Gauge and Lorentz transformation placed on the same foundation

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Abstract

In this note we show that a “dynamical” interaction for arbitrary spin can be constructed in a straightforward way if gauge and Lorentz transformations are placed on the same foundation. As Lorentz transformations act on space-time coordinates, gauge transformations are applied to the gauge field. Placing these two transformations on the same ground means that all quantized field like spin-$1/2$ and spin-$3/2$ spinors are functions not only of the coordinates but also of the gauge field components. This change of perspective solves a couple of problems occuring for higher spin fields like the loss of causality, bad high-energy properties and the deviation of the gyromagnetic ratio from its constant value $g = 2$ for any spin, as caused by applying the minimal coupling. Starting with a “dynamical” interaction, a non-minimal coupling can be derived which is consistent with causality, the expectation for the gyromagnetic ratio, and well-behaved for high energies. As a consequence, on this stage the (electromagnetic) gauge field has to be considered as classical field. Therefore, standard quantum field theory cannot be applied. Despite this inconvenience, such a common ground is consistent with an old dream of physicists almost a century ago. Our approach, therefore, indicates a straightforward way to realize this dream.
1 Introduction

After the formulation of general relativity which explained forces on a geometric ground, physicists and mathematicians tried to incorporate the electromagnetic interaction into this geometric picture. Hermann Weyl claimed that the action integral of general relativity is invariant not only under space-time Lorentz transformations but also under the gauge transformation, if this is incorporated consistently [1]. However, the theories at that time were not ready to incorporate this view. Nowadays, we see more clearly that all physical variables (like position, momentum, etc.), quantum wave functions and fields transform as finite-dimensional representations of the Lorentz group. The reason is that interactions between fundamental particles (as irreducible representations of the Poincaré group) are most conveniently formulated in terms of field operators (i.e., finite-dimensional representations of the Lorentz group) if the general requirements like covariance, causality, etc. are to be incorporated in a consistent way. The relation between these two groups and their representations is given by the Lorentz-Poincaré connection [2]. In this note we show that if gauge transformation is put on the same foundation, the resulting non-minimal “dynamical” interaction obeys all necessary symmetries which for higher spins are broken if the interaction is introduced by the usual minimal coupling.

In Sec. 2 we explain details of the Poincaré group which are necessary in the following. In Sec. 3 we deal with linear wave equations as objects to the Lorentz transformation. In Sec. 4 we introduce the external electromagnetic field by a nonsingular transformation. In Sec. 5 we specify the nonlinear transformation by the claim of gauge invariance of the Poincaré algebra. Finally, in Sec. 6 we give our conclusions.

2 The Poincaré group

Relativistic field theories are based on the invariance under the Poincaré group $\mathcal{P}_{1,3}$ (known also as inhomogeneous Lorentz group $\mathcal{I}\mathcal{L}$ [3, 4, 5, 6, 7, 8, 9, 2, 10]). This group is obtained by combining Lorentz transformations $\Lambda$ and space-time translations $a_T$,

$$
(a, \Lambda) \equiv a_T\Lambda: E_{1,3} \ni x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + a^\mu \in E_{1,3}.
$$

The group’s composition law $(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$ generates the semidirect structure of $\mathcal{P}_{1,3}$,

$$
\mathcal{P}_{1,3} = \mathcal{T}_{1,3} \circ \mathcal{L}
$$

where $\mathcal{T}_{1,3}$ is the abelian group of space-time translations (i.e. the additive group $\mathbb{R}^4$) and $\mathcal{L} = \{\Lambda: \det \Lambda = +1, \Lambda^0_0 \geq 1\}$ is the proper orthochronous Lorentz group acting on the Minkowski space $E_{1,3}$ with metric

$$
\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1).
$$

The condition of the metric to be invariant under Lorentz transformations $\Lambda$ takes the form

$$
\Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\sigma = \eta_{\rho\sigma}.
$$
Under the Lorentz transformation $\Lambda \in \mathcal{L}$ the transformation of the covariant functions $\psi$ according to a representation $\tau(\Lambda)$ of the Lorentz group is determined by the commutative diagram

\[
\begin{array}{ccc}
\psi: & x \in \mathbb{E}_{1,3} & \longrightarrow & \psi(x) \\
\tau(\Lambda) \downarrow & \Lambda & \downarrow T(\Lambda) \\
\tau(\Lambda)\psi: & \Lambda x & \longrightarrow & T(\Lambda)\psi(x)
\end{array}
\]

i.e.

