The Transition from Brownian Motion to Boom-and-Bust Dynamics in Financial and Economic Systems

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Quasi-equilibrium models for aggregate variables are widely-used throughout finance and economics. The validity of such models depends crucially upon assuming that the systems’ participants behave both independently and in a Markovian fashion.

We present a simplified market model to demonstrate that herding effects between agents can cause a transition to boom-and-bust dynamics at realistic parameter values. The model can also be viewed as a novel stochastic particle system with switching and reinjection.

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I. INTRODUCTION

In the physical sciences using a stochastic differential equation (SDE) to model the effect of exogenous noise upon an underlying ODE is often straightforward. The noise consists of many uncorrelated effects whose cumulative effect is well-approximated by a Brownian process $B_s$, $s \geq 0$ and the ODE $df = a(f, t) \, dt$ is replaced by an SDE $df = a(f, t) \, dt + b(f, t) \, dB_t$.

However, in financial and socio-economic systems the inclusion of exogenous noise (ie new information entering the system) is more problematic — even if the noise can be legitimately modeled as a Brownian process. This is because such systems are themselves the aggregation of many individuals or trading entities (referred to as agents) who typically

a) have differing interpretations of the new information, 
b) act differently depending upon their own recent history (ie non-Markovian behaviour), and 
c) may not act independently of each other.

The standard approach in neoclassical economics and modern finance is simply to ‘average away’ these awkward effects by assuming the existence of a single representative agent as in macroeconomics [1], or by assuming that the averaged reaction to new information is correct/rational, as in microeconomics and finance [2, 3]. In both cases, the possibility of significant endogenous dynamics is removed from the models resulting in unique, Markovian, (quasi)-equilibrium solutions.

In reaction to this, many Heterogeneous Agent Models (HAMs) have been developed [4] that simulate the agents directly. These have demonstrated that it is relatively easy to generate aggregate output data, such as the price of a traded asset, that approximate reality better than the standard averaging-type models. In particular the seemingly universal ‘stylized facts’ [5, 6] of financial markets such as heteroskedasticity (volatility clustering) and leptokurtosis (fat-tailed price-return distributions resulting from booms-and-busts) have been frequently reproduced. However, the effects of such research upon mainstream modeling have been minimal perhaps, in part, because some HAMs require fine tuning of important parameters, others are too complicated to analyze, and the plethora of different HAMs means that many are mutually incompatible.

The purpose of this report is to investigate the range of validity of the quasi-equilibrium solutions obtained by ignoring endogenous effects such as a), b) and c) above. We do this by introducing a simplified version of the modeling framework introduced in [7, 8] that can also be described as a particle system in two dimensions (Figure 1).

II. A STOCHASTIC PARTICLE SYSTEM WITH REINJECTION AND SWITCHING

We define the open set $D \subset \mathbb{R}^2$ by $D = \{(x, y) : -y < x < y, y > 0\}$. There are $M$ signed particles (with states +1 or −1) that move within $D$ subject to three different motions. Firstly there is a bulk Brownian forcing $B_t$ in
In the continuum limit the particles are re-
defined probability measure. Finally, when a particle does 
switch the position of the other particles is kicked in the
x-direction that acts upon every particle. Secondly, 
each particle has its own independent two-dimensional 
diffusion process. Thirdly, for agents in the minority state only, there is a downward (negative y-direction) 
1 state whenever the price crosses either threshold, i.e 
state = +1 and negative if the switch is in the opposite 
direction. Note that the particles do not interact locally 
or collide with one another. 

A. Financial market interpretation

We take as our starting point the standard geometric 
Brownian motion (gBm) model of an asset price \( p_t \) at 
time \( t \) with \( p_0 = 1 \). It is more convenient to use the log-
price \( r_t = \ln p_t \) for constant drift \( a \) and volatility 
\( b \) is given by the solution \( r_t = at + bB_t \) to the SDE

\[
dr_t = a \, dt + b \, dB_t, \tag{1}
\]

