

1. Introduction

In mathematics, the Ornstein-Uhlenbeck process (named after Leonard Ornstein and George Eugene Uhlenbeck joint celebrated work [13]), is a Gauss - Markov stochastic process (see, for example, [4], [20]) that describes the velocity of a massive Brownian particle under the influence of friction. Over time, this process tends to drift towards its long-term mean: such a process is called mean-reverting (in this context, see, for example, [17], [16]).

The process can be considered to be a modification of the random walk in continuous time, or Wiener process, in which the properties of the process have been changed so that there is a tendency of the walk to move back towards a central location, with a greater attraction when the process is further away from the center. The Ornstein-Uhlenbeck process can also be considered as the continuous-time analogue of the discrete-time process.

In recent years, however, the Ornstein-Uhlenbeck process has appeared in finance as a model of the volatility of the underlying asset price process (see, for example, [14], [15]).

Note that the Ornstein-Uhlenbeck process, $x_t$, satisfies the following stochastic differential equation:

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t$$  \hspace{1cm} (1.1)

where $\theta > 0$, $\mu \in R$ and $\sigma > 0$ are parameters and $W_t$ denotes the Wiener process.

The solution of the stochastic differential equation (1.1) has the following form

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s,$$  \hspace{1cm} (2.2)

where $x_0$ is assumed to be constant.

The parameters in (2.2) have the following sense:

(i) $\mu$ represents the equilibrium or mean value supported by fundamentals (in other words, the central location);

(ii) $\sigma$ is the degree of volatility around it caused by shocks;

(iii) $\theta$ is the rate by which these shocks dissipate and the variable reverts towards the mean;

(iv) $x_0$ is the underlying asset price at moment $t = 0$ (the underlying asset initial price);

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(v) $x_t$ is the underlying asset price at moment $t > 0$.

There are various scientific papers devoted to estimate of parameter $\mu$, $\sigma$ and $\theta$ (see, for example [16, 17]). There least-square minimization and maximum likelihood estimation techniques are used for the estimating parameters $\sigma$ and $\mu$ which work successfully. The same we can not say concerning the estimating the parameter $\theta$ (see, for example, [16]).

The purpose of the present paper is to introduce a new approach which by use values $(z_k)_{k \in \mathbb{N}}$ of corresponding trajectories at a fixed positive moment $t$, will allows us to construct a consistent estimate for each unknown parameter of the Ornstein-Uhlenbeck’s stochastic process under an assumption that all another parameters are known.

The rest of the present paper is the following:
In Section 2 we consider some auxiliary notions and facts from the theory of stochastic differential equations and mathematical statistics.
In Section 3 we present the constructions of consistent estimates for unknown parameters of the Ornstein-Uhlenbeck’s stochastic process.
In Section 4 we present simulation of the Ornstein-Uhlenbeck’s stochastic process and some computation results.

2. Some auxiliary facts from the theory of stochastic differential equations and mathematical statistics

2.1. Some auxiliary facts from the mathematical statistics. We begin this subsection by the following definition.

**Definition 2.1.1** ([19]) A sequence $(x_k)_{k \in \mathbb{N}}$ of real numbers from the interval $(a, b)$ is said to be equidistributed or uniformly distributed on an interval $(a, b)$ if for any subinterval $[c, d]$ of $(a, b)$ we have

$$
\lim_{n \to \infty} n^{-1} \#(\{x_1, x_2, \ldots, x_n\} \cap [c, d]) = (b - a)^{-1} (d - c),
$$

where $\#$ denotes a counting measure.

**Definition 2.1.2** Let $\mu$ be a probability Borel measure on $\mathbb{R}$ and $F$ be it’s corresponding distribution function. A sequence $(x_k)_{k \in \mathbb{N}}$ of elements of $\mathbb{R}$ is said to be $\mu$-equidistributed or $\mu$-uniformly distributed on $\mathbb{R}$ if for every interval $[a, b]$ ($-\infty \leq a < b \leq +\infty$) we have

$$
\lim_{n \to \infty} n^{-1} \#([a, b] \cap \{x_1, \ldots, x_n\}) = F(b) - F(a).
$$

**Lemma 2.1.1** Let $(x_k)_{k \in \mathbb{N}}$ be $\ell_1$-equidistributed sequence on $(0, 1)$, $F$ be a strictly increasing continuous distribution function on $\mathbb{R}$ and $p$ be a Borel probability measure on $\mathbb{R}$ defined by $F$. Then $(F^{-1}(x_k))_{k \in \mathbb{N}}$ is $p$-equidistributed on $\mathbb{R}$.

**Proof.** We have

$$
\lim_{n \to \infty} n^{-1} \#(\{F^{-1}(a), \ldots, F^{-1}(b)\}) = F(b) - F(a),
$$

$$
\lim_{n \to \infty} n^{-1} \#([F(a), F(b)] \cap \{x_1, \ldots, x_n\}) = F(b) - F(a).
$$

□

**Corollary 2.1.1** Let $F$ be a strictly increasing continuous distribution function on $\mathbb{R}$ and $p$ be a Borel probability measure on $\mathbb{R}$ defined by $F$. Then for a set $D_F \subseteq \mathbb{R}^N$ of all $p$-equidistributed sequences on $\mathbb{R}$ we have:

(i) $D_F = \{(F^{-1}(x_k))_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in D\}$;
(ii) $p^N(D_F) = 1$.

Let $\{\mu_\theta : \theta \in \mathbb{R}\}$ be a family Borel probability measures in $\mathbb{R}$. By $\mu_\theta^N$ we denote the $N$-power of the measure $\mu_\theta$ for $\theta \in \mathbb{R}$.
Definition 2.1.3 A Borel measurable function $T_n : R^n \to R$ ($n \in N$) is called a consistent estimator of a parameter $\theta$ (in the sense of convergence almost everywhere) for the family $(\mu^N_\theta)_{\theta \in R}$ if the following condition

$$
\mu^N_\theta((\{x_k\}_{k \in N} : (x_k)_{k \in N} \in R^N \& \lim_{n \to \infty} T_n(x_1, \ldots, x_n) = \theta)) = 1
$$

holds true for each $\theta \in R$.

