ON THE BEHAVIOR OF FORMAL NEIGHBORHOODS IN THE
NASH SETS ASSOCIATED WITH TORIC VALUATIONS: A
COMPARISON THEOREM

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Abstract. We show that there exists a strong connection between the generic
formal neighborhood at a rational arc lying in the Nash set associated with
a toric divisorial valuation on a toric variety and the formal neighborhood at
the generic point of the same Nash set. This may be interpreted as the fact
that analytically along such a Nash set the arc scheme of a toric variety is a
product of a finite dimensional singularity and an infinite dimensional affine
space.

1. Introduction

1.1. This work establishes, in the case of toric singularities, a comparison result
between two kinds of formal neighborhoods in arc schemes, which until now had
been studied independently, in particular in terms of motivations and involved
techniques.

1.2. The first class of formal neighborhoods we shall consider are those of some
finite-codimensional points of the arc scheme known as stable points. Their study
was motivated by the infamous Nash problem ([29]), which, loosely speaking, may
be understood as the problem of describing the connection between the resolution
of the singularities of a variety and the geometry of the arc space associated with
the variety. An approach to this problem for surfaces was proposed by Lejeune-
Jalabert in terms of a problem of lifting of wedges ([25]). Then Reguera extended
the approach in higher dimensions, putting forward the relevance of the study of the
formal neighborhoods of the generic points of the so-called Nash sets associated with
divisorial valuations, which are the prototypical examples of stable points ([32, 33]).
She showed in particular that these formal neighborhoods are noetherian (later in
[17], a new proof of the result, as well as a proof of the converse statement, were
given), allowing her to establish a curve selection lemma for arc spaces which was
crucially used in subsequent works on the Nash problem ([20, 16, 26]). She pointed
out that there should exist a strong connection between the algebraic properties
of the formal neighborhood of the generic point of the Nash set associated with a
valuation and the geometric properties and invariants of the valuation (see e.g. [33,
Corollary 5.12]).

Later in [28] Mourtada and Reguera showed that the embedding dimension of
such a formal neighborhood equals the Mather discrepancy of the valuation; they
also pointed out the interest and the difficulty of computing the dimension of these
formal neighborhoods.

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1.3. The second kind of formal neighborhoods we are interested in are those of non-degenerate \textit{rational} arcs (here non-degenerate means not entirely contained in the singular locus of the variety). Contrarily to the aforementioned case of stable points, these neighborhoods are not noetherian in general. Yet Grinberg and Kazdhan (when the base field is the field of complex numbers, \cite{21}) and then Drinfeld (over any field, \cite{18}) established a striking finiteness property in this context, showing that any formal neighborhood of this kind may be written as the product of an infinite smooth factor (\textit{i.e.} countably many copies of the formal disk) with the formal neighborhood of a rational point of a scheme of finite type (a "finite formal model" of the arc under consideration). The work of Drinfeld, Grinberg and Kazdhan was motivated by geometric representation theory and Langlands program; subsequent works in this direction include \cite{9, 8, 30}. On the other hand, the first and last named authors of the present paper suggested in \cite{2} that the finite formal models of a rational arc could be interpreted in terms of the singularity at the origin of the arc. In \cite{5} they showed that the origin of the arc is smooth if and only if it admits a trivial finite formal model, supporting the fact that the finite formal models should provide a sensible measure of the complexity of the involved singularities. In \cite{4} the case of monomial plane curves singularities is explored whereas in \cite{6} finite formal models of rational arcs on toric varieties are studied. Here it is also worth mentioning the recent work \cite{13} of Chiu, de Fernex and Docampo showing in particular that the non-degeneracy condition in Drinfeld-Grinberg-Kazdhan theorem is also necessary (previous related works are \cite{3} and \cite{14}).

1.4. Summing up, the two classes of formal neighborhoods (non-degenerate stable points / rational arcs) on the arc space of an algebraic variety share two common features: a finiteness property and interesting (and potential in full generality) connections with the singularities of the variety. Yet for each class the manifestation of these properties seems to take rather different forms. It is natural to ask whether the phenomena observed for both classes are the emanation of a common mechanism.

The aforementioned work \cite{6} provides the first piece of evidence that there may exist a strong connection between the two classes of formal neighborhoods, at least in the case of toric singularities. On one hand, the formula obtained there for the embedding dimension of the minimal formal model of a non-degenerate arc on a toric variety (Theorem 1.4 of \textit{op.cit.}) presents a striking similarity with the one obtained more generally by Reguera and Mourtada in the context of stable points (\cite{28}, Theorem 3.4). On the other hand, the fact that the formal neighborhood of a generic rational arc of the Nash set associated with a toric valuation is constant (\cite{6, Theorem 1.3}) makes it senseful to ask whether such a formal neighborhood can be compared with the formal neighborhood of the generic point of the Nash set; the question was explicitly raised in \cite{7, Question 7.13}.

1.5. In the present work, we answer positively the question, thus establishing for the first time a strong direct connection between the two important classes of formal neighborhood discussed before. More precisely, we show the following result.

\textbf{Theorem 1.6.} Let \(k\) be a field of characteristic zero. Let \(V\) be an affine normal toric \(k\)-variety. Let \(v\) be a toric divisorial valuation on \(V\) and \(\mathcal{N}_v\) be the associated Nash set. Let \(\eta_v\) be the generic point of \(\mathcal{N}_v\) and \(\kappa_v\) be the residue field of \(\eta_v\).

For a general \(k\)-rational arc \(\alpha \in \mathcal{N}_v\) there exists an isomorphism of \(\kappa_v\)-formal schemes between \(\mathcal{L}_\infty(V)_{\eta_v} \hat{\otimes}_{\kappa_v} \kappa_v[[T_i]_{i \in \mathbb{N}}]\) and \(\mathcal{L}_\infty(V)_{\alpha} \hat{\otimes}_{k} \kappa_v\).

Geometrically, this might be interpreted as the fact that \(\mathcal{L}_\infty(V)\) is analytically along \(\mathcal{N}_v\) a product of a finite dimensional singularity and an infinite dimensional affine space.
1.7. We stress that theorem 1.6 is a direct consequence of theorem 6.18 stated and proved below, which is our main statement and gives a more precise result. In particular, it provides an explicit description of the formal neighborhood of the generic point of the Nash set associated with a toric valuation in terms of a noetherian formal scheme associated with the same valuation, which had been introduced in the previous work [6]. Some geometric consequences of this description are explained below.

1.8. We now describe the strategy and techniques involved in the proof of theorem 6.18. First we insist on the fact that although the results of [6] inspired the present work, there only the case of formal neighborhood of rational arcs was treated. The main point of the present article is to show that the result obtained there agrees, in some sense, with the one obtained for generic points of toric Nash sets. It is important to note that the techniques in op.cit. can not be directly imported. The reason is that one of the crucial ingredients in op.cit. is the interpretation of the formal neighborhood of a rational arc as a parameter space for the infinitesimal deformations of the arc, allowing in particular to take full advantage of the Weierstrass preparation and division theorems. This point of view, already used by Drinfeld in [18], provides a very efficient tool in the context of rational arcs, but to the best of our knowledge no sensible analog exists in the context of stable points, and this constitutes a major technical issue for the proof of the comparison theorem.

The important steps in the proof of our result are the following.

1. As a preliminary step, we obtain a sensible presentation of the formal neighborhood of the generic point of a toric Nash set; this is done in section 4 in an “abstract” way and some extra work is needed in section 6 in order to apply the results to the toric context. This kind of computation was considered more generally by Reguera in [33, 34] and Reguera and Mourtada in [28]. However, the results as given there are not adapted to our purposes, and our approach, based on a direct application of a version of the Hensel’s lemma for an infinite set of variables, is somewhat different. We hope that this kind of approach may shed some new light on the computation of formal neighborhoods of stable points.

2. We compare the obtained presentation with to the one obtained in [6] for rational arcs. The basic idea is that despite no interpretation in terms of deformations of the presentation exists, we are still able to transform it using the Weierstrass factorization and division theorems in a rather intricate way, ending up with the conclusion that in some sense it coincides with the finite model obtained in op.cit.. The technical core of the argument is in section 5, the results of which are applied in section 6 to the toric context.

1.9. As a direct by-product of our main result, we obtain a formula for the dimension and the number of irreducible components of the formal neighborhood of the generic point of a Nash toric set (see corollary 6.21 for the precise statement).

This in turn implies the following statement. Recall that a divisorial valuation \( v \) on an algebraic variety \( V \) is essential if for every resolution \( W \to V \) of the singularities of \( V \), the center of \( v \) on \( W \) is an irreducible component of the exceptional locus of the resolution. Here we say that such a divisorial valuation is strongly essential if for every resolution \( W \to V \) of the singularities of \( V \), the center of \( v \) on \( W \) is an irreducible component of codimension 1 of the exceptional locus of the resolution.

**Theorem 1.10.** Let \( V \) be a toric variety and \( v \) a divisorial toric valuation on \( V \), centered in the singular locus of \( V \). Let \( \eta_v \) be the generic point of the Nash set associated with \( v \). Then the formal neighborhood \( \mathcal{L}_\infty(V)_{\eta_v} \) of \( \eta_v \) in the arc scheme
\( \mathcal{L}_\infty(V) \) associated with \( \nu \) is of dimension 1 if and only if \( \nu \) is strongly essential. Moreover, in this case, \( \mathcal{L}_\infty(V)_{\eta_\nu} \) is irreducible and the associated reduced formal scheme is a formal disk.

Recall that Reguera showed that in general, if \( \eta_\nu \) is the generic point of the Nash set associated with an essential divisorial valuation \( \nu \) on an algebraic variety \( V \), then \( \mathcal{L}_\infty(V)_{\eta_\nu} \) is irreducible of dimension 1 if and only if \( V \) satisfies a property of lifting wedges centered at \( \eta_\nu \), and that this condition implies that \( \nu \) is a Nash valuation (see [33, Corollary 5.12]). Note that the latter property of lifting wedges is stronger than the one considered in [32, Section 5], which was shown to be equivalent to the fact that \( \nu \) is a Nash valuation.

In view of theorem 1.10 and of the results of [6], the following question seems natural.

**Question 1.11.** Let \( \eta_\nu \) is the generic point of the Nash set associated with an essential divisorial valuation \( \nu \) on an algebraic variety \( V \). Are the following conditions equivalent?

1. The formal neighborhood \( \mathcal{L}_\infty(V)_{\eta_\nu} \) is irreducible of dimension 1.
2. The valuation \( \nu \) is Nash and strongly essential.
3. The minimal formal model of a generic rational arc the Nash set associated with \( \nu \) is irreducible of dimension 0

Our results show that the answer is positive in case \( \nu \) is a toric valuation on a toric variety. Note also that if \( \nu \) is a terminal valuation (hence strongly essential), \( \nu \) is a Nash valuation and it is known that (1) holds (see [16] and [28, Corollary 4.3]).

1.12. Of course, it is natural to ask whether there exist other classes of varieties for which the formal neighborhood of a rational non-degenerate arc is generically constant on Nash sets, and, if it is so, whether a comparison result akin to theorem 6.15 still holds. In [7], it is observed that for curve singularities the genericity property holds. For this class of singularities, the first and second authors of the present work will address the question of the validity of the comparison theorem in a forthcoming paper. For normal varieties equipped with a “big” action of an algebraic group (typically for spherical varieties), it is very likely that at least the genericity property holds. It would also be interesting to be able to connect this kind of property to the status of the Nash problem (which is known to admit a positive answer in particular for curve and toric singularities).

1.13. It is our hope that understanding more precisely the connection between formal neighborhoods of rational and stable points on arc spaces could open the way to the study of stable points via the deformation-theoretic point of view which is so useful in the context of rational points. At the end of the paper, we give an example of a computation for a toric valuation illustrating how our comparison theorem provides a more tractable result than a direct approach.

## 2. General conventions and notation

2.1. Throughout the whole article, we designate by \( k \) a field of characteristic zero. \( \text{Alg}_k \) (resp. \( \text{Sch}_k \)) is the category of \( k \)-algebras (resp. of \( k \)-schemes). If \( K \) is a field extension of \( k \), we denote by \( \mathcal{L}_c\text{Cpl}_K \) the category of complete local \( k \)-algebras with residue field \( k \)-isomorphic to \( K \). For any category \( \mathcal{C} \) and any objects \( A, B \in \mathcal{C} \), we denote by \( \text{Hom}_\mathcal{C}(A, B) \) the set of morphisms from \( A \) to \( B \) in the category.
2.2. A $k$-variety is a $k$-scheme of finite type. The non-smooth locus of the structural morphism of a $k$-variety $V$ is the singular locus of $V$ and its associated reduced $k$-scheme is denoted by $\text{NSm}(V)$. If $V$ is an affine $k$-variety and $f$ is a regular function on $V$, we denote by $\{f \neq 0\}$ the distinguished open subset of $V$ where $f$ does not vanish and by $\{f = 0\}$ the closed set $V \setminus \{f \neq 0\}$.

