Abstract

Although the post-Newtonian Lagrangian formalism is widely used in relativistic dynamical and statistical studies of test bodies moving around arbitrary mass distributions, the corresponding general Hamiltonian formalism is still relatively uncommon, being restricted basically to the case of N-body problems. Here, we present a consistent Hamiltonian formalism for the dynamics of test particles in spacetimes with arbitrary energy-momentum distributions in the first post-Newtonian (1PN) approximation. We apply our formalism to orbital motion in stationary axisymmetric spacetimes and obtain the 1PN relativistic corrections to the radial and vertical epicyclic frequencies for quasi-circular equatorial motion, a result potentially interesting for galactic dynamics. For the case of razor-thin disk configurations, we obtain an approximated third integral which could be used to determine analytically the envelope of nearly equatorial orbits. One of the main advantages of this 1PN analysis is the explicit presence of frame-dragging effects in all pertinent expressions, allowing some qualitative predictions in rotating spacetimes. We finish discussing the 1PN collisionless Boltzmann equation in terms of the Hamiltonian canonical variables and its relation with previous results in the literature.

1 Introduction

The dynamics of test bodies around extended mass distributions is a fundamental issue in many astrophysical scenarios. In the context of weak fields in General Relativity (GR), it is customary to employ the so-called first post-Newtonian (1PN) approximation for the description of the motion of test bodies with relatively low velocities. Apart from the usual Newtonian gravitational potential \( \Phi \), the 1PN approximation includes two additional quantities, the scalar potential \( \Psi \) and the three-vector \( \vec{\zeta} \), representing, respectively, a relativistic correction to the Newtonian potential and frame-dragging effects. In this approximation, the non-vanishing spacetime metric components can be written as [1]

\[
g_{00} = - \left( 1 + 2 \frac{\Phi}{c^2} + 2 \frac{\Phi^2}{c^4} + 2 \frac{\Psi}{c^4} \right), \\
g_{ij} = \delta_{ij} \left( 1 - 2 \frac{\Phi}{c^2} \right), \\
g_{0i} = \frac{\zeta_i}{c^3},
\]  

(1)

where \( c \) stands for the speed of light. Latin indices are assumed to vary from 1 to 3 and correspond to spatial directions, whereas Greek indices vary from 0 to 3, ranging over \((ct, x^i)\). In the 1PN approximation, all analyses and pertinent expressions are restricted to terms up to \( \frac{1}{c^2} \)-order. The field equations for \( \Phi \),

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\( \Psi \) and \( \vec{\zeta} \) in the 1PN approximation are \([1,2,3]\)

\[
\nabla^2 \Phi = 4\pi G T^{00},
\]

\[
\nabla^2 \Psi - \frac{\partial^2 \Phi}{\partial t^2} = 4\pi G c^2 \left( T^{00} + \sum_i T^{0i} \right),
\]

\[
\nabla^2 \zeta^i = 16\pi G c T^{0i},
\]

where \( T^{\mu\nu} \) stands for the \( \frac{1}{c^2} \)-order term of the energy-momentum tensor \( T^{\mu\nu} \). The differential operators such as \( \nabla \) in (2)-(4) are the usual flat-space ones, and indices are raised and lowered with the Minkowski metric \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \), allowing us to use the usual three-dimensional vector operations for the spatial quantities. Furthermore, it is usually assumed in the 1PN approximation the so-called harmonic coordinate gauge, \( g^{\mu\nu} \Gamma^\delta_{\mu\nu} = 0 \), which gives us

\[
4 \frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{\zeta} = 0.
\]

