Smooth over-parameterized solvers for non-smooth structured optimization

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Abstract
Non-smooth optimization is a core ingredient of many imaging or machine learning pipelines. Non-smoothness encodes structural constraints on the solutions, such as sparsity, group sparsity, low-rank and sharp edges. It is also the basis for the definition of robust loss functions and scale-free functionals such as square-root Lasso. Standard approaches to deal with non-smoothness leverage either proximal splitting or coordinate descent. These approaches are effective but usually require parameter tuning, preconditioning or some sort of support pruning. In this work, we advocate and study a different route, which operates a non-convex but smooth over-parameterization of the underlying non-smooth optimization problems. This generalizes quadratic variational forms that are at the heart of the popular Iterative Reweighted Least Squares. Our main theoretical contribution connects gradient descent on this reformulation to a mirror descent flow with a varying Hessian metric. This analysis is crucial to derive convergence bounds that are dimension-free. This explains the efficiency of the method when using small grid sizes in imaging. Our main algorithmic contribution is to apply the Variable Projection method which defines a new formulation by explicitly minimizing over part of the variables. This leads to a better conditioning of the minimized functional and improves the convergence of simple but very efficient gradient-based methods, for instance quasi-Newton solvers. We exemplify the use of this new solver for the resolution of regularized regression problems for inverse problems and supervised learning, including total variation prior and non-convex regularizers.

Keywords Sparsity · Low-rank · Compressed sensing · Variable projection · Mirror descent · Non-convex optimization
1 Introduction

This paper introduces and studies a new class of solvers for a general set of sparsity-regularized problems. It leverages two key ideas: a smooth over-parameterization of the initial non-smooth problem and a bi-level variable projection to enhance its conditioning and cope with analysis-type priors. We first present these two points before relating them to previous works.

1.1 Non-convex parameterizations

Structured non-smooth optimization problems Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $L : \mathbb{R}^n \to \mathbb{R}^p$ be linear operators. We consider the following non-smooth optimization problem

$$\min_{x \in \mathbb{R}^n} \Phi(x) \triangleq ||Lx||_{1,2} + F_0(Ax).$$

(1)

Here $A \in \mathbb{R}^{m \times n}$ plays the role of the imaging operator in inverse problems or the design matrix for supervised learning, while $F_0 : \mathbb{R}^m \to [0, \infty]$ is a proper, lower semi-continuous convex loss function. A guiding example is the $\ell_2$ loss $F_0(z) = \frac{1}{2\lambda}||z - y||^2$ where $y \in \mathbb{R}^m$ represents the given data and $\lambda > 0$ is a regularization parameter. The regularization is induced by a group sparsity norm

$$||z||_{1,2} \triangleq \sum_{g \in \mathcal{G}} ||z_g||_2 = \sum_{g \in \mathcal{G}} \left( \sum_{k \in g} z_k^2 \right)^{\frac{1}{2}},$$

(2)

where $\mathcal{G}$ is a partition of $\{1, \ldots, p\}$. The simplest setup is $L = \text{Id}_{n \times n}$, so that (1) is a group-lasso problem inducing direct group-sparsity of $x$ [74]. The sub-case where the group have size 1 is the classical Lasso [68], which is useful to perform feature selection in learning, regularized inverse problems in imaging [67] and for compressed sensing [19]. Using more general $L$ operators leads to more complex regularization priors. A popular case is when $L$ is a finite difference discretization of the gradient operator, so that $||Lx||_{1,2}$ is the total variation semi-norm, favoring piecewise constant signals in 1-D [48] and cartoon images in 2-D [64]. Another example is when $Lx = (x_g)_{g \in \mathcal{G}}$ extract (possibly overlapping) blocks $g$ (so that $\mathcal{G}$ is in general not a partition) to favor possibly complex block patterns [4].

Hadamard over-parameterization The goal of this paper is to study the application on (1) of the Hadamard parameterization of $\| \cdot \|_{1,2}$, which reads

$$\|z\|_{1,2} = \min_{u \oplus v = z} \frac{1}{2} \|u\|_2 + \frac{1}{2} \|v\|_2,$$

(3)
Smooth over-parameterized solvers for non-smooth...

where the minimization is over vectors $u \in \mathbb{R}^p$, $v \in \mathbb{R}^{|G|}$ and $u \odot v \triangleq (u_g, v_i)_{i \in [N]}$ where $G \triangleq \{g_1, \ldots, g_N\}$. Thanks to this “over-parameterization”, problem (1) can be equivalently written as

$$\min_{x \in \mathbb{R}^n} \Phi(x) = \min_{v \in \mathbb{R}^{|G|}} \min_{u \in \mathbb{R}^p} G(u, v) \quad (4)$$

where $G(u, v) \triangleq \min_x \left\{ \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 + F_0(Ax) : Lx = u \odot v \right\}$. (5)

In the case where $L = \text{Id}$, this problem can be written as

$$\min_{v \in \mathbb{R}^{|G|}} \min_{u \in \mathbb{R}^p} \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 + F_0(A(u \odot v)) \quad (6)$$

and this is a smooth albeit nonconvex optimization problem provided that $F_0$ is smooth. This idea has been previously studied in [44], and more recently, there has been much interest in such parameterizations for the study of implicit regularization in simple gradient descent algorithms [41], with in particular a sparsity enforcing behavior for this Hadamard parameterization [72, 76]. Moreover, in this case, the nonconvexity is harmless in the sense that all saddle points are strict and one can guarantee global convergence with certain gradient-based algorithms [60]. In Sect. 4, we provide connections of gradient descent on this reparameterized form to mirror descent and show how such a parameterization leads to dimension-independent convergence rates.

**Variable projection (VarPro) reduction**

In the case $L = \text{Id}$, it is tempting to directly use smooth optimization methods to solve (6), but as exposed in our previous work [60], it makes sense to improve its conditioning by the so-called “variable projection” (VarPro) technique. In the more complicated case where $L \neq \text{Id}$ and is not invertible, the Hadamard parameterization looks at first sight unhelpful as we have simply added in (6) the difficulty of non-convexity without alleviating the non-smoothness issue. We thus propose to replace (6) by the following bilevel program

$$\min_{v \in \mathbb{R}^{|G|}} f(v) \quad \text{where} \quad f(v) = \min_{u \in \mathbb{R}^p} G(u, v). \quad (7)$$

This idea of marginalizing on one variable is called variable projection (VarPro) and is a well-known technique [39, 40]. One of its advantages is that splitting into a bilevel problem leads to better problem conditioning. In particular, in certain situations when $G$ is smooth, minimizing $f$ instead of $G$ is preferable because the condition number of the Hessian of $f$ can be shown to be no worse (and often substantially better) than that of $G$ [65]. Moreover, as we see below, while $G$ is not differentiable, the function $f$ is differentiable. The motivations for the VarPro formulation are thus two-fold: first, it is essential to obtain a smooth optimization problem when $L$ is not invertible; second, even in the case where $L = \text{Id}$, while both optimization of $G$ and $f$ improve over standard algorithms for handling (1), the improvement in conditioning in the VarPro formulation can further lead to substantial numerical gains over directly optimizing $G$. 

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Remark 1 (More general settings) For the sake of clarity, we mostly focus in this article on this $\ell^1 - \ell^2$ functional (2). The methods and algorithm that we introduce can be extended to any regularizer that admits a quadratic variational form, including nuclear norm and $\ell_q$ regularization with $q < 2$; and also handle non-smooth convex losses including the $\ell_1$-loss and the constrained setting. Such extensions are discussed in Sect. 5.

1.2 Previous works

Lasso solvers The case $L = \text{Id}$ is arguably simpler, and can be tackled using a flurry of non-smooth optimization solvers. The simplest one is the forward-backward algorithm [47], which is known as the iterative soft thresholding (ISTA) algorithm [27]. Its convergence is relatively slow, and assuming the dimension is a fixed constant, it enjoys a $1/t$ rate in worse case. This rate is improved to $1/t^2$ using Nesterov acceleration [55] and leads to the FISTA algorithm [9]. In practice, the speed of the algorithm is improved using adaptive stepsizes and restarting strategies [58] as well as quasi-Newton and variable metric methods [10, 26]. To better cope with fine grid settings in imaging, and obtain better dimension-free analysis, it is possible to replace Euclidean metrics by mirror descent methods [54]. We give some details about this line of ideas in Sect. 4 since this is closely related to the Hadamard parameterization. For problems with very sparse solutions, algorithms leveraging coordinate descent strategies are often more efficient [33]. These schemes are typically combined with support pruning schemes [37, 50, 53].

Analysis-type priors and non-smooth loss Problems where $L \neq \text{Id}$ cannot be treated directly using primal descent methods, and require some form of primal-dual reformulation. Proximal splitting schemes can be applied, such as Alternating Direction Method of Multipliers (ADMM) [14], Douglas–Rachford algorithms [29] and primal-dual algorithms [22] These schemes are popular due to their relatively low per iteration complexity. They usually exhibit slow sublinear convergence rates in general, with linear convergence under strong convexity and sharpness assumptions [46]. The performance of these methods is improved by using adaptive step size selection and preconditioning [16]. These solvers can also be used for non-smooth loss functions (as detailed in Sect. 5) such as the square root lasso problem [11], TV-$\ell^1$ [56] and matrix-regularizers such as nuclear norm [61]. Similarly to the forward-backward algorithm, mirror geometry can be introduced in these primal-dual solvers to better cope with sparsity and positivity constraints [66].

The quadratic variational formulation and IRLS As explained in [60], over-parametrization formula of the form (3) are equivalent (up to a change of variable) to so-called quadratic variational formulations. For the case of the $\ell^1 - \ell^2$ norm, writing $v_i = \sqrt{\eta_i}$ and $u_{gi} = z_{gi}/\sqrt{\eta_i}$ for $\eta_i \in \mathbb{R}_+$, the non-convex smooth formulation (3) is re-written as the convex but non smooth over-parametrization
\[ \|z\|_{1,2} = \frac{1}{2} \min_{\eta \in \mathbb{R}^{|G|}} \sum_i \left( \frac{\|z_{gi}\|^2}{\eta_i} + \eta_i \right). \] (8)

These formulations can be traced back to early computer vision works such as [34, 35]. A detailed account for these variational formulations can be found in [51], and further studied in the monograph [4] under the name of subquadratic norms. Such quadratic variational formulations are useful to derive, in the case \( L = \text{Id} \), the celebrated iterative reweighted least squares (IRLS) algorithm, which alternatively minimize on \( z \) and \( \eta \). In this basic form, IRLS fails to converge in general because of the non-smoothness of (8).

One popular approach is to add a regularization penalty \( \frac{\varepsilon}{2} \sum_i \eta_i^{-1} \) to the formulation (8) as detailed in [28]. A nuclear norm version of IRLS has been used in [2] where an alternating minimization algorithm was introduced. Instead of this IRLS convex optimization strategy, another route is to use alternating minimization directly on the non-convex \((u, v)\) parameterization (6), see for instance [43, 44, 49, 62] for the case of the \( \ell_1 \) and nuclear norms.

**Variable projection** These alternating minimization methods, either on \((u, v)\) or on \((x, \eta)\) are quite slow in practice because of the poor conditioning of the resulting over-parameterized optimization problem. As explained in [39, 65] the variable projection reformulation (7) provably improves the conditioning of the Hessian of the functionals involved, we refer also to [45, 75] for more recent studies. This approach is classical (see for instance [63, Chap.10] for some general theoretical results on reduced gradients), and was introduced initially for solving nonlinear least squares problems.

**Gradient flow and fine grid analysis** The convergence speed of first order non-smooth methods in general degrades as the dimension increases. This is the case in particular in imaging problems (such as deconvolution or super-resolution problems) as the grid size goes to zero, which corresponds to a setting where the object of interest is a stream of Dirac masses and one seeks to estimate their precise positions [31]. While dedicated solvers have been developed to alleviate this issue and can even cope with “off-the-grid” formulations (without explicit discretization) [15, 17], ISTA and related forward-backward solvers are still the most popular. Chizat proposed in [25] an analysis of rate of convergence of forward-backward when the grid size is arbitrary small, in which case \( O(1/t) \) rate does not holds, and one obtains slower \( O(1/t^{4/4}) \) rates (where \( d \) is the ambient dimension, e.g. \( d = 2 \) for images). These rates can be improved to \( O(\log(t)/t) \) by replacing Euclidean proximal operators by more general mirror operators, as detailed in Sect. 4. In our work, we relate the Hadamard over-parameterization to this mirror flow, which partly explains its efficiency, and is useful to derive convergence bounds.

### 1.3 Contributions

Our first set of contributions is the derivation and the analysis in Sect. 2 of a VarPro reduced method in the general case of an analysis sparsity prior. The main contribution is the proof in Theorem 1 that the resulting functional is differentiable, and an explicit formula for the gradient. Our second set of contributions is a study of the benefits
that VarPro offers in terms of problem conditioning in Sect. 3. In the case of the Lasso ($L = \text{Id}$) with regularization parameter $\lambda > 0$, we provide a theoretical result which shows that the ratio in condition number $\text{Cond}(\nabla^2 f) / \text{Cond}(\nabla^2 G) = O(\lambda)$, so the smaller $\lambda$ is, the larger the improvement in conditioning. In the case of $A = \text{Id}$ and $L$ being the gradient matrix, we present a numerical study to highlight the improved convergence behaviour of VarPro. Our third set of contributions is the proof, in Sect. 4, that the gradient descent on the Hadamard formulation (3) is equivalent to a mirror-flow with a time-varying entropy function. This shows that while the descent is computed in an over-parameterized domain $(u, v)$, it is still equivalent to a classical flow on the initial variable $x$, and that this flow should be understood for a non-Euclidean, time-varying, Hessian-type metric. This analysis is leveraged to derive dimension-free (i.e. insensitive to the grid step size) convergence bound for the gradient descent on the Hadamard formulation. Most notably, we show in Proposition 6 a $1/\sqrt{t}$ convergence bound on the gradient of the minimized energy, and Proposition 8 shows that the convergence in function’s value is controlled by the convergence of the gradients. Lastly, Sect. 5 focuses on more practical considerations, by explaining how to extend our approach to non-smooth loss functions and non-convex regularizers. These extensions are exemplified with numerical simulations on imaging problems.

Connection with previous works This work builds on our initial work [60], which derived and studied the VarPro method in the case $L = \text{Id}$. The case of an arbitrary $L$ is more involved because of the lack of smoothness of the Hadamard parameterization, which fortunately is to a large extent absorbed by the VarPro reduction. Beside this extension to analysis-type priors, this work also proposes a novel mirror-type analysis of the Hadamard formulation.

2 Hadamard and VarPro parameterizations

We now give a detailed analysis of the Hadamard formulation (4) and its associated VarPro marginalization (7), which is crucial to ensure differentiability of the function to be minimized.

Remark on notation For $n \in \mathbb{N}$, let $[n] \triangleq \{1, 2, \ldots, n\}$. When we write $a \gtrless b$ (resp. $a \lesssim b$) in a statement, we mean that there is a constant $C > 0$, independent of all other variables being considered, such that $a \gtrsim Cb$ (resp. $a \lesssim Cb$). Let $C_{1,1}$ denotes the set of continuously differentiable functions with Lipschitz gradient. Given a block-matrix $H = [A, B; B^\top, D] \in \mathbb{R}^{n \times n}$, its Schur complement with respect to $D$ is denoted $H/D \triangleq A - B B^\top D^{-1} B$. We write $\|x\|_q \triangleq \left(\sum_{j=1}^n |x_j|^q\right)^{1/q}$ and we will sometimes drop the subscript $\| \cdot \|_q \triangleq \| \cdot \|_2$ for the Euclidean. Given a matrix $A$, $\|A\|_{p \rightarrow q} \triangleq \sup_{\|x\|_p \leq 1} \|Ax\|_q$ and we drop the subscripts for the spectral norm $\|A\| \triangleq \|A\|_{2 \rightarrow 2}$. For $G \triangleq \{g_1, \ldots, g_N\}$ (where each $g_i$ is a set of indices), the Hadamard product for $u \in \mathbb{R}^P$ and $v \in \mathbb{R}^{[G]}$ is defined as $u \odot v \triangleq (u_{g_i} v_i)_{i \in [N]}$. When $u, v \in \mathbb{R}^P$ are of the same length (i.e. trivial group structure), we write $u \cdot v \triangleq (u_i v_i)_{i}$.
to denote pointwise multiplication and \( u/v \triangleq (u_i/v_i) \) to denote pointwise division.

We will also use \( u^2 \triangleq u \cdot u \).

### 2.1 Dual formulation

Since for a generic \( L \), the Hadamard formulation (4) involves the resolution of a constrained problem, analyzing the differentiability of \( f \) requires studying a dual formulation. The VarPro formulation (4) has the form of a bi-level program

\[
f(v) \triangleq \min_{u \in \mathbb{R}^p} G(u, v) \quad \text{and} \quad u(v) = \arg\min_u G(u, v).
\]

For such problem, it tempting to compute the gradient of \( f \) by applying the chain rule:

\[
\nabla f(v) = \partial_v G(u(v), v) + \partial_u G(u(v), v) \nabla u(v) = \partial_v G(u(v), v)
\]

where we used the fact \( \partial_u G(u(v), v) = 0 \) due to optimality of \( u(v) \). This is of course only a formal argument, and in particular, this requires \( \partial_v G(u(v), v) \) to be well-defined. For the VarPro problem (7), it is not immediately clear that \( \partial_v G(u(v), v) \) is well-defined since the variable \( v \) appears inside a linear constraint. However, the following proposition shows how the inner optimization problem in (7) can be written as a concave maximization problem involving a dual function \( \varphi(v, \alpha, \xi) \) that is differentiable with respect to \( v \). Note also that due to the \( \|\alpha \odot v\|^2 \) term, given a maximizer \((\alpha, \xi)\) to the inner problem, \( v \odot \alpha \) is uniquely defined and \( \partial_v \varphi(v, \alpha, \xi) = -\alpha^2 \odot v \) is thus well-defined. Precise regularity properties of the function \( f \) is studied in the following section, but from this proposition, one can at least formally expect \( f \) to be differentiable.

**Proposition 1** (Dual formulation of the inner problem) The function \( f \) defined in (7) can be written as

\[
f(v) = \min_{u \in \mathbb{R}^p} G(u, v) = \max_{\xi \in \mathbb{R}^m, \alpha \in \mathbb{R}^p} \frac{1}{2} \|v\|^2 + \varphi(v, \alpha, \xi)
\]

where \( \varphi(v, \alpha, \xi) \triangleq -\frac{1}{2} \|\alpha \odot v\|^2 - F_0^*(\xi) - \iota_K(\xi, \alpha), \) (9)

where \( \iota_K \) denotes the indicator function on the set \( K \triangleq \{ (\xi, \alpha) : \ L^\top \alpha = -A^\top \xi \} \). Moreover, the optimal \( \alpha, \xi \) satisfy \( Ax \in \partial F_0^*(\xi) \) and \( Lx = \alpha \odot v^2 \) for some \( x \in \mathbb{R}^n \).