2.3. Let $R$ be a ring, let $i$ be an ideal of $R$ and $f \in R$. We denote by $R_I$ the localization of $R$ with respect to the multiplicative subset $\{f^r : r \in \mathbb{N}\}$. We denote by $i : f^\infty$ the ideal $\{g \in R : f^rg \in i \text{ for some } r \in \mathbb{N}\}$. Let $R'$ be another ring and $\vartheta : R \to R'$ a morphism of rings. For the sake of easy reading and abusing notation, the extension ideal of $i$ in $R'$ via the morphism $\vartheta$ is denoted by $\vartheta(i)$, or even by $i$ if the involved morphism $\vartheta$ is clear from the context (for example if $R$ is a subring of $R'$).

2.4. Let $R[X,\omega; \omega \in \Omega]$ be a polynomial ring and $f \in R$. Let $S$ be a $R$-algebra and $\{s_\omega\}_{\omega \in \Omega}$ a collection of elements in $S$. Then we denote by $f|_{X=\omega} \in S$ the image of $f$ by the unique morphism of $R$-algebra $R[X,\omega] \to S$ mapping $X$ to $s_\omega$ for each $\omega \in \Omega$.

2.5. Let $(A,\mathfrak{m}_A)$ be a complete local ring. An element $f = \sum_{i \in \mathbb{N}} f_it_i^i \in A[[t]]$ is regular if $f \notin \mathfrak{m}_A[[t]]$. Its order is $\inf \{i \in \mathbb{N}, f_i \notin \mathfrak{m}_A\}$. Let $d \in \mathbb{N}$. A Weierstrass polynomial of order $d$ is a monic polynomial of degree $d$, whose order as a regular element of $A[[t]]$ is $d$. We shall make a crucial use of the following classical results (the Weierstrass division and preparation theorems, see e.g. [24] Theorems 9.1 & 9.2). Let $f \in A[[t]]$ be a regular element of order $d$. Then:

(i) There exists a unique pair $(p(t), u(t)) \in A[[t]]^2$ such that $f(t) = p(t)u(t)$, $p(t)$ is a Weierstrass polynomial of degree $d$ and $u(t)$ is a unit in $A[[t]]$.

(ii) Let $g \in A[[t]]$; then there exists a unique pair $(q(t), r(t)) \in A[[t]]^2$ such that $g(t) = f(t)q(t) + r(t)$ and $r(t)$ is a polynomial of degree $< d$.

Note that in particular any regular element of $A[[t]]$ is not a zero divisor in $A[[t]]$.

3. Recollection on arc scheme and toric varieties

The crucial objects of our study are arc schemes and toric varieties. For the convenience of the reader, we give in this section an overview of the main definitions and properties that we will use in the article. Along the way, we fix some notation and state and prove some technical lemmas useful for the sequel.

3.1. Since we are only interested in local properties of arc schemes, we limit ourselves to the case of arc schemes associated with affine varieties. Proofs as well as more details on the general theory of arc schemes are to be found e.g. in [12, 1].

To every affine $k$-variety $V$ one attaches its arc scheme $\mathcal{L}_\infty(V)$ which is an affine $k$-scheme characterized by the fact that for every $k$-algebra $R$ one has a functorial bijection

$$\text{Hom}_{\text{Sch}_k}(\text{Spec}(R), \mathcal{L}_\infty(V)) \cong \text{Hom}_{\text{Sch}_k}(\text{Spec}(R[[t]]), V).$$

(3.1)

A point of $\mathcal{L}_\infty(V)$ is called an arc. The above functorial bijection and the $k$-algebra morphism $R[[t]] \to R$ mapping $t$ to $0$ induces a morphism of $V$-scheme $\mathcal{L}_\infty(V) \to V$ which sends an arc $\alpha$ to its base-point $\alpha(0)$.

We will need explicit equations of the affine scheme $\mathcal{L}_\infty(V)$ in terms of equations of the affine $k$-variety $V$. We begin with the case of the affine space. Let $Z = \{Z_1, \ldots, Z_n\}$ be a finite set of indeterminates. Consider the ring $k[Z_\infty] := k[Z_i, j : i \in \{1, \ldots, n\}, j \in \mathbb{N}]$ and the $k$-algebra morphism $\varphi : k[Z] \to k[Z_\infty]$ mapping $Z_i$ to $Z_i,0$. Then the affine $k$-scheme $\mathcal{L}_\infty(\mathbb{A}^n_k)$ is isomorphic to $\text{Spec}(k[Z_\infty])$. The morphism $\varphi : k[Z] \to k[Z_\infty]$ induces a morphism $O(\mathbb{A}^n_k) \to O(\mathcal{L}_\infty(\mathbb{A}^n_k))$ dual to the morphism $\alpha \mapsto \alpha(0)$. 
For every \( F \in k[\mathcal{Z}] \), define \( \{F_i\}_{i \in \mathbb{N}} \in k[\mathcal{Z}_\infty]^{[\mathbb{N}}\) by the following identity in \( k[\mathcal{Z}_\infty][t]\):

\[
F|_{\mathcal{Z},s}^{\iota} = \sum_{s \in \mathbb{N}} F_at^s.
\]  

(3.2)

Note that HS: \( F \mapsto \sum_{s \in \mathbb{N}} F_at^s \) is a morphism of \( k \)-algebras \( k[\mathcal{Z}] \to k[\mathcal{Z}_\infty][t] \). If \( i \) is an ideal of \( k[\mathcal{Z}] \), generated by a family \( \{F_\delta : \delta \in \Delta\} \), the ideal \( \langle F_{\delta,s} : \delta \in \Delta, s \in \mathbb{N}\rangle \) does not depend on the choice of the generating family. We denote it by \([i]\). (This notation is borrowed from differential algebra; for more information on the link between differential algebra and algebra schemes, see e.g. [1].) The following lemma will be useful.

**Lemma 3.2.** Let \( i \) be an ideal of \( k[\mathcal{Z}] \). Let \( d \leq h \) and \( F \in k[\mathcal{Z}] \) such that \( F \) lies in the quotient ideal \( \iota : \left( \prod_{i=1}^{l} \mathcal{Z}_i \right)^\infty \). Let \( (k_i) \in \mathbb{N}^d \) and \( a \) be the ideal \( \langle \mathcal{Z}_{i,s} : 1 \leq i \leq h, 0 \leq s \leq k_i \rangle \) of \( k[\mathcal{Z}_\infty] \). Let \( G = \prod_{i=1}^{h} \mathcal{Z}_{i,k_i} \). Then in the localization \( k[\mathcal{Z}_\infty]_{G} \), the ideal \([F]\) is contained in the ideal \([i] + \mathfrak{a}\).

**Proof.** Let \( H \in i \) and \( N \in \mathbb{N} \) such that \( \left( \prod_{i=1}^{l} \mathcal{Z}_i \right)^{N}F = H \). Applying HS and using the very definition of \( \mathfrak{a} \), one obtains the relation

\[
\prod_{i=1}^{d}(t^k[Z_{i,k_i} + t(\ldots)])^N \text{HS}(F) = \text{HS}(H) \pmod{\mathfrak{a}[t]}.\]

Thus, setting \( K := \sum_{i=1}^{d} Nk_i \), for \( s < K \) one has \( H_s \in \mathfrak{a}[t] \) and one may write

\[
\prod_{i=1}^{d}(Z_{i,k_i} + t(\ldots))^N \text{HS}(F) = \sum_{s \geq 0} H_{s+k}t^s \pmod{\mathfrak{a}[t]}.\]

By the definition of \( G \), the series \( \prod_{i=1}^{d} [Z_{i,k_i} + t(\ldots)]^N \) is invertible in \( k[\mathcal{Z}_\infty]_{G}[t] \). That concludes the proof. \( \square \)

Now if \( i \) is an ideal of \( k[\mathcal{Z}] \) and the affine \( k \)-scheme \( V \) is presented as \( \text{Spec}(k[\mathcal{Z}]/[i]) \), then the affine \( k \)-scheme \( \mathcal{L}_\infty(V) \) is isomorphic to \( \text{Spec}(k[\mathcal{Z}_\infty]/[i]) \). The morphism \( \phi : k[\mathcal{Z}] \to k[\mathcal{Z}_\infty] \) induces a morphism \( \mathcal{O}(V) \to \mathcal{O}(\mathcal{L}_\infty(V)) \) dual to the morphism \( \phi \cdot \alpha \to \alpha(0) \).

The morphism \( \text{HSV} : \mathcal{O}(V) \to \mathcal{O}(\mathcal{L}_\infty(V))[t] \), dual to the so-called universal arc on \( V \). If \( R \) is a \( k \)-algebra and \( \alpha^* : \mathcal{O}(\mathcal{L}_\infty(V)) \to R \) is an \( R \)-point of \( \mathcal{L}_\infty(V) \), inducing a \( k \)-algebra morphism \( \alpha^*[t] : \mathcal{O}(\mathcal{L}_\infty(V))[t] \to R[t] \), the corresponding \( R[t][t] \) point of \( \mathcal{O}(V) \) by bijection (3.1) is \( \alpha^*[t] \circ \text{HSV} \).

Let \( W \) be a closed \( k \)-subscheme of \( V \) and \( j = (G_\gamma : \gamma \in \Gamma) \) be an ideal of \( k[\mathcal{Z}] \) such that \( W \cong \text{Spec}(k[\mathcal{Z}]/[i] + j) \). Then

\[\mathcal{L}_\infty(W) \cong \text{Spec}(k[\mathcal{Z}_\infty]/[i] + [j])\]

identifies with a closed subscheme of \( \mathcal{L}_\infty(V) \) and the open subset \( \mathcal{L}_\infty(V) \setminus \mathcal{L}_\infty(W) \) of \( \mathcal{L}_\infty(V) \) is the union of the distinguished open subsets \( \{G_\gamma, \alpha \neq 0\} \) for \( \gamma \in \Gamma \) and \( \alpha \in \mathbb{N} \).

An element of \( \mathcal{L}_\infty(V) \setminus \mathcal{L}_\infty(n\text{Sm}(V)) \) is called a non-degenerate arc.

3.3. (See, e.g., [19][22][23].) Let \( \alpha \) be an arc of \( V \) with residue field \( \kappa(\alpha) \), inducing a \( \kappa(\alpha)[t] \)-point of \( V \). Composing the morphism \( \mathcal{O}(V) \to \kappa(\alpha)[t] \) with the \( t \)-valuation defines a semivaluation \( \text{ord}_\alpha : \mathcal{O}(V) \to \mathbb{N} \cup \{+ \infty\} \). Now let \( v \) be a divisorial valuation over \( V \). The associated Nash set, or maximal divisorial set, is the closure in \( \mathcal{L}_\infty(V) \) of the set \( \{\alpha \in \mathcal{L}_\infty(V) \mid \text{ord}_\alpha = v\} \). It is an irreducible subset of \( \mathcal{L}_\infty(V) \), denoted by \( \mathcal{N}_v \).
3.4. From now on, we introduce some notation and basic facts on normal toric varieties. (For further details, e.g., see [15] Sections 1.1 and 1.2.) Since we are studying local properties, in this article we can restrict ourselves to the case of affine normal toric varieties.

Let $d$ be a positive integer and $T$ a split algebraic $k$-torus of dimension $d$. Let $N := \text{Hom}(\mathbb{G}_{m,k}, T)$ be the group of its cocharacters which is a free $\mathbb{Z}$-module of rank $d$ (i.e., a lattice isomorphic to $\mathbb{Z}^d$) and $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual $\mathbb{Z}$-module (i.e., the group of characters of $T$). We denote by $N_\mathbb{R} = N \otimes \mathbb{R}$ (resp. $M_\mathbb{R} = M \otimes \mathbb{R}$) the $\mathbb{R}$-vector space of dimension $d$ associated with $N$ (resp. $M$). We have a $\mathbb{R}$-bilinear canonical map $\langle , \rangle : M_\mathbb{R} \times N_\mathbb{R} \rightarrow \mathbb{R}$. The points of the lattices $N$ and $M$, considered as points of the associated vector spaces, are called their integral points. We will simply call a cone of $N_\mathbb{R}$ a strongly convex rational polyhedral cone of the vector space $N_\mathbb{R}$ (i.e. a convex cone generated by finitely many elements of $N$, which moreover does not contain any line).

3.5. Let $\sigma$ be a cone of $N_\mathbb{R}$. By the Gordan Lemma (e.g., [15] Proposition 1.2.17]), the semigroup $S_\sigma := \sigma^\vee \cap M$ is finitely generated. The spectrum of the $k$-algebra $k[S_\sigma]$ associated with the semigroup $S_\sigma$ then defines a normal affine toric variety $V_\sigma$ with torus $T$. Note that every affine normal toric variety with torus $T$ is of the form $V_\sigma$ for $\sigma$ a cone of $N$ (see e.g., [15] Theorem 1.3.5]). For every $m \in S_\sigma$, we denote by $\chi^m$ the regular function on $V_\sigma$ defined by $m$. Recall that for every $k$-algebra $A$, the set $\text{Hom}_{\text{Alg}}(k[S_\sigma], A)$ is in natural bijection with the set of semigroup morphism $S_\sigma \rightarrow A$, where the semigroup structure on $A$ is induced by the multiplication.

