An Extension of the Classical Gauss Series-product Identity by Fermionic Construction of $\hat{sl}_n$

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Abstract

The main result of this paper is two infinity classes of series-product identities which is based on classical Gauss identity and two different interpretations of character formula for irreducible highest weight modules of affine Lie algebras.

1 INTRODUCTION

It is well known that celebrated Macdonald identities (and especially the Jacobi triple product identity) are nothing else than the denominator identity for affine Kac-Moody Lie algebras. Moreover, some specializations of the denominator
identity give interesting series-product identities. For instance, the following classical identities

\[ \varphi(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(3n^2+n)/2} \quad (Euler), \]

\[ \varphi(q)^3 = \sum_{n \in \mathbb{Z}} (4n+1)q^{2n^2+n} \quad (Jacobi), \]

\[ \frac{\varphi(q)^2}{\varphi(q^2)} = \sum_{n \in \mathbb{Z}} (-1)^n q^n \quad (Gauss), \]

and in particular

\[ \frac{\varphi(q^2)^2}{\varphi(q)} = \sum_{n \in \mathbb{Z}} q^{2n^2+n} \quad (Gauss), \quad (1.1) \]

where \( \varphi(q) = \prod_{j \geq 1} (1 - q^j) \) is Euler’s product function, can be also expressed following the same approach (see [1], Exercise 12.4, pp. 241). It is also quite interesting that the classical Gauss identity (1.1) arises from the first concrete computations of characters of nontrivial modules for affine Lie algebras (see [2]).

As Victor Kac writes in his book [1] (pp. 216) the basic idea of this approach is very simple: "one gets an interesting identity by computing the character of integrable representation in two different ways and equating the results. In particular, Macdonald identities are deduced via trivial representation."

Following Kac’s inventive consideration the central object of the observation in this article will be a character formula of an irreducible highest weight module \( L(\Lambda) \) of the affine Lie algebra \( \hat{\mathfrak{sl}}_n \), where \( \Lambda \) can be any fundamental weight, not just a trivial representation. The above observation results with two infinity classes of series-product identities. If we denote by \( m \) the arbitrary positive integer and by \( \kappa \) the following polynomial of the several variables

\[ \kappa(k_1, \ldots, k_{4m-1}) = k_1^2 + k_2^2 + \cdots + k_{4m-1}^2 - k_1k_2 - k_2k_3 - \cdots - k_{4m-2}k_{4m-1} \]

than the first class of series-product identities looks like

\[ \frac{\varphi(q^{4m-1})\varphi(q^{2m})^2}{\varphi(q)\varphi(q^m)} = \sum_{k_1, \ldots, k_{4m-1} \in \mathbb{Z}} q^{(4m-1)\kappa(k_1, \ldots, k_{4m-1}) + \text{lin}(k_1, \ldots, k_{4m-1})}, \quad (1.2) \]
where
\[
\text{lin}(k_1, \ldots, k_{4m-1}) = (2m-1)k_1 - k_2 - \cdots - k_{3m-1} + (4m-2)k_{3m} - k_{3m+1} - \cdots - k_{4m-1}.
\]

The another class
\[
\frac{\varphi(q^{3m})}{\varphi(q)^2} \frac{\varphi(q^2)^2}{\varphi(q^2)} = \sum_{k_1, \ldots, k_{4m-1} \in \mathbb{Z}} q^{3m\text{lin}(k_1, \ldots, k_{4m-1}) + \text{lin}(k_1, \ldots, k_{4m-1})} \tag{1.3}
\]
holds for
\[
\text{lin}(k_1, \ldots, k_{4m-1}) = -3k_1 - \cdots - 3k_{m-1} + (3m-2)k_m - k_{m+1} - \cdots - k_{4m-2} + (3m-1)k_{4m-1}.
\]

Since the identity (1.1) will be essentially involved in the proof of mentioned classes (1.2) and (1.3) we can interpret it as an extension of this classical Gauss identity.

As we say above, we looked at the mentioned object from two different points of view. One point of view is based on the character formula
\[
\text{ch } L(\Lambda) = e^{\frac{1}{2}|\Lambda|^2} \sum_{\gamma \in \mathbb{Z}^+} e^{\Lambda_0 + \gamma - \frac{1}{2} |\gamma|^2} \prod_{j \geq 1} (1 - e^{-j\delta})^{\text{mult j}\delta}. \tag{1.4}
\]
for any dominant integral weight \(\Lambda\) in the special case of the affine Lie algebras of type \(A_l^{(1)}, D_l^{(1)}, E_l^{(1)}\) (see [1] or [3] and [4]).

Another point of view is based on a boson-fermionic realization of \(L(\Lambda)\) for affine Lie algebra \(\widehat{\mathfrak{sl}}_n\) (see [5]), parameterized by a partition of a number \(n = \{n_1, n_2, \ldots, n_r\}\). The corresponding character formula for the affine Lie algebra \(\widehat{\mathfrak{sl}}_n\) in the original notation [5] is
\[
\text{Trace}_{L(\Lambda_k)}(q) = q^{\text{const}} \frac{\varphi(q)}{\prod_{i=1}^r \varphi(q^{1/n_i})} \sum_{k_1 + k_2 + \cdots + k_r = k} q^\frac{k^2}{n_1^2} + \frac{k_2^2}{n_2^2} + \cdots + \frac{k_r^2}{n_r^2}. \tag{1.5}
\]

The connection between two different points of view is made by a particular specialization \(\mathcal{F}_s : \mathbb{C}[[e^{\alpha_0}, e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_l}]] \to \mathbb{C}[[q]]\) of type \(s = (s_0, s_1, s_2, \ldots, s_l)\) which satisfies the following formula
\[
\mathcal{F}_s(\text{ch } L(\Lambda_k)) = q^{\text{const}} \text{Trace}_{L(\Lambda_k)}(q^N). \tag{1.6}
\]
The \( n \)-tuple \( s \) and positive integer \( N \) are generated (we shall provide details later) from the same partition \( n \).

