RESTRICTION OF HECKE EIGENFORMS TO HOROCYCLES

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Abstract. We prove a sharp upper bound on the $L^2$-norm of Hecke eigenforms restricted to a horocycle, as the weight tends to infinity.

1. Introduction

A central problem in “quantum chaos” is to understand the limiting behavior of eigenfunctions. An important example that has attracted a lot of attention is that of Maass cusp forms with large Laplace eigenvalue on the modular surface $X = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Let $\phi$ denote such a Maass form, with eigenvalue $\lambda$, and normalized to have $L^2$-norm 1: that is, $\int_X |\phi(z)|^2 \frac{dx\,dy}{y^2} = 1$. Then the Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak \cite{RS} states that the measure $\mu_\phi = |\phi(z)|^2 \frac{dx\,dy}{y^2}$ tends to the uniform measure on $X$ as $\lambda \to \infty$. If $\phi$ is also assumed to be an eigenfunction of all the Hecke operators, then QUE holds by the work of Lindenstrauss \cite{L}, with a final step on escape of mass provided by Soundararajan \cite{S}. Thus, the measure $\mu_\phi$ does not concentrate on subsets of $X$ with small measure, but is uniformly spread out. A finer problem is to understand how much the measure can concentrate on sub-manifolds; for example, on a geodesic, or a closed horocycle, or even at just a point (that is, bounding the $L^\infty$ norm). The letter of Sarnak to Reznikov \cite{Sarn} draws attention to such restriction problems, and these problems (and generalizations) have been studied extensively in recent years, see for example \cite{Be}, \cite{Be2}, \cite{Bu}, \cite{Hu}, \cite{HS}, \cite{R}, \cite{R2}, \cite{R3}.

This note is concerned with a related question for holomorphic modular forms for $\text{SL}_2(\mathbb{Z})$ that are also eigenfunctions of all Hecke operators, when the weight $k$ becomes large. Let $f$ be a Hecke eigenform of weight $k$ on the modular surface $X$, with $L^2$-norm 1: that is,

$$\int_X y^k |f(z)|^2 \frac{dx\,dy}{y^2} = 1.$$ 

To $f$, we associate the measure $\mu_f = y^k |f(z)|^2 \frac{dx\,dy}{y^2}$. The analog here of QUE states that $\mu_f$ tends to the uniform measure $\frac{2}{\pi} \frac{dx\,dy}{y^2}$ as $k \to \infty$, and this is known to hold by the work of Holowinsky and Soundararajan \cite{HS}. As with Maass forms, one may now ask for finer restriction theorems for holomorphic Hecke eigenforms. We study the problem of bounding the $L^2$-norm of Hecke eigenforms on a fixed horocycle, and establish the following uniform bound.

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Theorem 1. Let $f$ be a Hecke eigenform of weight $k$ on $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$ with $L^2$-norm normalized to be 1. Let $\delta > 0$ be fixed. Uniformly in the range $1/k \leq y \leq k^{1/2-\delta}$ we have

$$\int_0^1 y^k |f(z)|^2 dx \leq C(\delta),$$

for some constant $C(\delta)$.

Our result gives a uniform bound for the $L^2$-norm restricted to horocycles, answering a question from Sarnak [16]. In the Maass form situation, Ghosh, Reznikov and Sarnak [4] establish weaker restriction bounds (of size $\lambda^c$) for the corresponding problem, and Sarnak [16] notes that uniform boundedness there follows from the Ramanujan conjecture and a sub-convexity bound (in eigenvalue aspect) for the Rankin-Selberg $L$-function $L(s, \phi \times \phi)$. One might hope to strengthen and extend Theorem 1 in the following two ways. First, Young [22, Conjecture 1.4] has conjectured that for any fixed $y > 0$, the restriction of $\mu_f$ to the horocycle $[0, 1]+iy$ still tends to the uniform measure, as $k \to \infty$: in particular, as $k \to \infty$

$$\int_0^1 y^k |f(z)|^2 dx \to \frac{3}{\pi}.$$

Second, one might expect that two different eigenforms $f$ and $g$ of weight $k$ are approximately orthogonal on the horocycle $[0, 1]+iy$, so that (as $k \to \infty$)

$$\int_0^1 y^k f(x+iy)g(x+iy) dx \to 0.$$

Our proof, which relies crucially on bounds for mean-values of non-negative multiplicative functions in short intervals, does not allow us to address these refined conjectures.