Let $\{m_1, \ldots, m_h\}$ be the minimal set of generators of the semigroup $S_\sigma$. We may and shall assume in the sequel that the set $\{m_1, \ldots, m_d\}$ is a $\mathbb{Z}$-basis of $M$. If we call $z_1, \ldots, z_h$ respectively $\chi^{m_1}, \ldots, \chi^{m_h}$, we deduce that $\mathcal{O}(V_\sigma) := k[S_\sigma] = k[z_1, \ldots, z_h]$. Moreover the closed subscheme defined by the ideal $\prod_{1 \leq i \leq h} (z_i)$ has for support the closed set $V_\sigma \setminus T$, and the same holds for the ideal $\prod_{1 \leq i \leq d} (z_i)$.

Now let $\mathcal{L}_\infty^\sigma(V_\sigma)$ be the open subset of $\mathcal{L}_\infty^\sigma(V_\sigma)$ defined by $\mathcal{L}_\infty^\sigma(V_\sigma) := \mathcal{L}_\infty^\sigma(V_\sigma) \setminus \mathcal{L}_\infty^\sigma(V_\sigma \setminus T)$. Thus, by (331) for any $\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma)$, one has $\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma)$ if and only if for every $m \in S_\sigma$, $\alpha^*(\chi^m) \neq 0$ if and only if for every $1 \leq i \leq d$ one has $\alpha^*(z_i) \in \kappa(\alpha)[[\mathbb{N}]] \setminus \{0\}$. Therefore one has

$$\mathcal{L}_\infty^\sigma(V_\sigma) = \bigcap_{i=1}^d \bigcup_{s \in \mathbb{N}} \{z_{i,s} \neq 0\}.$$

3.6. Let $n$ be a point of $\sigma \cap N$. For $f \in \mathcal{O}(V_\sigma)$, set

$$\text{ord}_n(f) = \langle n, f \rangle := \inf_{\chi^m \in f} \langle m, n \rangle$$

(here $\chi^m \in f$ means that the coefficient of $\chi^m$ in the decomposition of $f$ is not zero). Then $\text{ord}_n$ is a divisorial toric valuation on $V_\sigma$, and one easily sees that $n \mapsto \text{ord}_n$ is a bijection between $\sigma \cap N$ and the set of divisorial toric valuations on $V_\sigma$. From now on we shall identify the later set with $N \cap \sigma$.

Let $\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma)$. The semigroup morphism $m \in S_\sigma \mapsto \text{ord}_n(\alpha^*(\chi^m))$ extends uniquely to a group morphism $n_\alpha : M \rightarrow \mathbb{Z}$ which is nonnegative on $S_\sigma$. In other words $n_\alpha$ is the unique element of $N \cap \sigma$ satisfying: for every $1 \leq i \leq h$, $\text{ord}_n(\alpha^*(z_i)) = \langle m_i, n_\alpha \rangle$. For every $n \in \sigma \cap N$, we set

$$\mathcal{L}_\infty^\sigma(V_\sigma)_n := \{\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma) ; n_\alpha = n\}$$

and

$$\mathcal{L}_\infty^\sigma(V_\sigma)_{\geq n} := \{\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma) ; n_\alpha \in n + \sigma\}.$$

Thus $\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma)_{\geq n}$ if and only if for every $m \in S_\sigma$ one has $\text{ord}_n(\alpha^*(\chi^m)) \geq \langle m, n \rangle$ if and only if for every $1 \leq i \leq h$ one has $\text{ord}_n(\alpha^*(z_i)) \geq \langle m_i, n \rangle$. If $\alpha \in \mathcal{L}_\infty^\sigma(V_\sigma)_{\geq n}$ and $\varphi_\alpha : S_\sigma \rightarrow \kappa(\alpha)[[\mathbb{N}]]$ is the associated semigroup morphism,
then $m \mapsto t^{-\langle m, n \rangle}\phi_{\alpha}$ defines a semigroup morphism $\psi_{\alpha}: S_{\sigma} \rightarrow (\kappa(\alpha))[t]$, and $n_\alpha = n$ if and only if $\psi_{\alpha}(S_{\sigma}) \subset (\kappa(\alpha))[t]$ if and only if for $1 \leq i \leq d$ one has $\text{ord}_{i}(\alpha^{i}(\langle m, n \rangle)) = (m_i, n)$. Note also that the element of $L^\infty(\kappa(t))(k)$ corresponding to the semigroup morphism $S_{\sigma} \rightarrow k[t]$, $m \mapsto t^{\langle m, n \rangle}$ lies in $L^\infty(\kappa(t))n$, which is therefore nonempty.

The following lemma will be useful for describing the generic points of the Nash sets associated with toric valuations.

**Lemma 3.7.** Let $n \in \sigma \cap N$.

(i) One has

$$L^\infty_{\infty}(V_{\sigma}) \geq n = L^\infty_{\infty}(V_{\sigma}) \cap \bigcap_{i=1}^{d} \bigcap_{s=0}^{h} \{z_{i,s} = 0\}.$$ 

(ii) One has

$$L^\infty_{\infty}(V_{\sigma})n = L^\infty_{\infty}(V_{\sigma}) \cap \left( \bigcap_{i=1}^{h} \bigcap_{s=0}^{(m_i, n) - 1} \{z_{i,s} = 0\} \right) \cap \bigcap_{i=1}^{d} \{z_{i,(m_i, n)} \neq 0\}.$$ 

(iii) The closure of $L^\infty_{\infty}(V_{\sigma})n$ coincides with the Nash set $N_{\sigma} = N_{\text{ord}_n}$ (see §3.3) associated with the toric valuation $n$.

(iv) One has

$$L^\infty_{\infty}(V_{\sigma})n = N_{\sigma} \cap L^\infty_{\infty}(V_{\sigma}) \cap \bigcap_{i=1}^{d} \{z_{i,(m_i, n)} \neq 0\}.$$ 

In particular $L^\infty_{\infty}(V_{\sigma})$ is a nonempty open subset of $N_{\sigma}$.

**Proof.** Assertion (i) and (ii) are nothing but a reformulation of the above descriptions of $L^\infty_{\infty}(V_{\sigma})n$ and $L^\infty_{\infty}(V_{\sigma}) \geq n$.

For a proof of (iii), see [23 Example 2.10].

Assertion (iv) is a straightforward topological consequence of (ii) and (iii). □

3.8. An explicit description of $V_{\sigma}$ as a closed subscheme of the affine space $\mathbb{A}^h$ will be useful in the sequel.

Recall that $\{m_1, \ldots, m_h\}$ is the minimal set of generators of $S_{\sigma}$, and that we may and shall assume that $\{m_1, \ldots, m_h\}$ is a $\mathbb{Z}$-basis of $M$. Let $\{e_i: i \in \{1, \ldots, h\}\}$ be the canonical basis of $\mathbb{Z}^h$. Being given $\ell = (\ell_1, \ldots, \ell_h) \in \mathbb{Z}^h$, we set

$$\ell^+ = \sum_{\ell_i \geq 0} \ell_i e_i \quad \text{and} \quad \ell^- = - \sum_{\ell_i < 0} \ell_i e_i,$$

which are both elements of $\mathbb{N}^h$. Note that $\ell = \ell^+ - \ell^-$. On the other hand, for $\ell \in \mathbb{N}^h$, set $Z_{\ell} := \prod_{i=1}^{h} Z_{\ell_i}^{\ell_i}$ and $F_{\ell} := Z_{\ell^+} - Z_{\ell^-}$.

Mapping $e_i$ to $\pi(e_i) := m_i$ induces an exact sequence of groups

$$0 \rightarrow L \rightarrow \mathbb{Z}^h \xrightarrow{\pi} M \rightarrow 0,$$

where $L$ is a subgroup of $\mathbb{Z}^h$. For $\ell \in L$ and $n \in N$, we set

$$\langle n, \ell \rangle := \langle n, \sum_{i=1}^{h} \ell_i m_i \rangle = - \langle n, \sum_{\ell_i < 0} \ell_i m_i \rangle.$$

By [15] Proposition 1.1.9, the ideal of $k[\mathbb{Z}]$ defining $V_{\sigma}$ is

$$i_{\sigma} := (F_{\ell}; \ell \in L).$$

(3.4)
The set \( \{m_1, \ldots, m_d\} \) being a \( \mathbb{Z} \)-basis of \( M \), for \( q \in \{d+1, \ldots, h\} \), we can write the element \( m_q \) as a linear combination with (possibly negative) integer coefficients of \( m_1, \ldots, m_d \). Thus we have in \( L \) an element \( \ell_q = (\ell_{q,1}, \ldots, \ell_{q,h}) \) such that \( \ell_{q,q} = 1 \) and \( \ell_{q,q'} = 0 \) for every \( q' \in \{d+1, \ldots, h\} \setminus \{q\} \). The element \( \ell_q \in L \) induces an element

\[
F_{\ell_q} = Z_q \prod_{i=1}^{d} Z_{i,q}^{\ell_{q,i}} - \prod_{i=1}^{d} Z_i^{\ell_{q,i}} = 0
\]

in the ideal \( i_\tau \). We observe that in the binomial \( F_{\ell_q} \) none of the variables \( Z_{d+1}, \ldots, Z_h \) appears, excepting \( Z_q \).

**Lemma 3.9.** Let \( j := \langle F_{\ell_q} : d+1 \leq q \leq h \rangle \).

(i) Set \( G_d := \prod_{i=1}^{d} Z_i \). For every \( \ell \in L \), \( F_{\ell} \) lies in the quotient ideal \( j : G_d^\infty \). In other words, the ideal \( i_\sigma \) vanishes in \( k[Z_{\infty}]_{G_d} / j \).

(ii) Let \( n \in \sigma \cap N \) and \( a_n \) be the ideal \( \langle Z_{i,n_i} : 1 \leq i \leq h, 0 \leq s_i < \langle n, m_i \rangle \rangle \) of \( k[Z_{\infty}] \). Let \( G_n := \prod_{i=1}^{d} Z_{i,n,m_i} \). Then the ideals \( [i_\sigma] + a_n \) and \( [j] + a_n \) coincide in the localization \( k[Z_{\infty}]_{G_n} \).

**Proof.** Set \( G_h := \prod_{i=1}^{h} Z_i \). Since \( \langle \ell_q : d+1 \leq q \leq h \rangle \) spans the lattice \( L \), \([3.5]\) Lemma 12.2 shows that \( i_\sigma \) vanishes in \( k[Z_{\infty}]_{G_d} / j \). But \([3.5]\) shows that the natural morphism \( k[Z_{G_d}] \to k[Z_{G_h}] \) induces an isomorphism \( k[Z_{G_d}] / j \cong k[Z_{G_h}] / j \). This shows (i).

By (i) and lemma \([3.2]\) in the localization \( k[Z_{\infty}]_{G_n} \), the ideal \( [i_\sigma] \) is contained in \( [j] + a_n \). Since the inclusion \( [j] \subset [i_\sigma] \) holds by definition, one deduces that (ii) also holds.

#### 4. Technical machinery for computing the formal neighborhood at the generic point of the Nash set

In this section we develop the technical results which we will use in section \([5]\) to obtain a convenient presentation of the formal neighborhood of the generic point of the Nash set associated with a divisorial toric valuation. The main result of this section is theorem \([4.1]\) whose hypotheses are formulated in a somewhat abstract form. In section \([6]\) we will verify that these hypotheses hold in the toric setting.

4.1. We first state a version of the Hensel’s lemma for an arbitrary set of variables. The proof is basically the same as in the case of a finite set of variables. Since we have not been able to find a convenient reference, we include it.

**Proposition 4.2.** Let \((A, \mathfrak{m}_A)\) be a complete local ring with residue field \( \kappa \). Let \( I \) be a set and \( \{Y_i\} \subset I \) be a collection of indeterminates. Let \( J \) be a set and \( \{F_j : j \in J\} \) be a collection of elements in \( A[Y] \). For \( y \in A^I \), we denote by \( J_y \) the \( A \)-linear application \( A^J \to A^I \) induced by the Jacobian matrix \([\partial_{F_j}F_j]|_{Y=y}\), and by \( F|_{Y=y} \in A^J \) the \( J \)-tuple \( \langle F_j|_{Y=y} : j \in J \rangle \).

We assume that there exists \( y^{(0)} \in A^I \) such that:

1. One has \( F|_{Y=y^{(0)}} = 0 \pmod{\mathfrak{m}_A} \).
2. The \( \kappa \)-linear application \( \kappa^J \to \kappa^I \) deduced from \( J_y^{(0)} \) by reduction modulo \( \mathfrak{m}_A \) is invertible.