In this article we would like to emphasize the importance of (1.6). Observe that both sides of (1.6) contain infinite sums (denoted by \( \sum_{\text{left}} \) and \( \sum_{\text{right}} \)) dependent on significantly different sizes of the set of indices. On the left hand side we have \( n - 1 \) indices, while on the right hand side we have \( r - 1 \). Very often we have the case (which is of particular interest to us) when

\[
n \gg r.
\]

In this particular case we obtain a significant reduction of the sum \( \sum_{\text{left}} \) by the sum \( \sum_{\text{right}} \) modulo some fraction of Euler’s product functions \( \varphi(q) \). Hypothetically, if we recognize, in particular cases, that the sum \( \sum_{\text{right}} \) is a one side of the some well known series-product identity we can express the sum \( \sum_{\text{left}} \), which depends on arbitrary indices using the fraction of Euler’s product functions \( \varphi(q) \).

Moreover, we would like to argue that our approach provides an algebraic method to reveal numerous important series-product identities; studied mainly by number-theorists. This is one of the main points of our article. We illustrate the power of our method by constructing two infinite classes of series-product identities; both are based on the classical Gauss series-product identity (1.1).

Finally, let us also mention that these two classes have only one common element which was the starting point of our research. We shall conclude this introduction by providing details on this particular example. This example provides a review of above observations and in the same time is a motivation for the construction of the mentioned two series-product identity classes.

We also believe that this example will guide our reader through the main body of the article.

Let \( g \) be a simple Lie algebra of the type \( A_3 \), i.e., \( g = \mathfrak{sl}_4 \). For the corresponding affine Lie algebra \( \hat{g} = \hat{\mathfrak{sl}}_4 \), the partition \( \underline{4} = \{1, 3\} \) and the fundamental
weight $\Lambda_3$ the character formula (1.4) looks like
\[
\chi_{L(\Lambda_3)} = e^{\frac{1}{2} |\gamma|^2 \delta} \sum_{k_1,k_2,k_3 \in \mathbb{Z}} e^{\Lambda_0 + (k_1 + 1/4) \alpha_1 + (k_2 + 2/4) \alpha_2 + (k_3 + 3/4) \alpha_3 - \frac{1}{2} |\gamma|^2 \delta} \prod_{n \geq 1} (1 - e^{-j \delta})^3
\]
where
\[
\frac{1}{2} |\gamma|^2 = (k_1 + 1/4)^2 + (k_2 + 2/4)^2 + (k_3 + 3/4)^2 - (k_1 + 1/4)(k_2 + 2/4) - (k_2 + 2/4)(k_3 + 3/4)
\]
and
\[
\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3
\]
is the corresponding imaginary root.

For above settings ($\mathfrak{g}_4$, $\mathfrak{d} = \{1,3\}$, $\Lambda_3$) the trace formula (1.5) has the following form
\[
\text{Trace}_{L(\Lambda_3)}(q) = q^{\text{const}} \frac{\varphi(q)}{\varphi(q^{1/3})^3} \sum_{k_1+k_2=3} q^{\frac{1}{2}(k_1^2 + k_2^2)}
\]
Now using the substitution of variables $k_2 = 3 - k_1$ and classical Gauss identity (1.1) in the sum
\[
\sum_{k_1+k_2=3} q^{\frac{1}{2}(k_1^2 + k_2^2)}
\]
we have
\[
\text{Trace}_{L(\Lambda_3)}(q) = q^{\text{const}} \frac{\varphi(q^{2/3})^2}{\varphi(q^{1/3})^2}.
\]

As outlined above, we now apply a particular specialization, in this case the specialization $\mathcal{F}_s : \mathbb{C}[[e^{\alpha_0}, e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3}]] \rightarrow \mathbb{C}[[q]]$ defined by parameters
\[
s = (s_0, s_1, s_2, s_3) = (2, -1, 1, 1)
\]
which are also generated by the partition $\mathfrak{d} = \{1,3\}$. Now, using (1.5) for $N = 3$ we obtain
\[
\varphi(q^{2/3})^2 \varphi(q^{3/3}) = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} q^{3(k_1^2 + k_2^2 + k_3^2 - k_1 k_2 - k_1 k_3 + k_1 - k_2 + 2k_3)}
\]
which seems to be a new series-product identity extended from the classical Gauss identity (1.1).
2 THE NOTATION AND BASIC SETTINGS

Let \( g = \mathfrak{sl}_n \) be a simple Lie algebra defined for the Dynkin diagram \( A_l \), where \( l = n - 1 \). Denote by \( \mathfrak{h} \) the corresponding Cartan subalgebra and by

\[
\Delta = (\alpha_1, ..., \alpha_l)
\]

the basis of the root system \( \mathcal{R}(\subset \mathfrak{h}^*) \). Besides, by \( \theta \) we denote the highest root of the root system \( \mathcal{R} \). It is well known that

\[
\mathcal{R} = \{ \pm(\varepsilon_i - \varepsilon_j) | 1 \leq i < j \leq l + 1 \} \\
\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, ..., \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l - \varepsilon_{l+1} \quad (2.1)
\]

\[
\theta = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_l = \varepsilon_1 - \varepsilon_{l+1}
\]

Let

\[
g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha
\]

be a root space decomposition of the simple Lie algebra. Denote by \( x_\alpha \in \mathfrak{g}_\alpha \) the root vector which satisfies

\[
[x_\alpha, x_{-\alpha}] = \alpha^\vee
\]

for the coroot \( \alpha^\vee \).

