathcal{T}(x', m') = \mathcal{T}(x', m_2)$. Thus, we have $\vdash_{m_2} x'$.

- Case $\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle c_1, m_1 \rangle$ and $\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle c_2, m_1 \rangle$:

$\vdash_{m_2}$ is trivial since $m_2 = m_1$. For types, from the typing rule (T-IF), we have $\Gamma \vdash e : \tau$, $\Gamma, pc \cup \tau \vdash c_1$ and $\Gamma, pc \cup \tau \vdash c_2$. Since $pc \subseteq pc \cup \tau$, we can derive from Lemma 9 that $\Gamma, pc \vdash c_1$ and $\Gamma, pc \vdash c_2$.

- Case $\langle \text{if} (\text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1) \rightarrow \langle [c_1], m_1 \rangle$, and $\langle \text{if} (\text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1) \rightarrow \langle [c_2], m_1 \rangle : \vdash m_2$ is trivial since $m_2 = m_1$. For types, from the typing rule (T-IF), we have $\Gamma \vdash e : \tau$, $\Gamma, pc \cup \tau \vdash c_1$ and $\Gamma, pc \cup \tau \vdash c_2$. From the assumption, we have $\langle e, m_1 \rangle \Downarrow \lceil n \rceil$. By Lemma 8, $\mathcal{V}(\tau, m_1) \not\subseteq L$. Hence, $\mathcal{T}(pc \cup \tau, m_1) \not\subseteq L$ due to Lemma 7. Thus, we can derive $\Gamma, pc \vdash [c_1]$ and $\Gamma, pc \vdash [c_2]$ by using $pc \cup \tau$ as the “$\tau$” in rule (T-BRACKETCMD).

- Case $\langle \text{while} (e) c, m_1 \rangle \rightarrow \langle \text{if} (\text{while} (e) c) \text{ else } \text{skip}, m_1 \rangle$:

$\vdash_{m_2}$ is trivial since $m_2 = m_1$. By the typing rule (T-WHILE), we have two assumptions $A = \Gamma \vdash e : \tau$ and $B = \Gamma, \tau \cup pc \vdash c$. So the program after evaluation can be typed as follows:

$$
\frac{A \quad B}{\Gamma, \tau \cup pc \vdash \text{while} (e) c} \quad \frac{\Gamma, \tau \cup pc \vdash \text{skip}}{\Gamma, pc \vdash \text{if} (e) \text{ then } (\text{while} (e) c) \text{ else } \text{skip}}
$$

□

Lemma 11 (High-Step) A command that type-checks in a high-pc context only modifies high variables.

$$\forall m_1, m_2, c, pc. \mathcal{V}(pc, m_1) \not\subseteq L \land \Gamma, pc \vdash c \land \vdash m_1 \land \langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle \implies m_1 \sim m_2$$

Proof. By induction on evaluation rules $\langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle$:

- Cases $\langle \text{skip}, m_1 \rangle \rightarrow \langle \text{skip}, m_1 \rangle$, $\langle \text{skip}, c, m_1 \rangle \rightarrow \langle c, m_1 \rangle$,

$\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle c_1, m_1 \rangle$, $\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle c_2, m_1 \rangle$,

$\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle [c_1], m_1 \rangle$, $\langle \text{if} (e) \text{ then } c_1 \text{ else } c_2, m_1 \rangle \rightarrow \langle [c_2], m_1 \rangle$,

$\langle \text{while} (e) c, m_1 \rangle \rightarrow \langle \text{if} (e) \text{ then } (\text{while} (e) c) \text{ else } \text{skip}, m_1 \rangle$:

Trivial since $m_2 = m_1$. 38
• Case \( \langle c_1; c_2, m_1 \rangle \rightarrow \langle c_1'; c_2, m_2 \rangle \): From the type rule (T-SEQ), we have \( \Gamma, pc \vdash c \). By induction hypothesis on the assumption \( \langle c_1, m_1 \rangle \rightarrow \langle c_1', m_2 \rangle \), we have \( m_1 \sim m_2 \).

• Case \( \langle [c], m_1 \rangle \rightarrow \langle [c'], m_2 \rangle \): From the evaluation rule, we have \( \langle c, m_1 \rangle \rightarrow \langle c', m_2 \rangle \). From the typing rule, we have \( \Gamma, \tau \vdash c \) for some \( \tau \) such that \( pc \sqsubseteq \tau \). Since by assumption, \( \mathcal{V}(pc, m_1) \not\sqsubseteq L \), we have \( \mathcal{V}(\tau, m_1) \not\sqsubseteq L \) by Lemma[7]. Hence, by induction hypothesis on the evaluation assumption, we have \( m_1 \sim m_2 \).

• Case \( \langle x := \eta \ e, m_1 \rangle \rightarrow \langle \text{skip, erase}(m', x) \rangle \): First, we can infer from the type rule that \( \Gamma \vdash e : \tau, \models \mathcal{P}(\setminus\eta) \Rightarrow \tau \sqcup pc \sqsubseteq \Gamma(x) \). Due to the correctness of predicate generator, we have \( \tau \sqcup pc \sqsubseteq \Gamma(x) \). Since \( \mathcal{V}(pc, m_1) \not\sqsubseteq L \), we have \( \mathcal{T}(x, m_1) \not\sqsubseteq L \) by Lemma[7]. From assumption \( \vdash m_1 \), we know \( x \) must hold a bracketed value in \( m_1 \). We also know that S-Asgn1 cannot be applied since it requires \( \mathcal{T}(x, m_1) \sqsubseteq L \). When S-Asgn2 or S-Asgn3 is applied, we have \( m' = m_1\{x \mapsto [n]\} \). Since there is no self-dependence, \( m_2(x) = m'(x) = [n] \). So \( m_2(x) \sim m_1(x) \). Next, we show that for variable \( x' \neq x \), we have \( m_1(x') \sim m_2(x') \):
  
  - Case \( x \notin \text{FV}(x') \): by definition of erase, \( m_2(x') = m'(x') = m_1(x') \). Hence, \( m_1(x') \sim m_2(x') \).
  