Then there exists a unique element \( Y = (Y_i) \in A^I \) such that:

1. One has \( F|_{Y=Y} = 0 \).
2. For every \( i \in I \), one has \( Y_i = y_i^{(0)} \pmod{\mathfrak{m}_A} \).
Proof. We begin with two remarks.

First, note that though in this context the Jacobian matrix may have an infinite number of rows and columns, each row has only a finite number of nonzero entries, thus $J_y$ is well defined for any $y$ in $A^l$. Also, by assumption, there exists an $A$-linear application

$$K_{y^{(0)}} : A^l \rightarrow A^l$$

such that $K_{y^{(0)}} J_{y^{(0)}} = \text{Id}_{A^l}$ (mod $\mathcal{M}_A$) and $J_{y^{(0)}} K_{y^{(0)}} = \text{Id}_{A^l}$ (mod $\mathcal{M}_A$).

Second, note that by the Taylor formula, for $y \in A^l$, there exists a family \{\(H_{i_1,i_2} : i_1,i_2 \in I\)\} of elements of $A[Y]^I$, depending on $y$ and the $F_j$'s, such that for every $j \in J$, $H_{i_1,i_2,j} = 0$ for all but finitely many $(i_1,i_2)$ and for every $z \in A^l$, one has

$$F|_{Y = y + z} = F|_{Y = y} + J_y(z) + \sum_{i_1,i_2 \in I} z_{i_1} z_{i_2} H_{i_1,i_2} \mid_{Y = y + z}. \quad (4.1)$$

Note that here and elsewhere the notation we use is a condensed form for writing a possibly infinite number of relations, each of them being easily verified.

We show by induction that for every $c \geq 0$, there exists a family $y^{(c)} = (y^{(c)}; i \in I)$ of elements of $A$, unique modulo $\mathcal{M}_A^{c+1}$, such that $y^{(c)} = y^{(0)}$ (mod $\mathcal{M}_A$) and $F|_{Y = y^{(c)}} = 0$ (mod $\mathcal{M}_A^{c+1}$). The case $c = 0$ is given by our assumptions.

Now take $c \in \mathbb{N}$ and assume that our induction statement holds for $c$. Consider the equation

$$F|_{Y = y^{(c)} + z} = 0 \pmod{\mathcal{M}_A^{c+2}} \quad (4.2)$$

with unknown $z = (z_i) \in A^l$ such that $z = 0 \pmod{\mathcal{M}_A^{c+1}}$. Since $y^{(c)} = y^{(0)}$ (mod $\mathcal{M}_A$), the Jacobian matrices $[\partial_H F_j]|_{Y = y^{(0)}}$ and $[\partial_H F_j]|_{Y = y^{(c)}}$ are equal modulo $\mathcal{M}_A$. Since $z = 0 \pmod{\mathcal{M}_A^{c+1}}$, one thus has

$$J_{y^{(c)}}(z) = J_{y^{(0)}}(z) \pmod{\mathcal{M}_A^{c+2}}.$$

Thus by (1.1) and using again $z = 0 \pmod{\mathcal{M}_A^{c+1}}$, equation (4.2) is equivalent to

$$J_{y^{(0)}}(z) = -F|_{Y = y^{(c)}} \pmod{\mathcal{M}_A^{c+2}}. \quad (4.3)$$

By assumption, $F|_{Y = y^{(c)}} = 0 \pmod{\mathcal{M}_A^{c+1}}$. Thus by the first remark above, and using $z = 0 \pmod{\mathcal{M}_A^{c+1}}$ one more time, (4.3) is equivalent to

$$z = -K_{y^{(0)}}(F|_{Y = y^{(c)}}) \pmod{\mathcal{M}_A^{c+2}}.$$

Since $F|_{Y = y^{(c)}} = 0 \pmod{\mathcal{M}_A^{c+1}}$, the latter expression gives indeed a solution $z$ such that $z = 0 \pmod{\mathcal{M}_A^{c+1}}$.

In order to show the uniqueness of the solution modulo $\mathcal{M}_A^{c+2}$, note that if $w \in A^l$ is such that $w = 0 \pmod{\mathcal{M}_A^{c+1}}$, one has by (1.1)

$$F|_{Y = y^{(c)} + w} = F|_{Y = y^{(c)}} + J_{y^{(c)}}(w) \pmod{\mathcal{M}_A^{c+2}}$$

thus

$$K_{y^{(0)}}(F|_{Y = y^{(c)} + w}) = K_{y^{(0)}}(F|_{Y = y^{(c)}}) + K_{y^{(0)}}(J_{y^{(c)}}(w)) \pmod{\mathcal{M}_A^{c+2}}$$

and finally

$$K_{y^{(0)}}(F|_{Y = y^{(c)} + w}) = K_{y^{(0)}}(F|_{Y = y^{(c)}}) + w \pmod{\mathcal{M}_A^{c+2}}.$$

$\square$
4.3. We consider the following general setting and notation for the rest of this section. Let $A$ be a $k$-algebra which is a domain. Let $\Omega$ be a finite set, $I$ be a set, $X = \{X_\omega \}_{\omega \in \Omega}$ and $Y = \{Y_i \}_{i \in I}$ be collections of indeterminates. Set
\[
A[Y] := A[[X_\omega]]_{\omega \in \Omega} \quad \text{and} \quad A[X, Y] := A[[X_\omega, Y_i]_{\omega \in \Omega}].
\]
We denote by $(X)$ the prime ideal $(X_\omega; \omega \in \Omega)$ of $A[X]$. In accordance with (4.3) for any $A[X]$-algebra $B$, we often still denote by $(X)$ the extension of the ideal $(X)$ to $B$.

4.4. The following lemma will be useful in the proof of theorem 4.7.

**Lemma 4.5.** Assume that we are in the setting described in \S 4.3. Let $h$ be an ideal of $A[X, Y]$ such that:

(i) One has $(X) + h = \langle X, Y \rangle$.
Assume moreover that there exists an $A[X]$-algebra morphism $\hat{\varepsilon} : A[X, Y] \rightarrow \mathrm{Frac}(A)[X]$ such that:

(ii) For every $i \in I$ one has $\hat{\varepsilon}(Y_i) = Y_i \pmod{h}$ in the ring $\mathrm{Frac}(A)[X][Y]$.
(iii) For every $i \in I$, one has $\hat{\varepsilon}(Y_i) \in (X)$.

Then the $(X)$-adic completion of the localization $(A[X, Y]/h)_{(X)}$ is isomorphic to $\mathrm{Frac}(A)[X][\hat{\varepsilon}(h)]$.

**Remark 4.6.** Assume that the hypotheses of the lemma hold. Let $g$ be any ideal of $A[X, Y]$ containing $h$ such that $\langle X \rangle + h = \langle X \rangle + g$ and $\hat{\varepsilon}(h) = \hat{\varepsilon}(g)$. Then $g$ also satisfies the hypotheses of the lemma, with the same morphism $\hat{\varepsilon}$. In particular the lemma shows that the $(X)$-adic completions of $(A[X, Y]/h)_{(X)}$ and $(A[X, Y]/g)_{(X)}$ are isomorphic.

**Proof.** Note that (iii) shows that $\hat{\varepsilon}(h)$ is contained in $\langle X \rangle$, thus $\mathrm{Frac}(A)[X][\hat{\varepsilon}(h)]$ is a complete noetherian local ring with maximal ideal $\langle X \rangle$. Moreover (i) and the fact that $A$ is a domain show that $\langle X \rangle$ is indeed a prime ideal of $A[X, Y]/h$.

Let $e \geq 1$. Let $\pi_e$ be the composition of $\hat{\varepsilon}$ with the quotient morphism
\[
\mathrm{Frac}(A)[X][\hat{\varepsilon}(h)] \rightarrow \mathrm{Frac}(A)[X]/(\hat{\varepsilon}(h) + \langle X \rangle^e).
\]
Thanks to (iii), any element of $A[X, Y]$ whose constant term is not zero is sent by $\hat{\varepsilon}$ to an invertible element of $\mathrm{Frac}(A)[X]$. Thus $\pi_e$ induces a morphism
\[
A[X, Y]_{(X, Y)} \rightarrow \mathrm{Frac}(A)[X]/(\hat{\varepsilon}(h) + \langle X \rangle^e)
\]
which in turn induces a morphism
\[
\hat{\pi}_e : A[X, Y]_{(X, Y)}/(h + \langle X \rangle^e) \rightarrow \mathrm{Frac}(A)[X]/(\hat{\varepsilon}(h) + \langle X \rangle^e).
\]
Note that since $h + \langle X \rangle = \langle X, Y \rangle$, one has $h + \langle X \rangle^e = h + \langle X, Y \rangle^e$. Thus in order to obtain the claimed isomorphism, it suffices to show that $\hat{\pi}_e$ is an isomorphism for any $e \geq 1$. Since the natural inclusion $A[X, Y]_{(X, Y)} \subset \mathrm{Frac}(A)[X, Y]_{(X, Y)}$ is an isomorphism, surjectivity is clear.

Let us show injectivity. This amounts to show that if $P \in \mathrm{Frac}(A)[X, Y]$ lies in $\ker(\pi_e)$, then $P \in h + \langle X \rangle^e$. By assumption (ii), for any $P \in A[X, Y]$, one has $\hat{\varepsilon}(P) = P \pmod{h}$ in the ring $\mathrm{Frac}(A)[X][Y]$. In particular in the ring $\mathrm{Frac}(A)[X, Y]$ one has
\[
\hat{\varepsilon}(P) + \langle X \rangle^e = P + h + \langle X \rangle^e \quad \text{and} \quad \hat{\varepsilon}(h) + \langle X \rangle^e \subset h + \langle X \rangle^e.
\]
Now if $P \in \mathrm{Frac}(A)[X, Y]$ lies in $\ker(\pi_e)$, then one has $\hat{\varepsilon}(P) + \langle X \rangle^e \subset \hat{\varepsilon}(h) + \langle X \rangle^e$. Therefore, by the above properties, one has $P + h + \langle X \rangle^e \subset h + \langle X \rangle^e$. Thus $P \in h + \langle X \rangle^e$. That concludes the proof.

Now we can state and prove the main result of the section.
Theorem 4.7. Assume that we are in the setting described in [4.3] we assume moreover that the set I is of the shape $\Gamma \times \mathbb{N}$ where $\Gamma$ is a finite set.

Let $\mathfrak{h}$ be an ideal of $A[X, Y]$ such that:

(A) The ideal $\mathfrak{h}$ contains a collection of elements $\{H_{\gamma,s}, \gamma \in \Gamma, s \in \mathbb{N}\}$ of the form $H_{\gamma,s} = Y_{\gamma,s}^+ + E_{\gamma,s}$ such that for every $\gamma \in \Gamma$ and every $s \in \mathbb{N}$:

(1) $U_{\gamma,s}$ is a unit in $A$.

(A2) There exists a family $(E_{\gamma,s,r}) \in A[X]^{[\mathbb{N}, \{-1\}] + 1}$ such that $E_{\gamma,s,r} \in \langle X \rangle$

for $r \geq s$, $E_{\gamma,s,r} = 0$ for all but a finite number of $r$, and one has

$$E_{\gamma,s} = E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r} \cdot Y_{\gamma,r}.$$

(B) Let $(y_{\gamma,s}) \in A^{\Gamma \times \mathbb{N}}$ be the unique family of elements of $A$ such that for every $\gamma \in \Gamma$ and $s \in \mathbb{N}$, one has $H_{\gamma,s} |_{Y_{\gamma,s} = y_{\gamma,s}} = 0$ (mod $\langle X \rangle$); then the ideal $\langle X \rangle + \mathfrak{h}$ is contained in the ideal $\langle X \rangle + \langle Y_{\gamma,s} - y_{\gamma,s}; \gamma \in \Gamma \times \mathbb{N} \rangle$.

Then there exists an $A[X]$-algebra morphism $\hat{\varepsilon} : A[X, Y] \to \text{Frac}(A)[[X]]$ such that:

(i) For every $(\gamma, s) \in \Gamma \times \mathbb{N}$ one has $\hat{\varepsilon}(H_{\gamma,s}) = 0$.

(ii) For every $(\gamma, s) \in \Gamma \times \mathbb{N}$, one has $\hat{\varepsilon}(Y_{\gamma,s}) = Y_{\gamma,s}$ (mod $\mathfrak{h}$) in the ring $\text{Frac}(A)[[X, Y]]$.

(iii) For every ideal $\mathfrak{g}$ containing $\mathfrak{h}$ such that $(X) + \mathfrak{h} = (X) + \mathfrak{g}$ and $\hat{\varepsilon}(\mathfrak{g}) = \hat{\varepsilon}(\mathfrak{h})$, the $(X)$-adic completion of the localization $A[X, Y][\mathfrak{g}][[X]]$ is $(X)$-adically isomorphic to $\text{Frac}(A)[[X]]/\hat{\varepsilon}(\mathfrak{h})$.

Assume moreover that:

(C) For every $\gamma \in \Gamma$, one has $E_{\gamma,0,-1} \in A[X] \setminus \langle X \rangle$.