Let

\[
\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus (\mathbb{C}c + \mathbb{C}d)
\]

and write \( x(i) = x \otimes t^i \) for \( x \in g \) and \( i \in \mathbb{Z} \). Then \( \hat{g} = \hat{\mathfrak{sl}}_n \) is an affine Lie algebra with

\[
[x(i), y(i)] = [x, y](i + j) + i\delta_{i+j,0}(x \mid y),
\]

where \( (x \mid y) \) is the Killing form for the simple Lie algebra \( g \), \( c \) being a central element

\[
c = \sum_{i=0}^{l} \alpha_i^\vee
\]
and $d$ a scaling element
\[ [d, x(i)] = ix(i) . \]

The Cartan subalgebra of $\hat{\mathfrak{g}}$ is given by
\[ \hat{\mathfrak{h}} = \mathfrak{h} \oplus (\mathbb{C} c + \mathbb{C} d) . \]

The corresponding Dynkin diagram is of type $A_l^{(1)}$ and related numerical labels are
\[ a_{A_l^{(1)}} = (1, 1, 1, ..., 1, 1, 1) . \] (2.2)

We denote by $\delta$ the linear functional on CSA $\hat{\mathfrak{h}}$ defined by
\[ \delta |_{\mathfrak{h} \oplus \mathbb{C} c} = 0 \quad \langle \delta, d \rangle = (\delta | d) = 1. \]

The affine Lie algebra root system $\hat{\mathcal{R}}(\subset \hat{\mathfrak{h}}^*)$ is composed of the real and imaginary roots
\[ \hat{\mathcal{R}} = \hat{\mathcal{R}}^{Re} \cup \hat{\mathcal{R}}^{Im} = \{\alpha + n\delta | \alpha \in \mathcal{R}, n \in \mathbb{Z}\} \cup \{n\delta | n \in \mathbb{Z} \setminus \{0\}\} . \]

If we denote by $\alpha_0$ the following root
\[ \alpha_0 = -\theta + \delta \] (2.3)
then
\[ \Delta = (\alpha_0, \alpha_1, ..., \alpha_l) \]
form the basis of the root system $\hat{\mathcal{R}}$ and $Q = \sum_{i=0}^{l} Z\alpha_i$ is the corresponding root lattice.

The imaginary root $\delta$, spanned in above basis, due to (2.2) and (2.3), looks like
\[ \delta = \sum_{i=0}^{l} a_i \alpha_i = \sum_{i=0}^{l} \alpha_i . \] (2.4)

It is well known that all affine Lie algebras are a Kac-Moody algebras $\mathfrak{g}(A)$ for a generalized Cartan matrix $A$ of the corank one. Since the affine Lie algebra $\hat{\mathfrak{sl}}_n$ is
also the Kac-Moody algebra then for every \( \Lambda \in \hat{\mathfrak{h}}^* \) the irreducible highest-weight module \( L(\Lambda) \) is uniquely defined. The number

\[
\Lambda(c) = \langle \Lambda, c \rangle
\]  

is called the level of the weight \( \Lambda \) or of the module \( L(\Lambda) \).

Denote by \( P(\Lambda) \) the set of all weights of the module \( L(\Lambda) \) and by \( \text{mult} \lambda \) the multiplicity of \( \lambda \in P(\Lambda) \). The set

\[
P = \{ \lambda \in \hat{\mathfrak{h}}^* | \lambda(\alpha_i) \in \mathbb{Z}, \ i = 0, 1, ..., n - 1 \}
\]

is called the weight lattice and weights from \( P \) are called integral weights. Integral weights from

\[
P_+ = \{ \lambda \in P | \lambda(\alpha_i) \geq 0, \ i = 0, 1, ..., n - 1 \}
\]

are called dominant. The weight lattice contains the root lattice \( Q \) and it is clear that

\[
P(\Lambda) \subset P
\]

if \( \Lambda \in P \). Besides, the irreducible highest-weight \( \mathfrak{g}(A) \)-module is integrable if and only if \( \Lambda \in P_+ \). The fundamental weights \( \Lambda_i \) for \( i = 0, 1, ..., n - 1 \) are defined by

\[
\Lambda_i(\alpha_j^\vee) = \delta_{ij}, \ j = 0, 1, ..., n - 1 \quad \text{and} \quad \Lambda_i(d) = 0 . \quad (2.6)
\]

It is obvious that fundamental weights are always dominant.

### 3 THE FIRST POINT OF VIEW

For a subset \( S \) of \( \hat{\mathfrak{h}}^* \) denote by \( \overline{S} \) the orthogonal projection of \( S \) on \( \mathfrak{h}^* \) by the extension of the Killing form from the simple \( \mathfrak{g} \) to the affine Lie algebra \( \hat{\mathfrak{g}} \). Then we have the following useful formula for \( \lambda \in \hat{\mathfrak{h}}^* \)

\[
\lambda = \overline{\lambda} + \lambda(c)\Lambda_0 + (\lambda | \Lambda_0)\delta
\]
where $\Lambda_0$ is the fundamental weight.

Especially for the $\mathfrak{sl}_n$ we have that

$$\sum_{i=1}^l Z\alpha_i$$

where $\Lambda_0 = 0$ and $\Lambda_1, ..., \Lambda_{n-1}$ are the fundamental weights of simple Lie algebra $\mathfrak{sl}_n$.

Since $\Lambda_i \in P_+$ then for all fundamental weights the irreducible highest-weight $\mathfrak{sl}_n$-module $L(\Lambda_i)$ is integrable.

