A note on the Bruhat decomposition of semisimple Lie groups

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Abstract

Let a split element of a connected semisimple Lie group act on one of its flag manifolds. We prove that each connected set of fixed points of this action is itself a flag manifold. With this we can obtain the generalized Bruhat decomposition of a semisimple Lie group by entirely dynamical arguments.

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0 Introduction

Let a split element $h$ of a connected semisimple Lie group $G$ act on one of its flag manifolds $F_{\Theta}$ (notation of semisimple Lie groups is recalled in Section 1). We prove that each connected set of fixed points of this action is itself a flag manifold, but a flag manifold of a semisimple Lie subgroup of $G$. This generalizes directly the fact that each connected fixed point set of a diagonable matrix acting on a projective space is given by a projective subspace. Apart for being interesting in itself, this result also allows us to obtain generalized Bruhat decomposition of a semisimple Lie group by entirely dynamical arguments, as we explain below.

Standard textbooks on semisimple Lie groups [2, 3] prove by entirely algebraic arguments what we will call the regular Bruhat decomposition of a connected semisimple Lie group $G$, namely

$$G = \bigsqcup_{w \in W} PwP = \bigsqcup_{w \in W} N^+ wP,$$

where $P$ is the minimal parabolic and $W$ the Weyl group of $G$. This decomposition corresponds to the regular Bruhat decomposition of the maximal flag manifold $F = \text{Ad}(G)p$ of $G$, given by

$$F = \bigsqcup_{w \in W} Pwp = \bigsqcup_{w \in W} N^+ wp,$$

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which can be seen as the decomposition of \( F \) into unstable manifolds of the action of a split-regular element \( h \in A^+ \): see Section 3 of [1] for a proof of this by entirely dynamical arguments (this is recalled at the end of Section 1). From this regular Bruhat decomposition on the maximal flag manifold \( F \), one readily obtains the regular Bruhat decomposition on the partial flag manifolds \( F_{\Theta} = \text{Ad}(G) p_{\Theta} \) given by

\[
F_{\Theta} = \bigcup_{w \in W/W_{\Theta}} P w p_{\Theta} = \bigcup_{w \in W/W_{\Theta}} N^+ w p_{\Theta},
\]

the argument goes as follows. Projecting the regular Bruhat decomposition of \( F \) onto \( F_{\Theta} \) one needs only to show the disjointedness of the above decomposition. If the unstable manifolds \( N^+ w p_{\Theta} \) and \( N^+ s p_{\Theta} \) meet, for \( s, w \in W \), then there exists \( n \in N^+ \) such that \( w p_{\Theta} = n s p_{\Theta} \). Taking the regular element \( h \in A^+ \) we have for \( k \in \mathbb{Z} \) that \( w p_{\Theta} \) is a fixed point so that

\[
w p_{\Theta} = h^{-k} w p_{\Theta} = h^{-k} n s p_{\Theta} \rightarrow s p_{\Theta},
\]

when \( k \to +\infty \). It follows that \( w p_{\Theta} = s p_{\Theta} \), so that \( s^{-1} w p_{\Theta} = p_{\Theta} \) which, by the Iwasawa decomposition of \( P_{\Theta} \), implies that \( s^{-1} w \in K_{\Theta} \cap M^* \) so that \( s^{-1} w \in W_{\Theta} \), that is, \( w \in s W_{\Theta} \), as claimed. The corresponding decomposition in \( G \) is the regular Bruhat decomposition

\[
G = \bigcup_{w \in W/W_{\Theta}} P w P_{\Theta} = \bigcup_{w \in W/W_{\Theta}} N^+ w P_{\Theta}.
\]

Usually much harder to obtain is what we will call the generalized Bruhat decomposition of \( G \), given by

\[
G = \bigcup_{w \in W_{\Delta} \setminus W/W_{\Theta}} P_{\Delta} w P_{\Theta}.
\]