\[T(\Lambda)\psi(x) = (\tau(\Lambda)\psi)(\Lambda x) \equiv \psi^\Lambda(\Lambda x).\] (3)

The map $T: \Lambda \to T(\Lambda)$ is a finite-dimensional representation of $\mathcal{L}$. If we parametrize the element $\Lambda \in \mathcal{L}$ by $\Lambda(\omega) = \exp(-\frac{1}{2}\omega_{\mu\nu}e^{\mu\nu})$ where the Lorentz generators are given by

\[e^{\mu\nu}\rho = -\eta_{\mu}\eta_{\nu} + \eta_{\mu\sigma}\eta_{\nu}\rho\]

and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are six independent parameters, the parametrization of $T$ reads

\[T(\Lambda(\omega)) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}s^{\mu\nu}\right).\] (4)

The Lorentz group $\mathcal{L}$ is non-compact. As a consequence, all unitary representations are infinite dimensional. In order to avoid this, we introduce the concept of $H$-unitarity (see e.g. Ref. \cite{8} and references therein). A finite representation $T$ is called $H$-unitary if there exists a nonsingular hermitian matrix $H = H^\dagger$ so that

\[T^\dagger(\Lambda)H = HT^{-1}(\Lambda) \quad \Leftrightarrow \quad s^\dagger_{\mu\nu}H = HS_{\mu\nu}.\] (4)

Notice that a $H$-unitary metric is always indefinite, so that the inner product $\langle \cdot, \cdot \rangle$ generated by $H$ is sesquilinear sharing the hermiticity condition $\langle \psi, \varphi \rangle = \langle \varphi, \psi \rangle^\ast$. The most famous case of $H$-unitarity is given in the Dirac theory of spin-1/2 particles where $H = \gamma^0$.

For an operator $\mathcal{O}$ \cite{16} \cite{17} acting on the $\psi$-space of covariant functions the transformation $\tau(\Lambda)$ in Eq. (3) is a covariant transformation if the diagramm

\[
\begin{array}{ccc}
\mathcal{O}\psi: & x & \longrightarrow & (\mathcal{O}\psi)(x) \\
\tau(\Lambda) \downarrow & \Lambda & \downarrow T(\Lambda) \\
\tau(\Lambda)(\mathcal{O}\psi): & \Lambda x & \longrightarrow & T(\Lambda)(\mathcal{O}\psi)(x)
\end{array}
\]

is commutative, i.e.

\[T(\Lambda)\mathcal{O}\tau^{-1}(\Lambda)(\Lambda x)(\tau(\Lambda)\psi)(\Lambda x) = T(\Lambda)\mathcal{O}(x)\psi(x).\] (5)

Using Eq. (3) we obtain

\[T(\Lambda)\mathcal{O}\tau^{-1}(\Lambda)(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)\mathcal{O}(x)\psi(x).\]
Notice that the covariance of the transformation embodies only the property of equivalence of reference systems. The covariant operator $O$ is invariant under the transformation if in addition $\tau(\Lambda)O\tau^{-1}(\Lambda) = O$. As a consequence we obtain the commutative diagram

$$
\begin{array}{ccc}
O\psi : & x & \longrightarrow & (O\psi)(x) \\
\tau(\Lambda) \downarrow & \downarrow & T(\Lambda) \\
O(\tau(\Lambda)\psi) : & \Lambda x & \longrightarrow & T(\Lambda)(O\psi)(x)
\end{array}
$$

(6)

or $O(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)O(x)\psi(x)$ which means

$$
O(\Lambda x)T(\Lambda) = T(\Lambda)O(x)
$$

(7)
on the $\psi$-space. The invariance is a symmetry of the physical system and implies the conservation of currents. In particular, the symmetry transformations leave the equations of motion form-invariant.