Note that the solution \( r_t \) depends only upon the value 
of the exogenous Brownian process \( B_t \) at time \( t \) and not 
upon \( \{B_s\}_{s \geq 0} \). This seemingly trivial observation im-
plies that \( r_t \) is Markovian and consistent with various 
notions of market efficiency. Thus gBm can be consid-
ered a paradigm for all economic and financial models in 
which the aggregate variables are in a quasi-equilibrium 
reacting adiabatically to new information. 
The instantaneous translation of new information \( B_t \) 
into price changes is effected by ‘fast’ agents who will not 
be modeled directly. However, we posit the existence of 
\( M \) ‘slow’ agents who are primarily motivated by price 
changes rather than new information and act over much 
longer timescales (weeks or months). At time \( t \) the \( i^{th} \) 
slow agent is either in state \( s_i(t) = +1 \) (owning the asset) 
or \( s_i(t) = -1 \) (not owning the asset) and the sentiment 
\( \sigma(t) \in [-1, 1] \) is defined as \( \sigma(t) = \sum_{i=1}^{M} s_i(t) \). The \( i^{th} \) slow agent is deemed to have an evolving strategy that at time 
\( t \) consists of an open interval \( (L_i(t), U_i(t)) \) containing the 
current log-price \( r_t \) (see Figure 2). The \( i^{th} \) agent switches 
state whenever the price crosses either threshold, i.e \( r_t = 
L_i(t) \) or \( U_i(t) \), and a new strategy interval is generated 
straddling the current price. 

We assume in addition that each threshold for every 
slow agent has its own independent diffusion with rate 
\( \alpha_i \) (corresponding to slow agents’ independently evolving 
strategies) and those in the minority (whose state differs 
from \( \text{sgn}(\sigma) \)) also have their lower and upper thresholds 
drift inwards each at a rate \( C_i |\sigma| \), \( C_i > 0 \). 

These herding constants \( C_i \) are crucial as they provide 
the only (global) coupling between agents. The inward 
drift of the minority agents’ strategies makes them more 
likely to switch to join the majority. Herding, and other 
mimetic effects, appear to be a common feature of financial 
and economic systems. Some causes are irrationally 
human while others may be rational responses by, for ex-
ample, fund managers not wishing to deviate too far from 
the majority opinion and thereby risk severely underper-
forming their average peer performance. The reader is 
directed to [8] for a more detailed discussion of these and 
other modeling issues. 

Finally, changes in the sentiment \( \sigma \) feed back into the 
asset price so that gBm \( \{B_t\} \) is replaced with

\[
dr_t = a \, dt + b \, dB_t + \kappa \Delta \sigma \tag{2}
\]

where \( \kappa > 0 \) and the ratio \( \kappa/b \) is a measure of the relative 
impact upon \( r_t \) of exogenous information versus endoge-
nous dynamics. Without loss of generality we let \( a = 0 \) 
and \( b = 1 \) by setting the risk-free interest rate to zero 
and rescaling time. 

One does not need to assume that all the slow agents 
are of equal size, have equal strategy-diffusion, and equal 
herding propensities. But if one does set \( \alpha_i = \alpha \) and 
\( C_i = C \, \forall \)i then one obtains the particle system simply 
by defining the position of the \( i^{th} \) particle as \( (x_i, y_i) = 
(L_i + L_{i-1})/2 - r_t, U_i - L_i/2) \). In other words, the bulk stochastic 
motion is due to exogenous noise; the individual diffu-
sions are caused by strategy-shifting of the slow agents; 
the downward drift of minority agents is due to herd-
ing effects; the reination and switching are the agents 
changing investment position; and the kicks that occur at 
switches are due to the change in sentiment affecting the 
asset price via a linear supply/demand-price assumption. 

B. Limiting values of the parameters

There are different parameter limits that are poten-
tially of interest. 
1) \( M \to \infty \). In the continuum limit the particles are re-
placed by a pair of evolving density functions \( \rho^{+}(x, y, t) \) 
and \( \rho^{-}(x, y, t) \) representing the density of each agent 
state on \( D \) — such a mesoscopic Fokker-Planck descrip-
tion of a related, but simpler, market model can be found in [11]. 
The presence of nonstandard boundary conditions, global coupling, and bulk stochastic motion present 
formidable analytic challenges for even the most basic 
questions of existence and uniqueness of solutions. How-
ever, numerical simulations strongly suggest that, minor
discretization effects aside, the behaviour of the system is independent of \( M \) for \( M \gtrsim 1000 \).