The 1PN equations of motion for massive test particles can be written in terms of their three-velocity \( \vec{v} \) as \([1]\)

\[
\frac{d\vec{v}}{dt} = -\nabla (\Phi + 2 \frac{\Phi^2}{c^2} + \Psi) + \frac{1}{c^2} \left( \vec{v} \times (\nabla \times \vec{\zeta}) - \frac{\partial \vec{\zeta}}{\partial t} + \left( \frac{3}{2} \frac{\partial \Phi}{\partial t} + 4(\vec{v} \cdot \nabla) \Phi \right) \vec{v} - v^2 \nabla \Phi \right),
\]

where \( v^2 = v_i v^i = |\vec{v}|^2 \). Equation (6) is known to be derived from the Lagrangian \([1]\)

\[
\mathcal{L} = \frac{1}{2} v^2 + \frac{1}{c^2} \left( \frac{3}{8} v^4 - \frac{3}{2} v^2 \Phi + \vec{\zeta} \cdot \vec{v} \right) - \Theta,
\]

where

\[
\Theta = \Phi + \frac{\Phi^2}{2c^2} + \frac{\Psi}{c^2},
\]

from which we have the following canonical momenta

\[
p_i = \frac{\partial \mathcal{L}}{\partial v^i} = v_i \left( 1 - \frac{3\Phi}{c^2} + \frac{v^2}{2c^2} \right) + \frac{\zeta_i}{c^2},
\]

and energy function

\[
E = \sum_i p_i v^i - \mathcal{L} = \frac{v^2}{2} + \frac{1}{c^2} \left( \frac{3}{8} v^4 - \frac{3}{2} v^2 \Phi \right) + \Theta,
\]

which is indeed conserved for stationary spacetimes since in this case we will have \( \frac{\partial \Theta}{\partial t} = 0 \). For further discussions on the Lagrangian formulation of this problem, see \([2,3]\) and references therein.

In the following sections, we will derive an 1PN Hamiltonian for the Lagrangian (7) and apply it to the study of orbits in stationary axisymmetric spacetimes. We first obtain the 1PN relativistic corrections to the radial and vertical epicyclic frequencies for quasi-circular equatorial motion. Furthermore, for the case of razor-thin disk configurations, we get an approximated third integral which could be explored to determine analytically the envelope of nearly equatorial orbits. We finish discussing the 1PN collisionless Boltzmann equation (CBE) in terms of the Hamiltonian canonical variables and its relation with some previous results in the literature.

2 The 1PN Hamiltonian formalism

The easiest and more direct way of obtaining the Hamiltonian formulation associated with the Lagrangian (7) is to perform a 1PN Legendre transformation from the expression for the canonical momenta (9). Keeping only terms up to \( \frac{1}{c^2} \)-order, one can invert (9) as

\[
v_i = p_i \left( 1 + \frac{3\Phi}{c^2} - \frac{p_i^2}{2c^2} \right) - \frac{\zeta_i}{c^2}.
\]
Since the 1PN Hamiltonian $H$ is essentially the function (10) written in terms of the canonical variables $(x^i, p_j)$, we have, keeping again only terms up to $\frac{1}{c^2}$-order,

$$H = \frac{1}{2} p^2 \left( 1 + \frac{3\Phi}{c^2} - \frac{\Phi^2}{4c^4} \right) - \frac{\zeta \cdot \vec{p}}{c^2} + \Theta. \quad (12)$$

One can check by direct calculations that the Hamilton’s equations obtained from (12) are equivalent to the Euler-Lagrange equations of (7).

We could also have obtained (12) by exploring the so-called isenergetic reduction formalism [4], which basically consists in considering the Hamiltonian constraint for timelike geodesics $g^{\mu\nu} \pi_\mu \pi_\nu = -c^2$, where $\pi_\mu = (-E/c, p_i)$. The reduced Hamiltonian associated with the time parameter $t = x^0/c$ is given simply by $H = E_0$ [4], which can be determined directly by solving the quadratic Hamiltonian constraint, leading to

$$\frac{H}{c} = \sqrt{\frac{c^2 + \Phi \pi_i \pi_i}{g^{00}}} + \frac{\phi_0 \pi_i}{g^{00}}, \quad (13)$$

where

$$\Phi^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}, \quad (14)$$

with the 1PN inverse of (1) given by [1]

$$g^{00} = -\left( 1 - \frac{2\Phi}{c^2} + \frac{2\Phi^2}{c^4} - \frac{2\Psi}{c^4} \right),$$

$$g^{ij} = \delta^{ij} \left( 1 + \frac{2\Phi}{c^2} \right), \quad g^{0i} = \frac{c_i}{c^3}. \quad (15)$$

