Quantum metastability in a class of moving potentials

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In this paper we consider quantum metastability in a class of moving potentials introduced by Berry and Klein. Potential in this class has its height and width scaled in a specific way so that it can be transformed into a stationary one. In deriving the non-decay probability of the system, we argue that the appropriate technique to use is the less known method of scattering states. This method is illustrated through two examples, namely, a moving delta-potential and a moving barrier potential. For expanding potentials, one finds that a small but finite non-decay probability persists at large times. Generalization to scaling potentials of arbitrary shape is briefly indicated.

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I. INTRODUCTION

An interesting issue in cosmology is the evolution of metastable states in the early universe according to the original version and its variants of the inflationary models [1, 2]. In these models inflation of the early universe is governed by a Higgs field trapped in a metastable state. Inflation ends when the metastable state decays to the true ground state of the universe. During inflation the universe expands exponentially. It is thus obvious that the metastable state of the Higgs field is trapped in a rapidly varying potential. The problem is therefore a truly time-dependent one. However, owing to the inherent difficulties of the problem, more often than not one considers the decay of the Higgs field in a quasi-stationary approximation, in which the decay is studied assuming a static potential [3]. Surely this approximation is hard to justify, but for the present one has to be content with it. Ultimately one hopes to be able to tackle the non-stationary case. To this end, it is desirable to gain some insights first by studying metastability in time-dependent potential in simple quantum-mechanical models.

Roughly speaking, time-dependent potentials can be divided into three classes. In the first class we have potentials with time-dependent strength. When the strength is small, the Schrödinger equation can be solved by time-dependent perturbation theory. Almost all textbook examples belong to this type. When the strength of the potential is not small, other methods of solution must be sought. For example, solutions of time-dependent harmonic oscillator [4] and time-dependent linear potential [5] can be obtained by the method of invariant. We note here that the interesting phenomenon of quantum tunneling induced by an externally driven field has also been examined experimentally and theoretically [6, 7, 8]. The second class of potentials involves time-dependent boundaries. Unlike the first class, this class of potentials attracts much less attention, and almost all previous works in this area concerned only the simplest of all cases, namely, an infinite potential well with a moving wall [9, 10]. The last class is the combination of the previous two classes.

We believe that the barrier potential in an inflationary universe is non-stationary, not only the barrier height but also the barrier width should be changing as time elapses. However, it will be extremely difficult to study metastability in such a time-dependent potential in full generality. Thus it would be helpful if the quantum tunneling effect could be studied in any class of moving potential, special though it is, as a step to understanding the decay of a non-stationary metastable system.

In this paper we consider quantum metastability in a class of scaling potentials which allows one to apply techniques used in the corresponding problem with stationary potentials. This class of potentials was introduced by Berry and Klein [11]. Potentials in this class have their heights and widths scaled in a specific way so that one can transform the potential into a stationary one.

The organization of the paper is as follows. In Sec. II, we give a general discussion of the solutions of the Schrödinger equation with the scaling form of time-dependent potential introduced in [11]. It is argued that the most suitable technique for studying quantum metastability in such kind of potential is the less known method of scattering states. Two simple examples of such metastable systems, a moving delta-potential and a moving square barrier, are investigated in Sect. III and IV, respectively. Generalization to arbitrary barrier is briefly discussed in Sec. V. Sec. IV concludes the paper.
II. SCHRÖDINGER EQUATION WITH A SCALING POTENTIAL

We shall consider the problem of quantum metastability of a particle of mass $m$ trapped in a moving potential $V(x,t)$. We assume that the potential $V(x,t)$ is of the scaling form proposed by Berry and Klein \[11\], namely,

$V(x,t) = \bar{V}(x/L(t))/L^2(t)$,

where $L(t)$ is a time-dependent scaling factor. The Schrödinger equation is

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{L^2(t)} \bar{V} \left( \frac{x}{L(t)} \right) \right] \Psi(x,t).$$

So far solution of Eq. (1) is restricted mostly to the special case in which $\bar{V}$ has the functional form of an infinite potential well, i.e. $V(x,t)$ is an infinite well with a moving wall \[9, 10\]. In this case the scaling factor $L^2(t)$ in front of $\bar{V}$ is immaterial.

For our purpose, we shall assume $\bar{V}$ to have the generic shape of a potential well which is impenetrable to the left and has a finite barrier to the right, much like that usually employed in the discussion of alpha-decay. We further assume $L(t)$ to be a linear function of time:

$$L(t) = L_0 + vt, \ v = \text{constant}.$$ 

Of course, for $v < 0$, the problem is meaningful only for time duration $0 < t < L_0/|v|$. Eq. (1) cannot be solved by separating the time and spatial coordinates. However, for the scaling form of $V(x,t)$ in Eq. (1) and the linear form of $L(t)$, separation of variables can be achieved through a series of transformations introduced in \[11, 12\] (see also: \[9, 10\]). One first transforms the coordinate frame into a rescaled frame with a rescaled coordinate $\bar{x}$ defined by

$$\bar{x}(t) \equiv \frac{x}{L(t)}.$$

In this frame the Schrödinger equation becomes

$$i\hbar \frac{\partial \Psi(\bar{x},t)}{\partial \tau} = \left[ -\frac{\hbar^2}{2mL^2} \frac{\partial^2}{\partial \bar{x}^2} + i\hbar \frac{v}{L} \frac{\partial}{\partial \bar{x}} + \bar{V}(\bar{x}) \right] \Psi(\bar{x},t).$$