  - Case \( x \in \text{FV}(x') \): by definition of erase, \( x' \) is erased to 0 or 0 according to \( \mathcal{T}(x', m') \). We already know that \( \mathcal{T}(x, m_1) \not\sqsubseteq L \). Since there is no self-dependence, \( \mathcal{T}(x', m') \not\sqsubseteq L \). Recall that \( x \in \text{FV}(x') \) implies \( \Gamma(x) \subseteq \Gamma(x') \). Hence, by Lemma[7] \( \mathcal{T}(x', m_1) \not\sqsubseteq L \) and \( \mathcal{T}(x', m') \not\sqsubseteq L \). By assumption \( \vdash m_1 \), \( x' \) holds a bracketed value in \( m_1 \). Moreover, by the erasure semantics, \( x' \) is erased to 0 after the assignment. So \( m_1(x') \sim m_2(x') \).

\[
\forall m_1, m_2, m_1 \sim m_2 \land \langle e, m_1 \rangle \Downarrow v_1 \implies \exists v_2, \langle e, m_2 \rangle \Downarrow v_2 \land v_1 \sim v_2
\]

**Proof.** By rule induction on the structure of expression \( e \).

- Case \( e = \eta \ e \) trivial since we have \( \langle e, m \rangle \Downarrow n \) for any memory, so \( v_1 = v_2 = n \).

- Case \( e = [n] \) trivial since we have \( \langle e, m \rangle \Downarrow [n] \) for any memory, so \( v_1 = v_2 = [n] \).

- Case \( e = x \) From \( m_1 \sim m_2 \), we have \( m_1(x) = m_2(x) \), thus, \( v_1 \sim v_2 \).

- Case \( e = e_1 \ op \ e_2 \): By the induction hypothesis, we have \( \langle e_1, m_1 \rangle \Downarrow v_3, \langle e_1, m_2 \rangle \Downarrow v'_3, \langle e_2, m_1 \rangle \Downarrow v_4, \langle e_2, m_2 \rangle \Downarrow v'_4 \), and \( v_3 \sim v'_3, v_4 \sim v'_4 \).
  
  - If both \( v_3, v_4 \) are non-bracketed values, then \( v_3 = v'_3 \) and \( v_4 = v'_4 \). Result is trivial.
  
  - If at least one of \( v_3 \) and \( v_4 \) holds bracketed value, we know that at least one of \( v'_3 \) and \( v'_4 \) holds a bracketed value. From the semantics we know \( e \) must be evaluated to bracketed values under both \( m_1 \) and \( m_2 \). Thus, \( v_1 \sim v_2 \).

\[
\square
\]
Lemma 13 (Unwinding)

\[ \forall c_1, c_2, m_1, m_2, m_3, m_4. \]
\[ \vdash c_1 \land \vdash c_2 \land c_1 \sim c_2 \land \vdash m_1 \land \vdash m_2 \land m_1 \sim m_2 \]
\[ \land \langle c_1, m_1 \rangle \rightarrow \langle c_3, m_3 \rangle \]
\[ \implies (\exists c_4, m_4. c_4 \sim c_3 \land \langle c_2, m_2 \rangle \rightarrow^* \langle c_4, m_4 \rangle \land m_3 \sim m_4) \lor ((c_2, m_2) \uparrow \land \exists c. c_2 = [c]) \]

Proof. By rule induction on \( \langle c_1, m_1 \rangle \rightarrow \langle c_3, m_3 \rangle \).

- Case \( \langle \text{skip}, m_1 \rangle \rightarrow \langle \text{skip}, m_1 \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 \) has the form of \([c_5]\), for some \( c_5 \). If \( c_2 \) diverges, we are done with \( c = c_5 \). Otherwise, we have \( \langle [c_5], m_2 \rangle \rightarrow^* \langle [\text{skip}], m_4 \rangle \rightarrow \langle \text{skip}, m_4 \rangle \). By induction on the number of steps using Lemma 11 we have \( m_4 \sim m_2 \sim m_1 \). So we can choose \( c_4 = \text{skip} \).

- Case \( \langle [c_5], m_1 \rangle \rightarrow \langle [c_6], m_3 \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 \) has the form of \([c_6]\) for some \( c_6 \). Hence, \( c_2 = [c_6] \sim [c_6] \). Moreover, by Lemma 11 \( m_3 \sim m_1 \). Hence, \( m_2 \sim m_3 \). Therefore, we can choose \( c_4 = c_2 \) and make zero step under \( m_2 \).

- Case \( \langle \text{skip}; c_5, m_1 \rangle \rightarrow \langle c_5, m_1 \rangle \): Command \( c_2 \) must also have the form of \( \text{skip}; c_6 \) where \( c_5 \sim c_6 \). So \( \langle \text{skip}; c_6, m_2 \rangle \rightarrow \langle c_6, m_2 \rangle \) preserves the equivalence on memory and command as required.

