Simply transitive geodesic ball packings to \( S^2 \times \mathbb{R} \) space groups generated by glide reflections

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Abstract

The \( S^2 \times \mathbb{R} \) geometry can be derived by the direct product of the spherical plane \( S^2 \) and the real line \( \mathbb{R} \). In \cite{1} J. Z. Farkas has classified and given the complete list of the space groups of \( S^2 \times \mathbb{R} \). The \( S^2 \times \mathbb{R} \) manifolds were classified by E. Molnár and J. Z. Farkas in \cite{2} by similarity and diffeomorphism. In \cite{12} we have studied the geodesic balls and their volumes in \( S^2 \times \mathbb{R} \) space, moreover we have introduced the notion of geodesic ball packing and its density and have determined the densest geodesic ball packing for generalized Coxeter space groups of \( S^2 \times \mathbb{R} \).

In this paper we study the locally optimal ball packings to the \( S^2 \times \mathbb{R} \) space groups having Coxeter point groups and at least one of the generators is a glide reflection. We determine the densest simply transitive geodesic ball arrangements for the above space groups, moreover we compute their optimal densities and radii.

The density of the densest packing is \( \approx 0.80407553 \), may be surprising enough in comparison with the Euclidean result \( \frac{\pi}{\sqrt{18}} \approx 0.74048 \). E.

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Molnár has shown in [4], that the homogeneous 3-spaces have a unified interpretation in the real projective 3-sphere $\mathcal{P}S^3(V^4, V_4, \mathbb{R})$. In our work we shall use this projective model of $S^2 \times \mathbb{R}$ geometry.

1 Introduction

$S^2 \times \mathbb{R}$ is derived as the direct product of the spherical plane $S^2$ and the real line $\mathbb{R}$. The points are described by $(P, p)$ where $P \in S^2$ and $p \in \mathbb{R}$. The isometry group $Isom(S^2 \times \mathbb{R})$ of $S^2 \times \mathbb{R}$ can be derived by the direct product of the isometry group of the sphere $Isom(S^2)$ and the isometry group of the real line $Isom(\mathbb{R})$.

\begin{equation}
Isom(S^2) := \{ A \in O(3) : S^2 \mapsto S^2 : (P, p) \mapsto (PA, p) \} \text{ for any fixed } p.
\end{equation}

\begin{equation}
Isom(\mathbb{R}) := \{ \rho : (P, p) \mapsto (P, \pm p + r) \}, \text{ for any fixed } P.
\end{equation}

here the "-" sign provides a reflection in the point $r/2 \in \mathbb{R}$, by the "+" sign we get a translation of $\mathbb{R}$.

The structure of an isometry group $\Gamma \subset Isom(S^2 \times \mathbb{R})$ is the following: $\Gamma := \{(A_1 \times \rho_1), \ldots, (A_n \times \rho_n)\}$, where $A_i \times \rho_i := A_i \times (R_i, r_i) := (g_i, r_i), \ (i \in \{1, 2, \ldots n\}$ and $A_i \in Isom(S^2)$, $R_i$ is either the identity map $1_\mathbb{R}$ of $\mathbb{R}$ or the point reflection $T_\mathbb{R}$. $g_i := A_i \times R_i$ is called the linear part of the transformation $(A_i \times \rho_i)$ and $r_i$ is its translation part. The multiplication formula is the following:

\begin{equation}
(A_1 \times R_1, r_1) \circ (A_2 \times R_2, r_2) = ((A_1A_2 \times R_1R_2, r_1R_2 + r_2).
\end{equation}

**Definition 1.1** A group of isometries $\Gamma \subset Isom(S^2 \times \mathbb{R})$ is called space group if the linear parts form a finite group $\Gamma_0$ called the point group of $\Gamma$, moreover, the translation parts to the identity of this point group are required to form a one dimensional lattice $L_{\Gamma}$ of $\mathbb{R}$.

