ALMOST METRIC VERSIONS OF
ZHONG’S VARIATIONAL PRINCIPLE

MIHAI TURINICI

Abstract. A refinement of Zhong’s variational principle [Nonlin. Anal., 29 (1997), 1421-1431] is given, in the realm of almost metric structures. Applications to equilibrium points are also provided.

1. Introduction

Let $M$ be some nonempty set; and $d : M \times M \to \mathbb{R}_+$, a metric over it (in the usual sense). Further, take some function $\varphi : M \to \mathbb{R} \cup \{\infty\}$ with

(a01) $\varphi$ is inf-proper ($\text{Dom}(\varphi) \neq \emptyset$ and $\varphi_* := \inf[\varphi(M)] > -\infty$).

The following 1979 statement in Ekeland [5] (referred to as Ekeland’s variational principle; in short: EVP) is our starting point. Assume that

(a02) $d$ is complete (each $d$-Cauchy sequence is $d$-convergent)

(a03) $\varphi$ is $d$-lsc ($\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$).

Theorem 1. Let these conditions hold. Then,

\begin{enumerate}[i)]
  \item for each $u \in \text{Dom}(\varphi)$ there exists $v = v(u) \in \text{Dom}(\varphi)$ with
    \[ d(u, v) \leq \varphi(u) - \varphi(v) \quad (\text{hence } \varphi(u) \geq \varphi(v)) \]  
    \[ d(v, x) > \varphi(v) - \varphi(x), \quad \text{for all } x \in M \setminus \{v\} \]  
  \item if $u \in \text{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \leq \rho$, then (1.1) gives
    \[ (\varphi(u) \geq \varphi(v) \text{ and}) \quad d(u, v) \leq \rho. \]
\end{enumerate}

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the quoted paper for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of EVP were proposed. For example, the (abstract) order one starts from the fact that, with respect to the (quasi-) order

(a04) $(x, y \in M) \ x \leq y \text{ iff } d(x, y) + \varphi(y) \leq \varphi(x)$

the point $v \in M$ appearing in (1.2) is maximal; so that, Theorem 1 is nothing but a variant of the Zorn maximality principle. The dimensional way of extension refers to the ambient space ($R$) of $\varphi(M)$ being substituted by a (topological or not) vector space; an account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [6, Ch 3]. Further, the (pseudo) metrical one consists in the conditions imposed to the almost metric over $M$ being relaxed. The basic result in this direction was obtained in 1992 by

2010 Mathematics Subject Classification. 54F05 (Primary), 47J20 (Secondary).

Key words and phrases. Inf-proper lsc function, variational principle, maximal element, almost metric, normal function, nonexpansiveness, equilibrium point.

1
Tataru [14], via Ekeland type techniques; subsequent extensions of it may be found in the 1996 paper by Kada, Suzuki and Takahashi [7]. Finally, we must add to this list the "functional" extension of EVP obtained in 1997 by Zhong [15] (and referred to as Zhong’s variational principle; in short: ZVP). Take a function \( t \mapsto b(t) \) from \( R_+ := [0, \infty] \) to itself, with the normality properties
\[
(b01) \quad b \text{ is decreasing and } b(R_+) \subseteq R^*_+ := ]0, \infty[.
\]
\[
(b02) \quad B(\infty) = \infty, \quad \text{where } B(t) = \int_0^t b(\tau) d\tau, \quad t \geq 0.
\]

**Theorem 2.** Let \( a \in M \) be given, and \( u \in \text{Dom}(\varphi) \), \( \rho > 0 \) be taken so as \( \varphi(u) - \varphi_* \leq B(d(a,u) + \rho) - B(d(a,u)) \). There exists then \( v = v(u) \) in \( \text{Dom}(\varphi) \) with
\[
d(a,v) \leq d(a,u) + \rho, \quad \varphi(u) \geq \varphi(v)
\]
\[
b(d(a,v))d(v,x) > \varphi(v) - \varphi(x), \quad \text{for each } x \in M \setminus \{v\}.
\]

Clearly, ZVP includes (for \( b = 1 \) and \( a = u \)) the local version of EVP based upon (1.3). The relative form of the same, based upon (1.4) also holds (but indirectly); cf. Bao and Khanh [2]. Summing up, ZVP includes EVP; but, the argument developed there is rather involved; this is equally true for another proof of the same, proposed by Suzuki [13]. A simplification of this reasoning was given in Turinici [16], by a technique due to Park and Bae [11]; note that, as a consequence of this, ZVP \( \iff \) EVP. It is our aim in the following to show that such a conclusion continues to hold - under general completeness conditions - in the almost metric framework; details will be given in Section 3. The basic tool for this is a pseudometric variational principle in Turinici [17], discussed in Section 2. Finally, in Section 4 and Section 5, an application of these facts to equilibrium points is considered.

