QUICKEST DETECTION OF A MINIMUM OF TWO POISSON DISORDER TIMES

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Abstract. A multi-source quickest detection problem is considered. Assume there are two independent Poisson processes $X^1$ and $X^2$ with disorder times $\theta_1$ and $\theta_2$, respectively; that is, the intensities of $X^1$ and $X^2$ change at random unobservable times $\theta_1$ and $\theta_2$, respectively. $\theta_1$ and $\theta_2$ are independent of each other and are exponentially distributed. Define $\theta = \theta_1 \wedge \theta_2 = \min\{\theta_1, \theta_2\}$. For any stopping time $\tau$ that is measurable with respect to the filtration generated by the observations define a penalty function of the form

$$R_\tau = P(\tau < \theta) + cE[(\tau - \theta)^+]$$

where $c > 0$ and $(\tau - \theta)^+$ is the positive part of $\tau - \theta$. It is of interest to find a stopping time $\tau$ that minimizes the above performance index. This performance random can be useful for example in the following scenario: There are two assembly lines producing $A$ and $B$, respectively. Assume that the malfunctioning (disorder) of the machines producing $A$ and $B$ are independent events. Later, the products $A$ and $B$ are to be put together to obtain another product $C$. A product manager who is worried about the quality of $C$ will want to detect the minimum of the disorder times (as accurately as possible) in the assembly lines producing $A$ and $B$. Another problem to which we can apply our framework is the internet surveillance problem: A router receives data from, say, $n$ channels. The channels are independent and the disorder times of channels are $\theta_1, \ldots, \theta_n$. The router is said to be under attack at $\theta = \theta_1 \wedge \cdots \wedge \theta_n$. The administrator of the router is interested in detecting $\theta$ as quickly as possible.

Since both observations $X^1$ and $X^2$ reveal information about the disorder time $\theta$, even this simple problem is more involved than solving the disorder problems for $X^1$ and $X^2$ separately. This problem is formulated in terms of a three dimensional sufficient statistic, and the corresponding optimal stopping problem is examined. The solution is characterized by iterating a suitable functional operator.

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1. Introduction. Consider two independent Poisson processes $X^i = \{X^i_t : t \geq 0\}$ $i \in \{1,2\}$ with the same arrival rate $\lambda$. At some random unobservable times $\theta_1$ and $\theta_2$, with distributions

$$P(\theta_i = 0) = \pi_i \quad P(\theta_i > t) = (1 - \pi_i) e^{-\lambda t} \text{ for } t \geq 0,$$

(1.1)

the arrival rates of the Poisson processes $X^1$ and $X^2$ change from $\lambda$ to $\alpha$, respectively, i.e.,

$$X^i_t = \int_0^t h^i_s \, ds, \quad t \geq 0, \ i = 1,2,$$

(1.2)

are martingales, in which

$$h^i(t) = [\beta 1_{\{s < \theta_i\}} + \alpha 1_{\{s \geq \theta_i\}}], \quad t \geq 0, \ i = 1,2.$$

(1.3)

Here $\alpha$ and $\beta$ are known positive constants. We seek a stopping rule $\tau$ that detects the instant $\theta = \theta_1 \wedge \theta_2$ of the first regime change as accurately as possible given the past and the present observations of the processes $X^1$ and $X^2$. More precisely, we wish to choose a stopping time $\tau$ of the history of the processes $X^1$ and $X^2$ that minimizes the following penalty function

$$R_\tau = P(\tau < \theta) + cE[(\tau - \theta)^+]$$

(1.4)

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The first term in (1.4) penalizes the frequency of false alarms, and the second term penalizes the detection delay. The disorder time demarcates two regimes, and in each of these regimes the decision maker uses distinctly different strategies. Therefore, it is in the decision maker’s interest to detect the disorder time as accurately as possible from its observations. Here, we are solving the case when a decision maker has two identical and independent sources to process. In the Section 5 we discuss how our analysis can be extended to non-identical sources.

Quickest detection problems arise in a variety of applications such as seismology, machine monitoring, finance, health, and surveillance, among others (see e.g. [1, 10, 7, 11] and [14]). Because Poisson processes are often used to model abrupt changes, Poisson disorder problems have potential applications e.g. to the effective control and prevention of infectious diseases, quickest detection of quality and reliability problems in industrial processes, and surveillance of Internet traffic to protect network servers from the attacks of malicious users. This is because the number of patients infected, number of defected items produced and number of packets arriving at a network node are usually modeled by Poisson processes. In these examples the disorder time corresponds to the time when an outbreak occurs, when a machine in an assembly line breaks down or when a router is under attack, respectively. The multi-source quickest detection problem considered here can be applied to tackle these problems when there are multiple sources of information. For example in the monitoring of industrial processes the minimum of disorder times represents the first time when one of many assembly lines in a plant breaks down during the production of a certain type of item. Let us be more specific: Assume that there are two assembly lines that produce products A and B, respectively. Assume also that the malfunctioning (disorder) of the machines producing A and B are independent events. Later, the products A and B are to be put together to obtain another product C. A product manager who is worried about the quality of C will want to detect the minimum of the disorder times (as accurately as possible) in the assembly lines producing A and B. The performance function (1.4) is an appropriate choice because the product manager will worry about the quality of the end product C, not of the individual pieces separately. Another problem to which we can apply our framework is the internet surveillance problem: A router receives data from, say, n channels. The channels are independent and the disorder times of channels are θ₁, · · · , θₙ. The router is said to be under attack at θ = θ₁ ∧ · · · ∧ θₙ. The administrator of the router is interested in detecting θ as quickly as possible.

The one dimensional Poisson disorder problem, i.e., the problem of detecting θ₁ as accurately as possible given the observations from the Poisson process X¹ has recently been solved (see [2], [3] and the references therein). The two-dimensional disorder problem we have introduced cannot be reduced to solving the corresponding one-dimensional disorder problems since both X¹ and X² reveal some information about θ whenever these processes jump. That is, if we take the minimum of the optimal stopping times that solve the one dimensional Poisson disorder problems, then we obtain a stopping time that is a sub-optimal solution to (1.4) (see Remark 4.1).

We will show that the quickest detection problem of (1.4) can be reduced to an optimal stopping problem for a three-dimensional piece-wise deterministic Markov process. Continuous-time Markov optimal stopping problems are typically solved by formulating them as free boundary problems associated with the infinitesimal generator of the Markov process. In this case, however the infinitesimal generator contains differential delay operators. Solving free boundary problems involving differential delay operators is a challenge even in the one dimensional case and the smooth fit principle is expected to fail (see [2], [3] and the references therein). Instead as in [1] and [6] we work with an integral operator, iteration of which generates a monotonically increasing sequence of functions converging exponentially to the value function of the optimal stopping problem. That is, using the integral operator we reduce the problem to a sequence of deterministic optimization problems. This approach provides a new numerical method for calculating and characterizing the value function and the continuation region in addition to providing information about the shape and the location of the optimal continuation region. Using the structure of the paths of the piece-wise deterministic Markov process we also provide a non-trivial bound on the optimal stopping time which can be used to obtain approximate stopping strategies.

The remainder of this paper is organized as follows. In Sections 2 and 3 we restate the problem of interest under a suitable reference measure P₀ that is equivalent to P. Working under the reference
measure \( P_0 \) reduces the computations considerably, since under this measure the observations \( X^1 \) and \( X^2 \) are simple Poisson processes that are independent of the disorder times. Here we show that the quickest detection problem reduces to solving an optimal stopping problem for a three-dimensional statistic. In Section 2 we analyze the path behavior of this sufficient statistic. In Section 3 we provide a tight upper bound on the continuation region of the optimal stopping problem, which can be used to determine approximate detection rules besides helping us to determine the location and the shape of the continuation region. Here, we also show that the smallest optimal stopping time of the problem under consideration has finite expectation. In Section 6 we convert the optimal stopping problem into sequences of deterministic optimal stopping problems using a suitably defined integral operator. In Section 7 we construct optimal stopping problems using a suitably defined integral operator. In Section 8 we discuss the structure of the optimal stopping regions. And finally, we discuss how to extend our approach to the case with more than two sources, and to the case when the jump sizes are random and the jump size distribution changes at the time of disorder.

2. Problem Description. Let us start with a probability space \((\Omega, \mathcal{F}, P_0)\) that hosts two independent Poisson processes \( X^1 \) and \( X^2 \), both of which have rate \( \beta \), as well as two independent random variables \( \theta_1 \) and \( \theta_2 \) independent of the Poisson processes with distributions

\[
P_0(\theta_i = 0) = \pi_i \quad \text{and} \quad P_0(\theta_i > t) = (1 - \pi_i)e^{-\lambda_i t}, \tag{2.1}
\]

for \( 0 \leq t < \infty, \, i \in \{1, 2\} \) and for some known constants \( \pi_i \in [0, 1) \) and \( \lambda > 0 \) for \( i \in \{1, 2\} \). We denote by \( \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t < \infty} \) the filtration generated by \( X^1 \) and \( X^2 \), i.e., \( \mathcal{F}_t = \sigma(X^1_s, X^2_s, 0 \leq s \leq t) \), and denote by \( \mathbb{G} = \{\mathcal{G}_t\}_{0 \leq t < \infty} \) the initial enlargement of \( \mathbb{F} \) by \( \theta_1 \) and \( \theta_2 \), i.e., \( \mathcal{G}_t = \sigma(\theta_1, \theta_2, X^1_s, X^2_s : 0 \leq s \leq t) \). The processes \( X^1 \) and \( X^2 \) satisfy (1.2) under a new probability measure \( P \), which is characterized by

\[
\frac{dP}{dP_0} \bigg|_{\mathcal{G}_t} \triangleq Z^1_t \triangleq Z^1_t Z^2_t, \tag{2.2}
\]

where

\[
Z^i_t \triangleq \exp \left( \int_0^t \log \left( \frac{h_i(s)}{\beta} \right) dX^i_s - \int_0^t [h_i(s) - \beta] ds \right), \tag{2.3}
\]

for \( t > 0 \) and \( i \in \{1, 2\} \) are exponential martingales (see e.g. [5]). Under this new probability measure \( P \) of (2.2), \( \theta_1 \) and \( \theta_2 \) have the same distribution as they have under the measure \( P_0 \), i.e., their distribution is given by (1.1). This holds because \( \theta_1 \) and \( \theta_2 \) are \( \mathcal{G}_0 \)-measurable and \( dP/dP_0 \big|_{\mathcal{G}_0} = 1 \), i.e., \( P \) and \( P_0 \) coincide on \( \mathcal{G}_0 \). Under the new probability measure \( P \) the processes \( X^1 \) and \( X^2 \) have measurable intensities \( h_1 \) and \( h_2 \) respectively. That is to say that (1.2) holds. In other words, the probability space \((\Omega, \mathcal{F}, P)\) describes the model posited in (1.1) and (1.2). Now, our problem is to find a quickest detection rule for the disorder times \( \theta_1 \wedge \theta_2 \), which is adapted to the history \( \mathcal{F} \) generated by the observed processes \( X^1 \) and \( X^2 \) because the complete information (concerning \( \theta_1 \) and \( \theta_2 \)) embodied in \( \mathcal{G} \) is not available. We will achieve our goal by finding an \( \mathcal{F} \) stopping time that minimizes (1.4).

