Toward an analytic determination of the deconfinement temperature in SU(2) L.G.T.

M. Billó\textsuperscript{a,b}, M. Caselle\textsuperscript{b,c}, A. D’Adda\textsuperscript{b,c}, and S. Panzeri\textsuperscript{d}

\textsuperscript{a} NORDITA, Blegdamsvej 17, København Ø, Denmark
\textsuperscript{b} Istituto Nazionale di Fisica Nucleare, Sezione di Torino
\textsuperscript{c} Dipartimento di Fisica Teorica dell’Università di Torino via P.Giuria 1, I-10125 Turin, Italy
\textsuperscript{d} SISSA, Via Beirut 2-4, 34013 Trieste, Italy

Abstract

We consider the SU(2) lattice gauge theory at finite temperature in \((d+1)\) dimensions, with different couplings \(\beta_t\) and \(\beta_s\) for timelike and spacelike plaquettes. By using the character expansion of the Wilson action and performing the integrals over space-like link variables, we find an effective action for the Polyakov loops which is exact to all orders in \(\beta_t\) and to the first non-trivial order in \(\beta_s\). The critical coupling for the deconfinement transition is determined in the \((3+1)\) dimensional case, by the mean field method, for different values of the lattice size \(N_t\) in the compactified time direction and of the asymmetry parameter \(\rho = \sqrt{\beta_t/\beta_s}\). We find good agreement with Montecarlo simulations in the range \(1 \leq N_t \leq 5\), and good qualitative agreement in the same range with the logarithmic scaling law of QCD. Moreover the dependence of the results from the parameter \(\rho\) is in excellent agreement with previous theoretical predictions.
1 Introduction

The aim of this paper is to obtain, by using only analytical methods, reliable estimates of the deconfinement temperature in the SU(2) pure gauge theory (namely without quarks) in (3+1) dimensions. The natural framework to pose this question is that of the finite temperature Lattice Gauge Theories (LGT). In this framework, during these last years, the best estimates of the deconfinement temperature have been obtained by means of Montecarlo simulations, which are certainly the most powerful tool to extract quantitative results from LGT. However we think that it is important in itself to have some independent analytical estimate of the location of the critical point, besides the outputs of the computer simulations, to reach a deeper theoretical understanding of the deconfinement transition. The attempts to obtain analytically the critical temperature have a rather long history, starting more than ten years ago [1, 2, 3, 4, 5]. However the strategy has always been essentially the same: first, construct an effective action in terms of the Polyakov loops (which, as we shall see below, are the relevant dynamical variables in the physics of deconfinement for pure gauge theories). Second, use a mean field approximation to extract the critical coupling. A common feature of all these attempts was that the effective actions were always constructed neglecting the spacelike part of the action. As a consequence it was impossible to reach a consistent continuum limit for the critical temperature.

The aim of this paper is to show that it is possible to overcome this problem. We shall construct in the SU(2) case an improved effective action which takes into account also the spacelike part of the original Wilson action and is exact to all orders in the timelike coupling. This is a rather non trivial result and we shall devote most of this paper to describe how it can be obtained. Moreover, as we shall see, our approach is a constructive one and can be extended in principle to all orders in the space-like couplings.

We decided in this paper to concentrate only on the gauge group SU(2) for simplicity reasons, but most of our results can be extended to SU(N) models with N > 2. Indeed, this paper can be considered as the natural continuation of [4] where these same techniques were applied to the $N \to \infty$ limit of LGT. Here we try to eliminate the large N approximation by looking directly at the $N = 2$ case.

This paper is organized as follows: after a short introduction to finite temperature lattice gauge theory (sect. 2), we shall devote sect. 3 to the construction of the effective action. In particular, sections 3.3.1 and 3.3.2 contain the computation of the first non-trivial contributions from the space-like part of the action; these sections are rather technical and the reader interested mainly in the results may wish to skip them, as the results are anyhow summarized in section 3.4. In sect. 4 we shall extract the critical deconfinement temperature with mean field techniques, we shall discuss our results in comparison with existing Montecarlo estimates and check their consistency in the case of asymmetric lattices with known theoretical results. Finally sect. 5 will be devoted to some concluding remarks. We shall try
to keep our formalism as general as possible, so we shall derive in sect. 2 and 3 the effective action for the Polyakov loop in a \((d + 1)\)-dimensional LGT with an arbitrary \(d\), and we shall fix \(d = 3\) only in sect. 4.

2 Finite Temperature LGT

2.1 General Setting

Let us consider a pure gauge theory with gauge group \(SU(2)\), defined on a \(d + 1\) dimensional cubic lattice. In order to describe a finite temperature LGT, we have to impose periodic boundary conditions in one direction (which we shall call from now on “time-like” direction), while the boundary conditions in the other \(d\) direction (which we shall call “space-like”) can be chosen freely. We take a lattice of \(N_t (N_s)\) spacings in the time (space) direction, and we work with the pure gauge theory, containing only gauge fields described by the link variables \(U_{n,i} \in SU(2)\), where \(n \equiv (\vec{x}, t)\) denotes the space-time position of the link and \(i\) its direction. It is useful to choose different bare couplings in the time and space directions. Let us call them \(\beta_t\) and \(\beta_s\) respectively. The Wilson action is then

\[
S_W = \sum_n \frac{1}{2} \left\{ \beta_t \sum_i \text{Tr}_f(U_{n,0i}) + \beta_s \sum_{i<j} \text{Tr}_f(U_{n,ij}) \right\},
\]

where \(\text{Tr}_f\) denotes the trace in the fundamental representation and \(U_{n,0i}\) (\(U_{n,ij}\)) are the time-like (space-like) plaquette variables, defined as usual by

\[
U_{n,ij} = U_{n,i}U_{n+i,j}U^\dagger_{n+j,i}U^\dagger_{n,ij}.
\]

In the following we shall call \(S_s\) (\(S_t\)) the space-like (time-like) part of \(S_W\).

Let us introduce an asymmetry parameter \(\rho\) defined by the relation: \(\beta_t/\beta_s \equiv \rho^2\). As \(\rho\) varies we have different, but equivalent, lattice regularization of the same model. This equivalence is summarized by the following equations, which can be obtained by taking the classical continuum limit of (3) and which relate \(\beta_s\) and \(\beta_t\) to the (bare) gauge coupling \(g\) and to the temperature \(T\):

\[
\frac{4}{g^2} = a^{3-d} \sqrt{\beta_s\beta_t}, \quad T = \frac{1}{N_t a} \sqrt{\frac{\beta_t}{\beta_s}}.
\]

Here \(a\) is the space-like lattice spacing and \(\frac{1}{N_t T}\) is the time-like spacing, hence \(\rho\) is the ratio between the two. From this last observation it is clear that equivalent regularizations with different values of \(\rho\) require different values of \(N_t\). Hence, to maintain the equivalence, \(N_t\) must be a function of \(\rho\): \(N_t(\rho)\).

Among all these equivalent regularizations a particular role is played by the symmetric one, which is defined by:

\[
\beta \equiv \frac{4}{g^2} a^{d-3}
\]
(from now on we shall distinguish the symmetric regularization from the asymmetric ones by eliminating the subscripts \( t \) and \( s \) in \( \beta \)). Comparing eqs. (3, 4) we see that all the regularizations are equivalent if the following relations hold:

\[
\beta = \rho \beta_s = \frac{\beta_t}{\rho}, \tag{5}
\]

\[
N_t(\rho) = \rho N_t(\rho = 1). \tag{6}
\]

Notice however that these equivalence relations are constructed in the *naive* or “classical” continuum limit. At the quantum level, in the (3+1) dimensional case these relations change slightly. These modifications have been studied by F.Karsch in \cite{7} and we shall discuss them in sect.4.

Let us now only anticipate that eq. (5) becomes:

\[
\beta_t = \rho(\beta + 4c_\tau(\rho)) \tag{7}
\]

\[
\beta_s = \frac{\beta + 4c_\sigma(\rho)}{\rho}, \tag{8}
\]

where the two functions \( c_\sigma(\rho) \) and \( c_\tau(\rho) \) can be found in \cite{7}. For our purposes we only need to know the first terms of their \( 1/\rho \) expansion in the \( \rho \to \infty \) limit. Let us define:

\[
4c_{\sigma,\tau} \equiv \alpha_{\sigma,\tau}^0 + \frac{\alpha_{\sigma,\tau}^1}{\rho} + \cdots. \tag{9}
\]

The \( \alpha \)'s can be calculated from the expression of \( c_{\sigma,\tau}(\rho) \) given in \cite{7}; they are:

\[
\alpha_\tau^0 = -0.27192; \quad \alpha_\tau^1 = 1/2; \quad \alpha_\sigma^0 = 0.39832; \quad \alpha_\sigma^1 = 0.
\]

These relations will play a major role in the following, since the invariance of our results as \( \rho \) changes is a crucial consistency check of all our approach. Actually it is a non trivial test, since these relations are the result of a weak coupling calculation, and only become manifest in the continuum limit of the model. Being able to reproduce them within the framework of a strong coupling calculation would be a remarkable and a priori unexpected result. This is actually the case, as discussed in detail in sect. 4.

A second reason for which it is important to have under control this \( \rho \) symmetry is that, at the end, we would like to compare our prediction with the Montecarlo simulations, which are all made on symmetric lattices. However, at the same time, it is only in the limit of highly asymmetric lattices (\( \rho \to \infty \)) that (as we shall see in more detail below) we can define in an unambiguous way our expansion of the spacelike part of the action, so it is somehow mandatory for us to be able to match these two different limits.

In a finite temperature discretization it is possible to define gauge invariant observables which are topologically non-trivial, as a consequence of the periodic
boundary conditions in the time directions. The simplest choice is the Polyakov loop defined in terms of link variables as:

\[ \hat{P}_x \equiv \text{Tr} \prod_{t=1}^{N_t} (V_{x,t}) \] (10)

where \( V_{x,t} \equiv U_{x,t,0} \) are the vertical link matrices. In the following we shall often use the untraced quantity \( P_x \), defined as:

\[ P_x \equiv \prod_{t=1}^{N_t} (V_{x,t}) \] (11)

which will be referred to as “Polyakov line”.

As it is well known, the finite temperature theory has a new global symmetry (unrelated to the gauge symmetry), with symmetry group the center \( C \) of the gauge group (in our case \( Z_2 \)). The Polyakov loop is a natural order parameter for this symmetry.

In \( d > 1 \), finite temperature gauge theories admit a deconfinement transition at \( T = T_c \), separating the high temperature, deconfined, phase \((T > T_c)\) from the low temperature, confining domain \((T < T_c)\). In the following we shall be interested in the phase diagram of the model as a function of \( T \), and we shall make some attempt to locate the critical point \( T_c \). The high temperature regime is characterized by the breaking of the global symmetry with respect to the center of the group. In this phase the Polyakov loop has a non-zero expectation value, and it is an element of the center of the gauge group (see for instance [8]).