It is easy to check that (12), up to a constant $c^2$ term, follows from (13) by using (15) and keeping only terms up to $\frac{1}{c^2}$-order. It is worth mentioning that both the 1PN Lagrangian (7) and Hamiltonian (12) correspond to the choice $t = x^0/c$ as the time-evolution parameter for the geodesics dynamics, breaking in this way the GR reparametrization invariance. Different choices for the time-evolution parameter may result in different Lagrangian and Hamiltonian functions, but nevertheless with fully equivalent dynamics. This can be seen by considering the action for a relativistic particle

$$S = \int L \, d\tau, \quad L = -c \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad (16)$$

where the dot denotes here derivation with respect to the evolution parameter $\tau$. Such action is invariant under reparametrizations $\tau \rightarrow f(\tau)$, with $f(\tau) > 0$, and as a consequence the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{c \dot{x}_\mu}{\sqrt{-g_{\mu\nu} \dot{x}^\nu}}, \quad (17)$$

obey the Hamiltonian constraint $\pi_\mu \pi^\mu = -c^2$. The reparameterization-invariant Lagrangian (16) is a well-known example of a singular system for which the Hamiltonian formulation can only be defined properly in the context of constrained systems, see [5] for further details. However, this problem can be circumvented if we fix a specific parametrization for the action (16), breaking in this way the reparametrization invariance of the problem. Choosing $\tau = t$, with $x^\mu = (ct, x^i)$, the Lagrangian (16) reads

$$L = -c \sqrt{(c^2 g_{00} + c g_{0i} \dot{x}^i + g_{ij} \dot{x}^i \dot{x}^j)}, \quad (18)$$

where the dot now denotes derivation with respect to $t$. With the choice $\tau = t$ we have effectively reduced the dimension of the system, from 4 spacetime dimensions to only 3 spatial ones. Moreover, the 1PN Lagrangian (7) follows, up to a constant term, from the expansion of (18) up to $\frac{1}{c^2}$-order. For the Hamiltonian, we can essentially follow the same steps of [4]. The canonical momenta associated to the reduced Lagrangian (18) are

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{c^2 g_{0i} + c g_{ij} \dot{x}^j}{\sqrt{-c^2 g_{00} + c g_{0i} \dot{x}^i + g_{ij} \dot{x}^i \dot{x}^j}}, \quad (19)$$

and the associated Hamiltonian function will be

$$\mathcal{H} = p_i \dot{x}^i - L = -c \frac{c^2 g_{00} + c g_{0i} \dot{x}^i}{\sqrt{-c^2 g_{00} + c g_{0i} \dot{x}^i + g_{ij} \dot{x}^i \dot{x}^j}} = -c \pi_0, \quad (20)$$
where the expressions for the momenta (19) were used in the last equation. We can now solve the Hamiltonian constraint \( \pi_\mu \pi^\mu = -c^2 \) and write \( \pi^0 \) in terms of the spatial momenta, and essentially we reproduce the isoenergetic reduction construction and get (13). The freedom in choosing different evolution parameters will be important in order to properly compare different Hamiltonian and Lagrangian formulations of the CBE in the following sections.

3 Stationary, axisymmetric configurations

Let us start by applying the 1PN Hamiltonian formalism to the dynamics of test particles in stationary, axially symmetric spacetimes. We will use the usual cylindrical coordinates \((ct, r, \varphi, z)\) and assume reflection symmetry with respect to the equatorial plane \(z = 0\). The corresponding gravitational potentials are independent of \(t\) and \(\varphi\) and, moreover, have well-defined parity properties with respect to \(z \to -z\) reflections. In cylindrical coordinates, we have

\[
p^2 = p_r^2 + p^2_z + \frac{L_z^2}{r^2}, \tag{21}
\]

where we have already used that \(\varphi\) is a cyclic coordinate and hence \(p_\varphi = L_z\) is a constant of motion. Since we are dealing with stationary and axially symmetric spacetimes, the post-Newtonian metric must be a special case of the Weyl-Lewis-Papapetrou metric, and therefore we must have \(g_{0r} = g_{0z} = 0\). The rotation term is, thus, purely centrifugal. Therefore, our vector potential must have the form \(\vec{\zeta} = \zeta_\varphi \hat{\varphi}\), where \(\zeta_\varphi\) is the azimuthal component of the vector \(\vec{\zeta}\) in the orthonormal frame \((\hat{r}, \hat{\varphi}, \hat{z})\).