Eq. (4) can be further simplified by the following transformation

$$\Psi(\bar{x},t) = \frac{1}{\sqrt{L(t)}} e^{im\frac{\hbar L}{v}x^2} \Phi(\bar{x},t),$$

and the introduction of a new time variable $\tau$

$$\tau = \int_0^t ds \frac{1}{L^2(s)} = \frac{t}{L_0 L(t)}.$$ 

After substituting Eq. (5) and (6) into (4), we obtain the equation

$$i\hbar \frac{\partial}{\partial \tau} \Phi(\bar{x},\tau) = -\frac{\hbar^2}{2mL^2} \frac{\partial^2}{\partial \bar{x}^2} \Phi(\bar{x},\tau) + \bar{V}(\bar{x}) \Phi(\bar{x},\tau),$$

which resembles the Schrödinger equation with a stationary potential. Eq. (7) can be solved by separation of variables

$$\Phi(\bar{x},\tau) = \Phi(\bar{x}) e^{-\frac{\hbar L}{v} t \bar{E}} \tau,$$

where $\Phi(\bar{x})$ satisfies the eigenvalue equation

$$\left[ -\frac{\hbar^2}{2mL^2} \frac{d^2}{d \bar{x}^2} + \bar{V}(\bar{x}) \right] \Phi_k(\bar{x}) = \bar{E}_k \Phi_k(\bar{x}).$$

Once Eq. (9) is solved exactly in the rescaled frame, the exact wave function in the original frame is then given by

$$\Psi_k(x,t) = \frac{1}{\sqrt{L(t)}} e^{im\frac{\hbar L}{v}x^2} e^{-\frac{\hbar L}{v} t \bar{E}_k} \Phi_k \left( \frac{x}{L} \right).$$
The set of solutions \( \{10\} \) is complete and orthonormal
\[
\langle \Psi_k(x,t) | \Psi_l(x,t) \rangle = \langle \Phi_k(\bar{x}) | \Phi_l(\bar{x}) \rangle = \delta_{kl} ,
\]
so using this set of solutions we can find a solution satisfying any initial condition. Furthermore, if an initial state \( \Psi(x,0) \) is expressible in the basis \( \{\Psi_k\} \) as
\[
\Psi(x,0) = \sum_k c_k \Psi_k(x,0) , \quad c_k = \langle \Psi_k(x,0) | \Psi(x,0) \rangle ,
\]
then at a later time \( t \) the state is
\[
\Psi(x,t) = \sum_k c_k \Psi_k(x,t) .
\]

We have now succeeded in transforming the original time-dependent Schrödinger equation into a time-independent one. The problem of calculating the decay probability of a particle confined in \( V(x,t) \) at time \( t \) is reduced to the corresponding problem with a static potential \( \bar{V}(\bar{x}) \). Hence techniques used in the time-independent potential for calculating decay rate can be borrowed.

However, there are some subtleties. Naively, one is tempted to employ the most well-known method, namely, the complex eigenvalue method, proposed by Gamow in his studies of the alpha-decay [13]. In this approach an “outgoing wave boundary condition” is imposed on the solutions of the Schrödinger equation for the particle trapped in the well. That means incoming plane wave solutions outside the potential well are discarded right from the beginning. This procedure naturally leads to an eigenvalue problem with complex energy eigenvalues. One then relates the imaginary parts of the energy to the decay rate. While the complex eigenvalue method is straightforward and physically reasonable, it suffers from some conceptual difficulties [14]. For example, how can energy eigenvalues be complex as we are dealing with a Hermitian Hamiltonian? Also, the eigenfunctions are not normalizable, a difficulty directly related to the eigenvalues being complex. Furthermore, the particle trapped in the well cannot be in an eigenstate of the system in the first place, since such states are not completely confined at \( t = 0 \).

Apart from the difficulties mentioned above, the complex eigenvalue method cannot be employed in our problem for other reasons. First, the problem we are interested in is an intrinsically time-dependent one, with a non-conservative Hamiltonian. Hence energy eigenvalues and eigenstates lose their meanings altogether (\( \bar{E} \) in Eq. \( 10 \) is not an energy eigenvalue). Second, the “outgoing wave boundary condition”, essential to Gamow’s method, cannot be imposed in our case. The reason is as follows. As we discussed before, in order to fix a moving potential we need to transform our problem to a corresponding static one in a rescaled frame. But in this frame the meaning of incoming or outgoing plane wave is rather obscure. In fact, it can be checked that an outgoing plane wave in the original \( x-t \) (rescaled \( \bar{x}-\tau \)) frame contains both “incoming” and “outgoing” components in the rescaled \( \bar{x}-\tau \) (original \( x-t \)) frame.