To show this [3] appeals to the use of Tits Systems (see Section 1.2 of [3]). Dynamically, this corresponds to the decomposition of the flag manifold \( F_{\Theta} \) into unstable manifolds of the action of an irregular split-element \( h \in \text{cl} A^+ \), \( h = \exp(H) \),

\[
F_{\Theta} = \bigcup_{w \in W_{\Delta} \setminus W/W_{\Theta}} P_{H} w p_{\Theta} = \bigcup_{w \in W_{\Delta} \setminus W/W_{\Theta}} N^+_H Z_H w p_{\Theta},
\]

where \( \Delta \) is the set of simple roots which annihilate \( H \). In this case the fixed points of \( h \) degenerate into fixed point manifolds \( Z_H w p_{\Theta} \). To show that these unstable manifolds are disjoint we can argue as above to get rid of the unstable part \( N^+_H \) so that the only difficulty is to show that fixed point manifolds are disjoint when we take \( w \in W_{\Delta} \setminus W/W_{\Theta} \). At this point of the argument [1] appeals to a general theorem of Borel-Tits (see Proposition 1.3 of [1]).

In this note we show the disjointedness of the above fixed point manifolds as a byproduct of showing that each of these fixed point manifold is itself equivariantly diffeomorphic to a flag manifold. With this, one can obtain the generalized Bruhat decomposition of a semisimple Lie group by entirely dynamical arguments: one follows Section 3 of [1] to
prove the regular Bruhat decomposition and then uses the result of this article to prove 
the generalized Bruhat decomposition.

In the first section we recall notation and preliminary results on semisimple Lie theory. 
In the second section we prove the main result Theorem 2.2 and deduce from it that the 
fixed point manifolds are disjoint.

1 Preliminaries on Semi-simple Lie Theory

For the theory of semi-simple Lie groups and their flag manifolds we refer to Duistermat-
Kolk-Varadarajan [1], Helgason [2] and Warner [3]. To set notation let $G$ be a connected 
noncompact semi-simple Lie group with Lie algebra $\mathfrak{g}$. We assume throughout that $G$ 
has finite center. Fix a Cartan involution $\theta$ of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The form 
$B_{\theta}(X,Y) = -\langle X, \theta Y \rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form of $\mathfrak{g}$, is an inner product.

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. We let $\Pi$ be the 
set of roots of $\mathfrak{a}$, $\Pi^+$ the positive roots corresponding to $\mathfrak{a}^+$, $\Sigma$ the set of simple roots in 
$\Pi^+$ and $\Pi^- = -\Pi^+$ the negative roots. For a root $\alpha \in \Pi$ we denote by $H_\alpha \in \mathfrak{a}$ its coroot 
so that $B_\theta(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$. The Iwasawa decomposition of the Lie algebra 
$\mathfrak{g}$ reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm$ with $\mathfrak{n}^\pm = \sum_{\alpha \in \Pi^\pm} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space associated to $\alpha$. As 
to the global decompositions of the group we write $G = K\mathfrak{S}$ and $G = K\mathfrak{A}\mathfrak{N}$ with 
$K = \exp \mathfrak{k}$, $S = \exp \mathfrak{s}$, $A = \exp \mathfrak{a}$ and $\mathfrak{N}^\pm = \exp \mathfrak{n}^\pm$.

The Weyl group $W$ associated to $\mathfrak{a}$ is the finite group generated by the reflections over 
the root hyperplanes $\alpha = 0$ in $\mathfrak{a}$, $\alpha \in \Pi$. $W$ acts on $\mathfrak{a}$ by isometries and can be alternatively 
be given as $W = M^*/M$ where $M^*$ and $M$ are the normalizer and the centralizer of $A$ in $K$, 
respectively. We write $\mathfrak{m}$ for the Lie algebra of $M$. There is an unique element $w^- \in W$ 
which takes the simple roots $\Sigma$ to $-\Sigma$, $w^-$ is called the principal involution of $W$.