- Case \( \langle c_5; c_6, m_1 \rangle \rightarrow \langle c'_6; c_6, m_3 \rangle \): Command \( c_2 \) must have the form of \( c_7; c_8 \) where \( c_5 \sim c_7, c_6 \sim c_8 \). We can infer from \( c_5 \sim c_7 \) and the evaluation assumption \( \langle c_5, m_1 \rangle \rightarrow \langle c'_5, m_3 \rangle \) that \( \langle c_7, m_2 \rangle \rightarrow^* \langle c'_7, m_5 \rangle \) and \( m_5 \sim m_3 \) and \( c'_7 \sim c'_5 \). Thus, we can derive \( \langle c_7; c_8, m_2 \rangle \rightarrow^* \langle c'_7; c_8, m_5 \rangle \). Therefore, we can choose \( c_4 = c'_7; c_8 \) and \( m_4 = m_5 \).

- Case \( \langle x :=_\eta e, m_1 \rangle \rightarrow \langle \text{skip}, \text{erase}(m'_1, x, \eta) \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 = (x :=_\eta e) \), and \( \langle x :=_\eta e, m_2 \rangle \rightarrow \langle \text{skip}, \text{erase}(m'_2, x, \eta) \rangle \). Trivially, \( \text{skip} \sim \text{skip} \). Next, we show \( \text{erase}(m'_1, x, \eta) \sim \text{erase}(m'_2, x, \eta) \).

From Lemma 12 we know that \( \langle e, m_1 \rangle \downarrow v_1, \langle e, m_2 \rangle \downarrow v_2 \), then \( v_1 \sim v_2 \). We know \( m'_1 = m_1 \{ x \mapsto v_1 \} \) and \( m'_2 = m_2 \{ x \mapsto v_2 \} \). Thus, \( m'_1 \sim m'_2 \) given \( m_1 \sim m_2 \). Since there is no self-dependence, we have \( m_3(x) = m'_1(x) \sim m'_2(x) = m_4(x) \). Next, we show for any \( x' \neq x \), we have \( m_3(x') \sim m_4(x') \):

  - Case \( x \notin \text{FV}(x') \): by definition of \( \text{erase} \), we have \( m_3(x') = m'_1(x') \sim m'_2(x') = m_4(x') \).

  - Case \( x \in \text{FV}(x') \): by definition of \( \text{erase} \), \( x' \) is erased to 0 or \([0]\) according to its type \( T(x', m'_1) \) or \( T(x', m'_2) \).

* Case S-Asgn1, we have \( \langle e, m_1 \rangle \downarrow n, T(x, m_1) \subseteq L \) and \( m'_1(x) = n \). By Lemma 12 and 8 \( \langle e, m_2 \rangle \downarrow n, T(x, m_2) \subseteq L \). So S-Asgn1 applies under \( m_2 \) as well, and \( m'_2(x) = n \land m'_1 \sim m'_2 \). When \( x' \) depends on no variable whose level is low under \( m_1 \), we have \( T(x', m'_1) = T(x', m'_2) \) since all variables it depends on must be identical under \( m'_1 \) and \( m'_2 \). So \( x' \) will be erased to the same value in this case. Otherwise, say \( x' \) depends on \( y \) such that \( T(y, m'_1) \not\subseteq L \). By Lemma 8 \( y \) has
a bracketed value under \( m'_1 \). Since \( m'_1 \sim m'_2 \), \( y \) has a bracketed value under \( m'_2 \) as well. By Lemma 8 again, \( \mathcal{T}(y, m'_2) \not\subseteq L \). Since \( \Gamma(y) \subseteq \Gamma(x') \) when \( x' \) depends on \( y \), we have \( \mathcal{T}(x', m'_1) \not\subseteq L \) and \( \mathcal{T}(x', m'_2) \not\subseteq L \) by Lemma 7. Hence, \( x' \) will be erased to \([0]\) under \( m'_1 \) and \( m'_2 \).

* Case S-Asgn2: we have \( \mathcal{T}(x, m_1) \not\subseteq L \). Since \( m_1 \sim m_2 \), we have \( \mathcal{T}(x, m_2) \not\subseteq L \) by Lemma 12. So S-Asgn2 applies under \( m_2 \) as well. Since there is no self dependence, we have \( \mathcal{T}(x, m') = \mathcal{T}(x, m_1) \not\subseteq L \) and \( \mathcal{T}(x, m_2) = \mathcal{T}(x, m_2) \not\subseteq L \). Since \( x' \) depends on \( x \), we must have \( \mathcal{T}(x', m'_1) \not\subseteq L \) and \( \mathcal{T}(x', m'_2) \not\subseteq L \). So by the definition of \( \text{erase} \), \( m_3(x') = m_4(x') = [0] \).

* Case S-Asgn3: we have \( \langle e, m_1 \rangle \downarrow [n_1] \) for some \( n_1 \). By Lemma 12 \( v_2 = [n_2] \) for some \( n_2 \). So S-Asgn3 applies under \( m_2 \) as well. Moreover, given \( \Gamma \vdash e : \tau \), \( \mathcal{V}(\tau, m_1) \not\subseteq L \) and \( \mathcal{V}(\tau, m_2) \not\subseteq L \) by Lemma 8. We also know from typing rule T-Assgn that \( | \mathcal{P}(\bullet q) | \Rightarrow \tau \sqcup pc \subseteq \Gamma(x) \). Due to the correctness of predicate generation, \( \tau \sqcup pc \subseteq \Gamma(x) \). By Lemma 7 \( \mathcal{T}(x, m_1) \not\subseteq L \land \mathcal{T}(x, m_2) \not\subseteq L \). Similar to case S-Asgn2, we know that \( m_3(x') = m_4(x') = [0] \) in this case.

