A REMARK ON GLOBAL WELL-POSEDNESS BELOW $L^2$ FOR THE GKDV-3 EQUATION

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Abstract. The I-method in its first version as developed by Colliander et al. in [2] is applied to prove that the Cauchy-problem for the generalised Korteweg-de Vries equation of order three (gKdV-3) is globally well-posed for large real-valued data in the Sobolev space $H^s(\mathbb{R} \rightarrow \mathbb{R})$, provided $s > -\frac{1}{42}$.

1. Introduction

In a recently published paper of Tao [12] concerning the Cauchy-problem for the generalised Korteweg-de Vries equation of order three (for short: gKdV-3), i.e.:

$$u_t + u_{xxx} \pm (u^4)_x = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

(1)

it was shown that this problem is locally well-posed for data $u_0$ in the critical Sobolev space $H^{-\frac{1}{6}}(\mathbb{R} \rightarrow \mathbb{C})$ and globally well-posed for data with sufficiently small $H^{-\frac{1}{6}}$-norm. Moreover, scattering results in $H^1 \cap H^{-\frac{1}{6}}(\mathbb{R} \rightarrow \mathbb{R})$ for the radiation component of a perturbed soliton were obtained. Tao’s local result improves earlier work of Kenig, Ponce, and Vega ($s \geq \frac{1}{12}$, see [9, Theorem 2.6]) and of the first author ($s > -\frac{1}{6}$, see [7]), while the global small data theory seems to be completely new in Sobolev spaces of negative index. For large real valued data $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$, $s \geq 0$, global well-posedness of (1) was obtained in [7] by combining the conservation of the $L^2$-norm with the local $L^2$-result, for $s \geq 1$ this was already in [9, Corollary 2.7], where the energy conservation was used.

Starting with Bourgain’s splitting argument [1] and followed by the “I-method” or “method of almost conservation laws” introduced and further refined by Colliander, Keel, Staffilani, Takaoka, and Tao in a series of papers - see e. g. [2], [3], [4], [5], [6] - effective techniques have been developed, which are capable to show large data global well-posedness below certain conserved quantities such as the energy or the $L^2$-norm. The question of whether and to what extent these methods apply to the Cauchy-problem for gKdV-3, was raised as well by Linares and Ponce [10, p.177 and p.183] as by Tao, see Remark 5.3 in [12]. In this note we establish global well-posedness of (1) for large data $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$, provided $s > -\frac{1}{12}$, thus giving a partial answer to this question. Our proof combines the first version of the I-method as in [2] with a sharp four-linear $X_{s,b}$-estimate exhibiting an extra gain of half a derivative.

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\[i.e. \, beyond \, the \, cancellation \, of \, the \, derivative \, in \, the \, nonlinearity\]
Before we turn to the details, let us point out that substantial difficulties appear, if we try to push the analysis further to lower values of \( s \); by following the construction of a sequence of “modified energies” in [5] we are led already in the second step to a Fourier multiplier, say \( \mu_8 \), corresponding to \( M_4 \) in [5], with a quadratic singularity, and the argument breaks down\(^2\). Our fruitless effort in this direction seems to confirm Tao’s remark, that “it is unlikely that these methods would get arbitrarily close to the scaling regularity \( s = -\frac{1}{6} \)” [12, Remark 5.3]

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2. A variant of local well-posedness, the decay estimate, and the main result

Here we follow the lines of [2]: The operator \( I_N \) is defined via the Fourier transform by

\[
\hat{I_N}u(\xi) := m\left(\frac{|\xi|}{N}\right)\hat{u}(\xi),
\]

where \( m: \mathbb{R}^+ \to \mathbb{R}^+ \) is a smooth monotonic function with \( m(x) = 1 \) for \( x \leq 1 \) and \( m(x) = x^s, x \geq 2 \). Here \( s < 0 \), so that \( 0 < m(x) \leq 1 \).

