Unextendible maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$

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We solved the unextendible maximally entangled basis (UMEB) problem in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d \neq d')$, the results turn out to be that there always exist a UMEB. In addition, there might be two sets of UMEB with different numbers. The main difficult is to prove the unextendibility of the set of states. We give an explicit construction of UMEB by considering the Schmidt number of the complementary space of the states we construct.

I. INTRODUCTION

Entanglement plays an important role in quantum information, such as teleportation, quantum error correction and quantum secret sharing[1–3]. The unextendible product basis (UPB) has been extensively investigated. Considerable elegant results have been obtained with interesting applications to the theory of quantum information [5, 6]. The UPB is a set of incomplete orthonormal product basis whose complementary space has no product states. It was also shown that the space complementary to a UPB contains bound entanglement [4]. Moreover, the states comprising a UPB are not distinguishable by local measurements and classical communication.

There it was shown that there are sets of orthogonal product vectors of a tensor product Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d \neq d')$ such that there are no further product states orthogonal to every state in the set, even though the size of the set is smaller than $dd'$.

Recently S. Bravyi and J. A. Smolin generalized the notion of the UPB to unextendible maximally entangled basis [7]; a set of orthonormal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ consisting of fewer than $d^2$ vectors which have no additional maximally entangled vectors orthogonal to all of them. In [7], the authors proved that there does not exist UMEBs for $d = 2$ and constructed a 6-member UMEB for $d = 3$ and a 12-member UMEB for $d = 4$. And the authors left us a question: whether the nonsquare UMEBs exist or not? In addition, there are some authors study the locally unextendible non-maximally entangled basis (LUNMEB)[9].

In [8], the authors studied the UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (\frac{d}{2} < d < d')$. They constructed a $d^2$-member UMEB and also left a question: whether there exist UMEB in other case $\frac{d}{2} \geq d$ or not.

In this paper, we study the UMEB in arbitrary bipartite spaces. We give an explicit construction of UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d < d')$. Hence we state that the UMEB exists if $d < d'$ and this gives an answer to the question asked in [7, 8]. We can give explicit expression of the vectors in the complementary space of the constructed states. Then we can state that there is no maximally entangled states in the complementary space of the constructed states by considering their Schmidt number.

II. UMEBS IN $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d > d)$

A $d^2$ member UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (\frac{d^2}{2} < d < d')$ has been constructed in [8] as the following.

$$|\phi_{mn}\rangle = \frac{1}{\sqrt{d'}} \sum_{p=0}^{d-1} \zeta_{d^2}^{np} |p \oplus m\rangle |p'\rangle$$

where $\zeta_d = e^{\frac{2\pi i}{d^2}}$, $k \oplus m$ denotes $(k + m) \mod d$

If we take a look at the condition $\frac{d^2}{2} < d < d'$ mentioned above, we can find that this condition is just equivalent with $d' = d + r, 0 < r < d$.

It’s no wonder that the problem of UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ with $d' = dq + r, 0 < r < d$ is a generalized problem.

Proposition 1. In $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, $d' = dq + r, 0 < r < d$, for any integer

$$m, n = 0, 1, \ldots, d - 1; l = 1, \ldots, q$$

we define

$$|\phi_{nm(l)}\rangle \triangleq \frac{1}{\sqrt{d'}} \sum_{p=0}^{d-1} \zeta_{d^2}^{np} |p \oplus m\rangle ((l - 1)d + p')$$

where $k \oplus m$ denotes $(k + m) \mod d.$
Then these $qd^2$ states are unextendible mutually orthogonal maximally entangled states.

Proof: (i) Orthogonality
\[
\langle \phi_{nm|l} | \phi_{mn|l} \rangle = \frac{1}{d} \sum_{p=0}^{d-1} \sum_{p=0}^{d-1} c_p^{n-m} |(\hat{l} - 1)d + \hat{p}|(\hat{p} \oplus m)|p \oplus m\rangle |(\hat{l} - 1)d + \hat{p}\rangle = \frac{1}{d} \sum_{p=0}^{d-1} \sum_{p=0}^{d-1} c_p^{n-m} |(\hat{l} - 1)d + \hat{p}|(\hat{p} \oplus m)|p \oplus m\rangle |(\hat{l} - 1)d + \hat{p}\rangle = \frac{1}{d} \sum_{p=0}^{d-1} (n-m)p |p \oplus m\rangle |p \oplus m\rangle = \delta_{mn} \delta_{n\hat{m}}
\]

(ii) Maximally entangled.
This can be easily checked by the definition of $|\phi_{mn|l}\rangle$.

(iii) Unextendible
Let $V_1$ denotes the subspace span by
\[
\{|\phi_{mn|l}\}, n, m = 0, 1, \cdots, d - 1; l = 1, \cdots, q\}.
\]
We have
\[
\text{Dim}(V_1) = qd^2,
\]
so
\[
\text{Dim}(V_1^\perp) = dd' - qd^2 = dr.
\]

Now let
\[
|\psi_{ij}\rangle = |ij\rangle, i = 0, 1, \cdots, d - 1; j = qd,\cdots, qd + r - 1
\]
Let $V_2$ denotes the subspace span by
\[
\{|\psi_{ij}\rangle, i = 0, 1, \cdots, d - 1; j = qd,\cdots, qd + r - 1\}.
\]
Following easily calculation, we have
\[
\langle \psi_{i'} j'| \phi_{i, j} \rangle = 0.
\]
Because $\text{Dim}(V_2) = dr$, so $V_1^\perp = V_2$, $\forall v \in V_2$ has the bellowing form
\[
v = \sum_{i=0}^{d-1} \sum_{j=0}^{r-1} a_{i,j}^{(i)} |q|d + j\rangle.
\]
So the Schmidt rank of every vector in $V_2$ is no higher than $r$ which is less than $d$, hence we derived that any state in $V_1^\perp$ is not maximally entangled.

