For a topological group $G$ the dual object $\hat{G}$ is defined as the set of equivalence classes of irreducible unitary representations of $G$ equipped with the Fell topology. It is well known that if $G$ is compact, $\hat{G}$ is discrete. In this paper, we investigate to what extent this remains true for precompact groups, that is, dense subgroups of compact groups. We show that: (a) if $G$ is a metrizable precompact group, then $\hat{G}$ is discrete; (b) if $G$ is a countable non-metrizable precompact group, then $\hat{G}$ is not discrete; (c) every non-metrizable compact group contains a dense subgroup $G$ for which $\hat{G}$ is not discrete. This extends to the non-Abelian case what was known for Abelian groups. We also prove that if $G$ is a countable Abelian precompact group, then $G$ does not have Kazhdan’s property (T), although $\hat{G}$ is discrete if $G$ is metrizable.

1. Introduction

For a topological group $G$ let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$. The set $\hat{G}$ is equipped with the so-called Fell topology [12], which can be defined on every set of equivalence classes of not necessarily irreducible representations of a given topological group $G$ (see Section 2 for a definition). We recall what it means in some familiar cases (see [23]):

(1) if $G$ is Abelian, then $\hat{G}$ is the standard Pontryagin-van Kampen dual group and the Fell topology on $\hat{G}$ is the usual compact-open topology;

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(2) when $G$ is compact, the Fell topology on $\hat{G}$ is the discrete topology;
(3) when $\hat{G}$ is neither Abelian nor compact, $\hat{G}$ usually is non-Hausdorff.

In general, little is known about the properties of the Fell topology. Its understanding heavily depends on Harmonic Analysis. For example, if $G$ is a second countable locally compact group, then the Fell topology on $\hat{G}$ satisfies the $T_1$ separation axiom if and only if $G$ is type I (see [3]).

It is possible to define a different topology on $\hat{G}$ that, when $G$ is compact, is *grosso modo* the natural quotient of the set of all irreducible representations equipped with the compact-open topology. However, this topology is less useful than the Fell topology because for every integer $n$ it makes the set of all $n$-dimensional representations closed, while it is often desirable to regard certain lower-dimensional representations as limits of higher-dimensional ones (see [3, 9, 15, 22, 23]).

A topological group $G$ is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group $H$ (we may assume that $G$ is dense in $H$). If $G$ is compact, then $\hat{G}$ is discrete. If $G$ is a dense subgroup of $H$, the natural mapping $\hat{H} \rightarrow \hat{G}$ is a bijection but in general need not be a homeomorphism. Following Comfort, Raczkowski and Trigos-Arrieta [7], we say that $G$ determines $H$ if $\hat{G}$ is discrete (equivalently, if the natural bijection $\hat{H} \rightarrow \hat{G}$ is a homeomorphism). A compact group $H$ is determined if every dense subgroup of $H$ determines $H$. In the present paper we investigate the following question: if $H$ is a compact group and $G$ is a dense subgroup of $H$, under what conditions does $G$ determine $H$? Equivalently, for what precompact groups $G$ is $\hat{G}$ discrete?

In the Abelian case, this question has been settled in the work of several authors. Aussenhofer [2] and, independently, Chasco [5] showed that every metrizable Abelian
compact group $H$ is determined. Comfort, Raczkowski and Trigos-Arrieta \cite{7} noted that the Aussenhofer-Chasco theorem fails for non-metrizable Abelian compact groups $H$. More precisely, they proved that every non-metrizable compact Abelian group $H$ of weight $\geq 2^\omega$ contains a dense subgroup that does not determine $H$. Hence, under the assumption of the continuum hypothesis, every determined compact Abelian group $H$ is metrizable. Subsequently, it was shown in \cite{16} that the result also holds without assuming the continuum hypothesis (see also \cite{8}).

Our goal is to extend the results quoted above to compact groups that are not necessarily Abelian. We now formulate our main results.

**Theorem 4.1** If $G$ is a precompact metrizable group, then $\hat{G}$ is discrete. Equivalently, every metrizable compact group is determined.

This extends the aforementioned results from \cite{2} and \cite{5} to non-Abelian compact metrizable groups.

A certain extension of the Aussenhofer-Chasco theorem to non-Abelian groups is due to Lukács in \cite{21}, where he considers the compact open topology on certain groups of continuous mappings. Theorem 4.1 does not follow from Lukács’ results because in general the Fell topology on the dual object is not reduced to the compact-open topology.

In this paper “countable” means “countable infinite”. Let $1_G$ be the class of the trivial representation.

**Theorem 5.1** If $G$ is a countable precompact non-metrizable group, then $1_G$ is not an isolated point in $\hat{G}$.

Theorem 5.1 also shows that if $G$ is a countable dense subgroup of a non-metrizable compact group $H$, then $G$ does not determine $H$. 
Theorem 5.2  If $H$ is a non-metrizable compact group, then $H$ has a dense subgroup $G$ such that $\hat{G}$ is not discrete.

Together with Theorem 4.1 this shows that a compact group is determined if and only if it is metrizable. This extends to non-Abelian compact groups the results given in [7, 16, 8] for Abelian compact groups.

The group $G$ has property (T) if $1_G$ is isolated in $R \cup \{1_G\}$ for every set $R$ of equivalence classes of unitary representations of $G$ without non-zero invariant vectors. This definition is equivalent to the definition of property (T) in terms of Kazhdan pairs which we recall below in Section 2 (see Proposition 1.2.3 in [3]). Compact groups have property (T), and we are interested in property (T) for precompact groups.

It follows from Theorem 5.1 that every countable precompact group with property (T) is metrizable. Furthermore, our proof also yields the following result of Wang [28]: if a discrete group $G$ has property (T), then its Bohr compactification $bG$ is metrizable. (see Corollary 5.7 below).

Theorem 6.1  If $G$ is a countable Abelian precompact group, then $G$ does not have property (T).

We do not know whether there exists a non-compact precompact Abelian group with property (T) (Question 7.1). According to Theorem 6.1 such a group must be uncountable.

2. Preliminaries: Fell topologies and property (T)

All topological groups are assumed to be Hausdorff. For a (complex) Hilbert space $\mathcal{H}$ the unitary group $U(\mathcal{H})$ of all linear isometries of $\mathcal{H}$ is equipped with the strong operator topology (this is the topology of pointwise convergence). With this topology, $U(\mathcal{H})$ is a topological group. If $\mathcal{H} = \mathbb{C}^n$, we identify $U(\mathcal{H})$ with the unitary
group $U(n)$ of order $n$, that is, the compact Lie group of all complex $n \times n$ matrices $M$ for which $M^{-1} = M^*$.

A unitary representation $\rho$ of the topological group $G$ is a continuous homomorphism $G \rightarrow U(H)$, where $H$ is a complex Hilbert space. A closed linear subspace $E \subseteq H$ is an invariant subspace for $S \subseteq U(H)$ if $ME \subseteq E$ for all $M \in S$. If there is a closed subspace $E$ with $\{0\} \subsetneq E \subsetneq H$ which is invariant for $S$, then $S$ is called reducible; otherwise $S$ is irreducible. An irreducible representation of $G$ is a unitary representation $\rho$ such that $\rho(G)$ is irreducible.

Two unitary representations $\rho : G \rightarrow U(H_1)$ and $\psi : G \rightarrow U(H_2)$ are equivalent if there exists a Hilbert space isomorphism $M : H_1 \rightarrow H_2$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$. The dual object of a topological group $G$ is the set $\hat{G}$ of equivalence classes of irreducible unitary representations of $G$.

