THE ROELCKE COMPACTIFICATION OF GROUPS OF HOMEOMORPHISMS

V. V. Uspenskij

Abstract. Let $X$ be a zero-dimensional compact space such that all non-empty clopen subsets of $X$ are homeomorphic to each other, and let $\text{Aut} \ X$ be the group of all self-homeomorphisms of $X$ with the compact-open topology. We prove that the Roelcke compactification of $\text{Aut} \ X$ can be identified with the semigroup of all closed relations on $X$ whose domain and range are equal to $X$. We use this to prove that the group $\text{Aut} \ X$ is topologically simple and minimal, in the sense that it does not admit a strictly coarser Hausdorff group topology. For $X = 2^\omega$ the last result is due to D. Gamarnik.

§ 1. Introduction

Let $G$ be a topological group. There are at least four natural uniform structures on $G$ which are compatible with the topology [4]: the left uniformity $\mathcal{L}$, the right uniformity $\mathcal{R}$, their least upper bound $\mathcal{L} \vee \mathcal{R}$ and their greatest lower bound $\mathcal{L} \wedge \mathcal{R}$. In [4] the uniformity $\mathcal{L} \wedge \mathcal{R}$ is called the lower uniformity on $G$; we shall call it the Roelcke uniformity, as in [6]. Let $\mathcal{N}(G)$ be the filter of neighbourhoods of unity in $G$. When $U$ runs over $\mathcal{N}(G)$, the covers of the form $\{xU : x \in G\}$, $\{Ux : x \in G\}$, $\{xU \cap Ux : x \in G\}$ and $\{UxU : x \in G\}$ are uniform for $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L} \vee \mathcal{R}$ and $\mathcal{L} \wedge \mathcal{R}$, respectively, and generate the corresponding uniformity.

All topological groups are assumed to be Hausdorff. A uniform space $X$ is precompact if its completion is compact or, equivalently, if every uniform cover of $X$ has a finite subcover. For any topological group $G$ the following are equivalent: (1) $G$ is $\mathcal{L}$-precompact; (2) $G$ is $\mathcal{R}$-precompact; (3) $G$ is $\mathcal{L} \vee \mathcal{R}$-precompact; (4) $G$ is a topological subgroup of a compact group. If these conditions are satisfied, $G$ is said to be precompact. Let us say that $G$ is Roelcke-precompact if $G$ is precompact with respect to the Roelcke uniformity. A group $G$ is precompact if and only if for every $U \in \mathcal{N}(G)$ there exists a finite set $F \subset G$ such that $UF = FU = G$. A group $G$ is Roelcke-precompact if and only if for every $U \in \mathcal{N}(G)$ there exists a finite $F \subset G$ such that $UFU = G$. Every precompact group is Roelcke-precompact, but not vice versa. For example, the unitary group of a Hilbert space or the group $\text{Symm}(E)$ of all permutations of a discrete set $E$, both with the pointwise convergence topology, are Roelcke-precompact but not precompact [6], [4]. Unlike the usual precompactness, the property of being Roelcke-precompact is not inherited by subgroups. (If $H$ is a subgroup of $G$, in general the Roelcke uniformity of $H$ is finer than the uniformity induced on $H$ by the Roelcke uniformity.)
of $G$.) Moreover, every topological group is a subgroup of a Roelcke-precompact group [7].

The Roelcke completion of a topological group $G$ is the completion of the uniform space $(G, \mathcal{L} \wedge \mathcal{R})$. If $G$ is Roelcke-precompact, the Roelcke completion of $G$ will be called the Roelcke compactification of $G$.

A topological group is minimal if it does not admit a strictly coarser Hausdorff group topology. Let us say that a group $G$ is topologically simple if $G$ has no closed normal subgroups besides $G$ and $\{1\}$. It was shown in [6], [7] that the Roelcke compactification of some important topological groups has a natural structure of an ordered semigroup with an involution, and that the study of this structure can be used to prove that a given group is minimal and topologically simple. In the present paper we apply this method to some groups of homeomorphisms.

A semigroup is a set with an associative binary operation. Let $S$ be a semigroup with the multiplication $(x, y) \mapsto xy$. We say that a self-map $x \mapsto x^*$ of $S$ is an involution if $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Every group has a natural involution $x \mapsto x^{-1}$. An element $x \in S$ is symmetrical if $x^* = x$, and a subset $A \subset S$ is symmetrical if $A^* = A$. An ordered semigroup is a semigroup with a partial order $\leq$ such that the conditions $x \leq x'$ and $y \leq y'$ imply $xy \leq x'y'$. An element $x \in S$ is idempotent if $x^2 = x$.