## 2. Pseudometric Ordering Principles

Let \( M \) be a nonempty set. By a pseudometric over \( M \) we shall mean any map \( e : M \times M \to R_+ \). Fix such an object; which in addition is triangular \( e(x, z) \leq e(x, y)+e(y, z), \forall x, y, z \in M \) and reflexive \( e(x, x) = 0, \forall x \in M \). Call the sequence \((x_n)\) in \( M, I \) strongly \( e\)-asymptotic (in short: \( e\)-strasy) provided [the series \( \sum_{n=1}^{\infty} e(x_n, x_{n+1}) \) converges] and \( II \) \( e\)-Cauchy when \( [\forall \delta > 0, \exists n(\delta), \text{ such that } n(\delta) \leq p \leq q \implies e(x_p, x_q) \leq \delta] \). By the triangular property, we have \([\forall \text{ sequence}]: \text{ \( e\)-strasy } \implies \text{ \( e\)-Cauchy}\); but the converse is not in general true. Further, define an \( e\)-convergence structure on \( X \) as: \( x_n \stackrel{e}{\to} x \) iff \( e(x_n, x) \to 0 \) as \( n \to \infty \); referred to as: \( x \) is an \( e\)-limit of \((x_n)\). The set of all these will be denoted \( \text{lim}_n(x_n) \); when it is nonempty, we call \((x_n)\), \( e\)-convergent. Note that, by the lack of symmetry, a relationship like \([\forall \text{ sequence}]: \text{ \( d\)-convergent } \implies \text{ \( d\)-Cauchy}\) is not in general true. Finally, let \( \varphi : M \to R \cup \{\infty\} \) be some inf-proper function. We consider the regularity condition

\[
(b01) \quad (e, \varphi) \text{ is weakly descending complete: for each } \text{ \( e\)-strasy sequence } \(x_n) \subseteq \text{Dom}(\varphi) \text{ with } \langle \varphi(x_n) \rangle \text{ descending there exists } x \in M \text{ with } x_n \stackrel{e}{\to} x \text{ and } \text{lim}_n \varphi(x_n) \geq \varphi(x).
\]

By the generic property above, it is implied by its (stronger) counterpart

\[
(b02) \quad (e, \varphi) \text{ is descending complete: for each } \text{ \( e\)-Cauchy sequence } \(x_n) \in \text{Dom}(\varphi) \text{ with } \langle \varphi(x_n) \rangle \text{ descending there exists } x \in M \text{ with } x_n \stackrel{e}{\to} x \text{ and } \text{lim}_n \varphi(x_n) \geq \varphi(x).
\]

The reciprocal inclusion also holds \([so, (b01) \iff (b02)]\) as it can be directly seen.

The following variational principle is our starting point.
Proposition 1. Assume that (b01)/(b02) holds. Then, for each \( v \in \text{Dom}(\varphi) \), there exists \( v = v(n) \in \text{Dom}(\varphi) \) satisfying (1.3) (with \( e \) in place of \( d \)) as well as
\[
i) \quad \varphi(v) = \varphi(x) \quad \text{and} \quad e(v,x) = 0.
\]
Consequently, the relations below hold
\[
ii) \quad e(v,x) \geq \varphi(v) - \varphi(x), \quad \text{for all} \quad x \in M
\]
\[
iii) \quad e(v,x) > \varphi(v) - \varphi(x), \quad \text{for each} \quad x \in M \quad \text{with} \quad e(v,x) > 0.
\]

The proof consists in applying Brezis-Browder’s ordering principle [4] to the triplet \((M,\preceq;\psi)\), where \( M = \{ x \in M : \varphi(x) \leq \varphi(u) \} \), \( (\preceq) \) stands for the quasi-order \((a04)\) (with \( e \) in place of \( d \)) and \( \psi(.) = \varphi(.) - \varphi_+ \); see Turinici [17].