Instanton method is another technique commonly used in the calculation of the decay rate of a metastable state [15]. This semiclassical method amounts essentially to finding the imaginary part of the ground state energy of the system. Again, it is not suitable for our case for the same first reason given above for the failure of the complex eigenvalue method.

Now that the two most common methods fail to suit our purpose, we have to look for alternatives. Fortunately, a different method exists, namely, the scattering state method (or virtual level method, as Fermi called it) [16]. This method is much less well known and seldom used in the literature [17]. However, it is conceptually the most satisfying one of all the methods. In this method, one first constructs the initial confining state, which is not viewed as an eigenstate, as a linear superposition of scattering states with real energies, and follows its evolution in time. In the course of this evolution, no energy will become complex. Unlike the Gamow states, the scattering states contain both incoming and outgoing components in the region into which the particle escapes. It is this feature of the method that makes it most suitable for our present problem. The method is easily adapted to Eq. \( 10 \) by taking the scattering states as the states \( \langle 5 \rangle \) with real values of \( \bar{E} \).

In the following two sections, we apply the scattering state method to two simple examples of the class of scaling potentials discussed in this section. As the scattering state method is not so well known in the literature, we think it appropriate to give some details in order to make our paper self-contained. We shall follow the procedures given in [14] which are slightly adapted to our needs.

III. MOVING DELTA-FUNCTION POTENTIAL

Our first example is a uniformly moving delta-function potential
\[
V(x,t) = \begin{cases} \frac{\bar{\nu} \delta(x - a(t))}{\bar{\nu} \delta \left( \frac{x}{\bar{\nu} t} - \bar{a} \right)} , & x \leq 0 ; \\
\frac{\bar{\nu} \delta(x - a(t))}{\bar{\nu} \delta \left( \frac{x}{\bar{\nu} t} - \bar{a} \right)} , & x > 0 , \end{cases}
\]
\( (14) \)
where \( a(t) = \tilde{a}L(t) > 0 \) gives the location of the delta-potential. This is of the class of potentials defined in the last section and corresponds in the rescaled frame to \( \tilde{V}(\tilde{x}) = \infty \) for \( \tilde{x} < 0 \) and \( \tilde{V}(\tilde{x}) = \tilde{V}_0(\tilde{x} - \tilde{a}) \). We must have \( \Phi(\tilde{x}) = 0 \) in the region \( \tilde{x} < 0 \). In the region \( \tilde{x} > 0 \), Eq. (13) is

\[
-\frac{\hbar^2}{2m} \frac{d^2\Phi(\tilde{x})}{d\tilde{x}^2} + \tilde{V}_0(\tilde{x} - \tilde{a})\Phi(\tilde{x}) = \tilde{E}\Phi(\tilde{x}) .
\]

Its general solutions are

\[
\Phi(\tilde{x}) = \begin{cases} \sin(\tilde{k}\tilde{x}) , & 0 < \tilde{x} < \tilde{a} ; \\ C \cos(\tilde{k}\tilde{x} + \theta) , & \tilde{a} < \tilde{x} , \end{cases}
\]

where \( \tilde{k} = \sqrt{2m\tilde{E}/\hbar} \), \( C \) is a real constant, and \( \theta \) a phase angle. Note that we have chosen the wave function to be real, and included incoming wave component in the region \( \tilde{x} > \tilde{a} \). This ensures that \( \tilde{k} \), and hence \( \tilde{E} \), is always real. The wave function and its first derivative satisfy the following boundary conditions at \( \tilde{x} = \tilde{a} \):

\[
\Phi(\tilde{x} = \tilde{a}^+) = \Phi(\tilde{x} = \tilde{a}^-) ,
\]

and

\[
\frac{d\Phi(\tilde{x})}{d\tilde{x}} \bigg|_{\tilde{x} = \tilde{a}^+} - \frac{d\Phi(\tilde{x})}{d\tilde{x}} \bigg|_{\tilde{x} = \tilde{a}^-} = \frac{2m}{\hbar^2} \tilde{V}_0 \Phi(\tilde{x} = \tilde{a}) .
\]

From these relations the coefficient \( C \) can be determined as a function of \( \tilde{k} \),

\[
C^2(\tilde{k}) = \sin^2(\tilde{k}\tilde{a}) + \left( \cos(\tilde{k}\tilde{a}) + \frac{2m\tilde{V}_0}{\hbar^2\tilde{k}\tilde{a}} \sin(\tilde{k}\tilde{a}) \right)^2.
\]

Physically, the value of \( C^2(\tilde{k}) \) can be interpreted as the ratio of the probability of finding particles in the region \( \tilde{x} > \tilde{a} \) to the probability within the confined region \( 0 < \tilde{x} < \tilde{a} \) for a particular \( \tilde{k} \). The general shape of \( C^2(\tilde{k}) \) is shown in Fig. 1, from which we can assert that the particle can be trapped within the confined region only when \( C^2 \) assumes one of its minima, which occur only in the neighborhood of some specific values of \( \tilde{k} \). In these regions the values of \( C^2(\tilde{k}) \) are extremely small. From Eq. (18) it is obvious that these minima will center around \( \tilde{k}_n = n\pi/\tilde{a} \) \( (n = 1, 2, \ldots) \) (i.e. \( \sin(k_n\tilde{a}) = 0 \)) as long as \( \tilde{V}_0 \) is large enough so that \( 2m\tilde{V}_0/\hbar^2 \gg n\pi \).