**Remark 2** (Example of quadratic loss) When \( F_0(z) = \frac{1}{2\lambda} \|z - y\|^2 \), \( F_0^*(\xi) = \frac{1}{2\lambda} \|\xi\|^2 + \langle \xi, y \rangle \), so

\[
\varphi(v, \alpha, \xi) = -\frac{1}{2} \|\alpha \odot v\|^2 - \frac{\lambda}{2} \|\xi\|^2 - \langle \xi, y \rangle.
\]
The maximization problem (9) is therefore a quadratic problem and given optimal solutions \((\alpha, \xi)\), there exists \(x\) such that
\[
\lambda \xi = Ax - y \quad \text{and} \quad Lx = \alpha \odot v^2 \quad \text{and} \quad L^\top \alpha + A^\top \xi = 0.
\]
This can be written as the linear system
\[
M_v \begin{pmatrix} \xi \\ \alpha \\ x \end{pmatrix} = \begin{pmatrix} -y \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad M_v \triangleq \begin{pmatrix} \lambda \Id & 0 & -A \\ 0 & -\text{diag}(\bar{v}^2) & L^\top \\ A^\top & L & 0 \end{pmatrix}.
\]
(10)
where \(\bar{v} \in \mathbb{R}^p\) is the extension of \(v\) such that \(\bar{v} \cdot \alpha = \alpha \odot v\) for all \(\alpha \in \mathbb{R}^p\). One can therefore handle the inner problem \(\max_{\alpha, \xi} \varphi(v, \alpha, \xi)\) by solving a linear system.

**Proof** We first write
\[
f(v) = \min_{u \in \mathbb{R}^p, x \in \mathbb{R}^n, w \in \mathbb{R}^m} \left\{ \frac{1}{2} \|v\|^2 + \frac{1}{2} \|u\|^2 + F_0(w) : Lx = u \odot v, \ Ax = w \right\}.
\]
Note that this is a convex optimization problem, and by considering its dual formulation, we have
\[
f(v) = \min_{u \in \mathbb{R}^p, x \in \mathbb{R}^n, w \in \mathbb{R}^m} \max_{\alpha \in \mathbb{R}^p, \xi \in \mathbb{R}^m} \frac{1}{2} \|v\|^2
\]
\[
+ \frac{1}{2} \|u\|^2 + F_0(w) + \langle \alpha, Lx - u \odot v \rangle + \langle \xi, Ax - w \rangle
\]
\[
= \max_{\alpha \in \mathbb{R}^p, \xi \in \mathbb{R}^m} \frac{1}{2} \|v\|^2 - \frac{1}{2} \|\alpha \odot v\|^2 - F_0^*(\xi) \quad \text{where} \quad L^\top \alpha = -A^\top \xi,
\]
with the optimal \(\alpha, \xi\) satisfying \(Ax \in \partial F_0^*(\xi)\) and \(Lx = \alpha \odot v^2\).

**2.2 Differentiability**

In this section, we consider the regularity properties of \(f\). We recall that \(F_0 : \mathbb{R}^m \to [0, \infty]\) is a proper, lower semi-continuous convex loss function with Lipschitz gradient.

**Proposition 2** (Well-posedness) Assume that \(F_0 \in C^{1,1}(\mathbb{R}^m; \mathbb{R})\) and recall the definition of \(f\) from (9) of Proposition 1. Then, \(\text{dom}(f) = \mathbb{R}^p\) and the set of maximizers in (9) is non-empty.

**Notation** We denote the range of a matrix \(L\) by \(\mathcal{R}(L)\). Given \(\alpha \in \mathbb{R}^n\) and \(S \subset \mathcal{G}\), we write \(\alpha_S\) to denote the restriction of \(\alpha\) to entries whose indices are in the groups defined by \(S\), that is, the vector taking values \(\alpha_j\) whenever there is a group \(g\) such that \(j \in g \in S\) and taking value 0 otherwise. \(L_S\) denotes the matrix \(L\) with columns restricted to those indexed by \(S\).
Proof First note that since \( F_0 \in C^{1,1}, \) \( F_0^* \) is strongly convex, and so, it is bounded from below. Also, there exist \( \alpha \) and \( \xi \) satisfying \( L^T \alpha + A^T \xi = 0 \) (take \( \alpha = 0 \) and \( \xi = 0 \)). So, for each \( v \), \( \max_{\alpha, \xi} \varphi(v, \xi, \alpha) \) exists and hence, \( f(v) < \infty \) for all \( v \).

To show that the set of maximizers is nonempty, first note that existence of the maximum along with strong concavity of \( \varphi(v, \alpha, \cdot) \) in the variable \( \xi \) imply that we can consider a maximizing sequence \((\alpha_n, \xi_n)\) with \( L^T \alpha_n + A^T \xi_n = 0 \), and \( \|\xi_n\| \leq C \) for some \( C > 0 \). It follows that there is a convergent subsequence \( \xi_{n_k} \to \xi^* \) for some \( \xi^* \in \mathbb{R}^m \). Moreover, denoting the range of \( L^T \) by \( \mathcal{R}(L^T) \), since \( (A^T \xi_{n_k}) \subset \mathcal{R}(L^T) \) is a convergent sequence and \( \mathcal{R}(L^T) \) is closed, \( A^T \xi^* \in \mathcal{R}(L^T) \). We also have \( v \odot \alpha_n \) is uniformly bounded, so, denoting \( S \triangleq \text{Supp}(v) \), \((\alpha_n)_S\) has a convergent subsequence. Since

\[
L^T \alpha_n = L^T (\alpha_n)_S + L^T (\alpha_n)_{S^c}
\]

and both \( A^T \xi_n = L^T \alpha_n \) and \( L^T (\alpha_n)_S \) converge up to a subsequence, \( L^T (\alpha_n)_{S^c} \) is also convergent up to a subsequence. It follows that there exists \( \alpha_* \) and \( \xi_* \) such that \( \xi_{n_k} \) converges to \( \xi^* \), \((\alpha_n)_S \) converges to \( (\alpha_*)_S \) and \( A^T \xi_* = L^T \alpha_* \). We then apply the fact that \( \varphi(v, \cdot, \cdot) \) is upper semi-continuous to deduce that \((\xi_*, \alpha_*)\) is a maximizer. \( \square \)

The case where \( F_0 \) is the quadratic loss and \( L = \text{Id} \) was investigated in [60] and it is straightforward in this case to see that \( f \) is smooth. For more general \( F_0 \) and \( L \neq \text{Id} \), we have the following regularity result, which implies the differentiability of \( f \).

**Theorem 1** (Differentiability) Assume that \( F_0 \in C^{1,1}(\mathbb{R}^m; \mathbb{R}) \). Then, \( f \) is differentiable for all \( v \in \mathbb{R}^m \) with \( \nabla f(v) = v - \tilde{\alpha}_v \cdot v \) where given \((\alpha_v, \xi_v) \in \arg\min_{\alpha, \xi} \varphi(v, \alpha, \xi)\), we define \( \tilde{\alpha}_v \triangleq (\| (\alpha_v)_g \|)_g \in \mathbb{R}^{\mathcal{G}} \). Moreover, if \( v_i \neq 0 \) for all \( i \), then \( f \) is strictly differentiable (see Appendix A for the definition).

Note that even through \( \varphi(v, \cdot, \cdot) \) does not necessarily have unique maximizers, \( \alpha_v \odot v \) is uniquely defined due to the quadratic term \( \| \alpha_v \odot v \|^2 \) in \( \varphi \), so \( \alpha_v \) is unique on the support of \( v \) and hence, the formula given for \( \nabla f \) in the above theorem is clearly well-defined.

From the above result, we see that the computation of \( \nabla f(v) \) simply requires solving the inner problem \( \max_{\alpha, \xi} \varphi(v, \alpha, \xi) \) to obtain \( \alpha_v \). Before proving this theorem in Sect. 2.4, we first make some remarks on the computation of the \( \alpha_v \) and provide numerical examples.

### 2.3 Squared Euclidean loss

The inner problem \( \max_{\alpha, \xi} \varphi(v, \alpha, \xi) \) is a convex optimization problem, which might require a dedicated inner solver. In the remaining part of this section, we focus on the setting where

\[
F_0(z) = \frac{1}{2\lambda} \| z - y \|^2, \quad \text{where} \quad y \in \mathbb{R}^m.
\]
In this case, the inner maximization problem is a least squares problem. In the case where \( v_i \neq 0 \) for all \( i \), this can be further simplified, by first rewriting (10) as

\[
\begin{pmatrix}
  A^T A & \lambda L^T \\
  L & -\text{diag}(\tilde{v}^2)
\end{pmatrix}
\begin{pmatrix}
  x \\
  \alpha
\end{pmatrix}
= \begin{pmatrix}
  A^T y \\
  0
\end{pmatrix}.
\] (11)

Equivalently, we have

\[
A^T Ax + \lambda L^T \text{diag}(1/\tilde{v}^2)Lx = A^T y
\] (12)

and let \( \tilde{v}^2 \cdot \alpha = Lx \) and \( \xi = \frac{1}{\lambda} (Ax - y) \).

### 2.3.1 Group-Lasso setting

In the case where \( L = \text{Id} \), which was studied in [60], the inner problem can be written as

\[
f(v) = \max_{\xi \in \mathbb{R}^m} \frac{1}{2} \|v\|^2 - \frac{1}{2} \|(A^T \xi) \odot v\|^2 - \frac{\lambda}{2} \|\xi\|^2 - \langle \xi, y \rangle.
\]

The maximizer \( \xi \) is the solution to the linear system

\[
(A \text{diag}(\tilde{v}^2) A^T + \lambda \text{Id}) \xi = -y,
\]

where we recall that \( \tilde{v} \) is the extension of \( v \) to \( \mathbb{R}^p \) (see the definition below equation (10)).

### 2.3.2 Proximal operators

When \( A = \text{Id} \), one has

\[
f(v) = \max_{\alpha \in \mathbb{R}^p} \frac{1}{2} \|v\|^2 - \frac{1}{2} \|\alpha \odot v\|^2 - \frac{\lambda}{2} \|\alpha\|^2 + \langle L^T \alpha, y \rangle
\]

The maximizer is the solution to the linear system

\[
(\lambda LL^T + \text{diag}(\tilde{v}^2)) \alpha = Ly \quad \text{and} \quad x = y - \lambda L^T \alpha.
\]

### 2.3.3 The overlapping group Lasso

Complex block-sparse patterns can be favored in the solution using an operator \( L \) which extracts blocks of the vector \( x \) [4]. We set \( L : x \mapsto (\sqrt{n_g} x_{I_g})_{g \in G} \) where \( I_g \subseteq \{1, \ldots, n\}, n_g = |I_g| \). This induces the regularizer \( \|Lx\|_{1,2} = \sum_{g \in G} \sqrt{n_g} \|x_{I_g}\| \). If the groups span the entire index set, that is \( \bigcup_{g \in G} I_g = \{1, \ldots, n\} \), then \( W \triangleq \)
Smooth over-parameterized solvers for non-smooth...

Fig. 1 Overlapping group lasso. Top row: $A \in \mathbb{R}^{300 \times 3000}$ is a random Gaussian matrix and the sought after vector has group overlaps of 5. Bottom row: $A$ is the the breast cancer data matrix of [71].

$L^\top \text{diag}(1/\bar{v}^2)L$ is a diagonal matrix with $W_{i,i} = \sum_{g \in G_i} v_g^{-2} n_g$ where $G_i = \{g \in G : i \in g\}$. We can therefore conveniently rewrite (12) leveraging

\[
(A^\top A + \lambda L^\top \text{diag}(1/\bar{v}^2)L)^{-1} = \frac{1}{\lambda} W^{-1} - \frac{1}{\lambda} W^{-1} A^\top \left(\lambda \text{Id}_m + AW^{-1}A^\top\right)^{-1} AW^{-1}.
\]

This formulation is advantageous in the under-determined setting where $m \ll n$.

The performance of VarPro is illustrated in Fig. 1, where we apply L-BFGS quasi-Newton to minimize $f$. The top row of Fig. 1 shows the results when $A \in \mathbb{R}^{m \times n}$ is a random Gaussian matrix with $m = 300$ and $n = 3000$. The groups are chosen such that they have an overlap of $s$ and the size of each group is chosen at random from 1 to 20. The results shown in the figure are for different regularization parameters. In the bottom row of Fig. 1, we show the results on the breast cancer dataset [71] commonly used for benchmarking the group Lasso [57]. We compare our method against ADMM with different parameters (see Section C in the Appendix).

2.3.4 Total variation regularization for image processing

We identify images $x \in \mathbb{R}^n$ with 2-D arrays $x \in \mathbb{R}^{d \times d}$ so that $n = d^2$. Horizontal and vertical derivative operators are defined as

\[
D^h x \triangleq (x_{i,j} - x_{i+1,j})_{i,j}, \quad \text{and} \quad D^v x \triangleq (x_{i,j} - x_{i,j+1})_{i,j}.
\]

and the 2 dimensional gradient operator acting on images $x \in \mathbb{R}^{d \times d}$ is

\[
L x \triangleq (D^h x, D^v x).
\]

---

1 We used the data matrix downloaded from https://github.com/samdavanloo/ProxLOG.
Given $T$ images $x = (x^t)^T_{t=1} \in \mathbb{R}^{d \times d \times T}$, we can consider the multi-channel total variation regularization function by defining

$$L : \mathbb{R}^{d \times d \times T} \rightarrow \mathbb{R}^{d \times d \times 2T}, \quad Lx = \left((D^h x^t, D^v x^t)^T_{t=1}\right)$$

and $R(x) = \|Lx\|_{1,2} = \sum_{i,j=1}^{d} \sum_{t=1}^{T} (D^h x^t)_{i,j}^2 + (D^v x^t)_{i,j}^2$.

Suppose also that $A : \mathbb{R}^{d \times d \times T} \rightarrow \mathbb{R}^{\sum_{t=1}^{T} m_t}$ is of the form $A(x) = (A_t(x_t))^T_{t=1}$ where $A_t : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m_t}$ is a linear operator and $y = (y_t)^T_{t=1}$ with $y_t \in \mathbb{R}^{m_t}$ are the observations. In this case, the function $f$ is defined over $v \in \mathbb{R}^{d \times d}$ and the linear system in (11) can be written as $T$ linear systems,

$$\begin{pmatrix} A_t^T A_t & \lambda L^T \\ -L & \begin{pmatrix} \text{diag}(v_t^2) & 0 \\ 0 & \text{diag}(v_t^2) \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_t \\ \alpha_t \end{pmatrix} = \begin{pmatrix} A_t^T y_t \\ 0 \end{pmatrix}.$$ 

These linear systems can be placed in the form (12) and solved simultaneously. Two popular settings for image processing with total variation is denoising where $A = \text{Id}$ and inpainting where $A$ is a masking operator.

We compare the VarPro formulation (using a BFGS solver) against primal-dual and ADMM for the following two problems:

(i) Total variation inpainting of color images, where we let $T = 3$ and for $x = (x^1, x^2, x^3) \in \mathbb{R}^{d \times d \times 3}$, each $x^i$ correspond to one of three color channels. The operator $Ax = x_J$ is a subsampling operator, where $J$ is an index set selecting 30% of the pixels at random. Figures 2 and 3 (rows 1 and 2), show convergence curves and examples of reconstructions.

(ii) Hyperspectral imaging. We consider total variation denoising on the Indian pines dataset. Here, $A = \text{Id}$ and the input data is normalized to take values between 0 and 1. This dataset is of size $145 \times 145$ with $T = 224$ spectral reflectance bands. Figures 2 and 3 (row 3), show convergence curves and examples of reconstructions.

One of the advantages of our method is that one can simply plug our gradient formula for $f$ into any gradient based method such as quasi-Newton BFGS, without the need to tune extra parameters. Thus, although ADMM and primal-dual do show favourable performance for certain parameter choices, we found that the VarPro is more straightforward to apply.

### 2.4 Proof of differentiability

In this section, we prove Theorem 1. In order to establish that $f$ is differentiable, we need to show that

$$f_0(v) \triangleq \max_{\alpha, \xi} \varphi(v, \xi, \alpha)$$

$^2$ Dataset downloaded from [http://www.ehu.eus/ccwintco/index.php/Hyperspectral_Remote_Sensing_Scenes](http://www.ehu.eus/ccwintco/index.php/Hyperspectral_Remote_Sensing_Scenes).
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Fig. 2 The top two rows show plots of the objective error against computational time in seconds for 2D vectorial TV color inpainting with Matlab stock images “pears” of 486 × 732 pixels (top row) and “Peppers” of 384 × 512 pixels (middle row). The regularization of vectorial TV of \( u = (u_1, u_2, u_3) \) is \( J(u) = \sum_{i,j} |\hat{u}_{ij}|^2 \) where \( \hat{u} = (D_v(u_k), D_h(u_k))_{k=1,2,3} \) and \( \hat{u}_{ij} \) is a vector of length 6, corresponding to the vertical and horizontal gradient of pixel \((i, j)\) of each of the 3 color channels. Comparisons are against ADMM and primal-dual (PD). For primal-dual, the matrix inversion is trivial. For ADMM, we need to invert \( M = A^\top A + \gamma L^\top L \), for this inversion, one Cholesky factorization is computed at the start of the iterations and is re-used throughout. The bottom row is a Hyperspectral imaging example. Here, we perform total variation denoising with \( A = \text{Id} \) on the Indian Pines dataset. This consists of images of size 145 by 145, with 224 spectral reflectance bands.

is differentiable, where \( \varphi \) is defined in Proposition 1. We first show that \( f_0 \) is strictly differentiable at \( v \) if \( v_i \neq 0 \) for all \( i \).

**Notation** Throughout the proofs, given an index set \( S \) and a vector \( v \), let \( v_S \) be the restriction of \( v \) to the index set, such that \((v_S)_i = v_i\) for all \( i \in S \) and \((v_S)_i = 0\) otherwise. Also, given a matrix \( A \), \( A_S \) denotes the restriction of the columns of \( A \) by the indices \( S \).

**Proposition 3** Let \( v \in \mathbb{R}^p \) and assume that \( F_0 \) is Lipschitz smooth. If \( v_i \neq 0 \) for all \( i \), then \( f_0 \) (defined in (13)) is strictly differentiable at \( v \).