In particular, condition (b01) is retainable under
\[
(b03) \quad (e,\varphi) \text{ is weakly complete: for each } e\text{-strasy sequence } (x_n) \text{ in } \text{Dom}(\varphi)
\]
there exists \( x \in M \) with \( x_n \overset{e}{\to} x \) and \( \liminf_n \varphi(x_n) \geq \varphi(x) \).

As a consequence, Proposition 1 incorporates the variational principle in Tataru [14]; see also Kang and Park [8].

Call the pseudometric \( e : M \times M \to R_+ \), an almost metric provided it is in addition triangular and reflexive sufficient \([e(x,y) = 0 \iff x = y]\). In this case, Proposition 1 yields the following practical statement.

Theorem 3. Let the almost metric \( e \) and the inf-proper function \( \varphi \) be as in (b02). Then, conclusions of Theorem 1 hold, with \( e \) in place of \( d \).

Now, evidently, (b02) is retainable whenever
\[
(b04) \quad (e,\varphi) \text{ is complete: for each } e\text{-Cauchy sequence } (x_n) \text{ in } \text{Dom}(\varphi)
\]
there exists \( x \in M \) with \( x_n \overset{e}{\to} x \) and \( \liminf_n \varphi(x_n) \geq \varphi(x) \).

If \( e \) is in addition symmetric \((e(x,y) = e(y,x), \forall x,y \in M)\) (hence, a metric over \( M \),
\( (b04) \) holds under \((a02)+(a03)\) (modulo \( e \)). This tells us that Theorem 3 includes EVP; it will be referred to as the almost metric version of EVP (in short: EVPam).
The reciprocal inclusion \((\text{EVP} \implies \text{EVPam})\) is open; we conjecture that the answer is positive. Some related aspects may be found in Turinici [15].

3. Zhong variational statements

(A) Let \( M \) be some nonempty set. Take a couple of almost metrics \( d,e \) on \( M \); we say that \( e \) is \( d\)-compatible provided
\[
(c01) \quad \text{each } e\text{-Cauchy sequence is } d\text{-Cauchy too}
\]
\[
(c02) \quad y \mapsto e(x,y) \text{ is } d\text{-lsc, for each } x \in M.
\]

Note that both these properties hold when \( e = d \). In fact, \((c01)\) is trivial; and \((c02)\) results from the triangular property of \( d \) (see Proposition 4 for details). Further, let \( \varphi : M \to R \cup \{\infty\} \) be an inf-proper function. The following fact will be useful.

Lemma 1. Suppose that \( e \) is \( d\)-compatible. Then, \([d,\varphi) = \text{descending complete}]\ implies \([e,\varphi) = \text{descending complete}]\.

Proof. Let \((x_n)\) be some \( e\)-Cauchy sequence in \( \text{Dom}(\varphi) \) with \( (\varphi(x_n)) \) descending. From \((c01)\), \((x_n)\) is \( d\)-Cauchy too; so, by (b02) (modulo \( d \)), there exists \( y \in X \) with \( x_n \overset{d}{\to} y \) and \( \lim_n \varphi(x_n) \geq \varphi(y) \). We claim that this is our desired point. In fact, let \( \gamma > 0 \) be arbitrary fixed. By the initial choice of \((x_n)\), there exists \( k = k(\gamma) \) so that:
\[
e(x_p,x_n) \leq \gamma, \quad \text{for each } p \geq k \text{ and each } n \geq p.
\]
Passing to limit upon \( m \) one gets (via \((c02)\)) \( e(x_p,y) \leq \gamma \), for each \( p \geq k \); and since \( \gamma > 0 \) was arbitrarily chosen, \( x_n \overset{e}{\to} y \). This (and the previous relation) gives the conclusion we want. \( \square \)
Now, by simply combining this with Theorem 3 one gets the following ”relative” type variational statement (involving these data):

**Theorem 4.** Let the pair \((d, \varphi)\) be descending complete; and \(e\) be \(d\)-compatible. Then, conclusions of Theorem 3 are retainable.

For the moment, Theorem 3 \(\Rightarrow\) Theorem 4. The reciprocal is also true; for (see above) \(e = d\) is allowed here; so, Theorem 3 \(\equiv\) Theorem 4.