Then one has in addition:

(iv) For every $\gamma \in \Gamma$, $\hat{\varepsilon}(Y_{\gamma,0})$ is a unit in $\text{Frac}(A)[[X]]$.

Proof. First note that for each $\gamma \in \Gamma$, the reduction of the $H_{\gamma,s}$’s modulo $\langle X \rangle$ gives a triangular and invertible $A$-linear system in the $Y_{\gamma,s}$’s. Thus the existence and uniqueness of $(y_{\gamma,s})$ in assumption (B) is a straightforward consequence of assumption (A). In fact, up to dividing $H_{\gamma,s}$ by $U_{\gamma,s}$ and modifying the $E_{\gamma,s,r}$’s, one may assume that for every $\gamma, s, r$ one has $E_{\gamma,s,r} \in \langle X \rangle$ and that for every $\gamma, s$ one has

$$H_{\gamma,s} = Y_{\gamma,s} - y_{\gamma,s} + \sum_{r=0}^{s-1} \alpha_{\gamma,r}(Y_{\gamma,r} - y_{\gamma,r}) + E_{\gamma,s,-1} + \sum_{r \in \mathbb{N}} E_{\gamma,s,r}(Y_{\gamma,r} - y_{\gamma,r}), \quad (4.4)$$

where the $\alpha_{\gamma,r}$’s are elements of $A$.

Applying proposition [4.2] with $A = \text{Frac}(A)[[X]]$ and $\{F_j; j \in J\} = \{H_{\gamma,s}; \gamma \in \Gamma, s \in \mathbb{N}\}$, this shows the existence of a family $\{\mathcal{V}_{\gamma,s}; \gamma \in \Gamma, s \in \mathbb{N}\}$ of elements of $\text{Frac}(A)[[X]]$ such that for every $(\gamma, s) \in \Gamma \times \mathbb{N}$ one has $\mathcal{V}_{\gamma,s} = y_{\gamma,s}$ (mod $\langle X \rangle$) and $H_{\gamma,s} |_{\mathcal{V}_{\gamma,s} = y_{\gamma,s}} = 0$. Thus mapping $Y_{\gamma,s}$ to $\mathcal{V}_{\gamma,s}$ defines an $A[X]$-algebra morphism $\hat{\varepsilon} : A[X, Y] \to \text{Frac}(A)[[X]]$ such that (i) holds.

For every $\gamma \in \Gamma$, $\hat{\varepsilon}(Y_{\gamma,0})$ and an induction on $s$ shows that for every $s$ one has $Y_{\gamma,s} - y_{\gamma,s} \in \langle X \rangle + \mathfrak{h}$. By assumption (B), one then has $(X) + \mathfrak{h} = (X) + \langle Y_{\gamma,s} - y_{\gamma,s}; \gamma \in \Gamma \times \mathbb{N} \rangle$.

Thus $\text{Frac}(A)[[X],[Y]]/\mathfrak{h}$ is a noetherian local ring with maximal ideal $\langle X \rangle$.

On the other hand, [4.3] shows that for every $(\gamma, s) \in \Gamma \times \mathbb{N}$, since $H_{\gamma,s} \in \mathfrak{h}$ and $\hat{\varepsilon}(H_{\gamma,s}) = 0$, one has in the ring $\text{Frac}(A)[[X],[Y]]$ the relation

$$Y_{\gamma,s} - y_{\gamma,s} = - \sum_{r=0}^{s-1} \alpha_{\gamma,r}(Y_{\gamma,r} - y_{\gamma,r}) + \sum_{r \geq 0} E_{\gamma,s,r}(Y_{\gamma,r} - y_{\gamma,r}) \quad (\text{mod } \mathfrak{h}).$$
Thus by a straightforward induction one gets that \( Y_{\gamma,s} - Y_{\gamma,s} \in \langle X \rangle^e + \mathfrak{h} \) for every \( \gamma,s \) and \( e \geq 1 \), and finally by the Krull’s intersection theorem \( Y_{\gamma,s} - Y_{\gamma,s} \in \mathfrak{h} \) for every \( \gamma,s \). Thus (ii) holds. Recalling that \( Y_{\gamma,s} - y_{\gamma,0} \in \langle X \rangle \), (iii) then follows from an application of lemma 5.5 (replacing \( Y_{\gamma,s} \) with \( Y_{\gamma,s} - y_{\gamma,0} \)) and remark 5.4.

Assumption (C) is equivalent to the property \( \mathfrak{g}_{\gamma,0} = 0 \) (mod \( \langle X \rangle \)), \( \mathfrak{g}_{\gamma,0} = 0 \) also is a unit, and (iv) holds.

\[ \square \]

Remark 4.8. In the statement of the theorem, if one assumes that (A) holds and that moreover \( \mathfrak{h} \) is generated by the \( H_{\gamma,s} \)'s and some elements of \( \langle X \rangle \), then (B) automatically holds. Indeed, the above proof shows that without changing the ideal generated by the \( H_{\gamma,s} \)'s, one may assume that for every \( \gamma,s \) one has \( H_{\gamma,s} = Y_{\gamma,s} - y_{\gamma,0} \) (mod \( \langle X \rangle \)).

5. Technical machinery for the comparison theorem

In this section we will obtain the crucial technical result (theorem 5.6) allowing to establish our comparison theorem in section 6. As for theorem 5.7, the hypotheses are formulated in a somewhat abstract form, and in section 6 we will verify that these hypotheses hold in the toric setting.

5.1. We begin with an elementary yet useful lemma.

**Lemma 5.2.** Let \( K \) be a field, \( A \) be an object of \( \mathcal{L} \), \( a \) and \( b \) two ideals of \( A \) such that for every object \( B \) of \( \mathcal{L} \) one has the inclusion

\[ \{ \varphi \in \text{Hom}(A,B) \mid b \subset \text{Ker}(\varphi) \} \subset \{ \varphi \in \text{Hom}(A,B) \mid a \subset \text{Ker}(\varphi) \} \]

Then one has the inclusion \( a \subset b \).

**Proof.** We apply the assumption with \( \varphi \) the quotient morphism \( A \rightarrow A/b \).

\[ \square \]

**Notation 5.3.** Let \( \Delta \) be a finite set and \( Y \) be the set of indeterminates \( \{ Y_\delta \mid \delta \in \Delta \} \). Let \( R \) be a ring. Let \( \mathcal{Y}(t) := \{ Y_\delta(t) : \delta \in \Delta \} \) be a family of elements in the power series ring \( R[t] \). Let \( P \in R[\mathcal{Y}] \). Then we define the family \( \{ P_s, \mathcal{Y}(t) : s \in \mathbb{N} \} \) of elements of \( R \) by the following equality in \( R[t] \):

\[ P|_{\mathcal{Y}(t)} = \sum_{s \in \mathbb{N}} P_s, \mathcal{Y}(t)^s. \]  

**Remark 5.4.** Keep the same notation as before. Let \( S \) be another ring, \( \varphi : R \rightarrow S \) is a ring morphism. We also denote by \( \varphi \) the induced morphisms \( R[\mathcal{Y}] \rightarrow S[\mathcal{Y}] \) and \( R[t] \rightarrow S[t] \) obtained by applying \( \varphi \) coefficientwise. Then for every \( s \in \mathbb{N} \) one has \( \varphi(P_s, \mathcal{Y}(t)) = \varphi(P_s, \varphi(\mathcal{Y})(t)) \).

5.5. Now we can state and prove the main result of the section.

**Theorem 5.6.** Let \( K \) be a field extension of \( k \), \( \Delta \) be a finite set and \( X \) be the set of indeterminates \( \{ X_\delta \mid \delta \in \Delta \} \). Let \( (d_\delta) \in \mathbb{N}^\Delta \) be a family of nonnegative integers. Let \( X \) be the set of variables \( \{ X_\delta,j \mid \delta \in \Delta, 0 \leq j < d_\delta \} \). We denote by \( \langle X \rangle \) the maximal ideal of the power series ring \( K[\mathcal{X}] \).

Let \( \Omega \) be a (possibly infinite) set, and let \( \{ P_\omega \}_{\omega \in \Omega} \) be a family of elements in the polynomial ring \( K[\mathcal{Y}] \) such that for every \( \omega \in \Omega \), one has:

1. One may write \( P_\omega = \prod_{\delta \in \Delta} Y_\delta^{u_\omega,\delta} - \prod_{\delta \in \Delta} Y_\delta^{u_\omega,\delta} \), where \( u_\omega,\delta, u_\omega,\delta \in \mathbb{N} \).

2. One has \( P_\omega|_{\mathcal{Y}(t)} = 0 \) in \( K[t] \), in other words

\[ \sum_{\delta \in \Delta} d_\delta u_\omega,\delta = \sum_{\delta \in \Delta} d_\delta u_\omega,\delta =: c_\omega. \]
Let \( \{x_{\delta,j} : \delta \in \Delta, \, j \geq d_{\delta}\} \) be a family of elements in \( K[[X]] \). For \( \delta \in \Delta \), set
\[
\mathcal{Y}_{\delta}(t) := \sum_{j=0}^{d_{\delta}-1} X_{\delta,j} t^j + \sum_{j \geq d_{\delta}} x_{\delta,j} t^j \in K[[X]][t]
\]
and
\[
\tilde{\mathcal{Y}}_{\delta}(t) := \sum_{j=0}^{d_{\delta}-1} X_{\delta,j} t^j + t^{d_{\delta}} \in K[[X]][t]
\]
We assume:

(a) for every \( \delta \in \Delta \), \( x_{\delta,d_{\delta}} \) is a unit;

(b) for every \( \omega \in \Omega \) and every \( s \geq c_{\omega} \), one has \( P_{\omega,s} \mathcal{Y}(t) = 0 \).

We consider the following ideals of \( K[[X]] : a := \langle \{P_{\omega,s} \mathcal{Y}(t) : \omega \in \Omega, \, s \in \mathbb{N}\} \rangle \) and \( b := \langle \{P_{\omega,s} \tilde{\mathcal{Y}}(t) : \omega \in \Omega, \, s \in \mathbb{N}\} \rangle \).

Then \( K[[X]]/a \) and \( K[[X]]/b \) are isomorphic objects of \( \mathfrak{LcCl}_K \).

Proof. By assumption (a), for every \( \delta \in \Delta \), the series \( \mathcal{Y}_{\delta}(t) \) is a \( d_{\delta} \)-regular element of \( K[[X]][t] \). Thus by the Weierstrass preparation theorem (see [23]), there exists a family \( \{x_{\delta,j} : \delta \in \Delta, \, 0 \leq j < d_{\delta}\} \) of elements of the maximal ideal \( \langle X \rangle \) of \( K[[X]] \) and a family \( \{U_{\delta,r} : \delta \in \Delta, \, r \in \mathbb{N}\} \) of elements of \( K[[X]] \) with \( U_{\delta,0} \) an unit, such that, setting
\[
W_{\delta}(t) := t^{d_{\delta}} + \sum_{j=0}^{d_{\delta}-1} X_{\delta,j} t^j \quad \text{and} \quad U_{\delta}(t) := \sum_{r \in \mathbb{N}} U_{\delta,r} t^r
\]
one has
\[
\mathcal{Y}_{\delta}(t) = W_{\delta}(t) U_{\delta}(t).
\]
Identifying the \( t \)-coefficients in the latter equation yields the following relations in \( K[[X]] \):
\[
X_{\delta,j} = X_{\delta,j} U_{\delta,0} + \sum_{r=0}^{j-1} X_{\delta,r} U_{\delta,j-r}, \quad 0 \leq j < d_{\delta}.
\]
Since \( U_{\delta,0} \) is a unit, we deduce that the element of \( \text{Hom}_{\mathfrak{LcCl}_K}(K[[X]], K[[X]]) \) sending \( X_{\delta,j} \) to \( X_{\delta,j} \) for \( \delta \in \Delta \) and \( 0 \leq j < d_{\delta} \) is an isomorphism.

Setting \( \epsilon := \langle \{P_{\omega,s} (W_{\delta}(t)) : \omega \in \Omega, \, s \in \mathbb{N}\} \rangle \), the above isomorphism shows that \( K[[X]][t]/\epsilon \) and \( K[[X]][t]/\mathfrak{c} \) are isomorphic objects in \( \mathfrak{LcCl}_K \). To conclude the proof, we show that \( a = \mathfrak{c} \), using lemma 5.2.2.

Let \( (B, \mathfrak{m}_B) \) be an object in \( \mathfrak{LcCl}_K \), and let \( \varphi \) be an element of \( \text{Hom}_{\mathfrak{LcCl}_K}(K[[X]], B) \). We still denote by \( \varphi \) the induced morphism \( K[[X]][t] \to B[t] \) obtained by applying \( \varphi \) coefficientwise.

Let us assume that for every \( \omega \in \Omega \) one has \( P_{\omega} |_{\mathcal{Y}_\delta} = \varphi(\mathcal{Y}_\delta(t)) = 0 \). One has to show that for every \( \omega \in \Omega \) one has \( P_{\omega} |_{\mathcal{Y}_\delta(t)} = \varphi(\mathcal{Y}_\delta(t)) = 0 \).