**Remark 1.2**

1. It can be proved that the space group $\Gamma$ has a compact fundamental domain $\mathcal{F}_\Gamma$.

2. If $\Gamma$ is not assumed to have a lattice, then it may have an infinite point group $\Gamma_0$. 


Definition 1.3 The $S^2 \times \mathbb{R}$ space groups $\Gamma_1$ and $\Gamma_2$ are geometrically equivalent, called equivariant, if there is a "similarity" transformation $\Sigma := S \times \sigma (S \in Isom(S^2), \sigma \in Sim(\mathbb{R}))$, such that $\Gamma_2 = \Sigma^{-1} \Gamma_1 \Sigma$. Here $\sigma(s, t) : p \mapsto p \cdot s + t$ is a similarity of $\mathbb{R}$, i.e. multiplication by $0 \neq s \in \mathbb{R}$ and then addition by $t \in \mathbb{R}$ for every $p \in \mathbb{R}$.

Remark 1.4 If $\Gamma_1$ and $\Gamma_2$ are equivariant space groups then the their factor groups $\Gamma_1/L_{\Gamma_1}$ and $\Gamma_2/L_{\Gamma_2}$ are also equivariant.

Thus the structure of the space group remains invariant under a similarity in the $\mathbb{R}$-component and the spherical part is uniquely determined up to an isometry of $S^2$.

We characterize the spherical plane groups by the Macbeath-signature (see [3], [12]).

In this paper we deal with such a $S^2 \times \mathbb{R}$ space group where the generators $g_i$, $(i = 1, 2, \ldots m)$ of its point group $\Gamma_0$ are reflections and at least one of the possible translation parts of the above generators unequal to zero. These groups are called glide reflection groups.

Remark 1.5 In [12] we have introduced the notion of generalized Coxeter group, if the generators $g_i$, $(i = 1, 2, \ldots m)$ of its point group $\Gamma_0$ are reflections with translation parts $\tau_i = 0$, $(i = 1, 2, \ldots m)$.

In this paper we deal with the glide reflection space groups in $S^2 \times \mathbb{R}$ space which are by denotation of [1]:

1. $\{+, 0, [\ldots \{(q, q)\}\ldots]\} \times 1_{\mathbb{R}}, \ q \geq 2$, $\Gamma_0 = (g_1, g_2, g_3 - g_1^2, g_2^2, (g_2g_3)^2, (g_1g_2)^2), (g_1g_3)^2)$,
2q. I. 2: $(\frac{1}{2}, \frac{1}{2})$; 2qe. I. 3: $(0, \frac{1}{2})$;

2. $\{+, 0, [\ldots \{(2, 2, q)\}\ldots]\} \times 1_{\mathbb{R}}, \ q \geq 2$, $\Gamma_0 = (g_1, g_2, g_3 - g_1^2, g_2^2, g_3^2, (g_1g_3)^2, (g_2g_3)^2, (g_1g_2)^2), (g_1g_3)^2)$,
4q. I. 2: $(0, 0, 0, 0); 4q. I. 3: (\frac{1}{2}, \frac{1}{2}, 0); 4q. I. 4: (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; 4qe. I. 5: $(0, \frac{1}{2}, 0); 4qe. I. 6: (0, \frac{1}{2}, \frac{1}{2})$;

3. $\{+, 0, [\ldots \{(2, 3, 3)\}\ldots]\} \times 1_{\mathbb{R}}, \ \Gamma_0 = (g_1, g_2, g_3 - g_1^2, g_2^2, g_3^2, (g_1g_2)^2, (g_1g_3)^2, (g_2g_3)^3)$, 11. I. 2: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$;
4. \((+, 0, [ ] \{(2, 3, 4)\}) \times 1\mathbb{R}\),
\[
\Gamma_0 = (g_1, g_2, g_3 - g_1^2, g_2^2, g_3^2, (g_1g_2)^2, (g_1g_3)^3, (g_2g_3)^4),
\]
12. I. 2: \((0, \frac{1}{2}, 0)\); 12. I. 3: \((\frac{1}{2}, 0, \frac{1}{2})\); 12. I. 4: \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\);