### 2.2 Svetitsky-Yaffe conjecture

The Svetitsky-Yaffe conjecture [8] is based on the idea that, if one were able to integrate out all the gauge degrees of freedom of the original \((d + 1)\)-dimensional model except those related to the Polyakov loops, then the resulting effective theory for the Polyakov loops would be a \(d\)-dimensional spin system with symmetry group \( C \). The deconfinement transition of the original model would become the order–disorder transition of the effective spin system. This effective theory would obviously have very complicated interactions, but Svetitsky and Yaffe were able to argue that all these interactions should be short ranged. As a consequence, if the transition point of the effective spin system is of second order, near this critical point, where the correlation length becomes infinite, the precise form of the short ranged interactions should not be important, and the universality class of the deconfinement transition should coincide with that of the simple spin model with only nearest neighbour interactions and the same global symmetry group. In particular the deconfinement transition of the \(d + 1\) dimensional SU(2) LGT in which we are interested should belong to the same universality class of the magnetization transition of the \(d\)-dimensional spin Ising model. Unfortunately this argument cannot
help to fix the critical temperature, which is not an universal result, but depends on the precise form of the action that we study, and hence of the short ranged interactions that we neglected above. In the next section we shall construct these correction terms explicitly.

2.3 Character expansion

An important role in the following analysis will be played by the character expansion, which in the SU(2) case is very easy to handle. Let us briefly summarize few results. The character of the group element $U$ in the $j^{th}$ representation is:

$$\chi_j(U) \equiv \text{Tr}_j(U) = \frac{\sin((2j+1)\theta)}{\sin(\theta)}$$

(12)

where $\text{Tr}_j$ denotes the trace in the $j^{th}$ representation and $\theta$ is defined according to the following parametrization of $U$ in the fundamental representation:

$$U = \cos(\theta) 1 + i \hat{n} \cdot \vec{\sigma} \sin(\theta).$$

(13)

where $\vec{n}$ is a tridimensional unit vector and $\sigma_i$ are the three Pauli matrices. Notice, as a side remark, that with this parametrization the Haar measure has the following form:

$$DU = \sin^2(\theta) \frac{d\theta d^2\vec{n}}{4\pi^2}$$

(14)

and the Polyakov loop becomes $\hat{P}_z = 2 \cos(\theta z)$

The following orthogonality relations between characters hold:

$$\int DU \chi_r(U) \chi_s(U) = \delta_{r,s}$$

(15)

$$\sum_r d_r \chi_r(U V^{-1}) = \delta(U, V)$$

(16)

where $d_r$ denotes the dimensions of the $r^{th}$ representation: $d_r = 2r + 1$. In the following we shall use two important properties of the characters:

$$\int DU \chi_r(U) \chi_s(U^{-1} V) = \delta_{r,s} \frac{\chi_r(V)}{d_r}$$

(17)

$$\int DU \chi_r(U V_1 U^{-1} V_2) = \frac{1}{d_r} \chi_r(V_1) \chi_r(V_2).$$

(18)

The character expansion of the Wilson action has a particularly simple form:

$$e^{\frac{\beta}{2} \text{Tr}(U)} = \sum_j 2(2j+1) \frac{I_{2j+1}(\beta)}{\beta} \chi_j(U), \quad j = 0, \frac{1}{2}, 1, \cdots$$

(19)

where $I_n(\beta)$ is the $n^{th}$ modified Bessel function. It is customary to collect in front of expression (19) a factor of $\frac{I_1(\beta)}{\beta^{1/2}}$, so that the expansion starts with 1.
3 Construction of the Effective Action

In this section our goal is to construct an effective action for the finite temperature LGT in terms of the Polyakov loops only. To do so one should be able to integrate exactly on the spacelike variables so that the only remaining degrees of freedom at the end are the Polyakov loops. Notice that in this way the resulting effective action would live in \( d \) dimensions (one dimension less than the starting model). This is exactly along the line of the original Svetitsky-Yaffe program. As already remarked in the introduction the early attempts to determine analytically the critical temperature were all based on the assumption that the deconfinement transition is dominated by the timelike plaquettes, and the contribution of the space-like plaquettes was consequently neglected. Although this approximation correctly predicts the existence of the deconfinement transition, a quantitative estimate of the critical temperature for large enough values of \( N_t \), namely near the continuum limit, requires the contribution of the space-like plaquettes to be taken into account. Accordingly, we shall treat the timelike part of the Wilson action \( S_t \) as a Born term and treat the spacelike part \( S_s \) as a perturbation; namely, we shall make a strong coupling expansion in \( \beta_s \), while the time-like part of action will be treated exactly. This means that order by order in \( \beta_s \) the dependence of the effective action from \( \beta_t \) will be exact, the only expansion parameter being thus \( \beta_s \). Of course, the zeroth order in \( \beta_s \) will contain the timelike plaquettes only. It is not at all obvious that the integration over the spacelike links could be done to all orders in \( \beta_t \), but it turns out to be the case in the framework of the characters expansion (see below) order by order in \( \beta_s \). Rather than a straightforward expansion in powers of \( \beta_s \) we shall use for each space-like plaquette a character expansion. Each representation \( j \) in the expansion gives a contribution proportional to a ratio of Bessel functions which is of order \( \beta_s^{2j} \), so that the character expansion and the expansion in powers of \( \beta_s \) coincide up to higher order terms arising from the power series expansion of the Bessel functions. As it will be discussed later in sect. 4 these higher order terms vanish anyway in the limit of highly asymmetric lattice. We shall consider in this paper only the zeroth order and the first non trivial order in \( \beta_s \), namely \( \beta_s^2 \). Terms of order \( \beta_s^2 \) come either from one space-like plaquette in the adjoint representation or from a couple of plaquettes in the fundamental representation. However it should be noted that there is no obstruction in principle to go to higher orders.

For any given order in \( \beta_s \) the result is given by an infinite sum of characters. Remarkably enough in the \( N_t = 1 \) case this series can be summed exactly and the result can be written in a closed form. This is essentially due to the fact the if \( N_t = 1 \) the same effective action can be obtained in a completely different way, using techniques typical of matrix models (see below), thus allowing a non trivial check of all our strong coupling results.
3.1 Expansion in $\beta_s$ of the effective action

The effective action $S_{\text{eff}}$ for the Polyakov lines $P_x \equiv \prod_{t=1}^{N_t} V_x$ is obtained by integrating over all the spacelike degrees of freedom in the action (1). As explained previously, our approach is to consider the contributions from the spacelike plaquettes up to a certain order in $\beta_s$ only. So, for our purposes, it will be convenient to expand separately the spacelike and the timelike part of the action (1):

$$\exp(S_{\text{eff}}) = \int \prod_{\vec{x},t,i} DU_{\vec{x},t,i} \exp S_W$$

$$= \int \prod_{\vec{x},t',i'} DU_{\vec{x},t',i'} \prod_{\vec{x}',t'',i''} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},t',i''};0) \right)$$

$$\times \prod_{\vec{x},t,i<j} \left( 1 + \sum_{t=\frac{1}{2}}^{\infty} d_t \frac{I_{2t+1}(\beta_s)}{I_1(\beta_s)} \chi_t(U_{\vec{x},t;i,j}) \right).$$

(20)

Specifically, we work out here the effective action up to $O(\beta_s^2)$. This means that in eq. (20) we must look only at the terms containing at most a single space-like plaquette in the adjoint representation, $\chi_1(U_{\vec{x},t;i,j})$, or two space-like plaquettes in the fundamental, $\chi_{\frac{1}{2}}(U_{\vec{x},t_1;i,j})\chi_{\frac{1}{2}}(U_{\vec{x},t_2;i,k})$. Due to the orthogonality relations for characters, it easy to convince oneself that a pair of plaquettes in the fundamental representation do actually contribute to the integral only if they appear in the same spatial position (at two different times $t_1$ and $t_2$); for the same reason a single fundamental plaquette cannot contribute. We are thus lead to the following expression:

$$\exp(S_{\text{eff}}) = \int \prod_{\vec{x},t',i'} DU_{\vec{x},t',i'} \prod_{\vec{x}',t'',i''} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},t',i''};0) \right)$$

$$\times \left( 1 + \sum_{\vec{x},t<i} \left[ \sum_{t=1}^{N_t} 3 \frac{I_3(\beta_s)}{I_1(\beta_s)} \chi_1(U_{\vec{x},t;i,j}) + \sum_{t_1<i,t_2<i} 4 \left( \frac{I_2(\beta_s)}{I_1(\beta_s)} \right)^2 \chi_{\frac{1}{2}}(U_{\vec{x},t_1;i,j})\chi_{\frac{1}{2}}(U_{\vec{x},t_2;i}) \right] \right).$$

(21)

In the next sections we will consider separately the three contributions appearing in (21). The first one that we will consider corresponds to the “1” in the second factor above, and gives the $O(\beta_s^0)$ result.

3.2 Zeroth order approximation

In the zeroth order approximation we have to consider only the timelike part of the Wilson action:

$$\exp(S_0) = \int \prod_{\vec{x},i,t} DU_{\vec{x},i,t} \left[ 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x},i,t} V_{\vec{x},i,t}^\dagger U_{\vec{x},i+1,t} V_{\vec{x},i+1,t}^\dagger) \right].$$

(22)
In this case we can easily integrate all the spacelike links. The reason is that each spacelike link only belongs to two timelike plaquettes; hence, by making a character expansion, it can be exactly integrated out. Let us do this integration in two steps, for future commodity. First let us integrate all the spacelike links except the lowermost ones (which, due to the periodic boundary conditions coincide with the uppermost ones). We obtain, using eq. (17):

$$\exp(S_0) = \prod_{\vec{x},i} \left( 1 + \sum_{j=1}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j \left( U_{\vec{x},i} P_{\vec{x}+i} U_{\vec{x},i}^\dagger P_{\vec{x}}^\dagger \right) \right), \quad (23)$$

where $P_{\vec{x}}$ is the open Polyakov line (whose trace is the Polyakov loop) in the site $\vec{x}$ and $U_{\vec{x},i}$ are the remaining lowermost spacelike links. Integrating also on $U_{\vec{x},i}$ using eq. (18) we end up with

$$\exp(S_0) = \prod_{\vec{x},i} \left( 1 + \sum_{j=1}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j(P_{\vec{x}+i}) \chi_j(P_{\vec{x}}^\dagger) \right). \quad (24)$$

Let us define, for future convenience, the link element of $\exp(S_0)$ as follows:

$$C^0_{\vec{x},i} \equiv \sum_{j=0}^{\infty} \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \chi_j(P_{\vec{x}+i}) \chi_j(P_{\vec{x}}^\dagger). \quad (25)$$

It is now evident that this basic element, which will be denoted also as $C^0_{\vec{x},i} = C^0(\theta_{\vec{x}}, \theta_{\vec{x}+i})$, depends only on $\theta_{\vec{x}}, \theta_{\vec{x}+i}$, which are the invariant angles for the Polyakov lines $P_{\vec{x}}, P_{\vec{x}+i}$ in the sites joined by the link. Indeed from now on we will always assume to have gauge-rotated the Polyakov lines to be diagonal:

$$P_{\vec{x}} = \begin{pmatrix} e^{i\theta_{\vec{x}}} & 0 \\ 0 & e^{-i\theta_{\vec{x}}} \end{pmatrix} \quad (26)$$

Notice that the zeroth-order action (24) is simply given by

$$\exp(S_0) = \prod_{\vec{x},i} C^0_{\vec{x},i} \quad (27)$$

### 3.3 First order approximation

The $O(\beta_s^0)$ effective action (27) contains just nearest-neighbour interactions between the Polyakov loops. As we shall see in the following, the net outcome to the effective action from the $O(\beta_s^2)$ terms in (21) is the addition of interaction terms involving more than two Polyakov loops. Specifically, the term of order $\beta_s^2$ contains

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2 The links we are referring to are those of the $d$-dimensional spatial lattice, corresponding to a space-like slice in the original $d+1$-dimensional lattice.
interactions among the invariant angles of all the Polyakov lines around a spatial plaquette.