While the Lorentz transformation $T(\Lambda)$ changes the wave function $\psi$ itself as well as the argument of this function (cf. Eq. (3)), the proper Lorentz transformation $\tau(\Lambda)$ causes a change of the wave function only. On the ground of infinitesimal transformations, this change is performed by the substancial variation. Starting from an arbitrary infinitesimal coordinate transformation $\Lambda(\delta\omega) : x^\mu \rightarrow x^\mu + \delta\omega^\mu x^\nu$, the substancial variation is given by Ref. [12]

$$
\delta_0\psi(x) \equiv \psi'(x) - \psi(x) = -\frac{i}{2} \delta\omega^\rho\sigma M^\rho\sigma \psi(x)
$$

where $M^\rho\sigma = \ell^\rho\sigma + s^\rho\sigma$, $\ell^\rho\sigma = i(x^\rho \partial^\sigma - x^\sigma \partial^\rho)$. The corresponding finite proper Lorentz transformation can be written as

$$
\tau(\Lambda(\omega)) = \exp \left( -\frac{i}{2} \omega^\mu\nu M^\mu\nu \right),
$$

and the multiplicative structure of the group generates the adjoint action

$$
\text{Ad}_{\tau(\Lambda)} : M^\mu\nu \rightarrow \tau^{-1}(\Lambda)M^\mu\nu\tau(\Lambda) = \Lambda^\rho^\mu \Lambda^\sigma^\nu M^\rho\sigma.
$$

(8)

Due to Eq. (4) the generators $s^\rho\sigma$ fulfill $s^\dagger_\rho\sigma H = H s^\rho\sigma$. They depend on the spin of the field but not on the coordinates $x^\mu$. Therefore, we have $[\ell^\mu\nu, s^\rho\sigma] = 0$. If a generic element of the translation group is written as

$$
\exp(+ia_\mu P^\mu),
$$

the commutator relations of the Lie algebra are given by

$$
[M^\mu\nu, M^\rho\sigma] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),
$$

$$
[M^\mu\nu, P^\rho] = i(\eta_{\rho\mu}P^\nu - \eta_{\rho\nu}P^\mu),
$$

$$
[P^\mu, P^\nu] = 0.
$$

(9)
The Casimir operators of the algebra are $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ where

$$W^\mu = + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_{\sigma}$$

is the Pauli-Lubanski pseudovector, $[P_\mu, W_\nu] = 0$. In coordinate representation we have $P_\mu = i \partial_\mu$, and the finite Poincaré transformation has the form

$$\tau(a, \Lambda) : \psi(x) \rightarrow (\tau(a, \Lambda)\psi)(x) = T(\Lambda)\psi \left( \Lambda^{-1}(x-a) \right).$$

(10)

This relation constitutes the Lorentz–Poincaré connection [2]. While the representation $T$ generally generates a reducible representation of $\mathcal{P}_{1,3}$, the spectra of the Casimir operators $P^2$ and $W^2$ determine the mass and spin content of the system.

3 The wave equations

As an operator $O$ in the above sense we consider the operator of the wave equation. The Dirac-type wave equation we will consider has the form

$$\mathcal{D}(\partial) \psi(x) \equiv (i\beta^\mu \partial_\mu - \rho)\psi(x) = 0$$

(11)

where $\psi$ is an $N$-component function, $\beta^\mu$ ($\mu = 0, 1, 2, 3$), and $\rho$ are $N \times N$ matrices independent of $x$. Following Bhabha’s conception [18], it is “... logical to assume that the fundamental equations of the elementary particles must be first-order equations of the form (11) and that all properties of the particles must be derivable from these without the use of any further subsidiary conditions.”