5. \((+, 0, [ ] \{(2, 3, 5)\}) \times 1\mathbb{R}\),
\[
\Gamma_0 = (g_1, g_2, g_3 - g_1^2, g_2^2, g_3^2, (g_1g_2)^2, (g_1g_3)^3, (g_2g_3)^5),
\]
13. I. 2: \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\);

2 Geodesic curve and balls in \(S^2 \times \mathbb{R}\) space

E. Molnár has shown in [4], that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere \(\mathcal{P}S^3(\mathbb{V}_4, \mathbb{V}_4, \mathbb{R})\). In our work we shall use this projective model of \(S^2 \times \mathbb{R}\) and the Cartesian homogeneous coordinate simplex \(E_0(e_0), E_1^\infty(e_1), E_2^\infty(e_2), E_3^\infty(e_3), \{e_i\} \subset \mathbb{V}_4\) with the unit point \(E(e = e_0 + e_1 + e_2 + e_3)\) which is distinguished by an origin \(E_0\) and by the ideal points of coordinate axes, respectively. Moreover, \(y = cx\) with \(0 < c \in \mathbb{R}\) (or \(c \in \mathbb{R} \setminus \{0\}\)) defines a point \((x) = (y)\) of the projective 3-sphere \(\mathcal{P}S^3\) (or that of the projective space \(\mathcal{P}^3\) where opposite rays \((x)\) and \((-x)\) are identified). The dual system \(\{(e^i)\} \subset \mathbb{V}_4\) describes the simplex planes, especially the plane at infinity \((e^0) = E_1^\infty E_2^\infty E_3^\infty\), and generally, \(v = u_c^\perp\) defines a plane \((u) = (v)\) of \(\mathcal{P}S^3\) (or that of \(\mathcal{P}^3\)). Thus \(0 = xu = yv\) defines the incidence of point \((x) = (y)\) and plane \((u) = (v)\), as \((x)I(u)\) also denotes it. Thus \(S^2 \times \mathbb{R}\) can be visualized in the affine 3-space \(\mathbb{A}^3\) (so in \(\mathbb{E}^3\)) as well.

In this context E. Molnár [4] has derived the well-known infinitezimal arc-length square at any point of \(S^2 \times \mathbb{R}\) as follows
\[
(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{x^2 + y^2 + z^2}.
\]  
(2.1)

We shall apply the usual geographical coordinates \((\phi, \theta), (-\pi < \phi \leq \pi, -\frac{\pi}{2} < \theta \leq \frac{\pi}{2})\) of the sphere with the fibre coordinate \(t \in \mathbb{R}\). We describe points in the above coordinate system in our model by the following equations:
\[
x^0 = 1, \quad x^1 = e^t \cos \phi \cos \theta, \quad x^2 = e^t \sin \phi \cos \theta, \quad x^3 = e^t \sin \theta.
\]  
(2.2)

Then we have \(x = \frac{x^1}{x^0} = x^1, y = \frac{x^2}{x^0} = x^2, z = \frac{x^3}{x^0} = x^3\), i.e. the usual Cartesian coordinates. We obtain by [4] that in this parametrization the infinitezimal arc-length square at any point of \(S^2 \times \mathbb{R}\) is the following
\[
(ds)^2 = (dt)^2 + (d\phi)^2 \cos^2 \theta + (d\theta)^2.
\]  
(2.3)
The geodesic curves of $S^2 \times \mathbb{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $\gamma(t(\tau), \phi(\tau), \theta(\tau))$ in our model can be determined by the general theory of Riemann geometry (see [12]).