If $G$ is a precompact group, the Peter-Weyl Theorem (see [18]) implies that all irreducible unitary representation of $G$ are finite-dimensional and determine an embedding of $G$ into the product of unitary groups $U(n)$.

If $\rho : G \rightarrow U(H)$ is a unitary representation, a complex-valued function $f$ on $G$ is called a function of positive type (or positive-definite function) associated with $\rho$ if there exists a vector $v \in H$ such that $f(g) = (\rho(g)v, v)$ (here $(\cdot, \cdot)$ denotes the inner product in $H$). We denote by $P_\rho'$ be the set of all functions of positive type associated with $\rho$. Let $P_\rho$ be the convex cone generated by $P_\rho'$, that is, the set of sums of elements of $P_\rho'$. Observe that if $\rho_1$ and $\rho_2$ are equivalent representations, then $P_{\rho_1}' = P_{\rho_2}'$ and $P_{\rho_1} = P_{\rho_2}$.

Let $G$ be a topological group, $R$ a set of equivalence classes of unitary representations of $G$. The Fell topology on $R$ is defined as follows: a typical neighborhood of
$[\rho] \in \mathcal{R}$ has the form

$$W(f_1, \cdots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \cdots, g_n \in P_\sigma \ \forall x \in C \ |f_i(x) - g_i(x)| < \epsilon\},$$

where $f_1, \cdots, f_n \in P_\rho$ (or $P'_\rho$), $C$ is a compact subspace of $G$, and $\epsilon > 0$. In particular, the Fell topology is defined on the dual object $\widehat{G}$. Even though we will not use the Jacobson topology in this paper, it is perhaps worth mentioning that, if $G$ is locally compact, the Fell topology on $\widehat{G}$ can be derived from the Jacobson topology on the primitive ideal space of $C^*\! (G)$, the $C^*$-algebra of $G$ [9, section 18], [3, Remark F.4.5].

As mentioned in the Introduction, the group $G$ has property (T) if the trivial representation $1_G$ is isolated in $\mathcal{R} \cup \{1_G\}$ for every set $\mathcal{R}$ of equivalence classes of unitary representations of $G$ without non-zero invariant vectors.

Let $\pi$ be a unitary representation of a topological group $G$ on a Hilbert space $\mathcal{H}$, $F \subseteq G$, and $\epsilon > 0$. A unit vector $v \in \mathcal{H}$ is called $(F, \epsilon)$-invariant if $\|\pi(g)v - v\| < \epsilon$ for every $g \in F$. A topological group $G$ has property (T) if and only if there is compact subset $Q$ of $G$ and $\epsilon > 0$ such that for every unitary representation $\rho$ with a $(Q, \epsilon)$-invariant vector has a non-zero invariant vector [3, Proposition 1.2.3]. The pair $(Q, \epsilon)$ is often referred to as a Kazhdan pair.

We refer to Fell’s papers [12, 13], the classical text by Dixmier [9] and the recent monographs by de la Harpe and Valette [15], and Bekka, de la Harpe and Valette [3] for basic definitions and results concerning Fell topologies and property (T).

It is well known that every compact group $H$ has a unique normalized Borel regular invariant measure, the Haar measure. We let $L^2(H)$ denote the Hilbert space that is constructed with the aid of this measure. If $H$ is metrizable, then it follows from the Peter–Weyl Theorem [18, 23] that, up to equivalence, there are countably many irreducible unitary representations of $H$. We henceforth enumerate them as $\rho_i,$
$i \in \mathbb{N}$, let $\chi_i = \text{Tr} (\rho_i)$ be their characters and $d_i$ their dimensions. Put $P'_i = P'_{\rho_i}$ be the corresponding set of functions of positive type, let $P_i$ be the convex cone generated by $P'_i$, and let $Q_i \subseteq C(H)$ be the linear space generated by $P_i$. Because $H$ is compact, the spaces $Q_i$ are finite-dimensional ($\dim Q_i = d_i^2$) and are pairwise orthogonal in the Hilbert space $L^2(H)$. Let $N_i = \{ f \in P_i : f(e) = 1 \}$ be the space of normalized functions in $P_i$. This is a compact subset of $Q_i$, (see Lemma 3.1 below). Set $h_i = \chi_i / d_i \in N_i$ be the normalized character. A basic and comprehensive reference for all notions and results mentioned here is the monograph by Alain Robert [23, Part I].

3. Precompact groups

In this section, we prove some general results about precompact groups that are essential for proving one of our main results (Theorem 4.1) in the next section. We assume that $G$ as a dense subgroup of a compact group $H$, and use the notation $\rho_i$, $P_i$, etc., introduced in the end of the previous section. We assumed there that $H$ is metrizable, and the index $i$ runs over the set $\mathbb{N}$ of positive integers; the results of this section remain true if we allow $H$ to be non-metrizable and accordingly allow $i$ to run over an uncountable index set $I$.

Integrals over $H$ will be taken with respect to the normalized Haar measure on $H$. If $X$ is compact, we consider the space $C(X)$ of continuous functions as a metric space, with the metric defined by the sup-norm.

Lemma 3.1. For every $i \in I$, the convex subset $N_i$ of the finite-dimensional vector space $Q_i$ is compact.

Of course, “compact” refers to the natural topology of the finite-dimensional vector space $Q_i$. For greater clarity, the space $Q_i$ is equipped with the topology of
$L^2(H)$, which coincides on $Q_i$ with the topology inherited from $C(H)$, because $Q_i$ is finite dimensional \cite[Theorem 1.21]{24}.

**Proof.** Consider the set $N'_i = \{ f \in P'_i : f(e) = 1 \}$. We claim that $N_i$ is the convex hull of $N'_i$. Indeed, given $f \in N_i$, write $f$ as a finite sum $f = \sum f_k$, where $f_k \in P'_i$. We may assume that $f_k(e) \neq 0$ for each $k$, since otherwise $f_k$ is identically zero and can be omitted. Put $g_k = f_k / f_k(e)$. Then $\sum f_k(e) = 1$, and $f$ is the convex combination $f = \sum f_k(e) g_k$ of the functions $g_k \in N'_i$.

Denote by $S$ the unit sphere (= the set of all vectors of length one) of the space of the representation $\rho_i$. Define a continuous onto mapping $S \rightarrow N'_i$ by sending every $v \in S$ to the function $f_v \in N'_i$ defined by $f_v(g) = (\rho_i(g)v, v)$. Since $S$ is compact, so is $N'_i$. It is well known that the convex hull of a compact set in a finite-dimensional vector space is compact \cite[Theorem 3.20(d)]{24}. Thus $N_i$ is compact. $\square$

Recall that a function on $G$ is *central* if it is constant on conjugacy classes (cf. \cite{23}). The next lemma is a straightforward consequence of the properties of characters in a compact group (see \cite[7.6]{23}). We include its proof here for the reader’s convenience.