The crucial nonlinear estimate in the proof of local well-posedness for (1) with \( H^s \)-data, \( s > -\frac{1}{6} \), is

\[
\|\partial_x^4 \prod_{i=1}^4 u_i\|_{X_{s,b'}} \lesssim \prod_{i=1}^4 \|u_i\|_{X_{s,b}},
\]

which holds true, whenever \( 0 \geq s > -\frac{1}{6}, -\frac{1}{2} < b' < s - \frac{1}{3} \) and \( b > \frac{1}{2} \), see [7, Theorem 1]. The \( X_{s,b} \)-norms used here are given by

\[
\|u\|_{X_{s,b}} = \left( \int d\xi d\tau (\tau - \xi^3)^{2b}(\xi)^{2s}|F u(\xi, \tau)|^2 \right)^{\frac{1}{2}},
\]

where \( F \) denotes the Fourier transform in both variables. Later on we shall also use the restriction norms \( \|v\|_{X_{s,b}(\delta)} = \inf \{ \|u\|_{X_{s,b}} : u|_{[0,\delta] \times \mathbb{R}} = v \} \). Applying the interpolation lemma [6, Lemma 12.1] to (2) we obtain, under the same assumptions on the parameters \( s, b' \) and \( b \),

\[
\|I_N \partial_x^4 \prod_{i=1}^4 u_i\|_{X_{0,b'}} \lesssim \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}},
\]

where the implicit constant is independent of \( N \). Now familiar arguments invoking the contraction mapping principle give the following variant of local well-posedness.

Lemma 1. For \( s > -\frac{1}{6} \) the Cauchy-problem (1) is locally well-posed for data \( u_0 \in (H^s, \|I_N \cdot\|_{L^2}) \). The lifespan \( \delta \) of the local solution \( u \) satisfies

\[
\delta \gtrsim \|I_N u_0\|_{L^2}^{-\frac{18}{2s+7}},
\]

and moreover we have for \( b = \frac{1}{2} \)

\[
\|I_N u\|_{X_{0,b}(\delta)} \lesssim \|I_N u_0\|_{L^2}.
\]

\(^2\)A similar problem was observed by Tzirakis for the quintic semilinear Schrödinger equation in one dimension, see the concluding remark in [13].
Replacing $u^2$ by $u^4$ in the calculation on p. 2 of [2], we obtain for a solution $u$ of (1)
\[
\|I_N u(\delta)\|_{L^2}^2 - \|I_N u(0)\|_{L^2}^2 \lesssim \|\partial_x (I_N u^4 - (I_N u)^4)\|_{X_{0,-b}^{\lambda}} \|I_N u\|_{X_{0,\lambda}^{\delta}}. \tag{6}
\]
The next section will be devoted to the proof that for $b > \frac{1}{2}$, $0 \leq s \leq -\frac{1}{3}$.
\[
\|\partial_x (I_N u^4 - (I_N u)^4)\|_{X_{0,-b}^{\lambda}} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|_{L^2}^4 \tag{7}
\]
(see Corollary [1] below), which together with (6) and (5) gives
\[
\|I_N u(\delta)\|_{L^2} - \|I_N u(0)\|_{L^2} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|_{L^2}^4. \tag{8}
\]
Now the decay estimate (5) allows us to prove our main result:

**Theorem 1.** Let $s > -\frac{1}{12}$ and $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$. Then the solution $u$ of (1) according to Lemma [1] extends uniquely to any time interval $[0, T]$ and satisfies
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \lesssim \langle T \rangle^{-\frac{5}{1+12s}} \|u_0\|_{H^s}. \tag{9}
\]