The proof of this result is a direct consequence of the following lemma.
Lemma 1 Let $B > 0$ and let

$$T \triangleq \left\{ (\xi, \alpha) \in \mathbb{R}^m \times \mathbb{R}^p : \|\alpha\| \leq B, \|\xi\| \leq B, \ L^\top \alpha = -A^\top \xi \right\}.$$  

The function

$$f_0(v) \triangleq \max_{\alpha, \xi \in T} \varphi(v, \xi, \alpha)$$  

is strictly differentiable with $\nabla f_0(v) = -\left(\| (\alpha_v)_{g} \|_{g \in G} \right) \cdot v$ where

$$(\alpha_v, \xi_v) \in \text{argmax}_{\alpha, \xi \in T} \varphi(v, \xi, \alpha).$$

Proof Note that $\varphi$ and $\partial_v \varphi(v, \xi, \alpha) = -\left(\| (\alpha_v)_{g} \|_{g \in G} \right) \cdot v$ are continuous on $\mathbb{R}^{|G|} \times T$. Moreover, due to the strongly concave $-\|\alpha \odot v\|^2$ term inside $\varphi$, $\alpha_v \odot v$ is unique if $(\xi_v, \alpha_v) \in \text{argmax}_{(\xi, \alpha) \in T} \varphi(v, \xi, \alpha)$, and so,

$$\left\{ \partial_v \varphi(v, \xi_v, \alpha_v) : (\xi_v, \alpha_v) \in \text{argmax}_{(\xi, \alpha) \in T} \varphi(v, \xi, \alpha) \right\}$$

is single-valued. It follows from Theorem 3 (see the Appendix) that $f_0$ is strictly differentiable with $\nabla f_0(v) = \partial_v \varphi(v, \xi_v, \alpha_v)$.

Proof of Proposition 3 By Lemma 1, to show that $f_0(v) = \max_{\alpha \in \mathbb{R}^p, \xi \in \mathbb{R}^m} \varphi(v, \xi, \alpha)$ is differentiable, it is sufficient to show that for each $v$, there exists a neighborhood of $v$ and a bounded set $T$ such that for all $v \in V$, $f_0(v) = \max_{(\alpha, \xi) \in T} \varphi(v, \xi, \alpha)$. 

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Suppose that \( v_i \neq 0 \) for all \( i \) and there is a neighborhood \( V \) around \( v \) on which \( |v_i'| > c > 0 \) for all \( v' \in V \). Then, for all \( v' \in V \), the optimal solution \( (\xi, \alpha) \) to
\[
\max_{\alpha, \xi} \varphi(v', \xi, \alpha)
\]
satisfies
\[
-\frac{1}{2} \|\alpha \odot v'\|^2_2 - F_0^* (\xi) \geq \varphi(v', 0, 0) - F_0^* (0) = -F_0^* (0).
\]

(14)

Moreover, since \( F_0^* \) is strongly convex, there exists \( \gamma > 0 \) such that for any \( z \in \partial F_0^* (0) \),
\[
F_0^* (\xi) - F_0^* (0) - \langle z, \xi \rangle \geq \gamma \|\xi\|^2
\]
and combining with (14), we obtain
\[
\langle z, \xi \rangle \geq \frac{1}{2} \|\alpha \odot v'\|^2_2 + \gamma \|\xi\|^2.
\]

(15)

Hence,
\[
\|\xi\|_2 \leq \frac{1}{\gamma'} \|z\| \quad \text{and} \quad c^2 \|\alpha\|^2_2 < \min_i |v_i'|^2 \|\alpha\|^2_2 \leq 2 \|z\| \|\xi\| \leq \frac{2}{\gamma'} \|z\|^2.
\]

We can therefore take \( B = \max \left( \frac{1}{\gamma'} \|z\|, \frac{\sqrt{\gamma}}{c \sqrt{\gamma'}} \right) \).

We now show that \( f_0 \) is differentiable for any \( v \in \mathbb{R}^p \).

**Proposition 4** Let \( F_0 \in C^{1,1} \) and \( v \in \mathbb{R}^p \). Then, given
\[
(\alpha_v, \xi_v) \in \arg\max_{\alpha, \xi} \varphi(v, \alpha, \xi),
\]
\( \alpha^2_v \odot v \) is uniquely defined, and \( f_0 \) is differentiable with \( \nabla f_0 (v) = -\left(\|\alpha_v\|^2\right)_{g \in G} \cdot v \).

**Proof** We break the proof into several steps.

**Step 1:** \( f_0 \) is strictly continuous when restricted to the support of \( v \) By strong convexity of \( F_0^* \), we know from (14) and (15) that any maximizer \( (\alpha, \xi) \) to \( \max_{\alpha, \xi} \varphi(v, \xi, \alpha) \) satisfy \( \|\xi\| \leq \|z\|/\gamma \), where \( \gamma \) is the strong-convexity constant of \( F_0^* \) and \( z \in \partial F_0^* (0) \). Also, letting \( S = \text{Supp}(v) \), we have \( \frac{1}{2} \|v_S \odot \alpha_S\|^2_2 \leq \|z\| \|\xi\| \leq \|z\|^2/\gamma \). So, there exists a constant \( C > 0 \) and a neighborhood \( V \) around \( v \) such that for all \( w \in V \), the maximizers to \( \max_{\alpha, \xi} \varphi(w, \xi, \alpha) \) satisfy
\[
\|\alpha_S\| \leq C \quad \text{and} \quad \|\xi\| \leq C.
\]

So, for all \( w \in V \), we can restrict the maximization of \( \varphi(w, \cdot, \cdot) \) to the set
\[
T \triangleq \left\{ (\xi, \alpha) : \|\xi\| \leq C, \|\alpha_S\| \leq C, A^\top \xi + L^\top \alpha = 0 \right\}
\]

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and write

\[ f_0(v) = \sup_{(\alpha, \xi) \in T} \varphi(v, \xi, \alpha). \]

If \( w, w' \in V \) and \( \text{Supp}(w) = \text{Supp}(w') = S \), since

\[ \forall (\xi, \alpha) \in T, \quad \varphi(w, \xi, \alpha) = \varphi(w', \xi, \alpha) + \frac{1}{2} \|w' \odot \alpha\|^2 - \frac{1}{2} \|w \odot \alpha\|^2, \]

we have

\[ |\varphi(w, \xi, \alpha) - \varphi(w', \xi, \alpha)| \leq \frac{1}{2} C^2 \max_{i \in S} |w_i^2 - w_i' | \leq C^2 \|w' - w\|, \]

where \( B = \sup \{\|w\| : w \in V\} \). Hence,

\[ |f_0(w) - f_0(w')| \leq C^2 B \|w' - w\|. \]

**Step 2: Continuity of maximizers** Let \( v \in \mathbb{R}^p \) and \( (\alpha_v, \xi_v) \in \text{argmax}_{\alpha, \xi} \varphi(v, \alpha, \xi) \). We will show that \( \alpha_v \) is continuous when restricted to the support of \( v \) which we denote by \( S \triangleq \text{Supp}(v) \). We shall see that this again follows because \( \varphi(w, \alpha, g) \) is strongly convex with respect to \( \alpha_S \) and \( \xi \). There is a neighborhood \( V \) around \( v \), such that for all \( w \in V \) with \( \text{Supp}(w) = S \),

\[ \varphi(w, \alpha_w, \xi_w) - \varphi(w, \alpha_v, \xi_v) \geq \|\alpha_v - \alpha_w\|_S^2 + \|\xi_v - \xi_w\|^2. \]

Indeed, since \( (\alpha_w, \xi_w) \) maximizes \( \varphi(w, \cdot, \cdot) \), there exists \( x \) such that \( Ax \in \partial F_0^*(\xi_w) \) and \( Ax = \omega^2 \alpha_w \) and for any \( z_w \in \partial F_0^*(\xi_w) \). Using the fact that \( F_0^* \) is \( \gamma \)-strongly convex,

\[ \varphi(w, \alpha_w, \xi_w) - \varphi(w, \alpha_v, \xi_v) = \frac{1}{2} \langle w^2, \alpha_v^2 - \alpha_w^2 \rangle - F_0^*(\xi_w) + F_0^*(\xi_v) \]

\[ \geq \frac{1}{2} \langle w^2, \alpha_v^2 - \alpha_w^2 \rangle + \langle \xi_v - \xi_w, z_w \rangle + \gamma \|\xi_v - \xi_w\|^2 \]

\[ = \langle w^2 \alpha_w, \alpha_v - \alpha_w \rangle + \langle z_w, \xi_v - \xi_w \rangle + \frac{1}{2} \|w \odot (\alpha_v - \alpha_w)\|^2 + \gamma \|\xi_v - \xi_w\|^2 \]

\[ = \langle x, L^\top \alpha_v - L^\top \alpha_w \rangle \]

\[ + \langle x, A^\top \xi_v - A^\top \xi_w \rangle + \frac{1}{2} \|w \odot (\alpha_v - \alpha_w)\|_S^2 + \gamma \|\xi_v - \xi_w\|^2 \]

\[ \geq \frac{1}{2} \|\alpha_v - \alpha_w\|_S^2 + \gamma \|\xi_v - \xi_w\|^2. \]

Moreover, for all \( w \in V \) with \( \text{Supp}(w) = \text{Supp}(v) \),

\[ |\varphi(v, \alpha_v, \xi_v) - \varphi(w, \alpha_v, \xi_v)| \leq \sum_i (\alpha_v)_i^2 |v_i^2 - w_i^2| \lesssim \|v - w\|. \]
It follows that for all $w \in V$ with $\text{Supp}(w) = S$,
\[
\| (\alpha_v - \alpha_w)_S \|^2 \lesssim \varphi(w, \alpha_w, \xi_w) - \varphi(w, \alpha_v, \xi_v) \\
\lesssim f_0(w) - f_0(v) + \| v - w \| \lesssim \| v - w \|,
\]
and also,
\[
\| \xi_v - \xi_w \| \lesssim \| v - w \|
\]
for all $w \in V$ with $\text{Supp}(w) = S$.

**Step 3: formula on directional derivatives** Let $v, w \in \mathbb{R}^p$. Let $t \in \mathbb{R}$ and $w_t \in \mathbb{R}^p$ be such that $w_t \to w$ as $t \to 0$. Given $\alpha_0, \xi_0 \in \text{argmax}_{\alpha, \xi} \varphi(v, \alpha, \xi)$,
\[
f_0(v) = \frac{1}{2} \| \alpha_0 \odot (v + t w_t) \|^2 - \frac{1}{2} \| \alpha_0 \odot v \|^2 + \varphi(v + t w_t, \alpha_0, \xi_0)
\]
\[
\leq t \langle \alpha_0 \odot v, \alpha_0 \odot w_t \rangle + \frac{t^2}{2} \| \alpha_0 \odot w_t \|^2 + f_0(v + t w_t)
\]
So,
\[
\frac{f_0(v + t w_t) - f_0(v)}{t} + \langle \alpha_0 \odot v, \alpha_0 \odot w_t \rangle \geq -\frac{1}{2} t^2 \| \alpha_0 \odot w_t \|^2.
\]

On the other hand, $S \triangleq \text{Supp}(v)$ and $(\alpha_t, \xi_t) \in \text{argmax}_{\alpha, \xi} \varphi(v + t(w_t)_S, \alpha, \xi)$, we have
\[
f_0(v + tw) \leq f_0(v + t(w_t)_S)
\]
\[
= -\frac{1}{2} \| \alpha_t \odot (v + t(w_t)_S) \|^2 + \frac{1}{2} \| \alpha_t \odot v \|^2 + \varphi(v, \alpha_t, \xi_t)
\]
\[
\leq -t \langle \alpha_t \odot v, \alpha_t \odot w_t \rangle - \frac{t^2}{2} \| \alpha_t \odot w_t \|^2 + f_0(v)
\]
Note that $\alpha_t \to \alpha_0$ as $t \to 0$ due to the continuity of maximizers proved in Step 2. It follows that
\[
\lim_{t \to 0} \frac{f_0(v + t w_t) - f_0(v)}{t} = -\langle \alpha_0 \odot v, \alpha_0 \odot w_0 \rangle.
\]
So, $f_0$ is semi-differentiable and since the directional derivative is linear with respect to $w$, it follows that $f_0$ is differentiable (see 7.21 and 7.22 of [63]).

**Additional claim: $f$ is continuous at $v$** Although we have completed the proof at this point, we conclude by showing an additional continuity property of $f$. Given $v \in \mathbb{R}^p$, let the neighborhood $V$ and set $T$ be as in Step 1. Now, for $w \in V$, let
\[(\alpha_1, \xi_1) \in \arg\max_{\alpha, \xi \in T} \varphi(w, \alpha, \xi)\] and let \((\alpha_0, \xi_0) \in \arg\max_{\alpha, \xi \in T} \varphi(v, \alpha, \xi)\). Note that

\[
f_0(w) = \begin{align*}
&= \frac{1}{2} \|(\alpha_1)_{S^c} \odot w_{S^c}\|^2 + \varphi(w, \alpha_1, \xi_1) = \varphi(w, \alpha_1, \xi_1) \\
&\geq \varphi(w, \alpha_0, \xi_0)
\end{align*}
\]

and let \((\alpha_0, \xi_0) \in \arg\max_{\alpha, \xi \in T} \varphi(v, \alpha, \xi)\). Note that \(\varphi(w, \alpha_0, \xi_0) = \varphi(v, \alpha_0, \xi_0)\). It follows that

\[
-f_0(w) \geq \begin{align*}
&\geq -\frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 - C \|w_S - v\| + \varphi(v, \alpha_0, \xi_0) \\
&= -\frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 - C \|w_S - v\| + f_0(v) \\
&\geq -\frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 - C \|w_S - v\| + \varphi(v, \alpha_1, \xi_1) \\
&= -\frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 - C \|w_S - v\| - \frac{1}{2} \sum_{i \in S} (v_i^2 - w_i^2) (\alpha_1)_i^2 + \varphi(w, \alpha_1, \xi_1) \\
&\geq -\frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 - 2C \|w_S - v\| + \varphi(w, \alpha_1, \xi_1)
\end{align*}
\]

where we made use of optimality of \((\alpha_0, \xi_0)\) for the second inequality. It follows that

\[
\|(\alpha_0)_{S^c} \odot w_{S^c}\|^2 + 2C \|w_S - v\| \geq \|(\alpha_1)_{S^c} \odot w_{S^c}\|^2
\]

and

\[
f_0(w) \geq f_0(v) - C \|w_S - v\| - \frac{1}{2} \|(\alpha_0)_{S^c} \odot w_{S^c}\|^2
\]

Finally, since for all \(\alpha, \xi\),

\[
\varphi(w, \alpha, \xi) \leq \varphi(w, \alpha, \xi)
\]

we have \(f_0(w) \leq f_0(w_S)\) and hence, \(f_0(w) \leq f_0(v) + C \|v - w_S\|\). It follows that

\[
|f_0(w) - f_0(v)| \leq \frac{1}{2} \|\alpha_0\|^2 \|w_{S^c}\|_2^2 + C \|v - w_S\|.
\]

In particular, \(f_0\) is calm (see Appendix A, this is a slightly weaker condition than locally Lipschitz, as we compared only against \(v\)). \(\Box\)
2.5 Differentiability for basis pursuit

Differentiability when $F_0$ is not $C^{1,1}$ is more delicate. We consider the case where $F_0(z) = \iota_{\{y\}}(x)$ here, which corresponds to the so-called basis pursuit problem, where one imposes exact reconstruction $Ax = y$. See Sect. 5.4 for additional examples of robust constraints.

In the case of basis pursuit, one has

$$\varphi(v, \xi, \alpha) = -\frac{1}{2} \|v \odot \alpha\|^2 - \langle \xi, y \rangle.$$ 

Observe that by setting $v = 0$, one can see that $\sup_{\alpha, \xi} \varphi(0, \xi, \alpha) = \sup_{L^\top \alpha = -A^\top \xi} - \langle \xi, y \rangle = +\infty$ and the domain of $f$ cannot be the entire space. The following result shows that $f$ is differentiable provided that the constraint is feasible and the support of $v$ contains the true support.

**Theorem 2** Let $F_0 = \iota_{\{y\}}$ and suppose that $y = Ax$ for some $x \in \mathbb{R}^n$. Suppose that $v \in \mathbb{R}^p$ satisfies $\text{Supp}(v) \supset \{i : (Lx)_{gi} \neq 0\}$. Then, $v \in \text{dom}(f)$, a maximizer to $\sup_{\alpha, \xi} \varphi(v, \alpha, \xi)$ exist, and $f$ is differentiable at $v$.

**Proof** Note that we can write

$$f_0(v) = \max_{L^\top \alpha \in \mathcal{R}(A^\top)} \psi(v, \alpha), \quad \text{where } \psi(v, \alpha) = -\frac{1}{2} \|v \odot \alpha\|^2 + \langle \alpha, Lx \rangle.$$ 

By our assumption on the relationship between $\text{Supp}(Lx)$ and $S \triangleq \text{Supp}(v)$,

$$\psi(v, \alpha) \leq \sum_{i \in S} \left( -\frac{1}{2} v_i^2 \|\alpha_i\|^2 + \|\alpha_i\| \|(Lx)_{gi}\| \right) \leq \sum_{i \in S} \frac{\|(Lx)_{gi}\|^2}{2|v_i|^2}.$$ 

So, $f_0(v) < \infty$. Let $\alpha_n$ be a maximizing sequence, note that $\{\alpha_n\}_n$ is uniformly bounded since $\psi(v, \cdot, \cdot)$ is strongly concave, so, there exists $\alpha_*$ such that $\alpha_n \rightarrow \alpha_*$. Since $L^\top (\alpha_n)_S \in \mathcal{R}([L^T_S, A])$ and is convergent, its limit is also in the $\mathcal{R}([L^T_S, A])$. That is, there exists $(\alpha_*)_S$ and $\xi_*$ with $L^\top (\alpha_*)_S = L^\top (\alpha_*)_S + A^\top \xi_*$. One can finally conclude from upper semicontinuity of $\psi(v, \cdot, \cdot)$ that $\alpha_*$ is a maximizer.

It remains to deduce that $f_0$ is differentiable at $v$. The proof is similar to that of Proposition 4, we first show that on a neighborhood $V$ of $v$, the mapping $v' \mapsto \alpha_{v'}$ is continuous when restricted to all $v' \in V$ with $\text{Supp}(v') = \text{Supp}(v) \triangleq S$. Indeed,

$$\psi(v, \alpha_{v'}) = -\frac{1}{2} \|\alpha_{v'} \odot v\|^2 + \langle \alpha_{v'}, Ax \rangle \geq \psi(v, 0) = 0,$$ 

which implies that

$$\frac{1}{2} \|\alpha_{v'} \odot v\|^2 \leq \|\alpha_{v'}\|_{\text{Supp}(Ax)} \|Ax\| \leq \|\alpha_v\|_S \|Ax\|.$$
since $\operatorname{Supp}(Ax) \subseteq S$. It follows that on a neighborhood around $v$, $v' \mapsto \|\alpha v\|$ is uniformly bounded, just like step 1 of the proof of Proposition 4. One can then show that $f_0$ is strictly continuous at $v$ when restricted to the support $S$. Strong concavity with respect to $\|\alpha \odot v\|^2$ also implies that $v \mapsto \alpha v$ is Lipschitz continuous. Using continuity of the maximizers, we can then compute the semiderivative of $f_0$ as in Step 3 of the proof of Prop 4 to deduce that $f$ is differentiable.

\[\Box\]

3 Conditioning improvement of VarPro

In this section we quantify the conditioning of the VarPro method, theoretically in the case of pure Lasso problems ($L = \operatorname{Id}$), and numerically for total variation denoising ($A = \operatorname{Id}$).