The frame-dragging term can be written as

\[
\vec{\zeta} \cdot \vec{p} = \zeta_\varphi \frac{L_z}{r}. \tag{22}
\]

Introducing the quantities

\[
A(r, z) = 1 + \frac{1}{c^2} \left( 3\Phi - \frac{L_z^2}{2r^2} \right), \tag{23}
\]

\[
B(p_r, p_z, r, z) = A(r, z) - \frac{1}{4c^2} (p_r^2 + p_z^2), \tag{24}
\]

\[
C(r, z) = 1 + \frac{1}{c^2} \left( 3\Phi - \frac{L_z^2}{4r^2} \right), \tag{25}
\]

and the effective potential

\[
V_{\text{eff}} = \Theta + C(r, z) \frac{L_z^2}{2r^2} - \frac{1}{c^2} \frac{\zeta_\varphi L_z}{r}, \tag{26}
\]

we can write the Hamiltonian (12) simply as

\[
H = \frac{1}{2} (p_r^2 + p_z^2) B(p_r, p_z, r, z) + V_{\text{eff}}(r, z). \tag{27}
\]

Notice that the positiveness of the kinetic term requires \(B > 0\), which is indeed verified if we consider the \(\frac{1}{c^2}\)-order terms as subdominant.

3.1 Quasi-circular motion

Circular orbits are given by the fixed points of the Hamiltonian when restricted to the equatorial plane \(z = 0\). They are given by the conditions \(p_r = p_z = 0\) and are the solutions for \(r\) of

\[
\frac{\partial V_{\text{eff}}}{\partial r} = \frac{\partial \Theta}{\partial r} - \frac{L_z^2}{r^2} + \frac{1}{c^2} \left[ \left( \frac{3}{2} \frac{\partial \Phi}{\partial r} - 3\Phi + \frac{L_z^2}{2r^2} \right) - \frac{L_z^2}{r^2} - L_z \frac{\partial}{\partial r} \left( \frac{\zeta_\varphi}{r} \right) \right] = 0 \tag{28}
\]

at \(z = 0\). In order to determine the angular momentum \(L_z\) of a circular orbit, we solve (28) explicitly for \(L_z\) up to \(\frac{1}{c^2}\)-order, obtaining

\[
L_z^2(r) = \left( L_z^N \right)^2 \left[ 1 - 2 \frac{r^2}{c^2} \left( \frac{\Phi - \left( L_z^N \right)^2}{r^2} \right) \right] + \frac{r^2}{c^2} \left( \frac{\partial \Phi}{\partial r} - L_z^N \frac{\partial}{\partial r} \left( \frac{\zeta_\varphi}{r} \right) \right). \tag{29}
\]
where \( L_z^N \) is the corresponding Newtonian angular momentum, which is known to be given by

\[
L_z^N = \pm \sqrt{r^3 \frac{\partial \phi}{\partial r}} .
\]

The \( L_z^N \) linear contribution in the last term of (29) explicitly exhibits a frame-dragging effect. The 1PN azimuthal angular momentum \( L_z^{1PN} \) for circular orbits, Equation (29), corresponds to a circular velocity \( \dot{v} = v_\dot{\phi} \hat{\phi} \), where \( v_\dot{\phi} = r \dot{\phi} \), which can be determined from (11), leading to

\[
v_\dot{\phi}^2 = r \frac{\partial \phi}{\partial r} + \frac{r}{c^2} \left[ 4 \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r} + r \left( \frac{\partial \phi}{\partial r} \right)^2 - \frac{L_z^N}{r} \left( \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial r} \right) \right] .
\]

Notice that the second term in the frame-dragging contribution is lacking in the equation (A5) of [3].