From our assumption and hypothesis (I) we deduce the following equality in \( B[t] \):
\[
\prod_{\delta \in \Delta} \varphi(\mathcal{Y}_\delta(t))^{u_{\omega,\delta}} = \prod_{\delta \in \Delta} \varphi(\mathcal{Y}_\delta(t))^{u_{\omega,\delta}}
\]
which can be rewritten, using equation 5.2, as
\[
\prod_{\delta \in \Delta} \varphi(W_{\delta}(t))^{u_{\omega,\delta}} \prod_{\delta \in \Delta} \varphi(U_{\delta}(t))^{u_{\omega,\delta}} = \prod_{\delta \in \Delta} \varphi(W_{\delta}(t))^{u_{\omega,\delta}} \prod_{\delta \in \Delta} \varphi(U_{\delta}(t))^{u_{\omega,\delta}}. \tag{5.3}
\]
Note that for every \( \delta \in \Delta \), \( \varphi(W_{\delta}(t)) \) is a Weierstrass polynomial of degree \( d_{\delta} \) in \( B[t] \) and \( \varphi(U_{\delta}(t)) \) is a unit in \( B[t] \), since \( \varphi(U_{\delta,0}) \) is.
By uniqueness of the Weierstrass factorization in $\mathcal{B}[t]$, one gets the equality
\[
\prod_{\delta \in \Delta} \varphi(W_{\delta}(t))^{u_{\delta, t}} = \prod_{\delta \in \Delta} \varphi(W_{\delta}(t))^{u_{\delta}}
\]  
which means exactly that (5.4) holds, and let us show that for every $\omega$ where for every $\ell$ where
\[
\omega = t^\ell \varphi(W_{\delta}(t))^{u_{\delta}} - \prod_{\delta \in \Delta} \varphi(U_{\delta}(t))^{u_{\delta}}
\]
By assumption (b), $P_{\omega}|_{\mathcal{Y}_{\omega}(\mathcal{Y}_{\omega}(t))} = 0$, is an element of the polynomial ring $\mathcal{B}[t]$ with degree less than $\omega$. By the uniqueness of the Weierstrass division by $\mathcal{W}_{\omega}(t)$ in $\mathcal{B}[t]$ one concludes that $P_{\omega}|_{\mathcal{Y}_{\omega}(\mathcal{Y}_{\omega}(t))} = 0$.  

6. A comparison theorem between formal neighborhoods

In this section we will make use of the results in sections 4 and 5 in order to obtain the main comparison theorem as an application of the results in those sections to the toric setting. It should be noted that our results provide basically two approaches for computing effectively the formal neighborhood of the generic point of the Nash set associated with a toric valuation. The first one is based on an effective implementation of the Hensel’s lemma crucially used in section 4. The second one takes advantage of the comparison theorem in order to use exactly the same techniques as in the case of rational arcs described in [6]. The latter seems to be much more efficient in practice. See §6.23 below for an explicit example of computation, as well as [27] for more details and explicit examples.

6.1. We retain the notation introduced in section 3. In particular, $V_\sigma$ is the affine toric $k$-variety of dimension $d$ associated with a cone $\sigma$ and presented as $k[Z]/i_\sigma$, where $Z = \{ Z_i : i \in \{1, \ldots, h\} \}$ and $i_\sigma$ is generated by the binomial elements $\{ F_\ell = Z^\ell L - Z^{\ell - \ell} : \ell \in L \}$, $L$ being a subgroup of $\mathbb{Z}^h$. Moreover, denoting by $Z_\infty$ the set of variables $\{ Z_{i,s} : i \in \{1, \ldots, h\}, s \in \mathbb{N} \}$, the arc scheme $\mathcal{L}_\infty(V_\sigma)$ associated with the affine toric variety $V_\sigma$ may be identified with the affine scheme $\text{Spec}(k[Z_\infty]/i_\sigma)$; the ideal $[i_\sigma]$ is generated by the elements $\{ F_\ell : \ell \in L, s \in \mathbb{N} \}$ for every $\ell \in L$, where the elements $F_{\ell,s} \in k[Z_\infty]$ may be characterized by the following equality in $k[Z_\infty][t]$: 
\[
F_\ell|_{Z_{i,s} = t^s} = \sum_{s \in \mathbb{N}} F_{\ell,s} t^s.
\]

6.2. The following proposition gives an explicit description of the generic point of the Nash set associated with a toric valuation.

**Proposition 6.3.** Let $n \in \sigma \cap N$ be a toric valuation, $N_n$ be the associated Nash set and $\eta_n$ be the generic point of $N_n$. Let $a_n$ be the ideal $\langle \{ Z_{i,s} : 1 \leq i \leq h, 0 \leq s_i \leq \langle n, m_i \rangle \} \rangle$ of $k[Z_\infty]$. Let $G_n := \prod_{i=1}^d Z_{i,(n,m_i)}$ and $g_n$ be the image of $G_n$ in $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$. Then:

(i) The prime ideal of $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$ corresponding with $\eta_n$ is the radical of the image of $a_n$ in $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))$. 

The point $\eta_n$ belongs to the distinguished open subset $\{g_n \neq 0\}$ of $\mathcal{L}_\infty(V_\sigma)$. The prime ideal of $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))_{g_n}$ corresponding with $\eta_n$ is the extension of $a_n$ to $\mathcal{O}(\mathcal{L}_\infty(V_\sigma))_{g_n}$.

Proof. Assertion (i) follows from lemma 6.4.

Let us now prove assertion (ii). By (i), it is enough to show that the $k$-algebra
$$R := k[Z_\infty|_{G_n}]/[\varepsilon] + a_n$$
is a domain. Let us show that its functor of points is isomorphic to the functor of points of the $k$-algebra $\mathcal{O}(\mathcal{L}_\infty(T))$, the latter being a domain since $T$ is a smooth irreducible variety.

Let $A$ be a $k$-algebra. By the very definition of $R$ and 6.3.1 and 6.3.2, the set $\text{Hom}_{\text{Alg}}(R, A)$ is in natural bijection with the set of semigroup morphisms $\varphi: S_\sigma \to A[[t]]$ such that for $1 \leq i \leq h$ one has $\omega_i(\varphi(m_i)) \geq (m_i, n)$ and for $1 \leq i \leq d$ one has $t^{-(m_i, n)} \varphi(m_i) \in (A[[t]])^\times$. The latter property also holds for $d + 1 \leq i \leq h$ since $m_i$ is a $\mathcal{Z}$-basis of $M$. In particular $m \mapsto t^{-(m, n)} \varphi(m)$ is a semigroup morphism $S_\sigma \to (A[[t]])^\times$. Thus the set $\text{Hom}_{\text{Alg}}(R, A)$ is in natural bijection with the set of group morphism $M \to (A[[t]])^\times$, which in turn is in natural bijection with $\text{Hom}_{\text{Alg}}(\mathcal{O}(\mathcal{L}_\infty(T)), A)$.

6.4. Recall that $j$ is the ideal $\{F_{\infty} : d + 1 \leq q \leq h\}$ of the ring $k[Z]$. Its defines an affine $k$-scheme $W := \text{Spec}(k[Z]/j)$ which contains $V_\sigma$ as a closed subscheme. Recall also that $\mathcal{L}_\infty(W)$ may be identified with $\text{Spec}(k[Z_\infty]/[j])$ and that $j = \{F_{h,s} : d + 1 \leq q \leq h, s \in \mathbb{N}\}$. The closed immersion $V_\sigma \to W$ induces a closed immersion $\mathcal{L}_\infty(V_\sigma) \to \mathcal{L}_\infty(W)$ between the corresponding arc schemes. For $n \in \sigma \cap N$, let $\eta'_n$ be the image of $\eta_n$ by this closed immersion. We shall reduce the computation of the formal neighborhood of $\mathcal{L}_\infty(V_\sigma)$ at $\eta_n$ to that of the formal neighborhood of $\mathcal{L}_\infty(W)$ at $\eta'_n$. We will say that we are in the toric setting in the former situation and (abusing terminology) in the complete intersection setting in the latter.

The following lemma is a straightforward consequence of the definition, lemma 6.3 ii) and proposition 6.3.

Lemma 6.5. Retain the notation and hypotheses of proposition 6.3. Let $g'_n$ be the image of $G_n$ in $\mathcal{O}(\mathcal{L}_\infty(W))$.

Then the point $\eta'_n$ belongs to the distinguished open subset $\{g'_n \neq 0\}$ of $\mathcal{L}_\infty(W)$, and the prime ideal of $\mathcal{O}(\mathcal{L}_\infty(W))_{g'_n}$ corresponding with $\eta'_n$ is the extension of $a_n$ to $\mathcal{O}(\mathcal{L}_\infty(W))_{g'_n}$.

Notation 6.6. For $q \in \{1, \ldots, h\}$ and $r \in \mathbb{N}^d$ we denote by $Z_{\leq q, \leq r}$ the set of variables $\{Z_{i,s} : 1 \leq i \leq q, 0 \leq s \leq r_i\}$. If $q = h$ we write $Z_{\leq r}$ instead of $Z_{\leq h,r}$. We define similarly $Z_{\geq q, \geq r}, Z_{\geq q, \leq r}$, and so on.

6.7. The following lemma shows that we can apply theorem 4.7 in the complete intersection setting.

Lemma 6.8. Let $n \in \sigma \cap N$ be a toric valuation.

Let $G_n := \prod_{i=1}^d Z_{i,(n,m_i)}$ and $A$ be the $k$-algebra $k[Z_{\leq d, \geq (n,m_i)}]|_{G_n}$.

Let $\Omega$ be the finite set $\{(i,s_i) : i \in \{1, \ldots, h\}, 0 \leq s_i < (n,m_i)\}$. For $\omega \in \Omega$, set $X_{\omega} := Z_{\omega}$. Set $\underline{\Gamma} = \{d+1, \ldots, h\}$. For $q \in \underline{\Gamma}$ and $s \in \mathbb{N}$, set $Y_{q,s} := Z_{q,(n,m_q)+s}$. Let $h$ be the extension of the ideal $[j]$ in $k[Z_\infty]|_{G_n}$. For $s \in \mathbb{N}$ and $q \in \underline{\Gamma}$, set $H_{q,s} := F_{q,s}(n_{t_s}^+ + s)$. (Recall from 4.2.8 the definition of $(\omega, \ell_q)$.) Then with this notation the hypotheses in theorem 4.7 hold true.

Proof. Note that with the notation of the statement, one has in particular $A[X,Y] = k[Z_\infty]|_{G_n}$ and the ideal $(X)$ corresponds to $a_n = (Z \neq (n,m_i))$.

Let us show that assumption (A) in theorem 4.7 holds. Pick up $q \in \{d+1, \ldots, h\}$. Set
$$\Lambda_q^+ := \{i \in \{1, \ldots, d\} : \ell_{q,i} > 0\}, \quad \Lambda_q^- := \{i \in \{1, \ldots, d\} : \ell_{q,i} < 0\}$$
\[ \Theta^+_{q,s} := \{(r_q, (r_{i,k} ; i \in \Lambda^+_q, 1 \leq k \leq \ell_q,i)) ; r_q, r_{i,k} \in \mathbb{N}, r_q + \sum_{i \in \Lambda^+_q} \sum_{k=1}^{\ell_q,i} r_{i,k} = s\} \]

and \[ \Theta^-_{q,s} := \{(r_{i,k} ; i \in \Lambda^-_q, 1 \leq k \leq -\ell_q,i) ; r_{i,k} \in \mathbb{N}, \sum_{i \in \Lambda^-_q} \sum_{k=1}^{-\ell_q,i} r_{i,k} = s\}. \]

Then by (3.5) and (3.2), the polynomial \( F_{\ell_q,s} \) has the following form:

\[
F_{\ell_q,s} = \sum_{(r_q, r_{i,k}) \in \Theta^+_{q,s}} Z_q r_q \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_q,i} Z_{r_{i,k}} - \sum_{(r_q, r_{i,k}) \in \Theta^-_{q,s}} \prod_{i \in \Lambda^-_q} \prod_{k=1}^{-\ell_q,i} Z_{r_{i,k}}. \tag{6.1}
\]

Note that setting \( r_q = \langle n, m_q \rangle + s \) and \( r_{i,k} = \langle n, m_i \rangle \) for \( 1 \leq k \leq \ell_q,i \) defines an element of \( \Theta^+_{q,\langle n, \ell_q \rangle + s} \). Set

\[ U_q := \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_q,i} Z_{r_{i,k}}. \]

By the definition of \( G_n \), \( U_q \) is an invertible element of \( k[Z_{\leq d, \geq \langle n, m_q \rangle}] \).