3.1 Conditioning in Lasso problems

Improved conditioning compared to the non-VarPro function As mentioned in the introduction, one of the advantages of VarPro is that it leads to better conditioning of the Hessian. The basic reason is that the Hessian of $f(v) \triangleq \min_u G(u, v)$ is the Schur complement of the Hessian of $G$, and the condition number of the Schur complement is always no worse than that of the original matrix [39], thanks to the interlacing property of eigenvalues: given a block-matrix $H = [A, B; B^\top, D] \in \mathbb{R}^{n \times n}$, its Schur complement $H/D$ for $D \in \mathbb{R}^{r \times r}$ satisfies

$$
\lambda_i(H) \leq \lambda_i(H/D) \leq \lambda_{i+n-r}(H),
$$

where $\lambda_i$ is the $i$th largest eigenvalue. Hence, $\operatorname{Cond}(H) \geq \operatorname{Cond}(H/D)$.

To quantify the improvement in conditioning, we consider the case of the quadratic loss here and contrast the Hessian of $G(u, v)$ with the Hessian of the VarPro function $f(v) \triangleq \min_u G(u, v)$.

We already know that all saddle points of $f$ are either global minimums or strict saddles (Hessian has a negative eigenvalue). In the following, we provide a bound on the ratio of condition numbers between $\nabla^2 f$ and $\nabla^2 G$ at saddle points that satisfy a nondegeneracy condition. This condition ensures that the Hessian of $f$ (and hence $G$) is positive definite and the saddle point is a global minimum. Since we have that the matrix $H$ defined below satisfies $\|H\| = \mathcal{O}(1)$ if the columns of $A$ are normalized, the following bound shows that this ratio is $\mathcal{O}(\lambda)$, so the smaller $\lambda$ is, the bigger the improvement in conditioning for VarPro.

**Proposition 5** Let $v$ be a saddle point of $f$ and $u \triangleq \operatorname{argmin}_u G(u, v)$. Let $J \triangleq \operatorname{Supp}(v)$ and assume that $x \triangleq u \odot v$ is nondegenerate, that is, for $\xi \triangleq A^\top (Au \odot v - y)/\lambda$, $\min_{i \notin J} (1 - \|\xi_i\|^2) > 0$. 

\[\text{ Springer}\]
Then, $\nabla^2 f(v)$ and $\nabla^2 G(u, v)$ are positive definite and the ratio of their condition numbers satisfy

$$\frac{\text{Cond}(\nabla^2 f)}{\text{Cond}(\nabla^2 G)} \leq \lambda \cdot \max \left\{ 4, \frac{\lambda + \sigma}{\sigma} \right\} \frac{\| H + \lambda E_J \|}{\| J + \lambda E_J \|}.$$

where $\sigma$ is the smallest eigenvalue of $(v_i^T A_{gi}^T A_{gj} v_j)_{i,j \in J}$, we define

$$H \triangleq \begin{pmatrix} (u_{gi}^T A_{gi}^T A_{gj} u_{gj})_{i,j \in J} & (u_{gi}^T A_{gi}^T A_{gj} v_j)_{i,j \in J} \\ (v_{gi} A_{gj}^T u_{gj})_{i,j \in J} & (v_{gi} A_{gj}^T v_j)_{i,j \in J} \end{pmatrix}.$$

and given an index set $S$,

$$E_S \triangleq \begin{pmatrix} \text{Id}_{|S|} & \text{diag}(\xi_{gi})_{i \in S} \\ \text{diag}(\xi_{gi})_{i \in S} & \text{Id}_{|S|} \end{pmatrix}.$$

**Proof** The Hessian of $G$ is $\nabla^2 G = \begin{pmatrix} \partial_{vv} & \partial_{vu} \\ \partial_{uv} & \partial_{uu} \end{pmatrix}$ where

$$\partial_{vv} = \lambda^{-1} \left( u_{gi}^T A_{gi}^T A_{gj} u_{gj} \right)_{i,j} + \text{Id} \quad \text{and} \quad \partial_{uu} = \lambda^{-1} \left( v_{gi}^T A_{gi}^T A_{gj} v_j \right)_{i,j} + \text{Id}$$

and

$$\partial_{vu} = \lambda^{-1} \left( u_{gi}^T A_{gi}^T A_{gj} v_j \right)_{i,j} + \text{diag} \left( \left( \xi_{gi}^T \right) \right)_{i,j},$$

where $\xi = A^T (A(u \odot v) - y)/\lambda$ is the dual solution to the group-Lasso problem. By rearranging the columns/rows of $\nabla^2 G(u, v)$ and denoting $J \triangleq \text{Supp}(v)$, we can write

$$\nabla^2 G(u, v) = \lambda^{-1} \begin{pmatrix} H & E_J \\ 0_{|J^c|} & E_{J^c} \end{pmatrix} + \lambda \begin{pmatrix} H & E_J \\ 0_{|J^c|} & E_{J^c} \end{pmatrix},$$

where the two matrices inside the square brackets are block-diagonal matrices, $0_n$ denotes an $n \times n$ matrix of zeros and $H$ is define in (17) and $E_J, E_{J^c}$ are defined as in (18). It follows that

$$\text{Cond}(\nabla^2 G) = \| \nabla^2 G \| (\nabla^2 G)^{-1} \| = \max \{ \| H + \lambda E_J \|, \lambda \| E_J \| \} \cdot \max \left\{ \| (H + \lambda E_J)^{-1} \|, \frac{1}{\lambda} \| E_{J^c}^{-1} \| \right\} \geq \frac{1}{\lambda} \| H + \lambda E_J \| \| E_{J^c}^{-1} \|.
As shown in Theorem 3 of [60], the Hessian of $f$ can be written as

$$\nabla^2 f(v) = \nabla v \nabla v^\top - \partial v u \partial u^\top - u u^\top \nabla v u$$

and at optimality, the eigenvalues of $\nabla^2 f(v)$ are at most 4 and at least

$$\min \left( 4(1 - \lambda / (\lambda + \sigma)), \min_{i \notin J} (1 - \|g_i\|^2) \right)$$

(20)

where $J = \text{Supp}(v)$ and $\sigma$ is the smallest eigenvalue of $(v_i^\top A_i A_j v_j)_{i,j \in J}$. So, $\|\nabla^2 f\| \leq 4$.

Note that

$$E_{J^c} \setminus \text{Id}_{J^c} = \text{diag}(1 - \|g_i\|^2)_{i \in J^c}$$

which, by the interlacing property of eigenvalues (16), implies that

$$\|E_{J^c}^{-1}\| \geq \left[ \min_{i \notin J} (1 - \|g_i\|^2) \right]^{-1} \geq 1.$$

So, there are two scenarios:

- if $\min_{i \notin J} (1 - \|g_i\|^2) \leq 4\sigma / (\lambda + \sigma)$, then

$$\min \text{Eig}(\nabla^2 f) \geq \min_{i \notin J} (1 - \|g_i\|^2) \geq \|E_{J^c}^{-1}\|^{-1} \implies \| (\nabla^2 f)^{-1} \| \leq \|E_{J^c}^{-1}\|,$$

and hence,

$$\text{Cond}(\nabla^2 f) \leq 4 \|E_{J^c}^{-1}\| \leq 4\lambda \frac{\text{Cond}(\nabla^2 G)}{\|H + \lambda E_J\|}.$$  

- If $\min_{i \notin J} (1 - \|g_i\|^2) > 4\sigma / (\lambda + \sigma)$, then $\| (\nabla^2 f)^{-1} \| \leq \frac{\lambda + \sigma}{4\sigma}$, and hence

$$\text{Cond}(\nabla^2 f) \leq \frac{\lambda + \sigma}{\sigma} \leq \lambda \cdot \frac{\lambda + \sigma}{\sigma} \cdot \frac{\text{Cond}(\nabla^2 G)}{\|H + \lambda E_J\|}.$$

Taking the maximum of these two bounds concludes the proof. \qed

**Conditioning of the inner and outer problems in VarPro** In this section, we numerically investigate the condition numbers of the inner linear system and the Hessian to the outer function $f$ change as we iterate through gradient descent in the case of an ill-conditioned data matrix $A$. Consider the Fourier matrix

$$A = \left( \exp(2\pi i k t_j) \right)_{k=0,\ldots,f_c \atop j=0,\ldots,s}, \text{ where } t_j \in [0, 1].$$

(21)

This is a highly ill-conditioned problem and of particular interest in super-resolution where one wishes to resolve, from low frequency data (up to cutoff $f_c$), point sources supported at positions $t_j$ as they collapse together. Singular values of such matrices are well studied in the literature, and in the regime where $\Delta$ the minimum separation
between $t_j$’s are below $1/f_c$ is the so-called sub-Rayleigh regime where the condition number of $A$ scales exponentially as $\left(\frac{1}{(f_c \Delta)}\right)^{2s+1}$ [6]. To numerically investigate the conditioning of VarPro for this problem, we let $t_j = j/N$, $s = f_c$ for fixed $f_c = 15$ and look at the conditioning as we increase $N$. We consider the range $N \in \{2^j : j = 4, \ldots, 8\}$.

3.2 Total variation denoising

Section 3.1 proves the conditioning improvement brought by VarPro in the simplified setting where $L = \text{Id}$ (pure Lasso problems). The general case $L \neq \text{Id}$ is much more intricate, and to gain some insight on the effect of VarPro, we consider the denoising problem ($A = \text{Id}$), that is

$$\min_{x \in \mathbb{R}^n} \|Lx\|_1 + \frac{1}{2\lambda} \|x - y\|_2^2,$$

(22)

where $Lx \triangleq (x_i - x_{i+1})_{i=1}^n$ is the gradient operator with periodic boundary condition (so we let $x_{n+1} \triangleq x_1$). This TV regularization is challenging because it leads to
piecewise constant solutions, and the precise positions of the jumps depend globally on the input signal \( y \).

One popular way of solving (22) is to apply the projected gradient descent method to the dual problem [20]. The dual of (22) is

\[
\sup_{\|z\|_\infty \leq 1} \langle y, L^\top z \rangle - \frac{\lambda}{2} \|L^\top z\|_2^2.
\]

The projected gradient descent (PGD) method computes the iterates

\[
\begin{align*}
z_{k+1} &= \text{Proj}_{\| \cdot \|_\infty \leq 1} \left( z_k - \tau \left( Ly - \lambda L L^\top z_k \right) \right), \\
\end{align*}
\]

where the stepsize \( \tau \) satisfies \( \tau \in (0, \frac{2}{\|L\|_F^2}) \) and \( \text{Proj}_{\| \cdot \|_\infty \leq 1} (u) = \left( \frac{u_i}{\max \{|u_i|, 1| \}} \right)_i \) is the projection onto the \( \ell_\infty \) ball. The primal iterates are taken to be \( x_k \triangleq y - \lambda L L^\top z_k \).

Following Sect. 2.3.2, the application of gradient descent applied to the VarPro function \( f \) is

\[
\begin{align*}
\alpha_k &= \left( \lambda L L^\top + \text{diag}(v_k^2) \right)^{-1} L y, \\
v_{k+1} &= v_k - \tau \left( v_k - \alpha_k^2 \cdot v_k \right),
\end{align*}
\]

and the primal iterates are \( x_k \triangleq y - \lambda L L^\top \alpha_k \). In Fig. 5, we show the iterations of (23) initialized with \( z_0 = 0_n \) and (24) initialized with \( v_0 = 1_n \) with \( f \) being a box function. The number of grid points is taken to be \( n = 512 \) and the same stepsize \( \tau = 1.8/\|L\|^2 \) is used for both methods.

Observe that assuming that \( v_k \) has all non-zero entries, we can also apply the Woodbury matrix inversion formula to write

\[
\alpha_k = \frac{1}{v_k^2} \cdot L (\text{Id} + \lambda L L^\top \text{diag}(1/v_k^2) L)^{-1} y
\]

so each step of (24) can be viewed as solving a weighted Poisson equation. In contrast to (23), the VarPro gradient is nonlocal which leads to the acceleration of the convergence, compared to PGD which is slow in diffusing across flat regions.

Note that similar ideas have been employed in [73] (see also [21] for a related IRLS like approach), where the gradient update is computed by solving a weighted Poisson equation but where \( v_k \) is taken to be \( \sqrt{|L x_k|^2 + \beta_k} \) for some \( \beta_k > 0 \) that needs to be tuned (i.e. this corresponds to some smoothing of the TV objective). Note that in our setting, the advantage is that this improvement in conditioning is automatically achieved without the need for additional smoothing parameters.

4 The fine grids settings

Particularly challenging settings correspond to cases where the columns of \( A \) are highly correlated. This is a typical situation for inverse problems in imaging sciences,
and in particular deconvolution-type problems [17, 25]. In these settings, $A$ arises from the discretization of some continuous operator, and the dimension $n$ grows as the grid refines. For the sake of concreteness, we consider an ideal low-pass filter in dimension $d$ (for instance $d = 2$ for images), which is equivalent to the computation of low Fourier frequencies, up to some cut-off frequency $p$. The rows $A_k \in \mathbb{R}^n$ of $A$ are indexed by $k = (k_i)_{i=1}^d \in [p]^d \triangleq \{0, \ldots, p\}^d$,

$$A_k = \varphi(\theta_k), \quad \text{where} \quad \theta_k \triangleq \frac{1}{p} (k_i)_{i=1}^d, \quad \varphi(\theta) \triangleq \frac{1}{md/2} \left( e^{2\pi \sqrt{-1} \langle \theta, \ell \rangle} \right)_{\ell \in [m/2]^d},$$

(25)

where $[m/2] \triangleq \{-m/2, \ldots, m/2\}$, and so, $A$ corresponds to the Fourier operator discretized on a uniform grid on $[0, 1]^d$. To better cope with the ill-conditioning of the resulting optimizations problem, it is possible to use a descent method according to some adapted metric. This can be conveniently achieved using so-called mirror descent scheme/Proximal Bregman Descent scheme, which we review below in Sect. 4.1, since this is closely linked to the Hadamard parameterization (as exposed in Sect. 4.4).

As discussed below, in the mirror descent scheme, the usual $\ell^2$ proximal gradient descent is retrieved when taking the potential function as the squared Euclidean function. This Euclidean scheme suffers from an exponential dependency on $d$ in the convergence rate. Using non-quadratic potential functions (such as the so-called hyperbolic entropy), together with a dimension-dependent parameter tuning, leads in sharp contrast to dimension independent rates [25]. After this review of mirror descent, we then analyse in Sect. 4.3 the performance of gradient descent on the Hadamard parameterized function $G(u, v)$ on the case of the Lasso. The key observation is that the Lipschitz constant of $G$ is independent of the grid size $n$ and hence, one can derive dimension-free convergence rates on the gradient. Moreover, we draw in Sect. 4.4 connections to mirror descent by showing that the continuous time limit (as the gradient descent stepsize tends to 0) corresponds to the mirror descent ODE with a hyperbolic entropy map whose parameter changes with time.
4.1 Overview of mirror descent

We consider a structured optimization problem of the form

$$\min_{x \in \mathbb{R}^n} \Phi(x) \triangleq R(x) + F(x)$$

(26)

where $R : \mathbb{R}^n \to [0, \infty]$ is a (nonsmooth) convex function and $F : \mathbb{R}^n \to \mathbb{R}$ is assumed to be convex and Lipschitz continuous on a closed convex set $\mathcal{X} \supset \text{dom}(R)$ with

$$\|\nabla F(x) - \nabla F(x')\|_{\mathcal{X}^*} \leq M \|x - x'\|_{\mathcal{X}}$$

(27)

where $\| \cdot \|_{\mathcal{X}}$ is some norm on $\mathcal{X}$ and $\| \cdot \|_{\mathcal{X}^*}$ is the dual norm. This includes in particular sparsity regularized problems of the form (1). A natural algorithm to consider is the Bregman proximal gradient descent method, of which the celebrated iterative soft thresholding algorithm is a special case. In this section, we provide a brief overview of this method and the associated convergence results.

Given a strictly convex function (called an potential function) $\eta : \mathcal{E} \to [-\infty, \infty)$ that is differentiable on an open set $\mathcal{E} \supset \text{int}(\mathcal{X})$, its associated Bregman divergence is defined to be

$$D_\eta(a, b) \triangleq \eta(a) - \eta(b) - \langle \eta'(b), a - b \rangle.$$  

(28)

By possibly rescaling $\eta$, we also assume that $\eta$ is strongly convex with respect to $\| \cdot \|_{\mathcal{X}}$ (although more general conditions are possible [7]):

$$D_\eta(a, b) \geq \frac{1}{2} \|a - b\|_{\mathcal{X}}^2.$$  

(29)

The Bregman proximal gradient descent method (BPGD) [69] is

$$x_{k+1} = \arg\min_x F(x_k) + \nabla F(x_k)^T (x - x_k) + R(x) + \frac{M}{2} D_\eta(x, x_k),$$

(30)

with corresponds to taking constant stepsize $1/M$.

Remark 3 The case of $R = 0$ corresponds to the mirror descent method, this dates back to [1, 13] and has in more recent years been revitalized by [8]. The proximal version given here is due to [69].

It is shown in [69] that this is a descent method with $\Phi(x_{k+1}) \leq \Phi(x_k)$ and for any $x \in \text{dom}(R)$,

$$\Phi(x_k) - \Phi(x) \leq \frac{1}{k} M D_\eta(x, x_0).$$

(31)

4.2 The Lasso ($\ell_1$) special case

The BPGD algorithm (30) is mainly interesting when the updated (the so-called proximal operator associated to $R$) step can be computed in closed form. This is not the case for an arbitrary operator $L$, and we thus focus on the setting $L = \text{Id}$. For the
sake of simplicity, we also consider the case where there is no group structure (Lasso), so that $R(x) = \|x\|_1$. The most natural norm to perform the convergence analysis is $\|\cdot\|_\infty$ so that $\|\nabla F(x) - \nabla F(x')\|_\infty \leq M_1 \|x - x'\|_1$. For this choice of $R$, (30) can be rewritten as

$$\nabla \eta(x_{k+1}) = T_{\tau} \left( \nabla \eta(x_k) - \frac{\tau}{n} \nabla F(x_k) \right),$$

(32)

where $T_{\tau}(z) = \max(|z| - \tau, 0) \odot \text{sign}(z)$ is the soft thresholding operator. Let us now single out notable choices of potential functions, in order to particularize the convergence bound (31):

- **The quadratic function** observe that $n\|x - x'\|^2 \geq \|x - x'\|_1^2$, so (29) holds by choosing $\eta(x) = \frac{n}{2}\|x\|^2$. For $\|x\|_1 \leq 1$ and $\|x_0\|_1 \leq 1$, we have $D(x, x_0) \leq n\|x\|^2 + n\|x_0\|^2 \leq 2n$. The error bound (31) is therefore $O\left(\frac{nM_1}{k}\right)$.

