COOPER PAIR BOX COUPLED TO A CURRENT-BIASED JOSEPHSON JUNCTION

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We study the dynamics of a quantum superconducting circuit which consists of a Josephson charge qubit, coupled capacitively to a current biased Josephson junction. Under certain conditions, the eigenstates of the qubit and the junction become entangled. We obtain the time evolution of these states in the limit of weak coupling. Rabi oscillations occur, as a result of the spontaneous emission and re-absorption of a single oscillation quantum in the junction. We discuss a possible way to experimentally determine the quantum state of the junction and hence observe the Rabi oscillations.

1 Introduction

Advances in quantum information theory\textsuperscript{2} are at the origin of the recent search for quantum mechanical two-level systems that can be used as quantum bits (qubits). In order to process quantum information in a useful way, certain operations on systems which consist of many qubits must be performed, such as the preparation and manipulation of, as well as a measurement on, entangled quantum states of coupled qubits. In this context, superconducting devices using small Josephson junctions are of particular interest. It has been experimentally demonstrated that a single Cooper pair box constitutes a two-level system which can be coherently controlled\textsuperscript{2,3,4}. Moreover, the use of a Cooper pair box as a qubit in the context of quantum computers was studied theoretically\textsuperscript{5,6,7}. But as of yet, the existence of entangled states, which are at the heart of quantum information processing, has not been demonstrated experimentally.

In this article we study one of the simplest superconducting circuits in which entangled states can be realized\textsuperscript{8}. It consists of a Cooper pair box coupled to a current-biased Josephson junction. Theoretically, this quantum circuit can be described by a two-level system coupled to a harmonic oscillator. After a description of the quantum circuit in the next Section, we discuss the time evolution of its eigenstates in Sec. 3. In particular, we demonstrate the existence of entanglement, very similar to the entanglement found for an atom in an electromagnetic cavity\textsuperscript{9}. In Sec. 4 we discuss a possible way to measure the Rabi oscillations, associated with the dynamics of the entangled states. The last Section contains a discussion of the feasibility of our proposal.

2 Superconducting circuit

The circuit that we will study is depicted in Fig. 1. It contains a quantum superconducting circuit connected to an electrodynamic environment. The quantum circuit itself consists of three elements, which we will discuss in some detail below.

2.1 Cooper pair box

The Cooper pair box consists of a small superconducting island, coupled capacitively to a gate voltage $V_g$ (gate capacitance $C_g$). The island is furthermore connected capacitively to a superconducting electrode via a Josephson junction (capacitance $C_j$, Josephson energy $E_j$). Using the basis states $|n\rangle$, where $n$ corresponds to the number of excess Cooper pairs on the island,
we can write the Hamiltonian for the box as

$$\hat{H}_{\text{box}} = E_{C,j} \sum_n (2n - N_g)^2 |n\rangle\langle n| - \frac{E_j}{2} \sum_n (|n + 1\rangle\langle n| + |n - 1\rangle\langle n|).$$

(1)

Here, $E_{C,j} = e^2/2C_{j,\text{eff}}$ is the total charging energy of the box ($C_{j,\text{eff}} = C_j + [1/C_r + 1/(C_c + C_g)]^{-1}$) and $N_g = -C_g V_g/e$. The first term of this Hamiltonian corresponds to the electrostatic energy of the box. Generally, this energy is minimal if the number $n$ of excess Cooper pairs on the island is an integer. However, if the dimensionless gate-voltage $N_g$ is an odd integer, $N_g = 2m + 1$, a degeneracy occurs, and the number $n$ of Cooper pairs fluctuates between $m$ and $m + 1$. The second term of the Hamiltonian, corresponding to the Josephson coupling between the island and the electrode, lifts this degeneracy. As a result, close to odd integer values of $N_g$, a gap opens up of order $E_j$ and the relevant charge states are superpositions of the charge states $|m\rangle$ and $|m + 1\rangle$. In particular, if the gate-voltage is such that $N_g \simeq 1$, the states with $n = 0$ and $n = 1$ are almost degenerate. At low energies, the Hamiltonian $\hat{H}_{\text{box}}$ involves only these two states, and thus can be written as a matrix

$$\hat{H}_{\text{box}} \simeq \begin{pmatrix} E_{C,j} N_g^2 & -E_j/2 \\ -E_j/2 & E_{C,j}(2 - N_g)^2 \end{pmatrix}. \quad (2)$$