**Lemma 3.2.** The only central functions in $Q_i$ are the functions $c\chi_i$, $c \in \mathbb{C}$.

**Proof.** Any central function $g \in Q_i$ is the sum in $L^2(H)$ of the series $\sum c_j \chi_j$, where $c_j = \int_H g \overline{\chi_j}$ \cite[7.6]{23}. Since $Q_i \perp Q_j$, we have $c_j = 0$ for all $j \neq i$. $\square$

**Lemma 3.3.** Let $X$ be compact, $D$ a dense subset of $X$ and $N$ a compact subset of $C(X)$. If $g \in C(X)$ is at the distance $> \epsilon$ from $N$, there exists a finite subset $F \subseteq D$ such that the distance from $g|_F$ to $N|_F$ in $C(F)$ is $> \epsilon$. 
Proof. For every \( f \in N \) there exists a point \( x = x(f) \in D \) such that \( |f(x) - g(x)| > \epsilon \). Pick a neighborhood \( O_f \) of \( f \) such that \( |h(x) - g(x)| > \epsilon \) for every \( h \in O_f \). Since \( N \) is compact, it is covered by a finite collection of such neighborhoods, say \( O_{f_1}, \ldots, O_{f_s} \). The set \( F = \{ x(f_1), \ldots, x(f_s) \} \) is as required. \( \square \)

**Lemma 3.4.** The space \( \widehat{G} \), equipped with the Fell topology, is \( T_1 \).

**Proof.** Let \( \rho_i \) and \( \rho_j \) be non-equivalent irreducible unitary representations of \( G \). We construct a neighborhood of \([\rho_i]\) in \( \widehat{G} \) which does not contain \([\rho_j]\). Consider the set \( K_j = \{ f \in P_j : f(e) \leq 2 \} \). This set is compact in the natural topology of the finite-dimensional space \( Q_j \) that contains \( P_j \) and \( K_j \). Indeed, \( K_j \) is the image of \([0, 2] \times N_j \) under the mapping \((t, f) \mapsto tf \) \((0 \leq t \leq 2, f \in N_j)\), and \( N_j \) is compact (Lemma 3.1). The normalized character \( h_i = \chi_i/d_i \) of \( \rho_i \) is orthogonal to \( Q_j \), hence \( \text{dist}(h_i, K_j) \geq \text{dist}(h_i, Q_j) > 0 \). It follows from Lemma 3.3 that there exist a finite set \( F \subseteq G \) and \( \epsilon > 0 \) such that \( \text{dist}(h_i|_F, K_j|_F) > \epsilon \). We may assume that \( e \in F \) and \( \epsilon \leq 1 \). We claim that the neighborhood \( W(h_i, F, \epsilon) \) of \([\rho_i]\) does not contain \([\rho_j]\). Equivalently, if \( f \in P_j \), then \( \text{dist}(h_i|_F, f|_F) \geq \epsilon \). Indeed, we have just seen that this is true if \( f \in K_j \).

If \( f \in P_j \setminus K_j \), we have \( f(e) > 2 \) and hence \( \text{dist}(h_i|_F, f|_F) \geq |f(e) - h_i(e)| = f(e) - 1 > 1 \geq \epsilon \). \( \square \)

The next lemma is a straightforward consequence of the classical Riemann-Lebesgue lemma for compact groups (see [17 28.40]). Again, we include its proof here for the reader’s convenience. If \( \{ x_i : i \in I \} \) is a family of numbers, the notation \( x_i \to 0 \) means that for every \( \epsilon > 0 \) the set \( \{ i \in I : |x_i| > \epsilon \} \) is finite.

**Lemma 3.5.** Let \( V \) be a measurable subset of \( H \). Then \( \int_V \chi_i \to 0 \) as \( i \in I \).
Proof. The integrals \( \int_V \chi_i \) are the scalar products of the characteristic function of \( V \) with the terms of the orthonormal sequence \( (\overline{\chi}_i) \) in \( L^2(H) \). \( \square \)

Lemma 3.6. Let \( V \subseteq H \) be a compact neighborhood of the identity \( e \) that is invariant under inner automorphisms of \( H \). Let \( f \in C(H) \) be a continuous central function. Then the distance from \( f|_V \) to \( N_i|_V \) in the space \( C(V) \) is attained at \( h_i|_V \), where \( h_i = \chi_i/d_i \in N_i \).

Proof. Let \( h \in N_i \) be such that \( h|_V \) is as close to \( f|_V \) as possible. Here we use the compactness of \( N_i \), see Lemma 3.1. If \( g \) is a function on \( H \) and \( y \in H \), let \( g_y \) be the function on \( H \) defined by \( g_y(x) = g(yxy^{-1}) \). The formula \( (\rho_i(x)v, v) = (\rho_i(yxy^{-1})\rho_i(y)v, \rho_i(y)v) \), where \( v \) is any vector in the space of the representation \( \rho_i \), shows that \( P'_i \) is invariant under the mapping \( h \mapsto h_y \), hence \( N_i \) is invariant as well. Let \( h' \) be the function on \( H \) defined by \( h'(x) = \int_{y \in H} h(yxy^{-1}) \); equivalently, \( h' = \int_{y \in H} h_y \), where we use vector-valued integrals, as in [24, Theorem 3.27]. According to the cited theorem, \( h' \in N_i \). For every \( y \in H \) the distance from \( h|_V \) to \( f|_V \) is the same as the distance from \( h_y|_V \) to \( f_y|_V \). Since \( f \) is central, we have \( f_y = f \). Thus all \( h_y|_V \) lie in the compact convex set of points of \( N_i|_V \) that are as close as possible to \( f|_V \). It follows that \( h'|_V = \int_{y \in H} h_y \) also lies in this set. Therefore the distance from \( f|_V \) to \( N_i|_V \) is attained at \( h'|_V \). It is easy to verify that \( h'_y = h' \) for every \( y \in H \), which means that \( h' \) is a central function. By Lemma 3.2 the only central function in \( N_i \) is \( h_i = \chi_i/d_i \). Thus \( h' = h_i \). \( \square \)

4. Precompact metrizable groups

The aim of this section is to prove the following result.

Theorem 4.1. If \( G \) is a precompact metrizable group, then \( \hat{G} \) is discrete.
The idea of the proof of Theorem 4.1 is as follows. We want to show that every point \([\rho] \in \hat{G}\) is isolated. Since \(\hat{G}\) is \(T_1\) (Lemma 3.4), it suffices to find a neighborhood \(W\) of \([\rho]\) which for some integer \(i_0\) does not contain any \([\rho_i]\) with \(i \geq i_0\). Our neighborhood will be of the form \(W = W(h, F, \varepsilon)\), where \(h\) is the normalized character of \([\rho]\) and \(F = \{e\} \cup \bigcup_{i \geq i_0} F_i\) is a compact subset of \(G\), where \((F_i)\) is a sequence of finite sets which converges to \(e\) in the sense that every neighborhood of \(e\) contains every \(F_i\) with a sufficiently large index. The finite set \(F_i\) “takes care” of \(\rho_i\), in the sense that it ensures that the neighborhood \(W\) does not contain \([\rho_i]\). We derive the existence of \(F_i\) from the orthogonality of characters. If \(V\) is a neighborhood of \(e\) on which \(h\) is close to 1, the orthogonality relations imply that \(\int_V \chi_i \to 0\) as \(i \to \infty\) (Lemma 3.5), which forces \(\chi_i\) to be close to 0 somewhere on \(V\) for \(i \geq i_0\). This implies that \(h\) and \(h_i\) are not close to each other on \(V\). With a little more work we show that \(h\) is not close to any element of \(P_i\), and that this is witnessed by a certain finite subset \(F_i\) of \(V\).

We now are ready to prove Theorem 4.1.

**Proof.** Recall that we view \(G\) as a dense subgroup of a compact metrizable group \(H\). Pick a bi-invariant (i.e., invariant under left and right translations, hence also under inner automorphisms) metric \(b\) on \(H\). We denote by \(V_\varepsilon\) the closed \(\varepsilon\)-ball (with respect to \(b\)) centered at the neutral element \(e\). We use the notation introduced above.