Now, this ”relative” variational statement may be viewed as an ”abstract” version of ZVP. To explain this, we need some constructions and auxiliary facts.

(B) Let \(b : R_+ \rightarrow R_+\) be some normal function. In particular, it is Riemann integrable on each compact interval of \(R_+\) and

\[
\int_p^q b(\xi)d\xi = (q - p)\int_0^1 b(p + \tau(q - p))d\tau, \quad 0 < p < q < \infty. \tag{3.1}
\]

Some basic facts involving the couple \((b, B)\) (where \(B : R_+ \rightarrow R_+\) is the one of (a06)) are being collected in

**Proposition 2.** The following are valid

i) \(B\) is a continuous order isomorphism of \(R_+\); hence, so is \(B^{-1}\)

ii) \(b(s) \leq (B(s) - B(t))/(s - t) \leq b(t), \forall t, s \in R_+, t < s\)

iii) \(B\) is almost concave: \(t \mapsto [B(t + s) - B(t)]\) is decreasing on \(R_+, \forall s \in R_+\)

iv) \(B\) is concave: \(B(t + \lambda(s - t)) \geq B(t) + \lambda(B(s) - B(t)), \forall t, s \in R_+\) with \(t < s\) and all \(\lambda \in [0, 1]\)

v) \(B\) is sub-additive (hence \(B^{-1}\) is super-additive).

The proof is immediate, by (3.1) above; so, we do not give details. Note that the properties in iii) and iv) are equivalent to each other, under i). This follows at once from the (non-differential) mean value theorem in Bantaş and Turinici [12].

(C) Now, let \(M\) be some nonempty set; and \(d : M \times M \rightarrow R_+\), an almost metric over it. Further, let \(\Gamma : M \rightarrow R_+\) be chosen as

(c03) \(\Gamma\) is almost \(d\)-nonexpansive \((\Gamma(x) - \Gamma(y)) + d(x, y) \geq 0, \forall x, y \in M\).

Define a pseudometric \(e = e(B, \Gamma; d)\) over \(M\) as

(c04) \(e(x, y) = B(\Gamma(x) + d(x, y)) - B(\Gamma(x)), \quad x, y \in M\).

This may be viewed as an ”explicit” formula; the implicit version of it is

(c05) \(d(x, y) = B^{-1}(B(\Gamma(x)) + e(x, y)) - \Gamma(x), \quad x, y \in M\).

We shall establish some properties of this map, useful in the sequel.

I) First, the ”metrical” nature of \((x, y) \mapsto e(x, y)\) is of interest.

**Proposition 3.** The pseudometric \(e(.,.)\) is an almost metric over \(M\).

*Proof.* The reflexivity and sufficiency are clear, by Proposition 2 i); so, it remains to establish the triangular property. Let \(x, y, z \in M\) be arbitrary fixed. The triangular property of \(d : M \times M \rightarrow R_+\) yields (via Proposition 2 i)) \(e(x, z) \leq B(\Gamma(x) + d(x, y) + d(y, z)) - B(\Gamma(x) + d(x, y)) + e(x, y)\). On the other hand, the almost \(d\)-nonexpansiveness of \(\Gamma\) gives \(\Gamma(x) + d(x, y) \geq \Gamma(y)\); so (by Proposition 2 iii)) \(B(\Gamma(x) + d(x, y) + d(y, z)) - B(\Gamma(x) + d(x, y)) \leq e(y, z)\). Combining with the previous relation yields our desired conclusion. \(\Box\)

II) By definition, \(e\) will be called the Zhong metric attached to \(d\) and the couple \((B, \Gamma)\). The following properties of \((d, e)\) are immediate (via Proposition 2):
Lemma 2. Under the prescribed conventions,

vi) $b(\Gamma(x) + d(x,y))d(x,y) \leq e(x,y) \leq b(\Gamma(x))d(x,y)$, for all $x, y \in M$ 

vii) $e(x,y) \leq B(d(x,y)), \forall x, y \in M$; hence $x_n \xrightarrow{d} x$ implies $x_n \xrightarrow{e} x$.

III) A basic property of $e(\cdot, \cdot)$ to be checked is $d$-compatibility.