We now consider razor-thin disks in the presence of the scalar potential \( \Phi \) and the vector potential \( \vec{\zeta} = \zeta \hat{\phi} \), a situation corresponding to the inclusion of \( \frac{1}{c^2} \) order frame-dragging effects. From the field equations (2)–(4), we see that the vector potential generates an azimuthal momentum flux, but it does not affect neither the energy density nor the principal pressures of the disk. Since the 1PN field equations (2)–(4) are linear in the fields \( \phi, \psi \) and \( \zeta \), we can linearly superpose their solutions in order to consider more general systems. In particular, we can superpose the razor-thin disk to axially symmetric structures such as a thick disk, a spheroidal bulge and a spheroidal (stellar + dark matter) halo, as in mass models.

\[
\vec{\zeta} = \vec{v} \hat{\phi} .
\]

\[
\phi = \hat{\phi} .
\]

\[
\partial \phi \partial r = \partial \phi \partial r .
\]

\[
\partial \phi \partial r = \partial \phi \partial r .
\]

\[
L_z = L_z(r) .
\]

\[
\kappa = A(r, 0) \frac{\partial \phi}{\partial r} = \kappa_N^2 + \kappa_{1PN}^2 ,
\]

\[
\nu^2 = A(r, 0) \frac{\partial \psi}{\partial z} = \nu_N^2 + \nu_{1PN}^2 ,
\]

evaluated at \( L_z = L_z(r) \), Equation (29). Here, \( \kappa_N \) and \( \nu_N \) stand for, respectively, the radial and vertical Newtonian epicyclic frequencies

\[
\kappa_N^2 = \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{3}{r} \frac{\partial \phi}{\partial r} \right) ,
\]

\[
\nu_N^2 = \frac{\partial^2 \phi}{\partial z^2} .
\]

The 1PN corrections for the epicyclic frequencies are

\[
\kappa_{1PN}^2 = \frac{1}{c^2} \left\{ 4 \phi \kappa_N^2 + 3 \frac{\partial \psi}{\partial r} \frac{\partial \phi}{\partial r} - \frac{L_z^N}{r^2} \left( \frac{\partial \phi}{\partial r} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) - 3 \frac{\partial \psi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{L_z^N}{r^2} \left( \frac{\partial \phi}{\partial r} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) \right\} ,
\]

and

\[
\nu_{1PN}^2 = \frac{1}{c^2} \left\{ 4 \phi \nu_N^2 + \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial \psi}{\partial z} \frac{\partial \phi}{\partial z} - \frac{L_z^N}{r^2} \left( \frac{\partial \phi}{\partial r} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) - L_z^N \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} \right) \right\} .
\]

One can appreciate again the frame-dragging effect in both radial and vertical epicyclic frequencies for rotating and counter-rotating equatorial circular orbits. All the above expressions may be relevant for post-Newtonian corrections to galactic dynamics. Equation (31) gives us the 1PN corrections to the rotation curves of spiral and elliptical galaxies, including the frame-dragging term, and Equations (37) and (38) give us the 1PN corrections to the epicyclic frequencies of quasi-circular motion, having impact for instance on the radii of Lindblad resonances (the angular speed is given by \( \Omega(r) = \nu_\phi/r \)). Moreover, the Hamiltonian (27) extends the corresponding Newtonian expression and may be useful to the study of chaos in 1PN configurations. These topics, however, are beyond the scope of the present work.

### 3.2 Approximate third integral of motion for razor-thin disks

We now consider razor-thin disks in the presence of the scalar potential \( \Psi \) and the vector potential \( \vec{\zeta} = \zeta \hat{\phi} \), a situation corresponding to the inclusion of \( \frac{1}{c^2} \) order frame-dragging effects. From the field equations (2)–(4), we see that the vector potential generates an azimuthal momentum flux, but it does not affect neither the energy density nor the principal pressures of the disk. Since the 1PN field equations (2)–(4) are linear in the fields \( \phi, \psi \) and \( \vec{\zeta} \), we can linearly superpose their solutions in order to consider more general systems. In particular, we can superpose the razor-thin disk to axially symmetric structures such as a thick disk, a spheroidal bulge and a spheroidal (stellar + dark matter) halo, as in mass models.
for spiral galaxies [6, 7, 8, 9, 10]. Anyway, we can always analyze the thin-disk contribution separately from the contribution due to the surrounding matter.