Below we shall restrict our discussions to the case of large \( \tilde{V}_0 \). In this case approximate analytic expressions can be obtained and compared with the corresponding results in the time-independent case \( \tilde{E}_n \). According to the scattering state method, one constructs confining states in the potential well by taking suitable superposition of the scattering states with \( \tilde{k} \) in the neighborhood of \( \tilde{k}_n \). To this end, let us first expand \( C^2(\tilde{E}) \) about \( \tilde{E}_n = \hbar^2 k_n^2/2m \) (we revert to the variable \( \tilde{E} \) below)

\[
C^2(\tilde{E}) \approx \left( \frac{m\tilde{a}}{\hbar^2 k_n} \right)^2 \left[ 1 + \left( \frac{2m\tilde{V}_0}{\hbar^2 k_n} \right)^2 \right] \left( \tilde{E} - \tilde{E}_n + \delta \right)^2 + \left[ 1 + \left( \frac{2m\tilde{V}_0}{\hbar^2 k_n} \right)^2 \right]^{-1} \]
\[
= G^2 (\Delta + \delta)^2 + F^2 ,
\]

where \( \Delta = \tilde{E} - \tilde{E}_n \) and the constants

\[
\delta = \frac{2\tilde{V}_0}{\tilde{a}} \left[ 1 + \left( \frac{2m\tilde{V}_0}{\hbar^2 k_n} \right)^2 \right]^{-1} ,
\]

\[
G^2 = \left( \frac{m\tilde{a}}{\hbar^2 k_n} \right)^2 \left[ 1 + \left( \frac{2m\tilde{V}_0}{\hbar^2 k_n} \right)^2 \right] ,
\]

\[
F^2 = \left[ 1 + \left( \frac{2m\tilde{V}_0}{\hbar^2 k_n} \right)^2 \right]^{-1} .
\]

The scattering states with \( \tilde{E} \) in the neighborhood of \( \tilde{E}_n \) can then be written as

\[
\psi_{\Delta}(\tilde{x}) = \begin{cases} \sqrt{\frac{2}{\hbar}} \cos(\tilde{k}\tilde{x} + \theta) , & \tilde{a} < \tilde{x} ; \\ \sqrt{\frac{2}{\hbar}} \cos(\tilde{k}\tilde{x} + \theta) , & \tilde{a} < \tilde{x} ; \\ \end{cases}
\]
Choosing from Eq. (10), (12) and (13), the solution at a later time \( \tau \) is

\[
\Phi(\bar{x}, \tau = 0) = \sum_{\Delta} c_\Delta \psi_{\Delta}(\bar{x}) = \begin{cases} 
\phi_n(\bar{x}), & \bar{x} < \bar{a}; \\
0, & \bar{x} > \bar{a}.
\end{cases}
\] (26)

The coefficient \( c_\Delta \) can be calculated from orthogonality of the states \( \psi_{\Delta}(\bar{x}) \),

\[
c_\Delta = \int_0^R \bar{d} \bar{x} \psi_{\Delta}(\bar{x}) \Phi(\bar{x}, 0) = \sqrt{\frac{2}{R}} \frac{1}{\sqrt{G^2(\Delta + \delta)^2 + F^2}} \int_0^{\bar{a}} \bar{d} \bar{x} \sin(\bar{k} \bar{x}) \phi_n(\bar{x}).
\] (27)

Choosing

\[
\phi_n(\bar{x}) \approx \sqrt{\frac{2}{\bar{a}}} \sin \left( \frac{n\pi \bar{x}}{\bar{a}} \right) ; \ n = 1, 2, 3, \ldots
\] (28)

we get

\[
c_\Delta \approx \frac{1}{\sqrt{Ra}} \sqrt{\frac{2}{G^2(\Delta + \delta)^2 + F^2}} \int_0^{\bar{a}} \bar{d} \bar{x} \sin(\bar{k} \bar{x}) \sin \left( \frac{n\pi \bar{x}}{\bar{a}} \right)
\] (29)

\[
\approx \sqrt{\frac{\bar{a}}{R}} \frac{1}{\sqrt{G^2(\Delta + \delta)^2 + F^2}}.
\] (30)

