Characterizing Oscillations in Heterogeneous Populations of Coordinators and Anticoordinators

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Abstract—Oscillations often take place in populations of decision-making individuals that are either a coordinator, who takes action only if enough others do so, or an anticoordinator, who takes action only if few others do so. Populations consisting of exclusively one of these types are known to reach an equilibrium, where every individual is satisfied with her decision. Yet it remains open whether and when oscillations take place in a population consisting of both types, and if they do, what features they share. We take the first step towards answering this question by simulating a well-mixed population of coordinators and anticoordinators, each associated with a possibly unique non-negative threshold and initialized with the strategy $A$ or $B$. We take the distribution of the actions $A$ over the thresholds as the state of the population dynamics. The dynamics in our example admit two minimally positively invariant sets, where the solution trajectory oscillates, and an equilibrium. We identify the basic properties of the dynamics, based on which, we introduce a class of sets that are positively invariant. Our results highlight the possibility of non-trivial, complex oscillations in the absence of noise and population structure and shed light on the reported oscillations in decision-making populations.

I. INTRODUCTION

Oscillatory phenomena are often observed in nature as well as human societies with their ability to adapt and make decisions: periodic patterns in biology, fluctuations in market values, unsettlements in social emotions, non-fixation in fashion trends, and twist of power among political parties [1], [2]. In most such situations, individuals have to choose between one out of two actions, and either pick an action only if enough others do so, which are referred to as coordinating individuals or simply coordinators, or pick an action only if few others do so, which are referred to as anticoordinating individuals or simply anticoordinators. Followers in technology markets avoid risk by producing common products, whereas innovators perceive their benefits in building up a monopoly by developing new or rare products [3], [4]. “Activator” and “repressor” cells in synthetic microbial consortia, respectively increase and decrease gene expression if the transcription is low and high, resulting in a positive and negative feedback loop [5]. Populations consisting exclusively of one of these two types of individuals are known to eventually reach an equilibrium state where individuals are satisfied with their decisions [6]. Therefore, it is only the coexistence of the two that may explain the observed oscillations. However, not every coexistence of the two leads to an oscillation, and even if it does, the characteristics of the oscillation is unclear.

Researchers have studied the asymptotic behavior of structured heterogeneous populations exclusively consisting of either of the two types. In [7], [8], networks of all coordinators or all anticoordinators who update their strategies synchronously, that is all at a time, are shown to reach a limit cycle of length at most two. In [6], any network of all coordinators or all anticoordinators who update their strategies asynchronously, that is one at a time, is guaranteed to reach an equilibrium state in finite time. Many real-world populations are well-mixed, where individuals know the total number of others who have chosen a particular strategy. Well-mixed heterogeneous populations of anticoordinators updating asynchronously reach a unique equilibrium state regardless of the initial condition [9]. Well-mixed heterogeneous populations of coordinators updating asynchronously also always reach an equilibrium state, yet it is not necessarily unique, even for the same initial condition [10]. For mixed networks of coordinators and anticoordinators, sufficient conditions for finite time convergence to equilibrium has been provided in [11], [12]. However, to best of our knowledge, there is no in-depth study on the non-converging and fluctuating behavior of mixed populations of both coordinators and anticoordinators, despite their inevitable coexistence in many real-world situations [13]. This missing piece in the literature is perhaps key in understanding many oscillatory phenomena.

We study a well-mixed heterogeneous population comprising both coordinators and anticoordinators who play either of the strategies $A$ or $B$ and update asynchronously over time. We take the distribution of $A$-players over the coordinators and anticoordinators with different types as the state of the system. First, via a numerical example, we show that a single population can both reach an equilibrium and undergo never-ending fluctuations, depending on the initial condition and the agents’ activation sequence. Next, based on the intuitions from the example, we reveal some useful properties of the coordinators and anticoordinators, based on which we characterize a set that is positively invariant, which is either a singleton, and hence, an equilibrium, or non-singleton, where the solution trajectory fluctuates. The results shed light on observed oscillations in populations of binary decision-makers, and serve as a stepping stone towards providing incentives or establishing regulatory polices in technology market, social and cultural learning, health plans [13–15], etc., to either settle or modify the oscillations.
II. MODEL