Proposition 4. The Zhong metric $e$ is $d$-compatible (cf. (c01)+(c02)).

Proof. We firstly check (c02); which may be written as

$$[e(x,y_n) \leq \lambda, \forall n] \text{ and } y_n \xrightarrow{d} y \implies e(x, y) \leq \lambda.$$  

So, let $x, (y_n), \lambda$ and $y$ be as in the premise of this relation. By Lemma 2 we have $y_n \xrightarrow{d} y$ as $n \to \infty$. Moreover (as $e$ is triangular) $e(x,y) \leq e(x,y_n) + e(y_n,y) \leq \lambda + e(x,y_n)$, for all $n$. It will suffice passing to limit as $n \to \infty$ to get the desired conclusion. Further, we claim that (c01) holds too, in the sense: 

[(for each sequence) $d$-Cauchy $\iff$ $e$-Cauchy]. The left to right implication is clear, via Lemma 2. For the right to left one, assume that $(x_n)$ is an $e$-Cauchy sequence in $M$. In particular (by the triangular property) $e(x_i, x_j) \leq \mu$, for all $(i,j)$ with $i \leq j$ and some $\mu \geq 0$. This, along with (c05), yields $d(x_0, x_i) = B^{-1}(B(\Gamma(x_0)) + e(x_0, x_i)) - \Gamma(x_0) \leq B^{-1}(B(\Gamma(x_0)) + \mu) - \Gamma(x_0), \forall i \geq 0$; wherefrom (cf. (c03)) $\Gamma(x_i) \leq \Gamma(x_0) + d(x_0, x_i) \leq B^{-1}(B(\Gamma(x_0)) + \mu)$ (hence $B(\Gamma(x_i)) \leq B(\Gamma(x_0) + \mu)$, for all $i \geq 0$. Putting these facts together yields (again via (c05)) $\Gamma(x_i) + d(x_i, x_j) = B^{-1}(B(\Gamma(x_i)) + e(x_i, x_j)) \leq \nu := B^{-1}(B(\Gamma(x_0)) + 2\mu)$, for all $(i,j)$ with $i \leq j$. And this, via Lemma 2 gives (for the same pairs $(i,j)$) 

$$e(x_i, x_j) \geq b(\Gamma(x_i) + d(x_i, x_j))d(x_i, x_j) \geq b(\nu)d(x_i, x_j).$$  

But then, the $d$-Cauchy property of $(x_n)$ is clear; and the proof is complete. 

(D) We are now in position to make precise our initial claim. Let the almost metric $d$ and the in-proper function $\phi$ be as in (b02) (modulo $d$). Further, take a normal function $b : R_+ \to R_+$; as well as an almost $d$-nonexpansive map $\Gamma : M \to R_+$. Finally, put $e = e(B, \Gamma; d)$ (the Zhong metric introduced by (c04)/(c05)).

Theorem 5. Let the conditions above be admitted. Then

viii) For each $u \in \operatorname{Dom}(\phi)$ there exists $v = v(u) \in \operatorname{Dom}(\phi)$ with

$$b(\Gamma(u) + d(u,v))d(u,v) \leq e(u,v) \leq \phi(u) - \phi(v) \tag{3.2}$$

$$b(\Gamma(v))d(v,x) \geq e(v,x) \geq \phi(v) - \phi(x), \quad \forall x \in M \setminus \{v\} \tag{3.3}$$

ix) For each $u \in \operatorname{Dom}(\phi)$, $\rho > 0$ with $\phi(u) - \phi(x) \leq B(\Gamma(u) + \rho) - B(\Gamma(u))$ the above evaluation (3.2) gives

$$d(u,v) \leq \rho; \quad \text{hence } \Gamma(v) \leq \Gamma(u) + \rho \tag{3.4}$$

$$b(\Gamma(u) + \rho)d(u,v) \leq \phi(u) - \phi(v) \quad (\text{hence } \phi(u) \geq \phi(v)). \tag{3.5}$$