In terms of the exponential likelihood processes

\[
L^i_t \triangleq \left( \frac{\alpha}{\beta} \right)^{X^i_t} \exp(-\alpha t) \exp(-\beta t), \quad t \geq 0, \, i \in \{1, 2\}, \tag{2.4}
\]

we can write

\[
Z^i_t = 1_{\{\theta_i > t\}} + 1_{\{\theta_i \leq t\}} \frac{L^i_t}{E_0[L^i_t]}, \tag{2.5}
\]

Let us introduce the posterior probability process

\[
\Pi_t \triangleq P(\theta \leq t | \mathcal{F}_t) = \frac{P_0 \left[ Z^1_t 1_{\{\theta \leq t\}} | \mathcal{F}_t \right]}{E_0 \left[ Z^1_t | \mathcal{F}_t \right]}, \tag{2.6}
\]
where the second equality follows from the Bayes formula (see e.g. [9]). Then it follows that from (2.5) and (2.6) that

\[ 1 - \Pi_t = \frac{(1 - \pi)e^{-2\lambda t}}{E_0[Z_i | F_t]}, \quad \text{where} \]

\[ \pi \triangleq 1 - (1 - \pi_1)(1 - \pi_2). \]  

(2.7)

Let us now introduce the odds-ratio process

\[ \Phi_t \triangleq \frac{\Pi_t}{1 - \Pi_t}, \quad 0 \leq t < \infty. \]  

(2.9)

Then observe from (2.6) and (2.7) that

\[ \mathbb{E}_0[Z_i 1_{\{\theta \leq t\}} | F_t] = (1 - \pi)e^{-\lambda t} \Phi_t, \quad t \geq 0. \]  

(2.10)

Now, we will write the penalty function of (1.4) in terms of the odds-ratio process.

\[
\mathbb{E}[(\tau - \theta)^+] = \mathbb{E}\left[\int_0^\infty I_{\{\tau > t\}}1_{\{\theta \leq t\}}dt\right] = \mathbb{E}_0\left[I_{\{\tau > t\}}\mathbb{E}_0\left[Z_i 1_{\{\theta \leq t\}} | F_t\right]\right]dt
\]

\[
= (1 - \pi)\mathbb{E}_0\int_0^\tau e^{-2\lambda t} \Phi_t dt.
\]  

(2.11)

Since \( \{\tau < \theta\} \in G_\theta \) we can write

\[
\mathbb{P}(\tau < \theta) = \mathbb{E}_0\left[Z_i 1_{\{\tau < \theta\}}\right] = \mathbb{P}_0(\tau < \theta) = (1 - \pi) \left(1 - \lambda \mathbb{E}_0\left[\int_0^\tau e^{-2\lambda t} dt\right]\right),
\]  

(2.12)

where the second equality follows since \( Z_\theta = 1 \) almost surely under \( P_0 \). Using (2.11) and (2.12) we can write the penalty function as

\[
R_\tau(\pi_1, \pi_2) = 1 - \pi + c(1 - \pi)\mathbb{E}_0\left[\int_0^\tau e^{-2\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right) dt\right].
\]  

(2.13)

On the other hand the following lemma obtains a representation for the odds ratio process \( \Phi \).

**Lemma 2.1.** Let us denote

\[
\Phi^i_t \triangleq \frac{e^{\lambda t}}{1 - \pi_i} \mathbb{E}_0\left[1_{\{\theta_i \leq t\}}\frac{L_i}{L_{\theta_i}} | F_t\right],
\]  

(2.14)

for \( t \geq 0 \) and \( i \in \{1, 2\} \). Then we can write the odds-ratio process \( \Phi \) as

\[
\Phi_t = \Phi^1_t + \Phi^2_t + \Phi^1_t \Phi^2_t, \quad t \geq 0
\]  

(2.15)

**Proof.** From (2.10)

\[
\Phi_t = \frac{e^{2\lambda t}}{(1 - \pi)} \mathbb{E}_0\left[Z_i 1_{\{\theta \leq t\}} | F_t\right],
\]

\[
= \frac{e^{2\lambda t}}{(1 - \pi)} \left\{ \mathbb{P}_0(\theta_1 > t)\mathbb{E}_0\left[1_{\{\theta_1 \leq t\}}\frac{L_1}{L_{\theta_1}} | F_t\right] + \mathbb{P}(\theta_2 > t)\mathbb{E}_0\left[1_{\{\theta_2 \leq t\}}\frac{L_2}{L_{\theta_2}} | F_t\right] \right\} + \mathbb{E}_0\left[1_{\{\theta_1 \leq t\}}\frac{L_1}{L_{\theta_1}} | F_t\right] \mathbb{E}_0\left[1_{\{\theta_2 \leq t\}}\frac{L_2}{L_{\theta_2}} | F_t\right],
\]  

(2.16)
The second equality follows from (2.18), (2.19) and the independence of the sigma algebras $\mathcal{F}_{t_i}^1$ and $\mathcal{F}_{t_i}^2$. Now the claim follows from (2.1), (2.8) and (2.14).

Using the fact that the likelihood ratio process $L^i$ is the unique solution of the equation
\[ dL^i_t = [(\alpha/\beta) - 1]L^i_{t-}(dX^i - \alpha \, dt), \quad L^i_0 = 1, \] (2.17)
(see e.g. [13]) and by means of the chain-rule we obtain
\[ d\Phi^i_t = (\lambda + (\lambda - \alpha + \beta)\Phi^i_t)dt + [(\alpha/\beta) - 1]\Phi^i_t dX^i_1, \quad \Phi^i_0 = \frac{\pi_i}{1 - \pi_i}, \] (2.18)
for $t \geq 0$ and $i \in \{1, 2\}$ (see [13]). If we let
\[ \Phi^+_i \triangleq \Phi^i_1 + \Phi^i_2, \quad \Phi^-_i \triangleq \Phi^i_1, \quad t \geq 0, \] (2.19)
then using a change of variable formula for jump processes gives
\[ d\Phi^-_t = [\lambda\Phi^-_t + 2a\Phi^-_t]dt + (\alpha/\beta - 1)\Phi^-_t d(X^1_1 + X^2_1), \] (2.20)
\[ d\Phi^+_t = [2\lambda + a\Phi^+_t]dt + ((\alpha/\beta - 1)\Phi^+_1 dX^1_1 + \Phi^+_2 dX^2_1] \]
with $\Phi^-_t = \pi_1 \pi_2 / (1 - \pi_1)(1 - \pi_2)$, and $\Phi^+_0 = \pi_1 / (1 - \pi_1) + \pi_2 / (1 - \pi_2)$, where $a \triangleq \lambda - \alpha + \beta$. Note that $X_i \triangleq X^1_i + X^2_i$, $t \geq 0$, is a Poisson process with rate $2\beta$ under $\mathbb{P}$.

It is clear from (2.18) and (2.20) that
\[ \Upsilon \triangleq (\Phi^-, \Phi^+, \Phi^1), \] (2.21)
is a piece-wise deterministic Markov process; therefore the original change detection problem with penalty function [1.4] has been reformulated as (2.13) and (2.19)-(2.21), which is an optimal stopping problem for a two dimensional Markov process driven by three dimensional piecewise-deterministic Markov process.

We will denote by $\mathcal{A}$ the infinitesimal generator of $\Upsilon$. Its action on a smooth test function $f : \mathbb{B}_+^3 \to \mathbb{R}$ is given by
\[ [\mathcal{A}f](\phi^x, \phi^+, \phi^1) = D_{\phi^x}f(\phi^x, \phi^+, \phi^1)[\lambda\phi^+ + 2a\phi^+] + D_{\phi^+}f(\phi^x, \phi^+, \phi^1)[2\lambda + a\phi^+] \]
\[ + D_{\phi^1}f(\phi^x, \phi^+, \phi^1)[\lambda + a\phi^1] + \beta \left[ f \left( \frac{\alpha}{\beta} \phi^x, \phi^+ + \left( \frac{\alpha}{\beta} - 1 \right) \phi^1, \phi^1 \right) - f(\phi^x, \phi^+, \phi^1) \right] \]
\[ + \beta \left[ f \left( \frac{\alpha}{\beta} \phi^x, \frac{\alpha}{\beta} \phi^+ - \left( \frac{\alpha}{\beta} - 1 \right) \phi^1, \phi^1 \right) - f(\phi^x, \phi^+, \phi^1) \right]. \] (2.22)

Let us denote
\[ \mathbb{B}_+^2 \triangleq \{(x, y) \in \mathbb{R}_+^2 : y \geq 2\sqrt{x}\} \quad \text{and} \quad (2.23) \]
\[ \mathbb{B}_+^3 \triangleq \{(x, y, z) \in \mathbb{R}_+^3 : y \geq 2\sqrt{x}, y \geq z\}. \] (2.24)

Now, for every $(\phi^x, \phi^+, \phi^1) \in \mathbb{B}_+^3$, let us denote by $x(t, \phi^x)$, $y(t, \phi^+)$ and $z(t, \phi^1)$, $t \in \mathbb{R}$, the solutions of
\[ \frac{d}{dt}x(t, \phi^x) = [\lambda y(t, \phi^+) + 2ax(t, \phi^x)]dt, \quad x(0, \phi^x) = \phi^x, \] (2.25)
\[ \frac{d}{dt}y(t, \phi^+) = [2\lambda + a y(t, \phi^+)]dt, \quad y(0, \phi^+) = \phi^+, \]
\[ \frac{d}{dt}z(t, \phi^1) = [\lambda + az(t, \phi^1)]dt, \quad z(0, \phi^1) = \phi^1. \]
The solutions of (2.25), when \( a \neq 0 \), are explicitly given by

\[
x(t, \phi^x) = \frac{\lambda^2}{a^2} + e^{2at} \left[ \phi^x - \frac{\lambda^2}{a^2} \right] + e^{2at} \left( 1 - e^{-at} \right) \frac{\lambda}{a} \left( \phi^+ + \frac{2\lambda}{a} \right),
\]

\[
y(t, \phi^+) = -\frac{2\lambda}{a} + e^{at} \left( \phi^+ + \frac{2\lambda}{a} \right),
\]

\[
z(t, \phi^1) = -\frac{\lambda}{a} + e^{at} \left( \phi^+ + \frac{\lambda}{a} \right).
\]

(2.26)

Otherwise, \( x(t, \phi^x) = \phi^x + \lambda t \phi^x + \lambda^2 t^2 \), \( y(t, \phi^+) = \phi^+ + 2\lambda t \), and \( z(t, \phi^1) = \phi^1 + \lambda t \). Note that the solution \((x, y, z)\) of the system of equations in (2.25) satisfy the semi-group property, i.e., for every \( s, t \in \mathbb{R} \)

\[
x(t + s, \phi_0) = x(s, x(t, \phi_0)), \quad y(t + s, \phi_1) = y(s, y(t, \phi_1)), \quad \text{and} \quad z(t + s, \phi_1) = z(s, z(t, \phi_1)).
\]

Note from (2.18), (2.20) and (2.26) that

\[
\Phi^x_t = x(t - \sigma_n, \Phi^x_{\sigma_n}), \quad \Phi^+_t = y(t - \sigma_n, \Phi^+_{\sigma_n}), \quad \Phi^1_t = z(t - \sigma_n, \Phi^1_{\sigma_n}) \quad \sigma_n \leq t < \sigma_{n+1}, \quad n \in \mathbb{N},
\]

(2.28)

and

\[
\Phi^x_{\sigma_{n+1}} = \alpha \Phi^x_{\sigma_{n+1}} - \beta \Phi^x_{\sigma_{n+1}}, \quad \Phi^+_{\sigma_{n+1}} = \alpha \Phi^+_{\sigma_{n+1}} - \beta \Phi^+_{\sigma_{n+1}}, \quad \text{and},
\]

\[
\Phi^1_{\sigma_{n+1}} = \left[ \alpha \Phi^1_{\sigma_{n+1}} - \beta \Phi^1_{\sigma_{n+1}} \right] 1_{\{X^1_{\sigma_{n+1}} \neq X^1_{\sigma_{n+1}}\}} + \left[ \alpha \Phi^+_{\sigma_{n+1}} - \beta \Phi^+_{\sigma_{n+1}} \right] 1_{\{X^2_{\sigma_{n+1}} \neq X^2_{\sigma_{n+1}}\}}.
\]

(2.29)

Here, for any function \( h \), \( h(t-) \overset{\Delta}{=} \lim_{t \downarrow s} h(t) \). Note that an observer watching \( \Upsilon \) is able to tell whenever the processes \( X^1 \) and \( X^2 \) jump (see (2.18), (2.20)), i.e., the filtration generated by \( \Upsilon \) is the same as \( \mathcal{F} \).