Consider a well-mixed population of \( n \geq 2 \) agents who play either of the two strategies A or B, and over a discrete time sequence \( t \in \mathbb{Z}_{\geq 0} \) revise their strategies based on their types which are either coordinating and anticoordinating. More specifically, each agent \( j \in \{1, \ldots, n\} \) has a temper \( \tau_j \in [0, n - 1] \). At each time step \( t \in \mathbb{Z}_{\geq 0} \), a single agent \( j \) becomes active to update her strategy at time \( t + 1 \) as follows. If agent \( j \) is a coordinator, then she plays A only if the number of other A-playing agents does not fall short of her temper:

\[
s_j(t + 1) = \begin{cases} A & A_j(t) \geq \tau_j \\ B & A_j(t) < \tau_j \end{cases},
\]

where \( A_j \) denotes the total number of A-playing agents in the remaining of the population, i.e., excluding agent \( j \). If agent \( j \) is an anticoordinator, then she plays A only if the number of other A-players does not exceed her temper, i.e.,

\[
s_j(t + 1) = \begin{cases} A & A_j(t) \leq \tau_j \\ B & A_j(t) > \tau_j \end{cases}.
\]

The agents may represent people aiming at establishing an NGO for a social service program. The individuals know how many others have volunteered (played A) and accordingly decide to also volunteer (play A) or not (play B). Some may consider the NGO a failure or burdensome if the volunteers are few, and hence, would only join if enough others have joined (coordinators). Others may find it a duty to join even if no one else does, but they also find it unnecessary for too many to join. Hence, they would join only if less than enough others have joined (anticoordinators). Another example is the co-existence of innovators and followers in the technology market [3]. Companies can either take an existing market from an entrenched technology (play A) or take on competitors with any form of innovation (play B). Some companies are conservative in the sense that enough number of other sectors focusing on a technology convinces them to follow (coordinators). Others tend to build up a monopoly, so they change their strategy to innovate upon the presence of rivals developing the same technology (anticoordinators).

We categorize all anticoordinators (resp. coordinators) with the same temper as the same type, and assume that there are all together \( b \geq 1 \) (resp. \( b' \geq 1 \)) different types of anticoordinating agents (resp. coordinators). We label the anticoordinator (resp. coordinator) types in the descending (resp. ascending) order of their tempers by \( 1, 2, \ldots, b \) (resp. \( 1', 2', \ldots, b' \)). So we have a total of \( b + b' \) types of agents. Denote the temper of an anticoordinator (resp. coordinator) of type \( i \) by \( \tau_i \) (resp. \( \tau'_i \)). Then we have

\[
\tau_1 > \tau_2 > \ldots > \tau_b, \quad \tau'_b > \tau'_{b - 1} > \ldots > \tau'_1.
\]

We take the distribution of the A-players among these types as the state of the system:

\[
x(t) \overset{\Delta}{=} (x_1(t), \ldots, x_b(t), x'_b(t), \ldots, x'_1(t)),
\]

where \( x_i \) (resp. \( x'_i \)) denotes the number of A-playing anticoordinators (resp. coordinators) of type \( i \). Clearly, \( x(t) \) lies in the state space

\[
X \overset{\Delta}{=} \{(x_1, \ldots, x_b, x'_b, \ldots, x'_1) \in \mathbb{Z}_{\geq 0}^{b+b'} \mid x_i \leq n_i \forall i \in \{1, \ldots, b\}, x'_i \leq n'_i \forall i \in \{1, \ldots, b'\}\},
\]

where \( n_i \geq 1 \) (resp. \( n'_i \geq 1 \)) is the number of anticoordinating (resp. coordinators) of type \( i \). Let \( A(x) \) denote the total number of A-players in the population at state \( x \). One can write update rules (1) and (2) with respect to \( A(x(t)) \) and the tempers of the types according to Lemma 3 in the appendix.