Definition 2.1.4 A Borel measurable function $T_n : R^n \to R$ ($n \in N$) is called a consistent estimator of a parameter $\theta$ (in the sense of convergence in probability) for the family $(\mu^N_\theta)_{\theta \in R}$ if for every $\epsilon > 0$ and $\theta \in R$ the following condition

$$
\lim_{n \to \infty} \mu^N_\theta((\{x_k\}_{k \in N} : (x_k)_{k \in N} \in R^N \& |T_n(x_1, \ldots, x_n) - \theta| > \epsilon)) = 0
$$

holds.

Definition 2.1.5 A Borel measurable function $T_n : R^n \to R$ ($n \in N$) is called a consistent estimator of a parameter $\theta$ (in the sense of convergence in distribution) for the family $(\mu^N_\theta)_{\theta \in R}$ if for every continuous bounded real valued function $f$ on $R$ the following condition

$$
\lim_{n \to \infty} \int_{R^N} f(T_n(x_1, \ldots, x_n)) d\mu^N_\theta((x_k)_{k \in N}) = f(\theta)
$$

holds.

Remark 2.1.1 Following [18] (see, Theorem 2, p. 272), for the family $(\mu^N_\theta)_{\theta \in R}$ we have:

(a) an existence of a consistent estimator of a parameter $\theta$ in the sense of convergence almost everywhere implies an existence of a consistent estimator of a parameter $\theta$ in the sense of convergence in probability;

(b) an existence of a consistent estimator of a parameter $\theta$ in the sense of convergence in probability implies an existence of a consistent estimator of a parameter $\theta$ in the sense of convergence in distribution.

Definition 2.1.6 Following [18], the family $(\mu^N_\theta)_{\theta \in R}$ is called strictly separated if there exists a family $(Z_\theta)_{\theta \in R}$ of Borel subsets of $R^N$ such that

(i) $\mu^N_\theta(Z_\theta) = 1$ for $\theta \in R$;

(ii) $Z_\theta \cap Z_{\theta'} = \emptyset$ for all different parameters $\theta_1$ and $\theta_2$ from $R$.

(iii) $\cup_{\theta \in R} Z_\theta = R^N$.

Definition 2.1.7 Following [18], a Borel measurable function $T : R^N \to R$ is called an infinite sample consistent estimator of a parameter $\theta$ for the family $(\mu^N_\theta)_{\theta \in R}$ if the following condition

$$
(\forall \theta)(\theta \in R \to \mu^N_\theta((\{x_k\}_{k \in N} : (x_k)_{k \in N} \in R^N \& T((x_k)_{k \in N}) = \theta)) = 1)
$$

holds.

Remark 2.1.2 Note that an existence of an infinite sample consistent estimator of a parameter $\theta$ for the family $(\mu^N_\theta)_{\theta \in R}$ implies that the family $(\mu^N_\theta)_{\theta \in R}$ is strictly separated. Indeed, if we set $Z_\theta = \{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \& T((x_k)_{k \in N}) = \theta\}$ for $\theta \in R$, then all conditions in Definition 2.1.6 will be satisfied.

In the sequel we will need the well known fact from the probability theory (see, for example, [20], p. 390).

Lemma 2.1.2 (Kolmogorov’s strong law of large numbers) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables defined on the probability space $(\Omega, F, P)$. If these random variables have a finite expectation $m$ (i.e., $E(X_1) = E(X_2) = \ldots = m < \infty$), then the following condition

$$
P(\{\omega : \lim_{n \to \infty} n^{-1} \sum_{k=1}^n X_k(\omega) = m\}) = 1
$$

was satisfied. Indeed, if we set $Z_\theta = \{(x_k)_{k \in N} : (x_k)_{k \in N} \in R^N \& T((x_k)_{k \in N}) = \theta\}$ for $\theta \in R$, then all conditions in Definition 2.1.6 will be satisfied.
holds true.

2.2. Some auxiliary facts from the theory of stochastic differential equations. By use approaches introduced in [6] one can get the validity of the following assertions.

**Lemma 2.2.1** Let’s consider an Ornstein-Uhlenbeck process \( x_t \) satisfies the following stochastic differential equation:

\[
\frac{dx_t}{dt} = \theta(\mu - x_t)dt + \sigma dW_t
\]

where \( \theta > 0, \mu \) and \( \sigma > 0 \) are parameters and \( W_t \) denotes the Wiener process. Then the solution of this stochastic differential equation (3.1.1) is given by

\[
x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)}dW_s,
\]

where \( x_0 \) is assumed to be constant.

**Proof.** The stochastic differential equation (3.1.1) is solved by variation of parameters. Changing variable

\[
f(x_t, t) = x_t e^{\theta t}
\]

we get

\[
df(x_t, t) = \theta x_t e^{\theta t}dt + e^{\theta t}dx_t = e^{\theta t} \theta \mu dt + \sigma e^{\theta t} dW_t.
\]

Integrating from 0 to \( t \) we get

\[
x_t e^{\theta t} = x_0 + \int_0^t e^{\theta s} \theta \mu ds + \int_0^t \sigma e^{\theta s} dW_s
\]

whereupon we see

\[
x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)}dW_s.
\]

\[\square\]

**Lemma 2.2.2** Under conditions of Lemma 2.2.1, the following equalities

(i) \( E(x_t) = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}); \)

(ii) \( \text{cov}(x_s, x_t) = \frac{\sigma^2}{2\theta} \left( e^{-\theta(t-s)} - e^{-\theta(t+s)} \right); \)

(iii) \( \text{var}(x_s) = \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta s} \right); \)

hold true.