3. An Optimal Stopping Problem. Let us denote the set of \( \mathcal{F} \) stopping times by \( \mathcal{S} \). The value function of the quickest detection problem

\[
U(\pi_1, \pi_2) \overset{\Delta}{=} \inf_{\tau \in \mathcal{S}} R_\tau(\pi_1, \pi_2)
\]

(3.1)

can be written as

\[
U(\pi_1, \pi_2) = (1 - \pi) \left[ 1 + c V \left( \frac{\pi_1 \pi_2}{1 - \pi}, \frac{\pi_1 + \pi_2 - 2\pi_1 \pi_2}{1 - \pi}, \frac{\pi_1}{1 - \pi} \right) \right],
\]

(3.2)

where \( V \) is the value function of the optimal stopping problem

\[
V(\phi^x, \phi^+, \phi^1) \overset{\Delta}{=} \inf_{\tau \in \mathcal{S}} \mathbb{E}_0(\phi^x, \phi^+, \phi^1) \left[ \int_0^\tau e^{-\lambda t} h(\Phi^x_t, \Phi^+_t) dt \right],
\]

(3.3)

in which \((\phi^x, \phi^+, \phi^1) \in \mathbb{D}^3_1\), and \( h(x, y) \overset{\Delta}{=} x + y - \lambda/c \). Here \( \mathbb{E}_0(\phi^x, \phi^+, \phi^1) \) is the expectation under \( \mathbb{P}_0 \) given that \( \Phi^x_0 = \phi^x, \Phi^+_0 = \phi^+, \Phi^1_0 = \phi^1 \).

It is clear from (2.20) that for both optimal stopping problems it is not optimal to stop before \((\Phi^x_t, \Phi^+_t)\), \( t \geq 0 \), leaves the \textit{advantageous region} defined by

\[
C_0 \overset{\Delta}{=} \{(\phi^x, \phi^+) \in \mathbb{B}^2_+ : \phi^x + \phi^+ \leq \lambda/c\}.
\]

(3.4)

Let us also denote

\[
C \overset{\Delta}{=} \{(\phi^x, \phi^+, \phi^1) \in \mathbb{B}^3_+ : \phi^x + \phi^+ \leq \lambda/c\}.
\]

(3.5)

Also note that the only reason not to stop at the time of the first exit from the region \( C_0 \) is the prospect of \((\Phi^x_t, \Phi^+_t)\), \( t \geq 0 \) returning to \( C_0 \) at a future time.

**Remark 3.1.** It is reasonable to question our choice of statistic, since it is clear that \((\Phi^1_t, \Phi^2_t)_{t \geq 0}\) contains all the information \( X \) has to offer. Our choice \((\Upsilon_t)_{t \geq 0}\), which is defined in (2.21), is motivated...
by the mere desire of having a concave value function $U$ and a convex optimal stopping region. The concavity is due to the linearity of the function $h$ (see Lemma 6.2 and its proof along with Lemmas 6.1 and 6.7, and Theorem 6.7).

If we had chosen to work with $(\Phi_1^+, \Phi_2^+)_{t \geq 0}$, then the relevant optimal stopping problem becomes

$$W(a, b) \equiv \inf_{\tau \in S} \mathbb{E}_0^{(a, b)} \left[ \int_0^T e^{-\lambda \tilde{h} (\Phi_1^+, \Phi_2^+)} dt \right], \quad (3.6)$$

in which

$$\tilde{h}(x, y) \equiv x + y + xy, \quad (x, y) \in \mathbb{R}_+^2. \quad (3.7)$$

Since $h(\cdot, \cdot)$ is non-linear the concavity of the value function, i.e. $W(\cdot, \cdot)$ is not concave. In fact, $W(x, y) = V(xy, x + y, x)$. The function $V$ is concave but $W$ is not. So there is a trade off between concavity and the dimension of the statistic to be used.

In what follows, for the sake of the simplicity of notation, when the meaning is clear, we will drop the superscripts of the expectation operators $\mathbb{E}_0^{(\phi^+, \phi^-, \phi^+)}$.

4. Sample Paths of $\Psi = (\Phi^X, \Phi^+)$. It is illustrative to look at the sample paths of the sufficient statistic $\Psi$, to understand the nature of the problem. Indeed, this way, for a certain parameter range, we will be able to identify the optimal stopping time without any further analysis.

From (2.26), we see that, if $a > 0$, then the paths of the processes $\Phi^X$ and $\Phi^+$ increase between the jumps, and otherwise the paths of the processes $\Phi^X$ and $\Phi^+$ mean-revert to the levels $2\lambda^2/a^2$ and $-2\lambda/a$ respectively. Also observe that $\Phi^X$, $\Phi^+$ increase (decrease) with a jump if $\alpha \geq \beta$ ($\beta > \alpha$). See (2.20).

Case I: $\alpha \geq \beta$, $a > 0$. The following theorem follows from the description of the behavior of the paths above.

**Theorem 4.1.** If $\alpha > \beta$ and $a > 0$, then the stopping rule

$$\tau_0 \equiv \inf \{ t \geq 0 : \Phi^X_t + \Phi^+_t \geq \lambda/c \} \quad (4.1)$$

is optimal for (3.3).

**Proof.** Under the hypothesis of the theorem and whenever a path of $(\Phi^X, \Phi^+)$ leaves $\mathbb{C}_0$ it never returns. \qed

In Section 5 we will identify another case (another range of parameters) in which the advantageous region $\mathbb{C}_0$ is the optimal continuation region and the stopping time $\tau_0$ is optimal (see Cases II-b-i-1 and II-b-ii-1).

**Remark 4.1.** Let $\kappa_i \equiv \inf \{ t \geq 0 : \Phi^i_t \geq \lambda/c \}$. If $\alpha \geq \beta$ and $a > 0$, then $\kappa_i$ is the optimal stopping time for the one dimensional disorder problem with disorder time $\theta_i$ (3.3). Let us define $\kappa \equiv \kappa_1 \land \kappa_2$. Since with this choice of parameters $\Phi_{\kappa}^X + \Phi_{\kappa}^+ > \lambda/c$, it follows that $\tau_0 < \kappa$ almost surely. Therefore, if we follow the rule dictated by the stopping time $\kappa$, then we pay an extra penalty for detection delay. This example illustrates that solving the two one dimensional quickest detection problems separately in order to minimize the penalty function of (1.4) is suboptimal.

In what follows, we will consider the remaining cases: $\alpha \geq \beta$ and $a < 0$; $\alpha < \beta$.

5. Construction of a Bound on the Continuation Region. In this section the purpose is to show that the continuation region of (3.3) is bounded. The construction of upper bounds is carried out in the next two theorems. These upper bounds are tight as the next theorem shows and might be used to construct useful approximations to the two optimal stopping times solving the problems defined in (3.3).

We will carry out the analysis for $a = \lambda - \alpha + \beta \neq 0$. A similar analysis for this case can be similarly performed. As a result of Theorem 5.3, we are also able to conclude that the (smallest) optimal stopping time has a finite expectation.
The first two theorems in this section assume that an optimal stopping time of \([3.3]\) exists and in particular the stopping time
\[
\tau^*(a, b, c) \triangleq \inf\{t \geq 0 : V(Y_t) = 0, \ Y_0 = (a, b, c)\},
\]
is optimal. In Section [4] we will verify that this assumption holds. We will denote by
\[
\Gamma \triangleq \{(a, b, c) \in \mathbb{R}^3_+ : v(a, b, c) = 0\}, \quad \mathbf{C} \triangleq \mathbb{B}^3_+ - \Gamma,
\]
the optimal stopping region and optimal continuation region of \([3.3]\) respectively.

**Theorem 5.1.** In this theorem the standing assumption is that \(\alpha \geq \beta\) and that \(a < 0\) (Case II).

(Case II-a) Let us further assume that \(\lambda/a^2 - 2/a \leq 1/c\) and denote
\[
D_0 \triangleq \{(x, y) \in \mathbb{B}_+^2 : x \left(1 - \frac{1}{2(\lambda - a)}\right) + y \left(\frac{3\lambda - 2a}{2(\lambda - a)(2\lambda - a)}\right) + k > 0\},
\]
in which
\[
k \triangleq \frac{\lambda}{2a^2} - \frac{1}{a} - \frac{1}{2c} + \frac{\lambda^2}{2a^2(\lambda - a)} + \frac{1}{2\lambda - a} + 2 \left(\frac{\lambda}{a} - \frac{\lambda^2}{a^2}\right).
\]
Let \((\phi_0, \phi_1) \in D_0 \cap (\mathbb{B}_+^2 - C_0)\). Then for any \(\phi_2 \leq \phi_1\), \((\phi_0, \phi_1, \phi_2)\) is in the stopping region of \([3.3]\).

(Case II-b) Assume that \(\lambda/a^2 - 2/a \geq 1/c\) (standing assumption in the rest of the theorem). Consider the four different possible ranges of parameters:

Case II-b-i: If \(\lambda + a \leq 0\)

- and if \(-a/c - 1 \leq 0\) (Case II-b-i-1), then \(C\) in \([3.3]\) is the optimal continuation region for \([3.3]\).
- Else if \(-a/c - 1 > 0\) (Case II-b-i-2), then a superset of the continuation region can be constructed as follows. Let us introduce the line segment
\[
C \triangleq \{(x, y) \in \mathbb{B}_+^2 : x + y = \lambda/c\}, \quad (5.5)
\]
and define
\[
C_1 \triangleq \{(x, y) \in \mathbb{B}_+^2 : x = x(t, x^*), y = y(t, 0) \text{ for } t \in [0, t^*]\}
\[
\bigcup \left\{(x, y) \in C : x < \frac{\lambda}{c} + \frac{2\lambda}{\lambda - a} \left(1 + \frac{a}{c}\right), y > -\frac{2\lambda}{\lambda - a} \left(1 + \frac{a}{c}\right)\right\}. \quad (5.6)
\]
Here, \(t^*\) is the solution of \(y(-t^*, -\frac{2\lambda}{\lambda - a} (1 + \frac{a}{c})) = 0\) and \(x^* = x(-t^*, \frac{\lambda}{c} + \frac{2\lambda}{\lambda - a} (1 + \frac{a}{c}))\). The curve \(C_1\) separates \(\mathbb{R}_+^2\) into two connected regions. Let us denote the region that lies above the curve \(C_1\) by
\[
D_1 \triangleq \{(x, y) \in \mathbb{B}_+^2 : \text{there exists a positive number } \tilde{y}(x) < y \text{ such that } (x, \tilde{y}(x)) \in C_1\}. \quad (5.7)
\]
Then \(((\mathbb{B}_+^2 - D_1) \times \mathbb{R}_+^2) \cap \mathbb{B}_+^3\) is an upper bound on the continuation region of \([3.3]\).

Case II-b-ii: If \(\lambda + a > 0\)

- and if \(-a/c - 1 < 0\) (Case II-b-ii-1), then \(C\) in \([3.3]\) is the optimal continuation region for \([3.3]\).
- Else if \(-a/c - 1 > 0\), then \(((\mathbb{B}_+^2 - D_1) \times \mathbb{R}_+^2) \cap \mathbb{B}_+^3\) is an upper bound on the continuation region of \([3.3]\).

Note that all the supersets of the continuation we constructed are bounded subsets of \(\mathbb{R}_+^3\).

**Proof.**

Note that
\[
\Phi_i^+ \geq y(t, \phi_1), \quad \Phi_i^- \geq x(t, \phi_0), \quad t \geq 0,
\]
(5.8)
almost surely if \( \Phi_0^+ = \phi_1 \) and \( \Phi_0^- = \phi_0 \). This is because \( \Phi^- \) and \( \Phi^+ \) increase with jumps.