Let $K$ be a compact space. A closed relation on $K$ is a closed subset of $K^2$. Let $E(K)$ be the compact space of all closed relations on $K$, equipped with the Vietoris topology. If $R, S \in E(K)$, then the composition of $R$ and $S$ is the relation $RS = \{(x, y) : \exists z((x, z) \in S \text{ and } (z, y) \in R)\}$. The relation $RS$ is closed, since it is the image of the closed subset $\{(x, z, y) : (x, z) \in S, (z, y) \in R\}$ of $K^3$ under the projection $K^3 \to K^2$ which is a closed map. If $R \in E(K)$, then the inverse relation $\{(x, y) : (y, x) \in R\}$ will be denoted by $R^*$ or by $R^{-1}$; we prefer the first notation, since we are interested in the algebraic structure on $E(K)$, and in general $R^{-1}$ is not an inverse of $R$ in the algebraic sense. The set $E(K)$ has a natural partial order. Thus $E(K)$ is an ordered semigroup with an involution. In general the map $(R, S) \mapsto RS$ from $E(K)^2$ to $E(K)$ is not (even separately) continuous.

For $R \in E(K)$ let $\text{Dom} \, R = \{x : \exists y((x, y) \in R)\}$ and $\text{Ran} \, R = \{y : \exists x((x, y) \in R)\}$. Put $E_0(K) = \{R \in E(K) : \text{Dom} \, R = \text{Ran} \, R = K\}$. The set $E_0(K)$ is a closed symmetrical subsemigroup of $E(K)$.

Denote by $\text{Aut}(K)$ the group of all self-homeomorphisms of $K$, equipped with the compact-open topology. For every $h \in \text{Aut}(K)$ let $\Gamma(h) = \{(x, h(x)) : x \in K\}$ be the graph of $h$. The map $h \mapsto \Gamma(h)$ from $\text{Aut}(K)$ to $E_0(K)$ is a homeomorphic embedding and a morphism of semigroups with an involution. Identifying every self-homeomorphism of $K$ with its graph, we consider the group $\text{Aut}(K)$ as a subspace of $E_0(K)$.

We say that a compact space $X$ is $h$-homogeneous if $X$ is zero-dimensional and all non-empty clopen subsets of $X$ are homeomorphic to each other.

1.1. Main Theorem. Let $X$ be an $h$-homogeneous compact space, and let $G = \text{Aut}(X)$ be the topological group of all self-homeomorphisms of $X$. Then:

1. $G$ is Roelcke-precompact; the Roelcke compactification of $G$ can be identified with the semigroup $E_0(X)$ of all closed relations $R$ on $X$ such that $\text{Dom} \, R = \text{Ran} \, R = X$;

2. $G$ is minimal and topologically simple.
In the case when \( X = 2^\omega \) is the Cantor set, the minimality of \( \text{Aut}(X) \) was proved by D. Gamarnik [3].

Let us explain how to deduce the second part of Theorem 1.1 from the first. Let \( G = \text{Aut}(X) \) be such as in Theorem 1.1, and let \( f : G \to G' \) be a continuous onto homomorphism. We must prove that either \( f \) is a topological isomorphism or \( |G'| = 1 \). Let \( \Theta = E_0(X) \). The first part of Theorem 1.1 implies that \( f \) can be extended to a map \( F : \Theta \to \Theta' \), where \( \Theta' \) is the Roelcke compactification of \( G' \).

Let \( e' \) be the unity of \( G' \), and let \( S = F^{-1}(e') \). Then \( S \) is a closed symmetrical subsemigroup of \( \Theta \). Let \( \Delta \) be the diagonal in \( X^2 \). The set \( \{ r \in S : \Delta \subset r \} \) has a largest element. Denote this element by \( p \). Then \( p \) is a symmetrical idempotent in \( \Theta \) and hence an equivalence relation on \( X \). The semigroup \( S \) is invariant under inner automorphisms of \( \Theta \), and so is the relation \( p \). But there are only two \( G \)-invariant closed equivalence relations on \( X \), namely \( \Delta \) and \( X^2 \). If \( p = \Delta \), then \( S \subset G, G = F^{-1}(G') \) and \( f \) is perfect. Since \( G \) has no non-trivial compact normal subgroups, we conclude that \( f \) is a homeomorphism. If \( p = X^2 \), then \( S = \Theta \) and \( G' = \{ e' \} \).

A similar argument was used in [7] to prove that every topological group is a subgroup of a Roelcke-precompact topologically simple minimal group, and in [6] to yield an alternative proof of Stoyanov’s theorem asserting that the unitary group of a Hilbert space is minimal [5], [2]. For more information on minimal groups, see the recent survey by D. Dikranjan [1].

We prove the first part of Theorem 1.1 in Section 2, and the second part in Section 4.

§ 2. Proof of Main Theorem, Part 1

Let \( X \) be an \( h \)-homogeneous compact space, and let \( G = \text{Aut}(X) \). Let \( \Theta = E_0(X) \) be the semigroup of all closed relations \( R \) on \( X \) such that \( \text{Dom } R = \text{Ran } R = X \). We identify \( G \) with the set of all invertible elements of \( \Theta \). We prove in this section that \( \Theta \) can be identified with the Roelcke compactification of \( G \).