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups 
(cf. [3], Section 1.2.4): We write $\mathfrak{g}(\Theta)$ for the (semi-simple) Lie subalgebra generated by 
$\mathfrak{g}_\alpha$, $\alpha \in \Theta$, and put $\mathfrak{k}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{k}$, $\mathfrak{a}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{a}$, and $\mathfrak{n}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{n}^\pm$. The simple roots of $\mathfrak{g}(\Theta)$ are given by $\Theta$, more precisely, by restricting the functionals of $\Theta$ 
to $\mathfrak{a}(\Theta)$. The coroots $H_\alpha$, $\alpha \in \Theta$, form a basis for $\mathfrak{a}(\Theta)$. Let $G(\Theta)$ and $K(\Theta)$ be the 
connected groups with Lie algebra $\mathfrak{g}(\Theta)$ and $\mathfrak{k}(\Theta)$, respectively. Then $G(\Theta)$ is a connected 
semisimple Lie group with finite center. Let $A(\Theta) = \exp \mathfrak{a}(\Theta)$, $\mathfrak{n}(\Theta) \equiv \exp \mathfrak{n}(\Theta)$. We 
have the Iwasawa decomposition $G(\Theta) = K(\Theta)A(\Theta)\mathfrak{N}(\Theta)$. Let $\mathfrak{a}_\Theta = \{ H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta \}$ 
be the orthocomplement of $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$ with respect to the $B_\theta$-inner product 
and put $A_\Theta = \exp \mathfrak{a}_\Theta$. The subset $\Theta$ singles out the subgroup $W_\Theta$ of the Weyl group 
which acts trivially $\mathfrak{a}_\Theta$. Alternatively $W_\Theta$ can be given as the subgroup generated by the 
reflections with respect to the roots $\alpha \in \Theta$. The restriction of $w \in W_\Theta$ to $\mathfrak{a}(\Theta)$ furnishes 
an isomorphism between $W_\Theta$ and the Weyl group $W(\Theta)$ of $G(\Theta)$.

The standard parabolic subalgebra of type $\Theta \subset \Sigma$ with respect to chamber $\mathfrak{a}^+$ is defined by 
$$p_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$ 
The corresponding standard parabolic subgroup $P_\Theta$ is the normalizer of $p_\Theta$ in $G$. It has
the Iwasawa decomposition \( P_\Theta = K_\Theta AN^+ \). The empty set \( \Theta = \emptyset \) gives the minimal parabolic subalgebra \( p = m \oplus a \oplus n^+ \) whose minimal parabolic subgroup \( P = P_\emptyset \) has Iwasawa decomposition \( P = MAN^+ \). Let \( \Delta \subset \Theta \subset \Sigma \). Then \( P_\Delta \subset P_\Theta \), also we denote by \( P(\Theta)_\Delta \) the parabolic subgroup of \( G(\Theta) \) of type \( \Delta \).

We let \( Z_\Theta \) be the centralizer of \( a_\Theta \) in \( G \) and \( K_\Theta = Z_\Theta \cap K \). We have that \( K_\Theta \) decomposes as \( K_\Theta = MK(\Theta) \) and that \( Z_\Theta \) decomposes as \( Z_\Theta = MG(\Theta)A_\Theta \) which implies that \( Z_\Theta = K_\Theta AN(\Theta) \) is the Iwasawa decomposition of \( Z_\Theta \) (which is a reductive Lie group). Let \( \Delta \subset \Sigma \), then\(^1\) \( a_\Theta \cap \Delta = a_\Theta + a_\Delta \). Thus it follows that \( Z_{\Theta \cap \Delta} = Z_\Theta \cap Z_\Delta, K_{\Theta \cap \Delta} = K_\Theta \cap K_\Delta \) and \( P_{\Theta \cap \Delta} = P_\Theta \cap P_\Delta \). For \( H \in \mathfrak{a} \) we denote by \( Z_H, W_H \) etc. the centralizer of \( H \) in \( G, W \) etc., respectively. When \( H \in \text{cla}^+ \) we put

\[
\Theta(H) = \{ \alpha \in \Sigma : \alpha(H) = 0 \},
\]

and we have that \( Z_H = Z_{\Theta(H)}, K_H = K_{\Theta(H)}, N^+_H = N^+(\Theta(H)) \) and \( W_H = W_{\Theta(H)} \).