Moreover, when $\Lambda \in P_1^+$ and the type of Dynkin diagram is equal to $A_i^{(1)}$, $D_i^{(1)}$ or $E_i^{(1)}$ the following formula

$$P(\Lambda) = \left\{ \Lambda_0 + \frac{1}{2}|\Lambda|^2 \delta + \alpha - \frac{1}{2} |\alpha|^2 + s \delta \mid \alpha \in \mathfrak{X}, s \in \mathbb{Z}_+ \right\}$$

explicitly describes the weights system $P(\Lambda)$. This result is proved in [3] or [6].

A weight $\lambda \in P(\Lambda)$ is called maximal if $\lambda + \delta / \notin P(\Lambda)$. Denote by $\max(\Lambda)$ the set of all maximal weights of $L(\Lambda)$. For $\lambda \in \max(\Lambda)$ the series

$$a^\Lambda_{\lambda} = \sum_{n=0}^{+\infty} \text{mult}_{L(\Lambda)}(\lambda - n\delta)e^{-n\delta}$$

is well defined. Using the result (3.2), the theory of the series $a^\Lambda_{\lambda}$ (which has been started by [2] and [3]) and the work [4] (or [1], Ch.12) the character formula of $L(\Lambda)$ can be written as

$$\text{ch}_{L(\Lambda)} = e^{\frac{1}{2}|\Lambda|^2 \delta} a^\Lambda_{\Lambda_0} \sum_{\gamma \in \mathfrak{Q} + \mathfrak{X}} e^{\Lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta} = e^{\frac{1}{2}|\Lambda|^2 \delta} \sum_{\gamma \in \mathfrak{Q} + \mathfrak{X}} e^{\Lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta} \prod_{n \geq 1} (1 - e^{-n\delta})\text{mult}_{n\delta}$$

Since the Dynkin diagram of affine Lie algebra $\mathfrak{sl}_n$ is equal to $A_i^{(1)}$ and all fundamental weights $\Lambda_k$ are level one integral dominant weights (see 3.1) then
it is obvious that the above character formula is appropriate for $\hat{sl}_n$. Finally, we finish this exposition by the character formula

$$ch_{L(Λ_k)} = e^{\frac{1}{2} |Λ_k|^2 δ} \sum_{γ \in \mathcal{Q} + Λ_k} e^{Λ_0 + γ - \frac{1}{2} |γ|^2 δ} \prod_{n \geq 1} (1 - e^{-nδ})^{\text{mult } nδ}$$

for the irreducible integrable highest weight $\hat{sl}_n$-module $L(Λ_k)$ where $k = 0, 1, \ldots, l$.

4 THE SECOND POINT OF VIEW

Many of the vertex operator constructions of integrable highest weight representations are based on an inequivalent Heisenberg subalgebras. The inequivalent Heisenberg subalgebras, as conjugacy classes in $S_n$, are parametrized by partitions of $n$. Denote by $\underline{n} = \{n_1, n_2, \ldots, n_r\}$ a partition of $n$ where

$$n_1 \leq n_2 \leq \ldots \leq n_r .$$

Moreover, the standard canonical base $\{E_{ij} | i, j = 1, 2, \ldots, n\}$ for $gl_n$ is also parametrized by the partition $\underline{n}$ (see [5]). The associated partition of $n \times n$ matrices is then given schematically by:

$$
\begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1s} \\
B_{21} & B_{22} & \cdots & B_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
B_{s1} & B_{s2} & \cdots & B_{ss}
\end{bmatrix}_{n \times n}
$$

where $B_{ij}$ is a block of size $n_i \times n_j$. With this blockform in mind the standard canonical base is remodeled with the set of matrices

$$\{E_{ij}^{pq} | p = 1, \ldots, n_i, q = 1, \ldots, n_j, i, j = 1, 2, \ldots, s\}$$

where

$$E_{ij}^{pq} = E_{n_1 + \cdots + n_{i-1} + p, n_1 + \cdots + n_{j-1} + q} .$$

Using just mentioned notation, in [5] authors give an explicit vertex operator constructions of level one irreducible integrable highest weight representation.
of \( \hat{gl}_n \) for all inequivalent Heisenberg subalgebras (i.e. for all partitions). The construction uses multicomponent fermionic fields and yields a correspondence between bosons (elements of Heisenberg subalgebra) and fermions. The mentioned construction in addition to [7] results with explicit "q-dimension" trace formula for \( \hat{gl}_n \)-module

\[
T \text{race}_{\Lambda} = q^{\frac{1}{2} |H_n|^2} \sum_{k_1+k_2+\cdots+k_r = k} q^{\frac{k}{2}(\frac{n_1^2}{n_1} + \frac{n_2^2}{n_2} + \cdots + \frac{n_r^2}{n_r})} \prod_{i=1}^{l} \prod_{j \geq 1} (1 - q^{\pi_j})
\]

where \( H_n \) is element of standard Cartan subalgebra which satisfies the following commutation relations

\[
\text{ad } H_n(E_{kl}) = [H_n, E_{kl}] = (\frac{l}{n_j} - \frac{k}{n_i} + \frac{1}{2n_i} - \frac{1}{2n_j})E_{kl}.
\] (4.2)

Finally, after restriction to \( \hat{sl}_n \) case (also [5]), the corresponding irreducible integrable highest weight module \( L(\Lambda_k) \) have the following trace formula

\[
T \text{race}_{L(\Lambda_k)}(q) = q^{\frac{1}{2} |H_n|^2} \prod_{j \geq 1} (1 - q^{\pi_j}) \sum_{k_1+k_2+\cdots+k_r = k} q^{\frac{k}{2}(\frac{n_1^2}{n_1} + \frac{n_2^2}{n_2} + \cdots + \frac{n_r^2}{n_r})} \prod_{i=1}^{l} \prod_{j \geq 1} (1 - q^{\pi_j})
\] (4.3)

