Resource-aware Exact Decentralized Optimization Using Event-triggered Broadcasting

Changxin Liu, Student Member, IEEE, Huiping Li, Member, IEEE, and Yang Shi, Fellow, IEEE

Abstract—This work addresses the decentralized optimization problem where a group of agents with coupled private objective functions work together to exactly optimize the summation of local interests. Upon modeling the decentralized problem as an equality-constrained centralized one, we leverage the linearized augmented Lagrangian method (LALM) to design an event-triggered decentralized algorithm that only requires light local computation at generic time instants and peer-to-peer communication at sporadic triggering time instants. The proposed method is universal in the sense that it perfectly accommodates all types of convex objective functions with tailored local implementations and competitive convergence rates, that is, a rate of $O\left(\frac{1}{k}\right)$ for nonsmooth objectives, the same rate for smooth objectives with a simplified iteration scheme, and a linear convergence rate for smooth and strongly convex objectives, provided that some mild conditions are satisfied. We examine the developed strategy in a decentralized logistic regression problem; comparison results illustrate its effectiveness and superiority in exploiting communication resources.

Index Terms—Decentralized optimization, event-triggered control, inexact method, augmented Lagrangian method.

I. INTRODUCTION

Decentralized optimization methods have received increasing attention recently due to its key role in advancing future developments of many engineering areas as diverse as wireless decentralized control systems, sensor networks, and decentralized machine learning [11], [2]. Usually, they involve a group of computing units that are connected via a communication network and rely on only local computation and peer-to-peer communication to cooperatively solve a large-scale optimization problem, where some coupling sources in the objective or/and constraint make the partition nontrivial.

This paper considers the case with coupled objective functions, i.e., the global objective is the sum of multiple private ones. In the literature, many efforts have been devoted to problems of this type. Depending on how they approach the solution, existing algorithms can be roughly categorized into three classes, that is, the primal method [3], the dual method [9], [10], and the primal-dual method [11], [13]–[15]. Amongst the three, the primal method seems the most popular. In typical primal methods, each computing unit directly seeks consensus on the primal decision variable by iteratively shifting its local estimate about the global minimizer in light of the local (sub)gradient and the information from its immediate neighbors. The convergence properties have been thoroughly investigated for this scheme with decaying and constant stepsizes in [3] and [3], respectively. It is worth mentioning that when using constant stepsizes with these methods, between the accumulation point and the global minimum, there is always an undesired gap whose magnitude is proportional to the stepsizes. To achieve exact convergence, the authors in [3] further added a cumulative correction term to the iteration rule of decentralized gradient descent (DGD) [5]. Note that there are other interpretations for the method in [3]; therein please find more details. Another remedy to this problem was reported in [4], [6], [7] where the local gradient used in DGD is replaced by an estimate of the global gradient supplied by the dynamic average consensus scheme. Although these methods share a similar iteration rule, the analyses are significantly different from one to another due to different network configurations. In marked contrast, the dual methods manage to agree on the estimate of the global gradient, and minimize the sum of a linear function characterized by this estimated gradient and a prox-function at each iteration to generate a sequence of local primal estimates [9], [10]. Upon treating the decentralized optimization problem as a linear equality-constrained centralized one, the dual decomposition [14], the augmented Lagrangian method [12], the alternating direction method of multipliers (ADMM) [11], the Bregman method [13], and other primal-dual methods [15] can be leveraged to design decentralized algorithms. For a recent overview of decentralized optimization, the interested readers are referred to [2].

In another line of research, event-triggered control emerges as a communication-efficient approach for large-scale network control systems [16]. The idea is to generate network transmission only when the information conveyed by the message is deemed innovative to the system, and whether or not it is essential is determined via an event-triggered function that takes the deviation between the actual system state and the state just broadcast as an argument. The hope of event-triggered control is to reduce the communication load while largely preserving the control performance.

Thanks to this attractive feature, event-triggered communication has been recently incorporated into decentralized optimization algorithms recently [17]–[22]. For example, the authors developed their event-triggered variants based on the standard decentralized optimization algorithm in [3] for convex functions. Although reductions in communication were observed in numerical experiments, the convergence rates are rather slow: $\frac{\log k}{\sqrt{k}}$ in [19] and $\log k$ in [17], where $k$ is the time counter, mainly due to the use of decaying stepsizes. To
speed up convergence, constant stepsizes were used in event-triggered DGD [20]. In particular, a linear convergence rate was secured for strongly convex and smooth functions provided that some reasonable conditions on the stepsize and the event-triggered function are met. However, similar to standard DGD, the algorithm does not ensure exact minimization but only yields an accumulation point in a neighborhood of the global minimizer. Based on [7], the authors in [18] solved this problem for strongly convex and smooth objective functions at the expense of maintaining an extra variable that tracks the global gradient using an event-triggered dynamic average consensus scheme. Recent work in [21] considered smooth and convex functions and presented an event-triggered decentralized ADMM that only requires each agent to route the decision variable to its neighbors and guarantees exact convergence. Convergence rates are further analyzed for special strongly convex and smooth objectives. Furthermore, it is remarked in [21] that the event-triggered zero-gradient-sum decentralized optimization method in [22] can be seen as an event-triggered version of dual decomposition that is empirically slower than ADMM. In these schemes, each agent at every generic time instant is required to exactly solve a subproblem, which may be not practical in most cases. Considering this, two questions naturally arise: 1) For general convex functions, is it possible to devise an event-triggered decentralized optimization algorithm that enjoys a competitive convergence rate even in the presence of node errors due to event-triggered communication? 2) If the objective functions exhibit some desired properties, e.g., smooth or/and strongly convex, is it possible to simplify the subproblem-solving process to simple algebraic operations without sacrificing the convergence rate?

We give affirmative answers to these questions in this work. First, the primal-dual methodology introduced earlier is used to tackle the decentralized optimization problem. More specifically, the linearized augmented Lagrangian method (LALM) in the recent work [29] with a specific pre-conditioning strategy is used to design a periodic decentralized algorithm. Then, each agent employs an event-triggered broadcasting strategy to communicate with its neighbors to avoid unnecessary network utilization. Since event-triggered communication essentially injects errors to computing units, a careful investigation of its effect on convergence rates for different types of objective functions is carried out. Compared to the state-of-the-art, the developed method features the following: 1) It ensures exact minimization with large constant stepsizes; 2) It is a universal method in the sense that it accommodates all kinds of convex functions with tailored local computation patterns and different yet competitive convergence guarantees. For example, when the objective function is nonsmooth, each agent involves solving a subproblem at each time instant for a convergence rate of $O(\frac{1}{k})$, while for smooth objectives only algebraic operations are required for the same rate. Furthermore, a linear convergence rate can be stated if the objective is further assumed to be strongly convex.

It is worth mentioning that there are also other attempts in the literature to develop communication-efficient algorithms. For instance, the authors in [23] considered DGD with random communication link failures, and established convergence rate and error bound for decaying and constant stepsizes, respectively. Using a similar idea, reference [24] presented an asynchronous DGD where only a randomized set of working agents choose to update their local variables. The authors proved that the local estimates converge to a neighborhood of the minimizer provided that the activation probability grows to 1 asymptotically. The works in [25], [26] considered asynchronous ADMM and established convergence rates. However, in these methods each agent is still dictated to exactly solving a subproblem at each iteration. Recently, reference [27] built an asynchronous decentralized consensus optimization algorithm based on [8] for a network of agents where communication delays may occur, and proved convergence. Another communication-efficient decentralized gradient method was reported in [28]; its novelty may lie in the use of only signs of relative state information between immediate neighbors. However, the convergence is rather slow, i.e., $\frac{\log k}{k}$, due to diminishing stepsizes. Note that these algorithms are significantly different from the proposed method in terms of communication patterns; the superiority of our algorithms over some of them will be demonstrated via an experiment study on decentralized logistic regression.