Then by (2.2) we get with $c = \sin v$, $\omega = \cos v$ the equation systems of a geodesic curve, visualized in Fig. 1 in our Euclidean model:

$$
\begin{align*}
x(\tau) &= e^\tau \sin v \cos (\tau \cos v), \\
y(\tau) &= e^\tau \sin v \sin (\tau \cos v) \cos u, \\
z(\tau) &= e^\tau \sin v \sin (\tau \cos v) \sin u, \\
-\pi < u &\leq \pi, \quad -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}.
\end{align*}
$$

(2.4)

**Remark 2.1** Thus we have harmonized the scales along the fibre lines.

**Definition 2.2** The distance $d(P_1, P_2)$ between the points $P_1$ and $P_2$ is defined by the arc length of the shortest geodesic curve from $P_1$ to $P_2$.

**Definition 2.3** The geodesic sphere of radius $\rho$ (denoted by $S_{P_1}(\rho)$) with centre at the point $P_1$ is defined as the set of all points $P_2$ in the space with the condition $d(P_1, P_2) = \rho$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in $S^2 \times \mathbb{R}$ space.

**Remark 2.4** We shall see that this last condition depends on radius $\rho$.

**Definition 2.5** The body of the geodesic sphere of centre $P_1$ and of radius $\rho$ in the $S^2 \times \mathbb{R}$ space is called geodesic ball, denoted by $B_{P_1}(\rho)$, i.e. $Q \in B_{P_1}(\rho)$ iff $0 \leq d(P_1, Q) \leq \rho$.

In [12] we have proved that $S(\rho)$ is a simply connected surface in $\mathbb{E}^3$ if and only if $\rho \in [0, \pi)$, because if $\rho \geq \pi$ then there is at least one $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ so that $y(\tau, v) = z(\tau, v) = 0$, i.e. selfintersection would occur (see (2.4)). Thus we obtain the following

**Proposition 2.6** The geodesic sphere and ball of radius $\rho$ exists in the $S^2 \times \mathbb{R}$ space if and only if $\rho \in [0, \pi]$.

We have obtained (see [12]) the volume formula of the geodesic ball $B(\rho)$ of radius $\rho$ by the metric tensor $g_{ij}$ and by the Jacobian of (2.4):
Theorem 2.7

$$Vol(B(\rho)) = \int_V \frac{1}{(x^2 + y^2 + z^2)^{3/2}} dx \, dy \, dz =$$

$$= \int_0^\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} |\tau \cdot \sin(\cos(v)\tau)|\, du \, dv \, d\tau =$$

$$= 2\pi \int_0^\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\tau \cdot \sin(\cos(v)\tau)|\, dv \, d\tau. \quad (2.5)$$

2.1 On fundamental domains

A type of the fundamental domain of a studied space group can be combined as a fundamental domain of the corresponding spherical group with a part of a real line segment. This domain is called $$S^2 \times \mathbb{R}$$ prism (see [12]). This notion will be important to compute the volume of the Dirichlet-Voronoi cell of a given space group because their volumes are equal and the volume of a $$S^2 \times \mathbb{R}$$ prism can be calculated by Theorem 2.8.

Figure 1: Prism-like fundamental domains

The $$p$$-gonal faces of a prism called cover-faces, and the other faces are the side-faces. The midpoints of the side edges form a ”spherical plane” denoted by $$\Pi$$. It can be assumed that the plane $$\Pi$$ is the base plane: in our coordinate system (see (2.2)) the fibre coordinate $$t = 0$$.

From [12] we recall

Theorem 2.8 The volume of a $$S^2 \times \mathbb{R}$$ trigonal prism $$\mathcal{P}_{B_0B_1B_2C_0C_1C_2}$$ and of a digonal prism $$\mathcal{P}_{B_0B_1C_0C_1}$$ in $$S^2 \times \mathbb{R}$$ (see Fig. 1.a-b) can be computed by the following
Geodesic ball packings in $S^2 \times \mathbb{R}$ space

formula:

$$\text{Vol}(P) = A \cdot h$$

(2.5)

where $A$ is the area of the spherical triangle $A_0 A_1 A_2$ or digon $A_0 A_1$ in the base plane $\Pi$ with fibre coordinate $t = 0$, and $h = B_0 C_0$ is the height of the prism.