The initial state is then given by

\[
\Phi(\bar{x}, \tau = 0) \approx \sqrt{\frac{\bar{a}}{R}} \sum_{\Delta} \frac{1}{\sqrt{G^2(\Delta + \delta)^2 + F^2}} \psi_{\Delta}(\bar{x}).
\] (31)

From Eq. (10), (12) and (13), the solution at a later time \( \tau \) is

\[
\Phi(\bar{x}, \tau) \approx \sqrt{\frac{\bar{a}}{R}} \sum_{\Delta} \frac{1}{\sqrt{G^2(\Delta + \delta)^2 + F^2}} \psi_{\Delta}(\bar{x}) e^{-\frac{\hbar}{\pi}(E_n + \Delta) \tau}.
\] (32)

As the system is quantized in the interval \([0, R]\), we have \( \bar{k} R = n' \pi / 2 \) where \( n' \) is a very large integer \( (n' \gg n) \). After replacing the sum by an integral

\[
\sum_{\Delta} \rightarrow \int d\Delta \frac{R}{\pi \hbar} \sqrt{\frac{2m}{E_n}},
\] (33)

Eq. (32) becomes

\[
\Phi(\bar{x}, \tau) \approx \frac{R}{\pi \hbar} \sqrt{\frac{2m}{E_n}} \frac{\bar{a}}{R} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\Delta \frac{1}{\sqrt{G^2(\Delta + \delta)^2 + F^2}} \psi_{\Delta}(\bar{x}) e^{-\frac{\hbar}{\pi}(E_n + \Delta) \tau}.
\] (34)

Substituting \( \psi_{\Delta}(\bar{x}) \) from Eq. (26), we obtain the approximate wave function of the state confined in the well \( (0 < \bar{x} < \bar{a}) \) as

\[
\Phi(\bar{x}, \tau) \approx \frac{2}{\pi \hbar} \sqrt{\frac{\bar{a} m}{E_n}} \sin(\bar{k} \bar{x}) e^{-\frac{\hbar}{\pi} E_n \tau} \int_{-\infty}^{\infty} d\Delta \frac{e^{-\frac{\hbar}{\pi} \Delta \tau}}{G^2(\Delta + \delta)^2 + F^2}
\]

\[
= \frac{2}{\hbar} \sqrt{\frac{\bar{a} m}{E_n} |FG|} \sin(\bar{k} \bar{x}) e^{-\frac{\hbar}{\pi} E_n \tau} e^{-\frac{\hbar}{\pi} |FG| \tau}.
\] (35)

For metastable system an important quantity is the non-decay probability \( P(t) \) that the particle is still in the well at time \( t \) if it is initially confined in the well at \( t = 0 \) \( (P(t = 0) = 1) \). In our case \( P(t) \) is defined as

\[
P(t) = \frac{\int_0^a |\Psi(x, t)|^2 dx}{\int_0^a |\Psi(x, 0)|^2 dx} = \frac{\int_0^a |\Phi(\bar{x}, \tau)|^2 d\bar{x}}{\int_0^a |\Phi(\bar{x}, 0)|^2 d\bar{x}}.
\] (36)
From Eq. (35) we find
\[ P(t) \sim \exp(-\gamma_n(t)) \, , \tag{37} \]
where
\[ \gamma_n(t) = \frac{2F}{G} \left| \frac{t}{L_0L(t)} \right| \quad \text{with} \quad \bar{a} = \frac{\hbar^2}{ma} \quad \text{rescaled frame}, \]
\[ = 2 \left( \frac{\hbar^2 \bar{k}_n}{ma} \right) \left[ 1 + \left( \frac{2mV_0}{\hbar^2 \bar{k}_n} \right)^2 \right]^{-1} \frac{t}{L_0L(t)} . \tag{39} \]

We have used Eq. (6), (23) and (24) to obtain the result (39). We note that for expanding potential \((v > 0)\)
\[ \gamma_n(t) \rightarrow 2 \left| \frac{F}{G} \right| \frac{1}{L_0V} \quad \text{as} \quad t \rightarrow \infty . \tag{40} \]

Unlike the stationary case \((v = 0)\), there is a small but finite probability that the particle does not tunnel out of the well. This result is reasonable, since as the barrier moves away from \(x = 0\) it leaves more room for the particle to stay within the well.

For \(V_0\) much larger than the characteristic value of \(\bar{k}_n\) of the escaping particles, \(\gamma(t)\) becomes
\[ \gamma_n(t) \rightarrow \left[ \frac{\hbar^6 \bar{k}^3_n}{2m^3aV_0^2} \right] \frac{t}{L_0L(t)} = \left[ \frac{\hbar^6(n\pi)^3}{2m^3a^4V_0^2} \right] \frac{t}{L_0L(t)} , \tag{41} \]
where \(\bar{k}_n = n\pi/\bar{a}\) has been substituted. It is proper to compare Eq. (41) with the corresponding result in the stationary case \((v = 0)\). In the limit \(v \rightarrow 0\), we have \(L(t) \rightarrow L_0, \bar{a} \rightarrow a/L_0, E_n \rightarrow E_n\bar{L}_0, \) and \(\bar{k}_n \rightarrow \bar{k}_nL_0\), where \(E_n\) is the corresponding energy in the static frame, and \(\bar{k}_n = \sqrt{2mE_n/\hbar}\). In this limit \(\gamma(v = 0)\) is directly proportional to the time \(t\). We can therefore define a decay rate by \(\Gamma_n = \gamma_n(v = 0)/t\), which in this case is
\[ \Gamma_n = \frac{\hbar^6 \bar{k}^3_n}{2m^3aV_0^2} = \frac{\hbar^6(n\pi)^3}{2m^3a^4V_0^2} . \tag{42} \]