The notations adopted in this work are explained as follows. We use $\mathbf{1}$ to denote a column vector with all entries being 1, where the dimension shall be understood from the context. For a set $A$, let $|A|$ denote its cardinality. Given a symmetric matrix $P$ and a column vector $x$, $P \succ 0$ ($P \succeq 0$) means that the matrix is positive (semi)definite; $\lambda_+ (P)$ and $\lambda_- (P)$ stand for $P$’s maximum and minimum eigenvalues, respectively; $\| x \|$ represents the Euclidean norm; $\| P \|$ is the corresponding induced norm; $\| x \|_P = \sqrt{x^T P x}$ denotes the $P$-weighted norm. Finally, the Kronecker product of two matrices is denoted by $\otimes$.

II. Problem Statement and Preliminaries

A. Problem statement

This work considers the large-scale optimization problem given by

$$\min_{\theta \in \mathbb{R}^m} \sum_{i=1}^{n} F_i(\theta)$$

where $F_i : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}, i \in \mathbb{N}_{[1,n]}$ represents the local objective function and $\theta$ the common decision variable. Assume there are a group of agents that are connected via a network to solve $\mathbf{1}$, each of which, say $i$, has only access to $F_i$. In particular, the communication network is characterized by a simple undirected graph $G = (V, \mathcal{E})$. Each node $i \in V$ and edge $(i,j) \in \mathcal{E}$ in $G$ stand for each agent $i$ and the communication channel between agents $i$ and $j$, respectively. Moreover, agent $j$ is said to be a neighbor of $i$ if $(i,j) \in \mathcal{E}$. We let $\mathcal{N}_i \subset V$ denote the set of neighbors of $i$. The standard definitions of three $n \times n$ matrices are recalled: The adjacency matrix $A = [a_{ij}]$ where each entry $a_{ij} = 1$ if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise, the diagonal degree matrix $D = \text{diag}(|\mathcal{N}_1| \cdots |\mathcal{N}_n|)$, and the graph Laplacian $L = D - A$. Note that, for undirected graphs, the matrix $L$ is ensured to be positive semidefinite.
In this work, each local objective further assumes the following structure:

\[ F_i(\theta) = f_i(\theta) + g_i(\theta), \quad i \in \mathbb{N}_{[1,n]} \]

where \( f_i : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \) is smooth and convex while \( g_i : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\} \) is convex but possibly non-differentiable. This problem, known as composite optimization, finds wide applications in signal processing and machine learning, e.g., the data fitting problem with \( f_i \) being the loss function and \( g_i \) the regularizer. By letting \( f_i(\theta) = 0 \) or \( g_i(\theta) = 0, \quad i \in \mathbb{N}_{[1,n]} \), the composite setting recovers the special nonsmooth and smooth optimization.

Formally, we make the following assumptions for the objective function and the communication graph, respectively.

**Assumption 1.** \( f_i \) is a convex Lipschitz differentiable function with parameter \( L_{f_i} \), i.e.,

\[ \|\nabla f_i(x) - \nabla f_i(y)\| \leq L_{f_i}\|x - y\|, \forall x, y \in \mathbb{R}^m. \]

**Assumption 2.** \( G = (\mathcal{V}, \mathcal{E}) \) is fixed and connected.

Our goal is to design a decentralized first-order method to solve the composite optimization problem in (1) and further resort to event-triggered broadcasting to develop a communication resource-aware version with convergence rate guarantees.

**B. Primal-dual formulation of decentralized optimization**

Define \( x = [x_1^T, \ldots, x_n^T]^T, F(x) = \sum_{i=1}^{n} F_i(x_i), \ f(x) = \sum_{i=1}^{n} f_i(x_i), \) and \( g(x) = \sum_{i=1}^{n} g_i(x_i). \) Since \( \text{null}(\sqrt{L} \otimes I_m) = \text{span}\{1 \otimes u | u \in \mathbb{R}^m\} \), the problem in (1) can be equivalently written as the following linear equality-constrained optimization problem

\[
\begin{align*}
\min_{x \in \mathbb{R}^m, t} & \quad F(x) \\
\text{s.t.} & \quad (\sqrt{L} \otimes I_m) x = 0. 
\end{align*}
\]

(2)

By Assumption 1 we readily have that \( f(x) \) is also convex and has Lipschitz continuous gradient with constant \( L_f = \max_{i \in [1,n]} L_{f_i} \).

The augmented Lagrangian for (2) is written as

\[ L_\beta(x, y) = F(x) - \langle y, (\sqrt{L} \otimes I_m) x \rangle + \frac{\beta}{2} \|x\|^2_{L \otimes I_m} \]

where \( y = [y_1^T, \ldots, y_n^T]^T \in \mathbb{R}^{mn} \) denotes the dual variable and \( \beta > 0 \) a designable parameter. The KKT conditions can be identified as

\[
\begin{align*}
0 & \in \partial F(x^*) - (\sqrt{L} \otimes I_m)y^* \\
0 & = (\sqrt{L} \otimes I_m)x^* 
\end{align*}
\]

(3a)

(3b)

where \((x^*, y^*)\) is an optimal primal-dual pair and \( \partial F(x^*) \) the set of all subgradients of \( F \) evaluated at \( x^* \). Note that every \( y \in \mathbb{R}^{mn} \) can be decomposed into two vectors that belong to \( \text{null}(\sqrt{L} \otimes I_m) \) and \( \text{span}\{1 \otimes u | u \in \mathbb{R}^m\} \), respectively. Clearly, the former does not contribute to the augmented Lagrangian. It is therefore, without loss of generality, to assume that the dual variable \( y \) only takes values in \( \text{span}\{1 \otimes u | u \in \mathbb{R}^m\} \), i.e., \( \langle y, 1 \otimes u \rangle = 0, u \in \mathbb{R}^m \).

Using the convexity of \( F \) and the KKT conditions, we obtain

\[
\begin{align*}
F(x) - F(x^*) - \langle y^*, (\sqrt{L} \otimes I_m) x \rangle & \geq \langle \nabla F(x^*), x - x^* \rangle - \langle y^*, (\sqrt{L} \otimes I_m)(x - x^*) \rangle \\
& = \langle \nabla F(x^*) - (\sqrt{L} \otimes I_m)y^*, x - x^* \rangle \\
& = 0, \forall x 
\end{align*}
\]

(4)

where \( \nabla F(x^*) \in \partial F(x^*) \).

In the remaining sections, we will focus on the case with \( m = 1 \) for ease of notation, i.e., \( 1 \otimes I_m = 1, L \otimes I_m = L \). The developed results can be extended to the case \( m > 1 \) without much efforts.

**III. ALGORITHM DEVELOPMENT**

In this section, we develop a new periodic decentralized optimization algorithm and an event-triggered form of it to achieve a better tradeoff between network utilization and convergence speed. Some discussions about how this work relates to some recent works are presented.