- Case \( \langle \text{if } (e) \text{ then } c_5 \text{ else } c_6, m_1 \rangle \rightarrow \langle c_5, m_1 \rangle \) and \( \langle \text{if } (e) \text{ then } c_5 \text{ else } c_6, m_1 \rangle \rightarrow \langle c_6, m_1 \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 \) must be \( \text{if } (e) \text{ then } c_5 \text{ else } c_6 \). This rule is applied only when \( e \)'s value under \( m_1 \) is not bracketed. By Lemma 12 \( e \)'s value is not bracketed under \( m_2 \) and \( e \) must evaluate to the same value under \( m_1 \) and \( m_2 \). Therefore, \( c_1 \) and \( c_2 \) must evaluate using the same rule. We can construct \( c_4 \) as the corresponding branch taken under \( m_1 \).

- Case \( \langle \text{if } (e) \text{ then } c_5 \text{ else } c_6, m_1 \rangle \rightarrow \langle [c_5], m_1 \rangle \) and \( \langle \text{if } (e) \text{ then } c_5 \text{ else } c_6, m_1 \rangle \rightarrow \langle [c_6], m_1 \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 \) must be \( \text{if } (e) \text{ then } c_5 \text{ else } c_6 \). This rule is applied when \( \langle e, m_1 \rangle \downarrow [n_1] \). By Lemma 12 we know that \( \langle e, m_2 \rangle \downarrow [n_2] \). We construct \( c_4 \) as the branch taken under \( m_2 \) in one step. According to the semantics, \( c_4 \) evaluates to either \([c_5]\) or \([c_6]\). That is, \( c_3 \) and \( c_4 \) evaluate to bracket commands. Hence, we have \( c_3 \sim c_4 \).

- Case \( \langle \text{while } (e) \text{ } c_5, m_1 \rangle \rightarrow \langle \text{if } (e) \text{ then } (\text{while } (e) \text{ } c_5) \text{ else } \text{skip}, m_1 \rangle \): From \( c_1 \sim c_2 \), we know \( c_2 \) has the form of \( \text{while } (e) \text{ } c_5 \). Therefore, we construct \( c_4 \) as \( \text{if } (e) \text{ then } (\text{while } (e) \text{ } c_5) \text{ else } \text{skip} \). Hence, \( m_2 \sim m_4 \) and \( c_3 \sim c_4 \).

\[ \square \]

**Theorem 5 (Soundness under Erasure Semantics)** Any transforms program that type checks satisfies noninterference:

\[
\forall C, m_1, m_2, m_3, m_4, \Gamma, \ell, A'. \\
\Gamma \vdash X \land m_1 \Rightarrow X \land m_2 \\
\langle c, m_1 \rangle \rightarrow^{\ast}_{\mathcal{E}(A')} \langle \text{skip}, m_3 \rangle \land \langle c, m_2 \rangle \rightarrow^{\ast}_{\mathcal{E}(A')} \langle \text{skip}, m_4 \rangle \\
\Rightarrow m'_1 \Rightarrow^{\ell}_{\Gamma} m'_2
\]

**Proof.** By the construction of the extended language, for any particular \( \ell \), the relation \( m_1 \Rightarrow^{\ell}_{\Gamma} m_2 \) is the same as \( \vdash m_1, \vdash m_2, m_1 \sim m_2 \) in the extended language. We proceed by induction on the number of steps in the execution under \( m_1 \).

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Case zero step: trivial since $c$ must be `skip`.

Case $N + 1$ steps: consider the first step of the execution under $m_1$. By Lemma 13, we know either $c$ diverges under $m_2$ or we can make multiple steps under $m_2$ and $m_3 \sim m_4$. By Lemma 10, we also have $\vdash m_3, \vdash m_4$ and the remaining programs $c_1$ and $c_2$ type-checks. Hence, result is true by the induction hypothesis. □

**Proof of Theorem 3**

\[
\forall c, m_1, m_2, m_3, m_4, \ell, A, A', \Gamma.
\langle c, A \rangle \Rightarrow \langle c, A' \rangle \land \vdash \Gamma \land \vdash c \land m_1 \approx^\ell_{\Gamma, A} m_2
\land \langle c, m_1 \rangle \rightarrow^* \langle \text{skip}, m_3 \rangle \land \langle c, m_2 \rangle \rightarrow^* \langle \text{skip}, m_4 \rangle
\implies m_3 \approx^\ell_{\Gamma, A'} m_4
\]

**Proof.** Trivial by Theorem 5 and Lemma 5, which states that the erasure semantics agrees with the standard semantics on active copies in $A'$.

**Proof of Theorem 2**

\[
\forall c, c, m_1, m_2, m_1', m_2', \ell, \Gamma, A, A'.
\langle c, A \rangle \Rightarrow \langle c, A' \rangle \land \vdash \Gamma \land \vdash c \land m_1 \approx^\ell_{\Gamma, A} m_2 \land
\langle c, m_1 \rangle \rightarrow^* \langle \text{skip}, m_1' \rangle \land \langle c, m_2 \rangle \rightarrow^* \langle \text{skip}, m_2' \rangle
\implies m_1' \approx^\ell_{\Gamma, A'} m_2'
\]

**Proof.** Trivial by Theorem 5 and Theorem 1, the correctness of program transformation. □
C Enabling Flow-Sensitivity with Program Transformation

To facilitate the proof, we say a typing environment $\Gamma'$ is an extension of $\Gamma$, written $\Gamma \preceq \Gamma'$, if $\forall x \in \text{Dom}(\Gamma). \ \Gamma(x) = \Gamma'(x)$. Easy to check that this relation is an partial order on environments (i.e., the relation satisfies reflexivity, antisymmetry and transitivity).