- **The hyperbolic entropy** introduced in [36], it is defined for $c > 0$ by

$$\eta_c(s) = s \cdot \text{arcsinh}(s/c) - \sqrt{s^2 + c^2} + c,$$

(33)

so that $\eta'_c(s) = \text{arcsinh}(s/c)$ and $\eta''_c(s) = \frac{1}{\sqrt{s^2 + c^2}}$. It is shown in [36] that $\eta_c$ satisfies $D_\eta(x, x') \geq \frac{1+cn}{2}\|x - x'\|_1^2$. In particular, (29) holds by choosing $c = \frac{1}{n}$ and rescaling $\eta$ by $\frac{1}{2}$. On the other hand, for $\|x\|_1, \|x'\|_1 \leq 1$, $D_\eta(x, x') = O(\log(n))$. The error bound (31) is therefore $O\left(\frac{\log(n)M_1}{k}\right)$.

**Grid-free convergence rates** The above results show that the error bound (31) in general has a dependency on $n$, either through $M$ or through $D_\eta(x, x_0)$. This is thus unable to cope with very fine grids, and the analysis breaks in the “continuous” (often called off-the-grid) setting where discrete vectors with bounded $\ell^1$ are replaced by measures with bounded total variation [15, 17]. To address this issue, a more refined analysis of BPGD is carried out in [25] and this lead to the first grid-free convergence rates for BPGD. In particular, it is shown that the objective for quadratic potential converges at rate $O(k^{-2/(d+2)})$, independent of grid size $n$ but dependent on the underlying dimension $d$. In contrast, BPGD with hyperbolic entropy satisfies $\Phi(x_k) - \min_x \Phi(x) = O(d \log(k)/k)$.

### 4.3 The Hadamard parameterization: grid-free convergence analysis

In this section, we show that gradient descent with fixed timestep on the Hadamard parameterization also leads to grid and dimension free convergence guarantees. The caveat is that due to the nonconvex nature of our problem, our convergence results are only for the gradient norm and thus weaker than the objective convergence results of [25]. The Hadamard parameterization of (26) in the group Lasso case (i.e. $L = \text{Id}$ in (1)) is

$$\min_{u, v} G(u, v) \quad \text{where} \quad G(u, v) \triangleq \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|v\|_2^2 + F(u \odot v)$$
where $F$ is differentiable with $M_F > 0$ such that

$$\| \nabla F(x) - \nabla F(x') \|_{\infty,2} \leq M_F \| x - x' \|_{1,2},$$

(34)

where $\| z \|_{\infty,2} \triangleq \max_{g \in \mathcal{G}} \| zg \|$ is the dual of the group norm $\| \cdot \|_{1,2}$ defined in (2). For a stepsize $\tau > 0$, the gradient descent iterations are

$$
\begin{align*}
    u_{k+1} &= u_k - \tau g_{u_k}, \quad \text{where} \quad g_{u_k} = u_k + v_k \odot \nabla F(u_k \odot v_k), \\
    v_{k+1} &= v_k - \tau g_{v_k}, \quad \text{where} \quad g_{v_k} = v_k + \left( (u_k)_{g}^\top \nabla F(u_k \odot v_k)_{g} \right)_{g \in \mathcal{G}}.
\end{align*}
$$

(35)

**Remark 4** In the Lasso setting where $u_k$ and $v_k$ have the same dimensions, if $u_0 = v_0$, then $u_k = v_k$ for all $k$, while if $u_0 = -v_0$, then $u_k = -v_k$ for all $k$. One should therefore initialize with $|u_0| \neq |v_0|$. In practice, we find that random initialization of $u_0$ and $v_0$ works well.

Since gradient descent is a descent method, one can assume that all iterates lie inside some ball, that is, all iterates satisfy

$$
\frac{1}{2} \left( \| u \|^2 + \| v \|^2 \right) \leq G(u_0, v_0) \triangleq \frac{1}{2} B^2.
$$

(36)

Suppose that

$$
\sup_{\| x \|_{1,2} \leq B^2/2} \| \nabla F(x) \|_{\infty,2} \leq K.
$$

(37)

Note that this implies $\sup_{\| u \|_{1,2}^2 + \| v \|_{1,2}^2 \leq B^2} \| \nabla F(u \odot v) \|_{\infty,2} \leq K$. Under these assumptions, the following Proposition shows that $\nabla G$ is Lipschitz with respect to the Euclidean norm, with a Lipschitz constant that depends only on $M_F, B, K$. This in turn ensure convergence rates for (35) which are dimension-free. Note that using this Hadamard parameterization, one considers descent on $u$ and $v$ with respect to the standard Euclidean metric. This convergence statement is thus a direct consequence of the standard descent lemma for gradient descent (Lemma 5).

**Proposition 6** Assume that $F \in C^{1,1}$ with Lipschitz gradient satisfying (34) and uniformly bounded gradient (37). Then, given $u_1, v_1, u_2, v_2$ satisfying (36), we have the following Lipschitz bound on $\nabla G$,

$$
\| \nabla G(u_1, v_1) - \nabla G(u_2, v_2) \| \leq M_G \| (u_1, v_1) - (u_2, v_2) \|,
$$

where $M_G \triangleq 2(K + M_FB^2)$. For stepsize $\tau = 1/M_G$, we have

$$
\min_{k \leq T} \| \nabla G(u_k, v_k) \|_2^2 \leq \frac{2M_G}{T} (G(u_0, v_0) - G(u_{T+1}, v_{T+1}))
$$

(38)
and if \( \lim_{j \to \infty} G(u_j, v_j) \) exists, then \( \{\|\nabla G(u_j, v_j)\|^2\}_j \) is summable with

\[
\sum_{j=k}^{\infty} \|\nabla G(u_j, v_j)\|^2 \leq 2M_G(G(u_k, v_k) - \lim_{j \to \infty} G(u_j, v_j)).
\] (39)

**Proof** Note that \( \nabla_u G(u, v) = u - v \odot \nabla F(u \odot v) \) and \( \nabla_v G(u, v) = v - (u^\top \nabla F(u \odot v)_g)_g \), so that

\[
\begin{align*}
\|\nabla_u G(u_1, v_1) - \nabla_u G(u_2, v_2)\| &\leq \|u_1 - u_2\| + \|v_1 \odot \nabla F(u_1 \odot v_1) - v_2 \odot \nabla F(u_2 \odot v_2)\| \\
&\leq \|u_1 - u_2\| + \|(v_1 - v_2) \odot \nabla F(u_1 \odot v_1)\| + \|v_2 \odot (\nabla F(u_1 \odot v_1) - \nabla F(u_2 v_2))\| \\
&\leq \|u_1 - u_2\| + \|v_1 - v_2\|\|\nabla F(u_1 \odot v_1)\|_{\infty,2} + \|v_2\|_{1,2}M_F\|u_1 \odot v_1 - u_2 \odot v_2\| \\
&\leq \|u_1 - u_2\| + K\|v_1 - v_2\| + M_F\|v_2\| (\|v_1\|_{1,2} + \|u_2\|_{1,2} + \|v_2\|_{1,2}) \\
&\leq (1 + M_F B^2)\|u_1 - u_2\| + (K + M_F B^2)\|v_1 - v_2\|.
\end{align*}
\]

The term \( \|\nabla_u G(u_1, v_1) - \nabla_u G(u_2, v_2)\| \) can be bounded in a similar way and the result follows. The final gradient bound is then a direct consequence of Lemma 5. \( \square \)

The crucial point which makes this Hadamard parameterization attractive is that in the fine grids setting, \( M_F \) and \( K \) typically have no dependence on the grid discretization or the underlying dimension. This implies that the Hadamard parameterization leads to grid-free and dimension-free convergence rates on the gradient. Consider the case of trivial groups \( G = \{\{j\}\}_{j=1}^n \), \( F(x) = F_0(Ax) \), \( \nabla F(x) = A^\top \nabla F_0(Ax) \) and

\[
M_F \leq \|A\|_{1 \to 2}^2 M_{F_0}
\]

where \( M_{F_0} \) is the Lipschitz constant of \( F_0 \) with respect to the Euclidean norm and \( \|A\|_{1 \to 2} \leq 1 \) if the columns of \( A \) are normalized. For the Fourier example mentioned in (25), \( F_0 \) is the quadratic function \( \| \cdot - y \|_2^2 \) and we can take \( M_F = 1 \).

### 4.4 The Hadamard flow: connection with mirror descent

In this section, we consider the case of \( \ell_1 \) regularization (4.2) (trivial group structure). The goal of this section is to highlight the connection to mirror descent (Proposition 7). Based on this connection, we show in Proposition 8 that convergence of the objective is controlled by the convergence of \( \nabla G \). This analysis does not carry over the group Lasso case, because the metric associated to the group-lasso over-parameterization can be shown to be associated to a Riemannian metric which is not of Hessian type [42]. Specialized to the case of \( L = \text{Id} \), the Hadamard parameterized function is

\[
G(u, v) = \min_{u, v \in \mathbb{R}^n} F(u \cdot v) + \frac{\lambda}{2} \|u\|_2^2 + \frac{\lambda}{2} \|v\|_2^2.
\]
Note that letting \( \tau \to 0 \), the continuous flow equations of (35) are
\[
\dot{u}(t) = -\lambda u(t) - v(t) \cdot \nabla F(u \cdot v), \\
\dot{v}(t) = -\lambda v(t) - u(t) \cdot \nabla F(u \cdot v).
\]

The following propositions show that the \( L_2 \) flow on \( G \) corresponds to “mirror descent” with the generalized hyperbolic entropy function \( \eta^{D(t)}_{\gamma(t)} \).

Proposition 7 Let \( x(t) \triangleq u(t) \cdot v(t) \) where \( u, v \) satisfy (40) and (41). For \( \gamma > 0 \), let \( \eta_{\gamma} \) denote the hyperbolic entropy function defined in (33). The following holds
\[
\frac{d}{dt} \nabla \eta_{\gamma(t)}(x(t)) = -2 \nabla F(x(t))
\]
where \( \gamma(t) = \frac{1}{2} |u(0)^2 - v(0)^2| \exp(-2\lambda t) \).

Proof From the flow equations (40) and (41),
\[
\dot{x}(t) = \dot{u}(t) \cdot v(t) + \dot{v}(t) \cdot u(t) = -(u(t)^2 + v(t)^2) \cdot \nabla F(x(t)) - 2x(t),
\]
Note that \( (u(t)^2 + v(t)^2)^2 - 4u(t)^2 \cdot v(t)^2 = (u(t)^2 - v(t)^2)^2 \) and
\[
\frac{d}{dt} [(u(t)^2 - v(t)^2)^2] = -4\lambda (u(t)^2 - v(t)^2)^2
\]
and so, \( (u(t)^2 - v(t)^2)^2 = \exp(-4\lambda t)(u(0)^2 - v(0)^2)^2 \) which implies that
\[
(u(t)^2 + v(t)^2)^2 = 4x(t)^2 + c^2 \exp(-4\lambda t) \quad \text{where} \quad c \triangleq |u(0)^2 - v(0)^2|.
\]
By denoting \( \gamma(t) \triangleq \frac{1}{2} c \exp(-2\lambda t) \), the equation (43) can be rewritten as
\[
\frac{(\dot{x}(t) + 2\lambda x(t))}{\sqrt{x(t)^2 + \gamma(t)^2}} = -2 \nabla F(x(t)).
\]
Finally,
\[
\frac{d}{dt} \nabla \eta_{\gamma(t)}(x(t)) = \frac{d}{dt} \nabla \eta_{c}(x(t) \exp(2\lambda t))
\]
\[
= \left( \eta_{c}''(x(t) \exp(2\lambda t)) \exp(2\lambda t) (\dot{x}(t) + 2\lambda x(t)) \right)
\]
\[
= \frac{(\dot{x}(t) + 2\lambda x(t))}{\sqrt{x(t)^2 + \gamma(t)^2}} = -2 \nabla F(x(t)).
\]
Remark 5 (Algorithmic regularization properties) Following [3], we make some informal comments on the significance of Proposition 7 when $\lambda = 0$, as inducing an “implicit bias” selecting a particular solution to the linear system $Ax = y$. Here, $\gamma(t) = c \triangleq \frac{1}{2}|u(0)^2 - v(0)^2|$ is constant for all $t$ and

$$\nabla \eta_c(x(t)) - \nabla \eta_c(x(0)) = -2A^\top r(t) \quad \text{where} \quad r(t) \triangleq \int_0^t Ax(s)ds - y,$$

which, assuming that $x(t)$ converges to $x_*$ such that $Ax_* = y$, is the optimality condition for

$$\min_x D_{\eta_c}(x, x(0)) \quad \text{s.t.} \quad Ax = y,$$

where $D_{\eta_c}(x, x')$ is the Bregman divergence associated to $\eta_c$. So, even without explicit regularization, the flow $x(t)$ defined by (40), (41) is regularized by $\eta_c$.

Remark 6 (Difference of squares parameterization) Another parameterization for the lasso is the squared-parameterization [25]: let $x = r \cdot r - s \cdot s$ and perform $L_2$ gradient flow on

$$F(r \cdot r - s \cdot s) + \lambda \|r\|^2 + \lambda \|s\|^2. \quad (45)$$

By writing $r = \frac{1}{2}(u + v)$ and $s = \frac{1}{2}(u - v)$, we have $r \cdot r - s \cdot s = u \cdot v$ and one can observe that this is equivalent to the Hadamard parameterization. Note however that this equivalence is only in the case of the Lasso and the Hadamard parameterization can be used to handle more complex regularizers such as the group $\ell_1$ norm.

We saw in Proposition 7 that in the continuous time limit, gradient descent on the Hadamard parameterization can be interpreted as mirror descent with a varying entropy function. For fixed timestep, one can write for $x_k \triangleq u_k \cdot v_k$

$$x_{k+1} = x_k - \tau H_k^{-1}\nabla F(u_k \cdot v_k) + \tau^2 g_{u_k} \cdot g_{v_k} \quad \text{where} \quad H_k = \text{diag}(1/(u_k^2 + v_k^2)).$$

Ignoring the $\tau^2$ term, one can view this as variable metric descent on $x_k$. By making use of this link, we can relate the convergence of the objective on $x_k$ to the convergence of the gradient of the overparameterized function $\nabla G$ as follows.

Proposition 8 Suppose that $\mathcal{G} = \{i\}_{j=1}^p$. Assume that $F$ satisfies (34) and (37). Let $\tau = 1/ (\kappa M_G)$ where $M_G$ upper bounds the Lipschitz constant of $\nabla G$ and $\kappa = \max(1, (1 + K^2)/M_G)$ (note that $M_G = O(K + MFB^2)$ by Proposition 6). Then,

$$\Phi(x_k) - \lim_{k \to \infty} \Phi(x_k) \leq C \sum_{j=k}^\infty \|\nabla G(u_j, v_j)\|^2 + C \rho^k.$$

with $\rho \triangleq 1 - \frac{1}{\kappa M_G}$. In particular, if $\|\nabla G(u_k, v_k)\| = O(1/k)$, then the objective converges at rate $O(1/k)$. 

\[ \square \] Springer
Fig. 6 Comparison of: ISTA, Gradient descent for Noncvx-pro, gradient descent on the Hadamard parameterization, Bregman projected gradient with the Hyperbolic entropy. The left figure shows the objective convergence against iterations $k$ with fixed stepsize, and the right figure corresponds to Barzilai–Borwein (BB) stepsize.

The proof of this proposition can be found in Sect. 4.5. This proposition along with (39) shows that the objective convergence rate is equivalent to the convergence of the tail sum of the gradients, but, at present, we do not have sufficiently strong convergence results on this gradient sum to obtain convergence rates.

Figure 6 provides some empirical finding suggesting that indeed, $\| \nabla G(u_k, v_k) \|$ is of the order $O(1/k)$. The problem considered is the Fourier system (25) with $n = 300$, $m = 2$, $\lambda = \lambda_{\text{max}}/10$ and the underlying signal to recover being 1-sparse (a single Dirac mass). For reference, the dashed lines show the $1/k$ and $1/k^{2/3}$ convergence lines. Moreover, “Hadamard Grad” on the left figure shows how $\| \nabla G(u_k, v_k) \|$ converges—it matches the $1/k$ line and converges at the same rate as the objective function. Note that both Hyperbolic entropy and the Hadamard flow exhibit $O(1/k)$ convergence, however, one practical advantage of Hadamard and Noncvx-pro is that since these methods are based on Euclidean geometry, one can apply standard tools for acceleration, such as Barzilai–Borwein (BB) stepsize [5]. Finally, observe that ISTA converges at rate $O(1/k^{2/3})$ as proved in [25], and as can be seen on the right figure, the use of BB stepsize also accelerates ISTA (although there is no theoretical proof of this).

4.5 Proof of Proposition 8

Proposition 8 is a direct consequence of the following stronger result, which is essentially a bound on the objective value error relative to a given reference point $\bar{x}$ since we have that $\Phi_k - \Phi(x_k)$ converges linearly (see the remarks immediately following the proposition.

Proposition 9 Suppose that we have a trivial group structure $G = \{\{j\}\}_{j=1}^p$ and assume that $F$ satisfies (34) and (37). Let $\tau = 1/M_G$ where $M_G$ upper bounds the
Lipschitz constant of $\nabla G$, note that $M_G = O(K + M_F B^2)$ by Proposition 6. Define $H_k = \text{diag}(1/(u_k^2 + v_k^2))$ and $|x|^2_H \triangleq \sqrt{\langle H_k x, x \rangle}$. Define $\Phi_k \triangleq F(x_k) + 2\|x_k\|^2_{H_k}$ and $\Phi(x) \triangleq F(x) + \|x\|_1$. To simplify the expression below, assume that $K, M_G, B \geq 1$. For any $\bar{x}$, we have

$$(\Phi_k - \Phi(\bar{x})) = O \left( C\|\nabla G(u_k, v_k)\|^2 + \sqrt{C}\|x_k - \bar{x}\|_{H_k} \|\nabla G(u_k, v_k)\| \right),$$

where $C = K M_G (G(u_0, v_0) - G(u_*, v_*))$.

