In this Letter, we suggest that the GWs from magnetar GFs last much longer than the X-QPOs if the GW frequency is close to the X-QPO frequency. So, we propose that GWs from magnetar GFs are promising candidates for the GW emission from magnetar GFs, and the GW emission from magnetar GFs is crucial for the GW detection. With the observed GW frequency and its evolution, we can probe the interior magnetic field strength of the magnetar GFs.
signal to noise ratio (S/N) even if we assume the minimum magnetic field decay required to supply energy for the observed emission from magnetars [see Eq. (3)].

In our picture, GWs are produced by the polar oscillations while X-QPOs are caused by the axial ones. This is implied by the amplitude of X-QPOs $\sim 0.1$ [18], which is much larger than the expected polar oscillation amplitude $\lesssim 10^{-4}$ [see Eq. (2)]. Thus, the GW searches beyond the X-QPO duration are meaningful.

To clarify the above statement, we first make an order-of-magnitude estimate before the detailed analysis. We take the fiducial minimum magnetic field inside the NS as $B = 10^{10}$G because the total magnetic energy $\sim (B^2/8\pi) \cdot (4\pi R^3/3) \sim 10^{40}$ erg should supply $\sim 10^{46}$ erg $\times 10^5$ yr $\times 10^2$ yr $\sim 10^{48}$ erg for the GF activities once per century (with three GFs out of $\sim 10$ sources within 30 years) during the lifetime of magnetars, $\tau_{\text{mag}} \sim 10^4$ yr, and the ionospheric observations suggest that $\sim 9$ times energy is released outside the X-γ ray band [21].

When the interior magnetic fields are deformed by the GFs, the induced difference in the moment of inertia changes by

$$\frac{\Delta I}{I} \equiv \epsilon \lesssim \epsilon_{\text{max}} \equiv \frac{(B^2/8\pi) \cdot (4\pi R^3/3)}{GM^2/R} \sim 3 \times 10^{-5} B_{16}^2,$$

(2)

where $I \approx 2MR^2/5$ is the moment of inertia of the magnetar, $B_{16} \equiv (B/10^{16}$G) is the interior magnetic field, and $M$, $R$ are the mass and the radius, respectively [14, 22]. Hereafter, we fix $R = 10^6$ cm and $M = 1.4M_\odot$. The oscillation energy or the GW energy is about the gravitational energy shift caused by the stellar deformation,

$$E_{\text{GW}} \approx (\epsilon^2/5) \cdot (GM^2/R) \lesssim 8 \times 10^{43} B_{16}^4 \text{ [erg]}. \quad (3)$$

The oscillation frequencies are determined by the Alfvén sound crossing,

$$f_a = \frac{v_a}{R} = \frac{1}{R} \cdot \frac{B}{\sqrt{4\pi \rho}} \sim 100 B_{16} \text{ [Hz]}, \quad (4)$$

where $v_a$ is the Alfvén velocity, and $\rho \approx M/(4\pi R^3/3)$ is the mass density. If these oscillations are polar mode [17], in which the moment of inertia changes, GWs are emitted. The luminosity of the GWs is estimated to be

$$L_{\text{GW}} \approx (G/5c^5) \cdot \langle \dot{f}_{ij} \dot{f}_{ij} \rangle \approx (G/5c^5) \cdot (\epsilon I(2\pi f_a)^3)^2 \lesssim 2 \times 10^{37} B_{16}^{10} \text{ [erg/sec]}. \quad (5)$$

Then the GW duration after the GF becomes

$$\tau_{\text{GW}} \approx E_{\text{GW}}/L_{\text{GW}} \sim 4 \times 10^6 B_{16}^{-6} \text{ [sec]}. \quad (6)$$

The characteristic GW amplitude $h_c$ [23] is given by

$$h_c \approx \frac{(G L_{\text{GW}}/\pi c^2 d_{10} f_a)^{1/2}}{\sqrt{f_a \tau_{\text{GW}}}} \lesssim 5 \times 10^{-23} B_{16}^{3/2} d_{10}^{-1}, \quad (7)$$

where $d_{10} = d/10$ kpc is the distance from the source. Therefore, if the GW energy is $\gtrsim 10^{44}$ erg (only a fraction of the GF energy $\sim 10^{44}$–$10^{46}$ erg) and the data taking lasts for a day to months, the GWs from magnetar GFs within 10 kpc are detectable with enough S/N by the next generation detectors. The event rate will be $\sim 1/10$ yr, and more for weaker flares. We should note that the GW energy or the deformation $\epsilon$ has a large uncertainty. The minimum deformation would be $\epsilon \sim \epsilon_{\text{max}}/100$ in Eq. (2) since $\sim 1\%$ of magnetic energy is released in a GF. We can rescale the following results with $S/N \propto h_c \propto \epsilon$. Meanwhile, Eqs. (4) and (5) do not depend on $\epsilon$, and thus are relatively robust.