**Proof.** The validity of the item (i) is obvious. In order to prove the validity of the items (ii)-(iii), we can use the Ito isometry to calculate the covariance function by

\[
\text{cov}(x_s, x_t) = E[(x_s - E[x_s])(x_t - E[x_t])] = E \left[ \int_0^s \sigma e^{\theta(u-s)}dW_u \int_0^t \sigma e^{\theta(v-t)}dW_v \right]
\]

\[
= \sigma^2 e^{-\theta(t+s)} E \left[ \int_0^s e^{\theta u}dW_u \int_0^t e^{\theta v}dW_v \right] = \frac{\sigma^2}{2\theta} e^{-\theta(t+s)} (e^{2\theta \min(s,t)} - 1).
\]

Thus if \( s < t \) (so that \( \min(s, t) = s \)), then we have

\[
\text{cov}(x_s, x_t) = \frac{\sigma^2}{2\theta} \left( e^{-\theta(t-s)} - e^{-\theta(t+s)} \right).
\]

Similarly, if \( s = t \) (so that \( \min(s, t) = s \)), then we have

\[
\text{var}(x_s) = \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta s} \right).
\]

\[\square\]
3. Estimation of the parameters of the Ornstein-Uhlenbeck stochastic model

3.1. Estimation of the the underlying asset initial price \( x_0 \) in an Ornstein-Uhlenbeck stochastic model. Let consider Ornstein-Uhlenbeck process

\[
x_t = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta (t-s)} dW_s,
\]

where \( \theta > 0, \mu \in R, \sigma > 0 \) and \( W_s \) is Wiener process.

Here

(i) \( \mu \) represents the equilibrium or mean value supported by fundamentals;
(ii) \( \sigma \) is the degree of volatility around it caused by shocks;
(iii) \( \theta \) is the rate by which these shocks dissipate and the variable reverts towards the mean
(iv) \( x_0 \) is the underlying asset price at initial moment \( t = 0 \)(the underlying asset initial price);
(v) \( x_t \) is the underlying asset price at moment \( t \) \( (t > 0) \).

**Lemma 3.1.1** For \( t > 0, x_0 \in R, \theta > 0, \mu \in R \) and \( \sigma > 0 \), let’s \( \gamma(t, x_0, \theta, \mu, \sigma) \) be a Gaussian probability measure in \( R \) with the mean \( m_1 = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) \) and the variance \( \sigma^2 = \frac{\sigma^2}{\theta} (1 - e^{-2\theta t}) \). Assuming that parameters \( t, \theta, \mu \) and \( \sigma \) are fixed, denote by \( \gamma_{x_0} \) the measure \( \gamma(t, x_0, \theta, \mu, \sigma) \). Let define the estimate \( T_n : R^n \to R \) by the following formula

\[
T_n((z_k)_{1\leq k \leq n}) = e^{\theta t} \sum_{k=1}^n \frac{z_k}{n} - \mu e^{\theta t} (1 - e^{-\theta t}).
\]

Then we get

\[
\gamma_{x_0}^{\infty}\{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \} \cap \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \}
\]

provided that \( T_n \) is a consistent estimator of the underlying asset price \( x_0 \in R \) in the sense of convergence almost everywhere for the family of probability measures \( (\gamma_{x_0})_{x_0 \in R} \).

**Proof.** Let’s consider probability space \( (\Omega, F, P) \), where \( \Omega = R^\infty, F = B(R^\infty), P = \gamma_{x_0} \).

For \( k \in N \) we consider \( k \)-th projection \( P_{R_k} \) defined on \( R^\infty \) by

\[
P_{R_k}((x_i)_{i \in N}) = x_k
\]

for \( (x_i)_{i \in N} \in R^\infty \).

It is obvious that \( (P_{R_k})_{k \in N} \) is sequence of independent Gaussian random variables with the mean \( m_1 = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) \) and the variance \( \sigma^2 = \frac{\sigma^2}{\theta} (1 - e^{-2\theta t}) \). By use Kolmogorov Strong Law of Large numbers we get

\[
\gamma_{x_0}^{\infty}\{(z_k)_{k \in N} \in R^\infty \} \cap \lim_{n \to \infty} \frac{\sum_{k=1}^n P_{R_k}((z_k)_{k \in N})}{n} = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) \}
\]

which implies

\[
\gamma_{x_0}^{\infty}\{(z_k)_{k \in N} \in R^\infty \} \cap \lim_{n \to \infty} \left( e^{\theta t} \sum_{k=1}^n \frac{z_k}{n} - e^{\theta t} \mu (1 - e^{-\theta t}) \right) = x_0 \}
\]

\[
= \gamma_{x_0}^{\infty}\{(z_k)_{k \in N} \in R^\infty \} \cap \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} = 1.
\]

**Remark 3.1.1** By use Remark 2.1.1 and Lemma 3.1.1 we deduce that \( T_n \) is a consistent estimator of the underlying asset price \( x_0 \in R \) in the sense of convergence in probability for the statistical structure \( (\gamma_{x_0})_{x_0 \in R} \) as well \( T_n \) is a consistent estimator of the underlying asset price \( x_0 \in R \) in the sense of convergence in distribution for the statistical structure \( (\gamma_{x_0})_{x_0 \in R} \).
Theorem 3.1.1 Suppose that the family of probability measures \((\gamma_{x_0}^\infty)_{x_0 \in \mathbb{R}}\) and the estimators \(T_n : \mathbb{R}^n \to \mathbb{R}(n \in \mathbb{N})\) come from Lemma 3.1.1. Then the estimators \(T^{(0)} : \mathbb{R}^\infty \to \mathbb{R}\) and \(T^{(1)} : \mathbb{R}^\infty \to \mathbb{R}\) defined by
\[
T^{(0)}((z_k)_{k \in \mathbb{N}}) = \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n})
\]
and
\[
T^{(1)}((z_k)_{k \in \mathbb{N}}) = \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}).
\]
are infinite-sample consistent estimators of the underlying asset price \(x_0\) for the family of probability measures \((\gamma_{x_0}^\infty)_{x_0 \in \mathbb{R}}\).

Proof. Note that we have
\[
\gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& T^{(0)}((z_k)_{k \in \mathbb{N}}) = x_0 \} \\
= \gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} \\
\geq \gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} = 1,
\]
which means that \(T^{(0)}\) is an infinite-sample consistent estimator of the underlying asset price \(x_0\) for the family of probability measures \((\mu_{x_0}^\infty)_{x_0 \in \mathbb{R}}\).