Note that $\Phi_k$ approximates $\Phi(x_k)$, indeed, $\Phi_k = 2\|x_k\|^2_{H_k} = \sum_i 2(u_k \odot v_k)_i^2/(u_k^2 + v_k^2)_i$. Since $a, b > 0$ and $a \geq b$ implies that

$$ab - \frac{2a^2b^2}{a^2 + b^2} = \frac{ab}{a^2 + b^2} (a - b)^2 \leq a^2 - b^2,$$

we have $\Phi(x_k) - \Phi_k \leq \|u_k\|^2 - \|v_k\|^2 = O(\rho^k)$. Finally, by plugging in $\bar{x} = x_{k+1}$ to the above proposition and summing over $k, k + 1, k + 2, \ldots$ yields Proposition 8. The rest of this section is devoted to proving Proposition 9.

We begin with two lemmas, the first Lemma will be used to show that $\Phi_k - \Phi(x_k)$ defined in Proposition 9 converges to 0 linearly, while the second lemma provides several useful bounds in terms of the gradient of $G$.

**Lemma 2** Assume that $F$ satisfies (34) and (37) holds for all $k$. Let $\kappa > 0$ be such that $\kappa \geq \frac{1}{M_G} (1 + K^2)$ and let $\tau = \frac{1}{\kappa M_G}$. Then,

$$\|u_k\|^2 - \|v_k\|^2 = O(\rho^k),$$

where $\rho \triangleq 1 - \frac{1}{\kappa M_G}$.

**Proof** Notice that

$$\|u_{k+1}\|^2 - \|v_{k+1}\|^2 = (1 - 2\tau)(\|u_k\|^2 - \|v_k\|^2) + \tau^2(\|g_u_k\|^2 - \|g_v_k\|^2) \leq \|u_k\|^2 - \|v_k\|^2 (1 - 2\tau + \tau^2(1 + \|\nabla F'(x_k)\|_\infty^2))$$

By choosing $\tau = 1/(\kappa M_G)$,

$$(1 - 2\tau + \tau^2 (1 + \|\nabla F'(x_k)\|_\infty^2)) \leq 1 - \frac{1}{\kappa M_G} \triangleq \rho$$

and so, $\|u_{k+1}\|^2 - \|v_{k+1}\|^2 \leq \rho \|u_k\|^2 - \|v_k\|^2 = O(\rho^{k+1})$. \hfill $\square$

**Lemma 3** Let $H_k = \text{diag}(1/(u_k^2 + v_k^2))$. We have the following bounds

- $\|u_k \cdot g_{v_k}\|^2_{H_k} \leq \|g_{v_k}\|^2$ and $\|v_k \cdot g_{u_k}\|^2_{H_k} \leq \|g_{u_k}\|^2$
- $\|g_{u_k} \cdot g_{v_k}\|^2_{H_k} \leq C_0 \|g_{v_k}\|^2$, where $C_0 \triangleq 2 \max(1, K^2)$.
- $\|x_k - x_{k+1}\|^2_{H_k} \leq \tau^2 C_1 \|\nabla G(u_k, v_k)\|^2$ where $C_1 \triangleq (2 + \tau + 2\tau(2 + \tau) \max(1, K))$.  

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Proof Clearly, \( \| u_k \cdot g_{v_k} \|_{H_k}^2 \leq \| g_{v_k} \|_2^2 \) and

\[
\| g_{u_k} \cdot g_{v_k} \|_{H_k}^2 = \sum_i \left( \frac{g_{u_k}}{u_k^2 + v_k^2} \right)_i (g_{v_k})_i \leq 2 \max(1, K^2) \| g_{v_k} \|_2^2
\]

since \( g_{u_k}^2 = u_k^2 + v_k^2 \cdot \nabla F(x_k)^2 + 2 x_k \cdot \nabla F(x_k) \leq 2(u_k^2 + v_k^2) \cdot \nabla F(x_k)^2 \)

\[ \leq 2 \max(1, \nabla F(x_k)^2) \cdot (u_k^2 + v_k^2). \]

Finally, \( \frac{1}{\tau} (x_{k+1} - x_k) = u_k \cdot g_{v_k} + v_k \cdot g_{u_k} - \tau g_{u_k} \cdot g_{v_k} \), so that

\[
\| u_k \cdot g_{v_k} + v_k \cdot g_{u_k} - \tau g_{u_k} \cdot g_{v_k} \|_{H_k}^2
= \| u_k \cdot g_{u_k} \|_{H_k}^2 + \| v_k \cdot g_{v_k} \|_{H_k}^2 + 2 \langle u_k \cdot g_{u_k}, v_k \cdot g_{v_k} \rangle_{H_k} + \tau^2 \| g_{u_k} \cdot g_{v_k} \|_{H_k}^2
- 2 \tau \langle u_k \cdot g_{u_k} + v_k \cdot g_{u_k}, g_{u_k} \cdot g_{v_k} \rangle_{H_k}
\leq (2 + \tau) \| u_k \cdot g_{u_k} \|_{H_k}^2 + (2 + \tau) \| v_k \cdot g_{v_k} \|_{H_k}^2 + (\tau^2 + 2\tau) \| g_{u_k} \cdot g_{v_k} \|_{H_k}^2,
\]

and the result follows by the preceding bounds. \( \square \)

Proof of Proposition 9 By multiplying together the two equations in (35), we first interpret (35) as variable metric descent on \( x_k = u_k \cdot v_k \):

\[
x_{k+1} = x_k - \tau u_k \cdot g_{v_k} - \tau v_k \cdot g_{u_k} + \tau^2 g_{u_k} \cdot g_{v_k}.
\]

Then,

\[
\| x_{k+1} - \bar{x} \|_{H_k}^2 = \| x_k - \bar{x} - \tau u_k \cdot g_{v_k} - \tau v_k \cdot g_{u_k} + \tau^2 g_{u_k} \cdot g_{v_k} \|_{H_k}^2
= \| x_k - \bar{x} \|_{H_k}^2 + T_1 + T_2 + T_3
\]

where \( T_1 \triangleq \| \tau u_k \cdot g_{v_k} + \tau v_k \cdot g_{u_k} - \tau^2 g_{u_k} \cdot g_{v_k} \|_{H_k}^2 \)

\( T_2 \triangleq 2\tau^2 \langle g_{u_k} \cdot g_{v_k}, H_k(x_k - \bar{x}) \rangle \)

\( T_3 \triangleq -2\tau \langle u_k \cdot g_{v_k} + v_k \cdot g_{u_k}, H_k(x_k - \bar{x}) \rangle. \)

The theorem is simply a consequence of bounding \( T_1, T_2, T_3 \) using Lemma 3 and summing the inequality over \( k = 0, \ldots, T \). Let \( C_0, C_1 \) be as in Lemma 3. Indeed, by Lemma 3, \( T_1 \leq \tau^2 C_1 \| \nabla G(u_k, v_k) \|_2^2 \). To bound \( T_2 \), observe that

\[
\langle g_{u_k} \cdot g_{v_k}, H_k(x_k - \bar{x}) \rangle \leq \| x_k - \bar{x} \|_{H_k} \| g_{u_k} \cdot g_{v_k} \|_{H_k} = \| x_k - \bar{x} \|_{H_k} \sqrt{C_0} \| g_{v_k} \|
\]

So,

\[
T_2 \leq 2\tau^2 \sqrt{C_0} \| x_k - \bar{x} \|_{H_k} \| \nabla G(u_k, v_k) \|.
\]

\( \square \) Springer
Consider the final term $T_3$: observe that

$$u_k \cdot g_{u_k} + v_k \cdot g_{v_k} = 2x_k + (u_k^2 + v_k^2) \cdot \nabla F(x_k)$$

and $\|x_k/(u_k^2 + v_k^2)\|_\infty \leq 1/2$. It follows by convexity of $F$ that

$$\langle u_k \cdot g_{v_k} + v_k \cdot g_{u_k}, H_k(x_k - \bar{x}) \rangle$$

$$= \left\langle \frac{2x_k}{u_k^2 + v_k^2} x_k - \bar{x}, + \langle \nabla F(x_k), x_k - \bar{x} \rangle \right\rangle$$

$$\geq (2\|x_k\|_H^2 - \|\bar{x}\|_1) + (F(x_k) - F(\bar{x})) = \Phi_k - \Phi_*.$$ 

So,

$$T_3 \leq -2\tau(\Phi_k - \Phi_*).$$

It follows that

$$2\tau(\Phi_k - \Phi_*) \leq \left(\|x_k - \bar{x}\|_H^2 - \|x_{k+1} - \bar{x}\|_H^2\right)$$

$$+ \tau C_1 \|\nabla G(u_k, v_k)\|^2 + 2\tau^2 \sqrt{C_0}\|x_k - \bar{x}\|_H \|\nabla G(u_k, v_k)\|.$$ 

Finally,

$$\|x_k - \bar{x}\|_H^2 - \|x_{k+1} - \bar{x}\|_H^2 = \langle x_k - x_{k+1}, x_k - \bar{x} \rangle_H + \langle x_k - x_{k+1}, x_{k+1} - \bar{x} \rangle_H$$

$$\leq \tau \sqrt{C_1} (\|x_k - \bar{x}\|_H + \|x_{k+1} - \bar{x}\|_H) \|\nabla G(u_k, v_k)\|,$$

and

$$\|x_{k+1} - \bar{x}\|_H \leq \|x_{k+1} - x_k\|_H + \|x_k - \bar{x}\|_H \leq \tau \sqrt{C_1} \|\nabla G(u_k, v_k)\| + \|x_k - \bar{x}\|_H.$$ 

\[\square\]

### 5 Nonsmooth robust losses

In this section, we describe some generalizations of our method to cope with nonsmooth robust losses in Sect. 5.2 and non-convex regularization functionals in Sect. 5.3. These generalizations leverage so-called quadratic variational forms, recalled in Sect. 5.1, which generalizes the overparameterization formula (3) beyond the $\ell^1 - \ell^2$ norm.
5.1 Quadratic variational forms

It is well known that quadratic variational forms exist for many nonsmooth regularizers, including nuclear norm, $\ell_q$ and also other nonconvex regularizers [12, 35]. In general, for a function $R : \mathbb{R}^n \to \mathbb{R}$ (see [60] for a proof), one has the equivalence between:

i) $R(x) = \varphi(x \odot x)$ where $\varphi$ is proper, concave and upper semi-continuous, with domain $\mathbb{R}^d_+$.

ii) There exists a convex function $\psi$ for which $R(x) = \inf_{z \in \mathbb{R}^n_+} \frac{1}{2} \sum_{i=1}^{n} z_i x_i^2 + \psi(z)$.

Furthermore, $\psi(z) = (-\varphi)^*(-z/2)$ is defined via the convex conjugate $(-\varphi)^*$ of $-\varphi$.

One particularly interesting class of functions which fit into the quadratic variational framework are (group) $\ell_q$ semi-norms for $q \in (0, 2)$.

Lemma 4 Let $\beta > 0$ and $q = 2\beta/(1 + \beta)$. Then, given a group decomposition $G = \{g_1, g_2, \ldots, g_N\}$, we have

$$\frac{1}{q} \sum_{g \in G} \|x_g\|^q = \min_{\eta \in \mathbb{R}_{d_1}^{|G|}} \frac{1}{2} \sum_{i=1}^{N} \frac{\|x_{g_i}\|^2}{\eta_i} + \frac{1}{2\beta} \sum_{i=1}^{N} \eta_i^\beta = \min_{x \in \mathbb{R} \otimes v} \frac{1}{2} \|u\|^2 + \frac{1}{2\beta} \sum_{i} |v_i|^{2\beta}. \tag{46}$$

In the remaining part of this section, we discuss two extensions of our VarPro approach: the first is where both the loss function and regularizer have quadratic variational forms, and the second is the use of non-convex functionals.

Remark 7 Lemma 4 is well known and is widely used in iterative reweighted algorithms [23, 24, 28]. However, for the purpose of the VarPro techniques in this paper, this result is mainly interesting when $q > 2/3$ so that $2\beta > 1$ as this ensures that the projected function $f$ is differentiable in $v$. See Sect. 5.3 for further details and a discussion on how to move below $q = 2/3$ using the VarPro approach.

5.2 Nonsmooth loss functions

In this section, we show how the VarPro technique can be applied to handle the case where the loss function is nonsmooth but has a quadratic variational form. Such losses are commonly considered in image processing problems, and some concrete examples can be found in Sects. 5.2.1 and 5.2.2.

Consider for $y \in \mathbb{R}^m$, $L \in \mathbb{R}^{p \times n}$ and $A \in \mathbb{R}^{m \times n}$,

$$\min_{x \in \mathbb{R}^n} \Phi(x) = R_1(Lx) + \frac{1}{\lambda} R_2(Ax - y)$$

where the $R_i$ functionals (for $i = 1, 2$) both have quadratic variational forms

$$R_i(z) = \min_{\eta \in \mathbb{R}^{d_i}_+} \frac{1}{2} \sum_{k=1}^{d_i} \frac{\|z_{g_{i,k}}\|^2}{\eta_k} + \sum_{k=1}^{d_i} h_i(\eta_k), \tag{46}$$

$\square$ Springer
where we assume that $h_1$ and $h_2$ are both differentiable functions, we have the partitions $\bigcup_{g \in \mathcal{G}_1} g = \bigcup_k g_{1,k} = \{1, \ldots, p\}$, $\bigcup_{g \in \mathcal{G}_2} g = \bigcup_k g_{2,k} = \{1, \ldots, m\}$ and $d_1 = |\mathcal{G}_1|$, $d_2 = |\mathcal{G}_2|$. The following proposition shows how $\Phi$ can be written as a differentiable function, whose gradient can be computed by solving a linear system.

**Proposition 10**  We have

$$
\min_x \Phi(x) = \min_{v \in \mathbb{R}^{d_1}, w \in \mathbb{R}^{d_2}} f(v, w) \text{ where } f(v, w) \triangleq h_1(v) + \frac{1}{\lambda} h_2(w) + \varphi(v, w),
$$

with

$$
\varphi(v, w) = \max_{\alpha \in \mathbb{R}^p, \xi \in \mathbb{R}^m} \left\{ -\frac{1}{2} \|v \odot \alpha\|^2 - \frac{\lambda}{2} \|w \odot \xi\|^2 + \langle \xi, y \rangle : L^\top \alpha + A^\top \xi = 0 \right\}.
$$

The optimal solutions satisfy

$$
Lx = -v^2 \alpha \text{ and } Ax = y - \lambda w^2 \xi.
$$

Moreover, the maximizer $\alpha, \xi$ to the inner problem $\varphi$ satisfy for some $x \in \mathbb{R}^n$

$$
\begin{pmatrix}
\text{diag}(\bar{v}^2) & 0 & L \\
0 & \lambda \text{ diag}(\bar{w}^2) & A \\
L^\top & A^\top & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\xi \\
x
\end{pmatrix} =
\begin{pmatrix}
y \\
0 \\
0
\end{pmatrix},
$$

(47)

where as before, $\bar{v}$ and $\bar{w}$ are the extensions of $v$ and $w$ so that $\bar{v} \cdot \alpha = \alpha \odot v$ and $\bar{w} \cdot \xi = \xi \odot w$.

**Proof** We can write $\min_{x \in \mathbb{R}^n} \Phi(x)$ as

$$
\min_{x, u, v, z, w} \left\{ \frac{1}{2} \|u\|^2 + h_1(v) + \frac{1}{2\lambda} \|z\|^2 + \frac{1}{\lambda} h_2(w) : u \odot v = Lx, z \odot w = Ax - y \right\}
$$

where the minimization is over the variables $x \in \mathbb{R}^m$, $v \in \mathbb{R}^{d_1}$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^{d_2}$ and $z \in \mathbb{R}^m$. This is a convex problem over the variables $u, z, x$, so by convex duality,

$$
\min_{v, w} \min_{x, u, v, z, w} \frac{1}{2} \|u\|^2 + h_1(v) + \frac{\|z\|^2}{2\lambda} + \frac{h_2(w)}{\lambda} + \langle \alpha, u \odot v - Lx \rangle + \langle \xi, z \odot w - Ax + y \rangle
$$

$$
= \min_{v, w} \max_{\alpha, \xi} \left\{ -\frac{\|\alpha \odot v\|^2}{2} + h_1(v) - \frac{\lambda}{2} \|\xi \odot w\|^2 + \frac{h_2(w)}{\lambda} + \langle \xi, y \rangle : L^\top \alpha + A^\top \xi = 0 \right\}
$$

where the dual variable are $\alpha \in \mathbb{R}^p$, $\xi \in \mathbb{R}^m$ and where the optimal solutions satisfy $u + \alpha \odot v = 0$, $z + \lambda \xi \odot w = 0$. 

\( \square \)
Remark 8  Note that if $A = \text{Id}$, then we can write (47) as

\[
x = y - \lambda w^2 \odot \xi \quad \text{and} \quad \xi = -L^\top \alpha
\]

and \[(\text{diag}(v^2) + \lambda L \text{diag}(w^2)L^\top)\alpha = -Ly\]

**Differentiability of $f$**  Formally, the gradient of $f$ is given as

\[
\partial_v f(v, w) = \nabla h_1(v) - \lambda \lambda \odot v \quad \text{and} \quad \partial_w f(v, w) = 1/\lambda \nabla h_2(w) - \lambda \xi \odot w
\]

where $(\alpha, \xi) \in \text{argmax} \varphi(v, w)$ solve the inner problem given in Proposition 10. Note that these formulas are well-defined for $v$ and $w$ such that the maximizers to the inner problem exist: in this case, $\alpha \odot v$ and $\xi \odot w$ are unique thanks to the quadratic terms $\|\alpha \odot v\|^2$ and $\|\xi \odot w\|^2$ in $\varphi$. It is straightforward to conclude using Theorem 3 that $f$ is differentiable whenever $v$ and $w$ have all nonzero entries. To see differentiability in general, one should assume that there exists $x_0, u_0, z_0$ such that

\[
Ax_0 - y = z_0 \odot w \quad \text{and} \quad u \odot v = Lx_0.
\]

Indeed, to establish the existence of maximizers, following (2.5), one first observes that $(0, 0)$ is a feasible point to the inner maximization problem in $\varphi(v, w)$, so $\varphi(v, w) \succeq 0$ and one can restrict the maximization problem to $\alpha, \xi$ such that $L^\top \alpha + A^\top \xi = 0$ and

\[
\langle \xi, y \rangle \succeq 1/2 \|\alpha \odot v\|^2 + \lambda \|\xi \odot w\|^2.
\]

From this, in general, it is not clear that one can extract uniformly bounded (and hence convergent up to a subsequence) maximising sequences $\alpha_n$ and $\xi_n$. However, if (49) holds, then

\[
\|v \odot \alpha\|_{u_0} + \|w \odot \xi\|_{z_0} \succeq \langle \xi, Ax_0 - z_0 \odot w \rangle \succeq 1/2 \|\alpha \odot v\|^2 + \lambda \|\xi \odot w\|^2
\]

from which it is clear that $\|\alpha \odot v\|$ and $\|\xi \odot w\|$ are uniformly bounded. One can then proceed as in Sect. 2.5 to extract maximising sequences to deduce the existence of maximizers and hence differentiability of $f$.