Eq. (42) is consistent with the result obtained by the complex eigenvalue method in [18] for a static delta-potential located at \(x = a\) with strength \(V_0/L_0\).

\[ \text{IV. MOVING SQUARE BARRIER POTENTIAL} \]

Next, we consider a moving barrier potential
\[ V(x,t) = \begin{cases} \infty, & x \leq 0 ; \\ \frac{1}{L^2(t)}V_0, & a(t) < x < b(t) ; \\ 0, & x > b(t) , \end{cases} \tag{43} \]
with \(a(t) = \bar{a}L(t)\) and \(b(t) = \bar{b}L(t)\) (\(\bar{a}\) and \(\bar{b}\) are two positive constants). When the problem is transformed to the rescaled frame, it is equivalent to solving Eq. (9) with a stationary potential
\[ \bar{V}(\bar{x}) = \begin{cases} \infty, & \bar{x} \leq 0 ; \\ \bar{V}_0, & \bar{a} < \bar{x} < \bar{b} ; \\ 0, & \bar{x} > \bar{b} . \end{cases} \tag{44} \]

The general solutions are
\[ \Phi(\bar{x}) = \begin{cases} \sin(\bar{k}\bar{x}), & 0 < \bar{x} < \bar{a} ; \\ Ae^{\bar{k}'\bar{x}} + Be^{-\bar{k}'\bar{x}}, & \bar{a} < \bar{x} < \bar{b} ; \\ C \cos(\bar{k}\bar{x} + \theta), & \bar{a} < \bar{x} . \end{cases} \tag{45} \]
Here $A, B$ and $C$ are real constants, and $\theta$ a phase angle. As before, we have set the solutions real in the whole region to ensure that $\tilde{k}$ is always real. We also require the solutions and their derivatives be continuous at the boundaries $\bar{x} = \bar{a}$ and $\bar{b}$. These boundary conditions determine the values of the coefficients $A, B$ and $C$ as functions of $\tilde{k}$:

\begin{align}
A(\tilde{k}) &= \frac{1}{2} e^{\tilde{k}'\bar{a}} \left[ \sin(\tilde{k}\bar{a}) + \frac{\tilde{k}}{\tilde{k}'} \cos(\tilde{k}\bar{a}) \right], \\
B(\tilde{k}) &= \frac{1}{2} e^{\tilde{k}'\bar{a}} \left[ \sin(\tilde{k}\bar{a}) - \frac{\tilde{k}}{\tilde{k}'} \cos(\tilde{k}\bar{a}) \right], \\
C^2(\tilde{k}) &= \left( 1 + \frac{\tilde{k}'^2}{\tilde{k}^2} \right) e^{2\tilde{k}'\bar{a}} A^2 + 2 \left( 1 - \frac{\tilde{k}'^2}{\tilde{k}^2} \right) AB + \left( 1 + \frac{\tilde{k}'^2}{\tilde{k}^2} \right) e^{-2\tilde{k}'\bar{b}} B^2.
\end{align}

The general shapes of $A(\tilde{k})$ and $C^2(\tilde{k})$ are shown in the Fig. 2. We see that metastable states of the system will occur only in a finite number of neighborhood of $\tilde{k}_n$ ($n = 1, 2, \ldots$) such that $A(\tilde{k}_n) = 0$. The roots $\tilde{k}_n$ satisfy

\[ \sin(\tilde{k}_n\bar{a}) + \frac{\tilde{k}_n}{\tilde{k}'} \cos(\tilde{k}_n\bar{a}) = 0. \]

Eq. (48) implies that $C^2(\tilde{k})$ is minimal at $\tilde{k}_n$. For a given $\bar{V}_0$, the number of roots $\tilde{k}_n$ is restricted by the condition that $\tilde{k}'_n$ in Eq. (49),

\[ \tilde{k}'_n = \sqrt{\frac{2m\bar{V}_0}{\hbar^2} - \tilde{k}^2_n} \]

must be real. Hence the possible values of $\tilde{k}_n$ can only lie in the interval $(0, \sqrt{\frac{2mV_0}{\hbar^2}})$. For instance, there are only two roots for the parameters assumed in Fig. 2.