We can identify the sets \(\hat{G}\) and \(\hat{H}\) (not taking the topology of these sets into account). Let \(\rho\) be an irreducible unitary representation of \(H\). We show that \([\rho]\) is an isolated point in \(\hat{G}\).

Let \(\chi = \chi_\rho\) be the character of \(\rho\), and \(d = d_\rho\) its dimension. Consider the normalized character \(h = \chi/d\). Pick \(\varepsilon > 0\) such that \(\text{Re} h(x) > 2/3\) for every \(x \in V_\varepsilon\). Put \(V = V_\varepsilon\). Since \(\int_V \chi_i \to 0\) (Lemma 3.5), there is an index \(i_0\) such that for all \(i \geq i_0\)
there exists a point \( x_i \in V \) such that \( \text{Re} \chi_i(x_i) \leq 1/3 \). Choose such an \( x_i \) as close to \( e \) as possible. In other words, for \( i \geq i_0 \) the point \( x_i \) is a point in the compact set \( \{ x \in V : \text{Re} \chi_i(x_i) \leq 1/3 \} \) closest to \( e \).

Let \( \epsilon_i = b(x_i, e) \). We claim that \( x_i \to e \) in the metric \( b \) (equivalently, that \( \epsilon_i \to 0 \)). Indeed, otherwise there exist a \( \delta > 0 \) and an infinite set \( A \) of integers \( \geq i_0 \) such that \( \epsilon_i > \delta \) for every \( i \in A \). Then \( \text{Re} \chi_i(x) > 1/3 \) for every \( x \in V_\delta \) and every \( i \in A \), since \( x \) is closer to \( e \) than \( x_i \). This contradicts the fact that \( \int_{V_\delta} \chi_i \to 0 \) (Lemma 3.3).

As before, let \( h_i = \chi_i/d_i \). Pick \( \epsilon'_i > \epsilon_i \) so that \( \epsilon'_i \to 0 \). Put \( V_i = V_{\epsilon'_i}, V'_i = V'_{\epsilon'_i} \). Note that \( \text{Re} h(x_i) > 2/3 \) and \( \text{Re} h_i(x_i) = \text{Re} \chi_i(x_i)/d_i \leq 1/3 \), so \( |h(x_i) - h_i(x_i)| > 1/3 \) and hence \( \text{dist} (h|_{V_i}, h_i|_{V_i}) > 1/3 \). It follows from Lemma 3.6 that \( \text{dist} (h|_{V_i}, N_i|_{V_i}) > 1/3 \).

We claim that there exists a finite set \( F_i \subseteq V'_i \cap G \) such that \( \text{dist} (h|_{F_i}, N_i|_{F_i}) > 1/3 \). This follows from Lemma 3.3, in which we make the following identifications: using the notation of Lemma 3.3 we set \( D = G \cap \text{Int} V'_i \), where \( \text{Int} \) denotes the interior with respect to \( H \); for \( X \) we take the closure of \( D \) in \( H \), which is the same as the closure of \( \text{Int} V'_i \); since \( G \) is dense in \( H \); and the role of the compact set \( N \subseteq C(X) \) is now played by \( N_i|_X \), where \( X \) is as above. The compactness of \( N_i \) was established in Lemma 3.1. Since \( V_i \subseteq \{ x \in H : b(x, e) < \epsilon'_i \} \subseteq \text{Int} V'_i \subseteq X \), we have \( \text{dist} (h|_X, N_i|_X) \geq \text{dist} (h|_{V_i}, N_i|_{V_i}) > 1/3 \), so the conditions of Lemma 3.3 are satisfied with \( \epsilon = 1/3 \).

Set \( F = \{ e \} \cup \bigcup_{i \geq i_0} F_i \). Since \( \epsilon'_i \to 0 \), \( F \) is a compact subset of \( G \). Observe that \( \text{dist} (h|_F, N_i|_F) \geq \text{dist} (h|_{F_i}, N_i|_{F_i}) > 1/3 \) for every \( i \geq i_0 \). We claim that the neighborhood \( W(h|_G, F, 1/6) \) of \([\rho] \) does not contain any \([\rho_i] \) with \( i \geq i_0 \). Indeed, let \( i \geq i_0 \) and \( f \in P_i \). We show that \( \text{dist} (h|_F, f|_F) \geq 1/6 \). Assume that \( \text{dist} (h|_F, f|_F) < 1/6 \). Put \( c = f(e) \) and \( g = f/c \in N_i \). Since \( e \in F \), we have \( |1-c| = |h(e)-f(e)| < 1/6 \). Since \( g \) is a function of positive type, we have \( |g(x)| \leq g(e) = 1 \) for every \( x \in H \).
Proposition C.4.2(ii). It follows that \(|f(x) - g(x)| = |cg(x) - g(x)| = |c-1| \cdot |g(x)| < 1/6\) for every \(x \in H\), hence \(\text{dist}(h|_F, g|_F) \leq \text{dist}(h|_F, f|_F) + \text{dist}(f|_F, g|_F) < 1/6 + 1/6 = 1/3\). Since \(g \in N_i\), this contradicts the inequality \(\text{dist}(h|_F, N_i|_F) > 1/3\).

We have proved that the neighborhood \(W(h|_G, F, 1/6)\) of \([\rho]\) contains only finitely many elements of \(\hat{G}\). Since \(\hat{G}\) is a T₁-space (Lemma 3.4), it follows that \(\hat{G}\) is discrete.

\(\square\)

The compact set \(F = F_\rho \subseteq G\) that we constructed in the proof above is the set of points of a sequence converging to \(e\). It depends on the point \([\rho] \in \hat{G}\) the neighborhood of which we are constructing. However, with some little more work, we can obtain a sequence which does not depend on the point we select in \(\hat{G}\).

**Corollary 4.2.** There exists a single compact subset \(Q \subseteq G\) such that for every \([\rho_i] \in \hat{G}\) the neighborhood \(W(h_i|_G, Q, 1/6)\) of \([\rho_i]\) in \(\hat{G}\) is finite.

**Proof.** Indeed, in our definition of \(F_\rho\) as the union \(\{e\} \cup \bigcup_{i \geq i_0} F_i\) we clearly could have replaced the index \(i_0\) by any larger integer, without losing the property that \(W(h|_G, F_\rho, 1/6)\) is finite. Therefore we may assume that the diameter of \(F_\rho\) is as small as we wish. Construct the sets \(F_{\rho_1}, F_{\rho_2}, \ldots\) so that their diameters tend to 0, and let \(Q\) be their union. If \(U\) is any neighborhood of \(e\), each set \(F_{\rho_i} \setminus U\) is finite, and only finitely many of them are non-empty. It follows that \(Q\) is the set of points of a sequence converging to \(e\) and hence compact. Clearly \(Q\) has the required property: for every \([\rho_i] \in \hat{G}\) the neighborhood \(W(h_i|_G, Q, 1/6)\) of \([\rho_i]\) in \(\hat{G}\) is finite, being a subset of the finite set \(W(h_i|_G, F_{\rho_i}, 1/6)\).

\(\square\)

**Remark 4.3.** The referee suggested the following questions:
(i) For a precompact group $G$, does the Fell topology on $\hat{G}$ contain the co-countable topology? That is, is every countable subset closed?

(ii) What if $G$ is pseudocompact?