2. Preliminaries

Let $f$ be a Hecke eigenform of weight $k$ on $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$. Write

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left( 1 - \frac{\alpha_p}{p^s} \right)^{-1} \left( 1 - \frac{\beta_p}{p^s} \right)^{-1},$$

where $\lambda_f(n)$ are the Hecke eigenvalues for $f$, and $\alpha_p, \beta_p = \alpha_p^{-1}$ are the Satake parameters. Our $L$-function has been normalized such that the Deligne bound reads $|\lambda_f(n)| \leq d(n)$ (the divisor function), or equivalently that $|\alpha_p| = |\beta_p| = 1$.

The symmetric square $L$-function $L(s, \text{sym}^2 f)$ is defined by

$$L(s, \text{sym}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} = \prod_p \left( 1 - \frac{\alpha_p^2}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^s} \right)^{-1} \left( 1 - \frac{\beta_p^2}{p^s} \right)^{-1}.$$

From the work of Shimura [17] we know that $L(s, \text{sym}^2 f)$ has an analytic continuation to the entire complex plane, and satisfies a functional equation connecting $s$ and $1-s$: namely, with $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$,

$$\Lambda(s, \text{sym}^2 f) = \Gamma_\mathbb{R}(s+1)\Gamma_\mathbb{R}(s+k-1)\Gamma_\mathbb{R}(s+k)L(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f).$$

Moreover, Gelbart and Jacquet [3] have shown that $L(s, \text{sym}^2 f)$ arises as the $L$-function of a cuspidal automorphic representation of $GL(3)$. Invoking the Rankin-Selberg $L$-function attached to $\text{sym}^2 f$, a standard argument establishes the classical zero-free region for $L(s, \text{sym}^2 f)$, with the possible exception of a real Landau-Siegel zero (see Theorem 5.42 of [8]). The work
of Hoffstein and Lockhart [5] (especially the appendix by Goldfeld, Hoffstein and Lieman) has ruled out the existence of Landau-Siegel zeroes for this family. Thus, for a suitable constant $c > 0$, the region

$$\mathcal{R} = \left\{ s = \sigma + it : \sigma \geq 1 - \frac{c}{\log k(1 + |t|)} \right\}$$

does not contain any zeroes of $L(s, \text{sym}^2 f)$ for any Hecke eigenform $f$ of weight $k$.

Lastly, we shall need a “log-free” zero-density estimate for this family, which follows from the work of Kowalski and Michel (see [9], and also the recent works of Lemke Oliver and Thorner [14], and Motohashi [13]).

**Lemma 2.** There exist absolute constants $B$, $C$, and $c$ such that for all $1/2 \leq \alpha \leq 1$, and any $T$ we have

$$\left| \left\{ \rho = \beta + i\gamma : L(\rho, \text{sym}^2 f) = 0, \beta \geq \alpha, |\gamma| \leq T \right\} \right| \leq C(T + 1)Bk^c(1 - \alpha).$$

The special value $L(1, \text{sym}^2 f)$ shows up naturally when comparing the $L^2$ normalization and Hecke normalization of a modular form. Suppose $f$ has been normalized in such a way that

$$\int_X y^k |f(z)|^2 dx \, dy = 1.$$

Then the Fourier expansion of $f(z)$ is given by (see, for example, Chapter 13 of [7])

$$f(z) = C_f \sum_{n=1}^{\infty} \lambda_f(n)(4\pi n)^{k-1/2} e(nz), \quad (1)$$

where

$$C_f = \left( \frac{2\pi^2}{\Gamma(k)L(1, \text{sym}^2 f)} \right)^{1/2}.$$

We can now state our main lemma, which refines Lemma 2 of [6], and allows us to estimate $L(1, \text{sym}^2 f)$ by a suitable Euler product. Below we use the notation $g \asymp h$ to denote $g \ll h$ and $h \ll g$.