The update rules (1) and (2) together with the activation sequence of the agents govern the dynamics of \( x(t) \), called the population dynamics. We do not impose any assumption on the activation sequence, although a realistic sequence might be generated by a random process. A positively invariant set under the population dynamics is a set \( O \subseteq X \) such that if \( x(0) \in O \), then \( x(t) \in O \) for all \( t \in \mathbb{Z}_{\geq 0} \) and under any activation sequence. Namely, the solution trajectory never leaves the set after entering it. Our goal is to identify these sets. If \( O \) is a singleton, then it consists of a single equilibrium state, where the solution trajectory settles. However, if \( O \) is a non-singleton and does not include an equilibrium, then the solution trajectory perpetually fluctuates between several states in \( O \), as shown in the following section.

III. EXAMPLE AND INTUITION

Example 1. Consider a population of 42 agents, consisting of four anticoordinating and five coordinating types with tempers

\[
(\tau_1, \tau_2, \tau_3, \tau_4, \tau'_5, \tau'_3, \tau'_2, \tau'_1) = (18, 9, 8, 7, 25, 19, 14, 10, 5).
\]

The distribution of the population over the types is given by

\[
(n_1, n_2, n_3, n_4, n'_5, n'_3, n'_2, n'_1) = (4, 3, 1, 3, 15, 1, 10, 2, 3).
\]

Figure 1 shows the evolution of the state \( x \) for the initial condition \( x(0) = (4, 3, 0, 0, 0, 0, 0, 0, 0, 3) \). All type-1 coordinators and anticoordinators have fixed their strategies to A, and all type-5, 4, and 3 coordinators have fixed their strategies to B. The state fluctuates in a positively invariant set, where only type-2, 3, and 4 anticoordinators and type-2 coordinators may switch strategies. The minimum number of k-players in this set, i.e., 7, exceeds the temper of type-1 coordinators plus one, i.e., \( \tau'_1 + 1 = 6 \), ensuring that they always play A. On the other hand, the maximum number of k-players equaling 12, ensures all type-1 anticoordinators play A and all type-3, 4, and 5 coordinators play B.

The evolution of \( x \) under a particular activation sequence is shown in Table 1. When all coordinators, except for those who have fixed their strategies to A, are playing B, some low-temper anticoordinators start switching to A. This causes some coordinators to switch to A, which in turn
makes the aforementioned anticoordinators switch back to B. The solution trajectory revisits the states in the positively invariant set, e.g., \( x(1) = x(5) = x(17) \).

Following the example, we expect the existence of a benchmark type \( p \in \{0, 1, \ldots, b \} \) of anticoordinators and a benchmark type \( p' \in \{0, 1, \ldots, b' \} \) of coordinators such that all type 1, 2, \ldots, \( p \) anticoordinators and all type 1, 2, \ldots, \( p' \) coordinators eventually fix their strategies to A, and the agents of no other type all fix their strategies to A. By \( p = 0 \) (resp. \( p' = 0 \)), we mean that the agents of no anticoordinating (resp. coordinating) type all fix their strategies to A. We also expect the existence of two other benchmark types \( q \in \{ p + 1, \ldots, b + 1 \} \) of anticoordinating and \( q' \in \{ p' + 1, \ldots, b' + 1 \} \) of coordinating such that all type \( q, q + 1, \ldots, b \) anticoordinators and all type \( q', q' + 1, \ldots, b' \) coordinators eventually fix their strategies to B, and the agents of no other type all fix their strategies to B. By \( q = b + 1 \) (resp. \( q' = b' + 1 \)), we mean that the agents of no anticoordinating (resp. coordinating) type all fix their strategies to B. We refer to those agents fixing their strategies to A as \( A \)-fixed, those fixing their strategies to B as \( B \)-fixed, and the remaining as \( \textit{wandering} \) agents. The above long-term behavior is what we anticipate when the solution trajectory enters a positively invariant set. Namely, for every positively invariant set \( \mathcal{O} \subseteq \mathcal{X} \), we expect the existence of benchmarks \( (p, q, q', p') \in \Omega \), such that \( \mathcal{O} \) is a subset of the set

\[
\mathcal{X}_{p,q,q',p} \triangleq \left\{ x \in \mathcal{X} \mid x_i = n_i \forall i \in \{1,\ldots,p\}, \quad x_i = n'_i \forall i \in \{1,\ldots,q\}, \quad x_i = 0 \forall i \in \{q,q'\}, \quad x_i = 0 \forall i \in \{q',b'\} \right\}.
\]