The space \( \Theta \), being compact, has a unique compatible uniformity. Let \( U \) be the uniformity that \( G \) has as a subspace of \( \Theta \). The first part of Theorem 1.1 is equivalent to the following:

**2.1. Theorem.** Let \( X \) be an \( h \)-homogeneous compact space, \( \Theta = E_0(X) \), and \( G = \text{Aut}(X) \). Identify \( G \) with the set of all invertible elements of \( \Theta \). Then:

1. \( G \) is dense in \( \Theta \);
2. the uniformity \( U \) induced by the embedding of \( G \) into \( \Theta \) coincides with the Roelcke uniformity \( \mathcal{L} \wedge \mathcal{R} \) on \( G \).

Let us first introduce some notation. Let \( \gamma = \{ U_\alpha : \alpha \in A \} \) be a finite clopen partition of \( X \). A \( \gamma \)-rectangle is a set of the form \( U_\alpha \times U_\beta, \alpha, \beta \in A \). Given a relation \( R \in \Theta \), denote by \( M(\gamma, R) \) the set of all pairs \( (\alpha, \beta) \in A \times A \) such that \( R \) meets the rectangle \( U_\alpha \times U_\beta \). Let \( \nu(\gamma, R) \) be the family \( \{ U_\alpha \times U_\beta : (\alpha, \beta) \in M(\gamma, R) \} \) of all \( \gamma \)-rectangles which meet \( R \). If \( r \) is a subset of \( A \times A \), put

\[ O_{\gamma, r} = \{ R \in \Theta : M(\gamma, R) = r \}. \]

The sets of the form \( O_{\gamma, r} \) constitute a base of \( \Theta \). Denote by \( E_0(A) \) the set of all relations \( r \) on \( A \) such that \( \text{Dom } r = \text{Ran } r = A \). A set \( O_{\gamma, r} \) is non-empty if and only if \( r \in E_0(A) \).
Let \( O_\gamma(R) \) be the set of all relations \( S \in \Theta \) which meet the same \( \gamma \)-rectangles as \( R \). We have \( O_\gamma(R) = O_{\gamma,r} \), where \( r = M(\gamma, R) \). The sets of the form \( O_\gamma(R) \) constitute a base at \( R \). If \( \lambda \) is another clopen partition of \( X \) which refines \( \gamma \), then 
\[
O_\lambda(R) \subset O_\gamma(R).
\]

**Proof of Theorem 2.1.** Our proof proceeds in three parts.

(a): We prove that \( G \) is dense in \( \Theta \).

Let \( \gamma = \{ U_\alpha : \alpha \in A \} \) be a finite clopen partition of \( X \) and \( r \in E_0(A) \). We must prove that \( O_{\gamma,r} \) meets \( G \). Decomposing each \( U_\alpha \) into a suitable number of clopen pieces, we can find a clopen partition \( \{ W_{\alpha,\beta} : (\alpha, \beta) \in r \} \) of \( X \) such that \( U_\alpha = \bigcup\{ W_{\alpha,\beta} : (\alpha, \beta) \in r \} \) for every \( \alpha \in A \). Similarly, there exists a clopen partition \( \{ W'_{\alpha,\beta} : (\alpha, \beta) \in r \} \) of \( X \) such that \( U_\beta = \bigcup\{ W'_{\alpha,\beta} : (\alpha, \beta) \in r \} \) for every \( \beta \in A \). Let \( f \in G \) be a self-homeomorphism of \( X \) such that \( f(W_{\alpha,\beta}) = W'_{\alpha,\beta} \) for every \( (\alpha, \beta) \in r \). The graph of \( f \) meets each rectangle of the form \( W_{\alpha,\beta} \times W'_{\alpha,\beta}, (\alpha, \beta) \in r \), and is contained in the union of such rectangles. It follows that \( M(\gamma, f) = r \) and 
\[
f \in G \cap O_{\gamma,r} \neq \emptyset.
\]

(b): We prove that the uniformity \( U \) is coarser than \( L \land R \).

This is a special case of the following:

**2.2. Lemma.** For every compact space \( K \) the map \( h \mapsto \Gamma(h) \) from \( \text{Aut}(K) \) to \( E_0(K) \) is \( L \land R \)-uniformly continuous.

**Proof.** It suffices to prove that the map under consideration is \( L \)-uniformly continuous and \( R \)-uniformly continuous. Let \( d \) be a continuous pseudometric on \( K \). Define \( d_2((x, y), (x', y')) = d(x, x') + d(y, y') \), and let \( d_H \) be the corresponding Hausdorff pseudometric on \( E_0(K) \). If \( R, S \in E_0(K) \) and \( a > 0 \), then \( d_H(R, S) \leq a \) if and only if each of the relations \( R \) and \( S \) is contained in the closed \( a \)-neighbourhood of the other with respect to \( d_2 \). The pseudometrics of the form \( d_H \) generate the uniformity of \( E_0(K) \).

