Some Singular Vector-valued Jack and Macdonald Polynomials

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Abstract
For each partition τ of N there are irreducible modules of the symmetric groups $S_N$ or the corresponding Hecke algebra $\mathcal{H}_N(t)$ whose bases consist of reverse standard Young tableaux of shape τ. There are associated spaces of nonsymmetric Jack and Macdonald polynomials taking values in these modules, respectively. The Jack polynomials are a special case of those constructed by Griffeth for the infinite family $G(n,p,N)$ of complex reflection groups. The Macdonald polynomials were constructed by Luque and the author. For both the group $S_N$ and the Hecke algebra $\mathcal{H}_N(t)$ there is a commutative set of Dunkl operators. The Jack and the Macdonald polynomials are parametrized by $\kappa$ and $(q,t)$ respectively. For certain values of the parameters (called singular values) there are polynomials annihilated by each Dunkl operator; these are called singular polynomials. This paper analyzes the singular polynomials whose leading term is $x_1^{\tau} \otimes S$, where $S$ is an arbitrary reverse standard Young tableau of shape τ. The singular values depend on properties of the edge of the Ferrers diagram of τ.

1 Introduction

For each partition τ of N there are irreducible modules of the symmetric groups $S_N$ and the corresponding Hecke algebra $\mathcal{H}_N(t)$, whose bases consist of reverse standard Young tableaux of shape τ. There are associated spaces of nonsymmetric Jack and Macdonald polynomials taking values in these modules, respectively. (In what follows the polynomials are always of the nonsymmetric type.) The Jack polynomials are a special case of those

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constructed by Griffeth [7] for the infinite family $G(n,p,N)$ of complex reflection groups. The Macdonald polynomials were constructed by Luque and the author [6]. The polynomials are the simultaneous eigenfunctions of the Cherednik operators, which form a commutative set. For both the group $S_N$ and the Hecke algebra $H_N(t)$ there is a commutative set of Dunkl operators, which lower the degree of a homogeneous polynomial by 1.

The Jack and the Macdonald polynomials are parametrized by $\kappa$ and $(q,t)$ respectively. For certain values of the parameters (called singular values) there are polynomials annihilated by each Dunkl operator; these are called singular polynomials. The structure of the singular polynomials for the trivial module corresponding to the partition $(N)$, that is the ordinary scalar polynomials, is more or less well understood by now. For the modules of dimension $\geq 2$ the singular polynomials are mostly a mystery. In [3] and [4] we constructed special singular polynomials which correspond to the minimum parameter values. To be specific denote the longest hook-length in the Ferrers diagram of $\tau$ by $h_\tau$ then any other singular value $\kappa$ satisfies $|\kappa| \geq \frac{1}{H_\tau}$ and if a pair $(q,t)$ such that $q^m t^n = 1$ provides a singular polynomial then $\left| \frac{m}{n} \right| \geq \frac{1}{H_\tau}$. The main topic of this paper is the determination of all the singular values for which the Jack or Macdonald polynomials with leading term $x^m_\tau \otimes S$ are singular, where $S$ is an arbitrary reverse standard Young tableau of shape $\tau$. The singular values depend on properties of the edge of the Ferrers diagram of $\tau$.

There is a brief outline of the needed aspects of the representation theory of $S_N$ and $H_N(t)$ in Section 2 focussing on the action of the generators on the basis elements. The important operators on scalar and vector-valued polynomials are defined in Section 3. Subsection 3.1 deals with the Cherednik-Dunkl and Dunkl operators on the vector-valued polynomials, introduces the Jack polynomials, and the key formulas for the action of Dunkl operators, in particular, when specialized to the polynomials with leading term $x^m_\tau \otimes S$. Subsection 3.2 contains the analogous results on Macdonald polynomials. Section 4 combines the previous results with analyses of the spectral vectors and a combinatorial analysis of the possible singular values, to prove our main results on Jack and Macdonald polynomials. Subsection 4.1 illustrates the representation-theoretic aspect of singular polynomials.

## 2 Representation Theory

The symmetric group $S_N$ is the group of permutations of $\{1,2,\ldots,N\}$. The transpositions $w = (i,j)$, defined by $w(i) = j, w(j) = i$ and $w(k) = k$
for \( k \neq i, j \) are fundamental tools in this study. The simple reflections \( s_i := (i, i + 1), 1 \leq i < N \), generate \( S_N \) and the group is abstractly presented by \( \{ s_i^2 = 1 : 1 \leq i < N \} \) and the braid relations:

\[
\begin{align*}
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}, 1 \leq i \leq N - 2, \\
    s_is_j &= s_js_i, 1 \leq i < j \leq N - 2.
\end{align*}
\]

The group algebra \( \mathbb{C}S_N \), namely the linear space \( \left\{ \sum_{w \in S_N} c_w w \right\} \), is of dimension \( N! \). The associated Hecke algebra \( H_N(t) \) where \( t \) is transcendental (formal parameter) or a complex number not a root of unity, is the associative algebra generated by \( \{ T_1, T_2, \ldots, T_{N-1} \} \) subject to the relations

\[
\begin{align*}
    (T_i + 1)(T_i - t) &= 0, \\
    T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, 1 \leq i \leq N - 2, \\
    T_i T_j &= T_j T_i, 1 \leq i < j \leq N - 2.
\end{align*}
\]

It can be shown that there is a linear isomorphism between \( \mathbb{C}S_N \) and \( H_N(t) \) based on the map \( s_i \to T_i \). When \( t = 1 \) they are identical.

The irreducible modules of these algebras correspond to partitions of \( N \) and are constructed in terms of Young tableaux. The descriptions will be given in terms of the actions of \( \{ s_i \} \) or \( \{ T_i \} \) on the basis elements (see [2]).

Let \( N_0 := \{ 0, 1, 2, 3, \ldots \} \) and denote the set of partitions \( N_0^{N,+} := \{ \lambda \in N_0^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \} \). Let \( \tau \) be a partition of \( N \) that is \( \tau \in N_0^{N,+} \) and \( |\tau| = N \). Thus \( \tau = (\tau_1, \tau_2, \ldots) \) (often the trailing zero entries are dropped when writing \( \tau \)). The length of \( \tau \) is \( \ell(\tau) := \max \{ i : \tau_i > 0 \} \).

There is a Ferrers diagram of shape \( \tau \) (given the same label), with boxes at points \((i, j)\) with \( 1 \leq i \leq \ell(\tau) \) and \( 1 \leq j \leq \tau_i \). A tableau of shape \( \tau \) is a filling of the boxes with numbers, and a reverse standard Young tableau (RSYT) is a filling with the numbers \( \{1, 2, \ldots, N\} \) so that the entries decrease in each row and each column. Denote the set of RSYT's of shape \( \tau \) by \( \mathcal{Y}(\tau) \) and let \( V_\tau = \text{span}_F \{ S : S \in \mathcal{Y}(\tau) \} \) with orthogonal basis \( \mathcal{Y}(\tau) \), (where \( F \) is some extension field of \( \mathbb{Q} \) containing the parameters \( \kappa \) or \( q, t \)). The dimension of \( V_\tau \), that is \( \#\mathcal{Y}(\tau) \), is given by the well-known hook-length formula. For \( 1 \leq i \leq N \) and \( S \in \mathcal{Y}(\tau) \) the entry \( i \) is at coordinates \((\text{row}(i, S), \text{col}(i, S))\) and the content of the entry is \( c(i, S) = \text{col}(i, S) - \text{row}(i, S) \). Each \( S \in \mathcal{Y}(\tau) \) is uniquely determined by its content vector \( [c(i, S)]_{i=1}^N \). For example let \( \tau = (4, 3) \) and \( S = \begin{bmatrix} 7 & 6 & 5 & 2 \\ 4 & 3 & 1 \end{bmatrix} \) then the content vector is \([1, 3, 0, -1, 2, 1, 0]\).