Similarly, we have
\[
\gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& T^{(1)}((z_k)_{k \in \mathbb{N}}) = x_0 \} \\
= \gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} \\
\geq \gamma_{x_0}^\infty \{ (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} = 1,
\]
which means that \(T^{(1)}\) is an infinite-sample consistent estimator of the underlying asset price \(x_0\) for the family of probability measures \((\gamma_{x_0}^\infty)_{x_0 \in \mathbb{R}}\).

Remark 3.1.2 By use Remark 2.1.2 we deduce that an existence of infinite sample consistent estimators \(T^{(0)}\) and \(T^{(1)}\) of the underlying asset price \(x_0\) for the family \((\gamma_{x_0}^\infty)_{x_0 \in \mathbb{R}}\) (cf. Theorem 3.1.1) implies that the family \((\gamma_{x_0}^\infty)_{x_0 \in \mathbb{R}}\) is strictly separated.

3.2. Estimation of the equilibrium \(\mu\) in an Ornstein - Uhlenbeck stochastic model. This subsection we begin by the following assertion.

Lemma 3.2.1 For \(t > 0, x_0 \in \mathbb{R}, \theta > 0, \mu \in \mathbb{R}\) and \(\sigma > 0\), let’s \(\gamma_{(t,x_0,\theta,\mu,\sigma)}\) be a Gaussian probability measure in \(R\) with the mean \(m_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})\) and the variance \(\sigma_t^2 = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\). Assuming that parameters \(x_0, t, \theta\) and \(\sigma\) are fixed, for \(\mu \in \mathbb{R}\) let’s denote by \(\gamma_\mu\) the measure \(\gamma_{(t,x_0,\theta,\mu,\sigma)}\). Let define the estimate \(T^*_n : \mathbb{R}^n \to \mathbb{R}\) by the following formula
\[
T^*_n((z_k)_{1 \leq k \leq n}) = \left( \frac{\sum_{k=1}^n z_k}{n} - x_0 e^{-\theta t} \right) / (1 - e^{-\theta t}).
\]
Then we get
\[
\gamma_\mu^\infty \{ (z_k)_{k \in \mathbb{N}} : (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{N \to \infty} T_n((z_k)_{1 \leq k \leq n}) = x_0 \} = 1,
\]
provided that \(T_n\) is a consistent estimator of the equilibrium \(\mu \in \mathbb{R}\) in the sense of convergence almost everywhere for the family of probability measures \((\gamma_\mu^\infty)_{\mu \in \mathbb{R}}\).

Proof. Let’s consider probability space \((\Omega, \mathcal{F}, P)\), where \(\Omega = \mathbb{R}^\infty, \mathcal{F} = B(\mathbb{R}^\infty), P = \gamma_\mu^\infty\).

For \(k \in \mathbb{N}\) we consider \(k\)-th projection \(P_{R^k}\) defined on \(\mathbb{R}^\infty\) by
\[
P_{R^k}((x_i)_{i \in \mathbb{N}}) = x_k
\]
for \((x_i)_{i \in \mathbb{N}} \in \mathbb{R}^\infty\).
It is obvious that \((Pr_k)_{k \in \mathbb{N}}\) is sequence of independent Gaussian random variables with the mean \(m_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})\) and the variance \(\sigma_t^2 = \frac{\sigma^2}{2\theta^2} (1 - e^{-2\theta t})\). By use Kolmogorov’s Strong Law of Large numbers we get
\[
\gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} \sum_{k=1}^{n} \frac{Pr_k((z_k)_{k \in \mathbb{N}})}{n} = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})\} = 1, \tag{3.2.4}
\]
which implies
\[
\gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} \left( \frac{\sum_{k=1}^{n} z_k}{n} - x_0 e^{-\theta t} \right) / (1 - e^{-\theta t}) \} = \mu\}
= \gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) = \mu\} = 1.
\]

Remark 3.2.1 By use Remark 2.1.1 and Lemma 3.2.1 we deduce that \(T^*_n\) is a consistent estimator of the equilibrium \(\mu \in \mathbb{R}\) in the sense of convergence in probability for the family of measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\) as well \(T^*_n\) is a consistent estimator of the equilibrium \(\mu \in \mathbb{R}\) in the sense of convergence in distribution for the family of measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\).

Theorem 3.2.1 Suppose that the family of probability measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\) and the estimators \(T^*_n : \mathbb{R}^n \to \mathbb{R}(n \in \mathbb{N})\) come from Lemma 3.2.1. Then the estimators \(T^{*(0)} : \mathbb{R}^\infty \to \mathbb{R}\) and \(T^{*(1)} : \mathbb{R}^\infty \to \mathbb{R}\) defined by
\[
T^{*(0)}((z_k)_{k \in \mathbb{N}}) = \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) \tag{3.2.5}
\]
and
\[
T^{*(1)}((z_k)_{k \in \mathbb{N}}) = \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}). \tag{3.2.6}
\]
are infinite-sample consistent estimators of the equilibrium \(\mu \in \mathbb{R}\) for the family of probability measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\).

Proof. Note that we have
\[
\gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& T^{*(0)}((z_k)_{k \in \mathbb{N}}) = \mu\}
= \gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) = \mu\}
\geq \gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) = \mu\} = 1,
\]
which means that \(T^{*(0)}\) is an infinite-sample consistent estimators of the equilibrium \(\mu \in \mathbb{R}\) for the family of probability measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\).

Similarly, we have
\[
\gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& T^{*(1)}((z_k)_{k \in \mathbb{N}}) = \mu\}
= \gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) = \mu\}
\geq \gamma^\infty \{(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \& \lim_{n \to \infty} T^*_n((z_k)_{1 \leq k \leq n}) = \mu\} = 1,
\]
which means that \(T^{*(1)}\) is an infinite-sample consistent estimators of the equilibrium \(\mu \in \mathbb{R}\) for the family of probability measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\).