Let \( d_s \) be the right-invariant pseudometric on \( \text{Aut}(K) \) defined by 
\[
d_s(f, g) = \sup\{ d(f(x), g(x)) : x \in K \}.
\]

The pseudometrics of the form \( d_s \) generate the right uniformity \( R \) on \( \text{Aut}(K) \). Since \( d_H(\Gamma(f), \Gamma(g)) \leq d_s(f, g) \), the map \( \Gamma : \text{Aut}(K) \to E_0(K) \) is \( R \)-uniformly continuous. For the left uniformity \( L \) we can either use a similar argument, or note that the involution on \( \text{Aut}(K) \) is an isomorphism between \( L \) and \( R \), and use the formula \( \Gamma(f) = \Gamma(f^{-1})^* \) to reduce the case of \( L \) to the case of \( R \).

(c): We prove that \( U \) is finer than \( L \land R \).

Let \( \gamma = \{ U_\alpha : \alpha \in A \} \) be a finite clopen partition of \( X \). Put \( V_\gamma = \{ f \in G : f(U_\alpha) = U_\alpha \text{ for every } \alpha \in A \} \). The open subgroups of the form \( V_\gamma \) constitute a base at unity of \( G \). We must show that if \( f, g \in G \) are close enough in \( \Theta \), then 
\[
f \in V_\gamma g V_\gamma.
\]

The set of all pairs \( (R, S) \in \Theta^2 \) such that \( M(\gamma, R) = M(\gamma, S) \) is a neighbourhood of the diagonal in \( \Theta^2 \) and therefore an entourage for the unique compatible uniformity on \( \Theta \). It suffices to prove that for every \( f, g \in G \) the condition \( M(\gamma, f) = M(\gamma, g) \) implies that \( f \in V_\gamma g V_\gamma \). Suppose that \( M(\gamma, f) = M(\gamma, g) = r \). The following conditions are equivalent for every \( \alpha, \beta \in A \): (1) \( f(U_\alpha) \cap U_\beta \neq \emptyset \); (2) \( g(U_\alpha) \cap U_\beta \neq \emptyset \); (3) \( (\alpha, \beta) \in r \). Pick \( u \in G \) such that \( u(f(U_\alpha) \cap U_\beta) = g(U_\alpha) \cap U_\beta \) for every \( (\alpha, \beta) \in r \). Such a self-homeomorphism \( u \) of \( X \) exists, since all non-empty clopen subsets of \( X \) are homeomorphic. Since for a fixed \( \beta \in A \) the sets \( f(U_\alpha) \cap U_\beta \), cover \( U_\beta \), we have \( u(U_\beta) = U_\beta \). Thus \( u \in V_\gamma \). It follows that 
\[
f \in V_\gamma g V_\gamma.
\]
that \( uf(U_\alpha) \cap U_\beta = u(f(U_\alpha) \cap U_\beta) = g(U_\alpha) \cap U_\beta \) for all \( \alpha, \beta \in A \) and hence \( uf(U_\alpha) = g(U_\alpha) \) for every \( \alpha \in A \). Put \( v = g^{-1}uf \). Since \( uf(U_\alpha) = g(U_\alpha) \), we have \( v(U_\alpha) = U_\alpha \) for every \( \alpha \in A \). Thus \( v \in V_\gamma \) and \( f = u^{-1}gv \in V_\gamma gV_\gamma \). \( \square \)

\[3.3. \text{Lemma.} \]

We modify the proof of Theorem 2.1. For every \( \gamma \in A \) let \( W_{\alpha,\gamma} = \{ (\alpha, \gamma) \in s \} \) of \( U_\gamma \). For every \( \gamma, \beta \in r \) put \( W_{\gamma,\beta} = \bigcup \{ V_{\alpha,\gamma,\beta} : (\alpha, \gamma) \in s \} \). For every \( \alpha, \gamma \in s \) put \( Y_{\alpha,\gamma} = \bigcup \{ V_{\alpha,\gamma,\beta} : (\gamma, \beta) \in r \} \). Take a clopen partition \( \{ W_{\gamma,\beta} : (\gamma, \beta) \in r \} \) of \( X \) such that for every \( \beta \in A \) we have \( U_\beta = \bigcup \{ fV_\gamma : (\gamma, \beta) \in r \} \).

§ 3. Continuity-like properties of composition

We preserve all the notation of the previous section. In particular, \( X \) is an \( h \)-homogeneous compact space, \( G = \text{Aut}(X) \), \( \Theta = E_0(A) \).