The principle of relativity states that a change of the reference frame cannot have implications for the motion of the system. This means that Eq. (11) is invariant under Lorentz transformations. Equivalently, the Lorentz symmetry of the system means the covariance and form-invariance of Eq. (11) under the transformation in Eq. (3), i.e. the transformed wave equation is equivalent to the old one. Therefore, we require that every solution $\psi^\Lambda(\Lambda x)$ of the transformed equation

$$\mathcal{D}^\Lambda(\Lambda \partial) \psi^\Lambda(\Lambda x) = 0$$

can be obtained as Lorentz transformation of the solution $\psi(x)$ of Eq. (11) in the original system and that the solutions in the original and transformed systems are in one-to-one correspondence. The explicit form of the covariance follows from Eq. (3),

$$\left( \tau(\Lambda) \mathcal{D} \tau^{-1}(\Lambda) \right) (\Lambda \partial) (\tau(\Lambda) \psi) (\Lambda x) = T(\Lambda) \mathcal{D}(\partial) \psi(x) = 0$$

(12)

and leads to the explicit Lorentz transformations

$$\beta^\Lambda \mu = \Lambda^\mu_\rho T(\Lambda) \beta^\rho T^{-1}(\Lambda), \quad \rho^\Lambda = T(\Lambda) \rho T^{-1}(\Lambda).$$
The Lorentz invariance is given by the substitution
\[ D(\partial)\psi(x) = 0 \quad \text{Eq. (3)} \quad D(\partial)\psi^\Lambda(x) = 0. \]

or
\[ T^{-1}(\Lambda)\beta^\mu T(\Lambda) = \Lambda^\mu_\rho \beta^\rho; \quad T^{-1}(\Lambda)\rho T(\Lambda) = \rho. \]

The difference of the original and transformed wave equation is given by the wave equation
where the wave function \( \psi \) is replaced by the substantial variation \( \delta_0 \psi \),
\[ D(\partial)\delta_0 \psi(x) = 0. \]

As a consequence we obtain \([D, M^{\rho\sigma}] = 0 \text{ or } [\beta^\mu, s^{\rho\sigma}] = i(\eta^{\mu\rho} \beta^\sigma - \eta^{\mu\sigma} \beta^\rho) \]
\[ \rho H = H \rho. \]

(14)

Writing \( \bar{\psi} = \psi^\dagger H \), one obtains the adjoint equation
\[ \bar{\psi} D(-\bar{\partial}) = \bar{\psi}(\bar{i} \beta^\mu \bar{\partial}_\mu - \rho) = (H D(\partial)\psi)^\dagger = 0. \]

(15)

4 Introduction of the external field

It may be reasonable to introduce an external field directly into the Poincaré algebra
which can be applied to classically understand the elementary particle. To do so one has
to transform the generators of the Poincaré group to be dependent on the external field in
such a way that the new, field-dependent generators obey the commutation relations (9).

As it was proposed by Chakrabarti \[23\] and Beers and Nickle \[24\], the simplest way to
build such a field dependent algebra is to introduce the external field \( A \) by a nonsingular
transformation
\[ \text{Ad}_{\mathcal{V}(A)} : p_{1,3} \rightarrow p_{1,3}^d(A) = \mathcal{V}(A)p_{1,3} \mathcal{V}^{-1}(A). \]

(16)

In case of a particular external electromagnetic field \( A \), the external field can be introduced
by using an evolution operator \( \mathcal{V}(A) \), called the “dynamical” representation \[25 \text{ [26]}\]. By
analogy with the free particle case one can realize this representation on the solution
space of relativistically invariant equations. Expressing the operators explicitly in terms
of free-field operators, one obtains the “dynamical” interaction. Applying for instance the
operator \( \mathcal{V}(A) \) to Eq. (11) one obtains
\[ \mathcal{V}(A) : D(\partial)\psi(x) = 0 \quad \rightarrow \quad D^d(\partial, A)\Psi(x, A) = 0 \]