We now exemplify this general formulation on two illustrative scenarios: TV-L1 and the square root Lasso. Discussions on the use of this formulation for matrix recovery problems can be found in Appendix D.

### 5.2.1 TV-L1

We consider the case where $R_1$ and $R_2$ are both $\ell_1$-norms, so that $h_1 = h_2 = 1/2 \| \cdot \|^2$. The use of the $\ell^1$ norm as a loss function is popular to cope with outliers and impulse noise, and a typical example is when $L$ is a finite difference approximation of the gradient operator (as defined in Sect. 2.3.4), corresponding to the so-called TV-L1
Fig. 7 Comparison of primal-dual and VarPro for TV-L₁ denoising with salt and pepper noise. The top row corresponds to the image “hestain” of size 227 x 303 and the bottom row corresponds to the image “peppers” of size 384 x 512. The figures show the objective error against computational time.

Given solutions \((\alpha, \xi, x)\) to the linear system (47) (note that \(\alpha\) and \(\xi\) are unique defined on the support of \(v\) and \(w\) respectively), the gradient of \(f\) is

\[ \partial_v f = v - \alpha^2 \odot v \quad \text{and} \quad \partial_w f = w - \xi^2 \odot w. \]

In Fig. 7, we show the results of denoising where \(A = \text{Id}, L\) and \(R_1\) correspond to group-TV as described in Sect. 2.3.4 and \(R_2\) is a group-\(\ell_1\) norm where each group is of size 3 corresponding to the 3 color channels. For each image, we corrupt 25% of the pixels with salt and pepper noise and show the objective convergence error for different regularization strengths \(\lambda\). We compare against primal-dual (whose implementation is as described in the Appendix) The outputs of VarPro are displayed in Fig. 8.

5.2.2 The square root Lasso

We now consider the case of \(L = \text{Id}, A \in \mathbb{R}^{m \times n}\) and \(y \in \mathbb{R}^m\), and \(R_i\) are the group-\(\ell_1\) norms. This corresponds to the square root lasso

\[ \min_x \|x\|_1 + \frac{1}{\lambda \sqrt{m}} \|Ax - y\|_2. \] (50)

This optimization problem is equivalent to the original Lasso problem but this equivalence requires to tune the multipliers \(\lambda\) involved in both problems, which depends on the input data \(y\) and the design matrix \(A\) [11]. An advantage of the square root Lasso
formulation is that, thanks to the 1-homogeneity of the functionals, the parameter $\lambda$ requires less tuning and is approximately invariant under modification of the noise level and number of observations. Another setting used in multitask learning is where the Loss function is the nuclear norm [70] (see Appendix D for remarks on the use of quadratic variational forms for this setting). The equivalent VarPro formulation is an optimization problem over $n + 1$ variables

$$
\min_{w \in \mathbb{R}, v \in \mathbb{R}^n} f(v, w),
$$

where $f(v, w) = \frac{1}{2} \|v\|^2 + \frac{1}{2\sqrt{m}} w^2 + \max_\xi - \frac{1}{2} \|v \odot (A^T \xi)\|^2 - \frac{\lambda}{\sqrt{m}} w^2 \|\xi\|^2 + \langle y, \xi \rangle$ with $x = (A^T \xi) \odot v^2$.

We consider two examples: (i) $A$ is a random Gaussian matrix with $m = 300$ and $n = 2000$ and $y = Ax_0 + w$ where $x_0$ is $s = 40$-sparse and the entries of $w$ are iid Gaussian with variance 0.01. (ii) $A$ is the MNIST dataset, 3 with $m = 60,000$ and $n = 683$. Note that the first order optimality condition of (50) is

$$
0 \in \partial \|x\|_1 + \frac{1}{\lambda \sqrt{m}} A^T \partial \|Ax - y\|_2 = 0
$$

and 0 is a solution if $\lambda \|y\| \sqrt{m} \geq \|A^T y\|_\infty$. We therefore define $\lambda_{\text{max}} \triangleq \frac{1}{\|y\| \sqrt{m}} \|A^T y\|_\infty$ and consider $\lambda = \frac{1}{p} \lambda_{\text{max}}$ for different $p > 1$.

Two popular approaches to solve this optimization problem in the literature are alternating minimization [38] and coordinate descent [11, 52]. The alternating approach iterates between the following steps [38]

$$
\eta_k \triangleq \|Ax_k - y\| \quad \text{and} \quad x_{k+1} \in \arg\min_{x} \lambda \|x\|_1 + \frac{1}{2\eta_k} \|Ax - y\|^2.
$$

3 https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/.
This is referred to as the scaled Lasso algorithm and requires solving the Lasso at each step. In Fig. 9, we compare against these two approaches—the scaled Lasso algorithm uses Varpro for the Lasso as the inner solver and the coordinate descent code can be downloaded online [11].\footnote{https://faculty.fuqua.duke.edu/~abn5/belloni-software.html.} One can observe that for coordinate descent, although it is very effective for large regularization parameters, its performance deteriorates for small $\lambda$. In contrast, our proposed method is robust to different regularization strengths.

5.3 Nonconvex regularizations

Let us consider the case where the regularizer is a group $\ell_q$ semi-norm, denote $\|x\|_{q,2} \triangleq \left(\sum_{g \in G} \|x_g\|^q\right)^{1/q}$ for $q \in (0, 1)$. Recall that in Lemma 4, if $q > 2/3$, then $2\beta > 1$, so when combined with smooth loss functions, the VarPro method corresponds to a differentiable optimization problem since the $\|v_i\|_{2\beta}^2$ term is differentiable (see [60]). To go below $p = 2/3$ and induce a stronger sparsity regularization while maintaining differentiability, we propose to introduce more over-parameterization. We expose the setting of a three-factors parameterization, but this is easily generalizable to more factors to further reduce the value of $q$. By doing so, this also raises the question of exploring multiple-levels optimizations (beyond simply bilevel programming).

In the following proposition, we see that if VarPro is applied with $w$ as the outermost variable, then $\|w\|_{2\beta}^2$ is differentiable when $2\beta > 1$ which would correspond to $q > 1/2$ (as opposed to $q > 2/3$) as before. Note also that even in the setup of Lemma 4, although VarPro can be applied to obtain a differentiable problem when $q > 2/3$, in practice, since the Lipschitz constant of the gradient deteriorates as $q$ approaches...
the optimization behaviour tends to deteriorate for values of \( q \) close to \( \frac{2}{3} \). So, the following formulation can be desirable not only because it allows for the use of \( q < \frac{2}{3} \), but also for improving the Lipschitz constants.

**Proposition 11** Let \( G = \{ g_1, g_2, \ldots, g_N \} \) and let \( q, \beta > 0 \) be such that \( \beta = q \left( \frac{2}{2-q} \right) \).

Then,

\[
\frac{1}{q} \sum_{i=1}^{N} |x_{g_i}|^q = \min \left\{ \frac{1}{2} \|u\|_2^2 + \frac{1}{2} \|v\|_2^2 + \frac{1}{2\beta} \sum_{i=1}^{N} |w_i|^{2\beta} : x = u \odot (v \cdot w) \right\},
\]

(51)

where the minimization is over \( u \in \mathbb{R}^p \) and \( v, w \in \mathbb{R}^N \).

**Proof** By applying Lemma 4 twice,

\[
\min_{x = u \odot (v \cdot w)} \frac{1}{2} \|u\|_2^2 + \frac{1}{2} \|v\|_2^2 + \frac{1}{2\beta} \sum_{i=1}^{N} |w_i|^{2\beta}
\]

\[
= \min_{x = u \odot (v \cdot w)} \frac{1}{2} \|u\|_2^2 + \min_{z = v \cdot w} \left( \frac{1}{2} \|v\|_2^2 + \frac{1}{2\beta} \sum_{i=1}^{N} |w_i|^{2\beta} \right)
\]

\[
= \min_{x = u \odot (v \cdot w)} \frac{1}{2} \|u\|_2^2 + \frac{1}{2r} \sum_{i=1}^{N} |z_i|^{2r} \text{ where } r = \beta/(1 + \beta)
\]

\[
= \frac{1}{q} \sum_{i=1}^{N} \|x_{g_i}\|^q \text{ where } q = \frac{2r}{1 + r} = \frac{2\beta}{1 + 2\beta},
\]

and equivalently, \( \beta = \frac{q}{(2-2q)} \).

We now consider the case \( q = \frac{2}{3} \), so that given a convex loss function \( F \), we re-write

\[
\min_{x} \frac{3}{2} \|x\|_2^{2/3} + F_0(Ax)
\]

equivalently as

\[
\min_{u, v, w} \frac{1}{2} \left( \|u\|_2^2 + \|v\|_2^2 + \|w\|_2^2 \right) + F_0(A(u \odot (v \cdot w))).
\]

(52)

There are many approaches for handling this optimization problem. Here, we discuss three ways: as an optimization over 3 variables; formulate a VarPro bilevel problem with two variables \((v, w)\) on the outer problem; formulate as a bilevel problem with one variable \( v \) on the outer problem. We first discuss differentiability issues of each of these three cases.

**Option 1: Optimization over 3 variables** One can directly optimize (52) over the three variables \( u, v, w \) and this is differentiable when \( F_0 \) is differentiable.
**Option 2: Two variables on outer problem** One rewrite (52) as \( \min_{v,w} f(v, w) \) where

\[
f(v, w) = \min_u \frac{1}{2} \left( \|u\|^2 + \|v\|^2 + \|w\|^2 \right) + F_0(A(u \odot (v \cdot w)))
= \frac{\|v\|^2 + \|w\|^2}{2} + \max_\alpha - \frac{1}{2} \left\| (A^\top \alpha) \odot (v \cdot w) \right\|^2 - F_0^*(\alpha),
\]

where we observe that the minimization problem over \( u \) is convex and the second line is the result of convex duality. Assuming that \( F_0 \) is convex, the inner problem over \( u \) is strongly convex and \( f \) is differentiable whenever \( F_0 \) is differentiable. Note however that even when \( F_0 \) is not differentiable, the solution to the inner problem is unique and one can write

\[
\partial_v f = v - v \cdot \left( w_i \|A_{gi}^\top \alpha\|^2 \right)_{i=1}^N \quad \text{and} \quad \partial_w f = w - w \cdot \left( v_i \|A_{gi}^\top \alpha\|^2 \right)_{i=1}^N,
\]

where \( \alpha \) is a dual solution to the inner problem. These formulas are well defined since \((A^\top \alpha) \odot (v \cdot w)\) is unique.

**Option 3: One variable on the outer problem** One can consider \( \min_v f(v) \) where

\[
f(v) = \min_{u,w} \frac{1}{2} \left( \|u\|^2 + \|v\|^2 + \|w\|^2 \right) + F_0(A(u \odot (v \cdot w)))
= \min \frac{1}{2} \|u\|^2 + \|z\|_{1,2} + F_0(A(z \odot v))
= \min \max_\alpha \frac{1}{2} \|u\|^2 + \|z\|_{1,2} + \langle \alpha, A(z \odot v) \rangle - F_0^*(\alpha).
\]

Note that the inner minimization problem is convex and provided that the inner problem has a unique solution \( z \) and \( F_0 \) is differentiable, the function \( f \) is differentiable with

\[
\nabla f(v) = v + (z_{gi}, A_{gi}^\top \alpha)_{i=1}^N
\]

Indeed, since \((A^\top \alpha) \odot v \in \partial \|z\|_{1,2}, (A^\top \alpha) \odot v\) is unique on the group support of \( z \) and since the group support of \( z \) is contained in the support of \( v \), \( z \odot A^\top \alpha \) is unique. One condition to ensure that \( z \) is unique is if \( F_0 \) is strongly convex and \( A \) is injective.

**Remarks on conditioning** This analysis raises the question as to which option to favor among the three. It is clear that Option 3 is computationally the most expensive, since one needs to solve an \( \ell_1 \) minimization problem to compute the gradient, while the gradient in Option 2 can be computed in closed form by inverting a linear system. For Option 3, the resolution of this inner \( \ell_1 \) problem can leverage any existing solvers, and it is possible to re-use another VarPro method, which corresponds to doing a three-level programming. We explore this option in the numerical examples below.
However, in terms of conditioning of the Hessian, Option 3 is the most desirable as we now explain.

Given \((u, v, w) \mapsto f(u, v, w)\) and denoting its Hessian by \(H\), the Hessian of \(u \mapsto \min_{v,w} f(u, v, w)\) is the Schur complement of \(H\) with respect to \(\partial^2_{vw} f\) and the Hessian of \((u, v) \mapsto \min_w f(u, v, w)\) is the Schur complement of \(H\) with respect to \(\nabla^2_w f(u, v, w)\), assuming that these Hessians exist. In addition to the interlacing property of eigenvalues mentioned in section (16), we also have the following interlacing property [32] for Schur complements: if \(H\) is symmetric semi-definite, \(\alpha' \subseteq \alpha \subseteq [n]\) and \(H[\alpha]\) is the submatrix indexed by \(\alpha\),

\[
\lambda_i \left( \frac{H}{H[\alpha']} \right) \leq \lambda_i \left( \frac{H}{H[\alpha]} \right) \leq \lambda_i + |\alpha| - |\alpha'| \left( \frac{H}{H[\alpha']} \right)
\]

so the condition number of \(\frac{H}{H[\alpha]}\) is no worse than the condition number of \(\frac{H}{H[\alpha']}\). Putting aside the difficulty that \(v \mapsto \min_u f(u, v, w)\) may not be differentiable, these interlacing properties of Schur complements suggest that this formulation is better conditioned than the alternative formulation of \((v, w) \mapsto \min_u f(u, v, w)\).

**Numerical illustrations**

The methods that we compare to are:

- **VarPro (1 inside).** This is minimising the function \(f\) defined in (53) where the inner problem is the solution to a linear system.
- **VarPro (2 inside).** This is minimising the function \(f\) defined in (54) where the inner problem is the solution to an \(\ell_1\) problem. For both versions of VarPro, We initialise with vectors whose entries are taking iid uniformly at random in \([0, 1]\).
- **Iterative reweighted least squares** (we followed the implementation as described in [23]). IRLS is initialised with the least squares solution \(A^\dagger y\).
- **Reweighted \(\ell_1\)** [24]. This is initialized with the solution to the Lasso (\(\ell_1\)-regularized problem).

Since the \(\ell^q\) regularizer we consider is non-convex, we analyze the performances of these algorithms according both to its ability to select “good” minimizers and its speed of convergence.

In order to assess the quality of the computed solutions, we consider a noiseless recovery problem from observations \(Y = Ax^*\) where \(x^*\) has \(s\) nonzero rows, and check whether the considered algorithms are able to recover \(x^*\) as a function of \(s\). We consider, for \(A \in \mathbb{R}^{m \times n}\), \(Y \in \mathbb{R}^{m \times T}\) and \(q = \frac{2}{3}\), the constrained problem

\[
\min_{x \in \mathbb{R}^{n \times T}} \sum_{i=1}^n \|x^{(i)}\|_2^q \quad \text{s.t.} \quad Ax = Y.
\]

Here, \(x^{(i)}\) denotes the \(i\)th row of \(x\). We consider the case of \(n = 256\), \(s = 40\) and \(T \in \{1, 50, 100\}\) and several values of \(m\). For each \(T\), we generate 100 random instances of \(A \in \mathbb{R}^{m \times n}\), \(x^* \in \mathbb{R}^{s \times T}\) which is \(s\)-row sparse and let \(y = Ax^*\). Then, for each value of \(m\), we take the first \(m\) rows of \(A\) and first \(m\) entries of \(y\), run our methods for this data, then count the number of times for which one has successful recovery (here, successful means that the relative error in 2-norm is less than 0.01). The results for \(T = 1, 50, 100\) are shown in Fig. 10. In terms of recovering the phase transition,
the observation is that when $T$ is large, VarPro obtains on-par or slightly better than IRLS and reweighted-$\ell_1$. One of the advantages of VarPro with only a linear solve on the inner problem is that it is substantially faster than the other methods (see the next paragraph). When $T = 1$, the performance is more varied, with IRLS performs the best, and Varpro with $\ell_1$ on the inside and reweighted $\ell_1$ both out perform Varpro with a linear solver on the inside. It should be noted that since IRLS is gradually decreasing a regularization parameter, it is a form of graduated non-convexity method, and this is likely to lead to better local minimums than directly solving the $\ell_{2/3}$ optimization problem. In practice, we find that VarPro performs more favorably for group problems $T > 1$, although we do not have a clear explanation for this.

To illustrate the time-performance of the different algorithms, we consider

$$\min_{x \in \mathbb{R}^{n \times T}} \sum_{i=1}^{n} \|x^{(i)}\|_{2}^{q} + \frac{1}{2\lambda} \|Ax - Y\|_{2}^{2}$$

for $\lambda = 0.1$, and where $A \in \mathbb{R}^{m \times n}$ is a random Gaussian matrix with $n = 256$ and $x_{\ast}$ is row-sparse with 40 nonzero rows. For $T = 100$ and $T = 50$, we set $m = 45$ and for $T = 1$, we set $m = 85$. In each case, we generate 20 random instances, and given each problem instance, we apply each method and record the running time against the objective error (this is the error to the best objective value found by the 4 methods). The results are show in Fig. 11. Note that the objective value error is calculated as $\Phi(x_{k}) - \Phi_{\ast}$ where $\Phi_{\ast}$ is the best objective value found across all 4 methods.

5.4 Robust constraints

Let $q \geq 1$ and consider

$$\min_{x} \|x\|_{1} \text{ such that } \|Ax - y\|_{q} \leq \varepsilon,$$
By convex duality, this can be written as

$$\min_{x,z} \max_{\alpha} \|x\|_1 + \|z\|_q \leq \varepsilon(z) + \langle \alpha, Ax - y - z \rangle$$

$$= \min_{x} \max_{\alpha} \|x\|_1 + \langle \alpha, Ax - y \rangle - \varepsilon\|\alpha\|_{q^*}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$. By making use of the quadratic variational form for $\| \cdot \|_1$, this is equivalent to $\min_v f(v)$ where

$$f(v) = \frac{1}{2} \|v\|_2^2 - \min_{\alpha} \left\{ \frac{1}{2} \|(A^\top \alpha) \cdot v\|_2^2 + \langle \alpha, y \rangle + \varepsilon\|\alpha\|_{q^*} \right\}. \quad (55)$$

The optimal variables are related by $x = -(A^\top \alpha) \cdot v^2$.