(iii) What if $G$ contains no infinite compact subsets?

(iv) What if $G$ contains many compact $G_δ$-subgroups?

Regarding the first three questions, let $G$ be an Abelian pseudocompact group without infinite compact subsets (see [1] [10] and the references there). The dual $\hat{G}$ coincides with the ordinary Pontryagin dual, which is a topological group. Furthermore, since $G$ has no infinite compact subsets, the compact open topology on $\hat{G}$ coincides with the pointwise convergence topology. It follows that $\hat{G}$ is a subgroup of a power of the circle group and hence is precompact. Let $H$ be the completion of $\hat{G}$. Then $H$ is a compact group. If $F$ is a countable subset of $\hat{G}$, then there is a point $p ∈ H$ that is an accumulation point of $F$. Therefore the identity of $\hat{G}$ is an accumulation point of the countable set $A = \{xy^{-1} : x, y ∈ F, x ≠ y\}$. Since the neutral element does not belong to $A$, this yields a countable subset of $\hat{G}$ which is not closed. Therefore the answer to the referee’s first three questions is “not always”.

Regarding the fourth question, Theorem 5.1 shows that the answer is also “not always”. Indeed, in a countable group every subgroup is a $G_δ$-set. Furthermore, we can take the product of a compact metrizable group and a countable precompact group of uncountable character in order to obtain a precompact group with many compact $G_δ$-subgroups such that its dual is not metrizable. On the other hand, Theorem 4.1 asserting that if $G$ is precompact metrizable group then $\hat{G}$ is discrete, remains true for almost metrizable groups (that is, groups $G$ containing a compact subgroup $K$ such
that \(G/K\) is a metrizable space). The proof extends the ideas used here but full details will appear elsewhere.

5. Countable precompact groups

In this section we prove the following:

**Theorem 5.1.** If \(G\) is a countable precompact non-metrizable group, then \(1_G\) is not an isolated point in \(\hat{G}\).

**Theorem 5.2.** If \(H\) is a non-metrizable compact group, then \(H\) has a dense subgroup \(G\) such that \(\hat{G}\) is not discrete.

For a topological group \(G\) let \(\hat{G}_n \subseteq \hat{G}\) be the set of classes of \(n\)-dimensional irreducible unitary representations. We denote by \(w(X)\) the weight (i.e., the least cardinality of a base) of a topological space \(X\). We allow the weight to be finite: if \(X\) a finite discrete space, \(w(X)\) equals the cardinality of \(X\). We first establish two lemmas that will be used in the proof of Proposition 5.5

**Lemma 5.3.** If a compact group \(H\) acts on a metric space \((M,d)\) by isometries, there is a natural metric on the quotient space \(M/H\) compatible with the quotient topology, and \(w(M/H) \leq w(M)\).

*Proof.* Let \(Hx\) and \(Hy\) be two \(H\)-orbits, where \(x, y \in M\). Set the distance between \(Hx\) and \(Hy\) (considered as points in \(M/H\)) to be the same as the distance between compact subsets \(Hx\) and \(Hy\) of the metric space \(M\), which is \(\min\{d(x', y') : x' \in Hx, y' \in Hy\}\). This number is also equal to \(\min\{d(x', y) : x' \in Hx\}\) and to \(\{d(x, y') : y' \in Hy\}\). It is easy to verify that we have defined a metric on \(M/H\) which is compatible with the
quotient topology. The inequality \( w(M/H) \leq w(M) \) follows from the fact that the quotient mapping \( M \to M/H \) is open.

\[ \square \]

**Lemma 5.4.** Let \( f : A \to M \) be a mapping from an infinite set \( A \) to a metric space \((M,d)\). Suppose that either \( w(M) < |A| \) or \( M \) is compact. Then for every \( \delta > 0 \) there exist distinct elements \( x,y \in A \) such that \( d(f(x), f(y)) < \delta \).

**Proof.** The conclusion clearly is true if \( f \) is not injective. Suppose that \( f \) is injective, and let \( \delta > 0 \) be given. If \( w(M) < |A| \), the subset \( f(A) \) of \( M \) cannot be discrete, hence it contains a pair of distinct \( \delta \)-close points. Similarly, if \( M \) is compact, the infinite set \( f(A) \) cannot be closed discrete, so again there exists a pair of distinct \( \delta \)-close points in \( f(A) \), and the lemma follows.

\[ \square \]

**Proposition 5.5.** Let \( G \) be a topological group. Suppose that there exists an integer \( n \) such that \( w(K) < |\hat{G}_n| \) for every compact subset \( K \) of \( G \). Then \( 1_G \) is not an isolated point in some \( \bigcup_{1 \leq m \leq k} \hat{G}_m \cup \{1_G\} \) with \( 1 \leq m \leq n^2 \).

**Proof.** Observe that our assumption implies that \( G \) and \( \hat{G}_n \) are infinite. Indeed, for every finite subset \( K \) of \( G \) we have \( |K| = w(K) < |\hat{G}_n| \). If \( |\hat{G}_n| \) is finite, it follows that so is \( G \), and \( |G| < |\hat{G}_n| \). This is not possible, since a finite group cannot have more irreducible unitary representations than its cardinality (\cite[Theorem 7]{23}).

Let \( k = n^2 \). It suffices to prove that \( 1_G \) is not isolated in \( \bigcup_{1 \leq m \leq k} \hat{G}_m \cup \{1_G\} \).

Let \( F \) be a compact subset of \( G \) and \( \epsilon > 0 \). We must prove that the neighborhood \( W(1,F,\epsilon) \) of the trivial representation meets \( \bigcup_{1 \leq m \leq k} \hat{G}_m \).

**Step 1.** It suffices to construct a non-trivial irreducible unitary representation \( \rho \) of \( G \) of dimension \( \leq k \) which has an \((F,\epsilon)\)-invariant unit vector \( v \). Indeed, the function \( f \) on \( G \) defined by \( f(g) = (\rho(g)v,v) \) is a function of positive type associated with \( \rho \),
and for every \( g \in F \) we have \(|f(g) - 1| = |(\rho(g)v - v, v)| \leq \|\rho(g)v - v\| < \epsilon \). Thus \([\rho] \in W(1, F, \epsilon) \cap \bigcup_{1 \leq m \leq k} \hat{G}_m \cup \{1_G\}\).

**Step 2.** It suffices to construct a \( k \)-dimensional unitary representation \( \tau \) of \( G \) without non-zero invariant vectors which has an \((F, \epsilon/n)\)-invariant unit vector \( v \). Indeed, writing the representation as a direct sum of \( s \) irreducible unitary representations \( \tau = \bigoplus_{i=1}^{s} \tau_i \), we can write \( v = v_1 + \cdots + v_s \), where \( s \leq k \) and the vectors \( v_i \) are \( \tau_i \)-invariant and pairwise orthogonal. If \( g \in F \), then \( \tau(g)v - v = \sum_{i=1}^{s} (\tau_i(g)v_i - v_i) \) is a sum of orthogonal vectors, hence \( \|\tau_i(g)v_i - v_i\| \leq \|\tau(g)v - v\| < \epsilon/n \) for every \( i \). Since \( \sum_{i=1}^{s} \|v_i\|^2 = 1 \), one of the vectors \( v_i \) has length \( \geq 1/\sqrt{s} \geq 1/\sqrt{k} = 1/n \). Suppose it is the vector \( v_1 \). Then \( u = v_1/\|v_1\| \) is a unit vector which is \((F, \epsilon)\)-invariant. Indeed, for every \( g \in F \) we have \( \|\tau_1(g)u - u\| = \|\tau_1(g)v_1 - v_1\|/\|v_1\| < \epsilon/(n\|v_1\|) \leq \epsilon \). We are in the situation of Step 1: \( \tau_1 \) is an irreducible representation of dimension \( \leq k \) for which \( u \) is an \((F, \epsilon)\)-invariant unit vector. Note that \( \tau_1 \) is non-trivial because \( \tau \) has no non-zero invariant vectors.