**A. Development of a periodic decentralized optimization algorithm**

Based on the above primal-dual formulation, we recruit the LALM [29] to solve the decentralized composite optimization problem in (1). For decentralized implementation, a particular pre-conditioning strategy is used in LALM:

\[
\begin{align*}
x_{k+1} & = \arg \min_x \langle \nabla f(x_k) - \sqrt{\eta} \gamma_k, x \rangle + g(x) \\
& + \frac{\beta}{2} \|x\|^2_L + \frac{1}{2} \|x - x_k\|^2_{\eta L - \beta L} \\
y_{k+1} & = y_k - \beta \sqrt{\eta} x_{k+1}.
\end{align*}
\]

(5)

Note that the weight matrix used for the quadratic approximation of \( f \) in (5) is chosen as \( \eta L - \beta L \) to avoid computing the inverse of \( L \) and thus circumvent the need of global information in each node. To see this, we first let

\[ z = \sqrt{L} y \]

(6)

and rewrite the iteration rule as

\[
\begin{align*}
x_{k+1} & = \arg \min_x \langle \nabla f(x_k) - z_k, x \rangle + g(x) \\
& + \frac{\beta}{2} \|x\|^2_L + \frac{1}{2} \|x - x_k\|^2_{\eta L - \beta L} \\
z_{k+1} & = z_k - \beta L x_{k+1}.
\end{align*}
\]

(7)

Then, consider the optimality condition for \( \eta I + \partial g \)

\[ 0 \in \nabla f(x_k) - z_k + \partial g(x_{k+1}) - (\eta I - \beta L)x_k + \eta x_{k+1}, \]

which is equivalent to

\[ x_{k+1} = (\eta I + \partial g)^{-1}(- \nabla f(x_k) + z_k + \eta x_k - \beta L x_k). \]

(8)

Element-wisely,

\[ x_{i,k+1} = (\eta + \partial g_i)^{-1} \times (- \nabla f_i(x_{i,k}) + z_{i,k} + \eta x_{i,k} - \beta \sum_{j \in N_i} (x_{i,k} - x_{j,k})). \]
By the equivalence of optimality conditions, the update rule for primal variable in (7) becomes

\[ x_{k+1} = \arg\min_x \left\{ \nabla f_i(x_k) - z_{i,k}, x \right\} + g_i(x) + \frac{\eta}{2} \| x - x_{i,k} + \frac{\beta}{\eta} \sum_{j \in \mathcal{N}_i} (x_{i,k} - x_{j,k}) \|^2 \]

The proposed decentralized optimization algorithm is summarized in Algorithm 1.

**Algorithm 1** Periodic decentralized optimization algorithm

1. Set \( k = 0 \); each agent \( i \in \mathcal{V} \) broadcasts \( x_{i,0} \) to its neighbors
2. for each agent \( i \in \mathcal{V} \) do
3. \hspace{1em} Update primal variable
4. \hspace{2em} \[ x_{i,k+1} = \arg\min_x \left\{ \nabla f_i(x_{i,k}) - z_{i,k}, x \right\} + g_i(x) + \frac{\eta}{2} \| x - x_{i,k} + \frac{\beta}{\eta} \sum_{j \in \mathcal{N}_i} (x_{i,k} - x_{j,k}) \|^2 \]
5. \hspace{1em} Broadcast \( x_{i,k+1} \) to \( j \in \mathcal{N}_i \);
6. \hspace{1em} Update dual variable
7. \hspace{2em} \[ z_{i,k+1} = z_{i,k} - \beta \sum_{j \in \mathcal{N}_i} (x_{i,k+1} - x_{j,k+1}) \]
8. end for
9. Set \( k = k + 1 \).

**B. Event-triggering in communication**

In Algorithm 1 each agent is dictated to broadcasting its local estimate at every generic time instant \( k \). This subsection exploits the fact that possibly at some time instants the progress locally made by each agent is not sufficiently significant to be sent out to save communication resources. Let \( \kappa = \{ k | k \in \mathbb{N} \} \) be the set of generic time instants and \( \kappa_i = \{ k_i | l \in \mathbb{N} \} \subseteq \kappa \) the set of triggering time instants for agent \( i \). At each time \( k \), each agent \( i \) maintains the following variables:

1. \( i \)'s local primal variable: \( x_{i,k} \);
2. \( i \)'s local dual variable: \( z_{i,k} \);
3. \( i \)'s local primal variable assumed by \( j \): \( \tilde{x}_{i,k}, j \in \mathcal{N}_i \);
4. \( j \)'s local primal variable assumed by \( i \): \( \tilde{x}_{j,k}, j \in \mathcal{N}_i \).

where the identity \( \tilde{x}_{j,k}, j \in \mathcal{N}_i \) is defined as

\[ \tilde{x}_{j,k} = \begin{cases} x_{j,k}, & k \in \kappa_j \\ \tilde{x}_{j,k-1}, & \text{otherwise.} \end{cases} \]

Let \( \tilde{k}_j \in \kappa_j \) be the last triggering time instant of agent \( j \) before \( k \). We readily have \( \tilde{x}_{j,k} = \tilde{x}_{j,k-1} = \cdots = \tilde{x}_{j,\tilde{k}_j} \) by the definition of \( \tilde{x}_{j,k} \).

As briefly explained earlier, the degree of innovation of \( x_{i,k} \) to the overall system is measured by the deviation between it and the estimate broadcast most recently, i.e., \( \tilde{x}_{i,k} \). If the gap is large enough, \( x_{i,k} \) will be deemed as novel and sent out. Specifically, the event-triggering time instant \( k_i^{l+1} \) is determined by

\[ k_i^{l+1} = \min \{ k \in \kappa_i | k > \tilde{k}_i = k_i^l, \| x_{i,k} - x_{i,\tilde{k}_i} \| > E_{i,k} \} \tag{9} \]

where \( E_{i,k} > 0 \) represents the triggering threshold. It can be verified from the definition that the deviation between \( x_{i,k} \) and \( \tilde{x}_{i,k} \) is always bounded from above by \( E_{i,k} \), that is,

\[ \| x_{i,k} - \tilde{x}_{i,k} \| \leq E_{i,k}. \]

For the triggering threshold, we make the following assumption.

**Assumption 3.** Define \( E_k = \max_{i \in [1, n]} E_{i,k} \) for all \( k \in \mathbb{N} \). \( E_k \) is non-increasing and summable, i.e., \( \sum_{k=0}^{\infty} E_k < \infty \).

It is worth to note that the non-increasing property of \( E_k \) can be relaxed to that \( E_k \) is upper bounded by a non-increasing and summable sequence without affecting the theoretical results given later. Examples of such sequences include \( \{ E_k \} \) and \( \{ E_0 \rho^k \} \) where \( \rho < 1 \).

Based on such a communication pattern, an event-triggered decentralized optimization algorithm is formulated in Algorithm 2. In Step 1, each agent automatically triggers an event to initialize the algorithm. The primary difference between Algorithms 1 and 2 is that in the latter the local estimate by each agent is not necessarily broadcast at every time instant but some particular triggering time instants; see Step 4. Without the event trigger, Algorithm 2 reduces to Algorithm 1.

**Algorithm 2** Event-triggered decentralized optimization algorithm

1. Set \( k = 0 \); each agent \( i \in \mathcal{V} \) broadcasts \( x_{i,0} \) to its neighbors
2. for each agent \( i \in \mathcal{V} \) do
3. \hspace{1em} Update primal variable
4. \hspace{2em} \[ x_{i,k+1} = \arg\min_x \left\{ \nabla f_i(x_{i,k}) - z_{i,k}, x \right\} + g_i(x) + \frac{\eta}{2} \| x - x_{i,k} + \frac{\beta}{\eta} \sum_{j \in \mathcal{N}_i} (x_{i,k} - x_{j,k}) \|^2 \]
5. \hspace{1em} if triggered then
6. \hspace{2em} Broadcast \( x_{i,k+1} \) to its neighbors;
7. \hspace{2em} end if
8. \hspace{1em} Update dual variable
9. \hspace{2em} \[ z_{i,k+1} = z_{i,k} - \beta \sum_{j \in \mathcal{N}_i} (x_{i,k+1} - x_{j,k+1}) \]
10. end for
11. Set \( k = k + 1 \).