Remark 3.2.2 Note that an existence of infinite sample consistent estimators \(T^{*(0)}\) and \(T^{*(1)}\) of the equilibrium \(\mu \in \mathbb{R}\) for the family of probability measures \((\gamma^\infty)_{\mu \in \mathbb{R}}\) (cf. Theorem 3.2.1) implies that the family \((\gamma^\infty)_{\mu \in \mathbb{R}}\) is strictly separated.
3.3. Estimation of the rate $\theta$ in Ornstein - Uhlenbeck stochastic model. We begin this subsection by the following lemma.

Lemma 3.3.1 For $t > 0$, $x_0 \in R$, $\theta > 0$, $\mu \in R$ and $\sigma > 0$, let’s $\gamma(t, x_0, \theta, \mu, \sigma)$ be a Gaussian probability measure in $R$ with the mean $m_t = x_0e^{-\theta t} + \mu(1 - e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$. Assuming that parameters $x_0, \theta$ and $\sigma$ are fixed such that $x_0 \neq \mu$, for $\theta > 0$, let’s denote by $\gamma_\theta$ the measure $\gamma(t, x_0, \theta, \mu, \sigma)$. Let define the estimate $T_{n}^{\ast*}: R^n \rightarrow R$ by the following formula

$$T_{n}^{\ast*}((z_k)_{1 \leq k \leq n}) = -\frac{1}{t} \ln \left( \frac{\sum_{k=1}^{n} z_k - \mu}{x_0 - \mu} \right). \quad (3.2.2)$$

Then we get

$$\gamma_\theta^\infty \{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \ & \ \text{and} \ \lim_{N \rightarrow \infty} T_{n}^{\ast*}((z_k)_{1 \leq k \leq n}) = \theta \} = 1, \quad (3.2.3)$$

provided that $T_{n}^{\ast*}$ is a consistent estimator of the rate $\theta > 0$ in the sense of convergence almost everywhere for the family of probability measures $(\gamma_\theta^\infty)_{\theta > 0}$.

Proof. Let’s consider probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_\theta^\infty$.

For $k \in N$ we consider $k$-th projection $Pr_k$ defined on $R^\infty$ by

$$Pr_k((x_i)_{i \in N}) = x_k \quad (3.2.4)$$

for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is sequence of independent Gaussian random variables with the mean $m_t = x_0e^{-\theta t} + \mu(1 - e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$. By use Kolmogorov’s Strong Law of Large numbers we get

$$\gamma_\theta^\infty \{(z_k)_{k \in N} \in R^\infty \ & \ \text{and} \ \lim_{n \rightarrow \infty} \sum_{k=1}^{n} Pr_k((z_k)_{k \in N}) = x_0e^{-\theta t} + \mu(1 - e^{-\theta t}) \} = 1, \quad (3.2.5)$$

which implies

$$\gamma_\theta^\infty \{(z_k)_{k \in N} \in R^\infty \ & \ \text{and} \ \lim_{n \rightarrow \infty} -\frac{1}{t} \ln \left( \frac{\sum_{k=1}^{n} z_k - \mu}{x_0 - \mu} \right) = \theta \} = \gamma_\theta^\infty \{(z_k)_{k \in N} \in R^\infty \ & \ \text{and} \ \lim_{n \rightarrow \infty} T_{n}^{\ast*}((z_k)_{1 \leq k \leq n}) = \theta \} = 1.$$

\[\square\]

Remark 3.3.1 By use Remark 2.1.1 and Lemma 3.3.1 we deduce that $T_{n}^{\ast*}$ is a consistent estimator of the equilibrium $\theta$ in the sense of convergence in probability for the family of probability measures $(\gamma_\theta^\infty)_{\theta > 0}$ as well $T_{n}^{\ast*}$ is a consistent estimator of $\theta$ in the sense of convergence in distribution for the same family of probability measures.

Theorem 3.3.1 Suppose that the family of probability measures $(\gamma_\theta^\infty)_{\theta > 0}$ and the estimators $T_{n}^{\ast*} : R^n \rightarrow R(n \in N)$ come from Lemma 3.3.1. Then the estimators $T_{n}^{\ast*}(0) : R^\infty \rightarrow R$ and $T_{n}^{\ast*}(1) : R^\infty \rightarrow R$ defined by

$$T_{n}^{\ast*}(0)((z_k)_{k \in N}) = \lim_{n \rightarrow \infty} T_{n}^{\ast*}((z_k)_{1 \leq k \leq n}) \quad (3.2.6)$$

and

$$T_{n}^{\ast*}(1)((z_k)_{k \in N}) = \lim_{n \rightarrow \infty} T_{n}^{\ast*}((z_k)_{1 \leq k \leq n}). \quad (3.2.7)$$

are infinite-sample consistent estimators of the rate $\theta$ for the family of probability measures $(\gamma_\theta^\infty)_{\theta > 0}$. 

Proof. Note that we have
\[
\gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, T^{(0)}_*(((z_k)_{k \in N}) = \mu) \\
= \gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, \lim_{n \to \infty} T^{(0)}_n((z_k)_{1 \leq k \leq n}) = \mu) \\
\geq \gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, \lim_{N \to \infty} T^{*\ast}_n((z_k)_{1 \leq k \leq n}) = \mu\} = 1,
\]
which means that $T^{(0)}_*$ is an infinite-sample consistent estimator of the parameter $\theta$ for the family of probability measures $(\gamma_0^\infty)_{\theta > 0}$.

Similarly, we have
\[
\gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, T^{(1)}_*(((z_k)_{k \in N}) = \mu) \\
= \gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, \lim_{n \to \infty} T^{(1)}_n((z_k)_{1 \leq k \leq n}) = \mu) \\
\geq \gamma_0^\infty \{(z_k)_{k \in N} \in \mathbb{R}^\infty \, \& \, \lim_{N \to \infty} T^{*\ast}_n((z_k)_{1 \leq k \leq n}) = \mu\} = 1,
\]
which means that $T^{(1)}_*$ is an infinite-sample consistent estimator of the parameter $\theta$ for the family of probability measures $(\gamma_0^\infty)_{\theta > 0}$.

\[\square\]

Remark 3.3.2 Note that an existence of infinite sample consistent estimators $T^{(0)}_*$ and $T^{(1)}_*$ of the rate $\theta$ for the family of probability measures $(\gamma_0^\infty)_{\theta > 0}$ (cf. Theorem 3.3.1) implies that the family $(\gamma_0^\infty)_{\theta > 0}$ is strictly separated.