Lemma 14

$$\langle [c], A \rangle \Rightarrow \langle c, A' \rangle \Rightarrow A' \subseteq (A \cup \text{FVars}(c))$$

Proof. By induction on the structure of $c$.

- skip: $A' \subseteq A \cup \text{FVars}(c)$ since $A = A'$ in this case.
- $x := e$: we have $A' \subseteq A \cup \text{FVars}(c)$ since $A' = \{x_i\} \cup (A - \{A(x)\})$.
- $c_1; c_2$: by the transformation rule we have $\langle [c_1], A \rangle \Rightarrow \langle c_1, A_1 \rangle$ and $\langle [c_2], A_1 \rangle \Rightarrow \langle c_2, A' \rangle$ for some $A_1$. By the induction hypothesis, we have
  $$A_1 \subseteq (A \cup \text{FVars}(c_1)) \land A' \subseteq (A_1 \cup \text{FVars}(c_2))$$

  Hence, we have
  $$A' \subseteq (A_1 \cup \text{FVars}(c_1)) \cup \text{FVars}(c_2) \subseteq (A \cup \text{FVars}(c_1) \cup \text{FVars}(c_2))$$

Therefore, $A' \subseteq (A \cup \text{FVars}(c_1; c_2))$.

- if $(e)$ then $c_1$ else $c_2$: by the transformation rule, we have $\langle e, A \rangle \Rightarrow e$ and $\langle [c_1], A \rangle \Rightarrow \langle c_1, A_i \rangle, i \in \{1, 2\}$. By the induction hypothesis, we have
  $$A_1 \subseteq (A \cup \text{FVars}(c_1)) \land A_2 \subseteq (A \cup \text{FVars}(c_2))$$

  The transformed branch has the form of $c_i; A_3 := A_i$ where $A_3 = \Phi(A_1, A_2)$. By the definition of $\Phi$, $A_3 \subseteq A_1 \cup A_2 \cup \text{FVars}(A_3 := A_1; A_3 := A_2) \subseteq A \cup \text{FVars}(c)$.

- while $(e) c$: by the transformation rule, we have
  $$\langle [c], A \rangle \Rightarrow \langle c_1, A_1 \rangle \langle [e], A_1 \rangle \Rightarrow \langle e, A_1 \rangle \Rightarrow e$$

  By induction on $\langle [c], A \rangle \Rightarrow \langle c_1, A_1 \rangle$, we have $A_1 \subseteq A \cup \text{FVars}(c_1)$. For a variable $v$ in $A_1$ but not in $A$, we know that $v \in \text{FVars}(A_1 := A)$ since by definition, $A_1 := A$ assigns to any $v \in A_1 \land v \not\in A$. Hence, $A_1 \subseteq A \cup \text{FVars}(c)$.

\[ \square \]

The next lemma shows that the extension of a typing environment $\Gamma$ subsumes $\Gamma'$:

Lemma 15

$$\Gamma \preceq \Gamma' \land \Gamma \vdash c \Rightarrow \Gamma' \vdash c$$
**Proof.** We first show $\Gamma \subseteq \Gamma' \land \Gamma \vdash e : \tau \Rightarrow \Gamma' \vdash e : \tau$ by induction on the structure of $e$. Then we can prove the lemma by induction on the structure of $c$. □

**Proof of Lemma 2**

$p c \vdash_{HS} \Gamma \{ c \} \Gamma' \wedge \langle \{ c \} , A \rangle \Rightarrow \langle c , A' \rangle \Rightarrow \forall v \in \text{Vars.} \ (A(v) = A'(v)) \Rightarrow (\Gamma(v) = \Gamma'(v))$

**Proof.** By induction on the structure of $c$.

- **skip:** trivial since $\Gamma = \Gamma'$.

- **$x := e$:** when $v = x$, $A'(v) = x_i \neq A(x)$ since $x_i$ is fresh. So the result is trivially true. For other variables, $\Gamma(v) = \Gamma'(v)$ by the HS typing rule (HS-ASSIGN).

- **$c_1; c_2$:** by the transformation rule and HS typing rule, we have $\langle \{ c_1 \} , A \rangle \Rightarrow \langle c_1 , A_1 \rangle$ and $\langle \{ c_2 \} , A_1 \rangle \Rightarrow \langle c_2 , A' \rangle$ for some $A_1$, as well as $p c \vdash_{HS} \Gamma \{ c_1 \} \Gamma_1$ and $p c \vdash_{HS} \Gamma \{ c_2 \} \Gamma'$ for some $\Gamma_1$. By Lemma 14, $A' \subseteq A_1 \cup \text{FVars}(c_2)$. So when $A(v) = A'(v)$, it must be true that $A_1(v) = A'(v)$ since otherwise, $A'(v)$ must be a fresh variable in $c_2$, and hence, cannot appear in $A$. Therefore, we have $A(v) = A'(v) = A_1(v)$. By the induction hypothesis, it must be true that $\Gamma(v) = \Gamma_1(v) = \Gamma'(v)$.