(17)

where \( D^d(\partial, A) = \mathcal{V}(A)D(\partial)\mathcal{V}^{-1}(A) \) and
\[ \Psi(x, A) = \mathcal{V}(A)\psi(x) \]

(18)
Having introduced the external gauge field $A$, we introduce gauge covariance on the same foundation as Lorentz covariance in Eq. (3), i.e. by claiming that the diagram

\[
\begin{array}{c}
\Psi : A \rightarrow \Psi(A) \\
g(\lambda) \downarrow \downarrow \lambda \\
\Psi^\lambda : A^\lambda = A + \partial^\lambda \rightarrow G(\lambda)\Psi(A)
\end{array}
\]

is commutative, i.e.

\[
\Psi^\lambda(A + \partial^\lambda) = G(\lambda)\Psi(A). \quad (19)
\]

According to Eq. (6), the “dynamical” interaction $D^d$ is gauge invariant under the gauge transformation $A \rightarrow A^\lambda \equiv A + \partial^\lambda$ if the diagram

\[
\begin{array}{c}
D^d\Psi : A \rightarrow D^d(A)\Psi(A) \\
\downarrow \downarrow \lambda \\
D^d\Psi^\lambda : A + \partial^\lambda \rightarrow G(\lambda)D^d(A)\Psi(A)
\end{array}
\]

is commutative, i.e.

\[
D^d(A + \partial^\lambda)\Psi^\lambda(A + \partial^\lambda) = G(\lambda)D^d(A)\Psi(A). \quad (20)
\]

Together with Eq. (19) we obtain $D^d(A + \partial^\lambda)G(\lambda)\Psi(A) = G(\lambda)D^d(A)\Psi(A)$ or

\[
D^d(A + \partial^\lambda)G(\lambda) = G(\lambda)D^d(A) \quad (21)
\]

on the $\psi$-space. Note that up to now we have not specified the explicit shape of the finite dimensional representation $G : \lambda \rightarrow G(\lambda)$ of the gauge group.

## 5 Specifying $V(A)$ by gauge invariance

At this point we specify $V(A)$ by two claims. Due to gauge symmetry as a fundamental principle the dynamical transformation $V$ has to be compatible with the gauge transformation. Therefore, we first claim the gauge invariance in Eq. (21) not only for the operator $D^d$ but for the whole dynamical Poincaré algebra $p^d_{1,3}(A)$,

\[
p^d_{1,3}(A + \partial^\lambda)G(\lambda) = G(\lambda)p^d_{1,3}(A). \quad (22)
\]

By using Eq. (19) and multiplying by $G(\lambda)^{-1}$ from the right we obtain

\[
V(A + \partial^\lambda)p_{1,3}V^{-1}(A + \partial^\lambda) = G(\lambda)V(A)p_{1,3}(G(\lambda)V(A))^{-1}. \quad (23)
\]

This means that the first claim is fulfilled if

\[
V(A + \partial^\lambda) = G(\lambda)V(A). \quad (24)
\]
On the other hand, with Eqs. (18) and (19) we obtain

\[ V^\lambda(A + \partial \lambda)\psi(x) = G(\lambda)V(A)\psi(x) \] (25)

and, therefore, \( V^\lambda = V \) on the \( \psi \)-space. To summarize, by the first claim the gauge symmetry determines the gauge properties of \( V(A) \) and, therefore, of the interacting field \( \Psi(A) \).