The two eigenstates $|\cdot\rangle$ and $|+\rangle$ are superpositions of the charge states $|0\rangle$ and $|1\rangle$,

$$|\cdot\rangle = \alpha|0\rangle + \beta|1\rangle$$

(3)

$$|+\rangle = \beta|0\rangle - \alpha|1\rangle$$

(4)

where $\alpha^2 = 1 - \beta^2 = [1 + \delta E_g/\sqrt{(\delta E_g)^2 + E_j^2}]/2$ with $\delta E_g = -4E_{C,j}\delta N_g$ and $\delta N_g = N_g - 1$. The corresponding eigenenergies are

$$E_{\mp} = E_{C,j}[1 + (\delta N_g)^2] \mp \frac{1}{2}\sqrt{(\delta E_g)^2 + E_j^2}. \quad (5)$$

We see that the Cooper pair box behaves as a quantum mechanical two-level system, usually referred to as a Josephson charge qubit

### 2.2 Current biased Josephson junction

A current biased Josephson junction can be characterized by two conjugate variables: the charge $\hat{Q}$ on the junction and the phase difference $\phi$ across it. Ignoring effects related to the presence
of the environment, we can write the Hamiltonian as
\[ \hat{H}_r = \frac{\hat{Q}^2}{2C_{r,\text{eff}}} + U(\phi), \] (6)
where \( C_{r,\text{eff}} \) is the effective capacitance of the junction in the circuit, \( C_{r,\text{eff}} = C_r + [1/C_j + 1/(C_c + C_g)]^{-1} \), and \( U(\phi) = -E_r \cos \phi - (\Phi_0/2\pi)I\phi \) is the well-known tilted washboard potential \((\Phi_0 = h/2e \) is the superconducting flux quantum). It depends on the Josephson energy \( E_r \) and the bias current \( I \). For bias currents below the Josephson critical current \( I_c = 2\pi E_r/\Phi_0 \), the potential \( U(\phi) \) contains local periodic minima as a function of \( \phi \). These minima are quadratic with characteristic frequency \( \omega_r = (\sqrt{8E_rE_{C,r}/h})(1 - (I/I_c)^2)^{3/4} \), where \( E_{C,r} = e^2/2C_{r,\text{eff}} \).

For very low bias currents \((I \ll I_c)\), the minima are separated by large potential barriers \( \Delta U \sim 2E_r - \Phi_0 I/2 \). As a result, in the quantum limit, various low-lying states can be found in each minimum if \( E_r \gg E_{C,r} \); the broadening of the energies of these states due to tunneling between neighboring minima can be ignored. Hence these states are well approximated by harmonic oscillator eigenstates, \( \langle \chi_k \rangle \), corresponding to the presence of \( k = 0, 1, 2, \ldots \) oscillation quanta in the junction. Thus, at low bias current, the junction behaves as a quantum mechanical superconducting resonator. Since in this limit the phase is localized in a well-defined minimum of the potential \( U(\phi) \), the time-averaged voltage \( V_m \) across the junction remains zero.

If the bias current is increased to values close to \( I_c \), such that \( I \lesssim I_c \), the potential \( U(\phi) \) is well approximated by a cubic potential with a barrier height given by \( \Delta U \sim (4\sqrt{2}/3)E_r(1 - I/I_c)^{3/2} \). Both \( \Delta U \) and \( h\omega_r \) decrease with increasing \( I \); they vanish when \( I = I_c \). The number of localized states in a given well decreases as \( \Delta U/h\omega_r \sim (1 - I/I_c)^{5/4} \). Moreover, the remaining levels broaden due to quantum tunneling out of the minima of the potential \( U(\phi) \). A tunneling event changes the state of the junction thereby leading to a finite voltage \( V_m \) across it that can be detected. The broadening of the ground state energy is given by the tunneling rate \( \Gamma_0 \sim h\omega_r \exp(-\Delta U/h\omega_r) \). Since the excited states are located closer to the top of the barrier, the tunneling rates increase with increasing level index \( k \): \( \Gamma_k/\Gamma_0 \sim (432\Delta U/h\omega_r)^k \). The tunneling rates for the first two states \( |\chi_0 \rangle \) and \( |\chi_1 \rangle \) are therefore very different. We will see in Section 3 how this property can be used in order to distinguish these two quantum states.