Recall that is a non-empty collection \( \mathcal{F} \) of non-empty subsets of a set \( Y \) is a filter base on \( Y \) if for every \( A, B \in \mathcal{F} \) there is \( C \in \mathcal{F} \) such that \( C \subset A \cap B \). If \( Y \) is a topological space, \( \mathcal{F} \) is a filter base on \( Y \) and \( x \in Y \), then \( x \) is a cluster point of \( \mathcal{F} \) if every neighbourhood of \( x \) meets every member of \( \mathcal{F} \), and \( \mathcal{F} \) converges to \( x \) if every neighbourhood of \( x \) contains a member of \( \mathcal{F} \). If \( \mathcal{F} \) and \( \mathcal{G} \) are two filter bases on \( G \), let \( \mathcal{F} \mathcal{G} = \{ AB : A \in \mathcal{F}, B \in \mathcal{G} \} \).

For every \( R \in \Theta \) let \( \mathcal{F}_R = \{ G \cap V : V \) is a neighbourhood of \( R \) in \( \Theta \} \). In other words, \( \mathcal{F}_R \) is the trace on \( G \) of the filter of neighbourhoods of \( R \) in \( \Theta \). We have noted that the multiplication on \( \Theta \) is not continuous. If \( R, S \in \Theta \), it is not true in general that \( \mathcal{F}_R \mathcal{F}_S \) converges to \( RS \). However, \( RS \) is a cluster point of \( \mathcal{F}_R \mathcal{F}_S \).

This fact will be used in the next section.

3.1. Proposition. If \( R, S \in \Theta \), then \( RS \) is a cluster point of the filter base \( \mathcal{F}_R \mathcal{F}_S \).

We need some lemmas. First we note that for any compact space \( K \) the composition of relations is upper-semicontinuous on \( E(K) \) in the following sense:

3.2. Lemma. Let \( K \) be a compact space, \( R, S \in E(K) \). Let \( O \) be an open set in \( K^2 \) such that \( RS \subset O \). Then there exist open sets \( V_1, V_2 \) in \( K^2 \) such that \( R \subset V_1 \), \( S \subset V_2 \), and for every \( R', S' \in E(K) \) such that \( R' \subset V_1 \), \( S' \subset V_2 \) we have \( R'S' \subset O \).

Proof. Consider the following three closed sets in \( K^3 \): \( F_1 = \{ (x, z, y) : (z, y) \in R \} \), \( F_2 = \{ (x, z, y) : (x, z) \in S \} \), \( F_3 = \{ (x, z, y) : (x, y) \notin O \} \). The intersection of these three sets is empty. There exist neighbourhoods of these sets with empty intersection. We may assume that the neighbourhoods of \( F_1 \) and \( F_2 \) are of the form \( \{ (x, z, y) : (z, y) \in V_1 \} \) and \( \{ (x, z, y) : (x, z) \in V_2 \} \), respectively, where \( V_1 \) and \( V_2 \) are open in \( K^2 \). The sets \( V_1 \) and \( V_2 \) are as required. \( \square \)

3.3. Lemma. Let \( \gamma = \{ U_\alpha : \alpha \in A \} \) be a finite clopen partition of \( X \). For every \( R, S \in \Theta \) we have \( M(\gamma, RS) \subset M(\gamma, R)M(\gamma, S) \) (the product on the right means the composition of relations on \( A \)).

Proof. Let \( (\alpha, \beta) \in M(\gamma, RS) \). Then \( RS \) meets the rectangle \( U_\alpha \times U_\beta \). Pick \( (x, y) \in RS \cap (U_\alpha \times U_\beta) \). There exists \( z \in X \) such that \( (x, z) \in S \) and \( (z, y) \in R \). Pick \( \delta \in A \) such that \( z \in U_\delta \). Then \( (x, z) \in S \cap (U_\alpha \times U_\delta) \), \( (z, y) \in R \cap (U_\delta \times U_\beta) \), hence \( (\alpha, \delta) \in M(\gamma, S) \) and \( (\delta, \beta) \in M(\gamma, R) \). It follows that \( (\alpha, \beta) \in M(\gamma, R)M(\gamma, S) \). \( \square \)

3.4. Lemma. Let \( \lambda = \{ U_\alpha : \alpha \in A \} \) be a finite clopen partition of \( X \), and let \( r, s \in E_0(A) \). There exist \( f, g \in G \) such that \( M(\lambda, f) = r \), \( M(\lambda, g) = s \) and \( M(\lambda, fg) = rs \).

