EXPLORING THE 4D SUPERCONFORMAL ZOO

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We discuss a new constraint for determining the superconformal $U(1)_R$ symmetry of 4d $N=1$ SCFTs: It is the unique one which locally maximizes $a(R) \equiv 3TrR^3 - TrR$. This constraint comes close to proving the conjectured “a-theorem” for $N=1$ SCFTs. Using this “a-maximization”, exact results can now be obtained for previously inaccessible 4d $N=1$ SCFTs. We apply this method to a rich class of examples: 4d $N=1$ SQCD with added matter chiral superfields in the adjoint representation. We classify a zoo of SCFTs, finding that Arnold’s ADE singularity classification arises in classifying these theories via all possible relevant Landau-Ginzburg superpotentials. We verify that all RG flows are indeed compatible with the “a-theorem” conjecture, $a_{IR} < a_{UV}$, in every case.

There is an intuitive picture of renormalization group (RG) flows from the UV to the IR as being like the flows of water down mountains, through valleys, and eventually ending up in lakes. The water height corresponds to the number of “degrees of freedom” of the theory, which is decreasing because of the coarse graining removal of UV degrees of freedom in RG flow to the IR. Zamolodchikov made this intuition precise in 2d by defining a $c$-function which counts the number of degrees of freedom of the theory, and proving that it monotonically decreases along RG flows to the IR. In the far IR, the RG can flow to a fixed point, which is a conformal field theory (CFT). The $c$-function is stationary at the CFT endpoint of the flow, and in the 2d case it becomes the central charge $c$ of the Virasoro algebra.

The intuition about course graining suggests that such a monotonically decreasing $c$-function should exist for RG flows in any space-time dimension, but a definitive proof above 2d has been elusive. Cardy has suggested that an appropriate central charge for 4d CFTs (i.e. only the endpoints of the RG flow) is the coefficient, conventionally called “a”, of the conformal anomaly $\langle T^\mu_\mu \rangle$ of the theory on a curved space-time background such as $S^4$.

On a general background $\langle T^\mu_\mu \rangle \sim a(Euler) + c(Weyl^2)$. It is known from counterexamples that the coefficient “c” of the Weyl curvature squared term does not obey a 4d analog
This proposal is the conjectured 4d “a-theorem”, that all 4d RG flows have $a_{IR} < a_{UV}$. All known 4d RG flows are compatible with this conjecture\(^b\).

We’ll discuss some new RG flows and fixed points which provide additional striking, non-trivial evidence for its validity.

4d asymptotically free gauge theories have $g(\mu) \to 0$ in the extreme UV, i.e. free quarks and gluons, with $g$ increasing in the flow to the IR. Eventually, non-perturbative phenomena can take over, e.g. confinement and a mass gap, and the theory in the far IR can be a free field theory of composites, e.g. pions, or have no massless degrees of freedom at all. Another possibility is a non-trivial IR fixed point, an interacting CFT (no mass gap). This happens if the RG flow drives the coupling $g$ to a critical value $g_*$, where the beta functions vanish, $\beta(g_*) = 0$. A scenario for realizing this phase was argued for long ago\(^5,6\): if the theory is just barely asymptotically free, $\beta(g)$ can have a non-trivial zero, $\beta(g_*) = 0$, for $g_* \neq 0$ which is sufficiently weak to justify the perturbative analysis.

Interacting 4d CFTs were once thought to be very rare and exotic, perhaps only occurring in contrived examples. However, we now know of many such interacting RG fixed points. In this talk, we’ll mention some new ones, which we’ve recently found and explored.

For the case of supersymmetric gauge theories, some exact results can be obtained, giving better insight into their non-perturbative dynamics. These studies have dovetailed with the use of other non-perturbative methods, which do not rely on supersymmetry, to explore more general strongly interacting gauge theories. A lesson learned from these studies is that the non-Abelian Coulomb phase exists somewhat generically, provided that there are sufficiently many massless matter fields (but not too many so as to spoil asymptotic freedom).