There are representations of \( S_N \) and \( H_N(t) \) on \( V_\tau \); each will be denoted by
\(\tau\). For each \(i\) and \(S\) (with \(1 \leq i < N\) and \(S \in \mathcal{Y}(\tau)\)) there are four different possibilities:

1) row \((i, S) = row (i + 1, S)\) (implying \(col (i, S) = col (i + 1, S) + 1\) and \(c(i, S) - c(i + 1, S) = 1\)) then

\[
S \tau (s_i) = S, \quad S \tau (T_i) = tS;
\]

2) \(\text{col}(i, S) = \text{col}(i + 1, S)\) (implying \(\text{row}(i, S) = \text{row}(i + 1, S) + 1\) and \(c(i, S) - c(i + 1, S) = -1\)) then

\[
S \tau (s_i) = -S, \quad S \tau (T_i) = -S;
\]

3) row \((i, S) \prec \text{row}(i + 1, S)\) and \(\text{col}(i, S) > \text{col}(i + 1, S)\). In this case

\[
c(i, S) - c(i + 1, S) = (\text{col}(i, S) - \text{col}(i + 1, S)) + (\text{row}(i + 1, S) - \text{row}(i, S)) \geq 2,
\]

and \(S^{(i)}\), denoting the tableau obtained from \(S\) by exchanging \(i\) and \(i + 1\), is an element of \(\mathcal{Y}(\tau)\) and

\[
S \tau (s_i) = S^{(i)} + \frac{1}{c(i, S) - c(i + 1, S)} S,
\]

\[
S \tau (T_i) = S^{(i)} + \frac{t - 1}{1 - \frac{t}{c(i + 1, S) - c(i, S)}} S;
\]

4) \(c(i, S) - c(i + 1, S) \leq -2\), thus row \((i, S) > \text{row}(i + 1, S)\) and \(\text{col}(i, S) < \text{col}(i + 1, S)\) then with \(b = c(i, S) - c(i + 1, S)\),

\[
S \tau (s_i) = \left(1 - \frac{1}{b^2}\right) S^{(i)} + \frac{1}{b} S,
\]

\[
S \tau (T_i) = \frac{t}{(b - 1)^2} \left(\frac{b + 1}{b^2} - 1\right) S^{(i)} + \frac{b}{b - 1} S.
\]

The formulas in (4) are consequences of those in (3) by interchanging \(S\) and \(S^{(i)}\) and applying the relations \(\tau (s_i)^2 = I\) and \((\tau (T_i) + I)(\tau (T_i) - tI) = 0\) (where \(I\) denotes the identity operator on \(V_\tau\)).

There is a commutative set of Jucys-Murphy elements in both \(\mathbb{Z} S_N\) and \(\mathcal{H}_N (t)\) and which are diagonalized with respect to the basis \(\mathcal{Y}(\tau)\) (with \(1 \leq i \leq N\) and \(S \in \mathcal{Y}(\tau)\))

\[
\omega_i := \sum_{j=i+1}^{N} (i, j), \quad S \tau (\omega_i) = c(i, S) S,
\]

\[
\phi_N = 1, \quad \phi_i = \frac{1}{t} T_i \phi_{i+1} T_i, \quad S \tau (\phi_i) = t^{c(i, S)} S.
\]

(1)
The representation \( \tau \) of \( \mathcal{S}_N \) is unitary (orthogonal) when \( V_\tau \) is furnished with the inner product

\[
\langle S, S' \rangle_0 := \delta_{S, S''} \times \prod_{1 \leq i < j \leq N, \ c(j, S) - c(i, S) \geq 2} \left( 1 - \frac{1}{(c(i, S) - c(j, S))^2} \right), \ S, S' \in \mathcal{Y}(\tau).
\]

The analogue for \( \mathcal{H}_N(t) \) is

\[
\langle S, S' \rangle_0 := \delta_{S, S''} \times \prod_{1 \leq i < j \leq N, \ c(j, S) - c(i, S) \geq 2} u \left( t^{c(i, S) - c(j, S)} \right), \ S, S' \in \mathcal{Y}(\tau),
\]

where

\[
u(z) := \frac{(t-z)(1-tz)}{(1-z)^2}.
\]

This form satisfies \( \langle f \tau(T_i), g \rangle_0 = \langle f, g \tau(T_i) \rangle_0 \) for \( f, g \in V_\tau \) and \( 1 \leq i < N \).

### 3 Representations and Operators on Polynomials

For \( N \geq 2, x = (x_1, \ldots, x_N) \in \mathbb{R}^N \). The cardinality of a set \( E \) is denoted by \( \#E \). For \( \alpha \in \mathbb{N}_0^N \) (a composition) let \( |\alpha| := \sum_{i=1}^N \alpha_i, \ x^\alpha := \prod_{i=1}^N x_i^{\alpha_i} \), a monomial of degree \( |\alpha| \). The spaces of polynomials, respectively homogeneous, polynomials are

\[
\mathcal{P} := \text{span}_\mathbb{F} \{ x^\alpha : \alpha \in \mathbb{N}_0^N \}, \ 
\mathcal{P}_n := \text{span}_\mathbb{F} \{ x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n \}, \ n \in \mathbb{N}_0.
\]

For \( \alpha \in \mathbb{N}_0^N \) let \( \alpha^+ \) denote the nonincreasing rearrangement of \( \alpha \). We use partial orders on \( \mathbb{N}_0^N \) : for \( \alpha, \beta \in \mathbb{N}_0^N \), \( \alpha \succ \beta \) (\( \alpha \) dominates \( \beta \)) means that \( \alpha \neq \beta \) and \( \sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i \) for \( 1 \leq j \leq N \); and \( \alpha \succ \beta \) means that \( |\alpha| = |\beta| \) and either \( \alpha^+ \succ \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \succ \beta \). Also there is the rank function:

\[
r_\alpha(i) := \# \{ j : 1 \leq j \leq i, \alpha_j \geq \alpha_i \} + \# \{ j : i < j \leq N, \alpha_j > \alpha_i \}, 1 \leq i \leq N;
\]

then \( r_\alpha \in \mathcal{S}_N \) and \( r_\alpha(i) = i \) for all \( i \) if and only if \( \alpha = \alpha^+ \).

The action of the symmetric group on polynomials is defined by

\[
x s_i = \left( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N \right)
\]

\[
p(x) s_i = p(xs_i), 1 \leq i < N.
\]
For arbitrary transpositions $x(i, j) = (\ldots, x_{j}, \ldots, x_{i}, \ldots)$ and $p(x)(i, j) = p(x(i, j))$. There is a subtlety (implicit inverse) involved due to acting on the right: for example $p(x)s_{1}s_{2} = p(x)s_{1}s_{2} = p((xs_{2})s_{1})$, that is, $p(x_{1}, x_{2}, x_{3})s_{1}s_{2} = p(x_{2}, x_{1}, x_{3})s_{2} = p(x_{3}, x_{1}, x_{2})$. In general $p(x)w = p(xw^{-1})$ where $(xw)_{i} = x_{w^{-1}(i)}$ for all $i$.

The action of the Hecke algebra on polynomials is defined by

$$p(x)T_{i} = (1 - t)_{x_{i+1}}x_{i}x_{i+1} + tp(x_{i})p(x).$$

The defining relations can be verified straightforwardly. There are special values: $x_{i}T_{i} = x_{i+1}, (x_{i} + x_{i+1})T_{i} = t(x_{i} + x_{i+1})$ and $(tx_{i} - x_{i+1})T_{i} = -(tx_{i} - x_{i+1})$. Also $pT_{i} = tp$ if and only if $ps_{i} = p$, because $tp - pT_{i} = tx_{i} - x_{i+1}(p - ps_{i})$.

For a partition $\tau$ of $N$ let $\mathcal{P}_{\tau} := \mathcal{P} \otimes V_{\tau}$. The set $\{x^{\alpha} \otimes S : \alpha \in \mathbb{N}_{0}^{N}, S \in \mathcal{Y}(\tau)\}$ is a basis of $\mathcal{P}_{\tau}$. The representations of $\mathcal{S}_{N}$ and $\mathcal{H}_{N}(t)$ on $\mathcal{P}_{\tau}$ are respectively defined by the linear extension from the action on generators by

$$s_{i} : p(x) \otimes S \rightarrow p(xs_{i}) \otimes S \tau(s_{i}),$$
$$T_{i} : p(x) \otimes S \rightarrow (1 - t)_{x_{i+1}}x_{i}x_{i+1} + p(x_{i}) \otimes S \tau(T_{i}),$$

for $p \in \mathcal{P}, S \in \mathcal{Y}(\tau)$ and $1 \leq i < N$. (For details and background for the vector-valued Macdonald polynomials see [6].)