Proof. We modify the proof of Theorem 2.1. For every \( \gamma \in A \) take a clopen partition \( \{ V_{\alpha,\gamma} : (\alpha, \gamma) \in s \} \) of \( U_\gamma \). For every \( (\gamma, \beta) \in r \) put \( W_{\gamma,\beta} = \bigcup \{ V_{\alpha,\gamma,\beta} : (\alpha, \gamma) \in s \} \). For every \( (\alpha, \gamma) \in s \) put \( Y_{\alpha,\gamma} = \bigcup \{ V_{\alpha,\gamma,\beta} : (\gamma, \beta) \in r \} \). Take a clopen partition \( \{ W'_{\alpha,\gamma} : (\alpha, \gamma) \in s \} \) of \( X \) such that for every \( \beta \in A \) we have \( U_\beta = \bigcup \{ fV_\gamma : (\gamma, \beta) \in r \} \).
α, γ \{ 6 \}

V. V. USPENSKIJ

for every

\[ M_{\alpha, \beta} \]

Let (\( \alpha, \beta \)) be a compact \( \Theta \)-homogeneous space, and let \( \Theta = E_0(X) \). We saw that \( G \) is Roelcke-precompact and that \( \Theta \) can be identified with the Roelcke compactification of \( G \). In this section we prove that \( G \) is minimal and topologically simple.

If \( H \) is a group and \( g \in H \), we denote by \( l_g \) (respectively, \( r_g \)) the left shift of \( H \) defined by \( l_g(h) = gh \) (respectively, the right shift defined by \( r_g(h) = hg \)).

4.1. Proposition. Let \( H \) be a topological group, and let \( K \) be the Roelcke completion of \( H \). Let \( g \in H \). Each of the following self-maps of \( H \) extends to a self-homeomorphism of \( K \): (1) the left shift \( l_g \); (2) the right shift \( r_g \); (3) the inversion \( g \mapsto g^{-1} \).

Proof. Let \( L \) and \( R \) be the left and right uniformity on \( H \), respectively. In each of the cases (1)–(3) the map \( f : H \to H \) under consideration is an automorphism of the uniform space \( (H, L \wedge R) \). This is obvious for the case (3). For the cases (1) and (2), observe that the uniformities \( L \) and \( R \) are invariant under left and right shifts, hence the same is true for their greatest lower bound \( L \wedge R \). It follows that in all cases \( f \) extends to an automorphism of the completion \( K \) of the uniform space \( (H, L \wedge R) \).

For \( g \in G \) define self-maps \( L_g : \Theta \to \Theta \) and \( R_g : \Theta \to \Theta \) by \( L_g(R) = gR \) and \( R_g(R) = Rg \).
4.2. Proposition. For every \( g \in G \) the maps \( L_g : \Theta \to \Theta \) and \( R_g : \Theta \to \Theta \) are continuous.

Proof. We have \( gR = \{ (x,g(y)) : (x,y) \in R \} \). Let \( \lambda = \{ U_\alpha : \alpha \in A \} \) be a clopen partition of \( X \). Let \( r = M(\lambda, gR) \), and let \( O_\lambda(gR) = \{ S \in \Theta : M(\lambda, S) = r \} \) be a basic neighbourhood of \( gR \). Let \( U \) be the set of all \( T \in \Theta \) such that \( T \) meets every member of the family \( \{ U_\alpha \times g^{-1}(U_\beta) : (\alpha, \beta) \in r \} \) and is contained in the union of this family. Then \( U \) is a neighbourhood of \( R \) and \( L_g(U) = O_\lambda(gR) \). Thus \( L_g \) is continuous. The argument for \( R_g \) is similar. \( \square \)

Let \( \Delta \) be the diagonal in \( X^2 \).

4.3. Proposition. Let \( S \) be a closed subsemigroup of \( \Theta \), and let \( T \) be the set of all \( p \in S \) such that \( p \supset \Delta \). If \( T \neq \emptyset \), then \( T \) has a greatest element \( p \), and \( p \) is an idempotent.

Proof. We claim that every non-empty closed subset of \( \Theta \) has a maximal element. Indeed, if \( C \) is a non-empty linearly ordered subset of \( \Theta \), then \( C \) has a least upper bound \( b = \overline{\cup C} \) in \( \Theta \), and \( b \) belongs to the closure of \( C \) in \( \Theta \). Thus our claim follows from Zorn’s lemma.

The set \( T \) is a closed subsemigroup of \( \Theta \). Let \( p \) be a maximal element of \( T \). For every \( q \in T \) we have \( pq \supset p\Delta = p \), whence \( pq = p \). It follows that \( p \) is an idempotent and that \( p = pq \supset \Delta q = q \). Thus \( p \) is the greatest element of \( T \). \( \square \)

An inner automorphism of \( \Theta \) is a map of the form \( p \mapsto gpg^{-1} \), \( g \in G \).

4.4. Proposition. There are precisely two elements in \( \Theta \) which are invariant under all inner automorphisms of \( \Theta \), namely \( \Delta \) and \( X^2 \).