3.4. Estimation of the square of the degree of volatility $\sigma$ around it caused by shocks in Ornstein - Uhlenbeck stochastic model. We begin this subsection by the following proposition.

Lemma 3.4.1 For $t > 0$, $x_0 \in R$, $\theta > 0$, $\mu \in R$ and $\sigma > 0$, let’s $\gamma_{t,x_0,\theta,\mu,\sigma}$ be a Gaussian probability measure in $R$ with the mean $m_t = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t})$ and the variance $\sigma^2_t = \sigma^2 \left(1 - e^{-2\theta t}\right)$. Assuming that parameters $x_0$, $t$, $\mu$ and $\theta$ are fixed. For $\sigma^2 > 0$, let’s denote by $\gamma_{\sigma^2}$ the measure $\gamma_{t,x_0,\theta,\mu,\sigma}$. Let define the estimate $T^{*\ast}_n : R^n \to R$ by the following formula
\[
T^{*\ast}_n((z_k)_{1 \leq k \leq n}) = \frac{2\theta \sum_{k=1}^n (z_k - x_0 e^{-\theta t} - \mu (1 - e^{-\theta t}))^2}{n (1 - e^{-2\theta t})}. \tag{3.4.1}
\]

Then we get
\[
\gamma_{\sigma^2}^\infty \{(z_k)_{k \in N} : (z_k)_{k \in N} \in R^\infty \, \& \, \lim_{N \to \infty} T^{*\ast}_n((z_k)_{1 \leq k \leq n}) = \sigma^2\} = 1, \tag{3.4.2}
\]
provided that $T^{*\ast}_n$ is a consistent estimator of the square of the degree of volatility $\sigma$ around it caused by shocks in the sense of convergence almost everywhere for the family of probability measures $(\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}$.

Proof. Let’s consider probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = R^\infty$, $\mathcal{F} = B(R^\infty)$, $P = \gamma_{\sigma^2}^\infty$.

For $k \in N$ we consider $k$-th projection $Pr_k$ defined on $R^\infty$ by
\[
Pr_k((x_i)_{i \in N}) = x_k \tag{3.4.3}
\]
for $(x_i)_{i \in N} \in R^\infty$.

It is obvious that $(Pr_k)_{k \in N}$ is sequence of independent Gaussian random variables with the mean $m_t = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t})$ and the variance $\sigma^2_t = \sigma^2 \left(1 - e^{-2\theta t}\right)$. By use Kolmogorov’s Strong Law of Large numbers for a sequence of independent identically distributed random variables $(X_n)_{n \in N}$, where
\[
X_n((z_j)_{j \in N}) = (Pr_n((z_j)_{j \in N}) - x_0 e^{-\theta t} - \mu (1 - e^{-\theta t}))^2
\]
for $n \in N$, we get
\[
\gamma^\infty_{\sigma^2}(\{z_j\}_{j \in N} \in R^\infty \& \lim_{n \to \infty} \sum_{k=1}^{n} \frac{(Pr_k((z_j)_{j \in N}) - x_0 e^{-\theta t} - \mu(1-e^{-\theta t}))^2}{n} = \frac{\sigma^2}{2\theta} (1 - e^{-2\theta s}) = 1,
\]
which implies
\[
\gamma_{\sigma^2}^\infty(\{z_j\}_{j \in N} \in R^\infty \& \lim_{n \to \infty} \frac{2\theta \sum_{k=1}^{n} (z_k - x_0 e^{-\theta t} - \mu(1-e^{-\theta t}))^2}{n (1 - e^{-2\theta s})} = \sigma^2
\]
\[
= \gamma_{\sigma^2}^\infty(\{z_j\}_{j \in N} \in R^\infty \& \lim_{n \to \infty} T^{**}_n ((z_j)_{1 \leq j \leq n}) = \sigma^2) = 1.
\]

\[\square\]

**Remark 3.4.1** By use Remark 2.1.1 and Lemma 3.4.1 we deduce that \(T^{**}_n\) is a consistent estimator of the parameter \(\sigma^2\) in the sense of convergence in probability for the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\) as well, \(T^{**}_n\) is a consistent estimator of the parameter \(\sigma^2\) in the sense of convergence in distribution for the same family of probability measures.

**Theorem 3.4.1** Suppose that the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\) and the estimators \(T^{**}_n : R^n \to R(n \in N)\) come from Lemma 3.4.1. Then the estimators \(T^{(0)}_n : R^\infty \to R\) and \(T^{(1)}_n : R^\infty \to R\) defined by
\[
T^{(0)}_n(\{z_k\}_{k \in N}) = \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n})
\]
and
\[
T^{(1)}_n(\{z_k\}_{k \in N}) = \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n}).
\]
are infinite-sample consistent estimators of the square of the degree of volatility \(\sigma\) around it caused by shocks for the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\).

**Proof.** Note that we have
\[
\gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& T^{(0)}_n(\{z_k\}_{k \in N}) = \sigma^2)
\]
\[
= \gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n}) = \sigma^2)
\]
\[
\geq \gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n}) = \sigma^2) = 1,
\]
which means that \(T^{(0)}_n\) is an infinite-sample consistent estimators of the parameter \(\sigma^2\) for the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\).

Similarly, we have
\[
\gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& T^{(1)}_n((z_k)_{k \in N}) = \sigma^2)
\]
\[
= \gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n}) = \sigma^2)
\]
\[
\geq \gamma_{\sigma^2}^\infty(\{z_k\}_{k \in N} \in R^\infty \& \lim_{n \to \infty} T^{**}_n((z_k)_{1 \leq k \leq n}) = \sigma^2) = 1,
\]
which means that \(T^{(1)}_n\) is an infinite-sample consistent estimators of the parameter \(\sigma^2\) for the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\).

\[\square\]

**Remark 3.4.2** Note that an existence of infinite sample consistent estimators \(T^{(0)}_n\) and \(T^{(1)}_n\) of the parameter \(\sigma^2\) for the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\) (cf. Theorem 3.4.1) implies that the family of probability measures \((\gamma_{\sigma^2}^\infty)_{\sigma^2 > 0}\) is strictly separated.
4. Simulation of the Ornstein-Uhlenbeck stochastic process and estimation it’s parameters

In this section we give a short explanation whether can be obtained the simulations of the Ornstein-Uhlenbeck process. Similar simulations can be found in [17].