Let us now compute the contributions from the adjoint space-like plaquettes and from the pairs of fundamental ones in eq. (21). As already said in the introduction, the next two subsections, containing these computations, are quite technical; the results are summarized in section 3.4.

3.3.1 The adjoint representation term

To calculate the contribution from the adjoint representation, we have to select from (21) the term:

\[ 3 \frac{I_3(\beta_s)}{I_1(\beta_s)} \int \prod_{\vec{x},t';i'} DU_{\vec{x},t',i'} \prod_{\vec{x},t'',i''} \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \chi_j(U_{\vec{x}',t'',0;i''}) \right) \times \sum_{\vec{x},i<j} \sum_{t=1}^{N_t} \chi_1(U_{\vec{x},t;i,j}). \]

(28)

Let us fix a spatial position \( \vec{x}, i < j \) in the above sum over the space-like plaquettes, and study the corresponding integral. The integration over all the link matrices not pertaining to the chosen spatial position can be performed exactly as in the \( O(\beta_s^0) \) case, giving as a result the product of all the link factors \( C^{0}_{\vec{x}',i} \) except those in the
chosen spatial position\(\vec{x}\) (i.e. except \(C_{0\vec{x},i}^0, C_{\vec{x}+i,j}^0, C_{\vec{x}+i+j,-i}^0, C_{\vec{x}+j,-j}^0\)). To treat the remaining non-trivial integrations, first we can note that all the spacelike plaquettes in the same spatial position give evidently the same contribution, regardless of the time \(t\); therefore the sum over the time positions in (28) results simply in a \(N_t\) factor. Secondly, it is convenient to use the following relation for the \(SU(2)\) characters:

\[
\chi_1 = (\chi_\frac{1}{2})^2 - 1.
\] (29)

The “\(-1\)” simply reproduces the zeroth order term, and gives a renormalization of order \(\beta_s^2\) to such contribution. The integral over the link variables along the plaquette can now be decoupled into products of integrals over single link matrices, by writing explicitly \[\chi_\frac{1}{2}(U_{\vec{x},t;ij})\] in term of traces of link variables in the fundamental representation. Thus eq.(28) can be rewritten in terms of the following integrals over the unitary spacelike link matrix \(U\):

\[
B_{\alpha\beta\gamma\delta}(P_{\vec{x}}, P_{\vec{x}+i}) = \int DU \left( 1 + \sum_{j=\frac{1}{2}}^{\infty} d_j \left[ \frac{I_{2j+1}(\beta_{t})}{I_1(\beta_{t})} \right] \chi_j(U P_{\vec{x}+i} U^\dagger P_{\vec{x}}^\dagger) \right) U_{\alpha\beta} U_{\gamma\delta}^\dagger
\] (30)

where \(\alpha, \ldots = 1, 2\) are the indices of the \(U\) matrix in the fundamental representation. Let us assume that the Polyakov lines \(P_{\vec{x}}\) in eq.(30) are already set in the diagonal form of eq.(26). Then the measure and the argument of \(\chi_j\) at the r.h.s. of eq.(30) are invariant under the transformations

\[
U_{\alpha\beta} \rightarrow \omega_{\alpha\alpha} U_{\alpha\beta}, \quad U_{\gamma\delta}^\dagger \rightarrow U_{\gamma\delta}^\dagger (\omega^{-1})_{\delta\delta}
\] (31)

and

\[
U_{\alpha\beta} \rightarrow U_{\alpha\beta} \omega_{\beta\beta}, \quad U_{\gamma\delta}^\dagger \rightarrow (\omega^{-1})_{\gamma\gamma} U_{\gamma\delta}^\dagger
\] (32)

where \(\omega\) is a diagonal \(SU(2)\) matrix. By using this invariance one can easily conclude that \(B_{\alpha\beta\gamma\delta} = 0\) unless \(\beta = \gamma\) and \(\delta = \alpha\). As a consequence, the integral (30) depends on the invariant angles of the Polyakov line only, and can be written as follows:

\[
B_{\alpha\beta\gamma\delta}(P_{\vec{x}}, P_{\vec{x}+i}) \equiv B_{\alpha\beta\gamma\delta}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) = \delta_{\beta\gamma} \delta_{\delta\alpha} C_{\alpha\beta}(\theta_{\vec{x}}, \theta_{\vec{x}+i})
\] (33)

(no summation over repeated indices). Moreover, it is not difficult to show that \(C_{\alpha\beta}\) is a real symmetric matrix. By using these facts, we can write the contribution (28) to the effective action in terms of the invariant angles of the Polyakov lines. The contribution to eq.(28) at fixed \(\vec{x}, i\) can be expressed as:

\[
3N_t \frac{I_3(\beta)}{I_1(\beta)} \left[ \prod_{\vec{x}',t'} C_{\vec{x}',t'}^0 \right] \left[ \text{Tr}[\hat{C}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) \hat{C}(\theta_{\vec{x}+i}, \theta_{\vec{x}+i+j}) \hat{C}(\theta_{\vec{x}+j}, \theta_{\vec{x}+j}) \hat{C}(\theta_{\vec{x}+j}, \theta_{\vec{x}})] - 1 \right]
\] (34)

\(^3\)Later on we will denote this product as \(\prod_{\text{link}\in\text{pl}} C_0(\text{link})\).
where the term (-1) in (34) corresponds to the term (-1) in (28) and the matrices \( C(\theta_{\bar{x}}, \theta_{\bar{x}+i}) \) are a normalized version of \( C \):

\[
\hat{C}_{\alpha,\beta}(\theta_{\bar{x}}, \theta_{\bar{x}+i}) = \frac{C_{\alpha,\beta}(\theta_{\bar{x}}, \theta_{\bar{x}+i})}{C_{\alpha,\beta}^0}
\]  

(35)

Indeed in writing eq.(34) we have multiplied and divided by \( C_{\alpha,\beta}^0 \cdot C_{\alpha,\beta}^0 \cdot C_{\alpha,\beta}^0 \) in order to collect the factor \( \prod_{\bar{x},i} C_{\alpha,\beta}^0 = \exp S_0 \).

The last step is the explicit evaluation of the matrix elements \( C_{\alpha,\beta} \). We set

\[
C_{\alpha,\beta} = \sum_{j=0}^{\infty} d_j \left[ \frac{I_{2j+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t} \alpha^{(j)}_{\alpha,\beta},
\]

(36)

so that \( \alpha^{(j)}_{\alpha,\beta} \) is defined as the contribution of the \( j^{th} \) representation in eq.(30):

\[
\int DU \alpha^{(j)}_{\alpha,\beta} U_{\alpha,\beta} (UP_{\bar{x}+i}U^\dagger P_{\bar{x}+i}^\dagger) \equiv \delta_{\alpha,\beta} \delta_{\beta,\beta} \alpha^{(j)}_{\alpha,\beta}.
\]

(37)

It follows from (37) that the non vanishing integrals at the l.h.s. depend only on \( |U_{\alpha,\beta}|^2 \), and hence that \( \alpha^{(j)}_{11} = \alpha^{(j)}_{22} \) and \( \alpha^{(j)}_{12} = \alpha^{(j)}_{21} \).

To compute the matrix elements \( \alpha^{(j)}_{\alpha,\beta} \), which is not a completely trivial task, we use the following strategy. We note that the matrix \( \alpha^{(j)} \) can be expressed in terms of the integral

\[
K^{(j)}(\theta_{\bar{x}}, \theta_{\bar{x}+i}) = \int DU X_j (UP_2 U^\dagger P_1^\dagger) = \frac{1}{d_j} \chi_j(P_2) \chi_j(P_1^\dagger)
\]

(38)

through a system of two linear Schwinger–Dyson-like equations. Indeed, considering the integral \( \int DU U_{\alpha,\beta} U_{\alpha,\beta}^\dagger \chi_j (UP_{\bar{x}+i}U^\dagger P_{\bar{x}+i}^\dagger) \) we easily find that

\[
\alpha^{(j)}_{11} + \alpha^{(j)}_{12} = K^{(j)}.
\]

(39)

To construct a second independent equation, let us consider the integral

\[
\int DU X_j (UP_{\bar{x}+i}U^\dagger P_{\bar{x}+i}^\dagger) X_j (UP_{\bar{x}+i}U^\dagger P_{\bar{x}+i}^\dagger).
\]

(40)

On one hand we can write the character \( X_j \) explicitly as a trace and express the integral in terms of the \( \alpha^{(j)}_{\alpha,\beta} \) by using eq.(37). On the other hand, the integral (40) can be written in terms of \( K^{(j)} \) functions by using the basic SU(2) Clebsch-Gordan relation: \( \chi_{\alpha,j} \chi_j = \chi_{\alpha,j + \frac{1}{2}} + \chi_{\alpha,j - \frac{1}{2}} \). The resulting equation is:

\[
2 \cos(\theta_{\bar{x}+i} - \theta_{\bar{x}}) \alpha^{(j)}_{11} + 2 \cos(\theta_{\bar{x}+i} + \theta_{\bar{x}}) \alpha^{(j)}_{12} = K^{(j - \frac{1}{2})} + K^{(j + \frac{1}{2})}
\]

(41)

There is another possible way to compute \( \alpha^{(j)}_{\alpha,\beta} \), based on the expression of the SU(2) characters as Tchebicheff polynomials of second kind: \( \chi_{\alpha,j}(\theta) = U_{\alpha,j}(\cos(\theta)) \), which was utilized in [9]. However this alternative technique cannot easily be applied to the case of a pair of fundamental plaquettes.
Eq.s (39) and (41) form a set of two linear equations in the two unknowns $C_{11}^{(j)}$ and $C_{12}^{(j)}$ whose solution is:

\[
C_{11}^{(j)} = \frac{K^{(j-\frac{1}{2})} - 2 \cos(\theta_{x+i} + \theta_x)K^{(j)} + K^{(j+\frac{1}{2})}}{4 \sin \theta_{x+i} \sin \theta_x}
\]

\[
C_{12}^{(j)} = -\frac{K^{(j-\frac{1}{2})} - 2 \cos(\theta_{x+i} - \theta_x)K^{(j)} + K^{(j+\frac{1}{2})}}{4 \sin \theta_{x+i} \sin \theta_x}
\]  

Eq. (42)

By inserting these results in eq. (36) we finally obtain the $C_{a\beta}$ coefficients. We choose to write the matrix $C$ in the form

\[
C = \frac{1}{2} \begin{pmatrix} C_0 + C_1 & C_0 - C_1 \\ C_0 - C_1 & C_0 + C_1 \end{pmatrix}
\]

which is consistent with the symmetries of $C$ and allows an easy evaluation of the trace contained in eq. (34). After some algebraic rearrangements it follows from eq. (42) that

\[
C_1(\theta_x, \theta_{x+i}) = \frac{1}{2 \sin \theta_{x+i} \sin \theta_x} \left\{ \sum_{j=\frac{1}{2}}^{\infty} \chi_j(\theta_{x+i}) \chi_j(\theta_x) \frac{I_{2j+2}(\beta_i)N_i - I_{2j}(\beta_i)N_i}{(2j + 1)I_1(\beta_i)N_i} \right. \\
+ 2 \cos [(2j + 1)\theta_{x+i}] \cos [(2j + 1)\theta_x] \left( \frac{I_{2j+1}(\beta_i)}{I_1(\beta_i)} \right)^{N_i} \\
+ 2 \cos \theta_{x+i} \cos \theta_x + \left( \frac{I_2}{I_1} \right)^{N_i} \right\},
\]

while $C_0(\theta_{x+i}, \theta_x)$ coincides with the link element $C_{x,i}^0$, as defined in eq. (25).