For example, in $N = 1$ supersymmetric QCD, Seiberg\(^7\) argued that the theory flows to an interacting 4d $N = 1$ SCFT for $N_f$ in the range $\frac{4}{2}N_c < N_f < 3N_c$. More generally, we suspect that $N = 1$ SCFTs are fairly generic for matter content in the range $T(G) < T(Matter) < 3T(G)$ of the $c$-theorem: there are RG flows with $c_{IR} > c_{UV}$\(^3\).

\(^{a}\)An upper conjectured $c$-function candidate is the coefficient of $T^4$ in the free-energy per unit volume of the theory at finite temperature\(^4\). This has some advantages over the conformal anomaly: it is not restricted to even space-time dimensions, and it places tighter bounds on the maximum number of massless flavors where chiral symmetry breaking can occur. But it has the disadvantage that it can only be readily computed in free field theories. In particular, supersymmetry does not help to compute it, so we will not discuss this thermal $c$-function candidate further here.
(there is also the more exotic possibility of the free-magnetic phase, as seen in SQCD\(^7\); see e.g. \(^9\) for a review). Here \(T(\text{Matter})\) and \(T(G)\) are the quadratic Casimirs of the matter and adjoint representations of the gauge group \(G\), with \(T(\text{Matter}) < 3T(G)\) for asymptotic freedom and \(T(\text{Matter}) > T(G)\) to avoid dynamically generated superpotentials, which would spoil conformal invariance.

Supersymmetry relates the stress tensor \(T_{\mu\nu}\) to a \(U(1)_R\) current \(R_\mu\); they are both in a current supermultiplet \(T_\alpha^{\dot{\alpha}}\) whose first component is the \(U(1)_R\) current (\(\alpha\) and \(\dot{\alpha}\) are spinor indices). Let’s call the charge associated with this \(U(1)_R\) current \(R_T\). Away from RG fixed points, \(R_T\) isn’t really conserved: its non-conservation is related by supersymmetry to non-zero \(T_{\mu\mu}\) and lack of scale invariance. As the theory RG flows to a SCFT, \(R_T \to R_*=\) which is conserved. 4d SCFTs necessarily have a conserved \(U(1)_R\) symmetry, with \(U(1)_R \subset SU(2,2|1)\), the 4d superconformal group. We refer to this \(U(1)_R\) as the superconformal R-symmetry. The supersymmetry relation \(R_\mu \leftrightarrow T_{\mu\nu}\) implies many powerful results, for example:

- \(\Delta(O) \geq \frac{3}{2}|R_*(O)|\) for all gauge invariant spin zero operators \(O\), with \(\Delta(O)\) the exact operator dimension and \(R_*(O)\) the operator’s \(U(1)_R\) charge. Chiral primary operators have \(\Delta(O) = \frac{3}{2}R_*(O)\), and additivity of the \(U(1)_R\) charge for composite operators implies that they form a closed OPE ring, with additive operator dimensions. A unitarity bound, \(\Delta(O) \geq 1\) for spin zero operators, implies that \(R_*(O) \geq \frac{2}{3}\) for spin zero chiral primary operators, with \(R_*(O) = \frac{2}{3}\) if and only if \(O\) is a decoupled free field.
- \(R_*\) is anomaly free precisely if the exact NSVZ\(^10\) beta function vanishes, \(\beta_{NSVZ}(g_*) = 0\), as is appropriate for a CFT.
- The conformal anomalies \(a\) and \(c\) can be exactly related\(^3\) to the superconformal R-symmetry’s ‘t Hooft anomalies:
  \[
  a = \frac{3}{32}(3\text{Tr}R^2_* - \text{Tr}R_*) \\
  c = \frac{1}{32}(9\text{Tr}R^2_* - 5\text{Tr}R_*).
  \]

This is extremely powerful, because ‘t Hooft anomaly matching\(^8\) implies that such ‘t Hooft anomalies are constant along the RG flows\(^c\) and can thus be evaluated in the weakly coupled UV limit. In what follows, we will rescale \(a\) to eliminate the factor of 3/32.