We are going to construct a unitary representation \( \tau \) with the properties described in Step 2. The idea of the construction is the following. Let \( \mathcal{F}_n \) be the set of classes of all \( n \)-dimensional unitary representations of \( G \) (which may be reducible). Since \( \hat{G}_n \subseteq \mathcal{F}_n \) is “big”, we can find two distinct classes \([\rho_1], [\rho_2] \in \hat{G}_n \) which are “almost the same on \( F \)”. We use \( \rho_1 \) and \( \rho_2 \) to construct a \( k \)-dimensional unitary representation \( \tau \) on the space of all \( n \times n \) complex matrices. We define \( \tau \) by \( \tau(g)A = \rho_1(g)A\rho_2^{-1}(g) \). Shur’s Lemma implies that \( \tau \) has no non-zero invariant vectors, and the identity matrix \( I \) will be \((F, \epsilon)\)-invariant. We now elaborate.
Step 3. Equip $U(n)$ with any compatible bi-invariant metric $b$, and equip $C(F, U(n))$ with the sup-metric. The compact group $U(n)$ isometrically acts on $C(F, U(n))$ by conjugation. Denote by $O_n$ the orbit space. According to Lemma 5.3, the space $O_n$ has a natural metric compatible with the quotient topology.

Consider the set $C$ of all continuous homomorphisms $G \to U(n)$ (we do not consider any topology on $C$). The group $U(n)$ acts on $C$ by conjugation, and the orbit space is the set $F_n$. There is a natural map $F_n \to O_n$ that is obtained from the restriction map $C \to C(F, U(n))$ by passing to quotients. (We do not claim that the mapping $F_n \to O_n$ is continuous if $F_n$ is equipped with the Fell topology.) The image of $[\rho] \in F_n$ in $O_n$ under this mapping will be denoted by $[\rho]|_F$. We have the following commutative diagram:

$$
\begin{array}{ccc}
C & \longrightarrow & C(F, U(n)) \\
\downarrow & & \downarrow \\
F_n & \longrightarrow & O_n
\end{array}
$$

The horizontal arrows are restriction mappings, the vertical arrows are quotients by the action of $U(n)$.

Step 4. We claim that for every $\delta > 0$ we can find two homomorphisms $\rho_1, \rho_2 : G \to U(n)$ which are $\delta$-close on $F$ and determine non-equivalent irreducible representations of $G$ on $\mathbb{C}^n$ (see [14]).

Consider the mapping $[\rho] \to [\rho]|_F$ from $\hat{G}_n \subseteq F_n$ to $O_n$. If $F$ is infinite, we have $w(C(F, U(n))) = w(F) < |\hat{G}_n|$ [11] Theorem 3.4.16 and $w(O_n) \leq w(C(F, U(n)))$ (Lemma 5.3), so $w(O_n) < |\hat{G}_n|$. If $F$ is finite, $C(F, U(n))$ and $O_n$ are compact. In either case we can apply Lemma 5.4 which implies that we can find distinct elements $[\rho_1], [\rho_2] \in \hat{G}_n$ such that $[\rho_1]|_F, [\rho_2]|_F$ are $\delta$-close in $O_n$. The distance between $[\rho_1]|_F$
and $[\rho_2]_F$ in $O_n$ is equal to

$$\inf \{ \text{dist} (\rho_1|_F, A\rho_2|_F A^{-1}) : A \in \mathcal{U}(n) \},$$

where dist refers to the distance in $C(F, \mathcal{U}(n))$. It follows that replacing if necessary $\rho_2$ by an equivalent representation, we may assume that $\rho_1$ and $\rho_2$ are $\delta$-close on $F$, as claimed.

**Step 5.** Let $E = \text{End} \mathbb{C}^n$ be the $n^2$-dimensional Hilbert space of endomorphisms of $\mathbb{C}^n$. The scalar product on $E$ is given by the formula $(A, B) = \text{Tr}(AB^*)/n$. If $\rho_1, \rho_2$ are two irreducible unitary representations of $G$ on $\mathbb{C}^n$, the formula $\tau(g)A = \rho_1(g)A\rho_2^{-1}(g)$ $(g \in G, A \in E)$ defines a unitary representation of $G$ on $E$ that does not contain non-zero invariant vectors. Indeed, if $A \in E$ is such that $\tau(g)A = A$ for all $g \in G$, then $A$ intertwines $\rho_1$ and $\rho_2$ and hence is zero by Shur’s Lemma.

If $I \in E$ is the identity mapping on $\mathbb{C}^n$, then $I$ is unit vector in $E$, since $(I, I) = \text{Tr}(II^*)/n = 1$. By the continuity of the map $A \mapsto \text{Tr}(A - I)(A - I)^*$, there exists $\delta > 0$ such that the following holds: if $A \in \mathcal{U}(n)$ and $A$ is $\delta$-close to $I$ (that is, $b(A, I) < \delta$), then

$$\text{Tr}(A - I)(A - I)^* < \epsilon^2/n.$$

Pick two homomorphisms $\rho_1, \rho_2 : G \to \mathcal{U}(n)$ which are $\delta$-close on $F$ and determine non-equivalent irreducible representations of $G$ on $\mathbb{C}^n$ (Step 4). Then $I \in E$ is an $(F, \epsilon/n)$-invariant vector for the representation $\tau$ on $E$ constructed from $\rho_1, \rho_2$ as above. Indeed, let $g \in F$. Put $A = \tau(g)I = \rho_1(g)\rho_2^{-1}(g)$. Since the operators $\rho_1(g)$ and $\rho_2(g)$ are $\delta$-close and the metric $b$ is bi-invariant, $A$ is $\delta$-close to $I$. Therefore we have

$$\|\tau(g)I - I\|^2 = \|A - I\|^2 = \text{Tr}(A - I)(A - I)^* / n < \epsilon^2 / n^2$$

and $\|\tau(g)I - I\| < \epsilon/n$. Thus $I$ is $(F, \epsilon/n)$-invariant.
We have constructed a $k$-dimensional unitary representation $\tau$ of $G$ without non-zero invariant vectors which has an $(F, \epsilon/n)$-invariant unit vector. According to Step 2, this completes the proof.

The dual $\hat{G}$ of a discrete group $G$ in general contains infinite-dimensional irreducible unitary representations (see [23]). We denote by $\bigcup_{n \in \mathbb{N}} \hat{G}_n$ the subset of $\hat{G}$ consisting of all finite dimensional unitary representations. Applying Proposition 5.5 to the finite subsets of a discrete group $G$, we obtain the following Corollary.

**Corollary 5.6.** Let $G$ be a discrete group such that $1_G$ is an isolated point in $\bigcup_{n \in \mathbb{N}} \hat{G}_n$. Then $\hat{G}_n$ is finite for all $n \in \mathbb{N}$.

Corollary 5.6 yields the following result of Wang [28]. (Remark that property (T) is treated in the next section).