The energy-momentum tensor for a razor-thin disk has the form \( T^{\mu\nu} \propto \delta(z) \), see for instance [11, 12, 13] and references therein. We can write it as [2, 3]

\[
T^{00} = n(\sigma(r)) \delta(z),
\]

\[
T^{ii} = \frac{\bar{P}_i(r)}{c^2} \delta(z),
\]

\[
T^{0\varphi} = \frac{\mu(r)}{c} \delta(z),
\]

where \( \sigma = \bar{\sigma} + \tilde{\sigma} \) is the surface mass density, \( \bar{P}_i \) are the principal pressures, and \( \mu(r) \) is the azimuthal surface momentum density of the disk. We can now evaluate the field equations (2)–(4) near the equatorial plane \( z = 0 \) and obtain

\[
\frac{\partial \Phi}{\partial |z|} = 2\pi G \bar{\sigma},
\]

\[
\frac{1}{c^2} \frac{\partial \Psi}{\partial |z|} = 2\pi G (\bar{\sigma} + \bar{P} / c^2),
\]

\[
\frac{1}{r} \frac{\partial \zeta_\varphi}{\partial |z|} = 8\pi G \mu,
\]

where \( \bar{P} = P_r + P_\varphi \) since razor-thin disks have no vertical pressure. Let us now consider the vertical stability of equatorial circular orbits in 1PN stationary razor-thin disks. From (26) and (42)–(44), we have

\[
\frac{1}{2\pi G} \frac{\partial V_{\text{eff}}}{\partial |z|} = \bar{\sigma} + \tilde{\sigma} + \frac{1}{c^2} \left( \bar{P} - 4\frac{L_z^2}{r} + \left( \Phi + \frac{3L_z^2}{2r^2} \right) \frac{\bar{\sigma}}{r} \right).
\]

The usual vertical stability criterion for a circular orbit is \( \frac{\partial V_{\text{eff}}}{\partial |z|} > 0 \) at \( z = 0 \) [11, 12], implying that, since \( \bar{\sigma} \) dominates the matter contributions, the Newtonian condition \( \bar{\sigma} > 0 \) is enough to guarantee vertical stability of equatorial circular orbits.

In order to obtain an approximate third integral of motion for nearly equatorial orbits (in addition to \( E \) and \( L_z \)), we follow closely [11, 12]. The approximate Hamiltonian \( H^{(a)} \) for the motion near the stable equatorial circular orbit of radius \( r_o \) is written as

\[
H^{(a)} = \frac{A(r_o, 0)}{2} \left[ p_r^2 + p_z^2 \right] + V_{\text{eff}}^{(a)},
\]

where the approximate effective potential is given by

\[
V_{\text{eff}}^{(a)} = V_{\text{eff}}(r, 0) + \omega_{r_o} |z|
\]

with

\[
\omega_{r_o} = \frac{\partial V_{\text{eff}}}{\partial |z|}(r_o, 0).
\]

One can now separate the Hamiltonian as \( H^{(a)} = H_r + H_z \), where

\[
H_z = \frac{A(r_o, 0)}{2} p_z^2 + \omega_{r_o} |z|.
\]

The corresponding action variable can be written as [11, 12]

\[
J_z = \frac{4}{3\pi} \sqrt{\frac{2 \omega_{r_o} Z^3}{A(r_o, 0)}},
\]

where \( Z \) is the vertical amplitude of the motion. Considering that \( r_o \) varies slowly in time due to off-equatorial motion, one can explore adiabatic invariance arguments for the action variable \( J_z \) and derive a relation between \( Z \) and \( r \), establishing in this way the orbit’s envelope. Introducing the quantity

\[
\tilde{\omega}(r) = \frac{1}{A(r, 0)} \frac{\partial V_{\text{eff}}}{\partial |z|}(r, 0)
\]
(see Equation (45)), we have that $\tilde{\omega}(r)Z^3$ is approximately a constant, which gives us the envelopes for off-equatorial orbits

$$Z(r) \propto \left[ \tilde{\omega}(r) \right]^{-1/3}.$$ 

(52)