As an example, consider \(SU(N_c)\) SQCD, with \(N_f\) flavors in the SCFT range\(^7\) \(\frac{3}{2}N_c < N_f < 3N_c\). This theory has a unique anomaly free \(U(1)_R\)

\(^c\)Here we are supposing the the \(R_*\) symmetry is conserved along the RG flow, rather than being an accidental symmetry of the IR fixed point. Away from the IR fixed point, \(R_*\) differs from the R-current in the \(T_{\alpha\dot{\alpha}}\) supermultiplet.
symmetry which commutes with all of the other flavor symmetries: the R-charges of the fundamentals are \( R(Q_i) = R(\tilde{Q}_i) = 1 - (N_c/N_f) \). This \( U(1)_R \) is conserved along the entire RG flow, from \( g = 0 \) in the UV to the \( g_* \) in the IR where \( \beta(g_*) = 0 \), so it’s natural to suppose that the superconformal \( U(1)_R \) of the IR SCFT is this unique candidate (rather than being an accidental symmetry). This yields the exact operator dimensions of chiral primary operators via \( \Delta = \frac{3}{2} R(M) = 3 \left( 1 - \left( \frac{N_c}{N_f} \right) \right) \). And, using 't Hooft anomaly matching, we exactly compute \( a \) and \( c \), even at strongly interacting RG fixed points, using the spectrum of the UV free field theory.

More generally, to make use of the above powerful supersymmetry relations, one must be able to identify precisely which possible R-symmetry is the special one, \( R_* \), of the superconformal algebra. It is generally not uniquely fixed by the symmetries, as it was in the SQCD example. Indeed, if the theory has a large group \( \mathcal{F} \) of non-R global flavor symmetries, with charges \( F_I \), we can make a general R-symmetry by combining any initial R-symmetry, \( R_0 \), with any linear combination of the flavor symmetries:

\[
R_t = R_0 + \sum_I s_I F_I,
\]

with the \( s_I \) real parameters. The question, then, is to determine the particular values of the \( s_I \) which produce the special \( U(1)_R \subset SU(2,2|1) \).

We recently found a simple solution for this problem\(^\text{12}\); the superconformal R-symmetry is the unique choice of the trial R-symmetry, \( R_t \) as given above, which locally maximizes

\[
a_{\text{trial}}(s_I) = 3Tr R_t^3 - Tr R_t.
\]

The value of \( a_{\text{trial}} \) at this local maximum is then the central charge \( a \) of the SCFT, so we refer to this procedure for finding the R-charge as “\( a \)-maximization.” (Because \( a_{\text{trial}} \) is a cubic function, there is a unique local maximum, but no global maximum.)

We proved the above by showing that the \( T_{\mu\nu} \leftrightarrow R_\mu \) supersymmetry relation implies that \( 9Tr R^2 F_I = Tr F_I \), from which it follows that the superconformal \( U(1)_R \) extremizes \( a_{\text{trial}} \), and also \( Tr R_* F_I F_J < 0 \), which implies that the extremum is a local maximum. The relation \( 9Tr R^2 F_I = Tr F_I \) is obtained by relating the \( F_I R^2 F_J \) three-current triangle anomaly to another triangle anomaly, with the \( R_* \) currents replaced by stress tensors. The relation \( Tr R_* F_I F_J < 0 \) comes from relating the \( F_IF_J R_* \) three-current triangle anomaly to the \( F_I, F_J, \) stress tensor three-point function, which is then related to the \( F_I, F_J \) current-current two point function by a general
relation for 4d CFTs. A unitarity condition, fixing the sign of the current-current two-point function, then implies $\text{Tr} R^*_i F_i F_j < 0$.

As a simple example to illustrate the a-maximization procedure, consider a free chiral superfield $\Phi$, with trial R-charge $r$. We compute $a_{\text{trial}} = 3(r-1)^3 - (r-1)$, which has a unique local maximum at $r = 2/3$, the correct value for a free field: $\Delta = 3R/2 = 1$. a-maximization for interacting theories is essentially just as simple as this free-field example, the only difference being that some constraints associated with the interactions must be imposed. For example, gauge interactions simply have the effect of imposing the condition that the R-symmetry must be anomaly free.