The simulation of the Ornstein-Uhlenbeck process can be obtained as follows:

\[
x_t = x_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} W_{c^2\theta t - 1},
\]

where \(W_t\) denotes Wiener process.

Wiener (1923) gave a representation of a Brownian path in terms of a random Fourier series. If \((\xi_n)_{n \in \mathbb{N}}\) is the sequence of independent standard Gaussian random variables, then

\[
W_t = \xi_0 t + \sqrt{2} \sum_{n=1}^{\infty} \xi_n \frac{\sin \pi nt}{\pi n}
\]

represents a Brownian motion on \([0,1]\).

Following Karhunen-Loeve well known theorem (see, [12], [11]), the scaled process

\[
\sqrt{c} W \left( \frac{t}{c} \right)
\]

is a Brownian motion on \([0,c]\).

Let \((y_n^{(k)})_{n \in \mathbb{N}}\) be a sequence of real numbers defined by \(y_n^{(k)} = n\sqrt{p_k} - \lfloor n\sqrt{p_k} \rfloor\) for \(k, n \in \mathbb{N}\), where \(\lfloor \cdot \rfloor\) denotes the integer part of the real number and \((p_n)_{n \in \mathbb{N}}\) denotes the set of all prime numbers. Note that this sequence is uniformly distributed in \((0,1)\) for each \(k \in \mathbb{N}\) (see, for example, [19]).

Let \(\Phi\) be a standard Gaussian distribution function in \(\mathbb{R}\). Then following Lemma 2.1.1, the sequence \((x_n^{(k)})_{n \in \mathbb{N}} = (\Phi^{-1}(y_n^{(k)}))_{n \in \mathbb{N}}\) will be \(\gamma\)-uniformly distributed in \(\mathbb{R}\) for each \(k \in \mathbb{N}\), where \(\gamma\) denotes a standard Gaussian measure in \(\mathbb{R}\).

In our simulation we use MatLab command \texttt{random(’Normal’, 0, 1, p, q)} which generates \(\gamma\)-uniformly distributed sequences \((x_n^{(k)})_{1 \leq n \leq q}\) \((1 \leq k \leq p)\).
Table 4.1. The value $z_k$ of the Ornstein-Uhlenbeck’s $k$-th trajectory at moment $t = 0.5$ when $\theta = 0.5$, $\sigma = 1$, $\mu = -3$ and $x_0 = 3$.

| $k$ | $z_k$ | $k$ | $z_k$ | $k$ | $z_k$ | $k$ | $z_k$ |
|-----|-------|-----|-------|-----|-------|-----|-------|
| 1   | 2.7082| 21  | 1.2571| 41  | 1.2185| 61  | 1.9426|
| 2   | 2.0594| 22  | 2.1261| 42  | 3.0131| 62  | 1.453 |
| 3   | 2.1303| 23  | 2.6017| 43  | 2.0324| 63  | 1.6666|
| 4   | 2.3939| 24  | 0.7975| 44  | 1.8216| 64  | 1.2806|
| 5   | 2.641 | 25  | 1.9225| 45  | 1.1374| 65  | 1.3268|
| 6   | 1.5119| 26  | 1.8187| 46  | 2.7327| 66  | 1.4312|
| 7   | 1.6549| 27  | 1.8187| 47  | 2.3649| 67  | 2.7034|
| 8   | 1.2017| 28  | 1.1202| 48  | 1.3785| 68  | 1.227 |
| 9   | 1.261 | 29  | 0.3467| 49  | 2.6211| 69  | 1.0065|
| 10  | 0.8576| 30  | 1.2734| 50  | 1.258 | 70  | 0.7277|
| 11  | 1.3968| 31  | 2.6075| 51  | 1.5606| 71  | 1.3227|
| 12  | 2.8304| 32  | 1.1872| 52  | 2.0278| 72  | 1.3528|
| 13  | 1.9669| 33  | 1.9519| 53  | 1.3095| 73  | 2.102 |
| 14  | 3.0409| 34  | 1.9615| 54  | 1.8024| 74  | 1.1705|
| 15  | 0.7784| 35  | 1.6775| 55  | 1.62  | 75  | 1.162 |
| 16  | 1.6111| 36  | 2.5195| 56  | 0.9569| 76  | 0.9056|
| 17  | 1.1053| 37  | 1.894 | 57  | 0.8123| 77  | 0.6306|
| 18  | 1.2955| 38  | 0.9174| 58  | 0.9781| 78  | 0.3304|
| 19  | 1.2756| 39  | 1.5291| 59  | 1.9541| 79  | 1.0314|
| 20  | 1.711 | 40  | 1.3806| 60  | 1.4921| 80  | 1.9173|

Note that

$$W_{e^{2\theta t} - 1}^{(k)} = \xi_0^{(k)}(e^{2\theta t} - 1) + \sqrt{2} \sum_{n=1}^{\infty} \left( \xi_n^{(k)} \right) \frac{\sin \pi n e^{2\theta t} - 1}{\pi n}$$

will be the value of the Wiener’s $k$-th trajectory at moment $e^{2\theta t} - 1$ for $k \in N$.

Hence the value of the Ornstein-Uhlenbeck’s $k$-th trajectory at moment $t$ will be

$$z_k = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} \left( \xi_0^{(k)}(e^{2\theta t} - 1) + \sqrt{2} \sum_{n=1}^{\infty} \left( \xi_n^{(k)} \right) \frac{\sin \pi n e^{2\theta t} - 1}{\pi n} \right)$$

for each $k \in N$.