Since the magnetic fields of magnetars are considered to decay significantly in the lifetime $\tau_{\text{mag}} \sim 10^4$ yr, the magnetic field are decaying at least at the rate

$$\dot{B} \approx -B/\tau_{\text{mag}} \sim -10^5 B_{16} \text{ [G/sec]} \quad (8)$$

This leads to the evolution of the Alfvén frequency

$$f_a \approx \frac{1}{R} \cdot \frac{\dot{B}}{\sqrt{4\pi \rho}} \sim -10^{-9} B_{16} \text{ [Hz/sec]} \quad (9)$$

During the GW emission, the oscillation frequency changes by

$$|\Delta f_a| \sim |\dot{f}_a \times \tau_{\text{GW}}| \sim 4 \times 10^{-3} B_{16}^{-5} \text{Hz}. \quad (10)$$

In principle, the frequency resolution of GW is given by the inverse of the observational time $T_{\text{obs}}$,

$$\Delta f_{\text{obs}} \sim T_{\text{obs}}^{-1} \sim \tau_{\text{GW}}^{-1} \sim 3 \times 10^{-7} B_{16}^6 \text{Hz} \quad (11)$$

for long enough data taking. Therefore, we could resolve the GW frequency evolution to prove the magnetic field decay rate inside the magnetar for $B_{16} \lesssim 3.0$. In other words, the 1FD term could be crucial for keeping S/N in the GW detection. If the magnetic field does not decay constantly in the GF phase, higher FD terms may be important, where the long-term frequency-time analysis [24] may be useful.

Next, we shall move on to more details with the matched filtering Fisher analyses. We only consider the $l = 2, m = 0$ mode as the magnetar oscillation because one-sided outflow has been confirmed by the radio observation of the GF from SGR 1806-20 [25]. This suggests that a GF is a pointlike energy release from a NS, which would mainly trigger the $l = 1, m = 0$ plus $l = 2, m = 0$ mode oscillations. We take $m = 0$ since there is no specific direction along the azimuthal angle and the rotation is slow ($\sim 0.1$ Hz) compared with the Alfvén oscillations. The $l = 1, m = 0$ mode is the dipole oscillation which does not contribute to the GW emission.

Taking a misalignment between the polar oscillation and the rotation axes as in Fig. 1 the waveform of the observed plus (+) and cross (×) GWs are obtained from
the quadrupole formula,
\[ h_+ + i h_\times = \frac{eI}{d} (A + iB) \cos \Psi(t) \exp \left(-\frac{t}{\tau_{GW}}\right), \] (12)
where
\[ A = \frac{2}{\sqrt{3}} \left( \frac{d\Psi}{dt} \right)^2 \{ (\cos^2 \theta + 1) \sin^2 \alpha \cos 2(2\pi vt + \phi) \\
+ \sin^2 \theta (3 \cos^2 \alpha - 1) + \sin 2\theta \sin 2\alpha \cos (2\pi vt + \phi) \}, \] (13)
\[ B = \frac{2}{\sqrt{3}} \left( \frac{d\Psi}{dt} \right)^2 \{ \cos \theta \sin \alpha \sin 2(2\pi vt + \phi) \\
+ \sin 2\alpha \sin (2\pi vt + \phi) \}. \] (14)

The stellar oscillation phase \( \Psi(t) \) can be Taylor expanded as Eq. (1). In this Letter, we stop the expansion at the 1FD term, and add \( \psi_0, f_a \) and \( f_a \) as the parameters for the GW waveform. We neglect the proper motion of the magnetar. As a whole, the parameters are the strength of deformation \( \epsilon \), the mean moment of inertia \( I \), the distance to the magnetar \( d \), the rotational frequency of the magnetar \( \nu \), the angle between the magnetar rotation and the oscillation axes \( \alpha \), the frequency of the oscillation \( f_a \), its evolution rate \( \dot{f}_a \), the initial phase of the oscillation \( \psi_0 \), the duration of the GW emission \( \tau_{GW} \), and the observational polar and azimuthal angle \( \theta \) and \( \phi \). The position \( (\theta, \phi) \) can be determined independently by the x-ray or radio observations. Here, \( I, d \) and \( \epsilon \) are completely degenerated. We fix \( I = 10^{45}\text{erg cm}^2\text{sec}^{-1}, d = 10\text{kpc} \) and leave \( \epsilon \) as a free parameter. According to Eqs. (9, 10, 11) and (12), we set \( f_a = 100B_{15}\text{Hz}, \dot{f}_a = -10^{-9}B_{15}\text{Hz/sec}, \tau_{GW} = 4 \times 10^{9}B_{15}^{-2}\text{sec} \) and \( \epsilon = 3 \times 10^{-5}B_{15}^2 \) (i.e., optimal case). We also set \( \alpha = \pi/4, \nu = 0.1\text{Hz}, \theta = \pi/2, \phi = \pi/4 \) and \( \psi_0 = \pi/4 \).