From this observation we obtain the following inequality

\[
\inf_{\tau \in S} E_0 \left[ \int_0^\tau e^{-2\lambda s h(\Phi_s^-, \Phi_s^+)ds} \right] \geq \inf_{\tau \in S} E_0 \left[ \int_0^\tau e^{-2\lambda s h(x(s, \phi_0), y(s, \phi_1))ds} \right]
\]

\[
= \inf_{t \in [0, \infty]} \int_0^t e^{-2\lambda s h(x(s, \phi_0), y(s, \phi_1))ds}. \tag{5.10}
\]

Note that if for a given \((\phi_0, \phi_1)\) the expression in \((5.10)\) is equal to zero, then the infimum on the left hand side of \((5.9)\) is attained by setting \(\tau = 0\). In what follows we will find a subset of the stopping region of the optimal stopping problem using this argument.

Case II-a: \( \lambda/a^2 - 2/a \leq 1/c \). In this case the mean reversion level of the path \((x(\cdot, \phi_0), y(\cdot, \phi_1)), (\phi_0, \phi_1) \in \mathbb{B}_+^2\), namely \((\lambda^2/a^2, -2\lambda/a)\), is inside the region \(\mathbb{C}_0\) which is defined in \((3.3)\). In this case, for any \((\phi_0, \phi_1) \in \mathbb{B}_+^2 - \mathbb{C}_0\) the minimizer \(t_{\text{opt}}(\phi_0, \phi_1)\) of the expression in \((5.10)\) is either \(0\) or \(\infty\) by the following argument. For any \((\phi_0, \phi_1) \in \mathbb{B}_+^2 - \mathbb{C}_0\) the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) is in the advantageous region \(\mathbb{C}_0\) all the time except for possibly a finite duration. Therefore if

\[
\int_0^\infty e^{-2\lambda s h(x(s, \phi_0), y(s, \phi_1))ds} < 0, \tag{5.11}
\]

then in order to minimize \((5.10)\) it is never optimal to stop. On the other hand if \((5.11)\) is positive, then it is not worth taking the journey into the advantageous region and it is optimal to stop immediately in order to minimize \((5.10)\).

We shall find the pairs \((\phi_0, \phi_1)\) for which \(t_{\text{opt}} = 0\). Using \((2.26)\) we can write

\[
\int_0^\infty e^{-2\lambda s h(x(s, \phi_0), y(s, \phi_1))ds} = \phi_0 \left( \frac{1}{\alpha - \beta} \right) + \phi_1 \left( \frac{a}{(\alpha - \beta)^2} \right) + k, \tag{5.12}
\]

where \(k\) is given by \((5.4)\). Note that if \((\phi_0, \phi_1) \in \mathbb{D}_0 \cap (\mathbb{B}_+^2 - \mathbb{C}_0)\), then by \((5.9)\) and \((5.10)\) we can see that the infimum in \((5.10)\) is equal to 0. Therefore \([(\mathbb{B}_+^2 - \mathbb{D}_0) \cup \mathbb{C}_0] \times \mathbb{R}_+ \cap \mathbb{B}_+^3\) is a superset of the optimal continuation region of \((3.3)\).

Case II-b: \( \lambda/a^2 - 2/a \geq 1/c \). In this case the mean reversion level of \(t \rightarrow (x(t, \phi_0), y(t, \phi_1))\) is outside \(\mathbb{C}_0\). Therefore, the minimizer of \((5.10)\) is \(t_{\text{opt}}(\phi_0, \phi_1) \in \{0, t_c(\phi_0, \phi_1), \infty\}\) where \(t_c(\phi_0, \phi_1)\) is the exit time of the path \((x(t, \phi_0), y(t, \phi_1))\) from \(\mathbb{C}_0\). The derivative

\[
\frac{d}{dt} |x(t, \phi_0) + y(t, \phi_1)| = (\lambda + a) y(t, \phi_1) + 2ax(t, \phi_1) + 2\lambda \tag{5.13}
\]

vanishes if \((x(t, \phi_0), y(t, \phi_1))\) meets the line segment

\[
L = \{(x, y) \in \mathbb{B}_+^2 : (\lambda + a)y + 2ax + 2\lambda = 0\}. \tag{5.14}
\]

Note that the mean reversion level belongs to \(L\), i.e.,

\[
\left( \frac{\lambda^2}{a^2}, -\frac{2\lambda}{a} \right) \in L. \tag{5.15}
\]

Case II-b-i: \( \lambda + a < 0 \). (In addition to \(\alpha > \beta\), \(a < 0\) and \(\lambda/a^2 - 2/a \geq 1/c\).) In this case the line \(L\) is decreasing (as a function of \(x\)).

Case II-b-ii: \(-a/c - 2 < 0\). (In addition to \(\alpha > \beta\), \(a < 0\), \(\lambda/a^2 - 2/a \geq 1/c\) and \(\lambda + a < 0\).) In this case the line segment \(C\) in \((5.5)\) lies entirely below \(L\). Assume that a path \((x(t, \phi_0), y(t, \phi_1))\) originating at \((\phi_0, \phi_1) \in \mathbb{B}_+^2 - \mathbb{C}_0\) enters \(\mathbb{C}_0\) at time \(t_0 > 0\). This path must leave \(\mathbb{C}_0\) at time \(t_1 < \infty\) since the mean reversion level \((\lambda^2/a^2, -\lambda/a) \notin \mathbb{C}_0\). This implies that for any \(t \in (t_0, t_1)\) \(x(t, \phi_0) + y(t, \phi_1) < \lambda/c\) and
\(x(t_0, \phi_0) + y(t_0, \phi_1) = \lambda/c.\) This yields a contradiction, because \(\lambda + a < 0\) together with (5.13) implies that \(t \rightarrow x(t, \phi_0) + y(t, \phi_1)\) is increasing below the line segment \(L.\) Therefore the minimizer \(t_{\text{opt}}(\phi_0, \phi_1)\) of (5.10) is equal to 0 if \((\phi_0, \phi_1) \not\in C_0,\) and it is equal to \(t_{c}(\phi_0, \phi_1)\) if \((\phi_0, \phi_1) \in C_0.\) From (5.9) we can conclude that \(C\) is equal to the optimal continuation region of (3.3).

Case II-b-i-2: \(-a/c - 1 > 0.\) In this case the line segments \(C\) and \(L\) intersect at \(I = (x', y') \triangleq \left(\frac{\lambda}{a} + \frac{\lambda}{a} - (1 + \frac{\lambda}{a})\right).\) By running the paths backward in time, we can find \(x^*\) such that

\[
(x^*, 0) = (x(-t^*, x^*), y(-t^*, y^*)).
\]  

(5.16)

By the semi-group property (2.27), we have

\[
x(t^*, x^*) = x(t^*, x(-t^*, x^*)) = x(t^* + (-t^*), x^*) = x(0, x^*) = x^*.
\]  

(5.17)

Similarly, \(y(t^*, 0) = y^*.\) The function \(t \rightarrow x(t, x^*) + y(t, 0)\) is decreasing on \((0, t^*)\) and increasing on \((t^*, \infty).\) It follows that the path \(t \rightarrow (x(t, x^*), y(t, 0))\) is tangential to \(C\) at \(I\) and lies above the region \(C_0.\)

We will now show that if a path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) originates in \(D_1,\) then it stays in \(D_1.\) Let us first consider a pair \((\phi_0, \phi_1) \in D_1\) such that \(\phi_1 < -2\lambda/a.\) Consider the curve

\[
P \triangleq \{(x, y) \in B_1^2 : x = x(t, x^*), y = y(t, 0) \text{ for } t \in [0, \infty).\}
\]  

(5.18)

The following remark will be useful in completing the proof.

**Remark 5.1.** The semi-group property in (2.27) implies that two distinct curves \((x(\cdot, \phi_0^a, y(\cdot, \phi_1^a)))\) and \((x(\cdot, \phi_0^b, y(\cdot, \phi_1^b)))\) do not intersect. If

\[
(x(t^a, \phi_0^a), y(t^a, \phi_1^a)) = (x(t^b, \phi_0^b), y(t^b, \phi_1^b)) = (\phi_0, \phi_1)
\]  

(5.19)

for some \(t^a, t^b \in \mathbb{R}\) then (2.27) implies that

\[
(x(t, \phi_0^a), y(t, \phi_1^a)) = (x(t + (t - t^a), \phi_0^a), y(t + (t - t^a), \phi_1^a))
\]  

\[
= (x(t - t^a, \phi_0^a), y(t - t^a, \phi_1^a)) = (x(t^b + (t - t^a), \phi_0^b), y(t^b + (t - t^a), \phi_1^b))
\]  

(5.20)

\[
= (x(t^b - t^a + t, \phi_0^b), y(t^b - t^a + t, \phi_1^b)), \text{ for all } t \in \mathbb{R},
\]

i.e., the two curves are identical after a reparametrization.

If the point \((\phi_0, \phi_1)\) lies above \(P,\) and we recall that \(P\) lies above \(C_0,\) then by Remark 5.1 the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) will lie above \(C_0.\) If the point \((\phi_0, \phi_1)\) lies between \(P\) and \(C_0,\) then the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) will lie below the line segment \(L.\) This observation together with the fact that \(\lambda + a < 0,\) implies (using (5.13)) that the function \(t \rightarrow x(t, \phi_0) + y(t, \phi_1)\) is increasing. Therefore the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) cannot intersect \(C_0.\)

Now let us consider a pair \((\phi_0, \phi_1) \in D_1\) such that \(\phi_1 > -2\lambda/a.\) If \((\phi_0, \phi_1)\) lies above \(L,\) then the function \(t \rightarrow x(t, \phi_0) + y(t, \phi_1)\) is decreasing and its range is \(\{\phi_0 + \phi_1, 2\lambda/a(\lambda/a - 1)\},\) which is always above \(\lambda/c,\) and therefore the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) does not enter \(C_0.\) If the point \((\phi_0, \phi_1)\) lies below \(L\) then \(t \rightarrow x(t, \phi_0) + y(t, \phi_1)\) is increasing. This monotonicity implies that the path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) cannot visit \(C_0.\) If \(\phi_1 = -2\lambda/a,\) then \(y(t, \phi_1) = -2\lambda/a\) for all \(t \geq 0.\) \((x(t, \phi_0))\) increases or decreases depending on whether \((\phi_0, -2\lambda/a)\) is below or above \(L.\) Therefore if \((\phi_0, -2\lambda/a) \not\in C_0,\) then \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) never visits \(C_0.\) These arguments show that if a path \((x(\cdot, \phi_0), y(\cdot, \phi_1))\) originates in \(D_1,\) then it stays in \(D_1.\) Therefore if \((\phi_0, \phi_1) \in D_1,\) then the infimum in (5.10) is equal to 0 (by (5.9) and (5.10)). Therefore \([(B_1^2 - D_1) \times \mathbb{R}_+] \cap B_1^2\) is a superset of the optimal continuation region of (3.3).

Case II-b-ii: \(\lambda + a > 0.\) In this case \(L\) is increasing (as a function of \(x\)). The function \(t \rightarrow x(t, \phi_0) + y(t, \phi_1)\) is increasing if \((\phi_0, \phi_1)\) lies above \(L,\) and it is decreasing otherwise.
Case II-b-ii-1: $-a/c - 1 \leq 0$. In this case the line segments $L$ and $C$ do not intersect. Let us first consider a pair $(φ₀, φ₁) \in B²⁺$ such that $φ₁ < -2λ/a$. If $(φ₀, φ₁) \notin C₀$ lies above the line segment $L$, then $t \rightarrow x(t, φ₀) + y(t, φ₁)$ is increasing and the path $(x(·, φ₀), y(·, φ₁))$ cannot enter $C₀$. Consider the curve

$$
\hat{P} \triangleq \left\{ (x, y) \in B²⁺ : x = t, \quad y = y(t, 0) \quad \text{for} \quad t \in [0, \infty) \right\},
$$

(5.21)

which starts at the intersection of $L$ with the $x$-axis. The semi-group property Remark 5.1 implies that no path starting to the right of $\hat{P}$ intersects $P$ and therefore lies to the right of the region $C₀$. Therefore, if $(φ₀, φ₁)$ is below the line segment $L$, then the path $(x(·, φ₀), y(·, φ₁))$ never visits the advantageous region $C₀$. (Note that if the path $(x(·, φ₀), y(·, φ₁))$ meets the line $L$ at time $t_L(φ₀, φ₁)$, then $t \rightarrow (x(t, φ₀) + y(t, φ₁))$ is increasing (decreasing) on $[0, t_L]$ ($[t_L, \infty)$.)