With the parametrization of the matrix $C$ given in eq. (43), the trace in eq. (34) simply reads:

\[
\text{Tr} \left[ C(\theta_x, \theta_{x+i}) C(\theta_{x+i}, \theta_{x+i+j}) C(\theta_{x+i+j}, \theta_{x+j}) C(\theta_{x+j}, \theta_x) \right]
\]

\[
= C_0(\theta_x, \theta_{x+i})C_0(\theta_{x+i}, \theta_{x+i+j})C_0(\theta_{x+i+j}, \theta_{x+j})C_0(\theta_{x+j}, \theta_x)
\]

\[
+ C_1(\theta_x, \theta_{x+i})C_1(\theta_{x+i}, \theta_{x+i+j})C_1(\theta_{x+i+j}, \theta_{x+j})C_1(\theta_{x+j}, \theta_x).
\]

By inserting this expression into eq. (34), the products of $C_0$’s cancel, and we are left with the explicit expression for the contribution of an adjoint plaquette:

\[
3N_i \frac{I_3(\beta_i)}{I_1(\beta_i)} \left[ \prod_{x,i} C_{x,i}^0 \right] \hat{C}_1(\theta_x, \theta_{x+i}) \hat{C}_1(\theta_{x+i}, \theta_{x+i+j}) \hat{C}_1(\theta_{x+i+j}, \theta_{x+j}) \hat{C}_1(\theta_{x+j}, \theta_x)
\]

\[
= C_1(\theta_x, \theta_{x+i}) \frac{C_0(\theta_x, \theta_{x+i})}{C_{x,i}^0}
\]

where

\[
\hat{C}_1(\theta_x, \theta_{x+i}) = \frac{C_1(\theta_x, \theta_{x+i})}{C_{x,i}^0}
\]

and $C_1(\theta_x, \theta_{x+i})$ is given by eq. (14).
3.3.2 Pair of fundamental representations

Let us go back to eq. (21), and consider the last type of contributions, namely the ones coming from two plaquettes in the fundamental representation. As in the previous case of the adjoint plaquettes, we consider the contribution of a single pair of plaquettes; that is, we fix the spatial position \( \vec{x}, i < j \) of the plaquettes. We can moreover fix the time position of one of these two plaquettes, say at \( t_1 = 0 \). Then the sum over \( t_1 \) just gives a factor of \( N_t \). The second spatial plaquette will be located at \( t_2 = M, (M = 1, \ldots, N_t - 1) \). Notice that the contribution from the fundamental plaquettes is not present when \( N_t = 1 \).

To perform the computation of this contribution, it is convenient to choose the gauge so that the Polyakov lines are concentrated for instance in the uppermost vertical links; this choice is always possible.

The integration of all the space-like links not involved in the two space-like plaquettes can be performed in the usual way. With notations similar to those of eq. (30) [see also Fig. 1] we are left with the expression:

\[
\prod_{\text{link} \notin \text{pl}} C_0(\text{link}) \times 4 \left( \frac{I_2(\beta_s)}{I_1(\beta_s)} \right)^2 N_t \times \sum_{M=1}^{N_t-1} \int \prod_{\vec{x} \neq \vec{y} \in \text{pl}} DU_{\vec{x},\vec{y}} D\tilde{U}_{\vec{x},\vec{y}} \left( 1 + \sum_{m=1}^{\infty} d_m \left[ \frac{I_{2m+1}(\beta_t)}{I_1(\beta_t)} \right]^M \chi_m(U_{\vec{x},\vec{y}} \tilde{U}_{\vec{x},\vec{y}}^{\dagger}) \right) \times \left( 1 + \sum_{n=1}^{\infty} d_n \left[ \frac{I_{2n+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_t-M} \chi_n(\tilde{U}_{\vec{x},\vec{y}} P_{\vec{x},\vec{y}} U_{\vec{x},\vec{y}}^{\dagger}) \right) \times \text{Tr}(U_{\vec{x},\vec{x}+i} U_{\vec{x}+i,\vec{x}+i+j} U_{\vec{x}+j,\vec{x},\vec{x}+i+j} U_{\vec{x}+i,\vec{x}+i+j} U_{\vec{x}+j,\vec{x},\vec{x}+i+j} U_{\vec{x},\vec{x}+i}) \right). \]

By writing explicitly the products in the traces, one can express eq. (48) in terms of integrals of the form

\[
\int DU D\tilde{U} U_{\alpha \beta} \tilde{U}_{\gamma \delta} \chi_m(U\tilde{U}^{\dagger}) \chi_n(\tilde{U} P_{\vec{x},\vec{x}+1} U^{\dagger} P_{\vec{x}}^{\dagger}) \equiv \delta_{\alpha \delta} \delta_{\beta \gamma} C_{\alpha \beta}^{(m,n)}(\theta_{\vec{x}}, \theta_{\vec{x}+i}).
\]

where the Kronecker deltas at the r.h.s. originate, as in the case of the adjoint representation, by the symmetry given in eq.s (31,32). We have assumed in eq. (49) to have diagonalized the Polyakov lines as in eq.(26). The original expression (48) can then be written as

\[
\prod_{\text{link} \notin \text{pl}} C_0(\text{link}) \times 4 N_t \left( \frac{I_2(\beta_s)}{I_1(\beta_s)} \right)^2 \times \sum_{M} \text{Tr}[C^{(M)}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) C^{(M)}(\theta_{\vec{x}+i, \theta_{\vec{x}+i+j}) C^{(M)}(\theta_{\vec{x}+i+j, \theta_{\vec{x}+j}) C^{(M)}(\theta_{\vec{x}+j, \theta_{\vec{x}}})].
\]

(50)
where the $2 \times 2$ matrices $C^{(M)}$ are given by

$$C^{(M)}_{\alpha\beta} = \sum_{m,n} d_m d_n \left[ \frac{I_{2m+1}(\beta t)}{I_1(\beta t)} \right]^M \left[ \frac{I_{2n+1}(\beta t)}{I_1(\beta t)} \right]^{N_t-M} C^{(m,n)}_{\alpha\beta}. \quad (51)$$

Unlike the $C^{(j)}_{\alpha\beta}$ defined in eq. (53), $C^{(m,n)}_{\alpha\beta}$ is not a real symmetric matrix. It rather satisfies the following properties:

$$[C^{(m,n)}_{11}]^* = C^{(m,n)}_{22} \quad ; \quad [C^{(m,n)}_{12}]^* = C^{(m,n)}_{21}. \quad (52)$$

Moreover, one can easily show from its definition (49) that

$$[C^{(m,n)}_{\alpha\beta}]^* = (P_{\bar{x}+i})_{\beta\alpha} P_{\bar{x}} C^{(m,n)}_{\alpha\beta} \quad (53)$$

and $C^{(m,n)}_{\alpha\beta}(\theta_\bar{x}, \theta_{\bar{x}+i}) = C^{(m,n)}_{\beta\alpha}(\theta_\bar{x}, -\theta_{\bar{x}+i})$.

In order to compute the matrix elements of $C^{(m,n)}$ we follow a strategy analogous to that utilized in the adjoint plaquette case. We show how the $C^{(m,n)}_{\alpha\beta}$, that have four independent real components, can be expressed in term of the integral

$$\int DU D\bar{U} \chi_m(U\bar{U}^\dagger)\chi_n(\bar{U}P_{\bar{x}+i}U^\dagger P_{\bar{x}}^\dagger) \equiv \frac{d_m}{d_n} K^{(n)} \quad (54)$$

by means of a set of four linear Schwinger–Dyson-like equations. Consider first the integral

$$\int DU D\bar{U} \chi_{\frac{1}{2}}(U\bar{U}^\dagger)\chi_m(U\bar{U}^\dagger)\chi_n(\bar{U}P_{\bar{x}+i}U^\dagger P_{\bar{x}}^\dagger). \quad (55)$$

On one hand we can write explicitly as a trace the $\chi_{\frac{1}{2}}$ factor and use the definition (19); on the other we can use the Clebsch-Gordan relation to rewrite the integral (53) as a combination of $K^{(n)}$ functions via eq. (54). We thus find:

$$C^{(m,n)}_{11} + C^{(m,n)}_{12} + C^{(m,n)}_{22} + C^{(m,n)}_{21} = (\delta_{m+\frac{1}{2},n} + \delta_{m-\frac{1}{2},n}) \frac{1}{d_m} K^{(n)}. \quad (56)$$

By considering an integral analogous to (53), but containing the trace $\chi_{\frac{1}{2}}(\bar{U}P_{\bar{x}+i}U^\dagger P_{\bar{x}}^\dagger)$ instead of $\chi_{\frac{1}{2}}(U\bar{U}^\dagger)$, we obtain that

$$e^{i(\theta_{x+i} - \theta_x)} C^{(m,n)}_{11} + e^{-i(\theta_{x+i} - \theta_x)} C^{(m,n)}_{22} + e^{i(\theta_{x} + \theta_{x+i})} C^{(m,n)}_{21} + e^{-i(\theta_{x} + \theta_{x+i})} C^{(m,n)}_{12} =$$

$$= (\delta_{m,n+\frac{1}{2}} + \delta_{m,n-\frac{1}{2}}) \frac{1}{d_m} K^{(m)}. \quad (57)$$

To determine all the components of $C^{(m,n)}_{\alpha\beta}$ we need two more relations, that can be
obtained by taking derivatives of eq.(54) with respect to the invariant angles\(^\dagger\):

\[
i(2n+1)[e^{i(\theta_x+\theta_{x+i})}C_{21}^{(m,n)}] - e^{-i(\theta_x+\theta_{x+i})}C_{12}^{(m,n)}] = \frac{1}{2d_m}(\delta_{m,n+\frac{1}{2}} - \delta_{m,n-\frac{1}{2}})\partial_+ K^{(m)} \tag{58}
\]

and

\[
i(2n+1)[e^{i(\theta_{x+i}-\theta_x)}C_{11}^{(m,n)}] - e^{-i(\theta_{x+i}-\theta_x)}C_{22}^{(m,n)}] = \frac{1}{2d_m}(\delta_{m,n+\frac{1}{2}} - \delta_{m,n-\frac{1}{2}})\partial_- K^{(m)}, \tag{59}
\]

where \(\partial_\pm\) stands for \(\frac{\partial}{\partial \theta_{x+i}} \pm \frac{\partial}{\partial \theta_x}\). We can now obtain the expression of \(C_{\alpha\beta}^{(m,n)}\) by solving the system formed by the four equations (56,57,58,59). This is more easily done in terms of the matrix \(\tilde{C}^{(m,n)}\), defined by \(\tilde{C}_{11}^{(m,n)} = e^{i(\theta_{x+i}-\theta_x)}C_{11}^{(m,n)}\) and \(\tilde{C}_{12}^{(m,n)} = e^{-i(\theta_{x+i}-\theta_x)}C_{12}^{(m,n)}\), i.e. by

\[
\tilde{C}_{\alpha\beta}^{(m,n)} = (P_{x+i})_{\beta\gamma}(P_x)_{\alpha\delta}C_{\alpha\delta}^{(m,n)} \tag{60}
\]