Let \( \mathfrak{n}^+_\Theta = \sum_{\alpha \in \Pi^+_{\Theta}} \mathfrak{g}_\alpha \) and \( N^+_\Theta = \exp(\mathfrak{n}^+_\Theta) \). Then \( N^+ \) decomposes as \( N^+ = N(\Theta)^\perp N^+_{\Theta} \) where \( N(\Theta)^\perp \) normalizes \( N^+_{\Theta} \) and \( N(\Theta)^\perp \cap N^+_\Theta = 1 \). We have that \( \mathfrak{g} = \mathfrak{n}^+_\Theta \oplus \mathfrak{p}_\Theta \), that \( N^+_{\Theta} \cap P_\Theta = 1 \) and also that \( P_\Theta \) is the normalizer of \( \mathfrak{n}^+_{\Theta} \) in \( G \). \( P_\Theta \) decomposes as \( P_\Theta = Z_\Theta N^+_{\Theta} \), where \( Z_\Theta \) normalizes \( N^+_{\Theta} \) and \( Z_\Theta \cap N^+_{\Theta} = 1 \). We write \( \mathfrak{p}^\perp_\Theta = \theta(\mathfrak{p}_\Theta) \) for the parabolic subalgebra opposed to \( \mathfrak{p}_\Theta \). It is conjugate to the parabolic subalgebra \( \mathfrak{p}_\Theta \), where \( \Theta^* = -(w^*)\Theta \) is the dual to \( \Theta \) and \( w^* \) is the principal involution of \( W \). More precisely, \( \mathfrak{p}^\perp_\Theta = k\mathfrak{p}_\Theta \), where \( k \in M^* \) is a representative of \( w^* \). If \( P_\Theta \) is the parabolic subgroup associated to \( \mathfrak{p}_\Theta \) then \( Z_\Theta = P_\Theta \cap P_\Theta^\perp \) and \( P_\Theta^\perp = Z_\Theta N^\perp_\Theta \), where \( Z_\Theta \) normalizes \( N^\perp_\Theta \) and \( Z_\Theta \cap N^\perp_\Theta = 1 \).

The flag manifold of type \( \Theta \) is the orbit \( F_\Theta = \text{Ad}(G)\mathfrak{p}_\Theta \), which identifies with the homogeneous space \( G/P_\Theta \). Since the center of \( G \) normalizes \( \mathfrak{p}_\Theta \), the flag manifold depends only on the Lie algebra \( \mathfrak{g} \) of \( G \). The empty set \( \Theta = \emptyset \) gives the maximal flag manifold \( F = F_\emptyset \). If \( \Delta \subset \Theta \) then there is a \( G \)-equivariant projection \( F_\Delta \to F_\Theta \) given by \( g\mathfrak{p}_\Delta \mapsto g\mathfrak{p}_\Theta, g \in G \).