5 THE CONNECTION BETWEEN STANDPOINTS

Let \( s = (s_0, s_1, \ldots, s_n) \) be a sequence of integers. Then the sequence \( s \) (under some assumptions) defines a homomorphism

\[
\mathcal{F}_s : \mathbb{C}[e^{-\alpha_0}, e^{-\alpha_1}, e^{-\alpha_2}, \ldots, e^{-\alpha_n}] \rightarrow \mathbb{C}[q]
\]

by

\[
\mathcal{F}_s(e^{-\alpha_i}) = q^{s_i} \quad (i = 0, 1, \ldots, n).
\]

This homomorphism is called the specialization of type \( s \).

Let \( N' \) be the least common multiple of \( n_1, n_2, \ldots, n_r \), then the integer \( N \) is defined by:

\[
N = \begin{cases} 
N' & \text{if } N'(\frac{1}{n_i} + \frac{1}{n_j}) \in 2\mathbb{Z} \quad \forall i, j \\
2N' & \text{if } N'(\frac{1}{n_i} + \frac{1}{n_j}) \notin 2\mathbb{Z} \text{ for a pair } (i, j)
\end{cases}
\] (5.1)
Following the boson-fermionic construction for $\hat{\mathfrak{gl}}_n$ from the paper [5] we can conclude that the connection between the mentioned two different points of view is made by particular specialization of type $s = (s_0, s_1, s_2, ..., s_n)$ where $s$ was parametrized with the partition $\underline{n} = \{n_1, n_2, ..., n_r\}$ and integer $N$. In fact we have the following proposition.

**Proposition 5.1** Let $\underline{n} = \{n_1, ..., n_r\}$ be a partition of $n$. Let $N$ be the corresponding integer defined by (5.1). For affine Lie algebra $\hat{\mathfrak{sl}}_n$ and for all fundamental weights $\Lambda_k k = 0, 1, ..., n - 1$ the next equation

$$F_s \left( \sum_{\gamma \in \mathbb{Z}^+} e^{\Lambda_0 + \gamma - 1/2|\gamma|^2} \prod_{j \geq 1} (1 - e^{-j\delta})^{\text{mult } j\delta} \right)$$

$$= q^{\text{const}} \prod_{j \geq 1} (1 - q^{jN}) \left( \sum_{k_1 + k_2 + ... + k_r = k} q^{\frac{k_1^2 + k_2^2 + ... + k_r^2}{2}} \prod_{i=1}^r \prod_{j \geq 1} (1 - q^{jN_{ni}}) \right)$$

holds for

$$s = N \left( \frac{n_1 + n_r}{2n_1 n_r}, \frac{1}{n_1}, ..., \frac{n_1 + n_2}{2n_1 n_2} - 1, \frac{1}{n_2}, ..., \frac{1}{n_r}, \frac{n_2 + n_3}{2n_2 n_3} - 1, ..., \frac{1}{n_{r-1}}, ..., \frac{n_{r-1} + n_r}{2n_{r-1} n_r} - 1, \frac{1}{n_r}, ..., \frac{1}{n_r} \right).$$

**PROOF.** Since the root subspaces $\hat{\mathfrak{g}}_\alpha$, for $\alpha \in \hat{\mathfrak{R}}^{Re}$, are one-dimensional we have unique $1 - 1$ correspondence

$$\alpha_0 \longleftrightarrow \hat{x}_- \theta \otimes t$$

$$\alpha_i \longleftrightarrow \hat{x}_{\alpha_i} \otimes 1 \quad i = 1, ..., l.$$ 

Hence, we have correspondence, based on (2.1), between the base $\Delta$ and the element of standard canonical base of $\mathfrak{gl}_n$ $\{E_{ij} | i, j = 1, ..., n\}$

$$\alpha_i \longleftrightarrow E_{i,i+1} \quad i = 1, ..., l.$$ 

From (2.3) and (2.4) it is evident

$$\alpha_0 \longleftrightarrow E_{n,1} \otimes t.$$ 

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The commutation relations (4.2) for $\text{ad}H_n$ and $E_{kl}^{ij}$ as indexed in (4.1) express degrees of eigenvectors $E_{kl}^{ij}$ by

$$\text{deg}E_{kl}^{ij} = N\left(\frac{l}{n_j} - \frac{k}{n_i} + \frac{1}{2n_i} - \frac{1}{2n_j}\right) \mod N.$$  \hspace{1cm} (5.6)

From (5.6) and from the exposition of [5] which lead to trace formula (4.3) follows that eigenvalues for the adjoint action of $\text{ad}H_n$ are pointers for the right specialization $s$. More precisely, using $1 - 1$ correspondence (5.4), (5.5) the sequence $s$ consists of the eigenvalues for eigenvectors

$$\{\alpha_0^\vee, \alpha_1^\vee, ..., \alpha_{n-1}^\vee\} = \{E_{n,1} \otimes t, E_{1,2} \otimes 1, E_{2,3} \otimes 1, ..., E_{n-1,n} \otimes 1\}.$$  \hspace{1cm} (5.7)

as shown in

$$s = (\text{deg}E_{n,1} + N, \text{deg}E_{1,2}, \text{deg}E_{2,3}, ..., \text{deg}E_{n-1,n}).$$  \hspace{1cm} (5.7)

Due to remodeling (4.1) of the standard canonical base by $\{E_{ij}^{pq}\}$ we conclude that sequence $s$ (5.7) is equal to (5.3), i.e. the equation (5.2) holds.