In terms of the original canonical variables we can write

$$I_3 = \left[ \tilde{\omega}(r) \right]^{-2/3} \left( \frac{1}{2} p_i^2 + \tilde{\omega}(r) z_i \right).$$

(53)

The quantity $I_3$ is therefore an approximate third integral of motion for our problem. The novelty here, with respect to the Newtonian [11] and the relativistic [12] cases, is that this integral and, consequently, the envelope of the orbits, exhibit explicitly the frame-dragging effect arising from the $L_2$ linear term in (45) and, consequently, in $\tilde{\omega}(r)$.

4 Post-Newtonian statistical mechanics

The distribution function $f(x^i, p_j, t)$ of a system of particles whose motion is described by the Hamiltonian flow of $H$, in the absence of collisions, satisfies the so-called collisionless Boltzmann equation (CBE)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{ f, H \} = 0,$$

(54)

with $\{ , \}$ standing for the usual Poisson brackets, where $(x^i, p_j)$ are canonical coordinates. In our case, the CBE reads

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} - \frac{\partial \Phi}{\partial x^i} \frac{\partial f}{\partial p_i} + \frac{1}{c^2} \left( \left( 3 \Phi - \frac{p^2}{2} \right) p^i - \zeta^i \right) \frac{\partial f}{\partial x^i}$$

$$- \frac{1}{c^2} \left( \Phi + \frac{3}{2} p^2 \right) \frac{\partial f}{\partial x^i} + \frac{\partial \Psi}{\partial x^i} - p^j \frac{\partial \zeta^j}{\partial x^i} \right) \frac{\partial f}{\partial p_i} = 0,$$

(55)

where summation over repeated indices is assumed. The formulation of the CBE in terms of canonical momenta may be useful for constructing self-consistent, stationary models of self-gravitating systems in the 1PN approximation as a complementary tool to the results developed in [2, 3]. It also gives us, in a direct way, the conserved quantities associated with symmetries of the dynamics, for instance energy in stationary spacetimes and angular momentum in spherically symmetric spacetimes. The expressions obtained are the same of [2].

Nevertheless, for sake of comparison with some previous results in the literature, let us consider the distribution function $\tilde{f}$ in the Lagrangian formulation, $f(x^i, p_j, t) = \tilde{f}(x^i, v^i(x, p, t), t)$. We have, up to $\frac{1}{c^2}$ order,

$$\frac{\partial \tilde{f}}{\partial x^i} = \frac{\partial \tilde{f}}{\partial x^i} + \frac{1}{c^2} \left( 3 v^i \frac{\partial \Phi}{\partial x^i} - \frac{\partial \zeta^i}{\partial x^i} \right) \frac{\partial \tilde{f}}{\partial v^i}$$

(56)

$$\frac{\partial \tilde{f}}{\partial p_i} = \frac{\partial \tilde{f}}{\partial v_i} + \frac{1}{c^2} \left[ \left( 3 \Phi - \frac{v^2}{2} \right) \frac{\partial \tilde{f}}{\partial v_i} - v^i \frac{\partial v^j}{\partial v^i} \right]$$

(57)

$$\frac{\partial \tilde{f}}{\partial t} = \frac{\partial \tilde{f}}{\partial x^i} + \frac{1}{c^2} \left( 3 v^i \frac{\partial \Phi}{\partial t} - \frac{\partial \zeta^i}{\partial t} \right) \frac{\partial \tilde{f}}{\partial v^i},$$

(58)

giving from (55) the following equation for the distribution function $\tilde{f}$,

$$\frac{\partial \tilde{f}}{\partial t} + v^i \frac{\partial \tilde{f}}{\partial x^i} + \dot{v}^i \frac{\partial \tilde{f}}{\partial v^i} = 0,$$