3 Ball packings

By remark (1.2) a $S^2 \times \mathbb{R}$ space group $\Gamma$ has a compact fundamental domain. Usually the shape of the fundamental domain of a group of $S^2$ is not determined uniquely but the area of the domain is finite and unique by its combinatorial measure. Thus the shape of the fundamental domain of a crystallographic group of $S^2 \times \mathbb{R}$ is not unique as well.

In the following let $\Gamma$ be a fixed glide reflection space group of $S^2 \times \mathbb{R}$. We will denote by $d(X, Y)$ the distance of two points $X, Y$ by definition (2.2).

Definition 3.1 We say that the point set

$$D(K) = \{X \in S^2 \times \mathbb{R} : d(K, X) \leq d(K^g, X) \text{ for all } g \in \Gamma\}$$

is the Dirichlet–Voronoi cell (D-V cell) to $\Gamma$ around the kernel point $K \in S^2 \times \mathbb{R}$.

Definition 3.2 We say that

$$\Gamma_X = \{g \in \Gamma : X^g = X\}$$

is the stabilizer subgroup of $X \in S^2 \times \mathbb{R}$ in $\Gamma$.

Definition 3.3 Assume that the stabilizer $\Gamma_K = 1$ i.e. $\Gamma$ acts simply transitively on the orbit of a point $K$. Then let $B_K$ denote the greatest ball of centre $K$ inside the D-V cell $D(K)$, moreover let $\rho(K)$ denote the radius of $B_K$. It is easy to see that

$$\rho(K) = \min_{g \in \Gamma \setminus 1} \frac{1}{2} d(K, K^g).$$

The $\Gamma$-images of $B_K$ form a ball packing $B_K^\Gamma$ with centre points $K^G$.

Definition 3.4 The density of ball packing $B_K^\Gamma$ is

$$\delta(K) = \frac{\text{Vol}(B_K)}{\text{Vol}D(K)}.$$
It is clear that the orbit $K^\Gamma$ and the ball packing $B^\Gamma_K$ have the same symmetry group, moreover this group contains the starting crystallographic group $\Gamma$:

$$SymK^\Gamma = SymB^\Gamma_K \geq \Gamma.$$  

**Definition 3.5** We say that the orbit $K^\Gamma$ and the ball packing $B^\Gamma_K$ is characteristic if $SymK^\Gamma = \Gamma$, else the orbit is not characteristic.

### 3.1 Simply transitive ball packings

*Our problem is* to find a point $K \in S^2 \times \mathbb{R}$ and the orbit $K^\Gamma$ for $\Gamma$ such that $\Gamma_K = I$ and the density $\delta(K)$ of the corresponding ball packing $B^\Gamma(K)$ is maximal. In this case the ball packing $B^\Gamma(K)$ is said to be *optimal*.

The lattice of $\Gamma$ has a free parameter $p(\Gamma)$. Then we have to find the densest ball packing on $K$ for fixed $p(\Gamma)$, and vary $p$ to get the optimal ball packing.

$$\delta(\Gamma) = \max_{K, p(\Gamma)} (\delta(K)) \quad (3.1)$$

Let $\Gamma$ be a fixed *glide reflection group*. The stabiliser of $K$ is trivial i.e. we are looking the optimal kernel point in a 3-dimensional region, inside of a fundamental domain of $\Gamma$ with free fibre parameter $p(\Gamma)$. It can be assumed by the homogeneity of $S^2 \times \mathbb{R}$, that the fibre coordinate of the center of the optimal ball is zero.