Let us now expand the coefficients $A(\tilde{k})$ and $B(\tilde{k})$ about $\bar{E}_n (= \hbar^2 \tilde{k}^2_n/2m)$

\begin{align}
A(\bar{E}) &\approx \left[ \frac{dA}{d\bar{E}} \right]_{\bar{E} = \bar{E}_n} (\bar{E} - \bar{E}_n), \\
B(\bar{E}) &\approx B(\bar{E}_n).
\end{align}

Inserting Eq. (51) and (52) into Eq. (48), and after some tedious calculations we find that $C^2(\bar{E})$ behaves in the neighborhood of $\bar{E}_n$ as

\[ C^2(\bar{E}) = G^2(\Delta + \delta)^2 + F^2, \]

where $\Delta = \bar{E} - \bar{E}_n$ as in the previous example, and the constants in the present case are

\begin{align}
G^2 &= \frac{1}{4} \left( \frac{m\bar{a}}{\hbar^2 \bar{k}_n} \right)^2 \left( 1 + \frac{\tilde{k}'^2_n}{\tilde{k}^2_n} \right) \left[ \cos(\tilde{k}_n\bar{a}) - \frac{\tilde{k}_n}{\tilde{k}'} \sin(\tilde{k}_n\bar{a}) \right]^2 e^{2\tilde{k}'_n(\bar{b} - \bar{a})}; \\
F^2 &= \left( 1 + \frac{\tilde{k}'^2_n}{\tilde{k}^2_n} \right)^{-1} \left[ \sin(\tilde{k}_n\bar{a}) - \frac{\tilde{k}_n}{\tilde{k}'} \cos(\tilde{k}_n\bar{a}) \right]^2 e^{-2\tilde{k}'_n(\bar{b} - \bar{a})}
\end{align}

and

\[ \delta = \left( \frac{\hbar^2 \tilde{k}_n}{m\bar{a}} \right) \left( \frac{\tilde{k}_n^2 - \tilde{k}'^2_n}{\tilde{k}^2_n} + \frac{\tilde{k}'^2_n}{\tilde{k}^2_n} \right) \left[ \frac{\tilde{k}_n}{\tilde{k}'} \sin(\tilde{k}_n\bar{a}) - \frac{\bar{a}}{\bar{b}} \cos(\tilde{k}_n\bar{a}) \right] e^{-2\tilde{k}'_n(\bar{b} - \bar{a})}. \]

Once we obtain the relation (53), we can construct the scattering states relevant to this metastable system following exactly the same procedures as those in the previous section. The non-decay probability $P(t)$ of finding the particle within the confined region ($0 < \bar{x} < \bar{a}$) at time $t$ is again of the from $P(t) \sim \exp(-\gamma_n(t))$, where $\gamma_n(t)$ is now given by

\begin{align}
\gamma_n(t) &= 2 \left| \frac{F}{G} \right| \frac{t}{\bar{L}_0 \bar{L}(t)} \\
&= \frac{8\hbar^2 \tilde{k}_n^3}{ma} \left( \frac{\tilde{k}'_n}{\tilde{k}^2_n + \tilde{k}'^2_n} \right)^2 e^{-2\tilde{k}'_n(\bar{b} - \bar{a})} \frac{t}{\bar{L}_0 \bar{L}(t)}.
\end{align}
where we have used Eqs. (53), (55) and (19). When the barrier height is much larger than the characteristic “energy” of the escaping particles ($\tilde{V}_0 \gg \tilde{E}_n$), which is equivalent to the relation $\tilde{k}_n' \gg \tilde{k}_n$, $\gamma_n(t)$ becomes

$$
\gamma_n(t) \rightarrow \frac{8\hbar^2}{ma} \frac{k_n^3}{k_n'^2} e^{-2k_n'(\tilde{b}-\tilde{a})} \frac{t}{L_0 L(t)} .
$$

(58)

As in the previous example, for positive $v > 0$ (the expanding case) one finds a small but finite probability that the particle does not tunnel out of the well at large time. In this case not only does the barrier leave more room for the particle to stay within the well as it moves away from the $x = 0$, but its width also become thicker, thus making tunneling difficult.

In order to transform the result (58) to the stationary one, we again make use of the same substitutions as given at the end of the last section, with the addition of $\tilde{b} \rightarrow \tilde{b}/L_0$ and $\tilde{k}_n' \rightarrow \tilde{k}_n' L_0$, where $k_n' = \sqrt{2m((\tilde{V}_0/L_0^2) - \tilde{E}_n)/\hbar}$. Once again $\gamma_n(v = 0)$ is directly proportional to the time $t$, in which case a decay rate can be defined: $\Gamma_n \equiv \gamma_n(v = 0)/t$.

For the present example we have

$$
\Gamma_n(v = 0) = \frac{8\hbar^2}{ma} \frac{k_n^3}{k_n'^2} e^{-2k_n'(\tilde{b}-\tilde{a})} ,
$$

(59)

which is the same as the result obtained by the complex eigenvalue method for a square barrier with width ($\tilde{b} - \tilde{a}$) and height $\tilde{V}_0/L_0^2$ [17].