### 3.1 Jack polynomials

The Dunkl $\{\mathcal{D}_{i}\}$ and Cherednik-Dunkl $\{\mathcal{U}_{i}\}$ operators on $\mathcal{P}_{\tau}$ for $p \in \mathcal{P}, S \in \mathcal{Y}(\tau)$ and $1 \leq i \leq N$, are defined by

$$(p(x) \otimes S)\mathcal{D}_{i} = \frac{\partial}{\partial x_{i}}p(x) \otimes S + \kappa \sum_{j \neq i, j = 1}^{N} \frac{p(x - p(x, j))}{x_{i} - x_{j}} \otimes S \tau((i, j)),$$

$$(p(x) \otimes S)\mathcal{U}_{i} = (x_{i}p(x) \otimes S) \mathcal{D}_{i} - \kappa \sum_{j < i} p(x_{i}, j) \otimes S \tau((i, j)).$$

Each of the sets $\{\mathcal{D}_{i}\}$ and $\{\mathcal{U}_{i}\}$ consists of pairwise commuting elements. There is a basis of $\mathcal{P}_{\tau}$ consisting of homogeneous polynomials each of which is a simultaneous eigenfunction of $\{\mathcal{U}_{i}\}$; these are the nonsymmetric Jack polynomials. For each $(\alpha, S) \in \mathbb{N}_{0}^{N} \times \mathcal{Y}(\tau)$ there is the polynomial

$$J_{\alpha, S} = x^{\alpha} \otimes S \tau(r_{\alpha}) + \sum_{\beta < \alpha} x^{\beta} \otimes v_{\alpha, \beta, S}(\kappa),$$

(4)
where \( v_{\alpha,\beta,S}(\kappa) \in V_\tau \); these coefficients are rational functions of \( \kappa \). These polynomials satisfy

\[
J_{\alpha,S}U_i = \zeta_{\alpha,S}(i) J_{\alpha,S},
\]

\[
\zeta_{\alpha,S}(i) := \alpha_i + 1 + \kappa c(\alpha(i), S), \quad 1 \leq i \leq N.
\]

The spectral vector is \([\zeta_{\alpha,S}(i)]_{i=1}^N\). For detailed proofs see [5].

We are concerned with the special case \( \alpha = (m, 0, \ldots, 0) \in \mathbb{N}_0^N \). We apply formulas from [3] to analyze \( J_{\alpha,S}D_i \).

**Proposition 1** ([3, Cor. 6.2]) Suppose \( (\beta,S) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) and \( \beta_j = 0 \) for \( j \geq k \) with some fixed \( k > 1 \) then \( J_{\beta,S}D_j = 0 \) for all \( j \geq k \).

The next result uses the inner product on Jack polynomials for partition labels \( \beta \). The Pochhammer symbol is \( (a)_n = \prod_{i=1}^n (a + i - 1) \).

**Proposition 2** Suppose \( \beta \in \mathbb{N}_0^{N,+} \) and \( S \in \mathcal{Y}(\tau) \) then

\[
\|J_{\beta,S}\|^2 = \langle S, S \rangle_0 \prod_{i=1}^N (1 + \kappa c(i, S))_{\beta_i} \times \prod_{1 \leq i < j \leq N} \prod_{\ell=1}^{\beta_i-\beta_j} \left(1 - \left(\frac{\kappa}{\ell + \kappa (c(i,S) - c(j,S))}\right)^2\right).
\]

**Corollary 3** Suppose \( \alpha = (m, 0, \ldots, 0) \) then

\[
\|J_{\alpha,S}\|^2 = \langle S, S \rangle_0 (1 + \kappa c(1, S)) \prod_{j=2}^N \prod_{t=1}^m \left(1 - \left(\frac{\kappa}{\ell + \kappa (c(1,S) - c(j,S))}\right)^2\right).
\]

These norm formulas are results of Griffeth [7] specialized to the symmetric groups. The final ingredient for the formula is a special case of [3, Thm. 6.3].

**Proposition 4** Suppose \( \alpha = (m, 0, \ldots, 0) \) and \( \hat{\alpha} = (m - 1, 0, \ldots, 0) \) then

\[
J_{\alpha,S}D_1 = \frac{\|J_{\alpha,S}\|^2}{\|J_{\hat{\alpha},S}\|^2} J_{\hat{\alpha},S} = (m + \kappa c(1, S)) \prod_{j=2}^N \left(1 - \left(\frac{\kappa}{m + \kappa (c(1,S) - c(j,S))}\right)^2\right) J_{\hat{\alpha},S}. \tag{5}
\]
Proof. The first line comes from \cite{3} Thm. 6.3. Then the norm ratios are computed, which involves much cancellation.

Denote the prefactor of $J\hat{\alpha}_S$ in equation (5) by $C_{S,m}(\kappa)$. Our interest is in the zeros of $C_{S,m}(\kappa)$ as a function of $\kappa$. We will see that $C_{S,m}(\kappa)$ depends only on $\tau$ and the location of 1 in $S$. The idea is to group entries of $S$ by row and use telescoping properties. There is a simple formula (proven inductively)

$$\prod_{i=a}^{b} \frac{g(i+1)g(i-1)}{g(i)^2} = \frac{g(a-1)g(b+1)}{g(a)g(b)}$$

where $g$ is a function on $\mathbb{Z}$ and $a \leq b$. For the present application set $g(i) = m + \kappa(c(1,S) - i)$.

Definition 5 The partition $\hat{\tau} \in \mathbb{N}_0^{N,+}$ is obtained from $\tau$ by removing the box $(\text{row } (1,S), \text{col } (1,S))$: for $1 \leq i \leq \ell(\tau)$ set $\hat{\tau}_i = \tau_i - 1$ if $\text{row } (1,S) = i$ otherwise set $\hat{\tau}_i = \tau_i$.

The part of the product in $C_{S,m}(\kappa)$ coming from row $\#i$ has $c(j, S)$ ranging from $1 - i$ to $\hat{\tau}_i - i$ so the corresponding subproduct is

$$\prod_{j=1-i}^{\hat{\tau}_i-i} \frac{g(j+1)g(j-1)}{g(j)^2} = \frac{g(-i)g(\hat{\tau}_i-i+1)}{g(1-i)g(\hat{\tau}_i-i)}.$$

Multiply these factors for $i = 1, 2, \ldots \ell(\tau)$; note that

$$\prod_{i=1}^{\ell(\tau)} \frac{g(-i)}{g(1-i)} = \frac{g(-\ell(\tau))}{g(0)} = \frac{m + \kappa(c(1,S) + \ell(\tau))}{m + \kappa(c(1,S))},$$

and thus

$$C_{S,m}(\kappa) = (m + \kappa(c(1,S) + \ell(\tau))) \prod_{i=1}^{\ell(\tau)} \frac{m + \kappa(c(1,S) - \hat{\tau}_i + i - 1)}{m + \kappa(c(1,S) - \hat{\tau}_i + i)}.$$ (6)

As stated before the formula depends only on $\tau$ and the location of 1 in $S$. More simplification is possible due to telescoping if some $\hat{\tau}_i$’s are equal.