**Corollary 5.7.** If a discrete group $G$ has property (T), then $\hat{G}_n$ is finite for all $n \in \mathbb{N}$.

**Proof.** If $G$ has property (T), then $1_G$ is isolated in $\hat{G} \cup \{1_G\}$. □

The **Bohr topology** on a topological group $G$ is the finest precompact group topology that is coarser than the given topology on $G$. The **Bohr compactification** of $G$ is the completion of $G$ equipped with the Bohr topology. Proposition 5.5 also implies the following.

**Theorem 5.8.** Let $G$ be topological group, $\kappa$ a cardinal such that $w(K) \leq \kappa$ for every compact subset of $G$. If $1_G$ is an isolated point in $\hat{G}_n \cup \{1_G\}$ for every $n$, then $w(bG) \leq \kappa$, where $bG$ is the Bohr compactification of $G$. 
Proof. According to Proposition 5.5, $|\hat{G}_n| \leq \kappa$ for every $n$. The compact group $bG$ therefore has $\leq \kappa$ non-equivalent irreducible unitary representations and hence has weight $\leq \kappa$ by the Peter–Weyl theorem. 

The case $\kappa = \omega$ of the previous theorem deserves to be explicitly stated:

**Corollary 5.9.** Let $G$ be topological group such that every compact subset of $G$ is metrizable. If $1_G$ is an isolated point in $\hat{G}_n \cup \{1_G\}$ for every $n$, then the Bohr compactification of $G$ is metrizable.

Note that this Corollary implies the following assertion which is stronger than Theorem 5.1: if $G$ is a non-metrizable precompact group such that all compact subsets of $G$ are metrizable, then $1_G$ is not an isolated point in $\hat{G}$.

Our proof of Theorem 5.2 is based on the following lemma. A proof can be found e.g. in [6]. For the reader’s convenience we provide the proof.

**Lemma 5.10.** Every compact group of weight $\omega_1$ has a dense countable subgroup.

Proof. A compact space $X$ is dyadic if there exists a mapping $2^\kappa \to X$ of a Cantor cube $2^\kappa$ onto $X$. All compact groups are dyadic (moreover, any compact $G_\delta$-subset of any topological group is dyadic, see [26, 27]), and any dyadic compact space of weight $\leq \mathfrak{c} = 2^\omega$ is separable, being an image of the separable space $2^\mathfrak{c}$. 

We now prove the following stronger version of Theorem 5.2: If $K$ is a non-metrizable compact group, then $K$ has a dense subgroup $G$ such that $1_G$ is not isolated in $\hat{G}$.

Embedding $K$ into the product of unitary groups, we see that there exists a continuous homomorphism $f : K \to K'$ onto a compact group of weight $\omega_1$. Let $G'$ be
a dense countable subgroup of $K'$ (Lemma 5.10). Put $G = f^{-1}(G')$. Since $f$ is open, $G$ is dense in $K$. Consider the continuous mapping $f^* : \hat{G}' \to \hat{G}$ dual to $f$. Since $f$ is onto, $f^*$ is injective. Therefore, $f^*$ sends $1_{G'}$ to $1_G$ and $\hat{G}' \setminus \{1_G\}$ to $\hat{G} \setminus \{1_G\}$. According to Theorem 5.1, $1_{G'}$ is in the closure of $\hat{G}' \setminus \{1_G\}$. It follows that $1_G$ is in the closure of $\hat{G} \setminus \{1_G\}$. Equivalently, $1_G$ is not isolated in $\hat{G}$.

6. The property (T)

The Kazhdan property (T) for topological groups was introduced in [19]. This property has several consequences on the structure of the groups that satisfy it. For $\sigma$-compact locally compact groups Property (T) is equivalent to the fixed point property for isometric affine actions on real Hilbert spaces [3, Theorem 2.12.4]. See [3, 9, 15] for further details on this important topic.

In the previous sections, we saw that for every metrizable precompact group $G$ the dual $\hat{G}$ is discrete (Theorem 4.1). In contrast, we have the following result.

**Theorem 6.1.** If $G$ is an Abelian, countable precompact group, then $G$ does not have property (T).

The result is no longer true if “Abelian” is dropped. Indeed, certain compact Lie groups admit dense countable subgroups which have property (T) as discrete groups [8, Theorem 6.4.4] and hence also as precompact topological groups.

We begin with the following characterization of property (T) for precompact groups.

**Theorem 6.2.** Let $G$ be a dense subgroup of a compact group $H$. Then $G$ has property (T) if and only if the following condition holds: there exist a compact subset $K \subseteq G$ and $\epsilon > 0$ such for every function $f$ of positive type with zero integral over $H$ (with
respect to the Haar measure) the distance from $f|_K$ to 1 in the space $C(K)$ (with respect to the sup-norm) is $\geq \epsilon$.

Proof. Functions of positive type on $H$ are the functions of the form $g \mapsto (\rho(g)v, v)$ associated with unitary representations $\rho$ of $H$ (here $v$ is a vector in the Hilbert space $\mathcal{H}$ of the representation). If a representation $\rho$ does not contain the trivial representation (that is, there are no non-zero invariant vectors), the integral of $(\rho(g)v, v)$ over $H$ is zero, for every $v \in \mathcal{H}$ (this follows, for example, from orthogonality relations). Conversely, if the integral of $f(g) = (\rho(g)v, v)$ is zero, then $v$ lies in $\mathcal{H}_1 = \mathcal{H}_2^\perp$, where $\mathcal{H}_2 \subseteq \mathcal{H}$ is the subspace of invariant vectors. Therefore $f$ is associated with a representation on $\mathcal{H}_1$ without non-zero invariant vectors.

Now suppose that $G$ has property (T). Let $(K, \delta)$ be a Kazhdan pair for $G$ and assume, without loss of generality, that $e \in K$. Suppose that $f$ is a function of positive type on $H$ with zero integral over $H$. Then $f(g) = (\rho(g)v, v)$ for some unitary representation $\rho$ of $H$ on a Hilbert space $\mathcal{H}$ without non-zero invariant vectors and some $v \in \mathcal{H}$. We first consider the case when $v$ is a unit vector, that is, $f(e) = 1$, where $e$ the neutral element of $G$. By the definition of a Kazhdan pair, there exists $g \in K$ such that $\|\rho(g)v - v\| \geq \delta$. Writing $f(g) = (\rho(g)v, v) = a + bi$, we have $\delta^2 \leq \|\rho(g)v - v\|^2 = 2 - 2a$ and thus $a \leq 1 - \delta^2/2$. It follows that $|f(g) - 1| \geq 1 - a \geq \delta^2/2$.

In the general case (when we no longer assume that $f(e) = 1$), we replace $f$ by $f/f(e)$ (we exclude the trivial case $f(e) = 0$). If there is no $\epsilon$ such that the pair $(K, \epsilon)$ satisfies the conditions of the theorem, there is a sequence $(f_n)$ of functions of positive type with zero integral over $H$ such that the sequence $(f_n|_K)$ uniformly converges to 1 on $K$. We may assume, without loss of generality, that $f_n(e) = 1$ for every $n$ (otherwise replace $f_n$ by $f_n/f_n(e)$). According to the last inequality of the previous paragraph,
the distance from \(f_n|_K\) to 1 in \(C(K)\) is at least \(\delta^2/2\). This contradicts the fact that the sequence \((f_n|_K)\) uniformly converges to 1 on \(K\) and proves the existence of a number \(\epsilon > 0\) such that the pair \((K, \epsilon)\) satisfies the conditions of the theorem.