We also expect certain patterns in the wandering agents’ strategies. In particular, starting from the state where all fixed agents have already fixed their strategies and all other agents are playing B, i.e.,

\[
(n_1, \ldots, n_p, 0, 0, 0, \ldots, n'_{q-1}, n'_q)
\]

as we update the wandering anticoordinators in the ascending order of their tempers, i.e., from type \( q - 1 \) to \( p + 1 \), then the total number of \( A \)-players never exceeds the temper of the active wandering anticoordinator. The intuition behind is that an anticoordinator choosing \( A \) does not make another anticoordinator with a higher temper to switch to \( B \). We expect this to be also true for every state \( x \) in the invariant set, i.e., for every \( i \in \{p + 1, \ldots, q - 1\} \),

\[
\sum_{k=1}^{p} n_k' + \sum_{k=p+1}^{q-1} n_k + \sum_{k=q}^{x_k} n_k \leq \lfloor \tau_i \rfloor + 1,
\]

Namely, for every type-i wandering anticoordinator, the total number of \( A \)-playing wandering anticoordinators with a non-less temper, together with the \( A \)-fixed agents, does not exceed the type-i temper plus one.

A similar condition should hold if we start from the state where all fixed agents have already fixed their strategies and all wandering agents are playing A, i.e.,

\[
(n_1, \ldots, n_{p-1}, 0, 0, 0, \ldots, 0, n'_{q-1}, \ldots, n'_q)
\]

Then by updating the wandering anticoordinators in the descending order of their tempers, i.e., from \( p + 1 \) to \( q - 1 \), the total number of \( A \)-players never falls short of the temper of the active wandering anticoordinator. The intuition behind is that an anticoordinator choosing \( B \) does not make another anticoordinator with a lower temper to switch to \( A \). We expect this to be also true for every state \( x \) in the invariant set, i.e., for every \( i \in \{p + 1, \ldots, q - 1\} \),

\[
\sum_{k=1}^{q-1} n_k' + \sum_{k=p+1}^{q-1} n_k + \sum_{k=q}^{x_k} n_k \geq \lfloor \tau_i \rfloor + 1.
\]
Given all this, for indices \((p, q, q', p') \in \Omega\), define
\[
I_{p,q,q',p'} \triangleq \left\{ x \in \mathcal{X}_{p,q,q',p'} \mid (\Omega, \mathcal{R}) \forall i \in \{p + 1, \ldots, q\}\right\}.
\]
For every positively invariant set \(\mathcal{O}\), we expect the existence of \((p, q, q', p') \in \Omega\) such that \(\mathcal{O} \subseteq I_{p,q,q',p'}\). The quadruple \((p, q, q', p')\) is not necessarily unique under the population dynamics, and hence, neither is the invariant set. Starting from different or even the same initial condition, we may end up at different positively invariant sets, based on the activation sequence. Apparently, the population in Example 1 admits another positively invariant set as well as an equilibrium point.

**Example 1 (continued).** When the state starts from the initial condition \(x(0) = (4,0,0,0,0,0,10,2,3)\), it wanders in a positively invariant set, different from the previous one (Figure 2). The set consists of the first 4 states in Table II. Interestingly, this time, the state transition in the table is the only possible transition for \(x(t)\). Namely, starting from any initial condition in this set and under any activation sequence, the state follows the same cycle of length 4 to return to that initial condition. This behavior is similar to that of a limit cycle with the difference that here, the returning time is not necessarily fixed. The reason is that if any agent other than those mentioned in the table become active, they will not switch, and hence, the state remains unchanged.