Set

\[ E_{q,s-1} := \sum_{(r_q, r_{i,k}) \in \Theta^+_{q,\langle n, m_q \rangle + s}} Z_q r_q \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_q,i} Z_{r_{i,k}} - \sum_{(r_q, r_{i,k}) \in \Theta^-_{q,\langle n, m_q \rangle + s}} \prod_{i \in \Lambda^-_q} \prod_{k=1}^{-\ell_q,i} Z_{r_{i,k}}. \]

For \( r \in \mathbb{N} \), set \( \delta_{r,s} = 1 \) if \( r = s \) and 0 otherwise, and

\[ E_{q,s,r} := -\delta_{r,s} U_q + \sum_{(r_q, r_{i,k}) \in \Theta^+_{q,\langle n, m_q \rangle + r}} \prod_{i \in \Lambda^+_q} \prod_{k=1}^{\ell_q,i} Z_{r_{i,k}}. \]

Thus by (6.1), one has

\[ F_{\ell_q,\langle n, \ell_q \rangle + s} = U_q Z_q \langle n, m_q \rangle + s + E_{q,s-1} + \sum_{r \in \mathbb{N}} E_{q,s,r} Z_q \langle n, m_q \rangle + r. \]

Since \( \Lambda^q \subset \{1, \ldots, d\} \), it is clear that for \( r \in \mathbb{N} \) one has \( E_{q,s,r} \in k[Z_{\leq d, \bullet}] \), and that \( E_{q,s-1} \in k[Z_{\leq d, \bullet} \cup Z_{s,\langle n, m_q \rangle}] \).

Thus (A1) is satisfied, and in order to show that (A2) also holds, it remains to prove that for any \( r \geq s \), each monomial of \( E_{q,s,r} \) contains a variable \( Z_{r_{i,k}} \) with \( i \in \{1, \ldots, d\} \) and \( r_i < \langle n, m_i \rangle \). Take \( (r_{i,k} ; i \in \Lambda^+_q, 1 \leq k \leq \ell_q,i) \) a family of nonnegative integers such that \( (\langle n, m_q \rangle + r, (r_{i,k})) \in \Theta^+_{q,\langle n, \ell_q \rangle + s} \), that is

\[ \langle n, m_q \rangle + r + \sum_{i \in \Lambda^+_q} \sum_{k=1}^{\ell_q,i} r_{i,k} = \langle n, \ell_q \rangle + s. \]

We have to show that either at least one of the \( r_{i,k} \)'s is \( < \langle n, m_i \rangle \) or \( r = s \) and \( r_{i,k} = \langle n, m_i \rangle \) for every \( i, k \). (The latter case corresponds to the monomial \( U_q Z_q \langle n, m_q \rangle + s \).) Assume \( r_{i,k} \geq \langle n, m_i \rangle \) for every \( i, k \). Then

\[ \langle n, \ell_q \rangle + s = \langle n, m_q \rangle + r + \sum_{i \in \Lambda^+_q} \sum_{k=1}^{\ell_q,i} r_{i,k} \geq r + \left( \langle n, m_q \rangle + \sum_{i \in \Lambda^+_q} \ell_q,i, m_i \right) = r + \langle n, \ell_q \rangle. \]

If \( r > s \) this is a contradiction. If \( r = s \), the first minoration must be an equality, which imposes \( r_{i,k} = \langle n, m_i \rangle \) for every \( i, k \).
Let us prove that (C) holds. We have to show that \( E_{q,0,-1} \) does not belong to the ideal \( \langle Z_\ast,<(n,m_i)\rangle \). By the definition of \( E_{q,0,-1} \) it is enough to show that
\[
\widehat{E}_{q,0,-1} := - \sum_{(r_i,k) \in \Theta_{q,(n,\ell_\ell)}} \prod_{i \in \Lambda_q^+} Z_{i,r_i,k}
\]
does not belong to the ideal \( \langle Z_\ast,<(n,m_i)\rangle \). But arguing similarly as above, one sees that the only monomial in \( \widehat{E}_{q,0,-1} \) not belonging to the above ideal corresponds to \( r_{i,k} = (n,m_i) \). Thus one has \( \widehat{E}_{q,0,-1} = \prod_{i \in \Lambda_q^+} Z_{i,\ell_\ell} \) (mod \( \langle Z_\ast,<(n,m_i)\rangle \)) which allows to conclude.

Let us show that (B) holds. Since \( \mathfrak{h} \) is the extension of the ideal \([i]\) in \( k[Z_\infty]_G, \) it is generated by the union of the families \( \{H_{q,s}; q \in \Gamma, s \in \mathbb{N}\} \) and \( \{F_{\ell,q}; q \in \Gamma, s \in \mathbb{N}, s < \langle n, \ell_\ell \rangle\} \).

Arguing similarly as above, one sees using\(^{[6.7]}\) that in case \( s < \langle n, \ell_\ell \rangle \) every monomial of \( F_{\ell,q} \) must contain a variable \( Z_{i,r} \) with \( r < \langle n, m_i \rangle \). Thus \( \mathfrak{h} \) is generated by some elements of \( (X) \) and the \( H_{q,s} \)'s. By remark 4.8 assumption (B) holds in this case. \( \square \)

6.9. Thanks to lemma 6.8 we can apply theorem 4.7 in the complete intersection setting. In the proof of the following corollary, we shall see that this also holds in the toric setting.

**Corollary 6.10.** Let \( n \) be a toric valuation of \( \sigma \cap N \). There exists a \( k[Z_{\leq d,\geq}(n,m_i)] \)-algebra morphism \( \hat{\varepsilon}: k[Z_{\infty}] \rightarrow k[Z_{\leq d,\geq}(n,m_i)]\langle Z_\ast,<(n,m_i)\rangle \) such that:

(i) The section ring of the formal neighborhood of \( \mathcal{L}_\infty(V_\ell) \) at \( n_\ell \) (resp. of \( \mathcal{L}_\infty(W) \) at \( n'_\ell \)) are both isomorphic to the complete noetherian local ring 
\[
k(Z_{\leq d,\geq}(n,m_i))\langle Z_\ast,<(n,m_i)\rangle \langle \hat{\varepsilon}([i]) \rangle.
\]

(ii) For every \( i \in \{1,\ldots,h\} \), \( \hat{\varepsilon}(Z_i,(n,m_i)) \) is invertible.

(iii) For every \( q \in \{d+1,\ldots,h\} \) and \( s \in \mathbb{N} \), we have \( \hat{\varepsilon}(F_{\ell,q,d}(n,\ell_\ell)) = 0 \).

**Proof of corollary 6.10.** By lemma 6.9(ii) and theorem 4.7(iii), it only remains to show that \( \hat{\varepsilon}([i]) = \hat{\varepsilon}(i_\ell) \). Recall from\(^{[6.7]}\) the definition of \( HS \). It is enough to show that for every element \( F \) of \( i_\ell \), one has \( \hat{\varepsilon}(HS(F)) = \hat{\varepsilon}([i])\langle t \rangle \). By lemma 3.9 there exists a positive integer \( N \) such that \( (\prod_{i=1}^d Z_i)^N \in F \). Thus
\[
\hat{\varepsilon}(HS((\prod_{i=1}^d Z_i)^N)) = \hat{\varepsilon}(HS((\prod_{i=1}^d Z_i)^N)) \in \hat{\varepsilon}([i])\langle t \rangle.
\]

Since for every \( i \in \{1,\ldots,d\} \), \( \hat{\varepsilon}(Z_i,(n,m_i)) \) is a unit, \( \hat{\varepsilon}(HS((\prod_{i=1}^d Z_i)^N)) \) is a regular element of \( k(Z_{\leq d,\geq}(n,m_i))\langle Z_\ast,<(n,m_i)\rangle \langle t \rangle \), as well as its projection to
\[
k(Z_{\leq d,\geq}(n,m_i))\langle Z_\ast,<(n,m_i)\rangle \langle \hat{\varepsilon}([i]) \rangle \langle t \rangle.
\]
Since a regular element is not a zero divisor, we infer that \( \hat{\varepsilon}(HS(F)) \in \hat{\varepsilon}([i])\langle t \rangle \). \( \square \)

6.11. Let us recall the definition of some objects in [6] Subsection 5.1, adapted to the notation in the present section. Denote by \( \hat{\varepsilon}: k[Z_{\infty}] \rightarrow k[Z_\ast,<(n,m_i)] \) the unique \( k \)-algebra morphism mapping, for every \( i \in \{1,\ldots,h\} \), \( Z_{i,s} \) to \( Z_{i,s} \) for \( s < \langle n, m_i \rangle \) and \( Z_{i,s} \) to 0 for \( s > \langle n, m_i \rangle \).

For \( L' \subseteq L \), let \( W(n,L') \) be the affine closed \( k \)-subscheme of the affine space \( \text{Spec}(k[Z_\ast,<(n,m_i)]) \) defined by the ideal \( \langle \hat{\varepsilon}(F_{\ell,s}) \rangle; \ell \in L', s \in \mathbb{N} \) and \( W(n,L') \) the formal completion of \( W(n,L') \) along the origin of \( \text{Spec}(k[Z_\ast,<(n,m_i)]) \).
Remark 6.12. Let \((A, \mathcal{M}_A)\) be an object of \(\mathfrak{LeCpl}_k\). Then \(\text{Hom}_{\mathfrak{LeCpl}_k}(W(n, L'), A)\) is in natural bijection with the set of families \(\{z_i; i \in \{1, \ldots, h\}, 0 \leq s < \langle n, m_i \rangle\}\) of elements of \(\mathcal{M}_A\) such that for every element \(\ell \in L'\) one has

\[
F_{\ell}|_{Z_i=\sum_{s=0}^{\langle n, m_i \rangle} z_i, t^i e(n, m_i)} = 0.
\]

The following result follows from \([6, \text{Theorem 5.2}]\).

**Theorem 6.13.** For an appropriate choice of \(L' \subseteq L\) such that \(\{\ell_q; d + 1 \leq q \leq h\} \subseteq L'\), for every toric valuation \(n \in N \cap \sigma\) and every arc \(\alpha \in \mathcal{L}_\infty(V_\sigma)_n^\circ(k)\), the formal neighborhood of \(\mathcal{L}_\infty(V_\sigma)\) at \(\alpha\) is isomorphic to \(W(n, L') \hat{\otimes}_k [T_i]_{i \in \mathbb{N}}\).

The following lemma shows that for the computation of formal neighborhoods of \(k\)-rational arcs on \(\mathcal{L}_\infty(V_\sigma)\), one may also reduce to the complete intersection setting.

**Lemma 6.14.** Let \(n \in \sigma \cap N\) be a toric valuation and \(L'\) be a subset of \(L\) such that \(\{\ell_q; d + 1 \leq q \leq h\} \subseteq L'\). Then \(W(n, L')\) is isomorphic, as a formal \(k\)-scheme, to \(W(n, \{\ell_q; d + 1 \leq q \leq h\})\).

Remark 6.15. Thanks to this lemma, for any \(L' \subseteq L\) such that \(\{\ell_q; d + 1 \leq q \leq h\} \subseteq L'\) and any \(n \in \sigma \cap N\), one may denote \(W(n, L')\) by \(W(n)\).

**Proof.** By remark 6.12 there is, for every object \((A, \mathcal{M}_A)\) of \(\mathfrak{LeCpl}_k\), a natural inclusion \(\text{Hom}_{\mathfrak{LeCpl}_k}(W(n, L'), A) \subset \text{Hom}_{\mathfrak{LeCpl}_k}(W(n, \{\ell_q; d + 1 \leq q \leq h\}), A)\). To conclude, it suffices to show that this is an equality. Let \(\{z_i; i \in \{1, \ldots, h\}, 0 \leq s < \langle n, m_i \rangle\}\) be a family of elements of \(\mathcal{M}_A\) such that, setting

\[
z_i(t) := \sum_{s=0}^{\langle n, m_i \rangle} z_i, t^i e(n, m_i),
\]

one has, for every \(\ell_q\) with \(d + 1 \leq q \leq h\), \(F_{\ell_q}|_{Z_i=\langle n, m_i \rangle} = 0\). Let \(\ell \in L'\). By lemma 6.13 there exists a positive integer \(N\) such that \((\prod_{i=1}^{h} z_i(t))^{\langle n, m_i \rangle} F_{\ell}(z_i(t)) = 0\).

Since \(z_i(t)\) is a Weierstrass polynomial in \(A[[t]]\), it is a non zero divisor (see \([6, \text{Remark}]\)). Thus one infers that \(F_{\ell}|_{Z_i=\langle n, m_i \rangle} = 0\). That concludes the proof.

The following proposition performs the aimed comparison in the complete intersection setting.

**Proposition 6.16.** Let \(n \in \sigma \cap N\) be a toric valuation. Let \(K := k(Z_{d+1} \geq (n, m_i))\). Then the residue field of \(\eta_n\) is isomorphic to \(K\) and the formal neighborhood of \(\mathcal{L}_\infty(W)\) at the point \(\eta_n\) is isomorphic, as a formal \(K\)-scheme, to \(K \hat{\otimes}_k W(n, \{\ell_q; d + 1 \leq q \leq h\})\).