**Lemma 3.** For any Hecke eigenform $f$ of weight $k$ for the full modular group, we have

$$L(1, \text{sym}^2 f) \asymp \exp \left( \sum_{p \leq k} \frac{\Lambda_f(p^2)}{p} \right).$$

Recall that $g \asymp h$ means $g \ll h$ and $h \ll g$.

**Proof.** Let $1 \leq \sigma \leq \frac{5}{4}$, and consider for some $c > 0$ and $x \geq 1$, the integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s, \text{sym}^2 f)(s + 1)\Gamma(s)x^s ds, \quad (2)$$

which we shall evaluate in two ways. Here we shall take $x = k^A$ for a suitably large constant $A$. On one hand, we write

$$-\frac{L'}{L}(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\Lambda_{\text{sym}^2 f}(n)}{n^s}$$

where $\Lambda_{\text{sym}^2 f}(n) = 0$ unless $n = p^k$ is a prime power, in which case

$$\Lambda_{\text{sym}^2 f}(p^k) = (\alpha_p^{2k} + 1 + \beta_p^{2k}) \log p,$$
so that $|\Lambda_{\text{sym}^2 f}(n)| \leq 3\Lambda(n)$ for all $n$. Using this in (2), and integrating term by term, using

$$\frac{1}{2\pi i} \int_{(c)} (s + 1)\Gamma(s) y^s ds = e^{-1/y} \left(1 + \frac{1}{y}\right),$$

we obtain

$$\frac{1}{2\pi i} \int_{(c)} -\frac{L'}{L}(s + \sigma, \text{sym}^2 f)(s + 1)\Gamma(s) x^s ds = \sum_{n=2}^{\infty} \frac{\Lambda_{\text{sym}^2 f}(n)}{n^{\sigma}} e^{-n/x} \left(1 + \frac{n}{x}\right). \quad (3)$$

On the other hand, shift the line of integration in (2) to $\Re(s) = -3/2$. We encounter poles at $s = 0$, and at $s = \rho - \sigma$ for non-trivial zeroes $\rho = \beta + i\gamma$ of $L(s, \text{sym}^2 f)$. Computing these residues, we see that (2) equals

$$-\frac{L'}{L}(\sigma, \text{sym}^2 f) - \sum_{\rho} x^{\rho-\sigma}(\rho-\sigma+1)\Gamma(\rho-\sigma) + \frac{1}{2\pi i} \int_{(-3/2)} -\frac{L'}{L}(s + \sigma, \text{sym}^2 f) x^s (s + 1)\Gamma(s) ds. \quad (4)$$

Differentiate the functional equation of $L(s, \text{sym}^2 f)$ logarithmically, and use Stirling’s formula. Thus with $s = -\frac{3}{2} + it$ we obtain that

$$-\frac{L'}{L}(s + \sigma, \text{sym}^2 f) \ll \log(k(1 + |t|)) + \left|\frac{L'}{L}(1 - s - \sigma, \text{sym}^2 f)\right| \ll \log(k(1 + |t|)).$$

Therefore the integral in (4) may be bounded by $O((\log k) x^{-3/2})$, and we conclude that

$$\sum_{n} \frac{\Lambda_{\text{sym}^2 f}(n)}{n^{\sigma}} e^{-n/x} \left(1 + \frac{n}{x}\right) = -\frac{L'}{L}(\sigma, \text{sym}^2 f) - \sum_{\rho} x^{\rho-\sigma}(\rho + 1 - \sigma)\Gamma(\rho - \sigma) + O(x^{-3/2} \log k). \quad (5)$$