**TABLE II**

| State of the system | Active agent | Number of \(k\)-players |
|---------------------|--------------|-------------------------|
| \(x(0) = (4,0,0,0,0,10,2,3)\) | type-4 coordinator | 19 |
| \(x(1) = (4,0,0,0,1,10,2,3)\) | type-1 anticoordinator | 20 |
| \(x(2) = (3,0,0,0,1,10,2,3)\) | type-4 coordinator | 19 |
| \(x(3) = (3,0,0,0,0,10,2,3)\) | type-1 anticoordinator | 18 |
| \(x(4) = (4,0,0,0,0,0,10,2,3)\) | type-1 anticoordinator | 19 |

Furthermore, if all coordinators play \(k\), they are enough to keep themselves doing so and stimulate all anticoordinators to play \(B\). Thus, the population also possesses the equilibrium point \(x^* = (0,0,0,0,15,1,10,2,3)\).

Identifying the benchmarks \(p, q, q', p'\) and \(p'\) is not straightforward. The number of \(A\)-fixed anticoordinators depends on the number of \(A\)-fixed coordinators and the maximum number of \(A\)-playing wandering agents. The number of \(A\)-fixed coordinators depends on the number of \(A\)-fixed anticoordinators and the minimum number of \(A\)-playing wandering agents. The number of wandering coordinators depends on the number of \(A\)-fixed agents and the minimum and maximum number of \(A\)-playing wandering anticoordinators, which in turn depend on the number of \(A\)-fixed agents and wandering coordinators. This results in several loops, complexifying the identification of the benchmarks as well as proving the invariance of the resulting set. In what follows, we show how to break the loops to find the benchmarks. This allows us to characterize a collection of positively invariant sets under the population dynamics. The collection also includes two positively invariant sets \(I_{1,5,3,1}\) and \(I_{0,2,5,3}\) that match precisely the first and second invariant sets in the example.

**IV. POSITIVELY INVARIANT SETS**

We provide a sufficient condition for the set \(I_{p,q,q',p'}\) to be positively invariant. More specifically, we introduce quadruples of benchmarks \((\zeta, \eta, \eta', \zeta') \in \Omega\) such that \(I_{\zeta, \eta, \eta', \zeta'}\) is positively invariant. The benchmarks depend on the maximum and minimum number of \(A\)-players that in the invariant set. In Subsection IV-A, we introduce activation sequences, under which the population can reach these extremum numbers of \(A\)-players. Then in Subsection IV-B, we introduce a collection of pairs \((\zeta, \delta)\), each results in a quadruple \((\zeta, \eta, \eta', \zeta')\). Finally, in Subsection IV-C, we state the invariance of \(I_{\zeta, \eta, \eta', \zeta'}\) in Theorem 1 and proceed to the proof.

Motivated by the structure of the invariant set in Example 1, given a state \(x \in \mathcal{X}\), we define the types \(x^a, x^b, x^c, x^d\) by
\[
(x^a, x^b, x^c, x^d) \triangleq \arg \min_{(p,q,q',p') \in \Omega} (-p + q + q' - p') \quad \text{s.t.} \quad \mathcal{X}_{p,q,p',q'} \supseteq x.
\]
Namely, \(\mathcal{X}_{x^a, x^b, x^c, x^d}\) is the smallest set in the form of \(\mathcal{X}_{p,q,p',q'}\) that contains \(x\). The special cases \(p = 0, p' = 0, q = b + 1, \) and \(q' = b' + 1\) are treated as in the previous section. In the absence of the special cases, \(x\) takes the general form
\[
x = (n_1, \ldots, n_r, x_{r+1}, \ldots, x_{s-1}, 0, \ldots, 0) \subseteq \left(0, \ldots, 0, x_{s'-1}, \ldots, x'_{r+1}, n'_{r'}, \ldots, n'_{s'}\right)
\]