**Proof:** We choose $\varepsilon_0$ so that Lemma [1] gives the lifespan $\delta = 1$ for all data $\phi \in H^s$ with $\|I_N \phi\|_{L^2} \leq 2\varepsilon_0$. Moreover we demand $16C_{\varepsilon_0} \leq 1$, where $C$ is the implicit constant in the decay estimate (5). Assuming without loss that $T \gg 1$, we fix parameters $C_1$, $N$ and $\lambda$ with
\[
2C_1^{-\frac{1}{2}}\varepsilon_0 \|u_0\|_{H^s} = \varepsilon_0,
\]
\[
N^{\frac{1}{2}} = \lambda^3 T,
\]
and
\[
\lambda = C_1 N^{-\frac{1}{4}+\varepsilon_0}.
\]
Then $N^\frac{1}{2} = \lambda^3 T$ and for $u_0^\lambda(x) = \lambda^{-\frac{1}{2}} u_0(\lambda x)$ it is easily checked that $\|I_N u_0^\lambda\|_{L^2} \leq \varepsilon_0$. For any $k \in \mathbb{N}$, repeated applications of Lemma [1] give a solution $u^\lambda$ of gKdV-3 with $u^\lambda(0) = u_0^\lambda$ on $[0, k]$, as long as
\[
\|I_N u^\lambda(j)\|_{L^2} \leq 2\varepsilon_0, \quad 1 \leq j < k. \tag{10}
\]
Since by (5) and the second assumption on $\varepsilon_0$
\[
\|I_N u^\lambda(j)\|_{L^2} \leq \varepsilon_0 + jCN^{-\frac{1}{2}}(2\varepsilon_0)^4 \leq \varepsilon_0 (1 + jN^{-\frac{1}{2}}),
\]
condition (10) is fulfilled for $j \leq N^\frac{1}{2} = \lambda^3 T$. Thus $u^\lambda$ is defined on $[0, \lambda^3 T]$, and with $u(x, t) = \lambda^{\frac{1}{2}} u^\lambda(\lambda x, \lambda^3 t)$ we obtain a solution of (1) on $[0, T]$. Finally we have
\[
\|u(t)\|_{H^s} \lesssim \|I_N u(t)\|_{L^2} \lesssim \lambda^\frac{1}{4} \|I_N u^\lambda(\lambda^3 t)\|_{L^2}
\]
with $\lambda^3 \approx T^{\frac{1}{1+12s}}$ and $\|I_N u^\lambda(\lambda^3 t)\|_{L^2}$ being bounded during the whole iteration process by $2\varepsilon_0 \lesssim \|u_0\|_{H^s}$. This gives the growth bound (9). \qed

### 3. The decisive four-linear estimate

Let us first recall several linear and bilinear Airy estimates (in their $X_{s,b}$-versions), which shall be used below: by interpolation between the sharp version of Kato’s smoothing effect (see [8] Theorem 4.1) and the maximal function estimate from [11] Theorem 3] we have
\[
\|J^s u\|_{L^2_t(L^2_x)} \lesssim \|u\|_{X_{0,b}^s}, \tag{11}
\]
whenever $b > \frac{1}{2}$, $-\frac{1}{4} \leq s \leq 1$ and $(\frac{1}{p}, \frac{1}{q}) = (\frac{1+2s}{p+2s}, \frac{1+6s}{q+12s})$. We will use (11) with $s = 0$, i.e.
\[
\|u\|_{L^2_t(L^2_x)} \lesssim \|u\|_{X_{0,1}^0}, \tag{12}
\]
and the dual version of (11) with $s = \frac{1}{2}$, which is
\[
\|u\|_{X_{s,-b}^\frac{1}{2}} \lesssim \|u\|_{L^2_t(L^\infty_x)(L^2)} \tag{13}
\]
Moreover we shall rely on the Strichartz type estimate
\[
\|u\|_{L^\infty_t(L^b_x)} \lesssim \|u\|_{X_{0,b}^s}, \quad (b > \frac{1}{2}) \tag{14}
\]
Lemma 2. Let $b > \frac{1}{2}$, and the bilinear estimate
\[
\|I^s I^s (u, v)\|_{L^2_{\xi}} \lesssim \|u\|_{X_{0,b}} \|v\|_{X_{0,b}}, \quad (b > \frac{1}{2})
\] (15)
from [7, Corollary 1]. Here $I^s$ denotes the Riesz (Bessel) potential operator of order $-s$ and $I^s$ is defined via the Fourier transform by
\[
\hat{I^s(f)}(\xi) := \int_{\xi_1 + \xi_2 = \xi} d\xi_1 |\xi_1 - \xi_2|^s \hat{f}(\xi_1) \hat{g}(\xi_2).
\]
Now we turn to the crucial four-linear $X_{s,b}$-estimate:

**Lemma 2.** Let $b > \frac{1}{2}$, $s_i \leq 0$, $1 \leq i \leq 4$, with $\sum_{i=1}^4 s_i = -\frac{1}{2}$. Then
\[
\left\| \partial_x \prod_{i=1}^4 v_i \right\|_{X_{0,b}} \lesssim \left\| \prod_{i=1}^4 v_i \right\|_{X_{s_i,b}}.
\] (16)