Proof. By Proposition 3, $e$ is an almost metric over $M$; and, by Proposition 4, it is $d$-compatible. Hence, Theorem 1 applies to such data. In this case, (3.2)+(3.3) are clear via Lemma 2. Moreover, if $u \in \operatorname{Dom}(\phi)$ is taken as in the premise of ix), then (cf. (3.2)) $e(u,v) \leq \phi(u) - \phi(v) \leq \phi(u) - \phi(x)$; wherefrom (by (c05)) 

$$d(u,v) \leq B^{-1}(B(\Gamma(u)) + \phi(u) - \phi(x)) - \Gamma(u) \leq \rho; \text{ and } (3.4)+(3.5) \text{ follow as well}. \quad \square$$

So far, Theorem 3 $\implies$ Theorem 4 $\implies$ Theorem 5. In addition, Theorem 5 $\implies$ Theorem 3 just take $b = 1$ (hence $B$-identity, $e = d$). Summing up, these three variational principles are mutually equivalent. On the other hand, Theorem 5 may
be also viewed as an extended (modulo $\Gamma$) version of ZVP. For, if $d$ is symmetric (hence a (standard) metric), (c03) becomes

\[ |\Gamma(x) - \Gamma(y)| \leq d(x, y), \text{ for all } x, y \in M \quad (\Gamma \text{ is } d\text{-nonexpansive}). \]

In addition, the choice

\[(c07) \quad \Gamma (x) = d(a, x), \quad x \in M, \quad \text{ for some } a \in M\]

is in agreement with it; hence the claim. For this reason, Theorem 5 will be referred to as the almost metric version of ZVP (in short: ZVPam). This inclusion is technically strict; because the conclusions involving the middle terms in (3.2)+(3.3) cannot be obtained in the way described by Zhong [18]. Some related aspects were delineated in Ray and Walker [12]; see also Suzuki [13].

4. Application (equilibrium points)

Let $M$ be some nonempty set; and $e : M \times M \to R_+$ be an almost metric over it. Any (extended) function $G : M \times M \to R \cup \{-\infty, \infty\}$ will be referred to as a relative generalized pseudometric on $M$. Given such an object, we say that $v \in M$ is an equilibrium point of $G$ when $G(v, x) \geq 0$, $\forall x \in M$. Note that, if $G = F + e$, where $F : M \times M \to R \cup \{-\infty, \infty\}$ is another relative generalized pseudometric on $M$, this definition becomes $e(v, x) \geq -F(v, x)$, $\forall x \in M$; which, under the choice (for some $\varphi : M \to R \cup \{\infty\}$)

\[(d01) \quad F(x, y) = \varphi(y) - \varphi(x), \quad x, y \in M \quad (\text{where } -\infty - \infty = 0)\]

tells us that the variational property of $v$ is "close" to the one in Theorem 3. So, existence of such points is deductible from the quoted result; to do this, one may proceed as follows. Assume that the relative generalized pseudometric $F$ is triangular $[F(x, z) \leq F(x, y) + F(y, z)$, whenever the right member exists] and reflexive [for all $x \in M$. Define the (extended) function $\mu(x) = \sup\{-F(x, y) : y \in M\}$, $x \in M$; clearly, $\mu(M) \subseteq R_+ \cup \{\infty\}$. The alternative $\mu(M) = \{\infty\}$ cannot be excluded; to avoid this, assume

\[(d02) \quad \mu \text{ is proper (Dom}(\mu) := \{x \in M ; \mu(x) < \infty\} \neq \emptyset).\]

For the arbitrary fixed $u \in \text{Dom}(\mu)$ put $F_u(.) = F(u, .)$. We have by definition

\[F_u(u) = 0; \quad F_u^* := \inf\{F_u(x) : x \in M\} = -\mu(u) > -\infty; \quad (4.1)\]

so, $F_u$ is inf-proper [and we say: $F$ is semi inf-proper]. Further, let $d$ be another almost metric on $M$ with

\[(d03) \quad (d, F) \text{ is semi descending complete:}\]
\[d(F_u) \text{ is descending complete, for each } u \in \text{Dom}(\mu).\]

**Theorem 6.** Let $(d02)+(d03)$ hold; and $e$ be $d$-compatible. Then, for each $u \in \text{Dom}(\mu)$ there exists $v = v(u)$ in $M$ such that

i) $e(u, v) \leq -F(u, v) \leq \mu(u)(< \infty)$

ii) $e(v, x) > -F(v, x)$, for all $x \in M \setminus \{v\}$.

Hence, in particular, $v$ is an equilibrium point for $G := F + e$.