Proof. A relation \( R \in \Theta \) is invariant under all inner automorphisms if and only if the following holds: if \( x, y \in X \) and \( (x, y) \in R \), then \( (f(x), f(y)) \in R \) for every \( f \in G \). Suppose that \( R \in \Theta \) has this property and \( \Delta \neq R \). Pick \( (x, y) \in R \) such that \( x \neq y \). We claim that the set \( B = \{ (f(x), f(y)) : f \in G \} \) is dense in \( X^2 \). Indeed, pick disjoint clopen neighbourhoods \( U_1 \) and \( U_2 \) of \( x \) and \( y \), respectively, such that \( X \) is not covered by \( U_1 \) and \( U_2 \). Given disjoint clopen non-empty sets \( V_1 \) and \( V_2 \), by \( h \)-homogeneity of \( X \) we can find an \( f \in G \) such that \( f(U_i) \subset V_i \), \( i = 1, 2 \). It follows that \( V_1 \times V_2 \) meets \( B \), hence \( B \) is dense in \( X^2 \). Since \( B \subset R \), it follows that \( R = X^2 \). \( \square \)

4.5. Proposition. The group \( G \) has no compact normal subgroups other than \( \{1\} \).

We shall prove later that actually \( G \) has no non-trivial closed normal subgroups.

Proof. Let \( H \neq \{1\} \) be a normal subgroup of \( G \). We show that \( H \) is not compact.

Let \( Y \) be the collection of all non-empty clopen sets in \( X \). Consider \( Y \) as a discrete topological space. The group \( G \) has a natural continuous action on \( Y \). Pick \( f \in H \), \( f \neq 1 \). Pick \( U \in Y \) such that \( f(U) \cap U = \emptyset \) and \( X \setminus (f(U) \cup U) \neq \emptyset \). Let \( Y_1 \) be the set of all \( V \in Y \) such that \( V \) is a proper subset of \( X \setminus U \). If \( V \in Y_1 \), there exists \( h \in G \) such that \( h(U) = U \) and \( h(f(U)) = V \). Put \( g = hfh^{-1} \). Then \( g(U) = V \). Since \( H \) is a normal subgroup of \( G \), we have \( g \in H \). It follows that the \( H \)-orbit of \( U \) contains \( Y_1 \). Since \( Y_1 \) is infinite, \( H \) cannot be compact. \( \square \)
4.6. Proposition. For every topological group $H$ the following conditions are equivalent:

1. $H$ is minimal and topologically simple;
2. if $f : H \rightarrow H'$ is a continuous onto homomorphism of topological groups, then either $f$ is a homeomorphism, or $|H'| = 1$. \(\square\)

We are now ready to prove Theorem 1.1, part (2):

For every compact $h$-homogeneous space $X$ the topological group $G = \text{Aut}(X)$ is minimal and topologically simple.

Proof. Let $f : G \rightarrow G'$ be a continuous onto homomorphism. According to Proposition 4.6, we must prove that either $f$ is a homeomorphism or $|G'| = 1$.

Since $G$ is Roelcke-precompact, so is $G'$. Let $\Theta'$ be the Roelcke compactification of $G'$. The homomorphism $f$ extends to a continuous map $F : \Theta \rightarrow \Theta'$. Let $e' \in \Theta'$ be the unity of $G'$, and let $S = F^{-1}(e') \subset \Theta$.

**Claim 1.** $S$ is a subsemigroup of $\Theta$.

Let $p, q \in S$. In virtue of Proposition 3.1, there exist filter bases $F_p$ and $F_q$ on $G$ such that $F_p$ converges to $p$ (in $\Theta$), $F_q$ converges to $q$ and $pq$ is a cluster point of the filter base $F_p F_q$. The filter bases $F_p = F(F_p)$ and $F_q = F(F_q)$ on $G'$ converge to $F(p) = F(q) = e'$, hence the same is true for the filter base $F_p F_q = F(F_p F_q)$. Since $pq$ is a cluster point of $F_p F_q$, $F(pq)$ is a cluster point of the convergent filter base $F(F_p F_q)$. A convergent filter on a Hausdorff space has only one cluster point, namely the limit. Thus $F(pq) = e'$ and hence $pq \in S$.

**Claim 2.** The semigroup $S$ is closed under involution.

In virtue of Proposition 4.1, the inversion on $G'$ extends to an involution $x \mapsto x^*$ of $\Theta'$. Since $F(p^*) = F(p)^*$ for every $p \in G$, the same holds for every $p \in \Theta$. Let $p \in S$. Then $F(p^*) = F(p)^* = e'$ and hence $p^* \in S$.

**Claim 3.** If $g \in G$ and $g' = f(g)$, then $F^{-1}(g') = gS = Sg$.

We saw that the left shift $h \mapsto gh$ of $G$ extends to a continuous self-map $L = L_g$ of $\Theta$ defined by $L(p) = gp$ (Proposition 4.2). According to Proposition 4.1, the self-map $x \mapsto g'x$ of $G'$ extends to self-homeomorphism $L'$ of $\Theta'$. The maps $FL$ and $L'F$ from $\Theta$ to $\Theta'$ coincide on $G$ and hence everywhere. Replacing $g$ by $g^{-1}$, we see that $FL^{-1} = (L')^{-1}F$. Thus $F^{-1}(g') = F^{-1}L'(e') = LF^{-1}(e') = gS$. Using right shifts instead of left shifts, we similarly conclude that $F^{-1}(g') = Sg$.