### 3.2 Optimal ball packing to space group 12. I. 3

Now we consider the following point group:

$$\Gamma_0 := \{g_1, g_2, g_3, g_1^2, g_2^2, g_3^2, (g_1 g_2)^2, (g_1 g_3)^3, (g_2 g_3)^4\}.$$

This is the full isometry group of the usual cube surface, generated by the three reflections $g_i, \ i = 1, 2, 3$. The possible translation parts of the generators of $\Gamma_0$ will be determined by (1.2) and by the defining relations of the point group. Finally, from the so-called Frobenius congruence relations we obtain the four non equivariant solutions:

$$(\tau_1, \tau_2, \tau_3) \cong (0, 0, 0), (0, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$
If \((\tau_1, \tau_2, \tau_3) \cong (\frac{1}{2}, 0, \frac{1}{2})\) then we get the \(S^2 \times \mathbb{R}\) space group 12. I. 3. The fundamental domain of its point group is a spherical triangle \(A_0A_1A_2\) with angles \(\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{4}\) lying in the base plane \(\Pi\) (see Fig. 2). It can be assumed by the homogeneity of \(S^2 \times \mathbb{R}\), that the fibre coordinate of the center of the optimal ball is zero and it is an interior point of \(A_0A_1A_2\) triangle.

We shall apply the Cartesian homogeneous coordinate system introduced in Section 2 (see Fig. 2) and the usual geographical coordiantes \((\phi, \theta)\), \((-\pi < \phi \leq \pi, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})\) of the sphere with the fibre coordinate \(t \in \mathbb{R}\) (see (2.2)).

We consider an arbitrary interior point \(K(x^0, x^1, x^2, x^3)\) of spherical triangle \(A_0A_1A_2\) in the above coordinate system in our model by the following equations:

\[
x^0 = 1, \quad x^1 = \cos \phi \cos \theta, \quad x^2 = \sin \phi \cos \theta, \quad x^3 = \sin \theta
\]  

(3.3)

Let \(\mathcal{B}_\Gamma(R)\) denote a geodesic ball packing of \(S^2 \times \mathbb{R}\) space with balls \(B(R)\) of radius \(R\) where their centres give rise to the orbit \(K^\Gamma\). In the following we consider to each ball packing the possible smallest translation part \(\tau(K, R)\) (see Fig. 2) depending on \(\Gamma, K\) and \(R\). A fundamental domain of \(\Gamma\) is its D-V cell \(D(K)\) around the kernel point \(K\). It is clear that the optimal ball \(\mathcal{B}_K\) has to touch some faces of its D-V cell. The volume of \(D(K)\) is equal to the volume of the prism which is given by the fundamental domain of the point group \(\Gamma_0\) of \(\Gamma\) and by the height \(2\tau(R, K)\). The images of \(D(K)\) by our discrete isometry group covers the
\( S^2 \times \mathbb{R} \) space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid \( D(K) \) (see Definition 3.4).

It is clear, that the densest ball arrangement \( B_{T}(R) \) of balls \( B(R) \) has to hold the following requirements:

\[
\begin{align*}
(a) \quad & d(K, K^{g_2}) = 2R = d(K, K^{\tau g_1}), \\
(b) \quad & d(K, K^{g_2}) = 2R = d(K, K^{\tau g_3}), \\
& \quad \quad (c) \quad d(K, K^{2\tau}) \geq 2R, \quad (3.5) \\
(d) \quad & \text{Balls of radius } R \text{ with centres } \\
& K, K^{g_2}, K^{\tau g_1}, K^{\tau g_3}, K^{2\tau} \text{ form a packing.}
\end{align*}
\]

Here \( d \) is the distance function in the \( S^2 \times \mathbb{R} \) space (see Definition 2.2). The equations (a) and (b) mean that the ball centres \( K^{\tau g_1} \) and \( K^{\tau g_3} \) lie on the equidistant geodesic surface of the points \( K \) and \( K^{2\tau} \) which is a sphere in our model in this case (see [7]).

We consider two main ball arrangements:

1. We denote by \( B_{T}(R_0, K_0) \) those packing where requirements (3.5) and \( d(K, K^{2\tau}) = 2R \) hold (see Fig. 3).