Some subalgebras of \( \mathfrak{g} \) which are defined by the choice of a Weyl chamber of \( \mathfrak{a} \) and a subset of the associated simple roots can be defined alternatively by the choice of an element \( H \in \mathfrak{a} \) as follows. First note that the eigenspaces of \( \text{ad}(H) \) in \( \mathfrak{g} \) are the weight spaces \( \mathfrak{g}_\alpha \), and that the centralizer of \( H \) in \( \mathfrak{g} \) is given by \( \mathfrak{z}_H = \sum \{ \mathfrak{g}_\alpha : \alpha(H) = 0 \} \), where the sum is taken over \( \alpha \in \mathfrak{a}^* \). Now define the negative and positive nilpotent subalgebras of type \( H \) given by

\[
\mathfrak{n}^-_H = \sum \{ \mathfrak{g}_\alpha : \alpha(H) < 0 \}, \quad \quad \mathfrak{n}^+_H = \sum \{ \mathfrak{g}_\alpha : \alpha(H) > 0 \},
\]

and the parabolic subalgebra of type \( H \) which is given by

\[
\mathfrak{p}_H = \sum \{ \mathfrak{g}_\alpha : \alpha(H) \geq 0 \},
\]

where in all cases \( \alpha \) runs through all the roots \( \Pi \). Then we have that

\[
\mathfrak{g} = \mathfrak{n}^-_H \oplus \mathfrak{z}_H \oplus \mathfrak{n}^+_H \quad \text{and} \quad \mathfrak{p}_H = \mathfrak{z}_H \oplus \mathfrak{n}^+_H,
\]

\(^1\)Using that \( a_\Theta = \mathfrak{a}(\Theta)^\perp \) and that \( \mathfrak{a}(\Theta) \cap \Delta = \mathfrak{a}(\Theta) \cap \mathfrak{a}(\Delta) \) this follows by taking perp on both sides and using that \( (V + W)^\perp = V^\perp \cap W^\perp \) for \( V, W \) linear subspaces.
and that
\[ n_{wH}^\pm = w n_H^\pm, \quad p_{wH} = w p_H. \]

Define the flag manifold of type \( H \) given by the orbit
\[ \mathcal{F}_H = \text{Ad}(G)p_H. \]

Now choose a chamber \( a^+ \) of \( a \) which contains \( H \) in its closure, consider the simple roots \( \Sigma \) associated to \( a^+ \) and take \( \Theta(H) \subset \Sigma \). Since a root \( \alpha \in \Theta(H) \) if, and only if, \( \alpha|_{a^+(H)} = 0 \), we have that
\[ \mathcal{F}_H = \mathcal{F}_{\Theta(H)}, \]
and that the isotropy of \( G \) in \( p_H \) is \( P_{\Theta(H)} = K_{\Theta(H)}AN^+ = K_HAN^+ \), since \( K_{\Theta(H)} = K_H \).

In particular we have that
\[ w_p_{\Theta(H)} = w p_H = p_{wH}. \]

We note that we can proceed reciprocally. That is, if \( a^+ \) and \( \Theta \) are given, we can choose an \( H \in \text{cla}^+ \) such that \( \Theta(H) = \Theta \) and describe the objects that depend on \( a^+ \) and \( \Theta \) by \( H \) (clearly, such an \( H \) is not unique.)

We recall the dynamics of a split element of \( G \) acting in the flag manifold \( \mathcal{F}_\Theta \) (see Section 3 of [I]). Let the split element \( H \in \text{cla}^+ \) act in \( \mathcal{F}_\Theta \) by
\[ t \cdot b = \exp(tH)b, \quad b \in \mathcal{F}_\Theta. \]

The connected sets of fixed point of the \( H \)-action are parametrized by \( w \in W \) and are given by
\[ \text{fix}(H, w) = Z_{Hw_p_{\Theta}} = K_{Hw_p_{\Theta}}, \]
so that they are in bijection with the double coset \( W_H \backslash W/W_\Theta \). It follows from \( Z_H = Z_{\Theta(H)} \) and from the decomposition \( Z_\Theta = G(\Theta)MA_\Theta \) that we have
\[ \text{fix}(H, w) = G(\Theta(H))w_p_{\Theta}. \]

Each \( w \)-fixed point set has stable/unstable manifold given respectively by
\[ \text{st}(H, w) = N_{\Theta(H)}^\pm \text{fix}(H, w) = P_{\Theta(H)}^\pm w_p_{\Theta}. \]