2) \( B \to 0 \) As the external information stream is reduced the system settles into a state where \( \sigma \) is close to either \( \pm 1 \). Therefore this potentially useful simplification is not available to us.

3) \( \alpha \to 0 \) or \( \infty \) In the limit \( \alpha \to 0 \) the particles do not diffuse i.e. the agents do not alter their thresholds between trades/switches. This case was examined in [11] and the lack of diffusion does not significantly change the boombust behaviour shown below. On the other hand, for \( \alpha \gg \max(1, C) \) the diffusion dominates both the exogenous forcing and the herding/drift ing and equilibrium-type dynamics is re-established. This case is unlikely in practice since slow agents will alter their strategies more slowly than changes in the price of the asset.

4) \( C \to 0 \) This limit is the focus of the report. When \( C = 0 \) the particles are uncoupled and if the system is started with approximately equal distributions of \( \pm 1 \) states then \( \sigma \) remains close to 0. Thus (2) reduces to (1) and the particle system becomes a standard equilibrium model — agents have differing expectations about the future which causes them to trade but on average the price remains 'efficient' [1]. In Section III we shall observe that endogenous dynamics arise as \( C \) is increased and the equilibrium solution is no longer stable.

5) \( \kappa \to 0 \) For \( \kappa > 0 \) one agent switching can cause an avalanche of similar switches, especially when the system is highly one-sided with \( |\sigma| \) close to 1. When \( \kappa = 0 \) the particles no longer provides kicks (or affect the price) when they switch although they are still coupled via \( C > 0 \). The sentiment \( \sigma \) can still drift between \(-1\) and \(+1\) over long timescales but switching avalanches and large, sudden, price changes do not occur.

III. PARAMETER ESTIMATION, NUMERICAL SIMULATIONS AND INSTABILITY

In all the simulations below we use \( M = 1000 \) and discretize using a timestep \( h = 0.000004 \) which corresponds to approximately 1/10 of a trading day if one assumes a daily standard deviation in prices of \( \approx 0.6\% \) due to new information. The price changes of 10 consecutive timesteps are then summed to give daily price return data making the difference between synchronous vs asynchronous updating relatively unimportant.

We choose \( \alpha = 0.2 \) so that slow agents’ strategies diffuse less strongly than the price does. A conservative choice of \( \kappa = 0.2 \) means that the difference in price between neutral (\( \sigma = 0 \)) and polarized markets \( \sigma = \pm 1 \) is, from (2), \( \exp(0.2) \approx 22\% \).

After switching, an agent’s thresholds are chosen randomly from a Uniform distribution to be within 5% and 25% higher and lower than the current price. This allows us to estimate \( \kappa \) by supposing that in a moderately polarized market with \( |\sigma| = 0.5 \) a typical minority agent (outnumbered 3–1) would switch due to herding pressure after approximately 80 trading days (or 3 months), a typical reporting period for investment performance [12]. The calculation 80\( C |\sigma| = |\ln(0.85)|/0.00004 \approx 100 \). Finally, we note that no fine-tuning of the parameters is required for the observations below.

Figure 3 shows the results of a typical simulation, started close to equilibrium with agents’ states equally mixed and run for 40 years. The difference in price history between the above parameters and the equilibrium gBm solution is shown in the top left. The sudden market reversals and over-reactions can be seen more clearly in the top right plot where the market sentiment undergoes sudden shifts due to switching cascades. These result in price returns (bottom left) that could quite easily bankrupt anyone using excessive financial leverage and gBm as an asset pricing model! Finally in the bottom right the number of days on which the magnitude of the price change exceeds a given percentage is plotted on log-log axes. It should be emphasized that this is a simplified version of the market model in [8] and an extra parameter that improves the statistical agreement with real price data (by inducing volatility clustering) has been ignored.