The matrix \(\tilde{C}^{(m,n)}\) enjoys the same symmetries (52) as \(C_{\alpha\beta}^{(m,n)}\), while eq.(53) is replaced by

\[
\tilde{C}_{\alpha\beta}^{(m,n)} = (P_{x+i})_{\beta\gamma}(P_x)_{\alpha\delta}C_{\alpha\delta}^{(m,n)} \tag{61}
\]

The symmetries (52) can be implemented by writing the 2 \(\times\) 2 matrix \(\tilde{C}^{(m,n)}\) as

\[
\tilde{C}^{(m,n)} = \frac{1}{2} \left( C^{(m,n)}_0 + C^{(m,n)}_1 - iB^{(m,n)}_+ - iB^{(m,n)}_- \right) \tag{62}
\]

The equations (56,57,58,59) imply that the only non-vanishing components of \(\tilde{C}^{(m,n)}\) are those with \(n = m \pm \frac{1}{2}\). The solution of the system can be expressed as follows:

\[
C_0^{(m,m+\frac{1}{2})} = C_0^{(m,m-\frac{1}{2})} = \frac{1}{2d_m}K^{(m)}
\]

\[
d_mC_0^{(m,m+\frac{1}{2})} = -\frac{1}{2(2m+2)}\partial_+ K^{(m)}; \quad d_mC_0^{(m,m-\frac{1}{2})} = \frac{1}{2(2m)}\partial_- K^{(m)}
\]

\[
d_mC_1^{(m,m+\frac{1}{2})} = \frac{\cos [(2m+1)\theta_{x+i}] \cos [(2m+1)\theta_{x}]}{(2m+2)\sin \theta_{x+i}\sin \theta_{x}}K^{(m)} - K^{(m+\frac{1}{2})} - K^{(m-\frac{1}{2})} \tag{63}
\]

\[
d_mC_1^{(m,m-\frac{1}{2})} = \frac{\cos [(2m+1)\theta_{x+i}] \cos [(2m+1)\theta_{x}]}{(2m)\sin \theta_{x+i}\sin \theta_{x}}K^{(m)} + K^{(m-\frac{1}{2})} \tag{63}
\]

\(^5\text{SU(2) characters satisfy} \quad \frac{\partial}{\partial \theta_{x+i}} \chi_{j+\frac{1}{2}}(\theta) - \chi_{j-\frac{1}{2}}(\theta) = (2j+1)\chi_j(\theta), \text{ as can be seen by expressing them as Tchebicheff polynomials:} \chi_j(\theta) \equiv U_{2j}(\cos(\theta)). \text{ One has then for instance:} \]

\[
\frac{\partial}{\partial \theta_{x+i}} \chi_{n+\frac{1}{2}}(\theta) - \chi_{n-\frac{1}{2}}(\theta) = (2n+1)\chi_n(\tilde{U}_{P_{x+i}}U^\dagger P_{x+i})\frac{\partial}{\partial \theta_{x+i}}\text{Tr}(\tilde{U}P_{x+i}U^\dagger P_{x+i}).
\]
In analogy to eq. (51) we can introduce a matrix $\tilde{C}^{(M)}_{\alpha\beta}$ defined by the relation

$$
\tilde{C}^{(M)}_{\alpha\beta}(\theta_{\vec{x}+i}, \theta_{\vec{x}}) = (P_{\vec{x}+i})_{\beta\alpha}(P_{\vec{x}}^\dagger)_{\alpha\alpha}C^{(M)}(\theta_{\vec{x}+i}, \theta_{\vec{x}})
$$

$$
= \sum_{m,n} d_m d_n \left[ \frac{I_{2m+1}(\beta_t)}{I_1(\beta_t)} \right]^M \left[ \frac{I_{2n+1}(\beta_t)}{I_1(\beta_t)} \right]^{N_1-M} \tilde{C}^{(m,n)}_{\alpha\beta}(\theta_{\vec{x}+i}, \theta_{\vec{x}}). \tag{64}
$$

We can now insert the matrix $\tilde{C}^{(M)}$ instead of $C^{(M)}$ in the expression (50), as the extra phases appearing in $\tilde{C}^{(M)}$ cancel in the trace.

The matrix $\tilde{C}^{(M)}_{\alpha\beta}$ can be written, in analogy with eq. (62), as

$$
\tilde{C}^{(M)} = \frac{1}{2} \left( C^{(M)}_0 + C^{(M)}_1 - iB^{(M)}_- - C^{(M)}_0 - C^{(M)}_1 + iB^{(M)}_+ \right). \tag{65}
$$

The explicit form of $C^{(M)}_0, C^{(M)}_1$ and $B^{(M)}_\pm$ can now be obtained from eqs. (64) and (65). The result is:

$$
C^{(M)}_0 = \frac{1}{2} \sum_{m=\frac{1}{2}}^{\infty} \left[ \frac{I_{2m+1}}{I_1} \right]^M \left( d_{m+\frac{1}{2}} \left[ \frac{I_{2m+2}}{I_1} \right]^{N_1-M} + d_{m-\frac{1}{2}} \left[ \frac{I_{2m}}{I_1} \right]^{N_1-M} \right) K^{(m)}(\theta_{\vec{x}+i}, \theta_{\vec{x}})
$$

$$
+ \frac{1}{2} \sum_{m=\frac{1}{2}}^{\infty} \left[ \frac{I_{2m+1}}{I_1} \right]^M \left( \left[ \frac{I_{2m+2}}{I_1} \right]^{N_1-M} - \left[ \frac{I_{2m}}{I_1} \right]^{N_1-M} \right) \theta_\pm K^{(m)}(\theta_{\vec{x}+i}, \theta_{\vec{x}})
$$

$$
C^{(M)}_1 = \frac{1}{2 \sin \theta_{\vec{x}+i} \sin \theta_{\vec{x}}} \left\{ \left[ \frac{I_2}{I_1} \right]^M + \left[ \frac{I_2}{I_1} \right]^{N_1-M} 2 \cos \theta_{\vec{x}+i} \cos \theta_{\vec{x}} \right.
$$

$$
\quad + \sum_{m=\frac{1}{2}}^{\infty} \left[ \frac{I_{2m+1}}{I_1} \right]^{N_1-M} \left[ \left[ \frac{I_{2m+2}}{I_1} \right]^{N_1-M} + \left[ \frac{I_{2m}}{I_1} \right]^{N_1-M} \right) K^{(m)}(\theta_{\vec{x}+i}, \theta_{\vec{x}}) +
$$

$$
\left. + \left[ \frac{I_{2m+1}}{I_1} \right]^M \left( \left[ \frac{I_{2m+2}}{I_1} \right]^{N_1-M} + \left[ \frac{I_{2m}}{I_1} \right]^{N_1-M} \right) \left( \cos [(2m+1)\theta_{\vec{x}+i}] \cos [(2m+1)\theta_{\vec{x}}] \right) \right\}. \tag{66}
$$

In the above formula, the argument of all the Bessel functions is the timelike-coupling $\beta_t$.

The explicit expression (66) of the matrix $\tilde{C}^{(M)}$ is quite complicated; but there are some non-trivial consistency checks that we can perform on it. First, it is not difficult to see that if we set $M = 0$ in the formulae (66), namely if we let the two fundamental plaquettes coincide at the same time position, we correctly reproduce
the results obtained in section 3.3.1 for the $\chi_1 \chi_1$ term in the adjoint plaquette:

\[
C^{(M)}_0(\theta_{\bar{x}}, \theta_{\bar{x}+i}) \rightarrow \infty \rightarrow C_0(\theta_{\bar{x}}, \theta_{\bar{x}+i})
\]

\[
C^{(M)}_1(\theta_{\bar{x}}, \theta_{\bar{x}+i}) \rightarrow \infty \rightarrow C_1(\theta_{\bar{x}}, \theta_{\bar{x}+i})
\]

\[
B^{(M)}_t(\theta_{\bar{x}}, \theta_{\bar{x}+i}) \rightarrow \infty \rightarrow 0.
\] (67)

Second, we can rearrange, through some algebraic manipulations, the expressions (66) in such a way that the symmetry under the exchange of $M$ with $N_t - M$:

\[
\tilde{C}^{(M)}_{\alpha \beta}(\theta_{\bar{x}}, \theta_{\bar{x}+i})^* = (P^\dagger_{x+i})_{\beta \alpha} C^{(N_t-M)}_{\alpha \beta}(\theta_{\bar{x}}, \theta_{\bar{x}+i}),
\] (68)

which comes from the analogous property (61) of $\vec{c}^{(m,n)}_{\alpha \beta}$, becomes manifest. It is possible indeed to write $\tilde{C}^{(M)}$ as a matrix in the following form:

\[
\tilde{C}^{(M)} = \frac{1}{4 \sin \theta \sin \theta + i} \left\{ \sum_{m=\frac{1}{2}}^{\infty} \left( \begin{array}{cc} e^{-i(2m+1)\theta_+} & -e^{-i(2m+1)\theta_-} \\ -e^{-i(2m+1)\theta_+} & e^{-i(2m+1)\theta_-} \end{array} \right) \left[ \begin{array}{c} I_{2m+1} \\ I_{I_1} \end{array} \right]^M \right\}
\]

\[
+ K^{(m)} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \left[ \begin{array}{c} I_{2m+1} \\ I_{I_1} \end{array} \right]^{N_t-M} \left( \begin{array}{c} I_{2m+2} \\ I_{I_1} \end{array} \right)^M \right) \right\}
\]

\[
+ K^{(m)} \left( \begin{array}{cc} e^{i\theta_+} & -e^{-i\theta_-} \\ -e^{i\theta_+} & e^{-i\theta_-} \end{array} \right) \left[ \begin{array}{c} I_{2m+1} \\ I_{I_1} \end{array} \right]^{M} \left( \begin{array}{c} I_{2m+2} \\ I_{I_1} \end{array} \right)^{N_t-M} \right) \right\}
\]

\[
+ K^{(m)} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \left[ \begin{array}{c} I_{2m+1} \\ I_{I_1} \end{array} \right]^{M} \right) \right\}
\]

Finally, having determined the matrix $\tilde{C}^{(M)}(\theta_{\bar{x}+i}, \theta_{\bar{x}})$ we can insert its expression into eq. (51), as we discussed above, to get the contribution of a pair of fundamental plaquettes to the effective action. The result of the trace is cumbersome and not very illuminating, so we shall not write it here; it will however be used in the mean field analysis of the following sections.