This \( H \)-action decomposes \( \mathcal{F}_\Theta \) in the disjoint union of stable/unstable manifolds
\[ \mathcal{F}_\Theta = \coprod_{W_H \backslash W/W_\Theta} P_{\Theta(H)}^\pm w_p_{\Theta}, \]
this is known as the Bruhat decomposition of \( \mathcal{F}_\Theta \). It follows from these considerations that the dynamics of the \( H \)-action in \( \mathcal{F}_\Theta \) depends only on the set of roots \( \Theta(H) \) which annihilate \( H \).
2 Fixed points as flag manifolds

Let $\pi_\Theta : a \rightarrow a(\Theta)$ be the orthogonal projection parallel to $a_\Theta$.

Lemma 2.1 The following assertions are true.

1. The projection by $\pi_\Theta$ of a regular element of $a$ is a regular element of $a(\Theta)$.

2. The projection by $\pi_\Theta$ of a chamber in $a$ is contained inside a chamber of $a(\Theta)$.

3. For $w \in W$ denote by $a(\Theta)^w$ the chamber of $a(\Theta)$ which contains the projection by $\pi_\Theta$ of the chamber $w a^+$. Then the nilpotent subalgebras $n^+, e n(\Theta)^w$ w.r.t. to the chambers $a^+$ and $a(\Theta)^w$ satisfy

$$n(\Theta)^w \subset w n^+.$$

Proof: We first observe that for $\alpha \in \Theta$, we have that $\alpha|_{a_\Theta} = 0$ so that for $H \in a$ we have $\alpha(\pi_\Theta(H)) = \alpha(H)$. From this it follows that if $H$ is regular in $a$, then $\pi_\Theta(H)$ is regular in $a(\Theta)$, which proves the first item. For the second item we observe that the projection of a chamber of $a$ is a convex set of $a(\Theta)$ which, by the first item, consists of regular elements of $a(\Theta)$, and hence it is contained in a chamber of $a(\Theta)$. For the third item let $\alpha \in \Theta$. If $\alpha > 0$ in $a(\Theta)^w$ then $\alpha > 0$ in $\pi_\Theta(w a^+)$ and hence, by the first remark of the proof, we have that $\alpha > 0$ in $w a^+$. It follows that

$$n(\Theta)^w = \sum \{ g_\alpha : \alpha|_{a(\Theta)^w} > 0, \alpha \in \langle \Theta \rangle \} \subset \sum \{ g_\alpha : \alpha|_{w a^+} > 0, \alpha \in \Pi \} = w n^+,$$

as desired. \qed

Theorem 2.2 Let $X \in \mathrm{cla}^+$, $\Theta \subset \Sigma$, $w \in W$. Consider $\Delta = \Theta(X)$ and $H_\Theta \in \mathrm{cla}^+$ such that $\Theta(H_\Theta) = \Theta$. Then the map

$$\text{fix}(X, w)_\Theta \rightarrow \mathbb{F}_{\pi_\Delta(w H_\Theta)}(g(\Delta)), \quad g p_{w H_\Theta} \mapsto g p_{\pi_\Delta(w H_\Theta)}; \quad g \in G(\Delta)$$

is a well defined $G(\Delta)$-equivariant diffeomorphism.