Now let us consider a point $(φ₀, φ₁) \in B²⁺ - C₀$ such that $φ₁ > -2λ/a$. Then $t \rightarrow (x(t, φ₀) + y(t, φ₁))$ is increasing on $[0, t_L(φ₀, φ₁)]$ and is decreasing on $(t_L(φ₀, φ₁), \infty)$ (it decreases to $-2λ/a + λ²/a² > λ/c$). And the monotonicity of $t \rightarrow x(t, φ₀) + y(t, φ₁)$ on $[0, t_L(φ₀, φ₁)]$ implies that $x(t, φ₀) + y(t, φ₁) > λ/c$ for $t \in [0, t_L(φ₀, φ₁)]$. If $φ₁ = -2λ/a$, then $y(t, φ₁) = -2λ/a$ for all $t \geq 0$, $x(t, φ₀)$ increases (decreases) depending on whether $(φ₀, -2λ/a)$ is above or below $L$. These arguments show that if a path $(x(·, φ₀), y(·, φ₁))$ originates in $B²⁺ - C₀$ then it stays in $B²⁺ - C₀$. Therefore the minimizer $tₚ(φ₀, φ₁)$ of (5.10) for any $φ₀, φ₁ \in B²⁺ - C₀$ is equal to zero. Now using (5.9) and (5.10) the optimal continuation region of (3.3) is equal to $C$.

Case II-b-ii-2: $-a/c - 1 > 0$. In this case the line segments $C$ and $L$ intersect at $I = (x₁, y₁)$. Arguments similar to those of Case II-b-i-2 show that $([B²⁺ - D₁] × R⁺] \cap B²⁺$, in which $D₁$ is defined in (5.7), is a superset of the optimal continuation region of (3.3)∎

**Theorem 5.2.** Let us assume that $α < β$ (Case III: $α < β$) and define

$$
D₂ \triangleq \left\{ (x, y) \in B²⁺ : x + y \geq \frac{λ + 2β}{c} \right\}.
$$

(5.22)

Then $([B²⁺ - D₁] × R⁺] \cap B²⁺$, which is a bounded region in $R⁴$, is an upper bound on the continuation region of (3.3).

**Proof.** Note that in this case $a > 0$. The paths of the processes $Φ^⁺, \Phi^−$ increase between the jumps and decrease with a jump. If $τ ∈ S$ then there is a constant $t ≥ 0$ such that $τ ∧ σ₁ = t ∧ σ₁$ almost surely. Hence we can write

$$
E₀ \left[ \int _0 ^{τ} e^{-λs} h(Ψ s) ds \right] = E₀ \left[ \int _0 ^{τ ∧ σ₁} e^{-λs} h(Ψ s) ds \right] + E₀ \left[ 1_{(t ≥ σ₁)} \int _{σ₁} ^{τ} e^{-λs} h(Ψ s) ds \right]
$$

$$
= E₀ \left[ \int _0 ^{τ ∧ σ₁} e^{-λs} h(Ψ s) ds \right] + E₀ \left[ 1_{(t ≥ σ₁)} \int _{σ₁} ^{τ} e^{-λs} h(Ψ s) ds \right]
$$

$$
≥ E₀ \left[ \int _0 ^{τ ∧ σ₁} e^{-λs} h(Ψ s) ds \right] - \frac{1}{c} E₀ \left[ 1_{(t ≥ σ₁)} e^{-λσ₁} \right]
$$

$$
= \int _0 ^{τ} e^{-(λ + 2β)s} \left[ h(x(s, φ₀), y(s, φ₁)) - \frac{2β}{c} \right] ds,
$$

(5.23)

using also the fact that $σ₁$ has exponential distribution with rate $2β$. From (5.23) it follows that if $x(s, φ₀) + y(s, φ₁) - (λ + 2β)/c > 0$, then $E₀ \left[ \int _0 ^{t} e^{-λs} h(Ψ s) ds \right] > 0$ for every stopping time $τ ≠ 0, τ ∈ S$. Since the paths $x(t, φ₀), y(t, φ₁)$ are increasing we can conclude that stopping immediately is optimal for (3.3). That is $τ = 0$ is optimal for (3.3) if $(φ₀, φ₁) ∈ D₂$ and $φ₂ ≤ φ₁$, in which $D₂$ is as in (5.22) ∎

Theorems 5.1 and 5.2 can be used to determine approximate detection rules besides helping us to determine the location and the shape of the continuation region. As we have seen in Cases II-b-i-1 and II-b-ii-1, these approximate rules turn out to be tight. The next theorem is essential in proving the fact that the smallest optimal stopping time of (3.3) has a finite expectation.

**Theorem 5.3.** Let $τ_D$ be the exit time of the process $Y$ from a bounded region $D \subset B²⁺$. Then $E₀^{φ^+, φ^-, φᵀ} [τ_D] < ∞$ for every $(φ^+, φ^-, φᵀ) \in B²⁺$. Hence $τ^*$ defined in (5.7) has a finite expectation.
On the other hand, the second term is greater than

\[ \xi = \phi^x + \phi^+ \]

in which \( \xi \) is the pointwise limit of \( \frac{\lambda}{\beta} \phi^+ + \frac{\alpha}{\beta} \phi^x + \left( \frac{\alpha}{\beta} - 1 \right) \phi^1 - \phi^x - \phi^+ \) for all \( (a, b, c) \in B^3_+ \). Since \( f \) is bounded on \( D \), and \( \tau_D \land t \) is a bounded \( \mathbb{F} \)-stopping time, we have

\[ \mathbb{E}_0 [ f(\Upsilon_{\tau_D \land t})] = f(\Upsilon_0) + \mathbb{E}_0 \left[ \int_0^{\tau_D \land t} [Af](\Upsilon_s) \, ds \right] \geq 2\lambda \mathbb{E}_0 [\tau_D \land t]. \] \hfill (5.25)

On the other hand

\[ \mathbb{E}_0 [ f(\Upsilon_{\tau_D \land t})] \leq \frac{\alpha}{\beta} \xi, \]

in which \( \xi = \min\{a \in \mathbb{R}_+ : \text{for any} (x, y, z) \in D, \max(x, y, z) \leq a\} < \infty \). An application of the monotone convergence theorem implies that \( \mathbb{E}_0 [\tau_D] < \infty \). \( \square \)

The results of this section can be used to determine approximate detection rules besides helping us to determine the location and the shape of the continuation region. As we have seen in Cases II-b-i-1 and II-b-ii-1, these approximate rules turn out to be tight.

### 6. Optimal Stopping with Time Horizon \( \sigma_n \)

In this section, we will first approximate the optimal stopping problem (3.3) by a sequence of optimal stopping problems. Let us denote

\[ V_n(a, b, c) \overset{\Delta}{=} \inf_{\tau \in \mathcal{S}} \mathbb{E}_0^{a, b, c} \left[ \int_0^{\tau \land \sigma_n} e^{\lambda h} (\Phi_t^x, \Phi_t^+) \, dt \right], \] \hfill (6.1)

for all \( (a, b, c) \in B^3_+ \) and \( n \in \mathbb{N} \). Here, \( \sigma_n \) is the \( n \)th jump time of the process \( X \).

Observe that \( (V_n)_{n \in \mathbb{N}} \) is a decreasing sequence and they each of its members satisfy \(-1/c < V_n < 0\). Therefore the pointwise limit \( \lim_n V_n \) exist. It can be shown that more is true using the fact that the function \( h \) is bounded from below and \( \sigma_n \) is a sum of independent exponential random variables.

**Lemma 6.1.** For any \( (a, b, c) \in B^3_+ \)

\[ 0 \leq V_n(a, b, c) - V(a, b, c) \leq \frac{1}{c} \left( \frac{2\beta}{2\beta + \lambda} \right)^n. \] \hfill (6.2)

**Proof.** For any \( \tau \in \mathcal{S} \),

\[ \mathbb{E}_0 \left[ \int_0^\tau e^{\lambda h} (\Phi_t^x, \Phi_t^+) \, dt \right] = \mathbb{E}_0 \left[ \int_0^{\tau \land \sigma_n} e^{\lambda h} (\Phi_t^x, \Phi_t^+) \, dt \right] + \mathbb{E}_0 \left[ \int_{\sigma_n}^\tau e^{\lambda h} (\Phi_t^x, \Phi_t^+) \, dt \right] \]

\hfill (6.3)

The first term on the right-hand-side of (6.3) is greater than \( V_n \). Since \( h(\cdot, \cdot) > -\lambda/c \) we can show that the second term is greater than

\[ -\frac{\lambda}{c} \mathbb{E}_0^{\phi_0, \phi_1} \left[ \int_{\sigma_n}^\tau e^{-\lambda s} \, ds \right] \geq \frac{-1}{c} \mathbb{E}_0^{\phi_0, \phi_1} \left[ e^{-\lambda \sigma_n} \right] \geq \frac{1}{c} \left( \frac{2\beta}{\lambda + 2\beta} \right)^n. \] \hfill (6.4)

To show the last inequality we have used the fact that \( \sigma_n \) is a sum of \( n \) independent and identically distributed exponential random variables with rate \( 2\beta \). Now, the proof of the lemma follows immediately. \( \square \)
As in [4] and [5] to calculate the value functions \( V_n \), iteratively we introduce the functional operators \( J, J_i \). These operators are defined through their actions on bounded functions \( g : \mathbb{R}^3 \to \mathbb{R} \) as follows:

\[
[Jg](t, a, b, c) \triangleq \mathbb{E}_0^{a, b, c} \left[ \int_0^{t} e^{-\lambda t} h(x_s, s) + 1(t \geq \sigma_1) e^{-\lambda \sigma_1} g(x_{\sigma_1}, \Phi_{\sigma_1, \Phi_{\sigma_1}^{\pm}}) \right],
\]

and,

\[
[J_i g](a, b, c) \triangleq \inf_{s \in [t, \infty]} [Jg](s, a, b, c), \quad t \in [0, \infty].
\]

Observe that

\[
\mathbb{E}_0 [1(t \geq \sigma_1) e^{-\lambda \sigma_1} g(x_{\sigma_1}, \Phi_{\sigma_1}^{\pm}, \Phi_{\sigma_1}^{\pm})] = \mathbb{E}_0 \left[ \left( g \left( \frac{\alpha}{\beta} \Phi_{\sigma_1}^{\pm}, \Phi_{\sigma_1}^{\pm} + \left( \frac{\alpha}{\beta} - 1 \right) \Phi_{\sigma_1}^{1} \right) \right) \right] \end{align*}

\[
+ g \left( \frac{\alpha}{\beta} \Phi_{\sigma_1}^{\pm} - \frac{\alpha}{\beta} \Phi_{\sigma_1}^{\pm} - \left( \frac{\alpha}{\beta} - 1 \right) \Phi_{\sigma_1}^{1} \right) 1\{x_{\sigma_1} \neq x_{\sigma_1}^{\pm}\} \right] \end{align*}

\[
= \frac{1}{2} \int_0^t 2 \beta e^{-\lambda t} g \left( \frac{\alpha}{\beta} x(s, a), y(s, b) + \left( \frac{\alpha}{\beta} - 1 \right) z(s, c), \frac{\alpha}{\beta} z(s, c) \right) ds
\]

\[
+ \frac{1}{2} \int_0^t 2 \beta e^{-\lambda t} g \left( \frac{\alpha}{\beta} x(s, a), \frac{\alpha}{\beta} y(s, b) - \left( \frac{\alpha}{\beta} - 1 \right) z(s, c), z(s, c) \right) ds.
\]

(6.6)

To derive (6.6) we used the fact that \( \sigma_1 \) has exponential distribution with rate \( 2\beta \), the dynamics in (2.20), and the fact that conditioned on the event that there is a jump it has \( 1/2 \) probability of coming from \( X_1 \) and

Using (6.6) and Fubini’s theorem we can write

\[
[Jg](t, a, b, c) = \int_0^t e^{-\lambda t} h(x(s, a), y(s, b), z(s, c)) ds,
\]

(6.7)

where

\[
F_i(a, b, c) = \left( \frac{\alpha}{\beta} a, \frac{\alpha}{\beta} b, -1 \right) (\frac{\alpha}{\beta} c, \frac{\alpha}{\beta} c, 2-3) \end{align*}

\[
i \in \{1, 2\}.
\]

(6.8)

Using (2.20) it can be shown that

\[
\lim_{t \to \infty} [Jg](t, a, b, c) = [Jg](\infty, a, b, c) < \infty.
\]

(6.9)

**Lemma 6.2.** For every bounded function \( f \), the mapping \( J_0 f \) is bounded. If \( f \) is a concave function, then \( J_0 f \) is also a concave function. If \( f_1 \leq f_2 \), then \( J_0 f_1 \leq J_0 f_2 \).