**Proof.** We still denote by \(\widehat{\varepsilon}\) the composition of the morphism defined in \([6, \text{11}]\) with the natural inclusion morphism \(k[[Z_{d+1} \geq (n, m_i)]] \rightarrow K[[Z_{d+1} \geq (n, m_i)]]\).

By corollary 6.10 and the very definition of \(W(n, \{\ell_q; d + 1 \leq q \leq h\})\), it is enough to show that the quotients of \(K[[Z_{d+1} \geq (n, m_i)]]\) by the ideals \(I^{(i)}(\ell_q)\) are isomorphic, for \(i \in \Delta := \{1, \ldots, h\}\), to \(I^{(i)}(\ell_q)\), for \(d + 1 \leq q \leq h, s \in N\) on one hand, \(I^{(i)}(\ell_q)\), for \(d + 1 \leq q \leq h, s \in N\) on the other hand, are isomorphic.

We aim to apply theorem 5.6. Set \(\Delta := \{1, \ldots, h\}\). For \(i \in \Delta\), set \(d_i := (n, m_i);\) for \(0 \leq s < (n, m_i)\) set \(X_i,s := Z_{i,s}\); for \(s \geq (n, m_i)\) set \(x_i,s := \varepsilon(Z_{i,s})\). For
6.15. the Nash set $N$ if it cannot be written indecomposable.

Then still by corollary 6.10, for every $i \in \Delta$, set

$$\mathcal{Y}_i(t) := \sum_{s \in \mathbb{N}} \hat{e}(Z_{i,s}) t^s = \sum_{s=0}^{(n, m_i)-1} X_{i,s} t^s + \sum_{s \geq (n, m_i)} x_{i,s} t^s$$

and \( \hat{Y}_i(t) := \sum_{s=0}^{(n, m_i)-1} X_{i,s} t^s + t^{(n, m_i)}. \)

For every $P \in k[Z]$, we then have (see notation [5.3] and remark [5.4]) $P_s \mathcal{Y}(t) = \hat{e}(P_s)$ and $P_s \hat{Y}(t) = \hat{e}(P_s)$.

Set $\Omega := \{d+1, \ldots, h\}$ and for $q \in \Omega$ set $P_q := F_{\xi_q}$.

Assumption (a) is a consequence of corollary 6.10.

With our identifications, the nonzero integer $c_q$ defined in the statement of theorem 6.16 is

$$c_q = \sum_{i=1}^{h} (\ell_q^i)_i \langle n, m_i \rangle = \langle n, \sum_{i=1}^{h} \ell_q^i m_i \rangle = \langle n, \ell_q \rangle.$$  

Then still by corollary 6.10 for every $s \in \mathbb{N}$ we have $\hat{e}(F_{\xi_q \ast (n, \ell_q) + s}) = 0$. Thus $F_{\xi_q \ast (n, \ell_q) + s} \mathcal{Y}(t) = 0$ and assumption (b) holds. That concludes the proof. \qed

6.17. Now one can state the main theorem of the article. It illustrates the striking fact that not only the isomorphism class of the formal neighborhood of a generic $k$-rational arc of the Nash set associated with a toric valuation is constant (as observed in [6]) but moreover the involved isomorphism class is encoded in some sense in the formal neighborhood of the generic point of the Nash set. This could be interpreted as the fact that the arc scheme of a toric variety is analytically a product along the Nash set associated with the toric valuation.

**Theorem 6.18.** Let $n \in \sigma \cap N$ be a toric valuation. Let $\eta_n$ be the generic point of the Nash set $N_n$. Let $W(n)$ be the noetherian formal $k$-scheme defined in remark 6.16.

Then there exists a nonempty open subset $U_n$ of the Nash set $N_n$ such that:

(i) The formal neighborhood of $L_{\infty}(V_\sigma)$ at $\eta_n$ is isomorphic, as a formal $k(\eta_n)$-scheme, to $k(\eta_n) \otimes k W(n)$. In particular it is isomorphic to the formal spectrum of the completion of an essentially of finite type local $k(\eta_n)$-algebra.

(ii) For any arc $\alpha \in U_n(k)$, the formal neighborhood of $L_{\infty}(V_\sigma)$ at $\alpha$ is isomorphic to $W(n) \otimes k[[T_i]]_{i \in \mathbb{N}}$.

**Proof.** One takes $U_n := L_{\infty}(V_\sigma)_{\eta_n}$ and one combines proposition 6.10 with theorem 6.13, lemma 6.14 and corollary 6.10(i). \qed

6.19. An element $n \in N \setminus \{0\}$ is said to be primitive if it can not be written as $dn'$ where $n' \in N$ and $d$ is an integer $> 1$. An element $n \in N \cap \sigma \setminus \{0\}$ is said to be indecomposable if it can not be written $n = n_1 + n_2$ with $n_1, n_2 \in N \cap \sigma \setminus \{0\}$. A decomposition of $n$ into indecomposable elements is a decomposition $n = \sum_{i=1}^{r} n_i$ where $r$ is a positive integer and the $n_i$'s are indecomposable elements in $N \cap \sigma \setminus \{0\}$; the length of such a decomposition is $r$.

6.20. Using results of our previous work [6], we deduce, as a straightforward byproduct of theorem 6.18, the following corollary. The result has been obtained independently by Reguera using a different approach (see [31]).

**Corollary 6.21.** Let $n \in \sigma \cap N$ be a toric valuation of $V_\sigma$ and $\eta_n$ be the generic point of the Nash set $N_n$. 


Then there is a natural bijection between the set of irreducible components of the formal neighborhood \( \mathcal{L}_\infty(V_\sigma)_{\eta_n} \) and the set of decompositions of \( n \) into a sum of indecomposable elements of the semigroup \( N \cap \sigma \). The dimension of the component corresponding to a given decomposition of \( n \) is the length of the decomposition. In particular the dimension of \( \mathcal{L}_\infty(V_\sigma)_{\eta_n} \) is equal to the maximal length of such a decomposition of \( n \).

**Proof.** The fact that the conclusion holds for \( \mathcal{W}(n) \) is shown in section (3) of the proof of [6, Theorem 6.3]. In the latter \( \mathcal{W}(n) \) is denoted by \( \mathcal{H}_n \). Though it is assumed in the statement of the theorem that \( n \) is primitive, this is not used in the aforementioned section of the proof. The corollary then follows from theorem 6.18.

6.22. It remains to explain why Theorem 1.10 stated in the introduction is a consequence of corollary 6.21. Let \( n \in N \cap \sigma \) be a primitive integral point representing a toric valuation of multiplicity 1 and assume that \( v \) is centered in the singular locus of \( V_\sigma \). Then, by [11, Theorem 1.10] and [10, Theorem 1.2], \( n \) is indecomposable if and only if \( v \) is a strongly essential valuation, in the sense given in the introduction. Thus theorem 1.10 is indeed a consequence of corollary 6.21 (using again the proof of [6, Theorem 6.3]).

6.23. We end this section with an explicit example of computation of the formal neighborhood of the generic point of the Nash set associated with a toric valuation. See [27] for more details.

Let \( N = M = \mathbb{Z}^2 \), \( \sigma \) be the cone of \( \mathbb{R}^2 \) generated by \((1,0)\) and \((1,2)\), and \( V_\sigma \) be the associated affine toric variety. The semigroup \( S_\sigma \) is minimally generated by \( m_1 = (0,1) \), \( m_2 = (1,0) \) and \( m_3 = (2,-1) \). We observe that \( m_1 \) and \( m_2 \) form a \( \mathbb{Z} \)-basis of \( M \) and the relation \( m_1 + m_3 = 2m_2 \) generates all the relations between elements of \( S_\sigma \). Thus, setting \( F := Z_1Z_3 - Z_2^2 \), the ideal of \( V_\sigma \) in \( k[Z_1,Z_2,Z_3] \) is the ideal generated by \( F \). The ideal of \( \mathcal{L}_\infty(V_\sigma) \) in the ring \( k[Z_\infty] = k[Z_1,s,Z_2,s,Z_3,s] ; \ s \in \mathbb{N} \) is generated by \( \{F_s ; s \in \mathbb{N}\} \) where \( F_s = \sum_{r=0}^{s-1} (Z_1,s-rZ_3,r - Z_2,s-rZ_2,r) \).

We now consider the toric valuation \( \text{ord}_n \) of \( V_\sigma \) corresponding to \( n = (1,1) \in \sigma \cap N \). The prime ideal of \( k[Z_\infty] \) corresponding to the generic point \( \eta_n \) of the Nash set associated with \( \text{ord}_n \) is the radical of the ideal \( (Z_1,0,Z_2,0,Z_3,0) \). The residue field of \( \eta_n \) is isomorphic to \( k := k(Z_{1,s},Z_{2,s} ; s \geq 1) \).

Denote by \( \{Z_{3,s} ; s \geq 1\} \) the unique family of elements of \( k \) such that for every \( s \geq 2 \), one has

\[
\sum_{r=1}^{s-1} (Z_{1,s-r}Z_{3,r} - Z_{2,s-r}Z_{2,r}) = 0.
\]

Note that the latter is a triangular invertible \( k \)-linear system in the \( Z_{3,s} \)’s.

Now let \( \{Z_{3,s} ; r \geq 1\} \) be the unique family of elements of \( k[Z_1,0,Z_2,0,Z_3,0] \) such that

1. for every \( s \geq 1 \), one has \( Z_{3,s} = z_{3,s} \) (mod \( \langle Z_{1,0},Z_{2,0},Z_{3,0}\rangle \));
2. for every \( s \geq 2 \), one has

\[
Z_{1,s}Z_{3,0} - Z_{2,0}Z_{2,s} + \sum_{r=1}^{s} (Z_{1,s-r}Z_{3,r} - Z_{2,s-r}Z_{2,r}) = 0.
\]
Explicit truncations of the series $Z_{3,s}$ may be obtained by applying effectively the Hensel’s lemma, in other words by successive approximations, though explicit computations quickly become cumbersome. For example one has
\[ Z_{3,1} = \frac{Z_{2,1}^2}{Z_{1,1}} + \frac{Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}} - \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,2}}{Z_{1,1}} \quad (\text{mod } \langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle^2) \]
and
\[
Z_{3,2} = \frac{Z_{1,2}Z_{2,1}}{Z_{1,1}^2} + \frac{2Z_{2,1}Z_{2,2}}{Z_{1,1}} - \frac{2Z_{1,0}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} + \frac{4Z_{1,0}Z_{1,2}Z_{2,1}Z_{2,2}}{Z_{1,1}^2} \\
+ \frac{Z_{1,0}Z_{1,3}Z_{2,1}}{Z_{1,1}^2} - \frac{Z_{1,0}Z_{2,2}}{Z_{1,1}^2} - \frac{2Z_{1,0}Z_{2,1}Z_{2,3}}{Z_{1,1}^2} + \frac{Z_{1,2}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{1,2}Z_{2,0}Z_{2,2}}{Z_{1,1}} \\
- \frac{Z_{1,1}Z_{3,0}}{Z_{1,1}} + \frac{2Z_{2,0}Z_{2,3}}{Z_{1,1}} \quad (\text{mod } \langle Z_{1,0}, Z_{2,0}, Z_{3,0} \rangle^2).
\]
Then the formal neighborhood of $\eta_n$ in $\mathcal{L}_\infty(V_\sigma)$ is isomorphic to the formal spectrum of
\[ K[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]/\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,1}Z_{3,0} + Z_{1,0}Z_{3,1} - 2Z_{2,0}Z_{2,1} \rangle.
\]
Note that it is not clear that the latter is the completion of an essentially of finite type local $K$-algebra.

6.24. Using our comparison theorem, the computation of the formal neighborhood of $\eta_n$ in $\mathcal{L}_\infty(V_\sigma)$ may also be done in the following much more straightforward way. First we compute the formal scheme $\mathcal{W}(n)$ defined in 6.11. We have the following equality in $k[Z_{1,0}, Z_{2,0}, Z_{3,0}, t]:$
\[ F|_{Z_j=t+Z_{3,0}} = (t+Z_{1,0})(t+Z_{3,0}) - (t+Z_{2,0})^2 = (Z_{1,0} + Z_{3,0} - 2Z_{2,0})t + Z_{1,0}Z_{3,0} - Z_{2,0}^2.
\]
We deduce that
\[ \mathcal{W}(n) = \text{Spf} \left( \frac{k[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0} \rangle} \right) \]
and that the formal neighborhood of $\eta_n$ in $\mathcal{L}_\infty(V_\sigma)$ is isomorphic to
\[ \text{Spf} \left( \frac{K[[Z_{1,0}, Z_{2,0}, Z_{3,0}]]}{\langle Z_{1,0}Z_{3,0} - Z_{2,0}^2, Z_{1,0} + Z_{3,0} - 2Z_{2,0} \rangle} \right). \]
In addition, it is not difficult to see that $\mathcal{W}(n)$ is isomorphic to $\text{Spf}(k[[Z_{1,0}, Z_{2,0}]]/\langle Z_{1,0}^2 \rangle)$.

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