![FIG. 1: The configuration of the magnetar and detectors.](image)

In the matched filtering analysis, we can compute the determination accuracy of these parameters using the Fisher matrix formalism with a waveform template in the frequency domain \( \tilde{h}(f) \) [24, 27]. The variance-covariance matrix of the parameter estimation error \( \Delta \gamma_i \) is given by the inverse of the Fisher information matrix \( \Gamma_{ij} \) as \( \langle \Delta \gamma_i \Delta \gamma_j \rangle = (\Gamma^{-1})_{ij} \). The Fisher matrix becomes
\[ \Gamma_{ij} = 4\text{Re} \sum_{\alpha=+,\times} \int \frac{df}{S_n(f)} \frac{\partial \tilde{h}_\alpha(f)}{\partial \tilde{h}_\alpha(f)} \] (15)
where \( S_n(f) \) is the noise spectrum. The accuracies of the interested parameters are described as \( \Delta B/B = \Delta f_a/f_a, \Delta B/B = \Delta f_a/f_a \) and \( \Delta E_{GW}/E_{GW} = 2\Delta \epsilon/\epsilon \). The resolution for the position of the magnetar is
\[ \Delta \Omega = 2\pi \sqrt{\langle \Delta \mu^2 \rangle \langle \Delta \varphi^2 \rangle - (\langle \Delta \mu \Delta \varphi \rangle)^2}, \] (16)
where \( \mu = \cos \theta \). The S/N is given by
\[ (S/N)^2 = 4 \sum_{\alpha=+,\times} \int \frac{df}{S_n(f)} |\tilde{h}_\alpha(f)|^2. \] (17)

We integrate the waveform in Eqs. (15) and (17) for the duration \( T_{obs} \). We refer advanced LIGO [1] (2nd generation) and ET [3] (3rd generation) for a detector noise spectrum \( S_n(f) \).

**TABLE I: S/N and parameter determination accuracy for the optimal case (\( \epsilon = \epsilon_{\text{max}} \))**

| detector | 2nd generation | 3rd generation |
|----------|----------------|----------------|
| \( B_{16} \) | 1.0 | 2.0 | 1.0 | 2.0 |
| \( T_{obs} (\gtrsim \tau_{GW}) \) | 4 months | 2 days | 4 months | 2 days |
| \( S/N \) | 11 | 23 | 170 | 350 |
| \( \Delta B/B \) | \( 6.4 \times 10^{-10} \) | \( 1.0 \times 10^{-8} \) | \( 4.0 \times 10^{-11} \) | \( 6.5 \times 10^{-10} \) |
| \( \Delta \nu/\nu \) | \( 4.2 \times 10^{-8} \) | \( 1.3 \times 10^{-6} \) | \( 2.7 \times 10^{-9} \) | \( 8.4 \times 10^{-8} \) |
| \( \Delta \alpha/\alpha \) | 0.18 | 0.090 | 1.2 \times 10^{-2} | 5.8 \times 10^{-3} |
| \( \Delta \Omega \) | 0.044 | 0.015 | 1.9 \times 10^{-3} | 9.8 \times 10^{-4} |

Table II shows the S/N and the parameter determination accuracy assuming enough data taking, \( T_{obs} \gtrsim \tau_{GW} \). For the optimal case (\( \epsilon = \epsilon_{\text{max}} \)), even 2nd generation detectors can detect the GWs with enough S/N. The detection of the GW is possible when \( \epsilon \gtrsim 0.1\epsilon_{\text{max}} \) and \( \epsilon \gtrsim 0.01\epsilon_{\text{max}} \) for 2nd and 3rd generation detectors, respectively. Thanks to the large Q value \( \sim f_a \times \tau_{GW} \sim 10^9 \), parameters related to the phase of the GW (i.e., \( f_a, \dot{f}_a \) and \( \nu \)) could be determined accurately. When fixing \( B \), errors scales as \( (S/N)^{-2} \) for \( \Delta \Omega \), and \( (S/N)^{-1} \) for other parameters.