To conclude we examine the stability of the equilibrium gBm solution using the herding parameter \( C \) as a bifurcation parameter. In order to quantify the level of disequilibrium in the system we record the maximum value of \( |\sigma| \) ignoring the first 10 years of the simulation (to remove any possible transient effects caused by the initial conditions) and average over 20 runs each for values of 0 \( \leq C \leq 40 \). All the other parameters and the initial conditions are kept unchanged.
The results in Figure 4 show that for values of \( C \) as low as \( 20 \) the deviations from the equilibrium solution are as large as the system will allow, with the large majority of agents being in the same state at some point during the simulation. This is lower than the value of \( C = 100 \) estimated above by a significant margin and it should be noted that there are other phenomena, such as new investors and money entering the asset market after a bubble has started, and localized interactions between certain subsets of agents that might cause further destabilization.

![Graph showing average maximum sentiment vs. coupling parameter C]

**FIG. 4.** A measure of disequilibrium \( |\sigma|_{\text{max}} \) averaged over 20 runs as the herding parameter \( C \) changes.

IV. CONCLUSIONS

Financial and economic systems are subject to many different kinds of inter-dependence between agents and potential positive feedbacks. However, even those mainstream models that attempt to quantify such effects\(^{12}\) assume that the result will be a shift of the equilibria to nearby values without qualitatively changing the nature of the system. However we have demonstrated that at least one such form of coupling (herding) results in disequilibrium. Furthermore the new dynamics occurs at realistic parameters and is clearly recognizable as ‘boom-and-bust’. It is characterized by long periods of low-level endogenous activity (long enough, certainly, to convince equilibrium-believers that the system is behaving adiabatically) followed by large, sudden, reversals involving cascades of switching agents triggered by price changes.

The model presented here is compatible with existing (non-mathematized) critiques of equilibrium theory by Minsky and Soros\(^{13, 14}\). Furthermore, work on related models to appear elsewhere shows that positive feedbacks can result in similar non-equilibrium dynamics in more general micro and macro-economic situations.

Finally, the model has interesting links to other areas of mathematics and physics. In \(^{8}\) it was shown that the switching cascades can be described using Queueing theory, with the price changes being equivalent to the busy-period of a queue. The model also shares similarities with other self-organizing systems such as the OFC earthquake model\(^{15}\). And if one considers agents to be evolving hysteretic operators then results concerning the interaction of stochastic processes and hysteresis\(^{16}\) and phase transitions may provide valuable insights into the price dynamics.

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[1] A. Kirman, Journal of Economic Perspectives 6, 117 (1992)
[2] J. Muth, Econometrica 6 (1961)
[3] E. Fama, Journal of Business 38, 34 (1965)
[4] C. H. Hommes (Elsevier, 2006) pp. 1109 – 1186, [http://www.sciencedirect.com/science/article/pii/S157400210502023X](http://www.sciencedirect.com/science/article/pii/S157400210502023X)
[5] R. Mantegna and H. Stanley, An Introduction to Econophysics (CUP, 2000)
[6] R. Cont, Quantitive Finance 1, 223 (2001)
[7] H. Lamba and T. Seaman, Phys. A 387, 3904 (2008)
[8] H. Lamba, Eur. Phys. J. B 77, 297 (2010)
[9] A web-based interactive simulation of the model can be found at [http://math.gmu.edu/~harbir/PRLmarket.html](http://math.gmu.edu/~harbir/PRLmarket.html)
[10] M. Grinfeld, H. Lamba, and R. Cross, “A mesoscopic market model with hysteretic agents,” To appear, Discr. Cont. Dyn. Sys. B
[11] R. Cross, M. Grinfeld, H. Lamba, and T. Seaman, Phys. A 354, 463 (2005)
[12] D. Scharfstein and J. Stein, The American Economic Review 80, 465 (1990), ISSN 00028282
[13] H. Minsky, The Jerome Levy Institute Working Paper 74 (1992)
[14] G. Soros, The alchemy of finance (John Wiley & Sons Inc, 1987)
[15] Z. Olami, H. Feder, and K. Christensen, Phys. Rev. Lett. 68, 1244 (1992)
[16] I. D. Mayergoyz and M. Dimian, Journal of Physics: Conference Series 22, 139 (2005), [http://stacks.iop.org/1742-6596/22/i=1/a=009](http://stacks.iop.org/1742-6596/22/i=1/a=009)