### 3.4 The effective action up to $O(\beta_8^2)$

Let us summarize here our results by reporting the form of the effective action for the Polyakov loops determined in the previous sections. To this action we will in the later sections apply standard and improved mean field techniques in order to extract the value of the critical coupling. We have:

\[
\exp S_{\text{eff}} = \exp(S_0 + S_1)
\] (70)

where

\[
\exp S_0 = \sum_{x,i} C_0(\theta_{\bar{x}}, \theta_{\bar{x}+i})
\] (71)
and

\[
S_1 = N_t \sum_{x,i<j} \left\{ 3 \frac{I_3(\beta_s)}{I_1(\beta_s)} \frac{C_1(\theta_{\vec{x}}, \theta_{\vec{x}+x})C_1(\theta_{\vec{x}+x}, \theta_{\vec{x}+x+i})C_1(\theta_{\vec{x}+x+i}, \theta_{\vec{x}+i})C_1(\theta_{\vec{x}+i}, \theta_{\vec{x}})}{C_0(\theta_{\vec{x}}, \theta_{\vec{x}+x})C_0(\theta_{\vec{x}+x}, \theta_{\vec{x}+x+i})C_0(\theta_{\vec{x}+x+i}, \theta_{\vec{x}+i})C_0(\theta_{\vec{x}+i}, \theta_{\vec{x}})} + 4 \left( \frac{I_2(\beta_s)}{I_1(\beta_s)} \right)^2 2^{N_t-1} \frac{\text{Tr} \left[ \bar{C}^{(M)}(\theta_{\vec{x}}, \theta_{\vec{x}+x}) \ldots \bar{C}^{(M)}(\theta_{\vec{x}+i}, \theta_{\vec{x}}) \right]}{C_0(\theta_{\vec{x}}, \theta_{\vec{x}+x}) \ldots C_0(\theta_{\vec{x}+i}, \theta_{\vec{x}})} \right\}.
\] (72)

The quantities in the above equations are defined as follows: \(C_0(\theta_{\vec{x}}, \theta_{\vec{x}+x}) \equiv C^0_{x,i}\) is given by eq. (23), \(C_1(\theta_{\vec{x}}, \theta_{\vec{x}+x})\) by eq. (14) and the matrix \(\bar{C}^{(M)}(\theta_{\vec{x}}, \theta_{\vec{x}+x})\) by equations (63) and (66) or, alternatively, by eq. (69).

\(S_0\) is the \(O(\beta_s^0)\) effective action. It contains a sum over the links \(\vec{x}, i\) in the \(d\)-dimensional spatial lattice and each term of the sum represents a nearest neighbour interaction between Polyakov loops. \(S_1\) describes the effect at \(O(\beta_s^2)\) of the space-like plaquettes, and it is given by a sum over the plaquettes \(\vec{x}, i < j\) in the \(d\)-dimensional lattice. Each term represents the interaction among the four Polyakov loops at the vertices of each plaquette. Notice that in eqs. (71,72) the contribution of the space-like plaquettes has been exponentiated, which is correct at the order \(O(\beta_s^2)\).

### 3.5 The \(N_t = 1\) case and the Kazakov-Migdal model

The interesting feature of the \(N_t = 1\) case is that the model that we are studying becomes a particular case of the Kazakov-Migdal model [10]. This connection was already noticed in [11,12] and was the origin of our previous analysis in the \(N \to \infty\) limit [9]. All the integrals that we have described in the previous sections can be directly evaluated in the \(N_t = 1\) case as particular instances of a nontrivial generalization of the so called Itzykson-Zuber integral [13], evaluated in [14]. This alternative derivation in the \(N_t = 1\) case provides another check of our computations (at least for the contribution of the adjoint plaquettes, as this is the only \(O(\beta_s^2)\) contribution when \(N_t = 1\)).

The basic integral used in the computation of \(S_0\) [see eq. (23)] coincides, when \(N_t = 1\), with the link integral

\[
\int dU_{\vec{x},i} \exp \left\{ \frac{\beta_t}{2} \text{Tr} \left[ V(\vec{x}) U_{\vec{x},i} V^\dagger(\vec{x} + i) U^\dagger_{\vec{x},i} \right] \right\} = e^{\beta_t \cos(\theta_{\vec{x}} - \theta_{\vec{x}+i})} - e^{\beta_t \cos(\theta_{\vec{x}} + \theta_{\vec{x}+i})} \frac{2 \beta_t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})}{2 \beta_t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})},
\] (73)

which was non-perturbatively computed in [14]. We can compare this expression with our general result (24). We find that, if \(N_t = 1\), the character expansion contained in eq. (24) can be summed exactly. In fact by inserting the explicit form (13) of the characters into eq. (24) and then using the relation

\[
2 \sin[(2r + 1)\theta_{\vec{x}}] \sin[(2r + 1)\theta_{\vec{x}+i}] = \cos[(2r + 1)(\theta_{\vec{x}} - \theta_{\vec{x}+i})] - \cos[(2r + 1)(\theta_{\vec{x}} + \theta_{\vec{x}+i})]
\] (74)
and the well known expansion
\[ e^{\beta \cos \theta} = I_0(\beta) + 2 \sum_{k=1}^{\infty} I_k(\beta) \cos(k\theta) \]  
(75)

it is easy to obtain:
\[ \exp(S_0) = \prod_{\vec{x}, i} \frac{e^{\beta t \cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})} - e^{\beta t \cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})}}{4I_1(\beta t) \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} \].  
(76)

This expression coincides with eq.(73), except for the irrelevant overall factor \( \frac{2t_i(\beta t)}{\beta t} \) [see the remark after eq.(79)].

In the \( N_t = 1 \) case, the first non-trivial contributions from the space-like plaquettes, i.e. those coming from an adjoint plaquette, can be extracted from the definition of the correlators given in [14]:
\[ \langle (U_{\vec{x}, i})_{\mu, \nu} (U_{\vec{x}+i})_{\mu, \nu} \rangle = \frac{\int dU_{\vec{x}, i} \exp \left\{ \frac{\beta t}{2} \text{Tr} \left( P_{\vec{x}} U_{\vec{x}, i} \rho V_{\vec{x}+i} U_{\vec{x}, i}^\dagger \right) \right\} (U_{\vec{x}, i})_{\mu, \nu} (U_{\vec{x}+i})_{\mu, \nu} \pi_{ij}}{\int dU_{\vec{x}, i} \exp \left\{ \frac{\beta t}{2} \text{Tr} \left( P_{\vec{x}} U_{\vec{x}, i} \rho V_{\vec{x}+i} U_{\vec{x}, i}^\dagger \right) \right\}} \]  
(77)

These correlators correspond to the \( B_{\mu\nu\rho\sigma} \) of eq.(30), divided by \( C_{\vec{x}, i}^0 \); they must be therefore diagonal, namely:
\[ \langle (U_{\vec{x}, i})_{\mu, \nu} (U_{\vec{x}+i})_{\mu, \nu} \rangle = \delta_{\mu}^\sigma \delta_{\nu}^\rho \hat{C}_{\mu, \nu}(\vec{x}; i) \]  
(78)

where the \( \hat{C}_{\mu, \nu}(\vec{x}; i) \) are equivalent, apart from the different normalization, to our \( C_{\mu, \nu} \) matrix elements. These correlators were calculated in [14]:
\[ \hat{C}_{1,1}(\vec{x}; i) = \hat{C}_{2,2}(\vec{x}; i) = \frac{2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i}) - (1 - e^{-2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})})}{(1 - e^{-2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})}) (2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i}))} \]
\[ \hat{C}_{1,2}(\vec{x}; i) = \hat{C}_{2,1}(\vec{x}; i) = \frac{1 - e^{-2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} (1 + 2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i}))}{(1 - e^{-2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})}) (2\beta t \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i}))} \]  
(79)

Again we can compare this result with our results for generic \( N_t \), expressed in section 3.3.1 in terms of character expansions. In the \( N_t = 1 \) case the sum over the representations can be performed exactly, and a closed expression for the \( C_{\alpha, \beta} \) coefficients can be obtained. This can be done by using the identity:
\[ I(\beta)_{n-1} - I(\beta)_{n+1} = 2n I(\beta)_n \]  
(80)

and eq.(73). The result is:
\[ C_{11}(N_t = 1) = \frac{e^{\beta t \cos(\theta_{\vec{x}} - \theta_{\vec{x}+i})} - e^{\beta t \cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})} - e^{\beta t \cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})}}{4I_1(\beta t) \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} \]
\[ C_{12}(N_t = 1) = \frac{e^{\beta t \cos(\theta_{\vec{x}} - \theta_{\vec{x}+i})} - e^{\beta t \cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})} - e^{\beta t \cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})}}{8I_1(\beta t) \sin^2(\theta_{\vec{x}}) \sin^2(\theta_{\vec{x}+i})} \]
\[ C_{12}(N_t = 1) = \frac{e^{\beta t \cos(\theta_{\vec{x}} - \theta_{\vec{x}+i})} - e^{\beta t \cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})} - e^{\beta t \cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})}}{4I_1(\beta t) \sin(\theta_{\vec{x}}) \sin(\theta_{\vec{x}+i})} \]  
(81)
which perfectly matches eq. (79), if we take into account the normalization by $C_{x,i}^0$ and the $\frac{2h_i(\beta t)}{\beta t}$ factor, as in eq. (76).

4 Mean Field computation of the critical coupling

The effective action obtained in the previous section describes a $d$ dimensional spin model with complicated interactions and cannot be solved exactly. However several features of the model can be figured out rather easily. First, it can be seen that all the interaction terms are even functions of the variables $\theta$, so that the model has a global $Z_2$ symmetry, in agreement with the Svetitsky-Yaffe conjecture. Here and in the following we shall assume to have fixed the asymmetry ratio $\rho$, and we shall study the phase diagram of the model in terms of only one coupling (for instance: $\beta_t$). However, as already anticipated, the $\rho$ dependence of the critical temperature will play a major role. For large values of the coupling $\beta_t$ the $Z_2$ symmetry is spontaneously broken, hence we expect a phase transition for some critical value of the coupling. The simplest way to estimate this critical coupling is certainly the mean field approximation. As it is well known this method gives in general rather rough estimates of the critical temperature, and much more refined techniques have been elaborated in these last years. In the section 4.4 we shall comment on this point in more detail and apply an improved version of the mean field approximation (Bethe approximation) to the $N_t = 1$ case. This test will make us confident of the the fact that the errors which we make by keeping a plain mean field approximation are of the order of the 10%-15%. Nevertheless, in this section we shall restrict ourselves to the plain mean field approximation. The reason is twofold: first, a relevant part of our results (and in particular the agreement with the weak coupling calculation of Karsch) rely on differences of critical couplings, and these differences are only slightly affected by the mean field approximation, which essentially affects the data in the sense of giving an overall systematic error. Second, as $N_t$ increases the error that we make by using the mean field approximation becomes smaller than that due to the truncation of the $\beta_s$ expansion. The huge amount of complexity needed to implement more refined approximations would be justified, and would become meaningful, only if higher orders in the expansion were taken into account. This could well be done in principle but, as we shall see, the results we obtain by keeping only the mean field result are already very interesting, since they clearly show the expected trend. Let us also mention that there is another situation in which more precise methods to estimate the critical coupling are justified, namely the $N_t = 1$ case, in which the diagrammatic entropy of higher order contributions is highly constrained and we can expect that the $\beta^2$ order alone gives already a very good approximation. This will be the subject of sect. 4.4.
4.1 Theoretical expectations.

The ultimate test of the correctness of any lattice regularization is that, as the continuum limit is approached, the various dimensional quantities in which one is interested follow the correct scaling behaviour. This scaling behaviour can be easily obtained by writing explicitly the dependence on the lattice spacing $a$ of the relevant (dimensional) observables. Let us study first the symmetric case $\beta_s = \beta_t \equiv \beta \equiv \frac{4}{g^2}$.