Definition 6 For $\hat{\tau}$ as in Definition 5 define the increasing sequence $\mathcal{I}(\hat{\tau}) = [i_1, i_2, \ldots, i_k]$ such that $i_1 = 1$, and $2 \leq s \leq k$ implies $\hat{\tau}_{i_s} < \hat{\tau}_{i_{s-1}}$ and $\hat{\tau}_j = \hat{\tau}_{i_{s-1}}$ for $i_{s-1} \leq j < i_s$. The last element $i_k = \ell(\tau) + 1$. Let $\mathcal{Z}(\hat{\tau}) = \{\hat{\tau}_{i_s} + 1 - i_s : 1 \leq s \leq k - 1\} \cup \{-\ell(\tau) : \hat{\tau}_{i_{\ell(\tau)}} \geq 1\}$ (the latter set is omitted when $\hat{\tau}_{i_{\ell(\tau)}} = 0$).
Example 7 Suppose \( \hat{\tau} = [5, 4, 4, 4, 3, 3, 2, 1] \) then \( \mathcal{I}(\hat{\tau}) = [1, 3, 6, 8, 9, 10] \), and \( \mathcal{Z}(\hat{\tau}) = \{5, 2, -2, -5, -7, -9\} \). If \( \hat{\tau} = [5, 4, 4, 4, 3, 3, 3, 0] \) then \( \mathcal{I}(\hat{\tau}) = [1, 3, 6, 9] \) and \( \mathcal{Z}(\hat{\tau}) = \{5, 2, -2, -8\} \).

Let \( \hat{S} \) denote the tableau formed by deleting the box \((\text{row } (1, S), \text{col } (1, S))\) from \( S \). The key property of \( \mathcal{I}(\hat{\tau}) \) is that it controls the possible locations where a box containing 1 could be adjoined to \( \hat{S} \) to form a RSYT. These locations are \( \{(1, \hat{\tau}_1 + 1), \ldots, (i_s, \hat{\tau}_i + 1), \ldots\} \). If \( \hat{\tau}_{\ell(\tau)} = 0 \) then the last location is \((\ell(\tau), 1)\) otherwise it is \((\ell(\tau) + 1, 1)\). Thus \( \mathcal{Z}(\hat{\tau}) \) is the set of contents of locations in the list. Evaluate the part of the product in formula (6) for the range \( i_s \leq j < i_{s+1} \) to obtain

\[
\prod_{j=i_s}^{i_{s+1}-1} \frac{m + \kappa(c(1, S) - \hat{\tau}_{i_s} + j - 1)}{m + \kappa(c(1, S) - \hat{\tau}_{i_s} + j)} = \frac{m + \kappa(c(1, S) - (\hat{\tau}_{i_s} + 1 - i_s))}{m + \kappa(c(1, S) - (\hat{\tau}_{i_s} + 1 - i_{s+1}))}.
\]

This completes the proof of the following:

Proposition 8 For \( \hat{\tau} \) and \( \mathcal{I}(\hat{\tau}) \) as in Definitions 3 and 6

\[
C_{S,m}(\kappa) = (m + \kappa(c(1, S) + \ell(\tau))) \prod_{s=1}^{k-1} \frac{m + \kappa(c(1, S) - (\hat{\tau}_{i_s} + 1 - i_s))}{m + \kappa(c(1, S) - (\hat{\tau}_{i_s} + 1 - i_{s+1}))},
\]

where \( i_k = \ell(\tau) + 1 \).

If \( \hat{\tau}_{\ell(\tau)} = 0 \) then the entry at \((\ell(\tau), 1)\) is 1, \( c(1, S) = 1 - \ell(\tau), i_{k-1} = \ell(\tau) \) and the last factor in the product (for \( s = k-1 \)) equals \( \frac{m}{m + \kappa} \); thus cancelling out the leading factor \( m + \kappa(c(1, S) + \ell(\tau)) = m + \kappa \).

Lemma 9 Suppose \( 1 \leq a, b \leq k-1 \) then \( \hat{\tau}_a - i_a \neq \hat{\tau}_b - i_{b+1} \).

Proof. By construction the sequence \( \{\hat{\tau}_a\}_{a \geq 1} \) is strictly decreasing and the sequence \( \{i_a\}_{a \geq 1} \) is strictly increasing. Suppose for some \( a, b \) the equation \( \hat{\tau}_a - i_a = \hat{\tau}_b - i_{b+1} \) holds, that is, \( i_a - i_{b+1} = \hat{\tau}_a - \hat{\tau}_b \). Clearly \( a = b \) or \( a = b + 1 \) are impossible. Suppose \( i_a - i_{b+1} > 0 \) then \( b < b+1 < a \) implying \( \hat{\tau}_a - \hat{\tau}_b < 0 \), a contradiction. Similarly suppose \( i_a - i_{b+1} < 0 \) then \( a < b+1 \), furthermore that \( a < b \) since \( a = b \) is impossible, thus \( \hat{\tau}_a - \hat{\tau}_b > 0 \), again a contradiction. This completes the proof.

Proposition 10 The set of zeros of \( C_{m,S}(\kappa) \) is

\[
\left\{-\frac{m}{c(1, S) - z}: z \in \mathcal{Z}(\hat{\tau}), z \neq c(1, S) \right\}.
\]
Proof. None of the numerator factors in the product are cancelled out due to Lemma 9. The only possible cancellation occurs for \( \hat{\tau}_\ell(\tau) = 0 \) when \( c(1, S) \) is the last entry in the list \( \mathcal{Z}(\hat{\tau}) \).

Example 11 Let \( N = 7, \tau = [5, 5, 5, 4, 4, 2, 2] \) and \( \text{row}(1, S), \text{col}(1, S) = (3, 5) \), then \( \hat{\tau} = [5, 5, 4, 4, 2, 2] \). The possible locations where the box containing 1 could be adjoined to \( \hat{\tau} \) are \( \{(1, 6), (3, 5), (6, 3), (8, 1)\} \) so that \( \mathcal{Z}(\hat{\tau}) = \{5, 2, -3, -7\} \) and

\[
C_{S,m}(\kappa) = \frac{m(m-3\kappa)(m+5\kappa)(m+9\kappa)}{(m-\kappa)(m+3\kappa)(m+7\kappa)}.
\]

Here is a sketch of \( \hat{\tau} \) marked by \( \Box \) and the possible cells for the entry 1

\[
\begin{array}{cccccc}
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\Box & \Box & \Box & \Box & \Box & 1 \\
\end{array}
\]

In a later section we examine the relation to singular polynomials of the form \( J_{\alpha, S} \).

3.2 Macdonald polynomials

Adjoin the parameter \( q \). To say that \((q, t)\) is generic means that \( q \neq 1, q^a t^b \neq 1 \) for \( a, b \in \mathbb{Z} \) and \(-N \leq b \leq N\). Besides the operators \( T_i \) defined in (3) we introduce (for \( p \in \mathcal{P}, S \in \mathcal{Y}(\tau) \))

\[
\omega := T_1 T_2 \cdots T_{N-1},
\]

\[
(p(x) \otimes S) w := p(qx_N, x_1, \ldots, x_{N-1}) \otimes \tau(\omega).
\]

The Cherednik \( \{\xi_i\} \) and Dunkl \( \{D_i\} \) operators, for \( 1 \leq i \leq N \), are defined by

\[
\xi_i := t^{i-N} T_{i-1}^{-1} \cdots T_1^{-1} w T_{N-1} \cdots T_i,
\]

\[
D_N := (1 - \xi_N)/x_N, \quad D_i := \frac{1}{t} T_i D_{i+1} T_i.
\]
These definitions were given for the scalar case by Baker and Forrester \[6\] and extended to vector-valued polynomials by Luque and the author \[1\]. The operators \{\xi_i : 1 \leq i \leq N\} commute pairwise, while the operators \{D_i : 1 \leq i \leq N\} commute pairwise and map \(P_n \otimes V_r\) to \(P_{n-1} \otimes V_r\) for \(n \geq 0\).

A polynomial \(p \in P_r\) is singular for some particular value of \((q, t)\) if \(pD_i = 0\), evaluated at \((q, t)\), for all \(i\). There is a basis of \(P_r\) consisting of homogeneous polynomials each of which is a simultaneous eigenfunction of \{\xi_i\}; these are the nonsymmetric Macdonald polynomials. For each \((\alpha, S) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau)\) there is the polynomial

\[M_{\alpha, S} = q^aq^b x^\alpha \otimes S \tau(R_{\alpha}) + \sum_{\beta < \alpha} x^\beta \otimes v_{\alpha, \beta, S}(q, t),\]

where \(v_{\alpha, \beta, S}(q, t) \in V_r\) for \(v_r := (T_i, T_{i_2} \cdots T_{i_m})^{-1}\) where \(\alpha.s_1 s_2 \cdots s_m = \alpha^+\) and there is no shorter product \(s_{j_1} s_{j_2} \cdots\) having this property (that is \(m = \# \{ (i, j) : i < j, \alpha_i < \alpha_j \}\), \(a, b \in \mathbb{Z}\) (see \[4\] p. 19) for the values of \(a, b,\) which are not needed here), and

\[\tilde{\zeta}_{\alpha, S}(i) = q^{\alpha_i} e^{(r_i(i), S)}, \quad 1 \leq i \leq N.\]

As before \([\tilde{\zeta}_{\alpha, S}(i)]_1 = \tilde{\zeta}_{\alpha, S}(i) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau)\) and \(\beta_j = 0\) for \(j \geq k\) with some fixed \(k > 1\) then \(M_{\alpha, S}D_j = 0\) for all \(j \geq k\).