A. **Left-to-right and right-to-left activation sequences**

The key property of coordinators is their joint switching to the same strategy: if a coordinator tends to switch
to A (resp. B), so do all other coordinators with a lower (resp. higher) temper. Consequently, starting from any initial condition, if all coordinators become active in the ascending order of their tempers, we will reach a state with a benchmark temper such that all coordinators with lower tempers play A and all coordinators with higher tempers play B. Formally, let $\overline{c} : \mathcal{X} \to \mathcal{X}$ be the function that maps the state $y \in \mathcal{X}$ to the resulting state after we consecutively update first all type-1, next all type-2, ..., and finally all type-$b'$ coordinators. That is, $\overline{c}(y) = x(t = \sum_{k=1}^{b'} n_k)$ when we start from $x(0) = y$ and follow the above activation sequence, which we refer to as the coordinating right-to-left activation (sequence). The state $\overline{c}(y)$ can be easily shown to take the structure

$$\overline{c}(y) = (\ast, \ldots, \ast, 0, 0, n'_1, n'_2, \ldots, n'_i),$$

where $i = \overline{c}^{-1}(y) = \overline{c}(y) - 1$. A similar behavior is seen when the coordinators become active in the descending order of their tempers. Correspondingly, we define the function $\overline{c}$ similar to $\overline{c}$ but when first all type-$b'$, next all type-$(b' - 1)$, ..., and finally all type-1 coordinators become active, which we refer to as the coordinating left-to-right activation.

The anticoordinators, however, do not exhibit such a conjoint switching to the same strategy. Let $\widehat{a} : \mathcal{X} \to \mathcal{Z}_{\geq 0}$ with $\widehat{a}(y) = A(x(\sum_{k=1}^{b} n_k))$ when we start from $x(0) = y$ and follow this activation sequence, which we refer to as the anticoordinating right-to-left activation. Clearly, $\overline{u}_i(y)$ then denotes the number of $A$-playing type-$i$ anticoordinators at the final state. For $i \in \{1, \ldots, b\}$, let $\overline{A}^i : \mathcal{X} \to \mathcal{Z}_{\geq 0}$ with $\overline{A}^i(y) = A(x(\sum_{k=1}^{b} n_k))$ given $x(0) = y$ and under the above activation sequence; that is, the total number of $A$-players after we consecutively update all type-$b$, type-$b_1 - 1$, ..., type-$i$ and none of the remaining anticoordinators in $y$.

The following result is straightforward.

**Lemma 1.** Let $x \in \mathcal{X}$. Under the anticoordinating right-to-left activation, for each $i \in \{1, \ldots, \overline{a}^b(x) - 1\}$, one or both of the followings hold:

$$\overline{u}_i(x) = n_i \quad \text{or} \quad \overline{A}^i(x) = \lfloor \tau_i \rfloor + 1.$$

We define the function $\overline{a}$ similar to $\overline{u}$, but when first all type-1, next all type-2, ..., and finally all type-$b$ anticoordinators become active, which we refer to as the anticoordinating left-to-right activation. The function $\overline{A}^i, i \in \{1, \ldots, b\}$, is defined correspondingly, and the following result is straightforward.

**Lemma 2.** Let $x \in \mathcal{X}$. Under the anticoordinating left-to-right activation, for each $i \in \{1, \ldots, \overline{a}^b(y) + 1, \ldots, b\}$, one or both of the followings hold:

$$\overline{d}_i(x) = 0 \quad \text{or} \quad \overline{A}^i(x) = \lfloor \tau_i \rfloor + 1.$$

It proves useful to also define the function $\overline{a}^i : \mathcal{X} \to \mathcal{X}, i \in \{1, \ldots, b\}$, similar to $\overline{a}$ but when we update only type-$i$, type-$(i + 1)$, ..., type-$b$ anticoordinators in the ascending order, which we refer to as the anticoordinating left-to-right activation starting from type $i$.