a-maximization also suggests a simple proof of the conjectured a-theorem, at least in the supersymmetric context. Consider a general RG flow, from some CFT in the UV to another CFT in the IR. Often the flavor symmetry group of the IR theory is a subgroup of that of the UV theory, $F_{\text{IR}} \subset F_{\text{UV}}$, because the relevant deformations of the UV CFT broke some of the flavor symmetries. It then follows from a-maximization that $a_{\text{IR}} < a_{\text{UV}}$, simply because maximizing over a subset leads to a smaller value. This proof is not fully general, because there are examples where $F_{\text{IR}}$ is not a subset of $F_{\text{UV}}$, because of accidental symmetries. In all such examples, $a_{\text{IR}} < a_{\text{UV}}$ is still satisfied, but this loophole in the proof needs filling. Another concern is that perhaps maximizing over a subspace need not lead to a smaller value, because it’s only a local maximum. This latter caveat can be dispensed with by a recent work by Kutasov, where the constraints associated with $F_{\text{IR}} \subset F_{\text{UV}}$ are imposed with Lagrange multipliers $\lambda$ and it’s shown that $a(\lambda)$ is monotonically decreasing between the UV and IR endpoints.

We also note that a-maximization always yields the R-charges as solutions of quadratic equations with rational coefficients. Thus, for any 4d $\mathcal{N} = 1$ SCFT, all of the operator R-charges, chiral primary operator dimensions, and central charges $a$ and $c$ are always algebraic numbers. Specifically, they’re solutions of quadratic equations with rational coefficients ("quadratic irrational numbers"). In particular, these quantities can’t depend on any continuous moduli.

Finally, a general caution: we must maximize $a_{\text{trial}} = 3\text{Tr} R^*_i R_i - \text{Tr} R_i$ over the complete space of all possible trial R-symmetries, including all accidental symmetries. One situation where accidental symmetries are readily apparent, and required, is when a gauge invariant chiral primary operator, e.g. $M = \bar{Q}Q$, hits or appears to violate the unitarity bound, $R(M) \geq 2/3$. $M$ then becomes a free field, with an accidental symmetry, $J_M$, under
which only $M$ is charged. The trial $R_t$ must then include mixing with $J_M$: $R_{t,new} = R_{t,old} + s_M J_M$, with the new parameter $s_M$ again fixed by maximizing $a_{\text{trial}}$. This has an important effect, leading to an additive correction to the quantity $a_{\text{trial}}$ to be maximized w.r.t. the other parameters $s_I$, as was first discussed by Kutasov, Parnachev, and Sahakyan (KPS)\(^\text{14}\).

$a$-maximization can be used to study previously mysterious theories. As an example, consider $SU(N_c)$ SQCD with $0 < N_f < 2N_c$ fundamental flavors, along with an extra matter chiral superfield, $X$, in the adjoint representation. This theory, with $W_{\text{tree}} = 0$, is believed to flow in the IR to an interacting $N = 1$ SCFT. This was argued for in\(^\text{9}\) by turning on $W_{\text{tree}} = \lambda \tilde{Q} X$, finding massless monopoles and dyons on submanifolds of a Coulomb branch, and noting that these non mutually-local fields all become massless at the origin upon taking $\lambda \to 0$. We refer to these IR SCFTs as the $\hat{A}(N_c, N_f)$ theories, or simply $\hat{A}$. We can also consider the theory with added superpotential $W_{\text{tree}} = Tr X^{k+1}$ which, for particular ranges of $N_f$ (depending on $k$), can drive a RG flow to different SCFTs in the IR, which we’ll refer to as the $A_k$ SCFTs. Finding the $U(1)_R$ symmetry of the $\hat{A}$ SCFTs requires $a$-maximization, while that of the $A_k$ theory is determined by $W_{\text{tree}}$ and the symmetries.

Let’s outline the salient features of the $\hat{A}$ and $A_k$ theories, obtained by KPS\(^\text{14}\). Write the trial $U(1)_R$ as $R(Q_i) = R(\tilde{Q}_i) \equiv y$, $R(X) = (1-y)/x$, with $x \equiv N_c/N_f$. To simplify things, take $N_c$ and $N_f$ large, holding $x > \frac{1}{2}$ fixed. For $x \approx \frac{1}{2}$, the theory is just barely asymptotically free, and the RG fixed point is of the “Banks-Zaks” type\(^\text{5,6}\), at weak gauge coupling $g_\ast \ll 1$. As we increase $x$, the RG fixed point moves to larger and larger values of the gauge coupling $g_\ast$. $a$-maximization can be used to determine the exact $R$-charges $y(x)$, for all $x$. As expected, $y(x)$ and $R(X)$ decrease with $x$, corresponding to the negative anomalous dimensions of gauge interactions.