- **if $(e)$ then $c_1$ else $c_2$:** by the HS typing rule, $p c \vdash_{HS} \Gamma \{ c_1 \} \Gamma_1 \land p c \vdash_{HS} \Gamma \{ c_2 \} \Gamma_2 \land \Gamma' = \Gamma_1 \cup \Gamma_2$. By the transformation rules, $\langle \{ c_i \} , A \rangle \Rightarrow \langle c_i , A_i \rangle, i \in \{ 1, 2 \}$.

By Lemma 14, $A_1 \subseteq A \cup \text{FVars}(c_1)$. So when $A(v) \neq A_1(v)$, $A_1(v)$ must be a fresh variable generated in $c_1$, and hence, cannot be in $A_2$. By the definition of $\Phi$, $A_3(v)$ must be fresh as well. This contradicts the assumption that $A(v) = A'(v)$. Hence, we have $A(v) = A_1(v)$ (and similarly, $A(v) = A_2(v)$). So $\Gamma(v) = \Gamma_1(v)$ by the induction hypothesis. Therefore, $\Gamma'(v) = \Gamma_1(v) \cup \Gamma_2(v) = \Gamma(v)$.

- **while $(e)$ $c$:** By rule (TRSF-WHILE), we have $\langle \{ c \} , A \rangle \Rightarrow \langle c , A \rangle$, where $A'$ is $A_1$ in this case. Hence, by the assumption, we have $A(v) = A_1(v)$. By rule (HS-WHILE), there is a sequence of environments $\Gamma'_i$, $\Gamma''_i$ such that $p c \sqcup \tau_i \vdash \Gamma'_i \{ c \} \Gamma''_i$. By the induction hypothesis, $\Gamma''_i(v) = \Gamma'(v)$. Since $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = \Gamma \cup \Gamma''_i$ in rule (HS-WHILE), we can further infer that $\Gamma'_{i+1}(v) = \Gamma''_i(v)$. Hence, we have $\Gamma'(v) = \Gamma_{n}(v) = \Gamma_0(v) = \Gamma(v)$.

□

**Lemma 16**

$\Gamma \vdash_{HS} e : \tau \land \langle e , A \rangle \Rightarrow e \land \Gamma = \Gamma^A \Rightarrow \Gamma \vdash e : \tau$

**Proof.** By rule induction on the transformation. □
Proof of Theorem 4. We prove a slightly stronger version of Theorem 4.

∀ c, ξ, pc, A, A', Γ, Γ', Γ.

\[(pc, Γ, A) \{ [c] \Rightarrow ξ \} (Γ', A') \Rightarrow Π \]

⇒ Dom(Π) ⊆ A \cup FVars(ξ) ∧ Γ = Γ^A ∧ Γ' = Γ'^A ∧ Γ, pc ⊢ ξ

Proof. By induction on the structure of c.

- **skip**: Γ, pc ⊢ ξ is trivial since skip can be type-checked with any pc, Γ. Other conditions are trivial.

- **x := e**: by the construction rule, we have Γ'(x) = τ for some τ, ξ has the form of \( x_i := \xi \), \( A'(x) = x_i \) and Γ = Γ_A ∪ \{ \{ x_i \mapsto τ \} \}.

  - Γ = Γ^A, Γ' = Γ^A': Since \( x_i \) is fresh, Γ = Γ^A. Moreover, Γ^A' = Γ\{ x \mapsto τ \} = Γ'.
  
  - Γ, pc ⊢ ξ: let Γ ⊢_HS e : τ_0. By the HS typing rule, we have τ = τ_0 \cup pc. Moreover, we have Γ ⊢ ξ : τ due to Lemma 16 and the fact Γ = Γ^A. By the construction, Γ(x_i) = τ. Hence, Γ, pc ⊢ x_i := ξ.

  - Dom(Π) ⊆ A ∪ FVars(ξ): By the construction, Dom(Π) = A ∪ \{ x_i \}.

- **c_1; c_2**: from the construction rule, we have \((pc, Γ, A) \{ [c_1] \Rightarrow ξ_1 \} (Γ'', A'') \Rightarrow Π_1 \) and \((pc, Γ'', A'') \{ [c_2] \Rightarrow ξ_2 \} (Γ', A') \Rightarrow Π_2 \) for some Γ'', A'', Γ_1 and Γ_2. By the induction hypothesis, we have

\[
\begin{align*}
Γ &= Γ_1^A ∧ Γ'' = Γ_1'' ∧ Dom(Π_1) \subseteq A ∪ FVars(ξ_1) \\
Γ'' &= Γ_2'' ∧ Γ' = Γ_2'' ∧ Dom(Π_2) \subseteq A'' ∪ FVars(ξ_2) \\
Γ_1, pc &\vdash c_1 ∧ Γ_2, pc \vdash c_2
\end{align*}
\]

- Γ = Γ^A, Γ' = Γ^A': By the construction, we have Γ_1 ≤ Γ. Next, we check Γ_2 ≤ Γ.

  Due to the results above on Dom(Π_1) and Dom(Π_2), we know that if \( v \in Dom(Π_1) \cap Dom(Π_2) \), then \( v \notin A'' \) since FVars(ξ_2) contain fresh variables generated in \( c_2 \) (hence, not in A and ξ_1). Therefore, \( Γ_1(v) = Γ_2(v) \) because \( Γ_1'' = Γ'' = Γ_2'' \) by the induction hypothesis. Therefore, Γ_2 ≤ Γ.

  Moreover, by the definition, Γ = Γ^A is equivalent to \( Γ_A ≤ Γ_1 \). So we have \( Γ_A ≤ Γ \) and hence, Γ = Γ^A. Similarly, we can prove that Γ' = Γ'^A.