It is worth to note that, if the overall objective has Lipschitz continuous gradients, e.g., \( g(x) = 0 \), the iteration for primal variable can be further simplified. To see this, we consider

\[ x_{i,k+1} = \arg\min_x \left\{ \nabla f_i(x_{i,k}) - z_{i,k}, x \right\} + \frac{\eta}{2} \| x - x_{i,k} + \frac{\beta}{\eta} \sum_{j \in \mathcal{N}_i} (x_{i,k} - x_{j,k}) \|^2 \]
and the optimality condition
\[ 0 = \nabla f_i(x_{i,k}) - z_{i,k} + \left( \eta x_{i,k+1} - \eta x_{i,k} + \beta \sum_{j \in N_i} (\tilde{x}_{i,k} - \tilde{x}_{j,k}) \right) \]
that is equivalent to
\[ x_{i,k+1} = x_{i,k} + \frac{1}{\eta} \left( z_{i,k} - \nabla f_i(x_{i,k}) - \beta \sum_{j \in N_i} (\tilde{x}_{i,k} - \tilde{x}_{j,k}) \right). \]

Clearly, this significantly lowers down the computation load required to exactly solve a minimization subproblem, and facilitates practical use when the objective is smooth. In addition, the algorithm also works for completely nonsmooth functions, i.e., \( f(x) = 0 \), with the following modifications in Step 3:

\[ x_{i,k+1} = \arg \min_x \left\{ -z_{i,k}, x \right\} + g_i(x) \]
\[ + \frac{\eta}{2} \left\| x - x_{i,k} - \frac{1}{\eta} \sum_{j \in N_i} (\tilde{x}_{i,k} - \tilde{x}_{j,k}) \right\|^2 \]
\[ = \arg \min_x g_i(x) \]
\[ + \frac{\eta}{2} \left\| x - x_{i,k} - \frac{1}{\eta} \sum_{j \in N_i} (\tilde{x}_{i,k} - \tilde{x}_{j,k}) \right\|^2. \]

These statements will become clearer in the next section, where the convergence rates are rigorously analyzed.

C. Connection with the event-triggered ADMM

The proposed method generalizes the recent work in [21]. To see this, we recall the decentralized ADMM in [11]
\[ x_{k+1} = (2cD + \partial F)^{-1} (c(D + A)x_k - z_k) \]
\[ z_{k+1} = z_k + c(D - A)x_{k+1} \]
and its event-triggered version in [21]
\[ x_{k+1} = (2cD + \partial F)^{-1} (c(D + A)\tilde{x}_k - z_k) \]
\[ z_{k+1} = z_k + c(D - A)\tilde{x}_{k+1} \]
where \( c > 0 \) is the stepsize. Since \( L = D - A \), we equivalently express (11) as
\[ x_{k+1} = (2cD + \partial F)^{-1} (c(2D - L)\tilde{x}_k - z_k) \]
\[ z_{k+1} = z_k + cL\tilde{x}_{k+1} \]
and therefore
\[ x_{k+1} = (cI_n + \frac{D^{-1}}{2} \partial F)^{-1} (c\tilde{x}_k - \frac{c}{2} D^{-1} L\tilde{x}_k - \frac{D^{-1}}{2} z_k) \]
\[ = (\frac{D^{-1}}{2} z_k + \frac{c}{2} D^{-1} L\tilde{x}_k + \frac{D^{-1}}{2} z_k). \]

By comparing (8) and (12), we can see that the event-triggered ADMM in [21] is very similar to the proposed method when the smooth part of the objective \( f(x) = 0 \), and the only difference is that the Laplacian matrix is normalized to \( \frac{1}{2} D^{-1} L \), the dual variable is scaled to \( \frac{D^{-1}}{2} z \), and the proximal parameter is changed to \( (cI_n + \frac{D^{-1}}{2} \partial F)^{-1} \). However, this work further establishes convergence rate for nonsmooth convex functions while [21] only proved convergence, and provides tailored implementations for smooth objectives to reduce computation load.

IV. CONVERGENCE RATE FOR COMPOSITE OBJECTIVE FUNCTIONS

This section examines the convergence rate for Algorithm 2 with Assumptions 13 satisfied.

Theorem 1. If Assumptions 13 hold and
\[ (\eta - L_f)I_n - \beta L \succ 0, \]
then
\[ \left\| \sqrt{L}\tilde{x}_t \right\| \leq \frac{\left\| (x_0 - x^*)\eta I_n - \beta L + \rho (L + \frac{11t}{4})^{-1} \right\| \left\| y^* \right\|}{2t (\rho - \left\| y^* \right\|)}, \]
and
\[ \left\| y^* \left\| (x_0 - x^*)\eta I_n - \beta L + \rho (L + \frac{11t}{4})^{-1} \right\| + \sqrt{2bA_t} \right\| \leq F(\tilde{x}_t) - F(x^*) \]
\[ \leq \frac{\left( \left\| x_0 - x^* \right\| \eta I_n - \beta L + \rho (L + \frac{11t}{4})^{-1} \right\| + \sqrt{2bA_t} \right)^2}{2t}, \]
where \( \tilde{x}_t = \frac{1}{t} \sum_{k=1}^t x_k \), \( \rho \succ \left\| y^* \right\| \). \( A_t = \frac{2a}{b} \sum_{k=1}^t E_{k-1} \), \( a = \max \{2(\eta - L_f), 1\} \), and \( b = \min \{L_f, \frac{1}{\lambda(L + \frac{11t}{4})^{-1}}\} \).

Remark 1. Theorem 2 states that both the consensus error \( \left\| \sqrt{L}\tilde{x}_t \right\| \) and the objective error \( F(\tilde{x}_t) - F(x^*) \) converge to zero at an ergodic convergence rate of \( O(\frac{1}{t}) \) if some reasonable assumptions hold true. It is worth to mention that the result remains valid for smooth objective functions, e.g., \( g(x) = 0 \), with a much simplified iteration rule in (10). For completely nonsmooth objective functions, e.g., \( f(x) = 0 \), the condition for stepsize to ensure the same convergence rate is further relaxed to
\[ \eta I_n - \beta L \succ 0, \]
which is in line with the condition developed in [27] for convergence.

Before developing the proof for Theorem 1, several useful technical lemmas are presented. Specifically, Lemma 3 characterizes the behavior of Algorithm 2 in one iteration. Lemma 4 establishes upper bounds for the primal-dual sequence, provided that the event-triggering threshold satisfies Assumption 3.

Lemma 1. If Assumption 2 holds, then for each \( y \in \text{span} \frac{1}{2} \), there exists a unique \( y' \in \text{span} \frac{1}{2} \) such that \( y = Ly' \) and vice versa.
Lemma 2. If all the conditions in Theorem 7 hold, then, for any \( x \in \text{null}(L) \) and \( z \in \text{span}^{\perp}L \),
\[
F(x_{k+1}) - F(x) - \langle y, \sqrt{L}x_{k+1} \rangle \\
\leq - \frac{1}{2}\|x_{k+1} - xk\|_{(\eta I_n - \beta L)}^2 \\
- \frac{1}{2}\|\eta I_n - \beta L\|\|x_{k+1} - xk\|_{(\eta I_n - \beta L)}^2 \\
- \frac{1}{2}\langle x_{k+1} - \beta Lxk + \eta I_n - \beta L \rangle \\
+ \frac{1}{2}\langle x_{k+1} - \beta L(e_k - e_k) \rangle + \langle e_k - e_k \rangle \\
- \frac{1}{2}\|z_k - zk + 1\|_{(\beta L + \Omega)}^2 \\
- \frac{1}{2}\|z_k - zk + 1\|_{(\beta L + \Omega)}^2 \\
\}
\]
where \( e_k = \tilde{x}_k - x_k \).