**Remark 5.2** In the paper [8] V.G.Kac showed that the automorphisms

$$\sigma_s(e_i) = e^{\frac{2\pi i s_i}{N}}e_i$$

$i = 0, 1, ..., l$

exhaust all $N$-th order automorphisms of $\mathfrak{g}$. By $\{e_i \mid i = 0, 1, ..., l\}$ are marked generators of $\mathfrak{g}$ and

$$s = (s_0, s_1, ..., s_l)$$

is a sequence of nonnegative relatively prime integers. The parameters $s_i$ are called the Kac parameters. Many of the vertex operator constructions of integrable highest weight representations and the corresponding gradations and specializations do not provide Kac parameters (see, in particular [5] and [7]).

Particularly, the sequence $s$ (5.3) and the associated specialization are determined by relatively prime integers, but all integers $N\left(\frac{n_i+n_{i+1}}{2n_i n_{i+1}} - 1\right)$ are negative. So, the specialization (5.2) is not parametrized by Kac parameters.
In the paper [9], it is given the exact algorithm for finding the Kac parameters $s^{Kac}$ of the sequence $s$ \[(5.3)\] and equation \[(5.2)\] holds for specialization by Kac parameters, too.

6 THE CLASS $GAUSS(\underline{n} = 1 + (4m - 1), \Lambda_{3m})$

Introduce the Euler product
$$\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

Denote by $\kappa$ the following polynomial of the several variables
$$\kappa(k_1, ..., k_l) = k_1^2 + k_2^2 + \cdots + k_l^2 - k_1k_2 - k_2k_3 - \cdots - k_{l-1}k_l$$
where $l = n - 1$. It is interesting to notice that above polynomial can be interpreted by the Killing form in such a way that
$$\kappa(k_1, ..., k_l) = \frac{1}{2} (k_1\alpha_1^\vee + \cdots k_l\alpha_l^\vee | k_1\alpha_1^\vee + \cdots k_l\alpha_l^\vee).$$

**Theorem 6.1** Let $n = 4m$ for an arbitrary positive integer $m$. Then the following series-product identity
\[\sum_{k_1, ..., k_{4m-1} \in \mathbb{Z}} q^{(4m-1)\kappa(k_1, ..., k_{4m-1}) + \text{lin}(k_1, ..., k_{4m-1})} = \frac{\varphi(q^{4m-1})^{4m-1} \varphi(q^{2m})^2}{\varphi(q) \varphi(q^{m})},\]
holds for
\[\text{lin}(k_1, ..., k_{4m-1}) = (2m - 1)k_1 - k_2 - \cdots - k_{3m-1} + (4m - 2)k_{3m} - k_{3m+1} - \cdots - k_{4m-1}.\]

**Proof.** First of all, the proof is based on the following setting:
\[\hat{g} = \hat{sl}_{4m},\]
\[\underline{n} = 4m = \{1, 4m - 1\}\]
\[\Lambda = \Lambda_{3m} = \Lambda_0 + \overline{\Lambda}_{3m} .\]
Since $\Lambda_{3m}$ is the corresponding fundamental weight of simple Lie algebra $\mathfrak{sl}_{4m}$ we can write (see [10], pp. 69):

$$\Lambda_{3m} = \frac{1}{4m} \left[ m\alpha_1 + 2m\alpha_2 + \cdots + (3m - 1)\cdot m\alpha_{3m-1} + 3m\cdot m\alpha_{3m} ight. $$

$$+ 3m \cdot (m - 1)\alpha_{3m+1} + 3m \cdot (m - 2)\alpha_{3m+2} + \cdots + 3m \cdot 2\alpha_{4m-2} + 3m \cdot 1\alpha_{4m-1} \right].$$

(6.4)

The numerator of the formula (3.3) looks like

$$e^{\Lambda_0 + \frac{1}{2}|\Lambda_{3m}|^2} \sum_{\gamma \in \mathbb{Z}^+ \Lambda_{3m}} e^{-\frac{1}{2}|\gamma|^2}$$

where

$$\gamma = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_{4m-1}\alpha_{4m-1} + \Lambda_{3m}.$$  

(6.5)

Using (6.4) the vector $\gamma$ is written down by the base $\Delta$

$$\gamma = (k_1 + \frac{1}{4})\alpha_1 + (k_2 + \frac{2}{4})\alpha_2 + \cdots + (k_{3m} + \frac{3m}{4})\alpha_{3m} $$

$$+(k_{3m+1} + \frac{3(m-1)}{4})\alpha_{3m+1} + (k_{3m+2} + \frac{3(m-2)}{4})\alpha_{3m+2} $$

$$+ \cdots + (k_{4m-2} + \frac{3 \cdot 2}{4})\alpha_{4m-2} + (k_{4m-1} + \frac{3 \cdot 1}{4})\alpha_{4m-1}.$$  

(6.6)

For the partition $4m = \{1, 4m - 1\}$ it is obvious (see 5.1) that the number $N$ is equal $4m - 1$. Besides, the mentioned partition implicates that the blockform of $4m \times 4m$ matrices looks like

$$\begin{bmatrix}
B_{11} & 1 \times 1 & B_{12} & 1 \times 4m-1 \\
B_{21} & 4m-1 \times 1 & B_{22} & 4m-1 \times 4m-1
\end{bmatrix}$$