2.3 Coupling capacitance

The capacitance \( C_c \) plays a crucial role in the circuit of Fig. 1 since it couples the charge qubit and the current-biased junction to each other. As a result, these two circuits are no longer independent and the system must be considered in its totality. Physically, the capacitance \( C_c \) couples the charge \( 2n - N_g \) on the qubit to the charge \( Q \) on the junction. Using the relation \( \hat{Q} = \sqrt{h\omega_r C_{r,\text{eff}}/2}(a - a^\dagger)/i \), the coupling Hamiltonian can be written as
\[ \hat{H}_c = -iE_{\text{coup}}(2n - N_g)(a^\dagger - a). \] (7)
Here we introduced the characteristic coupling energy \( E_{\text{coup}} = \sqrt{h\omega_r E_{C,r} E_{C,c}/2} \), where \( E_{C,c} = e^2/2C_{c,\text{eff}} \), with \( C_{c,\text{eff}} = C_c + C_g + (1/C_j + 1/C_r)^{-1} \). We will see in the next section how this coupling energy leads to entanglement between the states of the qubit and the junction.

3 Operation at low bias current: entanglement

In this section we discuss the behaviour of the circuit for values of the bias current away from the critical current, \( I \ll I_c \). In this limit, the system can be considered as a two-level system
(the qubit) coupled to a harmonic oscillator (the current biased junction). The dynamics of such a system has been discussed in the past. More recent applications are concerned with quantum optics or Josephson junctions. Here we will show how entangled states can be obtained which involve both the qubit and the junction.

Throughout this section we will be interested in the situation $N_g \simeq 1$, such that we have to consider the qubit states $|\rangle$ and $|\rangle$ only. Furthermore, as far as the junction is concerned, we will limit the discussion to the ground state $|\rangle$ and the first excited state $|\rangle$. Hence, in the absence of coupling, $E_{\text{coupl}} = 0$, the four lowest energy eigenstates are $|\rangle$, $|\rangle$, $|\rangle$, and $|\rangle$. Let us now consider the special case $\hbar \omega = E_j = \bar{E}$, i.e., the states $|\rangle$ and $|\rangle$ are degenerate for $N_g = 1$. Their energy is $\bar{E}$ with respect to the energy of the ground state $|\rangle$. In the limit of weak coupling, $E_{\text{coupl}} \ll \bar{E}$, this degeneracy is lifted: the states $|\rangle$ and $|\rangle$ get mixed and hence give rise to the two entangled states $|\rangle$ and $|\rangle$.

$$|\rangle = \left[|\rangle + i|\rangle\right]/\sqrt{2},$$
$$|\rangle = \left[|\rangle - i|\rangle\right]/\sqrt{2},$$
corresponding to the energies $\bar{E} - E_{\text{coupl}}$ and $\bar{E} + E_{\text{coupl}}$, respectively.

Suppose that the system has been prepared in the state $|\psi(t = 0)\rangle = |\rangle$ at time $t = 0$. This can be achieved by a suitable manipulation of the gate voltage $V_g$ at times prior to $t = 0$. If we keep $V_g$ fixed such that $N_g = 1$ at times $t > 0$, the time evolution of $|\psi(t)\rangle$ is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}i} \left[ e^{-i(\bar{E} - E_{\text{coupl}})/\hbar}|\rangle - e^{-i(\bar{E} + E_{\text{coupl}})/\hbar}|\rangle \right].$$

We see that the state $|\psi(t)\rangle$ oscillates coherently between $|\rangle$ and $|\rangle$. In fact, these so-called quantum Rabi oscillations can be interpreted as the spontaneous emission and re-absorption of one oscillation quantum by the junction. An interesting quantity is the probability $P_{\chi_0}(t)$ to find the junction in the state $|\rangle$ (no oscillation quanta) after a certain time $t$. This probability shows Rabi oscillations as a function of $t$ with frequency $2E_{\text{coupl}}/\hbar$,

$$P_{\chi_0}(t) = |\langle \rangle| \langle \rangle|^2 = \frac{1}{2} \left[ 1 + \cos(2E_{\text{coupl}}/\hbar) \right].$$

Since these Rabi oscillations are characteristic for the entanglement realized in the circuit, their measurement would provide direct evidence of the presence of the entangled states $|\rangle$ and $|\rangle$. We will discuss the possibility to perform such a measurement in the next Section.