Proof: Since we have $\text{fix}(X, w)_\Theta = G(\Delta)p_{w H_\Theta}$ it follows that the above map, let us call it $\psi$, is defined in all of its domain and it is $G(\Delta)$-equivariant. It remains to prove that $\psi$ is well defined and is injective both of which will follow if we show that the isotropy of $p_{w H_\Theta}$ in $G(\Delta)$ coincides with the isotropy of $p_{\pi_\Delta(w H_\Theta)}$ in $G(\Delta)$. For this, let $a(\Delta)^w$ the chamber of $a(\Delta)$ which contains the projection by $\pi_\Delta$ of the chamber $w a^+$. Consider the Iwasawa decomposition of $G$ and $P_{w H_\Theta}$ w.r.t. to the chamber $w a^+$, and the Iwasawa decomposition of $G(\Delta)$ w.r.t. to the chamber $a(\Delta)^w$

$$G = K A w N^+ w^{-1}, \quad P_{w H_\Theta} = K_{w H_\Theta} A w N^+ w^{-1}, \quad G(\Delta) = K(\Delta) A(\Delta) N(\Delta)^w,$$
where, by item (3) of the previous Lemma, we have that
\[ N(\Delta)^w \subset wN^+w^{-1}. \]

Thus, by the uniqueness of the Iwasawa decomposition of \( G \), it follows that the isotropy of \( p_{wH_\Theta} \) in \( G(\Delta) \) is given by
\[ G(\Delta) \cap P_{wH_\Theta} = (K(\Delta) \cap K_{wH_\Theta})A(\Delta)N(\Delta)^w. \]

The first term in the right hand side can be written as
\[ K(\Delta) \cap K_{wH_\Theta} = K(\Delta)_{wH_\Theta} = K(\Delta)_{\pi_\Delta(wH_\Theta)}, \]
where in the last equality we used that \( K(\Delta) \) already centralizes \( a_\Delta \). Since \( \pi_\Delta(wH_\Theta) \) lies in the closure of the chamber \( a(\Theta) \), it follows that
\[ G(\Delta) \cap wP_\Theta w^{-1} = K(\Delta)_{\pi_\Delta(wH_\Theta)}A(\Delta)N(\Delta)^w = P_{\pi_\Delta(wH_\Theta)}(\Delta), \]
which is precisely the isotropy of \( p_{\pi_\Delta(wH_\Theta)} \) in \( G(\Delta) \). It is then immediate that the inverse of \( \psi \) is given by \( g_{p_{\pi_\Delta(wH_\Theta)}} \mapsto g_{p_{wH_\Theta}}, g \in G(\Delta) \), which shows that \( \psi \) is a diffeomorphism. \( \square \)

**Corollary 2.3** If \( \text{fix}(X, w')_\Theta \cap \text{fix}(X, w)_\Theta \neq \emptyset \) then \( w' \in W_X wW_\Theta \).

**Proof:** Here we will adopt the notation of Theorem 2.2, denoting by \( \psi \) the diffeomorphism of that theorem. If \( \text{fix}(X, w')_\Theta \cap \text{fix}(X, w)_\Theta \neq \emptyset \) then there exists \( g \in G(\Delta) \) such that \( w'p_\Theta = gw p_\Theta \). Take a regular \( h \in A(\Delta) \), using the \( G(\Delta) \)-equivariance of \( \psi \) for \( k \in \mathbb{Z} \) we have that
\[ w'p_\Theta = h^kgw p_\Theta = h^kgw p_{wH_\Theta} = \psi^{-1}(h^kgp_{\pi_\Delta(wH_\Theta)}) = (\ast). \]

By the regular Bruhat decomposition of the flag manifold \( \mathbb{F}_{\pi_\Delta(wH_\Theta)}(g(\Delta)) \) (cf. Theorem ??), letting \( k \to \infty \) we have that there exists \( s \in W(\Delta) = W_\Delta = W_X \) such that
\[ (\ast) \to \psi^{-1}(sp_{\pi_\Delta(wH_\Theta)}) = sp_{wH_\Theta} = sw p_\Theta. \]

It follows that \( w^{-1}s^{-1}w'p_\Theta = p_\Theta \), so that \( w^{-1}s^{-1}w' \in M^* \cap K_\Theta \), which implies that \( w^{-1}s^{-1}w \in W_\Theta \). Hence \( w \in swW_\Theta \subset W_X wW_\Theta \), as desired. \( \square \)

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