2. We denote by \( B_{T}(R_1, K_1) \) those packing where requirements (3.5) and \( d(K^{\tau g_1}, K^{\tau g_3}) = 2R \) hold (see Fig. 4).
Geodesic ball packings in $S^2 \times \mathbb{R}$ space

First we determine the coordinates of the points $K_i$, $(i = 1, 2)$ ($K_i$ is given by (3.3) with parameters $\phi$ and $\theta$), the radius $R$ of the ball, the volume of a ball $B(R)$ and the density of the packing in both main cases. We get the following solutions by systematic approximation, where the computations were carried out by Maple V Release 10 up to 30 decimals:

$$
\phi_0 \approx 0.24389626, \quad \theta_0 \approx 0.20663860, \quad R_0 \approx 0.23860571,
\text{Vol}(B(R_0)) \approx 0.05668684, \quad \delta(R_0, K_0) \approx 0.45373556. \quad (3.5)
$$

$$
\phi_1 \approx 0.30773985, \quad \theta_1 \approx 0.17313169, \quad R_1 \approx 0.30299179,
\text{Vol}(B(R_1)) \approx 0.11580359, \quad \delta(R_1, K_1) \approx 0.44472930. \quad (3.6)
$$

We obtain by careful investigation of the density function $\delta(R, K)$ ($R \in [R_0, R_1]$) of the considered ball packing the following:

**Theorem 3.6** The ball arrangement $B_\Gamma(R_0, K_0)$ (see Fig. 3) provides the densest simply transitive ball packing belonging to the $S^2 \times \mathbb{R}$ space group 12.1.3.

### 3.3 The densest simply transitive ball packing

We consider the following point group:

$$
(+, \ 0, \ [ \ ] \ \{ (q, q) \}) \times \mathbb{1}_{\mathbb{R}}, \quad q \geq 2;
\Gamma_0 = (\mathfrak{g}_1, \mathfrak{g}_2 - \mathfrak{g}_1^2, \mathfrak{g}_2^2, (\mathfrak{g}_1 \mathfrak{g}_2)^q).
$$
This point group is generated by two reflections $g_i$, $i = 1, 2, 3$. The possible translation parts of the generators of $\Gamma_0$ will be determined by (1.2) and by the defining relations of the point group. Finally, we obtain from the so-called Frobenius congruence relations three non equivariant solutions:

$$(\tau_1, \tau_2) \cong (0, 0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right).$$

If $(\tau_1, \tau_2) \cong \left(\frac{1}{2}, \frac{1}{2}\right)$ then we have obtained the $S^2 \times \mathbb{R}$ space group $\text{2q. I. 2}$. 

The fundamental domain of the point group of the considered space group is a spherical digon $A_0A_1$ with angle $\frac{\pi}{4}$ in the base plane $\Pi$. Similarly to the above section can be assumed, that the fibre coordinate of the center of the optimal ball is zero and it is an interior point of $A_0A_1$ digon (see Fig. 5).

In the following we consider ball packings belonging to $q = 2$. We use also the above introduced Cartesian homogeneous coordinate system and the usual geographical coordinates $(\phi, \theta)$, $(-\pi < \phi \leq \pi$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$ of the sphere with the fibre coordinate $t \in \mathbb{R}$ (see (2.2)).

We consider an arbitrary interior point $K(1, x^1, x^2, x^3) = K(\phi, \theta)$ of spherical digon $A_0A_1$ in the above coordinate system in our model (see Fig. 5).

Our aim is to determine the maximal radius $R$ of the balls, and the maximal density $\delta(K, R)$.