Proof of Lemma 2. By the smoothness and convexity of \( f \), we have
\[
f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L_f}{2}\|x_{k+1} - x_k\|^2, \]
and
\[
f(x_{k+1}) = f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L_f}{2}\|x_{k+1} - x_k\|^2.
\]
Subtracting \( f(x) + \langle y, \sqrt{L}x_{k+1} \rangle \) on both sides and using the definition of \( z \) in (13) yield
\[
f(x_{k+1}) - f(x) - \langle y, \sqrt{L}x_{k+1} \rangle \\
\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L_f}{2}\|x_{k+1} - x_k\|^2 - \langle z_k, z_k \rangle.
\]
From the iteration rule, we have
\[
0 = \nabla f(x_k) - z_k + \nabla g(x_{k+1}) - \eta x_k + \beta L\tilde{x}_k + \eta x_{k+1} \\
0 = \beta L\tilde{x}_{k+1} + z_{k+1} - z_k
\]
where \( \nabla g(x_{k+1}) \) is a subgradient of \( g \) evaluated at \( x_{k+1} \). This implies
\[
0 = \nabla f(x_k) + \nabla g(x_{k+1}) - z_{k+1} + (\eta I_n - \beta L)(x_{k+1} - x_k) \\
+ \beta L(e_k - e_k) \\
\]
Calculating the inner products of \( x_{k+1} - x \) with both sides gives rise to
\[
\langle x_{k+1} - x, \nabla f(x_k) \rangle + \langle x_{k+1} - x, (\eta I_n - \beta L)(x_{k+1} - x_k) \rangle \\
+ \langle x_{k+1} - x, \beta L(e_k - e_k) \rangle \\
= 0.
\]
for any \( x \in \text{null}(L) \) and \( z \in \text{span}^{\perp} \). We then can obtain from Lemma 1 that
\[
\langle x_{k+1}, z - z_{k+1} \rangle \\
= \langle \tilde{x}_{k+1}, z - z_{k+1} \rangle - \langle \eta I_n - \beta L \rangle(x_{k+1} - x_k) \\
= \beta L\tilde{x}_{k+1} + z' - z'_{k+1} - \langle \eta I_n - \beta L \rangle(x_{k+1} - x_k) \\
= z_k - z_{k+1} - z'_{k+1} - \langle \eta I_n - \beta L \rangle(x_{k+1} - x_k) \\
= \beta L(z_k' - z_{k+1}') - \langle \eta I_n - \beta L \rangle(x_{k+1} - x_k) \\
\]
By convexity of \( g \),
\[
\langle x_{k+1} - x, \nabla g(x_{k+1}) \rangle \geq g(x_{k+1}) - g(x) \]
Plugging equations (17) and (18) into (16) leads to
\[
\langle x_{k+1} - x, \nabla f(x_k) \rangle - \langle y, \sqrt{L}x_{k+1} \rangle \\
\leq -\langle x_{k+1} - x, (\eta I_n - \beta L)(x_{k+1} - x_k) \rangle - g(x_{k+1}) + g(x) \\
- \langle \beta L(z_k' - z_{k+1}'), z' - z'_{k+1} \rangle + \langle e_k - e_k \rangle + \langle e_k - e_k \rangle \\
- \langle x_{k+1} - x, \beta L(e_k - e_k) \rangle,
\]
which in conjunction with (14) gives
\[
F(x_{k+1}) - F(x) - \langle y, \sqrt{L}x_{k+1} \rangle \\
\leq \frac{L_f}{2}\|x_{k+1} - x_k\|^2 - \langle x_{k+1} - x, (\eta I_n - \beta L)(x_{k+1} - x_k) \rangle \\
- \langle \eta I_n - \beta L \rangle(x_{k+1} - x_k) \\
\]
where we make use of the fact that
\[
\eta I_n - \beta L \geq L_f I_n \succ 0, \beta L \succeq 0
\]
and the following identity
\[
2\langle Wv, v \rangle = \|u\|^2_W + \|u\|^2_W - \|u - v\|^2_W, \forall u, v \in \mathbb{R}^n, W \succeq 0.
\]
Since \( z', \tilde{z}' \in \text{span}^{\perp} \) and therefore
\[
z_k - z = \beta L(z_k' - z_{k+1}') = (\beta L + \frac{11T}{n})(z_k' - z_{k+1}')
we have
\[ \|z_k-z^*\|_{\beta L}^2 = \|z_k-z\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}, \]
which together with (19) gives the desired inequality. \(\square\)

**Lemma 3.** If all the conditions in Theorem 1 hold, then, for \(k \leq t\),
\[
\|x_k-x^*\| + \|z^*-z_k\| \leq 2A_t + \sqrt{\frac{2}{b}} \left( \sum_{t=0}^{\infty} \|x_0-x^*\|_{\eta I_n-\beta L} + \|z_0-z^*\|_{(\beta L+\frac{11\tau}{n})^{-1}} \right)
\]
where \(b\), and \(A_t\) are defined in Theorem 7

**Proof of Lemma 3** Let \(x = x^*\) and \(y = y^*\). From Lemma 2 we have
\[
F(x_{k+1}) - F(x^*) - \left\langle y^*, \sqrt{L}x_{k+1} \right\rangle
\leq -\frac{1}{2} \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L}
- \frac{1}{2} \left| \left\langle x_{k+1} - x_k, x^* - x^* \right\rangle \right|_{\eta I_n - \beta L}^2
- \left\langle x_{k+1} - x^*, \beta L(e_k - e_{k-1}) \right\rangle + \left\langle z_k, z^* - z_{k-1} \right\rangle
+ \frac{1}{2} \left\| z^* - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}} - \frac{1}{2} \left\| z_k - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
\]
Summing the above inequality over \(k\) from 0 to \(t-1\) yields
\[
0 \leq \sum_{k=0}^{t-1} \left( F(x_{k+1}) - F(x^*) - \left\langle y^*, \sqrt{L}x_{k+1} \right\rangle \right)
\leq -\frac{1}{2} \sum_{k=0}^{t-1} \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L}
- \frac{1}{2} \sum_{k=0}^{t-1} \left| \left\langle x_{k+1} - x_k, x^* - x^* \right\rangle \right|^2_{\eta I_n - \beta L}
- \sum_{k=0}^{t-1} \left( \left\langle x^* - x_{k+1}, \beta L(e_k - e_{k-1}) \right\rangle + \left\langle z^* - z_{k+1}, z_k \right\rangle \right)
- \sum_{k=0}^{t-1} \left\| z_k - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
\]
Since \((\eta - L_f)I_n - \beta L > 0\) and \((\beta L+\frac{11\tau}{n})^{-1} > 0\), it then holds that
\[
\frac{1}{2} \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L} + \frac{1}{2} \left\| z^* - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
\leq \frac{1}{2} \left\| x_0 - x^* \right\|^2_{\eta I_n - \beta L} + \frac{1}{2} \left\| z_0 - z^* \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
+ \sum_{k=1}^{t} \left( \left\langle x^* - x_k, \beta L(e_k - e_{k-1}) \right\rangle + \left\langle z^* - z_k, e_{k} \right\rangle \right)
\]
By the monotonicity of \(E_k\) and the Cauchy-Schwarz inequality, we further have
\[
\frac{1}{4} \min \left\{ L_f, \frac{1}{\sqrt{(\beta L+\frac{11\tau}{n})}} \right\} \left( \left\| x_t - x^* \right\|^2 + \left\| z^* - z_t \right\|^2 \right) \leq \frac{1}{2} \left\| x_t - x^* \right\|^2_{\eta I_n - \beta L} + \frac{1}{2} \left\| z^* - z_t \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
+ \sum_{k=1}^{t} \left( \left\langle x^* - x_k, \beta L(e_k - e_{k-1}) \right\rangle + \left\langle z^* - z_k, e_{k} \right\rangle \right)
\]
+ \sum_{k=1}^{t} \left\| z_0 - z^* \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
+ \sum_{k=1}^{t} \max \left\{ 2(\eta - L_f), 1 \right\} \sqrt{nE_{k-1}} \left( \left\| x^* - x_k \right\| + \left\| z^* - z_k \right\| \right).
\]
Upon using Lemma 1 in (30), we obtain
\[
\left\| x_{t+1} - x^* \right\| + \left\| z^* - z_t \right\| \leq A_t
+ \left( \frac{2 \left\| x_0 - x^* \right\|^2_{\eta I_n - \beta L} + \left\| z_0 - z^* \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}}{b} + A_t^2 \right)^{1/2}
\]
where \(b\) and \(A_t\) are defined in the statement of Theorem 1. By the monotonicity and positivity of \(A_t\), the desired result follows. \(\square\)