**Proof.** The third assertion of the lemma directly follows from the representation (6.7). The first assertion holds since \( h \) is bounded from below and \( J_0 f(a, b, c) \leq J f(0, a, b, c) = 0 \). The second assertion follows from the linearity of the functions \( x(t, \cdot), y(t, \cdot), h(\cdot, \cdot), F_1(\cdot, \cdot, \cdot) \) and \( F_2(\cdot, \cdot, \cdot) \).

Using Lemma 6.2, we can prove the following corollary:

**Corollary 6.1.** Let us define a sequences of function \((v_n)_{n \in \mathbb{N}}\) by

\[
v_0 \triangleq 0, \quad v_n \triangleq J_n v_{n-1}.
\]

(6.10)

Then every \( n \in \mathbb{N}, v_n \) is bounded and concave; and \( v_{n+1} \leq v_n \). Therefore, the limit \( v = \lim_n v_n \) exist, and is bounded and concave. Moreover, \( v_n \), for all \( n \in \mathbb{N} \) and \( v \) are increasing in each of their arguments.

**Proof.** The proof of the first part directly follows from Lemma 6.2. That \( v_n \) for all \( n \in \mathbb{N} \) and \( v \) are increasing in each of their arguments follows from the fact that these functions are bounded from above and below and that they are concave.
We will need the following lemma to give a characterization of the stopping times of the filtration $\mathbb{F}$ (see [5]).

**Lemma 6.3.** For every $\tau \in \mathcal{S}$, there are $\mathcal{F}_{\sigma_n}$ measurable random variables $\xi_n : \Omega \to \infty$ such that $\tau \wedge \sigma_n+1 = (\sigma_n + \xi_n) \wedge \sigma_n+1 \mathbb{P}_0$ almost surely on $\{\tau \geq \sigma_n\}$. The main theorem of this section can be proven by induction using Lemma 6.3 and the strong Markov property.

**Theorem 6.1.** For every $n \in \mathbb{N}$, $v_n$ defined in Corollary 6.1 is equal to $V_n$. For $\varepsilon \geq 0$, let us denote

$$r_\varepsilon^n(a, b, c) \triangleq \inf\{t \in (0, \infty) : [Jv_n](t, (a, b, c)) \leq [J_0v_n](a, b, c) + \varepsilon\}. \quad (6.11)$$

And let us define a sequence of stopping times by $S_\varepsilon^n \triangleq r_\varepsilon^n(\Upsilon_0) \wedge 1$ and

$$S_\varepsilon^{n+1} = \begin{cases} r_\varepsilon^{n/2}(\Upsilon_0) & \text{if } \sigma_1 \geq r_\varepsilon^{n/2}(\Upsilon_0) \\ \sigma_1 + S_\varepsilon^n \circ \theta_{\sigma_1} & \text{otherwise.} \end{cases} \quad (6.12)$$

Here $\theta_s$ is the shift operator on $\Omega$, i.e., $X_t \circ \theta_s = X_{s+t}$. Then $S_\varepsilon^n$ is an $\varepsilon$-optimal stopping time of (6.1), i.e.,

$$\mathbb{E}^{a, b, c}_{\varepsilon} \left[ \int_0^{S^n_\varepsilon} e^{-\lambda t} h(\Psi_t) dt \right] \leq v_n(a, b, c) + \varepsilon, \quad (6.13)$$

in which $\Psi_t = (\Phi^\varepsilon_t, \Phi_t)$, $t \geq 0$.

**Proof.** See Appendix. □

Theorem 6.1 shows that the value function $V_n$ of the optimal stopping problem defined in (6.1) and the function $v_n$ introduced in Corollary 6.1 by an iterative application of the operator $J_0$ are equal. This implies that the value function of the optimal stopping problem of (6.1) can be found by solving a sequence of deterministic minimization problems.

7. Optimal Stopping Time. **Theorem 7.1.** $\tau^*$ defined in (5.7) is the smallest optimal stopping time for (6.3).

This theorem shows that $\Gamma$ defined in (6.2) is indeed an optimal stopping region. We will divide the proof of this theorem into several lemmas.

The following dynamic programming principle can be proven by the special representation of the stopping times of a jump process (Lemma 6.3) and the strong Markov property.

**Lemma 7.1.** For any bounded function $g : \mathbb{B}_+^3 \to \mathbb{R}$ we have

$$[J_g](a, b, c) = [Jg](t, a, b, c) + e^{-(\lambda+2\beta)t}[J_0g](x(t, a), y(t, b), z(t, c)). \quad (7.1)$$

Let us denote

$$r_n(a, b, c) \triangleq r_0^n(a, b, c), \quad (7.2)$$

which is well defined because of (6.9) and the continuity of the function $t \to [Jf](t, a, b, c)$, $t \geq 0$. (See 6.11 for the definition of $r_0^n$.)

Let us also denote

$$r(a, b, c) \triangleq \inf\{t \geq 0 : [JV](t, a, b, c) = J_0V(a, b, c)\}. \quad (7.3)$$

**Corollary 7.1.** The functions $r_n$ and $r$ defined by (7.2) and (7.3) respectively, satisfy

$$r_n(a, b, c) = \{t \geq 0 : \varepsilon_{n+1}(x(t, a), y(t, b), z(t, c)) = 0\} \quad (7.4)$$

$$r(a, b, c) = \{t \geq 0 : \varepsilon(x(t, a), y(t, b), z(t, c)) = 0\}.$$
with the convention that inf \( \emptyset = 0 \). Together with \( (7.4) \), Corollary \( (6.1) \) implies that \( r_n(a, b, c) \uparrow r(a, b, c) \) as \( n \uparrow \infty \).

**Proof.**

Suppose that \( r_n(a, b, c) < \infty \). Then from \( (6.11) \) it follows that

\[
[J v_n](r_n(a, b, c), a, b, c) = [J_0 v_n](a, b, c) = [J_{r_n(a, b, c)} v_n](a, b, c) = [J_0 v_n](a, b, c) + e^{-(\lambda + 2\beta)} v_{n+1}(x(r_n(a, b, c), a), y(r_n(a, b, c), b), z(r_n(a, b, c), c)).
\]

(7.5)

Here the second equality follows from the definition of the operator \( J_1 \) in \( (6.5) \) and the fact that \( J_0 v_n = v_{n+1} \).

From \( (7.5) \) it follows that \( v_{n+1}(x(r_n(a, b, c), a), y(r_n(a, b, c), b), z(r_n(a, b, c), c)) < 0 \) for every \( t \in (0, r_n(a, b, c)) \) which can be shown using same arguments as above. Therefore \( \{ t > 0 : v_{n+1}(x(t, a), y(t, b), z(t, c)) = 0 \} = \emptyset \) and \( (7.3) \) holds.

The proof for the representation of \( r \) can be proven using the same line of argument and the fact that \( J_0 V = V \). The fact that \( J_0 V = V \) can be proven by the dominated convergence theorem, since the sequences \( (v_n(a, b, c))_{n \geq 0} \) and \( ([J v_n](t, a, b, c))_{n \geq 0} \) are decreasing, and since \( v_n \) is bounded function for all \( n \in \mathbb{N} \).

In the next lemma we construct optimal stopping times for the family of problems introduced in \( (6.1) \).

**Lemma 7.2.** Let us denote \( S_n \equiv S_n^0 \), where \( S_n^\varepsilon \) is defined in Theorem \( (6.1) \) for \( \varepsilon > 0 \). Then the sequence \( (S_n)_{n \in \mathbb{N}} \) is an almost surely increasing sequence. Moreover \( S_n < \tau^* \) almost surely for all \( n \).

**Proof.** Since \( r_1 > 0 \), using Corollary \( (6.1) \) we can write

\[
S_2 - S_1 = \begin{cases} r_1 - r_0, & \text{if } \sigma_1 > r_1 \\ \sigma_1 - r_0 + S_1 \circ \theta_{\sigma_1}, & \text{if } r_0 < \sigma_1 \leq r_1 \\ S_1 \circ \theta_{\sigma_1}, & \text{if } \sigma_1 \leq r_0 \end{cases} > 0.
\]

(7.7)

Now, let us assume that \( S_n - S_{n-1} > 0 \) almost surely. From Lemma \( (6.1) \) we have that \( r_n > r_{n-1} \). Using this fact and the induction hypothesis we can write

\[
S_{n+1} - S_n = \begin{cases} r_n - r_{n-1}, & \text{if } \sigma_1 > r_n \\ \sigma_1 - r_{n-1} + S_n \circ \theta_{\sigma_1}, & \text{if } r_{n-1} < \sigma_1 \leq r_n \\ (S_n - S_{n-1}) \circ \theta_{\sigma_1} & \text{if } \sigma_1 \leq r_{n-1} \end{cases} > 0,
\]

(7.8)

which proves the first assertion of the lemma.

From Corollary \( (6.1) \) and the definition of \( \tau^* \) it follows that \( \tau^* \wedge \sigma_1 = r \wedge \sigma_1 \). Therefore \( \tau^* \wedge \sigma_1 > r_0 \wedge \sigma_1 = S_1 \), since \( r_0 < r \). Now, we will assume that \( S_n < \tau^* \) and show that \( S_{n+1} < \tau^* \). On \( \{ \sigma_1 \leq r_n \} \) we have that

\[
S_{n+1} = \sigma_1 + S_n \circ \theta_{\sigma_1} < \sigma_1 + \tau^* \circ \theta_{\sigma_1}.
\]

(7.9)

Since \( \tau^* \wedge \sigma_1 = r \wedge \sigma_1 \) and \( r > r_n \), if \( \sigma_1 \leq r_n \), then \( \tau^* \wedge \sigma_1 = \sigma_1 \). Because \( \tau^* \) is a hitting time, on the set \( \{ \sigma_1 \leq r_n \} \subset \{ \sigma_1 \leq \tau^* \} \) the following holds

\[
S_{n+1} \leq \sigma_1 + \tau^* \circ \theta_{\sigma_1} = \tau^*.
\]
On the other hand if $\sigma_1 > r_n$, then $\tau^* \wedge \sigma_1 = r \wedge \sigma_1 > r_n$. Therefore on $\{\sigma_1 > r_n\}$, $S_{n+1} = r_n < \tau^*$. This concludes the proof of the second assertion. \(\Box\)

**Lemma 7.3.** Let us denote $\Psi_t = (\Phi_t^x, \Phi_t^y)$, $t \geq 0$. If $S^* \triangleq \lim_n S_n$, then $S^* = \tau^*$ almost surely. Moreover, $\tau^*$ is an optimal stopping time, i.e.,

$$V(a, b, c) = \mathbb{E}_{0}^{a, b, c} \left[ \int_{0}^{S^*} e^{-\lambda s} h(\Psi_s) ds \right].$$

**Proof.** The limit $S^* \triangleq \lim_n S_n$ exists since $(S_n)_{n \in \mathbb{N}}$ is increasing and $S_n \leq \tau^* < \infty$ (as a corollary of Theorem 5.3) for all $n$. Let us show that $S^*$ is optimal.

$$\mathbb{E}_0 \left[ \lim_n \int_{0}^{S_n} e^{-\lambda t} h(\Psi_t) dt \right] \leq \liminf_n \mathbb{E}_0 \left[ \int_{0}^{S_n} e^{-\lambda t} h(\Psi_t) dt \right] = \lim_n V_n(a, b, c) = V(a, b, c).$$

(7.10)

The first inequality follows from Fatou’s Lemma, which we can apply since

$$\int_{0}^{S_n} e^{-\lambda t} h(\Psi_t) dt \geq \int_{\infty}^{0} e^{-\lambda t} h(\Psi_t) dt \geq -\frac{\sqrt{2}}{c}, \text{ a.s.}$$

The first equality in (7.10) follows from Theorem 6.1. Now it can be seen from (7.10) that $S^*$ is an optimal stopping time. Taking the limit of (6.12) as $n \to \infty$ and using Corollary 7.1 we conclude that $\tau^* = S^*$. \(\Box\)

**Proof of Theorem 7.1** The proof of the optimality of $\tau^*$ follows directly from Lemma 7.3. We will show that $\tau^*$ is the smallest optimal stopping time.