### 4 Operation at bias current close to $I_c$: quantum measurement

In order to measure the probability $P_{\chi_0}(t)$ to find the junction in the state $|\rangle$ at time $t$, it is necessary to perform a series of measurements. Each of these measurements consists of the preparation of the state $|\psi(t = 0)\rangle = |\rangle$ at time $t = 0$ and the subsequent coherent evolution until the time $t$, when the actual measurement of the state of the junction is performed. The statistics of the ensemble of measurements then yields $P_{\chi_0}(t)$.

The proposed measurement procedure consists in transforming the two distinct quantum states $|\rangle$ and $|\rangle$ into two different stable classical states which are easily measurable: the zero-voltage and the finite voltage state of the current-biased junction. This measurement is a one shot method which gives binary information on the initial quantum state of the junction.

In order to measure the state of the junction at a certain time $t$, we propose to proceed as follows. Starting at time $t$, the bias current $I$ is increased during a short ramping time $\delta t$ to a value $I_m$ close to $I_c$. The bias current is kept at $I_m$ for a time $\Delta t$ and then switched back to a low value away from $I_c$. Finally, the DC-voltage across the junction is measured.
The increase of the bias current to $I_m$ drastically increases the tunneling rates $\Gamma_0$ and $\Gamma_1$ of the two possible states of the junction. During the time $\Delta t$, the junction can transit into a finite voltage state as a result of a tunneling process. However, as we have seen, the escape time depends strongly on the initial quantum state. For instance, $\Gamma_1 \sim 1000\Gamma_0$ if $\Delta U/\hbar \omega_r > 3$.

Therefore, if the junction is in the state $|\chi_1\rangle$, a DC voltage starts to develop after a time $1/\Gamma_1$, whereas this will happen after a much longer time $1/\Gamma_0 \gg 1/\Gamma_1$ if the junction is in the state $|\chi_0\rangle$. Hence if we switch the bias current $I$ back to a value away from $I_c$ after a time $\Delta t$ such that $1/\Gamma_1 \ll \Delta t \ll 1/\Gamma_0$, the junction remains in the zero-voltage state if and only if the junction is in the state $|\chi_0\rangle$. It is during the time $\Delta t$ that the two quantum states bifurcate into two different voltage states, hence we refer to this time as a measuring time. After $\Delta t$, the current is switched back to a lower value. Since the $I-V$ characteristics of the junction are hysteretic, the zero voltage and finite voltage states remain dynamically stable. Thus we have a relatively long time to perform the actual voltage measurements and distinguish the two junction states.

5 Discussion

The functioning of the quantum circuit presented in this paper relies on various assumptions. Below we will discuss these assumptions for each part of the circuit in some detail.

Josephson charge qubit. One of the main assumptions concerning the qubit is the absence of quasiparticles. If a quasiparticle tunnels onto the island, the state of the Cooper pair box changes: it no longer behaves as a quantum mechanical two-level system. Thus we need the superconducting gap to be sufficiently large, $\Delta > E_{C,j} + E_j/2$. We furthermore need the box to be quantum mechanically coherent during the experiment. This is one more reason to impose the absence of quasiparticles. But in addition it means that the coupling to the outside world must be sufficiently weak. Generally, the main source of decoherence is formed by time-dependent fluctuations of the effective gate charge $N_g$, either induced by fluctuations of the gate-voltage $V_g$, or by residual dynamics of charged impurities close to the qubit. However, if the qubit is operated close to the degeneracy point $N_g = 1$, the effect of this noise will be weak, since the energies $E_{\pm}$ depend only quadratically on the amplitude of the fluctuations.

Current biased Josephson junction. When using the current biased Josephson junction as a quantum mechanical resonator, it is essential for the states $|\chi_0\rangle$ and $|\chi_1\rangle$ to be well-defined, i.e., the broadening $\Gamma_{0,1}$ of the corresponding energy levels must be small compared to $E_{\text{coupl}}$. For this reason we need to work at low bias current, such that $\Gamma_1 \sim 1000\Gamma_0 \sim 1000\sqrt{E_r E_{C,r}} \exp \sqrt{E_r/E_{C,r}} \ll E_{\text{coupl}}$. In addition, decoherence to the external circuit should not occur on a time scale of the order of $\hbar/E_{\text{coupl}}$. This means that the impedance should be sufficiently high. For a purely resistive environment for instance ($R$ finite and $L = 0$ in Fig. [3]), the typical relaxation time is $RC_r$. We do require this time to exceed the Rabi period $\hbar/E_{\text{coupl}}$.