Then the dependence of the lattice spacing on $\beta$ is known in the continuum limit in form of the renormalization group equation:

$$a \Lambda_L = (b_0 g^2)^{-\frac{b_1}{2b_0}} \exp \left( -\frac{1}{2b_0 g^2} \right),$$

(82)

where $\Lambda_L$ is the lattice scale parameter (in units of which we must measure any dimensional quantity on the lattice) and $b_0, b_1$ are the first two coefficients of the Callan-Symanzik equation:

$$b_0 = \frac{11N}{48\pi^2}, \quad b_1 = \frac{34}{3} \left( \frac{N}{16\pi^2} \right)^2,$$

(83)

with $N = 2$ in our case.

Here and in the following we have fixed the spacetime dimensions to be $(3+1)$. This is a particularly important remark since it is only in $(3+1)$ dimensions that the coupling constant $\beta$ is adimensional and the renormalization group equations have this peculiar exponential behaviour.

Plugging eq.(82) into the definition of critical temperature:

$$T_c = \frac{1}{a N_t}$$

(84)

we find:

$$\frac{T_c}{\Lambda_L} = \frac{1}{N_t} \left( \frac{6\pi^2 \beta}{11} \right)^{-\frac{b_1}{2b_0}} \exp \left( \frac{3\pi^2 \beta}{11} \right).$$

(85)

If the continuum limit is correctly reached then the ratio $T_c/\Lambda_L$ should approach for large enough values of $\beta$ (hence, in our case, also for large values of $N_t$) a constant value. Plugging this constant into eq. (85) we immediately recover the well known (approximate) logarithmic growth of $T_c$ as a function of $N_t$, which is the typical signature of the correct continuum limit behaviour of the deconfinement temperature in a $(3+1)$ dimensional LGT. It is important at this point to stress that this logarithmic behaviour is a non trivial requirement for any effective theory approach to the deconfinement transition. For instance, in the zeroth order approximation of the effective action, the effective coupling constant is written as a combination of modified Bessel functions raised to the $N_t$ power (see eq.(24)). Since the large $\beta$ asymptotic behaviour of the Bessel function $I_n(\beta)$ is

$$I_n(\beta) \sim \frac{e^\beta}{\sqrt{2\pi\beta}} \left[ 1 - \frac{4n^2 - 1}{8\beta} + \cdots \right]$$

(86)
then the effective couplings scales as:

\[
\left[ \frac{I_n(\beta)}{I_1(\beta)} \right]^{N_t} \sim 1 - \frac{(n^2 - 1)N_t}{2\beta} + \cdots,
\]

which implies a linear scaling of \( \beta_c \) as a function of \( N_t \).

The lack of logarithmic scaling in the zeroth order approximation is one of the main reasons which motivated us to look at higher order corrections in the effective action.

### 4.2 Asymmetric Lattices

The fact that we have in general asymmetric couplings \( \beta_s \neq \beta_t \) adds some further complication to the previous discussion, but has some very important consequences. For each value of \( \rho \) we have a new independent regularization scheme, with an independent renormalization group equation. This means that, if we define the coupling \( g \) according to eq.(3) as

\[
\frac{a}{\rho} = \sqrt{\beta_s \beta_t},
\]

we must substitute eq.(82) with:

\[
a\Lambda(\rho) = \left( b_0 g^2 \right)^{-\frac{b_1}{2b_0 g^2}} \exp \left( -\frac{1}{2b_0 g^2} \right).
\]

In other words we have that now the \( \Lambda \) parameter is also a function of \( \rho \). The last step, in order to obtain meaningful results in the continuum limit is then to relate the scale \( \Lambda(\rho) \) to the \( \Lambda \) parameter of the symmetric lattice regularization. This problem was studied in [7] where the ratio \( \Lambda(\rho) / \Lambda_L \) (with \( \Lambda_L \equiv \Lambda(1) \) denoting the scale of the symmetric case \( \beta_s = \beta_t \) discussed above) was evaluated explicitly at one loop. It was found to be a universal function of \( \rho \):

\[
\frac{\Lambda(\rho)}{\Lambda_L} = \exp \left( -\frac{c_\sigma(\rho) + c_\tau(\rho)}{4b_0} \right),
\]

where \( c_\sigma \) and \( c_\tau \) are the functions already introduced in eqs (7) and (8). It is easy to see that this result is completely equivalent to the shift in \( \beta \) of eqs (7) and (8). It gives us a tool to better understand eq.(7), which simply encodes the effect of the quantum fluctuations at one loop.

This result is particularly interesting for our purposes, since it allows to extend our analysis also to asymmetric lattices, taking into account also the quantum effects. Let us discuss in detail this point.

Consider an asymmetric regularization, with \( \beta_t = \rho^2 \beta_s \), \( \rho > 1 \), defined on a lattice with temporal extension \( \tilde{N}_t \). By using eq.(3) and (8) we see that this regularization is (classically) equivalent to that on a symmetric lattice with \( \beta \equiv \tilde{\beta}_t = \beta_s \rho \) and with temporal extension \( N_t = \frac{\tilde{N}_t}{\rho} \). Hence \( \rho \) must be a rational number; we shall choose in general \( \rho \) to be an integer number, so that \( \tilde{N}_t \) will be a multiple of \( N_t \).
At the quantum level we can still show the equivalence of the symmetric and asymmetric lattice regularizations provided we modify the previous relations according to eq. (7) and (8). By using the explicit knowledge of the $\rho \to \infty$ expansion of $c_\sigma$ and $c_\tau$, we obtain, for large enough values of $\rho$, the following scaling behaviour:

$$\beta_{t,c}(\rho) = (\beta_c + \alpha^0_t + \alpha^1_t)$$

(90)

where $\beta_{t,c}(\rho)$ is the critical coupling on the asymmetric lattice and $\beta_c$ the critical coupling on the equivalent symmetric lattice. The numerical values of $\alpha^0_t$ and $\alpha^1_t$ are reported in sect. 2, following eq. (9). Higher order corrections to eq. (90) vanish as $\rho \to \infty$.

The limit $\rho \to \infty$ is particularly interesting because some relevant simplifications occur in that limit in the effective action. Let us consider first the contributions of the timelike plaquettes in the zeroth order approximation given in eq. (24). We have now to perform in (24) the following substitution:

$$N_t \to \tilde{N}_t = \rho N_t, \quad \beta_t \to \beta_t(\rho)$$

(91)

with $\beta_t(\rho)$ given in eq. (91), and finally take the limit $\rho \to \infty$. By using the asymptotic expansion of the Bessel functions (86) one obtains in that limit that the effective couplings at the r.h.s. of eq. (24) become

$$\left[\frac{I_{2j+1}(\beta(\rho))}{I_1(\beta(\rho))}\right]^{\tilde{N}_t} \rho \to \infty \exp\left(-\frac{2j(j+1)N_t}{\beta + \alpha^0_t}\right),$$

(92)

where $\beta$ is the coupling of the corresponding symmetric lattice. One can easily recognize the quadratic Casimir of the representation $j$ at the exponent, and one immediately realizes that the action for the time-like plaquettes becomes the Heat-Kernel action in the $\rho \to \infty$ limit, namely in the hamiltonian limit.

The couplings of the spacelike plaquette in the adjoint representation and of the pair of plaquettes in the fundamental representation are respectively proportional to $3\tilde{N}_t I_{1}(\beta(\rho))$ and $4\tilde{N}_t^2 I_{1}(\beta(\rho))^2$ (see eq. (72)). We replace $\tilde{N}_t$ as in (94) and $\beta_s(\rho)$ by $\beta + \alpha^0_s$ and take the limit $\rho \to \infty$. The contribution of the adjoint representation is of order $1/\rho$ and hence it vanishes in this limit, due essentially to its zero measure in an infinite lattice. On the contrary the coupling of the pair of plaquettes in the fundamental representation is finite in the limit $\rho \to \infty$ and its limiting value is $N_t^2(\beta + \alpha^0_s)^2/4$. Notice that all powers of $\beta_s(\rho)$ higher than $\beta_s(\rho)^2$ in the power expansion of the Bessel functions give a vanishing contribution in the limit $\rho \to \infty$. This means that, as already remarked at the beginning of sect. 3, the character expansion and the power expansion coincide in this limit, thus removing any possible ambiguity. Notice also that in spite of the vanishing of $\beta_s(\rho)$ in the asymmetric limit the effective expansion parameter $N_t(\beta + \alpha^0_s)$ is never very small and it does indeed increase as we approach the continuum limit.
Tab. I  The critical coupling \( \beta_{t,c} \) as a function of \( \rho \). In the second column we have reported the temporal extension \( \tilde{N}_t = \rho N_t \) of the asymmetric lattices that we used. In the fourth column we have reported the values of \( \beta_{s,c} \equiv \beta_{t,c}/\rho^2 \). In the last two columns we have reported the values of \( \gamma \) and \( \delta \) (see text for explanation). The values of \( \delta \) reported in Tab. I are obtained by applying the definition (93) with \( \rho^2 = \rho \) and \( \rho_1 \) equal to the value of \( \rho \) in the previous row.

| \( \rho \) | \( \tilde{N}_t \) | \( \beta_{t,c} \) | \( \beta_{s,c} \) | \( \delta \) | \( \gamma \) |
|------|-------|-------|-------|-------|-------|
| 1    | 6     | 3.216 | 3.216 |       |       |
| 2    | 12    | 5.882 | 1.471 | 2.666 | 0.55  |
| 3    | 18    | 8.576 | 0.953 | 2.694 | 0.49  |
| 4    | 24    | 11.282| 0.705 | 2.706 | 0.46  |
| 5    | 30    | 13.993| 0.560 | 2.711 | 0.45  |
| 6    | 36    | 16.707| 0.464 | 2.714 | 0.42  |
| 7    | 42    | 19.422| 0.396 | 2.715 | 0.42  |
| 8    | 48    | 22.138| 0.346 | 2.716 | 0.41  |
| 10   | 60    | 27.571| 0.276 | 2.717 | 0.39  |
| 20   | 120   | 54.753| 0.137 | 2.718 | 0.39  |

4.3 Results.

In order to extract reliable estimates for the critical couplings, we must first discuss the behaviour of our results as functions of the asymmetry parameter \( \rho \).

4.3.1 \( \rho \) dependence of the results

It is very interesting to study the \( \rho \) dependence of our results, to see if eq.(90) is fulfilled. In Tab. I we have reported, as an example, the \( \rho \) behaviour of a set of asymmetric regularizations which are all equivalent to the symmetric \( N_t = 6 \) case. For each pair of (subsequent) values of \( \beta_{t,c}(\rho) \) we have constructed the two quantities

\[
\delta = \frac{\beta_{t,c}(\rho_2) - \beta_{t,c}(\rho_1)}{\rho_2 - \rho_1} \quad (\rho_2 > \rho_1) \tag{93}
\]

\[
\gamma = \beta_{t,c}(\rho_2) - \delta \rho_2 \tag{94}
\]

which, if eq.(93) is fulfilled, should provide a good estimate of \( \beta_c + \alpha_t^0 \) and \( \alpha_t^1 \) respectively. Their values together with those of \( \beta_{t,c} \) are reported in Tab. I.

It can be seen that the data follow very well the expected law. In particular it is clearly visible the \( 1/\rho \) quantum correction, which is definitely different from zero...
and smoothly approaches for large values of $\rho$ the value $\gamma \sim 0.39$, which is not too far from the expected value $\alpha^1_{t} = 0.5$.