**Claim 4.** $S$ is invariant under inner automorphisms of $\Theta$.

We have just seen that $gS = Sg$ for every $g \in G$, hence $gS g^{-1} = S$.

Let $T = \{ r \in S : r \supset \Delta \}$. According to Proposition 4.3, there is a greatest element $p$ in $T$. Claim 4 implies that $p$ is invariant under inner automorphisms. In virtue of Proposition 4.4, either $p = \Delta$ or $p = X^2$. We shall show that either $f$ is a homeomorphism or $|G'| = 1$, according to which of the cases $p = \Delta$ or $p = X^2$ holds.

First assume that $p = \Delta$.

**Claim 5** ($p = \Delta$). All elements of $S$ are invertible in $\Theta$.

Let $r \in S$. Then $r^* r \in S$ and $rr^* \in S$, since $S$ is a symmetrical semigroup. Since $\text{Dom} \ r = \text{Ran} \ r = X$, we have $r^* r \supset \Delta$ and $rr^* \supset \Delta$. The assumption $p = \Delta$ implies that every relation $s \in S$ such that $s \supset \Delta$ must be equal to $\Delta$. Thus $rr^* = r^* r = \Delta$ and $r$ is invertible.

**Claim 6** ($p = \Delta$). $|S| = 1$.
Claim 5 implies that $S$ is a subgroup of $G$. This subgroup is normal (Claim 4) and compact, since $S$ is closed in $\Theta$. Proposition 4.5 implies that $|S| = 1$.

Claim 7 ($p = \Delta$). $f : G \to G'$ is a homeomorphism.

Claims 6 and 3 imply that $G = F^{-1}(G')$ and that the map $f : G \to G'$ is bijective. Since $F$ is a map between compact spaces, it is perfect, and hence so is the map $f : G = F^{-1}(G') \to G'$. Thus $f$, being a perfect bijection, is a homeomorphism.

Now consider the case $p = X^2$.

Claim 8. If $p = X^2 \in S$, then $G' = \{e'\}$.

Let $g \in G$ and $g' = f(g)$. We have $gp = p \in S$. On the other hand, Claim 3 implies that $gp \in gS = F^{-1}(g')$. Thus $g' = F(gp) = F(p) = e'$. □

§ 5. Remarks

The group $\text{Aut}(K)$ is Roelcke-precompact also for some compact spaces $K$ which are not zero-dimensional. For example, let $I = [0, 1]$ and $G = \text{Aut}(I)$. Identify $G$ with a subspace of $E(I)$, as above. The Roelcke compactification of $G$ can be identified with the closure of $G$ in $E(I)$. Let $G_0$ be the subgroup of all $f \in G$ which leave the end-points of the interval $I$ fixed. The closure of $G_0$ in $E(I)$ is the set of all curves $c$ in the square $I^2$ such that $c$ connects the points $(0, 0)$ and $(1, 1)$ and has the following property: there are no points $(x, y) \in c$ and $(x', y') \in c$ such that $x < x'$ and $y > y'$. This can be used to yield an alternative proof of D. Gamarnik’s theorem saying that $G$ is minimal [3].

Let $K = I^\omega$ be the Hilbert cube and $G = \text{Aut}(K)$. I do not know if $G$ is minimal or Roelcke-precompact in this case.

References

[1] D. Dikranjan, Recent advances in minimal topological groups, Topology Appl. 85 (1998), 53–91.
[2] D. Dikranjan, I. Prodanov and L. Stoyanov, Topological groups: characters, dualities and minimal group topologies, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 130, Marcel Dekker Inc., New York–Basel, 1989.
[3] D. Gamarnik, Minimality of the group $\text{Aut}(C)$, Serdika 17 (1991), no. 4, 197–201.
[4] W. Roelcke, S. Dierolf, Uniform structures on topological groups and their quotients, McGraw-Hill, 1981.
[5] L. Stoyanov, Total minimality of the unitary groups, Math. Z. 187 (1984), 273–283.
[6] V.V. Uspenskij, The Roelcke compactification of unitary groups, Abelian group, module theory, and topology: proceedings in honor of Adalberto Orsatti’s 60th birthday (D. Dikranjan, L. Salce., eds.), Lecture notes in pure and applied mathematics; V. 201, Marcel Dekker, New York e. a., 1998, pp. 411–419.
[7] V.V. Uspenskij, On subgroups of minimal topological groups, http://xxx.lanl.gov/abs/math.GN/0004119.

Department of Mathematics, 321 Morton Hall, Ohio University, Athens OH 45701, USA
E-mail address: uspensk@bing.math.ohiou.edu