The second claim is that the dynamical transformation operator \( V(A) \) should be of Lorentz type, i.e. for the generators \( s_{\mu\nu} \) of the Poincaré algebra \( p_{1,3} \) one has

\[ V(A)s^{\mu\nu}V^{-1}(A) = V_{\mu\rho}(A)V^\nu_\sigma(A)s^{\rho\sigma} \] (26)

which is a local extension of Eq. (8). \( V(A) = V(x, A) \) is the local Lorentz transformation generated by the external field \( A \) and obeying

\[ V_{\mu\rho}(A)V^{\nu\sigma}(A) = \eta_{\mu\sigma}. \] (27)

If such a local Lorentz transformation exists, the problem is solved. Therefore, in the following we make the attempt to find explicit realizations of the local Lorentz transformation \( V_{\mu\nu}(A) \). It is hard to find the Lorentz transformation \( V_{\mu\nu}(A) \) in general. However, as first shown by Taub [27], in the case of a plane-wave field we obtain

\[ V_{\mu\nu}(A) = \eta_{\mu\nu} - \frac{q}{k_P}G_{\mu\nu} - \frac{q^2}{2k_P^2}A^2k_\mu k_\nu \] (28)

where \( q \) is the electric charge of the particle and \( G_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu \). The plane wave field \( A_\mu = A_\mu(\xi), \xi = kx \) is characterized by its lightlike propagation vector \( k_\mu, k^2 = 0 \), and its polarization vector \( a^\mu \) such that \( a^2 = -1 \) and \( ka = 0 \). The operator \( k_P \equiv k_\mu P^\mu \) commutes with any other and has a special role in the theory. For particles with nonzero mass one has \( k_\mu P^\mu \neq 0 \). Therefore, for the plane wave the differential operator \( 1/k_P \) is local and well-defined for the plane-wave solution \( \psi_P \) of the Klein–Gordon equation. In all other cases, \( 1/k_P \) is assumed to exist.

Note that the plane-wave solution of the Dirac equation was found more than 70 years ago by Volkov [28] and extended later on to a field of two beams of electromagnetic radiation [29, 30]. However, these approaches did not make use of the nonsingular transformation \( V(A) \). The realization of \( V(A) \) can be achieved by the nonsingular transformation \( V(A) = V_0(A)V_s(A) \) where

\[ V_0(A) = \exp \left\{ -i \int \frac{d\xi}{2k_P} (2q(AP) - q^2A^2) \right\}, \]

\[ V_s(A) = \exp \left\{ -\frac{iq}{2k_P}G_{\mu\nu}s^{\mu\nu} \right\}. \] (29)

It has to be mentioned that the evolution operator \( V(A) \) may be chosen to be \( H \)-unitary according to the representation \( T \) in Eq. (4), i.e.

\[ V^\dagger(A)H = HV^{-1}(A). \]
Considering the nonsingular transformation of Dirac-type wave equation

$$\mathcal{V}(A) : (\beta^\mu P_\mu - m)\psi = 0 \rightarrow (\Gamma^\mu(A)\Pi_\mu(A) - m)\Psi(A) = 0,$$  
(30)

with the help of Eq. (29) the “dynamical” counterparts to the operator $P_\mu = i\partial_\mu$ can be calculated to be $\Pi_\mu(A) = \mathcal{V}(A)\gamma_\mu\mathcal{V}_\mu^{-1}(A)$,

$$P_\mu \rightarrow \Pi_\mu(A) = P_\mu + k_\mu\frac{q}{2k_\mu}(qA^2 - 2AP - F),$$  
(31)

$$P^2 \rightarrow \Pi^2(A) = (P - qA)^2 - qF$$  
(32)

($F \equiv s^\mu F_{\mu\nu}$) while the “dynamical” counterpart to $\beta_\mu$ is given by $\Gamma^\mu(A) = \mathcal{V}(A)\beta_\nu\mathcal{V}_\nu^{-1}(A),$ 

$$\Gamma^\mu(A) = V_{\mu\nu}(A)\beta^\nu = \beta_\mu - \frac{q}{k_\mu}\left(\frac{q}{2k_\mu}A^2k_\mu k_\nu + G_{\mu\nu}\right)\beta^\nu.$$  
(33)

In terms of $\Pi_\mu(A)$ and $\Gamma^\mu(A)$ we have

$$D^d(A)\Psi(A) = (\Gamma^\mu(A)\Pi_\mu(A) - m)\Psi(A) = 0.$$  
(34)

However, expressed in terms of $D_\mu = P_\mu - qA_\mu$ and $\beta^\mu$, we obtain

$$D^d(A)\Psi(A) \equiv \left(\beta^\mu D_\mu - \frac{q}{2k_\mu}\frac{k_\nu}{k}F - m\right)\Psi(A) = 0$$  
(35)

where $k_\mu \equiv \beta^\mu k_\mu$. This interaction is non-minimal. However, as we have shown before, it is determined completely by the claim of gauge invariance.