V. GENERAL SCALING POTENTIALS

We have calculated the non-decay probabilities of two non-stationary metastable systems explicitly. The potential barriers in the rescaled frame considered in these systems assumed the form of a delta-function and a square barrier. These calculations can be immediately generalized to barriers with more general shapes. Without giving further examples, what we would like to do here is to discuss briefly a close connection between the the non-decay probability $P(t)$ of a particle in a metastable scaling potential $V(x,t) = \tilde{V}(x/L(t))/L^2(t)$ and the decay rate $\Gamma$ of the same particle if it were instead confined in a static potential well $V(x) = \tilde{V}(x)$.

From the discussions and examples in the previous sections, we know that the calculations of $P(t)$ is reduced to the corresponding computations in a static potential $\tilde{V}(\tilde{x})$ in the rescaled frame. Now the later task would be exactly the same as that carried out in the potential $V(x) = \tilde{V}(x)$ in ordinary coordinates. The only difference, as seen from the previous two examples, is that all ordinary parameters, such as $\tilde{E}$, $k$, $k'$, $t$, $a$, etc, are replaced by the corresponding rescaled ones, i.e. $\tilde{E}$, $\tilde{k}$, $\tilde{k}'$, $\tau$, $\tilde{a}$, etc. Application of the scattering state method to the general alpha-decay type of potential $V(x)$ in normal coordinates has been given in [14], and can be carried over directly. Following [14] the important step is to determine the discrete values $\tilde{E}_n$ (or equivalently $k_n$) that minimize the amplitude $C$ of the wave function in the region outside the well. Consider a confining state constructed with $E$ centered around a specific $\tilde{E}_n$. Minimization of $C$ then gives the two functions $F(\tilde{E}_n)$ and $G(\tilde{E}_n)$ (other parameters in $F$ and $G$ are not indicated). The non-decay probability in $\tilde{V}(\tilde{x})$ is then given by $\exp(-\Gamma_n t)$, where the decay rate $\Gamma_n$ is

$$
\Gamma_n(\tilde{E}_n) = 2 \left| \frac{F(\tilde{E}_n)}{G(\tilde{E}_n)} \right| .
$$

(60)

Suppose all these computations have been done in ordinary coordinates. Then one can immediately write down the expression of the non-decay probability $P(t) \sim \exp(-\gamma_n(t))$ for the scaling potential $V(x,t)$ as

$$
\gamma_n(t) = 2 \left| \frac{F(\tilde{E}_n)}{G(\tilde{E}_n)} \right| \frac{t}{L_0 L(t)}
= \Gamma_n(\tilde{E}_n) \frac{t}{L_0 L(t)} .
$$

(61)

Here the functional form of the decay rate $\Gamma_n$ is taken over directly, but with all the parameters replaced by the corresponding rescaled ones. Eq. (61) gives the connection between the non-decay probability in $V(x,t) = \tilde{V}(x/L(t))/L^2(t)$ and the decay rate in $V(x) = \tilde{V}(x)$. Finally, we note here that, in the non-moving limit $v = 0$, $V(x,t)$ becomes $V(x) = \tilde{V}(x/L_0)/L_0^2$. Setting $v = 0$ in Eq. (61) then gives the decay rate in this potential: $\Gamma_n(\tilde{E}_n)/L_0^2$, as we had seen in the previous cases.
VI. CONCLUSION

In this paper we consider quantum metastability in a class of moving potentials introduced by Berry and Klein. Potential in this class has its height and width scaled in a specific way so that it can be transformed into a stationary one. In deriving the non-decay probability of the system, we employed a method which is less well known but conceptually more satisfactory, namely, the method of scattering states. Non-decay probabilities in a moving delta-potential and a moving square barrier potential were derived. We also give a connection between the non-decay probability in a general scaling potential and the decay rate in a related static potential.

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**Figure Captions**

**Figure 1.** The shape of $\ln C^2(\bar{k}a)$ in Eq.(19) as a function of $\bar{k}\bar{a}$ for $2m\bar{V}_0/\hbar^2 = 10$ (dotted line) and 200 (solid line). The minima of $\ln C^2(\bar{k}a)$ will center around $\bar{k}_n\bar{a} = n\pi$ ($n = 1, 2, \ldots$) for large values of $2m\bar{V}_0/\hbar^2$.

**Figure 2.** The shapes of $\ln C^2(\bar{k}a)$ (solid line) in Eq.(48) and $5e^{\bar{k}'\bar{a}}A(\bar{k}a)/2$ (dotted line) in Eq.(46) as function of $\bar{a}$ for $\bar{b} = 2\bar{a}$ and $2m\bar{V}_0\bar{a}^2/\hbar^2 = 40$. It shows that the minima of $\ln C^2(\bar{k}a)$ only occur in a finite number of neighborhood of $\bar{k}_n$ ($n = 1, 2, \ldots$) such that $A(\bar{k}_n\bar{a}) = 0$. 