In contrast to the Lasso, the inner optimization problem over $\alpha$ does not have a closed form solution. To apply gradient based approaches to minimize $f$, we need to compute

$$\nabla f(v) = v - (A^\top \alpha) \cdot v$$

where $\alpha$ solves (55). Note two particularly interesting cases in practice are $q = \infty$ and $q = 2$, in which case, we have $q^* = 1$ and $q^* = 2$ respectively. For both these cases, the inner problem can either be solved via IRLS, or we can place this into a VarPro form as described in this paper. The case of $q = q^* = 2$ is easier to handle as we describe next.

### 5.4.1 Handling the subproblem (55)

We make some comments about the VarPro formulation of (55) in the case of $q = q^* = 2$. Consider

$$\min_{\alpha \in \mathbb{R}^m} \|\alpha\| + \frac{1}{2} \langle B\alpha, \alpha \rangle + \langle \alpha, y \rangle,$$
where $B \in \mathbb{R}^{m \times m}$ and $y \in \mathbb{R}^m$. By making use of the quadratic variational form for $\|\alpha\|$, this problem is equivalent to solving

$$\min_{t \in \mathbb{R}} h(t), \quad \text{where} \quad h(t) = \frac{1}{2} \|t\|^2 + \min_{w \in \mathbb{R}^m} \left( \frac{1}{2} \|w\|^2 + \frac{1}{2} t^2 \langle Bw, w \rangle + t \langle w, y \rangle \right),$$

and letting $\alpha \triangleq tw$. Note that $h$ is differentiable with $\nabla h(t) = t + \langle w, y \rangle + t \langle Bw, w \rangle$ and $w$ solves the linear system

$$\left( \frac{1}{t^2} \text{Id} + B \right) w = -\frac{1}{t} y. \quad (56)$$

This linear system can be solved efficiently at each iteration if the SVD decomposition of $B = U \Sigma U^\top$ is available (note that $U$ is unitary and $\Sigma$ is a diagonal matrix): by applying $U^\top$ to both sides of $(56)$, one obtains

$$\left( \frac{1}{t^2} + \Sigma \right) U^\top w = -\frac{1}{t} U^\top y.$$

Setting $z \triangleq U^\top w$ thus leads to $\nabla h(t) = t + \langle z, U^\top y \rangle + t \langle \Sigma z, z \rangle$, which can be computed in linear time.

**Conclusion**

We have presented a generic and versatile class of optimization methods, which can cope with a wide range of non-smooth losses and regularization functionals. An appealing feature of these approaches is that they rely on smooth optimization techniques and can thus leverage standard efficient solvers such as quasi-Newton. On the theoretical side, we highlighted that handling generalized sparse regularizers such as total variation is more intricate than the Lasso case, and in particular differentiability requires a greater care. We also draw connections with mirror-descent methods, which leads to constants being independent of the grid-size. Unfortunately, although we were able to partly lift difficulties due to non-convexity, a full convergence analysis is still beyond reach with our proof techniques.

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**Declarations**

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A Envelope theorem

In this section, we recall some useful definitions and results from [63].

**Definition 1** We say that $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is calm at $\bar{x}$ if there exists $\kappa > 0$ and a neighborhood $\mathcal{N}_{\bar{x}}$ around $x$ such that

$$|f(x) - f(\bar{x})| \leq \kappa \|x - \bar{x}\|, \quad \forall x \in \mathcal{N}_{\bar{x}}.$$

We say that $f$ is strictly continuous at $\bar{x}$ if there exists $\kappa > 0$ and a neighborhood $\mathcal{N}_{\bar{x}}$ around $\bar{x}$ such that

$$|f(x) - f(x')| \leq \kappa \|x - x'\|, \quad \forall x, x' \in \mathcal{N}_{\bar{x}}.$$

We say that $f$ is strictly differentiable at $\bar{x}$ if $f(\bar{x})$ is finite and there is a vector $v$, which is the gradient $\nabla f(\bar{x})$, such that

$$\lim_{x, x' \to x} \frac{f(x') - f(x) - \langle v, x' - x \rangle}{\|x - x'\|} = 0, \quad \text{with} \quad x' \neq x.$$

**Remark 9** We follow the terminology of [63]. Note that strict continuity of a function is also known as being locally Lipschitz continuous, while being calm at $\bar{x}$ is a slightly weaker condition since it looks at $\bar{x}$ and $x \in \mathcal{N}_{\bar{x}}$, rather than all pairs in $\mathcal{N}_{\bar{x}}$. Note that strict differentiability is stronger that simply differentiability and if there is an open set $\mathcal{O}$ on which $f$ is finite, then $f$ is strictly differentiable on $\mathcal{O}$ is equivalent to $f$ is $C^1$ on $\mathcal{O}$ [63, Cor 9.19].

**Definition 2** Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. A function $f : \mathcal{O} \to \mathbb{R}$ is said to be lower-$C^1$ on $\mathcal{O}$, if for all $x \in \mathcal{O}$, there exists a neighborhood $\mathcal{N}$ of $x$ with the representation

$$f(x) = \max_{t \in T} \varphi(t, x)$$

in which the functions $\varphi(t, \cdot)$ are $C^1$, the set $T$ is compact, and $\varphi(t, x)$ and $\nabla_x \varphi(t, x)$ depend continuously on $(t, x) \in T \times V$.

**Theorem 3** [63, Theorem 10.31] Suppose $f : \mathcal{O} \to \mathbb{R}$ is lower-$C^1$ on an open set $\mathcal{O}$, then $f$ is strictly differentiable on a set $D$ with $\mathcal{O} \setminus D$ negligible. Moreover, $f$ is differentiable at $x$ if $\{\nabla f(t, x) : t \in T(x)\}$ is single-valued.

**Proof** This is a direct consequence of [63, Thm 10.31] where it is shown that if $f$ is a lower-$C^1$ function, then:

- It is strictly continuous and regular on $\mathcal{O}$ and is semidifferentiable.
- $f$ is strictly differentiable on a set $D$ with $\mathcal{O} \setminus D$ negligible.
- $\partial f(x) = \text{con} \{\nabla f_t(x) : t \in T(x)\}$

Finally, by [63, Theorem 9.18], strict differentiability at $x$ is equivalent to $f$ strictly continuous and regular with $\partial f(x)$ single-valued. □

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We recall the notion of semiderivatives (note that this is different from directional derivatives, since we take the limit along all \( w' \) converging to \( w \)).

**Definition 3** [63, Def 7.20] Let \( f : \mathbb{R}^n \to \bar{\mathbb{R}} \) and suppose that \( \bar{x} \in \text{dom}(f) \). If

\[
\lim_{\tau \to 0, \quad w' \to w} \left( \frac{f(\bar{x} + \tau w') - f(\bar{x})}{\tau} \right)
\]

exists, it is the semiderivative of \( f \) at \( \bar{x} \) for \( w \), and \( f \) is semidifferentiable at \( \bar{x} \) for \( w \). If this holds for every \( w \), \( f \) is semidifferentiable at \( \bar{x} \).

In [63, Cor 7.22], we have the following result:

**Proposition 12** A function \( f : \mathbb{R}^n \to \bar{\mathbb{R}} \) is differentiable at \( \bar{x} \), a point with \( f(\bar{x}) \) finite, if and only if \( f \) is semidifferentiable at \( \bar{x} \) and the semiderivative for \( w \) depends linearly on \( w \).

In certain cases, we can show that the inner problem of our VarPro function can be restricted to a compact set and directly apply Theorem 3 to deduce differentiability. However, in cases where this is not possible, we will directly compute the semiderivative and apply Proposition 12.

**B Standard gradient descent results**

**Lemma 5** (Gradient bound) Suppose that \( f \in C^{1,1} \) and \( \nabla f \) being \( L_2 \)-Lipschitz with respect to the Euclidean norm. Let

\[
w^{k+1} = w^k - \frac{1}{L_2} \nabla f(w^k).
\]

For all \( T \in \mathbb{N} \) and \( j \in [1, 2] \), then for \( C \triangleq 2(f(w^0) - f(w^*))L_2 \),

\[
\sum_{k \leq T} \| \nabla f(x^k) \|^2 \leq C \quad \text{and} \quad \min_{k \leq T} \| \nabla f(x^k) \|^2 \leq \frac{C}{T}.
\]

**Proof of Lemma 5** By Taylor expansion, for all \( v, w \),

\[
f(v) \leq f(w) + \nabla f(w)^\top (v - w) + \frac{L_2}{2} \| v - w \|^2.
\]

So,

\[
f(w^{k+1}) \leq f(w^k) - \frac{1}{2L_2} \| \nabla f(w^k) \|^2, \quad (57)
\]

Note also that

\[
\| \nabla f(w^k) \|^j \leq \left( (2L_2)(f(w^k) - f(w^{k+1})) \right)^{j/2}
\]
So, for $j \in [1, 2],$
\[
\sum_{k \leq T} \| \nabla f(w^k) \|_j \leq (2L_2)^{j/2} \sum_{k \leq T} \left( f(w^k) - f(w^{k+1}) \right)^{j/2} \\
\leq (2L_2)^{j/2} \left( \sum_{k \leq T} \left( f(w^k) - f(w^{k+1}) \right) \right)^{T(2-j)/2} \\
\leq (2L_2)^{j/2} \left( f(w^0) - f(w^*) \right)^{j/2} T^{(2-j)/2}
\]
and
\[
\min_{k \leq T} \| \nabla f(w^k) \|^2 \leq \frac{2L_2}{T} \left( f(w^0) - f(w^*) \right).
\]

\[
\square
\]

C ADMM and primal-dual

In the following, we compare against ADMM [14] and primal-dual splitting [22].

**ADMM** ADMM [14] seeks to minimize for $\tau > 0,$
\[
\min \max_{\psi} \frac{1}{2\lambda} \left\| Ax - y \right\|^2_2 + \left\| z \right\|_1 + \langle \psi, z - Lx \rangle + \frac{\tau}{2} \left\| z - Lx \right\|^2_2
\]
by the iterations
\[
\begin{align*}
x_{k+1} &= \arg\min_x \frac{1}{2\lambda} \left\| Ax - y \right\|^2_2 - \langle \psi_k, Lx \rangle + \frac{\tau}{2} \left\| z_k - Lx \right\|^2_2 \\
z_{k+1} &= \arg\min_z \left\| z \right\|_1 + \langle \psi, z \rangle + \frac{\tau}{2} \left\| z - Lx \right\|^2_2 \\
\psi_{k+1} &= \psi_k + \tau (z_k - Lx_k)
\end{align*}
\]
The update on $x_k$ requires solving the linear system
\[
(A^T A + \lambda \tau L^T L)x = A^T y + \lambda L^T \psi_k + \lambda \tau L^T z_k
\]
For this step, we carry out a reordering of the columns and a cholesky factorization (to improve the sparsity of the factorization) which is reused throughout the iterations when carrying out the matrix inversion.

**Primal-dual** primal-dual splitting [22] solves
\[
\min \sup_{x, z} (Kx, z) - F^*(z) + G(x)
\]
The iterations are, for $\sigma, \tau > 0$ such that $\sigma \tau \leq 1/\|K\|^2$,

$$
\begin{align*}
\bar{z}^{n+1} &= \text{Prox}_{\sigma F^*}(z^n + \sigma \bar{K} \bar{x}^n) \\
x^{n+1} &= \text{Prox}_{\tau G}(x^n - \tau K^* \bar{z}^{n+1}) \\
\bar{x}^{n+1} &= x^{n+1} + \theta(x^{n+1} - x^n)
\end{align*}
$$

For $\min_x \|Lx\|_1 + \frac{1}{2\lambda} \|Ax - y\|_2^2$, we can take $K = L, F(z) = \|z\|_1$ and $G(x) = \frac{1}{2\lambda} \|Ax - y\|_2^2$. In this case,

$$
\text{Prox}_{\tau G}(z) = \left( \text{Id} + \frac{\tau}{\lambda} A^\top A \right)^{-1} \left( \frac{\tau}{\lambda} A^\top y + z \right).
$$

For the case where $A$ is a masking operation, the matrix inversion in the update of $x_k$ is a simple rescaling operation. Where the matrix $A$ does not admit an efficient inversion formula (e.g. random Gaussian matrices), we carry out a one-time Cholesky factorization that is used throughout the iterations.

For $\min_x \|Lx\|_1 + \frac{1}{2} \|Ax - y\|_1$ where $A \in \mathbb{R}^{m \times n}, L \in \mathbb{R}^{p \times n}$ and $y \in \mathbb{R}^m$, we write this as

$$
\begin{align*}
\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^{p+m}} \|z\|_1 \quad \text{s.t.} \quad (Lx, Ax - y) = z \\
\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^{p+m}} \max_{\xi \in \mathbb{R}^{p+m}} \|z\|_1 + \langle K \begin{pmatrix} x \\ z \end{pmatrix}, \xi \rangle - \langle \begin{pmatrix} 0 \\ y \end{pmatrix}, \xi \rangle
\end{align*}
$$

where

$$
K = \begin{pmatrix} L & -\text{Id}_p \\ A & 0 \\ 0 & -\text{Id}_m \end{pmatrix}.
$$

We let $G((x, z)) = \|x\|_1 + \|z\|_1$ and $F^*(\xi) = \langle \begin{pmatrix} 0 \\ y \end{pmatrix}, \xi \rangle$.

### D Quadratic variational form for matrices

Quadratic variational forms are also widely used for optimization on matrices (see for example, [43, 59]). One example is the nuclear norm (sum of the singular values of a matrix) and this is known to have a variational form [4], for $X \in \mathbb{R}^{n_1 \times n_2}$,

$$
\|X\|_* = \min_{\Sigma \in \mathbb{R}^{n_1 \times n_1}} \frac{1}{2} \langle \Sigma X, X \rangle + \frac{1}{2} \text{tr}(\Sigma)
= \min_{U \in \mathbb{R}^{n_1 \times n_1}, V \in \mathbb{R}^{n_1 \times n_2}} \left\{ \frac{1}{2} \|U\|_F^2 + \frac{1}{2} \|V\|_F^2 : X = UV \right\}.
$$
In Sect. 5.2, we discussed the use of VarPro in the case where one optimizes two nonsmooth terms, each with quadratic variational forms. In this section, we explain how to handle the gradient computations for two popular settings that involve matrices.

**Multitask learning** We consider the following multitask problem [70], for $A \in \mathbb{R}^{m \times n}$, $Y \in \mathbb{R}^{m \times T}$, and $X \in \mathbb{R}^{n \times T}$

$$\min_{X} \frac{1}{\lambda} \|AX - Y\|_* + \|X\|_{1,2}, \quad (58)$$

where $\|X\|_{1,2} = \sum_i \|X_i\|_2$ with $X_i$ denoting the $i$th row of $X$ and the loss function is taken to be the nuclear norm.

By making use of the quadratic variational forms for the group $\ell_1$ and nuclear norm, we can rewrite (58) as

$$\min_{v \in \mathbb{R}^n, U \in \mathbb{R}^{n \times T}, W \in \mathbb{R}^{m \times m}, Z \in \mathbb{R}^{m \times T}} \left\{ \frac{\|v\|^2}{2} + \frac{\|U\|^2_F}{2} + \frac{\lambda}{2} \|W\|^2_F + \frac{\lambda}{2} \|Z\|^2_F : A(\text{diag}(v)U) - Y = WZ \right\}$$

$$= \min_{v, W} f(v, W) \triangleq \max_{\alpha \in \mathbb{R}^{m \times T}} \left( -\frac{1}{2} \|v\|^2 - \frac{1}{2} \|A^T \alpha \|_F^2 \right)$$

$$+ \frac{\lambda}{2} \|W\|^2_F - \frac{1}{2\lambda} \|W^T \alpha\|^2_F - \langle \alpha, Y \rangle.$$

Note that the inner problem is a least squares problem with solution $\alpha$ satisfying

$$\left( A \text{diag}(v^2) A^T + \frac{1}{\lambda} WW^T \right) \alpha = -Y.$$

The outer problem has gradient

$$\partial_v f(v, W) = v - \left( v_i A_i^T \alpha \right)_{i=1}^n$$

and

$$\partial_W f(v, W) = \lambda W - \frac{1}{\lambda} \alpha \alpha^T W.$$

**Structured matrix decomposition** One popular setting is to recover a matrix $X$ such that $AX = Y$ where $y$ is given vector and $A$ is a given linear operator, under the assumption that $X = L + S$ where $L$ is a low rank matrix and $S$ is a sparse matrix [18]. One is then led to consider the optimization problem

$$\min_{L, S} \lambda_1 \|L\|_* + \lambda_2 \|S\|_1 + \frac{1}{2} \|A(L + S) - y\|_2^2. \quad (59)$$
By making use the quadratic variational formulations for the nuclear norm and the $\ell_1$ norm, we can rewrite (59) as
\[
\frac{1}{2} \min_{Z,V} \min_{W,U} \lambda_1 \left( \|U\|_F^2 + \|V\|_F^2 \right) + \lambda_2 \left( \|Z\|_F^2 + \|W\|_F^2 \right) + \|A(UV + W \cdot Z) - y\|^2
\]
(60)
Note that $W \cdot Z$ is the pointwise product of $W$ and $Z$, whereas $UV$ is the matrix product of $U$ and $V$. By placing this into a bilevel form with $Z$, $V$ as the outer variables, we have
\[
\min_{Z,V} f(Z, V) \triangleq \frac{\lambda_1}{2} \|V\|_F^2 + \frac{\lambda_2}{2} \|Z\|_F^2 + \varphi(Z, V)
\]
\[
\varphi(Z, V) \triangleq \frac{1}{2} \min_{W,U} \lambda_1 \|U\|_F^2 + \lambda_2 \|W\|_F^2 + \|A(UV + W \cdot Z) - y\|^2.
\]
(61)
Note that the inner problem is a least squares problem with unique minimizers that can be computed by solving a linear system, and $f$ is differentiable with
\[
\partial_Z f(Z, V) = \lambda_2 Z + W \cdot A^\ast (A(UV + W \cdot Z) - y),
\]
\[
\partial_V f(Z, V) = \lambda_1 V + V^\top A^\ast (A(UV + W \cdot Z) - y),
\]
where $A$ is the adjoint operator to $A$.

**Remark 10** In [30], VarPro was also applied to handle (59) by using only the quadratic variational formula for the nuclear norm. There, they considered
\[
\min_{U,V} f(U, V) \triangleq \frac{\lambda_1}{2} \left( \|U\|^2 + \|V\|^2 \right) + \varphi(U, V)
\]
(62)
where
\[
\varphi(U, V) = \min_S \lambda_2 \|S\|_1 + \frac{1}{2} \|A(UV + S) - y\|^2.
\]
In contrast to the approach described here where the gradient computation has a closed form expression as the solution to a linear system, computation of the gradient of $f$ in (62) cannot be carried out in closed form and requires solving a Lasso problem. Note also that while our function in (61) is always differentiable, differentiability of the function in (62) can only be guaranteed if the inner Lasso problem has a unique solution.

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