**Proof.** From Theorem 4 it follows that, for the starting $u \in \text{Dom}(\mu)$ (hence $u \in \text{Dom}(F_u)$) there must be another point $v \in \text{Dom}(F_u)$ with the properties I) $e(u, v) \leq F_u(u) - F_u(v)$ and II) $e(v, x) > F_u(v) - F_u(x)$, $\forall x \in M \setminus \{v\}$. The former of these is just i), by the reflexivity of $F$. And the latter yields ii); for (by the triangular property) $F(u, v) - F(u, x) \geq F(u, v) - (F(u, v) + F(v, x)) = -F(v, x)$. \qed
Now, a basic particular choice of $e(.,.)$ is related to the constructions in Section
3. Precisely, let the function $b : R_+ \to R_+$ be normal; and $\Gamma : M \to R_+$ be almost
$d$-nonexpansive. Let $e = e(B, \Gamma; d)$ stand for the Zhong metric given by (c04)/(c05).
By Theorem [5] we then have

**Theorem 7.** Let (d02)+(d03) hold. Then, for each $u \in \text{Dom}(\mu)$ there exists
$v = v(u) \in M$ such that

(iii) $b(\Gamma(u) + d(u,v))d(u,v) \leq e(u,v) \leq -F(u,v) \leq \mu(u)$

(iv) $b(\Gamma(v))d(v,x) \geq e(v,x) \geq -F(v,x), \forall x \in M \setminus \{v\}$.

Hence, in particular, $v$ is an equilibrium point for $G(x,y) = F(x,y) + b(\Gamma(x))d(x,y)$,
$x,y \in M$. Moreover, $u \in \text{Dom}(\mu)$ whenever

(d04) $\mu(u) \leq B(\Gamma(u) + \rho) - B(\Gamma(u)), \text{ for some } \rho > 0$;

and then (as $F_u(u) - F_u^* \equiv \mu(u)$), iii) gives (2.4) and

v) $b(\Gamma(u) + \rho)d(u,v) \leq -F(u,v)$ (hence $F(u,v) \leq 0$).

Some remarks are in order. Let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper function.
The relative (generalized) pseudometric $F$ over $M$ given as in (d01) is reflexive,
triangular and fulfills (d02); because $\mu(.) = \varphi(.) - \varphi_\ast$ (hence $\text{Dom}(\mu) = \text{Dom}(\varphi)$).
In addition, as $F_u(\cdot) = \varphi(.) - \varphi(u)$, $u \in \text{Dom}(\varphi)$, (d03) is identical with (b02)
(modulo $d$). Putting these together, it follows that Theorems [5] and [7] include
Theorems [3] and [5] respectively. The reciprocal inclusions are also true, by the very
argument above; so that Theorem [0] $\iff$ Theorem [3] and Theorem [7] $\iff$ Theorem
[5] In particular, when $\Gamma$ is taken as in (e07), Theorem [7] yields the main result in
Zhu, Zhong and Cho [10]; see also Bao and Khanh [2].

5. The BKP approach

Let $(M,d)$ be a complete metric space. By a relative pseudometric over $M$ we
mean any map $g : M \times M \to R$. Given such an object, remember that $v \in M$ is an
equilibrium point of it when $g(v,x) \geq 0, \forall x \in M$. Note that, if $g = f + d$, where $f : M \times M \to R$ is another relative pseudometric on $M$, this writes $d(v,x) \geq -f(v,x)$,
$\forall x \in M$; so that, under the choice (d01) of $f$ (where $\varphi : M \to R$), the variational
property of $v$ is ”close” to the one in EVP. The following 2005 result in the area
due to Bianchi, Kassay and Pini [3] (in short: BKP) is available.

**Theorem 8.** Suppose that $f$ is triangular reflexive and

(e01) $f(a,.)$ is bounded from below and lsc, for each $a \in M$.

Then, for each $u \in M$, there exists $v = v(u) \in M$ such that

i) $d(u,v) \leq -f(u,v)$; ii) $d(v,x) > -f(v,x), \forall x \in M \setminus \{v\}$.

Hence, in particular, $v$ is an equilibrium point for $g := f + d$.