Figs. 2 and 3 show the dependence of the S/N and the accuracies on the data taking time \( T_{obs} \). We can clearly
find that the long-term data taking for a day (∼10^5 sec) to months (∼10^7 sec) is necessary for high S/N and parameter accuracies, especially \(B\).

Since a NS has various oscillation modes, it is hard, at this stage, to answer which oscillation mode is mainly excited at the GF. Also, we should consider the mode coupling between the polar Alfvén mode and other modes. If the coupling is strong, the polar Alfvén mode oscillation cannot last so long as ∼ a day to months even if the GF energy are initially injected to the polar Alfvén mode, and the dependence of Alfvén mode frequencies on the magnetic field may become different from Eq. (4). The effects of the crust and superfluidity are also to be investigated in the future.

The change in the rotational frequency \(\dot{\nu}\) could become important as the 1FD. Although the magnetar spindown rate is usually small ∼\(-10^{-12}\)Hz/sec compared with Eq. (9), the sudden increase in the GF phase \(\Delta \nu / \nu \sim 10^{-4}\) has been reported in the August 27, 1998 event from SGR 1900+14 [28]. Since \(\nu \neq \nu_0\), we could also measure \(\dot{\nu}\) independently.

Finally, we should mention the strategic change in the analysis of the GW waveform. When discussing GWs from a single NS or black hole, the waveform usually has a short damping time \(\lesssim 10\text{ sec}\) [29, 30], hence the evolution of the waveform due to the daily and yearly motion of the earth could be neglected. However in the case of the long-term oscillations with \(\gtrsim 1\) day, we have to take into account these effects as in the analysis of GW from a pulsar [31, 32].

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[1] Harry, G.M., (for the LIGO Scientific Collaboration) 2010, Class. Quantum. Grav. 27, 084006.
[2] wwwcascina.virgo.infn.it/advirgo/
[3] Kuroda, K.,(on behalf of the LCGT Collaboration) 2010, Class. Quantum Grav. 27, 084004.
[4] Hild, S. et al, 2010, arXiv:1012.0908
[5] Andersson, N. and Kokkotas, K. D., 1998, MNRAS, 299, 1059.
[6] K.D. Kokkotas and B.G. Schmidt, Living Rev. Relativity 2, 2 (1999), http://www.livingreviews.org/lrr-1999-2
[7] Thompson, C. and Duncan, R. C., 1992, ApJ, 392, L9.
[8] Thompson, C. and Duncan, R. C., 1995, MNRAS, 275, 255.
[9] Duncan, R. C., 1998, ApJ, 498, L45.
[10] Mazets, E. P. et al., 1979, Nature, 282, 587.
[11] Hurley, K. et al., 1999, Nature, 397, 41.
[12] Hurley, K. et al., 2005, Nature, 434, 1098.
[13] Chatterjee, P., Hernquist, L. and Narayan, R. 2000, ApJ, 534, 373.
[14] Ioka, K., 2001, MNRAS, 327, 639.
[15] Corsi, A. and Owen, B. J., arXiv:1102.3421
[16] Abadie, B. P. et al., (The LIGO Science Collaboration), arXiv:1011.4079
[17] Sotani, H. & Kokkotas, K. D., 2009, MNRAS, 395, 1163.
[18] Israel, G. L. et al., 2005, ApJ, 628, L53.
[19] Sotani, H., Kokkotas, K. D. & Stergioulas, N., 2008, MNRAS, 385, 2161.
[20] Lee, U., 2010, MNRAS, 405, 1444.
[21] Inan, U. S. et al., 1999, Geophys.Res.Lett, 26,3357.
[22] Ioka, K. and Sasaki, M. 2004, ApJ, 600, 296; 2003, Phys. Rev. D, 67, 124026.
[23] Thorne, K. S., 1987, 300 Years of Gravitation. Cambridge Univ. Press, Cambridge, UK.
[24] Thrane, E. et al., 2010, arXiv:1012.2150
[25] Taylor, G. B. et al., 2005, ApJ, 634, L93.
[26] Finn, L. S., 1992, Phys. Rev. D, 46, 5236.
[27] Cutler, C., 1998, Phys. Rev. D, 57, 7089.
[28] Woods, P. M. et al., 1999, ApJ, 524, L55.
[29] Tunesada, Y. et al., 2005, Phys. Rev. D, 71, 103005.
[30] Abbott, B. P. et al., (LIGO Science Collaboration), 2009, Phys. Rev. D, 80, 102001.
[31] Niebauer, T. M. et al., 1993, Phys. Rev. D, 47, 3106.
[32] Akutsu, T. et al., 2008, Class. Quantum Grav. 25, 184013