From (i)(ii)(iii) and $qd^2 < dd'$, we can conclude that the states $\{|\phi_{ij}\rangle\}$ consist of a $qd^2$ member UMEB in $C^d \otimes C^{d'}$.

In [8], the authors ask whether the set of $d^2$ member orthonormal maximally entangled states they constructed is unextendible or not in the case of $d \leq \frac{d'}{7}$. Proposition 1 give a deny answer to this question when $d' = qd + r, q > 1, 0 < r < d$.

Example 1. We find an 8 member UMEB in $C^2 \otimes C^5$, this example is not satisfied the condition in [8]. Each row represent a states but not normal for the purpose of the conise notation.

| TABLE I: 8-member UMEB in $C^2 \otimes C^5$ |
|-----------------|---|---|---|---|---|---|---|---|
| 00              | 01 | 10 | 11 | 02 | 03 | 12 | 13 |
| 00              | 01 | 10 | 11 | 02 | 03 | 12 | 13 |
| 00              | 01 | 10 | 11 | 02 | 03 | 12 | 13 |
| 00              | 01 | 10 | 11 | 02 | 03 | 12 | 13 |

From the table above, we can observe that the four states can be look as the Bell states in $C^2 \otimes C^2$ with the base $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and the last four states also can be seen as the Bell states with the base $|02\rangle, |03\rangle, |12\rangle, |13\rangle$.

Now we notice that the above construction can not be efficient if $r = 0$, for $qd^2 = dd'$ in this case. So we give another construction to find the UMEB for other cases. A nature question arise when we consider the UMEB problem. Are there two sets of UMEB with different number? The Proposition below gives a positive answer to this problem when $(d \geq 3, d' - d \geq 2)$.

Proposition 2. In $C^d \otimes C^{d'}$ $(d < d')$, for any integer
\[
m \in \left\{ \{d' - 1, d' - 2, \cdots, d' - d + 1\} \right. d' \geq 2d
\]
\[
\{d' - 1, d' - 2, \cdots, d' - r\} d' < 2d, d' = d + r
\]
let
\[
|\phi_{ij}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{2\pi i k/m}\langle k | \langle k \oplus j\rangle,
\]
\[
i = 0, ..., d - 1; j = 0, ..., m - 1.
\]
where $k \oplus j$ means $k + j \mod m$.

Then $\{|\phi_{ij}\rangle\}$ is a $dm$ member UMEB.

Proof: (i) Orthogonality
\[
\langle \phi_{i' j'} | \phi_{i, j} \rangle = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{k'=0}^{d-1} e^{2\pi i (k - k') i} \langle k' | \langle k' \oplus j' \rangle | k \oplus j\rangle
\]
\[
\delta_{j,j'}\delta_{i,i'}
\]

(ii) Maximally entangled

By the definition of \(\kappa, \kappa'\), the question asked by [7, 8]. Compared with proposition 2, we can solve the case when the question asked by [7, 8].

Example 2. Now we give a 6 member UMEB in \(\mathbb{C}^2 \otimes \mathbb{C}^4\). (see Table II)

| TABLE II: 6-member UMEB in \(\mathbb{C}^2 \otimes \mathbb{C}^4\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 00  01  02  03  10  11  12  13  | 01  00  00  00  00  01  00  00  |
| 01  00  00  00  00  01  00  00  10  |
| 00  01  00  00  00  00  00  00  00  |
| 00  01  00  00  00  00  00  00  00  |
| 00  00  00  00  00  00  00  00  00  |
| 00  00  00  00  00  00  00  00  00  |
| 00  00  00  00  00  00  00  00  00  |

Example 3. Considering the UMEB in \(\mathbb{C}^3 \otimes \mathbb{C}^6\), we can be chosen to be 4 or 5 by proposition 2, so there exist a 12 member and a 15 member UMEB. Now we list the 15 member UMEB. (see Table III)

| TABLE III: 15-member UMEB in \(\mathbb{C}^3 \otimes \mathbb{C}^6\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 00  01  02  03  04  10  11  12  13  14  20  21  22  23  24  25  | 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |
| 01  00  00  00  00  00  00  00  00  00  00  00  00  00  00  00  |

III. CONCLUSION

We have studied the UMEB in \(\mathbb{C}^d \otimes \mathbb{C}^d\) \((d \neq d')\). We conclude that there always exist a UMEB in this case. Moreover, there might be two or more sets of UMEB with different numbers by proposition 2. Meanwhile, we also give an answer to the question in [8], when \(d = qd + r\), \(q > 1, 0 < r < d\), \(d = qd + r\). We extend them by some more \((q-1)d^2\) states to form a \(qd^2\) member UMEB. The main difficult for proving a set to be UMEB is the unextendible condition. Different with the paper [8], we just use some basic knowledge of the advance algebra to calculate the complementary space.

The result is just similar with the UPB problem in \(\mathbb{C}^d \otimes \mathbb{C}^d\) \((d \neq d')\), we can always find a UMEB. In the meantime, we also state that there are \(d-1\) different sets of UMEB if \(d' \geq 2d\).

Acknowledgments: We thank professor Shao-Ming Fei from Capital Normal University provided us this question and gave us some helpful advices.
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