The values of $\beta_{s,c} \equiv \beta_{t,c}/\rho^2$, that we have reported in the fourth column of Tab. I, give an idea of the reliability of our strong coupling expansion in $\beta_s$.

We repeated the same analysis for all the values of $N_t$ for which Montecarlo data are known (the MC data are reported in Tab. IV). Our results are collected in Tab. II where we have reported the asymptotic (large $\rho$) values of $\gamma$ and $\delta$ as well as of another quantity $\epsilon$ that should provide us with an estimate of $\alpha^0_{t}$. This is defined as $\epsilon = \delta - \beta_c$, where $\beta_c$ is the critical coupling obtained with a symmetric lattice of size $N_t$ by keeping strictly only the terms of order $\beta^2_{s}$ in the character expansion. In other words, in computing $\beta_s$ we neglect all contributions that vanish in the limit $\rho \to \infty$, thus keeping the same contributions in the symmetric and asymmetric lattice. Tab. II shows that, in the range $N_t = 2 - 16$, the agreement with the theoretically expected values of $\alpha^0_{t}$ and $\alpha^1_{t}$ is really remarkable. Let us stress again that this agreement is highly non trivial since $\alpha^0_{t}$ and $\alpha^1_{t}$ were obtained with a weak coupling calculation, while our effective action is the result of a strong coupling expansion. The reason of this success is very likely related to the fact that we have been able to sum to all orders in $\beta_t$ the timelike contribution of the effective action.

### 4.3.2 Scaling behaviour.

The agreement between the $\rho$ dependence of our mean field results and the theoretical expectations allows us to be confident on their consistency, also at the quantum level. So in order to extract our best estimate for the critical coupling we

| $N_t$ | $\delta$ | $\epsilon$ | $\gamma$ |
|---|---|---|---|
| 2  | 1.554 | -0.184 | 0.414 |
| 3  | 1.971 | -0.210 | 0.375 |
| 4  | 2.259 | -0.221 | 0.373 |
| 5  | 2.500 | -0.235 | 0.372 |
| 6  | 2.718 | -0.249 | 0.389 |
| 8  | 3.114 | -0.271 | 0.413 |
| 16 | 4.443 | -0.327 | 0.508 |
|   | -0.27192 |   | 0.50 |

**Tab. II** Values of $\delta$, $\epsilon$ and $\gamma$ as functions of $N_t$ (see the text for the definitions of these three quantities). $\epsilon$ and $\gamma$ are estimators of $\alpha^0_{t}$ and $\alpha^1_{t}$ whose theoretical values are reported, for comparison, in the last row of the table.
Values of the critical coupling $\beta_c$ are plotted for different values of the number of time-like links $N_t$. Results obtained with Montecarlo simulations, which are denoted by *, are compared with those obtained with our mean field analysis: $\triangle$ represents the data for $\beta_c|_0$, the critical coupling in the zeroth order approximation and $\square$ the data for $\beta_c|_{1, \rho \to \infty}$, the critical coupling including the effect of the space-like plaquettes at the lowest non trivial order, calculated in the limit $\rho \to \infty$ and reduced to the value $\rho = 1$ as described in the text.

The values obtained in this way for the critical couplings are reported in Tab. III, and plotted in Fig. 2, where they are also compared with the Montecarlo results (extracted from [15]), which are reported in Tab. IV. It is impressive to see how the contribution of the space-like plaquettes, although taken into account at the order $\beta_s^2$ only, improves the agreement of the results of the mean field method with the ones of the Montecarlo simulations. To the order $\beta_s^2$ the mean field method gives results which are in reasonably good agreement with the Montecarlo for $N_t$’s as high as 5, displaying in the range $1 \leq N_t \leq 5$ a behaviour which is compatible with the logarithmic scaling predicted by the renormalization group. In contrast the model with only time-like plaquettes shows from $N_t = 1$ a linear scaling behaviour in con-

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Let us notice, as a side remark, that the difference between the values of $\beta_c$ obtained in this way and those which one would obtain with the naive procedure of choosing right from the beginning a symmetric lattice is not very large, but nevertheless it is not negligible and could become very important if further orders in the $\beta_s$ expansion were to be added to the effective action.
| $N_t$ | $\beta_c|_0$ | $\beta_c|_{\rho \to \infty}$ |
|------|------------|----------------------------|
| 2    | 1.957      | 1.723                      |
| 3    | 2.957      | 2.089                      |
| 4    | 3.876      | 2.366                      |
| 5    | 4.763      | 2.606                      |
| 6    | 5.636      | 2.826                      |
| 8    | 7.369      | 3.227                      |
| 16   | 14.262     | 4.572                      |

Tab. III  The critical coupling $\beta_c$ as a function of the lattice size $N_t$ in the $t$ direction. In the second column the values obtained with the zeroth order approximation and in the third column those obtained at the order $\beta_s^2$ by rescaling the values obtained in highly asymmetric lattices $\rho \to \infty$ as explained in the text.

It must be stressed however that the almost perfect agreement of our results with the Montecarlo simulations for small $N_t$’s (especially for $N_t$ in the range $3-4$) is most probably only apparent, resulting from a compensation between an expected 10-15% systematic error due to the mean field approximation and the effects of higher orders contributions.

4.4 $N_t = 1$ again: improving the mean field approximation.

As we remarked in sect. 3.5, the most interesting feature of the $N_t = 1$ case is that, due to the remarkable simplifications which occur in this case, the character expansion of the effective action can be resummed exactly. This allows a much simpler implementation of improved versions of the mean field approximation, and we decided to use this case as a laboratory to test these improved estimators and above all to have a hint of the magnitude of the systematic errors involved in the plain mean field approximation that we used in the previous section. Notice also that for this particular $N_t = 1$ case it does exist a very precise Montecarlo estimate of the critical coupling, namely $\beta_c = 0.8730(2)$ [16], and we shall use this value as a reference point to compare our predictions.

Let us notice, as a preliminary remark, that in the $N_t = 1$ case the $\beta_s^2$ contribution is given by the adjoint plaquette term only, since there is no room to locate a pair of plaquettes in the fundamental representation. This fact has two consequences: first, the magnitude of the correction to the critical temperature due to the spacelike contribution is much smaller than in the $N_t > 1$ case; second, it has the opposite sign, namely $\beta_{t,c}$ increases as a consequence of the spacelike term.
The simplest way to improve the mean field approximation is to consider larger and larger clusters of spins (see Fig. 3). It can be shown that this modification indeed allows a more and more precise determination of the critical coupling. The first improvement (step 2 in the notation of Fig. 3) is also known as the Bethe approximation [17]. In Tab. V we report the results of our analysis. It is easy to see that both with and without the spacelike plaquettes the plain mean field approximation is affected by an error which ranges from 10% to 15% (depending on the role which is played by the spacelike contribution in the Montecarlo estimate), and that almost half of this gap is filled by the Bethe approximation.

We will show in a forthcoming publication [18], that the results of next order approximation (step 3 in the notation of Fig. 3) differ from the exact result by only a few per cent. Notice however that the step 3 approximation is very hard to handle and requires several technical manipulations. In this forthcoming paper, we will also show that a result which is very close to the Montecarlo one given in last column of Tab. V can be obtained by comparing the different steps of Bethe approximations in the present model with the corresponding ones in the Ising model, whose critical coupling is known with high precision.

An alternative approach one can follow is to identify explicitly the Ising model which is hidden in the effective action (following the Svetitsky–Yaffe analysis) and then use again our knowledge of the critical coupling of the three dimensional Ising model. This approach has been developed for the $N_t = 1$ case in [3] and it will also be fully exploited in [18].

| $N_t$ | $\beta_c$     | $T_c/\Lambda$ |
|-------|---------------|---------------|
| 2     | 1.8800(30)    | 29.7(2)       |
| 3     | 2.1768(30)    | 41.4(3)       |
| 4     | 2.2986(6)     | 42.1(1)       |
| 5     | 2.3726(45)    | 40.6(5)       |
| 6     | 2.4265(30)    | 38.7(3)       |
| 8     | 2.5115(40)    | 36.0(4)       |
| 16    | 2.7395(100)   | 32.0(8)       |

Tab. IV The critical coupling $\beta_c$ and the corresponding deconfinement temperature $T_c/\Lambda$ as a function of the lattice size in the $t$ direction, $N_t$, in the (3+1) dimensional $SU(2)$ LGT. The data are taken from [13].
Fig. 3 Progressively more precise Bethe-like approximations

|         | step 1 | step 2 | MonteCarlo |
|---------|--------|--------|------------|
|          | 0.7702 | 0.8139 |            |
|          | 0.7705 | 0.8145 | 0.8730(2)  |

Tab. V The critical coupling $\beta_c$ for the action $S_0$ (first row) and $S_0 + S_1$ (second row) in the mean field approximation (step 1) and the Bethe approximation (step 2). In the last column the Montecarlo result of ref. [10] (including full contribution of space-like plaquettes).

5 Conclusions

The main result of our paper is the construction of the effective action for the Polyakov loops at the first non trivial order in the spacelike coupling, which is exact to all orders in the timelike coupling.

Extracting the critical temperature from this effective action, and comparing our results with those of the Montecarlo simulation, we have seen that the effective action describes rather well (even within the mean field approximation) the full theory up to a lattice size in the compactified time direction of $N_t \sim 5$. This is an impressive improvement with respect to the previous studies, in which the spacelike contributions were neglected and which were constrained to $N_t = 1$. We see in principle no obstruction to extend our analysis to higher orders and to reach a better and better agreement with the renormalization group expectations. Such a result would obviously be very interesting, since it would be the first time that one can reach the scaling regime of a dimensional quantity (as the critical temperature) by using only analytical tools. From this point of view the critical temperature
seems indeed in a better position with respect for instance to the string tension, for which, due to the roughening transition, a strong coupling approach has very few chances to reach the scaling region. It is also remarkable the agreement between our results, based essentially on a strong coupling expansion, and the weak coupling calculations of Karsch, regarding the consistency of the theory with respect to lattice deformations described by the asymmetry parameter $\rho$.

Moreover our approach can be straightforwardly extended to the SU(3) gauge model, which is obviously more interesting from a phenomenological point of view.

Besides these obvious remarks there are two more reasons of interest of our result, which we think should deserve some attention. First, it was noticed in [15] that the ratio $T_c/\sqrt{\sigma}$ shows a very precocious scaling, and is essentially stable already for lattices as small as $N_t = 4$. This value lies inside the region where we have seen we can trust our expansion. Therefore, if one were able to extract the string tension in our framework, it would be possible to reach the continuum limit value of $T_c/\sqrt{\sigma}$ already within our (relatively simple) effective action.

A second possible interesting application is in the study of LGT with suitable generalizations of the Wilson action. In particular, much interest has been recently devoted to the so called fundamental-adjoint SU(2) LGT (see [19] and references therein) in which a term proportional to the plaquette in the adjoint representation is added to the ordinary Wilson action. This introduces a new coupling $\beta_A$. In [19] the behaviour of the deconfinement transition in the extended coupling plane ($\beta, \beta_A$) was studied with a strong coupling effective action truncated to the first order and for $N_t = 2$ and $N_t = 4$, hence in a region in which our approach seems to have a good behaviour. Since our action is written in terms of a character expansion, it should be rather straightforward to generalize it to include a coupling $\beta_A$ different from zero.

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