We now bound the sum over zeros in (5). Write $\rho = \beta + i\gamma$, and split into terms with $n \leq |\gamma| < n + 1$, where $n = 0, 1, 2, \ldots$. If $n \leq |\gamma| < n + 1$, we may check using the exponential decay of the $\Gamma$-function that

$$|\rho - \sigma + 1||\Gamma(\rho - \sigma)| \ll (\sigma - \beta)^{-1} e^{-n}.$$ 

Therefore the contribution of zeros from this interval is

$$\ll \sum_{n \leq |\gamma| < n + 1} \frac{x^{\beta-\sigma}}{\sigma - \beta} e^{-n}.$$ 

Splitting the zeros further based on $1 - (j + 1)/\log k \leq \beta < 1 - j/\log k$ (and using the zero free region, so that $\sigma - \beta \gg (j + 1)/\log k$) the above is

$$\ll e^{-n} \sum_{j=0}^{\log k} \frac{x^{1-\sigma-j/\log k}}{(j + 1)/\log k} \{\beta + i\gamma : 1 - (j + 1)/\log k \leq \beta < 1 - j/\log k, \ n \leq |\gamma| < n + 1\}.$$ 

Now using the log-free zero density estimate from Lemma 2 and recalling that $x = k^A$, the quantity above is

$$\ll e^{-n} x^{1-\sigma} \log k \sum_{j=0}^{\log k} e^{-jA} \frac{1}{j + 1} (n + 1)^{Bc(j + 1)/\log k} \ll (n + 1)^{B} e^{-n} x^{1-\sigma} \log k,$$

provided $A \geq c + 1$ is large enough. Now summing over $n$, we conclude that the sum over zeros in (5) is $\ll x^{1-\sigma} \log k$. 


Use this bound in (5), and integrate that expression over $1 \leq \sigma \leq 5/4$. It follows that

$$\log L(1, \text{sym}^2 f) = \sum_{n=2}^{\infty} \frac{\Lambda_{\text{sym}^2 f(n)}}{n \log n} e^{-n/x} \left(1 + \frac{n}{x}\right) + O(1) = \sum_{p \leq x} \frac{\lambda_f(p^2)}{p} + O(1),$$

since the contribution of prime powers above is easily seen to be $O(1)$, and since

$$\sum_{p \leq x} \frac{1}{p} \left|1 - e^{-p/x} \left(1 + \frac{p}{x}\right)\right| + \sum_{p > x} \frac{1}{p} e^{-p/x} \left(1 + \frac{p}{x}\right) = O(1).$$

Exponentiating, we obtain

$$L(1, \text{sym}^2 f) \asymp \exp \left(\sum_{p \leq x} \frac{\lambda_f(p^2)}{p}\right) \asymp \exp \left(\sum_{p \leq k} \frac{\lambda_f(p^2)}{p}\right),$$

since $x = k^4$, and $\sum_{k<p\leq k^4} 1/p \ll 1$. This concludes our proof. \qed

3. PROOF OF THEOREM 1

The Fourier expansion (1) and the Parseval formula give

$$\int_0^1 y^k |f(z)|^2 dx = \frac{C^2}{4\pi} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n} (4\pi ny)^k e^{-4\pi ny} \ll \frac{1}{\Gamma(k) L(1, \text{sym}^2 f)} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n} (4\pi ny)^k e^{-4\pi ny}. \quad (6)$$

For $\xi \geq 0$, note that

$$\frac{\xi^k e^{-\xi}}{\Gamma(k)} \asymp \sqrt{k} \left(\frac{\xi}{k}\right)^k e^{-\xi} \ll \begin{cases} \sqrt{k} \exp(- (k - \xi)^2/(4k)) & \text{if } \xi \leq 2k \\ \sqrt{k} \exp(- k/2) & \text{if } \xi > 2k, \end{cases} \quad (7)$$

where the first bound follows because $\log(1+t) \leq t - t^2/4$ for $|t| \leq 1$ (with $t = (\xi - k)/k$), and the second bound from $\log(1+t) \leq t \log 2$ for $t \geq 1$.