Following (4.2), (5.6) and (5.7) we can conclude that the specialization $\mathcal{F}_s$, defined by

$$s = (\text{deg}E_{4m-1,1}^{21}, \text{deg}E_{1,1}^{12}, \text{deg}E_{1,2}^{22}, \ldots, \text{deg}E_{4m-2,4m-1}^{22})$$

$$= (2m, -2m + 1, 1, \ldots, 1),$$

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is the specialization for the connection between two standpoints. More explicitly the specialization is given by

\[
e^{-\alpha_0} \leftrightarrow q^{2m} \\
e^{-\alpha_1} \leftrightarrow q^{-2m+1} \\
e^{-\alpha_2} \leftrightarrow q^1 \\
\vdots \\
e^{-\alpha_{4m-1}} \leftrightarrow q^1 \\
e^{-\delta} \leftrightarrow q^{4m-1}.
\]

After calculation

\[
|\gamma|^2 = (\gamma | \gamma) = (6.3) = 2\kappa(k_1, \ldots, k_{4m-1}) + 2k_{3m} + \frac{9m}{4}
\]

and the fact that \( \text{mult \ } n\delta \) always equals \( \text{dim} \ h = 4m - 1 \) the left side of the formula (5.2) has the following form

\[
\mathcal{F}_s\left(\sum_{\gamma \in \mathbb{Q} \times \mathbb{X}} e^{A_0 + \gamma - \frac{1}{2}|\gamma|^2\delta} \prod_{n \geq 1} (1 - e^{-n\delta})^{\text{mult} \ n\delta}\right)
\]

\[= q^{\text{const}} \sum_{k_1, \ldots, k_{4m-1}} q^{(4m-1)\kappa(k_1, \ldots, k_{4m-1}) + \text{lin}(k_1, \ldots, k_{4m-1})} \frac{[\varphi(q^{4m-1})]^{4m-1}}{[\varphi(q)]^{4m-1}} \]  

(6.6)

where

\[\text{lin}(k_1, \ldots, k_{4m-1}) = (2m-1)k_1 - k_2 - \cdots - k_{3m-1} + (4m-2)k_{3m} - k_{3m+1} - \cdots - k_{4m-1}.
\]

The right hand side of the formula (5.2) for the mentioned settings (6.3) looks like

\[q^{\text{const}} \prod_{j \geq 1} (1 - q^{(4m-1)j}) \sum_{k_1+k_2=3m} q^{\frac{4m-1}{2}(k_1^2 + k_2^2)} \prod_{j \geq 1} (1 - q^{(4m-1)j}) \prod_{j \geq 1} (1 - q^{(4m-1)j}) = \]

\[= q^{\text{const}} \sum_{k_1+k_2=3m} q^{\frac{1}{2}[(4m-1)k_1^2+k_2^2]} \frac{\varphi(q)}{\varphi(q)}.
\]

After the substitution \( k_2 = 3m - k_1 \) the calculations

\[\sum_{k_1+k_2=3m} q^{\frac{1}{2}[(4m-1)k_1^2+k_2^2]} = \sum_{k_1 \in \mathbb{Z}} q^{\frac{1}{2}[(4m-1)k_1^2+(3m-k_1)^2]} \]
implicate that the right hand side of the formula (5.2) has the form
\[ q^{\text{const}} \sum_{k_1 + k_2 = 3m} q^{\frac{1}{2}((4m-1)k_1^2 + k_2^2)} \varphi(q)^2 = q^{\text{const}} \frac{\varphi(q^{2m})^2}{\varphi(q) \varphi(q^m)}. \] (6.7)

Now, from (6.6) and (6.7) it is obvious that the series-product identity (6.1) holds for (6.2).

7 THE CLASS \textit{GAUSS}(\( n = m + 3m, \Lambda_{4m-1} \))

Denote again by \( \kappa \) the following polynomial of several variables
\[ \kappa(k_1, \ldots, k_l) = k_1^2 + k_2^2 + \cdots + k_l^2 - k_1 k_2 - k_2 k_3 - \cdots - k_{l-1} k_1. \]

**Theorem 7.1** Let \( n = 4m \) for an arbitrary positive integer \( m \). Then the following series-product identity
\[ \sum_{k_1, \ldots, k_{4m-1} \in \mathbb{Z}} q^{3m \kappa(k_1, \ldots, k_{4m-1}) + \text{lin}(k_1, \ldots, k_{4m-1})} = \frac{\varphi(q^{3m})^4 \varphi(q^2)^2}{\varphi(q)^2 \varphi(q^2)} \] (7.1)
holds for
\[ \text{lin}(k_1, \ldots, k_{4m-1}) = -3k_1 - \cdots - 3k_{m-1} + (3m - 2)k_m \]
\[ -k_{m+1} - \cdots - k_{4m-2} + (3m - 1)k_{4m-1}. \] (7.2)

**Proof.** First of all, the proof is based on the following setting:
\[ \hat{g} = \hat{a}_{4m} \]
\[ \mathbf{z} = 4m = \{m, 3m\} \]  \hspace{1cm} (7.3)
\[ \Lambda = \Lambda_{4m-1} = \Lambda_0 + \overline{\Lambda_{4m-1}} \, . \]

Now, it is well known (see [10]) that
\[ \overline{\Lambda_{4m-1}} = \frac{1}{4m} \left[ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + (4m - 2)\alpha_{4m-2} + (4m - 1)\alpha_{4m-1} \right] \, . \]  \hspace{1cm} (7.4)

The numerator of the formula (3.3) looks like
\[ e^{\Lambda_0 + \frac{1}{4m} \left| \Lambda_{4m-1} \right|^2} \delta \sum_{\gamma \in Q + \overline{\Lambda_{4m-1}}} e^{-\frac{1}{4m} \left| \gamma \right|^2} \]
where
\[ \gamma = k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_{4m-1} \alpha_{4m-1} + \overline{\Lambda_{4m-1}} \, . \]  \hspace{1cm} (7.5)

Using (7.4) the vector \( \gamma \) is written down by the base \( \Delta \)
\[ \gamma = (k_1 + \frac{1}{4m})\alpha_1 + (k_2 + \frac{2}{4m})\alpha_2 + \cdots + (k_{4m-1} + \frac{4m - 1}{4m})\alpha_{4m-1} \]

Now, the number \( N \) is equal \( 3m \) when the partition is \( 4m = \{m, 3m\} \) (see 5.1).