Given any $F$-stopping time $\tau < \tau^*$, let us define

$$\tilde{\tau} \triangleq \tau + \tau^* \circ \theta_{\tau} \quad (7.11)$$

Then the stopping time $\tilde{\tau}$ satisfies

$$\mathbb{E}_0^{a, b, c} \left[ \int_{0}^{\tilde{\tau}} e^{-\lambda s} h(\Psi_s) ds \right] = \mathbb{E}_0^{a, b, c} \left[ \int_{0}^{\tau} e^{-\lambda s} h(\Psi_s) ds + \int_{\tau}^{\tilde{\tau}} e^{-\lambda s} h(\Psi_s) ds \right]$$

$$= \mathbb{E}_0^{a, b, c} \left[ \int_{0}^{\tau} e^{-\lambda s} h(\Psi_s) ds + e^{-\lambda \tau} \int_{0}^{\tau^* \circ \theta_{\tau}} e^{-\lambda s} h(\Psi_{s+\tau}) ds \right]$$

$$= \mathbb{E}_0^{a, b, c} \left[ \int_{0}^{\tau} e^{-\lambda s} h(\Psi_s) ds + e^{-\lambda \tau} V(\Upsilon_{\tau}) \right]$$

$$< \mathbb{E}_0^{a, b, c} \left[ \int_{0}^{\tau} e^{-\lambda s} h(\Psi_s) ds \right].$$

(7.12)

Here the third equality follows from the strong Markov property of the process $\Upsilon$ and the inequality follows since $V(\Upsilon_{\tau}) < 0$. Equation (7.12) shows that any optimal stopping time $\tau < \tau^*$ cannot be optimal.

\(\Box\)

**8. Structure of the Continuation and Stopping Regions.** Let us recall (5.2) and denote

$$\Gamma_n \triangleq \{(a, b, c) \in \mathbb{B}_+^3 : v_n(a, b, c) = 0\}, \quad \mathcal{C}_n \triangleq \mathbb{B}_+^3 - \Gamma_n. \quad (8.1)$$
We have shown in Theorem 4.1 that that $\Gamma$ of (6.3) is the optimal stopping region for (5.3) and the first hitting time $\tau^*$ of $\Upsilon$ to this set is optimal. On the other hand although $\Gamma_n$ is an optimal stopping region for (6.1), the description of the optimal stopping times $S_0^n$ (see (6.12)) is more involved. These optimal stopping times are not hitting times of the sets $\Gamma_n$. $S_0^n$ prescribe to stop if $\Upsilon$ hits $\Gamma_n$ before it jumps. Otherwise if there is a jump before $\Upsilon$ reaches $\Gamma_n$, then $S_0^n$ prescribes to stop when the process hits $\Gamma_{n-1}$ before the next jump, and so on.

Theorem 6.1 shows that $V_n$ of (6.1) and the functions $v_n$ introduced in Corollary 6.1 are equal. Therefore, their respective limits $V$ and $v$ are also equal. Recall that $V^n$ converges to $V$ uniformly and the convergence rate is exponential (see Lemma 6.1). Since $(v_n)_{n \in \mathbb{N}}$ is a decreasing sequence with limit $v$ the stopping regions in (8.1) are nested and satisfy $\Gamma \subseteq \cdots \subseteq \Gamma_{n-1} \subseteq \cdots \Gamma_1$ and $\Gamma = \cap_{n=1}^\infty \Gamma_n$.

By Corollary 6.1 we know that each $v_n$ is concave and bounded, which also implies that the limit $v$ is concave and bounded. This in turn implies that the stopping regions $\Gamma_n$ and $\Gamma$ are convex and closed. Since we show in Section 5 that the continuation region is bounded, it can readily be shown that the stopping regions $\Gamma_n$ and $\Gamma$ are the epigraphs of some mappings $\gamma_n$ and $\gamma$ which are convex and strictly decreasing and the numbers $x_n \triangleq \inf\{y \in \mathbb{R}^+ : \gamma_n(y) = 0\}$ and $x \triangleq \inf\{y \in \mathbb{R}^+ : \gamma(y) = 0\}$ are finite.

9. Extensions.

9.1. Non-identical Sources. Consider two independent Poisson processes $X^1$ and $X^2$ with arrival rates $\beta_1$ and $\beta_2$ respectively. At some random unobservable times $\theta_1$ and $\theta_2$, with distributions

$$P(\theta_i = 0) = \pi_i, \quad P(\theta_i > t) = (1 - \pi_i)e^{-\lambda_i t} \text{ for } t \geq 0,$$

(9.1)

the arrival rates of the Poisson processes $X^1$ and $X^2$ change from $\beta_i$ to $\alpha_i$, respectively, i.e.,

$$X_i^t - \int_0^t h_i(s)ds, \quad t \geq 0, \quad i = 1, 2,$$

(9.2)

are martingales, in which

$$h_i(t) = [\beta_i 1_{t < \theta_i} + \alpha_i 1_{t \geq \theta_i}], \quad t \geq 0, \quad i = 1, 2.$$

(9.3)

Here $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are known positive constants. Then Equation the dynamics of $\Phi^x$ defined in (9.19) becomes

$$d\Phi^x_t = [\lambda_2 \Phi^1_t + \lambda_1 \Phi^2_t + (a_1 + a_2) \Phi^x_t]dt + \Phi^x_t \left[((\alpha_1/\beta_1) - 1)dX^1_t + ((\alpha_2/\beta_2) - 1)dX^2_t\right],$$

(9.4)

in which $a_i = \lambda_i - \alpha_i + \beta_i$, $i \in \{1, 2\}$. Let us introduce

$$x(t, \phi_0) = e^{(a_1 + a_2)t} \phi_0 + \int_0^t e^{(a_1 + a_2)(t-u)} (\lambda_2 y(u, \phi_1) + \lambda_1 z(u, \phi_2))du, \quad \text{in which} \quad y(t, \phi_1) = -\frac{\lambda_1}{a_1} + e^{a_1t} \left(\phi_1 + \frac{\lambda_1}{a_1}\right), \quad z(t, \phi_2) = -\frac{\lambda_2}{a_2} + e^{a_2t} \left(\phi_2 + \frac{\lambda_2}{a_2}\right).$$

(9.5)

Then $\Phi^x_t$, $\Phi^1_t$ and $\Phi^2_t$, $t \geq 0$ can be written as

$$\Phi^x_t = x(t - \sigma_n, \Phi^x_{\sigma_n}), \quad \Phi^1_t = y(t - \sigma_n, \Phi^1_{\sigma_n}), \quad \Phi^2_t = z(t - \sigma_n, \Phi^2_{\sigma_n}), \quad \sigma_n \leq t < \sigma_{n+1}, \quad n \in \mathbb{N},$$

(9.6)

and

$$\Phi^x_{\sigma_{n+1}} = \left(\frac{\alpha_1}{\beta_1} 1_{(x^1_{\sigma_{n+1}} \neq x^1_{\sigma_{n+1} -})} + \frac{\alpha_2}{\beta_2} 1_{(x^2_{\sigma_{n+1}} \neq x^2_{\sigma_{n+1} -})}\right) \Phi^x_{\sigma_{n+1} -},$$

$$\Phi^1_{\sigma_{n+1}} = \left(\frac{\alpha_1}{\beta_1} 1_{(x^1_{\sigma_{n+1}} \neq x^1_{\sigma_{n+1} -})} \Phi^1_{\sigma_{n+1} -}\right), \quad \Phi^2_{\sigma_{n+1}} = \left(\frac{\alpha_2}{\beta_2} 1_{(x^2_{\sigma_{n+1}} \neq x^2_{\sigma_{n+1} -})} \Phi^2_{\sigma_{n+1} -}\right).$$

(9.7)

Choosing $\Upsilon_t = (\Phi^x_t, \Phi^1_t, \Phi^2_t), \ t \geq 0$ as the Markovian statistic to work with, we can extend our analysis to deal with non-identical sources.
9.2. When there are more than two sources. We have solved a two-source quickest detection problem in which the aim is to detect the minimum of two disorder times. Our approach can easily be generalized to problems including several dimensions. To clarify how this generalization works, let us show what the sufficient statistics are when there are three independent sources. Assume that the observations come from the independent sources $X^1, X^2$ and $X^3$. Let $\Phi_i$ be the odds ratio defined in (9.6). Then
\[
\Phi_i = \Phi_i^1 + \Phi_i^2 + \Phi_i^3 + \Phi_i^1 \Phi_i^2 + \Phi_i^1 \Phi_i^3 + \Phi_i^2 \Phi_i^3 + \Phi_i^1 \Phi_i^2 \Phi_i^3,
\]
in which $\Phi_i^i$, $i \in \{1, 2, 3\}$ is defined as in (2.13). Let us denote $\Phi_i^{(i,j)} \triangleq \Phi_i^i \Phi_j^j$, $i,j \in \{1, 2, 3\}$ and $\Phi_i^{(2)} \triangleq \Phi_i^1 \Phi_i^2 \Phi_i^3$, $t \geq 0$. The dynamics of these processes can be written as
\[
\begin{align*}
&d\Phi_i^{(i,j)} = [\lambda(\Phi_i^i + \Phi_j^j) + 2(\lambda - \alpha + \beta)\Phi_i^{(i,j)}]dt + \left(\frac{\alpha}{\beta} - 1\right)\Phi_i^{(i,j)}d(X_i^i + X_j^j), \\
&d\Phi_i^{(2)} = \left[\lambda \left(\Phi_i^{(1,2)} + \Phi_i^{(1,3)} + \Phi_i^{(2,3)}\right) + 3(\lambda - \alpha + \beta)\Phi_i^{(2)}\right]dt + \left(\frac{\alpha}{\beta} - 1\right)\Phi_i^{(2)}d(X_i^1 + X_i^2 + X_i^3).
\end{align*}
\]

We can see from (2.14) and (2.11) that $\Upsilon \triangleq (\Phi_1^1, \Phi_2^2, \Phi_3^3, \Phi_1^{(1,2)}, \Phi_1^{(1,3)}, \Phi_2^{(2,3)}, \Phi_3^{(3)})$ is a 7 dimensional Markovian statistic whose natural filtration is equal to the filtration generated by $X^1, X^2$ and $X^3$. From this one can see that the results of Sections 6 and 7 can be extended to the three dimensional case since these results rely only on the fact that the sufficient statistic $\Upsilon$ is a strong Markov process. The boundedness of the continuation region can also be shown as in Section 5 since these results can be derived from the sample path properties of the sufficient statistic.