(59)

with $\dot{v}^i$ given by (6), confirming again the consistency of our results. Equation (60) can be compared with the IPN Liouville equation presented in [2], their equation (9). For a better comparison, let us write (59) explicitly with all the spatial terms,

$$\frac{\partial \tilde{f}}{\partial t} + v^i \frac{\partial \tilde{f}}{\partial x^i} - \partial \Theta \frac{\partial \tilde{f}}{\partial \Theta} + \frac{1}{c^2} \left( 4 v_i v_j \frac{\partial \Phi}{\partial x^j} - v^2 \frac{\partial \Phi}{\partial x^i} + 3 v_i \frac{\partial \Phi}{\partial t} + v^j \left( \frac{\partial \zeta^j}{\partial x^i} - \frac{\partial \zeta^i}{\partial x^j} \right) \right) = 0.$$  

(60)

Although they seem to be different at first look, one may check that the difference arises because of a global multiplying factor $U^0/c = dt/d\tau$ in their equation, where $\tau$ is the proper time along the trajectory.
More specifically, writing Eq. (60) as \(\mathcal{L}_v f = 0\), where \(\mathcal{L}_v\) is the Liouville operator, it follows that the Liouville operator \(\mathcal{L}_U\) of [2] corresponds to \(\mathcal{L}_U = \left(\frac{U^0}{c}\right) \mathcal{L}_v\). The equations are dynamically equivalent; their difference in form can be attributed to different choices for the time-evolution parameter. One advantage of the present formulation is that the 1PN correction to the Newtonian case appears only in the term \(\frac{\partial f}{\partial v}\), showing explicitly the post-Newtonian correction in the Lagrangian formulation of the CBE.

5 Conclusions

We presented here, for the first time, the Hamiltonian formalism for the dynamics of test particles in spacetimes with arbitrary energy-momentum distributions in the first post-Newtonian (1PN) approximation. This construction is consistent with the corresponding Lagrangian formalism up to 1PN order and therefore gives an alternative way to deal with particle dynamics in 1PN self-gravitating systems, with the advantage of presenting explicitly the symplectic nature of the dynamics.

The expressions obtained for physical observables in (stationary) axially symmetric configurations, such as the 1PN circular speed and (radial and vertical) epicyclic frequencies of quasi-circular equatorial motion in the presence of frame-dragging effects, are useful for further studies including 1PN terms in galactic dynamics, giving us relativistic corrections to rotation curves of spiral and elliptical galaxies, and to the positions of Lindblad resonances. Moreover, the stationary, axially symmetric Hamiltonian (27), or even the general expression (12), permits us to perform systematic studies of chaos in 1PN self-gravitating systems. One application of this formalism to the dynamics of disk-crossing orbits was presented here, the approximated third integral of motion for stationary, axially symmetric razor-thin disk systems which also determines analytically the envelope \(Z(r)\) of nearly equatorial orbits. We emphasize that in all our results we could identify the explicit presence of frame-dragging effects.

We also obtained the 1PN collisionless Boltzmann equation in terms of the canonical variables \((x^i, p_j)\), and of the Hamiltonian (12). This form of the CBE may be useful for constructing self-gravitating models for the distribution function of gravitating stellar systems in terms of canonical momenta and of action variables when GR corrections are taken into account, as a complementary method to the results of [2,3] based on the Lagrangian formalism. We point out that the CBE (55), when written in velocity space \((x^i, v^j)\), Equation (60), gives a different result from [2]. This difference, however, is due solely to a global multiplying factor in the Liouville operator, giving dynamically equivalent results. One advantage of our formalism is that, when written in terms of \((x^i, v^j)\), the post-Newtonian corrections appear only in the \(\frac{\partial f}{\partial v}\) term, as expected from the 1PN corrections to the equations of motion in the Lagrangian formalism.

One potentially interesting application of the results presented here is the construction of 1PN spherical self-consistent models and of distribution functions for 1PN razor-thin disks, possibly in a more direct way than in the Lagrangian formalism [2,3], helping to extend the results presented in those works.

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