When using the junction as a detector it is important that it changes its state before relaxation processes induced by the environment occur. Thus we need to increase the current $I$ sufficiently close to $I_c$ such that $1/\Gamma_1$ is much smaller than the relaxation time.

Apart from increasing the tunneling rates $\Gamma_k$, an increase of the bias current $I$ has another important effect. Since the frequency $\omega_r$ depends on $I$, the resonance condition $\hbar \omega_r = E_j$ is no longer satisfied after the increase. As soon as the difference $|\hbar \omega_r - E_j|$ exceeds $E_{\text{coupl}}$, the qubit and the junction are no longer entangled and the states $|-,\chi_1\rangle$ and $|+,\chi_0\rangle$ become again eigenstates of the system. The probability to be in either of these states $|\chi_0\rangle$ or $|\chi_1\rangle$ ceases to evolve in time. This way, the probability $P_{x_0}$ remains "frozen" into its value at the time $t$ of the measurement (as long as relaxation phenomena can be neglected). In order for $t$ to be well-defined, the ramping time $\delta t$ must be short compared to the period $\hbar/E_{\text{coupl}}$ of the Rabi oscillations. On the other hand, the ramping itself should not induce transitions in the junction, thus we need to impose the condition $\omega_r \delta t \gg 1$. 
In most experiments\textsuperscript{10,14}, the environment is dominated by a resistor $R$. On the one hand, $R$ should be large in order for the relaxation time to be long. On the other hand, $R$ should be low enough to avoid heating at low temperature due to the bias current flowing through the resistor. In order to avoid these conflicting conditions, we propose to use a large on-chip inductor in series with the junction (see Fig. 1). To minimize the effect of the environment, the inductor has to be much larger than $R/\omega_r$ and $1/(C\omega_r^2)$. Then the relaxation time is given by $(L\omega_r)^2C/R$, which can be much larger than typical $RC_r$-times\textsuperscript{10,14}, the characteristic frequency of the junction is not affected by the environment. Moreover, if the temperature $T$ of the environment satisfies the condition $R/L \ll 2kT/\bar{\hbar} \ll \omega_r$, the escape rate is given by \[
abla_0(T)/\nabla_0 = (9/\pi \sqrt{2})(\hbar/e^3)(C_r/Lc)(1-I_m/I_c)^{1/2}kT,\]
where $\nabla_0$ is the zero-temperature rate for quantum tunneling. The excess escape rate due to thermal fluctuations in the resistor is, surprisingly, independent of the resistance $R$ of the environment.

For the numerical estimates presented below we will consider parameters of typical superconducting circuits\textsuperscript{10,4}. For the Josephson charge qubit we have chosen $E_j = 26.1\mu eV$ and $E_{C,j} = 70\mu eV$. The coupling capacitance is chosen to be of the order of $C_j$, $C_c = 0.5fF$, yielding $E_c = 256neV$. For a Josephson junction with a capacitance of $6.35pF$ and a critical current of $10\mu A$, the characteristic frequency is about $13GHz$ at zero bias current.

The resonant condition $\hbar \omega_r = E_j$ for the charge qubit and the Josephson resonator is obtained at a bias current of $9.4\mu A$. The period of the Rabi oscillations is given by $T_{Rabi} \approx 8 ns$. Taking into account the environment of Fig. 1 with $R = 8\Omega$ and $L=10nH$, the relaxation time is about $136ns$ and thus longer than the period of the Rabi oscillations.

During the measurement process, the bias current is chosen to be $I_m = 9.91\mu A$, corresponding to a characteristic frequency of $4GHz$. The relaxation time is $51ns$. For $T = 20mK$, we estimate the escape time to the voltage state to be $6\mu s$ from the state $|\chi_0\rangle$ and $7ns$ from the state $|\chi_1\rangle$. Taking $\Delta t = 50ns$, we expect the junction to bifurcate into the voltage state with a probability larger than $99.9\%$ for the state $|\chi_1\rangle$ and smaller than $0.1\%$ for the state $|\chi_0\rangle$.

In conclusion, we proposed an efficient procedure for single-shot quantum measurements of entangled states in a Cooper pair box coupled to a current-biased Josephson junction.

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