- Γ, pc ⊢ ξ: we know that Γ, pc ⊢ c_1 ∧ Γ, pc ⊢ c_2 by Lemma 15. So Γ, pc ⊢ c_1; c_2.

- Dom(Π) ⊆ A ∪ FVars(ξ): by the construction, Dom(Π) = Dom(Π_1) ∪ Dom(Π_2). The result is true since A'' ⊆ A ∪ FVars(ξ_1) by Lemma 1 and Lemma 14.

- **if (e) then c_1 else c_2**: from the construction rule, we have \((pc \cup τ, Γ, A) \{ [c_1] \Rightarrow ξ_1 \} (Γ_i, A_i) \Rightarrow Π_i \) for \( i \in \{ 1, 2 \} \), and \( Φ(A_1, A_2) ⇒ A_3 \), where Γ ⊢_HS e : τ. By the induction hypothesis, we have

\[
\begin{align*}
Γ &= Γ_1^A ∧ Γ_1 = Γ_1'' ∧ Γ_i, pc ∪ τ \vdash c_i ∧ Dom(Π_i) \subseteq A ∪ FVars(ξ_i)
\end{align*}
\]
\( \Gamma_1 \preceq \Gamma, \Gamma_2 \preceq \Gamma \): By the construction, we have \( \Gamma_1 \preceq \Gamma \). Next, we check \( \Gamma_2 \preceq \Gamma \).

Since \( \text{Dom}(\Gamma_i) \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \), if \( \nu \in \text{Dom}(\Gamma_1) \cap \text{Dom}(\Gamma_2) \), then it must be true that \( \nu \notin A \). Since we know that \( \Gamma = \Gamma_1^{A} \) and \( \Gamma = \Gamma_2^{A} \) from the induction hypothesis, \( \Gamma(\nu) = \Gamma_1(\nu) = \Gamma_2(\nu) \) for such variables. Hence, we have \( \Gamma_2 \preceq \Gamma \).

\( \Gamma = \Gamma^{A_1}, \Gamma' = \Gamma^{A_2} \): We have \( \Gamma = \Gamma^{A_1} \) since \( \Gamma = \Gamma_1^{A} \) and \( \Gamma_1 \preceq \Gamma \). To check \( \Gamma' = \Gamma^{A_2} \), consider \( \nu \in A_3 \cap (\text{Dom}(\Gamma_1) \cup \text{Dom}(\Gamma_2)) \). We use \( \nu \) to denote \( \nu \).

* When \( \nu \in \text{Dom}(\Gamma_1) \): we have \( \nu \in A \) or \( \nu \in \text{FVars}(\underline{\varepsilon}) \) since \( \text{Dom}(\Gamma_1) \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \).

In the latter case, we have \( A_1(\nu) \neq A_2(\nu) \) since \( A_2 \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \) by Lemma [1] and Lemma [14]. So \( A_3(\nu) \) must be a fresh variable by the definition of \( \Phi \), and hence \( \nu \notin A_3 \). Contradiction.

So \( \nu \in A \). By the assumption that \( \nu \in A_3 \) and the definition of the \( \Phi \) function, we have \( \nu \in A_1 \) and \( \nu \in A_2 \). That is, \( A(\nu) = A_1(\nu) = A_2(\nu) \). By Lemma [2] we have \( \Gamma_1(\nu) = \Gamma(\nu) = \Gamma_2(\nu) \). Hence, \( \Gamma'(\nu) = \Gamma_1(\nu) \sqcup \Gamma_2(\nu) = \Gamma(\nu) = \Gamma(\nu) \), where the last equation is true since \( \nu \in A \cap \Gamma^{A_2} = \Gamma \).

* When \( \nu \in \text{Dom}(\Gamma_2) \): same argument as the case above.

Therefore, for any \( \nu \in A_3 \cap (\text{Dom}(\Gamma_1) \cup \text{Dom}(\Gamma_2)) \), \( \Gamma(\nu) = \Gamma'(\nu) \). For other variables, \( \Gamma(\nu) = \Gamma'(\nu) \) is trivial from the construction. So \( \Gamma' = \Gamma^{A_2} \).

\( \Gamma, pc \vdash \varepsilon \): We know that \( \langle e, A \rangle \Rightarrow \varepsilon \) in the transformation assumption. Given \( \Gamma^{A_1} = \Gamma \), we have \( \Gamma \vdash \varepsilon : \tau \) by Lemma [16]. Moreover, \( \Gamma, pc \sqcup \tau \vdash \varepsilon \) since \( \Gamma \preceq \Gamma, \Gamma_2 \preceq \Gamma \).

Furthermore, both \( A_3 := A_1 \) and \( A_3 := A_2 \) type-checks by our construction since: 1) by the definition, the extra assignments only assign to fresh variables in \( A_3 \) (i.e., \( \nu \notin A_1 \cap A_2 \)); 2) for those variables, we have \( \Gamma(\nu) = \Gamma'(\nu) \) by the construction; 3) the HS system ensures that \( \Gamma_i(\nu) \sqsubseteq \Gamma'(\nu) \) for \( i \in \{1, 2\} \) and \( pc \sqcup \tau \sqsubseteq \Gamma_i(\nu) \) for \( i = 1 \) or \( 2 \) (\( \nu \) must be assigned to under some branch since \( \nu \notin A_1 \cap A_2 \)). Therefore, we have \( pc \sqcup \tau \sqsubseteq \Gamma'(\nu) \) and \( \Gamma(A_i(\nu)) = \Gamma_i(\nu) \sqsubseteq \Gamma'(\nu) = \Gamma(\nu) \).