The estimate (7) with $\xi = 4\pi ny$ shows that the sum in (6) is concentrated around values of $n$ with $|4\pi ny - k|$ about size $\sqrt{k}$. To flesh this out, let us first show that the contribution to (6) from $n$ with $4\pi ny \geq 2k$ is negligible. Using the second bound in (7), such terms $n$ contribute (using that $L(1, \text{sym}^2 f) \gg (\log k)^{-1}$, which follows from Lemma 3 or [5])

$$\ll \frac{1}{L(1, \text{sym}^2 f)} \sum_{n \geq k/(2\pi y)} \frac{\lambda_f(n^2)}{n} \sqrt{k} \exp(- k/2) \ll \sqrt{k} \log k \sum_{n \geq k/(2\pi y)} \frac{\lambda_f(n^2)}{n} \frac{1}{n} e^{-k/10} \ll e^{-k/20}.$$  

This contribution to (6) is clearly negligible.

It remains to handle the contribution from those $n$ with $4\pi ny \leq 2k$. Divide such $n$ into intervals of the form $j\sqrt{k} \leq |4\pi ny - k| < (j+1)\sqrt{k}$, where $0 \leq j \ll \sqrt{k}$. We use the first bound in (7) with $\xi = 4\pi ny$, and in the range $j\sqrt{k} \leq |4\pi ny - k| < (j+1)\sqrt{k}$ this gives

$$\frac{1}{\Gamma(k)} \frac{(4\pi ny)^k}{n} e^{-4\pi ny} \ll \frac{\sqrt{k} e^{-j^2/4}}{n} \ll \frac{y}{\sqrt{k}} e^{-j^2/8},$$

where $0 \leq j \ll \sqrt{k}$.
provided $y \geq 1/k$ say. Thus the contribution from the terms $j\sqrt{k} \leq |4\pi ny - k| < (j + 1)\sqrt{k}$ is
\[
\ll \frac{ye^{-j^2/8}}{\sqrt{k}L(1,\text{sym}^2 f)} \sum_{j\sqrt{k} \leq |4\pi ny - k| < (j + 1)\sqrt{k}} \lambda_f(n)^2.
\] (8)

At this stage, we appeal to a result of Shiu (see Theorem 1 of [18]) bounding averages of non-negative multiplicative functions in short intervals.

**Lemma 4.** Let $g$ be a non-negative multiplicative function with (i) $g(p^l) \leq A^l$ for some constant $A$, and (ii) $g(n) \ll \epsilon^\epsilon n^\epsilon$ for any $\epsilon > 0$. Then for any $\delta > 0$, if $x^\delta \leq z \leq x$, we have
\[
\sum_{x < n \leq x + z} g(n) \ll_{A,\delta} \frac{z}{\log x} \exp \left( \sum_{p \leq x} \frac{g(p)}{p} \right).
\]

Applying this lemma in (8), in the range $y \leq k^{1/2-\delta}$, we may bound that quantity by
\[
\ll \frac{ye^{-j^2/8}}{\sqrt{k}L(1,\text{sym}^2 f)} \frac{\sqrt{k}}{y \log k} \exp \left( \sum_{p \leq k} \frac{\lambda_f(p)^2}{p} \right).
\]

Since $\lambda_f(p)^2 = \lambda_f(p^2) + 1$, the above bound when combined with Lemma 3 yields $\ll e^{-j^2/8}$, and summing this over all $j$ gives $\ll 1$. Thus we conclude that the quantity in (6) is bounded, completing the proof of our theorem.

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