Proof: We write
\[
\left\| \partial_x \prod_{i=1}^4 v_i \right\|_{X_{0,b}} = c \|\xi\| (\xi - \xi^3)^{-b} \int d\nu \prod_{i=1}^4 F v_i (\xi_i, \tau_i) \|_{L^2_{\nu,\tau}},
\]
where $d\nu = d\xi_1 \cdots d\xi_4 dv_1 \cdots dv_4$ and $\sum_{i=1}^4 (\xi_i, \tau_i) = (\xi, \tau)$, and divide the domain of integration into three regions $A$, $B$ and $C = (A \cup B)^c$. In region $A$ we assume that $|\xi_{\text{max}}| \leq 1$ and hence $|\xi| \leq 4$, so for this region we get the upper bound
\[
\left\| \prod_{i=1}^4 J^{s_i} v_i \right\|_{L^2_{\nu,\tau}} \lesssim \left\| \prod_{i=1}^4 J^{s_i} v_i \right\|_{L^2_{\nu,\tau}} \lesssim \left\| \prod_{i=1}^4 v_i \right\|_{X_{s_i,b}},
\]
where in the last step we have used the $L^2_{\nu,\tau}$-Strichartz-type estimate (14). Concerning the region $B$ we shall assume - besides $|\xi_{\text{max}}| \geq 1$, implying $|\xi_{\text{max}}| \lesssim |\xi_{\text{max}}|$ - that
\[
\text{i) } |\xi_{\text{min}}| \leq 0.99|\xi_{\text{max}}| \quad \text{or}
\]
\[
\text{ii) } |\xi_{\text{min}}| > 0.99|\xi_{\text{max}}|, \quad \text{and there are exactly two indices } i \in \{1, 2, 3, 4\} \text{ with } \xi_i > 0.
\]
Then the region $B$ can be split further into a finite number of subregions, so that for any of these subregions there exists a permutation $\pi$ of $\{1, 2, 3, 4\}$ with
\[
|\xi| \lesssim |\xi|^{\frac{1}{4}} |\xi_{\pi(1)} + \xi_{\pi(2)}|^{\frac{1}{4}} |\xi_{\pi(1)} - \xi_{\pi(2)}|^{\frac{1}{4}} \prod_{i=1}^4 (\xi_i)^{s_i}.
\]
Assume $\pi = \text{id}$ for the sake of simplicity now. Then we get the upper bound
\[
\left\| (I^s I^s (J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \right\|_{X_{0,-b}} \lesssim \left\| (I^s I^s (J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \right\|_{L^2_{\nu,\tau}} \lesssim \left\| (I^s I^s (J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \right\|_{L^2_{\nu,\tau}} \lesssim \left\| (I^s I^s (J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \right\|_{L^2_{\nu,\tau}} \lesssim \left\| \prod_{i=1}^4 v_i \right\|_{X_{s_i,b}}.
\]
Here we have applied the estimates (13), Hölder, (15) and (12). Finally we consider the remaining region $C$: Here the $|\xi_i|$, $1 \leq i \leq 4$, are all very close together and $\gtrsim (\xi_i)$. Moreover, at least three of the variables $\xi_i$ have the same sign. Thus for

\[
|\xi| \lesssim |\xi|^{\frac{1}{4}} \prod_{i=1}^4 (\xi_i)^{s_i}.
\]

\[\begin{align*}
\text{Here } \xi_{\text{max}} \text{ is defined by } |\xi_{\text{max}}| = \max_{i=1}^4 |\xi_i|, \text{ similarly } \xi_{\text{min}}.
\end{align*}\]
the quantity c.q. controlled by the expressions $\langle \tau - \xi^3 \rangle$, $\langle \tau_i - \xi_i^3 \rangle$, $1 \leq i \leq 4$, we have in this region:

$$c.q. := |\xi^3 - \sum_{i=1}^{4} \xi_i^3| \geq \sum_{i=1}^{4} |\xi_i|^3 \geq \langle \xi \rangle^3.$$ 