Conversely, suppose that \(G\) does not have property \((T)\). We must prove that for every compact subset \(K \subseteq G\) and \(\epsilon > 0\) there exists a function \(f\) of positive type with zero integral over \(H\) such that \(|f(g) - 1| < \epsilon\) for every \(g \in K\). Since \((K, \epsilon)\) is not a Kazhdan pair, there exist a unitary representation \(\rho\) of \(G\) on a Hilbert space \(\mathcal{H}\) without invariant vectors and a unit vector \(v \in \mathcal{H}\) which is \((K, \epsilon)\)-invariant, that is, \(|\rho(g)v - v| < \epsilon\) for every \(g \in K\). The representation \(\rho\) of \(G\) can be extended to a unitary representation \(\overline{\rho}\) of \(H\) on \(\mathcal{H}\). Since \(\rho\) admits no invariant vectors, the same is true for \(\overline{\rho}\). Let \(f\) be the function on \(H\) defined by \(f(g) = (\overline{\rho}(g)v, v)\). The function \(f\) is as required: it is of positive type with zero integral over \(H\), and for every \(g \in K\) we have \(|f(g) - 1| = |(\rho(g)v - v)| \leq \|\rho(g)v - v\| \cdot \|v\| < \epsilon\). □

**Corollary 6.3.** Let \(G\) be a dense subgroup of a compact group \(H\). If \(G\) has property \((T)\), then there exist a compact subset \(K \subseteq G\), a real (signed) measure \(\mu\) on \(K\) and a real number \(c\) such that for every real function \(f\) of positive type on \(H\) with \(f(e) = 1\) and zero integral over \(H\) we have

\[
\int_K f \mu < c < \int_K 1 \mu.
\]

**Proof.** Consider the convex set \(P\) of all real functions \(f\) of positive type on \(H\) such that \(\int_H f = 0\) and \(f(e) = 1\). Let \(K \subseteq G\) be a compact subset with the property described in Theorem 6.2. According to the theorem, the image of \(P\) under the restriction map \(f \mapsto f|_K\) is at a positive distance from 1. The Hanh-Banach theorem implies that 1 can be separated from the convex set \(P|_K\) by a linear functional. □
Following Lubotzky and Zimmer [20], if \( R \) is a set of classes of unitary representations of \( G \), we say that \( G \) has Property (T) with respect to \( R \) if \( 1_G \) is isolated in \( R \cup \{1_G\} \).

**Remark 6.4.** Let \( G \) be a precompact group, and let \( G_d \) denote the same group with the discrete topology. Let \( R \) be the set of equivalence classes of finite-dimensional unitary representations of \( G \) without non-zero invariant vectors. It is readily seen that the proof of Theorem 6.2 shows that \( G_d \) has property (T) with respect to \( R \) if and only if there exist a finite subset \( K \subseteq G \) and \( \epsilon > 0 \) such that for every function \( f \) of positive type with zero integral over the completion of \( G \) the distance from \( f|_K \) to 1 in the space \( C(K) \) (with respect to the sup-norm) is \( \geq \epsilon \).

**Theorem 6.5.** Let \( G \) be a precompact group such that every compact subset of \( G \) is countable, and let \( G_d \) be the same group equipped with the discrete topology. Let \( R \) be the set of equivalence classes of finite-dimensional unitary representations of \( G \) without non-zero invariant vectors. The following assertions are equivalent:

(a) \( G \) has property (T);

(b) \( G_d \) has property (T) with respect to \( R \).

**Proof.** (b) \( \Rightarrow \) (a) is clear.

(a) \( \Rightarrow \) (b). Let \( H \) be the completion of \( G \). Suppose that \( G \) has property (T) but \( G_d \) does not have property (T) with respect to \( R \). Let \( P \) be the convex set that was used in the proof of Corollary 6.3. \( P \) is the set of all real-valued functions \( f \) of positive type on \( H \) with zero integral and such that \( f(e) = 1 \). Using Corollary 6.3 find a compact set \( K \subseteq G \) such that \( 1|_K \) is not in the weak closure of the convex set \( P|_K \) in the Banach space \( C(K) \) (the weak closure and the norm closure are the same because
of convexity, see [24, Theorem 3.12]). Since all compact subsets of $G$ are countable, we can write $K$ as the set of points of a sequence $(x_n)$. According to Remark 6.4, for every finite subset $F$ of $G$ and $\epsilon > 0$ there exists $f \in P$ such that the distance from $f|_F$ to $1|_F$ in the space $C(F)$ is $< \epsilon$. For every positive integer $n$ apply this to $F = \{x_i : i \leq n\}$ and $\epsilon = 1/n$. We obtain a function $f_n \in P$ such that $|f_n(x_i) - 1| < 1/n$ for every $i \leq n$. The sequence $(f_n)$ pointwise converges to 1 on $K$. All functions in $P$ are uniformly bounded by 1, so Lebesgue’s dominated convergence theorem implies that the sequence $(f_n|_K)$ weakly converges to 1 in $C(K)$. Therefore, $1|_K$ is in the weak closure of $P|_K$, contrary to our assumption. □

We are now in a position to prove Theorem 6.1:

If $G$ is an Abelian, countable precompact group, then $G$ does not have property (T).

Proof. Let $H$ be the completion of $G$. By Theorem 6.5 we must verify that $G_d$ does not have property (T) with respect to $\mathcal{R}$, where $\mathcal{R}$ is the same as in Theorem 6.5.

It suffices to prove that for every finite set $F \subseteq G$ and $\epsilon > 0$ there is a continuous character $\chi : H \to \mathbb{U}(1)$, $\chi \neq 1$, such that $|\chi(x) - 1| < \epsilon$ for every $x \in F$.

Assume the contrary. Let $F \subseteq G$ be a finite set such that for some $\epsilon > 0$ and every $\chi \in \hat{H} \setminus \{1\}$ the restriction $\chi|_F$ is at the distance $\geq \epsilon$ from $1|_F$. Consider the restriction homomorphism $\varphi : \hat{H} \to \mathbb{U}(1)^F$ defined by $\varphi(\chi) = \chi|_F$. According to our assumption, $\varphi$ has a trivial kernel and hence is injective. Also, $1|_F$ is at the distance $\geq \epsilon$ from $\varphi(\hat{H} \setminus \{1\})$. It follows that $1|_F$ is an isolated point in the group $\varphi(\hat{H})$. Since every topological group which has an isolated point is discrete, we conclude that $\varphi(\hat{H})$ is a discrete subgroup of $\mathbb{U}(1)^F$. This is a contradiction, since the compact group $\mathbb{U}(1)^F$ cannot have infinite discrete subgroups. □
7. Questions

**Question 7.1.** Does there exist a non-compact precompact Abelian group with property \((T)\)?

By Theorem 6.1 such a group must be uncountable.

The definition of the Fell topology, given in [3] and used in the present paper, is not the same as the definition given in [22]. Using the notation of Section 2, the difference is the following. In [22], functions \(f_1, \ldots, f_n \in P'_\rho\) of positive type are approximated by functions \(g_1, \ldots, g_n \in P'_\sigma\) of positive type rather than by their sums in \(P_\rho\) that we allowed. For locally compact groups \(G\) the two definitions of the Fell topology on \(\hat{G}\) agree, according to [3, Proposition F.1.4].

**Question 7.2.** Do the two definitions of the Fell topology coincide on \(\hat{G}\): (a) for every topological group? (b) for every precompact group?

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