**B. Identifying positively invariant sets**

We define the indices $\zeta$, $\eta$, $\eta'$, and $\zeta'$ and provide the intuition behind the definitions, but note that the indices are rigorously defined and do not depend on what we claim to be intuitively true. As mentioned in Section III, the indices $\zeta, \eta, \eta'$, and $\zeta'$ are related recursively. This hinders any of the indices to be defined independently from the others. The difficulty is due to the fact that index $\zeta$, for example, is the greatest $A$-fixed anticoordinating type. If, instead, we search over every arbitrary (yet possibly close to the greatest) $A$-fixed anticoordinating type $\zeta$, it no longer has to depend on the other benchmarks, or may only depend on a single benchmark. More specifically, given $\zeta \in \{0, \ldots, b\}$ and $\delta \in \{0, \ldots, b'\}$, define the state

$$y^{\zeta, \delta} \triangleq (n_1, \ldots, n_\zeta, 0, 0, \ldots, 0, n'_1, n'_2, \ldots, n'_{b'}).$$

We approximate $\zeta'$ by $\delta$ and later tighten it up. Representing $A$-fixed agents, all type-1, ..., type-$\zeta$ anticoordinators must tend to play $A$ at the state $y^{\zeta, \delta}$, because the total number of $A$-players in the invariant set is always non-less than the number of $A$-players at $y^{\zeta, \delta}$. Now, although restrictive, we also force all type-1, ..., type-$\delta$ coordinators to tend to play $A$ at $y^{\zeta, \delta}$, a cost we pay to obtain the approximation. Therefore, given $\zeta \in \{0, \ldots, b\}$, we define the set of “acceptable” $\delta$’s by

$$\Delta(\zeta) \triangleq \{\delta \in \{0, \ldots, b\} \mid \tau_0 + 1 \leq \sum_{i=1}^{\zeta} n_i + \sum_{i=1}^{\delta} n'_i \leq \tau_{\zeta+1}\},$$

where we define $\tau_0 = -2$ and $\tau_0 = n$. The $A$-fixed anticoordinators should be resistant to not only the minimum number of $A$-players in the invariant set, but also the maximum. The maximum number of $A$-players that the solution trajectory can reach from $y^{\zeta, \delta}$ is obtained by applying an anticoordinating right-to-left activation to reach $\overline{a}(y^{\zeta, \delta})$, followed by a coordinating right-to-left activation to reach $\overline{c}(\overline{a}(y^{\zeta, \delta}))$. We then expect all type-1, ..., type-$\zeta$ anticoordinators to play $A$, implying $\overline{c}(\overline{a}(y^{\zeta, \delta})) \leq \tau_{\zeta+1}$ in view of Lemma 3. Moreover, we want $\zeta$ to be the maximum type that satisfies the inequality, implying $\tau_{\zeta+1} \leq \overline{c}(\overline{a}(y^{\zeta, \delta}))$. Thus, we end up at the following “acceptable” pairs of $(\zeta, \delta)$:

$$\Psi \triangleq \{(\zeta, \delta) \mid \zeta \in \{0, \ldots, b\}, \delta \in \Delta(\zeta),\,$$

$$\tau_{\zeta+1} < \overline{c}(\overline{a}(y^{\zeta, \delta})) \leq \tau_{\zeta+1} + 1\}. \quad (4)$$

Consider the case when $\Psi \neq \emptyset$, and let $(\zeta, \delta) \in \Psi$. Intuitively, the $B$-fixed anticoordinators, are those who did not switch their strategies to $B$ at the state $\overline{a}(y^{\zeta, \delta})$ under the above procedure. We, therefore, define the benchmark $\eta$ by

$$\eta \triangleq \overline{a}^b(y^{\zeta, \delta}). \quad (5)$$
We drop the argument of the functions when it is clear from the context, e.g., we use \( \eta \) instead of \( \eta(\zeta, \delta) \). We later show that the number of \( A \)-playing anticoordinators in the invariant set does not exceed that in \( \tilde{y}^{\zeta, \delta} \triangleq \tilde{c}(\tilde{a}(y^{\zeta, \delta})) \), which is in the form of

\[
\begin{pmatrix}
\eta, \ldots, \eta, 0, \ldots, 0, \ldots, 0, n_{\eta', \ldots, n_{\eta'}}
\end{pmatrix}_{\text{anticoordinating}}^{\text{coordinating}}.
\]