Note that due to the antimutation of the $\gamma$-matrices, in the spin-1/2 case the dynamical interaction in Eq. (35) reduces to the minimal coupling. However, in order to obtain the correct values of the gyromagnetic factor, in some cases the (phenomenological) Pauli term $\gamma_\mu\gamma_\nu F_{\mu\nu}$ has to be added by hand to the minimal coupling of the Dirac equation (see also Ref. [31], p. 109). In case of plane waves the exact solution of this (supplemented) Dirac equation as given by Chakrabarti [23] obeys the same gauge invariance condition $\Psi(A + \partial\lambda) = G(\lambda)\Psi(A)$. This property is found also in the book by Fried [32].

Finally, as a consequence of the explicit form (29), the associated transformation of the evolution operator $\mathcal{V}(A)$ under the local gauge transformation for the plane wave field,

$$A_\mu(\xi) \rightarrow A_\mu(\xi) + \partial_\mu\lambda(\xi)$$  
(36)

becomes

$$\mathcal{V}(A) \rightarrow \mathcal{V}(A + \partial\lambda) = e^{-iq\lambda}\mathcal{V}(A).$$  
(37)

As an example of higher spin, the spin-3/2 case is considered in detail in Ref. [33]. As it turns out, the Rarita–Schwinger spin-3/2 equation on the presence of a “dynamical” interaction is algebraically consistent and causal.
6 Conclusions

As a consequence of gauge invariance and Lorentz type of $V(A)$ we obtain

1. the invariance of the wave function under gauge transformations,

$$\Psi^\lambda(A + \partial \lambda) = V^\lambda(A + \partial \lambda)\psi = V(A + \partial \lambda)\psi = \Psi(A + \partial \lambda)$$

i.e. $\Psi^\lambda = \Psi$,

2. the explicit shape of $G(\lambda)$ in Eq. (19),

$$\Psi^\lambda(A + \partial \lambda) = V(A + \partial \lambda)\psi = e^{-iq\lambda}V(A)\psi = e^{-iq\lambda}\Psi(A),$$

i.e. $G(\lambda) = e^{-iq\lambda}$,

3. the invariance of $D^d$ under gauge transformations from Eq. (20) and

$$D^d(A + \partial \lambda)\Psi^\lambda(A + \partial \lambda) = D^d(A + \partial \lambda)e^{-iq\lambda}\Psi(A),$$

i.e. $D^d(A + \partial \lambda)G(\lambda) = G(\lambda)D^d(A)$ on the $\psi$-space,

4. the “dynamical” interaction for any spin as given by

$$D^d(A)\Psi(A) = \left(\beta^\mu D_\mu - \frac{q}{2k_p}k\cdot F - m\right)\Psi(A) = 0$$

being non-minimal but completely determined by gauge invariance, thereby causing Poincaré symmetry,

5. as a consequence of Eq. (32), the gyromagnetic factor in the presence of a “dynamical” interaction as being $g = 2$ for any spin [26].

Let us close again with Hermann Weyl. In Ref. [1] he honestly confessed: “Die entscheidenden Folgerungen in dieser Hinsicht verschanzen sich aber noch hinter einem Wall mathematischer Schwierigkeiten, den ich bislang nicht zu durchbrechen vermoch.” ( “However, the crucial consequences in this respect entrench oneself still behind a bank of mathematical difficulties which up to now I am not able to penetrate.”) We hope that our work breaks a small bay into this mathematical bank.

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