We are now in a position to present the proof for Theorem 1

**Proof of Theorem 1** We begin by rewriting (21) as
\[
\frac{1}{2} \sum_{k=0}^{t-1} \left( \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L} + \left\| z_k - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}} \right)
\leq -\frac{1}{2} \left\| x_t - x^* \right\|^2_{\eta I_n - \beta L} + \frac{1}{2} \left\| x_0 - x^* \right\|^2_{\eta I_n - \beta L}
- \frac{1}{2} \left\| z^* - z_t \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}} + \frac{1}{2} \left\| z_0 - z^* \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
+ \sum_{k=0}^{t-1} \left( \left\langle x^* - x_{k+1}, \beta L(e_k - e_{k-1}) \right\rangle + \left\langle z^* - z_{k+1}, e_{k+1} \right\rangle \right).
\]
Dropping the negative terms in the right-hand side and using the similar procedure as in (22) yield
\[
\frac{1}{2} \sum_{k=0}^{t-1} \left( \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L} + \left\| z_k - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}} \right)
\leq \frac{1}{2} \left\| x_0 - x^* \right\|^2_{\eta I_n - \beta L} + \frac{1}{2} \left\| z_0 - z^* \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}}
+ \left( \left\| x^* - x_t \right\| + \left\| z^* - z_t \right\| \right) \sum_{k=1}^{t} a \sqrt{nE_{k-1}}
\]
where \(a\) is defined in Theorem 1. In light of Lemma 3, we have that if \(E_{k-1}\) is summable, then
\[
\sum_{k=0}^{\infty} \left( \left\| x_{k+1} - x_k \right\|^2_{(\eta - L_f)I_n - \beta L} + \left\| z_k - z_{k+1} \right\|^2_{(\beta L+\frac{11\tau}{n})^{-1}} \right) < \infty.
\]
Since \((\eta - L_f)I_n - \beta L > 0\), \((\beta L+\frac{11\tau}{n})^{-1} > 0\), we further have
\[
\lim_{k \to \infty} (x_{k+1}, z_{k+1}) - (x_k, z_k) = 0.
\]
Denote the limit point of \( \{ (x_k, z_k) \}_{k \geq 1} \) by \( (x_\infty, z_\infty) \). Note that \( \lim_{k \to \infty} E_k = 0 \) by assumptions. From
\[
\beta L(e_{k+1} + x_{k+1}) = z_k - z_{k+1},
\]
and
\[
0 = \nabla F(x_{k+1}) - z_k - \eta x_k + \beta L(e_k + x_k) + \eta x_k + 1
\]
where \( \nabla F(x_{k+1}) \) is a subgradient of \( F \) evaluated at \( x_{k+1} \), we obtain \( L x_\infty = 0 \) and \( \nabla F(x_\infty) - z_\infty = 0 \), respectively. This implies that \( (x_\infty, y_\infty) \) is a KKT point. Again, from (21), we have
\[
\sum_{k=0}^{t-1} \left( F(x_{k+1}) - F(x^*) - \left\langle y^*, \sqrt{L} x_{k+1} \right\rangle \right)
\leq \frac{1}{2} \| x_0 - x^* \|_\eta I_{n-\beta L}^2 + \frac{1}{2} \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1
\]
\[
\quad + \left( \| x^* - x \| + \| z^* - z \| \right) \sum_{k=0}^{t} a \sqrt{L} x_{k+1} - 1
\]
\[
\leq \frac{1}{2} \| x_0 - x^* \|_\eta I_{n-\beta L}^2 + \frac{1}{2} \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 + \frac{b}{2} A_{t+1}
\]
\[
\left( 2A_t + \sqrt{\frac{2}{b}} \left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right) \right)
\]
\[
\left( \frac{\| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1}{\sqrt{2}} + \sqrt{6} A_{t+1} \right)^2,
\]
which in conjunction with
\[
\sum_{k=0}^{t-1} \left( F(x_{k+1}) - F(x^*) - \left\langle y^*, \sqrt{L} x_{k+1} \right\rangle \right)
\leq \frac{1}{2} \| x_0 - x^* \|_\eta I_{n-\beta L}^2 + \frac{1}{2} \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1
\]
\[
\quad + \left( \| x^* - x \| + \| z^* - z \| \right) \sum_{k=0}^{t} a \sqrt{L} x_{k+1} - 1
\]
\[
\leq \frac{1}{2} \| x_0 - x^* \|_\eta I_{n-\beta L}^2 + \frac{1}{2} \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 + \frac{b}{2} A_{t+1}
\]
\[
\left( 2A_t + \sqrt{\frac{2}{b}} \left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right) \right)
\]
\[
\left( \frac{\| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1}{\sqrt{2}} + \sqrt{6} A_{t+1} \right)^2,
\]
gives
\[
F(x_t) - F(x^*) - \left\langle y^*, \sqrt{L} x_t \right\rangle
\leq \frac{1}{2} \left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right)
\]
\[
+ \frac{1}{2} \left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| y^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right) + \sqrt{6} A_{t+1} \right)^2.
\]
Finally, we consider
\[
F(x_t) - F(x^*) + \rho \sqrt{L} x_t
\leq \sup_{\| y^* \| \leq \rho} \frac{\left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| z_0 - z^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right)}{2t} + \sqrt{6} A_{t+1} \right)^2
\]
\[
\leq \frac{\left( \| x_0 - x^* \|_\eta I_{n-\beta L} + \| y^* \|_\beta L + 1 + 1 \eta I_{n-\beta L} \|_1 - 1 \right)}{2t} + \sqrt{6} A_{t+1} \right)^2.
\]
By (4), it holds that
\[
F(x_t) - F(x^*) \geq - \| y^* \| \sqrt{L} x_t,
\]
which together with (24) completes the proof.

V. LINEAR CONVERGENCE RATE FOR STRONGLY CONVEX AND SMOOTH OBJECTIVE FUNCTIONS

This section considers strongly convex and smooth objective functions, for which stronger convergence results can be expected. Formally, the following assumption is made for the objective functions.

Assumption 4. For all \( i \in \mathbb{N}_{[1,n]} \), \( g_i(\theta) = 0 \) and \( f_i \) is strongly convex with parameter \( \mu_{f_i} \), i.e.,
\[
\| \nabla f_i(x) - \nabla f_i(y) \| \geq \mu_{f_i} \| x - y \|, \forall x, y \in \mathbb{R}^m.
\]

As a direct consequence, \( f(x) \) is strongly convex with modulus
\[
\mu_{f} = \min_{i \in \mathbb{N}_{[1,n]}} \mu_{f_i}.
\]

As mentioned earlier, for smooth objective functions, the step 3 in Algorithm 2 reduces to (10).