So the contribution of the subregion, where $\langle \tau - \xi^3 \rangle \geq \max_{i=1}^{4} \langle \tau_i - \xi_i^3 \rangle$, is bounded by

$$\| \prod_{i=1}^{4} J^{\tau_i} v_i \|_{L_{xt}^{2}} \leq \prod_{i=1}^{4} \| J^{\tau_i} v_i \|_{L_{xt}^{2}} \lesssim \prod_{i=1}^{4} \| v_i \|_{X_{s_i, b}},$$

where (13) was used again. On the other hand, if $\langle \tau_i - \xi_i^3 \rangle$ is dominant, we write $\Lambda^\frac{1}{2} = F^{-1} \langle \tau - \xi^3 \rangle \frac{1}{2} \mathcal{F}$ and obtain the upper bound

$$\| (\Lambda^\frac{1}{2} J^{\tau_i} v_i) \prod_{i=2}^{4} J^{\tau_i} v_i \|_{X_{0, -b}} \lesssim \| (\Lambda^\frac{1}{2} J^{\tau_i} v_i) \prod_{i=2}^{4} J^{\tau_i} v_i \|_{L_{xt}^{2}} \lesssim \prod_{i=1}^{4} \| v_i \|_{X_{s_i, b}}.$$ 

Here the dual version $X_{0, -b} \supset L_{xt}^{2}$ of the $L_{xt}^{2}$ estimate was used first, followed by Hölder’s inequality and the estimate itself. The remaining subregions, where $\langle \tau_k - \xi_k^3 \rangle$, $2 \leq k \leq 4$, are maximal, can be treated in precisely the same manner.

\[ \square \]

**Corollary 1.** Let $b > \frac{1}{2}$ and $0 \geq s \geq -\frac{1}{8}$. Then

$$\| \partial_x (I_N (\prod_{i=1}^{4} u_i) - \prod_{i=1}^{4} I_N u_i) \|_{X_{0, -b} (\delta)} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^{4} \| I_N u_i \|_{X_{0, b} (\delta)}. \quad (17)$$

Especially, if $u$ is a solution of gKdV-3 according to Lemma 7 with $u(0) = u_0$, then

$$\| \partial_x (I_N u^4 - (I_N u)^4) \|_{X_{0, -b} (\delta)} \lesssim N^{-\frac{1}{2}} \| I_N u_0 \|_{L^2}. \quad (18)$$

Proof: By (5) the estimate (17) implies (18). Thus it suffices to show

$$\| \partial_x (I_N (\prod_{i=1}^{4} u_i) - \prod_{i=1}^{4} I_N u_i) \|_{X_{0, -b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^{4} \| I_N u_i \|_{X_{0, b}}. \quad (19)$$

Now let $\xi_i$ denote the frequencies of the $u_i$, $1 \leq i \leq 4$. If all the $|\xi_i| \leq N$, then either $|\xi| \leq N$, such that there’s no contribution at all, or we have $|\xi| \geq N$, so that at least, say, $|\xi| \geq \frac{N}{4}$. In this case, by Lemma 2 the norm on the left of (13) is bounded by

$$\| \partial_x \prod_{i=1}^{4} u_i \|_{X_{0, -b}} \lesssim \| u_1 \|_{X_{\frac{1}{2}, b}} \prod_{i=2}^{4} \| u_i \|_{X_{0, b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^{4} \| I_N u_i \|_{X_{0, b}}.$$ 

Otherwise there are $k$ large frequencies for some $1 \leq k \leq 4$. By symmetry we may assume that $|\xi_1|, \ldots, |\xi_k| \geq N$ and $|\xi_{k+i}|, \ldots, |\xi_4| \leq N$. Then we have, again by Lemma 2

$$\| \partial_x \prod_{i=1}^{4} I_N u_i \|_{X_{0, -b}} \lesssim \prod_{i=1}^{k} \| I_N u_i \|_{X_{\frac{1}{2}, b}} \prod_{i=k+1}^{4} \| I_N u_i \|_{X_{0, b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^{4} \| I_N u_i \|_{X_{0, b}}.$$
as well as
\[ \| \partial_x I_N \prod_{i=1}^4 u_i \|_{X_{0,-b}} \lesssim \prod_{i=1}^k \| u_i \|_{X_{-\frac{1}{2}+b}} \prod_{i=k+1}^4 \| u_i \|_{X_{0,b}}. \]
Since for any \( s_1 \leq s \) and any \( v \) with frequency \( |\xi| \geq N \) it holds that
\[ \| v \|_{X_{s_1,b}} \lesssim N^{s_1-s} \| v \|_{X_{s,b}} \sim N^{s_1} \| I_N v \|_{X_{0,b}}, \]
the latter is again bounded by the right hand side of (19). \( \square \)

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