Adapting the proof of \[3\] Lemma 5] we show (recall \[2\] u \((z) = \frac{(t-z)(1-tz)}{(1-z)^r}\)):

**Proposition 12** (\[4\] Prop. 12) Suppose \((\beta, S) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau)\) and \(\beta_j = 0\) for \(j \geq k\) with some fixed \(k > 1\) then \(M_{\beta, S}D_j = 0\) for all \(j \geq k\).

Proposition 13 Let \(\alpha = (m, 0, \ldots, 0), \alpha' = (0, 0, \ldots, m)\) and \(S \in \mathcal{Y}(\tau)\) then

\[M_{\alpha, S}D_1 = t^{1-N} \prod_{j=1}^{N-1} u \left(q^m t e^{(1, S)} - c(j+1, S)\right) M_{\alpha', S}D_N T_{N-1} \cdots T_1.\]

The other ingredient is the affine step (from the Yang-Baxter graph, see \[6, 4\] (3.14)): for \(\beta \in \mathbb{N}_0^N\) set \(\beta \Phi := (\beta_2, \beta_3, \ldots, \beta_N, \beta_1 + 1)\) then \(M_{\beta \Phi, S} = x_N (M_{\beta, S} w).\) The spectral vector of \(\beta \Phi\) is \([\tilde{\zeta}_{\beta, S}(2), \ldots, \tilde{\zeta}_{\beta, S}(N), q \tilde{\zeta}_{\beta, S}(1)\].\) Observe \(\tilde{\alpha} \Phi = \alpha'\) for \(\tilde{\alpha} = (m - 1, 0, \ldots)\). By definition

\[M_{\alpha', S}D_N = \frac{1}{x_N} \{M_{\alpha', S}(1 - \xi_N)\} = \frac{1}{x_N} \left(1 - \tilde{\zeta}_{\alpha', S}(N)\right) M_{\tilde{\alpha}, S} = \left(1 - \tilde{\zeta}_{\alpha', S}(N)\right) M_{\tilde{\alpha}, S} w = \left(1 - q \tilde{\zeta}_{\tilde{\alpha}, S}(1)\right) M_{\tilde{\alpha}, S} w.\]
Furthermore \((1 - q^{c(1,S)})\) and \(wT_{N-1} \cdots T_1 = t^{N-1} \xi_1\) so that \(M_{\alpha,S} \xi_1 = q^{m-1}t^{c(1,S)}M_{\alpha,S}\).

**Proposition 14** Let \(\alpha = (m,0,\ldots), \hat{\alpha} = (m-1,0,\ldots,0)\) and \(S \in \mathcal{Y} (\tau)\) then

\[
M_{\alpha,S} D_1 = q^{m-1}t^{c(1,S)} \left(1 - q^{m}t^{c(1,S)}\right) \prod_{j=2}^{N} u \left(q^{m}t^{c(1,S)-c(j,S)}\right) M_{\alpha,S}. \tag{7}
\]

This is very similar to the Jack case \((5)\) and the same telescoping argument will be used. Denote the factor of \(M_{\alpha,S}\) in \((7)\) by \(C_{S,m} (q,t)\). Set \(g(i) = 1 - q^{m}t^{c(1,S)} - i\) for \(i \in \mathbb{Z}\) then

\[
u \left(q^{m}t^{c(1,S)-c(j,S)}\right) = t \frac{g(c(j,S) - 1)g(c(j,S) + 1)}{g(c(j,S))^2}.
\]

With the same notation for \(\tilde{T}\) as in Definition \((5)\)

\[
C_{S,m} (q,t) = q^{m-1}t^{c(1,S)+N-1} \left(1 - q^{m}t^{c(1,S)}\right) \prod_{i=1}^{\ell(\tau)} \frac{g(-i)}{g(1-i)} \frac{g(\tau_i - i + 1)}{g(\tilde{\tau}_i - i)}
= q^{m-1}t^{c(1,S)+N-1} \left(1 - q^{m}t^{c(1,S)}\right) \frac{g(-\ell(\tau))}{g(0)} \prod_{i=1}^{\ell(\tau)} \frac{g(\tau_i - i + 1)}{g(\tilde{\tau}_i - i)}
= q^{m-1}t^{c(1,S)+N-1} \left(1 - q^{m}t^{c(1,S)+\ell(\tau)}\right) \prod_{i=1}^{\ell(\tau)} \frac{g(\tau_i - i + 1)}{g(\tilde{\tau}_i - i)}.
\]

The same computational scheme as in Proposition \((8)\) proves the following:

**Proposition 15** For \(\tilde{T}\) and \(\mathcal{I} (\tau)\) as in Definitions \((5)\) and \((6)\)

\[
C_{S,m} (q,t) = q^{m-1}t^{c(1,S)+N-1} \left(1 - q^{m}t^{c(1,S)+\ell(\tau)}\right) \prod_{s=1}^{k-1} \frac{1 - q^{m}t^{c(1,S)-\ell(\tau)_s+1-\ell(\tau)_s}}{1 - q^{m}t^{c(1,S)-\ell(\tau)_s+1-\ell(\tau)_{s+1}}},
\]

where \(i_k = \ell(\tau) + 1\).

If \(\tilde{T}_{\ell(\tau)} = 0\) then the entry at \((\ell(\tau), 1)\) is 1, \(c(1,S) = 1 - \ell(\tau)\), \(i_{k-1} = \ell(\tau)\) and the last factor in the product (for \(s = k - 1\)) equals \(\frac{1 - q^{m}}{1 - q^{m}t}\) thus cancelling out the leading factor \(1 - q^{m}t^{c(1,S)+\ell(\tau)} = 1 - q^{m}t\).
Proposition 16 The set of zeros of $C_{m,S}(q,t)$ is
\[ \{q^m t^{c(1,S)} - z = 1 : z \in \mathcal{Z}(\bar{\tau}), z \neq c(1,S)\} \].

Proof. None of the numerator factors in the product are cancelled out due to Lemma 9. The only possible cancellation occurs for $\bar{\tau}_\ell(\tau) = 0$ when $c(1,S)$ is the last entry in the list $\mathcal{Z}(\bar{\tau})$. ■

Example 17 Let $N = 7, \tau = [5, 5, 4, 4, 3, 2]$ and $(\text{row}(1,S), \text{col}(1,S)) = (6,3)$, then $\bar{\tau} = [5, 5, 4, 4, 2, 2]$. This is the same $\bar{\tau}$ as in Example 11 and $\mathcal{Z}(\bar{\tau}) = \{5, 2, -3, -7\}$. The same diagram applies here. Then
\[
C_{S,m}(q,t) = q^{m-1} t^4 (1 - q^m) \frac{(1 - q^m t^{-8})(1 - q^m t^{-5})(1 - q^m t^4)}{(1 - q^m t^{-6})(1 - q^m t^{-2})(1 - q^m t^2)}.
\]

In the next section we will see under what conditions $M_{\alpha,S}$ is singular.