\( \text{Dom}(\underline{\varepsilon}) \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \): by the construction, \( \text{Dom}(\underline{\varepsilon}) = \text{Dom}(\underline{\varepsilon}) \cup \text{Dom}(\underline{\varepsilon}) \cup A_3 \).

By Lemma [14], \( A_3 \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \). Result is true since we also have \( \text{Dom}(\underline{\varepsilon}) \subseteq A \cup \text{FVars}(\underline{\varepsilon}) \).

\( \text{while } (e) \text{ c:} \) by the construction rule, we have

\( \Gamma' \vdash_{\text{HS}} e : \tau, (\tau \sqcup pc, \Gamma', A_1) \{[\underline{e}] \Rightarrow \varepsilon'\}(\Gamma', A_2) \leadsto \Gamma_0 \)

By the induction hypothesis, we have

\( \Gamma' = \Gamma_0^{A_1}, \Gamma' = \Gamma_0^{A_2} \wedge \Gamma_0, pc \sqcup \tau \vdash \varepsilon' \wedge \text{Dom}(\Gamma_0) \subseteq A_1 \cup \text{FVars}(\varepsilon') \)

\( \Gamma = \Gamma^{A_1}, \Gamma' = \Gamma^{A_1} \): We have \( \Gamma_0 \preceq \Gamma \) by the construction. Given \( \Gamma' = \Gamma_0^{A_1} \), we know that \( \Gamma' = \Gamma_0^{A_2} \). Next, we show that \( \Gamma = \Gamma^{A_1} \).

* When \( \nu \in A \cap \text{Dom}(\Gamma_0) \) (we use \( \nu \) to denote \( \nu \)). Since \( \text{Dom}(\Gamma_0) \subseteq A_1 \cup \text{FVars}(\varepsilon') \) from the induction hypothesis, we have \( \nu \in A_1 \) or \( \nu \) is fresh in \( \varepsilon' \). But the latter must be false since \( \nu \in A \). So \( \nu \in A_1 \).
From the construction rule assumption, we have \( pc \vdash_{\text{HS}} \Gamma \{ \text{while } (e) \} \Gamma' \). By Lemma 1, we have \( \langle \text{while } (e) \rangle, \mathcal{A} \rangle \Rightarrow \langle e, \mathcal{A}_1 \rangle \). In addition, we just showed that \( v \in \mathcal{A} \) and \( v \in \mathcal{A}_1 \). Hence, by Lemma 2, \( \Gamma(v) = \Gamma'(v) \).

Because \( \mathcal{A} \subseteq \mathcal{A}_1 \), we have \( \mathcal{A}(v) = \mathcal{A}_1(v) \). Therefore, \( \Gamma^\mathcal{A}(v) = \Gamma(A(v)) = \Gamma(A_1(v)) = \Gamma'(v) \) (since \( \Gamma' = \Gamma^{A_1} = \Gamma(v) \)).

* When \( v \in A - \text{Dom}(\Gamma_0) \), the result is trivial by the construction.

- \( \Gamma, pc \vdash c : \tau \): we have showed that \( \Gamma^{A_1} = \Gamma' \). In the transformation assumption, we have \( \langle e, \mathcal{A}_1 \rangle \Rightarrow e \). So by Lemma 16, \( \Gamma \vdash e : \tau \). Moreover, we have \( \Gamma, pc \sqcup \tau \vdash c' \) by Lemma 15 and the induction hypothesis \( \Gamma_0, pc \sqcup \tau \vdash c' \). Next, we show that the extra assignments (i.e., \( \mathcal{A}_1 := \mathcal{A} \) and \( \mathcal{A}_1 := \mathcal{A}_2 \)) type-check.

Since \( \Gamma' = \Gamma^{A_1} \land \Gamma' = \Gamma^{A_2} \) by the induction hypothesis, we have \( \Gamma^{A_1} = \Gamma^{A_2} \). Moreover, by the same argument as in the “if” case, the LHS of \( \mathcal{A}_1 := \mathcal{A}_2 \) must have a level that is higher than \( pc \sqcup \tau \) in the HS system. So \( \Gamma, pc \sqcup \tau \vdash \mathcal{A}_1 := \mathcal{A}_2 \). Moreover, we know that \( \forall v \in \text{Vars}. \Gamma(v) \subseteq \Gamma'(v) \) in the HS system. Since \( \Gamma^\mathcal{A} = \Gamma \) and \( \Gamma^{A_1} = \Gamma' \), we have \( \Gamma, pc \vdash \mathcal{A}_1 := \mathcal{A} \).

- \( \text{Dom}(\Gamma) \subseteq \mathcal{A} \cup \text{FVars}(\mathcal{C}) \): by the construction, \( \text{Dom}(\Gamma) = \text{Dom}(\Gamma_0) \cup \mathcal{A} \). From the induction hypothesis, we have \( \text{Dom}(\Gamma_0) \subseteq \mathcal{A}_1 \cup \text{FVars}(\mathcal{C}') \). Hence, \( \text{Dom}(\Gamma_0) \subseteq \mathcal{A}_1 \cup \text{FVars}(\mathcal{C}) \). We know from Lemmas 1 and 14 that \( \mathcal{A}_1 \subseteq \mathcal{A} \cup \text{FVars}(\mathcal{C}) \). Hence, \( \text{Dom}(\Gamma) \subseteq \mathcal{A} \cup \text{FVars}(\mathcal{C}) \).