Theorem 2. If Assumptions 1,2 hold and
\[
(\eta - \frac{L^2}{k_1}) I_n - \beta L > 0
\]
for some \( 0 < k_1 < 2\mu_f \), then there exists some positive \( \sigma \) such that
\[
\frac{1}{2} \| x_k - x^* \|_\eta I_{k_1(2\mu_f - k_1)} I_n - \beta L + \frac{1}{2} \| z_k - z^* \|_\beta L + 1 + 1 \eta I_{k_1(2\mu_f - k_1)} I_n - \beta L
\]
\[
+ \| z^* - z_{k+1} \|_\beta L + 1 + 1 \eta I_{k_1(2\mu_f - k_1)} I_n - \beta L
\]
\[
(\eta + k_1(2\mu_f - k_1)) I_n - \beta L > 0,
\]
where the constant
\[
C = \frac{2(\kappa + k_1 - 1)(\kappa + k_5) \sqrt{\beta L^2} + \kappa (\beta L + 1 + 1 \eta I_{k_1(2\mu_f - k_1)} I_n - \beta L)}{2k_5 \sqrt{\beta L^2}} + \kappa \beta L + \kappa \frac{1}{1 + \eta I_{k_1(2\mu_f - k_1)} I_n - \beta L}
\]
\[
0 < k_2, 2 < k_3, 0 < k_4 < 1, \text{and} 0 < k_5.
\]

Remark 2. It is revealed in Theorem 2 that if the objective function is further assumed to be strongly convex and the stepsize satisfies a relatively stricter condition then a much faster convergence rate can be obtained. In particular, if \( E_k \) linearly converges then we obtain a linear convergence rate for the primal-dual residual. And if the base for a linearly convergent \( E_k \) is smaller than \( \sqrt{\frac{1}{1+\eta I_{k_1(2\mu_f - k_1)} I_n - \beta L}} \), then the convergence of the primal-dual residual is linear with constant \( \frac{1}{1+\eta I_{k_1(2\mu_f - k_1)} I_n - \beta L} \) as in a periodic algorithm.

Proof Theorem 2. Recall (13)
\[
0 = \nabla f(x_{k+1}) - z_{k+1} + (\eta I_n - \beta L)(x_{k+1} - x_k) + \beta L(e_k - e_{k+1}),
\]
and the KKT condition
\[
0 = \nabla f(x^*) - z^*.
\]
We then have
\[
0 = \nabla f(x_k) - \nabla f(x^*) + z^* - z_{k+1} + \left((\eta I_n - \beta L) (x_{k+1} - x_k) \right)
+ \beta L (e_k - e_{k+1}).
\]
(26)

As in the proof of Lemma 2, we consider the inner products of \(x_{k+1} - x^*\) with both sides of the above equality
\[
\left< x_{k+1} - x^*, \nabla f(x_k) - \nabla f(x^*) \right>
+ \left< x_{k+1} - x^*, (\eta I_n - \beta L)(x_{k+1} - x_k) \right>
+ \left< x_{k+1} - x^*, \beta L(e_k - e_{k+1}) \right>
+ \left< x_{k+1} - x^*, z^* - z_{k+1} \right> = 0.
\]
(27)

By strong convexity and smoothness of \(f\), it holds that
\[
\left< x_{k+1} - x^*, \nabla f(x_k) - \nabla f(x^*) \right>
= \left< x_{k+1} - x^*, \nabla f(x_k) - \nabla f(x_{k+1}) + \nabla f(x_{k+1}) - \nabla f(x^*) \right>
\geq \mu_f \|x_{k+1} - x^*\|^2 - \frac{1}{2k_1} \|\nabla f(x_k) - \nabla f(x_{k+1})\|^2
- \frac{k_1}{2}\|x_{k+1} - x^*\|^2
\geq (\mu_f - \frac{k_1}{2})\|x_{k+1} - x^*\|^2 - \frac{L_f^2}{2k_1}\|x_k - x_{k+1}\|^2
\]
for \(0 < k_1 < 2\mu_f\). Using (26) allows us to obtain
\[
\left< x_{k+1} - x^*, (\eta I_n - \beta L) (x_{k+1} - x_k) \right>
= \frac{1}{2} \left\|x_{k+1} - x^*\right\|^2 \eta I_n - \beta L
+ \left\|x_{k+1} - x_k\right\|^2 \eta I_n - \beta L
- \left\|x_k - x^*\right\|^2 \eta I_n - \beta L
\]
With the same reasoning in (17), we have
\[
\left< x_{k+1} - x^*, z^* - z_{k+1} \right>
= \frac{1}{2} \left\|z^* - z_{k+1}\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
+ \frac{1}{2} \left\|z_k - z_{k+1}\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
- \frac{1}{2} \left\|z_k - z^*\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
- \left< e_k + 1, z^* - z_{k+1} \right>.
\]
(28)

Combining Eqs. (27)-(28) yields
\[
0 \geq (\mu_f - \frac{k_1}{2})\|x_{k+1} - x^*\|^2 - \frac{L_f^2}{2k_1}\|x_k - x_{k+1}\|^2
+ \frac{1}{2} \left\|x_{k+1} - x^*\right\|^2 \eta I_n - \beta L
+ \frac{1}{2} \left\|x_k - x_{k+1}\right\|^2 \eta I_n - \beta L
- \frac{1}{2} \left\|x_k - x^*\right\|^2 \eta I_n - \beta L
+ \left< x_{k+1} - x^*, \beta L(e_k - e_{k+1}) \right>
+ \frac{1}{2} \left\|z^* - z_{k+1}\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
+ \frac{1}{2} \left\|z_k - z_{k+1}\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
- \frac{1}{2} \left\|z_k - z^*\right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
- \left< e_k + 1, z^* - z_{k+1} \right>,
\]
which is equivalent to
\[
\left\| (\eta I_n - \beta L) (x_{k+1} - x_k) \right\|^2
= \left\| \nabla f(x_k) - \nabla f(x^*) + z^* - z_{k+1} + \beta L(e_k - e_{k+1}) \right\|^2
\geq (1 - k_2 - k_3) \left\| \nabla f(x_k) - \nabla f(x^*) \right\|^2
+ (1 - \frac{2}{k_3}) \left\| z^* - z_{k+1} \right\|^2
+ (1 - \frac{k_2}{k_3}) \left\| \beta L(e_k - e_{k+1}) \right\|^2
\geq (1 - k_2 - k_3) L_f^2 \|x_k - x^*\|^2 + (1 - \frac{2}{k_3}) \left\| z^* - z_{k+1} \right\|^2
+ (1 - \frac{k_2}{k_3}) \left\| \beta L(e_k - e_{k+1}) \right\|^2
\]
for any \(k_2 > 0\) and \(k_3 > 2\), where we use the inequality
\[
2 \left< u, v \right> \geq -w \left< u \right> \left< \frac{1}{w} \right> v, \forall u, v \in \mathbb{R}^n, w > 0
\]

obtain the first inequality and the Lipschitz differentiability of \(f\) the second. This implies, for some \(\sigma > 0\) and \(k_3 > 0\),
\[
\frac{\sigma + k_3}{2} \left\| z^* - z_{k+1} \right\|^2 \left(\beta L + \frac{L_f}{2}\right)^{-1}
\leq \frac{\sigma + k_5}{1 - \frac{k_2}{k_3}} \left( \frac{1}{1 - \frac{k_2}{k_3}} \right) \frac{L_f^2}{\|x_k - x_{k+1}\|^2 \eta I_n - \beta L}
- \frac{1 - k_2 - k_3}{1 - \frac{k_2}{k_3}} \left\| \beta L(e_k - e_{k+1}) \right\|^2.
\]
If \(\sigma + k_5\) is sufficiently small such that
\[
\frac{\sigma + k_3}{2} (\eta I_n - \beta L)^2 \leq \frac{L_f^2}{\|x_k - x_{k+1}\|^2 \eta I_n - \beta L}
\]
\[
\frac{\sigma + k_5}{2} \left( \frac{1}{1 - \frac{k_2}{k_3}} \Delta (\beta L + \frac{L_f}{2}) \right) \leq k_4 (\mu_f - k_1)
\]
\[
(\sigma + k_5) \left( \frac{1}{1 - \frac{k_2}{k_3}} \Delta (\eta I_n - \beta L) \right) \leq (1 - k_4)(\mu_f - k_1)
\]
for some $0 < k_4 < 1$, then

\[
\frac{1}{2} \left\| x_{k+1} - x^* \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 + \frac{1}{2} \left\| x_{k+1} - x_k \right\|^2 + \frac{1}{2} \left\| x_{k+1} - x_k \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 \geq \frac{\sigma + k_5}{2} \left( \left\| x_{k+1} - x^* \right\|^2 + \left\| x_k - x^* \right\|^2 + \left\| x_{k+1} - x_k \right\|^2 + \left\| x_k - x^* \right\|^2 \right) - \frac{\sigma + k_5}{2} \left\| x_{k+1} - x_k \right\|^2.
\]

Plugging (30) into (29) leads to

\[
\frac{1}{2} \left\| x_{k+1} - x^* \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 + \frac{1}{2} \left\| x_k - x^* \right\|^2 \geq \frac{\sigma + k_5}{2} \left( \left\| x_{k+1} - x^* \right\|^2 + \left\| x_k - x^* \right\|^2 + \left\| x_{k+1} - x_k \right\|^2 + \left\| x_k - x^* \right\|^2 \right) - \frac{\sigma + k_5}{2} \left\| x_{k+1} - x_k \right\|^2.
\]

By the monotonicity of $E_k$ and the inequality

\[
\left\langle u, v \right\rangle \leq \frac{k_4}{2} \left\| u \right\|^2 + \frac{1}{2k_4} \left\| v \right\|^2, \forall u, v \in \mathbb{R}^n, P > 0,
\]

we arrive at (25). This completes the proof.