In our simulation we consider

$$z_k = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} \left( \xi_0^{(k)}(e^{2\theta t} - 1) + \sqrt{2} \sum_{n=1}^{800} \left( \xi_n^{(k)} \right) \frac{\sin \pi n e^{2\theta t} - 1}{\pi n} \right)$$

for $1 \leq k \leq 100$.

Below we present some numerical results obtaining by using MatLab and Microsoft Excel. In our simulation

(i) $n$ denotes the number of trials;
(ii) $x_0 = 3$ is the underlying asset initial price;
(iii) $\mu = -3$ is the equilibrium or mean value supported by fundamentals;
(iv) $\sigma = 1$ is the degree of volatility around it caused by shocks;
(v) $\theta = 0.5$ is the rate by which these shocks dissipate and the variable reverts towards the mean;
(vi) $t = 0.5$ is the moment of the observation on the Ornstein-Uhlenbeck’s trajectories;
(vii) \( z_k \) is the value of the Ornstein-Uhlenbeck’s \( k \)-th trajectory at moment \( t = 0.5 \) (see, Figure 1 and Table 4.1).

Table 4.2. The value of the statistic \( T_n \) for the sample \( (z_k)_{1 \leq k \leq n} (n = 5i : 1 \leq i \leq 20) \) from the Table 4.1 in the Ornstein-Uhlenbeck’s stochastic model when \( \theta = 0.5, \sigma = 1, \mu = -3 \) and \( x_0 = 3 \).

| \( n \) | \( T_n \) | \( x_0 \) | \( n \) | \( T_n \) | \( x_0 \) |
|---|---|---|---|---|---|
| 5  | 3.916479949 | 3  | 55 | 3.050042943 | 3  |
| 10 | 3.171013312 | 3  | 60 | 2.999422576 | 3  |
| 15 | 3.189798604 | 3  | 65 | 2.985749187 | 3  |
| 20 | 3.053011377 | 3  | 70 | 2.963503799 | 3  |
| 25 | 3.059916865 | 3  | 75 | 2.944638775 | 3  |
| 30 | 2.933186125 | 3  | 80 | 2.891140712 | 3  |
| 35 | 2.980208972 | 3  | 85 | 2.951454785 | 3  |
| 40 | 2.978720876 | 3  | 90 | 2.918940576 | 3  |
| 45 | 3.005595171 | 3  | 95 | 2.936244133 | 3  |
| 50 | 3.056170079 | 3  |100 | 2.90958959  | 3  |

Remark 4.1 By use results of calculations placed in the Table 4.2, we see that the consistent estimator \( T_n \) works successfully.

Table 4.3. The value of the statistic \( T^*_n \) for the sample \( (z_k)_{1 \leq k \leq n} (n = 5i : 1 \leq i \leq 20) \) from the Table 4.1 in the Ornstein-Uhlenbeck’s stochastic model when \( \theta = 0.5, \sigma = 1, \mu = -3 \) and \( x_0 = 3 \).

| \( n \) | \( T^*_n \) | \( \mu \) | \( n \) | \( T^*_n \) | \( \mu \) |
|---|---|---|---|---|---|
| 5  | 0.226753293 | -3 | 55 | -2.823808222 | -3 |
| 10 | -2.397894335 | -3 | 60 | -3.002033002 | -3 |
| 15 | -2.331754861 | -3 | 65 | -3.05017443  | -3 |
| 20 | -2.813356927 | -3 | 70 | -3.12849625  | -3 |
| 25 | -2.789044002 | -3 | 75 | -3.194916445 | -3 |
| 30 | -3.235239072 | -3 | 80 | -3.38327305  | -3 |
| 35 | -3.06968049  | -3 | 85 | -3.170918559 | -3 |
| 40 | -3.074919788 | -3 | 90 | -3.285394965 | -3 |
| 45 | -2.980300456 | -3 | 95 | -3.224472402 | -3 |
| 50 | -2.80223573  | -3 |100 | -3.318318028 | -3 |

Remark 4.2 By use results of calculations placed in the Table 4.3, we see that the consistent estimator \( T^*_n \) works successfully.

Table 4.4. The value of the statistic \( T^{**}_n \) for the sample \( (z_k)_{1 \leq k \leq n} (n = 5i : 1 \leq i \leq 20) \) from the Table 4.1 in the Ornstein-Uhlenbeck’s stochastic model when \( \theta = 0.5, \sigma = 1, \mu = -3 \) and \( x_0 = 3 \).
Remark 4.3 By use results of calculations placed in the Table 4.4, we see that the consistent estimator $T_{n}^{\ast\ast}$ works successfully.

Table 4.5. The value of the statistic $T_{n}^{\ast\ast\ast}$ for the sample $(z_{k})_{1 \leq k \leq n}$ ($n = 5i : 1 \leq i \leq 20$) from the Table 4.1 in the Ornstein-Uhlenbeck’s stochastic model when $\theta = 0.5$, $\sigma = 1$, $\mu = -3$ and $x_{0} = 3$.

| $n$ | $T_{n}^{\ast\ast\ast}$ | $\sigma^{2}$ | $n$ | $T_{n}^{\ast\ast\ast}$ | $\sigma^{2}$ |
|-----|-----------------|----------|-----|-----------------|----------|
| 5   | 1.468059434     | 1        | 55  | 1.019738082     | 1        |
| 10  | 1.071475        | 1        | 60  | 1.013011822     | 1        |
| 15  | 1.448094876     | 1        | 65  | 0.950520018     | 1        |
| 20  | 1.168384329     | 1        | 70  | 0.979072751     | 1        |
| 25  | 1.145106531     | 1        | 75  | 0.944824889     | 1        |
| 30  | 1.174454042     | 1        | 80  | 1.011196765     | 1        |
| 35  | 1.098947717     | 1        | 85  | 1.03807104     | 1        |
| 40  | 1.053231322     | 1        | 90  | 1.030042542     | 1        |
| 45  | 1.074061413     | 1        | 95  | 1.088067168     | 1        |
| 50  | 1.106961814     | 1        | 100 | 1.070420297     | 1        |

Remark 4.3 By use results of calculations placed in the Table 4.5, we see that the consistent estimator $T_{n}^{\ast\ast\ast}$ works successfully.

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