4 Singular Polynomials

For $\alpha = (m, 0, \ldots) \in \mathbb{N}_0^{N,+}, \bar{\alpha} = (m-1, 0, \ldots)$ and $S \in \mathcal{Y}(\tau)$ we have shown
\[
J_{\alpha,S} D_i = 0, M_{\alpha,S} D_i = 0, \ 2 \leq i \leq N;
J_{\alpha,S} D_1 = C_{S,m}(\kappa) J_{\bar{\alpha},S},
M_{\alpha,S} D_1 = C_{S,m}(q,t) M_{\bar{\alpha},S},
\]
and we determined the zeros of $C_{S,m}(\kappa)$ and $C_{S,m}(q,t)$. But not all zeros lead to singular polynomials because, in general the coefficients of $J_{\beta,S}$ (with respect to the monomial basis $\{x^\gamma \otimes S'\}$) have denominators of the form $a + b\kappa$ and the coefficients of $M_{\beta,S}$ have denominators of the form $1 - q^a t^b$ where $a, b \in \mathbb{Z}$ and $|b| \leq N$. Thus to be able to substitute $\kappa = \kappa_0$, a zero of $C_{S,m}(\kappa)$, or $(q,t) = (q_0, t_0)$, a zero of $C_{S,m}(q,t)$, in equations (5) and (7) to conclude that $J_{\alpha,S}$ or $M_{\alpha,S}$ are singular it is necessary to show that neither $J_{\alpha,S}$ or $J_{\bar{\alpha},S}$ have a pole at $\kappa = \kappa_0$; the analogous requirement applies to $M_{\alpha,S}$ and $M_{\bar{\alpha},S}$. From the triangularity of $J_{\beta,S}$ and $M_{\beta,S}$ with respect to the monomial basis we can deduce that
\[
x^\lambda \otimes S = J_{\lambda,S} + \sum_{\gamma < \lambda, S' \in \mathcal{Y}(\tau)} b_{\lambda,\gamma,S,S'}(\kappa) J_{\gamma,S'},
\]
\[
x^\lambda \otimes S = c M_{\lambda,S} + \sum_{\gamma < \lambda, S' \in \mathcal{Y}(\tau)} b_{\lambda,\gamma,S,S'}(q,t) M_{\gamma,S'},
\]
13
where \( \lambda \in \mathbb{N}_0^{N,+} \), the coefficients \( b_{\lambda;S,S'}(\kappa) \), \( b_{\lambda;S,S'}(q,t) \) are rational functions of \( \kappa \), \( q,t \) respectively and \( c = q^at^b \) for some integers \( a,b \). If one can show that for each \((\gamma,S')\) with \( \gamma < \lambda \) that the spectral vector is distinct from that of \((\lambda,S)\), that is, \[ [\zeta_{\gamma,S'}(i)]_{i=1}^N \neq [\zeta_{\lambda,S}(i)]_{i=1}^N \] when evaluated at the specific values of \( \kappa \) or \((q,t)\) (with \( \zeta \)) then \( J_{\lambda,S}, \) respectively \( M_{\lambda,S}, \) do not have a pole there. The following is a device for analyzing possibly coincident spectral vectors.

**Definition 18** Let \((\beta,S),(\gamma,S') \in \mathbb{N}_0^N \times \mathcal{F}(\tau)\) such that \( \beta \triangleright \gamma \), and let \( m,n \in \mathbb{Z}\) with \( m \geq 1, n \neq 0 \). Then \([(\beta,S),(\gamma,S')]\) is an \((m,n)\)-critical pair if there is \( v \in \mathbb{Z}^N \) such that \( \beta_i - \gamma_i = mv_i \) and \( c(r_{\beta}(i),S) - c(r_{\gamma}(i),S') = nv_i \) for \( 1 \leq i \leq N \).

**Lemma 19** Let \((\beta,S),(\gamma,S') \in \mathbb{N}_0^N \times \mathcal{F}(\tau)\) such that \( \beta \triangleright \gamma \) and \( \zeta_{\beta,S}(i) = \zeta_{\gamma,S'}(i) \) for all \( i \) when \( \kappa = -\frac{m}{n} \), with \( \gcd(m,n) = 1 \), then \([(\beta,S),(\gamma,S')]\) is an \((m,n)\)-critical pair.

**Proof.** By hypothesis \( (1 + \beta_i - \frac{m}{n}c(r_{\beta}(i),S)) = (1 + \gamma_i - \frac{m}{n}c(r_{\gamma}(i),S')) \) for \( 1 \leq i \leq N \); thus
\[
\beta_i - \gamma_i = \frac{m}{n} (r_{\beta}(i),S) - c(r_{\gamma}(i),S')), \quad n(\beta_i - \gamma_i) = m(r_{\beta}(i),S) - c(r_{\gamma}(i),S')).
\]
From \( \gcd(m,n) = 1 \) it follows that \( \beta_i - \gamma_i = mv_i \) for some \( v_i \in \mathbb{Z} \) and thus \( r_{\beta}(i),S) - c(r_{\gamma}(i),S')) = nv_i \). \( \blacksquare \)

Now we specialize to \( \alpha = (m,0,\ldots) \) as in Subsection 1.1.1 and \( n \) satisfying \( C_{S,m}(-\frac{m}{n}) = 0 \). By Proposition 1.10 this is equivalent to \( n = c(1,S) - z \) with \( z \in \mathbb{Z}((\tau)) \).

**Proposition 20** There are no \((m,n)\)-critical pairs \([(\alpha,S),(\gamma,S')]\).

**Proof.** Suppose that \( \gamma \leq \alpha \) and \( \alpha_i - \gamma_i = mv_i, c(i,S) - c(r_{\gamma}(i),S') = nv_i \) with \( v_i \in \mathbb{Z} \), and \( 1 \leq i \leq N \). From \( |\gamma| = |\alpha| = m \) and \( \alpha_j = m \) or \( = 0 \) it follows that \( \gamma_k = m \) for some \( k \) and \( \gamma_i = 0 \) for \( i \neq k \). If \( k = 1 \) then \( z_i = 0 \) for all \( i \) and \( c(i,S) = c(r_{\gamma}(i),S') = c(i,S') \), because \( \gamma \in \mathbb{N}_0^{N,+} \). The content vector determines \( S' \) uniquely and thus \( S' = S \) and \( \gamma = \alpha \). Now suppose \( k > 1 \) then \( v_1 = 1, v_k = -1 \) and \( v_i = 0 \) otherwise. The respective content vectors are
\[
[c(i,S)]_{i=1}^N = [c(1,S),c(2,S),\ldots,c(k,S),c(k+1,S),\ldots,c(N,S)], \quad [c(r_{\gamma}(i),S')]_{i=1}^N = [c(2,S'),c(3,S')\ldots,c(1,S'),c(k+1,S'),\ldots,c(N,S')] .
\]
The hypothesis on \( \gamma \) implies \( c(i, S') = c(i-1, S) \) for \( i = 3 \leq i \leq k \), \( c(i, S') = c(i, S) \) for \( k + 1 \leq i \leq N \), and \( c(2, S') = c(1, S) - n \), \( c(1, S') = c(k, S) + n \). Since \( S \) and \( S' \) are both of shape \( \tau \) the two content vectors are permutations of each other. The list of values \([c(3, S'), \ldots , c(N, S')]\) agrees with \( [c(2, S), \ldots , c(k-1, S), c(k+1, S), \ldots , c(N, S)]\). Thus \( [c(1, S), c(k, S)] \) and \( [c(1, S'), c(2, S')] \) contain the same two numbers. Since \( c(2, S') = c(1, S) - n \neq c(1, S) \) the equation \( c(1, S) = c(1, S') \) must hold. The possible locations of the entry 1 in a RSYT must have different contents (else they would be on the same diagonal \( \{(i, j) : j - i = c(1, S)\}\)). Thus \( (\text{row}(1, S'), \text{col}(1, S')) = (\text{row}(1, S), \text{col}(1, S)) \) and \( S \) and \( S' \) lead to the same \( \tilde{\tau} \) (the partition formed by removing the cell of 1 from \( \tau \)). By construction \( n = z \) for some \( z \in \mathbb{Z}(\tilde{\tau}) \), and \( z \) determines a cell \( (i_s, \tilde{\tau_i} + 1) \) where 1 can be attached to the part of \( S' \) containing \( \{2, 3, \ldots , N\} \) to form a new RSYT \( S'' \). By construction \( c(1, S'') = z = c(1, S) - n = c(2, S') = c(2, S'') \). It is impossible for \( c(1, S'') = c(2, S'') \) for any RSYT, thus \( \gamma \neq \alpha \) cannot occur.