\section{VI. Simulation Studies}

\subsection{A. Simulation setup}

This section examines the effectiveness of the proposed algorithms by applying them to the logistic regression, a powerful probabilistic classification model for predicting class labels. We consider a decentralized configuration where each agent $i$ holds its local training data containing both the input features $f_i \in \mathbb{R}^m$ and the class labels $y_i \in \{-1, 1\}$ with $j = 1, \ldots, m_i$ to learn the parameters of a global sigmoid function. Note that we set the last element of the feature vector $f_i \in \mathbb{R}^m$ to 1 as in standard logistic regression, then the last element of the decision variable $\theta$ becomes the adjustable bias of the logistic regression model. A medium-sized network of $n = 50$ agents is considered, the number of samples for each agent $i$ is $m_i = 8$, and the dimension for decision variable is $m = 10$. All the 400 samples are generated randomly in the simulation. The global optimization problem becomes

\[
\min_{\theta} \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m_i} \ln \left( 1 + \exp \left( -y_i^j (M_i^T \theta) \right) \right) \right\}.
\]

Throughout the simulation, the communication topology is characterized by a fixed connected small world graph with connectivity ratio $r = 0.5$ [31]. A ground true logistic classifier $x^* = 1 \otimes \theta^*$ obtained offline by running the centralized gradient descent is used as a benchmark. The local initial guesses of primal and dual variables for each agent are set as 0. We evaluate the performance by considering the residual $\frac{\left\| x_n - x^* \right\|^2}{2}$ over the number local iteration and communication times of the first agent.

\subsection{B. Comparison results}

The Lipschitz constant is estimated based on the randomly generated samples as $L = 50$, and the largest eigenvalue of $L$ is identified as $X(L) = 7.7839$. Then we set $\eta = 130$ and $\beta = 1$ so that $(\eta - L)I_n - \beta L > 0$. With these parameters, we test the periodic decentralized optimization method in Algorithm [1] and the event-triggered variant in Algorithm [2] with three different event triggering thresholds, that is, $E_k = 0.9^{0.1k}$, $E_k = 0.9^{0.05k}$, and $E_k = 0.9^{0.01k}$. For comparison, the algorithms in recent works [20, 24] are reproduced. Since the consensus mechanism therein relies on a doubly stochastic matrix, the Metropolis-Hasting weights are used. According to their methods, the critical stepsize in [20] is derived as 0.0131. We set $\frac{1}{\eta} = 0.1$ and use the tightest triggering threshold bound $E_k = 0.9^{0.1k}$ for [20]. For the method in [24], to make the smallest eigenvalues positive, the averaging matrix is further modified to $\frac{1}{2}I + \frac{1}{2}P$ where $P$ is the mixing matrix constructed by the Metropolis-Hastings weights. However, the stepsize is still aggressively selected as $\frac{1}{\eta} = 0.01$ for comparison reasons. The activation probability is set as $1 - 0.09^{0.008k}$ where $k$ is the time counter.

The results are reported in Figs. [1] and [2]. First of all, they suggest that both of the algorithms achieve communication reductions to some extent. However, the proposed methods with three different triggering threshold bounds exactly converge to the reference logistic classifier $x^*$, while the methods in [20, 24] do not, which is in line with our theoretical results. For the proposed methods, it is interesting to see that the strategies with $E_k = 0.9^{0.1k}$ and $E_k = 0.9^{0.05k}$ enjoy almost the same (or even faster) convergence speed as their periodic counterpart. This may be because that differently using past primal information to update the dual variable results in acceleration, as in inertial methods. While superb performances are exhibited, the reductions in communication are also significant. To be specific, 4075 rounds of communication are required to achieve an accuracy of $10^{-5}$ for $E_k = 0.9^{0.1k}$ and 2004 for $E_k = 0.9^{0.05k}$, while 5305 is needed for the periodic algorithm, meaning that 23.19% and 62.22% reductions in network utilization are achieved, respectively. The case with the loosest bound $E_k = 0.9^{0.01k}$ however, suffers from a slow convergence rate evaluated over both the iteration and communication times. It is primarily because that the bound may be too loose for each agent to transmit information timely. In practice, the triggering threshold bound should be properly chosen for reduced network utilization and reasonable local computation loads.

To get a feel of how the stepsize influences the triggering behavior, we run the proposed algorithm with $E_k = 0.9^{0.1k}$ and $E_k = 0.9^{0.05k}$ under different choices of $\frac{1}{\eta}$. It is worth to mention that the proposed algorithm converges as long as $(\eta - L)I_n - \beta L$ is positive definite for smooth functions, which
can be verified that the stabilizing stepsize range is larger than 
[13], [20], [24]. The iteration steps and communication times 
for an accuracy of $10^{-5}$ with respect to $\eta$ are plotted in Figs. 
3 and 4. From them, we see that significant communication 
reductions are achieved under every choice of $E_k$ and $\eta$. More 
interestingly, when the stepsize is large, i.e., $\eta$ is small, a better 
tradeoff is sought by the strategy with a tighter bound $E_k = 0.9^{0.05k}$ 
in the sense that convergence rate over iteration steps is 
not compromised and the number of communication times is 
greatly reduced. As $\eta$ grows, the strategy with $E_k = 0.9^{0.05k}$ 
becomes more efficient. Specifically, when $\eta = 140$, the 
convergence speed over local iteration is completely reserved 
while the network utilization reduces to one-third of that in a 
periodic algorithm. The main reason for this phenomenon is 
that the local estimate becomes less informative to the overall 
system when the stepsize becomes smaller and therefore a 
looser bound can better exploits this fact, and vice versa. 
This qualitative analysis may shed light on how to choose 
an efficient event-triggering threshold in practice.

In summary, the proposed algorithms can better exploit 
the communication resources in decentralized optimization than 
existing periodic and communication-efficient decentralized 
gradient methods, and more importantly, guarantee exact minimization. For practical use, the triggering threshold bound 
should be properly selected to minimize the network utilization 
while retaining the performance in periodic methods. An 
aggressive choice of triggering threshold bound may suit better 
for the case with smaller stepsizes, where the local estimates 
at each step are generally less informative.

VII. CONCLUSION

In this work, we have designed a periodic exact decentralized 
algorithm for large-scale convex optimization problems 
with coupled cost functions and its event-triggered version. 
Competitive convergence rates of the proposed event-triggered
algorithm over existing periodic algorithms have been established for different types of objectives. Comprehensive numerical experiments demonstrated that the proposed method with proper event-triggered thresholds significantly lowers network utilization while almost preserving the performance in a periodic algorithm. The relation between the efficiency of event-triggered broadcasting strategies and the stepsize has been revealed via experiment results and discussed, shedding light on how to choose a useful threshold in practice.

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