As $x$ increases, generalized mesons of the form $M_j = QX^j Q$ successively hit, and then appear to violate, the unitarity bound $R(O) \geq 2/3$. Each time this happens, there is an associated accidental flavor symmetry which must be included in the trial $R_t$, leading to added contributions to the quantity $a_{\text{trial}}$ which is to be maximized. Because of this, it’s best to let a computer solve the required $a$-maximizations, numerically obtaining $y(x)$. The $x \to \infty$ limit can be treated analytically, with the result that $R(Q) \to (\sqrt{3} - 1)/3 \approx 0.244$, and $R(X) \to 0$ in the limit\(^\text{14}\).

Suppose that we sit at the $\hat{A}$ RG fixed point, and then perturb the theory by adding a superpotential $W_{A_k} = \lambda Tr X^{k+1}$. This deformation is relevant if $\Delta(W_{\text{tree}}) < 3$, i.e. if $R(X) < 2/(k+1)$. Using the $a$-maximization...
results, it is seen that \( W_{A_k} \) can be relevant for any value of \( k \), arbitrarily large, provided that \( x \) is sufficiently large, \( x > x_{A_k}^{\text{min}} \), with \( x_{A_k}^{\text{min}} \) determined by solving \( R(X) = (1 - y(x_{A_k}^{\text{min}}))/x_{A_k}^{\text{min}} = 2/(k + 1) \). For any \( x > x_{A_k}^{\text{min}} \), \( W_{A_k} \) is a relevant deformation of the \( \hat{\Lambda} \) SCFT, driving a RG flow away from the \( \hat{\Lambda} \) SCFT to a new SCFT, which we call \( A_k \). The \( A_k \) SCFTs exist for \( x \) in the range \( x_{A_k}^{\text{min}} < x < x_{A_k}^{\text{max}} = k \), with \( x_{A_k}^{\text{max}} = k \) the stability bound. This \( x \) range is non-empty, \( x_{A_k}^{\text{min}} < x < x_{A_k}^{\text{max}} \), for all \( k \).

The results of KPS also clarified the meaning of the duality found in the asymptotically free range for all RG flows. E.g. we can flow from the UV to the IR as: free UV \( \to \hat{\Lambda} \to A_k \to A_{k'} \), with \( k' < k \). The a-theorem would then predict \( a_{\text{free IR}}(x) > a_{\hat{\Lambda}}(x) > a_{A_{k'}}(x) > a_{A_k}(x) \) had some apparent violations for small \( x \). But using the new a-maximization results for \( x_{A_k}^{\text{min}} \) it is now seen that there never actually is a violation of \( a_{\text{IR}} < a_{\text{UV}} \) in any of those examples. The apparent violation of \( a_{\text{IR}} < a_{\text{UV}} \) always occurred for \( x \) outside of the range \( x > x_{A_k}^{\text{min}} \) needed for the \( A_k \) SCFT to exist, so there’s no a-theorem violating flow after all.

In our work, we generalized these examples by considering SQCD with \( N_f \) fundamental flavors, \( Q_i \), and \( \bar{Q}_i \), along with \( N_a = 2 \) adjoint matter flavors, \( X \) and \( Y \). As in, considering deformations suggests that this theory, with \( W_{\text{tree}} = 0 \), flows to an interacting \( \mathcal{N} = 1 \) SCFT for all \( N_f \) in the asymptotically free range \( 0 \leq N_f < N_c \), which we write as \( x = N_c/N_f > 1 \). We’ll call these SCFTs \( \hat{O}(N_c, N_f) \), and again consider \( N_c \) and \( N_f \) large, with \( x \) fixed, to simplify things. For \( x = 1 \), the \( \hat{O} \) IR fixed point is at weak coupling, and as \( x \) increases, the \( \hat{O} \) IR fixed point is at stronger coupling. We use a-maximization to solve for the R-charges for all \( x \), taking the anomaly free trial R-symmetry to be: \( R_t(Q) = R_t(\bar{Q}) \equiv y \) and \( R_t(X) = R_t(Y) = \frac{1}{2}(1 + \frac{y}{x}) \). Plugging this into \( a_{\text{trial}} = 3 T \tau R_t^2 - T \tau R_t \) and maximizing w.r.t. \( y \) gives \( y(x) = 1 + (3(8x^2 - 1))^{-1}(3x - 2x\sqrt{26x^2 - 1}) \). We obtain a relatively simple closed form expression for \( y(x) \) for the \( \hat{O} \)
SCFTs because no operators hit the unitarity bound, so no additional contributions to \( a_{\text{trial}} \), associated with accidental symmetries are needed (though we can’t definitively rule them out). We thus find that \( R(Q) \) and \( R(X) \) and \( R(Y) \) are decreasing functions of \( x \), and for \( x \to \infty \) we obtain \( R(Q) \to 1 - \sqrt{26/12} \approx 0.575 \), and \( R(X) = R(Y) \to \frac{1}{2} \).