The same problem for \( \widehat{\alpha} = (m - 1, 0, \ldots) \) is almost trivial.

**Lemma 21** Suppose \( S' \in \mathcal{Y}(\tau) \), \( |\gamma| = m - 1 \) and \( \widehat{\alpha}_i - \gamma_i = m v_i \), \( c(i, S) - c(r_\gamma(i), S') = n v_i \) with \( v_i \in \mathbb{Z} \), and \( 1 \leq i \leq N \). Then \( (\widehat{\alpha}, S) = (\gamma, S') \).

**Proof.** The hypothesis \( |\gamma| = m - 1 \) implies \( \gamma_i \leq m - 1 \) and thus \( |\widehat{\alpha}_i - \gamma_i| \leq m - 1 \) for all \( i \). This implies \( v_i = 0 \) for all \( i \) implying \( \gamma = \widehat{\alpha} \) and \( c(j, S) = c(j, S') \) for all \( j \), thus \( S = S' \).

**Proposition 22** Suppose \( (\beta, S) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \), \( \gcd(m, n) = 1 \) and there are no \((m, n)\)-critical pairs \( \{(\beta, S), (\gamma, S')\} \) then \( J_{\beta, S} \) has no poles at \( \kappa = -\frac{m}{n} \).

**Proof.** By the triangularity of formula \( \text{(4)} \) there is an expansion

\[
x^\beta \otimes S \tau (r_\beta) = J_{\beta, S} + \sum_{\gamma < \beta, S' \in \mathcal{Y}(\tau)} b_{\beta, \gamma, S, S'}(\kappa) J_{\gamma, S'}.
\]

By Lemma [19] for each \( \gamma < \beta, S' \in \mathcal{Y}(\tau) \) there is at least one \( i = i[\gamma, S'] \) such that \( \zeta_{\beta, S}(i) - \zeta_{\gamma, S'}(i) \neq 0 \) when \( \kappa = -\frac{m}{n} \). Define an operator

\[
\mathcal{T} := \prod_{\gamma < \beta, S' \in \mathcal{Y}(\tau)} \frac{U_{i[\gamma, S']}(\zeta_{\beta, S}(i[\gamma, S']) - \zeta_{\gamma, S'}(i[\gamma, S']))}{\zeta_{\beta, S}(i[\gamma, S']) - \zeta_{\gamma, S'}(i[\gamma, S'])}.
\]

Then \( J_{\beta, S} \mathcal{T} = J_{\beta, S} \) and each \( J_{\gamma, S'} \) (with \( \gamma < \beta \)) is annihilated by at least one factor of \( \mathcal{T} \). Thus \( J_{\beta, S} = (x^\beta \otimes S \tau (r_\beta)) \mathcal{T} \), a polynomial whose coefficients
have denominators which are factors of
\[ \prod_{\gamma \neq \beta, S' \in \mathcal{Y}(\tau)} (\zeta_{\beta,S}(i [\gamma, S']) - \zeta_{\gamma,S'}(i [\gamma, S'])). \]
By construction of \{i [\gamma, S']\} this product does not vanish at \( \kappa = -\frac{m}{n} \). □

We are ready for the main result on Jack polynomials.

**Theorem 23** Suppose \( \alpha = (m, 0, \ldots), S \in \mathcal{Y}(\tau) \) and \( \mathcal{Z}(\tilde{\tau}) \) is as in Definition 6. Further suppose \( z \in \mathcal{Z}(\tilde{\tau}) \), \( n := c(1, S) - z \neq 0 \) and \( \gcd(m, n) = 1 \) then \( J_{\alpha,S} \) is a singular polynomial for \( \kappa = -\frac{m}{n} \).

**Proof.** From Proposition 1 \( J_{\alpha,S}D_j = 0 \) for \( 2 \leq j \leq N \) and \( J_{\alpha,S}D_1 = C_{S,m}(\kappa)J_{\tilde{\alpha},S} \), where \( \tilde{\alpha} = (m - 1, 0, \ldots) \). By Propositions 20 and Lemma 21 \( J_{\alpha,S} \) and \( J_{\tilde{\alpha},S} \) do not have poles at \( \kappa = -\frac{m}{n} \). Furthermore \( C_{S,m}(\frac{-m}{n}) = 0 \) and thus \( J_{\alpha,S}D_1 = 0 \) at \( \kappa = -\frac{m}{n} \). □

To set up the analogous results for Macdonald polynomials consider the differences between two spectral vectors: \( \tilde{\zeta}_{\beta,S}(i) - \tilde{\zeta}_{\gamma,S'}(i) = q^{\tilde{\alpha}}t^{r_{\beta}(i), S} - q^{\tilde{\gamma}}t^{r_{\gamma}(i), S'} = q^{\tilde{\alpha}}t^{c(r_{\beta}(i), S)} (q^{\tilde{\gamma}} - q^{\tilde{\gamma}}t^{c(r_{\gamma}(i), S') - c(r_{\gamma}(i), S) - 1}). \) To relate this to \((m, n)\)-critical pairs we specify a condition on \((q, t)\) which implies \( a = mv \) and \( b = nv \) for some \( v \in \mathbb{Z} \) when \( q^a t^b = 1 \).

**Definition 24** Suppose \( m, n \) are integers such \( m \geq 1, n \neq 0 \) and \( \gcd(m, n) = g \geq 1 \). Let \( u \in \mathbb{C} \setminus \{0\} \) such that \( u \) is not a root of unity and \( \omega = \exp\left(\frac{2\pi i k}{m}\right) \) with \( \gcd(k, g) = 1 \). Define \( \varpi = (q, t) = (\omega u^{-n/g}, u^{m/g}) \).

**Lemma 25** Suppose \( a, b \) are integers such that \( q^a t^b = 1 \) at \((q, t) = \varpi \) then \( a = mv, b = nv \) for some \( v \in \mathbb{Z} \).

**Proof.** By hypothesis
\[ 1 = \left(\omega u^{-n/g}\right)^a \left(u^{m/g}\right)^b = \omega^a u^{(-na+mb)/g}. \]
Since \( u \) is not a root of unity it follows that \(-a \left(\frac{n}{g}\right) + b \left(\frac{m}{g}\right) = 0 \) but \( \gcd\left(\frac{n}{g}, \frac{m}{g}\right) = 1 \) and thus \( \frac{m}{g} \) divides \( a \). Write \( a = \left(\frac{m}{g}\right)c \) for some integer \( c \) then \( 1 = \omega^a = \exp\left(\frac{2\pi i k c}{m}\right) = \exp\left(\frac{2\pi i k}{g}\right). \) This implies \( c = v g \) with \( v \in \mathbb{Z} \) because \( \exp\left(\frac{2\pi i k}{g}\right) \) is a primitive \( g^{th} \) root of unity. Thus \( a = \left(\frac{m}{g}\right)v g = mv \) and \( b = \frac{m}{g}a = nv \). □

**Remark 26** All the possible values of \( \varpi \) are included when (1) \( g > 1 \) and \( \omega = \exp\left(\frac{2\pi i k}{m}\right) \) with \( \gcd(k, g) = 1 \) and \( 1 \leq k < g \) (2) \( g = 1 \) and \( \omega = 1 \). To
prove this let \( u = \phi v \) with \( \phi = \exp \left( \frac{2\pi il}{m} \right) \) and \( l \in \mathbb{Z} \) so that \( u^{m/g} = v^{m/g} \). Then \( q = \omega \phi^{-n/g} v^{-n/g} = \exp \left( \frac{2\pi i}{m} \right) v^{-n/g} \). Since \( \gcd(m,n) = g \) there are integers \( s,s' \) such that \( s'm + sn = g \). Set \( l = s''s \) (with \( s'' \in \mathbb{Z} \)) then \( k - nl = k - s''g - s''s'm; \) thus \( \omega \phi^{-n/g} = \exp \left( \frac{2\pi i}{m} \right) \). If \( g > 1 \) then let \( s'' = \left\lfloor \frac{k}{g} \right\rfloor + 1 \) implying \( 1 \leq k - s''g < g \), while if \( g = 1 \) set \( s'' = k \).