Consequently, the maximum number of \( A \)-playing coordinators appears in \( \tilde{y}^{\zeta, \delta} \triangleq \tilde{c}(\tilde{a}(y^{\zeta, \delta})) \). Hence, we define

\[
\eta' \triangleq \tilde{c}(\tilde{a}(y^{\zeta, \delta})).
\]

The state \( \tilde{y}^{\zeta, \delta} \) will, therefore, take the form

\[
\begin{pmatrix}
\eta, \ldots, \eta, 0, \ldots, 0, \ldots, 0, n_{\eta', \ldots, n_{\eta'}}
\end{pmatrix}_{\text{anticoordinating}}^{\text{coordinating}}.
\]

To find the other coordinating benchmark type \( \zeta' \), we consider the state where all wandering agents are playing \( A \):

\[
z^{\zeta, \delta} \triangleq \begin{pmatrix}
n_1, \ldots, n_{\zeta}, 0, \ldots, 0, \ldots, 0, n_{\eta', \ldots, n_{\eta'}}
\end{pmatrix}_{\text{anticoordinating}}^{\text{coordinating}}
\]

and perform an anticoordinating left-to-right activation starting from type \( \zeta + 1 \) to type \( \zeta' \), which takes the following form:

\[
\begin{pmatrix}
\eta, \ldots, \eta, 0, \ldots, 0, \ldots, 0, n_{\eta', \ldots, n_{\eta'}}
\end{pmatrix}_{\text{anticoordinating}}^{\text{coordinating}}.
\]

The \( A \)-fixed coordinators will not switch to \( B \) even when the population reaches its minimum number of \( A \)-players. On the other hand, we later show that the number of \( A \)-playing anticoordinators in the invariant set does not fall short of that at \( \tilde{y}^{\zeta, \delta} \). Hence, the minimum number of \( A \)-playing coordinators is obtained by performing a coordinating right-to-left activation to reach \( \tilde{z}^{\zeta, \delta} \triangleq \tilde{c}(\tilde{z}^{\zeta, \delta}) \), which takes the following form:

\[
\begin{pmatrix}
\eta, \ldots, \eta, 0, \ldots, 0, \ldots, 0, n_{\eta', \ldots, n_{\eta'}}
\end{pmatrix}_{\text{anticoordinating}}^{\text{coordinating}}.
\]

The following is the main result of the paper. Recall that \( \Psi \) is defined in (4), based on which \( \eta, \eta' \), and \( \zeta' \) are defined in (5) to (7). Correspondingly, the set \( I_{\zeta, \eta, \eta', \zeta'} \) and hence, \( I_{\zeta, \delta} \) is defined in (8).

**Theorem 1.** Given \((\zeta, \delta) \in \Psi\), the set \( I_{\zeta, \delta} \) is positively invariant.

The proof is presented in [16], which we skip here due to the space limit.

**APPENDIX**

The following results are straightforward. We simplify the notation \( A(x(t)) \) to \( A(t) \).

**Lemma 3.** A type-\( i \) anticoordinator playing \( A \) tends to play \( B \) (equivalently, does not tend to play \( B \)) at time \( t + 1 \) iff \( A(t) \leq \tau_i + 1 \) and tends to play \( B \) iff \( A(t) > \tau_i + 1 \). A type-\( i \) coordinating anticoordinator playing \( B \) tends to play \( B \) iff \( A(t) > \tau_i \), and tends to play \( k \) iff \( A(t) \leq \tau_k \). A type-\( i \) coordinating anticoordinator playing \( k \) tends to play \( k \) iff \( A(t) \geq \tau_k + 1 \), and tends to play \( B \) iff \( A(t) < \tau_k + 1 \). Type-\( i \) coordinating anticoordinator playing \( B \) tends to \( B \) if \( A(t) \geq \tau_k' \), and tends to play \( B \) if \( A(t) < \tau_k' \).

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