The ball arrangement $B_{\text{opt}}(K, R)$ is defined by the following equations:

\begin{align*}
(a) \quad & d(K, K^{\tau g_1}) = 2R = d(K, K^{\tau g_2}), \\
(b) \quad & d(K^{\tau g_1}, K^{\tau g_2}) = 2R,
\end{align*}

(3.7)
Geodesic ball packings in $S^2 \times \mathbb{R}$ space

We can determine the coordinates of the point $K$, the radius $R$ of the ball, the volume of a ball $B(R)$ and the density of this packing:

\[
\phi = \frac{\pi}{4} \approx 0.78539816, \quad \theta = 0, \quad R = \frac{\pi}{2} \approx 1.57079633,
\]

\[
Vol(B(R)) \approx 13.74539472, \quad \delta(R, K) \approx 0.80407553.
\] (3.8)

Similarly to the above Section we can prove the following theorem:

**Theorem 3.7** The ball arrangement $B_{\text{opt}}(R, K)$ (see Fig. 6) provides the densest simply transitive ball packing belonging to the $S^2 \times \mathbb{R}$ space group 2q. I. 2.

Considering all in this paper investigated space groups and in [12] discussed generalized Coxeter $S^2 \times \mathbb{R}$ space groups we get the next theorem:

**Theorem 3.8** The ball arrangement $B_{\text{opt}}(R, K)$ provides the densest simply transitive ball packing belonging to the generalized Coxeter and glide reflections generated $S^2 \times \mathbb{R}$ space groups.

By Theorems 2.7 and 2.8 and by Definitions 3.3 and 3.4, similarly to the above space groups, we have determined the data (radii, densities and volumes of optimal balls) of the optimal simply transitive ball packings to each glide reflections generated $S^2 \times \mathbb{R}$ space group which are summarized in Table 1.
Remark 3.9 The space groups $2q. I. 2$, $2qe. I. 3$, $4q. I. 2$, $4q. I. 3$, $4q. I. 4$, $4qe. I. 5$, $4qe. I. 6$ depend on parameter $q$ thus their optimal ball packings depend also on $q$ but in the Table 1 we give only the data of the densest ball packing indicating its $q$ parameter to each considered space group.

| Space group | $R$ | $\text{Vol}(B_K(R))$ | $\delta$ |
|-------------|-----|----------------------|---------|
| $2q. I. 2$, $q = 2$ | $\frac{\pi}{2} \approx 1.57079633$ | $\approx 13.74539472$ | $\approx 0.80407553$ |
| $2qe. I. 3$, $q = 2$ | $\frac{\pi}{2} \approx 1.57079633$ | $\approx 13.74539472$ | $\approx 0.69634983$ |
| $4q. I. 2$, $q = 2$ | $\approx 0.64360446$ | $\approx 1.08624788$ | $\approx 0.53722971$ |
| $4q. I. 3$, $q = 2$ | $\approx 0.67517586$ | $\approx 1.25058159$ | $\approx 0.58958340$ |
| $4q. I. 4$, $q = 2$ | $\approx 0.95531662$ | $\approx 3.43551438$ | $\approx 0.74837055$ |
| $4qe. I. 5$, $q = 2$ | $\approx 0.64360446$ | $\approx 1.08624788$ | $\approx 0.53722971$ |
| $4qe. I. 6$, $q = 2$ | $\approx 0.67517586$ | $\approx 1.25058159$ | $\approx 0.58958340$ |
| $11. I. 2$ | $\approx 0.46364761$ | $\approx 0.41154972$ | $\approx 0.5861600$ |
| $12. I. 2$ | $\approx 0.22770028$ | $\approx 0.04928081$ | $\approx 0.41334779$ |
| $12. I. 3$ | $\approx 0.23860571$ | $\approx 0.05668684$ | $\approx 0.4537556$ |
| $12. I. 4$ | $\approx 0.31004511$ | $\approx 0.12404486$ | $\approx 0.53597559$ |
| $13. I. 2$ | $\approx 0.18705243$ | $\approx 0.02735051$ | $\approx 0.49222087$ |

It is timely to arising the above question for further space groups in $S^2 \times \mathbb{R}$ space.

In this paper we have mentioned only some problems in discrete geometry of $S^2 \times \mathbb{R}$ space, but we hope that from these it can be seen that our projective method suits to study and solve similar problems ([7], [11], [12]). Analogous questions in other homogeneous Thurston geometries are interesting.

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Geodesic ball packings in $S^2 \times \mathbb{R}$ space

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