This partition implicates the blockform of \( 4m \times 4m \) matrices
\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\]
\[ B_{11} \, m \times m \quad B_{12} \, m \times 3m \]
\[ B_{21} \, 3m \times m \quad B_{22} \, 3m \times 3m \]

Following again (4.2), (5.6) and (5.7) we can concluded that the demanded specialization \( F_s \) is defined by
\[ s = (\text{deg} E_{3m,1}^{21}, N, \text{deg} E_{1,2}^{11}, ..., \text{deg} E_{m-1,m}^{11}, \text{deg} E_{m,1}^{12}, \text{deg} E_{1,2}^{22}, ..., \text{deg} E_{3m-1,3m}^{22}) \]
\[ = (2, 3, ..., 3, 2 - 3m, 1, ..., 1) \, . \]

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More explicitly
\[ e^{-\alpha_0} \leftrightarrow q^2 \]
\[ e^{-\alpha_1} \leftrightarrow q^3 \]
\[ \vdots \]
\[ e^{-\alpha_{m-1}} \leftrightarrow q^3 \]
\[ e^{-\alpha_m} \leftrightarrow q^{2-3m} \]
\[ e^{-\alpha_{m+1}} \leftrightarrow q^1 \]
\[ \vdots \]
\[ e^{-\alpha_{4m-1}} \leftrightarrow q^1 \]
\[ e^{-\delta} \leftrightarrow q^{3m} \].

After calculation
\[ |\gamma|^2 = (\gamma | \gamma) = (\gamma | \gamma) = (7.5) = 2^\kappa(k_1, \ldots, k_{4m-1}) + 2^k_{4m-1} + 4^{m-1} \]

and the fact that \( \text{mult } n\delta \) always equals \( \text{dim } h = 4m - 1 \) the left side of the formula (5.2) has the following form
\[ \mathcal{F} \left( \sum_{\gamma \in Q + \Lambda} e^{\lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta} \prod_{\alpha \geq 1} (1 - e^{-n\delta})^{\text{mult } \alpha} \right) = q_{\text{const}} \sum_{k_1, \ldots, k_{4m-1}} q^{(3m)\kappa(k_1, \ldots, k_{4m-1}) + \text{lin}(k_1, \ldots, k_{4m-1})} / [\varphi(q^{3m})]^{4m-1} \]

where
\[ \text{lin}(k_1, \ldots, k_{4m-1}) = -3k_1 - \cdots - 3k_{4m-1} + (3m-2)k_{m} - k_{m+1} - \cdots - k_{4m-2} + (3m-1)k_{4m-1} \]

The right hand side of the formula (5.2) for the mentioned settings (7.3) looks like
\[ q_{\text{const}} \prod_{j \geq 1} (1 - q^{(3m)j}) / \prod_{j \geq 1} (1 - q^{3j}) \prod_{j \geq 1} (1 - q^{m+1}) = q_{\text{const}} \varphi(q^{3m}) / \varphi(q) \varphi(q^3) \]
After the substitution $k_2 = (4m - 1) - k_1$ the calculations

$$
\sum_{k_1 + k_2 = 4m - 1} q^{\frac{1}{2}(3k_1^2 + k_2^2)} = \sum_{k_1 \in \mathbb{Z}} q^{\frac{1}{2}(3k_1^2 + ((4m - 1) - k_1)^2)}
$$

$$
= q^{(4m - 1)^2} \sum_{k_1 \in \mathbb{Z}} q^{\frac{1}{2}(4k_1^2 - 2(4m - 1)k_1)}
$$

$$
= q^{(4m - 1)^2} \sum_{k_1 \in \mathbb{Z}} q^2(k_1 - m)^2 + (k_1 - m) - 2m^2 + m
$$

$$
= q^{(4m - 1)^2} - 2m^2 + m \sum_{k_1 \in \mathbb{Z}} q^2(k_1 - m)^2 + (k_1 - m)
$$

$$
(Gauss \ref{eq:1.1}) = q^{\frac{4m^2 - 7m + 1}{2}} \frac{\varphi(q^2)^2}{\varphi(q) \varphi(q^3)} \varphi(q) \varphi(q^3) = q^{\frac{4m^2 - 7m + 1}{2}} \frac{\varphi(q)^2}{\varphi(q^3) \varphi(q^2)}.
$$

Implicate that the right hand side of the formula \ref{eq:5.2} has the form

$$
q^{\text{const.}} \varphi(q^{3m}) \sum_{k_1 + k_2 = 4m - 1} q^{\frac{1}{2}(3k_1^2 + k_2^2)} \varphi(q) \varphi(q^3) = q^{\text{const.}} \frac{\varphi(q^{3m}) \varphi(q^2)^2}{\varphi(q^3) \varphi(q^2)}.
$$

Now, from \ref{eq:7.6} and \ref{eq:7.7} it is obvious that the series-product identity \ref{eq:7.1} holds for \ref{eq:7.2}. \qed

Remark 7.2 Notice that for $m = 1$ series-product identities \ref{eq:1.2} and \ref{eq:1.3} (i.e. \ref{eq:6.1} and \ref{eq:7.1}) are the same.

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