**Example 27** Suppose \( m = 8 \) and \( n = -12 \), then \( g = 4 \) and the possible values of \( \varpi \) are \( \exp \left( \frac{\pi i}{2} \right) u^3, u^2 \) and \( \exp \left( \frac{3\pi i}{4} \right) u^3, u^2 \) where \( u \) is not a root of unity.

We will use this result to produce singular polynomials \( M_{\alpha,S} \) for \( (q,t) = \varpi \).

**Lemma 28** Let \( (\beta,S), (\gamma,S') \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) such that \( \beta \triangleright \gamma \) and \( \tilde{\zeta}_{\beta,S}(i) = \tilde{\zeta}_{\gamma,S'}(i) \) for all \( i \) when \( (q,t) = \varpi \) then \( [(\beta,S), (\gamma,S')] \) is an \( (m,n) \)-critical pair.

**Proof.** The equation \( \tilde{\zeta}_{\beta,S}(i) = \tilde{\zeta}_{\gamma,S'}(i) \) is \( q^{\beta_i}t^{c(r_{\gamma}(i),S)} = q^{\gamma_i}t^{c(r_{\gamma}(i),S')} \), that is, \( q^{\beta_i - \gamma_i}t^{c(r_{\gamma}(i),S') - c(r_{\gamma}(i),S)} = 1 \) at \( (q,t) = \varpi \). By Lemma 27 there is an integer \( \gamma \) such that \( \beta_i - \gamma_i = mv_i \) and \( c(r_{\gamma}(i),S) - c(r_{\gamma}(i),S') = nv_i \). This argument applies to all \( i \).

**Proposition 29** Suppose \( (\beta,S) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) and there are no \( (m,n) \)-critical pairs \( [(\beta,S), (\gamma,S')] \) then \( M_{\beta,S} \) has no poles at \( (q,t) = \varpi \).

**Proof.** The proof is essentially identical to that of Proposition 22 There replace \( x^\beta \otimes \mathcal{T}(r_{\alpha}) \) by \( q^{\beta_i}t^{c(r_{\gamma}(i),S') - c(r_{\gamma}(i),S)} \), \( J \) by \( M \), \( \zeta \) by \( \tilde{\zeta} \), \( U \) by \( \xi \). The formula shows that \( M_{\beta,S} \) is a polynomial, the denominators of whose coefficients are products of factors with the form \( q^{\beta_i}t^b - q^{\gamma_i}t^b \), and none of these vanish at \( (q,t) = \varpi \).

This is our main result for the Macdonald polynomials.

**Theorem 30** Suppose \( \alpha = (m,0,\ldots), S \in \mathcal{Y}(\tau) \) and \( \mathcal{Z}(\hat{\tau}) \) is as in Definition 4. Further suppose \( z \in \mathcal{Z}(\hat{\tau}), n := c(1,S) - z \neq 0 \) then \( M_{\alpha,S} \) is a singular polynomial for \( (q,t) = \varpi \).

**Proof.** From Proposition 12 \( M_{\alpha,S} \mathcal{D}_j = 0 \) for \( 2 \leq j \leq N \) and \( M_{\alpha,S} \mathcal{D}_1 = C_{S,m}(q,t) M_{\tilde{\alpha},S} \), where \( \tilde{\alpha} = (m-1,0,\ldots) \). By Propositions 20, 29 and Lemma 21 \( M_{\alpha,S} \) and \( M_{\tilde{\alpha},S} \) do not have poles at \( (q,t) = \varpi \). Furthermore \( C_{S,m} \left( \omega u^{-n/g}, u^{m/g} \right) = 0 \) (due to the factor \( 1 - q^{m}t^{c(1,S) - z} \), Proposition 16) and thus \( M_{\alpha,S} \mathcal{D}_1 = 0 \) at \( (q,t) = \varpi \). ■
4.1 Isotype of Singular Polynomials

The following discussion is in terms of Macdonald polynomials. It is straightforward to deduce the analogous results for Jack polynomials. Suppose $\sigma$ is a partition of $N$. A basis $\{p_S : S \in \mathcal{Y}(\sigma)\}$ of an $\mathcal{H}_N(t)$-invariant subspace of $\mathcal{P}_\tau$ is called a basis of isotype $\sigma$ if each $p_S$ transforms under the action of $T_i$ defined in Section 2 with $\sigma(T_i)$ replaced by $T_i$. For example if row $(i, S) = row (i + 1, S)$ then $p_S(xs_i) = p_S(x)$, equivalently $p_ST_i = tp_S$, or if col $(i, S) = col (i + 1, S)$ then $p_ST_i = -p_S$. There is a strong relation to singular polynomials.

Proposition 31: A polynomial $p \in \mathcal{P}_\tau$ is singular for a specific value of $(q,t) = \psi$ if and only if $p_{\xi_i} = p_{\phi_i}$ for $1 \leq i \leq N$, evaluated at $\psi$.

Proof. Recall the Jucys-Murphy elements $\{\phi_i\}$ from (1). By definition $pD_N = 0$ if and only if $p_{\xi_N} = p_{\phi_N}$. Proceeding by induction suppose that $pD_j = 0$ for $i < j \leq N$ if and only if $p_{\xi_j} = p_{\phi_j}$ for $i < j \leq N$. Suppose

$$0 = pD_i = \frac{1}{t}pD_{i+1}T_i \iff pT_iD_{i+1} = 0 \iff pT_i\xi_{i+1} = pT_i\phi_{i+1}$$

$$\iff pT_i\xi_{i+1}T_i = pT_i\phi_{i+1}T_i \iff tp\xi_i = tp\phi_i.$$ 

This completes the proof. □

With $M_{\alpha,S}$ and $n$ as in Theorem 30, the spectral vector $[\tilde{\zeta}_{\alpha,S}(i)]^N_{i=1} = [q^{n}t^{c(1,S)}, t^{c(2,S)}, \ldots, t^{c(N,S)}]$. Specialized to $(q,t) = \varpi$ the polynomial $M_{\alpha,S}$ is singular and $q^{n}t^{c(1,S)} = t^{-n+c(1,S)}$. Recall $n = z$ for some $z \in \mathcal{Z}(\hat{\tau})$, and $z$ determines a cell $(i_s, \hat{\tau}_{i_s} + 1)$. In terms of Ferrers diagrams let $\sigma = \hat{\tau} \cup (i_s, \hat{\tau}_{i_s} + 1)$, that is $\sigma_{i_s} = \tau_{i_s} + 1$. Let $S'$ denote the RSYT formed from the cells of $\tau$ containing the numbers $2, \ldots, N$ and the cell $(i_s, \hat{\tau}_{i_s} + 1)$ containing 1. Then $c(i, S') = c(i, S)$ for $2 \leq i \leq N$ and $c(1, S') = c(1, S) - n$. Thus the spectral vector of $M_{\alpha,S}$ evaluated at $(q,t) = \varpi$ is $[t^{c(i,S')}]^N_{i=1}$. This implies that $M_{\alpha,S}$ is (a basis element) of isotype $\sigma$. The other elements of the basis corresponding to $\mathcal{Y}(\sigma)$ are obtained from $M_{\alpha,S}$ by appropriate transformations using $\{T_i\}$.

5 Concluding Remarks

We have shown the existence of singular vector-valued Jack and Macdonald polynomials for the easiest possible values of the label $\alpha$, that is, $(m,0,\ldots,0)$. 

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The proofs required some differentiation formulas and combinatorial arguments involving Young tableaux. The singular values were found to have an elegant interpretation in terms of where another cell can be attached to an RSYT. It may occur that a larger set of parameter values, say gcd \((m, n) > 1\), or even \(\frac{m}{n} \notin \mathbb{Z}\), still leads to singular Jack polynomials but our proof techniques do not seem to cover these. One hopes that eventually a larger class of examples (more general labels in \(\mathbb{N}^N_0\)) will be found, with a target of a complete listing as is already known for the trivial representation \(\tau = (N)\). It is suggestive that the isotype \(\sigma\) of the singular polynomial \(M_{\alpha,\sigma}\) is obtained by a reasonably natural transformation of the partition \(\tau\).

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