Using the above result, we can classify the relevant superpotential deformations of the \( \hat{O} \) SCFT, which could drive RG flows from \( \hat{O} \) to various new SCFTs in the IR. Let’s consider superpotentials involving only the adjoints (there are additional ones using operators which include the quarks). Using our result for the \( \hat{O} \) SCFT that \( R(X) = R(Y) > \frac{1}{2} \) for all \( x \), we see that \( W = \text{Tr} X^{k+2} Y^\ell \) is only a relevant deformation of the \( \hat{O} \) SCFT if \( k + \ell \leq 3 \). The complete list of such relevant superpotential deformations (modulo field redefinitions) of \( \hat{O} \) is thus \( \hat{W}_A = \text{Tr} Y^2 \), \( \hat{W}_D = \text{Tr} X Y^2 \), and \( \hat{W}_E = \text{Tr} Y^3 \). Each of these drives \( \hat{O} \) to new SCFTs, which we name \( \hat{A}, \hat{D}, \) and \( \hat{E} \) respectively, for all \( N_f \) in the range \( 0 < N_f < N_c \). The \( \hat{A} \) case is that considered by \( ^{14} \) and reviewed above.

We can now use \( a \)-maximization to analyze the new \( \hat{D} \) and \( \hat{E} \) SCFTs. E.g. for the \( \hat{D} \) RG fixed points, we impose \( R(X Y^2) = 2 \), and the trial \( U(1)_R \) charges are \( R(Q) = R(\hat{Q}) \equiv y, R(X) = 2(1-y)/x, R(Y) = 1 + (y-1)/x \). Here we find that operators do hit the unitarity bound, so their accidental symmetries must be accounted for in \( a_{\text{trial}} \), in analogy with the \( \hat{A} \) case of \( ^{14} \). Because of this, we solved for \( y(x) \) numerically. The asymptotic values in the limit \( x \to \infty \) are \( y \to -1/8, R(Y) \to 1, \) and \( R(X) \to 0 \).

Using these results for the \( \hat{D} \) SCFT, we can now classify its relevant superpotential deformations. Since \( R(X) \to 0 \) for \( x \to \infty \), we find that \( \Delta W = \text{Tr} X^{k+1} \) can be a relevant deformation of the \( \hat{D} \) SCFT, for arbitrarily large \( k \), provided that \( x \) is sufficiently large, \( x > x_{\text{min}}^{\hat{D}_{k+2}} \). We solved for \( x_{\text{min}}^{\hat{D}_{k+2}} \) by using our results for the \( \hat{D} \) SCFT to determine the \( x \) where \( R(X) = 2/(k+1) \): we find \( x_{\text{min}}^{\hat{D}_2} = 2.09, x_{\text{min}}^{\hat{D}_3} = 3.14, \) etc., with \( x_{\text{min}}^{\hat{D}_{k+2}} \to k/k \) for \( k \to \infty \). When \( x > x_{\text{min}}^{\hat{D}_{k+2}} \), the relevant \( \Delta W \) drives a RG flow from the \( \hat{D} \) SCFTs to new SCFTs, which we name \( \hat{D}_{k+2} \). More generally, we can get to the \( D_{k+2} \) SCFTs, provided that \( x > x_{\text{min}}^{\hat{D}_{k+2}} \), by perturbing \( \hat{O} \) by \( W_{\hat{D}_{k+2}} = \lambda_1 \text{Tr} X^{k+1} + \lambda_2 \text{Tr} X Y^2 \). (The \( D_{k+2} \) SCFT also must satisfy \( x < x_{\text{max}}^{\hat{D}_{k+2}} = 3k \), to prevent generating a \( \hat{W}_{\text{dyn}} \).)