As a result, our results are applicable for decision making with large-scale distributed networks of information sources. In the future, using the techniques developed here, we would like to solve a multi-source detection problem where the observations come from correlated sources. We also would like to extend our results and develop change detection algorithms that can be applied effectively to multiple source data that involves both continuous and discrete event phenomena.

9.3. When the jump size of the observations are random. Consider two independent compound Poisson processes $X^i = \{X_i^i : t \geq 0\}$, $i \in \{1, 2\}$, where
\[
X_i^i = X_i^i + \sum_{j=1}^{N_i} Y_j^i,
\]
in which $N_i^i$, $i \in \{1, 2\}$ are two independent Poisson processes whose common rate $\beta > 0$ changes to $\alpha$ at some random unobservable times $\theta_i$, $i \in \{1, 2\}$, respectively. The random variables $Y_j^i \in \mathbb{R}^d$, $i \in \{1, 2\}$, which are also termed as ‘marks’, are independent and identically distributed with a common distribution, $\nu$, which is called as the ‘mark distribution’. At the change time $\theta_i$ the mark distribution of the process $X^i$ changes from $\nu$ to $\mu$. We will assume that $\mu$ is absolutely continuous with respect to $\nu$ and denote the Radon-Nikodym derivative by $r(y) \triangleq \frac{d\mu}{d\nu}(y)$, $y \in \mathbb{R}^d$. In this case $L_i^i$ in (2.4) becomes
\[
L_i^i = e^{-(\alpha - \beta)t} \prod_{k=1}^{N_i} \frac{\alpha}{\beta} r(Y_k^i)
\]
(9.9)
The likelihood ratio process $L_i^i$ is the unique solution of the stochastic differential equation (see e.g. [8])
\[
dL_i^i = L_i^i \left(-\alpha - \beta dt + \int_{y \in \mathbb{R}^d} \left(\frac{\alpha}{\beta} r(y) - 1\right) p(dy)\right), \quad L_i^i = 1,
\]
(9.12)
where $p$ is a random measure that is defined as
\[
p^i((0, t] \times A) \triangleq \sum_{k=1}^{\infty} 1_{\{\sigma_k^i \leq t\}} 1_{\{Y_k^i \in A\}}, \quad t \geq 0.
\]
(9.13)
and for any $A$ that is a Borel measurable subset of $\mathbb{R}^d$. Here $\sigma_k^i$ is the $k^{th}$ jump time of the process $X^i$. Now using the change of variable formula for semi-martingales (see e.g. [12]), we can write
\[
d\Phi_i^i = (\lambda + (\lambda - \alpha + \beta)\Phi_i^i)dt + \Phi_i^i \int_{y \in \mathbb{R}^d} \left(\frac{\alpha}{\beta} r(y) - 1\right) p^i(dy), \quad \Phi_i^0 = \frac{\pi_i}{1 - \pi_i}.
\]
(9.14)
for \( t \geq 0 \), and \( i \in \{1, 2\} \). Note that \( \Phi^i \equiv \oplus_r r(Y^n) \Phi^i_{\sigma_n} \) at the \( n \)th jump time of the process \( X^n \). Using a change of variable formula for semi-martingales, the dynamics of \( \Phi^+ \) and \( \Phi^- \) in (2.19) can be written as

\[
d\Phi_t^+ = \left[ \lambda \Phi_t^+ + a \Phi_t^+ \right] dt + \Phi_t^- \int_{y \in \mathbb{R}^d} \left( \frac{\alpha}{\beta} r(y) - 1 \right) \left( p^1 + p^2 \right) (dt dy),
\]

\[
d\Phi_t^- = \left[ 2 \lambda + 2 \Phi_t^+ \right] dt + \Phi_t^- \int_{y \in \mathbb{R}^d} \left( \frac{\alpha}{\beta} r(y) - 1 \right) \left( 1 + p^2 \right) (dt dy)
\]

with initial conditions \( \Phi_0^+ = \pi_1 \pi_2 / (1 - \pi_1)(1 - \pi_2) \), and \( \Phi_0^- = \pi_1 / (1 - \pi_1) + \pi_2 / (1 - \pi_2) \).

The bounds on the continuation region constructed for the simple Poisson disorder problem in Section 5 can also be shown to bound the continuation region of the compound Poisson disorder problem. On the other hand the results in Sections 6 and 7 can be shown to hold. The only change will be the form of the operator \( J \) in (6.7).

But this new operator can be shown to share the same properties as its counterpart for the un-marked case.

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10. Appendix. Proof of Theorem 6.1 We will prove only that \( V_n = v_n \) and \( S^\varepsilon_n \) is an \( \varepsilon \)-optimal stopping time of \( \mathbb{Z}_t \).

The proof will be carried out in three steps.

(i) First we will show that \( V_n \geq v_n \). To establish this fact, it is enough to show that for any stopping time \( \tau \in \mathcal{S} \)

\[
\mathbb{E}_0^{a,b,c} \left[ \int_{0}^{\tau \wedge \sigma_\varepsilon} e^{\lambda t} h(\Psi_t) (dt) \right] \geq v_n(a, b, c).
\]

(10.1)

In order to prove (10.1) we will show that

\[
\mathbb{E}_0 \left[ \int_{0}^{\tau \wedge \sigma_\varepsilon} e^{\lambda t} h(\Psi_t) (dt) \right] \geq \mathbb{E}_0 \left[ \int_{0}^{\tau \wedge \sigma_\varepsilon} e^{\lambda t} h(\Psi_t) (dt) + 1_{\{\tau \geq \sigma_n + 1\}} e^{-\lambda \sigma_n - 1} v_{k-1}(Y_{\sigma_n+1}) \right],
\]

(10.2)

for \( k \in \{1, 2, \ldots, n + 1\} \). Note that (10.1) follows from (10.2) if we set \( k = n + 1 \). In what follows we will to show (10.2) by induction.

When \( k = 1 \), (10.2) is satisfied since \( v_0 = 0 \). Assume that (10.2) holds for \( 1 \leq k \leq n + 1 \). Let us denote the right-hand-side of (10.2) by \( \rho_{k-1} \). We can write \( \rho_{k-1} = \rho_{k-1}^1 + \rho_{k-1}^2 \), where

\[
\rho_{k-1}^1 \triangleq \mathbb{E}_0 \left[ \int_{0}^{\tau \wedge \sigma_{n-k}} e^{\lambda t} h(\Psi_t) (dt) \right]
\]

and

\[
\rho_{k-1}^2 \triangleq \mathbb{E}_0 \left[ 1_{\{\tau \geq \sigma_{n-k}\}} \left( \int_{\sigma_{n-k}}^{\tau \wedge \sigma_{n-k+1}} e^{\lambda t} h(\Psi_t) (dt) + 1_{\{\tau \geq \sigma_{n-k+1}\}} e^{-\lambda \sigma_n - 1} v_{k-1}(Y_{\sigma_n+1}) \right) \right].
\]

(10.3)

Now by Lemma 6.3 there exists an \( \mathcal{F}_{\sigma_{n-k}} \)-measurable random variable \( \xi_n \) such that

\[
\tau \wedge \sigma_{n-k+1} = (\sigma_{n-k} + \xi_n) \wedge \sigma_{n-k+1}, \quad \text{almost surely on } \{\tau \geq \sigma_{n-k}\}.
\]

(10.4)

Equation (10.4) together with the strong Markov property of \( Y \) (with respect to the filtration \( \mathbb{F} \)) implies that

\[
\rho_{k-1}^2 = \mathbb{E}_0 \left[ 1_{\{\tau \geq \sigma_{n-k}\}} e^{-\lambda \sigma_n - 1} e^{-\lambda \sigma_{n-k} - 1} f_{k-1}(\xi_n, Y_{\sigma_{n-k}}) \right],
\]

(10.5)

in which

\[
f_{k-1}(r, (a, b, c)) \triangleq \mathbb{E}_0^{a,b,c} \left[ \int_{0}^{\tau \wedge \sigma_1} e^{\lambda t} h(\Psi_t) (dt) + 1_{\{\tau \geq \sigma_1\}} e^{-\lambda \sigma_1} v_{k-1}(Y_{\sigma_1}) \right]
\]

(10.6)

in which the second equality and the first inequality follow from (6.3) and the last equality follows from (6.10). Therefore

\[
\rho_{k-1}^2 \geq \mathbb{E}_0 \left[ 1_{\{\tau \geq \sigma_{n-k}\}} e^{-\lambda \sigma_n - 1} v_{k}(Y_{\sigma_n}) \right],
\]

(10.7)
Now using (10.2), (10.3) and (10.7) we obtain that (10.2) holds when $k$ is replaced by $k + 1$. At this point we have proved by induction that (10.2) holds for $k = 1, 2, ..., n + 1$.

(ii) The converse of (i), $V_n \leq v_n$, follows from (6.13), since $S^n_0 \leq \sigma_n$ by construction (see (6.12)).

(iii) What is left to prove is (6.13). If $n = 1$, then the left-hand-side of (6.13) becomes

$$
\mathbb{E}_0^{a,b,c} \left[ \int_0^{r_0^{\varepsilon}(a,b,c)\wedge \sigma_1} e^{-\lambda t} h(\Psi_t) \, dt \right] = Jv_0(r_0^{\varepsilon}(a,b,c), a,b,c) 
\leq J_0v_0(a,b,c) + \varepsilon = v_1(a,b,c) + \varepsilon. \tag{10.8}
$$

Now, suppose that (6.13) holds for all $\varepsilon > 0$ for some $n$. Using the fact that $S^n_{n+1} \wedge \sigma_1 = r^n_{n+1} \wedge \sigma_1$ almost surely and the strong Markov property of $\Upsilon$, we can write

$$
\mathbb{E}_0 \left[ \int_0^{S^n_{n+1} + 1} e^{-\lambda t} h(\Psi_t) \, dt \right] = \mathbb{E}_0 \left[ \int_0^{S^n_{n+1} + \sigma_1} e^{-\lambda t} h(\Psi_t) \, dt + 1(S^n_{n+1} \geq \sigma_1) \int_{\sigma_1}^{S^n_{n+1} + 1} e^{-\lambda t} h(\Psi_t) \, dt \right] \tag{10.9}
= \mathbb{E}_0 \left[ \int_0^{r^n_{n+1/2}(a,b,c) \wedge \sigma_1} e^{-\lambda t} h(\Psi_t) \, dt \right] + \mathbb{E}_0 \left[ 1(r^n_{n+1/2}(a,b,c) \geq \sigma_1) e^{-\lambda \sigma_1} g_0(\Upsilon_{\sigma_1}) \right],
$$

in which

$$
g_0(a,b,c) \triangleq \mathbb{E}_0^{a,b,c} \left[ \int_0^{S^n_{n+1/2}} e^{-\lambda t} h(\Psi_t) \, dt \right] \leq v_n(a,b,c) + \varepsilon/2. \tag{10.10}
$$

The inequality in (10.10) follows from the induction hypothesis. Using (10.10) we can write (10.3) as

$$
\mathbb{E}_0^{a,b,c} \left[ \int_0^{S^n_{n+1} + 1} e^{-\lambda t} h(\Psi_t) \, dt \right] \leq \mathbb{E}_0^{a,b,c} \left[ \int_0^{r^n_{n+1/2}(a,b,c) \wedge \sigma_1} e^{-\lambda t} h(\Psi_t) \, dt + 1(r^n_{n+1/2}(a,b,c) \geq \sigma_1) e^{-\lambda \sigma_1} v_0(\Upsilon_{\sigma_1}) \right] + \varepsilon/2 
= Jv_n(r^n_{n+1/2}(a,b,c), a,b,c) + \varepsilon/2 \leq v_{n+1}(a,b,c) + \varepsilon. \tag{10.11}
$$

This proves (6.13) when $n$ is replaced by $n + 1$. \hfill \Box

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