Note that this result is obtainable from Theorem [5] by simply taking $d = e$. On the other hand, under the same choice (d01) for $f$, (e01) becomes

(e02) $\varphi$ is bounded from below and lsc;

and Theorem [3] is just EVP. So, we may ask whether this extension is effective.
The answer is negative; i.e., Theorem [5] is deductible from (hence equivalent with)
EVP. This will follow from

**Proof.** (Theorem [8]) Define a new function $h : M \to R$ as $h(x) = f(u,x), x \in M$.
From (e01), EVP is applicable to $(M,d)$ and $h$; wherefrom, for the starting $u \in M$
there exists $v \in M$ with I) $d(u,v) \leq h(u) - h(v)$, II) $d(v,x) > h(v) - h(x), \forall x \in$
$\mathcal{M}\setminus \{v\}$. The former of these gives i), in view of $h(u) = 0$. And the latter one gives ii); because (from the triangular property) $h(v) - h(x) \geq -f(v, x)$, for all such $x$. Hence the conclusion.

This argument (taken from the 2003 paper due to Bao and Khanh [2]) tells us that Theorem 8 is just a formal extension of EVP. This is also true for the 1993 statement in the area due to Oettli and Thera [10]. In fact, the whole reasoning developed in [3] for proving Theorem 8 is, practically, identical with the one of this last paper. Further aspects may be found in Lin and Du [9].

References

[1] G. Bantaş and M. Turinici, Mean value theorems via division methods, An. Şt. Univ. ”A. I. Cuza” Iaşi (S. I-a, Mat.), 40 (1994), 135-150.
[2] T. Q. Bao and P. Q. Khanh, Are several recent generalizations of Ekeland’s variational principle more general than the original principle?, Acta Math. Vietnamica, 28 (2003), 345-350.
[3] M. Bianchi, G. Kassay and R. Pini, Existence of equilibria via Ekeland’s principle, J. Math. Analysis Appl., 305 (2005), 502-512.
[4] H. Brezis and F. E. Browder, A general principle on ordered sets in nonlinear functional analysis, Advances Math., 21 (1976), 355-364.
[5] I. Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.
[6] A. Goepfert, H. Riahi, C. Tammer and C. Zălinescu, Variational Methods in Partially Ordered Spaces, Canad. Math. Soc. Books in Math. vol. 17, Springer, New York, 2003.
[7] O. Kada, T. Suzuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japonica, 44 (1996), 381-391.
[8] B. G. Kang and S. Park, On generalized ordering principles in nonlinear analysis, Nonlinear Analysis, 14 (1990), 159-165.
[9] L. J. Lin and W. S. Du, Ekeland’s variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, J. Math. Analysis Appl., 323 (2006), 360-370.
[10] W. Oettli and M. Thera, Equivalents of Ekeland’s principle, Bull. Austral. Math. Soc., 48 (1993), 385-392.
[11] S. Park and J. S. Bae, On the Ray-Walker extension of the Caristi-Kirk fixed point theorem, Nonlinear Analysis, 9 (1985), 1135-1136.
[12] W. O. Ray and A. Walker, Mapping theorems for Gateaux differentiable and accretive operators, Nonlinear Analysis, 6 (1982), 423-433.
[13] T. Suzuki, Generalized distance and existence theorems in complete metric spaces, J. Math. Analysis Appl., 253 (2001), 440-458.
[14] D. Tataru, Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms, J. Math. Analysis Appl., 163 (1992), 345-392.
[15] M. Turinici, Metric variants of the Brezis-Browder ordering principle, Demonstr. Math., 22 (1989), 213-228.
[16] M. Turinici, Zhong’s variational principle is equivalent with Ekeland’s, Fixed Point Theory, 6 (2005), 133-138.
[17] M. Turinici, Function variational principles and coercivity over normed spaces, Optimization, 59 (2010), 199-222.
[18] C. K. Zhong, A generalization of Ekeland’s variational principle and application to the study of the relation between the weak P.S. condition and coercivity, Nonlinear Analysis, 29 (1997), 1421-1431.
[19] J. Zhu, C. K. Zhong and Y. J. Cho, Generalized variational principle and vector optimization, J. Optim. Th. Appl., 106 (2000), 201-217.

"A. Myller" Mathematical Seminar; "A. I. Cuza" University; 11, Copou Boulevard; 700506 Iaşi, Romania
E-mail address: mturi@uaic.ro