Continuing this process of using \( a \)-maximization to analyze each SCFTs, classifying their relevant superpotential deformations, and then using these deformations to flow to new SCFTs, we obtain a classification of the SCFTs that are obtainable from \( \hat{O} \) via the following superpotential deformations.
(again, there are additional ones making use of operators involving the quarks): $W_\hat{O} = 0$, $W_\hat{A} = \text{Tr}Y^2$, $W_\hat{D} = \text{Tr}XY^2$, $W_\hat{E} = \text{Tr}Y^3$, $W_{A_k} = \text{Tr}(X^{k+1} + Y^2)$, $W_{D_{k+2}} = \text{Tr}(X^{k+1} + XY^2)$, $W_{E_6} = \text{Tr}(Y^3 + X^4)$, $W_{E_7} = \text{Tr}(Y^3 + YX^3)$, and $W_{E_8} = \text{Tr}(Y^3 + X^5)$. Each of these LG superpotentials drives a RG flow to a corresponding interacting SCFT in the IR, with the $A_k$, $D_{k+2}$, $E_6$, $E_7$, and $E_8$ cases existing only if $x$ is sufficiently large.

We note that the above superpotential classification agrees with Arnold’s singularity classification. That classification also appeared, among other contexts, in the classification of 2d SCFTs having $c < 3^{23,24}$. The appearance of Arnold’s ADE in our 4d case is a new guise, which perhaps has a deeper connection to other occurrences of the ADE series in string theory and mathematics. This is a topic for future exploration.

The map of the possible flows between these SCFTs is:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{The map of possible flows between fixed points. Dotted lines indicate flow to a particular value of $k$.}
\end{figure}

Every arrow in the diagram corresponds to a RG flow, from a SCFT in the UV to another SCFT in the IR. For each of these flows, we used $a$-maximization to exactly compute $a_{UV}$ and $a_{IR}$, and we verified that $a_{IR} < a_{UV}$ is indeed satisfied for every such RG flow. E.g. $a_{E_6}(x) < a_{E_7}(x) < a_{E_8}(x) < a_{\hat{O}}(x) < a_{\hat{E}}(x) < a_{\text{free 
UV}}(x)$. There are some apparent violations for small $x$, e.g. it appears that $a_{E_7}(x) < a_{E_8}(x)$ for $x < 3.16$, but
but there is actually no violation of the a-theorem conjecture afterall, because we find that that $E_7$ RG fixed point only exists if $x > x_{E_7}^{\text{min}} \approx 4.12$. Likewise, all other such potential violations occur outside of the range where the RG flow can exist.

Our new results also give some insight into a proposed duality $^{18}$ for the theory with $W_{D_{k+2}} = \text{Tr}(X^{k+1} + XY^2)$, relating the original “electric” $SU(N_c)$ theory to a “magnetic” $SU(3kN_f - N_c)$ dual. We find different phases depending on $x \equiv N_c/N_f$. The duality is most useful in the $D_{k+2}$ conformal window: $x_{D_{k+2}}^{\text{min}} < x < 3k - x_{D_{k+2}}^{\text{min}}$, where $W_{D_{k+2}}$ is relevant in both the electric and magnetic theories. It’s interesting that this range is non-empty for all $k$, e.g. for large $k$ we find $x_{D_{k+2}}^{\text{min}} \approx \frac{9}{8} k$ and $3k - x_{D_{k+2}}^{\text{min}} \approx 1.8962k$. Both dual descriptions are good in the conformal window. Outside of the conformal window, the theory can instead flow to the electric or magnetic $\hat{D}$ SCFT, and the other dual description is then not useful.

The a-maximization methods used here can be applied to many other 4d supersymmetric gauge theories, and there are many other superconformal zoos remaining to be explored.

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